Chern character for twisted complexes

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In memory of Sasha Reznikov

1. Introduction

The Chern character from the algebraic K theory to the cyclic homology of associative algebras was defined by Connes and Karoubi \cite{C}, \cite{K}, \cite{L}. Goodwillie and Jones \cite{Go}, \cite{J} defined the negative cyclic homology and the Chern character with values there. In this paper we generalize this Chern character to the K theory of twisted modules over twisted sheaves of algebras.

More precisely, we outline the construction of the Chern character of a perfect complex of twisted sheaves of modules over an algebroid stack $\mathcal{A}$ on a space $M$. This includes the case of a perfect complex of sheaves of modules over a sheaf of algebras $\mathcal{A}$. In the latter case, the recipient of the Chern character is the hypercohomology of $M$ with coefficients in the sheafification of the presheaf of negative cyclic complexes. The construction of the Chern character for this case was given in \cite{BNT1} and \cite{K}. In the twisted case, it is not a priori clear what the recipient should be. One can construct \cite{K2}, \cite{MC} the Chern character with values in the negative cyclic homology of the category of perfect complexes (localized by the subcategory of acyclic complexes); the question is, how to compute this cyclic homology, or perhaps how to map it into something simpler.

Ideally, the recipient of the Chern character would be the hypercohomology of $M$ with coefficients in the negative cyclic complex of a sheaf of associative algebras. We show that this is almost the case. We construct associative algebras that form a presheaf not exactly on $M$ but rather on a first barycentric subdivision of the nerve of a cover of $M$. These algebras are twisted matrix algebras. We used them in \cite{BGNT} and \cite{BGNT1} to classify deformations of algebroid stacks.

We construct the Chern characters

\[ K_{\bullet}(\text{Perf}(\mathcal{A})) \to \mathbb{H}^\ast_{-}(M, \text{CC}_{\bullet}(\text{Matr}_{tw}(\mathcal{A}))) \] (1.1)

\[ K_{\bullet}(\text{Perf}_{Z}(\mathcal{A})) \to \mathbb{H}^\ast_{Z}(M, \text{CC}_{\bullet}(\text{Matr}_{tw}(\mathcal{A}))) \] (1.2)
where $K_\bullet(\mathrm{Perf}(\mathcal{A}))$ is the $K$ theory of perfect complexes of twisted $\mathcal{A}$-modules, $K_\bullet(\mathrm{Perf}_{Z}(\mathcal{A}))$ is the $K$ theory of perfect complexes of twisted $\mathcal{A}$-modules acyclic outside a closed subset $Z$, and the right hand sides are the hypercohomology of $M$ with coefficients in the negative cyclic complex of twisted matrices, cf. Definition 3.4.2.

Our construction of the Chern character is more along the lines of [K] than of [BNT1]. It is modified for the twisted case and for the use of twisted matrices. Another difference is a method that we use to pass from perfect to very strictly perfect complexes. This method involves a general construction of operations on cyclic complexes of algebras and categories. This general construction, in partial cases, was used before in [NT], [NTT] as a version of noncommutative calculus. We recently realized that it can be obtained in large generality by applying the functor $\mathbb{C}C^\bullet$ to the categories of $A_\infty$ functors from [BLM], [K1], [Ko], [Lu], and [Ta].

The fact that these methods are applicable is due to the observation that a perfect complex, via the formalism of twisting cochains of O’Brien, Toledo, and Tong, can be naturally interpreted as an $A_\infty$ functor from the category associated to a cover to the category of strictly perfect complexes. The fourth author is grateful to David Nadler for explaining this to him.

In the case when the stack in question is a gerbe, the recipient of the Chern character maps to the De Rham cohomology twisted by the three-cohomology class determined by this gerbe (the Dixmier-Douady class). A Chern character with values in the twisted cohomology was constructed in [MaS], [BCMMS], [AS] and generalized in [MaS1] and [TX]. The $K$-theory which is the source of this Chern character is rather different from the one studied here. It is called the twisted $K$-theory and is a generalization of the topological $K$-theory. Our Chern character has as its source the algebraic $K$-theory which probably maps to the topological one.

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2. Gerbes and stacks

2.1.

Let $M$ be a topological space. In this paper, by a stack on $M$ we will mean an equivalence class of the following data:

1. an open cover $M = \bigcup U_i$;
2. a sheaf of rings $\mathcal{A}_i$ on every $U_i$;
3. an isomorphism of sheaves of rings $G_{ij} : \mathcal{A}_j|_{(U_i \cap U_j)} \cong \mathcal{A}_i|_{(U_i \cap U_j)}$ for every $i, j$;
4. an invertible element $c_{ijk} \in \mathcal{A}_i(U_i \cap U_j \cap U_k)$ for every $i, j, k$ satisfying

$$G_{ij}G_{jk} = \text{Ad}(c_{ijk})G_{ik}$$

(2.1)

such that, for every $i, j, k, l$,

$$c_{ijkl}c_{ijl} = G_{ij}(c_{jkl})c_{ijkl}$$

(2.2)

To define equivalence, first recall the definition of a refinement. An open cover $\mathcal{U} = \{V_j\}_{j \in J}$ is a refinement of an open cover $\Omega = \{U_i\}_{i \in I}$ if a map $f : J \to I$ is given, such that $V_j \subset U_{f(j)}$. Open covers form a category: to say that there is a morphism from $\mathcal{U}$ to $\mathcal{V}$ is the same as to say that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Composition corresponds to composition of maps $f$.

Our equivalence relation is by definition the weakest for which the two data $(\{U_i\}, \mathcal{A}_i, G_{ij}, c_{ij})$ and

$$(\{V_p\}, \mathcal{A}_{f(p)}|_{V_p}, G_{f(p)f(q)}, c_{f(p)f(q)}f(\tau))$$

are equivalent whenever $\{V_p\}$ is a refinement of $\{U_i\}$ (the corresponding map $\{p\} \to \{i\}$ being denoted by $f$).

