Some results on the second relative homology of Leibniz algebras

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Abstract

In this paper, the structure of the second relative homology and the relative stem cover of the direct sum of two pairs of Leibniz algebras are determined by means of the non-abelian tensor product of Leibniz algebras. We also characterize all pairs of finite dimensional nilpotent Leibniz algebras such that \( \dim(n) = n, \dim(g/n) = m \) and \( \dim(\text{HL}_2(g, n)) = n(n + 2m) - 3 \).

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1 Introduction

Leibniz algebras have been initially appeared in the papers of Bloh [5, 6, 7] as a non skew-symmetric analogue of Lie algebras which satisfying a certain condition the so-called Leibniz rule. They further explored by Loday [24, 25] for constructing a new (co)homology theory for Lie algebras, the so-called Leibniz (co)homology (see also [27]). Leibniz algebras are naturally applied to several areas of mathematics and physics such as differential geometry, homological algebra, algebraic topology, algebraic K-theory and non-commutative geometry.

Although some theoretic results in Leibniz algebras are straightforward parallel to Lie algebras, however, there are some complex difficulties in generalization of some topics, specially the classification problem of Leibniz algebras, one can see some distinguished references like [1, 2, 3, 12]. Also, see [13, 23] to compare the vastity of the classes of low dimensional nilpotent non-Lie Leibniz algebras.

The homology theory of Leibniz algebras was flourished at the beginning of the development of these algebras in a paper by Loday and Pirashvili [27], and immediately several concepts related

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to the theory of abelian extensions tied to low dimensional homologies, see [9]. From the practical standpoint to the theory of Baer-invariants, the relative Lie-central extension, Lie-cover and Lie-multiplier of Leibniz algebras with respect to the subcategory of Lie algebras were studied in a series of papers, see for instance [11, 8]. The experience of working with Lie algebras inspired us to employ the second homology of Leibniz algebras (which is called the Schur multiplier) and some related tools to achieve a new points of view to the characterization of Leibniz algebras, notably for nilpotent ones, see [15, 16, 22]. In [20], the authors introduced the second relative homology of a pair \((g, n)\) as a term of a spectral sequence, here \(n\) is an ideal of the Leibniz algebra \(g\). We also presented the structure of relative stem cover, a version of the Hopf’s formula and a certain upper bound for \(HL_2(g, n)\). In addition, we classified all pairs of finite dimensional nilpotent Leibniz algebras that have one or two steps distance to this upper bound. Recent further development has been carried out by Biyogman and Safa [4], with emphasis on the universal relative central extension of a perfect pair of Leibniz algebras \((g, n)\) in which \(n\) admits a complement.

This paper is devoted to derive some fruitful results about the second relative homology of a pair of Leibniz algebras. In the first step, as a generalization of the Künneth formula for the second Leibniz homology (see [26]), we determine the structure of the second relative homology of the pair of Leibniz algebras \((g_1 \oplus g_2, n_1 \oplus n_2)\) in terms of the second relative homologies of the pairs \((g_1, n_1)\) and \((g_2, n_2)\). As a consequence, for an arbitrary pair of finite dimensional nilpotent Leibniz algebras \((g, n)\), we obtain an inequality for the dimension of \(HL_2(g, n)\) in terms of the dimension of the commutator subalgebra and quotient algebras. This inequality improves the result of [4] in the case that \(n\) admits a complement in \(g\). Also, we develop the classification obtained in [20], where all pairs of finite dimensional nilpotent Leibniz algebras in which have three steps distance to the maximal possible bound for \(HL_2(g, n)\) are determined. Finally, we close this paper by describing the structure of the relative stem cover of the direct sum of two pairs of Leibniz algebras.

**Notations.** Throughout this paper, by a Leibniz algebra we mean a right Leibniz algebra over some fixed field \(F\). We write \(\otimes_F\) for the usual tensor product of vector spaces over \(F\). If \(a\) and \(b\) are subspaces of a vector space \(\mathfrak{k}\) for which \(\mathfrak{k} = a \oplus b\) and \(a \cap b = 0\), we will write \(\mathfrak{k} = a \downarrow b\).

## 2 The non-abelian exterior product of Leibniz algebras

This section is devoted to the study of the properties of the non-abelian tensor and exterior products of Leibniz algebras. We begin by recalling these concepts.

Let \(m\) and \(g\) be two Leibniz algebras. By an action of \(g\) on \(m\) we mean a couple of \(F\)-bilinear maps \(g \times m \to m, (x, m) \mapsto ^xm\) and \(m \times g \to m, (m, x) \mapsto m^x\), satisfying the axioms
can form the tensor products $c \in m \otimes x \in g$. Note that if $g$ is a subalgebra of some Leibniz algebra $p$ and $m$ is an ideal in $p$, then the Leibniz product in $p$ induces an action of $g$ on $m$ given by $x \star m = [x, m]$ and $m \star x = [m, x]$. In particular, there is an action of $p$ on itself given by the Leibniz product in $p$.

A (Leibniz) crossed module is a homomorphism of Leibniz algebras $\mu : m \to g$ together with an action of $g$ on $m$ such that $\mu(xm) = [x, \mu(m)]$, $\mu(m^x) = [\mu(m), x]$ and $\mu(m)x = \mu(m)x + [m, \mu(m)]$, for all $x \in g$, $m, m' \in m$. Plainly, if $m$ is an ideal of $g$, then the inclusion map $m \to g$ is a crossed module.

Let $\mu : m \to g$ and $\lambda : n \to g$ be two crossed modules of Leibniz algebras. Then $m$ and $n$ act on each other via the action of $g$. The non-abelian tensor product $m * n$ is defined in [19] as the Leibniz algebra generated by the symbols $m * n$ and $n * m$ ($m \in m$, $n \in n$), subject to the relations

\begin{align*}
(1a) \quad & cm * n = cm * n = m * cn, & (1b) \quad & cn * m = cn * m = n * cm, \\
(2a) \quad & (m + m') * n = m * n + m' * n, & (2b) \quad & (n + n') * m = n * m + n' * m, \\
(2c) \quad & m * (n + n') = m * n + m * n', & (2d) \quad & n * (m + m') = n * m + n * m', \\
(3a) \quad & m * [n, n'] = m^n * n' - m^n' * n, & (3b) \quad & n * [m, m'] = n^m * m' - n^m' * m, \\
(3c) \quad & [m, m'] * n = m^m * n' - m^n * m', & (3d) \quad & [n, n'] * m = n^m * n' - n^m' * m, \\
(4a) \quad & m * m'^n = -m * m'^n, & (4b) \quad & n * n'^m = -n * n'^m, \\
(5a) \quad & m^n * m'^n = [m * n, m' * n'] = m^n * m'^n, & (5b) \quad & n^m * m'^m = [n * m, n' * m'] = n^m * m'^m, \\
(5c) \quad & m^n * m'^m = [m * n, n' * m'] = m^n * m'^m, & (5d) \quad & n^m * m'^m = [n * m, n' * m'] = m^n * m'^m,
\end{align*}

for all $c \in \mathbb{F}$, $m, m' \in m$, $n, n' \in n$. Note that the identity map $id_g : g \to g$ is a crossed module, so we can form the tensor products $g * m$, $g * n$ and $g * g$.

Let $m \subseteq n$ be the vector subspace of $m * n$ spanned by the elements $m * n' - n * m'$ with $\mu(m) = \lambda(n)$ and $\mu(m') = \lambda(n')$. One easily gets that $m \subseteq n$ lies in the center of $m * n$. The non-abelian exterior product $m \wedge n$ is defined to be the quotient $(m * n)/(m \subseteq n)$. We write $m \wedge n$ and $n \wedge m$ to denote the images in $m \wedge n$ of the generators $m * n$ and $n * m$, respectively.

The following results give some information and properties of the above notions that will be needed.

**Lemma 2.1** ([14]). Let $g$ be a Leibniz algebra with ideals $n$ and $\mathfrak{k}$.

(i) If $n$ and $\mathfrak{k}$ act trivially on each other, then $n \otimes \mathfrak{k} \cong n^a \otimes \mathfrak{k}^b \cong (n^a \otimes \mathfrak{k}^b) \oplus (\mathfrak{k}^a \otimes \mathfrak{n}^b)$, where $n^a = n/n^2$ and $\mathfrak{k}^a = \mathfrak{k}/\mathfrak{k}^2$.

