A Supersymmetric Grand Unified Theory of Flavour with $PSL_2(7) \times SO(10)$

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Abstract

We construct a realistic Supersymmetric Grand Unified Theory of Flavour based on $PSL_2(7) \times SO(10)$, where the quarks and leptons in the $16$ of $SO(10)$ are assigned to the complex triplet representation of $PSL_2(7)$, while the flavons are assigned to a combination of sextets and anti-triplets of $PSL_2(7)$. Using a $D$-term vacuum alignment mechanism, we require the flavon sextets of $PSL_2(7)$ to be aligned along the 3-3 direction leading to the third family Yukawa couplings, while the flavon anti-triplets describe the remaining Yukawa couplings. Other sextets are aligned along the neutrino flavour symmetry preserving directions leading to tri-bimaximal neutrino mixing via a type II see-saw mechanism, with predictions for neutrinoless double beta decay and cosmology.

Keywords: discrete family symmetries, tri-bimaximal mixing, GUT

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1 Introduction

The discovery of neutrino mass and approximately tri-bimaximal (TB) lepton mixing \([1]\) suggests some kind of a non-Abelian discrete family symmetry \(G_f\) might be at work, at least in the lepton sector. In the neutrino flavour basis (i.e. diagonal charged lepton mass basis), it has been shown that the TB neutrino mass matrix is invariant under \(S,U\) transformations, \(M_{\nu}^{TB} = S M_{\nu}^{TB} S^T = U M_{\nu}^{TB} U^T\) \([2]\). A very straightforward argument shows that this neutrino flavour symmetry group has only four elements corresponding to Klein’s four-group \(Z^S_2 \times Z^U_2\). By contrast the diagonal charged lepton mass matrix (in this basis) satisfies a diagonal phase symmetry \(T\). The matrices \(S,T,U\) form the generators of the group \(S_4\) in the triplet representation, while the \(A_4\) subgroup is generated by \(S,T\).

The observed neutrino flavour symmetry corresponding to the two generators \(S,U\) may arise either directly or indirectly from a range of discrete symmetry groups \([3]\). Examples of the direct approach, in which one or more generators of the discrete family symmetry appears in the neutrino flavour group, are typically based on \(S_4\) \([4]\) or a related group such as \(A_4\) \([5,6]\) or \(PSL_2(7)\) \([2]\). Models of the indirect kind, in which the neutrino flavour symmetry arises accidentally, include also \(A_4\) \([7]\) and \(S_4\) \([8]\) as well as \(\Delta_{27}\) \([9]\) and the continuous flavour symmetries like, e.g., \(SO(3)\) \([10]\) or \(SU(3)\) \([11]\) which accommodate the discrete groups above as subgroups \([12]\). For an incomplete list of models with family symmetries see \([13–27]\) and the reviews \([28,29]\).

A desirable feature of a complete model of quark and lepton masses and mixing angles is that it should be consistent with an underlying Grand Unified Theory (GUT) structure, either at the field theory level or at the level of the superstring. The most ambitious models which have been built to achieve this are based on an underlying \(SO(10)\) structure. This is very constraining because it requires that all the 16 spinor components of a single family should have the same family charge, comprising the left-handed fermions \(\psi\) and the CP conjugates of the right-handed fermions \(\psi^c\), including the right-handed neutrino. Although it seems somewhat \textit{ad hoc} to add right-handed neutrino singlets to \(SU(5)\), once they are present it is straightforward to construct models of lepton masses and TB mixing that are consistent with quark masses and mixing, and there are several successful models of this kind for example based on \(A_4 \times SU(5)\) \([23]\).

Despite the theoretical attractiveness of \(SO(10)\), there are very few \(SO(10)\) models capable of enforcing TB mixing by means of a family symmetry. It is desirable that such models contain complex triplet representations of the family symmetry and examples of such models based on the Pati-Salam subgroup of \(SO(10)\) have been constructed where the family group is \(SU(3)_f\) \([11]\) or \(\Delta_{27}\) \([9]\), with Yukawa couplings arising from operators of the form \(\bar{\phi}^i \psi_i \bar{\phi}^j \psi^c_j\). If the triplet representations were taken to be real rather than complex then this would allow an undesirable alternative contraction of the indices in the Yukawa operator namely \(\psi_i \psi^c_i \bar{\phi}^i \bar{\phi}^i H\) leading to a Yukawa matrix proportional to the unit matrix which would tend to destroy any hierarchies in the Yukawa matrix. For real representations, there is no symmetry at the effective operator level that could forbid such a trivial contraction of the two triplet fermion fields. However in principle it is possible to appeal to the details of the underlying theory in order to forbid the trivial contraction. This involves a discussion of the heavy messenger states whose exchange generates the operator, for example as was done recently in the \(S_4 \times SO(10)\) model in \([8]\).
A general problem with all $SO(10)$ models in which the quarks and leptons form triplet representations of the family symmetry (necessary to account for TB mixing) is that the top quark Yukawa coupling arises from a double flavon suppressed operator of the form $\bar{\phi}^i \psi_i \bar{\phi}^j \psi^c_j$. The situation improves somewhat if one considers flavons also in sextet representations, as pointed out in [2]. For example, introducing a two index anti-sextet of $SU(3)_f$, $\hat{\chi}^{ij}$, the lowest order Yukawa operators become, $\hat{\chi}^{ij} \psi_i \psi^c_j$. If the anti-sextet flavon $\hat{\chi}$ has a VEV aligned along the 3-3 direction $\langle \hat{\chi}^{ij} \rangle = V \delta_{i3} \delta_{j3}$, then this operator with a coefficient $y/M$ would imply a third family Yukawa coupling of $0.5$ implying $V/M \approx 0.5$, which has acceptable convergence properties.

Flavon sextets are therefore well motivated as the origin of the third family Yukawa couplings in $SO(10)$ models. Although this is possible in the case where the family group is $SU(3)_f$ [11], it is not possible for the discrete groups $\Delta_{27}$ [9] or $S_4$ [8] for the simple reason that these groups do not admit sextet representations. The smallest simple discrete group which contains complex triplets and sextet representations is $PSL_2(7)$, which is the projective special linear group of two dimensional matrices over the finite Galois field of seven elements. $PSL_2(7)$ contains 168 elements and is sometimes written as $\Sigma(168)$ [30]. The relationship of $PSL_2(7)$ to some other family symmetries that have been used in the literature is discussed in [26,31,32].

In a recent paper [2] we developed the representation theory of $PSL_2(7)$ for triplets and sextets in a convenient basis suitable for applications of $PSL_2(7)$ as a family symmetry capable of describing quark and lepton masses and mixing angles in the framework of $SO(10)$ models. We showed how the triplet representation given in terms of the standard generators $A,B$ in [31] may be related to four $PSL_2(7)$ generators $S,T,U,V$. In such a basis the subgroup structure $PSL_2(7) \supset S_4 \supset A_4$ just corresponds to the respective generators being $S,T,U,V \supset S,T,U \supset S,T$.

The purpose of the present paper is to construct a realistic Supersymmetric Grand Unified Theory of Flavour based on $PSL_2(7) \times SO(10)$. We require the sextets to be aligned along the 3-3 direction to account for the large third family Yukawa couplings, while we shall make use of anti-triplet flavons, whose VEVs are aligned along the columns of the TB mixing matrix, to account for the first and second family quark and lepton masses and mixings. It turns out that in $PSL_2(7)$ it is easier to obtain the sextet flavon vacuum alignments first using the $D$-term approach to vacuum alignment discussed in [3], but here applied to flavon sextets rather than flavon triplets. In this way we can obtain sextet vacuum alignments along the 3-3 direction, suitable for the top quark Yukawa coupling, if we assume two relations (possibly arising by virtue of a higher symmetry) amongst different quartic sextet combinations appearing in the flavon potential. Sextet vacuum alignments along the $S,U$ preserving directions, suitable for reproducing the neutrino flavour symmetry of TB mixing in a direct way as a subgroup of $PSL_2(7)$, can be naturally obtained. Once the sextet flavons have been aligned, the anti-triplet flavons are then aligned against the pre-aligned sextet flavons, in particular using the $S,U$ preserving sextet flavons, leading to triplet flavon alignments along the columns of the TB mixing matrix as mentioned.

In general the neutrino mass and mixing in this model can arise from either the type I see-saw or the type II see-saw or both. The discussion of the type I see-saw...
Figure 1: Scatter plots for $m_{ee}$ and $\sum_i |m_i|$ against $m_{\text{min}}$ for the type II see-saw model based on $PSL_2(7) \times SO(10)$.

The approach follows along the lines of the models in [9, 11] based on constrained sequential dominance [33], leading to the indirect type of models, since the triplet flavons aligned along the third column of the TB mixing matrix breaks both $S$ and $U$. Here we shall focus on the new possibility offered by the $PSL_2(7) \times SO(10)$ model of the type II see-saw mechanism where the sextet flavons aligned along the $S,U$ preserving directions enter the neutrino sector, thereby preserving these generators leading to the neutrino flavour symmetry being reproduced in a direct way. As a preview of our results, we shall find that the type II see-saw model leads to TB mixing in the neutrino sector with a mass spectrum spanning the range from hierarchical or inverse hierarchical, up to the quasi-degenerate region. The resulting predictions for neutrinoless double beta decay mass parameter $m_{ee}$ and the total sum of physical neutrino masses $\sum_i |m_i|$ relevant for cosmological hot dark matter are both shown in Fig. 1 as a function of the lightest physical neutrino mass $m_{\text{min}}$ using a double logarithmic scale. These results are in sharp contrast to the case of the type I see-saw possibility with constrained sequential dominance which would lead to a strong neutrino mass hierarchy with $m_{\text{min}} = |m_1| \sim 0$. Later on we shall analyse the type II see-saw results in more detail using linear scales.

