Solutions of nonlinear Sobolev-Burgers PDEs.

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Abstract

Nonlinear Sobolev-Burgers PDEs are considered. Their solutions are investigated. A technique of noncommutative line integration is utilized for their description. A new method of PDEs solution with the help of Cayley-Dickson algebras is developed in the article. Moreover, random operator valued measures are studied and applied to solutions of PDEs.

1 Introduction.

Studies of nonlinear PDEs compose an extensive part of nonlinear analysis and PDE (see, for example, [1, 20, 24, 40, 41, 45] and references therein). There are linear and nonlinear PDEs of particular types, for example, wave PDEs, heat PDEs, diffusion PDEs, Schrödinger PDEs to each of which more than a thousand articles and books are devoted. Though about an existence of solutions and their numerical simulations a lot of works is written, their integration remains a hard or an unsolved problem for many types of PDEs.

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On the other side, random functions methods based on Gaussian measures appeared to be useful for solutions of diffusion PDEs and Schrödinger PDEs (see, for example, [12, 13, 21, 36, 39] and references therein).

Among nonlinear PDEs the Sobolev type PDEs, which include the Burgers PDE, have important applications in physics and hydrodynamics [45]. They describe two-dimensional motions of a stratified rotating liquid, electromagnetic fields in crystals, internal gravitational waves, non stationary filtration process of liquid in a fissure porous medium, dissipation process, cold plasma, two temperature plasma in an external magnetic field, etc. (see [45] and references therein). There were works about numerical computation of some PDE solutions and an existence of solutions. Nevertheless needs arise to integrate such type and more general PDEs or their systems. In many cases it is also necessary to analyze properties of solutions. Therefore analytic approaches apart from that of numerical provide in this respect many advantages.

Such problems led to a new main stream of developing hypercomplex analysis for the PDE theory needs (see [5] - [11], [15] - [18], [27] - [34] and references therein). It was begun mainly in the years 1990-th over quaternions and Clifford algebras. Later on since the years 2010-th it was begun over octonions and more general Cayley-Dickson algebras. A reason for such activity is in an enlargement of possibilities: some PDEs which are not integrable over the complex field $\mathbb{C}$ appear to be integrable over the aforementioned algebras so that a suitable Clifford or a Cayley-Dickson algebra can be chosen for a given PDE.

Previously a new approach of noncommutative integration of nonlinear PDEs was investigated over hypercomplex numbers [32, 10]. It was applied to PDEs used in hydrodynamics such as the non-isothermal flow of a non-compressible Newtonian liquid PDE and the Korteweg-de-Vries PDE. In this paper also another new mathematical tool is developed and applied to solutions of PDEs, which consists in random operator valued measures.

Some readers of previous articles asked how the Cayley-Dickson algebras can be used for solutions of general type PDEs. To this problem Section 2 is devoted. In this paper such theory is developed further for other types of
PDEs which were not yet treated. Procedures permitting to write equivalent problems over quaternions, octonions or Cayley-Dickson algebras instead of PDEs over \( \mathbb{R} \) or \( \mathbb{C} \) are described. The corresponding Theorems 1, 2 and Proposition 3 are proved. The method is illustrated on Sobolev-Burgers PDEs in Section 3. There solutions of PDEs are investigated. For this purpose random operator valued measures are studied. The noncommutative line integration is utilized for their description. Solutions of the Sobolev-Burgers PDEs are investigated (see Theorems 6, 23 and 24 and Subsection 25 below).

All main results of this paper are obtained for the first time. They can be used for further studies and integrations of linear and nonlinear PDEs.

## 2 Solution of PDEs over octonions.

Frequently PDEs are given over the real field or the complex field. For an application of the noncommutative integration technique of PDEs it may be necessary at first to present the corresponding PDEs over octonions or Cayley-Dickson algebras. Henceforth notations and definitions of the work [32] are used.

1. **Theorem.** To each scalar or vector PDE

   \[(1.1) \ P(A_1, \ldots, A_v, u) = g \text{ over } \mathbb{R} \text{ which can be nonlinear relative to an unknown scalar or vector function } u = (u_1, \ldots, u_m), \text{ where } A_1, \ldots, A_v \text{ are linear PDO with real coefficients, } g = (g_1, \ldots, g_k), g_s \text{ and } u_j \text{ are real functions for each } s = 1, \ldots, k \text{ and } j = 1, \ldots, m, 1 \leq k \in \mathbb{N}, 1 \leq m \in \mathbb{N}, P = (P_1, \ldots, P_k) \text{ and } g \text{ are given, } P_s \text{ is a polynomial (or power series) with real coefficients for each } s, \text{ a domain } U \text{ is open in a canonically closed subset } V \text{ in } \mathbb{R}^n, \text{ can be posed a PDE}
   \]

   \[(1.2) \ Q(\hat{A}_1, \ldots, \hat{A}_v, \hat{u}) = \hat{g} \text{ over the Cayley-Dickson algebra } \mathbb{A}_r \text{ such that a bijective correspondence between solutions } u \text{ of (1.1) and } \hat{u} \text{ of (1.2) exists, where } \hat{A}_1, \ldots, \hat{A}_n \text{ are PDO over the Cayley-Dickson algebra, } Q \text{ is a polynomial (or power series) with } \mathbb{A}_r \text{ coefficients, } \hat{g} \text{ and } \hat{u} \text{ are } \mathbb{A}_r\text{-valued functions.}
   \]

**Proof.** I. Let PDE (1.1) be in a variable \( x = (x_1, \ldots, x_n) \) belonging to a domain \( U \) in the Euclidean space \( \mathbb{R}^n \), where each variable \( x_1, \ldots, x_n \) belongs
to the real field $\mathbb{R}$, where $U$ is open in $V$ by the conditions of this theorem. The subset $V$ in $\mathbb{R}^n$ is canonically closed, which means by the definition that the closure $cl(Int(V))$ of the interior $Int(V)$ of $V$ coincides with $V$. To this variable $x = (x_1, ..., x_n) \in U$ we pose a variable $z = z(x)$ by the formula

\[(1.3) \quad z = x_1i_1 + ... + x_ni_n \in V \text{ with } l_s \neq l_p \text{ for each } s \neq p, \text{ where } l_1, ..., l_n \text{ are fixed nonnegative integers}, \text{ V notates the corresponding domain in the Cayley-Dickson algebra } \mathcal{A}_t \text{ with } n \leq 2^t, \quad 2 \leq t. \] The family $\{i_0, i_1, ..., i_{2^t-1}\}$ denotes the standard basis of the Cayley-Dickson algebra $\mathcal{A}_t$ over $\mathbb{R}$ such that $i_0 = 1$, $i_j^2 = -1$, $i_ji_k = -i_ki_j$ for each $j \geq 1$ and $k \geq 1$ with $j \neq k$. Thus to each $x \in U$ a unique $z = z(x)$ is posed so that $V = \{z \in \mathcal{A}_t : z = z(x), x \in U\}$. Vise versa to each $z \in V$ a unique $x \in U$ corresponds.

\[(1.4) \quad x_j = \pi_j(z) \text{ for each } j, \text{ where } \pi_j : \mathcal{A}_t \to \mathbb{R} \text{ is an } \mathbb{R}-\text{linear operator prescribed by the formulas:} \]

\[(1.5) \quad \pi_j(z) = z_j = (-zi_j + i_j(2^t - 2)^{-1}\{-z + \sum_{k=1}^{2^t-1} i_k(z_i_k^*\})}/2 \text{ for each } j = 1, 2, ..., 2^t - 1, \]

\[(1.6) \quad \pi_0(z) = z_0 = (z + (2^t - 2)^{-1}\{-z + \sum_{k=1}^{2^t-1} i_k(z_i_k^*)\})}/2, \]

where $2 \leq t \in \mathbb{N}$, $z$ is a Cayley-Dickson number in $\mathcal{A}_t$ presented as

\[(1.7) \quad z = z_0i_0 + z_1i_1 + ... + z_{2^t-1}i_{2^t-1} \in \mathcal{A}_t, \quad z_j \in \mathbb{R} \text{ for each } j, i_j^* = i_k = -i_k \text{ for each } k > 0, \quad i_0 = 1, \quad z^* = z_0i_0 - z_1i_1 - ... - z_{2^t-1}i_{2^t-1} \text{ (see Formulas II(1.1) – (1.3) in [29]).} \]

Thus to each basic vector $e_j = (0, ..., 0, 1, 0, ...,)$ in $\mathbb{R}^n$ with 1 at $j$-th place the basic generator $i_{l_j}$ in the Cayley-Dickson algebra $\mathcal{A}_t$ is counterposed according to the mapping $\mathbb{R}^n \ni x \mapsto z(x) \in \mathcal{A}_t$. Therefore \((x, y) = Re(z(x)z^*(y))\) for each $x$ and $y$ in $\mathbb{R}^n$, where \((x, y) = \sum_{j=1}^{n} x_jy_j\) is the scalar product in the Euclidean space $\mathbb{R}^n$, $Re(w) = (w + w^*)/2$ is the real part of $w$ for each $w \in \mathcal{A}_t$. Particularly, for $n = 3$ the vector product $x \times y$ can be expressed as $Im(z(x)z(y))$ with $1 \leq l_j$ for each $j = 1, 2, 3$ and with $i_1i_2i_3 = i_{l_3}$, for example, $l_1 = 1, l_2 = 2, l_3 = 3$, where $Im(w) = w - Re(w)$ denotes the imaginary part of a Cayley-Dickson number $w \in \mathcal{A}_t$.

II. To each function $f : U \to \mathbb{R}$ a unique function $h^f(z)$ corresponds such that $h : V \to \mathbb{R}$ and

\[(1.8) \quad h^f(z(x)) = f(x) \text{ for each } x \in U. \]

For sufficiently times differentiable function $f$ to each partial derivative
by induction to each PDO \( Af(x) = \sum_{\alpha} c_{\alpha}(x) \partial^{\alpha} f(x)/\partial x_1^{\alpha_1}...x_m^{\alpha_m} \) the PDO \( \hat{A} f'(z) = \sum_{\alpha} h^{\alpha}(z) \partial^{\alpha} h'(z)/\partial z_1^{\alpha_1}...z_m^{\alpha_m} \) corresponds with the help of Formula (1.8), where \( c_{\alpha}(x) \) are coefficients, \( \alpha = (\alpha_1, ..., \alpha_n), |\alpha| = \alpha_1 + ... + \alpha_n, \alpha_j \) is a nonnegative integer for each \( j \).

III. To a function \( g(x) \) with values in \( \mathbb{R}^k \) we pose a function \( \hat{g}(z(x)) = g_1(x)i_0 + ... + g_k(x)i_{k-1} \) with values in \( \mathcal{A}_t \), where \( 2^{t-1} < k \leq 2^t \). Then to \( u \) we pose a function \( \hat{u}(z(x)) = u_1(x)i_{q_1} + ... + u_m(x)i_{q_m} \) having values in \( \mathcal{A}_t \) with fixed nonnegative integers \( q_1, ..., q_m \) such that \( q_s \neq q_p \) for each \( s \neq p \). Next we choose \( r \geq \max(t, t_1, t_2, 2) \), hence \( \mathcal{A}_t \subset \mathcal{A}_r \) and similar embeddings are for \( t_1 \) and \( t_2 \) instead of \( t \) also. For example, \( q_j = j - 1 \) and \( l_j = j - 1 \) can be taken for each natural number \( j = 1, 2, ... \).

