Finite size corrections in two dimensional gauge theories
and a quantitative chiral test of the overlap.

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Abstract

An argument is presented for a certain universality of finite size corrections in two
dimensional gauge theories. In the abelian case a direct calculation is carried out for a
particular chiral model. The analytical result confirms the above universality and that the
't Hooft vertex previously measured using the overlap smoothly approaches the correct
continuum limit within statistical errors.

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This note contains two main parts: In the first we apply some analytical methods to the problem of finite size effects in two dimensional gauge theories. In the second part the theoretical results are exploited to interpret some numerical data.

1. Quite recently it was argued that two dimensional gauge theories exhibit a dynamical decoupling of the Hilbert space into a conformal theory and a massive sector [1]. The global “flavor” symmetries of the original model get elevated to Kac–Moody algebras represented within the massless sector. Moreover, different theories can have identical massive sectors, the latter being sensitive to only some very general properties of the model.

Here our concern with two dimensional gauge theories is limited to their usability as benchmark cases against which proposals to regulate a chiral gauge theory in any dimension should be tested. Such numerical tests are carried out in a finite Euclidean volume, typically, a torus. It is therefore useful to obtain as much exact information about finite size effects in the continuum as possible.*

Although the arguments of Kutasov and Schwimmer apply more directly at infinite volume, it is plausible that with judicious choices of boundary conditions, their conclusions hold exactly also for selected theories defined on finite tori. Assume we are in such a situation and that we are computing the expectation value of some one-point observable. The observable can be factorized into an operator acting within the massless sector and another acting within the massive sector. Attaching the right power of the gauge coupling constant the massless factor can be made dimensionless, so its expectation value is a size independent pure number because of conformal invariance. Therefore, any finite size correction must come from the massive factor. Then, the Kutasov–Schwimmer universality extends to the finite size correction, meaning that it can be computed in any of a class of theories. In particular, one can always find a vector representative of the massive sector. We conclude that finite size corrections for any chiral model can be evaluated by looking at appropriate finite size corrections in an associated vector model.

The above holds equally in the abelian and non-abelian case. Actually, in the abelian case the decoupling and the finite size effects are easier to analyze. In previous work [2,3] we concentrated on a particular abelian model called the 11112 model for its fermion content. The action of the 11112 model in Euclidean space is:

\[
S = \frac{1}{4e_0^2} \int d^2x F_{\mu\nu}^2 - \frac{4}{\sigma_1} \int d^2x \bar{\chi}_k \sigma_\mu (\partial_\mu + iA_\mu) \chi_k - \int d^2x \bar{\psi} \sigma_\mu (\partial_\mu + 2iA_\mu) \psi,
\]

where \( \sigma_1 = 1, \sigma_2 = i \) and \( \mu = 1, 2 \). In addition to this, we have to specify the boundary conditions on the fermions. We can always choose one of the fermions to obey periodic conditions.

* This may have some applications to finite temperature problems.
boundary conditions by a suitable redefinition of the gauge field. We will assume this is
done and choose $\psi$ to obey periodic boundary conditions. The four $q = 1$ fermions obey
the following boundary conditions:

$$\chi_f(x_\mu + l\hat{\mu}) = e^{2\pi i b^f_\mu} \chi_f(x_\mu).$$  (2)

We restrict the $b^f_1$ to the interval $[-1/2, 1/2)$ and the $b^f_2$ to the interval $(-1/2, 1/2]$. Physically, $b^f_\mu$ differing by integers mean exactly the same thing, but in all subsequent formulae
the symbols $b^f_\mu$ are assumed to reside within the above ranges.

Due to the abelian $U(1)$ anomaly and instantons the dimensionless operator

$$V(x) = \frac{\pi^2}{e_0^4} \chi_1(x)\chi_2(x)\chi_3(x)\chi_4(x)\bar{\psi}(x)(\sigma \cdot \partial)\psi(x),$$  (3)

which is the 't Hooft vertex in this model, gets a nonzero expectation value. Assuming
clustering at infinite volume we evaluated this expectation value in [2], obtaining

$$\langle V \rangle = \frac{e^{4\gamma}}{4\pi^3} \approx 0.081$$  (4)

In this case the factorization of $V$ could be seen explicitly quite easily in a formal
bosonized operator solution at infinite volume. To get a rigorous formula in a finite volume
one would need to worry about boundary conditions and topological effects in the operator
formalism and this we have not done. Our objective here is to show by direct computation
of the path integral that the finite size corrections to $\langle V \rangle_l$ measured on a torus of physical
size $l \times l$ indeed are universal in the sense explained above. In particular we wish to see
that these finite size effects are identical to the ones in a four flavor vector Schwinger
model, which happens to be the simplest associated model. We should emphasize that the
pertinent 't Hooft vertex operators are quite different in the two models: In the chiral case
we have six fermions and a derivative while in the vector case we have eight fermions. In
the chiral case the operator does not commute with fermion number but in the vector case
it does.

