The size of maximal systems of brick islands

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Abstract

For integers $m_1, \ldots, m_d > 0$ and a cuboid $M = [0, m_1] \times \ldots \times [0, m_d] \subset \mathbb{R}^d$, a brick of $M$ is a closed cuboid whose vertices have integer coordinates. A set $H$ of bricks in $M$ is a system of brick islands if for each pair of bricks in $H$ one contains the other or they are disjoint. Such a system is maximal if it cannot be extended to a larger system of brick islands. Extending the work of Lengvárszky, we show that the minimum size of a maximal system of brick islands in $M$ is $\sum_{i=1}^d m_i - (d-1)$. Also, in a cube $C = [0, m]^d$ we define the corresponding notion of a system of cubic islands, and prove bounds on the sizes of maximal systems of cubic islands.

1 Introduction

The concept of a system of rectangular islands was introduced by Czédli in [1]. In [4], Pluhár generalised this concept to that of a system of brick islands in higher dimensions, a direction mentioned both in [1] and by Lengvárszky in [2]. To introduce these concepts, let $M = [0, m_1] \times \ldots \times [0, m_d] \subset \mathbb{R}^d$ be a closed cuboid. Then a brick of $M$ is a set of the form $[a_1, b_1] \times \ldots \times [a_d, b_d]$, where for each $1 \leq i \leq d$ we have $0 \leq a_i < b_i \leq m_i$, and $a_i, b_i \in \mathbb{Z}$. A system of brick islands in $M$ is a set $H$ of bricks in $M$ such that whenever $M_1, M_2 \in H$, either $M_1 \subseteq M_2$, $M_2 \subseteq M_1$ or $M_1 \cap M_2 = \emptyset$. We denote the set of systems of brick islands in $M$ by $I_M$, and the maximal elements of $I_M$ with respect to inclusion by $\text{Max}(I_M)$. When $M$ is 2-dimensional, a system of brick islands can also be called a system of rectangular islands.

A related concept is that of a system of square islands, introduced by Lengvárszky in [3]. For $m$ a positive integer, let $S = [0, m] \times [0, m]$ be a closed square in the plane. A system of square islands in $S$ is a system of rectangular islands $H$ with every rectangle in $H$ being a square. We denote the set of these systems by $I_S$, and the maximal elements of $I_S$ with respect to inclusion by $\text{Max}(I_S)$.

We define the higher dimensional analogue of systems of square islands, as suggested in [3]. Let $C = [0, m]^d$ be a closed cube in $d$-dimensional space. Then we define a system of cubic islands in $C$ to be a system $H$ of brick islands in $C$ such that each brick in $H$ is a cube. We denote the set of these systems by $I_C$, and the maximal elements of $I_C$ with respect to inclusion by $\text{Max}(I_C)$.

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We shall be concerned with the possible cardinalities of maximal systems of brick and cubic islands. For a cuboid \( M = [0, m_1] \times \ldots \times [0, m_d] \), we define
\[
f_d(m_1, \ldots, m_d) = \max \{|H| : H \in Max(IM)\}
\]
and
\[
g_d(m_1, \ldots, m_d) = \min \{|H| : H \in Max(IM)\}.
\]
Similarly, for a cube \( C = [0, m]^d \), we define
\[
f'_d(m) = \max \{|H| : H \in Max(IC)\}
\]
and
\[
g'_d(m) = \min \{|H| : H \in Max(IC)\}.
\]

2 Earlier work

We summarise the main results of Czédi [1], Lengvárszky [2, 3] and Pluhár [4] that relate to our work. All these results concern the possible cardinalities of maximal systems of rectangular, square or brick islands. In [1], Czédi proved
\[
f_2(m_1, m_2) = \left\lfloor \frac{m_1 m_2 + m_1 + m_2 - 1}{2} \right\rfloor.
\]
In [2] and [3], Lengvárszky proved
\[
g_2(m_1, m_2) = m_1 + m_2 - 1,
\]
and for systems of square islands in \( S = [0, m] \times [0, m] \),
\[
g'_2(m) = m, \text{ and } f'_2(m) \leq \frac{m(m + 2)}{3}
\]
with equality in the last being achieved for \( k \) a positive integer and \( m = 2^k - 1 \). In [4], Pluhár proved that for systems of brick islands in \( M = [0, m_1] \times \ldots \times [0, m_d] \),
\[
\frac{m_1 m_2 \ldots m_d + \sum m_{j_1} \ldots m_{j_d-1}}{2^{d-1}} - 1 \leq f_d(m_1, \ldots, m_d) \leq \left(\frac{m_1 + 1}{2^{d-1}}\right) \ldots \left(\frac{m_d + 1}{2^{d-1}}\right) - 1
\]
where the sum in the lower bound runs over the \( d-1 \) element subsets of \( \{1, \ldots, d\} \).

