NON-RELATIVISTIC GLOBAL LIMITS OF THE ENTROPY SOLUTIONS TO THE RELATIVISTIC EULER EQUATIONS WITH $\gamma$-LAW

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(Communicated by Gui-Qiang Chen)

ABSTRACT. We analyze the limit as the speed of light $c \to \infty$ of the global entropy solutions of the $2 \times 2$ relativistic Euler equations for the state $p = \kappa^2 \rho^\gamma$ $(1 < \gamma < 2)$, and find that the limit is the entropy solution of the corresponding non-relativistic Euler equations.

1. Introduction. The relativistic Euler equations for a perfect fluid in two-dimensional Minkowski space-time have the form (cf. [3, 4, 14, 21, 22, 23, 24, 25]):

\[
\begin{cases}
\partial_t \left( \frac{(p + \rho c^2) v^2}{c^2 - v^2} + \rho \right) + \partial_x \left( \frac{(p + \rho c^2) v}{c^2 - v^2} \right) = 0, \\
\partial_t \left( \frac{(p + \rho c^2) v}{c^2 - v^2} \right) + \partial_x \left( \frac{(p + \rho c^2) v^2}{c^2 - v^2} + p \right) = 0,
\end{cases}
\]

(1.1)

where $v = v(x, t)$ is the classical velocity of the fluid, $\rho = \rho(x, t)$ is the mass-energy density of the fluid, $p = p(\rho)$ is the pressure, and $c$ is the speed of light.

The equation of state is

\[ p = p(\rho), \]

where $p(\rho)$ is a smooth function of $\rho$. For a perfect gas,

\[ p(\rho) = \kappa^2 \rho^\gamma \quad \gamma \geq 1, \tag{1.2} \]

where $\kappa > 0$ is constant, $\gamma$ is the adiabatic exponent, $\gamma = 1$ models a barotropic (or isothermal) gas, and $\gamma > 1$ a polytropic gas.

The corresponding physical region is

\[ V = \{ U = (\rho, v) : 0 \leq \rho < \rho_{\text{max}}, |v| < c \}, \tag{1.3} \]

where

\[ \rho_{\text{max}} = \sup \{ \rho : p'(\rho) \leq c^2 \}, \]

which means that the sound speed $\sqrt{p'(\rho)}$ is less than the light speed $c$. For perfect gases governed by (1.2), $\rho_{\text{max}} = \infty$ for $\gamma = 1$ and $\rho_{\text{max}} = \left( \frac{c^2}{\gamma \kappa^2} \right)^{\frac{1}{\gamma - 1}} < \infty$ for $\gamma > 1$. 

2000 Mathematics Subject Classification. Primary: 35B40, 35A05, 76Y05; Secondary: 35B35, 35L65, 85A05.

Key words and phrases. Relativistic Euler equations, Riemann solutions, uniqueness, Lorentz transformation.

This research was partially supported by the Chinese National Natural Science Foundation under Grant #10571120 and Shanghai Natural Science Foundation under Grant #04ZR14090.
For the Cauchy problem of system (1.1) with initial data
\[ t = 0 : \rho = \rho_0(x), v = v_0(x), \] (1.4)
Smoller-Temple [21] established the existence of global BV solutions when \( \gamma = 1 \) for any large initial data with bounded variation. Chen [4] established the existence of global BV solutions when \( \gamma > 1 \) for the initial data satisfying
\[ (\gamma - 1)TV\{\rho_0, v_0\} \leq C, \]
for some \( C > 0 \) independent of \( \gamma \). For other related problems about system (1.1), we refer the reader to [1, 2, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18] and the references cited therein.

On the other hand, the classical non-relativistic isentropic Euler system is
\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p) &= 0.
\end{align*}
\] (1.5)
One of the motivations of this paper is that the classical mechanics is regarded as the limit of the relativistic mechanics when \( c \to \infty \), and in particular, it is easy to check that the relativistic Euler system (1.1) reduces formally to system (1.5) when \( c \to \infty \). However, rigorous mathematical proof of the singular limit for global weak solutions has been an open challenging mathematical problem. Min-Ukai [15] discussed the global convergence of weak solutions of (1.1) when \( \gamma = 1 \) as \( c \to \infty \), and proved that the limit is a weak solution of (1.5). The purpose of this paper is to establish the same results for entropy solutions when \( \gamma > 1 \).

Our main result is

**Main Theorem.** Suppose \( \rho_0(x) \) and \( v_0(x) \) are independent of \( c \) and satisfy
\[ 0 < \underline{\rho} \leq \rho_0(x) \leq \overline{\rho} < \infty, \quad |v_0(x)| < c. \]
Then there exists a constant \( M_0 > 0 \) such that, when
\[ (\gamma - 1)TV\{\rho_0, v_0\} < M_0, \]
there exists \( c_0 > 0 \) such that, for any \( c \geq c_0 \), there is an \( \mathcal{L}^\infty \) entropy solution \((\rho^c, v^c)\) of (1.1) and (1.4) satisfying
\[ TV\{\rho^c(\cdot, t)\} + TV\{v^c(\cdot, t)\} \leq M, \]
for all \( t \geq 0 \), where \( M \) is a constant depending only on the initial data \((\rho_0, v_0)\) but independent of \( c > c_0 \). Moreover, there exists a subsequence \( \{c_k\} \), \( c_k \to \infty \) as \( k \to \infty \), such that
\[ \rho^{c_k} \to \rho, \quad v^{c_k} \to v \] (1.6)
strongly in \( L^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \) as \( k \to \infty \), and the limit \((\rho, v)\) is an entropy solution of the Cauchy problem (1.5) and (1.4).

The key point in the proof of the main theorem is based on the total variation estimates, independent of large \( c \), on the approximate solutions constructed by the Glimm scheme. To achieve the desired estimates, we need to improve the estimates in [4] for the wave length in the approximate solutions, since the bounds in [4] for the total variations of the approximate solutions depend on the light speed \( c \).

We organize this paper as follows. In \( \S 2 \), we review some basic and important properties of the system. In \( \S 3 \), we study the Riemann problem, analyze the global geometric behavior of nonlinear waves, and obtain some estimates for wave lengths independent of large \( c \). In \( \S 4 \) and \( \S 5 \), we define the approximate solutions of the
Cauchy problem based on the Glimm scheme and establish some essential estimates independent of $c \geq c_0$ on the approximate solutions. Finally, we use the $BV$-compactness to get the convergence, thus to prove the main theorem.

2. The Relativistic Euler System. In this section we review some basic properties and important features of system (1.1), and introduce the notion of entropy solutions.

