THE ORBIFOLD TRANSFORM AND ITS APPLICATIONS

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Abstract. We discuss the notion of the orbifold transform, and illustrate it on simple examples. The basic properties of the transform are presented, including transitivity and the exponential formula for symmetric products. The connection with the theory of permutation orbifolds is addressed, and the general results illustrated on the example of torus partition functions.

1. Introduction

In the last few decades, Conformal Field Theory (CFT) \cite{9, 20} and the closely related String Theory \cite{22, 28} have had many fruitful interactions with different branches of mathematics, ranging from algebraic topology through differential geometry to the theory of modular forms, not to mention the intimately related theories of Vertex Operator Algebras \cite{21, 23} and Modular Tensor Categories \cite{31, 1}.

Orbifold constructions, i.e. the gauging of discrete internal symmetries, have played an important role in Conformal Field Theory for quite some time \cite{16, 17, 15}. Among these, the theory of permutation orbifolds \cite{25, 10, 2, 5} is a most interesting chapter: not only does it provide a general procedure for constructing new Conformal Field Theories from known ones, with pretty good control over the structure of the resulting theory, but through the so-called Orbifold Covariance Principle \cite{3} one has a very effective tool for the study of deeper aspects of CFT, which led ultimately to a proof of the Congruence Subgroup conjecture for Rational CFT \cite{6} (a related proof in the context of nets of subfactors has been recently provided in \cite{32}). Symmetric product orbifolds, i.e. permutation orbifolds of the full symmetric group play a basic role in the description of second quantized strings \cite{13, 12, 7, 26}.

Many important aspects of permutation orbifolds can be understood to a large extent with the help of a general group theoretic construct that we term the orbifold transform. In special instances one gets back well known classical concepts, like the cycle indicator polynomial of finite permutation groups, while in other special cases the transform describes correlation and partition functions of permutation orbifolds. The aim of the present paper is describe this general notion and to discuss its properties, as well as its applications to CFT.

In the next section, we’ll introduce the concept of the orbifold transform, and illustrate it in some simple cases. Section 3 is devoted to the statement and proof of a most important property of the orbifold transform, namely transitivity, which describes the result of successive applications of the transform. Section 4 is concerned with the proof and applications of a general combinatorial identity satisfied by the orbifold transform, which plays a fundamental role in the theory of symmetric products. In Section 5 we sketch the connection of the orbifold transform with

Key words and phrases. finitely generated groups, permutation orbifolds, wreath products.

Work supported by grants OTKA T047041, T043582, the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and EC Marie Curie MRTN-CT-2004-512194.
the theory of permutation orbifolds. Finally, we conclude by indicating possible extensions, which might prove usefull in describing such concepts as discrete torsion. A short appendix treats some combinatorial results used in the text.

2. THE ORBIFOLD TRANSFORM

To start with, let’s recall the following basic facts about finitely generated groups and their finite index subgroups [27,29]:

- a finite index subgroup of a finitely generated group is itself finitely generated (this follows from the Reidemeister-Schreier theorem);
- a finite index subgroup of a finite index subgroup is again of finite index (by Lagrange’s theorem), and the intersection of two finite index subgroups is again of finite index (by a theorem of Poincaré);
- there are only finitely many different homomorphisms from a finitely generated group into a finite group (since a homomorphism is determined by the images of the generators, and there are only finitely many possibilities for them);
- a finitely generated group has only finitely many conjugacy classes of subgroups of a given finite index (because each conjugacy class corresponds to a transitive permutation action of degree equal to the index, and these form a finite set by the above), and each such conjugacy class contains finitely many different subgroups (because the normalizer of a finite index subgroup is obviously also of finite index).

Let $G$ be a finitely generated group, and let $\mathcal{L}(G)$ denote the set of finite index subgroups of $G$. Let $R$ be a commutative $\mathbb{Q}$-algebra, i.e. a commutative ring with identity that contains an isomorphic copy of the field of rational numbers. By a class function with values in $R$ we shall mean a mapping $Z : \mathcal{L}(G) \to R$ which is constant on conjugacy classes of subgroups, i.e. such that

\[ Z(g^{-1}Hg) = Z(H) \quad (1) \]

holds for all $g \in G$ and $H \in \mathcal{L}(G)$. We shall denote by $\mathcal{C}(G,R)$ the set of such class functions with values in $R$.

Given a permutation group $\Omega < S_X$ acting on the finite set $X$, we can associate to any $Z \in \mathcal{C}(G,R)$ a new map $Z : \mathcal{L}(G) \to R$

\[ Z : \Omega : \mathcal{L}(G) \to R \quad (2) \]

\[ H \mapsto \frac{1}{|\Omega|} \sum_{\phi : H \to \Omega} \prod_{\xi \in O(\phi)} Z(H_{\xi}) \cdot \]

Let’s see the different ingredients entering this definition! First, the summation runs over all homomorphisms $\phi : H \to \Omega$ mapping the subgroup $H$ into $\Omega$: since $H$ is finitely generated and $\Omega$ is finite, the sum is finite. For each homomorphism $\phi : H \to \Omega$, the image $\phi(H)$ – being a subgroup of $\Omega$, hence of $S_X$ – is a permutation group acting on the set $X$, and we denote by $O(\phi)$ the set of its orbits: there are only finitely many of them, since $X$ is finite. Finally, for a given orbit $\xi \in O(\phi)$ we denote by $H_{\xi}$ any of its point stabilizers, more precisely

\[ H_{\xi} = \{ g \in H \mid \xi^* \text{ is fixed by } \phi(g) \} \quad (3) \]

for some representative $\xi^* \in \xi$ chosen at will: $H_{\xi}$ is a finite index subgroup, its index in $H$ being equal to the length $|\xi|$ of the orbit $\xi$. Note that, although the subgroup $H_{\xi}$ does depend on the choice of the representative $\xi^*$, the value $Z(H_{\xi})$ doesn’t, since stabilizers of points on the same orbit are conjugate subgroups and $Z$ is a class function.
In summary, every term in Eq. (2) makes sense: $Z(H_\xi)$ exists and is independent of the chosen representative $\xi^* \in \xi$; the sum and the product are both finite; and we can divide by the order $|\Omega|$ of the permutation group $\Omega$, since $R$ is a $\mathbb{Q}$-algebra. Consequently, Eq. (2) gives a well defined map $Z:\mathcal{L}(G) \to R$, which we call the orbifold transform of the class function $Z$.