3 Our results

Using methods similar to those employed in [2] and [3], we shall prove the following theorems about the possible cardinalities of maximal systems of brick and cubic islands.
Theorem 1. Let $M = [0,m_1] \times \ldots \times [0,m_d] \subset \mathbb{R}^d$ be a cuboid. Then the minimal size of a maximal system of cuboid islands in $M$, is given by

$$g_d(m_1, \ldots, m_d) = \sum_{i=1}^{d} m_i - (d-1).$$

Theorem 2. Let $C = [0,m]^d$ be a cube in $d$-dimensional space. Then the minimal size of a maximal system of cubic islands in $C$ is given by

$$g'_d(m) = m.$$

Theorem 3. Let $C = [0,m]^d$ be a cube in $d$-dimensional space. Then the maximal size of a system of cubic islands in $C$, is bounded by

$$f'_d(m) \leq \frac{(m+1)^d - 1}{2^d - 1}.$$  

Moreover, equality can be achieved when $m = 2^k - 1$ for some positive integer $k$.

The rest of this paper will be organised as follows. In section 4, we shall prove the upper bound for Theorem 1. In section 5, we shall make some preliminary observations which will help us in the proof of the lower bound. Then we shall prove the lower bound, in section 6. In section 7, we shall classify the minimal members of $\text{Max}(I_M)$ for a cuboid $M$. In sections 8 and 9, we shall prove Theorems 2 and 3 respectively.

4 The upper bound in Theorem 1

To establish one direction of Theorem 1, we show that

$$g_d(m_1, \ldots, m_d) \leq \sum_{i=1}^{d} m_i - (d-1)$$

by exhibiting a system of brick islands of this size. Indeed, for $M = [0,m_1] \times \ldots \times [0,m_d]$, define a set of bricks $H$ by

$$H = \{ [0,m_1] \times \ldots \times [0,m_{i-1}] \times [0,n_i] \times [0,1] \times \ldots \times [0,1] : 1 \leq i \leq d, \ 1 \leq n_i \leq m_i \}.$$  

This defines a system of $\sum_{i=1}^{d} m_i - (d-1)$ nested bricks. Since each of these bricks extends the last by 1 in one dimension, $H$ is a maximal system of brick islands in $M$, which establishes our upper bound on $g_d$.

5 Preliminary results

Working towards the lower bound for $g_d$, we shall start with some observations about maximal systems of brick islands. For $M = [0,m_1] \times \ldots \times [0,m_2]$, we define an elementary cube in $M$ to be a cube of the form $[a_1, a_1 + 1] \times \ldots \times [a_d, a_d + 1]$, where for each $1 \leq i \leq d$, $a_i \in \mathbb{Z}$ and $0 \leq a_i \leq m_i - 1$. We call this cube the elementary cube based at $(a_1, a_2, \ldots, a_d)$. 

3
**Observation 1.** Suppose $I \subset [n]$ and $A$ is an elementary cube based at $(a_1, \ldots, a_d)$ such that $a_i = 0$ whenever $i \in I$. Suppose $A'$ is another elementary cube, based at $(a'_1, \ldots, a'_d)$. If $a'_i = a_i$ for $i \notin I$, and $a'_i \in \{0, 1\}$ for $i \in I$, then every elementary cube that intersects $A$ also intersects $A'$.

For a system of brick islands $H$ in $M$, let $Max(H)$ be the set of maximal elements of $H \setminus \{M\}$ with respect to inclusion.

**Corollary 1.** Let $M = [0, m_1] \times \cdots \times [0, m_d]$ and let $H \in Max(I_M)$. Suppose that $|Max(H)| > 1$ and $R = [r_{1,1}, r_{1,2}] \times \cdots \times [r_{d,1}, r_{d,2}] \in Max(H)$. Then no $r_{i,1}$ is 1, and no $r_{i,2}$ is $m_i - 1$.

