RADIATIVE TAU DECAYS
WITH ONE PSEUDOSCALAR MESON

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Abstract

We have calculated the decay $\tau \to \nu \pi(K)\gamma$. We present the photon energy spectrum, the meson-photon invariant mass spectrum and the integrated rate as a function of a photon energy cut or an invariant mass cut. Both the internal bremsstrahlung and the structure dependent radiation have been taken into account. To this aim we have parametrized the form factors $F_V$ and $F_A$, which determine the structure dependent radiation. Observables especially suited for the measurement of the structure dependent form factors are found and implications on the width of the $a_1$ discussed.
1 Introduction

Present and future high luminosity electron positron colliders and especially tau-charm factories will produce very large samples of tau pairs in the near future. This will give the opportunity to observe rare tau decays, for example radiative semihadronic modes, the simplest of which are $\tau \to \nu \pi \gamma$ and $\tau \to \nu K \gamma$, which we will therefore calculate in this paper.

The decay $\tau \to \nu \pi \gamma$ is related by crossing symmetry to the radiative pion decay ($\pi \to e \nu \gamma$), where the electron plays the role of the tau lepton. There is, however, a crucial difference, namely that the momentum transfer squared $t$ between the leptonic and the pion-photon system, which is almost zero in the case of the pion decay, can take any value up to $m_\tau^2$ in the radiative tau decay. Several resonances, which can contribute only as small virtual corrections in the pion decay, can now be produced as real particles and duly have to be taken into account.

The decay amplitude can be divided into internal bremsstrahlung, where the pion behaves like a pointlike particle and which therefore is completely determined by quantum electrodynamics, and the structure dependent part, which is governed by strong interaction effects and is parametrized by two formfactors $F_V(t)$ and $F_A(t)$. So the main task is to obtain these formfactors. We will assume them to be dominated by several Breit Wigner type resonances and determine their parameters by employing different constraints, both theoretical and experimental ones. For instance normalizations at $t = 0$ are fixed by the Adler-Bell anomaly ($F_V$) and the $\pi \to e \nu \gamma$ measurement ($F_A$), respectively. On the other hand the relative strengths of the different resonances are taken from a high energy QCD prediction and experimental data on the formfactors at finite values of $t$.

This paper is organized as follows: In Sec. 2 we shortly discuss the kinematics, the differential decay rate and lay down our conventions. In Sec. 3 the general form of the matrix element, photon spectrum and the decay rate of $\tau \to \nu \pi \gamma$ are calculated. Their dependence on the two formfactors $F_V(t)$ and $F_A(t)$ are worked out. In Sec. 4 these formfactors are parametrized along the lines indicated above. In Sec. 5 the numerical results for the photon and invariant mass spectra and the decay rate are presented. In Sec. 6 we estimate the decay $\tau \to \nu K \gamma$ and in Sec. 7 we discuss our results.

2 Kinematics and the Decay Rate

For definiteness and without loss of generality we consider the decay of the negatively charged tau into a pion, a neutrino and a photon

$$\tau^- (s) \to \nu_\tau (q) \pi^- (p) \gamma (k)$$

only and transcribe the formulae to the case of the decay into a kaon in Sec. 7. The kinematics of this decay is equivalent to that of the radiative pion decay [1]. We can write the differential decay rate for an unpolarized tau in such a way that it depends on two Lorentz scalars, taken to be $x$ and $y$:

$$x := \frac{2s \cdot k}{m_\tau^2} \quad y := \frac{2s \cdot p}{m_\tau^2}$$

In the tau rest frame $x$ and $y$ are the energies $E_\gamma$ and $E_\pi$ of the photon and the pion, respectively, expressed in units of $m_\tau/2$:

$$E_\gamma = \frac{m_\tau}{2} x \quad E_\pi = \frac{m_\tau}{2} y$$

The kinematical boundaries for $x$ and $y$ are given by

$$0 \leq x \leq 1 - r^2$$
\[ 1 - x + \frac{r^2}{1 - x} \leq y \leq 1 + r^2 \quad (4) \]

where
\[ r^2 := \left( \frac{m_{\pi}}{m_\tau} \right)^2 = 6.17 \times 10^{-3} \ll 1 \quad (5) \]

using the central value of the new \( \tau \) mass experiments [3]
\[ m_\tau = 1777 \text{ MeV.} \quad (6) \]

Let us quote some useful formulae which relate different kinematical expressions:
\[ p \cdot k = \frac{m_\tau^2}{2} (x + y - 1 - r^2) \quad (7) \]
\[ t := (s - q)^2 = (k + p)^2 = m_\tau^2 (x + y - 1) \quad (8) \]

Now we are in position to present the formula for the decay rate of an unpolarized tau decaying into \( |\nu_\tau \pi \gamma\rangle \) The differential decay rate is given by
\[ d\Gamma(\tau \to \nu_\tau \pi \gamma) = \frac{1}{512 \pi^5 E_\gamma} \delta^{(4)}(k + p + q - s) |M|^2 \frac{d^3 \bar{q}}{E_\gamma E_\pi E_\nu} \quad (9) \]

where the bar over the matrix element denotes summing over the photon polarization and neutrino spin and averaging over the tau spin.

Choice of the tau rest frame, integration over the neutrino momentum and the remaining angles and introduction of the dimensionless variables \( x \) and \( y \) lead to
\[ \frac{d^2 \Gamma}{dx dy} = \frac{m_\tau}{256 \pi^3} |M|^2 \quad (10) \]

The integration over \( y \) yields the photon spectrum
\[ \frac{d\Gamma}{dx} = \int_{1 - x + \frac{r^2}{1 - x}}^{1 + r^2} dy \frac{d^2 \Gamma}{dx dy} \quad (11) \]

Because of the infrared divergence of the internal bremsstrahlung a low energy cut must be introduced for the photon energy, eg. by requiring
\[ x \geq x_0 \quad (12) \]

The integrated decay rate is then obtained by
\[ \Gamma(x_0) = \Gamma(E_0) = \int_{x_0}^{1 - r^2} \frac{d\Gamma}{dx} dx \quad (13) \]

and does depend on the cut in the photon energy \( (E_0 = \frac{m_\tau}{2} x_0) \).

Instead of the invariants \( x \) and \( y \) we will also use \( x \) and \( z \) where \( z \) is the scaled momentum transfer squared:
\[ z = \frac{t}{m_\tau^2} = x + y - 1 \quad (14) \]
Their kinematical boundaries are given by

\[
\begin{align*}
z - r^2 & \leq x \leq 1 - \frac{r^2}{z} \\r^2 & \leq z \leq 1
\end{align*}
\] (15)

Integration of \(d^2\Gamma/dx\,dy\) over \(x\) then yields the spectrum in \(z\), i.e. the spectrum in the invariant mass of the pion-photon system:

\[
d\Gamma dz(z) = d\Gamma dz(\sqrt{t}) = \int_{z-r^2}^{1-r^2/z} dx\,d^2\Gamma dx\,dy(x, y = z - x + 1)
\] (16)

The integrated rate for events with \(t \geq t_0\) is then given by

\[
\Gamma(z_0) = \Gamma(\sqrt{t_0}) = \int_{z_0}^{1} dz\,d\Gamma(z)
\] (17)

where, of course, \(z_0 = \frac{t_0}{m_\tau^2}\). Note that the cut in \(t\) acts as both an infrared and a collinear cut off.

