QUASI-ISOLATED ELEMENTS IN REDUCTIVE GROUPS

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Abstract. A semisimple element $s$ of a connected reductive group $G$ is said quasi-isolated (respectively isolated) if $C_G(s)$ (respectively $C_G^o(s)$) is not contained in a Levi subgroup of a proper parabolic subgroup of $G$. We study properties of quasi-isolated semisimple elements and give a classification in terms of the affine Dynkin diagram of $G$. Tables are provided for adjoint simple groups.

1. Preliminaries and notation

1.A. Notation. Let $F$ be an algebraically closed field. Let $p$ denote its characteristic. By a variety (respectively an algebraic group), we mean an algebraic variety (respectively an algebraic group) over $F$. We denote by $Z(p)$ the localization of $Z$ at the prime ideal $pZ$ (in particular, if $p = 0$, then $Z(p) = Q$).

We fix a connected reductive group $G$. We denote by $Z(G)$ its center and $D(G)$ its derived subgroup.

If $g \in G$, we denote by $g_s$ (respectively $g_u$) its semisimple (respectively unipotent) part, $C_G(g)$ its centralizer and $C_G^o(g)$ the neutral component of $C_G(g)$. We denote by $o(g) \in \{1, 2, 3, \ldots\} \cup \{\infty\}$ the order of $g$.

1.B. Isolated and quasi-isolated elements. The element $g \in G$ is said quasi-isolated (respectively isolated) if $C_G^o(g_s)$ (respectively $C_G^o(g_u)$) is not contained in a Levi subgroup of a proper parabolic subgroup of $G$. If there is some ambiguity, we will speak about $G$-isolated or $G$-quasi-isolated elements to refer to the ambient group. Of course, an isolated element is quasi-isolated.

The isolated elements are present in many different papers while the quasi-isolated ones are not often mentioned (see [Bon, §4.5]). One reason might be the following: if the derived group of $G$ is simply connected, then centralizers of semisimple elements are connected (by a theorem of Steinberg [S, Theorem 8.1], see also [Bou, Chapter VI, §2, Exercise 1]) so the notions of isolated and quasi-isolated elements coincide. Another possible reason is that the notion of isolated element depends only on the Dynkin diagram of $G$, by opposition to the notion of quasi-isolated element (see Proposition 2.3 and Example 2.4).

Whenever the derived group of $G$ is not simply connected, a quasi-isolated element might not be isolated. The following extreme case can even happen: there exist quasi-isolated semisimple elements $s$ which are regular (that is, such that $C_G^o(s)$ is a maximal torus), as it is shown by the following example.

Example 1.1 - Let $n \geq 2$ be a natural number invertible in $F$. Let us assume in this example that $G = PGL_n(F)$. Let $\zeta$ be a primitive $n$-th root of unity in $F$ and let $s$ be the image of diag$(1, \zeta, \zeta^2, \ldots, \zeta^{n-1}) \in GL_n(F)$ in $G$. Then $C_G^o(s)$ is the maximal torus consisting of the image of diagonal matrices in $G$ (in particular, $s$ is regular, so it is not isolated) but $C_G(s)/C_G^o(s)$ is cyclic of order $n$: it is generated by a Coxeter element of the Weyl group of $G$ relatively to $C_G^o(s)$. Therefore, $s$ is quasi-isolated. □

1.C. Root system. The notions of isolated and quasi-isolated elements involve only the semisimple part, so we will focus on semisimple elements. For this reason, we fix once and for all a maximal torus of
If determining if an element of this torus is quasi-isolated or not can be done thanks to the root system or the Weyl group relatively to this torus.

Let $B$ be a Borel subgroup of $G$ and let $T$ be a maximal torus of $B$. Let $W$ be the Weyl group and let $\Phi$ be the root system of $G$ relatively to $T$. Let $\Phi^+$ (respectively $\Delta$) denote the positive root system (respectively the basis) of $\Phi$ associated to $B$.

We fix once and for all an element $s \in T$. We denote by $\Phi(s)$ and by $W^s$ (respectively $\Delta(s)$) denote the positive root system (respectively the basis) of $\Phi$ associated to $B$.

**Example 1.2** - $C^s_B(s)$ is a Borel subgroup of $C^s_B(s)$ containing $T$. If $B(s) = C^s_B(s)$, then $\Phi^+(s) = \Phi^+ \cap \Phi(s)$. □

We gather some elementary facts:

**Proposition 1.3.** Let $s \in T$. Then:

(a) $\Phi(s) = \{ \alpha \in \Phi \mid \alpha(s) = 1 \}$.
(b) $W(s)$ is the Weyl group of $C_G(s)$ relatively to $T$.
(c) $W(s) = A(s) \rtimes W^o(s)$.
(d) $A(s) \simeq C_G(s)/C_G^o(s)$.

**Corollary 1.4.** Let $s \in T$. Then:

(a) $s$ is isolated (respectively quasi-isolated) if and only if $W^o(s)$ (respectively $W(s)$) is not contained in a proper parabolic subgroup of $W$.
(b) The following are equivalent:
   (1) $s$ is isolated;
   (2) $\Phi(s)$ is not contained in a proper parabolic subsystem of $\Phi$;
   (3) $|\Delta(s)| = |\Delta|$.

**Proposition 1.5.** Let $s \in T$. Then there exists an element $s' \in T$, of finite order, such that $C_G(s) = C_G(s')$.

**Proof** - Let $S$ denote the Zarisky closure of the group generated by $s$. Then $S/S^o$ is generated by the image of $s$, so it is cyclic. Moreover, $C_G(s) = C_G(S)$. Therefore, Proposition 1.5 follows immediately from the following easy lemma:

**Lemma 1.6.** Let $D$ be a diagonalizable group acting on an affine variety $X$. We assume that $D/D^o$ is cyclic. Then there exists an element $t \in D$ of finite order such that $X^D = X^t$.

**Proof of Lemma 1.6** - We first prove the following statement:
Remark - D remain valid if of tori on affine varieties (see for instance [DM, Proposition 0.7]). Moreover, Lemma 1.6 does not

Let \(G\) be a simply connected covering of the derived group of \(G\) and if \(Im\) \(\chi\) has coprime order, we have \(X^{td} = (X^t)^d = (X^{D^o})^d = X^D.\)

Remark - Lemma 1.6 and statement (*) are slight refinements of a well-known lemma on the action of tori on affine varieties (see for instance [DM, Proposition 0.7]). Moreover, Lemma 1.6 does not remain valid if \(D/D^o\) is not cyclic. For instance, assume that \(p = 2\) and consider the action of \(D = \{1, -1\} \times \{1, -1\}\) on \(A^3(F)\) by \((\varepsilon, \varepsilon', (x, y, z)) = (\varepsilon x, \varepsilon' y, \varepsilon' z).\)

2. Isotypic morphisms

2.A. Definition. A morphism \(\pi : \tilde{G} \to G\) is said isotypic if \(\tilde{G}\) is a connected reductive group, if \(\ker\) \(\pi\) is central in \(G\) and if \(\text{Im}\) \(\pi\) contains the derived group of \(G\).

Example and Notation - Let \(\pi_{sc} : G_{sc} \to G\) be a simply connected covering of the derived group of \(G\). Let \(G_{ad}\) denote the adjoint group of \(G\) and let \(\pi_{ad} : G \to G_{ad}\) be the canonical surjective morphism. Then \(\pi_{sc}\) and \(\pi_{ad}\) are isotypic morphisms. We set \(B_{ad} = \pi_{ad}(B)\) and \(T_{ad} = \pi(T)\). Then \(B_{ad}\) is a Borel subgroup of \(G_{ad}\) and \(T_{ad}\) is a maximal torus of \(B_{ad}\). Moreover, if \(t \in T\), we set \(\tilde{t} = \pi_{ad}(t) \in T_{ad}.\)

We fix in this section an isotypic morphism \(\pi : \tilde{G} \to G\). Let \(\ker'\) \(\pi = D(\tilde{G}) \cap \ker\) \(\pi\). It must be noticed that \(\ker'\) \(\pi\) is a finite abelian group of order prime to \(p\). Let \(\tilde{B} = \pi^{-1}(B)\) and \(\tilde{T} = \pi^{-1}(T).\) Then \(B\) is a Borel subgroup of \(G\) and \(T\) is a maximal torus of \(B\). We will identify the Weyl group of \(G\) relatively to \(T\) with \(W\) through the morphism \(\pi\). Let \(\Phi\) denote the root system of \(\tilde{G}\) relatively to \(\tilde{T}\). Then the morphism \(\pi^* : X(T) \to X(T)\) induced by \(\pi\) provides a bijection \(\tilde{\Phi} \leftrightarrow \Phi.\)

Since \(\text{Im}\) \(\pi\) contains \(D(G)\), we have \(\pi(\tilde{G})Z(G)^o = G\). We fix once and for all in this section an element \(s \in T\) such that \(\pi(s) \in sZ(G)\). Then

\[
\pi(C_{G}(s)).Z(G)^o \subset C_{G}(s).
\]

Moreover, by Proposition 1.3 (a), we have

\[
\pi(C_{\tilde{G}}(\tilde{s})).Z(\tilde{G})^o = C_{\tilde{G}}(\tilde{s}).
\]

Therefore, \(W(s) \subset W(s)\) and \(W^o(\tilde{s}) = W^o(s)\). Moreover, \(A(s) \subset A(s)\) (if we choose \(\tilde{B}(\tilde{s}) = \pi^{-1}(B(s))\)).

These remarks have the following consequences:

Proposition 2.3. With the above notation, we have:

(a) If \(\tilde{s}\) is quasi-isolated in \(\tilde{G}\), then \(s\) is quasi-isolated in \(G\).
(b) \( \tilde{s} \) is isolated in \( \tilde{G} \) if and only if \( s \) is isolated in \( G \).