**Proof.** Indeed, suppose $r_{1,1} = 1$. Let $R'$ be $[0, r_{1,2}] \times \cdots \times [r_{d,1}, r_{d,2}]$. From the observation above, any elementary cube intersecting $R'$ intersects $R$, and hence no element of $Max(H) \setminus \{R\}$ intersects $R'$. The brick $R'$ cannot be in $H$ already as then we would have $R' = M$ and $|Max(H)| = 1$. This shows that $H$ is not maximal, since we can add $R'$ to it, which is a contradiction.

**Corollary 2.** Let $M = [0, m_1] \times \cdots \times [0, m_d]$ and let $H \in Max(I_M)$. If $|Max(H)| > 1$, every vertex of $M$ is occupied by a member of $Max(H)$.

**Proof.** Given a vertex $v$ of the cuboid $M$, let $C_v$ be the elementary cube which contains $v$. As $H$ is maximal, $Max(H)$ contains some brick $R$ which intersects $C_v$. By the previous result, $R$ must contain $C_v$.

**Corollary 3.** Let $M = [0, m_1] \times \cdots \times [0, m_d]$ and let $H \in Max(I_M)$. Suppose $|Max(H)| > 1$ and $R_1, R_2$ are bricks in $Max(H)$ which intersect an edge $E$ of the cube. There is some section of $E$ between the intersections of $R_1$ and $R_2$ with $E$ - we shall call this the gap between $R_1$ and $R_2$ on $E$. Suppose that no other member of $Max(I_M)$ intersects this gap. Then the length of the gap is at most 2. Further, if the length of the gap is exactly 2, neither of $R_1$, $R_2$ is an elementary cube.

**Proof.** We may assume that $E = \{(x, 0, \ldots, 0) : 0 \leq x \leq m_1\}$. Suppose there is a gap of at least 3 between $R_1$, $R_2$ on $E$ - so no member of $Max(H)$ intersects $\{(x, 0, 0, \ldots, 0) : a \leq x \leq a + 3\}$, for some integer $1 \leq a \leq n - 4$. Then, by applying Corollary 1 three times, we see that the elementary cube based at $(a + 1, 0, 0, \ldots, 0)$ intersects no member of $Max(H)$ - otherwise this member of $Max(H)$ would also intersect the gap between $R_1$ and $R_2$ on $E$. This gives rise to a contradiction - $H$ is not maximal, as we can add this elementary cube to it.

Now, suppose we have a gap of length 2 between $R_1$ and $R_2$ on $E$ - so that $(a, 0, 0, \ldots, 0) \in R_1$, $(a + 2, 0, 0, \ldots, 0) \in R_2$, and no member of $Max(H)$ intersects $\{(x, 0, 0, \ldots, 0) : a \leq x \leq a + 2\}$, for some integer $1 \leq a \leq n - 3$. If $R_1$ was an elementary cube, then we extend it to $R'_1$ by adding in the elementary cube based at $(a, 0, 0, \ldots, 0)$. By Corollary 1 applied to this elementary cube the elementary cube based at $(a + 1, 0, 0, \ldots, 0)$, $R'_1$ intersects no elements of $Max(H)$ other than $R$. This shows that we can add $R'_1$ into $H$, contradicting its maximality.

**Observation 2.** Let $M = [0, m_1] \times \cdots \times [0, m_d]$, $H \in Max(I_M)$, and $R \in H$. Then those bricks in $H$ which are contained in $R$ form a set in $Max(I_R)$. Also,
$M$ itself must be in $H$. In particular, if $R_1, ..., R_k$ are members of $\text{Max}(H)$, where $R_i$ has side length $r_{ij}$ in dimension $j$, then

$$|H| \geq 1 + \sum_{i=0}^{k} g_d(r_{i1}, ..., r_{id})$$

6 Proof of the lower bound in Theorem 1

Now we are ready to prove the main theorem. Given $H \in \text{Max}(H)$, our task is to show that $|H| \geq \sum_{i=1}^{d} m_i - (d - 1)$. We shall proceed by induction on $d$, and within this by induction on $\sum_{i=1}^{d} m_i$. First, we establish a slightly stronger result for $d = 1$.

