A SYSTEMATIC CONSTRUCTION OF FINITE ELEMENT COMMUTING EXACT SEQUENCES

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Abstract. We present a systematic construction of finite element exact sequences with a commuting diagram for the de Rham complex in one-, two- and three-dimensional domains. We apply the construction in two-space dimensions to rediscover two families of exact sequences for triangles and three for squares, and to uncover one new family of exact sequence for squares and two new families of exact sequences for general polygonal elements. We apply the construction in three-space dimensions to rediscover two families of exact sequences for tetrahedra, three for cubes, and one for prisms; and to uncover four new families of exact sequences for pyramids, three for prisms, and one for cubes.

Key words. finite elements, commuting diagrams, exact sequences, polyhedral elements

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1. Introduction. We give a systematic construction of finite element exact sequences with a commuting diagram property for the de Rham complex in one-, two- and three-dimensional domains. The construction of these commuting exact sequences for the three-dimensional case relies on the construction of commuting exact sequences for the two dimensional case, which in turn relies on the construction of commuting exact sequences in one-space dimension. For each dimension, the construction is carried out on a single element $K$ in such way that proper continuity properties hold which guarantee that the commutativity of the diagram holds when $K$ is replaced by the triangulation of the domain $Ω$, $Ω_h := \{K\}$. In two dimensions, the elements $K$ can be polygons of arbitrary shape, and in three-dimensions, polyhedra of arbitrary shape.

Roughly speaking, by a commuting exact sequence on a polyhedral domain $Ω$ in $\mathbb{R}^3$, we mean that, the mappings in the diagram

$$
\begin{align*}
C^\infty(Ω) & \xrightarrow{\nabla} C^\infty(Ω) & \xrightarrow{\nabla} C^\infty(Ω) & \xrightarrow{\nabla} C^\infty(Ω) \\
\downarrow \Pi_H & \downarrow \Pi_E & \downarrow \Pi_V & \downarrow \Pi_W \\
H(Ω_h) & \xrightarrow{\nabla} E(Ω_h) & \xrightarrow{\nabla} V(Ω_h) & \xrightarrow{\nabla} W(Ω_h)
\end{align*}
$$

commute in the sense that

$$
\begin{align*}
\nabla \Pi_H u &= \Pi_E \nabla u & \forall u & \in C^\infty(Ω), \\
\nabla \times \Pi_E u &= \Pi_V \nabla \times u & \forall u & \in C^\infty(Ω), \\
\nabla \cdot \Pi_V u &= \Pi_W \nabla \cdot u & \forall u & \in C^\infty(Ω).
\end{align*}
$$

The importance of commuting exact sequences, for the devising of stable finite element methods, has been amply discussed in [3, 4] and the references therein.

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Most of the previous work on the construction of commuting exact sequences (in three-space dimensions) focuses on the explicit construction of shape functions on one of four particular reference polyhedra, namely, the reference tetrahedron, hexahedron (cube), prism, and pyramid. See [20, 21] for sequences on the reference tetrahedron and reference hexahedron, [2, 15] on the reference hexahedron, [25] on the reference prism, and [22, 23] on the reference pyramid. All of these spaces in these sequences are spanned by polynomial shape functions, except those in [22, 23] which also contain rational shape functions.

Currently, there are two ways of constructing commuting exact sequences on general polyhedral meshes. The first is provided by the Virtual Element Methods (VEM); see the 2013 paper [5] and the 2015 paper [6]. These methods define the basis functions of their local spaces on each polyhedral element in terms of solutions to certain partial differential equations. The explicit form of these basis functions, usually not computable, is not needed by the methods, but a set of unisolvent degrees of freedom which can be used to exactly compute integrals related to the polynomial parts of the basis functions needs to be constructed. In our construction, we avoid basis functions defined as solutions to certain partial differential equations.

The second way is provided by the so-called Finite Element System (FES) developed in 2011 in [10, Section 5]; see also the recent papers [11, 9]. Therein, the notion of a compatible FES was introduced, see [10, Definition 5.12], which was then proven to be equivalent to the existence of a commuting diagram, see [10, Proposition 5.44]. The construction of a FES with a commuting diagram was thus reduced to the construction of a compatible FES. In [10, Example 5.29], the authors obtained compatible FESs on a general (n-dimensional) polytope mesh via element agglomeration from (available) compatible FESs defined on a refined (simplicial) mesh. However, the resulting compatible FES on the polytope mesh is not particularly interesting since it either provides identical spaces on the refined (simplicial) mesh or requires finding a subsystem of locally harmonic forms which seems to be a very hard task.

Furthermore, in [9, Corollary 3.2], a criterion for finding a compatible FES, containing certain prescribed functions, with the smallest possible dimension was presented given thus rise to the concept of a minimal compatible FES (mcFES). The minimality was proven for three finite element systems, namely, the trimmed polynomial differential forms [3] on a simplicial mesh (in three-dimensions this is the exact sequence due to Nedélec [20]), the serendipity elements [2] on a cubic mesh, and the TNT elements [15] on a cubic mesh; these seem to be the only mcFESs available in the literature. Although a simple dimension count equation in [9, Corollary 3.2] can be used to check whether a compatible FES is minimal or not, a practical construction of an mcFES was not provided in [9].

Our work can be naturally considered as a practical way to construct an mcFES in one-, two- and three-space dimensions. We develop a systematic construction of commuting exact sequences on a general polytope in one-, two- and three-space dimensions, and then (in three-space dimensions) apply the construction to each of the above-mentioned four reference polyhedra to discover and rediscover concrete examples of commuting exact sequences. In particular, we show that all the results in [20, 21, 2, 15, 25] on a tetrahedron, cube, and prism fit nicely within our construction. Moreover, a significant dimension reduction is obtained on the pyramidal commuting exact sequence in comparison with the exact sequence proposed in [22, 23]. For a general polyhedron, our construction of of high-order commuting exact sequences is significantly more difficult than that of the cases already mentioned. It will be carried
out elsewhere.

Let us now roughly describe the main steps of the systematic construction in three-dimensions. We proceed as follows. First, we pick $K$ to be a specific polyhedron. Then, we take the set of commuting exact sequences on the faces of $K$,

$$S_2(\partial K) := \{S_2(f) : f \in \mathcal{F}(K)\}.$$ 

Each of the commuting exact sequences

$$S_2(f) : H_2(f) \xrightarrow{\nabla} E_2(f) \xrightarrow{\nabla \times} W_2(f),$$

was previously obtained by applying the systematic construction in the two dimensional case. Moreover, we require a compatibility condition on the edges, namely, that, if the faces $f_1$ and $f_2$ share the edge $e$, the trace on the edge $e$ of $H_2(f_1)$ must coincide with that of $H_2(f_2)$, and that the tangential trace on the edge $e$ of $E_2(f_1)$ must coincide with that of $E_2(f_2)$. Next, we consider a candidate for the commuting exact sequence we are seeking of the form

$$S_3^g(K) : H_3^g(K) \xrightarrow{\nabla} E_3^g(K) \xrightarrow{\nabla \times} V_3^g(K) \xrightarrow{\nabla \cdot} W_3^g(K),$$

We require $S_3^g(K)$ to be exact; to have, for each face $f$ of $K$, the traces of $H_3^g(K)$ on the face $f$ in the space $H_2(f)$, the traces of $E_3^g(K)$ on the face $f$ in the space $E_2(f)$, and the traces of $V_3^g(K)$ on the face $f$ in the space $W_2(f)$; and to have the constant functions in the space $W_3^g(K)$.

Finally, we characterize the spaces $\delta H_3(K)$ and $\delta E_3(K)$ as the spaces with smallest dimension such that

$$H_3^g(K) \oplus \delta H_3(K) \xrightarrow{\nabla} E_3^g(K) \oplus \delta E_3(K) \xrightarrow{\nabla \times} V_3^g(K) \oplus \delta E_3(K) \xrightarrow{\nabla \cdot} W_3^g(K)$$

is a commuting exact sequence for which, for each face $f$ of $K$, the traces of $H_3^g(K) \oplus \delta H_3(K)$ on the face $f$ constitute the space $H_2(f)$, the tangential traces of $E_3^g(K) \oplus \delta E_3(K)$ on the face $f$ constitute the space $E_2(f)$, and the normal traces of $V_3^g(K) \oplus \delta E_3(K)$ on the face $f$ constitute the space $W_2(f)$.

This completes the rough description of the systematic construction. Note that the spaces $\delta H_3(K)$ and $\delta E_3(K)$ are not necessarily unique even though they do have their dimension is uniquely determined. Note also that the construction of the space $\delta E_3(K)$, which is the most difficult part of the general construction, is essentially a particular case of the construction of $M$-decompositions for mixed methods, see [14, 12, 13]. Thus, the present construction can be considered to be an extension of the $M$-decomposition approach to the setting of exact sequences for the de Rham complex.

Let us now comment on the actual commuting exact sequences we obtain here. We apply the systematic construction to explicitly obtain fourteen families of commuting exact sequences: on the reference tetrahedron (2 sequences), reference cube (4 sequences), reference prism (4 sequences), and reference pyramid (4 sequences). In Table 1.1 we indicate if these are known or new commuting exact sequences. Let us note that, on a reference prism, there is an additional family of commuting exact
sequence, namely, the one proposed in [25]; it uses the $H(\text{div})$ and $L^2$ spaces obtained in [8]. The sequence introduced in [18] is a slight modification of this one. On a reference pyramid, there are two additional family of exact sequences in [22] [23]. Our spaces are significantly smaller.

Our fourteen commuting exact sequences can be gathered into four groups of sequences each of which is displayed in a row in Table 1.1. These four groups of sequences are such that the $H^1$, $H(\text{curl})$, and $H(\text{div})$-trace spaces on similar faces are the same. As a consequence, each of the four groups of sequences can be patched into a hybrid polyhedral mesh $\Omega_h = \{K\}$ where the elements $K$ are suitably defined by affine mappings of the four reference polyhedral elements. This way of regrouping the sequences is motivated by the work proposed in [18], where one group of sequences that can be used on a hybrid mesh (with a more complicated pyramidal sequence from [22]) was recently carefully studied to construct “orientation embedded high-order shape functions”.

The rest of the paper is organized as follows. In Section 2, we present our main results on the systematic construction of commuting exact sequences in one-, two- and three-space dimensions. Then in Section 3 we apply the systematic construction to explicitly obtain commuting exact sequences on the reference interval in one-space dimension; on the reference triangle, reference square, and on a general polygon in two-space dimensions; and on the above-mentioned four reference polyhedra in three-space dimensions. Section 4 is devoted to the proofs of the results in Section 3. We end in Section 6 with some concluding remarks.

### 2. A systematic construction of commuting exact sequences

In this section, we introduce the notation used throughout the paper. We then define the concept of a compatible exact sequence in one-, two- and three-space dimensions, which, in differential form language, is nothing but the compatible FES introduced in [10, Definition 5.12]. Let us recall that, the theory of FESs introduced in [10, Proposition 5.44], establishes the equivalence of a compatible exact sequence and a sequence admitting a commuting diagram. As a consequence, the construction of a sequence admitting a commuting diagram is reduced to the construction of a compatible exact sequence. We use this powerful result and devote ourselves to developing a systematic construction of compatible exact sequences in one-, two- and three-space dimensions.

In Section 5 this approach is applied to obtain many compatible exact sequences with explicitly-defined shape functions for elements of various shapes.
2.1. Notation. Here we introduce the notation we use for the rest of the paper.

Geometry. We denote by $K \subset \mathbb{R}^d$, a segment if $d = 1$, a polygon if $d = 2$, and a polyhedron if $d = 3$. We denote its boundary by $\partial K$, the set of its vertices by $\mathcal{V}(K)$, the set of its edges (for $d = 1, 2$) by $\mathcal{E}(K)$, and the set of its faces (for $d = 3$) by $\mathcal{F}(K)$.

Trace operators. For a scalar-valued function $v$ on $K$ with sufficient regularity, we denote by $\text{tr}^\gamma_H v := v|_{\partial K}$ the trace of $v$ on a vertex $v \in \mathcal{V}(K)$, by $\text{tr}^\gamma_E v := v|_e$ the trace of $v$ on an edge $e \in \mathcal{E}(K)$, by $\text{tr}^\gamma_F v := v|_f$ the trace of $v$ on a face $f \in \mathcal{F}(K)$, and by $\text{tr}^\gamma_{\partial K} v := v|_{\partial K}$ the trace of $v$ on the whole boundary $\partial K$.

If $d \geq 2$, for a $d$-dimensional vector-valued function $v$ with sufficient regularity, we denote by $\text{tr}^\gamma_E v := (v \cdot t_e)|_e$, where $t_e$ is the unit vector in the direction of the edge $e$, the tangential trace of $v$ on an edge $e \in \mathcal{E}(K)$. We denote by $\text{tr}^\gamma_F v := (n_f \times (v \times n_f))|_f$, where $n_f$ is the unit outward normal to the face $f$, the tangential trace of $v$ on a face $f \in \mathcal{F}(K)$. Finally, we denote by $\text{tr}^\gamma_{\partial K} v := \begin{cases} (v \cdot t_{\partial K})|_{\partial K} & \text{if } d = 2, \\ (n_{\partial K} \times (v \times n_{\partial K}))|_{\partial K} & \text{if } d = 3, \end{cases}$ the tangential trace of $v$ on the whole boundary $\partial K$. Here, $t_{\partial K}$ on the edge $e$ is nothing but $t_e$. Similarly, $n_{\partial K}$ on the face $f$ is nothing but $n_f$.

Finally, if $d = 3$, for a $d$-dimensional vector-valued function $v$ with sufficient regularity, we denote by $\text{tr}^\gamma_{\partial K} v := (v \cdot n_{\partial K})|_{\partial K}$, the normal trace of $v$ on a face $f \in \mathcal{F}(K)$, and by $\text{tr}^\gamma_{\partial K} v := (v \cdot n_{\partial K})|_{\partial K}$ the normal trace of $v$ on the whole boundary $\partial K$.

Differential operators. If $d = 1$, we denote by $x_1$ the coordinate of a point on $K$, if $d = 2$, we denote by $(x_1, x_2)$ the coordinate of a point on $K$, and if $d = 3$, we denote by $(x_1, x_2, x_3)$ the coordinate of a point on $K$. We define the gradient operator $\nabla$, curl operator $\nabla \times$ (for $d \geq 2$), and divergence operator $\nabla \cdot$ (for $d = 3$) on the element $K$ as follows.

For a scalar function $v$ on $K$ with sufficient regularity, we set

$$\nabla v := \begin{cases} \partial_{x_1} v & \text{if } d = 1, \\ (\partial_{x_1} v, \partial_{x_2} v)^t & \text{if } d = 2, \\ (\partial_{x_1} v, \partial_{x_2} v, \partial_{x_3} v)^t & \text{if } d = 3. \end{cases}$$

For a $d$-dimensional vector function $v$ on $K$ with sufficient regularity ($d \geq 2$), we set

$$\nabla \times v := \begin{cases} -\partial_{x_2} v_1 + \partial_{x_1} v_2 & \text{if } d = 2, \\ -\partial_{x_3} v_2 + \partial_{x_2} v_3 & \\ -\partial_{x_1} v_3 + \partial_{x_3} v_1 & \\ -\partial_{x_1} v_1 + \partial_{x_2} v_2 & \text{if } d = 3, \end{cases}$$

where $v = (v_1, v_2)^t$ if $d = 2$ and $v = (v_1, v_2, v_3)^t$ if $d = 3$.

