It is easy to construct classical 2-state systems illustrating the behavior of the short-lived and long-lived neutral $K$ mesons in the limit of CP conservation. The emulation of CP violation is more tricky, but is provided by the two-dimensional motion of a Foucault pendulum. Analogies are drawn between the pendulum and observables in neutral kaon decays. An emulation of CP- and CPT-violation using electric circuits is also discussed.

I. INTRODUCTION

Two-state systems abound in quantum mechanics, and have the virtue of permitting exact solutions \[1\]. One such system is that of the neutral kaons $K^0$ and $\bar{K}^0$ \[2\], which mix with one another \[3\] to form short-lived and long-lived mass eigenstates. Previously \[1\] we have shown that one can emulate this system using coupled resonant circuits, with the violation of CP invariance \[3\] exhibited in kaon decays corresponding to asymmetric coupling between the circuits.

We sought to implement the suggestion of Ref. \[1\] in a physical device, such as a pair of coupled pendula which may be used to illustrate the CP-conserving limit \[3\]. In the course of this activity, we came upon a familiar system which has many of the features of the neutral kaon system, including the asymmetric coupling between two oscillation modes which leads to the phenomenon of CP violation. This system is the Foucault pendulum. In the present article we explore the parallels between the Foucault pendulum and the neutral kaon system, showing how one might construct a pendulum with the closest possible relation to the actual short-lived and long-lived neutral kaon states, and discussing not only CP, but T and CPT violation as well.

\[1\]To be submitted to Am. J. Phys.
This article is arranged as follows. We first recall the motion of the pendulum in Section II, review the neutral kaons in Section III (partly a recapitulation of results from Ref. [4]), and explore the parallels in Section IV. Some suggestions for modifying the basic pendulum to make it emulate the actual kaon system are made in Section V, while Section VI concludes. An Appendix deals with the CPT-violating case.

II. FOUCAULT PENDULUM

A. Equations of motion in a rotating frame

The motion of a body subject to a force \( F \) in a system rotating with constant angular velocity \( \Omega_E \) is described by

\[
m\ddot{r} = F - 2m(\Omega_E \times \dot{r}) - m\Omega_E \times (\Omega_E \times r) - 2m\beta \dot{r},
\]

where the second term is the Coriolis force, the third term is the centripetal acceleration, and the last term has been inserted to describe damping due to air resistance.

For the case in question, \( \Omega_E \) will be a vector pointing toward the Earth’s North Pole, with magnitude \( \frac{2\pi}{d-1} \).

Consider a pendulum with a spherically symmetric support point so that it is free to move in two directions. Denote the corresponding axes by \( \hat{x} \) (East), \( \hat{y} \) (North), and \( \hat{z} \) (up, i.e., perpendicular to the surface of the Earth). The components of \( \Omega_E \) are

\[
\Omega_{Ex} = 0, \quad \Omega_{Ey} = \Omega_E \cos \theta, \quad \Omega_{Ez} = \Omega_E \sin \theta,
\]

where \( \theta \) is the latitude (positive for North latitude). The components of \( \Omega_E \times \dot{r} \) are

\[
(\Omega_E \times \dot{r})_x = -\Omega_E \sin \theta \dot{y}, \quad (\Omega_E \times \dot{r})_y = \Omega_E \sin \theta \dot{x}, \quad (\Omega_E \times \dot{r})_z = -\Omega_E \cos \theta \dot{x},
\]

where we have neglected \( \dot{z} \) for small oscillations of the pendulum.

The centripetal acceleration term \( -m\Omega_E \times (\Omega_E \times r) \) in Eq. (1) has magnitude \( m\Omega_E^2 a \cos \theta \) and acts in the direction \( -\dot{z} \cos \theta + \dot{y} \sin \theta \), where \( a \) is the radius of the Earth and \( \Omega_E^2 a = 3.38 \text{ cm s}^{-2} \). It thus changes the local acceleration of gravity slightly, \( g \rightarrow g_{\text{eff}} \), and displaces the equilibrium position of the pendulum. We shall take account of these effects by redefining \( g \equiv g_{\text{eff}} \) and setting \( x = y = 0 \) to be the equilibrium position.

We then define \( \omega_0^2 \equiv g/l \), where \( l \) is the length of the pendulum, and \( \Omega \equiv \Omega_E \sin \theta \), write Eq. (1) in component form, and cancel a factor of \( m \). The result is

\[
\ddot{x} = -\omega_0^2 x + 2\Omega \dot{y} - \beta \dot{x}, \quad \ddot{y} = -\omega_0^2 y - 2\Omega \dot{x} - \beta \dot{y}.
\]

The coupled equations (1) can be solved for periodic motion by substituting \( x = x_0 \exp(-i\omega t), y = y_0 \exp(-i\omega t) \), leading to an eigenvalue equation for \( \omega^2 \). Expanding around \( \omega = \omega_0 \) and dividing by \( 2\omega_0 \), the result is

\[
M \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \omega \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},
\]

This represents an oscillation in both \( x \) and \( y \) with angular frequency \( \omega \).
where

\[ M \equiv \begin{bmatrix} \omega_0 - i\beta & -i\Omega \\ i\Omega & \omega_0 - i\beta \end{bmatrix}. \tag{6} \]

Eq. (6) is very close to the result which one would obtain for mixing of a neutral-kaon system in which the $K^0$ is represented by the basis vector $\hat{x}$ while the $\bar{K}^0$ is represented by the basis vector $\hat{y}$. We shall explore this parallel presently. First, however, we show that Eq. (6) leads to the familiar behavior of the Foucault pendulum in which the plane of linear oscillations precesses by a daily amount which depends on the latitude.

