DEFORMING A CONVEX HYPERSURFACE BY ANISOTROPIC CURVATURE FLOWS

HONGJIE JU, BOYA LI, AND YANNAN LIU

Abstract. In this paper, we consider a fully nonlinear curvature flow of a convex hypersurface in the Euclidean \( n \)-space. This flow involves \( k \)-th elementary symmetric function for principal curvature radii and a function of support function. Under some appropriate assumptions, we prove the long-time existence and convergence of this flow. As an application, we give the existence of smooth solutions to the Orlicz-Christoffel-Minkowski problem.

1. Introduction

Let \( M_0 \) be a smooth, closed, strictly convex hypersurface in the Euclidean space \( \mathbb{R}^n \), which encloses the origin and is given by a smooth embedding \( X_0 : S^{n-1} \to \mathbb{R}^n \). Consider a family of closed hypersurfaces \( \{M_t\} \) with \( M_t = X(S^{n-1}, t) \), where \( X : S^{n-1} \times [0, T) \to \mathbb{R}^n \) is a smooth map satisfying the following initial value problem:

\[
\frac{\partial X}{\partial t}(x, t) = \frac{1}{f(\nu)} \sigma_k(x, t) \varphi(\langle X, \nu \rangle \langle X, \nu \rangle) \eta(t) \nu - X,
\]

\[
X(x, 0) = X_0(x).
\]

(1)

Here \( f \) is a given positive and smooth function on the unit sphere \( S^{n-1} \), \( \nu \) is the unit outer normal vector of \( M_t \) at the point \( X(x, t) \). \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^n \), \( \varphi \) is a positive smooth function defined in \( (0, +\infty) \), \( \eta \) is a scalar function to be specified later, and \( T \) is the maximal time for which the solution exists. We use \( \{e_{ij}\}, 1 \leq i, j \leq n - 1 \) and \( \nabla \) for the standard metric and Levi-Civita connection of \( S^{n-1} \) respectively. Principal radii of curvature are the eigenvalues of the matrix

\[
b_{ij} := \nabla_i \nabla_j h + e_{ij} h
\]

with respect to \( \{e_{ij}\} \). \( \sigma_k(x, t) \) is the \( k \)-th elementary symmetric function for principal curvature radii of \( M_t \) at \( X(x, t) \) and \( k \) is an integer with \( 1 \leq k \leq n - 1 \). In this paper, \( \sigma_k \) is normalized so that \( \sigma_k(1, \ldots, 1) = 1 \).

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Geometric flows with speed of symmetric polynomial of the principal curvature radii of the hypersurface have been extensively studied, see e.g. [35, 8, 11, 37].

On the other hand, anisotropic curvature flows provide alternative methods to prove the existences of elliptic PDEs arising from convex geometry, see e.g. [3, 4, 5, 6, 21, 26, 27, 28, 33].

A positive homothetic self-similar solution of (1), if exists, is a solution to the following fully nonlinear equation

\[ c \varphi(h) \sigma_k(x) = f(x) \text{ on } S^{n-1} \]

for some positive constant \( c \). Here \( h \) is the support function defined on \( S^{n-1} \). We are concerned the existence of smooth solutions for equation (2).

When \( k = n-1 \), equation (2) is just the smooth case of Orlicz-Minkowski problem. The Orlicz-Minkowski problem is a basic problem in the Orlicz-Brunn-Minkowski theory in convex geometry. It is a generalization of the classical Minkowski problem which asks what are the necessary and sufficient conditions for a Borel measure on the unit sphere \( S^{n-1} \) to be a multiple of the Orlicz surface area measure of a convex body in \( \mathbb{R}^n \). In [15], Haberl, Lutwak, Yang & Zhang studied the even case of the Orlicz-Minkowski problem. After that, the Orlicz-Minkowski problem attracted great attention from many scholars, see for example [9, 10, 19, 22, 34, 36, 39].

When \( \varphi(s) = s^{1-p}, k = n-1 \), Eq. (2) reduces to the \( L_p \)-Minkowski problem, which has been extensively studied, see e.g. [2, 7, 16, 18, 20, 23, 24, 29, 30, 31, 38].

When \( 1 \leq k < n-1 \), Eq. (2) is so-called the Orlicz-Christoffel-Minkowski problem. For \( \varphi(s) = s^{1-p}, 1 \leq k < n-1 \), Eq. (2) is known as the \( L_p \)-Christoffel-Minkowski problem and is the classical Christoffel-Minkowski problem for \( p = 1 \). Under a sufficient condition on the prescribed function, existence of solution for the classical Christoffel-Minkowski problem was given in [12].

The \( L_p \)-Christoffel-Minkowski problem is related to the problem of prescribing \( k \)-th \( p \)-area measures. Hu, Ma & Shen in [17] proved the existence of convex solutions to the \( L_p \)-Christoffel-Minkowski problem for \( p \geq k + 1 \) under appropriate conditions. Using the methods of geometric flows, Ivaki in [21] and then Sheng & Yi in [33] also gave the existence of smooth convex solutions to the \( L_p \)-Christoffel-Minkowski problem for \( p \geq k + 1 \). In case \( 1 < p < k + 1 \), Guan & Xia in [13] established the existence of convex body with prescribed \( k \)-th even \( p \)-area measures.

In this paper, we study the long-time existence and convergence of flow (1) for strictly convex hypersurfaces and the existence of smooth solutions to the Orlicz-Christoffel-Minkowski problem (2).

The scalar function \( \eta(t) \) in (1) is usually used to keep \( M_t \) normalized in a certain sense, see for examples [4, 21, 33]. In this paper, \( \eta \) is given by

\[ \eta(t) = \frac{\int_{S^{n-1}} hf(x)/\varphi(h) \, dx}{\int_{S^{n-1}} h \sigma_k \, dx}, \]

where \( h(\cdot, t) \) is the support function of the convex hypersurface \( M_t \). It will be proved in section 2 that \( \int_{S^{n-1}} h \sigma_k \, dx \) is non-decreasing along the flow under this choice of \( \eta \).
To obtain the long-time existence of flow (1), we need some constraints on $\varphi$.

**A**: $\varphi(s)$ is a positive and continuous function defined in $(0, +\infty)$ such that $\varphi > \alpha s^{-k-\varepsilon}$ for some positive constants $\varepsilon$ and $\alpha$ for $s$ near $0$ and $\phi(s) = \int_0^s \frac{1}{\varphi(\tau)} d\tau$ is unbounded as $s \to +\infty$. Here $k$ is the order of $\sigma_k$.

The main results of this paper are stated as follows.

**Theorem 1.** Assume $M_0$ is a smooth, closed and strictly convex hypersurface in $\mathbb{R}^n$. Suppose $k$ is an integer with $1 \leq k < n - 1$ and $\varphi \in C^\infty(0, +\infty)$ satisfying **A**. Moreover, for any $s > 0$,

$$\frac{\partial}{\partial s} \left( s \frac{\partial}{\partial s} (\log \varphi(s)) \right) \geq 0 \quad \text{and} \quad -a \leq s \frac{\partial}{\partial s} (\log \varphi(s)) \leq -1,$$

where $a$ is a positive constant. Suppose $f$ is a smooth function on $S^{n-1}$ such that $(k + 1)f^{\frac{1}{k+a}}e_{ij} + (k + a)\nabla_i \nabla_j (f^{\frac{1}{k+a}})$ is positive definite. Then flow (1) has a unique smooth solution $M_t$ for all time $t > 0$. Moreover, when $t \to \infty$, a subsequence of $M_t$ converges in $C^\infty$ to a smooth, closed, strictly convex hypersurface, whose support function is a smooth solution to equation (2) for some positive constant $c$.