For a 3-dimensional vector function $v = (v_1, v_2, v_3)^t$ on $K$ with sufficient regularity, we set

$$\nabla \cdot v := \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3.$$
dimensional space are defined on a domain of $\mathbb{R}^d$, we use the subscript $d$. For any face $f$, we define its sequence of traces on $f$ as

$$\text{tr}_f^i(S(K)) : 0 \rightarrow \mathbb{R} \xrightarrow{i} H_3(K) \xrightarrow{\nabla} E_3(K) \xrightarrow{\nabla \times} V_3(K) \xrightarrow{\nabla} W_3(K) \xrightarrow{\sigma} 0,$$

and, for any edge $e \in \mathcal{E}(K)$, we define its sequence of traces on $e$ by

$$\text{tr}_e^i(S(K)) : 0 \rightarrow \mathbb{R} \xrightarrow{i} H_3(K) \xrightarrow{\nabla} E_3(K) \xrightarrow{\nabla \times} V_3(K) \xrightarrow{\nabla} W_3(K) \xrightarrow{\sigma} 0,$$

Sequence of traces on an edge for a two-dimensional sequence is defined in the same way.

**Definition 2.3 (Bubble spaces).** Let $K \subset \mathbb{R}^3$ be a polyhedron. For any sequence

$$S(K) : \mathbb{R} \xrightarrow{i} H_3(K) \xrightarrow{\nabla} E_3(K) \xrightarrow{\nabla \times} V_3(K) \xrightarrow{\nabla} W_3(K) \xrightarrow{\sigma} 0,$$

we define the related $H^1$, $H(\text{curl})$, $H(\text{div})$, and $L^2$-bubble spaces as

$$\overset{\sigma}{\tilde{H}}_3(K) := \{ v \in H_3(K) : \text{tr}_Hv = 0 \},$$

$$\overset{\sigma}{\tilde{E}}_3(K) := \{ v \in E_3(K) : \text{tr}_Ev = 0 \},$$

$$\overset{\sigma}{V}_3(K) := \{ v \in V_3(K) : \text{tr}_Vv = 0 \},$$

$$\overset{\sigma}{W}_3(K) := \{ v \in W_3(K) : \int_K v = 0 \},$$

respectively.

Similar, obvious definitions for bubble spaces hold for the two- and one-dimensional cases.

From now on, we remove the first two and last terms in the definition of the exact sequence to simplify the notation. For example, we simply write $H_1(\Omega) \xrightarrow{\nabla} W_1(\Omega)$, instead of writing $0 \rightarrow \mathbb{R} \xrightarrow{i} H_1(\Omega) \xrightarrow{\nabla} W_1(\Omega) \xrightarrow{\sigma} 0$.

Finally, let us emphasize that all the spaces in the sequences considered below will have finite dimension.
Polynomial spaces. We denote the polynomial space of degree at most $p$ with argument $(x, y, z) \in \mathbb{R}^3$ by
\[
P_p(x, y, z) := \text{span}\{x^i y^j z^k : i, j, k \geq 0, i + j + k \leq p\},
\]
and we denote the homogeneous polynomial space of total degree $p$ by
\[
\tilde{P}_p(x, y, z) := \text{span}\{x^i y^j z^k : i, j, k \geq 0, i + j + k = p\}.
\]
We denote the tensor-product polynomial space of degree at most $p$ by
\[
Q_p(x, y, z) := P_p(x) \otimes P_p(y) \otimes P_p(z) = \text{span}\{x^i y^j z^k : 0 \leq i, j, k \leq p\}.
\]
We also denote the polynomial space of degree at most $p$ in the $(x, y)$ variable and of degree at most $p$ in the $z$ variable by
\[
P_{p|p}(x, y, z) := P_p(x, y) \otimes P_p(z).
\]
Similar definitions hold in the two- and one-dimensional cases.

Given an element $K \subset \mathbb{R}^d$, we denote $P_p(K)$ to be the space of polynomials with degree at most $p$ defined on $K$. Similarly for $\tilde{P}_p(K), Q_p(K), P_{p|p}(K)$. We denote by $P_p(K)$, respectively, $\tilde{P}_p(K), Q_p(K),$ and $P_{p|p}(K)$, the vector-valued functions whose components lie in $P_p(K)$, respectively, $\tilde{P}_p(K), Q_p(K),$ and $P_{p|p}(K)$.

Whenever there is no possible confusion, we write $P_p$ instead of $P_p(K)$.

2.2. Compatible exact sequences. Here we introduce the concept of compatible exact sequences in one-, two- and three-space dimensions, which is just a reformulation, in our notation, of the compatible FES in differential form language introduced in [10, Definition 5.12].

The main result of a compatible FES in [10, Proposition 5.44], see also [11, Proposition 2.8], provides the equivalence of a compatible FES with a FES admitting a commuting diagram. This powerful result reduces the search for a commuting diagram to that of a compatible FES (or compatible exact sequence in our notation). In it, the harmonic interpolator, a generalization of the projection-based interpolation operator proposed in [16, 17], was used to obtain the commuting diagram; we reformulate these harmonic interpolators in one-, two- and three-space dimensions using our notation in the Appendix A.

For a comprehensive theory of FESs, we refer to [10, Section 5] and [11]. See also [9] where the concept of a minimal compatible FES was introduced.

**Definition 2.4** (One-dimensional compatible exact sequence). Let $K$ be a segment. Consider the finite dimensional exact sequence
\[
S_1(K) : \quad H_1(K) \xrightarrow{\nabla} W_1(K).
\]
We say that the sequence $S_1(K)$ is a compatible exact sequence if
\[
(i) \quad \dim \text{tr}_{H_1(K)} = \sum_{v \in \nabla(K)} 1 = 2.
\]

**Definition 2.5** (Two-dimensional compatible exact sequence). Let $K$ be a polygon. Consider the finite dimensional exact sequence
\[
S_2(K) : \quad H_2(K) \xrightarrow{\nabla} E_2(K) \xrightarrow{\nabla \times} W_2(K),
\]

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and, for every edge \( e \in \mathcal{E}(K) \), its sequence of traces

\[
\text{tr}^e(S_2(K)) : \quad H_1(e) \xrightarrow{\nabla} W_1(e),
\]

where \( H_1(e) := \text{tr}_H^e H_2(K) \) and \( W_1(e) := \text{tr}_E^e E_2(K) \). We say that the sequence \( S_2(K) \) is a compatible exact sequence if

(i) For each edge \( e \in \mathcal{E}(K) \), the sequence \( \text{tr}^e(S_2(K)) \) is a (one-dimensional) compatible exact sequence,

(ii) \[
\begin{cases}
\text{dim tr}_H(H_2(K)) = \sum_{v \in \mathcal{V}(K)} 1 + \sum_{e \in \mathcal{E}(K)} \text{dim} \overset{\circ}{H}_1(e), \\
\text{dim tr}_E(E_2(K)) = \sum_{e \in \mathcal{E}(K)} \text{dim} W_1(e),
\end{cases}
\]

**DEFINITION 2.6** (Three-dimensional compatible exact sequence). Let \( K \) be a polyhedron. Consider the exact sequence

\[
S_3(K) : \quad H_3(K) \xrightarrow{\nabla} E_3(K) \xrightarrow{\nabla \times} V_3(K) \xrightarrow{\nabla} W_3(K)
\]

and its sequences of traces for all faces \( f \in \mathcal{F}(K) \)

\[
\text{tr}^f(S_3(K)) : \quad H_2(f) \xrightarrow{\nabla} E_2(f) \xrightarrow{\nabla \times} W_2(f)
\]

and all edges \( e \in \mathcal{E}(K) \)

\[
\text{tr}^e(S_3(K)) : \quad H_1(e) \xrightarrow{\nabla} W_1(e),
\]

where \( H_2(f) \times E_2(f) \times W_2(f) := \text{tr}_H^f H_3(K) \times \text{tr}_E^f E_3(K) \times \text{tr}_V^f V_3(K) \), and \( H_1(e) \times W_1(e) := \text{tr}_H^e H_3(K) \times \text{tr}_E^e E_3(K) \).

We say that the sequence \( S_3(K) \) is a compatible exact sequence if

(i) For each face \( f \), the sequence \( \text{tr}^f(S_3(K)) \) is a (two-dimensional) compatible exact sequence.

(ii) \[
\begin{cases}
\text{dim tr}_H H_3(K) = \sum_{v \in \mathcal{V}(K)} 1 + \sum_{e \in \mathcal{E}(K)} \text{dim} \overset{\circ}{H}_1(e) + \sum_{f \in \mathcal{F}(K)} \text{dim} \overset{\circ}{H}_2(f), \\
\text{dim tr}_E E_3(K) = \sum_{e \in \mathcal{E}(K)} \text{dim} W_1(e) + \sum_{f \in \mathcal{F}(K)} \text{dim} \overset{\circ}{E}_2(f), \\
\text{dim tr}_V V_3(K) = \sum_{f \in \mathcal{F}(K)} \text{dim} W_2(f).
\end{cases}
\]

The following result on the equivalence of compatible exact sequence and compatible FES is trivial to verify by definition; we omit its proof.

**PROPOSITION 2.7.** Let \( K \in \mathbb{R}^d \) (\( d = 1, 2, 3 \)) be a \( d \)-dimensional polytope. Then, with a change of scalar/vector fields to differential forms, a compatible exact sequence on \( K \) defined above is the corresponding compatible FES on \( \alpha(K) \), the set that contains the element \( K \) and its vertices, edges (if \( d \geq 2 \)), and faces (if \( d = 3 \)).

The following result is a direct consequence of [11] Proposition 2.8.

**PROPOSITION 2.8.** Let \( K \in \mathbb{R}^d \) (\( d = 1, 2, 3 \)) be a \( d \)-dimensional polytope. The following statements are equivalent:

- \( S(K) \) is a compatible exact sequence on \( K \).
- \( S(K) \) admits a commuting diagram.

Note that, in the proof of [11] Proposition 2.8, the so-called harmonic interpolator is used to obtain the commuting diagram. We give a reformulation of this concept in our notation in the Appendix A.
2.3. The construction of compatible exact sequences. Here, we give our main results on the systematic construction of compatible exact sequences. Again, for each space dimension, we have a specific construction. The corresponding proofs are given in Section 4. The resulting compatible exact sequences are all minimal sequences containing a prescribed exact sequence in the sense of [9, Corollary 3.2].

The one-dimensional case. In one space dimension, the construction of compatible exact sequences is fairly simple.

**Theorem 2.9.** Let $K$ be a segment. Let any given exact sequence
\[ S_1^0(K) : H_1^g(K) \xrightarrow{\nabla} W_1^g(K) \]
be such that
\[ \mathcal{P}_0(K) \subset W_1^g(K). \]
Then, it is compatible.

The two-dimensional case. In two-space dimensions, the systematic construction of compatible exact sequences is more involved than in the one-dimensional case, not only because of the geometry but also because we seek a compatible exact sequence with certain given traces on each of the edges of the element. Those traces are compatible exact sequences (we found while dealing with the one-dimensional case) which we gather in the set $S_1(\partial K)$. The sequence with which we begin the construction must then be what we call $S_1(\partial K)$-admissible. We define this term next.

**Definition 2.10 ($S_1(\partial K)$-admissible exact sequence).** Let
\[ S_1^0(\partial K) := \{ S_1(e) : H_1^g(e) \xrightarrow{\nabla} W_1^g(e) \quad \forall e \in \mathcal{E}(K) \}, \]
be any set of one-dimensional exact sequences. Then, we say that a given two-dimensional exact sequence
\[ H_2^g(K) \xrightarrow{\nabla} E_2^g(K) \xrightarrow{\nabla^\times} W_2^g(K) \]
is $S_1(\partial K)$-admissible if
(i) $\text{tr}^\times_H(H_2^g(K)) \times \text{tr}^\times_E(E_2^g(K)) \subset H_1(e) \times W_1(e) \quad \forall e \in \mathcal{E}(K)$,
(ii) $\mathcal{P}_0(K) \subset W_2^g(K)$.

The next theorem is the main result of the two-dimensional case. It shows how to construct a compatible exact sequence by suitably enriching a given $S_1(\partial K)$-admissible exact sequence. The compatible exact sequence we seek is such that its traces on the edges coincide with the exact sequences of the set $S_1(\partial K)$.

**Theorem 2.11.** Let $K$ be a polygon, let
\[ S_1(\partial K) := \{ S_1(e) : H_1(e) \xrightarrow{\nabla} W_1(e) \quad \forall e \in \mathcal{E}(K) \}, \]
be a set of compatible exact sequences, and let
\[ H_2^g(K) \xrightarrow{\nabla} E_2^g(K) \xrightarrow{\nabla^\times} W_2^g(K) \]
be a given $S_1(\partial K)$-admissible exact sequence. Let the space $\delta H_2^g(K) \subset H^1(K)$ satisfy the following properties:
(i) $\text{tr}^\times_H(\delta H_2^g(K)) \subset H_1(e)$ for all edges $e \in \mathcal{E}(K)$. 

(ii) \( \delta H^2_2(K) \cap H^2_3(K) = \{0\} \).

(iii) \( \{v \in H^2_3(K) \oplus \delta H^2_3(K) : \text{tr}_\mathcal{H} v = 0\} = H^2_3(K) \).

(iv) \( \dim \delta H^2_2(K) = \sum_{v \in \nabla(K)} 1 + \sum_{e \in \mathcal{E}(K)} \dim \delta H_2(e) + \dim H^2_3(K) - \dim H^2_2(K) \).

Then, the following sequence

\[
S_2(K) : H^2_3(K) + \delta H^2_3(K) \stackrel{\nabla}{\rightarrow} E^2_2(K) + \nabla \delta H^2_2(K) \stackrel{\nabla \times}{\rightarrow} W^2_2(K)
\]

is a compatible exact sequence. Moreover, it is also a minimal compatible exact sequence containing the exact sequence

\[
H^2_2(K) \stackrel{\nabla}{\rightarrow} E^2_2(K) \stackrel{\nabla \times}{\rightarrow} W^2_2(K).
\]

Let us relate this result with the theory of \( M \)-decompositions developed in [14]. Such theory has to do with the right-most part of the commuting diagram. Since the operator \( \nabla \times \) was replaced by the divergence operator \( \nabla \cdot \) and since

\[
\nabla \times (v_1, v_2) = -\partial_2 v_1 + \partial_1 v_2 = \nabla \cdot (v_2, -v_1) = \nabla \cdot (v_1, v_2)_{\text{rot}},
\]

we have that \( \nabla \times E_2(K) = \nabla \cdot E_2(K) \), with the obvious notation. In [14] Proposition 5.1 is was shown that \( (E^2_2(K) \oplus \nabla \delta H^2_2(K))_{\text{rot}} \times W^2_2(K) \) is the smallest space containing \( (E^2_2(K))_{\text{rot}} \times W^2_2(K) \) which admits an \( M(\partial K) \)-decomposition with the trace space

\[
M(\partial K) := \{ \mu \in L^2(\partial K) : \mu|_e \in W_1(e) \text{ } \forall e \in \mathcal{E}(K) \}.
\]

This result implies that the right-most part of the diagram commutes.

The three-dimensional case. Now, we present our most involved result in three-space dimensions. As we did in the two-dimensional case, we have to provide a set of compatible exact sequences on the faces of the element \( K \), \( S_2(\partial K) \); they were previously obtained when dealing with the two-dimensional case. Then, we have to start from certain exact sequences we call \( S_2(\partial K) \)-admissible.

