IGUSA AND DENEF-SPERBER CONJECTURES ON NONDEGENERATE $p$-ADIC EXPONENTIAL SUMS

by

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Abstract. — We prove the intersection of Igusa’s Conjecture of [Igusa, J., Lectures on forms of higher degree, Lect. math. phys., Springer-Verlag, 59 (1978)] and the Denef - Sperber Conjecture of [Denef, J. and Sperber, S., Exponential sums mod $p^n$ and Newton polyhedra, Bull. Belg. Math. Soc., suppl. (2001) 55-63] on nondegenerate exponential sums modulo $p^m$.

1. Introduction

Let $f$ be a polynomial over $\mathbb{Z}$ in $n$ variables. Consider the “global” exponential sum

$$S_f(N) := \frac{1}{N^n} \sum_{x \in \{0, \ldots, N-1\}^n} \exp(2\pi i \frac{f(x)}{N}),$$

where $N$ varies over the positive integers. In order to bound $|S_f(N)|$ in terms of $N$, it is enough to bound

$$|S_f(p^m)|$$
in terms of $m > 0$ and prime numbers $p$. When $f$ is nondegenerate in several senses related to its Newton polyhedron, specific bounds which depend uniformly on $m$ and $p$ have been conjectured by Igusa and by Denef - Sperber.

We prove these bounds, thus solving a conjecture by Denef and Sperber from a 1990 manuscript [7] (published in 2001 [8]), and the nondegenerate case of Igusa’s conjecture for exponential sums from the introduction of his book [10].

One of the main points of this article is that, while for finite field exponential sums like $S_f(p)$ one knows that the weights and Betti numbers have some uniform behaviour for big $p$, for $p$-adic exponential sums $S_f(p^m)$ one does not yet completely

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know what the analogues of the weights and Betti numbers are, let alone that they have some uniform behaviour in $p$.

**Notation.** — Let $f$ be a nonconstant polynomial over $\mathbb{Z}$ in $n$ variables with $f(0) = 0$.[1] Write $f(x) = \sum_{i \in \mathbb{N}^n} a_i x^i$ with $a_i \in \mathbb{Z}$. The global Newton polyhedron $\Delta_{\text{global}}(f)$ of $f$ is the convex hull of the support $\text{Supp}(f)$ of $f$, with

$$\text{Supp}(f) := \{i \mid i \in \mathbb{N}^n, a_i \neq 0\}.$$ 

The Newton polyhedron $\Delta_0(f)$ of $f$ at the origin is

$$\Delta_0(f) := \Delta_{\text{global}}(f) + \mathbb{R}^n_+$$

with $\mathbb{R}^n_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$ for $A, B \subset \mathbb{R}^n$. For a subset $I$ of $\mathbb{R}^n$ define

$$f_I(x) := \sum_{i \in I \cap \mathbb{N}^n} a_i x^i.$$ 

By the faces of $I$ we mean $I$ itself and each nonempty convex set of the form

$$\{x \in I \mid L(x) = 0\}$$

where $L(x) = a_0 + \sum_{i=1}^n a_i x_i$ with $a_i \in \mathbb{R}$ is such that $L(x) \geq 0$ for each $x \in I$. By the proper faces of $I$ we mean the faces of $I$ that are different from $I$. For $I$ a collection of subsets of $\mathbb{R}^n$, call $f$ nondegenerate with respect to $I$ when $f_I$ has no critical points on $(\mathbb{C}^\times)^n$ for each $I$ in $\mathcal{I}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. For $k \in \mathbb{R}^n_+$ put

$$\nu(k) = k_1 + k_2 + \ldots + k_n,$$

$$N(f)(k) = \min_{i \in \Delta_0(f)} k \cdot i,$$

$$F(f)(k) = \{i \in \Delta_0(f) \mid k \cdot i = N(f)(k)\},$$

where $k \cdot i$ is the standard inproduct on $\mathbb{R}^n$. Denote by $F_0(f)$ the smallest face of $\Delta_0(f)$ which has nonempty intersection with the diagonal $\{(t, \ldots, t) \mid t \in \mathbb{R}\}$ and let $(1/\sigma(f), \ldots, 1/\sigma(f))$ be the intersection point. If there is no confusion, we write $\sigma$ instead of $\sigma(f)$, $N(k)$ instead of $N(f)(k)$, and $F(k)$ for $F(f)(k)$. Let $\kappa$ be the codimension of $F_0(f)$ in $\mathbb{R}^n$.