Then particularly
\[
\sum_j \partial \hat{u}_j(z)/\partial z_{l_j} = Re(\sigma \hat{u}^*(z)) \quad \text{corresponds to}
\]
\[
div u(x) = (\nabla, u)(x) = \sum_j \partial u_j(x)/\partial x_j, \quad \text{where}
\]
\[
\sigma \hat{f}(z) = \sum_j (\partial \hat{f}(z)/\partial z_j)i_{q_j}; \quad \text{also}
\]
\[
\sigma \hat{u}_s(z) \quad \text{to}
\]
\[
grad u_s(x) = \nabla u_s(x) = \sum_j (\partial u_s(x)/\partial x_j)e_j. \quad \text{In particular, for } n = 3
\]
the operator \(-Im(\sigma \hat{u}(z)) \) to \( \text{rot } u(x) = \nabla \times u(x) \) corresponds with \( q_1 = 1, q_2 = 2, q_3 = 3 \) or more generally with natural numbers \( q_j \geq 1 \) such that \( i_{q_1}i_{q_2} = i_{q_3} \).

Therefore taking \( Q(\hat{A}_1, ..., \hat{A}_v, \hat{u}) = \hat{P}(\hat{A}_1, ..., \hat{A}_v, (\pi_{q_1} \hat{u}, ..., \pi_{q_m} \hat{u})) \) we get PDE (1.2) instead of (1.1), since the Cayley-Dickson algebra \( \mathcal{A}_r \) is power associative: \( z^\phi z^\psi = z^{\phi + \psi} \) for each natural numbers \( \phi \) and \( \psi \), where an order of multiplications in \( Q \) over \( \mathcal{A}_r \) is essential (see also Section 2 in [27]). The correspondences described above between domains, functions and classes of differentiable functions are bijective. Moreover, the mappings \( U \ni x \mapsto z(x) \in V, g \mapsto \hat{g}, P \mapsto \hat{P}, f \mapsto h^f, u \mapsto \hat{u} \) are bijective isometries. Thus PDE (1.1) and (1.2) are equivalent: to each solution of (1.1) whenever it exists a unique solution of (1.2) corresponds and vice versa, since \( P(A_1, ..., A_v, u) \) and \( Q(\hat{A}_1, ..., \hat{A}_v, \hat{u}) \) exist and converge simultaneously.

2. Theorem. Suppose that conditions of Theorem 1 are fulfilled with \( P \) being a polynomial in PDOs \( A_1, ..., A_v \), suppose also that each PDO \( A_s \) is
a linear combination of elliptic operators of finite orders. Then Dirac-type operators $\Upsilon_j, j = 1, 2, ..., l$ exist such that the principal symbol of the PDO $Q(\hat{A}_1, ..., \hat{A}_v, \cdot)$ is $S(\Upsilon_1, ..., \Upsilon_l, \cdot)$, where $S$ is a polynomial in $\Upsilon_1, ..., \Upsilon_l$ over the Cayley-Dickson algebra $A_p$ with $p \geq r \geq 2$.

**Proof.** In view of Theorem 1 PDE (1.1) is equivalent to (1.2). Applying Theorem 2.1 in [34] we get a decomposition of each PDO $\hat{A}_s$ with the help of Dirac-type operators $\Upsilon_j, j = 1, 2, ..., l$, where $l$ is a natural number. Each PDO $\hat{A}_s$ has a decomposition over the Cayley-Dickson algebra $A_{t_s}$ with $t_s \geq r \geq 2$. Taking $p = \max(t_s : s = 1, ..., v)$ we get the decomposition over $A_p$ for all PDOs $\hat{A}_1, ..., \hat{A}_v$. Therefore, the principal symbol of the PDO $Q(\hat{A}_1, ..., \hat{A}_v, \hat{u})$ acting on a function $\hat{u}$ becomes a polynomial $S(\Upsilon_1, ..., \Upsilon_l, \hat{u})$ in $\Upsilon_j$ with $j = 1, 2, ..., l$.

**3. Proposition.** If conditions of Theorem 1 are satisfied and each operator $A_s$ is of the second order with constant coefficients, $s = 1, ..., v$, then Dirac-type operators $\Upsilon_j$ with $j = 1, 2, ..., l$ exist such that

$$Q(\hat{A}_1, ..., \hat{A}_v, \hat{u}) = S(\Upsilon_1, ..., \Upsilon_l, \hat{u}),$$

where $S$ is a polynomial (or power series) in $\Upsilon_1, ..., \Upsilon_l$ and $\hat{u}$ over the Cayley-Dickson algebra $A_p$ with $p \geq r \geq 2$.

**Proof.** By virtue of Theorem 2.1 and Example 2.6 in [34] each PDO $\hat{A}_s$ of the second order with constant coefficients is a polynomial of the second order in a finite number of Dirac-type operators $\Upsilon_j$ over the Cayley-Dickson algebra $A_{t_s}$ with $t_s \geq r \geq 2$. Taking $Q$ given by Theorem 1 and substituting PDOs $\hat{A}_1, ..., \hat{A}_v$ with polynomials of the second order in Dirac-type operators we deduce that $Q(\hat{A}_1, ..., \hat{A}_v, \hat{u}) = S(\Upsilon_1, ..., \Upsilon_l, \hat{u})$, where $S$ is a polynomial (or power series) in $\Upsilon_1, ..., \Upsilon_l$ and $\hat{u}$ over the Cayley-Dickson algebra $A_p$ with $p = \max(t_s : s = 1, ..., l)$.

### 3 Nonlinear Sobolev-Burgers PDE.

#### 4. Sobolev-Burgers PDEs. A more general Sobolev type PDE is with $Q(\frac{\partial}{\partial t})$ instead of $\frac{\partial}{\partial t}$ in the Burgers PDE, where $Q(t) = t^m + c_{m-1}t^{m-1} + ... + c_0$ is a polynomial with real or complex coefficients $c_0, ..., c_{m-1}$, where $m \geq 1$ is
a natural number. We take them in the form:

\( Q(\frac{\partial}{\partial t})(-\Delta_x^2 + \alpha \Delta_x + \beta I)u(t, x) + \gamma \frac{\partial u^2(t, x)}{\partial x_1} + \varsigma u^2(t, x) = 0, \)

where \( \alpha \neq 0, \beta, \gamma \) and \( \varsigma \) are real or complex constants, \( |\gamma| + |\varsigma| > 0, \ t \geq 0, \ x \in \mathbb{R}^n, \ x = (x_1, ..., x_n), \)

\[ \Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \]

denotes the Laplace operator, \( n \geq 2, \ I \) is the unit operator (see Ch. 3, Sect. 6 in [15]). Classically real constants \( \alpha > 0, \beta, \gamma > 0 \) and \( \varsigma \) are considered. In the latter case by changing the variables \( t \) and \( x \) and with the help of the dilation \( u \mapsto bu, \) where \( b \) is a real constant, one can choose \( \alpha = 1 \) and \( \gamma = 1/2. \) Particularly if \( \beta = 0 \) and \( \varsigma = 0 \) and \( m = 1, \) this corresponds to the Burgers PDE.

It is useful to take the complexified Cayley-Dickson algebra \( A_{r,C} = A_r \oplus (A_i, i) \), where \( i^2 = -1, \ ib = bi \) for each \( b \in A_r, \ 2 \leq r < \infty. \) That is any complexified Cayley-Dickson number \( z \in A_{r,C} \) has the form \( z = x + iy \) with \( x \) and \( y \) in \( A_r, \ x = x_0i_0 + x_1i_1 + ... + x_{2^r-1}i_{2^r-1}, \) while \( x_0, ..., x_{2^r-1} \) are in \( \mathbb{R} \) (see also the notation in Subsection 1). The real part of \( z \) is \( Re(z) = x_0 = (z + z^*)/2, \) the imaginary part of \( z \) is defined as \( Im(z) = z - Re(z), \) where the conjugate of \( z \) is \( z^* = \bar{z} = Re(z) - Im(z), \) that is \( z^* = x^* - iy \) with \( x^* = x_0i_0 - x_1i_1 - ... - x_{2^r-1}i_{2^r-1}. \) Then put \( |z|^2 = |x|^2 + |y|^2 \), where \( |x|^2 = xx^* = x_0^2 + ... + x_{2^r-1}^2. \) Mention that the operator \( \pi_j \) (see Formulas (1.5), (1.6) above) has the natural \( \mathbb{C} \)-linear extension: \( \pi_j(au + bz) = a\pi_j(u) + b\pi_j(z) \) for every \( a \) and \( b \) in \( \mathbb{C} \) and \( v \) and \( z \) in \( A_{r,C}. \) In another words the operator \( \pi_j \) is \( \mathbb{C} \)-homogeneous and \( A_{r,C} \)-additive.

Using Theorem 1 of the preceding section we write the corresponding PDE over the Cayley-Dickson algebra \( A_r \) with \( 2 \leq r \) and \( 2^r > n. \) Put \( x_j = z_j \) for each \( j \geq 1, \) where \( z_0i_0 + z_1i_1 + z_2i_2 + ... = z \in A_r, \) then

\[ \Delta_z = -\sigma_z^2, \ \text{where} \ \sigma_z f(z) = \sum_{j=1}^{n} i_j^* \frac{\partial f(z)}{\partial z_j}, \]

\[ \frac{\partial}{\partial x_1} = \pi_1(\sigma_z), \ \text{where} \ \pi_j \ \text{is provided by Formula (1.5) for} \ j \geq 1. \) Functions \( u \) and \( f \) are supposed to be sufficient times differentiable by the corresponding variables, so that \( u \) and \( f \) more generally may have values not only in \( \mathbb{R}, \) but in \( \mathbb{C} \) also.
Let $\Omega$ be a set supplied with an algebra $\mathcal{F}$ of its subsets and let $\mu : \mathcal{F} \to [0, 1]$ be a probability measure and let $\mathcal{F}$ be $\mu$-complete. We take the PDE:

$$E \{Q(\frac{\partial}{\partial t})(-\Delta_x + \alpha \Delta_x + \beta I)u(t, x; \omega) + \gamma \frac{\partial u^2(t, x; \omega)}{\partial x_1} + \varsigma u^2(t, x; \omega) \} = 0,$$

where $\omega \in \Omega$, $u(t, x; \omega)$ is a random function, $Eg$ denotes a mean value (expectation) of a random variable $g$ whenever it exists:

$$Eg = \int_{\Omega} g(\omega)\mu(d\omega).$$

In order to use the noncommutative integral over $A_r$ approach the following PDE generalizing (4.2) is written in the form:

$$E \{Q(\frac{\partial}{\partial t})S_0u(t, x, y; \omega) + \gamma \pi_1(\sigma_x + \sigma_y)(u^2(t, x, y; \omega)) + \varsigma u^2(t, x, y; \omega) \} |_{x=y} = 0,$$

where $x$ and $y$ are in $V \subset A_r$, $V = Ri_1 \oplus ... \oplus Ri_n$,

$$S_0 = - (\sigma_x^2 + \sigma_y^2)^2 + a(\sigma_x^2 + \sigma_y^2) + bI,$$

$a \in \mathbb{C} \setminus \{0\}, \ b \in \mathbb{C}$, the Dirac operator in (4.5) is more general:

$$\sigma_x f(x) = \sum_{j=0}^{2^r-1} i_j^* (\partial f(x)/\partial x_{\xi(j)}) \psi_j$$

with real constants $\psi_j \in \mathbb{R}$ so that $\psi_0^2 + ... + \psi_{2^r-1}^2 > 0$. Particularly it is convenient to choose $\psi_j = 2^{-1/2}$ for each $j = 1, ..., n$; $\psi_j = 0$ otherwise for $j = 0$ or $j > n$; $\xi(j) = j$ for each $j = 0, ..., 2^r - 1$. In this particular case $-\frac{1}{2} \Delta_x = \sigma_x^2$.

