Thomae type formula for $K3$ surfaces given by double covers of the projective plane branching along six lines

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February 3, 2010

Dedicated to Professor Takayuki Oda on his sixtieth birthday

Abstract

In this paper, we give Thomae type formula for $K3$ surfaces $\mathcal{X}$ given by double covers of the projective plane branching along six lines. This formula gives relations between theta constants on the bounded symmetric domain of type $I_{22}$ and period integrals of $\mathcal{X}$. Moreover, we express the period integrals by using the hypergeometric function $F_S$ of four variables. As applications of our main theorem, we define $\mathbb{R}^4$-valued sequences by mean iterations of four terms, and express their common limits by the hypergeometric function $F_S$.

MSC2000: Primary 33C70; Secondary 11F55.
Keywords: Hypergeometric Functions, Theta Functions.

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1
1 Introduction

Let us consider period integrals

\[ \omega_A(\lambda) = \int_1^{1/\lambda} \frac{dt}{\sqrt{t(1-t)(1-\lambda t)}}, \quad \omega_B(\lambda) = \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-\lambda t)}}, \]

of an elliptic curve \( s^2 = t(1-t)(1-\lambda t) \) with \( \lambda \in \mathbb{C} - \{0, 1\} \). If \( \lambda \) belongs to the open interval \((0, 1)\), then they are expressed by the Gauss hypergeometric function \( F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \):

\[ \omega_A(\lambda) = i\pi F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right), \quad \omega_B(\lambda) = \pi F\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right), \]

where \( i = \sqrt{-1} \). The function \( \tau = \omega_A(\lambda)/\omega_B(\lambda) \) of \( \lambda \) is continued to a map

\[ \text{per} : \tilde{X} \to \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \]

from the universal covering \( \tilde{X} \) of \( \mathbb{C} - \{0, 1\} \) to \( \mathbb{H} \), which is called the period map. The inverse of the map \( \text{per} \) can be described as

\[ \lambda = \vartheta^4_{[10]}(\tau)/\vartheta^4_{[00]}(\tau), \]
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where
\[ \vartheta_{[ab]}(\tau) = \sum_{n \in \mathbb{Z}} \exp\{\pi i ((n + \frac{a}{2})^2 \tau + (n + \frac{a}{2})b)\} \quad ([ab] = [00], [01], [10]) \]
is Jacobi’s theta constant. Under these correspondences of variables $\lambda \in \mathbb{C} - \{0, 1\}$ and $\tau \in \mathbb{H}$, the theta constant and the elliptic integral are related as
\[ \vartheta^4_{[ab]}(\tau) = \frac{\Lambda_{[ab]}}{\pi^2} \omega_B(\lambda)^2, \quad (2) \]
where
\[ \Lambda_{[00]} = 1, \quad \Lambda_{[01]} = 1 - \lambda, \quad \Lambda_{[10]} = \lambda. \]
The identity (2) is called Jacobi’s formula. On the other hand, we have the $2\tau$-formulas for the theta constants
\[ \vartheta^2_{[00]}(2\tau) = \frac{\vartheta^2_{[00]}(\tau) + \vartheta^2_{[01]}(\tau)}{2}, \quad \vartheta^2_{[01]}(2\tau) = \vartheta_{[00]}(\tau)\vartheta_{[01]}(\tau). \]

These formulas are applied to the study of the arithmetic-geometric mean as follows. Let $c_1, c_2 \in \mathbb{R}_+^\times$ be positive real numbers. We define vector valued sequence $\{m^n(c_1, c_2)\}_{n \in \mathbb{N}}$ by
\[ m^n(c_1, c_2) = m \circ \cdots \circ m(c_1, c_2) \]
where the map $m : (\mathbb{R}_+^\times)^2 \to (\mathbb{R}_+^\times)^2$ is
\[ m(u_1, u_2) = (\frac{u_1 + u_2}{2}, \sqrt{u_1u_2}). \]

Both components have a common limit and it is called the arithmetic-geometric mean and denoted by $m^\infty(c_1, c_2)$. Using Jacobi’s formula and $2\tau$-formulas, we have a relation between the arithmetic-geometric mean and the hypergeometric function:
\[ m^\infty(c_1, c_2) = \frac{c_1}{F(\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{(c_2)^2}{c_1})}. \]

By the above relation and the invariance property $m^\infty(m(c_1, c_2)) = m^\infty(c_1, c_2)$, we have the Gauss transformation formula
\[ F(\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{4z}{(1+z)^2}) = \frac{1+z}{2} F(\frac{1}{2}, \frac{1}{2}, 1, 1 - z^2). \quad (3) \]
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Thomae studies period integrals of a hyperelliptic curve of arbitrary genus and generalizes Jacobi’s formula to Thomae’s formulas in [To]. Based on $2\tau$-formulas of theta constants defined on the Siegel upper half space $\mathbb{H}_2$ of degree 2, Borchardt introduces a vector valued sequence $\{m^n(c_1, \ldots, c_4)\}_{n \in \mathbb{N}}$ with initial $(c_1, \ldots, c_4) \in (\mathbb{R}_+^\times)^4$ given by the iteration of the map

$$m: (\mathbb{R}_+^\times)^4 \ni u = (u_1, \ldots, u_4) \mapsto (m_1(u), \ldots, m_4(u)) \in (\mathbb{R}_+^\times)^4,$$

where

$$m_1(u) = \frac{u_1 + u_2 + u_3 + u_4}{4}, \quad m_2(u) = \frac{\sqrt{u_1u_2} + \sqrt{u_3u_4}}{2},$$
$$m_3(u) = \frac{\sqrt{u_1u_3} + \sqrt{u_2u_4}}{2}, \quad m_4(u) = \frac{\sqrt{u_1u_4} + \sqrt{u_2u_3}}{2}.$$

By using Thomae’s formulas, he expresses the common limit of the components of $\{m^n(c_1, \ldots, c_4)\}_{n \in \mathbb{N}}$ by period integrals of a genus 2 hyperelliptic curve. For related studies, refer to [B], [MT] and [Me].

In this paper, we give Thomae type formula for K3 surfaces which are double covers of the complex projective plane $\mathbb{P}^2$ branching along normal crossing six lines. The configurations of normal crossing six lines are parametrized by $3 \times 6$ matrices $x$ and the corresponding K3 surface is denoted by $X(x)$. Period integrals of $X(x)$ are expressed in terms of two kinds of hypergeometric functions $F_S$ and $F_T$ of four variables defined in (8) and (9), respectively. In §3.1 we define a normalized period matrix $\tau$ of $X(x)$ in the 4-dimensional bounded symmetric domain $\mathbb{D}$ of type $I_{22}$. Let $P_{3,3}$ be the set of unordered pair $\langle J \rangle = (J, J^c)$, such that $\#J = \#J^c = 3$ and $J \cup J^c = \{1, \ldots, 6\}$. Then we have $\#P_{3,3} = 10$. To state the main theorem, we introduce the following notations:

1. $\Theta_{\langle J \rangle}(\tau)$ is theta functions on $\mathbb{D}$ evaluated at the normalized period matrix $\tau$ indexed by $\langle J \rangle \in P_{3,3}$,

2. $x\langle J \rangle$ is the product of two $3 \times 3$-minors of a $3 \times 6$-matrix $x$ in the configuration space $X(3, 6)$ also indexed by $\langle J \rangle \in P_{3,3}$,

3. $\omega_{34}(x)$ is the period integral of the K3 surface $X(x)$ given in (6).

Then the main theorem is the identity

$$\Theta^2_{\langle J \rangle}(\tau) = \frac{1}{4\pi^4} x\langle J \rangle \omega_{34}(x)^2 \quad (4)$$
for any \( \langle J \rangle \in P_{3,3} \).

The subfamily consisting of Kummer varieties of principally polarized abelian varieties is called the Kummer locus. This locus corresponds to the Siegel upper half space \( \mathbb{H}_2 \) realized as a closed subdomain of \( \mathbb{D} \). Our identity becomes Thomae’s formula for genus two curves on this locus.

In the paper [MSY] and [Ma], they prove that the point \( [\Theta_{\langle J \rangle}(\tau)]_{\langle J \rangle \in P_{3,3}} \) in \( \mathbb{P}^9 \) is equal to \( [x_{\langle J \rangle}]_{\langle J \rangle \in P_{3,3}} \). The key for our proof of the main theorem is the study of the relation between a period of \( X \) and the automorphic factor of \( \Theta_{\langle J \rangle} \) by the action of the monodromy group of \( \text{per} \) via the isomorphism between \( D \) and \( D_H \) defined in §3.1.

As an application of our main theorem, we study a vector valued sequence obtained by mean iteration of a map from \( \mathbb{R}_+^4 \) to \( \mathbb{R}_+^4 \) which is different from that defined by Borchardt in §5.1. We show that this vector valued sequence has a common limit and that it can be expressed by the hypergeometric function \( F_S \). This formula is obtained by the main theorem and \( 2\tau \) formulas for the theta functions \( \Theta_{\langle J \rangle} \) in Theorem 1. We also give an explanation on the relation between Borchardt’s arithmetic-geometric mean \( m_\infty^*(c_1, \ldots, c_4) \) and the hypergeometric function \( F_S \) in §5.2. In the last section, we prove several functional equations of the hypergeometric function \( F_S \) arising from the invariance property for vector valued mean iterations. These are analogs of the Gauss transformation formula [3] for the hypergeometric function \( F_S \).

2 Certain family of K3 surfaces

2.1 Double coverings of \( \mathbb{P}^2 \) branching along 6 lines

Let \( M^\times(3,6) \) be the open subset of \( M(3,6) \) defined by

\[
M^\times(3,6) = \left\{ x = (\ell_1, \ldots, \ell_6) \in M(3,6) \mid \text{the determinants of (3,3)-minors are non-zero} \right\}
\]

For \( \ell_i = (\ell_{0i}, \ell_{1i}, \ell_{2i}) \), we define a linear function \((t, \ell_i)\) by \( \sum_{j=1}^{3} \ell_j t_j \). Let \( \tilde{\mathcal{X}}^* \) be the variety defined by

\[
\tilde{\mathcal{X}}^* = \left\{ (t : y) \times x \in \mathbb{P}(1,1,1,3) \times M^\times(3,6) \mid y^2 = (t, \ell_1) \cdot \cdots \cdot (t, \ell_6) \right\},
\]

where \( (t : y) = (t_0 : t_1 : t_2 : y) \), \( x = (\ell_1, \ldots, \ell_6) \) and \( \mathbb{P}(1,1,1,3) \) is the weighted projective space of weight \( (1,1,1,3) \). Then by the natural map
pr$_1 : \hat{\mathcal{X}}^* \to \mathbb{P}^2 \times M^\times(3, 6)$, $\hat{\mathcal{X}}^*$ is a family of branched covering of $\mathbb{P}^2$ over $M^\times(3, 6)$. By resolving singularities, we have a family of K3 surfaces pr$_2 : \tilde{\mathcal{X}} \to M^\times(3, 6)$ on $M^\times(3, 6)$.