A detailed understanding of finite size corrections in the vector case would lead one to
guess that the above universality holds even without employing the Kutasov–Schwimmer
argument directly. Thus, in our previous work we already used the four flavor Schwinger
model as a source for an estimate of the finite size effects in the chiral case. Here we shall
present explicit proof and validate our previous procedure. Moreover, by adopting the
Kutasov–Schwimmer logic it appears that a similar approach to finite size effects will hold
also in the non-abelian case briefly mentioned in [2].
The boundary conditions for the fermion fields in the path integral have to be chosen with care; this was discussed at length in ref [3] from a different point of view. Here we do not wish to get into a general discussion of all possible different boundary conditions. But, we shall see that the “good” choice we adopted before, namely,

\[ b_1^1 = 0; \quad b_2^1 = 0; \quad b_3^1 = -\frac{1}{2}; \quad b_4^1 = -\frac{1}{2}; \quad b_1^2 = 0; \quad b_2^2 = \frac{1}{2}; \quad b_3^2 = 0; \quad b_4^2 = \frac{1}{2}, \]

indeed is also “good” in that it leads to an answer compatible with a clustering vacuum in the limit \( l \to \infty \). However, the calculation is done for arbitrary boundary conditions.

Our calculation proceeds similarly to one done for the vector Schwinger model, presented in great detail in ref [4]. Since ours is a chiral case, we have to worry about phase choices in the definition of various fermionic determinants, and the generalization of [4] we need is not entirely trivial. To carry out the calculation, we have to resolve several ambiguities on the way.

Following [4], we decompose the gauge fields as

\[ A_1 = \frac{\pi k}{l^2} x_2 + \partial_2 \phi + \frac{2\pi}{l} h_1 + ig^{-1} \partial_1 g; \quad A_2 = -\frac{\pi k}{l^2} x_1 - \partial_1 \phi + \frac{2\pi}{l} h_2 + ig^{-1} \partial_2 g \]  

The terms dependent on \( g \) represent the gauge degree of freedom. \( g \) is a periodic function on the torus and takes values in \( U(1) \). \( g \) will disappear from the calculation because of gauge invariance. \( \phi \) is periodic and has no zero momentum component - it describes the non-uniform components of the electric field. The uniform components of the vector potential are represented by the constants \( h_\mu \). The uniform component of the electric field is described by the \( k \) dependent terms which are a symmetric gauge representation of a configuration carrying flux \( k \). The space of gauge potentials falls into classes labeled by the integer \( k \).

In [4] the various needed “vectorial” determinants were given definite values using \( \zeta \)-function regularization. For chiral fermions we need to define complex square roots of these quantities. At \( k = 0 \) the required phase (a function of the \( h_\mu \)) is determined by the imposition of several symmetries [3,5,6]. At \( k = 1 \) the fermionic zero modes have to be separated out, and the remaining vectorial determinant depends only on the volume of the system, which gives it the right dimension. Now the phase freedom can be associated with the zero modes and our choice will be explained below.

In what follows, it will be useful to define the function

\[ \hat{\theta}(\alpha_1, \alpha_2; \tau) = \sum_{n=-\infty}^{\infty} \exp(-\pi \tau(n + \alpha_2)^2 + i2\pi n \alpha_1 + \pi i \alpha_1 \alpha_2) \]  

4
\( \hat{\theta} \) is closely related to the usual \( \Theta \) function \([4,5,6]\). The chiral determinant and the zero modes are expressible in terms of \( \hat{\theta} \).

