Selfsimilar Hessian and conformally Kähler manifolds

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December 15, 2021

Abstract

Let \((M, \nabla, g)\) be a Hessian manifold. Then the total space of the tangent bundle \(TM\) can be endowed with a Kähler structure \((I, g^r)\). We say that a homogeneous Hessian manifold is a Hessian manifold \((M, \nabla, g)\) endowed with a transitive action of a group \(G\) preserving \(\nabla\) and \(g\). We construct by a Hessian (special Kähler) structure on a simply connected manifold with a certain condition a Kähler (hyper-Kähler) structure on the tangent (cotangent) bundle. A selfsimilar Hessian (Kähler) manifold is a Hessian manifold endowed with a homothetic vector field \(\xi\). We construct by a selfsimilar Hessian (Kähler) structure on a simply connected manifold with a certain condition a conformally Kähler (hyper-Kähler) structure on the tangent (cotangent) bundle.

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1 Introduction

An open cone \(V \subset \mathbb{R}^n\) is called regular if it does not contain any straight full line. The tube neighborhood \(V + \sqrt{-1} \mathbb{R}^n \subset \mathbb{C}^n\) is biholomorphic to a bounded complex domain. All complex domains arising by this way are called Siegel domains of the first kind. All complex affine

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\ †Pavel Osipov is partially supported by the HSE University Basic Research Program, Simons Foundation, and by the contest “Young Russian Mathematics”.

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automorphisms of $V + \sqrt{-1} \mathbb{R}^n \subset \mathbb{C}^n$ can be written in the form $x + \sqrt{-1} y \rightarrow A x + \sqrt{-1} (A x + b)$ where $A$ is a linear automorphism of $V$ and $b \in \mathbb{R}^n$. We consider linear automorphisms of cones and complex affine automorphisms of Siegel domains of the first kind. In these assumptions, $V + \sqrt{-1} \mathbb{R}^n$ is homogeneous if and only if $V$ is homogeneous. Any homogeneous Siegel domain of the first kind admits an invariant Kähler structure (see [VGP] or [C]).

In this paper we modify the construction of invariant Kähler structures on Siegel domains of the first kind and get a construction of a certain class of homogeneous conformally Kähler manifolds. In particular, we confirm that homogeneous Siegel domains of the first kind admit invariant conformally Kähler structures.

A flat affine manifold is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all translation maps between charts are affine transformations (see [FGH] or [Sh]). A Hessian manifold is an affine manifold with a Riemannian metric which is locally equivalent to a Hessian of a function. Any Kähler metric can be locally defined as a complex Hessian $\partial \bar{\partial} \varphi$. Thus, the Hessian geometry is a real analogue of the Kähler one.

Hessian manifolds have many different applications: in supersymmetry ([CMMS1], [CM], [AC]), in convex programming ([N], [NN]), in the Monge-Ampère Equation ([F1], [F2], [Gu]), in the WDVV equations ([T]).

A Kähler structure $(I, g^{\ast})$ on $TM$ can be constructed by a Hessian structure $(\nabla, g)$ on $M$ (see [Sh]). The correspondence

$r : \{\text{Hessian manifolds}\} \rightarrow \{\text{Kähler manifolds}\}$

$$(M, \nabla, g) \rightarrow (TM, I, g^{\ast})$$

is called the (affine) r-map. In particular, this map associates some special Kähler manifolds to special real manifolds (see [AC]). In this case, r-map describes a correspondence between the scalar geometries for supersymmetric theories in dimension 5 and 4. See [CMMS1] for details on the r-map and supersymmetry.

Any regular cone admits a function $\varphi$ called characteristic function such that $g_{\text{can}} = \text{Hess} (\ln \varphi)$ is a Hessian metric which is invariant with respect to all automorphisms of the cone. ([V]) The r-map constructs an invariant Kähler structure $(I, g_{\text{can}})$ on $TV \simeq V \oplus \sqrt{-1} \mathbb{R}^n$. Thus, any homogeneous Siegel domain of the first kind admits an invariant Kähler structure. The Kähler potential of $g_{\text{can}}^{\ast}$ equals $4 \pi^{n} (\ln \varphi)$.

