Internal clocks in timeless universe

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Abstract. The Hamiltonian formalism of general relativity involves a Hamiltonian constraint. Attempts at quantisation of the Hamiltonian constraint formalism face an obstacle associated with the lack of predetermined time parameter whose existence is assumed in usual quantisation prescriptions. A way to deal with it is to employ an arbitrary internal degree of freedom as the internal clock to describe the evolution of a gravitational system. We use the so called reduced phase space approach in which the choice of internal clock is made prior to quantisation. We discuss the construction of reduced phase spaces based on the essential role played by internal clocks. Then we introduce the so called extended transformations which extend the well-known notion of canonical transformations. The extended transformations elucidate the relation between the canonical structure of the reduced phase space and the internal clock. Quantisation of reduced phase spaces and respective Hamiltonians reveals the relation between quantum dynamics and the choice of internal clock. Finally, we discuss concrete examples of canonical formalisms of the Friedmann-Lemaître universe and Bianchi type I universe. We find that many physical features of the quantum dynamics depend on the choice of internal clock. In the Conclusions we speculate about the possible physical meaning of the result.

1. Introduction

In this short contribution we summarise our previous work [1–3] on the structure of timeless quantum mechanics. This kind of quantum mechanics does not involve a fixed, predetermined time. Instead, it involves clocks which are internal degrees of freedom in a given system and which are chosen ambiguously to parametrise the evolution. The immediate motivation to study such a formalism comes from attempts at quantisation of spacetime. The distinctive feature of general relativistic models of spacetime is the lack of the concept of predetermined time. They undergo an evolution but it is often expressed in terms of an auxiliary and ambiguous parameter. Attempts at quantisation of general relativity lead to an unresolved conundrum called the problem of time.

We will make use of the canonical formalism and canonical quantisation to investigate this problem. The canonical formalism for general relativistic models was formulated by Arnowitt, Deser and Misner [4] and its restriction to cosmological model can be found e.g. in [5]. The Hamiltonians of gravitational systems turn out to be constraints (we discuss this point below). In general, the Hamiltonian constraint of general relativity involves four terms and each of them corresponds to a generator of spacetime diffeomorphisms. In cosmological models, the underlying spatial symmetry enables to disregard spatial diffeomorphisms and formulate the spacetime dynamics in terms of a mechanical system with a single-term Hamiltonian constraint, a generator of time-like diffeomorphisms. Our analysis is going to be confined to this relatively simple case.
2. Hamiltonian constraint

The canonical formalism of general relativity involves a Hamiltonian which is a constraint. In other words, the Hamiltonian’s role is twofold in this formalism. Namely, the Hamiltonian constraint, \( C \), generates the motion through the Poisson bracket:

\[
\frac{d\mathcal{O}}{d\tau} = \{\mathcal{O}, C\},
\]

where \( \mathcal{O} \) is any observable and it constrains the space of physical states to a sub-manifold in the phase space:

\[ C = 0. \]

It can be seen that rescaling the Hamiltonian constraint with any non-vanishing function \( N \),

\[
\frac{d\mathcal{O}}{d\tau'} = \{\mathcal{O}, NC\} = N \cdot \{\mathcal{O}, C\} + C \cdot \{\mathcal{O}, N\},
\]

does not change the dynamics since the term \( C \cdot \{\mathcal{O}, N\} = 0 \) vanishes. Nevertheless, it redefines the time parameter, \( \tau \mapsto \tau' \), in such a way that:

\[ N d\tau' = d\tau. \]

The parameters \( \tau \) and \( \tau' \) are auxiliary and have no physical meaning. The phase space on which the Hamiltonian constraint \( C \) is defined is called the kinematical phase space. The Hamiltonian constraint confines the physical motion to an odd dimensional surface \( C = 0 \) in the kinematical phase space. Some of the functions on the constraint surface could be employed to measure the evolution. Contrary to time parameters, constraint surface functions carry physical meaning as they are parameters in the sub-manifold of physical states. A simple counting shows that the remaining physical parameters in the constraint surface must form an even dimensional space.

As we will see they are endowed with a natural canonical structure and form the so-called reduced phase space.

