New Parameterization in Muon Decay
and the Type of Emitted Neutrino

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Normal muon decay, \( \mu^+ \to e^+ \nu_e \nu_\mu \), is studied as a tool to discriminate between the Dirac and Majorana types of neutrinos and to survey the structure of the weak interaction. It is assumed that massive neutrinos mix with one another and that the interaction Hamiltonian consists of the \( V^- A \) and \( V^+ A \) charged currents. A new set of parameters used in place of the Michel parameters is proposed for the positron distribution. Explicit forms of these new parameters are obtained by assuming that the masses are less than 10 eV for light neutrinos and sufficiently large for heavy Majorana neutrinos, which are not emitted in the muon decay. It is shown that a possible method to discriminate between the Dirac and Majorana cases is to use a characterization given by the \( \chi^2 \) fitting of their spectra. It is also confirmed that the theoretical predictions in the Majorana neutrino case are almost the same as those obtained from the standard model. Indeed, their differences cannot be distinguished within the present experimental precision.

§1. Introduction

The structure of the leptonic charged weak interaction provides us with an important source of information that may lead to a unified theory beyond the standard model. Normal muon decay is a pure leptonic process accessible to precise measurements of this structure with high statistics, because it is free from the complications of the strong interaction and hadronic structure.

Experimental data have been analyzed by employing the helicity-preserving four fermion weak interaction with \( (S \pm P) \), \( (V \pm A) \) and \( T \) forms, because this arrangement allows one to make direct contact with specific models. The Michel parameters have been used to obtain some information concerning the structure of the weak interaction under the assumption that the neutrinos are massless and the lepton number is conserved. Recent experimental data have exhibited smaller deviations from the predictions based on the standard model.2–4)

The neutrino emitted in the annihilation of negatively charged leptons has been

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regarded as a particle, while the neutrino created together with the negatively charged leptons has been assigned to an anti-particle. In this assignment, in which there is a distinction between the neutrino particle and its anti-particle, the neutrino is of the Dirac type, and the lepton number is conserved in the weak interaction. In the other case, in which there is no such distinction, the neutrino is of the Majorana type. In the Majorana case, the lepton number is not conserved.

The dominant interaction responsible for muon decay has a $V-A$ structure, and the standard model is constructed on this footing. In this model, the left-handed neutrino field $\nu_L$ is assigned to a member of a doublet of the $SU(2)_L \times U(1)$ group, and no right-handed field $\nu'_R$ is present. The neutrino is assumed to be massless ($m_\nu = 0$), and the charged current weak interaction takes place via the exchange of the left-handed weak gauge boson $W_L$. The neutrino and anti-neutrino have the definite helicities $h = -1/2$ and $h = +1/2$, respectively, and we cannot distinguish between the Dirac and Majorana neutrino within the standard model.

Now, however, it has been established through the discovery of neutrino oscillation that neutrinos have finite masses and mix with one another. Thus, it is now known that the neutrino cannot be in a definite helicity state. For this reason, it is possible to discriminate between neutrinos of the Dirac and Majorana types. It is an important and fundamental question to determine whether the neutrino is of the Dirac or Majorana type.

There are two other unsolved problems regarding leptons. One is to determine why the observed mass differences among the three neutrinos are so small in comparison with charged leptons and quarks. The other is to understand why the left-handed $V-A$ interaction is favored over the right-handed $V+A$ interaction which has not been detected definitively. In the framework of gauge theory, it seems natural that the $V-A$ interaction is favored, as a result of the spontaneous breakdown of left-right symmetry, which is believed to be satisfied at sufficiently high energy.

One appealing way to solve these problems simultaneously is to use the idea of the seesaw mechanism, through which the right-handed neutrino field $\nu'_R$ is introduced. Let us explain the scenario of this mechanism for the one generation case for simplicity. In this scenario, the terms related to the neutrino mass can have the Majorana-type mass term $\langle \nu'_R \rangle^2 M_R \nu'_R$, in addition to the Dirac-type mass term, $\nu'_R M_D \nu_L$. The Majorana neutrino field is defined after diagonalizing the mass matrix, see Eq. (A.3) in Appendix A). The left-handed Majorana neutrino can have a small mass, of the order of $M_D^2/M_R$, while the right-handed Majorana neutrino can have a large mass, of the order of $M_R$, provided that the condition $M_D \ll M_R$ is satisfied. This situation leads to various extensions of the standard model, such as the $SU(2)_L \times SU(2)_R \times U(1)$ model. If this scenario is indeed valid, then we can conclude that there exists a right-handed gauge boson $W_R$ and that the muon decay receives contributions from interactions whose structure is somewhat different from that of the standard model.

It is not yet known whether the neutrino is of the Dirac or Majorana type and, further, what structure the weak interaction has beyond the standard model. For this reason, it is necessary to construct a method that provides some information
concerning these points. Neutrinoless double beta decay which violates lepton number conservation is the only presently known possible way to directly determine the type of the neutrino. However, this decay process requires very high resolution experimentally, because of the long half-lives, due to the tiny neutrino mass and/or the small contribution from the $V + A$ current. Thus, this method is yet to provide a decisive conclusion.

Muon decay takes place irrespective of the type of neutrino involved and precise data have been accumulated and analyzed to investigate the structure of the weak interaction by assuming the neutrino to be massless and of the Dirac type. It is also meaningful, as a complementary study, to survey the possibility of whether muon decay can be used as a tool to determine the type of the neutrino. With this in mind, we note that there is a difference between the spectra of emitted positrons in the Dirac and Majorana neutrino cases.

The aim of this paper is to propose a new parameterization of muon decay that is suitable for analyzing the type of neutrinos and the structure of the weak interaction. We adopt a Hamiltonian consisting of both $V - A$ and $V + A$ currents, which is inspired by the $SU(2)_L \times SU(2)_R \times U(1)$ gauge model, and present a method to analyze the implications of the experimental data.

In §2, we summarize the general framework of our study and discuss kinematical effects on the emitted $e^\pm$ due to finite neutrino mass ($m_\nu \neq 0$). There, our assumptions and the approximation adopted in our analysis are discussed. In §3, the $e^\pm$ energy spectrum is surveyed in detail. We propose some parameterizations to discriminate between the types of neutrinos and discuss their experimental feasibility. The polarization of the $e^\pm$ is discussed in §4. A summary and conclusion are given in §5. In Appendix A, to make the paper self-contained, features of the lepton mixing matrix for a model with left- and right-handed neutrinos are summarized. Also there, the details of the coupling constants for the weak interaction Hamiltonian based on the $SU(2)_L \times SU(2)_R \times U(1)$ gauge model are presented for convenience. In Appendix B, definitions of various coefficients are listed, and their explicit forms under certain conditions are given for the Dirac and Majorana neutrino cases separately.

§2. General framework

We assume the following form of the effective weak interaction Hamiltonian for the $\mu^\pm$ decay: \(^7\)

$$H_W(x) = \frac{G_F}{\sqrt{2}} \left\{ j_{\ell L\alpha} j_{\mu L}^\alpha + \lambda j_{e R\alpha} j_{\mu R}^\alpha + \eta j_{e R\alpha} j_{e L}^\alpha + \kappa j_{e L\alpha} j_{\mu R}^\alpha \right\} + \text{H.c.},$$

(2.1)

where $G_F$ is the Fermi coupling constant. The left-handed and right-handed charged weak lepton currents, $j_{\ell L}$ and $j_{\ell R}$, are defined as

$$j_{\ell L\alpha}(x) = \bar{E_\ell}(x)\gamma_\alpha(1 - \gamma_5)\nu_{\ell L}(x) \quad \text{and} \quad j_{\ell R\alpha}(x) = \bar{E_\ell}(x)\gamma_\alpha(1 + \gamma_5)\nu_{\ell R}(x).$$

(2.2)

Here $E_\ell$, $\nu_{\ell L}$ and $\nu_{\ell R}$ are the weak eigenstates of the charged lepton, left-handed neutrinos and right-handed neutrinos, with flavors $\ell = e$ and $\mu$. The interaction
in Eq. (2.1) is a general form of the four fermion, derivative-free Lorentz-invariant interaction, which consists of the $V - A$ and $V + A$ currents.

The weak eigenstates of the charged leptons ($E_\ell$) and neutrinos ($\nu_{\ell L}$ and $\nu_{\ell R}$) are defined, respectively, as the superpositions of the mass eigenstate charged leptons $E_\ell$ and neutrinos $N_j$ with mass $m_j$. Then, the charged currents are expressed as

$$j_{\ell L\alpha}(x) = \sum_{j=1}^{2n} E_\ell(x) \gamma_\alpha (1 - \gamma_5) U_{\ell j} N_j(x),$$

$$j_{\ell R\alpha}(x) = \sum_{j=1}^{2n} E_\ell(x) \gamma_\alpha (1 + \gamma_5) V_{\ell j} N_j(x),$$

for the case of the $n$ generations. Here $U_{\ell j}$ and $V_{\ell j}$ are, respectively, the left-handed and right-handed lepton mixing matrices.

The weak interaction given in Eq. (2.1) is naturally expected from the gauge models that contain the left-handed and right-handed weak gauge bosons, $W_L$ and $W_R$. In these models, the appearance of the coupling constant $\lambda$ is due to $W_R$, while terms with $\eta$ and $\kappa$ come from the possible mixing between $W_L$ and $W_R$. As a typical example with right-handed interactions, we consider the $SU(2)_L \times SU(2)_R \times U(1)$ model in Appendix A. If this model is assumed, then the coupling constants $\kappa$ and $\eta$ in Eq. (2.1) can be taken as identical, as shown in Eq. (A.19). However, they are treated as independent constants in this paper in order to allow comparison with the more general case without a restriction from the gauge theory (see, e.g., Ref. 1). The structures and magnitudes of the lepton mixing matrices $U$ and $V$ are also briefly summarized in Appendix A. We do not take account of the mirror lepton currents and Higgs boson exchange, for simplicity.

Now we study the normal muon decay $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ (or $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$). In the framework of the effective weak interaction Hamiltonian given in Eq. (2.1), $\mu^\pm$ decay takes place as

$$\mu^\pm \rightarrow e^\pm + N_j + \bar{N}_k,$$

where $\bar{N}_k$ represents an antineutrino for the Dirac neutrino case, but it should be understood as $N_k$ for the Majorana neutrino case.

