SMOOTH SOLUTIONS TO THE EVEN DUAL ORLICZ-MINKOWSKI PROBLEMS

LI CHEN, YAN-NAN LIU, JIAN LU, AND NI XIANG

ABSTRACT. In this paper we study a normalised anisotropic Gauss curvature flow of strictly convex, smooth closed hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. We prove that the flow exists for all time and converges smoothly to the unique, strictly convex solution of a Monge-Ampère type equation and we establish a new existence result of solutions to the Dual Orlicz-Minkowski problem for even smooth measure.

Keywords: Gauss curvature flow, convex hypersurface, Monge-Ampère type equation.

MSC: Primary 53C44, Secondary 35K96.

1. INTRODUCTION

In recent years, the Orlicz-Brunn-Minkowski theory, arose from works of Ludwig [21], Ludwig, Reitzner [22], and Lutwak, Yang, Zhang [24, 25], has been built up gradually and is developed rapidly. It can be viewed as the recent development of the classical Brunn-Minkowski theory and its extension $L_p$ Brunn-Minkowski theory. The first one was developed by Minkowski, Aleksandrov and the later one was brought to fruition by germinating a seed planted by Firey, Lutwak [23]. More and more scholars pay attention to these relevant problems, see for example [7, 8, 9, 10, 11, 12].

In the Brunn-Minkowski theory, it is well known that the classical Minkowski problem is of central importance, and has many applications. In the new Orlicz-Brunn-Minkowski theory, the corresponding Minkowski problem is called the Orlicz-Minkowski problem.

The first Orlicz version of the Minkowski problem appeared in [13], at the inception of the Orlicz-Brunn-Minkowski theory in 2010, see [15, 17] for recent work on this problem. Later, the dual Minkowski problem was extended to the Orlicz setting and partially solved in [32, 33]; here, the $q$-th dual volume in [16] was replaced by certain dual Orlicz quermassintegrals, while $L_0$ addition was retained. A common
A generalization of the problems in [26, 32, 33] was proposed in [10], in which the $q$-th dual volume was replaced by a very general notion of dual volume denoted by $\tilde{V}_\psi(K)$, and simultaneously $L_0$ addition was replaced by an extension of $L_p$ addition called Orlicz addition. By combining the general dual volume $\tilde{V}_\psi(K)$ with Orlicz addition, a general dual Orlicz curvature measure of $K$ with the origin in its interior denoted by $\tilde{C}_{\varphi, \psi}(K, \cdot)$ is defined in [10], see also [11, 12].

Let $\varphi : (0, +\infty) \to (0, +\infty)$ and $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty)$ be two given continuous functions. For a convex body $K \in \mathcal{K}_0^{n+1}$, where $\mathcal{K}_0^{n+1}$ denotes all the convex bodies in $\mathbb{R}^{n+1}$ with the origin in its interior, the dual Orlicz curvature measure in [10] is defined as

$$\tilde{C}_{\varphi, \psi}(K, E) = \frac{1}{n+1} \int_{\alpha_K^*(E)} \frac{\varphi(u_K(\alpha_K(\xi)))\psi(r_K(\xi)\xi)r_{K+1}^{n+1}(\xi)}{u_K(\alpha_K(\xi))} d\xi$$

for each Borel set $E \subset S^n$, where $\alpha_K$ is the radial Gauss, $\alpha_K^*$ is the reverse radial Gauss image on $S^n$, $r_K$ is the radial function of $K$, $d\xi$ is the spherical measure on $S^n$. In [10], the following Minkowski problem was stated:

**Problem 1.1 (Dual Orlicz-Minkowski problem).** For which nonzero finite Borel measures $\mu$ on $S^n$ and continuous functions $\varphi : (0, +\infty) \to (0, +\infty)$ and $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty)$, do there exist $c \in \mathbb{R}^{n+1}$ and $K \in \mathcal{K}_0^{n+1}$ such that $\mu = c\tilde{C}_{\varphi, \psi}(K, \cdot)$?

In [11], they prove the existence of convex polytopes containing the origin in their interiors solving the Minkowski problem above for the discrete measures $\mu$. For general (not necessarily discrete) measures $\mu$, they use an approximation argument. When the measures $\mu$ are even, solutions that are origin symmetric convex bodies, are also provided under some mild conditions on $\varphi$ and $\psi$ in [11]. Their results generalize several previous works and provide more precise information about the solutions of the Minkowski problem when $\mu$ is discrete or even.

When the given measure $\mu$ has a density $f$, this Dual Orlicz Minkowski problem becomes the following Monge-Ampère type equation on $S^n$:

$$c\varphi(u)\psi(Du + ux) \cdot \det(u_{ij} + u \delta_{ij}) = f(x) \quad \text{on} \quad S^n.$$  

(1.1)

When $\varphi(t) = t^{1-p}$ and $\psi(z) = |z|^{q-n}$, the equation above becomes

$$\frac{u^{1-p}}{(|Du|^2 + u^2)^{\frac{n+1}{2}}} \cdot \det(u_{ij} + u \delta_{ij}) = f(x) \quad \text{on} \quad S^n.$$
which is the equation of the well-known $L_p$ dual Minkowski problem derived in [26]. The existence of smooth solutions to the $L_p$ Dual Orlicz Minkowski problem is proved in [14] for $p > q$, and in [3] for $p = q$, in [2] for $pq \geq 0$ and even $f$. Recently, Liu-Lu [19] prove the existence of smooth solutions to this Dual Orlicz Minkowski problem (1.1) for the non-even smooth function $f$ which recovers the existence of smooth solutions to the $L_p$ Dual Orlicz Minkowski problem is proved in [14] for $p > q$. In this paper, we extend the result in [2] to the dual Orlicz-Minkowski problem.