**Definition 2.12** \( (S_2(\partial K))-\text{admissible exact sequence} \). Let

\[
S_2(\partial K) := \{ S_2(f) : H_2(f) \stackrel{\nabla}{\rightarrow} E_2(f) \stackrel{\nabla \times}{\rightarrow} W_2(f), f \in \mathcal{F}(K) \}\]

be any set of two-dimensional exact sequences. We say that a given three-dimensional exact sequence

\[
H^2_2(K) \stackrel{\nabla}{\rightarrow} E^2_2(K) \stackrel{\nabla \times}{\rightarrow} V^\theta_2(K) \stackrel{\nabla}{\rightarrow} W^\theta_2(K)
\]

is \( S_2(\partial K) \)-admissible if

(i) \( \text{tr}_H H_2(K) \times \text{tr}_E E^2_2(K) \times \text{tr}_V V^\theta_2(K) \subset H_2(f) \times E_2(f) \times W_2(f) \forall f \in \mathcal{F}(K) \),

(ii) \( \mathcal{P}_0(K) \subset W^\theta_2(K) \).

The following theorem is our main result. It shows how to enrich an \( S_2(\partial K) \)-admissible exact sequence to get a commuting exact sequence. Note also that the traces on the faces of the sequence we seek must coincide with the sequence of traces in the set \( S_2(\partial K) \).
Theorem 2.13. Let $K$ be a polyhedron, let

$$S_2(\partial K) := \{S_2(f) : H_2(f) \xrightarrow{\nabla} E_2(f) \xrightarrow{\nabla \times} W_2(f), \ f \in \mathcal{F}(K)\}$$

be a set of compatible exact sequences satisfying the following compatibility condition

$$\text{tr}_E^\nu H_2(f_1) \times \text{tr}_E^\nu E_2(f_1) = \text{tr}_E^\nu H_2(f_2) \times \text{tr}_E^\nu E_2(f_2) =: H_1(e) \times W_1(e),$$

for any faces $f_1, f_2 \in \mathcal{F}(K)$ sharing an edge $e := \mathcal{E}(f_1) \cap \mathcal{E}(f_2)$. Let

$$H_3^\delta(K) \xrightarrow{\nabla} E_3^\delta(K) \xrightarrow{\nabla \times} V_3^\delta(K) \xrightarrow{\nabla} W_3^\delta(K)$$

be a given $S_2(\partial K)$-admissible exact sequence.

Let the spaces $\delta H_3^\delta(K) \times \delta E_3^\delta(K) \subset H^1(K) \times H(\text{curl}, K)$ satisfy the following properties:

**Properties of $\delta H_3^\delta(K)$**

(i) $\text{tr}_E^\nu \delta H_3^\delta(K) \subset H_2(f)$ for all faces $f \in \mathcal{F}(K)$.

(ii) $\delta H_3^\delta(K) \cap H_3^0(K) = \{0\}$.

(iii) $\{v \in \delta H_3^\delta(K) \oplus \delta H_3^\delta(K) : \text{tr}_E v = 0\} = H_3^\delta(K)$.

(iv) $\dim \delta H_3^\delta(K) = \sum_{v \in \mathcal{V}(K)} 1 + \sum_{e \in \mathcal{E}(K)} \dim \partial H_1(e) + \sum_{t \in \mathcal{T}(K)} \dim \partial H_2(t) + \dim \partial H_3(K) - \dim H_3^\delta(K)$.

**Properties of $\delta E_3^\delta(K)$**

(i) $\text{tr}_E^\nu \delta E_3^\delta(K) \subset E_2(f)$ for all faces $f \in \mathcal{F}(K)$.

(ii) $\nabla \times \delta E_3^\delta(K) \cap V_3^\delta(K) = \{0\}$.

(iii) $\{v \in V_3^\delta(K) + \nabla \times \delta E_3^\delta(K) : \text{tr}_E v = 0, \nabla \cdot v = 0\} = \{v \in V_3^\delta(K) : \nabla \cdot v = 0\}$.

(iv) $\dim \delta E_3^\delta(K) = \dim \nabla \times \delta E_3^\delta(K) = \sum_{t \in \mathcal{T}(K)} \dim W_2(t) + \dim \partial W_3^\delta(K) + \dim \{v \in V_3^\delta(K) : \nabla \cdot v = 0\} - \dim V_3^\delta(K)$.

Then, the sequence

$$S_3(K) : H_3^\delta(K) \xrightarrow{\nabla} E_3^\delta(K) \xrightarrow{\nabla \times} V_3^\delta(K) \xrightarrow{\nabla} W_3^\delta(K) \oplus \delta H_3^\delta(K) \oplus \nabla \delta H_3^\delta(K) \oplus \delta E_3^\delta(K) \oplus \nabla \times \delta E_3^\delta(K)$$

is a commuting exact sequence. Moreover, it is a minimal commuting exact sequence containing the exact sequence

$$H_3^\delta(K) \xrightarrow{\nabla} E_3^\delta(K) \xrightarrow{\nabla \times} V_3^\delta(K) \xrightarrow{\nabla} W_3^\delta(K).$$

Note that, unlike the two-dimensional case, we are requiring the sequences of the set $S_2(\partial K)$ to satisfy a compatibility condition on each of the edges of the polyhedral. Such compatibility condition was automatically satisfied, and hence was not required, in the two-dimensional case.
Let us relate this result with the theory of $M$-decompositions introduced in [14]. As for the two-dimensional case, the part of this theory concerned with mixed methods is associated with the right-most side of the diagram. Indeed, in [14, Proposition 5.1] it is shown that the space $(V^g_3(K) \oplus \nabla \times \delta E^g_3(K)) \times W^g_3(K)$ is the smallest one containing $V^g_3(K) \times W^g_3(K)$ and admitting an $M(\partial K)$-decomposition with the trace space

$$M(\partial K) := \{ \mu \in L^2(\partial K) : \mu|_f \in W_2(f) \ \forall f \in \mathcal{F}(K) \}.$$  

3. Applications. Now, we apply our main results on the systematic construction of compatible exact sequences in Section 2 to explicit construct them on various element shapes in one-, two- and three-space dimensions. To emphasize on the existence of a commuting diagram for compatible exact sequence, we denote such sequence as commuting exact sequence.

3.1. The one-dimensional case. In one-space dimension, we consider the element $K \subset \mathbb{R}^1$ to be the reference interval $\{x : 0 < x < 1\}$. The most useful commuting exact sequence on $K$ is given below.

**Theorem 3.1.** Let $K \subset \mathbb{R}^1$ be the reference interval with coordinate $x$. Then, the following sequence on $K$ is a commuting exact sequence for $k \geq 0$.

$$P_{k+1}(x) \xrightarrow{\nabla} P_k(x)$$

3.2. The two-dimensional case. In two-space dimensions, we proceed as follows. We first consider the element $K$ to be either the reference triangle $\{(x, y) : x > 0, y > 0, x + y < 1\}$, or the reference square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$. We present two commuting exact sequences on the reference triangle and four on the reference square. All the sequences, except the second one on the reference square, are known and its spaces are all spaces of polynomials.

Then, we consider the case in which $K$ is a general polygon, and present two new commuting exact sequences which contain non-polynomial functions; this is based on the results in [12] on the construction of $M$-decompositions in two dimensions. All these commuting exact sequences are the smallest ones, as stated in Theorem 2.11 that contain certain given exact sequence and have a certain prescribed sequence of traces on each edge.

We end by obtaining commuting exact sequences on two-dimensional polygonal meshes.

**Triangle.**

**Theorem 3.2.** Let $K$ be the reference triangle with coordinates $(x, y)$. Then, the following two sequences on $K$ are commuting exact sequences for $k \geq 0$.

$$S^1_{1,k}(K) : \ P_{k+2}(x, y) \xrightarrow{\nabla} P_{k+1}(x, y) \xrightarrow{\nabla \times} P_k(x, y),$$

$$S^2_{2,k}(K) : \ P_{k+1}(x, y) \xrightarrow{\nabla} P_k(x, y) \oplus x \times \tilde{P}_k(x, y) \xrightarrow{\nabla \times} P_k(x, y).$$

Here $x \times p = (yp, -xp)^t$ for a scalar function $p$.

Moreover, the sequence of traces for $S^1_{1,k}$ on an edge $e \in \mathcal{E}(K)$ is

$$\text{tr}^e (S^1_{1,k}(K)) : P_{k+2}(e) \rightarrow P_{k+1}(e),$$

and that for $S^2_{2,k}$ is

$$\text{tr}^e (S^2_{2,k}(K)) : P_{k+1}(e) \rightarrow P_k(e).$$

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These two sequences are well-known. Indeed, the first sequence $S_{1,k}^1$ is mainly due to Brezzi, Douglas, and Marini [7] since its $H(\text{curl})$ space is a ninety-degree rotation of the $H(\text{div})$-space, usually called the BDM space, obtained in [7], of degree $k+1$. Its $H^1$ and $L^2$ spaces are the Lagrange polynomial spaces of degree $k+2$ and discontinuous polynomial space of degree $k$. The second sequence $S_{2,k}^1$ is mainly due to Raviart and Thomas [24] since its $H(\text{curl})$ space is a ninety-degree rotation of the $H(\text{div})$ space, usually called the RT space, obtained in [24], of degree $k$. Its $H^1$ and $L^2$ spaces are the serendipity polynomial spaces of degree $k+1$ and discontinuous polynomial space of degree $k$.

**Square.**

**Theorem 3.3.** Let $K$ be the reference square with coordinates $(x,y)$. Then, the following four sequences are commuting exact sequences for $k \geq 0$:

$$ S_{1,k}^1(K) : \begin{array}{c} \mathcal{P}_{k+2}^0(x,y) \quad \nabla \quad \mathcal{P}_{k+1}^0(x,y) \quad \nabla \quad \mathcal{P}_k(x,y), \\ \oplus \delta H_{k+2}^2 \\ \mathcal{P}_k(x,y) \quad \nabla \quad \mathcal{P}_{k+1}(x,y) \quad \nabla \quad \mathcal{P}_k(x,y) \\ \oplus \nabla \delta H_{k+2}^2 \\ \mathcal{P}_k(x,y) \quad \nabla \quad \mathcal{P}_{k+2}^0(x,y) \quad \nabla \quad \mathcal{P}_k(x,y) \\ \oplus \delta H_{k+1}^2 \\ \mathcal{P}_k(x,y) \quad \nabla \quad \mathcal{P}_{k+1}(x,y) \quad \nabla \quad \mathcal{P}_k(x,y) \quad \nabla \quad \mathcal{P}_k(x,y) \\ \oplus \nabla \delta H_{k+1}^2 \\ \mathcal{P}_k(x,y) \quad \nabla \quad \mathcal{P}_{k+2}^0(x,y) \quad \nabla \quad \mathcal{P}_k(x,y) \\ \oplus \delta H_{k+1}^2 \end{array} $$

Here the additional space $\delta H_{k}^{2,I}$, for $k \geq 1$, takes the following form:

$$ \delta H_{k}^{2,I} := \text{span}\{x^k, y^k\}.$$ 

Moreover, the sequence of traces for $S_{1,k}^1$ on an edge $e \in \mathcal{E}(K)$ is

$$ \text{tr}^{e}(S_{1,k}^1(K)) : \mathcal{P}_{k+2}(e) \longrightarrow \mathcal{P}_{k+1}(e), $$

and that for $S_{i,k}^1$ with $i \in \{2,3,4\}$ is

$$ \text{tr}^{e}(S_{i,k}^1(K)) : \mathcal{P}_{k+2}(e) \longrightarrow \mathcal{P}_{k+1}(e). $$

Note that for $k = 0$, the last three sequences are exactly the same. Here the second sequence is new and the other three are well-known. The first sequence $S_{1,k}$ is mainly due to Brezzi, Douglas, and Marini [7] since its $H(\text{curl})$ space is a ninety-degree rotation of the $H(\text{div})$ space, usually called the BDM space, obtained in [7], of degree $k+1$ on the square. Its $H^1$ and $L^2$ spaces are the serendipity polynomial spaces of degree $k+2$ and discontinuous polynomial space of degree $k$. The second sequence $S_{2,k}$ is a new one resulting a new family of $H(\text{curl})$ spaces. The third sequence is the TNT sequence [15] on the square. And the last one is mainly due to Raviart and Thomas [24] since its $H(\text{curl})$ space is a ninety-degree rotation of the $H(\text{div})$ space, usually called the RT space, obtained in [24], of degree $k$ on the square. Its $H^1$ and $L^2$ spaces are the tensor-product Lagrange polynomial space of degree $k+1$ and discontinuous tensor product polynomial space of degree $k$. 

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Polygon.

The explicit construction of (high-order) commuting exact sequences on a general
polygon $K$ is not known. Here we fill this gap by presenting two families of commuting
exact sequences by applying Theorem 2.11. To do so, we take advantage of the recent
results on constructing $M$-decompositions in [12] to deal with the right-most is of the
diagram.

To state our result, we need to introduce some notation. Let $\{v_i\}_{i=1}^{ne}$ be the set of
vertices of the polygonal element $K$ which we take to be counter-clockwise ordered.
Let $\{e_i\}_{i=1}^{ne}$ be the set of edges of $K$ where the edge $e_i$ connects the vertices $v_i$ and
$v_{i+1}$. Here the subindexes are integers module $ne$. We also define, for $1 \leq i \leq ne$, $\lambda_i$
to be the linear function that vanishes on edge $e_i$ and reaches maximum value 1 in
the closure of the element $K$. To each vertex $v_i$, $i = 1, \ldots, ne$, we associate a function
$\xi_i$ satisfying the following conditions:

(L.1) $\xi_i \in H^1(K)$,
(L.2) $\xi_i|_{e_j} \in P_1(e_j)$, $j = 1, \ldots, ne$,
(L.3) $\xi_i(v_j) = \delta_{i,j}$, $j = 1, \ldots, ne$,

where $\delta_{i,j}$ is the Kronecker delta. Note that conditions (L.2) and (L.3) together ensure
that the trace of $\xi_i$ on the edges is only no-zero at $e_i$ and $e_{i+1}$, where they are linear.
These functions, with examples given in [12], are not polynomials if the polygon $K$ is
not a triangle or a parallelogram.

Now, we are ready to state the result.

**Theorem 3.4.** Let $K$ be a polygon of $ne$ edges with coordinates $(x, y)$ that does
not have hanging nodes. Then, the following two sequences on $K$ are commuting exact
sequences for $k \geq 0$,

$$S^\text{poly}_{1,k}(K) : \begin{array}{ll}
P_{k+2}(x, y) & \xrightarrow{\nabla} P_{k+1}(x, y) \\
\oplus \delta H^2_{k+2} & \oplus \nabla \delta H^2_{k+2}
\end{array} \xrightarrow{\nabla \times} P_k(x, y),$$

$$S^\text{poly}_{2,k}(K) : \begin{array}{ll}
P_{k+1}(x, y) & \xrightarrow{\nabla} P_k(x, y) \\
\oplus \delta H^2_{k+1} & \oplus \nabla \delta H^2_{k+1}
\end{array} \xrightarrow{\nabla \times} P_k(x, y).$$

Here the additional space $\delta H^2_{k+1}$, for $k \geq 1$, takes the following form:

$$\delta H^2_{k+1} = \bigoplus_{i=3}^{ne} \Psi_{i,k},$$

where

$$\Psi_{i,k} = \begin{cases} 
\text{span}\{\xi_{i+1} \lambda_a^{i+1} : \max\{k+3-i,0\} \leq a \leq k-1\} & \text{if } 3 \leq i \leq ne-1, \\
\text{span}\{\xi_{i+1} \lambda_a^{i+1} : \max\{k+4-i,1\} \leq a \leq k-1\} & \text{if } i = ne,
\end{cases}$$

and the functions $\{\xi_i\}_{i=1}^{ne}$ are assumed to satisfy conditions (L).

