0. Introduction

0.1. Our motivation for writing this paper was twofold. On the one hand, we are trying to make sense of the notion of D-module on some infinite-dimensional variety attached to an algebraic group, and on the other hand, we study the structure of a certain interesting representation of the corresponding affine algebra.

First, we will explain the representation-theoretic side of the picture. Thus, let $G$ be an affine algebraic group with Lie algebra $\mathfrak{g}$. For simplicity, in the introduction we will assume that $G$ is unimodular.

Consider the group-scheme $G[[t]]$, which classifies maps of the formal disc $D = \text{Spf}(\mathbb{C}[[t]])$ into $G$. In addition, given an invariant symmetric form $Q : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ one can consider the affine Lie algebra $\widetilde{\mathfrak{g}}_Q := \mathfrak{g}((t)) \oplus \mathbb{C}$. As a subalgebra it contains $\mathfrak{g}[[t]]$, which is the Lie algebra of $G[[t]]$, in particular, the ring of regular functions $\mathcal{O}_{G[[t]]}$ is a $\mathfrak{g}[[t]]$-module, via the action by left-invariant vector fields. Let us consider the induced module over the affine algebra $V_{G,Q} := \text{Ind}_{\mathfrak{g}[[t]]}^{\widetilde{\mathfrak{g}}_Q} \mathfrak{g}((t)) \oplus \mathbb{C}(\mathcal{O}_{G[[t]]})$.

It turns out that, in addition to being just a $\widetilde{\mathfrak{g}}_Q$-module, $V_{G,Q}$ possesses several other structures and the goal of this paper is to study them.

The initial observation is that $V_{G,Q}$ has a natural structure of a vertex operator algebra. In this paper we will adopt the language of chiral algebras, rather than vertex operator algebras, and the above assertion reads as follows: on every curve $X$ there exists a chiral algebra $\mathcal{D}_{G,Q}$ and for a point $x \in X$, there is an isomorphism between $V_{G,Q}$ and the fiber of $\mathcal{D}_{G,Q}$ at $x$ for every choice of a local parameter at $x$.

In fact, the $\widetilde{\mathfrak{g}}_Q$-module structure on $\mathcal{D}_{G,Q}$ comes from an embedding of chiral algebras $\mathfrak{l} : A_{\mathfrak{g},Q} \to \mathcal{D}_{G,Q}$, where $A_{\mathfrak{g},Q}$ is the (Kac-Moody) chiral algebra canonically attached to the pair $(\mathfrak{g}, Q)$, cf. Sect. 3.1.

Here comes a crucial observation, which is the main result of this paper:

It turns out that in addition to $\mathfrak{l}$, there is another embedding of chiral algebras $\mathfrak{r} : A_{\mathfrak{g},Q'} \to \mathcal{D}_{G,Q}$, where $Q' = -Q - Q_0$, where $Q_0$ is the Killing form.

Moreover, the embeddings $\mathfrak{l}$ and $\mathfrak{r}$ commute with one another in the appropriate sense (in the chiral terminology, they *-commute). On the level of representation theory, this means that on $V_{G,Q}$ there is an action of $\widetilde{\mathfrak{g}}_{Q'}$, which commutes with the initial $\widetilde{\mathfrak{g}}_Q$-action; the induced action of the subalgebra $\mathfrak{g}[[t]] \subset \widetilde{\mathfrak{g}}_{Q'}$ comes from the natural $\mathfrak{g}[[t]]$-action on $\mathcal{O}_{G[[t]]}$ by right-invariant vector fields.

0.2. Now, let us explain the algebro-geometric meaning of the module $V_{G,Q}$ and of our constructions.
Let \( Z \) be an arbitrary affine smooth variety and let \( Z((t)) \) be the ind-scheme that classifies maps of the formal punctured disc to \( Z \).

One may wonder whether it is possible to define the notion of D-module on \( Z((t)) \). The general answer is still not clear, however, a partial solution has been proposed by Beilinson and Drinfeld:

One can single out a class of chiral algebras, which can be called \textit{chiral algebras of differential operators} (CADO) corresponding to \( Z \) (cf. Sect. 2). Given a CADO \( \mathcal{D}_Z \), one can consider the category of chiral modules over it, which will be a candidate for the sought-for category of D-modules. For example, the vacuum module, i.e. the fiber of \( \mathcal{D}_Z \) at some point \( x \in X \), corresponds to the \( \delta \)-function on the subscheme \( Z[[t]] \) inside \( Z((t)) \).

We remark that the same class of chiral algebras was independently discovered in \[3\].

A priori, there is more than one CADO that corresponds to \( Z \) (or maybe none) and one thus obtains non-equivalent theories of D-modules. Note that even in the finite-dimensional case, along with ordinary D-modules one can consider twisted D-modules, and the corresponding categories would not be equivalent in general. The essential complication of the infinite-dimensional situation (which we view in terms of chiral algebras) is that there is no preferred (i.e. zero) twisting.

Now let us take \( Z = G \). In this case, it turns out that a choice of a form \( Q : g \otimes g \to \mathbb{C} \) determines a specific CADO, which will be our \( \mathcal{D}_{G,Q} \). In particular, in Sect. 6 we will show that when \( Q = 0 \), the category of chiral \( \mathcal{D}_{G,Q} \)-modules is indeed close to what one might call the category of D-modules on \( G((t)) \).

Let us now explain the geometric meaning of the chiral algebra maps \( I \) and \( r \) mentioned above. The embedding \( I \) corresponds to the embedding of \( \mathfrak{g}((t)) \), identified with left-invariant vector fields on \( G((t)) \), into "the ring of differential operators", and its existence follows from the construction of \( \mathcal{D}_{G,Q} \).

The embedding \( r \) corresponds, on the heuristic level, to the embedding of \( \mathfrak{g}((t)) \) into differential operators by means of right-invariant vector fields. However, the fact that \( \mathcal{D}_{G,Q} \) is attached to \( G \) non-canonically results in an anomaly: in the formula for \( r \) a non-trivial correction term appears; in particular, the initial quadratic form \( Q \) gets shifted by the Killing form.

0.3. What we said above summarizes the main results and ideas of this paper. Finally, let us mention one more property of \( \mathcal{V}_{G,Q} \), which has to do with the semi-infinite cohomology of \( \mathfrak{g}((t)) \).

Consider the affine algebra \( \mathfrak{g}_{-Q_0} \) corresponding to the Killing form of \( \mathfrak{g} \). Recall that if \( N \) is a module over \( \mathfrak{g}_{-Q_0} \) (we are assuming that \( 1 \in \mathbb{C} \subseteq \mathfrak{g}_{-Q_0} \) acts on \( N \) as identity), it makes sense to consider its \textit{semi-infinite} cohomology with respect to \( \mathfrak{g}((t)) \), denoted \( H^{\infty+k}_\mathfrak{g} \mathfrak{g}((t)), N \).

In particular, if \( M \) is a module over \( \mathfrak{g}_Q \) for some \( Q \), the embedding \( r \) defines on the tensor product \( N := M \otimes \mathcal{V}_{G,Q} \) a structure of a module over \( \mathfrak{g}_{-Q_0} \), via the diagonal action. In addition, this tensor product has a commuting \( \mathfrak{g}_Q \)-module structure by letting \( \mathfrak{g}_Q \) act only on \( \mathcal{V}_{G,Q} \) via \( I \).
Therefore, the semi-infinite cohomology $H_{∞}^{+k}(\mathfrak{g}(t), M \otimes \mathcal{V}_{G,Q})$ is naturally a $\tilde{\mathfrak{g}}_{Q}$-module. In the last section we prove that, under the assumption that $M$ is $G[[t]]$-integrable, there is a canonical isomorphism of $\tilde{\mathfrak{g}}_{Q}$-modules:

$$H_{∞}^{+k}(\mathfrak{g}(t), M \otimes \mathcal{V}_{G,Q}) \simeq M \otimes H^{k}(\mathfrak{g}, \mathbb{C}).$$

Therefore, up to the ordinary Lie algebra cohomology, $\mathcal{V}_{G,Q}$ behaves "as a regular representation" with respect the operation $M \mapsto H_{∞}^{+k}(\mathfrak{g}(t), M \otimes \mathcal{V}_{G,Q})$.

Let us remark in conclusion, that by taking instead of the vacuum module $\mathcal{V}_{G,Q}$ another chiral module over $D_{G,Q}$ (corresponding to the Iwahori subgroup of $G((t))$, rather than to $G[[t]]$), one could eliminate the appearance of $H^{k}(\mathfrak{g}, \mathbb{C})$ in the above formula. In the terminology suggested by B. Feigin, a $\tilde{\mathfrak{g}}_{Q}-\tilde{\mathfrak{g}}_{Q}'$ bimodule with this property should be called a semi-regular module over the affine algebra.

0.4. Let us briefly explain how the paper is organized:

In Sect. 1 we review the basics of the theory of chiral algebras.

In Sect. 2 we introduce the Beilinson-Drinfeld formalism of chiral differential operators.

In Sect. 3 we present the construction of our main object of study, i.e. the chiral algebra $D_{G,Q}$.

In Sect. 4 we prove the main theorem about the existence of the embedding $\tau$.

In Sect. 5 we discuss issues related to semi-infinite cohomology.

In the Appendix (Sect. 6) we study the relation between the category of chiral $D_{G,Q}$-modules and "actual" D-modules on the loop group $G((t))$.

0.5. Acknowledgements. We would like to express our deep gratitude to B. Feigin for explaining to us his idea that the semi-regular module over the affine algebra should possess a structure of vertex algebra.

All we know about chiral algebras we learned from A. Beilinson. In particular, our paper uses the unpublished treatise on chiral algebras [3], and in the Appendix we use another unpublished work of Beilinson and Drinfeld, namely [3, 4].

When this paper was in preparation, we received a preprint by V. Gorbunov, F. Malikov and V. Schechtman, in which results similar to ours were obtained; we thank V. Schechtman for this communication.

0.6. Notation and conventions. For a scheme $Z$, $\mathcal{O}_X$ (resp., $T_Z$, $\Omega_Z$, $D_Z$) will denote the structure sheaf (resp., the sheaf of 1-forms, the sheaf of differential operators) on $Z$.

We will work with a fixed smooth algebraic curve $X$ over $\mathbb{C}$. For $n \geq 2$, $j_n$ will denote the (open) embedding of the complement of the diagonal divisor in $X^n$; $\Delta_n$ will denote the (closed) embedding of the main diagonal $X \to X^n$; the subscript $n$ will sometimes be omitted when $n = 2$ and no confusion is likely to occur.

Our main objects of study will be D-modules on $X$ or $X^n$. Normally, we will work with right D-modules; for a right D-module $M$ on $X$, we will denote by $M^l$ the corresponding left D-module, i.e. $M \simeq M^l \otimes \Omega_X$. For two right D-modules $M_1$ and $M_2$, 

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1 Although unpublished, the above manuscripts are available electronically, cf. our list of references.
we will denote by \( M_1 \otimes M_2 \) their D-module tensor product, i.e. as an \( \mathcal{O}_X \)-module \( M_1 \otimes M_2 \simeq M_1 \otimes M_2 \otimes \Omega_X^{-1} \).

**De Rham cohomology.** Let \( \mathcal{M} \) be a right D-module on \( X^0 \), where \( X^0 \) is either an affine curve, or a formal (resp., formal punctured) disc. It then makes sense to consider the 0-th De Rham cohomology of \( \mathcal{M} \) on \( X^0 \). By definition, \( DR(X^0, \mathcal{M}) \) is a vector space equal to \( \mathcal{M}/\mathcal{M} \cdot T_{X^0} \). We will denote by \( h \) the natural projection \( h : \Gamma(X^0, \mathcal{M}) \to DR(X^0, \mathcal{M}) \).

Let \( \mathcal{M} \) be a right D-module on \( X \) and consider its direct image \( \Delta_t(\mathcal{M}) \) under \( \Delta : X \to X \times X \). Then the \( \mathcal{O} \)-module direct image of \( \Delta_t(\mathcal{M}) \) under \( p_2 : X \times X \to X \) is naturally a right D-module on \( X \) and it surjects by means of the residue map onto its D-module direct image, the latter being isomorphic to \( \mathcal{M} \) itself. Since this map is a version of the De Rham projection, we will denote it by \((h \boxtimes \text{id}) : \Delta_t(\mathcal{M}) \to \mathcal{M} \).

### 1. Chiral algebras

1.1. **Definition.** We begin by recalling some basic definitions from the theory of chiral algebras.

A chiral algebra over \( X \) is a right D-module \( \mathcal{A} \) endowed with a chiral bracket, i.e. a map

\[
\{ \cdot, \cdot \} : j_\ast j^\ast (\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_t(\mathcal{A})
\]

which is antisymmetric and satisfies the Jacobi identity in the following sense:

Let \( \mathcal{M} \) be a \( D \)- (or \( \mathcal{O} \))-module on \( X^n \) equivariant with respect to the action of the symmetric group \( S^n \). (The examples are \( \mathcal{A} \boxtimes \mathcal{A}^\infty_n \), \( j_n^\ast j_n^\ast (\mathcal{A} \boxtimes \mathcal{A}^\infty_n) \), \( \Delta_n(\mathcal{A}) \), etc.) For an element \( \sigma \in S_n \), we will denote by the same symbol \( \sigma \) its action on the space \( \Gamma(X^n, \mathcal{M}) \) of sections of \( \mathcal{M} \).

Let us denote by \( \{ \{ \cdot, \cdot \}, \cdot \} \) the map

\[
j_3^\ast j_3^\ast (\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}) \{ \cdot, \cdot \} \boxtimes \text{id} \to (\Delta_2 \boxtimes \text{id})_!(j_2^\ast j_2^\ast (\mathcal{A} \boxtimes \mathcal{A})) \{ \cdot, \cdot \} \boxtimes \text{id} \circ \Delta_2(\mathcal{A}) \simeq \Delta_3(\mathcal{A}).
\]

We must have:

- Let \( \sigma \) be the transposition acting on \( X \times X \). Then for a (local) section \( a \) of \( j_\ast j^\ast (\mathcal{A} \boxtimes \mathcal{A}) \),
  \[
  \sigma(\{ \cdot, \cdot \}(a)) = -\{ \cdot, \cdot \}(\sigma(a)).
  \]

- Let \( \sigma \) be a cyclic permutation \( \sigma \) acting on \( X \times X \times X \). Then for a (local) section \( a \) of \( j_3^\ast j_3^\ast (\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}) \),
  \[
  \{ \{ \cdot, \cdot \}, \cdot \}(a) + \sigma^{-1}(\{ \{ \cdot, \cdot \}, \cdot \}(\sigma(a))) + \sigma^{-2}(\{ \{ \cdot, \cdot \}, \cdot \}(\sigma^2(a))) = 0 \in \Delta_t(\mathcal{A})
  \]

A unit in a chiral algebra \( \mathcal{A} \) is a map \( \Omega_X \to \mathcal{A} \) such that the composition

\[
j_\ast j^\ast (\Omega_X \otimes \mathcal{A}) \to j_\ast j^\ast (\mathcal{A} \otimes \mathcal{A}) \to \Delta_t(\mathcal{A})
\]

is the canonical map \( j_\ast j^\ast (\Omega_X \otimes \mathcal{A}) \to \Delta_t(\Omega_X \otimes \mathcal{A}) \simeq \Delta_t(\mathcal{A}) \).

As in the case of associative algebras, a unit, if it exists, is unique. In all our examples, chiral algebras will have a unit.
For a point $x \in X$, let $DR(\mathcal{D}_x, \mathcal{A})$ (resp., $DR(\mathcal{D}_x^*, \mathcal{A})$) be the De Rham cohomology of $\mathcal{A}$ over the formal (resp., formal punctured) disc around $x$. Both $DR(\mathcal{D}_x, \mathcal{A})$ and $DR(\mathcal{D}_x^*, \mathcal{A})$ are topological Lie algebras. The latter is sometimes called the local completion of $\mathcal{A}$ at $x$, denoted $\mathcal{A}_{x,loc}$ and it contains the former as a subalgebra (provided that $\mathcal{A}$ is $X$-flat).