A basic property of the orbifold transform is that $Z \iota \Omega$ is itself a class function. To see this, let’s consider a conjugate $K = H^g$ of the subgroup $H \in \mathcal{L}(G)$, and let’s determine the value of $Z \iota \Omega$ on $K$: according to Eq. (2), this is given by

$$\frac{1}{|\Omega|} \sum_{\phi:K \to \Omega} \prod_{\xi \in \mathcal{O}(\phi)} Z(K_{\xi}) .$$

But to each $\phi : H \to \Omega$ we can associate a homomorphism $\phi^g : K \to \Omega$ via the rule $\phi^g(h^g) = \phi(h)$, and this correspondence is clearly one-to-one. Moreover, the image of $\phi^g$ equals that of $\phi$, so they have the same orbits, and the stabilizers of the orbit $\xi \in \mathcal{O}(\phi) = \mathcal{O}(\phi^g)$ are conjugate subgroups: $K_{\xi} = g^{-1}H_{\xi}g$. Since $Z$ is a class function, we conclude that $Z \iota \Omega$ agrees on $H$ and $K$, i.e. it is a class function too, as claimed.

In summary, for each permutation group $\Omega$ of finite degree the orbifold transform provides a map

$$\mathcal{C}(G,R) \to \mathcal{C}(G,R)$$

$$Z \mapsto Z \iota \Omega .$$

To get some familiarity with this map, let’s see how it looks like in simple cases!

Of course, the simplest case is when $G$ is trivial, i.e. consists of the identity element solely. There is just one subgroup of $G = \{1\}$, namely $G$ itself, so a class function is nothing but an element of $R$. There is just one homomorphism $\phi : G \to \Omega$ for any permutation group $\Omega$, namely the one that takes the identity of $G$ to the identity of $\Omega$, and each point of $X$ forms an orbit in itself, whose stabilizer is the whole of $G$. Consequently, in this particular case the orbifold transform takes the form

$$Z \iota \Omega = \frac{1}{|\Omega|} Z^d ,$$

where $d$ denotes the degree of $\Omega$, i.e. the cardinality of $X$.

One can analyze in a similar fashion the case of any finite $G$: the orbifold transform gives then – for each permutation group $\Omega$ of finite degree – a polynomial map $R[t_1,\ldots,t_n] \to R[t_1,\ldots,t_n]$, where $n$ denotes the number of conjugacy classes of subgroups of $G$, whose explicit expression depends on both $G$ and $\Omega$.

Let’s turn to a less trivial case, when $G = \mathbb{Z}$ is infinite cyclic! Since $G$ is Abelian, each conjugacy class consists of one subgroup; moreover, for each positive integer $n$ there is exactly one subgroup of index $n$, namely $G_n = n\mathbb{Z}$, and these are all infinite cyclic. This means that a class function $Z \in \mathcal{C}(G,R)$ may be viewed as an infinite sequence $z_1, z_2, \ldots$ of elements of $R$, where $z_n = Z(G_n)$. Since $G_n$ is generated by one element, a homomorphism $\phi : G_n \to \Omega$ is specified by giving the image of the generator, which could be an arbitrary element $x$ of $\Omega$: the image of $\phi$ is then the cyclic subgroup of $\Omega$ generated by $x$, and the orbits of the image are nothing but the orbits of $x$. Finally, the stabilizer of a given orbit $\xi$ – which is independent of the choice of the representative $\xi^* \in \xi$, since the subgroup $G_n$ is Abelian – is $\{g \in G_n | g^{[\xi]} = 1\}$, which is nothing else but the subgroup $G_n[\xi]$ of $G$.

Summarizing all this, we have

$$Z \iota \Omega : G_n \to \frac{1}{|\Omega|} \sum_{x \in \Omega} \prod_{\xi \in \mathcal{O}(x)} Z(G_n[\xi]) .$$
We see that the orbifold transform maps the infinite sequence \(z_1, z_2, \ldots\) into another infinite sequence \(z_\Omega^1, z_\Omega^2, \ldots\) \(\in R\), whose elements are finite polynomials in the \(z_i\). Actually, this map has a classical interpretation: to see this, recall from enumerative combinatorics \([30]\) the notion of the cycle indicator of the permutation group \(\Omega\), which is the multivariate polynomial

\[
P_\Omega(t_1, \ldots, t_d) = \frac{1}{|\Omega|} \sum_{x \in \Omega} \prod_{\xi \in \Omega(x)} t_{|\xi|},
\]

where \(d\) denotes the degree of \(\Omega\) and the \(t_i\) are indeterminates. With the help of the cycle indicator, we can recast Eq.\((5)\) for the orbifold transform into

\[
z_\Omega^n = P_\Omega(z_n, z_{2n}, \ldots, z_{dn}).
\]

