STEADY-STATE MODE INTERACTIONS OF RADially SYMMETRIC MODES FOR THE LUGIATO-LEFEVER EQUATION ON A DISK

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. We study a nonlinear Schrödinger equation with damping, detuning, and spatially homogeneous input terms, which is called the Lugiato-Lefever equation, on the unit disk with the Neumann boundary conditions. We aim at understanding bifurcations of a so-called cavity soliton which is a radially symmetric stationary spot solution. It is known by numerical simulations that a cavity soliton bifurcates from a spatially homogeneous steady state. We prove the existence of the parameter-dependent center manifold and a branch of radially symmetric steady state in a neighborhood of the bifurcation point. In order to capture further bifurcations of the radially symmetric steady state, we study a degenerate bifurcation for which two radially symmetric modes become unstable simultaneously, which is called the two-mode interaction. We derive a vector field on the center manifold in a neighborhood of such a degenerate bifurcation and present numerical simulations to demonstrate the Hopf and homoclinic bifurcations of bifurcating solutions.

1. Introduction. We study the Lugiato-Lefever equation (LLE) on the unit disk:
\[
\begin{aligned}
\partial_t E &= - \left\{ 1 + i \left( \theta - |E|^2 \right) \right\} E + i b^2 \Delta E + E_{\text{in}}, \quad t > 0, \, x \in \Omega, \\
\nabla E \cdot n &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

where \( i = \sqrt{-1}, \Delta \) is the two-dimensional Laplace operator, \( \Omega \subset \mathbb{R}^2 \) is the unit disk, and \( n \) is an outward normal unit vector on \( \partial \Omega \). In a physical context, \( E \) denotes the slowly varying envelope of the electric field and \( E_{\text{in}} \) is the spatially homogeneous input field. The symbols \( \theta \) and \( b \) are real numbers corresponding to detuning and diffraction parameters, respectively. LLE is a nonlinear Schrödinger equation with damping, detuning and driving force. It was proposed by Lugiato and Lefever as a model equation for pattern formation in the ring cavity with the Kerr medium [11].

A brief review of LLE and related topics from the point of view of nonlinear optics is found in [2]. Scroggie et al. have studied pattern formation of the two-dimensional

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LLE by numerical simulation [16]. Colet et al. have studied the bifurcation of a spatially localized and radially symmetric solution, which is called cavity soliton [3]. They have aimed at understanding the property of cavity soliton on the whole plane which is free from the effect of the boundary, and they have studied the equation in a sufficiently large disk by numerical simulation.

Colet et al. [3] have found the following bifurcation scenario. A branch of unstable cavity solitons bifurcates from the spatially homogeneous steady state as $E_{\text{in}}$ increases. It undergoes a saddle-node bifurcation, and a stable cavity soliton appears. The stable cavity soliton undergoes a Hopf bifurcation resulting in a spatially localized and temporally periodic solution. The periodic solution disappears due to a homoclinic bifurcation. Our aim is to understand the origin of such a bifurcation scenario from a viewpoint of mathematics. It is thought that the two-mode interaction plays an important role in this bifurcation process. By the two-mode interaction, we mean the degenerate bifurcation for which two modes become unstable simultaneously.

In the context of mathematics, the well-posedness and the existence of the global attractor for a damped-driven nonlinear Schrödinger equation have been studied on some regions: a finite interval [7, 19], open bounded subset of $\mathbb{R}^2$ [1], on a unit disk of $\mathbb{R}^2$ [17], and $\mathbb{R}^N$ with $N \leq 3$ [10]. Steady-state bifurcation for LLE on $\mathbb{T}^1$ and $\mathbb{R}^1$ [12] and the stability of bifurcating solution in $L^2(\mathbb{T}^1)$ [13] have been studied by the present authors.

In particular, the authors have proved the existence of secondary bifurcation from the spatially homogeneous steady state by studying interactions of $n$-th and $(n+1)$-th Fourier modes near a bifurcation point [12]. It seems natural to extend these results to the two-dimensional case, though it is not easy. The unit disk is one of the simplest examples of two-dimensional domains, whose eigenfunctions are not trigonometric functions unlike the two-dimensional torus. In this paper, we adopt a similar approach to [12] for understanding bifurcations of radially symmetric solutions for the two-dimensional LLE. We consider the LLE on a disk and study the steady-state bifurcation of a spatially homogeneous steady state through the center manifold theory. Below, we prove the existence of center manifold (Theorem 1.2) and a primary branch of nontrivial solutions (Theorem 1.3). In addition, we derive a vector field on the center manifold near a bifurcation point at which two radially symmetric modes become unstable simultaneously (Theorem 1.4). It is expected that this vector field exhibits a Hopf bifurcation. However, it is too hard to prove it by hand. We carry out a computer-aided analysis for investigating bifurcations for the vector field.

A spatially homogeneous steady state $E_S$ satisfies

$$E_S = \frac{E_{\text{in}}}{1 + i(\theta - \alpha)}, \quad \alpha = |E_S|^2.$$  

We regard $\alpha$ as a parameter, and we parametrize $E_{\text{in}}$ by

$$E_{\text{in}}^2 = \alpha \left\{ 1 + (\theta - \alpha)^2 \right\}.$$  

Introducing two real-valued functions $u_1$ and $u_2$ by $E = E_S(1 + u_1 + iu_2)$, we obtain the equations for $u_1$ and $u_2$:

$$\partial_t u = F(u, \lambda) \equiv L(\lambda)u + g(u, \lambda)$$ (2)
with the homogeneous Neumann boundary conditions on \( \partial \Omega \), where \( u = (u_1, u_2)^T \), \( \lambda = (\alpha, \theta, b) \), and

\[
L(\lambda) = \begin{pmatrix}
-1 & \theta - \alpha - b^2 \Delta \\
3\alpha - \theta + b^2 \Delta & -1
\end{pmatrix},
\]

\[
g(u, \lambda) = \begin{pmatrix}
-2u_1u_2 - u_2(u_1^2 + u_2^2) \\
3u_1^2 + u_2^2 + u_1(u_1^2 + u_2^2)
\end{pmatrix},
\]

\( E_S \) corresponds to the trivial equilibrium \((u_1, u_2) = (0, 0)\) of (2). In this study, we consider the bifurcation of the trivial equilibrium (2).

Since the domain is a disk, it is convenient to use the polar coordinates. We denote the radial and angular coordinates by \( r \) and \( \phi \), respectively. Let \( L^2(\Omega) \) be the space of square-integrable functions equipped with the inner product defined by

\[
(f, g)_{L^2} = \int_0^{2\pi} d\phi \int_0^1 f(r, \phi) \overline{g(r, \phi)} r dr, \quad f, g \in L^2(\Omega),
\]

where \( \overline{z} \) means the complex conjugate of \( z \in \mathbb{C} \). We choose the function spaces

\[
X = \left\{ u = (u_1, u_2)^T \in H^2(\Omega)^2 \mid \nabla u_1 \cdot n = \nabla u_2 \cdot n = 0 \text{ on } \partial \Omega \right\},
\]

\[
Z = L^2(\Omega)^2.
\]

The boundary conditions are included in the definition of \( X \). \( Z \) is equipped with the inner product defined by

\[
(\mathbf{f}, \mathbf{g}) = (f_1, g_1)_{L^2} + (f_2, g_2)_{L^2},
\]

where \( \mathbf{f} = (f_1, f_2)^T \) and \( \mathbf{g} = (g_1, g_2)^T \) are arbitrary elements of \( Z \). \( X \) is compactly embedded in \( Z \). We identify \( Z \) with its dual \( Z^* \) by the Riesz representation theorem. Thus we have the triplet \( X \subset Z \equiv Z^* \subset X^* \), where \( X^* \) is the dual of space \( X \).

