Relativistic Collective Coordinate System of Solitons and Spinning Skyrmion

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We consider constructing the relativistic system of collective coordinates of a field theory soliton on the basis of a simple principle: The collective coordinates must be introduced into the static solution in such a way that the equation of motion of the collective coordinates ensures that of the original field theory. As an illustration, we apply this principle to the quantization of spinning motion of the Skyrmion by incorporating the leading relativistic correction to the rigid body approximation. We calculate the decay constant and various static properties of nucleons, and find that the relativistic corrections are in the range of 5% - 20%. We also examine how the baryons deform due to the spinning motion.

Subject Index: 135, 160

§1. Introduction

Collective coordinates of a field theory soliton play a central role in the study of the soliton dynamics (see, e.g., Ref. 1) for a review). They represent the motion of the soliton as a whole in the symmetry directions of the original field theory. Since the soliton breaks these symmetries, the collective coordinates are related with the zero-mode fluctuations around the soliton. The simplest way to construct the system of collective coordinates is to promote the (originally constant) free parameters of the static soliton associated with symmetries to time-dependent dynamical variables by discarding the fluctuation of non-zero modes.

As an example, let us consider the collective coordinate of space-translation of a static soliton $\phi_{\text{cl}}(x)$ in a scalar field theory in 1 + 1 dimensions with Lagrangian density $L = -(1/2)(\partial_\mu \phi)^2 - U(\phi)$. In the simple treatment, we take

$$\phi(x,t) = \phi_{\text{cl}}(x - X(t)),$$

as the soliton field with a dynamical variable of the center-of-mass $X(t)$. Plugging this into the Lagrangian density and carrying out the space integration, we get the following Lagrangian of $X(t)$:

$$L = -m + \frac{1}{2}m\dot{X}(t)^2,$$

with $m$ being the energy of the static soliton. Somehow, we have obtained the non-relativistic Lagrangian of a point particle with coordinate $X(t)$, although we started with a relativistic scalar field theory. We need an improved treatment of collective

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coordinates which leads to the relativistic Lagrangian
\[ L = -m\sqrt{1 - \dot{X}^2}. \] (1.3)

The purpose of this paper is to propose a simple and general principle of constructing the system of collective coordinates of a field theory soliton, by which we can obtain the complete and relativistic Lagrangian of collective coordinates.

First, let us state our principle of introducing the collective coordinates into a static soliton solution: Collective coordinates must be introduced into the static solution in such a way that the equation of motion (EOM) of the collective coordinates ensures that of the original field theory. We call this the “EOM principle” hereafter. Although the original field theory has an infinite number of degrees of freedom, the reduced system of collective coordinates has only a finite number of ones. Therefore, even if the EOM of collective coordinates holds, it does not necessarily imply that the field theory EOM holds. Our EOM principle demands in a sense that there should be no mismatch between the field theory dynamics and the collective coordinate dynamics. If the original field theory is a relativistic one, our EOM principle should automatically lead to a relativistic Lagrangian of the collective coordinates.

Practically, the introduction of collective coordinates by the EOM principle must be carried out in a self-consistent manner. Starting with a suitably chosen dependence of the soliton field on the collective coordinates, we obtain the Lagrangian and consequently the EOM of the collective coordinates by plugging the soliton field into the field theory Lagrangian. Besides this, we plug the assumed soliton field into the field theory EOM, and examine whether it holds under the EOM of collective coordinates. We tune the dependence of the soliton field on the collective coordinates so that the EOM principle is realized.

In the above example of the center-of-mass in a scalar field theory in 1 + 1 dimensions, the field theory EOM is in fact broken by the term \( \dot{X}^2 \partial''_{\mathrm{cl}}(x - X) \) if we take (1.1) as the soliton field (see Eq. (1.6)). Let us apply our EOM principle to this system to resolve the breaking of field theory EOM successively in the number of time-derivatives. As an improved soliton field, we take \( \phi(x, t) = \phi_{\mathrm{cl}}(y(x, t)) \) with
\[ y(x, t) = (x - X(t)) \left(1 + c_2 \dot{X}(t)^2 + c_4 \dot{X}(t)^4 + \cdots\right), \] (1.4)
where \( c_2 \) and \( c_4 \) are constants to be determined by the EOM principle. The corresponding Lagrangian of \( X(t) \) reads
\[ L = \int dx \mathcal{L}_{\phi(x,t)=\phi_{\mathrm{cl}}(y(x,t))} = m \left\{ -1 + \frac{1}{2} \dot{X}^2 + \frac{1}{2} c_2(1 - c_2) \dot{X}^4 + \cdots \right\}, \] (1.5)
up to time-derivative terms. Therefore, the EOM of \( X(t) \) remains unchanged from that in the simple treatment: \( \ddot{X} = 0 \). On the other hand, for the field theory EOM, we have
\[ -\partial_{\mu}^2 \phi(x, t) + U'(\phi(x, t)) = \left[ (1 - 2c_2) \dot{X}^2 + (2c_2 - c_2^2 - 2c_4) \dot{X}^4 + \cdots \right] \phi''_{\mathrm{cl}}(y), \] (1.6)
up to terms containing \( \dot{X} \). From this, we find that the EOM principle is satisfied (up to \( \dot{X}^6 \)) by taking \( c_2 = 1/2 \) and \( c_4 = 3/8 \). Then, \( y(x, t) \) (1.4) and \( L \) (1.5) are nothing
but the first three terms of the expansion in powers of $\dot{X}^2$ of the Lorentz boost

$$y(x, t) = \frac{x - X(t)}{\sqrt{1 - \dot{X}^2}}, \quad (1.7)$$

and the relativistic Lagrangian (1.3), respectively.\(^\dagger\)

Here, we should mention another and conventional way of constructing the system of collective coordinates. It is, starting with suitably introduced collective coordinates (for example, (1.1) for the center-of-mass), to integrate over the non-zero modes around the soliton. Equivalently, we solve the EOM of the non-zero modes to express them in terms of the collective coordinates. Since the set of EOMs of both the zero and non-zero modes are equivalent to the field theory EOM, integration over the non-zero modes leads to a system of collective coordinates whose EOM implies the field theory EOM. In fact, the relativistic energy of the center-of-mass motion, $E = \sqrt{P^2 + m^2}$, has been obtained by this method for a scalar field theory in 1 + 1 dimensions.\(^2\)–\(^4\) Looking back the procedure of (1.4), (1.5) and (1.6), the EOM of the non-zero modes implies that they are equal to zero if we adopt the elaborated way of introducing the center-of-mass coordinate $X(t)$ by using $y(x, t)$ of (1.4).

In the case of the center-of-mass coordinate of a soliton, we know its relativistic Lagrangian (1.3) from the start. However, for the collective coordinate of the spinning motion of a soliton, it is quite a non-trivial problem to obtain its relativistic description, and this is our main concern of this paper. Concretely, we consider the spinning collective coordinate of the Skyrmion, which is a soliton representing a baryon in the low energy effective theory of pions (the Skyrme-model).\(^5\) Let $U_{cl}(x)$ be the static Skyrmion solution. In the simple treatment, we take $U(x, t) = U_{cl}(R^{-1}(t)x)$ as the spinning Skyrmion field with the rotation matrix $R(t)$ of spinning motion. This leads to the Hamiltonian of a spinning spherical rigid body:

$$H_{\text{rigid body}} = M_{cl} + \frac{1}{2}I\Omega^2 = M_{cl} + \frac{1}{2}I\Gamma^2, \quad (1.8)$$

where $M_{cl}$, $I$, $\Omega$ and $I(= I\Omega)$ are the energy of the static solution, moment of inertia, angular velocity in isospace (body-fixed frame) and the isospin operator, respectively. ($H_{\text{rigid body}}$ also has an equivalent expression with $\Omega$ and $I$ replaced with the angular velocity $\omega$ in real space and the spin operator $J(= I\omega)$, respectively.) However, this rigid body description is certainly a non-relativistic one valid only for a slow spinning motion. For a large angular velocity, the Skyrmion should intuitively deform from its original spherical shape, and this should lead to corrections with higher powers of $\Omega$ to the rigid body Hamiltonian. Therefore, it is an interesting theoretical subject to obtain relativistic (i.e., higher time-derivative) corrections to the rigid body approximation (1.8), or, if possible, to find the fully relativistic description which is the spinning motion counterpart of (1.3).

Relativistic corrections to the rigid body approximation (1.8) may be important also phenomenologically for the Skyrmion describing baryons. Identifying the eigen-

\(^\dagger\) For a constant $\dot{X}$, $y(x, t)$ of (1.7) obviously leads to the relativistic Lagrangian (1.3) to all orders in $\dot{X}^2$. However, for a generic $X(t)$, we have to add (1.7) corrections of $O(\partial_t^3)$ starting with $\frac{2}{3}(x - X(t))^3(\dot{X}^2)\dot{X}^2$ for obtaining (1.3).
values of the Hamiltonian (1.8) with the masses of the nucleon (\( M_N = 939 \) MeV, \( I^2 = J^2 = 3/4 \)) and \( \Delta \) (\( M_\Delta = 1232 \) MeV, \( I^2 = J^2 = 15/4 \)), we find the following:

\[
M_{cl} = 866 \text{ MeV}, \quad \mathcal{I} = 0.00512 \text{ MeV}^{-1}, \quad |\Omega_N| = 169 \text{ MeV}, \quad |\Omega_\Delta| = 378 \text{ MeV}.
\] (1.9)

This implies that, (i) about 8% (30%) of the total mass of the nucleon (\( \Delta \)) comes from the rotational energy, and (ii) the rotational velocity of the nucleon (\( \Delta \)) at the radius of 1 fm is 86% (190% !) of the light velocity. These facts suggest that relativistic corrections are non-negligible especially for \( \Delta \), and that we should reexamine the analysis of Refs. 6) and 7) based on the rigid body approximation by incorporating the effects of the corrections. The limitations of the rigid body approximation were shown numerically without relying on the expansion in powers of angular velocity in the case of the \((2 + 1)\)-dimensional baby version of the Skyrme theory.8)

In this paper, we derive the leading relativistic correction to the rigid body Lagrangian of the spinning Skyrmion by taking the EOM principle as the basic principle of constructing the Lagrangian of spinning collective coordinate. We follow the procedure starting with (1.4) for the center-of-mass motion. Namely, assuming the spinning Skyrmion field of the form \( U(x,t) = U_{cl}(y(x,t)) \) with \( y(x,t) = R(t)^{-1}x + (\text{relativistic correction term}) \), we determine the correction term from the EOM principle. Note that the relativistic correction to \( U(x,t) \) is made only to the argument of the static solution. In the case of the center-of-mass motion, the first relativistic correction is specified by the constant coefficient \( c_2 \). For the spinning Skyrmion, we have to determine three functions \((A(r), B(r), C(r))\) of the radial coordinate \( r \) appearing in \( y(x,t) \) (see Eq. (3.2)) by solving their differential equations derived from the EOM principle. Then, from \((A(r), B(r), C(r))\), the \( \Omega^4 \) correction term to the rigid body Lagrangian is obtained (see Eq. (4.1)). From the function \( y(x,t) \), we can also learn how the static Skyrmion of spherical shape deforms due to the spinning motion.

Then, we repeat the analysis of Refs. 6) and 7) for the decay constant \( f_\pi \) and various static properties of nucleons, such as charge radii, magnetic moments and axial vector coupling, by including the relativistic corrections. We find that the contribution of the relativistic corrections is in the range of 5% to 20%. However, the comparison with the experimental data is a rather disappointing one: Although the value of \( f_\pi \) is shifted closer to the experimental one due to the relativistic correction, it makes the theoretical value move away from the experimental one for most of the static properties. However, this should not be regarded as a problem of our EOM principle. Either higher order relativistic corrections are important, or we cannot expect the Skyrme model to reproduce the baryon sector so precisely.

As we mentioned before, there is another way of constructing the system of collective coordinates. This is, starting with a simply introduced collective coordinates (such as in the rigid body approximation), to solve the EOM of the massive modes to express them in terms of the collective coordinates. Though there has appeared no explicit analysis of the spinning Skyrmion by this method, the present one using the EOM principle should have some advantages: Firstly, ours is simpler since we are not bothered by the non-zero modes, and secondly we can directly know how the
soliton deforms due to its fast spinning motion since the collective coordinates are introduced by deforming the coordinate of the static solution. We should mention that there have appeared many attempts to improve the rigid body approximation of Refs. 6) and 7). They include the papers 9)–29). We wish to emphasize that our method based on the EOM principle gives a systematic and self-consistent way of analyzing the spinning motion of a field theory soliton without any ad hoc physical pictures.

Finally, we remark that the EOM principle has already been used for solitons in gauge theories even in the “rigid body approximation” before introducing the relativistic corrections. Namely, the dependence of the gauge field on the collective coordinates must be determined from the EOM of $A_0$, the Gauss law, which is non-trivial even in the rigid body approximation. The examples are the magnetic monopoles in gauge-Higgs systems (see Ref. 30) for a review) and the baryon solution$^{31),32)}$ in the Sakai-Sugimoto model.$^{33),34)}$

The organization of the rest of this paper is as follows. In §2, we summarize the rigid body approximation to the spinning Skyrmion and show how our EOM principle is violated. In §3, we introduce the improved spinning Skyrmion field with the leading relativistic correction to obtain, on the basis of the EOM principle, the differential equations and the boundary conditions of the functions $(A(r), B(r), C(r))$ specifying the correction. We also consider the deformation of the spinning Skyrmion caused by the relativistic correction. The Lagrangian of the spinning Skyrmion with the leading relativistic correction is given in §4. In §5, we determine the values of the parameters $(f_\pi, e)$ of the Skyrme theory from the masses of the nucleon, $\Delta$ and the pion. The profiles of the functions and the integrands related with the corrections are given. The deformations of the baryons are also pictorially shown there. Then, we present the analysis of the static properties of nucleons. The final section (§6) is devoted to a summary and future problems. In the Appendices (A–E), we present technical details and complicated equations used in the text.

The present paper is a detailed version of Ref. 35), where the EOM principle was first proposed and applied to the quantization of the spinning motion of the Skyrmion.

§2. Skyrmion in the rigid body approximation and its limitation

In this section, we summarize the Skyrmion and the quantization of its spinning degrees of freedom in the rigid body approximation.$^{6),7)}$ We also explain how the EOM principle is violated in this approximation. This is the starting point of our relativistic extension. Another purpose of this section is to fix our notations and conventions.

We consider the $SU(2)$ Skyrme model$^5)$ described by the chiral Lagrangian with the Skyrme term:

$$L = \text{tr} \left\{ -\frac{f_\pi^2}{16} L_\mu^2 + \frac{1}{32e^2}[L_\mu, L_\nu]^2 + \frac{f_\pi^2}{8} m_\pi^2 (U - 1_2) \right\}, \quad (2.1)$$
where $U(x)$ is an $SU(2)$-valued scalar field, $L_\mu$ is defined by

$$L_\mu = -iU \partial_\mu U^\dagger,$$

and $e$ is a dimensionless parameter. Our flat space-time metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, though not written explicitly in (2.1). As we will see later in §4, the inclusion of non-zero pion mass $m_\pi$ is indispensable for our relativistic corrections to be finite. The EOM of the Lagrangian (2.1) reads

$$\partial_\mu \left( L_\mu - \frac{1}{f_\pi^2 e^2} [L_\mu, [L_\mu, L^\nu]] \right) - im_\pi^2 \left( U - \frac{1}{2} \text{tr} U \right) = 0.$$  \hspace{1cm} (2.3)

The Skyrme model has a topologically conserved current $J^\mu_B$:

$$J^\mu_B = -\frac{i}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(L_\nu L_\rho L_\sigma),$$  \hspace{1cm} (2.4)

with $\epsilon^{0123} = 1$. The conserved charge is identified with the baryon number $N_B$:

$$N_B = \int d^3x J^0_B.$$  \hspace{1cm} (2.5)

Next, the diagonal $SU(2)$ symmetry of the theory gives the conserved isospin current:

$$J^\mu_{V,a} = \text{tr} \left[ (J^\mu_L + J^\mu_R) \tau_a \right],$$  \hspace{1cm} (2.6)

where the left and right chiral currents $J^\mu_{L,R}$ are defined by

$$J^\mu_L = -\frac{f_\pi^2}{16} \left( L_\mu - \frac{1}{f_\pi^2 e^2} [L_\mu, [L_\mu, L^\nu]] \right), \quad J^\mu_R = J^\mu_L |_{L_\mu \text{ replaced with } R_\mu},$$  \hspace{1cm} (2.7)

with $R_\mu = -iU^\dagger \partial_\mu U$. On the other hand, the conservation of the axial $SU(2)$ current,

$$J^\mu_{A,a} = \text{tr} \left[ (J^\mu_L - J^\mu_R) \tau_a \right],$$  \hspace{1cm} (2.8)

is broken due to the pion mass term; $\partial_\mu J^\mu_{A,a} = -i(f_\pi^2 m_\pi^2/8) \text{tr} (U\tau_a)$.