The construction of the invariant Kähler structure on $V \oplus \sqrt{-1} \mathbb{R}^n$ is well known (see [VGP] or [C]). We modify this construction. A (globally) conformally Kähler manifold $(M, I, \omega)$ is a complex manifold endowed with a Riemannian metric $g$ which is (globally) conformally equivalent to a Kähler one. We consider the metric $g_{\text{can}} = \text{Hess} \varphi$ on a regular homogeneous cone $V$. This metric is invariant under $\text{Aut}(V) \cap \text{SL}(\mathbb{R}^n)$ and coincides with $g_{\text{can}}$ on the hypersurface $\{ \varphi(x) = 1 \}$. The dilation $x \mapsto qx$ acts on $g_{\text{can}}$ by $\lambda^2 q^{n} g_{\text{can}} = \lambda^{n} q^{-n} g_{\text{can}}$. The Kähler metric $g_{\text{can}}^{\ast}$ on $V \oplus \sqrt{-1} \mathbb{R}^n$ constructed by the r-map is not invariant but it is conformally equivalent to the invariant Riemannian metric $r^{-2} g_{\text{can}}^{\ast}$ on the homogeneous domain $V \times \sqrt{-1} \mathbb{R}^n$. Thus, Siegel domains of the first kind admit two different invariant structures: Kähler and conformally Kähler.

We generalize this construction to selfsimilar Hessian manifolds. A selfsimilar Hessian manifold is a Hessian manifold endowed with a homothetic vector field generating a flow of affine automorphisms. For example, an $n$-dimansional regular convex cone $(V, g_{\text{can}})$ endowed with a field $-\frac{2}{n} \sum x^{i} \frac{\partial}{\partial x^{i}}$ is a selfsimilar Hessian manifold.

The main result of the paper is the following.
Theorem 1.1. Let \((M, \nabla, g, \xi)\) be a simply connected selfsimilar Hessian manifold such that \(\xi\) is complete and \(G\) be the group of affine isometries of \((M, \nabla, g)\) preserving \(\xi\). Suppose that \(G\) acts simply transitively on the level line \(\{g(\xi, \xi) = 1\}\). Then \(TM\) admits a homogeneous conformally Kähler structure.

Also, we adapt our construction for the c-map. A **special Kähler manifold** \((M, I, g, \nabla)\) is a Kähler manifold \((M, I, g)\) endowed with a torsion free symplectic connection \(\nabla\) such that such that \(g\) is a Hessian metric with respect to \(g\) (see [ACD]).

A hyper-Kähler structure \((I_1, I_2, I_3, g^c)\) on \(T^*M\) can be constructed by a special Kähler structure \((I, g, \nabla)\) on \(M\). The correspondence
c : \{Special Kähler manifolds\} \rightarrow \{hyper-Kähler manifolds\}

\((M, I, g, \nabla) \rightarrow (T^*M, I_1, I_2, I_3, g^c)\)
is called the **(affine) c-map**. r-map describes a correspondence between the scalar geometries for supersymmetric theories in dimension 4 and 3. See [CMMS2] and [ACM] for details on the r-map.

A **selfsimilar special Kähler manifold** is a special Kähler manifold \((M, I, g, \nabla)\) endowed with an affine vector field \(\xi\) satisfying \(L_\xi g = 2g\). For example, any conic special Kähler manifold is selfsimilar special Kähler (see [ACM] for the definition).

Theorem 1.2. Let \((M, I, g, \nabla, \xi)\) be a selfsimilar special Kähler manifold, such that \(\xi\) is complete and \(G\) the group of affine holomorphic isometries of \((M, I, g, \nabla)\) preserving \(\xi\). Suppose that \(G\) acts simply transitively on the level line \(\{g(\xi, \xi) = 1\}\). Then \(T^*M\) admits a homogeneous conformally hyper-Kähler structure.

2 Hessian and Kähler structures

**Definition 2.1.** A flat affine manifold is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all translation maps between charts are affine transformations (see [FGH]).

**Definition 2.2.** A Riemannianian metric \(g\) on a flat affine manifold \((M, \nabla)\) is called to be a **Hessian metric** if \(g\) is locally expressed by a Hessian of a function \(g = \text{Hess} \varphi = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i dx^j\), where \(x^1, \ldots, x^n\) are flat local coordinates. A **Hessian manifold** \((M, \nabla, g)\) is a flat affine manifold \((M, \nabla)\) endowed with a Hessian metric \(g\). (see [Sh]).

Let \(U\) be an open chart on a flat affine manifold \(M\), functions \(x^1, \ldots, x^n\) be affine coordinates on \(U\), and \(x^1, \ldots, x^n, y^1, \ldots, y^n\) be the corresponding coordinates on \(TU\). Define the complex structure \(I\) by \(I\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}\). Corresponding complex coordinates are given by \(z^i = x^i + \sqrt{-1} y^i\). The complex structure \(I\) does not depend on a choice of flat coordinates on \(U\). Thus, in this way, we get a complex structure on the \(TM\).