When $f \equiv 1$, we have the following result.

**Theorem 2.** Assume $M_0$ is a smooth, closed and strictly convex hypersurface in $\mathbb{R}^n$. If $f \equiv 1$, $\varphi \in C^\infty(0, +\infty)$ satisfying **A**, and $k$ is an integer with $1 \leq k < n - 1$. Moreover, for any $s > 0$,

$$\frac{\partial}{\partial s} \left( s \frac{\partial}{\partial s} (\log \varphi(s)) \right) \geq 0 \quad \text{and} \quad s \frac{\partial}{\partial s} (\log \varphi(s)) \leq -1.$$

Then flow (1) has a unique smooth solution $M_t$ for all time $t > 0$. Moreover, when $t \to \infty$, a subsequence of $M_t$ converges in $C^\infty$ to a smooth, closed, strictly convex hypersurface, whose support function is a smooth solution to equation (2) for some positive constant $c$.

As an application, we have

**Corollary 1.** Under the assumptions of Theorem 1 or Theorem 2, there exists a smooth solution to equation (2) for some positive constant $c$.

From the proof of Lemma 7 in section 3, we will see if $\frac{\varphi'(s)s}{\varphi(s)} = a_0$ for some negative constant $a_0$, then convexity condition on $f$ reduces to $f^{\frac{1}{k+a}}e_{ij} + \nabla_j \nabla_j (f^{\frac{1}{k+a}})$ being positive definite. Hence when $\varphi(s) = s^{1-p}$ for $p \geq k + 1$ with above condition on $f$, our conclusion recovers the existence results to the $L_p$-Christoffel-Minkowski problem which have been obtained in [17], [21] and [33].

This paper is organized as follows. In section 2, we give some basic knowledge about the flow (1) and evolution equations of some geometric quantities. In section
3, the long-time existence of flow (1) will be obtained. First, under assumption (A), uniform positive upper and lower bounds for support functions of \( \{ M_t \} \) are derived. Based on the bounds of support functions, we obtain the uniform bounds of principal curvatures by constructing proper auxiliary functions. The long-time existence of flow (1) then follows by standard arguments. In section 4, by considering a related geometric functional, we prove that a subsequence of \( \{ M_t \} \) converges to a smooth solution to equation (2), completing the proofs of Theorem 1 and Theorem 2.

2. Preliminaries

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space, and \( \mathbb{S}^{n-1} \) be the unit sphere in \( \mathbb{R}^n \). Assume \( M \) is a smooth closed strictly convex hypersurface in \( \mathbb{R}^n \). Without loss of generality, we may assume that \( M \) encloses the origin. The support function \( h \) of \( M \) is defined as

\[
h(x) := \max_{y \in M} \langle y, x \rangle, \quad \forall x \in \mathbb{S}^{n-1},
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^n \).

Denote the Gauss map of \( M \) by \( \nu_M \). Then \( M \) can be parametrized by the inverse Gauss map

\[
X(x) = \nu_M^{-1}(x).
\]

The support function \( h \) of \( M \) can be computed by

\[
h(x) = \langle x, X(x) \rangle, \quad x \in \mathbb{S}^{n-1}.
\]

Note that \( x \) is just the unit outer normal of \( M \) at \( X(x) \). Differentiating (4), we have

\[
\nabla_i h = \langle \nabla_i x, X(x) \rangle + \langle x, \nabla_i X(x) \rangle.
\]

Since \( \nabla_i X(x) \) is tangent to \( M \) at \( X(x) \), we have

\[
\nabla_i h = \langle \nabla_i x, X(x) \rangle.
\]

It follows that

\[
X(x) = \nabla h + hx.
\]

By differentiating (4) twice, the second fundamental form \( A_{ij} \) of \( M \) can be computed in terms of the support function, see for example [35],

\[
A_{ij} = \nabla_{ij} h + h e_{ij},
\]

where \( \nabla_{ij} = \nabla_i \nabla_j \) denotes the second order covariant derivative with respect to \( e_{ij} \).

The induced metric matrix \( g_{ij} \) of \( M \) can be derived by Weingarten’s formula,

\[
e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = A_{im} A_{lj} g^{ml}.
\]

The principal radii of curvature are the eigenvalues of the matrix \( b_{ij} = A^{ik} g_{jk} \). When considering a smooth local orthonormal frame on \( \mathbb{S}^{n-1} \), by virtue of (6) and (7), we have

\[
b_{ij} = A_{ij} = \nabla_{ij} h + h \delta_{ij},
\]

We will use \( b^{ij} \) to denote the inverse matrix of \( b_{ij} \).
From the evolution equation of $X(x, t)$ in flow (1), we derive the evolution equation of the corresponding support function $h(x, t)$:

\begin{equation}
\frac{\partial h(x, t)}{\partial t} = \frac{1}{f(x)} \sigma_k(x, t) \varphi(h) h(x, t) \eta(t) - h(x, t).
\end{equation}

The radial function $\rho$ of $M$ is given by

$$
\rho(u) := \max \{ \lambda > 0 : \lambda u \in M \}, \quad \forall \ u \in S^{n-1}.
$$

Note that $\rho(u)u \in M$.

From (5), $u$ and $x$ are related by

\begin{equation}
\rho(u)u = \nabla h(x) + h(x)x
\end{equation}

and

$$
\rho^2 = |\nabla h|^2 + h^2.
$$

In the rest of the paper, we take a local orthonormal frame $\{e_1, \cdots, e_{n-1}\}$ on $S^{n-1}$ such that the standard metric on $S^{n-1}$ is $\{\delta_{ij}\}$. Double indices always mean to sum from 1 to $n - 1$. We denote partial derivatives $\frac{\partial \sigma_k}{\partial b_{ij}}$ and $\frac{\partial^2 \sigma_k}{\partial b_{pq} \partial b_{mn}}$ by $\sigma^{ij}_k$ and $\sigma^{pq,mn}_k$ respectively. For convenience, we also write

$$
N = \frac{1}{f(x)} \varphi(h) h,
$$

$$
F = N \sigma_k \eta(t).
$$

Now, we can prove that the mixed volume $\int_{S^{n-1}} h(x, t) \sigma_k(x, t) \, dx$ is non-decreasing along the flow (1).