B. Solution of equations of motion

The normalized eigenmodes of the system (6) are

\[ |R\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad |L\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \tag{7} \]

with eigenvalues

\[ \mu_R = \omega_0 - i\beta + \Omega, \quad \mu_L = \omega_0 - i\beta - \Omega. \tag{8} \]

An arbitrary two-component solution $x(t)$ describing motion in the $x - y$ plane can then be written as a linear combination of $|R\rangle$ and $|L\rangle$ as

\[ x(t) = c_R|R\rangle e^{-i\mu_R t} + c_L|L\rangle e^{-i\mu_L t}. \tag{9} \]

The initial conditions on $x(0)$ and $\dot{x}(0)$ then permit us to specify the complex quantities $c_R$ and $c_L$, by imposing the condition

\[ \text{Re} \ x(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}, \quad \text{Re} \ \dot{x}(0) = \begin{bmatrix} \dot{x}(0) \\ \dot{y}(0) \end{bmatrix} \tag{10} \]

on the solution (9) at $t = 0$.

Let us now assume the pendulum is initially displaced in the $\hat{x}$ (East) direction, with zero velocity, so that $x(0) = x_0$, $\dot{x}(0) = y(0) = \dot{y}(0) = 0$. We then find

\[ \text{Re} \ c_R = \frac{x_0 \omega_0 - \Omega}{\sqrt{2} \omega_0}, \quad \text{Re} \ c_L = \frac{x_0 \omega_0 + \Omega}{\sqrt{2} \omega_0}, \tag{11} \]

\[ \text{Im} \ c_R = \text{Im} \ c_L = \frac{\beta x_0}{\sqrt{2} \omega_0}. \tag{12} \]

The solution for the motion of the pendulum as a function of time is

\[ x(t) = x_0 e^{-\beta t} \left( \cos \omega_0 t + \frac{\beta}{\omega_0} \sin \omega_0 t \right) \cos \Omega t + \frac{\Omega}{\omega_0} \sin \omega_0 t \sin \Omega t \],

\[ y(t) = x_0 e^{-\beta t} \left( \cos \omega_0 t + \frac{\beta}{\omega_0} \sin \omega_0 t \right) \sin \Omega t - \frac{\Omega}{\omega_0} \sin \omega_0 t \cos \Omega t. \tag{13} \]
For $\Omega \ll \omega_0$, one sees by making the replacement $\Omega t \rightarrow \Omega t + \frac{\pi}{2}$ and comparing the terms $\sim \sin \Omega t$ and $\cos \Omega t$ in Eq. (13) that the plane of oscillation rotates with angular frequency $-\Omega = -\Omega_E \sin \theta$. For $\theta = 0$ there is no precession of the plane since at the Equator the only effect of the Earth’s rotation to lowest order is a change in the effective value of $g$. At the North or South pole the pendulum oscillates in a fixed plane in an inertial frame, and thus its plane of oscillation rotates with respect to the frame fixed with respect to the Earth with angular velocity $\pm \Omega_E$.

**III. THE TWO-STATE KAON SYSTEM**

The neutral kaon $K^0$ was first identified in cosmic radiation in the late 1940’s \[8\], via its decay to a pair of charged pions $\pi^+\pi^-$ in a cloud chamber. It was “strange” because its production occurred much more rapidly than its decay. Gell-Mann and Nishijima \[9\] explained this feature by assigning the $K^0$ a “strangeness” quantum number $S = 1$, conserved in production processes but violated in decays. A typical production process, for example, would be

$$\pi^- \ (S = 0) + p \ (S = 0) \rightarrow K^0 (S = 1) + \Lambda (S = -1) \ , \quad (14)$$

where $\Lambda$ is another “strange” particle first observed in the late 1940’s. Strong interactions were assumed to conserve strangeness, while weak interactions (such as those responsible for $K^0 \rightarrow \pi \pi$) could violate it by a maximum of one unit.