Let \(\pi : TM \rightarrow M\) be a natural projection. Consider a Riemannian metric \(g\) on \(M\) given locally by

\[ g_{i,j} dx^i dx^j. \]
Define a bilinear form $g^r$ on $TM$ by

$$g^r = \pi^* g_{i,j} \left( dx^i dx^j + dy^i dy^j \right)$$

or, equivalently,

$$g^r(X,Y) = (\pi^* g)(X,Y) + (\pi^* g)(IX, IY),$$

for any $X, Y \in T(TM)$.

**Proposition 2.3** ([Sh], [AC]). Let $M$ be a flat affine manifold, $g$ and $g^r$ as above. Then the following conditions are equivalent:

(i) $g$ is a Hessian metric.

(ii) $g^r$ is a Kähler metric.

Moreover, if $g = \text{Hess} \varphi$ locally then $g^r$ is equal to a complex Hessian

$$g^r = \text{Hess}_C(4\pi^* \varphi) = \partial \bar{\partial} (4\pi^* \varphi).$$

**Definition 2.4.** The metric $g^r$ is called a Kähler metric associated to $g$. The correspondence which associates the Kähler manifold $(TM, g^r)$ to a Hessian manifold $(M, g)$ is called the (affine) r-map (see [AC]).

### 3 The r-map for homogenous manifolds

**Definition 3.1.** Let $G$ be a group of affine automorphisms of a simply connected affine manifold $M$ and $\phi$ be the corresponding action. Then there is an affine action $\theta$ of the group $G$ on $\mathbb{R}^n$ and a $G$-equivariant local affine diffeomorphism

$$\text{dev}: M \to \mathbb{R}^n$$

called the development map [Go]. The $G$-equivariance means that for any $g \in G$ we have the following diagram

$$\begin{array}{ccc}
M & \xrightarrow{\text{dev}} & \mathbb{R}^n \\
\downarrow \phi(g) & & \downarrow \theta(g) \\
M & \xrightarrow{\text{dev}} & \mathbb{R}^n
\end{array}$$

Note that the development map can be not injective: if $\text{dev}(p) = \text{dev}(q)$ then for any $g \in G$ we have $\text{dev}(\phi_g(p)) = \text{dev}(\phi_g(q))$.

A flat connection $\nabla$ on a simply connected manifold $M$ sets a trivialization $TM \simeq M \times \mathbb{R}^n$. Since the development map is defined we can consider any tangent vector from $T(TM) = T(M \times \mathbb{R}^n)$ as a vector $X \times Y \in \mathbb{R}^n \times \mathbb{R}^n$. The complex structure described in the previous section acts by the rule $I(X \times Y) = -Y \times X$.

Define an action $\psi$ of $G \ltimes \mathbb{R}^n$ on $TM \simeq M \times \mathbb{R}^n$ by

$$\psi_{a \times u}(m \times v) = \phi_a(m) \times (\theta_a(v) + u),$$

where $a \in G, m \in M$, and $u, v \in \mathbb{R}^n$. 

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Proposition 3.2. Let $G, M, \psi, \theta$ be as above. Then the complex structure $I$ on $TM$ constructed from the affine structure on $M$ is $G \ltimes \theta \mathbb{R}^n$-invariant.

Proof. Since there is the affine immersion $\text{dev} : M \to \mathbb{R}^n$, it is enough to prove the proposition for $M \subset \mathbb{R}^n$. In this case, action $\phi$ and $\theta$ define the same affine automorphism of $\mathbb{R}^n$ for any $g \in G$. The action $\psi$ of $G$ on any tangent vector $X \times Y \in TM \times \mathbb{R}^n$ is defined by

$$
\psi_g(X \times Y) = A_gX \times A_gY,
$$

where $A_g$ is a linear part of an affine action $\phi_g$ on $\mathbb{R}^n$. The complex structure $I$ is defined by $I(X \times Y) = -Y \times X$. Therefore, the action $\phi$ preserves the complex structure.

Theorem 3.3. Let $(M, \nabla, g)$ be a simply connected Hessian manifold, $(I, g^f)$ the corresponding Kähler structure on $TM = M \times \mathbb{R}^n$, $G$ the group of affine isometries of $(M, \nabla, g)$, $\theta$ the action of $G$ on $\mathbb{R}^n$, and $\psi$ be the action of $G \ltimes \theta \mathbb{R}^n$ as above. Then the Kähler structure $(I, g^f)$ is invariant under the action $\psi$ of $G \ltimes \theta \mathbb{R}^n$. In particular, if $(M, \nabla, g)$ is a homogeneous Hessian manifold for the group $G$ then $TM$ is a homogeneous Kähler manifold for the group $G \ltimes \theta \mathbb{R}^n$.

