Notes on gravitational wave detection

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Abstract

We propose a novel approach to detect gravitational waves. It relies on simple ideas and allows us to redesign the interferometer setup shortening the Fabry-Perot cavity. As a consequence the number of bounces could be increased and the signal enhanced.
1 Motivation

Gravitational dynamics has been quite quantitatively well tested in the weak-field approximation, i.e. perturbatively close to flat space \([1]\). Due to the steadily increasing number of gravitational wave observation from coalescing binaries \([2]\) new techniques to tackle the strong field regimen are also been developed \([3]\). Besides these efforts and achievements there are some fundamental issues that are left aside in these developments \([4]\). Although gravitational waves is a relativistic effect, its detection could be understood as non-relativistic, depending on the way we measure distances, essentially whether or not we use light.

In the detection zone, we can find a vacuum solution \(\phi\) from the non-relativistic Poisson equation \(\Delta \phi = 4\pi G \rho\) in the form

\[
\phi = \phi_x(t) x y + \phi_+(t) (x^2 - y^2). \tag{1}
\]

From the equation of motion and given the initial data \((x_0, y_0, z_0 = 0)\) test particles trajectories take the form

\[
\begin{align*}
x &= x_0 + \frac{1}{2} (h_+(t)x_0 + h_x(t)y_0), \\
y &= y_0 + \frac{1}{2} (h_x(t)x_0 - h_+(t)y_0), \\
z &= 0.
\end{align*} \tag{2}
\]

The above expression, \((2)\), is just identical to that derived in GR within the linear approximation in the metric perturbation, restricted to zero order in \(c^{-1}\) and taking \(t\) as the proper time at origin \([5]\). It takes into account how a free test mass is affected when gravitational waves interact with it and is precisely what is actually measured. If this is so, to what extent are we not merely measuring Newtonian effects on test particles? \([6]\). In fact, nowadays experiments make use of light interferometry. Is this – the presence of light– the reason why the detection of gravitational waves should be considered relativistic? In this note we will try to clarify these questions.

To begin with in section \(\S\) we briefly explain two alternatives frames for GR: the Gaussian and the Fermi. In section \(\S\) we express the linear gravitational wave to be detected in a suitable way accordingly to the accuracy of the detection and the possible design of the detection device. In section \(\S\) we find the observable time delay \(\delta t\) of bouncing photons between mirrors caused by the gravitational wave. This time delay is calculated in relation to two possible designs of the detection device. Finally, in sections \(\S\) and \(\S\) we analyze the results with respect to the optimal number of round trips.
2 General Relativity in two frames

Due to general covariance, the metric in a general space-time can be parametrized in terms of ten potentials. Out of these only six are independent once coordinate transformations are taken into account. We will highlight two formulations that directly use these:

The Gaussian frame: The widest known form of GR is the Gaussian formulation. We can always describe the proper time and space coordinates of a given congruence of free particles as \( \{T, X^i = \text{constant}\} \). Within this coordinate system the metric can be cast as

\[
dT^2 = dT^2 - \frac{1}{c^2} g_{ij} dX^i dX^j ,
\]

where \( g_{ij} \) contains the six independent potentials. Thus an observer can measure the motion of any particle referring to the given congruence of moving clocks. An equivalent Newtonian formulation can be obtained considering \( c \to \infty \) over the test particle and field equations derived from \([3, 7]\).

The Fermi frame: As Fermi showed, along any timelike geodesic, we can choose a set of coordinates such that the metric coincides with Minkowski up to quadratic order in the space coordinates \( x^i \). There are many exact space-time coordinate systems that led to the above Fermi condition \([8]\). We shall chose those relative to a geodesic, with proper time and position given by \( (\lambda, 0) \) that belong to a geodesic congruence with proper time and velocity field \( \{\tau(\lambda, x^i) = \lambda + \mathcal{O}(x^2), V^i(\lambda, x^i)\} \). Under these conditions it is possible to write the metric as

\[
dT^2 = \Phi^2 d\lambda^2 - \frac{1}{c^2} \left( 2 K_i d\lambda \, d\sigma(i, j) + \gamma_{ij} dX^i dX^j \right) , \quad \gamma_{ij} := \frac{1}{\Phi^2} \delta_{ij} - \sigma, i \sigma, j ,
\]

being \((t, x^i)\) the Fermi coordinates with \( t = \tau(\lambda, x^i)\)\([9]\). The six independent potentials are \( \Phi, K_i, \mathcal{H} \) and \( \sigma \). Notice that \([4]\) is covariant under space transformations that leave shape invariant the space slices \((d\lambda = 0) \gamma_{ij} \). As above the standard Newtonian formulation can be obtained considering \( c \to \infty \).