2. The main results

From here up to section 9 let $f$ be a nonconstant polynomial over $\mathbb{Z}$ in $n$ variables with $f(0) = 0$ (the adaptation to the case $f(0) \neq 0$ is easy).

**2.1. Theorem.** — Suppose that $f$ is homogeneous and nondegenerate w.r.t. the faces of $\Delta_0(f)$. Then there exists $c > 0$ and $M > 0$ such that

$$|S_f(p^m)| \leq cp^{-\sigma m} m^{\kappa - 1}$$

(1) When $f(0) \neq 0$, then there is no harm in replacing $f$ by $f - f(0)$: all corresponding changes in the paper are easily made.
for all primes \( p > M \) and all integers \( m > 0 \), with \( \sigma = \sigma(f) \) and \( \kappa \) as in the section on notation. Moreover, \( c \) can be taken depending on \( \Delta_0(f) \) only.

One sees that the dependence on \( p \) and \( m \) is very simple. Since moreover for each \( p \) there exists \( c_p > 0 \) such that for all \( m > 0 \)

\[
|S_f(p^m)| \leq c_p p^{-\sigma m} m^{n-1},
\]

by \([10],[9],[3]\), and properties of toric resolutions, and since \( \kappa \leq n \) one finds the following.

**2.2. Corollary.** — With \( f \) as in Theorem 2.1 there exists \( c > 0 \) such that for all primes \( p \) and all integers \( m > 0 \),

\[
|S_f(p^m)| \leq cp^{-\sigma m} m^{n-1}.
\]

Denef and Sperber \([8]\) prove Theorem 2.1 under the extra condition that no vertex of \( F_0(f) \) belongs to \( \{0,1\}^n \). Corollary 2.2 is the nondegenerate case of Igusa’s conjecture \([10]\) on exponential sums for toric resolutions of \( f \). The exponent \( \sigma \) in the bounds of Theorem 2.1 is conjectured to be optimal for infinitely many \( p \) and \( m \) by Denef and Sargos \([5],[6]\), in analogy to conjectures on the real case. When no vertex of \( F_0(f) \) belongs to \( \{0,1,2\}^n \), the bounds in Theorem 2.1 are shown to be optimal for infinitely many \( p \) and \( m \) in \([8]\), Theorem (1.3).

### 3. Denef - Sperber Formula for \( S_f(p^m) \) for big \( p \)

The following proposition has the same proof as Proposition (2.1) of \([8]\), but is slightly more general. We give the proof for the convenience of the reader.

**3.1. Proposition.** — Suppose that \( f \) is nondegenerate w.r.t. (all) the faces of \( \Delta_0(f) \). Then there exists \( M > 0 \) such that

\[
S_f(p^m) = (1 - p^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} (A(p, m, \tau) + E(p, f, \tau)B(p, m, \tau))
\]

for all primes \( p > M \) and all integers \( m > 0 \), with

\[
A(p, m, \tau) := \sum_{k \in \mathbb{N}^n \atop F(f)(k) = \tau \atop N(f)(k) \geq m} p^{-\nu(k)},
\]

\[
B(p, m, \tau) := \sum_{k \in \mathbb{N}^n \atop F(f)(k) = \tau \atop N(f)(k) = m - 1} p^{-\nu(k)}.
\]
and

\[(3.1.2) \quad E(p, f_\tau) := \frac{1}{(p-1)^n} \sum_{x \in \{1, \ldots, p-1\}^n} \exp \left( \frac{2\pi i}{p} f_\tau(x) \right). \]

Proof. — Writing

\[ S_f(p^m) = \int_{\mathbb{Z}_p^m} \exp \left( \frac{2\pi i}{p^m} f(x) \right) |dx|, \]

with \(|dx|\) the normalized Haar measure on \(\mathbb{Q}_p^m\), we deduce

\[ S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{k \in \mathbb{N}^n} \int_{\ord x = k, F(f)(k) = \tau} \exp \left( \frac{2\pi i}{p^m} f(x) \right) |dx|. \]