The strangeness scheme implied that the $K^0$ could not be its own antiparticle (in contrast to some other neutral particles like the photon and the neutral pion $\pi^0$), since it carried an additive quantum number $S = 1$. There then had to exist a neutral kaon with $S = -1$, the $K^0$. When Gell-Mann proposed this scheme, Enrico Fermi asked him what would distinguish the $K^0$ from the $\bar{K}^0$, since they would have equal masses and each could decay to $\pi \pi$. The solution, proposed by Gell-Mann and Pais, \[3\], was that one linear combination of the $K^0$ and $\bar{K}^0$, namely $K_1 \equiv (K^0 + \bar{K}^0)/\sqrt{2}$, would be able to decay to $\pi \pi$, while the orthogonal combination, $K_2 \equiv (K^0 - \bar{K}^0)/\sqrt{2}$, would be forbidden from decaying to $\pi \pi$ and thus would be longer-lived. This particle was searched for and found \[10\]. Its main decay modes are $3\pi$, $\pi\nu\bar{\nu}$, and $\pi\mu\nu$. These three-body decays have smaller phase space and thus occur with a smaller rate than the $2\pi$ process.

The Gell-Mann – Pais scheme was originally based on the assumed invariance of the weak (decay-causing) interactions under parity (P), or mirror reflection. When the weak interactions were found in 1957 to violate parity invariance, the argument was recast in terms of the product CP of charge reflection (C) and parity, since CP was thought at the time to be the symmetry obeyed by the weak interactions. The $K^0$ and $\bar{K}^0$ have spin zero. A spin-zero final state of $\pi \pi$ necessarily has CP eigenvalue equal to +1, since in the center-of-mass frame inverting space and reversing the signs of all charges restores the original system. Thus, if CP is conserved, it is the CP-even linear combination of $K^0$ and $\bar{K}^0$ which decays to $\pi \pi$. With a phase convention for the neutral kaons and the charge-conjugation operator chosen such that $CP|K^0) = |\bar{K}^0)$, this is just the combination $K_1$ defined above.
Thus, in the limit of CP conservation, the neutral kaon states with definite mass and lifetime are

\[ K_1 = \frac{K^0 + \overline{K}^0}{\sqrt{2}}, \quad K_2 = \frac{K^0 - \overline{K}^0}{\sqrt{2}}. \] (15)

These may be emulated with a pair of pendula of equal frequencies by coupling them with a spring which also dissipates energy. One pendulum may be thought of as the \( K^0 \) and the other as the \( \overline{K}^0 \). In the absence of coupling, there are two degenerate modes for which any two orthogonal linear combinations of \( K^0 \) and \( \overline{K}^0 \) can serve as an acceptable basis. When the pendula are coupled to one another via the spring, however, the linear combination in which the pendula are moving with equal and opposite displacements will be shifted in frequency and damped. This will be the mode \( K_1 \) defined above if the \( K^0 \) and \( \overline{K}^0 \) are defined with appropriate phases. The mode in which the pendula are moving in phase with one another will correspond to the \( K_2 \) and will be unshifted in frequency and undamped. The common damping term for \( K^0 \) and \( \overline{K}^0 \) corresponding to air resistance will imply that the mode with the pendula moving in phase with one another is also damped, but much longer-lived than the mode which excites the spring. Thus, damping of both eigenmodes permits a meaningful parallel with the neutral kaon system. A simple laboratory demonstration can be constructed which illustrates the problem up to this point.

The coupling and damping of a two-state system in quantum mechanics can be described by a “mass matrix” \( \mathcal{M} \) very similar to that discussed in Sec. II A [11]:

\[ i \frac{\partial}{\partial t} \begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix} = \mathcal{M} \begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix}; \quad \mathcal{M} = M - i \Gamma/2, \] (16)

where \( M \) and \( \Gamma \) are Hermitian. The eigenstates of \( \mathcal{M} \) evolve in proper time \( t \) as \( e^{-i \mu_i t} \), where \( \mu_i \) (\( i = 1, 2 \)) are the (complex) eigenvalues of \( \mathcal{M} \). Each \( \mu_i \) may be expressed as \( \mu_i = m_i - i \gamma_i/2 \), where \( m_i \) is the mass of the eigenstate \( i \) and \( \gamma_i \) denotes its decay width. Here and in what follows we are using units with \( \hbar = c = 1 \). The lifetime of the eigenstate \( i \) is then \( \tau_i = 1/\gamma_i \).

In the CP-invariant case when the eigenstates of \( \mathcal{M} \) are given by (13), a little algebra shows that one must have

\[ \mathcal{M}_{11} = \mathcal{M}_{22}; \quad \mathcal{M}_{21} = \mathcal{M}_{12}. \] (17)

This form is only consistent with the case of the Foucault pendulum discussed in Sec. II when the \( \Omega \) term in Eq. (3) vanishes, i.e., when there is no rotation. We will see that rotation of the reference frame in classical mechanics is analogous to CP violation in the neutral kaon system.

When CP is violated [3], the states of definite mass and lifetime \( K_S \) (for “short-lived”) and \( K_L \) (for “long-lived”) are eigenstates of a more general mass matrix no longer obeying \( \mathcal{M}_{21} = \mathcal{M}_{12} \). However, it turns out that invariance under the product CPT, where \( T \) denotes time reversal, still guarantees \( \mathcal{M}_{11} = \mathcal{M}_{22} \). In words, this just says that the quantum-mechanical amplitudes for \( K^0 \to K^0 \) and \( \overline{K}^0 \to \overline{K}^0 \) are equal. CPT invariance is a very general property of quantum field theories [12] and
we shall assume its validity here for the moment. We shall see in the Appendix how to relax the constraint $M_{11} = M_{22}$.

