ON SELF-AVOIDING POLYGONS AND WALKS: COUNTING, JOINING AND CLOSING

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Abstract. For $d \geq 2$ and $n \in \mathbb{N}$, let $c_n = c_n(d)$ denote the number of length $n$ self-avoiding walks beginning at the origin in the integer lattice $\mathbb{Z}^d$, and, for even $n$, let $p_n = p_n(d)$ denote the number of length $n$ self-avoiding polygons in $\mathbb{Z}^d$ up to translation. Then the probability under the uniform law $W_n$ on self-avoiding walks $\Gamma$ of any given odd length $n$ beginning at the origin that $\Gamma$ closes – i.e., that $\Gamma$’s endpoint is a neighbour of the origin – is given by $W_n(\Gamma \text{ closes}) = 2(n + 1)p_{n+1}/c_n$. The polygon and walk cardinalities share a common exponential growth: $\lim_{n \to \infty} c_n^{1/n} = \lim_{n \to \infty} p_n^{1/n} = \mu$ (where the common value $\mu \in (0, \infty)$ is called the connective constant). Madras [26] has shown that $p_n \leq Cn^{-1/2}\mu^n$ in dimension $d = 2$, while the closing probability was recently shown in [13] to satisfy $W_n(\Gamma \text{ closes}) \leq n^{-1/4+o(1)}$ in any dimension $d \geq 2$.

Here we establish that

- $W_n(\Gamma \text{ closes}) \leq n^{-1/2+o(1)}$ for any $d \geq 2$;
- $W_n(\Gamma \text{ closes}) \leq n^{-6/11+o(1)}$ for a subsequence of odd $n$, if $d = 2$;
- $p_n \leq Cn^{-1}\mu^n$ for a subsequence of even $n$ when $d = 2$.

We also argue that the closing probability is bounded above by $n^{-2/3+o(1)}$ when $d = 3$ for a certain variant of self-avoiding walk.

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1. Introduction

Self-avoiding walk was introduced in the 1940s by Flory and Orr [16, 32] as a model of a long polymer chain in a system of such chains at very low concentration. It is well known among the basic models of discrete statistical mechanics for posing problems that are simple to state but difficult to solve. Two recent surveys are the lecture notes [3] and [23, Section 3].

1.1. The model. Let \( d \geq 2 \). For \( u \in \mathbb{R}^d \), let \( ||u|| \) denote the Euclidean norm of \( u \). A walk of length \( n \in \mathbb{N}^+ \) is a map \( \gamma : \{0, \ldots, n\} \rightarrow \mathbb{Z}^d \) such that \( ||\gamma(i+1) - \gamma(i)|| = 1 \) for each \( i \in \{0, \ldots, n-1\} \). An injective walk is called self-avoiding. Let \( \text{SAW}_n \) denote the set of self-avoiding walks of length \( n \) that start at 0. We denote by \( W_n \) the uniform law on \( \text{SAW}_n \). The walk under the law \( W_n \) will be denoted by \( \Gamma \).

A walk \( \gamma \in \text{SAW}_n \) is said to close (and to be closing) if \( ||\gamma(n)|| = 1 \). When the missing edge connecting \( \gamma(n) \) and \( \gamma(0) \) is added, a polygon results.

**Definition 1.1.** Let \( \gamma : \{0, \ldots, n-1\} \rightarrow \mathbb{Z}^d \) be a closing self-avoiding walk. For \( 1 \leq i \leq n-1 \), let \( u_i \) denote the unordered nearest neighbour edge in \( \mathbb{Z}^d \) with endpoints \( \gamma(i-1) \) and \( \gamma(i) \). Let \( u_n \) denote \( \gamma \)'s missing edge, with endpoints \( \gamma(n-1) \) and \( \gamma(0) \). We call the collection of edges \( \{u_i : 1 \leq i \leq n\} \) the polygon of \( \gamma \). A self-avoiding polygon in \( \mathbb{Z}^d \) is defined to be any polygon of a closing self-avoiding walk in \( \mathbb{Z}^d \). The polygon's length is its cardinality.

We will usually omit the adjective self-avoiding in referring to walks and polygons. Recursive and algebraic structure has been used to analyse polygons in such domains as strips, as [6] describes.

Note that the polygon of a closing walk has length that exceeds the walk's by one. Polygons have even length and closing walks, odd.

1.2. Main results. The closing probability is \( W_n(\Gamma \text{ closes}) \). In [13], an upper bound on this quantity of \( n^{-1/4+o(1)} \) was proved in general dimension.

In this paper, we use a variety of approaches to prove several closing probability upper bounds.

First, we revisit the closing probability upper bound method of [13], styling it the snake method via Gaussian pattern fluctuation. This technique cannot hope to show that the closing probability decays faster than \( n^{-1/2} \). We show that this decay can be proved in general dimension.

**Theorem 1.2.** Let \( d \geq 2 \). For any \( \varepsilon > 0 \) and \( n \in 2\mathbb{N} + 1 \) sufficiently high,

\[
W_n(\Gamma \text{ closes}) \leq n^{-1/2+\varepsilon}.
\]
Define the polygon number \( p_n \) to be the number of length \( n \) polygons up to translation, and the walk number \( c_n \) to be the number of length \( n \) walks beginning at the origin. As we shall soon review, the limiting exponential growth rates \( \lim_{n \in \mathbb{N}} c_n^{1/n} \) and \( \lim_{n \in 2\mathbb{N}} p_n^{1/n} \) exist and coincide. Writing \( \mu \) for the common value, we define real-valued polygon and walk growth deviation exponents \( \theta_n \) and \( \xi_n \) according to the formulas

\[
p_n = n^{-\theta_n} \cdot \mu^n \quad \text{for} \quad n \in 2\mathbb{N}.
\]

and

\[
c_n = n^{\xi_n} \cdot \mu^n \quad \text{for} \quad n \in \mathbb{N}.
\]

The closing probability may be written in terms of the polygon and walk numbers. There are \( 2^n \) closing walks whose polygon is a given polygon of length \( n \), since there are \( n \) choices of missing edge and two of orientation. Thus,

\[
W_n (\Gamma \text{ closes}) = \frac{2(n+1)p_{n+1}}{c_n},
\]

for any \( n \in \mathbb{N} \) (but non-trivially only for odd values of \( n \)).

As we will shortly review, it is a straightforward fact that each \( \theta_n \) and \( \xi_n \) is non-negative in any dimension \( d \geq 2 \). Hara and Slade [20, Theorem 1.1] used the lace expansion to prove that \( c_n \sim C\mu^n \) when \( d \geq 5 \), while Madras and Slade [28, Theorem 6.1.3] have proved that \( \theta_n \geq d/2 + 1 \) in these dimensions for spread-out models, in which the vertices of \( \mathbb{Z}^d \) are connected by edges below some bounded distance. Thus, the sharp conclusion that the closing probability decays as fast as \( n^{-d/2} \) has been reached for such models.

Madras [29] has proved a bound on the moment generation function of the sequence \( \{p_n : n \in 2\mathbb{N}\} \) which when \( d = 3 \) would assert \( \lim_{n \in 2\mathbb{N}} \theta_n \geq 1 \) were this limit known to exist. More relevantly for us, he has shown in [26] using a polygon joining technique that \( \theta_n \geq 1/2 \) for \( d = 2 \). We develop this technique to prove a stronger lower bound valid on certain subsequences.

**Definition 1.3.** The limit supremum density of a set \( A \) of even, or odd, integers is

\[
\limsup_n \frac{|A \cap [0, n]|}{|2\mathbb{N} \cap [0, n]|} = \limsup_n n^{-1} |A \cap [0, 2n]|.
\]

The limit infimum density is defined analogously.

**Theorem 1.4.** Let \( d = 2 \). On a set of \( n \in 2\mathbb{N} \) of positive limit infimum density, \( \theta_n \geq 1 \).

As we rework the method of [13] to prove Theorem 1.2, we take the opportunity to present the technique in a general guise. The snake method is a proof-by-contradiction technique for deriving closing probability upper bounds that involves constructing sequences of laws of self-avoiding walks conditioned on increasingly severe avoidance events. We exploit it in a new way to prove closing probability upper bounds below \( n^{-1/2} \) in two dimensions.
Theorem 1.5. Let $d = 2$.

(1) Let $\varepsilon > 0$. On a set of $n \in 2\mathbb{N} + 1$ of positive limit supremum density,
$$W_n(\Gamma \text{ closes}) \leq n^{-6/11+\varepsilon}.$$ 

(2) Suppose that the limit $\theta := \lim_{n \in 2\mathbb{N}} \theta_n$ exists with $\theta < \infty$ and that $\sup_{n \in \mathbb{N}} \xi_n < \infty$. Then $\theta \geq 5/3$ and, for any $\varepsilon > 0$ and $n \in 2\mathbb{N} + 1$ sufficiently high,
$$W_n(\Gamma \text{ closes}) \leq n^{-2/3+\varepsilon}.$$ 

In our view, Theorem 1.5(1) is the most conceptually interesting result in this paper. Its proof brings together many of the ideas harnessed here, using the snake method and the polygon joining technique at once. Theorem 1.5(2) is only a conditional result, but it serves a valuable expository purpose: its proof is that of the theorem’s first part with certain technicalities absent.

It is vaguely plausible that all of the paper’s results may be generalized to higher dimensions, though some quite cumbersome technical obstacles would have to be overcome. We have chosen to keep an intent focus on the two-dimensional case, because we believe that very little is lost conceptually by this focus. Beyond Theorem 1.2 whose proof has almost no extra obstacles in general dimension, we present one further result, in three dimensions. We select a variant model tailored to eliminate technical difficulties.

Definition 1.6. The maximal edge local time of a nearest neighbor walk $\gamma : \{0, \ldots, n\} \to \mathbb{Z}^d$ is the maximum number of times that $\gamma$ traverses an edge of $\mathbb{Z}^d$; more formally, it is the maximum cardinality of a subset $I \subseteq \{0, \ldots, n-1\}$ such that the unordered sets $\{\gamma(i), \gamma(i+1)\}$ and $\{\gamma(j), \gamma(j+1)\}$ coincide for each pair $(i, j) \in I^2$. Call $\gamma$ $k$-edge self-avoiding if the maximal edge local time is at most $k \in \mathbb{N}$.

When considering (as we will) 3-edge self-avoiding walks, we say that a walk $\gamma$ as above closes if $||\gamma(n)|| = 0$. Note that this definition entails that closing walks have even length.

Write $W_n^{3,E}$ for the uniform law on the set $\text{SAW}_n^{3,E}$ of length $n$ 3-edge self-avoiding walks beginning at 0.

Theorem 1.7. Let $d = 3$. For any $\varepsilon > 0$, the set of $n \in 2\mathbb{N}$ for which
$$W_n^{3,E}(\Gamma \text{ closes}) \leq n^{-2/3+\varepsilon}$$
has positive limit supremum density.

Theorem 1.7’s proof is a three-dimensional analogue of an alternative, polygon joining, derivation of Theorem 1.2 that works along positive limsup density subsequences of $n \in 2\mathbb{N}$ (see Corollary 3.4). In general dimension, the exponent would be $1 - 1/d$, but the lace expansion anyway provides sharp conclusions for high
dimension for spread-out models (and is the only known technique for deriving
sharp results for these dimensions on the nearest neighbour lattice).

The upper critical dimension of self-avoiding walk on $\mathbb{Z}^d$ is $d = 4$. The lace expan-
sion does not apply here. The continuous-time weakly self-avoiding walk in $d = 4$
has been the subject of an extensive investigation of Bauerschmidt, Brydges and
Slade. In [1], a decay of $||x||^{-2}$ is proved for the critical two-point function; in [2], a
log $^{1/4}$ correction to the susceptibility is derived. Both of these works rely on a rig-
orous renormalization group method developed in a five-paper series [7, 8, 4, 9, 10].

In the next section, we will discuss Madras’ polygon joining technique and use to
it derive heuristically a well-known hyperscaling relation that may be expressed in
terms of the closing probability. This exposition presents a useful framework for
discussing many of the paper’s ideas, and we defer further overview of the paper’s
structure or concepts until the end of Section 2.

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closing probability.

2. Polygon number bounds via joining: an heuristic prelude

2.1. Some general notation and tools. We present our proofs in the body of
the paper for the case $d = 2$, since we hope that doing so will encourage the reader
to visualise examples of the constructs being discussed. For this reason, some of
the notation we now introduce is specifically adapted to the two dimensional case.

2.1.1. Multi-valued maps. For a finite set $B$, let $P(B)$ denote its power set. Con-
sider a multi-valued map $\Psi : A \rightarrow P(B)$, where $A$ is also a finite set. An arrow is a
pair $(a, b) \in A \times B$ for which $b \in \Psi(a)$; such an arrow is said to be outgoing from $a$
and incoming to $b$. We consider multi-valued maps in order to find lower bounds
on $|B|$, and for this, we need upper (and lower) bounds on the number of incoming
(and outgoing) arrows. For example, setting $m$ to be the minimum over $a \in A$ of
the number of arrows outgoing from $a$, and $M$ to be the maximum over $b \in B$ of
the number of arrows incoming to $b$, we have the bound $|B| \geq \frac{mM}{M}|A|$, since $M|B|
and m|A|$ are upper and lower bounds on the total number of arrows.

2.1.2. Denoting walk vertices and subpaths. For $i, j \in \mathbb{N}$ with $i \leq j$, we write $[i, j]$ for
$k \in \mathbb{N} : i \leq k \leq j$. For a walk $\gamma : [0, n] \rightarrow \mathbb{Z}^d$ and $j \in [0, n]$, we write $\gamma(j)$ for
place of $\gamma(j)$. For $0 \leq i \leq j \leq n$, $\gamma[i, j]$ denotes the subpath $\gamma[i, j] : [i, j] \rightarrow \mathbb{Z}^d$ given
by restricting $\gamma$. 
Definition 2.1. The Cartesian unit vectors are denoted by \( e_1 \) and \( e_2 \) and the coordinates of \( u \in \mathbb{Z}^2 \) by \( x(u) \) and \( y(u) \). For a finite set of vertices \( V \subseteq \mathbb{Z}^2 \), we define the northeast vertex \( \text{NE}(V) \) in \( V \) to be that element of \( V \) of maximal \( e_2 \)-coordinate; should there be several such elements, we take \( \text{NE}(V) \) to be the one of maximal \( e_1 \)-coordinate. That is, \( \text{NE}(V) \) is uppermost element of \( V \), and the rightmost among such uppermost elements if there is more than one. Using the four compass directions, we may similarly define eight elements of \( V \), including the lexicographically minimal and maximal elements of \( V \), \( \text{WS}(V) \) and \( \text{EN}(V) \). We extend the notation to any self-avoiding walk or polygon \( \gamma \), writing for example \( \text{NE}(\gamma) \) for \( \text{NE}(V) \), where \( V \) is the vertex set of \( \gamma \). For a polygon or walk \( \gamma \), set \( y_{\max}(\gamma) = y(\text{NE}(\gamma)) \), \( y_{\min}(\gamma) = y(\text{SE}(\gamma)) \), \( x_{\max}(\gamma) = x(\text{EN}(\gamma)) \) and \( x_{\min}(\gamma) = x(\text{WN}(\gamma)) \). The height \( h(\gamma) \) of \( \gamma \) is \( y_{\max}(\gamma) - y_{\min}(\gamma) \) and its width \( w(\gamma) \) is \( x_{\max}(\gamma) - x_{\min}(\gamma) \).

2.1.4. Polygons with northeast vertex at the origin. For \( n \in 2\mathbb{N} \), let \( \text{SAP}_n \) denote the set of length \( n \) polygons \( \phi \) such that \( \text{NE}(\phi) = 0 \). The set \( \text{SAP}_n \) is in bijection with equivalence classes of length \( n \) polygons where polygons are identified if one is a translate of the other. Thus, \( p_n = |\text{SAP}_n| \).

We write \( P_n \) for the uniform law on \( \text{SAP}_n \). A polygon sampled with law \( P_n \) will be denoted by \( \Gamma \), as a walk with law \( W_n \) is.

There are \( 2n \) ways of tracing the vertex set of a polygon \( \phi \) of length \( n \): \( n \) choices of starting point and two of orientation. We now select one of these ways. Abusing notation, we may write \( \phi \) as a map from \( [0,n] \) to \( \mathbb{Z}^2 \), setting \( \phi_0 = \text{NE}(\phi) \), \( \phi_1 = \text{NE}(\phi) - e_1 \), and successively defining \( \phi_j \) to be the previously unselected vertex for which \( \phi_{j-1} \) and \( \phi_j \) form the vertices incident to an edge in \( \phi \). Note that \( \phi_n = \text{NE}(\phi) - e_2 \), so that this abuse of notation identifies the polygon \( \phi \) with the closing walk beginning at \( \text{NE}(\phi) \) formed by omitting the edge \([\text{NE}(\phi) - e_2, \text{NE}(\phi)]\) from the edge-set of \( \phi \).

2.1.5. Cardinality of a finite set \( A \). This is denoted by either \( \#A \) or \( |A| \).

2.1.6. Plaquettes. The shortest non-empty polygons contain four edges. Certain such polygons play an important role in several arguments and we introduce notation for them now.

Definition 2.2. A plaquette is a polygon with four edges. Let \( \phi \) be a polygon. A plaquette \( P \) is called a join plaquette of \( \phi \) if \( \phi \) and \( P \) intersect at precisely the two horizontal edges of \( P \). Note that when \( P \) is a join plaquette of \( \phi \), the operation of removing the two horizontal edges in \( P \) from \( \phi \) and then adding in the two vertical edges in \( P \) to \( \phi \) results in two disjoint polygons whose lengths sum to the length of \( \phi \). We use symmetric difference notation and denote the output of this operation by \( \phi \Delta P \).
The operation may also be applied in reverse: for two disjoint polygons $\phi^1$ and $\phi^2$, each of which contains one vertical edge of a plaquette $P$, the outcome $(\phi^1 \cup \phi^2) \Delta P$ of removing $P$’s vertical edges and adding in its horizontal ones is a polygon whose length is the sum of the lengths of $\phi^1$ and $\phi^2$.

2.2. Polygon superadditivity: polygon joining in its simplest guise. Here are some very classical facts regarding the growth of walk and polygon numbers.

**Proposition 2.3.** Consider any dimension $d \geq 2$.

1. For $n, m \in \mathbb{N}$, $c_{n+m} \leq c_n c_m$.

2. For $n, m \in 2\mathbb{N}$, $p_{n+m} \geq \frac{1}{d-1} p_n p_m$.

3. The limits $\mu_1 := \lim_{n \to \infty} c_n^{1/n}$ and $\mu_2 := \lim_{n \to \infty} \frac{p_{n+1/2n}}{2n}$ exist; moreover, $c_n \geq \mu_1^n$ and $p_{2n} \leq \mu_2^{2n}$.

4. The limits are equal: $\mu_1 = \mu_2$.

The common exponential growth constant $\mu_1 = \mu_2$ is called the connective constant and denoted by $\mu$. It is easily seen that $\mu \in [d, 2d - 1]$. Duminil-Copin and Smirnov [11] have proved using parafermionic observables that $\mu$ equals $\sqrt{2 + \sqrt{2}}$ for the honeycomb lattice. These observables have also been used in [5] and [17], the second of which computes the connective constant for a weighted walk model.

Polygon joining arguments are fundamental to the ideas in this paper. Perhaps the most basic such argument proves Proposition 2.3(2). We take the opportunity to present this joining idea in a simple guise.

**Proof of a portion of Proposition 2.3.** (1). A walk $\gamma$ of length $n + m$ with $\gamma_0 = 0$ can be severed at $\gamma_n$ to form two subpaths, $\gamma_{[0,n]}$ and $\gamma_{[n,n+m]}$. The first of these (called $\gamma^a$) is a length $n$ walk beginning at 0. The second becomes a length $m$ walk (called $\gamma^b$) from 0 after translation by $-\gamma_n$ (and reindexing of the domain from $[n, n+m]$ to $[0, m]$). The application $\gamma \to (\gamma^a, \gamma^b)$ is injective. Thus, $c_{n+m} \leq c_n c_m$.

(2). Polygons cannot be severed, but they can be joined in pairs. We consider only the case of $d = 2$ and equal polygon length $m = n \in 2\mathbb{N}$. Consider a pair $\phi, \phi' \in \text{SAP}_n$. Relabel $\phi'$ by translating it so that $\text{WN}(\phi')$ is one unit to the right of $\text{EN}(\phi)$. The plaquette $P$ whose upper left vertex is $\text{EN}(\phi)$ has one vertical edge in $\phi$ and one in $\phi'$. Note that $\chi := (\phi \cup \phi') \Delta P$ belongs to $\text{SAP}_{2n}$. Moreover, the application $(\phi, \phi') \to \chi$ is injective, because from $\chi$ we can detect the location of the plaquette $P$. The reader may wish to confirm this property, using that $\phi$ and $\phi'$ have the same length. This injectivity implies that $p_{2n} \geq p_n^2$. See [28, Theorem 3.2.3] for a complete proof.

(3). This is Fekete’s lemma [33, Lemma 1.2.1].
Lemma 2.4. There is a constant $c > 0$ such that, for $n \in 2\mathbb{N} + 1$,
\[
W_n(\Gamma \text{ closes}) \geq \exp\left\{ -cn^{1/2} \right\}.
\]

Proof. The Hammersley-Welsh lower bound [19] on the number of self-avoiding bridges in terms of walk number is a classical unfolding argument that is recounted in [28, Chapter 3]. It implies that there exists a constant $c_{HW} > 0$ such that, for all $n \in \mathbb{N}$,
\[
c_n \leq e^{c_{HW}n^{1/2}} \mu^n.
\]
We may now use the bound $p_n \geq e^{-cn^{1/2}} \mu^n$ in [22, Theorem 3] and (1.3) to conclude. Alternatively (but similarly), the lemma is proved from (2.1) by further unfolding in [12]: see Lemma 5 and the proof of Proposition 3 in that paper. \qed

2.3. The polygon deviation exponent via a hyperscaling relation. The limiting value $\lim_n \theta_n$ is predicted to exist and to satisfy a relation with the Flory exponent $\nu$ for mean-squared radius of gyration. We hypothesise the existence of the limit
\[
\theta := \lim_n \theta_n
\]
and suppose also that $\nu$ exists, given by the formula
\[
E_{W_n} ||\Gamma_n||^2 = n^{2\nu + o(1)},
\]
where $E_{W_n}$ denotes the expectation associated with $W_n$; we may expect that $||\Gamma_n||$ is typically of order $n^\nu$.

The hyperscaling relation between $\theta$ and $\nu$ is
\[
\theta = 1 + d\nu
\]
for any dimension $d \geq 2$. In $d = 2$, $\nu = 3/4$ and thus $\theta = 5/2$ is expected. That $\nu = 3/4$ was predicted by the Coulomb gas formalism [30, 31] and then by conformal field theory [14, 15]. Hara and Slade [20] used the lace expansion to show that $\nu = 1/2$ when $d \geq 5$ by demonstrating that, for some constant $D \in (0, \infty)$, $E_{W_n} ||\Gamma_n||^2 - Dn$ is $O(n^{-1/4+o(1)})$. This value of $\nu$ is anticipated in four dimensions as well, since $E_{W_n} ||\Gamma_n||^2$ is expected to grow as $n\left( \log n \right)^{1/4}$; see [2] for a recent advance in a closely related model.

We review the standard heuristic derivation of this relation; see equation (1.4.14) of [28] for a representation of (2.4) written in terms of the exponent $\alpha_{\text{sing}} = 3 - \theta$, and Section 2.1 of this text for a slightly more detailed presentation of the derivation.

First recall from (1.2) the walk counterpart $\xi_n$ to the polygon deviation exponent, $\theta_n$. Note that the absence of a minus sign in (1.2) compared to (1.1). Note also that Proposition 2.3(1) and (2) imply that $\xi_n \geq 0$ and $\lim inf \theta_n \geq 0$ for $n \in \mathbb{N}$. 

(4). This follows directly from the following lower bound on the closing probability (specifically, from the resulting subexponential decay). \qed
Observe that the proof of Proposition 2.3(1) in fact shows that $c_n + m$ equals the probability that independent samples of $W_n$ and $W_m$ avoid each other, i.e., intersect only at the origin. Supposing the existence of the limit

$$\xi := \lim_n \xi_n,$$  \hspace{1cm} (2.5)

we see that $\xi$ would have the interpretation

$$\left( W_n \times W_n \right) \left( \Gamma^1 \text{ avoids } \Gamma^2 \right) = n^{-\xi + o(1)}.$$  \hspace{1cm} (2.6)

Assuming (2.3), the endpoint $\Gamma_n$ under $W_n$ presumably adopts a location that is fairly uniform over a ball about the origin of radius of order $n^\nu$. For generic locations $x$ in this ball, we infer that $W_n(\Gamma_n = x) = n^{-d\nu + o(1)}$. For $x$ close to the origin, however, this probability is in addition penalized since $\{\Gamma_n = x\}$ entails that the initial and final segments of $\Gamma$ come close without touching one another. When $||x|| = 1$, the order of this additional penalty is given by (2.6). Thus, we expect

$$W_n(\Gamma \text{ closes}) = n^{-d\nu - \xi + o(1)},$$

as $n \to \infty$ through odd values. Using (1.3) and our hypotheses (2.2) and (2.5), we also find that

$$W_n(\Gamma \text{ closes}) = n^{1 - \theta - \xi + o(1)}.$$  \hspace{1cm} \text{The } \xi \text{ terms cancel and we obtain (2.4).}

We mention that $\xi = 11/32$ is expected when $d = 2$; in light of the $\theta = 5/2$ prediction and (1.3), $W_n(\Gamma \text{ closes}) = n^{-\psi + o(1)}$ with $\psi = 59/32$ is expected. The 11/32 value was predicted by Nienhuis in [30] and can also be deduced from calculations concerning SLE$_{8/3}$; see [24, Prediction 5]. We will refer to $\psi$ as the closing exponent. Incidentally, the possibility that the half-integer value of $\theta$ in $d = 2$ may indicate that polygons are more tractable than walks has been mooted in Tony Guttmann’s survey [18]: certainly the present article builds on the theme of [13] to provide further evidence that polygons, and in particular the closing probability, may be some of the more tractable aspects of the theory of self-avoiding walk.

### 2.4. The hyperscaling relation argued by polygon joining.

We now present another heuristic derivation of the hyperscaling relation (2.4). The new argument takes longer to present than the first and is no more convincing. However, it is conceptually central to this paper, and it provides a useful framework for understanding some of the main ideas in the proofs of our principal results.

In fact, these concepts are illustrated by rederiving merely the lower bound

$$\theta \geq 1 + 2\nu$$  \hspace{1cm} (2.7)

when $d = 2$. We will derive (2.7) in three steps, arguing that $\theta \geq \nu$ and $\theta \geq 1 + \nu$ in the first and second steps. In light of the three step derivation, it is of interest to reflect on why we may also expect $\theta \leq 1 + 2\nu$; but we do not do so here. We also mention that essentially the same heuristics point to $\theta \geq 1 + d\mu$ in...
dimension \( d \geq 2 \), with steps one and two reaching the conclusions that \( \theta \geq (d-1)\mu \) and \( \theta \geq 1 + (d-1)\mu \).

**Step one.** This step is Madras’ argument in [26] that \( \theta_0 \geq 1 - 1/d \) when \( d = 2 \). The argument holds for any \( d \geq 3 \) provided that some conceptually minor technical obstacles may be overcome. Assuming (2.2), of course, we would have \( \theta \geq 1 - 1/d \).

We recall Madras’ argument in outline when \( d = 2 \). When the value of \( \nu \) is hypothesised by (2.3), his conclusion in \( d = 2 \) is \( \theta \geq \nu \) (and this is what we claim in the first step). His argument develops the polygon joining argument in the proof of Proposition 2.3(2). The length \( n \) polygons \( \phi \) and \( \phi' \) were joined only in one alignment, after displacement of \( \phi' \) to a given location. Madras argues under (2.3) that there are at least \( n\nu \) locations to which \( \phi' \) may be translated and then attached to \( \phi \). A total of \( n\nu \) distinct length \( 2n \) polygons result, and we learn that

\[
p_{2n} \geq n\nu p_n^2,
\]
whence \( \theta \geq \nu \). Where are these new locations for joining? Orient \( \phi \) and \( \phi' \) so that the height of each is at least its width; thus each height is at least of order \( n\nu \). Translate \( \phi' \) vertically so that some vertices in \( \phi \) and \( \phi' \) share their \( y \)-coordinate, and horizontally so that \( \phi' \) is to the right of \( \phi \). Push \( \phi' \) to the left stopping just before the polygons overlap. Then try to find a plaquette \( P \) whose left and right vertical edges are occupied by \( \phi \) and \( \phi' \). The operation in the proof of Proposition 2.3(2) applied with plaquette \( P \) then yields a length \( 2n \) polygon. There are \( n\nu \) different heights that may be used, and the bound (2.8) results.