If the radiative corrections are not included, the differential decay rate for polarized $e^\pm$ in the rest frame of polarized $\mu^\pm$ is expressed as

$$\frac{d^2\Gamma(\mu^\pm \rightarrow e^\pm \nu\bar{\nu})}{dx\,d\cos\theta} = \left( \frac{m_\mu G_F^2 W^4}{6 \cdot 4 (\pi)^3} \right) \sqrt{x^2 - x_0^2} A D(x, \theta) [1 + P_\mu(x, \theta) \cdot \hat{\zeta}],$$

where

$$x = \frac{E}{W} \quad \text{and} \quad W = \frac{m_\mu^2 + m_e^2}{2 m_\mu} = 52.8 \ \text{MeV}.$$
the finite neutrino mass \( m_\nu \neq 0 \), the allowed range of \( x \) is limited kinematically as

\[
x_0 \leq x \leq x_{\text{max}} = (1 - r_{jk}^2),
\]

where

\[
x_0 = \frac{m_e}{W} = 9.65 \cdot 10^{-3} \quad \text{and} \quad r_{jk}^2 = \frac{(m_j + m_k)^2}{2m_\mu W} = 3.63 \cdot 10^{-14}.
\]

Here \( m_j \) and \( m_k \) are masses of the neutrinos emitted in the muon decay and should satisfy the relation

\[
m_j + m_k < (m_\mu - m_e).
\]

In Eq. (2.8), the neutrino masses have been taken as \( m_j = m_k = 10 \text{ eV} \), in order to obtain a rough idea of the magnitude of \( r_{jk}^2 \).

The constant \( A \) in Eq. (2.5) is introduced to simplify the expression for the differential decay rate by using the arbitrariness of its normalization. It is referred to as a normalization factor in this paper. It is shown in §3 that there are various possibilities for the choice of \( A \), when experimental data are analyzed, although these choices differ only by rearrangements of the terms in the theoretical expression. This constant \( A \) plays the role of the leading term in the overall normalization related to the muon lifetime.

In the differential decay rate given in Eq. (2.5), \( D(x, \theta) \) is the \( e^\pm \) energy spectrum part, expressed as

\[
D(x, \theta) = [N(x) \pm P_\mu \cos \theta P(x)],
\]

where \( P_\mu = |\vec{P}_\mu| \), and the functions \( N(x) \) and \( P(x) \) are, respectively, the isotropic and anisotropic parts of the \( e^\pm \) energy spectrum. Their details are discussed in §3. The plus (minus) sign in Eq. (2.10) corresponds to \( \mu^+ (\mu^-) \) decay. The vector \( \vec{P}_e(x, \theta) \) in Eq. (2.5) is a polarization vector of \( e^\pm \), and \( \zeta \) is the directional vector of the measurement of the \( e^\pm \) spin polarization. We discuss \( \vec{P}_e(x, \theta) \) in §4.

The isotropic and anisotropic parts of the energy spectrum consist of various terms, each of which has a different \( x \) dependence taking a complicated form and includes some combination of the lepton mixing matrices. Therefore, in order to extract useful information, it is desirable, as the first step, to investigate the characteristic features of their \( x \) dependences and to simplify them.

As an example, let us consider only two terms of \( N(x) \),

\[
N(x) = \left( \frac{1}{A} \right) \left[ a_+ (3x - 2x^2 - x_0^2) + (b_+ - a_+) (2x - x^2 - x_0^2) \right],
\]

where

\[
a_+ = (a + \lambda^2 \hat{a}), \quad b_+ = (b + \lambda^2 \hat{b}),
\]

\[
a = \sum_j \sum_k' \left\{ F_{jk}^{3/2} |U_{ej}|^2 |U_{\mu k}|^2 \right\}, \quad b = \sum_j \sum_k' \left\{ F_{jk}^{1/2} G_{jk} |U_{ej}|^2 |U_{\mu k}|^2 \right\},
\]

\[
\hat{a} = \sum_j \sum_k' \left\{ F_{jk}^{3/2} |V_{ej}|^2 |V_{\mu k}|^2 \right\}, \quad \hat{b} = \sum_j \sum_k' \left\{ F_{jk}^{1/2} G_{jk} |V_{ej}|^2 |V_{\mu k}|^2 \right\}.
\]
Here, the primed sum represents the sum taken over neutrinos whose emission in the \( \mu^\pm \) decay is allowed by the restriction given in Eq. (2.9).

Two kinematical factors, \( F_{jk} \) and \( G_{jk} \), emerge from the phase space integral of the emitted neutrinos in both the Dirac and Majorana neutrino cases. They represent the additional kinematical effect due to the finite mass of the neutrino (\( m_\nu \neq 0 \)), and their explicit forms are given in §2.1 [see Eqs. (2.15) and (2.16)]. The treatment of the lepton mixing matrices, \( U_{\ell j} \) and \( V_{\ell j} \), is considered in §2.2.

2.1. Kinematical factors due to the non-vanishing neutrino mass · · · Condition (A)

First, it is worthwhile to note that all terms in the spectrum \( D(x, \theta) \) and the polarization \( \tilde{P}_e(x, \theta) \) in Eq. (2.5) are proportional to \( F_{jk}^{1/2} \), for example, as shown in Eq. (2.13). These terms include some combinations of the following kinematical factors:

\[
F_{jk} = \left[ 1 - \left( \frac{r_{jk}^2}{1 - x} \right) \right] \left[ 1 - \left( \frac{r_{jk}^2}{1 - x} \right) (1 - 4 \mu_{jk}) \right], \tag{2.15}
\]

\[
G_{jk} = \left[ 1 - \left( \frac{r_{jk}^2}{1 - x} \right) \right] \left[ 1 + 2 \left( \frac{r_{jk}^2}{1 - x} \right) (1 - \mu_{jk}) \right] + 6 \left( \frac{r_{jk}^2}{1 - x} \right)^2 \mu_{jk}, \tag{2.16}
\]

where \( \mu_{jk} \) is defined by

\[
\mu_{jk} = \frac{m_j m_k}{(m_j + m_k)^2}. \tag{2.17}
\]

If \( m_\nu = 0 \), both \( F_{jk} \) and \( G_{jk} \) are unity for the entire range of \( x \), as seen from Eqs. (2.15) and (2.16). However, if \( m_\nu \neq 0 \), they display the following different types of behavior near \( x_{\text{max}} \):

\[
F_{jk} \to 0 \quad \text{and} \quad G_{jk} \to 6 \mu_{jk} \quad \text{in the limit} \ x \to x_{\text{max}} = (1 - r_{jk}^2). \tag{2.18}
\]

This implies that the spectrum and polarization of \( e^\pm \) tend to zero suddenly near \( x_{\text{max}} \).

** Here we state the reason why \( F_{jk} \) suddenly becomes zero near \( x_{\text{max}} \) for \( m_\nu \neq 0 \), in spite of the fact that \( F_{jk} = 1 \) for \( m_\nu = 0 \). It is convenient to introduce the momentum transfer squared, \( \Delta^2 = (q_j + q_k)^2 \), where \( q_j \) is the 4-dimensional moment of \( N_j \). Because the \( e^\pm \) energy \( E \) is given by \( E = (m_j^2 + m_k^2 - \Delta^2)/2m_\mu \), the maximum energy is realized when \( \Delta^2 \) takes the minimum value \( \Delta^2 = (m_j + m_k)^2 \), as shown in Eq. (2.7). This minimum is realized under the following two conditions: (i) Both of the neutrinos are emitted in the direction opposite to \( e^\pm \) in order to satisfy the momentum conservation, namely, \( q_j^2 + q_k^2 + p_e^2 = 0 \). (ii) Each neutrino has a definite momentum, say \( q_j = p_e m_j/(m_j + m_k) \). Therefore, there is no freedom to assign arbitrary neutrino momentum. In other words, the density in the phase space is zero; that is, we have \( F_{jk} = 0 \). A similar situation exists in the case that the mass of one neutrino is zero. Contrastingly, if \( m_j = m_k = 0 \), we have to choose \( \Delta^2 = 0 \). This situation is allowed for various combinations of two neutrino momenta, since only the total momentum of the neutrinos is fixed under the condition (i). Thus, there is no special restriction in the phase space compared to the general case for an arbitrary \( e^\pm \) energy; that is, we have \( F_{jk} = 1 \).
different behavior for the massive and massless neutrino cases.

Fortunately, the \( x \) dependences of \( F_{jk} \) and \( G_{jk} \) appear significantly only in a very tiny range near \( x_{\text{max}} \), say, for \( x > (1 - 10^8 r_{jk}^2) \sim (1 - 10^{-6}) \) if the required numerical accuracy of the experiment is of order \( 10^{-6} \). Thus, practically, there seems to be no problem in treating \( F_{jk} \) and \( G_{jk} \) in \( D(x, \theta) \) and \( \vec{P}_e(x, \theta) \) as independent of \( x \). More specifically, if we assume that the emitted neutrinos have small masses, i.e., at most of the order of \( 10 \text{ eV} \), the following approximations yield very good accuracy:

\[
F_{jk} = 1 \quad \text{and} \quad G_{jk} = 1. \tag{2.19}
\]

Hereafter, this approximation is referred to as Condition (A).

Under Condition (A), the second term of \( N(x) \) in Eq. (2.11) yields no contribution. Hence, we need not consider this term, except for the negligible tiny range near \( x_{\text{max}} \). We omit such terms in this paper without any loss of practical accuracy.

2.2. Contributions from the lepton mixing matrices \( U \) and \( V \) ... Condition (B)

The sum over the square (or product) of the lepton mixing matrix elements appears in the spectrum \( D(x, \theta) \) and polarization \( \vec{P}_e(x, \theta) \) appearing in Eq. (2.5), for example as seen from Eqs. (2.13) and (2.14). If Condition (A) is accepted, we are able to take the sum over neutrino indices under some assumptions.

In the Dirac neutrino case, it is assumed that all neutrinos can be emitted in the \( \mu^\pm \) decay. Then we have the following properties from the unitarity conditions of \( U \) and \( V \):

\[
\sum_j |U_{\ell j}|^2 = \sum_j |V_{\ell j}|^2 = 1. \tag{2.20}
\]

By contrast, in the Majorana neutrino case, we assume the existence of heavy Majorana neutrinos, which are not emitted in the \( \mu^\pm \) decay. Then, there is a different situation, in which we have

\[
\sum_j' |U_{\ell j}|^2 = 1 - \overline{u}_{\ell}^2 \quad \text{and} \quad \sum_j' |V_{\ell j}|^2 = \overline{v}_{\ell}^2, \tag{2.21}
\]

where the primed sums are taken over only the light neutrinos. Here \( \overline{u}_{\ell}^2 \) and \( \overline{v}_{\ell}^2 \) are the representatives of small deviations from unitarity due to these heavy Majorana neutrinos. [The details are given in Eqs. (A-6) – (A-8), (A-15) and (A-16).] In what follows, the explicit forms of these matrices \( U \) and \( V \) are not needed.

In addition, in the Majorana neutrino case, the following products of \( U \) and \( V \) appear:

\[
\overline{w}_{e\mu} \equiv \sum_j' U_{ej} V_{\mu j} \quad \text{and} \quad \overline{w}_{e\mu h} \equiv \sum_k' V_{ek} U_{\mu k}. \tag{2.22}
\]

An example in which \( \overline{w}_{e\mu} \) appears is mentioned in §2.3. The quantities \( \overline{w}_{e\mu} \) and \( \overline{w}_{e\mu h} \) are also small, as shown in Appendix A and Eq. (4.14). The assumptions Eqs. (2.20) – (2.22) are referred to as Condition (B).