To state our theorem, we need the following assumption (with a small modification) which is proposed by Haberl, Lutwak, Yang and Zhang in their study about the even Orlicz-Minkowski problem [13].

**Assumption 1.2.** $\phi : (0, +\infty) \to (0, +\infty)$ is a continuous function such that
\[
\Phi(t) = \int_{\epsilon}^{t} \frac{1}{\phi(s)} \, ds
\]
is unbounded as $t \to +\infty$, where $\epsilon$ is a positive constant.

And we also need the following assumption of $\psi$.

**Assumption 1.3.** $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty)$ is a continuous function such that
\[
\Psi(t, \xi) = \int_{0}^{t} \psi(s\xi) s^n \, ds
\]
exists for every $t > 0$ and $\xi \in S^n$.

We get the following existence of solutions to the dual Orlicz-Minkowski problem (1.1).

**Theorem 1.4.** Assume that $f \in C^\infty(S^n)$ is a positive smooth function, $\phi : (0, +\infty) \to (0, +\infty)$ and $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty)$ are two smooth functions. Under the Assumptions 1.2 and 1.3, if $f$ is in addition even function, then there exists a positive constant $c$ and a smooth even function $u$ satisfying the equation (1.1).

**Remark 1.5.** For the special case $\phi(u) = u^{1-p}$ and $\psi(r\xi) = r^{q-n-1}$, the Assumptions 1.2 and 1.3 are equivalent to $p > 0$ and $q > 0$ respectively, thus Theorem 1.4 recovers the existence of smooth solutions to the $L_p$ dual Minkowski problem which proven in [2] for $p > 0$ and $q > 0$.

The main idea is to find a suitable functional and a suitable anisotropic Gauss curvature flow which converges to the solution of the equation (1.1) as $t \to +\infty$. Let
\( \mathcal{M}_0 \) be a smooth, closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) enclosing the origin given by
\[
X_0 : \mathcal{M} \to \mathbb{R}^{n+1},
\]
where \( \mathcal{M} \) is an \( n \)-dimensional closed smooth Riemannian manifold. We consider the long-time behavior of the following normalised anisotropic Gauss curvature flow which is a family of hypersurfaces \( \mathcal{M}_t = X(\mathcal{M}, t) \) given by smooth maps \( X : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+1} \) satisfying the initial value problem
\[
\begin{align*}
\frac{\partial X}{\partial t} &= -\theta(t)f(\nu)\frac{\langle X, \nu \rangle}{\varphi(\langle X, \nu \rangle)}K\nu + X, \quad \text{on } \mathcal{M} \times [0, T), \\
X(\cdot, 0) &= X_0, \quad \text{on } \mathcal{M},
\end{align*}
\]
where \( \nu \) is the unit outer vector of \( \mathcal{M}_t \) at \( X \), \( K \) denotes the Gauss curvature of \( \mathcal{M}_t \) at \( X \), \( f \in C^\infty(\mathbb{S}^n) \) with \( f > 0 \), \( \varphi : (0, +\infty) \to (0, +\infty) \) and \( \psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty) \) are two smooth functions, and
\[
\theta(t) = \int_{\mathbb{S}^n} \psi(r(\xi, t)\xi) r^{n+1}(\xi, t) \, d\xi \left[ \int_{\mathbb{S}^n} \frac{u(x, t)}{\varphi(u(x, t))} f(x) \, dx \right]^{-1}.
\]
Here we parametrize the radial function \( r = |X| \) as a function from \( \mathbb{S}^n \) to \( \mathbb{R} \). Both \( d\xi \) and \( dx \) are the spherical measures on \( \mathbb{S}^n \).

As a natural extension of Gauss curvature flows, anisotropic Gauss curvature flows have attracted considerable attention and they provide alternative proofs for the existence of solutions to elliptic PDEs arising in geometry and physics, especially for the Minkowski-type problem. For example a alternative proof based on the logarithmic Gauss curvature flow was given by Chou-Wang in [5] for the classical Minkowski problem, in [31] for a prescribing Gauss curvature problem. Bryan-Ivaki-Scheuer in [1] have given a unified flow approach to smooth, even \( L_p \)-Minkowski problems. Using a contracting Gauss curvature flow, Li-Sheng-Wang [15] have provided a parabolic proof in the smooth category for the classical Aleksandrov and dual Minkowski problems. Recently, two kinds of normalised anisotropic Gauss curvature flow are used to prove the \( L_p \) dual Minkowski problems by Chen-Huang-Zhao [2] and Chen-Li [3], respectively. These results are major source of inspiration for us.

We obtain the following results for the flow (1.2) which implies Theorem 1.4.

**Theorem 1.6.** Assume that \( f \in C^\infty(\mathbb{S}^n) \) is a positive smooth function and \( \varphi : (0, +\infty) \to (0, +\infty) \) and \( \psi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty) \) are two smooth functions. Let
Let $\mathcal{M}_0 \subset \mathbb{R}^{n+1}$ be a strictly convex, closed hypersurface which contains the origin in its interior.