Put \(x_j = p^{k_j}u_j\) for \(k \in \mathbb{N}^n\). Then \(|dx| = p^{-\nu(k)}|du|\) and

\[ f(x) = p^{N(k)}(f_{F(f)}(u) + p(...)), \]

where the dots take values in \(\mathbb{Z}_p\) and where \(N(k) = N(f)(k)\). Hence,

\[(3.1.3) \quad S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{k \in \mathbb{N}^n} p^{-\nu(k)} \int_{u \in (\mathbb{Z}_p^m)^n} \exp \left( \frac{2\pi i}{p^m} (f_\tau(u) + p(...)) \right) |du|, \]

where \(\mathbb{Z}_p^m\) denotes the group of \(p\)-adic units. Because of the nondegenerated assumptions, for \(\tau\) a face of \(\Delta_0(f)\) and \(p\) a big enough prime, the reduction \(f_\tau \mod p\) has no critical points on \((\mathbb{F}_p \times \mathbb{Z}_p^m)^n\). Hence, the integral in \((3.1.3)\) is zero whenever \(m - N(k) \geq 2\). When \(m - N(k) \leq 0\), the integral over \((\mathbb{Z}_p^m)^n\) in \((3.1.3)\) is just the measure of \((\mathbb{Z}_p^m)^n\) and thus equals \((1 - p^{-1})^n\). When \(m - N(k) = 1\) the integral over \((\mathbb{Z}_p^m)^n\) in \((3.1.3)\) equals \(p^{-n}(p - 1)^n E(p, f_\tau)\). The Proposition now follows from \((3.1.3)\). \(\square\)

4. Lower bounds for \(\nu(k)\)

The main result of this section is:

4.1. Theorem. — Let \(\tau\) be a face of \(\Delta_0(f)\). Then one has for all \(k\) with \(F(f)(k) = \tau\) that

\[(4.1.1) \quad \nu(k) \geq \sigma(f)(N(f)(k) + 1) - \sigma(f_\tau), \]

where \(\sigma(f)\) and \(\sigma(f_\tau)\), as well as \(\nu(k)\) and \(N(f)(k)\) are as in the section on notation.

The main points are that one subtracts \(\sigma(f_\tau)\) instead of \(\sigma(f)\) and that \(\sigma(f_\tau) \leq \sigma(f)\). Subtracting \(\sigma(f)\) would yield trivial bounds since one has \(\nu(k) \geq \sigma(f) N(k)\) for all \(k \in \mathbb{R}_+^n\). The Theorem’s proof is based on two facts:
4.2. Lemma. — Let \( \tau \) be a face of \( \Delta_0(f) \), and let \( R_j \in \mathbb{R}^n \) be finitely many points belonging to \( \tau \). Let \( \beta_j \geq 0 \) satisfy
\[
\sum_j \beta_j R_j \leq (1/\sigma, \ldots, 1/\sigma),
\]
where \( a \leq b \) for \( a, b \in \mathbb{R}^n \) means \( a_i \leq b_i \) for all \( i \). Then
\[
\sum \beta_j \leq 1.
\]

Proof. — Clearly there is no point \( S \) in the interior of \( \Delta_0(f) \) that satisfies \( S \leq (1/\sigma, \ldots, 1/\sigma) \). When \( \sum_j \beta_j > 1 \), then \( \sum_j \beta_j R_j \) lies in the interior of \( \Delta_0(f) \).

