Machine learning techniques in joint default assessment

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Abstract

This paper studies the consequences of capturing non-linear dependence among the covariates that drive the default of different obligors and the overall riskiness of their credit portfolio. Joint default modeling is, without loss of generality, the classical Bernoulli mixture model. Using an application to a credit card dataset we show that, even when Machine Learning techniques perform only slightly better than Logistic Regression in classifying individual defaults as a function of the covariates, they do outperform it at the portfolio level. This happens because they capture linear and non-linear dependence among the covariates, whereas Logistic Regression only captures linear dependence. The ability of Machine Learning methods to capture non-linear dependence among the covariates produces higher default correlation compared with Logistic Regression. As a consequence, on our data, Logistic Regression underestimates the riskiness of the credit portfolio.

**keywords:** Bernoulli mixture model, credit risk, default risk, ML methods, Logistic Regression, credit cards.
1 Introduction

The prediction of default probability both of single and groups of obligors remains one of the main issues in Financial Risk Management. The classification of single obligors is traditionally performed using first order Logistic Regression (LR). Machine Learning (ML) techniques have only recently been considered as alternatives. Most of the literature so far has searched for the most accurate machine learning method for single obligor defaults.

The novel contribution of this paper is to empirically analyse the impact of introducing ML techniques to include linear and non-linear dependencies among obligors' characteristics in the risk evaluation of a homogeneous credit portfolio. This task cannot be accomplished by a simple comparison of the actual, observed number of double, triple and n-tuple defaults versus the theoretical default, since, as in many empirical cases, no data on multiple defaults is available. It is therefore performed using quantile-based indicators.

We consider credit card clients, whose joint - and not only single - default matters to the card issuing company. The credit card data gives us a snapshot of a number of covariates at a specific point in time and the default indicator of each obligor one unit of time later. The covariates include socio-economic indicators as well as present and past bill and payment values. The commonality of the observable covariates across credit card holders lets us represent them as a homogeneous group of obligors or credit portfolios. We argue below that, without loss of generality, a homogeneous credit portfolio can be modelled using an exchangeable Bernoulli mixture model, provided that the marginal default probability is assumed to be a function of the set of observable common covariates. The resulting function is then used as a mixing variable, or, equivalently, the individual default probabilities represent the realizations of the mixing distribution. The mixing distribution fully determines the estimate of the credit card issuer’s loss, with no need for additional assumptions such as copulas.

We proceed as follows: we estimate the individual default probability of each member of the observation sample using the LR and three ML approaches: Random Forest, K-Nearest Neighbor and Ada Boost. We show that, on this data, the ML methods provide a slightly better fit of individual default probabilities. Then we move to a homogeneous portfolio: in addition to the probability distribution of the portfolio loss, which is numerically challenging in high dimensions, we compute its parametric (beta-binomial) approximation for the LR and each ML method. We accompany the computation with a measure of the fit, for the goodness of the approximation. Using a non-parametrical distribution for small portfolios and a parametrical one for large ones (6000 obligors in our case) we show that LR underestimates the high quantiles of the portfolio loss. This happens as a result of the fact that ML approaches capture both linear and non-linear dependence among the covariates - and consequently among defaults - while LR captures
linear dependence only. We conclude that joint default prediction using ML approaches is more accurate, also at joint level.

The paper is organized as follows: Section 2 illustrates the background literature in univariate ML default prediction and mixing reduced-form approaches for joint default assessment. Section 3 sets up the multivariate model consistently. Section 4 specifies the research design. Section 5 illustrates the credit card data and its basic statistics. Section 6 explains how we fit the individual and joint default probabilities in the LR and in the three ML cases, and compares their fit. Section 7 studies the corresponding measures of the high quantiles of the portfolio loss distribution. Section 8 concludes and outlines future research.

2 Background literature

The assignment of a credit scoring and a default probability have long been a task of credit risk. Classic instruments for probability prediction are the LR and discriminant analysis. More recently, ML algorithms have been used. First, individual classifiers were used (see for instance Yeh and Lien (2009)). Later on, ensemble methods were also adopted (see for instance Chen et al. (2021)). Individual algorithms include for instance K-Nearest Neighbors, naive Bayesian, neural networks and classification trees, while ensemble include random forest and support vector machines (for an explanation of their main features and differences see for instance Kenett and Zacks (2021)). Lately, the boosting techniques have been adopted, leading to the development of the Ada Boost and GBoost or XGBoost techniques (see for instance Chen et al. (2021)).

We adopt individual, ensemble and boosted approaches, together with the first order LR method, and describe their specifications below. In particular, we use K-Nearest Neighbors, Random Forest and Ada Boost, because all of them are consistent (see Devroye et al. (1994), Scornet et al. (2015), Bartlett and Traskin (2006)). K-Nearest Neighbors in the form of closest neighbor was used for the first time in the credit risk domain by Chatterjee and Barcun (1970), who pointed at its non-parametric, simple applicability to credit scoring, and computed the probability of misclassification. It was then used also for the computation of the probability of default. Random forest, with the specific purpose of computing the default probability, appears in Kruppa et al. (2013). Tsai et al. (2014) presented bagging and boosting methods for different ensemble techniques and analyzed their performance in bankruptcy prediction.

The literature has had difficulty in assessing whether one of these approaches performs better than the others. Desai et al. (1996), an early contribution in the field, simply compares neural networks and LR in credit scoring. They consider only the ability to separate "bad" and "good" loans, instead of predicting the probability of default,
and arrive at mixed conclusions.

Yeh and Lien (2009) instead examine the default probability and compare the ability to predict it of different individual algorithms and LR. They conclude that neural networks do outperform LR.

Lessmann et al. (2015), who include individual and ensemble methods, and compare 41 criteria, point at their overall superiority with respect to the LR and find that the best performer is the random forest.

Wang et al. (2018) find that GBoost outperforms random forests without boosting.

Chen et al. (2021), who again consider logistic methods, individual and ensemble ones, show that there is no unique best predictor among the machine learning ones. Sometimes, even the LR is preferable to the ML approaches. The only paper, up to our knowledge, which includes in the LR also higher order terms and interaction terms, and compares their predictive accuracy on individual defaults, is Dumitrescu et al. (2022). In spite of the Authors including polynomial terms to improve the fit of the LR, the latter remains less accurate than the ML techniques.