3 General Structure of the Matrix Element and the Decay Rate

In complete analogy to the case of the radiative pion decay [3], the matrix element for the decay \(\tau^- \rightarrow \nu\pi\gamma\) can be written as the sum of four contributions:

\[
\mathcal{M}[\tau^-(s) \rightarrow \nu\pi^- (p)\gamma(k)] = \mathcal{M}_{IB\tau} + \mathcal{M}_{IB\pi} + \mathcal{M}_V + \mathcal{M}_A
\] (18)

with

\[
\begin{align*}
\mathcal{M}_{IB\tau} &= -G_F \cos \theta_C e f_\pi p_\mu \epsilon_\nu(k) L^{\mu\nu} \\
\mathcal{M}_{IB\pi} &= G_F \cos \theta_C e f_\pi \epsilon_\nu(k) \left( \frac{2p_\nu(k + p)_\mu}{m_\tau^2 - t} - g_{\mu\nu} \right) L^\mu \\
\mathcal{M}_V &= \frac{G_F \cos \theta_C e F_V(t)}{\sqrt{2}} \frac{1}{m_\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_\nu(k) k^\rho p^\sigma L^\mu \\
\mathcal{M}_A &= -\frac{G_F \cos \theta_C e F_A(t)}{\sqrt{2}} \frac{1}{m_\pi} \epsilon_\nu(k) \left( (p_\nu k_\mu - (p \cdot k) g_{\mu\nu}) \right) L^\mu
\end{align*}
\] (19)

where \(G_F\) is the Fermi constant, \(\theta_C\) the Cabibbo angle, \(e\) the electric charge and \(\epsilon_\nu\) the polarisation vector of the photon. For our numerical results we use the pion coupling constant \(f_\pi\) to be \[4, 5\]

\[
f_\pi = 92.5 \text{ MeV}
\] (20)

\(F_V\) and \(F_A\) are the so called structure dependent form factors which are parametrized in the next section. Note that our definition implies that these form factors are dimensionless. Finally \(L^\mu\) and \(L^{\mu\nu}\) are leptonic currents defined by

\[
\begin{align*}
L^\mu &= \bar{u}(q) \gamma^\mu \gamma^- u(s) \\
L^{\mu\nu} &= \bar{u}(q) \gamma^\mu \gamma^- \frac{k^\nu - q^\nu - m_\tau}{(k - s)^2 - m_\tau^2} u(s)
\end{align*}
\] (21)

In fact the four terms correspond to the Feynman diagrams in Fig. 1:
\( \mathcal{M}_{IB} \) is the bremsstrahlung off the tau (a),
\( \mathcal{M}_{IB_{\pi}} \) the sum of the pion bremsstrahlung (b) and the seagull (c),
\( \mathcal{M}_V \) is the structure dependent vector (e) and
\( \mathcal{M}_A \) the structure dependent axial contribution (f).

Note that all ambiguities due to the strong interactions have been parametrized in terms of the two form factors \( F_A \) and \( F_V \). In fact these form factors are the same functions of the momentum transfer \( t \) as those in the radiative pion decay, the only difference being that \( t \) now varies from 0 up to \( m_\tau \) rather than just up to \( m_\pi \).

The two matrix elements \( \mathcal{M}_{IB_{\tau}} \) and \( \mathcal{M}_{IB_{\pi}} \) are not separately gauge invariant, but their sum, i.e. the (total) matrix element for internal bremsstrahlung IB

\[
\mathcal{M}_{IB} = \mathcal{M}_{IB_{\tau}} + \mathcal{M}_{IB_{\pi}} \tag{22}
\]

is, as are \( \mathcal{M}_V \) and \( \mathcal{M}_A \). We also define the (total) structure dependent radiation SD, viz. by

\[
\mathcal{M}_{SD} = \mathcal{M}_V + \mathcal{M}_A \tag{23}
\]

Using standard technics the matrix elements may be simplified, the main rearrangement being

\[
- \bar{u}(q)\gamma_+\gamma_5 \frac{k' \cdot q - m_\tau}{(k - s)^2 - m_\tau^2} u(s) = \bar{u}(q)\gamma_+ \left[ m_\tau \frac{k' \cdot q}{2s \cdot k} - m_\tau \frac{s \cdot \epsilon}{s \cdot k} + \frac{q}{2} \right] u(s) \tag{24}
\]

We obtain

\[
\mathcal{M}_{IB} = G_F \cos \theta_c e f_\pi m_\tau \bar{u}(q)\gamma_+ \left[ \frac{p \cdot \epsilon}{p \cdot k} + \frac{k' \cdot q}{2s \cdot k} - \frac{s \cdot \epsilon}{s \cdot k} \right] u(s)
\]

\[
\mathcal{M}_{SD} = \frac{G_F \cos \theta_c e}{\sqrt{2}} \left\{ i\epsilon_{\mu\rho\sigma}L^\mu \epsilon^\nu k^\rho p^\sigma \frac{F_V(t)}{m_\pi} + \bar{u}(q)\gamma_+ \left[ (p \cdot k)q - (\epsilon \cdot p)k' \right] u(s) \frac{F_A(t)}{m_\pi} \right\} \tag{25}
\]

The square of the matrix element is then given by

\[
|\mathcal{M}|^2 = |\mathcal{M}_{IB}|^2 + 2\text{Re}(\mathcal{M}_{IB}\mathcal{M}_{SD}^*) + |\mathcal{M}_{SD}|^2, \tag{26}
\]

where — as stated in the previous section — the bar denotes summing over the photon polarization and neutrino spin and averaging over the tau spin.

In order to make our results transparent we have divided the decay rate into several parts:

- the internal bremsstrahlung part \( \Gamma_{IB} \) arising from \(|\mathcal{M}_{IB}|^2\),
- the structure dependent part \( \Gamma_{SD} \) arising from \(|\mathcal{M}_{SD}|^2\)
- the interference part \( \Gamma_{INT} \) arising from \(2\text{Re}(\mathcal{M}_{IB}\mathcal{M}_{SD}^*)\).

Furthermore we have subdivided \( \Gamma_{SD} \) in the obvious way into \( \Gamma_{VV} \) arising from the vector matrix element squared, \( \Gamma_{AA} \) form the axial part and the vector-axial interference term \( \Gamma_{VA} \) and similarly \( \Gamma_{INT} \) gets divided into the internal bremsstrahlung-vector interference \( \Gamma_{IB-V} \) and the internal bremsstrahlung-axial interference \( \Gamma_{IB-A} \).
So our master formulae are:

\[
\begin{align*}
\Gamma_{\text{total}} &= \Gamma_{IB} + \Gamma_{SD} + \Gamma_{\text{INT}} \\
\Gamma_{SD} &= \Gamma_{VV} + \Gamma_{VA} + \Gamma_{AA} \\
\Gamma_{\text{INT}} &= \Gamma_{IB-V} + \Gamma_{IB-A}
\end{align*}
\] (27)

One finds for the differential decay rate

\[
\begin{align*}
\frac{d^2\Gamma_{IB}}{dx\,dy} &= \frac{\alpha}{2\pi} f_{IB}(x,y,r^2) \frac{\Gamma_{\tau\to\nu\pi}}{(1-r^2)^2} \\
\frac{d^2\Gamma_{SD}}{dx\,dy} &= \frac{\alpha}{16\pi} \frac{m^4}{f^2m^2_{\pi}} \left[ |F_V|^2 f_{VV}(x,y,r^2) + 2\text{Re}(F_V F_A^{*}) f_{VA}(x,y,r^2) + |F_A|^2 f_{AA}(x,y,r^2) \right] \frac{\Gamma_{\tau\to\nu\pi}}{(1-r^2)^2} \\
\frac{d^2\Gamma_{\text{INT}}}{dx\,dy} &= \frac{\alpha}{2\sqrt{2} \pi} \frac{m^2_{\tau}}{f_{\pi} m_{\pi}} \left[ f_{IB-V}(x,y,r^2)\text{Re}(F_V) + f_{IB-A}(x,y,r^2)\text{Re}(F_A) \right] \frac{\Gamma_{\tau\to\nu\pi}}{(1-r^2)^2}
\end{align*}
\]