**Lemma 1.** $\int_{S^{n-1}} h(x, t) \sigma_k(x, t) \, dx$ is non-decreasing along the flow (1).

**Proof.** According to the evolution equation of $h$ in (9), we get

$$
\partial_t \sigma_k = \sigma^{ij}_k \partial_t (\nabla_{ij} h + \delta_{ij} h)
$$

$$
= \sigma^{ij}_k \nabla_{ij} (\partial_t h) + \sigma^{ij}_k \delta_{ij} \partial_t h
$$

$$
= \sigma^{ij}_k \nabla_{ij} F - \sigma^{ij}_k \nabla_{ij} h + \sigma^{ij}_k \delta_{ij} F - \sigma^{ij}_k \delta_{ij} h
$$

$$
= \sigma^{ij}_k \nabla_{ij} F + \sigma^{ij}_k \delta_{ij} F - k \sigma_k.
$$
the last equality holds because $\sigma_k$ is homogeneous of degree $k$ and $\sigma_k^{ij}b_{ij} = k\sigma_k$. Hence,

\[
\partial_t \int_{S^{n-1}} h\sigma_k \, dx \\
= \int_{S^{n-1}} (\partial_t\sigma_k)h \, dx + \int_{S^{n-1}} \sigma_k \partial_t h \, dx \\
= \int_{S^{n-1}} (h\sigma_k^{ij}\nabla_ij F + h\sigma_k^{ij} \delta_{ij} F - kh\sigma_k) \, dx + \int_{S^{n-1}} F\sigma_k \, dx - \int_{S^{n-1}} h\sigma_k \, dx \\
= (k + 1) \int_{S^{n-1}} F\sigma_k \, dx - (k + 1) \int_{S^{n-1}} h\sigma_k \, dx + \int_{S^{n-1}} (h\sigma_k^{ij}\nabla_ij F - F\sigma_k^{ij} \nabla_ij h) \, dx \\
= (k + 1) \int_{S^{n-1}} F\sigma_k \, dx - (k + 1) \int_{S^{n-1}} h\sigma_k \, dx,
\]

where in the last equality we use the fact $\sum_i \nabla_i (\sigma_k^{ij}) = 0$.

By Hölder’s Inequality, we have

\[
\frac{1}{k+1} \partial_t \int_{S^{n-1}} h\sigma_k \, dx \\
= \int_{S^{n-1}} \frac{1}{f(x)}\sigma_k^2 \varphi(h)h \eta \, dx - \int_{S^{n-1}} h\sigma_k \, dx \\
= \frac{1}{\int_{S^{n-1}} h\sigma_k \, dx} \left[ \int_{S^{n-1}} \frac{1}{f(x)}\sigma_k^2 \varphi(h)h \, dx \int_{S^{n-1}} \frac{h}{\varphi(h)}f(x) \, dx - \left( \int_{S^{n-1}} h\sigma_k \, dx \right)^2 \right] \\
\geq 0,
\]

and the equality holds if and only if

\[
c\varphi(h)\sigma_k(x) = f(x)
\]

for some positive constant $c$. \hfill \Box

By the flow equation \(\Box\), we can derive evolution equations of some geometric quantities.
Lemma 2. The following evolution equations hold along the flow (11).

\[ \partial_t b_{ij} - N\eta(t)\sigma^p_k\nabla_{pq}b_{ij} \]
\[ = (k + 1)N\eta(t)\sigma_k\delta_{ij} - N\eta(t)\sigma^p_k\delta_{pq}b_{ij} + N\eta(t)(\sigma^p_k b_{jp} - \sigma^j_k b_{ip}) \]
\[ + N\eta(t)\sigma^p,q_{k,mn}\nabla_j b_{pq} \nabla_i b_{mn} + \eta(t)(\sigma_k \nabla_{ij} N + \nabla_j \sigma_k \nabla_i N + \nabla_i \sigma_k \nabla_j N) - b_{ij} \]
\[ \partial_t b^{ij} - N\eta(t)\sigma^p_k \nabla_{pq} b^{ij} \]
\[ = -(k + 1)N\eta(t)\sigma_k b^{ip}b^{jp} + N\eta(t)\sigma^p_k \delta_{pq} b^{ij} - N\eta(t)b^{ip}b^{jq}(\sigma^r_k b_{rq} - \sigma^r_k b_{rp}) \]
\[ - N\eta(t)b^{ij}\delta^{js}(\sigma^p_k \sigma^m_{pq} + 2\sigma^m_p b^{pq})\nabla_i b_{pq} \nabla_s b_{mn} \]
\[ - \eta(t)b^{ip}b^{jq}(\sigma_k \nabla_{ij} N + \nabla_j \sigma_k \nabla_i N + \nabla_i \sigma_k \nabla_j N) + b^{ij} \]
\[ \partial_t \left( \frac{\rho^2}{2} \right) - N\eta(t)\sigma_k^{ij} \nabla_{ij} \left( \frac{\rho^2}{2} \right) \]
\[ = (k + 1)h\eta(t)\sigma_k - \rho^2 + \eta(t)\sigma_k \nabla_i h \nabla_i N - N\eta(t)\sigma_k^{ij}b_{mi}b_{mj}. \]

Proof. From (11),

\[ \partial_t \nabla_{ij} h = \nabla_{ij}(\partial_t h) = \eta(t)(\sigma_k \nabla_{ij} N + \nabla_j \sigma_k \nabla_i N + \nabla_i \sigma_k \nabla_j N) + N\eta(t)\nabla_{ij} \sigma_k - h_{ij}, \]

where

\[ \nabla_{ij} \sigma_k = \sigma^p_{k,mn} \nabla_j b_{pq} \nabla_i b_{mn} + \sigma^p_{k} \nabla_{ij} b_{pq}. \]

By Gauss equation,

\[ \nabla_{ij} b_{pq} = \nabla_{pq} b_{ij} + \delta_{ij} \nabla_{pq} h - \delta_{pq} \nabla_{ij} h + \delta_{iq} \nabla_{pj} h - \delta_{pj} \nabla_{iq} h. \]

Hence

\[ \partial_t h_{ij} = N\eta(t)\sigma^p_k \nabla_{pq} b_{ij} + kN\eta(t)\sigma_k \delta_{ij} - N\eta(t)\sigma^p_k \delta_{pq} b_{ij} + N\eta(t)(\sigma^p_k b_{jp} - \sigma^j_k b_{ip}) \]
\[ + N\eta(t)\sigma^p,q_{k,mn} \nabla_j b_{pq} \nabla_i b_{mn} + \eta(t)(\sigma_k \nabla_{ij} N + \nabla_j \sigma_k \nabla_i N + \nabla_i \sigma_k \nabla_j N) - h_{ij}. \]

This together with (8) gives the evolution equation of \( b^{ij} \). The evolution equation of \( b^{ij} \) then follows from

\[ \partial_t b^{ij} = -b^{ij} \delta^{kl} \partial_t b_{kl}. \]

For more details of computations about the evolution equations of \( b_{ij} \) and \( b^{ij} \), one can refer to [8, 35].