The Skyrme model has static soliton solutions which are called Skyrmion and represent the baryons.$^5$ The simplest one takes the hedgehog form:

$$U_{cl}(x) = \exp \left( i\hat{x} \cdot \tau F(r) \right) = \cos F(r) + i\hat{x} \cdot \tau \sin F(r),$$  \hspace{1cm} (2.9)

with $r = |x|$ and $\hat{x} = x/|x|$. The function $F(r)$ is subject to the following differential equation:

$$\frac{d^2 F}{dr^2} + \frac{2}{r} \frac{dF}{dr} - \frac{\sin 2F}{r^2} + \frac{4}{f_\pi^2 e^2} \left[ \frac{2 \sin^2 F}{r^2} \frac{d^2 F}{dr^2} + \frac{\sin 2F}{r^2} \left( \frac{dF}{dr} \right)^2 - \frac{\sin^2 F \sin 2F}{r^4} \right] - m_\pi^2 \sin F = 0,$$  \hspace{1cm} (2.10)
which is obtained by substituting (2.9) into the EOM (2.3), or equivalently, by minimizing the mass (energy) $M_{\text{cl}}$ of the configuration (2.9):

$$M_{\text{cl}} = \frac{\pi f_\pi^2}{2} \int_0^\infty dr \, r^2 \left\{ (F')^2 + \frac{2 \sin^2 F}{r^2} + \frac{4}{f_\pi^2} \frac{\sin^2 F}{r^2} \left[ 2(F')^2 + \sin^2 F \right] + 2m_\pi^2 (1 - \cos F) \right\},$$

(2.11)

with the prime on $F$ denoting a differentiation with respect to $r$. For the hedgehog (2.9) to be non-singular both at the origin and the infinity and for the mass (2.11) to be finite, we must have $F(0) = n\pi$ and $F(\infty) = 0 \pmod{2\pi}$ with the integer $n$ being equal to the baryon number $N_B$. Restricting ourselves to the solution with unit baryon number $N_B = 1$, the behavior of $F(r)$ near the origin is given by

$$F(r) = \pi - \kappa r + O(r^3), \quad (r \to 0)$$

(2.12)

while that near the infinity reads

$$F(r) = \frac{a}{r} \left( 1 + \frac{1}{m_\pi r} \right) e^{-m_\pi r} + O(e^{-2m_\pi r}), \quad (r \to \infty)$$

(2.13)

with $\kappa$ and $a$ being constants. In particular, the power part multiplying $e^{-m_\pi r}$ in (2.13) is exact. The hedgehog solution (2.9) specified by a single function $F(r)$ represents a spherically symmetric extended object; the energy density depends only on $r$.

The hedgehog solution (2.9) has two kinds of collective coordinates; the center-of-mass motion and the space (isospace) rotation. Note that, for the hedgehog (2.9), the space rotation is equivalent to the isospace rotation. Namely, for any orthogonal matrix $O$, we have

$$U_{\text{cl}}(O^{-1}x) = WU_{\text{cl}}(x)W^{-1},$$

(2.14)

where $W$ is the $SU(2)$ matrix corresponding to $O$; $O_{ab}\tau_b = W\tau_a W^{-1}$. In the quantization of the spinning collective coordinate in the rigid body approximation, we assume that the spinning Skyrmion field $U(x,t)$ is simply given by

$$U(x,t) = U_{\text{cl}}(R^{-1}(t)x),$$

(2.15)

where the orthogonal matrix $R(t)$ is the dynamical variable to be quantized. Since the classical solution $U_{\text{cl}}(x)$ is of spherical shape, so does the Skyrmion field of (2.15). An important property of $U(x,t)$ (2.15) is that the left and right $SO(3)$ transformations on the matrix $R(t)$ correspond to the rotations in the real space and the isospace, respectively:

$$U(x,t)\big|_{R \to O_{\text{real}}^{-1}tO_{\text{iso}}} = W_{\text{iso}} U(O_{\text{real}}x,t)W_{\text{iso}}^{-1}. $$

(2.16)

Therefore, we write the components of $R(t)$ by $R_{ia}(t)$, with $i$ and $a$ being the real and isospace indices, respectively.
The dynamics of $R(t)$ is governed by its Lagrangian obtained by inserting (2.15) into the Skyrme model Lagrangian density (2.1) and carrying out the space integration. It turns out to be the Lagrangian of a spherical rigid rotor:

$$L_{\text{rigid body}}(R, \dot{R}) = \int d^3x \left. \mathcal{L} \right|_{U(x,t) = U_{\text{cl}}(R^{-1}x)} = -M_{\text{cl}} + \frac{1}{2} \mathcal{I} \Omega^2,$$

(2.17)

where

$$\mathcal{I} = \frac{2\pi f_\pi^2}{3} \int_0^\infty dr r^2 \sin^2 F \left\{ 1 + \frac{4}{f_\pi^2 e^2} \left[ (F')^2 + \frac{\sin^2 F}{r^2} \right] \right\},$$

(2.18)

is the moment of inertia, and

$$\Omega_a = \frac{1}{2} \epsilon_{abc} (R^{-1} \dot{R})_{bc},$$

(2.19)

is the angular velocity in isospace (which is equal to the angular velocity in the body-fixed frame). The angular velocity $\omega_i$ in real space is given by

$$\omega_i = -\frac{1}{2} \epsilon_{ijk} (R^{-1})_{jk} = \Omega_i \big|_{R \rightarrow R^{-1}} = -R_{ia} \Omega_a.$$

(2.20)

Note that we have

$$\Omega^2 = \omega^2 = -\frac{1}{2} \text{Tr}(R^{-1} \dot{R})^2.$$

(2.21)

The Lagrangian (2.17) describes baryons with unit baryon number $N_B = 1$ and zero center-of-mass momentum. They are specified by the quantum numbers of the spin $J = \mathcal{I} \omega$ and the isospin $I = \mathcal{I} \Omega$, which are the Noether charges corresponding to the symmetry transformations of the Lagrangian (2.17):

$$R(t) \rightarrow O^{-1}_\text{real} R(t) O_{\text{iso}}.$$

(2.22)

Since $I$ and $J$ are related by $J_i = I_i \big|_{R \rightarrow R^{-1}} = -R_{ia} I_a$ in the present system, we have $I^2 = J^2$, which is consistent with the real baryon spectrum. The decay constant $f_\pi$ and various static properties of nucleons including charge radii, magnetic moments and axial vector coupling have been calculated in Refs. 6) ($m_\pi = 0$ case) and 7) by using the masses of the nucleon, $\Delta$ and the pion as inputs. The results agree with the experimental values within about 30% for most of the quantities.

However, the quantization of the spinning collective coordinate explained above is not a fully satisfactory and consistent one as we mentioned in the Introduction. Here, we recapitulate the reasons:

1) Firstly, the Lagrangian (2.17) is nothing but that of a spherical rigid body. However, the rotating soliton should intuitively deform from the spherical shape, and the Lagrangian of $R$ should contain terms which automatically incorporate such deformation.

2) By identifying the eigenvalues of the rigid body Hamiltonian (1.8) corresponding to the Lagrangian (2.17) with the experimental masses of the nucleon and $\Delta$, we saw that about 8% (30%) of the total mass of the nucleon ($\Delta$) comes from the rotational energy, and that the spinning velocity at the radius of 1 fm is nearly (or
over) the light velocity. Therefore, the validity of the rigid body (i.e., non-relativistic) approximation for baryons is a non-trivial problem.

3) The Skyrme field (2.15) does not satisfy the field theory EOM (2.3) even if we use the EOM of \( R(t) \) obtained from the Lagrangian (2.17):

\[
\frac{d\Omega}{dt} = \frac{d}{dt} R^{-1} \dot{R} = 0, \tag{2.23}
\]

which is equivalent to

\[
\frac{d\omega}{dt} = \frac{d}{dt} \dot{R} R^{-1} = 0. \tag{2.24}
\]

In fact, the EOM (2.3) is violated by terms of \( O(\Omega^2) \):

\[
\text{LHS of (2.3)} |_{U(x,t) = U_{cl}(y)} = \left( L^a_{cl} - \frac{1}{f^2 \pi^2} \left[ L^b_{cl}, [L^a_{cl}, L^b_{cl}] \right] \right) (y) \left[ \left( \frac{d}{dt} R^{-1} \dot{R} \right) \tilde{y} \right]_a + \frac{2}{f^2 \pi^2} \left\{ (1 - \cos 2F) \left( \frac{F''}{r^2} + \frac{2}{r^2} (2 - \sin 2F) \right) + 2 \sin 2F (F')^2 \right\} \frac{1}{r^3} (R^{-1} \tilde{\dot{y}})^2 (\tilde{y} \cdot \tau) + \left\{ 1 + \frac{4}{f^2 \pi^2} \left( (F')^2 - \frac{1 - \cos 2F}{r^2} \right) \right\} \times \left\{ \frac{\sin 2F}{2r} \left( (R^{-1} \tilde{\dot{y}})^2 \right)_a \tau_a - \frac{1 - \cos 2F}{2r^2} \left[ \tilde{y} \times \left( (R^{-1} \tilde{\dot{y}})^2 \right)_a \tau_a \right] \right\}, \tag{2.25}
\]

where \( \tilde{y} \) and \( L^a_{cl}(y) \) are defined by

\[
\tilde{y} = \tilde{y}(x, t) = R^{-1}(t) x, \tag{2.26}
\]

\[
L^a_{cl}(\tilde{y}) = -i U_{cl}(\tilde{y}) \frac{\partial}{\partial y_a} U_{cl}(y)^\dagger, \tag{2.27}
\]

and we have used the EOM (2.3) for \( U_{cl} \). (For the derivation of (2.25), see Appendix A, where we present the derivation (3.17) with relativistic corrections.) Therefore, the quantization of spinning motion starting with (2.15) is valid only for slowly rotating baryons with small angular velocity \( \Omega \).

In the subsequent sections, we study relativistic corrections to the rigid body approximation on the basis of the EOM principle stated in the Introduction.

§3. Realizing the EOM principle to \( O(\partial^2_t) \)

As we saw in the previous section, the spinning Skyrmion field given by (2.15) does not satisfy our EOM principle for introducing the collective coordinates: The field theory EOM (2.3) is broken by terms of \( O(\Omega^2) = O(\partial^2_t) \) after using the EOM of \( R(t) \) (see (2.25)). Let us therefore try to resolve this \( O(\partial^2_t) \) breaking of field theory EOM to shift it to the next higher order of \( O(\partial^4_t) \). Although we are still assuming that the rotation is not so fast, our analysis in this paper must prove an important step toward the complete realization of our EOM principle.
3.1. Improved spinning Skyrmion field

Our proposal for the improved Skyrmion field with the collective coordinate $R(t)$ of spinning motion is that all the $R$-dependence is contained in the argument of $U_{\text{cl}}$ as in (2.15) in the rigid body approximation:

$$U(x, t) = U_{\text{cl}}(y(x, t)), \quad (3.1)$$

where $y(x, t)$ is given by

$$y(x, t) = [1 + A(r)(\dot{R}R^{-1}x)^2 + B(r)r^2 \text{Tr}(R^{-1}\dot{R})^2 + C(r)r^2(R^{-1}\dot{R})^2] R^{-1}x \quad (3.2)$$

with $r = |x|$. Compared with (2.15) in the rigid body approximation, we have introduced terms of $O(\Omega^2)$ in the argument of $U_{\text{cl}}$. The present $y(x, t)$ is a spinning motion analogue of (1.4) for the center-of-mass motion. The three functions $A(r)$, $B(r)$ and $C(r)$ in (3.2) should be determined to fulfill our EOM principle. However, it is quite non-obvious at this stage whether the improvement can be accomplished by (3.1) and (3.2), namely, whether we can consistently determine $(A(r), B(r), C(r))$ so that our EOM principle is satisfied to $O(\Omega^2)$.

The form of $y(x, t)$ (3.2) is the most general one containing at most two time-derivatives and satisfying the following requirements:

• $y(x, t)$ is odd under $x \to -x$,

$$y(-x, t) = -y(x, t), \quad (3.3)$$

and is even under the time-inversion of $R(t)$:

$$y(x, t)|_{R(t) \to R(-t)} = y(x, -t). \quad (3.4)$$

• $y(x, t)$ has the following property under the constant left and right $SO(3)$ transformations of $R(t)$:

$$y(x, t)|_{R(t) \to O_{\text{real}}^{-1}R(t)O_{\text{iso}}} = O_{\text{iso}}^{-1}y(O_{\text{real}}x, t). \quad (3.5)$$

This implies that our improved $U(x, t)$ of (3.1) keeps the property (2.16).

• $y(x, t)$ does not contain $\dot{R}(t)$. This is necessary for the Lagrangian of $R(t)$ to consist of $R$ and $\dot{R}$ without $\ddot{R}$.

As we will see later in §4, the EOM of $R(t)$ corresponding to the improved $U(x, t)$ (3.1) remains the same as (2.23). Under the EOM (2.23), our Skyrmion field (3.1) is indeed spinning both in space and isospace with angular velocities $\omega$ (2.20) and $\Omega$ (2.19), respectively. This fact may precisely be stated as follows. First, $y(x, t)$ of (3.2) has the following property:

$$\left(\frac{\partial}{\partial t} + (\dot{\Omega} \times x)\frac{\partial}{\partial x_i}\right) y(x, t)|_{\text{R-EOM}} = 0, \quad (3.6)$$

where "$|_{\text{R-EOM}}$" means "upon using the EOM (2.23) of $R(t)$". Since $U(x, t)$ (3.1) is a function of $y$ only, it obeys the same equation as (3.6):

$$\left(\frac{\partial}{\partial t} + (\Omega \times x)\frac{\partial}{\partial x_i}\right) U(x, t)|_{\text{R-EOM}} = 0. \quad (3.7)$$
This implies that \( U(x, t) \) is spinning in real space with angular velocity \( \omega \) when the EOM (2.23) holds. Next, \( y(x, t) \) also obeys

\[
\frac{\partial y(x, t)}{\partial t} \bigg|_{\text{R-EOM}} = -R^{-1} \dot{R} y = \Omega \times y. \tag{3.8}
\]

This together with the hedgehog property (2.14) means that \( U(x, t) \) is spinning in isospace with angular velocity \( \Omega \).

Finally in this subsection, we consider the geometrical meaning of \( y(x, t) \) (3.2), and, in particular, the spatial shape represented by our spinning Skyrmion (3.1). Below we assume that \( r^2 \Omega^2 \) is sufficiently small. Let \( u \) be the displacement vector, namely, the difference of \( y \) (3.2) and \( \tilde{y} = R^{-1} x \) (2.26):

\[
y = \tilde{y} + u. \tag{3.9}
\]

Taking, in the \( \tilde{y} \)-space, the polar coordinate system \((r = |\tilde{y}| = |x|, \theta, \varphi)\) with \( z\)-axis in the \( \Omega \) direction, \( u \) has the following decomposition in terms of the unit vectors \( e_r = \tilde{y}/r \) and \( e_\theta \) (see Fig. 1):

\[
u = -\left\{ \frac{2}{3} \left[ Y(r) + (A(r) - C(r)) P_2(\cos \theta) \right] e_r + C(r) \sin \theta \cos \theta e_\theta \right\} r^3 \Omega^2, \tag{3.10}
\]

where the function \( Y(r) \) is defined by

\[
Y(r) = -A(r) + 3B(r) + C(r), \tag{3.11}
\]

and \( P_2 \) is a Legendre polynomial:

\[
P_2(z) = \frac{1}{2} (3z^2 - 1). \tag{3.12}
\]

Note that the functions \((A, B, C)\) appear in \( u \) as another three independent combinations; \( Y, A - C \) and \( C \), which also appear in other places below. The decomposition (3.10) of \( u \) is restated as the following relation between the polar coordinate \((|y|, \theta_y, \varphi)\) of \( y \) and \((r, \theta, \varphi)\) of \( \tilde{y} \) (the \( \varphi \) coordinate is common since there is no \( e_\varphi \) term in (3.10)):

\[
|y| = \left\{ 1 - \frac{2}{3} \left[ Y(r) + (A(r) - C(r)) P_2(\cos \theta) \right] r^2 \Omega^2 \right\} r, \tag{3.13}
\]

\[
\tan \theta_y = (1 - C(r) r^2 \Omega^2) \tan \theta, \tag{3.14}
\]

where \( \theta_y \) is the angle between \( y \) and \( \Omega \).

The (classical) shape of our spinning Skyrmion (3.1) in \( x \)-space for a given value of \( \Omega^2 \) may have various definitions; for example, surfaces of constant energy density,
constant baryon number density, and so on.\textsuperscript{1}) Here, noting that $U_{\mathrm{cl}}(y)$ represents a spherically symmetric object in $y$-space, let us take the simplest one; surfaces of constant $|y|$. From (3.13), the surface of $|y| = a (= \text{const})$ is mapped to the $x$-space as

$$\frac{a}{r} = 1 + \left[ (A(a) - C(a)) \sin^2 \theta - 2B(a) \right] a^2 \Omega^2 + O(a^4 \Omega^4).$$

(3.15)

Here, we have $r = |x|$, and $\pi - \theta$ is the angle between $\omega$ and $x$. Equation (3.15) represents an ellipse on the $(r, \theta)$ plane, and hence our spinning Skyrmion (3.1) has a shape of spheroid, as is intuitively expected.

3.2. Differential equations and boundary conditions of $(A, B, C)$

The three functions $(A, B, C)$ in (3.2) must be determined in such a way that the field theory EOM (2.3) with the spinning Skyrmion field (3.1) substituted holds to $O(\Omega^2)$ upon using the EOM of the collective coordinate $R(t)$. Namely, the contribution of the $\Omega^2$ terms in (3.2) to the field theory EOM must cancel the $\Omega^2$ terms in (2.25).