Finally, let’s consider a most interesting case for applications in the theory of permutation orbifolds, when \(G = \mathbb{Z} \oplus \mathbb{Z}\) is free Abelian of rank two (this is the fundamental group of a two-dimensional torus, explaining its relevance to Conformal Field Theory). Again, such a \(G\) is Abelian, so each conjugacy class contains just one subgroup. Finite index subgroups of \(\mathbb{Z} \oplus \mathbb{Z}\) are all isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}\), and they are in one-to-one correspondence with 2-by-2 integer matrices in Hermite normal form (HNF) \([11]\). Recall, that such a matrix has the form

\[
H = \begin{pmatrix} \mu & \kappa \\ 0 & \lambda \end{pmatrix},
\]

where \(\mu\) and \(\lambda\) are positive integers, while \(\kappa\) is a nonnegative integer less than \(\lambda\), i.e. \(0 \leq \kappa < \lambda\). If \(G = \mathbb{Z} \oplus \mathbb{Z}\) is generated by \(a\) and \(b\) (note that \(a\) and \(b\) commute), then the finite index subgroup corresponding to \(H\) is generated by \(a^\lambda\) and \(a^\kappa b^\mu\), and its index equals the determinant of \(H\). From now on we shall freely identify the matrix \(H\) with the corresponding subgroup, and view class functions \(Z \in \mathcal{C}(G, R)\) as defined on the set of matrices in HNF.

A homomorphism \(\phi : \mathbb{Z} \oplus \mathbb{Z} \to \Omega\) is specified by a pair \((x, y)\) of commuting elements of \(\Omega\) (the images of the generators): the image of such a homomorphism is the subgroup of \(\Omega\) generated by \(x\) and \(y\), and we shall denote by \(\mathcal{O}(x, y)\) the orbits of this subgroup. The stabilizer of each orbit \(\xi \in \mathcal{O}(x, y)\) – which is once again independent of the choice of the representative \(\xi^* \in \xi\), since the group is Abelian – is a finite index subgroup of \(\mathbb{Z} \oplus \mathbb{Z}\), so it corresponds to a matrix

\[
H_\xi = \begin{pmatrix} \mu_\xi & \kappa_\xi \\ 0 & \lambda_\xi \end{pmatrix}
\]

in HNF; here, \(\lambda_\xi\) denotes the common length of all \(x\) orbits contained in \(\xi\) and \(\mu_\xi\) denotes their number, while \(\kappa_\xi\) characterizes the ‘skewness’ of the orbit \(\xi\). Taking all this into account, the orbifold transform reads

\[
Z \wr \Omega : H \mapsto \frac{1}{|\Omega|} \sum_{x, y \in G} \sum_{\xi \in \mathcal{O}(x, y)} \prod_{\xi \in \mathcal{O}(x, y)} Z(H_\xi H).
\]

When suitably interpreted, the above formula gives the torus partition function of the permutation orbifold with twist group \(\Omega\) \([2]\).

Hopefully, the above examples were able to give an impression of the general notion of the orbifold transform and its relation to some classical notions. But it should be stressed that this construct works for any finitely generated group, leading to some genuinely new structures in general.
3. Transitivity of the Orbifold Transform

By far the most important property of the orbifold transform is transitivity, which describes the result of successive applications of the transform, and is closely related to the corresponding property of permutation orbifolds [25]. Since the formulation of this property, as well as its proof, relies strongly on the theory of wreath products, let’s begin by recalling some facts about the latter [14] [24].

Consider two permutation groups $\Omega_1 < S_X$ and $\Omega_2 < S_Y$ acting on the finite sets $X$ and $Y$. To any $\lambda \in \Omega_1^2$ — mapping $Y$ to $\Omega_1$ — and a permutation $\omega \in \Omega_2$ one can associate a permutation $\lambda \ast \omega$ of $X \times Y$ via the rule

$$
\lambda \ast \omega : (x, y) \mapsto (\lambda (y) x, \omega y)
$$

The important observation is that the product of two permutations of this type is again of this type, to wit

$$
(\lambda_1 \ast \omega_1) (\lambda_2 \ast \omega_2) = (\lambda_1 \ast \omega_2 \lambda_2) \ast (\omega_1 \omega_2),
$$

where $\lambda_1 \ast \omega_2 \lambda_2 : Y \to \Omega_1$ is given by

$$
\lambda_1 \ast \omega_2 \lambda_2 : y \mapsto \lambda_1 (\omega_2 y) \lambda_2 (y).
$$

In short, the permutations in $\Omega_1 \wr \Omega_2 = \{ \lambda \ast \omega \mid \lambda \in \Omega_1^Y, \omega \in \Omega_2 \}$ form a group that acts on the Cartesian product $X \times Y$, called the wreath product of $\Omega_1$ with $\Omega_2$. Clearly, the degree of $\Omega_1 \wr \Omega_2$ is the product of the degrees of its factors:

$$
deg (\Omega_1 \wr \Omega_2) = \deg (\Omega_1) \deg (\Omega_2),
$$

while the order of the wreath product is given by

$$
|\Omega_1 \wr \Omega_2| = |\Omega_1|^{\deg(\Omega_2)} |\Omega_2|.
$$

The orbits of the wreath product are easy to describe: each orbit of $\Omega_1 \wr \Omega_2$ is of the form $\xi \times \eta$, where $\xi$ is an orbit of $\Omega_1$ on $X$ while $\eta$ is an orbit of $\Omega_2$ on $Y$.

Wreath products are associative but not commutative, in the sense that the permutation groups $(\Omega_1 \wr \Omega_2) \wr \Omega_3$ and $\Omega_1 \wr (\Omega_2 \wr \Omega_3)$ are always equivalent, but $\Omega_1 \wr \Omega_2$ and $\Omega_2 \wr \Omega_1$ fail to be equivalent in general.