Let $\mathcal{A}_x$ denote the D-module fiber of $\mathcal{A}$ at $x$. Then $\mathcal{A}_x$ is naturally a module over $DR(\mathcal{D}_x^*, \mathcal{A})$, called the vacuum module. Note that we have a short exact sequence of vector spaces

$$0 \to DR(\mathcal{D}_x, \mathcal{A}) \to DR(\mathcal{D}_x^*, \mathcal{A}) \to \mathcal{A}_x \to 0,$$

where the last arrow is given by the action of $DR(\mathcal{D}_x^*, \mathcal{A})$ on the unit element $1_x \in \mathcal{A}_x$.

Loosely speaking, the chiral algebra structure on $\mathcal{A}$ is completely determined by the action of $DR(\mathcal{D}_x^*, \mathcal{A})$ on $\mathcal{A}_x$.

A chiral module over a chiral algebra $\mathcal{A}$ is a right D-module $\mathcal{M}$ endowed with an action map $act : j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \to \Delta_l(\mathcal{M})$ such that:

- Let $\sigma$ be the transposition of the first two factors in $X \times X \times X$. Then for a (local) section $m$ of $j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{M})$

  $$act \circ (id \boxtimes act)(a) - \sigma^{-1}(act \circ (id \boxtimes act)(\sigma(a))) = act \circ (\{ \cdot, \cdot \} \boxtimes id)(a) \in \Delta_3(\mathcal{M}).$$

- The composition $j_*j^*(\Omega_X \boxtimes \mathcal{M}) \to j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \to \Delta_l(\mathcal{M})$ is the canonical map $j_*j^*(\Omega_X \boxtimes \mathcal{M}) \to \Delta_l(\mathcal{M})$.

For $x \in X$, let $\mathcal{M}_x$ denote the D-module fiber of $\mathcal{M}$ at $x$. Then $DR(\mathcal{D}_x^*, \mathcal{A})$ (and hence $DR(\mathcal{D}_x, \mathcal{A})$) acts on $\mathcal{M}_x$ in a natural way and this action is continuous with respect to the (only natural) discrete topology on $\mathcal{M}_x$.

Note, that a chiral module need not be flat as on $\mathcal{O}_X$-module. In fact, we will be particularly interested in chiral modules supported at a point $x \in X$. If $\mathcal{M}$ is such a module, its D-module fiber at $x$ is of course 0, and instead we will denote by $\mathcal{M}_x$ the vector space such that $\mathcal{M} = i_x!(\mathcal{M}_x)$, where $i_x$ denotes the embedding of the point.

1.2. Commutative chiral algebras. Let $\mathcal{B}^l$ be a left D-module endowed with a commutative and associative multiplication map $\mathcal{B}^l \otimes \mathcal{B}^l \to \mathcal{B}^l$ and a unit $\mathcal{O}_X \to \mathcal{B}^l$, both respecting the D-module structure. Such $\mathcal{B}^l$ is called a (commutative associative) D-algebra.

Let $\mathcal{B}$ be the corresponding right D-module, i.e. $\mathcal{B} = \mathcal{B}^l \otimes \Omega_X$. We obtain a map

$$\{ \cdot, \cdot \} : j_*j^*(\mathcal{B} \boxtimes \mathcal{B}) \to \Delta_l(\mathcal{B} \otimes \mathcal{B}) \to \Delta_l(\mathcal{B})$$

and we leave it to the reader to check (or, alternatively, consult [2]) that it satisfies the axioms of a chiral bracket.

Moreover, in loc. cit. it is explained that $\mathcal{B}^l \to \mathcal{B}$ establishes an equivalence between the category of (com. assoc.) D-algebras and the subcategory of chiral algebras for which the chiral bracket $\{ \cdot, \cdot \}$ factors as

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_l(\mathcal{A} \otimes \mathcal{A}) \to \Delta_l(\mathcal{A}).$$

Note that the latter condition is equivalent to the fact that the composition $\mathcal{A} \boxtimes \mathcal{A} \to j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\{ \cdot, \cdot \}} \Delta_l(\mathcal{A})$ vanishes. Such chiral algebras will be henceforth referred to
as commutative. (More generally, for an arbitrary chiral algebra \( \mathcal{A} \) one says that two sections \( a_1, a_2 \in \mathcal{A} \) *-commute if \( \{\cdot, \cdot\}(a_1 \otimes a_2) = 0 \).

The simplest way to produce a D-algebra (and hence a commutative chiral algebra) is as follows: Let \( \mathcal{M}^l \) be a left D-module and let \( \mathcal{B}^l := \text{Sym}_{\mathcal{O}_X}(\mathcal{M}^l) \). We will say that a D-algebra is finitely generated if it is a quotient of a one of the above form for a finitely generated D-module \( \mathcal{M}^l \).

Let \( \mathcal{C} \) be a quasi-coherent sheaf of (com. assoc.) algebras on \( X \). We define a D-algebra \( J(\mathcal{C})^l \) as a quotient of \( \text{Sym}_{\mathcal{O}_X}(\mathcal{D}_X \otimes \mathcal{C}) \) by the ideal generated by \((1 \otimes c_1) \cdot (1 \otimes c_2) - (1 \otimes c_1 \cdot c_2) - (1 \otimes 1) - 1\). This \( J(\mathcal{C})^l \) is called the jet construction of \( \mathcal{C} \) and it has the following universal property: for a D-algebra \( \mathcal{B}^l \),

\[ \text{Hom}_{D-alg}(J(\mathcal{C})^l, \mathcal{B}^l) = \text{Hom}_{\mathcal{O}-alg}(\mathcal{C}, \mathcal{B}^l). \]

In other words, \( J \) is the left adjoint to the forgetful functor from the category of D-algebras to the category of quasi-coherent sheaves of algebras. It is easy to see that \( J(\mathcal{C})^l \) is finitely generated if \( \mathcal{C} \) is.

Let \( \mathcal{B}^l \) be a D-algebra and \( x \in X \) a point (we will denote by \( k_x \) the residue field of \( X \) at \( x \)). Let \( \mathcal{B}_x \) be the D-module fiber of \( \mathcal{B}^l \) at \( x \). By definition, as a set \( \text{Spec}(\mathcal{B}_x) \) consists of all \( \mathcal{O}_x \)-algebra homomorphisms \( \mathcal{B}^l \to k_x \). It is well-known that for a left D-module \( \mathcal{M}^l \) on \( X \),

\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{M}^l, k_x) = \text{Hom}_{\mathcal{D}_X}(\mathcal{M}^l, \hat{\mathcal{O}}_x). \]

Analogously, we have:

\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{B}^l, k_x) = \text{Hom}_{\mathcal{D}_X}(\mathcal{B}^l, \hat{\mathcal{O}}_x), \]

as algebras. Therefore, \( \text{Spec}(\mathcal{B}_x) \) consists of all flat sections of \( \text{Spec}(\mathcal{B}^l) \) over \( \mathcal{D}_x \).

In particular, if \( \mathcal{B}^l = J(\mathcal{C})^l \), by the universal property, \( \text{Spec}(J(\mathcal{C})_x) \) consists of all sections of \( \mathcal{C} \) over \( \mathcal{D}_x \), which explains the name “jets”.

1.3. Lie-* algebras. To proceed further, we need to recall the definition of Lie-* algebras. These objects are basic tools for constructing non-commutative chiral algebras.

By definition, a Lie-* algebra is a right D-module \( L \) on \( X \) endowed with a Lie-* bracket \( \{\cdot, \cdot\} : L \otimes L \to \Delta_!(L) \), which is antisymmetric and satisfies the Jacobi identity in the same sense as in the definition of a chiral bracket.

A Lie-* algebra is called commutative if its Lie-* bracket is trivial, i.e. equals 0.

A Lie-* module over a Lie-* algebra \( L \) is a right D-module \( \mathcal{M} \) endowed with a map:

\[ \text{act} : L \otimes \mathcal{M} \to \Delta_!(\mathcal{M}), \]

which satisfies the condition identical to condition (1) in the definition of chiral modules over a chiral algebra.

If \( L \) is a Lie-* algebra, the vector spaces \( DR(\mathcal{D}_x, L) \) and \( DR(\mathcal{D}^*_x, L) \) are naturally topological Lie algebras. If \( \mathcal{M} \) is a Lie-* module over \( L \), then its fiber \( \mathcal{M}_x \) is a continuous module over \( DR(\mathcal{D}_x, L) \).
Here is a simplest example of a Lie-* algebra. Let \( g \) be a Lie algebra and let us consider \( L_g = g \otimes D_X \). Then the Lie bracket on \( g \) induces a map
\[
g \otimes D_X \boxtimes g \otimes D_X \to \Delta_1(g \otimes D_X),
\]
which is easily seen to satisfy the properties of a Lie-* bracket.

1.4. The universal enveloping chiral algebra. There exists an obvious forgetful functor from the category of chiral algebras to that of Lie-* algebras: we compose the chiral bracket \( \{ \cdot , \cdot \} : j_* j^* (A \boxtimes A) \to \Delta_1(A) \) with the embedding \( A \boxtimes A \to j_* j^*(A \boxtimes A) \). (Note that the resulting Lie-* algebra is commutative if and only if \( A \) is a commutative chiral algebra.)

It is a basic fact that the above functor admits a left adjoint \( L \mapsto \mathcal{U}(L) \). In other words, for a chiral algebra \( A \) there is a functorial isomorphism
\[
\text{Hom}_{\text{Lie-*}}(L, A) = \text{Hom}_{\text{Chiral}}(\mathcal{U}(L), A).
\]
This \( \mathcal{U}(L) \) is called the universal enveloping chiral algebra of \( L \).

We refer the reader to [2] or [4] for the proof of the existence of this functor and of its basic properties, some of which are reviewed below. In what follows we will assume that \( L \) is flat as an \( \mathcal{O}_X \)-module.

By the universal property, we have a map \( L \to \mathcal{U}(L) \). Hence, if \( M \) is a chiral module over \( \mathcal{U}(L) \) supported at \( x \in X \), then \( M_x \) is a continuous module not only over \( DR(\mathcal{D}_x, L) \), but over the whole \( DR(\mathcal{D}_x^*, L) \).

**Lemma 1.5.** The above functor establishes an equivalence

Chiral modules over \( \mathcal{U}(L) \) supported at \( x \) \( \leftrightarrow \) Continuous modules over \( DR(\mathcal{D}_x^*, L) \).

In particular, \( \mathcal{U}(L)_x \) is a \( DR(\mathcal{D}_x^*, L) \)-module.

**Lemma 1.6.** We have:
\[
\mathcal{U}(L)_x \simeq \text{Ind}_{DR(\mathcal{D}_x^*, L) / DR(\mathcal{D}_x, L)}^{DR(\mathcal{D}_x^*, L)}(\mathcal{C}),
\]
where \( \mathcal{C} \) is the trivial module over \( DR(\mathcal{D}_x, L) \), in such a way that

a) The generator in \( \text{Ind}_{DR(\mathcal{D}_x^*, L) / DR(\mathcal{D}_x, L)}^{DR(\mathcal{D}_x^*, L)}(\mathcal{C}) \) corresponds to the unit \( 1_x \in \mathcal{U}(L)_x \).

b) The embedding \( L \to \mathcal{U}(L) \) corresponds to
\[
L_x = DR(\mathcal{D}_x^*, L) / DR(\mathcal{D}_x, L) \to \text{Ind}_{DR(\mathcal{D}_x^*, L) / DR(\mathcal{D}_x, L)}^{DR(\mathcal{D}_x^*, L)}(\mathcal{C}).
\]

Here is an additional property of the chiral algebra \( \mathcal{U}(L) \) ([2, 4]).

**Proposition 1.7.** As a \( D \)-module, \( \mathcal{U}(L) \) carries a unique increasing filtration \( \mathcal{U}(L) = \bigcup_{i \geq 0} \mathcal{U}(L)_i \) such that

a) \( \mathcal{U}(L)_0 = \Omega_X \)

b) The embedding \( L \to \mathcal{U}(L) \) induces an isomorphism \( \mathcal{U}(L)_1 \simeq \Omega_X \oplus L \)

c) It is compatible with the chiral bracket in the sense that
\[
\{ \cdot , \cdot \} : j_* j^*(\mathcal{U}(L)_i \boxtimes \mathcal{U}(L)_j) \mapsto \Delta_1(\mathcal{U}(L)_{i+j})
\]
\[
\{ \cdot , \cdot \} : \mathcal{U}(L)_i \boxtimes \mathcal{U}(L)_j \mapsto \Delta_1(\mathcal{U}(L)_{i+j-1}).
\]
In particular, the above proposition implies that \( \text{gr}(\mathfrak{u}(L)) := \bigoplus_{i \geq 0} \mathfrak{u}(L)_i/\mathfrak{u}(L)_{i-1} \) is a commutative chiral algebra. The map \( L \to \mathfrak{u}(L) \) induces a map \( \text{Sym}(L) \to \text{gr}(\mathfrak{u}(L)) \) and it is shown in [2] that an analog of the PBW theorem holds:

**Theorem 1.8.** \( \text{Sym}(L) \cong \text{gr}(\mathfrak{u}(L)) \).

On the level of fibers, this filtration corresponds to the standard filtration on the induced module

\[
U(\text{DR}(D^*_x, L)) \otimes U(\text{DR}(D^*_x, L)) \simeq \text{Ind}_{\text{DR}(D^*_x, L)}^{\text{DR}(D^*_x, L)}(C) \simeq U(L)_x,
\]

coming from the filtration on the usual universal enveloping algebra \( U(\text{DR}(D^*_x, L)) \).

## 2. Chiral differential operators

### 2.1. The case of an affine space.

As was explained in the introduction, the object of study of this paper is a chiral algebra \( D_{G,Q} \) attached to a group \( G \) and a form \( Q \) on \( g \). This chiral algebra will be constructed as a chiral algebra of differential operators on \( G \). In this section we will recall the theory of chiral differential operators, developed by Beilinson-Drinfeld and Malikov-Schechtman-Vaintrob, cf. [2], Sect. 3.9 and [6].

Let \( Z \) be a smooth affine algebraic variety and let \( C \) be the \( O_X \)-algebra \( O_Z \otimes O_X \).

Consider \( Z(\hat{O}_x) := \text{Spec}(J(C)_x) \). This is a scheme, which is a projective limit of finite-dimensional schemes \( Z_i := Z(\hat{O}_x/m^i_x) \), where \( m_x \subset \hat{O}_x \) is the maximal ideal.

Since the reduction maps \( Z(\hat{O}_x/m^i_x) \to Z(\hat{O}_x/m^{i-1}_x) \) are smooth, one can easily make sense of the category of left \( D \)-modules on \( Z(\hat{O}_x) \): by definition, this category is \( \lim_{\to} D\text{-mod}(Z_i) \).

In addition, one can define the ind-scheme \( Z(\hat{K}_x) \), where \( K_x \) is the fraction field of \( O_x \) (cf. [4]). In general, if one has an ind-scheme of ind-finite type (i.e. representable as a union of closed subschemes, each of which is of finite type) one can define the notion of a right \( D \)-module on it. However, \( Z(\hat{K}_x) \) is not of ind-finite type and it is a priori not clear that the theory of \( D \)-modules on it exists at all.

Thus, the first obstacle in defining \( D \)-modules on \( Z(\hat{K}_x) \) is the fact that one cannot pass between left and right \( D \)-modules in the infinite-dimensional setting. Therefore, we will start by considering the example of \( Z = \mathbb{A}^n \), in which case this difficulty can be eliminated, and which will suggest the form of the solution in general.

We emphasize that material that we are reviewing in this section is fully contained in [2] and in [3].