4.3. Corollary. — Let \( \tau \), \( R_j \), and \( \beta_j \) be as in Lemma 4.2. Then
\[
\sum \beta_j \leq \frac{\sigma(f_\tau)}{\sigma}.
\]

Proof. — Since \( \sum_j \beta_j R_j \leq (1/\sigma, \ldots, 1/\sigma) \) one has
\[
\frac{\sigma}{\sigma(f_\tau)} \sum_j \beta_j R_j \leq (1/\sigma(f_\tau), \ldots, 1/\sigma(f_\tau)).
\]

Lemma 4.2 thus implies
\[
\frac{\sigma}{\sigma(f_\tau)} \sum_j \beta_j \leq 1.
\]

Proof of Theorem 4.1 — Since \( (1/\sigma, \ldots, 1/\sigma) \) lies in the interior of \( F_0(f) \), by convexity one can write
\[
(1/\sigma, \ldots, 1/\sigma) = \sum_i \alpha_i P_i + \sum_j \beta_j R_j
\]
for some \( \alpha_i \geq 0 \) and \( \beta_j \geq 0 \) with \( \sum_i \alpha_i + \sum_j \beta_j = 1 \) and with \( P_i \) finitely many integral points of \( F_0(f) \setminus \tau \) and \( R_j \) finitely many integral points of \( \tau \). For \( k \in \mathbb{N}^k \) with \( F(f)(k) = \tau \) calculate
\[
\nu(k) = \sigma(1/\sigma, \ldots, 1/\sigma) \cdot k
\]
\[
= \sigma \left( \sum_i \alpha_i P_i + \sum_j \beta_j R_j \right) \cdot k
\]
\[
= \sigma \left( \sum_i \alpha_i P_i \cdot k + \sum_j \beta_j R_j \cdot k \right)
\]
\[
\geq \sigma \left( \sum_i \alpha_i (N(k) + 1) + \sum_j \beta_j N(k) \right)
\]
\[
= \sigma \left( \sum_i \alpha_i + \sum_j \beta_j \right)(N(k) + 1) - \sum_j \beta_j)
\]
\[
= \sigma \left( (N(k) + 1) \right) - \sum_j \beta_j)
\]
\[
\geq \sigma \left( (N(k) + 1) - \frac{\sigma(f_\tau)}{\sigma} \right)
\]
where (4.3.5) follows from \(k \cdot R_j = N(k)\) and \(k \cdot P_i \geq N(k) + 1\) which is true by definition of \(N(k)\), and where (4.3.8) follows from Corollary 4.3.

5. Upper bounds for \(A(p, m, \tau)\) and \(B(p, m, \tau)\)

We recall one result from [8].

5.1. Lemma ([8, Lemma (3.3)]). — Let \(C\) be a convex polyhedral cone in \(\mathbb{R}_+^n\) generated by vectors in \(\mathbb{N}^n\), and let \(L\) be a linear form in \(n\) variables with coefficients in \(\mathbb{N}\). We denote by \(C^{\text{int}}\) the interior of \(C\) in the sense of Newton polyhedra. Let \(\sigma > 0\) and \(\gamma \geq 0\) be real numbers satisfying

\[
(5.1.1) \quad \nu(k) \geq L(k)\sigma + \gamma, \quad \text{for all } k \in C^{\text{int}} \cap \mathbb{N}^n.
\]

Put

\[
e = \dim\{k \in C \mid \nu(k) = L(k)\sigma\}.
\]

Then there exists a real number \(c > 0\) such that for all \(m \in \mathbb{N}\) and for all \(p \in \mathbb{R}\), with \(p \geq 2\),

\[
(5.1.2) \quad \sum_{k \in C^{\text{int}} \cap \mathbb{N}^n \atop L(k) = m} p^{-\nu(k)} \leq cp^{-m\sigma-\gamma} (m + 1)^{\max(0,e-1)}.
\]

From this Lemma and from Theorem 4.1 follows:

5.2. Corollary. — Let \(f\), \(A(p, m, \tau)\), and \(B(p, m, \tau)\) be as in Proposition 3.1. Then there exists a real number \(c > 0\) such that for all integers \(m > 0\), for all faces \(\tau\) of \(\Delta_0(f)\), and for all big enough primes \(p\)

\[
(5.2.1) \quad A(p, m, \tau) \leq cp^{-m\sigma} m^{\kappa-1}
\]

and

\[
(5.2.2) \quad B(p, m, \tau) \leq cp^{-m\sigma+\sigma(f_\tau)} m^{\kappa-1}.
\]

Proof. — To derive (5.2.1) from Lemma 5.1 note that \(\nu(k) \geq N(k)\sigma\) for any \(k \in \mathbb{N}^n\), and that \(\kappa = \dim\{k \in \mathbb{R}_+^n \mid \nu(k) = N(k)\sigma\} \geq 1\).