There is a technical difficulty in implementing this argument: in some configurations, no such plaquette \( P \) exists. Madras develops a different local joining procedure which we will also use. His procedure will be reviewed shortly, in Section 3.1.

**Step two.** The next step in this second derivation of (2.4) is to argue that

\[
\theta \geq 1 + \nu.
\]

We do so by arguing heuristically in favour of a strengthening of (2.8),

\[
p_{2n} \geq n\nu \sum_{j=-n/2}^{n/2} p_{n-j} p_{n+j}.
\]

Expressed using the polygon deviation exponents, we would then have \( n^{-\theta_{2n}} \geq n\nu \sum_{j=-n/2}^{n/2} (n-j)^{-\theta_{n-j}} (n+j)^{-\theta_j} \). Using (2.2), the bound (2.9) results.

To argue for (2.10), note that, in deriving \( p_{2n} \geq n\nu p_n^2 \), each polygon pair \((\phi, \phi') \in \text{SAP}_n \times \text{SAP}_n \) resulted in \( n\nu \) distinct length \( 2n \) polygons. The length pair \((n, n)\) may
be varied to be of the form \((n - j, n + j)\) for any \(j \in [0, n/2]\). We are constructing a multi-valued map
\[
\Psi : \bigcup_{|j| \leq n/2} \text{SAP}_{n-j} \times \text{SAP}_{n+j} \to \mathcal{P}(\text{SAP}_{2n})
\]
to the power set of \(\text{SAP}_{2n}\) which associates to each polygon pair \((\phi, \phi')\) in the domain an order of \(n^\nu\) elements of \(\text{SAP}_{2n}\). Were \(\Psi\) injective, we would obtain (2.10). (The term \textit{injective} is being misused: we mean that no two arrows of \(\Psi\) are incoming to the same element of \(\text{SAP}_{2n}\).) The map is not injective but it is plausible that it only narrowly fails to be so: that is, abusing notation in a similar fashion, for typical \(\chi \in \text{Range}(\phi)\), the cardinality of \(\Psi^{-1}(\chi)\) is at most \(n^{o(1)}\). A definition is convenient before we argue this.

\textbf{Definition 2.5.} Let \(\phi\) be a polygon, and let \(P\) be one of \(\phi\)'s join plaquettes. Let \(\phi^1\) and \(\phi^2\) denote the disjoint polygons of which \(\phi \Delta P\) is comprised. If each of \(\phi^1\) and \(\phi^2\) has at least one quarter of the length of \(\phi\), then we call \(P\) a macroscopic join plaquette.

To see that \(\Psi\) is close to being injective, note that each pre-image of \(\chi \in \text{Range}(\phi)\) under \(\Psi\) corresponds to a macroscopic join plaquette of \(\chi\). Each macroscopic join plaquette entails a probabilistically costly macroscopic four-arm event, where four walks of length of order \(n\) must approach the plaquette without touching each other. That \(\chi\) belongs to \(\text{Range}(\phi)\) amounts to saying that \(\chi\) is an element of \(\text{SAP}_{2n}\) having at least one macroscopic join plaquette. The four-arm costs make it plausible that a typical such polygon has only a few such plaquettes, gathered together in a small neighbourhood. Thus, \(\Psi\) is plausibly close to injective so that (2.10), and (2.9), results.

\textbf{Step three.} A further term of \(\nu\) is now sought to move from (2.9) to (2.7). Let \(\text{SAP}^{mj}_{2n}\) denote the subset of \(\text{SAP}_{2n}\) whose elements have a macroscopic join plaquette. Note that \(\Psi\) is a map into the power set of \(\text{SAP}^{mj}_{2n}\), so that we have derived
\[
\text{P}^{mj}_{2n} \geq n^\nu \sum_{j=-n/2}^{n/2} \text{P}_{n-j} \text{P}_{n+j},
\]
(2.11)
where \(\text{P}^{mj}_{2n} = |\text{SAP}^{mj}_{2n}|\). Since \(\text{P}_{2n}(\Gamma \in \text{SAP}^{mj}_{2n}) = \text{P}^{mj}_{2n} / \text{P}_{2n}\), the following, alongside (2.9), will imply (2.7):
\[
\text{P}_{2n}(\Gamma \in \text{SAP}^{mj}_{2n}) \leq Cn^{-\nu}.
\]
(2.12)
To argue for this, consider a typical polygon \(\phi \in \text{SAP}_{2n}\). It occupies horizontal coordinates including 0 in an interval of length of order \(n^\nu\). Suppose that this interval contains \([-5n^\nu/4, 5n^\nu/4]\) (where \(5/4\) may be any constant exceeding one). The polygon crosses the strip \([-n^\nu, n^\nu]\) at least twice. Let \(\psi^1\) and \(\psi^2\) denote the subpaths of \(\phi\) consisting of the uppermost and lowermost of these crossings. Suppose as we may (for it is plausibly a positive probability event) that
the origin, which is \( \text{NE}(\phi) \), lies in \( \psi^1 \). Now conduct an experiment where the law \( \mathbb{P}_{2n} \) is resampled by first realizing the law, then forgetting about all information except:

- the subpath \( \psi^1 \); and the subpath \( \psi^2 \), up to vertical translation.

Finally, the law \( \text{SAP}_{2n} \) is resampled by reconstructing the forgotten data from its conditional law given the above data. We argue for the bound (2.12) by deriving it for the resampled copy of the law \( \mathbb{P}_{2n} \). What happens to the subpaths \( \psi^1 \) and \( \psi^2 \) during the reconstruction step? The path \( \psi^1 \) stays in place, while \( \psi^2 \) undergoes a random vertical shift, as Figure 1 illustrates. It is consistent with the hypothesis (2.3) that the height of typical polygons under \( \text{SAP}_{2n} \) is of order \( n^{\nu} \). It is also a natural hypothesis, akin to the Russo-Seymour-Welsh theorem for critical percolation (which has been recently recounted in [34]), that any macroscopic subpath of such a polygon has comparable probability of moving in one or other of the four compass directions by an amount of the order of the polygon’s diameter. These suppositions point to the conclusion that the random vertical shift experienced by \( \psi^2 \) during the reconstruction step has order of magnitude \( n^{\nu} \) and that any admissible shift in this range has probability of order \( n^{-\nu} \). The highest such shift is the only one that may bring the translate of \( \psi^2 \) within distance one of the upper subpath \( \psi^1 \). This translate may result in a macroscopic join plaquette with \( x \)-coordinate in \( [-n^{\nu}, n^{\nu}] \), but all the others may not. Thus, there is probability at most a constant multiple of \( n^{-\nu} \) that the reconstructed polygon has a macroscopic join plaquette. This rests the case for (2.12) and thus for (2.7).

2.5. The polygon deviation exponent as the sum of three terms. We now formulate some notation that will permit us to describe the upcoming proofs in terms of the preceding three step derivation. We informally specify three exponents \( \theta^{(1)} \), \( \theta^{(2)} \) and \( \theta^{(3)} \) (and suppose they exist).
• Let the number of ways of joining a typical pair of length \( n \) polygons in step one scale as \( n^{\theta(1)} \).

• Let the number of lengths \( k \in [n/2, n] \) for which typically polygon pairs of lengths \( k \) and \( 2n - k \) may be joined in about \( n^{\theta(1)} \) ways scale as \( n^{\theta(2)} \).

• Finally, suppose that \( P_{2n}(\Gamma \in \text{SAP}_{2n}^{\text{nj}}) \) scales as \( n^{-\theta(3)} \).

Then we have seen that we should expect that \( \theta \geq \theta(1) + \theta(2) + \theta(3) \) (and in fact equality should hold), and that \( \theta(1) = \nu, \theta(2) = 1 \) and \( \theta(3) = \nu \) in \( d = 2 \). Madras has proved \( \theta \geq 1/2 \) in this dimension by arguing that \( \theta(1) \) indeed is at least \( \nu \) and using the trivial bound \( \nu \geq 1/2 \); we may say that he carried out a \((1/2, 0, 0)\)-argument, in that these are the proven lower bounds on the respective exponents.

2.6. Structure of the paper. Next we implement step two via polygon joining arguments: Section 3 presents two rigorous interpretations of (2.10), Propositions 3.2 and 3.5, the first of which will be ultimately used to prove the sub-\( n^{-1/2} \) closing probability Theorem 1.5. After setting out further general notation in Section 4, we turn to a presentation of the general apparatus of the snake method in Section 5. The method via Gaussian pattern fluctuation is used in Section 6 to prove the \( n^{-1/2 + o(1)} \) closing probability Theorem 1.2. The snake method is allied with the polygon joining technique in Section 7 to yield Theorem 1.5. The final Section 8 treats details specific to three dimensions and gives the proof of Theorem 1.7.

See Figure 2 for another record of the paper’s arguments. Two caveats should be noted.

• The language of the \((\theta(1), \theta(2), \theta(3))\)-argument is a mnemonic, and we make no attempt to specify these exponents precisely.

• All such arguments except Madras’ and Corollary 3.3 are made assuming some hypothesis regarding the closing probability. In these cases, the inference regarding this probability (in the fourth bar) is unconditional (except for Theorem 1.5(2)), but this is not true for the values of \((\theta(1), \theta(2), \theta(3))\).

3. Polygon number bounds via joining: a rigorous treatment

Step two of the hyperscaling relation lower bound gives an impression of being much easier to make rigorous than either step three or than showing \( \theta(1) > 1/2 \). We now endeavour to make step two rigorous: that is, to carry out a \((1/2, 1, 0)\)-argument. We succeed only by making Hypothesis PCP, that the closing probability is often of polynomial decay.
Figure 2. Results for \( d = 2 \) are summarised. Polygon joining arguments may be described as \((\theta^{(1)}, \theta^{(2)}, \theta^{(3)})\)-arguments, in the language of Section 2.5. For each result where the polygon joining technique is applied, the first three vertical bars indicate the values for which a \((\theta^{(1)}, \theta^{(2)}, \theta^{(3)})\)-argument is undertaken. The bold fourth bar indicates the lower bound established on the closing exponent. The acronyms PJ and SM indicate that the polygon joining technique or the snake method is used. When S or C appears in square brackets, the fourth bar assertion concerns a subsequence of \( \{\theta_n : n \in 2\mathbb{N}\} \) or is conditional.

Definition 3.1. For \( \zeta > 0 \), define the set of indices \( \text{HCP}_\zeta \subseteq 2\mathbb{N} \) of \( \zeta \)-high closing probability,

\[
\text{HCP}_\zeta = \left\{ n \in 2\mathbb{N} : W_{n-1}(\Gamma \text{ closes}) \geq n^{-\zeta} \right\}.
\]

Hypothesis PCP (Polynomial Closing Probability). There exists a constant \( \zeta > 0 \) such that the set \( 2\mathbb{N} \setminus \text{HCP}_\zeta \) has limit supremum density less than \( 1/100 \).

The assumption that \( d = 2 \) is in force throughout Section 3.
3.0.1. The \((1/2, 1, 0)\) argument. The next two propositions state the conclusion of our \((1/2, 1, 0)\)-argument. They are the rigorous counterparts to the informal (2.10) and (2.9).

Proposition 3.2. For any \(\zeta > 0\), there is a constant \(C_1 = C_1(\zeta) > 0\) such that, for \(n \in 2N \cap \text{HCP}_\zeta\),

\[
p_n \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j \in 2N \cap [2^{i-1}, 2^i]} p_j p_{n-j},
\]

where \(i \in N\) is chosen so that \(n \in 2N \cap [2^{i+1}, 2^{i+4}]\).

Proposition 3.3. Suppose that Hypothesis PCP holds. For any \(\delta > 0\),

\[
\left| \left\{ j \in 2N \cap [2^i, 2^{i+2}] : \theta_j \geq \frac{3}{2} - \delta \right\} \right| \geq \frac{1}{20} \cdot |2N \cap [2^i, 2^{i+2}]|
\]

for all but finitely many values of \(i \in N\).

The real role of Proposition 3.3 will be in Section 7 when we prove the sub-

\(n^{-1/2}\)

bounds on closing probability in Theorem 1.5. For now, the two preceding results may seem conditional. We prove a corollary which shows that this is not the case.

Corollary 3.4. Let \(d = 2\). For any \(\varepsilon > 0\), the set of \(n \in 2N\) such that \(W_{n-1}(\Gamma \text{ closes})\) is at most \(n^{-1/2+\varepsilon}\) has positive limit supremum density.

Proof. If Hypothesis PCP fails, then \(W_{n-1}(\Gamma \text{ closes})\) decays faster than \(n^{-C}\) on a subsequence of \(n \in 2N\) of positive limsup density, whatever the value of \(C > 0\). Turning to the case that the hypothesis holds, note that

\[
W_{n-1}(\Gamma \text{ closes}) = 2np_n/c_{n-1} \leq 2\mu n^{1-\theta_n},
\]

since \(c_{n-1} \geq \mu^{n-1}\). Thus, Proposition 3.3 completes the proof. □

Note that, since \(d = 2\) is needed in Corollary 3.4, the result is weaker than Theorem 1.2. However, the exponent is best seen as \(1 - 1/d\): the corollary’s three-dimensional counterpart in this paper is Theorem 1.7 which does go beyond Theorem 1.2.

3.0.2. The \((1/2, 1/2, 0)\) argument. This argument is carried out unconditionally, leading to Theorem 1.4 which is a counterpart to (2.9). The argument’s conclusion counterpart to (2.10) is now stated.

Proposition 3.5. There is a constant \(c > 0\) such that, for \(n \in 2N\),

\[
p_n \geq c \sum_{j \in 2N \cap [2^{i-1}, 2^i]} p_j p_{n-j},
\]

where \(i \in N\) is chosen so that \(n \in 2N \cap [2^{i+1}, 2^{i+4}]\).
The rest of Section 3 is devoted to proving Propositions 3.2, 3.3 and 3.5, and deriving Theorem 1.4 from Proposition 3.5. We mention that it may well be that further arguments in the style of the proof of Proposition 3.3 would improve the conclusion of Theorem 1.4 so that it holds on a set of density one. However, it is clear that Proposition 3.5 is inadequate for proving the conclusion \( \theta_n \geq 1 \) for all \( n \in 2\mathbb{N} \), because this tool permits occasional spikes in the value of the \( p_n \), as the sequence

\[
\theta_n = \begin{cases} 
  1/2 & \text{if } n \text{ is a power of } 2, \\
  1 + \frac{1}{100} & \text{if } n \in 2\mathbb{N} \text{ is otherwise,}
\end{cases}
\]

demonstrates.

3.1. Madras’ polygon joining procedure. Making steps one and two in the derivation of the hyperscaling relation bound (2.7) rigorous involves joining pairs of polygons that almost touch but for which there is not necessarily a plaquette whose vertical edges are divided between the two elements of the pair. Madras developed a local surgery technique for making such joinings. We will use this technique and recall it now.

Consider two polygons \( \tau \) and \( \sigma \) of lengths \( n \) and \( m \) for which the intervals

\[
[y_{\min}(\tau) - 1, y_{\max}(\tau) + 1] \text{ and } [y_{\min}(\sigma) - 1, y_{\max}(\sigma) + 1]
\]

intersect. (3.1)

Madras’ procedure joins \( \tau \) and \( \sigma \) to form a new polygon of length \( n + m + 16 \) in the following manner.

First translate \( \sigma \) to the right by far enough that the \( x \)-coordinates of the vertices of this translate are all strictly greater than all of those of \( \tau \). Now shift \( \sigma \) to the left step by step until the first time at which there is a pair of vertices, one in \( \tau \) and the other in the \( \sigma \)-translate, that share an \( x \)-coordinate and whose \( y \)-coordinates differ by at most two; such a moment necessarily occurs, by the assumption (3.1). Write \( \sigma' = \sigma + T_1e_1 \) (with \( T_1 \in \mathbb{Z} \)) for this particular horizontal translate of \( \sigma \). There is at least one vertex \( z \in \mathbb{Z}^2 \) such that the set \( \{z - e_2, z, z + e_2\} \) contains a vertex of \( \tau \) and a vertex of \( \sigma' \). The set of such vertices contains at most one vertex with any given \( y \)-coordinate. Denote by \( Y \) the vertex \( z \) with the maximal \( y \)-coordinate. Madras now defines a modified polygon \( \tau_{\text{mod}} \), which is formed from \( \tau \) by changing its structure in a neighbourhood of \( Y \in \mathbb{Z}^2 \). Depending on the structure of \( \tau \) near \( Y \), either two edges are removed and ten edges added to form \( \tau_{\text{mod}} \) from \( \tau \), or one edge is removed and nine are added. As such, \( \tau_{\text{mod}} \) has length \( n + 8 \). The rule that specifies \( \tau_{\text{mod}} \) is recalled from [26] in Figure 3.

A modified polygon formed from \( \sigma' \) is also defined. Rotate \( \sigma' \) about the vertex \( Y \) by \( \pi \) radians to form a new polygon \( \sigma'' \). Form \( \sigma''_{\text{mod}} \) according to the same rules, recalled in Figure 3. Then rotate back the outcome by \( \pi \) radians about \( Y \) to produce the modification of \( \sigma' \), which to simplify notation we denote by \( \sigma'_{\text{mod}} \).

Writing \( Y = (Y_1, Y_2) \), note that no vertex of \( \tau \) belongs to the right corridor \( \{Y_1 + 1, Y_2 + 2, \ldots\} \times \{Y_2 - 1, Y_2, Y_2 + 1\} \), the region that lies strictly to the right of
\(\{Y - e_2, Y, Y + e_2\}\). Equally, no vertex of \(\sigma'\) belongs to the left corridor \(\{\cdots, Y_1 - 2, Y_1 - 1\} \times \{Y_2 - 1, Y_2, Y_2 + 1\}\).

Note that the polygon \(\tau_{\text{mod}}\) extends \(\tau\) to the right of \(Y\) by either two or three units inside the right corridor (by two in case IIa, IIci or IIIci and by three otherwise). Likewise \(\sigma_{\text{mod}}\) extends \(\sigma'\) to the left of \(Y\) by either two or three units in the left corridor (by two when \(\sigma''\) satisfies case IIa, IIci or IIIci and by three otherwise).

Note from Figure 3 that, in each case, \(\tau_{\text{mod}}\) contains two vertical edges that cross the right corridor at the maximal \(x\)-coordinate adopted by vertices in \(\tau_{\text{mod}}\) that lie in this corridor. Likewise, \(\sigma_{\text{mod}}\) contains two vertical edges that cross the left corridor at the minimal \(x\)-coordinate adopted by vertices in \(\sigma_{\text{mod}}\) that lie in the left corridor.

Translate \(\sigma_{\text{mod}}\) to the right by \(T_2\) units, where \(T_2\) equals

- five when one of cases IIa, IIci and IIIci obtains for both \(\tau\) and \(\sigma''\);
- six when one of these cases obtains for exactly one of these polygons;
- seven when none of these cases holds for either polygon.

Note that \(\tau_{\text{mod}}\) and \(\sigma_{\text{mod}} + T_2 e_1\) are disjoint polygons such that, for some pair of vertically adjacent plaquettes \((P_1, P_2)\) in the right corridor (whose left sides have \(x\)-coordinate either \(Y_1 + 2\) or \(Y_1 + 3\)), the edge-set of \(\tau_{\text{mod}}\) intersects the plaquette pair on the two left sides of \(P_1\) and \(P_2\), while the edge-set of \(\sigma_{\text{mod}} + T_2 e_1\) intersects this pair on the two right sides of \(P_1\) and \(P_2\). Let \(P_1\) denote the upper element of this plaquette pair.

The polygon that Madras specifies as the join of \(\tau\) and \(\sigma\) is given by \((\tau_{\text{mod}} \cup (\sigma_{\text{mod}} + T_2 e_1)) \Delta P_1\). Note that, to form the join polygon, \(\sigma\) is first horizontally translated by \(T_1\) units to form \(\sigma'\), modified locally to form \(\sigma_{\text{mod}}\), and then further horizontally translated by \(T_2\) units to produce the polygon \(\sigma_{\text{mod}} + T_2 e_1\) that is joined onto \(\tau_{\text{mod}}\). Thus, \(\sigma\) undergoes a horizontal shift by \(T_1 + T_2\) units as well as a local modification before being joined with \(\tau_{\text{mod}}\).

**Definition 3.6.** For two polygons \(\tau \in \text{SAP}_n\) and \(\sigma \in \text{SAP}_m\) satisfying (3.1), define the Madras join polygon

\[
J(\tau, \sigma) = (\tau_{\text{mod}} \cup (\sigma_{\text{mod}} + T_2 e_1)) \Delta P_1 \in \text{SAP}_{n+m+16}.
\]

The plaquette \(P_1\) will be called the junction plaquette.

Such polygons \(\tau\) and \(\sigma\) are called Madras joinable if \(T_1 + T_2 = 0\): that is, no horizontal shift is needed so that \(\sigma\) may be joined to \(\tau\) by the above procedure. Note that the modification made is local in this case: \(J(\tau, \sigma) \Delta (\tau \cup \sigma)\) contains at most twenty edges. See Figure 4.
Figure 3. Changes made near the vertex $Y$ in a polygon $\tau$ to produce $\tau_{\text{mod}}$ are depicted. The second column depicts $\tau$ around $Y$, with $Y$ indicated by a large disk or circle; a disk denotes a vertex that belongs to $\tau$ and a circle one that does not; the line segments denote edges of $\tau$. In the third column, the modified polygon $\tau_{\text{mod}}$ is shown in the locale of $Y$. The vertex $Y$ is shown as a black disk. Black line segments are edges in $\tau$ or $\tau_{\text{mod}}$ and dotted line segments on the right are edges in $\tau$ that are removed in the formation of $\tau_{\text{mod}}$. Several cases are not depicted. These may be labelled cases IIIa, IIIb, IIIci and IIIcii. In each case, the picture of $\tau$ and $\tau_{\text{mod}}$ is formed by reflecting the counterpart case II picture horizontally through $Y$. 
3.2. **Global join plaquettes are few.** Recall that in step two of the derivation of (2.7), the near injectivity of the multi-valued map $\Psi$ was argued as a consequence of the sparsity of macroscopic join plaquettes. We now present in Corollary 3.9 a rigorous counterpart to this sparsity assertion. In the rigorous approach, we use a slightly different definition to the notion of macroscopic join plaquette.

**Definition 3.7.** For $n \in 2\mathbb{N}$, let $\phi \in \text{SAP}_n$. A join plaquette $P$ of $\phi$ is called global if the two polygons comprising $\phi \Delta P$ may be labelled $\phi^\ell$ and $\phi^r$ in such a way that

- every vertex that is rightmost in the union $\phi^\ell \cup \phi^r$ belongs to $\phi^r$;
- the northeast vertex NE($\phi$) of $\phi$ belongs to $\phi^\ell$.

Write $\text{GJ}_\phi$ for the set of global join plaquettes of the polygon $\phi$.

**Proposition 3.8.** There exists $c > 0$ such that, for $n \in 2\mathbb{N}$ and any $k \in \mathbb{N}$,

$$|\text{SAW}_n| \geq c \exp \left\{ \frac{1}{2d}(2d - 1)^{-2}k \right\} \cdot \# \left\{ \phi \in \text{SAP}_n : |\text{GJ}_\phi| \geq k \right\}.$$ 

**Corollary 3.9.**

1. For all $\zeta > 0$ and $C_2 > 0$, there exists $C_3 > 0$ such that, for each $n \in \text{HCP}_\zeta$,

$$P_n \left( |\text{GJ}_\phi| \geq C_3 \log n \right) \leq n^{-C_2}.$$ 

2. There exists a constant $C_4 > 0$ such that, for each $n \in 2\mathbb{N}$,

$$P_n \left( |\text{GJ}_\phi| \geq C_4 n^{1/2} \right) \leq \exp \left\{ -n^{1/2} \right\}.$$
Definition 3.10. Let $\phi^1$ and $\phi^2$ denote two polygons. The two polygons are said to be strongly joinable if

- the pair $(\phi^1, \phi^2)$ is Madras joinable;
- every vertex that is uppermost in $\phi^1 \cup \phi^2$ is vertex of $\phi^1$;
- the value $y(ES(\phi^1))$ is the $y$-coordinate of some vertex in $\phi_2$.

The right-hand polygon pair in Figure 4 is strongly joinable but the left-hand pair is not.

The very short proof of the next lemma is omitted.

Lemma 3.11. Let $(\phi_1, \phi_2)$ be a pair of strongly joinable polygons of lengths $n$ and $m$. In the Madras join polygon $J(\phi_1, \phi_2) \in SAW_{n+m+16}$, the junction plaquette is a global join plaquette.

The next lemma will be used in the proof of Proposition 3.8.

Lemma 3.12. Let $n \in 2\mathbb{N}$ and $\phi \in SAP_n$. The upper and lower edges of any $P \in GJ_\phi$ belong to $\phi$ and the left and right edges do not. Writing $j \in [0,n]$ so that $ES(\phi) = \phi_j$, consider the two subpaths $\phi_{[0,j]}$ and $\phi_{[j,n]}$, the first starting at $NE(\phi) = 0$ and the second ending there. Each of these paths contains precisely one of the upper and lower edges of $P$.

Proof. Since $P \in GJ_\phi$, we may decompose $\phi \Delta P$ as $\phi^\ell \cup \phi^r$ in accordance with Definition 3.7. We then have that $NE(\phi) = \phi_j$ and $ES(\phi) = \phi_j$. The path $[0,n] \rightarrow \mathbb{Z} : j \rightarrow \phi_j$ leaves 0 to follow $\phi^\ell$ until passing through an edge in $P$ to arrive in $\phi^r$, tracing this polygon until passing back through the other horizontal edge of $P$ and following $\phi^\ell$ until returning to $NE(\phi)$. It is during the trajectory in $\phi^r$ that the visit to $ES(\phi)$ is made. \qed

Proof of Proposition 3.8. Let $\phi \in SAP_n$. Again setting $j \in [0,n]$ so that $ES(\phi) = \phi_j$, write $\phi^1 = \phi_{[0,j]}$ and $\phi^2 = \phi_{[j,n]}$. Writing $\mathcal{R}_z^2$ for reflection in the vertical ($e_2$-directed) line that passes through $z \in \mathbb{Z}^2$, define a map $\mathcal{S} : SAP_n \rightarrow SAW_n$ to be the concatenation

$$\mathcal{S}(\phi) = \phi^1 \circ \mathcal{R}_z^2(\phi^2).$$

(We have not defined concatenation but hope that the meaning is evident.) By Lemma 3.12, each of $\phi^1$ and $\phi^2$ traverses precisely one horizontal edge of each of $\phi$'s global join plaquettes. Set $r = \#GJ_\phi$ and enumerate $GJ_\phi$ by the sequence $(P^1, \ldots, P^r)$ (in an arbitrary order; for example, in the order in which $\phi^1$ traverses an edge of each plaquette). For each $j \in \{1, \ldots, r\}$, let $(s_j, f_j)$ denote the unique edge in $P^j$ traversed by $\phi^2$. Consider the path formed by modifying $\phi^2$ so that the one-step subpath $(s_j, f_j)$ is replaced by a three-step subpath from $s_j$ to $f_j$ that traverses the plaquette $P^j$ using its three edges other than $(s_j, f_j)$. The modification may be made iteratively for several choices of $j \in \{1, \ldots, r\}$, and the
outcome is independent of the order in which the modifications are made. In this way, we may define a modified path $\phi^{2,\kappa}$ for each $\kappa \subseteq \{1, \ldots, r\}$, under which the modified route is taken along plaquettes $P_j$ precisely when $j \in \kappa$. Note that $\phi^{2,\kappa}$ is a self-avoiding walk whose length exceeds $\phi^2$'s by $2|\kappa|$; it is self-avoiding because this walk differs from $\phi^2$ by several disjoint replacements of one-step subpaths by three-step alternatives, and, in each case, the two new vertices visited in the alternative route are vertices in $\phi^1$, and, as such, cannot be vertices in $\phi^2$.

Note further that the intersection of the edge-sets of $\phi^1$ and $\phi^{2,\kappa}$ equals $\cup_{j \in \kappa} E(P_j)$ (where the sets in the union are each singletons).