(28)

where

\[
\begin{align*}
 f_{IB}(x,y,r^2) &= \frac{[r^4(x+2) - 2r^2(x+y) + (x+y-1)(2-3x+x^2+xy)](r^2-y+1)}{(r^2-x-y+1)^2x^2} \\
 f_{VV}(x,y,r^2) &= -[r^4(x+y) + 2r^2(1-y)(x+y) + (x+y-1)(-x+x^2+y+y^2)] \\
 f_{AA}(x,y,r^2) &= f_{VV}(x,y,r^2) \\
 f_{VA}(x,y,r^2) &= [r^2(x+y) + (1-x-y)(y-x)](r^2-x-y+1) \\
 f_{IB-V}(x,y,r^2) &= -\frac{(r^2-x-y+1)(r^2-y+1)}{r^2-x-y+1} \\
 f_{IB-A}(x,y,r^2) &= \frac{[r^4 - 2r^2(x+y) + (1-x+y)(x+y-1)](r^2-y+1)}{(r^2-x-y+1)x}
\end{align*}
\] (29)

In the approximation \( r^2 \approx 0 \) (vanishing pion mass) the formulae simplify to

\[
\begin{align*}
 f_{IB}(x,y,0) &= \frac{[1 + (1-x)^2-x(1-y)](1-y)}{(x+y-1)x^2} \\
 f_{VV}(x,y,0) &= (x-x^2+y-y^2)(x+y-1) \\
 f_{AA}(x,y,0) &= (x+y-1)^2(y-x) \\
 f_{VA}(x,y,0) &= (x+y-1)(1-y) \\
 f_{IB-V}(x,y,0) &= \frac{(x+y-1)(1-y)}{x} \\
 f_{IB-A}(x,y,0) &= \frac{(x-y-1)(1-y)}{x}
\end{align*}
\] (30)

Note that we expressed the radiative decay rate in terms of the rate of the non-radiative decay \((\tau \to \nu_{\tau} \pi)\):

\[
\Gamma_{\tau\to\nu\pi} = \frac{G_F^2 \cos^2 \theta_c f_{\pi}^2 m^2_{\tau}(1-r^2)^2}{8\pi}
\] (31)

Although the factor \( m^2_{\tau}/(f_{\pi} m_{\pi}) \) in front of \( F_A \) and \( F_V \) is rather large a number (\( \approx 240 \)), we should mind that the form factors \( F_A \) and \( F_V \) themselves are of the order of \( 10^{-2} \) (see next section), so that we expect the different contributions to be of the same order of magnitude. Note also that the internal bremsstrahlung and the interference term are infrared divergent in the limit \( x \to 0 \).
(vanishing photon energy). Furthermore there is an enhancement of the internal bremsstrahlung (but not of the interference term) in the region
\[(x + y - 1) \rightarrow r^2 \iff t \rightarrow m^2_π\]  
(32)

This enhancement is due to large logs of the pion mass, the usual collinear divergencies for massless particles.

Next we would like to make a short comment on the internal bremsstrahlung part of the decay rate. Let us first give the corresponding expression in the radiative pion decay \[\Pi\] (neglecting \(m_e\))
\[
\frac{d^2\Gamma_{IB}(\pi \rightarrow e\nu\gamma)}{dx\,dy} = \frac{\alpha}{2\pi}\Gamma(\pi \rightarrow e\nu)IB(x, y)
\]  
(33)

with
\[
IB(x, y) = \frac{(1 - y)[1 + (1 - x)^2]}{x^2(x + y - 1)}
\]  
(34)

where now, however, \(x\) and \(y\) are the photon and electron energies in units of \(m_π/2\):
\[
E_γ = \frac{m_π}{2}x \quad E_e = \frac{m_π}{2}y
\]  
(35)

Expressed in this form eq.(34) is identical with eq.(30), apart from the \(x(1 - y)\) term. On the other hand keeping in mind the definition of \(x\) we note that \(m_τ/2\) and \(m_π/2\), respectively, set the scale for photons to be considered as “hard” or “soft”. This means that the formulae for internal bremsstrahlung should be similar for radiative tau and pion decay, once they are expressed in terms of \(x\) and \(y\), which is indeed the case. So photons from the \(τ\)- or \(π\)- decays of comparable “softness”, e.g. comparable \(x\), have very different energies.

Finally we present analytical expressions for the invariant mass spectrum:
\[
\begin{align*}
\frac{dΓ_{IB}}{dz} &= \frac{\alpha}{2\pi} \left[r^4(1 - z) + 2r^2(z - z^2) - 4z + 5z^2 - z^3 + \right. \\
&\left. + (r^4z + 2r^2z - 2z - 2z^2 + z^3) \ln z\right] \frac{1}{z^2} \frac{Γ_{τ→νπ}}{r^2(1 - r^2)^2} \\
\frac{dΓ_{VV}}{dz} &= \frac{\alpha}{48π} \left[m^2_τ(z - 1)^2(z - r^2)^3(1 + 2z)\right] |F_V|^2 \frac{Γ_{τ→νπ}}{r^2(1 - r^2)^2} \\
\frac{dΓ_{VA}}{dz} &= 0 \\
\frac{dΓ_{AA}}{dz} &= \frac{\alpha}{48π} \left[m^2_τ(z - 1)^2(z - r^2)^3(1 + 2z)\right] |F_A|^2 \frac{Γ_{τ→νπ}}{r^2(1 - r^2)^2} \\
\frac{dΓ_{IB-V}}{dz} &= \frac{\alpha}{2\sqrt{2}π} \frac{m^2_τ}{f_πm_π} \left[r^2(1 - z) - 1 - z + 2z^2 + \right. \\
&\left. + (r^2z - 2z - z^2) \ln z\right] \frac{z - r^2}{z} \Re(F_V) \frac{Γ_{τ→νπ}}{r^2(1 - r^2)^2} \\
\frac{dΓ_{IB-A}}{dz} &= \frac{\alpha}{2\sqrt{2}π} \frac{m^2_τ}{f_πm_π} \left[r^2(1 - z) - 1 - z + 2z^2 + \right. \\
&\left. + (r^2z - 2z - z^2) \ln z\right] \frac{z - r^2}{z} \Re(F_A) \frac{Γ_{τ→νπ}}{r^2(1 - r^2)^2}
\end{align*}
\]  
(36)

Note that the interference terms IB-V and IB-A are now finite in the limit \(z \rightarrow r^2\), which proves that their infrared divergencies are integrable.
4 Parametrization of the Form Factors

4.1 The Axial Form Factor

The axial form factor $F_A(t)$ fulfills a once subtracted dispersion relation. Since this form factor is measured at $t = 0$ in the radiative pion decay we will use this value to fix the subtraction constant. Furthermore it is supposed to be dominated by the $a_1$ meson. The following form includes these constraints:

$$F_A(t) = F_A(0) \text{Res}_{a_1}(t)$$

(37)

where $F_A(0)$ is taken from The Particle Data Group [3]

$$F_A(0)_{exp} = -0.0116 \pm 0.0016.$$  

(38)

and $\text{Res}_{a_1}(t)$ is a normalized Breit Wigner resonance factor

$$\text{Res}_{a_1}(t = 0) = 1$$

(39)

Note that most experiments on the radiative pion decay have been performed with pions at rest. These experiments are sensitive mainly to $(1 + \gamma)^2$ only, where

$$\gamma = \frac{F_A(0)}{F_V(0)}$$

(40)

So they also allow for a positive sign solution for $F_A(0)$:

$$F_A(0) = +0.0422$$

(41)

Measurements performed with pions in flight support our value in (38). However, we will also comment on implications of the other solution.