With CPT invariance assumed but CP violated, the eigenstates of the mass matrix $M$ can then be written as

$$|S\rangle = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left[ (1 + \epsilon)|K^0\rangle + (1 - \epsilon)|\bar{K}^0\rangle \right],$$

$$|L\rangle = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left[ (1 + \epsilon)|K^0\rangle - (1 - \epsilon)|\bar{K}^0\rangle \right],$$

where $S, L$ denote $K_S, K_L$, and $\epsilon$ is related to $M$ by

$$\epsilon \equiv \frac{\sqrt{M_{12}} - \sqrt{M_{21}}}{\sqrt{M_{12}} + \sqrt{M_{21}}} \approx \frac{M_{12} - M_{21}}{4\sqrt{M_{12}M_{21}}} \approx \frac{\text{Im}(\Gamma_{12}/2) + i \text{Im}M_{12}}{\mu_S - \mu_L}. \tag{20}$$

Here the eigenvalues $\mu_{S,L}$ of $M$ are related to its elements by

$$\mu_S = M_{11} + \sqrt{M_{12}M_{21}} ; \quad \mu_L = M_{11} - \sqrt{M_{12}M_{21}}. \tag{21}$$

The eigenstates $|S\rangle$ and $|L\rangle$ can be parametrized approximately as

$$|S\rangle \approx |K_1\rangle + |\epsilon|K_2\rangle , \quad |L\rangle \approx |K_2\rangle + |\epsilon|K_1\rangle. \tag{22}$$

These states have a scalar product $\langle L|S \rangle \approx 2 \text{Re} \epsilon$, which is no longer necessarily zero.

Both the magnitude and phase of $\epsilon$ can be measured accurately. The squared magnitude of $\epsilon$ is measured by comparing the rates for $K_L \to \pi\pi$ and $K_S \to \pi\pi$; the result is $|\epsilon| \approx (2.28 \pm 0.02) \times 10^{-3}$. The phase of $\epsilon$ can be measured by experiments in which decays of $K_S$ and $K_L$ produced in a known relative phase add coherently; the result is $\text{Arg} \epsilon \approx 43^\circ \pm 3^\circ$. Given what we know about the masses and lifetimes of $K_S$ and $K_L$ [13], this phase agrees with what one can derive from Eq. (20) [14]. Specifically, one can show [13] that $|\text{Im}\Gamma_{12}/2| \ll |\text{Im}M_{12}|$. This result then implies that

$$\text{Arg} \epsilon \approx \begin{cases} 90^\circ & \text{for } \text{Im}M_{12} > 0 \\ 270^\circ & \text{for } \text{Im}M_{12} < 0 \end{cases} \tag{23}$$

The observed masses $m_{S,L}$ and decay widths $\gamma_{S,L}$ in $\mu_{S,L} = m_{S,L} - i\gamma_{S,L}/2$ are such as to imply $\text{Arg} \epsilon = (43.5 \pm 0.1)^\circ$ for $\text{Im} M_{12} < 0$, and $\text{Arg} \epsilon = \pi + (43.5 \pm 0.1)^\circ$ for $\text{Im} M_{12} > 0$.

Equation (24) implies that $\epsilon \neq 0$ arises from $M_{12} \neq M_{21}$. As we have seen, the Foucault pendulum with $\Omega \neq 0$ in Eq. (6) provides a mechanical illustration of this behavior. In the next section we shall explore this parallel a bit further. First, we discuss one quantity sensitive to $\epsilon$ in the neutral kaon system. This is the asymmetry in the semileptonic decay of a neutral kaon beam to those final states which can arise from $K^0$ or $\bar{K}^0$. We introduce this topic because it will turn out to have a parallel in the case of the Foucault pendulum.
We shall assume, in accord with predictions at the quark level, that the allowed processes are $K^0 \to \pi^- \ell^+ \nu_\ell$ and $\bar{K}^0 \to \pi^+ \ell^- \bar{\nu}_\ell$, where $\ell = e$ or $\mu$. Suppose, for example, that all the $K_S$ in a neutral kaon beam have decayed away, leaving pure $K_L$. The leptonic asymmetry

$$A_\ell \equiv \frac{\Gamma(K^0 \to \pi^- \ell^+ \nu_\ell) - \Gamma(\bar{K}^0 \to \pi^+ \ell^- \bar{\nu}_\ell)}{\Gamma(K^0 \to \pi^- \ell^+ \nu_\ell) + \Gamma(\bar{K}^0 \to \pi^+ \ell^- \bar{\nu}_\ell)}$$

is then just $A_\ell = 2 \text{Re} \, \epsilon$. Measurements of this quantity are consistent with the measured magnitude and phase of $\epsilon$.

