HARDY’S INEQUALITY IN A LIMITING CASE ON GENERAL BOUNDED DOMAINS

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ABSTRACT. In this paper, we study Hardy’s inequality in a limiting case:
\[
\int_{\Omega} |\nabla u|^N dx \geq \frac{C_N(\Omega)}{N} \int_{\Omega} \frac{|u(x)|}{|x|^N} \left( \log \frac{R}{|x|} \right)^N dx
\]
for functions \( u \in W^{1,N}_0(\Omega) \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( R = \sup_{x \in \Omega} |x| \). We study the attainability of the best constant \( C_N(\Omega) \) in several cases. We provide sufficient conditions that assure \( C_N(\Omega) > C_N(B_R) \) and \( C_N(\Omega) \) is attained, here \( B_R \) is the \( N \)-dimensional ball with center the origin and radius \( R \). Also we provide an example of \( \Omega \subset \mathbb{R}^2 \) such that \( C_2(\Omega) > C_2(B_R) = 1/4 \) and \( C_2(\Omega) \) is not attained.

1. INTRODUCTION

The classical Hardy inequality in one space dimension states that
\[
\int_0^\infty |u'(t)|^p \, dt \geq \left( \frac{p-1}{p} \right)^p \int_0^\infty \frac{|u(t)|^p}{t^p} \, dt
\]
holds for all \( u \in W^{1,p}_0(0, +\infty) \) where \( 1 < p < \infty \). This scaling invariant inequality is now very classical and there are wonderful treatises [15], [27], [28] on further generalizations of the inequality (1). It is also known that the constant \( \left( \frac{p-1}{p} \right)^p \) is best possible and it is not achieved by any function in \( W^{1,p}_0(0, +\infty) \). The inequality (1) has been generalized to higher dimensions in two directions: one is to replace the function \( t \) in the right-hand side by the distance to the origin, and the other is to replace it by the distance to the boundary.

For the former direction, let \( \Omega \) be a domain with \( 0 \in \Omega \) in \( \mathbb{R}^N \) (\( N \geq 2 \)) and let \( p \geq 1 \). Then the classical \( L^p \)-Hardy inequality states that
\[
\int_{\Omega} |\nabla u|^p \, dx \geq \left( \frac{N-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \, dx
\]
holds for all \( u \in W^{1,p}_0(\Omega) \) when \( 1 \leq p < N \), and for all \( u \in W^{1,p}_0(\Omega \setminus \{0\}) \) when \( p > N \). It is known that for \( p > 1 \), the best constant \( \left( \frac{N-p}{p} \right)^p \) is never attained in \( W^{1,p}_0(\Omega) \) when \( p < N \), or in \( W^{1,p}_0(\Omega \setminus \{0\}) \) when \( p > N \), respectively. After the pioneering work of Brezis and Vázquez [7], which showed that the inequality can be improved on bounded domains when \( p < N \), there are many papers that treat the improvements of the inequality (2) (see [1], [4], [5], [8], [12], [13], [14], [20], the recent book [15] and the reference therein.)

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For the latter direction, let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary and define $d(x) = \text{dist}(x, \partial \Omega)$. Then, a version of Hardy inequalities, called “geometric type”, states that for any $p > 1$, there exists $c_p(\Omega) > 0$ such that the inequality

$$\int_{\Omega} |\nabla u|^p \, dx \geq c_p(\Omega) \int_{\Omega} \frac{|u|^p}{(d(x))^p} \, dx$$

holds for all $u \in W_0^{1,p}(\Omega)$. For this inequality, refer to [2], [4], [6], [12], [19], [21], [26], [32], [33], the recent book [3] and the references therein. In [26], it is proved that $c_p(\Omega) = \left( \frac{p-1}{p} \right)^p$ is the best constant on any convex domain $\Omega$, that is,

$$c_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} \frac{|u(x)|^p}{(d(x))^p} \, dx} = \left( \frac{p-1}{p} \right)^p.$$

In [4], [33], the authors obtained an additional extra term on the right-hand side of (3), which means that the best constant $c_p(\Omega)$ is never attained on any convex domain $\Omega$. When $\Omega$ is the half-space $\mathbb{R}^N_+ = \{ x = (x_1, \cdots, x_N) | x_N > 0 \}$, the inequality (3) has the form

$$\int_{\mathbb{R}^N_+} |\nabla u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{\mathbb{R}^N_+} \frac{|u|^p}{x_N^p} \, dx$$

and the best constant $\left( \frac{p-1}{p} \right)^p$ is never attained by functions in $W_0^{1,p}(\mathbb{R}^N_+)$. On the other hand, let $\Omega$ be a bounded domain with $C^{1,\gamma}$ boundary for some $\gamma \in (0,1)$. Then it is proved by Marcus, Mizel, and Pinchover in [24] that there exists a minimizer of $C_2(\Omega)$ if and only if $C_2(\Omega) < 1/4$. See also [21], [25], [19] for the corresponding results for $1 < p < \infty$. So the compactness of any minimizing sequence fails only at the bottom level $\left( \frac{p-1}{p} \right)^p$.

In the critical case $p = N$, the weight $|x|^{-N}$ is too singular for the same type of inequality as (2) to hold true for functions in $W_0^{1,N}(\Omega)$. Instead of (2), it is known that the following Hardy inequality in a limiting case

$$\int_{\Omega} |\nabla u|^N \, dx \geq \left( \frac{N-1}{N} \right)^N \int_{\Omega} \frac{|u|^N}{|x|^N (\log \frac{R}{|x|})^N} \, dx$$

holds for all $u \in W_0^{1,N}(\Omega)$ where $R = \sup_{x \in \Omega} |x|$; refer to [22], [20], [11], [16], [31] and references therein. Note that the additional log term weakens the singularity of $|x|^{-N}$ at the origin, however, the weight function

$$W_R(x) = \frac{1}{|x|^N \left( \log \frac{R}{|x|} \right)^N}$$

becomes singular also on the boundary $\partial \Omega$ since $R = \sup_{x \in \Omega} |x|$. Indeed, since

$$|x|^N \left( \log \frac{R}{|x|} \right)^N = (R - |x|)^N + o((R - |x|)^N)$$

as $|x| \to R$, $W_R$ has a similar effect of $(1/d(x))^N$ near the boundary. In this sense, the critical Hardy inequality (4) has both features of the inequalities (2) and (3).
Note that (6) is invariant under the scaling
\begin{equation}
    u_\lambda(x) = \lambda^{\frac{N-1}{N}} u \left( \frac{|x|}{R} \right)^{\lambda - 1} x
\end{equation}
for \( \lambda > 0 \),
which is different from the usual scaling \( u_\lambda(x) = \lambda^{\frac{N-p}{p}} u(\lambda x) \) for (2) when \( \Omega = \mathbb{R}^N \) and \( p < N \). (However recently, a relation of both scaling transformations is obtained, see [30]).

Let \( C_N(\Omega) \) be the best constant of the inequality (6):
\begin{equation}
    C_N(\Omega) = \inf_{u \in W_0^{1,N}(\Omega), u \not\equiv 0} \frac{\int_{\Omega} |\nabla u|^N \, dx}{\int_{\Omega} |u(x)|^N (\log \frac{R}{|x|})^\lambda \, dx}.
\end{equation}
By this definition and (6), we see \( C_N(\Omega) \geq \left( \frac{N-1}{N} \right)^N \) for any bounded domain \( \Omega \subset B_R \) with \( R = \sup_{x \in \Omega} |x| \). Here and henceforth, \( B_R \) will denote the \( N \)-dimensional ball with radius \( R \) and center 0.

In [10], the authors proved that \( C_N(\Omega) = \left( \frac{N-1}{N} \right)^N \) and \( C_N(\Omega) \) is never attained by any function in \( W_0^{1,N}(B_R) \). See also [10], [11]. Let us recall the arguments in [10]. First, the authors of [10] prove that, if the infimum \( C_N(\Omega) \) is attained by a radially symmetric function \( u \in W_0^{1,N}(B_R) \), then \( u \in C^1(B_R \setminus \{0\}) \), \( u > 0 \) and \( u \) is unique up to multiplication of positive constants. By using these facts and the scaling invariance (5), the authors prove that \( C_N(\Omega) \) is not attained by radially symmetric functions. Indeed, by the scaling invariance (5) and the uniqueness up to multiplication of positive constants, the possible radially symmetric minimizer has the form \( C(\log \frac{R}{|x|})^{\frac{N-1}{N}} \) which is not in \( W_0^{1,N}(B_R) \). Finally, they prove that if there exists a minimizer of \( C_N(\Omega) \), then there exists also a radially symmetric minimizer. The argument of this part is elementary and the proof of the non-attainability of \( C_N(\Omega) \) is established.