After a tedious but straightforward calculation by using

$$L_0(x, t) = \frac{\partial y_a(x, t)}{\partial t} L_{\mathrm{cl}}^a(y), \quad L_i(x, t) = \frac{\partial y_a(x, t)}{\partial x_i} L_{\mathrm{cl}}^a(y),$$

(3.16)

with $L_{\mathrm{cl}}^a(y) = -iU_{\mathrm{cl}}(y)(\partial/\partial y_a)U_{\mathrm{cl}}(y)^\dagger$ defined by (2.27), we get, instead of (2.25), the following result (we summarize the derivation in Appendix A):

$$\text{LHS of (2.3)} \mid_{U(x, t) = U_{\mathrm{cl}}(y)} = \left[ L_{\mathrm{cl}}^a - \frac{1}{f^2 e^2} \left[ L_{\mathrm{cl}}^b, [L_{\mathrm{cl}}^c, L_{\mathrm{cl}}^b] \right] \right](y) \left[ \left( \frac{d}{dt} R^{-1} \dot{R} \right) y \right]_a$$

$$+ r^2 \text{Tr}(R^{-1}\dot{R})^2(y \cdot \tau) \times \text{EQ}_1 + (R^{-1}\dot{R}y)^2(y \cdot \tau) \times \text{EQ}_2$$

$$+ r^2 \left[ (R^{-1}\dot{R}^2y) \cdot \tau \times \text{EQ}_3 + r \left[ y \times (R^{-1}\dot{R}^2y) \cdot \tau \times \text{EQ}_4 \right] + O(\partial_t^4), \right. \tag{3.17}$$

where $\text{EQ}_n$ ($n = 1, 2, 3, 4$) are linear in $(A, B, C)$ and their first and second derivatives with respect to $r$ with coefficients given in terms of $F$ and its $r$-derivatives. The concrete expressions of $\text{EQ}_n$ are very lengthy, and they are presented in Appendix B. As we have already mentioned in §3.1, and as we will see in the next section, the EOM of the collective coordinate $R(t)$ derived from its Lagrangian, which is obtained by substituting (3.1) into the Skyrme Lagrangian density (2.1) and carrying out the space integration, remains unchanged from (2.23) in the rigid body approximation. This implies that the first term on the RHS of (3.17) vanishes upon using the EOM of $R(t)$, and our EOM principle is satisfied to $O(\Omega^2)$ if the three functions $(A, B, C)$ satisfy four differential equations,

$$\text{EQ}_n = 0. \quad (n = 1, 2, 3, 4) \tag{3.18}$$

This is apparently overdetermined, but, fortunately, $\text{EQ}_3$ and $\text{EQ}_4$ are not independent. We have

$$\text{EQ}_3 = 2 \cos F \times \text{EQ}_3 \text{4}, \quad \text{EQ}_4 = -2 \sin F \times \text{EQ}_4 \text{4}, \tag{3.19}$$

\textsuperscript{1}) The isoscalar quadrupole moment operator, which vanishes identically in the rigid body approximation, can be non-trivial due to present relativistic correction and is given by (E.38) in Appendix E. However, its nucleon expectation value is equal to zero.
with \( \text{EQ}_{34} \) given by (B·5). Therefore, \((A, B, C)\) are determined by solving three inhomogeneous differential equations of second order, \( \text{EQ}_1 = \text{EQ}_2 = \text{EQ}_{34} = 0 \).

As we can guess from the fact that \( Y(r) \) (3·11) is the coefficient of the lowest mode \( 1 = P_0(\cos \theta) \) in (3·13), we can extract the differential equation for a single function \( Y(r) \) by taking a suitable linear combination of \( \text{EQ}_{1,2,3} \). In fact, the special combination,

\[
\text{EQ}_Y = -3\text{EQ}_1 + \text{EQ}_2 - \text{EQ}_3. \tag{3·20}
\]

consists only of \( Y(r) \) and its derivatives as given in (B·6). Note also that \( B(r) \) and its derivatives are missing from \( \text{EQ}_2 \) (B·2) and \( \text{EQ}_{34} \) (B·5). Therefore, for practical purposes, it is convenient to solve \( \text{EQ}_Y = 0 \) for \( Y(r) \), and \( \text{EQ}_2 = \text{EQ}_{34} = 0 \) for \((A(r), C(r))\).

For solving the differential equations for \((A, B, C)\), we have to specify the boundary conditions of the three functions at \( r = 0 \) and \( r = \infty \). First, substituting the approximate expression of \( F(r) \) near the origin given by (2·12) into \( \text{EQ}_2, \text{EQ}_{34} \) and \( \text{EQ}_Y \), we obtain the following differential equations for \((A, C)\) and \( Y \) valid for \( r \sim 0 \):

\[
(1 + 8\mu^2)r^2\frac{d^2A}{dr^2} - 4\mu^2r^2\frac{dC}{dr^2} + 8(1 + 7\mu^2)r\frac{dA}{dr} - 12\mu^2r\frac{dC}{dr} = 0,
\]

\[
(1 + 4\mu^2)r^2\frac{d^2C}{dr^2} - 8\mu^2r\frac{dA}{dr} + 8(1 + 5\mu^2)r\frac{dC}{dr} - 4(1 + 14\mu^2)A + (10 + 56\mu^2)C = 1 - 4\mu^2,
\]

\[
r^2\frac{d^2Y}{dr^2} + 8r\frac{dY}{dr} + 10Y = \frac{1 - 4\mu^2}{1 + 8\mu^2}, \tag{3·21}
\]

with dimensionless \( \mu \) defined by \( \mu = \kappa/ef_s \). From this we find that \((A, C)\) and \( Y \) near the origin is generically given in terms of a particular solution and six independent modes:

\[
\begin{align*}
(A(r)) &= \frac{1 - 4\mu^2}{10(1 + 8\mu^2)} \begin{pmatrix} 4\mu^2 \\ 1 + 4\mu^2 \end{pmatrix} \\
&\quad + c_0 \begin{pmatrix} 5 + 28\mu^2 \\ 2 + 28\mu^2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1/r \end{pmatrix} + c_5 \begin{pmatrix} -6\mu^2 \\ 1 + 4\mu^2 \end{pmatrix} \frac{1}{r^5} + c_7 \begin{pmatrix} 5 \\ 2 \end{pmatrix} \frac{1}{r^7},
\end{align*}
\]

\[
Y(r) = \begin{pmatrix} A(r) \\ C(r) \end{pmatrix} = \begin{pmatrix} 1 - 4\mu^2 \\ 10(1 + 8\mu^2) \end{pmatrix} + c_2 \frac{1}{r^2} + c_5 \frac{1}{r^5}, \quad (r \sim 0) \tag{3·22}
\]

where \( c_n \) \((n = 0, 2, 5, 7)\) and \( c_n^Y \) \((n = 2, 5)\) are constants. Each term on the RHSs of (3·22), for example, \((5/2) (1/r^7)\), means a series in \( r^2 \) starting with this term.

On the other hand, plugging the \( r \rightarrow \infty \) behavior of \( F(r) \) given by (2·13) into \( \text{EQ}_2, \text{EQ}_{34} \) and \( \text{EQ}_Y \), we obtain the differential equations for \((A, C)\) and \( Y \) approximated near the infinity. They are lengthy and are given by (B·7) in Appendix B by using the dimensionless variable \( s = m_x r \). The Skyrme term does not contribute to these differential equations. From (B·7), we find that \((A, C)\) and \( Y \) near the infinity are given as the following sum of a particular solution and six independent modes:

\[
\begin{pmatrix} A(r) \\ C(r) \end{pmatrix} = -\frac{1}{2s} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d_2 \begin{pmatrix} 1 \\ 1/s^2 \end{pmatrix} + d_3 \begin{pmatrix} 1 \\ 1/s^3 \end{pmatrix} + \left\{ f_2 \begin{pmatrix} 1 \\ 1/s^2 \end{pmatrix} + f_3 \begin{pmatrix} 3 \\ 2/s^3 \end{pmatrix} \right\} e^{2s},
\]
\( y(r) = \frac{1}{2s^2} + d_3^y \frac{1}{s^4} + f_3^y \frac{e^{2s}}{s^4}, \quad (r \to \infty) \)  

(3.23)

where \( d_{2,3}, f_{2,3}, d_3^y, f_3^y \) are constants, and each term on the RHSs shows the first term of the series in \( 1/s \). Concrete expressions keeping terms up to \( 1/s^5 \) for the power parts are given in (B.8) in Appendix B.

For solving the differential equations for \((A, B, C)\) globally in the range \( 0 \leq r < \infty \), we impose the following boundary conditions in terms of the coefficients appearing in (3.22) and (3.23):

\[
\begin{align*}
  c_5 &= c_5^y = c_7 = 0, \quad (3.24) \\
  f_2 &= f_3 = f_3^y = 0. \quad (3.25)
\end{align*}
\]

Namely, we choose the “mildest” boundary conditions both at the origin and the infinity. As we will see later, the most important functional of \((A, B, C)\) is \( J \) given by (4.2). It is the coefficient of the \( \Omega^4 \) term of the Lagrangian (4.1) of \( R(t) \) and hence appears in various conserved charge operators such as Hamiltonian, angular momentum and isospin. Owing to our boundary conditions, (3.24) and (3.25), the \( r \)-integration for \( J \) is convergent at both ends, \( r = 0 \) and \( r = \infty \). This is the case also for other physical quantities which we will discuss later in §5.3. Moreover, the deformation vector \( u(x, t) \) (3.10) is non-divergent at \( r = 0 \). It is also finite in the limit \( r \to \infty \) if we fix \( r^2 \Omega^2 = \text{finite} \). These properties look natural for our improvement to make sense.

Having fixed the boundary conditions for \((A, B, C)\), we can numerically solve the differential equations for them by the shooting method. Namely, setting the boundary condition (3.24) at \( r = 0 \), we tune the three parameters \((c_0, c_2, c^y_2)\) in (3.22) so that the condition (3.25) at \( r = \infty \) is satisfied. This task of solving the differential equations for \((A, B, C)\) will be done in §5.2 after we obtain the Lagrangian and the Hamiltonian for \( R(t) \).

Finally, we add that our EOM principle implies that the spinning Skyrmion field (3.1) with a constant \( \Omega \) is nothing but a classical spinning solution of the Skyrme field theory up to \( O(\Omega^4) \).

§4. Lagrangian of \( R(t) \)

The Lagrangian of \( R(t) \) in our formalism with relativistic correction is given, as in the rigid body approximation of (2.17), by substituting the improved Skyrmion field (3.1) into the Lagrangian density (2.1) and carrying out the space integration. The corrections we have added in (3.2) contain two time-derivatives, and hence we can consider terms with four time-derivatives in the Lagrangian. We find that it is given by

\[
L(R, \dot{R}) = \int d^3 x \mathcal{L}|_{U(x, t) = U_{cl}(y)} = -M_{cl} + \frac{1}{2} I \Omega^2 + \frac{1}{4} J \Omega^4,
\]

(4.1)

where the rest mass \( M_{cl} \) and the moment-of-inertia \( I \) are the same as in the rigid body approximation, (2.17), and the coefficient \( J \) of the newly added \( \Omega^4 \) term is
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\[ J = \frac{4\pi f_2^2}{15} \int_0^\infty dr \ r^4 \sin^2 F \times \left\{ rZ' + 5Z - C + \frac{4}{f_2^2 e^2} \left[ \frac{\sin^2 F}{r^2} (rZ' + 3Z + 2C) - (F')^2 (rZ' + Z + C) \right]\right\}, \]  
(4.2)

with \( Z(r) \) defined by\(^*\)

\[ Z = -2A + 5B + 2C = \frac{5}{3}Y - \frac{1}{3}(A - C). \]  
(4.3)

The Lagrangian (4.1) should be regarded as the first three terms of the rotational motion counterpart of the relativistic Lagrangian of the center-of-mass coordinate \( X(t) \): \(-M_{cl}\sqrt{1 - V^2} = -M_{cl} + (1/2)M_{cl}V^2 + (1/8)M_{cl}V^4 + \ldots\). The EOM of \( R(t) \) derived from the Lagrangian (4.1) reads

\[ \frac{d}{dt} \left[ (J + J \Omega^2) \Omega \right] = 0. \]  
(4.4)

This generically implies \( \dot{\Omega} = 0 \). Namely, the EOM of \( R(t) \) remains unchanged from (2.23) in the rigid body approximation, which we already used deriving the differential equations for \( A, B, C \) in §3.2.

Our result (4.1), in particular, \( J \) of (4.2), can be obtained by a rather lengthy calculation with use of (3.16) for \( L_{\mu} \). The outline of the derivation of (4.2) is given in Appendix C. Here, we explain a number of points used in deriving (4.1) and (4.2).

First, note that Lagrangian (4.1) consists only of the angular velocity \( \Omega \); \( \ddot{R}(t) \) and hence \( \ddot{\Omega} \) are missing. It is due to this fact that the EOM of \( R(t) \) remains essentially the same as in the rigid body approximation. The reason why the higher time-derivative of \( R(t) \) does not appear in the Lagrangian is as follows. Since the Lagrangian of \( R(t) \) has an invariance under (2.22) due to the property (3.5) and hence (2.16), and since \( \mathbf{y}(x, t) \) (3.2) does not contain \( \dot{R} \) nor terms with a single \( R^{-1}\dot{R} \), the possible terms in the Lagrangian of \( R(t) \) containing \( \dot{R} \) and at most four time-derivatives in total are \( \text{Tr}(R^{-1}\dot{R}(d/dt)R^{-1}\dot{R}) \) and \( \text{Tr}((R^{-1}\dot{R})^2(d/dt)R^{-1}\dot{R}) \). The origin of these terms is the part quadratic in \( L_0 \) in the Skyrme model Lagrangian density (2.1). However, neither of these two terms can actually exist: The former term with three time-derivatives cannot appear in the Lagrangian owing to another property (3.4) of \( \mathbf{y}(x, t) \) (this term is in any case a time-derivative term and can be discarded even if it does exist). The latter term vanishes identically since \( R^{-1}\dot{R} \) is an anti-symmetric matrix.

\(^*\) The function \( Z(r) \) is extracted from the coefficient of \( e_r \) in (3.10) by carrying out the \( \cos \theta \)-integration with weight function \( \sin^2 \theta \):

\[ \frac{1}{2} \int_{-1}^1 d(\cos \theta) \left[ Y + (A - C)P_2(\cos \theta) \right] \sin^2 \theta = \frac{2}{3}Z. \]
Now, the Lagrangian of $R(t)$ takes symbolically the form $L = 1 + \Omega^2 + \Omega^4$. To consider the coefficients of the three terms, we divide the Lagrangian into the “kinetic part” $T$ (the part with $L_0$) and the “potential part” $V$ (the part without $L_0$), and write $L = T - V$ with

$$T = \int d^3x \text{ tr} \left\{ \frac{j^2}{16} L_0^2 - \frac{1}{16 \pi^2} [L_0, L_i]^2 \right\} \bigg|_{U(x,t)=U_{cl}(y)},$$

$$V = \int d^3x \text{ tr} \left\{ \frac{j^2}{16} L_i^2 - \frac{1}{32 \pi^2} [L_i, L_j]^2 - \frac{f^2}{8} m_0^2 (U - 1) \right\} \bigg|_{U(x,t)=U_{cl}(y)}.$$

Then, $T$ and $V$ have the following expressions:

$$T = \frac{1}{2} \mathcal{I} \Omega^2 + \frac{1}{4} \mathcal{J}_1 \Omega^4 + O(\partial^6_t),$$

$$V = M_{cl} - \frac{1}{2} \Delta \mathcal{I} \Omega^2 - \frac{1}{4} \mathcal{J}_2 \Omega^4 + O(\partial^6_t),$$

and hence

$$L = -M_{cl} + \frac{1}{2} \mathcal{I} + \Delta \mathcal{I} \right\} \Omega^2 + \frac{1}{4} \left( \mathcal{J}_1 + \mathcal{J}_2 \right) \Omega^4.$$

The origin of each term in (4.7) and (4.8) is explained as follows. Since $y = R^{-1} x + u$ of (3.2) is the sum of the rigid body term $R^{-1} x$ and the correction term $u$ which are quadratic in $R^{-1} \dot{R}$, the coefficients of the lowest order terms in (4.7) and (4.8), namely, $\mathcal{I}$ and $M_{cl}$, are the same as those in the rigid body approximation. Next, the $\mathcal{J}_1 \Omega^4$ and the $\Delta \mathcal{I} \Omega^2$ terms are the leading terms due to our relativistic correction $u$, and hence they are linear in $(A, B, C)$. Finally, the $\mathcal{J}_2 \Omega^4$ term in (4.8) is quadratic in $u$ and therefore is quadratic in $(A, B, C)$. One might wonder that, if we have added the $O(\partial^4_t)$ terms to $y$, they would also contribute to the $\Omega^4$ term in $V$. However, this is not the case as we will see below.