The linearized stability of the trivial equilibrium of (2) is determined by the eigenvalues of \( L(\lambda) \). Let \( \sigma(L(\lambda)) \) be the spectrum of \( L(\lambda) \). By using the eigenpairs of the Laplacian, we can reduce the eigenvalue problem for \( L(\lambda) \) to a set of the eigenvalue problems for \( 2 \times 2 \) matrix \( L_{mn}(\lambda) \) given by

\[
L_{mn}(\lambda) = \begin{pmatrix}
-1 & \theta - \alpha + b^2 k_{mn}^2 \\
3\alpha - \theta - b^2 k_{mn}^2 & -1
\end{pmatrix},
\]

(3)

where \( k_{00} = 0 \), and \( k_{mn} \) is the \( n \)-th positive root of the derivative of the \( m \)-th order Bessel function of first kind for non-negative integer \( m \) and positive integer \( n \). Let \( \mathbb{N} \) be the set of all positive integers, and let \( \mathbb{N}_0 \) be the set of all nonnegative integers.

Let \( \Lambda \) be given by \( \Lambda = \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \).

Definition 1.1. For each \((m, n) \in (\mathbb{N}_0 \times \mathbb{N}) \cup \{(0, 0)\}\), we define a subset of \( \Lambda \) by

\[
S_{mn} = \left\{ \lambda \in \Lambda \mid \det(L_{mn}(\lambda)) = 0 \right\}.
\]

We refer \( S_{mn} \) as a neutral stability surface of \((m, n)\)-mode.

We now state three theorems on the center manifold, which we prove in Sections 3, 4 and 5, respectively. Furthermore, we present numerical simulations for the mode-interaction in Section 6.

Theorem 1.2 (Center manifold). Suppose that we have \( \lambda_c = (\alpha_c, \theta_c, b_c) \), \( m \in \mathbb{N}_0 \), and \( n \in \mathbb{N} \) satisfying \( \det(L_{mn}(\lambda_c)) = 0 \). Let \( A = L(\lambda_c) \). Let \( \pi_c \in \mathcal{L}(Z; X) \) be the spectral projection onto \( \ker(A) \) and let \( X_c := \pi_c(Z) \). Then there exists a
neighborhood $\mathcal{O}_u \times \mathcal{O}_\lambda$ of $(0, \lambda_\ast)$. Let

$$M_c(\lambda) = \{ u_c + V(u_c, \lambda) \mid u_c \in X_c \}$$

has the following properties: $M_c(\lambda)$ is locally invariant and contains the set of

bounded solutions of (2) staying in $\mathcal{O}_u$ for all $t \in \mathbb{R}$.

$M_c(\nu)$ is called a (parameter-dependent) center manifold.

**Theorem 1.3.** There exist $\alpha_\ast > 1$, $b > 0$, $\sqrt{3} > \theta > 0$, $\eta > 0$, $c > 0$, $n \in \mathbb{N}$, and $u_0 \in \mathbb{R}^2 \setminus \{0\}$ such that $(\alpha_\ast, \theta, b) \in S_{0n} \cap S_{0n}$ and (2) has a family of stationary solutions

$$\{ u(\alpha) = c(\alpha - \alpha_\ast)\varphi_{0n}u_0 + R(\alpha) \mid |\alpha - \alpha_\ast| < \eta \}$$

satisfying

$$\|R(\alpha)\|_{H^2} = o(|\alpha - \alpha_\ast|) \quad (\alpha \to \alpha_\ast),$$

$$R(\alpha) \perp \text{span}\{\varphi_{0n}u_0\},$$

provided that $\int_0^1 (J_0(k_{0n}r))^3rdr \neq 0$, where $\varphi_{0n}$ is defined as in (5) below.

The following theorem gives us the normal form in a neighborhood of a bifurcation point where the two-mode interaction is expected to take place.

**Theorem 1.4.** Let $m, n \in \mathbb{N}$, $m < n$ be arbitrarily fixed. Let $\lambda_d = (\alpha_d, \theta_d, b_d) \in S_{0m} \cap S_{0n}$. Then there exists two-dimensional parameter-dependent center manifold in a neighborhood of $(u, \lambda) = (0, \lambda_d)$. The Taylor expansion of the reduced vector field on the center manifold is given by

$$\begin{align*}
\dot{z}_1 &= \Gamma_1(z, \nu) = a_1\nu z_1 + l_1\nu z_1 + q_{10}z_1^2 + q_{11}z_1z_2 + q_{02}z_2^2 + O(|z|^3), \\
\dot{z}_2 &= \Gamma_2(z, \nu) = a_2\nu z_2 + l_2\nu z_2 + r_{20}z_2^2 + r_{11}z_1z_2 + r_{02}z_2^2 + O(|z|^3),
\end{align*}$$

(4)

where $z = (z_1, z_2)^T \in \mathbb{R}^2$ is the coordinates on the center manifold and $\nu = (\nu_1, \nu_2)^T = (\alpha - \alpha_d, \theta - \theta_d)^T$ is the difference of parameters from the bifurcation point. The constants $a_1, a_2, l_1, l_2, q_{10}, q_{11}, q_{02}, r_{20}, r_{11}$ and $r_{02}$ are explicitly given by formulas (17) below.

The rest of this paper is organized as follows. We consider the linearized eigenvalue problem of the trivial equilibrium for (2) in the next section. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3. We derive the normal form of bifurcation caused by two-mode interaction in Section 5, that is, we prove Theorem 1.4. We present some numerical simulation for the mode-interaction in Section 6. Finally, Section 7 is devoted to concluding remarks based on the numerical results.

2. The linearized eigenvalue problem. In this section, we consider the condition for $\lambda = (\alpha, \theta, b)$ such that $L(\lambda)$ has a nontrivial kernel, which is a necessary condition of a bifurcation. Let $k_{00} = 0$ and let $k_{mn}$ be the $n$-th positive root of the derivative of the $m$-th order Bessel function of first kind $J_m(r)$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. We define functions $\varphi_{00}$ and $\varphi_{mn}$ in the following manner.

$$\varphi_{00}(r, \phi) = \frac{1}{\sqrt{\pi}},$$

$$\varphi_{mn}(r) = \begin{cases} 
\left[2\pi \int_0^1 J_0(k_{0n}r)^2rdr \right]^{-\frac{1}{2}} J_0(k_{0n}r), & \text{if } m = 0, \\
\left[\pi \int_0^1 J_m(k_{mn}r)^2rdr \right]^{-\frac{1}{2}} J_m(k_{mn}r), & \text{if } m > 0.
\end{cases}$$

(5)
Then the spectrum of $-\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition is
\[ \{k_{mn}^2 \mid m, n = 0, 1, 2, \ldots, n = 1, 2, \ldots \} \cup \{0\}. \]

$k_{00}^2 = 0$ is a simple eigenvalue with an eigenvector $\varphi_{00}$. For all $n \in \mathbb{N}$, $k_{0n}^2$ is a simple eigenvalue with an eigenfunction $\varphi_{0n}$. For all $m, n \in \mathbb{N}$, $k_{mn}^2$ is a semisimple eigenvalue with two eigenfunctions

\[ \varphi_{mn}(r) \cos(m\phi), \quad \varphi_{mn}(r) \sin(m\phi), \]

that is, the algebraic and geometric multiplicities of $k_{mn}^2$ are two. By the standard theory, the set
\[ \{ \varphi_{mn} \mid n \in \mathbb{N}_0 \} \cup \{ \varphi_{mn} \cos(m\phi), \varphi_{mn} \sin(m\phi) \mid m, n \in \mathbb{N} \} \]
forms an orthonormal basis of $L^2(\Omega)$. Therefore, we can reduce the eigenvalue problem of $L(\lambda)$ to that of $L_{mn}(\lambda)$ defined by (3).

**Remark 1.** For any $(m, n) \in \mathbb{N}_0 \times \mathbb{N}$ and any $\lambda \in \Lambda = \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{>0}$, we have $\text{tr}(L_{mn}) = -2$. It implies that there is no purely imaginary eigenvalue, whence there is no Hopf bifurcation of the trivial equilibrium.