The mixed results we have on the performance of the different forecasting methods are due to the fact that the performance is not uniform over measurement tools and data.

In Section 6 below we show that, as known in the ML literature, the appropriateness of each method for single default prediction really depends on whether the researcher is more interested in false negative, false positive, or combinations of the two with respect to true predictions. These features are measured by selected metrics, such as precision, sensitivity, specificity, the area below the so-called ROC curve, and the so-called F1-score.

The data on which the ability of the different prediction approaches is tested is also important for the result. Below, we examine a publicly available dataset on Taiwanese credit card holders, which has already been examined with selected ML algorithms, including K-Nearest Neighbors and LR, in Yeh and Lien (2009). Unlike us, the authors do not use the F1-score or the area under the ROC curve. They test the predictive accuracy by regressing the real on the forecasted probability, where the real probability is produced using a ”sorting smoothing method”. They reach the conclusion that the artificial neural network outperforms other individual methods. We depart from their conclusions, when examining the predictive accuracy of ML methods for single defaults, because the set of ML algorithms we use is different from their own, and because the metrics for assessing the performance is not a regression of their type. We use the standard metrics of ML mentioned above, in particular the F1-score and the area under the ROC curve.

Other studies that use ML to predict default in credit cards are Bellotti and Crook.
use a different credit-card dataset, namely a proprietary dataset of 25000 credit card clients of a private institution, with contracts opened in March 2004. They show that on their data support vector machines, another ML approach, which we do not use, is superior to the traditional approaches they adopt, namely LR and discriminant analysis. Bellotti and Crook focus on the analysis of the covariates that are most important in driving defaults. We are not interested in understanding the importance of the single covariates in order to determine the default of single obligors. We are interested in their dependence because it is that which determines joint defaults.

Foster and Stine (2004) use a credit card dataset from the Wharton Financial Institutions Center, with more than 280,000 observations from several lenders in a 12-month period between 1996 and 1997. They compare the ability of the least square stepwise regression vs data mining techniques (C4.5 and C5 or boosted C4, as in Salzberg (1994)) to predict personal bankruptcy and find that modified regressions perform at least as well as non parametric methods.

The reason why also the inclusion of higher order terms in the LR does not make it as good in prediction as ML is that, in any case, polynomial terms provide an approximation to the actual, non-linear relationship among the covariates. A paper which testifies that on a different, but huge dataset of mortgages (120 million) is Sadhwani et al. (2021): they recognize that, being ML entirely dictated by the data themselves, it minimizes the model mis-specification and the bias of the variable weights’ estimates inherent in - even higher order - LR. That is why below we stop to the first order LR. Still, we challenge the latter at the multivariate level using quantiles, as suggested by Dumitrescu et al. (2022).

As for the modelling approach for joint defaults, we use a typical, reduced form approach, the mixing-variable approach. A standard reference for mixture models, and the particular one we use, the Bernoulli one, is McNeil et al. (2005).

Given the reduced-form nature of default prediction at the univariate level, both with the LR and ML methods, we indeed take a reduced-form approach also at the multivariate level. Once we account for the dependence of single defaults on a set of common covariates, through LR or ML, single defaults are the realizations of the mixing distribution in an exchangeable Bernoulli mixture model. This approach - as we recall below, in Section 3.1 - has the advantage of providing the probability of any number of joint defaults in quasi-closed form and, in the end, to give a one-to-one mapping between the moments of the mixing variable and the unconditional probabilities of any order of the number of defaults. Also, it is the counterpart of any structural threshold model in which defaults are conditionally independent given the value of some common factors, as proved in Frey and McNeil (2001). This means that its adoption can occur without loss of generality, from the point of view of the economic interpretation. From the numerical
point of view, a reduced form mixture representation lends itself to easier Monte Carlo simulation and simpler asymptotic results - when the dimension of the credit portfolio becomes large - than a structural one (see McNeil et al. (2005)).

We differ from the literature on Bernoulli mixture models in the formulation of the link between the marginal and joint unconditional default probabilities and the covariates, which here relies on ML.

3 Modelling dependent defaults

Let the random vector \( Y = (Y_1, \ldots, Y_d) \) be the vector of default indicators of a set of \( d \) obligors or credit card owners over a fix time horizon \( T \), that in our case is one month. Let \( P = (w_1, \ldots, w_d) \) be the percentage weights which represent a credit risk portfolio at time \( T \) associated to the \( d \) obligors, where \( w_i \in (0, 1] \) and \( \sum_{i=1}^d w_i = 1 \). To model the loss of the portfolio \( P \), we consider the sum of the percentage individual losses \( L \), given by:

\[
L = \sum_{i=1}^d w_i Y_i, \tag{3.1}
\]

In this paper we consider the case \( w_i = \frac{1}{d}, \; i = 1, \ldots, d \). The extension to unequal weights can be done numerically or by simulation. A relevant variable is the number of defaults,

\[
S = \sum_{i=1}^d Y_i, \tag{3.2}
\]

that fully characterizes the loss in the case of equal weights, where \( L = S/d \).

To represent \( Y \) and \( S \) we use a Bernoulli mixture model. We do this without loss of generality, because all threshold credit risk models can be represented as Bernoulli mixture models (Basoğlu et al. (2018)), see also Frey and McNeil (2001).

**Definition 3.1.** Given some \( n < d \) and a \( n \)-dimensional random vector \( \psi = (\psi_1, \ldots, \psi_n) \), the random vector \( Y = (Y_1, \ldots, Y_d)' \) follows a Bernoulli mixture model with factor vector \( \psi \), if there are functions \( q_i : \mathbb{R}^p \rightarrow [0, 1] \; 1 \leq i \leq d \), such that conditional on \( \psi \) the default indicator \( Y \) is a vector of independent Bernoulli random variables with \( \mathbb{P}(Y_i = 1|\psi) = q_i(\psi) \).

In a mixture model the default probability of an obligor is assumed to depend on a set of economic factors \( \psi \): given \( \psi \) defaults of different obligors are independent.
3.1 Exchangeable Bernoulli mixture model

Usually, \( \psi \) is a latent variable and has a factor structure to model common and idiosyncratic factors that affect defaults. Here we depart from this model because we can observe the factors driving default probability. We suppose that the distributions of individual defaults, \( q_i(\psi) \) in Definition 3.2, are functions of a set of observable covariates \( X \), thus we write \( q_i(X) \). Since the vector of covariates \( X \) is common to all obligors we assume that there is only one distribution of default probabilities, i.e. \( q_i(X) = h(X) \). Because default probabilities are conditionally independent given a single common mixing variable \( Q = h(X) \), that represents the distribution of individual default probabilities of obligors, we assume the Bernoulli mixture model defined below.

