An Evolutionary Optimization Approach to Risk Parity Portfolio Selection

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Abstract

In this paper we present an evolutionary optimization approach to solve the risk parity portfolio selection problem. While there exist convex optimization approaches to solve this problem when long-only portfolios are considered, the optimization problem becomes non-trivial in the long-short case. To solve this problem, we propose a genetic algorithm as well as a local search heuristic. This algorithmic framework is able to compute solutions successfully. Numerical results using real-world data substantiate the practicability of the approach presented in this paper.

1 Introduction

The portfolio selection problem is concerned with finding an optimal portfolio $x$ of assets from a given set of $n$ risky assets out of a pre-specified asset universe such that the requirements of the respective investor are met. In general, investors seek to optimize their portfolio in regard of the trade-off between return and risk, such that the meta optimization problem can be formulated as shown in Eq. (1).

$$\begin{align*}
\text{minimize} & \quad \text{Risk}(x) \\
\text{maximize} & \quad \text{Return}(x)
\end{align*}$$

This bi-criteria optimization problem is commonly reduced to a single-criteria problem by just focusing on the risk and constraining the required mean, i.e. the investor sets a lower expected return target $\mu$, which is shown in Eq. (2).

$$\begin{align*}
\text{minimize} & \quad \text{Risk}(x) \\
\text{subject to} & \quad \text{Return}(x) \geq \mu
\end{align*}$$

Markowitz [10] pioneered the idea of risk-return optimal portfolios using the standard deviation of the portfolios profit and loss function as risk measure. In this case, the optimal portfolio $x$ is computed by solving the quadratic optimization problem shown in Eq. (3). The investor needs to estimate a vector of expected returns $r$ of the assets under consideration as well as the covariance matrix $C$. Finally the minimum return target $\mu$ has to be defined. Any standard quadratic programming solver can be used to solve this problem numerically.
minimize \quad x^T C x
subject to \quad r \times x \geq \mu
\sum x = 1

While this formulation has been successfully applied for a long time, criticism has sparked recently. This is especially due to the problem of estimating the mean vector. To overcome this problem one seeks optimization model formulations that solely depend on the covariance matrix. Sometimes even simpler approaches are favored, e.g. the 1-over-N portfolio, which equally weights every asset under consideration. It has been shown that there are cases, where this simple strategy outperforms clever optimization strategies, see e.g. DeMiguel et al. [7].

Of course, the Markowitz problem can be simplified to a model without using returns easily by dropping the minimum return constraint. In this case one receives the Minimum Variance Portfolio (MVP), which is overly risk-averse.

One important technique used for practical portfolio purposes are risk-parity portfolios, where the assets are weighted such that they equally contribute risk to the overall risk of the portfolio. The properties of such portfolios are discussed by Maillard et al. [9] and alternative solution approaches are shown by Chaves et al., see [5] and [6], as well as Bai et al. [4].

In this paper, an evolutionary optimization approach to compute optimal risk parity portfolios will be presented. Evolutionary optimization approaches have been shown to be useful for solving a wide range of different portfolio optimization problems, see e.g. [8] and the references therein.

This paper is organized as follows. Section 2 describes the risk-parity problem in detail, Section 3 presents the evolutionary algorithm developed for solving the problem, and Section 4 presents numerical results. Finally, Section 5 concludes the paper.

2 Risk Parity Portfolio Selection

The type of risk-parity portfolios discussed in this paper are also called Equal Risk Contribution (ERC) portfolios. The idea is to find a portfolio where the assets are weighted such that they equally contribute risk to the overall risk of the portfolio.

We follow Maillard et al. [9] in their definition of risk contribution, i.e. reconsider the above mentioned portfolio \( x = (x_1, x_2, \ldots, x_n) \) of \( n \) risky assets. Let \( C \) be the covariance matrix, \( \sigma_i^2 \) the variance of asset \( i \), and \( \sigma_{ij} \) the covariance between asset \( i \) and \( j \). Let \( \sigma(x) \) be the risk (i.e. standard deviation) of the portfolio as defined in Eq. (4).

\[
\sigma(x) = \sqrt{x^T C x} = \sum_i x_i^2 \sigma_i^2 + \sum_{i \neq j} x_i x_j \sigma_{ij}.
\]