If two data $(\{U_i\}, \mathcal{A}_i', G_{ij}', c_{ij}')$ and $(\{U''_i\}, \mathcal{A}''_i, G_{ij}'', c_{ij}'')$ are given on $M$, define an isomorphism between them as follows. First, choose an open cover $M = \cup U_i$ refining both $\{U'_i\}$ and $\{U''_i\}$. Pass from our data to new, equivalent data corresponding to this open cover. An isomorphism is an equivalence class of a collection of isomorphisms $H_i : \mathcal{A}_i' \cong \mathcal{A}''_i$ on $U_i$ and invertible elements $b_{ij}$ of $\mathcal{A}_i'(U_i \cap U_j)$ such that

$$G''_{ij} = H_i \text{Ad}(b_{ij})G'_{ij}H_j^{-1}$$

(2.3)

and

$$H^{-1}_i(c''_{ijk}) = b_{ij}G''_{ij}(b_{jk})c'_{ijk}b^{-1}_{ik}$$

(2.4)

If $\{V_p\}$ is a refinement of $\{U_i\}$, we pass from $(\{U_i\}, \mathcal{A}_i, G_{ij}, c_{ij})$ to the equivalent data $(\{V_p\}, \mathcal{A}_{f(p)f(q)}, c_{f(p)f(q)}f(\tau))$ as above. We define the equivalence relation to be the weakest for which, for all refinements, the data $(H_i, b_{ij})$ and $(H_{f(p)}, b'_{f(p)f(q)})$ are equivalent.

Define composition of isomorphisms as follows. Choose a common refinement $\{U_i\}$ of the covers $\{U'_i\}$, $\{U''_i\}$, and $\{U'''_i\}$. Using the equivalence relation, identify all the stack data and all the isomorphism data with the data corresponding to the cover $\{U_i\}$. Define $H_i = H'_i \circ H''_i$ and $b_{ij} = H^{-1}_i(b'_{ij})b''_{ij}$. It is easy to see that this composition is associative and is well defined for equivalence classes.

Now consider two isomorphisms $(H'_i, b'_{ij})$ and $(H''_i, b''_{ij})$ between the stacks $(\{U'_i\}, \mathcal{A}'_i, G_{ij}'', c_{ij}')$ and $(\{U''_i\}, \mathcal{A}''_i, G_{ij}'', c_{ij}'')$. We can pass to a common refinement, replace our data by equivalent data, and assume that $\{U'_i\} = \{U''_i\} = \{U'''_i\}$. A two-morphism between the above isomorphisms is an equivalence class of a collection of invertible elements $a_i$ of $\mathcal{A}'_i(U_i)$ such that $H''_i = H'_i \circ \text{Ad}(a_i)$ and $b''_{ij} = a_i^{-1}b'_{ij}G_{ij}'(a_j)$. The equivalence relation is the weakest for which, whenever $\{V_p\}$ is a refinement of $\{U_i\}$, $\{a_i\}$ is equivalent to $\{a_{f(p)}\} : (H'_{f(p)}, b'_{f(p)f(q)}) \to (H''_{f(p)}, b''_{f(p)f(q)})$. The composition between $\{a'_i\}$ and $\{a''_i\}$ is defined by $a_i = a'_ia''_i$. This operation is well-defined at the level of equivalence classes.
With the operations thus defined, stacks form a two-groupoid.

A gerbe on a manifold $M$ is a stack for which $\mathcal{A}_i = \mathcal{O}_{U_i}$ and $G_{ij} = 1$. Gerbes are classified up to isomorphism by cohomology classes in $H^2(M, \mathcal{O}_M^*)$.

For a stack $\mathcal{A}$ define a twisted $\mathcal{A}$-module over an open subset $U$ as an equivalence class of a collection of sheaves of $\mathcal{A}_i$-modules $\mathcal{M}_i$ on $U \cap U_i$, together with isomorphisms $g_{ij} : \mathcal{M}_j \rightarrow \mathcal{M}_i$ on $U \cap U_i \cap U_j$ such that $g_{ik} = g_{ij} G_{ij}(g_{jk})c_{jk}$ on $U \cap U_i \cap U_j \cap U_k$. The equivalence relation is the weakest for which, if $\{V_p\}$ is a refinement of $\{U_i\}$, the data $(\mathcal{M}_{f(p)}; g_{f(p)f(q)})$ and $(\mathcal{M}_i; g_{ij})$ are equivalent.

We leave it to the reader to define morphisms of twisted modules. A twisted module is said to be free if the $\mathcal{A}_i$-module $\mathcal{M}_i$ is.

### 2.2. Twisting cochains

Here we recall the formalism from [TT], [OTT], [OB], generalized to the case of stacks. For a stack $\mathcal{A}$ on $M$ as above, by $\mathcal{F}$ we will denote a collection $\{\mathcal{F}_i\}$ where $\mathcal{F}_i$ is a graded sheaf which is a direct summand of a free graded $\mathcal{A}_i$-module of finite rank on $U_i$. A $p$-cochain with values in $\mathcal{F}$ is a collection $\alpha_{i_0...i_p} \in \mathcal{F}_{i_0}(U_{i_0} \cap \ldots \cap U_{i_p})$; for two collections $\mathcal{F}$ and $\mathcal{F}'$ as above, a $p$-cochain with values in $\text{Hom}(\mathcal{F}, \mathcal{F}')$ is a collection $\alpha_{i_0...i_p} \in \text{Hom}_{\mathcal{A}_{i_0}}(\mathcal{F}_{i_0}, \mathcal{F}'_{i_0})(U_{i_0} \cap \ldots \cap U_{i_p})$ (the sheaf $\mathcal{A}_{i_0}$ acts on $\mathcal{F}_{i_0}$ via $G_{i_0i_p}$). Define the cup product by

$$ (a \cup b)_{i_0...i_p+q} = (-1)^{|a_{i_0...i_p}|} a_{i_0...i_p} G_{i_{p+1}...i_{p+q}}(b_{i_{p+1}...i_{p+q}}) c_{i_0...i_{p+q}} $$

and the differential by

$$ (\partial a)_{i_0...i_{p+1}} = \sum_{k=1}^{p+1} (-1)^k \alpha_{i_0...\hat{i}_k...i_{p+1}} $$

Under these operations, $\text{Hom}(\mathcal{F}, \mathcal{F})$-valued cochains form a DG algebra and $\mathcal{F}$-valued cochains a DG module.

