Thermal form-factor approach to dynamical correlation functions of integrable lattice models

Frank Göhmann,† Michael Karbach,† Andreas Klümper,† Karol K. Kozlowski,* and Junji Suzuki‡

†Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42097 Wuppertal, Germany
*Univ Lyon, ENS de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France
‡Department of Physics, Faculty of Science, Shizuoka University, Ohya 836, Suruga, Shizuoka, Japan

Abstract

We propose a method for calculating dynamical correlation functions at finite temperature in integrable lattice models of Yang-Baxter type. The method is based on an expansion of the correlation functions as a series over matrix elements of a time-dependent quantum transfer matrix rather than the Hamiltonian. In the infinite Trotter-number limit the matrix elements become time independent and turn into the thermal form factors studied previously in the context of static correlation functions. We make this explicit with the example of the XXZ model. We show how the form factors can be summed utilizing certain auxiliary functions solving finite sets of nonlinear integral equations. The case of the XX model is worked out in more detail leading to a novel form-factor series representation of the dynamical transverse two-point function.
1 Introduction

The goal of this work is to design a method for the calculation of dynamical correlation functions at finite temperature in integrable lattice models of Yang-Baxter type. This was attempted before by K. Sakai [33], but the multiple-integral formula he obtained turned out to be computationally inefficient. Still, the basic idea in Sakai’s work, which was to use the ‘solution of the quantum inverse problem’ [24] twice, for the usual transfer matrix and for the quantum transfer matrix, seems very natural and is awaiting to be used in a more efficient way. Here we combine this idea with the thermal form-factor expansion introduced by two of the authors in [4]. This leads to a form-factor series of the same degree of complexity as in the static case. In particular, only a single (rather than a double) sum over excited states is involved.

We shall put some emphasis on the general formalism which applies to a large class of integrable lattice models, namely to those with an $R$-matrix which turns into the transposition matrix for certain values of the spectral parameters. For all integrable lattice models in this class we derive a form factor series for their dynamical correlation functions at finite temperature in the first part of this work. In order to evaluate the form-factor series, which is the subject of the second part of this work, the form factors should be known in a form which admits to take the Trotter limit. Until recently this was only the case for the form factors of the XXZ chain and of models directly related to it as limiting cases. For the XXZ chain useful determinant representations of the form factors [4, 17, 20] were derived on the basis of Slavnov’s scalar product formula [34]. The representations obtained in [4] apply to the quantum transfer matrix and can be used to take the Trotter limit. Rather recently determinant formulae for form factors of local operators for the $gl(3)$ and $gl(2|1)$ models were derived within an algebraic Bethe Ansatz approach [13, 30, 31], and important progress toward a generalization to $gl(m|n)$ for generic values of $m$ and $n$ was made in [14]. These determinant formulae may become the starting point for taking the Trotter limit and for setting at work our novel form factor series in more general higher-rank cases.

In order to make the form factor series efficient a partial summation over classes of excitations seems necessary. Such partial summation may be achieved by means of (multiple-) integration over ‘auxiliary functions’ for ‘higher-level Bethe equations’ which turns sums into integrals. This idea was developed in [7] in the context of the usual transfer matrix for ground state correlation functions of the XXZ model in the antiferromagnetic massive regime and was first applied to thermal form-factor series of the same model in [6, 8]. In this work we suggest how to perform a similar partial summation of the thermal form-factor series for the XXZ chain in the dynamical case. This partial summation is quite different from the partial summation employed in the analysis of the long-time large-distance asymptotics of the form factor series of the ground state correlation functions of the XXZ chain in the critical regime [21, 22], which relied on ‘restricted sums’ rather than on contour integration. By way of contrast the auxiliary function techniques developed in [6–8] are closer in spirit to those of [10, 23].

Explicit limiting cases of our form factor series, in particular the high- and low-temperature asymptotics, will be worked out in future publications. In order to demonstrate that our form-factor series can be efficient we focus here on the special case of the XX model [28]. For the longitudinal two-point functions our form factor series reduces to a simple explicit formula equivalent to the classical result of Niemeijer [19, 29]. In the transverse case we obtain a novel form factor series which seems to be rather promising for addressing some open questions concerning the long-time and large distance asymptotics.
of the dynamical two-point functions at finite temperature [15, 16, 18].

2 Foundations

As far as the general formalism is concerned we shall work within the setting of fundamental integrable models with unitary $R$-matrix. In particular, no crossing symmetry will be required.

2.1 Fundamental integrable lattice models

Such models are defined in terms of their $R$-matrices $R : \mathbb{C}^2 \mapsto \text{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$ which are solutions of the Yang-Baxter equation

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu).$$

The subscripts refer to the pairs of spaces in the triple tensor product $(\mathbb{C}^d)^{\otimes 3}$ on which the corresponding $R$-matrix is acting nontrivially. We shall assume that the $R$-matrix has the following additional properties:

- regularity $R(\lambda, \lambda) = P$,  
- unitarity $R_{12}(\lambda, \mu) R_{21}(\mu, \lambda) = id$,  
- symmetry $R^t(\lambda, \mu) = R(\lambda, \mu)$.

Here the superscript $t$ denotes matrix transposition and $P$ is the permutation matrix defined by $P(x \otimes y) = (y \otimes x)$ for all $x, y \in \mathbb{C}^d$.

With a given $R$-matrix, which has the above properties, we associate an integrable lattice model in the standard way. We define a (‘row-to-row’) monodromy matrix

$$T_{\perp,a}(\lambda) = R_{aL}(\lambda, 0) \ldots R_a(\lambda, 0)$$

on $L$ lattice sites and the corresponding transfer matrix

$$t_{\perp}(\lambda) = \text{tr}_a\{T_{\perp,a}(\lambda)\}.$$  

Then typically a constant $h_R \in \mathbb{C}$ exists such that, for an appropriate choice of the parameters in the $R$-matrix,

$$H_0 = h_R \ell_\perp(0) t_{\perp}^{-1}(0)$$

can be interpreted as a local lattice Hamiltonian, which is Hermitian on $\text{End}(\mathbb{C}^d)^{\otimes L}$. The locality is obvious from the explicit expression

$$H_0 = h_R \sum_{j=1}^{L} \partial_{\lambda}(PR)_{j-1,j}(\lambda, 0)|_{\lambda=0},$$

where periodic boundary conditions, $(PR)_{0,1} = (PR)_{L,1}$, are understood. Note that the constant $h_R$ depends on the respective $R$-matrix.
2.2 Transfer matrix realization of statistical operator and time evolution operator

In order to be able to calculate correlation functions we need to realize the exponential of $H_0$ in terms of transfer matrices. For this purpose we introduce

$$t_{\perp}(\lambda) = \text{tr}_a \{ T^{-1}_{\perp,a}(\lambda) \} = \text{tr}_a \{ R_{1a}(0, \lambda) \ldots R_{La}(0, \lambda) \} ,$$

where the second equation holds due to the unitarity condition. Then it follows that

$$t_{\perp}(0) = t_{\perp}^{-1}(0) , \quad H_0 = -h_R t_{\perp}(0)t_{\perp}'(0) .$$

Combining (5) and (8) we obtain

$$t_{\perp} \left( -\frac{h_R}{NT} \right) t_{\perp} \left( \frac{h_R}{NT} \right) = \text{id} - 2H_0 \frac{1}{NT} + O(N^{-2}) .$$

For every even $N$ let

$$\rho_{N,L}(1/T) = \left( t_{\perp} \left( -\frac{h_R}{NT} \right) \right)^N \left( t_{\perp} \left( \frac{h_R}{NT} \right) \right)^N .$$

Then (9) implies that

$$\lim_{N \to \infty} \rho_{N,L}(1/T) = e^{-H_0/T} .$$

For finite $N$ the product of transfer matrices $\rho_{N,L}(1/T)$ is an approximation to the statistical operator $e^{-H_0/T}$, where $T$ is the temperature. We shall call $N$ the Trotter number and the limit $N \to \infty$ the Trotter limit.

Remark. In previous work we used to define $\rho_{N,L}(1/T)$ with the opposite order of factors. The order is irrelevant in (11), but the present order turns out to be slightly more convenient, when we consider dynamical correlation functions below.

For the evaluation of correlation functions we will also have to express the action of a local operator $x \in \text{End}(\mathbb{C}^d)$ on the first site of our quantum chain in terms of monodromy and transfer matrix. For this purpose we shall employ the `solution of the quantum inverse problem’ formula, whose significance was first understood in [24],

$$x_1 = t_{\perp}(0) \text{tr}_a \{ x_a T_{\perp,a}^{-1}(0) \} = \lim_{\varepsilon \to 0} t_{\perp}(-\varepsilon) \text{tr}_a \{ x_a T_{\perp,a}^{-1}(\varepsilon) \} .$$

Following [33] we have introduced a regularization parameter $\varepsilon$. The regularization is trivial for the row-to-row transfer matrix. It becomes important only later when we apply a variant of the above formula to the quantum transfer matrix introduced in the next section.

2.3 Quantum transfer matrix

Now we introduce the central notion of our formalism, the quantum transfer matrix [35], which was previously used in order to obtain efficient formulae for the free energy per lattice site [25] and for static correlation functions of integrable lattice models [10]. We shall see that, in a slightly generalized form, it can be also used to study dynamical correlation functions at finite temperature. This was already recognized in [33], but the
formula obtained by that time do not seem to be efficient for a numerical or asymptotic analysis.

We would like to include a simple class of external fields which do not break the integrability of the model. They enter the formalism in the following way. Let $\hat{\varphi} \in \text{End}(\mathbb{C}^d)$ be a local operator, let $\Theta(\alpha) = e^{\alpha \hat{\varphi}}$, and assume that

$$[R_{12}(\lambda, \mu), \Theta_1(\alpha)\Theta_2(\alpha)] = 0. \quad (13)$$

Then

$$[t(\lambda_{\perp}, \Theta_1(\alpha) \ldots \Theta_L(\alpha)] = [t_{\perp}(\lambda), \Theta_1(\alpha) \ldots \Theta_L(\alpha)] = 0. \quad (14)$$

Define

$$\hat{\Phi} = \sum_{j=1}^{L} \hat{\varphi}_j. \quad (15)$$

Then (14) implies that

$$[H_0, \hat{\Phi}] = 0. \quad (16)$$

Using $\Theta(\alpha)$ we define the staggered, twisted and inhomogeneous monodromy matrix acting on ‘vertical spaces’ with site indices $1, \ldots, 2N + 2$,

$$T_a(\lambda|\alpha) = \Theta_a(\alpha)R_{2N+2,a}^{1}(\nu_{2N+2}, \lambda)R_{a,2N+1}(\lambda, \nu_{2N+1}) \ldots R_{2,a}^{L}(\nu_{2}, \lambda)R_{a,1}(\lambda, \nu_{1}). \quad (17)$$

Here the superscript $t_1$ denotes transposition with respect to the first space $R$ is acting on, and $\nu_1, \ldots, \nu_{2N+2}$ are $2N + 2$ arbitrary complex ‘inhomogeneity parameters’. The corresponding transfer matrix

$$t(\lambda|\alpha) = \text{tr}_a\{T_a(\lambda|\alpha)\} \quad (18)$$

is called the inhomogeneous quantum transfer matrix of the model.

The most important property of the staggered monodromy matrix (17) is that it provides a representation of the Yang-Baxter algebra,

$$R_{ab}(\lambda, \mu)T_a(\lambda|\alpha)T_b(\mu|\alpha) = T_b(\mu|\alpha)T_a(\lambda|\alpha)R_{ab}(\lambda, \mu). \quad (19)$$

This follows from (1), (13) and from the equation

$$R_{ab}(\lambda, \mu)R_{j\alpha}^{1}(\nu, \lambda)R_{j\beta}^{1}(\nu, \mu) = R_{j\beta}^{1}(\nu, \mu)R_{j\alpha}^{1}(\nu, \lambda)R_{ab}(\lambda, \mu) \quad (20)$$

which is obtained from the Yang-Baxter equation (1) by taking the transpose with respect to the first space, permuting the indices and redefining the spectral parameters. In many cases the Yang-Baxter algebra can be used in order to diagonalize the quantum transfer matrix.

A crucial formula for our derivation of a form-factor series for dynamical correlation functions at finite temperature is a generalized form of the inversion formula [11, 24].

**Lemma.** *Solution of the quantum inverse problem for the inhomogeneous quantum transfer matrix.* Let $j \in \{1, \ldots, 2N + 2\}$ be odd. Then, for any $x \in \text{End}(\mathbb{C}^d)$,

$$x_j = t(\nu_1|\alpha)t^{-1}(\nu_2|\alpha) \ldots t^{-1}(\nu_{j-1}|\alpha)$$

$$\times \text{tr}\{xT(\nu_j|\alpha)\}t^{-1}(\nu_j|\alpha)t(\nu_{j-1}|\alpha) \ldots t^{-1}(\nu_1|\alpha), \quad (21)$$

provided that $t(\nu_k|\alpha)$ is invertible for $k = 1, \ldots, j$. 

A proof of this formula is given in Appendix A. In general invertibility will require that the inhomogeneity parameters on odd and even lattice site are mutually distinct. Otherwise for some \( k \) and \( \ell \) the determinant of \( t(\nu_k | \alpha) \) will contain a factor \( \det(P_{k,\ell}) = (\det(P^1))^{dN} \), where the determinant on the left hand side is evaluated in \( (\mathbb{C}^d)^{\otimes 2N+2} \) and the determinant on the right hand side is evaluated in \( \mathbb{C}^d \otimes \mathbb{C}^d \). But

\[
P^{11}(x \otimes y) = P^{11}(y \otimes x),
\]

implying that \( P^{11} \) has a nontrivial kernel and that \( \det(t(\nu_k | \alpha)) = 0 \). For our main example, the XXZ chain, we shall provide sufficient conditions for \( t(\nu_k | \alpha) \) to be invertible in Appendix B. Note that the \( \epsilon \)-regularization in (12) was introduced to avoid invertibility problems in the intermediate calculations.

### 2.4 Correlation functions

We would like to calculate correlation functions of integrable lattice models with Hamiltonian

\[
H = H_0 - \alpha \Phi,
\]

where \( H_0 \) is defined in (6) and \( \Phi \) in (15). We restrict ourselves to the dynamical two-point functions of two operators \( x, y \in \text{End}(\mathbb{C}^d) \) defined by

\[
\langle x_1 y_{m+1}(t) \rangle_T = \lim_{L \to \infty} \frac{\text{tr}_{1,\ldots,L} \{ e^{-H/T} x_1 e^{itH} y_{m+1} e^{-itH} \}}{\text{tr}_{1,\ldots,L} \{ e^{-H/T} \} } = \lim_{L \to \infty} \frac{\text{tr}_{1,\ldots,L} \{ e^{-(1/T+it)H} x_1 e^{itH} y_{m+1} \}}{\text{tr}_{1,\ldots,L} \{ e^{-H/T} \} }.
\]

Here we have denoted the spatial distance on the lattice by \( m \) and the time by \( t \). The indices \( 1, \ldots, L \) in (24) indicate that the traces are computed in \( (\mathbb{C}^d)^{\otimes L} \) which is the space of states of the Hamiltonian (23).

Our goal is to express the right hand side of (24) in terms of the quantum transfer matrix (18) and the entries of the corresponding monodromy matrix (17), then to simplify the resulting expression using the spectral decomposition of the quantum transfer matrix. Inserting (10) and (11) into the right hand side of (24) we obtain

\[
\langle x_1 y_{m+1}(t) \rangle_T = \lim_{L \to \infty} \lim_{N \to \infty} \frac{\text{tr}_{1,\ldots,L} \{ \rho_{N,L}(1/T + it) e^{(1/T+it)\alpha \Phi} x_1 e^{-it\alpha \Phi} \rho_{N,L}(-it) y_{m+1} \}}{\text{tr}_{1,\ldots,L} \{ \rho_{N,L}(1/T + it) \rho_{N,L}(-it) \} } = \lim_{L \to \infty} \lim_{N \to \infty} \frac{\text{tr}_{1,\ldots,L} \{ e^{i\alpha \hat{\Phi}} \rho_{N,L}(1/T + it) e^{it\alpha \hat{\Phi}} x_1 e^{-it\alpha \hat{\Phi}} \rho_{N,L}(-it) y_{m+1} \}}{\text{tr}_{1,\ldots,L} \{ \rho_{N,L}(1/T + it) \rho_{N,L}(-it) \} }.
\]

We assume that the adjoint action of \( e^{i\alpha \hat{\phi}} \) can be diagonalized. This is rather natural, since the operator \( \hat{\phi} \) is typically a Cartan element of a Lie algebra acting as a local symmetry. Then, without loss of generality, one may assume that \( x \) is an eigenvector under the adjoint action of \( \hat{\phi} \), implying that

\[
e^{it\alpha \hat{\phi}} x_1 e^{-it\alpha \hat{\phi}} = e^{it\alpha(x)} x_1,
\]

(26)
where $s(x)$ is the eigenvalue corresponding to $x$. We first of all concentrate on the numerator on the right hand side of (25). Inserting (26) and using (12) we find

\[
\text{tr}_{1,\ldots,L}\left\{ e^{\alpha\hat{\phi}/T} \rho_{N,L}(1/T + it) e^{i\alpha\hat{\phi}/T} x_1 e^{-i\alpha\hat{\phi}/T} \rho_{N,L}(-it) y_{m+1} \right\} = e^{i\alpha s(x)}
\]

\[
\times \lim_{\epsilon \to 0} \text{tr}_{1,\ldots,L}\left\{ e^{\alpha\hat{\phi}/T} \rho_{N,L}(1/T + it) t_\perp (-\epsilon) \text{tr}_a\left\{ x_a T_{\perp,a}^{-1}(\epsilon) \right\} \rho_{N,L}(-it) y_{m+1} \right\} .
\]

This is now of a form that allows us to write it in terms of the quantum transfer matrix and its monodromy matrix. The easiest way to proceed is to represent the right hand side graphically (see Fig. 1) and re-express it in terms of column-to-column rather than row-to-row monodromy matrices. The column-to-column monodromy matrix at hand is a special case of the staggered monodromy matrix defined in (17). The inhomogeneities can, for instance, be fixed to

\[
\nu_{2k-1} = -\nu_{2k} = \begin{cases} 
-\frac{t_R}{N} & k = 1, \ldots, \frac{N}{2} \\
\epsilon & k = \frac{N}{2} + 1 \\
\frac{t_R + h_R/T}{N} & k = \frac{N}{2} + 2, \ldots, N + 1,
\end{cases}
\]

where

\[
t_R = i h_R t.
\]

Then

\[
\text{tr}_{1,\ldots,L}\left\{ e^{\alpha\hat{\phi}/T} \rho_{N,L}(1/T + it) t_\perp (-\epsilon) \text{tr}_a\left\{ x_a T_{\perp,a}^{-1}(\epsilon) \right\} \rho_{N,L}(-it) y_{m+1} \right\} = \text{tr}_{T,\ldots,2N+2}\left\{ x_{N+1} t^m(0|\kappa) \text{tr}\{ y T(0|\kappa) \} t^{L-m-1}(0|\kappa) \right\}
\]

\[
= \text{tr}_{T,\ldots,2N+2}\left\{ \left[ t \left( -\frac{t_R}{N} |\kappa \right) t_\perp^{-1} \left( \frac{t_R}{N} |\kappa \right) \right]^\frac{N}{2} \text{tr}\{ x T(\epsilon|\kappa) \} t^{-1}(\epsilon|\kappa) \right. \\
\times \left. \left[ t^{-1} \left( -\frac{t_R}{N} |\kappa \right) t \left( \frac{t_R}{N} |\kappa \right) \right]^\frac{N}{2} t^m(0|\kappa) \text{tr}\{ y T(0|\kappa) \} t^{L-m-1}(0|\kappa) \right\} .
\]

Figure 1: Graphical representation of (27), the unnormalized finite Trotter number approximant to the dynamical two-point function.
where we wrote $\kappa = \alpha / T$ for short and used (21) in the second equation. We reinsert (30) back into (27), (25) and take the limit $L \to \infty$ first, exploiting the fact that there is a single dominant eigenvalue $\Lambda_0$ with eigenvector $|\Psi_0\rangle$ of the quantum transfer matrix. The interchangeability of the limits was discussed in (35), albeit for a slightly differently defined quantum transfer matrix. The existence of a dominant eigenvalue at finite Trotter number is at least clear at high enough temperature. We obtain the following representation for the two-point functions in the thermodynamic limit,

$$\langle x_1 y_{m+1}(t) \rangle_T = \lim_{N \to \infty} \lim_{\varepsilon \to 0} e^{i t \alpha s(x)} \left( \frac{\Lambda_0(0)^N}{\Lambda_0(|\varepsilon|)^N} \right) \sum_n \frac{\langle \Psi_n | X(\varepsilon | \kappa) | \Psi_n \rangle}{\Lambda_n(\varepsilon | \kappa)} \frac{\langle \Psi_n | Y(\varepsilon | \kappa) | \Psi_0 \rangle}{\Lambda_0(\varepsilon | \kappa)} \frac{\langle |\Psi_0\rangle | \Psi_n \rangle}{\langle |\Psi_0\rangle | \Psi_0 \rangle} \left( \frac{\Lambda_n(0 | \kappa)}{\Lambda_0(0 | \kappa)} \right)^N \left( \frac{\Lambda_n(t \varepsilon | \kappa)}{\Lambda_0(0 | \kappa)} \right)^N \right).$$

(31)

Here we have introduced the notation

$$X(\lambda | \kappa) = \text{tr} \{ x T(\lambda | \kappa) \}, \quad Y(\lambda | \kappa) = \text{tr} \{ y T(\lambda | \kappa) \}.$$  

(32)

We also took the liberty to change the spectral parameter from zero to $\varepsilon$ on one of the vertical lines by replacing $Y(0 | \kappa)$ by $Y(\varepsilon | \kappa)$ and by performing a similar replacement in the denominator. This will produce more symmetric and slightly more convenient expressions later on.

