Finite System-size Effects in Self-organized Criticality Systems

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Abstract

We explore upper limits for the largest avalanches or catastrophes in nonlinear energy dissipation systems governed by self-organized criticality. We generalize the idealized “straight” power-law size distribution and Pareto distribution functions in order to accommodate incomplete sampling, limited instrumental sensitivity, finite system-size effects, and “Black Swan” and “Dragon King” extreme events. Our findings are as follows. (i) Solar flares show no finite system-size limits up to $L \lesssim 200$ Mm, but solar flare durations reveal an upper flare duration limit of $\lesssim 6$ hr. (ii) Stellar flares observed with Kepler exhibit inertial ranges of $E \approx 10^{34}–10^{37}$ erg, finite system-size ranges of $E \approx 10^{37}–10^{38}$ erg, and extreme events at $E \approx (1–5) \times 10^{38}$ erg. (iii) The maximum flare energies of different spectral type stars (M, K, G, F, A, giants) reveal a positive correlation with the stellar radius, which indicates a finite system-size limit imposed by the stellar surface area. Fitting our finite system-size models to terrestrial data sets (earthquakes, wildfires, city sizes, blackouts, terrorism, words, surnames, web links) yields evidence (in half of the cases) for finite system-size limits and extreme events, which can be modeled with dual power-law size distributions.

Unified Astronomy Thesaurus concepts: Solar flares (1496); Stellar flares (1603); Stellar phenomena (1619); Astrostatistics distributions (1884)

1. Introduction

At the time of writing this paper, while the author worked in smoke-filled poor air caused by large wildfires around San Francisco, The New York Times (2020 September) reported that the number of observed wildfires in California doubled in number as well as size this year, most likely caused by the global warming that contributes to drier conditions of forests and thus their enhanced flammability. An obvious question arises, therefore, as to what maximum size the largest possible wildfire could reach. The size of wildfires is generally measured by the burned area (in units of acres). In the USA, of the 1.4 million wildfires that have occurred since 2000, 197 exceeded 100,000 acres, and 13 exceeded 500,000 acres, according to a report of the Interagency Fire Center (https://fas.org/sgp/irs/misc/IF10244.pdf).

Although wildfires represent just one phenomenon of nonlinear energy dissipation events that have been modeled with self-organized criticality (SOC; Drossel & Schwabl 1992; Malamud et al. 1998; Zinck & Grimm 2008; Hergarten 2013), a fundamental question is the determination of finite system-size effects, which constrain the largest possible event in a size distribution. Here we investigate finite system-size effects in large astrophysical data sets (with $\gtrsim 10^3$ events), which mostly apply to solar and stellar flare statistics (with 16 data sets). In the spirit of interdisciplinary research, we also subject eight empirical data sets to our analysis, selected by Clauset et al. (2009), covering information technology (words, surnames, web links), geophysics (earthquakes, wildfires), and human activities (city sizes, blackouts, terrorism statistics).

The SOC concept predicts power law–like size distributions of nonlinear energy dissipation events (also called avalanches or catastrophes), as well as scaling laws in the form of power-law relationships (Aschwanden 2020). However, although power laws are the hallmark of SOC avalanches, many observed size distributions exhibit significant deviations from ideal power-law functions, caused by critical thresholds, instrumental sensitivity limitations, background subtraction, incomplete sampling, finite system-size effects, and extreme event outliers. In this study, we demonstrate that all of these effects can be built into generalized power-law models that fit the data within acceptable values of least-squares fits ($\chi^2 < 2$). Alternative size distribution functions have also been tested, such as Poisson, lognormal, exponential, stretched exponential, or power law–plus–cutoff (Clauset et al. 2009). The modeled size distributions may show large parameter ranges with ideal power-law behavior for some size distributions on one side but also curved power laws without any straight power-law segments in other cases, which triggered some researchers to question whether power-law size distribution functions exist at all (Stumpf & Porter 2012). Nevertheless, we find that straight power laws still exist for large inertial ranges (of $\approx 2–5$ decades) but are strongly convoluted in data sets with small inertial ranges (of $\approx 1–2$ decades). The promise of this study is the detectability of finite system sizes in the form of exponential-like cutoffs near the upper end of an observed size distribution, while a straight power law all the way to the largest observed event indicates a lack of finite system-size effects; thus, even larger events are expected when the statistics is prolonged in time. In other words, we attempt to discriminate between a temporary maximum size (for short-term data sets) and an absolute maximum size (for long-term data sets). There is also the notion of “Dragon King” and “Black Swan” events, which represent outliers from a canonical power-law size distribution and thus may be produced by an alternative physical mechanism altogether (Sornette 2009; Sornette & Ouillon 2012; measured for solar and stellar flares for the first time in Aschwanden 2019). The Black Swan theory is essentially a metaphor for rare events that cannot be predicted by standard statistics (Taleb 2007). Dragon King events, defined by Sornette (2009) in financial risk management, are fundamentally the same: an unusually extreme event that was unforeseen by almost everyone, even when examining the far right tail of the loss distribution. For astrophysical applications, Black Swan or Dragon King events are produced by different physical
mechanisms than events in the left tail of the power-law distributions. The precise model for such outliers is of course unknown, but we identify them by an excess in the size distribution above the exponential-like cutoff of the finite system-size limit. It has been shown that the solar flare events observed in the past four centuries have not exceeded the level of the largest flares observed in the space era, and that there is at most a 10% chance of a flare larger than about X30 (GOES class) in the next 30 yr (Schrijver et al. 2012).