Lemma 1. Let $M = [0, m] \subset \mathbb{R}$ be a line segment. Then every maximal system of cuboid islands in $M$ has size $m$. In particular, $g_1(m) = m$.

Proof. We prove this by induction on $m$. For $m = 1$, the result is trivial. For $m \geq 2$ suppose that $H \in \text{Max}(I_M)$. There are two different forms that $\text{Max}(H)$ can take - either it is a single interval $[0, m-1]$, or $\text{Max}(H) = \{[0, a], [a+1, m]\}$ for some integer $1 \leq a \leq m - 1$. In either case, we use Observation 2 and apply the induction hypothesis to the members of $\text{Max}(H)$. This shows that $H$ consists of $m - 1$ elements contained in one of the members of $\text{Max}(H)$, together with $M$ itself, and so $|H| = m$.

In $d$ dimensions, our base case is when any side length $m_i$ of $M$ is 1. In this case, the problem reduces immediately to the $(d - 1)$-dimensional case. Using this, we shall assume that $m_i \geq 2$ for all $i$, and that the theorem holds whenever $\sum_{i=1}^{d} m_i$ is reduced. We shall now proceed in three different ways, depending on the configuration of $\text{Max}(H)$ inside $M$. The first two cases deal with special configurations which can arise when $|\text{Max}(H)|$ is small.

6.1 Case 1: $|\text{Max}(H)| = 1$

Without loss of generality,

$$\text{Max}(H) = \{[0, m_1 - 1] \times [0, m_2] \times ... \times [0, m_d]\}$$

Applying the induction hypothesis to the sole member of $\text{Max}(H)$, and using Observation 2, we find that

$$|H| \geq 1 + g_d(m_1 - 1, m_2, ..., m_d) = \sum_{i=1}^{d} m_i + (d - 1)$$

We note that in this case we can get equality.

6.2 Case 2: $|\text{Max}(H)| > 1$, and $\text{Max}(H)$ has an element which divides $M$ into 2 or more regions

Let $R$ be a member of $\text{Max}(H)$ which divides $M$, with

$$R = [r_1, r_2] \times [0, m_2] \times ... \times [0, m_d].$$
Then by Corollary 1, \( r_1 \neq 1 \) and \( r_2 \neq m_1 - 1 \). If \( r_1 = 0 \), then we use observation 2 and apply the induction hypothesis in \( R \) and in \( R = [r_2 + 1, m_1] \times [m_2] \times \ldots [m_d] \), which must be the sole other member of \( \text{Max}(H) \). This gives

\[
|H| \geq 1 + \left( \sum_{i=1}^{d} m_i - m_1 + r_2 - (d - 1) \right) \\
+ \left( \sum_{i=1}^{d} m_i - m_1 + (m_1 - r_2 - 1) - (d - 1) \right) \\
= \left( \sum_{i=1}^{d} m_i - (d - 1) \right) + \left( \sum_{i=1}^{d} m_i - m_1 - (d - 1) \right). 
\]

As every \( m_i \) is at least 2, this shows that \( |H| \) larger than we claim for Theorem 1 and so equality cannot hold in this case.

If, on the other hand, \( 1 < r_1 < r_2 < n - 1 \), then we must have that \( \text{Max}(H) = \{ R, [r_2 + 1, m_1] \times [m_2] \times \ldots [m_d], [0, r_1 - 1] \times [m_2] \times \ldots [m_d] \} \)

Using Observation 2 and applying the induction hypothesis to each of these three bricks, we find that

\[
|H| \geq \left( \sum_{i=1}^{d} m_i - (d - 1) \right) + \left( \sum_{i=1}^{d} m_i - m_1 - (d - 1) \right) + \left( \sum_{i=1}^{d} m_i - 2m_1 - 2(d - 1) - 1 \right). 
\]

Again, using the fact that each \( m_i \) is at least 2, this gives the bound we require for \( |H| \) with strict inequality.

