Two-time distribution functions in the Gaussian model of randomly forced Burgers turbulence

Victor Dotsenko

LPTMC, Université Paris, 4 Place Jussieu, 75252 Paris, France
L D Landau Institute for Theoretical Physics, ulitsa Kosygina, 2, 119334 Moscow, Russia
E-mail: dotsenko@lptmc.jussieu.fr

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Abstract. The problem of randomly forced Burgers turbulence (‘Burgulence’) is considered in terms of the toy Gaussian Larkin model of directed polymers. In terms of the replica technique the explicit expression for the two-time four-point free energy distribution function is obtained which makes it possible to derive the exact result for the two-time velocity distribution function in the corresponding Burgulence problem.

Keywords: cavity and replica method, disordered systems (theory), fluctuations (theory)
1. Introduction

The KPZ problem [1] describing the growth in time of an interface in the presence of noise has been the subject of intense investigations during the past almost three decades [2–13]. During last five years major progress in solving this problem has been achieved which resulted in derivation of the exact solutions for various types of the free energy probability distribution function (PDF) [14–33].

In this paper I would like to consider the possibility to apply the ideas and technique developed in the studies of the KPZ problem for the formally equivalent problem of the randomly forced Burgers turbulence (the ‘Burgulence’ problem). Here one considers a velocity field \( v(x, t) \) governed by the Burgers equation

\[
\partial_t v(x, t) + v(x, t) \partial_x v(x, t) = \nu \partial_x^2 v(x, t) + \zeta(x, t)
\]

where the parameter \( \nu \) is the viscosity and \( \zeta(x, t) \) is the Gaussian distributed random force which is \( \delta \) -correlated in time and which is characterized by finite correlation length \( \xi \) in space: \( \zeta(x, t) \zeta(x', t') = u \delta(t - t') U[(x - x')/\xi] \). Here \( U(x) \) is a smooth function decaying to zero fast enough at large arguments and the parameter \( u \) is the injected energy density. In this problem one would like to derive e.g. the probability distribution functions of the velocity gradients \( P[\partial_x v(x, t)] \) or two-points distribution function \( P[v(x, t), v(x', t)] \) at scales smaller than the length scale \( \xi \) of the stirring force \( \zeta \) (see e.g. [3, 34–36] and references therein).

Formally the above problem is equivalent to the KPZ equation as well as to the \((1+1)\) directed polymers. Indeed, redefining \( v(x, t) = -\partial_x F(x, t) \) and \( \zeta(x, t) = -\partial_x V(x, t) \) one gets the KPZ equation for the interface profile \( F(x, t) \) (which is the free energy of \((1+1)\) directed polymers):

\[
\partial_t F(x, t) = \frac{1}{2} (\partial_x F(x, t))^2 - \frac{1}{2} T \partial_x^2 F(x, t) - V(x, t)
\]

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where $T = 2\nu$ is the temperature parameter of the directed polymer problem and $V(x, t)$ is the Gaussian distributed random potential.

The idea of a new approach to the Burgulence problem which I would like to demonstrate in this paper is in the following. According to the above definitions the velocity field can be represented as

$$v(x, t) = \lim_{\epsilon \to 0} \frac{F(x + \epsilon, t) - F(x, t)}{\epsilon}$$

(3)

Thus, deriving the four-point KPZ probability distribution function $P[F(x + \epsilon, t), F(x, t), F(x', t'), F(x', t')]$ and taking the limit $\epsilon \to 0$ one could hopefully obtain the result for $P[v(x, t), v(x', t')]$. The only ‘little problem’ is that unlike the usual KPZ studies operating with the random potential $\delta$-correlated in space, in the Burgulence problem one is mainly interested in the spatial scales comparable to or much smaller than the random potential correlation length $\xi$. In other words, in this approach, first one has to study the KPZ problem with random potentials having finite correlation length. In the present study (as a matter of ‘warming up’ exercise) I’m going to consider another ‘extreme case’ in which the random potential $V(x, t)$ of the KPZ problem is changed by its linear approximation: $V(x, t) \rightarrow \zeta(t)x$ where $\zeta(t)$ is Gaussian distributed random force. In this case we obtain the model introduced by Larkin [37,38] a long time ago to study small scale displacements of directed polymers. In this approximation the model becomes Gaussian and therefore exactly solvable. Nevertheless, the statistical properties of its free energy (as well as some other quantities) turn out to be rather non-trivial (see e.g. [39–41]). For that reason this model hopefully could serve as a good ground for testing various approaches developed in the recent KPZ studies.

