MSSM-like from $SU_5 \times D_4$ Models

R.Ahl Laamara$^{1,2}$, M. Miskaoui$^{1,2}$, E.H Saidi$^{2,3}$

1. LPHE-Modeling and Simulations, Faculty Of Sciences, Mohamed V University, Rabat, Morocco
2. Centre of Physics and Mathematics, CPM- Morocco
3. International Centre for Theoretical Physics, Miramare, Trieste, Italy

April 7, 2016

Abstract

Using finite discrete group characters and symmetry breaking by hyperflux as well as constraints on top- quark family, we study minimal low energy effective theory following from $SU_5 \times D_4$ models embedded in F-theory with non abelian flux. Matter curves spectrum of the models is obtained from $SU_5 \times S_5$ theory with monodromy $S_5$ by performing two breakings; first from symmetric group $S_5$ to $S_4$ subsymmetry; and next to dihedral $D_4$ subgroup. As a consequence, and depending on the ways of decomposing triplets of $S_4$, we end with three types of $D_4$- models. Explicit constructions of these theories are given and a MSSM-like spectrum is derived.

Key words: F-GUT models with discrete symmetries, Characters of discrete groups, $SU_5 \times D_4$ models; MSSM like.

1 Introduction

Recently, there has been an increasing interest in building $SU_5 \times \Gamma$ GUT models, with discrete symmetries $\Gamma$, embedded in Calabi-Yau compactification of F-theory down to 4d space time [1]-[11]; and in looking for low energy minimal prototypes with broken monodromies [12]-[19]. This class of supersymmetric GUTs with discrete groups lead to quasi-realistic field spectrum having quark and lepton mass matrices with properties fitting with MSSM requirements. In the geometric engineering of these F-GUTs, splitting spectral cover method together with Galois theory tools are used to generate appropriate matter curves spectrum [20]-[25]; and a geometric $Z_2$ parity has been also introduced to

*e-mail: h-saidi@fsr.ac.ma
suppress unwanted effects such as exotic couplings and undesired proton decay operators \[26, 27, 28, 29\].

In this paper, we develop another manner to deal with monodromy of F-GUT that is different from the one proposed first in \[18\], and further explored in \[27, 30, 31\], where matter curves of the same orbit of monodromy are identified. In our approach, we use the non abelian flux conjecture of \[15, 16\] to think of monodromy group of F- theory \(SU_5\) models as a non abelian flavor symmetry \(\Gamma\). Non trivial irreducible representations of the non abelian discrete group \(\Gamma\) are used to host the three generations of fundamental matter; a feature that opens a window to build semi-realistic models with matter curves distinguished from each other in accord with mass hierarchy and mixing neutrino physics \[32, 33, 34\].

In this work, we study the family of supersymmetric \(SU_5 \times \Gamma_p \times U(1)^{5-p}\) models in the framework of F-theory GUT; with non abelian monodromies \(\Gamma_p\) contained in the permutation group \(S_5\) \[30]-[42\]; and analyse the realisation of low energy constraints under which one can generate an effective field spectrum that resembles to MSSM. A list of main constraints leading to a good low energy spectrum are described in section 5; it requires amongst others a tree- level Yukawa coupling for top-quark family. To realise this condition with non abelian \(\Gamma_p\), we consider the case where \(\Gamma_p\) is given by the order 8 dihedral group \(D_4\); this particular non abelian discrete symmetry has representations which allow more flexibility in accommodating matter generations. Recall that the non abelian alternating \(A_4\) group has no irreducible doublet as shown on the character relation \(12 = 3^2 + 1^2 + 1^2 + 1^2\); and the irreducible representation of non abelian \(S_4\) and \(S_3\), which can be respectively read from \(24 = 3^2 + 3^2 + 2^2 + 1^2 + 1^2\) and \(6 = 2^2 + 1^2 + 1^2\), have a doublet and two singlets. The non abelian dihedral group \(D_4\) however has representations \(R_i\) with dimensions, that can be read from \(8 = 2^2 + 1^2 + 1^2 + 1^2 + 1^2\), seemingly more attractable phenomenologically; it has 5 irreducible \(R_i\)’s; four singlets, indexed by their basis characters as \(1_{++}, 1_{+-}, 1_{-+}, 1_{--}\); and an irreducible doublet \(2_{00}\); offering therefore several pictures to accommodate the three generations of matter of the electroweak theory; in particular more freedom in accommodating top quark family.

To deal with the engineering of \(SU_5 \times D_4\)- models, we develop a new method based on finite discrete group characters \(\chi_{R_i}\); avoiding as a consequence the complexity of Galois theory approach. The latter is useful to study F- theory models with the dihedral \(D_4\) and the alternating \(A_4\) subgroups of \(S_4\) as they are not directly reached by the standard splitting spectral cover method; they are obtained in Galois theory by putting constraints on the discriminant of underlying spectral covers; and introducing other monodromy invariant of the covers such a resolvent \[14, 15, 29\].

To derive the \(D_4\)- matter curves spectrum in \(SU_5 \times D_4\)- models, we think of it in terms
of a two steps descent from $S_5$- theory; a first descent down to $S_4$; and a second one to $D_4$ by turning on appropriate flux that will be explicitly described in this work; see also appendix C. By studying all scenarios of breaking the triplets $S_4$- theory in terms of irreducible $D_4$- representations, we end with three kinds of $D_4$- models; one having a field spectrum involving all $D_4$- representations including doublet $2_{00}$ \textit{(model I)}; the second theory \textit{(model II)} has no doublet $2_{00}$ nor the singlet $1_{--}$; and the third model has no $2_{00}$; but does have $1_{--}$. We have studied the curves spectrum of the three $D_4$-models; and we have found that only model III allows a tree level 3-couplings and exhibits phenomenologically interesting features.

The presentation is as follows: In section 2, we study the $SU_5 \times S_5$ model; and describes the picture of the two steps breaking $S_5 \rightarrow S_4 \rightarrow S_3$ by using standard methods. In section 3, we introduce our method; and we revisit the construction of the $S_4$- and $S_3$- models from the view of discrete group characters. In section 4, we use character group method to build three $SU_5 \times D_4 \times U_1^+$ models. In section 5, we solve basic conditions for deriving MSSM like spectrum from $SU_5 \times D_4 \times U_1^+$ models. In section 6, we conclude and make discussions. Last section is devoted to three appendices: In appendix A, we give relations regarding group characters. In appendix B, we report details on other results obtained in this study; and in appendix C we exhibit the link between non abelian monodromies and flavor symmetry.

2 Spectral Covers in $SU_5 \times \Gamma$ models

In F-GUT models with $SU_5$ gauge symmetry, matter curves carry quantum numbers in $SU_5 \times SU_5^\perp$ bi-representations following from the breaking of $E_8$ as given below

$$
248 \rightarrow (24, 1_\perp) \oplus (1, 24_\perp) \oplus \\
(10, 5_\perp) \oplus (\overline{10}, \bar{5}_\perp) \oplus \\
(5, 10_\perp) \oplus (\overline{5}, \overline{10}_\perp)
$$

(2.1)

In this $SU_5$ theory, the perpendicular $SU_5^\perp$ is restricted to its Cartan-Weyl subsymmetry $(U_1^+)^4$, see appendix C for some explicit details; and the matter content of the model is labeled by five weights $t_i$ like

$$
10_{t_i}, \overline{10}_{-t_i}, \bar{5}_{t_i+t_j}, \ 5_{-t_i-t_j}, \ 1_{t_i-t_j}
$$

(2.2)

with traceless condition

$$
t_1 + t_2 + t_3 + t_4 + t_5 = 0
$$

(2.3)

The components of the five 10-plets $10_{t_i}$ and those of the ten 5-plets $\bar{5}_{t_i+t_j}$ are related to each other by monodromy symmetries $\Gamma$; offering a framework of approaching GUT
- models with discrete symmetries originating from geometric properties of the elliptic
Calabi-Yau fourfold CY4 which, naively, can be thought of as given by the 4- dim
complex space
\[ CY4 \sim E \times B_3 \] (2.4)
In this fibration, the complex 3- dim base \( B_3 \) contains the complex GUT surface \( \mathcal{S}_{GUT} \)
wrapped by 7-brane; and the complex elliptic curve E fiber is as follows
\[ y^2 = x^3 + b_5 xy + b_4 x^2 z + b_3 y z^2 + b_2 x z^3 + b_0 z^5 \] (2.5)
where the homology classes \([x]\), \([y]\), \([z]\) and \([b_k]\); associated with the holomorphic sections
\( x, y, z \) and \( b_k \), are expressed in terms of the Chern class \( c_1 = c_1 (\mathcal{S}_{GUT}) \) of the tangent
bundle of the \( \mathcal{S}_{GUT} \) surface; and the Chern class \( -t \) of the normal bundle \( N_{\mathcal{S}_{GUT}|B_3} \) like
\[
[y] = 3 (c_1 - t) \quad , \quad [z] = -t \\
[x] = 2 (c_1 - t) \quad , \quad [b_k] = (6c_1 - t) - kc_1 \] (2.6)

2.1 Matter curves in \( SU_5 \times S_5 \) model
Matter curves of \( SU_5 \times U(1)^{5-k} \times \Gamma_k \) models live on GUT surface \( \mathcal{S}_{GUT} \) with monodromy
symmetries \( \Gamma_k \) contained in \( S_5 \), the Weyl group of \( SU_5^w \); see eq(9.8) of appendix C. In the
case of \( \Gamma_5 = S_5 \); these curves organise into reducible multiplets \( \mathbf{1} \) of \( S_5 \) with the following
characteristic properties

| matters curves | weights | \( S_5 \) repres | homology classes | holomorphic sections |
|---------------|---------|----------------|-----------------|---------------------|
| \( 10_{t_i} \) | \( t_i \) | 5 | \( \eta - 5c_1 \) | \( b_5 = b_0 \prod_{i=1}^{5} t_i \) |
| \( 5_{t_i+t_j} \) | \( t_i + t_j \) | 10 | \( \eta' - 10c_1 \) | \( d_{10} = d_0 \prod_{j>i=1}^{5} T_{ij} \) |
| \( 1_{t_i-t_j} \) | \( t_i - t_j \) | 20 | \( \eta'' - 20c_1 \) | \( g_{20} = g_0 \prod_{i \neq j=1}^{5} S_{ij} \) |

where the \( t_i \)'s as above; \( T_{ij} = t_i + t_j \) with \( i < j \); and \( S_{ij} = t_i - t_j \) with \( i \neq j \). These
\( t_i \)'s, \( T_{ij} \)'s; and \( S_{ij} \)'s are respectively interpreted as the simple zeros of the spectral covers
\( C_5 = 0 \) describing ten-plets, \( C_{10} = 0 \) describing five-pelts and \( C_{20} = 0 \) for flavon singlets

\[ ^1 \text{An equivalent spectrum can be also given by using irreducible representations of } S_5 \text{ and their characters; to fix ideas see the analogous } S_4- \text{ and } S_3- \text{ models studied in section 3.} \]
The homology classes of the complex curves in (2.7) are nicely obtained by defining the spectral covers in terms of the usual holomorphic sections; for the 5-sheeted covering of $S_{GUT}$, we have

$$C_5 = b_0\prod_{i=1}^{5} (s - t_i) \equiv b_0 \prod_{i=1}^{5} s_i$$

$$C_{10} = d_0 \prod_{j>i=1}^{5} (s - T_{ij}) \equiv d_0 \prod_{j>i=1}^{5} s_{ij}$$

$$C_{20} = g_0 \prod_{i\neq j}^{5} (s - S_{ij}) \equiv g_0 \prod_{i\neq j}^{5} s'_{ij}$$

(2.8)

with

$$b_1 = 0$$

due to traceless condition; and homology classes of the complex holomorphic sections $b_k$ as follows

| holomorphic sections | homology classes |
|---------------------|-----------------|
| $s$                 | $-c_1$          |
| $b_k$               | $\eta - k c_1$ |

(2.10)

with canonical homology class $\eta$ given by

$$\eta = 6c_1 - t$$

(2.11)

with $c_1$ and $-t$ as in eqs (2.6). From these relations, the homology class $[10_n] = [C_5]_{s=0}$ is given by $[b_5]$; by using $b_5 = b_0 \prod_{i=1}^{5} t_i$, we have $[b_5] = \eta - 5c_1$ in agreement with (2.6). For the 10-sheeted covering, we have

$$C_{10} = \sum_{k=0}^{10} d_k s^{10-k}$$

(2.12)

and leads to the homology class $[d_{10}] = \eta' - 10c_1$ where, due to $d_0 = b_0^3$, the class $\eta'$ can be related to the canonical $\eta$ of the 5-sheeted cover like $3\eta$. Similar relation can be written down for singlets

$$C_{20} = \sum_{k=0}^{20} g_k s^{20-k}$$

(2.13)

leading to $[g_{20}] = \eta'' - 20c_1$ with the property $\eta'' = 9\eta$.

For later use, we consider together with (2.7) the so called geometric $Z_2$ parity of [19];
but as approached in [14, 15] in dealing with local models. For simplicity, we use a short way to introduce this parity by requiring, up to an overall phase, invariance of $C_5 = 0$, $C_{10} = 0$, $C_{20} = 0$ under the following transformations along the spectral fiber; see [14, 15, 16] for explicit details,

\begin{align}
    s_i' &= e^{-i\phi}s_i \\
    b_k' &= e^{i(\beta+(5-k)\phi)}b_k \\
    d_k' &= e^{i(\gamma+(10-k)\phi)}d_k \\
    g_k' &= e^{i(\delta+(20-k)\phi)}g_k
\end{align} \tag{2.14}

Under this phase change, the spectral covers eqns transform like

\begin{align}
    C_5' &= e^{i\beta}C_5 \\
    C_{10}' &= e^{i\gamma}C_{10} \\
    C_{20}' &= e^{i\delta}C_{20}
\end{align} \tag{2.15}

Focussing on 10-plets, and equating above $C_5'$ with the one deduced from construction of [16] namely $C_5' = e^{i(\zeta-\phi)}C_5$; we learn that we should have $\beta = \zeta - \phi$; and therefore $b_k' = e^{i(\zeta+(k-6)\phi)}b_k$. For the particular choice $\phi = \pi$, we have $s_i' = -s_i$ and

\begin{align}
    b_k' &= (-)^k e^{i\zeta}b_k
\end{align} \tag{2.16}

If we put $\zeta = 0$, we get $(b_0', b_5') = (+b_0, -b_5)$; while by taking $\zeta = \pi$, we have $(b_0', b_5') = (-b_0, +b_5)$; below we set $\zeta = \pi$. To get the parity of the holomorphic sections $d_k$ and $g_k$ of eqs (2.8), we use their relationships with the $b_k$ coefficients. By help of the relations $d_{10} = b_4^5b_4 - b_2b_3b_5 + b_0b_5^2$ and $g_{20} = 256b_5^4b_0^4 + \ldots$, it follows that $Z_2(d_{10}) \sim Z_2(b_4^5b_4)$ and $Z_2(g_{20}) = Z_2(b_5^4b_0^4)$; so we have [27, 30, 31]

\begin{align}
    Z_2(d_{10}) &= -1 , \quad Z_2(g_{20}) = -1 , \quad Z_2(b_5) = +1 \\
    Z_2(d_0) &= -1 , \quad Z_2(g_0) = -1 , \quad Z_2(b_0) = -1
\end{align} \tag{2.17}

in agreement with the homology class properties $\eta' = 3\eta$ and $\eta'' = 9\eta$.

## 2.2 Models with broken $S_5$

To engineer matter curves with monodromy $\Gamma_k \subset S_5$; we generally use spectral cover splitting method combined with constraints inspired from Galois theory [14, 15, 16, 26, 27]. In this study, we develop a new method without need of the involved tools of Galois group theory; our approach uses characters $\chi_R(g)$ of discrete group representations; and relies directly the roots of the spectral covers. To illustrate the method; but also for later use, we first study the two interesting cases by using the standard method:
• $\Gamma_4 = S_4 \subset S_5$,
• $\Gamma_3 = S_3 \subset S_5$.

The case $\Gamma_4 = D_4$ requires more tools; it will be studied later after revisiting $S_4$- and $S_3$-models from the view of characters of their representations.

2.2.1 $S_4$- model in standard approach

To engineer the breaking of $S_5$ down to $S_4$, we proceed as follows: First, we use $S_5$-invariance to rewrite the holomorphic polynomial $C_5$ like

$$C_5 = \frac{b_0}{5!} \sum_{\sigma \in \Gamma_5} \prod_{i=1}^{5} (s - t_{\sigma(i)})$$

(2.18)

and similarly for $C_{10}$ and $C_{20}$. To break $S_5$ down to $S_4$, we impose a condition fixing one of the weight $[51]$; for example

$$\sigma(t_5) = t_5 \iff \sigma(5) = 5$$

(2.19)

This requirement breaks $S_5$ down to one of the five possible $S_4$ subgroups living inside $S_5$; and leads to the following features:

(a) the traceless condition (2.3) of the orthogonal $SU_5^1$ is solved as $t_5 = -(t_1 + t_2 + t_3 + t_4)$; it is manifestly $S_4$- invariant. To deal with this $t_5$ weight, we shall think about the breaking of $S_5$ down to $S_4$ in terms of the descent of the symmetry $SU_5 \times U(1)^{5-k} \times \Gamma_k$ from $k=5$ to $k=4$ as follows [58, 59]

$$SU_5 \times U(1)^{5-5} \times S_5 \to SU_5 \times U(1)^{5-4} \times S_4$$

$$\sim SU_5 \times S_4 \times U(1)$$

(2.20)

(b) the spectral covers $C_5$ and $C_{10}$ split as the product of two factors: (α) the spectral cover $C_5$ factorises like $C_4 \times C_1$ with

$$C_4 = A_0 \prod_{i=1}^{4} (s - t_i), \quad C_1 = a_0 (s - t_5)$$

(2.21)

and

$$b_0 = A_0 \times a_0$$

$$b_5 = A_4 \times a_1$$

(2.22)

together with the transformations following from (2.14,2.15). Notice that the above factorisations put conditions on the field $K$ where live the holomorphic sections; a feature

---

2 The holomorphic sections $A_i$ and $a_m$ eqs (2.21) are directly derived by expanding the factorised forms of the spectral covers $C_4$ and $C_1$; we will not give these details here; for example the relevant $A_4$ and $a_1$ are given by $A_4 = A_0 \prod_{i=1}^{4} t_i$ and $a_1 = -a_0 t_5$. 

