Quantum-classical duality for Gaudin magnets with boundary

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Abstract

We establish a remarkable relationship between the quantum Gaudin models with boundary and the classical many-body integrable systems of Calogero-Moser type associated with the root systems of classical Lie algebras (B, C and D). We show that under identification of spectra of the Gaudin Hamiltonians $H^G_j$ with particles velocities $\dot{q}_j$ of the classical model all integrals of motion of the latter take zero values. This is the generalization of the quantum-classical duality observed earlier for Gaudin models with periodic boundary conditions and Calogero-Moser models associated with the root system of the type A.

1 Introduction: quantum-classical duality

The quantum-classical duality for integrable systems is a certain relation between the spectrum of an inhomogeneous quantum spin chain (or its limit to a model of the Gaudin type) and intersection of two Lagrangian submanifolds in the $2N$-dimensional phase space of a classical relativistic $N$-body integrable system of the Ruijsenaars-Schneider type (or its non-relativistic limit which is the Calogero-Moser system). Namely, the first Lagrangian manifold is the $N$-dimensional hyperplane corresponding to fixing all coordinates $q_j$ of the classical particles, and...
the second one is the level set of the $N$ integrals of motion in involution. Since their dimensions are complimentary, they intersect in a finite number of points. The essence of the quantum-classical duality is that the values of the particles velocities $\dot{q}_j$ at the intersection points provide spectra of the quantum Hamiltonians of the inhomogeneous spin chain (or the Gaudin model). Different intersection points correspond to different eigenstates of the commuting Hamiltonians.

The relation of this type was independently observed several times from several different viewpoints and using different approaches including the ones related to quantum cohomologies [11], the Knizhnik-Zamolodchikov equations [18], the (modified) Kadomtsev-Petviashvili hierarchy [1] and the quiver gauge theories [9]. For quantum models with rational $R$-matrices, the quantum-classical duality in its most general form was formulated and proved in the paper [12] by means of the algebraic Bethe ansatz method and certain relations for characteristic polynomials of the Lax matrices of the classical integrable many-body systems. In this paper we follow the latter approach.

Let us consider, in more detail, the simplest example illustrating the duality. On the quantum side, we have the $\mathfrak{gl}_2$ Gaudin XXX magnet [10]. The quantum $\mathfrak{gl}_2$ rational Gaudin model describes the spin-exchange type long-range interaction of $N$ spins 1/2 on the complex plane (more precisely, on $\mathbb{CP}^1$). It is defined by the set of commuting Hamiltonians

$$H^G_i = w^{(i)} + \hbar \sum_{k \neq i} \frac{P_{ik}}{z_i - z_k}, \quad i = 1, \ldots, N, \quad (1.1)$$

where $P_{ik}$ is the permutation operator of two complex linear spaces $V_i \cong \mathbb{C}^2$ and $V_k \cong \mathbb{C}^2$ written in either the standard matrix basis $E_{ab}$ ($a, b = 1, 2$) in $\text{Mat}(2, \mathbb{C})$ ($E_{ab}E_{cd} = \delta_{ac}\delta_{bd}$) or in the Pauli matrices basis $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, as follows:

$$P_{ik} = \sum_{a,b=1}^{2} E^{(i)}_{ab} E^{(k)}_{ba} = \frac{1}{2} \sum_{a=0}^{3} \sigma_{a}^{(i)} \sigma_{a}^{(k)}, \quad E^{(i)}_{ab} = 1 \otimes 1 \ldots 1 \otimes E_{ab} \otimes 1 \ldots 1 \otimes 1. \quad (1.2)$$

It acts non-trivially on the $i$-th and $k$-th components of the tensor product $\mathcal{H} = \otimes_{i=1}^{N} V_i, V_i \cong \mathbb{C}^2$, which is the Hilbert space of states of the quantum model. The term $w^{(i)}$ in (1.1) is a constant matrix $w = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ acting non-trivially in the $i$-th tensor component. It is called the twist matrix and $\omega$ is called the twist parameter. Due to commutativity of the Gaudin Hamiltonians, $[H^G_i, H^G_j] = 0$, the eigenvalue problems

$$H^G_i \psi = H^G_i \psi, \quad \psi \in \mathcal{H}, \quad i = 1, \ldots, N \quad (1.3)$$

have a complete set of common solutions. The algebraic Bethe ansatz provides the following answer for the spectrum $H^G_i$ of the Hamiltonians (1.1):

$$H^G_i = \omega + \sum_{k \neq i}^{N} \frac{\hbar}{z_i - z_k} + \sum_{\gamma=1}^{M} \frac{\hbar}{\mu_{\gamma} - z_i}, \quad i = 1, \ldots, N, \quad (1.4)$$

where the set of Bethe roots $\{\mu_{\alpha}, \alpha = 1, \ldots, M\}$ is a solution of the system of $M$ Bethe equations (BE):

$$2\omega + \hbar \sum_{k=1}^{N} \frac{1}{\mu_{\alpha} - z_k} = 2\hbar \sum_{\gamma \neq \alpha}^{M} \frac{1}{\mu_{\alpha} - \mu_{\gamma}}, \quad \alpha = 1, \ldots, M. \quad (1.5)$$
The integer parameter $M \leq \lfloor N/2 \rfloor$ is the number of the overturned spins in the eigenstate $\psi$. For example, if the vacuum vector is chosen to consist of all spins looking up, then $M$ is the number of spins looking down in the state vector $\psi$. More precisely, since the Gaudin Hamiltonians commute with the operator

$$M = \sum_{i=1}^{N} E_{22}^{(i)} = \frac{1}{2} \sum_{i=1}^{N} (\sigma_{0}^{(i)} - \sigma_{3}^{(i)}),$$

the state vector $\psi$ which solves (1.3) is also an eigenvector of the operator $M$, then $M$ is the corresponding eigenvalue: $M \psi = M \psi$.

On the classical side, we deal with the $N$-body rational $gl_N$ Calogero-Moser model [4] (associated with the root system $A_{N-1}$). It is described by the Hamiltonian

$$H_{CM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \sum_{i<j} g \frac{2}{(q_i - q_j)^2} (1.6)$$

with the canonical Poisson brackets between $p_i$ and $q_j$, where $g \in \mathbb{C}$ is the coupling constant. The equations of motion are

$$\dot{q}_i = p_i$$

and

$$\dot{p}_i = \ddot{q}_i = -\sum_{k \neq i} \frac{g^2}{(q_i - q_k)^2} , \quad i = 1, \ldots , N. \quad (1.7)$$

The model is known to be integrable and possessing Lax representation $\hat{L}^{CM} = [L^{CM}, M^{CM}]$. The Lax matrix $L^{CM}$ of size $N \times N$ is [19]

$$L_{ij}^{CM} (\{\dot{q}_j\}, \{q_j\}, g) = \delta_{ij} \dot{q}_i + g \frac{1 - \delta_{ij}}{q_i - q_j}, \quad i, j = 1, \ldots , N. \quad (1.8)$$

Its eigenvalues $\text{Spec}(L^{CM}) = (I_1, \ldots , I_N)$ are integrals of motion. The higher Hamiltonians in involution are given by $H_{CM}^k = \frac{1}{k} \text{tr} (L^{CM})^k = \frac{1}{k} \sum_{i=1}^{N} I_i^k$. In particular, $H_{CM}^1 = H_{CM}^2$. The $M$-matrix is given by

$$M_{ij}^{CM} = \delta_{ij} \sum_{k \neq i} \frac{g}{(q_i - q_k)^2} - (1 - \delta_{ij}) \frac{g}{(q_i - q_j)^2}, \quad i, j = 1, \ldots , N. \quad (1.9)$$