This paper contains theoretical definitions of power law–like size distributions (Section 2), data analysis of 24 astrophysical and empirical data sets with discussion (Section 3), and conclusions (Section 4).

2. Theory and Method

Our goal is the determination of the power-law slope \( \alpha_x \) in power law–like size distributions by modeling a number of known effects, such as the (astrophysical) background subtraction (Section 2.1), incomplete sampling, instrumental sensitivity limitations, and detection thresholds, which affect differential size distributions (Sections 2.2) and require thresholded power-law distributions (Section 2.3), finite system-size effects (Section 2.4), and extreme (Dragon King) events (Section 2.5). The least-squares optimization criterion is described in Section 2.6.

2.1. Empirical Background Subtraction

We start with a data sample \( X_i, i = 1, \ldots, n_{ev} \) that contains some size parameter \( X_i \) with \( n_{ev} \) events. In astrophysical data sets, observables for the size parameter \( X_i \) are most often measured in terms of peak fluxes \( P_i \), time-integrated fluences, intensities, luminosities, or energies \( E_i \); and event time durations \( T_i \). The observable \( X_i \) refers to the size of an event, pulse amplitude, or flare volume. A first correction that needs to be applied to the size value \( X_i \) is the background subtraction,

\[
x_i = X_i - B_i, \quad i = 1, \ldots, n_{ev},
\]

where \( B_i \) represents an event-unrelated flux or fluence, for instance, the galactic quiescent soft X-ray flux that is irradiated from the same source direction as the soft X-ray flux of a solar flare. If the flux time profile \( F_i(t) \) of an event \( i \) is available, the background flux can be determined from the preflare flux \( B_i \), and the corrected peak flux is \( P_i = \max[F_i(t)] - B_i \). Since the background correction affects the power-law slope \( \alpha_p \) of the peak flux (see Figure 5 in Aschwanden 2011b), it is advisable to at least apply a mean estimated background correction \( B = \langle B_i \rangle \).

2.2. Differential Size Distribution

A size distribution of events, \( N(x)dx \), also called a differential occurrence frequency distribution, can ideally be approximated by a power-law function as a function of some size parameter \( x \), quantified by three observables \( (x_1, x_2, n_0) \) and one variable \( \alpha_x \),

\[
N(x)dx = n_0 x^{-\alpha_x}dx, \quad x_1 \leq x \leq x_2,
\]

where \( x_1 \) and \( x_2 \) are the lower and upper bounds of the power-law inertial range, \( n_0 \) is a normalization constant, and \( \alpha_x \) is the power-law slope. Uncertainties in previous calculations of the power-law slope mostly resulted from the arbitrary choice of fitting ranges \( [x_1, x_2] \), which can be largely corrected by a generalized power-law function that includes undersampling at the lower end and finite system-size effects at the upper end of the inertial range \( [x_1, x_2] \).

2.3. Thresholded Power-law Distribution

The straight power-law function (Equation (2)) can be generalized with a threshold parameter \( x_0 \). This generalized function is called the Lomax distribution (Lomax 1954), the generalized Pareto distribution (Hosking & Wallis 1987), or the thresholded power-law size distribution (Aschwanden 2015),

\[
N(x)dx = n_0(x_0 + x)^{-\alpha_x}dx,
\]

where \( n_0 \) is a normalization constant. This threshold parameter \( x_0 \) models three different features: truncation effects due to incomplete sampling of events below a threshold (if \( x_0 > 0 \)), incomplete sampling due to instrumental sensitivity limitations (if \( x_0 > 0 \)), and subtraction of event-unrelated background (if \( x_0 < 0 \)), as is common in astrophysical data sets (Aschwanden 2015, 2019).

2.4. Finite System-size Effects

The size distribution can be uniquely approximated with a classical power-law function if the lower \( (x_1) \) and upper \( (x_2) \) bounds are well defined. In practice, however, the lower bound is flattened by undersampling, detection thresholds, or instrumental sensitivity limitations, while the upper bound typically shows a gradual steepening due to finite system-size effects (Pruessner 2012). Ignoring these effects leads to power-law fits with arbitrary inertial ranges \( [x_1, x_2] \), which affects the accuracy of the determined power-law slope and its uncertainty. Finite system-size effects are modeled here with an exponential cutoff function, which we can combine with the generalized Pareto distribution (Equation (3)),

\[
N(x)dx = n_0(x_0 + x)^{-\alpha_x} \exp \left( \frac{-x}{x_0} \right)dx,
\]

which is quantified with an exponential function \( \exp(-x/x_0) \) at a characteristic size \( x \approx x_0 \). The inclusion of finite system-size effects steepens the power-law slope asymptotically to infinity at the upper end of the distribution function.