**IV. PARALLELS: BASIC PENDULUM**

The eigenstates $R$ and $L$ of the Foucault pendulum problem, which are eigenvectors of $\mathbf{H}$, can be put into correspondence with neutral-kaon eigenstates $S$ and $L$ by the correspondence $R \leftrightarrow S$, $L \leftrightarrow L$. Aside from overall phases assigned to the states, one then sees from Eq. (20) that $\epsilon = -i \text{sgn}(\Omega)$. The eigenstates have different oscillation frequencies, $\mu_S - \mu_L = 2\Omega$, but their lifetimes are the same. Since $\epsilon$ is imaginary, the eigenstates $S$ and $L$ are orthogonal to one another, just as in the CP-conserving case.

The $\pi\pi$ mode is excited by the decay of $K_1$ in (15), which corresponds in the Foucault pendulum case to an oscillation along the line $x = y$. Both the eigenmodes (24) have a component along this direction. Since the plane of oscillation of the Foucault pendulum is always rotating (as long as $\Omega \neq 0$), the excitation of the $\pi\pi$ mode will undergo variations in time as this plane rotates with angular frequency $\Omega$.

The analogue of the leptonic asymmetry $A_\ell$ discussed at the end of the previous Section is the asymmetry of the projection of oscillations of the $L$ eigenmode onto the $\hat{x} \leftrightarrow K^0$ and $\hat{y} \leftrightarrow \bar{K}^0$ directions. As one sees from (24), there is no asymmetry, only a phase difference, with respect to oscillations in the $\hat{x}$ and $\hat{y}$ directions. The same conclusion can be drawn from the fact that Re $\epsilon = 0$ in this example.

**V. MODIFICATIONS TO ILLUSTRATE ACTUAL KAON SYSTEM**

One needs eigenmodes with vastly different lifetimes in order to emulate the true neutral-kaon system, since $\Gamma_S/\Gamma_L = \tau_L/\tau_S \simeq 579$. It is not difficult to simulate such eigenmodes in a two-state system [4, 6]. For example, as mentioned in Sec. III, two pendula of the same natural frequency can be coupled to one another through a dissipative spring [6], leading to a difference in both frequency and lifetime between the eigenmodes in which the pendula oscillate in or out of phase with respect to each other.

For the spherical pendulum, one can simply introduce damping in one of the two directions, for example by using a permanent magnet as the pendulum bob, and placing a flat coil with windings oriented along one direction just beneath the pendulum. The coil should be connected to a dissipative load (e.g., a resistor). A coil with its windings along the $\hat{x}$ direction will damp oscillations in the $\hat{y}$ direction. However, if we were to treat the $\hat{x}$ and $\hat{y}$ damping differently, we would be violating CPT invariance, since then $M_{22} \neq M_{11}$. 

7
In the present Section we wish to emulate a CPT-preserving system, which is most convenient in a new basis. In the \((K^0, \bar{K}^0)\) basis a CPT-preserving mass matrix has the form

\[
\mathcal{M} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{11}
\end{bmatrix}.
\] (25)

Thus one must have equal natural frequencies and damping terms for oscillations in the \(\hat{x}\) and \(\hat{y}\) directions. However, one can transform [15] to the basis \((K_1, K_2)\) corresponding to oscillations in the \((K^0 \pm \bar{K}^0)/\sqrt{2} = (\hat{x} \pm \hat{y})/\sqrt{2}\) directions. In this basis the mass matrix is

\[
\mathcal{N} = U \mathcal{M} U^\dagger,
\]

where

\[
U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U^\dagger.
\] (26)

One finds

\[
\mathcal{N} = \begin{bmatrix}
N_{11} & N_{12} \\
-N_{12} & N_{22}
\end{bmatrix},
\] (27)

where \(N_{11} = (M_{11} + (M_{12} + M_{21})/2), N_{22} = (M_{11} - (M_{12} + M_{21})/2), \) and \(N_{12} = (M_{21} - M_{12})/2.\) Thus, in this basis, one can have a mass matrix of the form

\[
\mathcal{N} = \begin{bmatrix}
\omega_1 - i\gamma_1 & i\Omega' \\
-i\Omega' & \omega_2 - i\gamma_2
\end{bmatrix},
\] (28)

where \(\Omega'\) is not necessarily real. The CPT-invariance is expressed in the \(K_1- K_2\) basis by the condition \(N_{21} = -N_{12}.\)

Physically one can emulate the system described by the matrix \(\mathcal{N}\) by having a Foucault pendulum with different damping in the \(K_1\) and \(K_2\) directions. Such damping could be implemented, for example, by the inductive setup noted above. Natural frequencies in two orthogonal directions can be made to differ using a hinged set of supports. One could also introduce different damping constants in two perpendicular directions through properties of the hinged joints themselves.

The Foucault pendulum case corresponds to real \(\Omega'\) for the new basis. Note that \(\Omega'\) in \(\mathcal{N}\) is then the same as \(\Omega\) in \(\mathcal{M};\) aside from a sign flip in the off-diagonal elements, \(\mathcal{N}\) and \(\mathcal{M}\) are the same matrix.