The statement of the quantum-classical duality is as follows. Given the Lax matrix (1.8), consider the substitution

$$q_j = z_j , \quad g = \hbar \quad \text{and} \quad \dot{q}_j = H_j^G , \quad j = 1, \ldots , N, \quad (1.10)$$

where $H_j^G$ are eigenvalues of the Gaudin Hamiltonians given by (1.4). Then, if the $M$ Bethe roots $\mu_\alpha$ satisfy the Bethe equations (1.5), the spectrum of the Lax matrix takes the form

$$\text{Spec} \ L^{CM} (\{H_j^G\}, \{z_j\}, \hbar) \bigg|_{BE} = \{ \omega, \ldots, \omega, -\omega, \ldots, -\omega \} \bigg|_{N-M}, \quad \{ M \} \quad (1.11)$$

that is the eigenvalues of the Lax matrix are equal to the elements of the diagonal twist matrix, with the multiplicities determined by the eigenvalue of the operator $M$ (the multiplicities are...
the numbers of spins looking up and down in the eigenstate $\psi$ of the Hamiltonians $H^G_j$ with the eigenvalues $H^G_j$. In particular, in the case when the twist is absent ($\omega = 0$), all eigenvalues of the Lax matrix are equal to zero, i.e., the matrix consists of Jordan cells. This means that in this case all the Hamiltonians take zero values. Therefore, we see that the spectrum of the Gaudin Hamiltonians $H^G_j$ is indeed reproduced as the $\dot{q}_j = \psi_j$-coordinates of the intersection points of the level zero set of the higher Calogero-Moser Hamiltonians in involution (an $N$-dimensional hypersurface embedded in the phase space) and the $N$-dimensional hyperplane corresponding to the fixed values of the coordinates $q_j = z_j$.

The proof of the quantum-classical duality (i.e., of the statement (1.11)) is technically based on a remarkable determinant identity [12]. Consider the pair of matrices

$$L_{ij} = \delta_{ij} \left( \omega + \sum_{k \neq i}^N \frac{\hbar}{q_i - q_k} + \sum_{\gamma = 1}^M \frac{\hbar}{\mu_{\gamma} - q_i} \right) + (1 - \delta_{ij}) \frac{\hbar}{q_i - q_j}, \quad i, j = 1, \ldots, N \quad (1.12)$$

and

$$\tilde{L}_{\alpha\beta} = \delta_{\alpha\beta} \left( \omega - \sum_{\gamma \neq \alpha}^M \frac{\hbar}{\mu_{\alpha} - \mu_{\gamma}} - \sum_{k = 1}^N \frac{\hbar}{q_k - \mu_{\alpha}} \right) + (1 - \delta_{\alpha\beta}) \frac{\hbar}{\mu_{\alpha} - \mu_{\beta}}, \quad \alpha, \beta = 1, \ldots, M \quad (1.13)$$

Then the following relation between the characteristic polynomials of these matrices holds:

$$\det_{N \times N} \left( L - \lambda I \right) = (\omega - \lambda)^N \det_{M \times M} \left( \tilde{L} - \lambda I \right). \quad (1.14)$$

(Here $I$ is the identity matrix.) The matrix $L$ is the Lax matrix (1.8) with the identification $\dot{q}_i = H^G_i$ (where the parameters $\{\mu_\gamma\}$ entering $H^G_i$ are regarded as free parameters, i.e., here they are not supposed to satisfy the Bethe equations). Identities of this type also find their applications in computation of the scalar products of Bethe vectors in quantum integrable models solved by the algebraic Bethe ansatz [3] (see also [6, 23], where similar relations are applied to the Gaudin models). The proof of the determinant identities uses factorization formulae for the Lax matrices [25] which take their origin as classical limits (and rational degenerations) of expressions for the factorized $L$-operators through intertwining vectors participating in the so-called vertex-IRF correspondence [13, 20]. Imposing the Bethe equations in the elements of the matrix $\tilde{L}$ and using the identity (1.14) again, one arrives at $\det \left( L - \lambda I \right) = (\omega - \lambda)^N (-\omega - \lambda)^M$ which is (1.11).

In this paper we suggest an extension of (1.12)–(1.14) for other root systems (see (4.10) and (4.15) below).

Let us remark that a natural way to understand the quantum-classical duality and to explain the identification (1.10) is to consider a “non-stationary version” of the Gaudin spectral problems which is the set of the Knizhnik-Zamolodchikov equations

$$\kappa \partial_{z_i} \Psi = H_i^G \Psi, \quad \Psi \in \mathcal{H}, \quad i = 1, \ldots, N \quad (1.15)$$

In the stationary (“quasiclassical”) limit $\kappa \to 0$ the solutions to (1.15) admit the expansion [21]: $\Psi = (\Psi_0 + \kappa \Psi_1 + \ldots) e^{S/\kappa}$ with some function $S = S(z_1, \ldots, z_N)$. Then, in the leading order in $\kappa$, we get the spectral problems (1.13) with $\dot{\psi} = \Psi_0$ and $H_i^G = \partial_{z_i} S$. The explicit constructions of
the Matsuo-Cherednik type\footnote{The Matsuo-Cherednik construction \cite{18} is a projection of solutions to the Knizhnik-Zamolodchikov equations \eqref{1.15} onto some special vector \( \langle \Omega \rangle \) in such a way that \( \langle \Omega \vert \Psi \rangle \) become eigenfunctions of the quantized version of the many-body Hamiltonian \eqref{1.6} of the Calogero-Moser type.} reproducing the quantum-classical duality in the stationary limit are discussed in \cite{8}.

**Purpose of the paper** is as follows. It is widely known that the Calogero-Moser model \eqref{1.6}–\eqref{1.8} is related to the root system \( A_{N-1} \) and that it can be extended to other root systems of simple Lie algebras \cite{19}. In this paper we show that the Calogero-Moser models associated with the other classical root systems, i.e. with the root systems of BCD types, are quantum-classically dual to the quantum Gaudin magnets with boundary. Note that magnets of this type previously appeared in the context of the Knizhnik-Zamolodchikov equations \cite{5}. The Matsuo-Cherednik construction provides a relation between the quantum Calogero-Moser models of BCD types and the boundary Knizhnik-Zamolodchikov equations. In this respect the relation between the classical Calogero-Moser models of BCD types and the quantum Gaudin magnets with boundary could be anticipated. Our aim is to describe it explicitly using the algebraic Bethe ansatz method and determinant identities of the type \eqref{1.14} for the classical Lax matrices.

Let us describe the structure of the paper. The Calogero-Moser models for the classical root systems B, C and D are discussed in Section 2. Next, in Section 3, we proceed to the quantum Gaudin magnets with boundary. We present their solutions following \cite{14}. The new result, the duality between the quantum Gaudin magnets with boundary and the classical Calogero-Moser models associated with the root systems B, C and D, is the subject of Section 4. This result is established by means of nontrivial determinant identities for characteristic polynomials of the Lax matrices of the Calogero-Moser models which generalize \eqref{1.14} to the other root systems. A sketch of proof of the determinant identities is given in the appendix. Section 5 contains concluding remarks and directions for further study.

## 2 Calogero-Moser models for classical root systems

Following \cite{19}, let us consider an extension of the Calogero-Moser model with the Hamiltonian

\[
H = \frac{1}{2} \sum_{a=1}^{N} p_a^2 - g_2^2 \sum_{a<b} \frac{1}{(q_a - q_b)^2} + \frac{1}{(q_a + q_b)^2} - g_3^2 \sum_{a=1}^{N} \frac{1}{(2q_a)^2} - g_1^2 \sum_{a=1}^{N} \frac{1}{q_a^2} \tag{2.1}
\]

depending on three coupling constants \( g_2, g_3 \) and \( g_1 \). Notice that in the rational case the parameters \( g_1 \) and \( g_3 \) can be unified so that the Hamiltonian \eqref{2.1} essentially depends on two continuous parameters rather than three. However, the Hamiltonian \eqref{2.1} with three coupling constants naturally appears as the rational limit of its trigonometric version in which all the three parameters are independent.