We rewrite system (1.1) in the general form of conservation laws

$$U_t + F(U)_x = 0,$$

with initial data

$$U(x, 0) = U_0(x),$$

by setting

$$U = (U_1, U_2)^\top = \left(\frac{(\kappa^2 \rho^\gamma + \rho c^2) v^2}{c^2(v^2 - c^2)} + \rho, \frac{(\kappa^2 \rho^\gamma + \rho c^2) v^2}{c^2 - v^2}\right)^\top,$$

$$F(U) = (F_1(U), F_2(U))^\top = \left(\frac{(\kappa^2 \rho^\gamma + \rho c^2) v^2}{c^2 - v^2}, \frac{(\kappa^2 \rho^\gamma + \rho c^2) v^2}{c^2 - v^2} + \kappa^2 \rho^\gamma\right)^\top,$$

and

$$U_0(x) = U(\rho_0(x), v_0(x)),$$

where we already plug $p = \kappa^2 \rho^\gamma$ into the equations in (1.1).

For the mapping $(\rho, v) \to (U_1, U_2)$, it is easy to have

**Lemma 2.1.** The mapping $(\rho, v) \to (U_1, U_2)$ is 1-1, and the Jacobian of the mapping is continuous and non-zero in the region $V \cap \{\rho > 0\}$. Moreover, the convergence

$$U \to (\rho, \rho v)^\top, \quad F(U) \to (\rho v, \rho v^2 + \kappa^2 \rho^\gamma)^\top \quad \text{as} \quad c \to \infty,$$

are uniform in any bounded region $\{0 < \rho < M; \; |v| < \varpi < c\}$, where $M$ and $\varpi$ are positive constants.

The eigenvalues of system (1.1) are real and distinct:

$$\lambda_1 = \frac{v - \kappa \sqrt{\gamma \rho^\gamma - 1}}{1 - \kappa \sqrt{\gamma \rho^\gamma - 1}}, \quad \lambda_2 = \frac{v + \kappa \sqrt{\gamma \rho^\gamma - 1}}{1 + \kappa \sqrt{\gamma \rho^\gamma - 1}},$$

and the corresponding right eigenvectors are

$$r_j(u) = \left(\frac{-1)^j}{c^2 - v^2}, \frac{\kappa \sqrt{\gamma \rho^\gamma - 1}}{c^2 \rho^\gamma + \rho c^2}\right)^\top, \quad j = 1, 2,$$

so system (1.1) is strictly hyperbolic and genuinely nonlinear.

The Riemann invariants of system (1.1) are defined as

$$s = \frac{1}{2} c \ln \frac{c + v}{c - v} + 2c \frac{\sqrt{\gamma}}{\gamma - 1} \arctan \frac{\kappa \rho^{(\gamma - 1)/2}}{c},$$

$$r = \frac{1}{2} c \ln \frac{c + v}{c - v} - 2c \frac{\sqrt{\gamma}}{\gamma - 1} \arctan \frac{\kappa \rho^{(\gamma - 1)/2}}{c}.$$
Here we remark that the Riemann invariants \( r \) and \( s \) defined by (2.7) and (2.8) are different from those in [4], so that the limit of our pair makes sense as \( c \to \infty \) and converges to the pair of Riemann invariants for system (1.5):

\[
\begin{align*}
    s' &= v + \frac{2\kappa \sqrt{\gamma}}{\gamma - 1} \rho^{(\gamma-1)/2}, \\
    r' &= v - \frac{2\kappa \sqrt{\gamma}}{\gamma - 1} \rho^{(\gamma-1)/2}.
\end{align*}
\]

(2.9) \hspace{1cm} (2.10)

It is also easy to see that

Lemma 2.2. The mapping \((\rho, v) \to (s, r)\) is \(1-1\), and the Jacobian of this mapping is nonzero in the region \( \mathcal{V} \cap \{ \rho > 0 \} \).

So we can choose \((\rho, v)\), or \((s, r)\), according to our convenience, as a coordinate system.

One of the important features of system (1.1) is its Lorentz invariance. If a barred coordinates \((\bar{t}, \bar{x})\) moves with velocity \(\tau\) as measured in the unbarred coordinates \((t, x)\), and if \(v\) denotes the velocity of a particle as measured in the unbarred frame, and \(\bar{v}\) denotes the velocity of a particle as measured in the barred frame, then under the Lorentz transformation (see [3]):

\[
\begin{align*}
    \bar{x} &= \frac{cx - ct \tau}{\sqrt{1 - \tau^2 c^2}}, \\
    \bar{t} &= \frac{ct - \tau c x}{\sqrt{1 - \tau^2 c^2}},
\end{align*}
\]

we have

\[
v = \frac{\tau + \bar{v}}{1 + \frac{\tau^2 c^2}{c^2}}.
\]

(2.11)

Furthermore, we have [4]

Lemma 2.3. The quantity \(\ln\left(\frac{c + v_0}{c - v_0}\right) - \ln\left(\frac{c + v_1}{c - v_0}\right)\) is invariant under (2.11).

We recall that an entropy-entropy flux pair for (2.1) is a pair of \(C^1\) functions \((\eta(U), q(U))\) satisfying

\[
\nabla \eta(U) \cdot \nabla F(U) = \nabla q(U).
\]

In particular, it is easy to check that the following pair \((\eta_*(U), q_*(U))\)

\[
\begin{align*}
    \eta_*(U) &= -\frac{c^3}{\sqrt{c^2 - v^2}} \exp\left(c^2 \int_1^\rho \frac{ds}{\kappa^2 s^\gamma + c^2 s}\right) + c^2 U_1, \\
    q_*(U) &= \eta_*(U) v + c^3 F_1(U),
\end{align*}
\]

is the physical entropy-entropy flux pair of system (1.1) and \(c \to \infty\)

\[
(\eta_*(U), q_*(U)) \to \left(\frac{1}{2} \rho v^2 + \frac{\kappa^2 \rho^\gamma}{\gamma - 1}, \frac{1}{2} \rho v^3 + \frac{\kappa^2 \rho^\gamma v}{\gamma - 1}\right).
\]

(2.12) \hspace{1cm} (2.13)

which is the mechanical energy-energy flux pair of the non-relativistic system (1.5).

A direct calculation shows that \(\eta_*(U)\) is strictly convex in \(U\) in any compact domain of \(\mathcal{V} \cap \{ \rho > 0 \}\).