In section 2 we introduce the model and formulate the main ideas of the replica approach which will be used in the further calculations of the two-time free energy distribution functions (for the directed polymer model) and the corresponding velocity distribution function (in the Burgulence problem). In section 3 as the matter of the demonstration of the replica technique the explicit expression for the two-time (two-point) free energy distribution function is derived. In section 4 the two-time four-point free energy distribution function is calculated and the corresponding two-time velocity distribution function $P[v(x, t), v(x', t')]$ of the Burgulence problem is obtained. Further perspectives of the present approach are discussed in section 5.

2. The model and the method

In this paper we consider the model of one-dimensional directed polymers defined in terms of an elastic string $\phi(\tau)$ directed along the $\tau$-axes within an interval $[0, t]$ which passes through a random medium described by a random potential $V(\phi, \tau) = \zeta(\tau)\phi$. The energy of a given polymer’s trajectory $\phi(\tau)$ is given by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^t d\tau \left\{ \frac{1}{2} \left[ \partial_\tau \phi(\tau) \right]^2 + \zeta(\tau)\phi(\tau) \right\};$$

(4)

where the random force $\zeta(\tau)$ is described by the Gaussian distribution with a zero mean $\zeta(\tau) = 0$ and the $\delta$-correlations:

$$\zeta(\tau)\zeta(\tau') = u\delta(\tau - \tau').$$

(5)

The parameter $u$ describes the strength of the disorder.

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For the fixed boundary conditions, \( \phi(0) = y; \phi(t) = x \), the partition function of the model (4) is

\[
Z(x|y; t) = \int_{\phi(0)=y}^{\phi(t)=x} D\phi(\tau) \, e^{-\beta H[\phi]} = \exp[-\beta F(x|y; t)]
\]

(6)

where \( \beta \) is the inverse temperature and \( F(x|y; t) \) is the free energy. In the replica approach one considers the average of the \( N \)th power of the above partition function:

\[
\overline{Z^N}(x|y; t) \equiv Z(N; x|y; t) = \exp[-\betaNF(x|y; t)]
\]

(7)

where \( \langle \ldots \rangle \) denotes the average over the random force \( \zeta \). Simple Gaussian averaging yields:

\[
Z(N; x|y; t) = \prod_{a=1}^{N} \left[ \int_{\phi_a(0)=y}^{\phi_a(t)=x} D\phi_a(\tau) \right] \exp\left[ -\beta H_N[\phi] \right]
\]

(8)

where

\[
H_N[\phi] = \frac{1}{2} \int_0^t d\tau \left( \sum_{a=1}^{N} \left[ \partial_\tau \phi_a(\tau) \right]^2 - \beta u \sum_{a,b}^{N} \phi_a(\tau) \phi_b(\tau) \right)
\]

(9)

is the Gaussian replica Hamiltonian.

Introducing the free energy distribution function, \( P_{x|y;t}(F) \), the relation (7) can be represented as follows:

\[
Z(N; x|y; t) = \int_{-\infty}^{+\infty} dF \, P_{x|y;t}(F) \exp[-\beta NF]
\]

(10)

which is the Laplace transform of the distribution function, \( P_{x|y;t}(F) \), with respect to the parameter \( \beta N \). In the lucky case when the moments of the partition function \( Z(N; x|y; t) \) allow an analytic continuation from integer to arbitrary complex values of the replica parameter \( N \) the above relation makes it possible to reconstruct the probability distribution function \( P_{x|y;t}(F) \) via the inverse Laplace transform:

\[
P_{x|y;t}(F) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} Z\left( \frac{s}{\beta}; x|y; t \right) \exp(sF).
\]

(11)

In the present model this distribution can be computed explicitly and the resulting function \( P_{x|y;t}(F) \) turns out to be rather non-trivial [39–41].