(ii) If $\mathfrak{k} \subseteq n$, then there is the following natural exact sequence of Leibniz algebras

$$
g \wedge \mathfrak{k} \xrightarrow{\alpha} g \wedge n \to \frac{g}{\mathfrak{k}} \wedge \frac{n}{\mathfrak{k}}.$$

Furthermore, if $g = n$ and $\mathfrak{k}$ has an ideal complement in $g$, then $\alpha$ is injective.
Proposition 2.2 ([21]). Let $g$ be any Lie algebra. Then there is an isomorphism of Lie algebras $\beta : g \times g \cong g \otimes g$ defined on generators by $\beta(x_1 \times x_2) = x_1 \otimes x_2$, where $\otimes$ denotes the non-abelian tensor product of Lie algebras, introduced in [18].

Proposition 2.3. Let $(g_1, n_1)$ and $(g_2, n_2)$ be arbitrary pairs of Leibniz algebras. Then

$$(g_1 \oplus g_2) \otimes (n_1 \oplus n_2) \cong (g_1 \otimes n_1) \oplus (g_2 \otimes n_2) \oplus \frac{(\pi_1 \ast \pi_2) \oplus (\pi_1 \ast \pi_2)}{a},$$

where $\pi_i = n_i/[g_i, n_i]$ and $\pi_i = g_i/[g_i, g_i]$ for $i = 1, 2$, and $a$ is the subalgebra generated by the elements $n_1 \ast n_2, -n_1 \ast n_2$ and $(n_2 \ast n_1, -n_2 \ast n_1)$ in which $n_1 \in n_1, n_2 \in n_2$ (here the bar $\bar{\cdot}$ denotes the equivalence class in each case).

Proof. Define the map $\varphi : (g_1 \oplus g_2) \otimes (n_1 \oplus n_2) \to (g_1 \otimes n_1) \oplus (g_2 \otimes n_2) \oplus \frac{(\pi_1 \ast \pi_2) \oplus (\pi_1 \ast \pi_2)}{a}$ on generators as follows:

$$\varphi((x_1, x_2) \otimes (n_1, n_2)) = (x_1 \otimes n_1, x_2 \otimes n_2, (\bar{n}_1 \ast \bar{x}_2, \bar{n}_2 \ast \bar{x}_1) + a),$$

$$\varphi((n_1, n_2) \otimes (x_1, x_2)) = (n_1 \otimes x_1, n_2 \otimes x_2, (\bar{n}_1 \ast \bar{x}_1, \bar{n}_2 \ast \bar{x}_2) + a),$$

for all $x_i \in g_i, n_i \in n_i, i = 1, 2$. It is not difficult to verify that $\varphi$ is well-defined and preserves the defining relations of the exterior product. For instance, we indicate that

$$\varphi([(x_1, x_2), (x_1', x_2')] \otimes (n_1, n_2)) = \varphi([(x_1, x_1'), [x_2, x_2')] \otimes (n_1, n_2))$$

$$= (\bar{x}_1 \otimes n_1, [x_2, x_2'] \otimes n_2, (\bar{n}_1 \ast \bar{x}_2, \bar{n}_2 \ast \bar{x}_1') + a)$$

$$= (\bar{x}_1 \otimes n_1, [x_2, x_2'] \otimes n_2, a).$$

On the other hand,

$$\varphi((x_1, x_2)(n_1, n_2) \otimes (x_1', x_2')) - \varphi((x_1, x_2) \otimes (n_1, n_2)(x_1', x_2'))$$

$$= (x_1 n_1 \otimes x_1' - x_1 \otimes n_1', x_2 n_2 \otimes x_2' - x_2 \otimes n_2', (\bar{x}_2 \ast \bar{n}_1, \bar{x}_2 \ast \bar{n}_2, \bar{x}_1 \ast \bar{n}_1') + a)$$

$$= (x_1 \otimes n_1, [x_2, x_2'] \otimes n_2, a),$$

because $x_1n_x, n_1' \in [g_1, n_1]$ and $x_2n_2, n_2' \in [g_2, n_2]$. Then the equality (1) holds. We now prove that $\varphi$ is an isomorphism by giving an inverse for it. To do this, let us first define the maps

$$\eta_1 : \pi_1 \ast \pi_2 \to (g_1 \otimes g_2) \otimes (n_1 \oplus n_2) \quad \text{and} \quad \eta_2 : \pi_1 \ast \pi_2 \to (g_1 \otimes g_2) \otimes (n_1 \oplus n_2)$$

by $\eta_1(\bar{n}_1 \ast \bar{x}_2) = (0, x_2) \otimes (n_1, 0)$, $\eta_1(\bar{x}_2 \ast \bar{n}_1) = (n_1, 0) \otimes (0, x_2)$ and $\eta_2(\bar{x}_1 \ast \bar{n}_2) = (0, n_2) \otimes (x_1, 0)$, $\eta_2(\bar{n}_2 \ast \bar{x}_1) = (l_1, 0) \otimes (0, n_2)$. For $n_1, n_1' \in n_1, x_2 \otimes x_2 \in g_2, y \in [g_1, n_1], z \in [g_2, g_2]$, if $n_1 = n_1' + y$ and $x_2 = x_2' + z$, then we have

$$\eta_1(\bar{n}_1 \ast \bar{x}_2) = (0, x_2' + z) \otimes (n_1', 0, y) = (0, x_2' \otimes (n_1', 0) + (0, x_2') \otimes (y, 0) + (0, z) \otimes (n_1' + y, 0).$$
But, assuming \( y = [a, b] \) and \( z = [c, d] \) for some \( a \in g_1, b \in n_1, c, d \in g_2 \), we have
\[
(0, x'_2) \wedge (y, 0) = (0, x'_2) \wedge (\langle a, b \rangle, 0) = (0, x'_2) (a, 0) \wedge (b, 0) - (0, x'_2) (0, 0) \wedge (a, 0) = 0,
\]
\[
(0, z) \wedge (n'_1 + y, 0) = (0, [c, d]) \wedge (n'_1 + y, 0) = (0, 0) \wedge (e, 0) - (0, c) \wedge (n'_1 + y, 0) (0, d) = 0,
\]
since \( g_1 \oplus 0 \) and \( 0 \oplus g_2 \) act trivially on each other. Consequently, \( \eta_1(n_1 + \bar{x}_2) = \eta_1(n'_1 + \bar{x}'_2) \) and, by an analogous argument, \( \eta_1(x_2 + \bar{n}_1) = \eta_1(x_2 + \bar{n}'_1) \). Therefore, \( \eta_1 \) and similarly, \( \eta_2 \) are well-defined. It is readily checked that \( \eta_1 \) and \( \eta_2 \) are Leibniz homomorphisms whose images are the abelian subalgebras of \((g_1 \oplus g_2) \cong (n_1 + n_2)\). Hence, we can obtain a Leibniz homomorphism \( \eta: (\bar{n}_1 \oplus \bar{g}_2) \oplus (\bar{g}_1 \oplus \bar{n}_2) \rightarrow (g_1 \oplus g_2) \oplus (n_1 \oplus n_2) \) defined by \( \eta(v, w) = \eta_1(v) + \eta_2(w) \). As \( a \) is annihilated by \( \eta \), this gives rise to a Leibniz homomorphism
\[
\eta: (\bar{n}_1 \oplus \bar{g}_2) \oplus (\bar{g}_1 \oplus \bar{n}_2) \rightarrow (g_1 \oplus g_2) \oplus (n_1 \oplus n_2).
\]
Now, if we consider the canonical homomorphisms
\[
\psi_1: g_1 \oplus n_1 \rightarrow (g_1 \oplus g_2) \oplus (n_1 \oplus n_2), \quad \text{x}_1 \oplus n_1 \rightarrow (x_1, 0) \oplus (n_1, 0), \quad n_1 \oplus x_1 \rightarrow (n_1, 0) \oplus (x_1, 0),
\]
\[
\psi_2: g_2 \oplus n_2 \rightarrow (g_1 \oplus g_2) \oplus (n_1 \oplus n_2), \quad \text{x}_2 \oplus n_2 \rightarrow (0, x_2) \oplus (0, n_2), \quad n_2 \oplus x_2 \rightarrow (0, n_2) \oplus (0, x_2),
\]
then the homomorphism \( \psi = \langle \psi_1, \psi_2, \bar{\eta} \rangle \) in the coproduct of vector spaces
\[
\psi: (g_1 \oplus n_1) \oplus (g_2 \oplus n_2) \oplus (\bar{n}_1 \oplus \bar{g}_2) \oplus (\bar{g}_1 \oplus \bar{n}_2) \rightarrow (g_1 \oplus g_2) \oplus (n_1 \oplus n_2),
\]
is evidently an inverse for \( \varphi \). This completes the proof. \( \square \)

Proposition 2.3 provides a description for the exterior product of the direct sum of Leibniz algebras.