Remark 3.1. Theorem 3.3 is known in the case of symmetric homogeneous Hessian manifold. Any simply connected symmetric homogeneous Hessian manifold is a direct product of flat space $\left( \mathbb{R}^k, \sum_{i=1}^k (dx^i)^2 \right)$ and a regular convex cone $(V, g_{can})$ (Sh). As we say in the introduction invariant Kähler metric on $TV \simeq V + \sqrt{-1} \mathbb{R}^n$ is described in [VGP].

Proof of Theorem 3.3. According to Proposition 3.2 the complex structure $I$ is $G \ltimes \theta \mathbb{R}^n$-invariant.

By Proposition 2.3, $g^f$ is is locally expressed by $\text{Hess}_\mathbb{C}(4 \pi^* \varphi)$. The subgroup $\{e\} \ltimes \mathbb{R}^n \subset G \ltimes \theta \mathbb{R}^n$ acts trivially on the first factor of the product $M \times \mathbb{R}^n$. The function $\pi^* \varphi$ depends only on the first factor of $M \times \mathbb{R}^n$. Hence, $g^f = \text{Hess}_\mathbb{C}(4 \pi^* \varphi)$ is invariant under the action of $\{e\} \ltimes \theta \mathbb{R}^n \subset G \ltimes \theta \mathbb{R}^n$.

By (2.1), $g^f(X, Y) = \pi^* g(X, Y) + \pi^* g(IX, IY)$. Moreover, $g$ is invariant under the action of $G = G \ltimes \theta 0$. Hence, $\pi^* g$ is invariant under the action of $G \ltimes 0 \subset G \ltimes \theta \mathbb{R}^n$. Thus, $g^f$ is $G \ltimes 0$-invariant. Therefore, $g^f$ is $G \ltimes \theta \mathbb{R}^n$-invariant.

Actions of subgroups $\{e\} \ltimes \theta \mathbb{R}^n$ and $G \ltimes \theta 0 \subset G \ltimes \theta \mathbb{R}^n$ preserve the Kähler structure $(I, g^f)$. Since these subgroups generate $G \ltimes \theta \mathbb{R}^n$, the Kähler structure $(I, g^f)$ is $G \ltimes \theta \mathbb{R}^n$-invariant.

4 Selfsimilar Hessian and conformally Kähler structures

Definition 4.1. A field $\xi$ on an affine manifold $(M, \nabla)$ is called affine if the flow along $\xi$ preserves the connection $\nabla$.

Proposition 4.2. Let $\xi$ be an affine field on a simply connected affine manifold $(M, \nabla)$. Then there exists a vector field $\hat{\xi}$ on $\mathbb{R}^n$ such that

$$
\text{dev}_\nabla \xi = \hat{\xi}.
$$

Proof. The field $\xi$ defines an infinitesimal affine automorphism $A$ of $M$. Since any affine automorphisms of $M$ sets an affine automorphism $B$ of $\mathbb{R}^n$ (see Definition 3.2), $A$ sets an infinitesimal affine automorphism of $\mathbb{R}^n$. Since the developing map is $\text{Aut}(M, \nabla)$-equivariant, the field $\hat{\xi}$ of the infinitesimal affine affine automorphism $B$ satisfies $\text{dev}_\nabla \xi = \hat{\xi}$. 

\[\hfill \square\]
Definition 4.3. A selfsimilar Hessian manifold \((M, \nabla, g, \xi)\) is a Hessian manifold \((M, \nabla, g)\) endowed with an affine field \(\xi\) satisfying \(\mathcal{L}_\xi g = 2g\).

Lemma 4.4. Let \((M, \nabla, g, \xi)\) be a simply connected selfsimilar Hessian manifold. Consider the field \(\hat{\xi}\) on \(\mathbb{R}^n\) defined in Proposition 4.2. Denote by \(\pi\) the natural projection from \(M \times \mathbb{R}^n\) to \(M\) and \(\xi_1 := \xi \times 0 \in TM \times T\mathbb{R}^n\), \(\xi_2 := 0 \times \xi \in TM \times T\mathbb{R}^n\). Then we have:

(i) \(\mathcal{L}_{\xi_1} \pi^* g = 2 \pi^* g\).

(ii) \(\mathcal{L}_{\xi_2} \pi^* g = 0\).

(iii) \(\mathcal{L}_{\xi_1 + \xi_2} I = 0\).

where \(\mathcal{L}\) is the Lie derivative.