The main purpose of this paper is to study the (non-)attainability of the infimum \( C_N(\Omega) \) for more general domains \( \Omega \subset B_R \). Some new phenomena will be shown in this paper. We first note that if \( C_N(\Omega) = \left( \frac{N-1}{N} \right)^N \), \( C_N(\Omega) \) is not attained. In fact, if \( C_N(\Omega) \) is attained by an element \( u \in W_0^{1,N}(\Omega) \), by a trivial extension of \( u \) as an element in \( W_0^{1,N}(B_R) \), \( C_N(\Omega) = \left( \frac{N-1}{N} \right)^N \) is attained by \( u \); this contradicts the result in [10] that \( C_N(\Omega) \) is not attained. In the following, we may not impose the assumption that \( 0 \in \Omega \). Since the weight function \( W_R(x) = (|x| (\log \frac{R}{|x|}))^{-\lambda} \) itself depends on the geometric quantity \( R \), it is not clear whether \( C_N(\Omega) \) has the same value as \( C_N(B_R) \) for all domains \( \Omega \subset B_R \) or not. Since \( W_R \) becomes unbounded around the origin and also around the set \( |x| = R \), it is plausible that minimizing sequences for \( C_N(\Omega) \) tend to concentrate on the origin or on the boundary portion \( \partial \Omega \cap \partial B_R \) in order to minimize the quotient
\[
    Q_R(u) = \frac{\int_{\Omega} |\nabla u|^N \, dx}{\int_{\Omega} W_R(x)|u(x)|^N \, dx}.
\]
This will result in that \( C_N(\Omega) = C_N(B_R) \) and \( C_N(\Omega) \) is not attained, if the origin is the interior point of \( \Omega \), or \( \Omega \) has a smooth boundary portion at a distance \( R \) to the origin (just like a ball \( B_R \)). We will prove later that these intuitions are true, see Theorem 1 and Theorem 2. However, when we treat a domain \( \Omega \subset B_R \) with \( R = \sup_{x \in \Omega} |x| \), which does not contain the origin in its interior, nor have
the smooth boundary portion \( \partial \Omega \cap B_R \), the situation is rather different. Actually, we provide a sufficient condition on \( \Omega \subset B_R \) which assures that \( C_N(\Omega) > C_N(B_R) \) (Theorem 4). Moreover, we prove that a stronger condition on \( \Omega \) than the sufficient condition assures that \( C_N(\Omega) \) is attained (Theorem 5). Finally, we provide an example of domain in \( \mathbb{R}^2 \) on which \( C_2(\Omega) > C_2(B_R) = 1/4 \) and \( C_2(\Omega) \) is not attained (Theorem 6). This is quite a contrast to the result for (4) in [24], which says that if \( C_2(\Omega) \) is strictly less than the critical number \( 1/4 \), the infimum \( c_2(\Omega) \) is attained.

The organization of this paper is as follows: In §2, we prove Theorem 1 which says that if \( 0 \in \Omega \), then \( C_N(\Omega) = \left( \frac{N-1}{N} \right)^N \) and the infimum is not attained. In §3, we prove Theorem 2 which says that if \( \partial B_R \cap \partial \Omega \) enjoys some regularity, then \( C_N(\Omega) = \left( \frac{N-1}{N} \right)^N \) and the infimum is not attained. In §4, we prove Theorem 3 which says that a strict inequality \( C_N(\Omega) > \left( \frac{N-1}{N} \right)^N \) holds under some condition on \( \Omega \) and Theorem 4 which says that under a stronger condition than the one in Theorem 3 the infimum is attained. Finally in §6, we prove Theorem 5 which says that the condition for the existence of a minimizer in Theorem 5 is optimal.

Now, we fix some notations and usages. For a bounded domain \( \Omega \subset \mathbb{R}^N \), the letter \( R \) will be used to denote \( R = \sup_{x \in \Omega} |x| \) throughout the paper. \( B_R \) will denote the \( N \)-dimensional ball with radius \( R \) and center 0. The surface area \( \int_{S^{N-1}} dS_x \) of the \((N-1)\) dimensional unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \) will be denoted by \( \omega_{N-1} \). \( S^{N-1}(r) \) will denote the sphere of radius \( r \) with center 0. Finally, the letter \( C \) may vary from line to line.

2. Hardy’s inequality for the case \( 0 \in \Omega \)

In this section, we treat the case when \( \Omega \subset B_R \) has the origin as an interior point of \( \Omega \). In this case, we prove the following theorem.

**Theorem 1.** For any bounded domain \( \Omega \subset \mathbb{R}^N \) with \( 0 \in \Omega \) and \( R = \sup_{x \in \Omega} |x| \),

\[
C_N(\Omega) = C_N(B_R) = \left( \frac{N-1}{N} \right)^N,
\]

and the infimum \( C_N(\Omega) \) is not attained.

**Proof.** Note that by the definition of \( R \), we have \( \Omega \subset B_R \). By a trivial extension of a function \( u \in W_0^{1,N}(\Omega) \) on \( B_R \) by \( u(x) = 0 \) for \( x \in B_R \setminus \Omega \), we see \( W_0^{1,N}(\Omega) \subset W_0^{1,N}(B_R) \) and thus

\[
C_N(\Omega) \geq C_N(B_R) = \left( \frac{N-1}{N} \right)^N.
\]

For the fact \( C_N(B_R) = \left( \frac{N-1}{N} \right)^N \), we refer to [16]. In [16], the authors prove this fact by using the test functions

\[
\psi_\beta(x) = \begin{cases} 
1, & 0 \leq |x| \leq \frac{R}{2}, \\
\left( \log \frac{R}{|x|} \right)^\beta, & \frac{R}{e} \leq |x| \leq R
\end{cases}
\]

for \( \beta > \frac{N-1}{N} \). Note that \( \{ \psi_\beta \} \) will concentrate on the boundary \( \partial B_R \) when \( \beta \downarrow \frac{N-1}{N} \). In our case, since \( 0 \in \Omega \) is an interior point, there exists a small \( c \in (0,1) \) such that


Then we see that origin and \( \phi \) exterior sphere condition is enough to obtain the result. Here we say that a point such that \( x \)

This proves that

\[
A \equiv \int_\Omega |\nabla \phi_\alpha|^N \, dx = \omega_{N-1} \int_0^{\frac{\pi}{2}} \left( \log \frac{R}{r} \right)^{\alpha - 1} \left( \frac{-1}{r} \right)^N r^{N-1} \, dr + O(1)
\]

\[
= \omega_{N-1} \alpha^N \int_0^{\frac{\pi}{2}} \left( \log \frac{R}{r} \right)^{N(\alpha - 1) + 1} \frac{1}{r} \, dr + O(1)
\]

\[
= \omega_{N-1} \alpha^N \left[ \frac{-1}{N(\alpha - 1) + 1} \left( \log \frac{R}{r} \right)^{N(\alpha - 1) + 1} \right]_0^{\frac{\pi}{2}} + O(1)
\]

\[
= \omega_{N-1} \alpha^N \left( \frac{-1}{N(\alpha - 1) + 1} \right) \log \frac{2}{c} + O(1).
\]

Since \( \alpha < \frac{N-1}{N} \), we have \( N(\alpha - 1) + 1 < 0 \). Thus \( |\nabla \phi_\alpha|^N \) is integrable near the origin and \( \phi_\alpha \in W^{1,N}_0(\Omega) \) for any \( \alpha \in (0, \frac{N-1}{N}) \). Also we see that

\[
B \equiv \int_\Omega \frac{|\phi_\alpha(x)|^N}{|x|^N} \left( \log \frac{R}{|x|} \right)^N r^{N-1} \, dr + O(1)
\]

\[
= \frac{\omega_{N-1}}{\omega_{N-1}} \int_0^{\frac{\pi}{2}} \left( \log \frac{R}{r} \right)^N \frac{1}{r} \, dr + O(1)
\]

\[
= \omega_{N-1} \left( \frac{-1}{N(\alpha - 1) + 1} \right) \log \frac{2}{c} + O(1).
\]