---
that is also predicted by Galois theory \cite{28, 29}. As a naive illustration, we use the comparison with arithmetics in the set of integers $\mathbb{Z}$; an integer number like 6 can be factorised in $\mathbb{Z}$ as $6 = 2 \times 3$; while a prime integer like 5 has no factorisation. By using $C'_5 = C'_4 \times C'_1$ and equating $e^{i(\zeta - \phi)} (C_4 \times C_1)$ with $(e^{i\xi}C_4) \times (e^{i\psi}C_1)$; it follows that $\zeta - \phi = \xi + \psi$; and

$$ A'_4 = e^{i\xi}A_4, \quad a'_1 = e^{i(\zeta - \xi - \phi)}a_1 $$

(2.23)

from which we learn that $A_4$ and $a_1$ sections transform differently; and then $Z_2 (b_4) = Z_2 (A_4) \times Z_2 (a_1)$. (\beta) the $C_{10}$ splits in turns like $\tilde{C}_6 \times \tilde{C}_4$ with

$$ C_6 = \tilde{A}_0 \prod_{j>i=1}^4 (s - T_{ij}), \quad C_4 = \tilde{a}_0 \prod_{i=1}^4 (s - T_{i5}) $$

(2.24)

and

$$ d_0 = \tilde{A}_0 \times \tilde{a}_0, \quad d_{10} = \tilde{A}_6 \times \tilde{a}_4 $$

(2.25)

as well as $\tilde{C}_6 = e^{2i\tilde{\xi}}\tilde{C}_6$ and $\tilde{C}_4 = e^{2i\tilde{\psi}}\tilde{C}_4$ with $\tilde{\xi} + \tilde{\psi} = \tilde{\zeta} - \phi$.

Under the above splitting, the spectrum (2.7) decomposes in terms of reducible $S_4$ multiplets as follows

| curves | weights | $S_4$ | $U^+$ | homology | sections | $Z_2$ | U$(1)_Y$ flux |
|--------|---------|------|-------|----------|----------|------|----------------|
| 10$t_i$ | $t_i$   | 4    | 0     | $\eta - 4c_1 + \chi$ | $A_4$ | $\varsigma_4$ | $N$ |
| 10$t_5$ | $t_5$   | 1    | 1     | $-\chi - c_1$ | $a_1$ | $\varsigma_1$ | $-N$ |
| 5$t_i + t_j$ | $t_i + t_j$ | 6 | 0 | $\eta' -6c_1 + \tilde{\chi}$ | $\tilde{A}_6$ | $\tilde{\varsigma}_6$ | $N$ |
| 5$t_i + t_5$ | $t_i + t_5$ | 4 | 1 | $-\tilde{\chi} - 4c_1$ | $\tilde{a}_4$ | $\tilde{\varsigma}_4$ | $-N$ |

(2.26)

with

$$ A_4 = A_0 \prod_{i=1}^4 t_i, \quad \tilde{A}_6 = \tilde{A}_0 \prod_{j>i=1}^4 T_{ij} $$

$$ a_1 = a_0 t_5, \quad \tilde{a}_4 = \tilde{a}_0 \prod_{i=1}^4 T_{i5} $$

(2.27)

and where $\varsigma_i$ and $\tilde{\varsigma}_k$ refer to $Z_2$ parities; for instance

$$ \varsigma_4 = Z_2 (A_4), \quad \tilde{\varsigma}_6 = Z_2 (\tilde{A}_6) $$

$$ \varsigma_1 = Z_2 (a_1), \quad \tilde{\varsigma}_4 = Z_2 (\tilde{a}_4) $$

$$ \varsigma_4 \varsigma_1 = Z_2 (b_5), \quad \tilde{\varsigma}_4 \tilde{\varsigma}_6 = Z_2 (d_{10}) $$

(2.28)

The last column of eq(2.26) refers to the hyperflux of the U$(1)_Y$ gauge field strength; it breaks SU$_5$ gauge symmetry down to standard model gauge invariance; and also pierces the matter curves of the model as shown on table.
2.2.2 $S_3$- model in standard approach

The breaking of $S_5$ down to $S_3$ may be obtained from above $S_4$ model by further breaking $S_4$ down to $S_3$; this corresponds to $SU_5 \times U(1)^{5-5} \times S_5 \to SU_5 \times U(1)^{5-3} \times S_3$. This can be realised by fixing one of the four $t_i$ roots; say $t_4$; so that the breaking pattern is given by

$$SU_5 \times U(1)^{5-5} \times S_5 \to SU_5 \times U(1)^{5-3} \times S_3$$

$$\sim SU_5 \times S_3 \times U(1)^2$$

(2.29)

Setting $U(1)^2 = U_{1}^{\perp} \times U_{1}^{\perp}$, the previous $S_4$ spectrum decomposes into reducible $S_3$ multiplets as follows,

| curves | $S_3$ | $U_{1}^{\perp} \times U_{1}^{\perp}$ | homology | section | $U(1)_Y$ flux |
|--------|------|----------------------------------|----------|---------|----------------|
| $10_{t_1}$ | 3 | (0, 0) | $\eta - 3c_1 - \chi - \chi'$ | $A'_3$ | $-N - P$ |
| $10_{t_4}$ | 1 | (1, 0) | $\chi' - c_1$ | $A'_1$ | $P$ |
| $10_{t_5}$ | 1 | (0, 1) | $\chi - c_1$ | $a_1$ | $N$ |
| $5_{t_1+t_2}$ | 3 | (0, 0) | $\eta' - 3c_1 - \tilde{\chi} - \tilde{\chi}'$ | $A''_3$ | $-N - P$ |
| $5_{t_1+t_3}$ | 3 | (1, 0) | $\tilde{\chi}' - 3c_1$ | $\tilde{A}'_3$ | $P$ |
| $5_{t_1+t_4}$ | 3 | (0, 1) | $\tilde{\chi} - 3c_1 - \tilde{\chi}'$ | $\tilde{a}'_3$ | $N - P$ |
| $5_{t_1+t_5}$ | 1 | (0, 1) | $\tilde{\chi}' - c_1$ | $\tilde{a}'_1$ | $P$ |

(2.30)

with

$$b_5 = (A'_3 A'_1) \times a_1$$

$$d_{10} = (\tilde{A}'_3 \tilde{A}'_3) \times (\tilde{a}'_3 \tilde{a}'_3)$$

(2.31)

where $A'_3$, $A'_1$, $a_1$ and $\tilde{A}'_3$, $\tilde{a}'_3$ are given by relations of form as in (2.27). An extra column for $Z_2$- parity can be also added as in (2.26) with the property

$$Z_2(b_5) = Z_2(A'_3) \times Z_2(A'_1) \times Z_2(a_1)$$

$$Z_2(d_{10}) = Z_2(\tilde{A}'_3) \times Z_2(\tilde{A}'_3) \times Z_2(\tilde{a}'_3) \times Z_2(\tilde{a}'_3)$$

(2.32)

Observe also that here we have two new homology class cycles $\chi$ and $\chi'$ with

$$\int_{\chi} F_X = N$$

$$\int_{\chi'} F_X = P$$

(2.33)

The non zero $P$ is responsible for the second splitting; this is because the breaking of $S_5$ down to $S_3$ has been undertaken into two stages: first $S_5 \to S_4$; and second $S_4 \to S_3$. In what follows we extend this idea to the breaking pattern of $S_5$ down to $D_4$.

3 Revisiting $S_4$ and $S_3$- models

In this section, we develop tools towards the study of the breaking of $S_5$ monodromy down to its $D_4$ sub-symmetry. To our knowledge these tools, have not been used before;
even for $S_n$ permutation groups; so we begin by revisiting the $S_4$- and $S_3$- models from the view of characters of their irreducible representations; and turn in next section to develop the $D_4$ theory.

3.1 $SU_5 \times S_4 \times U_1^\perp$ model

In the canonical $t_i$-weight basis, the matter spectrum of $S_4$- model is given by (2.26); there matter curves are organised into reducible multiplets of $S_4 \times U_1^\perp$. Below, we give another manner to approach the spectrum of $S_4$- model. By help of the standard relation $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$ showing that $S_4$ has 5 irreducible representations $R_i$ and 5 conjugacy classes $C_i$ [39, 40, 41, 42]; and by using properties of the irreducible $R_i$ representations of $S_4$ given in appendix; eq(2.26) may be expressed in terms of the $R_i$’s and their $\chi_R$ characters as follows

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{curves} & \text{weights} & \text{Irrep} S_4 & \chi_{R_i}^{(a,b,c)} & U_1^\perp & \text{homology} & U(1)_Y \text{ flux} \\
\hline
10_{x_i} & x_i & 3 & (1,0,-1) & 0 & \eta - 3c_1 & 0 \\
10_{x_4} & x_4 & 1 & (1,1,1) & 0 & \chi - c_1 & N \\
10_{t_5} & t_5 & 1 & (1,1,1) & 1 & -\chi - c_1 & -N \\
5_{x_{ij}} & X_{ij} & 3' & (-1,0,1) & 0 & \eta' - 3c_1 & 0 \\
5_{x_{i4}} & X_{i4} & 3 & (1,0,-1) & 0 & -3c_1 + \chi' & N \\
5_{x_{i5}} & X_{i5} & 3 & (1,0,-1) & 0 & -3c_1 - \chi' & -N \\
5_{x_{45}} & X_{45} & 1 & (1,1,1) & 1 & -c_1 & 0 \\
\hline
\end{array}
\]

(3.1)

Notice that $S_4$ has three generators denoted here by $(a,b,c)$ and chosen as given by 2-, 3- and 4-cycles; they obey amongst others the cyclic properties $a^2 = b^3 = c^4 = I_{id}$; these three generators are non commuting permutation operators making extraction of full information from them a difficult task; but part of this information is given their $\chi_{R_i}^{(a,b,c)}$s; these characters are real numbers as collected in following table [39, 40, 41, 42],

\[
\begin{array}{|c|c|c|c|c|}
\hline
\chi_{ij} & \chi_I & \chi_{a'} & \chi_2 & \chi_3 \\
\hline
a & 1 & -1 & 0 & 1 & -1 \\
b & 1 & 0 & -1 & 0 & 1 \\
c & 1 & 1 & 0 & -1 & -1 \\
\hline
\end{array}
\]

(3.2)

Notice also that the 4- and 6- representations of $S_4$, which have been used in the canonical formulation of section 2, are decomposed in (3.1) as direct sums of irreducible components as follows:

\[
\begin{align*}
4_{(2,1,0)} & = 1_{(1,1,1)} \oplus 3_{(1,0,-1)} \\
6_{(0,0,0)} & = 3_{(1,0,-1)} \oplus 3'_{(-1,0,1)}
\end{align*}
\]

(3.3)
Notice moreover that the previous $t_i$-weights are now replaced by new quantities $x_i$ given by some linear combinations of the $t_i$'s fixed by representation theory of $S_4$. One of these weights; say $x_4$, is given by the usual completely $S_4$-symmetric term

$$x_4 \sim (t_1 + t_2 + t_3 + t_4) \quad (3.4)$$

transforming in the trivial representation of $S_4$; the three other $x_i$ are given by some orthogonal linear combinations of the four $t_i$'s that we express as follows

$$x_i = \alpha_i t_1 + \beta_i t_2 + \gamma_i t_3 + \delta_i t_4 \quad (3.5)$$

These three weights transform as an irreducible triplet of $S_4$; but seen that we have two kinds of 3-dim representations in $S_4$ namely $3$ and $3'$, the explicit expressions of (3.5) depend in which of the two representations the $x_i$'s are sitting; details are reported in appendix where one also finds the relationships $t_\mu = U_{\mu\rho} x_\rho$ and $t_\mu \pm t_\nu = (U_{\mu\rho} \pm U_{\nu\rho}) x_\rho$.

Notice finally that the explicit expressions of $X_{\mu\nu}$ weights in (3.1) are not needed in our approach; their role will be played by the characters of the representations.

### 3.2 $SU_5 \times S_3 \times (U_1^\perp)^2$ model

The spectrum of GUT- curves of the $SU_5 \times S_3 \times (U_1^\perp)^2$ model follows from the spectrum of the $SU_5 \times S_5$ theory by using splitting spectral method. By working in the canonical basis for $t_i$-weights, this spectrum, expressed in terms of reducible multiplets, is given by (2.30). Here, we revisit the $SU_5 \times S_3 \times (U_1^\perp)^2$ curves spectrum by using irreducible representations of $S_3$ and their characters.

We start by recalling that $S_3$ has three irreducible representations as shown of the usual character relation $6 = 1^2 + 1^2 + 2^2$ linking the order of $S_3$ to the squared dimensions of its irreducible representations; these irreducible representations are nicely described in terms of Young diagrams [42]

$$1: \begin{array}{ccc} \, & \, & \, \\ \, & \, & \, \\ \, & \, & \, \end{array}, \quad 2: \begin{array}{c} \, \\ \, \\ \, \end{array}, \quad 1': \begin{array}{c} \, \\ \, \end{array} \quad (3.6)$$

The group $S_3$ is a non abelian discrete group; it has two non commuting generators $(a, b)$ satisfying $a^2 = b^3 = 1$ with characters as follows

$$\begin{array}{c|c|c|c} \chi_R & \chi_t & \chi_2 & \chi_3 \\ \hline a & 1 & 0 & -1 \\ b & 1 & -1 & 1 \end{array} \quad (3.7)$$

11
The spectrum of matter curves in the $S_3$-model is obtained here by starting from the $S_4$ spectrum $(t_1, t_2, t_3)$ \[2.30\]; and then breaking $S_4$ monodromy to $S_3 \times S_1$. We find

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{curves} & \text{weights} & \text{Irrep } S_3 & \chi_R^{(a,b)} & U_1^\perp & \text{homology} & U(1)_Y \text{ flux} \\
\hline
10_{x_i} & x_i & 2 & (0, -1) & 0 & \eta - 2c_1 - \chi' & -P \\
10_{x_3} & x_3 & 1 & (1, 1) & 0 & -\chi - c_1 & -N \\
10_{x_4} & x_4 & 1 & (1, 1) & 0 & \chi' - c_1 & P \\
10_{t_5} & t_5 & 1 & (1, 1) & 1 & \chi - c_1 & N \\
5_{X_{ij}} & X_{ij} & 2 & (0, -1) & 0 & \eta' - 2c_1 & 0 \\
5_{X_{i3}} & X_{i3} & 1 & (-1, 1) & 0 & -c_1 - \chi' - \chi & -P - N \\
5_{X_{i4}} & X_{i4} & 2 & (0, -1) & 0 & -2c_1 & 0 \\
5_{X_{34}} & X_{34} & 1 & (1, 1) & 0 & \chi' - c_1 & P \\
5_{X_{i5}} & X_{i5} & 2 & (0, -1) & 0 & -2c_1 & 0 \\
5_{X_{35}} & X_{35} & 1 & (1, 1) & 0 & -c_1 - \chi' + \chi & N - P \\
5_{X_{45}} & X_{45} & 1 & (1, 1) & 1 & \chi' - c_1 & P \\
\hline
\end{array}
\]

where the integers $P$ and $N$ are as in eq\[2.33\].

### 4 $SU_5 \times D_4$ models

First notice that the engineering of the $SU_5 \times D_4 \times U_1^\perp$ theory has been recently studied in \[16\] by using Galois theory; but here we use a method based on characters of the irreducible representations of $D_4$; and finds at the end that there are in fact three kinds of $SU_5 \times D_4 \times U_1^\perp$ models; they are explicitly constructed in this section. To that purpose, we first review useful aspects on characters of the dihedral group; then we turn to construct the three $D_4 \times U_1^\perp$ models.