In order to be able to deal with the Trotter limit $N \to \infty$ we insert a complete set of eigenstates $|\Psi_n\rangle$ of $t(\lambda | \kappa)$ with corresponding eigenvalues $\Lambda_n(\lambda | \kappa)$. This brings us to our main result.

**Theorem.** With the definitions above the dynamical two-point functions of two local operators $x$ and $y$ have the form-factor series expansion

$$\langle x_1 y_{m+1}(t) \rangle_T = \lim_{N \to \infty} \lim_{\varepsilon \to 0} e^{i t \alpha s(x)} \sum_n \frac{\langle \Psi_n | X(\varepsilon | \kappa) | \Psi_n \rangle}{\Lambda_n(\varepsilon | \kappa)} \frac{\langle \Psi_n | Y(\varepsilon | \kappa) | \Psi_0 \rangle}{\Lambda_0(\varepsilon | \kappa)} \frac{\langle |\Psi_0\rangle | \Psi_n \rangle}{\langle |\Psi_0\rangle | \Psi_0 \rangle} \left( \frac{\Lambda_n(0 | \kappa)}{\Lambda_0(0 | \kappa)} \right)^N \left( \frac{\Lambda_n(t \varepsilon | \kappa)}{\Lambda_0(0 | \kappa)} \right)^N \right).$$

(33)

**Remark 1.** Setting $t = 0$ we formally recover the known form-factor series expansion of the static correlation functions of integrable lattice models [4].

**Remark 2.** The Trotter limit can be performed for the individual terms occurring under the sum,

$$A_n = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \frac{\langle \Psi_n | X(\varepsilon | \kappa) | \Psi_n \rangle}{\Lambda_n(\varepsilon | \kappa)} \frac{\langle \Psi_n | Y(\varepsilon | \kappa) | \Psi_0 \rangle}{\Lambda_0(\varepsilon | \kappa)} \frac{\langle |\Psi_0\rangle | \Psi_n \rangle}{\langle |\Psi_0\rangle | \Psi_0 \rangle},$$

(34)

$$\rho_n(\lambda) = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \frac{\Lambda_n(\lambda | \kappa)}{\Lambda_0(\lambda | \kappa)}.$$  

(35)

With this we obtain the formal series expansion

$$\langle x_1 y_{m+1}(t) \rangle_T = \sum_n A_n \rho_n^m(0) e^{i t \alpha s(x) + h R \rho_n(0) / \rho_n(0)}.$$  

(36)
from \textsuperscript{(33)}. We would like to emphasize, however, that the Trotter limit in \textsuperscript{(33)} has to be dealt with with care, taking into account the peculiarities of the given model under consideration. The specific example of the XXZ chain will be considered in the following section.

Remark 3. Another way of looking at \textsuperscript{(36)} is by introducing the correlation lengths $\xi_n$ and the phase velocity $v_n$, setting

$$\rho_n(\lambda) = e^{-\frac{1}{\xi_n(\lambda)}}, \quad v_n = \frac{\xi_n'(0)}{\xi_n(0)},$$

Then

$$\langle x_1 y_{m+1}(t) \rangle_T = \sum_n A_n \exp \left\{ -m - v_n \frac{t}{\xi_n(0)} + it\alpha s(x) \right\}.$$  \hspace{1cm} (38)

Remark 4. In our derivation we can easily change the Trotter decomposition \textsuperscript{(28)}. This allows us to modify the Hamiltonian and to replace it by any linear combination of local conserved quantities generated by the row-to-row transfer matrix of the model \textsuperscript{(26)}.

3 The XXZ chain as an example

3.1 Hamiltonian and R-matrix

In this section we shall explore some of the features of our approach using the example of the XXZ chain. Only for this most elementary model the amplitudes occurring in the form-factor expansion of the two-point functions have been worked out in sufficient detail.

The Hamiltonian of the spin-$\frac{1}{2}$ XXZ chain in a magnetic field of strength $\mathcal{h}$ is defined by the local action of Pauli matrices $\sigma^\alpha$, $\alpha = x, y, z$, on a chain of spins on $L$ lattice sites,

$$H_{XXZ} = J \sum_{j=1}^{L} \left( \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta (\sigma_{j-1}^z \sigma_j^z - 1) \right) - \frac{\hbar}{2} \sum_{j=1}^{L} \sigma_j^z.$$ \hspace{1cm} (39)

Here $\Delta = (q + q^{-1})/2$ is the anisotropy parameter and $J > 0$ is the strength of the exchange interaction. In the following we shall restrict ourselves to $q = e^{-i\gamma}$, $\gamma \in (0, \pi/2]$ for simplicity, implying that $0 \leq \Delta < 1$.

The Hamiltonian $H_{XXZ}$ can be obtained from the $R$-matrix

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$ \hspace{1cm} (40a)

$$b(\lambda, \mu) = \frac{\text{sh}(\lambda - \mu)}{\text{sh}(\lambda - \mu - i\gamma)}, \quad c(\lambda, \mu) = \frac{\text{sh}(-i\gamma)}{\text{sh}(\lambda - \mu - i\gamma)}.$$ \hspace{1cm} (40b)

which is regular, unitary and symmetric (see \textsuperscript{(2)}). It has a $U(1)$ symmetry of the form \textsuperscript{(13)} with $\Theta(\alpha) = q^{2\alpha z}$ or $\hat{\phi} = -i\gamma \sigma^z$.\footnote{Note that $\sigma^z$ and $\sigma^\pm$ are eigenvectors under the adjoint action of $\hat{\phi}$ with eigenvalues $s(\sigma^z) = 0$ and $s(\sigma^\pm) = \mp 2i\gamma$.} Thus, we may construct $H$ as in \textsuperscript{(23)}. This Hamiltonian turns into $H_{XXZ}$ if we choose

$$\mathcal{h}_R = -2iJ \sin(\gamma), \quad \alpha = \frac{i\hbar}{2\gamma}.$$ \hspace{1cm} (41)
We consider the inhomogeneous monodromy matrix (17) and write it as a matrix in Appendix B).

The pseudo vacuum is an eigenvector of the operators $A$, $B$, $C$, and $D$.

This defines the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$. The operators $B(\lambda)$ generate the eigenstates of the quantum transfer matrix $t(\lambda|\kappa) = A(\lambda) + D(\lambda)$ by acting on a pseudo vacuum $|0\rangle$ defined by $C(\lambda)|0\rangle = 0$. In our case the pseudo vacuum is

$$|0\rangle = \left( \frac{1}{0} \otimes \frac{0}{1} \right) \otimes (N+1).$$

The pseudo vacuum is an eigenvector of the operators $A(\lambda)$, with eigenvalue $a(\lambda)$, and $D(\lambda)$, with eigenvalue $d(\lambda)$. The eigenvalues are readily calculated using the explicit form of the $R$-matrix and its transposed with respect to the first space,

$$a(\lambda) = q^n \prod_{k=1}^{N+1} b(\nu_{2k}, \lambda), \quad d(\lambda) = q^{-n} \prod_{k=1}^{N+1} b(\lambda, \nu_{2k-1}).$$

In the context of the quantum transfer matrix formalism it has turned out to be useful to describe the Bethe Ansatz solution in terms of certain auxiliary functions. For $M = 0, \ldots, 2N + 2$ define a family of functions

$$a(\lambda|\{\lambda_k\}_{k=1}^M, \kappa) = \frac{d(\lambda)}{a(\lambda)} \prod_{k=1}^{M} \frac{\sinh(\lambda - \lambda_k - i\gamma)}{\sinh(\lambda - \lambda_k + i\gamma)}$$

depending meromorphically on $M$ complex parameters $\lambda_j$. The equations

$$a(\lambda_j|\{\lambda_k\}_{k=1}^M, \kappa) = -1, \quad j = 1, \ldots, M$$

3.2 Algebraic Bethe Ansatz

In order to set our form-factor series (33) at work we first of all need to know the eigenvalues and eigenvectors of the quantum transfer matrix. For the XXZ chain the eigenvectors can be constructed by means of the algebraic Bethe Ansatz. This has been described at many instances (see e.g. [27]). Here we only have to adapt the known result to our conventions.

We consider the inhomogeneous monodromy matrix (17) and write it as a 2 × 2 matrix in ‘auxiliary space’ $\alpha$,

$$T_\alpha(\lambda|\kappa) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \quad (42)$$

This defines the operators $A(\lambda), \ldots, D(\lambda)$. The operators $B(\lambda)$ generate the eigenstates of the quantum transfer matrix $t(\lambda|\kappa) = A(\lambda) + D(\lambda)$ by acting on a pseudo vacuum $|0\rangle$ defined by $C(\lambda)|0\rangle = 0$. In our case the pseudo vacuum is

$$|0\rangle = \left( \frac{1}{0} \otimes \frac{0}{1} \right) \otimes (N+1). \quad (43)$$

The pseudo vacuum is an eigenvector of the operators $A(\lambda)$, with eigenvalue $a(\lambda)$, and $D(\lambda)$, with eigenvalue $d(\lambda)$. The eigenvalues are readily calculated using the explicit form of the $R$-matrix and its transposed with respect to the first space,

$$a(\lambda) = q^n \prod_{k=1}^{N+1} b(\nu_{2k}, \lambda), \quad d(\lambda) = q^{-n} \prod_{k=1}^{N+1} b(\lambda, \nu_{2k-1}). \quad (44)$$

In the context of the quantum transfer matrix formalism it has turned out to be useful to describe the Bethe Ansatz solution in terms of certain auxiliary functions. For $M = 0, \ldots, 2N + 2$ define a family of functions

$$a(\lambda|\{\lambda_k\}_{k=1}^M, \kappa) = \frac{d(\lambda)}{a(\lambda)} \prod_{k=1}^{M} \frac{\sinh(\lambda - \lambda_k - i\gamma)}{\sinh(\lambda - \lambda_k + i\gamma)} \quad (45)$$

depending meromorphically on $M$ complex parameters $\lambda_j$. The equations

$$a(\lambda_j|\{\lambda_k\}_{k=1}^M, \kappa) = -1, \quad j = 1, \ldots, M \quad (46)$$

The completeness of the system of eigenvectors obtained through the Bethe Ansatz is a separate issue. It was shown in [36] that all solutions to the Bethe equations with pairwise distinct Bethe roots provide a complete set of eigenvectors of the quantum transfer matrix if the magnetic field and the inhomogeneities are generic. In such a situation the Bethe vectors $|\Psi_n\rangle$ are all well-defined and, in particular, have non-vanishing ‘norm’ $\langle \Psi_n|\Psi_n\rangle$. For simplicity we shall work with the specific choice of the inhomogeneities $\nu_k$ as given in [28]. If any problem related to completeness or vanishing of the norm $\langle \Psi_n|\Psi_n\rangle$ should arise, it would be enough to slightly perturb the $\nu_k$ in the intermediate calculations and then send them to the values [28] in the end, on the level of the final formula, where the limit is already regular.
are called the Bethe Ansatz equations. Their solutions \( \{ \lambda_j^{(n)} \}_{j=1}^M \) are sets of ‘Bethe roots’. We have supplied a superscript ‘\((n)\)’ to distinguish the different solutions at fixed \( M \). With every set of Bethe roots we associate its corresponding auxiliary function

\[
a_n(\lambda|\kappa) = a(\lambda|\{\lambda_k^{(n)}\}_{k=1}^M, \kappa) .
\]  

(47)

Sets of Bethe roots \( \{ \lambda_j^{(n)} \}_{j=1}^M \) parameterize the solutions of the eigenvalue problem of the quantum transfer matrix. The eigenvalues can be written as

\[
\Lambda_n(\lambda|\kappa) = a(\lambda) \prod_{j=1}^M \frac{\text{sh}(\lambda - \lambda_j^{(n)} + i\gamma)}{\text{sh}(\lambda - \lambda_j^{(n)})} + d(\lambda) \prod_{j=1}^M \frac{\text{sh}(\lambda - \lambda_j^{(n)} - i\gamma)}{\text{sh}(\lambda - \lambda_j^{(n)})} .
\]  

(48)

The eigenvectors and their ‘duals’ take the form

\[
|\Psi_n\rangle = B(\lambda_1^{(n)}) \cdots B(\lambda_N^{(n)}|0\rangle , \langle 0|C(\lambda_1^{(n)}) \cdots C(\lambda_N^{(n)}) .
\]  

(49)

### 3.3 Nonlinear integral equations for the auxiliary functions and integral representations of the eigenvalue ratios

Let us write the auxiliary function for the general inhomogeneous quantum transfer matrix of the XXZ chain explicitly,

\[
a_n(\lambda|\kappa) = (-1)^s q^{-2k} \left[ \prod_{k=1}^{N+1} \frac{\text{sh}(\lambda - \nu_{2k-1})}{\text{sh}(\lambda - \nu_{2k})} \frac{\text{sh}(i\gamma + \lambda - \nu_{2k})}{\text{sh}(i\gamma - \lambda + \nu_{2k})} \right] \prod_{k=1}^M \frac{\text{sh}(i\gamma - \lambda + \lambda_k^{(n)})}{\text{sh}(i\gamma + \lambda - \lambda_k^{(n)})} ,
\]  

(50)

where

\[
s = N + 1 - M
\]  

(51)

is the (pseudo) spin of the excitation.

The reasoning that brings us a rough understanding of the structure of the solutions of the Bethe Ansatz equations (46) is the same as in the staggered case. The Bethe roots \( \lambda_j^{(n)} \) are located on the level curve \( |a_n(\lambda|\kappa)| = 1 \). If the \( \nu_j \) are mutually distinct and of order \( 1/N \), then \( a_n(\lambda|\kappa) \) has \( N + 1 \) simple poles at \( \nu_{2k} \) and \( N + 1 \) simple zeros at \( \nu_{2k-1} \), both close to \( \lambda = 0 \). If we group the poles and zeros in close-by pairs and join each pair by a straight line, then the function \( |a_n(\lambda|\kappa)| \) takes on the value 1 at some point on each of these lines. The contour \( |a_n(\lambda|\kappa)| = 1 \) is connecting these points. Because of the many poles and zeros the phase of \( a_n(\lambda|\kappa) \) strongly varies along this contour, meaning that close to zero there must be many points on the contour, where \( a_n(\lambda|\kappa) = -1 \).

Let us first focus on the case \( M = N + 1 \). We shall look for a special solution \( \{ \lambda_j^{(0)} \}_{j=1}^{N+1} \) to the Bethe equations for which all \( \lambda_j^{(0)} , j = 1, \ldots, N + 1 \), are of the form \( \lambda_j^{(0)} = \Theta(j/(TN)) \). For such a solution

\[
\prod_{k=1}^{N+1} \frac{\text{sh}(\lambda - \nu_{2k} + i\gamma)}{\text{sh}(\lambda - \nu_{2k-1} + i\gamma)} \frac{\text{sh}(\lambda - \lambda_k^{(0)} - i\gamma)}{\text{sh}(\lambda - \lambda_k^{(0)} + i\gamma)} = e(\lambda) = 1 + e^{O(1/T)}
\]  

(52)
as long as \( \lambda \) stays away from the poles close to \( \pm i\gamma \). Setting \( z = e^{2\lambda} \) and \( z_k = e^{2\nu_k} \) we see that the equation \( a_0(\lambda|\kappa) = -1 \) is equivalent to

\[
p(z) = c(\lambda(z)) e^{-2\kappa - h_R/T} \prod_{k=1}^{N+1} (z - z_{2k-1}) + \prod_{k=1}^{N+1} (z - z_{2k}) = 0 ,
\]

if

\[
\sum_{k=1}^{N+1} (\nu_{2k-1} - \nu_{2k}) = \frac{h_R}{T} .
\]

Note that our Trotter decomposition (28) satisfies the latter condition for \( \varepsilon \to 0 \). Sticking with this example we see that for \( |h_R/T|, |t_R| \to 0 \) we have \( z_k \to 1 \) which should define the high-temperature regime also in the general inhomogeneous case, whence

\[
p(z) \to (e^{-2\kappa} + 1)(z - 1)^{N+1}
\]

in the high-temperature limit. Thus, we have an \((N + 1)\)-fold zero at \( z = 1 \), which, for small \( |h_R/T|, |t_R| \), is split into \( N + 1 \) zeros close to \( z = 1 \), corresponding to \( N + 1 \) Bethe roots \( \lambda_i^{(0)} \) close to zero. These form a self-consistent high-temperature solution of the Bethe Ansatz equations. Inserting the solution back into (50) we see that the other zeros of \( \lambda \rightarrow 1 \) close to zero. These form a self-consistent high-temperature solution of the Bethe Ansatz equations. Inserting the solution back into (50) we see that the other zeros of \( 1 + a_0 \) must be located close to the poles at \( \pm i\gamma \). More precisely, we expect \( N + 1 \) of them close to \( i\gamma \) and \( N + 1 \) close to \( -i\gamma \).