**Corollary 2.4.** For any two Leibniz algebras \( g_1 \) and \( g_2 \), there is a Leibniz isomorphism
\[
(g_1 \oplus g_2) \wedge (g_1 \oplus g_2) \cong (g_1 \wedge g_1) \oplus (g_2 \wedge g_2) \oplus (g_1 \wedge g_2).
\]
*Proof.* It suffices to note that, in this case, \( a = ((a, -a) \mid a \in g_1^{ab} \oplus g_2^{ab}) \). \( \square \)

Combining the above corollary with Lemma 2.1(i) and Proposition 2.2, we obtain the following important result, which was already proved in Ellis (1991) using another technique.

**Corollary 2.5.** For any two Lie algebras \( g_1 \) and \( g_2 \), there is a Lie isomorphism
\[
(g_1 \oplus g_2) \otimes (g_1 \oplus g_2) \cong (g_1 \otimes g_1) \oplus (g_2 \otimes g_2) \oplus (g_1 \otimes g_2) \oplus (g_2 \otimes g_1).
\]

Let \( (g, n) \) be a pair of Lie algebras. In general, the Lie algebras \( g \wedge n \) and \( g \otimes n \) are not isomorphic. For example, if \( (g, n) \) is a pair of abelian Lie algebras with \( \dim(n) = n \) and \( \dim(g/n) = m \), then \( \dim(g \wedge n) = n(n + 2m) \), while \( \dim(g \otimes n) = n(n + m) \). However, the following proposition provides, under some condition, a precise relationship between these Lie algebras.

**Proposition 2.6.** Let \( (g, n) \) be a pair of Lie algebras. If \( n \) has an ideal complement in \( g \), then
\[
g \wedge n \cong (g \otimes n) \oplus (\frac{n}{[g, n]} \otimes_{\Lambda} \frac{g}{g^2 + n}).
\]
In particular, if the pair \( (g, n) \) is perfect (that is, \( [g, n] = n \)), then \( g \wedge n \cong (g \otimes n) \).
The following proposition gives some different descriptions of see [28, Page 47] for more details. The abelian Leibniz algebra relative homology

Let \( \pi \) be a Leibniz algebra over an arbitrary field \( \mathbb{F} \). The Leibniz homology of \( \pi \) with trivial coefficients, denoted by \( HL_\pi(\pi) \), is the homology of the chain complex \( (CL_n(\pi) = \pi \otimes \ldots \otimes \pi \otimes \pi, \partial_n) \), \( n \geq 0 \), such that the boundary map \( \partial_n : CL_n(\pi) \to CL_{n-1}(\pi) \) is a linear map of vector spaces given by

\[
\partial_n(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1 \otimes \cdots \otimes x_i \otimes x_j \otimes \cdots \otimes x_n), \quad n \geq 1,
\]

where the notation \( \hat{x}_j \) means that the variable \( x_j \) is omitted. It is easily seen that \( HL_\pi(\pi) \cong \pi^{ab} \) and by [13] Theorem 3.16, \( HL_\pi(\pi) \cong \ker(\pi \otimes \pi \to \pi) \). Let \( \mathfrak{n} \) be an ideal of \( \pi \), then the natural epimorphism \( \pi : \pi \to \pi/\mathfrak{n} \) induces a morphism of chain complexes \( \pi : (CL_n(\pi), \partial_n) \to (CL_n(\pi/\mathfrak{n}), \partial_n) \).

Let \( (M_\pi, \delta_\pi) = (M, \delta) \) be the mapping cone of \( \pi \), that is, \( M_\pi = CL_{n-1}(\pi) \oplus CL_n(\pi/\mathfrak{n}) \) and \( \delta_\pi(a, b) = \partial_n(a, b) + \pi_n(a) \).

Setting \( (K^+_\pi, \partial^+_\pi) \) to be the complex \( (CL_\pi(\pi), \partial_\pi) \) with the dimensions all raised by one and the sign of the boundary changed, that is \( (K^+_\pi)_n = CL_{n-1}(\pi) \), then there exists a short exact sequence of complexes \( 0 \to (CL_\pi(\pi), \partial_\pi) \to (M_\pi, \delta_\pi) \to (K^+_\pi, \partial^+_\pi) \to 0 \). This sequence induces a long exact sequence of homology groups

\[
\cdots \to HL_n(\pi/\mathfrak{n}) \to H_n(M_\pi, \delta_\pi) \to HL_{n-1}(\pi) \to HL_{n-1}(\pi/\mathfrak{n}) \to \cdots, \tag{2}
\]

see [28] Page 47 for more details. The abelian Leibniz algebra \( H_{n+1}(M_\pi, \delta_\pi) \) is called the \( n \)-th relative homology of the pair \( (\pi, \mathfrak{n}) \) and denoted by \( HL_n(\pi, \mathfrak{n}) \). It was shown in [10] Propositions 2.4 that \( HL_1(\pi, \mathfrak{n}) \cong \mathfrak{n}/[\pi, \mathfrak{n}] \) and if \( \mathfrak{n} \) is a central ideal of \( \pi \), then \( HL_2(\pi, \mathfrak{n}) \cong \text{Coker}(\tau) \), where \( \tau : \mathfrak{n} \otimes \mathfrak{n} \to (g / g^2 \otimes \mathfrak{n}) \oplus (\mathfrak{n} \otimes g / g^2) \) is defined by \( \tau(n_1 \otimes n_2) = (\tilde{n}_1 \otimes n_2, -n_1 \otimes \tilde{n}_2) \) for all \( n_1, n_2 \in \mathfrak{n} \).

The following proposition gives some different descriptions of \( HL_2(\pi, \mathfrak{n}) \).

**Proposition 3.1.** Let \((\pi, \mathfrak{n})\) be a pair of Leibniz algebras. Then

(i) \( HL_2(\mathfrak{n}, \mathfrak{n}) \) is isomorphic to the kernel of the commutator map \( \mathfrak{g} \otimes \mathfrak{n} \xrightarrow{\cdot} \mathfrak{g} \).
We denote an extra special Leibniz algebra by $g$. Under the assumptions of Proposition Corollary 3.2, Leibniz algebras with derived subalgebra of dimension 1.

The following proposition gives the dimension of the second homology of finite dimensional nilpotent $g$.

Proof. Parts (i) and (iv) are proved in [14, Proposition 4.3] and [20, Proposition 2.1], respectively. Part (ii) is a consequence of part (i) and Lemma 2.1(i). Part (iii) is obtained from the sequence (2).

By Propositions 2.6 and 3.1(i), we have the following which generalizes [14, Proposition 6.2] slightly.

**Corollary 3.2.** Under the assumptions of Proposition 2.6, there is an isomorphism of vector spaces

$$HL_2(g, n) \cong \ker(g \otimes n \to g) \oplus \left(\frac{n}{[g, n]} \otimes_F \frac{g}{g \cap n}\right).$$

In particular, if the pair $(g, n)$ is perfect, then $HL_2(g, n) \cong H_2(g, n)$, where $H_2(g, n)$ denotes the second relative Chevalley-Eilenberg homology of the pair $(g, n)$.

Let $g_1, g_2$ be two Leibniz algebras. The Künneth-style formula for the homology of direct sum of Leibniz algebras $g_1, g_2$, was extended in [26] by Loday. He proved that there is a canonical isomorphism of graded modules $HL_*(g_1 \oplus g_2) \cong HL_*(g_1) \ast HL_*(g_2)$ (here the symbol “*” means the non-commutative tensor product of graded modules). In the special case, for the second degree, theorem states that $HL_2(g_1 \oplus g_2) \cong HL_2(g_1) \oplus HL_2(g_2) \oplus (g_1^{ab} \otimes_F g_2^{ab}) \oplus (g_2^{ab} \otimes_F g_1^{ab})$. Note that one can use Lemma 2.1(i) to re-arrange the formula as

$$HL_2(g_1 \oplus g_2) \cong HL_2(g_1) \oplus HL_2(g_2) \oplus (g_1^{ab} \ast g_2^{ab}). \quad (3)$$

As an immediate consequence of Propositions 2.3 and 3.1(i), we can obtain the following generalization of the formula (3).