2.5. Extreme Events

Sometimes extreme events at the upper end of the size distribution function cannot be fitted with the exponential cutoff function, as expected from finite system-size effects.

These extreme events that deviate from a standard power-law distribution function have also been dubbed Black Swan and Dragon King events (alluding to their extremely rare appearance) by Sornette (2009) and Sornette & Ouillon (2012), who suggested that they are generated by a different physical mechanism. Detections of such extreme event outliers have occasionally been noted in astrophysical data sets (Aschwanden 2019).

To accommodate these extreme events, we define a size distribution with a second power-law component, where the first power-law distribution includes the exponential function (Equation (4)) with amplitude \( (1 - q_{pow}) \), while the second
power-law distribution with amplitude \((q_{\text{pow}})\) extends without a cutoff all the way to the largest event (with an identical power-law slope, in order to minimize the number of free parameters),

\[
N(x)dx = n_0(x_0 + x)^{-\alpha} \left(1 - q_{\text{pow}}\right)\exp\left(-\frac{x}{x_e}\right) + q_{\text{pow}}\right)dx,
\]

where we define the exponential cutoff energy \(x_e = x_2q_{\text{exp}}\), with \(q_{\text{exp}}\) being a free parameter. This combinatory definition of the size distribution converges to the canonical finite system-size distribution as defined in Equation (4) for \(q_{\text{pow}} \to 0\) and the previously defined Pareto distribution function for \(q_{\text{pow}} \to 1\).

A synopsis of the three models is shown in Figure 1, depicting the Pareto distribution model (PM), the finite system-size effect model (FM), and the combined extreme event component model (EM). Note that the most general model (EM) has three observables \((n_0, x_0, x_2)\) and three variables \((\alpha, q_{\text{exp}}, q_{\text{pow}})\).

### 2.6. Fitting Method

Our power-law fitting procedure is straightforward. As indicated in Figure 1, the total range of the size distribution is bound by the smallest \((x_0 = \min[x_i])\) and largest \((x_2 = \max[x_i])\) event size. Before any data binning is done, we first apply the empirical background subtraction (Section 2.1). Then we bin the number or counts of events per bin, \(C_j = N_j\Delta x_j\), with logarithmic steps \(\Delta x_j\) and determine the threshold \(x_0\) from the maximum of the binned count rate, \(\max[C_i] = C_i(x_0)\). The size \(x_0\) represents a threshold in the Pareto distribution that roughly demarcates the range \([x_1, x_0]\) of incomplete sampling, while the range \([x_0, x_2]\) represents the inertial range, where the size distribution is fitted. The lower bound \(x_0\) of the inertial range is adjusted when multiple peaks are present (see also Monte Carlo simulations in Aschwanden 2011b, 2015).

The fitting of any of the three models of the differential occurrence size distribution \(N_{\text{diff}}^\text{theo}(x)\) to an observed (binned) size distribution \(N_{\text{diff}}^\text{obs}(x)\) is performed with a standard least-squares \(\chi^2\)-criterion (i.e., reduced \(\chi^2\)),

\[
\chi_{\text{diff}} = \frac{1}{(n_{\text{bin}} - n_{\text{par}})} \sum_{j=1}^{n_{\text{obs}}} \left[\frac{N_{\text{diff}}^\text{theo}(x_j) - N_{\text{diff}}^\text{obs}(x_j)}{\sigma_{\text{diff},j}^2}\right]^2,
\]

where \(x_p, j = 1, \ldots, n_j\) are the counts per bin width, \(n_j\) is the number of bins, and \(n_{\text{par}}\) is the number of parameters (variables) of the fitted model functions. For the number of bins, we use six bins per decade. The estimated uncertainty of counts per bin, \(\sigma_{\text{diff},j}\) (Equation (6)), is according to Poisson statistics,

\[
\sigma_{\text{diff},j} = \sqrt{N_j\Delta x_j},
\]

where \(\Delta x_j\) is the (logarithmic) bin width. The goodness of fit \(\chi_{\text{diff}}\) quantifies which model size distribution is consistent with the (observed) data.

Finally, the uncertainty \(\sigma_\alpha\) of the best-fit power-law slope \(\alpha\) is estimated to be

\[
\sigma_\alpha = \frac{\alpha}{\sqrt{n_{\text{ev}}}},
\]

with \(n_{\text{ev}}\) the total number of events in the entire size distribution (or fitted range) according to Monte Carlo simulations with least-squares fitting (Aschwanden 2011b). A slightly different estimate of \(\sigma_\alpha = (\alpha - 1)/\sqrt{3}\) is calculated in Clauset et al. (2009) based on the Hill estimator (Hill 1975) for the maximum-likelihood estimator of the scaling parameter.