**Definition 3.2.** Given a random variable \( Q \), the random vector \( Y = (Y_1, \ldots, Y_d)' \) follows an exchangeable Bernoulli mixture model with mixing variable \( Q \) with support on \([0, 1]\), if conditional on \( Q \) the default indicator \( Y \) is a vector of independent Bernoulli random variables with \( \mathbb{P}(Y_i = 1 | Q) = Q \).

Usually the mixing variable is assumed to be beta distributed and it is not directly observable. Since \( Q = Q_h = h(X) \) is function of a vector of observable covariates \( X \), the realizations of \( Q_h \) are functions of the realizations of \( X \), \( q_i = h(x_i) \) and therefore the conditional default probability is

\[
P(Y_i = 1 | Q_h = h(x_i)) = h(x_i)
\]

(3.3)

The unconditional marginal default probability becomes

\[
P(Y_i = 1) = \int_0^1 q dG_h(q),
\]

(3.4)

where \( G_h(q) \) is the distribution of \( Q_h \), and the unconditional probability mass function (pmf) \( p_Y(y) \) of \( Y \) becomes:

\[
p_Y(y) = \mathbb{P}(Y = y) = \int_0^1 q^{\sum_{i=1}^d y_i} (1 - q)^{d - \sum_{i=1}^d y_i} dG_h(q).
\]

(3.5)

In this case it is easy to see that the Bernoulli mixture model \( Y \) is exchangeable; this is the situation that models an homogeneous group of obligors. We introduce the following simple notation for the cross moments of \( Y \):

\[
\pi_k = E[Y_{i_1} \cdots Y_{i_k}], \quad \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}, \quad 1 \leq k \leq d.
\]

(3.6)

Notice that \( \pi_1 = P(Y_i = 1) \) is the marginal default probability and we call it \( p := \pi_1 \). We can observe that \( \pi_k \), the \( k \)-th order joint default probability is the probability that
an arbitrary selected subgroup of \( k \) obligors defaults in \([0, T]\). We can easily compute the following relevant quantities:

\[
E[Y_i] = p, \quad \text{Var}(Y_i) = p - p^2
\]
\[
Cov(Y_i, Y_j) = p - p^2, \quad \rho = \rho(Y_i, Y_j) = \frac{\pi_2 - p^2}{p(1 - p)} \quad i \neq j.
\] (3.7)

The distribution \( S \) of the number of defaults can be computed in terms of the \( \pi_k \)'s in the following way (see Frey and McNeil (2001)):

\[
P(S = k) = \sum_{i=0}^{d-k} (-1)^i \frac{d!}{i!(d - k - i)!} \pi_{k+i}.
\] (3.8)

It is evident that the distribution of the number of defaults is entirely determined by the joint distributions of the default indicators. This is a consequence of the exchangeability of the vector \( Y \). In fact if \( Y \) is exchangeable there is a one to one correspondence between the distribution of the number of defaults and the joint distribution of defaults (Fontana et al. (2021)).

The unconditional distribution \( p_S(k) \) of the number of defaults \( S \) becomes:

\[
p_S(k) = \mathbb{P}(S = k) = \binom{d}{k} \int_0^1 q^k (1 - q)^{d-k} dG(q).
\] (3.9)

In an exchangeable Bernoulli mixture model it can be proved that the cross moments of \( Y \) are the moments of the mixing variable \( Q_h \), formally:

\[
\pi_k = E[Q_h^k],
\] (3.10)

in particular \( \pi_1 = E[Q_h] = p \). Furthermore, for \( i \neq j \) \( \text{Cov}(Y_i, Y_j) = \text{Var}(Q_h) \). Comparing (3.10) with (3.6) and (3.8) it follows that the moments of the mixing variable \( Q_h \) completely determine the joint distribution of defaults and consequently the distribution of the number of defaults. Therefore, using sample moments of \( Q_h \), that is a function of observable covariates, we can in principle estimate the distribution of \( S \). This can be done in practice for low dimensions, since the non parametrical distribution in (3.8) exhibits numerical problems from \( m \sim 30 \) onwards. The specification of a parametrical distribution for \( Q_h \) allows us to work also in high dimensions.

The estimated individual default probabilities are realizations of \( Q_h \), and we use them to estimate the moments of \( Q_h \) or its parameters, if we assume a parametrical distribution for \( Q_h \). The distribution of \( Q_h \) depends on the distribution of the covariates, thus we do not have any information about its parametrical form. The model risk associated to the parametrical form of the distribution of \( Q_h \) has been discussed in McNeil et al. (2005), Section 8.4.6. In particular it is considered the model risk associated to the parametrical
mixing distribution under the constraint that the default probability $p$ and the default correlation $\rho$ (or the second order moment $\pi_2$) are known and fixed (see [1]). Proposition 8.16 in [McNeil et al. (2005)] proves that the tail distribution of the number of defaults is determined by the tail distribution of $Q_h$. Nevertheless, [McNeil et al. (2005)] conclude that, if the first two moments are fixed, this seems to be significant only after the 0.99 quantile of the distribution. From a practical point of view the more relevant quantities are the marginal default probability and the default correlation. For this reason, the particular parametrical form of the mixing variable is less important than the estimates of $p$ and $\rho$. This means that in this model the tail is particularly sensitive to $p$ and $\rho$.

We consider a beta distribution with parameters $a$ and $b$ for $Q_h$, i.e. $Q \sim \beta(a,b)$, that gives rise to an exchangeable Bernoulli mixture model frequently used in practice, i.e. the beta mixing-model. If $Q \sim \beta(a,b)$ then the number of defaults $S$ follows a so-called beta-binomial distribution of parameters $d$, $a$ and $b$ (see [McNeil et al. (2005)]), where $d$ is the dimension of $Y$ or the number of obligors. The beta and the beta-binomial distributions are recalled in Appendix A.

4 Research design

The individual obligors considered here are a single homogeneous group of credit card owners. In this case, the set of covariates $X$ are obligors' characteristics, such as age and income.

