On Rings whose Maximal Essential Ideals are Pure

Raida D. Mahmood               Awreng B. Mahmood
raida.1961@uomosul.edu.iq       awring2002@yahoo.com
College of Computer sciences and Mathematics
University of Mosul, Iraq
Received on: 06/04/2006  Accepted on: 25/06/2006

ABSTRACT
This paper introduces the notion of a right MEP-ring (a ring in which every maximal essential right ideal is left pure) with some of their basic properties; we also give necessary and sufficient conditions for MEP-rings to be strongly regular rings and weakly regular rings.

Keywords: MEP-rings, pure ideals, weakly regular ring.

1- Introduction
An ideal I of a ring R is said to be right (left) pure if for every a ∈ I, there exists b ∈ I such that a = ab (a = ba). [1], [2].

Throughout this paper, R is an associative ring with unity.

Recall that:
1) R is called reduced if R has no non-zero nilpotent elements.
2) For any element a in R we define the right annihilator of a by r(a) = { x ∈ R : ax = 0 }, and likewise the left annihilator l(a).
3) R is strongly regular [4], if for every a ∈ R, there exists b ∈ R such that a = a^2 b.
4) \(Z,Y,J(R)\) are respectively the left singular ideal right singular ideal and the Jacobson radical of \(R\).

5) A ring \(R\) is said to be semi-commutative if \(xy=0\) implies that \(xRy=0\), for all \(x,y \in R\). It is easy to see that \(R\) is semi-commutative if and only if every right (left) annihilator in \(R\) is a two-sided ideal \([8]\)

2-MEP-Rings:

In this section we introduce the notion of a right MEP-ring with some of their basic properties;

**Definition 2.1:**

A ring \(R\) is said to be right MEP-ring if every maximal essential right ideal of \(R\) is left pure.

Next we give the following theorem which plays the key role in several of our proofs.

**Theorem 2.2:**

Let \(R\) be a semi commutative, right MEP-ring. Then \(R\) is a reduced ring.

**Proof:**

Let \(a\) be a non zero element of \(R\), such that \(a^2 = 0\) and let \(M\) be a maximal right ideal containing \(r(a)\). We shall prove that \(M\) is an essential ideal. Suppose that \(M\) is not essential, then \(M\) is a direct summand, and hence there exists \(0 \neq e = e^2 \in R\) such that \(M = r(e)\) (Lemma 2-3, of \([8]\)). Since \(R\) is semi commutative and \(a \in M\), then \(e\ a = 0\) and this implies that \(e \in r(a) \subseteq M = r(e)\).

Therefore \(e=0\), is a contradiction. Thus \(M\) is an essential right ideal. Since \(R\) is a right MEP-ring, then \(M\) is left pure for every \(a \in M\). Hence there exists \(b \in M\) such that \(a = ba\) implies that \((1-b) \in l(a) = r(a) \subseteq M\), so \(1 \in M\) and this implies that \(M=R\), which is a contradiction. Therefore \(a=0\) and hence \(R\) is a reduced ring. \(\Box\)

**Theorem 2.3:**

If \(R\) is a semi commutative, right MEP-ring, then every essential right ideal of \(R\) is an idempotent.
Proof:
Let \( I = bR \) be an essential right ideal of \( R \). For any element \( b \in I \), \( Rb + r(b) \) is essential in \( R \) (Proposition 3 of [5]).

If \( Rb + r(b) \neq R \), let \( M \) be a maximal right ideal containing \( Rb + r(b) \). Since \( R \) is MEP-ring, then there exists \( a \in M \) such that \( b = ab \) and \( (1-a) \in l(b) = r(b) \subseteq M \). So \( 1 \in M \) is a contradiction.

Thus \( Rb + r(b) = R \), and \( 1 = u + d \), \( u \in Rb \subseteq I \), \( d \in r(b) \).
Hence \( b = bu \). Therefore \( I = I^2 \) (Lemma 3 of [7]). \( \Box \)

Proposition 2.4:
Let \( R \) be a semi commutative, right MEP-ring. Then the \( J(R) = (0) \).

Proof:
Let \( 0 \neq a \in J(R) \). If \( aR + r(a) \neq R \), then there exists a maximal right ideal \( M \) containing \( aR + r(a) \). Since \( a \in M \) and \( r(a) \subseteq M \), then by a similar method of proof used in Theorem (2.2) \( M \) is an essential ideal.

Since \( R \) is MEP-ring, then there exists \( b \in M \), such that \( a = ba \), but \( a \in J(R) \subseteq M \) so \( 1 \in M \), is a contradiction. Therefore \( aR + r(a) = R \) (Proposition 5 of [8]) and \( ar + d = 1 \), for some \( r \in R \) and \( d \in r(a) \), this implies that \( a = a^2 r \).

Since \( a \in J \), then there exists an invertible element \( v \) in \( R \) such that \( (1-ar) v = 1 \), so \( (a-a^2 r) v = a \), yields \( a = 0 \). This proves that \( J(R) = (0) \). \( \Box \)

Recall that a ring \( R \) is said to be MERT-ring [7], if every maximal essential right ideal of \( R \) is a two-sided ideal.

Theorem 2.5:
If \( R \) is MERT, MEP-ring, then \( Y(R) = (0) \).