**Corollary 3.3.** (The generalization of Künneth-Loday formula) Let $(g_1, n_1)$ and $(g_2, n_2)$ be arbitrary pairs of Leibniz algebras. Then

$$HL_2(g_1 \oplus g_2, n_1 \oplus n_2) \cong HL_2(g_1, n_1) \oplus HL_2(g_2, n_2) \oplus (g_1^{ab} \ast g_2^{ab}) \oplus (g_2^{ab} \ast g_1^{ab}).$$

To state another interesting consequence of Proposition 3.1 we first need the following.

A finite dimensional nilpotent Leibniz algebra $g$ is called extra special if $\dim(Z(g)) = \dim(g^2) = 1$. We denote an extra special Leibniz algebra by $e$ and an abelian Leibniz algebra of dimension $q$ by $a(q)$.

The following proposition gives the dimension of the second homology of finite dimensional nilpotent Leibniz algebras with derived subalgebra of dimension 1.
Proposition 3.4 ([20]). (i) \( \dim(HL_2(\mathfrak{g})) = (\dim(\mathfrak{g}) - 1)^2 - 1 + t \), for some integer \( t \leq 2\dim(\mathfrak{g}) \). In particular, if the ground field is algebraically closed of characteristic different from two, then \( t = 1 \) if \( \mathfrak{g} = \mathfrak{J}_1 \) or \( \mathfrak{g} = \mathfrak{J}_2 \), where \( \mathfrak{J}_1 = \langle x, y \mid [x, x] = y \rangle \) and \( \mathfrak{J}_2 = \langle x, y, z \mid [x, y] = z \rangle \); and \( t = 2 \) if \( \mathfrak{g} = \mathfrak{H}_1 \), where \( \mathfrak{H}_1 = \langle x, y, z \mid [x, y] = -[y, x] \rangle \); and \( t = 0 \) otherwise.

(ii) If \( \mathfrak{g} \) is a finite dimensional nilpotent Leibniz algebra with \( \dim(\mathfrak{g}^2) = 1 \), then \( \mathfrak{g} = \mathfrak{e} \oplus \mathfrak{a}(q) \) for some \( q \geq 0 \), and so \( \dim(HL_2(\mathfrak{g})) = (\dim(\mathfrak{g}) - 1)^2 - 1 + t \).

In the following corollary, we get the dimension of the second relative homology of some known pairs of Leibniz algebras.

Corollary 3.5. (i) If \( \mathfrak{n} \) is an \( n \)-dimensional ideal of \( \mathfrak{a}(q) \), then \( \dim(HL_2(\mathfrak{a}(q), \mathfrak{n})) = n(2q - n) \).

(ii) For any finite dimensional nilpotent Leibniz algebra \( \mathfrak{g} \) with \( \dim(\mathfrak{g}^2) = 1 \), \( \dim(HL_2(\mathfrak{g}, \mathfrak{g}^2)) = 2\dim(\mathfrak{g}^{ab}) \).

(iii) For any non-zero ideal \( \mathfrak{n} \) of \( \mathfrak{J}_2 \) or \( \mathfrak{H}_1 \), we have

\[
\dim(HL_2(\mathfrak{J}_2, \mathfrak{n})) = \begin{cases} 
3 & \text{if } \dim(\mathfrak{n}) = 2 \\
4 & \text{otherwise} 
\end{cases}, \quad \dim(HL_2(\mathfrak{H}_1, \mathfrak{n})) = \begin{cases} 
5 & \text{if } \mathfrak{n} = \mathfrak{H}_1 \\
4 & \text{otherwise} 
\end{cases}.
\]

Proof. (i) It follows from Proposition 3.1(iii) and the fact that any ideal of an abelian Leibniz algebra has a complement.

(ii) It is a straightforward consequence of Proposition 3.1(ii).

(iii) We only prove the case \( \mathfrak{H}_1 \), the proof of the other case is similar. According to Proposition 3.4(i), \( \dim(HL_2(\mathfrak{H}_1, \mathfrak{H}_1)) = \dim(HL_2(\mathfrak{H}_1)) = 5 \) and by part (ii), \( \dim(HL_2(\mathfrak{H}_1, \mathfrak{H}_1^2)) = 4 \). Now, suppose that \( \mathfrak{n} \) is a two-dimensional ideal of \( \mathfrak{H}_1 \). Then \( \mathfrak{n} \) has a one-dimensional complement, say \( \mathfrak{k} \), in \( \mathfrak{H}_1 \) and hence, owing to Proposition 3.1(iii), \( \dim(HL_2(\mathfrak{H}_1, \mathfrak{n})) = \dim(HL_2(\mathfrak{H}_1)) - \dim(HL_2(\mathfrak{k})) = 4 \). \( \square \)

It is proved in [20] that for any pair \( (\mathfrak{g}, \mathfrak{n}) \) of finite dimensional nilpotent Leibniz algebras, if \( \mathfrak{n} \) has a complement in \( \mathfrak{g} \), then

\[
\dim(HL_2(\mathfrak{g}, \mathfrak{n})) \leq \dim(HL_2(\frac{\mathfrak{g}}{\mathfrak{g}^2 \cap \mathfrak{n}}, \frac{\mathfrak{n}}{\mathfrak{g}^2 \cap \mathfrak{n}})) + 2\dim([\mathfrak{g}, \mathfrak{n}])(d(\frac{\mathfrak{g}}{Z(\mathfrak{g}, \mathfrak{n})}) - 1) + \dim([\mathfrak{g}, \mathfrak{n}]),
\]

where \( Z(\mathfrak{g}, \mathfrak{n}) = Z(\mathfrak{g}) \cap \mathfrak{n} \) and \( d(\mathfrak{g}) \) is the cardinal of a minimal generating set of \( \mathfrak{g} \). Note that in this case, \( \mathfrak{g}^2 \cap \mathfrak{n} = [\mathfrak{g}, \mathfrak{n}] \). Using Corollary 3.3, we prove a similar result for any pair of finite dimensional nilpotent Leibniz algebras.

Theorem 3.6. Let \( \mathfrak{g} \) be a finite dimensional nilpotent Leibniz algebra with an ideal \( \mathfrak{n} \). Then

\[
\dim(HL_2(\mathfrak{g}, \mathfrak{n})) \leq \dim(HL_2(\frac{\mathfrak{g}}{\mathfrak{g}^2 \cap \mathfrak{n}}, \frac{\mathfrak{n}}{\mathfrak{g}^2 \cap \mathfrak{n}})) + 2\dim(\mathfrak{g}^2 \cap \mathfrak{n})d(\frac{\mathfrak{g}}{Z(\mathfrak{g}, \mathfrak{n})}).
\]

Our proof of the above theorem requires the following proposition.
**Proposition 3.7.** Let $\mathfrak{g}$ be a Leibniz algebra with ideals $\mathfrak{n}$ and $\mathfrak{k}$ such that $\mathfrak{k} \subseteq Z(\mathfrak{g}, \mathfrak{n})$. Then there is an exact sequence of Leibniz algebras

$$\mathfrak{g} \otimes \mathfrak{k} \longrightarrow HL_2(\mathfrak{g}, \mathfrak{n}) \longrightarrow HL_2(\frac{\mathfrak{g}}{\mathfrak{k}} \otimes \frac{\mathfrak{n}}{\mathfrak{k}}) \rightarrow [\mathfrak{g}, \mathfrak{n}] \cap \mathfrak{k}.$$ 

**Proof.** By Lemma 2.1(ii), we have the following diagram with exact rows

$$\begin{array}{ccc}
\mathfrak{g} \otimes \mathfrak{k} & \longrightarrow & \mathfrak{g} \otimes \mathfrak{n} \\
\downarrow \, [\ , ]_1 & & \downarrow \, [\ , ]_2 \\
[\mathfrak{g}, \mathfrak{n}] \cap \mathfrak{k} & \longrightarrow & [\mathfrak{g}, \mathfrak{n}] \\
\end{array}
$$

where the vertical arrows are the commutator maps. In this diagram, the right-hand-side square is always commutative. Since $\mathfrak{k}$ is central, the commutator map $[\ , ]_1$ is equal to the zero map and so the left-hand-side square is also commutative. The Snake Lemma yields the following exact sequence

$$\ker([\ , ]_1) \longrightarrow \ker([\ , ]_2) \longrightarrow \ker([\ , ]_3) \rightarrow \coker([\ , ]_1).$$

The result now follows from Proposition 3.1(i). \qed

We get the following corollary from the previous proposition.