Recall that if \( V \) is a finite-dimensional vector space, then the category of left (resp., right) \( D \)-modules on \( V \) viewed as an algebraic variety is naturally equivalent to the category of left (resp., right) modules over the Weyl algebra, \( W(V) \). By definition, \( W(V) \) is generated by elements \( v \in V, v^* \in V^* \) which satisfy the relations

\[
\begin{align*}
v_1 \cdot v_2 &= v_2 \cdot v_1, \quad v^*_1 \cdot v^*_2 &= v^*_2 \cdot v^*_1 \\
v \cdot v^* - v^* \cdot v &= \langle v, v^* \rangle.
\end{align*}
\]
This definition can be generalized to the case when $V$ is no longer finite-dimensional, but rather is a topological vector space, e.g. $V := \mathbb{A}^n \otimes \mathcal{K}_x$. Note that in the latter case, the topological dual of $V$ can be identified with the space of 1-forms on $D_x^*$ with values in $\mathbb{A}^n$, i.e. with $\mathbb{A}^n \otimes \Omega_{\mathcal{K}_x}$.

Now, we claim that the category of continuous left modules over $W(\mathbb{A}^n \otimes \mathcal{K}_x)$ is naturally equivalent to the category of modules supported at $x$ over an explicit chiral algebra, called the Weyl algebra.

Consider the Heisenberg Lie-* algebra

$$\mathfrak{H}^n = \Omega_X \oplus D_X^{\otimes_n} \oplus (\Omega_X \otimes D_X)^{\otimes n},$$

where the only non-trivial component of the Lie-* bracket,

$$D_X^{\otimes n} \otimes (\Omega_X \otimes D_X)^{\otimes n} \to \Delta_t(\Omega_X) \text{ and } (\Omega_X \otimes D_X)^{\otimes n} \otimes D_X^{\otimes n} \to \Delta_t(\Omega_X),$$

are given by means of the canonical element in

$$\operatorname{Hom}_{D_X}(D_X \otimes (\Omega_X \otimes D_X), \Delta_t(\Omega_X)) \simeq \operatorname{Hom}_{\Omega_X}(\mathcal{O}_X \otimes \Omega_X, \Delta_t(\Omega_X)).$$

Note that $DR(D_X^*, \mathfrak{H}^n)$ is the usual Heisenberg Lie algebra

$$(\mathbb{A}^n \otimes \mathcal{K}_x) \oplus (\mathbb{A}^n \otimes \Omega_{\mathcal{K}_x}) \oplus \mathbb{C},$$

where $DR(D_X^*, \Omega_X)$ is identified with $\mathbb{C}$ via the residue map.

Consider the universal enveloping chiral algebra $\mathcal{U}(\mathfrak{H}^n)$. We have two different embeddings of the D-module $\Omega_X$ into $\mathcal{U}(\mathfrak{H}^n)$. One is the unit in $\mathcal{U}(\mathfrak{H}^n)$ and another comes from $\Omega_X \subset \mathfrak{H}^n$. Let $I$ be the ideal in $\mathcal{U}(\mathfrak{H}^n)$ generated by the anti-diagonal copy of $\Omega_X$.

Set $\mathcal{W}^n = \mathcal{U}(\mathfrak{H}^n)/I$. We have the following:

**Proposition 2.2.** The category of chiral $\mathcal{W}^n$-modules supported at $x$, is equivalent to the category of modules over the Weyl algebra $W(\mathbb{A}^n \otimes \mathcal{K}_x)$.

**Proof.** Using Lemma 1.5, we obtain that the category of $\mathcal{W}^n$-modules supported at $x$, is equivalent to the category of continuous modules over $DR(D_X^*, \mathfrak{H}^n)$, on which $1 \in \mathbb{C}$ acts as the identity. However, these are the same as modules over $W(\mathbb{A}^n \otimes \mathcal{K}_x)$. $\square$

To conclude the discussion of the “flat” case (i.e. $Z = \mathbb{A}^n$), let us make the following observation: $\Omega_X \otimes D_X^{\otimes n}$ is contained in $\mathfrak{H}^n$ as a Lie-* subalgebra; hence, we obtain a map of chiral algebras $\mathcal{U}(\Omega_X \otimes D_X^{\otimes n}) \to \mathcal{W}^n$. However, since $\Omega_X \otimes D_X^{\otimes n}$ is commutative, $\mathcal{U}(\Omega_X \otimes D_X^{\otimes n}) \simeq \text{Sym}(D_X^{\otimes n}) \otimes \Omega_X$, the latter being isomorphic to the jet construction of $\mathcal{C} = \mathcal{O}_{\mathbb{A}^n} \otimes \Omega_X$. In addition, $\mathfrak{H}^n$ contains as a Lie-* subalgebra $D_X^{\otimes n}$, and $\mathcal{W}^n$ is generated by $D_X^{\otimes n}$ and $J(\text{Sym}(D_X^{\otimes n}))$. This will be the form of the answer in general.
2.3. The Lie-* algebra of vector fields. Let $\mathcal{C}$ be a locally finitely generated quasi-coherent sheaf of algebras over $X$, such that $\text{Spec}(\mathcal{C})$ is smooth over $X$. (In practice, we will take $\mathcal{C} = \mathcal{O}_X \otimes \mathcal{O}_X$.) Consider $\mathcal{B} = J(\mathcal{C})^l$. In principle, our discussion applies to a more general D-algebra, but for simplicity, we will consider the above case only.

Let $T_\mathcal{C}$ be the $\mathcal{C}$-module of vertical vector fields on $\text{Spec}(\mathcal{C})$. Consider the tensor product

$$\Theta_\mathcal{C} := T_\mathcal{C} \otimes (J(\mathcal{C})^l \otimes D_X).$$

This is a right D-module on $X$ and it carries an action of the D-algebra $J(\mathcal{C})^l$. In particular, we obtain a map

$$j_*j^*(J(\mathcal{C}) \boxtimes \Theta_\mathcal{C}) \to \Delta_l(J(\mathcal{C}) \otimes \Theta_\mathcal{C}) \to \Delta_l(\Theta_\mathcal{C}),$$

which makes it a module over $J(\mathcal{C})$, viewed as a chiral algebra.

In addition, we claim that $\Theta_\mathcal{C}$ has a natural structure of a Lie-* algebra over which $J(\mathcal{C})$ is a module:

**Lemma 2.4.** There exists a unique Lie-* bracket $\{ \cdot, \cdot \} : \Theta_\mathcal{C} \boxtimes \Theta_\mathcal{C} \to \Delta_l(\Theta_\mathcal{C})$ and an action map $\text{act} : \Theta_\mathcal{C} \boxtimes J(\mathcal{C}) \to \Delta_l(J(\mathcal{C}))$ such that

a) The induced maps

$$T_\mathcal{C} \boxtimes T_\mathcal{C} \to \Theta_\mathcal{C} \boxtimes \Theta_\mathcal{C} \to \Delta_l(\Theta_\mathcal{C})$$

and

$$T_\mathcal{C} \boxtimes (\mathcal{C} \otimes \Omega_X) \to \Theta_\mathcal{C} \boxtimes J(\mathcal{C}) \to \Delta_l(J(\mathcal{C}))$$

factor, respectively, through the natural maps

$$T_\mathcal{C} \boxtimes T_\mathcal{C} \xrightarrow{\text{Lie bracket}} T_\mathcal{C} \to \Delta_l(\Theta_\mathcal{C})$$

and

$$T_\mathcal{C} \boxtimes (\mathcal{C} \otimes \Omega_X) \xrightarrow{\text{Lie der.}} \mathcal{C} \otimes \Omega_X \to \Delta_l(J(\mathcal{C})).$$

b) They satisfy the Leibnitz rule in the natural sense.

Let us try to give an intuitive view point on $\Theta_\mathcal{C}$. Consider a point in the fiber of $\text{Spec}(J(\mathcal{C})^l)$ over $x \in X$. As was explained in Sect. 1.2, it corresponds to a section $\phi' : \mathcal{D}_x \to \text{Spec}(\mathcal{C})$, which is the same as a flat section $\phi : \mathcal{D}_x \to \text{Spec}(J(\mathcal{C})^l)$. We can view $\Theta_\mathcal{C}$ as a coherent sheaf with a right $D_X$-action on $\text{Spec}(J(\mathcal{C})^l)$ and let us consider its pull-back with respect to $\phi$ as a right D-module on $\mathcal{D}_x$.

**Lemma 2.5.** There is a natural isomorphism $\text{DR}(\mathcal{D}_x, \phi^*(\Theta_\mathcal{C})) \simeq \Gamma(\mathcal{D}_x, \phi^*(T_\mathcal{C}))$.

Note that the RHS of the isomorphism of the lemma is canonically isomorphic to the vertical tangent space to $\text{Spec}(J(\mathcal{C})^l)$ at our chosen point in the spectrum.

2.6. The Beilinson-Drinfeld definition of chiral differential operators. For a smooth algebraic variety $Z$, let $D_Z$ denote the algebra of differential operators. By definition, $D_Z$ carries a canonical filtration $D_Z = \bigcup_{i \geq 0} D^i_Z$ such that

1. For $a \in D^i_Z$, $b \in D^j_Z$, we have:
   $$a \cdot b \in D^{i+j}_Z, \quad [a, b] \in D^{i+j-1}_Z.$$
2. $D^0_Z = \mathcal{O}_Z$ as algebras
3. We have an isomorphism $D^1_Z / D^0_Z \simeq T_Z$, which respects a) the Lie algebra structure, b) the structure of an $\mathcal{O}_Z$-module on $T_Z$, c) The adjoint action of $T_Z$ on $\mathcal{O}_Z$. 
4. We have a splitting $T_Z \to D^1_Z$, which respects the Lie-algebra and the $O_Z$-module structure, where on $D^1_Z$, $O_Z$ acts by left multiplication.

5. The natural map $\text{Sym}_{O_Z}(T_Z) \to \text{gr}(D_Z) := \bigoplus D^i_Z / D^{i-1}_Z$ is an isomorphism.

In [1] Beilinson and Bernstein introduced the notion of twisted differential operators (TDO). By definition, a TDO on $Z$ is a quasi-coherent sheaf $D'_Z$ of associative algebras over $Z$ endowed with an increasing filtration $D'_Z = \bigcup_{i \geq 0} (D'_Z)^i$ such that properties (1), (2), (3) and (5) hold. In particular, the untwisted differential operators are distinguished by the embedding of point (4) above.

The idea of the approach of Beilinson and Drinfeld is construct a suitable chiral algebra $D_Z$, which we will call the chiral algebra of differential operators (CADO), and to define $D$-modules on $Z(\hat{k})$ as chiral $D_Z$-modules supported at $x \in X$.

When $Z \simeq \mathbb{A}^n$, the Weyl chiral algebra $W^n$ will be an example of a CADO. In the Appendix, Sect. 6.10 we will show that when $Z$ is an algebraic group, the category of chiral $D_Z$-modules is indeed a reasonable approximation to the ill-defined category of $D$-modules on $G(\hat{k})$.

However, it turns out that there is no CADO canonically associated with a variety $Z$. In particular, it does not always exist, and when it does, it is not unique. Moreover, the category of possible CADOs on $Z$ forms a gerbe over an explicit Picard category.

The need for $D_Z$ was independently realized by the authors of [6], where this object was studied in the language of vertex algebras, under the name “chiral structure sheaf”. In particular, the authors of [6] calculated the obstruction for the existence of a CADO on a given $Z$.

Here is the definition. Let $\mathcal{C}$ be as in the previous subsection. A chiral algebra $D_{\mathcal{C}}$ is called a CADO on $\text{Spec}(\mathcal{C})$ if as a $D$-module on $X$ it is endowed with an increasing filtration $D_{\mathcal{C}} = \bigcup_{i \geq 0} D^i_{\mathcal{C}}$ with the following properties:

1. $\{ \cdot, \cdot \} : j_* j^*(D^i_{\mathcal{C}} \boxtimes D^j_{\mathcal{C}}) \to \Delta_!(D^{i+j}_{\mathcal{C}})$
   $\{ \cdot, \cdot \} : D^i_{\mathcal{C}} \boxtimes D^j_{\mathcal{C}} \to \Delta_!(D^{i+j-1}_{\mathcal{C}})$.

2. $D^0_{\mathcal{C}} \simeq J(\mathcal{C})$ as chiral algebras.

3. We have an homomorphism of $\Theta_{\mathcal{C}} \to D^1_{\mathcal{C}} / D^0_{\mathcal{C}}$, which respects a) the Lie-* algebra structure on both sides, b) the structure of a chiral $J(\mathcal{C})$-module on both sides, c) The structure on $J(\mathcal{C})$ of a Lie-* module over $\Theta_{\mathcal{C}}$ and $D^1_{\mathcal{C}} / D^0_{\mathcal{C}}$.

4. The natural map of D-algebras $\text{Sym}_{J(\mathcal{C})}(\Theta^!_{\mathcal{C}}) \to \text{gr}(D_{\mathcal{C}})^! := \bigoplus (D^!_{\mathcal{C}})^l / (D^{l-1}_{\mathcal{C}})^l$ is an isomorphism. (The superscript "$l$" denotes, as usual, the corresponding left D-module on $X$.)

The reason for the non-existence of a canonical CADO is that one cannot require the existence of an embedding $\Theta_{\mathcal{C}} \to D^1_{\mathcal{C}}$ which is compatible with the structure of a chiral $J(\mathcal{C})$-module. The existence of such an embedding would contradict other axioms.

To conclude this section, let us go back to the example of $Z = \mathbb{A}^n$ and $\mathcal{C} = O_Z \otimes O_X$. The construction of the canonical CADO $W^n$ in this case is explained by the fact that
we demanded an extra rigidity. We have:

$$\Theta_C \simeq J(\mathcal{C})^l \otimes D^\otimes_X$$

and it contains $D^\otimes_X$ as a Lie-* subalgebra. Our extra rigidity is the embedding of $D^\otimes_X$ into $\mathcal{D}^l_C := \mathcal{W}^n$ as a Lie-* algebra.

In our case of interest, when $\mathbb{A}^n$ is replaced by an arbitrary algebraic group, we will use a similar idea. We should warn the reader that the filtration on $\mathcal{W}^n$ which comes from the universal enveloping algebra construction is different from the canonical CADO filtration.

### 3. The case of an algebraic group

#### 3.1. Affine Lie-* algebras

Let $G$ be a linear algebraic group and let $\mathfrak{g}$ denote its Lie algebra. In Sect. 1.3 we showed that the $D$-module $L_\mathfrak{g} := \mathfrak{g} \otimes D_X$ has a natural structure of a Lie-* algebra.

Let now $Q : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be a $G$-invariant symmetric form. Consider the $D$-module $\tilde{L}_{\mathfrak{g},Q} = \Omega_X \oplus L_\mathfrak{g}$. We define on it a Lie-* bracket with the following components:

- $L_\mathfrak{g} \otimes L_\mathfrak{g} \to \Delta(!)(L_\mathfrak{g})$ is as before
- $\Omega_X \otimes L_{\mathfrak{g},Q} \to \Delta(!)(L_{\mathfrak{g},Q})$ vanishes, and
- $L_\mathfrak{g} \otimes L_\mathfrak{g} \to \Delta(!)(\Omega_X)$ is defined as follows:

To specify such a map is the same as to give a map $\mathfrak{g} \otimes \mathfrak{g} \to \Gamma(X \times X, \Delta(!)(\Omega_X))$. Let $1' \in \Gamma(X \times X, \Delta(!)(\Omega_X))$ be the canonical section, killed by the square of the equation of the diagonal. (In coordinates, $1'$ is the image of $\frac{dx \otimes dy}{(x-y)^2}$ under $j_\ast j^* (\Omega_X \otimes \Omega_X) \to \Delta(!)(\Omega_X)$).

We set

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\mathfrak{g} \otimes \mathfrak{g}} \mathbb{C} \xrightarrow{1'} \Gamma(X \times X, \Delta(!)(\Omega_X)).$$

Consider the universal enveloping chiral algebra $\mathcal{U}(\tilde{L}_{\mathfrak{g},Q})$. We define the affine chiral algebra $\mathcal{A}_{\mathfrak{g},Q}$ as the quotient of $\mathcal{U}(\tilde{L}_{\mathfrak{g},Q})$ by the ideal generated by the antidiagonal copy of $\Omega_X \to \mathcal{U}(\tilde{L}_{\mathfrak{g},Q})$, as in the definition of $\mathcal{W}^n$, cf. Sect. 2.1.