2.3. Terms characteristic of the Majorana-type neutrino

Finally, let us explain why the Majorana-type neutrino offers different information from the Dirac-type neutrino. We emphasize that some contributions that are
specific to the Majorana neutrino case are added to the decay rate. Among these, one appears even in the \((V - A)\) interaction alone if \(m_\nu \neq 0\). The others appear if the \((V + A)\) interaction exists in addition to \((V - A)\), even in the case \(m_\nu = 0\). We study two examples.

As the first example, let us consider the mode in which both the \((e^+, N_j)\) and \((\mu^+, \bar{N}_k)\) vertices are of the \((V - A)\) type. This assignment is referred to as the ordinary mode \((A)\). If \(m_\nu \neq 0\), \(N_j\) has a small component of helicity \(h = +1/2\), whose magnitude is proportional to \(m_j/\omega_j\), where \(\omega_j\) is the energy of \(N_j\). Therefore, in the Majorana neutrino case, in which this \(N_j\) cannot be distinguished from the \(N_j\) associated with \(\mu^+\), there is a new cross mode \((B)\) for which \(N_j\) \((= \bar{N}_j)\) comes from the annihilation of \(\mu^+\) and \(\bar{N}_k\) \((= N_k)\) is emitted with \(e^+\). The interference between the ordinary mode \((A)\) and the cross mode \((B)\) appears in the decay probability. It becomes proportional to the product of the neutrino masses, \(m_j m_k\). Of course, there is no such possibility in the Dirac neutrino case, where \(N_j \neq \bar{N}_j\).

Now, if the \((V + A)\) interaction is added to \((V - A)\), the situation changes greatly. As a second example, let us consider the mode for which the \((e^+, N_j)\) vertex is of the \((V - A)\) type, but the \((\mu^+, \bar{N}_k)\) vertex is of the \((V + A)\) type. In this case, the helicity of \(\bar{N}_k\) is \(h = -1/2\), the same as that of \(N_j\), even in the limit \(m_\nu \to 0\). We call this case the mode \((C)\). In the Majorana neutrino case, it is possible to have another cross emitting mode \((D)\) with the \((e^+, N_k)\) and \((\mu^+, N_j)\) vertices, in addition to the mode \((C)\). There arises interference between the \((C)\) and \((D)\) modes in the decay probability. The leading term of this interference is not proportional to \(m_\nu\), but it includes the coupling constant \(\kappa^2\) with lepton mixing matrices \(|\Sigma_j' U_{ej} V_{\mu j}|^2\). This results from the equality of the helicities, mentioned above. The mode \((D)\) does not exist in the Dirac neutrino case. [For details, see Fig. (2) and Table I of Ref. 9.]

Many terms proportional to \(m_\nu\) exist in the decay rate in both the Dirac and Majorana neutrino cases, because of the small component of the helicity that is proportional to \(m_\nu\). However, these terms are negligibly small and are not taken into account in this paper. The complete decay formulae including these terms are given in Refs. 10) and 11).

§3. Energy spectrum of \(e^\pm\)

The isotropic part \(N(x)\) and anisotropic part \(P(x)\) of the \(e^\pm\) energy spectrum appearing in Eq. (2-10) are, respectively, expressed as follows under Conditions \((A)\) and \((B)\) given in §2:

\[
N(x) = \left(\frac{1}{A}\right) \left[ a_+ (3x - 2x^2 - x_0^2) + 12 (k_{+c} + \varepsilon_m k_{+m}) x (1 - x) \right. \\
+ 6 \varepsilon_m \lambda d_e x_0 (1 - x) \left. \right],
\]  

\[
P(x) = \left(\frac{1}{A}\right) \sqrt{x^2 - x_0^2} \left[ a_+ (-1 + 2 x - r_0^2) \right. \\
+ 12 (k_{-c} + \varepsilon_m k_{-m}) (1 - x) \left. \right].
\]
Here, the decay formulae for the Dirac and Majorana neutrinos are obtained by setting \( \varepsilon_m = 0 \) and \( \varepsilon_m = 1 \), respectively. The original forms of \( N(x) \) and \( P(x) \) before adopting Conditions (A) and (B) are presented in Eqs. (B.2) and (B.3) of Appendix B.

The first terms of \( N(x) \) and \( P(x) \) represent their \( x \) dependences obtained from the standard model, where \( A = a_\pm = 1 \), and all other coefficients \( (k_{\pm c}, k_{\pm m} \text{ and } d_r) \) are zero. The quantity \( r_0^2 \) in \( P(x) \) is defined as follows:

\[
 r_0^2 = \frac{m_e^2}{m_\mu W} = \left( 1 - \sqrt{1 - \frac{x_0^2}{r_0^2}} \right) = 4.66 \cdot 10^{-5}.
\] (3.3)

Before listing the details of these coefficients, we note here that the suffix \( c \) of \( k_\pm \) indicates contributions that are the same for the Dirac and Majorana neutrino cases, whereas the suffix \( m \) indicates coefficients associated with the Majorana neutrino only.

In the Dirac neutrino case, these coefficients are expressed as follows:

\[
 a_\pm = \left( 1 \pm \lambda^2 \right) \quad \text{and} \quad k_{\pm c} = \left( \frac{1}{2} \right) \left( \kappa^2 \pm \eta^2 \right).
\] (3.4)

Of course, in this case there are no contributions from \( k_{\pm m} \) and \( d_r \). Note that the final expressions for \( N(x) \) and \( P(x) \) in the Dirac neutrino case are the same as those obtained by assuming \( m_\nu = 0 \) for one generation if Conditions (A) and (B) in \( \S 2 \) are both satisfied.

In the Majorana neutrino case, these coefficients have the following complicated forms:

\[
 a_\pm = \left[ \left( 1 - \bar{w}_{e\mu}^2 \right) \left( 1 - \bar{w}_{e\mu}^2 \right) \pm \lambda^2 \bar{v}_{e}^2 \bar{v}_{\mu}^2 \right],
\] (3.5)

\[
 k_{\pm c} = \left( \frac{1}{2} \right) \left[ \kappa^2 \left( 1 - \bar{w}_{e\mu}^2 \right) \bar{v}_{\mu}^2 \pm \eta^2 \bar{v}_{e}^2 \left( 1 - \bar{w}_{e\mu}^2 \right) \right],
\] (3.6)

\[
 k_{\pm m} = \left( \frac{1}{2} \right) \left[ \kappa^2 \left| \bar{w}_{e\mu} \right|^2 \pm \eta^2 \left| \bar{w}_{e\mu h} \right|^2 \right],
\] (3.7)

\[
 d_r = \left( \frac{1}{2} \right) \text{Re}(\bar{w}_{e\mu} \ast \bar{w}_{e\mu h}).
\] (3.8)

Here, \( \bar{w}_{e\ell}^2, \bar{v}_{e\ell}^2, \bar{w}_{e\mu} \) and \( \bar{w}_{e\mu h} \) are small quantities in general, as mentioned with regard to Eqs. (2.21) and (2.22). It should be noted that \( k_{\pm c} \) has the same order of magnitude as \( k_{\pm m} \), in contrast to the Dirac neutrino case. It is worthwhile noting that the final \( N(x) \) and \( P(x) \) in the Majorana neutrino case exhibit small deviations from those of standard model independently of the magnitudes of the coupling constants \( \lambda, \kappa \) and \( \eta \).

Since 1985, experimental data have been analyzed using expressions based on the well-known Michel parameters. Expressing \( N(x) \) and \( P(x) \) in terms of the Michel parameters using our notation, we have

\[
 N(x) = 6 \left[ x(1 - x) + \frac{2}{5} \rho_M \left( 4x^2 - 3x - x_0^2 \right) + \eta_M x_0 (1 - x) \right],
\] (3.9)

\[
 P(x) = 2 \xi_M \sqrt{x^2 - x_0^2} \left[ (1 - x) + \frac{2}{3} \delta_M \left( 4x - 3 - r_0^2 \right) \right].
\] (3.10)
These expressions are obtained, for example, from Ref. 1) or Eqs. (31) and (32) of Ref. 12).

Hereafter, these forms are referred to as the Michel parameterization. They are presented for the Dirac neutrino case with $m_\nu = 0$. In the standard model, these parameters take definite values: $\rho_M = \delta_M = 0.75$, $\xi_M = 1$, and $\eta_M = 0$. In our model, there is no $\eta_M$ term, even in the massive Dirac neutrino case, if we ignore terms proportional to $m_\nu$. The reason why the $\eta_M \neq 0$ term appears in the Michel parameterization of Ref. 1) is that it comes from the interference between the $(V \pm A)$ and $(S \pm P)$ (or $T$) interactions.

At first glance, the relation between $N(x)$ in Eq. (3.1) and that in Eq. (3.9) may not be clear. It is shown in §3.1 that the Michel parameterization given in Eq. (3.9) is a special case of our expression appearing in Eq. (3.1) in which some appropriate constant value is chosen for the normalization factor $A$. A similar situation is demonstrated for $P(x)$ in §3.2.

3.1. Isotropic part of the spectrum: $N(x)$

First, let us introduce the following quantity for the normalization factor $A$:

$$A_{n,\ell} = (a_+ + 2n k_{c+} + 2\ell \varepsilon_m k_{m+}) > 0,$$

(3.11)

where $n$ and $\ell$ are some integers. Next, we verify that Eq. (3.9) is a special case of Eq. (3.1) with the choice of either $A = A_{10}$ or $A_{11}$.

We first consider the case with $A = A_{10}$, in which terms characteristic of the Majorana neutrino case appear explicitly. Then, the following expression is derived:

$$N(x) = \left(\frac{1}{A_{10}}\right) \{ (a_+ + 2k_{c+})(3x - 2x^2 - x_0^2) + k_{c+}[12 x (1 - x) - 2(3x - 2x^2 - x_0^2)] + \varepsilon_m [12 k_{m+} x (1 - x) + 6 \lambda d_r x_0 (1 - x)] \},$$

(3.12)

$$= \left(3x - 2x^2 - x_0^2 + 2 \rho_c (3x - 4x^2 + x_0^2) + 12 \varepsilon_m \rho_m x (1 - x) + 6 \varepsilon_m \eta_m x_0 (1 - x) \right).$$

(3.13)

Here, the parameters are defined as follows:

$$\rho_c = \left(\frac{k_{c+}}{A_{10}}\right) > 0, \quad \rho_m = \left(\frac{k_{m+}}{A_{10}}\right) > 0 \quad \text{and} \quad \eta_m = \left(\frac{\lambda d_r}{A_{10}}\right).$$

(3.14)

The fact that $\rho_c$ and $\rho_m$ are positive is clear from Eqs. (3.4) – (3.7) and (3.11). Now, it is easy to confirm that the Michel parameterization, Eq. (3.9), can be obtained from Eq. (3.13) by introducing the relation

$$2 \rho_c = \left(1 - \frac{4}{3} \rho_M \right),$$

(3.15)

*Our definitions and those in Ref. 1) are related as $N(x) = 6F_{iS}(x)$ and $P(x) = 6F_{AS}(x)$. The amplitudes $g^{\gamma}_{\mu}$ defined in Ref. 1) correspond to our coupling constants as follows: $g^{V}_{LL} = 1$, $g^{V}_{RR} = \lambda$, $g^{V}_{LR} = \kappa$, $g^{V}_{RL} = \eta$, with all other amplitudes set to zero in the present paper.
because, for this confirmation, it is not necessary to take account of the terms $\rho_m$ and $\eta_m$ appearing only in the Majorana neutrino case. Note that the relation $A = 1$ is required in the Michel parameterization [see Eq. (44) of Ref. 12)].