The following example shows that the converse to statement (a) of Proposition 2.3 is not true in general.

**Example 2.4** - Keep here the hypothesis and notation of Example 1.1. Assume that \( \tilde{G} = \mathbf{GL}_n(\mathbb{F}) \) and that \( \pi : G \to G \) is the canonical morphism. Let \( \tilde{s} = \text{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1}) \). Then \( \tilde{s} \) is not quasi-isolated in \( \tilde{G} \) since \( C_G(\tilde{s}) \) is a maximal torus. But \( s = \pi(\tilde{s}) \) is quasi-isolated in \( G \) as it is shown in Example 1.1. \( \square \)

**Remark 2.5** - If \( \pi \) is injective, then the inclusion 2.1 is an equality. So \( s \) is quasi-isolated in \( G \) if and only if \( \tilde{s} \) is quasi-isolated in \( \tilde{G} \). \( \square \)

**2.B. The groups \( A(s) \) et \( A(\tilde{s}) \).** We will compare here the groups \( A(s) \) and \( A(\tilde{s}) \) in order to obtain general properties of the group \( A(s) \). Most of the results of this subsection are well-known, particularly the Corollary 2.9 (see [S, lemme 9.2] and [BM, lemme 2.1]) but they are rarely stated in the whole generality of this subsection.

Let \( \text{Com}(G) \) denote the set of couples \((x, y) \in G \times G\) such that \( xy = yx \). This is a closed subvariety of \( G \times G \). If \((x, y) \in \text{Com}(G)\), we denote by \( \omega(x, y) \) the element \([\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} \in \tilde{G} \) where \( \tilde{x} \in \tilde{G} \) and \( \tilde{y} \) are two elements of \( G \) such that \( \pi(\tilde{x}) \in x\mathbf{Z}(G) \) and \( \pi(\tilde{y}) \in y\mathbf{Z}(G) \). It is easily checked that \( \omega(x, y) \) depends only on \( x \) and \( y \) and does not depend on the choice of \( \tilde{x} \) and \( \tilde{y} \). Moreover, \( \pi([\tilde{x}, \tilde{y}]) = [x, y] = 1 \) so \( \omega(x, y) \in \text{Ker}^\prime \pi \).

**Lemma 2.6.** Let \( x, x', y \) and \( y' \) be four elements of \( G \) such that \( xy = yx \), \( x'y = yx' \) and \( xy' = y'x \). Then :

\[
\omega(x', yy') = \omega(x, y)\omega(x, y'),
\]

\[
\omega(xx', y) = \omega(x, y)\omega(x', y),
\]

and

\[
\omega(x, y) = \omega(x, y)^{-1}.
\]

**Proof** - Let us show the first equality (the second can be shown similarly and the third one is obvious).

Let \( \tilde{x}, \tilde{y} \) and \( \tilde{y}' \) be three elements of \( \tilde{G} \) such that \( \pi(\tilde{x}) \in x\mathbf{Z}(G) \), \( \pi(\tilde{y}) \in y\mathbf{Z}(G) \) and \( \pi(\tilde{y}') \in y'\mathbf{Z}(G) \). Then \( \pi(\tilde{y}'\tilde{y}) \in yy'\mathbf{Z}(G) \) and so

\[
\omega(x, yy') = [\tilde{x}, \tilde{y}'\tilde{y}]
\]

\[
= \tilde{x}\tilde{y}'\tilde{x}^{-1}\tilde{y}'^{-1}\tilde{y}^{-1}
\]

\[
= \tilde{x}\tilde{y}'\tilde{x}^{-1}\tilde{y}'\tilde{x}^{-1}\tilde{y}^{-1}\tilde{y}^{-1}
\]

\[
= \tilde{x}\tilde{y}'\tilde{x}^{-1}\omega(x, y'yy^{-1})
\]

where the last equality follows from the fact that \( \omega(x, y') \) is central in \( \tilde{G} \). \( \blacksquare \)

Let \( \omega_s : C_G(s) \to \text{Ker}^\prime \pi, g \mapsto \omega(g, s) \). The Lemma 2.6 shows that \( \omega_s \) is a morphism of groups.

**Lemma 2.7.** \( \text{Ker} \omega_s = \pi(C_G(\tilde{s})).\mathbf{Z}(G)^o \).

**Proof** - Let \( g \in \text{Ker} \omega_s \). There exists \( \tilde{g} \in \tilde{G} \) such that \( \pi(\tilde{g}) \in g\mathbf{Z}(G)^o \). Since \( \omega_s(g) = [\tilde{g}, \tilde{s}] = 1 \), we have \( \tilde{g} \in C_G(\tilde{s}) \). \( \blacksquare \)

**Corollary 2.8.** We have :

(a) \( \omega_s \) induces a morphism \( \tilde{\omega}_s : A(s) \to \text{Ker}^\prime \pi \). We have \( \text{Ker} \tilde{\omega}_s = A(\tilde{s}) \) and \( \text{Im} \tilde{\omega}_s = \text{Im} \omega_s = \{z \in \text{Ker} \pi \mid \tilde{s} \text{ and } \tilde{s}z \text{ are conjugated in } \tilde{G}\} \).

(b) \( |A(s)/A(\tilde{s})| \) is a finite abelian group of order dividing \( |\text{Ker}^\prime \pi| \) (so prime to \( p \)).
2.6, we have \( g \) denote the order of \( \bar{n} \) of isolated element depends only on the isogeny class \( G \). The Proposition 2.3 shows that the notion of quasi-isolated elements.

2.C. Isotypic morphisms and quasi-isolated elements. The Proposition 2.3 shows that the notion of isolated element depends only on the isogeny class \( G \). On the other hand, the Example 2.4 shows that the notion of quasi-isolated element does not behave so nicely. We will use the morphism \( \omega_s \) to study a weak converse to the statement (a) of Proposition 2.3. This weak converse will also be used to obtain some classification result for quasi-isolated elements.

Let \( e^s_x \) denote the exponent of the group \( A(s)/A(\bar{s}) \) (recall that \( e^s_x \) divides the exponent of \( \text{Ker} \pi \) and the order of \( \bar{s} \) in \( G_{\text{ad}} \)). A result analogous to the following has been shown in [Bon, preuve du corollaire 4.5.3].

Proposition 2.10. The group \( C_G(s) \) is contained in \( \pi(C_G(\bar{s}^{e^s_x})).Z(G)^\circ \).

Proof - Let \( g \in C_G(s) \). Then \( \omega_s(g)^{e^s_x} = 1 \). But, by Lemma 2.6, we have \( \omega_s(g)^{e^s_x} = \omega_s(g)^{e^s_x} \). This shows that \( g \in \text{Ker} \omega_{e^s_x} = \pi(C_G(\bar{s}^{e^s_x})).Z(G)^\circ \) (see Lemma 2.7).

Corollary 2.11. If \( s \) is quasi-isolated in \( G \), then \( \bar{s}^{e^s_x} \) is quasi-isolated in \( \bar{G} \).

Corollary 2.12. Let \( e \) be the exponent of \( \text{Ker} \pi_{\text{sc}} \). If \( s \) is quasi-isolated in \( G \), then \( s^e \) is isolated in \( G \).

Proof - Once again, the group \( \bar{G} \) is not involved in this statement, so we can assume here that \( \pi = \pi_{\text{sc}} \). Then \( e^{s^e} \) divides \( e \), by Corollary 2.11, \( s^e \) is quasi-isolated in \( \bar{G} = G_{\text{sc}} \). But, since \( \bar{G} \) is simply connected, \( \bar{s}^{e^x} \) is isolated in \( \bar{G} \). Therefore, by Proposition 1.4 (a), \( s^e \) is isolated in \( G \).

Corollary 2.13. If \( s \) is quasi-isolated, then \( \bar{s} \) has finite order.
3. Semisimple elements of finite order

We will describe in this subsection the possible structure of the centralizer of a semisimple element in \( \mathbf{G} \). By Proposition 1.5, we can focus on semisimple elements of finite order. For this, we fix an injective morphism \( i : (\mathbb{Q}/\mathbb{Z})_{p'} \rightarrow \mathbb{F}^\times \) and we denote by \( i : \mathbb{Q} \rightarrow \mathbb{F}^\times \) the composition of the morphisms \( \mathbb{Q} \rightarrow (\mathbb{Q}/\mathbb{Z})_{p'} \rightarrow \mathbb{F}^\times \). Finally, we set \( \overline{i}_T : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}(\mathbf{T}) \rightarrow \mathbf{T}, r \otimes \lambda \mapsto \lambda(\overline{i}(r)) \). The image of \( \overline{i}_T \) is the torsion subgroup of \( \mathbf{T} \).

To understand the structure of \( C_\mathbf{G}(s) \), then, by Proposition 1.5 and by Remark 2.5, it is sufficient to work under the following hypothesis:

**Hypothesis** - From now on, and until the end of this paper, we assume that \( \mathbf{G} \) is semisimple and that \( s \) has finite order.

**Remarque** - It must be noticed that, in view of classifying quasi-isolated semisimple elements, this hypothesis is not restrictive (see Remark 2.5 and Corollary 2.13). \( \Box \)

3.A. Preliminaries. Let \( V \) be the \( \mathbb{Q} \)-vector space \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}(\mathbf{T}) \) and let \( V^* \) be its dual, identified with \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}(\mathbf{T}) \). We denote by \( \langle, \rangle : V^* \times V \rightarrow \mathbb{Q} \) the canonical perfect pairing between \( V \) and \( V^* \). Then \( \mathbf{Y}(\mathbf{T}_{sc}) \) may be identified with \( \langle, \rangle \) and \( \mathbf{X}(\mathbf{T}_{ad}) \) may be identified with \( \langle, \rangle \). Since \( \mathbf{G} \) is semisimple, \( \Delta \) is a basis of \( V^* \). Let \( (v^\alpha)_{\alpha \in \Delta} \) be its dual basis. Then \( \mathbf{Y}(\mathbf{T}_{ad}) \) may be identified with \( \oplus_{\alpha \in \Delta} \mathbb{Z}v^\alpha \). As expected, we have \( \mathbf{Y}(\mathbf{T}_{ad}) \subset \mathbf{Y}(\mathbf{T}) \subset \mathbf{Y}(\mathbf{T}_{ad}) \subset \mathbf{V} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{Y}(\mathbf{T}_{sc}) \). If \( v \in \mathcal{V} \), let \( \tau_v : \mathcal{V} \rightarrow \mathcal{V}, x \mapsto x + v \) denote the translation by \( v \).