**Corollary 3.8.** With the assumptions of Proposition 3.7 if $\mathfrak{g}$ is finite dimensional, then

$$\dim(HL_2(\frac{\mathfrak{g}}{\mathfrak{k}}, \frac{\mathfrak{n}}{\mathfrak{k}})) \leq \dim(HL_2(\mathfrak{g}, \mathfrak{n})) + \dim([\mathfrak{g}, \mathfrak{n}] \cap \mathfrak{k}).$$

In particular, if the ideal $\mathfrak{n}$ is central, then $\dim(HL_2(\mathfrak{g}/\mathfrak{k}, \mathfrak{n}/\mathfrak{k})) \leq \dim(HL_2(\mathfrak{g}, \mathfrak{n})).$

**Proof of Theorem 3.6.** We proceed by induction on $\dim(\mathfrak{g})$. The result certainly holds when $\mathfrak{g}$ is abelian. Suppose that $\dim(\mathfrak{g}) \geq 2$ and the result is true for all nilpotent Leibniz algebras of dimension smaller than $\dim(\mathfrak{g})$. We distinguish the following two cases.

**Case 1.** $Z(\mathfrak{g}, \mathfrak{n}) \subseteq \mathfrak{g}^2$. Choose a one-dimensional ideal $\mathfrak{k}$ of $Z(\mathfrak{g}, \mathfrak{n})$. Then, invoking Proposition 3.7 and using the induction hypothesis, we have

$$\dim(HL_2(\mathfrak{g}, \mathfrak{n})) \leq \dim(HL_2(\frac{\mathfrak{g}}{\mathfrak{k}}, \frac{\mathfrak{n}}{\mathfrak{k}})) + \dim(\mathfrak{g} \otimes \mathfrak{k})$$

$$\leq \dim(HL_2(\frac{\mathfrak{g}}{(\mathfrak{g}^2 \cap \mathfrak{n}) + \mathfrak{k}}, \frac{\mathfrak{n}}{(\mathfrak{g}^2 \cap \mathfrak{n}) + \mathfrak{k}})) + 2(\dim(\mathfrak{g}^2 \cap \mathfrak{n}) - 1)d(\frac{\mathfrak{g}}{Z(\mathfrak{g}/\mathfrak{k}, \mathfrak{n}/\mathfrak{k})}) + 2\dim(\frac{\mathfrak{g}}{\mathfrak{g}^2})$$

$$\leq \dim(HL_2(\frac{\mathfrak{g}}{(\mathfrak{g}^2 \cap \mathfrak{n}) + \mathfrak{k}}, \frac{\mathfrak{n}}{(\mathfrak{g}^2 \cap \mathfrak{n}) + \mathfrak{k}})) + 2\dim(\mathfrak{g}^2 \cap \mathfrak{n})d(\frac{\mathfrak{g}}{Z(\mathfrak{g}, \mathfrak{n})}) + 2(d(\frac{\mathfrak{g}}{\mathfrak{g}^2}) - d(\frac{\mathfrak{g}}{Z(\mathfrak{g}, \mathfrak{n}))})).$$

Since $\dim(\mathfrak{g}/\mathfrak{g}^2) = d(\mathfrak{g}/Z(\mathfrak{g}, \mathfrak{n})), the result follows in this case.

**Case 2.** $Z(\mathfrak{g}, \mathfrak{n}) \not\subseteq \mathfrak{g}^2$. Suppose that $\mathfrak{a}$ is a vector space complement of $Z(\mathfrak{g}, \mathfrak{n}) \cap \mathfrak{g}^2$ in $Z(\mathfrak{g}, \mathfrak{n})$. Then $\mathfrak{(g, n)} = (\mathfrak{k} \otimes \mathfrak{a}, (\mathfrak{k} \cap \mathfrak{n}) \otimes \mathfrak{a})$, for some ideal $\mathfrak{k}$ of $\mathfrak{g}$. It is straightforward to check that $\mathfrak{g}^2 \cap \mathfrak{n} = \mathfrak{g}^2 \cap \mathfrak{n}$ and $\mathfrak{g}/Z(\mathfrak{g}, \mathfrak{n}) = \mathfrak{k}/Z(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{n})$. Putting $\mathfrak{k} \cap \mathfrak{n} = (\mathfrak{k} \cap \mathfrak{n})/[(\mathfrak{k} \cap \mathfrak{n}]$ and $\mathfrak{k} = \mathfrak{k}/\mathfrak{k}^2$, we let $\mathfrak{B}$ be the quotient
of \((((\mathfrak{g} \cap n) * \mathfrak{a}) \oplus (\mathfrak{g} * \mathfrak{a})) by the ideal generated by the elements \((\mathfrak{g} * \mathfrak{a}, -\mathfrak{g} * \mathfrak{a}) and (\mathfrak{a} * \mathfrak{g}, -\mathfrak{a} * \mathfrak{g})\), in which \(n \in \mathfrak{g} \cap n\) and \(a \in \mathfrak{a}\). It follows from Corollary 3.3 and the induction hypothesis that

\[
\dim(\text{HL}_2(\mathfrak{g}, n)) = \dim(\text{HL}_2(\mathfrak{t} \oplus \mathfrak{a}, (\mathfrak{t} \cap n) \oplus \mathfrak{a})) = \dim(\text{HL}_2(\mathfrak{t}, \mathfrak{t} \cap n)) + \dim(\text{HL}_2(\mathfrak{a})) + \dim(B)
\]

\[
\leq \dim(\text{HL}_2(\frac{\mathfrak{t}}{\mathfrak{t}^2 \cap n}, \frac{\mathfrak{t} \cap n}{\mathfrak{t}^2 \cap n})) + 2 \dim(\mathfrak{t} \cap n) \cdot \dim(B) + \dim(\text{HL}_2(\mathfrak{a})) + \dim(B)
\]

\[
= \dim(\text{HL}_2(\frac{\mathfrak{t}}{\mathfrak{t}^2 \cap n}, \frac{\mathfrak{t} \cap n}{\mathfrak{t}^2 \cap n})) + 2 \dim(\mathfrak{g} \cap n) \cdot \dim(B) + \dim(\text{HL}_2(\mathfrak{a})) + \dim(B).
\]

But, repeated application of Corollary 3.3 deduces

\[
\dim(\text{HL}_2(\frac{\mathfrak{g}}{\mathfrak{g}^2 \cap n}, \frac{n}{\mathfrak{g}^2 \cap n})) = \dim(\text{HL}_2(\frac{\mathfrak{t}}{\mathfrak{t}^2 \cap n} \oplus \mathfrak{a}, \frac{\mathfrak{t} \cap n}{\mathfrak{t}^2 \cap n} \oplus \mathfrak{a}))
\]

\[
= \dim(\text{HL}_2(\frac{\mathfrak{t}}{\mathfrak{t}^2 \cap n}, \frac{\mathfrak{t} \cap n}{\mathfrak{t}^2 \cap n})) + \dim(\text{HL}_2(\mathfrak{a})) + \dim(B),
\]

and thus the result holds, as required.

Taking \(n = \mathfrak{g}\) in Theorem 3.6, we get the following

**Corollary 3.9.** Let \(\mathfrak{g}\) be a finite dimensional nilpotent Leibniz algebra. Then

\[
\dim(\text{HL}_2(\mathfrak{g})) \leq (\dim(\mathfrak{g}) - \dim(\mathfrak{g}^2))^2 + 2 \dim(\mathfrak{g}^2) \cdot \dim(\frac{\mathfrak{g}}{Z(\mathfrak{g})}).
\]

In particular, the equality holds if and only if \(\mathfrak{g}\) is abelian.