Under the Assumptions 1.2 and 1.3, if $f$ is in addition even function and the initial hypersurface $\mathcal{M}_0$ is origin-symmetric, then the normalised flow (1.2) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. For each $t \in [0, \infty)$, $\mathcal{M}_t = X(\mathcal{M}, t)$ is a closed, smooth, strictly convex and origin-symmetric hypersurface which also contains the origin in its interior, and the support function $u(x, t)$ of $\mathcal{M}_t = X(\mathcal{M}, t)$ converges smoothly, as $t \to \infty$, to the unique positive, smooth, strictly convex and even solution of the equation (1.1) with $\frac{1}{c} = \lim_{t_i \to \infty} \theta(t_i) > 0$.

Deriving the long-time existence of the flow (1.2) is the key to our theorems. Thus, we need to obtain uniform positive upper and lower bounds for the support function and principal curvatures along the flow. The difficulty of these a priori estimates for the flow (1.2) lies in the inhomogeneous term $\varphi$ and $\psi$. So we need to choose proper auxiliary functions and do more delicate computations. Then the long-time existence follows by standard arguments.

The organization of this paper is as follows. In Sect. 2 we start with some preliminaries. In Sect. 3 we obtain $C^0$ and $C^1$ estimates. The $C^2$ estimates are given in Sect. 4. In Sect. 5 we prove Theorem 1.6.

2. Preliminaries

2.1. Basic properties of convex hypersurfaces. We first recall some basic properties of convex hypersurfaces in $\mathbb{R}^{n+1}$, see [30] for details. Let $\mathcal{M}$ be a smooth, closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$. Assume that $\mathcal{M}$ is parametrized by the inverse Gauss map

$$X : \mathbb{S}^n \to \mathcal{M} \subset \mathbb{R}^{n+1}.$$ 

Without loss of generality, we may assume that $\mathcal{M}$ encloses the origin. The support function $u : \mathbb{S}^n \to \mathbb{R}$ of $\mathcal{M}$ is defined by

$$u(x) = \langle x, X(x) \rangle. \quad (2.1)$$

Evidently we also have

$$u(x) = \sup \{ \langle x, y \rangle : y \in \mathcal{M} \}.$$
The supreme is attained at a point $y$ such that $x$ is the outer normal of $\mathcal{M}$ at $X$. It is easy to check that
\[ X = u(x)x + Du(x), \]
where $D$ is the covariant derivative with respect to the standard metric $\sigma_{ij}$ of the sphere $\mathbb{S}^n$. Hence
\[ (2.2) \quad r = |X| = \sqrt{u^2 + |Du|^2}. \]
and
\[ (2.3) \quad u = \frac{r^2}{\sqrt{r^2 + |Dr|^2}}. \]
The second fundamental form of $\mathcal{M}$ is given by
\[ (2.4) \quad h_{ij} = u_{ij} + \sigma_{ij}, \]
where $u_{ij} = D_i D_j u$ denotes the second order covariant derivative of $u$ with respect to the spherical metric $\sigma_{ij}$. By Weingarten’s formula,
\[ (2.5) \quad \sigma_{ij} = \frac{\partial \nu}{\partial x^i} \cdot \frac{\partial \nu}{\partial x^j} = h_{ik} g^{kl} h_{lj}, \]
where $g_{ij}$ is the metric of $\mathcal{M}$ and $g^{ij}$ is its inverse. It follows from (2.4) and (2.5) that the principal radii of curvature of $\mathcal{M}$, under a smooth local orthonormal frame field $\{e_1, e_2, ..., e_n\}$ on $\mathbb{S}^n$, are the eigenvalues of the matrix
\[ b_{ij} = u_{ij} + u \delta_{ij}. \]
In particular, the Gauss curvature is given by
\[ K = \frac{1}{\det(u_{ij} + u \delta_{ij})}. \]

2.2. Geometric flow and its associated functional. For reader’s convenience, the associated Monge-Ampère type equation (1.1) is restated here,
\[ c \varphi(u) \psi(Du + ux) \cdot \det(u_{ij} + u \delta_{ij}) = f(x) \quad \text{on} \quad \mathbb{S}^n. \]
Recall the normalised anisotropic Gauss curvature flow (1.2)
\[ \begin{cases} \frac{\partial X}{\partial t} = -\theta(t) f(\nu) \frac{\langle X, \nu \rangle}{\varphi(\langle X, \nu \rangle) \psi(X)} K \nu + X, & \text{on} \quad \mathcal{M} \times [0, T), \\ X(\cdot, 0) = X_0, & \text{on} \quad \mathcal{M}, \end{cases} \]
where
\[ \theta(t) = \int_{S^n} \psi(r(\xi, t)\xi) r^{n+1}(\xi, t) d\xi \left[ \int_{S^n} \frac{u(x, t)}{\varphi(u(x, t))} f(x) dx \right]^{-1}. \]

By the definition of support function \( u(x, t) = \langle x, X(x, t) \rangle \), we obtain

\[
\begin{align*}
\partial_t u(x, t) &= -\theta(t) \frac{f(x)u}{\varphi(u)\psi(Du + ux)} K + u(x, t), \quad \text{on } S^n \times [0, T), \\
u(\cdot, 0) &= u_0, \quad \text{on } S^n.
\end{align*}
\]