Therefore, we conclude that

\[
\frac{A}{B} = \frac{\omega_{N-1} \alpha^N}{\omega_{N-1} \frac{1}{N(\alpha - 1) + 1}} \log \frac{2}{c} + O(1) = \frac{\alpha^N + O(1)(N(\alpha - 1) + 1) + O(1)(N(\alpha - 1) + 1)}{1 + O(1)(N(\alpha - 1) + 1)}
\]

\[
\rightarrow \left( \frac{N-1}{N} \right)^N \quad \text{as } \alpha \uparrow \frac{N-1}{N}.
\]

This proves that

\[
C_N(\Omega) = \left( \frac{N-1}{N} \right)^N,
\]

thus the infimum \( C_N(\Omega) \) is not attained; see Introduction. \( \square \)

### 3. Hardy’s Inequality for Smooth Domains

In this section, we prove that \( C_N(\Omega) \) equals to \( \left( \frac{N-1}{N} \right)^N \) if the domain has a smooth boundary portion on \( \partial B_R \). For the smoothness on the boundary, the interior sphere condition is enough to obtain the result. Here we say that a point \( x_0 \in \partial \Omega \cap \partial B_R \) satisfies an interior sphere condition if there is an open ball \( B \subset \Omega \) such that \( x_0 \in \partial B \). The idea here is to construct a (non-convergent) minimizing
sequence $\{u_n\}$ for $C_N(\Omega)$ for which the value of $Q_R(u_n)$ goes to $(\frac{N-1}{N})^N$, by modifying a minimizing sequence for the best constant of Hardy’s inequality on the half-space [3] when $p = N$:

\[
\inf_{u \in C_N^0(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} |\nabla u|^N dx = (\frac{N-1}{N})^N.
\]

This is possible since the weight function $W_R(x)$ can be considered as $(1/d(x))^N$ near the smooth boundary portion $\partial \Omega \cap \partial B_R$.

**Theorem 2.** For a bounded domain $\Omega$, we assume that there exists a point $x_0 \in \partial \Omega \cap \partial B_R$ satisfying an interior sphere condition. Then

\[
C_N(\Omega) = \left( \frac{N-1}{N} \right)^N
\]

and the infimum $C_N(\Omega)$ is not attained.

**Proof.** The following proof is inspired by [24]. We write $x = (x_1, \cdots, x_{N-1}, x_N) = (x', x_N)$ for $x \in \mathbb{R}^N_+$. Fix $\varepsilon > 0$ arbitrary. By (11), we may take $v_\varepsilon \in C_0^\infty(\mathbb{R}^N_+)$ such that

\[
\int_{\mathbb{R}^N_+} |v_\varepsilon|^N dx = 1, \quad \text{and} \quad \int_{\mathbb{R}^N_+} |\nabla v_\varepsilon|^N dx \leq \left( \frac{N-1}{N} \right)^N + \varepsilon.
\]

Since $\text{supp}(v_\varepsilon)$ is compact, we may assume that

$\text{supp}(v_\varepsilon) \subset \{x = (x', x_N) \in \mathbb{R}^N_+ \mid |x'|^2 < Ax_N, \ x_N < B\}$

if we take $A, B > 0$ sufficiently large depending on $\varepsilon$. We think $v_\varepsilon$ is 0 outside of its support and is defined on the whole $\mathbb{R}^N_+$. For $l \in \mathbb{N}$, we define $v_\varepsilon^l(x) = v_\varepsilon(lx)$. Note that for each $l > 0$, we have

\[
\int_{\mathbb{R}^N_+} |\nabla v_\varepsilon^l|^N dx = \int_{\mathbb{R}^N_+} |\nabla v_\varepsilon|^N dx, \quad \int_{\mathbb{R}^N_+} \left| \frac{v_\varepsilon^l}{x_N} \right|^N dx = \int_{\mathbb{R}^N_+} \left| \frac{v_\varepsilon}{x_N} \right|^N dx
\]

and

$\text{supp}(v_\varepsilon^l) \subset \left\{ (x', x_N) \in \mathbb{R}^N_+ \mid |x'|^2 < A^l x_N, \ x_N < B^l \right\}$.

By a rotation, we may assume that $x_0 = (-R)e_N \in \partial \Omega \cap \partial B_R$ satisfies an interior sphere condition, where $e_N = (0, \cdots, 0, 1)$. Then we see that for some $A', B' > 0$,

$\{ (x', x_N) \in \mathbb{R}^N_+ \mid |x'|^2 < A' x_N, \ x_N < B' \} \subset \Omega + Re_N$

Since (7) holds for small $R - |x|$, we see that

\[
|x|^N \left( \log \frac{R}{|x|} \right)^N \leq (x_N + R)^N + o((x_N + R)^N)
\]

for $x \in \Omega$ with small $x_N + R$. Now we define

$u_\varepsilon^l(x) \equiv v_\varepsilon^l(x + Re_N)$

for $x \in \Omega$. Then, for large $l > 0$, we see that $u_\varepsilon^l \in C_0^\infty(\Omega)$ and

$\text{supp}(u_\varepsilon^l) \subset \Omega \cap \{ x \in B_R \mid x_N + R < B/l \}.$
Now \[12\] implies that

\[
\int_{\Omega} \frac{|u^l_*(x)|^N}{|x|^N \left( \log \frac{R}{|x|} \right)^N} \, dx \geq \int_{\Omega} \frac{|u^l_*(x)|^N}{(x_N + R)^N} \, dx + o_l(1) = \int_{\Omega + R \in N} \frac{|v^l_*(y)|^N}{|y|^N} \, dy + o_l(1)
\]

where \(o_l(1) \to 0\) as \(l \to \infty\), and

\[
\int_{\Omega} |\nabla u^l_*(x)|^N \, dx = \int_{\Omega + R \in N} |\nabla v^l_*(y)|^N \, dy \leq \int_{\Omega + R \in N} |\nabla v^l_*(y)|^N \, dy.
\]

Thus we have

\[
\frac{\int_{\Omega} |\nabla u^l_*(x)|^N \, dx}{\int_{\Omega} \frac{|u^l_*(x)|^N}{|x|^N \left( \log \frac{R}{|x|} \right)^N} \, dx} \leq \frac{\int_{\Omega + R \in N} |\nabla v^l_*(y)|^N \, dy}{\int_{\Omega + R \in N} \frac{|v^l_*(y)|^N}{|y|^N} \, dy} + o_l(1) \leq \left( \frac{N - 1}{N} \right)^N + \varepsilon + o_l(1).
\]

This implies that

\[
\inf_{u \in W_{0, N}^{1, N}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^N \, dx}{\int_{\Omega} \frac{|u(x)|^N}{|x|^N \left( \log \frac{R}{|x|} \right)^N} \, dx} \leq \left( \frac{N - 1}{N} \right)^N.
\]

Since \(C_N(\Omega) \geq C_N(B_R) = \left( \frac{N - 1}{N} \right)^N\) by \([10]\), we conclude the equality. This again implies that the infimum \(C_N(\Omega)\) is not attained.

\[\square\]

4. Hardy’s inequality for nonsmooth domains

In this section, first we provide a sufficient condition to assure the strict inequality \(C_N(\Omega) > C_N(B_R)\) for bounded domains \(\Omega\) with \(R = \sup_{x \in \Omega} |x|\).

First, we recall the notion of spherical symmetric rearrangement. Let \(B_r(p, s)\) denote the geodesic open ball in \(S^{N-1}(r)\) with center \(p \in S^{N-1}(r)\) and geodesic radius \(s\). Then for each \(r \in (0, R)\), there exists a constant \(a(r) \geq 0\) such that the \((N - 1)\)-dimensional measure of the geodesic open ball \(B_r(re_N, a(r))\) with center \(re_N = (0, \ldots, 0, r)\) and radius \(a(r)\) equals to \(\mathcal{H}^{N-1}(\Omega \cap S^{N-1}(r))\), here \(\mathcal{H}^{N-1}\) denotes the \((N - 1)\)-dimensional Hausdorff measure. Define the spherical symmetric rearrangement \(\Omega^*\) of a domain \(\Omega \subset B_R\) by

\[
\Omega^* \equiv \bigcup_{r \in (0, R)} B_r(re_N, a(r))
\]

and the spherical symmetric rearrangement \(u^*\) of a function \(u\) on \(\Omega\) by

\[
u^*(x) \equiv \sup\{t \in \mathbb{R} \mid x \in \{x \in \Omega \mid u(x) \geq t\}^*\}, \quad x \in \Omega^*,
\]

see Kawohl \([17]\) p.17. Note that this is an equimeasurable rearrangement with \(u^*\) rotationally symmetric around the positive \(x_N\)-axis, and there hold that the Polya-Szegö type inequality

\[
\int_{\Omega} |\nabla u|^p \, dx \geq \int_{\Omega^*} |\nabla u^*|^p \, dx
\]

for \(u \in W_{0, N}^{1, p}(\Omega)\) with \(p > 1\), and the Hardy-Littlewood inequality

\[
\int_{\Omega} u(x)v(x) \, dx \leq \int_{\Omega^*} u^*(x)v^*(x) \, dx
\]

for nonnegative functions \(u, v\) on \(\Omega\), see \([17]\) pages 21, 23, and 26.
In the sequel, we use the Poincaré inequality on a subdomain of spheres of the following form:

**Proposition 3.** Let $S^n$ denote an $n$-dimensional unit sphere and $U \subset S^n$ be a relatively compact open set in $S^n$. For any $1 \leq p < \infty$, there exists $C > 0$ depending on $p$ and $n$ such that the inequality

$$
\int_U |\nabla S^n u|^p dS_\omega \geq C|U|^{-p/n} \int_U |u|^p dS_\omega
$$

holds for any $u \in W^{1,p}_0(U)$. Here $|U|$ denotes the $n$-dimensional measure of $U \subset S^n$.

**Proof.** The inequality $\int_U |\nabla S^n u|^p dS_\omega \geq C(U,p) \int_U |u|^p dS_\omega$ holds, see for example, [29] pp. 86. The constant $C(U,p)$ is bounded from below by the first Dirichlet eigenvalue $\lambda_p(U)$ of the $p$-Laplacian $-\Delta_p$ on the sphere, and the estimate

$$
\lambda_p(U) \geq C(n,p) |U|^{-p/n}
$$

can be seen, for example, in [23] or [18] when the ambient space is $\mathbb{R}^n$. Indeed, the lower bound of the first Dirichlet eigenvalue is also obtained on spheres. By spherically symmetric rearrangement, we have the Faber-Krahn type inequality

$$
\lambda_p(U) \geq \lambda_p(U^*)
$$

where $U^* \subset S^n$ be a geodesic ball with $|U| = |U^*|$. Also we have a scaling property $\lambda_p(rU) = r^{-p} \lambda_p(U)$ for the first eigenvalue of the $p$-Laplacian. Since $U^* = rB_1$ for some $r > 0$ where $B_1$ denotes the geodesic ball of radius 1, we have $|U| = |U^*| = r^n |B_1|$, which implies $r = (|U|/|B_1|)^{1/n}$. Thus we have

$$
\lambda_p(U) \geq \lambda_p(U^*) = \lambda_p(rB_1) = r^{-p} \lambda_p(B_1) = \left(\frac{|U|}{|B_1|}\right)^{-p/n} |B_1|.
$$

Define

$$
m(r) = \mathcal{H}^{N-1}\left(\{x \in \Omega \mid |x| = r\}\right) = \mathcal{H}^{N-1}(\Omega \cap S^{N-1}(r))
$$

for $r \in (0, R)$. Then we have the following.

**Theorem 4.** If

$$
m_0 \equiv \limsup_{r \to 0} m(r)/r^{N-1} < \omega_{N-1}
$$

and

$$
m_R \equiv \limsup_{r \to R} m(r)/(R - r)^{N-1} < \infty,
$$

it holds that

$$
C_N(\Omega) > \left(\frac{N - 1}{N}\right)^N.
$$

**Proof.** If $0 \in \Omega$, then $m(r) = r^{N-1}\omega_{N-1}$ for any small $r > 0$. Thus under the assumption (14), the origin must not be interior of $\Omega$.

We assume the contrary and suppose that there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in $C_0^\infty(\Omega) \setminus \{0\}$ such that

$$
\lim_{n \to \infty} \frac{\int_\Omega |\nabla \phi_n|^N dx}{\int_\Omega \frac{\phi_n(x)^N}{|x|^N (\log |x| + 1)^N} dx} = C_N(\Omega) = \left(\frac{N - 1}{N}\right)^N.
$$
Let $\phi_n^*$ be the spherical symmetric rearrangement of $\phi_n$. Then by the above remarks, it follows that
\[
\lim_{n \to \infty} \frac{\int_{\Omega^*} |\nabla \phi_n^*|^N dx}{\int_{\Omega^*} |\phi_n^*(x)|^N |x|^{N-1} dx} = C_N(\Omega^*) = \left(\frac{N-1}{N}\right)^N.
\]
Since $\text{supp}(\phi_n^*)$ is compact in $\Omega^*$, we find positive constants $R_n$ and $\delta_n$ with $\lim_{n \to \infty} R_n = R$ and $\lim_{n \to \infty} \delta_n = 0$ such that $\text{supp}(\phi_n^*) \subset B_{R_n} \setminus \overline{B_{\delta_n}}$. We define
\[
\Omega_n^* = \Omega^* \cap (B_{R_n} \setminus \overline{B_{\delta_n}}).
\]
Since the weight function $W_R$ is bounded from above and below by positive constants on $\Omega_n^*$, there exists a minimizer $\psi_n \in W^{1,N}_0(\Omega_n^*)$ of
\[
c_n \equiv \inf \left\{ \int_{\Omega_n^*} |\nabla \psi_n|^N dx \mid \int_{\Omega_n^*} |\psi(x)|^N|\log \frac{R}{|x|}|^N dx = 1, \psi \in W^{1,N}_0(\Omega_n^*) \right\}.
\]
We may assume $\psi_n \geq 0$, $\psi_n$ satisfies
\[
div(|\nabla \psi_n|^N \nabla \psi_n) + c_n \frac{\psi_n(x)^{N-1}}{|x|^N |\log \frac{R}{|x|}|^N} = 0 \quad \text{in } \Omega_n^*,
\]
and $\psi_n$ is rotationally symmetric with respect to $x_N$-axis. We think that $\psi_n$ is defined on $\Omega^*$ by extending by zero. Then we see
\[
\int_{\Omega^*} |\nabla \psi_n|^N dx = c_n \to \left(\frac{N-1}{N}\right)^N
\]
as $n \to \infty$. Since $(\frac{N-1}{N})^N$ is not attained by any element in $W^{1,N}_0(\Omega^*)$, elliptic estimates imply that for any small $R' > 0$ and any $R' \Subset R$ sufficiently close to $R$, $\psi_n$ converges uniformly to 0 on $\Omega^* \cap (B_R \setminus \overline{B_{R'}})$ and $\psi_n$ converges weakly to 0 in $W^{1,N}_0(\Omega^*)$ as $n \to \infty$. We denote
\[
\Omega^*(r) \equiv \{ \omega \in S^{N-1} \mid r\omega \in \Omega^* \} \subset S^{N-1},
\]
so $m(r) = r^{N-1}H^{N-1}(\Omega^*(r))$. Then we note that
\[
1 = \int_{\Omega^*(r)} \frac{|\psi_n(x)|^N}{|x| |\log \frac{R}{|x|}|^N} dx = \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r\omega)|^N}{r |\log \frac{R}{r}|^N} dS_\omega dr
\]
\[
= \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(x)|^N}{r |\log \frac{R}{r}|^N} dS_\omega dr + \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r\omega)|^N}{r |\log \frac{R}{r}|^N} dS_\omega dr + o_n(1)
\]
as $n \to \infty$.