For each $\kappa \subseteq \{1, \ldots, r\}$, define $\mathcal{J}_\kappa(\phi) \in \text{SAW}_{n+2|\kappa|}$,

$$\mathcal{J}_\kappa(\phi) = \phi^1 \circ \mathcal{R}^{2}_{\text{ES}(\phi)}(\phi^{2,\kappa}).$$

Let $\delta \in (0, 1)$ be a parameter to be determined shortly, and let $k \in \mathbb{N}$. Consider the multi-valued map

$$\Psi : \left\{ \phi \in \text{SAP}_n : |\text{GJ}_\phi| \geq k \right\} \rightarrow \mathcal{P}(\text{SAW}_{n+2|\delta k|})$$

that associates to each $\phi \in \text{SAP}_n$ with $|\text{GJ}_\phi| \geq k$ the set

$$\Psi(\phi) = \left\{ \mathcal{J}_\kappa(\phi) : \kappa \subseteq \text{GJ}_\phi, \; |\kappa| = \lceil \delta k \rceil \right\},$$
where here we abuse notation and identify a subset of \( G_J \phi \) with its set of indices under the given enumeration of \( G_J \).

Note that, for some constant \( c = c(\delta) \) and for all \( k \in \mathbb{N} \),
\[
|\Psi(\phi)| = \left( \frac{|G_J|}{\delta k} \right) \geq \left( \frac{k}{\delta k} \right) \geq c \delta^{-\delta k}.
\]

Note that, for any \( \gamma \in \text{SAW}_{n+2[\delta k]} \), the preimage \( \Psi^{-1}(\gamma) \) is either the empty-set or a singleton. Indeed, if \( \gamma \in \Psi(\phi) \) for some \( \phi \in \text{SAP}_n \) with \( |G_J| \geq k \), then \( \phi \) may be recovered from \( \gamma \) as follows:

- the coordinate \( x(\text{ES}(\phi)) \) equals \( x(\gamma_{n+2[\delta k]})/2 \);
- the vertex \( \text{ES}(\phi) \) is the lowest among the vertices of \( \gamma \) having the above \( x \)-coordinate;
- setting \( j \in [0,n] \) so that \( \gamma_j \) is this vertex, consider the non-self-avoiding walk \( \gamma_{[0,j]} \circ \mathcal{R}_j \gamma_{[j,n]} \). This walk begins and ends at 0. There are at least \( k \) instances where the walk traverses an edge twice. In each case, the three-step journey that the walk makes in the steps preceding, during and following the second crossing of the edge follow three edges of a plaquette. Replace this journey by the one-step journey across the remaining edge of the plaquette, in each instance. The result is \( \phi \).

We may thus use the bound given in Subsection 2.1.1 to find that
\[
c_{n+2[\delta k]} = \#\text{SAW}_{n+2[\delta k]} \geq \delta^{-\delta k} \cdot \# \left\{ \phi \in \text{SAP}_n : |G_J| \geq k \right\}.
\]

By Proposition 2.3(1) and (3), there exists \( C > 0 \) such that \( c_{n+2[\delta k]}/c_n \leq C (2\mu)^2[\delta k] \) for all \( n, k \in \mathbb{N} \). Using also the connective constant bound \( \mu \leq 2d-1 \) and adjusting the value of \( c > 0 \), we find that
\[
c_n \geq c 4(2d-1)^2 \delta^{-\delta k} \cdot \# \left\{ \phi \in \text{SAP}_n : |G_J| \geq k \right\}.
\]

Setting \( \delta = (2e)^{-1}(2d-1)^{-2} \) completes the proof of Proposition 3.8.

**Proof of Corollary 3.9** Using
\[
P_n \left( |G_J| \geq k \right) = p_n^{-1} \cdot \# \left\{ \phi \in \text{SAP}_n : |G_J| \geq k \right\},
\]
alongside Proposition 3.8 and \( c_{n-1} \geq c_n/2d \), we find that
\[
W_{n-1}(\Gamma \text{ closes}) = \frac{2np_n}{c_{n-1}} \leq \frac{4dnc^{-1} \exp \left\{ -\frac{1}{2d}(2d-1)^{-2} k \right\}}{P_n \left( |G_J| \geq k \right)}.
\]

Since \( n \in \text{HCP}_\zeta \), \( W_{n-1}(\Gamma \text{ closes}) \geq n^{-\zeta} \). Thus,
\[
P_n \left( |G_J| \geq k \right) \leq 4d^{-1} n^{\zeta+1} \exp \left\{ -\frac{1}{2d}(2d-1)^{-2} k \right\}.
\]

□
Set \( C_3 = 2e(2d - 1)^2(C_2 + \zeta + 2). \) Then \( n \geq 2 \) implies that, for \( k \geq C_3 \log n, \)
\[
P_n(\|G \phi\| \geq k) \leq n^{-C_2}.
\]
Thus, we obtain Corollary 3.9(1). We use Lemma 2.4 in place of \( n \in \text{HCP}_\zeta \) to obtain Corollary 3.9(2).

\[\square\]

### 3.3. Left and right polygons

Let \( \phi \) be a polygon. Recall from Definition 2.1 the notation \( y_{\max}(\phi) \) and \( y_{\min}(\phi) \), as well as the height \( h(\phi) \) and width \( w(\phi) \).

Again we employ the notationally abusive parametrization \( \phi : [0, n] \rightarrow \mathbb{Z}^d \), where \( n \) is the length of \( \phi \), such that \( \phi_0 = \phi_n = \text{NE}(\phi) \) and \( \phi_1 = \text{NE}(\phi) - e_1 \). If \( j \in [0, n] \) is such that \( \phi_j = \text{SE}(\phi) \), note that \( \phi \) may be partitioned into two paths \( \phi_{[0,j]} \) and \( \phi_{[j,n]} \). (This is not the division into two paths used in the proof of Proposition 3.8, the outward journey is now to \( \text{SE}(\phi) \), not \( \text{ES}(\phi) \).) The two paths are edge-disjoint, and the first lies to the left of the second; indeed, writing \( H \) for the horizontal strip \( \{(x, y) \in \mathbb{R}^2 : y(\text{SE}(\phi)) \leq y \leq y(\text{NE}(\phi))\} \), the set \( H \setminus \bigcup_{i=0}^{n-1}[\phi_i, \phi_{i+1}] \) has two connected components, one of these, the right component, contains \( H \cap \{(x, y) \in \mathbb{R}^2 : x \geq C\} \) for large enough \( C \); and the union of the edges \( [\phi_i, \phi_{i+1}], j \leq i \leq n - 1 \), excepting the points \( \phi_j = \text{SE}(\phi) \) and \( \phi_n = \text{NE}(\phi) \), lies in the right component. It is thus natural to call \( \phi_{[0,j]} \) the left path, and \( \phi_{[j,n]} \), the right path. We call \( \phi \) left-long if \( j \geq n/2 \) and right-long if \( j \leq n/2 \).

For \( n \in 2\mathbb{N} \), let \( \text{SAP}^\text{left}_n \) denote the set of left polygons \( \phi \in \text{SAP}_n \) such that

- \( h(\phi) \geq w(\phi) \) (and thus, by a trivial argument, \( h(\phi) \geq n^{1/2} \)),
- \( \phi \) is left-long,
- and \( y(\text{ES}(\phi)) \leq \frac{1}{2}(y_{\min}(\phi) + y_{\max}(\phi)) \).

Let \( \text{SAP}^\text{right}_n \) denote the set of right polygons \( \phi \in \text{SAP}_n \) such that

- \( h(\phi) \geq w(\phi) \).

(The handedness notion refers not to properties of the defined sets \( \text{per se} \), but to the use we will shortly make of them: in Lemma 3.14, for example, we will consider the Madras join of an element of \( \text{SAP}^\text{left}_n \) and certain translates of \( \text{SAP}^\text{right}_m \). In this construction, we may think of the latter polygons as being joined to the former on the right.)

**Lemma 3.13.** For \( n \in 2\mathbb{N} \),

\[
|\text{SAP}^\text{left}_n| \geq \frac{1}{5} \cdot |\text{SAP}_n| \quad \text{and} \quad |\text{SAP}^\text{right}_n| \geq \frac{1}{7} \cdot |\text{SAP}_n|.
\]

**Proof.** For an element \( \phi \in \text{SAP}_n \) to satisfy \( \phi \in \text{SAP}^\text{left}_n \), it must satisfy three requirements. These may be satisfied as follows.

- The first property may be ensured by a right-angled counterclockwise rotation if it does not already hold.
• It is easy to verify that, when a polygon is reflected in a vertical line, the right path is mapped to become a sub-path of the left path of the image polygon. Thus, a right-long polygon maps to a left-long polygon under such a reflection. A polygon’s height and width are unchanged by either horizontal or vertical reflection, so the first property is maintained if a reflection is undertaken at this second step.

• The third property may be ensured if necessary by reflection in a horizontal line. The first property’s occurrence is not disrupted for the reason just noted. Could the reflection disrupt the second property? When a polygon is reflected in a horizontal line, NE and SE in the domain map to SE and NE in the image. The left path maps to the reversal of the left path, and similarly for the right path. Thus, any left-long polygon remains left-long when it is reflected horizontally. We see that the second property is stable under the reflection in this third step.

Each of the three operations has an inverse, and thus \( \#SAP_{n}^{\text{left}} \geq \frac{1}{8} \#SAP_{n} \). Clearly the second assertion in the lemma holds by considering only the first step above. □

**Lemma 3.14.** Let \( n, m \in 2\mathbb{N} \) and let \( \phi^{1} \in SAP_{n}^{\text{left}} \) and \( \phi^{2} \in SAP_{m}^{\text{right}} \).

Every value
\[
k \in \left[y(ES(\phi^{1})) - y_{\max}(\phi^{2}), y(ES(\phi^{1})) - y_{\max}(\phi^{2}) + \min\{n^{1/2}/2, m^{1/2}\} - 1\right]
\]
is such that \( \phi^{1} \) and some horizontal shift of \( \phi^{2} + ke_{2} \) is strongly joinable.

Write \( SJ(\phi^{1}, \phi^{2}) \) for the set of \( \bar{u} \in \mathbb{Z}^{2} \) such that the pair \( \phi^{1} \) and \( \phi^{2} + \bar{u} \) is strongly joinable. Then
\[
|SJ(\phi^{1}, \phi^{2})| \geq \min\{n^{1/2}/2, m^{1/2}\}.
\]

**Proof.** Note that whenever \( k \in \mathbb{Z} \) is such that the two intervals
\[
[y_{\min}(\phi^{1}), y_{\max}(\phi^{1})] \quad \text{and} \quad k + [y_{\min}(\phi^{2}), y_{\max}(\phi^{2})]
\]
intersect, there is some horizontal displacement \( j \in \mathbb{Z} \) such that \( \phi^{1} \) and \( \phi^{2} + (j, k) \) are Madras joinable. Note that \( \phi^{1} \in SAP_{n}^{\text{left}} \) satisfies \( h(\phi^{1}) \geq n^{1/2} \). Choices of \( k \in [y_{\max}(\phi^{1}) - y_{\max}(\phi^{2}) - n^{1/2}, y_{\max}(\phi^{1}) - y_{\max}(\phi^{2}) - 1] \) thus produce polygon pairs \( (\phi^{1}, \phi^{2} + ke_{2}) \) whose first element contains vertices that are more northerly than any of the second, and which are Madras joinable after a horizontal shift of the second element of the pair. Also owing to \( \phi^{1} \in SAP_{n}^{\text{left}} \), \( y(ES(\phi^{1})) \) is at most \( y_{\max}(\phi^{1}) - n^{1/2}/2 \). Thus, we obtain the lemma’s first assertion. The second follows immediately. □

We mention that it is irrelevant to this proof that we demand that elements of \( SAP_{n}^{\text{left}} \) be left-long. Indeed, Proposition 3.2 may be derived without using this property. The property will play an important role later, however, when we prove Theorem 1.5 using regulation global join polygons (which we now introduce).
3.4. Regulation global join polygons. We make the next definition in light of Lemma 3.14.

Definition 3.15. Let $k, \ell \in 2\mathbb{N}$ satisfy $k/2 \leq \ell \leq 35k$. Let $\text{RGJ}_{k,\ell}$ denote the set of regulation global join polygons (with length pair $(k, \ell)$), whose elements are formed by Madras joining the polygon pair $(\phi_1, \phi_2 + \vec{u})$, where

1. $\phi_1 \in \text{SAP}^\text{left}_k$;
2. $\phi_2 \in \text{SAP}^\text{right}_\ell$;
3. $y_{\max}(\phi^2 + \vec{u}) \in [y(\text{ES}(\phi^1)), y(\text{ES}(\phi^1)) + k^{1/2}/10 - 1]$;
4. and $(\phi_1, \phi_2 + \vec{u})$ is Madras joinable.

Let $\text{RGJ}$ denote the union of the sets $\text{RGJ}_{k,\ell}$ over all such choices of $(k, \ell) \in 2\mathbb{N} \times 2\mathbb{N}$.

Lemma 3.16. For such $k$ and $\ell$,

$$|\text{RGJ}_{k,\ell}| = \frac{1}{40} k^{1/2} |\text{SAP}^\text{left}_k| |\text{SAP}^\text{right}_\ell|.$$
Figure 7. Suppose that two global join plaquettes $P^1$ and $P^2$ of a polygon $\phi$ are such that, in the outward journey along $\phi$, from $\text{NE}(\phi)$ to $\text{ES}(\phi)$, $P^1$ is encountered before $P^2$, and that the order is maintained on the return. Then the return journey must visit some vertex twice. The outward journey is depicted in bold. The crossing of an edge of $P^1$ by the return forces the dashed future of the return into a bounded component of the complement of the existing path.

Proof. It suffices to argue that, for any $\phi \in \text{RGJ}_{k,\ell}$, there is a unique choice of $\phi^1 \in \text{SAP}^\text{left}_k$, $\phi^2 \in \text{SAP}^\text{right}_\ell$ and $\vec{u} \in \mathbb{Z}^2$ for which $\phi = J(\phi^1, \phi^2 + \vec{u})$. It is enough to determine the junction plaquette associated to the Madras join that forms $\phi$. By Lemma 3.14, any such polygon pair $(\phi^1, \phi^2 + \vec{u})$ is strongly joinable. Thus, by Lemma 3.11, the associated junction plaquette is a global join plaquette of $\phi$. By Lemma 3.12, each global join plaquette has one vertical edge traversed in the outward journey along $\phi$ from $\text{NE}(\phi)$ to $\text{ES}(\phi)$, and one traversed on the return journey. A short exercise discussed in Figure 7 that invokes planarity shows that the set of $\phi$’s global join plaquettes is totally ordered under a relation in which the upper element is both reached earlier on the outward journey and later on the return. From this, we infer that the map that sends a global join plaquette $P$ of $\phi$ to the length of the polygon in $\phi \Delta P$ that contains $\text{NE}(\phi) = 0$ is injective. Since this length must equal $k+8$ for any admissible choice of $\phi^1 \in \text{SAP}^\text{left}_k$, $\phi^2 \in \text{SAP}^\text{right}_\ell$ and $\vec{u} \in \mathbb{Z}^2$, this choice is unique, and we are done. \qed

Note further that the set of join locations stipulated by the third condition in Definition 3.15 is restricted to an interval of length of order $k^{1/2}$. The restriction causes the formula in Lemma 3.16 to hold. It may be that many elements of $\text{SAP}^\text{left}_k$ and $\text{SAP}^\text{right}_\ell$ have heights much exceeding $k^{1/2}$, so that, for pairs of such polygons, there are many more than an order of $k^{1/2}$ choices of translate for the second element that result in a strongly joinable polygon pair. The term regulation has been attached to indicate that a specific rule has been used to produce elements.
of RGJ and to emphasise that such polygons do not exhaust the set of polygons that we may conceive as being globally joined.

3.5. Polygon joining is almost injective. We begin proving Proposition 3.2 by reducing the result to the counterpart where \( p_{n+16} \) is bounded below instead of \( p_n \).

Lemma 3.17. For any \( \zeta > 0 \), there is a constant \( C_1 = C_1(\zeta) > 0 \) such that, for \( n \in 2\mathbb{N} \cap \text{HCP}_\zeta \),

\[
p_{n+16} \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j \in 2\mathbb{N} \cap [2^{i-1}, 2^i]} p_j p_{n-j},
\]

where \( i \in \mathbb{N} \) is chosen so that \( n \in 2\mathbb{N} \cap [2^{i+1}, 2^{i+4}] \).

Proof of Proposition 3.2. By [28, Theorem 7.3.4(c)], \( \lim_{n \to \infty} p_{n+2}/p_n = \mu^2 \). Thus, \( p_n \geq \mu^{-16} p_{n+16}/2 \) for all \( n \) sufficiently high, so that Lemma 3.17 implies Proposition 3.2 if the term \( C_1 \) is increased by a factor of \( 2 \mu^{16} \) (or possibly by a greater factor, so that the required bound is valid for all \( n \in 2\mathbb{N} \)).

Proof of Lemma 3.17. First of all, we mention that, though we hypothesised that \( n \in \text{HCP}_\zeta \), we will actually make use of \( n + 16 \in \text{HCP}_\zeta \). We may do so by increasing the value of \( \zeta > 0 \) by an arbitrarily small amount in light of (1.3) and the bounds \( p_{n+16}/p_n \geq \mu^{16}/2 \) and \( c_{n+16}/c_n \leq 2\mu^{16} \) that follow from Theorem 7.3.4(a) and (c) in [28].

Consider the multi-valued map \( \Psi : A \to \mathcal{P}(B) \), where

\[
A = \bigcup_{j \in 2\mathbb{N} \cap [2^{i-1}, 2^i]} \text{SAP}_j \times \text{SAP}_{n-j}^{\text{right}}, \quad \text{and} \quad B = \text{SAP}_{n+16} \cap \text{RGJ},
\]

that associates to each \( (\phi^1, \phi^2) \in \text{SAP}_j \times \text{SAP}_{n-j}^{\text{right}}, \ j \in 2\mathbb{N} \cap [2^{i-1}, 2^i] \), the collection of length-\( (n + 16) \) polygons formed by Madras joining \( \phi^1 \) and \( \phi^2 + \vec{u} \) as \( \vec{u} \) ranges over the subset of \( \text{SJ}_{(\phi^1, \phi^2)} \) specified in Definition 3.15.

Since \( n \geq 2^{i+1} \) and \( 2^{i-1} \leq j \leq 2^i \), we find from the third condition in this definition that

\[
|\Psi(\phi^1, \phi^2)| = |\text{SJ}_{(\phi^1, \phi^2)}| \geq \frac{1}{10} 2^{(i-1)/2}.
\]

Applying Lemma 3.13, we learn that the number of arrows in \( \Psi \) is at least \( \frac{1}{10} 2^{(i-1)/2} \cdot 2^{-4} \sum_{j=2^{i-1}}^{2^i} p_{n-j} p_j \). Note that the index set of the sum in this expression is actually \( 2\mathbb{N} \cap [2^{i-1}, 2^i] \), because \( p_k \) is zero for any odd index \( k \).

For a constant \( C_5 > 0 \) to be shortly specified, denote by

\[
H_{n+16} = \left\{ \phi \in \text{SAP}_{n+16} : \left| \text{GJ}_\phi \right| \geq C_5 \log n \right\}
\]

the set of length-\( (n + 16) \) polygons with a high number of global join plaquettes.
By Lemma 3.11 for \( \phi \in \textup{SAP}_{n+16} \), \(|\Psi^{-1}(\phi)| \leq |\textup{GJ}_\phi|\). Since \(|\textup{GJ}_\phi| \leq n + 16\) for all \( \phi \in \textup{SAP}_{n+16} \), we find that

\[
\max \{ \Psi^{-1}(\phi) : \phi \in \textup{SAP}_{n+16} \} \leq n + 16.
\]

The proof of Proposition 3.2 will be completed by considering two cases.

In the first case,

- at least one-half of the arrows in \( \Psi \) point to elements of \( H_{n+16} \);

in the second case, then, at least one-half of these arrows point to elements of \( \textup{SAP}_{n+16} \setminus H_{n+16} \).

**The first case.** The inequality

\[
|H_{n+16}| \cdot \max \left\{ |\Psi^{-1}(\phi)| : \phi \in H_{n+16} \right\} \geq \frac{1}{2} \cdot 2^{(i-1)/2 - 4} \frac{1}{10} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}
\]

holds because the left-hand side is an upper bound on the number of arrows in \( \Psi \) arriving in \( H_{n+16} \), which is at least one-half of the total number of arrows; and the latter quantity is at least the right-hand side. Since \( 16 \leq n \leq 2^{i+2} \), we learn that

\[
|H_{n+16}| \geq \frac{1}{10} 2^{-15/2} \cdot n^{-1/2} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}.
\]

Let \( C_6 \) be a constant satisfying \( C_6 \geq 4\zeta \). Noting that

\[
\Pr_{n+16} \left( |\textup{GJ}_\Gamma| \geq C_5 \log n \right) = \frac{|H_{n+16}|}{|\textup{SAP}_{n+16}|},
\]

we recall that \( n + 16 \in \textup{HCP}_\zeta \) and use Corollary 3.9(1) with \( C_2 = C_6/2 \) (insisting that \( C_5 > 0 \) equal the value of \( C_3 \) determined by the choice \( C_2 = C_6/2 \) and the value of \( \zeta \)) to find that

\[
|H_{n+16}| \leq (n + 16)^{-C_6/2} |\textup{SAP}_{n+16}|.
\]

If \( C_6 \geq 1 \), we find then that

\[
|\textup{SAP}_{n+16}| \geq n^{(C_6-1)/2} \cdot \frac{1}{10} 2^{-15/2} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}.
\]

We complete the proof of Lemma 3.17 in the first case by further insisting that \( C_6 > 2 \).

**The second case.** Note that, for \( \phi \in \textup{SAP}_{n+16} \setminus H_{n+16} \), \(|\Psi^{-1}(\phi)| \leq C_5 \log n\). The multi-valued map principle thus implies that

\[
\left| \textup{SAP}_{n+16} \cap \textup{RGJ} \setminus H_{n+16} \right| \cdot \max \left\{ |\Psi^{-1}(\phi)| : \phi \in \textup{SAP}_{n+16} \cap \textup{RGJ} \setminus H_{n+16} \right\}
\]
is at least \(2^{i/2 - 5} \frac{1}{10} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}\); using \(n \leq 2^{i+4}\), we find that
\[
p_{n+16} \geq \frac{1}{10} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j},
\]
and thus complete the proof of Lemma 3.17 in the second case also. \(\square\)

**Proof of Proposition 3.5.** Repeat the proof of Proposition 3.2 via Lemma 3.17, redefining the set \(H_{n+16}\) to be
\[
H_{n+16} = \left\{ \phi \in \text{SAP}_{n+16} : |GJ_{\phi}| \geq C_4 n^{1/2} \right\}.
\]
In the first case, apply Corollary 3.9(2) to find that
\[
|H_{n+16}| \leq \exp \left\{-n^{1/2}\right\} \cdot |\text{SAP}_{n+16}|,
\]
and
\[
|\text{SAP}_{n+16}| \geq \exp \left\{n^{1/2}\right\} \cdot \frac{1}{10} 2^{-7} n^{-1/2} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}.
\]
In the second case, (3.2) becomes
\[
|\text{SAP}_{n+16}| \geq \frac{1}{10} 2^{-6} C_4^{-1} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j}.
\]
The proof concludes as before. \(\square\)

Some explanation is needed as to why we called this proof a \((1/2, 1/2, 0)\)-argument, as we did in stating Proposition 3.5. It is more accurate to say that we carried out a \((1/2, 1, 0)\)-argument in which a one-half of value was lost in implementing step two because it was used that typical preimages under \(\Psi\) have cardinality at most of order \(n^{1/2}\). Using a slight variation of this informal notation, we may say that we performed a \((1/2, 1 - 1/2, 0)\)-argument.

### 3.6. Bounds on polygon number: the proof of Proposition 3.3

Whenever \(i \in \mathbb{N}\) and \(n \in [2^{i+1}, 2^{i+2}] \cap \text{HCP}_\zeta\),
\[
p_n \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j=2^{i-1}}^{2^i} p_j p_{n-j},
\]
by Proposition 3.2.

For \(R \in \mathbb{N}\) to be specified shortly, let \(b_i \in \mathbb{R}\) be given by
\[
\sup \left\{ b \in \mathbb{R} : \# \left\{ j \in 2\mathbb{N} \cap [2^{i-1}, 7 \cdot 2^{i-1}] : \theta_j \geq b \right\} \geq 2^{-R} \cdot |2\mathbb{N} \cap [2^{i-1}, 7 \cdot 2^{i-1}]| \right\}.
\]
It is readily checked that, to prove Proposition 3.3, it is enough to show that, for any \(\delta > 0\), the bound \(b_i \leq 3/2 - \delta\) holds for at most finitely many \(i \in \mathbb{N}\).
Note that
\[ \# \left\{ j \in 2\mathbb{N} \cap [2^{i-1}, 7 \cdot 2^{i-1}] : \theta_j \leq b_i \right\} \geq (1 - 2^{-R}) |2\mathbb{N} \cap [2^{i-1}, 7 \cdot 2^{i-1}]| \, . \tag{3.3} \]

Now fix \( i \in \mathbb{N} \) for which \( b_i \leq 3/2 - \delta \). It will be mildly convenient to be able to use \( b_i \geq 0 \). Proposition 2.3(3) for \( d = 2 \) implies this bound.

For \( n \in 2\mathbb{N} \cap [2^{i+1}, 2^{i+2}] \), let \( J_n \subseteq 2\mathbb{N} \) be given by
\[ J_n = \left\{ j \in 2\mathbb{N} \cap [2^{i-1}, 2^i] : \theta_j \leq b_i, \theta_{n-j} \leq b_i \right\} \, . \]

Note that, if \( R \geq 2 \),
\[ \# J_n \geq |2\mathbb{N} \cap [2^{i-1}, 2^i]| - 2 \cdot 2^{-R} |2\mathbb{N} \cap [2^{i-1}, 7 \cdot 2^{i-1}]| \geq 2^{i-2} - 3 \cdot 2^{i-R} \, . \]

Note that, if \( 2^{i+1} \leq n \leq 2^{i+2} \) and \( 2^{i-1} \leq j \leq 2^i \), then \( n - j \leq 2^{i+2} - 2^{i-1} \). Thus, using \( b_i \geq 0 \), we find that the bound
\[ p_n \geq \mu^n \frac{n^{1/2}}{C_1 \log n} \sum_{j \in J_n} (j(n - j))^{-b_i} \]

satisfied for any \( n \in [2^{i+1}, 2^{i+2}] \cap \text{HCP}_\zeta \) also implies that
\[ p_n \geq \mu^n \frac{n^{1/2}}{C_1 \log n} \left(2^{i-2} - 3 \cdot 2^{i-R}\right) \left(7 \cdot 2^{i-1} \cdot 2^i\right)^{-b_i} \, . \]

Assuming now that \( R \geq 4 \) and using \( 2^{i+1} \leq n \leq 2^{i+2} \), we find that
\[ p_n \geq \mu^n \frac{n^{1/2}}{C_1 \log n} \left(\frac{4}{11} \cdot \frac{n}{4}\right) \left(7 \cdot \frac{4}{11} \cdot \frac{n}{4}\right)^{-b_i} \geq \mu^n \frac{n^{3/2 - 2b_i}}{64C_1 \log n} \, . \]

where the second inequality invoked \( b_i \geq 0 \).

Recalling that \( p_n = n^{-\theta_n} \mu^n \), we learn that, for all \( n \in [2^{i+1}, 2^{i+2}] \cap \text{HCP}_\zeta \),
\[ \theta_n \leq -\frac{3}{2} + 2b_i + \frac{\log(64C_1) + \log \log n}{\log n} \, . \]

Since \( b_i \leq 3/2 - \delta \), note that \( \theta_n \leq b_i \) for all such \( n \).

Note that we have proved the assumption of the next claim when \( \alpha = 2 \).

**Claim.** Suppose that, for some \( \alpha \in [2, 31/2] \), it is known that \( \theta_n \leq b_i \) for all \( n \in [2^{i+1}, \alpha \cdot 2^{i+1}] \cap \text{HCP}_\zeta \). Then, whenever \( n \in [\alpha \cdot 2^{i+1}, (\alpha + 1/2) \cdot 2^{i+1}] \cap \text{HCP}_\zeta \), we have that \( \theta_n \leq b_i - \delta/2 \).

To demonstrate this, consider \( n \in 2\mathbb{N} \cap [\alpha \cdot 2^{i+1}, (\alpha + 1/2) \cdot 2^{i+1}] \cap \text{HCP}_\zeta \). When \( 2^i \leq j \leq 2^{i+1} \), note that \((\alpha - 1) \cdot 2^{i+1} \leq n - j \leq \alpha \cdot 2^{i+1} \).