#### 4.1 Characters in $D_4$ models

The dihedral $D_4$ is an order 8 subgroup of $S_4$ with no 3-cycles; there are three kinds of such subgroups inside $S_4$; an example of $D_4$ subgroup is the one having the following elements

\[
I_{id}, \quad \begin{pmatrix} 24 \\ 13 \end{pmatrix}, \quad \begin{pmatrix} 13 \\ 24 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 34 \end{pmatrix}, \quad \begin{pmatrix} 14 \\ 23 \end{pmatrix}, \quad \begin{pmatrix} 1234 \\ 1432 \end{pmatrix}
\]

(4.1)
with non commuting generators \( a = \langle (24) \rangle \) and \( b = \langle (1234) \rangle \) satisfying \( a^2 = b^4 = I \) and \( aba = b^3 \). The two other \( \mathbb{D}_4' \) and \( \mathbb{D}_4'' \) have similar contents; but with other transpositions and 4-cycles. In terms of \((a, b)\) generators, the eight elements (4.1) of the dihedral \( \mathbb{D}_4 \) reads as

\[
I_{id}, \quad a, \quad b, \quad a^2, \quad b^2, \quad a^3, \quad b^3, \quad b^4
\]

they form 5 conjugacy classes as follows

\[
\mathcal{C}_1 = \{I_{id}\}, \quad \mathcal{C}_2 = \{b^2\}, \quad \mathcal{C}_3 = \{b, b^3\}, \\
\mathcal{C}_4 = \{a\}, \quad \mathcal{C}_5 = \{ab\} \tag{4.3}
\]

The dihedral group \( \mathbb{D}_4 \) has also 5 irreducible representations \( R_i \); this can be directly learnt on the character formula \( 8 = 1^2_1 + 1^2_2 + 1^2_3 + 1^2_4 + 2^2 \), linking the order of \( \mathbb{D}_4 \) with the sum of \( d_i^2 \), the squares of the dimensions \( d_i \) of the irreducible \( R_i \) representations of \( \mathbb{D}_4 \). So, the order 8 dihedral group has four irreducible representations with 1-dim; and a fifth irreducible \( \mathbb{D}_4 \)- representation with 2-dim \([42]\). The character table of \( \mathbb{D}_4 \) representations is given by

| \( \mathcal{C}_i \) \( \chi_{R_i} \) | \( \chi_1 \) | \( \chi_{1_2} \) | \( \chi_{1_3} \) | \( \chi_{1_4} \) | \( \chi_2 \) | number |
|---|---|---|---|---|---|---|
| \( \mathcal{C}_1 \) | 1 | 1 | 1 | 1 | 2 | 1 |
| \( \mathcal{C}_2 \) | 1 | 1 | 1 | 1 | -2 | 1 |
| \( \mathcal{C}_3 \) | 1 | 1 | -1 | -1 | 0 | 2 |
| \( \mathcal{C}_4 \) | 1 | -1 | 1 | -1 | 0 | 2 |
| \( \mathcal{C}_5 \) | 1 | -1 | -1 | 1 | 0 | 2 |

(4.4)

from which we learn the following characters of the \((a, b)\) generators

| \( \chi_{ij}^{(g)} \) | \( \chi_1 \) | \( \chi_{1_2} \) | \( \chi_{1_3} \) | \( \chi_{1_4} \) | \( \chi_2 \) |
|---|---|---|---|---|---|
| \( a \) | 1 | -1 | 1 | -1 | 0 |
| \( b \) | 1 | 1 | -1 | -1 | 0 |

(4.5)

For other features see [41]. With these tools at hand, we turn to engineer the \( SU_5 \times \mathbb{D}_4 \times U_1^\perp \) models with dihedral monodromy symmetry.

### 4.2 Three \( \mathbb{D}_4 \)- models

As in the case of \( S_3 \) monodromy, the breaking of \( S_4 \) down to \( \mathbb{D}_4 \) is induced by non zero flux piercing the curves of the \( SU_5 \times S_4 \times U_1^\perp \) model. Using properties from the
character table of $D_4$, we distinguish three kinds of models depending on the way the $S_4$-irreducible triplets have been pierced; there are three possibilities and are as described in what follows:

4.2.1 First case: $3 = 1_+ , - \oplus 2_{0,0}$

In this model, the various irreducible triplets of $S_4$; in particular those involved in:

(i) the five 10-plets namely $5 = 1 \oplus 3 \oplus 1_{t_5}$, and

(ii) the ten 5-plets which includes the four 10-plets charged under $U_{1}^{\perp}$ namely $4_{t_5} = 1_{t_5} \oplus 3_{t_5}$, and the six uncharged 10-plets given by $6 = 3 \oplus 3'$;

are decomposed as sums of two singlets $1_{p,q} + 1_{p',q'}$ and a doublet $2_{0,0}$. The character properties of the $D_4$- representations indicate that the decompositions of the triplets should be as

$$3|_{D_4} = 1_{+,-} \oplus 2_{0,0}$$

$$3'|_{D_4} = 1_{-,+} \oplus 2_{0,0}$$ (4.6)

By substituting these relations back into the restricted spectrum resulting from (3.1), we end with the following $SU_5 \times D_4 \times U_{1}^{\perp}$ spectrum

- five 10-plets

| curves | weights | $D_4$ | $\chi_R^{(a,b)}$ | $U_{1}^{\perp}$ homology | $U(1)_Y$ flux |
|--------|---------|-------|----------------|--------------------------|----------------|
| $10_{y_1}$ | $y_1$ | 2 | $(0, 0)$ | $\eta - 2c_1 - \varphi$ | $-N - P$ |
| $10_{y_3}$ | $y_3$ | 1 | $(1, -1)$ | $-c_1$ | 0 |
| $10_{y_4}$ | $y_4$ | 1 | $(1, 1)$ | $\chi' - c_1$ | $P$ |
| $10_{t_5}$ | $t_5$ | 1 | $(1, 1)$ | $\chi - c_1$ | $N$ |

(4.7)

where $\chi_R^{(a,b)}$ stands for the character of the generators in the $R$ representation; $\varphi = \chi + \chi'$, and the integers $N$ and $P$ as in eqs(2.33). Notice that the multiplets $10_{y_4}$ and $10_{t_5}$ transform in the same trivial $D_4$- representation; but having different $t_5$- charges; the $10_{y_3}$ transforms also as a singlet; but with character $(1, -1)$; it is a good candidate for accommodating the top-quark family.

14
• **ten 5-plets**

| curves    | weight | $\mathbb{D}_4$ | $\chi^{(a,b)}_R$ | $U^+_1$ homology | $U(1)_Y$ flux |
|------------|--------|----------------|-------------------|-------------------|----------------|
| $5Y_{13}$  | $Y_{13}$ | 2              | $(0,0)$           | 0                 | $\eta' - 2c_1 + \varphi$ | $N + P$         |
| $5Y_{12}$  | $Y_{12}$ | 1              | $(-1,1)$          | 0                 | $-\chi - c_1$             | $-N$            |
| $5Y_{i4}$  | $Y_{i4}$ | 2              | $(0,0)$           | 0                 | $-\chi' - c_1$           | $-P$            |
| $5Y_{34}$  | $Y_{34}$ | 1              | $(1,-1)$          | 0                 | $-2c_1$                  | 0               |
| $5Y_{i5}$  | $Y_{i5}$ | 2              | $(0,0)$           | 0                 | $-\chi' - c_1$           | $-P$            |
| $5Y_{35}$  | $Y_{35}$ | 1              | $(1,-1)$          | 0                 | $-\chi - c_1$             | $-N$            |
| $5Y_{45}$  | $Y_{45}$ | 1              | $(1,1)$           | 0                 | $\varphi - 2c_1$         | $N + P$         |

where we have set $\varphi = \chi + \chi'$. From this table, we learn that among the ten 5-plets, two sit in the $1_{+, -}$ representation with character $(1, -1)$; but with different $t_5$ charges; one in $1_{-, +}$ with character $(-1, 1)$ with no $t_5$ charge; and a fourth in the trivial representation of $\mathbb{D}_4$ with a unit $t_5$ charge.

**flavons**

Among the 24 flavons of the $SU_5 \times S_4 \times U_1^+$ model, there are 20 ones charged under $\mathbb{D}_4$ monodromy symmetry; but because of hermitic feature, they can be organised into $10 \oplus 10'$ subsets with opposite $\mathbb{D}_4$ characters and opposite $t_5$ charges. Moreover due to reducibility of the 10-dim multiplet as $10 = 4_{t_5} \oplus 6$, which is also equal to $(1_{t_5} \oplus 3_{t_5}) \oplus (3 + 3')$; and therefore to the direct sum $1_{t_5}^{1+} \oplus (1_{t_5}^{1-} \oplus 2_{0,0}^{t_5}) + (1_{-}^{t_5} \oplus 2_{0,0}^{t_5}) + (1_{-}^{t_5} \oplus 2_{0,0}^{t_5})$; one ends with: (a) flavons doublets $\vartheta_i, \vartheta'_i$, having character $(0,0)$ with and without $t_5$ charges; and (b) flavon singlets having characters $(\pm 1, \pm 1)$ with and without $t_5$ charges; they are as collected below.

| curves    | weights | $\mathbb{D}_4$ irrep | $\chi^{(a,b)}_R$ character | $t_5$ charge |
|------------|---------|----------------------|-----------------------------|--------------|
| $1_{\pm Z_{i3}}$ | $\pm Z_{i3}$ | 2                     | $(0,0)$                     | 0            |
| $1_{\pm Z_{i2}}$ | $\pm Z_{i2}$ | 1                     | $\pm (-1,1)$                | 0            |
| $1_{\pm Z_{i4}}$ | $\pm Z_{i4}$ | 2                     | $(0,0)$                     | 0            |
| $1_{\pm Z_{34}}$ | $\pm Z_{34}$ | 1                     | $\pm (1, -1)$               | 0            |
| $1_{\pm Z_{i5}}$ | $\pm Z_{i5}$ | 2                     | $(0,0)$                     | $\mp 1$      |
| $1_{\pm Z_{35}}$ | $\pm Z_{35}$ | 1                     | $\pm (1, -1)$               | $\mp 1$      |
| $1_{\pm Z_{45}}$ | $\pm Z_{45}$ | 1                     | $\pm (1, 1)$                | $\mp 1$      |
4.2.2 Second case: $3 = 1_{+,-} \oplus 1_{+,-} \oplus 1_{-,+}$

This is a completely reducible model; under restriction to dihedral sub-symmetry, the $3$ and $3'$ triplets of $S_4$ are decomposed as follows

$$
3\big|_{D_4} = 1_{+,-} \oplus 1_{+,-} \oplus 1_{-,+} \\
3'\big|_{D_4} = 1_{-,+} \oplus 1_{-,+} \oplus 1_{+,-}
$$

(4.10)

by substituting these decompositions back into the spectrum of $SU_5 \times S_4 \times U_1^\perp$ theory given by (3.11), we obtain the curves spectrum of the second $SU_5 \times D_4 \times U_1^\perp$ model:

- **five 10-plets**
  The spectrum of the 10-plets in the $D_4$- model II can be also deduced from (4.7) by splitting the $2_{0,0}$ doublet as $1_{+,-} \oplus 1_{-,+}$; we have

| curves | $D_4$ irrep | character | $U_1^\perp$ | homology | $U(1)_Y$ flux |
|--------|------------|-----------|-------------|-----------|--------------|
| $10_{+,-}$ | $1_{+,-}$ | $(1,-1)$ | 0 | $\eta - c_1 - \chi - \chi'$ | $-N - P$ |
| $10_{-,-}$ | $1_{-,-}$ | $(-1,1)$ | 0 | $-c_1$ | 0 |
| $10_{+,-}$ | $1_{+,-}$ | $(1,-1)$ | 0 | $-c_1$ | 0 |
| $10_{+,+}$ | $1_{+,+}$ | $(1,1)$ | 0 | $\chi' - c_1$ | $P$ |
| $10_{5,+,+}$ | $1_{+,+}$ | $(1,1)$ | 1 | $\chi - c_1$ | $N$ |

(4.11)

Here we have two matter multiplets namely $10_{+,-}$ and $10_{5,+,+}$; they transform in the same trivial $D_4$- representation with character $(1,1)$; but having different $t_5$-charges. We also have two $10_{+,-}$ multiplets transforming in $1_{+,+}$ with character $(1,-1)$; but with different fluxes; and one multiplet $10_{-,+}$ with character $(-1,1)$; it will be interpreted in appendix B as the one accommodating the top-quark family.

- **ten 5-plets**

| curves | $D_4$ irrep | $\chi_R^{(a,b)}$ | $U_1^\perp$ | homology | $U(1)_Y$ flux |
|--------|------------|----------------|-------------|-----------|--------------|
| $5_{+,+}$ | $1_{+,+}$ | $(1,-1)$ | 0 | $\eta' - c_1 + \chi + \chi'$ | $N + P$ |
| $5_{-,-}$ | $1_{-,-}$ | $(-1,1)$ | 0 | $-c_1$ | 0 |
| $5_{+,+}$ | $1_{-,+}$ | $(-1,1)$ | 0 | $-\chi - c_1$ | $-N$ |
| $5_{+,+}$ | $1_{+,+}$ | $(1,-1)$ | 0 | $-\chi' - c_1$ | $-P$ |
| $5_{-,+}$ | $1_{-,+}$ | $(-1,1)$ | 0 | $-c_1$ | 0 |
| $5_{+,+}$ | $1_{+,+}$ | $(1,1)$ | 1 | $-2c_1 + \chi + \chi'$ | $N + P$ |

(4.12)
where $N$ and $P$ as in eqs (2.33).

In this model, there is no flavon doublets; there are only singlet flavons transforming in the representations $1_{+,+}$, $1_{-,+}$, $1_{+,+}$, $1_{-,+}$ with and without $t_5$ charges; they are denoted in what follows as $\vartheta_{p,q}$ and $\vartheta^{\pm t_5}_{p,q}$ with $p, q = \pm 1$.

### 4.2.3 Third case: $3 = 1_{+,+} \oplus 1_{+,+} \oplus 1_{+,+}$

This $D_4$-model differs from the previous one by the characters of the singlets; since in this case the $S_4$-triplets $3|_{S_4}$ and $3'|_{S_4}$ are decomposed in terms of irreducible representations of $D_4$ like

$$3|_{D_4} = 1_{+,+} \oplus 1_{+,+} \oplus 1_{+,+} \oplus 1_{+,+} \oplus 1_{+,+}$$

Substituting these relationships back into (3.1), we get the curve spectrum of the third model namely:

- **five 10-plets**

| curves | $D_4$ irrep | character | $U_1^+$ | homology | $U(1)_Y$ flux |
|--------|-------------|-----------|---------|-----------|---------------|
| $10_{+,+}$ | $1_{+,+}$ | $(1, 1)$ | 0 | $\eta - c_1 - \varphi$ | $-N - P$ |
| $10_{+,+}$ | $1_{+,+}$ | $(1, 1)$ | 0 | $c_1$ | 0 |
| $10_{+,+}$ | $1_{+,+}$ | $(1, 1)$ | 0 | $c_1$ | 0 |
| $10_{+,+}$ | $1_{+,+}$ | $(1, 1)$ | 0 | $\chi' - c_1$ | $P$ |
| $10_{+,+}$ | $1_{+,+}$ | $(1, 1)$ | 1 | $\chi - c_1$ | $N$ |

Here we have three $10_{p,q}$ matter multiplets in the trivial $D_4$-representation with character $(p, q) = (1, 1)$; one of them namely $10^{t_5}_{+,+}$ having a $t_5$ charge and the two others not. A fourth curve $10_{+,+}$ in $1_{+,+}$ without $t_5$ charge nor a flux; and a fifth $10_{-,+}$ in $1_{-,+}$ with no $t_5$ but carrying a flux.
• ten 5-plets

| curves | \( \mathbb{D}_4 \) irrep | \( \chi_R^{(a,b)} \) | \( U_1^{\perp} \) | homology | U(1)_Y flux |
|--------|-----------------|----------------|-----------------|-----------|-------------|
| 5_{-,-} | 1_{-,-} | (-1, -1) | 0 | \( \eta' - c_1 + \chi + \chi' \) | \( N + P \) |
| 5_{-,+} | 1_{-,+} | (-1, 1) | 0 | \( -\kappa_1 \chi' - c_1 \) | \( -\kappa_1 P \) |
| 5_{+,+} | 1_{+,+} | (1, 1) | 0 | \( -\chi_1 - c_1 \) | \( -N \) |
| 5_{-,-} | 1_{-,-} | (-1, -1) | 0 | \( -c_1 \) | \( 0 \) |
| 5_{+,+} | 1_{+,+} | (1, -1) | 0 | \( -c_1 \) | \( 0 \) |
| 5_{-,-} | 1_{-,-} | (-1, 1) | 1 | \( -\kappa_1 \chi' - c_1 \) | \( -\kappa_1 P \) |
| 5_{+,+} | 1_{+,+} | (1, 1) | 1 | \( -\kappa_2 \chi' - c_1 \) | \( -\kappa_2 P \) |
| 5_{+,+} | 1_{+,+} | (1, 1) | 1 | \( -\chi_1 - c_1 \) | \( -N \) |

with \( \kappa_1 + \kappa_2 = 1 \) whose values will be fixed by the derivation of MSSM. The ten 5-plets \( 5_{p,q} \) splits as follows: 4 with \( p = q = 1 \); the two \( 5_{t,+,+} \) having a \( t_5 \) charge and the two others \( 5_{+,+,+} \) chargeless; the \( U_1^{\perp} \) charges and the \((N, P)\) fluxes allow to distinguish the four. There are also 3 types of \( 5_{-,+} \)-plets; two \( 5_{+,+} \) and one \( 5_{-,+} \). This model has no flavon doublets; there are only singlet flavons \( \vartheta_{p,q} \) and \( \vartheta_{p,q}^{\pm} \) with \( p, q = \pm 1 \).

## 5 MSSM like spectrum

First, we describe the breaking of the \( SU_5 \times \mathbb{D}_4 \times U_1^{\perp} \) theory down to supersymmetric standard model; then we study the derivation of the spectrum of MSSM like model with \( \mathbb{D}_4 \) monodromy; and where the heaviest top-quark family is singled out.

### 5.1 Breaking gauge symmetry

Gauge symmetry is broken by \( U(1)_Y \) hyperflux; by assuming doublet- triplet splitting produced by \( N \) units of \( U(1)_Y \), but still preserving \( \mathbb{D}_4 \times U_1^{\perp} \), the 10-plets and 5-plets get decomposed into irreducible representations of standard model symmetry. The 5-plets of the \( SU_5 \times \mathbb{D}_4 \times U_1^{\perp} \) models with multiplicity \( M_5 \) split as \([60, 61]\)

\[
\begin{align*}
n_{(3,1)_{-1/3}} - n_{(3,1)_{1/3}} &= M_5 \\
n_{(1,2)_{-1/2}} - n_{(1,2)_{1/2}} &= M_5 + N
\end{align*}
\]

(5.1)
leading to a difference between number of triplets and doublets in the low energy MSSM effective theory. These two relations are important since for $N \neq 0$ the correlation is somehow relaxed; by choosing

$$M_5^{\text{(Higgs)}} = 0$$

(5.2)

the coloured triplet-antitriplet fields $(3, 1)_{-1/3}$ and $(\bar{3}, 1)_{+1/3}$ in the Higgs matter curve come in pair that form heavy massive states; which decouple at low energy. Moreover, by making particular choices of the $M_5^{\text{(matter)}}$ multiplicities, we can also have the desired matter curve properties for accommodating fermion families; in particular the chirality property $n_{(1,2)_{+1/2}} \neq n_{(1,2)_{-1/2}}$ which is induced by hyperflux. Furthermore, due to the flux, we also have different numbers of down quarks $d_L$ and lepton doublets $L$.