First, let us assume
\[
\lim_{n \to \infty} \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r\omega)|^N}{r |\log \frac{R}{r}|^N} dS_\omega dr \geq C
\]
for some $C > 0$. Since $m_0 < \omega_{N-1}$ by assumption, $\Omega^*(r)$ is a proper subset of $S^{N-1} \setminus \{ -e_N \} \simeq \mathbb{R}^{N-1}$ for any small $r > 0$. Thus there exists a constant $C > 0$ independent of small $r > 0$ and $n \in \mathbb{N}$ such that the Poincaré inequality in Proposition 3 (with $U = \Omega^*(r)$, $p = N$, $n = N - 1$)
\[
\int_{\Omega^*(r)} |\nabla_{S^{N-1}} \psi_n(r\omega)|^N dS_\omega \geq C \int_{\Omega^*(r)} |\psi_n(r\omega)|^N dS_\omega
\]
holds true. Note that
\[ \nabla \psi_n = \frac{x}{|x|} \frac{\partial \psi_n}{\partial r} + \frac{1}{r} \nabla_{S^{N-1}} \psi_n, \quad |\nabla \psi_n|^N \geq \left| \frac{\partial \psi_n}{\partial r} \right|^N + \frac{1}{r^N} |\nabla_{S^{N-1}} \psi_n|^N. \]

Then for each small \( R' > 0 \), we have
\[ \int_{\Omega^*} |\nabla \psi_n|^N dx = \int_0^R \int_{\Omega^*(r)} \nabla \psi_n(r \omega)^N r^{N-1} dS_dr \]
\[ \geq \int_0^{R'} \int_{\Omega^*(r)} \frac{1}{r^N} |\nabla_{S^{N-1}} \psi_n|^N r^{N-1} dS_dr \]
\[ \geq C \int_0^{R'} \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r} dS_dr \]
by the Poincaré inequality (19). On the other hand, since
\[ \int_0^{R'} \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r} dS_dr \geq \left( \log \frac{R}{R'} \right)^N \int_0^{R'} \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r} dS_dr, \]
we have by (18),
\[ \left( \frac{N-1}{N} \right)^N + o_n(1) = \int_{\Omega^*} |\nabla \psi_n|^N dx \geq \frac{C}{2} \left( \log \frac{R}{R'} \right)^N \]
as \( n \to \infty \). This inequality is invalid if \( R' \) is very small. Thus (18) cannot happen and
\[ \lim_{n \to \infty} \int_0^{R'} \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r} dS_dr = 0 \]
under the assumption (14).
Therefore by (17), we have
\[ \lim_{n \to \infty} \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r \left( \log \frac{R}{r} \right)^N} dS_dr = 1. \]

Next, we will prove that (22) cannot occur under the assumption (15). In fact, we see by (22) and (17) that
\[ 1 + o_n(1) = \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{r \left( \log \frac{R}{r} \right)^N} r^{N-1} dS_dr \]
\[ = (1 + o(1)) R^{N-1} \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{(R-r)^N} dS_dr, \]
where \( o_n(1) \to 0 \) as \( n \to \infty \) and \( o(1) \to 0 \) as \( \tilde{R} \to R \). Thus we have
\[ \lim_{n \to \infty} \int_0^R \int_{\Omega^*(r)} \frac{|\psi_n(r \omega)|^N}{(R-r)^N} dS_dr = (1 + o(1)) R^{-(N-1)} \]
as $\tilde{R} \to R$. On the other hand, since $\psi_n(r\omega)|_{r=\tilde{R}} = 0$, we can apply the one-dimensional Hardy inequality

$$
(24) \quad \left( \frac{N-1}{N} \right)^N \int_{\tilde{R}}^R \frac{|\psi_n(r\omega)|}{(r-R)^N} dr \leq \int_{\tilde{R}}^R \frac{|\partial \psi_n(r\omega)|}{\partial r} |d r|
$$

to $\psi_n(r\omega)$. Note that the best constant $\left( \frac{N-1}{N} \right)^N$ in the inequality $24$ is the same as, by assumption, the value of $C_N(\Omega^*)$. Then $24$ implies

$$
\left( \frac{N-1}{N} \right)^N \int_{\tilde{R}}^R \int_{\Omega^*(r)} |\psi_n(r\omega)|^N (r-R)^N dS_\omega dr \leq \int_{\tilde{R}}^R \int_{\Omega^*(r)} \left| \frac{\partial \psi_n(r\omega)}{\partial r} \right|^N dS_\omega dr = (1 + o(1)) R^{-(N-1)} \int_{\Omega^*} \left| \frac{\partial \psi_n}{\partial r} (x) \right| dx.
$$

The above inequality, $23$ and $C_N(\Omega^*) = \left( \frac{N-1}{N} \right)^N = \lim_{n \to \infty} \int_{\Omega^*} |\nabla \psi_n (x)|^N dx$ by 16 imply that

$$
\lim_{n \to \infty} \int_{\Omega^*} |\nabla \psi_n|^N dx \leq \lim_{n \to \infty} \int_{\Omega^*} \left| \frac{\partial \psi_n}{\partial r} (x) \right|^N dx.
$$

The converse inequality holds trivially, thus we see that

$$
\lim_{n \to \infty} \int_{\Omega^*} |\nabla \psi_n|^N dx = \lim_{n \to \infty} \int_{\Omega^*} \left| \frac{\partial \psi_n}{\partial r} \right|^N dx,
$$

which implies

$$
(25) \quad \lim_{n \to \infty} \int_{r \Omega^*} |\nabla_{S^{N-1}(r)} \psi_n(\sigma)|^N d\sigma_r = 0,
$$

here $\sigma = r \omega \in S^{N-1}(r)$, $d\sigma_r = r^{N-1} dS_\omega$ is a volume element of a geodesic ball $r \Omega^*(r)$ with center $Rr$ in $S^{N-1}(r)$, and $\nabla_{S^{N-1}(r)} = (1/r) \nabla_{S^{N-1}}$.

From the assumption $m_R < \infty$ in 15, there exists a constant $C > 0$ independent of $r \in (\tilde{R}, R)$ and $n$ such that

$$
r^{N-1} H^{N-1}(\Omega^*(r)) \leq C (R-r)^{N-1}
$$

holds true. This implies that

$$
(\mathcal{H}^{N-1}(r \Omega^*(r)))^{-N/(N-1)} \geq D (R-r)^{-N},
$$

where $D = C^{-N/(N-1)} > 0$ independent of $r \in (\tilde{R}, R)$ and $n$. Then, by the Poincaré inequality in Proposition 3 ($n = N-1$, $p = N$) on the spherical cap $U = r \Omega^*(r) \subset S^{N-1}(r)$,

$$
(26) \quad \int_{r \Omega^*} |\nabla_{S^{N-1}(r)} \psi_n(\sigma)|^N d\sigma_r \geq D \int_{r \Omega^*} \left| \psi_n(\sigma) \right|^N d\sigma_r dr
$$

holds true. Combining $25$ and $26$, we have

$$
a_n(1) = \int_{\tilde{R}}^R \int_{r \Omega^*(r)} |\nabla_{S^{N-1}(r)} \psi_n(\sigma)|^N d\sigma_r dr \geq D \int_{\tilde{R}}^R \int_{r \Omega^*(r)} \left| \psi_n(\sigma) \right|^N d\sigma_r dr
$$

$$
= (1 + o(1)) R^{N-1} \int_{r \Omega^*(r)} \frac{|\psi_n(r\omega)|}{|r-R|^N} dS_\omega dr
$$
where \( o_n(1) \to 0 \) as \( n \to \infty \) and \( o(1) \to 0 \) as \( \tilde{R} \to R \). Combining this to (23) and letting \( n \to \infty \), we see
\[
0 = D(1 + o(1))R^{N-1} \times (1 + o(1))R^{-(N-1)} = D + o(1)
\] as \( \tilde{R} \to R \). This is a contradiction and we complete the proof.

Next, we prove that a condition on \( \Omega \) stronger than that of in Theorem 4 assures the attainability of \( C_N(\Omega) \). The condition below implies that the boundary point \( x \in \partial B_R \cap \partial \Omega \), if it existed, must be cuspidal, but the origin, if \( 0 \in \partial \Omega \), may be a Lipschitz continuous boundary point.

**Theorem 5.** For \( r \in (0, R) \), let \( m(r) \) be defined as (13). If
\[
m_0 \equiv \limsup_{r \to 0} m(r)/r^{N-1} < \omega_{N-1}
\]
and
\[
m_R \equiv \limsup_{r \to R} m(r)/(R-r)^{N-1} = 0,
\]
then
\[
C_N(\Omega) > \left( \frac{N-1}{N} \right)^N
\]
and \( C_N(\Omega) \) is attained.

