A MEAN VALUE OF THE REPRESENTATION FUNCTION
FOR THE SUM OF TWO PRIMES IN ARITHMETIC PROGRESSIONS

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Abstract. In this note, assuming a variant of the Generalized Riemann Hypothesis, which
does not exclude the existence of real zeros, we prove an asymptotic formula for the mean
value of the representation function for the sum of two primes in arithmetic progressions.
This is an improvement of the result of F. Rüppel in 2009, and the generalization of the
result of A. Languasco and A. Zaccagnini concerning the ordinary Goldbach problem in
2012.

1. Introduction

In this note, we consider the sum of two primes in arithmetic progressions. For the
conventional studies on this additive problem, for example, see Lavrik [8] or Liu and Zhan
[9]. They gave some estimates on the exceptional set for this problem. In the following,
although it is rather an indirect way, we shall consider this problem in some average sense
as Rüppel did in [11, 12].

For this additive problem, the usual weighted representation function is given by

\[ R(n, q_1, a_1, q_2, a_2) := \sum_{\substack{m_1 + m_2 = n \atop m_i \equiv a_i \pmod{q_i}}} \Lambda(m_1) \Lambda(m_2), \]

where \( \Lambda(n) \) is the von Mangoldt function and \( a_1, a_2, q_1, q_2, n \) are positive integers satisfying
\( (a_1, q_1) = (a_2, q_2) = 1 \). Let us also introduce an abbreviation

\[ R(n, q, a, b) := R(n, q, a, q, b) \]

for positive integers \( a, b, q \) satisfying \( (ab, q) = 1 \). In 2009, Rüppel [11] studied the mean
value of this representation function, i.e.

\[ \sum_{n \leq X} R(n, q, a, b). \]

In particular, she obtained, under a weakened variant of the Generalized Riemann Hy-
pothesis, an asymptotic formula for the mean value [11]. More precisely, she assumed
the Generalized Riemann Hypothesis (GRH) except the existence of real zeros, i.e. she
assumed[1]

Hypothesis (GRH with real zeros). Every complex non-trivial\(^2\) zeros of Dirichlet L func-
tions lie on the critical line \( \sigma = 1/2 \).

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\(^1\) Her hypothesis was actually somewhat stronger than ours.
\(^2\) Here we say that a zero \( \rho \) of a Dirichlet L function is non-trivial if \( 0 < \text{Re} \rho < 1 \), and complex if \( \text{Im} \rho \neq 0 \).
In this note, we also assume this hypothesis following her and we call this hypothesis GRHR shortly. Note that the Riemann zeta function has no real non-trivial zero. Assuming this hypothesis GRHR, Rüppel [11] proved
\[
\sum_{n \leq X} R(n, q, a, b) = \frac{X^2}{2 \varphi(q)^2} + O\left( X^{1 + \delta} (\log q)^2 \right),
\]
where \( \delta = 1/2 \) unless real zeros exist for the modulus \( q \), in which case we let \( \delta \) be the largest one among these real zeros.

She considered the mean value (1.1), but we can also obtain the corresponding result for the mean value
\[
\sum_{n \leq X} R(n, q_1, a_1, q_2, a_2)
\]
through the same method. Moreover, her method can be used to prove the asymptotic formula
\[
\sum_{n \leq X} R(n, q, a, b) = \frac{X^2}{2 \varphi(q)^2} - \sum_{\chi \pmod{q}} \frac{\overline{\chi}(a) \overline{\psi}(b)}{\varphi(q)^2} \sum_{\beta \chi} \frac{X^{\beta+1}}{\beta \chi (\beta \chi + 1)} + O\left( X^{3/2} (\log q)^2 \right),
\]
where \( \beta \chi \) runs through all real zeros of \( L(s, \chi) \) with \( \beta \chi \geq 1/2 \).

These results correspond to the result of Fujii [3] for the ordinary Goldbach Problem. Fujii [3] proved, for the representation function
\[
R(n) = \sum_{k+l=n} \Lambda(k) \Lambda(l)
\]
of the ordinary Goldbach problem, an asymptotic formula
\[
\sum_{n \leq X} R(n) = \frac{X^2}{2} + O\left( X^{3/2} \right)
\]
under the Riemann Hypothesis (RH). This was improved by Fujii [4] himself to
\[
\sum_{n \leq X} R(n) = \frac{X^2}{2} - 2 \sum_{\rho} \frac{X^{\rho+1}}{\rho (\rho + 1)} + O\left( (X \log X)^{4/3} \right)
\]
under RH, where \( \rho \) runs through all non-trivial zeros of Riemann zeta function. After this pioneering work of Fujii, the error term on the right-hand side of (1.5) was improved to
\[
\ll X (\log X)^5 \quad \text{(by Bhowmik and Schlage-Puchuta [1])},
\]
\[
\ll X (\log X)^3 \quad \text{(by Languasco and Perelli [6])}.
\]
Of course we assume RH in all of these results. We note that Bhowmik and Schlage-Puchuta proved also the omega result
\[
= \Omega (X \log \log X)
\]
for this error term. This omega result is independent from RH or GRH.

\(^3\) This is somewhat different from what Rüppel claimed, but she essentially proved this.
In this note, we improve Rüppel’s result \([1,2]\) up to the accuracy of the result of Languasco and Zaccagnini \([6]\). Let us introduce

\[
W(X, z, w) := \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)} \frac{X^{z+w}}{z + w},
\]

\[
G^\ast(X, \chi) := \sum_{\rho} W(X, \rho, \chi), \quad G(X, \chi) = G^1(X, \chi),
\]

\[
G^\ast(X, q, a) := \sum_{\chi \mod q} \chi(a)G^\ast(X, \chi), \quad G(X, q, a) = G^1(X, q, a),
\]

where \(\chi \mod q\) are Dirichlet characters, \(\rho\) runs through all non-trivial zeros of Dirichlet \(L\) function \(L(s, \chi)\) including real zeros, and \(\kappa > 0\). Then our result is:

**Theorem 1.** Assume GRHR. For \(X \geq 2\) and positive integers \(a_1, a_2, q_1, q_2\) with \((a_1, q_1) = 1\) and \((a_2, q_2) = 1\), we have

\[
\sum_{n \leq X} R(n, q_1, a_1, q_2, a_2) = \frac{1}{\varphi(q_1)\varphi(q_2)} \left( \frac{X^2}{2} - G(X, q_1, a_1) - G(X, q_2, a_2) + H(X) \right) + E(X)
\]

with\(^4\)

\[
H(X) = \sum_{\chi_1 \mod q_1} \overline{\chi_1(a_1)}\overline{\chi_2(a_2)} \left( \sum_{\beta_1} G^{\beta_1}(X, \chi_1) + \sum_{\beta_2} G^{\beta_2}(X, \chi_2) - \sum_{\beta_1, \beta_2} W(X, \beta_1, \beta_2) \right),
\]

\[E(X) \ll X(\log X)(\log q_1X)(\log q_2X),\]

where \(\beta_i\) runs through all real non-trivial zeros \(^5\) of \(L(s, \chi_i)\) with \(\beta_i \geq 1/2\) for \(i = 1, 2\), and the implicit constant in the error term is absolute.