The normalised flow (1.2) can be also described by the following scalar equation for the radial function \( r(\cdot, t) \) with \( X = r(\xi, t)\xi \)

\[
\begin{align*}
\partial_t r(\xi, t) &= -\theta(t) \frac{f(x)r}{\varphi(u)\psi(r\xi)} K + r(\xi, t), \quad \text{on } S^n \times [0, T) \\
r(\cdot, 0) &= r_0, \quad \text{on } S^n,
\end{align*}
\]

in view of

\[ \frac{1}{r(\xi, t)} \frac{\partial r(\xi, t)}{\partial t} = \frac{1}{u(x, t)} \frac{\partial u(x, t)}{\partial t}, \]

see Section 3 in [3] for the proof. For a convex body \( \Omega \subset \mathbb{R}^{n+1} \) which contains the origin in its interior, we define

\[ V_\psi(\Omega) = \int_{S^n} d\xi \int_0^{r(\xi)} \psi(s\xi)s^n ds, \]

where \( r \) is the radial function of \( \Omega \) (see Definition 2.5 at the end of this subsection). When \( \psi(r\xi) = r^{q-n-1} \), \( V_\psi(\Omega) \) be the \( q \)-volume of the convex body \( \Omega \subset \mathbb{R}^{n+1} \), see [2, 3]. We show below that \( V_\psi(\Omega_t) \) is unchanged under the flow (1.2), where \( \Omega_t \) is a compact convex body in \( \mathbb{R}^{n+1} \) with the boundary \( \mathcal{M}_t \).

**Lemma 2.1.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.2) which encloses the origin for \( t \in [0, T) \), then we obtain

\[ V_\psi(\Omega_t) = V_\psi(\Omega_0). \]
Proof. By a direction calculation, we have by (2.7)
\[
\frac{d}{dt} V_\psi(\Omega_t) = \int_{\Sigma^n} \psi(r\xi) r^n \frac{\partial}{\partial t} d\xi = \int_{\Sigma^n} \psi(r\xi) r^n \left( -\theta(t) \frac{f(x)r}{\varphi(u)} \psi(r\xi) K + r(\xi,t) \right) d\xi
\]
\[
= -\theta(t) \int_{\Sigma^n} \frac{f(x)r^{n+1}}{\varphi(u)} K d\xi + \int_{\Sigma^n} \psi(r\xi) r^{n+1} d\xi
\]
\[= 0,
\]
where we use
\[
\frac{dx}{d\xi} = \frac{r^{n+1} K}{u},
\]
see e.g. [3, 16]. \qed

Next, we define the entropy functional of the flow (1.2) as
\[
J_{\varphi, \psi}(X(\cdot, t)) = \int_{\Sigma^n} f(x) dx \int_{\epsilon}^{u(x,t)} \frac{1}{\varphi(s)} ds,
\]
where \( \epsilon \) is a positive constant. The following lemma shows that the functional \( J_{\varphi, \psi} \) is non-increasing along the flow (1.2).

**Lemma 2.2.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.2) which encloses the origin for \( t \in [0, T) \). For any \( \varphi, \psi \geq 0 \), the functional is non-increasing along the flow (1.2). In particular,
\[
\frac{d}{dt} J_{\varphi, \psi}(X(\cdot, t)) \leq 0
\]
and the equality holds if and only if \( X(\cdot, t) \) satisfies the elliptic equation (1.1) with \( \frac{1}{\epsilon} = \theta(t) \).
Proof.

\[
\frac{d}{dt} J_{\varphi, \psi}(X(\cdot, t)) = \int_{S^n} \frac{1}{\varphi(u)} \frac{\partial u(x, t)}{\partial t} f(x) dx
\]

\[
= \int_{S^n} \frac{1}{\varphi(u)} \left( -\theta(t) \frac{f(x)u}{\varphi(u)\psi(Du + ux)} K + u(x, t) \right) f(x) dx
\]

\[
= \left[ \int_{S^n} \frac{u}{\varphi(u)} f(x) dx \right]^{-1} \left\{ -\int_{S^n} \frac{u}{K} f(x) dx \int_{S^n} \frac{uK}{\varphi^2} f^2 dx + \int_{S^n} \frac{u}{\varphi(u)} f dx \int_{S^n} \frac{u}{f} f dx \right\}
\]

\[
\leq 0
\]

in view of

\[
\left( \int_{S^n} d\sigma \right)^2 \leq \int_{S^n} \frac{f^2}{fK} d\sigma \int_{S^n} \frac{Kf}{\varphi^2} f d\sigma,
\]

which is implies by Hölder inequality, where \(d\sigma = \frac{u}{\varphi(u)} f(x) dx\). Clearly, the equality holds if and only if

\[
\frac{f(x)K}{\varphi \psi} = \frac{1}{c(t)}.
\]

Thus, \(X(\cdot, t)\) satisfies the elliptic equation \((1.1)\) with \(c(t) = \theta(t)\). \(\square\)

Before closing this section, we prove the following basic properties for any given \(\Omega \in \mathcal{K}_{0}^{n+1}\), while smoothness of \(\partial \Omega\) is not required. First, we introduce the following Lemma for convex bodies, see Lemma 2.6 in [3] for details.