Besides the replica partition function equation (8) it is convenient to introduce the \( N \)-particle ‘wave function’:

\[
\Psi_N[x|y; t] = \prod_{a=1}^{N} \left[ \int_{\phi_a(0)=y_a}^{\phi_a(t)=x_a} D\phi_a(x) \right] \exp\left[ -\beta H_N[\phi] \right]
\]

(12)

where the Hamiltonian \( H_N[\phi] \) is given in equation (9). In the present model the function \( \Psi_N[x|y; t] \) can be computed explicitly (see e.g. [41]):

\[
\Psi_N[x|y; t] = C(N,t) \exp\left\{ -\beta \frac{1}{2t} \sum_{a=1}^{N} (x_a - y_a)^2 - \beta \frac{1}{2t} A(N,t) \sum_{a=1}^{N} (x_a - y_a)^2 + \beta \frac{1}{t} B(N,t) \sum_{a,b=1}^{N} x_ay_b \right\}
\]

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where

\[ C(N, t) = \left( \frac{\beta}{2\pi t} \right)^{N/2} \sqrt{\frac{\sqrt{\beta Nu_t^2}}{\sin(\sqrt{\beta Nu_t^2})}} \]  
\[ A(N, t) = \frac{1}{N} \left( \frac{\sqrt{\beta Nu_t^2}}{\tan(\sqrt{\beta Nu_t^2})} - 1 \right) \]  
\[ B(N, t) = \frac{1}{N} \frac{\sqrt{\beta Nu_t^2}}{\sin(\sqrt{\beta Nu_t^2})} \left[ 1 - \cos(\sqrt{\beta Nu_t^2}) \right]. \]  

Note that the above expression for the wave function equation (13) is valid only at finite time interval: \( t < \frac{\pi}{2} (\beta Nu)^{-1/2} \equiv t_c(N) \). The reason is that due to the specific form of the interaction potentials in the replica Hamiltonian (9) the directed polymers’ trajectories go to infinity at finite time \( t_c(N) \). For the original physical system this phenomenon is explained by the presence of the slowly decaying left tail [39–41] of the free energy distribution function \( P_{x|y}(F) \) which (according to the relation (10)) results in the divergence of all partition function moments \( Z(N; x|y; t) \) with \( N > N_c(t) = \frac{\pi^2}{4} (\beta ut)^{-1} \).

3. Two-time free energy distribution function

For simplicity, in this section we consider the directed polymer problem with the zero boundary conditions: \( \phi(0) = \phi(t) = 0 \). For a given realization of the random function \( \zeta(\tau) \) let us denote by \( F_1 \) the free energy of the directed polymers with the (zero) ending point at time \( t \) and by \( F_2 \) that of the directed polymers with the (zero) ending point at time \( t + \Delta t \). According to the definition, equation (6), for the difference of these free energies, \( F = F_2 - F_1 \), one has:

\[ \exp\{-\beta F\} = Z^{-1}(0|0; t) Z(0|0; t + \Delta t). \]  

Taking the \( n \)th power of the above relation and performing the averaging over quenched randomness we get

\[ \exp\{-\beta n F\} = Z^{-n}(0|0; t) Z^n(0|0; t + \Delta t). \]  

Introducing the two-time free energy difference probability distribution function \( P_{t,\Delta t}(F) \) in terms of the replica approach the above relation can be represented as follows:

\[ \int_{-\infty}^{+\infty} dF P_{t,\Delta t}(F) \exp\{-\beta n F\} = \lim_{N \to 0} \frac{Z^{N-n}(0|0; t) Z^n(0|0; t + \Delta t)}{\sqrt{\beta Nu_t^2} \sin(\sqrt{\beta Nu_t^2})} \]  

where according to the usual replica formalism, in the first step, the r.h.s. of the above relation is computed for an arbitrary integer \( N \) and then, at the second step, the obtained result is analytically continued for arbitrary real \( N \), and finally the limit \( N \to 0 \) is taken.

For an integer \( N > n \), according to the definitions (6), (7) and (12), the object \( \frac{Z^{N-n}(0|0; t) Z^n(0|0; t + \Delta t)}{\sqrt{\beta Nu_t^2} \sin(\sqrt{\beta Nu_t^2})} \) is defined by the path integral over \( N \) directed polymer trajectories such that \((N - n)\) of them have the boundary conditions \( \phi_a(0) = \phi_a(t) = 0 \) \((a = 1, \ldots, (N - n))\), while the other \( n \) trajectories have the boundary conditions \( \phi_b(0) = \phi_b(t + \Delta t) = 0 \) \((b = 1, \ldots, n)\). Using the identity

\[ Z(0|0; t + \Delta t) = \int_{-\infty}^{+\infty} dx \ Z(0|x; t) Z^*(x|0; t|(t + \Delta t)) \]