Next, if the case with $A = A_{11}$ is chosen, we again obtain the form given in Eq. (3.9) if we replace $\rho_c$ in Eqs. (3.13) and (3.15) by $(\rho_c + \varepsilon_m \rho_m)$, because the third term in Eq. (3.13) is absorbed into the second term through this replacement, and the fourth term plays the role of the $\eta_M$ parameter. Of course, the denominator $A_{11}$ of Eq. (3.14) should be replaced by $A_{10}$.

Thus, it is clear that there is no advantage of introducing the Michel parameter at present, although it was very useful in determining the type of the weak interaction. In fact, in the Michel parameterization defined in Ref. 1), the form $(\rho_M - 3/4)$, like Eq. (3.15) [and $(\delta_M - 3/4)$, like Eq. (3.20)] is used. Therefore, instead of measuring the deviation from $\rho_M = 0.75$, it seems desirable to directly determine $\rho_c$ and $\rho_m$ themselves, which indicate the deviations from the standard model. Furthermore, it is worthwhile to note that we are interested in estimating the $x$ dependence of the difference between the experimental data and the prediction of the standard model, namely, $(3x - 2x^2 - x_0^2)$ appearing in Eq. (3.1), through use of the $\chi^2$-fitting in the experimental analysis.

Recently, the TWIST group reported a precise experimental result for $\rho_M$:

$$\rho_M = 0.75080 \pm 0.00032\text{(stat)} \pm 0.00097\text{(syst)} \pm 0.00023,$$

where the third error comes from the ambiguity in $\eta_M$ appearing in Eq. (3.9). This ambiguity is due to the fact that various values for $\eta_M$ have been used within the uncertainty of the accepted average value $\eta_M = (-7 \pm 13) \cdot 10^{-3}$. Assuming the Dirac-type neutrino, this group obtained the result $|\tan \zeta| < 0.030$ by combining Eqs. (3.15), (3.14) and (3.4) with Eq. (A.19), where $\zeta$ is the $W_L - W_R$ mixing angle defined in Eqs. (A.13) and (A.14).

Although the TWIST group analyzed their data using Eq. (3.9), their results can be interpreted in our analysis as follows. If neutrinos are of the Dirac type, there is no difference between the choices $A = A_{10}$ and $A = A_{11}$, because in this case $\rho_m = \eta_m = 0$. For the Majorana neutrino case, we choose $A = A_{11}$. Then, the restriction on $(\rho_c + \varepsilon_m \rho_m)$ can be obtained from their result for $\rho_M$ by using the relation in Eq. (3.15), and some information regarding $\eta_m$ can be extracted from their interpretation of the $\eta_M$ term.

It is useful to note that we have $\rho_M < 0.75$ in our model with the choice $A = A_{11}$ for either neutrino type. This follows from the relation $(\rho_c + \varepsilon_m \rho_m) > 0$, as seen in Eqs. (3.14) and (3.15). But the mean value in Eq. (3.16) is $\rho_M = 0.75080$. Accordingly, this implies $(\rho_c + \varepsilon_m \rho_m) < 0$, although $\rho_M < 0.75$ is satisfied within experimental uncertainty.*

Thus, we cannot distinguish from this experimental result whether the neutrino is of the Dirac or Majorana type. Although the $\eta_m$ parameter in Eq. (3.13) is characteristic of the Majorana neutrino, this parameter cannot be used for this

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* In order to account for the old result, $\rho_M = 0.7518 \pm 0.0026$, larger values of $|g_{LL}^Z|$ and $|g_{RL}^Z|$ have been obtained in Ref. 1).
Table I. The $x$ dependence of the term including $\rho_c$ for the cases of various $A_{n0}$.

| $n$ | $A_{n0}$ | Term including $\rho_c$ |
|-----|----------|--------------------------|
| 0   | $A_{00}$ | $12x(1 - x)$             |
| 1   | $A_{10}$ | $2x(3 - 4x) + 2x_0^2$    |
| 2   | $A_{20}$ | $-4x^2 + 4x_0^2$         |
| 3   | $A_{30}$ | $-6x + 6x_0^2$           |
| 4   | $A_{40}$ | $-4x(3 - x) + 8x_0^2$    |

distinction, because not only is it multiplied by the small coefficient $x_0$, but also $\eta_m$ itself takes a very small value in our model.

However, there is a method that might make it possible to distinguish between the two neutrino types. This method makes use of the different $x$ dependences of the coefficients for $\rho_c$ and $\rho_m$, as seen from Eq. (3.13). For example, suppose we analyze experimental data by using Eq. (3.13) with $\rho_m = 0$ and obtain some $\chi^2$ value, say, $\chi^2_m$ for the Majorana neutrino case. Then, suppose we repeat a similar analysis using Eq. (3.13) with $\rho_m = \eta_m = 0$ and thereby determine $\chi^2_d$ for the Dirac neutrino case. Indeed, such a $\chi^2_d$ was already determined by the TWIST group. In any case, if $\chi^2_m$ is much smaller than $\chi^2_d$, we can conclude that there is a higher probability that neutrinos are of the Majorana type.

As an aside, it is worthwhile noting that $\rho_c$ and $\rho_m$ appear symmetrically. For example, we have the same coefficient for $k_{-c}$ and $k_{-m}$ in Eq. (3.1), which corresponds to the choice $A = A_{00}$. Thus, the choice $A = A_{n0}$ is not of interest to us, because it does not discriminate between the Dirac and Majorana neutrino cases. On the other hand, if we choose $A = A_{10}$, we have an expression similar to Eq. (3.13) for $A = A_{10}$. That is to say, the roles of $\rho_c$ and $\rho_m$ are exchanged. Of course, theoretically, $A_{10}$ and $A_{01}$ are different, but it is not easy to distinguish them experimentally from the total decay rate, because their deviations from unity seem to be small.

We now discuss an important property useful for choosing $A$ when we try to compare $\chi^2_m$ with $\chi^2_d$. It is desirable to make the difference between $\chi^2_m$ and $\chi^2_d$ as large as possible. Because this difference depends on the different $x$ dependences of terms including the $\rho_c$ and $\rho_m$ parameters, this situation can be realized by choosing $A_{n\ell}$ with $n \neq \ell$, as seen in the derivation of Eq. (3.12). Let us fix $\ell = 0$ for simplicity. Then, the $x$ dependence of the term including $\rho_m$ is $12x(1 - x)$, as shown in Eq. (3.13), while the $x$ dependence of the term including $\rho_c$ is listed for various $n$ in Table I. Therefore, we have the freedom to minimize the $\chi^2$ value by choosing $A_{n\ell}$ according to the pattern of data distribution. Despite this fact, hereafter, we use the choice $A = A_{10}$ in this and subsequent sections to simplify our description.

3.2. Anisotropic part of the spectrum: $P(x)$

Next let us examine the features of $P(x)$ and introduce the following quantity for the common factor:

$$B_{n\ell} = (a_- + 2n k_{-c} + 2\ell \varepsilon_m k_{-m}).$$

We now show that Eq. (3.10) is a special case of Eq. (3.2) with the choice of
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either $B_{30}$ or $B_{33}$. First, in the case of $B_{30}$, we obtain the expression
\[ P(x) = \xi \sqrt{x^2 - x_0^2} \left\{ (-1 + 2x - r_0^2) + 6 \delta_c (3 - 4x + r_0^2) + 12 \varepsilon_m \delta_m (1 - x) \right\}, \]
(3.18)

where the parameters are defined as follows:
\[ \xi = \left( \frac{B_{30}}{A_{10}} \right), \quad \delta_c = \left( \frac{k_c}{B_{30}} \right) \quad \text{and} \quad \delta_m = \left( \frac{k_m}{B_{30}} \right). \]
(3.19)

It is easy to obtain Eq. (3.10) from Eq. (3.18) by introducing the relations
\[ 6 \delta_c = \left( 1 - \frac{4}{3} \delta_M \right) \quad \text{and} \quad \xi = \xi_M, \]
(3.20)

because we do not take account of the term $\delta_m$ appearing only in the Majorana neutrino case.

Next, in the case with $B_{33}$, we obtain the very same form as Eq. (3.10), with $\delta_c$ in Eqs. (3.18) and (3.20) replaced by $(\delta_c + \varepsilon_m \delta_m)$. Of course, the constant $B_{30}$ in Eq. (3.19) should be replaced by $B_{33}$.

It is worthwhile noting that $\delta_c$ and $\delta_m$ also appear symmetrically. Therefore, the situation is quite similar to that with the choice of the normalization factor $A = A_{n \ell}$. It is possible to obtain different $x$ dependences of the terms including $\delta_c$ and $\delta_m$ by choosing a common constant $B_{n \ell}$ with $n \neq \ell$. By again fixing $\ell = 0$ for simplicity, the $x$ dependence of the term including $\delta_c$ is tabulated for various values of $n$ in Table II. However, this property is not so effective in this case of $P(x)$, in contrast to the case of $N(x)$, because $\delta_c$ and $\delta_m$ themselves become zero or very small, as discussed in the next paragraph.

If the $SU(2)_L \times SU(2)_R \times U(1)$ model is assumed, then the relation
\[ \kappa = \eta \]
(3.21)

is obtained from Eq. (A.19). Therefore, we can conclude that in the Dirac neutrino case, we have
\[ \delta_c = 0, \]
(3.22)

as seen from Eqs. (3.19) and (3.4). On the other hand, in the Majorana neutrino case, the parameters $(\delta_c$ and $\delta_m$) include some very small contributions coming from

| $n$ | $B_{n0}$ | Term including $\delta_c$ |
|-----|----------|--------------------------|
| 0   | $B_{00}$ | $12(1 - x)$               |
| 1   | $B_{10}$ | $2(7 - 8x + r_0^2)$       |
| 2   | $B_{20}$ | $4(4 - 5x + r_0^2)$       |
| 3   | $B_{30}$ | $6(3 - 4x + r_0^2)$       |
| 4   | $B_{40}$ | $4(5 - 7x + 2r_0^2)$      |
the differences between small values of the lepton mixing matrices, as seen from Eqs. (3.6) and (3.7). Thus in this case, although strictly speaking they are non-zero, for practical purposes we can approximate them as zero:

\[ \delta_c \simeq 0 \quad \text{and} \quad \delta_m \simeq 0. \quad (3.23) \]

From these relations, we can conclude within the $SU(2)_L \times SU(2)_R \times U(1)$ model that we have the following expression:

\[ P(x) = \xi \sqrt{x^2 - x_0^2}(-1 + 2x - r_0^2). \quad (3.24) \]

If we set $\xi = 1$, this $P(x)$ itself is that obtained from the standard model. Although this form is common to the Dirac and Majorana neutrino cases, it should be noted that the definition of $\xi$ differs in the two cases, as seen from Eqs. (3.19), (3.4) – (3.6), namely,

\[ \xi = \frac{(1 - \lambda^2)}{(1 + \lambda^2 + 2\eta^2)} \quad \text{for the Dirac neutrino case}, \quad (3.25) \]

\[ \xi \simeq 1 \quad \text{for the Majorana neutrino case}. \quad (3.26) \]

Here, the deviation from $\xi = 1$ for the Majorana neutrino case is not expressed explicitly, because it cannot be measured within the present experimental precision. This definition of $\xi$ is independent of the choice of $B_{\mu \ell}$ in this $SU(2)_L \times SU(2)_R \times U(1)$ model.