Let us recall the following elementary fact:

**Lemma 3.1.** The map \( \mathbf{Y}(\mathbf{T}_{ad}) \rightarrow \mathbf{T}, \lambda \mapsto \overline{i}_T(\lambda) \) induces an isomorphism \( (\mathbf{Y}(\mathbf{T}_{ad})/\mathbf{Y}(\mathbf{T}))_{p'} \simeq \mathbb{Z}(\mathbf{G}) \).

The map \( \mathbf{Y}(\mathbf{T}) \rightarrow \mathbf{T}_{sc}, \lambda \mapsto \overline{i}_T(\lambda) \) induces an isomorphism \( (\mathbf{Y}(\mathbf{T})/\mathbf{Y}(\mathbf{T}_{sc}))_{p'} \simeq \operatorname{Ker} \pi_{sc} \).

If \( \lambda \in \mathcal{V} \), we set

\[
\Phi(\lambda) = \{ \alpha \in \Phi \mid \langle \alpha, \lambda \rangle \in \mathbb{Z} \}
\]

and

\[
W_G(\lambda) = \{ w \in W \mid w(\lambda) - \lambda \in \mathbf{Y}(\mathbf{T}) \}.
\]

We denote by \( o_{sc}(\lambda) \) (respectively \( o_{ad}(\lambda) \), respectively \( o_G(\lambda) \)) the order of the image of \( \lambda \) in \( \mathbf{V}/\mathbf{Y}(\mathbf{T}_{sc}) \) (respectively \( \mathbf{V}/\mathbf{Y}(\mathbf{T}_{ad}) \), respectively \( \mathbf{V}/\mathbf{Y}(\mathbf{T}) \)). Let \( W^\circ(\lambda) \) denote the Weyl group of the closed subsystem \( \Phi(\lambda) \) of \( \Phi \). Then \( W^\circ(\lambda) \) is a normal subgroup of \( W_G(\lambda) \). If we fix a positive root system \( \Phi^+(\lambda) \) in \( \Phi(\lambda) \), then we can define

\[
A_G(\lambda) = \{ w \in W_G(\lambda) \mid w(\Phi^+(\lambda)) = \Phi^+(\lambda) \}.
\]

Then

\[
W_G(\lambda) = A_G(\lambda) \times W^\circ(\lambda).
\]

The next lemma shows that, in order to understand the structure of \( C_\mathbf{G}(s) \), it is necessary and sufficient to understand the structure of \( W(\lambda), W^\circ(\lambda) \) and \( A_G(\lambda) \).

**Lemma 3.2.** Let \( \lambda \in \mathbb{Z}(\mathbf{p}) \otimes_{\mathbb{Z}} \mathbb{Z}(\mathbf{T}_{sc}) \subset \mathbf{V} \) and let \( s = i_T(\lambda) \). Then

(a) \( o_G(\lambda) \) is the order of \( s \);
(b) $\Phi(\lambda) = \Phi(s)$ so $W^\circ(\lambda) = W^\circ(s)$.
(c) $W_G(\lambda) = W(s)$ and, if $\Phi^+(\lambda) = \Phi^+(s)$, then $A_G(\lambda) = A(s)$.

By analogy, we say that $\lambda$ is $G$-isolated (respectively $G$-quasi-isolated) if $W^\circ(\lambda)$ (respectively $W(\lambda)$) is not contained in a proper parabolic subgroup of $W$.

Let $W_{\text{aff}} = W \ltimes Y(T_{sc})$ denote the affine Weyl group of $\Phi$. If $\lambda \in V$, we set

$$W_{\text{aff}}(\lambda) = \{ w \in W_{\text{aff}} \mid w(\lambda) = \lambda \}.$$

For the proof of the next proposition, see [DM, Lemme 13.14 and Remark 13.15 (i)] and [Bou, Chapter VI, §2, Exercise 1].

**Proposition 3.3.** Let $\lambda \in V$. Then

(a) $W_{\text{aff}}(\lambda)$ is generated by affine reflections. Its image in $W$ is $W^\circ(\lambda)$.
(b) $W^\circ(\lambda)$ is the kernel of the map $W_G(\lambda) \to Y(T)/Y(T_{sc})$, $w \mapsto w(\lambda) - \lambda + Y(T_{sc})$.
(c) The exponent of $A_G(\lambda)$ divides $o_G(\lambda)$.

**Remark** - By Proposition 1.5, by Lemma 3.2 and by Proposition 3.3 we get that the centralizer of a semisimple element in a simply connected group is connected (Steinberg’s Theorem).

**3.B. Affine Dynkin diagram.** We recall here some results from [Bou, Chapter VI, §2] concerning the affine Dynkin diagram associated to a root system. We denote by $\Phi_1, \Phi_2, \ldots, \Phi_r$ the distinct irreducible components of $\Phi$.

Let us fix $i \in \{1, 2, \ldots, r\}$. Let $V_i = \mathbb{Q} \otimes \mathbb{Z} < \Phi_i >$. Let $W_i$ denote the Weyl group of $\Phi_i$. We set $\Delta_i = \Delta \cap \Phi_i$, $\Phi_i^+ = \Phi_i^+ \cap \Phi_i$. Then $V_i = \bigoplus_{\alpha \in \Delta_i} \mathbb{Q} \gamma^\vee_\alpha$. We denote by $\tilde{\alpha}_i$ the highest root of $\Phi_i$ (with respect to the height defined by $\Delta_i$). We write

$$\tilde{\alpha}_i = \sum_{\alpha \in \Delta_i} n_{\alpha} \alpha,$$

where the $n_{\alpha}$ are non-zero natural numbers ($\alpha \in \Delta_i$). By convention, we set $\gamma^\vee_{\tilde{\alpha}_i} = 0$, $n_{\tilde{\alpha}_i} = 1$.

Let $\tilde{\Delta}_i = \Delta \cup \{-\tilde{\alpha}_i\}$, $\Delta_{i, \text{min}} = \{ \alpha \in \Delta_i \mid n_\alpha = 1 \}$ and $\tilde{\Delta}_{i, \text{min}} = \Delta_{i, \text{min}} \cup \{-\tilde{\alpha}_i\}$. If $\alpha \in \Delta_{i, \text{min}}$, we denote by $\Phi_\alpha$ the parabolic subsystem of $\Phi_i$ with basis $\Delta_i - \{\alpha\}$ (for instance, $\Phi_{-\tilde{\alpha}_i} = \Phi_i$) and we set $\Phi^+_\alpha = \Phi_i^+ \cap \Phi_\alpha$. Let $W_\alpha$ denote the Weyl group of the root system $\Phi_\alpha$ and $w_\alpha$ its unique element such that $w_\alpha(\Phi_{\tilde{\alpha}_i}^+) = -\Phi_{\tilde{\alpha}_i}^+$. We set $z_\alpha = w_\alpha w_{-\tilde{\alpha}_i} \in W_i$ (note that $z_{-\tilde{\alpha}_i} = 1$) and

$$\text{Aut}_{W_i}(\tilde{\Delta}_i) = \{ z \in W_i \mid z(\tilde{\Delta}_i) = \tilde{\Delta}_i \}.$$

By [Bou, chapter VI, §2, Proposition 6], we have

$$\text{Aut}_{W_i}(\tilde{\Delta}_i) = \{ z_\alpha \mid \alpha \in \tilde{\Delta}_{i, \text{min}} \}.$$

If $\alpha \in \Delta_i$, we set $m_\alpha = 0$. We also set $m_{-\tilde{\alpha}_i} = -1$. Now, let $C_i$ denote the alcove

$$C_i = \{ \lambda \in V_i \mid \forall \alpha \in \Delta_i, \ < \alpha, \lambda > \geq m_\alpha \}$$

and $C_i = \{ \lambda \in V_i \mid (\forall \alpha \in \Delta_i, \ < \alpha, \lambda > \geq 0) \text{ and } < \tilde{\alpha}_i, \lambda > \leq 1 \}.$

Then $C_i$ is a fundamental domain for the action of the affine Weyl group $W_{i, \text{aff}} = W_i \ltimes < \Phi_i^\vee >$ on $V_i$. Moreover, $C_i$ is a closed simplex with vertices $(\gamma^\vee_\alpha/n_\alpha)_{\alpha \in \Delta_i}$.