For the 10-plets of the GUT- model with multiplicity $M_{10}$, we have the following decompositions [27, 62, 63]

$$n_{(3,2)_{+1/6}} - n_{(3,2)_{-1/6}} = M_{10}$$
$$n_{(3,1)_{-2/3}} - n_{(3,1)_{+2/3}} = M_{10} - N$$
$$n_{(1,1)_{+1}} - n_{(1,1)_{-1}} = M_{10} + N$$

(5.3)

The first relation with $M_{10} \neq 0$ generates up-quark chirality since the number $n_{(3,2)_{+1/6}}$ of $Q_L = (3, 2)_{+1/6}$ representations differs from the number $n_{(3,2)_{+1/6}}$ of $\bar{Q}_L = (\bar{3}, 2)_{-1/6}$. With non zero units of hyperflux, the two extra relations leads to the other desired splitting; the second relation leads for $N \neq 0$ to lifting the multiplicities between $Q = (3, 2)_{+1/6}$ and $u^c = (3, 1)_{-2/3}$ while the third relation ensures the chirality property of $e^c_L$.

In what follows, we study the derivation of an effective matter curve spectrum that resembles to the field content of MSSM. In addition to three families and

$$\sum M_{10} + \sum M_5 = 0$$

(5.4)

as well as total hyperflux conservation

$$\sum_{\text{fluxes}} N_i = 0$$

(5.5)

we demand the following:

- only a tree- level Yukawa coupling is allowed; and is given by the top-quark family,
- the heaviest third generation is the least family affected by hyperflux,
- MSSM matter generations are in $\mathbb{D}_4 \times U_1$ representations,
- no dimension 4 and 5 proton decay operators are allowed,
- no $\mu$- term at a tree level,
- two Higgs doublets $H_u$ and $H_d$ as required by MSSM.
5.2 Building the spectrum

Seen that there are three possible $SU_5 \times D_4 \times U_1^\perp$ models, we focus on the first model with curve spectrum given by eqs (4.7-4.8); and consider first the 10-plets; then turn after to 5-plets. Results regarding the two other models II and III are reported in appendix B.

5.2.1 Ten-plets sector in $D_4$- model I

The five 10-plets of the $D_4$ model carry different quantum numbers with respect to $D_4 \times U_1^\perp$ representations, different hyperflux units $(N, P)$; and different $M_{10}^{(n)}$ multiplicities satisfying the properties (5.3). By thinking about $\sum M_{10}$ as given by the number of MSSM generations

$$\sum M_{10} = 3 \quad (5.6)$$

and taking into account that the two components of the 10- doublet are monodromy equivalent; it follows that one of the five 10-plets should be disregarded; at least at a tree level analysis. Moreover, using the property that top- quark 10-plet should be a $D_4$- singlet; one may choose the $M_{10}^{(n)}$’s as in following table,

| curves | $D_4$ irrep | $U_1^\perp$ | U(1)$_Y$ flux | multiplicity |
|--------|-------------|-------------|----------------|--------------|
| 10$_i$ | 2$_{0,0}$   | 0           | $-N - P$       | $M_{10}^{(n)}$ | 2            |
| 10$_3$ | 1$_{+,+}$   | 0           | 0              | $M_{10}^{(3)}$  | 1            |
| 10$_4$ | 1$_{++}$    | 0           | $P$            | $M_{10}^{(4)}$  | 0            |
| 10$_5$ | 1$_{++}$    | 1           | $N$            | $M_{10}^{(5)}$  | 0            |

where chiral modes of 10$_4$ have been ejected ($M_{10}^{(4)} = 0$). Notice that the top- quark generation can a priori be taken in any one of the three $D_4$- singlets; that is either 10$_3$ or 10$_4$; or 10$_5$; the basic difference between these $D_4$- singlets is given by $t_5$ charge and hyperflux. But the choice of the 10$_3$-multiplet looks be the natural one as it is unaffected by hyperflux, a desired property for MSSM and beyond; and has no $t_5$ charge

$$10_3 = (Q_L, U^c_L, e^c_L) \equiv 10_{+,+} \quad (5.8)$$

This multiplet captures also an interesting signature of $D_4$ monodromy in the sense it behaves as a $D_4$-singlet 1$_{+,+}$ with non trivial character $(+1, -1)$. The importance of this feature at modeling level is twice: (i) first it fixes the quantum numbers of the $5_{H_u}$ Higgs representation as a $D_4$- singlet $5_{p,q}$ as shown on the tree level top- quark Yukawa coupling

$$10_{+,+} \otimes 10_{+,+} \otimes 5_{H_u} \quad (5.9)$$
Monodromy invariance of (5.9) under $\mathbb{D}_4 \times U_1^\perp$ requires $5_{H_u}$ in the trivial representation with no $t_5$ charge; i.e. $5_{H_u} \sim 1_{++,}$. However, an inspection of the characters of the $U_1^\perp$ chargeless 5-plets reveals that there is no $(5_{++})_{t_5=0}$ in the spectrum of the the $\mathbb{D}_4 \times U_1^\perp$ models I and II constructed above. To bypass this constraint, we realise the role of the Higgs $5_{H_u}$ by allowing VEVs to come from flavons as well; in other words by thinking of $5_{H_u}$ as follows

$$5_{H_u} \rightarrow 5_{p,q} \otimes \vartheta_{p',q'} \quad with \quad pp' = 1, \quad qq' = 1 \quad (5.10)$$

where $\vartheta_{p',q'}$ stands for a flavon in the representation $1_{p',q'}$.

(ii) second it gives an important tool to distinguish between matter and Higgs in the 5-plets sector as manifestly exhibited by the tri-coupling $10_{++,} \otimes 5_M \otimes 5_{H_d}$. This interaction requires matter $5_M$ and Higgs $5_{H_d}$ to be in different $\mathbb{D}_4$-singlets $1_{p,q}$ and $1_{p',q'}$ with $pp' = 1$ and $qq' = -1$; see discussion given later on.

By choosing the hyperflux units as $N = P = 1$; and using (5.3) we obtain the matter content

| curves | $\mathbb{D}_4$ | $U_1^\perp$ flux | matter content | $Z_2$ parity |
|--------|----------------|------------------|----------------|-------------|
| $10_3$ | 2_{0,0}        | 0                | $2Q_L \oplus 4e^c_L$ | $\chi_{43} = -$ |
| $10_4$ | 1_{++,}        | 0                | $Q_L \oplus U^c_L \oplus e^c_L$ | $\chi_{42} = -$ |
| $10_5$ | 1_{++,}        | 1                | $U^c_L \oplus e^c_L$ | $\chi_{41} = +$ |

Notice that by following [16] using Galois theory, the 10-plets have been attributed $Z_2$ parity charges as reported by the last column of above table. In our formulation these parities correspond to $s_i \rightarrow -s_i$ and $\chi_1$ and $\chi_4 = \chi_{41}\chi_{42}\chi_{43}$ as in eq(2.28); by help of (2.14) and (2.22) we obtain

$$Z_2 (b_5) = +1 \quad , \quad Z_2 (b_0) = -1$$
$$Z_2 (d_{10}) = -1 \quad , \quad Z_2 (d_0) = -1 \quad (5.12)$$

in agreement with (2.17).

### 5.2.2 Five-plets sector

Like for 10-plets, the ten 5-plets carry different quantum numbers of $\mathbb{D}_4 \times U_1^\perp$ representations, hyperflux units $(N, P)$ and $M_5^{(n)}$ multiplicities as in (5.1). To have a matter curve spectrum that resembles to MSSM, we choose the $M_5^{(n)}$’s and the hyperflux as
where \( \chi' \) and \( \xi' \) are two classes playing similar role as in the case of breaking \( S_5 \) monodromy down to \( S_3 \). By using (5.4-5.6), we have

\[
\sum M_5 = - \sum M_{10} = -3 \tag{5.14}
\]

and thinking of this number as \( \sum M_5 = 3 - 6 \), a possible configuration for a MSSM like spectrum is given by

\[
\begin{align*}
M_5^{(1)} &= 2 \\
M_5^{(2)} &= 0 \\
M_5^{(3)} &= -4 \\
M_5^{(4)} &= 0 \\
M_5^{(6)} &= 1 \\
M_5^{(7)} &= -2
\end{align*} \tag{5.15}
\]

By choosing the hyperflux as \( N = P = 1 \), and putting back into above table, we obtain, after relabeling, the 5-plets

\[
\begin{align*}
\left( 5^M_5 \right)_0 & \quad 2_{0,0} \quad 0 \quad -2 \quad 2 \quad 2\tilde{d}_L^c \\
\left( 5^H_u \right)_0 & \quad 1_{-,+} \quad 0 \quad 1 \quad 0 \quad H_u \\
\left( 5^M_{+,-} \right)_0 & \quad 1_{+,-} \quad 0 \quad 1 \quad -4 \quad -4\tilde{d}_L^c - 3\tilde{L} \\
\left( 5^H_d \right)_{+,-} & \quad 1_{+,+} \quad -1 \quad 1 \quad 0 \quad -H_d \\
\left( 5^M_{+,-} \right)_{-t_5} & \quad 1_{+,-} \quad -1 \quad 1 \quad 1 \quad \tilde{d}_L^c \\
\left( 5^M_{+,-} \right)_{-t_5} & \quad 2_{0,0} \quad -1 \quad -2 \quad -2 \quad -2\tilde{d}_L^c \\
\left( 5^M_i \right)_{-t_5} & \quad 2_{0,0} \quad 0 \quad 0 \quad 0 \quad 0 \\
\left( 5^i_5 \right)_0 & \quad 2_{0,0} \quad 0 \quad 0 \quad 0 \quad 0
\end{align*} \tag{5.16}
\]

From this table we learn that the up-Higgs 5-plet \( \left( 5^H_u \right)_{-+,+} \) has a character equal to \((-1,+1)\) and no \( t_5 \) charge; by substituting in (5.10), we obtain \( 5_H \sim \left( 5^H_u \right)_{0} \otimes (\vartheta_{-+,0}). \)
We also learn that the 5-plet $(5^M_{+, -})_0$ is the least multiplet affected by hyperflux; and because of our assumptions, it is the candidate for matter $\bar{5}^M_3$; the partner of $10_3$ in the underlying SO$_{10}$ GUT-model. With this choice, the down-type quarks tri-coupling for the third family, namely $10_3 \otimes \bar{5}^M_5 \otimes 5^H_d$; and which we rewrite like

$$10_{+,-} \otimes \bar{5}^M_{p,q} \otimes \bar{5}^H_{d'} \quad \text{with} \quad pp' = 1, qq' = -1$$ (5.17)

This coupling requires the matter $\bar{5}^M_3$ and the down-Higgs $\bar{5}^H_d$ multiplets to belong to different $\mathbb{D}_4$ singlets seen that $10_3$ is in $1_{+,-}$ representation. However, the candidates $(\bar{5}^H_{d,-})_{t_5}$ and $\bar{5}^M_3 \equiv (\bar{5}^M_{-,+})_0$ are ruled out because of the non conservation of $t_5$ charge. Nevertheless, a typical diagonal mass term of third family may be generated by using a flavon $\vartheta^+_{t_5}$ carrying $-1$ unit charge under $U^+_1$ and transforming as a trivial $\mathbb{D}_4$ singlet. This leads to the realisation $\bar{5}^H_d \sim (\bar{5}^H_{d,-})_{t_5}(\vartheta^+_{++, -})_{t_5}$; and then to

$$ (10_{+,-})_0 \otimes (\bar{5}^M_{-,+})_0 \otimes (\bar{5}^H_{d,-})_{t_5} \otimes (\vartheta^+_{++, -})_{t_5}$$ (5.18)

Non diagonal 4-order coupling superpotentials with one $(10_{+,-})_0$ are as follows: $^3$

$$ (10_{+,-})_0 \otimes (10_{+,+})_0 \otimes (\bar{5}^H_{d,+})_0 \otimes (\vartheta^+_{-, -})_0 $$

$$ (10_{+,-})_0 \otimes (10_{+,+})_{t_5} \otimes (\bar{5}^H_{d,+})_0 \otimes (\vartheta^+_{-, -})_{t_5} $$

$$ (10_{+,-})_0 \otimes (\bar{5}^H_{d,+})_0 \otimes (10^0_{0,0})_0 \otimes (\vartheta^i_{0, 0})_0 $$

$$ (10_{+,-})_0 \otimes (10_{+,+})_0 \otimes (\bar{5}^H_{d,+})_0 \otimes (\vartheta^i_{0, 0})_0 $$

$$ (10_{+,+})_0 \otimes (\bar{5}^H_{d,+})_0 \otimes (10^0_{0,0})_0 \otimes (\vartheta^i_{0, 0})_0 $$

Below, we discuss some properties of these couplings.

### 5.3 More on couplings in $\mathbb{D}_4$ model I

First, we study the quark sector; and turn after to the case of leptons.

#### 5.3.1 Quark sector

From the view of supersymmetric standard model with $SU(3) \times SU_L(2) \times U_Y(1)$ gauge symmetry; and denoting the triplet and doublet components of the Higgs 5-plets $5^H_s = 3^H_s \oplus 2^H_s$ respectively like $D_x \oplus H_x$, the usual tree level up/down-type Yukawa couplings in $SU_5$ model split like

$$10^M \cdot 10^M \cdot 5^H_u \rightarrow Qu^c H_u + u^c e^c D^c_u + QQ D^c_u $$

$$10^M \cdot \bar{5}^M \cdot \bar{5}^H_d \rightarrow Qd^c H_d + e^c LH_d + QD^c_d L$$

$^3$A complete classification requires also use $Z_2$ parity; see [10].
They involve up/down Higgs triplets $D_u$ and $D_d$, which are exotic to MSSM; but with the hyperflux $U_V$ (1) choice we have made in $SU_5 \times \mathbb{D}_4 \times U_1^\perp$ model \[5.14\], they are removed; therefore we have

\begin{align*}
10^M.10^M.5^{H_u} & \rightarrow Qu^cH_u \\
10^M.5^{M}.5^{H_d} & \rightarrow Qd^cH_d + e^cLH_d
\end{align*}

(5.21)

with right hand sides capturing same monodromy representations as left hand sides; that is $Q$, $u^c$ same $\mathbb{D}_4 \times U_1^\perp$ representations as $10^M$; and so on. In what follows, we study each of these terms separately by taking into account $\theta_{p,q}$ flavon contributions up to order four couplings; some of these flavons are interpreted as right neutrinos; they will be discussed at proper time.

- **Up-type Yukawa couplings**

Because of the $\mathbb{D}_4 \times U_1^\perp$ monodromy charge of the up-Higgs 5-plet like $(5_{-+}^{H_u})_0$, there is no monodromy invariant 3-coupling type $10^M.10^M.5^{H_u}$. As shown by eq\[5.10\], one needs to go to higher orders by implementing flavons with quantum numbers depending on the monodromy representation of the 10-plets. Indeed, by focussing on the third generation $10^M_3 \equiv (10_{+,-})_0$; we can distinguish diagonal and non diagonal interactions; an inspection of $\mathbb{D}_4$ quantum numbers of matter and Higgs multiplets reveals that we need $\mathbb{D}_4$- charged flavons to have monodromy invariant superpotentials as shown below

\[W_\text{top}^{(4)} = \alpha_3Tr[(10_{+,-})_0 \otimes (10_{+,-})_0 \otimes (5_{-+}^{H_u})_0 \otimes (\vartheta_{-+})_0]\]

(5.22)

By restricting to VEVs $\langle \vartheta_{-+} \rangle = \rho_0$ and $\langle H_u \rangle = v_u$; this non renormalisable coupling leads to the top quark mass term $m_t Q_3 u_3^c$ with $m_t$ equal to $\alpha_3 v_u \rho_0$. Such a term should be thought of as a particular contribution to a general up-quark mass terms $u_i^c M^{ij} u_j$ with $3 \times 3$ mass matrix as follows

\[M_{u,c,t} = v_u \begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & \alpha_3 \rho_0
\end{pmatrix}\]

(5.23)

where the (*)’s refer to contributions coming from other terms including non diagonal couplings; one of them is

\[Tr[(10_{+,-})_0 \otimes (5_{-+}^{H_u})_0 \otimes (10_{0,0})_0 \otimes (\vartheta_{0,0})_0']\]

(5.24)

it involves a 10-plet doublet $(10_{0,0})_0 \equiv (10_i)_0$ and a flavon doublet $(\vartheta_{0,0})'_0 \equiv (\vartheta_i)'_0$ with VEVs $(\rho_1, \rho_2)$; the latter $(\vartheta_{0,0})'_0$ will be combined the 10-plet doublet like $(10_{0,0})_0 \otimes (\vartheta_{0,0})'_0$
to make a scalar. Indeed, the tensor product can be reduced as direct sum over irreducible representations of $\mathbb{D}_4$ having amongst others the $\mathbb{D}_4$-component

$$S_{--} = (10_{0,0})_0 \otimes (\vartheta_{0,0})_0'_{--}$$

(5.25)

with $(-,-)$ charge character. This negative charge is needed to compensate the $(-,-)$ charge coming from $(10_{+,--})_0 \otimes (5_{-+})_0$. Restricting to quarks, this reduction corresponds to $(10_{0,0})_0 \otimes (\vartheta_{0,0})_0' \to Q_i \otimes \rho_i$ with

$$Q_i \otimes \rho_i|_{(--)} = Q_1 \rho_2 - Q_2 \rho_1$$

(5.26)