5. Auxiliary PDE. Consider the auxiliary PDE

$$\{S_{2,a}v(x, y) + q_1 \pi_1(\sigma_x + \sigma_y)(v^2(x, y)) + q_2 v^2(x, y) \} |_{x=y} = 0,$$

where $S_{2,a} = a_1(\sigma_x^2 + \sigma_y^2)^2 + a_2(\sigma_x^2 + \sigma_y^2) + a_3 I$,

$a_1 \in \mathbb{C} \setminus \{0\}, \ a_2 \in \mathbb{C}$ and $a_3 \in \mathbb{C}$ are constants, $a_1$ is nonzero, $a = (a_1, a_2, a_3)$, $\mathbb{C} = \mathbb{R} \oplus Ri$; $q_1 = -2a_1 p_1$, $q_2 = -2a_1 p_2$, $p_1$ and $p_2$ are in $\mathbb{C}$ with $|p_1| + |p_2| > 0$; where $i_j i_k = i_k i_j$ for each $j, \ i^2 = -1$. Remind that the family $\{i_0, i_1, ..., i_{2^r-1}\}$ denotes the standard basis of the Cayley-Dickson algebra $A_r$ over $\mathbb{R}$ such that $i_0 = 1$, $i_j^2 = -1$, $i_j i_k = -i_k i_j$ for each $j \geq 1$ and $k \geq 1$ with $j \neq k$. For its solution take two PDOs:
(5.3) \( S_1 = \sigma_x^2 - \sigma_y^2 \)
and \( S_2 = S_{2,a} \), where \( a = (a_1, a_2, a_3) \). Suppose that a function \( F(x, y) = F_a(x, y) \) satisfies the conditions

(5.4) \( S_j F(x, y) = 0 \) for \( j = 1 \) and \( j = 2 \). Put also

(5.5) \( K(x, y) = F(x, y) + AK(x, y) \) with

(5.6) \[
AK(x, y) = p_1 \pi_1(\sigma \int^{\infty}_w (\sigma \int^{\infty}_w F(z, v)K(w, z)dz)dv)dw
+ p_2 \sigma \int^{\infty}_w (\sigma \int^{\infty}_{w_0} (\sigma \int^{\infty}_w F(z, v)K(w, z)dz)dv)dw)ds,
\]
where \( p_1 \) and \( p_2 \) are real or complex parameters with \(|p_1| + |p_2| > 0\); \( K = K_{a,q}, \quad a = (a_1, a_2, a_3), \quad q = (q_1, q_2), \quad w_0 \) is marked point in a domain \( \mathcal{U} \) satisfying Conditions 3.1(D1) and (D2) in \([32]\).

According to Formulas (5.3) – (5.6) it is convenient to consider the functions \( F \) and \( K \) of the form:

(5.7) \( F(x, y) = F(\frac{x+y}{2}) \) and \( K(x, y) = K(\frac{x+y}{2}) \).

(5.8) Let \( F \) and \( K \) be with values in \( \mathbb{R} \) or \( \mathbb{C} \) and let they satisfy other Conditions of Proposition 3.4 in \([32]\) with \( m = 4 \).

Apparently any pair of operators from the family \( \{S_1, S_{2,a}, bI, \pi_j : b \in \mathbb{C}; j = 0, ..., 2^{r-1}\} \) commutes, since \( S_1 \) and \( S_{2,a} \) are PDOs with constant real or complex coefficients, where \( I \) notates the unit operator. From this proposition and Formula (3.2) in \([32]\) we infer that

\[
(S_{2,a} - a_3 I)K(x, y) = (S_{2,a} - a_3 I)F(x, y) + 
\]
\[
p_1 \pi_1(S_{2,a} - a_3 I)((\sigma \int^{\infty}_w (\sigma \int^{\infty}_w F(z, v)K(w, z)dz)dv)dw))
+ p_2 (S_{2,a} - a_3 I)(\sigma \int^{\infty}_w (\sigma \int^{\infty}_w F(z, v)K(w, z)dz)dv)dw)ds
= -a_3 F(x, y) + p_1 \pi_1(\{a_1(\sigma_x^2 + \sigma_y^2) + a_2 I\}(\sigma \int^{\infty}_x F(z, y)K(x, z)dz)
+ p_2 a_1(\sigma_x + \sigma_y)(\sigma \int^{\infty}_x F(z, y)K(x, z)dz)
+ p_2 a_2(\sigma \int^{\infty}_x F(z, y)K(x, z)dz)dv
= -a_3 F(x, y) + p_1 \pi_1(\{a_1(2\sigma_x^2 + 2\sigma_y^2) + a_2 I\}(\sigma \int^{\infty}_x F(z, y)K(x, z)dz)
+ p_1 \pi_1(a_1(A_2(F, K)(x, y) + B_2(F, K)(x, y))
+ p_2 a_1(\sigma_x^2 + \sigma_y^2)\int^{\infty}_x F(z, y)K(x, z)dz)
+ p_2 a_1(A_1(F, K)(x, y) + B_1(F, K)(x, y))
\]
Therefore

\[ S_{2,a}K(x,y) = \frac{2}{a} S_{2,a} p_1 \pi_1 ((\sigma \int_{w_0}^y (\sigma \int_{x}^\infty F(z_v)K(x,z)dz)dv)) + p_1 \pi_1 a_1 (A_2(F,K)(x,y)+B_2(F,K)(x,y))+p_2 a_1 (A_1(F,K)(x,y)+B_1(F,K)(x,y)). \]

By virtue of Corollary 3.6 in [32]

\[ A_2(F,K)(x,y) + B_2(F,K)(x,y) = -\sigma_x [F(x,y)K(x,x)] - 2\sigma_x [F(z,y)K(x,z)]|_{z=x} \]
\[ -1\sigma_z [F(z,y)K(x,z)]|_{z=x} + 2\sigma_z [F(z,y)K(x,z)]|_{z=x} \]

\[ A_1(F,K)(x,y) + B_1(F,K)(x,y) = -2F(x,y)K(x,x). \]

Moreover, this operator \( A \) restricted on any compact domain \( V \) in \( \mathcal{U} \) is compact due to Proposition 4.1 in [32] and Formula (5.6). The PDOs \( S_j \) commute with \( \sigma_x \) and \( \sigma_y \) for \( j = 1 \) and \( j = 2 \), since \( \text{ii}_k = i_ki \) for each \( k \). Thus from Formulas (3.2) in [32] and (5.9) and (5.10) given above the theorem follows.

6. Theorem. Let suppositions (5.8) be fulfilled, then the PDE (5.1) has a solution \( K_{\alpha,\beta} \) provided by Formulas (5.4) – (5.6) on \( (\mathcal{U} \cap V)^2 \) for a sufficiently small \( 0 < |p_1| + |p_2| \).

Below a random operator valued measure approach to a solution of Sobolev-Burgers PDEs is described. To avoid a misunderstanding we first recall necessary definitions and describe a notation and provide necessary statements.

7. Definition. Orthogonal random operator valued measure.

For Banach spaces \( X \) and \( Y \) both over \( \mathbf{F} \) by \( \mathcal{L}(X,Y) \) is denoted the space of all bounded linear operators \( J \) from \( X \) into \( Y \), where either \( \mathbf{F} = \mathbb{C} \) or \( \mathbf{F} = \mathbb{R} \). If it is supplied with the operator norm topology \( \tau_{||\cdot||} \), then \( (\mathcal{L}(X,Y),\tau_{||\cdot||}) \) is a Banach space, where \( |J| = \sup_{x \in X, \ 0 < |x| \leq 1} |Jx|_Y/|x|_X \) for any \( J \in \mathcal{L}(X,Y) \), \( \cdot \) \( \cdot \) denotes a norm on \( X \). A strong operator topology \( \tau_s \) on \( \mathcal{L}(X,Y) \) possesses a base of neighborhoods of 0 of the form \( Z_s(\epsilon,x) := \{ J \in \mathcal{L}(X,Y) : |Jx|_Y < \epsilon \} \), where \( x \in X, \ 0 < \epsilon \). A weak operator topology \( \tau_w \) is induced by a base of neighborhoods of zero \( Z_w(\epsilon,x,y') := \{ J \in \mathcal{L}(X,Y) : |y'(Jx)| < \epsilon \} \), where \( x \in X, \ y' \in Y', \ 0 < \epsilon \). \( Y' \) is a
topological dual space of all continuous linear functionals $y' : Y \to F$. For short the topological locally convex vector space $(\mathcal{L}(X,Y), \tau_\kappa)$ will also be denoted by $\mathcal{L}(X,Y)_\kappa$, where $\kappa \in \{| \cdot |, s, w\}$.

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, where $\nu$ is a nonnegative $\sigma$-finite and $\sigma$-additive measure on a $\sigma$-algebra $\mathcal{F}$ of a set $\Omega$. For the Banach space $Y$ over $F$ we notate a Banach space of all $(\mathcal{F}, B(Y))$-measurable functions $f : \Omega \to Y$ such that

$$\tag{7.1} |f|_v := \left[ \int_{\Omega} |f(\omega)|^v \nu(d\omega) \right]^{1/v} < \infty$$

by $L^v(\Omega, \mathcal{F}, \nu, Y)$, where $1 \leq v < \infty$, while $B(Y)$ is the Borel $\sigma$-algebra on $Y$.

Then $L^v(\Omega, \mathcal{F}, \nu; \mathcal{L}(X,Y)_\kappa)$ denotes a linear topological space of all $(\mathcal{F}, B(\mathcal{L}(X,Y)_\kappa))$ measurable functions $G : \Omega \to \mathcal{L}(X,Y)_\kappa$ such that either for $\kappa = | \cdot |$

$$\tag{7.2} |G|_v := \left[ \int_{\Omega} |G(\omega)|^v \nu(d\omega) \right]^{1/v} < \infty$$

or for $\kappa = s$ and each $x \in X$

$$\tag{7.3} \rho(G)_{v,x} := \left[ \int_{\Omega} |G(\omega)x|^v \nu(d\omega) \right]^{1/v} < \infty$$

or for $\kappa = w$ and every $x \in X$ and $y' \in Y'$

$$\tag{7.4} \rho(G)_{v,x,y'} := \left[ \int_{\Omega} |y'(G(\omega)x)|^v \nu(d\omega) \right]^{1/v} < \infty.$$
\( \hat{m}(M_1, M_2) := E(H^*(M_1)(\omega)H(M_2)(\omega)) \) exists and belongs to \( \mathcal{L}(X, X_w) \), where \( \kappa \in \{ \cdot | \cdot, s \} \); \( H^*(M) \) is an adjoint operator, that is \( (H^*(M)y, x)_X = (y, H(M)x)_Y \) for each \( x \in \mathcal{D}(H(M)) \) and \( y \in \mathcal{D}(H^*(M)) \), where \( \mathcal{D}(H^*(M)) \subset Y \), since \( Y \) is the Hilbert space and hence \( Y' \) is isomorphic with \( Y \), \( H^*(M) : \mathcal{D}(H^*(M)) \to X \). Shortly \( \hat{m}(M) \) also will be written instead of \( \hat{m}(M, M) \).

(7.8) \( \hat{m}(M_1, M_2) = 0 \) if \( M_1 \cap M_2 = \emptyset \).

The family of random operators \( \{H(M) : M \in \mathcal{M}\} \) satisfying Conditions (7.5) – (7.7) is called an elementary random operator valued measure and \( \hat{m} \) is called its structural function. It is called orthogonal if (7.8) also is valid.