Since \( 2 \leq \alpha \leq 31/2 \), we have that \( 2^{i+2} \leq n \leq 2^{i+5} \), and thus, by Proposition 3.2
\[ p_n \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j=2^i}^{2^{i+1}} p_j p_{n-j} \, . \]
Let \( J_{n,i+1} \) and \( L_{n,i+1} \) be given by
\[
J_{n,i+1} = \left\{ j \in 2N \cap [2^i, 2^{i+1}], \theta_j \leq b_i \right\}
\]
and
\[
L_{n,i+1} = \left\{ j \in 2N \cap [2^i, 2^{i+1}], n - j \in \text{HCP}_c \right\}.
\]
Note that, by (3.3), if \( R \geq 2 \),
\[
|2N \cap [2^i, 2^{i+1}] \setminus J_{n,i+1}| \leq 2^{-R}|2N \cap [2^{i-1}, 7 \cdot 2^{i-1}]| \leq 3 \cdot 2^{i-1-R},
\]
which is to say that
\[
|2N \cap [2^i, 2^{i+1}] \setminus J_{n,i+1}| \leq 3 \cdot 2^{-R} \cdot |2N \cap [2^i, 2^{i+1}]|.
\]
Thus, \( \#J_{n,i+1} \geq 2^{i-2} \) provided that \( R \geq 3 \).

Note that
\[
\#L_{n,i+1} = \left| \text{HCP}_c \cap [n - 2^{i+1}, n - 2^i] \right|.
\]
Note that \( \text{HCP}_c \) occupies a proportion of \( 2N \cap [0, n - 2^i] \) that exceeds \( 1 - 1/100 \) (by Hypothesis PCP), while the set \( 2N \cap [n - 2^{i+1}, n - 2^i] \) constitutes a proportion \( \frac{2^i}{n - 2^i} \geq 1/31 \) of the same set (since \( n \leq 2^{i+5} \)). Thus \( \text{HCP}_c \) occupies a proportion of \( 2N \cap [n - 2^{i+1}, n - 2^i] \) which is at least \( 1 - 31/100 \geq 1/2 \). Thus,
\[
\#L_{n,i+1} \geq \frac{1}{2} \left| 2N \cap [2^i, 2^{i+1}] \right|.
\]
Choosing \( R = 4 \), we see that
\[
|J_{n,i+1} \cap L_{n,i+1}| \geq \frac{5}{10} \left| 2N \cap [2^i, 2^{i+1}] \right|.
\]
For \( j \in J_{n,i+1}, \theta_j \leq b_i \). On the other hand, if \( j \in L_{n,i+1} \), then \( \theta_{n-j} \leq b_i \). Indeed, such \( j \) satisfies \( (\alpha - 1) \cdot 2^{i+1} \leq n - j \leq \alpha \cdot 2^{i+1} \); and, since \( \alpha - 1 \geq 1 \) and \( n - j \in \text{HCP}_c \), we see that the assumption of the claim applies to the index \( n - j \), and thus \( \theta_{n-j} \leq b_i \).

Using that \( b_i \geq 0 \), and using \( n \geq \alpha \cdot 2^i \geq 2^{i+1} \), we find that
\[
p_n \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j \in J_{n,i+1} \cap L_{n,i+1}} p_j p_{n-j}
\]
\[
\geq \mu^n \cdot \frac{n^{1/2}}{C_1 \log n} \sum_{j \in J_{n,i+1} \cap L_{n,i+1}} j^{-b_i} (n-j)^{-b_i}
\]
\[
\geq \mu^n \cdot \frac{n^{1/2}}{C_1 \log n} \cdot \frac{5}{10} \left| 2N \cap [2^i, 2^{i+1}] \right| \cdot (2^{i+1} \cdot \frac{31}{2} \cdot 2^{i+1})^{-b_i}
\]
\[
\geq \mu^n \frac{n^{3/2-2b_i}}{C_1 \log n} \cdot 2^{-8-4b_i}.
\]
Again using that \( b_i \leq 3/2 - \delta \), we see that
\[
\theta_n \leq -3/2 + 2b_i + \frac{\log \log n + \log C_1 + 14 \log 2}{\log n} \\
\leq b_i - \delta + \frac{\log \log n + \log C_1 + 14 \log 2}{\log n}.
\]

Thus, \( \theta_n \leq b_i - \delta/2 \). This completes the proof of the claim.

We may now harness the claim, because we know that its assumption when \( \alpha = 2 \) is satisfied. Since \( b_i \in [0, 3/2 - \delta] \), we learn that \( \theta_n \leq b_i - \delta/2 \) for all \( n \in 2N \cap [2^{i+2}, 2^{i+5}] \cap \text{HCP}_\zeta \). Note that
\[
\left| \frac{\text{HCP}_\zeta \cap [0, 7 \cdot 2^{i+2}]}{2N \cap [0, 7 \cdot 2^{i+2}]} \right| \geq 1 - \frac{1}{100} \quad \text{and} \quad \left| \frac{2N \cap [2^{i+2}, 7 \cdot 2^{i+2}]}{2N \cap [0, 7 \cdot 2^{i+2}]} \right| = \frac{6}{7} + O(2^{-i}),
\]
and so
\[
\left| \frac{\text{HCP}_\zeta \cap [2^{i+2}, 7 \cdot 2^{i+2}]}{2N \cap [2^{i+2}, 7 \cdot 2^{i+2}]} \right| \geq 1 - \left( \frac{7}{6} + O(2^{-i}) \right) \frac{1}{100} \geq 1 - \frac{1}{80}.
\]

Thus, \( b_{i+3} \leq b_i - \delta/2 \).

Note that, if this inequality holds for one sufficiently high choice of \( i \in \mathbb{N} \), it may be bootstrapped along the sequence \( \{ i + 3k : k \in \mathbb{N} \} \). A term for which \( b_{i+3k} \) is negative then arises, contradicting the positivity of this sequence. (Alternatively, we might remove the uses of \( b_i \geq 0 \) from the proof, and continue the bootstrapping to find that \( \limsup \frac{\log \log n \log \theta_j}{\log j} > 0 \), which amounts to an absurd superexponential growth for \( p_n \) on a subsequence.) Thus, \( b_i \leq 3/2 - \delta \) for at most finitely many values of \( i \in \mathbb{N} \), as it sufficed to show.

**Proof of Theorem 1.4.** We may mimic the proof of Proposition 3.3 to show that Proposition 3.5 implies that, for any \( \delta > 0 \), \( b_i \leq 1 - \delta \) for at most finitely many values of \( i \in \mathbb{N} \). We omit the details.

### 4. Some Further Generalities

#### 4.1. Notation

4.1.1. **Path reversal.** For \( n \in \mathbb{N} \) and a length \( n \) walk \( \gamma : [0, n] \to \mathbb{Z}^d \), the reversal \( \overleftarrow{\gamma} : [0, n] \to \mathbb{Z}^d \) of \( \gamma \) is given by \( \overleftarrow{\gamma}_j = \gamma_{n-j} \) for \( j \in [0, n] \).

4.1.2. **Walk length notation.** We write \( |\gamma| = n \) for the length of any \( \gamma \in \text{SAW}_n \).
4.1.3. The two-part decomposition. For certain purposes, it will be convenient to regard a walk $\gamma$ as composed of two walks emanating from $\text{NE}(\gamma)$ that are disjoint except at $\text{NE}(\gamma)$. As long as $\text{NE}(\gamma)$ is not the starting point or the endpoint of $\gamma$, then one of the two walks leaves $\text{NE}(\gamma)$ leftwards for $\text{NE}(\gamma) - e_1$. We label this path as the first of the two parts. We call this description the two-part decomposition of $\gamma$. We will use this decomposition for any dimension $d \geq 2$ in proving Theorem 1.2. Everything essential appears in the two-dimensional version that we now formulate (and which Figure 8 illustrates). Discussion of the variation in higher dimensions is deferred until Section 8.2.

\begin{definition}
Let $n \in \mathbb{N}$ and let $\gamma$ be a walk of length $n$. Suppose that $\gamma_j = \text{NE}(\gamma)$ with $j \in [0, n]$, $j \neq 0$, and further that $\gamma_{j-1} = \text{NE}(\gamma) - e_1$. Then define $\gamma^1$, the first part of $\gamma$, to be the element of $\text{SAW}_j$ given by the reversal of $\gamma_{[0,j]}$. Thus, $\gamma^1_0 = \text{NE}(\gamma)$, $\gamma^1_1 = \text{NE}(\gamma) - e_1$, and $\gamma^1_j = \gamma_0$. Define $\gamma^2$, the second part of $\gamma$, to be $\gamma_{[j,n]}$, and note that $\gamma^2_0 = \text{NE}(\gamma)$, $\gamma^2_1 = \text{NE}(\gamma) - e_2$ and $\gamma^2_{n-j} = \gamma_n$.

Suppose instead that $\gamma_j = \text{NE}(\gamma)$ with $j \in [0, n]$, $j \neq n$, with $\gamma_{j+1} = \text{NE}(\gamma) - e_1$. The reversal $\gamma^\rightarrow$ then satisfies the conditions of the preceding case. We define $\gamma^1$ and $\gamma^2$ to be the first and second parts of $\gamma^\rightarrow$.

If $\gamma_0 = \text{NE}(\gamma)$ and $\gamma_1 \neq \text{NE}(\gamma) - e_1$ (which in fact implies that $\gamma_1 = \text{NE}(\gamma) - e_2$), then set $\gamma^1 = \emptyset$ and $\gamma^2 = \gamma$. If $\gamma_n = \text{NE}(\gamma)$ and $\gamma_{n-1} \neq \text{NE}(\gamma) - e_1$, then set $\gamma^1 = \gamma^\rightarrow$ and $\gamma^2 = \emptyset$.

We write $\gamma = [\gamma^1, \gamma^2]$ to denote the two-part decomposition of $\gamma$.
\end{definition}

4.1.4. Polygonal invariance. The following trivial lemma will play an essential role. It is an important indication as to why polygons can be more tractable than walks.

\begin{lemma}
For $n \in 2\mathbb{N} + 1$ and $j \in [1, n-1]$, let $\chi : [0, n] \to \mathbb{Z}^d$ be a closing walk, and let $\chi'$ be the closing walk obtained from $\chi$ by the cyclic shift $\chi'(i) = \chi(j + i}$
\[ \text{mod } n + 1, \ i \in [0, n] \]. Then
\[ W_n \left( \Gamma \text{ is a translate of } \chi \right) = W_n \left( \Gamma \text{ is a translate of } \chi' \right). \]

4.1.5. Notation for walks not beginning at the origin. Let \( n \in \mathbb{N} \). We write \( \text{SAW}_n^* \) for the set of self-avoiding walks \( \gamma \) of length \( n \) (without stipulating the location \( \gamma_0 \)). We further write \( \text{SAW}_n^0 \) for the subset of \( \text{SAW}_n^* \) whose elements have northeast vertex at the origin. Naturally, an element \( \gamma \in \text{SAW}_n^0 \) is said to close (and be closing) if \( ||\gamma_n - \gamma_0|| = 1 \). The uniform law on \( \text{SAW}_n^0 \) will be denoted by \( W_n^0 \).

4.2. First parts and closing probabilities.

4.2.1. First part lengths with low closing probability are rare.

Lemma 4.3. Let \( n \in 2\mathbb{N} + 1 \) be such that, for some \( \alpha > 0 \), \( W_n \left( \Gamma \text{ closes} \right) \geq n^{-\alpha} \).
Then the set of \( i \in [0, n] \) for which
\[ \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = i \right\} \geq n^{\alpha + \delta} \cdot \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = i, \gamma \text{ closes} \right\} \]
has cardinality at most \( 2n^{1-\delta} \).

Proof. Note that \( W_n \left( \Gamma \text{ closes} \right) \geq n^{-\alpha} \) implies that
\[ |\text{SAW}_n| \leq n^\alpha \cdot \# \left\{ \gamma \in \text{SAW}_n : \gamma \text{ closes} \right\}. \]
Note also that this inequality holds when \( \text{SAW}_n \) is replaced by \( \text{SAW}_n^0 \).
We have that
\[ \# \left\{ \gamma \in \text{SAW}_n^0 : \gamma \text{ closes} \right\} = \sum_{j=0}^{n} \# \left\{ \gamma \in \text{SAW}_n^0 : \gamma \text{ closes, } |\gamma^1| = j \right\} \]
where, by Lemma 4.2, each term on the right-hand side has equal cardinality.
Write \( Q = Q_\delta \subseteq \{0, \ldots, n\} \) for the index set in the lemma’s statement. Note that
\[ |\text{SAW}_n^0| \geq |Q| \cdot n^{\alpha+\delta} \cdot \frac{1}{n+1} \# \left\{ \gamma \in \text{SAW}_n^0 : \gamma \text{ closes} \right\} \]

\[ \geq |Q| \cdot \frac{1}{2} n^{\alpha-1+\delta} \# \left\{ \gamma \in \text{SAW}_n^0 : \gamma \text{ closes} \right\}. \]
Thus, \( |Q| \cdot \frac{1}{2} n^{\alpha-1+\delta} \leq n^\alpha \). \( \square \)

4.2.2. Possible first parts and their conditional closing probabilities. For \( n \in \mathbb{N} \), let \( \Phi_n \subseteq \text{SAW}_n \) denote the set of walks \( \gamma : [0, n] \to \mathbb{Z}^2 \) that satisfy

- \( \gamma_0 = 0 \) and \( \gamma_1 = -e_1 \);
- \( y(\gamma_i) \leq 0 \) for all \( i \in [0, n] \);
- \( \gamma_i \notin \mathbb{N} \times \{0\} \) for any \( i \in [1, n] \).
Note that $\Phi_n$ may be characterized as the set of length $n$ walks $\gamma$ for which $\text{NE}(\gamma) = \gamma_0 = 0$ and $\gamma_1 = -e_1$. We wish to view $\Phi_n$ as the set of possible first parts of walks $\phi \in \text{SAW}_m$ of some length $m$ that is at least $n$. However, as Figure 9 illustrates, for given $m > n$, only some elements of $\Phi_n$ appear as such first parts, and we now record notation for the set of such elements. Write $\Phi_{n,m} \subseteq \Phi_n$ for the set of $\gamma \in \Phi_n$ for which there exists an element $\phi \in \text{SAW}_{m-n}$ with $\text{NE}(\phi) = 0$ such that $[\gamma, \phi]$ is the two-part decomposition of some element $\chi \in \text{SAW}_m$ with $\text{NE}(\chi) = 0$.

In this light, we now define the conditional closing probability $q_{n,m} : \Phi_{n,m} \to [0,1]$, 

\[ q_{n,m}(\gamma) = W^0_m \left( \Gamma \text{ closes} \left| \Gamma^1 = \gamma \right. \right), \]

where here $m, n \in \mathbb{N}$ satisfy $m > n$; note that since $\gamma \in \Phi_{n,m}$, the event in the conditioning on the right-hand side occurs for some elements of $\text{SAW}_m$, so that the right-hand side is well-defined.

We also identity a set of first parts with high conditional closing probability: for $\alpha > 0$, we write 

\[ \text{Hi} \Phi^\alpha_{n,m} = \left\{ \gamma \in \Phi_{n,m} : q_{n,m}(\gamma) > m^{-\alpha} \right\}. \]

5. **Proving closing probability upper bounds: the snake method**

In this section, we present in a general form a proof-by-contradiction technique, which we call the *snake method*, that was already employed in [13]. We will later use it in two different ways, to prove Theorem 1.2 and Theorem 1.5.

The snake method is used to prove upper bounds on the closing probability, and assumes to the contrary that to some degree this probability has slow decay. For the technique to be used, two ingredients are needed.

1. A charming snake is a walk or polygon $\gamma$ many of whose subpaths beginning at $\text{NE}(\gamma)$ have high conditional closing probability, when extended by some common length. It must be shown that charming snakes are not too atypical.
A charming snake is then shown to generate huge numbers of alternative self-avoiding walks. These alternatives overwhelm the polygons in number and show that the closing probability is very small, contradicting the assumption.

The technique used to carry out the first step may depend strongly on the context, while the second step may be performed using a general tool, valid in any dimension \( d \geq 2 \), that we present in this section. Indeed, the snake method including this second step tool was used in [13] to show that the closing probability is bounded above by \( n^{-1/4 + o(1)} \). The second step draws inspiration from the notion that reflected walks offer alternatives to closing (or near closing) ones that appears in Madras’ derivation [27] of lower bounds of moments of the endpoint distance under \( W_n \).

5.1. The general apparatus of the method.

5.1.1. Parameters. The snake method has three exponent parameters:
- the inverse charm \( \alpha > 0 \);
- the snake length \( \beta \in (0,1) \);
- and the charm deficit \( \eta \in (0,\beta) \).

It has two index parameters:
- \( n \in 2\mathbb{N} + 1 \) and \( \ell \in \mathbb{N} \), with \( \ell \leq n \).

5.1.2. Charming snakes. Here we define these creatures.

**Definition 5.1.** Let \( \alpha > 0 \), \( n \in 2\mathbb{N} + 1 \), \( \ell \in [0,n] \), \( \gamma \in \Phi_{\ell,n} \), and \( k \in [0,\ell] \) with \( \ell - k \in 2\mathbb{N} \). We say that \( \gamma \) is \((\alpha, n, \ell)\)-charming at index \( k \) if

\[
W_{k+n-\ell}^0 \left( \Gamma \text{ closes} \left| |\Gamma^1| = k, \Gamma^1 = \gamma_{[0,k]} \right. \right) > n^{-\alpha}.
\]

(The event that \(|\Gamma^1| = k\) in the conditioning is redundant and is recorded for emphasis.) Note that an element \( \gamma \in \Phi_{\ell,n} \) is \((\alpha, n, \ell)\)-charming at index \( k \) if in selecting a length \( n-\ell \) walk beginning at 0 uniformly among those choices that form the second part of a walk whose first part is \( \gamma_{[0,k]} \), the resulting \( (k+n-\ell) \)-length walk closes with probability exceeding \( n^{-\alpha} \). (Since we insist that \( n \) is odd and that \( \ell \) and \( k \) share their parity, the length \( k+n-\ell \) is odd; the condition displayed above could not possibly be satisfied if this were not the case.) Note that, for any \( \ell \in [0,n] \), \( \gamma \in \Phi_{\ell,n} \) is \((\alpha, n, \ell)\)-charming at the special choice of index \( k = \ell \) precisely when \( \gamma \in \text{Hi} \Phi_{\ell,n}^\alpha \).

For \( n \in 2\mathbb{N} + 1 \), \( \ell \in [0,n] \), \( \alpha, \beta > 0 \) and \( \eta \in (0,\beta) \), define the charming snake set

\[
\mathcal{CS}_{\beta,\eta}^{\alpha,\ell,n} = \left\{ \gamma \in \Phi_{\ell,n} : \gamma \text{ is } (\alpha, n, \ell)\text{-charming} \right\}
\]
for at least $n^{\beta-\eta}/4$ values of $j \in [\ell - n^\beta, \ell]$. 

For any element of $\gamma \in \Phi_{\ell,n}$, think of an extending snake consisting of $n^\beta + 1$ terms $(\gamma_{[0,\ell-n^\beta]}, \gamma_{[\ell-n^\beta+1]}, \cdots, \gamma_{[0,\ell]})$. If $\gamma \in \mathbb{CS}_{\beta,\eta}^{\alpha,\ell,n}$, then there are many charming terms in this snake.

5.1.3. A general tool for the method’s second step. For the snake method to produce results, we must work with a choice of parameters for which $\beta - \eta - \alpha > 0$. (The method could be reworked to handle equality in some cases.) Here we present the general tool for carrying out the method’s second step. This technique was already presented in [13, Lemma 5.8], and our treatment differs only by using notation adapted for the snake method with general parameters.

The tool asserts that, if $\beta - \eta - \alpha > 0$ and even a very modest proportion of snakes are charming, then the closing probability drops precipitously.

**Theorem 5.2.** Let $d \geq 2$. Set $c = 2^{\frac{1}{5(4d+1)}} > 1$. If $\delta = \beta - \eta - \alpha > 0$ and

$$P_{n+1} \left( \Gamma_{[0,\ell]} \in \mathbb{CS}_{\beta,\eta}^{\alpha,\ell,n} \right) \geq c^{-n^{\delta}/2}, \tag{5.1}$$

then

$$W_n \left( \Gamma \text{ closes} \right) \leq 2(n+1)c^{-n^{\delta}/2}.$$ 

Note that since the closing probability is predicted to have polynomial decay, the hypothesis (5.1) is never satisfied in practice. For this reason, the snake method will always involve argument by contradiction, with (5.1) being established under a hypothesis that the closing probability is, to some degree, decaying slowly.

5.2. A charming snake creates huge numbers of reflected walks. Here is the principal component of the proof of Theorem 5.2.

**Proposition 5.3.** Let $d \geq 2$. Set $\delta = \beta - \eta - \alpha$ and suppose that $\delta > 0$. Let $\phi \in \mathbb{CS}_{\beta,\eta}^{\alpha,\ell,n}$. Again writing $c = 2^{\frac{1}{5(4d+1)}} > 1$, we have that

$$\# \left\{ \gamma \in \mathbb{SAW}_n^* : \gamma_{[0,\ell]} = \phi \right\} \geq c^{n^{\delta}} \cdot \# \left\{ \gamma \in \mathbb{SAW}_n^* : \text{NE}(\gamma) = 0, \gamma^1 = \phi \right\}.$$ 

Note here that walks beginning with the reversal of an element $\phi \in \mathbb{SAW}_\ell$ will necessarily not begin at the origin, and thus we employ the notation introduced in Subsection 4.1.5.

**Proof of Proposition 5.3.** Let $W$ denote the set of walks $\gamma$ of length $n - \ell$ that originate at 0 and for which $\text{NE}(\gamma) = 0$. This set is not the same as $\Phi_{n-\ell}$, because we do not stipulate that $\gamma_1 = -e_1$: indeed, we will be using that $W$ contains all possible length $n - \ell$ walks that form the second (rather than the first) part of the two-part decomposition of some walk of at least this length. Let $P$ denote the uniform measure on the set $W$. We will denote by $\Gamma$ a random variable distributed
according to $P$. In particular, $\Gamma$ is contained in the lower half-space including the origin.

We now extend the notion of closing walk by saying that $\gamma'$ closes $\gamma$ if $\gamma_0 = \gamma_0'$ and the endpoints of $\gamma$ and $\gamma'$ are adjacent. We say that $\gamma'$ avoids $\gamma$ if no vertex except $\gamma_0$ is visited by both $\gamma'$ and $\gamma$.

We are given $\phi \in \Phi_{\ell,n}$ such that $\phi \in C_{\alpha,\ell,n}^{\alpha,\ell,n}$. By definition, we may find indices $j_1 < j_2 < \ldots < j_{n^\beta - \eta/4}$ lying in $[\ell - n^\beta, \ell]$ at each of which $\phi$ is $(\alpha, n, \ell)$-charming.

For $1 \leq i \leq n^{\beta - \eta}/4$, define the events

$$A_i = \left\{ \Gamma \text{ avoids } \phi_{[0,j_i]} \right\} \quad \text{and} \quad C_i = \left\{ \Gamma \text{ closes } \phi_{[0,j_i]} \right\}.$$ 

Also, define the set $A = \{ \gamma \in W : \gamma \text{ avoids } \phi_{[0,\ell]} \}$. Since $\phi$ is $(\alpha, n, \ell)$-charming at index $j_i$,

$$P\left( \Gamma \text{ closes } \phi_{[0,j_i]} \mid \Gamma \text{ avoids } \phi_{[0,j_i]} \right) = P(C_i \mid A_i) > n^{-\alpha} . \quad (5.2)$$

Write $k = \lceil 4d n^\alpha \rceil$ and note that $k \leq n^{1/2}/2$. Any realization $\Gamma \in W$ is in at most $2d$ events $C_i$. Hence, by [5.2] and the $A_j$ being decreasing,

$$2d \geq \sum_{i=1}^{k} P(C_i) \geq \sum_{i=1}^{k} P(C_i \mid A_i) \cdot P(A_k) \geq 4d P(A_k) .$$

Therefore, $P(A_k) \leq \frac{1}{2}$. If the procedure is repeated for indices between $k + 1$ and $2k$, one obtains

$$2d \geq \sum_{i=k+1}^{2k} P(C_i \mid A_k) \geq \sum_{i=k+1}^{2k} P(C_i \mid A_i) \cdot P(A_{2k} \mid A_k) \geq 4d P(A_{2k} \mid A_k) ,$$

and thus $P(A_{2k} \mid A_k) \leq 1/2$. Since $A_{2k} \subset A_k$, we find

$$P(A_{2k}) = P(A_k)P(A_{2k} \mid A_k) \leq \frac{1}{4} .$$

In these inequalities, we see the powerful bootstrap mechanism at the heart of the snake method, demonstrating that $P(A_{i+1})$ is at most one-half of $P(A_i)$. The mechanism works because the method’s definitions imply that all walk extensions are of common length $n - \ell$, and the avoidance conditions are monotone (i.e., the events $A_i$ are decreasing).

Indeed, the procedure may be repeated $\lfloor \frac{n^{\beta - \eta}}{4k} \rfloor \geq \frac{n^{\beta - \eta} - \alpha}{4(4d+1)}$ times. Recalling that $\phi = \phi_{[0,\ell]}$, we obtain

$$\frac{|A|}{|W|} = P(\Gamma \text{ avoids } \phi) \leq P\left( A_{n^{\beta - \eta/4}} \right) \leq 2^{\lfloor \frac{n^{\beta - \eta}}{4k} \rfloor} \leq 2^{\frac{n^{\beta - \eta} - \alpha}{4(4d+1)}},$$

for $n$ high enough.
The set $A$ contains all length $n - \ell$ walks $\gamma$ for which $[\phi, \gamma]$ is the two-part decomposition of a walk of length $n$ with $\text{NE} = 0$. Thus,

$$|A| = \# \left\{ \gamma \in \text{SAW}_n^* : \text{NE}(\gamma) = 0, |\gamma| = \ell, \gamma^1 = \phi \right\}.$$ 

On the other hand, for $\gamma \in W$, consider the walk obtained by concatenating three paths (and illustrated in Figure 10): the reversal $\leftarrow \phi$ of $\phi$; the edge $e_2$; and the $e_2$-translation of the reflection of $\gamma$ in the horizontal axis. The walk that results has length $n + 1$ and is self-avoiding. By deleting the last edge of such walks, we obtain at least $|W|/2d$ walks of length $n$, each of which follows $\leftarrow \phi$ in its first $\ell$ steps. Thus,

$$\# \left\{ \gamma \in \text{SAW}_n^* : \gamma_{[0, \ell]} = \leftarrow \phi \right\} \geq |W|/2d.$$ 

The three preceding displayed equations combine to complete the proof of Proposition 5.3. \hfill \Box

**Proof of Theorem 5.2.** Write $CS = CS_{\phi, \eta}^{n, \ell, n}$. Now using Proposition 5.3 in the second and (5.1) in the fourth inequalities,

$$|\text{SAW}_n| \geq \sum_{\phi \in CS} \# \left\{ \gamma \in \text{SAW}_n : \gamma_{[0, \ell]} = -\phi + \leftarrow \phi \right\}.$$
\[= \sum_{\phi \in \text{CS}} \#\left\{ \gamma \in \text{SAW}^n_\phi : [0, \ell_1] = \phi \right\} \]
\[\geq c^{n^\beta} \cdot \#\left\{ \gamma \in \text{SAW}^n_1 : \text{NE}(\gamma) = 0, \gamma_1 \in \text{CS} \right\} \]
\[\geq c^{n^\beta} \cdot \#\left\{ \gamma \in \text{SAW}^n_1 : \text{NE}(\gamma) = 0, \gamma_1 \in \text{CS}, \gamma \text{ closes} \right\} \]
\[= c^{n^\beta} \cdot \#\left\{ \gamma \in \text{SAP}_{n+1} : [0, \ell_1] \in \text{CS} \right\} \geq c^{n^\beta/2} \cdot |\text{SAP}_{n+1}|. \]

Thus, \(p_{n+1}/c_n \leq e^{-n^\delta/2}.\) We find that
\[W_n(\Gamma \text{ closes}) = 2(n+1)p_{n+1}/c_n \leq 2(n+1)e^{-n^\delta/2}. \]

\[\square\]

6. The snake method applied via Gaussian pattern fluctuation

In \[\text{[14, Theorem 1.1]}, \] it is proved that \(W_n(\Gamma \text{ closes}) \leq n^{-1/4+o(1)}.\) The technique used, which is the snake method with the first step carried out using Gaussian pattern fluctuation, clearly cannot prove faster decay on the closing probability than \(n^{-1/2}.\) We now rework the method to prove the assertion \(\text{Theorem 1.2(1)}\) that this faster decay rate can be obtained. The argument is not dimension-dependent, and the assumption that \(d \geq 2\) is in force throughout this section.