If we compare our result with Rüppel’s result, then we find that in her asymptotic formula \([1,2]\), the terms correspond to the oscillating terms

\[G(X, q_1, a_1), \quad G(X, q_2, a_2), \quad H(X)\]

of Theorem\(^\[1\]\) are included in the error term \(O(X^{1+\epsilon}(\log q)^2)\).

Rüppel obtained the meromorphic continuation of the function

\[
\Phi_2(s, a, b, q) = \sum_{n=1}^{\infty} \frac{R(n, a, q, b, q)}{n^s} = \sum_{k,m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)}{(k+m)^s}
\]

\(\mod (q_1)\) and \(\mod (q_2)\),

to the half plane \(\text{Re } s > 1\). This analytic approach to the problem which we are considering is initiated by Egami and Matsumoto \([2]\) in 2007. Meromorphic continuation of the above type relates to the mean value of the same type as \([1,3]\) through Perron’s formula. Such a correspondence also exist\(^6\) between Rüppel’s continuation and Theorem\(^\[1\]\).

Our proof of Theorem\(^\[1\]\) follows the argument of Languasco and Zaccagnini \([6]\) except that we need to treat the real zeros which we permit. And in order to treat these zeros, we have to modify the estimate of Languasco and Perelli \([5]\), i.e. we have to treat the terms arising from real zeros like the main term.

\(^4\)Rigorously speaking, these functions should be written as \(H(X, q_1, a_1, q_2, a_2)\) and \(E(X, q_1, a_1, q_2, a_2)\), but in the following, we use such abbreviations if there is no possibility of confusion.

\(^5\)Therefore, if there are no real zeros, then the sum over \(\beta_i\) turns to be empty.

\(^6\)See the Remark at the last of this note.
2. Notations

Here we briefly summarize the notations which we use in this note. Some exceptional notations are given at each occurrence.

\[ x, \alpha \] : real numbers,
\[ X, T \geq 2 \] : real numbers,
\[ N, M \geq 2 \] : positive integers,
\[ a_1, q_1, a_2, q_2 \] : positive integers satisfying \((a_1, q_1) = 1\) and \((a_2, q_2) = 1\),
\[ a, b, q \] : positive integers satisfying \((ab, q) = 1\),
\[ m, n \] : integers (We impose \(m, n \geq 1\) when these are used as summation variables.),
\[ p \] : prime numbers,
\[ \varphi(q) \] : Euler totient function,
\[ \Lambda(n) \] : von Mangoldt function,
\[ e(x) := \exp(2\pi i x) \],
\[ \chi \mod q \] : Dirichlet characters of modulus \(q\),

Chebyshev functions:
\[ \psi(x) := \sum_{n \leq x} \Lambda(n), \quad \psi(x, q, a) := \sum_{n \leq x \mod q} \Lambda(n), \quad \psi(x, \chi) := \sum_{n \leq x} \chi(n)\Lambda(n). \]

For Dirichlet characters \(\chi \mod q\), we define
\[ E(\chi) = \begin{cases} 0 & \text{(when } \chi \text{ is non-principal)}, \\ 1 & \text{(when } \chi \text{ is principal)}. \end{cases} \]

We regard the arithmetic function whose value is always constant 1 as the primitive Dirichlet character of modulus 1, and the other principal Dirichlet characters of modulus \(q \geq 2\) as imprimitive characters. For an imprimitive character \(\chi \mod q\), we denote by \(\chi^* \mod q^*\) the primitive character which induces \(\chi \mod q\). We use a complex variable \(s = \sigma + it\) where \(\sigma\) is the real part of \(s\) and \(t\) is the imaginary part of \(s\).

We say that a zero \(\rho\) of Dirichlet \(L\) functions is non-trivial if \(\rho\) satisfies \(0 < \Re \rho < 1\), and we will denote by \(\rho\) with imaginary part \(\gamma\) the non-trivial zeros of \(L(s, \chi)\). If we should show explicitly to which character these zeros correspond, then we attach the character as its suffix, e.g. \(\rho_\chi\). Moreover, \(\beta\) denotes real zeros of Dirichlet \(L\) functions, and if \(\beta\) is used as a summation variable, then it runs through all of the real zeros of a given Dirichlet \(L\) function with \(\geq 1/2\). We rigorously distinguish two terms real zero and Siegel zero, see Section 3.

3. Classical results and remarks on the Siegel zeros

In this section, we recall some classical results on Siegel zeros. We also introduce notations and conventions for real and Siegel zeros.

It is well-known that there exists an absolute constant
\[ c_1 > 0 \]
satisfying the following theorem of Landau. We use this letter \(c_1\) always as the same meaning throughout this paper. First we introduce the following definition.
Definition 1. Let \( q \geq 2 \) and \( \chi ( \text{mod} \ q) \) be a Dirichlet character. A real non-trivial zero \( \beta \) of \( L(s, \chi) \) is called a Siegel zero if
\[
\beta > 1 - \frac{c_1}{\log q}.
\]
If \( L(s, \chi) \) has a Siegel zero, then we say \( \chi ( \text{mod} \ q) \) has a Siegel zero.

Then the well-known theorem of Landau\(^7\) is the following [10, Corollary 11.8].

Theorem 2. Let \( q \) be a positive integer. Then among Dirichlet characters \( \chi ( \text{mod} \ q) \) of the modulus \( q \), there is at most one character which has a Siegel zero. Moreover, if such a character \( \psi ( \text{mod} \ q) \) with Siegel zero exists, then \( L(s, \psi) \) has only one Siegel zero even counting with multiplicity.