For a Hilbert space \( X \) over \( \mathbb{C} \) by \( (x, y)_X = (x, y) \) is denoted a scalar product on \( X \) with values in \( \mathbb{C} \) so that \( (\beta x, y) = \beta (x, y) \) and \( (x, y + u) = (x, y) + (x, u) \) and \( (x, y) = (y, x) \) for every \( x \) and \( y \) and \( u \) in \( X \) and \( \beta \in \mathbb{C} \), where \( \bar{\beta} \) is the complex conjugated number of \( \beta \). An induced norm is \( |x|_X = \sqrt{(x, x)_X} \) for each \( x \in X \). Particularly, if \( X \) is a Hilbert space over \( \mathbb{R} \), then \( (x, \beta y) = \beta (x, y) \) and \( (x, y) = (y, x) \in \mathbb{R} \), where \( \beta \in \mathbb{R} \).

8. Lemma. Suppose that \( H \) is an elementary orthogonal random operator valued measure with a structural function \( \hat{m} \). If \( M_1 \) and \( M_2 \) are in \( \mathcal{M} \), then

(8.1) \( \hat{m}(M_1, M_2) = \hat{m}(M_1 \cap M_2) \).

For each \( M \in \mathcal{M} \) an operator \( \hat{m}(M) \) is nonnegative definite:

(8.2) for every \( c_1, ..., c_n \in \mathbb{C} \) and \( x_1, ..., x_n \in X \)

\[
\sum_{j,k=1}^{n} c_j \bar{c}_k (\hat{m}(M)x_j, x_k)_X \geq 0.
\]

If \( M_1 \cap M_2 = \emptyset \), then

(8.3) \( \hat{m}(M_1 \cup M_2) = \hat{m}(M_1) + \hat{m}(M_2) \).

For every \( x \) and \( y \) in \( X \) and \( N \in \mathcal{M} \)

(8.4) \( |\hat{m}_{x,y}(N)| \leq \sqrt{\hat{m}_{x,x}(N) \cdot \hat{m}_{y,y}(N)} \),

where \( \hat{m}_{x,y}(N) := (\hat{m}(N)x, y)_X \).

Proof. Assertions (8.1) and (8.2) follow from (7.5) – (7.8), since \( M_1 \setminus M_2 \in \mathcal{M} \) and \( M_2 \setminus M_1 \in \mathcal{M} \), \( M_1 \cap (M_2 \setminus M_1) = \emptyset \) and \( M_2 \cap (M_1 \setminus M_2) = \emptyset \) and

\[
\hat{m}(M_1, M_2) = EH^*(M_1 \cap M_2)H(M_1 \cap M_2) + EH^*(M_1 \setminus M_2)H(M_1 \cap M_2) + EH^*(M_1 \cap M_2)H(M_2 \setminus M_1) + EH^*(M_1 \setminus M_2)H(M_2 \setminus M_1) = \hat{m}(M_1 \cap M_2).
\]

Therefore, if \( M_1 \cap M_2 = \emptyset \), then
\( \hat{m}(M_1 \cup M_2) = EH^*(M_1)H(M_1) + EH^*(M_1)H(M_2) + EH^*(M_2)H(M_1) + EH^*(M_2)H(M_2) = \hat{m}(M_1) + \hat{m}(M_2). \) Using (7.5) and (7.7) one gets that

\[
\sum_{j,k=1}^n c_j \hat{c}_k(\hat{m}(M)x_j, x_k) = E(H(M)x, H(M)x) \geq 0,
\]

where \( x = c_1 x_1 + ... + c_n x_n. \) Then from

\[
|\hat{m}_{x,y}(N)| = |E(H(N)x, H(N)y)| \leq E(|H(N)x| \cdot |H(N)y|) \\
\leq \sqrt{E|H(N)x|^2} \cdot E|H(N)x|^2,
\]

Inequality (8.4) follows.

9. Definition. Let \( L^0(\mathcal{M}, X) \) denote a linear space of all step functions \( f : \Lambda \rightarrow X \) such that \( f(\lambda) = \sum_{k=1}^n \chi_{M_k}(\lambda) a_k \), where \( X \) is a Hilbert space over \( \mathbf{F} \), \( M_k \in \mathcal{M}, \ a_k \in X \) for each \( k = 1, ..., \iota; \ i \in \mathbb{N}; \ \chi_{M}(\lambda) \) is the characteristic function of a set \( M \), that is \( \chi_{M}(\lambda) = 1 \) for each \( \lambda \in M \), whilst \( \chi_{M}(\lambda) = 0 \) for each \( \lambda \notin M \).

For each \( f \in L^0(\mathcal{M}, X) \) the integral relative to \( H \) is defined:

\[
(9.1) \quad \psi(f) = \int_{\Lambda} H(d\lambda)f(\lambda) := \sum_{k=1}^\iota H(M_k)a_k.
\]

By \( L^0(H, Y) \) we denote the family of all random vectors \( \eta = \psi(f) \) of the form (9.1).

A sequence of random vectors \( h_i \in L^2(\Omega, \mathcal{F}, \mu, Y) \) mean square converges to \( h \) if \( \lim_{i \to \infty} |h - h_i|_2 = 0 \) and this is denoted by \( l.m.s. \lim_{i \to \infty} h_i = h. \)

10. Remark. If \( f \) and \( g \) are in \( L^0(\mathcal{M}, X) \), then \( \iota \in \mathbb{N} \) and \( N_k \in \mathcal{M} \) for each \( k = 1, ..., \iota \) can be chosen such that \( f(\lambda) = \sum_{k=1}^\iota \chi_{N_k}(\lambda)a_k \) and \( g(\lambda) = \sum_{k=1}^\iota \chi_{N_k}(\lambda)b_k \) for each \( \lambda \in \Lambda \), where \( a_k \) and \( b_k \) are in \( X \) for each \( k \).

Therefore from Lemma 8 and Condition (7.7) it follows that

\[
(10.1) \quad E(\int_{\Lambda} H(d\lambda)f(\lambda), \int_{\Lambda} H(d\lambda)g(\lambda)) = \sum_{k=1}^\iota (\hat{m}(N_k)a_k, b_k)_X.
\]

The space \((L(X, Y), \tau_{|\iota|})\) is Banach. By virtue of the Banach-Steinhaus theorem (11.6.1) in [RS] or 7.1.3 in [Z] \((L(X, Y), \tau_\kappa)\) is complete as the topological locally convex vector space also for \( \kappa = s \) and \( \kappa = w. \)

Remind that a measure \( \hat{m} \) on \( \mathcal{M} \) with values in \((L(X, X), \tau_\kappa)\) is called \( \sigma \)-finite, if \( \Lambda = \bigcup_{k=1}^\infty M_k \), where \( M_k \in \mathcal{M} \) for each \( k \in \mathbb{N} \). If a finitely additive \( \sigma \)-finite nonnegative definite \((L(X, X), \tau_w)\) valued measure \( \hat{m} \) satisfies...
This implies that the semi-additivity condition:

\[
(10.2) \quad \hat{m}(N) \leq \sum_{k=1}^{\infty} \hat{m}(N_k)
\]

for every \(N\) and \(N_k\) in \(\mathcal{M}\) fulfilling the inclusion \(N \subseteq \bigcup_{k=1}^{\infty} N_k\), that is

\[
\hat{m}_x(N) := (\hat{m}(N)x,x)_X \leq \sum_{k=1}^{\infty} (\hat{m}(N_k)x,x)_X
\]

for each \(x \in X\), then \(\hat{m}\) has a unique extension on a minimal \(\sigma\)-algebra \(\sigma \mathcal{M}\) generated by \(\mathcal{M}\), since \(\hat{m}_x\) is with values in \([0, \infty)\) and has a unique extension on \(\sigma \mathcal{M}\) for each \(x \in X\) (see Theorem II.2.3 in [12]). In this case the structural function \(\hat{m}\) will be called a structural operator valued measure.

Notice that \(\forall M \in \sigma \mathcal{M} \ ((\hat{m}(M) = 0) \iff (\forall x \in X \ \hat{m}_x(M) = 0)))\). Due to the scalar product properties \(\hat{m}_{x,y}(M) := (\hat{m}(M)x,y)_X\) can be expressed through a linear combination of \(\hat{m}_{x,z} = \hat{m}_z\) for suitable vectors \(z \in \{x,y,x\pm y\}\) over \(F = \mathbb{R}\) or \(z \in \{x,y,x \pm y,x \pm iy\}\) over \(F = \mathbb{C}\). Therefore \(\hat{m}\) can be extended to a complete \(\sigma\)-additive measure on \(\mathcal{B} = \mathcal{B}_m(\Lambda)\), where a \(\sigma\)-algebra \(\mathcal{B}\) is the completion of \(\sigma \mathcal{M}\) by \(\hat{m}\)-null sets: \(\hat{m}(N) = 0\) if \(N \subseteq M\) and \(M \in \sigma \mathcal{M}\) and \(\hat{m}(M) = 0\). We denote by \(L^2(\Lambda, \mathcal{B}, \hat{m}, X)\) the completion of \(L^0(\mathcal{B}, X)\) relative to a norm \(|f|_{2,\hat{m}} = \sqrt{(f,f)_{\hat{m}}}\) induced by a scalar product

\[
(10.3) \quad (f,g)_{\hat{m}} := \int_{\Lambda} (\hat{m}(d\lambda)f(\lambda),g(\lambda))_X.
\]

This implies that \(L^0(\mathcal{M}, X)\) is a linear subspace in \(L^2(\Lambda, \mathcal{B}, \hat{m}, X)\), hence a closure \(L^2(\mathcal{M}, X)\) of \(L^0(\mathcal{M}, X)\) in \(L^2(\Lambda, \mathcal{B}, \hat{m}, X)\) exists. The closure of \(L^0(H,Y)\) in \(L^{v}(\Omega, \mathcal{F}, \mu, Y)\) will be denoted by \(L^{v}(H,Y)\), where \(1 \leq v < \infty\).

Formulas (9.1) and (10.1) induce a linear isometry \(\psi\) from \(L^2(\mathcal{M}, X)\) into \(L^2(H,Y)\). Hence this integral has an extension

\[
(10.4) \quad \int_{\Lambda} H(d\lambda)f(\lambda) := \psi(f)
\]

for each \(f \in L^2(\mathcal{M}, X)\).

From Lemma 8 and Remark 10 assertions of Theorem 11 follow.

11. **Theorem.** If Conditions (7.5) – (7.8) and (10.2) are satisfied, then

\[
(11.1) \quad \int_{\Lambda} H(d\lambda)(af(\lambda) + bg(\lambda)) = a\int_{\Lambda} H(d\lambda)f(\lambda) + b\int_{\Lambda} H(d\lambda)g(\lambda);
\]

for every \(f\) and \(g\) in \(L^2(\Lambda, \mathcal{B}, \hat{m}, X)\) and \(a\) and \(b\) in \(\mathbb{C}\).
(11.2) if a sequence \( f_i \) converges to \( f \) in \( L^2(\Lambda, \mathcal{B}, \hat{m}, X) \), then
\[
\int_{\Lambda} H(d\lambda) f(\lambda) = \lim_{i \to \infty} \int_{\Lambda} H(d\lambda) f_i(\lambda).
\]

12. Remark. Extension of an elementary orthogonal random operator valued measure. We put
\[
(12.1) \mathcal{B}_{0,\kappa} := \{ N \in \mathcal{B} : H(N) \in L^2(\Omega, \mathcal{F}, \mu, \mathcal{L}(X,Y)_\kappa) \}
\]
& \( \hat{m}(N) \in L(X,X)_w \}, \text{ where } \kappa \in \{|·|, s\}.
\]
Let \( \hat{H}(N) = \int_{\Lambda} H(d\lambda) \chi_N(\lambda) \), then
\[
(12.2) \hat{H} \text{ is defined on } \mathcal{B}_{0,\kappa} \text{ and}
\]
\[
(12.3) \text{If } N_k \in \mathcal{B}_{0,\kappa} \text{ for each } k = 0, 1, 2, \ldots \text{ and } N_0 = \bigcup_{k=1}^{\infty} N_k \text{ with } N_k \cap N_l = \emptyset \text{ for each } k \neq l, \text{ then } \hat{H}(N_0) = \sum_{k=1}^{\infty} \hat{H}(N_k) \text{ and this series converges in } L^2(\Omega, \mathcal{F}, \mu, \mathcal{L}(X,Y)_s) \text{ according to Theorem 11, that is for each } x \in X
\]
\[
\hat{H}(N_0)x = \lim_{i \to \infty} \sum_{k=1}^{\infty} \hat{H}(N_k)x
\]
(see Definition 9) and
\[
(12.4) \forall N \in \mathcal{B}_{0,\kappa} \quad \forall M \in \mathcal{B}_{0,\kappa} \quad E\hat{H}^*(N)\hat{H}(M) = \hat{m}(N \cap M) \quad \text{and}
\]
\[
(12.5) \forall M \in \mathcal{M} \quad \hat{H}(M) = H(M).
\]

13. Definition. A random function \( \hat{H} \) satisfying Conditions (12.2) – (12.4) is called an orthogonal random operator valued measure.