The layout of the remainder of the paper is as follows. In Section 2 we present the symmetries and superfield content of the SUSY $PSL_2(7) \times SO(10)$ model and discuss the desired vacuum alignments and the resulting fermion mass matrices. In Section 3 we discuss the type II neutrino phenomenology in the model. Section 4 is devoted to a detailed analysis of the $D$-term vacuum alignment for the sextet and ant-triplet flavons. Section 5 concludes the paper.
2 Fermion masses in the $PSL_2(7) \times SO(10)$ model

2.1 The desired vacuum alignments

As discussed in the Introduction, the sextet flavons are used in two ways in the model. The sextet $\hat{\chi}^{ij}_{\text{top}}$ aligned along the 3-3 direction is responsible for the third family Yukawa couplings, including that of the top quark, via operators like (dropping the Higgs fields)

$$\psi^c_i \hat{\chi}^{ij}_{\text{top}} \psi^c_j.$$  \hspace{1cm} (2.1)

The sextets $\hat{\chi}^{ij}_{\text{TB}}$ aligned along the $S$ and $U$ preserving directions are responsible for the effective Majorana couplings, via operators like (again dropping the Higgs fields)

$$\psi^c_i \hat{\chi}^{ij}_{\text{TB}} \psi^c_j + \psi^c_i \hat{\chi}^{ij}_{\text{TB}} \psi^c_j,$$ \hspace{1cm} (2.2)

where the first term above contributes to the type II see-saw mass while the second term is responsible for the heavy right-handed neutrino masses. As discussed in [2], the sextet flavon 3 by 3 matrices $\hat{\chi}$ which enter the above couplings to quarks and leptons, are related to the six component column vector

$$\chi = (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6)^T$$ \hspace{1cm} (2.3)

by,

$$\hat{\chi} = -\frac{(1+i)}{6\sqrt{2}} \left[ \chi_1 \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} - i\sqrt{3} \chi_2 \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix} - i\sqrt{3}b_7 \chi_3 \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
+ \sqrt{3}b_7 \chi_4 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \sqrt{2} \chi_5 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - i\sqrt{6} \bar{b}_7 \chi_6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right].$$ \hspace{1cm} (2.4)

where we adopt the notation of the “Atlas of finite groups” [34] which defines

$$b_7 = \frac{1}{2}(-1 + i\sqrt{7}), \quad \bar{b}_7 = \frac{1}{2}(-1 - i\sqrt{7}).$$ \hspace{1cm} (2.5)

Mass matrices which are of the tri-bimaximal form are obtained from alignments of the form

$$\langle \chi_{\text{TB}} \rangle = (0, 0, 0, \alpha_4, \alpha_5, \alpha_6)^T.$$ \hspace{1cm} (2.6)

The fully realistic type II model will require three flavons of this type which we label as $\chi^{[p]}_{\text{TB}}$, where $p = 0, 1, 2$. Explicitly their alignments read

$$\langle \chi^{[0]}_{\text{TB}} \rangle \propto (0, 0, 0, 0, 0, 1)^T,$$ \hspace{1cm} (2.7)

$$\langle \chi^{[1]}_{\text{TB}} \rangle \propto \frac{1}{6} \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, -1)^T,$$ \hspace{1cm} (2.8)

$$\langle \chi^{[2]}_{\text{TB}} \rangle \propto \frac{1}{6} \cdot (0, 0, 0, -\sqrt{14}, \sqrt{21}, -1)^T.$$ \hspace{1cm} (2.9)
In order to generate a Yukawa matrix which gives mass to only the third generation (top quark, bottom quark or tau lepton), we need the alignment
\[
\langle \chi_{\text{top}} \rangle \propto \frac{(1 - i)}{3\sqrt{2}} \cdot (1, i\sqrt{3}, -i\sqrt{3}/b_7, -\sqrt{3}/b_7, -\sqrt{2}, 0)^T .
\]
To obtain the alignments of Eqs. (2.7,2.10) it is necessary to study the \( PSL_2(7) \) symmetric potential for the flavon sextet, which we postpone to Section 4.

In addition the model also relies on the anti-triplet flavon fields \( \bar{\phi}_{23} \) and \( \bar{\phi}_{123} \) whose VEVs become aligned along the directions
\[
\langle \bar{\phi}_{23} \rangle \propto \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \langle \bar{\phi}_{123} \rangle \propto \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]
where again we postpone discussion of these alignments until later.

### 2.2 The symmetries and operators of the model

Having indicated the desired vacuum alignment of the flavon fields \( \chi_{\text{top}}, \chi_{TB}^{[p]}, \bar{\phi}_{23}, \bar{\phi}_{123} \), we can now formulate an \( SO(10) \) model of fermion masses using the \( PSL_2(7) \) family symmetry. Table 1 lists the particle content together with all transformation properties.

Here \( \psi \) denotes the \( 16 \) of \( SO(10) \) which incorporates the SM fermions. Even though our model is based on \( SO(10) \), it is convenient to distinguish the left-handed and the right-handed components by showing the decomposition into Pati-Salam representations. Using the order \( SU(4) \times SU(2)_L \times SU(2)_R \) we can write
\[
16 \rightarrow \underbrace{4, 2, 1}_\psi + \underbrace{4, 1, 2}_\psi .
\]

\( H_{10} \) and \( H_{126} \) are the \( SO(10) \) Higgs fields whose \( SU(2)_L \) doublet components enter the Yukawa couplings. In Pati-Salam language the relevant components that acquire an electroweak VEV are
\[
H_{10} \rightarrow (1, 2, 2) , \quad H_{126} \rightarrow (15, 2, 2) .
\]
In Pati-Salam language, $SU(4) \times SU(2)_L \times SU(2)_R$, the relevant components of the $SO(10)$ representations are:

- $\psi \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1})$
- $\psi^c \rightarrow (\mathbf{4}, \mathbf{1}, \mathbf{2})$
- $H_{10} \rightarrow (\mathbf{1}, \mathbf{2}, \mathbf{2})$
- $H_{126} \rightarrow (\mathbf{15}, \mathbf{2}, \mathbf{2})$

The latter gives rise to the Georgi-Jarlskog (GJ) factor of 3 in the (2,2) entry of the lepton mass matrix \[29, 35\]. With the above specified fields, the leading Yukawa operators take the form,

$$
\mathcal{L}_{Yuk} \sim \frac{1}{M_H} \psi_i \tilde{\chi}_{\text{top}} \psi^c_j H_{10} + \frac{1}{M^2} \psi_i \tilde{\phi}_{23} \tilde{\phi}_{23} \psi^c_j H_{126} + \frac{1}{M^3} \psi_i \langle \tilde{\phi}_{23} \tilde{\phi}_{123} + \tilde{\phi}_{23} \psi^c_j \xi H_{10} \rangle,
$$

where we have only given the $PSL_2(7)$ indices $i,j$. The corresponding diagrams are depicted in Fig. 2. Note that we have inserted a $PSL_2(7)$ singlet flavon $\xi$ in order to additionally suppress the third term. Inserting the flavon VEVs, the first term (a) fills in the (3,3) entry of the Yukawa matrix, the second (b) generates non-vanishing entries in the 2-3 block, and the third (c) enters everywhere in the Yukawa matrix except for the (1,1) component. Including sextet and anti-triplet flavons, we find the following structure of the Yukawa matrices,

$$
Y_{u,d,e,\nu} \sim \frac{\langle \chi_{\text{top}} \rangle}{M_H} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{10} + \frac{|\langle \tilde{\phi}_{23} \rangle|^2}{M^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} H_{126} + \frac{|\langle \tilde{\phi}_{23} \rangle| \langle \langle \tilde{\phi}_{123} \rangle \rangle}{M^3} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} H_{10}. \quad (2.15)
$$

To identify the Yukawa matrices for the different sectors $Y_{u,d,e,\nu}$ we note that the Higgs representations $H_{10}$ and $H_{126}$ each have two Higgs doublets in them, according to Eq. (2.13). The two MSSM doublets $H_u, H_d$ originate below the GUT scale, from one linear combination of these up-type doublets and one linear combination of the down-type ones which remain almost massless. The orthogonal linear combinations acquire GUT scale masses just as the colour triplets and other non-MSSM states. Electroweak symmetry is broken after the light MSSM doublets $H_u, H_d$ acquire VEVs $v_{u,d}$ and they then generate the fermion masses. Since the dominant contribution to the 2-3 block of the Yukawa matrices arises from the components of $H_u, H_d$ coming from $H_{126}$, these entries will receive a relative Clebsch factor of 3 for the leptons as compared to the quarks. Thus, ignoring...
m^{u,d,e,\nu} \sim \left( \begin{array}{ccc} 0 & \epsilon^3 & -\epsilon^3 \\ \epsilon^3 & a \epsilon^2 & -a \epsilon^2 \\ -\epsilon^3 & -a \epsilon^2 & 1 \end{array} \right) \frac{|\langle \chi_{\text{top}} \rangle|}{M_H} v_{u,d} , \quad \text{with} \quad \begin{cases} a = 1 & \text{for } u, d \\ a = -3 & \text{for } e, \nu \end{cases}, \quad (2.16)

where, for simplicity, we have chosen

\frac{|\langle \chi_{\text{top}} \rangle|}{M_H} \sim 0.5 , \quad \frac{|\langle \phi_{23} \rangle|^2}{M^2} \sim \epsilon^2 , \quad \frac{|\langle \phi_{23} \rangle| \cdot |\langle \phi_{123} \rangle| \langle \xi \rangle}{M^3} \sim \epsilon^3 . \quad (2.17)