Hence in this note we do not call all real non-trivial zeros of Dirichlet \( L \) function Siegel zero, although R"uppel used such a terminology. We denote by \( \beta_0 \) the Siegel zeros. If we should show explicitly \( \beta_0 \) is the one corresponds to the character \( \chi ( \text{mod} \ q) \), then we write like \( \beta_0 (\chi) \). In the following of this paper, there are some terms containing Siegel zeros. We ignore such terms if corresponding Siegel zeros do not exist.

We also recall the following approximation [10, Theorem 11.4] of \( L'/L(1, \chi) \).

Lemma 1. For a non-principal character \( \chi ( \text{mod} \ q) \), we have
\[
\frac{L'}{L}(1, \chi) = \frac{1}{1 - \beta_0} + O(\log q),
\]
where \( \beta_0 \) is the Siegel zero of \( L(s, \chi) \).

4. A modification of the estimate of Languasco and Perelli with real zeros

We consider the following generating functions
\[
\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)e^{-n/N}e(n\alpha),
\]
\[
\tilde{S}(\alpha, \chi) = \sum_{n=1}^{\infty} \chi(n)\Lambda(n)e^{-n/N}e(n\alpha).
\]

If we introduce the argument \( z = 1/N - 2\pi i\alpha \), then we can express them as
\[
\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)e^{-n}, \quad \tilde{S}(\alpha, \chi) = \sum_{n=1}^{\infty} \chi(n)\Lambda(n)e^{-n}.
\]

Let us first recall the following explicit formula for these series.

Lemma 2. For \( N \geq 2 \), \( \alpha \in [-1/2, 1/2] \) and a primitive character \( \chi ( \text{mod} \ q) \), we have
\[
\tilde{S}(\alpha) = \frac{1}{z} - \sum_{\rho} \frac{z^{-\rho}}{\Gamma(\rho)} + O(1), \quad q = 1, \quad \tag{4.1}
\]
\[
\tilde{S}(\alpha, \chi) = - \sum_{\rho} \frac{z^{-\rho}}{\Gamma(\rho)} + \frac{L'}{L}(1, \chi) + O(\log qN), \quad q \geq 2, \tag{4.2}
\]
where \( \rho \)'s are the non-trivial zeros of \( L(s, \chi) \) including real zeros.

\(^7\)Of course, we weaken the original Landau’s theorem up to the form we need.
This is an easy application of the Mellin-Cahen formula and we can obtain this explicit formula unconditionally. For the proof, see [7, Lemma 2].

The modified version of the estimate of Languasco and Perelli is the following.

**Theorem 3.** Assume GRHR. For \( N \geq 2, 0 < \xi \leq 1/2 \), and a Dirichlet character \( \chi \) (mod \( q \)), we have

\[
\int_{-\xi}^{\xi} \left| \tilde{S}(\alpha, \chi) - \frac{E(\chi)}{z} + \sum_{\beta} \frac{\Gamma(\beta)}{\pi^\beta} \right|^2 \, d\alpha \ll N\xi (\log qN)^2
\]

where \( \beta \) runs through the all real non-trivial zeros of \( L(s, \chi) \) satisfying \( \geq 1/2 \).

For the proof of this theorem, we need to modify only the preliminary calculations in the original proof of Languasco and Perelli. The remaining and most important part of the proof is exactly the same one which Languasco and Perelli give. Hence we see only how the preliminary calculations should be modified.

**Proof.** At first we assume that \( \chi \) is primitive. In the case where \( \chi \) is principal, i.e. the case \( q = 1 \), there is no real zeros. So the theorem reduces to the original estimate of Languasco and Perelli. Next we consider the case where \( \chi \) is non-principal or \( q \geq 2 \). Then the integral we are considering becomes

\[
(4.3) \quad \int_{-\xi}^{\xi} \left| \tilde{S}(\alpha, \chi) + \sum_{\beta} \frac{\Gamma(\beta)}{\pi^\beta} \right|^2 \, d\alpha.
\]

We shall substitute the explicit formula (4.2) into this integral. Before this substitution, we reform (4.2) slightly. By the usual approximation for \( L'(s, \chi) \), we get

\[
L'(1, \chi) = \sum_{|\gamma|<1} \frac{1}{1-\rho} + O(\log q).
\]

By GRHR, we can estimate this as

\[
L'(1, \chi) = \sum_{\beta} \frac{1}{1-\beta} + O(\log q),
\]

because there is at most \( O(\log q) \) zeros in the region \(|\gamma|<1\). Let us substitute this expression into the explicit formula (4.2). We obtain

\[
\tilde{S}(\alpha, \chi) = - \sum_{\rho} \frac{1}{1-\rho} \Gamma(\rho) + \sum_{\beta} \frac{1}{1-\beta} + O(\log qN)\]

where \( \Sigma' \) is the sum excluding real zeros. Now by the existence of the pole 0 of the gamma function, if \( 3/4 \leq \beta < 1 \), we have

\[
\frac{\Gamma(1-\beta)}{\pi^\beta} - \frac{1}{1-\beta} = \frac{1}{1-\beta} \left( \frac{1}{1-\beta} - 1 \right) + O\left( \frac{1}{|z|^{1-\beta}} \right)
\]

\[
= - \log z \int_0^{1-\beta} z^{-\sigma} d\sigma + O\left( \frac{1}{|z|^{1-\beta}} \right) \ll N^{1/4}(\log N) \ll N^{1/2},
\]
Therefore our integral (4.3) is reduced to
\[ \int S(\alpha, \chi) \, d\alpha \ll N. \]
for all primitive characters \( \chi \) (mod \( q \)).

Moreover, we have
\[ S(\alpha, \chi) - \tilde{S}(\alpha, \chi^\star) \ll \sum_{\rho \neq \chi^\star} (\log \rho) \sum_{k=1}^\infty e^{-\rho/k} \ll (\log q)(\log N) \ll N^{1/2}(\log q). \]

Therefore, we obtain the theorem for all Dirichlet characters.
5. Detection and Calculations of the Main and Oscillating Terms

Let us now consider the integrals

\[
\int_{-1/2}^{1/2} \frac{T(y, -\alpha)}{z^\mu} d\alpha, \quad \int_{-1/2}^{1/2} \frac{\overline{T(y, -\alpha)}}{z^\mu} d\alpha,
\]

where \( \mu \) is some positive constant and \( T(y, \alpha) \) is given by

\[
T(y, \alpha) = \sum_{m \leq N} e^{i m \alpha}.
\]

We begin with the following integral formula.