Note that the Michel parameter $\delta_M$ should be

\[ \delta_M = 0.75, \quad (3.27) \]

within the present experimental precision, as seen from Eq. (3.20), because $\delta_c \simeq 0$ in this model. The values $\delta_M = 0.75$ and $\xi = 1$ seem to be consistent with the recent experimental results obtained by the TWIST group;\(^3\)

\[ \delta_M = 0.74964 \pm 0.00066 \pm 0.00112, \quad (3.28) \]

\[ 0.9960 < P_\mu \xi \leq \xi < 1.0040. \quad (3.29) \]

Under the assumption that neutrinos are of the Dirac type, this group estimated $\lambda_c < (80.425/420)^2 = 3.7 \cdot 10^{-2}$ from Eqs. (3.29) and (3.25), where $\lambda_c$ is approximately equal to the ratio of the mass of $W_L$ to the mass of $W_R$ squared [see Eq. (A.18)].

3.3. Summary for the spectrum

Let us summarize our expression for the spectrum. Because it consists of many terms, we ignore some small terms, like $x_0$ and $r_0^2$, in order to see the essential features. Furthermore, in order to see its characteristic features, we assume the $SU(2)_L \times SU(2)_R \times U(1)$ model; in other words, we ignore the $\delta_c$ and $\delta_m$ parameters in Eq. (3.18). Then, we have the expression

\[ D(x, \theta) = x \left[ (3 - 2x) + 2 \rho_c (3 - 4x) + 12 \varepsilon_m \rho_m (1 - x) \pm P_\mu \xi \cos \theta (-1 + 2x) \right]. \quad (3.30) \]
where the normalization factor $A = A_{10}$ has been used. If another $x$ dependence of the term including the $\rho_c$ (or $\rho_m$) parameter is chosen according to Table I, all $A_{10}$ in this section should be replaced by the corresponding $A_{n\ell}$.

Finally, we mention the theoretical expression for the experiment which determined the following quantity:

$$\omega \equiv P_{\mu} \lim_{x \to 1} \left[ \frac{P(x)}{N(x)} \right]. \quad (3.31)$$

The experimental result

$$\omega > 0.99682 \quad (3.32)$$

was reported by Jodidio et al.\textsuperscript{13)}

Here, it should be noted that the real allowed range of $x$ is limited by $x_{\text{max}} = (1 - r_{jk}^2)$, as shown in Eq. (2.7). Also, both $N(x)$ and $P(x)$ become zero at $x_{\text{max}}$, as mentioned in Eq. (2.18). However, since these restrictions are only effective in a very tiny range, it is understood that this definition of $\omega$ is a theoretical result obtained by taking the extrapolation from the allowed range of $x$. In addition, because the radiative correction is known to be larger for $x > 0.9$, as shown in Ref. 8), it is assumed that the experimental results are adjusted by taking this radiative correction into account.

Because this $\omega$ is defined by taking the ratio of two parts of the $e^\pm$ spectrum, it is independent of the choice of the normalization factor $A_{n\ell}$. In other words, its theoretical expression is obtained from Eqs. (3.1) and (3.2) as follows:

$$\omega = P_{\mu} \frac{a_-}{a_+} = P_{\mu} \frac{\xi_M \delta_M}{\rho_M}, \quad (3.33)$$

where the coefficients $a_\pm$ are defined in Eqs. (3.4) and (3.5), and the last expression is obtained from the Michel parameterization given in Eqs. (3.9) and (3.10). For simplicity, the contribution from the electron mass is not included here by setting $x_0^2 = r_0^2 = 0$.

For convenience, we now present explicit forms of $\omega$ in our model. In the Dirac neutrino case, it is

$$\omega = P_{\mu} \frac{1 - \lambda^2}{1 + \lambda^2}, \quad (3.34)$$

while in the Majorana neutrino case, it becomes

$$\omega = P_{\mu} \frac{(1 - \bar{u}_e^2)(1 - \bar{u}_\mu^2) - \lambda^2 \bar{v}_e^2 \bar{v}_\mu^2}{(1 - \bar{u}_e^2)(1 - \bar{u}_\mu^2) + \lambda^2 \bar{v}_e^2 \bar{v}_\mu^2} \simeq P_{\mu}. \quad (3.35)$$

Here, the last approximation is good in practice, because $\bar{v}_e^2$ and $\bar{v}_\mu^2$ seem to be very small, as shown in Eq. (2.21). In other words, no deviation from the standard model can be expected again in the Majorana neutrino case, within the present experimental precision.
§4. Polarization of $e^{\pm}$

We now define three components of the spin polarization of $e^{\pm}$, $\vec{P}_e(x, \theta)$ in Eq. (2.5). Its longitudinal component along the momentum direction ($\hat{p}_e$) is expressed as $P_L(x, \theta)$. In order to separate its transverse components, we choose the decay plane defined by this $\hat{p}_e$ and the muon polarization vector ($\hat{P}_\mu$). The components of the transverse polarization within and perpendicular to this decay plane are, respectively, expressed as $P_{T1}(x, \theta)$ and $P_{T2}(x, \theta)$. Mathematically, these three components are expressed as follows:

$$\vec{P}_e(x, \theta) = P_L(x, \theta)\hat{p}_e + P_{T1}(x, \theta)\frac{(\hat{p}_e \times \hat{P}_\mu) \times \hat{p}_e}{|p_e \times \hat{P}_\mu|} + P_{T2}(x, \theta)\frac{\hat{p}_e \times \hat{P}_\mu}{|p_e \times \hat{P}_\mu|}.$$  \hspace{1cm} (4.1)

It is convenient to separate the $x$-dependent parts of these components from the emission angle of $e^{\pm}$, namely, $\cos \theta = (\hat{p}_e \cdot \hat{P}_\mu)$. Therefore, we introduce the quantities

$$P_L(x, \theta) = \frac{\pm Q(x) + P_\mu \cos \theta S(x)}{D(x, \theta)},$$  \hspace{1cm} (4.2)

$$P_{T1}(x, \theta) = \frac{P_\mu \sin \theta R(x)}{D(x, \theta)},$$  \hspace{1cm} (4.3)

$$P_{T2}(x, \theta) = \frac{P_\mu \sin \theta T(x)}{D(x, \theta)},$$  \hspace{1cm} (4.4)

where the denominator $D(x, \theta)$ is given in Eq. (3.30), if the $SU(2)_L \times SU(2)_R \times U(1)$ model is assumed.

The explicit expressions of these $Q(x)$, $S(x)$, $R(x)$ and $T(x)$ are presented in terms of the parameters defined in §3. Strictly speaking, if $A = A_{n\ell}$ is chosen in the case of $N(x)$, $A_{10}$ appearing in this section should be replaced by $A_{n\ell}$.

4.1. Longitudinal polarization: $Q(x)$ and $S(x)$

The isotropic part, $Q(x)$, and anisotropic part, $S(x)$, of the longitudinal polarization are, respectively, expressed as follows under Conditions (A) and (B) given in §2:

$$Q(x) = \xi \sqrt{x^2 - x_0^2} \left[(3 - 2x - r_0^2) - 6 \delta_c(1 - r_0^2) + 12 \varepsilon_m \delta_m (1 - x)\right],$$  \hspace{1cm} (4.5)

$$S(x) = \left[(-x + 2x^2 - x_0^2) + 2\rho_c (7x - 8x^2 + x_0^2) + 12 \varepsilon_m \rho_m x(1 - x) - 2 \varepsilon_m \eta_m x_0 (1 - x)\right].$$  \hspace{1cm} (4.6)

The parameters ($\xi$, $\delta_c$ and $\delta_m$) in $Q(x)$ are defined in Eq. (3.19) for the case of $P(x)$. In other words, the common factor $B_{30}$ has been used.\footnote{This function $Q(x)$ is equal to $6 F_{1P}(x)$ of Ref. 1, where one new parameter, $\xi_M$, was introduced, because the common factor $B_{10}$ was used instead of our $B_{30}$, and the interference term between the $(S \pm P)$ and $T$ interactions was included in $\xi_M$ in order to obtain a compact expression. Therefore, $F_{1P}(x)$ has a different $x$ dependence.} In addition, the parameters
(\rho_c, \rho_m \text{ and } \eta_m) in S(x) are defined in Eq. (3.14) for the case of N(x).\textsuperscript{1)} Of course, the corresponding results from the standard model are obtained by setting \( \xi = 1 \) and all other parameters to zero. The original forms of \( Q(x) \) and \( S(x) \) before adopting Conditions (A) and (B) are presented in Eqs. (B.4) and (B.5) of Appendix B.

If the \( SU(2)_L \times SU(2)_R \times U(1) \) model is assumed, \( Q(x) \) is reduced to the following simple expression corresponding to Eq. (3.24):

\[
Q(x) = \xi \sqrt{x^2 - x_0^2} (3 - 2x - r_0^2). \tag{4.7}
\]

Thus, the longitudinal polarization is expressed as follows, if some small terms, like \( x_0, r_0^2, \delta_c \text{ and } \delta_m \), are ignored in order to see the essential features:

\[
P_L(x, \theta) = \left( \frac{x}{D(x, \theta)} \right) \{ \pm \xi (3 - 2x) - P_\mu \cos \theta [(1 - 2x) - 2 \rho_c (7 - 8x) - 12 \varepsilon_m \rho_m (1 - x)] \}. \tag{4.8}
\]

Therefore, the longitudinal polarization of \( e^\pm \) from an unpolarized muon (or \( \theta = \pi/2 \)) is expressed as follows in the \( SU(2)_L \times SU(2)_R \times U(1) \) model:

\[
P_L(x, \pi/2) = \pm \xi \left[ 1 - \frac{2\rho_c(3 - 4x)}{(3 - 2x) + 2\rho_c(3 - 4x)} \right] \text{ for the Dirac neutrino,} \tag{4.9}
\]

\[
P_L(x, \pi/2) = \pm \xi \left[ 1 - \frac{2\rho_c(3 - 4x)}{(3 - 2x) + 2\rho_c(3 - 4x)} \right] \text{ for the Majorana neutrino,} \tag{4.10}
\]

where \( \xi \) is given in Eqs. (3.25) and (3.26) for the respective cases, and the additional terms due to the parameters \( \rho_c \text{ and } \rho_m \) have been ignored for the Majorana neutrino case, because of their smallness in comparison with the present experimental precision.