Then the marginal risk contributions \( \partial_{x_i} \sigma(x) \) of each asset \( i \) are defined as follows

\[
\partial_{x_i} \sigma(x) = \frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sum_{j \neq i} x_j \sigma_{ij}}{\sigma(x)}.
\]

If we are considering long-only portfolios then the optimal solution of can be written as an optimization problem containing a logarithmic barrier term which is shown in Eq. (5) and where \( c \) is an arbitrary positive constant. See e.g. also [13] for an alternative formulation. In this long-only case, a singular optimal solution can be computed.
$$\text{minimize} \quad x^T C x - e \sum_{i=1}^{n} \ln x_i$$
$$\text{subject to} \quad x_i > 0.$$ (5)

However, if we want to include short positions then we need to find solutions in other orthants than in the non-negative orthant. See Bai et al. [1] for a log-barrier approach in this case, which is shown in Eq. (6).

$$\text{minimize} \quad x^T C x - e \sum_{i=1}^{n} \ln \beta_i x_i$$
$$\text{subject to} \quad \beta_i x_i > 0,$$ (6)

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \{-1,1\}^n$ defines the orthant where the solution should be computed. For each choice of $\beta$ the above optimization problem is convex and can be solved optimally. However, as shown in [1] there are $2^n$ different solutions. Investors may add additional constraints to specify their needs, however this cannot be modeled as one convex optimization problem, which is why an evolutionary approach is presented here. The general formulation of the long-short risk parity portfolio problem can be formulated as Eq. (7) as shown in [2].

$$\text{minimize} \quad \sum_{i=1,j=1}^{n}(x_i(C x)_i - x_j(C x)_j)^2$$
$$\text{subject to} \quad a_i \leq x_i \leq b_i,$$
$$\sum_{i=1}^{n} x_i = 1.$$ (7)

3 Implementation

The solution is computed in two steps. First, a genetic algorithm will be employed and afterwards a local search algorithm will be applied.

3.1 Genetic Algorithm

We are using a standard genetic algorithm to compute risk-parity optimal portfolios. The algorithm was implemented using the statistical computing language R [11].

The fitness definition in the risk-parity setting is given by the deviance of each risk contribution from the mean of all risk contributions. Let us use the shorthand notation of $\Delta_i = \partial_x \sigma(x)$, so we compute the expectation $\Delta = E(\Delta_i)$ and define the fitness $f$ as the sum of the quadratic distance of each risk contribution from the mean. This non-negative fitness value has to be minimized,

$$f = \sum_{i}(\Delta_i - \Delta)^2$$

We use a genotype-phenotype equivalent formulation, i.e. we use chromosomes of length $n$ which contain the specific portfolio weights of the $n$ risky assets. Thus, an important operator is the repair operator, i.e. the sum of the portfolio is normalized to 1 after each operation.

The genetic operators used in the algorithm can be summarized as follows:

- Elitist selection: The best $n_{ES}$ chromosomes of each population are kept in the population.
Table 1: Parameters for the Genetic Algorithms.

| Parameter              | Value |
|------------------------|-------|
| Initial population size | 200   |
| Maximum iterations     | 300   |
| Elitist selection      | 10    |
| Random addition        | 50    |
| Mutation               | 100   |
| Intermediate crossover | 100   |

- Mutation: A random selection of \( n_M \) chromosomes of the parent population will be mutated. Up to five chromosomes will be changed to a random value between the portfolio bounds.
- Random addition: \( n_R \) random chromosomes are added to each new population.
- Intermediate crossover: Two chromosomes from the parent population will be randomly selected for an intermediate crossover. The mixing parameter between the two chromosomes will be chosen randomly. \( n_{IC} \) crossover children will be added to the next population.

3.2 Local Search

In a second step, a local search algorithm is applied to the best solution of the genetic algorithm. Thereby, within each iteration of the algorithm each asset weight of the \( n \) assets of the portfolio is increased or decreased by a factor \( \epsilon \). Each of this \( 2 \times n \) new portfolios is normalized and if one exhibits a lower fitness value then this new portfolio will be used subsequently. The algorithm terminates if no local improvement is possible anymore or the maximum number of iterations has been reached.