The Lax pair for the model \eqref{2.1} is the following pair of \((2N+1) \times (2N+1)\) block-matrices:

\[
L = \begin{pmatrix}
P + A & B & C \\
-B & -P - A & -C \\
-C^T & -P - A & -C^T 
\end{pmatrix}, \quad M = \begin{pmatrix}
\hat{A} + d & \hat{B} & \hat{C} \\
-\hat{B} & -\hat{A} + d & -\hat{C} \\
-\hat{C}^T & -\hat{A}^T & d_0
\end{pmatrix}. \tag{2.2}
\]

Here \( P, A, B \) are matrices of size \( N \times N \), \( C \) is a column of length \( N \) with the entries

\[
P_{ab} = \dot{q}_a \delta_{ab}, \quad A_{ab} = \frac{g_2(1 - \delta_{ab})}{q_a - q_b}, \quad (C)_a = \frac{g_1}{q_a}, \quad B_{ab} = \frac{g_2(1 - \delta_{ab})}{q_a + q_b} + \frac{g_4 \sqrt{2} \delta_{ab}}{2q_a} \tag{2.3}
\]
and
\[ \hat{A}_{ab} = -\frac{g_2(1 - \delta_{ab})}{(q_a - q_b)^2}, \quad (\hat{C})_a = -\frac{g_1}{q_a^2}, \quad \hat{B}_{ab} = -\frac{g_2(1 - \delta_{ab})}{(q_a + q_b)^2} - \frac{g_4\sqrt{2}\delta_{ab}}{2q_a^2}, \]
(2.4)
where \(a, b = 1, \ldots, N\). In the \(M\)-matrix one also has the diagonal matrix \(d_{ab} = \delta_{ab}d_a\) with
\[ d_a = \frac{g_2}{2} \frac{1}{q_a^2} + g_4\sqrt{2} \frac{1}{(2q_a)^2} + g_2 \sum_{c \neq a} \left( \frac{1}{(q_a - q_c)^2} + \frac{1}{(q_a + q_c)^2} \right), \quad d_0 = 2g_2 \sum_{c=1}^{N} \frac{1}{q_c^2}. \]
(2.5)
The superscript \(T\) denotes transposition, and the checks in the \(M\)-matrix mean the derivatives of the corresponding functions entering the Lax matrix.

The Lax equation \(\dot{L} = [L, M]\) with the Lax pair (2.2)–(2.5) is equivalent to the equations of motion generated by the Hamiltonian (2.1) \(H = (1/4)\text{tr}L^2\) if the additional constraint for the coupling constants \(g_2, g_4\) and \(g_1\) holds true:
\[ g_1(g_1^2 - 2g_2^2 + \sqrt{2}g_2g_4) = 0. \]
(2.6)
The higher Hamiltonians in involution are \(H_k = \frac{1}{2k} \text{tr}L^k\).

The following special cases satisfying (2.6) correspond to the models associated with the classical root systems:

- \(B_N (so_{2N+1})\): \(g_4 = 0, g_1^2 = 2g_2^2\), the size of the Lax matrix is \((2N + 1) \times (2N + 1)\);
- \(C_N (sp_{2N})\): \(g_1 = 0, \) the size of the Lax matrix is \(2N \times 2N\); \(\quad \) (2.7)
- \(D_N (so_{2N})\): \(g_1 = 0, g_4 = 0\), the size of the Lax matrix is \(2N \times 2N\).

Notice that in the \(C_N\) and \(D_N\) cases the Lax matrix has effective size \(2N \times 2N\), since in these cases we can remove the row and the column proportional to \(g_1\). The redefinition \(M \rightarrow M - d_01_{2N+1}\) (here \(1_{2N+1}\) is the matrix with the only nonzero element 1 at the south-east corner) which does not spoil the Lax equation makes the effective size of the \(M\)-matrix the same \((2N \times 2N)\).

Let us also mention that the Hamiltonian (2.1) can be straightforwardly generalized to the trigonometric (and elliptic) case \([19]\), where the constants \(g_1\) and \(g_4\) are not unified into a single combination \(g_1^2 + g_4^2/4\). The model (2.1) is associated with the \(BC_N\) root system, and the classical root systems (2.7) appear as its particular cases: \(g_4 = 0\) for \(B_N\), \(g_1 = 0\) for \(C_N\) and \(g_1 = g_4 = 0\) for \(D_N\). Moreover, the model (2.1) is integrable for all arbitrary constants without restriction (2.6), see [7]. In this paper we deal with the Lax representation (2.2) (which requires (2.6)) since we found the determinant identities for these Lax matrices. This is why the case \(B_N\) requires separate consideration (the Lax matrix is of size \((2N + 1) \times (2N + 1)\)), while \(D_N\) can be considered as a particular case of \(C_N\) model with \(g_4 = 0\).

**Factorization formulae.** In what follows we need the factorization formulae for the Lax matrix (2.2). Let us briefly recall the corresponding results from [25].

Factorization for the \(C_N\) and \(D_N\) root systems. Introduce a set of \(2N \times 2N\) matrices. The first one is the diagonal matrix
\[ D_{ij}^0 = \delta_{ij} \left\{ \begin{array}{ll}
2q_i \prod_{k \neq i}^{N} (q_i - q_k)(q_i + q_k), & i \leq N, \\
-2q_{i-N} \prod_{k \neq i-N}^{N} (q_{i-N} - q_k)(q_{i-N} + q_k), & N + 1 \leq i \leq 2N,
\end{array} \right. \]
(2.8)
the next one is the Vandermonde matrix
\[
V_{ij} = \begin{cases} 
q_i^{j-1}, & i \leq N, \\
(q_i-N)^{j-1}, & N+1 \leq i \leq 2N,
\end{cases}
\tag{2.9}
\]
and finally we introduce two nilpotent matrices
\[
(C_0)_{ij} = \begin{cases} 
\delta_{j,1}, & i = 2N+1, \\
j - 1 & \text{if } j = i + 1, \\
0 & \text{otherwise,}
\end{cases}
\quad (\tilde{C})_{ij} = \begin{cases} 
1 & \text{if } j = i + 1 \text{ and } j \text{ even,} \\
0 & \text{otherwise.}
\end{cases}
\tag{2.10}
\]
They are strictly triangular matrices with zeros on the main diagonal. Consider the $2N \times 2N$ matrix $L'$ obtained from the Lax matrix \(2.2\) in the $C_N$ case \(2.7\) as follows. Set the coupling constants $g_1 = 0$, $g_2 = \hbar$, $g_4 = \sqrt{2}\hbar\xi$ and make the substitutions
\[
\dot{q}_i \to \frac{\xi \hbar}{q_i} + \sum_{k \neq i}^{N} \left( \frac{\hbar}{q_i - q_k} + \frac{\hbar}{q_i + q_k} \right), \quad i = 1, \ldots, N.
\tag{2.11}
\]
Then the matrix $L'$ is represented in the form
\[
L' = \hbar (D^0)^{-1} V(C_0 - (1 - 2\xi)\tilde{C}) V^{-1} D^0.
\tag{2.12}
\]
For $\xi = 0$ \(2.12\) yields the $D_N$ case.
Factorization for the $B_N$ root system. Introduce the following set of $(2N+1) \times (2N+1)$ matrices:
\[
D^0_{ij} = \delta_{ij} \begin{cases} 
\sqrt{2}q_i^2 \prod_{k \neq i}^{N} (q_i - q_k)(q_i + q_k), & i \leq N, \\
\sqrt{2}q_i^{2N} \prod_{k \neq i-N}^{N} (q_i-N - q_k)(q_i-N + q_k), & N+1 \leq i \leq 2N, \\
\prod_{k=1}^{N} (-q_k^2), & i = 2N+1
\end{cases}
\tag{2.13}
\]
and
\[
V_{ij} = \begin{cases} 
q_i^{j-1}, & i \leq N, \\
(q_i-N)^{j-1}, & N+1 \leq i \leq 2N, \\
\delta_{j,1}, & i = 2N+1
\end{cases}
\tag{2.14}
\]
Consider the $(2N+1) \times (2N+1)$ matrix $L''$ obtained from the Lax matrix \(2.2\) in the $B_N$ case \(2.7\) as follows. Set the coupling constants $g_1 = \sqrt{2}\hbar$, $g_2 = \hbar$, $g_4 = 0$ and make the substitutions
\[
\dot{q}_i \to \frac{2\hbar}{q_i} + \sum_{k \neq i}^{N} \left( \frac{\hbar}{q_i - q_k} + \frac{\hbar}{q_i + q_k} \right), \quad i = 1, \ldots, N.
\tag{2.15}
\]
Then the matrix $L''$ is represented in the factorized form:
\[
L'' = \hbar (D^0)^{-1} V(C_0 + \tilde{C}) V^{-1} D^0,
\tag{2.16}
\]
where $C_0$ and $\tilde{C}$ are the matrices defined in \(2.10\) but of the size $(2N+1) \times (2N+1)$.
\[\]
The proofs of the factorization formulae \(2.12\), \(2.16\) can be found in Appendix B of the paper [25].
3 Gaudin model with boundary