Proof. i) Since \(\pi^* g\) depends only on the projection \(\pi\) on \(M\), we have

\[
\mathcal{L}_{\xi_1} \pi^* g = \pi^* (\mathcal{L}_{\pi^* \xi_1} g) = \pi^* \mathcal{L}_\xi g = 2 \pi^* g.
\]

ii) Since \(\pi^* g\) depends only on the projection \(\pi\) on \(M\) and \(\pi^* \xi_2 = 0\), we have

\[
\mathcal{L}_{\xi_2} \pi^* g = \pi^* (\mathcal{L}_{\pi^* \xi_2} g) = 0.
\]

iii) Since the condition is local, we can suppose that \(M \subset \mathbb{R}^n\). The field \(\xi\) is locally defines a 1-parameter family of actions \(A_t = \text{exp} t\xi\). Then the field \(\xi_1 + \xi_2\) locally defines 1-parameter family of actions \(A_t \times A_t\) on \(M \times \mathbb{R}^n\). The action \(A_t \times A_t\) preserves the complex structure. Thus \(\xi_1 + \xi_2 = \frac{d}{dt} (A_t \times A_t)|_{t=0}\) is holomorphic. \(\square\)

Definition 4.5. A selfsimilar Kähler manifold \((M, I, g, \xi)\) is a Kähler manifold \((M, I, g)\) endowed with a holomorphic vector field \(\xi\) satisfying

\[
\mathcal{L}_\xi g = 2g.
\]

Proposition 4.6. Let \((M, \nabla, g, \xi)\) be a simply connected selfsimilar Hessian manifold, \(I, g^r, \xi_1, \xi_2\) as above. Then \((M \times \mathbb{R}^n, I, g^r, \xi_1 + \xi_2)\) is a selfsimilar Kähler manifold.

Proof. By Lemma 4.4 we have

\[
\mathcal{L}_{\xi_1 + \xi_2} \pi^* g = 2 \pi^* g \quad \text{and} \quad \mathcal{L}_{\xi_1 + \xi_2} I = 0.
\]

Combining this with (2.1), we obtain that

\[
\mathcal{L}_{\xi_1 + \xi_2} g^r = 2 g^r.
\]

By the item (iii) of Lemma 4.4, \(\xi_1 + \xi_2\) is a holomorphic vector field. Thus, \((M \times \mathbb{R}^n, I, g^r, \xi_1 + \xi_2)\) is a selfsimilar Kähler manifold. \(\square\)

Lemma 4.7. Let \((M, \nabla, g, \xi)\) be a simply connected selfsimilar Hessian manifold, \((I, g^r)\) the corresponding Kähler structure on \(TM = M \times \mathbb{R}^n\), and \(G\) the group of affine isometries of \(M\) preserving \(\xi\). Then \(\pi^* g(\xi_1, \xi_1)\) is a \(G \ltimes \mathbb{R}^n\)-invariant function on \(M \times \mathbb{R}^n\) satisfying

\[
\mathcal{L}_{\xi_1 + \xi_2} (\pi^* g(\xi_1, \xi_1)) = 2 \pi^* g(\xi_1, \xi_1)
\]

at any point.
Proof. Since $\pi^*g$ and $\xi$ are $G \ltimes \theta \mathbb{R}^n$-invariant, the function $\pi^*g(\xi_1, \xi_1)$ is $G \ltimes \mathbb{R}^n$-invariant. A Lie derivative of a metric if defined by

$$(L_X g)(X, Y) = (L_X) g(X, Y) - g(L_X Y, Z) - g(Y, L_X Z).$$

We have $[\xi_1, \xi_2] = 0$. Therefore,

$$L_{\xi_1+\xi_2} (\pi^*g(\xi_1, \xi_1)) = (L_{\xi_1+\xi_2} \pi^*g) \pi^*g(\xi_1, \xi_1).$$

Combining this with items (i) and (ii) of Lemma 4.4, we get

$$L_{\xi_1+\xi_2} (\pi^*g(\xi_1, \xi_1)) = 2\pi^*g(\xi_1, \xi_1).$$

\[\square\]

Definition 4.8. A (globally) conformally Kähler structure on a manifold $N$ is a collection $(I, f\omega)$, where $(I, \omega)$ is a Kähler structure and $f$ is a positive definite function on $N$.

Let $G$ be the group of affine isometries of a selfsimilar affine manifold $(M, \nabla, \xi, g)$ preserving $\xi$ and $\xi$ is complete. Let $\rho_1 = (\exp t\xi)$ be an affine action of $\mathbb{R}$ on $M$. The action of $G$ preservers $\xi$. Therefore, this action commutes with $\rho$. Thus, there is an action of $\mathbb{R} \times G$ on $M$ defined by $\theta_{1\times g}(m) = \rho_1(\theta_g(m))$.