Putting back into (5.24), and thinking of $S_{--}$ in terms of the linear combination $\alpha_2(Q_1 \rho_2 - Q_2 \rho_1)$ of quarks, we obtain $\alpha_2 \nu_u (Q_1 \rho_2 - Q_2 \rho_1) u^c_3$; which can be put into the form $u^c_i M^{ij} u_j$ with mass matrix as

$$M_{u,c,t} = v_u \begin{pmatrix} * & * & \alpha_2 \rho_2 \\ * & * & -\alpha_2 \rho_1 \\ * & * & \alpha_3 \rho_0 \end{pmatrix}$$

(5.27)

One can continue to fill this mass matrix by using the VEV’s of other flavons; however to do that, one needs to rule out couplings with those flavons describing right neutrinos $\nu^c_i$. Extending ideas from [16], the 3 generations of the right handed neutrinos $\nu^c_i$ in $SU_5 \times D_4 \times U_1^\perp$ model should be as

$$\nu^c_3 \rightarrow (\vartheta_{+-})_0$$

(5.28)

$$\begin{pmatrix} \nu^c_1, \nu^c_2 \end{pmatrix}^\dagger \rightarrow (\vartheta_{0,0})_0$$

with the following features among the set of 15 flavons of the model

| flavons | $SU_5$ | $D_4$ irrep | $U_1^\perp$ | $Z_2$ Parity | VEV |
|---------|--------|-------------|-------------|--------------|-----|
| $(\vartheta_{0,0})_0'$ | 1 | 2 | 0 | + | $(\rho_1, \rho_2)^\dagger$ |
| $(\vartheta_{-,+})_0$ | 1 | 1 | 0 | + | $\rho_0$ |
| $(\vartheta_{0,0})_{\pm t_5}$ | 1 | 2 | $\pm 1$ | + | $(\sigma_1, \sigma_2)^\dagger$ |
| $(\vartheta_{+,-})_{\pm t_5}$ | 1 | 1 | $\pm 1$ | $\mp$ | $\omega$ |
| $(\vartheta_{0,0})_0 = (\nu^c_1, \nu^c_2)^\dagger$ | 1 | 2 | 0 | $-$ | $-$ |
| $(\vartheta_{+,--})_0 = \nu^c_3$ | 1 | 1 | 0 | $-$ | $-$ |

(5.29)

Therefore, the contribution to (5.27) coming from the diagonal couplings of the doublets $(10_{0,0})_0$ follows from

$$W^{(4)} = Tr[(5_{-+})_0 \otimes [(10_{0,0})_0 \otimes (10_{0,0})_0]|_{\nu^c} \otimes (\vartheta_{-+})_0]$$

(5.30)
However, though monodromy invariant, this couplings cannot generate the mass term \( mQ_{1,2}u^c_{1,2} \) since the matter curve \((10_{0,0})_0 \) don’t contain the quark \( u^c_{1,2} \); so the mass matrix (5.27) for the up-type quarks is

\[
M_{u,c,t} = v_u \begin{pmatrix}
0 & 0 & \alpha_2 \rho_2 \\
0 & 0 & -\alpha_2 \rho_1 \\
0 & 0 & \alpha_3 \rho_0
\end{pmatrix}
\]

(5.31)

it is a rank one matrix; it gives mass to the third generation (top-quark); while the two first generations are massless.

**masses for lighter families**

The rank one property of above mass matrix (5.31) is a known feature in GUT models building including F-Theory constructions; see for instance [36, 43, 44, 64]. To generate masses for the up- quarks in the first two generations, different approaches have been used in literature: (i) approach based on flux corrections using non perturbative effects [20] or non commutative geometry [21]; and (ii) method using \( \delta W \) deformations of the GUT superpotential \( W \) by higher order chiral operators [14, 43, 44, 64, 65, 66]. Following the second way of doing, masses to the two lighter families are generated by higher dimensional operators corrections that are invariant under \( \mathbb{D}_4 \) symmetry and \( Z_2 \) parity. This invariance requirement leads to involve 6- and 7-dimensional chiral operators which contribute to the up- quark mass matrix as follows

\[
\delta W = \sum_{i=1}^{5} x_i \delta W_i
\]

with

\[
\begin{align*}
\delta W_1 &= (10^i_{0,0})_0 \otimes (10_{+,+})_0 \otimes (5_{-,-})_0 \otimes (\vartheta^i_{0,0})_{-t_5} \otimes (\vartheta_{++,})_{t_5} \\
\delta W_2 &= (10^i_{0,0})_0 \otimes (10_{++,})_{t_5} \otimes (5_{-,-})_0 \otimes (\vartheta^i_{0,0})_{-t_5} \otimes (\vartheta_{++,})_{t_5} \\
\delta W_3 &= (10^i_{0,0})_0 \otimes (10_{++,})_{t_5} \otimes (5_{-,-})_0 \otimes (\vartheta^i_{0,0})_{-t_5} \otimes (\vartheta_{++,})_{t_5}
\end{align*}
\]

(5.33)

and

\[
\begin{align*}
\delta W_4 &= (10_{++,})_0 \otimes (10_{++,})_0 \otimes (5_{-,-})_0 \otimes (\vartheta_{++,})_{t_5} \otimes (\vartheta_{++,})_{t_5} \\
\delta W_5 &= (10_{++,})_0 \otimes (10_{++,})_{t_5} \otimes (5_{-,-})_0 \otimes (\vartheta^i_{0,0})_{-t_5} \otimes (\vartheta_{++,})_{t_5}
\end{align*}
\]

(5.34)

Notice that the adjunction of \( (\vartheta_{++,})_{t_5} \) chiral superfield is required by invariance under \( Z_2 \) parity. Using this deformation, a higher rank up- quark mass matrix is obtained as usual by giving VEVs to flavons as in (5.29) and \( \langle (\vartheta_{++,})_{-t_5} \rangle = \varphi \). By calculating the product of the operators in eqs (5.33, 5.34) using \( \mathbb{D}_4 \) fusion rules, we obtain

\[
\begin{align*}
x_1 \delta W_1 &= (10^i_{0,0})_0 \otimes (10_{++,})_0 \otimes (5_{-,-})_0 \otimes (\vartheta^i_{0,0})_{-t_5} \otimes (\vartheta_{++,})_{t_5} \\
&= x_1 v_u (Q_1 \sigma_1 - Q_2 \sigma_2) u^c_{2,3}
\end{align*}
\]
and
\[
x_2 \delta W_2 = \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_{t_5} \otimes (5^H_u)_{0} \otimes \left(\varphi^i_{0,0}\right)_{-t_5} \otimes (\varphi_{+,-})_{-t_5} \otimes (\varphi_{+,+})_{t_5}
\]
\[
= x_2 v_u (Q_1 \sigma_2 - Q_2 \sigma_1) u^c_i \omega \varphi
\]
The operator
\[
(10^i_{0,0})_0 \otimes (10_{+,+})_{t_5} \otimes (5^H_u)_{0} \otimes \left(\varphi^i_{0,0}\right)_{-t_5} \otimes (\varphi_{+,-})_{-t_5} \otimes (\varphi_{+,+})_{t_5}
\]
contributes in the up-quark mass matrix [5.31] as a correction to the matrix elements \(m_{1,1}\) and \(m_{1,2}\); it has the same role as the higher operator (5.35); so we will not take it into account in the quark mass matrix. Expanding the remaining operators by help of the \(D_4\) rules, we have
\[
x_4 \delta W_4 = (10_{+,-})_0 \otimes (10_{+,+})_0 \otimes (5^H_u)_{0} \otimes (\varphi_{+,-})_0 \otimes (\varphi_{+,+})_{t_5} \otimes (\varphi_{+,+})_{t_5}
\]
\[
= x_4 v_u \rho_0 \varphi \omega Q_3 u^c_i
\]
and
\[
x_5 \delta W_5 = (10_{+,-})_0 \otimes (10_{+,+})_{t_5} \otimes (5^H_u)_{0} \otimes (\varphi_{+,-})_0 \otimes (\varphi_{+,+})_{t_5} \otimes (\varphi_{+,+})_{t_5}
\]
\[
= x_5 v_u \omega Q_3 u^c_i (\sigma_1 \sigma_2 - \sigma_2 \sigma_1) = 0
\]
Summing up all contributions, we end with the following up-quark matrix
\[
M_{u,c,t} = v_u \begin{pmatrix}
x_2 \sigma_1 \omega \varphi & x_1 \omega & \alpha_2 \rho_2 \\
-x_2 \sigma_1 \omega \varphi & -x_1 \omega & -\alpha_2 \rho_1 \\
0 & x_4 \rho_0 \varphi \omega & \alpha_4 \rho_0
\end{pmatrix}
\]
\[
(5.36)
\]
• **Down-type Yukawa**
Following the same procedure as in up-Higgs type coupling, we can build invariant operators for the down-type Yukawa
\[
(10_{+,-})_0 \otimes (5^M_{+,-})_0 \otimes (3^M_{+,+})_{t_5} \otimes (\varphi_{+,+})_{-t_5}
\]
\[
(5^M_{+,-})_0 \otimes (3^M_{+,+})_{t_5} \otimes (10_{0,0})_0 \otimes (\varphi_{0,0})_{-t_5}
\]
Restricting VEV of down Higgs \(\langle H_d \rangle = v_d\), and using the flavons VEVs as in [5.29] as well as taking into account multiplicities, the first coupling gives a mass term of the form \(m_i d^c_i Q_3\) with \(m_i = \omega v_d y_{3,i}\) where \(y_{3,i}\) are coupling constants. For the second term, we need to reduce \((10_{0,0})_0 \otimes (\varphi_{0,0})_{-t_5}\) into irreducible \(D_4\) representations; and restricts to the component \(S_{(+,-)} = Q_i \otimes \sigma_{i(+,-)}\) with
\[
S_{(+,-)} = Q_1 \sigma_1 + Q_2 \sigma_2
\]
\[
(5.38)
\]
So the couplings in eqs (5.37) may expressed like
\[
y_{3,i} Q_3 d^c_i \omega v_d + y_{1,i} (Q_1 \sigma_1 + Q_2 \sigma_2) d^c_i v_d
\]
\[
(5.39)
\]
leading to the mass matrix

\[
m_{d,s,b} = v_d \begin{pmatrix}
y_{1,1}\sigma_1 & y_{1,2}\sigma_1 & y_{1,3}\sigma_1 \\
y_{1,1}\sigma_2 & y_{1,2}\sigma_2 & y_{1,3}\sigma_2 \\
y_{3,1}\omega & y_{3,2}\omega & y_{3,3}\omega \\
\end{pmatrix}
\] (5.40)

### 5.3.2 Lepton sector

First we consider the charged leptons; and then turn to neutrinos.

- **Charged leptons**

Charged leptons masses are determined by the same operators used in the case of the down quark sector \(10^M \otimes \overline{5}^M \otimes \overline{5}^{H_d}\); using spectrum eqs (5.11–5.16), the appropriate operators which provide mass to charged leptons are

\[
(10_{+,-})_0 \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}_{+,-})_{t_5} \otimes (\vartheta_{+,+})_{-t_5}
\] (5.41)

giving the lepton mass term \(m^{ij}e_i^c L_j\) with mass matrix

\[
m_{e,\mu,\tau} = v_d \begin{pmatrix}
z_{1,1}\sigma_1 & z_{1,2}\sigma_1 & z_{1,3}\sigma_1 \\
z_{1,1}\sigma_2 & z_{1,2}\sigma_2 & z_{1,3}\sigma_2 \\
z_{3,1}\omega & z_{3,2}\omega & z_{3,3}\omega \\
\end{pmatrix}
\] (5.42)

- **Neutrinos**

Right handed neutrinos are as in eq (5.28), they have negative R-parity. Dirac neutrino term is embedded in the coupling \(\nu_i^c \otimes \overline{5}^M \otimes \overline{5}^{H_u}\) where the right neutrino \(\nu_i^c\) is an \(SU_5\) singlet; it allows a total neutrino mass matrix using see-saw I mechanism [18]. The invariant operators that give the Dirac neutrino in \(SU_5 \times \mathbb{D}_4 \times U_1^\perp\) model are

\[
x_{1,i} (\vartheta_{+,+})_0 \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}^{H_u})_0 \otimes (\vartheta_{+,+})_0
\]

\[
x_{2,i} (\vartheta_{0,0})_0 \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}^{H_u})_0 \otimes (\vartheta_{0,0})_0
\]

(5.43)

Using the \(\mathbb{D}_4\) algebra rules and flavon VEV’s, these couplings lead to

\[
x_{1,i} v_u \rho_0 L_i \nu_3^c
\]

\[
x_{2,i} v_u \rho_2 L_i \nu_1^c - x_{2,i} v_u \rho_1 L_i \nu_2^c
\]

(5.44)

and then to a Dirac neutrino mass matrix as

\[
m_D = v_u \begin{pmatrix}
x_{2,1}\rho_2 & -x_{2,1}\rho_1 & x_{1,1}\rho_0 \\
x_{2,2}\rho_2 & -x_{2,2}\rho_1 & x_{1,2}\rho_0 \\
x_{2,3}\rho_2 & -x_{2,3}\rho_1 & x_{1,3}\rho_0 \\
\end{pmatrix}
\] (5.45)
The Majorana neutrino term is given by $\nu_i \otimes \nu_j^c$; by using eqs (5.11, 5.16), the Majorana neutrino couplings in $SU_5 \times D_4 \times U_{1}^{\uparrow}$ model are as follows

$$(\vartheta_{+-0} \otimes \vartheta_{+-0})$$

$$(\vartheta_{00} \otimes \vartheta_{00})_0$$

$$(\vartheta_{+-0} \otimes \vartheta_{00})_0 \otimes (\vartheta_{00})_0$$

we can also add the singlet $(\vartheta_{-,+})_0$ as a correction of the last two operators. The operators in above (5.46) lead to

$$m \nu_3^c \nu_3^c, \quad M \nu_1^c \nu_2^c, \quad \lambda \nu_3^c (\nu_1^c \rho_1 + \nu_2^c \rho_2)$$

and ends with a Majorana neutrino mass matrix like

$$m_M = \begin{pmatrix} 0 & M & \lambda \rho_1 \\ M & 0 & \lambda \rho_2 \\ \lambda \rho_1 & \lambda \rho_2 & m \end{pmatrix}$$

The general neutrino mass matrix is calculated using see-saw I mechanism; it reads as $M_\nu = -m_D m_M^{-1} m_D^\top$; and leads to the following effective neutrino mass matrix

$$M_\nu \simeq \xi_0 \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{1,2} & m_{2,2} & m_{2,3} \\ m_{1,3} & m_{2,3} & m_{3,3} \end{pmatrix}$$

with

$$m_{1,1} = \lambda^2 x_{2,1} \rho_1^4 - 2 x_{2,1}^2 \rho_1 \rho_2 m M + 2 \lambda x_{2,1} \rho_2^2 (\lambda x_{2,1} \rho_1^2 - x_{1,1} M \rho_0) + (\lambda x_{2,1} \rho_1^2 + x_{1,1} M \rho_0)^2$$

$$m_{2,2} = \lambda^2 x_{2,2} \rho_2^4 - 2 x_{2,2}^2 \rho_1 \rho_2 m M + 2 \lambda x_{2,2} \rho_2^2 (\lambda x_{2,2} \rho_1^2 - x_{1,2} M \rho_0) + (\lambda x_{2,2} \rho_1^2 + x_{1,2} M \rho_0)^2$$

$$m_{3,3} = \lambda^2 x_{2,3} \rho_3^4 - 2 x_{2,3}^2 \rho_1 \rho_2 m M + 2 \lambda x_{2,3} \rho_2^2 (\lambda x_{2,3} \rho_1^2 - x_{1,3} M \rho_0) + (\lambda x_{2,3} \rho_1^2 + x_{1,3} M \rho_0)^2$$

and

$$m_{1,2} = \lambda^2 x_{2,1} x_{2,2} \rho_1^4 - 2 x_{2,1} x_{2,2} \rho_1 \rho_2 m M + (\lambda x_{2,1} \rho_1^2 + x_{1,1} \rho_0 M)(\lambda x_{2,2} \rho_1^2 + x_{1,2} \rho_0 M) + \lambda \rho_2^2 [2 \lambda x_{2,1} x_{2,2} \rho_1^2 - \rho_0 M (x_{1,1} x_{2,2} + x_{2,1} x_{1,2})]$$

$$m_{1,3} = \lambda^2 x_{2,1} x_{2,3} \rho_1^4 - 2 x_{2,1} x_{2,3} \rho_1 \rho_2 m M + (\lambda x_{2,1} \rho_1^2 + x_{1,1} \rho_0 M)(\lambda x_{2,3} \rho_1^2 + x_{1,3} \rho_0 M) + \lambda \rho_2^2 [2 \lambda x_{2,1} x_{2,3} \rho_1^2 - \rho_0 M (x_{1,1} x_{2,3} + x_{2,1} x_{1,3})]$$

$$m_{2,3} = \lambda^2 x_{2,2} x_{2,3} \rho_2^4 - 2 x_{2,2} x_{2,3} \rho_1 \rho_2 m M + (\lambda x_{2,2} \rho_2^2 + x_{1,2} \rho_0 M)(\lambda x_{2,3} \rho_1^2 + x_{1,3} \rho_0 M) + \lambda \rho_2^2 [2 \lambda x_{2,2} x_{2,3} \rho_1^2 - \rho_0 M (x_{1,3} x_{2,2} + x_{2,3} x_{1,2})]$$
and where we have set
\[ \xi_0 = \frac{\nu_2^2}{M(mM - 2\lambda^2\rho_1\rho_2)} \] (5.52)

To obtain neutrino mixing compatible with experiments we need a particular parametrization and some approximations on \( M_\nu \). To that purpose, recall that there are three approaches to mixing using: (i) the well known Tribimaximal (TBM) mixing matrix, (ii) Bimaximal (BM) and (iii) Democratic (DC); all of the TBM, BM and DC mixing matrices predict a zero value for the angle \( \theta_{13} \). However recent results reported by MINOS \[24\], Double Chooz \[25\], T2K \[54\], Daya Bay \[55\], and RENO \[56\] collaborations revealed a non-zero \( \theta_{13} \); such non-zero \( \theta_{13} \) has been recently subject of great interest; in particular by perturbation of the TBM mixing matrix \[57\].