This application of the snake method is probabilistic, with the closing probability being discussed without reference to the polygon and walk numbers and the relating equation \([1.3].\) As such, the argument does not fit in the framework of \((\theta^{(1)}, \theta^{(2)}, \theta^{(3)})\)-arguments introduced in Section 2.5.

As we have stated, we will prove the result by assuming that its conclusion fails and seeking a contradiction. By a relabelling of \(\varepsilon > 0\), we may express the premise that the conclusion fails in the form that, for some \(\varepsilon > 0\) and infinitely many \(n \in 2\mathbb{N} + 1,\)
\[W_n(\Gamma \text{ closes}) \geq n^{-1/2+4\varepsilon}. \tag{6.1}\]

Henceforth, suppose that the pair \((n, \varepsilon)\) satisfies \((6.1).\) We may further assume that \(n\) satisfies lower bounds determined by \(\varepsilon.\)

The snake method will be applied with parameter choices \(\alpha = 1/2 - 2\varepsilon, \beta = 1/2\) and \(\eta = 0.\) One index parameter \(n \in 2\mathbb{N} + 1\) has just been set, and the other \(\ell \in \mathbb{N}\) will be specified momentarily. With these choices, we will argue that the hypothesis \((5.1)\) is comfortably satisfied, with charming snakes being the norm.

**Proposition 6.1.** There exist positive constants \(C\) and \(c\) such that, if \(n \in 2\mathbb{N} + 1\) and \(\varepsilon > 0\) satisfy \((6.7),\) then
\[P_{n+1}(\Gamma [0, \ell] \in \text{CS}^{1/2,2,\ell,n}) \geq 1 - C \exp \left\{ -c(\log n)^{1/2} \right\}. \]
Proof of Theorem 1.2. Note that $\delta = \beta - \eta - \alpha$ equals $2\varepsilon$ and is indeed positive. The conclusion of Theorem 5.2 contradicts (6.1) if $n$ is high enough. Thus (6.1) is false for all $n$ sufficiently high. Since $\varepsilon > 0$ may be chosen arbitrarily small, we are done.

6.1. Setting the snake method parameter $\ell$. Applying Lemma 4.3 with $\alpha = 1/2 - 4\varepsilon$ and $\delta = \varepsilon$, and noting that the assumption $n \geq 4^{1/\varepsilon}$ ensures that $2n^{1-\varepsilon} < \# [n/4, 3n/4]$, we find that we may select $\ell$ to lie in $[n/4, 3n/4]$ and to satisfy

$$\# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell \right\} < n^{1/2-3\varepsilon} \cdot \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell, \gamma \text{ closes} \right\},$$

or equivalently

$$W_n^0 \left( \Gamma \text{ closes} \left| |\Gamma^1| = \ell \right) > n^{-1/2+3\varepsilon}.$$

Lemma 6.2.

Let $n+1 \left( \Gamma_{[0,\ell]} \not\in \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon} \right)$ \leq n^{-\varepsilon}.

Proof. Note that $\Gamma_{[0,\ell]}$ under $P_{n+1}$ shares its law with the first part $\Gamma^1$ under $W_n^0(\cdot \left| \Gamma \text{ closes}, |\Gamma^1| = \ell \right)$. For this reason, the statement may be reformulated

$$W_n^0 \left( \Gamma^1 \not\in \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon} \left| |\Gamma^1| = \ell, \Gamma \text{ closes} \right) \right) \leq n^{-\varepsilon}. \quad (6.2)$$

To derive (6.2), set $p$ equal to its left-hand side. Note that

$$\# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell \right\} \geq \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell, \gamma^1 \not\in \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon} \right\} = \sum_{\phi \in \Phi_{\ell,n} \setminus \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon}} \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \phi \right\} \geq \sum_{\phi \in \Phi_{\ell,n} \setminus \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon}} n^{1/2-2\varepsilon} \cdot \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \phi, \gamma \text{ closes} \right\} = n^{1/2-2\varepsilon} \cdot \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell, \gamma^1 \not\in \text{Hi} \Phi_{\ell,n}^{1/2-2\varepsilon}, \gamma \text{ closes} \right\} = n^{1/2-2\varepsilon} \cdot p \cdot \# \left\{ \gamma \in \text{SAW}_n^0 : |\gamma^1| = \ell, \gamma \text{ closes} \right\},$$

whence

$$n^{1/2-2\varepsilon} \cdot p \leq W_n^0 \left( \Gamma \text{ closes} \left| |\Gamma^1| = \ell \right) \right)^{-1},$$

whose right-hand side we know to be at most $n^{1/2-3\varepsilon}$. Thus, $p \leq n^{-\varepsilon}$, and we have verified (6.2).
Figure 11. An example of type I and II patterns for \( d = 2 \).

6.2. Patterns and shells. Patterns are local configurations in self-avoiding walks that are the subject of a famous theorem \[22\] due to Kesten that we will shortly state. For our present purpose, we identify two particular patterns.

**Definition 6.3 (Type I/II patterns).** A pair of type I and II patterns is a pair of self-avoiding walks \( \chi^I, \chi^{II} \), both contained in the cube \([0,3]^d\), with the properties that

- \( \chi^I \) and \( \chi^{II} \) both visit all vertices of the boundary of \([0,3]^d\),
- \( \chi^I \) and \( \chi^{II} \) both start at \((1,3,1,\cdots,1)\) and end at \((2,3,1,\cdots,1)\),
- the length of \( \chi^{II} \) exceeds that of \( \chi^I \) by two.

Figure 11 contains examples of such patterns for \( d = 2 \). The existence of such pairs of walks for any dimension \( d \geq 2 \) may be easily checked, and no details are given here. Fix a pair of type \( I \) and \( II \) patterns henceforth.

A pattern \( \chi \) is said to occur at step \( k \) of a walk \( \gamma \) if \( \gamma_{[k,k+|\chi|]} \) is a translate of \( \chi \) (where recall that \(|\chi|\) is the length of \( \chi \)). A slot of \( \gamma \) is any translate of \([0,3]^d\) containing \( \gamma_{[k,k+|\chi|]} \) where a pattern \( \chi \) of type \( I \) or \( II \) occurs at step \( k \) of \( \gamma \). Note that the slots of \( \gamma \) are pairwise disjoint.

In \[13\], the notion of shell was introduced. A shell is an equivalence class of self-avoiding walks under the relation that two walks are identified if one may be obtained from the other by changing some patterns of type \( I \) to be of type \( II \) and vice versa. The walks in a given shell share a common set of slots, but they are of varying lengths. The shell of a given walk \( \gamma \) is denoted \( \varsigma(\gamma) \).

Consider a vertex \( v \in \mathbb{Z}^d \) that does not lie in the vertex set of any walk having a certain shell (nor elsewhere in any slot of this shell). For a walk having this shell, we may consider the collection of nearest neighbour paths emanating from \( v \) whose only intersection with the vertex set of the walk is at the path’s endpoint. The subset of the walk vertex set given by vertices that are the endpoints of such paths is independent of the walk in the given shell. This observation has the following basic consequence, which is crucial to our reasons for considering shells.

Consider two walks \( \gamma^1 \) and \( \gamma^2 \) with \( \gamma^1_0 = \gamma^2_0 \). Recall that \( \gamma^2 \) avoids \( \gamma^1 \) if the two walks both visit no other vertex.

**Lemma 6.4.** For some \( m \in \mathbb{N} \), let \( \gamma \in \text{SAW}_m \) and let \( \gamma' \in \varsigma(\gamma) \). A walk beginning at \( \gamma_0 \) avoids \( \gamma \) if and only if it avoids \( \gamma' \).
The reader may now wish to view Figure 12 and its caption for an expository overview of the snake method via Gaussian pattern fluctuation. We mention also that this Gaussian fluctuation has been utilized in [21] to prove a $n^{1/2-o(1)}$ lower bound on the absolute value of the writhe of a typical length $n$ polygon.

We will make some use of the notion of shell, but will predominantly consider a slightly different definition, the $(n+1)$-local shell, which we now develop. This new notion concerns polygons rather than walks.

Let $n \in 2\mathbb{N} + 1$. (The definitions may of course be formulated for any such $n$, but they have been arranged to suit our application, in which $n$ is the given snake method parameter.) Define an equivalence relation $\sim$ on $\text{SAP}_{n+1}$ as follows. For any $\gamma \in \text{SAP}_{n+1}$, let $\gamma^{\text{empty}}$ denote the polygon in the shell of $\gamma$ that has no type $\text{II}$ patterns, (formed by switching every type $\text{II}$ pattern of $\gamma$ into a type $\text{I}$ pattern). Thus, $\gamma^{\text{empty}} \in \text{SAP}_{n+1-2T_{\text{II}}(\gamma)}$, where $T_{\text{II}}(\gamma)$ denotes the total number of type $\text{II}$ patterns in $\gamma$. A type $\text{II}$ pattern contains thirteen edges (in $d = 2$; at least this number in higher dimensions), and these patterns are disjoint, so $T_{\text{II}}(\gamma) \leq (n+1)/13$. Thus, the length of $\gamma^{\text{empty}}$ is at least $11(n+1)/13$. Let $S_1$ denote the set of slots in $\gamma$ that are slots in $\gamma_{(0, (n+1)/10)}^{\text{empty}}$ and, writing $f^{\text{empty}}$ for the length of $\gamma^{\text{empty}}$, let $S_2$ denote the set of slots in $\gamma$ that are slots in $\gamma_{[f^{\text{empty}}-(n+1)/10, f^{\text{empty}}]}$. Note that $S_1$ and $S_2$ are disjoint. We further write $N_{\text{I}}(\gamma)$ and $N_{\text{II}}(\gamma)$ for the number of patterns of the given type in the slots $S_1 \cup S_2$, and $N_{\text{I}}^1(\gamma)$ and $N_{\text{II}}^2(\gamma)$ for the number of type $\text{I}$ patterns occupying slots in $S_1$ and in $S_2$; and similarly for $N_{\text{I}}^1(\gamma)$ and $N_{\text{II}}^2(\gamma)$.

For $\gamma, \gamma' \in \text{SAP}_{n+1}$, we say that $\gamma \sim \gamma'$ if $\gamma'$ may be obtained from $\gamma$ by relocating the type $\text{II}$ patterns of $\gamma$ contained in the set of slots $S_1 \cup S_2$ for $\gamma$ to another set of locations among these slots. The relation $\sim$ is an equivalence relation, because the polygon $\gamma^{\text{empty}}$ formed by filling all the slots of $\gamma$ with type $\text{I}$ patterns is shared by related polygons, so that the value of $S_1 \cup S_2$ is equal for such polygons. We call the equivalence classes $(n+1)$-local shells: the parameter $n+1$ appears to denote the common length of the member polygons, and the term local is included to indicate that members of a given class may differ only in locations that are close to the origin (in the chemical distance, along the polygon). Complementing the notation $c(\gamma)$ for the shell of $\gamma$, write $c_{n+1}^{\text{loc}}(\gamma)$ for the $(n+1)$-local shell of $\gamma \in \text{SAP}_{n+1}$.

For $\delta > 0$, write $\mathcal{G}_{n+1, \delta}$ for the set of $(n+1)$-local shells $\sigma \subseteq \text{SAP}_{n+1}$ such that each of the quantities $|S_1(\sigma)|$, $|S_2(\sigma)|$, $N_{\text{I}}$ and $N_{\text{II}}$ is at least $\delta (n+1)$. Such “good” shells are highly typical if $\delta > 0$ is small, as we know see.

**Lemma 6.5.** There exist constants $c > 0$ and $\delta > 0$ such that $\Pr_{n+1}(c_{n+1}^{\text{loc}}(\Gamma) \in \mathcal{G}_{n+1, \delta}) \geq 1 - e^{-cn}$.

**Proof.** By Kesten’s pattern theorem [22] Thm. 1] (and Lemma 24 in order that we may state the result for the polygon measure), there exist constants $c > 0$ and
In this figure, we explain in outline the method. Given $n \in 2\mathbb{N} + 1$, the index $\ell \in [n/4, 3n/4]$ has been fixed so that the bound (6.2) holds. As we have seen, the vast majority of indices in this range satisfy this bound. This means that when we draw a length $n + 1$ polygon $\gamma$ and mark with a black dot each vertex $\gamma_j$, $n/4 \leq j \leq 3n/4$, with the property that $W^0_n(\Gamma \text{ closes } |\Gamma^1| = j, \Gamma^1 = \gamma_{[0,j]}) \geq n^{-1/2+2\varepsilon}$, most such $\gamma$ will appear with black dots in most of the available spots. The left-hand sketch represents such a typical $\gamma$ and three of its many black dots. The pointed second part shows a sample of the law $W^0_n(\cdot |\Gamma^1| = j_k, \Gamma^1 = \gamma_{[0,j_k]})$, with $k = 3$, one that happens to close $\gamma_{[0,j_3]}$. The second part being sampled has length $n - j_3$. If we sample instead this law with $k = 2$, then the second part has a greater length, $n - j_2$, which equals $n - j_3 + 2$ in this instance. However, to construct a charming snake, we want this length to stay the same as we move from one black dot to the next. Pattern exchange is the mechanism that achieves this. By turning one type I pattern into a pattern of type II, we push two units of length into the first part, so that, in the middle sketch, the random second part has the original length $n - j_3$. The process is iterated in the right-hand sketch. The first part is akin to a belayer who takes in rope, storing it in accumulating type II patterns, so that the second part climber maintains a constant length of rope. This process of pattern exchange can be maintained for an order of $n^{1/2}$ steps, because the Gaussian fluctuation between the two types of pattern means that the process of artificially altering pattern type does not push the system out of its rough equilibrium when the number of changes is of this order. In this way, black dots also mark charming snake terms for a snake of length of order $n^{1/2}$.

\[ \delta > 0 \text{ such that, for any odd } n \geq d3^d, \]
\[ P_{n+1}(T_I(\Gamma) \leq \delta m) \leq e^{-cn} \quad \text{and} \quad P_{n+1}(T_{II}(\Gamma) \leq \delta m) \leq e^{-cn}. \]  
(6.3)
Note that every slot in $S_1$ belongs to $\Gamma_{[0,(n+1)/4]}$, because such slots belong to $\Gamma_{[0,(n+1)/10+2T_{1II}(\Gamma)]}$ and $T_{1II}(\Gamma) \leq (n+1)/13$. Thus,

$$P_{n+1}\left(S_1(\Gamma) \text{ contains fewer than } \delta(n+1) \text{ type I patterns}\right) \leq P_{n+1}\left(\Gamma'_{[0,\frac{n+1}{4}]} \text{ contains fewer than } \delta(n+1) \text{ type I patterns}\right).$$

There exist constants $\delta, c > 0$ such that

$$P_{n+1}\left(\Gamma'_{[0,\frac{n+1}{4}]} \text{ contains fewer than } \delta(n+1) \text{ type I patterns}\right) \leq \frac{c(n+1)^4 c_3(n+1)/4}{p_{n+1}} \cdot W_{(n+1)/4}(\Gamma \text{ contains fewer than } \delta(n+1) \text{ type I patterns}) \leq e^{(2c\sqrt{w+c})\sqrt{n}} \cdot W_{(n+1)/4}(\Gamma \text{ contains fewer than } \delta(n+1) \text{ type I patterns}) < e^{-cn},$$

where the second inequality comes from the Hammersley-Welsh bound \[2.1\] and Lemma 2.4, and the third from (6.3) (with a relabelling of $c > 0$). Thus, $P_{n+1}(N_I^1 \leq \delta(n+1)) \leq e^{-cn}$. The same holds for the quantity $N_{II}^1$. Considering $\Gamma'_{[\frac{2(n+1)}{4},n+1]}$ in place of $\Gamma_{[0,\frac{n+1}{4}]}$, the same conclusion may be reached about $N_I^2$ and $N_{II}^2$. It follows that

$$P_{n+1}\left(\min\{|S_1|,|S_2|,N_I,N_{II}\} < \delta(n+1)\right) < 4e^{-cn}.$$

This completes the proof. \(\square\)

In the next lemma, we see how, for any $\gamma \in \mathcal{G}_{n+1,\delta}$, the mixing of patterns that occurs when an element of $\varsigma_{n+1}^{\text{loc}}(\gamma)$ is realized involves an asymptotically Gaussian fluctuation in the pattern number $N_I^1$. The statement and proof are minor variations of those of [13, Lemma 3.5].

**Lemma 6.6.** For any $\delta > 0$, there exists $c > 0$ and $N \in \mathbb{N}$ such that, for $n \geq N$ and $\gamma \in \mathcal{G}_{n+1,\delta},$

1. if $k \in \mathbb{N}$ satisfies $\left|k - \frac{T_{1I}|S_1|}{|S_1| + |S_2|}\right| \leq n^{1/2}(\log n)^{1/4}$, then

$$P_{n+1}\left(N_I^1(\Gamma) = k \mid \Gamma \in \varsigma_{n+1}^{\text{loc}}(\gamma)\right) \geq n^{-1/2} \exp\left\{-c(\log n)^{1/2}\right\};$$

2. and, for any $g \in [1/8,1/2]$, \n
$$P_{n+1}\left(\left|N_I^1(\Gamma) - \frac{N_I|S_1|}{|S_1| + |S_2|}\right| \geq n^{1/2}(\log n)^g \mid \Gamma \in \varsigma_{n+1}^{\text{loc}}(\gamma)\right) \leq \exp\left\{-c(\log n)^{2g}\right\}.$$

**Proof.** If $\Gamma$ is distributed according to $P_{n+1}\left(\cdot \mid \varsigma_{n+1}^{\text{loc}}(\gamma) = \sigma\right)$, then $N_I$ type I patterns and $N_{II}$ type II patterns are distributed uniformly in the slots of $S_1 \cup S_2$. Thus, for $k \in \{0, \ldots, |S_1|\}$,

$$P_{n+1}\left(N_I^1(\Gamma) = k \mid \Gamma \in \varsigma_{n+1}^{\text{loc}}(\gamma)\right) = \frac{\binom{|S_1|}{k} \binom{|S_2|}{N_I - k}}{\binom{|S_1| + |S_2|}{N_I}}. \quad (6.4)$$
Write \( m = |S_1| + |S_2| \), \( |S_1| = \alpha m \) and \( N_I = \beta m \). By assumption \( \alpha, \beta \in [\delta, 1 - \delta] \) and \( m \geq 2\delta(n + 1) \). Let \( Z = \frac{N_I}{\alpha \beta m} - 1 \). Under \( P_{n+1}(\cdot \mid c_{n+1}^{\text{loc}}(\Gamma) \in \gamma) \), \( Z \) is a random variable of mean 0, such that \( \alpha \beta (1 + Z)m \in \mathbb{Z} \cap [0, \min\{\{S_1\}, T_I\}] \).

First, we investigate the case where \( Z \) is close to its mean. By means of a computation which uses Stirling’s formula and (6.4), we find that

\[
P_{n+1}(Z = z \mid \varsigma(\Gamma) = \sigma) = (1 + o(1)) \frac{\exp \left( -\frac{\alpha \beta}{2(1-\alpha)(1-\beta)} mz^2 \right)}{\sqrt{2\pi \alpha \beta (1-\alpha)(1-\beta)m}} ,
\]

where \( o(1) \) designates a quantity tending to 0 as \( n \) tends to infinity, uniformly in the acceptable choices of \( \sigma, S_1, S_2 \) and \( z \), with \( |z| \leq \frac{2^{n/2}(\log n)^{1/2+\varepsilon}}{\alpha \beta m} \). We have obtained Lemma 6.6(1).

We now turn to the deviations of \( Z \) from its mean. From (6.4), one can easily derive that \( P_{n+1}(Z = z \mid \varsigma(\Gamma) = \sigma) \) is unimodal in \( z \) with maximum at the value closest to 0 that \( Z \) may take. (We remind the reader that \( Z \) takes values in \( \frac{1}{\alpha \beta m} Z - 1 \), which contains 0 only if \( \alpha \beta m \in \mathbb{Z} \).) The asymptotic equality (6.5) thus implies the existence of constants \( c_0, c_1 > 0 \) depending only on \( \delta \) such that, for \( |z| \geq \frac{n^{1/2}(\log n)^g}{\alpha \beta m} \) and \( n \) large enough,

\[
P_{n+1}(Z = z \mid \varsigma(\Gamma) = \sigma) \leq c_1^{-1} n^{-1/2} \exp \left\{ -c_0 (\log n)^{2g} \right\} ;
\]

while for given \( \varepsilon > 0, |z| \geq \frac{n^{1/2}(\log n)^{1/2+\varepsilon}}{\alpha \beta m} \) and \( n \) large enough,

\[
P_{n+1}(Z = z \mid \varsigma(\Gamma) = \sigma) \leq c_1^{-1} n^{-1/2} \exp \left\{ -c_0 (\log n)^{1+2\varepsilon} \right\} \leq n^{-2} .
\]

Since \( T^1_I \) takes no more than \( n + 1 \) values, (6.6) and (6.7) imply Lemma 6.6(2).

**6.3. Mixing patterns by a random resampling.** Consider a random resampling experiment whose law we will denote by \( P_{\text{res}} \). First an input polygon \( \Gamma^\text{in} \) is sampled according to the law \( P_{n+1} \). Then the contents of the slots in \( S_1 \cup S_2 \) are forgotten and independently resampled to form an output polygon \( \Gamma^\text{out} \in \text{SAP}_{n+1} \). That is, given \( \Gamma^\text{in} \), \( \Gamma^\text{out} \) is chosen uniformly among the set of polygons \( \gamma \in \text{SAP}_{n+1} \) for which \( \gamma \in c_{n+1}^{\text{loc}}(\Gamma^\text{in}) \). Explicitly, if there are \( j \) type II patterns among \( k \) slots in \( S_1 \cup S_2 \) in \( \Gamma^\text{in} \) (so that \( k \geq j \)), the polygon \( \Gamma^\text{out} \) is formed by choosing uniformly at random a subset of cardinality \( j \) of these \( k \) slots and inserting type II patterns into the chosen slots.

Note the crucial property that \( \Gamma^\text{out} \) under \( P_{\text{res}} \) has the law \( P_{n+1} \): the resampling experiment holds the length \( n + 1 \) random polygon at equilibrium. We mention in passing that a basic consequence of this resampling is a delocalization of the walk midpoint.
Proposition 6.7 ([13] Proposition 1.3]). Let $d \geq 2$. There exists $C > 0$ such that, for $m \in \mathbb{N}$,
\[ \sup_{x \in \mathbb{Z}^d} W_m \left( \Gamma_{[m/2]} = x \right) \leq C m^{-1/2}. \]

It may be instructive to consider how to proof this result using the resampling experiment (or in fact a similar one involving walks rather than polygons) and Lemma 6.5, a proof using such an approach is given in [13, Section 3.2].

Any element $\phi \in \gamma_{n+1}^{\text{loc}}(\gamma)$ begins by tracing a journey over the region where slots in $S_1$ may appear, from the origin to $\gamma_{(n+1)/10}^{\text{empty}}$; it then follows its middle section, the trajectory of $\gamma$ from $\gamma_{(n+1)/10}^{\text{empty}}$ until $\gamma_{(n+1)/10}^{\text{empty}}$ (where $l' = l^{\text{empty}}$); and it ends by moving from this vertex back to the origin, through the territory of slots in $S_2$. Note that, in traversing the middle section, $\phi$ is exactly following a sub-walk of $\gamma$, because no pattern changes have been made to this part of $\gamma$. The timing of the middle section of this trajectory is advanced or retarded according to how many type $I$ patterns are placed in the slots in $S_1$. Each extra such pattern retards the schedule by two units. When $\phi$ has the minimum possible number $m := \max\{0, |T_{II}(\gamma)| - |S_2|\}$ of type $I$ patterns in the slots of $S_1$, the middle section is traversed by $\phi$ as early as possible, the journey taking place during $[(n + 1)/10 + 2m, n - 2(T_{II}(\gamma) - m) - (n + 1)/10]$. When $\phi$ has the maximum possible number $M := \min\{|S_1|, T_{II}(\gamma)|$ of type $I$ patterns in the slots of $S_1$, this traversal occurs as late as possible, during $[(n + 1)/10 + 2M, (n + 1) - (n + 1)/10 - 2(T_{II}(\gamma) - M)]$. Since $M \leq |S_1| \leq (n + 1)/13$ and $T_{II}(\gamma) - m \leq |S_2| \leq (n + 1)/13$, $\phi_j$ necessarily lies in the middle section whenever $j \in [(n + 1)/10 + 2(n + 1)/13, n + 1 - (n + 1)/10 - 2(n + 1)/13]$. Since the snake method parameter $\ell$ has been set to belong to the interval $[n/4, 3n/4]$, we see that $\phi_j$ always lies in the middle section whenever $j \in [\ell - n^{1/2}, \ell + n^{1/2}]$.

Taking $\gamma \in \text{SAP}_{n+1}$ and conditioning $P_{\text{res}}$ on $\Gamma_{\text{in}} = \gamma$, note that the mean number of type $I$ patterns that end up in the slots in $S_1$ under $\Gamma_{\text{out}}$ is given by $T(\gamma) \cdot \frac{|S_1(\gamma)|}{|S_1(\gamma)| + |S_2(\gamma)|}$, because this expression is the product of the number of type $I$ patterns that are redistributed and the proportion of the available slots that lie in $S_1$.

Consider now a polygon $\phi \in \gamma_{n+1}^{\text{loc}}(\gamma)$ that achieves as closely as possible the mean value for the number of type $I$ patterns among the slots in $S_1$; that is, $T(\phi)$ equals $\lfloor T(\gamma) \cdot \frac{|S_1(\gamma)|}{|S_1(\gamma)| + |S_2(\gamma)|} \rfloor$. As we have noted, $\phi_\ell$ is always reached during the middle section of $\phi$’s three-stage journey. Define the middle index $l_{\text{mid}} = l_{\text{mid}}(\gamma)$ so that $\phi_\ell = \gamma_{l_{\text{mid}}}$.

6.4. Snakes of walks with high closing probability are typical. To prove Proposition 6.1, take $\gamma \in \text{SAP}_{n+1}$ and define NoCharm$(\gamma)$ to be the set
\[ \left\{ j \in (\ell - 2\mathbb{N}) \cap \left[ l_{\text{mid}}(\gamma) - 2n^{1/2}(\log n)^{1/4}, l_{\text{mid}}(\gamma) + 2n^{1/2}(\log n)^{1/4} \right] \right\}. \]
Lemma 6.8. The events \( \{ \Gamma^\text{out} \text{ is charming at } \ell \} \) and \( \{ \Gamma^\text{in} \text{ is charming at } L \} \) coincide.

Proof. Note that the shells of \( \varsigma(\Gamma^\text{in}_{[0, L]}) \) and \( \varsigma(\Gamma^\text{out}_{[0, \ell]}) \) coincide, because \( \Gamma^\text{out}_{[0, \ell]} \) may be obtained from \( \Gamma^\text{in}_{[0, L]} \) by modifying the I/II-status of some of its slots (these being certain slots in \( S_1 \)). Thus, Lemma 6.4 implies the statement. \( \square \)

Lemma 6.9.

\[
\Pr_{\text{res}} \left( |\text{NoCharm}(\Gamma^\text{in})| \geq n^{1/2-\epsilon/6} \right) \leq n^{-\epsilon/6}.
\]

Proof. Choosing \( \delta > 0 \) small enough and abbreviating \( G = G_{n+1, \delta} \), Lemma 6.6(1) implies that, for each \( \gamma \in G \) and \( k \in \left[ -n^{1/2}(\log n)^{1/4}, n^{1/2}(\log n)^{1/4} \right] \),

\[
\Pr_{\text{res}} \left( L = l_{\text{mid}}(\Gamma^\text{in}) + 2k \; \bigg| \; \Gamma^\text{in} = \gamma \right) \geq n^{-1/2} \exp \left\{ -c(\log n)^{1/2} \right\}.
\]

Thus, again taking any \( \gamma \in G \),

\[
\Pr_{\text{res}} \left( \Gamma^\text{out} \text{ is not charming at } \ell \; \bigg| \; \Gamma^\text{in} = \gamma \right) \\
\geq \sum_{k=-n^{1/2}(\log n)^{1/4}}^{n^{1/2}(\log n)^{1/4}} \Pr_{\text{res}} \left( \Gamma^\text{out} \text{ is not charming at } \ell \; \bigg| \; L = l_{\text{mid}}(\Gamma^\text{in}) + 2k \; \bigg| \; \Gamma^\text{in} = \gamma \right) \\
\geq \sum_{k=-n^{1/2}(\log n)^{1/4}}^{n^{1/2}(\log n)^{1/4}} \Pr \left( \gamma \text{ is not charming at } l_{\text{mid}}(\Gamma^\text{in}) + 2k \; , \; L = l_{\text{mid}}(\Gamma^\text{in}) + 2k \; \bigg| \; \Gamma^\text{in} = \gamma \right) \\
\geq n^{-1/2} \exp \left\{ -c(\log n)^{1/2} \right\} \sum_{k=-n^{1/2}(\log n)^{1/4}}^{n^{1/2}(\log n)^{1/4}} \mathbb{1} \gamma \text{ is not charming at } l_{\text{mid}}(\gamma) + 2k \\
\geq n^{-1/2} \exp \left\{ -c(\log n)^{1/2} \right\} \cdot |\text{NoCharm}(\gamma)|,
\]

where the second inequality made use of Lemma 6.8.