By a simple scaling argument, we can get useful relationships among the coefficients in (4.9). Let us replace $(A, B, C)$ in $y$ of (3.2) by $\lambda(A, B, C)$ with $\lambda$ being a constant parameter, and denote this modified $y$ by $y_\lambda$; namely, $y_\lambda = R^{-1} x + \lambda u$. Then, using the fact that $\Delta \mathcal{I}$ and $\mathcal{J}_1$ are linear in $(A, B, C)$, while $\mathcal{J}_2$ is quadratic in them, the Lagrangian of $R(t)$ corresponding to $U(x, t) = U_{cl}(y_\lambda(x, t))$ reads, instead of (4.9), as follows:

$$L_\lambda = -M_{cl} + \frac{1}{2} \mathcal{I} + \lambda \Delta \mathcal{I} \right\} \Omega^2 + \frac{1}{4} \left( \lambda \mathcal{J}_1 + \lambda^2 \mathcal{J}_2 \right) \Omega^4.$$

At this point we should recall our EOM principle of introducing collective coordinates; $U(x, t) = U_{cl}(y_{\lambda=1})$ satisfies the field theory EOM, namely, it is the extrema of the field theory action if $R(t)$ is subject to its EOM, $\dot{\mathcal{O}} = 0$. This implies, in particular, that $L_\lambda$ (4.10) is stationary at $\lambda = 1$ for any constant $\Omega$:

$$0 = \frac{d}{d\lambda} L_\lambda \bigg|_{\lambda=1} = \frac{1}{2} \Delta \mathcal{I} \Omega^2 + \frac{1}{4} (\mathcal{J}_1 + 2 \mathcal{J}_2) \Omega^4. \quad (\text{for } \forall \Omega)$$

Therefore, we get

$$\Delta \mathcal{I} = 0, \quad \mathcal{J}_1 + 2 \mathcal{J}_2 = 0.$$
By the same rescaling argument (by using another parameter), we can show that the addition of $O(\partial^4_t)$ terms to $u$ does not affect the coefficient of $\Omega^4$ in (4.9) as we mentioned above.\footnote{The same kinds of relationships also hold in the case of the relativistic correction to the center-of-mass motion described by (1-4), (1-5) and (1-6): In the Lagrangian (1-5), $c_2$ does not affect the coefficient of the $\dot{X}^2$ term, the relation $c_2 + 2(-c_2^2) = 0$ holds for the coefficient $c_2 - c_2^2$ of $\dot{X}^4$ (since we have $c_2 = 1/2$), and $c_4$ does not enter the coefficient of the $\dot{X}^4$ term.}

Owing to the second relation of (4.12), the coefficient $J$ of $\Omega^4$ in (4.9) can be expressed only in terms of $J_1$:

$$J = J_1 + J_2 = \frac{1}{2} J_1.$$  \hfill (4-13)

This is very beneficial for obtaining $J$ since $J_2$, which is quadratic in $(A, B, C)$, is harder to evaluate. In Appendix C, we present the derivation of $J$ (4.2) by using (4-13).

Some comments are in order concerning our results (4.1) and (4.2) and the boundary conditions (3.24) and (3.25) we have chosen for $(A, B, C)$. First, from the boundary behaviors of $F(r)$ given by (2-12) and (2-13), the $r$-integration of $J$ (4.2) is approximated near $r = 0$ and $r = \infty$ by $\int_{r=0}^{\infty} dr r^6 [r Z' + 5 Z - C + (2 \kappa / e f_\pi)^2 (2 Z + C)]$ and $\int_{r=\infty}^{r=0} dr r^2 e^{-2m_\pi r} (r Z' + 5 Z - C)$, respectively. They are both convergent due to the boundary conditions $c_7 = 0$ at $r = 0$ and (3-25) at $r = \infty$.

Our second comment is on the necessity of the non-zero pion mass. If the pion mass $m_\pi$ is zero, the exponential falloff (2.2) of $F(r)$ near $r = \infty$ is changed to the power one $F(r) \sim 1/r^2$. Correspondingly, choosing the mildest boundary condition at $r = \infty$, we have the following leading behaviors near $r = \infty$; $(A, C) \sim (-3/4, -1/2)$ and $Y \sim 1/4$. This implies that the $r$-integration (4.2) for $J$ is linearly divergent at $r = \infty$ in the case of $m_\pi = 0$. This is the case also for relativistic corrections to other physical quantities which we discuss in §5.3. Therefore, the inclusion of non-zero pion mass is inevitable for our analysis in contrast to the case of the rigid body approximation.\footnote{The same kinds of relationships also hold in the case of the relativistic correction to the center-of-mass motion described by (1-4), (1-5) and (1-6): In the Lagrangian (1-5), $c_2$ does not affect the coefficient of the $\dot{X}^2$ term, the relation $c_2 + 2(-c_2^2) = 0$ holds for the coefficient $c_2 - c_2^2$ of $\dot{X}^4$ (since we have $c_2 = 1/2$), and $c_4$ does not enter the coefficient of the $\dot{X}^4$ term.}

Next, as described in Appendix C, the expression (4.2) for $J$ has been obtained by changing the variables of integration in (4-1) from $x$ to $y$, and carrying out the integration over the solid-angle of $y$. Therefore, $r$ in (4.2) is the length of $y$: $r = |y|$. This way of switching to the $y$-integration is easier than to consider the original $x$-integration. If we persist in carrying out the $x$-integration, we obtain the expression of $J$ which differs from (4-2) by a surface term:

$$J |_{\text{by } x\text{-integration}} = \text{Eq. (4.2)} - \frac{4 \pi f_\pi^2}{15} \left[ r^5 \sin^2 F \left\{ 1 + \frac{4}{f_\pi^2 e^2} \left( (F')^2 + \frac{\sin^2 F}{r^2} \right) \right\} Z \right]_{r=0}^{r=\infty},$$

with $r = |x|$ on the RHS. From (2-12) and (2-13) for $F(r)$, the surface terms of (4-14) at $r = 0$ and $r = \infty$ are (up to constants) $\lim_{r\to0} r^7 Z(r)$ and $\lim_{r\to\infty} r^3 e^{-2m_\pi r} Z(r)$, respectively. Fortunately, they both vanish safely owing to the boundary conditions (3-24) and (3-25) of $(A, B, C)$.
Our last comment is on $\Delta I$ in (4.9). Explicit calculation by using the $y$-integration and the EOM (2.10) of $F(r)$ leads to the following expression of $\Delta I$ as a surface term:

$$\Delta I = \frac{2\pi f_2^2}{3} \left[ r^5 \left\{ (F')^2 - \frac{2 \sin^2 F}{r^2} + \frac{4 \sin^2 F}{f_2^2 e^2} \left( 2(F')^2 - \frac{\sin^2 F}{r^2} \right) \right\} Y \right]_{r=0}^{r=\infty}$$

$$= \frac{2\pi f_2^2}{3} \left\{ \kappa^2 \left( 1 - \frac{4\kappa^2}{f_2^2 e^2} \right) \lim_{r \to 0} r^5 Y(r) + a_0^2 \lim_{r \to \infty} r^3 e^{-2m_\pi r} Y(r) \right\}. \quad (4.15)$$

This also vanishes owing to the boundary conditions $c^Y_0 = f^Y_3 = 0$ of $Y(r)$.

In §3.1, we saw that our improved spinning Skyrmion field (3.1) takes the form of a spheroid. In the rest of this section, let us consider how this deformation can be read off from the Lagrangian (4.1). One might think it strange that the Lagrangian (4.1), which is rotationally symmetric, represents an anisotropic rotor. The system of $R(t)$ must have rotational symmetry since the original Skyrmion field theory does. However, we should notice that the deformation which breaks the rotational symmetry can be induced “spontaneously” by the spinning motion. To see this explicitly, let us consider a stationarily rotating Skyrmion with a constant angular velocity $\Omega_0$ and add a perturbation to its spinning motion. Namely, we consider $R(t) = R_0(t)O(t)$ with orthogonal matrices $R_0(t)$ and $O(t) = \exp(\zeta(t)) \ (i\zeta = -\tilde{\zeta})$ representing the stationary rotation with a constant angular velocity $\Omega_0$ and the small perturbation $\zeta(t)$ on it, respectively. Then, using

$$\Omega = O^{-1} \left( \Omega_0 + \dot{\zeta} - \zeta \times \dot{\Omega} + O(\zeta^3) \right), \quad (4.16)$$

with $\xi_a = (1/2)\epsilon_{abc}\tilde{\zeta}^{bc}$, we find that the Lagrangian (4.1) is expanded up to $O(\zeta^3)$ and time-derivative terms as

$$L(R, \dot{R}) = L(R_0, \dot{R}_0) + \frac{1}{2} I^{\text{eff}}_{ab}(\Omega_0) \dot{\xi}_a \dot{\xi}_b + (I + J \Omega_0^2) \left( \zeta \times \Omega_0 \right) \cdot \dot{\Omega}, \quad (4.17)$$

with $I^{\text{eff}}(\Omega_0)$ defined by

$$I^{\text{eff}}_{ab}(\Omega_0) = (I + J \Omega_0^2) \delta_{ab} + 2J(\Omega_0)_a(\Omega_0)_b. \quad (4.18)$$

Let us restrict ourselves to the time region where the perturbation $\dot{\xi}$ has just started and is still sufficiently small; $\xi \ll \dot{\xi}/\Omega_0$. Then, the last term of (4.17) can be dropped, and $I^{\text{eff}}_{ab}(\Omega_0)$ of (4.18) may be regarded as the effective moment-of-inertia tensor for the perturbation. (Note that, for a rigid rotor with a generic moment-of-inertia $I_{ab}$, our $I^{\text{eff}}_{ab}$ is equal to the original $I_{ab}$.) This effective moment-of-inertia is in fact an anisotropic one. If the initial angular velocity $\Omega_0$ is along the third-axis, we have

$$I^{\text{eff}}(\Omega_0) = \text{diag}(I + J \Omega_0^2, I + J \Omega_0^2, I + 3J \Omega_0^2), \quad (4.19)$$

which represents a deformed rotor expanded in the direction perpendicular to $\Omega_0$. 

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In this section, starting with the Lagrangian (4.1) of $R(t)$ with the relativistic correction term $(1/4)J\Omega^4$, we first determine the parameters $(f_\pi, e)$ of the Skyrme model from the masses of the lightest baryons, the nucleon and $\Delta$, as well as $m_\pi$. This task includes the determination of the functions $F(r)$ and the newly introduced $(A(r), B(r), C(r))$. Then, we calculate various static properties of nucleons and compare them with the experimental values and also with the results of Ref. 7) without relativistic correction.

5.1. Hamiltonian and spin/isospin operators

Let us consider the standard canonical quantization of the system described by the Lagrangian (4.1) with dynamical variable $R(t) \in SO(3)$. For this, we take three independent variables $\xi_A(t) (A = 1, 2, 3)$ (for example, the Euler angles) which parametrize the $SO(3)$ matrix $R(t); R = R(\xi^A(t))$. Then, we have

$$\Omega_a = \frac{1}{2} \epsilon_{abc}(R^{-1}\dot{R})_{bc} = \dot{\xi}^A \varpi_{Aa}, \quad \Omega^2 = g_{AB}(\xi)\dot{\xi}^A\dot{\xi}^B,$$

with $\varpi_{Aa}$ and $g_{AB}(\xi)$ defined by

$$\left(R(\xi)^{-1}\frac{\partial}{\partial \xi^A}R(\xi)\right)_{ab} = \varpi_{Ac}(\xi)\epsilon_{abc}, \quad g_{AB}(\xi) = \varpi_{Aa}(\xi)\varpi_{Ba}(\xi).$$

The canonical momentum $\pi_A$ conjugate to $\dot{\xi}^A$ is

$$\pi_A = \frac{\partial L}{\partial \dot{\xi}^A} = (I + J\Omega^2)g_{AB}(\xi)\dot{\xi}^B,$$

and the Hamiltonian of the present system is given by

$$H = \pi_A\dot{\xi}^A - L = M_{cl} + \frac{1}{2}I\Omega^2 + \frac{3}{4}J\Omega^4.$$
Then, the equation relating $\Omega^2$ with the spin and the isospin is

$$I^2 = J^2 = (I + J\Omega^2)^2 \Omega^2. \quad (5.9)$$

The Hamiltonian and the (iso)spin charges above have been obtained from the Lagrangian (4.1) of $R(t)$. One may, however, wonder whether these conserved charges in the quantum mechanical system of $R$ agree with those from the Skyrme field theory, namely, whether $H$, $I_\alpha$ and $J_i$ above agree with those obtained by substituting $U(x, t) = U_{\text{cl}}(y)$ (3.1) into the corresponding Noether charges from the Lagrangian density (2.1) of the Skyrme model. This agreement indeed holds owing to our EOM principle (see Appendix D for a proof). Furthermore, by using the equality of the isospins $I_\alpha$ obtained in two ways, we can get the expression (4.2) of $J$ directly from the calculation in the field theory side without relying on the relation (4.13). This is explained in Appendix E (see the paragraph below (E.13)).

5.2. Determining $(f_\pi, e)$ and $(A, B, C)$ from $(M_N, M_\Delta, m_\pi)$

Now, let us consider determining $(f_\pi, e)$ from the Hamiltonian (5.4) together with the relation (5.9) by taking the masses of the nucleon, $\Delta$ and the pion, $M_N = 939 \, \text{MeV}, \quad M_\Delta = 1232 \, \text{MeV}, \quad m_\pi = 138 \, \text{MeV}, \quad (5.10)$
as inputs. Concrete procedure is as follows. We rewrite the differential equations for $F$ and $(A, B, C)$ and the integrations for $I$ (2.18) and $J$ (4.2) in terms of the dimensionless variable $\rho = e f_\pi r$. Then, the differential equations and the integrations for the dimensionless quantities $(\widehat{M}_{\text{cl}}, \widehat{I}, \widehat{J})$ defined by

$$\widehat{M}_{\text{cl}} = \frac{e}{f_\pi} M_{\text{cl}}, \quad \widehat{I} = e^3 f_\pi I, \quad \widehat{J} = e^5 f_\pi^3 J, \quad (5.11)$$

are characterized by a single parameter $\beta = m_\pi/(e f_\pi)$. For a given value of $\beta$, we first solve the differential equation for $F$ and then those for $(A, B, C)$ to obtain $(\widehat{M}_{\text{cl}}, \widehat{I}, \widehat{J})$. Next, by identifying the Hamiltonian (5.4) for the nucleon ($I^2 = 3/4$) and $\Delta$ ($I^2 = 15/4$) with the inputs $M_N$ and $M_\Delta$, respectively, we get the corresponding $(f_\pi, e, m_\pi)$. Namely, we solve

$$M = \frac{f_\pi}{e} \left( \widehat{M}_{\text{cl}} + \frac{1}{2} e^4 \widehat{I} \widehat{\Omega}^2 + \frac{3}{4} e^8 \widehat{J} \widehat{\Omega}^4 \right), \quad (5.12)$$

$$I^2 = \left( \widehat{I} + e^4 \widehat{J} \widehat{\Omega}^2 \right)^2 \widehat{\Omega}^2, \quad (5.13)$$

with dimensionless $\widehat{\Omega}^2$ defined by

$$\widehat{\Omega}^2 = \frac{1}{e^6 f_\pi^2} \Omega^2, \quad (5.14)$$

for both the nucleon and $\Delta$ to obtain $(f_\pi, e)$ and $m_\pi = e f_\pi \beta$. We tune $\beta$ to reproduce the experimental value of $m_\pi$. Our result obtained this way is

$$f_\pi = 125 \, \text{MeV}, \quad e = 5.64, \quad (5.15)$$
which is realized at $\beta = 0.196$. Compared with the experimental value $f^{(\text{exp})}_\pi = 186 \text{ MeV}$, our $f^\pi = 108 \text{ MeV}$, without relativistic correction. The dimensionless quantities corresponding to the result (5.15) are given by
\begin{equation}
(\hat{M}_\text{cl}, \hat{I}, \hat{J}) = (37.9, 70.2, 279), \tag{5.16}
\end{equation}
and the angular velocities of the nucleon and $\Delta$ are as follows:
\begin{align}
|\Omega_N| & = 206 \text{ MeV} \quad (\hat{\Omega}_N = 8.44 \times 10^{-5}), \quad |\Omega_\Delta| = 330 \text{ MeV} \quad (\hat{\Omega}_\Delta^2 = 2.17 \times 10^{-4}). \tag{5.17}
\end{align}
Note that the angular velocity of the nucleon ($\Delta$) has become larger (smaller) than that in the rigid body approximation (see Eq. (1.9)).

In the rest of this subsection, we present graphically various numerical results corresponding to $(f_\pi, e)$ of (5.15).

**Profiles of $(A, B, C)$**

First, in Figs. 2 and 3, the profiles of $(A, B, C)$ as functions of $\psi = \tan^{-1} \rho$ ($0 \leq \psi < \pi/2$) are given. Figure 2 shows the profiles of $(A, B, C)$ themselves, and Fig. 3 shows those of $\Omega^2 r^2 (C - A, Y, C)$, which are functions appearing in (3.10) and are directly relevant to the deformation of baryons. As $\Omega^2$ we have taken the nucleon angular velocity $\Omega_N^2 = (206 \text{ MeV})^2$.

**Profiles of the integrands of $(M_{\text{cl}}, I, J)$**

Next, Figs. 4 and 5 show the profiles of the integrands of the dimensionless quantities ($\hat{M}_{\text{cl}}, \hat{I}, \hat{J}$) constituting the Hamiltonian. They are given as $\rho$-integrations; (2.11), (2.18) and (4.2) with the replacements $r \to \rho$, $d/dr \to d/d\rho$, $m_\pi \to \beta$ and $(f_\pi, e) \to (1, 1)$. In Fig. 4, we show the profiles of their integrands as the $\psi$-
Fig. 4. (color online) Profiles of the integrands of $\psi$-integrations for (i) $\hat{M}_{cl}$ (black dash-dotted curve), (ii) $\hat{I}$ (blue broken one), and (iii) $(1/4)\hat{J}$ (red solid one), at $\beta = 0.196$. Note that the profile for $\hat{J}$ is multiplied by $(1/4)$. The horizontal coordinate is $\psi = \tan^{-1} \rho$.