Let $T$ be a torus defined by

$$T = \{ \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_6) \in \mathbb{C}^6 \times \mathbb{C}^\times \mid \lambda_0^2 = \lambda_1 \cdot \cdots \lambda_6 \}.$$ 

The group $GL_3(\mathbb{C}) \times T$ acts on $\tilde{\mathcal{X}}$ by

$$t \mapsto tg^{-1}, \quad x \mapsto g \cdot x \cdot \text{diag}(\lambda_1, \ldots, \lambda_6), \quad y \mapsto \lambda y$$

for $(g, \lambda) \in GL_3(\mathbb{C}) \times T$ and it induces an action of $GL_3(\mathbb{C}) \times T$ on $M^\times(3, 6)$. The quotients of $\tilde{\mathcal{X}}$ and $M^\times(3, 6)$ by $GL_3(\mathbb{C}) \times T$ are denoted by $\mathcal{X}$ and $\mathcal{X}$, respectively. The natural map $pr_2 : \tilde{\mathcal{X}} \to M^\times(3, 6)$ induces a map $\mathcal{X} \to \mathcal{Y}$, which is also denoted by $pr_2$. The variety $\mathcal{X}$ is equal to the double coset space:

$$\mathcal{X} = X(3, 6) = GL_3(\mathbb{C}) \backslash M^\times(3, 6)/(\mathbb{C}^\times)^6,$$

which is called the configuration space. The fiber of $\mathcal{X}$ at $x \in X$ is denoted by $\mathcal{X}(x)$.

There are 15 rational curves $\tilde{l}_{jk}(x)$ ($1 \leq j < k \leq 6$) in $\mathcal{X}(x)$ coming from the resolutions of nodes at $l_{jk}(x) = l_j(x) \cap l_k(x)$, where $l_i(x)$ is the line defined by $(t, \ell_i) = 0$. Let $\tilde{l}_0(x)$ be a pull back of a generic line in $\mathbb{P}^2$ by $pr_1$. Let $\mathcal{S}(x)$ be the subgroup of $H_2(\mathcal{X}(x), \mathbb{Z})$ generated by algebraic cycles $\tilde{l}_{jk}(x)$ and $\tilde{l}_0(x)$. Its orthogonal complement $T(x)$ in $H_2(\mathcal{X}(x), \mathbb{Z})$ with respect to the intersection form $(\cdot, \cdot)$ is called the transcendental lattice of $X$ and its rank is $22 - 16 = 6$. In §2.3 we give a basis of $T(x)$ and its dual in $H_2(\mathcal{X}(x), \mathbb{Z})$. These bases are slightly different from those defined in [MSY] and [Y].

### 2.2 Relative invariants and a global two form

The characters

$$\rho : GL_3(\mathbb{C}) \times \mathbb{T} \ni (g, \lambda) \mapsto \text{deg}(g)\lambda_0 \in \mathbb{C}^\times :$$

and $\rho^2$ define linearizations of $GL_3(\mathbb{C}) \times \mathbb{T}$ of $\mathcal{O}_{M^\times(3, 6)}$ and $\mathcal{O}_{\tilde{\mathcal{X}}}$. The invariant line bundles on $\mathcal{X}$ and $\mathcal{X}$ under these actions are denoted by $\mathcal{L}$ and $\mathcal{M}$, respectively. We have $\mathcal{L}^\otimes 2 = pr_2^*\mathcal{M}$. 

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We construct elements of $H^0(X, \mathcal{M})$. Let $J$ be a subset of the set $\{1, \ldots, 6\}$ with cardinality 3 and $J^c$ be its complement. By reordering elements, we may write $J$ and $J^c$ as

$$J = \{j_1, j_2, j_3\}, \quad j_1 < j_2 < j_3; \quad J^c = \{j_4, j_5, j_6\}, \quad j_4 < j_5 < j_6.$$ 

A pair $\langle J \rangle = (J, J^c) = (J^c, J)$ of $J$ and $J^c$ is called a $(3,3)$-partition of the set $\{1, \ldots, 6\}$. The set of $(3,3)$-partitions is denoted by $P_{3,3}$. Note that $\#P_{3,3} = 10$. For $x = (x_{ij}) \in M(3,6)$ and $\langle J \rangle \in P_{3,3}$, we set

$$x_{\langle J \rangle} = \det(x_{i,jk})_{1 \leq i,k \leq 3} \det(x_{i,jk+3})_{1 \leq i,k \leq 3}.$$ 

Then $x_{\langle J \rangle}$ is an element of $H^0(X, \mathcal{M})$. By Plücker relations, we have the following.

**Lemma 1** Let $St$ be the set of $(2,2,2)$-standard tableaux i.e. $(J, J^c) = (\{j_1, j_2, j_3\}, \{j_4, j_5, j_6\})$ with

$$j_1 < j_4 \quad \wedge \quad \wedge \quad j_2 < j_5 \quad \wedge \quad \wedge \quad j_3 < j_6.$$ 

Then $\#St = 5$ and $\{x_{\langle J \rangle} \mid \langle J \rangle \in St\}$ forms a basis of a linear system in $H^0(X, \mathcal{M})$ generated by the polynomials $x(ijk)$ ($1 \leq i < j < k \leq 6$).

Let $\widehat{pl}$ be the map from $M^\times(3,6)$ to $\mathbb{C}^{10}$ defined by

$$\widehat{pl} : M^\times(3,6) \ni x \mapsto (\ldots, x_{\langle J \rangle}, \ldots)_{\langle J \rangle \in P_{3,3}} \in \mathbb{C}^{10},$$ 

where we arrange $x_{\langle J \rangle}$ lexicographically for $J = \{j_1, j_2, j_3\}$ with $j_3 \leq 5$. By Lemma 1, the image of $\widehat{pl}$ is contained in a 5-dimensional linear subspace of $\mathbb{C}^{10}$. The map $X \to \mathbb{P}^4$ induced from $\widehat{pl}$ is denoted by $pl$.

The space of relative global holomorphic 2-forms $H^0(\hat{X}, \Omega_{\hat{X}/\hat{M}^\times(3,6)}^2)$ is generated by

$$\varphi = \frac{t_0 dt_1 \wedge dt_2 - t_1 dt_0 \wedge dt_2 + t_2 dt_0 \wedge dt_1}{y}.$$ 

**Proposition 1** The form $\varphi$ satisfies the equality:

$$(g, \lambda_i, \lambda)^* \varphi = \rho^{-1}(g, \lambda_i, \lambda) \varphi.$$ 

Therefore it defines a global section of $H^0(\hat{X}, \Omega_{\hat{X}/X}^2 \otimes L^{-1})$. 

2.3 Topological cycles at a reference point

We take a reference point \( \hat{x} \) in \( M^*(3, 6) \) as

\[
\hat{x} = \begin{pmatrix}
\frac{p_1^2}{p_1} & \frac{p_2^2}{p_1} & \frac{p_3^2}{p_1} & \frac{p_4^2}{p_1} & \frac{p_5^2}{p_1} & \frac{p_6^2}{p_1} \\
-1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
\]

with \( p_1 = -3, \ p_2 = -2, \ p_3 = -1, \ p_4 = 1, \ p_5 = 2, \ p_6 = 3 \). We consider the affine coordinates \( s_1 = t_1/t_0 \) and \( s_2 = t_2/t_0 \) of \( \mathbb{P}^2 \). We construct topological 2-cycles of \( \mathcal{X}(\hat{x}) \) using the isomorphism of the Kummer surface of \( C \) and \( \mathcal{X}(\hat{x}) \) given in [Te].

Let \( C \) be a hyperelliptic curve defined by

\[
u^2 = \prod_{i=1}^{6} (w - p_i),
\]

and \( C_1, C_2 \) be copies of \( C \). Let \( \text{sym} : C_1 \times C_2 \to \mathcal{X}(\hat{x}) \) be a map defined by

\[
((w_1, u_1), (w_2, u_2)) \mapsto (s_1, s_2, y) = (w_1 + w_2, w_1w_2, u_1u_2).
\]

For \( a < b \in \mathbb{R} \), the 1-chain in \( \mathbb{P}^1 \) defined by the segment from \( a \) to \( b \) is denoted by \( (a, b) \). We define chains \( A_1', A_2', B_1' \) and \( B_2' \) in \( C \) by the liftings of \( (p_1, p_2), (p_5, p_6), (p_2, p_3) \) and \( (p_4, p_5) \) on which \( \frac{1}{u} \) is in \( i\mathbb{R}_+, i\mathbb{R}_+, \mathbb{R}_+ \) and \( \mathbb{R}_+ \). Then \( A_i = A_i' - \sigma(A_i') \) and \( B_i = B_i' - \sigma(B_i') \) become 1-cycles on \( C \).

We set \( A_1 = \gamma_1, A_2 = \gamma_2, B_1 = \gamma_3 \) and \( B_2 = \gamma_4 \). We define a topological cycle \( \overline{\tau}_{ij}^* \) by \( \text{sym}^*(\overline{\tau}_{ij} \times \overline{\tau}_{ij}) \). The proper inverse image of \( \overline{\tau}_{ij}^* \) in \( \mathcal{X}(\hat{x}) \) is denoted by \( \overline{\tau}_{ij} \). Then \( \gamma_{ij} = \overline{\tau}_{ij}/2 \) is an element in \( H_2(\mathcal{X}(\hat{x}), \mathbb{Z}) \). Let \( \gamma_{ij} \) be the orthogonal projection of \( 2\gamma_{ij} \) to \( T(x) \). Then \( \{\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}\} \) is a basis of \( T(x) \). Since

\[
\text{sym}^*(\varphi(\hat{x})) = \frac{(w_1 - w_2)dw_1 \wedge dw_2}{u_1u_2},
\]

we have

\[
\iint_{\gamma_{12}} \varphi(\hat{x}) \in -\mathbb{R}^\times_+, \quad \iint_{\gamma_{23}} \varphi(\hat{x}) \in \mathbb{R}^\times_+,
\]

\[
\iint_{\gamma_{13}} \varphi(\hat{x}) \in \mathbb{R}^\times_+, \quad \iint_{\gamma_{14}} \varphi(\hat{x}) \in i\mathbb{R}^\times_+,
\]

\[
\iint_{\gamma_{24}} \varphi(\hat{x}) \in \mathbb{R}^\times_+, \quad \iint_{\gamma_{24}} \varphi(\hat{x}) \in -i\mathbb{R}^\times_+.
\]
Proposition 2 We set
\[\gamma = t(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}), \quad \gamma' = t(\gamma'_{12}, \gamma'_{13}, \gamma'_{14}, \gamma'_{23}, \gamma'_{24}, \gamma'_{34}).\]
Then the intersection matrix is equal to
\[(\gamma \cdot ^t\gamma') = H, \quad (\gamma' \cdot ^t\gamma') = 2H,
\]
where
\[H = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]
Thus the lattice structure of \(T(x)\) is equal to \(U(2) \oplus U(2) \oplus A_1(-1) \oplus A_1(-1)\).
(For details, see [MSY] and [Y], Chapter VIII.)