To derive (5.2.2) from Lemma 5.1 and Theorem 4.1 use for \(C\) the topological closure of the convex hull of \(\{0\} \cup \{k \in \mathbb{N}^n \mid F(k) = \tau\}\), and note that \(C^{\text{int}} \cap \mathbb{N}^n = \{k \in \mathbb{N}^n \mid F(k) = \tau\}\). Clearly \(\kappa \geq 1\) and \(\kappa \geq \dim\{k \in C \mid \nu(k) = N(k)\sigma\}\). By (4.1.1), \(\nu(k) \geq N(k)\sigma + \sigma - \sigma(f_\tau)\) for all \(k \in C^{\text{int}} \cap \mathbb{N}^n\). \(\square\)
6. Upper bounds for $\sigma(f)$ and $E(p, f_\tau)$

By Theorem 4 of Katz \[11\], for $f(x)$ a nonconstant homogeneous polynomial in $n$ variables over $\mathbb{Z}$, and $d$ the dimension of grad $f = 0$ in $\mathbb{A}^n_\mathbb{C}$, there exists $c$ such that for all big enough $p$ one has

\begin{equation}
| \sum_{x \in \mathbb{A}^n_F} \exp \left( \frac{2\pi i}{p} f(x) \right) | \leq cp^{\frac{n+d}{2}}.
\end{equation}

Moreover, $c$ can be taken depending on the degree of $f$ only. This implies:

6.1. **Corollary.** — Suppose that $f(x)$ is homogeneous of degree $\geq 2$ and let $d$ be the dimension of grad $f = 0$ in $\mathbb{A}^n_\mathbb{C}$. Then there exists $c$ such that for all $p$ big enough

\begin{equation}
| \sum_{x \in \mathbb{G}^n_m(F_p)} \exp \left( \frac{2\pi i}{p} f(x) \right) | < cp^{\frac{n+d}{2}}
\end{equation}

and hence, for some $c'$ one has, for all big enough $p$,

\begin{equation}
| E(p, f) | < c'p^\frac{n+d}{2}
\end{equation}

with $E(p, f)$ as defined by (3.1.2). Moreover, $c$ and $c'$ can be taken depending on $\Delta_0(f)$ only.

**Proof.** — Let $f_0(x_2, \ldots, x_n)$ be the polynomial $f(0, x_2, \ldots, x_n)$. Clearly $f_0$ is homogeneous in $n - 1$ variables. By Katz’ result (6.0.3) it is enough to show that $n - 1 + d(f_0) \leq n + d$, with $d(f_0)$ the dimension of grad $f_0 = 0$ in $\mathbb{A}^{n-1}_\mathbb{C}$. This inequality follows from writing

$$f(x) = x_1 g(x) + f_0(x_2, \ldots, x_n)$$

with $g$ a polynomial in $x$, and comparing grad $f$ with grad $f_0$. \qed

6.2. — Let $\{(N_i, \nu_i)\}_{i \in I}$ be the numerical data of a resolution $h$ of $f$ with normal crossings (that is, if $\pi_f : Y \to \mathbb{A}^n_\mathbb{C}$ is an embedded resolution of singularities with normal crossings of $f = 0$, then, for each irreducible component $E_i$ of $\pi_{f}^{-1} \circ f^{-1}(0)$, $i \in I$, let $N_i$ be the multiplicity of $E_i$ in div$(f \circ \pi_f)$, and $\nu_i - 1$ the multiplicity of $E_i$ in the divisor associated to $\pi_{f}^* (dx_1 \wedge \ldots \wedge dx_n)$, cf. \[3\]). The essential numerical data of $\pi_f$ are the pairs $(N_i, \nu_i)$ for $i \in J$ with $J = I \setminus I'$ and where $I'$ is the set of indices $i$ in $I$ such that $(N_i, \nu_i) = (1, 1)$ and such that $E_i$ does not intersect another $E_j$ with $(N_j, \nu_j) = (1, 1)$. Define $\alpha(\pi_f)$ as

\begin{equation}
\alpha(\pi_f) = - \min_{i \in J} \frac{\nu_i}{N_i}
\end{equation}

when $J$ is nonempty and define $\alpha(\pi_f)$ as $-2n$ otherwise.