If $\mathcal{U}$ is a refinement of $\mathcal{A}$ then cochains with respect to $\mathcal{U}$ map to cochains with respect to $\mathcal{A}$. For us, the space of cochains will be always understood as the direct limit over all the covers.

A twisting cochain is a $\text{Hom}(\mathcal{F}, \mathcal{F})$-valued cochain $\rho$ of total degree one such that

$$ \partial \rho + \frac{1}{2} \rho \sim \rho = 0 $$

A morphism between twisting cochains $\rho$ and $\rho'$ is a cochain $f$ of total degree zero such that $\partial f + f' \sim f = f \sim \rho = 0$. A homotopy between two such morphisms $f$ and $f'$ is a cochain $\theta$ of total degree $-1$ such that $f - f' = \partial \theta + \rho' \sim \theta + \theta \sim \rho$. More generally, twisting cochains form a DG category. The complex $\text{Hom}(\rho, \rho')$ is the complex of $\text{Hom}(\mathcal{F}, \mathcal{F}')$-valued cochains with the differential

$$ f \mapsto \partial f + \rho' \sim f - (-1)^{|f|} f \sim \rho. $$

There is another, equivalent definition of twisting cochains. Start with a collection $\mathcal{F} = \{\mathcal{F}_i\}$ of direct summands of free graded twisted modules of finite rank on $U_i$ (a twisted module on $U_i$ is said to be free if the corresponding $\mathcal{A}_i$-module is).
Define Hom(\(\mathcal{F}, \mathcal{F}'\))-valued cochains as collections of morphisms of graded twisted modules \(a_{i_0...i_p}: \mathcal{F}_{i_p} \to \mathcal{F}'_{i_0}\) on \(U_{i_0} \cap \ldots \cap U_{i_p}\). The cup product is defined by

\[
(a \smile b)_{i_0...i_p+q} = (-1)^{|a_{i_0...i_p}|} a_{i_0...i_p} b_{i_{p+1}...i_{p+q}}
\]

and the differential by \((2.6)\). A twisting cochain is a cochain \(\rho\) of total degree 1 satisfying \((2.7)\).

If one drops the requirement that the complexes \(\mathcal{F}\) be direct summands of graded free modules of finite rank, we get objects that we will call weak twisting cochains. A morphism of (weak) twisting cochains is a quasi-isomorphism if \(f_i\) is for every \(i\). Every complex \(\mathcal{M}\) of twisted modules can be viewed as a weak twisting cochain, with \(\mathcal{F}_i = \mathcal{M}\) for all \(i\), \(\rho_{ij} = \text{id}\) for all \(i, j\), \(\rho_i\) is the differential in \(\mathcal{M}\), and \(\rho_{i_0...i_p} = 0\) for \(p > 2\). We denote this weak twisting cochain by \(\rho_0(\mathcal{M})\). By \(\rho_0\) we denote the DG functor \(\mathcal{M} \mapsto \rho_0(\mathcal{M})\) from the DG category of perfect complexes to the DG category of weak twisting cochains.

If \(\{V_i\}\) is a refinement of \(\{U_i\}\), we declare twisting cochains \((\mathcal{F}_i, \rho_{i_0...i_p})\) and \((\mathcal{F}_{(s)}(V_{i_0}, \ldots, V_{i_p}))\) equivalent. Similarly for morphisms.

A complex of twisted modules is called perfect (resp. strictly perfect) if it is locally isomorphic in the derived category (resp. isomorphic) to a direct summand of a bounded complex of finitely generated free modules. A parallel definition can of course be given for complexes of modules over associative algebras.

**Lemma 2.2.1.** Let \(\mathcal{M}\) be paracompact.

1. For a perfect complex \(\mathcal{M}\) there exists a twisting cochain \(\rho\) together with a quasi-isomorphism of weak twisting cochains \(\rho \cong \rho_0(\mathcal{M})\).

2. Let \(f: \mathcal{M}_1 \to \mathcal{M}_2\) be a morphism of perfect complexes. Let \(\rho_i, \phi_i\) be twisting cochains corresponding to \(\mathcal{M}_i\), \(i = 1, 2\). Then there is a morphism of twisting cochains \(\phi(f)\) such that \(\phi_2 \phi(f)\) is homotopic to \(f \phi_1\).

3. More generally, each choice \(\mathcal{M} \mapsto \rho(\mathcal{M})\) extends to an \(A_\infty\) functor \(\rho\) from the DG category of perfect complexes to the DG category of twisting cochains, together with an \(A_\infty\) quasi-isomorphism \(\rho \to \rho_0\). (We recall the definition of \(A_\infty\) functors in [3.7] and that of \(A_\infty\) morphisms of \(A_\infty\) functors in [3.2].)

**Sketch of the proof:** We will use the following facts about complexes of modules over associative algebras.

1) If a complex \(\mathcal{F}\) is strictly perfect, for a quasi-isomorphism \(\psi: \mathcal{M} \to \mathcal{F}\) there is a quasi-isomorphism \(\phi: \mathcal{F} \to \mathcal{M}\) such that \(\psi \circ \phi\) is homotopic to the identity.

2) If \(f: \mathcal{M}_1 \to \mathcal{M}_2\) is a morphism of perfect complexes and \(\phi_i: \mathcal{F}_i \to \mathcal{M}_i\), \(i = 1, 2\), are quasi-isomorphisms with \(\mathcal{F}_i\) strictly perfect, then there is a morphism \(\phi(f): \mathcal{F}_1 \to \mathcal{F}_2\) such that \(\phi_2 \phi(f)\) is homotopic to \(f \phi_1\).

3) If \(\mathcal{F}\) is strictly perfect and \(\phi: \mathcal{F} \to \mathcal{M}\) is a morphism which is zero on cohomology, then \(\phi\) is homotopic to zero.

Let \(\mathcal{M}\) be a perfect complex of twisted modules. Recall that, by our definition, locally, there is a chain of quasi-isomorphisms connecting it to a strictly
perfect complex $F$. Let us start by observing that one can replace that by a quasi-isomorphism from $F$ to $M$. In other words, locally, there is a strictly perfect complex $F$ and a quasi-isomorphism $\phi : F \to M$. Indeed, this is true at the level of germs at every point, by virtue of 1) above. For any point, the images of generators of $F$ under morphisms $\phi$, resp. under homotopies $s$, are germs of sections of $M$, resp. of $F$, which are defined on some common neighborhood. Therefore quasi-isomorphisms and homotopies are themselves defined on these neighborhoods.