For any \(x \in M^\times(3, 6)\), take a path \(\rho_x\) in \(M^\times(3, 6)\) connecting \(\dot{x}\) and \(x\), and define bases \(\gamma_{ij}(x)\) and \(\gamma'_{ij}(x)\) as the continuations of \(\gamma_{ij}(\dot{x})\) and \(\gamma'_{ij}(\dot{x})\) along the path \(\rho_x\) by the local triviality. They depend only on the homotopy class of \(\rho_x\). The intersection matrix for \((\gamma'(x) \cdot ^t\gamma'(x))\) are equal to that in Proposition 2. Let \(\omega\) be a vector defined by
\[
\omega(x) = t(\omega_{12}(x), \omega_{13}(x), \omega_{14}(x), \omega_{23}(x), \omega_{24}(x), \omega_{34}(x)),
\]
\[
\omega_{ij}(x) = \int_{\gamma_{ij}(x)} \varphi(x).
\]
The map
\[\tilde{\text{per}} : \tilde{X} \ni x \mapsto [\omega(x)] \in \mathbb{P}^5,\]
is called the period map, where \(\tilde{X}\) is the universal covering of \(X\).

2.4 Period integrals and Hypergeometric functions

We define two hypergeometric series \(F_S^\alpha(z)\) and \(F_T^\alpha(z)\) of variables \(z = \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\) with parameters \(\alpha = (\alpha_1, \ldots, \alpha_6)\) satisfying \(\sum_{j=1}^6 \alpha_j = 3\) as
\[
F_S^\alpha(z) = \sum_{n \in \mathbb{N}^4} \frac{(1 - \alpha_1)_{n_1 + n_3}(1 - \alpha_2)_{n_2 + n_4}(\alpha_5)_{n_1 + n_2}(\alpha_6)_{n_3 + n_4}}{(2 - \alpha_1 - \alpha_3)_{n_1 + n_3}(2 - \alpha_2 - \alpha_4)_{n_2 + n_4}n_1!n_2!n_3!n_4!} z^n,
\]
\[
F_T^\alpha(z) = \sum_{n \in \mathbb{N}^4} \frac{(1 - \alpha_1)_{n_1 + n_3}(1 - \alpha_2)_{n_2 + n_4}(\alpha_5)_{n_1 + n_2}(\alpha_6)_{n_3 + n_4}}{(3 - \alpha_1 - \alpha_2 - \alpha_3)_{n_1 + n_2 + n_3 + n_4}n_1!n_2!n_3!n_4!} z^n.
\]
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where \( N = \{0, 1, 2, \ldots\} \), \( z^n = z_1^{n_1}z_2^{n_2}z_3^{n_3}z_4^{n_4} \) for \( n = (n_1, \ldots, n_4) \), \( (\alpha_j)_{n_j} = \alpha_j(\alpha_j + 1) \cdot \cdot (\alpha_j + n_j - 1) = \Gamma(\alpha_j + n_j)/\Gamma(\alpha_j) \), and we assume that

\[
\begin{align*}
\alpha_1 + \alpha_3 - 2, \quad \alpha_2 + \alpha_4 - 2 & \notin \mathbb{N} \quad \text{for} \quad F^\alpha_S(z), \\
\alpha_1 + \alpha_2 + \alpha_3 - 3 & \notin \mathbb{N} \quad \text{for} \quad F^\alpha_T(z).
\end{align*}
\]

They absolutely converge on the domain

\[
\{z \in \mathbb{C}^4 \mid |z_1| + |z_2| < 1, \quad |z_3| + |z_4| < 1\}.
\]

By the standard argument for Euler type integrals, we have the following Proposition.

**Proposition 3** The hypergeometric series \( F^\alpha_S \) and \( F^\alpha_T \) admit the integral representations:

\[
\begin{align*}
F^\alpha_S(z) &= \frac{1}{B(1-\alpha_1,1-\alpha_3)B(1-\alpha_2,1-\alpha_4)} \int_0^1 \int_0^1 L^\alpha_S(z,s) \, ds_1 ds_2, \\
F^\alpha_T(z) &= \frac{\Gamma(3-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)} \int_\Delta L^\alpha_T(z,s) ds_1 ds_2,
\end{align*}
\]

where \( B \) denotes the beta function,

\[
\begin{align*}
L^\alpha_S(z,s) &= s^{-\alpha_1}_1 s^{-\alpha_2}_2 (1-s_1)^{-\alpha_3}_1 (1-s_2)^{-\alpha_4}_1 \\
&\quad \times (1-z_1s_1-z_2s_2)^{-\alpha_5}_1 (1-z_3s_1-z_4s_2)^{-\alpha_5}_1, \\
L^\alpha_T(z,s) &= s^{-\alpha_1}_1 s^{-\alpha_2}_2 (1-s_1-s_2)^{-\alpha_3}_1 \\
&\quad \times (1-z_1s_1-z_2s_2)^{-\alpha_5}_1 (1-z_3s_1-z_4s_2)^{-\alpha_5}_1, \\
\Delta &= \{(s_1, s_2) \in \mathbb{R}^2 \mid s_1 > 0, s_2 > 0, s_1 + s_2 < 1\},
\end{align*}
\]

\( \arg(s_j) = \arg(1-s_j) = \arg(1-s_1-s_2) = 0 \) on each interior area of the integrations, \( \arg(1-z_1s_1-z_2s_2), \arg(1-z_3s_1-z_4s_2) \) become 0 at \((s_1, s_2) = (0, 0)\), and we assume that

\[
\begin{align*}
\text{Re}(\alpha_1), \text{Re}(\alpha_2), \text{Re}(\alpha_3), \text{Re}(\alpha_4) &< 1 \quad \text{for} \quad F^\alpha_S(z), \\
\text{Re}(\alpha_1), \text{Re}(\alpha_2), \text{Re}(\alpha_3) &< 1 \quad \text{for} \quad F^\alpha_T(z).
\end{align*}
\]

From now on, we put \( \alpha = (1/2, \ldots, 1/2) \) and we set

\[
F_S(z) = F^\alpha_S(z), \quad F_T(z) = F^\alpha_T(z).
\]

We can regard \( x(J)\omega_{ij}(x)^2 \) as a multivalued function on \( X \) for any \( \langle J \rangle \). Proposition 4 and 3 imply the following.
Proposition 4 For a fixed \( \langle J \rangle \in \mathbb{P}_{3,3} \), the product \( x(J) \cdot \omega_{ij}(x)^2 \) is invariant under the action of \( \text{GL}_3(\mathbb{C}) \times \mathbb{T} \), i.e.,

\[
(g, \lambda, \lambda)^* (x(J) \cdot \omega_{ij}^2) = x(J) \cdot \omega_{ij}^2.
\]

As a consequence, if \( x \) is in a neighborhood of \( \hat{x} \in M^\times(3, 6) \), \( \omega_{ij}(x)^2 \) can be expressed as

\[
\omega_{ij}(x)^2 x(J) = \begin{cases} 
4\pi^4 F_S(\zeta_{ij}(x))^2 \nu_{ij}(J), & (i, j) = (1, 2), (1, 4), (2, 3), (3, 4), \\
16\pi^2 F_T(\zeta_{ij}(x))^2 \nu_{ij}(J), & (i, j) = (1, 3), (2, 4),
\end{cases}
\]

where we set \( \zeta_{ij}(x) = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \) so that the following \( \nu_{ij}, \ldots, \nu_{34} \in M^\times(3, 6) \) are equivalent to \( x \) as elements of \( X \):

\[
\nu_{12} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -z_1 & -z_3 & 0 & 0 \\ 0 & 0 & -z_2 & -z_4 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \\
\nu_{13} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -z_1 & -z_3 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -z_1 & -z_2 & -z_4 & 1 & 1 \\ -z_2 & -1 & -z_4 & 0 & 0 \end{pmatrix}, \\
\nu_{14} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ -1 & 1 & -z_1 & 0 & 0 & -z_3 \\ 0 & 0 & -z_2 & 1 & -1 & -z_4 \\ -z_1 & 0 & 0 & -z_3 & 1 & 1 \\ -z_2 & -1 & 1 & -z_4 & 0 & 0 \end{pmatrix}, \\
\nu_{23} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & z_1 & -z_3 & -1 & 1 & 0 \\ 0 & -z_2 & -z_4 & -1 & 1 & 0 \\ -z_2 & -1 & 1 & 0 & 0 & -z_3 \\ -z_2 & 0 & 0 & 1 & -1 & -z_4 \end{pmatrix}.
\]

2.5 Preparation for the association involution

Let \( as \) be an automorphism of \( M^\times(3, 6) \) given by

\[
as : M^\times(3, 6) \ni (y_1, y_2) \mapsto (y_1^{-1}y_2y_1, y_1) \in M^\times(3, 6),
\]

where \( y_1, y_2 \in \text{GL}_3(\mathbb{C}) \). By a straightforward calculation, we have

\[
\tilde{p}l \circ as = \tilde{p}l, \quad as^2(y_1, y_2) = (y_1^{-1}y_2y_1y_2^{-1}y_1, y_1^{-1}y_2y_1) = y_1^{-1}y_2y_1y_2^{-1}(y_1, y_2).
\]

Therefore \( as \) induces an involution on \( X \), which is called the association involution and also denoted by \( as \). By the above equality, we have an induced morphism

\[
pl^* : X/\langle as \rangle \rightarrow \mathbb{P}^4.
\]
Proposition 5 (Chapter VII of [Y]) The morphism \( pl^* \) is an open immersion.

Let \( \overline{X} \) be the normalization of \( \mathbb{P}^4 \) in \( X \). Then we have the diagram:

\[
\begin{array}{ccc}
X & \rightarrow & \overline{X} \\
\downarrow & & \downarrow \\
X/\langle as \rangle & \rightarrow & \mathbb{P}^4.
\end{array}
\]

The induced map \( \overline{X} \rightarrow \mathbb{P}^4 \) is denoted as \( \overline{pl} \). Let \( x = (x_{ij})_{ij} \in M^x(3,6) \). We define the following polynomials

\[
Q = \det(x_{1i}^2, x_{2i}^2, x_{3i}^2, x_{2i}x_{3i}, x_{3i}x_{1i}, x_{1i}x_{2i})_{i=1,\ldots,6},
\]

\[
D(ijk) = \det(x_{pi}, x_{pj}, x_{pk})_{p=1,\ldots,3},
\]

\[
\{ij;kl\} = D(ijm)D(ijn)D(mkl)D(nkl),
\]

\[
T(ijklmn) = D(ijk)D(ikm)D(mln)D(njl)
\]

for \( \{i,j,k,l,m,n\} = \{1,\ldots,6\} \). Then we have \( \{ij;kl\} = \pm x(ijm)x(ijn) \) and \( as(Q) = -Q \). We give an explicit description of the normal form

\[
\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & x_1 & y_1 \\
0 & 0 & 1 & 1 & x_2 & y_2
\end{array}\right)
\]

of the inverse image under \( \overline{pl} \). By Lemma A6.8 of [MSY], we have

\[
x_2 = \frac{D(125)D(234)}{D(235)D(124)} = \frac{T(125364)}{\{35;14\}}.
\]

We have \( as(T(125364)) = T(364125) \) and by Lemma A7.3 of [MSY], we have

\[
T(125364) + as(T(125364)) = \{14;53\} - \{52;16\} + \{63;54\} - \{23;15\} + \{24;56\}
\]

\[
T(125364) \cdot as(T(125364)) = \{16;23\} \cdot \{12;36\}
\]

Therefore \( T(125364) \) is defined by a quadratic equation with the coefficients in polynomials of \( x(ijk) \)'s. The values \( x_1, y_2, y_2 \) can be obtained by substituting indices \( 2 \leftrightarrow 3, 5 \leftrightarrow 6 \).