### 6.3 Case 3: \(|\text{Max}(H)| > 1, \text{ and no element of } \text{Max}(H) \text{ divides } M \text{ into 2 regions}\)

We define a path \( P \) around some edges of the cuboid by

\[
P = \{ (0, 0, \ldots, 0, x_i, m_{i+1}, \ldots, m_d) : 1 \leq i \leq d, 0 \leq x_i \leq m_i \} \\
\cup \{ (m_1, m_2, \ldots, m_{i-1}, x_i, 0, \ldots, 0) : 1 \leq i \leq d, 0 \leq x_i \leq m_i \}
\]

We note that \( P \) has two edges in each direction, and that these edges are diametrically opposite each other in \( M \). Hence no brick in \( \text{Max}(H) \) intersects both of these edges, or else it would divide \( M \). The length of \( P \) is \( 2 \sum_{i=1}^{d} m_i \), and \( P \) has \( 2d \) corners with 2 edges incident at each.

Now, consider all the members of \( \text{Max}(H) \) which intersect \( P \). Suppose there are \( k \) of them, \( A_1, \ldots, A_k \), with the \( j \)th dimension edge length of \( A_i \) being denoted \( a_{ij} \). By Corollary 3 the gaps between consecutive bricks on \( P \) are at most 2. Writing \( n_2 \) as the number of gaps of length 2, Corollary 3 tells us that at least \( n_2 \) of the \( A_i \) are not elementary cubes (e.g. the ones after the gaps of length 2). Now, the edges of the bricks which lie on \( P \) have total length

\[
2 \sum_{i=1}^{d} m_i - k - n_2
\]
Also, there are \( k + 2d \) such edges (as there are \( 2d \) corners in \( P \)). Hence the \( A_i \) have between them \( k(d - 1) - 2d \) edges which are not on \( P \) - and so we have that

\[
\sum_{i=1}^{k} \sum_{j=1}^{d} a_{ij} \geq 2 \sum_{i=1}^{d} m_i - k - n_2 + k(d - 1) - 2d
\]

Now, using Observation 2 and applying the inductive hypothesis in each \( A_i \), we obtain

\[
|H| \geq 1 + \sum_{i=1}^{k} g_d(a_{i1}, ..., a_{id}) \geq \sum_{i=1}^{k} \sum_{j=1}^{d} a_{ij} - k(d - 1) + 1
\]

\[
\geq 2 \sum_{i=1}^{d} m_i - k - n_2 - 2d + 1
\]

\[
= \left( \sum_{i=1}^{d} m_i - d + 1 \right) + \left( \sum_{i=1}^{d} m_i - k - n_2 - d \right).
\]

Since the first bracket is the bound we wish to establish for \( H \), this establishes the theorem unless

\[
k + n_2 > \sum_{i=1}^{d} m_i - d.
\]

In this case, we observe that \( H \) contains each of the \( k \) bricks \( A_i \), at least one further brick contained in each \( A_i \) which is not an elementary cube, and \( M \) itself. Since there are at least \( n_2 \) bricks \( A_i \) which are not elementary cubes, we get that

\[
|H| \geq k + n_2 + 1 \geq \sum_{i=1}^{d} m_i - d + 2.
\]

This shows that in this final case Theorem 1 holds with strict inequality.

7 Classification of extremal examples for Theorem 1

When we showed the upper bound for \( g_d(m_1, ..., m_d) \), we gave one example of a smallest possible maximal system. In this section we classify all such systems.

Lemma 2. Let \( M = [0, m_1] \times ... \times [0, m_d] \) and let \( H \in Max(I_M) \). If \( |H| \) is minimal among members of \( Max(I_M) \), \( d \geq 2 \) and \( m_i \geq 2 \) for at least 2 choices of \( i \), then \( |Max(H)| = 1 \).

Proof. We first note that if \( m_d = 1 \), maximal systems of brick islands in \( M \) are precisely those in \( M' = [0, m_1] \times ... \times [0, m_{d-1}] \). Using this, we can work instead in the cuboid given by projecting in all dimensions where the side length of \( M \) is 1. So we shall assume that \( m_i \geq 2 \) for each \( 1 \leq i \leq d \). When \( d = 2 \), this was proved by Lengvárszky [2]. Examining the proof of Theorem 1 we note that for equality to hold we must have the following constraints on \( H \):

1
\[ \text{Max}(H) = \{ A_i : 1 \leq i \leq k \} \]

- Every \( A_i \) is an elementary cube or a brick with all sides of length 1 except for one side of length 2.

- If some side length \( a_{ij} \) of \( A_i \) is greater than 1, then some side of \( A_i \) that lies along \( P \) must be in direction \( j \).