We get a cover $\{U_i\}$, strictly perfect complexes $F_i$ with differentials $\rho_i$, and quasi-isomorphisms $\phi_i : F_i \to M$ on $U_i$. Now observe that, at any point of $U_{ij}$, the morphisms $\rho_{ij}$ can be constructed at the level of germs because of 2). As above, we conclude that each of them can be constructed on some common neighborhood. Therefore quasi-isomorphisms and homotopies are themselves defined on these neighborhoods.

Remark 2.2.2. One can assume that all components of a twisting cochain $\rho(M)$, of the $A_\infty$ functor $\rho$, and of the $A_\infty$ morphism of $A_\infty$ functors $\rho \to \rho_0$.

2.3. Twisted matrix algebras

For any $p$-simplex $\sigma$ of the nerve of an open cover $M = \cup U_i$ which corresponds to $U_{i_0} \cap \ldots \cap U_{i_p}$, put $I_\sigma = \{i_0, \ldots, i_p\}$ and $U_\sigma = \cap_{i \in I_\sigma} U_i$. Define the algebra $\text{Matr}^\tau_{tw}(A)$ whose elements are finite matrices

$$\sum_{i,j \in I_\sigma} a_{ij} E_{ij}$$

such that $a_{ij} \in (A_i(U_\sigma))$. The product is defined by

$$a_{ij}E_{ij} \cdot a_{lk}E_{lk} = \delta_{jl}a_{ij}G_{ij}(a_{jk})c_{ijk}E_{ik}$$

For $\sigma \subset \tau$, the inclusion

$$i_{\sigma\tau} : \text{Matr}^\tau_{tw}(A) \to \text{Matr}^\tau_{tw}(A),$$

$$\sum a_{ij}E_{ij} \mapsto \sum (a_{ij}|U_\tau)E_{ij},$$

is a morphism of algebras (not of algebras with unit). Clearly, $i_{\tau\rho}i_{\sigma\tau} = i_{\sigma\rho}$. If $\mathcal{G}$ is a refinement of $\mathfrak{U}$ then there is a map

$$\text{Matr}^\sigma_{tw}(A) \to \text{Matr}^{f(\sigma)}_{tw}(A)$$

which sends $\sum a_{ij}E_{ij}$ to $\sum (a_{f(i)f(j)}|V_{f(\sigma)})E_{f(i)f(j)}$. 

Remark 2.3.1. For a nondecreasing map $f : I_\sigma \rightarrow I_\tau$ which is not necessarily an inclusion, we have the bimodule $M_f$ consisting of twisted $|I_\sigma| \times |I_\tau|$ matrices. Tensoring by this bimodule defines the functor 

$$f_* : \text{Matr}_{tw}^\tau(A) - \text{mod} \rightarrow \text{Matr}_{tw}^\tau(A) - \text{mod}$$

such that $(fg)_* = f_*g_*$. 

3. The Chern character

3.1. Hochschild and cyclic complexes

We start by recalling some facts and constructions from noncommutative geometry. Let $A$ be an associative unital algebra over a unital algebra $k$. Set 

$$C_p(A, A) = C_p(A) = A \otimes (p+1).$$

We denote by $b : C_p(A) \rightarrow C_{p-1}(A)$ and $B : C_p(A) \rightarrow C_{p+1}(A)$ the standard differentials from the Hochschild and cyclic homology theory (cf. [C], [L], [T]). The Hochschild chain complex is by definition $(C_\bullet(A), b)$; define 

$$CC_\bullet(A) = (C_\bullet(A)[[u]], b + uB);$$

$$CC_{\text{per}}_\bullet(A) = (C_\bullet(A)[[u, u^{-1}], b + uB);$$

$$CC_\bullet(A) = (C_\bullet(A)[[u, u^{-1}]/uC_\bullet(A)[[u]], b + uB).$$

These are, respectively, the negative cyclic, the periodic cyclic, and the cyclic complexes of $A$ over $k$.

We can replace $A$ by a small DG category or, more generally, by a small $A_\infty$ category. Recall that a small $A_\infty$ category consists of a set $\text{Ob}(C)$ of objects and a graded $k$-module of $C(i, j)$ of morphisms for any two objects $i$ and $j$, together with compositions 

$$m_n : C(i_n, i_{n-1}) \otimes \ldots \otimes C(i_1, i_0) \rightarrow C(i_n, i_0)$$

of degree $2 - n$, $n \geq 1$, satisfying standard quadratic relations to which we refer as the $A_\infty$ relations. In particular, $m_1$ is a differential on $C(i, j)$. An $A_\infty$ functor $F$ between two small $A_\infty$ categories $C$ and $D$ consists of a map $F : \text{Ob}(C) \rightarrow \text{Ob}(D)$ and $k$-linear maps 

$$F_n : C(i_n, i_{n-1}) \otimes \ldots \otimes C(i_1, i_0) \rightarrow D(Fi_n, Fi_0)$$

of degree $1 - n$, $n \geq 1$, satisfying another standard relation. We refer the reader to [K] for formulas and their explanations.

For a small $A_\infty$ category $C$ one defines the Hochschild complex $C_\bullet(C)$ as follows: 

$$C_\bullet(C) = \bigoplus_{i_0, \ldots, i_n \in \text{Ob}(C)} C(i_1, i_0) \otimes C(i_2, i_1) \otimes \ldots \otimes C(i_n, i_{n-1}) \otimes C(i_0, i_n)$$
(the total cohomological degree being the degree induced from the grading of $C(i, j)$ minus $n$). The differential $b$ is defined by
\[ b(f_0 \otimes \ldots \otimes f_n) = \sum_{j,k} \pm m_k(f_{n-j+1}, \ldots, f_0, \ldots, f_{k-1-j}) \otimes f_k \otimes \ldots \otimes f_{n-j} \]
\[ + \sum_{j,k} \pm f_0 \otimes \ldots \otimes f_j \otimes m_k(f_{j+1}, \ldots, f_{j+k}) \otimes \ldots \otimes f_n \]
The cyclic differential $B$ is defined by the standard formula with appropriate signs; cf. [G].