Notice the zero in the (1,1) entry whose presence allows to accommodate the phenomenologically successful Gatto-Sartori-Tonin relation \[36\]. Moreover, the up and the down quark sectors can have independent messenger masses \( M \rightarrow M^u, M^d \), where \( M^u \approx 3M^d \) so that two different expansion parameters \( \epsilon \rightarrow \epsilon_u, \epsilon_d \) with \( \epsilon_u \approx \epsilon_d/3 \) are introduced as in \[11\]. Numerically we need \( \epsilon_u \approx 0.05 \) and \( \epsilon_d \approx 0.15 \). See \[37\] for a detailed discussion of the numerics including \( \chi^2 \) fits. Thus the following mass matrix structures for the quarks are quite elegantly achieved:

\begin{align*}
m^u & \sim \left( \begin{array}{ccc} 0 & \epsilon^3 & -\epsilon^3 \\ \epsilon^3 & \epsilon_u^2 & -\epsilon_u^2 \\ -\epsilon^3 & -\epsilon_u^2 & 1 \end{array} \right) \frac{|\langle \chi_{\text{top}} \rangle|}{M_H} v_u , & m^d & \sim \left( \begin{array}{ccc} 0 & \epsilon^3 & -\epsilon^3 \\ \epsilon_d^3 & \epsilon_d^2 & -\epsilon_d^2 \\ -\epsilon^3 & -\epsilon_d^2 & 1 \end{array} \right) \frac{|\langle \chi_{\text{top}} \rangle|}{M_H} v_d ,
\end{align*}

with similar considerations in the lepton sector leading to \( m^{e,\nu} \) being the same as \( m^{d,u} \) apart from the GJ factors of 3 in the 2-3 block, as in Eq. \(2.16\). The (left-handed) quark mixing angles in the up and the down sector are then calculated as \( \theta_{12}^{u,d} \sim \epsilon_{u,d}, \theta_{23}^{u,d} \sim \epsilon_{u,d}^2, \) and \( \theta_{13}^{u,d} \sim \epsilon_{u,d}^3 \), so that the down sector gives the dominant contribution to a viable CKM matrix. Regarding the leptons, the mixing angles are changed by GJ factors of 3 as follows \( \theta_{12}^{e,\nu} \sim \epsilon_{d,u}/3, \theta_{23}^{e,\nu} \sim 3\epsilon_{d,u}^2, \) and \( \theta_{13}^{e,\nu} \sim \epsilon_{d,u}^3 \). The effect of the charged lepton mixing angles on the PMNS matrix is small so that tri-bimaximal mixing is only perturbed within the experimentally allowed range \[38\].

In this model the successful mass matrices in Eq. \(2.18\) are achieved in a very natural way since, with the inclusion of the singlet flavon \( \xi \), the first row and column is cubic in the messenger mass, while the 2-3 block is quadratic and the 3-3 element involves the Higgs messenger mass \( M_H \) and so is universal.

Turning to the Majorana sector different components of the field \( \Delta_{\mathbf{126}} \) are responsible for the masses of the left-handed and the right-handed neutrinos. Analogous to the convention used to distinguish the left-handed from the right-handed doublet components of \( \psi \), we introduce a notation that allows us to tell apart the left-handed and the right-handed triplet components of \( \Delta_{\mathbf{126}} \), i.e.

\[ \mathbf{126} \rightarrow \underbrace{\mathbf{10} \times \mathbf{3} \times \mathbf{1}}_{\Delta_{\mathbf{126}}} + \underbrace{\mathbf{10} \times \mathbf{1} \times \mathbf{3}}_{\Delta_{\mathbf{126}}} + \cdots, \quad (2.19) \]

where the ellipsis denotes the rest of the decomposition of the \( \mathbf{126} \) of \( SO(10) \) into representations of \( SU(4) \times SU(2)_L \times SU(2)_R \). With these remarks, the leading Majorana operators take the form

\[ \mathcal{L}_{\text{Maj}} \sim \frac{1}{M} \sum_{p=0}^{2} \left( \psi_i \chi^{[p]}_{TB} \psi_j \Delta_{\mathbf{126}} + \psi_i \chi^{c [p]}_{TB} \psi_j \Delta^c_{\mathbf{126}} \right), \quad (2.20) \]
Figure 3: The diagrams which generate the Majorana couplings. In Pati-Salam language, $SU(4) \times SU(2)_L \times SU(2)_R$, the relevant components of the $SO(10)$ representations are: (a) $\psi \rightarrow (4, 2, 1)$, $\Delta_{126} \rightarrow (10, 3, 1)$ as well as (b) $\psi^c \rightarrow (\bar{4}, 1, 2)$, $\Delta_{126}^c \rightarrow (10, 1, 3)$.

with the corresponding diagrams given in Fig. 3.

While $\Delta_{126}^c$ gets a GUT scale VEV in the triplet component of $SU(2)_R$, $\Delta_{126}$ acquires a small induced VEV \[\langle \Delta_{126} \rangle \sim \frac{v^2_u}{M}, \tag{2.21}\]
in the triplet component of $SU(2)_L$, for details see Appendix A. The first term in Fig. 3(a) then effectively generates a type II see-saw contribution to the physical neutrino mass when the $\chi_{TB}^{[p]}$ and the $SU(2)_L$ triplet component of $\Delta_{126}$ get their VEVs. The resulting light neutrino mass matrix reads

$$ m_{\nu\text{type II}} \sim \sum_{p=0}^{2} \langle \chi_{TB}^{[p]} \rangle \cdot \frac{v^2_u}{M^2}, \tag{2.22} $$

with the flavour structure encoded in the three matrices $\langle \chi_{TB}^{[p]} \rangle$, $p = 0, 1, 2$.

On the other hand, the second term of Eq. (2.20) which is depicted in Fig. 3(b) gives rise to the mass of the right-handed neutrinos if $\Delta_{126}^c$ acquires a VEV in the direction of the right-handed triplet. In addition, the flavon $\chi_{TB}^{[p]}$ which is required for $PSL_2(7) \times U(1)$ invariance also needs to get a VEV. The existence of heavy right-handed neutrinos unavoidably leads to a type I see-saw contribution to the light neutrino masses. With the neutrino Dirac mass matrix $m_D^\nu$ of Eq. (2.16), we find

$$ m_{\nu\text{type I}} \sim - m_D^\nu \cdot \left( \sum_{p=0}^{2} \langle \chi_{TB}^{[p]} \rangle \langle \Delta_{126}^c \rangle \right)^{-1} \cdot \frac{m_D^\nu T}{M}. \tag{2.23} $$

The resulting effective light neutrino mass matrix then takes the form

$$ m_{\nu}^{\text{eff}} = m_{\nu\text{type I}} + m_{\nu\text{type II}}. \tag{2.24} $$

In our $PSL_2(7)$ model of flavour we want the type II contribution to dominate over type I. It is clear from Eqs. (2.22,2.23) how this can be achieved. With $\langle \chi_{TB}^{[p]} \rangle \sim \langle \Delta_{126}^c \rangle \sim M$ both contributions would be more or less equally important. Increasing the scale of $\langle \chi_{TB}^{[p]} \rangle$ slightly, the type II see-saw contribution increases while, at the same time, the type I
see-saw gets suppressed. From now on we will assume dominance of the type II over the type I see-saw. The type II see-saw mechanism has the additional advantage of avoiding the use of operators with Clebsch zeros in the neutrino direction, a requirement which is known to be essential in the type I see-saw mechanism as applied to these models in order to provide the necessary suppression required for TB mixing \[11\].

3 Type II neutrino phenomenology

In order to extract the neutrino mass spectrum that arises from the first term of Eq. (2.20), we need to insert the VEVs of the flavon sextets $\chi^{[p]}_{TB}$, $p = 0, 1, 2$ given in Eqs. (2.7,2.9) into Eq. (2.4). Each of the three flavons comes with its own coupling coefficient $c^{[p]}$ so that we have to diagonalise

$$m^{\nu}_{\text{eff}} = \frac{v_u^2}{M^2} \cdot \sum_{p=0}^{2} c^{[p]} \langle \chi^{[p]}_{TB} \rangle .$$

As each term is of tri-bimaximal form, we can diagonalise them individually using the tri-bimaximal mixing matrix

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} .$$

This leads to

$$m^{\nu}_{\text{diag}} = U_{TB}^T m^{\nu}_{\text{eff}} U_{TB} = \frac{(i - 1)\bar{b}_7}{2\sqrt{3}} \cdot \frac{v_u^2}{M^2} \times$$

\[
\times \left\{ c^{[0]} |\langle \chi^{[0]}_{TB} \rangle| \cdot \text{Diag}(1, 1, -1) \\
+ c^{[1]} |\langle \chi^{[1]}_{TB} \rangle| \cdot \text{Diag}(1, e^{i\varphi}, -e^{-i\varphi}) \\
+ c^{[2]} |\langle \chi^{[2]}_{TB} \rangle| \cdot \text{Diag}(e^{-i\varphi}, e^{i\varphi}, -1) \right\} ,
\]

where the phase factor $e^{i\varphi}$ is fixed by the $PSL_2(7)$ group specific parameters $b_7$ and $\bar{b}_7$ as

$$e^{i\varphi} = \frac{\bar{b}_7}{b_7} = \frac{(-3 + i\sqrt{7})}{4} ,$$

numerically corresponding to a phase of $\varphi \approx 138.6^\circ$. Then, after dropping a global phase, the three (complex) light neutrino masses $m_i$ are each calculated as the sum of three mass parameters

$$m^{[p]} = \frac{1}{\sqrt{3}} \cdot \frac{v_u^2}{M^2} \cdot c^{[p]} |\langle \chi^{[p]}_{TB} \rangle| ,$$

\[1\]We implicitly absorb the potentially non-zero overall phases of the matrices $\langle \chi^{[p]}_{TB} \rangle$ into a redefinition of the couplings $c^{[p]}$. 