Note that $\tilde{\mathfrak{g}}_Q := DR(\mathcal{D}_X, \tilde{L}_{\mathfrak{g},Q}) \simeq (\mathfrak{g} \otimes \check{\mathfrak{g}}_x) \oplus \mathbb{C}$ is the standard affine Lie algebra $\mathfrak{g} \otimes \check{\mathfrak{g}}_x$ corresponding to $Q$. In what follows, by a representation of $\tilde{\mathfrak{g}}_Q$ we will mean a continuous representation, on which $1 \in \mathbb{C} \subset \tilde{\mathfrak{g}}_Q$ acts as identity. From Lemma 1.5 we obtain

**Lemma 3.2.** The category of chiral $\mathcal{A}_{\mathfrak{g},Q}$-modules concentrated at $x$ is naturally equivalent to the category of continuous representations of $\tilde{\mathfrak{g}}_Q$.

#### 3.3. Construction of the CADO

From now on we will specialize to the case $Z = G$. We take $\mathcal{C} = \mathcal{O}_G \otimes \mathcal{O}_X$; however, we shall slightly abuse the notation and will write $J(G)$ (resp., $\Theta_G, \mathcal{D}_G$) instead of $J(\mathcal{C})$ (resp., $\Theta_C, \mathcal{D}_C$).

First, let us observe that we have the map $\mathfrak{g} \to T_G$ which corresponds to left invariant vector fields on $G$. It is easy to see that it extends to a map of Lie-* algebras $L_\mathfrak{g} \to \Theta_G$.

Moreover, the above it induces an isomorphism

$$J(G)^l \otimes L_\mathfrak{g} \to \Theta_G.$$
Theorem 3.4. To a fixed form \( Q \) as above, there corresponds a canonical CADO, \( \mathcal{D}_{G,Q} \), with the following extra structure: there exists an embedding of Lie-* algebras

\[
\mathfrak{l} : \tilde{L}_{g,Q} \to \mathcal{D}_{G,Q}^{1},
\]

such that \( \Omega_X \subset \tilde{L}_{g,Q} \) maps identically to \( \Omega_X \subset J(G) = \mathcal{D}_{G,Q}^{0} \) and the composition

\[
\tilde{L}_{g,Q} \to \mathcal{D}_{G,Q}^{1}/\mathcal{D}_{G,Q}^{0} \simeq \Theta_G
\]
equals the above canonical map \( \tilde{L}_{g,Q} \to L_g \to \Theta_G \).

Let us first explain the idea of the proof. The D-module \( L_g \subset \mathcal{D}_{G,Q} \) should be thought of as corresponding to “left-invariant vector fields on \( G(\mathbb{K}_x) \)”. Hence our task is to reconstruct the “ring of differential operators” by knowing functions and left-invariant vector fields.

In the finite-dimensional situation we would proceed as follows: the direct sum \( \mathcal{O}_G \oplus \mathfrak{g} \) has a natural structure of a Lie algebra. Consider the universal enveloping algebra \( U(\mathcal{O}_G \oplus \mathfrak{g}) \). As a subalgebra it contains \( U(\mathcal{O}_G) \simeq \text{Sym}(\mathcal{O}_G) \). However, since \( \mathcal{O}_G \) is already an algebra, we have a natural map \( \text{Sym}(\mathcal{O}_G) \to \mathcal{O}_G \). Let \( I \) denote the ideal generated by the kernel of this map inside \( U(\mathcal{O}_G \oplus \mathfrak{g}) \).

We have: \( D_G = U(\mathcal{O}_G \oplus \mathfrak{g})/I \). In the chiral setting the idea of the proof is exactly the same.

**Proof.** Consider the D-module \( J(G) \oplus L_g \). We claim that it has a natural Lie-* algebra structure, whose components are as follows:

\[
\begin{align*}
L_g \otimes L_g &\to \Delta_!(L_g) \text{ is the old bracket on } L_g \\
J(G) \otimes J(G) &\to \Delta_!(J(G)) \text{ vanishes} \\
L_g \otimes L_g &\to \Delta_!(J(G)) \text{ equals } L_g \otimes L_g \overset{\mathfrak{g}}{\to} \Delta_!(\Omega_X) \to \Delta_!(J(G)) \\
L_g \otimes J(G) &\to \Delta_!(J(G)) \text{ is the action map introduced in Sect. 2.3.}
\end{align*}
\]

Consider the universal enveloping chiral algebra \( \mathfrak{u}(J(G) \oplus L_g) \). It contains as a chiral subalgebra \( \mathfrak{u}(J(G)) \), where \( J(G) \) is regarded as a Lie-* algebra with a trivial bracket. However, since \( J(G) \) is already a chiral algebra, we have a homomorphism

\[
\mathfrak{u}(J(G)) \to J(G).
\]

Let \( I \) denote the ideal inside \( \mathfrak{u}(J(G) \oplus L_g) \) generated by the kernel of this map. We set

\[
\mathcal{D}_{G,Q} := \mathfrak{u}(J(G) \oplus L_g)/I
\]

and we claim that it satisfies all the requirements.

We define the filtration on \( \mathcal{D}_{G,Q} \) by declaring that \( \mathcal{D}_{G,Q}^{1} \) is the image under the chiral bracket of \( j_* \mathfrak{l}^*(J(G) \otimes \mathfrak{u}(J(G) \oplus L_g)) \), where \( \mathfrak{u}(J(G) \oplus L_g) \) is the corresponding term of the filtration of the universal enveloping chiral algebra, cf. Sect. 1.4.

From this definition it is easy to see that the embedding \( J(G) \to \mathcal{D}_{G,Q}^{1} \) induces an isomorphism \( J(G) \simeq \mathcal{D}_{G,Q}^{0} \). Property 1 of the definition of CADO holds in view of Proposition 1.7(c), since \( J(G) \) is commutative.
The chiral bracket induces a map
\[ j_\ast j^\ast(L_\mathfrak{g} \boxtimes J(G)) \to \Delta_!(\mathcal{D}^1_{G,Q}). \]
When we compose it with the projection \( \mathcal{D}_{G,Q} \to \mathcal{D}_{G,Q}^1/\mathcal{D}_{G,Q}^0 \), we obtain a map, which factors as
\[ j_\ast j^\ast(L_\mathfrak{g} \boxtimes J(G)) \to \Delta_!(L_\mathfrak{g} \otimes J(G)) \to \Delta_!(\mathcal{D}_{G,Q}^1/\mathcal{D}_{G,Q}^0). \]
However, \( L_\mathfrak{g} \otimes J(G) \simeq \Theta_G \), hence we obtain a map \( \Theta_G \to \mathcal{D}_{G,Q}^1/\mathcal{D}_{G,Q}^0 \), which is easily seen to satisfy the three conditions of point 3.

The map
\[ \text{Sym}_{J(G)}(\Theta_G) \to \text{gr}(\mathcal{D}c)^l \]
is a surjection, as follows from Theorem 1.8. Since the LHS is \( X \)-flat, to prove that this map is an isomorphism, it is enough to do so on the level of fibers.

Let \( \mathcal{O}_{G(\hat{O}_x)} \) denote the ring of functions on the group-scheme \( G(\hat{O}_x) \). It follows from Lemma 1.6 that
\[ (\mathcal{D}_{G,Q})_x = \text{Ind}_{\mathfrak{g} \otimes \hat{O}_x}^{\mathfrak{g} \otimes \hat{O}_x} (\mathcal{O}_{G(\hat{O}_x)}), \]
where \( \mathcal{O}_{G(\hat{O}_x)} \) is a \( \mathfrak{g} \otimes \hat{O}_x \)-module via the action by left-invariant vector fields. (In the above formula the induction is \( U'(\mathfrak{g}_Q) \otimes \mathcal{O}_{G(\hat{O}_x)} \), where \( U'(\mathfrak{g}_Q) \) is the quotient of the universal enveloping algebra by the standard relation that \( 1 \in \mathbb{C} \subset \mathfrak{g}_Q \) equals the identity.)

The filtration induced on \( (\mathcal{D}_{G,Q})_x \) coincides with the standard filtration on the induced module. Therefore, as in Theorem 1.8,
\[ \text{gr}(\mathcal{D}_{G,Q})_x \simeq \text{gr}(\text{Ind}_{\mathfrak{g} \otimes \hat{O}_x}^{\mathfrak{g} \otimes \hat{O}_x} (\mathcal{O}_{G(\hat{O}_x)})) \simeq \mathcal{O}_{G(\hat{O}_x)} \otimes \text{Sym}(\mathfrak{g} \otimes \hat{K}_x/\mathfrak{g} \otimes \hat{O}_x). \]
Therefore, the map
\[ \text{Sym}_{J(G)}(\Theta_G)_x \simeq \text{Sym}_{\mathcal{O}_{G(\hat{O}_x)}}((\mathfrak{g} \otimes \hat{K}_x/\mathfrak{g} \otimes \hat{O}_x) \otimes \mathcal{O}_{G(\hat{O}_x)}) \to \text{gr}(\mathcal{D}_{G,Q})_x \]
is an isomorphism.

In the course of the proof we have shown the following

**Corollary 3.5.** We have an isomorphism of \( \mathfrak{g}_Q \)-modules:
\[ \mathcal{V}_{G,Q} := (\mathcal{D}_{G,Q})_x \simeq \text{Ind}_{\mathfrak{g} \otimes \hat{O}_x}^{\mathfrak{g} \otimes \hat{O}_x} (\mathcal{O}_{G(\hat{O}_x)}). \]

3.6. **The main result.** As we have explained before, the \( D \)-module \( L_\mathfrak{g} \) embedded into \( \mathcal{D}_{G,Q} \) corresponds to left-invariant vector fields on the loop group. It is, therefore, natural to ask whether there exists another embedding \( \tau : L_\mathfrak{g} \to \mathcal{D}_{G,Q} \), which \(*\)-commutes with \( l(L_\mathfrak{g}) \) and corresponds to right-invariant vector fields. In particular, one would expect the module \( \text{Ind}_{\mathfrak{g} \otimes \hat{O}_x}^{\mathfrak{g} \otimes \hat{O}_x} (\mathcal{O}_{G(\hat{O}_x)}) \), which corresponds to the “\( \delta \)-function on \( G(\hat{O}_x) \) inside \( G(\hat{K}_x) \)” to be in fact a bi-module over the affine algebra.

The answer to this question is positive. However, this second embedding of \( L_\mathfrak{g} \) develops an anomaly. The precise statement is given below.
First, let us introduce the following canonical extension $\mathcal{F}$ of $\mathcal{O}_X$-modules on $X$:

$$0 \to \Omega_X \to \mathcal{F} \to \mathcal{O}_X \to 0.$$ 

We take $\mathcal{F} := p_2^*(\mathcal{O}_X \boxtimes \Omega_X(\Delta)/\mathcal{O}_X \boxtimes \Omega_X(-\Delta))$, where $p_2 : X \times X \to X$ is the second projection. Note that this extension is globally non-trivial, i.e. when $X$ is compact its class is the canonical element in $H^1(X, \Omega_X)$.

Let $\mathfrak{g}$ be a Lie algebra and let $\rho : \mathfrak{g} \to \mathbb{C}$ be its modular character (i.e. the character by which $\mathfrak{g}$ acts on $\Lambda^{\operatorname{top}}(\mathfrak{g}^*)$). We define the D-module $(\mathfrak{g} \otimes D_X)'$ as the quotient of $\mathfrak{g} \otimes (\mathcal{F} \otimes D_X)$ by the kernel of the map $\mathfrak{g} \otimes (\Omega_X \otimes D_X) \xrightarrow{\rho} \Omega_X \otimes D_X \xrightarrow{\cdot} D_X$.

By construction, $(\mathfrak{g} \otimes D_X)'$ is a D-module extension

$$0 \to \Omega_X \to (\mathfrak{g} \otimes D_X)' \to \mathfrak{g} \otimes D_X \to 0$$

and we have a splitting $[\mathfrak{g}, \mathfrak{g}] \otimes D_X \to (\mathfrak{g} \otimes D_X)'$, since $\rho$ is trivial on $[\mathfrak{g}, \mathfrak{g}]$.

Given an invariant form $Q$ on $\mathfrak{g}$ we obtain a Lie-* algebra structure on $(\mathfrak{g} \otimes D_X)'$, which we will denote by $\tilde{\mathfrak{L}}_{\mathfrak{g},Q}'$, defined in the same way as in the case of $\tilde{\mathfrak{L}}_{\mathfrak{g},Q}$.

We define the involution $Q' : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ by $Q' = -Q_0 - Q$.

**Theorem 3.7.** Let $\mathcal{D}_{G,Q}$ be as in Theorem 3.4. There exists a canonical Lie-* algebra embedding $\mathfrak{r} : \tilde{\mathfrak{L}}_{\mathfrak{g},Q}' \to \mathcal{D}_{G,Q}$ with the following properties:

a) $\Omega_X \subset \tilde{\mathfrak{L}}_{\mathfrak{g},Q}'$ maps identically onto $\Omega_X \subset \mathcal{D}_{G,Q}$.

b) The composition

$$\tilde{\mathfrak{L}}_{\mathfrak{g},Q} \boxtimes \tilde{\mathfrak{L}}_{\mathfrak{g},Q}' \xrightarrow{\boxtimes \mathfrak{r}} \mathcal{D}_{G,Q} \boxtimes \mathcal{D}_{G,Q} \xrightarrow{\cdot \cdot} \Delta!(\mathcal{D}_{G,Q})$$

vanishes.

c) The image of $\mathfrak{r}$ lies in $\mathcal{D}_{G,Q}^1$ and the resulting map

$$L_{\mathfrak{g}} \to \mathcal{D}_{G,Q}^1/\mathcal{D}_{G,Q}^0 \simeq \Theta_G$$

corresponds to the embedding of $\mathfrak{g}$ into $T_G$ by means of right-invariant vector fields.

**4. PROOF OF THE MAIN THEOREM**

4.1. **The formula.** Recall (cf. Sect. 0.6) that for a right D-module $\mathcal{M}$ on $X$ we have the De Rham projection $(h \boxtimes \operatorname{id}) : \Delta!(\mathcal{M}) \to \mathcal{M}$. Let $d_X$ denote the usual De Rham differential $\mathcal{M}^l \to \mathcal{M}$.

Consider the composition

$$\mathcal{O}_G \to J(G)^l \xrightarrow{d_X} J(G).$$

By construction, $J(G)$ is an $\mathcal{O}_G$-module and the above map is a derivation. Hence, we obtain a canonical $\mathcal{O}_G$-linear map $\eta : \Omega_G^l \to J(G)$.