**Proof.** The strict inequality \( C_N(\Omega) > (\frac{N-1}{N})^N \) was proved in Theorem 4.

For each positive integer \( n \), we define
\[
\Omega_n \equiv \Omega \cap (B_{R-1/n} \setminus B_{1/n}).
\]
Then, since the weight function \( W_R(x) \) is bounded on \( \Omega_n \), there exists a minimizer \( \psi_n \) of
\[
d_n \equiv \inf \left\{ \int_{\Omega_n} |\nabla \psi|^N \ dx \mid \int_{\Omega_n} \frac{|\psi(x)|^N}{|x|^N \left( \log \frac{R}{|x|} \right)^N} \ dx = 1, \psi \in W_0^{1,N}(\Omega_n) \right\}.
\]

We may assume \( \psi_n \geq 0 \) and \( \psi_n \) satisfies
\[
\text{div}(|\nabla \psi_n|^{N-2} \nabla \psi_n) + d_n \frac{\psi_n(x)^{N-1}}{|x|^N \left( \log \frac{R}{|x|} \right)^N} = 0 \quad \text{in} \quad \Omega_n.
\]

We note that
\[
\int_{\Omega_n} |\nabla \psi_n|^N \ dx = d_n \to C_N(\Omega) \quad \text{as} \quad n \to \infty.
\]

Let \( u \) be a weak limit of the sequence \( \{\psi_n\}_{n \in \mathbb{N}} \) in \( W_0^{1,N}(\Omega) \). Then, we see that for each positive integer \( n_0 \), \( \psi_n \) converges uniformly to \( u \) in \( C^1(\Omega_{n_0}) \), and that
\[
\text{div}(|\nabla u|^{N-2} \nabla u) + C_N(\Omega) \frac{|u(x)|^{N-1}}{|x|^N \left( \log \frac{R}{|x|} \right)^N} = 0, \quad u \geq 0 \quad \text{in} \ \Omega.
\]

Now it suffices to prove that \( u \neq 0 \) in \( \Omega \), then \( u \) becomes a minimizer for \( C_N(\Omega) \).

To the contrary, we assume that \( u \equiv 0 \). Then, we see that for each positive integer \( n_0 \), \( \psi_n \) converges uniformly to \( 0 \) on \( \Omega_{n_0} \). We denote
\[
\Omega(r) \equiv \{ \omega \in S^{N-1} \mid r\omega \in \Omega \} \subset S^{N-1}.
\]
Since $m_0 < \omega_{N-1}$, by the spherical symmetric rearrangement, Polyá-Szegő and the Poincaré inequality, we see there exists a constant $C > 0$, independent of small $r > 0$ and $n \in \mathbb{N}$, such that

$$\int_{\Omega(r)} |\nabla S^{N-1} \psi_n |^N dS_\omega \geq C \int_{\Omega(r)} |\psi_n|^N dS_\omega,$$

see the proof of Theorem 4.1. Then, we see that for each large positive integer $n_0$,

$$\int_{\Omega} |\nabla \psi_n|^N dx \geq \int_{0}^{1/n_0} \int_{\Omega(r)} |\nabla S^{N-1} \psi_n (r\omega)|^N r^{-1} dS_\omega dr$$

(28) \[ \geq C \int_{0}^{1/n_0} \int_{\Omega(r)} |\psi_n (r\omega)|^N r^{-1} dS_\omega dr. \]

Put $f_n(r) = \int_{\Omega(r)} |\psi_n (r\omega)|^N / r (\log \frac{R}{2})^N dS_\omega$. Then we have

$$1 = \int_{\Omega} \frac{|\psi_n(x)|^N}{|x| \log \frac{R}{|x|}} dx = \int_{0}^{R} \int_{\Omega(r)} \frac{|\psi_n (r\omega)|^N}{r (\log \frac{R}{2})^N} dS_\omega dr$$

$$= \int_{0}^{1/n_0} f_n(r) dr + \int_{1/n_0}^{R-1/n_0} f_n(r) dr + \int_{R-1/n_0}^{R} f_n(r) dr,$$

and that

$$\int_{0}^{1/n_0} \int_{\Omega(r)} \frac{|\psi_n (r\omega)|^N}{r (\log \frac{R}{2})^N} dS_\omega dr \leq \left( \log \frac{R}{1/n_0} \right)^{-N} \int_{0}^{1/n_0} \int_{\Omega(r)} \frac{|\psi_n (r\omega)|^N}{r} dS_\omega dr.$$

Then, (28) implies that for each large positive integer $n_0$,

$$\int_{0}^{1/n_0} \int_{\Omega(r)} \frac{|\psi_n (r\omega)|^N}{r (\log \frac{R}{2})^N} dS_\omega dr \leq \left( \log \frac{R}{1/n_0} \right)^{-N} \frac{d_n}{C}.$$

The right-hand side of the above inequality can be arbitrarily small if $n_0$ large, thus we have $\lim_{n \to \infty} \int_{0}^{1/n_0} f_n (r) dr = 0$. Since $\lim_{n \to \infty} \int_{1/n_0}^{R-1/n_0} f_n (r) dr = 0$, we deduce that for each large positive integer $n_0$,

$$\lim_{n \to \infty} \int_{R-1/n_0}^{R} f_n (r) dr = 1.$$

Now, as in the proof of Theorem 4.1, let $\Omega^*(r) \subset S^{N-1}$ be a geodesic ball with the center $e_N$ such that the $(N-1)$-dimensional measure of $\Omega^*(r)$ equals to that of $\Omega(r)$. Let $\psi_n^*$ be the spherical symmetric rearrangement of $\psi_n$ and put $f_n^*(r) = \int_{\Omega^*(r)} |\psi_n^*(r\omega)|^N r (\log \frac{R}{2})^N dS_\omega$. Since $r \log(R/r) = (R-r) + o(1)$ for small $R - r > 0$, we see that

$$f_n^*(r) = \int_{\Omega(r)} \frac{|\psi_n^*(r\omega)|^N}{r (\log \frac{R}{2})^N} dS_\omega = R^{N-1} \int_{\Omega^*(r)} \frac{|\psi_n^*(r\omega)|^N}{(R-r)^N} dS_\omega + o(1)$$

(29) \[ \text{for small } R - r > 0. \]

On the other hand, by the assumption $m_R = 0$, there exists $h(r) > 0$ with $h(r) \to 0$ as $r \to R$ such that $H^{N-1}(r \Omega^*(r)) \leq h(r)(R-r)^{N-1}$. Thus

$$(H^{N-1}(\Omega^*(r)))_{-N/(N-1)} \geq r^N (h(r))^{-N/(N-1)} (R-r)^{-N}.$$

Here, $\Omega = \{ x \in \mathbb{R}^n : |x| \leq 1 \}$ is the unit ball in $\mathbb{R}^n$, $\psi_n = \sum_{k=1}^{n} |x|^k \phi_n(x)$ is the spherical symmetric rearrangement of $\phi_n$, and $\phi_n = |x|^k \phi_n(x)$ is the spherical symmetric rearrangement of $\phi_n$.
Put \( g(r) = r^N (h(r))^{-N/(N-1)} \). Then \( \lim_{r \to R} g(r) = \infty \) and the Poincaré inequality in Proposition 3 (with \( U = \Omega^* (r) \), \( p = N \), \( n = N - 1 \))

\[
(30) \quad \int_{\Omega^* (r)} |\nabla_S N - 1 \psi_n^*(r \omega)|^N dS_\omega \geq C g(r) \int_{\Omega^* (r)} |\psi_n^*(r \omega)|^N |R - r|^N dS_\omega
\]

holds. Here \( C = C(N) > 0 \) is an absolute constant. Then by (24) and (30), we see

\[
\int_{\Omega^* (r)} |\nabla_S N - 1 \psi_n^*(r \omega)|^N dS_\omega \geq \frac{C g(r)}{2} \frac{f^*_n (r)}{R^{N-1}}
\]

and we may apply Polyá-Szegö inequality

\[
\int_{\Omega^* (r)} |\nabla_S N - 1 \psi_n^*(r \omega)|^N dS_\omega \geq \int_{\Omega^* (r)} |\nabla_S N - 1 \psi_n^*(r \omega)|^N dS_\omega.
\]

Then for large \( n_0 \), we have

\[
\int_{\Omega} |\nabla \psi_n|^N dx \geq \int_{R - 1/n_0} R \int_{\Omega^* (r)} |\nabla_S N - 1 \psi_n^*(r \omega)|^N dS_\omega dr
\]

\[
\geq \int_{R - 1/n_0} R \frac{C g(r)}{2} \frac{f^*_n (r)}{R^{N-1}} dr \geq \frac{C g(r^*)}{2 R^{N-1}} \int_{R - 1/n_0} R f^*_n (r) dr = \frac{C g(r^*)}{2 R^{N-1}} (1 + o_n (1))
\]

where \( r^* \) is a number with \( r^* \in (R - 1/n_0, R) \). Since \( g(r^*) \to \infty \) as \( n_0 \to \infty \), we conclude that \( \lim_{n \to \infty} \int_{\Omega} |\nabla \psi_n|^N dx = \infty \). This is a contradiction; thus \( C_N (\Omega) \) is attained.