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using different combinations of relative phase factors
\[
\begin{align*}
m_1 &= m^0 + m^1 + e^{-i\varphi} m^2, \\
m_2 &= m^0 + e^{i\varphi} m^1 + e^{i\varphi} m^2, \\
m_3 &= -m^0 - e^{-i\varphi} m^1 - m^2. 
\end{align*}
\] (3.30)

In the special case where the parameters \(m^{[p]}\) are all real, it is possible to find a very simple relation with the solar and the atmospheric neutrino mass squared differences,
\[
\begin{align*}
\Delta m_{\text{sol}}^2 &\equiv |m_2|^2 - |m_1|^2 = \frac{7}{2} \cdot m^1 (m^2 - m^0), \\
\Delta m_{\text{atm}}^2 &\equiv |m_3|^2 - |m_1|^2 = \frac{7}{2} \cdot m^0 (m^2 - m^1).
\end{align*}
\] (3.31) (3.32)

As \(\Delta m_{\text{sol}}^2\) and \(|\Delta m_{\text{atm}}^2|\) have been measured within certain error bars, it is straightforward to express \(m^1\) and \(m^2\) as functions of \(m^0\). Note that the sign ambiguity in \(\Delta m_{\text{atm}}^2\) leads to a case distinction between normal and inverted neutrino mass ordering. With the (complex) masses \(m_i\) taken from Eq. (3.30) we can calculate the effective neutrino mass relevant for neutrinoless double beta decay,
\[
m_{ee} = \left| \sum_i (U_{TB ei})^2 m_i \right| = \left| \frac{2}{3} m_1 + \frac{1}{3} m_2 \right|.
\] (3.33)

In the case of real \(m^{[p]}\) they will depend on \(m^0\) only. Furthermore we can easily determine the smallest (real-valued) neutrino mass ordering which, in the case of real \(m^{[p]}\), is again only a function of \(m^0\). We can thus plot \(m_{ee}\) against \(m_{\text{min}}\), as shown in Fig. 4.

Switching on the complex phases of the parameters \(m^{[p]}\), the dependence of \(m_{ee}\) on \(m_{\text{min}}\) becomes much more fuzzy. However, in order to illustrate the phenomenological

Figure 4: \(m_{ee}\) plotted against \(m_{\text{min}}\) for the case of a normal mass ordering (left panel, in red) and an inverted mass ordering (right panel, in green) assuming for simplicity real parameters \(m^{[p]}\).
consequences arising from the general structure of Eq. (3.30) we have performed a scan over the parameters $m^{[p]} = |m^{[p]}|e^{i\varphi^{[p]}}$ which we have taken to be within the interval $0 \text{ eV} \leq |m^{[p]}| \leq 0.5 \text{ eV}$ with arbitrary phases $\varphi^{[p]}$. We have used equal distribution in both the absolute values of the masses as well as the phases. With such randomly generated input parameters it is straightforward to calculate the resulting complex masses $m_i$ which in turn can be converted into atmospheric and solar mass squared differences. Keeping only those sets of parameters $m^{[p]}$ which lie within the $3\sigma$ intervals arriving

\begin{align}
2.07 \times 10^{-3} \text{ eV}^2 & \leq |\Delta m^2_{\text{atm}}| = \left| |m_3|^2 - |m_1|^2 \right| \leq 2.75 \times 10^{-3} \text{ eV}^2, \quad (3.35) \\
7.05 \times 10^{-5} \text{ eV}^2 & \leq \Delta m^2_{\text{sol}} = \left| |m_2|^2 - |m_1|^2 \right| \leq 8.34 \times 10^{-5} \text{ eV}^2, \quad (3.36)
\end{align}

we have calculated $m_{ee}$ and $m_{\text{min}}$ in order to generate the scatter plot version of Fig. 4. The result is shown in Fig. 5. We see that allowing for complex phase factors fills in the gaps between the branches depicted in Fig. 4. In addition we also show the dependence on $m_{\text{min}}$ of the sum of all light neutrino masses $\sum_i |m_i|$ which is relevant for cosmology, where the current cosmological limit is about $\sum_i |m_i| < 1.0 \text{ eV}$ [40].

The histograms in Fig. 6 show the distribution of the numbers of points as a function of $m_{\text{min}}$ (upper panels) and as a function of $m_{ee}$ (lower panels) corresponding to the scatter plots in Fig. 5, for the case of a normal mass ordering (left panels, in red) and an inverted mass ordering (right panels, in green). All the distributions exhibit a very broad peak at about $\sim 0.05 \text{ eV}$ with significant tails out to about $\sim 0.4 \text{ eV}$ in both $m_{\text{min}}$ and $m_{ee}$. 

Figure 5: Scatter plots for $m_{ee}$ (upper panels) and $\sum_i |m_i|$ (lower panels) against $m_{\text{min}}$ for the realistic case of complex parameters $m^{[p]}$ for the case of a normal mass ordering (left panels, in red) and an inverted mass ordering (right panels, in green).
Figure 6: These histograms show the distribution of the numbers of points as a function of $m_{\text{min}}$ (upper panels) and as a function of $m_{\text{ee}}$ (lower panels) corresponding to the scatter plots in Fig. 5 for the case of a normal mass ordering (left panels, in red) and an inverted mass ordering (right panels, in green). These histograms all exhibit a very broad peak at about 0.05 eV with significant tails out to about 0.4 eV.

4 Vacuum Alignment in the $PSL_2(7) \times SO(10)$ model

We now return to the question of vacuum alignment in the $PSL_2(7) \times SO(10)$ model, using the $D$-term method and starting with the alignment of the flavon sextets. It turns out that, for $PSL_2(7)$ triplets and anti-triplets, the $D$-term vacuum alignment method reviewed recently in [3] does not work since the invariants involving triplets are the same as in $SU(3)$ and the new invariants required for triplet flavon vacuum alignment are not present. However, as we shall discuss below, there are new invariants involving sextets so the $D$-term method of vacuum alignment is well suited to aligning the flavon sextets. Once the sextet flavons are properly aligned, the anti-triplet flavons may then be aligned by coupling them to the sextet flavons, as we shall also show. This implies that $PSL_2(7)$ sextets play the crucial role in vacuum alignment quite independently of the crucial role that they play in generating third family Yukawa couplings and Majorana masses.

4.1 Invariants with sextets

We determine the quartic invariants for the sextet flavon $\chi$. Although the sextet irrep of $PSL_2(7)$ is real, the physical field need not be real. In general we therefore consider invariants of type $\chi_i^\dagger \chi_j^\dagger \chi_k^\dagger \chi_l$, where we shall assume that such terms involve only sextet
flavons of a particular type, i.e. only purely $\chi^{[p]}_{TB}$ (for a particular choice of $p$) or the flavon $\chi_{top}$ but not operators involving a mixture of different flavon types. This may be enforced by assuming a particular messenger sector and introducing a $U(1)'$ symmetry as discussed in Appendix B. Since we are interested in operators involving only one flavon type, we have to consider only the symmetric combinations

$$ (6 \otimes 6)_s = 1 \oplus 6 \oplus 6 \oplus 8. $$

For $\chi$ given in the real sextet basis of [2], the representations on the right-hand side can be written as

1: $\Upsilon = \sum_{i=1}^{6} \chi_i \chi_i$, \hspace{1cm} (4.1)

6: $\Theta = \begin{pmatrix}
-2\sqrt{21}\chi_1\chi_5 - 6\chi_1\chi_6 \\
2\sqrt{21}\chi_2\chi_5 - 6\chi_2\chi_6 \\
2\sqrt{14}\chi_3\chi_4 + 8\chi_3\chi_6 \\
\sqrt{14}\chi_3^2 - \sqrt{14}\chi_4^2 + 8\chi_4\chi_6 \\
-\sqrt{21}\chi_1^2 + \sqrt{21}\chi_2^2 - 6\chi_5\chi_6 \\
-3\chi_1^2 - 3\chi_2^2 + 4\chi_3^2 + 4\chi_4^2 - 3\chi_5^2 + \chi_6^2
\end{pmatrix}$, \hspace{1cm} (4.2)

6: $\Theta' = \begin{pmatrix}
2\sqrt{21}\chi_2\chi_3 + 2\sqrt{7}\chi_1\chi_4 - 2\sqrt{2}\chi_1\chi_6 \\
2\sqrt{21}\chi_1\chi_3 + 2\sqrt{7}\chi_2\chi_4 - 2\sqrt{2}\chi_2\chi_6 \\
2\sqrt{21}\chi_1\chi_2 + 2\sqrt{7}\chi_3\chi_4 - 2\sqrt{2}\chi_3\chi_6 \\
\sqrt{7}\chi_1^2 + \sqrt{7}\chi_2^2 + \sqrt{7}\chi_3^2 - \sqrt{7}\chi_4^2 - 2\sqrt{7}\chi_5^2 - 2\sqrt{7}\chi_6^2 \\
-4\sqrt{7}\chi_4\chi_5 - 2\sqrt{2}\chi_5\chi_6 \\
-\sqrt{21}\chi_1^2 - \sqrt{21}\chi_2^2 - \sqrt{21}\chi_3^2 - \sqrt{21}\chi_4^2 - \sqrt{21}\chi_5^2 + 5\sqrt{2}\chi_6^2
\end{pmatrix}$, \hspace{1cm} (4.3)