The expectation value of \( V \) is the ratio of two path integrals, one with \( V \) inserted and the other with no insertion. The path integral with no insertions gets a contribution only from the \( k = 0 \) sector and gives the partition function. After a change of variables on the fermions and a computation of the fermion determinant in the background of a constant gauge potential \([5,6]\), the partition function acquires the following form:

\[
Z_0 = \frac{1}{\eta^5(1)} \int d^2 h \left\{ \prod_f \hat{\theta}(h_1 + b_1^f + \frac{1}{2}, h_2 + b_2^f - \frac{1}{2}; 1) \right\} \left\{ \hat{\theta}^*(2h_1 + \frac{1}{2}, 2h_2 - \frac{1}{2}; 1) \right\} \int D\phi e^{-\Gamma(\phi)}; \\
\Gamma(\phi) = \frac{1}{2e_0} \int \phi(\Delta^2 - m_\gamma^2) \phi; \quad m_\gamma^2 = \frac{4e_0^2}{\pi}; \quad \eta(\tau) = \exp(-\frac{1}{12} \pi \tau) \prod_{n=1}^{\infty} [1 - \exp(-2\pi \tau n)].
\]

The integral over \( h_\mu \) represents a sum over saddles in an otherwise Gaussian integral. The integral over \( \phi \) contains the massive sector of the theory and is identical to an integral that would appear in a four flavor vector Schwinger model. The integrand in (8) is invariant under the gauge transformation that takes \( h_\mu \rightarrow h_\mu + 1 \) as long as

\[
| \sum_{f} b_1^f | = | \sum_{f} b_2^f | = 1
\]

Clearly, (5) obeys this requirement. In [5] it was shown that if anomalies cancel, invariance under \( h_\mu \rightarrow h_\mu + 1 \) is guaranteed as long as all fermions obey antiperiodic boundary conditions. Here this observation is slightly generalized to other boundary conditions. It should be noted however that not all boundary conditions are allowed. This restriction on boundary conditions is distinct from the one discussed in [3].

To compute \( \langle V \rangle \) we need to also compute the path integral with the insertion of \( V \). It gets a contribution only from the \( k = 1 \) sector. In this sector there are fermionic zero modes and we need their explicit form. The zero modes at arbitrary \( \phi \) and \( g \) are simply related to those presented below for \( \phi = 0 \) and \( g \equiv 1 \).

For the \( q = 1 \) fermions, the equation

\[
\sigma_\mu(\partial_\mu + iA_\mu)\chi^0_f(x_1, x_2) = 0
\]

has one solution. Normalizing, it takes the form

\[
\chi^0_f(x_1, x_2) = \frac{2^{\frac{5}{2}}}{l^2} e^{\pi i\left[(b_1^f h_1 + h_2^f - h_2)(z_2 + h_1) + \frac{1}{2}\right]} \hat{\theta}(h_1 + b_1^f + z_2, h_2 + b_2^f - z_1; 1)
\]
where \( z_\mu = \frac{x_\mu}{\ell} \). For the \( q = 2 \) fermion, there are two linearly independent solutions to the equation

\[
\sigma_\mu (\partial_\mu + 2i A_\mu) \psi^0_p(x_1, x_2) = 0; \quad p = 0, 1
\]

which we pick (again normalized) of the form

\[
\psi^0_p(x_1, x_2) = \sqrt{\frac{2}{l}} e^{\pi i \left[ 2h_1 (h_2 - z_1) + (p - 2h_2)(z_2 + h_1) + \frac{1}{8} \right]} \hat{\theta}(2h_1 + 2z_2, h_2 - z_1 + \frac{p}{2}; 2)
\]

The above zero modes contain some phase choices we were free to make. Since the phase choices contain an undetermined dependence on the \( h_\mu \) what we choose has a nontrivial effect on the final answer. We recall that the \( h_\mu \) label different saddles. As such they play the role of collective coordinates the zero modes depend on. From the vector case we know that their role is to restore translational invariance of the one point vertex we are computing. We extend this role also to the phase choice. The \( h_\mu \) dependent phase factors are thus added so as to make the solution a function of \((z_1 - h_2)\) and \((z_2 + h_1)\). This leaves us with only one free constant which we pick so that the final answer corresponds to a vanishing \( \theta \)-parameter. With all this in place, the path integral in the \( k = 1 \) sector is

\[
Z_1 = \frac{1}{l^4} e^{-\frac{2\pi^2}{l^2}} \int d^2 h e^{\pi i \left[ (z_1 - h_2) \sum_f b_1^f + (z_2 + h_1) \sum_f b_2^f + \frac{1}{4} \right]} \left\{ \prod_f \hat{\theta}(h_1 + b_1^f + z_2, h_2 + b_2^f - z_1; 1) \right\}
\]