3.2. Categories of $A_\infty$ functors
For two $DG$ categories $\mathcal{C}$ and $\mathcal{D}$ one can define the $DG$ category $\text{Fun}_\infty(\mathcal{C}, \mathcal{D})$. Objects of $\text{Fun}_\infty(\mathcal{C}, \mathcal{D})$ are $A_\infty$ functors $\mathcal{C} \rightarrow \mathcal{D}$. The complex $\text{Fun}_\infty(\mathcal{C}, \mathcal{D})(F, G)$ of morphisms from $F$ to $G$ is the Hochschild cochain complex of $\mathcal{C}$ with coefficients in $\mathcal{D}$ viewed as an $A_\infty$ bimodule over $\mathcal{C}$ via the $A_\infty$ functors $F$ and $G$, namely
\[ \prod_{i_0, \ldots, i_n \in \text{Ob}(\mathcal{C})} \text{Hom}(\mathcal{C}(i_0, i_1) \otimes \ldots \otimes \mathcal{C}(i_n-1, i_n), \mathcal{D}(Fi_0, Gi_n)) \]
The $DG$ category structure on $\text{Fun}_\infty(\mathcal{C}, \mathcal{D})$ comes from the cup product. More generally, for two $A_\infty$ categories $\mathcal{C}$ and $\mathcal{D}$, $\text{Fun}_\infty(\mathcal{C}, \mathcal{D})$ is an $A_\infty$ category. For a conceptual explanation, as well as explicit formulas for the differential and composition, cf. [Lu], [BLM], [K1].

Furthermore, for $DG$ categories $\mathcal{C}$ and $\mathcal{D}$ there are $A_\infty$ morphisms
\[ \mathcal{C} \otimes \text{Fun}_\infty(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D} \] (3.1)
(the action) and
\[ \text{Fun}_\infty(\mathcal{D}, \mathcal{E}) \otimes \text{Fun}_\infty(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}_\infty(\mathcal{C}, \mathcal{E}) \] (3.2)
(the composition). This follows from the conceptual explanation cited below; in fact these pairing were considered already in [Ko]. As a consequence, there are pairings
\[ CC^\bullet(\mathcal{C}) \otimes CC^\bullet(\text{Fun}_\infty(\mathcal{C}, \mathcal{D})) \rightarrow CC^\bullet(\mathcal{D}) \] (3.3)
and
\[ CC^\bullet(\text{Fun}_\infty(\mathcal{D}, \mathcal{E})) \otimes CC^\bullet(\text{Fun}_\infty(\mathcal{C}, \mathcal{D})) \rightarrow CC^\bullet(\text{Fun}_\infty(\mathcal{C}, \mathcal{E})) \] (3.4)
Recall that Getzler and Jones constructed an explicit $A_\infty$ structure on the negative cyclic complex of an associative commutative algebra. The formulas involve the shuffle product and higher cyclic shuffle products; cf. [GJ], [L]. When the algebra is not commutative, the same formulas may be written, but they do not satisfy the correct identities. One can, however, define external Getzler-Jones products for algebras and, more generally, for $DG$ categories by the same formulas. One gets maps
\[ CC^\bullet(\mathcal{C}_1) \otimes \ldots \otimes CC^\bullet(\mathcal{C}_n) \rightarrow CC^\bullet(\mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n)[2-n] \]
which satisfy the usual $A_\infty$ identities. To get (3.3) and (3.4), one combines these products with (3.1) and (3.2).
Example 3.2.1. Let $F$ be an $A_{\infty}$ functor from $C$ to $D$. Then $\text{id}_F$ is a chain of $CC^{-}(\text{Fun}_{\infty}(C,D))$ (with $n=0$). The pairing (3.3) with this chain amounts to the map of the negative cyclic complexes induced by the $A_{\infty}$ functor $F$:

$$f_0 \otimes \ldots \otimes f_n \mapsto \sum \pm F_{k_0}(\ldots f_0 \ldots) \otimes F_{k_1}(\ldots) \otimes \ldots \otimes F_{k_m}(\ldots)$$

The sum is taken over all cyclic permutations of $f_0, \ldots, f_n$ and all $m, k_0, \ldots, k_m$ such that $f_0$ is inside $F_{k_0}$.

Remark 3.2.2. The action (3.1) and the composition (3.1) are parts of a very nontrivial structure that was studied in [Ta].

As a consequence, this gives an $A_{\infty}$ category structure $CC^{-}(\text{Fun}_{\infty})$ whose objects are $A_{\infty}$ categories and whose complexes of morphisms are negative cyclic complexes $CC^{-}(\text{Fun}_{\infty}(D,E))$.

From a less conceptual point of view, pairings (3.3) and (3.4) were defined, in partial cases, in [NT1] and [NT]. The $A_{\infty}$ structure on $CC^{-}(\text{Fun}_{\infty})$ was constructed (in the partial case when all $f$ are identity functors) in [TT]. Cf. also [T1] for detailed proofs.

3.3. The prefibered version

We need the following modification of the above constructions. Let $\mathcal{B}$ be a category. Consider, instead of a single DG category $D$, a family of DG categories $D_i, i \in \text{Ob}(\mathcal{B})$, together with a family of DG functors $f^*: D_i \leftarrow D_j, f \in \mathcal{B}(i,j)$, satisfying $(fg)^* = g^*f^*$ for any $f$ and $g$. In this case we define a new DG category $D$:

$$\text{Ob}(D) = \coprod_{i \in \text{Ob}(\mathcal{B})} \text{Ob}(D_i)$$

and, for $a \in \text{Ob}(D_i), b \in \text{Ob}(D_j)$,

$$D(a,b) = \oplus_{f \in \mathcal{B}(i,j)} D_i(a,f^*b).$$

The composition is defined by

$$(\varphi,f) \circ (\psi,g) = (\varphi \circ f^*\psi, f \circ g)$$

for $\varphi \in D_i(a,f^*b)$ and $\psi \in D_j(b,g^*c)$.

We call the DG category $D$ a DG category over $\mathcal{B}$, or, using the language of [Gl], a prefibered DG category over $\mathcal{B}$ with a strict cleavage. There is a similar construction for $A_{\infty}$ categories.