Burkard et al.\textsuperscript{14)} reported the following experimental result for \( e^+ \) from an unpolarized muon by assuming \( \rho_c = 0 \) for the Dirac neutrino case:

\[
P_L(x, \pi/2) = 0.998 \pm 0.045. \tag{4.11}
\]

The present average value given in the Particle Data Group is \( P_L = 1.00 \pm 0.04. \textsuperscript{1}) \)

4.2. Transverse polarization within the decay plane: \( R(x) \)

Under Conditions (A) and (B) presented in §2, we have

\[
R(x) = \left[ -(1 - 14 \rho_c - 12 \varepsilon_m \rho_m) x_0 (1 - x) - 2 \varepsilon_m \eta_m (x - x_0^2) \right]. \tag{4.12}
\]

The result of the standard model is obtained by setting \( \rho_c = \rho_m = \eta_m = 0 \). The original form before adopting Conditions (A) and (B) is presented in Eq. (B.6).\textsuperscript{**}

In the Dirac neutrino case, we cannot expect any useful information from this measurement, because the first main term obtained from the standard model is

\textsuperscript{1)} This function \( S(x) \) is equal to 6 \( F_{AP}(x) \) of Ref. 1, where one new parameter, \( \xi''_M \), was introduced, because the common factor \( A_{10} \) was used instead of our \( A_{10} \), and also the interference term between the \((S \pm P)\text{ and } T\text{ interactions was included in } \xi''_M \). Therefore, \( F_{AP}(x) \) has a different \( x \) dependence than our \( S(x) \).

\textsuperscript{**} Our \( R(x) \) is equal to 6 \( F_{T1}(x) \) of Ref. 1.
already proportional to the small $x_0$. By contrast, the Majorana parameter $\eta_m$ is accompanied by the coefficient $(x - x_0^2)$, which is significantly larger than the coefficient $x_0(1 - x)$ in $N(x)$ [cf. Eq. (3-13)]. However, $\eta_m$ itself is small, as discussed in the next paragraph. Therefore, it is rather difficult to obtain definite information regarding $\eta_m$ from this measurement.

Let us estimate the order of magnitude of the parameter $\eta_m$, which includes the following combination of lepton mixing matrices:

$$\lambda d_r = \lambda \text{Re} \sum_{jk} (U_{ej}^* V_{ck} V_{\mu j}^* U_{\mu k}) = \lambda \text{Re}(\bar{w}_{e\mu} \bar{w}_{e\mu h}), \quad (4.13)$$

where Eqs. (3.14), (3.8) and (2.22) have been used.

Since it can be naturally assumed that there are no contributions from the heavy Majorana neutrinos, we are able to express $\bar{w}_{e\mu}$ as follows by omitting the second component of the neutrino mixing matrices in Eqs. (A.15) and (A.16):

$$\bar{w}_{e\mu} = \sum_j' U_{ej} V_{\mu j} = \sum_j' \left(U^\dagger E^{(1)}_\nu V^{(1)}_\nu\right)_{ej} e^{-i\phi} \left(U^\dagger E^{(1)}_\nu V^{(1)}_\nu\right)_{\mu j}. \quad (4.14)$$

In this expression, the first matrix element, $U_{ej}$, is known to be of order unity from neutrino oscillation experiments, while, concerning the second element $V_{\mu j}$, there is no reliable information at present. But if we assume the seesaw mechanism, we must consider $(V^{(1)}_\nu)_{j'j}$ to have a very small value, as shown in Eq. (A.6). We may get some rough idea of its order of magnitude from the neutrinoless double beta decay, which gives the following upper bound for a similar quantity:

$$\langle \lambda \rangle = \lambda \sum_j' U_{ej} V_{\mu j} (\cos \theta'_c / \cos \theta_c) < O(10^{-6}). \quad (4.15)$$

The primed sum in this case represents a sum that extends over only the light neutrinos ($m_j < 10$ MeV), so that contributions from heavier neutrinos are ignored in comparison with the virtual Majorana neutrino momentum. Here, $\theta_c$ and $\theta'_c$ are, respectively, the Cabibbo-Kobayashi-Maskawa mixing angles for the left-handed and right-handed $d$ and $s$ quarks. The order of magnitude of $\lambda \bar{w}_{e\mu}$ in Eq. (4.13) seems to be less than $10^{-6}$, although the suffix $\mu$ in Eq. (4.14) is replaced by the suffix $e$ in Eq. (4.15), and there are some quantities related to the quark sector. Thus the order of magnitude of $\eta_m$ seems to be much smaller than $10^{-6}$, because of the additional factor of $\bar{w}_{e\mu h}$ in Eq. (4.13).

Recently, Danneberg et al. reported the following energy averaged value for $e^+$ in the direction $\theta = \pi/2$:

$$\langle P_{T1}(x, \theta = \pi/2) \rangle = (6.3 \pm 7.7 \pm 3.4) \cdot 10^{-3}. \quad (4.16)$$

This experimental result is of the same order of magnitude as the prediction of the standard model,

$$P_{T1}(x, \theta = \pi/2) = -P_{\mu} \frac{x_0(1 - x)}{x(3 - 2x)}, \quad (4.17)$$

where $x_0$ is defined in Eq. (2-8).
4.3. Transverse polarization perpendicular to the decay plane: $T(x)$

A non-zero value of $T(x)$ implies the existence of a non-zero Majorana CP violation phase in our model. It is expressed as

$$T(x) = 2\varepsilon_m \sqrt{x^2 - x_0^2} \eta_{mi} (1 - r_0^2),$$

(4.18)

where the parameter $\eta_{mi}$ is defined as follows, using Eqs. (B.7) and (B.25):

$$\eta_{mi} = \varepsilon_m \left( \frac{\lambda d_i}{A_{10}} \right) = \varepsilon_m \left( \frac{\lambda}{A_{10}} \right) \text{Im}(\bar{w}_{e\mu} w_{e\mu h}).$$

(4.19)

There is no corresponding term in either the standard model or our model for the Dirac neutrino. The parameter $\eta_{mi}$ is obtained by taking the imaginary part instead of the real part in Eq. (4.13). Therefore, $\eta_{mi}$ is proportional to the sin term of the CP violating phases appearing in the lepton mixing matrices. As we can imagine from Eq. (4.15), we cannot expect to measure $\eta_{mi}$ in practice, because its value seems to be too small, as predicted by our model (for details, see §3 of Ref. 9).

Recently, Danneberg et al. reported the following energy averaged value for $e^+$ in the direction $\theta = \pi/2$:

$$\langle P_{T2}(x, \theta = \pi/2) \rangle = (-3.7 \pm 7.7 \pm 3.4) \cdot 10^{-3}.$$  

(4.20)

A smaller value of $\langle P_{T2}(x, \theta = \pi/2) \rangle$ is expected if the neutrino is of the Majorana type and the CP-violating phase exists.

§5. Concluding remarks

It was shown in §3 that the Michel parameterization, which has been used by an experimental group, is a special case of the more general form to investigate the deviation from the standard model. We propose a new parameterization that directly represents deviations from the standard model. In general, there is the freedom to choose the normalization factor $A_{n, \ell}$ in Eq. (3.11). Then, we can find the most effective $x$ dependence of the term including that parameter, whose value is determined by analyzing experimental data. The $SU(2)_L \times SU(2)_R \times U(1)$ model has been used to elucidate this feature in the simplified expression given in Eq. (3.30). Concerning the spectrum, it has been confirmed that there is no significant deviation from the standard model, except that the mean value of $\rho_M$ determined experimentally is slightly larger than 0.75, as shown in Eq. (3.16), although we have $\rho_M < 0.75$ in our model.

Also, it was shown in §4 that the polarization of $e^\pm$ can be expressed in terms of the same parameters introduced to analyze the spectrum, in contrast to Ref. 1), in which several new parameters are introduced. The predictions for the polarization obtained from the standard model are also consistent with the recent experimental results.

*) Our $T(x)$ is equal to $6 F_{T2}(x)$ of Ref. 1). Their $F_{T2}(x)$ includes the CP violating term even in the massless Dirac neutrino case, because it comes from the interference between the $(V \pm A)$ and $(S \pm P)$ (or $T$) interactions.
It is an important problem to investigate whether the neutrino is of the Dirac or Majorana type, as mentioned in §1. In normal muon decay, there are three theoretically possible subjects for this purpose within the framework of gauge theory, as we now discuss.

The first subject is to measure the transverse polarization of $e^\pm$ perpendicular to the decay plane, namely $T(x)$ in Eq. (4.18). The reason for this is because this polarization exists in neither the standard model nor the massive Dirac neutrino case. However, the theoretical estimate of $T(x)$ is very small, as explained below Eq. (4.19). We cannot expect to obtain any useful information from this measurement within the present experimental precision.

The second subject is the transverse polarization in the decay plane, $R(x)$ appearing in Eq. (4.12). In this case, the term associated with $\eta_m$ characteristic of the Majorana neutrino has the larger $x$ dependence, but this $\eta_m$ itself is also too small, as mentioned below Eq. (4.13). It seems difficult to derive any definite conclusion from this measurement.

The remaining possibility is to take advantage of the different $x$ dependences of the terms including the parameters $\rho_c$ and $\rho_m$ in the energy spectrum Eq. (3.30) by comparing the $\chi^2$ values for the Dirac-type neutrino with those for the Majorana-type neutrino, as mentioned below Eq. (3.16). This may provide a test to determine the type of neutrino, although it is indirect.

Finally, let us summarize the general features. In the Dirac neutrino case, there is no important effect due to the lepton mixing matrices under Condition (B) in Eq. (2.20). In other words, we can use the theoretical expressions obtained by assuming massless neutrinos. In addition, we point out that it is useful to choose a different $n$ for the normalization factor $A_{n0}$ in Eq. (3.6) in order to minimize $\chi^2$. Thus, we can find some constraints on the coupling constants ($\lambda$, $\eta$ and $\kappa$) in principle by combining other information from various decay processes. We also note that deviations from the standard model become smaller as these coupling constants become smaller.

In the Majorana neutrino case, it is very difficult to find any deviation from the standard model under Condition (B) given in Eqs. (2.21) and (2.22). This is because all parameters include the small components of the lepton mixing matrices. This feature is independent of the values of $\lambda$, $\eta$ and $\kappa$.

**Appendix A**

--- Summary of Various Mixing Matrices ---

For the purpose of making this paper self-contained, here we summarize the theoretical foundations of this work, even though they have been discussed many times in the literature already. Many theoretical gauge models beyond the standard model have been proposed to analyze normal muon decay. Among them, let us consider a model that consists of $V - A$ and $V + A$ currents.