Recalling that \( \rho^2 = h^2 + |\nabla h|^2 \), we have

\[ \partial_t \left( \frac{\rho^2}{2} \right) - N\eta(t)\sigma_k^{ij} \nabla_{ij} \left( \frac{\rho^2}{2} \right) \]
\[ = h\partial_t h + \nabla_i h \nabla_j \partial_t h - N\eta(t)\sigma^i_j (h \nabla_{ij} h + \nabla_i h \nabla_j h + \nabla_m h \nabla_j \nabla_m h + \nabla_m h \nabla_{mj} h) \]
\[ = h\partial_t h + \nabla_i h \nabla_j (N\eta(t)\sigma_k - h) \]
\[ - N\eta(t)\sigma_k^{ij} \left[ \nabla_i h \nabla_j h + \nabla_m h \nabla_j (b_{mi} - h\delta_{mi}) \right] - N\eta(t)\sigma_k^{ij} h(b_{ij} - h\delta_{ij}) \]
\[ - N\eta(t)\sigma_k^{ij} (b_{mi} - h\delta_{mi})(b_{mj} - h\delta_{mj}) \]
\[ = (k + 1)hN\eta(t)\sigma_k - \rho^2 + \eta(t)\sigma_k \nabla_i h \nabla_i N - N\eta(t)\sigma_k^{ij}b_{mi}b_{mj}. \]
3. The long-time existence of the flow

In this section, we will give a priori estimates about support functions and curvatures to obtain the long-time existence of flow \((1)\) under assumptions of Theorem 1 and Theorem 2.

In the rest of this paper, we assume that \(M_0\) is a smooth, closed, strictly convex hypersurface in \(\mathbb{R}^n\) and \(h : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}\) is a smooth solution to the evolution equation \((9)\) with the initial \(h(\cdot, 0)\) the support function of \(M_0\). Here \(T\) is the maximal time for which the solution exists. Let \(M_t\) be the convex hypersurface determined by \(h(\cdot, t)\), and \(\rho(\cdot, t)\) be the corresponding radial function.

We first give the uniform positive upper and lower bounds of \(h(\cdot, t)\) and \(\rho(\cdot, t)\) for \(t \in [0, T)\).

**Lemma 3.** Let \(h\) be a smooth solution of \((9)\) on \(\mathbb{S}^{n-1} \times [0, T)\), \(f\) be a positive, smooth function on \(\mathbb{S}^{n-1}\) and \(\varphi \in C^\infty(0, +\infty)\) be a decreasing function satisfying \((A)\). Then

\[
\begin{align*}
\frac{1}{C} & \leq h(x, t) \leq C, \\
\frac{1}{C} & \leq \rho(u, t) \leq C,
\end{align*}
\]

where \(C\) is a positive constant independent of \(t\).

**Proof.** Let \(J(t) = \int_{\mathbb{S}^{n-1}} \phi(h(x, t)) f(x) \, dx\). We claim that \(J(t)\) is unchanged along the flow \((1)\). It is because

\[
J'(t) = \int_{\mathbb{S}^{n-1}} \phi'(h) \partial_t h f(x) \, dx
\]

\[
= \int_{\mathbb{S}^{n-1}} \frac{f(x)}{\varphi(h)} \partial_t h \, dx
\]

\[
= \int_{\mathbb{S}^{n-1}} \frac{f(x)}{\varphi(h)} \left( \frac{1}{f(x)} \sigma_k(x) \varphi(h) h \eta(t) - h \right) \, dx
\]

\[
= \int_{\mathbb{S}^{n-1}} \sigma_k(x) h \eta(t) \, dx - \int_{\mathbb{S}^{n-1}} \frac{h}{\varphi(h)} f(x) \, dx
\]

\[
= 0.
\]

For each \(t \in [0, T)\), suppose that the maximum of radial function \(\rho(\cdot, t)\) is attained at some \(u_t \in \mathbb{S}^{n-1}\). Let

\[
R_t = \max_{u \in \mathbb{S}^{n-1}} \rho(u, t) = \rho(u_t, t)
\]

for some \(u_t \in \mathbb{S}^{n-1}\). By the definition of support function, we have

\[
h(x, t) \geq R_t(u_t) , \quad \forall x \in \mathbb{S}^{n-1}.
\]
Denote the hemisphere containing $u_t$ by $S^+_u = \{ x \in \mathbb{S}^{n-1} : \langle x, u_t \rangle > 0 \}$. Since $\phi'(h) = \frac{1}{\varphi(h)} > 0$ implies that $\phi(h)$ is strictly increasing about $h$, we have

$$J(0) = J(t) \geq \int_{S^+_u} \phi(h(x,t)) f(x) \, dx \geq \int_{S^+_u} \phi(R_t \langle x, u_t \rangle) f(x) \, dx \geq f_{\min} \int_{S^+_u} \phi(R_t \langle x, u_t \rangle) \, dx = f_{\min} \int_{S^+_u} \phi(R_t x_1) \, dx,$$

where $S^+ = \{ x \in \mathbb{S}^{n-1} : x_1 > 0 \}$.

Denote $S_1 = \{ x \in \mathbb{S}^{n-1} : x_1 \geq 1/2 \}$, then

$$J(0) \geq f_{\min} \int_{S_1} \phi(R_t/2) \, dx = f_{\min} \phi(R_t/2)|S_1|,$$

which implies that $\phi(R_t/2)$ is uniformly bounded from above. By assumption (A), $\phi(s)$ is strictly increasing and tends to $+\infty$ as $s \to +\infty$. Thus $R_t$ is uniformly bounded from above.

Now we can prove that $\eta(t)$ has positive lower bound. Since mixed volumes are monotonic increasing, see [32, page 282], we have for each $t \in [0, T)$

$$h_{\min}^{k+1}(t) \leq \int_{\mathbb{S}^{n-1}} \frac{h \sigma_k}{\omega_{n-1}} \, dx \leq h_{\max}^{k+1}(t),$$

here $h_{\min}(t) = \min_{x \in \mathbb{S}^{n-1}} h(x, t)$ and $h_{\max}(t) = \max_{x \in \mathbb{S}^{n-1}} h(x, t)$.

This together with Lemma 1 and the upper bound of $h$ implies that there exist positive constants $c_1$ and $c_2$ such that

$$\int_{\mathbb{S}^{n-1}} h \sigma_k \, dx \leq c_1$$

and

$$h_{\max}(t) \geq c_2.$$
Recalling the definition of $\eta(t)$ and noticing that $\frac{1}{\varphi(s)}$ is an increasing function, we have

$$\eta(t) = \frac{\int_{S^{n-1}} hf(x)/\varphi(h) \, dx}{\int_{S^{n-1}} h\sigma_k \, dx}$$

$$\geq \frac{1}{c_1} \int_{S^{n-1}_+} hf(x)/\varphi(h) \, dx$$

$$\geq \frac{1}{c_1} \int_{S^{n-1}_+} R_t(x, u_t) f_{\min} \frac{1}{\varphi(R_t(x, u_t))} \, dx$$

$$= \frac{1}{c_1} f_{\min} \int_{S^+} R_t x \frac{1}{\varphi(R_t x)} \, dx$$

$$\geq \frac{1}{c_1} f_{\min} |S_1| \frac{1}{2} R_t \frac{1}{\varphi(\frac{1}{2} R_t)}$$

$$\geq c_3,$$

where $c_3$ is a positive constant independent of $t$.