Moreover, the sequence of traces for $S^\text{poly}_{1,k}$ on an edge $e \in \mathcal{E}(K)$ is

$$\text{tr}^e \left(S^\text{poly}_{1,k}(K)\right) : P_{k+2}(e) \longrightarrow P_{k+1}(e),$$

and that for $S^\text{poly}_{2,k}$ is

$$\text{tr}^e \left(S^\text{poly}_{2,k}(K)\right) : P_{k+1}(e) \longrightarrow P_k(e).$$
Note that in the notation of \cite{12}, we have $\delta V_{\text{fill}} := \nabla \times \delta H^{2,l+1}_k$ is the filling space to guarantee $M$-decompositions for the pair $V \times W := P_k \oplus \delta V_{\text{fill}} \times P_k$ for the trace space

$$M(\partial K) := \{ \mu \in L^2(\partial K) : \mu|_e \in P_k(e) \quad \forall e \in \mathcal{E}(K) \}.$$ 

When $K$ is a convex quadrilateral, construction of $H(\text{div})$-conforming spaces of the form $P_k(K) \oplus \nabla \times \delta H(K)$ was also presented in \cite{11}. While the filling space $\nabla \times \delta H(K)$ share the same dimension (1 if $k = 0$ and 2 if $k \geq 1$) as those in \cite{12} on a convex quadrilateral, they were not constructed directly on the quadrilateral element $K$ as was done in \cite{12} but by Piola mapping two (one if $k = 0$) divergence-free polynomial functions from the reference square to $K$, resulting rational functions. Moreover, $H(\text{div})$-conforming finite element shape functions were also created \cite{11} Section 5.

Let us now briefly comment on the special cases when $K$ is a triangle or a parallelogram. In these cases, the element $K$ can be considered as a physical element obtained from an affine mapping of the reference triangle or the reference square. We can easily obtain \textit{mapped} commuting exact sequences on the physical element $K$ from those in Theorem 3.2 on the reference triangle, and those in Theorem 3.3 on the reference square via proper linear mapping functions. To simplify the notation, we still denote the mapped sequences on a physical triangle $K$ as $S_{1,k}(K)$ and $S_{2,k}(K)$, and those on the physical parallelogram as $S_{1,k}(K)$ for $i \in \{1, 2, 3, 4\}$. These mapped sequences have an advantage over those in Theorem 3.4 namely, that numerical integration only needs to be done on the reference elements.

**Whole mesh.** Using the sequences in Theorem 3.2 and Theorem 3.3 we can readily obtain \textit{mapped} commuting exact sequences on a hybrid mesh, $\Omega_h := \{K\}$, of a polygonal domain $\Omega$, where each physical element $K$ is an affine mapping of the reference triangle or the reference square. We obtain two families of \textit{mapped} sequences with the following form,

$$S_{2d}(\Omega_h) : \quad H_2(\Omega_h) \xrightarrow{\nabla} E_2(\Omega_h) \xrightarrow{\nabla \times} W_2(\Omega_h),$$

where $H_2(\Omega_h) \times E_2(\Omega_h) \times W_2(\Omega_h) \subset H^1(\Omega) \times H(\text{curl}, \Omega) \times L^2(\Omega)$. The restriction on an element $K$ of the first family of sequences is $S_{1,k}(K)$ if $K$ is a triangle, and is $S_{\text{fill}}(K)$ if $K$ is a parallelogram; and the restriction on an element $K$ of the second family of sequences is $S_{2,k}(K)$ if $K$ is a triangle, and is any of $S_{\mu,h}(K)$ for $i \in \{2, 3, 4\}$ if $K$ is a parallelogram. The reason we are able to do so is due to the trace compatibility of the sequences $S_{\mu,k}(K)$ and $S_{\mu,h}(K)$, and the trace compatibility of the sequences $S_{2,k}(K)$ and $S_{\mu,k}$ for $i \in \{2, 3, 4\}$ in Theorem 3.2 and Theorem 3.3.

On the other hand, using the sequences in Theorem 3.4 we can readily obtain two families of \textit{non-mapped} commuting exact sequences on a more general polygonal mesh, $\Omega_h := \{K\}$, of a polygonal domain $\Omega$, where each physical element $K$ is a polygon, and the restriction on an element $K$ of the first family of sequences is $S_{\text{poly},k}(K)$, and that for the second is $S_{2,k}^\text{poly}(K)$.

**3.3. The three-dimensional case.** In three-space dimensions, we first consider the element $K \subset \mathbb{R}^3$ to be either of the following four reference polyhedra:
Then, the following four sequences are exact for $k \geq 0$:

- **tetrahedron**: $\{(x, y, z) : 0 < x, 0 < y, 0 < z, x + y + z < 1\}$,
- **hexahedron**: $\{(x, y, z) : 0 < x < 1, 0 < y < 1, 0 < z < 1\}$,
- **prism**: $\{(x, y, z) : 0 < x, 0 < y, 0 < z < 1, x + y < 1\}$,
- **pyramid**: $\{(x, y, z) : 0 < x, 0 < y, 0 < z, x + z < 1, y + z < 1\}$.

We present two commuting exact sequences on the reference tetrahedron and four on the reference hexahedron, prism, and pyramid. All these commuting exact sequences are the smallest ones, as described in Theorem 3.13 which contain certain given exact sequence and have a certain prescribed sequence of traces on each face. We then obtain commuting exact sequences on polyhedral meshes made of mapped tetrahedra, hexahedra, prisms, and pyramids.

**Tetrahedron.**

**Theorem 3.5.** Let $K$ be the reference tetrahedron with coordinates $(x, y, z)$. Then, the following two sequences on $K$ are exact for $k \geq 0$,

\[
\begin{align*}
S_{1,k}^\bullet: & \quad \mathcal{P}_{k+3} \xrightarrow{\nabla} \mathcal{P}_{k+2} \xrightarrow{\nabla \times} \mathcal{P}_{k+1} \xrightarrow{\nabla} \mathcal{P}_{k}, \\
S_{2,k}^\bullet: & \quad \mathcal{P}_{k+1} \xrightarrow{\nabla} \mathcal{P}_{k} \oplus x \times \mathcal{P}_{k} \xrightarrow{\nabla \times} \mathcal{P}_{k} \oplus x \mathcal{P}_{k} \xrightarrow{\nabla \times} \mathcal{P}_{k}.
\end{align*}
\]

Here $x = (x, y, z)^T$ and $\mathcal{P}_{k}$ denotes $\mathcal{P}(x, y, z)$.

Moreover, the sequence of traces on a face $f$ for $S_{1,k}^\bullet$ is the sequence $S_{1,k+1}(f)$, and that for $S_{2,k}^\bullet$ is the sequence $S_{2,k+1}(f)$.

Both of these sequences are well-known. The first sequence $S_{1,k}^\bullet$ is mainly due to Nédélec [21] since its $H(\text{curl})$ and $H(\text{div})$ spaces are Nédélec’s edge and face spaces of second kind of degree $k + 2$ and $k + 1$, respectively. Its $H^1$ and $L^2$ spaces are the Lagrange polynomial space of degree $k + 3$, and the discontinuous polynomial space of degree $k$. The second sequence $S_{2,k}^\bullet$ is mainly due to Nédélec [20] since its $H(\text{curl})$ and $H(\text{div})$ spaces are Nédélec’s edge and face spaces of first kind of degree $k$, respectively. Its $H^1$ and $L^2$ spaces are the Lagrange polynomial space of degree $k + 1$, and the discontinuous polynomial space of degree $k$, respectively.

**Hexahedron.**

**Theorem 3.6.** Let $K$ be the reference hexahedron with coordinates $(x, y, z)$. Then, the following four sequences are exact for $k \geq 0$:

\[
\begin{align*}
\begin{align*}
S_{1,k}^\bullet: & \quad \mathcal{P}_{k+3} \xrightarrow{\nabla} \mathcal{P}_{k+2} \xrightarrow{\nabla \times} \mathcal{P}_{k+1} \xrightarrow{\nabla \times} \mathcal{P}_{k}, \\
& \quad \mathcal{Q}_{k} \oplus \begin{cases} 
\mathcal{P}_{k+1}^x \oplus \delta H^{\text{III}}_{k+1} \\
\mathcal{P}_{k+1}^y \oplus \delta H^{\text{III}}_{k+1} \\
\mathcal{P}_{k+1}^z \oplus \delta H^{\text{III}}_{k+1} 
\end{cases} \\
& \quad \mathcal{Q}_{k} \oplus x \times \mathcal{P}_{k} \xrightarrow{\nabla \times} \mathcal{Q}_{k} \oplus x \mathcal{P}_{k} \xrightarrow{\nabla \times} \mathcal{Q}_{k}.
\end{align*}
\end{align*}
\]

Here the additional spaces $\delta H^{\text{III}}_{k+1}$, $\delta H^{\text{III}}_k$, and $\delta E^{\text{III}}_k$ for $k \geq 1$ takes the fol-
following forms:
\[
\delta H^3 : = \text{span}\left\{ x y z^k, y z^k, z x^k, x y^k, x z^k, y x^k, z y^k \right\},
\]
\[
\delta H^3 : = \text{span}\left\{ x y z^k, y z^k, z x^k, x y^k, x z^k, y x^k, z y^k \right\},
\]
\[
\delta L^3 : = \text{span}\left\{ x(y^k \nabla z - z^k \nabla y), y(z^k \nabla x - x^k \nabla z), z(x^k \nabla y - y^k \nabla x) \right\}
\]
\[
\delta E^3 : = \text{span}\left\{ x(y^k \nabla z - z^k \nabla y), y(z^k \nabla x - x^k \nabla z), z(x^k \nabla y - y^k \nabla x) \right\}
\]

Moreover, the sequence of traces on a square face \( f \) for \( \mathbb{S}_{1,k}^\Phi \) is the sequence \( \mathbb{S}_{1,k+1}^\Phi(f) \), and that for \( \mathbb{S}_{1,k}^\Phi \) is the sequence \( \mathbb{S}_{1,k}^\Phi(f) \) for \( i \in \{2, 3, 4\} \).

Note that for \( k = 0 \), the last three sequences are exactly the same. Here the second sequence is new; the other three are known. The first sequence \( \mathbb{S}_{1,k}^\Phi \) is the serendipity sequence of Arnold and Awanou [2]. The second sequence is a slight variation of the first one. It displays a new family of \( H(\text{curl}) \) and \( H(\text{div}) \) spaces. The third sequence \( \mathbb{S}_{1,k}^\Phi \) is the TNT sequence of Cockburn and Qiu [15] (with a slight variation in the space representation). The last sequence is mainly due to Nédélec [20] since its \( H(\text{curl}) \) and \( H(\text{div}) \) spaces are Nédélec’s edge and face spaces of first kind of degree \( k \) on the cube, respectively. Its \( H^1 \) and \( L^2 \) spaces are the tensor-product Lagrange polynomial space of degree \( k + 1 \), and the discontinuous tensor-product polynomial space of degree \( k \), respectively.

**Prism.**

**Theorem 3.7.** Let \( K \) be the reference prism with coordinates \( (x, y, z) \). Then, the following four sequences are exact for \( k \geq 0 \):

\[
\begin{align*}
\mathbb{S}^\Phi_{1,k} & : \mathcal{P}_{k+1} + \delta H^3 \rightarrow \mathcal{P}_{k+2} + \mathcal{P}_{3} + \delta E^3, \\
\mathbb{S}^\Phi_{2,k} & : \mathcal{P}_{k+1} + \delta H^3 \rightarrow \mathcal{P}_{k+2} + \mathcal{P}_{3} + \delta E^3, \\
\mathbb{S}^\Phi_{3,k} & : \mathcal{P}_{k+1} + \delta H^3 \rightarrow \mathcal{P}_{k+2} + \mathcal{P}_{3} + \delta E^3, \\
\mathbb{S}^\Phi_{4,k} & : \mathcal{P}_{k+1} + \delta H^3 \rightarrow \mathcal{P}_{k+2} + \mathcal{P}_{3} + \delta E^3.
\end{align*}
\]

Here, we have
\[
\mathbb{N}_k(x, y) := \mathcal{P}_k(x, y) \oplus \left( \frac{-y}{x} \right) \mathcal{P}_k(x, y), \quad \mathbb{R}T_k(x, y) := \mathcal{P}_k(x, y) \oplus \left( \frac{x}{y} \right) \mathcal{P}_k(x, y),
\]

\[17\]
and the additional spaces $\delta H^{3,III}_k$ and $\delta E^{3,III}_k$ for $k \geq 1$ takes the following forms:

$$\delta H^{3,III}_k := \text{span} \left\{ z^k \mathcal{P}_1(x, y), z \mathcal{P}_k(x, y) \right\},$$

$$\delta E^{3,III}_k := \text{span} \left\{ z^k (x \nabla y - y \nabla x), z \mathcal{P}_{k-1}(x, y)(x \nabla y - y \nabla x) \right\}.$$

Moreover, the sequence of traces for these four sequences are

$$\text{tr}^f \left( S^{\Phi}_{1,k} \right) = \begin{cases} S_{1,k+1}^{{\Phi}}(f) & \text{triangle f,} \\ S_{1,k+1}^{{\mathcal{O}}} & \text{square f,} \end{cases} \quad \text{tr}^f \left( S^{\mathcal{O}}_{2,k} \right) = \begin{cases} S_{2,k}^{{\mathcal{O}}} & \text{triangle f,} \\ S_{2,k}^{{\mathcal{O}}} & \text{square f,} \end{cases}$$

where the sequence $S^{\mathcal{O}}_{3,k}(f)$ on the square face with coordinates $(\xi, z)$ is a slight modification of the sequence $S^{\Phi}_{3,k}(f)$, with the same $H^1$ and $L^2$ spaces, but a different $H(\text{curl})$ space, namely, $\mathcal{Q}_k(\xi, z) + \xi^{k+1} z^k \nabla z \otimes \nabla \{ \xi z^{k+1}, z \xi^{k+1} \}$. 

Note that for $k = 0$, the sequences $S^{\mathcal{O}}_{2,k}$ and $S^{\Phi}_{3,k}$ are exactly the same; they are slightly different from the sequence $S^{\Phi}_{3,k}$. Note also that the $H(\text{curl})$ space for $S^{\Phi}_{3,k}(f)$ is not invariant under the coordinate permutation $(\xi, z) \rightarrow (z, \xi)$, where $(\xi, z)$ are the coordinates on $f$. We did not find a prismatic sequence with trace space on a square face exactly equal to $S^{\mathcal{O}}_{3,k}(f)$. This brings about a small complication for constructing the $H(\text{curl})$-conforming finite element spaces on a hybrid mesh; see the discussion on commuting exact sequences on the whole polyhedral mesh below.

Finally, note that, while the fourth sequence is known, see [18], the other three are new. These four sequences can be considered as the extensions of the related sequences on the reference cube to the reference prism.