Theorem 4.9. Let $(M, \nabla, g, \xi)$ be a simply connected selfsimilar Hessian manifold, $(I, \omega)$ the corresponding Kähler structure on $M \times \mathbb{R}^n$, $G$ the group of affine isometries of $(M, \nabla, g)$ preserving $\xi$, the actions $\theta$ and $\tilde{\theta}$ as above, $\pi : M \times \mathbb{R}^n \rightarrow M$ the natural projection and $\omega_{c.K.} = \pi^*(g(\xi, \xi)^{-1}) \omega$. Then the following conditions are satisfies:

i) The pair $(I, \omega_{c.K.})$ is a $G \ltimes \theta \mathbb{R}^n$-invariant conformally Kähler structure on $M \times \mathbb{R}^n$.

ii) If the field $\xi$ is complete then $(I, \omega_{c.K.})$ is $(\mathbb{R} \times G) \ltimes \theta \mathbb{R}^n$-invariant.

iii) If the field $\xi$ is complete and $G$ acts transitively on the level surface $S = \{g(\xi, \xi) = 1\} \subset M$ then $(M \times \mathbb{R}^n, I, \omega_{c.K.})$ is homogeneous conformally Kähler manifold for the group $(\mathbb{R} \times G) \ltimes \theta \mathbb{R}^n$.

Proof. i) According to Lemma 4.7, the function $(\pi^*g(\xi_1, \xi_1))^{-1}$ is $G \ltimes \mathbb{R}^n$-invariant. According to Theorem 4.3, $\omega$ is $G \ltimes \mathbb{R}^n$-invariant. Hence, $\omega_{c.K.}$ is $G \ltimes \mathbb{R}^n$-invariant. By Proposition 3.2, the complex structure $I$ is $G \ltimes \mathbb{R}^n$-invariant. Thus, the conformally Kähler structure $(I, \omega_{c.K.})$ is $G \ltimes \mathbb{R}^n$-invariant.

ii) The action of a subgroup

$$(\mathbb{R} \times 0) \ltimes \theta 0 \subset (\mathbb{R} \times G) \ltimes \theta \mathbb{R}^n.$$

is generated by the vector field $\xi_1 + \xi_2$. By Lemma 4.7, we have

$$L_{\xi_1+\xi_2} (\pi^*g(\xi_1, \xi_1))^{-1} = (\pi^*g(\xi_1, \xi_1))^{-2} L_{\xi_1+\xi_2} (\pi^*g(\xi_1, \xi_1))^{-1} = -2 (\pi^*g(\xi_1, \xi_1))^{-1}. $$

Therefore,

$$L_{\xi_1+\xi_2 \omega_{c.K.}} = L_{\xi_1+\xi_2} (\pi^*(g(\xi, \xi)^{-1}) \omega) = L_{\xi_1+\xi_2} (\pi^*g(\xi_1, \xi_1))^{-1} \omega + L_{\xi_1+\xi_2} \omega = -2\omega + 2\omega = 0.$$
Hence, if the field $\xi$ is complete then $\omega_{c,K}$ is invariant under the action of the subgroup $(\mathbb{R} \times 0) \rtimes \hat{\theta}$. By Lemma 4.4, the complex structure $I$ is invariant under the action of $(\mathbb{R} \times 0) \rtimes \hat{\theta}$. As we showed above, the c.K. structure $(I, \omega_{c,K})$ is invariant under the action of the group $G \rtimes \hat{\theta} \mathbb{R}^n = (0 \times G) \rtimes \hat{\theta} \mathbb{R}^n$.

Therefore, $(I, \omega_{c,K})$ is $(\mathbb{R} \rtimes G) \rtimes \hat{\theta} \mathbb{R}^n$-invariant.

iii) The subgroups $(\mathbb{R} \times \{e\}) \rtimes \hat{\theta} \{0\} \subset (\mathbb{R} \rtimes G) \rtimes \hat{\theta} \mathbb{R}^n$ act on the function $\pi^*g(\xi, \xi)$ be the rule

$$\psi_{(r \times e)\rtimes \hat{\theta} \{0\}}^* g(\xi, \xi) = r^2 \pi^* g(\xi, \xi).$$

Thus, we can can send any point of $M \times \mathbb{R}^n$ by the action of this group to a point of $S \times \mathbb{R}^n$. The group $(\{0\} \rtimes G) \rtimes \hat{\theta} \mathbb{R}^n = G \rtimes \hat{\theta} \mathbb{R}^n$ acts transitively on $S \times \mathbb{R}^n$. Therefore, $(\mathbb{R} \times G) \rtimes \hat{\theta} \mathbb{R}^n$ acts transitively on $M \times \mathbb{R}^n$.