For \(\omega_1 - i\gamma_1 \neq \omega_2 - i\gamma_2\) and \(\Omega \ll \omega_{1,2},\) the eigenvectors and their corresponding eigenvalues are approximately

\[
\begin{align*}
|S\rangle &= \begin{bmatrix} 1 \\ \epsilon_S \end{bmatrix}, & \mu_S &= \omega_1 - i\gamma_1 + \delta_S, \\
\epsilon_S &= \frac{-i\Omega}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}, & \delta_S &= \frac{-\Omega^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}, \\
|L\rangle &= \begin{bmatrix} \epsilon_L \\ 1 \end{bmatrix}, & \mu_L &= \omega_2 - i\gamma_2 + \delta_L, \\
\epsilon_L &= \frac{-i\Omega}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} = \epsilon_S \equiv \epsilon, & \delta_L &= -i\Omega\epsilon = -\delta_S.
\end{align*}
\] (29) (30) (31) (32)
Let us now discuss the time-evolution of states which are initially $|S\rangle$ and $|L\rangle$:

$$
|S\rangle = \left[ \begin{array}{c} 1 \\ \epsilon \end{array} \right] \rightarrow \left[ \begin{array}{c} 1 \\ \epsilon \end{array} \right] e^{-i\mu_S t}, \quad |L\rangle = \left[ \begin{array}{c} \epsilon \\ 1 \end{array} \right] \rightarrow \left[ \begin{array}{c} \epsilon \\ 1 \end{array} \right] e^{-i\mu_L t}.
$$

(33)

Defining $\phi \equiv \text{Arg} \epsilon$, these lead to the time-dependences

$$
\text{Re} \ x_S(t) = x_0 \text{Re} \left[ \begin{array}{c} 1 \\ \epsilon \end{array} \right] e^{-i\mu_S t} \simeq e^{-\gamma_1 t} \left[ \begin{array}{c} \epsilon \cos(\omega_1 t - \phi) \\ |\epsilon| \cos(\omega_1 t) \end{array} \right],
$$

(34)

$$
\text{Re} \ x_L(t) = x_0 \text{Re} \left[ \begin{array}{c} \epsilon \\ 1 \end{array} \right] e^{-i\mu_L t} \simeq e^{-\gamma_2 t} \left[ \begin{array}{c} |\epsilon| \cos(\omega_2 t - \phi) \\ \cos\omega_2 t \end{array} \right],
$$

(35)

where contributions of order $\delta_{S,L}$ have been omitted. Thus both $S$ and $L$ eigenstates correspond to orbits which are slightly rotated to favor an orientation in the direction of $K^0$-like decays by an amount proportional to $\epsilon$. The Coriolis force in this example induces an effect which is akin to the leptonic asymmetry parameter $A_\ell$ discussed at the end of Sec. III. Moreover, the decay to $\pi\pi$ in the $K_1-K_2$ basis is like the $x$-projection of the eigenstate. Although the dominant damping is along the $\hat{x}$-direction (we assume $\gamma_1 \gg \gamma_2$), the presence of the Coriolis force causes the eigenstate $L$ to have a projection proportional to $\epsilon$ along $\hat{x}$, i.e., the $K_L$ does decay to $\pi\pi$.

VI. CONCLUSIONS

We have shown that the motion of a Foucault pendulum has many features in common with the CP-violating neutral kaon system. When the natural frequencies for oscillation in two perpendicular directions and the damping terms for these directions are equal, the parameter $\epsilon$ describing the eigenstates $|K_S\rangle \simeq |K_1\rangle + \epsilon|K_2\rangle$ and $|K_L\rangle \simeq |K_2\rangle + \epsilon|K_1\rangle$ takes on the special value $\epsilon = -i$. In order to simulate the physical situation in which $|\epsilon| = \mathcal{O}(2 \times 10^{-3})$ and $\text{Arg}(\epsilon) \simeq \pi/4$, one must construct a Foucault pendulum with slightly different natural frequencies in two perpendicular directions, and with vastly different damping constants in these directions. While the practical realization of such a construction sounds challenging, it is interesting that, at least in principle, it seems feasible entirely within the realm of classical mechanics.

As we show in the Appendix, the phenomenon of CPT violation, with preservation of $T$ and violation of $CP$, can be emulated by coupled resonant circuits, building upon the results of Ref. [4]. The program set forth in that work is still incomplete; we would be delighted to see an implementation of $CP$ and $T$ violation, with CPT conservation, through the asymmetric coupling of two resonant circuits with equal frequencies.

The classical emulation of $CP$ violation via the Foucault pendulum leaves us with one big puzzle. In the classical system, the asymmetry in coupling between the two modes is imposed from the outside, so to speak, through the Earth’s rotation. In the neutral-kaon system, the corresponding asymmetry in $K^0-\bar{K}^0$ mixing is thought to arise from a complex phase in the Cabibbo-Kobayashi-Maskawa matrix [19] describing the charge-changing weak transitions of quarks. Is that phase an indication of a new fundamental asymmetry arising from physics beyond the Standard Model?
ACKNOWLEDGMENTS

We thank Bruno Carneiro da Cunha for the original suggestion which led to this investigation. This work was supported in part by the United States Department of Energy under Grant No. DE FG02 90ER40560.