**Pyramid.**

**Theorem 3.8.** Let $K$ be the reference pyramid with coordinates $(x, y, z)$. Then, the following four sequences are exact for $k \geq 0$:

\[
\begin{align*}
S^{\Phi}_{1,k} & : \quad P_{k+3} \quad \nabla \quad P_{k+2} \quad \nabla \times \quad P_{k+1} \quad \nabla \cdot \quad P_{k}, \\
P_{k+3} \oplus \delta H_{k+3}^{3,IV} & \quad \oplus \nabla \delta H_{k+3}^{3,IV} \quad \oplus \nabla \times \delta E_{k+2}^{3,IV} \\
S^{\mathcal{O}}_{2,k} & : \quad P_{k+1} \quad \nabla \quad P_{k} \quad \nabla \times \quad P_{k} \quad \nabla \cdot \quad P_{k}, \\
P_{k+1} \oplus \delta H_{k+1}^{3,IV} & \quad \oplus \nabla \delta H_{k+1}^{3,IV} \quad \oplus \nabla \times \delta E_{k+1}^{3,IV}
\end{align*}
\]

Here the additional spaces $\delta H_k^{3,IV}$ and $\delta E_k^{3,IV}$ for $k \geq 1$.
takes the following forms:

\[
\begin{align*}
\delta H^3_{k} & := \text{span} \left\{ \frac{x^y z}{1-z}, \frac{x^y z}{1-z} k_2(x, z), \frac{x^y z}{1-z} k_2(y, z) \right\}, \\
\delta H^3_{k} & := \delta H^3_{k} \oplus \text{span} \left\{ \frac{x^y z}{(1-z)^{\min\{\alpha, \beta\}}} : \alpha = 1, \beta = k \text{ or } \alpha = k, \beta = 1 \right\}, \\
\delta H^3_{k} & := \delta H^3_{k} \oplus \text{span} \left\{ \frac{x^y z}{(1-z)^{\min\{\alpha, \beta\}}} : \alpha \leq k - 1, \beta \leq k - 1, k + 1 \leq \alpha + \beta \right\}, \\
\delta H^3_{k} & := \delta H^3_{k} \oplus \text{span} \left\{ \frac{x^y z}{(1-z)^{\min\{\alpha, \beta\}}} : \alpha \leq k, \beta \leq k, k + 1 \leq \alpha + \beta \right\}, \\
\delta E^3_{k} & := \text{span} \left\{ \frac{x^y z}{1-z} \nabla z, \frac{x^y z}{1-z} \nabla x \right\}, \\
\delta E^3_{k} & := \delta E^3_{k} \oplus \text{span} \left\{ \frac{x^y z}{(1-z)^{\min\{\alpha, \beta\}}} \nabla x : \alpha \leq k, \beta \leq k, k + 1 \leq \alpha + \beta \right\}.
\end{align*}
\]

Moreover, the sequence of traces for these four sequences are

\[
\begin{align*}
\text{tr}^f \left( S_{1,k}^\Phi \right) &= \begin{cases} S_{1,k+1}(f) & \text{if } f \text{ is a triangle}, \\
S_{1,k+1}(f) & \text{if } f \text{ is the base square}, \\
S_{2,k}(f) & \text{if } f \text{ is a triangle}, \\
S_{2,k}(f) & \text{if } f \text{ is the base square}, \\
\end{cases} \\
\text{for } i \in \{2, 3, 4\}.
\end{align*}
\]

All the four sequences are new and contain certain rational functions. They can be considered as the extensions of the related sequences on the reference cube to the reference pyramid. Note that, unlike all the previous sequences, the function spaces here includes certain rational functions.

We remark that similar serendipity-type pyramidal \(H^1\)-conforming space as that in \(S_{1,k}^\Phi\) (with the same space dimension) was recently introduced in \([19]\). The serendipity space in \([19]\), containing \(P_k(K)\), was obtained by mapping certain rational functions from the infinite pyramid to the reference pyramid, extending similar results in \([22, 23]\) with a significant dimension reduction. A set of degrees of freedom, similar as those in \([2]\) on cubes, was also identified. We believe the \(H^1\)-conforming space in \([19]\) (obtained from the infinite pyramid) and that in \(S_{1,k}^\Phi\) (obtained directly from the reference pyramid) should be closely related since both enrich same amount of rational functions (whose precise definition seems to be different) to \(P_k(K)\) to achieve conformity.

Let us point out that our fourth sequence \(S_{4,k}^\Phi\) is significantly smaller than the related pyramidical sequence presented in \([18]\), originally obtained in \([22]\).

**Whole mesh.** Using the sequences in Theorem 3.3 to Theorem 3.8 we can readily obtain mapped commuting exact sequences on a hybrid mesh, \(\Omega_h := \{K\}\), of a polyhedral domain \(\Omega\), where each physical element \(K\) is an affine mapping of any of the four reference polyhedra. We obtain four family of mapped sequences with the following form,

\[\begin{align*}
S_{3\Phi}(\Omega_h) : \quad & H_3(\Omega_h) \xrightarrow{\nabla} E_3(\Omega_h) \xrightarrow{\nabla} V_3(\Omega_h) \xrightarrow{\nabla} W_3(\Omega_h),
\end{align*}\]

where \(H_3(\Omega_h) \times E_3(\Omega_h) \times V_3(\Omega_h) \times W_3(\Omega_h) \subset H^1(\Omega) \times H(\text{curl, } \Omega) \times H(\text{div, } \Omega) \times L^2(\Omega)\).

The restriction on an element \(K\) of the first family of sequences is \(S_{1,k}^\Phi(K)\) if \(K\) is a tetrahedron, \(S_{1,k}^\Phi(K)\) if \(K\) is a parallelepiped, \(S_{1,k}^\Phi(K)\) if \(K\) is a parallel prism, and \(S_{1,k}^\Phi(K)\) if \(K\) is a pyramid with a parallelogram base.
The restriction on an element $K$ of the second family of sequences is $S_{2,k}^\phi(K)$ if $K$ is a tetrahedron, $S_{2,k}^\phi(K)$ if $K$ is a parallelepiped, $S_{2,k}^\phi(K)$ if $K$ is a parallel prism, and $S_{2,k}^\phi(K)$ if $K$ is a pyramid with a parallelogram base.

The restriction on an element $K$ of the third family of sequences, defined on a hybrid mesh without prisms, is $S_{2,k}^\phi(K)$ if $K$ is a tetrahedron, $S_{3,k}^\phi(K)$ if $K$ is a parallelepiped, and $S_{3,k}^\phi(K)$ if $K$ is a pyramid with a parallelogram base.

The restriction on an element $K$ of the fourth family of sequences is $S_{2,k}^\phi(K)$ if $K$ is a tetrahedron, $S_{4,k}^\phi(K)$ if $K$ is a parallelepiped, $S_{4,k}^\phi(K)$ if $K$ is a parallel prism, and $S_{4,k}^\phi(K)$ if $K$ is a pyramid with a parallelogram base. Note that this family of sequences is a modification of the one considered in [18] on a hybrid mesh with a smaller pyramidal sequence.

The reason we are able to do so is due to the trace compatibilities in Theorem 3.5 to Theorem 3.3. In particular, we mention that we exclude prisms in the hybrid mesh of the third family of sequences mainly due to the incompatibility of the $H$(curl) trace of $S_{3,k}^\phi$ and that of $S_{3,k}^\phi$, which differ by a single function.

4. Proofs of the main results in Section 2

4.1. Proof of Theorem 2.9. Proof. Suppose the segment $K$ has coordinate $x$. By the exactness of the sequence $S_1^\phi(K)$, we have $R := \ker H_1^\phi(K)$ (and hence $1 \in H_1^\phi(K)$, and $\mathcal{P}_0(K) \subset W_0^2(K) = \nabla H_1(K)$ (and hence $x \in H_1^\phi(K)$). This implies that $\mathcal{P}_1(K) \subset H_1^\phi(K)$, and hence $\dim tr_H(H_1^\phi(K)) = \dim tr_H(\mathcal{P}_1(K)) = 2$. So $S_1^\phi(K)$ is a compatible exact sequence.

4.2. Proof of Theorem 2.11. Proof. To simplify the notation, we set $H_2(K) := H_2^\phi(K) + \delta H_2^\phi(K)$, $E_2(K) := E_2^\phi(K) + \nabla \delta H_2^\phi(K)$, and $W_2(K) := W_2^\phi(K)$.

The exactness of the sequence $S_2(K)$ follows directly from the exactness of the given sequence

$$H_2^\phi(K) \xrightarrow{\nabla} E_2^\phi(K) \xrightarrow{\nabla \times} W_2^\phi(K).$$

It is easy to show that for any exact sequence $H_2(K) \rightarrow E_2(K) \rightarrow W_2(K)$, we have

$$\dim H_2(K) - \dim E_2(K) + \dim W_2(K) = 1,$$

$$\nabla \overset{\circ}{H}_2 = \{ v \in E_2 : \nabla \times v = 0 \},$$

$$\nabla \times E_2(K) \subset \overset{\circ}{E}_2(K).$$

In view of Definition 2.4 on a compatible exact sequence, we need to prove that the two dimension count identities of Property (ii) are satisfied, and that the sequence of traces on each edge is a compatible exact sequence.

We have

$$\dim tr_H H_2(K) = \dim H_2(K) - \dim \overset{\circ}{H}_2(K) = \dim H_2^\phi(K) + \dim \delta H_2^\phi(K) - \dim \overset{\circ}{H}_2(K)$$

$$= \dim H_2^\phi(K) + \dim \delta H_2^\phi(K) - \dim \overset{\circ}{H}_2(K) = \sum_{e \in E(K)} (\dim \overset{\circ}{H}_1(e) + 1),$$
where the second equality is due to property (ii) of $\delta H^2_g(K)$, the third one is due to property (iii), and the last one is due to property (iv). Now, by property (i) of the Definition 2.10 of an $S_1(\partial K)$ admissible exact sequence, and by property (i) of $\delta H^2_g(K)$, we have

$$
\text{tr}_H^e H_2(K) \subset H_1(e) \quad \forall e \in \mathcal{E}(K).
$$

On the other hand, since

$$
\text{tr}_H H_2(K) \mid_e \subset \text{tr}_H^e H_2(K) \quad \forall e \in \mathcal{E}(K),
$$

we have

$$
\dim \text{tr}_H H_2(K) \leq \sum_{e \in \mathcal{E}(K)} \dim \text{tr}_H^e H_2(K) + \sum_{v \in \mathcal{V}(K)} 1 \leq \sum_{e \in \mathcal{E}(K)} (\dim \text{H}_1(e) + 1).
$$

By (4.2), we have the above inequalities are indeed equalities, and

$$
\text{tr}_H^e H_2(K) = H_1(e) \quad \forall e \in \mathcal{E}(K). \quad (4.3)
$$

This proves the first dimension count identity of Property (ii) for a compatible exact sequence.

Next, let us prove the second dimension count identity. We have that

$$
\dim \text{tr}_E E_2(K) = \dim E_2(K) - \dim \tilde{E}_2(K) = \dim H_2(K) + \dim W_2(K) - 1
$$

$$
- \dim \nabla \times \tilde{E}_2(K) - \dim \{v \in \tilde{E}_2(K) : \nabla \times v = 0\}
$$

$$
= \dim H_2(K) + \dim \tilde{W}_2(K) - \dim \nabla \times \tilde{E}_2(K) - \dim \tilde{H}_2(K)
$$

$$
= \sum_{e \in \mathcal{E}(K)} (\dim \text{H}_1(e) + 1) + \dim \tilde{W}_2(K) - \dim \nabla \times \tilde{E}_2(K)
$$

$$
\geq \sum_{e \in \mathcal{E}(K)} (\dim \text{H}_1(e) + 1)
$$

$$
= \sum_{e \in \mathcal{E}(K)} \dim W_1(e),
$$

where the second equality is due to (4.1a) of an exact sequence, the third equality is due to (4.11) and Property (ii) of Definition 2.10, the fourth equality is due to (4.12), the fifth inequality is due to (4.12), and the last equality is due to the exactness of the sequences $S_1(e)$. Now, by property (i) of Definition 2.10 for an $S_1(\partial K)$ admissible exact sequence, we have

$$
\text{tr}_E^e E_2^g(K) \subset W_1(e) \quad \forall e \in \mathcal{E}(K),
$$

and by property (i) of $\delta H^2_g(K)$, we have

$$
\text{tr}_E \nabla \delta H^2_g(K) = \nabla \text{tr}_H^e \delta H^2_g(K) \subset \nabla H_1(e) = W_1(e) \quad \forall e \in \mathcal{E}(K).
$$

Hence,

$$
\text{tr}_E^e E_2(K) \subset W_1(e) \quad \forall e \in \mathcal{E}(K).
$$
On the other hand, since
\[ \text{tr}_E E_2(K)|_e \subset \text{tr}^E_2 E_2(K) \quad \forall e \in \mathcal{E}(K), \]
we have
\[
\dim \text{tr}_E E_2(K) \leq \sum_{e \in \mathcal{E}(K)} \dim \text{tr}^E_2 E_2(K) \leq \sum_{e \in \mathcal{E}(K)} \dim W_1(e).
\]
By (4.4), we have the above inequalities are indeed equalities. Hence,
\[
\text{tr}^E_2 E_2(K) = W_1(e) \quad \forall e \in \mathcal{E}(K).
\] (4.5)
This completes the proof of the second dimension count identity of Property (ii) for a compatible exact sequence in Definition 2.5. The equalities (4.3) and (4.5) ensure that the sequence of traces \( \text{tr}^e(S_2(K)) = S_1(e) \) is a compatible exact sequence, for all edges \( e \in \mathcal{E}(K) \). So \( S_2(K) \) is a compatible exact sequence.

Finally, invoking [9, Corollary 3.2], we get the minimality of the sequence \( S_2(K) \) by the simple observation that
\[
\delta H_2^g(\mathcal{K}) = \delta E_2^g(\mathcal{K}), \quad E_2^g(\mathcal{K}) = W_2^g(\mathcal{K}).
\]
The proof of Theorem 2.11 is now complete. \( \square \)

4.3. Proof of Theorem 2.13.
Proof. To simplify the notation, we set
\[
\begin{align*}
H_3^g(\mathcal{K}) &:= H_3^g(\mathcal{K}) \oplus \delta H_3^g(\mathcal{K}), \\
E_3^g(\mathcal{K}) &:= E_3^g(\mathcal{K}) \oplus \nabla \delta H_3^g(\mathcal{K}) \oplus \delta E_3^g(\mathcal{K}), \\
V_3^g(\mathcal{K}) &:= V_3^g(\mathcal{K}) \oplus \nabla \delta E_3^g(\mathcal{K}), \\
W_3^g(\mathcal{K}) &:= W_3^g(\mathcal{K}).
\end{align*}
\]
The exactness of the sequence \( S_3(\mathcal{K}) \) comes directly from the exactness of the given sequence
\[
\begin{align*}
H_3^g(\mathcal{K}) &\xrightarrow{\nabla} E_3^g(\mathcal{K}) & \xrightarrow{\nabla \times} V_3^g(\mathcal{K}) & \xrightarrow{\nabla} W_3^g(\mathcal{K}).
\end{align*}
\]
In view of Definition 2.6 of a compatible exact sequence, we need to prove that the three dimension count identities of Property (ii) are satisfied, and that the sequence of traces on any face \( f \in \mathcal{F}(\mathcal{K}) \) is a compatible exact sequence.

The proof of the first dimension count identity for the \( H^1 \)-trace space \( \text{tr}_H H_3(\mathcal{K}) \) and, consequently, that of the identity
\[ \text{tr}_H H_3(\mathcal{K}) = H_2(\mathcal{K}) \quad \forall f \in \mathcal{F}(\mathcal{K}) \] (4.6)
are omitted because they are very similar to those of the two-dimensional case; see details in the proof of Theorem 2.11.

Now, let us prove the third dimension count identity for the \( H(\text{div}) \)-trace space \( \text{tr}_V V_3(\mathcal{K}) \) of Property (ii) in Definition 2.6.