Following Isham [6] there are three approaches to quantise Hamiltonian constraint systems: (I) time is introduced before quantisation, (II) time is introduced after quantisation or (III) time is not introduced at all and a timeless interpretation of quantum theory is proposed. Our basic tenet is that a selected degree of freedom is needed to parametrise the evolution of such systems, which is in compliance with the approaches (I) and (II). They both, however, suffer from the ambiguity in the choice of evolution parameter. To emphasise the identification of the evolution parameter with an internal degree of freedom, we will use the word ‘clock’ or ‘internal clock’ rather than ‘time’. The approach (II) is based on the following constraint operator equation:

\[
\hat{C}\Psi = 0,
\]

where \( \hat{C} \) is a quantised constraint. The physical state \( \Psi(q_1, \ldots, q_n) \) has no time-dependence. However, it is assumed that one of its independent variables \( q_1, \ldots, q_n \) (or, a combination of them) can be set as internal clock to describe a unitary evolution of \( \Psi \) understood as a wavefunction of the remaining variables, e.g.:

\[
\hat{C}\Psi = \left(i \frac{\partial}{\partial q_1} - \hat{H}\right)\Psi = 0,
\]

by means of the Schrödinger equation based on a new Hamiltonian \( \hat{H} \) that generates evolution with respect to \( q_1 \).
The aim of our analysis is to understand how much predictions of a quantum theory depend on the employed clock. To address this problem we need to study the quantum dynamics of a given system with respect to various clocks. For the sake of simplicity we will employ the approach (I) which we describe in detail below. Although it may produce quantitatively different results from the approach (II), the addressed problem concerns them both and we do not expect to find any qualitative differences in this regard between these two approaches. The approach (I) entails first defining the reduced phase space equipped with a non-vanishing Hamiltonian generating physical dynamics with respect to a selected clock and next quantising it by usual procedures.

3. Ambiguities in quantum dynamics

The first explanation of the origin of ambiguous predictions produced by quantum theories based on different clocks was provided by Isham [6] and Kuchar [7]. They claimed that the relation between classical formalisms based on different internal clocks can be understood in terms of canonical transformations in the kinematical phase space. The latter relate different canonical coordinate systems and each of them includes a distinguished clock coordinate. Since many canonical transformations do not have a unitary representation they concluded that quantum theories based on different clocks must be in general unitarily inequivalent. Thus, they must produce, to some extent, different predictions.

It is known now that the above reasoning is mistaken. The real reason for unitary non-equivalence between different clock-based quantum theories lies elsewhere, namely in canonical non-equivalence of different reduced phase spaces associated with different clocks. This idea can be already found in the paper of Hajicek [8]. He shows by an explicit example that a gravitational model reduced and parametrised with different clocks entails different Poisson brackets between some metric components of the field. We will follow this observation and give it a more precise formulation below. The basic tool for this purpose is the idea of the so-called extended transformations that extend the notion of canonical transformations and which was already considered by Arnowitt, Deser and Misner in [9]. Some interesting considerations of explicit, though very limited, transformations of clock variable in the case of a spherically symmetric dust shell can be found in [10].

The notion of extended transformations (called pseudo-canonical transformations therein) was thoroughly investigated in the paper [3]. The extended transformations are defined in the reduced phase space rather than the kinematical phase space. They transform both canonical variables and internal clock. Although there is no longer Hamiltonian constraint in the reduced phase space, the role of extended transformations is to reflect the fact that the reduced phase space is descendent from the kinematical phase space and that its definition is tied to the ambiguous definition of the clock.

4. Internal clock

In this section we define the reduced phase space and show how it is related to the choice of clock. We introduce the extended transformations and discuss how they break at the quantum level.

4.1. Reduced phase space

Let us discuss how the reduced phase space is obtained. Solving a given Hamiltonian constraint $C = 0$ in the kinematical phase space $P$ of dimension $2n + 2$ and equipped with a symplectic form $\Omega$ leads to a $2n + 1$-dimensional sub-manifold $\mathcal{M}_C \subset P$ equipped with a degenerate two-form $\Omega|_{C=0}$ induced from $\Omega$. A very important property of $\Omega|_{C=0}$ is that a field of its null vectors $\{v_C \in T^*_\mathcal{M}_C : \Omega|_{C=0}(v_C) = 0\}$ generates a $2n$-dimensional space $S$ of curves in $\mathcal{M}_C$. These curves represent physical motion of the model. The key element of the construction is the choice of internal clock $t : \mathcal{M}_C \rightarrow \mathbb{R}$ which is a real function on the constraint surface and whose level
sets \( \{ p \in \mathcal{M}_C : t(p) = t_{\text{fixed}} \} \) cross each curve in \( \mathcal{M}_C \) at most once. Therefore, the internal clock provides a foliation of \( \mathcal{M}_C \),

\[
(t, S) \mapsto \mathcal{M}_C.
\]