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(where $Z^*(x|0; t|t + \Delta t)$ is the ‘backward’ partition function defined by the trajectories starting at zero at time moment $(t + \Delta t)$ and coming to the point $x$ at the time moment $t$) the product of the partition functions in the r.h.s. of equation (19) can be represented as follows:

$$Z^{N-n}(0|0; t|t) = Z^{n}(0|0; t+\Delta t) \equiv Z(N, n; t, \Delta t)$$

$$= \int_{-\infty}^{+\infty} dx_1 \ldots dx_n \Psi_N[0, \ldots, 0, x_1, \ldots, x_n|0; t] \Psi_n^*[x_1, \ldots, x_n|0; \Delta t]$$

where the second (conjugate) wave function represents the ‘backward’ propagation from the time moment $(t + \Delta t)$ to the previous time moment $t$. Schematically the above expression is represented in figure 1.

Substituting equation (13) into equation (20) we get:

$$Z(N, n; t, \Delta t) = C(N, t)C(n, \Delta t) \times \int_{-\infty}^{+\infty} dx_1 \ldots dx_n \exp \left\{ -\frac{1}{2} \beta \sum_{a,b=1}^n \left[ \frac{1}{t} A(N, t) + \frac{1}{\Delta t} A(n, \Delta t) \right] x_a x_b \right\}.$$  (21)

Simple integration yields:

$$Z(N, n; t, \Delta t) = C(N, t)C(n, \Delta t) \left[ \frac{2\pi t \Delta t}{\beta(t + \Delta t)} \right]^{n/2} \times \left[ 1 + n \frac{t \Delta t}{t + \Delta t} \left( \frac{1}{t} A(N, t) + \frac{1}{\Delta t} A(n, \Delta t) \right) \right]^{-1/2}.$$  (22)

Substituting here the explicit expressions (14) and (15), and taking the limit $N \to 0$ we obtain

$$\lim_{N \to 0} Z(N, n; t, \Delta t) = \left( \frac{t}{t + \Delta t} \right)^{n/2} \frac{\sqrt{\beta n u(\Delta t)^2(t + \Delta t)}}{\sin(\sqrt{\beta n u(\Delta t)^2}) \left[ \Delta t - \frac{1}{3} \beta n u(\Delta t)^2 t^2 + t \frac{\sqrt{\beta n u(\Delta t)^2}}{\tan(\sqrt{\beta n u(\Delta t)^2})} \right]}$$

then substituting equation (23) into equation (19) and redefining:

$$\Delta t = \xi t$$

$$\beta n u(\Delta t)^2 = \omega$$

$$F = u\xi^2 t^2 f$$

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we get the following relation for the probability distribution function \( P_{\xi}(f) \) of the rescaled free energy \( f \):

\[
\int_{-\infty}^{+\infty} df \, P_{\xi}(f) \exp\{-\omega f\} = (1 + \xi)^{-\frac{\omega}{2(\sqrt{\xi} + \sqrt{-\omega})}} \frac{\omega^{1/4} \sqrt{1 + \xi}}{\sqrt{\xi - \frac{\omega}{i\xi}}} \sin(\sqrt{\omega}) + \sqrt{\omega} \cos(\sqrt{\omega}).
\]  

(27)

By inverse Laplace transform in the limit when both \( t \to \infty \) and \( \Delta t \to \infty \) (such that the parameter \( \xi = \Delta t/t \) remains finite) we get the following universal result for the limiting two-time free energy distribution function:

\[
\lim_{t \to \infty} P_{\xi}(f) \equiv P(t) = \sqrt{1 + \xi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{\omega^{1/4} \exp\{\omega f\}}{\sqrt{\xi - \frac{\omega}{i\xi}}} \sin(\sqrt{\omega}) + \sqrt{\omega} \cos(\sqrt{\omega}).
\]  

(28)

It is interesting to note that this function (like its one-time counterpart [39–41]) is identically equal to zero at \( f > 0 \). Indeed, since at \( f > 0 \) the function under the integral in the r.h.s. of equation (28) quickly goes to zero at \( w \to -\infty \), the contour of integration in the complex plane can be safely shifted to \(-\infty\), which means that \( P(t) \equiv 0 \).