By properties (ii), (iii), and (iv) of \( \delta E_3^g(\mathcal{K}) \), we have
\[
\dim \text{tr}_V V_3(\mathcal{K}) = \dim V_3(\mathcal{K}) - \dim V_3^g(\mathcal{K})
\]
\[= \dim V_3^g(\mathcal{K}) + \dim \delta E_3^g(\mathcal{K}) - \dim \nabla \times V_3^g(\mathcal{K})
\]
\[= \sum_{f \in \mathcal{F}(\mathcal{K})} \dim W_2(f) + \dim V_3^g(\mathcal{K}) - \dim \nabla \times V_3^g(\mathcal{K})
\]
\[\geq \sum_{f \in \mathcal{F}(\mathcal{K})} \dim W_2(f),
\] (4.7)
where in the last inequality, we used the fact that
\[ \nabla \cdot \circ V_3(K) \subset W_3(K) = W_3^0(K). \]

Now, by Property (i) of Definition 2.12 of an \( S_2(\partial K) \)-admissible exact sequence we have \( \text{tr}_V^I V_3^0(K) \subset W_2(f) \), and by property (i) of \( \delta E_3^0(K) \) and the exactness of the sequence \( S_2(f) \), we have
\[ \text{tr}_V^I \nabla \times \delta E_3^0(K) = \nabla \times \text{tr}_E^I \delta E_3^0(K) \subset \nabla \times E_2 = W_2(f). \]

Hence,
\[ \text{tr}_V^I V_3(K) \subset W_2(f) \quad \forall f \in \mathcal{F}(K). \]

This implies that
\[ \dim \text{tr}_V V_3(K) \leq \sum_{f \in \mathcal{F}(K)} \text{tr}_V^I V_3(K) \leq \sum_{f \in \mathcal{F}(K)} W_2(f). \]

By (4.7), the above inequalities are indeed equalities, and we have
\[ \text{tr}_V^I V_3(K) = W_2(f) \quad \forall f \in \mathcal{F}(K), \quad (4.8) \]
\[ \nabla \cdot \circ V_3(K) = W_3(K). \quad (4.9) \]

This completes the proof of the third dimension count identity of Property (ii) in Definition 2.6.

Now, let us prove the second dimension count identity for the \( H(\text{curl}) \)-trace space \( \text{tr}_E E_3(K) \) of Property (ii) in Definition 2.6. We use the following results of an exact sequence:
\[ \dim H_3(K) - \dim E_3(K) + \dim V_3(K) - \dim W_3(K) = 1, \quad (4.10a) \]
\[ \text{tr}_E^I E_3(K) \subset E_2(f) \quad \forall f \in \mathcal{F}(K), \quad (4.10b) \]
\[ \nabla \circ H_3(K) = \{ v \in \circ E_3(K), \nabla \times v = 0 \}, \quad (4.10c) \]
\[ \nabla \times \circ E_3(K) \subset \{ v \in \circ V_3 : \nabla \cdot v = 0 \}. \quad (4.10d) \]

Their proofs are trivial and hence omitted. The inclusion (4.10b) implies that
\[ \dim \text{tr}_E E_3(K) \leq \sum_{e \in \mathcal{E}(K)} \text{tr}_E^e E_3(K) + \sum_{v \in \mathcal{V}(K)} \circ \text{tr}_E^v E_3(K) \]
\[ \leq \sum_{e \in \mathcal{E}(K)} W_1(e) + \sum_{v \in \mathcal{V}(K)} \circ E_2(f). \]

On the other hand, we have, by (4.10a) of an exact sequence, the equalities (4.9),
By the first and third dimension count identities of Property (ii) in Definition 2.6, we have the right-hand side of the above inequality is equal to

\[
\sum_{v \in V(K)} 1 + \sum_{e \in \mathcal{E}(K)} (\dim \overset{\circ}{H}_1(e)) + \sum_{f \in \mathcal{F}(K)} (\dim \overset{\circ}{H}_2(f) + \dim \overset{\circ}{W}_2(f)) - 2
\]

\[
= \sum_{v \in V(K)} 1 + \sum_{e \in \mathcal{E}(K)} (\dim \overset{\circ}{W}_1(e) - 1) + \sum_{f \in \mathcal{F}(K)} (\dim \overset{\circ}{H}_2(f) + \dim \overset{\circ}{W}_2(f) + 1) - 2
\]

\[
= \sum_{e \in \mathcal{E}(K)} \dim \overset{\circ}{W}_1(e) + \sum_{f \in \mathcal{F}(K)} \dim \overset{\circ}{E}_2(f) - I,
\]

where

\[
I := \sum_{v \in V(K)} 1 - \sum_{e \in \mathcal{E}(K)} 1 + \sum_{f \in \mathcal{F}(K)} 1 - 2 = 0
\]

by Euler’s polyhedral formula. Hence,

\[
\dim \text{tr}_E E_3(K) \geq \sum_{e \in \mathcal{E}(K)} \dim \overset{\circ}{W}_1(e) + \sum_{f \in \mathcal{F}(K)} \dim \overset{\circ}{E}_2(f).
\]

And the above inequality is indeed an equality by (4.11). Moreover, the inclusions in (4.10) are also equalities:

\[
\text{tr}_E E_3(K) = E_2(f) \quad \forall f \in \mathcal{F}(K), \quad (4.12a)
\]

\[
\nabla \times \overset{\circ}{E}_3(K) = \{ v \in \overset{\circ}{V}_3 : \nabla \cdot v = 0 \}, \quad (4.12b)
\]

and this completes the proof of the second dimension count identity of Property (ii) in Definition 2.6. The equalities (4.6), (4.8), and (4.12a) imply that the sequence of traces \( \text{tr}_E^f S_3(K) \) is a compatible exact sequence. Hence, \( S_3(K) \) is a compatible exact sequence.

Finally, invoking [9, Corollary 3.2], we get the minimality of the sequence \( S_3(K) \) by the simple observation that

\[
\overset{\circ}{H}_3 = H_3^g, \quad \overset{\circ}{E}_3 = E_3^g, \quad \overset{\circ}{V}_3 = V_3^g, \quad \overset{\circ}{W}_3 = W_3^g.
\]

This completes the proof of Theorem 2.13.
5. Proofs of the applications in Section 3. In this section, we prove all the results of Section 3 by applying Theorem 2.9 (for one dimension), Theorem 2.11 (for two dimensions), and Theorem 2.13 (for three dimensions).

5.1. The one-dimensional case. The proof of Theorem 3.1 is a direct application of Theorem 2.9.

5.2. The two-dimensional case. We first present the following result on exact sequences on the whole space $\mathbb{R}^2$. We give its proof in Appendix B.

**Lemma 5.1.** The following four sequences on $\mathbb{R}^2$ with coordinates $(x, y)$ are exact for $k \geq 0$:

- $S_{2d, 1}^k: \mathcal{P}_{k+2} \xrightarrow{\nabla} \mathcal{P}_{k+1} \xrightarrow{\nabla} \mathcal{P}_k,$
- $S_{2d, 2}^k: \mathcal{P}_{k+1} \xrightarrow{\nabla} \mathcal{P}_k \oplus x \times \tilde{\mathcal{P}}_k \xrightarrow{\nabla} \mathcal{P}_k,$
- $S_{2d, 3}^k: \Omega_k \oplus \{x^{k+1}, y^{k+1}\} \xrightarrow{\nabla} \Omega_k \oplus x \times \{x^k y^k\} \xrightarrow{\nabla} \Omega_k,$
- $S_{2d, 4}^k: \Omega_{k+1} \xrightarrow{\nabla} \Omega_k \oplus \left( \frac{y^{k+1} \mathcal{P}_k(x)}{x^{k+1} \mathcal{P}_k(y)} \right) \xrightarrow{\nabla} \Omega_k.$

The third sequence $S_{2d, 3}^k$ is new to the best of the authors’ knowledge; the other three are well-known. Note that all these four sequences have good “symmetry” in the sense that they are invariant under the coordinate permutation $(x, y) \rightarrow (y, x)$.

Now, we are ready to prove the two-dimensional results of Theorem 3.2, Theorem 3.3, and Theorem 3.4 by applying Theorem 2.11.

**Proof of Theorem 3.2.** Let us fit the sequences $S_{1,k}^1$ and $S_{2,k}^2$ into the framework of Theorem 2.11.

For the first sequence $S_{1,k}^1$, we have that the set of complete trace sequences is

$$S_1(\partial K) := \{S_1(e): \mathcal{P}_{k+2}(e) \rightarrow \mathcal{P}_{k+1}(e) \forall e \in \mathcal{E}(K)\},$$

and the $S_1(\partial K)$-admissible sequence is $S_{1,k}^1$ itself. We also have

$$\dim \delta H^2_2 = 3(k + 2) + k(k + 1)/2 - (k + 3)(k + 4)/2 = 0,$$

hence $\delta H^2_2(K) = \{0\}$. So, $S_{1,k}^1$ is a commuting exact sequence. Similarly, we conclude that $S_{2,k}^2$ is also a commuting exact sequence. This completes the proof of Theorem 3.2.

**Proof of Theorem 3.3.** Let us fit the sequences $S_{1,k}^0$, $S_{2,k}^2$, $S_{3,k}^3$, and $S_{4,k}^4$ into the framework of Theorem 2.11.

For the first sequence $S_{1,k}^0$, we have that the set of complete trace sequences is

$$S_1(\partial K) := \{S_1(e): \mathcal{P}_{k+2}(e) \rightarrow \mathcal{P}_{k+1}(e) \forall e \in \mathcal{E}(K)\},$$

and the $S_1(\partial K)$-admissible sequence is $S_{1,k}^0$. We also have the space $\delta H^2_2 := \delta H^2_{k+2}$ satisfy all the four properties of $\delta H^2_2$ in Theorem 2.11. In particular, we have

$$\dim \delta H^2_2 = 4(k + 2) + (k - 1)k/2 - (k + 3)(k + 4)/2 = 2.$$

Hence, $S_{1,k}^0$ is a commuting exact sequence.
The proof for the second sequence $S_{2,k}^{\delta}$ is identical to that for the first one. For the third sequence $S_{3,k}^{\delta}$, we have that the set of complete trace sequences is

$$S_1(\partial K) := \{ S_1(e) : P_{k+1}(e) \rightarrow P_k(e) \ \forall e \in E(K), \}$$

and the $S_1(\partial K)$-admissible sequence is $S_{3,k}^{ad}$ in Lemma 5.1. We also have the space $\delta H_2^q := \delta H_{k+1}^{2,q}$ satisfying all the four properties of $\delta H_2^q$ in Theorem 2.11 in particular, we have

$$\dim \delta H_2^q = 4(k+1) + (k-1)^2 - ((k+1)^2 + 2) = 2.$$ 

Hence, $S_{3,k}^{\delta}$ is a commuting exact sequence.

For the last sequence $S_{4,k}^{\delta}$, we have the set of complete trace sequences is

$$S_1(\partial K) := \{ S_1(e) : P_{k+2}(e) \rightarrow P_{k+1}(e) \ \forall e \in E(K), \}$$

and the $S_1(\partial K)$-admissible sequence is $S_{4,k}^{ad}$ in Lemma 5.1. We also have the space $\delta H_2^q := \{0\}$ satisfy all the four properties of $\delta H_2^q$ in Theorem 2.11 since

$$\dim \delta H_2^q = 4(k+1) + k^2 - (k+2)^2 = 0.$$ 

Hence, $S_{4,k}^{\delta}$ is also a commuting exact sequence. This completes the proof of Theorem 3.4. 

**Proof of Theorem 3.4**  
Proof. Let us fit the sequences $S_{1,k}^{\text{poly}}$ and $S_{2,k}^{\text{poly}}$ into the framework of Theorem 2.11.  

For the first sequence $S_{1,k}^{\text{poly}}$, we have the set of complete trace sequences is

$$S_1(\partial K) := \{ S_1(e) : P_{k+2}(e) \rightarrow P_{k+1}(e) \ \forall e \in E(K), \}$$

and the $S_1(\partial K)$-admissible sequence is $S_{1,k}^{ad}$ in Lemma 5.1. The proof of the space $\delta H_2^q := \delta H_{k+1}^{2,q}$ satisfying all the four properties of $\delta H_2^q$ in Theorem 2.11 is not trivial. It is given in the proof of [12] Theorem 2.6).  

The proof for the second sequence is identical to that for the first one. This completes the proof of Theorem 3.4.

**5.3. The three-dimensional case.** As we did for the two-dimensional case, we first present the following result on exact sequences on the whole space $\mathbb{R}^3$. We give its proof in Appendix B.

**Lemma 5.2.** The following six sequences on $\mathbb{R}^3$ are exact for $k \geq 0$. The first two sequences are the famous polynomial de Rham sequences that contain polynomials of certain degree:

$$S_{1,k}^{3d} : \ P_{k+3} \xrightarrow{\nabla} \ P_{k+2} \xrightarrow{\nabla x} \ P_{k+1} \xrightarrow{\nabla} \ P_k, \quad S_{3,k}^{3d} : \ P_{k+1} \xrightarrow{\nabla} \ P_k \oplus \mathbf{x} \times \mathbf{P}_k \xrightarrow{\nabla x} \ P_{k+1} \xrightarrow{\nabla} \ P_k.$$ 

The next two sequences have spaces containing tensor product polynomials of certain degree.

$$S_{4,k}^{3d} : \ Q_{k+1} \xrightarrow{\nabla} \mathbf{Q}_k \oplus \mathbf{x} \times \mathbf{Q}_k \xrightarrow{\nabla x} \mathbf{Q}_k \xrightarrow{\nabla} \mathbf{Q}_k,$$

$$S_{5,k}^{3d} : \mathbf{Q}_{k+1} \xrightarrow{\nabla} \left( \begin{array}{c} P_{k+1,k+1} \\ P_{k+1,k+1} \\ P_{k+1,k+1} \\ P_{k+1,k+1} \\ P_{k+1,k+1} \end{array} \right) \xrightarrow{\nabla x} \left( \begin{array}{c} P_{k+1,k+1} \\ P_{k+1,k+1} \\ P_{k+1,k+1} \\ P_{k+1,k+1} \end{array} \right) \xrightarrow{\nabla} \mathbf{Q}_k.$$
The last two sequences contain polynomials of certain degree in the \((x, y)\)-plane, and have some spaces with tensor product structure in the \((x, z)\)– and \((y, z)\)-plane.

\[
S_{3,k}^{\delta} : \mathcal{P}_{k,3} \rightarrow \mathcal{P}_{k,3} \oplus \mathcal{T}_{k+1} \oplus \{x^{k+1}\} \rightarrow \mathcal{P}_{k,3} \oplus \mathcal{T}_{k+1} \oplus \{x^{k+1}\}
\]

The third sequence \(S_{5,k}^{3}\) and the fifth sequence \(S_{5,k}^{5}\) are new to the best of the authors’ knowledge; the other four are well-known. Note that all these six sequences, except the fifth, have good “symmetry” in the sense that they are invariant under any coordinate permutation. The fifth sequence is only invariant under the coordinate permutation \((x, y, z) \rightarrow (y, x, z)\).

To further simplify the notation, we use the so-called \(M\)-index introduced in [14] and used in [13] to obtain various finite element spaces admitting \(M\)-decompositions in three dimensions. The definition of an \(M\)-index is given as follows.

**Definition 5.3 (The \(M\)-index).** The \(M\)-index of the space \(V(K) \times W(K) \subset H(\text{div}, K) \times H^{1}(K)\) is the number

\[
I_{M}(V \times W) := \dim M(\partial K) - \dim \text{tr}_{\mathcal{V}}\{v \in V(K) : \nabla \cdot v = 0\}
- \dim \text{tr}_{\mathcal{H}}\{w \in W(K) : \nabla w = 0\},
\]

where

\[
M(\partial K) = \{\mu \in L^{2}(\partial K) : \mu|_{f} \in M(f) \ \forall f \in \mathcal{F}(K)\}
\]

is a finite element space defined on the boundary \(\partial K\) of a polyhedron \(K\).