As mentioned above the classical model specification for the function $h$ to estimate individual default probabilities $q_i$, and consequently for the estimates of the moments of $Q_h$, is a first order LR. We compare it with different ML techniques as model specifications $h$ to estimate $Q_h$. The advantage of ML methods over the traditional LR is their ability to discover complex structures and interactions between covariates that were not specified in advance. These complex structures are reflected in the moments of $Q_h$, and therefore in the estimate of the relevant quantities for the number of default distributions, that are $p$ and $\rho$. We consider three ML methods: Random Forest (RF), Ada Boost (AB) and K-Nearest Neighbors (KNN), reviewed in Appendix B. We have a total of four models to compare, corresponding to $h = LR$, RF, AB, KNN.

Our aim is to measure the risk associated to the choice of a ML technique instead of the classical LR to estimate $Q_h$. We therefore consider the risk associated to the corresponding distribution of number of defaults. As a measure of risk we consider the Value at Risk (VaR), that is the most used measure of risk by financial institutions, due to regulatory requirements. We recall its definition for a general random variable $Y$.

**Definition 4.1.** Let $Y$ be a random variable representing a loss with finite mean. Then
the VaR at level $\alpha$ is defined by

$$\text{VaR}_\alpha(Y) = \inf\{y \in \mathbb{R} : P(Y \leq y) \geq \alpha\}$$

The VaR is essentially the $\alpha$ quantile of the distribution of $S$. As discussed in the previous section the relevant factors affecting the VaR are the first two moments of the mixing variable. Different choices of the function $h$, i.e. of the ML technique, give different estimates for the conditional default probabilities, because they imply different estimates for the first and second order moment of $Q_h$.

To work in high dimensions we need to specify the beta parametrical distribution for $Q_h$. In this case we have to take into account the risk associated to a $\beta$ specification. As discussed in Section 3.1 in practice a misspecification of the parametrical distribution of $Q_h$ is less important than the estimate of its first two moments. Nevertheless, we compare the VaR of the number of defaults of the beta binomial model with the VaR of the non parametrical Bernoulli mixture model for each choice of $h$. This can be done for low dimension portfolios; we choose $d = 25$ that is the computational limit for the non parametrical distribution of $S$.

Staring from a sample of size $n$ of obligors with $m$ covariates, the analysis proceeds according to the following steps.

1. For each choice of $h = \text{LR}, \text{RF}, \text{AB}, \text{KNN}$, we estimate $q^h_i = h(x_i)$, where $x_i$ is the $m$-dimensional vector of covariates realizations, and find a sample of estimated conditional default probabilities $\hat{q}^h = (\hat{q}^h_1, \ldots, \hat{q}^h_n)$.

2. For each choice of $h = \text{LR}, \text{RF}, \text{AB}, \text{KNN}$ we compute the marginal default probability $p$ and the equicorrelation among default indicators. Then, we estimate the distribution of the number of default $S$. We follow two different approaches, in the first one we assume that defaults come from a Bernoulli mixture model, but we do not specify the parametrical distribution of $Q_h$, in the second we assume that $Q_h \sim \beta(a_h, b_h)$. Non parametrical approach:

(a) Since $E[Q^k_h] = \pi^k_h$, we use the sample moments $\hat{Q}^k_h$ of $Q_h$ to estimate the moments $\hat{\pi}^k_h$. Thus, $\hat{\pi}^k_h = \hat{Q}^k_h$, with $k \leq n$.

(b) The non parametrical distribution of the number of defaults $S_h$ is then given by (3.8).

Parametrical approach:

(a) We use moments method to estimate the beta parameters under the assumption $Q_h \sim \beta(a_h, b_h)$. The choice of the moments method derives from the fact that, as discussed above, it is more important to estimate properly the
first two moments of $Q_h$ than its distribution, because $p$ and $\rho$ are the most relevant quantities for $S$;

(b) The estimated distribution of $S$ is then given by a beta-binomial distribution, that is implemented in R.

3. For each choice of $h=L$R, $R$, AB, KNN we compare the non parametrical distribution in (3.8) of a low dimensional ($d = 25$) portfolio with his theoretical beta-binomial estimated distribution. We use the Kullback-Leibler (KL) distance.

4. For each choice of $h=L$R, $R$, AB, KNN we compute the VaR of the number of defaults. For a low dimensional portfolio $d = 25$ we use both the non parametrical and the parametrical distribution of $S$, while for a high dimensional portfolio ($d = 6000$) we use the parametrical distribution that best fits, according to the KL distance, the non parametrical one.

5 Data description

We use the Kaggle database on credit card defaulters\footnote{https://www.kaggle.com/uciml/default-of-credit-card-clients-dataset}, which is a collection of data from 30000 clients of a bank issuing credit cards in Taiwan, from April to September 2005. Both the descriptive analysis of this Section and most of the results obtained in the next Section are obtained using the Sklearn library in Python.

For each client the dataset contains the values of $m = 24$ covariates $X$, which are listed in Appendix C. Some covariates, such as the monthly repayment status, the past payments, the past bill amounts, are lagged values of the same economic variable. As a first step in the analysis, we investigate whether there is multicollinearity. To envisage it, we build the correlation matrix of the 24 covariates, which is presented here as Figure 1. Figure 1 shows that the variables representing the payment status in the previous and next period, as well as the variables representing the bill amount up to 6 months before and after the current date, are highly correlated. Nevertheless, we do not remove them in the LR model in order to keep the same set of variables for all the models. This is not a limitation for our study since multicollinearity does not affect the overall fit of the model, but only the regression coefficients that are not of interest to our purposes.

The second step consists in detecting whether the dataset is balanced or not. As one can easily argue, because the predicted event is default, and credit card clients are chosen so as to have a minimum credit standing, the dataset is unbalanced, with an higher number of non defaulters than defaulters, when it comes to the variable $Y$ representing default in the next period of time. We illustrate this circumstance in Figure 2. The unbalanced nature of the dataset is not extreme and we do not correct it.
Figure 1: Correlation matrix of the variables.

Figure 2: Unbalancing of the dataset. Default=1, non-default=0

The third step consists in dividing the dataset into a train set and a test one. The criterion adopted to divide the dataset is the following: 2/3 belong to the training set and 1/3 to the test. As a consequence, our training set contains 24000 observations of the 24 covariates $X$ and the outcome $Y$, while the test one contains 6000 of them. Thus in our case $n = 6000$.