### 4. Two-time velocity distribution function

Velocity in the Burgers problem is given by the derivative of the free energy of the directed polymer problem:

\[
v(x, t) = -\partial F(x, t)/\partial x = -\lim_{\epsilon \to 0} \frac{F(x + \epsilon) - F(x, t)}{\epsilon}.
\]  

(29)

Thus, to compute the two-point velocity distribution function in terms of the directed polymers, first, keeping \( \epsilon \) finite we consider a specially constructed four-point object (see below), and only in the final stage of calculations we take the limit \( \epsilon \to 0 \).

According to the relation (6)

\[
\exp\{-\beta[F(x + \epsilon, t) - F(x, t)]\} = \exp\{\beta \epsilon v(x, t)\} = \frac{Z(x + \epsilon|0; t)}{Z(x|0; t)}.
\]  

(30)

Following the procedure described in the previous section we have:

\[
\lim_{N_1, N_2 \to 0} \frac{Z^{n_1}(x_1 + \epsilon|0; t) Z^{n_2}(x_2 + \epsilon|0; t + \Delta t) Z^{N_1-n_1}(0; t) Z^{N_2-n_2}(0; t + \Delta t)}{Z^{n_1}(x_1 + \epsilon|0; t) Z^{n_2}(x_2 + \epsilon|0; t + \Delta t) Z^{N_1-n_1}(0; t) Z^{N_2-n_2}(0; t + \Delta t)}.
\]

Introducing two-time velocity distribution function \( P_{x_1, x_2, \Delta t}(v_1, v_2) \) the above relation can be represented as follows:

\[
\int \int_{-\infty}^{+\infty} dv_1 dv_2 \, P_{x_1, x_2, \Delta t}(v_1, v_2) \exp\{\beta n_1 \epsilon v_1 + \beta n_2 \epsilon v_2\}
\]

\[
= \lim_{N_1, N_2 \to 0} Z_e(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t)
\]

(32)

where

\[
Z_e(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t)
\]

\[
= \frac{Z^{n_1}(x_1 + \epsilon|0; t) Z^{n_2}(x_2 + \epsilon|0; t + \Delta t) Z^{N_1-n_1}(0; t) Z^{N_2-n_2}(0; t + \Delta t)}{Z^{n_1}(x_1 + \epsilon|0; t) Z^{n_2}(x_2 + \epsilon|0; t + \Delta t) Z^{N_1-n_1}(0; t) Z^{N_2-n_2}(0; t + \Delta t)}.
\]

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Figure 2. Schematic representation of the directed polymer paths corresponding to equation (34).

In terms of the wave function, equations (12) and (13),

\[ Z_\epsilon(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = \int_{-\infty}^{+\infty} Dz \int_{-\infty}^{+\infty} Dz \Psi_{N_1+N_2}[x_1, \ldots, x_1, x_1+\epsilon, \ldots, x_1, y_1, \ldots, y_{N_2-n_2} \ldots, y_{N_2-n_2}, z_1, \ldots, z_{n_2} \mid 0; t] \]

\[ \times \Psi_{N_2}[y_1, \ldots, y_{N_2-n_2}, z_1, \ldots, z_{n_2} \mid x_2, \ldots, x_2, x_2+\epsilon, \ldots, x_2+\epsilon; \Delta t] \] (34)

where

\[ \int_{-\infty}^{+\infty} Dy \equiv \prod_{a=1}^{N_2-n_2} \int_{-\infty}^{+\infty} dy_a \]

\[ \int_{-\infty}^{+\infty} Dz \equiv \prod_{a=1}^{n_2} \int_{-\infty}^{+\infty} dz_a . \]

Schematically the expression in equation (34) is represented in figure 2.

Substituting the explicit expressions for the wave function (13) into equation (34) we get:

\[ Z_\epsilon(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = C(N_1 + N_2, t) C(N_2, \Delta t) \]

\[ \times \int_{-\infty}^{+\infty} Dz \int_{-\infty}^{+\infty} Dz \exp \left\{ -\frac{\beta}{2\epsilon} \left[ (N_1 - n_1)x_1^2 + n_1(x_1+\epsilon)^2 + \sum_{a=1}^{N_2-n_2} y_a^2 + \sum_{a=1}^{n_2} z_a^2 \right] \right\} \]

\[ -\frac{\beta}{2\epsilon} A(N_1 + N_2, t) \left[ (N_1 - n_1)x_1 + n_1(x_1+\epsilon) + \sum_{a=1}^{N_2-n_2} y_a + \sum_{a=1}^{n_2} z_a \right]^2 \]