\[
\left\{ \hat{\theta}^* (2h_1 + 2z_2, h_2 - z_1; 2) (\partial_1 + i \partial_2) \left[ e^{-\pi i (z_2 + h_1)} \hat{\theta}^* (2h_1 + 2z_2, h_2 - z_1 + \frac{1}{2}; 2) \right] \right.
\]

\[
- (\partial_1 + i \partial_2) \left[ \hat{\theta}^* (2h_1 + 2z_2, h_2 - z_1; 2) e^{-\pi i (z_2 + h_1)} \hat{\theta}^* (2h_1 + 2z_2, h_2 - z_1 + \frac{1}{2}; 2) \right] \right\}
\]

\[
\int D\phi e^{-\Gamma(\phi) - 8\phi(x)}
\]

where \( z_\mu = \frac{x_\mu}{\ell} \) and \( \partial_\mu \) means derivative with respect to \( z_\mu \). \( Z_1 \) factorizes in a way similar to \( Z_0 \). The dependence on \( m_\gamma \) comes in through a factor which would be the same had we computed the ’t Hooft vertex in the four flavor Schwinger model.

Using an identity proven in the Appendix we can relate the \( q = 2 \) contribution at
\[ k = 0 \text{ with no insertion to the } q = 2 \text{ contribution at } k = 1 \text{ in the presence of the insertion:} \]

\[
\left\{ \hat{\theta}^*(2h_1 + 2z_2, h_2 - z_1; 2)(\partial_1 + i\partial_2) \left[ e^{-\pi i(z_2 + h_1)} \hat{\theta}^*(2h_1 + 2z_2, h_2 - z_1 + \frac{1}{2}; 2) \right] \right. \\
- \left( \partial_1 + i\partial_2 \right) \left[ \hat{\theta}^*(2h_1 + 2z_2, h_2 - z_1; 2) e^{-\pi i(z_2 + h_1)} \hat{\theta}^*(2h_1 + 2z_2, h_2 - z_1 + \frac{1}{2}; 2) \right] \\
= 4\pi \eta^3(1) e^{\pi i(z_1 - h_2 - z_2 - h_1 - \frac{1}{4})} \hat{\theta}^*(2h_1 + 2z_2 - \frac{1}{2}, 2h_2 - 2z_1 + \frac{1}{2}; 1) \\
\]

We then obtain the expectation value of the 't Hooft vertex in our finite volume:

\[
\langle V \rangle_l = \frac{64\pi}{(lm_{\gamma})^4} T_n \exp \left[ -\frac{4\pi}{l m_{\gamma}} \text{coth} \left( \frac{1}{2} l m_{\gamma} \right) \right] e^{4F(lm_{\gamma}) - 8H(lm_{\gamma}, 1)}.
\]

The new \( lm_{\gamma} \) dependent factors come from the integration over \( \phi \). The functions \( F(x) \) and \( H(x, \tau) \) are given by

\[
F(x) = \sum_{n>0} \left[ \frac{1}{n} - \frac{1}{\sqrt{n^2 + \left( x/2\pi \right)^2}} \right] \\
H(x, \tau) = \sum_{n>0} \frac{1}{\sqrt{n^2 + \left( x/2\pi \right)^2}} \frac{1}{e^{\tau \sqrt{(2\pi n)^2 + x^2}} - 1}.
\]

\( T_n \) and \( T_d \) are given by

\[
T_n = \int d^2h \left\{ \prod_f \hat{\theta}(h_1 + b_1^f + z_2, h_2 + b_2^f - z_1; 1) \right\} \left\{ \hat{\theta}^*(2h_1 + 2z_2 - \frac{1}{2}, 2h_2 - 2z_1 + \frac{1}{2}; 1) \right\} \\
e^{\pi i \left[ \sum_f b_1^{f+1}(z_1 - h_2) + \sum_f b_1^f - 1 \right] (z_2 + h_1)}
\]

\[
T_d = \int d^2h \left\{ \prod_f \hat{\theta}(h_1 + b_1^f + \frac{1}{2}, h_2 + b_2^f - \frac{1}{2}; 1) \right\} \left\{ \hat{\theta}^*(2h_1 + \frac{1}{2}, 2h_2 - \frac{1}{2}; 1) \right\}.
\]