Suppose the minimum of $h(x, t)$ is attained at a point $(x_t, t)$. At $(x_t, t)$, $\nabla_i h$ is non-negative. It follows that

$$\sigma_k(x_t, t) \geq h_{\min}^k(t).$$

Then in the sense of the lim inf of difference quotient, see [14], we have

$$\frac{\partial h_{\min}(t)}{\partial t} \geq \frac{1}{f_{\max}} h_{\min}(t)[\eta(t)h_{\min}^k(t)\varphi(h_{\min}(t)) - f(x)]$$

$$\geq \frac{1}{f_{\max}} h_{\min}(t)[c_3 h_{\min}^k(t)\varphi(h_{\min}(t)) - f_{\max}].$$

If $\varphi(s) > \alpha s^{-k-\varepsilon}$ for some positive constant $\varepsilon$ for $s$ near 0, then

$$\frac{\partial h_{\min}(t)}{\partial t} \geq \frac{1}{f_{\max}} h_{\min}(t)(h_{\min}^{-\varepsilon}(t)\alpha c_3 - f_{\max}).$$

The right hand of the above inequality is positive for $h_{\min}(t)$ small enough and the lower bound of $h_{\min}(t)$ follows from the maximum principle in [14].

□

By the equality $\rho^2 = h^2 + |\nabla h|^2$, we can obtain the gradient estimate of support function from Lemma 3.

**Corollary 2.** Under the assumptions of Lemma 3, we have

$$|\nabla h(x, t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),$$

where $C$ is a positive constant depending only on constants in Lemma 3.

The uniform bounds of $\eta(t)$ can be derived from Lemmas 1 and 3.
Lemma 4. Under the assumptions of Lemma 3, \( \eta(t) \) is uniformly bounded above and below from zero.

Proof. In term of the proof of Lemma 3, \( \eta(t) \) has uniform positive lower bound. From Lemma 1, we know that \( \int_{S^{n-1}} h \sigma_k \, dx \) is monotonic decreasing about \( t \), which give a positive lower bound on \( \int_{S^{n-1}} h \sigma_k \, dx \). This together with the uniform bounds of \( h(x, t) \) implies that \( \eta(t) \) is bounded from above.

\[ \square \]

To obtain the long-time existence of the flow \( (1) \), we need to establish the uniform bounds on principal curvatures. By Lemma 3, for any \( t \in [0, T) \), \( h(\cdot, t) \) always ranges within a bounded interval \( I' = [1/C, C] \), where \( C \) is the constant in Lemma 3. First, we give the estimates of \( \sigma_k \).

Lemma 5. Under the assumptions of Lemma 3
\[ \sigma_k(x, t) \geq C, \quad \forall (x, t) \in S^{n-1} \times [0, T), \]
where \( C \) is a positive constant independent of \( t \).

Proof. Consider the auxiliary function \( Q = \log M - A \frac{\rho^2}{2} \), where \( M = N\sigma_k = \frac{1}{f(x)} \varphi(h) h \sigma_k \) and \( A \) is a positive constant to be determined later. The evolution equation of \( M \) is given by
\[ \partial_t M = N \partial_t \sigma_k + \sigma_k \partial_t N \]
\[ = N(\sigma_k^{ij} \nabla_{ij} F + \sigma_k^{ij} \delta_{ij} F - k \sigma_k) + \frac{M}{h} \left( 1 + \frac{\varphi'h}{\varphi} \right) (F - h) \]
\[ = N \sigma_k^{ij} \nabla_{ij} F + N \sigma_k^{ij} \delta_{ij} F - k M + \frac{M^2}{h} \eta(t) \left( 1 + \frac{\varphi'h}{\varphi} \right) - M \left( 1 + \frac{\varphi'h}{\varphi} \right) \]
\[ = N \eta(t) \sigma_k^{ij} \nabla_{ij} M + MN \eta(t) \sigma_k^{ij} \delta_{ij} - M \left( k + 1 + \frac{\varphi'h}{\varphi} \right) + \frac{M^2}{h} \eta(t) \left( 1 + \frac{\varphi'h}{\varphi} \right). \]

It is easy to compute
\[ \nabla_i Q = \frac{\nabla_i M}{M} - A \nabla_i \left( \frac{\rho^2}{2} \right), \]
\[ \nabla_{ij} Q = \frac{\nabla_{ij} M}{M} - \frac{1}{M^2} \nabla_i M \nabla_j M - A \nabla_{ij} \left( \frac{\rho^2}{2} \right). \]

Due to the evolution equation of \( \frac{\rho^2}{2} \) in Lemma 3, the evolution equation of \( Q \) is
\[ \partial_t Q - N \eta(t) \sigma_k^{ij} \nabla_{ij} Q \]
\[ = \frac{1}{M^2} N \eta(t) \sigma_k^{ij} \nabla_i M \nabla_j M + N \eta(t) \sigma_k^{ij} \delta_{ij} - \left( k + 1 + \frac{\varphi'h}{\varphi} \right) + \frac{M}{h} \eta(t) \left( 1 + \frac{\varphi'h}{\varphi} \right) \]
\[ - (k + 1) Ah N \eta(t) \sigma_k + Ap^2 - A \eta(t) \sigma_k \nabla_i h \nabla_i N + AN \eta(t) \sigma_k b_{mi} b_{mj}. \]
For fixed $t$, at a point where $Q$ attains its spatial minimum, we have
\[
\partial_t Q \geq A \rho^2 - \left( k + 1 + \frac{\varphi' h}{\varphi} \right) + \frac{M}{h} \eta(t) \left( 1 + \frac{\varphi' h}{\varphi} \right) \\
- (k + 1)AhN \eta(t) \sigma_k - A \eta(t) \sigma_k \nabla_i h \nabla_i N \\
= \frac{1}{2} A \rho^2 - \left( k + 1 + \frac{\varphi' h}{\varphi} \right) + \frac{1}{h} e^{Q + A \rho^2} \eta(t) \left( 1 + \frac{\varphi' h}{\varphi} \right) \\
+ AN \eta(t) \sigma_k \left( \frac{\rho^2}{2e^{Q + A \rho^2}} \eta(t) - h(k + 1) - \frac{1}{N} \nabla_i h \nabla_i N \right).
\]

Now we choose $A > \frac{2}{\rho^2}(k + 1)$. Notice that $\varphi$ is a monotonic decreasing, positive function and we have obtained uniform bounds of $h, \rho, |\nabla h|$ and $\eta(t)$. If $Q$ is negatively large enough, the right-hand side is positive and the lower bound of $Q$ follows. □

**Lemma 6.** Under the assumptions of Lemma 3,
\[
\sigma_k \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),
\]
where $C$ is a positive constant independent of $t$.