**Lemma 3.** For integers \( N, n \) with \( N \geq 2 \) and a positive real number \( \mu > 0 \), we have

\[
\int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^\mu} d\alpha = \begin{cases} 
\frac{e^{-n/N} \cdot n^{\mu-1}}{\Gamma(\mu)} + O\left(\frac{2^\mu}{|n|}\right) & \text{(when } n > 0), \\
O\left(\frac{2^\mu}{|n|}\right) & \text{(when } n < 0), \\
O(\log N) & \text{(when } n = 0, 0 < \mu \leq 1). 
\end{cases}
\]

**Proof.** We first consider the case \( n \neq 0 \). For arbitrary \( Y > X \geq 0 \), by integration by parts, we have

\[
\int_X^Y \frac{e(-n\alpha)}{z^\mu} d\alpha = \int_X^Y \frac{e(-n\alpha)}{(1/N - 2\pi a)^\mu} d\alpha
\]

\[
= -\frac{1}{2\pi in} \left[ \frac{e(-n\alpha)}{(1/N - 2\pi a)^\mu} \right]_X^Y + \frac{\mu}{n} \int_X^Y \frac{e(-n\alpha)}{(1/N - 2\pi a)^\mu+1} d\alpha
\]

\[
\ll \frac{1}{|n|^\mu} + \frac{\mu}{|n|} \int_X^Y \frac{d\alpha}{\alpha^{\mu+1}} \ll \frac{1}{|n|^\mu}.
\]

Therefore we can extend the integral with the following error term:

\[
\int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^\mu} d\alpha = \int_{-\infty}^{\infty} \frac{e(-n\alpha)}{z^\mu} d\alpha + O\left(\frac{2^\mu}{|n|}\right).
\]

Now we rewrite this completed integral into the complex integral as

\[
\int_{-\infty}^{\infty} \frac{e(-n\alpha)}{z^\mu} d\alpha = \frac{e^{-n/N}}{2\pi i} \int_{1/N+i\infty}^{1/N-i\infty} \exp(ns) \frac{1}{s^\mu} ds.
\]

Besides \( n \neq 0 \), let us also assume \( n > 0 \). Let \( T, K > 0 \) be positive real numbers, and consider the contours of integration

- \( C_1 \): The line segment from \( 1/N - iT \) to \(-K - iT\),
- \( C_2 \): The line segment from \(-K - iT \) to \(-K - i/n\),
- \( C_3 \): The line segment from \(-K + i/n \) to \(-K + iT\),
- \( C_4 \): The line segment from \(-K + iT \) to \(1/N + iT\),
- \( H_1(K) \): The line segment from \(-K - i/n \) to \(-i/n\),
- \( H_2(K) \): The half circle from \(-i/n \) to \(i/n\) with center 0 and radius \(1/n\) in the positive direction,
- \( H_3(K) \): The line segment from \(i/n \) to \(-K + i/n\),

and

\[
\mathcal{H}(K) = H_1(K) + H_2(K) + H_3(K), \quad \mathcal{H} = \mathcal{H}(\infty).
\]
We shift the contour to the left to obtain
\[
e^{-n/N} 2\pi i \int_{1/N-i\infty}^{1/N+i\infty} \frac{\exp(ns)}{s^\mu} ds = e^{-n/N} 2\pi i \int_{\mathcal{H}(K)} \frac{\exp(ns)}{s^\mu} ds + e^{-n/N} 2\pi i \int_{C_1+C_2+C_3+C_4} \frac{\exp(ns)}{s^\mu} ds.
\]
Here we have
\[
\int_{C_1+C_4} \frac{\exp(ns)}{s^\mu} ds \ll nK \frac{1}{T^\mu}.
\]
Hence letting \( T \to \infty \), we get
\[
e^{-n/N} 2\pi i \int_{1/N-i\infty}^{1/N+i\infty} \frac{\exp(ns)}{s^\mu} ds = e^{-n/N} 2\pi i \int_{\mathcal{H}(K)} \frac{\exp(ns)}{s^\mu} ds + e^{-n/N} 2\pi i \left( \int_{-K-i\infty}^{-K+\infty} + \int_{-K+i\infty}^{-K-i\infty} \right) \frac{\exp(ns)}{s^\mu} ds.
\]
For one of the last two integrals, we get
\[
\int_{-K-i\infty}^{-K+\infty} \frac{\exp(ns)}{s^\mu} ds = 2\pi i e^{-nk} n \int_{1/2\pi n}^{\infty} \frac{e(-\alpha)}{(-K-2\pi i\alpha)^\mu} d\alpha
\]
\[
= -\frac{e^{-nk}}{n} \left[ \frac{e(-\alpha)}{(-K-2\pi i\alpha)^\mu} \right]_{1/2\pi n}^{\infty} + 2\pi i e^{-nk} n \int_{1/2\pi n}^{\infty} \frac{e(-\alpha)}{(-K-2\pi i\alpha)^{\mu+1}} d\alpha
\]
\[
\ll \frac{1}{K^\mu} + \frac{\mu}{|\mu|} \int_{0}^{\infty} \frac{1}{(K^2 + \alpha^2)^{\mu+1/2}} d\alpha \ll \frac{1}{K^\mu}.
\]
The other case can be estimated similarly. Hence letting \( K \to \infty \), we get
\[
e^{-n/N} 2\pi i \int_{1/N-i\infty}^{1/N+i\infty} \frac{\exp(ns)}{s^\mu} ds = e^{-n/N} 2\pi i \int_{\mathcal{H}} \frac{\exp(ns)}{s^\mu} ds
\]
and by the Hankel integral formula for the gamma function, we have
\[
e^{-n/N} 2\pi i \int_{1/N-i\infty}^{1/N+i\infty} \frac{\exp(ns)}{s^\mu} ds = n^{\mu-1} e^{-n/N} 2\pi i \int_{\mathcal{H}} \frac{\exp(s)}{s^\mu} ds = e^{-n/N} n^{\mu-1} \frac{1}{\Gamma(\mu)}.
\]
Therefore we have proved
\[
\int_{-1/2}^{1/2} \frac{e(-\alpha)}{\alpha^\mu} d\alpha = e^{-n/N} n^{\mu-1} \frac{1}{\Gamma(\mu)} + O\left( \frac{2^\mu}{n} \right)
\]
for the case \( n > 0 \). For the case \( n < 0 \), we can prove the lemma by shifting the contour of integration to the right and arguing as above.