Fig. 5. (color online) Profiles of the angle averaged densities in real space of (i) $\hat{M}_{cl}$ (black dash-dotted curve), (ii) $\hat{I}$ (blue broken one), and (iii) $(1/4)\hat{J}$ (red solid one), at $\beta = 0.196$. The horizontal coordinate is $\psi = \tan^{-1} \rho$.

Fig. 6. (color online) The spheroids (3.15) of the nucleon (red broken ellipse curve) and $\Delta$ (blue dash-dotted one) spinning around the vertical axis and corresponding to the original sphere (black solid circle) with radius $a = 1$ fm. The units on the horizontal and vertical axes are fm.

Fig. 7. (color online) Deformation vectors $-u$ (3.10) of the nucleon at various angles $\theta$. The red solid arrows and blue broken ones are $-u$ on $r = 0.5$ fm and $r = 1$ fm, respectively. The units on the horizontal and vertical axes are fm.

integrations. Namely, giving $O = \hat{M}_{cl}, \hat{I}, (1/4)\hat{J}$ as $O = \int_{0}^{\pi/2} d\psi \ f_{O}(\psi)$, Fig. 4 shows the profiles of the three $f_{O}(\psi)$ (for the sake of visualization, we take $(1/4)\hat{J}$ instead of $\hat{J}$ itself). On the other hand, expressing $O$ differently as $O = 4\pi \int_{0}^{\pi/2} d\rho \ \rho^{2} g_{O}(\psi)$, Fig. 5 shows the profiles of $g_{O}(\psi)$, which are the angle averaged densities in real space. We have $f_{O}(\psi) = 4\pi (\sin^{2} \psi / \cos^{4} \psi) g_{O}(\psi)$. 
**Deformation of baryons**

As we explained in §3.1, the spinning Skyrmion field (3.1) describes a spheroid (3.15). In Fig. 6, we show the spheroids (ellipses) which represent the nucleon and Δ and correspond to the sphere of radius $a = 1$ fm without spinning motion. Namely, they are the spheroids (3.15) with $\Omega^2$ given by $\Omega_N^2$ and $\Omega_{\Delta}^2$ of (5.17), respectively, and with $a = 1$ fm. In Fig. 7, we show the deformation vectors $-u$ (3.10) of the nucleon representing the shift of each point on the original sphere due to the spinning motion at radii 0.5 fm and 1 fm. Note that $y$ is on the sphere and $\tilde{y} = y - u$ is on the spheroid. In both the figures, the angular velocity points to the vertical direction.

### 5.3. Static properties of nucleons

Having determined the parameters $(f, e)$ of the Skyrme model from the experimental values of $(M_N, M_{\Delta}, m_\pi)$, let us calculate the various static properties of nucleons which were analyzed in the rigid body approximation in Refs. 6) and 7). Concretely, we consider the charge radii, magnetic moments, magnetic charge radii, and axial vector coupling of nucleons. We are of course interested in how the relativistic correction modifies the numerical results from those of Ref. 7).

First, we need the analytic expressions of these static properties derived from the Lagrangian density (2.1) of the Skyrme model and the spinning Skyrmion field (3.1). In Appendix E, we present the definitions, outline of derivations and the final analytic expressions of the static properties. The analytic expressions are as follows:

- Isoscalar mean square charge radius $\langle r^2 \rangle_{I=0}$: Eq. (E.4).
- Isovector mean square charge radius $\langle r^2 \rangle_{I=1}$: Eq. (E.11).
- Isoscalar mean square magnetic radius $\langle r^2 \rangle_{M,I=0}$: Eq. (E.20).
- Isovector mean square magnetic radius $\langle r^2 \rangle_{M,I=1}$: Eq. (E.28). This is in fact equal to the electric one $\langle r^2 \rangle_{I=1}$.
- Isoscalar $g$-factor $g_{I=0}$: Eq. (E.19).
- Isovector $g$-factor $g_{I=1}$: Eq. (E.26) together with (E.27). It has another and simpler expression (E.30).
- Axial vector coupling constant $g_A$: Eq. (E.34) together with (E.35) and (E.36). These quantities are all given as integrations over the dimensionless radial coordinate $\rho = ef_\pi r$. The mean square charge radii are given in units of $1/f^2_{\pi} e^2 = (0.280 \text{ fm})^2$. In every quantity, the relativistic correction part is multiplied by $e^4 \tilde{\Omega}_N^2$ with $\tilde{\Omega}_N^2$ being the dimensionless angular velocity of the nucleon (5.17).

In Table I, we summarize the numerical values of the static properties obtained by using $(e, f_\pi, \tilde{\Omega}_N^2)$ and the functions $F$ and $(A, B, C)$ determined in §5.2. Instead of the $g$-factors, we present the nucleon magnetic moments $\mu_p$ and $\mu_n$ in units of Bohr magneton:

$$
\mu_p = \frac{1}{4} (g_{I=0} + g_{I=1}), \quad \mu_n = \frac{1}{4} (g_{I=0} - g_{I=1}).
$$

(5.18)

In the table, we also present the predictions of Ref. 7) without relativistic correction as well as the experimental values.

As seen from the table, the difference between the prediction of this paper and that of Ref. 7) is in the range of 5% to 20%. (Note that each of our predictions is
Table I. The static properties of nucleons. Prediction of this paper and that of Ref. 7) in the rigid body approximation both use the experimental values of \((M_N, M_\Delta, m_\pi)\) as inputs. We follow the notations of Ref. 6).

| Prediction (this paper) | Prediction (Ref. 7)) | Experiment |
|-------------------------|----------------------|------------|
| \(f\) | \(125 \text{ MeV} \) | \(108 \text{ MeV} \) | \(186 \text{ MeV} \) |
| \(\langle r^2 \rangle_{1/2}^{1/2} \) | 0.59 fm | 0.68 fm | 0.81 fm |
| \(\langle r^2 \rangle_{1/2} \) | 1.17 fm | 1.04 fm | 0.94 fm |
| \(\langle r^2 \rangle_{1/2}^{1/2} \) | 0.85 fm | 0.95 fm | 0.82 fm |
| \(\langle r^2 \rangle_{m,J=0}^{1/2} \) | 1.17 fm | 1.04 fm | 0.86 fm |
| \(\langle r^2 \rangle_{m,J=1} \) | 1.65 | 1.97 | 2.79 |
| \(\mu_p \) | -0.99 | -1.24 | -1.91 |
| \(\mu_n \) | 1.67 | 1.59 | 1.46 |
| \(|\mu_p/\mu_n|\) | 0.58 | 0.65 | 1.24 |

not given simply by adding the \(\Omega^2\) correction to the result of Ref. 7) since the values of \(f\) and \(e\) are also changed.) Although the fundamental parameter of the theory, \(f\), has been improved, the relativistic correction makes the value of theoretical prediction further away from the experimental one for most of the static properties of nucleons (the only exception is \(\langle r^2 \rangle_{M,I=0}\)). However, this should not be regarded as a manifestation of problems of our basic EOM principle for introducing collective coordinates. There would be two possibilities for this unwelcome result. One is that, since the relativistic correction is rather large, we have to take into account the contributions from still higher orders in \(\Omega^2\) for obtaining better results of the static properties of nucleons. Another possibility would be that we cannot expect the Skyrme theory, which is merely a low energy effective theory, to reproduce precisely the physics of the baryon sector even if the full relativistic treatment of rotational collective coordinate is carried out. Concerning the first possibility, we should note that the ratio of the contributions of the three terms of the Hamiltonian (5.4) to the baryon masses is 89 : 7 : 4 for the nucleon and 68 : 14 : 18 for \(\Delta\). This suggests to us that the expansion in powers of the angular velocity \(\Omega\) is not a good approximation especially for \(\Delta\) with a larger (iso)spin.

§6. Summary and outlook

In this paper, we proposed the EOM principle for constructing the relativistic system of collective coordinates of a field theory soliton, and applied it to the “next to the rigid body approximation” analysis of the spinning Skyrmion. The relativistic correction was successfully incorporated in the argument of the static Skyrmion solution in terms of the three functions \((A(r), B(r), C(r))\) obeying a consistent set of differential equations demanded by the EOM principle. We computed the decay constant and various static properties of nucleons with leading relativistic corrections by taking the masses of the nucleon, \(\Delta\) and the pion as inputs. The relativistic corrections to the results in the rigid body approximation are found to be in the
range of 5% - 20%. Though the value of the decay constant has become closer to the experimental one due to the correction, the results are not good for most of the static properties of nucleons. We also studied how the baryons deform from the spherical shape to the spheroidal one due to the spinning motion.

Since we have introduced the relativistic corrections in a proper manner based on the EOM principle, our unwelcome result on the static properties of nucleons is an inevitable nature of the baryon sector of the Skyrme model: It should not be regarded as a problem of our basic EOM principle. If the simple Skyrme model can reproduce the real baryon physics more precisely, a remedy for the unwelcome results should be looked for in our use of expansion in powers of the angular velocity. Though we took into account only the first relativistic correction of $O(\Omega^4)$ in this paper, it would be necessary to consider higher order corrections, or to devise an approximation valid for an extreme large angular velocity, at least for analyzing $\Delta$. For this, we have to consider a systematic generalization of the expressions (3-13) and (3-14) in terms of the Legendre polynomials. The analysis$^8$ of the spinning baby Skyrmion on the two-dimensional plane may be helpful for considering the extreme case in the $3 + 1$ dimensional model. With a better (or the full) realization of the EOM principle, we can give more reliable predictions for the Skyrmion. It is also an important subject to apply our method to the collective coordinate quantization of other interesting physical systems.

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Appendix A

Derivation of Eq. (3.17)

In this appendix, we sketch the derivation of (3.17), namely, the LHS of the Skyrme field theory EOM (2.3) with the improved spinning Skyrmion field (3-1) substituted. In this derivation, the most important formula is (3-16) for $L_\mu(x,t)$
with \( L_a^{\mathrm{cl}}(y) \) \((a = 1, 2, 3)\) given by

\[
L_a^{\mathrm{cl}}(y) = -iU_a^{\mathrm{cl}}(y) \frac{\partial}{\partial y_a} U_a^{\mathrm{cl}}(y)^\dagger
= -\frac{1}{2r} \left\{ \sin 2F\delta_{ab} + (2rF' - \sin 2F) \hat{y}_a \hat{y}_b - (1 - \cos 2F) \epsilon_{abc} \hat{y}_c \right\} \tau_b, \tag{A.1}
\]

where \( \hat{y}_a \) is a unit vector \( \hat{y}_a = y_a/|y| \). In this appendix, \( r \) denotes the length of \( y \), and the argument \( r \) of the function \( F \) and its derivatives is omitted:

\[
r = |y|, \quad F = F(r), \quad F' = \frac{dF(r)}{dr}. \tag{A.2}
\]

As for the argument \( r \) of \((A, B, C)\), the difference between \( r = |x| \) and \( r = |y| \) does not matter since \((A, B, C)\) are already multiplied by the \( O(r^2) \) quantities in (3.2).

### Necessary formulas

Let us present the formulas of the differentiations of \( y(x, t) \) (3.2) with respect to \( x \) and \( t \). First, the time-derivatives are given by

\[
\frac{\partial y_a(x, t)}{\partial t} = -(R^{-1} \dot{R} y)_a + 2 \left[ A (R^{-1} \dot{R} y)_b (Wy)_b + r^2 B \operatorname{Tr} (WR^{-1} \dot{R}) \right] y_a \\
+ r^2 C (\{W, R^{-1} \dot{R}\} y)_a + O(\partial_t^4), \tag{A.3}
\]

\[
\frac{\partial^2 y_a(x, t)}{\partial t^2} = -(Wy)_a + ((R^{-1} \dot{R})^2 y)_a + O(\partial_t^4), \tag{A.4}
\]

where \( W \) is defined by

\[
W = \frac{d}{dt} R^{-1} \dot{R}. \tag{A.5}
\]

Note that, in (A.3) and (A.4), we have presented their expressions in terms of \( y \), which are obtained by rewriting the original ones in terms of \( \hat{y} \) by using (3.2). The counting of the order of time-derivatives \( \partial_t \) is of course defined by regarding \( x \) (and not \( y \)) as a zero-th order quantity. Though we need only the first term on the RHS of (A.3) in this appendix, we have presented the expression valid to \( O(\partial_t^4) \). This will be used in deriving the coefficient \( J \) of (4.2) in Appendix C.

Next, for the \( x \)-derivatives, we have

\[
\frac{\partial y_a(x, t)}{\partial x_i} = R_{ib} (\delta_{ab} + \Upsilon_{ab}), \tag{A.6}
\]

\[
\frac{\partial^2 y_a(x, t)}{\partial x_i \partial x_j} = R_{ib} R_{jc} \Gamma_{bc}^{\alpha} \tag{A.7}
\]

where \( \Upsilon_{ab} \) and \( \Gamma_{bc}^{\alpha} \) are given by

\[
\Upsilon_{ab} = A \left[ (R^{-1} \dot{R} y)^2 \delta_{ab} - 2y_a ((R^{-1} \dot{R})^2 y)_b \right] + \frac{1}{r} \frac{dA}{dr} (R^{-1} \dot{R} y)^2 y_a y_b \\
+ \operatorname{Tr} (R^{-1} \dot{R})^2 \left[ r^2 B \delta_{ab} + \left( \frac{dB}{dr} + 2B \right) y_a y_b \right].
\]
and
\[
\Gamma_{bc}^{a} = \Gamma_{cb}^{a} = \frac{\partial Y_{ab}}{\partial y_{c}} = -2A \left[ \delta_{ab}((R^{-1}\dot{R})^{2}y)^{c} + \delta_{ac}((R^{-1}\dot{R})^{2}y^{b}) + y_{a}((R^{-1}\dot{R})^{2}y)_{bc} \right] + \frac{1}{r} \frac{dA}{dr} \left[ (R^{-1}\dot{R}y)^{2}(\delta_{ab}y_{c} + \delta_{ac}y_{b} + y_{a}(\delta_{bc} - \hat{y}_{b}\hat{y}_{c})) \right] - 2y_{a}((R^{-1}\dot{R}^{2}y)^{b}y_{c} + ((R^{-1}\dot{R}^{2}y)^{c}y_{b})) + \frac{d^{2}A}{dr^{2}}(R^{-1}\dot{R}y)^{2}y_{a}\hat{y}_{b}\hat{y}_{c} + \left( 2B + r \frac{dB}{dr} \right) \text{Tr}(R^{-1}\dot{R})^{2}(\delta_{ab}y_{c} + \delta_{ac}y_{b} + y_{a}\delta_{bc}) + \left( 3 \frac{dC}{dr} + 2 \frac{dB}{dr} \right) \text{Tr}(R^{-1}\dot{R})^{2}y_{a}y_{b}y_{c} + \left( 2 + r \frac{dC}{dr} \right) \left[ (R^{-1}\dot{R})^{2}_{ab}y_{c} + ((R^{-1}\dot{R})^{2})_{ac}y_{b} + ((R^{-1}\dot{R})^{2})_{a}y_{bc} \right] + \left( 3 \frac{dC}{dr} + 2 \frac{dB}{dr} \right) \left( (R^{-1}\dot{R})^{2}y^{a}y_{b}y_{c} + O(\partial_{t}^{4}) \right). \tag{A.9}
\]

In particular, we have
\[
\Upsilon_{ab} + \Upsilon_{ba} = 2 \left[ A(R^{-1}\dot{R}y)^{2} + r^{2}B \text{Tr}(R^{-1}\dot{R})^{2} \right] \delta_{ab} + 2 \left[ \frac{1}{r} \frac{dA}{dr}(R^{-1}\dot{R}y)^{2} + \left( 2 + r \frac{dB}{dr} + 2B \right) \text{Tr}(R^{-1}\dot{R})^{2} \right] y_{a}y_{b} + \left( -2A + 2C + r \frac{dC}{dr} \right) \left[ (R^{-1}\dot{R})^{2}y_{a}y_{b} + ((R^{-1}\dot{R})^{2}y)^{b}y_{a} \right] + 2r^{2}C((R^{-1}\dot{R})^{2})_{ab} \tag{A.10}
\]

and
\[
\Gamma_{bb}^{a} = \left( \frac{d^{2}A}{dr^{2}} + \frac{8}{r} \frac{dA}{dr} \right) (R^{-1}\dot{R}y)^{2}y_{a} + \left( r^{2} \frac{d^{2}B}{dr^{2}} + 8r \frac{dB}{dr} - 2A + 10B \right) \text{Tr}(R^{-1}\dot{R})^{2}y_{a} + \left( r^{2} \frac{d^{2}C}{dr^{2}} + 8r \frac{dC}{dr} - 4A + 10C \right) (R^{-1}\dot{R}^{2}y)_{a}. \tag{A.11}
\]

We also need formulas concerning $L_{a}^{cl}(y)$. First, we have
\[
\frac{\partial L_{b}^{cl}(y)}{\partial y_{a}} = \frac{1}{r^{2}} \left\{ \left[ -r^{2}F'' + (2 + \cos 2F) rF' - \frac{3}{2} \sin 2F \right] \hat{y}_{a}\hat{y}_{b}\hat{y}_{c} + \left( \frac{1}{2} \sin 2F - \cos 2F \right) \hat{y}_{a}\delta_{bc} + \left( \frac{1}{2} \sin 2F - rF' \right) (\hat{y}_{b}\delta_{ac} + \hat{y}_{c}\delta_{ab}) + (1 - \cos 2F - \sin 2F rF') \hat{y}_{a}\epsilon_{bdc}\hat{y}_{d} + \frac{1}{2} (1 - \cos 2F) \epsilon_{abc} \right\} \tau_{c}. \tag{A.12}
\]
Next, the double commutator of $L_a^{cl}(y)$ is given by
\[
T_{abc}(y) = -T_{acb}(y) \equiv [L_a^{cl}(y), [L_b^{cl}(y), L_c^{cl}(y)]]
\]
\[
= -\frac{1}{r} \left[ 2(F')^2 - \frac{1 - \cos 2F}{r^2} \right]
\times \hat{y}_a \left\{ \sin 2F \left( \hat{y}_b \delta_{cd} - \hat{y}_c \delta_{bd} \right) + (1 - \cos 2F) \left( \hat{y}_b \epsilon_{ced} - \hat{y}_c \epsilon_{bed} \right) \hat{y}_e \right\} \tau_d
\]
\[
-\frac{1 - \cos 2F}{r^3} \left\{ \sin 2F \left( \delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} \right) + (2rF' - \sin 2F) \left( \delta_{ab} \hat{y}_c - \delta_{ac} \hat{y}_b \right) \hat{y}_d
\right\} \tau_d.
\] (A.13)

Upon contraction, we have
\[
T_{bab}(y) = \left[ \frac{2}{r} \left( F' \right)^2 + \frac{1 - \cos 2F}{r^3} \right] \left( \sin 2F \delta_{ab} + (1 - \cos 2F) \epsilon_{abc} \hat{y}_c \right) \tau_b
\]
\[
- \left[ 2 \frac{2F}{r} \left( F' \right)^2 - \frac{4 - \cos 2F}{r^2} F' + \frac{2F(1 - \cos 2F)}{r^3} \right] \hat{y}_a \hat{y}_b \tau_b.
\] (A.14)

Finally, the orthogonality,
\[
(R^{-1} \hat{R} y)_a y_a = 0,
\] (A.15)
which is due to the anti-symmetric nature of the matrix $R^{-1} \hat{R}$, often simplifies the calculations in many places.