Consider a homomorphism $\phi : G \to \Omega_1 \wr \Omega_2$ from an arbitrary group $G$ into the wreath product $\Omega_1 \wr \Omega_2$. Such a $\phi$ assigns to each element $g \in G$ a permutation $\phi (g) = \lambda (g) \ast \omega (g)$. This means that $\phi$ can be described by the pair $(\lambda, \omega)$, where $\lambda$ maps $G$ into $\Omega_1^Y$, while $\omega$ maps $G$ into $\Omega_2$. Taking into account Eq. (11), one sees that actually $\omega : G \to \Omega_2$ is a homomorphism, while $\lambda : G \to \Omega_1^Y$ is a $(\omega)$-crossed homomorphism, i.e. a map that satisfies

$$
\lambda (gh) = \lambda (g) \omega (h) \lambda (h).
$$

To formulate the next result that classifies crossed homomorphisms, let’s recall that for an orbit $\xi \in O (\omega)$ we denote by $\xi^*$ a representative point of $\xi$, and by $G_\xi = \{ g \in G \mid \omega (g) \xi^* = \xi^* \}$ the stabilizer of $\xi^*$. To each point $y \in \xi$ we associate a suitable element $\gamma_y \in G$, such that $\xi^*$ is mapped to $y$ by the permutation $\omega (\gamma_y)$ (of course, such a $\gamma_y$ is far from unique, any element of the coset $\gamma_y G_\xi$ would do the job).

**Lemma 1.** For a given homomorphism $\omega : G \to \Omega_2$, the crossed homomorphisms $\lambda : G \to \Omega_1^Y$ are in one-to-one correspondence with pairs $(\Phi_\xi, \varphi_\xi)$, one for each orbit $\xi \in O (\omega)$, where $\Phi_\xi : G_\xi \to \Omega_1$ is a homomorphism, while $\varphi_\xi : \xi \to \Omega_1$ is an arbitrary map for which $\varphi_\xi (\xi^*) = \Phi_\xi (\gamma_y)$.

**Proof.** To begin with, let’s note that any crossed homomorphism $\lambda : G \to \Omega_1^Y$ may be viewed as a map $\lambda : G \times Y \to \Omega_1$ that satisfies

$$
\lambda (gh, y) = \lambda (g, hy) \lambda (h, y).
$$
for all $g, h \in G$ and $y \in Y$, where for simplicity $hy$ denotes the image of $y$ under the permutation $\omega(h) \in \Omega_2$. By Eq. (15), the map

$$\Phi_\xi : G_\xi \to \Omega_1$$

is a group homomorphism for each orbit $\xi \in O(\omega)$.

Let’s substitute $y = \xi^*$ and $h = \gamma_z$ into Eq. (15) to get

$$\lambda(g, z) = \lambda(g \gamma_z, \xi^*) = \lambda(\gamma_z, \xi^*)^{-1}$$

for $z \in \xi$. By definition, $(g \gamma_z) \xi^* = g \xi = \gamma_g \xi^*$, in other words $\gamma_g \xi^* = \gamma_z \xi^* \in G_\xi$. Taking this into account, Eq. (15) gives

$$\lambda(g \gamma_z, \xi^*) = \lambda(\gamma_g \xi^*) \lambda(\gamma_g^{-1} \gamma_z, \xi^*)$$

If we introduce for each $\xi \in O(\omega)$ the maps

$$\varphi_\xi : \xi \to \Omega_1$$

$$y \mapsto \lambda(\gamma_g \xi^*)$$

then Eqs. (17) and (18) lead to

$$\lambda(g, y) = \varphi_\xi (gy) \Phi_\xi (\gamma_g^{-1} g \gamma y) \varphi_\xi (y)^{-1}$$

for all $g \in G$ and $y \in \xi$. Note that, since $\gamma_z \in G_\xi$, one has

$$\varphi_\xi (\xi^*) = \lambda(\gamma_z \xi^*) = \Phi_\xi (\gamma_z \xi^*)$$

Eq. (20) means that, given a choice of the coset representative $\xi^*$ and of the elements $\gamma_z$ mapping $\xi^*$ to $y \in \xi$, the pair $(\Phi_\xi, \varphi_\xi)$ determines completely the values $\lambda(g, y)$ for all points $y \in \xi$. Since the orbits $\xi$ partition the set $Y$, it follows that the collection of such pairs determines the crossed homomorphism $\lambda$.

**Lemma 2.** With the notations of Lemma 1, the orbits of the image of the homomorphism $\phi : G \to \Omega_1 \wr \Omega_2$ are of the form $\langle \xi, \eta \rangle = \{(\varphi_\xi (y) x, y) \mid x \in \xi, y \in \eta\}$, where $\eta \in O(\omega)$ and $\xi \in O(\Phi_\eta)$. Moreover, the stabilizer of such an orbit is given by $G_{\langle \xi, \eta \rangle} = \{g \in G_\eta \mid \Phi_\eta (g) \xi^* = \xi^*\}$.