Defining the canonical contour \( \mathcal{C}_0 \) in the usual way as \((-i\gamma/2 + i\delta - \infty, -i\gamma/2 + i\delta + \infty) \cup (i\gamma/2 - i\delta + \infty, i\gamma/2 - i\delta - \infty)\), where \( \delta > 0 \) is small, we observe that for \( N \) large enough the Bethe roots of the above solution are inside the contour, while all other zeros of \( 1 + a_0 \) remain outside. This is enough information to derive a nonlinear integral equation for the auxiliary function associated with this specific solution of the Bethe Ansatz equations.

Note that for our choice of parameters the function \( \ln(\text{sh}(i\gamma - \lambda + \mu)/\text{sh}(i\gamma + \lambda - \mu)) \), where ‘\( \ln \)’ denotes the principal branch of the logarithm, is analytic inside \( \mathcal{C}_0 \). We fix a point \( x_0 \in \mathcal{C}_0 \) and, for every \( \lambda \in \mathcal{C}_0 \), define a contour \( \mathcal{C}_x \) starting at \( x_0 \) and running along \( \mathcal{C}_0 \) up to the point \( \lambda \). This enables us to define

\[
\ln(1 + a_0)(\lambda|\kappa) = \int_{\mathcal{C}_x} \frac{d\mu}{2\pi i} \ln \frac{\text{sh}(i\gamma - \lambda + \mu)}{\text{sh}(i\gamma + \lambda - \mu)} \frac{a_0(\mu|\kappa)}{1 + a_0(\mu|\kappa)} .
\]

It follows that

\[
\int_{\mathcal{C}_x} \frac{d\mu}{2\pi i} \ln \left( \frac{\text{sh}(i\gamma - \lambda + \mu)}{\text{sh}(i\gamma + \lambda - \mu)} \right) \frac{a_0(\mu|\kappa)}{1 + a_0(\mu|\kappa)} = - \int_{\mathcal{C}_x} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a_0)(\mu|\kappa) = \ln(a_0(\lambda|\kappa)) - 2i\gamma\kappa + h_R e_N(\lambda)/T ,
\]

where

\[
K(\lambda) = \cth(\lambda + i\gamma) - \cth(\lambda - i\gamma) ,
\]

\[
e_N(\lambda) = \frac{T}{h_R} \sum_{k=1}^{N+1} \ln \left( \frac{\text{sh}(\lambda - \nu_{2k})}{\text{sh}(\lambda - \nu_{2k-1})} \frac{\text{sh}(\lambda - \nu_{2k-1} - i\gamma)}{\text{sh}(\lambda - \nu_{2k} - i\gamma)} \right) .
\]
In (57) we have used the fact that $1 + a_0$ has as many poles as zeros inside $\mathcal{C}_0$ when we performed the partial integration in the first equation. In the second equation we have used our knowledge about the location of the poles and zeros of $1 + a_0$ and the explicit representation (50) of the auxiliary function. Equation (57) can be read as a nonlinear integral equation for the auxiliary function $a_0$,

$$\ln \left( a_0(\lambda|\kappa) \right) = 2i\gamma\kappa - h_\text{Re}N(\lambda)/T - \int_{\mathcal{C}_0} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a_0)(\mu|\kappa).$$  (59)

This equation is an inhomogeneous version of the well-known \[10,25\] nonlinear integral equation for the auxiliary function of the dominant eigenvalue of the quantum transfer matrix. Our derivation shows that it holds for any inhomogeneous Trotter decomposition if $|\nu_j|, j = 1, \ldots, 2N + 2$, is small enough uniformly in $j$. It holds in particular for our Trotter decomposition (28) for finite but small enough $|h_R/T|, |t_R|$. Our experience with the numerical solution of (59) suggests that it has a unique solution. In forthcoming work we shall show that this is indeed the case if the $|\nu_j|$ are small enough. This implies, at least for $M = N + 1$, that all other solutions to the Bethe equations, which correspond to ‘excited states’ must contain roots which are away from the origin if the $|\nu_j|$ are small.

Numerical studies also suggest that the contour $\mathcal{C}_0$ is independent of the Trotter number implying that the Trotter limit $N \to \infty$ only affects the term $e_N(\lambda)$. For our Trotter decomposition (28) we also have to send $\varepsilon$ to zero. Then

$$\lim_{N \to \infty, \varepsilon \to 0} e_N(\lambda) = e(\lambda) = \text{cth}(\lambda) - \text{cth}(\lambda - i\gamma)$$  (60)

which is the bare energy function. With this the nonlinear integral equation for the auxiliary function of the dominant state in the Trotter limit $a_0^{\text{lim}}$ takes its familiar form \[10,25\]

$$\ln \left( a_0^{\text{lim}}(\lambda|\kappa) \right) = 2i\gamma\kappa - h_\text{Re}e(\lambda)/T - \int_{\mathcal{C}_0} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a_0^{\text{lim}})(\mu|\kappa).$$  (61)

The function $a_0^{\text{lim}}$ determines the thermodynamic properties and the static temperature dependent correlation functions of the XXZ chain at all finite temperatures \[10,25\] as it parameterizes the integral representations of the dominant eigenvalue and of the reduced density matrix of the model. The amazing fact, which we would like to emphasize and which was already observed by Sakai [33], is that no dependence on time $t$ has remained in the Trotter limit.

The thermal form factors and eigenvalue ratios in (33) are parameterized by the auxiliary functions connected with excited states of the quantum transfer matrix. For these auxiliary functions several alternative descriptions are available. The formally simplest one uses equivalence classes of contours $\mathcal{C}_n$ in order to classify the excitations (see e.g. [5]). For any given auxiliary function $a_n$ there exists a contour $\mathcal{C}_n$ which encircles all the Bethe roots, but no other zeros of $1 + a_n$ and no other poles of this function than those at $\nu_{2k}, k = 1, \ldots, N + 1$. We shall assume that we can shape the contour $\mathcal{C}_n$ in such a way that $\lambda - \mu \pm i\gamma$ remains outside for $\lambda, \mu \in \mathcal{C}_n$. For simplicity we also assume that $\mathcal{C}_n$ contains all Bethe roots of the dominant state and no additional pole or zero of $1 + a_0$ as compared to $\mathcal{C}_0$.

‡For more general inhomogeneous Trotter decompositions see [26].
With these prerequisites the only difference in the derivation of a nonlinear integral equation is that the partial integration performed in \( (57) \) produces additional boundary terms due to the fact that for an excitation with \( M \) Bethe roots

\[
\int \frac{d\mu}{\mathcal{C}_n} \frac{a_n'(\mu|\kappa)}{2\pi i (1 + a_n(\mu|\kappa))} = M - N - 1 = -s. \tag{62}
\]

As in case of the dominant state we fix a point \( x_n \) on \( \mathcal{C}_n \) and define the sub-contour \( \mathcal{C}_n^\lambda \) connecting \( x_n \) and \( \lambda \in \mathcal{C}_n \) in positive direction along \( \mathcal{C}_n \). Then \( \ln(1 + a_n)(\lambda|\kappa) \) can be defined in analogy with \( (56) \). Using this determination of the logarithm partial integration gives us

\[
\int \frac{d\mu}{\mathcal{C}_n} \ln \left( \frac{\sh(i\gamma - \lambda + \mu)}{\sh(i\gamma + \lambda - \mu)} \right) \frac{a_n'(\mu|\kappa)}{1 + a_n(\mu|\kappa)} = - \int \frac{d\mu}{\mathcal{C}_n} K(\lambda - \mu) \ln(1 + a_n)(\mu|\kappa) - s \ln \left( \frac{\sh(i\gamma - \lambda + x_n)}{\sh(i\gamma + \lambda - x_n)} \right). \tag{63}
\]

Assuming that the contour \( \mathcal{C}_n \) is such that we can send \( \Re x_n \rightarrow -\infty \) we obtain the nonlinear integral equation

\[
\ln(a_n(\lambda|\kappa)) = i\pi s + 2i\gamma(s - \hbar\epsilon e_N(\lambda)/T - \int \frac{d\mu}{\mathcal{C}_n} K(\lambda - \mu) \ln(1 + a_n)(\mu|\kappa)) \tag{64}
\]

for the excited states of the quantum transfer matrix in a form parameterized by the contour \( \mathcal{C}_n \). Solutions of \( (64) \) are classified by equivalence classes of contours \( \mathcal{C}_n \), two contours being equivalent if they admit the same solution \( a_n(\cdot|\kappa) \). Another form of equations is obtained if we deform all contours for a given value of \( s \) to some reference contour \( \mathcal{C}_{0,s} \). In the process of the deformation the contour will cross branch points of \( \ln(1 + a_n) \) which will appear as ‘particle and hole parameters’ in the additional driving terms generated on the right hand side of \( (64) \). This possibility will be further elaborated in the next section, where we discuss the summation of the form-factor series in the general case.

About the Trotter limit of the functions \( a_n \) the same can be said as in case of the dominant state. Our experience tells us that the contour \( \mathcal{C}_n \) becomes independent of \( N \) in the Trotter limit. The only part which depends on \( N \) and \( \epsilon \) is the function \( e_N \). In the limit \( N \rightarrow \infty, \epsilon \rightarrow 0 \) it turns into the bare energy \( (60) \), and no dependence on \( t \) is remaining. Now all functions appearing in the form-factor expansion are parameterized by the auxiliary functions \( a_0 \) and \( a_n \) and by the corresponding contours. Their dependence on \( a_0 \) and \( a_n \) is always the same independent of the underlying Trotter decomposition. If we choose our Trotter decomposition \( (28) \) they will all be time dependent, but the time dependence will vanish in the Trotter limit. At this point we understand that we can fully resort to the results of \( [4] \) for the eigenvalue ratios and the amplitudes \( A_n \) in the form-factor series \( (33) \).

Let us recall the expressions for the eigenvalue ratios. Following \( [2] \) we shall consider ratios of eigenvalues with different values of the magnetic field. We will use the function

\[
z_n(\lambda|\kappa, \kappa') = \frac{\ln(1 + a_0)(\lambda|\kappa) - \ln(1 + a_n)(\lambda|\kappa')}{2\pi i} \tag{65}
\]

in order to have more compact expressions. Then

\[
r_n(\lambda|\kappa, \kappa') = \frac{\Lambda_n(\lambda|\kappa')}{\Lambda_0(\lambda|\kappa)} = q^{s+\kappa'-\kappa} \exp \left\{ - \int_{\mathcal{C}_n} d\mu \epsilon(\mu - \lambda) z_n(\lambda|\kappa, \kappa') \right\}. \tag{66}
\]
The eigenvalue ratios appearing in the form-factor series are recovered by setting \( \kappa' = \kappa \).

In \([4]\) we derived expressions for the amplitudes in the form-factor series of a generating function of the longitudinal two-point functions and for the amplitudes that determine the transverse correlation functions \( \langle \sigma_1^+ \sigma_{n+1}^- (t) \rangle_T \). Let us recall only the transverse case as an example here. In this case the relevant amplitudes needed in \((33)\) are

\[
\langle \Psi_0 \vert B(\xi \vert \kappa) \vert \Psi_n \rangle \langle \Psi_n \vert C(\xi \vert \kappa) \vert \Psi_0 \rangle = \frac{\lambda_n(\xi \vert \kappa)(\Psi_0 \vert \Psi_0)\lambda_0(\xi \vert \kappa)(\Psi_n \vert \Psi_n)}{\lambda_n(\xi \vert \kappa)(\Psi_0 \vert \Psi_0)\lambda_0(\xi \vert \kappa)(\Psi_n \vert \Psi_n)} = \lim_{\kappa' \to \kappa} A_{n}^{+}(\xi \vert \kappa, \kappa') , \tag{67}
\]

where

\[
A_{n}^{+}(\xi \vert \kappa, \kappa') = \frac{\overline{G}^{+}(\xi)\overline{G}^{+}(\xi)}{(q^{1+\kappa'-\kappa} - q^{1-\kappa'+\kappa})(q^{1+\kappa'-\kappa} - q^{1-\kappa'+\kappa})} \times \exp \left\{ \int_{\mathbb{C}_n} d\lambda \, \ln(\rho_n(\lambda \vert \kappa, \kappa')) \partial_{\lambda} \left( \overline{G}^{+}(\xi \vert \kappa, \kappa') \right) \right\} \times \frac{\det_{dm_{-}, \mathcal{C}_{n}} \{ 1 - \overline{K}_{1-\kappa'+\kappa} \} \det_{dm_{-}, \mathcal{C}_{n}} \{ 1 - \overline{K}_{1+\kappa'-\kappa} \}}{\det_{dm_{0}, \mathcal{C}_{n}} \{ 1 - \overline{K} \} \det_{dm_{0}, \mathcal{C}_{n}} \{ 1 - \overline{K} \}}. \tag{68}
\]

Here, for \( \sigma = \pm \),

\[
\overline{G}^{\pm}(\lambda, \xi) = \lim_{\text{Re} \lambda \to \pm \infty} \overline{G}_{\sigma}(\lambda, \xi) \tag{69}
\]

and \( \overline{G}_{\pm}(\lambda, \xi) \) is the solution of the linear integral equation

\[
\overline{G}_{\pm}(\lambda, \xi) = - \csc h(\lambda - \xi) + q^{\kappa'-\kappa + 1} \rho_{\pm 1}(\xi \vert \kappa', \kappa) \csc h(\lambda - \xi + i\gamma) + \int_{\mathbb{C}_{n}} dm_{\pm}(\mu) \overline{G}_{\pm}(\mu, \xi) K_{\kappa'-\kappa + 1}(\mu - \lambda) \tag{70}
\]

with deformed kernel

\[
K_{\kappa}(\lambda) = q^{-\kappa} \csc h(\lambda + i\gamma) - q^{\kappa} \csc h(\lambda - i\gamma). \tag{71}
\]

Note that \( \overline{G}_{-}^{+} \) is analytic in \( \kappa' - \kappa \) and that \( \overline{G}_{-}^{+}(\kappa' = \kappa) = 0 \) which implies that the limit \( \kappa' \to \kappa \) exists in \((68)\) \([3]\).

The ‘measures’ \( dm_{\epsilon} \), \( \epsilon = - , 0 , + \), and \( dm \) are defined by

\[
dm_{-}(\lambda) = \frac{d\lambda \, \rho_{- 1}(\lambda \vert \kappa', \kappa')}{2\pi i(1 + a_{0}(\lambda \vert \kappa))} , \quad dm_{+}(\lambda) = \frac{d\lambda \, \rho_{n}(\lambda \vert \kappa, \kappa')}{2\pi i(1 + a_{n}(\lambda \vert \kappa'))} , \tag{72a}
\]

\[
\dm(\lambda) = \frac{d\lambda}{2\pi i(1 + a_{0}(\lambda \vert \kappa))} , \quad \dm(\lambda) = \frac{d\lambda}{2\pi i(1 + a_{n}(\lambda \vert \kappa'))}. \tag{72b}
\]

The determinants in \((68)\) are Fredholm determinants of integral operators defined by the respective kernels and measures and by the integration contours \( \mathcal{C}_{n} \) (see \([4]\) for more details). Note that the representation \((68)\) is well-defined in the sense that for Bethe states with non-vanishing norm \( \langle \Psi_{n} \vert \Psi_{n} \rangle \neq 0 \) – which is what we assume to hold and what is always achievable for generic inhomogeneities – all factors, and particularly the Fredholm determinants in the denominator, are finite.
3.4 On the summation of the form-factor series

The form-factor series (33) is a sum over all excitations of the quantum transfer matrix. But typically the matrix elements satisfy ‘selection rules’ connected with conservation laws that will make many of them disappear. In case of the XXZ chain the selection rule that has to be obeyed is related to the pseudo-spin conservation. For the example of the transverse correlation functions considered above pseudo-spin conservation implies that we may restrict ourselves to excitations with \( s = 1 \). Generically it will be enough to consider excitations with fixed \( s \in \mathbb{Z} \).

The issue to be discussed in this section is the partial summation of the form-factor series by means of multi-dimensional residue calculus. For this purpose the above description of the form factors and eigenvalue ratios based on equivalence classes of contours is not convenient. We shall rather deform those contours into reference contours \( \mathcal{C}_{0,s} \) which brings about an explicit dependence on particle and hole parameters. The contours \( \mathcal{C}_n \) were characterized by the fact that all Bethe roots of a given state are located inside the contour while all other zeros of \( 1 + a_n \) are outside. After the deformation some of the Bethe roots will be outside the new reference contour \( \mathcal{C}_{0,s} \), some zeros of \( 1 + a_n \) which are no Bethe roots will be inside. We shall call the former particles, the latter holes and denote their numbers by \( n_p, n_h \), respectively.

There is a degree of arbitrariness in the choice of the reference contour. A choice that might appear natural would be a contour that contains all Bethe roots and no other zeros of the auxiliary function pertaining to an eigenvalue of largest modulus in the pseudo-spin-\( s \) sector. Due to (62) this choice implies that the function \( \ln(1 + a_n) \) has nontrivial monodromy along such contour unless \( s = 0 \). Our experience with the case \( s = 1 \) in the low-temperature limit \([5]\) and with \( \Delta = 0 \) (see below) suggests that we obtain simpler formulae if we slightly deform the reference contour such as to include \( s \) more holes, if \( s \) is positive, or to exclude \(-s\) more particles, if \( s \) is negative. Then the equation

\[
\int_{\mathcal{C}_{0,s}} \frac{d\mu}{2\pi i} \frac{a'_n(\mu|\kappa)}{1 + a_n(\mu|\kappa)} = n_h - n_p - s = 0
\]

connects the numbers of particles and holes defined with respect to \( \mathcal{C}_{0,s} \) with the pseudo-spin. In the following we shall assume for simplicity that (73) is satisfied. We emphasize, however, that this is mostly motivated by our aim to give the formulae a simpler appearance and that this assumption is inessential for the argument that will allow us to partially sum the form-factor series below.

All equations and expressions considered in the previous subsection can be rewritten with respect to the reference contour \( \mathcal{C}_{0,s} \). The auxiliary functions \( a_n \), the eigenvalue ratios \( \rho_n \), and the amplitudes \( A_n \) then become explicit functions of the particle and hole roots. Instead of equivalence classes of contours \( \mathcal{C}_n \) we then use sets of holes \( \{x_j^{(n)}\}_{j=1}^{n_h} \) and particles \( \{y_k^{(n)}\}_{k=1}^{n_p} \) to classify the solutions, meaning that we have altogether three equivalent parameterizations: by sets of Bethe roots, by equivalence classes of contours, or by sets of particles and holes associated with a reference contour \( \mathcal{C}_{0,s} \).