Definition 2.1. A bounded measurable function \(U(t, x)\) is an entropy solution of (1.1) and (1.4) in \(\Pi_T := [0, T) \times \mathbb{R}\) if \(U(t, x) \in \mathcal{V} \cap \{ \rho > 0 \}\) satisfies the following:

(i). Equations in (1.1) hold in the weak sense in \(\Pi_T\), i.e., for any \(\phi \in C^1_0(\Pi_T)\),

\[
\int_{\Pi_T} (U \partial_t \phi + F(U) \partial_x \phi) \, dx \, dt + \int_{-\infty}^\infty U_0(x) \phi(0, x) \, dx = 0;
\]

(2.14)
The shock curves

Lemma 3.1. \[ \rho, v \] by \((\eta_\ast(U)\partial_t\phi + q_\ast(U)\partial_x\phi) \) \ dx \ dt + \int_{-\infty}^{\infty} \eta_\ast(U_0(x))\phi(0, x) \ dx \geq 0, \] (2.15)

for the \( C^2 \) convex entropy pair \((\eta_\ast(U), q_\ast(U))\) defined by (2.12).

3. Riemann problem. The Riemann problem is a special Cauchy problem when the initial data \( U_0(x) \equiv U(\rho_0(x), v_0(x)) \) consists of constant states \( U_- \equiv U(\rho_-, v_-) \) and \( U_+ \equiv U(\rho_+, v_+) \) separated by a jump discontinuity at \( x = 0 \),

\[
U_0(x) = \begin{cases} U_-, & x < 0, \\ U_+, & x > 0, \end{cases}
\]

where

\[
(\rho_0(x), v_0(x)) = \begin{cases} (\rho_-, v_-), & x < 0, \\ (\rho_+, v_+), & x > 0. \end{cases}
\] (3.2)

We notice that, in view of Lemmas 2.1 and Lemma 2.2, \( U_-, U_+, (s_-, r_-) \), and \((s_+, r_+)\) of system (1.1) are uniquely determined by \((\rho_-, v_-)\) and \((\rho_+, v_+)\).

The Riemann problem can be solved in the class of functions consisting of constant states, separated by discontinuities determined by both Rankine-Hugoniot jump conditions

\[ \sigma(U) = [F], \]

and the Lax entropy conditions

\[ \sigma_1 < \lambda_1(U_l), \quad \lambda_1(U_r) < \sigma_1 < \lambda_2(U_r) \quad \text{on 1-shocks}, \]

\[ \lambda_1(U_l) < \sigma_2 < \lambda_2(U_l) \quad \sigma_2 > \lambda_2(U_r) \quad \text{on 2-shocks}. \]

(3.4) (3.5)

Here \([f] = f(U_l) - f(U_r)\) denotes the jump of the function \(f(U)\) between the left and right hand states along the curve of discontinuity in the \( x-t \) plane, while \(\sigma_1\) and \(\sigma_2\) represent the shock speed of 1-shock and 2-shock, respectively. Rarefaction waves are continuous solutions of the form \(U(x/t)\).

Due to the Lorentz invariance, we can always, without loss of generality, assume that the velocity state on the left-hand side of the shock wave is \(v_l = 0\). We denote by \((\rho, v)\) the state on the right-hand side \((\rho_r, v_r)\). The Lax entropy inequality and the Rankine-Hugoniot conditions imply that (see e.g. [3, 4, 12])

Lemma 3.1. The shock curves \(S_1\) and \(S_2\) are given by

\[ S_1 : \quad v = -c^2 \sqrt{\frac{(p - p_l)(\rho - \rho_l)}{(p + \rho_l c^2)(\rho_l + p c^2)}}, \quad \rho > \rho_l, \quad v < v_l = 0, \] (3.6)

\[ S_2 : \quad v = -c^2 \sqrt{\frac{(p - p_l)(\rho - \rho_l)}{(p + \rho_l c^2)(\rho_l + p c^2)}}, \quad 0 < \rho < \rho_l, \quad v < v_l = 0, \] (3.7)

where \(p = \kappa^2 \rho^\gamma\).

The rarefaction wave curves \(R_1\) and \(R_2\) are

\[ R_1 : \quad \frac{1}{2} \ln \frac{c + v}{c - v} + 2c \sqrt{\frac{\kappa \rho^{\gamma - 1/2}}{\gamma - 1}} \arctan \frac{c}{\sqrt{\gamma - 1}} = \text{const}, \quad 0 < \rho < \rho_l, \quad v > v_l = 0, \] (3.8)

\[ R_2 : \quad \frac{1}{2} \ln \frac{c + v}{c - v} - 2c \sqrt{\frac{\kappa \rho^{\gamma - 1/2}}{\gamma - 1}} \arctan \frac{c}{\sqrt{\gamma - 1}} = \text{const}, \quad \rho > \rho_l, \quad v > v_l = 0. \] (3.9)
Furthermore

\[
\frac{dv}{d\rho} < 0 \quad \text{on} \quad S_1, \quad \frac{dv}{d\rho} > 0 \quad \text{on} \quad S_2;
\]

\[
\frac{dv}{d\rho} < 0 \quad \text{on} \quad R_1, \quad \frac{dv}{d\rho} > 0 \quad \text{on} \quad R_2.
\]

The wave curves are sketched in Figure 1.

We further describe the nonlinear waves in the \( r - s \) plane. By definition (2.7)-(2.8) of the Riemann invariants, we know that, \( s \) is constant along the 1-rarefaction wave curve and \( r \) is constant along the 2-rarefaction wave curve. The geometric behavior of shock curves can be expressed for general pressure \( p \) with \( p' (\rho) > 0 \) and \( p'' (\rho) > 0 \).

**Lemma 3.2.** [4] For shock curves, it holds that

\[
0 \leq \frac{ds}{dr} < 1 \quad \text{on} \quad S_1; \quad 0 \leq \frac{dr}{ds} < 1 \quad \text{on} \quad S_2.
\]

We can depict the shock curves and the rarefaction wave curves in Fig. 2 in the \( r - s \) plane.

A standard analysis leads to the following existence theorem for the Riemann problem of system (1.1).

**Theorem 3.1.** [3, 4] Assume that \( p' > 0, p'' > 0 \) and that we are given initial data \((\rho_l, v_l)\) and \((\rho_r, v_r)\) where \( \rho_l > 0, \rho_r > 0 \) and \( -c < v_l, v_r < c \) for the Riemann problem of system (1.1). Let

\[
r_{\min} = \min (r(\rho_l, v_l), r(\rho_r, v_r)) \quad \text{and} \quad s_{\max} = \max (s(\rho_l, v_l), s(\rho_r, v_r)).
\]

If \( s_{\max} - r_{\min} < 2 \int_0^{s_{\max}} \sqrt{\frac{p'(s)}{p(s)+s}} ds \) and \( s(\rho_l, v_l) \geq r(\rho_r, v_r) \), then there exists a solution of the Riemann problem for system (1.1). The solution is unique in the class of constant states separated by rarefaction waves and shock waves.
The following lemma will be useful.