**Lemma 2.1.** For any $(m, n) \in (\mathbb{N}_0 \times \mathbb{N}) \cup \{(0, 0)\}$, $S_{mn}$ is written by
\[ S_{mn}^\pm = \left\{ \left( \alpha_{mn}^\pm (\theta, b), \theta, b \right) \mid \theta \in \mathbb{R}, b > 0, (\theta + b^2k_{mn}^2) \geq \sqrt{3} \right\}, \quad (6) \]
\[ \alpha_{mn}^\pm (\theta, b) = \frac{1}{3} \left[ 2(\theta + b^2k_{mn}^2) \pm \sqrt{(\theta + b^2k_{mn}^2)^2 - 3} \right]. \quad (7) \]

In addition, $\alpha_{mn}^\pm$ satisfies $1 \leq \alpha_{mn}^- (\theta, b) \leq \alpha_{mn}^+ (\theta, b)$ for any $\theta, b$ with $(\theta + b^2k_{mn}^2) \geq \sqrt{3}$.

**Proof.** It follows by solving $\det(L_{mn}(\lambda)) = 0$ for $\lambda \in \mathbb{R}_{\geq 0}$. \hfill \Box

**Lemma 2.2.** Let $(m, n), (p, q) \in (\mathbb{N}_0 \times \mathbb{N}) \cup \{(0, 0)\}, (m, n) \neq (p, q)$ be arbitrarily fixed. Then $S_{mn} \cap S_{pq}$ is written by
\[ S_{mn} \cap S_{pq} = \left\{ (\alpha, \theta_d(\alpha), b_d(\alpha)) \mid \alpha > 1 \right\}, \]
where
\[ \theta_d(\alpha) = 2\alpha - \frac{k_{pq}^2 + k_{mn}^2}{|k_{pq}^2 - k_{mn}^2|} \sqrt{\alpha^2 - 1}, \quad b_d(\alpha) = \sqrt{\frac{2(2\alpha - \theta_d(\alpha))}{|k_{pq}^2 - k_{mn}^2|}}. \quad (8) \]

**Proof.** It follows by solving $\det(L_{mn}(\lambda)) = \det(L_{pq}(\lambda)) = 0$ for $\theta$ and $b$. \hfill \Box

**Remark 2.** Obviously, $b_d(\alpha)$ is positive for any $\alpha > 1$. Let $\kappa = (k_{pq}^2 + k_{mn}^2)/|k_{pq}^2 - k_{mn}^2|$. If $\kappa^2 \geq 2$, then $\theta_d(\alpha)$ is monotone decreasing in $\alpha > 1$, $\theta_d(1) = 2$, and $\theta_d(\alpha) \to -\infty$ as $\alpha \to \infty$. If $0 < \kappa^2 < 2$, then $\theta_d(\alpha)$ has a unique minimal at $\alpha = (2/(2 - \kappa^2))^{-1/2}$ for $\alpha > 1$, and $\theta_d(\alpha)$ tends to $\infty$ as $\alpha \to \infty$. The minimal value $(2\sqrt{2 - \kappa^2})(2 - \kappa^2)^{-1/2}$ is greater than $\sqrt{3}$.

**Remark 3.** Let $(m, n), (m', n'), (m'', n'')$ be three different elements in $(\mathbb{N}_0 \times \mathbb{N}) \cup \{(0, 0)\}$. Without loss of generality, we can assume that $k_{mn} < k_{m'n'} < k_{m''n''}$ holds. Then, $S_{mn} \cap S_{m'n'} \cap S_{m''n''}$ is an empty set. Indeed, if there exists $(\alpha, \theta, b) \in S_{mn} \cap S_{m'n'} \cap S_{m''n''}$, then we obtain
\[ \frac{2(2\alpha - \theta)}{b^2} = k_{mn}^2 + k_{m'n'}^2 = k_{mn}^2 + k_{m''n''}^2 = k_{m'n'}^2 + k_{m''n''}^2. \]
However, it contradicts with $k_{mn} < k_{m'n'} < k_{m''n''}$.

We summarize all possible cases of nontrivial kernel of $L(\lambda)$.

**Proposition 1.** For each $(m,n) \in (N_0 \times N) \cup \{(0,0)\}$, we denote $\tilde{S}_{mn} = S_{mn} \setminus \bigcup_{(p,q) \neq (m,n)} S_{pq}$.

1. Let $(m,n) \in (N_0 \times N) \cup \{(0,0)\}$ be arbitrarily fixed.
   - (a) If $m = 0$ and $n \in N \cup \{0\}$, then $\dim (\ker(L(\lambda))) = 1$ for any $\lambda \in \tilde{S}_{0n}$.
   - (b) If $m \neq 0$ and $n \in N$, then $\dim (\ker(L(\lambda))) = 2$ for any $\lambda \in \tilde{S}_{mn}$.

2. Let $(m,n), (p,q) \in (N_0 \times N) \cup \{(0,0)\}, (m,n) \neq (p,q)$ be arbitrarily fixed.
   - (a) If $m = p = 0$, then $\dim (\ker(L(\lambda))) = 2$ for any $\lambda \in S_{0n} \cap S_{0p}$.
   - (b) If $m = 0$ and $p \neq 0$, then $\dim (\ker(L(\lambda))) = 3$ for any $\lambda \in S_{0n} \cap S_{pq}$.
   - (c) If $m \neq 0$ and $p = 0$, then $\dim (\ker(L(\lambda))) = 4$ for any $\lambda \in S_{mn} \cap S_{pq}$.

**Proof.** We have reduced the eigenvalue problem of $L(\lambda)$ to that of $L_{mn}(\lambda)$. Suppose that $\lambda$ lies on a neutral stability curve. Then there exists $(m,n)$ such that the matrix $L_{mn}(\lambda)$ has a simple zero eigenvalue with an associated eigenvector $u_{mn}$. If $m = 0$, then $\varphi_{on}u_{0n}$ is an eigenfunction of $L(\lambda)$ associated with the zero eigenvalue. If $m \neq 0$, then $\varphi_{on}\cos(m\phi)u_{mn}$ and $\varphi_{on}\sin(m\phi)u_{mn}$ are linearly independent eigenfunctions of $L(\lambda)$ associated with the zero eigenvalue. Since at most two modes become critical at the same parameter value, $\dim(\ker(L(\lambda)))$ is at most four. Taking all possible combinations into account, we obtain this proposition. \qed

3. **Proof of Theorem 1.2.** We show that our system satisfies a sufficient condition for the center manifold theorem that is given in [18, Section 3]. The discussion below is almost analogous to that for a damped nonlinear wave equation described in [18, pp. 154–156]. Let $\lambda_c = (\alpha_c, \theta_c, b_c) \in \bigcup_{(m,n)} S_{mn}$, and let $A = L(\lambda_c)$.

Let $\lambda_e = (\alpha_e, \theta_e, b_e) \in \bigcup_{(m,n)} S_{mn}$ be arbitrarily fixed. Then the kernel of $L(\lambda_e)$ is nontrivial. Let us define new bifurcation parameter $\nu = (\nu_1, \nu_2)$ by

$$
\nu_1 = \alpha - \alpha_e, \quad \nu_2 = \theta - \theta_e.
$$

We can rewrite (2) by

$$
\partial_t u = Au + h(u, \nu),
$$

where $A = L(\lambda_e)$ and

$$
h(u, \nu) = \begin{pmatrix}
(\nu_2 - \nu_1)u_2 - (\alpha_e + \nu_1) \left[ 2u_1 u_2 + u_2 (u_1^2 + u_2^2) \right]

(3\nu_1 - \nu_2)u_1 + (\alpha_e + \nu_1) \left[ 3u_1^2 + u_2 + u_1 (u_1^2 + u_2^2) \right]
\end{pmatrix}.
$$

$h$ is $C^k$ map from $X \times \mathbb{R}^2$ to $X$ for any $k \geq 1$. Let us denote the spectrum of $A$ by $\sigma(A)$. We have $0 \in \sigma(A)$. Let $\pi_c$ be the spectral projection defined by

$$
\pi_c = \frac{1}{2\pi i} \int_{\Gamma_c} (\lambda - A)^{-1} \, d\lambda,
$$

where $\Gamma_c$ is a closed path in $\mathbb{C}$ surrounding 0 and not including any other element of $\sigma(A)$. $\pi_c \in \mathcal{L}(Z; X)$ is a projection onto $X_c := \pi_c(Z)$, which is the kernel of $A$. Let $I_Z$ be the identity map on $Z$. Let $\pi_{h} := I_Z - \pi_c$ and $X_h := \pi_h(X)$. Let $C^k_b(X_c \times \mathbb{R}^2; X_h)$ be the space of $C^k$ mapping from $X_c \times \mathbb{R}^2$ to $X_h$ whose sup-norms of $j$-th derivatives are bounded for any $0 \leq j \leq k$.