**Lemma 2.3.** Let \(\Omega \in \mathcal{K}_{0}^{n+1}\). Let \(u\) and \(r\) be the support function and radial function of \(\Omega\) respectively, and \(x_{\text{max}}\) and \(\xi_{\text{min}}\) be two points such that \(u(x_{\text{max}}) = \max_{S^n} u\) and \(r(\xi_{\text{min}}) = \min_{S^n} r\). Then

\[
\max_{S^n} u = \max_{S^n} r \quad \text{and} \quad \min_{S^n} u = \min_{S^n} r,
\]

\[
u(x) \geq x \cdot x_{\text{max}} u(x_{\text{max}}), \quad \forall x \in S^n,
\]

\[
r(\xi) \xi \cdot \xi_{\text{min}} \geq r(\xi_{\text{min}}), \quad \forall \xi \in S^n.
\]

Let \(\mathcal{K} = \{ K | K \text{ is a convex body in } \mathbb{R}^{n+1} \}\). We recall Blaschke selection theorem theorem (see also Theorem 1.8.7 in [27]).
Theorem 2.4. If $K_i \in \mathcal{K}$ for each $i \in \mathbb{N}$ and there exists a constant $R > 0$ such that $K_i \subset B_R(0)$ for all $i \in \mathbb{N}$, where $B_R(0)$ is a ball in $\mathbb{R}^{n+1}$ with radius $R$ and center at $0$, then there exists a subsequence $K_{i_j}$ and $K_0 \in \mathcal{K}$ such that

$$K_{i_j} \to K_0 \text{ in the Hausdorff metric,}$$

as $i_j \to +\infty$.

To statement the following theorem, we first recall the definition of the radial function of a convex body. (see also [27]).

Definition 2.5. Let $K \in \mathcal{K}$, $0 \in K$, a radial function $r_K : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is defined as

$$r_K(x) = \max\{r \geq 0 | rx \in K\}.$$

Now, the convergence of convex bodies imply the uniform convergence of the corresponding radial functions.

Theorem 2.6. Let $K_0, K_i \in \mathcal{K}$, $0 \in \text{int}K_0$ and $K_i \to K_0$ in the Hausdorff metric as $i \to +\infty$, then $r_{K_i} \Rightarrow r_{K_0}$ as $i \to +\infty$.

For the proof of the theorem above, see [27].

3. $C^0$ and $C^1$-estimates

In this section, we will derive $C^0$ and $C^1$-estimates of the flow (1.2). The key is the lower bound of $u$. The difficulty of the proof lies in dealing with the inhomogeneous term $\varphi$ and $\psi$.

Lemma 3.1. Under the assumptions of Theorem 1.6, let $X(\cdot, t)$ be a strictly convex solution to the flow (1.2) which encloses the origin for $t \in [0, T)$, then we have

$$\frac{1}{C} \leq u(x, t) \leq C, \quad \forall (x, t) \in S^n \times [0, T),$$

and

$$|Du|(x, t) \leq C, \quad \forall (x, t) \in S^n \times [0, T).$$
Proof. Assume that \( x_t \) is a point at where \( u(\cdot,t) \) attains its spatial maximum. Then, we know from Lemma 2.2

\[
\int_{\mathbb{S}^n} f(x) dx \int_{\epsilon}^1 \frac{1}{\varphi(y)} dy \geq \int_{\mathbb{S}^n} f(x) dx \int_{\epsilon}^1 \frac{1}{\varphi(y)} dy
\]

\[
\geq \int_{\{x \in \mathbb{S}^n : x \cdot x_t > 1/2\}} f(x) dx \int_{\epsilon}^1 \frac{1}{\varphi(y)} dy
\]

\[
\geq C \int_{\epsilon}^1 \frac{1}{\varphi(y)} dy,
\]

which implies together with Assumption 1.2

\[
C \geq \max_{\mathbb{S}^n} u(\cdot,t).
\]

This yields the inequality (3.1). By the relation (2.2) and Lemma 2.3 we conclude

\[
\max_{\mathbb{S}^n} |Du(\cdot,t)|^2 \leq \max_{\mathbb{S}^n} r^2(\cdot,t) = \max_{\mathbb{S}^n} u^2(\cdot,t).
\]

so

\[
\max_{\mathbb{S}^n} |Du(\cdot,t)| \leq \max_{\mathbb{S}^n} u(\cdot,t),
\]

we obtain (3.2). Now We only need prove the first inequality in (3.1). Here we use the idea in [2] to complete our proof by contradiction. Assume \( r(\xi,t) \) is not uniformly bounded away from 0 which means there exists \( t_i \)

\[
\inf_{\mathbb{S}^n} r(\xi,t_i) \to 0
\]

as \( i \to \infty \), where \( t_i \in [0,T) \). Since \( f \) and \( u_0 \) are even, \( r(\xi,t) \) is even. Thus, \( \Omega_t \) is a origin-symmetric body, where \( \Omega_t \) is the convex body containing the origin and \( \partial \Omega_t = M_t \). Thus, using Theorem 2.4 we have \( \Omega_{t_i} \) (after choosing a subsequence) converges to a origin-symmetric convex body \( \Omega_0 \). Then, we have by Theorem 2.6

\[
\inf_{\xi \in \mathbb{S}^n} r_{\Omega_0}(\xi) = 0.
\]

So, there exists \( \xi_0 \in \mathbb{S}^n \) such that \( r_{\Omega_0}(\xi_0) = 0 \) and thus \( r_{\Omega_0}(-\xi_0) = 0 \), which implies \( \Omega_0 \) contained in a lower-dimensional subspace. This means that

\[
r(\xi,t_i) \to 0
\]
as \( i \to \infty \) almost everywhere with respect to the spherical Lebesgue measure. Combined with bounded convergence theorem, we conclude by Lemma 2.1

\[
\int_{S^n} d\xi \int_0^{r(\xi,0)} \psi(s\xi)s^n ds = \int_{S^n} d\xi \int_0^{r(\xi,t)} \psi(s\xi)s^n ds \to 0
\]
as \( i \to \infty \), which is a contraction. So, we complete our proof. \( \square \)

\( C^0 \) and \( C^1 \) estimates of \( u \) imply the corresponding \( C^0 \) and \( C^1 \) estimates of \( r \) by using (2.3) and Lemma 2.3.