Since the decay of the $a_1$ is strongly dominated by its three pion decay we approximate in the Breit Wigner the total width by the three pion width. Furthermore we will use a energy dependent width as given in [3]

$$\text{Res}_{a_1}(t) = \frac{m_{a_1}^2}{m_{a_1}^2 - t - im_{a_1} \Gamma_{ks}(t)}$$

(42)

where

$$\Gamma_{ks}(t) = \frac{g(t)}{g(m_{a_1}^2)} \Gamma_{a_1}$$

(43)

with

$$g(t) = \begin{cases} 0 & \text{if } t < 9m_{a_1}^2 \\ 4.1(t - 9m_{a_1}^2)^3[1 - 3.3(t - 9m_{a_1}^2) + 5.8(t - 9m_{a_1}^2)^2] & \text{if } 9m_{a_1}^2 \leq t < (m_\rho + m_\pi)^2 \\ t(1.623 + 10.38/t^2 - 9.32/t^4 + 0.65/t^6) & \text{else} \end{cases}$$

(44)

(all numbers in appropriate powers of GeV).

We have studied the influence of a energy-dependent width by comparing our results to those obtained from

$$\text{Res}_{a_1}(t) = \frac{m_{a_1}^2 - im_{a_1} \Gamma_{a_1}}{m_{a_1}^2 - t - im_{a_1} \Gamma_{a_1}}$$

(45)

with a constant width $\Gamma_{a_1}$.  

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In order to check our form factor we use it to compute the radiative decay of the \( a_1 \). Indeed in the narrow width approximation, \( F_A(m_{a_1}^2) \) can be related to \( \Gamma_{a_1 \rightarrow \pi \gamma} \), once the transition amplitude for \( \tau \rightarrow \nu a_1 \) is known. Let this transition be parametrized by the coupling constant \( f_{a_1} \)

\[
\mathcal{M}(\tau \rightarrow \nu a_1) = G_F \cos \theta \epsilon_\nu f_{a_1} \epsilon_\mu k^\mu L^\mu
\]  

(46)

Then

\[
\Gamma_{a_1 \rightarrow \pi \gamma} = \frac{\alpha}{48} \frac{m_{a_1}^5 \Gamma_{a_1}^2}{m_\pi^2 f_{a_1}^2} |F_A(m_{a_1}^2)|^2 \left( 1 - \frac{m_\pi^2}{m_{a_1}^2} \right)^3
\]  

(47)

Using equation (37), this can be reexpressed in terms of \( F_A(0) \) as

\[
\Gamma_{a_1 \rightarrow \pi \gamma} = \frac{\alpha}{48} \frac{m_{a_1}^7 |F_A(0)|^2}{m_\pi^2 f_{a_1}^2} \left( 1 - \frac{m_\pi^2}{m_{a_1}^2} \right)^3
\]  

(48)

From the experimental result with (49)

\[
(\Gamma_{a_1 \rightarrow \pi \gamma})_{\text{exp}} = 640 \pm 246 \text{ keV}
\]  

we obtain for the central values

\[
f_{a_1} = 0.09 \text{ GeV}^2
\]  

(50)

which has to be compared with

\[
f_{a_1} = f_\rho = \sqrt{2} m_\rho f_\pi = 0.10 \text{ GeV}^2
\]  

(51)

from chiral symmetry and the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation or with \( f_{a_1} \) extracted from \( \tau \rightarrow a_1 \nu \) data:

\[
f_{a_1}^2 = \frac{m_{a_1}^2 m_\pi^2}{24 \pi^2 \cos^2 \theta_C} \frac{1}{\left( 1 - \left( \frac{m_{a_1}}{m_\pi} \right)^2 \right)^2 \left( 1 + 2 \left( \frac{m_{a_1}}{m_\pi} \right)^2 \right) \frac{BR(\tau \rightarrow a_1 \nu)}{BR(\tau \rightarrow e^- \nu \bar{\nu})}}
\]  

(52)

(where \( BR \) denotes branching ratio). With present data, \( BR(\tau \rightarrow \nu a_1) = 6.8\% \) we obtain

\[
f_{a_1} = 0.131 \text{ GeV}^2
\]  

(53)

from the direct measurement.

So we observe a reasonable agreement between the predicted and measured radiative decay width of the \( a_1 \).

### 4.2 The Vector Form Factor

The hypothesis of the conserved vector current (CVC) relates the vector formfactor \( F_V \) to the formfactor \( F_{\pi \gamma} \) of the vertex \( \pi^0 \gamma \gamma^* \) by

\[
F_V(t) = \frac{m_\pi F_{\pi \gamma}(t)}{\sqrt{2}}
\]  

(54)

where \( F_{\pi \gamma}(t) \) is defined by

\[
\mathcal{M}[\pi^0(p) \rightarrow \gamma(k) \gamma^*(q)] = i e^2 F_{\pi \gamma}(q^2) \epsilon_{\mu \nu \rho \sigma} k^\mu q^\nu \epsilon^\rho(k) \epsilon^\sigma(q)
\]  

(55)

The case \( q^2 = 0 \) corresponds to the decay of a pion into two real photons.

There are three constraints for the form factor \( F_{\pi \gamma} \):

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• there is a low energy theorem for $F_{\pi\gamma}(0)$,
• the slope of $F_{\pi\gamma}(t)$ at $t = 0$ has been measured and
• there is a perturbative QCD theorem for $q^2 \to -\infty$.

In good agreement with the measured rate for the decay $\pi^0 \to \gamma\gamma$, $F_{\pi\gamma}(0)$ is predicted by the Adler-Bell-Jackiw anomaly [10] as

$$F_{\pi\gamma}(0) = \frac{1}{4\pi^2 f_\pi}$$  \hspace{1cm} (56)

This yields

$$F_V(0) = -0.0270,$$  \hspace{1cm} (57)

which is almost within one standard deviation of the experimental value [5] measured in the radiative pion decay (where $t \approx 0$), which in our phase convention reads

$$F_V(0)_{\text{exp}} = -0.017 \pm 0.008$$  \hspace{1cm} (58)

Perturbative QCD predicts [11]

$$F_{\pi\gamma}(q^2 \to -\infty) \to -\frac{2f_\pi}{q^2}$$  \hspace{1cm} (59)

Furthermore the slope of $F_{\pi\gamma}$ at $t = 0$ has been measured. Using the definition

$$F_{\pi\gamma}(t) = F_{\pi\gamma}(0) \left[ 1 + a \frac{t}{m_{\pi^0}^2} + O(t^2) \right]$$  \hspace{1cm} (60)

the experimental value [5] is

$$a_{\text{exp}} = +0.0326 \pm 0.0026.$$  \hspace{1cm} (61)

Assuming $F_{\pi\gamma}$ to be dominated by $\rho$ like resonances, we will try to parameterize $F_{\pi\gamma}(t)$ in accordance with the constraints described above using the following models:

• monopole

$$F_{\pi\gamma}(t) = \frac{1}{4\pi^2 f_\pi} \text{Res}_X(t)$$  \hspace{1cm} (62)

• dipole

$$F_{\pi\gamma}(t) = \frac{1}{4\pi^2 f_\pi} \left[ \text{Res}_\rho(t) + \lambda \text{Res}_\rho'(t) \right] \frac{1}{1 + \lambda}$$  \hspace{1cm} (63)