**Proof.** By Lemma 3, there exists a positive constant $B$ such that
\[
B < \rho^2 < \frac{1}{B}
\]
for all $t > 0$. Define
\[
P(x, t) = \frac{\varphi \sigma_k}{f(1 - B \rho^2)} = \frac{M}{h} \frac{1}{1 - B \rho^2}.
\]
By the evolution equation of $M$ in Lemma 5 we have
\[
\partial_t \frac{M}{h} - N \eta(t) \sigma_k \nabla_i \nabla_j \frac{M}{h} = -\frac{M}{h} \left( k + \frac{\varphi' h}{\varphi} \right) + \frac{M^2}{h^2} \eta(t) \left( k + \frac{\varphi' h}{\varphi} \right) + \frac{2N}{h} \eta(t) \sigma_k \nabla_i h \nabla_j \frac{M}{h}.
\]
Hence
\[
\partial_t P - N \eta(t) \sigma_k \nabla_i \nabla_j P = \frac{1}{1 - B \rho^2} \left[ -\frac{M}{h} \left( k + \frac{\varphi' h}{\varphi} \right) + \frac{M^2}{h^2} \eta(t) \left( k + \frac{\varphi' h}{\varphi} \right) + \frac{2N}{h} \eta(t) \sigma_k \nabla_i h \nabla_j \frac{M}{h} \right] \\
+ \frac{MB}{h(1 - B \rho^2)^2} [(k + 1)Nh \eta(t) \sigma_k - \rho^2 + \eta(t) \sigma_k \nabla_i h \nabla_i N - N \eta(t) \sigma_k b_{ij} b_{ij}] \\
- \frac{2B}{1 - B \rho^2} N \eta(t) \sigma_k \nabla_i \frac{\rho^2}{2} \nabla_j P.
\]
At a point where \( P(\cdot, t) \) attains its maximum, we have

\[
\nabla_j \frac{M}{h} = -\frac{M}{h} \frac{B \nabla_j \varphi^2}{1 - \frac{B \varphi^2}{2}} = -\frac{M}{h} \frac{B b_{jm} h_m}{1 - \frac{B \varphi^2}{2}}.
\]

Due to the inverse concavity of \((\sigma_k)^{\frac{1}{k}}\), we have from Lemma 5 in [1],

\[
\left( (\sigma_k)^{\frac{1}{k}} \right)^{ij} b_{im} b_{jm} \geq (\sigma_k)^{\frac{2}{k}},
\]

which means

\[
\sigma_k^{ij} b_{im} b_{jm} \geq k(\sigma_k)^{1+\frac{1}{k}}.
\]

Then at the point where \( P(\cdot, t) \) attains its maximum, we have

\[
\partial_t P \leq c_1 P + c_2 P^2 - c_3 P^{2 + \frac{1}{k}}.
\]

By maximum principle, we see that \( P(x, t) \) is uniformly bounded from above. The upper bound of \( \sigma_k \) follows from the uniform bounds on \( h \) and \( \rho \). \( \square \)

Now we can derive the upper bounds of principal curvatures \( \kappa_i(x, t) \) of \( M_t \) for \( i = 1, \cdots, n - 1 \).

**Lemma 7.** Under the assumptions of Theorem 1, we have

\[
\kappa_i \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),
\]

where \( C \) is a positive constant independent of \( t \).

**Proof.** By rotation, we assume that the maximal eigenvalue of \( b^{ij} \) at \( t \) is attained at point \( x_t \) in the direction of the unit vector \( e_1 \in T_{x_t} S^{n-1} \). We also choose orthonormal vector field such that \( b^{ij} \) is diagonal. By the evolution equation of \( b^{ij} \) in Lemma 2...
we get
\[
\partial_t \frac{b_{11}}{h} - N\eta(t)\sigma^p_k \nabla_{pq} \frac{b_{11}}{h} \\
= \frac{2}{h} N\eta(t)\sigma^p_k \nabla_p \frac{b_{11}}{h} \nabla_q h + \frac{N}{h^2} \eta(t) b_{11} \sigma^p_k \nabla_{pq} h \\
- (k + 1) \frac{N}{h} \eta(t) \sigma_k (b_{11})^2 + \frac{N}{h} \eta(t) \sigma^p_k \delta_{pq} b_{11} \\
- \frac{N}{h^2} \eta(t) (b_{11})^2 (\sigma^p_{k,mm} + 2 \sigma^p_k^m b^m_{pq}) \nabla_1 b_{pq} \nabla_1 b_{mn} \\
- \frac{\eta(t)}{h} (b_{11})^2 (\nabla_{11} N \sigma_k + 2 \nabla_1 \sigma_k \nabla_1 N) - \frac{b_{11}}{h^2} N \eta(t) \sigma_k + \frac{2 b_{11}}{h}.
\]

According to inverse concavity of \((\sigma_k)^2\), we obtain by \(35\) or \(1\)
\[(\sigma^p_k^m + 2 \sigma^p_k^m b^m_{pq}) \nabla_1 b_{pq} \nabla_1 b_{mn} \geq \frac{k + 1}{k} \frac{(\nabla_1 \sigma_k)^2}{\sigma_k}.
\]

On the other hand, by Schwartz inequality, the following inequality holds
\[2|\nabla_1 \sigma_k \nabla_1 N| \leq \frac{k + 1}{k} \frac{N (\nabla_1 \sigma_k)^2}{\sigma_k} + \frac{k}{k + 1} \frac{\sigma_k (\nabla_1 N)^2}{N}.
\]

Hence, we have at \((x_t, t)\)
\[
\partial_t \frac{b_{11}}{h} \leq - \frac{(b_{11})^2}{h} \sigma_k \eta(t) \left( \nabla_{11} N - \frac{k}{k + 1} \frac{(\nabla_1 N)^2}{N} + (k + 1) N + (1 - k) \frac{N b_{11}}{h} \right) + \frac{2 b_{11}}{h}.
\]

Let \(\tau\) be the arc-length of the great circle passing through \(x_t\) with the unit tangent vector \(e_1\). Notice that
\[
\nabla_{11} N = \frac{k}{k + 1} \frac{(\nabla_1 N)^2}{N} + (k + 1) N = (k + 1) N \frac{h}{\varphi} \left( N \frac{h}{\varphi} + (N \frac{h}{\varphi})_{\varphi} \right).
\]

Since
\[
N_\tau = (f^{-1})_\tau \varphi h + f^{-1} \varphi h_\tau \left( 1 + \frac{\varphi' h}{\varphi} \right),
\]
\[
N_{\tau \tau} = (f^{-1})_{\tau \tau} \varphi h + 2 (f^{-1})_\tau \varphi h_\tau \left( 1 + \frac{\varphi' h}{\varphi} \right) + f^{-1} \varphi^2 h_\tau^2 \left( 1 + \frac{\varphi' h}{\varphi} \right) + f^{-1} \varphi h_\tau \left( 1 + \frac{\varphi' h}{\varphi} \right),
\]

\[\text{where} \quad \frac{h'}{h} = \frac{\varphi'}{\varphi} \quad \text{and} \quad h' \varphi' = \varphi' h.
\]
here $f^{-1}$ is $\frac{1}{f}$.