With the above notation, we have:

$$W = W_1 \times W_2 \times \cdots \times W_r,$$

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

$$Y(T_{sc}) = \bigoplus_{i=1}^r \left( V_i \cap Y(T_{sc}) \right)$$

and

$$Y(T_{ad}) = \bigoplus_{i=1}^r \left( V_i \cap Y(T_{ad}) \right).$$
We set \( \tilde{\Delta} = \tilde{\Delta}_1 \cup \tilde{\Delta}_2 \cup \cdots \cup \tilde{\Delta}_r \). Now, let
\[
\mathcal{A} = \{ z \in W \mid z(\tilde{\Delta}) = \tilde{\Delta} \}.
\]
In other words, \( \mathcal{A} \) is the automorphism group of the affine Dynkin diagram of \( G \) induced by an element of \( W \). We have
\[
\mathcal{A} = \text{Aut}_{W_1}(\tilde{\Delta}_1) \times \text{Aut}_{W_2}(\tilde{\Delta}_2) \times \cdots \times \text{Aut}_{W_r}(\tilde{\Delta}_r).
\]
If \( z = (z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_r}) \in \mathcal{A} \), with \( \alpha_i \in \tilde{\Delta}_{i,\text{min}} \), we set
\[
\varpi^\vee(z) = \varpi^\vee_{\alpha_1} + \varpi^\vee_{\alpha_2} + \cdots + \varpi^\vee_{\alpha_r}.
\]
Finally, let
\[
\mathcal{C} = \{ \lambda \in V \mid \forall \alpha \in \tilde{\Delta}, \langle \alpha, \lambda \rangle \geq m_\alpha \}
\]
Then \( \mathcal{C} \) is a fundamental domain for the action of \( W_{\text{aff}} \) in \( V \). Then, by [Bou, Chapter VI, §2], we have, for every \( z \in \mathcal{A} \),
\[
(3.5) \quad z(\mathcal{C}) + \varpi^\vee(z) = \mathcal{C}
\]
and the map
\[
(3.6) \quad \varpi^\vee : \mathcal{A} \rightarrow Y(T_{\text{ad}})/Y(T_{\text{sc}})
\]
\[
z \mapsto \varpi^\vee(z) + Y(T_{\text{sc}})
\]
is an isomorphism of groups. If \( z = (z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_r}) \in \mathcal{A} \), with \( \alpha_i \in \tilde{\Delta}_{i,\text{min}} \), and if \( \alpha \in \tilde{\Delta}_i \), then
\[
(3.7) \quad n_{z(\alpha)} = n_\alpha
\]
and
\[
(3.8) \quad z(\frac{1}{n_\alpha}\varpi^\vee_{\alpha}) + \varpi^\vee_{\alpha} = \frac{1}{n_\alpha}\varpi^\vee_{z(\alpha)}.
\]
Since we will be working with the affine Weyl group of \( W_{\text{aff}} \), it will be convenient to work with “affine coordinates”. More precisely, if \( \lambda \in V \), we will denote by \( (\lambda_\alpha)_{\alpha \in \tilde{\Delta}} \) the unique family of rational numbers such that
\[
(1) \quad \forall i \in \{1, 2, \ldots, r\}, \sum_{\alpha \in \tilde{\Delta}_i} \lambda_\alpha = 1 ;
\]
\[
(2) \quad \lambda = \sum_{\alpha \in \tilde{\Delta}} \frac{\lambda_\alpha}{n_\alpha} \varpi^\vee_{\alpha}.
\]
Note that \( \lambda \in \mathcal{C} \) if and only if \( \lambda_\alpha \geq 0 \) for every \( \alpha \in \tilde{\Delta} \). Then, we have, for every \( \alpha \in \tilde{\Delta} \),
\[
(3.9) \quad \langle \alpha, \lambda \rangle = \frac{\lambda_\alpha}{n_\alpha} + m_\alpha.
\]
**Proof of 3.9** - Recall that \( m_\alpha \) has been defined in §3.A. If \( \alpha \in \Delta \), then \( m_\alpha = 0 \) and, by (3), \( \langle \alpha, \lambda \rangle = \lambda_\alpha/n_\alpha \). On the other hand, if \( \alpha \in \tilde{\Delta} - \Delta \), then \( m_\alpha = -1 \) and there exists a unique \( i \in \{1, 2, \ldots, r\} \) such that \( \alpha = -\alpha_i \). Therefore, by (2) and (3),
\[
\langle \alpha, \lambda \rangle = -\sum_{\beta \in \Delta_i} \lambda_\beta = \lambda_\alpha - 1.
\]
Moreover, it follows from 3.8 that, for every \( z \in \mathcal{A} \),
\[
(3.10) \quad z(\lambda) + \varpi^\vee(z) = \sum_{\alpha \in \tilde{\Delta}} \frac{\lambda_{z^{-1}(\alpha)} - 1}{n_\alpha} \varpi^\vee_{\alpha}
\]
In other words, \( (z(\lambda) + \varpi^\vee(z))_\alpha = \lambda_{z^{-1}(\alpha)} \).

**3.C. Orbits under the action of \( W \ltimes Y(T) \)**. Let \( \mathcal{A}_G \) be the subgroup of \( \mathcal{A} \) defined to be the inverse image of \( Y(T)/Y(T_{\text{sc}}) \) under the isomorphism \( \varpi^\vee \). Since \( \mathcal{C} \) is a fundamental domain for the action of \( W_{\text{aff}} \), it will be interesting to understand whenever two elements of \( \mathcal{C} \) are in the same orbit under \( W \ltimes Y(T) \). The answer is given by the following proposition.
Proposition 3.11. Let \( \lambda \) and \( \mu \) be two elements of \( \mathcal{C} \) and let \( w \in W \). If \( \mu - w(\lambda) \in Y(T) \), then there exists \( \omega_\alpha \in W^0(\lambda) \) and \( z \in A_G \) such that \( w = zw_\alpha \). Moreover, if \( d \) is a common multiple of \( o(\lambda) \) and \( o(\mu) \), then \( z^d = 1 \).

**Proof** - Assume that \( \mu - w(\lambda) \in Y(T) \). Then there exists \( z \in A_G \) and \( u \in Y(T_{sc}) \) such that \( w(\lambda) - \mu = -\omega_\alpha(z) + u \). But, \( (\tau_{-u}\mu)(\lambda) = \mu - \omega_\alpha(z) \in C - \omega_\alpha(z) = z(C) \) (see 3.5). Therefore, \( (z^{-1}\tau_{-u}w)(\lambda) \in C \). Since \( C \) is a fundamental domain for the action of \( W_{aff} \) on \( V \) and since \( z^{-1}\tau_{-u}w \in W_{aff} \), we deduce that \( z^{-1}\tau_{-u}w(\lambda) = \lambda \). So, by Proposition 3.3 (a), \( z^{-1}w \in W^0(\lambda) \), as expected.

For the last assertion, note that the hypothesis implies that \( d(w(\lambda) - \mu) \in Y(T_{sc}) \). Therefore, \( d\omega_\alpha(z) \in Y(T_{sc}) \). Since the map 3.6 is an isomorphism, we get that \( z^d = 1 \). \( \blacksquare \)

Corollary 3.12. Let \( \lambda \) and \( \mu \) be two elements of \( \mathcal{C} \). Then the following assertions are equivalent:

1. \( \lambda \) and \( \mu \) are in the same \( W \rtimes Y(T) \)-orbit.
2. There exists \( z \in A_G \) such that \( z(\lambda) - \mu \in Y(T) \).

**Proof** - Clear. \( \blacksquare \)

3.D. The group \( W_G(\lambda) \). Let us now come back to the aim of this section, namely the description of the group \( W_G(\lambda) \). Since \( C \) is a fundamental domain for the action of \( W_{aff} \) in \( V \), it is sufficient to understand the structure of \( W_G(\lambda) \) whenever \( \lambda \in C \).

Proposition 3.13. Let \( \lambda \in \mathcal{C} \). We set \( I_\lambda = \{ \alpha \in \tilde{\Delta} \mid \lambda_\alpha = 0 \} = \{ \alpha \in \tilde{\Delta} \mid < \alpha, \lambda > = m_\alpha \} \). Then:

(a) \( I_\lambda \) is a basis of \( \Phi(\lambda) \).

(b) If \( \Phi^+(\lambda) \) is the positive root system of \( \Phi(\lambda) \) associated to the basis \( I_\lambda \), then

\[ A^G(\lambda) = \{ z \in A_G \mid \forall \alpha \in \tilde{\Delta}, \lambda_{z(\alpha)} = \lambda_\alpha \} \]

**Proof** - (a) For \( \alpha \in \tilde{\Delta} \), let \( H_\alpha = \{ v \in V \mid < \alpha, v > = m_\alpha \} \). Then \( (H_\alpha)_{\alpha \in \tilde{\Delta}} \) is the family of walls of the alcove \( C \). Moreover, \( W_{aff} \) is generated by the affine reflections with respect to the walls of \( C \) which contains \( \lambda \) (see [Bou, ??]). Therefore, \( W^0(\lambda) \) is generated by the reflections \( (s_\alpha)_{\alpha \in I_\lambda} \). Since \( < \alpha, \beta > = 0 \) for every \( \alpha, \beta \in \tilde{\Delta} \), this implies that \( I_\lambda \) is a basis of \( \Phi(\lambda) \).

(b) Let \( A = \{ z \in A_G \mid \forall \alpha \in \tilde{\Delta}, \lambda_{z(\alpha)} = \lambda_\alpha \} \). Then \( A \) stabilizes \( I_\lambda \) by construction and, for every \( z \in A \), \( z(\lambda) - \lambda = \omega_\alpha(z)Y(T) \) by 3.10. So \( A \subset A_G(\lambda) \). Let us prove now the reverse inclusion.

First, let us prove that \( A_G(\lambda) \subset A_G \). Let \( z \in A_G(\lambda) \). By Proposition 3.11, there exists \( a \in A_G \) and \( w^0 \in W^0(\lambda) \) such that \( z = aw^0 \). So \( a \in W(\lambda) \) and \( a(I_\lambda) \subset \Delta \). In particular, \( a(I_\lambda) \subset \Phi(\lambda) \cap \tilde{\Delta} \). But \( \Phi(\lambda) \cap \tilde{\Delta} = I_\lambda \) by (a). So \( a(I_\lambda) = I_\lambda \). Moreover, \( z(I_\lambda) = I_\lambda \) by definition of \( A_G(\lambda) \). So \( w^0(I_\lambda) = I_\lambda \) and \( w^0 \in W^0(\lambda) \), which implies that \( w^0 = 1 \), that is \( z = a \). This shows that \( z \in A_G \).