APPENDIX. CPT-VIOLATING CASE

The matrices $\mathcal{M}$ and $\mathcal{N}$ are arbitrary when CPT is violated. The relation between them is

\[
\mathcal{N}_{11} = (\mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22})/2, \quad \mathcal{N}_{12} = (\mathcal{M}_{11} - \mathcal{M}_{12} + \mathcal{M}_{21} - \mathcal{M}_{22})/2,
\]
\[
\mathcal{N}_{21} = (\mathcal{M}_{11} + \mathcal{M}_{12} - \mathcal{M}_{21} - \mathcal{M}_{22})/2, \quad \mathcal{N}_{22} = (\mathcal{M}_{11} - \mathcal{M}_{12} - \mathcal{M}_{21} + \mathcal{M}_{22})/2.
\]

A simple example of a CPT-violating mass matrix $\mathcal{N}$ involves coupling between two resonant circuits, as discussed in Ref. [4]. If the circuits are taken to have different frequencies, the matrix takes the form

\[
\mathcal{N} = \begin{bmatrix}
\omega_1 - i\gamma_1 & \alpha \\
\alpha & \omega_2 - i\gamma_2
\end{bmatrix},
\]

in which the off-diagonal elements are equal (rather than equal and opposite as in the CPT-preserving case). An analysis parallel to that for the eigenstates $S$ and $L$ performed in the previous Section leads to the results (for $|\alpha| \ll \omega_{1,2}$)

\[
|S\rangle = \begin{bmatrix} 1 \\ \epsilon_S \end{bmatrix}, \quad |L\rangle = \begin{bmatrix} \epsilon_L \\ 1 \end{bmatrix},
\]

with

\[
\epsilon_S = \frac{\alpha}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} = -\epsilon_L \equiv \bar{\epsilon},
\]
\[
\mu_S = \omega_1 - i\gamma_1 + \frac{\alpha^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}, \quad \mu_L = \omega_2 - i\gamma_2 - \frac{\alpha^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}.
\]

The eigenstates are thus

\[
|S\rangle \simeq |1\rangle + \bar{\epsilon}|2\rangle, \quad |L\rangle \simeq |2\rangle - \bar{\epsilon}|1\rangle.
\]

This case (see, e.g., Ref. [14]) corresponds to invariance with respect to time-reversal, so that CP and CPT are violated. Both (22) and (41) can be written in the more general form

\[
|S\rangle \simeq |1\rangle + \epsilon_S|2\rangle, \quad |L\rangle \simeq |2\rangle + \epsilon_L|1\rangle.
\]

The CPT-preserving, CP-violating case corresponds to $\epsilon_S = \epsilon_L = \epsilon$, while the case (41) corresponds to $\epsilon_S = -\epsilon_L = \bar{\epsilon}$. To lowest order in $\epsilon_{S,L}$, one finds

\[
|K^0\rangle = \frac{1}{\sqrt{2}}(|S\rangle(1 - \epsilon_L) + |L\rangle(1 - \epsilon_S)), \quad |\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(|S\rangle(1 + \epsilon_L) - |L\rangle(1 + \epsilon_S)).
\]
Since the states $|S, L\rangle$ evolve with proper time $t$ as $|S, L\rangle \rightarrow e^{-i\mu_{S,L}t}|S, L\rangle$, a short calculation shows that

\[
|K^0\rangle \rightarrow |K^0\rangle[f_+(t) + (\epsilon_S - \epsilon_L)f_-(t)] + |\bar{K}^0\rangle[1 - (\epsilon_S + \epsilon_L)]f_-(t) ,
\]

\[
|\bar{K}^0\rangle \rightarrow |\bar{K}^0\rangle[f_+(t) + (\epsilon_L - \epsilon_S)f_-(t)] + |K^0\rangle[1 + (\epsilon_S + \epsilon_L)]f_-(t) ,
\]

(44)

where $f_\pm(t) \equiv \frac{\exp(-i\mu_St) \pm \exp(-i\mu_Lt)}{2}$.

For $\epsilon_L = -\epsilon_S$, the evolution of $K^0$ into $\bar{K}^0$ is the same as that for $\bar{K}^0$ into $K^0$, corresponding to a time-reversal-invariant situation. However, the amplitudes for $K^0 \rightarrow K^0$ and $\bar{K}^0 \rightarrow \bar{K}^0$ differ from one another, corresponding to CPT violation.

For $\epsilon_L = \epsilon_S = \epsilon$, the amplitudes for $K^0 \rightarrow K^0$ and $\bar{K}^0 \rightarrow \bar{K}^0$ are the same, corresponding to $CPT$ invariance, while those for $K^0 \rightarrow \bar{K}^0$ and $\bar{K}^0 \rightarrow K^0$ differ from one another, corresponding to T violation. In this case the terms $|f_-(t)|^2$ cancel in the rate asymmetry

\[
A_T \equiv \frac{\Gamma[K^0(0) \rightarrow \bar{K}^0(t)] - \Gamma[\bar{K}^0(0) \rightarrow K^0(t)]}{\Gamma[K^0(0) \rightarrow \bar{K}^0(t)] + \Gamma[\bar{K}^0(0) \rightarrow K^0(t)]} \ ,
\]

(45)

and to lowest order one finds [17] $A_T = 4 \text{ Re } \epsilon$. This relation has recently passed an experimental test at CPLEAR [18].