**Corollary 3.1.** Under the assumptions in Theorem 1.6, if \( X(\cdot, t) \) is a strictly convex solution to the flow (1.2) which encloses the origin for \( t \in [0, T) \), then we have

\[
\frac{1}{C} \leq r(\xi, t) \leq C, \quad \forall (\xi, t) \in S^n \times [0, T);
\]

\[
|Dr|(\xi, t) \leq C, \quad \forall (\xi, t) \in S^n \times [0, T),
\]

and

\[
\frac{1}{C} \leq \theta(t) \leq C, \quad \forall t \in [0, T).
\]

4. \( C^2 \)-Estimates

In this section we establish the uniformly upper bound of Gauss curvature, and uniformly positive lower bounds for the principle curvatures for the normalised flow (1.2). We first use the technique that was introduced by Tso [28] to derive the upper bound of the Gauss curvature along the flow (1.2), see also the similar proofs of Lemma 4.1 in [18] and Lemma 5.1 in [2].

**Lemma 4.1.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.2) which encloses the origin for \( t \in [0, T) \). Then, there exists a positive constant \( C \) depending only \( \varphi \), \( \max_{S^n \times [0,T)} u \) and \( \min_{S^n \times [0,T)} u \), such that

\[
\max_{S^n} K(\cdot, t) \leq C, \quad \forall t \in [0, T).
\]

**Proof.** We apply the maximum principle to the following auxiliary function defined on the unit sphere \( S^n \),

\[
W(x, t) = \frac{1}{\theta(t)} \frac{-u_t + u}{u - \varepsilon_0} = \frac{f(x)u}{\varphi(u)\psi(Du + ux)} \frac{K}{u - \varepsilon_0},
\]

where

\[
\varepsilon_0 = \frac{1}{2} \min_{(x,t) \in S^n \times [0,T)} u(x,t) > 0.
\]
At the maximum \( x_0 \) of \( W \) for any fixed \( t \in [0, T) \), we have at \( (x_0, t) \)
\[
0 = \theta(t)W_i = \frac{-u_{ti} + u_i}{u - \varepsilon_0} + \frac{u_t - u}{(u - \varepsilon_0)^2}u_i,
\]
and
\[
0 \geq \theta(t)D^2_{ij}W = \frac{-u_{tij} + u_{ij}}{u - \varepsilon_0} + \frac{(u_t - u)u_{ij}}{(u - \varepsilon_0)^2},
\]
where (4.1) was used in deriving the second equality above. The inequality (4.2) should be understood in sense of positive semi-definite matrix. Hence,
\[
u_{tij} + u_t\delta_{ij} \geq \theta(t)(-b_{ij} + \varepsilon_0\delta_{ij})W + b_{ij}.
\]
Thus,
\[
K_t = -Kb^{ij}(u_{tij} + u_t\delta_{ij}) \leq -nK - \theta(t)KW(-n + \varepsilon_0H),
\]
where \( H \) denotes the mean curvature of \( X(\cdot, t) \). Noticing that \( H \geq nK^{\frac{1}{n}} \), we obtain
\[
K_t \leq CW(1 + W) - CW^{2+\frac{1}{n}}.
\]
Using the equation (2.6) and the inequality above, we have
\[
W_t = \left[ \frac{f(x)u}{\varphi(u)\psi(Du + ux)} \frac{1}{u - \varepsilon_0} \right]K + \left[ \frac{f(x)u}{\varphi(u)\psi(Du + ux)} \frac{1}{u - \varepsilon_0} \right]K_t \leq CW^2 + CW - CW^{2+\frac{1}{n}},
\]
in view of
\[
u_t \approx CW + C,
\]
and
\[
\psi_t = \langle \nabla \psi, X_t \rangle = -\frac{\theta(t)f(\nu)u}{\varphi\psi} \langle \nabla \psi, K \rangle + \langle \nabla \psi, X \rangle \approx CW + C.
\]
Without loss of generality we assume that \( K \approx W \gg 1 \), which implies that
\[
W_t \leq 0.
\]
Therefore, we arrive at \( W \leq C \) for some constant \( C > 0 \) depending on the \( C^1 \)-norm of \( r \) and \( \varepsilon_0 \). Thus, the upper bound of \( K \) follows consequently. \( \square \)

Now, we show the principle curvatures of \( X(\cdot, t) \) are bounded from below along the flow (1.2). The proof is similar to Lemma 4.2 in [18] and Lemma 5.1 in [2].
Lemma 4.2. Let $X(\cdot,t)$ be a strictly convex solution to the flow \[1.2\] which encloses the origin for $t \in [0,T)$. Then, there exists a positive constant $C$ depending only $\varphi$, $\max_{S^n \times [0,T]} u$ and $\min_{S^n \times [0,T]} u$, such that the principle curvatures of $X(\cdot,t)$ are bounded from below
\begin{equation}
\kappa_i(x,t) \geq C, \quad \forall (x,t) \in S^n \times [0,T), \quad i = 1, 2, \ldots, n.
\end{equation}

Proof. We consider the auxiliary function
\begin{equation}
\tilde{\Lambda}(x,t) = \log \lambda_{\max}\{b_{ij}\} - A \log u + B |Du|^2,
\end{equation}
where $A$ and $B$ are positive constants which will be chosen later, and $\lambda_{\max}\{b_{ij}\}$ denotes the maximal eigenvalue of $\{b_{ij} = u_{ij} + u\delta_{ij}\}$. For convenience, we write $\{b^{ij}\}$ for $\{b_{ij}\}^{-1}$.