### 3. Data Analysis and Discussion

Theoretical models of SOC systems often involve a large 1D, 2D, or 3D lattice grid that has a finite system size. As long as avalanches evolve inside the lattice grid, we expect a power-law distribution of avalanche sizes, while avalanches propagating to the edge of the lattice grid have a quenched or reduced size that manifests itself as an exponential-like cutoff in the size distribution (Pruessner 2012). The presence of this suppression effect enables a measurement of the system size. Such a measurement, however, requires sufficiently long sampling times, otherwise one cannot distinguish between a power law with a sharp cutoff and an exponential cutoff. Fortunately, the data set of stellar flares observed with Kepler provides sufficient statistics to infer two power-law components in the size distribution. In the following, we investigate the presence or absence of finite system-size limits in solar and stellar flare data.

#### 3.1. Performance of the Three Models

Ultimately, the aim of this analysis is accurate modeling of the observed power-law size distributions, to which we apply three different theoretical models: (i) the Pareto distribution model, which shows a flattening at the lower end due to incomplete sampling and a straight power law at the upper end (PM); (ii) the Pareto distribution model, with an exponentially dropping cutoff at the upper end that is caused by finite system-size effects (FM); and (iii) an additional power-law component that extends above the exponential cutoff caused by extreme events, also called the Dragon King extreme event model (EM). These three models are depicted in Figure 1. The three models have an increasing number of free parameters (one, two, and...
Table 1
Observed Phenomena with Power-law Slopes, Goodness of Fit, and Models

| Observable                  | Number of Events | Background Level | Power Law Slope | Goodness of Fit | Model |
|-----------------------------|------------------|------------------|-----------------|----------------|-------|
| Peak flux HXRBS             | 11,352           | 1                | 1.75 ± 0.02     | 0.88            | EM    |
| Peak flux BATSE             | 7245             | 1                | 1.76 ± 0.02     | 1.87            | EM    |
| Peak flux RHESSI            | 7998             | 1                | 1.85 ± 0.02     | 1.41            | PM    |
| Counts HXRBS                | 11,550           | 1                | 1.57 ± 0.01     | 1.52            | PM    |
| Counts BATSE                | 3425             | 1                | 1.65 ± 0.03     | 0.90            | PM    |
| Counts RHESSI               | 11,549           | 0                | 1.74 ± 0.02     | 1.51            | PM    |
| Duration HXRBS              | 11,549           | 0                | 1.81 ± 0.02     | 1.81            | EM    |
| Duration BATSE              | 7243             | 0                | 1.99 ± 0.02     | 1.04            | FM    |
| Duration RHESSI             | 11,525           | 0                | 1.90 ± 0.02     | 1.40            | FM    |
| Kepler (catalog)            | 162,264          | 1                | 1.813 ± 0.002   | 1.45            | EM    |
| Words                       | 9695             | 1                | 2.12 ± 0.02     | 1.85            | EM    |
| Surnames                    | 2753             | 1                | 2.58 ± 0.05     | 1.72            | PM    |
| Web links                   | 8658             | 1                | 1.49 ± 0.02     | 1.94            | PM    |
| Earthquakes                 | 17,452           | 1                | 1.71 ± 0.01     | 1.98            | EM    |
| City sizes                  | 19,447           | 1                | 1.59 ± 0.01     | 1.57            | EM    |
| Wildfires                   | 56,052           | 1                | 1.77 ± 0.01     | 1.56            | FM    |
| Blackouts                   | 213              | 1                | 1.35 ± 0.10     | 0.73            | EM    |
| Terrorism                   | 4303             | 1                | 2.64 ± 0.04     | 1.26            | FM    |

three), where the models with a larger number of free parameters represent generalizations of the models with fewer parameters. The Pareto distribution, the simplest of our three size distribution models, has three observable parameters—the threshold $x_0$, maximum value $x_2$, and normalization constant $n_0$ (Equation (3))—and one variable, the power-law slope $\alpha_e$. The finite system-size model contains an additional size parameter $x_e = q_e x_2$ that marks the size of the exponential cutoff (Equation (4)). The extreme event model contains an additional parameter $q_{pow}$ that expresses the fraction of extreme events (Equation (5)).

The best-fitting cases of the 24 analyzed data sets with a goodness of fit $\chi^2 \lesssim 2$ (fitted over the inertial range) are listed in Table 1, containing the number of events ($n_{ev}$) per data set, the power-law slope $\alpha_e$, the goodness of fit $\chi^2$, and the best-fit model (PM, FM, EM). According to this evaluation, listed in Table 1, we find that six data sets are best fitted with the Pareto model (PM), four data sets are best fitted with the finite system-size model (FM), and eight data sets are best fitted with the extreme event model (EM). Thus, each of the three models has its own merit to fit real-world data. In the following, we analyze the finite system-size effects and deviations from ideal power-law distributions.