Let $\{k_n^2\}_{n=0}^{\infty} \subset \mathbb{R}$ be eigenvalues of $-\Delta$ on $\Omega$ with Neumann boundary conditions, which satisfies

$$
0 = k_0^2 < k_1^2 \leq \cdots \leq k_n^2 \leq \cdots.
$$
Let \( \varphi_n \) be an eigenfunction associated with \( k_n^2 \). The eigenvalue problem for \( A \) is reduced to the set of eigenvalue problems for 2 by 2 matrices,

\[
A_n \mathbf{v}_n = \mu_n \mathbf{v}_n, \quad A_n = \begin{pmatrix}
-1 & \zeta_n \\
2\alpha_c - \zeta_n & -1
\end{pmatrix},
\]

where \( \zeta_n = \theta_c - \alpha_c + t_n^2 k_n^2 \). It is easy to see that the eigenvalues \( \mu_n^\pm \) and associated eigenfunctions \( \mathbf{v}_n^\pm \) of \( A_n \) are given by

\[
\mu_n^\pm = -1 \pm \sqrt{\zeta_n(2\alpha_c - \zeta_n)}, \quad \mathbf{v}_n^\pm = \begin{pmatrix} \zeta_n \\ \pm \sqrt{\zeta_n(2\alpha_c - \zeta_n)} \end{pmatrix}.
\]

For any \( n \in \mathbb{N}_0 \), the real part of \( \mu_n^- \) is negative. If \( \zeta_n < 0 \) or \( \zeta_n > 2\alpha_c \), then \( \mu_n^- \) and \( \mu_n^+ \) are not real, \( \mu_n^- = \overline{\mu_n^+} \), and \( \text{Re} \mu_n^\pm = -1 \). If \( 0 < \zeta_n < 2\alpha_c \), then \( \mu_n^- \) and \( \mu_n^+ \) are real numbers. Moreover, if \( \alpha_c - \sqrt{\alpha_c^2 - 1} \leq \zeta_n \leq \alpha_c + \sqrt{\alpha_c^2 - 1} \), then \( \mu_n^+ \) is nonnegative. Remark that there are at most finite number of \( n \) such that \( \mu_n^+ \) is nonnegative. Let \( N_{cu} \) and \( N_s \) be defined by

\[
N_{cu} = \{ n \in \mathbb{N}_0 \mid \text{Re} \mu_n^+ \geq 0 \}, \quad N_s = \mathbb{N}_0 \setminus N_{cu}.
\]

We denote the resolvent set of \( A \) by \( \rho(A) \). Let \( \sigma_{cu}(A) \) and \( \sigma_s(A) \) be defined by

\[
\sigma_{cu}(A) = \{ \mu \in \sigma(A) \mid \text{Re} \mu \geq 0 \}, \quad \sigma_s(A) = \sigma(A) \setminus \sigma_{cu}(A).
\]

Remark that there exists a \( \delta > 0 \) such that

\[
\delta = -\max \{ \text{Re} \mu \mid \mu \in \sigma_s(A) \}.
\]

We define a spectral projection \( \pi_{cu} \in \mathcal{L}(Z; X) \) by

\[
\pi_{cu} = \frac{1}{2\pi i} \int_{\Gamma_{cu}} (\mu - A)^{-1} d\mu,
\]

where \( \Gamma_{cu} \) is a closed path in \( \mathbb{C} \) surrounding \( \sigma_{cu}(A) \) and not including any element in \( \sigma_s(A) \). Let \( X_{cu} = \pi_{cu}(Z) \), \( \pi_s = I_Z - \pi_{cu} \), and \( Z_s = \pi_s(Z) \). Then we have \( Z = X_{cu} \oplus Z_s \). Let \( A_{cu} = A|_{X_{cu}} \) and \( A_s = A|_{Z_s} \).

We prove that the following conditions are satisfied ([18, p.148]):

1. \( \text{Re} \mu \geq 0 \) for all \( \mu \in \sigma(A_{cu}) \), and \( \sigma(A_{cu}) \cap i\mathbb{R} \) consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalized eigenspace;
2. \( A_s \) is the infinitesimal generator of a strongly continuous semigroup \( \{ e^{A_s t} \mid t \geq 0 \} \) of bounded linear operator on \( Z_s \), satisfying

\[
\| e^{A_s t} \|_{\mathcal{L}(Z_s)} \leq M e^{-\beta t}, \quad t \geq 0,
\]

for some \( M \geq 1 \) and \( \beta > 0 \).

Since the first condition is satisfied, we focus on the second condition. In fact, it can be verified by the spectral mapping theorem for the semigroups on the Hilbert space (see the papers by [6, the formula on line 1, page 390] and Priess [15, Theorem 4 on page 853 and Corollary 5 on page 855]). But we give a proof for the reader’s sake.) As shown in [8, IX.4], the second condition is equivalent to the condition that \( \{ \mu \in \mathbb{R} \mid \mu > -\beta \} \subset \rho(A_s) \) and

\[
\| (\mu - A_s)^{-n} \|_{\mathcal{L}(Z_s)} \leq M(\mu + \beta)^{-n}, \quad n = 1, 2, \ldots
\]

(9)

When \( M = 1 \) it is sufficient that (9) holds for \( n = 1 \).

For any \( \mathbf{u} \in Z_s \) there exists \( \{ a_n \}_{n \in N_{cu}} \subset \mathbb{R} \) and \( \{ u_n \}_{n \in N_s} \subset \mathbb{R}^2 \) such that

\[
\mathbf{u} = \sum_{n \in N_{cu}} a_n \mathbf{v}_n^\varphi_n + \sum_{n \in N_s} u_n \varphi_n
\]
From (13) and (14), there exists \( \tilde{w} \) where
\[
|\tilde{w}|_2^2 = \sum_{n \in N_u} a_n^2 + \sum_{n \in N_s} |u_n|^2.
\]
We denote \( u_n = (u_{n1}, u_{n2})^T \). For each \( n \in N_s \) we set
\[
u_n = w_{n1} v_n^- + w_{n2} v_n^+,
\]
whose components satisfy
\[
u_n = \zeta_n (w_{n1} + w_{n2}), \quad u_n = \sqrt{\zeta_n (2\alpha_c - \zeta_n)} (-w_{n1} + w_{n2}).
\]
By solving (11), we obtain
\[
\begin{align*}
w_{n1} &= \sqrt{\zeta_n (2\alpha_c - \zeta_n)} u_{n1} - \zeta_n u_{n2}, \\
w_{n1} &= \sqrt{\zeta_n (2\alpha_c - \zeta_n)} (-w_{n1} + \zeta_n u_{n2}).
\end{align*}
\]
From (11), it is easy to see
\[
|\nu_n|^2 \leq \chi_n |w_n|^2,
\]
where \( w_n = (w_{n1}, w_{n2})^T \) and
\[
\chi_n = \frac{\zeta_n^2}{\sqrt{\zeta_n (2\alpha_c - \zeta_n)}} + \frac{\zeta_n^2}{\sqrt{\zeta_n (2\alpha_c - \zeta_n)}}
\]
for each \( n \in N_s \). On the other hand, (12) implies that there exists \( C > 0 \) such that
\[
|w_n| \leq C |u_n|.
\]
It should be noted that the matrix \( A_n \) is assumed to be diagonalizable in the above argument. Even for the case where \( A_n \) has a Jordan block, we can obtain the same inequalities as (13) and (14) with a slight modification for the expression of \( \chi_n \) by taking generalized eigenvectors associated with the multiple eigenvalue \(-1\) as \( v_n^- \) and \( v_n^+ \) in (10).