From these last two constraints we can deduce that every elementary cube contained in some \( A_i \) lies on an edge of \( P \). If \( d \geq 3 \) and \( m_i \geq 3 \) for all \( 1 \leq i \leq d \), let \( v \) be some vertex of \( M \) which is not on \( P \), and \( C_v \) be the elementary cube which contains it. Then no brick \( A_i \) intersects \( C_v \). However, by the first constraint there are no other members of \( \text{Max}(H) \); hence we can add \( C_v \) to \( H \), contradicting the maximality of \( H \). This contradiction establishes the lemma whenever \( d \geq 3 \) and \( m_i \geq 3 \) for all \( 1 \leq i \leq d \).

So we may assume that \( d \geq 3 \), and that \( m_d = 2 \). In this case we define sets of bricks \( H_1, H_2, H_{12} \) in \( d - 1 \) dimensions by writing

\[
H = \{ R \times [0,1] : R \in H_1 \} \cup \{ R \times [1,2] : R \in H_2 \} \cup \{ R \times [0,2] : R \in H_{12} \}
\]

We note that no element of \( H_1 \) intersects an element of \( H_2 \). Now, \( H_1 \cup H_2 \cup H_{12} \) is a maximal system of brick islands in the \((d - 1)\)-dimensional cuboid \( M' = [0,m_1] \times \ldots \times [0,m_{d-1}] \), and so by Theorem 1

\[
|H_1 \cup H_2 \cup H_{12}| \geq \sum_{i=1}^{d-1} m_i - d + 2 = \sum_{i=1}^{d} m_i - d.
\]

Thus if we have equality in Theorem 1 for \( H \), then there is at most one intersection between any of \( H_1, H_2 \) and \( H_{12} \). We observe that any minimal member of \( H_{12} \) must be in \( H_1 \cup H_2 \), and any maximal member of \( H_1 \cup H_2 \) must be in \( H_{12} \). So for equality to hold, \( H_{12} \) has a unique minimal element \( R \), which is also the unique maximal element of \( H_1 \cup H_2 \). We also know that \( H_{12} \) has a unique maximal element \( M' \) corresponding to \( M \in H \), and so the bricks in \( H_{12} \) must be nested. If \(|H_{12}| \geq 2 \), then the second largest element of \( H_{12} \) corresponds in \( H \) to the unique element of \( \text{Max}(M) \); if \(|H_{12}| = 1 \), then \( R \) in \( H_1 \cup H_2 \) corresponds in \( H \) to the unique element of \( \text{Max}(H) \). □

Using this lemma, we can classify the minimal elements of \( \text{Max}(I_M) \). A system of brick islands is a minimal element of \( \text{Max}(I_M) \) if and only if it can be obtained by the following procedure:

- Take any brick \( R \) in \( M \) such that every side length of \( R \) is 1 except for one dimension, in which it is \( r \).

- Take any system of \( r \) brick islands in \( R \) (the largest of which is \( R \) itself).

- All other bricks are nested, with \( R \) being the smallest and \( M \) being the largest, such that each brick extends the last by 1 in one direction.

We count the bricks in such a system. There are \( r \) bricks within \( R \), and \( \sum_{i=1}^{d} m_i - (r + d - 1) \) to extend each dimension to \( m_i \), giving a system of the required size. We prove that these are all the minimal elements of \( \text{Max}(I_M) \) by induction on \( \sum_{i=1}^{d} m_i \). The base case is when \( M \) has at most one dimension of size at least 2, in which case we can take \( R = M \). If \( m_i > 1 \) holds for at least 2 of the \( m_i \), then \( \text{Max}(H) \) has a unique element \( H_{\text{max}} \) by lemma 2 □. Applying the induction hypothesis in \( H_{\text{max}} \), we obtain the result for \( M \).
8 Proof of Theorem 2

Before we prove our results about systems of cubic islands, we observe the obvious analogue of Observation 2 for cubic islands.