The mass term of leptons in the Lagrangian with $n$ left-handed and $n$ right-
handed lepton doublets is generally defined by

\[
\mathcal{L}_M = -\bar{\mathcal{E}}_R M_E \mathcal{E}_L - \frac{1}{2} \left( (\nu_L)^c, (\nu_R'^c) \right) \mathcal{M} \left( \frac{\nu_L}{(\nu_R')^c} \right) + \text{H.c.},
\]

where \( \mathcal{E}, \nu_L \) and \( \nu_R'^c \) are, respectively, the weak eigenstates of the charged leptons and left-handed and right-handed neutrinos. Explicitly, we write \( \mathcal{E}_L = (e_L^t, \mu_L^t, \cdots), \nu_L^T = (\nu_e, \nu_\mu, \cdots) \) and \( \nu_R'^T = (\nu_e', \nu_\mu', \cdots) \). Note also that \( (\nu_{LL(R)})^c = C \nu_{LL(R)}^T \), where \( C \) is the charge conjugation operator. Here \( M_E \) is the \( n \times n \) mass matrix for charged leptons and \( \mathcal{M} \) is the \( 2n \times 2n \) neutrino mass matrix defined by

\[
\mathcal{M} = \begin{pmatrix}
M_L & M_D^T \\
M_D & M_R
\end{pmatrix},
\]

where \( M_D, M_L \) and \( M_R \) are, respectively, the Dirac-type and left-handed and right-handed Majorana type \( n \times n \) mass matrices for neutrinos. Here, the identity \( \nu_R^T M_D \nu_L = (\nu_L)^c M_D^c (\nu_R'^c)^c \) has been used.

Let us first examine the case in which the Majorana-type mass terms exist. Since \( M_L \) and \( M_R \) are symmetric matrices, \( \mathcal{M} \) is also symmetric and can be diagonalized by some orthogonal matrix in principle to determine the neutrino masses.\(^{18}\) However, we use the \( 2n \times 2n \) unitary matrix \( \mathcal{U}_\nu \) in order to obtain positive values for masses.\(^{7,19}\)

\[
\mathcal{U}_\nu^T \mathcal{M} \mathcal{U}_\nu = \mathcal{D}_\nu.
\]

Here, \( \mathcal{D}_\nu \) is a diagonal matrix whose \( 2n \) elements represent the masses of the Majorana-type neutrinos. Therefore, the weak eigenstates of neutrinos are expressed as superpositions of the mass eigenstate Majorana neutrinos \( N_j \) as follows:

\[
\begin{pmatrix}
(\nu_L)^c \\
(\nu_R'^c)
\end{pmatrix} = \mathcal{U}_\nu N_L = \begin{pmatrix}
U_\nu \\
V_\nu^*
\end{pmatrix} N_L = \begin{pmatrix}
U_\nu^{(1)} & U_\nu^{(2)} \\
V_\nu^{(1)*} & V_\nu^{(2)*}
\end{pmatrix} \begin{pmatrix}
N_{1L} \\
N_{1R}
\end{pmatrix}.
\]

Thus, we have the \( 2n \) mass eigenstate Majorana neutrinos, \( (N_1)^T = (N_1, N_2, \cdots, N_n) \) and \( (N_{11})^T = (N_{n+1}, N_{n+2}, \cdots, N_{2n}) \). Here, the \( n \times 2n \) neutrino mixing matrices \( \mathcal{U}_\nu \) and \( \mathcal{V}_\nu \), which are expressed as \( \mathcal{U}_\nu = (U_\nu^{(1)}, U_\nu^{(2)}) \) and \( \mathcal{V}_\nu = (V_\nu^{(1)}, V_\nu^{(2)}) \), are introduced as

\[
\nu_{1L} = \sum_{j=1}^{2n} (U_\nu)_{\ell j} N_{jL} \quad \text{and} \quad \nu_{1R}' = \sum_{j=1}^{2n} (V_\nu)_{\ell j} N_{jR}
\]

for the case of the \( n \) generations.\(^7\)

In this scenario, the small masses of the left-handed Majorana-type neutrinos \( (N_1) \) are naturally explained by the seesaw mechanism under the assumption that the right-handed Majorana neutrinos \( (N_{11}) \) have large masses. Thus, the elements of both \( U_\nu^{(1)} \) and \( V_\nu^{(2)} \) are of order one, while \( U_\nu^{(2)} \) and \( V_\nu^{(1)} \) are of order \( m_{\nu_D}/m_{\nu_R} \), which seems to be very small. Here the quantities \( m_{\nu_D} \) and \( m_{\nu_R} \) represent the orders
of the matrices $M_D$ and $M_R$, respectively:

$$U^{(1)}_{\nu} = O(1), \quad U^{(2)}_{\nu} = O(m_{\nu D}/m_{\nu R}),$$
$$V^{(1)}_{\nu} = O(m_{\nu D}/m_{\nu R}), \quad V^{(2)}_{\nu} = O(1). \quad (A-6)$$

From the unitarity condition for $U_{\nu}$, the matrices $U^{(1)}_{\nu}, U^{(2)}_{\nu}, V^{(1)}_{\nu}$ and $V^{(2)}_{\nu}$ should satisfy the relations

$$U^{(1)}_{\nu}U^{(1)\dagger}_{\nu} + U^{(2)}_{\nu}U^{(2)\dagger}_{\nu} = 1, \quad (A-7)$$
$$V^{(1)}_{\nu}V^{(1)\dagger}_{\nu} + V^{(2)}_{\nu}V^{(2)\dagger}_{\nu} = 1, \quad (A-8)$$
$$U^{(1)}_{\nu}V^{(1)T}_{\nu} + U^{(2)}_{\nu}V^{(2)T}_{\nu} = 0, \quad \text{etc.} \quad (A-9)$$

Note that $U^{(1)}_{\nu}$ and $V^{(1)}_{\nu}$ themselves are not unitary.

We now note that the reason for the smallness of the quantities $u_{\ell 2}$, $v_{\ell 2}$, $w_{e\mu}$ and $w_{e\mu h}$ introduced in Eqs. (2.21) and (2.22) is that they include, respectively, the elements of the neutrino mixing matrix products $(U^{(2)}_{\nu}U^{(2)\dagger}_{\nu})_{\ell\ell}$, $(V^{(1)}_{\nu}V^{(1)\dagger}_{\nu})_{\ell\ell}$, $(U^{(1)}_{\nu}V^{(1)T}_{\nu})_{e\mu}$ and $(U^{(1)}_{\nu}V^{(1)T}_{\nu})_{\mu e}$. In fact, the elements like $U^{(2)}_{\nu}$ and $V^{(1)}_{\nu}$ in them seem to be of order $m_{\nu D}/m_{\nu R}$, as shown in Eq. (A-6).

Next, let us consider the other simplified scenario, in which $M_L = M_R = 0$. Namely, there exists only the Dirac-type neutrino mass matrix $M_D$. Because there is no theoretical restriction on $M_D$ to be symmetric, it can be diagonalized by two $n \times n$ unitary matrices $U_{\nu}$ and $V_{\nu}$ as

$$V^{\dagger}_{\nu}M_DU_{\nu} = D'_{\nu}. \quad (A-10)$$

Here, $D'_{\nu}$ is a diagonal matrix whose $n$ elements represent the masses of the Dirac-type neutrinos. Therefore, the weak eigenstates of neutrinos in this scenario are expressed as superpositions of $n$ mass eigenstate Dirac-type neutrinos $N_j$, as shown in Eq. (A-5), although the upper limit of the sum over $j$ is restricted to $n$.

In order to avoid the complication of needing to treat the Dirac and Majorana neutrino cases separately, we use the following convention in this paper: In the Dirac neutrino case, only $n$ mass eigenstate neutrinos $N_I$ exist, while $N_{II}$ do not exist. Thus we can set $U^{(2)}_{\nu} = 0$ and $V^{(2)}_{\nu} = 0$. Therefore, $U^{(1)}_{\nu}$ and $V^{(1)}_{\nu}$ are both unitary matrices, and they are of order one. This situation can be expressed simply as follows:

$$U^{(1)}_{\nu} = O(1), \quad U^{(2)}_{\nu} = 0, \quad V^{(1)}_{\nu} = O(1), \quad V^{(2)}_{\nu} = 0. \quad (A-11)$$

We have considered the simplified case of introducing the Dirac neutrino fields. However, there is another possible scenario. In that scenario, one Dirac neutrino field can be expressed as a superposition of two Majorana neutrino fields with equal masses.\footnote{In this case, one Majorana neutrino field is chosen from $N_I$ and the other field from $N_{II}$; that is, we have $m_{j+k} = m_j$ for $1 \leq j(k) \leq n$.}

As a typical example of gauge theory with left- and right-handed weak gauge bosons, we consider the $SU(2)_L \times SU(2)_R \times U(1)$ model.\footnote{7,17} In this framework,
the charged weak current interaction is written in the bases of mass eigenstates of
the charged leptons $E_\ell$ and the neutrino $N_j$ after diagonalizing the charged lepton
and neutrino mass matrices:

$$\mathcal{L}_{CC} = -\frac{g_L}{2\sqrt{2}} \sum_{\ell,j} \overline{E_\ell} \gamma_\alpha (1 - \gamma_5) (U_E^\dagger U_\nu)_{\ell j} N_j L W^\alpha_L$$

$$- \frac{g_R}{2\sqrt{2}} \sum_{\ell,j} \overline{E_\ell} \gamma_\alpha (1 + \gamma_5) (V_E^\dagger V_\nu)_{\ell j} N_j R W^\alpha_R + \text{H.c.} \quad (A.12)$$

Here, $g_L$ and $g_R$ are the real gauge coupling constants for the left- and right-handed
weak gauge bosons, $W_L$ and $W_R$, respectively, and $U_E$ and $V_E$ are unitary matrices
that diagonalize the charged lepton mass matrix $M_E$ as $V_E^\dagger M_E U_E = D_E$, in analogy
to Eq. (A.10).

The weak gauge bosons $W_L$ and $W_R$ are expressed in terms of the mass eigenstate
gauge bosons $W_1$ and $W_2$ as

$$W_L = W_1 \cos \zeta + e^{i\varphi} W_2 \sin \zeta, \quad (A.13)$$

$$W_R = -e^{-i\varphi} W_1 \sin \zeta + W_2 \cos \zeta, \quad (A.14)$$

where $\zeta$ is a $W_L$-$W_R$ mixing angle and $\varphi$ is a $CP$-violating phase. The phase factor
$e^{-i\varphi}$ comes from the VEV of the $(1/2, 1, 1)$ Higgs field through the diagonalization of
the mass matrix of the charged weak gauge bosons.

In obtaining the effective current-current interaction, the left- and right-handed
Maki-Nakagawa-Sakata (MNS) lepton mixing matrices $U$ and $V$ are defined as

$$U = U_E^\dagger U_\nu = \left( U_E^{(1)} U_\nu^{(1)}, U_E^{(2)} U_\nu^{(2)} \right), \quad (A.15)$$

$$V = e^{-i\varphi} V_E^\dagger V_\nu = e^{-i\varphi} \left( V_E^{(1)} V_\nu^{(1)}, V_E^{(2)} V_\nu^{(2)} \right). \quad (A.16)$$

It should be noted that, although the lepton mixing matrices $U$ and $V$ themselves
are unitary in the Dirac neutrino case, the $2n \times 2n$ lepton mixing matrix $\left( \begin{array}{c} U \\ V^* \end{array} \right)$ is
a unitary matrix in the Majorana neutrino case.