### 4 The main results

Let \((\mathfrak{g}, n)\) be a pair of Leibniz algebras such that \(\dim(n) = n\) and \(\dim(\mathfrak{g}/n) = m\). In our recent paper [20], we obtained an upper bound \(n(n + 2m)\) for the dimension of \(\text{HL}_2(\mathfrak{g}, n)\). We also determined the structure of all pairs of finite dimensional nilpotent Leibniz algebras with \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m) - k\), for which \(k = 0, 1, 2\).

**Theorem 4.1** ([20]). With the above assumptions and notations, if \(\mathfrak{g}\) is nilpotent, then

(i) \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m)\) if and only if \(\mathfrak{g}\) is abelian.

(ii) \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m) - 1\) if and only if \((\mathfrak{g}, n) = (\mathfrak{c} \oplus \mathfrak{a}(q), \mathfrak{c}\)), for some \(q \geq 0\).

(iii) \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m) - 2\) if and only if \(\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}\), where \(\mathfrak{m}\) is a nilpotent Leibniz algebra with \(\dim(\mathfrak{m}^2) = 1\) and \(\mathfrak{n}\) is a one-dimensional central ideal.

In the following theorem, we characterize all pairs of finite dimensional nilpotent Leibniz algebras with \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m) - 3\).

**Theorem 4.2.** With the above assumptions and notations, \(\dim(\text{HL}_2(\mathfrak{g}, n)) = n(n + 2m) - 3\) if and only if one of the following cases occurs.

(a) \(\mathfrak{g}\) is nilpotent with \(\dim(\mathfrak{g}^2) = 2\) and \(\mathfrak{n}\) is a one-dimensional ideal of \(Z(\mathfrak{g}) \cap \mathfrak{g}^2\).

(b) \(\mathfrak{g}\) is nilpotent with \(\dim(\mathfrak{g}^2) = 1\) and \(\mathfrak{n}\) is a two-dimensional ideal of \(Z(\mathfrak{g})\) containing \(\mathfrak{g}^2\).
To prove this, we use the following proposition.

**Proposition 4.3.** Let $g = \mathfrak{e} \oplus a(q)$ and $n$ be an ideal of $g$. Then

(i) If $\mathfrak{e}^2 \subseteq n \subseteq Z(g)$, then $\dim(HL_2(g, n)) = 2\dim(g)\dim(n) - (\dim(n) + 1)^2 + 2$.

(ii) If $\mathfrak{e} = J_1, J_2$ or $H_1$ and $\dim(n) = 2$, then

\[
\dim(HL_2(J_1 \oplus a(q), n)) = \begin{cases} 
4q & \text{if } n \subseteq a(q), \\
4q + 1 & \text{if } J_1^2 \subseteq n \subseteq Z(g), \\
2q + 1 & \text{otherwise.}
\end{cases}
\]

\[
\dim(HL_2(J_2 \oplus a(q), n)) = \begin{cases} 
4q + 1 & \text{if } n \subseteq a(q), \\
4q + 5 & \text{if } J_2^2 \subseteq n \subseteq Z(g), \\
2q + 3 & \text{otherwise.}
\end{cases}
\]

\[
\dim(HL_2(H_1 \oplus a(q), n)) = \begin{cases} 
4q + 5 & \text{if } H_1^2 \subseteq n \subseteq Z(g), \\
2(q + 2) & \text{otherwise.}
\end{cases}
\]

**Proof.** (i) Evidently, $n$ can be represented as the direct sum $\mathfrak{e}^2 \oplus i$, where $i$ is an ideal of $a(q)$. Hence, applying Corollary 3.3, there is an isomorphism

\[HL_2(g, n) \cong HL_2(\mathfrak{e}, \mathfrak{e}^2) \oplus HL_2(a(q), i) \oplus (e^a * i) \oplus \frac{\mathfrak{e}^2 * a(q)}{\langle x * y, y * x | x \in \mathfrak{e}^2, y \in i \rangle} \] 

Since $\mathfrak{e}^2 * a(q)$ is abelian, the denominator of the above factor is isomorphic to $\mathfrak{e}^2 * i$. Now, by considering the split exact sequence of abelian Leibniz algebras $\mathfrak{e}^2 * i \to \mathfrak{e}^2 * a(q) \to \mathfrak{e}^2 * a(q)/i$, and using Corollary 3.5(i),(ii), we get the required result.

(ii) We only prove the case $\mathfrak{e} = H_1$, the other two are verified similarly. If $n \subseteq H_1$ or $a(q)$, then invoking Corollaries 3.3 and 3.5(ii), respectively, we have

\[
\dim(HL_2(g, n)) = \dim(HL_2(H_1, n)) + \dim(HL_2(a(q), 0)) + \dim(\frac{n}{H_1, n} * a(q)) = 2(q + 2),
\]

\[
\dim(HL_2(g, n)) = \dim(HL_2(H_1, 0)) + \dim(HL_2(a(q), n)) + \dim(H_1^{ab} * n) = 4q + 1.
\]

Also, if $H_1^2 \subseteq n \subseteq Z(g)$, then part (i) implies that $\dim(HL_2(g, n)) = 4q + 5$. Finally, we assume $n \nsubseteq Z(g)$ and $n = \langle z, c_1 x + c_2 y + t \rangle$, where $t \in a(q)$, $c_1, c_2 \in \mathbb{F}$ and at least one of the scalars, say $c_1$, is non-zero. If we set $\mathfrak{t}$ to be the semidirect sum of $a(q)$ by $\langle x \rangle$, it is readily checked that $\mathfrak{t}$ is an abelian complement of $n$ in $g$ and so, by Propositions 3.1(iii) and 3.4, $\dim(HL_2(g, n)) = \dim(HL_2(g)) - \dim(HL_2(g/n)) = 2(q + 2)$.

Now, we equipped to prove the theorem.
Proof. Assume \( \dim(HL_2(\mathfrak{g}, \mathfrak{n})) = n(n + 2m) - 3 \). By Theorem 4.1(i), \( \mathfrak{g} \) is non-abelian and by the nilpotency of \( \mathfrak{g}, \mathfrak{n} \cap Z(\mathfrak{g}) \neq 0 \). Pick non-zero elements \( x \in \mathfrak{g}^2 \) and \( e_1 \in \mathfrak{n} \cap Z(\mathfrak{g}) \), \( e_2, \ldots, e_n \in \mathfrak{n} \) and \( f_1, \ldots, f_m \in \mathfrak{g} - \mathfrak{n} \) such that the set \( A = \{ e_1, \ldots, e_n, f_1, \ldots, f_m \} \) is a basis of \( \mathfrak{g} \) and \( x \in A \). Using the relations (2a)-(2d), it is routine to check that \( B = \{ e_i \wedge e_j, e_i \wedge f_l, f_l \wedge e_i \mid 1 \leq i, j \leq n, 1 \leq l \leq m \} \) is a generating set for the Leibniz algebra \( \mathfrak{g} \wedge \mathfrak{n} \). We divide the rest of the proof into two steps.

Step 1. We claim that \( \mathfrak{n} \) is central.