Now we define another norm on \( Z_s \) by
\[
|\|u\||^2_{Z_s} = \sum_{n \in N_u} a_n^2 + \sum_{n \in N_s} \chi_n |w_n|^2.
\]
From (13) and (14), there exists \( \hat{C} > 0 \) such that
\[
|\|u\||_{Z_s} \leq |\|u\||_{Z_s} \leq \hat{C} |\|u\||_{Z_s}
\]
for any \( u \in Z_s \).

For given \( \mu > -\delta \) and \( y \in Z_s \), there exists \( u \in Z_s \cap X \) such that \( (\mu - A_s)u = y \). By expanding \( u \) and \( y \)
\[
\begin{align*}
u = \sum_{n \in N_u} a_n \varphi_n v_n^- + \sum_{n \in N_s} (w_{n1} v_n^- + w_{n2} v_n^+) \varphi_n, \\
y = \sum_{n \in N_u} c_n \varphi_n v_n^- + \sum_{n \in N_s} (y_{n1} v_n^- + y_{n2} v_n^+) \varphi_n,
\end{align*}
\]
we obtain
\[
\begin{align*}
a_n &= (\mu - \mu_n^-)^{-1} c_n, & n & \in N_u, \\
w_{n1} &= (\mu - \mu_n^-)^{-1} y_{n1}, & w_{n2} &= (\mu - \mu_n^+)^{-1} y_{n2}, & n & \in N_s.
\end{align*}
\]
Since \( |\mu - \mu_n| \geq \mu + \delta \) for \( \mu_n \in \sigma(A_s) \), we obtain
\[
|\|u\||^2_{Z_s} \leq (\mu + \delta)^{-2} |\|y\||^2_{Z_s}.
\]
It implies that
\[
|||(\mu - A_s)^{-1}|||_{\mathcal{L}(Z_s)} \leq (\mu + \delta)^{-1}, \quad \mu > -\delta
\]
and that \(\mathcal{A}_s\) generates a strongly continuous semigroup satisfying
\[
|||e^{\mathcal{A}_s t}|||_{\mathcal{L}(Z_s)} \leq \exp(-\delta t), \quad \forall t \geq 0.
\]
Therefore the equivalence of the norms implies that there exists \(M > 0\) satisfying
\[
|||e^{\mathcal{A}_s t}|||_{\mathcal{L}(Z_s)} \leq M \exp(-\delta t), \quad \forall t \geq 0.
\]
Thus we can apply the center manifold theory to prove Theorem 1.2.

4. **Proof of Theorem 1.3.** In this section, we prove Theorem 1.3, which is a bifurcation theorem for a radially symmetric mode. Let \(n \in \mathbb{N}\) and fix \(\lambda_s = (\alpha_s, \theta_s, b_s) \in S_{0n} \setminus \bigcup_{(p,q) \neq (0,n)} S_{pq}\) arbitrarily. From Proposition 1, the kernel of \(L(\lambda_s)\) is one-dimensional. In this case, we have \(\det (L_{0n}(\lambda_s)) = 0\). Let \(\zeta = \theta_s - \alpha_s + b_s^2 k_{0n}^2\). It is easy to see that \(\zeta^{-1} = 3\alpha_s - \theta_s - b_s^2 k_{0n}^2\) and \(1 + \zeta^2 = 2\alpha_s \zeta\) hold. In addition, \(\zeta\) is positive. We can choose the basis of \(\ker(L(\lambda_s))\) and \(\ker(L(\lambda_s)^*)\) as
\[
\psi = \frac{1}{\sqrt{2\alpha_s \zeta}} \left( \begin{array}{c} \zeta \\ 1 \end{array} \right) \varphi_{0n}, \quad \psi^* = \sqrt{\frac{\alpha_s}{2\zeta}} \left( \begin{array}{c} 1 \\ \zeta \end{array} \right) \varphi_{0n},
\]
respectively. Then \(\langle \psi, \psi^* \rangle = 1\) holds, and the spectral projection is given by \(\pi_{s} u = \langle u, \psi^* \rangle \psi\).

We fix the values of \(\theta\) and \(b\) and vary only \(\alpha\) to study a codimension one bifurcation near \((u, \lambda) = (0, \lambda_s)\). Theorem 1.2 implies that there exists a parameter-dependent center manifold, which is parametrized by a single parameter \(\nu = \alpha - \alpha_s\). Theorem 1.2 implies that we can approximate the solution to (2) by
\[
u u = z \psi + \mathcal{R},
\]
where \(\mathcal{R}\) is the remainder term and is a function of \(z\) and \(\nu\) at least in a small neighborhood of \((u, \nu) = (0, 0)\). Therefore the bifurcation problem is reduced to that of the equation for \(z\). As \(z = \langle u, \psi^* \rangle\), we have \(\dot{z} = \langle F(u, \lambda), \psi^* \rangle\). The Taylor expansion of the equation for \(z\) is given by
\[
\dot{z} = \Gamma(z, \nu) \equiv a
\]
where \(\Gamma(0, \nu) = 0\) for any \(\nu\), and the coefficients are determined by
\[
a = \langle (D_{u\alpha} F)^o \psi, \psi^* \rangle, \quad q = \langle (D_{u\theta}^2 F)^o (\psi, \psi^*), \psi^* \rangle,
\]
where \((D_{u\alpha} F)^o, (D_{u\theta} F)^o, (D_{u\theta}^2 F)^o\) are the Fréchet derivatives at \((u, \lambda) = (0, \lambda_s)\). Since we have
\[
(D_{u\alpha} F)^o \psi = \frac{1}{\sqrt{2\alpha_s \zeta}} \left( \begin{array}{c} -1 \\ 3\zeta \end{array} \right) \varphi_{0n}, \quad (D_{u\theta}^2 F)^o (\psi, \psi^*) = \left( \begin{array}{c} -2 \\ 3\zeta + \zeta^{-1} \end{array} \right) \varphi_{0n}^3,
\]
a and \(q\) are given by
\[
a = \frac{1}{2} (3\zeta - \zeta^{-1}), \quad q = \sqrt{2\alpha_s \zeta} a \int_{\Omega} \varphi_{0n}^3.
\]
By Lemma 2.1, we obtain
\[
a = \begin{cases} -\sqrt{(\theta_s + b_s^2 k_{0n}^2)^2 - 3} \leq 0 & \text{if } \lambda_s \in S_{0n}^+, \\
\sqrt{(\theta_s + b_s^2 k_{0n}^2)^2 - 3} \geq 0 & \text{if } \lambda_s \in S_{0n}^-. 
\end{cases}
\]
We represent a solution to (2) by
\[ \int \]  
Therefore, \( a \) is nonzero unless the radicand in the formula above is zero. If 
\[ \int_0^1 (J(x_0, r))^q \, dr \neq 0, \]  
then \( q \neq 0 \) holds and Theorem 1.3 follows immediately from (15).

5. Proof of Theorem 1.4. Next, we consider Theorem 1.4. Let \( m, n \in \mathbb{N}, m < n \) be arbitrarily fixed. Let \( \lambda_d = (\alpha_d, \theta_d, b_d) \in S_{0m} \cap S_{0n} \). Then, the kernel of \( L(\lambda_d) \) is two-dimensional. In this case, we can choose the basis of \( \ker(L(\lambda_d)) \) and \( \ker(L(\lambda_d)^*) \) as
\[
\psi_1 = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \varphi_{0m}, \quad \psi_2 = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} \zeta \\ 1 \end{pmatrix} \varphi_{0n},
\]
\[
\psi_1^* = \frac{\alpha_d}{2\zeta} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \varphi_{0m}, \quad \psi_2^* = \frac{\alpha_d}{2\zeta} \begin{pmatrix} \zeta \\ 1 \end{pmatrix} \varphi_{0n},
\]
where \( \zeta = \theta_d - \alpha_d + b_d^2 k_d^2 \). We have \( \langle \psi_1, \psi_1^* \rangle = \delta_{ij} \) with the Kronecker delta.