3.2. Solar Flare Finite Size Limit

The first six analyzed data sets were compiled from some $n_{ev} \lesssim 10^5$ solar flares, detected with three different instruments: the Hard X-ray Burst Spectrometer (HXRBS) on board the Solar Maximum Mission (SMM), the Burst And Transient Source Experiment (BATSE) on board the Compton Gamma Ray Observatory (CGRO), and RHESSI, where each instrument was operated during a different solar cycle (Aschwanden 2015, 2019, and references therein). The first three data sets were measured from the flare peak count rate $P$ (Figures 2(a)–(c)), while the next three sets were measured from the time-integrated counts (or fluxes) $F$ of hard X-ray photons with energies of $E \gtrsim 20$ keV (Figures 2(d)–(f)). Each histogram shown in Figure 2 contains in the uppermost bin the absolute maximum size $x_2$ of the largest event. The fact that the size distributions of these six data sets observed in solar flares do not reveal any recognizable finite system-size limit leaves the progression of the power-law distribution at the upper end open, implying that larger “superevents” are possible beyond the observed maximum event size $x_2$. This brings us to the question of what the size limits are in solar flares. One of the largest observed spatial scales of a solar flare (or active region) was measured during the “Bastille Day” flare, amounting to about $L \approx 200$ Mm $\approx 0.3$ solar radius (Aschwanden & Alexander 2001). Interestingly, the depth of the solar convection zone has a similar spatial scale in the vertical direction above a radius of $R \approx 0.7 R_\odot$. This coincidence between the maximum size of the horizontal flare area and the vertical depth of the convection zone may play a role in predicting the depths of stellar convection zones. The finding of a correlation between the stellar radius and flare energy as shown in Figure 7 is an encouraging result along this line.

3.3. Solar Flare Finite Duration Limit

The size distribution fits of solar flare durations $T$ are shown in Figures 3(a)–(c). All three cases can be fitted with an exponential cutoff over an inertial range of $\gtrsim 2$ decades, covering an inertial range of $T \approx 10^2$–$10^4$ s, or from $\approx 2$ minutes to 3 hr. The inferred power-law slopes have a mean of $\alpha_T = 1.90 \pm 0.09$ (i.e., the average of the three cases shown in Figure 3), which is close to the prediction of SOC models ($\alpha_T = 2.0$; Aschwanden & Freeland 2012). The meaning of a finite system size in the time domain of this data set of flare durations $T$ is a temporal limit that cannot be exceeded by a coherent flare process. We see that the longest flare durations are $T_2 \approx 5.6$ hr for HXRBS data (Figure 3(a)), $T_2 \approx 2.2$ hr for BATSE data (Figure 3(b)), and $T_2 \approx 0.8$ hr for RHESSI data (Figure 3(c)). Since each data set is recorded by a different instrument and during different (nonoverlapping) time epochs, the differences in the maximum event duration $T_2$ could result from different definitions of the flare duration or detection threshold. Nevertheless, all three data sets seem to indicate an exponential cutoff, which implies an upper limit of the flare duration, on the order of $T_2 \lesssim 6$ hr. The exponential cutoff
suggests that we are unlikely to observe solar flares with longer durations. We could not make such a statement if the size distribution of time durations showed a straight power law all the way to the upper end.

We may ask why flare durations (Figure 3) do not show power laws extending over 3–5 decades, as we find for peak fluxes and fluences (Figure 2). We note that the inertial range with straight power-law behavior is only 1–2 decades in size distributions of event durations, which makes it harder to separate it from the flattening at the lower end \((x \lesssim x_0)\) and the exponential cutoff at the upper end \((x \gtrsim x_3)\). However, at this time, it is not entirely understood why event durations have a pronounced exponential cutoff (Figure 3), in contrast to event peak rates and fluences (Figure 2). It appears that finite system-size effects are different in the time domain (e.g., flare durations) and spatial domain (see Figure 10 and Appendix B).

3.4. Stellar Flare Finite Energy Limits

The size distribution of stellar flares according to the entire Kepler (Borucki et al. 2010) flare event catalog (Davenport 2016; Yang & Liu 2019) includes a total of 162,264 flare events from different stars (normalized to the same stellar distance) and is shown in Figure 4. This size distribution describes the largest available solar and stellar flare data set and

![Figure 2. Differential occurrence size distributions of solar hard X-ray peak fluxes (left panels) and total counts (right panels), observed with three different instruments (HXRBS, BATSE, and RHESSI; binned histograms) and fitted with the model functions (solid curves) over an inertial range of \([x_0, x_3]\). For comparison, the power-law slope is indicated with a dotted line.](image)
extends over more than 4 decades; thus, it can be modeled with much higher statistical accuracy than even the solar data sets (Figure 2). The size distribution of the Kepler flare luminosities has a power-law slope of $\alpha_F = 1.813 \pm 0.004$, which is similar to the hard X-ray fluxes of solar flares (with $\alpha_F = 1.79 \pm 0.06$, averaged from Figures 2(b) and (c)). It is gratifying to find similar power-law slopes for solar and stellar flare fluxes, which can be interpreted in terms of an identical physical process operating in both cases (such as the magnetic reconnection process).