In order to understand this in more detail we shall consider a nonlinear integral equation in which the driving terms depend explicitly on the particles and holes. We will use the shorthand notation

\[
\theta(\lambda) = -i \ln \left( \frac{\text{sh}(i\gamma + \lambda)}{\text{sh}(i\gamma - \lambda)} \right).
\]

(74)
With respect to this contour we first define a multi-parametric function \( a \) (not to be confused with the function defined in (43)) as the solution of the nonlinear integral equation

\[
\ln\left( a(\lambda|\{u\}, \{v\}, \kappa) \right) = i\pi s + 2i\gamma\kappa - \frac{hR e_N(\lambda)}{T} + i \sum_{j=1}^{n_h} \theta(\lambda - u_j) - i \sum_{k=1}^{n_p} \theta(\lambda - v_k) - \int_{\mathcal{C}_{0,s}} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a(\mu|\{u\}, \{v\}, \kappa)). \tag{75}
\]

Here \( \ln(1 + a) \) is defined along the contour \( \mathcal{C}_{0,s} \) in a similar way as in (56) and is further required to have trivial monodromy along this contour. The function \( a(\lambda|\{u\}, \{v\}, \kappa) \) depends holomorphically on two sets of variables \( \{u\} = \{u_j\}_{j=1}^{n_h} \) and \( \{v\} = \{v_k\}_{k=1}^{n_p} \), where the \( u_j \) take values inside \( \mathcal{C}_{0,s} \) and the \( v_k \) take values outside. Then solutions \( \{x\} = \{x_j\}_{j=1}^{n_h}, \{y\} = \{y_k\}_{k=1}^{n_p} \) of the ‘subsidiary conditions’

\[
a(x_j|\{x\}, \{y\}, \kappa) = a(y_k|\{x\}, \{y\}, \kappa) = -1, \quad j = 1, \ldots, n_h, \quad k = 1, \ldots, n_p, \tag{76}
\]

define sets of hole and particle roots which are in one-to-one correspondence with solutions \( \{\lambda_j^{(n)}\}_{j=1}^{n} \) of the Bethe Ansatz equations (46) and with the above defined contours \( \mathcal{C}_n \). Thus, we may label them by the same superscript ‘(n)’ implying that

\[
a_n(\lambda|\kappa) = a(\lambda|\{x^{(n)}\}, \{y^{(n)}\}, \kappa). \tag{77}
\]

The advantage of the ‘particle-hole formulation’ of the excitations is, as we shall see, that within this formulation all terms in the form-factor series (33) can be interpreted as multi-dimensional residues. Using the properties of the function \( a(\lambda|\{u\}, \{v\}, \kappa) \) the series can then be turned into a sum over multiple integrals.

The terms in the form-factor series are composed of eigenvalue ratios and amplitudes. If we deform the contours into \( \mathcal{C}_{0,s} \), the eigenvalue ratios take the form

\[
\rho_n(\lambda|\kappa, \kappa') = q^{e' - s} \left[ \prod_{j=1}^{n_h} \frac{\text{sh}(\lambda - x_j^{(n)})}{\text{sh}(\lambda - x_j^{(n)} + i\gamma)} \right] \left[ \prod_{k=1}^{n_p} \frac{\text{sh}(\lambda - y_k^{(n)} + i\gamma)}{\text{sh}(\lambda - y_k^{(n)})} \right] \times \exp\left\{ - \int_{\mathcal{C}_{0,s}} d\mu \, e(\mu - \lambda) z_n(\mu|\kappa, \kappa') \right\}. \tag{78}
\]

Note that the function \( z_n \) under the integral depends on \( a_n \) and is also parameterized by sets of particles and holes. We could proceed with the different factors appearing in the representation (68) of the amplitudes and make the dependence on the particle and hole parameters explicit. The exponential term was treated in general in [5]. So far the remaining factors were only considered in the low-temperature limit [4][5][8]. But the case of arbitrary temperature is no more difficult. The important point is that the argument that follows below is based on the fact that these terms can be parameterized in terms of particles and holes, but does not depend on the details of the parameterization.

For this reason we refrain here from working out all details but rather concentrate on the determinant \( \det_{d_{\mathcal{M}}, e_n} \{ 1 - \tilde{K} \} \) in the denominator. As we shall see, a Jacobian can
be factored out from this term, whose structure suggests to use multiple-residue calculus for the summation over the excitations with a fixed number of particles and holes. This was observed before in the analysis of the low-temperature limit of the static two-point functions in the massive regime $\Delta > 1$ \cite{8} and even earlier in the analysis of the form factor of the usual transfer matrix for $\Delta > 1$ in \cite{7}. In Appendix C we show that such a Jacobian appears in general. We derive the identity

$$\det_{\text{dm} \in \mathcal{C}} \{1 - \tilde{K}\} = \det_{\text{dm} \in \mathcal{C}_0 \times \mathcal{A}_n} \left\{1 - \tilde{K}\right\} \prod_{j=1}^{n_h} \frac{1}{a'_n(x^{(n)}_j)} \prod_{j=1}^{n_p} \frac{1}{a'_n(y^{(n)}_j)} \times \det \left| \begin{array}{cc} \partial u_j a(u_j | \{u\}, \{v\}, \kappa) & \partial v_k a(v_k | \{u\}, \{v\}, \kappa) \\ \partial u_j a(v_j | \{u\}, \{v\}, \kappa) & \partial v_k a(v_j | \{u\}, \{v\}, \kappa) \end{array} \right|_{\{u\} = \{x^{(n)}\}, \{v\} = \{y^{(n)}\}}. \quad (79)$$

Here the products over reciprocals of $a'_n$ will be canceled by corresponding terms originating from the exponential factor in (68).

The determinant on the right hand side of equation (79) is exactly what is needed (see \cite{1, 7, 8, 32}) to transform a sum over solutions of the subsidiary conditions (76) into a multiple-contour integral over ‘particle and hole variables’ $u_j$ and $v_j$. It may be interpreted as the Jacobian $\partial (U, V) / \partial (u, v)$ of a transformation $\mathbb{C}^{n_h + n_p} \mapsto \mathbb{C}^{n_h + n_p}$, $(u, v) \mapsto (U, V)$, where

$$U_j(u, v) = 1 + a(u_j | \{u\}, \{v\}, \kappa), \quad j = 1, \ldots, n_h, \quad (80a)$$

$$V_k(u, v) = 1 + a(v_k | \{u\}, \{v\}, \kappa), \quad k = 1, \ldots, n_p. \quad (80b)$$

This transformation maps solutions to the subsidiary conditions (76) to the origin in $\mathbb{C}^{n_h + n_p}$,

$$(x^{(n)}, y^{(n)}) \mapsto (U, V) = (0, 0). \quad (81)$$

We shall assume that the map is invertible in the neighbourhood of $(x^{(n)}, y^{(n)})$, viz. that the Jacobian

$$\left[ \frac{\partial (U, V)}{\partial (u, v)} \right]^{(n)} = \frac{\partial (U, V)}{\partial (u, v)} \big|_{(u, v) = (x^{(n)}, y^{(n)})} \quad (82)$$

is non-vanishing. Let

$$D^{(n)}_{\epsilon, \eta} = \left\{ (u, v) \in \mathbb{C}^{n_h + n_p} \mid |U_j(u, v)| < \epsilon, \ |V_k(u, v)| < \epsilon \right\}, \quad (83)$$

where $\epsilon$ and $\eta$ are sufficiently small so that $D^{(n)}_{\epsilon, \eta}$ is included in the domain where the map (80) is invertible and holomorphic. Then, for any function $f(u, v)$ which is holomorphic in all $u_j$ and $v_k$, we obtain the local residue

$$\int_{S^{(n)}_{\epsilon, \eta}} \frac{du^{n_h}}{(2\pi i)^{n_h}} \frac{dv^{n_p}}{(2\pi i)^{n_p}} \frac{f(u, v)}{\prod_{j=1}^{n_h} U_j(u, v) \prod_{k=1}^{n_p} V_k(u, v)} = \frac{f(x^{(n)}, y^{(n)})}{\left[ \frac{\partial (U, V)}{\partial (u, v)} \right]^{(n)}}. \quad (84)$$

Here

$$S^{(n)}_{\epsilon, \eta} = \left\{ (u, v) \in \mathbb{C}^{n_h + n_p} \mid |U_j(u, v)| = \epsilon, \ |V_k(u, v)| = \epsilon \right\}. \quad (85)$$
If we now make the dependence on the particle and hole parameters explicit in every term and use (79), the summands in (33) take the form

$$F^{-+}(\{x^{(n)}\}|\{y^{(n)}\}) = \mathcal{A}^{-+}(\{x^{(n)}\}|\{y^{(n)}\}) \rho_n^{m_\ell}(0|\kappa, \kappa) \rho_n^{N}(t_R/N|\kappa, \kappa) \rho_n^{-N}(-t_R/N|\kappa, \kappa)$$

(87)

and where the ‘amplitude density’ $\mathcal{A}^{-+}(\{x^{(n)}\}|\{y^{(n)}\})$ is what we obtain when we make the dependence on the particle and hole parameters explicit in (68) and extract the Jacobian.

We now replace the particle and hole parameters $\{y^{(n)}\}$ and $\{x^{(n)}\}$ in (87) by complex variables $\{u\}$ and $\{v\}$ which we do not require to satisfy the subsidiary conditions (76). This means that $a_n(\lambda|\kappa)$ is replaced by $a(\lambda|\{u\}, \{v\}, \kappa)$ everywhere, implying that we obtain

$$z(\lambda|\{u\}, \{v\}, \kappa) = \frac{\ln(1 + a_0(\lambda|\kappa)) - \ln(1 + a(\lambda|\{u\}, \{v\}, \kappa))}{2\pi i}$$

(88)

instead of $z_n(\lambda|\kappa, \kappa)$ and

$$\rho(\lambda|\{u\}, \{v\}, \kappa) = \left[\prod_{j=1}^{n_h} \frac{\text{sh}(\lambda - u_j)}{\text{sh}(\lambda - u_j + i\gamma)}\right] \left[\prod_{k=1}^{n_p} \frac{\text{sh}(\lambda - v_k + i\gamma)}{\text{sh}(\lambda - v_k)}\right]$$

\[
\times \exp\left\{ - \int_{e_0, \kappa} d\mu e(\mu - \lambda) z(\mu|\{u\}, \{v\}, \kappa) \right\}.
\]

(89)

instead of $\rho_n(\lambda|\kappa, \kappa)$. Consequentially, $\mathcal{A}^{-+}(\{x^{(n)}\}|\{y^{(n)}\})$ is replaced by a function $\mathcal{A}^{-+}(\{x^{(n)}\}|\{y^{(n)}\})$ and $F^{-+}(\{x^{(n)}\}|\{y^{(n)}\})$ by a function $F^{-+}(\{u\}|\{v\})$.

Thus, (84) allows us to recast each individual term in the series (33) – specialised to the XXZ chain setting – as a multi-dimensional local residue integral. By the introduction of a suitable function holomorphic in all $\{u\}$ and $\{v\}$, one may expect a representation of the sum over all solutions to the subsidiary conditions in the form of a multi-dimensional residue integral. This idea is made precise, under certain reasonable hypotheses, in Appendix D. The resultant multi-dimensional integrations should be preformed over a skeleton (a distinguished boundary) defined in (D.16). By analogy with one-dimensional residue calculus, one may expect that the skeleton can be deformed into $e^{\ell}_{0,\kappa} \times e^{\ell-1}_{0,\kappa}$, where the contour $\overline{e}_{0,\kappa}$ encloses all particle roots. Taking into account that $n_p + 1 = n_h$ (which follows from (73) since $s = 1$) we then obtain the following representation for the transverse two-point functions

$$\langle \sigma_m^{-} \sigma_m^{-}(t) \rangle = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-i\lambda t}}{n!} \int_{e^{\ell}_{0,\kappa}} \frac{du}{(2\pi i)^\ell} \int_{e^{\ell-1}_{0,\kappa}} \frac{dv}{(2\pi i)^{\ell-1}} \mathcal{A}^{-+}(\epsilon|\{u\}, \{v\}, \kappa)$$

\[
\times \rho(\epsilon|\{u\}, \{v\}, \kappa) \rho^{N/2}(t_R/N|\{u\}, \{v\}, \kappa) \rho^{-N/2}(-t_R/N|\{u\}, \{v\}, \kappa)
\]

\[
\prod_{j=1}^{\ell} \left(1 + a(\{u\}, \{v\}, \kappa)\right) \prod_{k=1}^{\ell-1} \left(1 + a(\{v\}, \{u\}, \kappa)\right). \tag{90}
\]
When calculating the integrals over the extended contours $\mathcal{C}_{0,1}, \mathcal{C}_{0,1}$ we will obtain each local residue with multiplicity $\ell!(\ell - 1)!$ due to the symmetry of the functions in the denominator under the integral, which is why we divided each summand by this factor. Note that the determination of $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,1}$ in a way suitable for numerical calculations may be a subtle issue.

Morally we can understand equation (90) as follows. By shrinking the contours $\mathcal{C}_{0,1}, \mathcal{C}_{0,1}$ one picks up the contributions of all solutions to the subsidiary conditions. In principle, one should then also pick up contributions originating from the poles of $F^{-+}([u]|v\})$. Indeed, this function is not a holomorphic function of the $u_j$ and $v_k$.

As can be seen from (89), the factor $\rho^{-N/2}(-tR/N|[u], \{v\}, \kappa)$ has $N/2$-fold poles at $u_j = -tR/N, j = 1, \ldots, n_h$, inside $\mathcal{C}_{0,s}$. However, these poles will be compensated by the functions $\mathfrak{a}(u_j|[u], \{v\}, \kappa) = 1/\mathfrak{a}(u_j|[u], \{v\}, \kappa)$ for $j = 1, \ldots, n_h$. In fact these functions as well have poles of order $N/2$ at $-tR/N$ as can be inferred from (28), (58b) and (75). Similarly, the factor $\rho^{-N/2}(-tR/N|[u], \{v\}, \kappa)$ has poles of order $N/2$ at $v_k = i\gamma - tR/N, k = 1, \ldots, n_p$, outside $\mathcal{C}_{0,s}$ close to where we expect the particles to be located. These poles will be canceled by the functions $\mathfrak{a}(v_k|[u], \{v\}, \kappa)$ as can be seen again from (28), (58b) and (75). Moreover, the amplitude densities $\mathcal{A}^{-+}(\xi|[u], \{v\})$ for $\xi = 0$ have simple poles at $u_j = 0, j = 1, \ldots, n_h$, and at $v_k = i\gamma, k = 1, \ldots, n_p$, which will be compensated by $\rho_{m}^0(0|\kappa, \kappa)$ if $m > 0$.

After rewriting the form-factor series as a sum over multiple integrals we may finally take the Trotter limit. For this purpose we introduce the function

$$E(\lambda) = \ln \left( \frac{\text{sh}(\lambda)}{\text{sh}(\lambda - i\gamma)} \right)$$ (91)

and remark that

$$\lim_{N \to \infty} \rho^m(\varepsilon|[u], \{v\}, \kappa) \rho^{N/2}(tR/N|[u], \{v\}, \kappa) \rho^{-N/2}(-tR/N|[u], \{v\}, \kappa)$$

$$= \exp \left\{ \sum_{j=1}^{n_h} (mE(u_j) - tRe(u_j)) - \sum_{j=1}^{n_p} (mE(v_j) - tRe(v_j)) \right\}$$

$$- \int_{\mathcal{C}_{0,s}} d\mu \frac{z^{\text{lim}}(\mu|[u], \{v\}, \kappa)(me(\mu) - tRe(\mu))}{1 + \mathfrak{a}(\mu|[u], \{v\}, \kappa)},$$ (92)

which follows from (89) and where the superscript ‘lim’ refers to the Trotter limit. Then we end up with the thermal form-factor series representation

$$\langle \sigma_{-}^{0} \sigma^{+}_{m+1}(t) \rangle_T = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \int_{\mathcal{C}_{0,1}} \frac{du^n}{(2\pi)^n} \int_{\mathcal{C}_{0,1}} \frac{dv^{n-1}}{(2\pi)^{n-1}}$$

$$\times \prod_{j=1}^{n} \frac{e^{mE(u_j) - tRe(u_j)}}{1 + \mathfrak{a}^{\text{lim}}(u_j|[u], \{v\}, \kappa)} \prod_{j=1}^{n-1} \frac{e^{-mE(v_j) + tRe(v_j)}}{1 + \mathfrak{a}^{\text{lim}}(v_j|[u], \{v\}, \kappa)}$$

$$\times \mathcal{A}^{-+}_{\text{lim}}(0|[u], \{v\}) e^{-ih(t - \int_{\mathcal{C}_{0,1}} d\mu \frac{z^{\text{lim}}(\mu|[u], \{v\}, \kappa)(me(\mu) - tRe(\mu))}{1 + \mathfrak{a}^{\text{lim}}(\mu|[u], \{v\}, \kappa)}}$$ (93)

for the transverse correlation functions of the XXZ chain. In this formula $\mathfrak{a}^{\text{lim}}$ denotes the Trotter limit of the function $\mathfrak{a}$ which is obtained by replacing $e_N$ with $e$ in equation (75). Similarly $z^{\text{lim}}$ and $\mathcal{A}^{-+}_{\text{lim}}$ are obtained from $z$ and $\mathcal{A}^{-+}$ by replacing $\mathfrak{a}$ with $\mathfrak{a}^{\text{lim}}$. 
It should be clear from our derivation that a similar form factor series representation can be also derived for the longitudinal correlation functions. In our derivation we used the implicit assumption that the reference contour $C_{0,1}$ and the contour $\tilde{C}_{0,1}$ can be chosen independently of the excitation. In the low-temperature limit \[4, 5, 8\] and for $\Delta = 0$ (see below) we know that this is possible. The general case will need further study. In fact, the main difficulty imposed by the series \(93\) is that we have insufficient knowledge about the general Bethe root patterns at generic temperature. Moreover, it is not always clear if one can choose $C_{0,1}$ and $\tilde{C}_{0,1}$ of an appropriate size and shape, which might be needed to perform e.g. numerical calculations without technical difficulties. An appropriate starting point for studying \(93\) may be the high-temperature limit, where certain simplifications are expected to occur. At least the longitudinal dynamical correlation functions remain nontrivial even at infinite temperature (see e.g. \[9, 29\] and our discussion of the XX case below).

### 3.5 The XX chain

The previous section shows that the summands in the form-factor series \(33\) can be calculated for the XXZ chain and that a summation for fixed numbers of particles and holes can be at least formally achieved by multiple-contour integration. A crucial question will be how efficient the formulae can be made. The problem with the general XXZ case is that the patterns of Bethe roots vary as functions of anisotropy parameter, magnetic field and temperature and that no general classification is known. The only case in which we fully understand where the Bethe roots are located at any temperature is the case of the XX chain. As a further test and in order to provide more explicit examples, we therefore proceed with the two-point functions of the XX chain.

By definition the XX Hamiltonian is the XXZ Hamiltonian \(39\) with $\Delta = 0$ corresponding to $\gamma = \pi/2$ in our parameterization. For this specific value of $\gamma$ we have

\[
 h_R = -2iJ , \quad \alpha = \frac{ih}{\pi} .
\]

Our basic bare functions become

\[
 e(\lambda) = \frac{2}{\text{sh}(2\lambda)} , \quad K(\lambda) = 0 .
\]

The fact that the kernel function $K$ is identically zero is the reason for the severe simplification that occur in this case.

Inserting $\gamma = \pi/2$ into the expression \(50\) for the auxiliary function $a_n$ at finite Trotter number we obtain

\[
 a_n(\lambda | \kappa) = (-1)^s q^{-2\kappa} \prod_{k=1}^{N+1} \frac{\text{th}(\lambda - \nu_{2k-1})}{\text{th}(\lambda - \nu_{2k})} .
\]

Unlike in the general XXZ case there is a large degeneracy among the auxiliary functions here. Any Bethe state corresponds to a set of roots of one of only two different auxiliary functions, since $a_n(\lambda | \kappa) = (-1)^s a_0(\lambda | \kappa)$, $s = 0, 1 \mod 2$, where $a_0$ is the auxiliary function of the dominant state.