Remark 4. In this case, since \( \det(L_{0m}(\lambda_d)) = \det(L_{0n}(\lambda_d)) = 0 \), we have
\[
L_{0m}(\lambda_d) = \begin{pmatrix} -1 & \zeta^{-1} \\ \zeta & -1 \end{pmatrix}, \quad L_{0n}(\lambda_d) = \begin{pmatrix} -1 & \zeta \\ \zeta^{-1} & -1 \end{pmatrix}.
\]
From (8), \( \zeta \) can be written as \( \zeta = \alpha_d + \sqrt{\alpha_d^2 - 1} \). It is easy to see that \( \zeta^{-1} = \alpha_d - \sqrt{\alpha_d^2 - 1}, \zeta > 0 \), and \( 1 + \zeta^2 = 2\alpha_d \zeta \).

Let \( \nu = (\nu_1, \nu_2) \) be defined by
\[
\nu_1 = \alpha - \alpha_d, \quad \nu_2 = \theta - \theta_d.
\]
We represent a solution to (2) by
\[
u = z_1 \psi_1 + z_2 \psi_2 + \mathcal{R},
\]
where \( z_1 \) and \( z_2 \) are real-valued functions of \( t \) and \( \mathcal{R} \) is the reminder term. Theorem 1.2 implies that \( \mathcal{R} \) is a function of \( z_1, z_2, \nu_1 \) and \( \nu_2 \) in a sufficiently small neighborhood of \( (u, \nu) = (0, 0) \) and the bifurcation problem of the trivial equilibrium of (2) is reduced to that of the equations for \( z_1 \) and \( z_2 \). As \( z_j = \langle u, \psi_j^* \rangle + \frac{d}{dt} \), we obtain \( z_j = \langle F(u, \lambda), \psi_j^* \rangle \) for \( j = 1, 2 \). It is easy to see by a simple calculation that the vector field on the center manifold takes the form of (4). The coefficients are determined by the following formulas:
\[
a_1 = \langle (D_{uu} F)^\circ \psi_1, \psi_1^* \rangle, \quad a_2 = \langle (D_{uu} F)^\circ \psi_2, \psi_2^* \rangle, \quad l_1 = \langle (D_{uu} F)^\circ \psi_1, \psi_1^* \rangle, \quad l_2 = \langle (D_{uu} F)^\circ \psi_2, \psi_2^* \rangle,
\]
\[
g_{20} = \langle (D_{u2}^2 F)^\circ \psi_1, \psi_1^* \rangle, \quad r_{20} = \langle (D_{u2}^2 F)^\circ \psi_1, \psi_1^* \rangle, \quad q_{11} = \langle (D_{u1}^2 F)^\circ \psi_1, \psi_2^* \rangle, \quad r_{11} = \langle (D_{u1}^2 F)^\circ \psi_1, \psi_2^* \rangle,
\]
\[
g_{02} = \langle (D_{u0}^2 F)^\circ \psi_2, \psi_2^* \rangle, \quad r_{02} = \langle (D_{u0}^2 F)^\circ \psi_2, \psi_2^* \rangle,
\]
where \( (D_{uu} F)^\circ \), \( (D_{uu} F)^\circ \), and \( (D_{u2}^2 F)^\circ \) are Fréchet derivatives at \( (u, \lambda) = (0, \lambda_d) \). We obtain
\[
(D_{uu} F)^\circ \psi_1 = \frac{1}{\sqrt{2\alpha_d \zeta}} \begin{pmatrix} \zeta \\ 3 \end{pmatrix} \varphi_{0m}, \quad (D_{uu} F)^\circ \psi_2 = \frac{1}{\sqrt{2\alpha_d \zeta}} \begin{pmatrix} -1 \\ 3 \zeta \end{pmatrix} \varphi_{0n}, \quad (D_{uu} F)^\circ \psi_1 = \frac{1}{\sqrt{2\alpha_d \zeta}} \begin{pmatrix} \zeta \\ -1 \end{pmatrix} \varphi_{0m}, \quad (D_{uu} F)^\circ \psi_2 = \frac{1}{\sqrt{2\alpha_d \zeta}} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \varphi_{0n}.\]
On the other hand, the quadratic terms are computed from
\[
(D^2_u F)\circ(\psi_1, \psi_1) = \left(\frac{-2}{\zeta + 3\zeta - 1}\right) \varphi_{0n}^2,
\]
\[
(D^2_u F)\circ(\psi_1, \psi_2) = \left(\frac{-2\alpha_d}{4}\right) \varphi_{0m}\varphi_{0n},
\]
\[
(D^2_u F)\circ(\psi_2, \psi_2) = \left(\frac{-2}{3\zeta + \zeta - 1}\right) \varphi_{2n}^2.
\]

As a result, we obtain the coefficients:
\[
a_1 = \alpha_d - 2\sqrt{\alpha_d^2 - 1}, \quad a_2 = \alpha_d + 2\sqrt{\alpha_d^2 - 1},
\]
\[
l_1 = \sqrt{\alpha_d^2 - 1}, \quad l_2 = -\sqrt{\alpha_d^2 - 1},
\]
\[
q_{20} = \sqrt{2\alpha_d\zeta - 1}a_1 \int_\Omega \varphi_{0m}^3, \quad r_{20} = \sqrt{2\alpha_d\zeta\alpha_d} \int_\Omega \varphi_{0m}\varphi_{0n}, \quad (17)
\]
\[
q_{11} = \sqrt{2\alpha_d\zeta a_1} \int_\Omega \varphi_{0m}\varphi_{0n}^2, \quad r_{11} = \sqrt{2\alpha_d\zeta - 1}a_2 \int_\Omega \varphi_{0m}\varphi_{0n}^2,
\]
\[
q_{02} = \sqrt{2\alpha_d\zeta - 1}\alpha_d \int_\Omega \varphi_{0m}\varphi_{0n}^2, \quad r_{02} = \sqrt{2\alpha_d\zeta a_2} \int_\Omega \varphi_{0n}^3.
\]

6. Numerical results.

6.1. Single mode bifurcation. In Theorem 1.3, we have left the proof of non-degeneracy of the quadratic term of the reduced vector field (15). Table 1 shows approximate values of \(\int_\Omega \varphi_{0n}^3\) for some \(n\). It is computed by using Ooura’s Mathematical Software Packages [14] for Bessel function and for numerical integration by DE formula. Numerical computation suggests that the integral is always positive. We conjecture that
\[
1. q \leq 0 \text{ if } \lambda_s \in S_{0n}^+, \text{ and}
\]
\[
2. q \geq 0 \text{ if } \lambda_s \in S_{0n}^-.
\]

If it is true, then \(a\) and \(q\) have the same sign. It suggests that (15) undergoes the transcritical bifurcation at \(\nu = 0\).

\[
\begin{array}{|c|c|c|c|}
\hline
n & \text{value} & n & \text{value} & n & \text{value} \\
\hline
1 & 4.934760E-01 & 6 & 1.820328E-01 & 11 & 1.382262E-01 \\
2 & 2.922274E-01 & 7 & 1.732584E-01 & 12 & 1.310071E-01 \\
3 & 2.668396E-01 & 8 & 1.590709E-01 & 13 & 1.271857E-01 \\
4 & 2.188215E-01 & 9 & 1.527833E-01 & 14 & 1.215772E-01 \\
5 & 2.052654E-01 & 10 & 1.430242E-01 & 15 & 1.184378E-01 \\
\hline
\end{array}
\]

6.2. Mode interactions. In this and the following subsections, we study steady-state mode interactions of two radially symmetric modes by using AUTO07-p [4] that is a software for bifurcation analysis developed by E. J. Doedel. We numerically compute bifurcation diagrams for the following system of equations
\[
\begin{aligned}
\dot{z}_1 &= a_1\nu z_1 + l_1\nu z_1 + q_{20} z_1^2 + q_{11} z_1 z_2 + q_{02} z_2^2, \\
\dot{z}_2 &= a_2\nu z_2 + l_2\nu z_2 + r_{20} z_2^2 + r_{11} z_1 z_2 + r_{02} z_2^2,
\end{aligned}
\]
\( (18) \)
which is the truncated system at quadratic terms of (4) near \((0, m)-(0, n)\) mode interaction. Hereafter we restrict ourselves to the linear subspace of radially symmetric functions. Obviously, the following three cases should be distinguished:

1. \((0, n)-(0, n+1)\) mode interaction as the primary bifurcation,
2. \((0, n)-(0, n+1)\) mode interaction as the secondary or later bifurcation,
3. \((0, m)-(0, n)\) mode interaction with \(n > m + 1\).