• tripole

$$F_{\pi\gamma}(t) = \frac{1}{4\pi^2 f_\pi} \left[ \text{Res}_\rho(t) + \lambda \text{Res}_\rho'(t) + \mu \text{Res}_\rho''(t) \right] \frac{1}{1 + \lambda + \mu}$$  \hspace{1cm} (64)

All these models automatically satisfy the low energy constraint, because the resonance factors are again normalized according to

$$\text{Res}_X(0) = 1$$  \hspace{1cm} (65)

Furthermore we suppose

$$\text{Res}_X(t \to -\infty) \to \frac{m_X^2}{t}$$  \hspace{1cm} (66)
This is fulfilled by
\[ \text{Res}_X(t) = \frac{m_X^2}{m_X^2 - t - im_X \Gamma_{pw}(t)} \] (67)
with an energy dependent width corresponding to the P-wave two body phase space
\[ \Gamma_{pw}(t) = \begin{cases} 0 & \text{if } t \leq 4m_X^2 \\ \frac{m_X}{\sqrt{t}} \left( \frac{\sqrt{t} - 4m_X^2}{\sqrt{m_X^2 - 4m_X^2}} \right)^3 \Gamma_X & \text{else} \end{cases} \] (68)

The form
\[ \text{Res}_X(t) = \frac{m_X^2 - im_X \Gamma_X}{m_X^2 - t - im_X \Gamma_X} \] (69)
with a fixed width $\Gamma_X$ will again be considered for comparison.

If we use the physical rho mass $m_\rho = 768$ MeV in the monopole parametrization, the high energy limit is 14% below the QCD prediction. Brodsky and Lepage note in [12] that the choice
\[ m_X = M_{BL} := 2\sqrt{2}\pi f_\pi = 822 \text{ MeV} \] (70)
satisfies the low and the high energy constraint simultaneously with a monopole formfactor. The slope parameter $a$ depends on the resonance mass as follows:
\[ a = \frac{m_{\pi^0}^2}{m_X^2} = \begin{cases} 0.0309 & \text{if } m_X = m_\rho \\ 0.0270 & \text{if } m_X = M_{BL} \end{cases} \] (71)

So $m_X = M_{BL}$ is not a very good approximation in the low energy region and therefore we will use
\[ m_X = m_\rho \] (72)
for the resonance factor of the monopole parametrization. This monopole version defines the model $N = 1$.

In the dipol parametrization the parameter $\lambda$ must be
\[ \lambda = \frac{M_{BL}^2 - m_\rho^2}{m_{\rho'}^2 - M_{BL}^2} = 0.0584 \] (73)
in order to satisfy the QCD theorem. (If not stated otherwise, particle parameters of [5] are used everywhere.) For the slope parameter this implies
\[ a = \frac{m_{\pi^0}^2}{1 + \lambda} \left( 1 + \lambda \left( \frac{1}{m_\rho^2} + \lambda \frac{m_{\rho'}^2}{m_\rho^2} \right) \right) = 0.0296 \] (74)
in satisfactory agreement with (61). The dipol formfactor defines the model $N = 2$.

In order to determine the parameters $\lambda$ and $\mu$ of the tripol model, two more constraints will be considered, viz. the experimental values for $\Gamma_{\rho \rightarrow \pi \gamma}$ and $\Gamma_{\rho' \rightarrow \pi \omega}$. Using the narrow width approximation, parametrizing
\[ \mathcal{M}(\tau \rightarrow \rho \nu) = G_F \cos \theta_c f_\rho \epsilon_\mu L^\mu \] (75)
and dividing $F_V$ into
\[ F_V(t) = F_V^{(\rho)} + F_V^{(\rho')} + F_V^{(\rho'')} \] (76)

in an obvious notation, $\Gamma_{\rho \to \pi \gamma}$ can be calculated in terms of the parameters of $F_V^{(\rho)}$. The result is

$$\Gamma_{\rho \to \pi \gamma} = \frac{\alpha}{1536\pi^4} m_\rho^7 \left(1 - \frac{m_\pi^2}{m_\rho^2}\right)^3 \frac{1}{(1 + \lambda + \mu)^2}. \tag{77}$$

We will use $f_\rho$ as given by the KSRF relation,

$$f_\rho = \sqrt{2} m_\rho f_\pi = 0.10 \text{ GeV}^2. \tag{78}$$

Equally well we could extract $f_\rho$ directly from $\tau \to \rho \nu$ data:

$$f_\rho^2 = \frac{m_\rho^2 m_\tau^2}{24\pi^2 \cos^2 \theta_C} \left(1 - \left(\frac{m_\nu}{m_\tau}\right)^2\right)^2 \left(1 + 2\left(\frac{m_\nu}{m_\tau}\right)^2\right) \frac{\text{BR}(\tau \to \rho \nu)}{\text{BR}(\tau \to e^- \bar{\nu} \nu)} \tag{79}$$

With present data \[ this is \]

$$f_\rho = 0.11 \text{ GeV}^2 \tag{80}$$

We have checked that the use of this value would not change the final result significantly.

So for $\lambda = \mu = 0$ (monopole) we obtain

$$(\Gamma_{\rho \to \pi \gamma})_{\text{monopole}} = 80.5 \text{ keV} \tag{81}$$

On the other hand the experimental number is \[ this is \]

$$(\Gamma_{\rho \to \pi \gamma})_{\text{exp}} = (68 \pm 7) \text{ keV} \tag{82}$$

and so

$$1 + \lambda + \mu = \sqrt{\frac{(\Gamma_{\rho \to \pi \gamma})_{\text{monopole}}}{(\Gamma_{\rho \to \pi \gamma})_{\text{exp}}}} = 1.088 \tag{83}$$

In order to relate $\Gamma_{\rho' \to \pi \omega}$ to the parameters of $F_V^{(\rho')}$, we assume $F_V^{(\rho')}$ to be given by the Feynman diagram of Fig. 2. So we need $f_{\rho'}$, $g_{\rho' \omega \pi}$ and $f_\omega$. $f_{\rho'}$ can be extracted from a fit to $e^+ e^- \to 2\pi$ data \[ which may proceed through $\rho$, $\rho'$ or $\rho''$ resonance channels (see Fig. 3). Neglecting the pion mass, $g_{\rho \pi \pi}$ and $g_{\rho' \pi \pi}$ are related by

$$\left|\frac{g_{\rho' \pi \pi}}{g_{\rho \pi \pi}}\right| = \sqrt{\frac{\Gamma_{\rho' \to 2\pi} m_\rho}{\Gamma_{\rho \to 2\pi} m_{\rho'}}} = 0.23 \tag{84}$$

where we used \[ this is \]

$$\Gamma_{\rho' \to 2\pi} = 0.24 \times 0.21 \times 310 \text{ MeV} \tag{85}$$

Let the fit to $e^+ e^- \to 2\pi$ be parametrized by

$$(\text{Res}_\rho(t) + \sigma \text{Res}_{\rho'}(t) + \rho \text{Res}_{\rho''}(t)) \frac{1}{1 + \sigma + \rho} \tag{86}$$

Then we have

$$m_\rho^2 : (\sigma m_{\rho'}^2) = (f_\rho g_{\rho \pi \pi}) : (f_{\rho'} g_{\rho' \pi \pi}) \tag{87}$$

Reference \[ gives (see their model $N = 2$, table 1)

$$\sigma = -0.103 \tag{88}$$
Therefore
\[
\left| \frac{f_{\rho'}}{f_{\rho}} \right| = \left| \frac{g_{\rho\pi\pi}}{g_{\rho'\pi\pi}} \right| \frac{m_{\rho'}^2}{m_{\rho}^2} \sigma = 1.63
\]  