6 Results

The first step of our analysis is to estimate $Q_h$ for each $h = \text{LR, RF, AB, KNN}$. The results are in the following Section.
6.1 Mixing default probability

The typical measurement of the predictive performance of ML algorithms is given by the confusion matrix, which gives both the number of true positive (defaults) and true negatives (survivals) on the main diagonal, as well as the number of false negatives and false positives, on the other diagonal. Because false negatives and false positives determine respectively the second and first type errors, an exam of the confusion matrix is surely important to assess which, among several criteria, has higher capacity to concentrate on true positives and true negatives. From the confusion matrix one can indeed produce several indices. Among these indices, we will use the precision, sensitivity or recall, the F1-score and the area under the ROC curve.

The first index, the precision index, counts the number of correct positive predictions, or true defaults, over the total number of positive predictions, or predicted defaults, namely true positives plus false positives. So, it gives an idea of the likelihood of a predicted default to be a true one, which represents the precision of the corresponding method. This index is relevant in credit risk because true positives produce losses, while false positives do not.

Another measure relevant in credit risk is the sensitivity or recall, which counts the true positive occurrences, or defaults which are predicted and do occur, over the total number of true predictions. Another way to test the predictive ability of a ML model, and to compare them is the F1-index, which is an harmonic average of recall and precision.

Plotting on the horizontal axis the number of predicted defaults which turned out in survivals - or false positive - over the total actual survivals, and on the vertical the recall, one gets the so-called ROC curve. The area under the ROC curve or AUC is an important indicator: the closer this area to one, the better the overall performance of the model, because the closer to zero is the number of false positives, namely predicted defaults which lead to survivorship, and the closer to one is the number of true positives, namely correctly predicted defaults.

For each of the candidate models we report in Table 3 the performance measures. Precision is highest for AB, which is literally the most precise method to predict default on this database. Recall is highest for AB, meaning that the latter is also the method which maximizes the true defaults, considering all true predictions, on this dataset. As a synthetic measure, consider the F1-score, which is again highest for AB. All the three indicators are slightly lower for LR.

To get an overall grasp of the performance of the four methods to estimate individual default probabilities, we consider in Figure 3 the ROC curve and the area under it, the AUC, which is the same for RF and AB.

Comparing the indices in Table 3 we can conclude that the ML methods perform
| Model | Precision | Recall | F1-score | AUC  |
|-------|-----------|--------|----------|------|
| LR    | 0.61      | 0.78   | 0.68     | 0.64 |
| RF    | 0.79      | 0.81   | 0.78     | 0.77 |
| AB    | 0.80      | 0.82   | 0.79     | 0.77 |
| KNN   | 0.78      | 0.81   | 0.78     | 0.72 |

Table 1: Performance measure for each model

Figure 3: ROC curves

better that the traditional LR, at least on this database. It is therefore important to compare the discrepancy in joint defaults that they entail.

Once we have predicted the probabilities of default using the four methods, we proceed to the second step, calibration, whose purpose is that of aligning predicted and actual probabilities in each ML model. We proceed with a calibration of the ML probabilities for all three ML algorithms using both the isonotonic regression and the Platt scaling. To this end, we further split the train set into two subsets, the first one of which - consisting of 75% of the original data in the training part - is used for the calibration, the second for its validation. We then compare the expected calibration error for RF, KNN and AB both after the Platt scaling and the isotonic regression, and choose the calibration method which minimizes the error for each specific ML approach. The resulting methods are Isotonic regression for RF and KNN, Platt scaling for AB. These determine the predicted probabilities that we work with.

After the calibration we estimate the relevant quantities for the joint defaults and for the number of defaults and then the parameters of the beta mixing variables.
| Model | Calibration method | expected calibration error |
|-------|--------------------|-----------------------------|
| RF    | Isotonic regression| 2.38%                       |
| AB    | Isotonic regression| 1.79%                       |
| KNN   | Platt              | 5.65%                       |

Table 2: Calibration errors.

6.2 Defaults in a credit risk portfolio.

Table 3 provides the first two empirical moments of $Q_h$ for each method $h$ and the resulting default correlations, for LR and for each ML model.

| Moments | LR      | RF      | AB      | KNN     |
|---------|---------|---------|---------|---------|
| $\mu$   | 0.2635  | 0.2232  | 0.2234  | 0.2209  |
| $\pi_2$ | 0.0883  | 0.0925  | 0.0901  | 0.1023  |
| $\rho$  | 0.0975  | 0.2462  | 0.2319  | 0.3108  |

Table 3: Moments estimated with different ML techniques.

The LR method seems to overestimate the marginal default probability and underestimate the second order moment, if compared with the ML methods. ML techniques are capable to capture higher correlations among default. This happens because by definition ML approaches incorporate linear and non-linear dependencies among covariates, and covariate dependencies are obviously relevant in default correlations. This is the main factor affecting the VaR of the number of defaults.

The following step in building a multivariate default distribution, or, more specifically, a distribution for the number of defaults in the credit card portfolio, starting from the estimates of the mixing variable, consists, according to the research agenda presented in Section 4, in calibrating the parameters of the beta mixing variables. Using the method of moments, we get the parameters in Table 4.

| $\beta$-parameters | LR | RF | AB | KNN |
|--------------------|----|----|----|-----|
| $a$                | 2.42 | 0.68 | 0.73  | 0.48 |
| $b$                | 6.78 | 2.38 | 2.57  | 1.72 |
| $\mu$              | 0.2630 | 0.2222 | 0.2212  | 0.2182 |
| $\rho$             | 0.0980 | 0.2463 | 0.2326  | 0.3125 |

Table 4: Estimated parameters of the beta distribution for each model

The Kolmogorov-Smirnov test of the four $\beta$ distributions reject their appropriateness to describe the mixing variable, since the $p$-value is close to zero in all cases. However,
Table 4 shows that the theoretical moments are very close to the non parametrical ones. This means that despite the misspecifications for the distribution of \( Q \), we expect that theoretical and empirical VaR are close, with a possible more significant discrepancy in correspondence to \( \alpha = 0.99 \), as discussed in Section 3. In Table 5 we compare the fit to the non parametrical distribution of \( Q_h \), \( h = \text{LR}, \text{RF}, \text{AB}, \text{KNN} \), of the corresponding beta distribution, using the Kullback-Leibler (KL) distance. Because we do not want to incur into the problems of computing high factorials, we do the comparison on a portfolio of \( d = 25 \) obligors. The conclusion from Table 5 is that the parametrical pmf is closer to the non parametrical one for the LR and AB. We therefore focus on these two cases to estimate the VaR for the large portfolio. We should notice that the overall fit of the beta distribution does not correspond to the higher correlations.

| KL distance | LR  | RF  | AB  | KNN |
|-------------|-----|-----|-----|-----|
|             | 0.0450 | 0.1153 | 0.0783 | 0.1557 |

Table 5: KL distance for each model.