By way of contradiction, suppose that \( \dim([\mathfrak{g}, \mathfrak{n}]) \geq 1 \). Then \( \dim(\mathfrak{g} \wedge \mathfrak{n}) \geq n(n + 2m) - 2 \). If \( \dim(Z(\mathfrak{g}) \cap \mathfrak{n}) \geq 2 \) and assume that \( e_i \in Z(\mathfrak{g}) \cap \mathfrak{n} \) for some \( 2 \leq i \leq n \), then using the formulas (3b) and (3d), one can easily see that \( e_i \wedge x = x \wedge e_i = e_i \wedge x = 0 \), which means that \( \dim(\mathfrak{g} \wedge \mathfrak{n}) \leq n(n + 2m) - 3 \), an impossibility. Hence, \( Z(\mathfrak{g}) \cap \mathfrak{n} \) and similarly, \( \mathfrak{g}^2 \) must be of dimension 1. It follows, in particular, that \( Z(\mathfrak{g}) \cap \mathfrak{n} = \text{span}\{e_1\} \) and \( \mathfrak{g}^2 = \text{span}\{x\} \). Since \( 0 \neq [\mathfrak{g}, \mathfrak{n}] \subseteq \mathfrak{n} \cap \mathfrak{g}^2 \) and \( \mathfrak{g} \) is nilpotent, we conclude that \( \mathfrak{g}^2 \cap Z(\mathfrak{g}) \cap \mathfrak{n} \neq 0 \). Then \( \mathfrak{g}^2 = Z(\mathfrak{g}) \cap \mathfrak{n} = [\mathfrak{g}, \mathfrak{n}] = \text{span}\{e_1\} \) and \( e_1 \wedge e_1 = 0 \). Also, according to Proposition 3.4(ii), \( \mathfrak{g} = \mathfrak{e} \oplus \mathfrak{a}(q) \), for some \( q \geq 0 \). First, assume that \( \mathfrak{n} \) is non-abelian. Then there exist elements \( a, b \in \mathfrak{n} \) (which we can assume that \( a, b \in A \)) such that \( e_1 = [a, b] \). It is readily verified that if \( a = b \), then \( y \wedge e_1 = y \wedge [a, a] = 0 \) for all \( y \in \mathfrak{g} \) and if \( a \neq b \), then the sets \( \{ a \wedge e_1, e_1 \wedge a, e_1 \wedge a \} \) and \( \{ b \wedge e_1, e_1 \wedge b, e_1 \wedge a \} \) are linearly dependent. But both of these modes are contradictory to the assumption that \( \text{card}(B) \geq n(n + 2m) - 2 \). So, \( \mathfrak{n} \) must be abelian. In particular, \( \mathfrak{n} \neq \mathfrak{g} \). We claim that \( \mathfrak{g} \) is a Lie algebra. Suppose that \( e_1 = [a, a] \) for some \( a \in \mathfrak{g} - \mathfrak{n} \), note that we can assume \( a \in A \). Since \( \dim(\mathfrak{n}) > 1 \) we can employ \( e_2 \) and relation (3a) implies \( e_2 \wedge [a, a] = 0 \). Also using (3c) the set \( \{ e_1 \wedge e_2, e_1 \wedge a, a \wedge e_1 \} \) is linearly dependent, which gives the impossibility \( \dim([\mathfrak{g}, \mathfrak{n}]) \leq n(n + 2m) - 2 \). This proves the claim and then \( \mathfrak{g} = H(k) \oplus \mathfrak{a}(q) \) for some \( k \geq 1 \) and \( q \geq 0 \), where \( H(k) \) is Heisenberg Lie algebra of dimension \( 2k + 1 \), see [15]. Recalling that \( [\mathfrak{g}, \mathfrak{n}] = \text{span}\{e_1\} \), we can assume that \( e_1 = [a, b] \) for some \( a \in \mathfrak{g} \) and \( b \in \mathfrak{n} \). Then image of the adjoint map \( ad_{a} : \mathfrak{g} \to \mathfrak{g} \) is of dimension 1 and \( \dim(C_{\mathfrak{g}}(a)) = \dim(\ker ad_{a}) = \dim(\mathfrak{g}) - 1 \). Analogously, \( \dim(C_{\mathfrak{g}}(b)) = \dim(\mathfrak{g}) - 1 \) and consequently, \( \dim(C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b)) = \dim(\mathfrak{g}) - 2 \). We now distinguish the following two cases.

Case 1. \( C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) \) is abelian. Since \( \mathfrak{g} = (C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b)) + \text{span}\{a, b\} \), it is inferred that \( Z(\mathfrak{g}) = C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) \) and in consequence, \( 1 + q = 2k + q + 1 - 2 \). Then \( \mathfrak{g} = H(1) \oplus \mathfrak{a}(q) \) and \( \dim(\mathfrak{n} + Z(\mathfrak{g})) = \dim(\mathfrak{n}) + \dim(Z(\mathfrak{g})) - \dim(Z(\mathfrak{g}) \cap \mathfrak{n}) \leq \dim(\mathfrak{g}) = q + 3 \), or equivalently, \( 2 \leq \dim(\mathfrak{n}) \leq 3 \). If \( \dim(\mathfrak{n}) = 2 \), then the non-collinearity of \( \mathfrak{n} \) together with Proposition 3.3(ii) yields that \( \dim(H_{2}(\mathfrak{g}, \mathfrak{n})) = 2(q + 2) \), which contradicts our assumption. If \( \dim(\mathfrak{n}) = 3 \), then \( \mathfrak{g} = \mathfrak{n} + Z(\mathfrak{g}) \) and since \( \mathfrak{n} \) is abelian, we conclude that \( \mathfrak{g} \) is abelian, which is again a contradiction.

Case 2. \( C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) \) is non-abelian. Then \( e_1 = [y, z] \) for some \( y, z \in C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) \) and so, \( e_1 \wedge b = b \wedge e_1 = 0_{\mathfrak{g}, \mathfrak{n}, \mathfrak{o}} \). As we can assume that \( b \in A \), the last results contradicts the assumption that \( \dim(\mathfrak{g} \wedge \mathfrak{n}) \geq n(n + 2m) - 2 \). We therefore conclude that \( \mathfrak{n} \) is central.
Step 2. Completion of the proof.

Since \( n \subseteq Z(g) \), it follows that \( \dim(g \wedge n) = n(n+2m) - 3 \) and \( n \wedge y = y \wedge n = \theta_{g \wedge n} \) for all \( n \in n \), \( y \in g^2 \). If \( n \cap g^2 \) is trivial, one sees that \( \dim(n) = \dim(g^2) = 1 \), forcing \( g = m \oplus n \), where \( m \) is nilpotent with \( \dim(m^2) = 1 \) and \( n \) is a one-dimensional central ideal of \( g \). But this contradicts Theorem 3.1(iii). Hence, assume that \( n \cap g^2 \neq 0 \). If \( \dim(n) \geq 3 \) or \( \dim(g^2) \geq 3 \), then we again observe that at least four elements of the generating set \( B \) can be zero. So, one of the following situations occurs.

(i) \( \dim(n) = \dim(g^2) = 1 \). Then by Proposition 3.4(ii), \( g = \epsilon \oplus a(q) \) for some \( q \geq 0 \) and \( n = \epsilon^2 \), which gives a contradiction to Theorem 4.1(ii).

(ii) \( \dim(n) = 1 \) and \( \dim(g^2) = 2 \). In this case, \( g \) is nilpotent with the derived subalgebra of dimension 2 and \( n \subseteq Z(g) \cap g^2 \).

(iii) \( \dim(n) = 2 \) and \( \dim(g^2) = 1 \). In this case, \( g \) is nilpotent with the derived subalgebra of dimension 1 and \( g^2 \subseteq n \subseteq Z(g) \).

(iii) \( \dim(n) = \dim(g^2) = 2 \). In this case, as before, we regard that at least four elements in the set \( B \) are vanished, which contradicts the fact that \( \dim(g \wedge n) = n(n+2m) - 3 \).

The converse of theorem follows from Propositions 3.1(ii) and 4.3(i).

Let \((g,n)\) be a pair of Leibniz algebras. A Leibniz crossed module \( \delta : m \to g \) is called a relative stem cover of \((g,n)\) if \( \delta(m) = n \), \( n \subseteq [H \subset g, m] \cap [g, m] \), where

\[
Z(g, m) = \{ m \in m | ^xm = m^x, \text{ for all } x \in g \} \quad \text{and} \quad [g, m] = \{ ^xm, m^x | x \in g, m \in m \}.
\]

One observes that the relative stem cover \( \delta : m \to g \) of the pair \((g, g)\) is the usual stem cover of \( g \), which was introduced by Casas and Ladra [9]. In this case, \( x \in g \) acts on \( m \in m \) by \( ^xm = [\bar{x}, m], m^x = [m, \bar{x}] \), where \( \bar{x} \) is any element in the pre-image of \( x \) via \( \delta \). They also showed that Leibniz algebras have at least one stem cover.

In continuation of the section, we will construct a relative stem cover for the direct sum of two pairs of Leibniz algebras in terms of given relative stem covers of them, which is a generalization of [20] Corollary 5.6 for Leibniz algebras. To do this, we need the following lemma.