Now we consider the remaining case \( n = 0 \) and \( 0 < \mu \leq 1 \). In this case, we can simply estimate
\[
\int_{-1/2}^{1/2} \frac{e(-\alpha)}{\alpha^\mu} d\alpha \ll N \int_{0}^{1/N} \frac{1}{\alpha} d\alpha + \int_{1/N}^{1/2} \frac{1}{\alpha} d\alpha \ll \log N
\]
as desired, and we finally get the lemma for all cases.

\[\square\]

The first integral of (5.11) can be calculated as follows.

**Lemma 4.** For any positive integer \( N \geq 2 \) and real numbers \( 2 < \gamma, \ 0 < \mu \leq 2 \), we have
\[
\int_{-1/2}^{1/2} \frac{T(y, -\alpha)}{\alpha^\mu} d\alpha = \frac{1}{\Gamma(\mu)} \sum_{m \geq 0} e^{-m/N} n^{\mu-1} + O(\log y).
\]
Proof. Interchanging the order of summation and integration, by Lemma 3 we have
\[
\int_{-1/2}^{1/2} T(y, -\alpha) \mathcal{S}(\alpha, \chi) \frac{1}{z^{\mu}} d\alpha = \sum_{m \le Y} \int_{-1/2}^{1/2} \frac{e(-ma)}{z^{\mu}} d\alpha
\]
\[
= \frac{1}{\Gamma(\mu)} \sum_{m \le Y} e^{-m/N} m^{\mu-1} + O \left( \sum_{m \le Y} \frac{1}{m} \right)
\]
\[
= \frac{1}{\Gamma(\mu)} \sum_{m \le Y} e^{-m/N} m^{\mu-1} + O(\log y).
\]

In order to study the second integral of (5.1), we introduce
\[
\psi_\mu(x, \chi) := \sum_{n \le x} \chi(n) \Lambda(n)(x - n)^{\mu-1}
\]
for Dirichlet character $\chi \pmod{q}$. Then the result is the following.

**Lemma 5.** For any Dirichlet character $\chi \pmod{q}$, any positive integer $N \geq 2$, any real numbers $2 < y$ and $0 < \mu \leq 1$, we have
\[
\int_{-1/2}^{1/2} T(y, -\alpha) \mathcal{S}(\alpha, \chi) \frac{1}{z^{\mu}} d\alpha = \frac{1}{\Gamma(\mu)} \sum_{m \leq Y} e^{-m/N} \psi_\mu(m, \chi) + O(N(\log y N) + y(\log N)).
\]

Proof. First we interchange the order of summation and integration
\[
\int_{-1/2}^{1/2} T(y, -\alpha) \mathcal{S}(\alpha, \chi) \frac{1}{z^{\mu}} d\alpha = \sum_{m \leq Y} \sum_{n=1}^{\infty} \chi(n) \Lambda(n)e^{-n/N} \int_{-1/2}^{1/2} \frac{e(-m - n)\alpha}{z^{\mu}} d\alpha.
\]

We apply Lemma 3 to these last integrals, and we have
\[
= \frac{1}{\Gamma(\mu)} \sum_{m \leq Y} e^{-m/N} \sum_{n=1}^{m-1} \chi(n) \Lambda(n)(m - n)^{\mu-1}
\]
\[
+ O \left( 2^\mu \sum_{m \leq Y} \sum_{n=1}^{\infty} \Lambda(n) \frac{e^{-n/N}}{|m-n|} + (\log N) \sum_{m \leq Y} \Lambda(m)e^{-m/N} \right).
\]

Here the error terms can be estimated as
\[
2^\mu \sum_{m \leq Y} \sum_{n=m+1}^{\infty} \Lambda(n) \frac{e^{-n/N}}{|m-n|} \ll \sum_{n=1}^{\infty} \Lambda(n)e^{-n/N} \sum_{m < n} \frac{1}{|m-n|}
\]
\[
\ll \sum_{n=1}^{\infty} \Lambda(n)(\log 2n)e^{-n/N} \ll N(\log N),
\]
\[
2^\mu \sum_{m \leq Y} \sum_{n=1}^{m-1} \Lambda(n) \frac{e^{-n/N}}{|m-n|} \ll \sum_{n=1}^{\infty} \Lambda(n)e^{-n/N} \sum_{n < m \leq Y} \frac{1}{|m-n|}
\]
\[
\ll (\log y) \sum_{n=1}^{\infty} \Lambda(n)e^{-n/N}
\]
\[
\ll N(\log y),
\]
and
\[(\log N) \sum_{m \leq Y} \Lambda(m)e^{-m/N} \ll \psi(y)(\log N) \ll y(\log N).\]

Therefore, we have
\[\int_{-1/2}^{1/2} T(y, -\alpha) \frac{S(\alpha, \chi)}{\zeta'} d\alpha = \frac{1}{\Gamma(\mu)} \sum_{m \leq Y} e^{-m/N} \psi(m, \chi) + O(N(\log y N) + y(\log N)).\]

We now calculate the main term of Lemma 5 explicitly.

**Lemma 6.** Let \(\chi \pmod{q}\) be a Dirichlet character, \(M \geq 2\) be a positive integer and \(1/2 < \mu \leq 1\) be a real number. Moreover let \(\chi^* \pmod{q}\) be the primitive character which induces \(\chi \pmod{q}\). Then we have
\[\sum_{n=1}^{M} \psi_{\mu}(m, \chi) = E(\chi) \frac{M^{\mu+1}}{\mu(\mu + 1)} - G^\mu(M, \chi) + \frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi^*) + O(M(\log 2q)(\log M)),\]
where the term
\[\frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi^*)\]
appears only if \(\chi \pmod{q}\) is non-principal.