8: $\Omega = \begin{pmatrix}
\sqrt{3}\chi_2\chi_3 + \chi_1\chi_4 - 2\sqrt{6}\chi_1\chi_5 + 2\sqrt{14}\chi_1\chi_6 \\
2\sqrt{14}\chi_2\chi_3 - 3\sqrt{7}\chi_1\chi_4 \\
-\sqrt{6}\chi_1^2 + \sqrt{6}\chi_2^2 - 2\chi_4\chi_5 + 2\sqrt{14}\chi_5\chi_6 \\
-\sqrt{2}\chi_1^2 - \sqrt{2}\chi_2^2 + 2\sqrt{2}\chi_3^2 - 2\sqrt{2}\chi_4^2 + 2\sqrt{2}\chi_5^2 - 2\sqrt{7}\chi_6^2 \\
-2\sqrt{21}\chi_1\chi_3 + 3\sqrt{7}\chi_2\chi_4 \\
\sqrt{3}\chi_1\chi_3 + \chi_2\chi_4 + 2\sqrt{6}\chi_2\chi_5 + 2\sqrt{14}\chi_2\chi_6 \\
-2\sqrt{21}\chi_3\chi_5 \\
2\sqrt{6}\chi_1\chi_2 - 4\sqrt{2}\chi_3\chi_4 + 2\sqrt{7}\chi_3\chi_6
\end{pmatrix}$, \hspace{1cm} (4.4)

It should be mentioned that the two sextets $\Theta$ and $\Theta'$ are not defined uniquely since any linear combination of $\Theta$ and $\Theta'$ transforms as a 6 as well. This ambiguity doesn't affect the set of quartic invariants even though the explicit form of a particular invariant may be
different. For the octet $\Omega$ we have chosen a basis where the generators $S^{[8]}, T^{[8]}, U^{[8]}, V^{[8]}$ are real with $S^{[8]}$ and $U^{[8]}$ diagonal. They can be found in Appendix C. The derived quartic invariant is necessarily independent of the choice for the octet basis.

The Kronecker product for the symmetric combinations $(6 \otimes 6)_s$ shows that there are six independent quartic invariants of type $\chi^\dagger_i \chi_j \chi^\dagger_k \chi_l$. They can be easily obtained from $\Upsilon, \Theta, \Theta', \Omega$ and their complex conjugates, denoted by $\bar{\Upsilon}, \bar{\Theta}, \bar{\Theta}', \bar{\Omega}$. Since we have chosen real bases, the quartic invariants are formed trivially:

$$I_1 = \bar{\Upsilon} \Upsilon, \quad I_2 = \sum_{i=1}^{6} \bar{\Theta}_i \Theta_i, \quad I_3 = \sum_{i=1}^{6} \bar{\Theta}'_i \Theta'_i, \quad (4.5)$$

$$I_4 = \frac{1}{\sqrt{2}} \sum_{i=1}^{6} (\bar{\Theta}'_i \Theta_i + \Theta'_i \bar{\Theta}_i), \quad I_5 = \frac{i}{\sqrt{2}} \sum_{i=1}^{6} (\bar{\Theta}'_i \Theta_i - \Theta'_i \bar{\Theta}_i), \quad (4.6)$$

$$I_6 = \sum_{a=1}^{8} \Omega_a \Omega_a. \quad (4.7)$$

Here we have chosen a convention in which all invariants are real. The way we constructed these six invariants obscures a trivial one, namely $\left(\sum_i \chi^\dagger_i \chi_i\right)^2$. It is related to the above invariants by

$$I_0 \equiv \left(\sum_i \chi^\dagger_i \chi_i\right)^2 = \frac{1}{6 \cdot 49} (49 I_1 + 5 I_2 + 5 I_3 - I_4 + 7 I_6). \quad (4.8)$$

It is therefore possible to replace $I_6$ in favour of $I_0$ in our set of independent quartic invariants.\footnote{In $SU(3)$ the product $(6 \otimes 6)_s \otimes (\bar{6} \otimes 6)_s$ yields only two independent invariants: $I_0$ and the sum $2I_2 + 2I_3 + I_4 - \sqrt{7}I_5$.}

### 4.2 A potential for obtaining sextet flavon alignments

Let us study a sextet potential of the form $\bar{3}$

$$V = -m^2 \cdot \sqrt{I_0} + \lambda \cdot I_0 + \lambda \cdot \sum_{\alpha=1}^{5} \kappa_{\alpha} I_{\alpha}$$

$$= -m^2 \cdot \sqrt{I_0} + (\lambda \cdot I_0) \cdot f, \quad (4.9)$$

where we have defined

$$f \equiv \left(1 + \sum_{\alpha=1}^{5} \kappa_{\alpha} \frac{I_{\alpha}}{I_0}\right). \quad (4.10)$$

In order for this potential to have a minimum, $\lambda \cdot f$ must be positive. Moreover, the factor $f$ is independent of the normalisation of $\langle \chi \rangle$ so that $f$ is extremised by an appropriate choice of the alignment. On the other hand, $I_0$ is independent of a particular
alignment. Therefore, denoting the extremum of $f$ by $f_0$, the overall scale of the minimum is determined from
\[
\sum_{i=1}^{6} \langle \chi_i^\dagger \rangle \langle \chi_i \rangle = \sqrt{\langle I_0 \rangle} = \frac{m^2}{2 \lambda \cdot f_0}.
\]
In the following we will assume $\lambda > 0$. So we need to find the minimum $f_0 > 0$ of $f$. However, since we already know the desired alignment vectors, our procedure will be to insert these into $f$ and find suitable coefficients $\kappa_\alpha$ such that $f$ becomes a (local) minimum.

### 4.2.1 $\chi^{[p]}_{TB}$ alignment

Consider the alignment of Eq. (2.7) which breaks $PSL_2(7)$ down to $S_4$
\[
\langle \tilde{\chi} \rangle \propto (0, 0, 0, 0, 0, 1)^T.
\]
We find vanishing first derivatives for each individual invariant,
\[
\frac{\partial (I_\alpha / I_0)}{\partial (\text{Re} \chi_i)} = \frac{\partial (I_\alpha / I_0)}{\partial (\text{Im} \chi_i)} = 0, \quad \forall \alpha = 1, ..., 5, \quad \forall i = 1, ..., 6.
\]
In order to get a minimum, the $12 \times 12$ matrix of second derivatives (the Hessian $H_\alpha$) needs to be positive-definite except for those two real directions which give zero since they correspond to the invariance of $f$ under the scaling $\chi_i \rightarrow s \cdot \chi_i$, with $s \in \mathbb{C}$. With the alignment $\langle \tilde{\chi} \rangle$, this is the $\chi_6$ direction. For the remaining 10 real dimensions, we obtain the following matrices $h_\alpha$
\[
\begin{align*}
h_1 &= -8 \cdot \text{Diag}(0, 0, 0, 0, 1, 1, 1, 1), \\
h_2 &= 4 \cdot \text{Diag}(14, 14, 35, 35, 14, 20, 20, 27, 27, 20), \\
h_3 &= -16 \cdot \text{Diag}(14, 14, 14, 14, 9, 9, 9, 9), \\
h_4 &= -4 \cdot \text{Diag}(14, 14, 7, 7, 14, -18, -18, 45, 45, -18), \\
h_5 &= \begin{pmatrix} 0 & \tilde{h}_5 \\ \tilde{h}_5 & 0 \end{pmatrix}, \quad \tilde{h}_5 = 28 \cdot \text{Diag}(2, 2, -3, -3, 2).
\end{align*}
\]

The condition for a minimum of the potential is that all the eigenvalues of the $10 \times 10$ matrix
\[
h = \sum_{\alpha=1}^{5} \kappa_\alpha h_\alpha,
\]
should be positive-definite for appropriate $\kappa_\alpha$. The explicit forms of $h_\alpha$ show that many potentials can be constructed that are minimised by the alignment of Eq. (4.11). For instance, one could choose
\[
-2 < \kappa_1 < 10, \quad \kappa_2 = 1, \quad \kappa_3 = \kappa_4 = \kappa_5 = 0.
\]
The upper bound on $\kappa_1$ arises because the sum $\kappa_1 h_1 + \kappa_2 h_2$ must be positive-definite. The lower bound, on the other hand, is related to the requirement of $f > 0$. Another possibility would be
\[
-\frac{1}{50} < \kappa_3 < 0, \quad \kappa_1 = \kappa_2 = \kappa_4 = \kappa_5 = 0.
\]
where the lower bound arises due to $f > 0$. We emphasise here that both examples do not rely on any tuning and the vanishing of the $\kappa$ coefficients, as assumed in these examples, is not a requirement to get a positive-definite $h$. It is not our intention to give the most general potential that is minimised by the alignment $\langle \tilde{\chi} \rangle$, but merely to show that a potential leading to such an alignment is quite plausible. These examples are sufficient to show this.

Due to the symmetry of such a potential under $PSL_2(7)$, there exists a discrete degeneracy of minima. Given a potential $V$ that is minimised by $\langle \tilde{\chi} \rangle$, also the alignment $G \cdot \langle \tilde{\chi} \rangle$, with $G$ denoting any element of $PSL_2(7)$ in the sextet representation, yields a minimum. In addition to $\langle \tilde{\chi} \rangle \equiv \langle \tilde{\chi}^{[0]} \rangle$, we find six new alignment vectors:

$$\langle \tilde{\chi}^{[1]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, -1)^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, \sqrt{21}, -1)^T,
\end{array} \right.$$ 

(4.17)

$$\langle \tilde{\chi}^{[2]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, \sqrt{21}, -1)^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, 0)^T,
\end{array} \right.$$ 

(4.18)

$$\langle \tilde{\chi}^{[3]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, \sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T,
\end{array} \right.$$ 

(4.19)

$$\langle \tilde{\chi}^{[4]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T,
\end{array} \right.$$ 

(4.20)

$$\langle \tilde{\chi}^{[5]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T,
\end{array} \right.$$ 

(4.21)

$$\langle \tilde{\chi}^{[6]} \rangle \propto \left \{ \begin{array}{c} 1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T, \\
1/6 \cdot (0, 0, 0, -\sqrt{14}, -\sqrt{21}, \sqrt{7}, 0, -\sqrt{2})^T.
\end{array} \right.$$ 

(4.22)

The alignment vectors $\langle \tilde{\chi}^{[p]} \rangle$ with $p = 0, 1, 2$ have the form of Eq. (2.6) and correspond to the vectors $\langle \chi^{[p]} \rangle$ of Eqs. (2.7-2.9). Therefore, three of the seven discrete minima lead to a tri-bimaximal structure.