Using the definition of an \(M\)-index, we have the number in the right hand side of property (iv) of \(\delta E^{3}_{3}(K)\) in Theorem 2.13 is nothing but \(I_{M}(V_{3}^{3} \times W_{3}^{3})\) with the trace space

\[
M(\partial K) := \{\mu \in L^{2}(\partial K) : \mu|_{f} \in W_{2}(f) \ \forall f \in \mathcal{F}(K)\}.
\]

That is,

\[
I_{M}(V_{3}^{3} \times W_{3}^{3}) = \sum_{f \in \mathcal{F}(K)} W_{2}(f) + \dim V_{3}^{3}(K)
\]

\[
+ \dim \left\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\right\} - \dim V_{3}^{3}(K).
\]

To see this, we have

\[
\sum_{f \in \mathcal{F}(K)} W_{2}(f) + \dim V_{3}^{3}(K) + \dim \left\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\right\} - \dim V_{3}^{3}(K)
\]

\[
= \dim M(\partial K) + \dim W_{3}^{3}(K) - 1 + \dim \left\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\right\}
- \dim \text{tr}_{\mathcal{V}}\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\} - \dim V_{3}^{3}(K)
\]

\[
= \dim M(\partial K) + \dim W_{3}^{3}(K) - 1 - \dim \text{tr}_{\mathcal{V}}\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\}
- \dim V_{3}^{3}(K)
\]

\[
= \dim M(\partial K) - \dim \text{tr}_{\mathcal{H}}\{w \in W_{3}^{3}(K) : \nabla w = 0\}
- \dim \text{tr}_{\mathcal{V}}\{v \in V_{3}^{3}(K) : \nabla \cdot v = 0\}
\]

\[
= I_{M}(V_{3}^{3} \times W_{3}^{3}).
\]
From now on, the trace space $M(\partial K)$ will always be of the form (5.1) where the spaces $W_2(f)$ on each face vary in different locations.

Since the computation of the $M$-index for various polynomial spaces on the four reference polyhedra was given in [13], we can directly use those results to verify property (iv) of $\delta E^g_3(K)$.

Now, we are ready to prove the three-dimensional results of Theorem 3.5 to Theorem 3.8 by applying the general result of Theorem 2.13.

**Proof of Theorem 3.5.** Proof. Let us fit the sequences $S_{1,k}^\partial$ and $S_{2,k}^\partial$ into the framework of Theorem 2.13. Here the element

$$K = \{(x, y, z) : 0 < x < 1, 0 < y < 1, 0 < z < 1, x + y + z < 1\}$$

is the reference tetrahedron with four square faces, six edges, and four vertices.

For the first sequence $S_{1,k}^\partial$, we have the set of complete trace sequences is

$$S_2(\partial K) := \{S_{1,k+1}^\partial(f) : \forall f \in \mathcal{F}(K)\},$$

and the given $S_2(\partial K)$-admissible sequence is $S_{3d}^\partial$ in Lemma 5.2. We also have

$$\dim \delta H_3^g = 4 + 6(k + 2) + 4(k + 1)(k + 2)/2 + k(k + 1)(k + 2)/6$$

$$- (k + 4)(k + 5)(k + 6)/6$$

$$= 0$$

$$\dim \delta E_3^g = I_{MF}(V_3^g \times W_3^g)$$

$$= 0,$$

hence $\delta H_3^g = \{0\}$, and $\delta E_3^g = \{0\}$. So, $S_{1,k}^\partial$ is a commuting exact sequence. Similarly, we conclude that $S_{2,k}^\partial$ is also a commuting exact sequence. This completes the proof of Theorem 3.5.

**Proof of Theorem 3.6.** Proof. Let us fit the sequences $S_{1,k}^\partial, S_{2,k}^\partial, S_{3,k}^\partial$, and $S_{4,k}^\partial$ into the framework of Theorem 2.13. Here the element

$$K = \{(x, y, z) : 0 < x < 1, 0 < y < 1, 0 < z < 1\}$$

is the reference cube with six square faces, twelve edges, and eight vertices.

For the first sequence $S_{1,k}^\partial$, we have the set of complete trace sequences is

$$S_2(\partial K) := \{S_{1,k+1}^\partial(f) : \forall f \in \mathcal{F}(K)\},$$

and the given $S_2(\partial K)$-admissible sequence is $S_{3d}^\partial$ in Lemma 5.2. It is then easy to verify that $\delta H_3^g := \delta H_{k+3}^{3,1}$ satisfies all the four properties of $\delta H_3^g$, and $\delta E_3^g := \delta E_{k+2}^{3,1}$ satisfies all the four properties of $\delta E_3^g$ in Theorem 2.13. In particular, we have

$$\dim \delta H_3^g = 8 + 12(k + 2) + 6k(k + 1)/2 + (k - 2)(k - 1)k/6$$

$$- (k + 4)(k + 5)(k + 6)/6$$

$$= 3(k + 4),$$

$$\dim \delta E_3^g = I_{MF}(V_3^g \times W_3^g) = 3(k + 2),$$

where the last equality is due to [13, Theorem 2.12].
For the sake of completeness, here we we present a proof of property (iii) for $\delta E^3_3(K)$, which is a bit more difficult to verify that the other properties. Given a function $p \in \mathcal{P}_{k+1}(y, z)$, we have

$$\nabla \times (x p (y \nabla z - z \nabla y)) = (k + 3)p x \nabla x - p y \nabla y - p z \nabla z.$$ 

This means that the function $\nabla \times (x p (y \nabla z - z \nabla y)) \in \nabla \times \delta E^3_3(K)$ has normal trace equal to zero on the three faces $x = 0$, $y = 0$, and $z = 0$ of the unit cube $K$, and has normal trace equals to $(k + 3)p \in \mathcal{P}_{k+1}(y, z)$ on the face $x = 1$. Similar results hold for

$$\nabla \times (y q (z \nabla x - x \nabla z)) = -q x \nabla x + (k + 3)q y \nabla y - q z \nabla z,$$

$$\nabla \times (z r (x \nabla y - y \nabla x)) = -r x \nabla x - r y \nabla y + (k + 3)r z \nabla z,$$

with $q \in \mathcal{P}_{k+1}(z, x)$, and $r \in \mathcal{P}_{k+1}(x, y)$. Using this fact, we have any function in $\nabla \times \delta E^3_3(K)$ has a normal trace equal to zero on one face and be a polynomial of degree $k + 1$ on its opposite (parallel) face for at least one pair of parallel faces. On the other hand, if a function in $V^3_3(K) = \mathcal{P}_{k+1}$ has normal trace equal to zero on one face, the normal trace on its opposite face must be a polynomial of degree no greater than $k$. Hence, $\text{tr}_V \nabla \times \delta E^3_3(K) \cap \text{tr}_V V^3_3(K) = \{0\}$. This implies property (iii) of $\delta E^3_3(K)$.

So, $S_{3,k}$ is a commuting exact sequence.

For the second sequence $S_{2,k}$, we have the set of complete trace sequences is

$$S_2(\partial K) := \{S^{\square}_{3,k}(f) : \forall f \in \mathcal{F}(K)\},$$

and the given $S_2(\partial K)$-admissible sequence is $S^{\square}_{3,k}$ in Lemma [5.2]. It is then trivial to verify that $\delta H^3_3 := \delta H^3_{k+1}$ satisfies all the four properties of $\delta H^3_3$, and $\delta E^3_3 := \delta E^3_{k+1}$ satisfies all the four properties of $\delta E^3_3$ in Theorem [2.13]. In particular, we have, for $k \geq 1$,

$$\dim \delta H^3_3 = 8 + 12k + 6(k - 1)^2 + (k - 1)^3 - (k + 1)^3 + 3 = 9,$$

$$\dim \delta E^3_3 = I_M(V^3_3 \times W^3_3) = 6,$$

where the last equality is due to [13, Theorem 2.11]. So, $S^{\square}_{3,k}$ is a commuting exact sequence.

For the last sequence $S_{4,k}$, we have the set of complete trace sequences is

$$S_2(\partial K) := \{S^{\square}_{3,k}(f) : \forall f \in \mathcal{F}(K)\},$$

and the given $S_2(\partial K)$-admissible sequence is $S^{\square}_{4,k}$ in Lemma [5.2]. We also have the space $\delta H^3_3 := \{0\}$ satisfies all the four properties of $\delta H^3_3$, and $\delta E^3_3 := \{0\}$ satisfies all the four properties of $\delta E^3_3$ in Theorem [2.13] since

$$\dim \delta H^3_3 = 8 + 12k + 6k^2 + k^3 - (k + 2)^3 = 0,$$

$$\dim \delta E^3_3 = I_M(V^3_3 \times W^3_3) = 0.$$

So, $S^{\square}_{4,k}$ is a commuting exact sequence. This completes the proof of Theorem [3.6]
Proof of Theorem 3.7. Proof. Let us fit the sequences $S_{1,k}$, $S_{2,k}$, $S_{3,k}$, and $S_{4,k}$ into the framework of Theorem 2.13. Here the element

$$K = \{ (x, y, z) : 0 < x, \ 0 < y, \ 0 < z < 1, \ x + y < 1 \}$$

is the reference prism with five faces (two triangular faces and three square faces), nine edges, and six vertices.

For the first sequence $S_{1,k}$, we have the set of complete trace sequences is

$$S_2(\partial K) := \left\{ S_2(f) : \begin{array}{ll} S_2(f) = S_{2,k+1}(f) & \text{if } f \text{ is a triangle}, \\
S_2(f) = S_{1,k+1}(f) & \text{if } f \text{ is a square} \end{array} \right\},$$

and the given $S_2(\partial K)$-admissible sequence is $S_{3d_1}$ in Lemma 5.2. It is then trivial to verify that $\delta H_3^g := \delta H_{3,k+1}^{3,III}$ satisfies all the four properties of $\delta H_3^g$, and $\delta E_3^g := \delta E_{3,k+1}^{3,III}$ satisfies all the four properties of $\delta E_3^g$ in Theorem 2.13. In particular, we have

$$\dim \delta H_3^g = 6 + 9(k+2) + (k+1)(k+2) + 3(k+1)/2$$
$$+ (k-1)k(k+1)/6 - (k+4)(k+5)(k+6)/6$$
$$= k + 6,$$

$$\dim \delta E_3^g = I_M(V_3^g \times W_3^g) = k + 3,$$

where the last equality is due to [13, Theorem 2.8]. So, $S_{1,k}$ is a commuting exact sequence.

The proof for the second sequence $S_{2,k}$ is identical to that for the first one.

For the third sequence $S_{3,k}$, we have the set of complete trace sequences is

$$S_2(\partial K) := \left\{ S_2(f) : \begin{array}{ll} S_2(f) = S_{3,k+1}(f) & \text{if } f \text{ is a triangle}, \\
S_2(f) = S_{3,k+1}(f) & \text{if } f \text{ is a square} \end{array} \right\},$$

and the given $S_2(\partial K)$-admissible sequence is $S_{5d_1}$ in Lemma 5.2. It is then trivial to verify that $\delta H_3^g := \delta H_{k+1}^{3,III}$ satisfies all the four properties of $\delta H_3^g$, and $\delta E_3^g := \delta E_{k+1}^{3,III}$ satisfies all the four properties of $\delta E_3^g$ in Theorem 2.13. In particular, we have, for $k \geq 1$,

$$\dim \delta H_3^g = 6 + 9k + (k-1)k + 3(k-1)^2 + (k-2)(k-1)^2/2$$
$$- ((k+1)^2(k+2)/2 + k + 3)$$
$$= k + 4,$$

$$\dim \delta E_3^g = I_M(V_3^g \times W_3^g) = k + 2,$$

where the last equality is due to [13, Theorem 2.7]. So, $S_{3,k}$ is a commuting exact sequence.

For the last sequence $S_{4,k}$, we have the set of complete trace sequences is

$$S_2(\partial K) := \left\{ S_2(f) : \begin{array}{ll} S_2(f) = S_{4,k+1}(f) & \text{if } f \text{ is a triangle}, \\
S_2(f) = S_{4,k+1}(f) & \text{if } f \text{ is a square} \end{array} \right\},$$

and the given $S_2(\partial K)$-admissible sequence is $S_{6d_1}$ in Lemma 5.2. We also have the space $\delta H_3^g := \{0\}$ satisfies all the four properties of $\delta H_3^g$, and $\delta E_3^g := \{0\}$ satisfies all
the four properties of $\delta E_g^3$ in Theorem 2.13 since
\[
\dim \delta H_g^3 = 6 + 9k + (k - 1)k + 3k^2 + (k - 1)k^2 / 2 \\
- (k + 2)^2(k + 3)/2
= 0,
\]
\[
\dim \delta E_g^3 = I_M(V_g^3 \times W_g^3) = 0.
\]
So, $S^\Phi_{4,k}$ is a commuting exact sequence. This completes the proof of Theorem 3.7.

**Proof of Theorem 3.8.** Let us fit the sequences $S^\Phi_{1,k}$, $S^\Phi_{2,k}$, $S^\Phi_{3,k}$, and $S^\Phi_{4,k}$ into the framework of Theorem 2.13. Here the element

\[ K = \{(x, y, z) : 0 < x < 1, 0 < y < 1, 0 < z < 1, x + z < 1, y + z < 1\} \]

is the reference prism with five faces (four triangular faces and one square face), eight edges, and five vertices.

For the first sequence $S^\Phi_{1,k}$, we have the set of complete trace sequences is

\[ S_2(\partial K) := \left\{ S_2(f) : \begin{array}{ll}
S_2(f) = S^\Phi_{1,k+1}(f) & \text{if } f \text{ is a triangle}, \\
S_2(f) = S^\Phi_{1,k+1}(f) & \text{if } f \text{ is a square}.
\end{array} \right\} \]

and the given $S_2(\partial K)$-admissible sequence is $S^3_{1,k}$ in Lemma 5.2. It is then trivial to verify that $\delta H_g^3 := \delta H_{k+3}^{IV}$ satisfies all the four properties of $\delta H_g^3$, and $\delta E_g^3 := \delta E_{k+2}^{IV}$ satisfies all the four properties of $\delta E_g^3$ in Theorem 2.13. In particular, we have
\[
\dim \delta H_g^3 = 5 + 8(k + 2) + 2(k + 1)(k + 2) + k(k + 1)/2 \\
+ (k - 1)k(k + 1)/6 - (k + 4)(k + 5)(k + 6)/6
= 2k + 5,
\]
\[
\dim \delta E_g^3 = I_M(V_g^3 \times W_g^3) = 3,
\]
where the last equality is due to [13, Theorem 2.6]. So, $S^\Phi_{1,k}$ is a commuting exact sequence.

The proofs for the other three sequences are similar to that for the first one, and hence are omitted. This completes the proof of Theorem 3.8.

**6. Conclusion.** We presented a systematic construction of commuting exact sequences on polygonal/polyhedral elements. The systematic construction is applied to the reference triangle, reference square, and general polygon in two dimensions, and the reference tetrahedron, reference cube, reference prism, and reference pyramid in three dimensions to obtain concrete commuting exact sequences. We obtain \textit{mapped} commuting exact sequences on a two-dimensional hybrid mesh consists of triangles and parallelograms, and \textit{non-mapped} commuting exact sequences on a general two-dimensional polygonal mesh. We also obtain \textit{mapped} commuting exact sequences on a three-dimensional hybrid mesh with elements obtained via affine mapping from one of the four reference polyhedra. The actual implementation of shape functions for our four family of commuting exact sequences on hybrid polyhedral meshes constitutes the subject of ongoing work.