Let $\mathcal{C}, \mathcal{D}$ be two DG categories over $\mathcal{B}$. An $A_{\infty}$ functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an $A_{\infty}$ functor over $\mathcal{B}$ if for any $a \in \text{Ob}(\mathcal{C})$ $Fa \in \text{Ob}(\mathcal{D})$, and for any $a_k \in \text{Ob}(\mathcal{C}_{ik})$,

$$(\varphi_k, f_k) \in C(a_k, a_{k-1}), k = 1, \ldots, n,$$

$$F_n((\varphi_n, f_n), \ldots, (\varphi_1, f_1)) = (\psi, f_1 \ldots f_n)$$

for some $\psi \in D_{jn}$. One defines a morphism over $\mathcal{B}$ of two $A_{\infty}$ functors over $\mathcal{B}$ by imposing a restriction which is identical to the one above. We get a DG category $\text{Fun}_{\infty}^B(\mathcal{C}, \mathcal{D})$. As in the previous section, there are $A_{\infty}$ functors

$$\mathcal{C} \otimes \text{Fun}_{\infty}^B(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$$

(3.5)
(the action) and
\[
\text{Fun}_\infty^B(D, \mathcal{E}) \otimes \text{Fun}_\infty^B(C, D) \to \text{Fun}_\infty^B(C, \mathcal{E}) \tag{3.6}
\]
(the composition), as well as
\[
\mathbb{C}C^{-\bullet}(C) \otimes \mathbb{C}C^{-\bullet}(\text{Fun}_\infty^B(C, D)) \to \mathbb{C}C^{-\bullet}(D) \tag{3.7}
\]
and
\[
\mathbb{C}C^{-\bullet}(\text{Fun}_\infty^B(D, \mathcal{E})) \otimes \mathbb{C}C^{-\bullet}(\text{Fun}_\infty^B(C, D)) \to \mathbb{C}C^{-\bullet}(\text{Fun}_\infty^B(C, \mathcal{E})) \tag{3.8}
\]

3.3.1. We need one more generalization of the above constructions. It is not necessary if one adopts the convention from Remark 2.2.2.

Suppose that instead of \(B\) we have a diagram of categories indexed by a category \(U\) (in other words, a functor from \(U\) to the category of categories. In our applications, \(U\) will be the category of open covers). Instead of a \(B\)-category \(D\) we will consider a family of \(B_u\)-categories \(D_u\), \(u \in \text{Ob}(U)\), together with a functor \(D_v \to D_u\) for any morphism \(u \to v\) in \(U\), subject to compatibility conditions that are left to the reader. The inverse limit of categories \(\lim\leftarrow U D_u\) is then a category \(\mathbb{C}C^{-\bullet}(\text{Fun}_\infty^B(C, D)) \to \lim\leftarrow U \text{Hom}(\mathbb{C}C^{-\bullet}(C_u), \mathbb{C}C^{-\bullet}(D_v)) \tag{3.9}
\]

3.4. The trace map for stacks

3.4.1. From perfect to very strictly perfect complexes. Let \(M\) be a space with a stack \(\mathcal{A}\). Consider an open cover \(\mathcal{U} = \{U_i\}_{i \in I}\) such that the stack \(\mathcal{A}\) can be represented by a datum \(\mathcal{A}_i, G_{ij}, c_{ijk}\). Let \(\mathcal{B}_U\) be the category whose set of objects is \(I\) and where for every two objects \(i\) and \(j\) there is exactly one morphism \(f : i \to j\). Put \(C_\mathcal{U} = k[\mathcal{B}_U]\), i.e. \((C_\mathcal{U})_i = k\) for any object \(i\) of \(\mathcal{B}_U\).

There is a standard isomorphism of the stack \(\mathcal{A}|U_i\) with the trivial stack associated to the sheaf of rings \(\mathcal{A}_i\). Therefore one can identify twisted modules on \(U_i\) with sheaves of \(\mathcal{A}_i\)-modules. We will denote the twisted module corresponding to the free module \(\mathcal{A}_i\) by the same letter \(\mathcal{A}_i\).

**Definition 3.4.1.** Define the category of very strictly perfect complexes on any open subset of \(U_i\) as follows. Its objects are pairs \((e, d)\) where \(e\) is an idempotent endomorphism of degree zero of a free graded module \(\sum_{a=1}^{N} \mathcal{A}_i[n_a]\) and \(d\) is a differential on \(\text{Im}(e)\). Morphisms between \((e_1, d_1)\) and \((e_2, d_2)\) are the same as morphisms between \(\text{Im}(e_1)\) and \(\text{Im}(e_2)\) in the DG category of complexes of modules.
A parallel definition can be given for the category of complexes of modules over an associative algebra.

Let \((\mathcal{D}_U)_i\) be the category of very strictly perfect complexes of twisted \(A\)-modules on \(U_i\). By \(U\) we denote the category of open covers as above.

Strictly speaking, our situation is not exactly a partial case of what was considered in \([3.3]\). First, \((\mathcal{D}_U)_i\) is a presheaf of categories on \(U_i\) (in the most naive sense, i.e. it consists of a category \((\mathcal{D}_U)_i(U)\) for any \(U\) open in \(U_i\), and a functor \(G_{UV} : (\mathcal{D}_U)_i(V) \to (\mathcal{D}_U)_i(U)\) for any \(U \subset V\), such that \(G_{UV}G_{VW} = G_{UW}\)). Second, \(f^*\) are defined as functors on the subset \(U_i \cap U_j\). Also, the pairing \([3.7]\) and its generalization \([3.9]\) are defined in a slightly restricted sense: they put in correspondence to a cyclic chain \(i_0 \to i_1 \to i_2 \to \ldots \to i_0\) a cyclic chain of the category of very strictly perfect complexes of \(A\)-modules on \(U_{i_0} \cap \ldots \cap U_{i_n}\). Finally, in the notation of \([3.3.1]\) for a morphism \(f : \mathcal{U} \to \mathcal{V}\) in \(U\) and an object \(j\) of \(I_{\mathcal{V}}\), the functor \((\mathcal{D}_\mathcal{V})_j \to (\mathcal{D}_\mathcal{U})_j\) induced by \(f\) is defined only on the open subset \(V_j\).