5. Nonexistence of a minimizer for a domain \( \Omega \) with \( C_2 (\Omega) > \frac{1}{4} \)

In this section, we provide a Lipschitz domain \( \Omega \) in \( \mathbb{R}^2 \) on which \( C_2 (\Omega) > 1/4 \) and \( C_2 (\Omega) \) is not attained. Recall Hardy’s inequality (11) when \( N = 2 \):

\[
\inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 dx \mid \int_{\mathbb{R}^2} \frac{u^2}{(x_2)^2} dx = 1, u \in W_0^{1,2} (\mathbb{R}^2) \right\} = \frac{1}{4},
\]

and the best constant 1/4 is not attained, where \( x = (x_1, x_2) \). For \( a \in [0, \pi/2) \), we define

\[
E(a) \equiv \sup \left\{ \int_0^\pi \frac{a (\phi \theta)^2 d\theta}{a (\phi^2 \sin^2 \theta) d\theta} \mid \phi \in C_0^\infty ((a, \pi - a)) \setminus \{0\} \right\}.
\]

From [9] Corollary 4.4, we see that

\[
(31) \quad E \equiv E(0) = \inf \left\{ \int_0^\pi \frac{a (\phi \theta)^2 d\theta}{a (\phi^2 \sin^2 \theta) d\theta} \mid \phi \in C_0^\infty ((0, \pi)) \setminus \{0\} \right\} = \frac{1}{4}
\]

and \( E \) is not achieved. We prove these facts in Appendix for the reader’s convenience. It is obvious that for \( a \in (0, \pi/2) \), \( E(a) \) is achieved by a positive function \( \varphi_a \) on \( (a, \pi - a) \). Since \( E(0) \) is not achieved in \( W_0^{1,2} (0, \pi) \), \( E(a) > E(0) = \frac{1}{4} \) for \( a \in (0, \pi/2) \).

**Theorem 6.** There exists a domain \( \Omega \subset B_1 \subset \mathbb{R}^2 \) such that \( C_2 (\Omega) > \frac{1}{4} \) and \( C_2 (\Omega) \) is not attained.
Proof. For \( a \in (0, \pi/2) \), we define a cone
\[
C_a = \left\{ (r \cos \theta, r \sin \theta) \in \mathbb{R}_+^2 \mid r \in (0, \infty), \theta \in (a, \pi - a) \right\} \subset \mathbb{R}_+^2.
\]
We define
\[
R(y_1, y_2) = \left( (y_1)^2 + (1 - y_2)^2 \right) \left( \log \frac{1}{((y_1)^2 + (1 - y_2)^2)^{1/2}} \right)^2
\]
\[
= \frac{1}{4} h(r, \theta) \log h(r, \theta)^2
\]
for \((y_1, y_2) = (r \cos \theta, r \sin \theta)\), where \( h(r, \theta) = r^2 - 2r \sin \theta + 1 \). Since
\[
\log h(r, \theta) = h(r, \theta) - 1 - \frac{(h(r, \theta) - 1)^2}{2} + O(r^3)
\]
as \( r \to 0 \), we have
\[
\lim_{y_2 \to 0, (y_1, y_2) \in C_a} \frac{R(y_1, y_2)}{(y_2)^2} = 1
\]
for each \( a > 0 \). From now on, we fix \( a \in (\pi/4, \pi/2) \). We define
\[
g(r) = \inf \left\{ \frac{R(y_1, y_2)}{(y_2)^2} \mid (y_1, y_2) \in C_a, y_1^2 + y_2^2 = r^2 \right\}.
\]
By (32), we see that \( \lim_{r \to 0} g(r) = 1 \). Further, we see that \( g(r) < 1 \) for small \( r > 0 \).
We take \( r_0 \in (0, 1/2) \) such that \( g(r) < 1 \) for any \( r \in (0, r_0) \). Note that \( E(a) \) is monotone non-decreasing with respect to \( a \in (0, \pi/2) \). Now for each \( r \in (0, r_0) \), we take \( a(r) \in (a, \pi/2) \) such that \( E(a)/E(a(r)) = g(r) \in (0, 1) \). Since \( \lim_{r \to 0} g(r) = 1 \), it follows that \( \lim_{r \to 0} a(r) = a \). Since \( E \) is continuous on \((0, \pi/2)\) and \( g \) on \((0, r_0)\), \( a(r) \) is continuous with respect to \( r \in (0, r_0) \). We define
\[
\tilde{\Omega} = \left\{ (r \cos \theta, r \sin \theta) \in \mathbb{R}_+^2 \mid r \in (0, r_0), \theta \in (a(r), \pi - a(r)) \right\}
\]
and
\[
\Omega = \left\{ (x_1, x_2) \in B_1 \mid (x_1, 1 - x_2) \in \tilde{\Omega} \right\} \subset B_1 \subset \mathbb{R}^2.
\]
We claim that \( C_2(\Omega) = E(a) > 1/4 \) and \( C_2(\Omega) \) is not attained.

For any \( u \in C_0^\infty(\Omega) \), we define \( \tilde{u}(y_1, y_2) = u(y_1, 1 - y_2) \) for \( y = (y_1, y_2) \in \tilde{\Omega} \).
Then, we see that \( \tilde{u} \in C_0^\infty(\tilde{\Omega}) \) and
\[
\int_{\Omega} |\nabla u|^2 dx_1 dx_2 = \int_{\tilde{\Omega}} |\nabla \tilde{u}|^2 dy_1 dy_2 = \int_0^{r_0} \int_{a(r)}^{\pi - a(r)} r(\tilde{u}_r)^2 + r^{-1}(\tilde{u}_\theta)^2 d\theta dr
\]
and
\[
\int_{\Omega} \frac{(u(x_1, x_2))^2}{|x|^2 (\log |x|)^2} dx_1 dx_2 = \int_{\tilde{\Omega}} \frac{(\tilde{u}(y_1, y_2))^2}{R(y_1, y_2)} dy_1 dy_2.
\]
First of all, we claim that \( C_2(\Omega) \leq E(a) \). To prove this, we note that for any \( a' \in (a, \pi/2) \), we can find \( \delta' \in (0, r_0) \) such that
\[
\{(r \cos \theta, r \sin \theta) \in \tilde{\Omega} \mid r \in (0, \delta'), \theta \in (a', \pi - a') \} \subset \tilde{\Omega}.
\]
For any small $\varepsilon, \delta > 0$ with $4\varepsilon < \delta < \delta'$, we find a Lipschitz continuous function $\psi_\varepsilon^\delta$ satisfying $\psi_\varepsilon^\delta(r) = 0$ for $r < \varepsilon$ or $r \geq \delta$, $\psi_\varepsilon^\delta(r) = 1$ for $2\varepsilon \leq r < \delta/2$, $|\psi_\varepsilon^\delta'(r)| = 1/\varepsilon$ for $r \in (\varepsilon, 2\varepsilon)$, and $|\psi_\varepsilon^\delta'(r)| = 2/\delta$ for $r \in (\delta/2, \delta)$. We define that for $y = (y_1, y_2) = (r \cos \theta, r \sin \theta) \in \Omega$ and $x = (x_1, x_2) \in \Omega$,

$$
\tilde{u}_\varepsilon^\delta(y_1, y_2) = \psi_\varepsilon^\delta(r, \theta) \varphi_{\alpha'}(\theta) \text{ and } u_\varepsilon^\delta(x_1, x_2) = \tilde{u}_\varepsilon^\delta(x_1, 1 - x_2).
$$