From Lemma 8 and Theorem 11 and Remark 12 the theorem follows.

14. Theorem. If a structural function \( \hat{m} \) of an elementary orthogonal random operator valued measure \( H \) is semi-additive (see Formula (10.2)), then an orthogonal random operator valued measure \( \hat{H} \) extension of \( H \) exists.

15. Corollary. If the conditions of Theorem 14 are satisfied, then
\( L^2(H,Y) \) is isomorphic with \( L^2(\hat{H},Y) \) and
\[
\forall f \in L^2(\hat{H},Y) \exists f_{\hat{H}} H(d\lambda)f(\lambda) = \int_{\Lambda} \hat{H}(d\lambda)f(\lambda).
\]

16. Remark. Let \( H \) be an orthogonal random operator valued measure and let \( \hat{m} \) be its complete structural operator valued measure. For short \( H \) will be written instead of \( \hat{H} \). For each \( N \in \mathcal{B} \) and \( g \in L^2(\Lambda, \mathcal{B}, \hat{m}, X) \) we put
\[
(16.1) \eta(N) := \int_{\Lambda} H(d\lambda)g(\lambda)\chi_N(\lambda), \text{ consequently,}
\]
\[
(16.2) E(\eta(N), \eta(M))_{Y^*} = \int_{N \cap M} (\hat{m}(d\lambda)g(\lambda), g(\lambda))_X
\]

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for all \( N \) and \( M \) in \( \mathcal{B} \). Then we put

\[
(16.3) \quad \hat{n}(N) = \int_N (\hat{m}(d\lambda)g(\lambda), g(\lambda))_X.
\]

Therefore \( \eta \) is an orthogonal random vector valued measure and \( \hat{n} \) is its complete structural measure such that \( \eta(N) \in L^2(\Omega, \mathcal{F}, \mu; Z) \) and \( \hat{n}(N) \geq 0 \) for each \( N \in \mathcal{B} \), where either \( Z = (Y', | \cdot |_{Y'}) \) if \( \tau = \tau |_1 \) or \( Z = (Y', \sigma(Y', Y)) \) if \( \tau = \tau_s \), since \( Y \) is the Hilbert space over \( F \) and \( Z = L(Y, F)_{\kappa} \), where \( \sigma(Y', Y) \) denotes the weak topology on the topological dual space \( Y' \).

Since \( Y \) is the Hilbert space, then \( Y' \) and \( Y \) are isomorphic. That is \( Y' \) can be replaced on \( Y \) in the notation.

17. Lemma. If \( f \in L^2(\Lambda, \mathcal{B}, \hat{n}, F) \) and the conditions of Remark 16 are fulfilled, then \( fg \in L^2(\Lambda, \mathcal{B}, \hat{m}, X) \) and

\[
(17.1) \quad \int_{\Lambda} f(\lambda)\eta(d\lambda) = \int_{\Lambda} H(d\lambda)f(\lambda)g(\lambda).
\]

Proof. Take a fundamental sequence of step functions \( f_\iota \) in \( L^2(\Lambda, \mathcal{B}, \hat{n}, F) \). Then

\[
E(\int_{\Lambda} (f_k(\lambda) - f_{k+l}(\lambda))\eta(d\lambda)^2) = \int_{\Lambda} |f_k(\lambda) - f_{k+l}(\lambda)|^2\hat{n}(d\lambda)
\]

for each pair of natural numbers \( k \) and \( l \), consequently, the sequence \( f_\iota g_\iota \) is fundamental in \( L^2(\Lambda, \mathcal{B}, \hat{m}, X) \), hence

\[
\exists \lim_{\iota \to \infty} \int_{\Lambda} f_\iota(\lambda)\eta(d\lambda) = \lim_{\iota \to \infty} \int_{\Lambda} H(d\lambda)f_\iota(\lambda)g(\lambda).
\]

Thus for \( f = \lim_{\iota \to \infty} f_\iota \) Formula (17.1) follows.

18. Remark. Consider an open or a canonically closed domain \( V \) in the Euclidean space \( \mathbb{R}^k \) or in the unitary space \( \mathbb{C}^k \) and let \( \mathbf{l} \) be a Lebesgue measure on it. Suppose that \( \mathcal{B}(V) \) is a \( \mathbf{l} \) complete \( \sigma \)-algebra containing the Borel \( \sigma \)-algebra \( B(V) \) of \( V \), \( B(X) \) is the Borel \( \sigma \)-algebra of \((X, |\cdot|_X)\), \( \sigma(\mathcal{B}(V) \times \mathcal{B}) \) denotes the minimal \( \sigma \)-algebra containing \( \mathcal{B}(V) \times \mathcal{B} \), \( \mathcal{B} \) is the complete \( \sigma \)-algebra on \( \Lambda \) as above.

Let \( g(\tau, \lambda) \) be a \((\sigma(\mathcal{B}(V) \times \mathcal{B}), B(X))\) measurable function from \( V \times \Lambda \) into a Hilbert space \( X \) over \( F \), \( g \in L^2(V \times \Lambda, \sigma(\mathcal{B}(V) \times \mathcal{B}), \mathbf{l} \times \hat{m}, X) \) and such that for each marked \( \tau \in V \) the vector valued function \( g(\tau, \lambda) \) considered in
the \( \lambda \) variable belongs to \( L^2(\Lambda, \mathcal{B}, \hat{m}, X) \), where \( \lambda \in \Lambda \). If \( H \) is an orthogonal random operator valued measure and \( \hat{m} \) is its complete structural operator valued measure, then the integral

\[
(18.1) \quad \xi(\tau) = \int_\Lambda H(d\lambda)g(\tau, \lambda)
\]

is defined for each \( \tau \in V \) and for \( \mu \)-a.e. \( \omega \in \Omega \) (see Theorem 14 and Corollary 15).

19. Lemma. If conditions of Remark 18 are satisfied, then the integral in Formula (18.1) as a function in \( \tau \) can be defined such that \( \xi(\tau) \) will be \( (\sigma(\mathcal{B}(V) \times \mathcal{F}), B(Y)) \) measurable.

Proof. If \( g \) is a step function

\[
(19.1) \quad g(\tau, \lambda) = \sum_{k=1}^\iota \chi_{N_k}(\tau)\chi_{M_k}(\lambda)x_k
\]

with \( N_k \in \mathcal{B}(V) \), \( M_k \in \mathcal{B} \), \( x_k \in X \) for \( k = 1, \ldots, \iota \), then

\[
\xi(\tau) = \sum_{k=1}^\iota H(M_k)\chi_{N_k}(\tau)x_k,
\]

consequently, \( \xi(\tau) = \xi(\tau)(\omega) \) is \( (\sigma(\mathcal{B}(V) \times \mathcal{F}), B(Y)) \) measurable, since \( Y' \) and \( Y \) are isomorphic. Choose a sequence of step functions \( g_\iota \) of the form (19.1) so that \( g_\iota \) converges to \( g \) in \( L^2(\mathcal{B}(V) \times \mathcal{B}), \mathcal{B}(Y)) \). Let \( \xi_\iota \) be defined by Formula (18.1) for \( g_\iota \), where \( \iota = 1, 2, \ldots \). The space \( L^2(\mathcal{B}(V) \times \mathcal{F}), \mathcal{B}(Y)) \) is complete, since \( Y \) is the Hilbert space over \( \mathbf{F} \). The sequence \( \xi_\iota \) is fundamental in \( \Gamma(Y) \), hence converges to some \( \eta \) in it. On the other hand,

\[
E\left(\int_V |\eta(\tau) - \xi_\iota(\tau)|^2 \mathbf{1}(d\tau)\right) = \int_\Lambda \int_V (\hat{m}(d\lambda)(g(\tau, \lambda) - g_\iota(\tau, \lambda), (g(\tau, \lambda) - g_\iota(\tau, \lambda)))\mathbf{1}(d\tau)
\]

for each \( \iota \). Taking the limit when \( \iota \) tends to the infinity we deduce that

\[
E(|\eta(\tau) - \xi(\tau)|^2) = 0
\]

for \( \mathbf{1} \)-a.e. \( \tau \) in \( V \). Modifying \( \xi \) in the following manner

\[
\hat{\xi}(\tau) = \eta(\tau) \text{ if } \mu\{\omega \in \Omega : \xi(\tau)(\omega) \neq \eta(\tau)(\omega)\} = 0,
\]

also \( \hat{\xi}(\tau) = \xi(\tau) \) if \( \mu\{\omega \in \Omega : \xi(\tau)(\omega) \neq \eta(\tau)(\omega)\} > 0 \), we get a \( (\sigma(\mathcal{B}(V) \times \mathcal{F}), B(Y)) \) measurable random function \( \hat{\xi}(\tau) \) with values in \( Y \). Two random functions \( \hat{\xi} \) and \( \xi \) differ on a set of \( \mathbf{1} \times \mu \) measure null, consequently, \( \hat{\xi} \) and \( \xi \) are randomly equivalent.
20. Remark. Using Lemma 19 we shall consider measurable random vector valued functions defined by the integral like in Formula (18.1).

21. Lemma. Suppose that the conditions of Remark 18 are fulfilled. If $g \in L^2(V \times \Lambda, \sigma(B(V) \times \mathcal{B}), 1 \times \dot{m}, X)$ and $h \in L^2(V, B(V), 1, F)$, then

$$\int_V \int_{\Lambda} H(d\lambda) h(\tau, \lambda) I(d\tau) = \int_{\Lambda} H(\lambda) f(\lambda),$$

where $f(\lambda) = \int_V h(\tau) g(\tau, \lambda) I(d\tau)$.

Proof. Equality (21.1) is valid for each step function $g$. Notice that

$$E(\int_{\Lambda} H(d\lambda) f(\lambda), \int_{\Lambda} H(d\lambda) f(\lambda)) = \int_{\Lambda} (\hat{m}(d\lambda) f(\lambda), f(\lambda))_X.$$

From the Cauchy-Bunyakowskii inequality and Lemma 8 it follows that

$$E(\int_{\Lambda} H(d\lambda) h(\tau, \lambda) I(d\tau)|^2) \leq (\int_{\Lambda} |h(\tau)|^2 I(d\tau))^2 \cdot (\int_{\Lambda} (\hat{m}(d\lambda) g(\tau, \lambda), g(\tau, \lambda))_X I(d\tau))^2.$$

Taking a sequence of step functions $g_i$ converging to $g$ in $L^2(V \times \Lambda, \sigma(B(V) \times \mathcal{B}), 1 \times \dot{m}, X)$ and using the equality (21.2) and the inequality (21.3) we infer Formula (21.1).