The size distribution of stellar flare luminosities observed by Kepler is best fitted with the extreme event model, exhibiting a goodness of fit of $\chi = 1.48$ (Figure 4). The inertial range $[x_0 = 7 \times 10^{33}, x_3 = 1.5 \times 10^{37}]$ erg is bracketing the power-law part, over which our model fits apply. Some exponential cutoff around $x_3 = q_{\exp} x_2 = 0.04$ $x_2 \approx 1.5 \times 10^{37}$ erg indicates finite system-size behavior, and the upper range range $[x_0, x_3]$ (with the maximum size value at $x_2 = 4 \times 10^{38}$ erg) indicates extreme events beyond the exponential cutoff, making up for a fraction of $q_{\text{pow}} = 0.11$. It appears that 11% of the flare sizes belong to a group of high-energy flare events with extreme energies of $x \lesssim x_3$, while the remaining 89% of the flares constitute a different group with low-energy events.

3.5. Stellar Spectral Types

Since the entire Kepler flare catalog contains at least six different stellar spectral types (Davenport 2016; Yang & Liu 2019), it is of interest to study which spectral types produce the most extreme events and whether finite system-size effects occur that give us some stellar diagnostics. For instance, is the depth of the stellar convection zone related to the finite system size inferred from size distributions?

We make use of the stellar spectral type classification of the Kepler stellar flare catalog (A, F, G, K, M, giants) for which the size distributions and power-law fits are presented in Figure 3 of Yang & Liu (2019), which is reproduced in Figure 5 here. A list of the power-law slopes $\alpha$ obtained by Yang & Liu (2019), with the power-law slopes $\alpha_F$, $\alpha_E$, and $\alpha_G$ of our three models and the mean values (of the three models), is provided in Table 2. The agreement of the power-law slopes $\alpha$ between the two studies is largely compatible. The differences are generally small: 14% for F-type stars, 9% for G-type, 1% for K-type, 16% for M-type, and 5% for giants, except for A-type stars (with a difference of 32%). We attribute this discrepancy of the power-law slope in A-type stars $\alpha = 1.65 \pm 0.07$ (Figure 5 here) to a flawed fit by an inadequate choice of the fitting range, which nullifies the claim of Yang & Liu (2019) that A-type stellar flares are produced by a different physical mechanism. The model fits shown in Figure 5 exhibit a straight power law for K-type stars only (Figure 5(d)), which also shows the best agreement (1%) between the two studies. All other spectral types exhibit significant deviations from straight power laws, which explains the discrepancies between the power-law fits of

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**Figure 3.** Differential occurrence size distributions of solar flare durations, observed with three different instruments (HXRBS, BATSE, and RHESSI; binned histograms) and fitted with the model function (solid curve) over an inertial range of $[x_0, x_3]$. The size $x_3$ refers to the finite system-size model.

**Figure 4.** Differential occurrence size distributions of stellar flare energy observed with Kepler (binned histograms) and fitted with the model function (solid curve) over an inertial range of $[x_0, x_3]$. The size $x_3$ refers to the finite system-size model. The power-law slope is indicated with a dotted diagonal.

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**Table 2.** The agreement of the power-law slopes $\alpha$ of Yang & Liu 2019 versus the power-law slopes $\alpha_F$ of our three models and the mean values (of the three models), except for A-type stars (with a difference of 32%).
Yang & Liu (2019) and our model fits. While arbitrary inertial ranges \([x_0, x_2]\) have been used in the past to fit a power law, which is ill-defined in the absence of straight power laws, we recommend to using the Pareto distribution model (Equation (3)) to accommodate the incomplete sampling of small events or, more adequately, the model with extreme events (Equation (5)). The goodness of fit \(\chi^2 < 2\) of these models is found to be acceptable for all stellar spectral types (Figure 5).

We show a synopsis of the flare energy size distributions for the six spectral types of stellar flares juxtaposed with the size distributions of all stellar and solar flare events in Figure 6. This diagram demonstrates that incomplete sampling occurs up to energies of \(E \lesssim 10^{35}\) erg, while the power law–like inertial ranges are confined to \(E \approx 10^{35}–10^{37}\) erg. The most energetic flare events occur on giant stars in the energy range of \(E \approx 10^{37}–10^{38}\) erg, perhaps capped by extreme events at energies of \(E \approx (1-5) \times 10^{38}\) erg. This upper limit is slightly higher than that derived from sparse statistics of stellar flares \((E \approx 10^{37})\) obtained before the Kepler mission (see Figure 3 in Schrijver et al. 2012).