Recalling that $\sum_{k=1}^{N+1} (\nu_{2k} - \nu_{2k-1}) = -h_R/T - 2\varepsilon$ for the Trotter decomposition \(28\) and that $\nu_k = \mathcal{O}(1/N)$ for $k \neq N + 1, N + 2$ we can calculate the limit

\[
 \lim_{N \to \infty} \lim_{\varepsilon \to 0} a_n(\lambda | \kappa) = (-1)^s e^{-\frac{\epsilon(\lambda)}{T}} , \quad \epsilon(\lambda) = h - \frac{4iJ}{\text{sh}(2\lambda)}
\]
directly from (96). This result is compatible with the general consideration of the previous section. Using (95) in (64) we obtain again (97).

**Remark.** In the XX limit the Bethe Ansatz solution of the eigenvalue problem of the quantum transfer matrix can be analyzed with full rigour. We have performed such an analysis for the Trotter decomposition \( \nu_k = (-1)^{k+1} h_R/(2NT) \), \( k = 1, \ldots, 2N \), i.e. for the usual temperature case with no dependence of the inhomogeneity parameters on time or on \( \varepsilon \). In this case the Bethe roots are solutions of the equations \( a_n(\lambda|\kappa) + 1 = 0 \), and the following can be shown

(i) If \( N > 2J/(\pi T) \), then all roots of \( a_n(\lambda|\kappa) + 1 = 0 \) are located inside the strip \( 0 < \text{Im} \lambda < \pi/2 \) modulo \( i\pi \).

(ii) The patterns of roots are point-symmetric about \( i\pi/4 \), i.e. if \( \lambda \) is a root then \( i\pi/2 - \lambda \) is a root as well.

(iii) The functions \( a_n(\lambda|\kappa) + 1 \) with \( s = \pm 1 \) have \( 2N \) roots each inside the strip \( 0 < \text{Im} \lambda < \pi/2 \). Denote the sets of these roots by \( S_{\pm} \). A subset of \( S_{\pm} \) containing \( M \) roots with \( N - M \) even or a subset of \( S_{\pm} \) containing \( M \) roots with \( N - M \) odd is called a set of Bethe roots. Sets of Bethe roots are in one-to-one correspondence with eigenvalues of the quantum transfer matrix. There are altogether \( 2^{2N} \) such states, called Bethe states.

(iv) All eigenvalues corresponding to Bethe states are mutually distinct, implying that the quantum transfer matrix has a simple spectrum and that 'the Bethe Ansatz is complete'.

(v) The dominant eigenvalue is the eigenvalue determined by the unique set of Bethe roots \( \{ \lambda_j^{(0)} \}_{j=1}^M \) which is contained in the strip \( 0 < \text{Im} \lambda < \pi/4 \) and for which \( M = N \).

(vi) In the Trotter limit \( a_0(\lambda) \to e^{-\frac{\lambda}{T}} \), and a pair of roots \( \lambda_F^\pm \) of \( a_0(\lambda|\kappa) - 1 \) is located on the line \( \text{Im} \lambda = \pi/4 \),

\[
\lambda_F^\pm = i\pi/4 \pm \frac{1}{2} \text{arch} \left( \frac{4J}{h} \right) .
\]

These roots will be called the Fermi rapidities.

In the following we will continue to work with the inhomogeneous model with Trotter decomposition (28). In particular, we will keep \( \varepsilon \) and the Trotter number finite until the very last stage of our calculation. We shall assume, however, that \( N \) is large enough such that the properties of our auxiliary functions are close to those described in the above remark. This means that we assume that the dominant state has exactly \( N + 1 \) Bethe roots located in the strip \( -\pi/4 < \text{Im} \lambda < \pi/4 \) and that the corresponding auxiliary function \( 1 + a_0 \) has no other zeros in that strip.

Because of the appearance of the Fermi rapidities some care is necessary when we introduce the canonical contour \( \mathcal{C} \). We define it as \((-\infty - i\pi/4 + i\delta, +\infty - i\pi/4 + i\delta) \cup (+\infty + i\pi/4 - i\delta, -\infty + i\pi/4 - i\delta)\), where \( \delta > 0 \) is small, but with four small deformations consisting of semicircles of radius \( 2\delta \), say, which are centered about the points \( \lambda_F^\pm - i\delta \) and \( -\lambda_F^\pm + i\delta \) in such a way that the upper part of \( \mathcal{C} \) bypasses \( \lambda_F^\pm \) from below and \( \lambda_F^\pm \) from...
above, while the lower part of $\mathcal{C}$ bypasses $-\lambda_F^+$ from above and $-\lambda_F^-$ from below (see Fig. 2). We shall call every zero of $a_0(\lambda|\kappa) \pm 1$ inside $\mathcal{C}$, which is not a Bethe root, a hole, while every Bethe root outside $\mathcal{C}$ will be called a particle. The numbers of particles and holes will be denoted $n_p$ and $n_h$, respectively. Then $M = N + 1 - n_h + n_p$, implying that

$$n_h - n_p = s .$$

(99)

Following the usual reasoning we obtain the following expressions for the logarithms of the eigenvalues of the quantum transfer matrix,

$$\ln(\Lambda_n(\lambda|\kappa)) = -i\pi\kappa/2 + \sum_{k=1}^{n_h} \ln(i \text{ th}(\lambda_k^h - \lambda)) - \sum_{k=1}^{n_p} \ln(i \text{ th}(\lambda_k^p - \lambda))$$

$$+ \int_0^\epsilon \frac{d\mu}{\pi i} \ln \left(1 + a_n(\mu|\kappa)\right) \frac{\sh(2(\mu - \lambda))}{\sh(2(\mu - \lambda))} .$$

(100)

From this formula we easily deduce the eigenvalue ratios and their logarithmic derivatives needed in the form-factor series.

### 3.5.1 Longitudinal case

Let us now first consider the example of the longitudinal two-point functions $\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T$. For these the operators $X(\xi|\kappa)$ and $Y(\xi|\kappa)$ in (34) are both equal to $A(\xi) - D(\xi) = 2A(\xi) - t(\xi|\kappa)$. Then

$$A_0 = \lim_{N \to \infty} \lim_{\epsilon \to 0} \left( \frac{\langle \Psi_0 | (A(\epsilon) - D(\epsilon)) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \right)^2 = 4m^2(T,h)$$

(101)
is four times the square of the magnetization, and it remains to calculate

\[
A_n = \lim_{\varepsilon \to 0} \frac{4 \langle \Psi_0 | A(\varepsilon) | \Psi_n \rangle \langle \Psi_n | A(\varepsilon) | \Psi_0 \rangle}{\langle \Psi_0 | \Lambda_n(\varepsilon|\kappa) | \Psi_n \rangle \langle \Psi_n | \Lambda_0(\varepsilon|\kappa) | \Psi_0 \rangle}
\]

(102)

for \( n \neq 0 \). This can be done by means of Slavnov’s scalar product formula. The calculation is rather straightforward but slightly technical. We show the details in Appendix E, where we arrive at

\[
\frac{\langle \Psi_0 | A(\xi) | \Psi_n \rangle \langle \Psi_n | A(\xi) | \Psi_0 \rangle}{\langle \Psi_0 | \Lambda_n(\xi|\kappa) | \Psi_n \rangle \langle \Psi_n | \Lambda_0(\xi|\kappa) | \Psi_0 \rangle} = \frac{e(\xi - \lambda^h) e(\xi - \lambda^p)}{a_0(\lambda^h|\kappa) a_0(\lambda^p|\kappa)},
\]

(103)

which is valid for the amplitudes at any finite Trotter number and \( \xi \) arbitrary inside the contour. We may set \( \xi = \varepsilon \) and take the Trotter limit and the limit \( \varepsilon \to 0 \) which are determined by equation (97).

In this very special case all excitations with non-vanishing amplitudes are parameterized by one particle and one hole rapidity (see Appendix E). For such excitations \( s = 0 \) due to (99). Using (100) we obtain

\[
\rho_n(0) = \frac{\text{th}(\lambda^h)}{\text{th}(\lambda^p)}, \quad \rho_n'(0) = e(\lambda^p) - e(\lambda^h).
\]

(104)

Thus, for the longitudinal correlation functions the formal series (36) can be cast into the form

\[
\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T = 4m^2(T, h) = 4 \left[ \sum_{\lambda^h} \frac{e(\lambda^h)(-i \text{th}(\lambda^h))^m e^{-tR e(\lambda^h)} e'((\lambda^h)/T)}{e'((\lambda^p)/T)} \right] \left[ \sum_{\lambda^p} \frac{e(\lambda^p)(-i \text{th}(\lambda^p))^{-m} e^{tR e(\lambda^p)} e'((\lambda^p)/T)}{e'((\lambda^h)/T)} \right].
\]

(105)

Here \( \exp\{t \text{Re}(\lambda)\} \) has an essential singularity at \( \lambda = 0 \) which prevents us from rewriting the series as integrals and hints that the series are not uniformly convergent in the excitations.

In order to write the longitudinal two-point functions as an integral we rather have to use equation (33). Then

\[
\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T = 4m^2(T, h) = \lim_{\varepsilon \to 0} \sum_{\lambda^h, \lambda^p} \frac{4e(\lambda^h)e(\lambda^p)}{a_0^2(\lambda^h|\kappa) a_0^2(\lambda^p|\kappa)} \left( \frac{\text{th}(\lambda^h)}{\text{th}(\lambda^p)} \right)^m \left( \frac{\text{th}(\lambda^h - tR/N) \text{th}(\lambda^p + tR/N)}{\text{th}(\lambda^h + tR/N) \text{th}(\lambda^p - tR/N)} \right)^{N/2}.
\]

(106)

The individual terms under the sum have \( N/2 \)-fold poles at \( \lambda^h = -tR/N \) and at \( \lambda^p = i\pi/2 - tR/N \). Fortunately, these can be canceled if we choose the auxiliary functions appropriately. Inserting (28) into (96) we obtain

\[
a_0(\lambda|\kappa) = q^{-2k} \frac{\text{th}(\lambda - \varepsilon)}{\text{th}(\lambda + \varepsilon)} \left[ \frac{\text{th}(\lambda + tR/N) \text{th}(\lambda - (tR + hR/T)/N)}{\text{th}(\lambda - tR/N) \text{th}(\lambda + (tR + hR/T)/N)} \right]^{N/2},
\]

(107)

from which we can see that \( a_0 \) has an \( N/2 \)-fold zero at \( -tR/N \) and an \( N/2 \)-fold pole at \( \lambda = i\pi/2 - tR/N \). Thus, \( 1 + 1/a_0 \) has an \( N/2 \)-fold pole at \( -tR/N \), while \( 1 + a_0 \) has an \( N/2 \)-fold pole at \( \lambda = i\pi/2 - tR/N \). Setting \( \bar{a}_0 = 1/a_0 \) it follows for \( m > 0 \) that
\[ \langle \sigma_1^z \sigma_{m+1}^z \rangle_T = 4m^2(T, h) = \lim_{N \to \infty} \lim_{\varepsilon \to 0} \left[ \int_{\varepsilon}^{\varepsilon+i\pi} \frac{d\lambda}{\pi} \frac{e(\lambda)}{1 + \varepsilon(\lambda)} \left( \frac{\th(\lambda - t_R/N)}{\th(\lambda + t_R/N)} \right)^N \right] \]

\[ \times \left[ \int_{\varepsilon+i\pi}^{\varepsilon+\pi} \frac{d\lambda}{\pi} \frac{e(\lambda)}{1 + \varepsilon(\lambda)} \left( \frac{\th(\lambda + t_R/N)}{\th(\lambda - t_R/N)} \right)^N \right]. \quad (108) \]

Here the Trotter limit and the limit \( \varepsilon \to 0 \) required in (33) can be taken. Using (97) we obtain

\[ \langle \sigma_1^z \sigma_{m+1}^z \rangle_T = 4m^2(T, h) \]

\[ - \left[ \int_{\varepsilon}^{\varepsilon+i\pi} \frac{d\lambda}{\pi} \frac{e(\lambda)}{1 + \varepsilon(\lambda)} \right] \left[ \int_{\varepsilon+i\pi}^{\varepsilon+\pi} \frac{d\lambda}{\pi} \frac{e(\lambda)}{1 + \varepsilon(\lambda)} \right] . \quad (109) \]

This can be transformed into a more familiar form by employing the \( i\pi \)-periodicity of the integrand in the second integral and by turning to momentum variables. The one-particle momentum is defined as

\[ p(\lambda) = -i \ln(-i \th(\lambda)) , \quad (110) \]

where we understand the logarithm as its principal value, meaning that we provide cuts in the complex plane from \(-i\pi/2\) to zero modulo \(i\pi\). The one-particle momentum is real on the lines \( \text{Im} \lambda = \pm \pi/4 \),

\[ p(\lambda) = \begin{cases} 
-\frac{\pi}{2} + 2 \text{arctg}(e^{-2\text{Re}\lambda}) & \text{if } \text{Im} \lambda = \pi/4 \\
-\pi \text{sign}(\text{Re}\lambda) + \frac{\pi}{2} - 2 \text{arctg}(e^{-2\text{Re}\lambda}) & \text{if } \text{Im} \lambda = -\pi/4 .
\end{cases} \quad (111) \]

Hence the assignment \( \lambda \mapsto p \) maps

\[ (-\infty - i\pi/4, +\infty - i\pi/4) \mapsto [-\pi, -\pi/2] \cup [\pi/2, \pi] , \quad (112a) \]

\[ (+\infty + i\pi/4, -\infty + i\pi/4) \mapsto [-\pi/2, \pi/2] . \quad (112b) \]

Then, since the regularizations of the contour play no role in (109),

\[ \langle \sigma_1^z \sigma_{m+1}^z \rangle_T = 4m^2(T, h) + \left[ \int_{-\pi}^{\pi} \frac{dp}{\pi} e^{i(mp - \varepsilon(p))} \right] \left[ \int_{-\pi}^{\pi} \frac{dp}{\pi} e^{-i(mp - \varepsilon(p))/T} \right] , \quad (113) \]

where we have introduced the energy function in momentum variables

\[ \varepsilon(p) = h - 4J \cos(p) . \quad (114) \]

For the sake of completeness let us also recall the expression

\[ m(T, h) = \int_{-\pi}^{\pi} \frac{dp}{4\pi} \th \left( \frac{\varepsilon(p)}{2T} \right) \]

for the magnetization as a function of temperature and magnetic field here.

Equation (113) is the final result for the longitudinal finite temperature dynamical two-point correlation function of the XX chain. Note that this beautifully simple formula
is different from those given in the classical papers \[19,29\] (where less natural parameterizations were used), but can be also obtained within an approach based on mapping the XX model to free spinless Fermions.

In our derivation of the form-factor series \([33]\) we required invertibility of the inhomogeneous shift operators. This means for the example at hand that we have implicitly assumed that \(t \neq 0\). Nevertheless, the known static case (see e.g. \([12]\)) is reproduced from \((113)\) for \(t \to 0\) using that \((1 + e^{-\varepsilon(p)/T})^{-1} = 1 - (1 + e^{\varepsilon(p)/T})^{-1}\) and that \(J^\pi \frac{dp}{2\pi} e^{-imp} = \delta_{m,0}^\varepsilon\). Then

\[
\langle \sigma^z_{m+1} \rangle_T = 4m^2(T, h) + \delta_{m,0}(2 - 4m(T, h)) - \left| \int_{-\pi}^\pi \frac{dp}{\pi} \frac{e^{imp}}{1 + e^{\varepsilon(p)/T}} \right|^2. \tag{116}
\]

The known high-temperature limit \([29]\) follows easily as well if we set \(1/T = 0\), implying that

\[
\langle \sigma^z_{m+1} \rangle_\infty = J_m^2(4Jt), \tag{117}
\]

where \(J_m, m \in \mathbb{N}\), is a Bessel function.

Note that \((113)\), even holds if \(t = 0, m = 0\), although we assumed \(m > 0\) in the derivation. Similarly, \((116)\) and \((117)\) remain valid for \(m = 0\). It seems that the validity in these limiting cases is assured by analytic continuation in \(m\) and \(t\).

### 3.5.2 Transverse case

While our study of the longitudinal case in the previous section basically showed that our form-factor formalism works and reproduces the known result, it brings about something new when we move on to the transverse case.

We shall consider the correlation function \(\langle \sigma^+_m \sigma^-_{m+1}(t) \rangle_T\). For this correlation function the operators \(X(\xi|\kappa)\) and \(Y(\xi|\kappa)\) in \([34]\) are equal to \(B(\xi)\) and \(C(\xi)\), respectively. In Appendix \([\text{E}]\) we calculate the finite Trotter number amplitudes

\[
A^{n-+}_n(\xi) = \frac{\langle \Psi_0|B(\xi)|\Psi_n\rangle \langle \Psi_n|C(\xi)|\Psi_0\rangle}{\langle \Psi_0|\Psi_0\rangle \Lambda_n(\xi|\kappa) \langle \Psi_n|\Psi_n\rangle \Lambda_0(\xi|\kappa)} \tag{118}
\]

for small finite \(\varepsilon\). They are non-zero only for (pseudo-) spin \(s = 1\) excitations with corresponding auxiliary functions \(a_n = -a_0\) and are parameterized by sets of hole rapidities \(\{\lambda^h_j\}_{j=1}^{n_h}\) and particle rapidities \(\{\lambda^p_j\}_{j=1}^{n_p}\). For our choice of contour the numbers of particle and hole rapidities are related by \([99]\). Hence, \(n_h = n_p + 1\), and we write \(n = n_h\) for short. We further introduce the functions

\[
z(\lambda) = \frac{1}{2\pi i} \ln \left( 1 + \frac{a_0(\lambda|\kappa)}{1 + a_n(\lambda|\kappa)} \right), \tag{119}
\]

\[
\Phi(x) = \frac{e(x)}{2} \times \exp \left\{ 2 \int_c d\mu \ \text{cth}(x - \mu) z(\mu) \right\}, \tag{120}
\]

\[
\mathcal{D}\left(\{x_j\}_{j=1}^{n_h}, \{y_k\}_{k=1}^{n_p}\right) = \frac{\prod_{1 \leq j < k \leq n_h} \text{sh}^2(x_j - x_k) \prod_{1 \leq j < k \leq n_p} \text{sh}^2(y_j - y_k)}{\prod_{j=1}^{n_h} \prod_{k=1}^{n_p} \text{sh}^2(x_j - y_k)} \tag{121}
\]

and the ‘prefactors’

\[
\mathcal{A} = \exp \left\{ 2 \int_c d\mu \ \text{cth}(2\mu) z(\mu) - \int_{c' < c} d\lambda \int_c d\mu \ \text{cth}'(\lambda - \mu) z(\lambda) z(\mu) \right\}, \tag{122a}
\]

\[
\mathcal{A}' = \exp \left\{ -\int_c d\mu \ \text{cth}(\mu) z(\mu) \right\}. \tag{122b}
\]
\[ \mathcal{A}(m, t) = \mathcal{A} \times \exp \left\{ - \int_\varepsilon^1 d\mu (\mu) \left[ m e(\mu) - t_R e'(\mu) \right] \right\}, \]  
(122b)

which depend parametrically on temperature and magnetic field as well. The contour \( \mathcal{C}' \) in (122a) is tightly enclosed by \( \mathcal{C} \).