(89)

The coupling \( g_{\rho'\omega\pi} \) is defined by
\[
\mathcal{M}(\rho'(q) \rightarrow \omega(k)\pi(p)) = g_{\rho'\omega\pi}\epsilon_{\mu
u\rho\sigma}q^\mu(p)\epsilon^\nu(k)p^\sigma
\]  

(90)

and can be deduced from
\[
\Gamma_{\rho'\rightarrow\pi\omega} = \frac{|g_{\rho'\pi\pi}|^2}{96\pi} m_{\rho'}^3 \left(1 - \frac{m_{\omega}^2}{m_{\rho'}^2}\right)^3 = 0.21 \times 310 \text{ MeV}
\]  

(91)

as
\[
|g_{\rho'\omega\pi}| = 4.1 \times 10^{-3} \text{ MeV}^{-1}
\]  

(92)

Assuming \( f_{\omega} = f_{\rho}/3 \), comparison of the diagram of Fig. 2 and \( F_V^{(\rho')} \) then gives
\[
|\lambda| = \frac{2\sqrt{2}}{3} (1 + \lambda + \mu) \frac{m_{\pi}^2 m_{\rho'}^2}{m_{\rho}^2 m_{\omega}^2} \frac{|g_{\rho'\omega\pi}|}{F_V(0)} \frac{f_{\rho'}}{f_{\rho}} = 0.136
\]  

(93)

If \( \lambda \) and \( 1 + \lambda + \mu \) are given with moderate errors, \( \mu \) can only be calculated with a large uncertainty. So instead we use the QCD limit to express \( \mu \) in terms of \( \lambda \) and \( 1 + \lambda + \mu \) as
\[
\mu = \frac{1}{m_{\rho'}^2} [ (1 + \lambda + \mu) M_{BL}^2 - m_{\rho}^2 - \lambda m_{\rho'}^2 ]
\]  

(94)

We find
\[
\lambda = +0.136 \Rightarrow \mu = -0.051 \Rightarrow 1 + \mu + \lambda = 1.085
\]  

(95)

So in order be consistent with \( 1 + \mu + \lambda = 1.088 \) from \( \Gamma_{\rho'\rightarrow\pi\gamma} \), \( \lambda \) must be +0.136. The slope parameter \( a \) then is
\[
a = \frac{m_{\pi_0}^2}{1 + \lambda + \mu} \left( \frac{1}{m_{\rho}^2} + \frac{\lambda}{m_{\rho'}^2} + \frac{\mu}{m_{\rho''}^2} \right) = 0.0291
\]  

(96)

in satisfactory agreement with equation (61).

So for the tripol model,
\[
\lambda = +0.136
\]
\[
\mu = -0.051
\]  

(97)

is a choice for the two parameters which is consistent with all the four constraints (the QCD limit, the slope \( a \), \( \Gamma_{\rho\rightarrow\pi\gamma} \) and \( \Gamma_{\rho'\rightarrow\pi\omega} \)) and defines the model \( N = 3 \).

### 5 Results for the Spectra and the Decay Rate

We consider as standard the following choices:
- energy dependent widths in both form factors,
- model \( N = 3 \) (tripol) for \( F_V \),
\[ F_A(0) = -0.0116, \text{ and} \]
\[ \Gamma_{a_1} = 0.4 \text{ GeV}. \]

In Fig. 4 (a)–(d) the resulting photon spectrum is displayed for \(0.1 \leq x \leq 1\). For "soft photons" \((x_0 \leq 0.2)\) the internal bremsstrahlung completely dominates (Fig. 4(a)). Note that for very soft photons the multi-photon production rate becomes important, and so our order \(\alpha\) results are not reliable too close to the infrared divergence \(x = 0\). We have checked our internal bremsstrahlung spectrum against the data produced from PHOTOS and we obtain excellent agreement, apart from the region of very large \(x\).

The spectrum is significantly enhanced by structure dependent effects for hard photons \((x_0 \geq 0.4)\) (Fig. 4(b)). In Fig. 4(c) we show that the vector resonances dominate the structure dependent part while in Fig. 4(d) we plot the interference between bremsstrahlung and structure dependent part.

While the integration over the internal bremsstrahlung needs an infrared cut-off, the structure dependent part does not. In order to get a feeling on its importance we have performed the integration over the complete phase space which yields

\[
\begin{align*}
\Gamma_{VV} &= 0.75(0.86) \cdot 10^{-3} \Gamma_{\tau \to \nu \pi} \\
\Gamma_{AA} &= 0.33 \cdot 10^{-3} \Gamma_{\tau \to \nu \pi} \\
\Gamma_{VA} &= 0 \\
\Rightarrow \Gamma_{SD} &= 1.08(1.20) \cdot 10^{-3} \Gamma_{\tau \to \nu \pi}
\end{align*}
\]  

(98)

using the tripol parametrization for \(F_V\). The numbers in parentheses are the predictions of the monopole fit.

With the value of \(\frac{4}{2}\) for the branching ratio \(BR(\tau \to \nu \pi)\) this translates into

\[
\begin{align*}
BR_{VV}(\tau \to \nu \pi \gamma) &= 0.87(1.00) \cdot 10^{-4} \\
BR_{AA}(\tau \to \nu \pi \gamma) &= 0.38 \cdot 10^{-4}
\end{align*}
\]  

(99)

These values may be compared to narrow width estimates: taking into account the lowest lying resonances \(\rho, a_1\) we obtain

\[
\begin{align*}
BR(\tau \to \nu \rho \to \nu \pi \gamma) &\approx BR(\tau \to \nu \rho) \times BR(\rho \to \pi \gamma) \approx 22\% \times 4.5 \cdot 10^{-4} = 0.99 \cdot 10^{-4} \\
BR(\tau \to \nu a_1 \to \nu \pi \gamma) &\approx BR(\tau \to \nu a_1) \times BR(a_1 \to \pi \gamma) \approx 6.8\% \times 1.6 \cdot 10^{-3} = 1.01 \cdot 10^{-4}
\end{align*}
\]  

(100)

Here we have used the central value

\[
BR(a_1 \to \pi \gamma) = \frac{640 \text{ keV}}{400 \text{ MeV}} = 1.6 \cdot 10^{-3}
\]  

(101)

So for both the vector and the axial vector channel the narrow width approximation and the picture of a decay chain \(\tau \to \nu \rho \to \nu \pi \gamma\) give reasonable results (do not forget the large error on \(BR(a_1 \to \pi \gamma))\).

In tables \(\text{I}^\text{I}\) and \(\text{I}^\text{I}\) we display for two values of the photon energy cut how the different parts contribute to the total rate (using the standard parameters). Note that the only term involving \(F_V\) which is of importance is \(\Gamma_{VV}\), while for \(F_A\) the internal bremsstrahlung-structure dependent interference part \(\Gamma_{IB-A}\) is of the same size as \(\Gamma_{AA}\). While the vector-axial interference \(\Gamma_{VA}\) can always be neglected, the internal bremsstrahlung-structure dependent interference \(\Gamma_{INT}\) is rather small but not negligible.
Table 1: Contribution of the different parts to the total rate, using $E_\gamma \geq 400 \text{ MeV}$

| Part       | Expression            | Value         |
|------------|-----------------------|---------------|
| $\Gamma_{IB}(x_0)$ | $1.48 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{VV}(x_0)$  | $0.55 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.22 \cdot 10^{-3}$ |
| $\Gamma_{VA}(x_0)$  | $0.01 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.10 \cdot 10^{-3}$ |
| $\Gamma_{AA}(x_0)$  | $0.31 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.08 \cdot 10^{-3}$ |
| $\Gamma_{SD}(x_0)$  | $1.08 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{IB-V}(x_0)$ | $0.07 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{IB-A}(x_0)$ | $0.35 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |

$\Rightarrow \Gamma_{\text{total}}(x_0) = 2.76 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$. 