As already pointed out in [13], finding stable pairs of finite element spaces defining mixed methods for the diffusion problem on general polyhedral elements is a very complicated task, mainly due to the need of characterizing the \textit{solenoidal bubble space}
\[
\{ v \in V_g^3(K) : \nabla \cdot v = 0 \},
\]
that is, the subspace of $V^g_3$ containing divergence-free functions with zero normal trace on the boundary. Finding exact sequences with a commuting diagram property shares exactly the same difficulty. However, Theorem 2.13 does shed light on a promising approach to carry out the construction. In particular, we can take the given exact sequence to be $S_{1,k}$, which is a $S_2(\partial K)$-admissible exact sequence for the set of commuting trace sequences

$$S_2(\partial K) := \{S^\text{poly}_{1,k}(f) : \forall f \in \mathcal{T}(K)\},$$

or to be $S^\text{poly}_{2,k}$, which is a $S_2(\partial K)$-admissible exact sequence for the set of commuting trace sequences

$$S_2(\partial K) := \{S^\text{poly}_{2,k}(f) : \forall f \in \mathcal{T}(K)\}.$$  

Then, the only task left is to find the spaces $\delta H^g_3(K)$ and $\delta E^g_3(K)$ that satisfy the properties described in Theorem 2.13. The actual construction of commuting exact sequences on a general polyhedron is currently under way.

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Appendix A: the harmonic interpolators. Here we use our notation to rewrite the harmonic interpolators (in one-, two- and three-space dimensions) introduced in [10]. To simplify the notation, we let the domain of these interpolations be space of smooth fields. However, we only need these spaces to be regular enough such that the related differential and traces operators used in the harmonic interpolators make sense; see [11, Section 2.3] for the related spaces with minimal regularity.

One-dimensional harmonic interpolators. Let $K \in \mathbb{R}$ be a segment. We denote $\Pi_H^1$ and $\Pi_{W^1}$ being the harmonic interpolators related to $H^1$- and $L^2$-fields. We denote

$$H_1(K) \rightarrow W_1(K)$$

as the related compatible exact sequence.

The harmonic interpolators $\Pi_H^1 \times \Pi_{W^1} : C^\infty(\tilde{K}) \times C^\infty(\tilde{K}) \rightarrow H_1(K) \times W_1(K)$ are the following ones satisfying

$$\Pi_H^1 u(v) = u(v) \quad \forall v \in \mathcal{V}(K),$$

$$(\nabla \Pi_H^1 u, v)_K = (\nabla u, v)_K \quad \forall v \in \nabla \tilde{H}_1(K)$$

and

$$(\Pi_{W^1} u, v)_K = (u, v)_K \quad \forall v \in \tilde{W}_1(K) \oplus \mathcal{P}_0(K).$$

Two-dimensional harmonic interpolators. Let $K \in \mathbb{R}^2$ be a segment. We denote $\Pi_H^2$, $\Pi_E^2$ and $\Pi_{W^2}$ being the harmonic interpolators related to $H^1$, $H(\text{curl})$, and $L^2$-fields. We denote

$$H_2(K) \rightarrow E_2(K) \rightarrow W_2(K)$$

as the related compatible exact sequence.
The harmonic interpolators $\Pi^3_H \times \Pi^3_E \times \Pi^3_W : C^\infty(\bar{K}) \times C^\infty(\bar{K}) \times C^\infty(\bar{K}) \to H_2(K) \times E_2(K) \times W_3(K)$ are the following ones satisfying

\[
\Pi^3_H u(v) = u(v) \quad \forall v \in \mathcal{V}(K),
\]

\[
(\nabla \operatorname{tr}^e_H(\Pi^3_H u), v)_e = (\nabla \operatorname{tr}^e_H(u), v)_e \quad \forall v \in \nabla \mathcal{H}^1(e), \forall e \in E(K),
\]

\[
(\nabla \Pi^3_H u, v)_K = (\nabla u, v)_K \quad \forall v \in \nabla \mathcal{H}^2_2(K),
\]

and

\[
(\operatorname{tr}^e_E(\Pi^3_E u), v)_e = (\operatorname{tr}^e_E(u), v)_e \quad \forall v \in W_1(e) \oplus \mathcal{P}_0(e), \forall e \in E(K),
\]

\[
(\nabla \times \Pi^3_E u, v)_K = (\nabla \times u, v)_K \quad \forall v \in \nabla \mathcal{E}_2(K),
\]

\[
(\Pi^3_E u, v)_K = (u, v)_K \quad \forall v \in \{ v \in \mathcal{E}_2(K) : \nabla \times v = 0 \},
\]

and

\[
(\Pi^3_W u, v)_K = (u, v)_K \quad \forall v \in \mathcal{W}_2(K) \oplus \mathcal{P}_0(K).
\]

**Three-dimensional harmonic interpolators.** Let $K \in \mathbb{R}^3$ be a polyhedron. We denote $\Pi^3_H$, $\Pi^3_E$, $\Pi^3_W$ and $\Pi^3_H$ being the harmonic interpolators related to $H^1$-, $H(\text{curl})$-, $H(\text{div})$- and $L^2$-fields. We denote

\[
H_3(K) \hookrightarrow E_3(K) \hookrightarrow V_3(K) \hookrightarrow W_3(K)
\]

as the related compatible exact sequence.

The harmonic interpolators $\Pi^3_H \times \Pi^3_E \times \Pi^3_W : C^\infty(\bar{K}) \times C^\infty(\bar{K}) \times C^\infty(\bar{K}) \to H_3(K) \times E_3(K) \times V_3(K) \times W_3(K)$ are the following ones satisfying

\[
\Pi^3_H u(v) = u(v) \quad \forall v \in \mathcal{V}(K),
\]

\[
(\nabla \operatorname{tr}^e_H(\Pi^3_H u), v)_e = (\nabla \operatorname{tr}^e_H(u), v)_e \quad \forall v \in \nabla \mathcal{H}^1(e), \forall e \in E(K),
\]

\[
(\nabla \operatorname{tr}^f_H(\Pi^3_H u), v)_f = (\nabla \operatorname{tr}^f_H(u), v)_f \quad \forall v \in \nabla \mathcal{H}^2_2(f), \forall f \in F(K),
\]

\[
(\nabla \Pi^3_H u, v)_K = (\nabla u, v)_K \quad \forall v \in \nabla \mathcal{H}^3_3(K),
\]

and

\[
(\operatorname{tr}^e_E(\Pi^3_E u), v)_e = (\operatorname{tr}^e_E(u), v)_e \quad \forall v \in W_1(e) \oplus \mathcal{P}_0(K), \forall e \in E(K),
\]

\[
(\nabla \times \Pi^3_E u, v)_f = (\nabla \times u, v)_f \quad \forall v \in \nabla \mathcal{E}_2(f), \forall f \in F(K),
\]

\[
(\operatorname{tr}^f_E(\Pi^3_E u), v)_f = (\operatorname{tr}^f_E(u), v)_f \quad \forall v \in \{ v \in \mathcal{E}_2(f) : \nabla \times v = 0 \}, \forall f \in F(K),
\]

\[
(\Pi^3_E u, v)_K = (u, v)_K \quad \forall v \in \nabla \mathcal{E}_3_3(K),
\]

and

\[
(\operatorname{tr}^f_W(\Pi^3_W u), v)_f = (\operatorname{tr}^f_W(u), v)_f \quad \forall v \in W_2(f) \oplus \mathcal{P}_0(f), \forall f \in F(K),
\]

\[
(\nabla \cdot \Pi^3_W u, v)_K = (\nabla \cdot u, v)_K \quad \forall v \in \nabla \mathcal{V}_3_3(K),
\]

\[
(\Pi^3_W u, v)_K = (u, v)_K \quad \forall v \in \{ v \in \mathcal{V}_3_3(K) : \nabla \cdot v = 0 \},
\]

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and

$$(\Pi^3_W u, v)_K = (u, v)_K \quad \forall v \in \hat{W}_3(K) \oplus \mathcal{P}_0(K).$$

**Appendix B: proofs of sequence exactness.**

**Proof of Lemma 5.1.** *Proof.* We only provide a sketch of the proof since it is very simple. The exactness of these four sequences can be proven using exactly the same argument by showing that the differential operators map the previous function space into the next one, the curl operator is surjective, and the following dimension count holds

$$\dim H_2(K) - \dim E_2(K) + \dim W_2(K) = 1,$$

where $H_2(K)$, $E_2(K)$, $W_2(K)$ are the related spaces for the sequence. This completes the sketch of the proof.

**Proof of Lemma 5.2.** *Proof.* The exactness of the first two sequences, $S_{2d}^1$ and $S_{2d}^2$, is well-known; see [3] for an elegant proof in arbitrary space dimensions which includes the three-dimensional result as a special case, and uses the Koszul complex and the homotopy formula.

Next, we prove that the last two sequences, $S_{5, k}^{2d}$ and $S_{6, k}^{2d}$, are exact. We omit the proofs for the third and fourth sequences since they are similar to those we are going to present now.

We prove the exactness by showing that the following three identities hold

$$\mathbb{R} = \text{Ker} \nabla H_3,$$

$$\nabla H_3 = \text{Ker} \nabla \times E_3,$$

$$\nabla \cdot V_3 = W_3,$$

and that the following dimension count identity holds

$$\dim H_3 - \dim E_3 + \dim V_3 - \dim W_3 = 1.$$  \hspace{1cm} (6.1d)

The other equality, namely, $\nabla \times E_3 = \text{Ker} \nabla \cdot V_3$, is a direct consequence of the above results. To see this, we have

$$\dim \nabla \times E_3 = \dim E_3 - \dim \text{Ker} \nabla \times E_3 = \dim E_3 - \dim \nabla H_3 \quad \text{by} \ (6.1b)$$

$$= \dim E_3 - \dim H_3 + 1 \quad \text{by} \ (6.1a)$$

$$= \dim V_3 - \dim W_3 \quad \text{by} \ (6.1d)$$

$$= \dim V_3 - \dim \nabla \cdot V_3 \quad \text{by} \ (6.1c)$$

$$= \dim \text{Ker} \nabla \cdot V_3.$$

We start with the verification for the sequence $S_{5, k}^{2d}$. The first equality (6.1a) is trivially satisfied. We also have

$$\nabla \cdot \begin{pmatrix} 0 \\ \mathcal{P}_k(x, y) \oplus \mathcal{P}_{k+1}(z) \end{pmatrix} = \mathcal{P}_k(x, y) \oplus \mathcal{P}_k(z),$$
which implies the third equality (6.1c). The dimension equality (6.1d) is also easy to verify by the fact that

\[
\dim H_3 = \dim \mathcal{P}_{k+1|k+1} = (k + 2)^2(k + 3)/2
\]

\[
\dim E_3 = \dim \mathcal{N}_k(x, y) \cdot \dim \mathcal{P}_{k+1}(z) + \dim \mathcal{P}_{k+1|k}
= 3(k + 1)(k + 2)(k + 3)/2
\]

\[
\dim V_3 = \dim \mathcal{RT}_k(x, y) \cdot \dim \mathcal{P}_k(z) + \dim \mathcal{P}_{k|k+1}
= (k + 1)^2(k + 3) + (k + 1)(k + 2)^2/2
\]

\[
\dim W_3 = \dim \mathcal{P}_{k|k} = (k + 1)^2(k + 2)/2.
\]

Now, we are left to prove the identity (6.1b). Since \( \nabla H_3 \subset \ker \nabla \times E_3 \), we just need to show that \( \ker \nabla \times E_3 \subset \nabla H_3 \). To this end, let \( p \) be a function in \( \ker \nabla \times E_3 \). Since \( E_3 \subset \mathcal{P}_{2k+2} \), we have \( p = \nabla q \) for a scalar polynomial function \( q \in \mathcal{P}_{2k+3} \). We have

\[
\begin{pmatrix}
\partial_x p \\
\partial_y p
\end{pmatrix} \in \mathcal{N}_k(x, y) \otimes \mathcal{P}_{k+1}(z), \quad \text{and} \quad \partial_z p \in \mathcal{P}_{k+1}(x, y) \otimes \mathcal{P}_k(z)
\]

Now, we write \( q \) in terms of a polynomial in \( z \) with its coefficients being polynomials in \( x \) and \( y \):

\[
q = \sum_{\alpha=0}^{2k+3} f_\alpha(x, y) z^\alpha.
\]

We have

\[
\partial_z q = \sum_{\alpha=1}^{2k+3} \alpha f_\alpha(x, y) z^{\alpha-1} \in \mathcal{P}_{k+1}(x, y) \otimes \mathcal{P}_k(z).
\]

This implies that \( f_\alpha(x, y) \in \mathcal{P}_{k+1}(x, y) \) for \( 1 \leq \alpha \leq k + 1 \), and \( f_\alpha(x, y) = 0 \) for \( \alpha \geq k + 2 \). Hence,

\[
q = f_0(x, y) + \sum_{\alpha=1}^{k+1} f_\alpha(x, y) z^\alpha.
\]

Since \( f_\alpha(x, y) \in \mathcal{P}_{k+1}(x, y) \) for \( 1 \leq \alpha \leq k + 1 \) and \( H_3 := \mathcal{P}_{k+1}(x, y) \otimes \mathcal{P}_{k+1}(z) \), we have

\[
\sum_{\alpha=1}^{k+1} f_\alpha(x, y) z^\alpha \in H_3.
\]

Then, we have

\[
\begin{pmatrix}
\partial_x f_0(x, y) \\
\partial_y f_0(x, y)
\end{pmatrix} \in \mathcal{N}_k(x, y),
\]

which implies \( f_0(x, y) \in \mathcal{P}_{k+1}(x, y) \), hence \( q \in H_3 \). This completes the proof of the equality (6.1d). Hence the sequence \( S_{6,k}^d \) is exact.
Now, we prove equalities (6.1) for the sequence $S_{5,k}^{2d}$. The first equality (6.1a) is trivially satisfied. The third equality is due to the fact that
\[
\nabla \cdot \begin{pmatrix}
P_k(x, y) \oplus P_k(z) \\
0 \\
P_k(x, y) \oplus P_{k+1}(z)
\end{pmatrix} = P_k(x, y) \oplus P_k(z).
\]
The dimension equality (6.1d) is also easy to verify by the fact that
\[
\begin{align*}
\dim H_3 &= k + 3 + \dim P_{k|k} \\
\dim E_3 &= 2k + 3 + \dim P_{k|k} \\
\dim V_3 &= k + 1 + \dim P_{k|k} \\
\dim W_3 &= \dim P_{k|k}.
\end{align*}
\]
Now, we are left to prove the identity (6.1b). Again, we prove that $\ker \nabla \times E_3 \subset \nabla H_3$.
Since the spaces in $S_{5,k}^{2d}$ is included in the related spaces in $S_{6,k}^{2d}$, which is an exact sequence, we have any function $p \in \ker \nabla \times E_3$ is a gradient of a function $q \in P_{k+1|k+1}$.

Now, we show that the function $q$ is actually a function in the space $H_3 = P_{k|k} \oplus P_{k+1|k+1}(x, y) \oplus \{z^{k+1}\}$.

Again, we express $q$ as a polynomial of the variable $z$ with coefficients polynomials of $x$ and $y$:
\[
q = \sum_{\alpha=0}^{k+1} f_{\alpha}(x, y) z^\alpha,
\]
where $f_{\alpha}(x, y) \in P_{k+1}(x, y)$. Using the fact that $\partial_z q \in P_{k|k} \oplus P_{k+1|k+1} z^k$, we immediately get $f_{\alpha}(x, y) \in P_k(x, y)$ for $1 \leq \alpha \leq k$. Moreover, since $\partial_q q \in P_{k|k} \oplus y P_k(x, y)$, we have $\partial_x f_{k+1}(x, y) = 0$. Similarly, $\partial_y f_{k+1}(x, y) = 0$. This implies that $f_{k+1}(x, y)$ is a constant. Hence, $q \in H_3$ as desired. This completes the proof that $S_{5,k}^{2d}$ is an exact sequence and completes the proof of Theorem 5.2.

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