7 VaR estimates

Risk measurement of the credit losses is done through their VaR, computed along Definition 4.1. Both on a small portfolio \( (d = 25) \) and to the overall portfolio \( (d = 6000) \), we compute the VaR at the levels of confidence \( \alpha = 0.9; 0.95; 0.99 \) searching for the corresponding quantiles of the distribution. In the small portfolio case we use both the non parametrical and the theoretical pmfs to do so, while in the large portfolio we use the theoretical pmfs.

7.1 Small portfolio

The VaRs using the non parametrical and the four beta binomial models are reported in Table 6.

Compare first the non parametrical vs the beta VaR for each model in Table 6. For the LR case the beta VaR is always bigger than the non parametrical one, showing that LR is not capable to capture heavy tails. This is more evident for \( \alpha = 0.99 \), where the importance of the tails of \( Q \) is higher. On the contrary, for all the ML techniques the non parametrical VaR is higher that the beta one. This is in line with the ability of these techniques to capture complex dependencies. However, for all the models the non parametrical and beta VaR are close, supporting that the estimates of \( p \) and \( \rho \) is more
Table 6: Non-parametrical vs beta-binomial VaRs of the distribution of number of defaults for a portfolio of 25 obligors. LR\_e, RF\_e, AB\_e, KNN\_e are the non-parametrical distributions for each model and LR\_m, RF\_m, AB\_m, KNN\_m are the parametrical distributions for each model.

| α    | LR\_e | LR\_m | RF\_e | RF\_m | AB\_e | AB\_m | KNN\_e | KNN\_m |
|------|-------|-------|-------|-------|-------|-------|--------|--------|
| 0.99%| 15    | 18    | 23    | 21    | 22    | 21    | 24     | 23     |
| 0.95%| 13    | 14    | 19    | 17    | 18    | 16    | 20     | 18     |
| 0.90%| 12    | 12    | 15    | 14    | 15    | 13    | 16     | 15     |

important than the parametrical specification of a distribution for $Q_h$. This allows us to switch to the beta distribution to work in high dimensions.

Compare then the VaRs obtained for the three ML methods with the VaR obtained with LR. The VaRs obtained with ML techniques are significantly higher that the ML VaR. LR underestimates the VaR of the loss if compared to the ML methods. So, risk measurement through ML is necessary to include risks due to higher order dependencies among covariates in the estimate of the quantile from the losses.

7.2 Large portfolio

We consider a large portfolio, $d = n = 6000$ and the beta binomial distributions estimated using LR, AB, RF and KNN - the parameters are in Table 4. Figure 4 shows the four estimated beta binomial pmfs. It is evident that the three ML methods provides similar densities, while the LR is significantly different, also in the tails. This is reflected in the VaRs.

We focus on AB as a ML technique, because it gives the best fit on our data. In Figure 5, this two pmfs are overlapped to exhibit the differences in the tail distribution.

The theoretical VaRs obtained from the beta-binomial models corresponding to the parameter $a$ and $b$ estimated using LR and AB and with $d = 6000$ are reported in Table 7. Computations are made with the support of $R$.

| α    | LR  | AB  |
|------|-----|-----|
| 0.99%| 3794| 4798|
| 0.95%| 3107| 3788|
| 0.90%| 2729| 3139|

Table 7: Beta-binomial VaR for large portfolios.

Also in this case the LR seems to underestimate the riskness of the loss. The difference between the two VaRs in Table 7 is higher for the quantile $\alpha = 0.99$, where the
Figure 4: Beta binomial pmf estimated with LR, AB RF and KNN.

higher order moments of $Q_h$ are more relevant.
8 Conclusion and further research

The novel contribution of this paper is to empirically analyse the impact of introducing ML techniques to include linear and non-linear dependencies among obligors’ characteristics in the credit risk evaluation of a portfolio. We perform our analysis on a set of credit card owners with observable characteristics. We treat them as a homogeneous group and use an exchangeable Bernoulli mixture model. Covariates are encoded in the mixing distribution. We compare the performance of three ML techniques with the traditional first order LR method to estimate the mixing distribution and conclude that ML methods are slightly superior to the first order LR method, according to traditional metrics.

In a Bernoulli mixture model the first two moments of the mixing distribution are the relevant quantities for the tail of the distribution of the loss, which is associated to the riskiness of a credit portfolio measured by the VaR. Our main result is that the ML techniques, whose moments are different from the LR ones, can model significantly higher default correlations, which are the main factors influencing the VaR of the loss. This means that non-linear dependence among covariates captures a higher default correlation and the heavier tails of the loss, compared with linear dependence alone. As a result, when it comes to evaluating joint defaults, LR performs poorly, because it underestimates the risk of portfolios of losses. In case non-linear dependencies are weak the quantiles from LR will be closer to Random Forest, KNN or AB, while we expect high non-linear
dependencies to boost the quantile departures.

One could argue that, in order to be comparable to ML ones, non-ML methods should describe and incorporate higher order dependencies - non-linear and interaction terms - among the covariates. We already know from the previous literature (see Dumitrescu et al. (2022)) that adding non-linear and interaction terms to the LR does not improve the prediction accuracy for single defaults. That is why we did not add them here.