4 Numerical Results

4.1 Financial Data and Setup

We use data from all stocks from the Dow Jones Industrial Average (DJIA) index using the composition of September 20, 2013, i.e. using the stocks with the ticker symbols AXP, BA, CAT, CSCO, CVX, DD, DIS, GE, GS, HD, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, NKE, PFE, PG, T, TRV, UNH, UTX, V, VZ, WMT, XOM.

Using the R package `quantmod` we may easily obtain data from Yahoo! Finance. We use data from the beginning of 2010 until the beginning of November 2014 to compute the Variance-Covariance matrix, i.e. the matrix is entirely based on historical data. The data is solely used for comparison purposes such that a clever approximation algorithm for the Variance-Covariance matrix is not necessary for the purpose of this study. However it should be note that the matrix is the single most input parameter for the calculation.

The parameters used for the genetic algorithm are shown in Tab. 1. The local search algorithm was started twice, once with \( \epsilon = 0.01 \) and afterwards with \( \epsilon = 0.001 \). The number of maximum local search steps has been set to 500.

4.2 Long-Only

First, we compute a set if various long-only portfolios without using expected returns, i.e. the Minimum Variance Portfolio (MVP), the 1/N portfolio as well as the risk-parity portfolio using
Table 2: DJIA - Long Only - MVP, 1/N, and Risk Parity.

| Item | x(MVP) | RCn(MVP) | x(1/N) | RCn(1/N) | x(RP) | RCn(RP) |
|------|--------|----------|--------|----------|--------|----------|
| AXP  | 0.0000 | 0.0408   | 0.0300 | 0.0444   | 0.0000 | 0.0404   |
| BA   | 0.0000 | 0.0374   | 0.0300 | 0.0411   | 0.0000 | 0.0366   |
| CAT  | 0.0000 | 0.0420   | 0.0300 | 0.0484   | 0.0000 | 0.0413   |
| CSCO | 0.0000 | 0.0338   | 0.0300 | 0.0382   | 0.0000 | 0.0329   |
| CVX  | 0.0000 | 0.0345   | 0.0300 | 0.0357   | 0.0000 | 0.0341   |
| DD   | 0.0000 | 0.0382   | 0.0300 | 0.0410   | 0.0000 | 0.0376   |
| DIS  | 0.0000 | 0.0383   | 0.0300 | 0.0394   | 0.0000 | 0.0384   |
| GE   | 0.0000 | 0.0395   | 0.0300 | 0.0416   | 0.0000 | 0.0395   |
| GS   | 0.0000 | 0.0370   | 0.0300 | 0.0451   | 0.0000 | 0.0356   |
| HD   | 0.0000 | 0.0323   | 0.0300 | 0.0319   | 0.0000 | 0.0328   |
| IBM  | 0.0207 | 0.0285   | 0.0300 | 0.0283   | 0.0000 | 0.0272   |
| INTC | 0.0000 | 0.0312   | 0.0300 | 0.0353   | 0.0000 | 0.0305   |
| JNJ  | 0.2015 | 0.0285   | 0.0300 | 0.0218   | 0.0376 | 0.0257   |
| JPM  | 0.0000 | 0.0424   | 0.0300 | 0.0502   | 0.0000 | 0.0417   |
| KO   | 0.0338 | 0.0285   | 0.0300 | 0.0255   | 0.0275 | 0.0334   |
| MCD  | 0.2421 | 0.0285   | 0.0300 | 0.0195   | 0.2333 | 0.0288   |
| MMM  | 0.0000 | 0.0345   | 0.0300 | 0.0359   | 0.0000 | 0.0340   |
| MRK  | 0.0000 | 0.0301   | 0.0300 | 0.0274   | 0.0000 | 0.0299   |
| MSFT | 0.0000 | 0.0308   | 0.0300 | 0.0327   | 0.0000 | 0.0307   |
| NKE  | 0.0000 | 0.0343   | 0.0300 | 0.0365   | 0.0000 | 0.0347   |
| PFE  | 0.0000 | 0.0306   | 0.0300 | 0.0289   | 0.0000 | 0.0300   |
| PG   | 0.1890 | 0.0285   | 0.0300 | 0.0187   | 0.3050 | 0.0322   |
| T    | 0.0745 | 0.0285   | 0.0300 | 0.0228   | 0.0330 | 0.0288   |
| TRV  | 0.0000 | 0.0317   | 0.0300 | 0.0308   | 0.0000 | 0.0322   |
| UNH  | 0.0000 | 0.0305   | 0.0300 | 0.0324   | 0.0000 | 0.0293   |
| UTX  | 0.0000 | 0.0364   | 0.0300 | 0.0382   | 0.0000 | 0.0361   |
| V    | 0.0000 | 0.0330   | 0.0300 | 0.0360   | 0.0000 | 0.0320   |
| VZ   | 0.0554 | 0.0285   | 0.0300 | 0.0222   | 0.1072 | 0.0304   |
| WMT  | 0.2130 | 0.0285   | 0.0300 | 0.0176   | 0.2565 | 0.0312   |
| XOM  | 0.0000 | 0.0325   | 0.0300 | 0.0326   | 0.0000 | 0.0323   |