**Lemma 3.3.** [13] The regions
\[ \sum(\tilde{s}_0, \tilde{r}_0) = \{(\rho, v) : s \leq \tilde{s}_0, r \geq \tilde{r}_0, s - r \geq 0 \} \]
are invariant regions of the Riemann problem (1.1). That is, if the Riemann data lies in \( \sum(\tilde{s}_0, \tilde{r}_0) \), then the corresponding solution of the Riemann problem also lies in \( \sum(\tilde{s}_0, \tilde{r}_0) \).

Next, for simplicity and convenience, we let \( \gamma = 1 + 2\epsilon \), such that \( 0 \leq \epsilon < 1/2 \).

**Theorem 3.2.** Let \( 0 \leq \epsilon < 1/2 \), and consider two \( S_1 \) curves originating at the points \( (r_0, s_1) = (r(\rho_0, v_0), s(\rho_1, v_1)) \) and \( (r_0, s_0) = (r(\rho_0, v_0), s(\rho_0, v_0)) \), which are continued to the points \( (r, s_2) \) and \( (r, s) \), respectively (see figure 3). Then there exists a constant \( c_0 > 0 \) such that, for \( c > c_0 \),
\[ 0 \leq s_0 - s - (s_1 - s_2) \leq C\epsilon(s_1 - s_2)(r_0 - r), \] (3.10)
where \( C \) depends only on \( \epsilon \) and \( \rho_0, \rho_1 \in [\rho, \overline{\rho}] \), but independent of \( c > c_0 \). A similar estimate holds for \( S_2 \).

**Remark 3.1.** In Theorem 3.2, we have a constant \( C \) independent of large \( c \), which is of vital importance for this paper. To prove this, we need to check the constants in the proof of the corresponding theorem in [4].

**Remark 3.2.** In the Glimm scheme, the initial data are constants in small segments, and two neighboring segments give rise to a Riemann problem. Thus, if we let \( s_{\text{max}} \) and \( r_{\text{min}} \) be the maximum value of \( s \) and minimum value of \( r \) in that Riemann problem in the Glimm scheme, and let
\[ s_{\text{sup}} = \sup\{s(0, x), x \in \mathbb{R}\}, \; r_{\inf} = \inf\{r(0, x), x \in \mathbb{R}\}, \]
then we have
\[ s \leq s_{\text{max}} \leq s_{\text{sup}}, \; r \geq r_{\min} \geq r_{\inf}, \] (3.11)
Now we sketch the proof of Theorem 3.2.

Proof. When the point \((r_0, s_0)\) is given, the shape of the shock curve depends on \(\epsilon\), \(s\) is determined by \(r\) and certainly is a function of \(s_1\). We can express this as

\[ s_0 - s = (s_1 - s_2) = f(\epsilon, s_1, r). \]

Then domain \(D(f)\) of the function \(f\) is

\[ 0 \leq \epsilon \leq 1/2, \quad 0 \leq s_1 - s_0 \leq s_{\text{max}} - r_{\text{min}}, \quad 0 \leq r - r_0 \leq s_{\text{max}} - r_{\text{min}}. \quad (3.12) \]

From the definition of Riemann invariants, we know that \(D(f)\) is independent of large \(c\). Next we define

\[ g(\epsilon, s_1, r) = \frac{f(\epsilon, s_1, r)}{\epsilon(s_1 - s_0)(r - r_0)}. \quad (3.13) \]

Since the only possible singular points of \(g(\epsilon, s_1, r)\) are \(\epsilon = 0, s_1 - s_0 = 0\) and \(r - r_0 = 0\), we only need to show that the following limits:

\[ \lim_{\epsilon \to 0} \frac{f(\epsilon, s_1, r)}{\epsilon}, \quad \lim_{s_1 \to s_0} \frac{f(\epsilon, s_1, r)}{s_1 - s_0}, \quad \lim_{r \to r_0} \frac{f(\epsilon, s_1, r)}{r - r_0} \]

are bounded continuous functions of \((\epsilon, s_1, r)\).

We first consider \(\lim_{\epsilon \to 0} \frac{f(\epsilon, s_1, r)}{\epsilon}\). We notice that \(\rho_2\) and \(\rho\) are functions of \(\epsilon\), and, at the case \(\epsilon = 0\), \(\rho/\rho_1\) is uniquely determined by \(\Delta r\), i.e.,

\[ \frac{\rho_2}{\rho_1} = \frac{\rho}{\rho_0}. \quad (3.14) \]