5 An example: homogeneous regular convex cones

Definition 5.1. A set $V \in \mathbb{R}^n$ is a cone if and only if for any $x \in V$ and $a \in \mathbb{R}^+0$ we have $ax \in V$. We say that a cone $V$ is regular if it does not contain any straight full line.

Theorem 5.2 $([V], [VGP])$. Let $V$ be a homogeneous convex cone. Then there exists a function $\varphi$ satisfying the following conditions.

(i) The bilinear form $g_{can} = \text{Hess}(\ln \varphi)$ is $\text{Aut}(V)$-invariant Hessian metric.

(ii) The level line $\varphi = 1$ is $\text{Aut}_{SL}(V)$-invariant.

(iii) The bilinear form $g'_{can}$ is $\text{Aut}(V) \rtimes \mathbb{R}^n$-invariant Kähler structure on $TM \simeq V \oplus \sqrt{-1} \mathbb{R}^n$.

(iv) The bilinear form $g_{con} = \text{Hess} \varphi$ is $\text{Aut}_{SL}(V)$-invariant Hessian metric satisfying the condition $\mathcal{L}_\rho g_{con} = -ng$, where $\rho = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the radiant vector field.

According to item (iv) of Theorem 5.2, $(V, g_{con}, -\frac{2}{n} \rho)$ is a selfsimilar Hessian manifold.

Definition 5.3. Let $V$ be a convex regular cone. Then $V \oplus \sqrt{-1} \mathbb{R}^n$ is biholomorphic to a bounded domain in $\mathbb{C}^n$. All complex domains arising by this way are called Siegel domain of the first kind.

The item (iii) of Theorem 5.2 states that any homogeneous Siegel domain of the first kind admits a homogeneous Kähler structure. Applying Theorem 4.9 to the manifold $V$ with a metric $g_{con}$ and action of $\text{Aut}_{SL}(V)$ we get a similar result.

Corollary 5.4. Any homogeneous Siegel domain of the first kind admits a homogeneous conformally Kähler structure.

6 The c-map for homogeneous manifolds

Definition 6.1. A special Kähler manifold $(M, I, g, \nabla)$ is a Kähler manifold $(M, I, g)$ endowed with a torsion free symplectic connection $\nabla$ such that such that $g$ is a Hessian metric with respect to $g$. 

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Theorem 6.2 ([CMMS2], [ACM]). Let \((M, I, g, \nabla)\) be a special Kähler manifold. Using the connection \(\nabla\) we can identify \(T(TM) = T^h(TM) \oplus T^v(TM) = \pi^*TM \oplus \pi^*TM\), where \(\pi : TM \to M\) is the canonical projection, \(T^h(TM)\) and \(T^v(TM)\) are the horizontal and vertical distribution defined by \(\nabla\). Using this identification we define

\[
g^c = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad I_1 = \begin{pmatrix} I & 0 \\ 0 & I^* \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad I_3 = I_1I_2.
\]

Then \((T^*M, I_1, I_2, I_3, g^c)\) is a hyper-Kähler manifold.

Definition 6.3. The correspondence which associates the hyper-Kähler manifold \((T^*M, I_1, I_2, I_3, g^c)\) to a special Kähler manifold \((M, I, g, \nabla)\) is called the (affine) c-map (see [AC]).

The flat connection \(\nabla\) sets a trivialization \(TM \simeq M \times \mathbb{R}^n\). An affine action \(\phi\) of \(G\) on \(M\) sets an affine action of \(G\) on \(\mathbb{R}^n\) (see section 3). Since the bilinear form \(\omega\) is \(\nabla\)-flat, we can consider \(\omega\) as a bilinear form on \(\mathbb{R}^n\). Hence, \(\omega\) sets an identification of \(\mathbb{R}^n\) and \((\mathbb{R}^n)^*\). Thus, we can consider the conjugate action \(\hat{\theta}\) of \(G\) on \((\mathbb{R}^n)^*\). Define the action \(\hat{\psi}\) of \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^*\) on \(T^*M \simeq M \times (\mathbb{R}^n)^*\) by

\[
\hat{\psi}_{\alpha \times \theta}^a(m \times v) = \phi_a(m) \times \left(\hat{\theta}_a(v) + u\right).
\]

The \(\nabla\)-flat bilinear form \(\omega\) sets an identification \(TM \simeq M \times \mathbb{R}^n \simeq M \times (\mathbb{R}^n)^* \simeq T^*M\). Thus, we can define an action of \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^* = G \ltimes_{\hat{\theta}} \mathbb{R}^n\) on \(T^*M \simeq M \times (\mathbb{R}^n)^*\). and fields \(\xi_1 \in T^h(TM), \xi_2 \in T^v(TM)\).