It is worth mentioning that the above vacuum alignments $\langle \tilde{\chi}^{[p]} \rangle$ can alternatively also be obtained from an $F$-term alignment mechanism. Consider a superpotential of the form

$$W = \mathcal{M} \chi_0 \chi + g \chi_0 (\chi \chi)_6 + g' \chi_0 (\chi \chi)'_6,$$ 

(4.23)

where $\chi_0$ is a driving sextet field. The parentheses denote the contraction to the two distinct sextet as shown in Eqs. (4.2-4.3). It is easy to show that the resulting $F$-term equations of $\chi_0$ are solved by the sextet alignment $\langle \tilde{\chi}^{[0]} \rangle = (0, 0, 0, 0, 0, v)^T$ if

$$v = - \frac{\mathcal{M}}{g + 5\sqrt{2}g'}.$$

Due to the symmetry of the superpotential under $PSL_2(7)$, the other six alignments $\langle \tilde{\chi}^{[p]} \rangle$ lead to vanishing $F$-terms as well. Thus the three sextets leading to a tri-bimaximal structure can be aligned alternatively through an $F$-term mechanism. In the case of charged sextets the mass parameter $\mathcal{M}$ must be generated dynamically by the VEV of some $PSL_2(7)$ singlet.
4.2.2 \( \chi_{\text{top}} \) alignment

We now turn to the study of the alignment of Eq. (2.10). In general we seek to solve ten non-trivial conditions with a set of five parameters \( \kappa_i \). Therefore it is by no means guaranteed that an extended parameter space of solutions to the minimisation conditions of the potential should exist. Remarkably, by plugging Eq. (2.10) into the potential in Eq. (4.9), we find that, for a certain choice of parameters discussed below, the first derivatives of \( f \) can be made to vanish. However, unlike the previous alignments, first derivatives of \( f \) do not vanish for all the invariants taken separately.

To be precise, a straightforward calculation shows that, in order to achieve the desired top alignment \( \langle \chi_{\text{top}} \rangle \), the vanishing of the first derivatives requires only two relations amongst the different sextet combinations which result from the symmetric combinations of two sextets, to wit \( \kappa_2 = \kappa_3 = \kappa_4 + \kappa_5 / \sqrt{7} \). Clearly the origin of these relations remains to be understood, and, for example, could result from some underlying higher symmetry, although this goes beyond the scope of the present \( PSL_2(7) \) discussion. However we emphasise that the fact that a successful potential can be found with particular values of parameters which can lead to the desired top alignment is highly non-trivial and this is not the case in general for other alignments.

Assuming the above two relations, the potential that has a chance of being minimised by \( \langle \chi_{\text{top}} \rangle \) includes an \( f \) of the form

\[
\left[ \begin{array}{c}
I_0 \\
I_1 \\
I_2 \\
I_3 \\
I_4
\end{array} \right] + \kappa \left[ \begin{array}{c}
I_0 \\
I_1 \\
I_2 \\
I_3 \\
I_4
\end{array} \right]
+ \kappa' \left( \begin{array}{c}
I_0 \\
I_1 \\
I_2 \\
I_3 \\
I_4
\end{array} \right) + \kappa'' \left( \begin{array}{c}
I_0 \\
I_1 \\
I_2 \\
I_3 \\
I_4
\end{array} \right). \tag{4.24}
\]

Since \( f \) has to be positive, we get the condition

\[
0 < f \bigg|_{\langle \chi_{\text{top}} \rangle} = 1 + 0 \cdot \kappa + \frac{70}{3} \cdot \kappa' + \frac{140}{3} \cdot \kappa''. \tag{4.25}
\]

On the other hand, the Hessian needs to be positive-definite. The calculation for all three independent contributions to \( f \) shows that the 10-dimensional Hessian \( h \) has two positive and eight zero eigenvalues, while \( h' \) has six positive and four negative ones, and finally \( h'' \) has ten negative eigenvalues. We therefore conclude that the choice

\[
\kappa = \kappa' = 0, \quad -\frac{3}{140} < \kappa'' < 0, \tag{4.26}
\]

leads to a potential \( V \) which is minimised by the alignment of Eq. (2.10). In other words, \( I'' \) must be suppressed compared to \( I_0 \). Other solutions are possible as well, but as the three Hessians cannot be diagonalised simultaneously, it is not easy to give a general expression. However we emphasise that \( \kappa \) and \( \kappa' \) need not be zero, for instance the choice

\[
\kappa = \kappa' = -\kappa'' = \frac{1}{100},
\]

To illustrate this, consider the alignment \( \langle \chi_{\text{alt}} \rangle \propto (-1, 0, 0, \sqrt{3}/b7, \sqrt{2}, 0)^T \) which leads to a Yukawa matrix with identical \((2,2)\) and \((3,3)\) elements and zero entries everywhere else. Requiring vanishing first derivatives fixes the parameters \( \kappa_i \) uniquely. Up to an overall scale we find \( \kappa_1 = 0, \kappa_2 = \kappa_3 = 2\kappa_4 = 2\kappa_5 / \sqrt{7} \). However, in this case, the matrix of second derivatives turns out to have positive and negative eigenvalues showing that no quartic potential exists which is minimised by \( \langle \chi_{\text{alt}} \rangle \).
leads to an acceptable potential as well.

Having illustrated that the alignment $\langle \chi_{\text{top}} \rangle$ can be obtained from reasonable potentials, we again find that a transformation of this alignment vector under $PSL_2(7)$ yields new minima. In this case, we get 168 different minima, including the one of Eq. (2.10).

All of them are physically identical since we can use the freedom to redefine our basis by applying a suitable $PSL_2(7)$ symmetry transformation.

### 4.3 Flavon anti-triplet vacuum alignment

As outlined in [2] we need to introduce anti-triplet flavon fields to generate the first and second family Yukawa couplings. The question arises how to align such anti-triplets $\bar{\phi}$ of $PSL_2(7)$. The immediate idea to study quartic terms of the form $\bar{\phi}_i \bar{\phi}_j \bar{\phi}_k \bar{\phi}_l$, which proves successful for flavour groups like $\Delta_{27}$ [9], $Z_7 \times Z_3$ [26], $A_4$ [7] and others [3], does not lead to an alignment of the anti-triplets because $PSL_2(7)$ is too big a symmetry so that the trivial $SU(3)$ invariant $\bar{\phi}_i \bar{\phi}_i \bar{\phi}_j \bar{\phi}_j$ is the only allowed quartic term of only anti-triplets.

However, due to the presence of the sextet representation, the possibility of aligning the anti-triplets by coupling them to the pre-aligned sextets arises. It turns out that the simplest operator discussed above does not work, in the following we discuss the next simplest possibility of coupling two anti-triplets $\bar{\phi}$ to the product of two sextet flavons that have the tri-bimaximal alignment, $\langle \chi_{TB} \rangle$ with $p = 0, 1, 2$. We then prevent the flavon anti-triplets from coupling to the sextet flavon $\chi_{\text{top}}$ by means of a particular messenger sector and a $U(1)$ symmetry as discussed in Appendix B. The relevant scalar potential for the anti-triplets then reads

$$V = -m^2 \sum_{i=1}^{3} \bar{\phi}_i \phi_i + \lambda \left( \frac{1}{3} \sum_{i=1}^{3} \bar{\phi}_i \phi_i \right)^2 + \sum_{\alpha=1}^{3} \sum_{i,j=1}^{3} \sum_{k,l=1}^{6} c^{\alpha,i}_{jkl} \bar{\phi}_i \phi_j \langle \chi_{TB} \rangle_k \langle \chi_{TB} \rangle_l \right). (4.1)$$