For any fixed $t \in [0,T)$, we assume the maximum $\tilde{\Lambda}$ is achieved at some point $x_0 \in S^n$. By rotation, we may assume $\{b^{ij}(x_0,t)\}$ is diagonal and $\lambda_{\max}\{b_{ij}\}(x_0,t) = b_{11}(x_0,t)$. Thus, it is sufficient to prove $b_{11}(x_0,t) \leq C$. Then, we define a new auxiliary function
\begin{equation}
\Lambda(x,t) = \log b_{11} - A \log u + B |Du|^2,
\end{equation}
which attains the local maximum at $x_0$ for fixed time $t$. Thus, we have at $x_0$
\begin{equation}
0 = D_i \Lambda = b^{11}b_{11;i} - A \frac{u_i}{u} + 2B \sum_k u_k u_{ki}
\end{equation}
and
\begin{equation}
0 \geq D_i D_j \Lambda = b^{11}b_{11;ij} - (b^{11})^2 b_{11;ij} - A \left( \frac{u_{ij}}{u} - \frac{u_i u_j}{u^2} \right) + 2B \sum_k \left( u_{kj} u_{ki} + u_k u_{kij} \right).
\end{equation}

We can rewrite the equation \[2.6\] as
\begin{equation}
\log(u - u_t) = - \log \det(b_{ij}) + \alpha(x,t),
\end{equation}
where
\begin{equation}
\alpha(x,t) = \log \left( \theta(t) \frac{f(x)u}{\varphi\psi} \right).
\end{equation}

Differentiating \[4.6\], we have
\begin{equation}
\frac{u_k - u_{kt}}{u - u_t} = -b^{ij} b_{ij;k} + D_k \alpha
\end{equation}
and

\[
\frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ii}b_{ii;11} + b^{ii}b^{jj}(b_{ij;1})^2 + D_1D_1\alpha.
\]

(4.8)

Recalling the Ricci identity

\[ b_{ii;11} = b_{11;ii} - b_{11} + b_{ii}, \]

by taking it into (4.8) we have

\[
\frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ii}b_{11;ii} + \sum_i b^{ii}b_{11} - n
+ b^{ii}b^{jj}(b_{ij;1})^2 + D_1D_1\alpha.
\]

(4.9)

So, we have

\[
\frac{\partial_t A}{u - u_t}
= b^{11}\left(\frac{u_{11t} - u_{11}}{u - u_t} + \frac{u_{11} + u - u + u_t}{u - u_t}\right) - A\frac{1}{u - u_t} + 2B\frac{u^k u_{kt}}{u - u_t}
+ \frac{1 - A}{u - u_t} + A\frac{\sum_k u_k u_{kt}}{u - u_t} + (n - 1)b^{11}.
\]

(4.10)

We know from (4.5) and (4.7)

\[
0 \geq b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{11}(b_{11;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_i u_i}{u^2}
+ 2B\left[ b^{ii}(b_{ii} - u)^2 + \sum_k u_k(D_k\alpha - \frac{u_k - u_{kt}}{u - u_t}) - b^{ii}u_i u_i \right]
\geq b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{jj}(b_{ij;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_i u_i}{u^2}
+ 2B\left[ \sum_i b^{ii}(b_{ii}^2 - 2ub_{ii}) + \sum_k u_k(D_k\alpha - \frac{u_k - u_{kt}}{u - u_t}) - b^{ii}u_i u_i \right]
\geq b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{jj}(b_{ij;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_i u_i}{u^2}
+ 2B\left[ \sum_i b_{ii} - 2nu + \sum_k u_k(D_k\alpha - \frac{u_k - u_{kt}}{u - u_t}) - b^{ii}u_i u_i \right].
\]
Thus, plugging the inequality above into (4.10) gives

\[
\frac{\partial_t \Lambda}{u - u_t} \leq -b^{11} D_1 D_1 \alpha - 2B \sum_k u_k D_k \alpha + \frac{1 - A + 2B |Du|^2}{u - u_t} \\
+ \frac{(n + 1)A}{u} + (n - 1)b^{11} + (2B|Du| - A - 1) \sum_i b^{ii} \\
- Ab^{ii} \frac{u_i u_i}{u^2} - 2B \sum_i b_{ii} + 4nBu.
\]

Now, we need estimate the first two terms in the inequality above. Clearly, a direct calculation results in

\[
\psi_i = \langle \nabla \psi, D_iX \rangle = \langle \nabla \psi, e_k \rangle b_{ki}
\]

and

\[
\psi_{11} = \nabla^2 \psi(e_k, e_l)b_{kl}b_{11} + \langle \nabla \psi, D_i e_k \rangle b_{ki} + \langle \nabla \psi, e_k \rangle b_{11,k} \\
= \nabla^2 \psi(e_1, e_1)(b_{11})^2 - \langle \nabla \psi, x \rangle b_{11} + \langle \nabla \psi, e_k \rangle b_{11,k}.
\]

in view of the Gauss formula on $\mathbb{S}^n$.