the algorithm developed in this paper and described above. The results is shown in Tab. 2. Please note that the risk contribution has been normalized to 1. The fitness of the 1/N portfolio is 0.002253031, while the MVP exhibits a fitness of 0.00057129. The algorithm managed to find the Risk Parity portfolio with a fitness of 0.0005019655. A lower fitness is not possible due to the long-only constraint.

Furthermore, the convergence results in the long-only case can be seen in Fig. 1. The left picture shows the best fitness over 300 iterations, while the right picture shows the mean of the population fitness. The middle line depicts the mean of 100 instances while the upper and the lower line depict the 5% as well as the 95% quantile of the instances.

In the long-only case, a simple random multi-start local search algorithm like the one described in Section 3.2 above leads to the same result. We tested this by running it 100 times and figured out that both the GA+Local as well as the Random+Local approach led to the same optimal portfolio in all cases. However, the optimal solution of the genetic algorithm needed significantly less iterations compared to starting from random solutions. A statistical t-test
4.3 Long-Short

In the long-short case, a random multi-start local search heuristic does not return any useful result. However, the evolutionary approach works well. The long-short result with a lower bound of $-0.2$ is shown in Tab. 3. The convergence results in the long-only case can be seen in Fig. 2.

4.4 Scalability

To test for scalability of the algorithm, we used stocks from the S&P 100 index as of March 21, 2014. Again, we use historical data from the beginning of 2010 until the beginning of November 2014 to compute our Variance-Covariance matrix. Four stocks have been excluded due to data issues, i.e. ABBV, FB, GM, and GOOG, such that the stocks with the following ticker symbols have been considered: AAPL, ABT, ACN, AIG, ALL, AMGN, AMZN, APA, APC, AXP, BA, BAC, BAX, BB, BMY, BRK.B, C, CAT, CL, CMCSA, COF, COP, COST, CSCO, CVS, CVX, DD, DIS, DOW, DYN, EAY, EMC, EMR, EXC, F, FCX, FDX, FOXA, GD, GE, GILD, GS, HAL, HD, HON, HPQ, IBM, INTC, JNJ, JPM, KO, LLY, LMT, LOW, MA, MCD, MDT, MDZ, MMM, MO, MON, MRK, MS, MSFT, NKE, NOV, NSC, ORCL, OXY, PEP, PFE,
Figure 2: Convergence of the genetic algorithm in the long-short case.

Figure 3: S&P 100 - portfolio (left) and risk contribution (right).

PG, PM, QCOM, RTN, SBUX, SLB, SO, SPG, T, TGT, TWX, TXN, UNH, UNP, UPS, USB, UTX, V, VZ, WAG, WFC, WMT, XOM.

The lower bound was set to $-0.2$. Fig. 3 shows the resulting portfolio as well as the risk contribution of the assets.

5 Conclusion

In this paper, we presented an evolutionary approach to compute optimal risk parity portfolios. This algorithm was designed to overcome the problem that only the long-only case can be solved conveniently using convex optimization models. A two-step approach using a genetic algorithm as well as a local search technique proved to be successful, especially in the long-short case. Another advantage is that further constraints can be integrated directly into the algorithm and this approach can be extended to other risk measures as well.

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