We start from the inhomogeneous open XXX spin chain with the Yang’s $R$-matrix $R^{\eta}_{12}(u) = 1 \otimes 1 + (\eta/u)P_{12}$ satisfying the Yang-Baxter equation

$$R^{\eta}_{12}(u_1 - u_2)R^{\eta}_{13}(u_1)R^{\eta}_{23}(u_2) = R^{\eta}_{23}(u_2)R^{\eta}_{13}(u_1)R^{\eta}_{12}(u_1 - u_2),$$  \hspace{1cm} (3.1)

where the both sides are matrices acting in the space $V_1 \otimes V_2 \otimes V_3 \cong (\mathbb{C}^2)^{\otimes 3}$. According to the general theory of integrable models with boundary [22], one should also consider $K$-matrices $K^\pm$ which solve the reflection equations

$$R^{\eta}_{12}(u_1 - u_2)K^-_1(u_1)R^{\eta}_{12}(u_1 + u_2)K^-_2(u_2) = K^-_2(u_2)R^{\eta}_{12}(u_1 + u_2)K^-_1(u_1)R^{\eta}_{12}(u_1 - u_2),$$

$$R^{\eta}_{12}(-u_1 + u_2)(K^+_1)^{T_1}(u_1)R^{\eta}_{12}(-u_1 - u_2 - 2\eta)(K^+_2)^{T_2}(u_2) =$$

$$= (K^+_2)^{T_2}(u_2)R^{\eta}_{12}(-u_1 - u_2 - 2\eta)(K^+_1)^{T_1}(u_1)R^{\eta}_{12}(-u_1 + u_2).$$  \hspace{1cm} (3.2)

Here $K^\pm_1 = K^\pm \otimes 1$, $K^\pm_2 = 1 \otimes K^\pm$ and $T_{1,2}$ mean transpositions in the corresponding tensor components. In this paper we consider diagonal $K$-matrices of the form

$$K^{-}(u) = \begin{pmatrix} 1 + \frac{\alpha \eta}{u} & 0 \\ 0 & -1 + \frac{\alpha \eta}{u} \end{pmatrix}, \quad K^{+}(u) = \begin{pmatrix} 1 - \frac{\beta \eta}{u + \eta} & 0 \\ 0 & -1 - \frac{\beta \eta}{u + \eta} \end{pmatrix}$$  \hspace{1cm} (3.3)

with arbitrary parameters $\alpha, \beta$. The transfer matrix of the integrable spin chain with boundaries with the Hilbert space of states $\mathcal{H} = \otimes_{i=1}^{N}V_i \cong (\mathbb{C}^2)^{\otimes N}$ is given by

$$T(u) = \text{tr}_0\left( K^+_0(u)R^\eta_{01}(u - z_1)...R^\eta_{0N}(u - z_N)K^-_0(u)R^\eta_{0N}(u + z_N)...R^\eta_{01}(u + z_1) \right),$$  \hspace{1cm} (3.4)

where the trace $\text{tr}_0$ is taken in the auxiliary space $V_0 \cong \mathbb{C}^2$. This transfer matrix differs from the Sklyanin’s one [22] by an inessential common scalar factor. The parameters $\{z_k\}$ in (3.4) are inhomogeneity parameters. The Yang-Baxter equation together with the reflection equations imply that $T(u)$ is a commutative family of operators: $[T(u), T(v)] = 0$ for any $u, v$.

The limit to the Gaudin model [14] is the limit $\varepsilon \to 0$ in (3.4) performed after the substitution $\eta = \varepsilon h$. As one can readily check, in this limit

$$T(u) = 2 + \varepsilon h\gamma(u) + \varepsilon^2 h^2T^G(u) + O(\varepsilon^3),$$  \hspace{1cm} (3.5)

where $\gamma(u) = \sum_i \left( \frac{1}{u - z_i} + \frac{1}{u + z_i} \right)$ is a scalar function and

$$T^G(u) = -\frac{2\alpha\beta}{u^2} + \frac{1}{h} \sum_{i=1}^{N} \left( \frac{H^G_{ii}}{u - z_i} - \frac{H^G_{ii}}{u + z_i} \right),$$  \hspace{1cm} (3.6)

where

$$\frac{1}{h} H^G_{ii} = \frac{\xi \sigma^{(i)}_3}{z_i} + \sum_{k \neq i}^{N} \left( \frac{P_{ik}}{z_i - z_k} + \frac{\sigma^{(i)}_3 P_{ik} \sigma^{(i)}_3}{z_i + z_k} \right), \quad \xi = \alpha - \beta$$  \hspace{1cm} (3.7)
are Hamiltonians of the Gaudin model with boundary. Note that the boundary parameters $\alpha$, $\beta$ enter here in the combination $\xi = \alpha - \beta$, so effectively there is only one boundary parameter in the Gaudin limit instead of two. More general Gaudin models with boundary are discussed in [15] and [16].

The commutativity of the transfer matrices implies that the Gaudin Hamiltonians commute: $[H^G_i, H^G_j] = 0$. Therefore, one can find a complete set of common solutions to the eigenvalue problems $H^G_i \psi = H^G_i \psi$ for the Hamiltonians (3.7). The algebraic Bethe ansatz provides the following solution:

$$\frac{1}{\hbar} H^G_i = \frac{1}{\hbar} H^G_i (\{z_k\}_N, \{\mu_\gamma\}_M, \xi)$$

$$= \frac{\xi}{z_i} + \sum_{k \not= i}^N \left( \frac{1}{z_i - z_k} + \frac{1}{z_i + z_k} \right) - \sum_{\gamma = 1}^M \left( \frac{1}{z_i - \mu_\gamma} + \frac{1}{z_i + \mu_\gamma} \right).$$

(3.8)

The solution depends on the set $\{z_k\}_N$ of $N$ inhomogeneity parameters and the set $\{\mu_\gamma\}_M$ of $M \leq [N/2]$ Bethe roots. (As in the quasiperiodic case, $M$ is equal to the number of overturned spins in the eigenstate $\psi$). The Bethe roots are solutions of the system of $M$ Bethe equations

$$\frac{2 \xi}{\mu_\gamma} + \sum_{k = 1}^N \left( \frac{1}{\mu_\gamma - z_k} + \frac{1}{\mu_\gamma + z_k} \right) = \frac{2}{\mu_\gamma} + \sum_{c \not= \gamma}^M \left( \frac{2}{\mu_\gamma - \mu_c} + \frac{2}{\mu_\gamma + \mu_c} \right), \quad \gamma = 1, \ldots, M.$$

(3.9)

Different solutions to this algebraic system correspond to different eigenstates of the Gaudin Hamiltonians.