Observation 3. Let $C = [0, m]^d$, $H \in \text{Max}(I_C)$ and $R \in H$. Then those bricks in $H$ which are contained in $R$ form a set in $\text{Max}(I_R)$. Also, $C$ itself must be in $H$. In particular, if $R_1, ..., R_k$ are the members of $\text{Max}(H)$, where $R_i$ has side length $r_i$, then

$$1 + \sum_{i=0}^{k} g_d'(r_i) \leq |H| \leq 1 + \sum_{i=0}^{k} f_d'(r_i)$$

Now we prove Theorem 2 on the minimal size of maximal systems of cubic islands. We wish to show that $g_d'(m) = m$. We first note that $g_d'(m) \geq m$, as a sequence of $m$ nested cubes is maximal in $C = [0, m]^d$. To prove the upper bound for $g_d'(m)$ we will proceed by induction on $m$. For $m \leq 2$, the theorem is trivial. For $d = 2$, the theorem was proved in [3]. So we shall assume $d \geq 3$ and $m \geq 3$, and that the theorem holds for all $m' \leq m$. Given $C = [0, m]^d$ and $H \in \text{Max}(I_C)$, our task is to show that $|H| \geq m$. We proceed in three different ways, depending on the size of the largest element of $H$.

8.1 Case 1: The system $H$ contains an element of size $m - 1$

In this case, the result follows immediately from the inductive hypothesis, together with Observation 3.

8.2 Case 2: The largest element of $H$ is of size $m - 2$

Denote the element of $H$ of size $m - 2$ by $R$. Consider the bottom left corner of $R$, with coordinates $(r_1, ..., r_d)$, with each of the $r_i$ in $\{0, 1, 2\}$. Note that they are not all 1 - if they were we could extend the system $H$ by adding in a cube of size $[0, m - 1]^d$. Now, for $1 \leq i \leq d$, set

$$a_i = \begin{cases} 0 \text{ if } r_i = 2 \\ m_i - 1 \text{ if } r_i = 0 \text{ or } 1 \end{cases}$$

Then we note that the elementary cube based at $(a_1, ..., a_m)$ does not intersect $R$. This shows that, by the maximality of $H$, there is at least one cube $R'$ other than $R$ in $\text{Max}(H)$. Thus $H$ contains $C$, $R'$ and $(m - 2)$ cubes contained in $R$ (by Observation 3 and the inductive hypothesis applied to $R$). Consequently $|H| \geq m$, as required.

8.3 Case 3: All elements of $H$ are of size at most $m - 3$

Consider the path $P$ as in the proof of Theorem 1;

$$P = \{(0, 0, ..., 0, x_i, m, ..., m) : 1 \leq i \leq d, 0 \leq x_i \leq m\} \cup \{(m, m, ..., m, x_i, 0, ..., 0) : 1 \leq i \leq d, 0 \leq x_i \leq m\}$$

Given two points $p_1$ and $p_2$ on $P$ which are seperated on $P$ by at least two vertices of $C$, and elementary cubes $C_1$ and $C_2$ containing $p_1$ and $p_2$ respectively,
we note that \( p_1 \) and \( p_2 \) differ by \( m \) in (at least) 1 dimension. Hence no cube of side at most \( m - 3 \) can intersect both \( C_1 \) and \( C_2 \).

Let \( A_1, \ldots, A_k \) be those cubes in \( \text{Max}(H) \) which contain a point of the form \( p + (c_1, \ldots, c_d) \), with \( p \in P \) and \( |c_i| \leq 1 \) for \( 1 \leq i \leq d \). We project \( A_i \) on to those points \( p \in P \) for which \( A_i \) has a point of this form. Then each \( A_i \) is projected onto at most 2 edges of \( P \) (which occurs precisely when \( A_i \) is at most 1 away from a corner of \( P \) in every direction). The gaps between adjacent projections are at most 2, very similarly to in the cuboid case - if there is a gap of 3, we can put an elementary cube on \( P \) in the middle of it to extend \( H \). We may also get extra gaps at the 2\( d \) corners of \( P \), as the cubes closest to the corners need not project into them. These gaps have size at most 2. Writing \( a_i \) for the side length of the cube \( A_i \), this gives

\[
2 \sum_{i=1}^{k} a_i + 2k + 4d \geq 2md
\]

Thus at least one of the following must hold:

\[
k \geq m - 1 \tag{1}
\]

\[
\sum_{i=1}^{k} a_i \geq m - 1 \tag{2}
\]

\[
2d - 4 \geq m(d - 2)
\]