Below applications to PDEs of orthogonal random operator valued measures are described.

22. Remark. PDEs. Henceforth the unitary space $\Lambda = C^{m+3}$ is considered together with a $\sigma$-algebra $\mathcal{B} = B(\Lambda)$ generated by some semi-ring $\mathcal{M}$ of sets contained in $\Lambda$, where $m$ is the degree of the polynomial $Q(t)$ (see Subsection 4). Take an orthogonal random operator valued measure $H : B \to L^2(\Omega, \mathcal{F}, \mu; L(X, Y)_c)$ such that $\mathcal{B}$ is $\dot{m}$-complete, where $(\Omega, \mathcal{F}, \mu)$ is a probability space, either $\tau_n = \tau_{\lambda^0}$ or $\tau_n = \tau_{\lambda^s}$ (see Subsection 7). It also will be supposed that a singleton $\{\lambda\}$ is in $\mathcal{B}$ for each $\lambda$ in $\Lambda$.

We consider the Sobolev space $W_{2,m,A}([0, T] \times V_t^2, \mathcal{B}_1, 1, C)$ of all complex valued functions $m$ times Sobolev in $t$ and 4 times Sobolev in the variables $x_1, ..., x_n, y_1, ..., y_n, x = (x_1, ..., x_n) \in V_1$ and $y \in V_1$, where $V_1$ is a domain in $\mathbb{R}^n$, $1$ is the Lebesgue measure restricted on $[0, T] \times V_t^2$ from that of on $\mathbb{R}^{2(n+1)}$, $\mathcal{B}_1$ denotes an $1$ complete $\sigma$-algebra on $[0, T] \times V_t^2$ (see also Ch. III, Section 4 in [37]). That is

$$|f|_{W_{s,m,k}} = \left( \sum_{m_0=0}^{m} \sum_{m_1+m_2 \leq k} \int_0^T \int_{V_1} \int_{V_1} \left| \frac{\partial^{m_0+m_1+m_2}}{\partial t^{m_0} \partial x_j^{m_1} \partial y_k^{m_2}} f(t,x,y) \right|^s dt dx dy \right)^{1/s} < \infty$$
for each \( f \in W_{s,m,k}([0,T] \times V_1^2, \mathcal{B}_1, 1, C) \), where \( 1 \leq s < \infty, k \in \mathbb{N} \). We omit \( \mathcal{B}_1 \) and \( 1 \) in order to shorten the notation.

Let \( 0 < T < \infty \) and a set \( V_1 \) be canonically closed and compact in \( \mathbb{R}^n \) such that to it a domain \( V \) in \( \mathcal{A}_r \) corresponds by Formula (1.4). Let also \( w_0 \in \text{Int}(V) \) (see \( w_0 \) in Formula (5.6)). The Lebesgue measure on \([0,T] \times V_1^2\) induces the Lebesgue measure on \([0,T] \times V^2\). Then Formula (1.8) provides the Sobolev space \( W_{2,m,4}([0,T] \times V^2, C) \), which is taken as a Hilbert space \( X \). Then \( L^2([0,T] \times V^2, C) \) is chosen as a Hilbert space \( Y \). So it can be taken a restriction from \( \mathcal{U} \) onto \( V, V \subset \mathcal{U} \), where \( \mathcal{U} \) is a domain as in Subsection 5.

Let a function \( \phi(t, \lambda) \) in \( t \) with a parameter \( \lambda \) be a solution of the Cauchy problem

\[
(22.1) \quad Q(\frac{d}{dt})\phi(t, \lambda') = \lambda_1 \phi^2(t, \lambda') \quad \text{for} \quad t \geq 0,
\]

\[
(22.2) \quad \phi(0, \lambda') = \lambda_2, \quad \left. \frac{d}{dt} \phi(t, \lambda') \right|_{t=0} = \lambda_3, \ldots, \left. \frac{d^{m-1}}{dt^{m-1}} \phi(t, \lambda') \right|_{t=0} = \lambda_{m+1},
\]

where \( \lambda' = (\lambda_1, \ldots, \lambda_{m+1}) \), \( \lambda_1 \neq 0 \), \( \lambda_j \) is a real or a complex constant for each \( j = 1, \ldots, m + 1 \). In view of Theorem 1 in Subsection 3.1.5 in [35] this Cauchy problem has a unique solution \( \phi \) belonging to \( C^m([0, \infty), C) \) in the variable \( t \in [0, \infty) \), where \( C^m([0, \infty), C) \) denotes the space of all \( m \) times continuously differentiable functions from \([0, \infty)\) into \( C \). For each \( 0 < T < \infty \) the mapping \( C^{m+1} \ni \lambda' \mapsto \phi(\cdot, \lambda') \in C^m([0, T], C) \) is continuous. If \( \lambda' \in \mathbb{R}^{m+1} \) and \( \lambda_1 > 0 \), then \( \phi \) is real valued. To simplify notations we write \( \phi(t, \lambda) \) instead of \( \phi(t, \lambda') \).

Let \( a_1(\lambda) = -\lambda_1 \), \( a_2(\lambda) = -\alpha \lambda_1 \) and \( a_3(\lambda) = \beta \lambda_1 \), \( q_1(\lambda) = 2\lambda_1 \lambda_{m+2} \), \( q_2(\lambda) = 2\lambda_1 \lambda_{m+3} \), \( \lambda_{m+2} = p_1 \), \( \lambda_{m+3} = p_2 \) and \( p_2 \gamma = p_1 \varsigma \), where \( \gamma \) and \( \varsigma \) belong to \( C \), \( |\gamma| + |\varsigma| > 0 \); \( p_1 = 0 \) if \( \gamma = 0 \), \( p_2 = 0 \) if \( \varsigma = 0 \), \( p_1 \neq 0 \) if \( \gamma \neq 0 \), \( p_2 \neq 0 \) if \( \varsigma \neq 0 \) (see Formulas (4.2) and (4.5) and (5.2) also).

Let \( X_0 = C^{m,4}([0,T] \times V^2, C) \) denote the Banach space of \( m \) times continuously differentiable functions \( f(t, z(x), z(y)) \) in \( t \in [0,T] \) and \((x_1, \ldots, x_n) \in V_1 \) and \((y_1, \ldots, y_n) \in V_1 \), where \( z(x) = x_1 i_1 + \ldots + x_n i_n \) (see also Formulas (1.3) and (1.4)). The mapping \( C^{m+3} \ni \lambda \mapsto F_{a(\lambda)} \in C^4(V^2, C) \) is continuous (see, for example, [19, 14, 37]), consequently, the mapping \( C^{m+3} \ni \lambda \mapsto K_{a(\lambda), q(\lambda)} \in C^4(V^2, C) \) also is continuous by Formulas (5.5) and (5.6). Then
we put

\[(22.3)\quad u(t, x, y; \omega) = \int_{\Lambda} H(d\lambda)(\omega)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(x, y).\]

It will be supposed that

\[(22.4)\quad H(\lambda)(\omega) \text{ commutes almost } \hat{m} \times \mu\text{-everywhere with } \frac{\partial m_0}{\partial m_0}\text{ and } \sigma^m_x\]

and \(bf\) for every \(m_0 = 1, \ldots, m\) and \(m_1 = 1, \ldots, 4\) and \(b\) in \(A_{r, c}\),

where \(\lambda \in \Lambda, \ \omega \in \Omega\). The space \(X_0\) is dense in \(X\) (see the notation above). If \(f\) and \(g\) belong to \(X_0\), then \(fg \in X_0\), where \((fg)(t, x, y) = f(t, x, y)g(t, x, y)\) for each \((t, x, y) \in [0, T] \times V^2\). Henceforward it also will be supposed that for each \(f \in X_0\)

\[(22.5)\quad E((H(d\lambda)(\omega)f)(H(d\phi)(\omega)f)) = \delta(\lambda - \vartheta)\xi(\lambda)E(H(d\lambda)(\omega)f^2)\]

\(\hat{m}\) almost everywhere, where \(\lambda\) and \(\vartheta\) are in \(\Lambda\), \(\delta(\lambda - \vartheta)\) denotes the \(\delta\) function relative to \(EH(d\lambda)(\omega)\). If \(\gamma \neq 0\) it will be taken \(\xi(\lambda) = \gamma/(2\lambda_1\lambda_{m+2})\) for each \(\lambda \in C^{m+3}\) with \(\lambda_1\lambda_{m+2} \neq 0\). If \(\varsigma \neq 0\) it will be chosen \(\xi(\lambda) = \varsigma/(2\lambda_1\lambda_{m+3})\) for each \(\lambda \in C^{m+3}\) with \(\lambda_1\lambda_{m+3} \neq 0\) (see the notations in Subsections 4 and 5). Let

\[(22.6)\quad \rho_s(\frac{\partial^l}{\partial t^{m_0}\partial x^{m_1}_j\partial y^{m_2}_k}u(t, x, y; \omega)) = E\left(\int_{\Lambda} \frac{\partial^l}{\partial t^{m_0}\partial x^{m_1}_j\partial y^{m_2}_k}H(d\lambda)(\omega)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(x, y)|^s\right) < \infty\]

converge uniformly in \((t, x, y)\) on each canonically closed compact subset in \([0, T] \times (\mathcal{U} \cap V)^2\) for every \(m_0 = 0, 1, \ldots, m, m_1 = 0, \ldots, 4, m_2 = 0, \ldots, 4\) and \(m_1 + m_2 \leq 4\) if \(s = 1\); \(m_0 = 0, 1, \ldots, m, m_1 = 0, 1, m_2 = 1, 0, m_1 + m_2 \leq 1\) if \(s = 2\); \(l = m_0 + m_1 + m_2, j = 1, \ldots, n, k = 1, \ldots, n, t \in [0, T], x\) and \(y\) in \(V \cap \mathcal{U}, 0 < T < \infty\). Hence the integral \((22.3)\) exists \(\mu\)-almost everywhere on \(\Omega\) for each \(t \in [0, T], x\) and \(y\) in \(V \cap \mathcal{U}\) and

\[(22.7)\quad \frac{\partial^l}{\partial t^{m_0}\partial x^{m_1}_j\partial y^{m_2}_k}u(t, x, y) = E\left(\int_{\Lambda} \frac{\partial^l}{\partial t^{m_0}\partial x^{m_1}_j\partial y^{m_2}_k}H(d\lambda)(\omega)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(x, y),\right),\]

where

\[(22.8)\quad u(t, x, y) = Eu(t, x, y; \omega)\]

(see Ch. V Section 1 in [12] and the Fubini Theorem in [3 6]). Then From \((22.4) - (22.6)\), Lemmas 8, 19 and 21 we deduce that

\[(22.9)\quad Eu^2(t, x, y; \omega) = \]
\[ \int_{\Lambda} \int_{\Lambda} E(H(d\lambda)(\omega)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(x, y)H(d\eta)(\omega)\phi(t, \eta)K_{a(\eta), q(\eta)}(x, y)) = E \int_{\Lambda} H(d\lambda)(\omega)\xi(\lambda)q^2(t, \lambda)K_{a(\lambda), q(\lambda)}^2(x, y). \]

On the other hand, from (22.5), (5.5) and (5.6) the equality follows

\[ (22.10) \quad H(d\lambda)(\omega)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(x, y) = H(d\lambda)(\omega)(\phi(t, \lambda)F_{a(\lambda)}(x, y)) + p_1\pi_1(\sigma \int_{w_0}^x (\sigma \int_{w_0}^y (\sigma \int_{w_0}^\infty H(d\lambda)(\omega)F_{a(\lambda)}(z, v)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(w, z))dz)dv)dw) + p_2 \sigma \int_{w_0}^y (\sigma \int_{w_0}^x (\sigma \int_{w_0}^\infty H(d\lambda)(\omega)F_{a(\lambda)}(z, v)\phi(t, \lambda)K_{a(\lambda), q(\lambda)}(w, z))dz)dv)dw)ds, \]

and \( Q\left(\frac{a_t}{a_T}S_0\right)(\phi(t, \lambda)F_{a(\lambda)}(x, y)) = 0. \)

Therefore utilizing Condition (22.4) and Formulas (22.9), (22.10), (5.1) and (5.9) – (5.11) we deduce that \( u(t, x; y; \omega) \) fulfills the PDE (4.4).