To estimate the proper masses of the \( M_\nu \) matrix; we diagonalise it by using the unitary \( U_{TBM} \) TBM mixing matrix; we use the \( \mu - \tau \) symmetry requiring \( m_{2,2} = m_{3,3}, m_{1,2} = m_{1,3} \); as well as the condition \( m_{2,3} = m_{1,1} + m_{1,2} - m_{2,2} \). So we have \( M_{\nu}^{diag} = U_{TBM}^\dagger M_\nu U_{TBM} \) with

\[
U_{TBM} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\] (5.53)

and therefore

\[
M_{\nu}^{diag} \simeq \xi_0 \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 \\
\end{pmatrix}
\] (5.54)

with eigenvalues as

\[
\lambda_1 = \xi_0(m_{1,1} - m_{1,2}) \\
\lambda_2 = \xi_0(m_{1,1} + 2m_{1,2}) \\
\lambda_3 = \xi_0(2m_{3,2} - m_{1,1} - m_{1,2})
\] (5.55)

6 Conclusion and discussions

In this paper, we have developed a method based on characters of discrete group representations to study \( SU_5 \times D_4 \times U_1^\perp \)- GUT models with dihedral monodromy symmetry. After having revisited the construction of \( SU_5 \times S_4 \times U_1^\perp \) and \( SU_5 \times S_3 \times (U_1^\perp)^2 \) models from the character representation view, we have derived three \( SU_5 \times D_4 \times U_1^\perp \) models (referred here to as I, II and III) with curves spectrum respectively given by eqs(4.7-4.8), (4.11-4.12) and (4.14-4.15). These models follow from the three different ways of decomposing the irreducible \( S_4 \)- triplets in terms of irreducible representations of \( D_4 \); see eqs (4.6,10,11,13); such richness may be interpreted as due to the fact that \( D_4 \) has four
kinds of singlets with generator group characters given by the \((p, q)\) pairs with \(p, q = \pm 1\). Then we have focussed on the curve spectrum \((4.7, 4.8)\) of the first \(SU_5 \times D_4 \times U_1^\perp\) model; and studied the derivation of a MSSM-like spectrum by using particular multiplicity values and turning on adequate fluxes. We have found that with the choice of: \((i)\) top-quark family \(10_3\) as \((10^{+}_{+, 0})_0\), transforming into a \(D_4\)-singlet with \(\chi^{(a,b)}\) character equal to \((1, -1)\); and \((ii)\) a \(5^H_u\) up-Higgs as \((5^{+, +})_0\), transforming into a different \(D_4\)-singlet with character equal to \((-1, 1)\); there is no tri-Yukawa couplings of the form

\[
(10^{+, -})_0 \otimes (10^{+, -})_0 \otimes (5^H_u)_{++}
\]

as far as \(D_4 \times U_1^\perp\) invariance is required; this makes \(SU_5 \times D_4 \times U_1^\perp\) model with two quark generations accommodated into a \(D_4\)-doublet non interesting phenomenologically.

Monodromy invariant couplings require implementation of flavons \(\vartheta_{p,q}\) by thinking of \(5^H_u \sim (5^{+, +})_0 \otimes (\vartheta^{(+,-)}_t)_0\) leading therefore to a superpotential of order 4. The same property appears with the down-Higgs couplings where \(D_4 \times U_1^\perp\) invariance of \(10^{+, -} \otimes 5^M_d \otimes 5^H_d\) requires: \((\alpha)\) a matter \(5^M_d \equiv (\vartheta^{(+,-)}_t)_0\) in a \(U_1^\perp\) chargeless \(D_4\)-singlet with character \((-1, 1)\); and \((\beta)\) a curve \(5^H_d\) with a \(D_4\)-character like \((\vartheta^{(+,-)}_t)_0\) composed with a charged flavon \((\vartheta^{(+,-)}_t)_0\); that is as

\[
(5^{+, -})_{t5} \otimes (\vartheta^{(+,+)}_t)_{-t5}
\]

By analysing the conditions that a \(D_4 \times U_1^\perp\)-spectrum has to fulfill in order to have a tri-Yukawa coupling for top-quark family \(10_3\), we end with the constraint that the character of \(5^H_u\) up-Higgs should be equal to \((1, 1)\) as clearly seen on \(10^{+, -} \otimes 10^{+, -} \otimes 5^H_u\). This constraint is valid even if \(10_3\) was chosen like \(10^{+, +}\). By inspecting the spectrum of the three studied \(SU_5 \times D_4 \times U_1^\perp\) models; it results that the spectrum of the third model given by eqs(4.14-4.15) which allow tri-Yukawa coupling; for details on contents and couplings of models II and III; see appendix B.

7 Appendix A: Characters in \(S_4\)-models

In this appendix, we give details on some useful properties of \(\Gamma\)-models studied in this paper; in particular on the representations of \(S_4\) and their characters.

7.1 Irreducible representations of \(S_4\)

First, recall that \(S_4\) has five irreducible representations; as shown on the character formula 
\[24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2;\] these are the 1-dim representations including the trivial \(1\) and the sign \(\epsilon = 1^4;\) a 2-dim representation \(2;\) and the 3-dim representations \(3\) and \(3',\) obeying some ”duality relation”. This duality may be stated in different manners; but,
in simple words, it may be put in parallel with polar and axial vectors of 3-dim euclidian space. In the language of Young diagrams; these five irreducible representations are given by
\[ 1: \quad \begin{array}{c} \hline \hline \hline \end{array}, \quad 2: \quad \begin{array}{c} \hline \hline \hline \end{array}, \quad 3: \quad \begin{array}{c} \hline \hline \end{array} \] (7.1)
and
\[ 3': \quad \begin{array}{c} \hline \hline \hline \hline \end{array}, \quad 1': \quad \begin{array}{c} \hline \hline \hline \hline \hline \end{array} \] (7.2)

This diagrammatic description is very helpful in dealing with \( S_4 \) representation theory; it teaches us a set of useful information; in particular helpful data on the three following:

\textit{i) Expressions of (3.5)}

In the representation 3 of the permutation group \( S_4 \), the three \( x_i \)- weights in (3.5) read in terms of the \( t_i \)'s as
\[ \bar{x} = \frac{1}{2} \begin{pmatrix} t_1 - t_2 - t_3 + t_4 \\ t_1 + t_2 - t_3 - t_4 \\ t_1 - t_2 + t_3 - t_4 \end{pmatrix} = \begin{pmatrix} x_4 - t_2 - t_3 \\ x_4 - t_3 - t_4 \\ x_4 - t_4 - t_2 \end{pmatrix} \] (7.3)

where \( x_4 = \frac{1}{2} (t_1 + t_2 + t_3 + t_4) \) is the completely symmetric term. The normalisation coefficient \( \frac{1}{2} \) is fixed by requiring the transformation \( x_i = U_{ij} t_j \) as follows
\[ U = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \det U = 1 \] (7.4)

For the the representation 3', we have
\[ \bar{x}' = \frac{1}{\sqrt{8}} \begin{pmatrix} t_1 - 3t_2 + t_3 + t_4 \\ t_1 + t_2 - 3t_3 + t_4 \\ t_1 + t_2 + t_3 - 3t_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_4 - 2t_2 \\ x_4 - 2t_3 \\ x_4 - 2t_4 \end{pmatrix} \] (7.5)

The entries of these triplets are cyclically rotated by the (234) permutation.

\textit{ii) \( S_4 \)- triplets as \( 3 \)-cycle (234)}

The \( \{|t_i\}\) and \( \{|x_i\}\) weight bases are related by the orthogonal 5×5 matrix
\[ \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}, \quad |x_i\rangle = U_{ij} |t_j\rangle \] (7.6)
with $U$ as in (7.4); and then
\[
\begin{align*}
t_1 &= \frac{1}{2} (x_4 + x_1 + x_2 + x_3) \\
t_2 &= \frac{1}{2} (x_4 - x_1 + x_2 - x_3) \\
t_3 &= \frac{1}{2} (x_4 - x_1 - x_2 + x_3) \\
t_4 &= \frac{1}{2} (x_4 + x_1 - x_2 - x_3)
\end{align*}
\]
(7.7)

From these transformations, we learn $t_i = U_{ki} x_k$; and then $t_i \pm t_j = (U_{ki} \pm U_{kj}) x_k$ which can be also expressed $t_i \pm t_j = V_{ij}^{\pm kl} X_{kl}^{\pm}$.

### 7.2 Characters

The discrete symmetry group $S_4$ model has 24 elements arranged into five conjugacy classes $C_1, ..., C_5$ as on table (7.8); it has five irreducible representations $R_1, ..., R_5$ with dimensions given by the relation $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$;

| $C_i \backslash \text{irrep} R_j$ | $\chi_i$ | $\chi_{y'}$ | $\chi_2$ | $\chi_3$ | $\chi_\epsilon$ | number |
|---|---|---|---|---|---|---|
| $C_1 \equiv e$ | 1 | 3 | 2 | 3 | 1 | 1 |
| $C_2 \equiv (\alpha \beta)$ | 1 | -1 | 0 | 1 | -1 | 6 |
| $C_3 \equiv (\alpha \beta)(\gamma \delta)$ | 1 | -1 | 2 | -1 | 1 | 3 |
| $C_4 \equiv (\alpha \beta \gamma)$ | 1 | 0 | -1 | 0 | 1 | 8 |
| $C_5 \equiv (\alpha \beta \gamma \delta)$ | 1 | 1 | 0 | -1 | -1 | 6 |

(7.8)

The $S_4$ group has 3 non commuting generators $(a, b, c)$ which can be chosen as given by the 2-, 3- and 4- cycles obeying amongst others the cyclic relations $a^2 = b^3 = c^4 = I_{id}$.

In our approach the character of these generators have been used in the engineering of GUT models with $S_4$ monodromy; they are as follows

| $\chi_{ij}$ | $\chi_i$ | $\chi_{y'}$ | $\chi_2$ | $\chi_3$ | $\chi_\epsilon$ |
|---|---|---|---|---|---|
| $a$ | 1 | -1 | 0 | 1 | -1 |
| $b$ | 1 | 0 | -1 | 0 | 1 |
| $c$ | 1 | 1 | 0 | -1 | -1 |

(7.9)

In the SU$_5 \times S_4$ theory considered in paper, the various curves of the spectrum of the GUT- model belong to $S_4$- multiplets which can be decomposed into irreducible representation of $S_4$. In doing so, one ends with curves indexed by the characters of the generators of $S_4$ as follows

\[
\begin{align*}
4 &= 1_{(1,1,1)} \oplus 3_{(1,0,-1)} \\
6 &= 3_{(1,0,-1)} \oplus 3'_{(1,0,1)}
\end{align*}
\]

(7.10)
8 Appendix B: Results on $SU_5 \times \mathbb{D}_4$ models II & III

In this appendix, we collect results regarding the $SU_5 \times \mathbb{D}_4 \times U_1^\perp$ models II and III of subsections § 4.2.2 and § 4.2.3. In addition to higher order terms, we also study when couplings like

| Couplings | $SU_5$ | $\mathbb{D}_4$ | $U_1^\perp$ | Parity |
|-----------|--------|----------------|-------------|--------|
| $10_i \otimes 10_j \otimes 5_{H_u}$ | 1 | $1_{++}$ | 0 | + |
| $10_i \otimes \overline{5}_j \otimes 5_{H_d}$ | 1 | $1_{++}$ | 0 | + |
| $\nu_i^c \otimes 5_M \otimes 5_{H_u}$ | 1 | $1_{++}$ | 0 | + |
| $m \nu_i^c \otimes \nu_j^c$ | 1 | $1_{++}$ | 0 | + |

(8.1)

can be generated.

8.1 $SU_5 \times \mathbb{D}_4$ model II

The spectrum of the $SU_5 \times \mathbb{D}_4 \times U_1^\perp$ model II under breaking $SU_5 \times \mathbb{D}_4 \times U_1^\perp$ to MSSM is given by:

| Curve in $D_4$ model II | $U_1^\perp$ | Spectrum in MSSM |
|-------------------------|-------------|------------------|
| $10_1 = 10_{+,-}$       | 0           | $M_1 Q_L + u_L^c (M_1 - N - P) + e_L^c (M_1 + N + P)$ |
| $10_2 = 10_{-,+}$       | 0           | $M_2 Q_L + u_L^c M_2 + e_L^c M_3$ |
| $10_3 = 10_{+,+}$       | 0           | $M_3 Q_L + u_L^c M_3 + e_L^c M_4$ |
| $10_4 = 10_{+,+}$       | 0           | $M_4 Q_L + u_L^c (M_4 + P) + e_L^c (M_4 - P)$ |
| $10_5 = 10_{+,+}$       | 1           | $M_5 Q_L + u_L^c (M_5 + N) + e_L^c (M_5 - N)$ |
| $5_1 = 5_{+,-}$         | 0           | $M_1 d_L^c + (M'_1 + N + P) \overline{L}$ |
| $5_2 = 5_{-,+}$         | 0           | $M_2 d_L^c + M_2 \overline{L}$ |
| $5_3 = 5_{-,+}$         | 0           | $M_3 d_d^c + (M'_3 - N) \overline{H}_d$ |
| $5_4 = 5_{+,-}$         | 0           | $M_4 d_u^c + (M'_4 - P) H_u$ |
| $5_5 = 5_{+,-}$         | 0           | $M_5 d_L^c + M_5 \overline{L}$ |
| $5_6 = 5_{-,+}$         | 0           | $M_6 d_L^c + M'_6 \overline{L}$ |
| $5_7 = 5_{-,+}^{5_{+,+}}$ | $-1$ | $M_7 d_L^c + (M'_7 - P) \overline{L}$ |
| $5_8 = 5_{-,+}^{5_{+,+}}$ | $-1$ | $M_8 d_L^c + M'_8 \overline{L}$ |
| $5_9 = 5_{+,+}^{5_{+,+}}$ | $-1$ | $M_9 d_L^c + (M'_9 - N) \overline{L}$ |
| $5_{10} = 5_{+,+}^{5_{+,+}}$ | $-1$ | $M_{10} d_L^c + (M'_{10} + N + P) \overline{L}$ |

(8.2)

To get 3 generations of matter curves and 2 Higgs doublets of MSSM, taking into account the constraints in subsection (5.1), we make the following choice of the flux parameters;
\[ P = -N = 1, \text{ and} \]
\[ M_1 = M_2 = M_3 = M_4 = -M_5 = 1 \]
\[ M'_1 = M'_3 = M'_4 = M'_8 = M'_{10} = 0 \]
\[ M'_2 = M'_5 = M'_6 = M'_9 = -M'_7 = -1 \]

Using the property \( \sum_i M'_i = - \sum_i M'_i = -3 \), the localization of Higgs curves are as \( 5^{H_u} = 5_{-+}, 5^{H_d} = 5_{+-} \); and the third generation like \( 10 = 10^3; \) and \( 5 = 5^3 \). The distribution of the matter curves is collected in the following table:

| Curve in \( D_4 \) model II | \( U^+_\perp \) | Spectrum in MSSM | \( Z_2 \) parity |
|-----------------------------|----------------|-----------------|-----------------|
| \( 10 = 10^3 = (10, -) \)  | 0              | \( Q_L + u^c_L + e^c_L \) | -               |
| \( 10 = (10, +) \)         | 0              | \( Q_L + u^c_L + e^c_L \) | -               |
| \( 10 = (10, -) \)         | 0              | \( Q_L + u^c_L + e^c_L \) | -               |
| \( 10 = (10, +) \)         | 0              | \( Q_L + u^c_L + e^c_L \) | -               |
| \( 5 = (5, +) \)           | 0              | \( -Q_L - 2u^c_L \)      | -               |
| \( 5 = (5, -) \)           | 0              | \( -Q_L - 2u^c_L \)      | +               |
| \( 5 = (5, +) \)           | 0              | \( H_u \)              | +               |
| \( 5 = (5, -) \)           | 0              | \( -H_d \)             | +               |
| \( 5 = (5, +) \)           | 0              | \( -d_L - \bar{L} \)    | -               |
| \( 5 = (5, -) \)           | 0              | \( -d_L - \bar{L} \)    | -               |

From this spectrum, we learn that we have three families of fermions, an extra vector like pairs, \( d^c_L + \bar{d}_L, Q_L + \bar{Q}_L \); and two \( 2(u^c_L + \bar{u}_L) \) which are expected to get a large mass if some of the singlet states acquire large VEV's. In this \( D_4 \) model; there are only singlet flavons transforming in the representations \( 1_{++}, 1_{+-}, 1_{-+} \); with and without \( t_5 \) charges, they are classified as \( (\vartheta_{p,q})_{0,\pm t_5} \) with \( p, q = \pm 1 \); they lead to the following order 4-couplings

- **Up-type quark Yukawa couplings**
The allowed Yukawa couplings that are invariant under $\mathbb{D}_4 \times U_1^\perp$ are:

\[
\begin{align*}
(10_{+,0})_0 \otimes (10_{+,0})_0 \otimes (5^H_{u,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{-,0})_0 \otimes (10_{-,0})_0 \otimes (5^H_{u,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{+,0})_0 \otimes (10_{+,0})_0 \otimes (5^H_{u,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{+,0})_0 \otimes (10_{-,0})_0 \otimes (5^H_{u,+})_0 \otimes (\vartheta_{+,0})_0 \\
(10_{-,0})_0 \otimes (10_{+,0})_0 \otimes (5^H_{u,+})_0 \otimes (\vartheta_{+,0})_0
\end{align*}
\]