We mainly focus on the first case because we are interested in the bifurcation which generates the primary branch of nontrivial solutions. There are parameter restrictions. Suppose that \(n \in \mathbb{N}\) and \(\lambda_d = (\alpha_d, \theta_d, b_d) \in S_{0,n} \cap S_{0,n+1}\). Remark that the first case occurs if and only if \(\alpha_d \leq 2/\sqrt{3}\), or, equivalently, the following relation holds:

\[
\sqrt{3} \leq \theta_d + b_d^2 k_{0,n}^2 < 2 < \theta_d + b_d^2 k_{0,n+1}^2 \leq \frac{5}{\sqrt{3}}.
\]

It can be checked by a straightforward calculation from Lemma 2.1.

In bifurcation diagrams, \(BP\) denotes Branch Points, \(LP\) denotes Limit Points (fold bifurcation points), and \(HB\) denotes Hopf Bifurcation points. \(SS\) and \(US\) are stable and unstable equilibria, respectively. \(SPO\) and \(UPO\) are stable and unstable periodic orbits (limit cycles).

6.3. **\((0, n)-(0, n+1)\) mode interaction as the primary bifurcation.** In this subsection, we show a typical example of the case where a mode interaction of adjacent radial modes occurs as the primary bifurcation. As an example, we consider \((0, 2)-(0, 3)\) mode interaction at \(\theta_d = 1.2\). Let \(\lambda_d = (\alpha_d, \theta_d, b_d) \in S_{02} \cap S_{03}\) and \(\theta_d = 1.2\). Then we have

\[
(\alpha_d, \theta_d, b_d) \approx (1.049896, 1.2, 0.108553).
\]

The coefficients of (18) are computed by the formulas in Theorem 1.4. We regard \(\nu_1\) as a main bifurcation parameter.

Figure 1(A) shows one-parameter bifurcation diagrams at \(\nu_2 = 0.05\). There are two branch points and a branch of nontrivial equilibria passing through them. Remark that all the points labeled \(BP\) are actually branch points; the branch of nontrivial equilibria intersects with that of trivial equilibria transversely in \((\nu_1, z_1, z_2)\)-space.

We can observe the Hopf bifurcation of nontrivial equilibrium. As shown in Fig. 2(A), the bifurcating limit cycle enlarges as \(\nu_1\) goes away from the Hopf bifurcation point \(HB\), and it collides with the trivial equilibrium at a certain value of \(\nu_1\), say \(\nu_c\). It can be thought that a homoclinic bifurcation occurs at \(\nu_1 = \nu_c\); the branch of limit cycles bifurcate from the homoclinic orbit of the trivial equilibrium [9, Theorem 6.1]. In other words, the period of a limit cycle tends to infinity as \(\nu_1 \to \nu_c\), and the cycle disappears after \(\nu_1\) exceeds \(\nu_c\). As shown in Fig. 2(B), the limit cycle satisfies a characteristic scaling law [5]. Let \(\lambda_s\) and \(\lambda_u\) be eigenvalues of the origin at \(\nu_1 = \nu_c\). Suppose that \(\lambda_s < 0 < \lambda_u\) holds. Close to the homoclinic bifurcation, the period \(T\) of the limit cycle obeys

\[
T \propto -\frac{1}{\lambda_m} \ln |\nu_1 - \nu_c|,
\]

where \(\lambda_m = \min(|\lambda_s|, |\lambda_u|)\).

Finally, Figure 1(B) is two-parameter bifurcation diagram. All the bifurcation points arise from the origin of \((\nu_1, \nu_2)\)-plane. For \(\nu_2 < 0\), we can obtain a similar
bifurcation diagram to the case of $\nu_2 > 0$. The difference is that the stability of limit cycle is opposite to the case of $\nu_2 > 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Bifurcation diagrams for (18) near (0,2)-(0,3) mode interactions at $\theta_d = 1.2$. (A) one-parameter bifurcation diagrams at $\nu_2 = 0.05$. (B) two-parameter bifurcation diagram.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{Limit cycles for (18) near (0,2)-(0,3) mode interactions at $\theta_d = 1.2$. (A) limit cycles for several values of $\nu_1$ at $\nu_2 = 0.05$. (B) plot of $|\nu_1 - \nu_c|$ versus the period of limit cycles, where $\nu_c$ is the homoclinic bifurcation point. The horizontal axis is plotted on a log scale. The dashed line is the graph of a function proportional to $-\ln|\nu_1 - \nu_c|/\lambda_m$.}
\end{figure}

A similar behavior can be observed for other values of $\theta_d$ and $n$. We computed bifurcation diagrams for $\theta_d = 0.0, 0.1, \ldots, 1.7$ and $n = 1, 2, \ldots, 15$. Figure 3 summarizes the result of numerical experiments. In the figure, a parameter value which is marked with a circle leads to the primary bifurcation, and the bifurcation diagrams are qualitatively similar to that for $(\theta_d, n) = (1.2, 2)$. Figure 4 shows one-and two-parameter bifurcation diagrams at $(\theta_d, n) = (1.7, 10)$, for example. The bifurcation diagrams are qualitatively similar to Fig. 1.
At the parameter value in the shaded region of Fig. 3, \((0, n)-(0, n+1)\) mode interaction does not lead to the primary bifurcation, and the bifurcation diagram is different from Fig. 1. We show the result in the next subsection. Steady-state mode interaction cannot occur at parameter values in the black region of Fig. 3.

**Figure 3.** Summary of numerical experiments for \((0, n)-(0, n+1)\) mode interactions.

**Figure 4.** Bifurcation diagrams for (18) near \((0, 10)-(0, 11)\) mode interactions at \(\theta_d = 1.7\). (A) one-parameter bifurcation diagrams at \(\nu_2 = 0.2\). (B) two-parameter bifurcation diagram.

6.4. \((0, n)-(0, n + 1)\) mode interaction as the secondary or further bifurcation. In this subsection, we consider \((0, n)-(0, n+1)\) mode interactions which do not lead to the primary bifurcation. There are two typical behaviors. First, we show the bifurcation diagrams near \((0, 1)-(0, 2)\) mode interaction at \(\theta_d = 1.2\). In this case, the bifurcation point \(\lambda_d \in S_{01} \cap S_{02}\) is given approximately by

\[
\lambda_d = (\alpha_d, \theta_d, b_d) \approx (1.180662, 1.2, 0.190651).
\]

Figure 5 shows one- and two-parameter bifurcation diagrams. In this case, the trivial equilibrium point is always unstable. There are two fold bifurcations on the branch of nontrivial equilibria. Again, we can observe the Hopf bifurcation and the homoclinic bifurcation. Unlike the previous case, the Hopf bifurcation point is located on the left branch of nontrivial equilibria. We obtained a similar diagram to Fig. 5 at almost all parameter values in shaded region of Fig. 3.
Next, we look at the case of \((0,1)-(0,2)\) mode interaction at \(\theta_d = 0.8\). In this case, the bifurcation point \(\lambda_d \in S_{01} \cap S_{02}\) is given approximately by
\[
\lambda_d = (\alpha_d, \theta_d, b_d) \approx (1.967066, 0.8, 0.313200).
\]

Figure 6 shows bifurcation diagrams. There are two fold bifurcations and two Hopf bifurcations. One fold bifurcation point is very close to the second branching point. The branch of unstable limit cycles terminates at the homoclinic orbit to the origin. On the other hand, the termination of the branch of stable limit cycles is the homoclinic orbit to a nontrivial equilibrium point. Note that \((0,1)-(0,2)\) mode interaction occurs only when \(\theta_d > 0.759\). A bifurcation diagram similar to Fig.6 is obtained for \(0.759 < \theta_d < 0.849\) and \(n = 1\).