For an element $v \in \mathfrak{g}$ let us denote by $v^l$ (resp., $v^r$) the corresponding left-invariant (resp., right-invariant) vector field on $G$. (Note that the value of $v^l$ (resp., $v^r$) at $1 \in G$
is $-v$ (resp., $v$). Given a quadratic form $Q$ we will denote by $\eta_Q : g \to J(G)$ the composition

$$g \xrightarrow{v \to v'} T_G \xrightarrow{Q} \Omega^1_G \xrightarrow{\eta} J(G).$$

The sought-for embedding $\tau : \tilde{L}_{g,Q'} \to \mathfrak{D}_{G,Q}$ is the sum of two D-module maps, $\tau_1$ and $\tau_2$. To define $\tau_1$, consider the $\mathcal{O}_X \times X$-module map $g \otimes (\mathcal{O}_X \boxtimes \Omega_X(\Delta)) \to \Delta_!(\mathfrak{D}_{G,Q})$ defined as follows:

Let $v \in g$ be an element and let $f(x,y) \cdot \omega(y) \in \mathcal{O}_X \boxtimes \Omega_X(\Delta)$ be a local section. Let us express the right-invariant vector field $v^r$ as

$$v^r = \Sigma_i g_i \cdot v_i^l, \quad g_i \in \mathcal{O}_G.$$

We set the image of $v \otimes f(x,y) \cdot \omega(y)$ in $\Delta_!(\mathfrak{D}_{G,Q})$ to be

$$\Sigma \{ \cdot, \cdot \}(f(x,y) \cdot l(v_i) \boxtimes \omega(y) \cdot g_i),$$

where $\omega \cdot g_i$ is understood as a section of $J(G) \subset \mathfrak{D}^1_{G,Q}$. It is easy to see that this formula yields 0 if $f(x,y) \in \mathcal{O}_X \boxtimes \mathcal{O}_X(\Delta)$. Therefore, by composing with $(h \boxtimes \text{id})$ we obtain an $\mathcal{O}_X$-module map $g \otimes \mathcal{F} \to \mathfrak{D}_{G,Q}$ and hence a D-module map

$$g \otimes (\mathcal{F} \otimes D_X) \to \mathfrak{D}_{G,Q}.$$

To show that it induces a well-defined map of D-modules $\tau_1 : \tilde{L}_{g,Q'} \to \mathfrak{D}_{G,Q}$, we must prove that the composition $g \otimes \Omega_X \to g \otimes \mathcal{F} \to \Delta_!(\mathfrak{D}_{G,Q})$ factors as

$$g \otimes \Omega_X \xrightarrow{\rho} \Omega_X \xrightarrow{\tau_1} J(G) \xrightarrow{\eta} \mathfrak{D}_{G,Q} \to \Delta_!(\mathfrak{D}_{G,Q}).$$

By property 3(c) of CADO, the above composition sends a section $g \otimes \omega$ to

$$\Sigma \{ \cdot, \cdot \}(l(v_i) \boxtimes g_i \cdot \omega) = (\Sigma_i \text{Lie}_{v_i}(g_i)) \cdot \omega.$$

We have the following simple lemma:

**Lemma 4.2.** For $v \in g$ and $v^r = \Sigma_i g_i \otimes v_i^l$ as above, the function $\Sigma_i \text{Lie}_{v_i}(g_i) \in \mathcal{O}_G$ is the constant function equal to $\rho(v)$.

Thus, we have defined the map $\tau_1$. The map $\tau_2$ is a “correction term”. It factors as

$$\tilde{L}_{g,Q'} \to g \otimes D_X \to J(G) \xrightarrow{\eta} \mathfrak{D}_{G,Q},$$

where the middle arrow corresponds to $\eta_{Q_0} : g \to J(G)$.

To prove the theorem, we must verify properties (a), (b) and (c), and most importantly, that $\tau_1 + \tau_2$ commutes with the Lie-* bracket. Note, however, that properties (a) and (c) are immediate from the construction.
4.3. **Proof of property** (b). For $u,v \in \mathfrak{g}$, let $g_{u,v}^Q : \mathcal{O}_G$ be the function defined by $g \mapsto Q(\text{Ad}_g(u),v)$. Our starting point is the following observation

**Lemma 4.4.** Let $v^r = \sum_i g_i \cdot v_i^r$, $g_i \in \mathcal{O}_G$ as above. Then

$$\text{Lie}_{[u,v]}(g_i) = g_{u,v}^Q g_i.$$  

Fix $f(\cdot,\cdot)$ and $\omega \in \Omega_X$ so that $f(x,y) \cdot \omega(y) \in \mathcal{O}_X \boxtimes \Omega_X(\Delta)$ projects to $1 \in \mathcal{O}_X \boxtimes \Omega_X(\Delta) \boxtimes \mathcal{O}_X \boxtimes \Omega_X$. We will use this choice to have a well-defined $\mathcal{O}_G \to J(G)^!$.

To prove that $\mathcal{I}$ and $\mathfrak{r}$ *-commute, it is enough to show that:

1. $\{\cdot,\cdot\}(l(u) \boxtimes \mathfrak{r}_1(v)) = d_1(g_{u,v}^Q l(u))$
2. $\{\cdot,\cdot\}(l(u) \boxtimes \mathfrak{r}_2(v)) = d_1(g_{u,v}^Q l(u))$

where we have used the natural map defined for any D-module $M$:

$$d_1 : M^l \to \Delta(M),$$

where $M^l$ is the corresponding left D-module and $d_1$ is the De Rham differential along the first variable. In our case $M = J(G)$ we compose it with the embedding $\mathcal{O}_G \to J(G)^!$.

Equation (2) follows from the next result:

**Lemma 4.5.** For $u \in T_G$ and $\omega \in \Omega^1_G$, the Lie-* action $\Theta_G \boxtimes J(G) \to \Delta(l(J(G)))$ yields

$$\{\cdot,\cdot\}(u \boxtimes \eta(\omega)) = \eta(\text{Lie}_u(\omega)) - d_1(\langle u,\omega \rangle),$$

where the first term belongs to $J(G) \subset \Delta(l(J(G)))$ and where $\langle u,\omega \rangle \in \mathcal{O}_G$ is the contraction of $u$ and $\omega$.

This follows immediately from the definitions. Now let us prove Equation (1). Consider the expression

$$\sum_i \{\cdot,\cdot\}(f(y,z)(l(u) \boxtimes l(v_i) \boxtimes \omega(z) \cdot g_i)) \in \Delta(l(\mathcal{O}_G,Q))$$

on $X \times X \times X$. The LHS of Equation (1) is by definition the De Rham under $\text{id} \boxtimes (h \boxtimes \text{id})$ of (3).

By applying the Jacobi identity to (3), we obtain that the LHS of Equation (1) can be rewritten as a sum of three terms:

- $a) \quad (\text{id} \boxtimes h)(\sum_i \{\cdot,\cdot\}(f(y,x) \cdot \omega(x) \cdot \text{Lie}_u(g_i) \boxtimes l(v_i))) \in D_G,Q \subset \Delta(l(D_G,Q))$
- $b) \quad \sum_i \{\cdot,\cdot\}(f(x,y) \cdot l([u,v_i]) \boxtimes \omega(y) \cdot g_i) \in \Delta(l(D_G,Q))$
- $c) \quad \sum_i Q(u,v_i)(\text{id} \boxtimes (h \boxtimes \text{id}))(\text{can}(f(y,z) \cdot l_{x,y} \boxtimes \omega(z) \cdot g_i)) \in \Delta(l(D_G,Q))$

where the notation in term (c) will be explained in Lemma 4.7.

Let us analyze this expression term-by-term. Term (a), which is scheme-theoretically supported on the diagonal $X \subset X \times X$ equals

$$(\text{id} \boxtimes h)(\sum_i \{\cdot,\cdot\}(f(y,x) \cdot \omega(x) \cdot g_i \boxtimes l([u,v_i]))) \in D_G,Q,$$
because right- and left-invariant vector field commute.

We have the following general assertion:

**Proposition 4.6.** Let \( a, b \) be two sections of a chiral algebra \( A \) such that \( \{ \cdot, \cdot \}(a \boxtimes b) \in \Delta_1(A) \) is scheme-theoretically supported on the diagonal. Let \( f(x, y) \) be a local section of \( \mathcal{O}_X \boxtimes \mathcal{O}_X(\Delta) \) and let \( \xi \) be the vector field on \( X \) equal to the image of \( f(x, y) \) under the natural projection \( \mathcal{O}_X \boxtimes \mathcal{O}_X(\Delta) \to T_X \). Then

\[
\{ \cdot, \cdot \}(f(x, y)(a \boxtimes b)) + (\text{id} \circ h)(\{ \cdot, \cdot \}(f(y, x)(b \boxtimes a))) = \{ \cdot, \cdot \}(a \boxtimes b)(\xi, 0),
\]

where \( (\xi, 0) \) is the corresponding vector field on \( X \times X \).

**Proof.** It suffices to show that \( \{ \cdot, \cdot \}(f(x, y)(a \boxtimes b)) - \{ \cdot, \cdot \}(a \boxtimes b)(\xi, 0) \in \Delta_1(A) \) is supported scheme-theoretically on the diagonal and verify that its De Rham projection under \( h \circ \text{id} \) equals \( -\text{id}(h \circ \text{id})(\{ \cdot, \cdot \}(\phi(x), \phi(y))) \). The latter fact is obvious from the anti-symmetry property of the chiral bracket.

Let us multiply \( \{ \cdot, \cdot \}(f(x, y)(a \boxtimes b)) - \{ \cdot, \cdot \}(a \boxtimes b)(\xi, 0) \) by a function on \( X \times X \) of the form \( \phi(x) - \phi(y) \). We obtain

\[
\{ \cdot, \cdot \}(f(x, y) \cdot (\phi(x) - \phi(y))(a \boxtimes b)) - \text{Lie}_\xi(\phi)(x) \cdot \{ \cdot, \cdot \}(a \boxtimes b).
\]

However, \( f(x, y) \cdot (\phi(x) - \phi(y)) = \text{Lie}_\xi(\phi) \mod \mathcal{O}_X \boxtimes \mathcal{O}_X(-\Delta) \), which implies our assertion.

By applying this proposition to \( I([u, v_i]) \boxtimes \omega(y) \cdot g_i \), we obtain that the sum of terms (a) and (b) equals

\[
(\sum_i \omega \cdot \text{Lie}_{[u, v_i]}(g_i))(\xi, 0) = -d_1(\text{Lie}_{[u, v_i]}(g_i)),
\]

which, according to Lemma 4.4, equals \( -d_1(S_{u, v}^Q) \).

It remains to analyze term (c). We need to show that it equals \( d_1(S_{u, v}^Q) \). This results from the following general assertion:

Let \( M \) be a right \( D \)-module on \( X \) and let \( M' \) be the corresponding left \( D \)-module. Consider the canonical map of \( D \)-modules on \( X \times X \times X \)

\[
\text{can} : (\Delta_2 \boxtimes \text{id})! : (\text{id} \boxtimes f_2^*)((\omega \boxtimes M)) \to \Delta_3!(M).
\]

Consider the following section in the RHS:

\[
(f(y, z) \cdot 1'_{x, y} \boxtimes \omega(z) \cdot m),
\]

where \( m \in M' \), \( f(\cdot, \cdot) \), and \( \omega \) are as above, and \( 1'_{x, y} \) is the section \( 1' \) defined in Sect. 3.1 in the first two variables.

**Lemma 4.7.** The \( \text{id} \boxtimes (h \boxtimes \text{id}) \) projection of \( \text{can}(f(y, z) \cdot 1'_{x, y} \boxtimes \omega(z) \cdot m) \) equals \( -d_1(m) \).

**Proof.** Let us multiply both sides by a function on \( X \times X \) of the form \( \phi(x) - \phi(y) \). In both cases we get \( d\phi \cdot m \in M \subset \Delta_1(M) \). Therefore, the expression

(4) \[
(\text{id} \boxtimes (h \boxtimes \text{id}))((\text{can}(f(y, z) \cdot 1'_{x, y} \boxtimes \omega(z) \cdot m)) + d_1(m)) \in \Delta_1(M)
\]

is killed by the equation of the diagonal. However, since

\[
(h \boxtimes \text{id}) \circ (\text{id} \boxtimes (h \boxtimes \text{id}))((\text{can}(f(y, z) \cdot 1'_{x, y} \boxtimes \omega(z) \cdot m)) = 0 = (h \boxtimes \text{id})(-d_1(m)),
\]

the result follows.
we obtain that \( \phi \) vanishes identically.

\[
\phi(x) = \phi(y) = 0,
\]

\[
\phi(x) - \phi(y) = 0.
\]

4.8. **Proof of the compatibility with the Lie-* bracket.** Let \( v, v' \in g \) be two elements. The expression \( \{ \cdot, \cdot \}(r(v) \boxtimes r(v')) \) is a well-defined section of \( \Delta(\mathfrak{D}_{G,Q}) \) and \( \{ \cdot, \cdot \}(v \boxtimes v') \) is a well-defined section of \( \Delta(\mathfrak{L}_{g,Q}) \). Our goal is to prove that

\[
\{ \cdot, \cdot \}(r(v) \boxtimes r(v')) - \{ \cdot, \cdot \}(v \boxtimes v') = 0 \in \Delta(\mathfrak{D}_{G,Q})
\]

First, we will prove that the RHS of (5) is killed by the equation of the diagonal. We must show that for a function \( \phi \in O_X \),

\[
(\phi(x) - \phi(y)) \cdot \{ \cdot, \cdot \}(r(v) \boxtimes r(v')) = d_X(\phi) \cdot (Q_0 - Q)(v, v').
\]

It is clear that \( \{ \cdot, \cdot \}(r_2(v) \boxtimes r_2(v')) \) equals zero. We can compute \( \{ \cdot, \cdot \}(r_1(v) \boxtimes r_2(v')) \) using Lemma 4.5. We obtain:

\[
(\phi(x) - \phi(y)) \cdot \{ \cdot, \cdot \}(r_1(v) \boxtimes r_2(v')) = (\phi(x) - \phi(y)) \cdot d_1((Q - Q_0)(v, v')).
\]

Similarly, \( (\phi(x) - \phi(y)) \cdot \{ \cdot, \cdot \}(r_2(v) \boxtimes r_1(v')) = d_X(\phi) \cdot (Q_0 - Q)(v, v') \). Thus, it remains to compute \( \{ \cdot, \cdot \}(r_1(v) \boxtimes r_1(v')) \).

The latter is the \((h \boxtimes \text{id}) \boxtimes (h \boxtimes \text{id})\) projection of

\[
\sum_{i,j} \{ \cdot, \cdot \}(f(x, y) \cdot l(v_i) \boxtimes \omega(y) \cdot g_i \boxtimes f(z, w) \cdot l(v_j) \boxtimes \omega(w) \cdot g_j)
\]

where \( f(\cdot, \cdot) \in O_X \boxtimes O_X(\Delta) \) and \( \omega \in \Omega_X \) are as before.

Using Jacobi identity, we can rewrite (5) as a sum over \( i, j \) of four terms:

\[
\sum_{i,j} \{ \cdot, \cdot \}(f(x, y) \cdot l(v_i) \boxtimes \omega(y) \cdot g_i \boxtimes f(z, w) \cdot l(v_j) \boxtimes \omega(w) \cdot g_j)
\]

where \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \in S^4 \) are the transpositions of the corresponding factors.

It is easy to see that term (b) vanishes identically and that term (a) is killed by \((\phi(y) - \phi(w)) \). The contribution of term (c), multiplied by \((\phi(y) - \phi(w)) \) consists of two parts. One is \( \text{id} \boxtimes (h \boxtimes \text{id}) \) applied to

\[
-\sum_{i,j} \{ \cdot, \cdot \}(f(y, x) \cdot (\phi(x) - \phi(z)) \cdot \omega(x) \cdot g_i \boxtimes l(v_i) \boxtimes f(y, z) \cdot \omega(z) \cdot g_j)
\]

and the other is \( (\text{id} \boxtimes h) \boxtimes (h \boxtimes \text{id}) \) applied to

\[
-\sum_{i,j} Q(v_i, v_j) \cdot \{ \cdot, \cdot \}(f(y, x) \cdot (\phi(x) - \phi(w)) \cdot \omega(x) \cdot g_i \boxtimes \text{can}(f(z, w) \cdot l_{y,z} \boxtimes \omega(w) \cdot g_j)),
\]

where ”can” is as in Lemma 4.7.
The first part can be computed using Lemma 4.4 and we obtain

\[-\sum_{i,j} \{ \cdot, \{ \cdot, \cdot \}\}(f(y, x) \cdot (\phi(x) - \phi(y)) \cdot \omega(x) \cdot g_i \boxtimes \text{L}(v_i, v'_j)) \boxtimes f(y, z) \cdot \omega(z) \cdot g_j' = \]

\[-\sum_{i,j} \{ \cdot, \{ \cdot, \cdot \}\}(f(y, x) \cdot (\phi(y) - \phi(z)) \cdot \omega(x) \cdot g_i \boxtimes \text{L}(v_i, v'_j)) \boxtimes f(y, z) \cdot \omega(z) \cdot g_j' = \]

\[\sum_{i,j} d_X(\phi) \cdot g_i \cdot \text{Lie}_{v_i, v'_j}'(g_j') + \sum_{i,j} d_X(\phi) \cdot \text{Lie}_{v'_j, v_i}'(g_i) \cdot g_j' = \]

\[d_X(\phi) \cdot (\sum_i g_i \cdot g_i^{Q_0} + \sum_j g_j' \cdot g_j^{Q_0}) = -2d_X(\phi) \cdot Q_0(v, v').\]

The second part of term (c) can be calculated using Lemma 4.7 and we obtain:

\[\sum_{i,j} d_X(\phi) \cdot g_i \cdot g_j' \cdot Q(v_i, v'_j) = d_X(\phi) \cdot Q(v, v').\]

Finally, term (d) multiplied by \((\phi(y) - \phi(w))\) gives

\[-\sum_{i,j} \{ \cdot, \{ \cdot, \cdot \}\}(d_X(\phi)(x) \cdot g_i \boxtimes f(z, y) \cdot \omega(y) \cdot \text{L}(v'_i, v'_j) \boxtimes f(v_j')) = \]

\[-d_X(\phi) \cdot \sum_{i,j} \text{Lie}_{v_i, v'_j}'(g_i) \cdot \text{Lie}_{v'_j, v_i}'(g_i) = d_X(\phi) \cdot \sum_{i,j} g_i \cdot \text{Lie}_{v_i, v'_j}'(g_i) = -d_X(\phi) \cdot Q_0(v, v').\]

By summing up, we obtain the desired

\[(\phi(x) - \phi(y)) \cdot \{ \cdot, \cdot \} (\tau_1(v) \boxtimes \tau_1(v')) = d_X(\phi) \cdot (-Q - Q_0)(v, v').\]

Thus, the LHS of (3) can be regarded as a map

\[\psi : g \otimes g \rightarrow \mathcal{D}_{G,Q} .\]

By property 3 of CADO, the image of \(\psi\) is contained in \(\mathcal{D}_{G,Q}^0 = J(G)\). In addition, according to property (b) of the map \(\mathfrak{t}\) that has already been checked, the image of \(\psi^*\)-commutes with \(\text{L} : \tilde{\text{L}}_{g,Q} \rightarrow \mathcal{D}_{G,Q}\).