Since

\[
T(125364) - as(T(125364)) = Q,
\]

if \( \{16;23\} \cdot \{12;36\} = 0 \), then \( Q = T(125364) \) or \( -T(364125) \) and the values \( x_2, x_1, y_2, y_1 \) are polynomials of \( x(ijk) \). Thus we have the following lemma.
Lemma 2 The inverse image of the divisor \( \{ x(164) = 0 \} \) on \( \mathbb{P}^4 \) under the map \( \pi \) consists of two irreducible components. On \( X \), we can express \( x_1, x_2, y_1, y_2 \) as rational functions of \( x(ijk) \) on each irreducible component.

For an explicit description of the inverse image, see Proposition 8.

3 Bounded symmetric domains and theta functions.

3.1 Period map and symmetric domains \( D_H \) and \( \mathbb{D} \)

In this section, we introduce two symmetric domains \( D_H \) and \( \mathbb{D} \). The target space for the natural period map for K3 surfaces is the symmetric domain \( D_H \) of type IV. We use theta functions on \( \mathbb{D} \) and an isomorphism between \( D_H \) and \( \mathbb{D} \) to construct automorphic functions on \( D_H \) by using results in \([Ma]\).

By Riemann bilinear relations for the K3 surface \( X(x) \), and the choice of orientations of \( T(x) \) in \([5]\), the class \([\omega(x)]\) of \([6]\) in \( \mathbb{P}^5 \) belongs to the subset

\[
D_H = \{ [w] \in \mathbb{P}^5 \mid {}^t \omega H \omega = 0, \quad \omega^* H \omega > 0, \quad \text{Im} \left( \frac{\omega_{14}}{\omega_{34}} \right) > 0 \}.
\]

Here \( y^* = {}^t \overline{y} \) denotes the adjoint of a matrix \( y \). Therefore we regard the map \( \widetilde{\text{per}} : \widetilde{X} \to D_H \).

Let \( \mathbb{D} \) be the bounded symmetric domain of type \( I_{22} \) defined by

\[
\mathbb{D} = \{ \tau \in M(2,2) \mid \frac{\tau - \tau^*}{2i} \text{ is positive definite} \}.
\]

In this subsection, we define an isomorphism \( \mathbb{D} \to D_H \). Let \( \tau \) be an element in \( \mathbb{D} \). We set

\[
\tilde{\tau} = \begin{pmatrix} \tau & E_2 \end{pmatrix}.
\]

Let \( \tilde{\tau}(i_1i_2) \) be the \((i_1, i_2) \times (1,2)\)-minor of the \( 4 \times 2 \) matrix \( \tilde{\tau} \). They satisfy the Plücker relation

\[
\tilde{\tau}(12)\tilde{\tau}(34) - \tilde{\tau}(13)\tilde{\tau}(24) + \tilde{\tau}(14)\tilde{\tau}(23) = {}^t v(\tilde{\tau}) H' v(\tilde{\tau}) = 0,
\]
Thomae type formula for K3 surfaces

where

$$H' = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad v(\tilde{\tau}) = \begin{pmatrix} \tilde{\tau}(12) \\ \tilde{\tau}(13) \\ \tilde{\tau}(14) \\ \tilde{\tau}(23) \\ \tilde{\tau}(24) \\ \tilde{\tau}(34) \end{pmatrix}. $$

Since the matrix $(\tau - \tau^*)/2i$ is positive definite, we have

$$v(\tilde{\tau})^* H v(\tilde{\tau}) > 0, \quad \text{Im}(\tilde{\tau} \langle 14 \rangle \tau) > 0.$$

We set

$$Q = \begin{pmatrix} 1 & \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{-1+i}{2} & 1 & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1+i}{2} & 1 \end{pmatrix}.$$ 

Then by the equality

$$Q^* H Q = H, \quad H' = 'Q H Q, \quad H = 'Q H' Q,$$

the class $[Qv(\tilde{\tau})]$ of $Qv(\tilde{\tau})$ in $\mathbb{P}^5$ is contained in $D_H$. Thus we have an isomorphism

$$p_{D_\mathbb{D}} : \mathbb{D} \ni \tau \mapsto [Qv(\tilde{\tau})] \in D_H. \quad (10)$$

We define the normalized period matrix of $X(x)$ by $\tau = \tau(x) = p_{D_\mathbb{D}}^{-1}(\omega(x)) \in \mathbb{D}$. Then we have

$$\tau(x) = \frac{1}{\omega_{34}(x)} \begin{pmatrix} \omega_{14}(x) & -\omega_{13}(x) \omega_{24}(x) \\ -\omega_{13}(x) \omega_{24}(x) & \omega_{13}(x) - \omega_{24}(x) \end{pmatrix} \begin{pmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \end{pmatrix}, \quad (11)$$

where $\omega_{ij}(x)$ are defined by (6). Thus we have a map

$$\bar{\rho} : \bar{X} \ni x \mapsto \tau(x) \in \mathbb{D}.$$ 

Since $\tilde{\tau}(34)(\tau) = 1$, we have

$$j_\mathbb{D}(\tau(x)) = \omega(x)/\omega_{34}(x) \quad (12)$$

as elements of $\mathbb{C}^6$. 
3.2 Homomorphisms of discrete groups

We define a discrete group $\Gamma_H$ in $GL_6(\mathbb{Z})$ by

$$\Gamma_H = \{ R \in GL_6(\mathbb{Z}) \mid {}^t RHR = H, \quad \text{Im}\left(\frac{[R\omega(\hat{x})]_{14}}{[R\omega(\hat{x})]_{34}}\right) > 0\},$$

where $[R\omega(\hat{x})]_{ij}$ denotes the $(ij)$-component of $R\omega(\hat{x})$. Its center consists of $\pm E_6$. The group $\Gamma_H$ acts on $D_H$ from the left. Since the monodromy action preserves the intersection forms, the monodromy group is contained in $\Gamma_H$.

We define $U_{22}(\mathbb{Z}[i])$ and the principal congruence subgroup of level $(1+i)$ by

$$U_{22}(\mathbb{Z}[i]) = \left\{ g \in GL_4(\mathbb{Z}[i]) \mid g I_{22} g^* = I_{22} \right\},$$

$$U_{22}(1+i) = \left\{ g \in U_{22}(\mathbb{Z}[i]) \mid g \equiv E_4 \mod (1+i) \right\}.$$

Then an element $g = \left(\begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array}\right)$ in $U_{22}(\mathbb{Z}[i])$ acts on $D$ by

$$g \cdot \tau = \left(\begin{array}{c} g_{11} \tau + g_{12} \\ g_{21} \tau + g_{22} \end{array}\right)^{-1},$$

where $I_{22} = \left(\begin{array}{cc} O & -E_2 \\ E_2 & O \end{array}\right)$ and $g_{ij} \in M(2,2)$.

In this subsection, we define a homomorphism $U_{22}(\mathbb{Z}[i]) \to \Gamma_H/\langle \pm 1 \rangle$ of discrete group which is compatible with the isomorphism of symmetric domains $\mathbb{D} \to D_H$ defined in (10). For $g = (g_{ij}) = \left(\begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array}\right) \in U_{22}(\mathbb{Z}[i])$, we set a $6 \times 6$ matrix $\wedge^2 g$ by

$$\wedge^2 g = \left(\begin{array}{cc} g_{i_1j_1} & g_{i_1j_2} \\ g_{i_2j_1} & g_{i_2j_2} \end{array}\right)_{(i_1,i_2),(j_1,j_2)},$$

where $1 \leq i_1 < i_2 \leq 4$, $1 \leq j_1 < j_2 \leq 4$, and they are arranged lexicographically. Then we have

$$\det(\wedge^2 g) = \det(g)^3,$$

and

$$\det(g_{21}\tau + g_{22})v(\tau') = v(g\bar{\tau}) = (\wedge^2 g)v(\bar{\tau})$$

as elements of $\mathbb{C}^6$, where $\tau \in \mathbb{D}$,

$$\tau' = g \cdot \tau = \left(\begin{array}{c} g_{11} \tau + g_{12} \\ g_{21} \tau + g_{22} \end{array}\right)^{-1}, \quad \bar{\tau'} = \left(\begin{array}{c} \tau' \\ E_2 \end{array}\right), \quad g\bar{\tau} = g \left(\begin{array}{c} \tau \\ E_2 \end{array}\right).$$
Thus we have
\[
\det(g_{21} \tau + g_{22})Qv(\tilde{\tau}') = \{Q(\wedge^2 g)Q^{-1}\}\{Qv(\tilde{\tau})\},
\]
and by the definition of \(j_D\), we have
\[
\det(g_{21} \tau + g_{22})j_D(g \cdot \tau) = Q(\wedge^2 g)Q^{-1}j_D(\tau).
\tag{13}
\]
The matrix \(Q(\wedge^2 g)Q^{-1}\) belongs to the orthogonal group with respect to the quadratic form \(H\). Moreover, a straightforward calculation shows
\[
Q(\wedge^2 g)Q^{-1} \in \begin{cases} SL_6(\mathbb{Z}) & \text{if } \det(g) = 1, \\ 1 SL_6(\mathbb{Z}) & \text{if } \det(g) = -1, \end{cases}
\]
for \(g \in U_{22}(\mathbb{Z}[i])\). We set
\[
R_g = \sqrt{\det(g)Q(\wedge^2 g)Q^{-1}},
\tag{14}
\]
which is determined modulo sign. Since \(\wedge^2 (1g) = -\wedge^2 g\), we have \(R_{ig} = -R_g\). Thus \(R_g\) defines a homomorphism
\[
U_{2,2}(\mathbb{Z}[i])/\langle iE_4 \rangle \to \Gamma_H/\langle \pm E_6 \rangle \cap Aut(D) \to Aut(D_H).
\]

### 3.3 Monodromy actions on the spaces \(D_H\) and \(D\)

Each center of \(U_{22}(\mathbb{Z}[i])\) and \(U_{22}(1 + 1)\) is the group \(\langle iE_4 \rangle\) generated by the scalar matrix \(iE_4\). We have
\[
t^t(\bar{g} \cdot \tau) = g \cdot t^t\tau
\]
for any \(g \in U_{22}(\mathbb{Z}[i])\) and \(\tau \in D\). Let \(tp\) be the transpose operator acting on \(D\) and \(\langle tp \rangle\) be the group generated by \(tp\). The fixed locus of \(tp\) is the Siegel upper half space \(\mathbb{H}_2 = \{\tau \in \mathbb{D} \mid t^t\tau = \tau\}\) of degree 2. We define \(U_{22}^t(\mathbb{Z}[i])\) acting on \(D\) as the group generated by \(U_{22}(\mathbb{Z}[i])/\langle iE_4 \rangle\) and \(\langle tp \rangle\) with relations
\[
(tp)g = \bar{g}(tp)
\]
for any \(g \in U_{22}(\mathbb{Z}[i])\). This group is a semi-direct product \((U_{22}(\mathbb{Z}[i])/\langle iE_4 \rangle) \rtimes (\langle tp \rangle).\) We set \(U_{22}^t(1 + 1) = (U_{22}(1 + 1)/\langle iE_4 \rangle) \rtimes (\langle tp \rangle).\)
Proposition 6 ([Ma], [KiM], [Y])

(1) We define the principally congruence subgroup \( \Gamma_H(2) \) of level 2 by

\[
\Gamma_H(2) = \{ R \in \Gamma_H \mid R \equiv E_6 \mod 2 \}.
\]

Then the monodromy group for \( \tilde{\text{per}} : \tilde{X} \to D_H \) is equal to \( \Gamma_H(2) \).