Using this notation all amplitudes can be written as

\[ A_n^+(0) = \mathcal{A} \times \left[ \prod_{\lambda^h \in \mathcal{H}} \frac{2 \Phi(\lambda^h)}{a_n^h(\lambda^h | \kappa)} \right] \left[ \prod_{\lambda^p \in \mathcal{P}} \frac{2 \Phi(\lambda^p)}{a_n^p(\lambda^p | \kappa)} \right] \mathcal{D}(\mathcal{H}, \mathcal{P}), \]  
(123)

where \( \mathcal{H} = \{ \lambda^h \}_{j=1}^n \) and \( \mathcal{P} = \{ \lambda^p \}_{k=1}^{n-1} \) are sets of hole and particle rapidities, i.e. sets of zeros of \( 1 + a_n \) located inside \( \mathcal{C} \) or \( \mathcal{C} + i\pi/2 \), respectively. For the eigenvalue ratios equation (100) implies that

\[ \rho_n(\lambda) = \prod_{\lambda^h \in \mathcal{H}} i \theta h(\lambda^h - \lambda) \prod_{\lambda^p \in \mathcal{P}} i \theta h(\lambda^p - \lambda) \exp \left\{ - \int_\varepsilon d\mu (\mu) e(\mu - \lambda) z(\mu) \right\}. \]  
(124)

Inserting (123) and (124) into (33) we obtain

\[ \langle \sigma_1 \sigma_{m+1}^+(t) \rangle_T = e^{-iht} \times \lim_{N \to \infty} \mathcal{A}(m, t) \sum_{\mathcal{H}, \mathcal{P}} \left[ \prod_{\lambda^h \in \mathcal{H}} \frac{2 \Phi(x)}{a_n^h(x | \kappa)} \left( i \theta h(x) \right)^m \left( \frac{\theta h(x - t_R/N)}{\theta h(x + t_R/N)} \right)^\frac{N}{2} \right] \times \left[ \prod_{\lambda^p \in \mathcal{P}} \frac{2 \Phi^{-1}(x)}{a_n^p(x | \kappa)} \left( i \theta h(x) \right)^{-m} \left( \frac{\theta h(x - t_R/N)}{\theta h(x + t_R/N)} \right)^{-\frac{N}{2}} \right] \mathcal{D}(\mathcal{H}, \mathcal{P}), \]  
(125)

where the sum is over all sets of particles and holes. This sum can be easily transformed into a sum over multiple integrals. The discussion about the singularities of the integrands parallels the discussion below equation (105). In particular, we shall assume that \( m > 0 \). Then we rewrite the sum in (125) as a sum over multiple integrals, introduce the one-particle energy (97) and momentum function (111) and finally perform the limits \( N \to \infty \) and \( \varepsilon \to 0 \). We arrive at

\[
\langle \sigma_1 \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{A}(m, t) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \int_{\varepsilon}^1 \frac{dx_r}{\pi i} \Phi_-(x_r) e^{i(mp(x_r) - te(x_r))} \frac{1 - e^{-e(x_r)}}{1 - e^{-\varepsilon(x_r)}} \times \int_{\varepsilon}^1 \prod_{s=1}^{n-1} \frac{dy_s}{\pi i} \Phi_-(y_s) \left[ 1 - e^{-e(y_s)/\varepsilon} \right] \mathcal{D}\left( \{x_r\}_{r=1}^n, \{y_s\}_{s=1}^{n-1} \right). \]  
(126)

By \( \Phi_- \) we mean the boundary values of \( \Phi \) from inside the contour. \( \mathcal{C} \) is the particle contour which can be chosen as \( \mathcal{C} = \mathcal{C} + i\pi/2 \). In this case the regularization introduced above is important. By \( \mathcal{A}(m, t) \) and \( \Phi_-(x) \) we now mean the functions obtained from (122b) and (120) in the Trotter limit, i.e. by replacing \( a_0(\lambda) \) by \( e^{-\varepsilon(\lambda)/T} \). Equation (126) provides a novel form-factor series for the transverse two-point functions of the XX chain. A detailed analysis of this series will be presented in a separate work.

4 Conclusions

We have devised a thermal form-factor approach to the dynamical correlation functions of fundamental integrable lattice models at finite temperature. This approach provides thermal
form-factor series expansions of the two-point correlation functions of these models. The summands in the series are determined by ratios of eigenvalues of the quantum transfer matrix and by amplitudes, which are products of two thermal form factors. For finite Trotter number both, the eigenvalue ratios and the amplitudes, depend on time in the dynamical case. But at least for the XXZ chain this time dependence vanishes in the Trotter limit in which eigenvalue ratios and amplitudes are given by the well-known expressions studied in [4,5,8].

Hence, for the XXZ chain, the remaining question is how to evaluate the series. With the simplest possible example, namely the two-point functions of the XX chain, we have shown that the summation can be efficiently performed. We have reproduced the existing results for the longitudinal case and have derived a novel form-factor series in the transverse case that will be the starting point for further studies. In the general XXZ case we have suggested how the sums in every sector of fixed particle and hole numbers can be rewritten as multiple-contour integrals. Our formula will remain a conjecture until we have gained deeper insight into the concrete construction of the integration contours.

In future work we plan to proceed with the general XXZ case. In the most generic situation of arbitrary times and distances at arbitrary temperatures we expect that some computer effort will remain. For small and large temperature we hope to obtain explicit results for the long-time and large-distance asymptotics. An important goal of our future work will be to develop a physical intuition for the behaviour of thermal correlation functions, particularly for long times and large distances.

Acknowledgments. FG, MK and AK acknowledge financial support by the DFG in the framework of the research group FOR 2316 and through grant number Go 825/9-1. FG wishes to thank the ENS de Lyon and Shizuoka University for hospitality. KKK is supported by CNRS. JS is grateful for support by a JSPS Grant-in-Aid for Scientific Research (C) No. 15K05208.
Appendix A: A proof of the inversion formulae for the quantum transfer matrix

In this appendix we provide a proof of equation (21). Without restriction of generality we may replace $N$ by $N - 1$ in the definition of the staggered monodromy matrix (17). For every odd $j \in \{1, \ldots, 2N\}$ we introduce cyclically reordered staggered monodromy matrices

$$T_{a;j,...,2N,1,...,j-1}(\lambda|\alpha) =$$

$$R^{t_1}_{j-1,a}(\nu_{j-1},\lambda) \cdots R_{a,1}(\lambda,\nu_1)\Theta_a(\alpha)R^{t_1}_{2N,a}(\nu_{2N},\lambda) \cdots R_{a,j}(\lambda,\nu_j), \quad (A.1a)$$

$$T_{a;j-1,...,2N,1,...,j-2}(\lambda|\alpha) =$$

$$R_{a,j-2}(\lambda,\nu_{j-2}) \cdots R_{a,1}(\lambda,\nu_1)\Theta_a(\alpha)R^{t_1}_{2N,a}(\nu_{2N},\lambda) \cdots R^{t_1}_{j-1,a}(\nu_{j-1},\lambda). \quad (A.1b)$$

Then $T_a(\lambda|\alpha) = T_{a;1,...,2N}$.

Step 1. Using (2a) we obtain

$$\text{tr}_a \{ x_a T_{a;j,...,2N,1,...,j-1}(\nu_j|\alpha) \} = x_j R^{t_1}_{j-1,j}(\nu_{j-1},\nu_j) \cdots R_{j,1}(\nu_j,\nu_1)\Theta_j(\alpha)R^{t_1}_{2N,j}(\nu_{2N},\nu_j) \cdots R^{t_1}_{j+1,j}(\nu_{j+1},\nu_j)$$

$$= x_j t(\nu_j|\alpha). \quad (A.2)$$

If $j = 1$, (A.2) reads

$$\text{tr}_a \{ x_a T_a(\nu_1|\alpha) \} = x_1 t(\nu_1|\alpha) \quad (A.3)$$

which proves (21) for $j = 1$.

Step 2. For $j$ odd and $j > 1$ we have

$$t(\nu_j|\alpha) = \text{tr}_a \{ R_{a,j-2}(\lambda,\nu_{j-2}) \cdots R_{a,1}(\lambda,\nu_1)\Theta_a(\alpha)$$

$$\times R^{t_1}_{2N,a}(\nu_{2N},\lambda) \cdots R^{t_1}_{j-1,a}(\nu_{j-1},\lambda) \} \bigg|_{\lambda = \nu_{j-1}}$$

$$= \text{tr}_a \{ R_{j-1,a}(\nu_{j-1},\lambda)R^{t_1}_{a,j}(\lambda,\nu_j) \cdots R_{2N,a}(\nu_{2N},\lambda)\Theta_a(\alpha)$$

$$\times R^{t_1}_{a,1}(\lambda,\nu_1) \cdots R^{t_1}_{a,j-2}(\lambda,\nu_{j-2}) \} \bigg|_{\lambda = \nu_{j-1}}$$

$$= R^{t_1}_{j-1,j}(\nu_{j-1},\nu_j) \cdots R_{2N,j-1}(\nu_{2N},\nu_{j-1})\Theta_j(\alpha)$$

$$\times R^{t_1}_{j-1,1}(\nu_{j-1},\nu_1) \cdots R^{t_1}_{j-1,j-2}(\nu_{j-1},\nu_{j-2}) \bigg|_{\lambda = \nu_{j-1}}. \quad (A.4)$$

Here we have used the cyclicity of the trace in the first equation, (2c) in the second equation and (2a) in the third equation. The invariance equation (13) implies that

$$\Theta_j(\alpha)R^{t_1}_{j-1,a}(\nu_{j-1},\lambda)\Theta_a(\alpha) = \Theta_a(\alpha)R^{t_1}_{j-1,a}(\nu_{j-1},\lambda)\Theta_j(\alpha). \quad (A.5)$$

Using (A.4), (A.5), (1), (2b) and (20) it is easy to see that

$$t(\nu_j|\alpha) T_{a;j,...,2N,1,...,j-1}(\lambda|\alpha) = T_{a;j-1,...,2N,1,...,j-2}(\lambda|\alpha) t(\nu_j|\alpha). \quad (A.6)$$

Similarly using the representation of $t(\nu_{j-2}|\alpha)$ that can be read off from (A.2) with $x = \text{id}$ as well as (1), (2b), (13) and (20) we obtain

$$T_{a;j-1,...,2N,1,...,j-2}(\lambda|\alpha) t(\nu_{j-2}|\alpha) = t(\nu_{j-2}|\alpha) T_{a;j-2,...,2N,1,...,j-3}(\lambda|\alpha). \quad (A.7)$$
The last two equations can be combined to

\[
T_{\alpha,j,\ldots,2N,1,\ldots,j-1}(\lambda|\alpha) = t(\nu_{j-2}|\alpha)t^{-1}(\nu_{j-1}|\alpha)T_{\alpha,j-2,\ldots,2N,1,\ldots,j-3}(\lambda|\alpha)t(\nu_{j-1}|\alpha)t^{-1}(\nu_{j-2}|\alpha).
\] (A.8)

It follows by iteration that

\[
T_{\alpha,j,\ldots,2N,1,\ldots,j-1}(\lambda|\alpha) = \prod_{k=1}^{(j-1)/2} t(\nu_{2k-1}|\alpha)t^{-1}(\nu_{2k}|\alpha)T_{\alpha}(\lambda|\alpha) \prod_{k=1}^{(j-1)/2} t(\nu_{2k}|\alpha)t^{-1}(\nu_{2k-1}|\alpha).
\] (A.9)

Inserting this into (A.2) and replacing \(N\) by \(N + 1\) we have established (21).

**Appendix B: Invertibility of the inhomogeneous shift operators for the XXZ chain**

In the previous appendix we have seen that for odd \(j\) the operators

\[
t(\nu_j|\alpha) = R_{j-1,j}(\nu_{j-1},\nu_j) \cdots R_{j,1}(\nu_j,\nu_1)\Theta_j(\alpha)R_{2N,j}(\nu_{2N},\nu_j) \cdots R_{j+1,j}(\nu_{j+1},\nu_j)
\] (B.1a)

\[
t(\nu_{j-1}|\alpha) = R_{j-1,j}(\nu_{j-1},\nu_j) \cdots R_{2N,j-1}(\nu_{2N},\nu_{j-1})\Theta_{j-1}(\alpha)
\times R_{j-1,1}(\nu_{j-1},\nu_1) \cdots R_{j-1,j-2}(\nu_{j-1},\nu_{j-2})
\] (B.1b)

shift the monodromy matrix indices cyclically if applied from the left or right, respectively. In order to establish sufficient criteria for the invertibility of the inhomogeneous shift operators for the XXZ chain we calculate their determinants using the specific form of \(\Theta(\alpha)\) and of the \(R\)-matrix \(\Box\). We will employ the formula \(\det(\text{id}_m \otimes A) = (\det(A))^m\), valid for \(A \in \text{End}(\mathbb{C}^n)\) if \(\text{id}_m\) is the identity in \(\text{End}(\mathbb{C}^n)\).

Taking the determinant in (B.1a) we obtain

\[
\det(t(\nu_j|\alpha)) = \det(\Theta_j(\alpha)) \prod_{k=1}^{N \atop k \neq (j+1)/2} \det(R_{j,2k-1}(\nu_j,\nu_{2k-1})) \times \prod_{k=1}^{N} \det(R_{2k,j}(\nu_{2k},\nu_j)).
\] (B.2)

The site indices can be shifted by means of permutation operators \(P_{jk} = e_{j}^{\beta}e_{k}^{\alpha}\), where \(e_{\alpha}^{\beta}\) are the canonical matrix units having a single non-zero matrix element one in the \(\alpha\)th row and \(\beta\)th column. The \(P_{jk}\) are invertible, since \(P_{jk}^2 = \text{id}\) which also implies that \(\det^2(P_{jk}) = 1\). It follows that

\[
\det(P_{jk}) = \det(P_{2N-1,2N}) = (\det(P))^{2N-2} = 1,
\] (B.3)

since \(\det P = -1\) and \(N\) is even. Hence,
\[
\det(t(\nu_j|\alpha)) = (\det(\Theta(\alpha)))^{2N-1} \left[ \prod_{k=1}^{N} \det(R(\nu_j,\nu_{2k-1})) \right]^{2N-2} \times \left[ \prod_{k=1}^{N} \det(R^t(\nu_{2k},\nu_j)) \right]^{2N-2}.
\] (B.4)

Now
\[
\det(\Theta(\alpha)) = 1, \quad \det(R(\lambda,\mu)) = \frac{\text{sh}(\lambda - \mu + i\gamma)}{\text{sh}(\lambda - \mu - i\gamma)}, \quad \text{(B.5)}
\]

\[
\det(R^t(\lambda,\mu)) = \frac{\text{sh}^3(\lambda - \mu)}{\text{sh}^4(\lambda - \mu - i\gamma)}. \quad \text{(B.6)}
\]

Thus,
\[
\det(t(\nu_j|\alpha)) = \left[ \prod_{k=1}^{N} \frac{\text{sh}(\nu_j - \nu_{2k-1} + i\gamma) \text{sh}^3(\nu_{2k} - \nu_j) \text{sh}(\nu_{2k} - \nu_j - 2i\gamma)}{\text{sh}(\nu_j - \nu_{2k-1} - i\gamma) \text{sh}^4(\nu_{2k} - \nu_j - i\gamma)} \right]^{22N-2}. \quad \text{(B.7)}
\]

Similarly
\[
\det(t(\nu_{j-1}|\alpha)) = \left[ \prod_{k=1}^{N} \frac{\text{sh}(\nu_{2k} - \nu_{j-1} + i\gamma) \text{sh}^3(\nu_{j-1} - \nu_{2k-1}) \text{sh}(\nu_{j-1} - \nu_{2k-1} - 2i\gamma)}{\text{sh}(\nu_{2k} - \nu_{j-1} - i\gamma) \text{sh}^4(\nu_{j-1} - \nu_{2k-1} - i\gamma)} \right]^{22N-2}. \quad \text{(B.8)}
\]

From the latter two equations we infer that
\[
|\nu_k| < \gamma/2, \quad k = 1, \ldots, 2N, \quad \text{(B.9a)}
\]

\[
\nu_{2j-1} \neq \nu_{2k}, \quad j, k = 1, \ldots, N, \quad \text{(B.9b)}
\]

is a set of sufficient conditions for all inhomogeneous shift operators connected with the inhomogeneous quantum transfer matrix of the XXZ model to be invertible.

**Appendix C: Deforming the contour in a norm determinant**

In order to deform the contour in \( \det_{d_{m_0,\mathcal{C}_n}} \{1 - \hat{K}\} \) to \( \mathcal{C}_{0,s} \) we consider the action of \( 1 - \hat{K} \) on a function \( f \) that is holomorphic on \( \mathcal{C}_n - \mathcal{C}_{0,s} \),

\[
(1 - \hat{K}) f(x) = f(x) - \int_{\mathcal{C}_n} \frac{dy}{2\pi i} K(x - y)f(y) \frac{1 + a_n(y|\kappa)}{-a_n(y|\kappa)}
\]

\[
= f(x) + \sum_{k=1}^{n_k} \frac{K(x - x_k)f(x_k)}{a'_n(x_k|\kappa)} - \sum_{k=1}^{n_p} \frac{K(x - y_k)f(y_k)}{a'_n(y_k|\kappa)} - \int_{\mathcal{C}_{0,s}} \frac{dy}{2\pi i} K(x - y)f(y) \frac{1 + a_n(y|\kappa)}{-a_n(y|\kappa)}. \quad \text{(C.1)}
\]

The latter equation allows us to interpret \( 1 - \hat{K} \) as a linear operator acting on functions supported on the set \( \mathcal{C}_{0,s} \cup \{x_j\}_{j=1}^{n_h} \cup \{y_j\}_{j=1}^{n_p} \). Then the Fredholm determinant can be interpreted as

\[
\det_{d_{m_0,\mathcal{C}_n}} \{1 - \hat{K}\} = \text{(C.2)}
\]
Appendix D: Details of the partial summation of the form-factor series for the transverse two-point functions of the XXZ chain

In this appendix we present a more detailed derivation of equation (90). Special attention will be payed to the cancellation of the singularities in the integrand and to the issue of spurious singularities related with possibly unphysical solutions of the subsidiary conditions. In order to achieve a cancellation of the singularities which is compatible with the usual formulation of the multiple-residue theorem [1][32] we shall explicitly extract the poles of the functions $1 + a(\cdot|\{u\}, \{v\}, \kappa)$ and $1 + \overline{a}(\cdot|\{u\}, \{v\}, \kappa)$. Concerning the issue of spurious singularities, so far we can only state a set of assumptions that would exclude them.

$$\ln(\overline{a}(\lambda|\{u\}, \{v\}, \kappa)) = -2i\gamma\kappa - i\pi s - hR e(-\lambda)/T - i \sum_{j=1}^{n_h} \theta(\lambda - u_j) + i \sum_{j=1}^{n_p} \theta(\lambda - v_j)$$

$$+ \int_{c_{0,s}} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + \overline{a}(\mu|\{u\}, \{v\}, \kappa))$$

for the reciprocal $\overline{a} = 1/a$ of the auxiliary function at $u_k = x_k$ and $v_k = y_k$ and comparing with (C.4). Then equation (79) of the main text follows from (C.5) and (C.6).