Now, by 3.10, we have

\[ z(\lambda) - \lambda = \omega_\alpha(z) = \sum_{i=1}^r \left( \sum_{\alpha \in \Delta} \frac{\lambda_{z^{-1}(\alpha)} - \lambda_\alpha}{n_\alpha} \omega_\alpha \right) \in Y(T) \subset Y(T_{ad}) \]

Since \( z \) stabilizes \( I_\lambda \), we have, for every \( \alpha \in \Delta \),

\[ \lambda_{z^{-1}(\alpha)} - \lambda_\alpha = 0 \Rightarrow \lambda_{z^{-1}(\alpha)} = \lambda_\alpha = 0 \]

Moreover, \( 0 \leq \lambda_\alpha \leq 1 \). Therefore, \( (\lambda_{z^{-1}(\alpha)} - \lambda_\alpha)/n_\alpha = 1/n_\alpha, 1 \). Moreover, \( (\omega_\alpha)_{\alpha \in \Delta} \) is a \( \mathbb{Z} \)-basis of \( Y(T_{ad}) \). So \( \lambda_{z^{-1}(\alpha)} = \lambda_\alpha \) for every \( \alpha \in \Delta \). Then, by condition (1), \( \lambda_{z^{-1}(\alpha)} = \lambda_\alpha \) for every \( \alpha \in \tilde{\Delta} \). \( \blacksquare \)

Remark 3.14 - Keep the notation of Proposition 3.13. Then it may happen that \( A_G(\lambda) \) is strictly contained in the stabilizer of \( I_\lambda \) in \( A_G \). Take for instance \( G = \text{PGL}_2(F) \) and \( \lambda = \omega_\alpha/3 \) where \( \alpha \) is the unique simple root of \( G \). \( \square \)
REM 3.15 - If $\lambda \in \mathcal{C}$, note that $I_\lambda \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1, 2, \ldots, r\}$. □

If $\lambda \in \mathcal{C}$, we will choose for $\Phi^+(\lambda)$ the positive root subsystem of $\Phi(\lambda)$ associated to the basis $I_\lambda$.

4. Classification of quasi-isolated elements

4.A. A characterization of quasi-isolated elements. If $I$ is a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1, 2, \ldots, r\}$, we denote by $\Phi_I$ the root subsystem of $\Phi$ with basis $I$ and by $W_I$ the Weyl group of $\Phi_I$. It must be noticed that $W_I$ is not necessarily a parabolic subgroup of $W$. The Proposition 3.13 shows that, whenever $\lambda \in \mathcal{C}$, $W_G(\lambda) = A \ltimes W_{I_\lambda}$ for some subgroup $A$ of $A$ stabilizing $I_\lambda$. To determine if such a subgroup is contained or not in a proper parabolic subgroup of $W$, we need to determine the dimension of its fixed-points space. This is done in general in the next lemma.

**Lemma 4.1.** Let $I$ be a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1, 2, \ldots, r\}$ and let $A$ be a subgroup of $A$ stabilizing $I$. Let $r'$ denote the number of orbits of $A$ in $\tilde{\Delta} - I$. Then $\dim Q V^{A \ltimes W_I} = r' - r$.

**Proof -** By taking direct products, we may assume that $\Phi$ is irreducible or, in other words, that $r = 1$. Let $V_I = \mathbb{Q} \otimes \mathbb{Z} < \Phi_I >$ and let $E_I$ be the orthogonal of $I$ in $V$. Then $V = V_I \oplus E_I$ and $AW_{I'}$ stabilizes $V_I$ and $E_I$. Moreover,

$$\{ v \in V_I \mid \forall w \in W_I, w(v) = v \} = \{ 0 \}$$

and $W_I$ acts trivially on $E_I$. Consequently,

$$V^{A \ltimes W_I} = E_I^A.$$ 

Let $\mathbb{Q}[\tilde{\Delta} - I]$ denote the $\mathbb{Q}$-vector space with basis $(e_\alpha)_{\alpha \in \tilde{\Delta} - I}$. This is a permutation $A$-module. Let $f : \mathbb{Q}[\tilde{\Delta} - I] \rightarrow E_I$ the $\mathbb{Q}$-linear map sending $e_\alpha$ on the projection of $\alpha$ in $E_I$ (for every $\alpha \in \tilde{\Delta} - I$). Then $f$ is a morphism of $QA$-modules, whose kernel has dimension 1 (because $|\tilde{\Delta}| = \dim V + 1$).

Let $M = \{ \mathbb{Q}[\tilde{\Delta} - I] \mid \forall v \in A, z(v) = v \}$. Then

$$\dim Q E_I^A = \dim Q M - \dim Q (M \cap \text{Ker } f).$$

Since $\dim Q M = r'$, we only need to show that $A$ acts trivially on $\text{Ker } f$. But $\sum_{\alpha \in \tilde{\Delta}} n_\alpha \alpha = 0$. So, by projection on $E_I$, we get that $\text{Ker } f$ is generated by $\sum_{\alpha \in \tilde{\Delta} - I} n_\alpha e_\alpha$. By equality 3.7, this element is invariant under the action of $A$. This completes the proof of Lemma 4.1. □

**Corollary 4.2.** Let $I$ be a subset of $\tilde{\Delta}$ such that $I \cap \tilde{\Delta}_i \neq \tilde{\Delta}_i$ for every $i \in \{1, 2, \ldots, r\}$ and let $A$ be a subgroup of $A$ stabilizing $I$. Then $AW_I$ is not contained in a proper parabolic subgroup of $W$ if and only if $A$ acts transitively on $\tilde{\Delta}_i - I$ for every $i \in \{1, 2, \ldots, r\}$.

**Proof -** This follows immediately from Proposition 4.1. □

**Corollary 4.3.** Let $\lambda \in \mathcal{C}$. Then :

(a) $\lambda$ is $G$-isolated if and only if $|\tilde{\Delta}_i - I_\lambda| = 1$ for every $i \in \{1, 2, \ldots, r\}$.

(b) $\lambda$ is $G$-quasi-isolated if and only if $A_G(\lambda)$ acts transitively on $\tilde{\Delta}_i - I_\lambda$ for every $i \in \{1, 2, \ldots, r\}$. 
4.B. Classification of quasi-isolated elements in $V$. We are now ready to complete the classification of conjugacy classes of $G$-quasi-isolated elements in $V$. Let $Q(G)$ denote the set of subsets $\Omega$ of $\tilde{\Delta}$, such that, for every $i \in \{1, 2, \ldots, r\}$, $\Omega \cap \tilde{\Delta}_i \neq \emptyset$ and the stabilizer of $\Omega_i$ in $A_G$ acts transitively on $\tilde{\Delta}_i$. If $\Omega$ is such a subset, we set

$$
\lambda_\Omega = \sum_{i=1}^{r} \left( \frac{1}{n_i(\Omega)} \lambda_{\Omega \cap \tilde{\Delta}_i} \right),
$$

where $n_i(\Omega)$ is equal to $n_\alpha$ for every $\alpha \in \Omega \cap \tilde{\Delta}_i$ (see equality 3.7). Note that $A_G$ acts on $Q(G)$. Moreover, by 3.8, we have, for every $z \in A_G$,

$$
z(\lambda_\Omega) + \varpi(\varpi(z)) = \lambda_z(\Omega).
$$

Finally, we denote by $o_G^G(\varpi)$ the number $o_G(\varpi)$ where $\alpha \in \Omega \cap \tilde{\Delta}_i$. Note that this number is constant on $\Omega \cap \tilde{\Delta}_i$. All the work done in this section shows that :

\begin{itemize}
  \item[(a)] The map $Q(G) \rightarrow \mathcal{C}, \Omega \mapsto \lambda_\Omega$ induces a bijection between the set of orbits of $A_G$ in $Q(G)$ and the set of $\mathcal{W} \times Y(\mathfrak{T})$-orbits of quasi-isolated elements in $V$.
  \item[(b)] Let $\Omega \in Q(G)$. Then :
    \begin{itemize}
      \item[(a)] $\mathcal{W}^\gamma(\lambda_\Omega) = W_{\tilde{\Delta} - \Omega}$ ;
      \item[(b)] $A_G(\lambda_\Omega) = \{ z \in A_G \mid z(\Omega) = \Omega \}$ ;
      \item[(c)] $o_G^G(\varpi)$ is the lowest common multiple of $(n_i(\Omega) o_G^G(\varpi) | \Omega \cap \tilde{\Delta}_i) | i \leq r$ ;
    \end{itemize}
    \item[(d)] $\lambda_\Omega$ is $G$-isolated if and only if $|\Omega_i| = 1$ for every $i \in \{1, 2, \ldots, r\}$.
\end{itemize}

4.C. Classification of quasi-isolated semisimple elements. Let $\tilde{\Delta}_{p'}$ denote the subset of elements $\alpha \in \tilde{\Delta}$ such that $\varpi/n_\alpha \in \mathbb{Z}_p \otimes \mathbb{Z} Y(\mathfrak{T}_{sc})$. Let $Q(G)_{p'}$ denote the set of $\Omega \in Q(G)$ such that $\Omega \subset \tilde{\Delta}_{p'}$ and, for every $i \in \{1, 2, \ldots, r\}$, $p$ does not divide $|\Omega \cap \tilde{\Delta}_i|$. If $\Omega \in Q(G)_{p'}$, we set $t_\Omega = \tilde{\tau}(\lambda_\Omega) \in \mathfrak{T}$.