\[ -\frac{\beta}{2\Delta t} \sum_{a=1}^{N_2-n_2} (y_a - x_2)^2 + \sum_{a=1}^{n_2} (z_a - x_2 - \epsilon)^2 \]

\[ -\frac{\beta}{2\Delta t} A(N_2, \Delta t) \left[ \sum_{a=1}^{N_2-n_2} (y_a - x_2) + \sum_{a=1}^{n_2} (z_a - x_2 - \epsilon) \right]^2 \]

\[ + \frac{\beta}{\Delta t} B(N_2, \Delta t) \left[ \sum_{a=1}^{N_2-n_2} y_a + \sum_{a=1}^{n_2} z_a \right] \left[ (N_2 - n_2)x_2 + n_2(x_2 + \epsilon) \right] \] (36)

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where $A(N, t)$, $B(N, t)$ and $C(N, t)$ are given in equations (14)–(16). Introducing $N_2$-component vector $\chi = \{y_1, ..., y_{N_2-n_2}, z_1, ..., z_{n_2}\}$ after simple algebra we get

$$Z_e(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = C(N_1 + N_2, t) C(N_2, \Delta t)$$

$$× \left[ \prod_{a=1}^{N_2} \int_{-\infty}^{+\infty} d\chi_a \right] \exp \left\{ -\frac{1}{2} \sum_{a,b=1}^{N_2} T_{ab} \chi_a \chi_b + \sum_{a=1}^{N_2} L_a \chi_a - \frac{1}{2} \beta G \right\} \quad (37)$$

where

$$G = \frac{1}{t} [n_1(x_1 + \epsilon)^2 + (N_1 - n_1)x_1^2] + \frac{1}{\Delta t} [n_2(x_2 + \epsilon)^2 + (N_2 - n_2)x_2^2]$$

$$+ \frac{1}{t} A(N_1 + N_2, t)(N_1 x_1 + n_1 \epsilon)^2 + \frac{1}{\Delta t} A(N_2, \Delta t)(N_2 x_2 + n_2 \epsilon)^2 \quad (38)$$

and

$$T_{ab} = \gamma \delta_{ab} + \kappa$$

$$\gamma = \frac{\beta t + \Delta t}{t \Delta t} \quad (39)$$

$$\kappa = \frac{\beta}{t} A(N_1 + N_2, t) + \frac{\beta}{\Delta t} A(N_2, \Delta t) \quad (40)$$

$$L_a = L + X_a$$

$$L = -\frac{\beta}{t} A(N_1 + N_2, t)(N_1 x_1 + n_1 \epsilon) + \frac{\beta}{\Delta t} [A(N_2, \Delta t) + B(N_2, \Delta t)](N_2 x_2 + n_2 \epsilon) \quad (41)$$

$$X_a = \begin{cases} x_2 + \epsilon, & \text{for } a = 1, ..., n_2 \\ x_2, & \text{for } a = n_2 + 1, ..., N_2. \end{cases} \quad (42)$$

Simple integration over $\chi$’s in equation (37) yields:

$$Z_e(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = C(N_1 + N_2, t) C(N_2, \Delta t)$$

$$× \exp \left\{ -\frac{1}{2} \beta G - \frac{1}{2} \text{Tr} \ln \hat{T} + \frac{1}{2} \sum_{a,b=1}^{N_2} L_a L_b \hat{T}_{ab}^{-1} \right\} \quad (43)$$

where

$$\text{Tr} \ln \hat{T} = N_2 \ln \gamma + \ln \left( 1 + N_2 \frac{\kappa}{\gamma} \right)$$

$$\hat{T}_{ab}^{-1} = \frac{1}{\gamma} \delta_{ab} - \frac{\kappa}{\gamma(\gamma + N_2 \kappa)} \quad (44)$$