Averaging over such \( \gamma \), we find that

\[
\Pr_{\text{res}} \left( \Gamma^\text{out} \text{ is not charming at } \ell \; \bigg| \; \Gamma^\text{in} \in G \right)
\]
\[
\geq cn^{-1/2} \exp \left\{ -c(\log n)^{1/2} \right\} \cdot E_{\text{res}} \left[ |\text{NoCharm}(\Gamma^\text{in})| \middle| \Gamma^\text{in} \in \mathcal{G} \right],
\]
where \(E_{\text{res}}\) denotes the expectation associated with the law \(P_{\text{res}}\).

Note that
\[
P_{\text{res}} \left( \Gamma^\text{out} \text{ is not charming at } \ell \middle| \Gamma^\text{in} \in \mathcal{G} \right) \\
\leq 2 P_{n+1} \left( \Gamma \text{ is not charming at } \ell \right) = 2 P_{n+1} \left( \Gamma_{[0, \ell]} \notin \text{Hi}\Phi_{\ell, n}^{1/2-2\varepsilon} \right) \leq 2n^{-\varepsilon},
\]
where in the first inequality we use that \(\Gamma^\text{out}\) under \(P_{\text{res}}\) has the law \(P_{n+1}\), and then apply Lemma 6.5 to find that \(P_{\text{res}}(\Gamma^\text{in} \in \mathcal{G}) = P_{n+1}(\Gamma \in \mathcal{G}) \geq 1 - e^{-cn} \geq 1/2\). The final inequality above used Lemma 6.2.

Thus,
\[
E_{\text{res}} \left[ |\text{NoCharm}(\Gamma^\text{in})| \middle| \Gamma^\text{in} \in \mathcal{G} \right] \leq n^{1/2-\varepsilon/2}.
\]
We find that \(E_{\text{res}}|\text{NoCharm}|\) is at most
\[
E_{\text{res}} \left[ |\text{NoCharm}(\Gamma^\text{in})| \middle| \Gamma^\text{in} \in \mathcal{G} \right] + \left( 2n^{1/2}(\log n)^{1/4} + 1 \right) P_{\text{res}} \left( \Gamma^\text{in} \notin \mathcal{G} \right) \\
\leq n^{1/2-\varepsilon/2} + \left( 2n^{1/2}(\log n)^{1/4} + 1 \right) e^{-cn} \leq n^{1/2-\varepsilon/3}.
\]
Applying Markov’s inequality yields Lemma 6.9. \(\square\)

**Proof of Proposition 6.1.** Lemma 6.6(2) implies that
\[
P_{n+1} \left( \ell \in \left[ \ell_{\text{mid}}(\Gamma) - n^{1/2}(\log n)^{1/4}, \ell_{\text{mid}}(\Gamma) + n^{1/2}(\log n)^{1/4} \right] \middle| \Gamma \in \mathcal{G}_{n, \delta} \right)
\]
is at least \(1 - \exp \left\{ -c(\log n)^{1/2} \right\}\). Note that the interval centred on \(\ell_{\text{mid}}(\Gamma)\) considered here is shorter than its counterpart in the definition of \(\text{NoCharm}(\gamma)\) for \(\gamma \in \text{SAP}_{n+1}\). Applying Lemmas 6.5 and 6.9, we find that
\[
P_{n+1} \left( \# \left\{ j \in \left[ \ell - n^{1/2}(\log n)^{1/4}, \ell + n^{1/2}(\log n)^{1/4} \right] : \Gamma \text{ is not charming at } j \right\} \geq n^{1/2-\varepsilon/6} \right) \leq n^{-\varepsilon/6} + e^{-c(\log n)^{1/2}} + e^{-cn}.
\]
When the complementary event occurs, \(\Gamma\) is charming at least one-quarter of the indices in \(\left[ \ell - n^{1/2}, \ell \right]\), so that \(\Gamma_{[0, \ell]}\) is an element of \(\text{CS}_{\ell, n, \theta}^{1/2-2\varepsilon}. \) (We write one-quarter rather than one-half here, because one-half of such indices are inadmissible due to their having the wrong parity.) \(\square\)

### 7. The closing exponent above one-half

Note that the relation (1.3) implies that when \(\theta_n \geq 3/2\), the closing probability is at most order \(n^{-1/2}\). This order is the limit of the snake method via Gaussian pattern fluctuation. This section is devoted to the proof of the two results that
push the closing probability upper bound below the level $n^{-1/2}$: Theorems 1.5(1) and (2).

In each case, our method will combine the two main approaches used thus far: it is the snake method implemented via a certain form of the polygon joining technique. It is the use of this technique which means that we work in dimension $d = 2$ throughout Section 7.

Since the length parameter $\beta$ in the snake method must certainly be at most one, the snake method may at best prove closing probability upper bounds of the order $n^{-1}$. Hypothesising inter alia the existence of the closing exponent $\theta = \lim_{n \to \infty} \theta_n$, Theorem 1.5(2) covers one-third of the interval from the solved $1/2$ to the possible-in-principle 1, to reach a bound of $2/3$. Despite its conditional nature, the result is attractive for expository reasons: the proof of Theorem 1.5(1) shares the same framework, but a number of technicalities that arise in its proof are absent in the conditional result’s, permitting a more attentive focus on the main concepts. For this reason, we present the latter proof first.

7.1. Relating polygon laws $P_n$ for distinct $n$ via global join polygons. A comparison of measure of the uniform polygon laws $P_n$ for distinct values of $n$ of the same order will be essential for our new rendering of the snake method. The comparison will be made via regulation global join polygons. We now present Lemma 7.3, the tool that we will use for the comparison, after some prerequisites.

**Definition 7.1.** Recall the set $\text{SAP}_n^{\text{left}} \subseteq \text{SAP}_n$ (with $n \in 2\mathbb{N}$) introduced in Section 3.3. Write $P_n^{\text{left}}$ for the uniform law on $\text{SAP}_n^{\text{left}}$.

The superscript $\text{left}$ has two meanings: elements of $\text{SAP}_n^{\text{left}}$ are joined on the left in regulation global polygon joining; and these elements are left-long.

**Lemma 7.2.** Consider a Madras joinable polygon pair $(\phi^1, \phi^2)$. Let $j \in \mathbb{N}$ be such that $\phi^1_j$ equals $\text{SE}(\phi^1)$. The join polygon $J(\phi^1, \phi^2)$ has the property that the initial subpaths $\phi^1_{[0,j]}$ and $J(\phi^1, \phi^2)_{[0,j]}$ coincide.

**Proof.** Review Figure 8 and suppose that some case other than IIcii or IIIcii occurs. Note by inspection that, when $J(\phi^1, \phi^2)$ is formed, either one or two edges are removed from $\phi^1$, and that every endpoint $v$ of these edges has the property that the horizontal line segment extending to the right from $v$ intersects the vertex set of $\phi^1$ only at $v$. Note however that every vertex $\phi^1_i$, $0 \leq i \leq j - 1$, (i.e., the vertices of the left path of $\phi^1$ except its endpoint $\text{SE}(\phi^1)$) has the property that the line segment extending rightwards from the vertex does have a further intersection with the vertex set of $\phi^1$. Thus, the above vertex $v$ must equal $\phi^1_i$ for some $i > j$ (i.e., must lie in the right path of $\phi^1$). The initial subpath $\phi^1_{[0,j]}$ is thus unchanged by the joining operation and is shared by $J(\phi^1, \phi^2)$.

Cases IIcii and IIIcii remain. In these cases, note that at least one endpoint $v$ of one of the two removed edges enjoys the above property and thus belongs to
the right path of $\phi^2$. Note by inspecting the figure that $SE(\phi^1)$ cannot lie in the vicinity of $v$ that is altered by the joining operation. Thus, once again, $\phi^1_{[0,j]}$ is shared by $J(\phi^1, \phi^2)$. □

Lemma 7.3. Let $k, \ell \in 2\mathbb{N}$ satisfy $k/2 \leq \ell \leq 35k$. Let $j \in \mathbb{N}$ satisfy $j \leq k/2$. Then

$$P_{k + \ell + 16}(\Gamma_{[0,j]} = \phi \mid \Gamma \in RGJ_k, \ell) = p^{\text{left}}_k(\Gamma_{[0,j]} = \phi).$$

Proof. By the uniqueness of the decomposition $\phi = J(\phi^1, \phi^2 + \vec{u})$ of any element $\phi \in RGJ_k, \ell$ established in the proof of Lemma 3.16, we find that the conditional distribution of $P_{k + \ell + 16}$ given that $\Gamma \in RGJ_k, \ell$ has the law of the output in this procedure:

- select a polygon according to the law $p^{\text{left}}_k$;
- select another uniformly among elements of $\text{SAP}^{\text{right}}_\ell$;
- choose one of the $\frac{1}{10} k^{1/2}$ locations specified in Definition 3.15 uniformly at random, and translate the latter polygon to this location;
- Madras join the first and the translate of the second polygon to obtain the output.

Denote by $\phi^1$ the element of $\text{SAP}^{\text{left}}_k$ selected in the first step. This polygon is left-long; let $m \geq k/2$ satisfy $\phi^1_m = \phi^1$. By Lemma 7.2, the output polygon has an initial subpath that coincides with the left path $\phi_{[0,m]}$ of $\phi^1$. Since $j \leq k/2 \leq m$, we see that the initial length $j$ subpath of the output polygon coincides with $\phi^1_{[0,j]}$. Thus, its law is that of $\Gamma_{[0,j]}$ where the polygon $\Gamma$ distributed according to $p^{\text{left}}_k$. □

7.2. Proving Theorem 1.5(2). The result is a consequence of an assertion which perhaps has a less pleasing appearance, but is more informative.

Proposition 7.4. Let $d = 2$. Assume that the limit $\theta := \lim_{n \in 2\mathbb{N}} \theta_n$ exists and is finite. If $\theta < 5/3$, then, for some $c > 1$ and $\delta > 0$, the set of $n \in 2\mathbb{N} + 1$ for which

$$W_n(\Gamma \text{ closes}) \leq c^{-n^{\delta}}$$

intersects the shifted dyadic scale $[2^i - 1, 2^{i+1} - 1]$ for all but finitely many $i \in \mathbb{N}$.

Proof of Theorem 1.5(2). Since $W_n(\Gamma \text{ closes}) = 2(n + 1)p_{n+1}/c_n$, our assumptions that $\theta = \lim_{n \in 2\mathbb{N}} \theta_n < \infty$ and that the quantity $\limsup_{n \in \mathbb{N}} \xi_n$ (that we here denote by $\xi$) is finite imply that this probability satisfies

$$W_n(\Gamma \text{ closes}) \geq n^{1 + \theta - \xi + o(1)}.$$ (7.1)

This polynomial decay contradicts Proposition 7.4. □

It remains of course to derive the proposition. Recall the heuristic derivation via polygon joining that $\theta \geq 5/2$ in $d = 2$, and the associated notion of a $(\theta^{(1)}, \theta^{(2)}, \theta^{(3)})$-argument from Section 2.3.
Assume throughout the derivation that the hypothesis that \( \theta = \lim_{n \to \infty} \theta_n \) exists and is finite is in force. We set \( \theta = 3/2 + \chi \). Note that, under the hypothesis, \( W_n(\Gamma \text{ closes}) \) has order \( n^{-1/2-\chi} \).

To prove Proposition 7.4 we suppose that \( \chi < 1/6 \). We then carry out a \((1/2, 1, \chi - \varepsilon)\)-argument for \( \varepsilon > 0 \) arbitrarily small.

**7.2.1. Regulation global join polygons are not rare.** Define \( R_i := 2N \cap [2^i, 2^{i+1}] \). Think of \( R_i \) as being a set of regular indices \( j \) in the interval \( 2N \cap [2^i, 2^{i+1}] \), those for which the polygon deviation exponents such as \( \theta_j \) are neither atypically high or low. Assuming as we do that \( \theta \) exists, we simply regard all indices as regular. We will respecify the regular index set non-trivially when we prove Theorem 1.5(1).

We begin by arguing that global join polygons are not atypical among polygons. The assumption that \( \theta \) exists is key here; in the language of the \((\theta^{(1)}, \theta^{(2)}, \theta^{(3)})\)-mnemonic, its existence implies that \( \theta^{(3)} \leq \chi + o(1) \).

**Lemma 7.5.** Let \( \varepsilon > 0 \). For any \( i \in \mathbb{N} \) sufficiently high, and \( n \in 2N \cap [2^{i+4}, 2^{i+5}] \)

\[
P_n\left( \Gamma \in \bigcup_{k \in R_{i+2}} \text{RGJ}_{k,n-16-k} \right) \geq n^{-\chi - \varepsilon}.
\]

(In due course, the snake method parameters, including \( n \), will be set for the proof of Proposition 7.4. Until then, we are treating \( n \) as a free variable.)

**Proof of Lemma 7.5.** Note that the probability in question is equal to the ratio of the cardinality of the union \( \bigcup_{k \in R_{i+2}} \text{RGJ}_{k,n-16-k} \) and the polygon number \( p_n \). Proposition 3.2 shows that

\[
\left| \bigcup_{k \in R_{i+2}} \text{RGJ}_{k,n-16-k} \right| \geq \mu^n \cdot \frac{1}{C_1 \log n} \sum_{k \in R_{i+2}} p_k p_{n-16-k},
\]

while the existence of \( \theta \) implies that \( p_n = n^{-\theta} \mu^n \leq n^{-\theta + \varepsilon} \mu^n \). Thus,

\[
P_n\left( \Gamma \in \bigcup_{k \in R_{i+2}} \text{RGJ}_{k,n-16-k} \right)
\geq \frac{n^{1/2}}{C_1 \log n} \sum_{k \in R_{i+2}} k^{-(\theta + \varepsilon)} (n-k)^{-(\theta + \varepsilon)} \cdot n^{\theta - \varepsilon} \geq \frac{n^{3/2-\theta-3\varepsilon}}{2^4 C_1 \log n},
\]

where we used \( \#R_{i+2} \geq 2^{i+1} \geq 2^{-4} n \). Recalling that \( \theta = 3/2 + \chi \) and relabelling \( \varepsilon > 0 \) completes the proof.

Figure 13 offers an outline of how we will exploit the ample supply of regulation polygons to implement the snake method and prove Proposition 7.4.

We also make a parenthetical comment before continuing. A word of caution is in fact needed when we describe Lemma 7.5 as saying that \( \theta^{(3)} \leq \chi + o(1) \), or indeed when we claim that our approach to deriving Proposition 7.4 is to make a
Figure 13. The snake method’s new guise: compare with Figure 12. Again black dots mark vertices of polygons \( \gamma \): fixing \( \tilde{n} \in 2\mathbb{N} \), assign a black dot at each vertex \( \gamma_j, \tilde{n}/4 \leq j \leq 3\tilde{n}/4 \), of a polygon \( \gamma \) of length of order \( \tilde{n} \), that has the property that SAW\(^0\) \( \tilde{n}+j \) \( (\Gamma \text{ closes } |\Gamma^1| = j, \Gamma^1 = \gamma_{[0,j]} ) \geq \tilde{n}^{-1/2-\chi-\epsilon} \). There is a difference to Figure 12 because polygons \( \gamma \) may differ in length. Our assumptions readily imply that most polygons \( \gamma \) with length \( n' \) of order say \( 2\tilde{n} \) will have a black dot at \( \gamma_{n'-\tilde{n}} \). What about at other locations \( \gamma_j \) where \( j \) is also of order \( \tilde{n} \)? A black dot is likely to appear at \( \Gamma_j \) when \( \Gamma \) is \( P_{j+\tilde{n}} \)-distributed. If we can make a comparison of measure, showing that the laws \( P_{j+\tilde{n}} \) and \( P_{n'} \) are to some degree similar, then the black dot will be known to also appear at \( \gamma_j \) in a typical sample of \( P_{n'} \). Applying this for many such \( j \), black dots will appear along the course of a typical length \( n' \) polygon. These black dots index charming snake terms and permit the use of the snake method. Comparison of measure will be undertaken in Proposition 7.11 by the use of polygon joining. Lemma 7.5 shows that a non-negligible proportion of polygons are regulation global join polygons; thus, such polygons themselves typically have black dots \( \tilde{n} \) steps from their end. The law of the left polygon under the uniform law on regulation polygons of a given length is largely unchanged as that total length is varied on a given dyadic scale, because the length discrepancy can be absorbed by altering the length of the right polygon. Indeed, in the above sketches, we deplete the length of the right polygon in a possible extension of the depicted first part, as the length of this first part shortens; in this way, we show that first parts typically arising in regulation polygons at one length are also characteristic in such polygons with lengths on the same scale.

\((1/2, 1, 1/6 - o(1))-\text{argument. We are interpreting the exponent } \theta^{(3)}, \text{ which was originally heuristically specified in Section 2.5, according to the formula}

\[ P_n \left( \Gamma \in \bigcup_{k \in \mathbb{R}_{i+2}} \text{RGJ}_{k, n-16-k} \right) = n^{-\theta^{(3)} + o(1)} \]
as \( n \in [2^{i+4}, 2^{i+5}] \to \infty \) through even values. Only order \( n^{1/2} \) ways of joining length order \( n \) polygons are admissible for regulation global join polygons. The positivity of \( \theta^{(3)} \) may arise as much from the typical diameter of an element of \( \text{SAP}_n \) growing more quickly than \( n^{1/2} \) (permitting more joinings than the regulation ones) as it does from the actual improbability of an element of \( \text{SAP}_{2n} \) being a Madras join of two length order \( n \) polygons. Indeed, with the above formula specifying \( \theta^{(3)} \), value is pushed from \( \theta^{(1)} \) to \( \theta^{(3)} \); we would expect in two dimensions that \( \theta^{(3)} = 5/2 \) with \((\theta^{(1)}, \theta^{(2)}, \theta^{(3)})\) equalling the vector \((1/2, 1, 1)\) rather than the vector \((3/4, 1, 3/4)\) that we discussed in Section 2.5. This change in point of view occurs because our notion of regulation polygon, while useful, does not acknowledge the anticipated possibilities of joining.

7.2.2. It is enough to show that individual snake terms are often charming. We set the snake method exponent parameters, taking

- the snake length \( \beta \) equal to one;
- the inverse charm \( \alpha \) equal to \( 1/2 + 2\chi + 4\varepsilon \);
- and the deficit \( \eta \) equal to \( \chi + 8\varepsilon \).

where henceforth in proving Proposition 7.4, \( \varepsilon > 0 \) denotes a given but arbitrarily small quantity. The quantity \( \delta = \beta - \eta - \alpha = 1/2 - 3\chi - 12\varepsilon \) must be positive if the snake method is to work, and thus we choose \( \varepsilon < (1/6 - \chi)/4 \).

We now reduce Proposition 7.4’s proof to that of the following assertion which is expressed in terms of the high conditional closing probability set notation \( \text{Hi} \Phi \) introduced in Subsection 4.2.2.

**Proposition 7.6.** For each \( i \in \mathbb{N} \) sufficiently high, there exists \( m \in 2\mathbb{N} \cap [2^{i+4}, 2^{i+5}] \) and \( m' \in [2^{i+3}, 2^{i+4}] \) such that, writing \( K \) for the set of values \( k \in \mathbb{N}, 1 \leq k \leq m - m' \), that satisfy

\[
P_m \left( \Gamma_{[0,k]} \in \text{Hi} \Phi_{\alpha}^{\alpha} \Phi_{k,m',-1} \right) \geq m^{-\eta + \varepsilon},
\]

we have that \( |K| \geq 2^{-9}m \).

Making the snake method work is a matter of verifying the assertion (5.1) that charming snakes are not highly atypical. We now use Proposition 7.4 to set the method’s two index parameters: for \( i \in \mathbb{N} \) given, we take \( n \) equal to \( m - 1 \), and \( \ell \) equal to \( m - m' \). Note then that max \( K \leq \ell \). Since \( n - \ell = m' - 1 \), we have that, for \( k \in [1, \ell] \), the walk \( \gamma \in \Phi_{\ell,n} \) is \((\alpha, n, \ell)\)-charming at index \( k \) if and only if

\[
\mathcal{W}_{k,m',-1}^{\alpha} \left( \Gamma \text{ closes } |\Gamma^1| = k \right) = \mathcal{W}_{k,m',-1}^{\alpha} \left( \Gamma \text{ closes } |\Gamma^1| = k \Rightarrow \Gamma_{[0,k]} = \gamma_{[0,k]} \right) > n^{-\alpha};
\]

note the displayed condition is almost the same as \( \gamma_{[0,k]} \in \text{Hi} \Phi_{k,m',-1}^{\alpha} \), the latter occurring when \( n^{-\alpha} \) is replaced by \((m' + k - 1)^{-\alpha} \), a change which involves only a bounded factor. Thus, Proposition 7.4 shows that individual snake terms are
charming with probability at least \( m^{-\chi-\alpha(1)} \), and indeed it is a simple matter to use the proposition to verify (5.1).

**Proof of Proposition 7.4.** For the given choice of \( i \in \mathbb{N} \) that determines the index parameters, let \( m' \) again be specified by Proposition 7.6. For \( \gamma \in \text{SAP}_{n+1} \), set

\[
X_{\gamma} = \sum_{k=0}^{\ell} \mathbb{I}_{\gamma[0,k] \in \text{Hi}\Phi_{k,m',k-1}^{\alpha}}.
\]

Recall that \( \gamma[0,\ell] \in \Phi_{\ell,n} \) is \((\alpha, n, \ell)\)-charming at an index \( k \in [1, \ell] \) if

\[
\text{SAW}_{k+n-\ell}^0 \left( \Gamma \text{ closes } |\Gamma^1| = k, \Gamma^1 = \gamma[0,k] \right) > n^{-\alpha};
\]

on the other hand, for given \( k \in [1, \ell] \), the property that such a \( \gamma \) satisfies \( \gamma[0,k] \in \text{Hi}\Phi_{k,n-\ell+k}^{\alpha} \) takes the same form with \( n^{-\alpha} \) replaced by the slightly larger quantity \( (n - \ell + k)^{-\alpha} \). Since \( n - \ell = m' - 1 \), we find that, for any \( \gamma \in \text{SAP}_{n+1} \),

\[
X_{\gamma} \geq n^{\beta-\eta/4} \quad \text{implies that} \quad \gamma[0,\ell] \in \text{CS}_{\alpha,\ell,n}^{\alpha,\ell,n}.
\]

We now show that

\[
P_{n+1} \left( X_{\gamma} \geq n^{1-\eta+\varepsilon/2} \right) \geq n^{-\eta+\varepsilon/2}.
\]

To derive this, consider the expression

\[
S = \sum_{\gamma \in \text{SAP}_{n+1}} \sum_{k=0}^{\ell} \mathbb{I}_{\gamma[0,k] \in \text{Hi}\Phi_{k,m',k}^{\alpha}}.
\]

Recall that \( p_{n+1} \) denotes \( \# \text{SAP}_{n+1} \); using Proposition 7.6 in light of \( n + 1 = m \),

\[
S = p_{n+1} \sum_{k=0}^{\ell} P_{n+1} \left( \Gamma[0,k] \in \text{Hi}\Phi_{k,m',k}^{\alpha} \right)
\]

\[
\geq p_{n+1} \cdot \#K \cdot (n + 1)^{-\eta+\varepsilon} \geq p_{n+1} \cdot 2^{-9n} \cdot 2^{-1} n^{-\eta+\varepsilon}.
\]

Let \( q \) denote the left-hand side of (7.3). Note that

\[
S \leq p_n \cdot \left( q(\ell + 1) + (1 - q)n^{1-\eta+\varepsilon/2} \right).
\]

From the lower bound on \( S \), and \( n \geq \ell \),

\[
q(n + 1) + n^{1-\eta+\varepsilon/2} \geq 2^{-10} n^{1-\eta+\varepsilon},
\]

which implies that \( q \geq n^{-\eta+\varepsilon}/2 \); in this way, we obtain (7.3).

It is unsurprising that \( n^{1-\eta+\varepsilon/2} \geq n^{1-\eta}/4 \) (for all \( n \in \mathbb{N} \), including our choice). Since the snake length exponent \( \beta \) is set to one, we learn from (7.2) and (7.3) that

\[
P_{n+1} \left( \Gamma[0,\ell] \in \text{CS}_{\alpha,\ell,n}^{\alpha,\ell,n} \right) \geq P_{n+1} \left( X_{\gamma} \geq n^{1-\eta+\varepsilon/2} \right) \geq n^{-\eta+\varepsilon/2}.
\]
Thus, if the dyadic scale parameter \( i \in \mathbb{N} \) is chosen so that \( n \geq 2^{i+4} \) is sufficiently high, the charming snake presence hypothesis (5.1) is satisfied. By Theorem 5.2 \( W_n(\Gamma \text{ closes}) \leq 2(n + 1)c^{-n^{4/3}} \). This deduction has been made for some value of \( n \in (2\mathbb{N} + 1) \cap [2^{i+4} - 1, 2^{i+5} - 1] \), where here \( i \in \mathbb{N} \) is arbitrary above a finite interval. Relabelling \( c > 1 \) to be any value in \((1, c^{1/2})\) completes the proof of Proposition 7.4.

7.3. An intermediate step. Advancing towards Proposition 7.6, we now state and prove an assertion that seems at first glance to be a very similar result. In fact, the next proposition is significantly easier to prove, being a fairly direct consequence of the existence of \( \theta \).

**Proposition 7.7.** For each \( i \in \mathbb{N} \), there exists \( m' \in 2\mathbb{N} \cap [2^{i+3}, 2^{i+4}] \) such that, writing \( K' \) for the set of values \( k \in \mathbb{N}, 2^i \leq k \leq 2^i + 2^{i-2} \), that satisfy

\[
P_{m'+k}(\Gamma[0,k] \in H_\alpha_{k,m'+k}) \geq 1 - 2^{-i\chi},
\]

we have that \( |K'| \geq 2^{-8}m' \).

The quantity \( m' \) is an analogue of \( \tilde{n} \) in the caption of Figure 13. The proposition is a precisely stated counterpart to the assertion in that caption that a black dot typically appears \( \tilde{n} \) steps from the end of a polygon drawn from \( P_{n'} \) where \( n' \) has order \( 2\tilde{n} \) (or say \( 2^{i+3} \)). Note the discrepancy that we have chosen \( \alpha \) to be marginally above \( 1/2 + 2\chi \) while in the caption a choice just above \( 1/2 + \chi \) was made. The extra margin permits \( 1 - 2^{-i\chi} \) rather than \( 1 - o(1) \) in the conclusion of the proposition. We certainly need the extra margin as we will later explain.

We also mention that we will not directly apply Proposition 7.7. The constructs in its proof are essential en route to Proposition 7.6 and we state it because its formal similarity to the latter proposition marks it as a natural half-way house.

Preparing to prove Proposition 7.7, fix a parameter \( \epsilon_1 > 0 \).

**Definition 7.8.** Define \( E \) to be the set of pairs \((k, j) \in \mathbb{N} \times 2\mathbb{N},
\)

- where \( j \in [2^{i+4}, 2^{i+5}] \) and \( k \in [2^i, 2^i + 2^{i-2}] \);
- and the pair \((k, j)\) is such that

\[
\# \left\{ \gamma \in \text{SAW}^0_{j-1} : |\gamma^1| = k \right\} < (j - 1)^{1/2 + \chi + \epsilon_1} \cdot \# \left\{ \gamma \in \text{SAW}^0_{j-1} : |\gamma^1| = k, \gamma \text{ closes} \right\}.
\]

By Lemma 4.3 for each \( j \in 2\mathbb{N} \cap [2^{i+4}, 2^{i+5}] \), the set of \( k \in [2^i, 2^i + 2^{i-2}] \) such that \((k, j) \notin E\) has cardinality at most \( 2j^{1-\epsilon_1} \leq 2(2^{i+5})^{1-\epsilon_1} \).