**Proof.** Consider an arbitrary point $(x, y) \in X \times Y$: the permutation $\phi(g)$ takes this to the point $(\lambda(g, y) x, gy)$: in other words, the pair $(w, z)$ lies in the orbit of $(x, y)$ if and only if there exists $g \in G$ such that $z = gy$ and $w = \lambda(g, y) x$. By Eq. (20), one has $\lambda(g, y) = \varphi_\eta (z) \Phi_\eta (\gamma_z^{-1} g \gamma y) \varphi_\eta (y)^{-1}$ if $y$ lies in the orbit $\eta \in O(\omega)$. Now, for fixed $y$ and $z$, the expression $\gamma_z^{-1} g \gamma y$ runs through all elements of $G_\eta$. This shows that $(w, z)$ lies in the same orbit as $(x, y)$ if and only if $z$ lies in the $\omega(\eta)$-orbit of $y$, and at the same time $\varphi_\eta (z)^{-1} w$ lies in the same $\Phi_\eta (G_\eta)$-orbit as $\varphi_\eta (y)^{-1} x$, which proves the claim about the structure of the orbits. The expression for the stabilizer is obvious.

Armed with the above results, we can now state and prove the fundamental transitivity property of the orbifold transform.

**Theorem 1.** Let $Z \in C(G, R)$ be a class function of the finitely generated group $G$, with values in the commutative $R$-algebra $R$, and let $\Omega_1, \Omega_2$ be two permutation groups of finite degree. Then

$$(Z \mid \Omega_1) \wr \Omega_2 = Z \mid (\Omega_1 \wr \Omega_2).$$

**Proof.** It is enough to prove that both sides of Eq. (22) assign the same value to the subgroup $H = G$. Indeed, for a nontrivial subgroup $H < G$, one may consider the restriction $Z_H \in C(H, R)$ of $Z$ to $\mathscr{L}(H)$: since $Z_H$ agrees with $Z$ for all finite index subgroups of $H$, it does so in particular for $H$ itself, so Eq. (22) for $H$ holds if and only if it holds for $Z_H$. 


Let’s first consider the rhs. of Eq. (22), which reads

\[
\frac{1}{|\Omega_1 \wr \Omega_2|} \sum_{\phi: G \to \Omega_1 \wr \Omega_2} \prod_{\zeta \in \mathcal{O}(\phi)} Z_G(\zeta) .
\]

By the previous arguments, we can associate to \(\phi\) the pair \((\lambda, \omega)\), where \(\omega: G \to \Omega_2\) is a homomorphism, while \(\lambda: G \times Y \to \Omega_1\) is a crossed homomorphism; moreover, \(\lambda\) itself may be described via pairs \((\Phi_\eta, \varphi_\eta)\), one for each orbit \(\eta \in \mathcal{O}(\omega)\), according to Lemma 1. Taking this into account, as well as the structure of the orbits of \(\phi\) as described by Lemma 2, the expression in Eq. (23) reads

\[
\frac{1}{|\Omega_1| |\Omega_2|} \sum_{\omega: G \to \Omega_2} \prod_{\eta \in \mathcal{O}(\omega)} \sum_{\varphi_\eta} \prod_{\zeta \in \mathcal{O}(\Phi_\eta)} Z_{G(\xi, \eta)} (\varphi_\eta).
\]

This may be rearranged into the more suggestive form

\[
\frac{1}{|\Omega_1| |\Omega_2|} \sum_{\omega: G \to \Omega_2} \prod_{\eta \in \mathcal{O}(\omega)} \left\{ \sum_{\varphi_\eta} \prod_{\Phi_\eta: G_\eta \to \Omega_1} \prod_{\zeta \in \mathcal{O}(\Phi_\eta)} Z_{G(\xi, \eta)} \right\}.
\]

Nothing in this expression depends explicitly on the maps \(\varphi_\eta\), so their contribution is simply to introduce a multiplicative factor \(|\Omega_1| |\eta| - 1\) equal to their number. Taking into account the structure of \(G(\xi, \eta)\) as described by Lemma 2, we recognize that the resulting expression is nothing but

\[
\frac{1}{|\Omega_2|} \sum_{\omega: G \to \Omega_2} \prod_{\eta \in \mathcal{O}(\omega)} (Z \wr \Omega_1) (G_\eta),
\]

which is just the lhs. of Eq. (22).

We note that it is this transitivity property Eq. (22) that motivates the wreath product notation \(Z \wr \Omega\) for the orbifold transform (besides the connection with permutation orbifolds).

4. THE EXPONENTIAL IDENTITY AND SYMMETRIC PRODUCTS

In Section 2, when discussing examples of the orbifold transform, we have fixed the finitely generated group \(G\), and let both the class function \(Z\) and the permutation group \(\Omega\) vary freely. Another possible approach is to fix the permutation group \(\Omega\), while leaving \(G\) and \(Z\) arbitrary: a most important case is when \(\Omega = S_n\), the symmetric group of degree \(n\), which is termed a symmetric product in the theory of permutation orbifolds. As it turns out, one has very good control over the orbifold transform in this case, thanks to a general combinatorial identity [7] that we are going to discuss.

First, let’s fix some notation. For a positive integer \(n\) and an arbitrary class function \(Z \in \mathcal{C}(G, R)\) of the finitely generated group \(G\), we define \(Z_n = Z \wr S_n\), and we set \(Z_0\) to the constant class function equal to 1: we call \(Z_n\) the \(n\)-th symmetric product of \(Z\).

**Theorem 2.** The following formal identity holds

\[
\sum_{n=0}^{\infty} Z_n (G) = \exp \left( \sum_{H \in \mathcal{P}(G)} \frac{Z(H)}{|G : H|} \right),
\]

where the exponential on the rhs. stands for its infinite power series.
Proof. Let’s write out the lhs. of Eq. (27): it reads

\[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\phi: G \to S_n} \prod_{\xi \in \mathcal{O}(\phi)} Z(G_{\xi}) \, . \]

Let’s first consider the sum over homomorphisms \( \phi: G \to S_n \) for a given \( n \): each such homomorphism is just a permutation action of \( G \) of degree \( n \), and \( \mathcal{O}(\phi) \) is the set of orbits of this action, while the \( G_{\xi} \) are the point stabilizers. The point is that the term corresponding to a given \( \phi \) does only depend on the equivalence class of this action, since there is a one-to-one correspondence between the orbits of equivalent actions, and the corresponding stabilizers are conjugate subgroups. This means that we can rewrite this sum over all permutation actions of degree \( n \) as a sum over equivalence classes of actions, provided we take into account the cardinality of each equivalence class.