At each level set of \( t \) a (non-degenerate) symplectic form may be introduced,

\[
\Omega|_{t=t_{\text{fixed}}} \quad (\text{2})
\]

induced from \( \Omega|_{C=0} \). The pair

\[
\{(t_{\text{fixed}}, S), \Omega|_{C=0, t=t_{\text{fixed}}}\}
\]

is called the reduced phase space. \(-\Omega|_{C=0, t=t_{\text{fixed}}} = \{\cdot, \cdot\}_{\text{phys}}\) is the physical Poisson bracket in \((t_{\text{fixed}}, S)\). It can be shown that the degenerate two-form can be decomposed as follows:

\[
\Omega|_{C=0} = \Omega|_{C=0, t=t_{\text{fixed}}} - \frac{\partial}{\partial t}dH, \quad (\text{4})
\]

where \( H \) is a function on \( \mathcal{M}_C \) called the Hamiltonian. Furthermore it can be shown that the vector field

\[
-\Omega|_{C=0, t=t_{\text{fixed}}} (\cdot, H) = \{\cdot, H\}_{\text{phys}} \quad (\text{5})
\]

generates the physical curves \( S \) in \( \mathcal{M}_C \) parametrised by \( t \). In other words, given any observable \( O : \mathcal{M}_C \rightarrow \mathbb{R} \), its evolution is given by means of the Hamilton equations:

\[
\frac{dO}{dt} = \frac{\partial O}{\partial t} + \{O, H\}_{\text{phys}}, \quad (\text{6})
\]

Clearly, the reduced phase space formalism is a reduction of the Hamiltonian constraint formalism to the unconstrained canonical formalism which includes a Hamiltonian \( H \), a time \( t \) and a Poisson bracket. The construction is depicted in figure (1).

The key difference between the reduced phase space formalism and the unconstrained canonical formalism is that the choice of internal clock is ambiguous whereas the time parameter of the usual canonical formalism is predetermined and completely fixed. The consequences are immediately seen in the above construction. Namely, the choice of \( \bar{t} \) instead of \( t \) as an internal clock leads to another foliation of the constraint sub-manifold, \((\bar{t}, S) \mapsto \mathcal{M}_C\), and another decomposition of \( \Omega|_{C=0} \),

\[
\Omega|_{C=0} = \Omega|_{C=0, \bar{t}=\bar{t}_{\text{fixed}}} - \frac{\partial}{\partial \bar{t}}d\bar{H}, \quad (\text{7})
\]

where \( \Omega|_{C=0, \bar{t}=\bar{t}_{\text{fixed}}} \neq \Omega|_{C=0, t=t_{\text{fixed}}} \) are inequivalent symplectic forms which, when inverted, give inequivalent physical Poisson brackets. In other words, different choices of internal clocks establish canonically inequivalent reduced phase spaces. The same physical curves are obtained with the respective Hamilton equations, though they are now parametrised by \( \bar{t} \).

4.2. Extended transformations

Let us introduce the so called extended transformations. By \((q, p)\) we denote the canonical coordinates which always exist by the virtue of Darboux theorem,

\[
\Omega|_{C=0, t=t_{\text{fixed}}} = dqdp. \quad (\text{8})
\]

We first recall the concept of canonical transformations. They are defined as such (time-dependent) transformations of canonical coordinates,

\[
t \times (q, p) \mapsto (\bar{q}, \bar{p}), \quad (\text{9})
\]
A foliation of the constraint surface \((t, S) \mapsto \mathcal{M}_C\) provides a decomposition of the degenerate form \(\Omega|_C\) into a symplectic term and another term proportional to the Hamiltonian. In this way a canonical formalism can be established for any choice of clock \(t\). However, different choices of clock lead to different symplectic terms and canonically inequivalent formalisms.