The rigid covariant form of the metric \([4]\) and the fact that the fulfillment of the Fermi condition only requires a change of time, \( t = \tau(\lambda, x^i) \), seems to point out that, under mild external perturbations of low frequencies, the points of a real (elastic) body can be described, with sufficient accuracy, using the coordinates \( x^i \), being \( x^i = \text{constant} \) the ones corresponding to a perfect rigid body. In section \( \S 4 \) for simplicity reasons, we will use a perfect rigid body, so \( x^i = \text{constant} \), but essentially the same reasoning will work using the Newtonian motion,
$x^i(t)$, of a sufficiently rigid body.

3 Gravitational waves in the detection zone

As an illustration of our proposal we apply it to the local detection of gravitational waves. There is a remarkable fact in this subject: the time delay of laser interferometers to gravitational waves has been calculated in the transverse-traceless, the so called Gaussian coordinate system, and in the local Lorentz gauge, which agrees with the Fermi condition, obtaining a mismatch in the result. In practice, this issue is rarely addressed in the literature [10].

We calculate the photon time delay, caused by the plus mode, $h_+ = h$, of an $h$-linear plane wave along $Z$, between two mirrors. Taking Gaussian coordinates (3), with $X^i = \{X,Y,Z\}$, leads to

$$dT^2 = dT^2 - \frac{1}{c^2} \left( \{1 + h(T - \frac{Z}{c})\} dX^2 + \{1 - h(T - \frac{Z}{c})\} dY^2 + dZ^2 \right).$$

(5)

To calculate the leading contributions to the time delay it is sufficient to consider (5) at order $c^{-2}$. We shall not take into account the orders coming through the source but those due to the propagation and therefore $h = h(T)$. On the other hand for this purpose only the Newtonian order of space coordinates for particles is needed.\(^1\)

The metric (5) takes the form (4) changing spatial coordinates to

$$X = x - \frac{1}{2} h(t)x, \quad Y = y + \frac{1}{2} h(t)y,$$

(6)

while the Fermi condition is meet after identifying the time coordinate $t$

$$T = t - \frac{1}{4c^2} \dot{h}(t) (x^2 - y^2),$$

(7)

where dot stands for the derivatives with respect to the argument. The explicit form of the metric after these changes, at order $c^{-2}$, is

$$dT^2 = \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \frac{1}{c^2} (dx^2 + dy^2 + dz^2),$$

(8)

with $\phi = -\frac{1}{4} \ddot{h}(x^2 - y^2)$. Notice that $t$ is the proper time at the origin $x^i = 0$, and can be identified with the Gaussian time $T$ at the same point. At Newtonian order the space

\(^1\)By Newtonian order we means zero order in $c^{-1}$. 

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coordinates for particle geodesics at rest when \( h = 0 \) are, from (6),
\[
x = x_0 + \frac{1}{2} h(t)x_0, \quad y = y_0 - \frac{1}{2} h(t)y_0, \quad z = z_0.
\]

4 Effects of gravitational waves on the time delay

To simplify the setup we shall consider a \((1 + 1)\) spacetime. We shall use a Gaussian \((T, X)\) and a Fermi \((t, x)\) coordinate systems, (5) and (8) respectively, both denoted generically by \((\lambda, z)\). We are after the round trip time delay of a photon traveling between two mirrors. In the forward direction \( z_+ \) (backward \( z_- \)) from mirror \( A \) located at \( z_A \) to a mirror \( B \) at \( z_B > z_A \), \(( z_B \) to \( z_A \)). The boundary conditions to be fulfilled for the trajectories \( z_{\pm} \) are

\[
\begin{align*}
\text{forward:} & \quad \begin{cases} z_+(\lambda) = z_A(\lambda), \\
z_+(\lambda + \Delta_+ \lambda) = z_B(\lambda + \Delta_+ \lambda).
\end{cases} \\
\text{backward:} & \quad \begin{cases} z_-(\lambda + \Delta_- \lambda) = z_B(\lambda + \Delta_+ \lambda), \\
z_-(\lambda + \Delta_+ \lambda + \Delta_- \lambda) = z_A(\lambda + \Delta_+ \lambda + \Delta_- \lambda),
\end{cases}
\end{align*}
\]

which gives the time delay
\[
\delta(\lambda) = \Delta_+ \lambda + \Delta_- \lambda - \frac{2}{c} (z_{B0} - z_{A0}),
\]

being \( z_{A0} \) and \( z_{B0} \) the mirrors locations in the absence of gravitational waves. Integrating \( dT^2 = 0 \) one obtains, up to order \( h \), the photon trajectory:

(a) In Gaussian coordinates
\[
X_{\pm}(T) = \pm c \left( T - \frac{1}{2} \int_0^T dT' h(T') \right) + K_{\pm}.
\]

(b) In Fermi coordinates
\[
x_{\pm}(t) = \pm c t + k_{\pm} - \frac{1}{4c} \left\{ c^2 \left[ \dot{h}(t) t^2 - 2 t h(t) + 2 \int_0^t dt' h(t') \right] \right. \\
\left. + 2 c k_{\pm} \left[ h(t) - h(0) - t \dot{h}(t) \right] + k_{\pm}^2 \left[ \dot{h}(t) - \dot{h}(0) \right] \right\},
\]

being \( K_{\pm} \) and \( k_{\pm} \) integration constants which can include a term of order \( h \).
(LG) Delay time for a LIGO like device:

For LIGO the mirror $A$ is located at the origin, which is both geodesic and rigid, i.e. $X = x = 0$. Times $T$ and $t$ will be the measured by the same clock at the origin. The mirror $B$ is free falling, see fig. 1. Either using (12) or (13) we find

$$\delta^{LG}(t, L) = \frac{1}{2} \int_{t}^{t + \frac{2L}{c}} h(t') \, dt',$$

in agreement with [10].

![Figure 1: A LIGO like device (LG) consisting of two free mirrors $A$ and $B$ located at $X_A = 0$ and $X_B = L$ in Gaussian coordinates (left) and at $x_A = 0$ and $x_B = L + \frac{1}{2} h(t)L$ in Fermi coordinates (right).](image)

(PR) Delay time for a pure rigid type device:

Before tackling the most interesting case we stop by in an extreme situation: the time delay for the photon traveling in between two mirrors at different rigid positions, see fig. 2-left. Solving (13) under conditions (10), with $\lambda \rightarrow t$, $z \rightarrow x$, $x_A = 0$ and $x_B = L$, one obtains

$$\delta^{PR}(t, L) = \delta^{LG}(t, L) - \frac{L}{c} h(t + \frac{L}{c}).$$

(15)

We can interpret this result as the measure of the radar-length variations of a rigid body due to gravitational waves.

(SR) Delay time for a semi-rigid type device:

Finally we envisage the most interesting case, where one end is rigid but not located at the origin and the other is free falling, fig. 2-right. Solving (13) under conditions (10), with $\lambda \rightarrow t$, $z \rightarrow x$, $x_A = R$, $x_B = L + \frac{1}{2} h(t)L$ and $x_B_0 - x_A_0 = L - R = \ell$, one gets

$$\delta^{SR}(t, \ell) = \left(1 - \frac{\omega^2 R^2}{2c^2}\right) \delta^{LG}(t, \ell) + \frac{R}{2c} \left[ h(t + \frac{2\ell}{c}) + h(t) \right],$$

(16)
that reduces to (14) in the limit $R \to 0$.

Figure 2: (left) A pure rigid type device (PR) consisting of two mirrors: $A$ free falling at $x_A = 0$ and $B$ fixed at $x_B = L$. The rigid bar has length $2L$. (right) A semi-rigid type device (SR) consisting of two mirrors: $A$ fixed at $x_A = R$ and $B$ free at $x_B = L + \frac{1}{2} h(t)L$. The rigid bar has length $2R$. We ignore the delay time due to the 0-$R$ strip. In both cases we use Fermi coordinates.

Since these findings may seem fanciful, it is worth to elaborate slightly on their outcomes.

5 Bouncing photons

To pursue further our analysis we need a few experimental inputs from the LIGO detectors. The interferometer main characteristic is its arm length $L \approx 4$ km and the laser wave length $\lambda_0 = 1064$ nm \cite{11}. Each laser beam bounces back and forth about $n = 280$ times before they are merged together again \cite{12}. Finally, the first black hole detected had an orbital frequency of $f = 75$ Hz, half of the gravitational wave frequency.