Any equivalence class of permutation actions may be decomposed into a sum of transitive classes, corresponding to the different orbits of the action:

\[ [\phi] = \oplus_i n_i \tau_i , \]

where the \( \tau_i \) denote the different transitives, and \( n_i \) is the multiplicity of the \( i \)-th transitive in \([\phi]\). For each given degree there are only finitely many transitives of that degree: indeed, transitives correspond to conjugacy classes of subgroups of finite index, the degree of the transitive being equal to the index of the subgroup.

Let \( G_i \) denote the stabilizer of the \( i \)-th transitive \( \tau_i \): in other words, \( \tau_i \) is the equivalence class of the action of \( G \) on the cosets of the subgroup \( G_i \). Then, for each permutation action in the equivalence class \([\phi]\), one has

\[ \prod_{\xi \in \mathcal{O}(\phi)} Z(G_{\xi}) = \prod_{i} Z(G_i)^{n_i} . \]

Denoting by \( \ell_i = [N_G(G_i) : G_i] \) the index of \( G_i \) in its normalizer, the cardinality of the equivalence class \([\phi]\) is given by

\[ \# [\phi] = \frac{n!}{\prod_i n_i \ell_i^{n_i}} , \]

where \( n \) is the degree of \([\phi]\) (see Appendix A for a detailed proof). Taking all this into account, the rhs. of Eq. (27) reads

\[ 1 + \sum_{[\phi]} \prod_{i} \frac{1}{n_i \ell_i^{n_i}} Z(G_i)^{n_i} , \]

where the summation runs over all equivalence classes \([\phi]\). But the multiplicities \( n_i \) may take on arbitrary nonnegative values, which leads to

\[ \sum_{n=0}^{\infty} Z_n(G) = \prod_i \left( \sum_{n_i=0}^{\infty} \frac{Z(G_i)^{n_i}}{n_i! \ell_i^{n_i}} \right) = \prod_i \exp \left( \frac{Z(G_i)}{\ell_i} \right) , \]

where the product runs over all transitives of finite degree, or – what is the same – over all conjugacy classes of finite index subgroups. Since the number of different conjugates of \( G_i \) equals

\[ [G : N_G(G_i)] = [G : G_i] \frac{[G : N_G(G)]}{\ell_i} , \]

and a product of exponentials is the exponential of the sum of the exponents, we get the assertion of the theorem. \( \square \)
The exponential identity is formal in the sense that convergence of the infinite sums is not guaranteed on either side. Of course, for suitable choice of the class function $Z$ one obtains an equality of convergent series.

Let us rewrite the exponential identity Eq.(27) into a slightly different form. To this end, we introduce the notation

$$(35) \quad Z^{[n]}(G) = \sum_{[G:H]=n} Z(H) ,$$

for a positive integer $n$, where the summation on the rhs. extends over all subgroups $H < G$ of index $n$. Moreover, we adjoin a formal variable $p$ to the ring $R$, and consider the class function (where $R\{p\}$ stands for the ring of formal power series in $p$ with coefficients from $R$)

$$\hat{Z} : \mathcal{L}(G) \to R\{p\}, \quad H \mapsto p^{[G:H]} Z(H) .$$

Applying the exponential identity Eq.(27) to the class function $\hat{Z}$, one gets the following:

$$(36) \quad \sum_{n=0}^{\infty} p^n Z_n(G) = \exp \left( \sum_{n=1}^{\infty} \frac{Z^{[n]}(G)}{n} \right).$$

Now, denoting by $P_n$ the cycle indicator of the symmetric group $S_n$ (a Schur-polynomial), one has the following well-known identity $[30]$:

$$(37) \quad 1 + \sum_{n=1}^{\infty} p^n P_n(t_1, \ldots, t_n) = \exp \left( \sum_{n=1}^{\infty} \frac{P^n(t_1, \ldots, t_n)}{n} \right),$$

which is actually a special case of Eq.(27) (for $G = \mathbb{Z}$, $R = \mathbb{Q}[t_1, t_2, \ldots]$ and $Z(k\mathbb{Z}) = t_k$, cf. Section 2). Comparing Eqs.(35) and (37), and equating the coefficients of equal powers of $p$, one arrives at

$$(38) \quad Z_n(G) = P_n \left( Z^{[1]}(G), \ldots, Z^{[n]}(G) \right) ,$$

providing an elegant closed formula for the symmetric products of $Z$.

5. The transform and permutation orbifolds

Consider a system that is made up of several identical, non-interacting subsystems, each of which may be described by a Conformal Field Theory. The dynamics of the whole system is again governed by a CFT, but what is more important, any permutation of the subsystems is a symmetry of this CFT, since the subsystems are indistinguishable: consequently, one may orbifoldize with respect to any group $\Omega$ of permutations of the subsystems. The resulting CFT, which is completely determined by the twist group $\Omega$ and the CFT $C$ describing the dynamics of the individual subsystems, is the permutation orbifold $C \rtimes \Omega$.