Theorem 6.4. Let \((M, I, g, \nabla)\) be a special Kähler manifold, \(G\) the group of affine holomorphic isometries of \((M, I, g, \nabla)\), \(\hat{\theta}\) the action of \(G\) on \((\mathbb{R}^n)^*\), \(\hat{\psi}\) the action of \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^*\) on \((\mathbb{R}^n)^*\) as above. Then the hyper-Kähler structure \((I_1, I_2, I_3, g^c)\) is invariant under the action \(\hat{\psi}\) of \(G \ltimes_{\hat{\theta}} \mathbb{R}^n\). In particular, if \((M, I, g, \nabla)\) is a homogeneous special Kähler manifold for a group \(G\) then \((T^*M, I_1, I_2, I_3, g^c)\) is a homogeneous hyper-Kähler manifold for the group \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^*\).

Proof. Tensors \(g^c, I_1, I_2, I_3\) depends only on the first factor of \(M \times (\mathbb{R}^n)^*\). The action \(\hat{\psi}\) acts on the first factor as \(\phi\). Tensors \(g, I, \omega\) on \(M\) are invariant with respect to the action \(\phi\) of the group \(G\). Hence, \(g^{-1}, I^*, -\omega^{-1}\) are invariant with respect to the action \(\phi\), too. Therefore, \(g^c, I_1, I_2, I_3\) are invariant with respect to the action \(\hat{\psi}\). Moreover, if the action \(\phi\) of \(G\) on \(M\) is transitive, then the action \(\psi\) of \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^*\) on \(T^*M \simeq M \times (\mathbb{R}^n)^*\) is transitive. Thus, if \((M, I, g, \nabla)\) is a homogeneous special Kähler manifold for a group \(G\) then \((T^*M, I_1, I_2, I_3, g^c)\) is a homogeneous hyper-Kähler manifold for the group \(G \ltimes_{\hat{\theta}} (\mathbb{R}^n)^*\). \(\square\)

7 Selfsimilar special Kähler and conformally hyper-Kähler manifolds

Definition 7.1. A (globally) conformally hyper-Kähler structure on a manifold \(N\) is a collection \((fg, I_1, I_2, I_3)\), where \((g, I_1, I_2, I_3)\) is a hyper-Kähler structure and \(f\) is a positive definite function on \(M\).

Definition 7.2. A selfsimilar special Kähler manifold \((M, I, g, \nabla, \xi)\) is a special Kähler manifold \((M, I, g, \nabla)\) endowed with an affine vector field \(\xi\) satisfying \(\mathcal{L}_\xi g = 2g\).
Theorem 7.3. Let $(M, I, g, \nabla, \xi)$ be a selfsimilar special Kähler manifold, $G$ the group of affine holomorphic isometries of $(M, I, g, \nabla)$ preserving $\xi$, $(I_1, I_2, I_3, g^c)$ the corresponding hyper-Kähler structure on $T^*M$, $\pi : M \times \mathbb{R}^n \to M$ the natural projection, and $g_{c,h.K} = \pi^* (g(\xi, \xi))^{-1} g^c$. Then the following conditions are satisfies:

i) The collection $(g_{c,h.K}, I_1, I_2, I_3)$ is $G \ltimes (\mathbb{R}^n)^*$-invariant hyper-Kähler structure on $T^*M$.

ii) If the field $\xi$ is complete then $(g_{c,h.K}, I_1, I_2, I_3)$ is $(\mathbb{R} \times G) \ltimes (\mathbb{R}^n)^*$-invariant.

iii) If the field $\xi$ is complete and $G$ acts transitively on the level surface $S = \{g(\xi, \xi) = 1\} \subset M$ then $(g_{c,h.K}, I_1, I_2, I_3)$ is homogeneous conformally hyper-Kähler manifold for the group $(\mathbb{R} \times G) \ltimes (\mathbb{R}^n)^*$.

Proof. By the same argument as in the Theorem 6.4, the hyper-Kähler structure $(I_1, I_2, I_3, g^c)$ is $G \ltimes (\mathbb{R}^n)^*$-invariant. Then the proof of Theorem 7.3 coincides with the proof of Theorem 4.9.

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