**Evaluation of each term in (2.3)**

Let us obtain the expression of each term in (2.3) in terms of $L_a^{cl}(y)$ and its derivative. We keep only terms containing at most two time-derivatives. First, for $\partial_{\mu} L^\mu = -\partial_0 L_0 + \partial_i L_i$, we have
\[
\partial_0 L_0(x, t) = \frac{\partial^2 y_a}{\partial t^2} L_a^{cl}(y) + \frac{\partial y_a}{\partial t} \frac{\partial y_b}{\partial t} (\partial_a L_b^{cl})(y)
\]
\[
= \left[ -(W y)_a + ((R^{-1} \hat{R})^2 y)_a \right] L_a^{cl}(y) + (R^{-1} \hat{R} y)_a(R^{-1} \hat{R} y)_b(\partial_a L_b^{cl})(y),
\] (A.16)
\[
\partial_i L_i(x, t) = \frac{\partial^2 y_a}{\partial x_i^2} L_a^{cl}(y) + \frac{\partial y_a}{\partial x_i} \frac{\partial y_b}{\partial x_i} (\partial_a L_b^{cl})(y)
\]
\[
= (\partial_a L_a^{cl})(y) + \gamma_{ab} L_b^{cl}(y) + (\gamma_{ab} + \theta_{ba}) (\partial_a L_b^{cl})(y),
\] (A.17)
with $(\partial_a L_a^{cl})(y) = (\partial / \partial y_a) L_a^{cl}(y)$.

Next, let us consider the part of EOM coming from the Skyrme term:
\[
\partial_{\mu} [L_{\nu}, [L^\mu, L^\nu]] = -\partial_0 [L_i, [L_0, L_i]] - \partial_i [L_0, [L_i, L_0]] + \partial_i [L_j, [L_i, L_j]].
\] (A.18)

For the first two terms on the RHS, we can substitute the lowest order expressions $L_0 = -(R^{-1} \hat{R} y)_a L_a^{cl}(y)$ and $L_i = R_{ia} L_a^{cl}(y)$ to get
\[
\partial_0 [L_i, [L_0, L_i]] = \left[ -(W y)_a + ((R^{-1} \hat{R})^2 y)_a \right] T_{bab}(y)
\]
This is nothing but the EOM of $U$ greatly simplifies our lengthy task.

The first two terms containing no time-derivatives on the RHS of (A.22) vanish:

$$\frac{\partial L_a^{cl}(y)}{\partial y_a} - im\pi \left( U_{cl}(y) - \frac{1}{2} \mathrm{tr} U_{cl}(y) \right) = 0.$$  \hfill (A.24)

This is nothing but the EOM of $U_{cl}$. The next term, $(\mathcal{W}y)_a L_a^{cl}$, gives the first term on the RHS of (3.17), and the remaining terms give the EOM-breaking part containing $\mathrm{EQ}_n$ ($n = 1, 2, 3, 4$). In the latter calculation, the orthogonality (A.15) greatly simplifies our lengthy task.

Finally, by dropping the terms containing $(A, B, C)$ in (A.22), namely, by removing $\Upsilon_{ab}$ and $\Gamma_{bc}^a$, we obtain (2.25) before introducing the improvement.

**Appendix B**

**Expressions of $\mathrm{EQ}_{n=1,2,3,4}$**

In this appendix, we present concrete (and lengthy) expressions of the four quantities $\mathrm{EQ}_n$ ($n = 1, 2, 3, 4$) appearing in (3.17). We also present the approximate
expressions of the differential equations for \((A, C)\) and \(Y\) near the infinity and their solutions. The primes on \(F\) denote differentiations with respect to \(r\). First, \(\text{EQ}_n\) are given as follows:

\[
\text{EQ}_1 = \frac{2}{r^2} F' A - F' \frac{d^2 B}{dr^2} - 2 \left( F'' + \frac{4}{r} F' \right) \frac{dB}{dr} - \frac{2}{r} \left( 3 F'' + \frac{7}{r} F' - \frac{1}{r^2} \sin 2F \right) B \\
- \frac{1}{r^2} \left( 2 F' - \frac{1}{r} \sin 2F \right) C + \frac{1}{f_2^2} \left( 4 r^4 (1 - \cos 2F) F' A - \frac{4}{r^2} (1 - \cos 2F) F' \frac{d^2 B}{dr^2} \right) \\
- \frac{8}{r^2} \left( 1 - \cos 2F \right) \left( F'' + \frac{3}{r} F' \right) + \sin 2F (F')^2 \frac{dB}{dr} \\
- \frac{8}{r^3} \left( 1 - \cos 2F \right) \left( 4 F'' + \frac{3}{r} F' - \frac{1}{r^2} \sin 2F \right) + 4 \sin 2F (F')^2 \right) B \\
- \frac{2}{r^3} (1 - \cos 2F) F' \frac{dC}{dr} \\
- \frac{4}{r^3} \left( 1 - \cos 2F \right) \left( F'' + \frac{2}{r} F' - \frac{1}{r^2} \sin 2F \right) + \sin 2F (F')^2 \right) C \right), \quad \text{(B.1)}
\]

\[
\text{EQ}_2 = -F' \frac{d^2 A}{dr^2} - 2 \left( F'' + \frac{4}{r} F' \right) \frac{dA}{dr} + \frac{2}{r} \left[ -3 F'' + \frac{1}{r} (\cos 2F - 3) F' + \frac{1}{r^2} \sin 2F \right] A \\
+ \left( F' - \frac{1}{2r} \sin 2F \right) \frac{d^2 C}{dr^2} + \left[ 2 F'' + \frac{1}{r} (7 - \cos 2F) F' - \frac{3}{r^2} \sin 2F \right] \frac{dC}{dr} \\
+ \frac{2}{r} \left[ 3 F'' + \frac{2}{r} (1 - \cos 2F) F' \right] C \\
+ \frac{1}{f_2^2} \left( 2 \left( 1 - \cos 2F \right) \left( F'' + \frac{2}{r} F' - \frac{2}{r^2} \sin 2F \right) + 2 \sin 2F (F')^2 \right) \\
- \frac{4}{r^2} (1 - \cos 2F) F' \frac{d^2 A}{dr^2} - \frac{4}{r^2} \left( 2 (1 - \cos 2F) \left( F'' + \frac{3}{r} F' \right) + \sin 2F (F')^2 \right) \frac{dA}{dr} \\
- \frac{4}{r^3} \left( 1 - \cos 2F \right) \left( 8 F'' + \frac{1}{r} (1 - 2 \cos 2F) F' - \frac{2}{r^2} \sin 2F \right) + 5 \sin 2F (F')^2 \right) A \\
+ \frac{1}{r^2} (1 - \cos 2F) \left( 4 F' - \frac{1}{r} \sin 2F \right) \frac{d^2 C}{dr^2} \\
+ \frac{2}{r^2} \left( 1 - \cos 2F \right) \left( 4 F'' + \frac{1}{r} (7 - 2 \cos 2F) F' - \frac{2}{r^2} \sin 2F \right) + 2 \sin 2F (F')^2 \right) \frac{dC}{dr} \\
+ \frac{4}{r^3} \left( 1 - \cos 2F \right) \left( 5 F'' + \frac{4}{r} (1 + \cos 2F) F' + \frac{2}{r^2} \sin 2F \right) + 4 \sin 2F (F')^2 \right) C \right) \right), \quad \text{(B.2)}
\]

\[
\text{EQ}_3 = 2 \cos F \times \text{EQ}_{34}, \quad \text{(B.3)}
\]

\[
\text{EQ}_4 = -2 \sin F \times \text{EQ}_{34}, \quad \text{(B.4)}
\]
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where EQ\textsubscript{34} in (B-3) and (B-4) is given by

\[
\text{EQ}_{34} = \frac{1}{2r^3} \sin F + \frac{2}{r^2} \cos F F' A - \frac{1}{2r} \sin F \frac{d^2C}{dr^2} - \frac{1}{r} \left( \cos F F' + \frac{3}{r} \sin F \right) \frac{dC}{dr} \\
- \frac{1}{r^2} \left( 4 \cos F F' + \frac{1}{r} \sin F \right) C + \frac{1}{f_{2\pi}^2} \sin F \left\{ \frac{2}{r^3} \left( (F')^2 - \frac{1}{r^2} (1 - \cos 2F) \right) \\
+ \frac{4}{r^2} (F')^2 \frac{dA}{dr} + \frac{4}{r^3} F' \left( 3F' + \frac{2}{r} \sin 2F \right) A \right. \\
- \frac{1}{r^3} \left( 1 - \cos 2F \right) \frac{d^2C}{dr^2} - \frac{4}{r^2} \left[ (F')^2 + \frac{1}{r} \sin 2F F' + \frac{1}{r^2} (1 - \cos 2F) \right] \frac{dC}{dr} \\
- \frac{4}{r^3} \left[ (F')^2 + \frac{4}{r} \sin 2F F' - \frac{1}{r^2} (1 - \cos 2F) \right] C \right\}. \quad (B.5)
\]

Next, the combination EQ\textsubscript{Y} (3-20) is given by

\[
\text{EQ}_Y = -3 \text{EQ}_1 + \text{EQ}_2 - \text{EQ}_3 \\
= \left( 1 + \frac{8}{f_{2\pi}^2} \frac{\sin^2 F}{r^2} \right) F' \frac{d^2Y}{dr^2} \\
+ \left\{ 2F'' + \frac{8}{r} F' + \left[ \frac{2}{f_{2\pi}^2} \left( \frac{2 \sin^2 F}{r^2} \left( F'' + \frac{3}{r} F' \right) + \frac{\sin 2F}{r^2} (F')^2 \right) \right] \frac{dY}{dr} \\
+ \left\{ \frac{6}{r} F'' + \frac{14}{r^2} F' - \frac{2 \sin 2F}{r^3} \\
+ \frac{16}{f_{2\pi}^2} \left[ \left( \frac{2 \sin^2 F}{r^3} \left( 4F'' + \frac{3}{r} F' - \frac{\sin 2F}{r^2} \right) + \frac{2 \sin 2F}{r^3} (F')^2 \right) \right] Y \\
- \frac{1}{r^3} \sin 2F + \frac{2}{f_{2\pi}^2} \left[ \frac{2 \sin^2 F}{r^3} \left( F'' + \frac{2}{r} F' - \frac{\sin 2F}{r^2} \right) + \frac{\sin 2F}{r^3} (F')^2 \right] \right\}. \quad (B.6)
\]

By using (2.13) for \( F(r) \), the differential equations EQ\textsubscript{2} = EQ\textsubscript{34} = 0 for \( A, C \) and EQ\textsubscript{Y} = 0 for \( Y \) are approximated near the infinity \( r \to \infty \) as follows:

\[
\left( 1 + \frac{2}{s} + \frac{2}{s^2} \right) \frac{d^2A}{ds^2} + 2 \left( -1 + \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) \frac{dA}{ds} - 2 \left( \frac{3}{s} + \frac{7}{s^2} + \frac{12}{s^3} + \frac{12}{s^4} \right) A \\
- \left( 1 + \frac{3}{s} + \frac{3}{s^2} \right) \frac{d^2C}{ds^2} + 2 \left( 1 - \frac{3}{s^2} - \frac{3}{s^3} \right) \frac{dC}{ds} + 6 \left( \frac{1}{s} + \frac{3}{s^2} + \frac{6}{s^3} + \frac{6}{s^4} \right) C = 0,
\]

\[
-4 \left( \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) A - \left( 1 + \frac{1}{s} \right) \frac{d^2C}{ds^2} + 2 \left( 1 - \frac{1}{s} - \frac{1}{s^2} \right) \frac{dC}{ds} + 2 \left( \frac{4}{s} + \frac{7}{s^2} + \frac{7}{s^3} \right) C \\
+ \frac{1}{s^2} + \frac{1}{s^3} = 0,
\]

\[
\left( 1 + \frac{2}{s} + \frac{2}{s^2} \right) \frac{d^2Y}{ds^2} + 2 \left( -1 + \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) \frac{dY}{ds} - 2 \left( \frac{3}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{2}{s^4} \right) Y \\
+ \frac{1}{s^3} + \frac{1}{s^4} = 0, \quad (B.7)
\]
where $s$ is the dimensionless variable $s = \frac{m_{\pi}}{r}$. The general solution to (B.7) is given symbolically by (3.23) as the sum of a particular solution and six independent modes. More concrete expression keeping terms up to $1/s^5$ for the power parts is follows:

$$
\begin{align*}
(A) &= -\frac{1}{2s} \left( 1 + \frac{3}{s^3} \right) + d_2 \frac{1}{s^2} \left( 1 - \frac{3}{s} \right) + d_3 \frac{3}{s^3} \\
\end{align*}
$$

where $\Omega$ follows:

$$
\begin{align*}
J &= 1 + \frac{17}{3s^2} - \frac{1}{s} \\
Y &= \frac{1}{2s^2} + d_3 \frac{1}{s^3} - \frac{3}{s^4} e^{2s}. \\
(r \to \infty)
\end{align*}
$$

### Appendix C

**Derivation of Eq. (4.2) for $J$**

In this appendix, we outline the derivation of Eq. (4.2) for $J$, the coefficient of the $\Omega^4$ term in the Lagrangian of $R(t)$. Here, as in Appendix A, $r$ denotes the length of $y$: $r = |y|$. As explained in §4, calculation of $J$ is reduced via (4.13) to that of $J_1$, the part of $J$ linear in $(A, B, C)$. It is convenient to use the expression of the Lagrangian $L$ as the integration over $y$ instead of $x$:

$$
L = \int d^3y \left. J L \right|_{U(x,t)=U_{cl}(y)},
$$

where $J$ is the Jacobian and is given via (A-6) in terms of $\Upsilon_{ab}$:

$$
J = \left| \det_{\{a,i\}} \left( \frac{\partial y_a}{\partial x_i} \right) \right|^{-1} = \left| \det_{\{a,b\}} (\delta_{ab} + \Upsilon_{ab}) \right|^{-1} = 1 - \text{Tr} \Upsilon + O(\delta^4).
$$

Explicitly, we have

$$
\text{Tr} \Upsilon = \Upsilon_{aa} = \left( 5A - 2C + r \frac{d}{dr} (A - C) \right) (R^{-1} \dot{R} y)^2 + r^2 \left( 5B + C + r \frac{dB}{dr} \right) \text{Tr}(R^{-1} \dot{R})^2.
$$

Let us identify the part of $J\mathcal{L}$ in (C.1) which is quartic in $\Omega$ and linear in $(A, B, C)$ (we call such part “QL-part” hereafter). We employ (3.16) for $L_\mu$ together with (A-3) and (A-6) for $\partial y_a/\partial t$ and $\partial y_a/\partial x_i$, respectively. First, for the $\text{tr} L_\mu^2$ part of $\mathcal{L}$ (2.1), we have

$$
J \text{ tr} L_0^2 |_{QL-part} = - \text{Tr} \Upsilon (R^{-1} \dot{R} y)_a (R^{-1} \dot{R} y)_b \text{ tr} (L_a^{cl}(y)L_b^{cl}(y)),
$$

$$
J \text{ tr} L_1^2 |_{QL-part} = 0.
$$

In obtaining (C.4), we have used the fact that the terms in (A-3) containing $W$ (A-5) do not contribute to the Lagrangian (C.1), which we showed in §4. This can of course
be confirmed by explicit calculation. For \( \text{C-(5)} \), we have used that the \( O(\partial^4_t) \) term of \( \Upsilon_{ab} \), which is not given in \( \text{A-(8)} \), is quadratic \((A, B, C)\) and hence cannot contribute to the QL-part. Similarly, for the Skyrme-term \( \text{tr}[L_\mu, L_\nu]^2 = -2 \text{tr}[L_0, L_i]^2 + \text{tr}[L_i, L_j]^2 \), we have

\[
J \text{tr}[L_0, L_i]^2 \bigg|_{\text{QL-part}} = - \left[ \delta_{cd} \text{Tr} \Psi - (\Upsilon_{cd} + \Upsilon_{dc}) \right] \\
\times (R^{-1}\dot{R}y)_a(R^{-1}\dot{R}y)_b \text{tr}([L_a^{\text{cl}}, L_b^{\text{cl}}][L_c^{\text{cl}}, L_d^{\text{cl}}])(y), \quad \text{(C-6)}
\]

\[
J \text{tr}[L_i, L_j]^2 \bigg|_{\text{QL-part}} = 0. \quad \text{(C-7)}
\]