(2) The monodromy group of \( \tilde{\text{per}} : \tilde{X} \to \mathbb{D} \) over \( X \) is equal to

\[
U_{22}^M (1 + 1) = \{ (g, tp^k) \in U_{22}^{tp} (1 + 1) \mid \det(g) = (-1)^k \}.
\]

We note that \( \det(g) \) is well defined on \( U_{22} (\mathbb{Z}[i]) / \langle 1E_4 \rangle \). The monodromy group over \( X / \langle as \rangle \) is equal to \( U_{22}^{tp} (1 + 1) \). The map \( \text{per} \circ \text{pl}^{-1} \) induces the isomorphism from \( \mathbb{P}^4 \) to the Satake compactification of the quotient \( \mathbb{D} / U_{22}^{tp} (1 + 1) \).

(3) Let \( \text{As} \) be an element in \( GL_6 (\mathbb{Z}) \) defined by

\[
\text{As}(\omega_{i,j}) = \left\{ \begin{array}{ll}
\omega_{ij} & (i,j) \neq (1,3), (2,4) \\
\omega_{13} & (i,j) = (2,4) \\
\omega_{24} & (i,j) = (1,3),
\end{array} \right.
\]

and as the association involution defined in [2.2]. Then we have

\[
\omega(\text{as}(x)) = \text{As}(\omega(x))
\]

for \( x \) in a small neighborhood of \( \hat{x} \in X \).

(4) Under the isomorphism \( \text{Aut}(D_H) \simeq \text{Aut}(\mathbb{D}) \), the matrix \( \text{As} \) defined in (3) corresponds to \( tp \). Therefore we have

\[
\tau(\text{as}(x)) = tp(\tau(x))
\]

for \( x \) in a small neighborhood of \( \hat{x} \in X \) and an isomorphism

\[
\tilde{\Gamma}_H(2) \simeq U_{22}^{tp} (1 + 1),
\]

where \( \tilde{\Gamma}_H(2) = \Gamma_H(2) / \langle \pm E_6 \rangle \cdot \langle \text{As} \rangle \).

By the above proposition, we have a map

\[
\text{per} : X \to D_H / \Gamma_H(2) \simeq \mathbb{D} / U_{22}^{M} (1 + 1).
\]
Since the last components \( j_\mathbb{D}(\tau) \) and \( j_\mathbb{D}(g \cdot \tau) \) are 1, we have
\[
\pm \det(g_{21} \tau(x) + g_{22}) = \pm \sqrt{\det(g)[R_g \omega(x)]_{34}} / \omega_{34}(x)
\]
by the equality (13) together with (12) and (14), where \([R_g \omega(x)]_{34}\) denotes the (34)-component of the column vector \( R_g \omega(x) \). By squaring this equality, we have
\[
\det(g) \det(g_{21} \tau(x) + g_{22})^2 = [R_g \omega(x)]_{34}^2 \omega_{34}(x)^2 \tag{18}
\]
for \( g \in U_{22}(\mathbb{Z}[i])/\langle iE_4 \rangle \).

### 3.4 Theta functions and their functional equations

The theta function \( \Theta_{ab} \) with characteristic \( a, b \) on \( \mathbb{D} \) is defined as
\[
\Theta_{ab}(\tau) = \sum_{n \in \mathbb{Z}[i]^2} e_2^{12}(n + a)\tau(n + a)^* + \text{Re}(n + a)b^*)], \tag{19}
\]
where \( x^* = \bar{x}, e[x] = \exp(2\pi i x), \tau \in \mathbb{D}, n = (n_1, n_2) \in \mathbb{Z}[i]^2, a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{Q}[i]^2 \).

**Remark 1** This \( \Theta_{ab} \) is different from that defined in \cite{MY} and \cite{Ma} by the factor of \( e[	ext{Re}(ab^*)] \). If \( \tau \) belongs to the Siegel upper half space \( \mathbb{H}_2 \) of degree 2, then \( \Theta_{ab} \) decomposes into the product of Riemann’s theta constants:

\[
\Theta_{ab}(\tau) = \vartheta_{\text{Re}(a)\text{Re}(b)}(\tau) \vartheta_{\text{Im}(a)\text{Im}(b)}(\tau),
\]
where
\[
\vartheta_{a'b'}(\tau) = \sum_{n \in \mathbb{Z}^2} e_2^{12}(n + a')\tau(n + a') b'
\]
for \( a', b' \in \mathbb{Q}^2 \).

This function satisfies
\[
\Theta_{ab}(t \tau) = \Theta_{a,b}(\tau), \quad \Theta_{a+n,b}(\tau) = \Theta_{ab}(\tau), \quad \Theta_{a,b+n}(\tau) = e[\text{Re}(an^*)] \Theta_{ab}(\tau),
\]
for any \( n \in \mathbb{Z}[i]^2 \). For \( a, b \in \mathbb{Z}[i]^2 \), \( \Theta_{a,b}(\tau) \) is denoted by \( \Theta_{[ab]}(\tau) \). Then \( \Theta_{[ab]}^2(\tau) \) depends only on the class of \( a \) and \( b \) in \( \mathbb{F}_2^2 \simeq (\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i])^2 \). We set
\[
Ev = \{(a, b) \in (\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i])^2 | \text{ab}^* = 0 \text{ mod } (1 + i)\}.\]
Thomae type formula for K3 surfaces

\[
\begin{align*}
\langle 123 \rangle & \leftrightarrow [1111] \\
\langle 124 \rangle & \leftrightarrow [0011] \\
\langle 125 \rangle & \leftrightarrow [0010] \\
\langle 134 \rangle & \leftrightarrow [0001] \\
\langle 135 \rangle & \leftrightarrow [0000] \\
\langle 145 \rangle & \leftrightarrow [1100] \\
\langle 234 \rangle & \leftrightarrow [1001] \\
\langle 235 \rangle & \leftrightarrow [1000] \\
\langle 245 \rangle & \leftrightarrow [0100] \\
\langle 345 \rangle & \leftrightarrow [0110]
\end{align*}
\]

Table 1: Correspondence between \( \langle J \rangle \) and \([ab]\)

Then we have \( \#Ev = 10 \) and \( \Theta_{[ab]}(\tau) = 0 \) if \((a, b) \notin Ev\). We identify the sets \( P_{3,3} \) and \( Ev \) by the rule given in Table 1. Under this correspondence, \( \Theta_{[ab]}^2(\tau) \) is denoted by \( \Theta_{\langle J \rangle}^2(\tau) \).

**Remark 2** The correspondence between \( P_{3,3} \) and \( Ev \) is different from that in \([M_\alpha]\), since the bases of the transcendental lattice \( T(x) \) are different.

**Proposition 7** (\([M_\alpha]\))

1. They satisfy
   \[
   \Theta_{[ab]}^2(\tau') = \Theta_{[ab]}^2(\tau),
   \Theta_{[ab]}^2(g \cdot \tau) = \det(g) \det(g_{21} \tau + g_{22})^2 \Theta_{[ab]}^2(\tau),
   \]
   for any \( g \in U_{22}(1 + i) \).

2. We define a map \( \theta : \mathbb{D}/U_{22}^0(1 + i) \to \mathbb{P}^9 \) by
   \[\tau \mapsto [\ldots, \Theta_{\langle J \rangle}^2(\tau), \ldots]_{\langle J \rangle \in P_{3,3}} \in \mathbb{P}^9.\]
   Then the map \( \text{pl} \) is equal to the composite \( \theta \circ \text{per} \) from \( X \) to \( \mathbb{P}^9 \).

**Theorem 1** (\(2\tau\)-formula) We have

\[
4 \Theta_{ab}(2\tau) = \sum_{q \in \mathbb{F}_2^2} e[-\text{Re}(aq^*)] \Theta_{(1+1)_{a+b+q}}(\tau),
\]

where \( q \) runs over the set \( \mathbb{F}_2^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). In particular,

\[
\begin{align*}
\Theta_{[0000]}(2\tau) &= \frac{1}{4} (\Theta_{[0000]}(\tau) + \Theta_{[0001]}(\tau) + \Theta_{[0010]}(\tau) + \Theta_{[0011]}(\tau)), \\
\Theta_{[0100]}(2\tau) &= \frac{1}{4} (\Theta_{[0000]}(\tau) - \Theta_{[0001]}(\tau) + \Theta_{[0010]}(\tau) - \Theta_{[0011]}(\tau)), \\
\Theta_{[1000]}(2\tau) &= \frac{1}{4} (\Theta_{[0000]}(\tau) + \Theta_{[0001]}(\tau) - \Theta_{[0010]}(\tau) - \Theta_{[0011]}(\tau)), \\
\Theta_{[1100]}(2\tau) &= \frac{1}{4} (\Theta_{[0000]}(\tau) - \Theta_{[0001]}(\tau) - \Theta_{[0010]}(\tau) + \Theta_{[0011]}(\tau)).
\end{align*}
\]
Thomae type formula for K3 surfaces

Proof. We consider the summation

\[ \sum_{n' \in L} \sum_{q \in \mathbb{F}_2^2} e^{\frac{1}{2} \Re (n' q^*)} \cdot e^{[(n' + a) \tau (n' + a)^* + \Re((n' + a)b^*)]}, \]

(20)

where \( L = \frac{1}{1+i} \mathbb{Z}[i]^2 \). Since \( \mathbb{Z}[i]^2 \subset L, L/\mathbb{Z}[i]^2 \simeq \mathbb{F}_2^2 \) and \( e^{\frac{1}{2} \Re (n' q^*)} \) \( (q \in \mathbb{F}_2^2) \) are the characters of the quotient group \( L/\mathbb{Z}[i]^2 \), this summation reduces to the four times of the summation over the subgroup \( \mathbb{Z}[i]^2 \):

\[ 4 \sum_{n \in \mathbb{Z}[i]} e^{((n + a) \tau (n + a)^* + \Re((n + a)b^*))} = 4 \Theta_{ab}(2\tau). \]