We have by direct computations

\[
1 + N^{-\frac{1}{k+1}} \left( \frac{N^\frac{1}{k+1}}{k+1} \right)_{\tau \tau}
\]

\[
= 1 + \frac{1}{k+1} N^{-1} N_{\tau \tau} - \frac{k}{(k+1)^2} N^{-2} N^2_{\tau}
\]

\[
= 1 + \frac{1}{k+1} f((f^{-1})_{\tau \tau} + \frac{2f}{(k+1)h} (f^{-1})_{\tau} h_{\tau} \left( 1 + \frac{\varphi' h}{\varphi} \right)
\]

\[
+ \frac{\varphi'}{(k+1)\varphi h^2} \left( 1 + \frac{\varphi' h}{\varphi} \right) + \frac{h_{\tau \tau}}{(k+1)h} \left( 1 + \frac{\varphi' h}{\varphi} \right)
\]

\[
+ \frac{h^2_{\tau}}{(k+1)2h^2} \left( 1 + \frac{\varphi' h}{\varphi} \right)^2 - \frac{k}{(k+1)^2} f^2 (f^{-1})_{\tau}^2
\]

\[
= 1 + \frac{1}{k+1} f((f^{-1})_{\tau \tau} + \frac{2f}{(k+1)^2 h} (f^{-1})_{\tau} h_{\tau} \left( 1 + \frac{\varphi' h}{\varphi} \right)
\]

\[
+ \frac{h_{\tau \tau}}{(k+1)h} \left( 1 + \frac{\varphi' h}{\varphi} \right) + \frac{h^2_{\tau}}{(k+1)h} \left( 1 + \frac{\varphi' h}{\varphi} \right) - \frac{k}{(k+1)^2} f^2 (f^{-1})_{\tau}^2
\]

\[
+ \frac{h^2_{\tau}}{(k+1)2h^2} \left( 1 + \frac{\varphi' h}{\varphi} \right) \left( \frac{\varphi' h}{\varphi} - k \right)
\]

\[
= 1 + \frac{\varphi' h}{\varphi} h_{\tau \tau} + h - \frac{h^2_{\tau}}{(k+1)h} \left( 1 + \frac{\varphi' h}{\varphi} \right)
\]

\[
- \frac{1 + \frac{\varphi' h}{\varphi}}{h(k+1)^2} f \left[ h_{\tau} \left( \frac{k - \varphi' h}{\varphi} \right)^\frac{1}{2} - (f^{-1})_{\tau} \left( \frac{hf}{k - \varphi' h} \right)^\frac{1}{2} \right] \right]^2
\]

\[
+ \frac{1}{k+1} \left[ (k - \varphi' h) - (f^{-1})_{\tau \tau} f^2 \left( \frac{k}{k+1} + \frac{1 + \varphi' h}{k+1} \varphi h - k \right) + (f^{-1})_{\tau \tau} f \right]
\]

\[
\geq \frac{1 + \frac{\varphi' h}{\varphi}}{k+1} h_{\tau \tau} + h
\]

\[
+ \frac{1}{k+1} \left[ (k - \frac{\varphi' h}{\varphi}) - (f^{-1})_{\tau \tau} f^2 \left( \frac{k}{k+1} + \frac{1 + \varphi' h}{k+1} \varphi h - k \right) + (f^{-1})_{\tau \tau} f \right],
\]
where in the last inequality we use the conditions $\frac{\varphi'h}{\varphi} \leq -1$ and $(\frac{\varphi'h}{\varphi})' \geq 0$. Since $(k + 1)f^{-\frac{1}{k+a}}e_{ij} + (k + a)(f^{-\frac{1}{k+a}})_{ij}$ is positive definite and $-a \leq \frac{\varphi'h}{\varphi} \leq -1$, thus we can estimate

$$
\left( k - \frac{\varphi'h}{\varphi} \right) - (f^{-1})^2 f^2 \frac{k - \frac{\varphi'h}{\varphi} - 1}{k - \frac{\varphi'h}{\varphi}} + (f^{-1})_{\tau\tau} f
\geq k + 1 - (f^{-1})^2 f^2 \frac{k + a - 1}{k + a} + (f^{-1})_{\tau\tau} f
= k + 1 + (k + a) f^{\frac{1}{k+a}} (f^{-\frac{1}{k+a}})_{\tau\tau}
= f^{\frac{1}{k+a}} \left[ (k + 1)f^{-\frac{1}{k+a}} + (k + a)(f^{-\frac{1}{k+a}})_{\tau\tau} \right]
\geq c_f,
$$

where $c_f$ is a positive constant depending on $f$ and the minimal eigenvalue of $(k + 1)f^{-\frac{1}{k+a}}e_{ij} + (k + a)(f^{-\frac{1}{k+a}})_{ij}$.

Now we can derive

$$
\partial_t \frac{b^{11}}{h} \leq - \left( \frac{b^{11}}{h} \right)^2 N \sigma_k \eta(t) (c_f h + (2 - a - k)b^{11}) + \frac{2b^{11}}{h}.
$$

By the uniform bounds on $h$, $f$, $\eta$ and $\sigma_k$, we conclude

$$
\partial_t \frac{b^{11}}{h} \leq -c_1 \left( \frac{b^{11}}{h} \right)^2 + c_2 \frac{b^{11}}{h}.
$$

Here $c_1$ and $c_2$ are positive constants independent of $t$. The maximum principle then gives the upper bound of $b^{11}$ and the result follows. □

When $f \equiv 1$, it can be seen from the proof of Lemma 7 that the conditions on $f$ and the lower bound of $\varphi'h$ can be removed.

**Corollary 3.** Under the assumptions of Theorem 2, we have

$$
\kappa_i \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),
$$

where $C$ is a positive constant independent of $t$.

Combining Lemma 5, Lemma 6 and Lemma 7, we see that the principal curvatures of $M_t$ has uniform positive upper and lower bounds. This together with Lemma 3 and Corollary 2 implies that the evolution equation (9) is uniformly parabolic on any finite time interval. Thus, the result of [25] and the standard parabolic theory show that the smooth solution of (9) exists for all time. And by these estimates again, a subsequence of $M_t$ converges in $C^\infty$ to a positive, smooth, strictly convex hypersurface $M_\infty$ in $\mathbb{R}^n$. To complete the proofs of Theorem 1 and Theorem 2, it only needs to check the support function of $M_\infty$ satisfies Eq. (2).
4. CONVERGENCE OF THE FLOW

By Lemma 1, Lemma 3 and Lemma 6, the functional

\[ V(t) = \int_{\mathbb{S}^{n-1}} h(x,t)\sigma_k(x,t) \, dx \]

is non-decreasing along the flow and

\[ V(t) \leq C, \quad \forall t \geq 0. \tag{13} \]

This tells that

\[ \int_0^t V'(t) \, dt = V(t) - V(0) \leq V(t) \leq C, \]

which leads to

\[ \int_0^\infty V'(t) \, dt \leq C. \]

This implies that there exists a subsequence of times \( t_j \to \infty \) such that

\[ V'(t_j) \to 0 \text{ as } t_j \to \infty. \]

By Lemma 1, we have

\[
(k + 1)V'(t_j)V(t_j)
\]

\[
= \int_{\mathbb{S}^{n-1}} \frac{1}{f(x)}\sigma_k^2(x)h\varphi(h) \, dx \int_{\mathbb{S}^{n-1}} \frac{h}{\varphi(h)} f(x) \, dx - \left( \int_{\mathbb{S}^{n-1}} h\sigma_k \, dx \right)^2.
\]

Since \( h \) and \( \sigma_k \) have uniform positive upper and lower bounds, by passing to the limit, we obtain

\[
\int_{\mathbb{S}^{n-1}} \frac{1}{f(x)}\tilde{\sigma_k}^2(x)\varphi(\tilde{h})\tilde{h} \, dx \int_{\mathbb{S}^{n-1}} \frac{\tilde{h}}{\varphi(\tilde{h})} f(x) \, dx = \left( \int_{\mathbb{S}^{n-1}} \tilde{h}\tilde{\sigma_k} \, dx \right)^2,
\]

where \( \tilde{\sigma_k} \) and \( \tilde{h} \) are the \( k \)-th elementary symmetric function for principal curvature radii and the support function of \( M_\infty \). According to the equality condition for the Hölder’s inequality, there exists a constant \( c \geq 0 \) such that

\[ c \varphi(\tilde{h})\tilde{\sigma_k}(x) = f \text{ on } \mathbb{S}^{n-1}. \]

Noticing that \( \tilde{h} \) and \( \tilde{\sigma_k} \) have positive upper and lower bounds, \( c \) should be positive. The proofs of Theorems 1 and 2 are finished.