**Proof.** First recalling \(1/2 < \mu \leq 1\) and using \(\rho(x) = \lfloor x \rfloor - 1/2\), we have
\[\sum_{n \leq x} n^{\mu-1} = \int_{1}^{x} u^{\mu-1} du - \int_{1-}^{x} u^{\mu-1} \rho(u) du = \int_{0}^{x} u^{\mu-1} du + (\mu - 1) \int_{1}^{x} u^{\mu-2} \rho(u) du + O(1) = \int_{0}^{x} u^{\mu-1} du + O(1).\]

Hence we get
\[\sum_{n=1}^{M} \psi_{\mu}(m, \chi) = \sum_{n=1}^{M} \sum_{n=1}^{n} \chi(n)\Lambda(n)(m-n)^{\mu-1} = \sum_{n=1}^{M} \chi(n)\Lambda(n) \sum_{m=1}^{M-n} (m-n)^{\mu-1} = \sum_{n=1}^{M-n} \chi(n)\Lambda(n) \sum_{m=1}^{M-n} m^{\mu-1} + O(M)\]

We can rewrite this as
\[= \sum_{n=1}^{M} \chi(n)\Lambda(n) \int_{n}^{M} (M-u)^{\mu-1} du + O(M) = \int_{0}^{M} \psi(u, \chi)(M-u)^{\mu-1} du + O(M).\]

Now we use the following explicit formula which can be derived from Theorem 12.5 and Theorem 12.10 of [10]:
\[\psi(u, \chi) = E(\chi)u - \sum_{|\rho| \leq T} \frac{u^\rho}{\rho} + \frac{L'}{L}(1, \chi^*) + O\left(\frac{M}{T}(\log qM)^2 + (\log 2q)(\log M)\right)\]
where \(2 \leq u \leq M, 2 < T \leq (qM)^2\), and \(\rho = \beta + iy\) runs through the non-trivial zeros of \(L(s, \chi)\) in \(0 < \sigma < 1\), and the term \(L'/L(1, \chi^*)\) appears only if \(\chi \pmod{q}\) is non-principal. Note that we have removed the restriction that \(\chi \pmod{q}\) is primitive as Theorem 12.5 and Theorem 12.10 of [10] at the cost of (See (12.13) of [10]):
\[(\log 2q)(\log N).\]
By this explicit formula, we have
\[
\sum_{m=1}^{\infty} \psi_p(m, \chi) = \sum_{m=1}^{\infty} \psi(u, \chi)(M - u)^{\mu - 1} du + O(M)
\]
\[
= E(\chi) \int_0^M u(M - u)^{\mu - 1} du - \sum_{\mu > 0} \frac{1}{\mu} \int_0^M u^2(M - u)^{\mu - 1} du
\]
\[
+ \frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi) + O\left( \frac{M^{\mu+1}}{T}(\log qM)^2 + M^\mu(\log 2q)(\log M) \right)
\]
\[
= E(\chi) \frac{M^{\mu+1}}{\mu(\mu + 1)} - \sum_{\mu > 0} \frac{M^{\mu+1}}{\rho + \mu} \cdot \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho + \mu)}
\]
\[
+ \frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi) + O\left( \frac{M^{\mu+1}}{T}(\log qM)^2 + M^\mu(\log 2q)(\log M) \right).
\]

Now we extend the sum over non-trivial zeros
\[
\sum_{\mu > 0} \frac{M^{\mu+1}}{\rho + \mu} \cdot \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho + \mu)}
\]
to the sum over all non-trivial zeros. In order to extend this sum, we recall the following estimate:
\[
|\Gamma(s)| \leq A \ |t|^{s-1/2} e^{-\pi|t|/2}, \quad |\sigma| \leq A, \quad |t| \geq 1
\]
which can be derived from Stirling’s formula. Then we can estimate each term of the extended part of the sum by
\[
\frac{M^{\mu+1}}{\rho + \mu} \cdot \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho + \mu)} \ll \frac{M^{\mu+1}}{|\gamma|^{\mu+1}}.
\]
Hence we can extend the sum with the error
\[
\ll M^{\mu+1} \sum_{\rho > 0, \rho \notin \chi} \frac{1}{\rho^{\mu+1}} = M^{\mu+1} \int_T^\infty \frac{1}{\rho^{\mu+1}} dN(t, \chi) \ll \frac{M^{\mu+1}}{T^{\mu}} \frac{L'}{L}(1, \chi).
\]
Summing up the above calculations, we have
\[
\sum_{m=1}^{\infty} \psi_p(m, \chi) = E(\chi) \frac{M^{\mu+1}}{\mu(\mu + 1)} - \sum_{\rho > 0} \frac{M^{\mu+1}}{\rho + \mu} \cdot \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho + \mu)}
\]
\[
+ \frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi) + O\left( \frac{M^{\mu+1}}{T}(\log qM)^2 + \frac{M^{\mu+1}}{T^{\mu}}(\log qM)^2 + M^\mu(\log 2q)(\log M) \right).
\]
Taking \( T = M(\log qM) \), we obtain
\[
\sum_{m=1}^{\infty} \psi_p(m, \chi) = E(\chi) \frac{M^{\mu+1}}{\mu(\mu + 1)} - \sum_{\rho > 0} \frac{M^{\mu+1}}{\rho + \mu} \cdot \frac{\Gamma(\rho)\Gamma(\mu)}{\Gamma(\rho + \mu)}
\]
\[
+ \frac{M^\mu}{\mu} \cdot \frac{L'}{L}(1, \chi) + O\left( M^\mu(\log 2q)(\log M) \right)
\]
as we claimed. \( \square \)
6. Proof of the main theorem

Now we prove the main theorem.

Proof of Theorem \[7\] Let us first define

\[
R(n, \chi_1, \chi_2) := \sum_{k_1, k_2 \equiv n} \chi_1(k_1)\Lambda(k_1)\chi_2(k_2)\Lambda(k_2)
\]

for Dirichlet characters \(\chi_1 \pmod{q_1}, \chi_2 \pmod{q_2}\) and consider the mean value of this function

\[
\sum_{n=1}^{N} R(n, \chi_1, \chi_2).
\]

By the Fourier coefficient formula, we have

\[
\sum_{m \leq y} e^{-m/N} R(m, \chi_1, \chi_2) = \int_{-1/2}^{1/2} \tilde{S}(\alpha, \chi_1)^2 \tilde{S}(\alpha, \chi_2) T(y, -\alpha) d\alpha
\]

for \(2 \leq y \leq N\). Here introducing

\[
\tilde{R}(\alpha, \chi) := \tilde{S}(\alpha, \chi) - \frac{E(\chi)}{z} + \sum_{\beta_i} \frac{\Gamma(\beta_i)}{z^{\beta_i}},
\]

we can expand the above integral as

\[
= \sum_{i,j} \left( I_{E, S_i} - \sum_{\beta_i} I_{\beta, S_i} + \sum_{\beta_i} I_{E, \beta_i} \right) - I_E - \sum_{\beta} I_{\beta, \beta_i} + I_R,
\]