Bearing this in mind let’s analyze the consequences of (14) and (16). We do not consider $\delta^{PR}(t, L)$ further since its contribution does not increases with the number of bounces. Assuming a gravitational perturbation of the type $h(t) = A\sin[\omega t]$ the delay time after $n$-bounces are

$$\delta^{LG}(t, n, L) = \frac{A}{2\omega} \left( \cos[\omega t] - \cos \left[ \omega \left\{ t + 2n \frac{L}{c} \right\} \right] \right) ,$$

$$\delta^{SR}(t, n, \ell) = \left( 1 - \frac{\omega^2 R^2}{2c^2} \right) \delta^{LG}(t, n, \ell) + \frac{AR}{c} \cot \left[ \frac{\ell}{c} \right] \sin \left[ \omega(1 + n) \frac{\ell}{c} \right] \sin \left[ \omega \left\{ t + (1 + n) \frac{\ell}{c} \right\} \right] .$$

There are several salient features in (17,18) of which we shall highlight three:

(i) For both, the optimal number of bounces to amplify the signal severely depend on the gravitational wave frequency. Without a priori knowledge this fact limits the experimental setup which must be sensitive to a range of frequencies. In addition the mirror plates separation plays a major role in this issue. For instance, with respect
to the number \( n \) of bounces, \((17)\) and the second term in \((18)\) reach their maximum respectively at

\[
n_{\text{max}}^{LG} = \left\lfloor \frac{\pi c}{2\omega L} \right\rfloor, \quad n_{\text{max}}^{\text{SR}} = \left\lfloor \frac{\pi c}{4\omega \ell} - 1 \right\rfloor.
\]

(19)

With the above data this implies \( n_{\text{max}}^{LG} \sim 125 \), stunningly less than half the number of bounces taken by LIGO (\( n = 280 \)). Thus LIGO has been using, for this frequency, a number of bounces beyond the maximum.

(ii) We can deal with two limiting cases: a) \( \ell \gg R \to 0 \), in this case there will be not substantial difference between the outcomes of \((17)\) and \((18)\) provided \( L \sim \ell \). b) If \( R \gg \ell \), there is a remarkable effect: the last contribution to \((18)\) becomes leading with respect to the first one and has no parallel in \((17)\), thus the approach described above is falseable experimentally: an essentially non null measure of \( \delta_{\text{SR}}(t, n, \ell) \) will support the rigid assumption.

(iii) In the construction of \((17)\), see fig. 1, the Michelson-Morley interferometer arms length and the Fabry-Perot cavity length, \( L \), are identify. The value of \( L \) is optimized if it is approximately half of the gravitational wave length \([6]\). Contrariwise \((18)\) contains as independent quantities both: the length of the Michelson-Morley interferometer arms, \( R + \ell \), and the size of the Fabry-Perot cavity, \( \ell \), fig. 2. Choosing \( \ell \) small enough, the first term in \((18)\) is negligible while the second is enhanced. In this case the second term is almost insensitive to the frequency of the signal and within a wide range it increases directly with number of bounces.

Summing up our findings, \((18)\) contains a contribution that is enhanced with an increasing number of bounces. This depends mainly on the Fabry-Perot cavity length \( \ell \) that is not related with the interferometer arm lengths \( R + \ell \) and can be made as small as desired. We explore next this possibility.

6 Analysis and Conclusions

We have looked for the number of bounces needed in \((18)\) as a function of the coordinate, \( R \), and the separation between the mirrors plates, \( \ell \), to match the outcome of \((17)\) with LIGO data, \( L = 4 \text{ km} \), \( \omega = 2\pi \times 150 \text{ s}^{-1} \) and \( n = 280 \) i.e. \( \delta_{\text{max}}^{LG} \approx 0.36 A/\omega \). We show in fig. 3-left this dependence. As it is evident for small \( R \) and large values of \( \ell \) the outcomes
of \((17)\) and \((18)\) are identical. What is less intuitive is that for a relatively short arms lengths \(R \approx 20\) m and Fabry-Perot cavities of less than 1 m one gets the very same results as LIGO but with a few thousand bounces instead of two hundred. This would not probably be a big deal if the Fabry-Perot cavity could not be shrank at our disposal. Reducing the distance \(\ell\) has two consequences: \(i)\) the number of bounces can be increased. This is feasible as high finesse micro-cavities can be constructed \([13]\). \(ii)\) An increasing number of bounces also has associated an increasing maximum value for the time delay, see fig. 3-right, which facilitates the detection. We want to stress that the same procedure does not hold with LIGO like configurations. Approaching the mirrors around \(L \approx 20\) m needs over 6000 bounces to obtain \(\delta_{\text{max}}^{\text{LG}}\). This translates on a quality factor of \(Q > 10^{13}\) which is several orders of magnitude higher than most low loss materials.

We have paved an interesting road to relook gravitational wave detectors. The approach only rely on an extensive use of the concept of rigidity. Despite the controversy that generates the adaptation of this concept to GR, these results seem clear enough to seriously consider their use at a practical level.

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