### Appendix D: Details of the partial summation of the form-factor series for the transverse two-point functions of the XXZ chain

When considering the partial summation of the form-factor series for the transverse two-point functions of the XXZ chain, we will be payed to the cancellation of the singularities in the integrand and to the issue of spurious singularities related with possibly unphysical solutions of the subsidiary conditions. In order to achieve a cancellation of the singularities which is compatible with the usual formulation of the multiple-residue theorem [1][32] we shall explicitly extract the poles of the functions $1 + a(\cdot|\{u\}, \{v\}, \kappa)$ and $1 + \overline{a}(\cdot|\{u\}, \{v\}, \kappa)$. Concerning the issue of spurious singularities, so far we can only state a set of assumptions that would exclude them.

$$\ln(\overline{a}(\lambda|\{u\}, \{v\}, \kappa)) = -2i\gamma\kappa - i\pi s - hR e(-\lambda)/T - i \sum_{j=1}^{n_h} \theta(\lambda - u_j) + i \sum_{j=1}^{n_p} \theta(\lambda - v_j)$$

$$+ \int_{c_{0,s}} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + \overline{a}(\mu|\{u\}, \{v\}, \kappa))$$

for the reciprocal $\overline{a} = 1/a$ of the auxiliary function at $u_k = x_k$ and $v_k = y_k$ and comparing with (C.4). Then equation (79) of the main text follows from (C.5) and (C.6).
In the following, the interior of the contours \( C_{0,s}, \overline{C}_{0,s} \) will be denoted \( \Omega = \text{Int} C_{0,s}, \overline{\Omega} = \text{Int} \overline{C}_{0,s}. \) It is convenient to combine the sets \( \{ u \} = \{ u_j \}_{j=1}^{n_u} \) and \( \{ v \} = \{ v_j \}_{j=1}^{n_v} \) with \( u_j \in \Omega, v_k \in \overline{\Omega} \) and the twist parameter \( \kappa' \) into triples \( M = (\{ u \}, \{ v \}, \kappa') \) and to write \( a(\cdot;\{ u \}, \{ v \}, \kappa') = a(\cdot;\{ M \}) \) for short. Similarly, solutions \( \{ x \}, \{ y \} \) of the subsidiary conditions (76) and \( \kappa' \) will be combined into \( \mathcal{Z} = (\{ x \}, \{ y \}, \kappa') \) such that \( a(\cdot;\{ x \}, \{ y \}, \kappa') = a(\cdot;\{ \mathcal{Z} \}). \) We also introduce the notation

\[
\chi(\lambda|\mathcal{M}) = - \int_{\mathcal{C}_{0,s}} \frac{\mathrm{d}\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a(\mu|\mathcal{M})) \quad \text{(D.1)}
\]

and restrict ourselves to \( s = 1 \) in the following.

Thanks to (75), the auxiliary function is presented as a ratio

\[
a(\lambda|\mathcal{M}) = \frac{e^{\chi(\lambda|\mathcal{M})} g(\lambda|\mathcal{M})}{h(\lambda|\mathcal{M})}, \quad \text{(D.2)}
\]

where the two functions

\[
g(\lambda|\mathcal{M}) = (-1)^{q-2\kappa'} \delta(\lambda) \left[ \prod_{u \in \{ u \}} \sh(\lambda - u + i\gamma) \right] \left[ \prod_{v \in \{ v \}} \sh^2(\lambda - v + i\gamma) \right], \quad \text{(D.3a)}
\]

\[
h(\lambda|\mathcal{M}) = \alpha(\lambda) \left[ \prod_{u \in \{ u \}} \sh(\lambda - u + i\gamma) \right] \left[ \prod_{v \in \{ v \}} \sh(\lambda - v + i\gamma) \sh(\lambda - v + i\gamma) \right] \quad \text{(D.3b)}
\]

are holomorphic in \( \mathbb{C}. \) Above,

\[
\alpha(\lambda) = \prod_{k=1}^{N+1} \sh(\lambda - \nu_{2k}) \sh(\lambda - \nu_{2k+1} - i\gamma), \quad \text{(D.4a)}
\]

\[
\delta(\lambda) = \prod_{k=1}^{N+1} \sh(\lambda - \nu_{2k-1}) \sh(\lambda - \nu_{2k} - i\gamma), \quad \text{(D.4b)}
\]

and \( \nu_k, k = 1, \ldots, 2N + 2, \) are defined according to (28).

After elementary manipulations based on (D.2), the amplitude \( A_n^{-+}(\xi|\kappa, \kappa') \) in (68) can be expressed as

\[
A_n^{-+}(\xi|\kappa, \kappa') = (-1)^{n_{\kappa}} \frac{\prod_{x \in \{ x^{(n)} \}} \delta(x) \prod_{y \in \{ y^{(n)} \}} \alpha(y)}{\det \begin{vmatrix} \partial_{u_j} y(u_j|\mathcal{M}) & \partial_{v_j} y(u_j|\mathcal{M}) \\ \partial_{u_k} y(v_k|\mathcal{M}) & \partial_{v_k} y(v_k|\mathcal{M}) \end{vmatrix}_{\mathcal{M} = \mathcal{Z}_n}} \quad \text{\mathcal{F}^{-+}(\mathcal{Z}_n).} \quad \text{(D.5)}
\]

Here \( \mathcal{Z}_n = (\{ x^{(n)} \}, \{ y^{(n)} \}, \kappa' \) is the triple associated with the solution \( \{ x^{(n)} \}, \{ y^{(n)} \} \) to the subsidiary condition (76). Furthermore, we have introduced

\[
y(\lambda|\mathcal{M}) = e^{\chi(\lambda|\mathcal{M})} g(\lambda|\mathcal{M}) + h(\lambda|\mathcal{M}) \quad \text{(D.6)}
\]

and the function \( \mathcal{F}^{-+}(\mathcal{Z}_n) \) is of the form

\[
\mathcal{F}^{-+}(\mathcal{Z}_n) = \left( \mathcal{H} \cdot \mathcal{D} \cdot \mathcal{W} \cdot \mathcal{E}_1 \cdot \mathcal{E}_2 \right)(\mathcal{Z}_n) \quad \text{(D.7)}
\]

with
\[ \mathcal{H}(M) = \frac{G^+(\xi)G^-(\xi)}{(q^{1+\kappa'}-q^{-1-\kappa'}) (q^{\kappa'-\kappa} - q^{-\kappa'+\kappa})} \times \frac{\det_{\mathcal{D}_{m+}} e_n \{1 - K_{1-\kappa'}\} \det_{\mathcal{D}_{m-}} e_n \{1 - K_{1+\kappa}\}}{\det_{\mathcal{D}_{m_0}} e_{n_0} \{1 - K\} \det_{\mathcal{D}_{m_0}} e_{n_0} \{1 - K\}} \prod_{x \in \{u\}} (-1)^s q^{-2\kappa'} \chi(x|\mathcal{M}) \]

(D.8)

and

\[ \mathcal{D}(M) = \left[ \frac{\prod_{u \neq u' \in \{u\}} \text{sh}(u - u')} {\prod_{u \in \{u\}} \prod_{v \in \{v\}} \text{sh}(u - v)} \right], \quad \text{(D.9a)} \]

\[ \mathcal{W}(M) = (-1)^{\alpha_p} \prod_{u \in \{u\}} \prod_{v \in \{v\}} \text{sh}^2(u - v + i\gamma) \text{sh}(v - u + i\gamma) \]

\[ \prod_{u, u' \in \{u\}} \text{sh}(u - u' + i\gamma). \quad \text{(D.9b)} \]

The functions \( G^\pm \) and the Fredholm determinants are defined similar to equations (68)-71 and (72a)-72b of the main text: one only has to replace \( \alpha(\cdot|\mathcal{Z}) \) by \( \alpha(\cdot|\mathcal{M}) \) and \( \rho_\kappa(\cdot|\mathcal{K}, \kappa') \) by \( \rho(\cdot|\mathcal{M}) \). Moreover, writing \( \mathcal{O} = (\emptyset, \emptyset, \kappa) \) and

\[ z(\lambda|\mathcal{M}) = \frac{\ln(1 + a)(\lambda|\mathcal{Z}) - \ln(1 + a)(\lambda|\mathcal{M})}{2\pi i} \]

we define

\[ \mathcal{E}_1(M) = (-1)^s q^{\kappa - \kappa'} \exp \left\{ - \int_{\mathcal{O}_{v_0}} d\lambda \int_{\mathcal{O}_{v_0}} d\mu \, e'(\lambda - \mu) z(\lambda|\mathcal{M}) z(\mu|\mathcal{M}) \right\} \]

\[ \times \prod_{\mu \in \{v\}} \exp \left\{ \int_{\mathcal{O}_{v_0}} d\lambda \left( e(\lambda - \mu) - e(\mu - \lambda) \right) z(\lambda|\mathcal{M}) \right\} \]

\[ \times \prod_{\mu \in \{u\}} \exp \left\{ - \int_{\mathcal{O}_{v_0}} d\lambda \left( e(\lambda - \mu) - e(\mu - \lambda) \right) z(\lambda|\mathcal{M}) \right\}, \quad \text{(D.11)} \]

where \( \mathcal{O}_{v_0} \subset \mathcal{O}_{v_0} \) infinitesimally, and

\[ \mathcal{E}_2(M) = \prod_{w \in \{u\} \cup \{v\}} \left[ \prod_{u \in \{u\}} \text{sh}(w - u + i\gamma) \right] \left[ \prod_{v \in \{v\}} \text{sh}(v - w + i\gamma) \right] \]

\[ - (-1)^s q^{2(\kappa - \kappa')} \text{e}^{\lambda(\alpha(\cdot|\mathcal{Z}) - \alpha(\cdot|\mathcal{M}))} \left[ \prod_{w \in \{u\}} \text{sh}(w - u + i\gamma) \right] \left[ \prod_{v \in \{v\}} \text{sh}(w - v + i\gamma) \right]. \]

(D.12)

Inserting (D.5) into our general expression (33) for the form factor series of the two-point functions we obtain

\[ \langle \sigma_1^- \sigma_{m+1}^+ (t) \rangle_T = \lim_{N \to \infty} \sum_{\ell=1}^{\infty} \sum_{z_n \in \{x^{(n)}\}} \left[ \delta(w) \prod_{w \in \{u\}} \alpha(w) \right] \rho_N^{\alpha}(0|M) \rho_N^{\alpha}(t_R/N|M) \rho_N^{\alpha}(-t_R/N|M) \left. \frac{\partial_{x_k} y(u_j |M) \partial_{x_k} y(v_j |M)}{\partial_{x_k} y(u_j |M) \partial_{x_k} y(v_j |M)} \right|_{M=Z_n} \right. \]

\[ \times \left[ \prod_{w \in \{u\}} \delta(w) \prod_{w \in \{v\}} \alpha(w) \right] \rho_N^{\alpha}(0|M) \rho_N^{\alpha}(t_R/N|M) \rho_N^{\alpha}(-t_R/N|M) \left. \right|_{M=Z_n}. \quad \text{(D.13)} \]
Above, the sum runs, for fixed $\ell$, over all solutions $\{x^{(n)}\}$, $\{y^{(n)}\}$ to the subsidiary conditions such that $|\{x^{(n)}\}| = |\{y^{(n)}\}| + 1 = \ell$. The Jacobian put aside, the remaining functions in the summand, viz. the factors

$$\mathcal{F}^{-}(M) \left[ \prod_{z \in \{u\}} \delta(z) \right] \left[ \prod_{z \in \{v\}} \alpha(z) \right] \rho^{m(0)|M|} \rho^{N} \left( t_{R}/N |M| \right) \rho^{-N} \left( -t_{R}/N |M| \right), \quad (D.14)$$

are already supposed to be analytic in $(u, v)$ belonging to the natural domains $\Omega^{n_{h}} \times \Omega^{n_{p}}$ for the hole-type variables $u = (u_{1}, \ldots, u_{n_{h}})$ and for the particle-type variables $v = (v_{1}, \ldots, v_{n_{p}})$, where $n_{p} = n_{h} - 1$. The explicit poles of $\rho \left( t_{R}/N |M| \right) \rho^{-1} \left( -t_{R}/N |M| \right)$ exist at $u_{j} = -t_{R}/N (1 \leq j \leq n_{h})$ and at $v_{k} = i\gamma - t_{R}/N (1 \leq k \leq n_{p})$. See (89). They are canceled by the zeros of $\alpha(v_{k}) \pm \delta(u_{j})$. The linear equation (70) tells that explicitly poles present in $\mathcal{G}^{+}_{0}(0)$, resp. $\mathcal{G}^{-}_{0}(0)$ are located at $u_{j} = 0$, resp. $v_{k} = i\gamma$. They are also canceled by the zeros of $\rho^{m(0)|M|}$, provided that $m > 0$. In principle, $\mathcal{F}^{-}(M)$ could also contain some singularities stemming from the Fredholm determinants occurring in $\mathcal{H}(M)$. However, we do not expect such kind of complication and simply assume that it does not occur.

The summation in (D.13) is equivalent to summing up all solutions of the equation $\mathcal{Y}(z|M) = 0$ with $z \in \{u\} \cup \{v\}$, provided that the two summands $e^{\lambda|\mathcal{M}|} g(\lambda|M)$ and $h(\lambda|M)$ do not vanish simultaneously on a solution. In principle, such a situation might occur, e.g. due to the presence of the common factor $\prod_{v \in \{v\}} \frac{1}{2} \frac{v - \gamma}{2\pi} h(\lambda + i\gamma)$ in $h$ and $g$. We will make however the assumption that, even if existing, such solutions do not contribute to the form factor series, for instance because these also correspond to zeros of $\mathcal{F}^{-}(M)$ which are not manifestly appearing in the formula or simply because these do not generate a multi-dimensional residue. Based on this assumption we can use the multi-dimensional residue formula (1132) to recast the sum into the form

$$\langle \sigma_{1}^{-} \sigma_{m+1}^{+}(t) \rangle_{T} = \lim_{\varepsilon \to 0} \sum_{\ell = 1}^{\infty} \frac{(-1)^{\ell} e^{-i\hbar t}}{\ell! (\ell - 1)!} \int_{S_{y}} \frac{d\ell d\ell - 1}{(2\pi)^{2\ell - 1}} \frac{\mathcal{F}^{-}(M)}{\prod_{w \in \{u\} \cup \{v\}} \mathcal{Y}(w|M)} \times \left[ \prod_{w \in \{u\}} \delta(w) \right] \left[ \prod_{w \in \{v\}} \alpha(w) \right] \rho^{m(0)|M|} \rho^{N} \left( t_{R}/N |M| \right) \rho^{-N} \left( -t_{R}/N |M| \right). \quad (D.15)$$

Here $\varepsilon > 0$, and $\delta_{y}^{(\ell, \varepsilon)}$ is the skeleton of $\mathcal{Y}$ defined as

$$\delta_{y}^{(\ell, \varepsilon)} = \left\{ (u, v) \in \Omega^{\ell} \times \Omega^{\ell - 1} | |\mathcal{Y}(u_{j}|M)| = |\mathcal{Y}(v_{k}|M)| = \varepsilon \right\}. \quad (D.16)$$

We assume that this skeleton can be deformed into $\mathcal{E}_{0,1}^{\ell, \varepsilon} \times \mathcal{E}_{0,1}^{\ell - 1, \varepsilon}$. Then the series representation (90) in the main text easily follows.

Appendix E: Details of the derivation of the form-factor expansions for the XX chain

For our derivation of form-factor formulae for the XX chain that are suitable for taking the Trotter limit we recall Slavnov’s scalar product formula for the XXZ model [34]. For any set of Bethe roots $\{\lambda_{j}^{(n)}\}_{j=1}^{M}$ and its associated auxiliary function $a_{n}$ it takes the form

$$\langle \Psi_{n}|B(\mu_{M}) \cdots B(\mu_{1})|0\rangle = \langle 0|C(\mu_{1}) \cdots C(\mu_{M})|\Psi_{n}\rangle$$
Thus, if \( \{ \mu_j \}_{j=1}^M \) is still arbitrary in this formula. Sending \( \mu_j \to \lambda_j^{(n)} \) we get the ‘norm formula’ for the eigenstates \( \langle 49 \rangle \).

### E.1 Longitudinal case

Equation (E.1) implies a formula for the ratio of the scalar product divided by the ‘square of the norm’ which in the XX limit simplifies due to \( [95] \).

\[
\frac{\langle \Psi_n | B(\mu_M) \ldots B(\mu_1) | 0 \rangle}{\langle \Psi_n | \Psi_n \rangle} = \left[ \prod_{j=1}^M \frac{a(\mu_j)}{a(\lambda_j^{(n)})} \frac{\prod_{k=1}^M \text{ch}(\lambda_j^{(n)} - \mu_k)}{\text{ch}(\lambda_j^{(n)} - \lambda_k^{(n)})} \right] \times \left[ \prod_{1 \leq j < k \leq M} \frac{\text{sh}(\lambda_j^{(n)} - \lambda_k^{(n)})}{\text{sh}(\mu_j - \mu_k)} \right] \times \det_M \left\{ \frac{2(1 + a_n(\mu | \kappa))}{\text{sh}(2(\mu - \lambda_j^{(n)}))} \right\}. \tag{E.2}
\]

If now \( B(\mu_M) \ldots B(\mu_1) | 0 \rangle \) is a Bethe vector as well, it has the same pseudo-spin \( s \) as \( \langle \Psi_n \rangle \), since it has the same number of Bethe roots. It follows that

\[
2(1 + a_n(\mu | \kappa)) = \lim_{\mu \to \mu_k} \frac{\text{sh}(2(\mu - \lambda_j^{(n)}))}{\text{sh}(2(\mu - \lambda_j^{(n)}))} = 0 \tag{E.3}
\]

unless \( \mu_k = \lambda_j^{(n)} \) for some \( j \), in which case

\[
\lim_{\mu \to \mu_k} \frac{2(1 + a_n(\mu | \kappa))}{\text{sh}(2(\mu - \lambda_j^{(n)}))} = a'_n(\lambda_j^{(n)} | \kappa). \tag{E.4}
\]

Thus, if \( \mu_k \notin \{ \lambda_j^{(n)} \}_{j=1}^M \), then a row in the determinant on the right hand side of equation (E.2) vanishes, and the normalized scalar product is equal to zero. It follows that \( \langle \Psi_n | \Psi_m \rangle = 0 \) for any two different sets of Bethe roots \( \{ \lambda_j^{(n)} \}_{j=1}^M \) and \( \{ \lambda_j^{(m)} \}_{j=1}^M \) (which proves completeness since the number of solutions equals the dimension of the Hilbert space and \( \langle \Psi_n | \Psi_n \rangle \neq 0 \)).