As is shown in the next section, this model is quantum-classically dual to the Calogero-Moser models associated with the root systems $C_N$ and $D_N$. In the case of the $B_N$ root system we consider the Gaudin model (3.7) with $N+1$ spins and the following conditions: $\xi = 0, z_{N+1} = 0$. Then the Gaudin Hamiltonians (3.7) take the form:

$$\frac{1}{\hbar} \tilde{H}^G_i = \frac{1}{\hbar} P_{i,N+1}^{\mu_1, \ldots, \mu_N} + \frac{1}{\hbar} \frac{\sigma_3^{(i)} P_{i,N+1} \sigma_3^{(i)}}{z_i} + \sum_{k \not= i}^N \left( \frac{P_{ik} \sigma_3^{(i)}}{z_i - z_k} + \frac{\sigma_3^{(i)} P_{ik} \sigma_3^{(i)}}{z_i + z_k} \right), \quad i = 1, \ldots, N.$$

(3.10)

In the case $z_{N+1} = 0$ the Hamiltonian $H^G_{i,N+1}$ disappears from the transfer matrix. For the eigenvalues of the Hamiltonians (3.10) we have:

$$\frac{1}{\hbar} \tilde{H}^G_i (\{z_k\}_N, \{\mu_\gamma\}_M) = \frac{2}{z_i} + \sum_{k \not= i}^N \left( \frac{1}{z_i - z_k} + \frac{1}{z_i + z_k} \right) - \sum_{\gamma = 1}^M \left( \frac{1}{z_i - \mu_\gamma} + \frac{1}{z_i + \mu_\gamma} \right),$$

(3.11)

and the Bethe equations are of the form

$$\sum_{k = 1}^N \left( \frac{1}{\mu_\gamma - z_k} + \frac{1}{\mu_\gamma + z_k} \right) = 2 \sum_{c \not= \gamma}^M \left( \frac{1}{\mu_\gamma - \mu_c} + \frac{1}{\mu_\gamma + \mu_c} \right), \quad \gamma = 1, \ldots, M.$$

(3.12)

\(^2\)For simplicity, we use the same notation for them as for the Gaudin Hamiltonians in the quasiperiodic case (1.1) discussed in the introduction. This can not lead to a misunderstanding since the latter will not appear in what follows.
4 Statement of duality

Now we are ready to formulate the main result of the paper.

**Theorem.** Let us identify the marked points \( z_i \) in the Gaudin model with coordinates of the Calogero-Moser particles

\[
z_j = q_j, \quad j = 1, \ldots, N
\]

and make the substitution

\[
\dot{q}_j = H^G_j \quad \text{or} \quad \dot{q}_j = \tilde{H}^G_j, \quad j = 1, \ldots, N
\]

in the Lax matrix (2.2), which we denote as \( L(\{\dot{q}_j\}, \{q_j\}| g_1, g_2, g_4) \). Here \( H^G_j \) and \( \tilde{H}^G_j \) are eigenvalues of the Gaudin Hamiltonians (3.8) and (3.11) respectively. For the classical root systems (2.7) set the coupling constants as follows:

- For \( B_N \): \( \tilde{H}^G_j \) from (3.11), \( g_1 = \sqrt{2}\hbar, \ g_2 = \hbar, \ g_4 = 0 \), the Lax matrix size is \((2N + 1) \times (2N + 1)\);
- For \( C_N \): \( H^G_j \) from (3.8), \( g_1 = 0, \ g_2 = \hbar, \ g_4 = \sqrt{2}\hbar \xi \), the Lax matrix size is \(2N \times 2N\);
- For \( D_N \): \( H^G_j \) from (3.8) with \( \xi = 0, \ g_1 = 0, \ g_2 = \hbar, \ g_4 = 0 \), the Lax matrix size is \(2N \times 2N\).

If, for any \( M \leq [N/2] \), the Bethe roots \( \{\mu_\gamma\} \) satisfy the Bethe equations (more precisely, (3.12) for the \( B_N \) case, (3.9) for the \( C_N \) case and (3.9) with \( \xi = 0 \) for the \( D_N \) case), i.e., \( H^G_j \) or \( \tilde{H}^G_j \) belong to the spectrum of the Gaudin model, then all eigenvalues of the Lax matrix \( L(\{H^G_j\}, \{q_j\}| g_1, g_2, g_4) \) or \( L(\{\tilde{H}^G_j\}, \{q_j\}| g_1, g_2, g_4) \) and, therefore, all the integrals of motion, are equal to zero.

This result means that the spectrum of the Hamiltonians of the quantum Gaudin magnets with boundary is reproduced from the intersection of two Lagrangian manifolds in the phase space of the \( N \)-body Calogero-Moser models associated with the classical root systems. One of these Lagrangian manifolds is the hyperplane corresponding to fixing all coordinates of the particles and the other one is the level zero set of \( N \) higher classical Hamiltonians in involution, i.e., the \( N \)-dimensional hypersurface obtained by putting all the integrals of motion equal to zero. This result looks similarly to the particular case for the \( A_{N-1} \) root system (the case when the twist matrix in the Gaudin model is absent) discussed in [18].

The proof of the quantum-classical duality between the Calogero-Moser models of the type (2.1) and the Gaudin models (3.7), (3.10) is based on the nontrivial determinant identities which are discussed below.

**The determinant identities.** The statement of the theorem is equivalent to the following relations which are valid assuming that the parameters \( \{\mu_\gamma\} \) satisfy the Bethe equations (i.e., that the Bethe vectors \( \psi \) are “on-shell”), so that \( H^G_j \) and \( \tilde{H}^G_j \) do belong to the spectrum of the Gaudin models:

\[
\det_{(2N+1) \times (2N+1)} \left[ L(\{H^G_j\}, \{q_j\}| \sqrt{2}\hbar, \hbar, 0) - \lambda I \right] = -\lambda^{2N+1} \tag{4.3}
\]

for the \( B_N \) case and

\[
\det_{2N \times 2N} \left[ L(\{H^G_j\}, \{q_j\}| 0, \hbar, \sqrt{2}\hbar \xi) - \lambda I \right] = \lambda^{2N} \tag{4.4}
\]
The coupling constants regarded here as free parameters (i.e., they are not supposed to satisfy the Bethe equations). In (4.3) $H_j^G$ is the set $H_j^G\{q\}_N, \{\mu\}_M$ from (3.11), and in (4.4) $H_j^G = H_j^G\{q\}_N, \{\mu\}_M, \xi$ from (3.8).

In order to prove (4.3)–(4.4) we need certain determinant identities.

The identity for the $B_N$ case. Let us introduce the notation

$$\mathcal{L} = L\left(\{\tilde{H}_j^G\}_N, \{q\}_N | \sqrt{2h}, \hbar, 0\right), \quad (4.5)$$

This is just the Lax matrix (2.2) of the $B_N$ type and of size $(2N + 1) \times (2N + 1)$ in which the set $\{\tilde{H}_j^G\}_N$ is taken from (3.11). Along with $\{q\}_N$, it depends on $M$ variables $\{\mu_\gamma\}$ which are regarded here as free parameters (i.e., they are not supposed to satisfy the Bethe equations). The coupling constants $g_1 = \sqrt{2}h, g_2 = h, g_4 = 0$ are the ones given in the argument in the r.h.s. of (4.5). Next, introduce the $2M \times 2M$ matrix

$$\tilde{\mathcal{L}} = L\left(\{-H_\alpha^G\} \{\mu_\gamma\}_M, \{q_j\}_N, \xi = -1\right) | 0, h, \sqrt{2h}\right), \quad (4.6)$$

where the arguments $\{q\}_N$ and $\{\mu\}_M$ are interchanged in the expression (4.8). More precisely, $\tilde{\mathcal{L}}$ is the block-matrix

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{B} & -\tilde{A} \end{pmatrix}, \quad \tilde{A}, \tilde{B} \in \text{Mat}(M, \mathbb{C}), \quad (4.7)$$

where for $i, j = 1, \ldots, M$

$$\tilde{A}_{ij} = \delta_{ij} \left[ \frac{h}{\mu_i} + \sum_{k=1}^{N} \left( \frac{h}{\mu_i - q_k} + \frac{h}{\mu_i + q_k} \right) - \sum_{\mu \neq \nu} \left( \frac{h}{\mu_i - \mu} + \frac{h}{\mu_i + \mu} \right) \right] + \frac{h(1 - \delta_{ij})}{\mu_i - \mu_j} \quad (4.8)$$

and

$$\tilde{B}_{ij} = \delta_{ij} \frac{h}{\mu_i} + (1 - \delta_{ij}) \frac{h}{\mu_i + \mu_j}. \quad (4.9)$$