We find that the median or maximum flare energy \(E_{\text{max}}\) in each stellar spectral type is related to the size of the star, based on the mean or maximum of the stellar radius \(R/R_{\odot}\) (Figure 7), defined by the Hertzsprung–Russell diagram of the Harvard spectral classification of main-sequence and giant stars:

- Giants, O types \(R \approx 6.6 R_{\odot}\),
- B types \(R = 1.8–6.6 R_{\odot}\),
- A types \(R = 1.4–1.8 R_{\odot}\),
- F types \(R = 1.15–1.4 R_{\odot}\),
- G types \(R = 0.96–1.15 R_{\odot}\),
- K types \(R = 0.7–0.96 R_{\odot}\),
- M types \(R \lesssim 0.7 R_{\odot}\).

We show this relationship for the median flare energy \(E_{\text{med}}\) (diamonds) in Figure 7, as well as for the maximum flare energy \(E_{\text{max}}\) (plus signs in Figure 7), sampled from the size distributions of the Kepler data (Figure 6). This relationship supports the notion that the maximum flare energy \(E_{\text{max}}\) scales...
with the spatial size (or stellar radius $R$), stellar surface area, or vertical depth of the stellar convection zone. In Figure 7, we also show the relationship $E \propto (R^2 + R_M^2)$, which expresses that the stellar area ($4\pi R^2$) is an upper limit of the flare energy for giants and A-type stars (since $R > R_M$), while M-, K-, F-, and G-type stars have flare energies with a lower limit at $R_M = 0.7 R_e$ (since $R < R_M$), possibly indicating a lower limit of the emission from the stellar corona, besides the chromospheric emission. Consequently, we can associate the largest flare energies in the stellar flare size distribution with the finite system-size limit.

4. Conclusions

Occurrence frequency distributions (or size distributions) come in all shapes and sizes. The most common case is the power-law function, which is easy to recognize by its straight line in a log(number)–log(size) representation, but one also finds curved functions that exhibit significant deviations from straight power laws and thus require refined models that can accommodate incomplete sampling, instrumental sensitivity limitations, event-unrelated background noise, truncation bias, finite system-size effects, and outliers of extreme events, possibly caused by multiple physical mechanisms, rather than a singular power-law function produced by a classical SOC model. In order to illustrate the meaning of finite system-size effects in the context of SOC models, see Appendix B and Figure 10. In this study, we focus on the existence or absence of finite system-size effects, which requires large statistics with $n_{\text{ev}} \gtrsim 10^4$ events in a sample. We apply our data analysis to solar and stellar data sets, as well as to other empirical “real-world”
data sets of interest to our human population (Appendix B). Our conclusions are summarized in the following.

1. We define three generalized power-law functions (Figure 1): (i) the Pareto distribution that shows a flattening at the lower end due to incomplete sampling and a straight power law in the tail model (PM); (ii) the Pareto distribution with an exponentially dropping cutoff in the tail that is caused by finite system-size effects (FM); and (iii) an additional power-law component that extends above the exponential cutoff, caused by outlier events that are also called the Dragon King extreme events model (EM). We find that each of the three model functions can fit most of the 24 observed size distributions analyzed here with a goodness of fit of $\chi^2 < 2$. This quantitative result is superior to previous straight power-law fits, where (i) arbitrary (inertial) fitting ranges are used, (ii) power laws are ill-defined for curved size distributions, and (iii) no goodness of fit is specified.

2. We revisit the solar flare size distributions of the hard X-ray peak flux, the hard X-ray fluence, and flare durations (Figures 2 and 3). We corroborate previous findings that the peak fluxes and fluences can be well fitted by Pareto distributions, with no sign of finite system-size effects in the tail, indicating that larger superflares are possible above the currently observed maximum size limit of $L \lesssim 200 \text{ Mm}$. For flare durations, we find that the best fits display an exponential cutoff in the tail of $T_2 \lesssim 6 \text{ hr}$, which suggests that solar flares cannot last longer in a coherent way. Size distributions of flare durations show no straight power-law part, partially because of the relatively smaller inertial ranges.

3. The Kepler mission provides the largest statistics ($n_{\text{obs}} \approx 2.6 \times 10^3$) of stellar flare energy size distributions and thus facilitates the most detailed modeling of size distributions (Figure 4). For the total of all Kepler observed flares, we find an approximate inertial range of $E \approx 10^{34} - 10^{37} \text{ erg}$, a finite system-size range of $E \approx 10^{37} - 10^{38} \text{ erg}$, and extreme events at $E \approx (1-5) \times 10^{38} \text{ erg}$. For smaller subsets of Kepler data (e.g., Figure 5), the detection of finite system-size effects is less feasible.