Here the index $\alpha$ labels the three different invariants of the $PSL_2(7)$ product

$$\left( \begin{array}{c} 3 \\ 1 \otimes 8 \end{array} \right) \otimes \left( \begin{array}{c} 6 \\ 1 \otimes 2 \otimes 6 \otimes 7 \otimes 2 \otimes 8 \end{array} \right),$$

with the index structure of $c^{\alpha,i}_{jkl}$ being defined by the Clebsch Gordan coefficients of the corresponding Kronecker products. The contraction to the singlet yields a term of the form

$$\Delta V_0 = \alpha_0 \sum_{i=1}^{3} \bar{\phi}_i \phi_i, (4.2)$$

Note, however, that it is possible to use similar couplings to align anti-triplets against the $T$ preserving sextet $\chi_T$ with an alignment of the form $\langle \chi_T \rangle \propto (\sqrt{2}, 0, 0, 0, 1, z)^T$, where $z$ remains undetermined. It turns out that the resulting anti-triplet alignment $\langle \phi \rangle \propto (1, 0, 0)$ preserves $T$ as well.
where \(\alpha_0\) includes the VEVs of the sextet flavons. This contribution to the potential does not constrain the anti-triplet alignment. The situation changes for the other two invariants obtained form the contractions to the octet. Inserting the alignments \(\langle \chi_{TB}^{p'} \rangle\) and \(\langle \chi_{TB}^{p} \rangle\), with \(p, p' = 0, 1, 2\), into Eqs. \((C.5, C.6)\) we find that, in general, only the third and the forth components of the octet of \(3 \otimes 3\) in Eq. \((C.7)\) can be projected out in the potential. However, with \(p = p'\), i.e. identical sextets entering in the potential, both terms vanish identically. Only in the case where \(p \neq p'\) we get non-zero contributions to the potential. Choosing for example \(p = 1\) and \(p' = 2\), the symmetric octet of Eq. \((C.5)\) is proportional to \((0, 0, 0, 1, 0, 0, 0, 0)\) while the antisymmetric octet of Eq. \((C.6)\) is proportional to \((0, 0, 1, 0, 0, 0, 0, 0)\). We thus find two additional independent terms

\[
\Delta V_s = \alpha_s \left[ \tilde{\phi}_1^t (\tilde{\phi}^2 + \tilde{\phi}^3) + \tilde{\phi}_2^t (\tilde{\phi}^1 + \tilde{\phi}^3) + \tilde{\phi}_3^t (\tilde{\phi}^1 + \tilde{\phi}^2) \right],
\]

\[
\Delta V_a = \alpha_a \left[ \tilde{\phi}_1^t (-2\tilde{\phi}^1 + \tilde{\phi}^2 + \tilde{\phi}^3) + \tilde{\phi}_2^t (\tilde{\phi}^1 + \tilde{\phi}^2 - 2\tilde{\phi}^3) + \tilde{\phi}_3^t (\tilde{\phi}^1 - 2\tilde{\phi}^2 + \tilde{\phi}^3) \right].
\]

Combining these two terms linearly with each other and with \(\Delta V_0\) we can define two new independent terms which determine the anti-triplet alignment\(^5\)

\[
\Delta V_1 = \alpha_1 (\tilde{\phi}_1^t + \tilde{\phi}_2^t + \tilde{\phi}_3^t) (\tilde{\phi}^1 + \tilde{\phi}^2 + \tilde{\phi}^3),
\]

\[
\Delta V_2 = \alpha_2 (\tilde{\phi}_2^t - \tilde{\phi}_3^t) (\tilde{\phi}^2 - \tilde{\phi}^3).
\]

These two terms of the scalar potential are at the core of the discussion of the anti-triplet alignment. Choosing the values of \(\alpha_1\) and \(\alpha_2\) appropriately gives rise to a potential which is minimised by an alignment of the anti-triplets required to generate the first and second family Yukawa couplings. As an aside we note that the two independent terms of Eqs. \((4.5, 4.6)\) can be obtained similarly from Eq. \((4.1)\) using tri-bimaximal flavon sextets with \(p = 0\) and \(p' = 1\) or 2.

Let us first consider the two corrections to the potential \textit{individually}. For positive \(\alpha_1\) the minimum of \(\Delta V_1\) is zero. This entails a partial alignment of the form

\[
\alpha_1 > 0 : \quad \langle \tilde{\phi} \rangle \propto \begin{pmatrix} x \\ y \\ -x - y \end{pmatrix},
\]

where \(x, y \in \mathbb{C}\) remain undetermined. In contrast, for negative \(\alpha_1\) the resulting alignment is completely fixed

\[
\alpha_1 < 0 : \quad \langle \tilde{\phi} \rangle \propto \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}.
\]

Similarly, the potential term \(\Delta V_2\) gives rise to the following structure of the anti-triplet VEVs

\[
\alpha_2 > 0 : \quad \langle \tilde{\phi} \rangle \propto \begin{pmatrix} x \\ y \\ y \end{pmatrix}, \quad \alpha_2 < 0 : \quad \langle \tilde{\phi} \rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.
\]

\(^5\)Explicitly, \(\Delta V_1/\alpha_1 = \Delta V_s/\alpha_s + \Delta V_0/\alpha_0\) and \(\Delta V_2/\alpha_2 = 1/3(\Delta V_a/\alpha_a - \Delta V_s/\alpha_s + 2\Delta V_0/\alpha_0)\).
Remarkably, the alignments derived from both terms $\Delta V_1$ and $\Delta V_2$ individually can be made compatible by choosing the signs of $\alpha_1$ and $\alpha_2$ according to the following combinations

$$\alpha_1 > 0, \alpha_2 < 0 : \langle \bar{\phi} \rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \alpha_1 < 0, \alpha_2 > 0 : \langle \bar{\phi} \rangle \propto \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad (4.7)$$

Thus it is possible to have an anti-triplet flavon field $\bar{\phi}_{23}$ whose VEV becomes aligned along $(0, 1, -1)$, while another flavon field $\bar{\phi}_{123}$ ends up with the alignment $(1, 1, 1)$.

5 Conclusion

In this paper we have constructed a realistic SUSY GUT of Flavour based on $PSL_2(7) \times SO(10)$, where the quarks and leptons in the 16 of $SO(10)$ are assigned to the complex triplet representation of $PSL_2(7)$, while the flavons are assigned to a combination of sextets and anti-triplets of $PSL_2(7)$. It represents the first model based on the finite group $PSL_2(7)$, which is the smallest simple group that contains both complex triplet and (real) sextet representations. This group seems particularly well suited to $SO(10)$ since the sextets may be used to provide the large third family Yukawa coupling, as well as type II neutrino masses. Furthermore $PSL_2(7)$ contains $S_4$ as a subgroup, and this allows the possibility of explaining TB neutrino mixing in a direct way, by preserving the generators $S, U$ of $PSL_2(7)$ in the neutrino sector, which become identified as the neutrino flavour symmetry. There are very few models that can account for TB neutrino mixing using $SO(10)$ and this is the first model which can do this directly.

Using a $D$-term vacuum alignment mechanism, we have shown how the flavon sextets of $PSL_2(7)$ can be aligned along the 3-3 direction leading to the third family Yukawa couplings. Such sextets aligned along the 3-3 direction are ideally suited for giving a universal contribution to the 3-3 Yukawa coupling at the lowest possible one-flavon order in $SO(10)$ models, allowing a sizeable universal top-bottom-tau Yukawa coupling. We emphasise that the fact that a successful potential can be found with particular values of parameters which can lead to the desired 3-3 vacuum alignment of the sextet flavons is highly non-trivial and this is not the case in general for other alignments. However, in order to realise the flavon sextet potential that yields an alignment along this 3-3 direction, it is necessary to assume certain relations amongst the parameters of the potential. These relations could in principle emerge from a higher symmetry, beyond $PSL_2(7)$, although this takes us beyond the scope of the present paper, though it should be the subject of future investigation.

Other sextets are aligned along the neutrino flavour symmetry preserving directions in an even more natural way, without requiring any relations between the parameters of the potential, and such alignments suggest the possibility of TB neutrino mixing via a type II see-saw mechanism. We have explored the phenomenological consequences of such a type II see-saw mechanism and obtained statistical predictions for neutrinoless double beta decay and neutrino masses in cosmology. The distributions of randomly generated points exhibit a very broad peak at about $\sim 0.05$ eV with significant tails out to about $\sim 0.4$ eV in both $m_{\text{min}}$ and $m_{ee}$.

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Anti-triplet flavons are also introduced and aligned against the pre-aligned TB sextet flavons, in order to give the remaining structure of the charged fermion mass matrices. In principle these may also be used to account for neutrino masses and TB mixing via a type I see-saw mechanism as in [11], but since this is well known we have focused on the new type II possibility opened up by having sextets in the model. This also avoids the use of operators with zero Clebsch factors in order to provide the necessary suppression in the type I Dirac neutrino sector [11]. Nevertheless, the anti-triplet flavons are instrumental in giving the successful mass matrices in Eq. (2.18) via the assumption of two different messenger mass scales in the up and down sectors. In this model the mass matrices are achieved in a very natural way since, with the inclusion of the singlet flavon $\xi$, the first row and column is cubic in the messenger mass, while the 2-3 block is quadratic and the 3-3 element involves the universal Higgs messenger mass $M_H$.

In conclusion the SUSY GUT of Flavour based on $PSL_2(7) \times SO(10)$ leads to a very elegant model, combining the mathematical beauty of $PSL_2(7)$ with the attractiveness of $SO(10)$ unification, and solving many of the problems related to achieving successful fermion masses and TB mixing in the $SO(10)$ framework. The SUSY flavour problem is also solved here exactly as in the $SU(3)$ model discussion in [11], since both models use identical anti-triplet flavon alignments. Finally we emphasise that the type II see-saw mechanism in this model is consistent with neutrinoless double beta decay right up to the limit of current experiments.

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**Appendix**

### A The induced VEV

In $SO(10)$ the induced VEV of the $\Delta_{126}$ field in Fig. 3(a) is obtained by replacing the $\Delta_{126}$ leg of the diagram by

![Diagram](image-url)
The relevant components of the $SO(10)$ representations are

\[ H_{10} \rightarrow (15,2,2), \quad \Xi_{126} \rightarrow (\overline{10},1,3), \quad (A.1) \]

with $H_{10}$ acquiring a VEV at the electroweak scale while $\Xi_{126}$ gets a GUT scale VEV. Assuming the messengers in the complete diagram to have a mass of order $M$, the induced VEV can be calculated to be

\[ \langle \Xi_{126} \rangle (10,1,3) \langle H_{10} \rangle (1,2,2) \langle H_{10} \rangle (1,2,2) \sim \frac{v^2}{M}, \quad (A.2) \]

where, for simplicity, we have taken $\langle \Xi_{126} \rangle (10,1,3) \sim M$. It is clear from Eq. (A.2) that the resulting contribution to the neutrino masses corresponds to the type II see-saw mechanism because the two left-handed doublets of the $H_{10}$ pair combine to an $SU(2)_L$ triplet, whereas the two right-handed doublets are contracted with the right-handed triplet of $\Xi_{126}$ to a singlet under $SU(2)_R$. The relevant component of the $\Delta_{126}$ messenger is thus an $SU(2)_L$ triplet.