Horizontal datasets - such as the credit card dataset - do not allow us to compare actual and predicted joint defaults, because they do not provide us with the former. We should compare the performance of LR - first or higher order and with interaction terms - versus ML techniques in predicting joint defaults, on a longitudinal dataset, which contains observations of joint defaults over time. The availability of such a dataset is the only obstacle to our progress in that direction. This will be the object of future research.
A Beta and Beta Binomial distributions

Let $Q$ have a Beta distribution of parameters $a$ and $b$, i.e. $Q \sim \beta(a, b)$, its density $g_Q$ is given by

$$g_Q(q) = \frac{1}{\beta(a, b)} q^{a-1}(1-q)^{b-1}, \quad a, b > 0, \quad 0 < q < 1$$  \hspace{1cm} (A.1)

where

$$\beta(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1}dz, \quad 0 < z < 1$$  \hspace{1cm} (A.2)

Standard calculations give:

$$\pi_k = \prod_{j=0}^{k-1} \frac{a+j}{a+b+j},$$  \hspace{1cm} (A.3)

in particular

$$\pi = \frac{a}{a+b}, \quad \text{and} \quad \rho_y = \frac{1}{a+b+1}.$$

A Beta Binomial random variable $S$ with parameters $a, b$ and $d$, $S \sim \beta Bin(a, b, d)$, has pmf $p_S(k)$ given by

$$p_S(k) = \mathbb{P}(S = k) = \binom{d}{k} \frac{\beta(a+k, b+d-k)}{\beta(a, b)}, \quad k \leq d.$$  \hspace{1cm} (A.4)

B Machine learning techniques

In this section we recall how conditional probabilities are estimated from covariates for each mode, i.e. LR, KNN, RF, AB. For a complete overview of this topic see Friedman et al. (2001).

B.1 Logistic Regression

The LR is a generalized linear model, where the individual conditional responses $Y_1, \ldots, Y_n$ are independent with distribution $B(q_i)$. In the LR model the conditional probabilities are given by:

$$q_i := \mathbb{P}(Y_i = 1|Q_h = h(x_i)) = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} = \frac{1}{1 + e^{-\beta^T x_i}}.$$  \hspace{1cm} (B.1)

Solving the previous equation for the exponential we gain a better insight into the meaning of vector parameter $\beta$:

$$e^{\beta^T x_i} = \frac{q_i}{1 - q_i}.$$  

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Then taking the logarithms of both sides, we obtain the statistical model

$$\log \left( \frac{q_i}{1 - q_i} \right) = \beta^T x_i.$$ 

The ratio $q_i/(1 - q_i)$ provides us with equivalent information in terms of odds. The logarithm of the odds is known as a logit function.

In order to find a way to fit the regression coefficients against a set of observations $x_i$ and $y_i \in \{0, 1\}$, $i = 1, \ldots, n$ the maximum-likelihood method is used.

The target variable $Y_i$, which can take values $y_i$ in the set $\{0, 1\}$, may be regarded as the realization of a Bernoulli variable, whose density probability function is

$$P(Y_i = y_i | Q_h = q_i) = q_i^{y_i} (1 - q_i)^{1 - y_i}.$$ 

From the expression B.1 for the probability $q_i$ we know that $q_i$ depends on the regressors $x_i$ and the vector of parameters $\beta$.

Assuming independence of errors, observations are independent as well, and the likelihood function is just the product of individual probabilities:

$$L = \prod_{i=1}^{n} p_i^{y_i} (1 - p_i)^{1 - y_i} = \prod_{i=1}^{n} \left( \frac{1}{1 + e^{-\beta^T x_i}} \right)^{y_i} \left( \frac{e^{-\beta^T x_i}}{1 + e^{-\beta^T x_i}} \right)^{1 - y_i}. \quad (B.2)$$

The task of maximizing $L$ with respect to the vector of coefficients $\beta$ can be simplified by taking its logarithm and by logarithm’s properties we obtain:

$$\mathcal{L} = \log L = \sum_{i=1}^{n} \log \left( \frac{1}{1 + e^{-\beta^T x_i}} \right) y_i + \sum_{i=1}^{n} \log \left( \frac{e^{-\beta^T x_i}}{1 + e^{-\beta^T x_i}} \right) (1 - y_i)$$

$$= \sum_{i=1}^{n} y_i \beta^T x_i - \log(1 + e^{\beta^T x_i}). \quad (B.3)$$

Thus the goal is to find the vector of coefficients $\beta$ that maximizes this log-likelihood function, namely to solve this optimization problem:

$$\beta = \arg \max_{\beta} \mathcal{L}. \quad (B.4)$$

### B.2 K-Nearest Neighbors

Nearest Neighbor (NN) algorithms are among the simplest of all machine learning algorithms. The idea behind the NN model is that the model needs to store all the training
examples and then it tries to predict the label of any new instance on the basis of the label of its closest neighbors in the training set. The rationale behind such a method is based on the assumption that the features that are used to describe the domain points are relevant to their labelings in a way that makes close-by points likely to have the same label.

Looking at the KNN algorithm from a generalized perspective, we can assume that our instance domain - the support of the covariate random vector $\mathbf{X}$, $\mathcal{X}$, is endowed with a metric function $\rho$. That is, $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a function that returns the distance between any two elements of $\mathcal{X}$. For example, if $\mathcal{X} = \mathbb{R}^d$ then $\rho$ can be the Euclidean distance. However, depending on the type of data we have, there are other popular distance measures, like the Hamming distance, which computes the distance between binary vectors, the Manhattan distance, which computes the distance between real vectors using the sum of their absolute difference and finally the Minkowski distance, which is a generalization of the previous ones.

In short, the concept of the KNN algorithm is as follows: given a positive integer $K$ and a test observation $\mathbf{x}_0$, the KNN classifier first identifies the $K$ points in the training data that are closest to $\mathbf{x}_0$, represented by $\mathcal{N}_0$. It then estimates the conditional probability for $Y_i$ to be in class $j$, $j = 0, 1$ as the fraction of points in $\mathcal{N}_0$ whose response values equal $j$:

$$
\mathbb{P}(Y_i = j | \mathbf{X} = \mathbf{x}_0) = \frac{1}{K} \sum_{i \in \mathcal{N}_0} I(y_i = j). \tag{B.5}
$$

Finally, KNN applies Bayes rule and classifies the test observation $\mathbf{x}_0$ to the class with the largest probability.

### B.3 Random Forest

A decision tree is a predictor

$$
h : \mathcal{X} \to \{0, 1\},
$$

that predicts the label associated with an instance $\mathbf{x}_i$ by traveling from a root node of a tree to a leaf. At each node of the root-to-leaf path, the successor is chosen on the basis of a splitting of the input space. Usually, the splitting is based on one of the features of $\mathbf{x}_i$ or on a predefined set of splitting rules and each leaf contains a specific label.