However, the last inequality does not hold for any pairs of integers \( m \geq 3 \) and \( d \geq 3 \). If (1) holds, then we note that \( |H| \geq k + 1 \), as each of the \( A_i \) and \( C \) itself are in \( H \). If (2) holds, we use Observation 3 and apply the inductive hypothesis to each \( A_i \) to obtain \( |H| \geq \sum_{i=1}^{k} a_i + 1 \). In either case, we get that \( |H| \geq m \). \( \square \)

9 Proof of Theorem 3

Finally, we prove Theorem 3 on the maximal size of a system of cubic islands. As in the setup of the theorem, let \( C = [m]^d \) and \( H \in \text{Max}(I_C) \). Then our task is to show that \( |H| \leq \frac{(m+1)^d - 1}{2^d - 1} \), and to demonstrate an \( H \) for which equality holds when \( m = 2^k - 1 \). We do the latter first. For \( C_k = [2^k - 1]^d \) we define systems of cubic islands \( H_k \) recursively. \( C_1 \) has \( H = \{C_1\} \). To form \( H_k \), divide \( C_k \) into \( 2^d \) subcubes of side \( 2^k - 1 \) using \( d \) hyperplanes passing through the middle of the cube. Place a copy of \( H_{k+1} \) in each of these subcubes, and add \( C_k \) to obtain \( H_k \). This gives

\[
|H_k| = 2^d |H_{k-1}| + 1
\]

\[
= 2^d \frac{2^{(k-1)d} - 1}{2^d - 1} + 1
\]

\[
= \frac{2^{kd} - 1}{2^d - 1}
\]

as required.

To show \( |H| \leq \frac{(m+1)^d - 1}{2^d - 1} \), we use induction on \( m \). The result is trivial for \( m = 1 \). Given \( m \geq 2 \), we split into 2 cases, depending on the size \( \text{Max}(H) \).
9.1 Case 1: $|Max(H)| = 1$

Here $Max(H)$ has a unique member $R$ of side length $m - 1$. Applying the induction hypothesis in $R$,

$$|H| \leq 1 + \frac{m^d - 1}{2^d - 1} \leq \frac{(m + 1)^d - 1}{2^d - 1}$$

implying the assertion of the theorem.

9.2 Case 2: $|Max(H)| \geq 2$

In this case, $Max(H)$ has no elements of side length greater than $m - 2$. Now, order the vertices of $C$ and consider each in turn. If the elementary cube $C_v$ which contains the vertex $v$ intersects no element of $Max(H)$, then add $C_v$ into $H$. If $C_v$ intersects some element $R$ of $Max(H)$ but is not contained in it, then move $R$ into the corner, together with every cube in $H$ that it contains. Note $R$ cannot have contained any other vertex of $C$, as it has side length at most $m - 2$. After applying this process to every vertex, we have a family $H'$ with $|H'| \geq |H|$, such that every vertex of $C$ is occupied by a different element of $Max(H')$. Hence $|Max(H')| \geq 2^d$. Now we use the same argument as applied in [1] and [3].

Suppose $A_1, ..., A_k$ are the elements of $Max(H')$. Then extend each $A_i$ by $\frac{1}{2}$ in every direction. The interiors of the extended cubes are disjoint, and are contained within an extended version of $C$ of side $m + 1$. Thus their total volume

$$\sum_{i=1}^{k} (a_i + 1)^d$$

is at most $(m + 1)^d$. We write $n(A_i)$ for the number of cubes of $H'$ which are contained in $A_i$, and $a_i$ for the side length of $A_i$. Using Observation 3 and applying the inductive hypothesis in each $A_i$, we get that

$$|H| \leq |H'| \leq 1 + \sum_{i=1}^{k} n(A_i) \leq 1 + \sum_{i=1}^{k} \frac{(a_i + 1)^d - 1}{2^d - 1}$$

$$\leq 1 + \frac{(m + 1)^d - 1}{2^d - 1} - \frac{2^d - 1}{2^d - 1} \leq \frac{(n + 1)^d - 1}{2^d - 1}$$

This is exactly the bound we want on $|H|$, and our proof is complete.

10 Further work

As mentioned in [3], we could consider the problem of cubic islands in a cuboid; the members of our system $H$ would be cubes, while $M$ remains a cuboid with arbitrary sides. While we have got a best polynomial upper bound on $f_d'(m)$, we have not found a reasonable lower bound, and this is another possible extension.

References

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