Therefore from Theorem 6, Lemmas 8, 19 and 21 we infer the following result.

**23. Theorem.** If conditions (5.8) and (22.4) – (22.6) are fulfilled, then the random function \( u(t, x; y; \omega) \) defined by Formula (22.3) is a solution of the PDE (4.4) on \( [0, T] \times (U \cap V)^2 \) and on the diagonal \( x = y \) it satisfies the PDE (4.2).

**24. Theorem.** Solutions of the Sobolev-Burgers PDE.

Let \( u(t, x) \in W_{2,m,4}([0, T] \times V_1, \mathbb{C}) \) be a solution of the Sobolev-Burgers PDE (4.1) on \( [0, T] \times V_1 \), where \( 0 < T < \infty \), \( V_1 \) is a canonically closed compact subset in \( \mathbb{R}^n \). Then an orthogonal random operator valued measure \( H \) exists such that \( u(t, x) = u(t, z(x), z(x)) \), where \( u(t, x, y) \) is given by Formulas (22.3) and (22.8), \( z(x) = x_1i_1 + ... + x ni_n \).

**Proof. I.** If in addition to conditions of Theorem 23

\[ (24.1) \quad E(u^2(t, x, y; \omega)) = u^2(t, x, y), \]

where \( u(t, x, y) \) is provided by Formulas (22.3) and (22.8), then \( u(t, x) = u(t, z(x), z(x)) \) is a solution of the PDE (4.1) (see also Formulas (1.3) and (1.4)).

From Subsections 4, 5 and 22 it follows that a domain

\[ (24.2) \quad \Upsilon := \{ \lambda \in \mathbb{C}^{m+3} : \exists v(t, x) = \phi(t, \lambda)K_{a(\lambda), q(\lambda)}(z(x), z(x)) \}

satisfying the PDE (4.1) on \( [0, T] \times V_1 \} \]

is a Borel set in \( \mathbb{C}^{m+3} \). We mean here the correspondence between PDEs in
variables belonging to domains in \( \mathbf{R}^n \) and in \( \mathcal{A}_r \) according to Subsections 1 and 4 and simplifying the notation.

II. It is sufficient to provide \( \Lambda(\lambda)(\omega) \) on \( Y \times \Omega \) and extend it by zero on \( (\mathbf{C}^{m+3} \setminus Y) \times \Omega \). We consider a family \( \mathcal{Z} \) of countable disjoint partitions \( \Phi = \{G_1, G_2, \ldots\} \) of \( \Lambda = \mathbf{C}^{m+3} \) possessing the following properties

\[
(24.3) \bigcup_{i=1}^{\infty} G_i = \Lambda, \quad G_i \cap G_j = \emptyset \text{ for each } i \neq j, \quad \sup_j (\text{diam}(G_j)) =: d(\Phi) < \infty, \quad \text{cl}(G_i) = \text{cl}(\text{Int}(G_i)), \quad \text{G}_i \text{ belongs to the Borel } \sigma\text{-algebra } B(\Lambda) \text{ of } \Lambda, \quad \text{where cl}(G_i) \text{ denotes the interior of } G_i, \quad \text{while cl}(G_i) \text{ denotes the}
\]

\[
\text{closure of } G_i, \quad \text{diam}(G_i) = \sup_{a,b \in G_i} |a - b|, \quad |a| = \sqrt{|a_1|^2 + \ldots + |a_{m+3}|^2} \text{ for each } a = (a_1, \ldots, a_{m+3}) \in \mathbf{C}^{m+3}. \quad \text{The family } \mathcal{Z} \text{ is partially ordered } \Phi_1 = \{G_{1,1}, G_{1,2}, \ldots\} < \Phi_2 = \{G_{2,1}, G_{2,2}, \ldots\} \text{ if and only if for each } G_{2,k} \text{ a natural number } j(k) \text{ exists such that } G_{2,k} \subset G_{1,j(k)}, \text{ that is a partition } \Phi_2 \text{ is finer than } \Phi_1.
\]

Evidently, the linear span \( \text{span}_C \{\phi(\cdot, \lambda) : \lambda \in \Lambda\} \), where \( \phi(\cdot, \lambda) \) is a solution of the Cauchy problem (22.1) and (22.2) is dense in \( W_{2,\infty}^Q := \{g \in W_{2,m}([0,T], \mathbf{C}) : g \notin \mathcal{N}(Q(\frac{Q}{\mathbf{d}t}))\} \), where \( \mathcal{N}(Q(\frac{Q}{\mathbf{d}t})) \) denotes the null-space of the differential operator \( Q(\frac{Q}{\mathbf{d}t}) \). Indeed, \( \phi(t, \lambda) \) is in \( C^m([0,T], \mathbf{C}) \) for each \( \lambda \in \Lambda \). A set \( \{\lambda_1 f^2 : f \in C^m([0,T], \mathbf{R}), \lambda_1 > 0\} \) has a non void interior in \( C^m([0,T], \mathbf{R}) \). On the other hand, \( C^2m([0,T], \mathbf{R}) \) and \( Q\mathbf{(}\frac{Q}{\mathbf{d}t})C^2m([0,T], \mathbf{R}) \) are dense in \( C^m([0,T], \mathbf{R}) \). Therefore, \( \Psi := \hat{\pi}_2 \{f, \lambda_1 f^2 = (g, Q(\frac{Q}{\mathbf{d}t})g) : f \in C^m([0,T], \mathbf{R}), \lambda_1 > 0, g \in C^{2m}([0,T], \mathbf{R})\} \) is dense in some open subset in \( \{g \in C^m([0,T], \mathbf{R}) : g \notin \mathcal{N}(Q(\frac{Q}{\mathbf{d}t}))\} \), where \( \hat{\pi}_2 \) denotes the linear projection \( \hat{\pi}_2(f, h) = h \) for all \( f \) and \( h \) in \( C^m([0,T], \mathbf{R}) \). The space \( C^m([0,T], \mathbf{C}) \) is dense in \( W_{2,m}([0,T], \mathbf{C}) \), hence the \( \mathbf{C} \) linear span of \( \Psi \) is dense in \( W_{2,\infty}^Q \).

The Sobolev spaces \( W_{2,m,4}([0,T] \times V^2, \mathbf{C}) = X, \ W_{2,m}([0,T], \mathbf{C}) \text{ and } W_{2,4}(V^2, \mathbf{C}) \text{ are separable. Suppose that } u(t, x, y) \text{ is in } W_{2,m,4}([0,T] \times V^2, \mathbf{C}) \text{ and } u(t, x, x) \text{ is a solution of the PDE (4.1). If } f \in X, \text{ then } Q(\frac{Q}{\mathbf{d}t})S_0f \in Y, \text{ where } Y = L^2([0,T] \times V^2, \mathbf{C}). \text{ On the other hand, the tensor product } L^2([0,T], \mathbf{C}) \otimes L^2(V^2, \mathbf{C}) \text{ is dense in } Y. \text{ For each } \epsilon > 0 \text{ there are non null functions } \phi_j(t) = \phi(t, \lambda(j)) \text{ and } f_j(x, y) \in W_{2,4}(V^2, \mathbf{C}) \text{ such that } \lambda(j) \in \Lambda \text{ for each } j = 1, \ldots, k \text{ and } |u - \sum_{j=1}^{k} \phi_j f_j|_{W_{2,m,4}} < \epsilon \text{ and } \lambda(j) \neq \lambda(i) \text{ for each } i \neq j, \text{ also functions } \{v_j := \phi_j f_j : j = 1, \ldots, k\} \text{ are linearly independent, where } k = k(\epsilon) \in \mathbf{N}. \text{ Thus } v_j \in X \subset Y \text{ for each } j. \]
Obviously $X$ and $Y$ considered as the $\mathbb{R}$ linear spaces are isomorphic with $X_\mathbb{R} \oplus X_\mathbb{R}$ and $Y_\mathbb{R} \oplus Y_\mathbb{R}$ respectively, where $X_\mathbb{R} = W_{2,m,4}([0,T] \times V^2, \mathbb{R})$ and $Y_\mathbb{R} = L^2([0,T] \times V^2, \mathbb{R})$. Let $\pi_{v_1,\ldots,v_k} : Y \rightarrow \text{span}_\mathbb{C}(v_1,\ldots,v_k)$ be a $\mathbb{C}$-linear projection operator, where

$$\text{span}_\mathbb{C}(v_1,\ldots,v_k) = \{v \in Y : v = c_1v_1 + \cdots + c_kv_k : c_1 \in \mathbb{C}, \ldots, c_k \in \mathbb{C}\}$$

denotes the linear span of vectors $v_1,\ldots,v_k$ over $\mathbb{C}$. Since $v_1,\ldots,v_k$ are linearly independent, then $\text{span}_\mathbb{C}(v_1,\ldots,v_k)$ is isomorphic with $\mathbb{C}^k$. If it is considered as the $\mathbb{R}$ linear space, then it is isomorphic with $\mathbb{R}^{2^k}$. Then a partition $\Phi = \Phi(v_1,\ldots,v_k)$ with $\Phi \in \mathcal{Z}$ exists so that

$$(24.4) \text{ for each } 1 \leq j \neq l \leq k \text{ there are unique } \iota(j) \neq \iota(l) \text{ with } \lambda(j) \in G_{\iota(j)} \text{ and } \lambda(l) \in G_{\iota(l)}.$$  

Let $\mathcal{M} = \mathcal{M}(v_1,\ldots,v_k)$ be a semi-ring of subsets in $\Lambda$ generated by a disjoint partition $\Phi = \{G_1,G_2,\ldots\}$ satisfying Conditions (24.3) and (24.4).

**III.** Take a probability space $(\Omega_{v_1,\ldots,v_k}, \mathcal{G}_{v_1,\ldots,v_k}, P_{v_1,\ldots,v_k})$ with

$$\Omega_{v_1,\ldots,v_k} = \text{span}_\mathbb{C}(v_1,\ldots,v_k), \text{ where } P_{v_1,\ldots,v_k} : \mathcal{G}_{v_1,\ldots,v_k} \rightarrow [0,1] \text{ is a probability, } \mathcal{G}_{v_1,\ldots,v_k} \text{ is a completion of the Borel } \sigma\text{-algebra } B(\Omega_{v_1,\ldots,v_k}) \text{ relative to } P_{v_1,\ldots,v_k}. \text{ We put } v(\lambda) = v_j \text{ for each } \lambda \in G_{\iota(j)} \text{ Choose an orthogonal random operator valued measure } H_{v_1,\ldots,v_k}(d\lambda) \text{ and a probability measure } P_{v_1,\ldots,v_k} \text{ such that to satisfy (22.4), (22.5) and (24.1) for } H_{v_1,\ldots,v_k}(d\lambda) \text{ restricted on } \text{span}_\mathbb{C}(v_1,\ldots,v_k) \text{ and for }$$

$$u_{v_1,\ldots,v_k} = \int_{\Lambda} H_{v_1,\ldots,v_k}(d\lambda)v(\lambda)$$

instead of $u$. This $P_{v_1,\ldots,v_k}$ can be taken as a Gaussian measure corresponding to a random vector $u_{v_1,\ldots,v_k}$ with a mean value $\theta_{v_1,\ldots,v_k}$ and a correlation operator $\mathcal{C}_{v_1,\ldots,v_k}$.