On the other hand, the summation (20) is

\[ \sum_{q \in \mathbb{F}_2^2} e^{\frac{1}{2} \Re (a q^*)} \cdot \sum_{n \in \mathbb{Z}[i]^2} e^{\frac{1}{2} (n + (1 + i)a) \tau (n + (1 + i)a)^*} \]

\[ \cdot e^{\Re((n + (1 + i)a)(b + q)^*)} \]

\[ = \sum_{q \in \mathbb{F}_2^2} e^{-\frac{1}{2} \Re (a q^*)} \Theta_{(1+i)a, \frac{b+i}{1+i}}(\tau). \]

For \( a \in \frac{1}{1+i} \mathbb{F}_2^2 \) and \( b = (0,0) \), we have the rests. \( \Box \)

Corollary 1

\[ \Theta_{[0001]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \frac{\Theta_{[0000]}(\tau) + \Theta_{[0010]}(\tau)}{2} = \frac{\Theta_{[0010]}(\tau) + \Theta_{[0011]}(\tau)}{2}, \]

(21)

\[ \Theta_{[0010]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \frac{\Theta_{[0000]}(\tau) + \Theta_{[0011]}(\tau)}{2} = \frac{\Theta_{[0001]}(\tau) + \Theta_{[0010]}(\tau)}{2}, \]

(22)

\[ \Theta_{[0011]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \frac{\Theta_{[0000]}(\tau) + \Theta_{[0011]}(\tau)}{2} = \frac{\Theta_{[0010]}(\tau) + \Theta_{[0011]}(\tau)}{2}. \]

Proof. By Proposition 7 and Plücker relations, we have

\[ \Theta_{[0001]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \Theta_{[0000]}^2(2\tau) - \Theta_{[0100]}^2(2\tau), \]

(23)

\[ \Theta_{[0010]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \Theta_{[0000]}^2(2\tau) - \Theta_{[0100]}^2(2\tau), \]

(24)

\[ \Theta_{[0011]}^2(2\tau) + \Theta_{[1111]}^2(2\tau) = \Theta_{[0000]}^2(2\tau) - \Theta_{[0100]}^2(2\tau) - \Theta_{[1000]}^2(2\tau) + \Theta_{[1100]}^2(2\tau). \]

By Theorem 9 we have the corollary. \( \Box \)
4 Thomae type formula for K3 surfaces

4.1 Main Theorem

**Theorem 2** Suppose that \( x \) is in a neighborhood \( U \) of our reference point \( \dot{x} \in M \times (3, 6) \). Let \( \tau \) be an element of \( \mathbb{D} \) defined in (17). Then we have

\[
\Theta^2_{(J)}(\tau) = \frac{1}{4\pi^2} x(J) \omega_{34}(x)^2
\]

for any \( \langle J \rangle \in P_{3,3} \)

**Remark 3** Using the notations in Proposition 4, the above value is equal to \( \nu_{34}(J) F_S(z)^2 \).

**Proof.** By the first statement of Proposition 4, \( x(J) \omega_{34}(x)^2 \) is a holomorphic function on \( D_H \). We use actions of \( \tilde{\Gamma}_H(2) \) (defined in Proposition 6) and \( U^{tp}_{22}(1+1) \) on the domains \( D_H \) and \( \mathbb{D} \) to compare two functions \( \Theta^2_{(J)}(\tau) \) and \( x(J) \omega_{34}(x)^2 \).

**Lemma 3** Let \( J_1(g, \tau) \) \((g \in U^{tp}_{22}(1+1), \tau \in \mathbb{D})\) and \( J_2(R, \omega) \) \((R \in \tilde{\Gamma}_H(2), \omega \in D_H)\) be two cocycles defined by

\[
J_1(g, \tau) = \frac{\Theta^2_{(J)}(g\tau)}{\Theta^2_{(J)}(\tau)}, \quad J_2(R, \omega) = \frac{R \omega^2_{34}}{\omega^2_{34}}.
\]

Then they coincide via the isomorphisms (10) and (17).

**Proof.** Since the group \( U^{tp}_{22}(1+1) \) is generated by \( U_{22}(1+1)/\langle 1E_4 \rangle \) and \( tp \), it is enough to show the identity (22) for

(1) \( g \in U_{22}(1+1)/\langle 1E_4 \rangle \) and \( R = R_g \), and

(2) \( g = tp \) and \( R = T \).

In the case (1), the statement follows from the equality (18), and that for (2) follows from Proposition 7 (1) and Proposition 6 (4). \( \square \)
By the above lemma, the function
\[ f(x) = \Theta_{(J)}^2(\tau(x)) / (x(J)\omega_{34}(x)^2) \]
becomes a function on \( \mathbb{D}/U^{tp}_{22} \simeq X/\langle as \rangle \). The space \( X/\langle as \rangle \) can be compactified by the embedding \( pl^*: X/\langle as \rangle \to \mathbb{P}^4 \). It is shown in [Ma] that the zero of \( \Theta_{(J)}(\tau(x))^2 \) coincides with that of \( x(J) \). Hence \( f(x) \) is a constant map. We evaluate this constant by taking the degeneration for \( z_2 \to 0, z_3 \to 0 \) in the affine open set of \( X \) defined by
\[
\left\{ \begin{array}{ccc}
-1 & 1 & 0 \\
-z_1 & 0 & 0 \\
-1 & 0 & 1 \\
-2 & 0 & 0 \\
-1 & -1 & -z_4 \\
-3 & 0 & 0 \\
\end{array} \right| z_1, \ldots, z_4 \in \mathbb{C} \}.
\]
Under this limit, we have \( \omega_{13}(x), \omega_{24}(x) \to 0 \) and
\[
\omega_{34} \to 2\omega_B(z_1)\omega_B(z_4), \quad \omega_{14} \to 2\omega_A(z_1)\omega_B(z_4), \quad \omega_{23} \to 2\omega_B(z_1)\omega_A(z_4),
\]
where \( \omega_A(\lambda), \omega_B(\lambda) \) are defined in [1]. We set \( \omega_{ij} = \lim_{z_2,z_3 \to 0} \omega_{ij}(x) \). Then we have
\[
\lim_{z_2,z_3 \to 0} \tau(x) = \text{diag}(\omega_{14}/\omega_{34}, \omega_{23}/\omega_{34}),
\]
and by \( x(135) \to 1 \),
\[
\lim_{z_2,z_3 \to 0} f(x) = \Theta_{[0000]}^2((\omega_{14}/\omega_{34}, \omega_{23}/\omega_{34}))/((x(135)\omega_{34}^2)) = \psi_{[00]}^4(\omega_A(z_1))\psi_{[00]}^4(\omega_A(z_4))/(4\omega_B(z_1)\omega_B(z_4)) = \frac{1}{4\pi^4}
\]
by Jacobi's formula [2].

5 Mean iterations

5.1 Mean iteration associated to \( D_4 \) degeneration

In this and next subsections, we apply the main identity [21] to the study of mean iterations. In this subsection, we consider configurations of six lines which contain three lines intersecting at one point. In this degeneration, three \( A_1 \) singularities on \( \hat{X}^* \) confluent to one \( D_4 \) singularity. This degeneration is obtained by taking the limit \( x(J) \to 0 \). We consider the case \( \langle J \rangle = \langle 123 \rangle \).
Proposition 8  The two preimages of the map \( pl \) on the subvariety defined by \( x(123) = 0 \) are expressed as

\[
\begin{pmatrix}
\frac{1}{-x(124)+x(134)+x(135)} & 1 & 0 & 0 & 1 & \frac{1}{-x(124)x(135)-x(125)x(134)} \\
-x(134)+x(135) & -1 & 1 & 0 & 0 & \frac{-x(134)x(125)+x(135)x(125)}{x(125)} \\
0 & 0 & 0 & 1 & -1 & \frac{-x(124)x(135)-x(125)x(134)}{x(125)}
\end{pmatrix},
\]

Let \( m \) be a map from \((\mathbb{R}_+^\times)^4\) to \((\mathbb{R}_+^\times)^4\) given by

\[
m : (\mathbb{R}_+^\times)^4 \ni u = (u_1, \ldots, u_4) \mapsto (m_1(u), \ldots, m_4(u)) \in (\mathbb{R}_+^\times)^4,
\]

where

\[
m_1(u) = \frac{u_1 + u_2 + u_3 + u_4}{4}, \quad m_2(u) = \frac{\sqrt{(u_1 + u_3)(u_2 + u_4)}}{2},
\]

\[
m_3(u) = \frac{\sqrt{(u_1 + u_2)(u_3 + u_4)}}{2}, \quad m_4(u) = \frac{\sqrt{u_1u_4 + u_2u_3}}{2}.
\]

For an element \( c = (c_1, \ldots, c_4) \in (\mathbb{R}_+^\times)^4 \) with \( c_1 > c_2 > c_3 > c_4 \), we define a vector valued sequence \( \{m^n(c) = (m_1^n(c), \ldots, m_4^n(c))\}_{n \in \mathbb{N}} \) by

\[
m^n(c) = m \circ \cdots \circ m(c).
\]

Lemma 4  \( 1 \) The components of the sequence \( \{(m_1^n(c), \ldots, m_4^n(c))\}_{n \in \mathbb{N}} \) converge and have a common limit \( m_\infty^*(c) \). The convergence is quadratic.

\[
(2) \quad \lim_{n \to \infty} \frac{m_1^n(c)^2 - m_2^n(c)^2}{m_3^n(c)^2 - m_4^n(c)^2} = \lim_{n \to \infty} \frac{m_1^n(c)^2 - m_3^n(c)^2}{m_2^n(c)^2 - m_4^n(c)^2} = 1.
\]

Proof.  \( 1 \) Since

\[
m_1(c)^2 - m_2(c)^2 = \frac{(c_1 - c_2 + c_3 - c_4)^2}{16},
\]

\[
m_2(c)^2 - m_3(c)^2 = \frac{(c_1 - c_4)(c_2 - c_3)}{4},
\]

\[
m_3(c)^2 - m_4(c)^2 = \frac{(c_1 - c_2)(c_3 - c_4)}{4},
\]
we have $m_1(c) > m_2(c) > m_3(c) > m_4(c)$ for $c_1 > c_2 > c_3 > c_4 > 0$. We can easily see that
\[ c_4 < m_4(c) < \cdots < m_4^n(c) < m_4^1(c) < \cdots < m_1(c) < c_1, \]
the sequences $\{m_4^n(c)\}$ and $\{m_4^1(c)\}$ converge. We set $\mu_1 = \lim_{n \to \infty} m_4^n(c)$ and $\mu_4 = \lim_{n \to \infty} m_4^1(c)$. Since
\[
\begin{align*}
m_1(c)^2 - m_4(c)^2 &= m_1(c)^2 - m_2(c)^2 + m_2(c)^2 - m_4(c)^2 \\
&= \frac{(c_1 - c_2 + c_3 - c_4)^2}{4} + \frac{(c_1 - c_3)(c_2 - c_4)}{4} \\
&< \frac{(c_1 - c_4)^2}{4} \left( \frac{1}{4} + \frac{(c_1 - c_4)^2}{2} \right), \tag{24}
\end{align*}
\]
we have $\mu_1^2 - \mu_4^2 \leq \frac{1}{2}(\mu_1 - \mu_4)^2$. If $\mu_1 > \mu_4$, then $\mu_1 + \mu_4 \leq \frac{1}{2}(\mu_1 - \mu_4)$, which implies $\mu_1 + 3\mu_4 \leq 0$. This is a contradiction.