So

\[ \ln \rho_2 - \ln \rho_1 = \ln \rho - \ln \rho_0. \quad (3.15) \]
Making a Taylor’s expansion of $f(\epsilon, s_1, r) = s_0 - s - (s_1 - s_2)$ at $\epsilon = 0$ gives
\[
f(\epsilon, s_1, r) = \frac{\kappa}{1 + (\frac{\kappa}{c})^2} \left( \ln(\rho_2(0)) - \ln(\rho_1) - \ln(\rho(0)) + \ln(\rho_0) \right) - \frac{1}{2} \kappa (\frac{\kappa}{c})^2 - 1 \left( \ln^2(\rho_2(0)) - \ln^2(\rho_1) - \ln^2(\rho(0)) + \ln^2(\rho_0) \right) \frac{\epsilon}{(1 + (\frac{\kappa}{c})^2)^2} \epsilon + O(\epsilon^2),
\]
where $0 < \kappa$ is a positive constant depending only on $\rho$. Then applying the fact that the numerator and denominator $| - \kappa^2(\frac{\kappa}{c})^2 - 1 | < \kappa$. Thus the coefficient of the second term is bounded and independent of $c$. Now we consider the coefficient of the third term:
\[
\frac{\kappa}{1 + (\frac{\kappa}{c})^2} \left( \frac{d\rho_2}{d\rho}(0) - \frac{d\rho_1}{d\rho}(0) \right).
\]
It is easy to see that $| - \frac{\kappa^2(\frac{\kappa}{c})^2 - 1}{1 + (\frac{\kappa}{c})^2} | < \kappa$. Next we consider $\frac{d\rho}{d\epsilon}/\rho$ at $\epsilon = 0$. We know from Figure 3 that $\Delta r$ is constant. Thus
\[
\frac{d\Delta r}{d\epsilon} = \frac{\partial \Delta r}{\partial \rho} \frac{d\rho}{d\epsilon} + \frac{\partial \Delta r}{\partial \epsilon} = 0,
\]
that is
\[
\frac{d\rho}{d\epsilon}|_{\epsilon=0} = - \frac{\partial \Delta r}{\partial \rho}|_{\epsilon=0}.
\]
We can calculate as in [4] that
\[
\frac{\partial \Delta r}{\partial \rho}|_{\epsilon=0} = - \frac{\kappa^2 \ln(\rho)\rho_0 + \kappa^2 \ln(\rho_0)\rho - c^2 \rho \ln(\rho) - c^2 \rho_0 \ln(\rho_0) \kappa^2}{(\kappa^2 + c^2)\sqrt{\kappa^2 \rho_0 + c^2 \rho_0} + c^2 \rho \sqrt{\kappa^2 + c^2}}
\]
\[
- \frac{\kappa^2}{2(c^2 + \kappa^2)^2} \ln \left( \frac{\rho}{\rho_0} \right) \left\{ (\kappa^2 - c^2) \ln(\rho_0) - 2(\kappa^2 + c^2) \right\}
\]
\[
\leq \mathcal{C}(\rho),
\]
where $\mathcal{C}$ is a positive constant depending only on $\rho$ but independent of $c$ when $c > c_0$ for some $c_0 > 0$ from Lemma 3.3 and the fact that the numerator and denominator have the same power of parameter $c$. Meanwhile,
\[
\frac{\partial \Delta r}{\partial \rho}|_{\epsilon=0} = \left\{ \frac{\partial}{\partial \rho} \left( \frac{1}{2} \frac{c-v}{c+v} \right) \frac{\gamma^{1/2}}{c} \frac{\partial}{\partial \rho} \left( \frac{\kappa^2 \gamma^\rho}{c^2} - \frac{\kappa^2 \gamma^\rho}{c^2} \right) \right\}|_{\epsilon=0}
\]
\[
= \left\{ \frac{\partial}{\partial \rho} \left( \frac{1}{2} \frac{c-v}{c+v} \right) \frac{\gamma^{1/2}}{c} \frac{\partial}{\partial \rho} \left( \frac{\kappa^2 \gamma^\rho}{c^2} - \frac{\kappa^2 \gamma^\rho}{c^2} \right) \right\}|_{\epsilon=0}
\]
\[
= \frac{c^2 v_\rho}{c^2 - v^2} + \frac{c^2 \sqrt{\kappa^2 \gamma^\rho c^2 + c^2 \rho}}{\kappa^2 \rho^2 + c^2 \rho}|_{\epsilon=0}
\]
\[
= \frac{c^2 \kappa}{\rho(\kappa^2 + c^2)} \geq \frac{\kappa}{2\rho},
\]
where we used Lemma 3.1. Thus we have
\[
\frac{d \rho(0)}{d \rho(0)} \leq \overline{C}(\overline{p}),
\]
where \(\overline{C}\) is a positive constant depending only on \(\overline{p}\) but independent of \(c\) when \(c > c_0\) for some \(c_0 > 0\). Therefore, \(\lim_{c \to 0} f(c, s_1, r)\) is a bounded continuous function of \(s_1\) and \(r\), and can be bounded by a constant which is independent of large \(c\).

Next we consider \(\lim_{s \to s_0} f(\varepsilon, s_1, r)\). We have
\[
\frac{f(s, s_1, r)}{s_s} = \frac{s_0 - s - (s_1 - s_2)}{s_1 - s_0} = -1 + \frac{s_2 - s}{s_1 - s_0}
\]
and \(r\) is constant, we have
\[
d_s = d \left\{ 2c \sqrt{\frac{\gamma}{\varepsilon}} \arctan \frac{\kappa \rho^{(\gamma-1)/2}}{c} \right\} = 2c^2 \sqrt{\frac{\kappa^2 \gamma \rho^{\gamma-1}}{\kappa^2 \rho^\gamma + c^2 \rho^0}} d\rho.
\]
Similar calculation gives
\[
ds_0 = 2c^2 \sqrt{\frac{\kappa^2 \gamma \rho^0^{\gamma-1}}{\kappa^2 \rho^0 + c^2 \rho^0}} d\rho_0.
\]
Thus
\[
\frac{d s}{d s_0} = \frac{\sqrt{\frac{\kappa^2 \gamma \rho^{\gamma-1}}{\kappa^2 \rho^\gamma + c^2 \rho^0}} d\rho}{\sqrt{\frac{\kappa^2 \gamma \rho^0^{\gamma-1}}{\kappa^2 \rho^0 + c^2 \rho^0}} d\rho_0}
\]
Since
\[
\Delta r = \frac{1}{2} \ln \left( \frac{c - v}{c + v} \right) - \frac{v}{2} \frac{c}{\varepsilon} \arctan \left( \frac{\kappa \rho}{c} \right) + \frac{\gamma^{1/2}}{2} \frac{c}{\varepsilon} \arctan \left( \frac{\kappa \rho^0}{c} \right)
\]
is constant,
\[
d \left\{ \frac{1}{2} \ln \left( \frac{c - v}{c + v} \right) - \frac{v}{2} \frac{c}{\varepsilon} \arctan \left( \frac{\kappa \rho}{c} \right) + \frac{\gamma^{1/2}}{2} \frac{c}{\varepsilon} \arctan \left( \frac{\kappa \rho^0}{c} \right) \right\} = 0,
\]
that is,
\[
\frac{c^2 v_\rho}{c^2 - v^2} d\rho + \frac{c^2 v_\rho^0}{c^2 - v^2} d\rho_0 - c^2 \kappa^2 \gamma \rho^{\gamma-1} \frac{1}{\kappa^2 \rho^\gamma + c^2 \rho^0} d\rho + c^2 \sqrt{\frac{\kappa^2 \gamma \rho^0^{\gamma-1}}{\kappa^2 \rho^0 + c^2 \rho^0}} d\rho_0 = 0,
\]
where we regard \(v\) as a function of \(\rho\) and \(\rho_0\). From this, we have
\[
\frac{d \rho}{d \rho_0} = \frac{c^2 v_\rho}{c^2 - v^2} + c^2 \sqrt{\frac{\kappa^2 \gamma \rho^0^{\gamma-1}}{\kappa^2 \rho^0 + c^2 \rho^0}}
\]
(3.19)
Thus,
\[
\frac{ds}{ds_0} = \frac{\sqrt{\kappa^2\gamma^2\rho^0 - 1}}{\kappa^2\rho^0 + c^2\rho} \left( \frac{c^2v_{p_0}}{c^2-v^2} + \frac{c^2\sqrt{\kappa^2\gamma^2\rho^0 - 1}}{\kappa^2\rho^0 + c^2\rho} \right). \tag{3.20}
\]
Since \(v_p < 0, v_{p_0} > 0\) on \(S_1\), we have
\[
0 < \frac{ds}{ds_0} < \frac{c^2v_{p_0} + c^2\sqrt{\kappa^2\gamma^2\rho^0 - 1}}{c^2\sqrt{\kappa^2\gamma^2\rho^0 - 1}} = 1 + \frac{v_{p_0}}{c^2 - v^2} \sqrt{\kappa^2\gamma^2\rho^0 - 1},
\]
whose second term has the same power of parameter \(c\) in the numerator and the denominator, then
\[
\lim_{s_1 \to s_0} \frac{f(\epsilon, s_1, r)}{s_1 - s_0} = -1 + \frac{ds}{ds_0} \tag{3.21}
\]
is a bounded continuous function, and can be bounded by a constant that is independent of large \(c\).