Then the identity is as follows:

$$\det_{(2N+1) \times (2N+1)} \left( \mathcal{L} - \lambda I \right) = -\lambda^{2N-2M+1} \det_{2M \times 2M} \left( \tilde{\mathcal{L}} - \lambda I \right). \quad (4.10)$$

**Example.** Let us give a simple example for $N = M = 1$:

$$\mathcal{L} = \begin{pmatrix} \tilde{H}_1^G\{q\}_1, \{\mu\}_1 & 0 & \sqrt{2h} \\ 0 & -\tilde{H}_1^G\{q\}_1, \{\mu\}_1 & -\sqrt{2h} \\ -\frac{\sqrt{2h}}{q_1} & \frac{\sqrt{2h}}{q_1} & 0 \end{pmatrix}, \quad (4.11)$$

where $\tilde{H}_1^G\{q\}_1, \{\mu\}_1 = 2\frac{h}{q_1} - \frac{h}{q_1 - \mu_1} - \frac{h}{q_1 + \mu_1}$.

$$\tilde{\mathcal{L}} = \begin{pmatrix} -H_1^G\{\mu\}_1, \{q\}_1, 1 & \frac{h}{\mu_1} & \frac{\sqrt{2h}}{q_1} \\ -\frac{h}{\mu_1} & H_1^G\{\mu\}_1, \{q\}_1, 1 & 0 \end{pmatrix}, \quad (4.12)$$
where \( H^G_1(\{\mu_1\}, \{q\}_1, -1) = -\frac{\hbar}{\mu_1} - \frac{\hbar}{\mu_1 - q_1} - \frac{\hbar}{\mu_1 + q_1} \). One can directly calculate the characteristic polynomials of these matrices and find that (4.10) holds true.

The identity for the \( C_N \) and \( D_N \) cases. Similarly, let us introduce the \( 2N \times 2N \) matrix

\[
\mathcal{L} = L(\{H_j^G\}_N(\{q\}_N, \{\mu\}_M, \xi), \{q\}_N|0, \hbar, \sqrt{2}\hbar \xi) ,
\]

(4.13)

which is the matrix (2.2) of the type \( C_N \) with the set \( \{H_j^G\}_N = \{H_j^G\}_N(\{q\}_N, \{\mu\}_M, \xi) \) from (3.8) depending on the \( M \) independent variables \( \{\mu\}_M \). The dual matrix \( \tilde{\mathcal{L}} \) has size \( 2M \times 2M \):

\[
\tilde{\mathcal{L}} = L(\{H_j^G\}_M(\{\mu\}_M, \{q\}_N, 1 - \xi), \{\mu\}_M|0, \hbar, \sqrt{2}\hbar (1 - \xi)) .
\]

(4.14)

The arguments of the eigenvalues of the Gaudin Hamiltonians are interchanged and the coupling constants are different. For the pair of matrices (4.13), (4.14) the identity is

\[
\det_{2N \times 2N}(\mathcal{L} - \lambda I) = \lambda^{2N-2M} \det_{2M \times 2M}(\tilde{\mathcal{L}} - \lambda I) .
\]

(4.15)

**Example.** Let us give a simple example for \( N = 2, M = 1 \). We have:

\[
\mathcal{L} = 
\begin{pmatrix}
H^G_1(\{q\}_2, \{\mu\}_1, \xi) & \frac{\hbar}{q_1 - q_2} & \frac{\hbar \xi}{q_1} & \frac{\hbar}{q_1 + q_2} \\
\frac{\hbar}{q_2 - q_1} & H^G_2(\{q\}_2, \{\mu\}_1, \xi) & \frac{\hbar}{q_1 + q_2} & \frac{\hbar \xi}{q_2} \\
-\frac{\hbar \xi}{q_1} & -\frac{\hbar}{q_1 + q_2} & -H^G_1(\{q\}_2, \{\mu\}_1, \xi) & \frac{\hbar}{q_2 - q_1} \\
-\frac{\hbar}{q_1 + q_2} & -\frac{\hbar \xi}{q_2} & \frac{\hbar}{q_1 - q_2} & -H^G_2(\{q\}_2, \{\mu\}_1, \xi)
\end{pmatrix}
\]

(4.16)

where

\[
H^G_1(\{q\}_2, \{\mu\}_1, \xi) = \frac{\xi \hbar}{q_1} + \frac{\hbar}{q_1 - q_2} + \frac{\hbar}{q_1 + q_2} - \frac{\hbar}{q_1 - \mu_1} - \frac{\hbar}{q_1 + \mu_1},
\]

\[
H^G_2(\{q\}_2, \{\mu\}_1, \xi) = \frac{\xi \hbar}{q_2} + \frac{\hbar}{q_2 - q_1} + \frac{\hbar}{q_2 + q_1} - \frac{\hbar}{q_2 - \mu_1} - \frac{\hbar}{q_2 + \mu_1},
\]

and

\[
\tilde{\mathcal{L}} = 
\begin{pmatrix}
H^G_1(\{\mu\}_1, \{q\}_2, 1 - \xi) & \frac{\hbar(1 - \xi)}{\mu_1} \\
-\frac{\hbar(1 - \xi)}{\mu_1} & -H^G_1(\{\mu\}_1, \{q\}_2, 1 - \xi)
\end{pmatrix},
\]

(4.17)

where

\[
H^G_1(\{\mu\}_1, \{q\}_2, 1 - \xi) = \frac{(1 - \xi) \hbar}{\mu_1} - \frac{\hbar}{\mu_1 - q_1} - \frac{\hbar}{\mu_1 - q_2} - \frac{\hbar}{\mu_1 + q_1} - \frac{\hbar}{\mu_1 + q_2}.
\]

One can directly verify that these matrices satisfy (4.15).
The proof of the identities (4.10) and (4.15) requires cumbersome calculations. In the $M = 0$ or $N = 0$ cases they follow from the factorization formulae for the Lax matrices (2.12), (2.16) [25]. In the general case the proof also uses the factorization formulae. It is similar to the one presented in [12] for (1.14) but the calculations are more complicated. We give a sketch of proof of (4.15) in the appendix. The identity (4.10) is proved in a similar way.

**Factorization formulae.** In the simplest cases when the set of the parameters $\{\mu\}$ is empty ($M = 0$) the determinant identities directly follow from the factorization formulae. Indeed, consider the $C_N$ and $D_N$ root systems. Then, for $M = 0$, the matrix (4.13) admits the factorization (2.12):

$$
\mathcal{L}
|_{M=0} = L \left( \{ H_j^G \}_N(\{q\}_N, \{\mu\}_0, \xi), \{q\}_N \right) 0, h, \sqrt{2} h \xi 
= h (D^0)^{-1} V(C_0 - (1 - 2\xi) \tilde{C}) V^{-1} D^0, \tag{4.18}
$$

where $\{\mu\}_0$ means that this set of parameters is empty. At $\xi = 0$ (4.18) reproduces the matrix for the $D_N$ case. Similarly, for the $B_N$ root system the matrix (4.15) is factorized as in (2.16). It can be rewritten as follows:

$$
\mathcal{L}
|_{M=0} = L \left( \{ \tilde{H}_j^G \}_N(\{q\}_N, \{\mu\}_0), \{q\}_N \right) \sqrt{2} h, h, 0 
= h(D^0)^{-1} V(C_0 + \tilde{C}) V^{-1} D^0, \tag{4.19}
$$

where $C_0$ and $\tilde{C}$ are taken from (2.12) but now they are of size $(2N + 1) \times (2N + 1)$. In both cases (4.18) and (4.19) the matrices are gauge equivalent (i.e. conjugated) to certain linear combinations of $C_0$ and $\tilde{C}$ which are nilpotent matrices (strictly triangular matrices with zeros on the main diagonal). In this way we get the identities (4.10) and (4.15) for $M = 0$:

$$
\det_{2N \times 2N} \left( \mathcal{L} - \lambda I \right)
|_{M=0} = \lambda^{2N} \quad \text{or} \quad \det_{(2N+1) \times (2N+1)} \left( \mathcal{L} - \lambda I \right)
|_{M=0} = -\lambda^{2N+1}.
$$