The extended transformations are such transformations of canonical coordinates and time coordinate,

\[
(q, p, t) \mapsto (\bar{q}, \bar{p}, \bar{t}),
\]

which preserve the form of the degenerate form \(\Omega|_C\),

\[
\Omega|_C = dqdp - dt dH = d\bar{q}d\bar{p} - d\bar{t}d\bar{H},
\]

but in general they do not preserve the symplectic form,

\[
\Omega|_{C,t=t_{fixed}} = dqdp \neq d\bar{q}d\bar{p} = \Omega|_{C,\bar{t}=\bar{t}_{fixed}}
\]

The above transformations do not only transform a canonical coordinate system of a given reduced phase space but they can also transform a given reduced phases space into another one. Canonical transformation are indeed a subgroup of extended transformations.

4.3. Quantum level
At the classical level, the many inequivalent reduced phase space formalisms are related by extended transformations. They constitute a symmetry of the canonical formalism with respect to which the physical motion represented by physical curves in \(\mathcal{M}_C\) is invariant. This is no longer true at the quantum level. The reason is that quantisation imposes uncertainty on canonical variables and the notion of canonical variables is tied to the choice of clock. Therefore, uncertainties are imposed differently for different internal clocks and the classical symmetry
breaks down, that is, the quantised motion cannot be invariant with respect to the choice of clock. How can we see this more closely?

The problem is that quantisation of different reduced phase spaces and the respective Hamiltonians may produce different results not because the canonical structure underlying quantisation is different but just because some usual quantisation ambiguities occur (choice of basic variables, orderings...). To ensure that any dissimilarities between quantised reduced phase spaces and respective dynamics are induced by differences in employed clocks rather then usual quantisation ambiguities we should quantise all reduced phase spaces in a unique way in some sense. The precise prescription of how to achieve it was given in [1–3]. In what follows we just sketch its main points.

The dimensionality of the reduced phase space is equal to the number of constants of motion (we assume an explicitly integrable system). It can be easily proven that the Poisson algebra of the constants of motion is independent of the choice of clock. Thus, a quantisation of constants of motion can be fixed for all reduced phase space formalisms. However, the Poisson algebra of dynamical observables differs from one clock to another and their quantum representation must differ from one clock to another too. Nevertheless, it can be shown that quantisation of all constants of motion determines quantisation of all observables including dynamical observables as well. Therefore, quantisation of both non-dynamical (constants of motion) and dynamical observables in all reduced phase spaces can be fixed with a unique quantum representation of constants of motion only.

One of the consequences of the described prescription is that constants of motion are given exactly the same operators for all choices of clock. Any state in the Hilbert space can be then decomposed in terms of invariant eigenstates of self-adjoint constants of motion operators. In other words, the physical interpretation of a state as a wave-function on the spectrum of quantised constants of motion is unique. On the other hand, the quantum representation of dynamical observables depends on the choice of clock. A fixed dynamical quantity is given a different operator for different clocks. Therefore, assuming they are self-adjoint, a state is given a different wave-function on the spectrum of a fixed quantised dynamical quantity for different clocks. In other words, the way in which quantum evolution is interpreted depends on the employed clock and we will see this in the examples discussed below.

5. Review of results

In this section we review some of the previous results on the relation between quantum theory and clocks, which rely on the notion of extended transformations which break down at the quantum level.

In the work [3] the definition of extended transformations (called therein pseudo-canonical transformations) and their relation to canonical transformations in the kinematical phase space were established. This work includes a careful discussion on the separation of usual quantisation ambiguities from clock-induced effects found in quantised reduced phase spaces. It was shown by a simple example that spectra of dynamical operators can change their character from continuous to discrete upon transforming the internal clock.