Then, the QL-part of the Lagrangian \( \text{C-(1)} \) is obtained by using \( \text{C-(4)} \)--\( \text{C-(7)} \), \( \text{A-(8)} \), \( \text{C-(3)} \) and

\[
\text{tr}(L_a^{\text{cl}}L_b^{\text{cl}})(y) = 2(F')^2\tilde{y}_a\tilde{y}_b + \frac{1 - \cos 2F}{r^2} (\delta_{ab} - \tilde{y}_a\tilde{y}_b), \quad \text{(C-8)}
\]

\[
\text{tr}([L_a^{\text{cl}}, L_c^{\text{cl}}][L_b^{\text{cl}}, L_d^{\text{cl}}])(y) = -2 \frac{1 - \cos 2F}{r^2} \delta_{ab} \left( (F')^2\tilde{y}_c\tilde{y}_d + \frac{1 - \cos 2F}{r^2} (\delta_{cd} - \tilde{y}_c\tilde{y}_d) \right) \\
+ 2 \left( \frac{1 - \cos 2F}{r^2} \right)^2 \delta_{ad}\delta_{bc} + \text{(terms containing } \tilde{y}_a \text{ and/or } \tilde{y}_b), \quad \text{(C-9)}
\]

and carrying out the solid-angle integration of \( y \). Denoting by overline the solid-angle average,

\[
\overline{\mathcal{O}} = \frac{1}{4\pi} \int d\Omega_y \mathcal{O}, \quad \text{(C-10)}
\]

the following formulas are of use:

\[
\overline{y_a y_b} = \frac{r^2}{3} \delta_{ab}, \quad \text{(C-11)}
\]

\[
\overline{y_a y_b y_c y_d} = \frac{r^4}{15} (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \quad \text{(C-12)}
\]

In particular, we have

\[
\overline{(R^{-1}\dot{R}y)^2} = -\frac{r^2}{3} \text{Tr}(R^{-1}\dot{R})^2 = \frac{2}{3} r^2 \Omega^2, \quad \text{(C-13)}
\]

\[
\overline{(R^{-1}\dot{R}y)^4} = \frac{r^4}{15} \left[ \left( \text{Tr}(R^{-1}\dot{R})^2 \right)^2 + 2 \text{Tr}(R^{-1}\dot{R})^4 \right] = \frac{8}{15} r^4 \Omega^4, \quad \text{(C-14)}
\]

where we have used

\[
\text{Tr}(R^{-1}\dot{R})^2 = -2 \Omega^2, \quad \text{(C-15)}
\]

\[
(R^{-1}\dot{R})^3 = \frac{1}{2} \text{Tr}(R^{-1}\dot{R})^2 R^{-1}\dot{R}. \quad \text{(C-16)}
\]

By identifying the QL-part of \( \text{C-(1)} \) with \( (1/4)\mathcal{J}_1 \Omega^4 \), we obtain \( \mathcal{J} = (1/2)\mathcal{J}_1 \) as given by \( (4\cdot2) \).
Appendix D

Equivalence of Charges from $L(R, \dot{R})$ and from the Skyrme Field Theory

In this appendix, we present a proof of the equivalence of the conserved charges (especially, the isospin $I$ and the Hamiltonian $H$) obtained in two different ways; one is from the Lagrangian $L(R, \dot{R})$ (4.1) of $R(t)$, and the other is by the substitution of $U(x, t) = U_{cl}(y)$ (3.1) into the Noether charges from the Lagrangian density (2.1) of the Skyrme field theory. We show that the two conserved charges agree with each other up to the EOM of $R(t)$ and that in the Skyrme field theory; the latter also vanishes upon using the EOM of $R(t)$ due to our principle.

Isospin charge $I_a$

First, let us consider the isospin charge $I_a$. For this, recall that the Lagrangian of $R$ is related to the Lagrangian density $\mathcal{L}(U(x, t))$ of the Skyrme model by

$$L(R, \dot{R}) = \int d^3 x \, \mathcal{L}_{|U(x, t) = U_{cl}(y)}.$$  \hfill (D.1)

The isospin charge $I_a$ on the $R$-system side is obtained by considering the following time-dependent transformation $\delta_\lambda$ acting on $L$:

$$\delta_\lambda R(t) = R(t)\lambda(t), \quad \delta_\lambda R^{-1}(t) = -\lambda(t)R^{-1}(t),$$  \hfill (D.2)

where $\lambda_{ab}(t)$ is an infinitesimal anti-symmetric matrix. We have

$$\delta_\lambda L(R, \dot{R}) = \lambda_a(t)I_a,$$  \hfill (D.3)

with $\lambda_a(t) = (1/2)\epsilon_{abc}\lambda_{bc}(t)$.

For relating the isospin charge of (D - 3) to the one on the field theory side, let us consider the response of $\mathcal{L}(U(x, t) = U_{cl}(y))$ under the transformation $\delta_\lambda$ of (D - 2). Since we have $\delta_\lambda(R^{-1}x) = -\lambda R^{-1}x$ and $\delta_\lambda(R^{-1}\dot{R}) = \dot{\lambda} + [R^{-1}\dot{R}, \lambda]$, the transformation of $y$ (3 - 2) consists of two parts:

$$\delta_\lambda y = -\lambda y + \dot{\lambda}_a Z_{ab}(x, t),$$  \hfill (D.4)

where $Z_{ab}$ is a three dimensional vector for each $(a, b)$:

$$(Z_{ab})_c = \left[ A(R^{-1}\dot{R}\tilde{y})_{ab} - Br^2(R^{-1}\dot{R})_{ab} \right] \tilde{y}_c + \frac{1}{2} C r^2 \left[ \delta_{ac}(R^{-1}\dot{R}\tilde{y})_{b} + (R^{-1}\dot{R})_{ca}\tilde{y}_b \right] - (a \leftrightarrow b),$$  \hfill (D.5)

with $\tilde{y} = R^{-1}x$. The first part $\delta_\lambda^{(0)} y = -\lambda y$ induces the standard isospin transformation on $U(x, t) = U_{cl}(y)$:

$$\delta_\lambda^{(0)} U_{cl}(y) = i \left[ U_{cl}(y), \lambda_a(t) \frac{T_a}{2} \right],$$  \hfill (D.6)

and therefore we have

$$\delta_\lambda^{(0)} \mathcal{L}(U(x, t) = U_{cl}(y)) = \lambda_a J_{\nu a}^{\mu=0} \bigg|_{U(x, t) = U_{cl}(y)}.$$  \hfill (D.7)
where $J_{V,a}^\mu$ is the isovector current (2.6) of the Skyrme field theory. (If $\lambda_a$ in (D-6) depends on both $t$ and $x$, we have $\delta^{(0)}\mathcal{L} = \partial_\mu \lambda_a(x,t) J_{V,a}^\mu$ instead of (D-7).)

The second part of the transformation (D-4), $\delta^{(1)}\mathbf{y} = \lambda_{ab} \mathbf{Z}_{ab}(x,t)$, needs careful treatment. Let us divide the Lagrangian density (2.1) into two parts, $\mathcal{L} = \mathcal{L}_0(L_\mu) + (f_\pi^2/8) m_\pi^2 \text{tr} (U - \mathbf{1}_2)$, and consider first the transformation of $\mathcal{L}_0(L_\mu)$ consisting only of $L_\mu = -iU_{cl}(y) (\partial/\partial x^\mu) U_{cl}^\dagger(y)$. Using

$$
\delta^{(1)} L_\mu = \frac{\partial \eta}{\partial x^\mu} + i [L_\mu(x,t), \eta],
$$

with $\eta$ defined by

$$
\eta = \lambda_{ab} (\mathbf{Z}_{ab}) c L_{cl}^{(1)}(y).
$$

This gives

$$
\delta^{(1)} \mathcal{L}_0(L_\mu) = \text{tr} \left( \frac{\partial \mathcal{L}_0}{\partial L_\mu} \delta^{(1)} L_\mu \right) = \text{tr} \left( \frac{\partial \mathcal{L}_0}{\partial L_\mu} \frac{\partial \eta}{\partial x^\mu} \right)
$$

with

$$
\frac{\partial \mathcal{L}_0}{\partial L_\mu} = -\frac{f_\pi^2}{8} \left( L_\mu - \frac{1}{f_\pi^2 e^2} [L_\nu, [L_\mu, L_\nu]] \right).
$$

In $\delta^{(1)} \int d^3x \mathcal{L}_0$, we carry out the integration-by-parts for the $\mu = 1, 2, 3$ parts of (D-11). For the $\mu = 0$ part, notice that $\partial/\partial t$ acting on $(\mathbf{Z}_{ab}) c L_{cl}^{(1)}(y)$ in (D-10), which consists of $\mathbf{y}$ and $R^{-1} \tilde{R}$, can be expressed as

$$
\frac{\partial}{\partial t} = -(R^{-1} \tilde{R})_i \frac{\partial}{\partial x_i} + (R\text{-EOM terms}),
$$

where (R-EOM terms) denote terms which vanish upon use of the EOM (2.23) of $R$ (cf., Eq. (3.6)). Since (D-13) holds also against $\partial \mathcal{L}_0/\partial \mathcal{L}_0$ and since we have $(\tilde{R}R^{-1})_{ii} = 0$, space-integration-by-parts for all $\mu$ in (D-11) is effectively allowed to give

$$
\delta^{(1)} \int d^3x \mathcal{L}_0 = (\bar{\lambda}\text{-term}) - \int d^3x \text{ tr} \left( \eta \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}_0}{\partial L_\mu} \right) + (R\text{-EOM terms}).
$$

The $\bar{\lambda}$-term is another contribution from the $\mu = 0$ term of (D-11); $\partial/\partial t$ acting on $\lambda_{ab}$ in $\eta$ (D-10). However, this $\bar{\lambda}$-term in fact vanishes. This is because, after carrying out the solid-angle integration of $\mathbf{y}$, the possible quantities multiplying anti-symmetric matrix $\lambda_{ab}$ are all symmetric ones, $\delta_{ab} \text{Tr}(R^{-1} \tilde{R})^2$ and $((R^{-1} \tilde{R})^2)_{ab}$, in our present approximation of keeping at most four time-derivatives in the Lagrangian (see the argument given in §4 for excluding $d/dt (R^{-1} \tilde{R})$ from the Lagrangian of $R$).

For the $\delta^{(1)}$ transformation of the remaining pion mass term in $\mathcal{L}$ (2.1), we have, using (D-8),

$$
\delta^{(1)} \text{ tr} (U - \mathbf{1}_2) = \lambda_{ab} (\mathbf{Z}_{ab}) c \text{ tr} \frac{\partial U_{cl}(y)}{\partial y_c} = -i \text{ tr} (\eta U_{cl}(y)).
$$

(D-15)
From this and (D.14), we have the following for the whole $\mathcal{L}$:

$$
\delta^{(1)}_{\lambda} \int d^3x \mathcal{L} = -\int d^3x \, \text{tr} \left\{ \eta \left[ \frac{\partial}{\partial x^\mu} \frac{\partial L_0}{\partial L_\mu} + i \frac{f_\pi^2}{8} m_\pi^2 \left( U - \frac{1}{2} \text{tr} U \right) \right] \right\}_{U=U_{cl}(y)} + (R\text{-EOM terms}).
$$

(D-16)

Note that the quantity inside the square bracket multiplying $\eta$ in the first term on the RHS is nothing but the EOM (2.3) of the Skyrme field theory. Due to our EOM principle, this term also vanishes upon use of the EOM of $R$.

We have calculated the $\delta_{\lambda}$ transformation of the both hand sides of (D.1); the transformation of the LHS is (D.3), while that of the RHS is the sum of (the space-integration of) (D.7) and (D.16). An important point is that the whole of (D.16) vanishes when the EOM (2.23) of $R$ holds. Therefore, comparing the coefficients of $\dot{\lambda}_a(t)$, we find that the isospin charge $I_a$ obtained from the Lagrangian (4.1) is equal to that in the Skyrme field theory up to the EOM of $R$:

$$
I_a = \int d^3x \, J^\mu_{\nu,a}(x,t) \Big|_{U(x,t)=U_{cl}(y)} + (R\text{-EOM terms}).
$$

(D.17)

For the spin $J_i$, we consider the infinitesimal left transformation on $R(t)$ instead of the right one (D.2). Then the equivalence of $J_i$ between the $R$ system and the Skyrme field theory can be shown quite similarly.

**Hamiltonian $H$**

Next, we show that the Hamiltonian (5.4) of the system of $R$ is equal to the Hamiltonian of the Skyrme field theory with the substitution (3.1) up to the EOM of $R$. The proof is quite similar to the above one for the isospin, and we will describe only the points of the proof.

By considering the time translation with infinitesimal time-dependent parameter $\varepsilon(t)$,

$$
\delta_{\varepsilon} R(t) = \varepsilon(t) \dot{R}(t),
$$

(D.18)

the Hamiltonian (5.4) of the $R$ system is obtained as

$$
\delta_{\varepsilon} L = \dot{\varepsilon} H + \partial_t (\varepsilon L).
$$

(D.19)

To relate this $H$ to the Hamiltonian of the Skyrme field theory, we consider the action of the same transformation (D.18) on the Lagrangian density on the RHS of (D.1). We have

$$
\delta_{\varepsilon} y(x,t) = \varepsilon(t) \partial_t y(x,t) + \dot{\varepsilon}(t) z(x,t),
$$

(D.20)

with

$$
z = 2 \left[ A(R^{-1} \dot{R} \tilde{y})^2 + B r^2 \text{Tr}(R^{-1} \dot{R})^2 + C r^2 (R^{-1} \dot{R})^2 \right] \tilde{y}.
$$

(D.21)

The first part of the transformation, $\delta^{(0)}_{\varepsilon} y = \varepsilon \partial_t y$, gives simply

$$
\delta^{(0)}_{\varepsilon} \int d^3x \mathcal{L}(U(x,t) = U_{cl}(y)) = \dot{\varepsilon} H_{\text{Skyrme}} \bigg|_{U(x,t)=U_{cl}(y)} + \partial_t (\varepsilon L),
$$

(D.22)
with $H_{\text{Skyrme}}$ being the Hamiltonian of the Skyrme field theory. For the second part of the transformation, $\delta^{(1)}_{\varepsilon}(y) = \dot{\varepsilon} z$, the same equation as (D-9) holds by replacing $\eta$ (D-10) with $\dot{\varepsilon} z L_a^\text{cl}(y)$. Then, the same argument by using (D-13) applies also to $\delta^{(1)}_{\varepsilon}$, leading again to (D-16). In this case, the absence of the $\ddot{\varepsilon}$ term is owing to the fact that the possible quantities multiplying $\dot{\varepsilon}$ are $\text{Tr}(R^{-1} \dot{R})^3$ and $\text{Tr}(R^{-1} \dot{R})^2 \text{Tr}(R^{-1} \dot{R})$, which however all vanish identically. In this way, we obtain the equivalence of $H$ and $H_{\text{Skyrme}} | U(x,t) = U_{\text{cl}}(y)$.

The equivalence of the Hamiltonians obtained by two ways, which holds owing to the EOM principle, provides us with another and simple derivation of the relation (4.12). First, the Hamiltonian $H$ corresponding to the Lagrangian (4.9) is given by

$$H = M_{\text{cl}} + \frac{1}{2} (I + \Delta I) \Omega^2 + \frac{3}{4} (J_1 + J_2) \Omega^4. \quad \text{(D.23)}$$

On the other hand, the Hamiltonian of the Skyrme field theory is obtained from the Lagrangian by inverting the sign of its “potential term” $V$ (4.8):

$$H_{\text{Skyrme}} \bigg|_{U(x,t) = U_{\text{cl}}(y)} = T + V = M_{\text{cl}} + \frac{1}{2} (I - \Delta I) \Omega^2 + \frac{1}{4} (J_1 - J_2) \Omega^4. \quad \text{(D.24)}$$

Comparing the above two, we reobtain (4.12).

### Appendix E

**Analytic Expressions of the Static Properties of Nucleons**

In this appendix, we outline the derivation of the analytic expressions of the various static properties of the nucleons. Throughout this appendix, $r$ denotes the length of $y$, $r = |y|$, except $\langle r^2 \rangle (= \langle x^2 \rangle)$ denoting the charge radii of various kinds.