Periodicity under \( h_\mu \rightarrow h_\mu + 1 \) (which are gauge transformations) of the integrand in \( T_n \) is assured by \( (11) \). On the other hand the integrand is only a function of the combinations \( h'_1 = h_1 + z_2 \) and \( h'_2 = h_2 - z_1 \), so it is actually periodic with unit period in the \( h'_\mu \). Since we integrate over a full fundamental domain the integral \( T_n \) becomes independent of the \( z_\mu \). We are then free to set \( z_\mu = \frac{1}{2} \) and observe that the integrand in \( T_n \) becomes identical to the integrand in \( T_d \) up to a phase. The boundary conditions in \( (5) \) make this phase vanish and imply \( T_n = T_d \). Actually, \( (5) \) also implies that in both integrands one can substitute for the \( q = 1 \) factor (represented by the product over “flavors” \( f \)) the complex
conjugate of the $q = 2$ factor, rendering the integrands non-negative. The relevant identity was given in [3].

However, there are solutions to (11) which differ from (5) and for which the phase in the integrand of $T_n$ does not vanish. In that case one has to carry out the integrals, which is not difficult, and one gets a $T_n/T_d$ ratio different from unity. Equation (22) below will change then and we shall have no agreement with clustering at infinite volume. Note that these complications regarding boundary conditions are absent in a vector theory. There flavor dependent boundary conditions have no effect on the vertex. Still, the finite size correction, as a multiplicative factor, is universal and can be obtained from the associated vector theory.

With our “good” boundary conditions, (5), $T_n = T_d$ and (18) gives us the final answer for the vertex in a finite volume. The infinite volume limit of (18) is

$$\langle V \rangle = \frac{e^{4\gamma}}{4\pi^3}$$

and agrees with the result in (4) quoted from [2] where it was obtained asserting clustering.

Up to now we worked on a torus of equal sides. The formulae can be generalized to the case of a torus of size $t \times l$ and the result is of potential use for future simulations. With the boundary conditions of the same type as in (5) the more general expression for the ’t Hooft vertex on the torus reads:

$$\langle V \rangle_{t \times l} = \frac{64\pi}{(tm_{\gamma})^4} \exp\left[ -\frac{4\pi}{tm_{\gamma}} \coth \left( \frac{1}{2}lm_{\gamma} \right) \right] e^{4F(tm_{\gamma})-8H(tm_{\gamma}, t)}.$$  \hspace{1cm} (23)

2. Now that we have the finite volume result we can look at the data in [2] closer since the only source of systematic error left is the finiteness of the UV cutoff. The size of the lattice is $L$ in each direction with $La = l$ where the lattice spacing is $a$. Our simulation was at constant $l$ (in terms of the gauge coupling) and we attempted to take $a$ to zero by letting $L$ grow.

We discussed in [2] the possible appearance of a Thirring term (of dimension 2) and tuned a certain free parameter in the overlap to make the induced Thirring coupling numerically negligible. So, we assume that there is no Thirring coupling and see if this assumption is consistent with our data and the exact result. The remaining finite $a$ corrections come in integral powers of $a$. Such corrections come from two sources: the operator and the action. The correction coming from the operator have been dealt with in [2] where we extracted a factor representing (quite sizable) UV corrections due to the lattice point split representation of the derivative in $V$. Thus we are left to worry only about corrections
coming from the action. Exact global chiral symmetries are preserved by the overlap and therefore only order $a^2$ corrections, coming from dimension four operators are allowed.*

In figure 1 we show a plot with our data (with full gauge invariance implemented). We have data for $L = 8, 10, 12, 14, 16$ and they are plotted against $1/L^2$ which is the same as $a^2$ for fixed $l$. To get a feel for how our numbers would extrapolate to the continuum we computed a simple least square linear fit to the last four points and show the straight line so obtained. The infinite $L$ limit is the continuum number computed from (18) and is also shown on the graph.

* If one applies the overlap to the simulation of four dimensional vector theories with massless quarks, one shall need no order $a$ improvement since chirality is exact. All errors are of order $a^2$. 

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Figure 1  Data for $L = 8, 10, 12, 14, 16$ versus $1/L^2 \propto a^2$, a linear fit to the points $L \geq 10$, the continuum result (rhombus at $a = 0$), and our estimate for the continuum result from the data (square with error bar at $a = 0$).
Our estimate, $.0376 \pm .0019$, for the continuum limit value of the ’t Hooft vertex in the volume we worked at $(lm, \gamma = 3)$ comes from the intercept of the linear fit. It falls sufficiently close to the exact result we get from (18), namely 0.0389, for us to be quite confident in our assumption about the smallness of the effective Thirring coupling. Realistically, we could have hardly expected a result more favorable to the overlap than the one we obtained.