Then we see that

$$
\int_\Omega |\nabla u_\varepsilon^\delta|^2 \, dx = \int_\Omega |\nabla \tilde{u}_\varepsilon^\delta|^2 \, dy = \int_0^\infty \int_{\alpha'}^{\pi - \alpha'} r((\tilde{u}_\varepsilon^\delta)_r)^2 + r^{-1}((\tilde{u}_\varepsilon^\delta)_\theta)^2 \, d\theta \, dr
$$

$$
= \left( \int_{\varepsilon}^{2\varepsilon} ((\psi_\varepsilon^\delta)'(r))^2 \, rdr + \int_{\delta/2}^{\delta} ((\psi_\varepsilon^\delta)'(r))^2 \, rdr \right) \int_{\alpha'}^{\pi - \alpha'} (\varphi_{\alpha'}(\theta))^2 \, d\theta
$$

$$
+ \int_{\varepsilon}^{\delta} \int_{\alpha'}^{\pi - \alpha'} r^{-1}((\psi_\varepsilon^\delta)(r))^2 \left( \frac{d\varphi_{\alpha'}}{d\theta} \right)^2 \, d\theta \, dr
$$

$$
= 3 \int_{\alpha'}^{\pi - \alpha'} (\varphi_{\alpha'}(\theta))^2 \, d\theta + \int_{\varepsilon}^{\delta} r^{-1}((\psi_\varepsilon^\delta)(r))^2 \, dr \int_{\alpha'}^{\pi - \alpha'} \left( \frac{d\varphi_{\alpha'}}{d\theta} \right)^2 \, d\theta
$$

and

$$
\int_\Omega \frac{(u_\varepsilon^\delta(x))^2}{|x|^2(\log |x|)^2} \, dx = \int_\Omega \frac{(\tilde{u}_\varepsilon^\delta(y))^2}{R(y_1, y_2)^2} \, dy = \int_{\varepsilon}^{\delta} \int_{\alpha'}^{\pi - \alpha'} \frac{(y_2)^2}{R(y_1, y_2)} r^{-1}((\psi_\varepsilon^\delta)(r))^2 \left( \frac{\varphi_{\alpha'}}{\sin \theta} \right)^2 \, d\theta \, dr.
$$

Since $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\delta} r^{-1}((\psi_\varepsilon^\delta)(r))^2 \, dr = \infty$ for each $\delta > 0$, we see that

$$
\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon^\delta|^2 \, dx}{\int_\Omega \frac{|u_\varepsilon^\delta|^2}{|x|^2(\log |x|)^2} \, dx} \leq E(a')(\min_{r \in [0, \delta]} g(r))^{-1}.
$$

Then, $C_3(\Omega) \leq E(a')$ for any $a' \in (a, \pi/2)$ since $\lim_{r \to 0} g(r) = 1$. This implies that $C_2(\Omega) \leq E(a)$.

Now for any $v \in W^{1,2}_0(\Omega)$ with $\tilde{v}(y_1, y_2) \equiv v(y_1, 1 - y_2) \in W^{1,2}_0(\bar{\Omega})$, we see that

$$
\int_\Omega |\nabla v|^2 \, dx_1 \, dx_2 \geq \int_0^{ro} \int_{a(r)}^{\pi - a(r)} r((\tilde{v}_r)_r)^2 + E(a(r))r^{-1}((\tilde{v}_r)/\sin \theta)^2 \, d\theta \, dr
$$

$$
= \int_0^{ro} \int_{a(r)}^{\pi - a(r)} \left[ (\tilde{v}_r)^2 + E(a(r)) \frac{(\tilde{v}_r)^2}{(y_2)^2} \right] \, d\theta \, dr
$$

$$
= \int_0^{ro} \int_{a(r)}^{\pi - a(r)} \left[ (\tilde{v}_r)^2 + E(a(r)) \frac{R(y_1, y_2)^2}{(y_2)^2} \frac{(\tilde{v}_r)^2}{R(y_1, y_2)} \right] \, d\theta \, dr
$$

$$
\geq \int_0^{ro} \int_{a(r)}^{\pi - a(r)} \left[ (\tilde{v}_r)^2 + E(a(r))g(r) \frac{(\tilde{v}_r)^2}{R(y_1, y_2)} \right] \, d\theta \, dr
$$

$$
= \int_0^{ro} \int_{a(r)}^{\pi - a(r)} (\tilde{v}_r)^2 r \, d\theta \, dr + E(a) \int_0^{ro} \frac{(\tilde{v}_r)^2}{R(y_1, y_2)} \, dy_1 \, dy_2
$$

$$
= \int_0^{ro} \int_{a(r)}^{\pi - a(r)} (\tilde{v}_r)^2 r \, d\theta \, dr + E(a) \int_\Omega \frac{(v(x))^2}{|x|^2(\log |x|)^2} \, dx.
$$
This implies that $C_2(\Omega) \geq E(a)$. Combining above upper and lower estimates, we see that $C_2(\Omega) = E(a) > \frac{1}{4}$.

From above estimate, we see that for any $u \in W_0^{1,2}(\Omega)$, we see that

$$\int_\Omega |\nabla u|^2 dx_1 dx_2 \geq \int_0^r \int_0^{\pi-a(r)} (\tilde{u}_r)^2 r d\theta dr + E(a) \int_\Omega \frac{(u(x_1, x_2))^2}{|x|^2 (\log |x|)^2} dx_1 dx_2. \tag{33}$$

If $C_2(\Omega)$ is attained by $u \in W_0^{1,2}(\Omega) \setminus \{0\}$, we see from Remark 7 that $\tilde{u}_r \equiv 0$ in $\tilde{\Omega}$. This contradicts to the fact $u \in W_0^{1,2}(\Omega)$. Thus we conclude that $C_2(\Omega)$ is not attained in $W_0^{1,2}(\Omega)$.

Remark 7. For the domain $\Omega$ in Theorem 4, let $P, Q$ be two points in $\partial \Omega \cap \partial B_r$ when $r$ is close to 1. Then $m(r)$ is the length of the arc $\tilde{P}Q$, which is larger than the length of the segment $PQ$. Thus it is easy to see that in this case, $m_0 = 0$ and $m_1 = \limsup_{r \to 1} m(r)/(1 - r) \geq 2 \cos a > 0$;

see Theorem 5.

Appendix A. Appendix

Here, we prove

$$E = \inf \left\{ \frac{\int_0^\pi (\phi')^2 d\theta}{\int_0^\pi (\phi/\sin \theta)^2 d\theta} \mid \phi \in W_0^{1,2}(0, \pi) \setminus \{0\} \right\} = \frac{1}{4}$$

and $E$ is not achieved.

Proof. For $u \in C_0^\infty((0, \pi))$, we compute

$$\left| \int_0^\pi \frac{u^2}{\sin^2 \theta} d\theta \right| = \left| \int_0^\pi \left( \frac{-\cos \theta}{\sin \theta} \right)' u^2 d\theta \right| = \left| \int_0^\pi \left( \frac{\cos \theta}{\sin \theta} \right) 2 uu' d\theta \right|$$

$$\leq 2 \left( \int_0^\pi \frac{u^2}{\sin^2 \theta} d\theta \right)^{\frac{1}{2}} \left( \int_0^\pi (u')^2 \cos^2 \theta d\theta \right)^{\frac{1}{2}} \leq 2 \left( \int_0^\pi \frac{u^2}{\sin^2 \theta} d\theta \right)^{\frac{1}{2}} \left( \int_0^\pi (u')^2 d\theta \right)^{\frac{1}{2}}.$$

Thus we have the inequality

$$\frac{1}{4} \int_0^\pi \frac{u^2}{\sin^2 \theta} d\theta \leq \int_0^\pi (u')^2 d\theta.$$

By density, this inequality holds for all $u \in W_0^{1,2}(0, \pi)$.

To see $E = 1/4$, test $E$ by functions $u_\alpha(\theta) = (\sin \theta)^\alpha$ for $\alpha > 1/2$. Then we find

$$\frac{\int_0^\pi (u_\alpha'(\theta))^2 d\theta}{\int_0^\pi (u_\alpha^2/\sin^2 \theta) d\theta} = \alpha^2 - \frac{\int_0^\pi (\sin \theta)^{2\alpha - 2} d\theta}{\int_0^\pi (u_\alpha^2/\sin^2 \theta) d\theta} \leq \alpha^2 - 1/4, \quad \alpha \downarrow 1/2.$$

To see that $E$ is not attained, we use the function $v(\theta) = u(\theta)/(\sin \theta)^{1/2}$ for $u \in W_0^{1,2}(0, \pi)$. Then a simple computation shows that

$$(u')^2 - \frac{1}{4} \frac{u^2}{\sin^2 \theta} = -\frac{u^2}{4} + (v')^2 \sin \theta + \left( \frac{u^2}{2} \right)' \cos \theta.$$
Integrating this on $[0, \pi]$, and noting that $\int_0^\pi (u''/2)\cos \theta d\theta = \int_0^\pi (u''/2) d\theta$ by integration by parts, we obtain

$$\int_0^\pi \left[ (u')^2 - \frac{1}{4} \frac{u^2}{\sin^2 \theta} \right] d\theta = \int_0^\pi \frac{u'^2}{4} d\theta + \int_0^\pi (v')^2 \sin \theta d\theta.$$ 

This implies that if $E$ is attained, then $u \equiv 0$ on $[0, \pi]$.

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