\begin{itemize}
  \item[(a)] The map $Q(G)_{p'} \rightarrow \mathcal{T}, \Omega \mapsto t_\Omega$ induces a bijection between the set of orbits of $(A_G)_{p'}$ in $Q(G)_{p'}$ and the set of conjugacy classes of quasi-isolated semisimple elements in $G$.
  \item[(b)] If $\Omega \in Q(G)_{p'}$ then :
    \begin{itemize}
      \item[(a)] $\mathcal{W}^\gamma(t_\Omega) = W_{\tilde{\Delta} - \Omega}$ ;
      \item[(b)] $A_G(t_\Omega) = \{ z \in A_G \mid z(\Omega) = \Omega \}$ ;
      \item[(c)] $o(t_\Omega)$ is the lowest common multiple of $(n_i(\Omega) o_G(\varpi) | \Omega \cap \tilde{\Delta}_i) | i \leq r$ ;
    \end{itemize}
    \item[(d)] $t_\Omega$ is $G$-isolated if and only if $|\Omega_i| = 1$ for every $i \in \{1, 2, \ldots, r\}$.
\end{itemize}

\begin{proof}
By Theorem 4.5 and Lemma 3.2, it is enough to show that the map $Q(G)_{p'} \rightarrow \mathcal{C}, \Omega \mapsto \lambda_\Omega$ induces a bijection between the set of orbits of $(A_G)_{p'}$ in $Q(G)_{p'}$ to the set of $W \times Y(\mathfrak{T})$-orbits of quasi-isolated elements $\lambda$ in $V$ such that $p$ does not divide $o_{sc}(\lambda)$. But this follows from Theorem 4.5 (b) (γ) and the last assertion of Proposition 3.11.
\end{proof}

Remark 4.7 - We recall that the prime number $p$ is said to be very good for $G$ if it does not divide the numbers $n_\alpha$ ($\alpha \in \Delta$) and $|A| = |Y(\mathfrak{T}_{sc})|/|Y(\mathfrak{T}_{ad})|$. We say here that $p$ is almost very good for $G$ if it does not divide the numbers $n_\alpha$ ($\alpha \in \Delta$) and $|A_G| = |Y(\mathfrak{T})|/|Y(\mathfrak{T}_{ad})|$. If $p$ is very good, then it is almost very good.

If $p$ is almost very good, then $\tilde{\Delta}_{p'} = \tilde{\Delta}$ and $(A_G)_{p'} = A_G$ so the set of $W \times Y(\mathfrak{T})$-orbits of $G$-quasi-isolated elements in $V$ is in natural bijection with the set of conjugacy classes of quasi-isolated semisimple elements in $G$ (through the map $\tilde{\tau}$). \qed
Example 4.8 - If all the irreducible components of \( \Phi \) are of type \( B, C \) on \( D \) and if \( p = 2 \), then \( \tilde{\Delta}_{p'} = \{ -\tilde{\alpha}_1, -\tilde{\alpha}_2, \ldots, -\tilde{\alpha}_r \} \). Therefore, 1 is the unique quasi-isolated element in \( G \). \( \Box \)

4.D. Simply connected groups. If \( G \) is simply connected then \( A_G = \{ 1 \} \). Therefore, we retrieve the well-known classification of isolated semisimple elements in \( G \):

**Proposition 4.9.** Assume that \( G \) is semisimple and simply connected. Then the map \( \tilde{\Delta}_{1,p'} \times \cdots \times \tilde{\Delta}_{r,p'} \rightarrow G, (\alpha_1, \ldots, \alpha_r) \mapsto \prod_{i=1}^r \tilde{\omega}_{\alpha_i} / n_{\alpha_i} \) induces a bijection between \( \tilde{\Delta}_{1,p'} \times \cdots \times \tilde{\Delta}_{r,p'} \) and the set of conjugacy classes of (quasi-)isolated elements in \( G \).

Example 4.10 - Assume here that \( p \neq 2 \) and that \( G = \text{Sp}(V) \) where \( V \) is an even-dimensional vector space endowed with a non-degenerate alternating form. Let \( \dim V = 2n \). Then \( G \) is simply connected, so \( A_G = \{ 1 \} \). Moreover, \( \tilde{\Delta}_{p'} = \tilde{\Delta} \). Let us write \( \alpha_0 = -\tilde{\alpha}_1 \) and let us number the affine Dynkin diagram \( \tilde{\Delta} \) of \( G \) as follows

\[
\begin{array}{cccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots \ldots & \alpha_{n-1} & \alpha_n \\
1 & 2 & 2 & \ldots & 2 & 1
\end{array}
\]

The natural numbers written inside the node \( \alpha_i \) is the number \( n_{\alpha_i} \). For \( 0 \leq i \leq n \), let \( \Omega_i = \{ \alpha_i \} \) and let \( t_i = \tilde{i}_p(\tilde{\omega}_{\alpha_i} / n_{\alpha_i}) \). Then \( \{ t_i \mid 0 \leq i \leq n \} \) is a set of representatives of conjugacy classes of isolated (i.e. quasi-isolated) elements in \( G \). Note that \( t_i \) is characterized by the following two properties:

\[ t_i^2 = 1 \quad \text{and} \quad \dim \ker(t_i + \text{Id}_V) = i. \]

This shows that an element \( s \) is isolated in \( G \) if and only if \( s^2 = 1 \). Finally, note that \( C_G(t_i) = \text{Sp}_{2i}(\mathbb{F}) \times \text{Sp}_{2(n-i)}(\mathbb{F}) \).

4.E. Special orthogonal groups. The case of special orthogonal groups in characteristic 2 has been treated in Example 4.8. In this subsection, we study the case of special orthogonal groups in good characteristic. We first adopt a naive point-of-view, using the natural representation of special orthogonal groups. At the end of this subsection, we will explain the link between this point-of-view and Theorem 4.6.

**Hypothesis:** Let us assume in this subsection, and only in this subsection, that \( p \neq 2 \) and that \( G = \text{SO}(V, \langle \cdot, \cdot \rangle) = \text{SO}(V) \) where \( V \) is a finite dimensional vector space over \( \mathbb{F} \) and \( \langle \cdot, \cdot \rangle \) is a non degenerate symmetric bilinear form on \( V \).

We denote by \( n \) the rank of \( G \). Then \( n = \left\lfloor \frac{\dim V}{2} \right\rfloor \), except whenever \( \dim V = 2 \) (in this case, \( n = 0 \)). If \( s^2 = 1 \), then \( \dim \ker(s + \text{Id}_V) \equiv 0 \mod 2 \) so \( \dim \ker(s - \text{Id}_V) \equiv \dim V \mod 2 \).

**Proposition 4.11.** With this hypothesis, we have:

(a) \( s \) is quasi-isolated if and only if \( s^2 = 1 \).

(b) If \( s^2 = 1 \), then \( s \) is isolated if and only if \( \dim \ker(s - \varepsilon \text{Id}_V) \neq 1 \) for every \( \varepsilon \in \{ 1, -1 \} \).

**Proof** - Assume first that there exists an eigenvalue \( \zeta \) of \( s \) such that \( \zeta^2 \neq 1 \). Let \( V_\zeta \) denote the \( \zeta \)-eigenspace of \( s \) in \( V \). Let \( E \) be the orthogonal subspace to \( V_\zeta \oplus V_{\zeta^{-1}} \). We have

\[ V = V_\zeta \oplus V_{\zeta^{-1}} \oplus E, \]

and this is an orthogonal decomposition. Therefore, the centralizer of \( s \) in \( G \) is contained in \( G \cap (\text{GL}(V_\zeta) \times \text{GL}(V_{\zeta^{-1}}) \times \text{GL}(E)) \), which is a Levi subgroup of a proper parabolic subgroup of \( G \). So \( s \) is not quasi-isolated.
Assume now that $s^2 = 1$. Then $V = V_1 \oplus V_{-1}$ and this decomposition is orthogonal. So

$$C_G(s) = (O(V_1) \times O(V_{-1})) \cap G \quad \text{et} \quad C_G^s(s) = SO(V_1) \times SO(V_{-1}).$$

So $s$ is quasi-isolated and it is isolated if and only if $\dim V_1 \neq 1$ and $\dim V_{-1} \neq 1$. ■

**Corollary 4.14.** Keep the hypothesis of this subsection. If $0 \leq i \leq n$, let $t_i$ denote a semisimple element of $G$ such that $t_i^2 = 1$ and $\dim \ker(t_i + \text{Id}_V) = 2i$. Then $\{t_i \mid 0 \leq i \leq n\}$ is a set of representatives of conjugacy classes of quasi-isolated elements in $G$. Moreover, $t_i$ is isolated if and only if $i \notin \{1, (\dim V)/2 - 1\}$.

Let us now compare the description given by Corollary 4.14 and the one given by Theorem 4.6. Since $p \neq 2$, we have $\tilde{\Delta}_{p'} = \tilde{\Delta}$ and $A_{p'} = A$ (indeed, $p$ is very good for $G$). For getting a uniform description, we assume that $\dim V \notin \{1, 2, 3, 4, 6\}$ (whenever $\dim V \in \{1, 2, 3, 4, 6\}$, then the reader can also check that Corollary 4.14 and Theorem 4.6 are still compatible!).

**4.E.1. Type B.** We assume here that $\dim V = 2n + 1$ and that $n \geq 2$. We set $\alpha_0 = -\tilde{\alpha}_1$. Then $A_G = A$ is of order 2. We denote by $\sigma$ its unique non-trivial element. We number the affine Dynkin diagram of $G$ as follows:

```
\sigma
\begin{array}{c}
\alpha_0 & 1 \\
\alpha_1 & 1 \\
\alpha_2 & 2 \\
\alpha_3 & 2 \\
\alpha_{n-1} & 2 \\
\alpha_n & 2 \\
\end{array}
```

The natural number written inside the node $\alpha_i$ is equal to $n_{\alpha_i}$. We have

$$\sigma = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-1}} s_{\alpha_n}.$$  

The action of $\sigma$ on $\tilde{\Delta}$ is given by the above diagram: $\sigma(\alpha_0) = \alpha_1$, $\sigma(\alpha_1) = \alpha_0$, $\sigma(\alpha_i) = \alpha_i$ for every $i \in \{2, 3, \ldots, n\}$.