where in what follows, \((i, j)\) take values \((1, 2)\) or \((2, 1)\), \(\sum_{i,j}\) is the sum over such \((i, j)\)'s, \(\beta_i\) runs through all real non-trivial zeros of \(L(s, \chi_i)\) with \(\beta_i \geq 1/2\) for \(i = 1, 2\), and

\[
I_{E, S_i} := E(\chi_i) \int_{-1/2}^{1/2} T(y, -\alpha) \frac{\tilde{S}(\alpha, \chi_i)}{z} d\alpha, \quad I_{\beta, S_i} := \Gamma(\beta_i) \int_{-1/2}^{1/2} T(y, -\alpha) \frac{\tilde{S}(\alpha, \chi_i)}{z^{\beta_i}} d\alpha,
\]

\[
I_{E, \beta_i} := E(\chi_i) \Gamma(\beta_i) \int_{-1/2}^{1/2} T(y, -\alpha) \frac{\tilde{S}(\alpha, \chi_i)}{z^{\beta_i}} d\alpha, \quad I_E := E(\chi_1) E(\chi_2) \int_{-1/2}^{1/2} T(y, -\alpha) \frac{\tilde{S}(\alpha, \chi_1) \tilde{S}(\alpha, \chi_2)}{z^2} d\alpha,
\]

\[
I_{\beta} := \Gamma(\beta_1) \Gamma(\beta_2) \int_{-1/2}^{1/2} T(y, -\alpha) \frac{\tilde{R}(\alpha, \chi_1) \tilde{R}(\alpha, \chi_2)}{z^2} d\alpha, \quad I_R := \int_{-1/2}^{1/2} T(y, -\alpha) \tilde{R}(\alpha, \chi_1) \tilde{R}(\alpha, \chi_2) d\alpha.
\]

First we calculate \(I_{E, S_i}, I_{\beta, S_i}\). These can be calculated by Lemma \[3\] as

\[
I_{E, S_i} = E(\chi_i) \sum_{m \leq y} e^{-m/N} \psi(m - 1, \chi_i) + O(N \log N),
\]

\[
I_{\beta, S_i} = \sum_{m \leq y} e^{-m/N} \psi_{\beta_i}(m, \chi_i) + O(N \log N).
\]

Next \(I_{E, \beta_i}, I_E, I_{\beta}\) are calculated by Lemma \[4\] as

\[
I_{E, \beta_i} = \frac{E(\chi_i)}{\beta_i} \sum_{m \leq y} e^{-m/N} m^{\beta_i} + O(\log N),
\]

\[
I_{E, E_2} = E(\chi_1) E(\chi_2) \sum_{m \leq y} e^{-m/N} m + O(\log N),
\]

\[
I_{\beta, \beta_i} = \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} \sum_{m \leq y} e^{-m/N} m^{\beta_1 + \beta_2 - 1} + O(\log N).
\]
We next estimate the error term \( I_R \). By the Cauchy-Schwarz inequality, we have

\[
I_R \ll \left( \int_{-1/2}^{1/2} |T(y, -\alpha)| \| \tilde{R}(\alpha, \chi_1) \|^2 \, d\alpha \right)^{1/2} \left( \int_{-1/2}^{1/2} |T(y, -\alpha)| \| \tilde{R}(\alpha, \chi_2) \|^2 \, d\alpha \right)^{1/2}.
\]

Let us define

\[
J_i := \int_{-1/2}^{1/2} |T(y, -\alpha)| \| \tilde{R}(\alpha, \chi_i) \|^2 \, d\alpha, \quad i = 1, 2.
\]

Note that

\[
T(y, -\alpha) \ll \min \left( y, \frac{1}{|\alpha|} \right)
\]

for \(|\alpha| \leq 1/2\). Then Theorem 3 gives the estimate

\[
J_i \ll y \int_{|\alpha| \leq 1/y} \| \tilde{R}(\alpha, \chi_i) \|^2 \, d\alpha + \int_{1/y < |\alpha| \leq 1/2} \frac{\| \tilde{R}(\alpha, \chi_i) \|^2}{\alpha} \, d\alpha
\]

\[
\ll N (\log q_i N)^2 + \sum_{k=0}^{O(\log y)} \frac{y}{2^k} \int_{2^k/y < |\alpha| \leq 2^{k+1}/y} \| \tilde{R}(\alpha, \chi_i) \|^2 \, d\alpha
\]

\[
\ll N (\log q_i N)^2 + \sum_{k=0}^{O(\log y)} \frac{y}{2^k} \cdot \frac{2^{k+1}}{y} N (\log q_i N)^2 \ll N (\log N)(\log q_i N)^2
\]

for \( i = 1, 2 \). Hence we get

\[
I_R \ll N (\log N)(\log q_1 N)(\log q_2 N).
\]

Let us introduce

\[
E(m, \chi_1, \chi_2) := \sum_{i,j} \left( E(\chi_i) \psi(m - 1, \chi_j) - \sum_{\beta_i} \psi_{\beta_i}(m, \chi_j) + \sum_{\beta_j} \frac{E(\chi_i)}{\beta_j} m^{\beta_j} \right)
\]

\[
- E(\chi_1) E(\chi_2) m - \sum_{\beta_1, \beta_2} \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} m^{\beta_1 + \beta_2 - 1}
\]

and

\[
\Delta(m, \chi_1, \chi_2) = R(m, \chi_1, \chi_2) - E(m, \chi_1, \chi_2).
\]

Then the above calculations give

\[
(6.1) \quad \sum_{m \leq y} e^{-m/N} \Delta(m, \chi_1, \chi_2) \ll N (\log N)(\log q_1 N)(\log q_2 N)
\]

for \( y \leq N \). By partial summation, we have

\[
\sum_{n \leq N} \Delta(m, \chi_1, \chi_2) = \sum_{n \leq N} e^{-n/N} \Delta(n, \chi_1, \chi_2) \left( e - \frac{1}{N} \int_n^N e^{y/N} \, dy \right)
\]

\[
= e \sum_{n \leq N} e^{-n/N} \Delta(n, \chi_1, \chi_2)
\]

\[
- \frac{1}{N} \int_1^N e^{y/N} \sum_{n \leq y} e^{-n/N} \Delta(n, \chi_1, \chi_2) \, dy
\]

\[
\ll N (\log N)(\log q_1 N)(\log q_2 N),
\]