**Proof of duality.** Having identities (4.10) and (4.15), one can prove the statements of the theorem (4.3), (4.4). Let us prove (4.4) corresponding to the $C_N$ root system. Plugging all the data from the theorem to the $2N \times 2N$ Lax matrix (2.2) and using (4.15) we have:

$$
\det_{2N \times 2N} \left[ L \left( (H^G(\{q\}_N, \{\mu\}_M, \xi) \right) 0, h, \sqrt{2} h \xi \right] 
= \lambda^{2N - 2M} \det_{2M \times 2M} \left[ L \left( (H^G(\{\mu\}_M, \{q\}_N, 1 - \xi) \right) 0, h, \sqrt{2} h (1 - \xi) \right] - \lambda I. \tag{4.20}
$$

Let us simplify the expression for the (eigenvalues of the) Gaudin Hamiltonians entering the determinant in the r.h.s. using the Bethe equations (3.9):

$$
H^G_1(\{\mu\}_M, \{q\}_N, 1 - \xi)
|_{BE} 
= \frac{h(1 - \xi)}{\mu_i} + \sum_{k \neq i} \left( \frac{h}{\mu_i - \mu_k} + \frac{h}{\mu_i + \mu_k} \right) - \sum_{l=1}^N \left( \frac{h}{\mu_i - q_l} + \frac{h}{\mu_i + q_l} \right) \tag{4.21}
$$

$$
= \frac{h(\xi - 1)}{\mu_i} - \sum_{k \neq i} \left( \frac{h}{\mu_i - \mu_k} + \frac{h}{\mu_i + \mu_k} \right) = -H^G_1(\{\mu\}_M, \{q\}_0, 1 - \xi).
Therefore, we can rewrite (4.20) as
\[
\lambda^{2N-2M} \det_{2M \times 2M} \left( L(-H^G(\{\mu\}_M, \{q\}_0, 1-\xi), 0, \sqrt{2\hbar}(1-\xi)) - \lambda I \right)
\]
\[
= \lambda^{2N-2M} \det_{2M \times 2M} \left( L^T(-H^G(\{\mu\}_M, \{q\}_0, 1-\xi), 0, -\sqrt{2\hbar}(1-\xi)) - \lambda I \right)
\]
\[
= \lambda^{2N-2M} \det_{2M \times 2M} \left( L(-H^G(\{\mu\}_M, \{q\}_0, 1-\xi), 0, -\sqrt{2\hbar}(1-\xi)) - \lambda I \right)
\]
\[
= \lambda^{2N}. \tag{4.22}
\]

The last equality is obtained using the identity (4.15) for $M = 0$. The proof for the $B_N$ case is similar. ■

## 5 Conclusion

We have shown that the quantum-classical duality for the BCD root systems works in much the same way as for the models associated with the $A_{N-1}$ root system including the existence of nontrivial determinant identities. We have proved that the rational quantum Gaudin models with boundary are dual to the classical rational Calogero-Moser models associated with the root systems of BCD types. However, an important difference with the $A$-case is the absence of the twist matrix in the quantum model.

We understand that the results obtained in this paper need further generalizations in several directions, including the extension to higher rank Gaudin magnets solved by means of the nested Bethe ansatz method, the supersymmetric extension (see [24] for the $A_{N-1}$ case), the trigonometric version of the duality [2] and the generalization to the level of spin chains and related Ruijsenaars-Schneider-van Diejen relativistic many-body systems.

Let us also note that on the classical side we have used the Lax representation [19] in which the coupling constants are not arbitrary (they are assumed to satisfy condition (2.6)). An alternative Lax representation in matrices of size $2N \times 2N$ is known [7], which do not require any additional constraints. It would be interesting to connect the spectrum of the Lax matrix from this representation to quantum integrable models.

We hope to study the extensions and generalizations mentioned above in future publications.

### Appendix: proof of the determinant identities

Here we give a proof for the determinant identity (4.15). The identity (4.10) is proved in a similar way. The proof of (4.15) is by induction in the number $M$ of the parameters $\{\mu\}$. The main idea is to compare the structure of poles and residues of both parts of (4.15) in all the variables $\{q\}$ and $\{\mu\}$. Before we proceed further let us formulate the following useful lemma.

**Lemma.** The l.h.s. of (4.15) does not have singularities at $q_i = \pm q_k$, and the r.h.s. does not have singularities at $\mu_a = \pm \mu_b$. 
The proof of the lemma is identical to the one given in [12] for the $A_{N-1}$ case. One should use the factorization formulae for the Lax matrices from [25].

Introduce the following notation:

\[
\mathcal{L}_N^M = L \left( \{ H_j \}_N \{ \mu \}_M, \xi \right), \quad \{ q \}_N | 0, \hbar, \sqrt{2} \hbar \xi, \]

\[
\tilde{\mathcal{L}}_M^N = L \left( \{ H_j \}_M \{ \mu \}_M, \{ q \}_N, 1 - \xi \right), \quad \{ \mu \}_M | 0, \hbar, \sqrt{2} \hbar (1 - \xi) .
\]

The lower index in $\mathcal{L}_N^M$ is the half of its size, and the upper one is the number of variables $\mu_\ell$ entering this matrix. Similarly, the lower index in $\tilde{\mathcal{L}}_M^N$ is the half of its size, and the upper one is the number of the variables $q_k$.

We prove the determinant identity by induction in the number $M$ of the parameters $\mu$. For $M = 0$ it holds true due to the factorization formula (4.18). Suppose that (4.15) is true for the $A_{N-1}$ case. One should consider the $A_N$ case. One should introduce the following notation:

\[
\det_{2N \times 2N} \left( \mathcal{L}_N^M - \lambda I \right) = a_0 + \sum_{k=1}^{N} \left( \frac{a_k^-}{\mu_1 - q_k} + \frac{a_k^+}{\mu_1 + q_k} \right) + \sum_{k=1}^{N} \left( \frac{c_k^-}{(\mu_1 - q_k)^2} + \frac{c_k^+}{(\mu_1 + q_k)^2} \right),
\]

\[
\det_{2M \times 2M} \left( \tilde{\mathcal{L}}_M^N - \lambda I \right) = \tilde{a}_0 + \sum_{k=1}^{N} \left( \frac{\tilde{a}_k^-}{\mu_1 - q_k} + \frac{\tilde{a}_k^+}{\mu_1 + q_k} \right) + \sum_{k=1}^{N} \left( \frac{\tilde{c}_k^-}{(\mu_1 - q_k)^2} + \frac{\tilde{c}_k^+}{(\mu_1 + q_k)^2} \right).
\]

Let us calculate all the coefficients in these expansions. For $a_0$ and $\tilde{a}_0$ we have:

\[
a_0 = \lim_{\mu_1 \to \infty} \left( \det_{2N \times 2N} \left( \mathcal{L}_N^M - \lambda I \right) \right) = \det_{2N \times 2N} \left( \mathcal{L}_N^{M-1} - \lambda I \right) = \lambda^{2N-2M+2} \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_M^{N-1} - \lambda I \right),
\]

\[
\tilde{a}_0 = \lim_{\mu_1 \to \infty} \left( \det_{2M \times 2M} \left( \tilde{\mathcal{L}}_M^N - \lambda I \right) \right) = \lambda^2 \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_M^{N-1} - \lambda I \right),
\]

which yields $a_0 = \lambda^{2N-2M} \tilde{a}_0$. This is exactly what we need.