Many important aspects of permutation orbifolds are pretty well understood: one knows how to classify their primary fields, one has elegant closed expressions for the genus one characters of these primaries, the modular transformations of the characters, the fusion rules, the partition functions, etc $[2, 5, 4]$. This is where the orbifold transform enters the picture: we’ll illustrate this point on the example of partition functions.

Among other things, a CFT assigns a number to each conformal equivalence class of two-dimensional metrics: this is what is called the partition function (more precisely, the generalized partition function, the usual partition function is obtained by restricting attention to metrics defined on tori). It is well known that for orientable
surfaces, conformal equivalence classes of metrics are in one-to-one correspondence with equivalence classes of complex structures, i.e. the partition function may be viewed as a function on the moduli space of Riemann surfaces. For simplicity, we'll restrict our attention to closed compact surfaces.

By the uniformization theorem [8, 18], every Riemann surface $\mathcal{S}$ may be obtained as a quotient
\begin{equation}
\hat{\mathcal{S}}/\Gamma_S,
\end{equation}
where $\hat{\mathcal{S}}$ is a simply connected Riemann surface (the universal cover of $\mathcal{S}$), while $\Gamma_S$, the uniformizing group of $\mathcal{S}$, is a discrete group of automorphisms of $\hat{\mathcal{S}}$, isomorphic to the fundamental group $\pi_1(\mathcal{S})$ of $\mathcal{S}$. Actually, the uniformizing group is only determined up to conjugacy in $\text{Aut}(\hat{\mathcal{S}})$, i.e. conjugate subgroups uniformize the same surface.

By the Riemann mapping theorem [8, 18], there are just three inequivalent simply connected Riemann surfaces: the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ and the upper half-plane $\mathbb{H} = \{z | \text{Im } z > 0\}$. Among compact closed surfaces, the only one whose universal cover is $\mathbb{C}P^1$ is the Riemann sphere itself, and the uniformizing group is trivial; $\mathbb{C}$ is the universal cover of tori, and in this case the uniformizing group is a group of complex translations isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$; finally, $\mathbb{H}$ is the universal cover of all other compact surfaces, and these are uniformized by Fuchsian groups. In terms of the genus $g$ of the surface $\mathcal{S}$, the above cases correspond respectively to $g = 0$, $g = 1$ and $g > 1$.

The theory of covering surfaces [19, 8] tells us that any finite index subgroup $H < \Gamma_S$ is again the uniformizing group of some compact Riemann surface $\hat{\mathcal{S}}/H$, which is a finite sheeted cover of $\mathcal{S}$. Since uniformizing groups are finitely generated (being isomorphic to the fundamental group of a surface of finite genus), given a CFT and compact Riemann surface $\mathcal{S}$, we can define a class function $Z_\mathcal{S} \in \mathcal{C}(\Gamma_S, \mathbb{C})$ by assigning to each subgroup $H \in \mathcal{L}(\Gamma_S)$ the value of the partition function on the surface $\hat{\mathcal{S}}/H$: the facts from complex analysis listed above ensure that this is well defined.

Going back to permutation orbifolds, the CFT $\mathcal{C}$ that describes the dynamics of the individual subsystems gives rise, according to the preceding discussion, to a class function $Z_\mathcal{S} \in \mathcal{C}(\Gamma_S, \mathbb{C})$ for any compact surface $\mathcal{S}$. The permutation orbifold $\mathcal{C} \bowtie \Omega$ leads to another class function $Z_\mathcal{S}^\Omega \in \mathcal{C}(\Gamma_S, \mathbb{C})$, and the question is whether these two class functions are related or not. It follows from the results of [4], that for any compact surface $\mathcal{S}$ the class function $Z_\mathcal{S}^\Omega$ is the orbifold transform of $Z_\mathcal{S}$:
\begin{equation}
Z_\mathcal{S}^\Omega = Z_\mathcal{S} \bowtie \Omega.
\end{equation}
This is the basic connection with permutation orbifolds. It should be stressed that this connection is not confined to partition functions, many other characteristic quantities of permutation orbifolds may be expressed as suitable orbifold transforms, e.g. the number of primary fields, the traces of mapping class group transformations, etc [5].

Of course, the result that the partition function of the permutation orbifold is the orbifold transform of the partition function of a single subsystem is far from being trivial. It is based on the physical picture that the dynamics of the orbifold on a given world sheet may be interpreted as the dynamics of a subsystem on suitable covers of the world sheet [4]. In this respect, the transitivity property Eq. (22) plays a decisive role: indeed, the corresponding property of permutation orbifolds is an immediate consequence of their definition [2], which has to manifest itself in any expression relating the quantities of the orbifold with those of the subsystems.
Actually, Eq. (22) would have been difficult to guess were it not for the connection with permutation orbifolds.

To conclude this section, let’s illustrate the above results on the simplest non-trivial case, the genus one partition function $Z$. It is defined on the moduli space of two dimensional tori, which is just the quotient of the upper half-plane $\mathbb{H}$ by the classical modular group $\text{SL}_2(\mathbb{Z})$. It is usually written in terms of the modular parameter $\tau \in \mathbb{H}$ of the corresponding torus, in terms of which it satisfies the functional equation (modular invariance)

$$Z \left( \frac{a\tau + b}{c\tau + d} \right) = Z (\tau),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

The universal cover of the torus with modular parameter $\tau$ is the complex plane, and its uniformizing group may be taken to be the group $G_\tau$ generated by the two translations

$$a : z \mapsto z + 1 \quad b : z \mapsto z + \tau.$$

Clearly, $G_\tau$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which is the last example discussed in Section 2. A finite index subgroup of $G_\tau$ corresponds to a matrix

$$H = \begin{pmatrix} \mu & \kappa \\ 0 & \lambda \end{pmatrix}$$

in HNF, and is generated by

$$a^\lambda : z \mapsto z + \lambda \quad a^\kappa b^\mu : z \mapsto z + \mu\tau + \kappa.$$

But this subgroup is conjugate in $\text{Aut}(\mathbb{C})$ to the group $G_{\tau(H)}$, where

$$\tau(H) = \frac{\mu\tau + \kappa}{\lambda},$$

i.e. it uniformizes a torus with modular parameter $\tau(H)$.