Finally, from
\[
\frac{f(\epsilon, s_1, r)}{r - r_0} = \frac{s_0 - s - (s_1 - s_2)}{r - r_0} = \frac{s - s_0}{r - r_0} + \frac{s_2 - s_1}{r - r_0}, \tag{3.22}
\]
and since the composite shock and rarefaction curve is \(C^2\) smooth, and \(s = \text{constant}\) on rarefaction wave curves, we obtain that, on shock curves,
\[
\lim_{r \to r_0} \frac{s - s_0}{r - r_0} = 0, \quad \lim_{r \to r_0} \frac{s_2 - s_1}{r - r_0} = 0,
\]
and so
\[
\lim_{r \to r_0} \frac{f(\epsilon, s_1, r)}{r_0 - r} = 0.
\]
Form Figure 3, it is obvious that
\[
0 < \frac{s - s_0}{r - r_0} < 1, \quad 0 < \frac{s_2 - s_1}{r - r_0} < 1, \tag{3.23}
\]
and so
\[
-1 < \frac{f(\epsilon, s_1, r)}{r_0 - r} < 1. \tag{3.24}
\]
Thus, \(f(\epsilon, s_1, r)/r_0 - r\) is bounded independent of \(c\).

4. The difference approximation. We now use Glimm’s scheme to construct an approximate solution \(U_{\Delta x}(x, t)\) for the problem (1.1) and (1.4), and derive some estimates on \(U_{\Delta x}(x, t)\), which will be used in the next section. Let \(\Delta x = l\) denote a mesh length in \(x\) and \(\Delta t = h\) a mesh length in \(t\), and let \(x_j = j\Delta x, t_n = n\Delta t\), denote the mesh points for the approximate solution. We start from
\[
U_{\Delta x}(x, 0) = U_j^0, \quad \text{for} \quad x_j \leq x < x_{j+1}, \tag{4.1}
\]
where \(U_j^0 = U_0(x_j+).\) For \(t_{n-1} < t < t_n\), let \(U_{\Delta x}(x, t)\) be the solution of the Riemann Problem posed at time \(t = t_{n-1}\). Then define
\[
U_{\Delta x}(x, t_n) = U_j^n, \quad \text{for} \quad x_j \leq x < x_{j+1}, \tag{4.2}
\]
where \(U_j^n = U_{\Delta x}(x_j + a_n\Delta x, t_n-\) for some random sequence \(a_n \in (0, 1)\), and use this as the initial data for the Riemann problem posed at \(t = t_n\). Thus, \(U_{\Delta x}(x, t)\)
can be defined for all \( x \in \mathbb{R} \) and \( t > 0 \) by induction, if the waves do not interact within one time step. It suffices to require that
\[
\frac{\Delta x}{2\Delta t} > c, \quad i = 1, 2. \tag{4.3}
\]

However, this does not make sense when we consider the limit \( c \to \infty \). Actually it suffices to choose \( \Delta x/(2\Delta t) \) to be larger than the eigenvalues of the system (1.1).

Next we study the interaction of the two families of waves. Thus if \( R_1 \) and \( R_2 \) are rarefaction waves corresponding to the first and second characteristic families, respectively, then we must study the following six nontrivial interactions (here the “first” wave is considered to be on the left of the “second” wave):

(i) \( S_2 \) interacts with \( S_1 \);
(ii) \( S_2 \) interacts with \( R_1 \) (or \( R_2 \) interacts with \( S_1 \));
(iii) \( S_2 \) interacts with \( S_2 \) (or \( S_1 \) interacts with \( S_1 \));
(iv) \( S_2 \) interacts with \( R_2 \) (or \( R_1 \) interacts with \( S_1 \));
(v) \( R_2 \) interacts with \( S_2 \) (or \( S_1 \) interacts with \( R_1 \));
(vi) \( R_2 \) interacts with \( R_1 \);

Also see Figure 4. The interactions obtained by interchanging the indices 1 and 2 can be treated similarly.

\[
\begin{align*}
\beta' & \quad \gamma' \\
\beta & \quad \gamma \\
J_2 & \quad J_1
\end{align*}
\]

Let \( J_2 \) and \( J_1 \) be mesh curves, with \( J_2 \) an immediate successor to \( J_1 \). Let \( \beta \) (resp. \( \gamma \)) be an \( S_1 \) (resp. \( S_2 \)) shock on \( J_1 \), and let \( \beta' \) (resp. \( \gamma' \)) denote an \( S_1 \) (resp. \( S_2 \)) shock on \( J_2 \). Let the absolute value in terms of the Riemann invariant \( r \) (resp. \( s \)), denote the strengths of the \( S_1 \) (resp. \( S_2 \)) shock. See Figure 5.
We denote by \( \gamma + \beta \rightarrow \beta' + \gamma' \) the interaction of an \( S_2 \) with an \( S_1 \) which produces an \( S_1 \) and an \( S_2 \); the other cases can be written in a similar way, while rarefaction waves are denoted by \( 0 \).

In order to investigate the convergence, we introduce the following functionals on mesh curves. If \( J \) is a mesh curve, we define

\[
\begin{align*}
L(J) &= \sum \{ |\alpha| : \alpha \text{ is a shock crossing } J \}, \\
Q(J) &= \sum \{ |\beta| |\gamma| : \beta \in S_1, \gamma \in S_2, \beta, \gamma \text{ cross } J \text{ and approach} \}, \\
F(J) &= L(J) + N Q(J),
\end{align*}
\]

(4.4)

where \( N \) will be suitably chosen.

Now we have the following interaction estimates.

**Lemma 4.1.** Assume that \( 0 \leq \epsilon < 1/2, 0 < \rho < \rho \), and that all waves considered below are contained in the strip \( \rho \in [\rho, \rho], \ |v| < c \), in the \( r - s \) plane.