One of the main advantages of decision trees is that the resulting classifier is very simple to understand and graphically interpret. The decision trees suffer from high variance in their results. This means that if we split the training data into two parts at random,
and fit a decision tree to both halves, the results that we get could be quite different. In contrast, a procedure with low variance will yield similar results if applied repeatedly to distinct data sets; for example, linear regression tends to have low variance if the ratio of $n$ (number of observations) and $d$ (number of features) is moderately large.

Bootstrap aggregation, or bagging, is a general-purpose procedure for reducing the variance of a statistical learning method. In fact we know that, given a set of $n$ independent observations $Z_1, \ldots, Z_n$, each with variance $\sigma^2$, the variance of the mean $\overline{Z}$ of the observations is given by $\sigma^2/n$. In other words, averaging a set of observations reduces variance. Hence a natural way to reduce the variance and then increase the prediction accuracy of a statistical learning method is to take repeated samples, we say $J$, from our training set, build a separate prediction model using each bootstrapped training set, and take a majority vote, since we are in the classification setting. In other words, we could calculate $\hat{h}_1(x_i), \hat{h}_2(x_i), \ldots, \hat{h}_J(x_i)$ using $J$ separate training sets, where $\hat{h}_j(x_i)$ is the classifier (in our case a classification tree) trained using the $j$-th bootstrapped training set, and then take a majority vote among all the $J$ predictions. This means that for a given test observation, we can record the class predicted by each of the $J$ trees, and take a majority vote: the overall prediction is the most commonly occurring class among the $J$ predictions.

Random forests provide an improvement over bagged trees by way of a small tweak that decorrelates the trees. As in bagging, we build a number of decision trees on bootstrapped training samples; but when building these decision trees, each time a split in a tree is considered, a random sample of $m$ predictors is chosen as split candidates from the full set of $d$ predictors. The split is allowed to use only one of those $m$ predictors. A fresh sample of $m$ predictors is taken at each split, and typically we choose $m \approx \sqrt{d}$. In other words, in building a random forests, at each split in the tree, the algorithm is not even allowed to consider a majority of the available predictors. In order to understand the reason of this concept, suppose that there is one very strong predictor in the data set, along with a number of moderately strong predictors. Then in the collection of bagged trees, most or all of the trees will use this strong predictor in the top split. Consequently, all of the bagged trees will look quite similar to each other. Hence the predictions from the bagged trees will be highly correlated. Unfortunately, taking a majority vote among many highly correlated quantities does not lead to as large of a reduction in variance as taking a majority vote among many uncorrelated quantities. In particular, this means that bagging will not lead to a substantial reduction in variance over a single tree in this setting.

Random forests overcome this problem by forcing each split to consider only a subset of the predictors. Therefore, on average $(d - m)/d$ of the splits will not even consider the strong predictor, and so other predictors will have more of a chance. We can think of this process as decorrelating the trees, thereby making the majority vote among the resulting trees less variable and hence more reliable.

The main difference between bagging and random forests is the choice of predictor subset
size \( m \): for instance, if a random forest is built using \( m = d \), then this amounts simply to bagging.

### B.3.1 Ada Boost

The AdaBoost algorithm has access to a weak learner and outputs a low empirical risk hypothesis that is a linear combination of simple hypotheses: we mean that it relies on the family of hypothesis classes obtained by composing a linear predictor on top of simple classes.

In more formal terms, the AdaBoost algorithm receives as input a training set of samples \( S = \{(x_1, y_1), \ldots, (x_m, y_m)\} \) where for each \( i \), \( y_i = f(x_i) \) for some labeling function \( f \).

The boosting process proceeds in a sequence of consecutive rounds. At round \( t \), the booster first defines a distribution over the samples in \( S \), denoted \( D^{(t)} \), that is such that \( D^{(t)} \in \mathbb{R}_+^m \) and \( \sum_{i=1}^{m} D_i^{(t)} = 1 \). Then, the booster passes the distribution \( D^{(t)} \) and the sample \( S \) to the weak learner, in such a way that the weak learner can construct i.i.d. samples according to \( D^{(t)} \) and \( f \). The weak learner is assumed to return a "weak" hypothesis \( h_t \), whose error

\[
\epsilon_t = \sum_{i=1}^{m} D_i^{(t)} I_{h_t(x_i) \neq y_i}, \quad (B.6)
\]

is at most \( 1/2 - \gamma \), where \( \gamma \) is a toleration parameter. Then, AdaBoost assigns a weight for \( h_t \) as follows: \( w_t = \frac{1}{\epsilon_t} \log \left( \frac{1}{\epsilon_t} - 1 \right) \). We can observe that the weight of \( h_t \) is inversely proportional to the error of \( h_t \). At the end of the round, AdaBoost updates the distribution so that the samples on which \( h_t \) errs will get a higher probability mass while samples on which \( h_t \) is correct will get a lower probability mass: intuitively, this will force the weak learner to focus on the problematic samples in the next round. The output of AdaBoost algorithm is a "strong" classifier that is based on a weighted sum of all the weak hypotheses.

### C  Covariates

For each client the dataset contains the values of 24 covariates, which include demographic variables, repayment status, past payments, past bill amount, and an indicator of default, namely:

- **ID**: ID of each client
- **LIMIT_BAL**: Amount of the given credit (NT dollar), which includes both the individual consumer credit and his/her family (supplementary) credit.
- **SEX**: Gender (1 = male; 2 = female)
• EDUCATION: (1 = graduate school; 2 = university; 3= high school; 4= others; 5= unknown; 6= unknown)

• MARRIAGE: Marital status (1 = married; 2 = single; 3 = others)

• AGE: Age in years

• PAY_0 to 6 (with 1 missing): Repayment status in September to April 2005 (-1=pay duly, 1=payment delay for one month, 2=payment delay for two months, ... 8=payment delay for eight months, 9=payment delay for nine months and above)

• BILL_AMT1 to 6: Amount of bill statement in September to April, 2005 (NT dollar)

• PAY_AMT1: Amount of previous payment in September to April, 2005 (NT dollar)

• default.payment.next.month: Default payment (1=yes, 0=no)

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