Fix a second parameter \( \epsilon_2 > \epsilon_1 \).
Lemma 7.9. If \((k, j) \in E\), then, for any \(a \geq 0\),
\[
P_j \left( \Gamma_{[0,k]} \not\in \text{Hi} \Phi_{k,j-1}^{1/2+(a+1)\chi+\epsilon_2} \right) \leq j^{-(a\chi+\epsilon_2-\epsilon_1)}.
\]

Proof. Recalling the notation \(W_{j-1}^0\) from Subsection 4.1.5, note that the assertion
\[
W_{j-1}^0 \left( \Gamma^1 \not\in \text{Hi} \Phi_{k,j-1}^{1/2+(a+1)\chi+\epsilon_2} \mid \Gamma \text{ closes } |\Gamma^1| = k \right) \leq j^{-(a\chi+\epsilon_2-\epsilon_1)} \tag{7.5}
\]
is a reformulation of the lemma. Note that
\[
\# \left\{ \gamma \in \text{SAW}_{j-1}^0 : |\gamma^1| = k, \gamma^1 \not\in \text{Hi} \Phi_{k,j-1}^{1/2+(a+1)\chi+\epsilon_2} \right\}
\geq (j-1)^{1/2+(a+1)\chi+\epsilon_2}, \quad \# \left\{ \gamma \in \text{SAW}_{j-1}^0 : |\gamma^1| = k, \gamma^1 \not\in \text{Hi} \Phi_{k,j-1}^{1/2+(a+1)\chi+\epsilon_2}, \gamma \text{ closes} \right\}.
\]
The left-hand side here is bounded above by \((7.4)\), and \((7.5)\) is obtained. □

Lemma 7.10. There exists \(k \in 2N \cap \left[2^{i+4}, 2^{i+4} - 2^{i-2}\right]\) for which there are at least \(2^{i-4}\) values of \(j \in 2N \cap [0, 2^{i-2}]\) with \((2^i + j, k + j) \in E\).

Proof. Note that
\[
\sum_{j \in 2N \cap [2^{i+4} + 2^{i-2}, 2^{i+4} + 2^i]} \sum_{\ell \in 2^i} \mathbb{I}_{(\ell,j) \in E} \geq \left(2^{i-2} + 1 - (2^{i+5})^{1-\epsilon_1}\right) \cdot \frac{1}{2} (2^i - 2^{i-2}).
\]
As Figure 4 illustrates, the left-hand side here is bounded above by
\[
\sum_{k \in 2N \cap [2^{i+4}, 2^{i+4} + 2^i]} \sum_{\ell = 2^i} 2^{i+2^{i-2}} \mathbb{I}_{(\ell,k+\ell-2^i) \in E}. \tag{7.6}
\]
There being \(2^{i-1} + 1 \leq 2^i\) indices \(k \in 2N \cap [2^{i+4}, 2^{i+4} + 2^i]\), one such \(k\) satisfies
\[
\sum_{\ell = 2^i} 2^{i+2^{i-2}} \mathbb{I}_{(\ell,k+\ell-2^i) \in E} \geq \frac{3}{8} \left(2^{i-2} + 1 - 2(2^{i+5})^{1-\epsilon_1}\right). \tag{7.7}
\]
Noting that \(\frac{3}{8}(2^{i+5})^{1-\epsilon_1} \leq \frac{1}{8} 2^{i-2}\) concludes the proof. □

Proof of Proposition 7.7. Let \(Q \in 2N \cap [2^{i+4}, 2^{i+4} + 2^i]\) be minimal such that such that there are at least \(2^{i-4}\) values of \(k \in 2N \cap [0, 2^{i-2}]\) with \((2^i + k, Q + k) \in E\).

Let \((k_1, \ldots, k_{2^{i-4}})\) be an increasing sequence of elements of \(2N \cap [0, 2^{i-2}]\) such that \((2^i + k_j, Q + k_j) \in E\) for each \(j \in [1, 2^{i-4}]\). Set \(m' = Q - 2^i\) and note that \(m' \in [2^{i+3}, 2^{i+4}]\). Take \(a = 1\) in Lemma 7.9 and recall that \(\epsilon_2 > \epsilon_1\) to find that, for all such \(j\), \(2^i + k_j \in K'\). Thus, \(|K'| \geq 2^{i-4} \geq 2^{-8} m'\).

This completes the proof of the proposition. □
7.4. Similarity of measure. Continuing to work with the constructed sequence, we set $r_j = 2^i + k$ and $s_j = Q + k$ for $j \in [1, 2^{i-4}]$. We also set $L = 2^{i-4}$ equal to the length of the sequence.

**Proposition 7.11.** For each $j \in [1, L]$, there is a subset $D_j \subseteq \Phi_{r_j}$ satisfying

$$P_{s_j}(\Gamma_{[0,r_j]} \in D_j) \geq 1 - s_j^{-5\epsilon}$$

such that, for each $\phi \in D_j$,

$$P_{s_L}(\Gamma_{[0,r_j]} = \phi) \geq \frac{1}{16} \left(\frac{3}{16}\right)^{3/2+\epsilon} s_L^{-4\epsilon} P_{s_j}(\Gamma_{[0,r_j]} = \phi, \Gamma \in \text{RGJ}_{k, s_j-16-k} \text{ for some } k \in R_{i+2}) .$$

Preparing for the proof of this proposition, we define, for each $\gamma \in \text{SAP}_{s_j}$, $1 \leq j \leq s_L$, the set of $\gamma$’s regulation global join indices

$$\tau_\gamma = \left\{ k \in R_{i+2} : \gamma \in \text{RGJ}_{k, s_j-16-k} \right\} .$$

For $\gamma \in \text{SAW}_{s_j}$, the map sending $k \in \tau_\gamma$ to the junction plaquette associated to the join of elements of $\text{SAP}_{k}^{\text{left}}$ and $\text{SAP}_{s_j-16-k}^{\text{right}}$ is an injective map into $GJ_\gamma$: see the proof of Lemma 3.16. Thus, $|\tau_\gamma| \leq |GJ_\gamma|$. As we saw in (7.1), our assumptions
imply that the closing probability has polynomial decay; thus, Corollary 3.9(1) implies that there exists a constant \( C_j > 0 \) such that

\[
P_{s_j} \left( |\tau| \geq C_j \log s_j \right) \leq s_j^{-10^j}. \tag{7.8}
\]

We make use of this constant to specify for each \( j \in [1, L] \) a set \( D_j \subseteq \Phi_{r_j} \) of length \( r_j \) first parts that, when extended to form a length \( s_j \) polygon, do not typically produce far-above-average numbers of regulation joinings. We define

\[ D_j = \left\{ \phi \in \Phi_{r_j} : P_{s_j} \left( |\tau| \leq C_j \log s_j \left| \Gamma_{[0,r_j]} = \phi \right. \right) \geq 1 - s_j^{-5^j} \right\}, \]

and argue that elements of \( D_j \) are rare.

**Lemma 7.12.** Let \( j \in [1, L] \). Then

\[
P_{s_j} \left( \Gamma_{[0,r_j]} \in D_j^c \right) \leq s_j^{-5^j}.
\]

**Proof.** Note that

\[
P_{s_j} \left( |\tau| \geq C_j \log s_j \right) \geq P_{s_j} \left( |\tau| \geq C_j \log s_j , \Gamma \in D_j^c \right)
\]

\[ = P_{s_j} \left( \Gamma \in D_j^c \right) P_{s_j} \left( |\tau| \geq C_j \log s_j \left| \Gamma \in D_j^c \right. \right) \geq s_j^{-5^j} P_{s_j} \left( \Gamma \in D_j^c \right). \]

The lemma thus follows from (7.8). \( \square \)

**Lemma 7.13.** Recall that \( L = 2^{i-4} \). Let \( j \in [1, L] \).

1. For \( \phi \in D_j \),

\[
P_{s_L} \left( \Gamma_{[0,r_j]} = \phi \right) \geq \frac{1}{160} s_L^{-2^j} \mu^{-16} \sum_{k \in R_{i+2}} \left( P_{k} \left( \Gamma_{[0,r_j]} = \phi \right) k^{1/2} p_k \mu^{-k} \right) \cdot
\]

2. For such \( \phi \),

\[
P_{s_j} \left( \Gamma_{[0,r_j]} = \phi, \Gamma \in \text{RGJ}_{k,s_j-16-k} \text{ for some } k \in R_{i+2} \right)
\]

\[
\leq \frac{1}{160} \left( \frac{13}{3} \right)^{3/2+\chi} \mu^{-16} s_j^{-2^j} \sum_{k \in R_{i+2}} \left( P_{k} \left( \Gamma_{[0,r_j]} = \phi \right) k^{1/2} p_k \mu^{-k} \right).
\]

**Proof:** (1). We begin by making the key observation that, for \( k \in R_{i+2}, \)

\[
P_{s_L} \left( \Gamma_{[0,r_j]} = \phi \left| \Gamma \in \text{RGJ}_{k,s_L-16-k} \right. \right) = P_{k} \left( \Gamma_{[0,r_j]} = \phi \right).
\]

Lemma 7.3 implies this, provided that its hypotheses that \( r_j \leq k/2 \) and \( k/2 \leq s_L - 16 - k \leq 35 k \) are valid. These bounds follow from \( 2^{i+3} \geq k \geq 2^{i+2}, 2^{i+5} \geq s_L \geq 2^{i+4} \) and \( i \geq 2 \).

By also using Lemmas 3.13 and 3.16, we learn that

\[
P_{s_L} \left( \Gamma_{[0,r_j]} = \phi \left| \Gamma \in \text{RGJ}_{k,s_L-16-k} \right. \right) = P_{k} \left( \Gamma_{[0,r_j]} = \phi \right) P_{s_L} \left( \Gamma \in \text{RGJ}_{k,s_L-16-k} \right)
\]
where we used \( 2 \)

Using \( p \)

\( \text{Proposition 7.11} \), at least

\( (2) \).

We find that

\[
\text{Proof of Proposition 7.6.} \quad \text{Note that the right-hand side of the bound}
\]

\[
\text{Proof of Proposition 7.11.} \quad \text{Using } p_{\text{sl}} \leq \mu_{\text{sl}} s_{\text{L}}^{-3/2-\epsilon+\epsilon}, \text{we obtain Lemma 7.13(1)}.
\]

\( (2) \). For \( \phi \in D_{j} \),

\[
\begin{align*}
\text{Lemma 7.11} & \quad \text{A consequence of Lemmas 7.12 and 7.13} \\
\text{Proof of Proposition 7.6} & \quad \text{Note that the right-hand side of the bound}
\end{align*}
\]
by Lemma 7.9 with \( a = 1 \), Lemma 7.5 with \( \varepsilon = \varepsilon_2/2 \), and Lemma 7.12. Thus,

\[
P_{s_L} \left( \Gamma_{[0,r_j]} \in \text{Hi}_{\Phi_1}^{1/2+2\chi+\varepsilon_2} \right) \geq c s_L^{-4\varepsilon} \left( s_j^{-(\chi+\varepsilon_2/2)} - s_j^{-(\chi+\varepsilon_2-\varepsilon_1)} - s_j^{-5\chi} \right).\]

The right-hand side is at least \( \frac{1}{2} s_L^{-\chi-\varepsilon_2/2-4\varepsilon} \) if we insist that \( 4\chi > \varepsilon_2 > 2\varepsilon_1 \).

Recalling that \( m' = Q - 2^i \), we set \( m = s_L \), and note that \( 2^{i+3} \leq m' \leq 2^{i+4} \leq m \leq 2^{i+5} \). Note that \( r_j \) belongs to the set \( K \) specified in Proposition 7.6 whenever \( j \in [1, L] \), because \( r_j = s_j - m' \leq m - m' \). Thus, \( \#K \geq 2^{i-4} \geq 2^{-9}m \). Set \( \varepsilon_2 = 4\varepsilon_1 \) and \( \varepsilon_1 = \varepsilon \), and Proposition 7.6 is proved.

We review our approach in the context of Figure 13. Note that, in the three bullet points above, the second line asserts that regulation global join polygons, while not necessarily typical, are not so rare under \( P_{s_j} \). The first line asserts that the initial subpath \( \Gamma_{[0,r_j]} \) is highly likely to have typical conditional closing probability when a length \( s_j - r_j \) extension is considered. Since \( s_j - r_j \) is independent of \( j \), we have labelled it, calling it \( m' \), and noted that it plays the role of \( \tilde{n} \) in the explanation that accompanies Figure 13. The first bullet point, similarly to Proposition 7.7, corresponds to the informal claim in the figure’s caption that black dots are very typically present at \( \tilde{n} \) steps from the end of a polygon. The extra margin in the choice of \( \alpha = 1/2 + 2\chi + o(1) \) that we alluded to after Proposition 7.7 ensures that such black dots are so highly typical that they remain typical even among regulation polygons; i.e., the error in the first bullet point is smaller than in the second. For these regulation polygons, we have implemented comparison of measure in Lemma 7.13. Pursuing the caption’s story: black dots make the jump as length changes from \( s_j \) to \( s_L \) with probability \( s_L^{-4\varepsilon} \) and are thus proved in Proposition 7.6 to have a non-negligible probability of appearing in generic locations in a uniform polygon of length \( m = s_L \).

7.5. **Proof of Theorem 1.5(1).** We begin by presenting statements and proofs for the principal changes needed to derive the theorem. First we present explicitly the assumption that we will invoke, using the notation of Definition 3.1.

**Hypothesis Closing Probability CP_\chi.** The set \( 2N \setminus \text{HCP}_{1/2} \) has limsup density in \( 2N \) less than \( 1/18200 \).

Here is the detailed version of Theorem 1.5(1), counterpart to Proposition 7.4. Note the evident contradiction concluded here.

**Proposition 7.14.** Let \( d = 2 \). Let \( \chi \in (0, 1/22) \), and assume Hypothesis CP_\chi. For some \( c > 1 \) and \( \delta > 0 \), the set of \( n \in 2N \cap \text{HCP}_\chi \) for which

\[
W_{n-1}(\Gamma \text{ closes}) \leq c^n \delta
\]

intersects the dyadic scale \([2^i, 2^{i+1}]\) for all but finitely many \( i \in \mathbb{N} \).
In the preceding section, we had the luxury of assuming the existence of \( \theta = \lim_{n \to \infty} \theta_n \). Now we must work with the information provided by Hypothesis \( \text{CP}_\chi \), that the closing probability is at least \( n^{-1/2 - \chi} \) on a positive proportion of indices \( n \).

In Proposition 7.15 we will show that this forces a positive proportion of the polygon deviation exponents \( \theta_n \) to lie in any given open set containing the interval \( [3/2 - \chi, 3/2 + \chi] \). In the preceding proof, we knew this for the one-point set \( \{3/2 + \chi\} \), for all high \( n \).

The weaker regularity leads to a counterpart to Lemma 7.5. Where before we found a lower bound on the probability that a polygon is regulation global join of the form \( n^{-\chi-o(1)} \), now we found only a bound \( n^{-3\chi-o(1)} \). The counterpart result is Lemma 7.17 its statement and proof follow Proposition 7.15's.

These changes explained, we reset the snake method’s parameters to handle the weaker information available. The mechanism of measure comparison for initial subpaths of polygons drawn from the laws \( P_n \) for differing lengths \( n \) is no longer made via all regulation global join polygons but rather via such polygons whose length index lies in a certain regular set. The changes to Lemma 7.13 are presented in Lemma 7.21. Using the new lemma, we then conclude the proof of Theorem 1.5(1).

7.6. Principal changes to the proof: weak regularity.

**Proposition 7.15.** Assume Hypothesis \( \text{CP}_\chi \). Let \( \epsilon \in (0,1) \). For all but finitely many values of \( i \in \mathbb{N} \),

\[
\# \{ j \in 2\mathbb{N} \cap [2^i, 2^{i+1}] : \frac{3}{2} - \chi - \epsilon \leq \theta_j \leq 3/2 + \chi + \epsilon \} \geq 2^{i-1} \left(1 - \frac{1}{300}\right).
\]

**Proof.** Assume Hypothesis \( \text{CP}_\chi \). Let \( \epsilon \in (0,1) \). For all but finitely many values of \( i \in \mathbb{N} \),

\[
\# \{ j \in 2\mathbb{N} \cap [2^i, 2^{i+1}] : \theta_j \leq 3/2 - \chi - \epsilon \} \leq \frac{1}{600} 2^{i-1}. \tag{7.9}
\]

Under the same hypothesis, and also for but finitely many \( i \),

\[
\# \{ j \in 2\mathbb{N} \cap [2^i, 2^{i+1}] : \theta_j > 3/2 + \chi + \epsilon \} \leq \frac{1}{600} 2^{i-1}. \tag{7.10}
\]

Of these, (7.10) is the simpler to derive. To do so, we make a

**Claim.** For any \( \delta > 0 \), the set

\[
\text{HCP}_{1/2+\chi} \setminus \{ n \in 2\mathbb{N} : \theta_n \leq 3/2 + \chi + \delta \}
\]

is finite.

(We have yet to respecify the snake method parameter \( n \), and have reverted to treating \( n \) as a free variable.)
To verify the claim, note that \(2np_n/c_{n-1} \geq n^{-1/2-\chi}\) whenever \(n \in \text{HCP}_{1/2+\chi}\). From \(c_{n-1} \geq \mu^{n-1}\) and \(p_n = \mu^n n^{-\theta_n}\), we learn that \(n^{-\theta_n} \geq 1/n^{3/2-\chi}\) for such \(n\), and the claim follows.

From the claim, it is immediate that, under Hypothesis CP\(_{\chi}\), the set of even integers for which \(\theta_n > 3/2 + \chi + \varepsilon\) has limsup density in \(2\mathbb{N}\) at most 1/18200. From this, [7.10] follows directly.

Define the set of \(\varepsilon\)-low \(\theta\) values on the \(i\)-th dyadic scale,

\[
\text{Low} \Theta_i^\varepsilon = \left\{ j \in 2\mathbb{N} \cap [2^i, 2^{i+1}] : \theta_j \leq \frac{3}{2} - \varepsilon \right\}.
\]

We now derive (7.9). Note that the inequality (7.9) says that \(\text{Low} \Theta_i^\varepsilon\) is at most \(1/600 \cdot 2^{i-1}\).

Let \(n \in 2\mathbb{N} \cap [2^{i+2}, 2^{i+3}] \cap \text{HCP}_{1/2+\chi}\), and use Proposition 3.3 to find that

\[
p_n \geq \frac{n^{1/2}}{C_1 \log n} \sum_{j=2^i}^{2^{i+1}} p_j p_{n-j} \geq \frac{\mu^n n^{1/2}}{C_1 \log n} \sum_{j \in \text{Low} \Theta_i^\varepsilon} j^{-\theta_j(n-j)} - \theta_{n-j}
\]

\[
\geq \frac{\mu^n n^{1/2}}{C_1 \log n} \cdot |\text{Low} \Theta_i^\varepsilon \cap (n - \text{HCP}_{1/2+\chi})| \cdot n^{-3/2+\chi+\varepsilon} n^{-3/2-\chi-\varepsilon/2}
\]

\[
= \mu^n \frac{n^{-5/2+\varepsilon/2}}{C_1 \log n} \cdot |\text{Low} \Theta_i^\varepsilon \cap (n - \text{HCP}_{1/2+\chi})|,
\]

where we used the claim with \(\delta = \varepsilon/2\) as well as \(j \leq n\) in the third inequality.

If \(|\text{Low} \Theta_i^{\chi+\varepsilon} \cap (n - \text{HCP}_{1/2+\chi})| > 1/60 \cdot 2^{i-1}\), then, since \(2^{i-1} > 2^{-4}n\),

\[
p_n \geq \mu^n \frac{n^{-3/2+\varepsilon/2}}{300 \cdot 2^6 C_1 \log n},
\]

and so

\[
\theta_n \leq \frac{3}{2} - \varepsilon/2 + \frac{\log(300C_1) + 6 \log 2 + \log \log n}{\log n}.
\]

We have this for all \(n\) belonging to the set \(2\mathbb{N} \cap [2^{i+2}, 2^{i+3}] \cap \text{HCP}_{1/2+\chi}\) which, by Hypothesis CP\(_{\chi}\), occupies a proportion of \(2\mathbb{N} \cap [2^{i+2}, 2^{i+3}]\) of at least \(1 - 1/9600\). By Proposition 3.3, this state of affairs may obtain for only finitely many values of \(i \in \mathbb{N}\). Thus, \(|\text{Low} \Theta_i^{\chi+\varepsilon} \cap (n - \text{HCP}_{1/2+\chi})| \leq \frac{1}{60} \cdot 2^{i-1}\) for all but finitely many \(i \in \mathbb{N}\). We find that \(|\text{Low} \Theta_i^{\chi+\varepsilon}|\) is at most

\[
|\text{Low} \Theta_i^{\chi+\varepsilon} \cap (n - \text{HCP}_{1/2+\chi})| + |(n - \text{HCP}_{1/2+\chi}^c) \cap 2\mathbb{N} \cap [2^i, 2^{i+1}]|
\]

\[
\leq \frac{1}{60} 2^{i-1} + \frac{1}{60} 2^{i-1},
\]

where we used \(n \leq 2^{i+3}\) in the form

\[
(n - \text{HCP}_{1/2+\chi}^c) \cap 2\mathbb{N} \cap [2^i, 2^{i+1}] = \text{HCP}_{1/2+\chi}^c \cap 2\mathbb{N} \cap [n - 2^{i+1}, n - 2^i] \subseteq \text{HCP}_{1/2+\chi}^c \cap 2\mathbb{N} \cap [0, 2^{i+3}]
\]
as well as Hypothesis CP\(\chi\) to find that
\[
\left| (n - HCP_{1/2+\chi}^\varepsilon) \cap 2N \cap \left[ 2^i, 2^{i+1} \right] \right| \leq \frac{1}{900} 2^{i+3}.
\]
That is, \(\left| \text{Low}\Theta_i^{\chi+\varepsilon} \right| \leq \frac{1}{900} 2^{i-1}\) – i.e., (7.10) holds – for all but finitely many values of \(i \in \mathbb{N}\).

This completes the derivation of Proposition 7.15. \(\square\)

During the proof of Theorem 1.5(2), in Subsection 7.2.1 we explained that we would respecify non-trivially the regular index set \(R_i\) that appears there. We will shortly define such a set \(R_{n,i}^{\chi+\varepsilon}\). It will be a subset of \(K_{n,i}^{\chi+\varepsilon}\) as now specified.

**Definition 7.16.** Let \(i \in \mathbb{N}\) and \(n \in 2N \cap \left[ 2^{i+2}, 2^{i+3} \right]\). For \(\varepsilon > 0\), set
\[
K_{n,i}^{\varepsilon} = \left\{ j \in 2N \cap \left[ 2^i, 2^{i+1} \right] : \max \left\{ \theta_j, \theta_{n-16-j} \right\} \leq \frac{3}{2} + \varepsilon \right\}.
\]

For \(i \in \mathbb{N}\), \(n \in 2N \cap \left[ 2^{i+2}, 2^{i+3} \right]\) and \(\varepsilon \in (0, 2)\), we claim that, under Hypothesis CP\(\chi\),
\[
\left| K_{n,i}^{\chi+\varepsilon} \right| \geq \frac{9}{16} 2^{i-1}.
\] (7.11)

Indeed, defining the set of \(\varepsilon\)-high \(\theta\) values on the \(i\)-th dyadic scale,
\[
\text{High}\Theta_i^{\varepsilon} = \left\{ j \in 2N \cap \left[ 2^i, 2^{i+1} \right] : \theta_j \geq \frac{3}{2} + \varepsilon \right\},
\]
and also setting \(S_{n,i}^{\varepsilon} = 2N \cap \left[ 2^i, 2^{i+1} \right] \setminus K_{n,i}^{\varepsilon}\), we have that, for any \(\varepsilon > 0\),
\[
S_{n,i}^{\varepsilon} \subseteq \text{High}\Theta_i^{\varepsilon} \cup \left\{ j \in 2N \cap \left[ 2^i, 2^{i+1} \right] : \theta_{n-16-j} \geq \frac{3}{2} + \varepsilon \right\};
\]
the latter event in the right-hand union is a subset of
\[
2N \cap \left[ 2^{i+1}-2^4, 2^{i+1} \right] \cup \left\{ j \in 2N \cap \left[ 2^i, 2^{i+1} - 2^4 \right] : n-16-j \in \text{High}\Theta_{i+1}^{\varepsilon} \cup \text{High}\Theta_{i+2}^{\varepsilon} \right\},
\]
because \(2^{i+1} \leq n - 16 - j \leq 2^{i+3} - 2^i \leq 2^{i+3}\) for \(j \in \left[ 2^i, 2^{i+1} - 2^4 \right]\). Thus,
\[
\# S_{n,i}^{\varepsilon} \leq \# \text{High}\Theta_i^{\chi+\varepsilon} + \# \text{High}\Theta_{i+1}^{\chi+\varepsilon} + \# \text{High}\Theta_{i+2}^{\chi+\varepsilon} + 9,
\]
so that (7.10) implies \(\left| S_{n,i}^{\varepsilon} \right| \leq \frac{7}{600} 2^{i-1} + 9 \leq \frac{1}{16} 2^{i-1}\) (for \(i \geq 8\)) and thus (7.11).

Here is the promised counterpart to Lemma 7.3.

**Lemma 7.17.** For \(i \in \mathbb{N}\), let \(n \in 2N \cap \left[ 2^{i+4}, 2^{i+5} \right]\) satisfy
\[
\theta_n \geq \frac{3}{2} - \chi - \varepsilon \quad \text{and} \quad W_{n-1}(\Gamma) \text{ closes} \geq n^{-1/2-\chi}.
\]
Then, for any \(R \subseteq K_{n,i+2}^{\chi+\varepsilon}\) such that \(\left| K_{n,i+2}^{\chi+\varepsilon} \setminus R \right| \leq 3 \cdot 2^{i-1}\), we have that
\[
P_n \left( \Gamma \in \text{RGJ}_{k,n-16-k} \text{ for some } k \in R \right) \geq \frac{c}{\log n} n^{-(3\chi+3\varepsilon)},
\]
where \(c > 0\) is a universal constant.
Proof. Similarly to before, this probability equals the ratio of the cardinality of \( \bigcup_{k \in R} RGJ_{k, n-16-k} \) and \( p_n \). In this case, we use the lower bound on the former quantity that is provided next in Lemma 7.18, and further note that

\[
\sum_{k \in R} p_k p_{n-16-k} \geq |R| \cdot \mu^n \cdot n^{-3-2\chi-2\varepsilon}
\]

(7.12)

\[
\geq \mu^n \left( \frac{9}{11} - \frac{1}{3} \right) 2^{i+1} \cdot n^{-3-2\chi-2\varepsilon} \geq \mu^n \cdot \frac{1}{110} n^{-2-2\chi-2\varepsilon},
\]

where we used (7.11), \( |K_{n+i+2}^\chi \cap R| \leq 3 \cdot 2^{i-1} \), and \( 2^{i+1} \geq 2^{-4} n \). We find a lower bound on the numerator of the ratio:

\[
\left| \bigcup_{k \in R} RGJ_{k, n-16-k} \right| \geq \mu^n \cdot \frac{c}{110 \log n} n^{-3/2-2\chi-2\varepsilon}.
\]

The hypothesis on the index \( n \) provides the upper bound on the ratio’s denominator in the form \( p_n \leq \mu^n n^{-3/2+\chi+\varepsilon} \). The lemma follows by relabelling \( c > 0 \).

Lemma 7.18. Let \( \varepsilon > 0 \), \( i \in \mathbb{N} \), and let \( n \in 2\mathbb{N} \cap [2^{i+4}, 2^{i+5}] \) satisfy

\[
\theta_n \geq \frac{3}{2} - \chi - \varepsilon \quad \text{and} \quad W_{n-1}(\Gamma \text{ closes}) \geq n^{-1/2-\chi}.
\]

We have that, for a universal constant \( c > 0 \),

\[
\left| \bigcup_{k \in R} RGJ_{k, n-16-k} \right| \geq \frac{c n^{1/2}}{\log n} \sum_{k \in R} p_k p_{n-16-k},
\]

where \( R \) once again denotes any subset of \( K_{n+i+2}^\chi \) for which \( |K_{n+i+2}^\chi \cap R| \leq 3 \cdot 2^{i-1} \).