in other words, we get

\[
(6.2) \quad \sum_{n \leq N} R(n, \chi_1, \chi_2) = \sum_{n \leq N} E(n, \chi_1, \chi_2) + O(N (\log N)(\log q_1 N)(\log q_2 N)).
\]
Next we shall calculate the right hand side of the last equation. By Lemma 6, we get

\[ E(\chi_1) \sum_{n=1}^{N} \psi(n-1, \chi_j) = E(\chi_1) \sum_{n=1}^{N} \psi(n, \chi_j) + O(N \log N) \]

\[ = E(\chi_1) E(\chi_2) \frac{N^2}{2} - E(\chi_2) G(N, \chi_1) \]

\[ + E(\chi_2) N \cdot \frac{L'}{L}(1, \chi_j) + O(N(\log N)(\log 2q_j)), \]

\[ \sum_{n=1}^{N} \sum_{\beta_j} \psi_{\beta_j}(n, \chi_j) = E(\chi_1) \sum_{\beta_j} \frac{N^{\beta_j+1}}{\beta_j(\beta_j + 1)} - \sum_{\beta_j} G^{\beta_j}(N, \chi_j) \]

\[ + \sum_{\beta_j} \frac{N^{\beta_j}}{\beta_j} \cdot \frac{L'}{L}(1, \chi_j) + O(N(\log N)(\log 2q_j)(\log 2q_j)). \]

Now let us recall that for \( 0 < d, \)

\[ \sum_{n \leq x} n^{d-1} = \frac{1}{d} x^d + O(1 + x^{d-1}) \]

holds. So we have

\[ \sum_{n=1}^{N} \sum_{\beta_j} \frac{E(\chi_1)}{\beta_j} n^{\beta_j} = \sum_{\beta_j} \frac{E(\chi_1)}{\beta_j} \frac{N^{\beta_j+1}}{\beta_j(\beta_j + 1)} + O(N(\log 2q_j)), \]

\[ \sum_{n=1}^{N} \sum_{\beta, \beta_2} \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} n^{\beta_1+\beta_2-1} = \sum_{\beta_1, \beta_2} W(N, \beta_1, \beta_2) + O(N(\log 2q_1)(\log 2q_2)), \]

and

\[ E(\chi_1) E(\chi_2) \sum_{n=1}^{N} n = E(\chi_1) E(\chi_2) \frac{N^2}{2} + O(N). \]

Summing up these formulae, we get

\[ \sum_{n \leq N} R(n, \chi_1, \chi_2) = E(\chi_1) E(\chi_2) \frac{N^2}{2} - E(\chi_2) G(N, \chi_1) - E(\chi_1) G(N, \chi_2) \]

\[ + \sum_{\beta_1} G^{\beta_1}(N, \chi_1) + \sum_{\beta_2} G^{\beta_2}(N, \chi_2) - \sum_{\beta_1, \beta_2} W(N, \beta_1, \beta_2) \]

\[ + E(\chi_2) N \cdot \frac{L'}{L}(1, \chi_1) + E(\chi_1) N \cdot \frac{L'}{L}(1, \chi_2) \]

\[ - \sum_{\beta_2} \frac{N^{\beta_1}}{\beta_2} \cdot \frac{L'}{L}(1, \chi_1) - \sum_{\beta_1} \frac{N^{\beta_1}}{\beta_1} \cdot \frac{L'}{L}(1, \chi_2) \]

\[ + O(N(\log N)(\log q_1 N)(\log q_2 N)). \]

Multiplying (6.3) by

\[ \frac{\chi_1(a_1) \chi_2(a_2)}{\phi(q_1) \phi(q_2)}, \]
and summing up over characters \( \chi_1 \mod q_1 \), \( \chi_2 \mod q_2 \), we have

\[
\sum_{n \leq N} R(n, a_1, q_1, a_2, q_2) = \frac{1}{\varphi(q_1)\varphi(q_2)} \left( \frac{N^2}{2} - G(N, q_1, a_2) - G(N, q_2, a_2) + H(N) + S(N) \right) + O(N(\log N)(\log q_1)(\log q_2 N)),
\]

where \( S(N) \) is defined by

\[
S(N) := N \sum_{\chi_1 \neq \chi_0 \mod q_1} \frac{L'}{L}(1, \overline{\chi_1}) + N \sum_{\chi_2 \neq \chi_0 \mod q_2} \frac{L'}{L}(1, \overline{\chi_2})
\]

\[
- \sum_{\chi_1 \neq \chi_0 \mod q_1} \frac{L'}{L}(1, \overline{\chi_1}) \sum_{\beta_1}^{N_{\beta_1}} + \sum_{\chi_2 \neq \chi_0 \mod q_2} \frac{L'}{L}(1, \overline{\chi_2}) \sum_{\beta_2}^{N_{\beta_2}}.
\]

These terms are affected by Siegel zeros but not much bigger than the error term. This can be estimated as

\[
S(N) \varphi(q_1)\varphi(q_2) \ll N(\log 2q_1)(\log 2q_2)
\]

by recalling Theorem \( 2 \), Lemma \( 1 \) the well-known estimate

\[
\varphi(q) \gg \frac{q}{\log \log 4q},
\]

and the famous fact \( 10 \), p.370, Corollary 11.12

\[
\frac{1}{1 - \beta_0(\chi_i)} \ll q_i^{1/2}(\log q_i)^2, \quad i = 1, 2.
\]

Substituting (6.5) into the above formula, we finally arrive at

\[
\sum_{n \leq N} R(n, a_1, q_1, a_2, q_2) = \frac{1}{\varphi(q_1)\varphi(q_2)} \left( \frac{N^2}{2} - G(N, q_1, a_2) - G(N, q_2, a_2) + H(N) \right) + O(N(\log N)(\log q_1)(\log q_2 N)).
\]

The restriction that our argument \( N \) is integer can be removed by considering the variation of each term while the argument varies over bounded intervals. \( \square \)

**Remark.** Our Theorem \( 1 \) conflicts with Rüppel’s continuation. The inconsistency happens on the residues of the poles

\[
\rho_\psi + \beta_1, \quad \rho_\chi + \beta_2
\]

of the function \( \Phi_2(s, a, b, q) \). The difference between our result and hers is only up to the sign and this happened because of her minor mistakes on the determination of the sign of residues

\[-\frac{L'}{L}(s - \rho_\psi, \chi)\]

at \( 11 \), p.30 or \( 12 \) p.142.

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THE REPRESENTATION FUNCTION FOR THE SUM OF TWO PRIMES IN A.P.

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