Then the following estimates are valid for the corresponding interactions:

(a). \( \gamma + \beta \rightarrow \beta' + \gamma' \) :

(a1). \( \beta' \leq |\beta| + C|\beta||\gamma|, \ |\gamma'| \leq |\gamma| + C|\beta||\gamma|, \) or there exist \( \eta, \varsigma \) such that

(a2). \( 0 \leq |\beta'| = |\beta| - \varsigma, \ |\gamma'| \leq |\gamma| + C|\beta||\gamma| + \eta, \) where \( 0 \leq \eta \leq g(|\beta|, \rho_0) \varsigma < \varsigma, \) or

(a3). \( 0 \leq |\gamma'| = |\gamma| - \varsigma, \ |\beta'| \leq |\beta| + C|\beta||\gamma| + \eta, \) where \( 0 \leq \eta \leq g(|\gamma|, \rho_0) \varsigma < \varsigma; \)

(b). \( \gamma + 0 \rightarrow 0 + \gamma' : \ |\gamma'| = |\gamma|; \)

(c). \( \gamma_1 + \gamma_2 \rightarrow 0 + \gamma' : \ |\gamma'| = |\gamma_1| + |\gamma_2|; \)

(d). \( \gamma + 0 \rightarrow \beta' + \gamma', \) or \( \gamma + 0 \rightarrow \beta' + 0 : \)

There exist \( \beta_0, \gamma_0 \) such that the interaction \( \gamma_0 + \beta_0 \rightarrow \beta' + \gamma' \) is the same as in (a) and \( |\beta_0| + |\gamma_0| \leq |\gamma| - C_0|\beta_0| \).

(e). \( 0 + \gamma \rightarrow \beta' + \gamma' \) or \( 0 + \gamma \rightarrow \beta' + 0 : \ |\beta'| + |\gamma'| \leq |\gamma| - C_0|\beta'|; \)

(f). \( 0 + 0 \rightarrow 0 + 0, \)

where \( C \) and \( C_0 \) are positive constants independent of \( \epsilon, \beta, \gamma, \rho \in [\rho, \rho], \) and \( c > c_0 \).

**Remark 4.1.** We remark that Lemma 3.2 and Theorem 3.2 play the key role in the process of finding the constants \( C \) and \( C_0 \). The proof is similar as the proof in [19].

**Remark 4.2.** It suffices to consider only shock waves in the above functionals since only shocks contribute to the decreasing variation of the solution across \( J \), and the total variation is controlled by twice the decreasing variation, plus the difference in the value of the functions at \( \pm \infty \).

We now give the following decreasing estimate without proof. The proof is similar as the proof in [19].

**Lemma 4.2.** If \( \epsilon F(0) \) is sufficiently small, then \( F(J_2) < F(J_1) \), where \( J_1 \) and \( J_2 \) are mesh curves and \( J_2 \) is an immediate successor to \( J_1 \).

Next we will estimate the total variation of \( v_{\Delta x} \) and \( \rho_{\Delta x} \).

**Theorem 4.1.** Suppose \( \rho_0(x) > 0 \) and \( v_0(x) \) are independent of \( \epsilon \), satisfying

\[
0 < \underline{\rho} \leq \rho_0(x) \leq \overline{\rho}, \ |v_0(x)| < c.
\]

Then there exists a constant \( M_0 > 0 \) such that, when

\[
(\gamma - 1) TV\{\rho_0, v_0\} < M_0,
\]
there exists a constant $c_0 > 0$ such that, for any $c > c_0$,
\[ \| v_{\Delta x}(\cdot, \cdot) \|_{L^\infty} + \| \rho_{\Delta x}(\cdot, \cdot) \|_{L^\infty} \leq M, \quad (4.5) \]
\[ TV\{ v_{\Delta x}(\cdot, t) \} + TV\{ \rho_{\Delta x}(\cdot, t) \} \leq M, \quad (4.6) \]
where $M > 0$ is a constant depending only on the initial data $(\rho_0, v_0)$, but independent of $c > c_0$.

**Proof.** The assumptions in the theorem imply that there exist states
\[ \rho_\pm = \lim_{x \to \pm \infty} \rho_0(x), \quad v_\pm = \lim_{x \to \pm \infty} v_0(x). \]

From the Glimm’s scheme, it is easy to see that $\lim_{x \to \pm \infty} \rho_{\Delta x}(x, t) = \rho_\pm$ and $\lim_{x \to \pm \infty} v_{\Delta x}(x, t) = v_\pm$.

From Lemma 4.2, the total variation of $s$ and $r$ along the mesh curve is bounded by $TV\{ c \ln \frac{c + v_{\Delta x}}{c - v_{\Delta x}} \} < L$, where we denote by $L > 0$ a generic constant depending on the initial data and independent of large $c$.

Similarly,
\[ | \ln \frac{c + v_1}{c - v_1} - \ln \frac{c + v_2}{c - v_2} | = \frac{2c^2}{c^2 - v^2} |v_1 - v_2| \geq 2|v_1 - v_2|, \]
where $v$ is between $v_1$ and $v_2$. Similarly,
\[ \frac{4c\sqrt{\gamma}}{\gamma - 1} \arctan \frac{\kappa \rho_1}{c} - \frac{4c\sqrt{\gamma}}{\gamma - 1} \arctan \frac{\kappa \rho_2}{c} \]
\[ \geq \frac{2\sqrt{\gamma}}{(1 + \frac{1}{\gamma}) \rho^{(3-\gamma)/2}} |\rho_1 - \rho_2|, \]
where $\rho$ is between $\rho_1$ and $\rho_2$. Thus,
\[ TV\{ v_{\Delta x} \} \leq L/2, \]
and,
\[ TV\{ \rho_{\Delta x} \} \leq \frac{(1 + \frac{1}{\gamma}) \rho^{(3-\gamma)/2}}{\kappa \sqrt{\gamma}} L. \]

**Remark 4.3.** Recall that Theorem 4.1 was established under the assumption that no waves interact within any one time step. However, the constants appearing in the previous estimates are all independent of $c > c_0$ and depend only on $(\rho_0, v_0)$. Now we can give the bounds independent of $c$ (when $c > c_0$) of the eigenvalues.
Suppose $|v_{\triangle x}| \leq M$ and $c \geq c_0$. Then the eigenvalues $\lambda_i(\rho, v)$ of the system (1.1) given in (2.6) satisfy

$$|\lambda_{1,2}| \leq \frac{M + \kappa \sqrt{\gamma M^{-1}}}{1 - \kappa M \sqrt{\gamma M^{-1}}}.$$  

(4.7)

Now we choose

$$\frac{\triangle x}{\triangle t} = \frac{2(M + \kappa \sqrt{\gamma M^{-1}})}{1 - \kappa M \sqrt{\gamma M^{-1}}}.$$  

(4.8)

Thus, our choice of $\triangle x/\triangle t$ is independent of $c$ for $c > c_0$, and we see that Theorem 4.1 holds with this choice. Moreover, it allows us to show that the approximate solutions $(\rho_{\triangle x}, v_{\triangle x})$ are $L^1$- Lipschitz continuous in $t$ through a standard procedure.