Proof. We rework the proof of Lemma 3.17, the central tool for proving Proposition 3.2. Set \( SAP_n^\varepsilon \subseteq SAP_n \),

\[
SAP_n^\varepsilon = \bigcup_{k \in R} RGJ_{k, n-16-k}.
\]

The multi-valued map is specified to be \( \Psi : A \to \mathcal{P}(B) \), where now

\[
A = \bigcup_{j \in R} SAP_{n-16-j}^\text{left} \times SAP_{n-16-j}^\text{right}, \quad \text{and} \quad B = SAP_n^\varepsilon,
\]

according to the same rule as the one used in the earlier proof. Note that, since \( R \) is a subset of \( 2\mathbb{N} \cap [2^{i+2}, 2^{i+3}] \), this choice of \( A \) belongs to a different dyadic scale to its counterpart in the earlier argument.

Set

\[
H_n^\varepsilon = \left\{ \phi \in SAP_n^\varepsilon : |GJ_\phi| \geq C_5 \log n \right\},
\]

and pursue the analysis in the proof of Lemma 3.17 with \( H_n^\varepsilon \) replacing \( H_{n+16} \).

The first case. We have that

\[
|H_n^\varepsilon| \cdot \max \left\{ |\Psi^{-1}(\phi)| : \phi \in H_n^\varepsilon \right\} \geq \frac{1}{2} \cdot 2^{i/2+1} \frac{1}{10} \sum_{j \in R} p_j p_{n-16-j},
\]
and that

\[ |H_n^\varepsilon| \geq \frac{1}{10} 2^{-5/2} \cdot n^{-1/2} \sum_{j \in \mathbb{R}} p_j p_{n-16-j}. \]

With the choice of constants in the earlier proof, it follows similarly as there that

\[ |H_n^\varepsilon| \leq n^{-C_6/2} |\text{SAP}_n|. \]

Since \( H_n^\varepsilon \subseteq H_n \) and \( |\text{SAW}_{n-1}| \geq 2n |\text{SAP}_n| \) by (1.3), we find that

\[ |\text{SAW}_{n-1}| \geq \frac{1}{10} 2^{-3/2} \cdot n^{C_6/2+1/2} \sum_{j \in \mathbb{R}} p_j p_{n-16-j}. \]

Using (7.12),

\[ c_{n-1} \geq \frac{1}{10} 2^{-3/2} \cdot n^{C_6/2+1/2} \cdot \mu n^\varepsilon \frac{1}{10} n^{-2-2\chi-2\varepsilon}. \]

Since \( W_{n-1} (\Gamma \text{ closes}) \geq n^{-1/2-\chi}, \) \( p_n > \mu n ^{10^{-4}} 2^{-5/2} \cdot n^{C_6/2-3/2-2\chi-2\varepsilon} \cdot n^{-\chi-3/2}. \)

However, this contradicts \( p_n \leq \mu n ^{n^{-3/2+\chi+\varepsilon}}. \)

**The second case.** Note that, for \( \phi \in \text{SAP}_{n+16}^\varepsilon \setminus H_{n+16}, \) \( |\Psi^{-1}(\phi)| \leq C_5 \log n. \)

Thus,

\[ |\text{SAP}_{n}^\varepsilon \setminus H_n^\varepsilon| \cdot C_5 \log n \geq 2^{i/2} \frac{1}{10} \sum_{j \in \mathbb{R}} p_j p_{n-16-j} \]

and, using \( n \leq 2^{i+5}, \) we find that

\[ |\text{SAP}_{n}^\varepsilon| \geq \frac{2^{-5/2} n^{1/2} \sum_{j \in \mathbb{R}} p_j p_{n-16-j}}{10C_5 \log n}. \]

\[ \square \]

**7.7. The snake method parameters, and similarity of measure revisited.**

The snake method’s exponent parameters are set so that

- \( \beta = 1; \)
- \( \alpha = 1/2 + 4\chi + 5\varepsilon; \)
- and \( \eta = 7\chi + 9\varepsilon. \)

Note that \( \delta = \beta - \eta - \alpha, \) which must be positive if the method to work, is equal to \( 1/2 - 11\chi - 14\varepsilon. \) Since \( \chi < 1/22, \) we may take \( \varepsilon \in (0, (1/2 - 11\chi)/14) \) to ensure that this is the case.

The index parameters were previously set \( n = m - 1 \) and \( \ell = m - m' \) where Proposition 7.6 provided \( m \) and \( m'. \) We now make the same choice, with the upcoming Proposition 7.22 providing the latter two quantities.

Fix a parameter \( \varepsilon_1 > 0. \) Recall the set \( E \) of index pairs \((k, j)\) specified in Definition 7.8. Let \( E' \) be the subset of \( E \) consisting of such pairs for which \( j \in \text{HCP}_{1/2+\chi} \) and

\[ 3/2 - \chi - \varepsilon \leq \theta_j \leq 3/2 + \chi + \varepsilon. \]
Proposition 7.15 implies that the set of $j \in 2\mathbb{N} \cap [2^{i+4}, 2^{i+5}]$ satisfying (7.13) — a set containing all second coordinates of pairs in $E'$ — has cardinality at least $(1 - \frac{1}{300})2^{i+3}$. Invoking Hypothesis CP$_x$, we see that, of these values of $j$, at most \(\frac{1}{960}2^{i+3}\) fail the test of membership of HCP$_{1/2+\chi}$. Lemma 7.10 is thus easily seen to hold when $E'$ replaces $E$.

We now specify the promised set $R_{s_j,i+2}^{\chi+\varepsilon}$ of regular indices that forms a non-trivial analogue of the set $R_{i+2}$ with which we worked in the proof of Theorem 1.5(2).

**Definition 7.19.** For $j \in [1, s_L]$, define $R_{s_j,i+2}^{\chi+\varepsilon}$ to be the set of $k \in 2\mathbb{N} \cap [2^{i+2}, 2^{i+3}]$ such that $k \in K_{s_j,i+2}^{\chi+\varepsilon}$ and

\[
\theta_{s_j-k} - k \geq \frac{3}{2} - \chi - \varepsilon, \quad \theta_{s_L-k} - k \leq \frac{3}{2} + \chi + \varepsilon.
\]

Note that $k \in K_{s_j,i+2}^{\chi+\varepsilon} \setminus R_{s_j,i+2}^{\chi+\varepsilon}$ implies that either $s_j - k$ belongs to the union up to index $i + 4$ of the sets in (7.9), or $s_L - k$ belongs to the comparable union of the sets in (7.10). Thus, $K_{s_j,i+2}^{\chi+\varepsilon} \setminus R_{s_j,i+2}^{\chi+\varepsilon}$ has cardinality at most $\frac{1}{12}2^{i+4} \leq 3 \cdot 2^{i-1}$. For this reason, the choice $R = R_{s_j,i+2}^{\chi+\varepsilon}$ that we will shortly make in Lemma 7.11 is an admissible one.

Proposition 7.11 becomes the next result.

**Proposition 7.20.** For each $j \in [1, L]$, there is a subset $D_j \subseteq \Phi_{r_j}$ satisfying

\[
P_{s_j}(\Gamma_{[0,r_j]} \subseteq D_j) \geq 1 - s_j^{-5}\chi
\]

such that, for each $\phi \in D_j$,

\[
P_{s_L}(\Gamma_{[0,r_j]} = \phi) \geq \frac{1}{16} \left(\frac{3}{16}\right)^{3/2} s_j^{-4\chi - 4\varepsilon} P_{s_j}(\Gamma_{[0,r_j]} = \phi, \Gamma \in R_{s_j,i+2}^{\chi+\varepsilon+\varepsilon_5} for some k \in R_{s_j,i+2}^{\chi+\varepsilon+\varepsilon_5}).
\]

**Proof.** Respecifying the index set $\tau_\gamma$ with $R_{s_j,i+2}^{\chi+\varepsilon+\varepsilon_5}$ replacing $R_{i+2}$, we invoke Corollary 3.9(1) using $s_j \in HCP_{1/2+\chi}$ in order to obtain (7.8). Thus, we obtain Lemma 7.12 again. We then use the next claim in place of Lemma 7.13.

**Lemma 7.21.** Let $j \in [1, L]$.

(1) For $\phi \in D_j$,

\[
P_{s_L}(\Gamma_{[0,r_j]} = \phi) \geq \frac{1}{100} s_L^{2\chi - 2\varepsilon} \mu^{-16} \sum_{k \in R_{s_j,i+2}^{\chi+\varepsilon}} P_{k}\left(\Gamma_{[0,r_j]} = \phi\right) k^{1/2} p_k \mu^{-k}.
\]

(2) For such $\phi$,

\[
P_{s_j}(\Gamma_{[0,r_j]} = \phi, \Gamma \in R_{s_j,i+2}^{\chi+\varepsilon+\varepsilon_5} for some k \in R_{s_j,i+2}^{\chi+\varepsilon+\varepsilon_5})
\]
Thus, fixing a parameter $\mu$ the constructed $m$ is, by Proposition 7.20, at least 

$$p_{s_L - 16 - k} \geq \mu^{s_L - 16 - k} (s_L - 16 - k)^{-3/2 - \epsilon} \geq \mu^{s_L - 16 - k} s_L^{-3/2 - \epsilon}.$$ 

The result follows from these bounds applied to the earlier argument.

(2) We apply $p_{s_j - k} \leq \mu^{s_j - k} (s_j - k)^{-3/2 + \epsilon}$ and $p_{s_j} \geq \mu^{s_j} s_j^{-3/2 - \epsilon}$ to the earlier proof. 

□

The new version of Proposition 7.6 differs from the original only in asserting that the constructed $m$ belongs to $\text{HCP}_{1/2 + \epsilon}$.

**Proposition 7.22.** For each $i \in \mathbb{N}$, there exists $m \in 2\mathbb{N} \cap [2^{i+4}, 2^{i+5}] \cap \text{HCP}_{1/2 + \epsilon}$ and $m' \in [2^{i+3}, 2^{i+4}]$ such that, writing $K$ for the set of values $k \in \mathbb{N}$, $1 \leq k \leq m - m'$, that satisfy

$$P_m \left( \Gamma_{[0,k]} \cap \text{Hi} \Phi_{k,k+m'-1}^0 \right) \geq m^{-\eta + \epsilon},$$

we have that $|K| \geq 2^{-9} m$.

**Proof.** Note that the right-hand side of the bound

$$P_{s_L} \left( \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon} \right) \geq P_{s_L} \left( \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \in D_j \right)$$

is, by Proposition 7.20 at least

$$\frac{1}{16} \left( \frac{1}{16} \right)^{3/2} s_j^{-4\epsilon - 4\epsilon} \left( \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \in D_j \right).$$

Fixing a parameter $\epsilon_2 > \epsilon_1$, note further that

- $P_{s_j} \left( \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \in D_j \right) \leq 1 - s_j^{-(3\epsilon + 3\epsilon)}$,
- $P_{s_j} \left( \Gamma_{[0,r_j]} \notin \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon} \right) \leq s_j^{-5\epsilon}$,
- $P_{s_j} \left( \Gamma_{[0,r_j]} \notin \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon}, \Gamma_{[0,r_j]} \notin D_j \right) \leq s_j^{-5\epsilon}$,

by Lemma 7.9 with $a = 3$, Lemma 7.17 (using that $s_j \in \text{HCP}_{1/2 + \epsilon}$), and Lemma 7.12.

Thus,

$$P_{s_L} \left( \Gamma_{[0,r_j]} \in \text{Hi} \Phi_{r_j,s_j}^{1/2 + 4\epsilon} \right) \geq \frac{1}{16} \left( \frac{1}{16} \right)^{3/2} s_j^{-4\epsilon - 4\epsilon} \left( \frac{c}{\log s_j} s_j^{-(3\epsilon + 3\epsilon)} - s_j^{-(3\epsilon + \epsilon_1)} - s_j^{-5\epsilon} \right).$$
The right-hand side is at least \( \frac{1}{32} \left( \frac{3}{16} \right)^{3/2} s_j^{-7\chi-7\varepsilon} \) if we set \( \varepsilon_2 = 5\varepsilon_1 \) and \( \varepsilon_1 = \varepsilon \) with \( \varepsilon \to 2\chi/3 \).

Recalling that \( m' = Q - 2^i \), we set \( m = s_L \), and note that \( 2^{i+3} \leq m' \leq 2^{i+4} \leq m \leq 2^{i+5} \). Note that \( r_j \) belongs to the set \( K \) specified in Proposition 7.6 whenever \( j \in [1, L] \), because \( r_j = s_j - m' \leq m - m' \). Thus, \( \#K \geq 2^{i-4} \geq 2^{-9}\mu \).

Finally, note that, since \( m = s_L \) is the second coordinate of a pair in the set \( E' \), we have that \( m \in \text{HCP}_{1/2+\chi} \). Proposition 7.22 is proved. \( \square \)

**Proof of Proposition 7.14.** The proof of Proposition 7.4 applies verbatim after recalling that the snake method’s parameter \( \delta = \beta - \eta - \alpha \) is positive, and noting that the snake index parameter \( n \) has been set equal to \( m = s_L \) and thus belongs to \( \text{HCP}_{1/2+\chi} \). \( \square \)

**Proof of Theorem 1.5(1).** Proposition 7.14’s conclusion contradicts its assumption, and the falseness of this assumption implies Theorem 1.5(1). \( \square \)

## 8. Variations in Three Dimensions

In stating Theorem 1.7 for the 3-edge self-avoiding walk model, we have chosen to illustrate the possibility of adapting the polygon joining technique to general dimension in a context comparatively unencumbered by technicalities. In this section, we present the proof of this result, and explain a very minor change needed to prove Theorem 1.2 when \( d \geq 3 \).

### 8.1. Proof of Theorem 1.7

The result is a three-dimensional analogue of Corollary 3.4 which follows from Proposition 3.2 via Proposition 3.3. We begin by stating an analogue of Proposition 3.2. For this, we need to define the notion of polygon for the 3-edge self-avoiding walk model.

**Definition 8.1.** Recall that a 3-edge self-avoiding walk \( \gamma : [0, n] \to \mathbb{Z}^d \) closes if \( ||\gamma_n|| = 0 \). Two such walks may be identified if they coincide after reparametrization by cyclic shift or reversal. A 3-edge self-avoiding polygon is an equivalence class under this relation on closing walks. The length of such a polygon is the length of any of its members (and is \( n \) if one of these members is \( \gamma \) as above).

Note that polygons have even length, just as before, but now, so do closing walks. For \( n \in 2\mathbb{N} \), let \( \mathcal{P}_n \) and \( \mathcal{P}_n^{3E} \) denote the number of, and the uniform law on, 3-edge self-avoiding polygons of length \( n \) up to translation.

Reformulating Definition 3.1 so that the law replaces \( \mathcal{W}_n^{3E} \) replaces \( \mathcal{W}_{n-1} \), we now state the counterpart to Proposition 3.2. The result is valid in any dimension \( d \geq 2 \) but we pursue only the three-dimensional case.
Proposition 8.2. Let $d = 3$. For any $\zeta > 0$, there is a constant $C_1 = C_1(\zeta) > 0$ such that, for $n \in 2\mathbb{N} \cap \text{HCP}_\zeta$,

$$\bar{p}_n \geq \frac{n^{1-1/d}}{C_1(\log n)^2} \sum_{j \in 2\mathbb{N} \cap [2^{i-1}, 2^i]} \bar{p}_j \bar{p}_{n-j},$$

where $i \in \mathbb{N}$ is chosen so that $n \in 2\mathbb{N} \cap [2^i, 2^{i+1}]$.

Note the presence of an extra logarithmic factor in the denominator.

Set $\mu$ equal to the connective constant $\lim_{n \to \infty} \frac{\log \bar{p}_n}{n}$, a limit whose existence is ensured by a slight variation of the proof of Proposition 2.3 (2). Further set $\bar{p}_n = n^{-\bar{V}_n} \mu^n$ for $n \in 2\mathbb{N}$.

Proposition 8.3. Let $d = 3$. Suppose that Hypothesis PCP holds. For any $\delta > 0$,

$$\left| \left\{ j \in 2\mathbb{N} \cap [2^i, 2^{i+1}] : \bar{V}_j \geq \frac{5}{3} - \delta \right\} \right| \geq \frac{1}{10} \cdot |2\mathbb{N} \cap [2^i, 2^{i+1}]|$$

for all but finitely many values of $i \in \mathbb{N}$.

Proof. This counterpart to Proposition 3.3 has a proof that contains no significant changes.

Proof of Theorem 1.7. In following the proof of Corollary 3.4, the only difficulty we encounter is to find an analogue of the walk number lower bound $c_n \geq \mu^n$. We need a counterpart to Proposition 2.3 for the 3-edge self-avoiding walk model. The proof given in this article, and the Hammersley-Welsh unfolding argument needed to show Proposition 2.3(4), can be adapted with only minor changes, a discussion of which we omit.

Definition 8.4. Let $n \in 2\mathbb{N}$ and let $\phi \in \text{SAP}_n^3$. A join edge of $\phi$ is a nearest neighbour edge in $\mathbb{Z}^d$ that is traversed precisely two times by $\phi$, with one crossing in each direction.

Note that when a join edge is removed from a polygon, two polygons result.

Definition 8.5. For $I$ any two element subset of $\{1, 2, 3\}$, let $\text{Proj}_I : \mathbb{Z}^3 \to \mathbb{Z}^3$ denote projection onto the axial plane containing the vectors $e_i$ for $i \in I$. Write $\text{Proj} = \text{Proj}_{2,3} : \mathbb{Z}^3 \to \{0\} \times \mathbb{Z}^2$.

We present an analogue of Madras’ local surgery procedure for polygon joining, respecifying the join $J(\phi, \phi')$ of two polygons. For $n, m \in 2\mathbb{N}$, let $\phi$ and $\phi'$ be polygons of lengths $n$ and $m$. Suppose that $\text{Proj}(\phi) \cap \text{Proj}(\phi') \neq \emptyset$. Translate $\phi'$ in the $e_1$-direction to the location, minimal in this coordinate, such that the vertex sets of $\phi$ and the $\phi'$-translate are disjoint. When two polygons have such a relative position, they are in a situation comparable to being Madras joinable; we call them simply joinable. Note that there exists an element in each vertex set that is an endpoint of an $e_1$-oriented edge of $\mathbb{Z}^3$. Let $e$ denote the maximal edge among
these (according to some fixed ordering of nearest neighbour edges of \( \mathbb{Z}^3 \)). Define \( J(\phi, \phi') \) to be the length \( n + m + 2 \) polygon formed by following the trajectory of \( \phi \) until an endpoint of \( e \) is encountered, crossing \( e \), following the whole trajectory of \( \phi' \), recrossing \( e \), and completing the trajectory of \( \phi \). The edge \( e \) will be called the *junction* edge in this construction.

Let the up vertex \( \text{Up}(\phi) \) of a polygon \( \phi \) be the vertex in \( \phi \) of maximal \( e_3 \)-coordinate that is the northeast element among such vertices. The up vertex is counterpart to the northeast vertex in the two-dimensional setting. For \( n \in 2\mathbb{N} \), we define \( \overline{\text{SAP}}^{3,E}_n \) to be the set of polygons of length \( n \) whose up vertex is the origin. Note that \( p_n = \# \overline{\text{SAP}}^{3,E}_n \). Let \( \overline{P}^{3,E}_n \) denote the uniform law on \( \overline{\text{SAP}}^{3,E}_n \).

**Definition 8.6.** Let \( n \in 2\mathbb{N} \) and let \( \phi \in \overline{P}^{3,E}_n \). A join edge \( e \) of \( \phi \) is called global if the two polygons comprising \( \phi \) without \( e \) may be labelled \( \phi^L \) and \( \phi^R \) in such a way that

- every vertex of maximal \( e_1 \)-coordinate in \( \phi^L \cup \phi^R \) belongs to \( \phi^R \);
- the up vertex \( \text{Up}(\phi) \) belongs to \( \phi^L \).

Write \( GJ_\phi \) for the set of global join edges of the polygon \( \phi \).

**Proposition 8.7.** There exists \( c > 0 \) such that, for \( n \in 2\mathbb{N} \) and any \( k \in \mathbb{N} \),

\[
|\text{SAW}^{3,E}_n| \geq c \exp \left\{ \frac{1}{22}(2d - 1)^{-2}k \right\} \cdot \# \left\{ \phi \in \overline{\text{SAP}}^{3,E}_n : |GJ_\phi| \geq k \right\}.
\]

**Proof.** Let \( \phi \in \overline{\text{SAP}}^{3,E}_n \). The right vertex \( \text{Right}(\phi) \) replaces \( \text{ES}(\phi) \) in this proof. It is defined to be any given vertex of maximal \( e_1 \)-coordinate that has minimal \( e_3 \)-coordinate among these. Set \( j \in [0, n] \) so that \( \phi_j = \text{Right}(\phi) \); write \( \phi^1 = \phi|_{[0,j]} \) and \( \phi^2 = \phi|_{[j,n]} \), and denote by \( R_z \) reflection in the vertical plane that passes through \( z \in \mathbb{Z}^3 \). Then set

\[
\mathcal{S}(\phi) = \phi^1 \circ R_{\text{Right}(\phi)}(\phi^2).
\]

The proof proceeds as Proposition 3.8’s did. A modification is made in specifying the alternative walks \( \mathcal{S}_\kappa(\phi) \), where \( \kappa \subseteq \{1, \ldots, r\} \). Denote the \( i \)-th global join edge \( e_i \in GJ_\phi \). This edge is traversed at most three times. Thus, either this edge is traversed at most once in \( \mathcal{S}(\phi) \), or its reflected image is. If the former applies, then the local modification for index \( i \) (analogous to the three step subpath around \( P^i \) in the original proof) takes the form of a two-step move, along \( e_i \) and straight back again, inserted after the last visit of \( \mathcal{S}(\phi) \) to an endpoint of \( e_i \). In the latter case, the same modification is made where the reflected image of \( e_i \) is used instead. Note that no edge is traversed more than three times in the resulting definition of \( \mathcal{S}_\kappa(\phi) \). (This property serves to explain our use of 3-edge self-avoiding walks.)

Write \( z(\vec{u}) = u_3 \) for \( \vec{u} \in \mathbb{Z}^3 \), and \( z_{\min}(\phi) \) and \( z_{\max}(\phi) \) for the minimal and maximal \( e_3 \)-coordinates occupied by the vertices of a polygon \( \phi \).
For \( n \in 2\mathbb{N} \), let \( \text{SAP}_{n}^{3,E,1} \) denote the set of left polygons \( \phi \in \text{SAP}_{n}^{3,E} \) such that \( |\text{Proj}(\phi)| \geq n^{2/3} \). Set \( \text{SAP}_{n}^{3,E,r} \), the set of right polygons, equal to \( \text{SAP}_{n}^{3,E,1} \).

**Lemma 8.8.** For \( n \in 2\mathbb{N} \),

\[
|\text{SAP}_{n}^{3,E,1}| \geq \frac{1}{6} \cdot |\text{SAP}_{n}^{3,E}|.
\]

**Proof.** We make use of the Loomis-Whitney inequality [25]: for any finite \( A \subseteq \mathbb{Z}^d \), then the product of the cardinality of the projections of \( A \) onto each of the \( d \) axial hyperplanes is at least \( |A|^{d-1} \). Also using the arithmetic-geometric mean inequality, one of the projections has cardinality at least \( |A|^{1/d} \). The lemma follows immediately. \( \square \)

**Proof of Proposition 8.2.** For polygons \( \phi \) and \( \phi' \), write \( \text{SJ}_{(\phi,\phi')} \) for the set of \( \bar{u} \in \mathbb{Z}^3 \) such that the pair \( (\phi, \phi' + \bar{u}) \) is strongly joinable, which is to say that

- \( (\phi, \phi' + \bar{u}) \) is simply joinable;
- the maximal \( e_3 \)-coordinate assumed among the vertices of \( \phi \) and \( \phi' + \bar{u} \) lies in \( \phi^1 \);
- the maximal such \( e_1 \)-coordinate lies in \( \phi^2 \).

We claim that, for \( n, m \in 2\mathbb{N} \), if \( \phi \in \text{SAP}_{n}^{3,E,1} \) and \( \phi' \in \text{SAP}_{m}^{3,E,r} \), then \( |\text{SJ}_{(\phi,\phi')}| \geq \frac{1}{2} \min \{ n^{2/3}, m^{2/3} \} \). The bound follows from noting that, whenever \( \text{Proj}(\phi' + \bar{u}) \) contains the vertex \( \text{Proj}(\text{Right}(\phi)) \), some \( e_1 \)-displacement of \( \bar{u} \) lies in \( \text{SJ}_{(\phi,\phi')} \).

We now proceed as in the proof of Lemma 3.17 considering the multi-valued map \( \Psi : A \rightarrow \mathcal{P}(B) \), where

\[
A = \bigcup_{j \in 2\mathbb{N} \cap [2^{-i-1}, 2^{-i}]} \text{SAP}_{n-j}^{3,E,1} \times \text{SAP}_{j}^{3,E,r}
\]

and \( B \) is, for example, the set of polygons of length \( n + 2 \) that contain the origin. The map \( \Psi \) associates to each \( (\phi^1, \phi^2) \in \text{SAP}_{n-j}^{3,E,1} \times \text{SAP}_{j}^{3,E,r} \), \( j \in 2\mathbb{N} \cap [2^{-i-1}, 2^{-i}] \), the set of length-\( (n + 2) \) polygons formed by simply joining \( \phi^1 \) and \( \phi^2 + \bar{u} \) for choices of \( \bar{u} \) in \( \text{SJ}_{(\phi^1,\phi^2)} \).

The proof follows the earlier one, invoking in place of Lemma 3.11 the readily verified statement that, in the join polygon \( J(\phi^1, \phi^2) \) of two polygons \( \phi^1 \) and \( \phi^2 \), the junction edge is a global join edge. Note that Corollary 3.9(1) may be invoked with obvious notational changes, because Proposition 3.7 replaces Proposition 3.8.

There is a notable difference compared to the argument for Lemma 3.17, however. Note that we did not take \( B \) to be \( \text{SAP}_{n+2}^{3,E} \), which is the obvious analogue of the earlier setup. The reason is that our new construction does not necessarily ensure that the origin is the up vertex of a polygon in \( \Psi(\phi^1, \phi^2) \).

To see how to overcome this difficulty, consider the two cases used in the proof.
In the first case, we learn that \( \#B \geq n^C \sum_{j=2^{-1}}^{2^{d-1}} \overline{p}_{n-j} \overline{p}_j \) for a large constant \( C \). Since any element of \( B \) contains at most \( n + 2 \) vertices, the element can be relocated so that its up vertex is the origin, and the bound \( \overline{p}_n \geq (n + 2)^{-1} \#B \), which is sufficient, results.

In the second case, consider for a given length \( n + 2 \) polygon \( \phi \) the set \( \text{UpGJ}(\phi) \) of up vertices of polygons formed by the removal of a global join edge from \( \phi \). In this case, we show that the set of length \( n + 2 \) polygons containing at most \( C \log n \) global join edges and for which \( 0 \in \text{UpGJ}(\phi) \) has cardinality at least \( c \frac{n^{1/2}}{\log n} \sum_{j=2^{-1}}^{2^{d-1}} \overline{p}_j \overline{p}_{n-j} \). The map that translates any such polygon so that its up vertex is the origin has preimage cardinality of at most \( 2C \log n \), because this quantity is an upper bound on \( \#\text{UpGJ}(\phi) \). Thus, \( \overline{p}_n \geq c \frac{n^{1/2}}{\log n} \sum_{j=2^{-1}}^{2^{d-1}} \overline{p}_j \overline{p}_{n-j} \), and we are done in the second case also.

\[ \square \]

8.2. **Theorem 1.2 when \( d \geq 3 \).** Beyond adopting a higher dimensional definition of type I and II pattern, which was already addressed in Section 5, the only change needed to prove Theorem 1.2 is to formulate correctly the two-part decomposition of a walk \( \gamma \in \text{SAW}_n \). By the correct ordering convention for axis coordinates and directions, the northeast vertex (in \( d = 2 \)) and the up vertex (in \( d = 3 \)) may be viewed as the lexicographically maximal vertices of a given polygon. If the visit of \( \phi \in \text{SAP}_n \) to this vertex is made at \( \phi_j \) with \( j > 0 \), use any given rule involving lexicographical comparison of \( \phi_{j-1} \) and \( \phi_j \) to determine whether \( \phi_{[0,j]} \), or the walk corresponding to the reversal of \( \phi_{[j,n]} \), is the first part of \( \phi \).

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