Hence, we obtain by Lemmas 3.1 and Corollary 3.1

\[
-b^{11} D_1 D_1 \alpha - 2B \sum_k u_k D_k \alpha \\
= -b^{11} \left[ \frac{f_{11}}{f} - \frac{f^2}{f^2} + \frac{u_{11}}{u} - \frac{u_1^2}{u^2} + \frac{(\varphi')^2 u_1^2}{\varphi^2} - \frac{\varphi'' u_1^2}{\varphi} - \frac{\varphi'}{\varphi} u_{11} \right] \\
- b^{11} \left[ \frac{(\nabla \psi, e_k b_{k1})^2}{\psi^2} - \frac{\nabla^2 \psi(e_1, e_1)(b_{11})^2}{\psi} + \frac{\nabla \psi, x b_{11}}{\psi} - \frac{\nabla \psi, e_k b_{11,k}}{\psi} \right] \\
- 2B \sum_k u_k \left( \frac{f_k}{f} + \frac{u_k}{u} - \frac{\varphi' u_k}{\varphi} - \frac{\nabla \psi, e_k b_{kl}}{\psi} \right) \\
\leq Cb^{11}(1 + b_{11} + b^2_{11}) + CB + \frac{\nabla \psi, e_k}{\psi}(b_{11} b_{11,k} + 2Bu b_{kl}) \\
\leq Cb^{11}(1 + b_{11} + b^2_{11}) + CB + \frac{\nabla \psi, e_k}{\psi}(b_{11} b_{11,k} + 2Bu u_{kl}).
\]
Then, using (4.4), we have
\[
-b^{11}D_1D_1\alpha - 2B \sum_k u_k D_k \alpha
\]
\[
\leq C b^{11} (1 + b_{11} + b_{11}^2) + CB + \frac{\langle \nabla \psi, e_k \rangle}{\psi} A \frac{u_k}{u}
\]
\[
\leq C b^{11} (1 + b_{11} + b_{11}^2) + CB + CA.
\]
Thus, using the inequality above, we conclude from (4.11)
\[
\frac{\partial_t \Lambda}{u - u_t} \leq C (b^{11} + 1 + b_{11}) + CB + CA + \frac{1 - A + 2B |Du|^2}{u - u_t} + \frac{(n + 1)A}{u}
\]
\[
+ (n - 1)b^{11} + (2B |Du| - A - 1) \sum_i b^{ii} - Ab^i u_i u_i - 2B \sum_i b^{ii} + 4nBu
\]
\[
< 0,
\]
provided \(b_{11} \gg 1\) and if we choose \(A \gg B\). So we complete the proof. \(\square\)

5. The convergence of the normalised flow

With the help of a prior estimates in the section above, we show the long-time existence and asymptotic behaviour of the normalised flow (1.2) which complete Theorem 1.6.

Proof. Since the equation (2.6) is parabolic, we have the short time existence. Let \(T\) be the maximal time such that \(u(\cdot, t)\) is a positive, smooth and strictly convex solution to (2.6) for all \(t \in [0, T)\). Lemmas 3.1, 4.1 and Corollary 3.1 enable us to apply Lemma 4.2 to the equation (2.6) and thus we can deduce a uniformly lower estimate for the biggest eigenvalue of \(\{(u_{ij} + u\delta_{ij})(x, t)\}\). This together with Lemma 4.2 implies
\[
C^{-1}I \leq (u_{ij} + u\delta_{ij})(x, t) \leq CI, \quad \forall (x, t) \in S^n \times [0, T),
\]
where \(C > 0\) depends only on \(n, \varphi, f\) and \(u_0\). This shows that the equation (2.6) is uniformly parabolic. Using Evans-Krylov estimates and Schauder estimates, we obtain
\[
|u|_{C^{l,m}_x(S^n \times [0, T])} \leq C_{t,m}
\]
for some \(C_{t,m}\) independent of \(T\). Hence \(T = \infty\). The uniqueness of the smooth solution \(u(\cdot, t)\) follows by the parabolic comparison principle.
By the monotonicity of $\mathcal{J}_\varphi$ (See Lemma 2.2), and noticing that

$$|\mathcal{J}_\varphi(X(\cdot,t))| \leq C, \quad \forall \ t \in [0, \infty),$$

we conclude that

$$\int_{0}^{\infty} \frac{d}{dt} |\mathcal{J}_\varphi(X(\cdot,t))| \leq C.$$  

Hence, there is a sequence $t_i \to \infty$ such that

$$\frac{d}{dt} \mathcal{J}_\varphi(X(\cdot,t_i)) \to 0.$$

In view of Lemma 2.2, we see that $u(\cdot,t_i)$ converges smoothly to a positive, smooth and strictly convex $u_\infty$ solving (1.1) with $\frac{1}{c} = \lim_{t_i \to \infty} \theta(t_i).$   \hspace{1cm} \Box

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Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan 430062, P.R. China
E-mail address: chernli@163.com

School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, P.R. China
E-mail address: liuyn@th.btbu.edu.cn

South China Research Center for Applied Mathematics and Interdisciplinary Studies, South China Normal University, Guangzhou 510631, P.R. China
E-mail address: jianlu@m.scnu.edu.cn, lj-tshu04@163.com

Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan 430062, P.R. China
E-mail address: nixiang@hubu.edu.cn