Let $\Omega_0 = \{\alpha_0\}$, $\Omega_1 = \{\alpha_0, \alpha_1\}$ and, for $2 \leq i \leq n$, let $\Omega_i = \{\alpha_i\}$. Then $\Omega_i \in \mathbb{Q}(G)$. Moreover, one can check that $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ is a set of representatives of $A_G$-orbits in $\mathbb{Q}(G)$. Then

$$t_{\Omega_i}^2 = 1 \quad \text{and} \quad \dim \ker(t_{\Omega_i} + \text{Id}_V) = i.$$

This shows that $t_{\Omega_i} = t_i$ : we retrieve Corollary 4.14.

**4.E.2. Type D.** We assume here that $\dim V = 2n$ and that $n \geq 4$. We set $\alpha_0 = -\tilde{\alpha}_1$. Then $A_G$ is of order 2. We denote by $\sigma$ its unique non-trivial element. Note that $A_G \neq A$. We number the affine Dynkin diagram of $G$ as follows:

```
\sigma
\begin{array}{c}
\alpha_0 & 1 \\
\alpha_1 & 1 \\
\alpha_2 & 2 \\
\alpha_3 & 2 \\
\alpha_{n-2} & 2 \\
\alpha_n & 2 \\
\end{array}
```

The natural number written inside the node $\alpha_i$ is equal to $n_{\alpha_i}$. We have

$$\sigma = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n-2} (s_{\alpha_{n-1}} s_{\alpha_n}).$$

The action of $\sigma$ on $\tilde{\Delta}$ is given by the above diagram: $\sigma(\alpha_0) = \alpha_1$, that $\sigma(\alpha_1) = \alpha_0$ and that $\sigma(\alpha_i) = \alpha_i$ for every $i \in \{2, 3, \ldots, n-2\}$, $\sigma(\alpha_{n-1}) = \alpha_n$ and $\sigma(\alpha_n) = \alpha_{n-1}$.

Let $\Omega_0 = \{\alpha_0\}$, $\Omega_1 = \{\alpha_0, \alpha_1\}$, $\Omega_i = \{\alpha_i\}$ (for $2 \leq i \leq n-2$), $\Omega_{n-1} = \{\alpha_{n-1}, \alpha_n\}$ and $\Omega_n = \{\alpha_n\}$. Then $\Omega_i \in \mathbb{Q}(G)$. Moreover, one can check that $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ is a set of representatives of $A_G$-orbits in $\mathbb{Q}(G)$. Then

$$t_{\Omega_i}^2 = 1 \quad \text{and} \quad \dim \ker(t_{\Omega_i} + \text{Id}_V) = i.$$
Moreover, for every \( \alpha \in A \), we have \( o_G(\omega'_\alpha) = 1 \). Therefore, the Theorem 4.6 can be stated as follows:

**Theorem 5.1.** Assume that \( G \) is adjoint and simple. Then \( Q(G)_\alpha' \) is the set of subsets \( \Omega \) in \( \tilde{\Delta}_\alpha' \) which are acted on transitively by their stabilizer in \( A \). If \( \Omega \in Q(G)_\alpha' \), let \( n_\Omega \) denote the number \( n_\alpha \) (for some \( \alpha \in \Omega \)). Then:

\[
t_\Omega = \frac{1}{n_\Omega |\Omega|} \sum_{\alpha \in \Omega} \omega'_\alpha.
\]

We have:

(a) The map \( Q(G)_\alpha' \to T, \Omega \mapsto t_\Omega \) induces a bijection between the set of orbits of \( A_\alpha' \) in \( Q(G)_\alpha' \) and the set of conjugacy classes of quasi-isolated semisimple elements in \( G \).

(b) Let \( \Omega \in Q(G)_\alpha' \). Then:

(a) \( W^\alpha(t_\Omega) = W_{\Delta_\alpha} \); 
(b) \( A_G(t_\Omega) = \{ z \in A_G \mid z(\Omega) = \Omega \} \); 
(c) \( o(t_\Omega) = n_\Omega |\Omega| \); 
(d) \( s_\Omega \) is \( G \)-isolated if and only if \( |\Omega| = 1 \).

This implies that the set of conjugacy classes of isolated semisimple elements in \( G \) is in bijection with the set of orbits of \( A \) in \( \Delta_\alpha' \).

**5.A. Classification by use of the affine Dynkin diagram.** We first set some notation. We denote by \( \alpha_0 \) the root \( -\hat{\alpha}_1 \) (recall that \( r = 1 \)). Let \( n \) denote the rank of \( G \) (i.e. \( n = |\Delta| \)). We write \( \Delta = \{ \alpha_0, \alpha_2, \ldots, \alpha_n \} \). If \( 0 \leq i \leq n \), we set \( n_{\alpha_i} = n_i \), \( z_{\alpha_i} = z_i \) and \( \varpi^\vee_{\alpha_i} = \varpi^\vee_i \). Note that \( z_i(\alpha_0) = \alpha_i \). The Table I gives the list of all the affine Dynkin diagrams together with the structure of \( A \) (see [Bou, Planches I-IX]).

We give in Table II the classification of conjugacy classes of quasi-isolated elements in adjoint classical groups. In Table III, we deal with the adjoint groups of exceptional type \( E_6 \) and \( E_7 \). We have not included adjoint groups of type \( E_8 \), \( F_4 \) and \( G_2 \) since they are also simply connected. Therefore, Proposition 4.9, Theorem 5.1 and Table I gives easily all informations concerning the (quasi-)isolated elements for these groups.

**5.B. Explicit descriptions for adjoint classical groups.** The case of special orthogonal groups was done in subsection 4.E. Therefore, we only have to investigate adjoint classical groups of type \( A, C \) and \( D \).

**5.B.1. Type \( A \).** Assume here that \( \tilde{G} = GL_{n+1}(\mathbb{F}) \), that \( G = PGL_{n+1}(\mathbb{F}) \) and that \( \pi : \tilde{G} \to G \) is the canonical morphism (here, \( n \) is a non-zero natural number). Let \( I_{n+1} \) denote the identity matrix of \( GL_{n+1}(\mathbb{F}) \). If \( d \) is a non-zero natural number invertible in \( \mathbb{F} \), we denote by \( \zeta_d \) a primitive \( d \)-th root of unity in \( \mathbb{F}^\times \) and we set \( J_d = \text{diag}(1, \zeta_d, \zeta_d^2, \ldots, \zeta_d^{d-1}) \in GL_d(\mathbb{F}) \).
Now, let $\text{Div}_p(n + 1)$ denote the set of divisors of $n + 1$ which are invertible in $\mathbb{F}$. If $d \in \text{Div}_p(n + 1)$, let $\bar{s}_{n+1,d}$ denote the matrix $I_{\frac{n+1}{d}} \otimes J_d \in G$. We set $s_{n+1,d} = \pi(\bar{s}_{n+1,d})$.

Note that $A_p'$ is cyclic of order $n_p' = |\Delta_p'|$ and that it acts transitively on $\Delta_p'$. If $d \in \text{Div}_p(n + 1)$, we denote by $\Omega_{n+1,d}$ the orbit of $\alpha_0$ under the unique subgroup of order $d$ of $A : \Omega_{n+1,d} = \{\alpha_j(n+1)/d \mid 0 \leq j \leq d - 1\}$.

**Proposition 5.2.** If $G = \text{PGL}_n(\mathbb{F})$, then the map $\text{Div}_p(n + 1) \rightarrow G$, $d \mapsto s_{n+1,d}$ is a bijection between $\text{Div}_p(n + 1)$ and the set of conjugacy classes of quasi-isolated semisimple elements in $G$. Through the parametrization of Theorem 5.1, this corresponds to the map $\text{Div}_p(n + 1) \rightarrow \mathbb{Q}(G)_{p'}$, $d \mapsto \Omega_{n+1,d}$.

If $d \in \text{Div}_p(n + 1)$, then $s_{n+1,d}$ has order $d$, $W^o(s) \simeq (\mathbb{S}^d_{n+1/d})^d$ and $A(s) \simeq (\mathbb{Z}/d\mathbb{Z})$ acts on $W^o(s)$ by permutation of the components. Moreover, $s_{n+1,d}$ is isolated if and only if $d = 1$.

**5.2. Type C.** We assume here that $p \neq 2$. Let $V$ be a 2$n$-dimensional vector space over $\mathbb{F}$, with $n \geq 2$. Let $\beta : V \times V \rightarrow \mathbb{F}$ be a non-degenerate skew-symmetric bilinear form. We assume here that $\mathbb{G} = \text{Sp}(V, \beta)$, that $G = \mathbb{G}/\{\text{Id}_V, - \text{Id}_V\}$ and that $\pi : \mathbb{G} \rightarrow G$ is the canonical morphism.

**Proposition 5.3.** Let $\bar{s} \in \mathbb{G}$ be semisimple and let $s = \pi(\bar{s})$.

(a) If $s$ is quasi-isolated, then $s^3 = 1$.

(b) If $s^2 = 1$, then $s$ is isolated.

(c) If $s^3 = 1$ and $s^2 \neq 1$, then $s$ is quasi-isolated if and only if $\dim \ker(\bar{s} - \text{Id}_V) = \dim \ker(\bar{s} + \text{Id}_V)$.

**Proof** - (a) follows immediately from Corollary 2.11 and Example 4.10. (b) and (c) follow from direct computation.

If $0 \leq i \leq n/2$, let $\bar{t}_i$ be an element of $\mathbb{G}$ such that $\bar{t}_i^2 = 1$ and $\dim \ker(\bar{t}_i + \text{Id}_V) = 2i$. If $0 \leq i < n/2$, let $\bar{s}_i$ be an element of $\mathbb{G}$ of order 4 such that $\dim \ker(\bar{s}_i - \text{Id}_V) = \dim \ker(\bar{s}_i + \text{Id}_V) = i$. We set $t_i = \pi(\bar{t}_i)$ and $s_i = \pi(\bar{s}_i)$.