We put \(\mathcal{B} = \varinjlim \mathcal{B}_U\) and \(\mathcal{D} = \varinjlim \mathcal{D}_U\).

Let \(\text{Perf}(A)\) be the DG category of perfect complexes of twisted \(A\)-modules on \(M\). We denote the sheaf of categories of very strictly perfect complexes on \(M\) by \(\text{Perf}^{\text{str}}(A)\). If \(Z\) is a closed subset of \(M\) then by \(\text{Perf}_Z(A)\) we denote the DG category of perfect complexes of twisted \(A\)-modules on \(M\) which are acyclic outside \(Z\).

**Definition 3.4.2.** Define

\[
\hat{\mathcal{C}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A))) = \lim_{\mathcal{U}} \prod_{\sigma_i \subset \sigma_1 \subset \ldots \subset \sigma_p} \text{CC}^{-\bullet}(\text{Matr}_{\tw}^{\sigma_p}(A))
\]

where \(\sigma_i\) run through simplices of \(\mathcal{U}\). The total differential is \(b + uB + \hat{\partial}\) where

\[
\hat{\partial}s_{\sigma_0} \ldots s_{\sigma_p} = \sum_{k=0}^{p-1} (-1)^k s_{\sigma_0} \ldots \hat{s}_{k} \ldots s_{\sigma_p} + (-1)^p s_{\sigma_0} \ldots s_{\sigma_{p-1}} |U_{\sigma_p}|
\]

For a closed subset \(Z\) of \(M\) define \(\hat{\mathcal{C}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A)))\) as

\[
\text{Cone}(\hat{\mathcal{C}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A)))) \to \hat{\mathcal{C}}^{-\bullet}(M \setminus Z, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A)))[-1].
\]

Let us construct natural morphisms

\[
\text{CC}^{-\bullet}(\text{Perf}(A)) \to \hat{\mathcal{C}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A))) \tag{3.10}
\]

\[
\text{CC}^{-\bullet}(\text{Perf}_Z(A)) \to \hat{\mathcal{C}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\tw}(A))) \tag{3.11}
\]

First, observe that the definition of a twisted cochain and Lemma \([2.4.1]\) can be reformulated as follows.

**Lemma 3.4.3.** 1. A twisting cochain is an \(A_{\infty}\) functor \(\mathcal{C} \to \mathcal{D}\) over \(\mathcal{B}\) in the sense of \([3.3]\).

2. There is an \(A_{\infty}\) functor from the DG category of perfect complexes to the DG category \(\text{Fun}_{A_{\infty}}(\mathcal{C}, \mathcal{D})\).
The second part of the Lemma together with (3.7) give morphisms
\[ \text{CC}^{-}_\bullet(\text{Perf}(A)) \to \text{CC}^{-}_\bullet(\text{Fun}^{\mathbb{R}}(\mathcal{C}, \mathcal{D})) \to \text{Hom}(\text{CC}^{-}_\bullet(\mathcal{C}), \text{CC}^{-}_\bullet(\mathcal{D})) \].

As mentioned above, the image of this map is the subcomplex of those morphisms that put in correspondence to a cyclic chain \( i_0 \to i_1 \to i_{n-1} \to \ldots \to i_0 \) a cyclic chain of the category of very strictly perfect complexes of \( \mathcal{A} \)-modules on \( U_{i_0} \cap \ldots \cap U_{i_n} \). We therefore get a morphism
\[ \text{CC}^{-}_\bullet(\text{Perf}(A)) \to \tilde{C}^{-\bullet}(M, \text{CC}^{-}_\bullet(\text{Perf}^{\text{str}}(A))) \]
Now replace the right hand side by the quasi-isomorphic complex
\[ \lim_{\mathcal{U}} \prod_{\sigma \subset \sigma_1 \subset \ldots \subset \sigma_p} \text{CC}^{-}_\bullet(\text{Perf}^{\text{str}}(A(U_{\sigma_p}))) \]
where \( \sigma_i \) run through simplices of \( \mathcal{U} \). There is a natural functor
\[ \text{Perf}^{\text{str}}(A(U_{\sigma_p})) \to \text{Perf}^{\text{str}}(\text{Matr}^{\sigma_p}_{\text{tw}}(A)) \]
where the right hand side stands for the category of very strictly perfect complexes of modules over the sheaf of rings \( \text{Matr}^{\sigma_p}_{\text{tw}}(A) \) on \( U_{\sigma_p} \). This functor acts as follows: to a twisted module \( \mathcal{M} \) it puts in correspondence the direct sum \( \oplus_{i \in I_{\sigma_p}} \mathcal{M}_i \); an element \( a_{ij}E_{ij} \) acts via \( a_{ij}g_{ij} \).

3.4.2. From the homology of very strictly perfect complexes to the homology of the algebra. Next, let us note that one can replace \( \text{CC}^{-}_\bullet(\text{Perf}^{\text{str}}(\text{Matr}^{\sigma_p}_{\text{tw}}(A))) \) by the complex \( \text{CC}^{-}_\bullet(\text{Matr}^{\sigma_p}_{\text{tw}}(A)) \); indeed, for any associative algebra \( A \) there is an explicit trace map
\[ \text{CC}^{-}_\bullet(\text{Perf}^{\text{str}}(A)) \to \text{CC}^{-}_\bullet(\text{Proj}(A)) \to \text{CC}^{-}_\bullet(\text{Free}(A)) \to \text{CC}^{-}_\bullet(A) \] (3.12)
Our construction of the trace map can be regarded as a modification of Keller’s argument from [K]. First, recall from (3.2) the internal Getzler-Jones products. The binary product will be denoted by \( \times \). We define the map (3.12) as a composition
\[ \text{CC}^{-}_\bullet(\text{Perf}^{\text{str}}(A)) \to \text{CC}^{-}_\bullet(\text{Proj}(A)) \to \text{CC}^{-}_\bullet(\text{Free}(A)) \to \text{CC}^{-}_\bullet(A); \]
the second DG category is the subcategory of complexes with zero differential; the third is the subcategory of complexes of free modules with zero differential. The morphism on the left is the exponential of the operator \( -(1 \otimes d) \times ? \) opposite to the operator of binary product with the one-chain \( 1 \otimes d \). The morphism in the middle is \( \text{ch}(e) \times ? \), the operator of binary multiplication by the Connes-Karoubi Chern character of an idempotent \( e \), cf. [L]. The morphism on the right is the standard trace map from the chain complex of matrices over an algebra to the chain complex of the algebra itself [L].