However, since the space of \(g \otimes \mathcal{O}_x\)-invariants in \(\mathcal{O}_{G(\hat{O}_x)}\) consists of constant functions, a section of \(J(G)\) \(\psi^*\)-commutes with the image of \(\mathfrak{t}\) if and only if it belongs to \(\Omega_X \subset J(G)\). Hence, \(\psi\) is a map \(g \otimes g \rightarrow \Omega_X\).

To prove that it vanishes we will use a "conformal dimension" argument:

Since our constructions are local, we can assume that \(X = \mathbb{C}\), and \(\psi\) is invariant with respect to the group of all automorphisms of \(\mathbb{C}\). Hence, the image of \(\psi\) belongs to the space of \(\text{Aut}(\mathbb{C})\)-invariant sections of \(\Omega_X\), but there are none.

5. Relation to semi-infinite cohomology

5.1. The BRST complex. Our goal in this section is to compute the semi-infinite cohomology of \(\mathcal{D}_{G,Q} = (\mathcal{D}_{G,Q})_x\) viewed as a module over the affine algebra. First, let us recall the basic definitions concerning semi-infinite cohomology in the context of chiral algebras, following [3], Section 3.8.

Let \(M\) be a finitely generated locally free right D-module and let

\[M^* := \text{Hom}(M, D_X \otimes \Omega_X)\]

be its Verdier dual. The direct sum \(\Omega_X \oplus (M \oplus M^*)\) has a natural structure of a (super) Lie-* algebra, where the Lie-* bracket vanishes on \(\Omega_X\) and \(M \otimes M^* \rightarrow \Delta(\Omega_X)\) is the canonical map, cf. [3], [4].
We define the Clifford (super) chiral algebra $\text{Cliff}(M)$ as the quotient of the universal enveloping chiral algebra $U(\Omega_X \oplus (M \oplus M^*))$ by the ideal generated by the anti-diagonal copy of $\Omega_X$. The $\mathbb{Z}_2$-grading on $\text{Cliff}(M)$ can be extended to a $\mathbb{Z}$-grading, by letting $M$ (resp., $M^*$, $\Omega_X$) have degree $-1$ (resp., 1, 0).

Consider the 0-th graded component of the second associated graded quotient with respect to the canonical filtration on $\text{Cliff}(M)$, $\text{gr}_2(\text{Cliff}(M))^0$. It has a natural structure of a Lie-* algebra isomorphic to $\text{End}(M) := M^! \otimes M^*$, cf. loc. cit. Hence, if we do not mod out by $\Omega_X \simeq \text{Cliff}(M)^0$, we obtain an extension of Lie-* algebras $0 \to \Omega_X \to \tilde{\text{End}}(M) \to \text{End}(M) \to 0$.

Now let us assume that $M = L$ is a Lie-* algebra, which is locally free as a $D$-module. As is explained in [2] or [4], the Lie-* bracket on $L$ yields a Lie-* algebra map $L \to \text{End}(L)$. In particular, from $\text{End}(L)$ we obtain a canonical central extension $0 \to \Omega_X \to \tilde{L}_{\text{can}} \to L \to 0$.

Let us denote by $\tilde{L}_{\text{can}}$ the Baer negative of the extension $\tilde{L}_{\text{can}}$. We will denote by $\tilde{\mathcal{U}}(L)_{\text{can}}$ (resp., $\tilde{\mathcal{U}}(L_{\text{can}})$) the quotient of $\mathcal{U}(\tilde{L}_{\text{can}})$ (resp., $\mathcal{U}(\tilde{L}_{\text{can}})$) by the antidiagonal copy of $\Omega_X$.

Let $N_x$ be a representation of the extension $0 \to \mathbb{C} \to DR(D^*_x, \tilde{L}_{\text{can}}) \to DR(D^*_x, L) \to 0$. According to Lemma 1.5, such $N_x$ is a vector space underlying a chiral module over $\tilde{\mathcal{U}}(L)_{\text{can}}$.

It is explained in loc. cit. that the tensor product $N_x \otimes \text{Cliff}(L)_x$, which is a $\mathbb{Z}$-graded vector space, acquires a differential $\delta$ (called BRST) of degree 1.

The semi-infinite cohomology of $N_x$ with respect to $L$ is defined as the cohomology of this complex:

$$H^{\mp +k}(L, N_x) := H^k(N_x \otimes \text{Cliff}(L)_x, \delta).$$

5.2. Identification of the canonical extension. From now on, we will take $L = L_\mathfrak{g} = \mathfrak{g} \otimes D_X$. According to the previous subsection, there exists a canonical central extension of Lie-* algebras $0 \to \Omega_X \to \tilde{L}_{\mathfrak{g}_{\text{can}}} \to L_\mathfrak{g} \to 0$.

Recall also that in Sect. 3.6 we defined another central extension of $L_\mathfrak{g}$ corresponding to an invariant form $Q$, namely $\tilde{L}'_{\mathfrak{g}, Q}$. In this subsection our goal will be to prove the following assertion:

**Theorem 5.3.** There is a canonical isomorphism of extensions $\tilde{L}'_{\mathfrak{g}, -Q_0} \simeq \tilde{L}_{\mathfrak{g}_{\text{can}}}$.

**Proof.** For $L = L_\mathfrak{g}$, $\text{End}(L_\mathfrak{g}) \simeq \text{End}(\mathfrak{g}) \otimes D_X$ and the Lie-* bracket on it comes from the usual Lie algebra structure on $\text{End}(\mathfrak{g})$. The map $L_\mathfrak{g} \to \text{End}(L_\mathfrak{g})$ comes from the action map $\mathfrak{g} \to \text{End}(\mathfrak{g})$.

To prove the theorem, it suffices to construct a Lie-* algebra map $t : \tilde{L}'_{\mathfrak{g}, -Q_0} \to \tilde{\text{End}}(\mathcal{L})$, where
such that the induced map from $\Omega_X \subset \tilde{L}_{g,-q_G}$ to $\Omega_X \subset \End(L) \subset \Cliff(L)$ is $-\text{id}$.

Let $v \in g$ and let $v^r = \sum_i g_i \cdot v_i^r$, $g_i \in \mathcal{O}_G$. Then the image of $v$ in $\End(g) \cong g \otimes g^*$ is the value at $1 \in G$ of the element

$$-\sum_i v_i \otimes \partial_G(g_i) \in g \otimes \Omega_G^*.$$

The required map $t$ is defined by a formula similar to the one of Sect. 4.1. Namely it sends a section $v \otimes f(x,y) \cdot \omega(y) \in g \otimes \mathcal{T} \otimes D_X$ to the $(h \otimes \text{id})$ projection of

$$\sum_i \{ \cdot, \} (f(x,y) \cdot \omega(y) \cdot v_i \otimes \partial_G(g_i)(1)) \in \Delta_1(\Cliff(L_g)).$$

The fact that $t|_{\Omega_X} = -\text{id}$ follows immediately from Lemma 1.2. To show that it commutes with the Lie-* algebra bracket, we proceed as in the proof of Theorem 3.7. Namely, by the same argument as in Sect. 4.8, it suffices to show that the difference

$$(7) \quad t(\{ \cdot, \} (v \otimes v')) - \{ \cdot, \} (t(v) \otimes t(v')) \in \Delta_1(\Cliff(L_g))$$

is killed by a function of the form $\phi(x) - \phi(y) \in \mathcal{O}_{X \times X}$.

By applying the (super-) Jacobi identity, we obtain that the second term in (7) is the projection under $(h \otimes \text{id}) \otimes (h \otimes \text{id})$ of the sum over $i$ and $j$ of four terms

$$\sigma_{3,4}\{ \cdot, \} (f(x,y) \cdot v_i \otimes \omega(y) \cdot \partial_G(g_i)(1) \otimes f(z,w) \cdot v'_j \otimes \omega(w) \cdot \partial_G(g'_j)(1))$$

$$\sigma_{1,2}\{ \cdot, \} (f(y,z) \cdot \omega(z) \cdot \partial_G(g'_j)(1) \otimes v_i \otimes f(z,w) \cdot v'_j \otimes \omega(w) \cdot \partial_G(g'_j)(1))$$

Let us analyze the above expression term-by-term. First of all, it is easy to see that the second and the third terms vanish. Secondly, the first term is killed by multiplication by $\phi(x) - \phi(y)$. The fourth term yields that:

$$\phi(x) - \phi(y) \cdot \{ \cdot, \} (t(v) \otimes t(v')) = -d_X(\phi) \cdot \sum_{i,j} (\text{Lie}_{v_i^r}^j(g'_j) \cdot \text{Lie}_{v'_j^r}(g_i))(1) =$$

$$-d_X(\phi) \cdot \sum_i g'_j \cdot \text{Lie}_{v_i^r}^j(g_i) = -d_X(\phi) \cdot \sum_i g'_j \cdot \mathcal{Q}^0_{v_i^r,v} = d_X(\phi) \cdot Q_0(v,v').$$

However, $(\phi(x) - \phi(y)) \cdot t(\{ \cdot, \} (v \otimes v')) = d_X(\phi) \cdot Q_0(v,v')$, which is what we had to prove.

5.4. Computation of $\Pi \tilde{\mathcal{F}}(\mathcal{D}_G,Q)$. In what follows, we will denote by $\tilde{g}_Q$ the affine algebra $\mathcal{D}_G(L_{x,\tilde{g}_Q})$. (Non-canonically, $\tilde{g}_Q$ and $\tilde{g}_Q'$ are isomorphic, but this isomorphism does not respect the action of the group Aut($\mathcal{D}_x$) of automorphisms of the formal disc, which acts on the whole picture.) In particular, if $M^1_x$ is a $\tilde{g}_Q^1$-representation and $M^2_x$ is a $\tilde{g}_Q^2$-representation, the tensor product $M^1_x \otimes M^2_x$ is a $\tilde{g}_Q^{1+2}$-representation via the diagonal action.

Let $M_x$ be a (continuous) module over the affine algebra $\tilde{g}_Q$ and let us consider the tensor product $M_x \otimes V_{G,Q}$. We let $\tilde{g}_Q$ act on it via the embedding $t : L_{g,Q} \to \mathcal{D}_G,Q$. However, since by Theorem 3.7 $V_{G,Q}$ is a bi-module over the affine algebra, the above
tensor product has an additional structure. Namely, by combining Theorem 3.7 and Theorem 5.3, we obtain on it an action of $\tilde{g}'_Q$.

Hence, it makes sense to consider $H^\infty_2(L_g, M_x \otimes \mathbb{V}_{G,Q})$, and will carry a continuous $\tilde{g}_Q$-action.

**Theorem 5.5.** Assume that $M_x$ is such that the action of $g \otimes \hat{O}_x \subset \tilde{g}_Q$ on it integrates to an $G(\hat{O}_x)$-action. Then there is a canonical isomorphism of $\tilde{g}_Q$-modules:

$$H^\infty_2(L_g, M_x \otimes \mathbb{V}_{G,Q}) \simeq M_x \otimes H^k(g, \mathbb{C}).$$

The proof will consist of two steps. First, we will consider the case $Q = 0$ and $M_x = \mathbb{C}$.

**Proposition 5.6.** There is a canonical isomorphism of $g \otimes \hat{K}_x$-modules

$$H^\infty_2(L_g, \mathbb{V}_{G,Q}) \simeq H^k(g, \mathbb{C}),$$

where the action of $g \otimes \hat{K}_x$ on the RHS is trivial.

**Proof.** We will use the following statement, valid for an arbitrary locally free Lie-* algebra:

Let $N_x$ be a module over $DR(\mathcal{D}_x^*, \tilde{L}_{can})$, which is induced from a $DR(\mathcal{D}_x, L)$-module, i.e.

$$N_x \simeq \text{Ind}_{DR(\mathcal{D}_x, L)}^{DR(\mathcal{D}_x^*, \tilde{L}_{can})} (\overline{N}),$$

where induction is understood in the restricted sense, i.e. $1 \in \mathbb{C} \subset DR(\mathcal{D}_x^*, \tilde{L}_{can})$ acts as identity.

**Lemma 5.7.** Under the above circumstances there is a canonical isomorphism

$$H^\infty_2(L, N_x) \simeq H^k(DR(\mathcal{D}_x, L), \overline{N}).$$

We will apply this lemma in the case $N_x \simeq \mathbb{V}_{G,0}$. It is easy to see, as in Lemma 3.3, that the embedding $J(G) \rightarrow \mathcal{D}_{G,0}$ induces an isomorphism of $\tilde{g}'_{Q_0}$-modules

$$\mathbb{V}_{G,0} \simeq \text{Ind}_{g \otimes \hat{O}_x}^{\tilde{g}'_{Q_0}} (\mathcal{O}_{G(\hat{O}_x)}).$$

Moreover, the restriction of the $g \otimes \mathcal{O}_x$-action on the LHS to $g \otimes \mathcal{O}_x$, that comes from the embedding $I : L_g \rightarrow \mathcal{D}_{G,0}$ coincides with the action that comes from the natural $g \otimes \mathcal{O}_x$-action on $\mathcal{O}_{G(\hat{O}_x)}$ by left-invariant vector fields. This follows from the fact that the embeddings $I$ and $*-$commute with one another.

Hence, we obtain an isomorphism $H^\infty_2(L, \mathbb{V}_{G,0}) \simeq H^k(g \otimes \mathcal{O}_x, \mathcal{O}_{G(\hat{O}_x)})$ and it respects the $g \otimes \mathcal{O}_x$-action on both sides. Since the kernel of the evaluation map $G(\hat{O}_x) \rightarrow G$ is pro-unipotent, we have an isomorphism of $g \otimes \hat{O}_x$-modules

$$H^k(g \otimes \mathcal{O}_x, \mathcal{O}_{G(\hat{O}_x)}) \simeq H^k(g, \mathcal{O}_x) \simeq H^k(g, \mathbb{C}),$$

where the $g \otimes \mathcal{O}_x$-action on the RHS is trivial.
To prove the proposition, it remains to show that the \( g \otimes \hat{\mathcal{K}}_x \)-action on \( H^k(\mathfrak{g}, \mathbb{C}) \) is trivial as well. Consider the group \( \text{Aut}(\mathcal{D}_x) \) of automorphisms of the formal disc. This group acts on the whole picture. In particular, our homomorphism

\[
g \otimes \hat{\mathcal{K}}_x \rightarrow \text{End}(H^k(\mathfrak{g}, \mathbb{C}))
\]

is \( \text{Aut}(\mathcal{D}_x) \)-equivariant. However, it is easy to see that any such homomorphism, which is, moreover, trivial on \( g \otimes \hat{\mathcal{O}}_x \subset g \otimes \hat{\mathcal{K}}_x \) is zero.