There exists some freedom in the treatment of the phase factor $e^{-i\varphi}$ mentioned
above. In our treatment, this phase factor is absorbed into $V$ of Eq. (A.16) in order
make the coupling constants $\eta$ and $\kappa$ real in the effective Hamiltonian of Eq. (2.1).
Contrastingly, Herczeg\textsuperscript{17} includes it in the definitions of $\kappa$ and $\eta$. In our convention,
all of the constants $G_F$, $\lambda$, $\eta$ and $\kappa$ are real:

$$\frac{G_F}{\sqrt{2}} = \frac{1}{8} \left( \frac{g_L \cos \zeta}{M_1} \right)^2 \left( 1 + \lambda_c \tan^2 \zeta \right), \quad (A.17)$$

$$\lambda = \left( \frac{g_R}{g_L} \right)^2 \frac{\lambda_c + \tan^2 \zeta}{1 + \lambda_c \tan^2 \zeta}, \quad (A.18)$$

$$\kappa = \eta = -\left( \frac{g_R}{g_L} \right) \frac{(1 - \lambda_c) \tan \zeta}{1 + \lambda_c \tan^2 \zeta}, \quad (A.19)$$

where $\lambda_c = (M_1/M_2)^2$, $M_i$ being the mass of $W_i$. Note that we have $\lambda \sim \kappa^2 = \eta^2$
for $\lambda_c \ll \tan^2 \zeta < 1$. 

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Appendix B

Definitions of Various Coefficients

Our model given by Eqs. (2.1) and (2.3) differs from the standard model because it takes account of the finite neutrino mass and the $V + A$ current. The complete decay formulas obtained by keeping both the neutrino masses ($m_j$ and $m_k$) and the $V + A$ parameters ($\lambda$ and $\kappa = \eta$), are given in Ref. 9).

Terms proportional to the neutrino masses can be omitted in practice, because the neutrino masses are very small in comparison with the energy scale ($W$):

$$\left(\frac{m_\nu}{W}\right) = 1.89 \cdot 10^{-7},$$

(B-1)

where $m_\nu$ represents $m_j$ and $m_k$, and is taken to be 10 eV in order to get a rough idea for the magnitude.

Then, we have the following results for various components of the decay rate:

$$N(x) = \left(\frac{1}{A}\right) \left[ a_+ (3x - 2x^2 - x_0^2) + (b_+ - a_+) (2x - x^2 - x_0^2) + 12k_+ x (1 - x) + 6\varepsilon_m \lambda d_r x_0 (1 - x) \right],$$

(B-2)

$$P(x) = \left(\frac{1}{A}\right) \sqrt{x^2 - x_0^2} \left[ a_- (-1 + 2x - r_0^2) + (b_- - a_-) (x - r_0^2) + 12k_- (1 - x) \right],$$

(B-3)

$$Q(x) = \left(\frac{1}{A}\right) \sqrt{x^2 - x_0^2} \left[ a_- (3x - 2x^2 - r_0^2) + (b_- - a_-) (2x - r_0^2) + 12k_- (1 - x) \right],$$

(B-4)

$$S(x) = \left(\frac{1}{A}\right) \left[ a_+ (-x + 2x^2 - x_0^2) + (b_+ - a_+) (x^2 - x_0^2) - 12k_+ (-x + x^2) - 2\varepsilon_m \lambda d_r x_0 (1 - x) \right],$$

(B-5)

$$R(x) = \left(\frac{1}{A}\right) \left[ -a_+ x_0 (1 - x) + 12k_+ x_0 (1 - x) - 2\varepsilon_m \lambda d_r (x - x_0^2) \right],$$

(B-6)

$$T(x) = \left(\frac{1}{A}\right) 2\varepsilon_m \sqrt{x^2 - x_0^2} \lambda d_i (1 - r_0^2).$$

(B-7)

Here, we keep all terms with respect to $\lambda$, $\kappa$ and $\eta$, whereas only their first order terms are kept in Ref. 9). The coefficients in these results are defined as follows:

$$a_\pm = (a \pm \lambda^2 \hat{a}), \quad b_\pm = (b \pm \lambda^2 \hat{b}),$$

(B-8)

$$k_\pm = (k \pm c \epsilon_m k \pm m),$$

(B-9)

The relations between the definitions in this paper and in Ref.9) are as follows: $AN(x) = (x/W)N(e)$, $AP(x) = -(xp_e/WE)P(e)$, $AQ(x) = (xp_e/WE)Q(e)$, $AS(x) = -(x/W)S(e)$, $AR(x) = -(x/W)R(e)$, $AT(x) = -(xp_e/WE)T(e)$. 

---

\(^{*1}\) The relations between the definitions in this paper and in Ref.9) are as follows: $AN(x) = (x/W)N(e)$, $AP(x) = -(xp_e/WE)P(e)$, $AQ(x) = (xp_e/WE)Q(e)$, $AS(x) = -(x/W)S(e)$, $AR(x) = -(x/W)R(e)$, $AT(x) = -(xp_e/WE)T(e)$.
where
\[ k_{\pm c} = \frac{1}{2} (\kappa^2 c \pm \eta^2 \hat{c}), \quad k_{\pm m} = \frac{1}{2} (\kappa^2 d \pm \eta^2 \hat{d}). \] \hspace{1cm} (B.10)

All of these coefficients are classified into two groups. One group consists of \(a, b, c, \hat{a}, \hat{b}\) and \(\hat{c}\), which are common to the Dirac and Majorana neutrino cases. The definitions of \(a, b, \hat{a}\) and \(\hat{b}\) are given in Eqs. (2.13) and (2.14), while \(c\) and \(\hat{c}\) are given by
\[ c = \sum_{jk} \frac{1}{3} F_{jk}^{1/2} (2F_{jk} + G_{jk}) |U_{ej}|^2 |V_{\mu k}|^2, \] \hspace{1cm} (B.11)
\[ \hat{c} = \sum_{jk} \frac{1}{3} F_{jk}^{1/2} (2F_{jk} + G_{jk}) |V_{ej}|^2 |U_{\mu k}|^2. \] \hspace{1cm} (B.12)

The other group is only for the Majorana neutrino case:
\[ d = \sum_{jk} \frac{1}{3} F_{jk}^{1/2} (2F_{jk} + G_{jk}) \text{Re}(U_{ej}^* U_{ek} V_{\mu j}^* V_{\mu k}), \] \hspace{1cm} (B.13)
\[ \hat{d} = \sum_{jk} \frac{1}{3} F_{jk}^{1/2} (2F_{jk} + G_{jk}) \text{Re}(V_{ej}^* V_{ek} U_{\mu j}^* U_{\mu k}), \] \hspace{1cm} (B.14)
\[ d_r = \sum_{jk} F_{jk} G_{jk} \text{Re}(U_{ej}^* V_{ek} V_{\mu j}^* U_{\mu k}), \] \hspace{1cm} (B.15)
\[ d_i = \sum_{jk} F_{jk} G_{jk} \text{Im}(U_{ej}^* V_{ek} V_{\mu j}^* U_{\mu k}). \] \hspace{1cm} (B.16)

Under Condition (A) in Eq. (2.19), we have \(a = b\) and \(\hat{a} = \hat{b}\), and accordingly
\[ a_{\pm} = b_{\pm}. \] \hspace{1cm} (B.17)

In this way, the spectrum and polarization given in Eqs. (B.2) – (B.5) are simplified.

B.1. Dirac neutrino case

According to Condition (B) presented in §2.2, the masses of the Dirac-type neutrinos are conjectured to be so small that all neutrinos are allowed to be emitted in muon decay. The finite neutrino masses give rise to slight deviations from unitarity characteristics, through the relations given in Eqs. (2.15) and (2.16). We express these small deviations as follows:
\[ a = 1 - \varepsilon_a (U^2 U^2) \Rightarrow 1, \quad b = 1 - \varepsilon_b (U^2 U^2) \Rightarrow a, \] \hspace{1cm} (B.18)
\[ \hat{a} = 1 - \varepsilon_{\hat{a}} (V^2 V^2) \Rightarrow 1, \quad \hat{b} = 1 - \varepsilon_{\hat{b}} (V^2 V^2) \Rightarrow \hat{a}, \] \hspace{1cm} (B.19)
\[ c = 1 - \varepsilon_c (U^2 V^2) \Rightarrow 1, \quad \hat{c} = 1 - \varepsilon_{\hat{c}} (V^2 U^2) \Rightarrow 1. \] \hspace{1cm} (B.20)

\(\text{Here we have introduced the new quantities } k_{\pm}, d_r \text{ and } d_i \text{ in place of } x_{\pm}, h \text{ and } h^I \text{ used in Ref. 10). The relations between them are } \kappa^2 x_{\pm} = 6(k_{\pm} + \varepsilon_m k_{\pm} m), h = \varepsilon_m d_r \text{ and } h^I = \varepsilon_m d_i.\)
Here, for example, $\varepsilon_c(U^2V^2)$ stands for a factor which is written as a product of $r_{jk}^2$ and $|U_{ej}|^2|V_{\mu k}|^2$, except in the narrow range near $x_{\text{max}}$. The magnitudes of these six factors are in general less than $10^{-13}$, as seen from Eq. (2.8). The arrows in the above expressions represent the limit taken under both Conditions (A) and (B) stated in §2. The values of $a, \hat{a}, b, \hat{b}, c$ and $\hat{c}$ can thus all be considered unity in practice.∗)

### B.2. Majorana neutrino case

In the Majorana neutrino case, it is assumed that there exist neutrinos with both small and large masses. Heavy neutrinos are forbidden energetically to be emitted in the muon decay we consider, and the primed sums are therefore only over light neutrinos. The coefficients are expressed as follows:

\begin{align*}
    a & \Rightarrow (1 - \overline{w}_{e}^2)(1 - \overline{w}_{\mu}^2), & b & \Rightarrow a, \\
    \hat{a} & \Rightarrow \overline{w}_{e}^2 \overline{w}_{\mu}^2, & \hat{b} & \Rightarrow \hat{a}, \\
    c & \Rightarrow (1 - \overline{w}_{e}^2) \overline{w}_{\mu}^2, & \hat{c} & \Rightarrow \overline{w}_{e}^2 (1 - \overline{w}_{\mu}^2), \\
    d & \Rightarrow |\overline{w}_{e\mu}|^2, & \hat{d} & \Rightarrow |\overline{w}_{e\mu h}|^2, \\
    d_r & \Rightarrow \text{Re}(\overline{w}_{e\mu}^* \overline{w}_{e\mu h}), & d_i & \Rightarrow \text{Im}(\overline{w}_{e\mu}^* \overline{w}_{e\mu h}).
\end{align*}

(B.21)
(B.22)
(B.23)
(B.24)
(B.25)

The quantities $\overline{w}_e$ and $\overline{w}_\mu$ here are defined in Eq. (2.21), and $\overline{w}_{e\mu}$ and $\overline{w}_{e\mu h}$ are defined in Eq. (2.22). The arrows represent the limit taken under both Conditions (A) and (B) given in §2. The quantities on the right-hand sides indicate the magnitudes of the coefficients in terms of the lepton mixing matrix elements.**

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