(8.5)

• *Down-type quark Yukawa couplings*

The Yukawa couplings down-type are:

\[
\begin{align*}
(10_{+,0})_0 \otimes (\bar{5}_{-,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{-,0})_0 \otimes (\bar{5}_{+,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{+,0})_0 \otimes (\bar{5}_{-,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{-,0})_0 \otimes (\bar{5}_{+,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{+,0})_0 \\
(10_{-,0})_0 \otimes (\bar{5}_{+,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{+,0})_0 \\
(10_{+,0})_0 \otimes (\bar{5}_{-,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{-,0})_0 \otimes (\bar{5}_{+,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{+,0})_0 \otimes (\bar{5}_{-,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{-,0})_0 \\
(10_{-,0})_0 \otimes (\bar{5}_{+,0})_0 \otimes (5^H_{d,+})_0 \otimes (\vartheta_{+,0})_0
\end{align*}
\]

(8.6)
8.2 $SU_5 \times D_4$ model III

The spectrum of the model $SU_5 \times D_4 \times U_1^\perp$ Model III is as follows

| Curves in $D_4$ model III | $U_1^\perp$ | Spectrum in MSSM |
|---------------------------|-------------|------------------|
| $10_1 = 10_{+,+}$         | 0           | $M_1Q_L + u_L^c (M_1 - N - P) + e_L^c (M_1 + N + P)$ |
| $10_2 = 10_{-,+}$         | 0           | $M_2Q_L + u_L^c M_2 + e_L^c M_2$ |
| $10_3 = 10_{+,+}$         | 0           | $M_3Q_L + u_L^c M_3 + e_L^c M_3$ |
| $10_4 = 10_{+,+}$         | 0           | $M_4Q_L + u_L^c (M_4 + P) + e_L^c (M_4 - P)$ |
| $10_5 = 10_{+,+}$         | 1           | $M_5Q_L + u_L^c (M_5 + N) + e_L^c (M_5 - N)$ |
| $5_1 = 5_{+,+}$           | 0           | $M_1^d L + (M_1' + N + P) \overline{L}$ |
| $5_2 = 5_{-,+}$           | 0           | $M_2^d L + (M_2' - \kappa_1 P) \overline{L}$ |
| $5_3 = 5_{-,+}$           | 0           | $M_3^d L + (M_3' - \kappa_2 P) \overline{L}$ |
| $5_4 = 5_{+,+}$           | 0           | $M_4^d L + (M_4' - \kappa_2 P) \overline{L}$ |
| $10_6 = 10_{+,+}$         | -1          | $M_6^d L + (M_6' - N) \overline{L}$ |
| $10_7 = 10_{+,+}$         | -1          | $M_7^d L + (M_7' - \kappa_1 P) \overline{L}$ |
| $10_8 = 10_{+,+}$         | -1          | $M_8^d L + (M_8' - \kappa_2 P) \overline{L}$ |
| $10_9 = 10_{+,+}$         | -1          | $M_9^d L + (M_9' - \kappa_2 P) \overline{L}$ |

The 3 generations of fermions and the 2 Higgs $H_u, H_d$ are obtained by taking the fluxes like $N = -P = -1$ with $\kappa_1 = 0, \kappa_2 = 1$; and

$$M_1 = M_2 = M_3 = M_4 = -M_5 = 1$$
$$M_1' = M_3' = M_4' = M_7' = M_{10}' = 0$$
$$M_2' = M_5' = M_8' = M_9' = -M_6' = -1$$

(8.8)

We choose the Higgs curves as $5^{H_u} = (5^{H_u}_{+,+})_0, 5^{H_d} = (5^{H_d}_{-,+})_0$ and the third $10^{M_3}, 5^{M_3}$ generation as follow.
Curves in $D_4$ model III | $U^\perp_1$ | Spectrum in MSSM | $Z_2$ parity
--- | --- | --- | ---
$10_1 = 10^M_3 = (10_{++})_0$ | 0 | $Q_L + u^c_L + e^c_L$ | −
$10_2 = (10_{--})_0$ | 0 | $Q_L + u^c_L + e^c_L$ | −
$10_3 = (10_{+-})_0$ | 0 | $Q_L + u^c_L + e^c_L$ | −
$10_4 = (10_{++})_0$ | 0 | $Q_L + 2e^c_L$ | +
$10_5 = (10_{++})_{t_5}$ | 1 | $-Q_L - 2e^c_L$ | −
$5_1 = (5_{++})_0$ | 0 | − | +
$5_2 = 5^M_3 = (5_{--})_0$ | 0 | $-\bar{d}_L - \bar{L}$ | −
$5_3 = (5^{H_u}_{++})_0$ | 0 | $-\bar{H}_d$ | +
$5_4 = (5^{H_u}_{++})_0$ | 0 | $H_u$ | +
$5_5 = 5^M_1 = (5_{--})_0$ | 0 | $-\bar{d}_L - \bar{L}$ | −
$5_6 = 5^M_2 = (5_{++})_0$ | 0 | $-\bar{d}_L - \bar{L}$ | −
$5_7 = (5_{++})_{-t_5}$ | −1 | − | +
$5_8 = (5_{--})_{-t_5}$ | −1 | $-\bar{d}_L$ | +
$5_9 = (5_{--})_{-t_5}$ | −1 | $\bar{d}_L$ | +
$5_{10} = (5_{++})_{-t_5}$ | −1 | − | +

- **Up-type quark Yukawa couplings**

The allowed Yukawa couplings that are invariant under $D_4 \times U^\perp_1$ and preserving parity symmetry are:

$$ (10_{+-})_0 \otimes (10_{+-})_0 \otimes (5^{H_u}_{++})_0 $$  \hspace{1cm} (8.10)

for third generation; and

$$ (10_{--})_0 \otimes (10_{--})_0 \otimes (5^{H_u}_{++})_0 $$ \hspace{1cm} (8.11)

- **Down-type quark Yukawa couplings**

The Yukawa coupling down-type are:

$$ (10_{++})_0 \otimes (5_{--})_0 \otimes (5^{H_d}_{--})_0 \otimes (\vartheta_{--})_0 $$  \hspace{1cm} (8.12)
for third generation; and

\[
\begin{align*}
(10_{+,+})_0 & \otimes (\overline{5}_{-,+})_0 \otimes (\overline{5}^{H}_{+,+})_0 \otimes (\overline{\vartheta}_{+,+})_0 \\
(10_{+,+})_0 & \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}^{H}_{-,+})_0 \otimes (\overline{\vartheta}_{-,+})_0 \\
(10_{-,+})_0 & \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}^{H}_{+,+})_0 \otimes (\overline{\vartheta}_{-},+)_0 \\
(10_{-,+})_0 & \otimes (\overline{5}_{-,+})_0 \otimes (\overline{5}^{H}_{-,+})_0 \otimes (\overline{\vartheta}_{-,+})_0 \\
(10_{+,+})_0 & \otimes (\overline{5}_{-},+)_0 \otimes (\overline{5}^{H}_{-,+})_0 \otimes (\overline{\vartheta}_{-},+)_0 \\
(10_{-,+})_0 & \otimes (\overline{5}_{+,+})_0 \otimes (\overline{5}^{H}_{+,+})_0 \\
(10_{+,+})_0 & \otimes (\overline{5}_{-},+)_0 \otimes (\overline{5}^{H}_{-,+})_0 \\
(10_{+,+})_0 & \otimes (\overline{5}_{-},+)_0 \otimes (\overline{5}^{H}_{-,+})_0 \\
\end{align*}
\]  

(8.13)

For the neutrino sectors in both models II and III, the couplings are embedded in the Dirac and Majorana operators as for model I; their mass matrix depend on the choice of the localization of right neutrino in the singlet curves \( \vartheta_{\pm,\pm} \).

9 Appendix C: Monodromy and flavor symmetry

We begin by recalling that in F-theory GUTs, quantum numbers of particle fields and their gauge invariant interactions descend from an affine \( E_8 \) singularity in the internal Calabi-Yau Geometry: \( CY_4 \sim \mathcal{E} \rightarrow \mathcal{B}_3 \). The observed gauge bosons, the 4D matter generations and the Yukawa couplings of standard model arise from symmetry breaking of the underlying \( E_8 \) gauge symmetry of compactification of F-theory to 4D space time. In this appendix, we use known results on F-theory GUTs to exhibit the link between non-abelian monodromy and flavor symmetry which relates the three flavor generations of SM. First, we briefly describe how abelian monodromy like \( \mathbb{Z}_p \) appear in F-GUT models; then we study the extension to non-abelian discrete symmetries such the dihedral \( \mathbb{D}_4 \) we have considered in present study.

9.1 Abelian monodromy

One of the interesting field realisations of the F-theory approach to GUT is given by the remarkable \( SU_5 \times SU_5^\perp \) model with basic features encoded in the internal geometry; in particular the two following useful ones: (i) the \( SU_5 \times SU_5^\perp \) invariance follows from a particular breaking way of \( E_8 \); and (ii) the full spectrum of the field representations of the model is as in eq(2.1). From the internal CY4 geometry view, \( SU_5 \) and \( SU_5^\perp \) have interpretation in terms of singularities; the \( SU_5 \) lives on the so called GUT surface \( \mathcal{S}_{GUT} \); it appears in terms of the singular locus of the following Tate form of the elliptic fibration

\[
y^2 = x^3 + b_5xy + b_4x^2z + b_3yz^2 + b_2xz^3 + b_0z^5;
\]

it is the gauge symmetry visible in 4D
space time of the GUT model. Quite similarly, the $SU_5^\perp$ may be also imagined to have an analogous geometric representation in the internal geometry; but with different physical interpretation; it lives as well on a complex surface $\mathcal{S}'$; another divisor of the base $\mathcal{B}_3$ of the complex four dimensional elliptic CY4 fibration. Obviously these two divisors are different, but intersect. Here, we want to focus on aspects of the representations of $SU_5^\perp$ appearing in eq(2.7) and too particularly on the associated matter curves $\Sigma_{t_i}$, $\Sigma_{t_i+t_j}$, $\Sigma_{t_i-t_j}$; which are nicely described in the spectral cover method using an extra spectral parameter $s$. If thinking of the hidden $SU_5^\perp$ in terms of a broken symmetry by an abelian flux or Higgsing down to its Cartan subgroup, the resulting symmetry of the GUT model becomes $U(1)^4 \times SU_5$ with

$$U(1)^4 = U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_4 = \prod_{i=1}^4 U(1)_i \quad (9.1)$$

The extra $U(1)$’s in the breaking $U(1)^4 \times SU_5$ put constraints on the superpotential couplings of the effective low energy model; the simultaneous existence of $U(1)^4$ is phenomenologically undesirable since it does not allow a tree-level Yukawa coupling for the top quark. This ambiguity is overcome by imposing abelian monodromies among the $U(1)$’s allowing the emergence of a rank one fermion mass matrix structure; see eqs(9.4-9.5) given below.

Following the presentation of section 2 of this paper, the spectral covers describing the above invariance are given by polynomials with an affine variable $s$ as in eq(2.8); see also (2.9, 2.12, 2.13). To fix the ideas, we consider monodromy properties of 10-plets $\Sigma_{t_i}$ encoded in the spectral cover equation

$$\mathcal{C}_5 : b_5 + b_3 s^2 + b_2 s^3 + b_4 s^4 + b_0 s^5 = 0 \quad (9.2)$$

The location of the seven branes on GUT surface associated to this $SU_5$ representation is given by $b_5 = 0$. Using the method of [18, 27, 30, 31], the possible abelian monodromies are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$; they lead to factorizations of the $\mathcal{C}_5$ spectral cover as

$$\mathcal{C}_2 \times (\mathcal{C}_1)^3 \quad \mathcal{C}_3 \times (\mathcal{C}_1)^2 \quad \mathcal{C}_4 \times \mathcal{C}_1 \quad \mathcal{C}_3 \times \mathcal{C}_2 \quad (\mathcal{C}_2)^2 \times \mathcal{C}_1 \quad (9.3)$$

and to the respective identification of the weights $\{t_1, t_2\}$, $\{t_1, t_2, t_3\}$, $\{t_1, t_2, t_3, t_4\}$, $\{t_1, t_2\} \cup \{t_3, t_4, t_5\}$ and $\{t_1, t_2\} \cup \{t_3, t_4\}$.

---

4 Recall the three useful relations: (a) Let $\tilde{H} = (H_1, ..., H_4)$ the generators of the $U(1)_i$ charge factors and $E_{\pm \alpha_i}$ the step operators associated with the simple roots $\alpha_i$, then we have $[E_{\pm \alpha_i}, E_{-\alpha_i}] = \delta_{\alpha_i, \tilde{H}}$. (b) If denoting by $|\tilde{\mu}\rangle$ a weight vector of the fundamental representation of $SU_5^\perp$, then we have $\tilde{\alpha}_i \tilde{H} |\tilde{\mu}\rangle = \lambda_i |\tilde{\mu}\rangle$ with $\lambda_i = \tilde{\alpha}_i \tilde{\mu}$. (c) Using the 4 usual fundamental weight vectors $\tilde{\omega}_i$ dual to the 4 simple roots, the 5 weight vectors $\{\tilde{\mu}_k\}$ of the representation are: $\tilde{\mu}_1 = \tilde{\omega}_1$, $\tilde{\mu}_2 = \tilde{\omega}_2 - \tilde{\omega}_1$, $\tilde{\mu}_3 = \tilde{\omega}_3 - \tilde{\omega}_2$, $\tilde{\mu}_4 = \tilde{\omega}_4 - \tilde{\omega}_3$, $\tilde{\mu}_5 = -\tilde{\omega}_4$. 

40
The algebraic equations for the matter curves $\Sigma_t$, $\Sigma_{t_1+t_2}$, $\Sigma_{t_1-t_2}$ in terms of the $t_i$ weights associated with the $SU_5^+$ fundamental representation are respectively given by $t_i = 0$; $(t_i + t_j)_{i<j} = 0$ and $\pm (t_i - t_j)_{i<j} = 0$; they are denoted like $10_{t_1}, 5_{t_1+t_2}$ and $1_{\pm(t_i-t_j)}$; see eq(2.2).

As a first step to approach non abelian monodromies we are interested in here, it is helpful to notice the two useful following things: (a) the homology 2-cycles in the CY4 underlying $SU_5 \times U(1)^4$ invariance has monodromies captured by a finite discrete group that can be used as a constraint in the modeling. (b) from the view of phenomenology, these monodromies must be at least $Z_2$ in order to have top- quark Yukawa coupling at tree level as noticed before. Notice moreover that under this $Z_2$, matter multiplets of the $SU_5$ model split into two $Z_2$ sectors even and odd; for example the two tenplets $\{10_{t_1}, 10_{t_2}\}$ are interchanged under $t_1 \leftrightarrow t_2$; the corresponding eigenstates are given by $10_{t_\pm}$ with eigenvalues $\pm 1$. By requiring the identification $t_1 \leftrightarrow t_2$, naively realised by setting $t_1 = t_2 = t$, matter couplings in the model get restricted; therefore the off diagonal tree level Yukawa coupling

$$10_{t_1}10_{t_2}5_{-t_1-t_2} \quad (9.4)$$

which is invariant under $SU_5 \times U(1)^4$, becomes after $t_1 \leftrightarrow t_2$ identification a diagonal top-quark interaction invariant under $Z_2$ monodromy. The resulting Yukawa coupling reads as follows [27, 30, 31]

$$10_{t_1}10_{t_2}5_{-2t} \quad (9.5)$$

the other diagonal coupling $10_{t_1}10_{t_2}5_{-2t}$ is forbidden by the $U(1)$ symmetry; see footnote 5. Notice that for bottom- quark the typical Yukawa coupling $10_{t_1}5_{t_1+t_2}5_{t_1+t_1}$ is allowed by $Z_2$ while $10_{t_1}5_{t_1+t_2}5_{t_1+t_1}$ is forbidden.

In this monodromy invariant theory, the symmetry of the model is given by $SU_5 \times U(1)^3 \times Z_2$; it may be interpreted as the invariance that remains after taking the coset with respect to $Z_2$; that is by a factorisation of type $G = H \times Z_2$ with $H = G/Z_2$. Indeed, starting from $SU_5 \times U(1)^4$ and performing the two following operations: (i) use the traceless property of the fundamental representation of $SU_5^+$ to think of (9.1) like

$$U(1)^4 = \left(\prod_{i=1}^{5} U(1)_{t_i}\right) / J \quad (9.6)$$

with $J = \{t_i \mid t_1 + t_2 + t_3 + t_4 + t_5 = 0\} \simeq U(1)_{diag}$; this property is a rephrasing of the usual $U(5)$ factorisation; i.e $SU(5) = \frac{U(5)}{U(1)}$. (ii) substitute the product $U(1)_{t_1} \times U(1)_{t_2}$ by the reduced abelian group $U(1)_t \times Z_2$ where monodromy group has been explicitly exhibited. In this way of doing, one disposes of a discrete group that may be

---

5 In general we have two $Z_2$ eigenstates: $t_\pm = \frac{1}{2} (t_1 \pm t_2)$ with eigenvalues $\pm 1$. While any function of $t_+$ is $Z_2$ invariant, only those functions depending on $(t_-)^2$ which are symmetric with respect to $Z_2$. 