The second step in the proof of Theorem 5.5 is the following result:

Let \( M_x \) be a \( \tilde{\mathfrak{g}}_Q \)-module as above and let us consider the tensor product \( M_x \otimes \mathbb{V}_{G,Q'} \), where \( Q' \) is another invariant form. It has a natural bi-module structure with respect to \( \tilde{\mathfrak{g}}_Q' \) and \( \tilde{\mathfrak{g}}_Q' - Q_0 - Q' \):

Consider now the tensor product \( M_x \otimes \mathbb{V}_{G,Q' - Q} \). It has a bi-module structure with respect to the same Lie algebras, where \( \tilde{\mathfrak{g}}_Q' \) acts diagonally (via \( l \)) and \( \tilde{\mathfrak{g}}_Q' - Q_0 - Q' \) acts only on \( \mathbb{V}_{G,Q' - Q} \) (via \( r \)).

**Theorem 5.8.** Assume that \( M_x \) is such that the action of \( g \otimes \hat{\mathcal{O}}_x \subset \tilde{\mathfrak{g}}_Q \) on it integrates to an \( \hat{G}(\hat{\mathcal{O}}_x) \)-action. Then the above bi-modules \( M_x \otimes \mathbb{V}_{G,Q'} \) and \( M_x \otimes \mathbb{V}_{G,Q' - Q} \) are canonically isomorphic.

It is clear that Theorem 5.8 combined with Proposition 5.6 imply Theorem 5.5, when one takes \( Q' = Q \).

**5.9. Proof of Theorem 5.8.** Recall that if \( A_1 \) and \( A_2 \) are chiral algebras, then their tensor product \( A_1 \otimes A_2 \) is also naturally a chiral algebra.

Thus, let us consider the chiral algebras \( A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q'} \) and \( A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q' - Q} \). We will denote by \( i \) the embedding of the Lie-* algebra \( \tilde{\mathbb{L}}_{\mathfrak{g},Q} \) into each of them along the first factor. Note that in addition, one has the natural (diagonal) maps

\[
l + i : \tilde{\mathbb{L}}_{\mathfrak{g},Q} \rightarrow A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q' - Q} \quad \text{and} \quad r + i : L'_{\mathfrak{g},Q' - Q_0 - Q'} \rightarrow A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q'}.
\]

**Proposition 5.10.** There exists a canonical isomorphism

\[
\phi : A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q'} \rightarrow A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q' - Q},
\]

such that

(a) \( \phi \) commutes with the embedding of \( J(G) \) into \( \mathbb{D}_{G,Q'} \subset A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q'} \) and \( \mathbb{D}_{G,Q' - Q} \subset A_{\mathfrak{g},Q} \otimes \mathbb{D}_{G,Q' - Q} \).

(b) \( \phi \circ l = l + i \).

(c) \( \phi \circ (r + i) = r \).
Proof. By construction, $A_{g,Q} \dagger \mathfrak{D}_{G,Q'}$ is a quotient of the universal enveloping chiral algebra of $\tilde{L}_{g,Q} \oplus J(G) \oplus L_g$. Therefore, $\phi$, if it exists, is uniquely described by a Lie-* algebra map

$$\tilde{L}_{g,Q} \oplus J(G) \oplus L_g \to A_{g,Q} \dagger \mathfrak{D}_{G,Q'} - Q.$$

On the second and the third summands $\phi$ is determined by conditions (a) and (b), respectively. The restriction of $\phi$ to $\Omega_X \subset \tilde{L}_{g,Q}$ is the identity map onto $\Omega_X$ in the target and the restriction to $g \otimes D_X \subset \tilde{L}_{g,Q}$ is determined by the following condition:

The adjoint action gives rise to a map $g \to g \otimes \mathfrak{O}_G$. From it we obtain a D-module map

$$g \otimes D_X \to (g \otimes D_X) \dagger J(G).$$

(On the level of fibers, this map is the co-action $g \otimes \hat{K}_x/\hat{O}_x \to g \otimes \hat{K}_x/\hat{O}_x \otimes \mathfrak{O}_{G(\hat{O}_x)}$.)

We need that the diagram

$$
\begin{align*}
g \otimes D_X &\longrightarrow (g \otimes D_X) \dagger J(G) \\
\downarrow &\quad \downarrow \\
\tilde{L}_{g,Q} &\longrightarrow \tilde{L}_{g,Q} \dagger \mathfrak{D}_{G,Q'} - Q
\end{align*}
$$

commutes.

The fact that the resulting map of D-modules commutes with the Lie-* bracket is straightforward. Thus, one obtains a map of chiral algebras

$$\mathcal{U}(\tilde{L}_{g,Q} \oplus J(G) \oplus L_g) \to A_{g,Q} \dagger \mathfrak{D}_{G,Q'} - Q,$$

and one easily checks that it factors through the natural surjection

$$\mathcal{U}(\tilde{L}_{g,Q} \oplus J(G) \oplus L_g) \to A_{g,Q} \dagger \mathfrak{D}_{G,Q'}.$$

The fact that condition (c) is satisfied follows from the formula for the embedding $\tau$, cf. Sect. [4.1]. The fact that $\phi$ is an isomorphism is easy, since one can construct its inverse by a similar procedure.

Now, let $M_x$ be as in the formulation of the theorem. The tensor products $M_x \otimes \mathbb{V}_{G,Q'}$ and $M_x \otimes \mathbb{V}_{G,Q'-Q}$ are chiral modules supported at $x \in X$ for $A_{g,Q} \dagger \mathfrak{D}_{G,Q'}$ and $A_{g,Q} \dagger \mathfrak{D}_{G,Q'-Q}$, respectively.

To prove the theorem, it suffices to construct an isomorphism

$$\phi_{M_x} : M_x \otimes \mathbb{V}_{G,Q'} \to M_x \otimes \mathbb{V}_{G,Q'-Q},$$

which covers the isomorphism $\phi$ of the above proposition.

However, it is easy to see that there exists a well-defined map $\phi_{M_x}$ such that its restriction to

$$M_x \otimes 1_x \subset M_x \otimes (\mathfrak{D}_{G,Q})_x \simeq M_x \otimes \mathbb{V}_{G,Q'}$$
is the map
\[ M_x \to M_x \otimes \mathcal{O}_{G(\check{O}_x)} = M_x \otimes J(G)_x \subset M_x \otimes (\mathcal{D}_{G,Q} - Q)_x = M_x \otimes \mathcal{V}_{G,Q} - Q, \]
where the first arrow is the co-action map corresponding to the \( G(\check{O}_x) \)-action on \( M_x \).

6. Appendix: From \( \mathcal{D}_{G,Q} \)-modules to D-modules

The discussion in this section is substantially based on the unpublished manuscript \[2\], Sect. 7, where the formalism of D-modules on ind-schemes is developed. For that reason, we call this section an Appendix, and its results and techniques are not directly connected to the contents of the main part of the paper.

6.1. A characterization of chiral \( \mathcal{D}_{G,Q} \)-modules. In this section the point \( x \in X \) will be fixed and we will denote \( \check{O}_x \) by \( \mathbb{C}[t] \) and \( \mathcal{K}_x \) by \( \mathbb{C}((t)) \), respectively. Moreover, all chiral \( \mathcal{D}_{G,Q} \)-modules we that will consider are supported at \( x \); therefore, by a slight abuse of notation, we will identify a chiral module \( M \) with the corresponding vector space \( \check{t}^*_x(M)[1] \).

Recall that \( G[[t]] \) has a canonical structure of a group-scheme and \( G((t)) \) of a group-indscheme. In particular, \( \mathcal{O}_{G((t))} \) is a topological commutative algebra, which is a continuous bi-module over \( g((t)) \).

We have the following proposition:

**Proposition 6.2.** To specify a structure of a chiral \( \mathcal{D}_{G,Q} \)-module on a vector space \( M \) is the same as to endow it with continuous (w.r. to the discrete topology on \( M \)) actions of \( \mathcal{O}_{G((t))} \) and \( \hat{\mathcal{B}}_Q \) compatible in the sense that for \( \xi \in \hat{\mathcal{B}}_Q, f \in \mathcal{O}_{G((t))} \) and \( m \in M \),
\[
\xi \cdot (f \cdot m) = f \cdot (\xi \cdot m) + \operatorname{Lie}_\xi(f) \cdot m,
\]
where \( \xi^t \) is the corresponding left-invariant vector field on \( G((t)) \).

**Proof.** First, let \( \mathcal{B} \) be a commutative chiral algebra. Consider the projective limit \( \hat{\mathcal{B}}_x := \lim \left( \mathcal{B}^t_x \right)_x \), where \( \mathcal{B}_t \) runs over the set of all chiral subalgebras \( \mathcal{B}_t \subset \mathcal{B} \) with \( \mathcal{B}_t \mid X - x \simeq \mathcal{B} \mid X - x \).

Then \( \hat{\mathcal{B}}_x \) is a commutative algebra, which carries a natural topology. Moreover, a structure of a chiral \( \mathcal{B} \)-module on a vector space \( M \) it amounts to a continuous \( \hat{\mathcal{B}}_x \)-action on \( M \).

When \( \mathcal{B} \) is of the form \( \mathcal{B} = J(\mathcal{C}) \), for a q.c. sheaf of \( \mathcal{O}_X \)-algebras \( \mathcal{C} \), \( \hat{\mathcal{B}}_x \) represents the functor on the category of \( \mathcal{C} \)-algebras given by \( A \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{C}, A((t))) \). In particular, \( \hat{J}(G)_x \simeq \mathcal{O}_{G((t))} \).

Let us suppose now that \( \mathcal{B} \) is a Lie-* module over a Lie-* algebra \( L \). Then we obtain a natural continuous action of \( DR(\mathcal{D}^*_x, L) \) on \( \hat{\mathcal{B}}_x \). It is easy to see that in the above example of \( \mathcal{B} = J(G) \) and the action \( g \otimes D_X \otimes J(G) \to \Delta_t(J(G)) \) of Sect. \[3\], the resulting \( g((t)) \)-action on \( \mathcal{O}_{G((t))} \) coincides with the natural action by left-invariant vector fields.

That said, the assertion of the proposition becomes a direct corollary of the construction of \( \mathcal{D}_{G,Q} \) as a quotient of the universal enveloping algebra of \( J(G) \oplus L_Q \) combined with Lemma \[1\].
Let us observe now that Theorem 3.7 implies that every chiral $\mathcal{D}_{G,Q}$-module $M$ carries also an action of $\tilde{g}'_Q$, which commutes with the initial $\tilde{g}_Q$-action. In what follows, we will refer to the above $\tilde{g}_Q$ and $\tilde{g}'_Q$-actions as “left” and “right”, respectively.

**Lemma 6.3.** For $M$ as above, the right $\tilde{g}'_Q$-action is compatible with the $\mathcal{O}_{G((t))}$-action in the sense that for $\xi' \in \tilde{g}'_Q$, $f \in \mathcal{O}_{G((t))}$ and $m \in M$,$$
abla \cdot (f \cdot m) = f \cdot (\nabla \cdot m) + \text{Lie}_{\xi'}(f) \cdot m,$$
where $\xi'$ is the corresponding right-invariant vector field on $G((t))$.

**Proof.** Let $L_0 \rightarrow \Theta_G$ be the Lie-* algebra map corresponding to the embedding $g \rightarrow T_G$ by means of right-invariant vector fields. Let us consider the corresponding Lie-* bracket $L_0 \otimes J(G) \rightarrow \Theta_G \otimes J(G) \rightarrow \Delta_l(J(G))$.

As in the proof of Proposition 6.2 above, we need to show that if we compose the above Lie-* bracket with the natural surjection $L'_0 \otimes J(G) \rightarrow L_0$, we obtain a commutative diagram

$$
\begin{array}{ccc}
L'_0 \otimes J(G) & \rightarrow & \Delta_l(J(G)) \\
\text{id} & \downarrow & \downarrow \\
\mathcal{D}_{G,Q} \otimes \mathcal{D}_{G,Q} & \rightarrow & \Delta_l(\mathcal{D}_{G,Q}).
\end{array}
$$

But this follows from the construction of the map $\tau$ and Lemma 2.4.

Since the roles of $\tilde{g}_Q$ and $\tilde{g}'_Q$ in $\mathcal{D}_{G,Q}$ are essentially symmetric, we have an analog of Proposition 6.2 with $\tilde{g}_Q$ replaced by $\tilde{g}'_Q$.

Let $K \subset G[[t]]$ be a normal group-subscheme of finite codimension and let $\mathfrak{k}$ denote its Lie algebra. By construction, the extension $0 \rightarrow \mathfrak{c} \rightarrow \tilde{g}'_Q \rightarrow g((t)) \rightarrow 0$ splits canonically over $g[[t]] \subset g((t))$. In particular, for every chiral $\mathcal{D}_{G,Q}$-module $M$ supported at the point $x$, we obtain a right $\mathfrak{k}$-action on the corresponding vector space $M_x$.

We let $\mathcal{D}^K_{G,Q}$-mod denote the full abelian subcategory of the category of chiral $\mathcal{D}_{G,Q}$-modules supported at $x$, for which the above right $\mathfrak{k}$-action integrates to an action of the group-scheme $K$.

Our goal now is to describe the category $\mathcal{D}^K_{G,Q}$-mod in geometric terms. To do that we need to make a digression on the theory of D-modules on ind-schemes. In what follows we put $Q = 0$ in order to deal with D-modules (and not with twisted D-modules). The generalization to the case of an arbitrary $Q$ is straightforward.

### 6.4. D-modules on ind-schemes

Consider the quotient $Y := K \backslash G((t))$, as a sheaf of sets on the category of quasi-compact schemes with respect to the faithfully flat topology.

It is known, cf. [3], Theorem 4.5.1, that $Y$ is in fact an ind-scheme of ind-finite type. This means that, as a functor, $Y$ is isomorphic to a direct limit $\lim_{\rightarrow} Y_i$, where $Y_i$ is
a functor representable by a scheme $Y_i$ and the maps $Y_i \to Y_j$ for $j \geq i$ are closed embeddings. The ind-finite type property means that the family $Y_i$ consists of schemes of finite type.

For an ind-scheme $Y$ one defines the category of $\mathcal{O}^l$-modules as the ind-completion of the direct limit category $\lim \mathcal{O}\text{-}\text{mod}(Y_i)$, where for $j \geq i$, the functor $\mathcal{O}\text{-}\text{mod}(Y_i) \to \mathcal{O}\text{-}\text{mod}(Y_j)$ is of course the direct image under the closed embedding $k_{i,j}$.

When $Y$ is of ind-finite type, one can define the category of right $D$-modules on it:

Recall first that if $Y$ is a scheme of finite type the category of right $D$-modules on it makes sense, cf. [3], Sect. 7.10. Moreover, we have a natural forgetful functor $F_Y : D\text{-}\text{mod}(Y) \to \mathcal{O}^l\text{-}\text{mod}(Y)$. (Note that the functor $F_Y$ is left exact but in general not right exact. If $Y$ is smooth, then it is right exact as well.) When $Y \to Y'$ is a closed embedding of schemes, there is a natural exact functor $k_t : D\text{-}\text{mod}(Y) \to D\text{-}\text{mod}(Y')$. Moreover, we have a natural transformation $k_s \circ F_Y \Rightarrow F_{Y'} \circ k_t$. This allows us to define the category of right $D$-modules on an ind-scheme of finite type. Namely, for $Y = \lim(Y_i)$ as above, we set $D\text{-}\text{mod}(Y)$ to be the ind-completion of the direct limit of abelian categories $\lim \mathcal{O}\text{-}\text{mod}(Y_i)$.