7. **Concluding remarks.** We have derived a vector field on the center manifold for (2) in a neighborhood of steady-state mode interactions of two radially symmetric modes, and we have analyzed numerically truncated systems by AUTO. As a result, we have observed Hopf and homoclinic bifurcations in a neighborhood of a steady-state mode interaction. Finally, we would like to compare the above results with...
the original LLE. Consider the LLE in the radial coordinate:

\[
\begin{align*}
    \partial_t u_1 &= -u_1 + (\theta - \alpha - b^2 \Delta_r)u_2 - \alpha \left[2u_1 u_2 + u_2(u_1^2 + u_2^2)\right], \\
    \partial_t u_2 &= (3\alpha - \theta + b^2 \Delta_r)u_1 - u_2 + \alpha \left[3u_1^2 + u_2^2 + u_1(u_1^2 + u_2^2)\right], \\
    \partial_r u_1 &= \partial_r u_2 = 0, \quad \text{at } r = 0, 1,
\end{align*}
\]

where \(\Delta_r = \partial_r^2 + r^{-1} \partial_r\), and \(r \in (0, 1)\). We approximate (19) by the second order finite difference method, and compute bifurcation diagrams by AUTO-07p. The domain is divided into 80 sub-intervals, that is, a system of 81 \(\times\) 2 dimensional ODEs is solved numerically. Although the resolution is not sharp, it will be enough for capturing qualitative properties of solutions under consideration.

We set \(b^2 = 0.117837\) and consider the bifurcation near \((0, 2)-(0, 3)\) mode interaction at \(\theta = 1.25\), which corresponds to the first examples in Section 6.3. Figure 7 shows one-parameter bifurcation diagram at \(\theta = 1.25\) and two-parameter bifurcation diagram. We can find fold, Hopf, and homoclinic bifurcations; these are nothing but the bifurcation scenario reported by Colet et al [3]. As shown in Fig. 7(B), the Hopf bifurcation curve arises from the fold bifurcation curve by the Bogdanov-Takens bifurcation point (BT). However, the fold and Hopf bifurcation curves are not continued from the intersection of neutral stability curves \(S_{02}\) and \(S_{03}\). Therefore steady-state mode interaction of two adjacent radial modes do not explain the occurrence of fold bifurcation of localized spot. It may be explained by taking the effect of higher-order terms of (4) into account. In order to prove it mathematically, it will be necessary to consider the degeneracy of quadratic terms. It requires the system to have another parameter. In addition, a local bifurcation associated with the \(2 \times 2\) zero matrix is codimension four [20, Chapter 3.1E]. Hence we need additional parameters in order to understand bifurcation of (4) completely.

On the other hand, the results in Section 6.3 suggests that there should be the Hopf bifurcation point on the branch of nontrivial solutions between the first and second branching points. However, there is no such bifurcation point in Fig. 7(A). The reason is that the parameter is far from \(S_{02}\)\(\cap\)\(S_{03}\). As shown in the left figure of Fig. 8, if \(\theta\) is close to 1.2, we can find the Hopf bifurcation of nontrivial equilibrium point. For the same reason the bifurcation diagram is not topologically equivalent to Fig. 1(A); there are two fold bifurcations in \(1.046 < \alpha < 1.048\). Of course, we cannot
Figure 8. (A) One-parameter bifurcation diagrams for second order FD approximation for (19) at $\theta_d = 1.205, b_d^2 = 0.0117837$. (B) Close-up of the right figure of Fig. 7. CP and GH mean the cusp bifurcation point and the generalized Hopf bifurcation point, respectively.

exclude a possibility that the higher order terms of the reduced vector field cause an additional bifurcation. Figure 8(B) is a close-up of Fig. 7(B). The Hopf bifurcation curve terminates at the Bogdanov-Takens point on a fold bifurcation curve which is continued from one of fold bifurcation points near $\alpha \approx 1.1$ in Fig. 7(A). By the generalized Hopf bifurcation (as known as Bautin bifurcation [9, Chapter 8.3]), the stability and the direction of bifurcation of limit cycles change. If $\theta$ is fixed below this point, the Hopf bifurcation leads to stable limit cycles and it agrees with the center manifold analysis.

In conclusion, a radially symmetric stationary solution bifurcates from the branch of trivial equilibria by a transcritical bifurcation. A steady-state mode interaction of two adjacent radially symmetric modes can produce a limit cycle and its homoclinic bifurcation. It should be noted that we can observe such bifurcations for wide range of parameters as shown in Fig. 3. Thus, the two-dimensional LLE has a nature to exhibit these bifurcations.

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REFERENCES

[1] M. Abounouh, Asymptotic behaviour for a weakly damped Schrödinger equation in dimension two, Appl. Math. Lett., 6 (1993), 29–32.
[2] T. Ackemann and W. J. Firth, Dissipative solitons in pattern-forming nonlinear optical systems, Lecture Notes in Phys., 661 (2005), 55–100.
[3] P. Colet, D. Gomila, A. Jacobo and M. A. Matía, Excitability mediated by dissipative solitons in nonlinear optical cavities, Lecture Notes in Phys., 751 (2008), 113–135.
[4] E. J. Doedel and B. E. Oldeman, AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations, Concordia University, Montreal, Canada, January 2012. Available from: http://cmvl.cs.concordia.ca/auto/.
[5] P. Gaspard, Measurement of the instability rate of a far-from-equilibrium steady state at an infinite period bifurcation, J. Phys. Chem., 94 (1990), 1–3.
[6] L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, Trans. Amer. Math. Soc., 236 (1978), 385–394.
[7] J.-M. Ghidaglia, Finite dimensional behavior for weakly damped driven Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5 (1988), 365–405.

[8] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.

[9] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Third edition, Springer-Verlag, New York, 2004.

[10] P. Laurençot, Long-time behaviour for weakly damped driven nonlinear Schrödinger equations in $\mathbb{R}^N$, $N \leq 3$, *NoDEA*, 2 (1995), 357–369.

[11] L. A. Lugiato and R. Lefever, Spatial dissipative structures in passive optical systems, *Phys. Rev. Lett.*, 58 (1987), 2209–2211.

[12] T. Miyaji, I. Ohnishi and Y. Tsutsumi, Bifurcation analysis to the Lugiato-Lefever equation in one space dimension, *Phys. D*, 239 (2010), 2066–2083.

[13] T. Miyaji, I. Ohnishi and Y. Tsutsumi, Stability of stationary solution for the Lugiato-Lefever equation, *Tohoku Math. J.*, 63 (2011), 651–663.

[14] T. Ooura, *Ooura’s mathematical software packages*, 2006. Available from: http://www.kurims.kyoto-u.ac.jp/ooura/index.html.

[15] J. Prüss, On the spectrum of $C_0$-semigroups, *Trans. Amer. Math. Soc.*, 284 (1984), 847–857.

[16] A. J. Scroggie, W. J. Firth, G. S. McDonald, M. Tlidi, R. Lefever and L. A. Lugiato, Pattern formation in a passive Kerr cavity, *Chaos Solitons Fractals*, 4 (1994), 1323–1354.

[17] N. Tzvetkov, Invariant measures for the nonlinear Schrödinger equation on the disc, *Dynamics of PDE*, 3 (2006), 111–160.

[18] A. Vanderbauwhede and G. Iooss, Center manifold theory in infinite dimensions, *Dynamics Reported New Series*, 1 (1992), 125–163.

[19] X. Wang, An energy equation for the weakly damped driven nonlinear Schrödinger equations and its application to their attractors, *Phys. D*, 88 (1995), 167–175.

[20] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.

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