\section*{B Controlling the flavon potential}

In Section 4 we discussed the terms of the flavon potential that are required to obtain the alignments that generate the Majorana and Yukawa couplings. There we encountered the following four types of couplings

\[ \chi^{\dagger}_{\text{top}} \chi_{\text{top}} \chi_{\text{top}}, \quad \chi^T_{\text{TB}} \chi^T_{\text{TB}} \chi^T_{\text{TB}} \chi^T_{\text{TB}}, \quad \phi_{23}^{\dagger} \phi_{23} \phi_{23}^{\dagger} \phi_{23}^{\dagger}, \quad \phi_{123}^{\dagger} \phi_{123} \phi_{123}^{\dagger} \phi_{123}^{\dagger}, \quad (B.1) \]

which involve four sextet and two anti-triplet flavon fields. As we have already discussed the contractions that yield $PSL_2(7)$ invariants, we suppress all indices in Eq. (B.1). At this point it is important to observe that the suggested vacuum alignment is based on the absence of terms like

\[ \chi^{\dagger}_{\text{top}} \chi_{\text{top}} \chi^T_{\text{TB}} \chi^T_{\text{TB}}, \quad \chi^T_{\text{TB}} \chi^T_{\text{TB}} \chi^T_{\text{TB}} \chi^T_{\text{TB}}, \quad \phi_{23}^{\dagger} \phi_{23} \phi_{23}^{\dagger} \phi_{23}^{\dagger}, \quad \phi_{123}^{\dagger} \phi_{123} \phi_{123}^{\dagger} \phi_{123}^{\dagger}, \quad (B.2) \]

that – at the effective level – cannot be forbidden by symmetries alone. It is therefore necessary to resort to a particular messenger sector. A neutral messenger would automatically give rise to diagrams such as

\[ \chi^T \chi \chi \]

\[ \chi' \chi' \phi \phi' \]

which cannot distinguish between the structure of the terms in Eq. (B.1) and Eq. (B.2). Therefore the messengers must be charged under additional symmetries. Accordingly, a possible way to forbid the terms of Eq. (B.2) is given by diagrams of type
where the charge of the respective messengers determines whether or not \( \chi' = \chi \) is allowed as well as which sextet flavons can couple to the anti-triplet flavons. Most notably we need to separate the top sextet from the sextets of tri-bimaximal type. This is achieved easily by assigning different \( U(1) \) charges \( q \) to the flavons \( \chi_\text{top} \) and \( \chi^{[p]}_{TB} \). Choosing for instance

\[
q(\chi_\text{top}) = -1 \ , \quad q(\chi^{[p]}_{TB}) = -2 \ , \quad q(\bar{\phi}_{23}) = -2 \ , \quad q(\bar{\phi}_{123}) = 4 \ , \quad (B.3)
\]

one can generate the operators of Eq. (B.1) using messengers with charges \( q = 2, 4, 4, -2 \), respectively, while the first, third and fourth term of Eq. (B.2) would require messengers with odd \( U(1) \) charge. In the absence of such messengers the top sextet flavon cannot mix with the tri-bimaximal one.

Furthermore, we also need to forbid the second term of Eq. (B.2), i.e. the mixing among the three tri-bimaximal flavon sextets. For this purpose we introduce a separate symmetry, \( U(1)' \), which distinguishes between \( \chi^{[p]}_{TB} \), with \( p = 0, 1, 2 \). One possible set of \( U(1)' \) charges could be

\[
q'(\chi^{[0]}_{TB}) = 2 \ , \quad q'(\chi^{[1]}_{TB}) = 3 \ , \quad q'(\chi^{[2]}_{TB}) = 5 \ . \quad (B.4)
\]

The second term of Eq. (B.1), corresponding to three distinct quartic operators, would arise from messengers with charges \( q' = -4, -6, -10 \), respectively. At the same time, the analogous mixing term of Eq. (B.2) would require messengers with either \( q' = -5, -7, -8 \) or \( q' = 1, 2, 3 \).

In the construction of the complete model it is necessary to introduce three \( PSL_2(7) \) singlet flavons \( \zeta^{[p]} \). These are associated with the three tri-bimaximal sextet flavons \( \chi^{[p]}_{TB} \) and carry opposite \( U(1)' \) charge. Then the \( \chi^{[p]}_{TB} \) legs of the diagrams in Fig. 3 need to be replaced by

\[
\begin{array}{c}
\zeta^{[p]} \\
\chi^{[p]}_{TB}
\end{array}
\]

With this completion, the only particles that are charged under the \( U(1)' \) symmetry are \( \chi^{[p]}_{TB}, \zeta^{[p]} \) and the above mentioned messengers with \( q' = -4, -6, -10 \).

\section{The octet representation}

Just like in \( SU(3) \), the octet can be constructed from the product \( 3 \otimes \overline{3} \), see for instance Ref. [31]. The resulting \( 8 \times 8 \) matrices for the \( PSL_2(7) \) generators are real, but none
is diagonal. Since we are interested in the combination \((6 \otimes 6) \rightarrow 8\), it is convenient to perform a similarity transformation to a basis in which – analogous to the sextet (see [2]) – the generators \(S^{[8]}\) and \(U^{[8]}\) are diagonal. We obtain

\[
S^{[8]} = \text{Diag}(-1, -1, 1, 1, -1, -1, 1, 1), \quad (C.1)
\]

\[
T^{[8]} = \frac{1}{2} \begin{pmatrix}
1 & 0 & \sqrt{2} & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & \sqrt{2} & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -\sqrt{3} \\
0 & -1 & 0 & 0 & 1 & 0 & -\sqrt{2} & 0 \\
-1 & 0 & \sqrt{2} & 0 & 0 & -1 & 0 & 0 \\
0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -1
\end{pmatrix}, \quad (C.2)
\]

\[
U^{[8]} = \text{Diag}(1, 1, 1, 1, -1, -1, -1, -1), \quad (C.3)
\]

\[
V^{[8]} = \frac{1}{4} \begin{pmatrix}
-3 & \sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{7} & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{3} & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & -\sqrt{14} \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{14} & \sqrt{2} \\
0 & 0 & 0 & 0 & -\sqrt{2} & -\sqrt{14} & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{14} & \sqrt{2} & 0 & 0
\end{pmatrix}. \quad (C.4)
\]

With the identification

\[
A = [T^{[8]} U^{[8]} S^{[8]} (T^{[8]})^2 U^{[8]}]^{-1} V^{[8]} [T^{[8]} U^{[8]} S^{[8]} (T^{[8]})^2 U^{[8]}], \quad B = T^{[8]},
\]

one can easily check that the presentation of \(PSL_2(7)\) is satisfied. The symmetric octet \(\Omega\) of Eq. (4.4) is given in the basis of Eqs. (C.1-C.4). More generally there are two independent octets derived from the product of two sextets which we take in the basis of [2]. They are

\[
\begin{pmatrix}
\sqrt{3}x_2x'_3 + x_1x'_4 - 2\sqrt{6}x_1x'_5 + 2\sqrt{14}x_1x'_6 \\
\sqrt{2}1x_2x'_3 - 3\sqrt{2}1x_1x'_4 \\
-\sqrt{6}x_1x'_4 + \sqrt{6}x_2x'_2 - 2x_4x'_5 + 2\sqrt{14}x_5x'_6 \\
-\sqrt{2}x_1x'_1 - \sqrt{2}x_2x'_2 + 2\sqrt{2}x_3x'_3 - 2\sqrt{2}x_4x'_4 + 2\sqrt{2}x_5x'_5 - 2\sqrt{7}x_4x'_6 \\
-\sqrt{2}1x_1x'_3 + 3\sqrt{2}1x_2x'_4 \\
\sqrt{3}x_1x'_3 + x_2x'_4 + 2\sqrt{6}x_2x'_5 + 2\sqrt{14}x_2x'_6 \\
-2\sqrt{2}1x_3x'_5 \\
2\sqrt{6}x_1x'_2 - 4\sqrt{2}x_3x'_4 + 2\sqrt{7}x_3x'_6
\end{pmatrix} + (\chi \leftrightarrow \chi'), \quad (C.5)
\]
Obviously, the antisymmetric one only exists for $\chi \neq \chi'$. Adopting the triplet basis of [2], the octet of the product $3 \otimes 3$ reads

$$
\left( -\sqrt{2} \chi_2 \chi_3 - \sqrt{7} \chi_1 \chi_4 - 2\sqrt{2} \chi_0 \chi_0' \right) \\
\left( \sqrt{3} \chi_2 \chi_3' - 3 \chi_1 \chi_4' - 2\sqrt{6} \chi_1 \chi_5' \\
-2\sqrt{3} \chi_1 \chi_5' - 2\sqrt{2} \chi_5 \chi_6' \\
-6 \chi_4 \chi_6' \\
-3 \chi_1 \chi_3' + 3 \chi_2 \chi_4' - 2\sqrt{6} \chi_2 \chi_5' \\
-\sqrt{2} \chi_1 \chi_5' - \sqrt{7} \chi_2 \chi_4' - 2\sqrt{2} \chi_5 \chi_6' \\
2\sqrt{6} \chi_1 \chi_2' + 2\sqrt{3} \chi_3 \chi_5' \\
6 \chi_3 \chi_6' \right) - (\chi \leftrightarrow \chi'). \tag{C.6}
$$

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