---

41
promoted to a symmetry of the fields spectrum. To that purpose, we need two more steps: first explore all allowed discrete monodromy groups; and second study how to link these groups to flavor symmetry. For the extension of above $\mathbb{Z}_2$, a similar method can be used to build other prototypes; in particular models with abelian discrete symmetries like $SU_5 \times U(1)^{5-k} \times \mathbb{Z}_k$ with $k = 3, 4, 5$; or more generally as

$$SU_5 \times U(1)^{5-p-q} \times \mathbb{Z}_p \times \mathbb{Z}_q$$

(9.7)

where $1 < p + q \leq 5$ and $\mathbb{Z}_1 \equiv I_{id}$, $\mathbb{Z}_0 \equiv I_{id}$. Notice that the discrete groups in eq (9.7) are natural extensions of those of the theories with $SU_5 \times U(1)^{5-k} \times \mathbb{Z}_k$ symmetry; and that the condition $p + q \leq 5$ on allowed abelian monodromies is intimately related with the Weyl symmetry $\mathcal{W}_{SU_5^\perp}$ of $SU_5^\perp$. Therefore, we end with the conclusion that the $\mathbb{Z}_p \times \mathbb{Z}_q$ abelian discrete groups in above relation are in fact particular subgroups of the non abelian symmetric group $\mathcal{W}_{SU_5^\perp} \simeq S_5$.

### 9.2 Non abelian monodromy and flavor symmetry

To begin notice that the appearance of abelian discrete symmetry in the $SU_5$ based GUT models with invariance (9.7) is remarkable and suggestive. It is remarkable because these finite discrete symmetries have a geometric interpretation in the internal CY4; and constitutes then a prediction of F- theory GUT. It is suggestive since such kind of discrete groups, especially their non abelian generalisation, are highly desirable in phenomenology; particularly in playing the role of a flavor symmetry. In this regards, it is interesting to recall that it is quite well established that neutrino flavors are mixed; and this property requires non abelian discrete group symmetries like the alternating $A_4$ group which has been subject to intensive research during last decade [32, 33, 34, 52, 53]. Following the conjecture of [15, 16], non-abelian discrete symmetries may be reached in F- theory GUT by assuming the existence of a non abelian flux breaking the $SU_5^\perp$ down to a non abelian group $\Gamma \subset \mathcal{W}_{SU_5^\perp}$. In this view, one may roughly think about the $\mathbb{Z}_p \times \mathbb{Z}_q$ group of (9.7) as special symmetries of a family of $SU_5$ based GUT models with invariance given by

$$SU_5 \times U(1)^{5-k} \times \Gamma_k$$

(9.8)

where now $\Gamma_k$ is a subgroup of $S_5$ that can be a non abelian discrete group. In this way of doing, one then distinguishes several $SU_5$ GUT models with non abelian discrete symmetries classified by the number of surviving $U(1)$’s. In presence of no $U(1)$ symmetry, we have prototypes like $SU_5 \times S_5$ and $SU_5 \times A_5$; while for a theory with one $U(1)$, we
have symmetries as follows

\[ SU_5 \times U(1) \times S_4 \]
\[ SU_5 \times U(1) \times A_4 \quad (9.9) \]
\[ SU_5 \times U(1) \times D_4 \]

where the alternating \( A_4 \) and dihedral \( D_4 \) are the usual subgroups of \( S_4 \) itself contained in \( S_5 \). In the case with two \( U(1) \)'s, monodromy gets reduced like \( SU_5 \times U(1)^2 \times S_3 \).

Moreover, by using non abelian discrete monodromy groups \( \Gamma_k \), one ends with an important feature; these discrete groups have, in addition to trivial representations, higher dimensional representations that are candidates to host more than one matter generation. Under transformations of \( \Gamma_k \); the generations get in general mixed. Therefore the non abelian \( \Gamma_k \)'s in particular those having 3- and/or 2-dimensional irreducible representations may be naturally interpreted in terms of flavor symmetry.

In the end of this section, we would like to add a comment on the splitting spectral cover construction regarding non abelian discrete monodromy groups like \( A_4 \) and \( D_4 \). In the models \( (9.9) \), the spectral cover for the fundamental \( C_5 \) is factorised like \( C_5 = C_4 \times C_1 \) and similarly for \( C_{10} \) and \( C_{20} \) respectively associated with the antisymmetric and the adjoint of \( SU_5^\perp \). In the \( C_4 \times C_1 \) splitting, we have

\[ C_4 = a_5 s^4 + a_4 s^3 + a_3 s^2 + a_2 s + a_1 \]
\[ C_1 = a_7 s + a_6 \quad (9.10) \]

where the \( a_i \)'s are complex holomorphic sections. For the generic case where the coefficients \( a_i \) are free, the splitted spectral cover \( C_4 \times C_1 \) has an \( S_4 \) monodromy. To have splitted spectral covers with monodromies given by the subgroups \( A_4 \) and \( D_4 \), one needs to put constraints on the \( a_i \)'s; these conditions have been studied in \[14, 16]\; they are non linear relations given by Galois theory. Indeed, starting from \( SU_5 \times SU_5^\perp \) model and borrowing tools from \[16\], the breaking of \( SU_5 \times SU_5^\perp \) down to \( SU_5 \times D_4 \times U(1) \) model considered in this paper may be imagined in steps as follows: first breaking \( SU_5^\perp \) to subgroup \( SU_4^\perp \times U(1) \) by an abelian flux; then breaking the \( SU_4^\perp \) part to the discrete group \( S_4 \) by a non-abelian flux as conjectured in \[15, 16\]; deformations of this flux lead to subgroups of \( S_4 \). To obtain the constraints describing the \( D_4 \) splitted spectral cover descending from \( C_4 \times C_1 \), we use Galois theory; they are given by a set of two constraints on the holomorphic sections of \( C_4 \times C_1 \); and are obtained as follows:

(i) the first constraint comes from the discriminant \( \Delta_{C_4} \) of the spectral cover \( C_4 \) which should not be a perfect square; that is \( \Delta_{C_4} \neq \delta^2 \). The explicit expression of the discriminant of \( C_4 \) has been computed in literature; so we have

\[ 108a_0(\lambda a_0^2 + 4a_1a_7)(\kappa^2 a_7^2 + a_0(\lambda a_0^2 + 4a_1a_7))^2 \neq \delta^2 \quad (9.11) \]
where dependence into $a_6$ and $a_7$ is due to solving the traceless condition $b_1 = 0$ in $C_5 = C_4 \times C_1$. (ii) the second constraint is given by a condition on the cubic resolvent which should be like $R_{C_4}(s)\big|_{s=0} = 0$. The expression of $R_{C_4}(s)$ is known; it leads to

$$a_2^2 a_7 = a_1 \left(a_0 a_6^2 + 4a_3 a_7\right)$$

(9.12)

where $a_0$ is a parameter introduced by the solving the traceless condition $b_1 = 0$; for explicit details see [16].

**Acknowledgement 1** Saidi thanks ICTP Trieste- Italy for kind hospitality where part of this work has been done.

**References**

[1] C. Vafa, *Evidence for F theory*, Nucl. Phys. B 469 (1996) 403 [arXiv:hep-th/9602022].

[2] R. Donagi and M. Wijnholt, *Model Building with F-Theory*, Adv. Theor. Math. Phys. 15 (2011) 1237 [arXiv:0802.2969 [hep-th]].

[3] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, JHEP 0901 (2009) 058 [arXiv:0802.3391 [hep-th]].

[4] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, JHEP 0901 (2009) 059 [arXiv:0806.0102 [hep-th]].

[5] T. Weigand, *Lectures on F-theory compactifications and model building*, Class. Quant. Grav. 27 (2010) 214004 [arXiv:1009.3497 [hep-th]].

[6] R. Donagi and M. Wijnholt, *Higgs Bundles and UV Completion in F-Theory*, Commun. Math. Phys. 326 (2014) 287 [arXiv:0904.1218 [hep-th]].

[7] R. Donagi and M. Wijnholt, *Breaking GUT Groups in F-Theory*, Adv. Theor. Math. Phys. 15 (2011) 1523 [arXiv:0808.2223 [hep-th]].

[8] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, Nucl. Phys. B 829 (2010) 325 [arXiv:0908.1784].

[9] J. Tate, “Algorithm for Determining the Type of a Singular Fiber in an Elliptic Pencil,” in Modular Functions of One Variable IV, Lecture Notes in Math. vol. 476, Springer-Verlag, Berlin (1975).

[10] Sven Krippendorf, Sakura Schafer-Nameki, Jin-Mann Wong, [arXiv:1507.05961].
[11] F. Denef, *Les Houches Lectures on Constructing String Vacua*, arXiv:0803.1194.

[12] J. Marsano, N. Saulina and S. Schafer-Nameki, *Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs*, JHEP **0908** (2009) 046 [arXiv:0906.4672 [hep-th]].

[13] Florent Baume, Eran Palti, Sebastian Schwieger, *On E8 and F-Theory GUTs*, arXiv:1502.03878.

[14] Athanasios Karozasy, Stephen F. King, George K. Leontaris and Andrew Meadowcroft, *Discrete Family Symmetry from F-Theory GUTs*, arXiv:1406.6290v4 [hep-th].

[15] I. Antoniadis and G. K. Leontaris, *Neutrino mass textures from F-theory*, Eur. Phys. J. C **73** (2013) 2670 [arXiv:1308.1581 [hep-th]].

[16] Athanasios Karozasy, Stephen F. King, George K. Leontaris, Andrew K. Meadowcroft, *Phenomenological implications of a minimal F-theory GUT with discrete symmetry*, [arXiv1505.009337v3 [hep-th]].

[17] Anshuman Maharana, Eran Palti, *Models of Particle Physics from Type IIB String Theory and F-theory: A Review*, arXiv:1212.0555.

[18] J. J. Heckman, A. Tavanfar and C. Vafa, *The Point of E(8) in F-theory GUTs*, JHEP **1008** (2010) 040 [arXiv:0906.0581 [hep-th]].

[19] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, Taizan Watari, HEP 1008:036, 2010, [arXiv:0910.2762 [hep-th]].

[20] A. Font, L. E. Ibanez, F. Marchesano and D. Regalado, *Non-perturbative effects and Yukawa hierarchies in F-theory SU(5) Unification*, JHEP **1303** (2013) 140 [Erratum-iibid. **1307** (2013) 036] [arXiv:1211.6529 [hep-th]].

[21] S. Cecotti, M. C. N. Cheng, J. J. Heckman and C. Vafa, arXiv:0910.0477[hep-th].

[22] L. Aparicio, A. Font, L. E. Ibanez and F. Marchesano, JHEP 1108 (2011)152 [arXiv:1104.2609[hep-th]].

[23] Asan Damanik, *Non zero θ13 and Neutrino Masses from Modified TBM*, EJTP 11, No. 31 (2014)125–130P.

[24] Adamson et. al. (MINOS Collab.), Phys. Rev. Lett. 107, 181802 (2011), arXiv:1108.0015[hep-ex]. doi: 10.1103/Phys Rev Lett.107.181802

45
[25] Y. Abe et al. [DOUBLE-CHOOZ Collaboration], Phys. Rev. Lett. 108, 131801 (2012) [arxiv: 1112.6353 [hep-ex]].

[26] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, *Flavor Structure in F-theory Compactifications*, JHEP 1008 (2010) 036 [arXiv:0910.2762 [hep-th]].

[27] I. Antoniadis and G. K. Leontaris, *[Building SO(10) models from F-theory]*, JHEP 1208 (2012) 001 [arXiv:1205.6930 [hep-th]].

[28] Patrick Morandi, *Field and Galois Theory*, Springer, 1996.

[29] Michael Artin, *Algebra*, Prentice-Hall Inc. 1991.

[30] J. C. Callaghan, S. F. King, G. K. Leontaris and G. G. Ross, *Towards a Realistic F-theory GUT*, [arXiv:1109.1297 [hep-th]].

[31] Emilian Dudas and Eran Palti, *On hypercharge flux and exotics in F-theory GUTs*, [arXiv:1007.1297 [hep-th]].

[32] S. F. King and C. Luhn, *Neutrino Mass and Mixing with Discrete Symmetry*, Rept. Prog. Phys. 76 (2013) 056201 [arXiv:1301.1340 [hep-ph]].

[33] Guido Altarelli, Ferruccio Feruglio, *Discrete Flavor Symmetries and Models of Neutrino Mixing*, [arXiv:1000.0211 [hep-th]].

[34] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, *Neutrino Mass and Mixing: from Theory to Experiment*, New J. Phys. 16 (2014) 045018 [arXiv:1402.4271 [hep-ph]].

[35] H. Hayashi, T. Kawano, R. Tatar and T. Watari, *Codimension-3 Singularities and Yukawa Couplings in F-theory*, [arXiv:0901.4941 [hep-th]].

[36] S. F. King, G. K. Leontaris and G. G. Ross, *Family symmetries in F-theory GUTs*, Nucl. Phys. B 838 (2010) 119 [arXiv:1005.1025 [hep-ph]].

[37] G. K. Leontaris, *Aspects of F-Theory GUTs*, PoS CORFU 2011 (2011) 095 [arXiv:1203.6277 [hep-th]].

[38] J. Marsano, *Hypercharge Flux, Exotics, and Anomaly Cancellation in F-theory GUTs*, Phys. Rev. Lett. 106 (2011) 081601 [arXiv:1011.2212 [hep-th]].

[39] Wiliam Fulton, Joe Harris, *Young Tabeaux with Applications to Representation Theory and Geometry*, Springer-Verlag (1991).
[40] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu and M. Tanimoto, *Non-Abelian Discrete Symmetries in Particle Physics*, [arXiv:1003.3552 [hep-th]].

[41] R. Ahl Laamara, M. Miskaoui, E.H Saidi, *Building SO_{10} models with D_4 symmetry*, [arXiv:1511.03166 [hep-th]].

[42] El Hassan Saidi, *On Building superpotentials in F- GUTs*, Prog. Theor. Exp. Phys. (2016) 013 B07, [arXiv:1512.02530]

[43] G. K. Leontaris and G. G. Ross, *Yukawa couplings and fermion mass structure F-theory GUTs*, JHEP 02 (2011) 108 [arXiv:1009.6000 [hep-ph]].

[44] C. Ludeling, H. P. Nilles and C. C. Stephan, *The Potential Fate of Local Model Building*, [arXiv:1101.3346 [hep-ph]].

[45] J. J. Heckman, *Particle Physics Implications of F-theory*, Ann. Rev. Nucl. Part. Sci. 60 (2010) 237 [arXiv:1001.0577 [hep-th]].

[46] T. W. Grimm, *The N=1 effective action of F-theory compactifications*, Nucl. Phys. B 845 (2011) 48 [arXiv:1008.4133 [hep-th]].

[47] C. M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, *Global SO(10) F-theory GUTs*, JHEP 1010 (2010) 057 [arXiv:1005.5735 [hep-th]].

[48] J. C. Callaghan and S. F. King, *E6 Models from F-theory*, JHEP 1304 (2013) 034 [arXiv:1210.6913 [hep-ph]].

[49] R. Tatar and W. Walters, *GUT theories from Calabi-Yau 4-folds with SO(10) Singularities*, JHEP 1212 (2012) 092 [arXiv:1206.5090 [hep-th]].

[50] J. C. Callaghan, S. F. King and G. K. Leontaris, *Gauge coupling unification in E6 F-theory GUTs with matter and bulk exotics from flux breaking*, JHEP 1312 (2013) 037 [arXiv:1307.4593 [hep-ph]].

[51] El Hassan Saidi, *Breaking discrete symmetries in F-GUT*, LPHE-MS -1511,

[52] Fredrik Björkeroth, Francisco J. de Anda, Ivo de Medeiros Varzielas, Stephen F. King, Journal-ref: JHEP 06 (2015) 141, [arXiv:1503.03306]

[53] S. F. King, *Neutrino mass models*, Rept. Prog. Phys. 67 (2004) 107 [hep-ph/0310204].

[54] K. Abe et al. (T2K Collab.), Phys. Rev. Lett. 107, 041801 (2011), [arXiv:1106.2822 [hep-ph]. doi: 10.1103/PhysRevLett.107.041801
[55] F. P. An et al, Phys. Rev. Lett. 108, 171803 (2012), arXiv:1203.1669v2 [hep-ex]. doi:10.1103/PhysRevLett.108.171803

[56] J. K. Ahn et al. (RENO Collab.), Phys. Rev. Lett. 108, 191802 (2012), arXiv:1204.0626v2 [hep-ex]. doi: 10.1103/PhysRevLett.108.191802

[57] S. Boudjemaa and S.F.King, Deviations from Tri-bimaximal Mixing: Charged Lepton Corrections and Renormalization Group Running, [arXiv:0808.2782v3 [hep-th]]

[58] J. Marsano, N. Saulina and Sakura Schafer Nameki, “Compact F-theory GUTs with $U(1)_{PQ}$,” [arXiv:0912.0272v2 [hep-th]].

[59] J. Marsano, N. Saulina and Sakura Schafer Nameki, “F-theory Compactifications for Supersymmetric GUTs,” [arXiv:0904.3032v3 [hep-th]],

[60] S. Krippendorf, D. K. M. Penaa, P. K. Oehlmann and F. Ruehle, “Rational F-theory GUTs without exotics,” [arXiv:1401.5084v1 [hep-th]],

[61] T. W. Grimm and H. Hayashi, “F-theory fluxes, Chirality and Chern-Simons theories,” [arXiv:1111.1232v2 [hep-th]],

[62] E. Dudas and E. Palti, “Froggatt-Nielsen models from E(8) in F-theory GUTs,” [arXiv:0912.0853 [hep-th]],

[63] M. Cvetic, T. W. Grimm and D. Klevers, “Anomaly Cancellation And Abelian Gauge Symmetries In F-theory,” [arXiv:1210.6034v2 [hep-th]].

[64] R. N. Mohapatra, Neutrino mass and Grand Unification of flavor, [arXiv:1007.1633 [hep-ph]].

[65] K. S. Babu and S. M. Barr, Natural Gauge Hierarchy in SO(10), [arXiv:9402291v1 [hep-ph]].

[66] Z. Berezhiani, M. Chianese, G. Mielec and S. Morisi, Chances for SUSY-GUT in the LHC Epoch, [arXiv:1505.04950 [hep-ph]].