For the coefficients in front of the second order poles we have:

\[
c_k^- = \lim_{\mu_1 \to q_k} \left( (\mu_1 - q_k)^2 \det_{2N \times 2N} \left( \mathcal{L}_N^M - \lambda I \right) \right) = -\hbar^2 \det_{(2N-2) \times (2N-2)} \left( \mathcal{L}_N^{M-1} - \lambda I \right) = -\hbar^2 \lambda^{2N-2M} \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_M^{N-1} - \lambda I \right),
\]

\[
\tilde{c}_k^- = \lim_{\mu_1 \to q_k} \left( (\mu_1 - q_k)^2 \det_{2M \times 2M} \left( \tilde{\mathcal{L}}_M^N - \lambda I \right) \right) = -\hbar^2 \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_M^{N-1} - \lambda I \right),
\]

which gives $c_k^- = \lambda^{2N-2M} \tilde{c}_k^-$. The proof of $c_k^+ = \lambda^{2N-2M} \tilde{c}_k^+$ is performed in the same way.
Calculation of the coefficients $a_k^\pm$ and $\tilde{a}_k^\pm$ is more complicated. They are residues of the determinants in the l.h.s. of (A.2). Consider the pole at $\mu_1 = q_1$. It follows from the explicit form of the matrices from (A.2) that the residue at $\mu_1 = q_1$ may come from the elements proportional to $\frac{1}{\mu_1 - q_1}$, which is present in $L_{11}$ and $L_{N+1,N+1}$ (and for the matrix $\tilde{L}$ such elements are in $\tilde{L}_{11}$ and $\tilde{L}_{M+1,M+1}$). Therefore, we have linear and quadratic terms in $\frac{1}{\mu_1 - q_1}$, and our purpose is to compute the residues. First, calculate the residues coming from the linear parts. The residue coming from $L^M_N$ is equal to

$$G = \hbar \det_{(2N-1)\times(2N-1)} \left( \begin{array}{ccc} A - \lambda I & X & B \\ -X^T & -F - \lambda I & -Y^T \\ -B & Y & -A - \lambda I \end{array} \right)$$

(A.5)

$$-\hbar \det_{(2N-1)\times(2N-1)} \left( \begin{array}{ccc} F - \lambda I & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{array} \right),$$

where $A$ and $B$ are the following matrices of size $(N-1) \times (N-1)$:

$$A_{ij} = \delta_{ij} \left( \frac{\hbar \xi}{q_i} + \sum_{k \neq 1, i}^N \left( \frac{\hbar}{q_i - q_k} + \frac{\hbar}{q_i + q_k} \right) - \sum_{l \neq 1}^M \left( \frac{\hbar}{q_i - \mu_l} + \frac{\hbar}{q_i + \mu_l} \right) \right) + (1 - \delta_{ij}) \frac{\hbar}{q_i - q_j}, \quad 2 \leq i, j \leq N,$n

(A.6)

$$B_{ij} = \delta_{ij} \frac{\hbar \xi}{q_i} + (1 - \delta_{ij}) \frac{\hbar}{q_i + q_j}, \quad 2 \leq i, j \leq N,$n

while $X$ and $Y$ are columns of size $N - 1$:

$$X_i = \frac{\hbar}{q_1 + q_i}, \quad 2 \leq i \leq N, \quad Y_i = \frac{\hbar}{q_1 - q_i}, \quad 2 \leq i \leq N.$$n

(A.7)

Finally, $F$ is the number

$$F = \frac{\hbar (\xi - \frac{1}{2})}{q_1} + \sum_{k \neq 1}^N \left( \frac{\hbar}{q_1 - q_k} + \frac{\hbar}{q_1 + q_k} \right) - \sum_{l \neq 1}^M \left( \frac{\hbar}{q_1 - \mu_l} + \frac{\hbar}{q_1 + \mu_l} \right).$$

(A.8)

Let us split (A.5) into parts proportional to the expressions containing $F$ and the other ones:

$$G = \hbar (-F - \lambda) \det_{(2N-2)\times(2N-2)} \left( \begin{array}{ccc} A - \lambda I & B \\ -B & -A - \lambda I \end{array} \right)$$

(A.9)

$$-\hbar (F - \lambda) \det_{(2N-2)\times(2N-2)} \left( \begin{array}{ccc} A - \lambda I & B \\ -B & -A - \lambda I \end{array} \right)$$

$$+\hbar \det_{(2N-1)\times(2N-1)} \left( \begin{array}{ccc} A - \lambda I & X & B \\ -X^T & 0 & -Y^T \\ -B & Y & -A - \lambda I \end{array} \right) - \hbar \det_{(2N-1)\times(2N-1)} \left( \begin{array}{ccc} 0 & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{array} \right).$$
Therefore,

\[
G = -2\hbar F \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-2) \times (2N-2)} + \hbar \frac{\det A - \lambda I \ X \ B}{(2N-1) \times (2N-1)} \left( \begin{array}{ccc} -X^T & 0 & -Y^T \\ -B & Y & -A - \lambda I \end{array} \right) \]

\[
= -\hbar \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-1) \times (2N-1)} \left( \begin{array}{ccc} 0 & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{array} \right).
\]

This is a part of the residue of \( \frac{\det \mathcal{L}_{N}^{M} - \lambda I}{2N \times 2N} \). To compute the full residue, we need to add the contribution coming from the second order pole at \( \mu_1 = q_1 \). Thereby, the full residue of \( \frac{\det \mathcal{L}_{N}^{M} - \lambda I}{2N \times 2N} \) at \( \mu_1 = q_1 \) is given by

\[
a_1^{-} = -2\hbar F \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-2) \times (2N-2)} - \hbar^2 \frac{\partial}{\partial \mu_1} \left[ \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-2) \times (2N-2)} \right]_{\mu_1 = q_1} - \hbar \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-1) \times (2N-1)} \left( \begin{array}{ccc} 0 & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{array} \right).
\]

For a while, let us forget about the first term in (A.11). We claim that the remaining part vanishes:

\[
Z = \hbar \frac{\det A - \lambda I \ X \ B}{(2N-1) \times (2N-1)} \left( \begin{array}{ccc} -X^T & 0 & -Y^T \\ -B & Y & -A - \lambda I \end{array} \right)
\]

\[
-\hbar \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-1) \times (2N-1)} \left( \begin{array}{ccc} 0 & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{array} \right)
\]

\[
-\hbar^2 \frac{\partial}{\partial \mu_1} \left[ \frac{\det \mathcal{L}_{N-1}^{M-1} - \lambda I}{(2N-2) \times (2N-2)} \right]_{\mu_1 = q_1} = 0.
\]

This can be verified by analyzing the structure of (A.12) as a rational function of \( q_1 \). As a function of \( q_1 \), \( Z \) has simple and second order poles at \( q_1 = \pm q_k \). Let us recall that in the first two determinants \( q_1 \) is contained only in the columns \( X \) and \( Y \). This instantly gives us \( \lim_{q_1 \to \infty} Z = 0 \). Then one has to analyze the second order poles of \( Z \) at \( q_1 = \pm q_k \). Notice that the last term in (A.12) contains only second order poles and does not contain simple poles. A direct calculation shows that the second order poles coming from the first two terms of (A.12) and the last one cancel identically. Thus we are left with simple poles that come only from the
first two terms

\[
\hbar \det_{(2N-1) \times (2N-1)} \begin{pmatrix} A - \lambda I & X & B \\ -X^T & 0 & -Y^T \\ -B & Y & -A - \lambda I \end{pmatrix}
\]

\[\text{(A.13)}\]

\[-\hbar \det_{(2N-1) \times (2N-1)} \begin{pmatrix} 0 & Y^T & X^T \\ -Y & A - \lambda I & B \\ -X & -B & -A - \lambda I \end{pmatrix}.
\]

Again, an accurate calculation provides their cancellation. Therefore, we find that \(Z = 0\) and so the expression for the residue (A.11) is significantly simplified:

\[a_1^- = -2\hbar F \det_{(2N-2) \times (2N-2)} \left( \mathcal{L}_{N-1}^M - \lambda I \right) = -2\hbar F \lambda^{2N-2M} \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_{M-1}^N - \lambda I \right), \quad (A.14)\]

where the last equality is true due to the induction hypothesis. Similar arguments applied to \(\tilde{a}_1^-\) yield

\[\tilde{a}_1^- = -2\hbar F \det_{(2M-2) \times (2M-2)} \left( \tilde{\mathcal{L}}_{M-1}^N - \lambda I \right), \quad (A.15)\]

which means that \(a_1^- = \lambda^{2N-2M} \tilde{a}_1^-\), and so the proof of the determinant identity (4.15) is complete.

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