Using this fact as well as the Yang-Baxter algebra relations \( [19] \) we conclude that

\[
\frac{\langle \Psi_n | A(\xi) | \Psi_m \rangle}{\langle \Psi_n | \Psi_n \rangle} = \sum_{j=1}^M a(\lambda_j^{(m)}) b(\xi, \lambda_j^{(m)}) \left[ \prod_{k=1}^M \frac{1}{b(\lambda_k^{(m)}, \lambda_j^{(m)})} \right] \langle \Psi_n | B(\xi) | \prod_{k=1, k\neq j}^M B(\lambda_k^{(m)}) | 0 \rangle \tag{E.5}
\]

if \( \{ \lambda_j^{(n)} \}_{j=1}^M \) and \( \{ \lambda_j^{(m)} \}_{j=1}^M \) are two different solutions of the Bethe Ansatz equations \( 1 + a_n(\lambda | \kappa) = 0 \). Following the same reasoning as above, each term on the right hand
we denote them by \( \lambda_{j}^{(m)} \) for \( j = 1, \ldots, M - 1 \). For such states (E.2) and (E.5) imply that

\[
\frac{\langle \Psi_n | A(\xi) | \Psi_m \rangle}{\langle \Psi_n | \Psi_n \rangle} = a(\lambda_{j}^{(m)}) \frac{e(\xi - \lambda^h)}{b(\xi, \lambda_{j}^{(m)})} \left[ \prod_{i=1}^{M-1} \frac{1}{b(\lambda_{j}^{(n)}, \lambda_{i}^{(m)})} \right] \frac{a(\lambda_{j}^{(m)}) a_0(\lambda_{j}^{(m)} | \kappa)}{a(\lambda_{j}^{(m)}) a_0(\lambda_{j}^{(m)} | \kappa)} a(\xi - \lambda^h) \left[ 1 + a_n(\xi | \kappa) \right].
\]  

(E.6)

In order to calculate the amplitudes \( A_n \) we consider two cases. First, \( \lambda_{j}^{(n)} \) are the Bethe roots of the dominant state. Then \( s = 0 \) and \( \lambda^p := \lambda_{j}^{(m)} \) has imaginary part larger than \( \pi/4 \), hence is a particle, while \( \lambda^h := \lambda_{j}^{(n)} \) is missing in the set \( \{ \lambda_{j}^{(m)} \}_{j=1}^{M} \) and is a hole. It follows that

\[
\frac{\langle \Psi_0 | A(\xi) | \Psi_n \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = a(\lambda_{j}^{(m)}) \frac{e(\xi - \lambda^h)}{a_0(\lambda^h | \kappa)} \frac{a(\lambda^p)}{a_0(\lambda^p | \kappa)} \left[ \prod_{j=1}^{M-1} \frac{-i e(\lambda_{j}^{(n)} - \lambda^h)}{e(\lambda_{j}^{(m)} - \lambda^h)} \frac{e(\lambda_{j}^{(n)} - \lambda^h)}{e(\lambda_{j}^{(m)} - \lambda^h)} \right].
\]  

(E.7)

In the second case we take \( \{ \lambda_{j}^{(m)} \}_{j=1}^{M} \) as the Bethe roots of the dominant state. Then \( \{ \lambda_{j}^{(n)} \}_{j=1}^{M} \) has to describe an excited state with one particle \( \lambda^p = \lambda_{j}^{(n)} \) and one hole \( \lambda^h = \lambda_{j}^{(m)} \). Thus,

\[
\frac{\langle \Psi_n | A(\xi) | \Psi_0 \rangle}{\langle \Psi_n | \Psi_n \rangle} = a(\lambda_{j}^{(m)}) \frac{e(\xi - \lambda^h)}{a_0(\lambda^h | \kappa)} \frac{a(\lambda^p)}{a_0(\lambda^p | \kappa)} \left[ \prod_{j=1}^{M-1} \frac{-i e(\lambda_{j}^{(n)} - \lambda^h)}{e(\lambda_{j}^{(m)} - \lambda^h)} \frac{e(\lambda_{j}^{(n)} - \lambda^h)}{e(\lambda_{j}^{(m)} - \lambda^h)} \right].
\]  

(E.8)

Multiplying (E.7) and (E.8) and using the formula (48) for the eigenvalues we arrive at

\[
\frac{\langle \Psi_0 | A(\xi) | \Psi_n \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n | A(\xi) | \Psi_0 \rangle}{\langle \Psi_n | \Psi_n \rangle} = \frac{e(\xi - \lambda^h)}{a_0(\lambda^h | \kappa)} \frac{a(\lambda^p)}{a_0(\lambda^p | \kappa)},
\]  

(E.9)

which is our final expression for the amplitudes at finite Trotter number. The Trotter limit and the limit \( \xi \to 0 \) are determined by equation (97).

E.2 Transverse case

In the transverse case we have to evaluate the amplitudes \( A_{n}^{-}(\xi) \) defined in equation (118) of the main text. Let us suppress superscripts for short in this section and denote the Bethe roots of the dominant state simply by \( \{ \lambda_j \}_{j=1}^{M} \), \( M = N + 1 \). The only excited states \( |\Psi_n \rangle \) that lead to non-zero amplitudes are those with \( M - 1 \) Bethe roots. In this section we denote them by \( \{ \mu_k \}_{k=1}^{M-1} \). They correspond to spin \( s = 1 \) (see (51)) and are zeros of
1 + a_n, where a_n = -a_0. Using the Slavnov formula (E.1) and the equation (48) for the
eigenvalues and setting γ = π/2 we find that

\[ A_n^{+, -}(\xi) = (-1)^{M-1} \frac{1 + a_0(\xi|\kappa)}{1 + a_n(\xi|\kappa)} \left[ \prod_{j=1}^{M} \frac{2}{a_0(\lambda_j|\kappa)} \right] \left[ \prod_{k=1}^{M-1} \frac{2}{a_n(\mu_k|\kappa)} \right] \]

\[ \times \prod_{j=1}^{M-1} \frac{\text{sh}(2(\xi - \mu_j))}{\text{sh}(2(\xi - \lambda_j))} \mathcal{D}(\{\lambda_j\}_{j=1}^{M}, \{\mu_k\}_{k=1}^{M-1}) \]  

(E.10)

where for any two mutually distinct sets of complex numbers \( \mathcal{D} \) is defined in (121).

We would like to rewrite (E.10) in a form that allows us to take the Trotter limit. Let us start with introducing some useful notation. Let

\[ \mathcal{B}_s = \left\{ \lambda \in \mathbb{C} \mid a_0(\lambda|\kappa) = (-1)^{s+1}, \text{Im} \lambda \leq \pi/4, \text{Re} \lambda < 0 \text{ if Im} \lambda = \pi/4 \right\} \]

(E.11)

where \( s = 0, 1 \). Then \( \mathcal{B}_0 = \{\lambda_j\}_{j=1}^{M} \) is the set of Bethe roots of the dominant state. We further define sets \( \mathcal{P} \) of ‘particles’ and \( \mathcal{H} \) of ‘holes’ by

\[ \mathcal{P} = \{\mu_k\}_{k=1}^{M-1} \setminus \{\mathcal{B}_1 \cap \{\mu_k\}_{k=1}^{M-1}\} \]

\[ \mathcal{H} = \mathcal{B}_1 \setminus \{\mathcal{B}_1 \cap \{\mu_k\}_{k=1}^{M-1}\} \]

(E.12)

All pseudo-spin 1 excitations are uniquely classified by set of particles and holes. Slightly
abusing the notion of the difference of two sets \( x \) and \( \gamma \) we introduce the notation

\[ \prod_{\lambda \in x \setminus \gamma} f(\lambda) = \frac{\prod_{\lambda \in x} f(\lambda)}{\prod_{\lambda \in \gamma} f(\lambda)} \]

(E.13)

which will turn out to be convenient in the following calculations. We shall also make use
of the function

\[ 1_{\text{condition}} = \begin{cases} 1 & \text{if condition is satisfied} \\ 0 & \text{else} \end{cases} \]

(E.14)

Using the above defined notation we separate the particles and holes from the products in (E.10),

\[ A_n^{+, -}(\xi) = (-1)^{M-1} \frac{1 + a_0(\xi|\kappa)}{1 + a_n(\xi|\kappa)} \left[ \prod_{\lambda \in \mathcal{B}_0} \frac{2}{a_0(\lambda|\kappa)} \right] \left[ \prod_{\lambda \in \mathcal{P} \setminus \mathcal{H}} \frac{2}{a'_n(\lambda|\kappa)} \right] \]

\[ \times \left[ \prod_{\lambda \in \mathcal{B}_1 \setminus \mathcal{B}_0} \text{sh}(2(\xi - \lambda)) \right] \left[ \prod_{\lambda \in \mathcal{P} \setminus \mathcal{H}} \text{sh}(2(\xi - \lambda)) \right] \left[ \prod_{\mu \in \mathcal{B}_1 \setminus \mathcal{B}_0} \text{sh}^2(\lambda - \mu) \right] \]

\[ \times \mathcal{D}(\mathcal{B}_0, \mathcal{B}_1) \mathcal{D}(\mathcal{P}, \mathcal{H}) \]  

(E.15)

Then logarithms of products over \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) can be transformed into integrals involving
the auxiliary functions \( a_0 \) and \( a_n \). We choose a point \( x_R \) on \( \mathcal{C} \) in such a way that \( z(\lambda) \)
defined in (119) is continuous on \( \mathcal{C} \) with the possible exception of \( \lambda = x_R \). It follows that

\[ \prod_{\lambda \in \mathcal{B}_1 \setminus \mathcal{B}_0} \text{sh}(2(\xi - \lambda)) = \text{sh}^{M_1 - M}(2(\xi - x_R)) \left( \frac{1 + a_n(\xi|\kappa)}{1 + a_0(\xi|\kappa)} \right)^{1_{\xi \in \text{Int}(\mathcal{C})}} \]
where \( M_1 = |B_1| \) is the number of zeros of \( 1 + a_n \) inside \( \mathcal{C} \) and \( z \) is inside \( \mathcal{C} \). By \( \text{Int}(\mathcal{C}) \) we mean the interior of the contour \( \mathcal{C} \). With the remaining products over \( B_0 \) and \( B_1 \) we proceed in a similar way. We have to be careful with the omissions though. To treat them properly we introduce a small regularization parameter \( \delta \in \mathbb{C} \). We have

\[
\prod_{\mu \in B_1 \setminus B_0} \text{sh}(\lambda - \mu + \delta) = \text{sh}^{M_1-M}(\lambda-x_R+\delta) \left(1 + a_n(\lambda + \delta|\kappa)\right) \frac{1}{1+a_0(\lambda + \delta|\kappa)}
\times \exp \left\{ -2 \int_\mathcal{C} d\mu \, \text{cth}(\lambda - \mu + \delta) z(\mu) \right\},
\]

(E.16)

and therefore

\[
\prod_{\lambda \in \mathcal{P}\setminus\mathcal{C}} \prod_{\mu \in B_1 \setminus B_0} \text{sh}^2(\lambda - \mu) = \lim_{\delta \to 0} \delta^{2|\mathcal{P}|} \prod_{\lambda \in \mathcal{P}\setminus\mathcal{C}} \prod_{\mu \in B_1 \setminus B_0} \text{sh}^2(\lambda - \mu + \delta)
\times \exp \left\{ -2 \int_\mathcal{C} d\mu \, \text{cth}(\lambda - \mu + \delta) z(\mu) \right\}
\]

\[
= \left[ \prod_{\lambda \in \mathcal{C}} \left( \frac{2}{a_n'(\lambda|\kappa)} \right)^2 \right] \prod_{\lambda \in \mathcal{P}\setminus\mathcal{C}} \text{sh}^{2(M_1-M)}(\lambda-x_R) \exp \left\{ -2 \int_\mathcal{C} d\mu \, \text{cth}(\lambda - \mu) z(\mu) \right\}.
\]

(E.17)

Similarly

\[
\mathcal{D}(B_0, B_1) = (-1)^{M_0} \lim_{\delta \to 0} \delta^{-M-M_1} \prod_{\lambda \in B_1 \setminus B_0} \prod_{\mu \in B_1 \setminus B_0} \text{sh}(\lambda - \mu + \delta)
= (-1)^{M_0} \lim_{\delta \to 0} \delta^{-M-M_1} \prod_{\lambda \in B_1 \setminus B_0} \text{sh}^{M_1-M}(\lambda-x_R+\delta) \left(1 + a_n(\lambda + \delta|\kappa)\right) \frac{1}{1+a_0(\lambda + \delta|\kappa)}
\times \exp \left\{ -\int_\mathcal{C} d\mu \, \text{cth}(\lambda - \mu + \delta) z(\mu) \right\}
\]

\[
= (-1)^{M_0} \left[ \prod_{\lambda \in B_0} \frac{a_0'(\lambda|\kappa)}{2} \right] \left[ \prod_{\lambda \in B_1} \frac{a_n'(\lambda|\kappa)}{2} \right] \left[ \prod_{\lambda \in \mathcal{C}} \text{sh}^{M_1-M}(\lambda-x_R) \right]
\exp \left\{ -\int_{\mathcal{C}'} d\lambda \int_\mathcal{C} d\mu \, \text{cth}'(\lambda - \mu) z(\mu) z(\lambda) + (M_1-M) \int_\mathcal{C} d\mu \, \text{cth}(\mu-x_R) z(\mu) \right\},
\]

(E.18)

where \( \mathcal{C}' \) is inside \( \mathcal{C} \) in such a way that it still encloses all \( \lambda \in B_0 \) and no other zeros of \( 1 + a_0 \), and where

\[
M_0 = \frac{1}{2} (M - M_1)(M + M_1 - 1) + MM_1.
\]

(E.19)
At least for large enough Trotter number the choice of the contour $\mathcal{C}$ as depicted in figure 2 assures that $M_1 = M$. It follows that $M - |\mathcal{C}| + |\mathcal{P}| = M - 1$ implying that $|\mathcal{C}| - |\mathcal{P}| = s = 1$. When $M_1 = M$, many of the above expressions simplify. Inserting the simplified expressions into (E.15) we arrive at equation (123) of the main text.

References

[1] I. A. Aizenberg and A. P. Yuzhakov, Integral representations and residues in multidimensional complex analysis, Translations of mathematical monographs, vol. 58, American Mathematical Soc., Providence, RI, 1983.

[2] H. Boos and F. Göhmann, On the physical part of the factorized correlation functions of the XXZ chain, J. Phys. A 42 (2009), 315001.

[3] M. Dugave, Formfaktorzugang zu thermischen Korrelationsfunktionen der Heisenbergkette, Ph.D. thesis, Bergische Universität Wuppertal, 2015.

[4] M. Dugave, F. Göhmann, and K. K. Kozlowski, Thermal form factors of the XXZ chain and the large-distance asymptotics of its temperature dependent correlation functions, J. Stat. Mech.: Theor. Exp. (2013), P07010.

[5] ______, Low-temperature large-distance asymptotics of the transversal two-point functions of the XXZ chain, J. Stat. Mech.: Theor. Exp. (2014), P04012.

[6] M. Dugave, F. Göhmann, K. K. Kozlowski, and J. Suzuki, Low-temperature spectrum of correlation lengths of the XXZ chain in the antiferromagnetic massive regime, J. Phys. A 48 (2015), 334001.

[7] ______, On form factor expansions for the XXZ chain in the massive regime, J. Stat. Mech.: Theor. Exp. (2015), P05037.

[8] ______, Thermal form factor approach to the ground-state correlation functions of the XXZ chain in the antiferromagnetic massive regime, J. Phys. A 49 (2016), 394001.

[9] K. Fabricius and B. M. McCoy, Spin diffusion and the spin-1/2 XXZ chain at $T = \infty$ from exact diagonalization, Phys. Rev. B 57 (1998), 8340.

[10] F. Göhmann, A. Klümper, and A. Seel, Integral representations for correlation functions of the XXZ chain at finite temperature, J. Phys. A 37 (2004), 7625.

[11] F. Göhmann and V. E. Korepin, Solution of the quantum inverse problem, J. Phys. A 33 (2000), 1199.

[12] F. Göhmann and A. Seel, XX and Ising limits in integral formulae for finite temperature correlation functions of the XXZ chain, Theor. Math. Phys. 146 (2006), 119.

[13] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, Form factors of the monodromy matrix entries in gl(2|1)-invariant integrable models, Nucl. Phys. B 911 (2016), 902.
[14] ______. _Norm of Bethe vectors in models with gl(m|n) symmetry_, preprint, arXiv:1705.09219, 2017.

[15] A. R. Its, A. G. Izergin, V. E. Korepin, and N. Slavnov, _Temperature correlations of quantum spins_, Phys. Rev. Lett. 70 (1993), 1704.

[16] A. G. Izergin, A. R. Its, V. E. Korepin, and N. Slavnov, _The matrix Riemann-Hilbert problem and differential equations for correlation functions of the XX0 Heisenberg chain_, St. Petersburg Math. J. 6 (1995), 315.

[17] A. G. Izergin, N. Kitanine, J. M. Maillet, and V. Terras, _Spontaneous magnetization of the XXZ Heisenberg spin-\(\frac{1}{2}\) chain_, Nucl. Phys. B 554 (1999), 679.

[18] X. Jie, _The large time asymptotics of the temperature correlation functions of the XX0 Heisenberg ferromagnet: The Riemann-Hilbert approach_, Ph.D. thesis, Indiana University Purdue University Indianapolis, 1998.

[19] S. Katsura, T. Horiguchi, and M. Suzuki, _Dynamical properties of the isotropic XY model_, Physica 46 (1970), 67.

[20] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, _On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain_, J. Math. Phys. 50 (2009), 095209.

[21] ______. _A form factor approach to the asymptotic behavior of correlation functions in critical models_, J. Stat. Mech.: Theor. Exp. (2011), P12010.

[22] ______. _Form factor approach to dynamical correlation functions in critical models_, J. Stat. Mech.: Theor. Exp. (2012), P09001.

[23] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, _Master equation for spin-spin correlation functions of the XXZ chain_, Nucl. Phys. B 712 (2005), 600.

[24] N. Kitanine, J. M. Maillet, and V. Terras, _Form factors of the XXZ Heisenberg spin-\(\frac{1}{2}\) finite chain_, Nucl. Phys. B 554 (1999), 647.

[25] A. Klümper, _Thermodynamics of the anisotropic spin-1/2 Heisenberg chain and related quantum chains_, Z. Phys. B 91 (1993), 507.

[26] A. Klümper and K. Sakai, _The thermal conductivity of the spin-1/2 XXZ chain at arbitrary temperature_, J. Phys. A 35 (2002), 2173.

[27] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, _Quantum inverse scattering method and correlation functions_, Cambridge University Press, 1993.

[28] E. H. Lieb, T. Schultz, and D. Mattis, _Two soluble models of an antiferromagnetic chain_, Ann. Phys. (N.Y.) 16 (1961), 407.

[29] T. Niemeijer, _Some exact calculations on a chain of spins \(\frac{1}{2}\)_, Physica 36 (1967), 377.

[30] S. Pakuliak, E. Ragoucy, and N. A. Slavnov, _GL(3)-based quantum integrable composite models. I. Bethe vectors_, SIGMA 11 (2015), 063, 20pp.
[31] ______, GL(3)-based quantum integrable composite models. II. Form factors of local operators, SIGMA 11 (2015), 064, 18pp.

[32] R. M. Range, Holomorphic functions and integral representations in several complex variables, second, revised ed., Graduate Texts in Mathematics, Springer Verlag, Berlin, 1998.

[33] K. Sakai, Dynamical correlation functions of the XXZ model at finite temperature, J. Phys. A 40 (2007), 7523.

[34] N. A. Slavnov, Calculation of scalar products of the wave functions and form factors in the framework of the algebraic Bethe ansatz, Teor. Mat. Fiz. 79 (1989), 232.

[35] M. Suzuki, Transfer-matrix method and Monte Carlo simulation in quantum spin systems, Phys. Rev. B 31 (1985), 2957.

[36] V. Tarasov and A. Varchenko, Completeness of Bethe vectors and difference equations with regular singular points, Int. Math. Res. Notices 13 (1995), 637.