In the work [2], a quantum theory of the Friedmann model of universe was studied. Initially, a variable associated with the perfect fluid filling the universe was set as the internal clock and a quantisation was performed. The quantum dynamics was shown to undergo a bounce which replaced the classical singularity. Then, extended transformations were applied and more reduced phase spaces were obtained and quantised. For simplicity, a method of phase space portraits based on coherent states was used. The approximate description of quantum dynamics in terms of phase space variables turned out to be enough to identify very strong dissimilarities between different clock-based quantum theories. They included the scale of the bounce, the number of bounces or the extent to which dynamics was symmetric with respect to the bounce.
Figure 2. Three phase space portraits of quantum dynamics of the Friedmann universe with a bounce based on coherent states (see [2] for details). The portraits are given in a fixed phase space parametrised by $q$, which denotes the volume of the universe, and $p$, which is a canonically conjugate variable. Each of the three portraits was derived with a distinct clock and presents a distinct behaviour. The existence of a bounce which replaces the classical singularity and the classical behaviour exhibited by the portraits away from the bounce are universal features which occur for all choices of clock. Whereas, the scale of the bounce, the number of bounces and the symmetry between expansion and contraction clearly depend on the employed clock.

The only confirmed invariant feature of the quantum dynamics was the existence of the bounce itself and of the classical limit for both the expanding and contracting branches. Figure (2) illustrates the effect of the choice of clock on the semiclassical dynamics of the Friedmann universe.

In the work [1], the quantum dynamics of the anisotropic Bianchi type I model with a perfect fluid was studied. This model includes the Friedmann model described above in the isotropy limit. The justification for extending the reduced phase space to include anisotropy variables was to investigate the classical limit of quantum theory in more detail than it was possible in [2]. Recall that classical solutions are parametrised by a collection of conserved quantities, including the Hamiltonian. The key question addressed in this work was following: does the asymptotic classical solution of a given quantum solution, expressed in terms of a complete set of conserved quantities, is the same for both the expanding and contracting phases and is independent of the employed clock? It was shown that indeed an anisotropy state in the quantum Bianchi type I model with a bounce is unique for a given quantum solution and independent of the employed clock if registered far enough from the bounce. Then a general analysis showed that the classical asymptotic limits, if they exist, must be invariant with respect to transformations of underlying reduced phase spaces.

6. Conclusions
By timeless quantum mechanics we mean quantum mechanics of systems without a fixed, predetermined time parameter which quantifies their evolution. In such a theory, we set a dynamical variable as the internal clock which measures the evolution of the remaining variables. This procedure is shown to establish a canonical formalism for the underlying classical theory. Different choices of internal clock lead to different underlying canonical formalisms. This ambiguity imprints on the effects of imposition of uncertainties on canonical variables when a given system is quantised. The ambiguity in the notion of quantum dynamics for such theories is unavoidable. However, it was shown that when a system becomes in some sense classical the mere choice of clock does not influence the dynamics. Let us speculate about the possible
physical meaning of the result.  

There are two important cases of states of a given system without a fixed notion of time. The first case is a state in which there are no degrees of freedom which could be assumed classical. It seems impossible then to apply the usual understanding of quantum mechanics. For example, given a fixed dynamical observable, the choice of internal clock determines the spectral decomposition of that state, i.e. the form of the wave-function in the given representation. This undermines the physical interpretation of that state. The choice-of-clock-invariant piece of information about the state can be read only from its spectral decomposition with respect to a conserved quantity. Therefore, no information about the evolution of the state can be provided.

The second case is a state in which some degrees of freedom can be assumed classical. Setting one of them as the internal clock seems to be a preferred choice. Moreover, extended transformations, which transform the formalism from one clock to another clock which can be also assumed classical in a given state, do not change the spectral representation of that state for any dynamical observable. This seems to be the case of the usual quantum mechanics and laboratory systems which the usual quantum mechanics describes. These systems are indeed only subsystems of a global system which encompasses both the system and its environment. Indeed, the measuring apparatus, including clocks, belongs to the classical environment. Therefore, we see that timeless quantum mechanics includes and extends the usual quantum mechanics.

When we deal with local quantum systems in laboratories, the time standard is provided by external degrees of freedom in the form of apparatus that measures the progress of classical ‘time’. But what should be the time standard for the evolution of global degrees of freedom like the ones we deal with in cosmological models? Clearly, there are no external degrees of freedom in this case and the only option seems to be choosing one of the global degrees of freedom as the internal clock. Then, however, anything we say about the quantum evolution of such a system strongly depends on the choice of clock we make. The only available and invariant information is time-independent. The quantum universe becomes truly timeless.

Acknowledgments
The author gratefully acknowledges support from Narodowe Centrum Nauki by decision No. DEC-2013/09/D/ST2/03714.

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