**Lemma 4.4.** Let \( \delta_i : m_i \to g_i, i = 1, 2, \) be two Leibniz crossed modules and \( a \) be an ideal of \((\bar{m}_1 \ast \bar{g}_2) \oplus (\bar{g}_1 \ast \bar{m}_2)\) generated by \((\bar{m}_1 \ast \bar{g}_2), -\delta_1(m_1) \ast \bar{m}_2\) and \((\bar{g}_1 \ast \bar{m}_2), -\bar{m}_2 \ast \delta_1(m_1)\), for all \( m_1 \in m_1, m_2 \in m_2 \). Set \( B = ((\bar{m}_1 \ast \bar{g}_2) \oplus (\bar{g}_1 \ast \bar{m}_2))/a \). Then

(i) \( m = m_1 + m_2 + B \) is a Leibniz algebra endowed with the Leibniz multiplication defined by

\[
[(m_1, m_2, z), (m'_1, m'_2, z')] = [(m_1, m'_1), [m_2, m'_2], (\bar{m}_1 \ast \bar{g}_2), (-\delta_1(m_1) \ast \bar{m}_2) + a]
\]

for all \( m_i, m'_i \in m_i, i = 1, 2, \) and \( z, z' \in B \).

(ii) There is an action of \( g := g_1 \oplus g_2 \) on \( m \) defined by

\[
(x_1, x_2)(m_1, m_2, z) = (x_1 m_1, x_2 m_2, (\bar{m}_2 \ast \bar{g}_2, -\bar{g}_1 \ast \bar{m}_2) + a),
\]

for all \( x_i \in g_i, i = 1, 2, \) and \( m_i \).
\[(m_1, m_2, z)^{(x_1, x_2)} = (m_1^{x_1}, m_2^{x_2}, (\overline{m}_1 * \overline{m}_2, -\overline{m}_2 * \overline{x}_1) + a),\]

for all \(m_i \in m_i, x_i \in g_i\) \((i = 1, 2)\) and \(z \in B\).

(iii) The homomorphism \(\delta : m \rightarrow g, (m_1, m_2, z) \mapsto (\delta_1(m_1), \delta_2(m_2))\), together with the action given in Part (ii) is a Leibniz crossed module.

(Here \(\overline{m}_i = m_i / [g_i, m_i], \overline{g}_i = g_i / g_i^2, i = 1, 2,\) and the symbol \(+\) denotes a direct sum of the underlying vector space structure.)

Proof. Parts (i) and (ii) follow straightforwardly. To prove the final part of the lemma, we first show that \(\delta^x m = [x, \delta(m)]\) for all \(m \in m, x \in g\). Suppose \(m = (m_1, m_2, z)\) and \(x = (x_1, x_2)\) for some \(m_i \in m_i, x_i \in g_i\) \((i = 1, 2)\) and \(z \in B\). Then we have

\[
\delta^{(x_1, x_2)}(m_1, m_2, z) = \delta^{(x_1m_1, x_2m_2, (\overline{m}_1 * \overline{x}_2, -\overline{m}_2 * \overline{x}_1) + a)} = (\delta_1^1(m_1), \delta_2^2(m_2))
\]

\[
= ([x_1, \delta_1(m_1)], [x_2, \delta_2(m_2)]) = [(x_1, x_2), \delta(m_1, m_2, z)].
\]

Analogously, one can check the accuracy of other relations and the proof is complete. \(\square\)

**Theorem 4.5.** Let \((g_i, n_i)\) be a pair of Leibniz algebras and \(\delta_i : m_i \rightarrow g_i\) be a relative stem cover of \((g_i, n_i)\) for \(i = 1, 2\). Then the Leibniz crossed module \(\delta : m \rightarrow g\) obtained in Lemma 4.4 is a relative stem cover of the pair \((g_1 \oplus g_2, n_1 \oplus n_2)\).

Proof. An easy verification shows that \(\ker \delta = \ker \delta_1 + \ker \delta_2 + B\) and \(\Im \delta = n_1 \oplus n_2\). As the actions of \(g_1\) on \(\overline{m}_1\) and \(\overline{g}_i\), \(i = 1, 2\), are trivial, it follows that \(g_1\) acts trivially on \(B\). Since a similar result holds for \(g_2\), we deduce that \(\ker \delta_1, \ker \delta_2\) and \(B\) act trivially on each other, whence \(\ker \delta = \ker \delta_1 \oplus \ker \delta_2 \oplus B\). Consequently, it can be inferred from Corollary 3.3 that \(\ker \delta \cong \mathcal{HL}_2(g_1, n_1) \oplus \mathcal{HL}_2(g_2, n_2) \oplus B \cong \mathcal{HL}_2(g_1 \oplus g_2, n_1 \oplus n_2)\). It remains to prove that \(\ker \delta \subseteq Z(g, m) \cap [g, m]\). Since \(\ker \delta_1 \subseteq Z(g_1, m_1) \cap [g_1, m_1]\) for \(i = 1, 2\), the action of \(g\) on \(\ker \delta\) is trivial and, moreover, any element of \(\ker \delta\) may be expressed as a finite linear combination of elements of the form \((y_1, y_2, z)\), where \(y_1 = x_1m_1\) or \(m_1^{x_1}, y_2 = x_2m_2\) or \(m_2^{x_2}\), and \(z = (\overline{m}_1' * \overline{x}_2', \overline{m}_2' * \overline{x}_1') + a, (\overline{m}_1' * \overline{x}_2', \overline{m}_2' * \overline{x}_1') + a, (\overline{m}_2' * \overline{m}_1', \overline{m}_2' * \overline{x}_1') + a, (\overline{m}_2' * \overline{m}_1', \overline{m}_2' * \overline{x}_1') + a\). It is seen that \((y_1, y_2, z) \in [g, m]\) for all possible cases of \(y_1, y_2\) and \(z\). For instance, we have

\[
(x_1, 0)(m_1, 0, 0) + (0, m_2, 0)(0, x_2) + (0, m_2', 0)(0, x_2') + (0, m_2', 0)(0, x_2') \in [g, m].
\]

The proof is now complete. \(\square\)

The following corollary is an immediate result of the above theorem. This result is similar to the works of Wiegold [31], and Salemkar and Edalatzadeh [30] in the cases of groups and of Lie algebras.

**Corollary 4.6.** Let \(g_1, g_2\) be any Leibniz algebras and \(m_1, m_2\) any covers of \(g_1, g_2\), respectively. Then \(m = m_1 + m_2 + m_1^{ab} \oplus m_2^{ab}\) with the Leibniz multiplication defined by

\[
(m_1, m_2, z)^{(x_1, x_2)} = (m_1^{x_1}, m_2^{x_2}, (\overline{m}_1 * \overline{m}_2, -\overline{m}_2 * \overline{x}_1) + a),
\]
for all \( m_i, m'_i \in m_i, i = 1, 2 \), and \( z, z' \in m^{ab}_1 \ast m^{ab}_2 \), is a cover of \( g_1 \ast g_2 \).

Theorem 4.5 may be useful to construct the relative stem covers of some pairs of Leibniz algebras.

**Example 4.7.** (i) With the assumptions of Theorem 4.5 if the Leibniz algebras \( g_1 \) and \( g_2 \) are perfect, then \( \delta : m_1 \ast m_2 \longrightarrow g_1 \ast g_2 \) is a relative stem cover of the pair \( (g_1 \ast g_2, n_1 \ast n_2) \).

(ii) Let \( g \) be a Leibniz algebra with ideals \( n \) and \( u \) such that \( g = n \ast u \) and let \( \delta_1 : m \longrightarrow n \) be a relative stem cover of the pair \((n,n)\). Since the map \( \delta_2 : 0 \longrightarrow u \) is a relative stem cover of the pair \((u,0)\), Theorem 4.5 yields that the map \( \delta : m + (m^{ab} \ast u^{ab}) \longrightarrow g, (m,z) \longmapsto \delta_1(m) \), is a relative stem cover of \((g,n)\).

(iii) Let \((g,n)\) be a pair of finite dimensional abelian Leibniz algebras. It is easy to see that 
\[
\delta : n + (n \ast n) \longrightarrow n, \quad (n,x) \longmapsto n,
\]
is a relative stem cover of \((g,n)\). But \( \dim((n \ast n) + (n \ast (g/n))) = (\dim(n))^2 + 2\dim(n)\dim(g/n) \) is equal to \( \dim(g \ast n) \), thanks to Corollary 3.5(i). It therefore follows that \((n \ast n) + (n \ast (g/n)) \cong g \ast n\) and \(\delta\) is a relative stem cover from \(n + (g \ast n)\) to \(g\) for the pair \((g,n)\).

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