4. Subdividing the Kepler flares into six stellar spectral types (Figures 5 and 6), we find that the maximum flare energy $E_{\text{max}}$ in each stellar spectral type is related to the stellar radius $R/R_*$. (Figure 7), based on the mean stellar radius defined by the Hertzsprung–Russell diagram of the Harvard spectral classification. This relationship supports the notion that the maximum flare energy may also scale with the vertical depth $z$ of the stellar convection zone, based on a convection cell geometry with compatible vertical and horizontal sizes, $z \approx 2R$. Consequently, we expect that the stellar surface area $(4\pi R^2)$ represents a finite system-size limit for flare energies in every stellar spectral type.

This study demonstrates that we can extract a large amount of new knowledge from model fitting of power law–like size distribution functions sampled from a host of interdisciplinary

![Figure 8](image-url)

**Figure 8.** Differential occurrence size distributions of empirical data sets compiled by Clauset et al. (2009; binned histograms) and fitted with the model functions (solid curves) over an inertial range of $[x_0, x_2]$. 

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data sets pertaining to nonlinear energy dissipation events in SOC models. Although power-law distributions are considered to be the hallmark of SOC models (Bak et al. 1987, 1988; Aschwanden 2011a, 2016), a large number of pertaining data sets exhibit significant deviations from straight power laws, which justifies the eloquent criticism of Stumpf & Porter (2012). We have demonstrated here that refined modifications of power-law–like functions, such as Pareto distribution functions, exponential cutoff functions, and dual (or multiple) power-law functions, are essential for more accurate fitting of observed size distributions. Future work may focus on finite system-size effects in solar and stellar flares, which may provide new knowledge on stellar convection and dynamos.

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Appendix A

Finite System-size Effects in Empirical Data

In the spirit of interdisciplinary research, we fit our power law–like size distribution functions to eight empirical data sets compiled by Clauset et al. (2009). Four of these eight cases show mostly Pareto distributions with a straight power-law function all the way to the upper end of the size distribution (Figure 8), covering information technology (word frequencies, surnames, and web links), as well as geophysics (earthquakes). Four other data sets (Figure 9) exhibit clear deviations from straight power laws, such as wildfires and human activities (city sizes, blackouts, terrorism statistics). Our size distribution models FM and EM indicate some sort of finite system-size effects, either in the form of an exponential cutoff (see wildfires in Figure 9(b)) or in the form of a secondary power-law component of extreme events beyond the cutoff (see city sizes, blackouts, and terrorism in Figure 9). The two power-law components in the size distribution may indicate some duality of scales; wildfires may be subdivided into natural (e.g., by lightning) and human-induced fires (Krenn & Hergarten 2009), and city sizes may be subdivided into two groups, depending on whether suburban agglomerations are merged with urban areas or not. The Zipf law does not fit as well with the data when using a traditional administrative definition of cities (Veneri 2013). Blackouts may be subdivided depending on whether a detected blackout produces a continent-wide ripple (cascading) or not (Shuvro et al. 2018). Acts of terrorism may have different statistics, depending on whether they are organized for economic or political causes (Kirk 1983). The identification of such dual behavior could give us a deeper understanding of complex size distribution functions that show significant deviations from ideal power laws.

Figure 9. Differential occurrence size distributions of empirical data compiled by Clauset et al. (2009; binned histograms) and fitted with the model functions (solid curves) over an inertial range of \([x_0, x_2]\).
Appendix B

Finite System-size Effects in SOC

All known SOC systems have a finite system size. For instance, solar and stellar flares are limited by their finite solar or stellar surface area. Earthquakes or wildfires are ultimately bound by the Earth’s continents, etc. In the original sandpile model (Bak et al. 1987, 1988), upper limits on avalanche sizes are given by the geometric size of lattice 1D, 2D, or 3D grids. The largest events span the distance from the central region to the remote boundaries of a sandpile. We illustrate the phenomenon of finite size effects in Figure 10. In the fractal-diffusive SOC model of a slowly driven SOC system (Aschwanden 2012), the centroid of an avalanche propagates according to a random walk within the boundaries of a finite lattice grid and stops at the edge of the lattice grid (thick solid line in Figure 10), while the itinerary can cross the lattice boundaries, return to the inside area of the lattice, or even cross the boundaries multiple times (thin solid line in Figure 10), performing a random walk pattern. The radial extent of the avalanche is also called the radius of gyration (Charbonneau et al. 2001), which obviously can have hugely different values for finite or infinite systems. Numerical simulations of avalanches with finite system-size effects have been carried out in a number of studies (Pruessner 2012), which mostly demonstrated that the size distribution can be approximated with an exponential cutoff function (Equation (4)). Alternatively, some simulations show an additional bump near the exponential cutoff (see p. 33 in Pruessner 2012).

Figure 10. Example of a 2D random walk (or classical diffusion) in a finite system-size lattice grid (trajectory with thick solid line) and an infinite system without boundaries (trajectory with thin solid line). Both travel paths start at the same location but end at a different location, either at the system boundary or outside of the finite system. The gyration radii of the two random walks are indicated with dashed circles, differing by a factor of about 2.
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