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References

[1] Andrews, B., McCoy, J. and Zheng, Y.: Contracting convex hypersurfaces by curvature, Calc. Var. Partial Diff. Equ., 47(2013): 611-665.
[2] Boroczky, K. J., Lutwak, E., Yang, D. and Zhang, G.: The logarithmic Minkowski problem, J. Amer. Math. Soc., 26 (2013): 831-852.
[3] Bryan, P., Ivaki, M. N. and Scheuer, J.: A unified flow approach to smooth, even $L_p$-Minkowski problems, Anal. PDE, 12 (2019): pp. 259–280.
[4] Chen, C. Q., Huang, Y. and Zhao, Y. M.: Smooth solutions to the $L_p$ dual Minkowski problems, Math. Ann., 373(2019): 953 - 976.
[5] Chen, H. and Li, Q. R.: The $L_p$ dual Minkowski problem and related parabolic flows, Preprint.
[6] Chou, K. S. and Wang, X. J.: A logarithmic Gauss curvature flow and the Minkowski problem, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000): 733–751.
[7] Chou, K. S. and Wang, X. J.: The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math., 205 (2006): 33-83.
[8] Chow, B. and Tsai, D. H.: Expansion of convex hypersurfaces by nonhomogeneous functions of curvature, Asian J. Math., 1(1997): 769-784.
[9] Gardner, R. J., Hug, D., Weil, W., Xing, S. and Ye, D.: General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I, Calc. Var. Partial Diff. Equ., 58 (2019): pp. Art. 12, 35 pp.
[10] Gardner, R. J., Hug, D., Weil, W., Xing, S. and Ye, D.: General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem II, Calc. Var. Partial Diff. Equ., 59 (2020): pp. Art. 15, 33 pp.
[11] Gerhardt, C.: Non-scale-invariant inverse curvature flows in Euclidean space, Calc. Var. Partial Diff. Equ., 49(2014): 471-489.
[12] Guan, P. F. and Ma, X. N.: The Christoffel-Minkowski problem. I. Convexity of solutions of Hessian equation, Invent. Math., 151(2003): 553-571.
[13] Guan, P. F. and Xia, C.: $L^p$ Christoffel-Minkowski problem: the case $1 < p < k + 1$. Calc. Var. Partial Diff. Equ., (2018): 57:69.
[14] Hamilton, R. S.: Four-manifolds with positive curvature operator, J. Diff. Geom., 24(1986): 153-179.
[15] Harbel, C., Lutwak, E., Yang, D. and Zhang, G.: The even Orlicz Minkowski problem, Adv. Math., 224(2010): 2485-2510.
[16] He, Y., Li, Q. R. and Wang, X. J.: Multiple solutions of the $L_p$-Minkowski problem, Calc. Var. Partial Diff. Equ., 55 (2016): Art. 117, 13 pp.
[17] Hu, C., Ma, X. N. and Shen, C.: On Christoffel-Minkowski problem of Firey’s $p$-sum, Calc. Var. Partial Diff. Equ., 21(2004): 137-155.
[18] Huang, Y., Liu, J. and Xu, L.: On the uniqueness of $L^p$-Minkowski problems: the constant $p$-curvature case in $R^3$, Adv. Math., 281 (2015): 906-927.
[19] Huang, Y., Lutwak, E., Yang, D. and Zhang, G.: Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems, Acta Math., 216 (2016): 325-388.
[20] Hug, D., Lutwak, E., Yang, D. and Zhang, G.: On the $L_p$ Minkowski problem for polytopes, Discrete Comput. Geom., 33 (2005): 699-715.
[21] Ivaki, M. N.: Deforming a hyper surface by principal radii of curvature and support function, Calc. Var. Partial Diff. Equ., 58(1)(2019): 2133-2165.
[22] Jian, H. Y. and Lu, J., Existence of solutions to the Orlicz-Minkowski problem, Adv. Math., 344(2019): 262-288.
[23] Jian, H. Y., Lu, J. and Wang, X. J.: Nonuniqueness of solutions to the $L_p$-Minkowski problem, Adv. Math., 281 (2015): 845-856.
[24] Jian, H. Y., Lu, J. and Zhu, G.: Mirror symmetric solutions to the centro-affine Minkowski problem, Calc. Var. Partial Diff. Equ., 55 (2016): Art. 41, 22 pp.

[25] Krylov, N. V. and Safonov, M. V.: A property of the solutions of parabolic equations with measurable coefficients, Izv. Akad. Nauk SSSR Ser. Mat., (44) 1980: 161–175, 239.

[26] Li, Q. R., Sheng, W. and Wang, X. J.: Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems, J. Eur. Math. Soc. (JEMS), 22 (2020): 893–923.

[27] Liu, Y. N. and Lu, J.: A flow method for the dual Orlicz-Minkowski problem, Trans. Amer. Math. Soc., 373 (2020): 5833–5853.

[28] Liu, Y. N. and Lu, J.: A generalized Gauss curvature flow related to the Orlicz-Minkowski problem. arXiv:2005.02376.

[29] Lu, J.: Nonexistence of maximizers for the functional of the centroaffine Minkowski problem. Sci. China Math., 61(2018): 511-516.

[30] Lu, J. and Wang, X. J.: Rationally symmetric solutions to the $L_p$-Minkowski problem, J. Diff. Equ., 254(2013): 983-1005.

[31] Lutwak, E.: The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom., 38 (1993): 131-150.

[32] Schneider, R.: Convex bodies, the Brunn-Minkowski theory, vol. 151 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, expanded, 2014.

[33] Sheng, W. M. and Yi, C. H.: A class of anisotropic expanding curvature flow. Disc. Conti. Dynam. Systems - A. 40(2020): 2017-2035.

[34] Sun, Y. and Long, Y.: The planar Orlicz Minkowski problem in the $L^1$ -sense, Adv. Math., 281(2015): 1364-1383.

[35] Urbas, J.: An expansion of convex hypersurfaces. J. Diff. Geom., 33(1991): 91-125.

[36] Xi, D., Jin, H. and Leng, G.: The Orlicz Brunn-Minkowski inequality, Adv. Math., 260(2014): 350-374.

[37] Xia, C.: Inverse anisotropic curvature flows from convex hypersurfaces, J. Geom. Anal., (27)2016: 1-24.

[38] Zhu, G.: The logarithmic Minkowski problem for polytopes. Adv. Math., 262 (2014): 909–931.

[39] Zou, D. and Xiong, G.: Orlicz-John ellipsoids, Adv. Math., 265(2014): 132-168.

HongJie Ju: School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, P.R. China

Email address: hjju@bupt.edu.cn

BoYa Li: School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, P.R. China

YanNan Liu: School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, P.R. China

Email address: liuyn@th.btbu.edu.cn