The Hilbert spaces $X$ and $Y$ are separable, consequently, they are isomorphic. If $\tilde{J} : Y \rightarrow X$ is an isomorphism, then $\tilde{J}H : X \rightarrow X$. By virtue of the Fubini theorem Conditions (22.5) and (24.1) mean that $\tilde{H}^2 = \Xi\tilde{H}$, where $\Xi = \tilde{J}\mathcal{E}H$. Here an operator $\Xi$ is the multiplication operator on a function $\xi(\lambda), \Xi y(\lambda) = \xi(\lambda)y(\lambda)$ for each $\lambda \in \Lambda$ and $y \in L^2(\Lambda,\mathcal{B},m,X)$. In order to describe an orthogonal random operator valued measure $\tilde{J}H_{v_1,\ldots,v_k}(M)$ for each $M \in \mathcal{M}(v_1,\ldots,v_k)$ it is sufficient to provide $\tilde{J}H_{v_1,\ldots,v_k}(G_j)$ for each $G_j$ of the disjoint partition $\Phi = \{G_1,G_2,\ldots\}$. For each $j = 1,2,\ldots$ an orthogonal
random operator valued measure \( \hat{J}H_{v_1,\ldots,v_k}(G_j) \) can be realized as a random normal operator and considering its representation with the help of an integral by a random projection operator valued measure on its spectrum (see Section 5.2 in [22], also [6, 12]).

Then we take a monotone decreasing sequence \( \epsilon_1 > \epsilon_2 > \ldots > 0 \) and choose \( v_1, v_2, \ldots \) such that

\[
(24.5) \quad |u - \sum_{j=1}^{k(i)} v_j|_{W_{2,m,4}} < \epsilon_i
\]

for each \( i = 1, 2, \ldots \), where \( k(i) < k(j) \) for each \( i < j \). Then for each \( j \) a countable partition \( \Phi_j \) of \( \Lambda \) can be chosen satisfying Conditions (24.3) and (24.4) and such that \( \Phi_i < \Phi_j \) for each \( i < j \) and \( \lim_{j \to \infty} d(\Phi_j) = 0 \).

Therefore it can be put \( \Omega = cl_Y(\text{span}_C(v_j : j \in \mathbb{N})) \), where the closure of a linear span is taken in \( Y \). There are natural projections \( \hat{\pi}_k : \Omega \to \Omega_{v_1,\ldots,v_k} \) and \( \hat{\pi}_k^l : \Omega_{v_1,\ldots,v_l} \to \Omega_{v_1,\ldots,v_k} \) for each \( l > k \).

A probability Gaussian measure \( P_{v_1,\ldots,v_k} \) on \( \mathcal{G}_{v_1,\ldots,v_k} \) induces a cylindrical measure \( \mu_{v_1,\ldots,v_k} \) on a cylindrical \( \sigma \)-algebra \( \mathcal{F}(v_1,\ldots,v_k) \) of all cylindrical sets \( (\hat{\pi}_{v_1,\ldots,v_k})^{-1}(N) \) in \( \Omega \) with \( N \) in \( \mathcal{G}_{v_1,\ldots,v_k} \). This family of measures \( \mu_{v_1,\ldots,v_k} \) can be chosen consistent with projections so that it induces a bounded cylindrical distribution on an algebra \( \bigcup_k \mathcal{F}(v_1,\ldots,v_k) \). Mean values \( \theta_{v_1,\ldots,v_k} \) and correlation operators \( \mathcal{C}_{v_1,\ldots,v_k} \) can be chosen so that \( \sup_k |\theta_{v_1,\ldots,v_k}|_X < \infty \) and \( \sup_k |\mathcal{C}_{v_1,\ldots,v_k}| < \infty \), where \( |\mathcal{C}_{v_1,\ldots,v_k}| \) denotes a norm of \( \mathcal{C}_{v_1,\ldots,v_k} \) for \( \text{span}_C(v_1,\ldots,v_k) \) embedded into \( (X,|\cdot|) \). The natural embedding operator \( \mathcal{J}_0 : X \hookrightarrow Y \) is nuclear (i.e. of trace class). Therefore, the bounded consistent family of measures induces a \( \sigma \)-additive Gaussian measure \( \mu \) on \( (\Omega,\mathcal{F}) \), where \( \mathcal{F} \) is a completion of \( \bigcup_k \mathcal{F}(v_1,\ldots,v_k) \) (see Section II.2 in [4]).

Then the family of orthogonal random operator valued measures \( H_{v_1,\ldots,v_k}(d\lambda) \) on subspaces \( \text{span}_C(v_1,\ldots,v_k) \) and subalgebras \( \mathcal{B}_k \) generated by \( \mathcal{M}(v_1,\ldots,v_k) \) induces an orthogonal random operator valued measure \( H(d\lambda) \) on \( X(v_j : j \in \mathbb{N}) := cl_X(\text{span}_C(v_j : j \in \mathbb{N})) \), since \( \mu((\hat{\pi}_k)^{-1}(N)) = \mu_{v_1,\ldots,v_k}(N) \) for each \( N \in \mathcal{G}_{v_1,\ldots,v_k} \). It has an extension by the identity operator \( I \) on the orthogonal complement \( X \ominus X(v_j : j \in \mathbb{N}) \). Naturally \( \mathcal{B} \) is the completion of \( \bigcup_k \mathcal{B}_k \), where \( \mathcal{B}_k \) denotes a completion of \( \mathcal{M}(v_1,\ldots,v_k) \), \( k = k(i), \ i = 1, 2, \ldots \). This provides properties (22.4), (22.5) and (24.1) for \( H(d\lambda) \) on \( X \). Taking a limit
of a fundamental sequence of step functions in \(L^2(\Lambda, \mathcal{B}, \mathbf{m}, \mathbf{C})\) gives a function \(v \in L^2(\Lambda, \mathcal{B}, \mathbf{m}, \mathbf{C})\) (see Subsection 10), where \(v(\lambda) = \phi(\cdot, \lambda)f(\cdot, \lambda)\) with \(f(\cdot, \lambda) \in W_{2,1}(V^2, \mathbf{C})\) for each \(\lambda\), also \(v(\lambda(j)) = v_j\) for each \(j\). By virtue of Lemma 21 we get

\[
u(t, x, y; \omega) = \int_{\Lambda} H(d\lambda)(\omega)v(\lambda).
\]

Using Formulas (5.5) and (5.6) define a function \(g(x, y, \lambda)\) related with \(f(x, y, \lambda)\) similarly to the pair of \(F_{\mathbf{m}(\lambda)}\) and \(K_{\mathbf{m}(\lambda), \mathbf{q}(\lambda)}\), where \(\lambda\) is the parameter in \(\Lambda\) so that \(\mathbf{m}(G) \neq 0\) for each \(j\) and \(G\) in \(\Phi_j\) with \(\lambda \in G\). Therefore, utilizing these properties together with the orthogonality of \(H\) we infer for a function \(h(t, x, y, \lambda) = \phi(t, \lambda)g(x, y, \lambda)\) that

\[
E(|\int_{\Lambda} H(d\lambda)Q(\frac{\partial}{\partial t})S_0 h(t, x, y, \lambda)|^2_{x=y}) = E(|Q(\frac{\partial}{\partial t})S_0 \int_{\Lambda} H(d\lambda)h(t, x, y, \lambda)|^2_{x=y}) = 0
\]

1 almost everywhere on \([0, T] \times V\), where \(a = -\alpha\) and \(b = \beta\) (see Formulas (4.1) and (4.5)). This implies that \(v(\lambda)(t, x, x) = \phi(t, \lambda)f(x, x, \lambda)\) satisfies the PDE (4.1) for \(\mathbf{m}\) almost all \(\lambda\) in \(\Lambda\). Thus \(f(\cdot, \lambda) = K_{\mathbf{m}(\lambda), \mathbf{q}(\lambda)}\) for \(\mathbf{m}\) almost all \(\lambda\) in \(\Lambda\).

25. Remark. The Laplacian \(\Delta_x\) is invariant under each orthogonal transformation \(T \in O(n)\), where \(O(n)\) notates the orthogonal group of the Euclidean space \(\mathbf{R}^n\). If there is a vector Sobolev-Burgers PDE with \(u \in \mathbf{R}^k\) and the scalar product \((u, u)\) instead of \(u^2\), where \(k \leq 4\), it is possible to consider its noncommutative analog for \(u\) with values in the quaternion skew field \(H = \mathcal{A}_2\). The following generalized PDE

\[
(25.1) \quad E\{Q(\frac{\partial}{\partial t})S_0 u(t, x, y; \omega) + (\sigma_x + \sigma_y)(u^2(t, x, y; \omega)q_1 + u^2(t, x, y; \omega)q_2)|_{x=y} = 0
\]

can be solved analogously to Subsection 4-6, 22, 23, where \(q_1\) and \(q_2\) are in the quaternion skew field \(\mathcal{A}_2 = H\) and \(|q_1| + |q_2| > 0\), \(x\) and \(y\) are in \(V\), \(2 \leq r \leq 3\) and \(2^r > n\). For this we put also

\[
(25.2) \quad K(x, y) = F(x, y) + AK(x, y)
\]

\[
(25.3) \quad AK(x, y) = (\sigma \int_{w_0}^{x} (\sigma \int_{w_0}^{y} (\sigma \int_{w_0}^{\infty} F(z, v)K(w, z)dz)dv)dw)p_1
\]

\[
+ (\sigma \int_{w_0}^{y} (\sigma \int_{w_0}^{x} (\sigma \int_{w_0}^{\infty} F(z, v)K(w, z)dz)dv)dw)p_2,
\]

where \(p_j \neq 0\) if and only if \(q_j \neq 0\), \(j \in \{1, 2\}\), \(K = K_{\mathbf{a}, \mathbf{q}}\); \(F = F_{\mathbf{a}}\); \(w_0\) is marked point in \(\mathcal{U}, \mathcal{U} \subset \mathcal{A}_3\), \(x\) and \(y\) are in \(V\) with \(V \subset \mathcal{U}\). Formulas for
solutions are similar, since the octonion algebra \( \mathcal{A}_3 = \mathbf{O} \) is alternative, while \( K \) and \( F \) have values in the quaternion skew field \( \mathbf{H} \).

### 26. Conclusion.

In this paper the Sobolev-Burgers PDEs were integrated using noncommutative line integral over Cayley-Dickson algebras and orthogonal random operator valued measures. The nonlinear problem was reduced to the linear PDEs for \( F \) and the \( \mathbb{C} \)-linear integral equation relating \( F \) and \( K \).

It is worth to mention that algorithms for numerical solutions of integral equations converge better than for PDEs. It extends previous approaches based on real and complex numbers, because each scalar or vector PDE over them can be reformulated over octonions and Cayley-Dickson algebras and new types of PDEs can be encompassed (see Section 2).

The obtained results can be used for further investigations of PDEs and properties of their solutions. For example, generalized PDEs including terms such as \( \Delta^p \) or \( \nabla^p \), \( \text{div}(|\nabla u|^p \nabla u) + \lambda |u|^p u \) for \( p > 0 \) can be investigated, for dynamical nonlinear processes, air target range radar measurements \([20, 24, 26, 45, 46]\), which have technical applications and in the sciences.

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