Let us explain in which sense do we apply the Getzler-Jones product. To multiply \( f_0 \otimes \ldots \otimes f_n \) by \( \text{ch}(e) \), recall that \( f_k : \mathcal{F}_{i_k} \to \mathcal{F}_{i_{k-1}}, \mathcal{F}_{i_k} \) are free of finite rank, \( e_k^2 = e_k \in \text{Hom}(\mathcal{F}_{i_k}, \mathcal{F}_{i_k}), \mathcal{F}_{i_{k-1}} = \mathcal{F}_{i_{k}}, e_{-1} = e_{n}, \) and \( f_k e_k = e_{k-1} f_k \). Write the usual formula for multiplication by \( \text{ch}(e) \), but, when a factor \( e \) stands between \( f_i \) and \( f_{i+1} \), replace this factor by \( e_i \). Similarly for the morphism on the left: if a
factor $d$ stands between $f_i$ and $f_{i+1}$, replace this factor by $d_i$ (the differential on the $i$th module). This finishes the construction of the morphism (3.10).

Next, we need to refine the map (3.12) as follows. Recall that for a DG category $D$ and for a full DG subcategory $D_0$ the DG quotient of $D$ by $D_0$ is the following DG category. It has same objects as $D$; its morphisms are freely generated over $D$ by morphisms $\epsilon_i$ of degree $-1$ for any $i \in \text{Ob}(D_0)$, subject to $d\epsilon_i = \text{id}_i$. It is easy to see that the trace map (3.12) extends to the negative cyclic complex of the Drinfeld quotient of $\text{Perf}^{\text{vstr}}(A)$ by the full DG subcategory of acyclic complexes. Indeed, a morphism in the DG quotient is a linear combination of monomials $f_0\epsilon_{i_0}f_1\epsilon_{i_1}...\epsilon_{i_{n-1}}f_n$ where $f_k : F_{i_k} \rightarrow F_{i_{k-1}}$ and $F_{i_k}$ are acyclic for $k = 0, \ldots, n - 1$. An acyclic very strictly perfect complex is contractible. Choose contracting homotopies $s_k$ for $F_{i_k}$. Replace all the monomials $f_0\epsilon_{i_0}f_1\epsilon_{i_1}...\epsilon_{i_{n-1}}f_n$ by $f_0s_0f_1s_1...s_{n-1}f_n$. Then apply the above composition to the resulting chain of $\text{CC}^{-\bullet}(\text{Perf}^{\text{vstr}}(A))$. We obtain for any associative algebra $A$

$$\text{CC}^{-\bullet}(\text{Perf}^{\text{vstr}}(A)_{\text{Loc}}) \rightarrow \text{CC}^{-\bullet}(A)$$

(3.13)

where $\text{Loc}$ stands for the Drinfeld localization with respect to the full subcategory of acyclic complexes.

To construct the Chern character with supports, act as above but define $D_i$ to be the Drinfeld quotient of the DG category $\text{Perf}^{\text{vstr}}(A(U_i))$ by the full subcategory of acyclic complexes. We get a morphism

$$\text{CC}^{-\bullet}(\text{Perf}(A)) \rightarrow \tilde{\mathbb{H}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Perf}^{\text{vstr}}(A)_{\text{Loc}})) \rightarrow \tilde{\mathbb{H}}^{-\bullet}(M, \text{CC}^{-\bullet}(A))$$

From this, and from the fact that the negative cyclic complex of the localization of $\text{Perf}_Z$ is canonically contractible outside $Z$, one gets easily the map (3.11).

### 3.5. Chern character for stacks

Now let us construct the Chern character

$$K^{-\bullet}(\text{Perf}(A)) \rightarrow \tilde{\mathbb{H}}^{-\bullet}(M, \text{CC}^{-\bullet}(\text{Matr}_{\text{tw}}(A)))$$

(3.14)

$$K^{-\bullet}(\text{Perf}_Z(A)) \rightarrow \tilde{\mathbb{H}}^{-\bullet}_Z(M, \text{CC}^{-\bullet}(\text{Matr}_{\text{tw}}(A)))$$

(3.15)

First, note that the $K$ theory in the left hand side can be defined as in [TV]; one can easily deduce from [MC] and [K2], section 1, the Chern character from $K^{-\bullet}(\text{Perf}(A))$ to the homology of the complex Cone($\text{CC}^{-\bullet}(\text{Perf}_{\text{ac}}(A)) \rightarrow \text{CC}^{-\bullet}(\text{Perf}(A))$). Here $\text{Perf}_{\text{ac}}$ stands for the category of acyclic perfect complexes.

Compose this Chern character with the trace map of 3.4. We get a Chern character from $K^{-\bullet}(\text{Perf}(A))$ to

$$\tilde{\mathbb{H}}^{-\bullet}(M, \text{Cone}(\text{CC}^{-\bullet}(\text{Perf}_{\text{ac}}^{\text{vstr}}(A)_{\text{Loc}}) \rightarrow \text{CC}^{-\bullet}(\text{Perf}^{\text{vstr}}(A)_{\text{Loc}})))$$

One gets the Chern characters (3.14), (3.15) easily by combining the above with (3.13).
3.6. The case of a gerbe

If $\mathcal{A}$ is a gerbe on $M$ corresponding to a class $c$ in $H^3(M, \mathcal{O}^*_M)$, then (in the $C^\infty$ case) the right hand side of (3.14) is the cohomology of $M$ with coefficients in the complex of sheaves

$$\Omega^{-\bullet}[[u]], ud_{\text{DR}} + u^2H \wedge$$

where $H$ is a closed three-form representing the three-class of the gerbe. In the holomorphic case, the right hand side of (3.14) is computed by the complex $\Omega^{-\bullet\bullet}[[u]], \partial + \alpha \wedge + u\partial$ where $\alpha$ is a closed $(2,1)$ form representing the cohomology class $\partial \log c$. This can be shown along the lines of [BGNT], Theorem 7.1.2.

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