References

[1] We thank N. F. Ramsey for a discussion on this point. See, e.g., J. J. Sakurai, *Modern Quantum Mechanics*, Revised Edition, Addison-Wesley, Reading, Mass. and Menlo Park, Calif., 1994, p. 320.

[2] G. Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1969).

[3] M. Gell-Mann and A. Pais, “Behavior of neutral particles under charge conjugation,” Phys. Rev. 97, 1387-1389 (1955).

[4] J. L. Rosner, “Table-top time-reversal violation,” Am. J. Phys. 64, 982-985 (1996).

[5] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, “Evidence for the $2\pi$ decay of the $K^0$ meson,” Phys. Rev. Lett. 13, 138-140 (1964).

[6] B. Weinstein, “CP violation,” in *Festi-Val – Festschrift for Val Telegdi*, ed. by K. Winter (Elsevier, Amsterdam, 1988), pp. 245-265.

[7] H. Goldstein, *Classical Mechanics*, Second Edition (Addison-Wesley, New York, 1980), p. 177.

[8] G. D. Rochester and C. C. Butler, “Evidence for the existence of new unstable elementary particles,” Nature 160, 855-857 (1947).
M. Gell-Mann, “Isotopic spin and new unstable particles,” Phys. Rev. 92, 833-834 (1953); “On the Classification of Particles,” 1953 (unpublished); M. Gell-Mann and A. Pais, in Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics, edited by E. H. Bellamy and R. G. Moorhouse (Pergamon, London and New York, 1955); M. Gell-Mann, “The interpretation of the new particles as displaced charge multiplets,” Nuovo Cim. 4, Suppl. 848-866 (1956); T. Nakano and K. Nishijima, “Charge independence for V-particles,” Prog. Theor. Phys. 10, 581-582 (1953); K. Nishijima, “Some remarks on the even-odd rule,” Prog. Theor. Phys. 12, 107-108 (1954); “Charge independence theory of V-particles,” Prog. Theor. Phys. 13, 285–304 (1955).

K. Lande, E. T. Booth, J. Impeduglia, and L. M. Lederman, “Observation of long-lived V particles,” Phys. Rev. 103, 1901-1904 (1956).

See, e.g., R. G. Sachs, The Physics of Time Reversal Invariance (University of Chicago Press, Chicago, 1988).

J. Schwinger, “The theory of quantized fields. II,” Phys. Rev. 91, 713-728 (1953) (see esp. p. 720 ff); “The theory of quantized fields. VI,” Phys. Rev. 94, 1362-1384 (1954) (see esp. Eq (54) on p. 1366 and p. 1376 ff); G. Lüders, “On the equivalence of invariance under time reversal and under particle-antiparticle conjugation for relativistic field theories,” Kong. Danske Vid. Selsk., Matt-fys. Medd. 28, No. 5, 1-17 (1954); “Proof of the TCP theorem,” Ann. Phys. (N.Y.) 2, 1-15 (1957); W. Pauli, “Exclusion principle, Lorentz group, and reflection of space-time and charge,” in W. Pauli, ed. Niels Bohr and the Development of Physics (Pergamon, New York, 1955), pp. 30-51.

Particle Data Group, C. Caso et al., “Review of particle physics,” Eur. Phys. J. C 3, 1-794 (1998).

T. P. Cheng and L. F. Li, Gauge Theory of Elementary Particles (Oxford University Press, 1984); K. Kleinknecht, “CP violation in the $K^0 - \bar{K}^0$ system,” in CP Violation, edited by C. Jarlskog (World Scientific, Singapore, 1989), pp. 41-104; B. Winstein and L. Wolfenstein, “The search for direct CP violation,” Rev. Mod. Phys. 63, 1113-1148 (1992); P. K. Kabir, The CP Puzzle (Academic Press, New York, 1968).

L. Wolfenstein, “Violation of CP invariance and the possibility of very weak interactions,” Phys. Rev. Lett. 13, 562-564 (1964).

R. H. Dalitz, “Kaon physics – the first 50+ years,” to be published in Kaon Physics, edited by J. L. Rosner and B. Winstein, University of Chicago Press, 2000.

P. K. Kabir, “What is not invariant under time reversal?”, Phys. Rev. D 2, 540-542 (1970).
[18] CPLEAR Collaboration, A. Angelopoulos et al., “First observation of time reversal noninvariance in the neutral kaon system,” Phys. Lett. B 444, 43-51 (1998).

[19] N. Cabibbo, “Unitary symmetry and leptonic decays,” Phys. Rev. Lett. 10, 531-532 (1963); M. Kobayashi and T. Maskawa, “CP violation in the renormalizable theory of the weak interaction,” Prog. Theor. Phys. 49, 652-657 (1973).