Simple calculations yield:

$$Z_e(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = C(N_1 + N_2, t) C(N_2, \Delta t)$$

$$× \exp \left\{ -\frac{1}{2} \beta G - \frac{1}{2} N_2 \ln \gamma - \frac{1}{2} \ln \left( 1 + N_2 \frac{\kappa}{\gamma} \right) - \frac{L^2 N_2}{2(\gamma + N_2 \kappa)} + \frac{\beta L}{\gamma \Delta t} (N_2 x_2 + n_2 \epsilon)$$

$$- \frac{\beta \kappa N_2 (N_2 x_2 + n_2 \epsilon)}{\gamma \Delta t (\gamma + N_2 \kappa)} - \frac{\beta^2 (N_2 x_2 + n_2 \epsilon)^2}{2 \gamma (\Delta t)^2 (\gamma + N_2 \kappa)} + \frac{\beta^2}{2 \gamma (\Delta t)^2} \left[ (N_2 - n_2) x_2^2 + n_2 (x_2 + \epsilon)^2 \right] \right\} \quad (45)$$

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The next step of the calculations is to take the limits $N_{1,2} \to 0$. Using explicit expressions (14)–(16), (38), (41) and (43), one easily finds:

$$\lim_{N \to 0} C(N, t) = 1$$  \hspace{0.5cm} (49)

$$\lim_{N \to 0} A(N, t) = -\frac{1}{3} \beta u t^2$$  \hspace{0.5cm} (50)

$$\lim_{N \to 0} B(N, t) = \frac{1}{2} \beta u t^2$$  \hspace{0.5cm} (51)

$$\lim_{N_{1,2} \to 0} \beta G = \frac{\beta n_1 \epsilon}{t} (2x_1 + \epsilon) + \frac{\beta n_2 \epsilon}{\Delta t} (2x_2 + \epsilon) - \frac{1}{3} (\beta n_1 \epsilon)^2 u t - \frac{1}{3} (\beta n_2 \epsilon)^2 u \Delta t$$  \hspace{0.5cm} (52)

$$\lim_{N_{1,2} \to 0} L = \frac{1}{3} \beta^2 n_1 \epsilon u t + \frac{1}{6} \beta^2 n_2 \epsilon u \Delta t$$  \hspace{0.5cm} (53)

$$\lim_{N_{1,2} \to 0} \kappa = -\frac{1}{3} \beta^2 u (t + \Delta t).$$  \hspace{0.5cm} (54)

Substituting the above limiting values into equation (48) we get

$$\lim_{N_{1,2} \to 0} Z_\epsilon(N_1, n_1, N_2, n_2, x_1, x_2, t, \Delta t) = Z_\epsilon(n_1, n_2, x_1, x_2, t, \Delta t)$$

$$= \exp \left\{ -\frac{\beta n_1 \epsilon}{2t} (2x_1 + \epsilon) - \frac{\beta n_2 \epsilon}{2(t + \Delta t)} (2x_2 + \epsilon) + \frac{1}{6} (\beta n_1 \epsilon)^2 u t + \frac{1}{6} (\beta n_2 \epsilon)^2 u (t + \Delta t) + \frac{(\beta n_1 \epsilon)(\beta n_2 \epsilon) u t^2}{3(t + \Delta t)} \right\}.$$  \hspace{0.5cm} (55)

Substituting the above result into equation (32) and introducing notations $\beta n_{1,2} \epsilon = s_{1,2}$ in the limit $\epsilon \to 0$ we obtain:

$$\int \int_{-\infty}^{+\infty} dv_1 dv_2 P_{x_1, x_2, t, \Delta t}(v_1, v_2) \exp \{ s_1 v_1 + s_2 v_2 \}$$

$$= \exp \left\{ -\frac{x_1}{t} s_1 - \frac{x_2}{t + \Delta t} s_2 + \frac{1}{6} u t s_1^2 + \frac{1}{6} u (t + \Delta t) s_2^2 + \frac{u t^2}{3(t + \Delta t)} s_1 s_2 \right\}.$$  \hspace{0.5cm} (56)

Redefining

$$s_1 = \sqrt{\frac{3}{u t}} \omega_1$$  \hspace{0.5cm} (57)

$$s_2 = \sqrt{\frac{3}{u(t + \Delta t)}} \omega_2$$  \hspace{0.5cm} (58)

$$v_1 = -\frac{x_1}{t} + \sqrt{\frac{1}{3}} u t \tilde{v}_1$$  \hspace{0.5cm} (59)

$$v_2 = -\frac{x_2}{t + \Delta t} + \sqrt{\frac{1}{3}} u (t + \Delta t) \tilde{v}_2$$  \hspace{0.5cm} (60)

$$\Delta t = \xi t$$  \hspace{0.5cm} (61)