Proof:
If \( Y(R) \neq 0 \), by Lemma (7) of [6], there exists \( 0 \neq y \in Y(R) \) with \( y^2 = 0 \). Let \( L \) be a maximal right ideal of \( R \), containing \( r(y) \).
We claim that \( L \) is an essential right ideal of \( R \).
Suppose this is not true, then there exists a non-zero ideal \( T \) of \( R \) such that \( L \cap T = (0) \). Then \( yRT \subseteq LT \subseteq L \cap T = 0 \) implies \( T \subseteq r(y) \subseteq L \), so \( L \cap T \neq 0 \). This contradiction proves that \( L \) is an essential right ideal.
Since \( R \) is an MEP-ring, then \( L \) is a left pure.
Thus for every $y \in L$, there exists $c \in L$ such that $y = cy$ (L is a left pure).
Since $R$ is MERT, then $cy \in L$ (two sided ideal) and thus $1 \in L$, is a contradiction. Therefore $Y(R) = (0)$. ∎

3- The connection between MEP-Rings and other rings

In this section, we study the connection between MEP-Rings and strongly regular rings, weakly regular rings.

Following [3], a ring $R$ is right (left) weakly regular if $I^2 = I$ for each right (left) ideal $I$ of $R$. Equivalently, $a \in aRaR$ ($a \in RaRa$) for every $a \in R$. $R$ is weakly regular if it’s both right and left weakly regular.

The following result is given in [3]:

**Lemma 3.1:**
A reduced ring $R$ is right weakly regular if and only if it is left weakly regular.

Next we give the following lemma:

**Lemma 3.2:**
If $R$ a semi-commutative ring then $RaR + r(a)$ is an essential right ideal of $R$ for any $a$ in $R$.

**Proof:**
Given $0 \neq a \in R$, assume that $[RaR + r(a)] \cap I = 0$, where $I$ is a right ideal of $R$. Then $I \subseteq I \cap RaR = 0$, and so $I \subseteq l(a) = r(a)$ ($R$ is semi commutative). Hence $I = 0$; whence $RaR + r(a)$ is an essential right ideal of $R$. ∎

**Theorem 3.3:**
If $R$ is a semi commutative, right MEP-ring, then $R$ is a reduced weakly regular ring.

**Proof:**
By Theorem (2.2), $R$ is a reduced ring. We show that $RaR + r(a) = R$, for any $a \in R$.

Suppose that $RaR + r(a) \neq R$, then there exists a maximal right ideal $M$ containing $RaR + r(a)$. By a similar method of proof used in Theorem (2.2), $M$ is an essential ideal.
Now \( R \) is MEP- ring, so \( a = ba \), for some \( b \in M \), hence 
\((1-b) \in 1(a) = r(a) \subseteq M\) and so \( 1 \in M \) which is a contradiction. Therefore 
\( M=R \) and hence \( RaR + r(a) = R \), for any \( a \in R \). In particular \( 1 = cab + d \),
for some \( c, b \in R, d \in r(a) \).

Hence \( a = acab \) and \( R \) is right weakly regular. Since \( R \) is reduced,
then by Lemma (3.1) \( R \) is a weakly regular ring. \( \square \)

Before closing this section, we give the following result.

**Theorem 3.4:**

A ring \( R \) is strongly regular if and only if \( R \) is a semi-commutative, 
MEP, MERT- ring.

**Proof:**

Assume that \( R \) is MEP, MERT-ring, let \( 0 \neq a \in R \), we shall prove
that \( aR + r(a) = R \). If \( aR + r(a) \neq R \), then there exists a maximal right
ideal \( M \) containing \( aR + r(a) \). Since \( M \) is essential, then \( M \) is left pure.
Hence \( a = ba \), for some \( b \in M \), so \( 1 \in M \), a contradiction. Therefore \( M=R \)
and hence \( aR + r(a) = R \). In particular \( ar+d = 1 \), for some \( r \in R, d \in r(a) \).
So \( a=a^2r \). Therefore \( R \) is strongly regular.

Conversely: Assume that \( R \) is strongly regular, then by [3], \( R \) is
regular and reduced. Also \( R \) is MEP and semi-commutative. \( \square \)
REFERENCES

[1] Al- Ezeh, H. (1989) “Pure ideals in commutative reduced Gelfand rings with unity”, *Arch. Math.*, Vol. 53, PP. 266 –269.

[2] Al- Ezeh, H. (1989) “On generalized PF-rings”, *Math J. Okayama Univ.* Vol.31, PP. 25 – 29.

[3] Ibraheem, Z. M. (1991) “On P-injective Modules”, M. Sc. Thesis, Mosul University.

[4] Ming, R. Y. C. (1974) “On simple P-injective modules”, *Math. Japonica*, 19, PP. 173-179.

[5] Ming, R. Y. C. (1976) “On VonNumann regular rings-II”, *Math.Scand.* 39, PP. 167-170.

[6] Ming, R.Y.C. (1983) “On quasi – injectivity and Von-Neumann regularity” ,Monatash. , *Math.* 95, PP. 25 – 32.

[7] Ming, R.Y.C. (1983) “Maximal ideals in regular rings”, *Hokkaido Math. J.* ,Vol. 12, PP. 119-128.

[8] Shuker, N. H. (2004) “On rings whose simple singular modules are GP-injective”, *Raf. J. Sci.*, Vol. 15, No. 1, PP. 37 – 40.