Table 2: Contribution of the different parts to the total rate, using $E_\gamma \geq 50 \text{ MeV}$

| Part       | Expression            | Value         |
|------------|-----------------------|---------------|
| $\Gamma_{IB}(x_0)$ | $13.1 \cdot 10^{-4} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{VV}(x_0)$  | $0.75 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.52 \cdot 10^{-3}$ |
| $\Gamma_{VA}(x_0)$  | $0.00 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.03 \cdot 10^{-3}$ |
| $\Gamma_{AA}(x_0)$  | $0.33 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ | $0.25 \cdot 10^{-3}$ |
| $\Gamma_{SD}(x_0)$  | $1.08 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{IB-V}(x_0)$ | $0.05 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |
| $\Gamma_{IB-A}(x_0)$ | $0.57 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$ |               |

$\Rightarrow \Gamma_{\text{total}}(x_0) = 14.8 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu \pi}$. 

15
Table 3: Dependence of the total rate on variation of the parameters. Standard choices are implied wherever not stated otherwise. \( E_\gamma \geq 400 \) MeV.

| Variation                  | \( \Gamma_{\text{total}} \left(10^{-3}\Gamma_{\tau\rightarrow\nu\pi}\right)\) |
|----------------------------|-------------------------------------------------|
| \( N = 1 \) (monopol)     | 2.88                                            |
| \( N = 2 \) (dipol)       | 2.77                                            |
| \( N = 3 \) (tripol)      | 2.76 (standard)                                 |
| fixed widths for the \( \rho \)'s | 2.80                                           |
| \( \Gamma_{a_1} = 0.6 \) GeV, variable width | 2.60                                           |
| \( \Gamma_{a_1} = 0.4 \) GeV, variable width | 2.76 (standard)                                |
| \( \Gamma_{a_1} = 0.25 \) GeV, variable width | 3.05                                           |
| \( \Gamma_{a_1} = 0.6 \) GeV, fixed width | 2.64                                           |
| \( \Gamma_{a_1} = 0.4 \) GeV, fixed width | 2.79                                           |
| \( \Gamma_{a_1} = 0.25 \) GeV, fixed width | 3.07                                           |
| \( F_A(0) = -0.0100 \)     | 2.63                                            |
| \( F_A(0) = -0.0116 \)     | 2.76 (standard)                                 |
| \( F_A(0) = -0.0132 \)     | 2.90                                            |
| \( F_A(0) = +0.0422 \)     | 4.94                                            |
| \( F_A(t) \equiv F_A(0), F_V(t) \equiv F_V(0) \) | 1.25                                           |

In table 3 the dependence of the total rate on the variation of the parameters is displayed for \( E_\gamma \geq 400 \) MeV. It turns out that the uncertainties in the vector form factor \( F_V \) (whether we use a monopole, dipole or tripole form and whether we assume fixed or energy dependent width) only have rather little impact on the total rate. Its uncertainty is dominated by the ambiguities of the axial form factor \( F_A \), in which connection the value for the \( a_1 \) width and the normalization error at \( t = 0 \) are of comparable importance.

Note that choice of the positive sign solution for \( F_A \) almost doubles the integrated rate, and so by measuring \( \tau \rightarrow \nu \pi \gamma \) a clear decision for one solution is possible.

In the last column of this table we display the result of an integration with the formfactors fixed at their values at \( t = 0 \) without Breit-Wigner resonances. It turns out that the Breit-Wigner enhancement substantially increases the total rate.

Fig. 5 displays the integrated decay rate in variation with the photon energy cut \( E_0 \) (\( E_\gamma \geq E_0 \)). We give the prediction using the standard parameter set, a lower and upper limit and the contribution from pure internal bremsstrahlung. The lower limit is obtained using

- \( N = 3 \), tripole form of \( F_V(t) \),
- \( \Gamma_{a_1} = 0.6 \) GeV,
- variable widths for the formfactors, and
- \( F_A(0) = -0.0100 \)

the upper limit corresponds to the following choice:

- \( N = 1 \), monopole form of \( F_V(t) \),
- \( \Gamma_{a_1} = 0.25 \) GeV,
Table 4: Contributions to the integrated rate $\Gamma(\sqrt{t_0})$ using the standard parameter set (all numbers
for the rate in units of $10^{-3}\Gamma_{\tau\rightarrow\nu\pi}$)

| $\sqrt{t_0}$ (MeV) | $\Gamma_{IB}$ | $\Gamma_{VV}$ | $\Gamma_{AA}$ | $\Gamma_{IB-V}$ | $\Gamma_{IB-A}$ | $\Gamma_{total}$ |
|---------------------|--------------|--------------|--------------|----------------|----------------|----------------|
| 400                 | 3.19         | 0.74         | 0.33         | 0.05           | 0.51           | 4.83           |
| 700                 | 0.77         | 0.61         | 0.33         | 0.13           | 0.33           | 2.17           |
| 1000                | 0.19         | 0.09         | 0.28         | 0.07           | 0.08           | 0.72           |

- fixed widths for the formfactors, and
- $F_A(0) = -0.0132$.

In Figs. 6 (a) — (d) we show the pion-photon invariant mass spectrum. We find that a much
better separation between internal bremsstrahlung and structure dependent effects is obtained here
(Fig. 6 (a) and (b)), as compared with the photon spectrum. So this spectrum (in the region
$\sqrt{t} \geq 700$ MeV) is an observable that is much better suited for studying the form factors than
the photon spectrum is. It turns out that with the $a_1$ width of $\Gamma_{a_1} = 400$ MeV, the $\rho$ is the only
resonance visible in the invariant mass spectrum. The total structure dependent (SD) spectrum is
the sum of the vector and axial vector Breit-Wigners (VV and AA), as the vector-axial interference
(VA) vanishes in the invariant mass spectrum after integration over the other kinematical variable.
So in the SD spectrum there is a very soft and broad $a_1$ resonance bump (Fig. 6 (c)). The internal
bremsstrahlung-structure dependent interference radiation (INT) near the $a_1$ is dominated by the
axial part (IB-A). It changes sign at the $a_1$ mass and rises strongly with decreasing $\sqrt{t}$ below $m_{a_1}$
(Fig. 6 (d)). The internal bremsstrahlung (IB) also rises below $m_{a_1}$, and so the $a_1$ bump is completely
erased in the total spectrum (Fig. 6 (b)).

In Fig. 7 we plot the spectrum in the $a_1$ mass region using $\Gamma_{a_1} = 250$, 400 and 600 MeV. The
result is that the $a_1$ resonance peak is visible if and only if the $a_1$ width is small ($\Gamma_{a_1} \approx 250$ MeV).

We close this section by discussing the integrated rate with an invariant mass cut. For this
invariant mass cut $\sqrt{t_0}$ we suggest three different values: The rather low value of 400 MeV, which
gives a rather high integrated rate, the value 700 MeV, which suppresses the internal bremsstrahlung
without loosing much structure dependent radiation and 1000 MeV in order to focus on the axial
channel (see Tab. 4).