It follows from \[2\], Theorem 5.1 and Corollary 3.4, that

\begin{equation}
\alpha(\pi_f) \geq \frac{-n + d}{2},
\end{equation}
with \( d \) the dimension of \( \text{grad } f = 0 \) in \( \mathbb{A}^n_C \), and where the empty scheme has dimension \(-\infty\).

**6.3. Lemma.** — Let \( f \) be homogeneous of degree \( \geq 2 \) and nondegenerate w.r.t. the faces of \( \Delta_0(f) \). Let \( d \) be the dimension of \( \text{grad } f = 0 \) in \( \mathbb{A}^n_C \). Then

\[
\sigma(f) \leq \frac{n - d}{2}.
\]

*Proof.* — By properties of a toric resolution \( \pi_f \) of \( f = 0 \), one has that

\[
\sigma(f) = -\alpha(\pi_f),
\]

with \( \alpha(\pi_f) \) as defined by (6.2.1). Now use (6.2.2).

\[ \square \]

From (6.3.1) and Corollary 6.1 applied to \( f_\tau \), follows:

**6.4. Corollary.** — Let \( f \) be a homogeneous polynomial of degree \( \geq 2 \) which is nondegenerate w.r.t. the faces of \( \Delta_0(f) \). Then there exists \( c \) such that for all faces \( \tau \) of \( \Delta_0(f) \) and all big enough primes \( p \)

\[
|E(p, f_\tau)| < cp^{-\sigma(f_\tau)}
\]

with \( E(p, f_\tau) \) as defined by (3.1.2). Moreover, \( c \) can be taken depending on \( \Delta_0(f) \) only.

**7. Proof of the main theorem**

*Proof of Theorem 2.1* — When the degree of \( f \) is \( \geq 2 \), use Proposition 3.1, Corollary 5.2, and (6.4.1). For linear \( f \) the theorem is trivial.

\[ \square \]

**8. Comparison with the Denef-Sperber approach**

As mentioned above, Denef and Sperber \cite{8} prove Theorem 2.1 under the extra condition that no vertex of \( F_0(f) \) belongs to \( \{0,1\}^n \). Key points in our proof of Theorem 2.1 are (4.1.1) (which implies Corollary 5.2 and (6.4.1). Instead of (4.1.1), Denef and Sperber used their result that, for similar \( k \) as in (4.1.1) but assuming the extra condition that no vertex of \( F_0(f) \) belongs to \( \{0,1\}^n \),

\[
\nu(k) \geq \sigma(f)(N(f)(k) + 1) - \frac{\dim \tau + 1}{2}.
\]

This often fails if one omits the extra condition, see Examples (1) and (2) below. Instead of (6.4.1), they used the Adolphson-Sperber \cite{1}, Denef-Loeser \cite{4} bounds

\[
|E(p, f_\tau)| < cp^{-\frac{\dim \tau - 1}{2}}
\]
which hold (in particular) under the same conditions as for (6.4.1), but which are sometimes not as good as the bounds (6.4.1).\footnote{Although (8.0.3) is sometimes sharper than (6.4.1) in cases where it does not matter for our course.}

We give two examples where our methods really make a difference with (8.0.2) and (8.0.3).

**Examples.**

1. First, for $f(x, y, z, u) = xy + zu$ and $\tau = F_0(f)$, one has $\dim \tau = 1$, $\sigma(f_\tau) = 2$, (8.0.2) does not hold and (8.0.3) is not optimal, while (6.4.1) yields the optimal $|E(p, f_\tau)| < cp^{-2}$.

2. Secondly, for $f(x, y, z, u) = xy + zu + xz + ayu$ with $a \in \mathbb{Z}$, $a \neq 1$, and $\tau = F_0(f)$, one has $\dim \tau = 2$, $\sigma(f_\tau) = 2$, (8.0.2) does not hold and (8.0.3) is not optimal, while (6.4.1) yields again the optimal $|E(p, f_\tau)| < cp^{-2}$ for big $p$.