**Isoscalar charge radius**

The isoscalar mean square charge radius $\langle r^2 \rangle_{I=0}$ is given in terms of the baryon number density $J_B^0(x,t)$ and a nucleon state $|N\rangle$ by

$$\langle r^2 \rangle_{I=0} = \langle N | \int d^3x \ x^2 J_B^0(x,t) |N\rangle. \quad \text{(E.1)}$$

From (2.4) and (3.16), we have

$$J_B^0(x,t) = -\frac{i}{24\pi^2} \varepsilon_{ijk} \text{tr} (L_i(x,t)L_j(x,t)L_k(x,t)) = \left| \det \left( \frac{\partial y_a}{\partial x_i} \right) \right| J_B^{\text{cl},0}(|y|), \quad \text{(E.2)}$$

where $J_B^{\text{cl},0}$ is the baryon number density of the static solution (2.9):

$$J_B^{\text{cl},0}(r) = -\frac{1}{4\pi^2} \frac{1 - \cos 2F}{r^2} \frac{dF(r)}{dr}. \quad \text{(E.3)}$$

Owing to the presence of the Jacobian in (E.2), it is convenient to switch to the $y$-integration to evaluate (E.1) as follows:

$$\langle r^2 \rangle_{I=0} = \langle N | \int d^3y \ y^2 J_B^{\text{cl},0}(|y|) |N\rangle.$$
\[ 4\pi \int_0^\infty dr \, r^4 \langle N | \left[ 1 - 2(A - C) (R^{-1} \hat{y})^2 - 2Br^2 \text{Tr}(R^{-1} \hat{R})^2 \right] | N \rangle J_B^{cl}(r) \]
\[ = \frac{4\pi}{f_\pi^2 e^2} \int_0^\infty d\rho \, \rho^4 \left( 1 + \frac{4}{3} e^4 \hat{\Omega}_N^2 \rho^2 Y \right) J_B^{cl}(\rho), \tag{E.4} \]
where we have used the relation
\[ x^2 = \left[ 1 - 2(A - C) (R^{-1} \hat{y})^2 - 2Br^2 \text{Tr}(R^{-1} \hat{R})^2 \right] r^2, \quad (r = |y|) \tag{E.5} \]
valid to \( O(\partial_i^2) \), and (C.13) for the angle average over \( y \). The final expression is given in terms of the dimensionless \( \rho = e_{\pi} r, \hat{\Omega}_N^2 \) (5.17) and the function \( Y(r) \) (3.11).

**Isovector charge radius**

To obtain the expression of the isovector charge radius, we first need that of the isospin charge density \( J_{\mu a}^0 \) of (2.6) with \( J_{L/R}^\mu \) defined by (2.7). Corresponding to (3.16) for \( L_\mu, R_\mu = -iU^\dagger \partial_\mu U \) for our spinning Skyrmion field (3.1) is given by
\[ R_0(x, t) = \frac{\partial y_a(x, t)}{\partial t} R_a^{cl}(y), \quad R_i(x, t) = \frac{\partial y_a(x, t)}{\partial x_i} R_a^{cl}(y), \tag{E.6} \]
with
\[ R_a^{cl}(y) = -iU_{cl}(y)^\dagger \frac{\partial}{\partial y_a} U_{cl}(y) \]
\[ = \frac{1}{2r} \left\{ \sin 2F \delta_{ab} + (2rF' - \sin 2F) \hat{y}_a \hat{y}_b + (1 - \cos 2F) \epsilon_{abc} \hat{y}_c \right\} \tau_b \]
\[ = -L_a^{cl}(-y). \tag{E.7} \]
Note that \( U_{cl}(y)^\dagger = U_{cl}(-y) \). Since we are considering the on-shell \( R(t) \) with \( \mathcal{W} = 0 \), (A.3) is reduced to \( \partial y_a(x, t)/\partial t = -(R^{-1} \hat{y})_a \) in the present calculation. This leads to
\[ L_0 + R_0 = \frac{1}{r} (1 - \cos 2F) \epsilon_{abc} (R^{-1} \hat{y})_a \hat{y}_b \hat{y}_c. \tag{E.8} \]
As for the part in \( J_0^\mu \) from the Skyrme term, we have
\[ [L_i, [L_0, L_i]] + (L_\mu \rightarrow R_\mu) = - (\delta_{ac} + \chi_{ac} + \chi_{ca}) (R^{-1} \hat{y})_b T_{abc} + (y \rightarrow -y), \tag{E.9} \]
with \( \chi_{ac} + \chi_{ca} \) and \( T_{abc} \) given by (A.10) and (A.13), respectively. From (E.8) and (E.9), we obtain
\[ 16e^2 J_0^0(x, t) = f_\pi^2 e^2 (L_0 + R_0) - \left( [L_i, [L_0, L_i]] + [R_i, [R_0, R_i]] \right) \]
\[ = 2 \left( 1 - \cos 2F \right) \left\{ \frac{1}{2} f_\pi^2 e^2 + 2(F')^2 + \frac{1 - \cos 2F}{r^2} \right\} \]
\[ + \left[ 4 \left( 3(A - C) + r \frac{d}{dr}(A - C) \right) (F')^2 + 2(A + C) \frac{1 - \cos 2F}{r^2} \right] (R^{-1} \hat{y})^2 \]
\[ + \left[ 4 \left( 3B + r \frac{dB}{dr} \right) r^2 (F')^2 + (2B + C)(1 - \cos 2F) \right] \text{Tr}(R^{-1} \hat{R})^2 \]
\[ \times \epsilon_{abc}(R^{-1} \hat{R} y)_a \hat{y}_b \tau_c. \quad (E.10) \]

Using this, we get the following expression for the isovector mean square charge radius of the nucleon:

\[
\langle r^2 \rangle_{I=1} = \frac{\langle N \rangle \int d^3x \, x^2 J^0_{V,a=3}(x,t) \, |N \rangle}{\langle N | I_3 | N \rangle} = \frac{4\pi}{3\sqrt{3}} \frac{\tilde{\Omega}_N}{f^2_{2e2}} \int_2^\infty d\rho \, \rho^4 \sin^2 F \left\{ 1 + 4(F')^2 + 4 \frac{\sin^2 F}{\rho^2} + \frac{2}{5} e^4 \tilde{\Omega}_N^2 \rho^2 \left[ \rho Z' + 7Z - C \right] + 4(F')^2 \left( -\rho Z' + Z - C \right) + 4 \frac{\sin^2 F}{\rho^2} \left( \rho Z' + 5Z + 2C \right) \right\}, \quad (E.11)
\]

with prime denoting \( d/d\rho \) and \( Z \) defined by (4.3). In deriving (E.11), it is convenient to switch to the \( y \)-integration by using \( d^3x = (1 - \text{Tr} \, \mathcal{Y}) \, d^3y \) with \( \text{Tr} \, \mathcal{Y} \) given by (C.3), and use (E.5) for \( x^2 \) and the following formulas for the angle averaging:

\[ \epsilon_{abc}(R^{-1} \hat{R} y)_a \hat{y}_c = \frac{1}{3} r \epsilon_{abc}(R^{-1} \hat{R})_c = \frac{2}{3} r \Omega_a, \]

\[ \epsilon_{abc}(R^{-1} \hat{R} y)^2(R^{-1} \hat{R} y)_a \hat{y}_c = -\frac{2}{15} r^3 \text{Tr}(R^{-1} \hat{R})^2 \epsilon_{abc}(R^{-1} \hat{R})_c = \frac{8}{15} r^3 \Omega^2 \Omega_a. \quad (E.12) \]

In rewriting \( \Omega_3 \) in the numerator of (E.11) in terms of \( I_3 \), we have used (5.13) for the nucleon, namely,

\[ (I + \mathcal{J} \Omega_N^2)^{-1} = \frac{2}{\sqrt{3}} |\Omega_N|. \quad (E.13) \]

Here, we comment on another derivation of (4.2) for \( \mathcal{J} \). In §5, the isospin charge \( I_a (5.5) \) was obtained from the Lagrangian (4.1) of \( R(t) \). As explained in Appendix D, the same \( I_a \) should also be obtained by integrating (E.10) over \( x \). By the latter way, we directly obtain (4.2) for \( \mathcal{J} \) (and also (2.18) for \( I \)) without using the relation (4.13).

**Isoscalar magnetic moment and magnetic charge radius**

Let us consider the isoscalar magnetic moment defined by

\[ \mu_{I=0} = \frac{1}{2} \int d^3x \, x \times J_B(x,t). \quad (E.14) \]

The space component of the baryon number current (2.4) is given by

\[ J^i_B(x,t) = \frac{i}{8\pi^2} \varepsilon^{ijk} \text{tr}(L_j L_k L_0) = (1 + \text{Tr} \, \mathcal{Y}) R_{ia} (\delta_{ab} - \gamma_{ab}) (R^{-1} \hat{R} y)_b J^0_B (|y|), \quad (E.15) \]

where we have used (3.16), (A.6), (A.3) with \( W = 0 \), and

\[ \text{tr} \left( L^a_c L^b_c L^c_a \right) = 4\pi^2 i \epsilon_{abc} J^0_B. \quad (E.16) \]

Again, (E.14) is most easily evaluated by switching to \( y \)-integration and expressing \( x \) in terms of \( y \) via

\[ \tilde{y} = R^{-1} x = \left[ 1 - A(R^{-1} \hat{R} y)^2 - B r^2 \text{Tr}(R^{-1} \hat{R})^2 \right] y - C r^2 (R^{-1} \hat{R})^2 y. \quad (E.17) \]
Then, the isoscalar $g$-factor $g_{I=0}$ of the nucleon defined by

$$
\mu_{I=0} = \frac{g_{I=0}}{2M_N} J,
$$

(E-18)
is found to be given by

$$
g_{I=0} = \frac{16\pi eM_N}{3\sqrt{3}} f_\pi |\hat{\Omega}_N| \int_0^\infty dp \rho^4 J_B^{cl\,0}(\rho) \left[ 1 + \frac{2}{5} e^4 \hat{\Omega}_N^2 \rho^2 (2Z + C) \right].
$$

(E-19)
The isoscalar mean square magnetic radius $\langle r^2 \rangle_{M,I=0}$, which is defined as (E-14) with extra weight $x^2$ divided by (E-14) itself, is given by

$$
f_\pi^2 e^2 \langle r^2 \rangle_{M,I=0} = \frac{\int_0^\infty dp \rho^5 J_B^{cl\,0}(\rho) \left[ 1 + \frac{2}{5} e^4 \hat{\Omega}_N^2 \rho^2 (4Z + C) \right]}{\int_0^\infty dp \rho^4 J_B^{cl\,0}(\rho) \left[ 1 + \frac{2}{5} e^4 \hat{\Omega}_N^2 \rho^2 (2Z + C) \right]}. 
$$

(E-20)

**Isovector magnetic moment and magnetic charge radius**

**Calculation of the isovector magnetic moment,**

$$
\mu_{I=1} = \int d^3x \mathbf{x} \times \mathbf{J}_{V,a=3}(\mathbf{x},t),
$$

(E-21)
is much more involved. Relaxing the $a = 3$ restriction in (E-21) to a generic $a$, the $i$-th component $(\mu_{I=1})_i$ is found to have three kinds of dependence on $(i,a)$; $R_{ia}$, $(R(R^{-1}\dot{R})^2)_{ia}$ and $\omega_i\Omega_a$. When we consider only the nucleon matrix elements, we have the following representation:

$$
J_i = \frac{1}{2} \sigma_i \otimes 1, \quad I_a = 1 \otimes \frac{1}{2} \tau_a, \quad R_{ia} = -\frac{1}{3} \sigma_i \otimes \tau_a.
$$

(E-22)

Then, owing to (E-13), we have

$$
\omega_i\Omega_a = (\mathcal{I} + \mathcal{J} \Omega_N^2)^{-2} \frac{1}{4} \sigma_i \otimes \tau_a = \frac{1}{3} \Omega_N^2 \sigma_i \otimes \tau_a,
$$

(E-23)
and consequently

$$
(R(R^{-1}\dot{R})^2)_{ia} = -\omega_i\Omega_a - \Omega_N^2 R_{ia} = 0,
$$

(E-24)
both for the nucleon states. Using these facts, the isovector $g$-factor $g_{I=1}$ of the nucleon defined by

$$
(\mu_{I=1})_i = \frac{g_{I=1}}{2M_N} \frac{\sigma_i}{2} \otimes \tau_3,
$$

(E-25)
is calculated to be given by

$$
g_{I=1} = \frac{8\pi M_N}{9 e^4 f_\pi g_{I=1}},
$$

(E-26)
with

$$
g_{I=1} = \frac{\tilde{g}_{I=1}}{2M_N} \left[ 1 + 4(F')^2 + 2\frac{\sin^2 \frac{F}{\rho^2} + \frac{2}{5} e^4 \hat{\Omega}_N^2 \rho^2 \left( \rho Z' + 5Z - C \right) \right].
$$
\[-4(F')^2 (\rho Z' + Z + C) + 4 \frac{\sin^2 F}{\rho^2} (\rho Z' + 3Z + 2C)\].

(E.27)

Similarly, the isovector mean square magnetic radius \(\langle r^2 \rangle_{M,I=1}\) is given by

\[
f^2 e^2 \langle r^2 \rangle_{M,I=1} = \frac{1}{g_{I=1}} \int_0^\infty d\rho \rho^4 \sin^2 F \left\{ 1 + 4(F')^2 + \frac{4 \sin^2 F}{\rho^2} + \frac{2}{5} e^4 \hat{\Omega}_N^2 \rho^2 \right. \\
\times \left[ \rho Z' + 7Z - C + 4(F')^2 (-\rho Z' + Z - C) + \frac{4 \sin^2 F}{\rho^2} (\rho Z' + 5Z + 2C) \right].
\]

(E.28)

Comparing (E.27) for \(\tilde{g}_{I=1}\) with (2.18) for \(I\) and (4.2) for \(J\), we find the following simple relationship:

\[
\tilde{g}_{I=1} = \frac{3}{2\pi} \left( \tilde{I} + e^4 \hat{\Omega}_N^2 \tilde{J} \right) = \frac{3\sqrt{3}}{4\pi} |\hat{\Omega}_N|^{-1},
\]

(E.29)

where the last equality is due to (E.13). This implies that \(g_{I=1}\) has a much simpler expression:

\[
g_{I=1} = \frac{2}{\sqrt{3}} \frac{M_N}{e^3 f_{\pi}} |\hat{\Omega}_N|^{-1},
\]

(E.30)

and that the magnetic radius \(\langle r^2 \rangle_{M,I=1}\) is in fact equal to the electric one \(\langle r^2 \rangle_{I=1}\) (E.11):

\[
\langle r^2 \rangle_{M,I=1} = \langle r^2 \rangle_{I=1}.
\]

(E.31)

The relations (E.30) and (E.31) are also valid in the rigid body approximation and should have a simple origin.

**Axial vector coupling**

Let us consider the axial vector current \(J^{i}_{A,a}\) (2.8). The axial vector coupling constant \(g_A = g_A(0)\) is obtained by identifying the nucleon matrix elements of

\[
\int d^3x J^i_{A,a}(x,t),
\]

(E.32)

with the \(q = p' - p \to 0\) limit of

\[
\langle N'(p') | J^i_{A,a}(0) | N(p) \rangle = \frac{1}{2} g_A(q^2) \langle N' | \sigma_i \otimes \tau_a | N \rangle,
\]

(E.33)

valid for nonzero \(m_\pi\). The evaluation of (E.32) is also very complicated. Using the relation (E.24), we obtain

\[
g_A = \frac{2\pi}{9e^2} \left( \tilde{g}^{(1)}_A + \tilde{g}^{(3)}_A \right),
\]

(E.34)
where \( \tilde{g}_A^{(1)} \) and \( \tilde{g}_A^{(3)} \), which are contributions from the part of \( J_{i,a}^3 \) linear and cubic in \( L_\mu \) or \( R_\mu \), respectively, are given by

\[
\tilde{g}_A^{(1)} = -\int_0^\infty d \rho \rho \left\{ \rho F' + \sin 2F + \frac{1}{5} e^4 \widehat{\Omega}_N^2 \rho^2 \left[ 2\rho F' + \sin 2F \left( 4 + \rho \frac{d}{d \rho} \right) \right] (2Z + C) \right\},
\]

(E.35)

\[
\tilde{g}_A^{(3)} = 4 \int_0^\infty d \rho \rho \sin 2F \left\{ -(F')^2 - F' \tan F - \sin^2 F \rho^2 \right. \\
\left. + \frac{1}{5} e^4 \widehat{\Omega}_N^2 \left[ -2\rho^2 (F')^2 (-\rho Z' + 5A - 10B - 4C) \\
+ \rho F' \tan F \left( \rho C' + 2A + 2C + 1 \right) + \sin^2 F \left( 4 - \left( 2 + \rho \frac{d}{d \rho} \right) (2Z + C) \right) \right] \right\}.
\]

(E.36)

**Isoscalar quadrupole moment**

The isoscalar quadrupole moment operator \( Q_{ij}^{l=0} \),

\[
Q_{ij}^{l=0} = \int d^3 x \left( x_i x_j - \frac{1}{3} \delta_{ij} x^2 \right) J_B^0(x,t),
\]

(E.37)

is a measure of the non-spherical deformation due to the spinning motion. \( Q_{ij}^{l=0} \) vanishes identically in the rigid body approximation where the baryon number density is spherically symmetric.\(^{36}\) In the present case with relativistic correction, it becomes non-trivial:

\[
f_\pi^2 e^2 Q_{ij}^{l=0} = \frac{8\pi e^4}{15} \int_0^\infty d \rho \rho^6 (2A - 5C) J_B^{cl 0}(\rho) \frac{\widehat{\Omega}_N^2}{J^2} \left( \frac{1}{2} \{ J_i, J_j \} - \frac{1}{3} \delta_{ij} J^2 \right).\]

(E.38)

However, the nucleon expectation value of the operator \( \frac{1}{2} \{ J_i, J_j \} - \frac{1}{3} \delta_{ij} J^2 \) and hence of (E.38) are equal to zero.

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