we get the following relation for the probability distribution function $P_{\xi}(\tilde{v}_1, \tilde{v}_2)$ for the rescaled velocities $\tilde{v}_1$ and $\tilde{v}_2$, equations (59) and (60):

$$\int \int_{-\infty}^{+\infty} d\tilde{v}_1 d\tilde{v}_2 P_{\xi}(\tilde{v}_1, \tilde{v}_2) \exp \{ \omega_1 \tilde{v}_1 + \omega_2 \tilde{v}_2 \} = \exp \left\{ \frac{1}{2} \omega_1^2 + \frac{1}{2} \omega_2^2 + \frac{\omega_1 \omega_2}{(1 + \xi)^{3/2}} \right\}.$$  \hspace{0.5cm} (62)
Performing simple inverse Laplace transformation

\[ \mathcal{P}_\xi(\tilde{v}_1, \tilde{v}_2) = \int \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi i)^2} \exp \left\{ \frac{1}{2} \omega_1^2 + \frac{1}{2} \omega_2^2 + \frac{\omega_1 \omega_2}{(1 + \xi)^3/2} - \omega_1 \tilde{v}_1 - \omega_2 \tilde{v}_2 \right\} \]  

one eventually obtains the following very simple result for the two-time velocity distribution function:

\[ \mathcal{P}_\xi(\tilde{v}_1, \tilde{v}_2) = \frac{1}{2\pi} \sqrt{\frac{(1 + \xi)^3}{(1 + \xi)^3 - 1}} \exp \left\{ -\frac{(1 + \xi)^3}{2[(1 + \xi)^3 - 1]} \left( \tilde{v}_1^2 - 2 \frac{\tilde{v}_1 \tilde{v}_2}{(1 + \xi)^{3/2}} + \tilde{v}_2^2 \right) \right\} \]  

where \( \xi = \Delta t/t \) is the reduced separation time parameter.

One can easily check that in the limit of infinite separation time, \( \xi \to \infty \), the distributions of two velocities become independent:

\[ \lim_{\xi \to \infty} \mathcal{P}_\xi(\tilde{v}_1, \tilde{v}_2) = \frac{1}{2\pi} \exp \left\{ -\frac{1}{2} \tilde{v}_1^2 - \frac{1}{2} \tilde{v}_2^2 \right\} \]  

while in the opposite limit of coinciding times, \( \xi \to 0 \), one finds

\[ \mathcal{P}_0(\tilde{v}_1, \tilde{v}_2) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \tilde{v}_1^2 \right\} \delta(\tilde{v}_1 - \tilde{v}_2) \]

as it should be.

Besides, using the exact result, equation (64), one can easily compute the time dependence of the two velocities’ correlation function:

\[ \langle \tilde{v}_1 \tilde{v}_2 \rangle = (1 + \xi)^{-3/2} \]  

as well as the probability distribution function for the velocities’ difference \( \tilde{v} \equiv \tilde{v}_2 - \tilde{v}_1 \):

\[ \mathcal{P}_\xi(\tilde{v}) = \sqrt{\frac{(1 + \xi)^{3/2}}{4\pi[(1 + \xi)^{3/2} - 1]}} \exp \left\{ -\frac{(1 + \xi)^{3/2}}{4[(1 + \xi)^{3/2} - 1]} \tilde{v}^2 \right\} \]  

5. Conclusions

In this paper we have considered the problem of velocity distribution functions in the Burgulence problem in terms of the toy Gaussian model of \((1 + 1)\) directed polymers. In particular the exact result for the two-time free energy, equation (28), and two-time velocity distribution functions, equation (64), has been derived. Of course the considered system is too far from the realistic one. Nevertheless, it has one important advantage: being exactly solvable, some of its statistical properties are rather non-trivial. All that, in my view, makes this model a rather useful tool for testing new ideas and various technical aspects of the calculations (like the replica technique considered in this paper).

Following the proposed route, the next step would be to consider the model with finite range correlations of the random potentials. Of course, one cannot hope to get exact results here. In terms of the replica approach, first of all, one is facing the problem of \(N\)-particle quantum bosons with attractive finite range interactions whose solution is not known. Nevertheless even the qualitative understanding of the structure of the \(N\)-particle wave function of this system (which at the qualitative level might be not so much different from that of the Bethe ansatz solution for the \(\delta\)-correlated potentials) could hopefully be sufficient to get some understanding of the velocity statistics in the Burgulence problem.
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