**Lemma 4.3.** Under the assumptions of Theorem 4.1, it holds that, for any $0 \leq t \leq t'$ and $c > c_0$, there exists a positive constant $M$ depending only on the initial data $(\rho_0, v_0)$ but independent of $c > c_0$ such that

$$\int_{-\infty}^{+\infty} (|v_{\triangle x}(x, t) - v_{\triangle x}(x, t')| + |\rho_{\triangle x}(x, t) - \rho_{\triangle x}(c, t')) \, dx \leq M|t - t'|,$$

(4.9)

where $c_0$ is the constant in Theorem 4.1.

5. **Convergence.** In this section we can complete the proof of the Main Theorem. First, for any initial data $(\rho_0, v_0)$ which satisfies the assumption of the Main Theorem, there exist positive constants $c_0$ and $M$ such that, for any $c > c_0$, the approximate solutions $(\rho_{\triangle x}^c, v_{\triangle x}^c)$ of (1.1) generated by the Glimm’s method satisfy the following (Theorem 4.1 and Lemma 3.2):

$$||v_{\triangle x}^c(\cdot, \cdot)||_{L^\infty} + ||\ln(\rho_{\triangle x}^c(\cdot, \cdot))||_{L^\infty} \leq M,$$

(5.1)

$$TV\{v_{\triangle x}^c(\cdot, t)\} + TV\{\ln(\rho_{\triangle x}^c(\cdot, t))\} \leq M,$$

(5.2)

$$||v_{\triangle x}^c(\cdot, t_1) - v_{\triangle x}^c(\cdot, t_2)||_{L^1} \leq M|t_1 - t_2|,$$

(5.3)

$$||\rho_{\triangle x}^c(\cdot, t_1) - \rho_{\triangle x}^c(\cdot, t_2)||_{L^1} \leq M|t_1 - t_2|,$$

(5.4)

where $M$ depends only on the initial data $(\rho_0, v_0)$ and is independent of $c > c_0$. For each $c > c_0$, we now apply Glimm’s theorem [6] to obtain the entropy solutions $(\rho^c, v^c)$ of system (1.1). Let $\alpha \equiv \{a_k\} \in A$ denote a (fixed) random sequence, $0 < a_k < 1$, $1 < k < \infty$, where $A$ denotes the infinite product of intervals $[0, 1]$ endowed with Lebesgue measure.

**Theorem 5.1.** Assume that the approximate solution $(\rho_{\triangle x}^c, v_{\triangle x}^c)$ of (1.1) satisfies (5.1)-(5.4). Then there exists a subsequence of mesh lengths

$$\triangle x_i \to 0,$$

(5.5)

such that

$$(\rho_{\triangle x}^c, v_{\triangle x}^c) \to (\rho^c, v^c),$$

(5.6)

where $(\rho^c, v^c)$ also satisfies (5.1)-(5.4). The convergence is pointwise a.e., and in $L^1_{loc}(\mathbb{R})$ at each time $t$, uniformly on bounded $x$ and $t$ sets. Moreover, there exists a set $N \subset A$ of Lebesgue measure zero such that, if $a \in A - N$, then $(\rho^c, v^c)$ is an entropy solution of the initial value problem (1.1)-(1.4).

Based on Theorem 5.1, we can consider the limit as $c \to \infty$. 
**Theorem 5.2.** Let \( \{(\rho^c, v^c)\} (c > c_0) \) be a family of solutions satisfying (5.1)-(5.4). Then there exists a subsequence \( \{c_k\} \) such that \( \{(\rho^{c_k}, v^{c_k})\} \) converges strongly to a pair of function \( (\rho, v) \) a.e. in \( L^1_{loc}(\mathbb{R}) \) at each time \( t \) and in \( L^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \). Moreover, \( (\rho, v) \) is an entropy solution satisfying (5.1)-(5.4) with the same constant.

**Proof.** For any \( \phi \in C^1_0(\mathbb{R} \times \mathbb{R}^+) \), we have
\[
\int_0^\infty \int_{-\infty}^\infty (U^c \phi_t + F(U^c) \phi_x) \, dx \, dt + \int_{-\infty}^\infty \phi(x,0)U^c_0(x) \, dx = 0, \tag{5.7}
\]
where \( U^c(x, t) \) is the weak solutions of (1.1)-(1.4) obtained from Theorem 5.1, which also satisfies (5.1)-(5.4). Let \( u = (\rho, \rho v)^\top \), \( f(u) = (\rho v, \rho v^2 + \kappa^2 \rho) \), what we need is the following:
\[
\int_0^\infty \int_{-\infty}^\infty (u \phi_t + f(u) \phi_x) \, dx \, dt + \int_{-\infty}^\infty \phi(x,0)u(x,0) \, dx = 0.
\]
Due to (5.1)-(5.4), there exists \( \{c_k\} \), such that
\[
\rho^{c_k} \to \rho, \quad v^{c_k} \to v,
\]
in \( L^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \) as \( c_k \to \infty \) \((k \to +\infty)\). Lemma 2.1 tells us that
\[
U \to u = (\rho, \rho v)^\top, \quad F(U) \to f(u) = (\rho v, \rho v^2 + \kappa^2 \rho) \to (\rho_0, \rho v_0)^\top,
\]
as \( c \to \infty \), and the convergence is uniform in the bounded region \( 0 < \rho < M, \quad |v| < \gamma \). From (5.7) we have
\[
0 = \lim_{k \to \infty} \left\{ \int_0^\infty \int_{-\infty}^\infty (U^{c_k} \phi_t + F(U^{c_k}) \phi_x) \, dx \, dt + \int_{-\infty}^\infty \phi(x,0)U^{c_k}_0(x) \, dx \right\}
\]
\[
= \int_0^\infty \int_{-\infty}^\infty \lim_{k \to \infty} (U^{c_k} \phi_t + F(U^{c_k}) \phi_x) \, dx \, dt + \int_{-\infty}^\infty \phi(x,0) \lim_{k \to \infty} U^{c_k}_0(x) \, dx
\]
\[
= \int_0^\infty \int_{-\infty}^\infty (u \phi_t + f(u) \phi_x) \, dx \, dt + \int_{-\infty}^\infty \phi(x,0)u(x,0) \, dx
\]
so \( (\rho, v) \) is the weak solution of (1.5),(1.4). Following the definition 2.1 and (2.13), we conclude that \( (\rho, v) \) is also the entropy solution of system (1.5) with initial data (1.4).

The Main Theorem follows immediately from Theorem 5.1 and Theorem 5.2.

**Remark 5.1.** With the initial data depending on \( c \), it is easy to see that the same conclusion holds if
\[
(\rho_0^c(x), v_0^c(x)) \to (\rho_0(x), v_0(x)) \tag{5.8}
\]
strongly in \( L^1_{loc} \) as \( c \to \infty \).

**Acknowledgements.** The authors would like to thank Professor Gui-Qiang Chen for his suggestions and encouragements.

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Received January 2006; revised May 2006.

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