We have the forgetful functor $F_Y : D\text{-}\text{mod}(Y) \to \mathcal{O}^l\text{-}\text{mod}(Y)$, determined uniquely by the property that for $S \in \mathcal{O}\text{-}\text{mod}(Y_i)$, $F_Y(S) = \lim_{j \geq i} F_Y(j(k_{i,j}(S)))$. If $Y$ is formally smooth, cf. [3], Sect. 7.11.1, the functor $F_Y$ is exact.

The main result of this section is the following theorem.\footnote{To distinguish the $\mathcal{O}$- and the $D$-module direct images, we denote the former by $k_*$ and the latter $k^*$. By $^*$ we will always mean the $\mathcal{O}$-module inverse image.}

**Theorem 6.5.** Let $Y \simeq K\backslash G((t))$ as above. There is an equivalence of categories $\text{Sec} : D\text{-}\text{mod}(Y) \to \mathcal{D}_{G,0}^K\text{-}\text{mod}$.

**6.6. Construction of the functor.** Let $\pi$ denote the natural projection $G((t)) \to Y$. For a (right) $D$-module $S$ on $Y$ consider the corresponding $\mathcal{O}^l$-module $F_Y(S)$. Then the pull-back $\pi^*(F_Y(S))$ is an $\mathcal{O}^l$-module on $G((t))$. Observe that since $G((t))$ is affine, an $\mathcal{O}^l$-module on it is the same a discrete continuous module over the topological algebra $\mathcal{O}_{G((t))}$.

We claim that the vector space $\text{Sec}(S) := \Gamma(G((t)), \pi^*(F_Y(S)))$ underlies a chiral $\mathcal{D}_{G,0}$-module supported at $x \in X$. First, by construction, $\text{Sec}(M)$ is a discrete $\mathcal{O}_{G((t))}$-module. Secondly, we endow it with a continuous action of $\mathfrak{g}((t))$ as follows:

Since the projection $\pi$ is right $G$-invariant, the right $D$-module structure on $S$ gives rise to the action of $\mathfrak{g}((t))$ on $\pi^*(S)$ by derivations of the $\mathcal{O}^l$-module structure, where $\xi \in \mathfrak{g}((t))$ goes to the vector field $-\xi^i$ on $G((t))$.

Therefore, $\text{Sec}(S)$ indeed corresponds to a chiral $\mathcal{D}_{G,0}$-module, in view of Proposition 6.2.\footnote{We would like to thank A.Beilinson once again for explaining that Theorem 6.5 should be a consequence of the construction of $\mathcal{D}_{G,Q}$.}
To complete the construction, it remains to show that the action of \( \mathfrak{f} \) coming from the embedding \( r \) is indeed integrable.

Obviously, for any \( \mathcal{O}^! \)-module \( S' \) on \( Y \), the pull-back \( \pi^!(S') \) is \( K \)-equivariant with respect to the \( K \)-action on \( G((t)) \) by left translations. We will prove the following proposition:

**Proposition 6.7.** For \( S \in \text{D-mod}(Y) \) the right action of \( \mathfrak{f} \) on \( \text{Sec}(S) \) coincides with the \( \mathfrak{f} \)-action coming from the \( K \)-equivariant structure on \( \pi^*(F_Y(S)) \).

**Proof.** Let us view \( \text{Sec} \) as a functor from \( \text{D-mod}(Y) \) to the category of \( \mathcal{D}_{G,0} \)-modules supported at \( x \).

The difference of the two actions of \( \mathfrak{f} \) for a given \( S \in \text{D-mod}(Y) \) is a map

\[
\mathbf{d} : \mathfrak{f} \to \text{End}_{\mathbb{C}}(\text{Sec}(S)),
\]

which commutes with both the left \( \mathfrak{g}(((t))) \) and the \( \mathcal{O}_{G((t))} \)-actions on \( \text{Sec}(S) \).

In other words, \( \mathbf{d} \) is a map from \( \mathfrak{f} \) to the endomorphism ring of the functor \( S \mapsto \text{Sec}(S) \). Our goal is to prove that \( \mathbf{d} \equiv 0 \).

Let us first consider the case when \( S \) is the \( \delta \)-function \( D \)-module \( \delta_1 \) on \( Y \), i.e. the direct image of \( \mathbb{C} \) under \( \text{pt} \to Y \) corresponding to the coset of \( 1 \in G((t)) \). It is easy to see that the corresponding \( \mathcal{D}_{G,0} \)-module identifies with \( \text{Ind}\mathfrak{g}((t))(\mathcal{O}_K) \) with the natural \( \mathfrak{g}((t)) \)- and \( \mathcal{O}_{G((t))} \)-actions, cf. Proposition 6.2. In particular, it is generated by a canonical element \( 1_{\text{can}} \in \text{Ind}\mathfrak{g}((t))(\mathcal{O}_K) \).

Therefore, the map \( \mathbf{d} : \mathfrak{f} \to \text{End}(\text{Sec}(S)) \) vanishes for \( S = \delta_1 \). Indeed it is enough to show that \( \mathbf{d}(1_{\text{can}}) = 0 \), but this follows immediately from the construction.

Observe now that the action of \( G((t)) \) by right translations induces endo-functors of \( \text{D-mod}(Y) \) and \( \mathcal{D}_{G,0} \)-mod, which commute with the functor \( \text{Sec} \) in the natural sense.

By construction, for every \( k \in \mathfrak{f} \), \( \mathbf{d}(k) \in \text{End}(\text{Sec}) \) commutes with this \( G((t)) \)-action.

In particular, since every \( \delta_y \) for \( y \in Y \), can be obtained from \( \delta_1 \) as a \( G \)-translate, we conclude that \( \mathbf{d} : \mathfrak{f} \to \text{End}(\text{Sec}(\delta_y)) \) vanishes for all \( y \in Y \).

The above observations imply what we need:

First, every subscheme of \( Y \) admits a Zariski-open cover over which the morphism \( \pi \) admits a section. Thus, let \( Y' \) be a locally closed subscheme of \( Y \) and \( s \) be a map \( Y' \to G((t)) \). Then, any \( D \)-module \( S \) on \( Y' \) is isomorphic as an \( O \)-module to \( s^*(\pi^*(S)) \).

In particular, \( \mathbf{d} \) defines a map from \( \mathfrak{f} \) to the endomorphism ring of the the forgetful functor \( F_{Y'} : \text{D-mod}(Y') \to \mathcal{O}^! \text{-mod}(Y') \), and it is known that \( \text{End}(F_{Y'}) \simeq \mathcal{O}_{Y'} \).

However, since \( \mathbf{d} \) “kills” all the \( D \)-modules of the form \( \delta_y \), the map \( \mathbf{d} \) vanishes identically.

\[ \square \]

6.8. **Examples.** Let us now describe the right \( D \)-module \( S_{\text{vac}} \) on \( Y \) corresponding to the vacuum representation \( V_{G,Q} \) (this was somewhat implicit in the proof of Proposition 6.2 above):

This \( S_{\text{vac}} \) will be the direct image of a certain right \( D \)-module on \( K \backslash G[[t]] \) under the closed embedding \( K \backslash G[[t]] \hookrightarrow Y \). The corresponding \( D \)-module on \( K \backslash G[[t]] \) is constructed as follows:
As an \( \mathcal{O} \)-module it is isomorphic to \( \mathcal{O}_{K \backslash G[[t]]} \) and for the left-invariant vector field \( \xi^l \in \text{Lie}(K \backslash G[[t]]) \) and \( f \in \mathcal{O}_{K \backslash G[[t]]} \) we set: \( f \cdot \xi^l := - \text{Lie}_{\xi^l}(f) \). This defines a D-module structure completely, since left-invariant vector fields and functions generate the ring of differential operators.

Note that in this construction it was crucial that \( K \backslash G[[t]] \) is a group: on an arbitrary variety \( Z \), \( \mathcal{O}_Z \) carries a canonical left D-module structure but, in general, no right D-module structure.

To produce another set of examples, let us recall, cf. [3], Sect. 7.11.6, that on any ind-scheme \( Y \) of ind-finite type, the forgetful functor \( F_Y : \text{D-mod}(Y) \to \mathcal{O}^l \text{-mod}(Y) \) has an exact left adjoint, which will denote by \( \text{Sec} : \mathcal{O}^l \text{-mod}(Y) \to \text{D-mod}(Y) \).

First, since \( F_Y \) is formally smooth, cf. [3], Theorem 4.5.1 and Proposition 7.11.8, \( \text{Sec}(I_Y(\mathcal{F})) \) is a group: on an arbitrary \( \mathcal{O}^l \)-module \( \mathcal{F} \) on \( Y \) and consider the corresponding \( \mathcal{D}_{G,0} \)-module \( \text{Sec}(I_Y(\mathcal{F})) \). We claim that it can be described as follows:

The pull-back \( \pi^*(\mathcal{F}) \) as an \( \mathcal{O}^l \)-module on \( G((t)) \), equivariant with respect to the \( K \)-action by left translations. We regard \( \Gamma(G((t)), \pi^*(\mathcal{F})) \) as a \( K \)-module and consider the induced \( \mathfrak{g}'_{Q'} \)-module \( \text{Ind}_{\mathfrak{k}} \mathfrak{g}'_{Q'}(\pi^*(\mathcal{F})) \). (Here \( Q' = -Q_0 \); recall also that the induction \( \text{Ind}_{\mathfrak{k}} \) is understood in the sense that \( 1 \in \mathbb{C} \subset \mathfrak{g}'_{Q'} \) acts on the induced module as the identity).

This module carries a compatible \( \mathcal{O}_G((t)) \)-action. Hence, as in Proposition 3.2, it is naturally a \( \mathcal{D}_{G,0} \)-module. The right \( K \)-action on it is integrable by construction and we claim that it can be canonically identified with \( \text{Sec}(I_Y(\mathcal{F})) \).

Indeed, we have a tautological map of \( \mathfrak{g}'_{Q'} \)-modules \( \text{Ind}_{\mathfrak{k}} \mathfrak{g}'_{Q'}(\pi^*(\mathcal{F})) \to \text{Sec}(I_Y(\mathcal{F})) \), which sends \( \pi^*(\mathcal{F}) \) identically to \( \pi^*(I_Y(\mathcal{F})) \subset \pi^*(I_Y(\mathcal{F})) \). To prove that it is an isomorphism we proceed as follows:

Since \( Y \) is formally smooth, cf. [3], Theorem 4.5.1 and Proposition 7.11.8, \( F_Y(\mathcal{F}) \) carries a canonical filtration such that the associated graded object identifies with \( \mathcal{F} \otimes \text{Sym}(T_Y) \), where \( T_Y \) is the tangent sheaf on \( Y \). The \( \mathfrak{g}'_{Q'} \)-module \( \text{Ind}_{\mathfrak{k}} \mathfrak{g}'_{Q'}(\pi^*(\mathcal{F})) \) carries a canonical filtration as well and the above map is compatible with filtrations.

To prove our assertion, it suffices to observe that the induced map of the graded objects is

\[
\text{gr}(\text{Ind}_{\mathfrak{k}} \mathfrak{g}'_{Q'}(\pi^*(\mathcal{F}))) \cong \Gamma(G((t)), \pi^*(\mathcal{F}) \otimes \text{Sym}(\mathfrak{g}((t))/\mathfrak{k})) \cong \Gamma(G((t)), \pi^*(\mathcal{F} \otimes \text{Sym}(T_Y))) \cong \text{gr}(\text{Sec}(I_Y(\mathcal{F}))).
\]

6.9. **Proof of Theorem 6.5.** First, since \( Y \) is formally smooth, the functor \( F_Y \) is exact and, hence, the functor \( \text{Sec} : \text{D-mod}(Y) \to \mathcal{D}_{G,0}^K \text{-mod} \) is exact as well. We claim now that it is fully-faithful:

According to [3], Sect. 7.11.8-9, we can think of D-modules on \( Y \), as of \( \mathcal{O}^l \)-modules endowed with a compatible right action of the tangent algebroid \( T_Y \). Since the natural map \( \mathfrak{g}((t)) \otimes \mathcal{O}_Y \to T_Y \) is surjective, the functor \( \text{Sec} \) is fully-faithful.
Hence, it remains to show that Sec is surjective on objects. However, every object of \( \mathcal{D}^K_{G,0} \)-mod can be represented as a quotient of an object of the form

\[
\text{Ind}_{\mathfrak{t}}^{\mathfrak{g}'}(N),
\]

where \( N \) is a right-\( K \)-integrable continuous \( \mathcal{O}_{G((t))} \)-module.

Since we know that all objects of this form are in the image of our functor, the theorem follows.

**Remark.** Let us denote by \( M \mapsto \text{Loc}(M) \) the quasi-inverse functor of Sec. Explicitly, it can be described as follows:

Let \( M \) be an object of \( \mathcal{D}^K_{G,0} \)-mod. Then \( M \) can be thought of as a \( K \)-equivariant \( \mathcal{O}_! \)-module on \( G((t)) \), and hence, gives rise to an \( \mathcal{O}_! \)-module on \( Y \). It acquires a natural (right) action of the Lie-algebroid \( \mathfrak{g}(t) \otimes \mathcal{O}_Y \) (from minus the left \( \mathfrak{g}(t) \)-action on \( M \)).

Hence, the content of Theorem 6.5 is that the above \( \mathfrak{g}(t) \otimes \mathcal{O}_Y \)-action factors through the action of the tangent algebroid \( T_Y \).

6.10. **The category of D-modules on** \( G((t)) \).

Let now \( K_1 \subset G[[t]] \) be another normal subgroup, contained inside \( K \). Denote \( Y_1 := K_1 \backslash G((t)) \). We have the natural exact pull-back functor

\[
\pi_{Y_1,Y}^* : \text{D-mod}(Y) \to \text{D-mod}(Y'),
\]

where \( \pi_{Y_1,Y} : Y_1 \to Y \) is the canonical projection.

Thus, one possible candidate for the category of D-modules on \( G((t)) \) is the ind-completion of the direct limit category

\[
\text{D-mod}(G((t))) := \text{ind. comp.} \left( \lim_{\longleftarrow \substack{K \subset G[t]}} (\text{D-mod}(Y)) \right).
\]

Note, however, that the functors \( F_{Y_1} \circ \pi_{Y_1,Y}^* \) and \( \pi_{Y_1,Y}^* \circ F_Y \) from \( \text{D-mod}(Y) \) to \( \mathcal{O}_! \)-mod\( (Y_1) \) differ by the determinant line: for a D-module \( S \) on \( Y \),

\[
F_{Y_1}(\pi_{Y_1,Y}^*(S)) \simeq \pi_{Y_1,Y}^*(F_Y(S)) \otimes \Lambda^\dim(t/t') \Lambda^\dim(t/t').
\]

Correspondingly, the functors Sec : \( \text{D-mod}(Y) \to \mathcal{D}^K_{G,0} \)-mod for different subgroups \( K \) are compatible only up to a twist by the above 1-dimensional vector space. Now, Theorem 6.5 suggests that the above category of chiral \( \mathcal{D}^K_{G,0} \)-modules is also a reasonable candidate to be called the category of D-modules on \( G((t)) \). However, as we have just seen, there is no canonical equivalence (and, probably, no equivalence at all) between it and the above category \( \text{D-mod}(G((t))) \).

**References**

[1] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, Adv. Sov. Math. 16, AMS, Providence (1993).

[2] A. Beilinson and V. Drinfeld, *Chiral algebras*, preprint (2000), available as a postscript file at www.math.uchicago.edu/~benzvi.
[3] A. Beilinson and V. Drinfeld, *Hitchin’s integrable system and Hecke eigensheaves*, preprint (2000), available as a postscript file at www.math.uchicago.edu/~benzvi.

[4] D. Gaitsgory, *Notes on 2D conformal field theory and string theory*, in: *Quantum fields and strings: a course for mathematicians*, AMS-IAS (1999), pp. 1017-1090.

[5] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, preprint math.AG/9912074 (1999).

[6] F. Malikov, V. Schechtman and A. Vaintrob, *Chiral De Rham complex*, Comm. Math. Phys. 204 (1999), pp. 439-473.

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