For the total rate we find

$$
\Gamma(\sqrt{t_0} = 400 \text{ MeV}) = 4.83 \times 10^{-3}\Gamma_{\tau\rightarrow\nu\pi} \\
\Gamma(\sqrt{t_0} = 700 \text{ MeV}) = 2.17 \times 10^{-3}\Gamma_{\tau\rightarrow\nu\pi} \\
\Gamma(\sqrt{t_0} = 1000 \text{ MeV}) = 0.72 \times 10^{-3}\Gamma_{\tau\rightarrow\nu\pi}
$$

(102)

where the central values correspond to the standard parameter set and the lower and upper limit
are obtained with the same parameter variations as above.

If we vary $\Gamma_{a_1}$ from 250 MeV to 600 MeV and keep the other parameters fixed at their standard
values, the total rate $\Gamma(\sqrt{t_0} = 1000 \text{ MeV})$ varies from 1.00 to 0.56 $10^{-3}\Gamma_{\tau\rightarrow\nu\pi}$, if we vary $F_A(0)$ from $-0.0100$ to $-0.0132$, the total rate varies from 0.64 to 0.81 $10^{-3}\Gamma_{\tau\rightarrow\nu\pi}$. So because of the
experimental uncertainty of $F_A(0)$ it is difficult to get much information on the $a_1$ width by measuring
the integrated rate and the invariant mass spectrum should be used, as discussed above.
6 Estimation of the Decay $\tau \rightarrow \nu K \gamma$

Compared with the pionic mode, the decay $\tau \rightarrow \nu K \gamma$ is both Cabbibo and phase space suppressed, i.e. it is even more rare. So a reasonable estimate for the total rate $\Gamma_K(x_0)$ will be sufficient.

For $\tau \rightarrow \nu K \gamma$, in Secs. 2 and 3 the following substitutions must be made:

\[
\begin{align*}
\cos \theta_c & \rightarrow \sin \theta_c \\
\cos \theta_c & \rightarrow \sin \theta_c \\
f_\pi & \rightarrow f_K = 113 \text{ MeV} \\
F_{V/A}(t) & \rightarrow F^{(K)}_{V/A}(t) \\
m_\pi & \rightarrow m_K \\
\Gamma_{\tau \rightarrow \nu \pi} & \rightarrow \Gamma_{\tau \rightarrow \nu K}.
\end{align*}
\]

Flavor symmetry implies the following relations for the formfactors at $t = 0$:

\[
\begin{align*}
F^{(K)}_A(0) &= \frac{m_K}{m_\pi} F_A(0) = -0.0410, \\
F^{(K)}_V(0) &= \frac{m_K}{m_\pi} F_V(0) = -0.0955. 
\end{align*}
\]

As far as the quantum numbers are concerned, the resonances $K^*(892)$, $K^*(1410)$ and $K^*(1680)$ could contribute in the vector channel and $K_1(1270)$ and $K_1(1400)$ in the axial one. The rates $\Gamma(K_1(1400) \rightarrow K\rho, K\omega)$ and $\Gamma(K^*(1410) \rightarrow K\rho)$, however, are compatible with zero, and so, assuming vector meson dominance, their contribution to $\tau \rightarrow \nu K \gamma$ may be neglected. In the case of the pionic mode, the contribution of the $\rho''$ was found to be small. Therefore we assume that in the kaonic mode the contribution of the $K^*(1680)$ is small also and approximate it by zero. So we are left with the $K^*(892)$ and the $K_1(1270)$ and a monopole form for both form factors:

\[
\begin{align*}
F^{(K)}_A(t) &= F^{(K)}_A(0) \text{Res}_{K_1(1270)}(t) \\
F^{(K)}_V(t) &= F^{(K)}_V(0) \text{Res}_{K^*(892)}(t).
\end{align*}
\]

For simplicity we use constant widths as in equations (45) and (69) for both the $K^*$ and the $K_1$.

The resulting integrated decay rate $\Gamma_K(E_0)$ is plotted in Fig. 8 in dependence on the photon energy cut $E_0$. Note that here the structure dependent effects are much more important than in the pionic case. This is understood easily, because the higher mass of the kaon suppresses the internal bremsstrahlung. So the measurement of structure dependent effects is fairly easy in this decay mode.

We have also performed the integration using a kaon-photon invariant mass cut. The results are:

\[
\begin{align*}
\Gamma_K(\sqrt{t_0} = 800 \text{ MeV}) &= 3.58 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu K} \\
\Gamma_K(\sqrt{t_0} = 1200 \text{ MeV}) &= 0.91 \cdot 10^{-3} \Gamma_{\tau \rightarrow \nu K}.
\end{align*}
\]

7 Summary and Discussion

We have calculated the radiative one pseudoscalar decay modes of the tau lepton. Because of the infrared divergence of the internal bremsstrahlung the integrated decay rates depend on a photon energy cut $E_0$ or equivalently on a meson-photon invariant mass cut $t_0$. The branching ratio for the pionic mode is found to be of the order of $10^{-3}$ ($10^{-4}$), using a rather low (high) value for $E_0$. For the kaonic mode, the branching ratio is of the order of $10^{-5}$. So at least the pionic radiative mode should be observable presently.
The rate for $\tau \to \nu \pi \gamma$ is dominated by the internal bremsstrahlung, and so, in order to see structure dependent effects, sufficient statistics and/or investigation of the invariant mass spectrum rather than the photon energy spectrum are needed. Because of the higher mass of the kaon, the observation of “structure” will be easier in the decay $\tau \to \nu K \gamma$.

Our basic assumption in the parametrization of the form factors $F_V$ and $F_A$ was that they are dominated by hadron resonances with suitable quantum numbers.

In order to parametrize $F_V$ in the pionic case, we then used the CVC-hypothesis to relate $F_V(t)$ to $F_{\pi \gamma}(t)$ and the QCD-theorem on $\lim_{t \to \infty} \left( t F_{\pi \gamma}(t) \right)$ and assumed a saturation of the QCD-theorem by the $\rho$-like resonances of lowest mass. While the other assumptions seem rather safe, there is no obvious a priori justification for the last one. We find, however, that the $\rho(770)$ alone saturates the QCD-theorem at the 86% level and that the inclusion of the $\rho'$ and the $\rho''$ changes the picture only little, which supports our assumption a posteriori. Then if this saturation is believed in, the remaining ambiguities in the parametrization of $F_V(t)$ have only little impact on the decay rate for $\tau \to \nu \pi \gamma$.

In the axial channel the $a_1$ is the only known resonance which has the correct quantum numbers. Still $F_A(t)$ dominates the uncertainty of the prediction for $\tau \to \nu \pi \gamma$, because neither $F_A(0)$ from radiative pion decay nor the width of the $a_1$ are known very well experimentally. In this paper we took the attitude that the negative sign solution for $F_A$ (corresponding to $\gamma \approx 1/2$) is the physical one. Measurements from pions at rest also allow for a solution with $\gamma \approx -2.5$. We showed that the prediction for $\tau \to \nu \pi \gamma$ changes very significantly if $\gamma \approx -2.5$, and so a measurement of this decay will allow for a clear independent check of $\gamma \approx 1/2$. Furthermore we showed that by measuring the pion-photon invariant mass spectrum in the $a_1$ region a decision for a small or large $a_1$ width can be made. So it will be very interesting to see whether the measurement of $\tau \to \nu \pi \gamma$ will support the large $\Gamma_{a_1}$ (of 400...600 MeV) as measured in the channel $\tau \to \nu 3\pi$ or a small $a_1$ width (of about 300 MeV), as prefered by hadronically produced $a_1$.

So we believe that once a high statistics sample of $\tau \to \nu \pi \gamma$ is available we will learn much about low energy physics.

In a forthcoming paper we will address the question of soft photons and present the total decay rate $\Gamma(\tau \to \nu \pi \gamma + \gamma)$.

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