By the inequality (24), the convergence of $m_1(c) - m_4(c) = \frac{m_1(c)^2 - m_4(c)^2}{m_1(c) + m_4(c)}$, is quadratic.

(2) We have
\[
\frac{m_1(c)^2 - m_2(c)^2}{m_3(c)^2 - m_4(c)^2} = \frac{1}{4} \left( \frac{c_1 - c_2}{c_3 - c_4} + 2 + \frac{c_3 - c_4}{c_1 - c_2} \right) \geq 1.
\]

We set
\[
s_n = \frac{m_4^n(c)^2 - m_2^n(c)^2}{m_3^n(c)^2 - m_4^n(c)^2}, \quad t_n = \frac{m_1^n(c) + m_2^n(c)}{m_3^n(c) + m_4^n(c)}, \quad f(s, t) = \frac{1}{4} \left( t + 2 + \frac{s}{t} \right).
\]

Then $s_n$ and $t_n$ satisfy
\[
s_{n+1} = f(s_n, t_n), \quad s_n, t_n \geq 1, \quad \lim_{n \to \infty} t_n = 1.
\]

Note that $f(hs, ht) = f(s, t)$ for any $h \in \mathbb{R}_+^\times$, $f(s, s) = 1$ and that
\[
f(s, 1) = f(1, s) = \frac{1}{4} (s + 2 + \frac{1}{s}) < \frac{1}{4} (s + 2s + s) < s
\]
for any $s > 1$. If $s_n > t_n$ then $(s_n/t_n) > 1$ and
\[
s_{n+1} = f(s_n, t_n) = f\left( \frac{s_n}{t_n}, 1 \right) < \frac{s_n}{t_n} < s_n.
\]
If \( s_n \leq t_n \) then \( (t_n/s_n) \geq 1 \) and
\[
s_{n+1} = f(s_n, t_n) = f\left(\frac{t_n}{s_n}, 1\right) \leq \frac{t_n}{s_n} \leq t_n.
\]
Thus we have \( s_{n+1} \leq \max(s_n, t_n) \). Since \( \lim_{n \to \infty} t_n = 1 \), for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( t_n < 1 + \varepsilon \) for any \( n > N \). If there exists \( n_0 > N \) such that \( s_{n_0} \leq t_{n_0} \), then \( s_n < 1 + \varepsilon \) for any \( n \geq n_0 \); this means \( \lim_{n \to \infty} s_n = 1 \).

Otherwise, i.e. \( s_n > t_n \) for any \( n > N \), then \( s_n \) is monotonously decreasing. Thus the limit \( s_n \) exists. Let \( n \to \infty \) for \( s_{n+1} = f(s_n, t_n) \), then we have
\[
\lim_{n \to \infty} s_n = 1.
\]
Similarly we can show \( \lim_{n \to \infty} m_n(c_2^2 - m_n(c_1)^2) = 1 \).

\textbf{Theorem 3} The common limit \( m^\infty(c) \) can be expressed as
\[
m^\infty(c) = \sqrt{\frac{c_2^2 - c_3^2}{c_2^3 - c_4^3} F_S(z)} = \sqrt{\frac{c_2^2 - c_3^2}{c_2^3 - c_4^3} F_S(w)},
\]
where \( z = \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \) and \( w = \begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} \) are given as
\[
z = \begin{pmatrix} 1 - \frac{c_3}{c_2} & 1 - \frac{c_1(c_2 - c_3)}{c_3(c_1 - c_2)} \\ 0 & 1 - \frac{c_1}{c_2} \end{pmatrix}, \quad w = \begin{pmatrix} 1 - \frac{c_3}{c_2} & 0 \\ \frac{c_1(c_2 - c_3)}{c_3(c_1 - c_2)} & 1 - \frac{c_1}{c_2} \end{pmatrix}.
\]

\textbf{Remark 4} For a given \( c = (c_1, \ldots, c_4) \), the hypergeometric series \( F_S \) in Theorem 3 may not converge. By Lemma 4 (2), there exists \( n \in \mathbb{N} \) such that it converges for \( m^n(c) \) instead of \( c \).

\textbf{Proof.} There exists \( \tau \in \mathbb{D} \) such that \( \Theta_{[1111]}(\tau) = 0 \) and
\[
\Theta_{[0000]}(\tau) : \Theta_{[0001]}(\tau) : \Theta_{[0010]}(\tau) : \Theta_{[0011]}(\tau) = c_1 : c_2 : c_3 : c_4.
\]

By Corollary 1, we have
\[
(\Theta_{[0000]}(2\tau), \Theta_{[0001]}(2\tau), \Theta_{[0010]}(2\tau), \Theta_{[0011]}(2\tau)) = m(\Theta_{[0000]}(\tau), \Theta_{[0001]}(\tau), \Theta_{[0010]}(\tau), \Theta_{[0011]}(\tau)),
\]
since \( \Theta_{[1111]}(\tau) = 0 \). By the homogeneity of \( m_1, \ldots, m_n, m^\infty \) satisfies
\[
m^\infty(c) = c_1 m^\infty(1, \frac{c_2}{c_1}, \frac{c_3}{c_1}, \frac{c_4}{c_1}).
\]
Thus

\[ m_\ast^\infty(c) = c_1 m_\ast^\infty(1, \frac{\Theta[0001](\tau)}{\Theta[0000](\tau)}, \frac{\Theta[0010](\tau)}{\Theta[0000](\tau)}, \frac{\Theta[0011](\tau)}{\Theta[0000](\tau)}) \]

\[ = \frac{c_1}{\Theta[0000](\tau)} m_\ast^\infty(\Theta[0000](\tau), \Theta[0001](\tau), \Theta[0010](\tau), \Theta[0011](\tau)) \]

\[ = \frac{c_1}{\Theta[0000](\tau)} m_\ast^\infty(m(\Theta[0000](\tau), \Theta[0001](\tau), \Theta[0010](\tau), \Theta[0011](\tau))) \]

\[ = \frac{c_1}{\Theta[0000](\tau)} m_\ast^\infty(\Theta[0000](2\tau), \Theta[0001](2\tau), \Theta[0010](2\tau), \Theta[0011](2\tau)) \]

\[ = \frac{c_1}{\Theta[0000](\tau)} m_\ast^\infty(\Theta[0000](2^n\tau), \Theta[0001](2^n\tau), \Theta[0010](2^n\tau), \Theta[0011](2^n\tau)) \]

\[ \to \frac{c_1}{\Theta[0000](\tau)} \text{ as } n \to \infty, \]

since \( \Theta_{ab}(2^n\tau) \) converge to 1 for \( a = (0, 0) \) and any \( b \in \mathbb{F}_2^2 \). By Proposition 8, the preimages of

\[ [x\langle 123 \rangle, x\langle 124 \rangle, x\langle 125 \rangle, x\langle 134 \rangle, x\langle 135 \rangle] = [0, c_4^2, c_3^2, c_2^2, c_1^2] \]

for the map \( \overline{pl} : \overline{X} \to \mathbb{P}^4 \subset \mathbb{P}^9 \) are given by

\[ x = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ -z_1 & -1 & 1 & 0 & 0 & -z_3 \\ 0 & 0 & 1 & -1 & 0 & -z_4 \end{pmatrix}, \]

\[ z_1 = 1 - \frac{c_2^2 - c_4^2}{c_1^2 - c_2}, \quad z_2 = 0, \quad z_3 = 1 - \frac{c_1^2(c_3^2 - c_4^2)}{c_3(c_1^2 - c_2^2)}, \quad z_4 = 1 - \frac{c_4^2}{c_3^2}, \]

with

\[ \sqrt{x\langle 135 \rangle} = \frac{c_1}{c_3} \sqrt{\frac{c_3^2 - c_4^2}{c_1^2 - c_2^2}}, \]

and

\[ x = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ -z_1 & -1 & 1 & 0 & 0 & 0 \\ -z_2 & 0 & 1 & -1 & 0 & -z_4 \end{pmatrix}, \]

\[ z_1 = 1 - \frac{c_2^2}{c_2^2}, \quad z_2 = 1 - \frac{c_1^2(c_2^2 - c_4^2)}{c_2(c_1^2 - c_3)}, \quad z_3 = 0, \quad z_4 = 1 - \frac{c_2^2}{c_1^2 - c_3^2}. \]
with
\[ \sqrt{x(135)} = \frac{c_1}{c_2} \sqrt{\frac{c_2^2 - c_4^2}{c_1^2 - c_3^2}}. \]

Theorem 2 implies this theorem. \[\square\]

5.2 Mean iteration associated to Kummer locus

The Borchardt’s mean iteration is obtained from the restriction of Thomae type formula for K3 surfaces to the Kummer locus. In this subsection, we explain how to recover limit formulas in [3], [MT] and [Me] from our main theorem.

**Proposition 9** Let \( c_1 > c_2 > c_3 > c_4 \) be real numbers such that \( c_1 - c_2 - c_3 + c_4 > 0 \). We set
\[ Q_1 = (c_1+c_2+c_3+c_4)(c_1+c_2-c_3-c_4)(c_1-c_2+c_3-c_4)(c_1-c_2-c_3+c_4), \]
and
\[
\begin{pmatrix}
  z_1 & z_3 \\
  z_2 & z_4
\end{pmatrix} = \begin{pmatrix}
  1 - \frac{c_4(c_1^2-c_2^2+c_3^2-c_4^2-\sqrt{Q_1})}{2c_3(c_1-c_2-c_3-c_4)} & 1 - \frac{c_1(c_4^2-c_2^2+c_3^2-c_4^2-\sqrt{Q_1})}{2c_3(c_1-c_2+c_3-c_4)} \\
  1 - \frac{c_1(c_4^2+c_2^2-c_3^2-c_4^2-\sqrt{Q_1})}{2c_3(c_1+c_2-c_3+c_4)} & 1 - \frac{c_4(c_1^2+c_2^2-c_3^2-c_4^2-\sqrt{Q_1})}{2c_3(c_1+c_2+c_3-c_4)}
\end{pmatrix}. \tag{25}
\]

Then
\[
x = \begin{pmatrix}
  1 & 1 & 0 & 0 & 1 & 1 \\
  -z_1 & -1 & 1 & 0 & 0 & -z_3 \\
  -z_2 & 0 & 0 & 1 & -1 & -z_4
\end{pmatrix} \tag{26}
\]
lies on the Kummer locus. In this case, we have
\[ [x(123) : x(135) : x(134) : x(125) : x(124)] = [c_0^2 : c_1^2 : c_2^2 : c_3^2 : c_4^2], \]
where
\[ c_0^2 = \frac{c_2^2 - c_4^2 - c_3^2 + c_1^2 + \sqrt{Q_1}}{2}. \]