Taking this into account, as well as Eqs. (9) and (40), we arrive at the following well-known expression for the torus partition function of permutation orbifolds

$$Z^\Omega (\tau) = \frac{1}{|\Omega|} \sum_{x,y} \prod_{\xi \in \mathcal{O}(x,y)} Z \left( \frac{\mu_\xi \tau + \kappa_\xi}{\lambda_\xi} \right),$$

where the integers $\lambda_\xi, \mu_\xi$ and $\kappa_\xi$ are numerical characteristics of the orbit $\xi \in \mathcal{O}(x,y)$, whose meaning is described just before Eq. (9).

6. Summary and outlook

The aim of this note was to introduce the notion of the orbifold transform, to illustrate it on some simple examples, and to discuss its most basic properties and some of its applications. The two major results, namely the transitivity property Eq. (22) of the transform and the exponential identity Eq. (27) already justify amply the consideration of this construct, and the connection with the theory of permutation orbifolds gives even more evidence of its importance.

One should mention that it is possible to generalize the transform by including suitable ‘cohomological twists’: these arise naturally in the CFT context, where they run under the name of discrete torsion. The resulting theory is pretty similar to the one described here, but we refrained from its exposition, since it would need a thorough treatment of the cohomology of wreath products, which would be beyond the scope of this note. While most results go over to this general case, some of
them get modified, e.g. the exponential formula Eq. (27): for those interested in this issue we mention [7], where the torus partition function of symmetric products in the presence of nontrivial discrete torsion is discussed.

**Appendix A.**

This appendix is devoted to the proof of Eq. (31), giving the number of different permutation actions in the same equivalence class. First, let’s fix the notation. Let \( \tau_i \) denote the different equivalence classes of transitive actions of \( G \); let \( G_i \) denote the stabilizer of some point of \( \tau_i \) (the stabilizers of different points form a conjugacy class of subgroups), so that \( \tau_i \) is equivalent to the action of \( G \) on the cosets of \( G_i \); finally, let \( L_i \) denote the factor group \( N_G(G_i)/G_i \).

**Lemma 3.** Let \( \phi : G \to S_n \) be a permutation action. The number of different permutation actions equivalent to \( \phi \) equals the index of the centralizer \( C[\phi] \) of the image of \( \phi \) in \( S_n \):

\[
\# [\phi] = [S_n : C[\phi]].
\]

**Proof.** The equivalence class \([\phi]\) consists of the conjugates

\[
\phi^\alpha : G \to S_n, \quad g \mapsto \alpha^{-1} \phi(g) \alpha
\]

of \( \phi \), where \( \alpha \in S_n \). Clearly, the assignment \( \phi \mapsto \phi^\alpha \) defines a permutation action of \( S_n \) on the set of actions of \( G \) of degree \( n \), and \([\phi]\) is the orbit of \( \phi \) under this action. The length of the orbit equals the index of the stabilizer of \( \phi \), but this stabilizer is nothing but the centralizer \( C[\phi] \) of the image of \( \phi \).

**Lemma 4.** If the permutation action \( \phi \) belongs to the equivalence class \( \oplus_i n_i \tau_i \), then the centralizer of its image is isomorphic to

\[
\times_i (L_i \wr S_{n_i}),
\]

in particular Eq. (31) holds.

**Proof.** By the definition of equivalence of permutation actions, one has a direct product decomposition

\[
C[\oplus_i n_i \tau_i] = \times_i C[n_i \tau_i].
\]

Now, the centralizer \( C[n_i \tau_i] \) is generated by two sorts of permutations: those that permute the transitive constituents (i.e. the orbits) en block, without permuting the points inside an orbit, which form a group isomorphic to \( S_{n_i} \); and those that do leave each orbit setwise fixed, but permute the points of the orbits while still commuting with the action. These two sort of permutations generate the wreath product

\[
C[n_i \tau_i] = C[\tau_i] \wr S_{n_i}.
\]

We have reduced the problem to that of determining \( C[\tau_i] \) for a transitive action. But such a transitive action is equivalent to the action of \( G \) on the cosets of \( G_i \), so we are looking for permutations \( \alpha \in S_{G/G_i} \) such that

\[
\alpha(gxG_i) = g\alpha(xG_i)
\]

holds for all \( g, x \in G \), which is just the condition \( \alpha \in C[\tau_i] \). Eq. (32) with \( x = 1 \) gives \( \alpha(gG_i) = g\alpha(G_i) \), i.e. the permutation \( \alpha \) is completely determined by the image \( \alpha(G_i) \) of the trivial coset. Since this image is itself a coset of \( G_i \), there exists some \( a \in G \) such that \( \alpha(G_i) = aG_i \). Now, Eq. (32) with \( g \in G_i \) yields

\[
aG_i = \alpha(G_i) = g\alpha(G_i) = gaG_i,
\]
in other words a should belong to the normalizer of $G_i$ in $G$ in order to get a permutation of the cosets. This means that

$$C[τ] ≅ N_G(G_i)/G_i.$$  

All-in-all, we arrive at Eq.(49). Combining this with Eq.(47), we get Eq.(51). □

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