In this example, $E(p, f_\tau)$ can be calculated by performing a transformation on $G_m^4$ coming from an element of $GL_n(\mathbb{Z})$ transforming $f(x)$ into $f(x', y', z', u') = x' + y' + z' + ax'y'z'^{-1}$; the bounds for $E(p, f_\tau)$ are surprisingly sharp compared, for example, to bounds for the resembling Kloosterman sums.

9. Analogues over finite extensions of $\mathbb{Q}_p$ and over $\mathbb{F}_q((t))$

For any nonarchimedean local field $K$ with valuation ring $\mathcal{O}_K$, write $\psi_K$ for an additive character

$$\psi_K : K \to \mathbb{C}^\times$$

that is trivial on $\mathcal{O}_K$ but nontrivial on some element of $K$ of order $-1$. Write $\text{ord}_K : K^\times \to \mathbb{Z}$ for the valuation, $| \cdot |_K : K \to \mathbb{R}$ for the norm on $K$, and $\bar{K}$ for its residue field, with $q_K$ elements. Let $k$ be a number field with ring of integers $\mathcal{O}_k$. In this section $f$ is a nonconstant polynomial over $\mathcal{O}_k[1/N]$ in $n$ variables, with $f(0) = 0$ and $N \in \mathbb{Z}$. For $K$ any nonarchimedean local field that is an algebra over $\mathcal{O}_k[1/N]$ and for $y \in K^\times$, consider the exponential integral

$$S_{f, K}(y) := \int_{\mathcal{O}_K^n} \psi_K(yf(x)) |dx|_K,$$

with $|dx|_K$ the normalized Haar measure on $K^n$. Note that $K$ may be of positive characteristic. Then the following generalization of Theorem 2.1 holds:

**9.1. Theorem.** — Suppose that $f$ is a homogeneous polynomial over $\mathcal{O}_k[1/N]$ which is nondegenerate w.r.t. the faces of $\Delta_0(f)$. Then there exist $c > 0$ and $M > N$ such that

$$|S_{f, K}(y)| \leq c |y|_K^\sigma |\text{ord}_K(y)|^{\kappa - 1}$$

for all nonarchimedean local fields $K$ that are algebras over $\mathcal{O}_k[1/N]$ and have residue characteristic $> M$, and all $y \in K^\times$ with $\text{ord}_K(y) < 0$, with $| \cdot |$ the complex norm. Moreover, $c$ can be taken depending on $\Delta_0(f)$ only.
Proof. — Same proof as of Theorem 2.1 using Proposition 9.2 instead of Proposition 3.1. □

9.2. Proposition. — Suppose that $f$ is nondegenerate w.r.t. (all) the faces of $\Delta_0(f)$. Then there exists $M > N$ such that

$$S_{f,K}(y) = (1 - q_K^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} (A(q_K, m, \tau) + E(\bar{K}, \tau, y)B(q_K, m, \tau))$$

for all nonarchimedean local fields $K$ that are algebras over $\mathcal{O}_k[1/N]$ and have residue characteristic $> M$ and all $y \in K^\times$ with $\text{ord}_K(y) \leq 0$.

In these formulas, $A(q_K, m, \tau)$ and $B(q_K, m, \tau)$ are as in Proposition 3.1, and

$$E(\bar{K}, \tau, y) := \frac{1}{(q_K - 1)^n} \sum_{u \in G^\times_m(\bar{K})} \psi_y(f_\tau(u)),$$

with $\psi_y$ a nontrivial additive character on $\bar{K}$ depending on $y$ and $\psi_K$.

Proof. — Same proof as of Proposition 3.1. □

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Footnotes:

(3) For such $K$ and for $y \in K^\times$ with $\text{ord}_K(y) \geq 0$, one has $S_{f,K}(y) = 1$.
(4) In fact, the character $\psi_y$ only depends on $\psi_K$ and on $\psi_K(y)$ for any multiplicative homomorphism $\psi: K^\times \to \bar{K}^\times$ extending the natural projection $\mathcal{O}_K^\times \to K^\times$. 

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