Let \( m \) be a map from \((\mathbb{R}_+^\times)^4\) to \((\mathbb{R}_+^\times)^4\) given by
\[
m : (\mathbb{R}_+^\times)^4 \ni u = (u_1, \ldots, u_4) \mapsto (m_1(u), \ldots, m_4(u)) \in (\mathbb{R}_+^\times)^4, \tag{27}
\]
where

\[
\begin{align*}
m_1(u) &= \frac{u_1 + u_2 + u_3 + u_4}{4}, \\
m_2(u) &= \frac{\sqrt{u_1u_2} + \sqrt{u_3u_4}}{2}, \\
m_3(u) &= \frac{\sqrt{u_1u_3} + \sqrt{u_2u_4}}{2}, \\
m_4(u) &= \frac{\sqrt{u_1u_4} + \sqrt{u_2u_3}}{2}.
\end{align*}
\]

Note that if \( u_1 > u_2 > u_3 > u_4 \) then

\[
m_1(u) > m_2(u) > m_3(u) > m_4(u).
\]

For an element \( c = (c_1, \ldots, c_4) \in (\mathbb{R}_+^*)^4 \) with \( c_1 > c_2 > c_3 > c_4 \), we define a vector valued sequence \( \{m^n(c) = (m^n_1(c), \ldots, m^n_4(c))\}_{n \in \mathbb{N}} \) by

\[
m^n(c) = m \circ \cdots \circ m(c).
\]

In [B] and [Me], they prove that the common limit \( m^\infty_*(c) \) is expressed in terms of period integrals of a hyperelliptic curve of genus 2. In [MJ], they give its expression in terms of the period integral \( \omega_{34}(x) \) of the K3 surface \( \mathcal{X}(x) \). Here, we give its expression by the hypergeometric series \( F_S \).

**Theorem 4** We can express the common limit \( m^\infty_*(c) \) by

\[
m^\infty_*(c) = \frac{4\sqrt{c_2c_3(c_1c_2 - c_3c_4)(c_1c_3 - c_2c_4)}}{(\sqrt{d_1d_2} - \sqrt{d_3d_4})(\sqrt{d_1d_3} - \sqrt{d_2d_4})} F_S(z)
\]

where

\[
\begin{align*}
z &= \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} = \begin{pmatrix} c_1 \sqrt{d_1d_3} - \sqrt{d_2d_4} & c_2 \sqrt{d_1d_2} + \sqrt{d_3d_4} \\ c_2 \sqrt{d_1d_3} + \sqrt{d_2d_4} & c_1 \sqrt{d_1d_2} - \sqrt{d_3d_4} \end{pmatrix}, \\
w &= \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} = \begin{pmatrix} c_1 \sqrt{d_1d_3} + \sqrt{d_2d_4} & c_2 \sqrt{d_1d_2} - \sqrt{d_3d_4} \\ c_2 \sqrt{d_1d_3} - \sqrt{d_2d_4} & c_1 \sqrt{d_1d_2} + \sqrt{d_3d_4} \end{pmatrix},
\end{align*}
\]

\[
d_1 = c_1 + c_2 + c_3 + c_4, \quad d_2 = c_1 + c_2 - c_3 - c_4, \\
d_3 = c_1 - c_2 + c_3 - c_4, \quad d_4 = c_1 - c_2 - c_3 + c_4.
\]
**Proof.** Let $c_1, \ldots, c_4$ be elements in $\mathbb{R}_+^4$ with $c_1 > \cdots > c_4$. Though the value $c_1 - c_2 - c_3 + c_4$ may be negative, by the inequality

$$m_1(c) - m_2(c) - m_3(c) + m_4(c) = \frac{1}{4}(\sqrt{c_1} - \sqrt{c_2} - \sqrt{c_3} + \sqrt{c_4})^2 > 0,$$

we may assume $c_1 - c_2 - c_3 + c_4 \geq 0$ by applying the map $m$. Let $x$ be $2 \times 2$ matrix defined as (25). Then $x$ defined in (26) lies on the Kummer locus. In this case, the theta constants $\Theta_{[a,b]}(\tau)$ coincide with the square of Riemann’s theta constants. Using $2\tau$-formulas for Riemann’s theta constants, and similar argument as in Theorem 3, we have

$$m_\infty^x(c) = \frac{c_1}{\Theta_{[0000]}(\tau)}.$$ 

By Theorem 2 and Proposition 4 we have the first expression of $m_\infty^x(c)$ by $F_S$. By putting

$$c_0^2 = \frac{c_1^2 - c_2^2 - c_3^2 + c_4^2 - \sqrt{Q_1}}{2},$$

we have the other expression of $m_\infty^x(c)$. \hfill \Box

### 6 Functional equations for $F_S$

The common limit $m_\infty^x(c)$ in Theorem 3 (resp. 4) satisfies

$$m_\infty^x(c) = m_\infty^x(m(c)).$$

This property implies functional equations for the hypergeometric function $F_S$.

**Theorem 5** We have the following functional equations for $F_S$:

$$F_S(m(z)) = \frac{(c_1 - c_2 + c_3 - c_4)(c_3 + c_4)}{4(c_1 - c_2)c_3}F_S(z),$$

$$F_S(m(w)) = \frac{(c_1 + c_2 - c_3 - c_4)(c_2 + c_4)}{4(c_1 - c_3)c_2}F_S(w),$$
where z and w are given in Theorem 3 and

\[
\begin{align*}
m(z) &= \left( \frac{(c_1-c_2-c_3-c_4)^2}{(c_1-c_2-c_3-c_4)^2} \right), \\
m(w) &= \left( \frac{2(c_1-c_2-c_3)(c_1-c_2-c_3-c_4)^2}{(c_1-c_2)(c_1-c_2)(c_1-c_2)(c_1-c_2-c_3-c_4)^2} \right).
\end{align*}
\]

**Theorem 6** We have the following functional equations for \( F_S \):

\[
\begin{align*}
F_S(m(z)) &= \frac{1}{16} \frac{\sqrt{d_2d_3}(\sqrt{d_1d_2} - \sqrt{d_3d_4})(\sqrt{d_1d_3} - \sqrt{d_2d_4})}{\sqrt{c_1c_2}(\sqrt{c_1c_2} - \sqrt{c_3c_4})(\sqrt{c_1c_3} - \sqrt{c_2c_4})} F_S(z), \\
F_S(m(w)) &= \frac{1}{4\sqrt{c_1c_3d_2d_3}} \frac{\sqrt{d_1d_2} + \sqrt{d_3d_4} + \sqrt{d_1d_3} + \sqrt{d_2d_4}}{\sqrt{c_1c_2}(\sqrt{c_1c_2} - \sqrt{c_3c_4})(\sqrt{c_1c_3} - \sqrt{c_2c_4})} F_S(w),
\end{align*}
\]

where z and w are given in Theorem 4 and

\[
\begin{align*}
m(z) &= \left( \frac{1 - \frac{2((\sqrt{c_1c_4} + \sqrt{c_2c_3})(\sqrt{c_1c_3} - \sqrt{c_2c_4})d_1}{(\sqrt{c_1c_2} + \sqrt{c_3c_4})d_2} \right), \\
m(w) &= \left( \frac{1 - \frac{2((\sqrt{c_1c_4} + \sqrt{c_2c_3})(\sqrt{c_1c_3} - \sqrt{c_2c_4})d_1}{(\sqrt{c_1c_2} + \sqrt{c_3c_4})d_2} \right).
\end{align*}
\]

**Proof.** To obtain the expression of \( m(z) \) and \( m(w) \), we use the equalities:

\[
m_1(c) + \epsilon_1m_2(c) + \epsilon_2m_3(c) + \epsilon_1\epsilon_2m_4(c) = \frac{1}{4}(\sqrt{c_1} + \epsilon_1\sqrt{c_2} + \epsilon_2\sqrt{c_3} + \epsilon_1\epsilon_2\sqrt{c_4})^2
\]

for \( \epsilon_1, \epsilon_2 = \pm 1 \) and

\[
m_i(c)m_j(c) - m_k(c)m_l(c) = \frac{1}{8}(\sqrt{c_ic_j} - \sqrt{c_kc_l})(c_i + c_j - c_k - c_l)
\]

for \((i, j, k, l) = (1, 2, 3, 4), (1, 3, 2, 4), (1, 4, 2, 3)\). \(\square\)

**References**

[B] C.W. Borchardt, Über das arithmetisch-geometrische Mittel aus vier Elementen, *Berl. Monatsber.*, 53 (1876), 611-621.
[F] E. Freitag, Modulformen zweiten Grades zum rationalen und Gaußschen Zahlkörper, *Sitzungsber. Heidelb. Akad. Wiss.*, 1 (1967), 1–49.

[I] J. Igusa, *Theta functions*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 194, Springer-Berlin-Heidelberg, New York, 1972.

[IKSY] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, *From Gauss to Painlevé*, Vieweg, Braunschweig, Wiesbaden, 1991.

[KY] M. Kita and M. Yoshida, Intersection theory for twisted cycles I, *Math. Nachr.* 166 (1994), 287–304.

[KaM] T. Kato and K. Matsumoto, The common limit of a quadruple sequence and the hypergeometric function $F_D$ of three variables, *Nagoya Math. J.* 195 (2009), 113–124.

[KiM] M. Kita and K. Matsumoto, Duality for hypergeometric functions and invariant Gauss-Manin systems, *Compositio Math.* 108 (1997), 77–106.

[MSY] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the Aomoto-Gel’fand hypergeometric function of type $(3,6)$, *Internat. J. of Math.*, 3 (1992), 1–164.

[MT] K. Matsumoto and T. Terasoma, Arithmetic-geometric means for hyperelliptic curves and Calabi-Yau varieties, to appear in *Internat. J. of Math.*

[MY] K. Matsumoto and M. Yoshida, Invariants for some real hyperbolic groups, *Internat. J. of Math.*, 13 (2002), 415–443.

[Ma] K. Matsumoto, Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of 4-parameter family of $K3$ surfaces, *Math. Ann.*, 295 (1993), 383–408.

[Me] J. Mestre, Moyenne de Borchardt et integrales elliptiques, *C. R. Acad. Sci. Paris Ser. I Math.* 313 (1991), no. 5, 273–276.

[Mu] D. Mumford, *Tata lectures on Theta I*, progress in Math 28. Birkhäuser, Boston-Basel-Berlin, 1983.
Thomae type formula for K3 surfaces

[Te] T. Terasoma, Exponential Kummer coverings and determinants of hypergeometric functions. *Tokyo J. Math.* 16 (1993), no. 2, 497–508.

[To] J. Thomae, Beitrag zur Bestimmung von $\theta(0, 0, \ldots, 0)$ durch die Klassenmoduln algebraischer Funktionen, *J. Reine Angew. Math.* 71 (1870), 201–222.

[Y] M. Yoshida, *Hypergeometric Functions, My Love*, Aspects of Mathematics, E32, Friedr Vieweg & Sohn, Braunschweig, 1997.

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\[ \Delta_{ij} = \Delta_i \cap \Delta_j \]