**Corollary 5.4.** The set $\{t_i \mid 0 \leq i \leq n/2\} \cup \{s_i \mid 0 \leq i < n/2\}$ is a set of representatives of quasi-isolated elements of $G$. The subset of $\Delta$ associated to $t_i$ (respectively $s_i$) through the parametrization of Theorem 5.1 is $\{\alpha_i\}$ (respectively $\{\alpha_i, \alpha_{n-i}\}$).

**5.3. Type D.** We assume here that $p \neq 2$. Let $V$ be a 2$n$-dimensional vector space over $\mathbb{F}$, with $n \geq 3$. Let $\beta : V \times V \rightarrow \mathbb{F}$ be a non-degenerate symmetric bilinear form. We assume here that $G = \text{SO}(V, \beta)$, that $G = \mathbb{G}/\{\text{Id}_V, - \text{Id}_V\}$ and that $\pi : \mathbb{G} \rightarrow G$ is the canonical morphism.

**Proposition 5.5.** Let $\bar{s} \in \mathbb{G}$ be semisimple and let $s = \pi(\bar{s})$.

(a) If $s$ is quasi-isolated, then $s^3 = 1$.

(b) If $s^2 = 1$, then $s$ is quasi-isolated. Moreover, $s$ is isolated if and only if $\dim \ker(\bar{s} - \text{Id}_V) \notin \{1, n - 1\}$.

(c) If $s^3 = 1$ and $s^2 \neq 1$, then $s$ is quasi-isolated if and only if $\dim \ker(\bar{s} - \text{Id}_V) = \dim \ker(\bar{s} + \text{Id}_V)$ and $\dim \ker(\bar{s} - \text{Id}_V) \neq 0$ if $n$ is odd.

**Proof** - (a) follows immediately from Corollary 2.11 and Proposition 5.5. (b) and (c) follow from direct computations.

If $0 \leq i \leq n/2$, let $\bar{t}_i$ be an element of $\mathbb{G}$ such that $\bar{t}_i^2 = 1$ and $\dim \ker(\bar{t}_i + \text{Id}_V) = 2i$. If $1 \leq i < n/2$, let $\bar{s}_i$ be an element of $\mathbb{G}$ of order 4 such that $\dim \ker(\bar{s}_i - \text{Id}_V) = \dim \ker(\bar{s}_i + \text{Id}_V) = i$. We set $t_i = \pi(\bar{t}_i)$ and $s_i = \pi(\bar{s}_i)$. Finally, there are two conjugacy classes of elements $\bar{s}$ of order 4 such that $\dim \ker(\bar{s} - \text{Id}_V) = \dim \ker(\bar{s} + \text{Id}_V) = 0$ (these two conjugacy classes are in correspondence through the non trivial automorphism of the Dynkin diagram of $G$) : we denote by $\bar{s}_0$ and $\bar{s}'_0$ some representatives of these two classes. We set $s_0 = \pi(\bar{s}_0)$ and $s'_0 = \pi(\bar{s}'_0)$. We set $E_n = \emptyset$ if $n$ is odd and $E_n = \{s_0, s'_0\}$ if $n$ is even.
Corollary 5.6. The set \( \{ t_i \} \cup \{ t_i \mid 2 \leq i \leq n/2 \} \) is a set of representatives of isolated elements of \( G \).

The subset of \( \hat{\Delta} \) associated to \( t_i \) through the parametrization of Theorem 5.1 is \( \{ \alpha_i \} \).

The set \( \{ t_i \} \cup \{ s_i \mid 1 \leq i < n/2 \} \cup E_n \) is a set of representatives of quasi-isolated but non-isolated elements of \( G \). Through the parametrization of Theorem 5.1, \( t_1 \) is associated to \( \{ \alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \} \) and \( s_1 \) is associated to \( \{ \alpha_1, \alpha_{n-1} \} \). If \( n \) is even, \( s_0 \) and \( s'_0 \) correspond to \( \{ \alpha_0, \alpha_{n-1} \} \) and \( \{ \alpha_0, \alpha_n \} \) (or conversely).

| Type of \( G \) | \( \hat{\Delta} \) | \( \Delta \) | \( |\Delta| \) |
|-----------------|-----------------|------------|----------|
| \( A_n \)       | ![Dynkin Diagram](image) |             |          |
| \( B_n \)       |                 |             |          |
| \( C_n \)       |                 |             |          |
| \( D_n \)       |                 |             |          |
| \( E_6 \)       |                 |             |          |
| \( E_7 \)       |                 |             |          |
| \( E_8 \)       |                 |             |          |
| \( F_4 \)       |                 |             |          |
| \( G_2 \)       |                 |             |          |

Table I. Affine Dynkin diagrams
| G    | \( \Omega \) | \( p \) | \( o(s_\Omega) \) | \( C_G(s_\Omega) \) | \( |A(s_\Omega)| \) | isolated ? |
|------|--------------|-------|-----------------|-----------------|-----------------|------------|
| \( A_n \) | \( \{\alpha_j(n+1)/d \mid 0 \leq j \leq d-1\} \) for \( d \mid n+1 \) | \( p \not\mid d \) | \( d \) | \( (A_{(n+1)/d} - 1)^d \) | \( d \) | iff \( d = 1 \) |
| \( B_n \) | \( \{\alpha_0\} \) | 1 | \( B_n \) | 1 | yes |
|      | \( \{\alpha_0, \alpha_1\} \) | \( p \not\equiv 2 \) | \( 2 \) | \( B_{n-1} \) | 2 | no |
|      | \( \{\alpha_d\}, 2 \leq d \leq n \) | \( p \not\equiv 2 \) | \( 2 \) | \( D_d \times B_{n-d} \) | 2 | yes |
| \( C_n \) | \( \{\alpha_0\} \) | 1 | \( C_n \) | 1 | yes |
|      | \( \{\alpha_d\}, 1 \leq d < n/2 \) | \( p \not\equiv 2 \) | \( 2 \) | \( C_d \times C_{n-d} \) | 1 | yes |
|      | \( \{\alpha_{n/2}\} \) (if \( n \) is even) | \( p \not\equiv 2 \) | \( 2 \) | \( C_{n/2} \times C_{n/2} \) | 2 | yes |
|      | \( \{\alpha_0, \alpha_n\} \) | \( p \not\equiv 2 \) | \( 2 \) | \( A_{n-1} \) | 2 | no |
|      | \( \{\alpha_d, \alpha_{n-d}\}, 1 \leq d < n/2 \) | \( p \not\equiv 2 \) | \( 4 \) | \( (B_d)^2 \times A_{n-2d-1} \) | 2 | no |
| \( D_n \) | \( \{\alpha_0\} \) | 1 | \( D_n \) | 1 | yes |
|      | \( \{\alpha_d\}, 2 \leq d < n/2 \) | \( p \not\equiv 2 \) | \( 2 \) | \( D_d \times D_{n-d} \) | 2 | yes |
|      | \( \{\alpha_{n/2}\} \) (if \( n \) is even) | \( p \not\equiv 2 \) | \( 4 \) | \( D_{n/2} \times D_{n/2} \) | 4 | yes |
|      | \( \{\alpha_d, \alpha_{n-d}\}, 2 \leq d < n/2 \) | \( p \not\equiv 2 \) | \( 4 \) | \( (D_d)^2 \times A_{n-2d-1} \) | 4 | no |
|      | \( \{\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n\} \) | \( p \not\equiv 2 \) | \( 4 \) | \( A_{n-3} \) | 4 | no |
|      | \( \{\alpha_0, \alpha_1\} \) | \( p \not\equiv 2 \) | \( 2 \) | \( D_{n-1} \) | 2 | no |
|      | \( \{\alpha_0, \alpha_{n-1}\} \) (if \( n \) is even) | \( p \not\equiv 2 \) | \( 2 \) | \( A_{n-1} \) | 2 | no |
|      | \( \{\alpha_0, \alpha_n\} \) (if \( n \) is even) | \( p \not\equiv 2 \) | \( 2 \) | \( A_{n-1} \) | 2 | no |

Table II. Quasi-isolated elements in adjoint classical groups
| $G$ | $\Omega$ | $p$ ? | $o(s_{11})$ | $C_G^\circ(s_{11})$ | $|A(s_{11})|$ | isolated ? |
|-----|---------|-----|---------|-------------|-------------|--------|
| $E_6$ | $\{\alpha_0\}$ | 1 | $E_6$ | 1 | yes |
|   | $\{\alpha_2\}$ | 2 | $A_5 \times A_1$ | 1 | yes |
|   | $\{\alpha_4\}$ | 3 | $A_2 \times A_2 \times A_2$ | 3 | yes |
|   | $\{\alpha_0, \alpha_1, \alpha_6\}$ | 3 | $D_4$ | 3 | no |
|   | $\{\alpha_2, \alpha_3, \alpha_5\}$ | 6 | $A_1 \times A_1 \times A_1 \times A_1$ | 3 | no |
| $E_7$ | $\{\alpha_0\}$ | 1 | $E_7$ | 1 | yes |
|   | $\{\alpha_1\}$ | 2 | $A_1 \times D_6$ | 1 | yes |
|   | $\{\alpha_2\}$ | 2 | $A_7$ | 2 | yes |
|   | $\{\alpha_3\}$ | 3 | $A_2 \times A_5$ | 1 | yes |
|   | $\{\alpha_4\}$ | 4 | $A_3 \times A_3 \times A_1$ | 2 | yes |
|   | $\{\alpha_0, \alpha_7\}$ | 2 | $E_6$ | 2 | no |
|   | $\{\alpha_1, \alpha_6\}$ | 4 | $D_4 \times A_1 \times A_1$ | 2 | no |
|   | $\{\alpha_3, \alpha_5\}$ | 6 | $A_2 \times A_2 \times A_2$ | 2 | no |

Table III. Quasi-isolated elements in adjoint groups of type $E_6$ and $E_7$

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