Therefore we would claim to have shown that the overlap quantitatively reproduces fermion number violation in the 11112 model.

**Appendix**

In this appendix we prove the identity written down in (17) which was used in our calculation of the ’t Hooft vertex.

Using the expression for the $\hat{\theta}$ function in (7) the LHS of the (17) becomes

$$\sum_{n,m} [4\pi(2m - 2n + 1)] \exp\left[ -2\pi(n^2 + m^2) - 4\pi(n + m)(h_2 - z_1) - 2\pi m - 2\pi(h_2 - z_1)^2 - 2\pi(h_2 - z_1 + \frac{1}{2})^2 - 4\pi i(n + m)(h_1 + z_2) - 2\pi i(h_1 + z_2)(2h_2 - 2z_1 + \frac{1}{2}) - \pi i(z_2 + h_1) \right]$$

(A.1)

We split the sum into cases. When $(n + m)$ is even, we write $n = k + l$ and $m = k - l$. When $(n + m)$ is odd, we write $n = k - l + 1$ and $m = k + l$. As $n$ and $m$ range over all integers, $k$ and $l$ also range over all integers. The LHS of (17) then becomes

$$- \sum_l 4\pi(4l - 1)e^{-4\pi(l - \frac{1}{4})^2} + \sum_k e^{-\pi(2k + 2h_2 - 2z_1 + \frac{1}{2})^2 - 2\pi i(2k)(2h_1 + 2z_2) - \pi i(2h_2 - 2z_1 + \frac{1}{2})(2h_1 + 2z_2) - \pi i(z_2 + h_1)}$$

$+ \sum_l 4\pi(4l - 1)e^{-4\pi(l - \frac{1}{4})^2} + \sum_k e^{-\pi(2k + 1 + 2h_2 - 2z_1 + \frac{1}{2})^2 - 2\pi i(2k + 1)(2h_1 + 2z_2) - \pi i(2h_2 - 2z_1 + \frac{1}{2})(2h_1 + 2z_2) - \pi i(z_2 + h_1)}$

(A.2)

where the first term corresponds to even $(n + m)$ and the second term corresponds to odd $(n + m)$. In both the terms in (A.2), the first factor is the same. The second factors in both the terms can be combined into one sum since the first one have $2k$ and the second
one has $2k + 1$. Then we can write (A.2) as

$$
\sum_l 16\pi \left( \frac{1}{4} - l \right) e^{-4\pi (l - \frac{1}{4})^2} \\
\sum_n e^{-\pi (n + 2h_2 - 2z_1 + \frac{1}{2})^2 - 2\pi in(2h_1 + 2z_2 - \frac{5}{4}) - \pi i(2h_2 - 2z_1 + \frac{1}{2})(2h_1 + 2z_2 - \frac{5}{4}) + \pi i(z_1 - h_2 - z_2 - h_1 - \frac{1}{4})} \\
= 4\pi \eta^3(1) \\
\sum_n e^{-\pi (n + 2h_2 - 2z_1 + \frac{1}{2})^2 - 2\pi in(2h_1 + 2z_2 - \frac{5}{4}) - \pi i(2h_2 - 2z_1 + \frac{1}{2})(2h_1 + 2z_2 - \frac{5}{4}) + \pi i(z_1 - h_2 - z_2 - h_1 - \frac{1}{4})} 
$$

(A.3) is obtained by employing the following identities

$$
\eta^3(1) = \frac{1}{2\pi} \theta'_1(0,1) = \sum_{n=\infty}^{\infty} (-1)^n (n + \frac{1}{2}) e^{-\pi (n + \frac{1}{2})^2} = 4 \sum_{l=\infty}^{\infty} \left( l + \frac{1}{4} \right) e^{-4\pi (l + \frac{1}{4})^2} 
$$

(A.4)

The first equality in (A.4) is a well known classical identity. For our purposes here, the second equality can be viewed simply as a definition of the quantity $\frac{1}{2\pi} \theta'_1(0,1)$. The last equality in (A.4) is obtained by splitting the sum over $n$ into even $n = 2l$ and odd $n = -(2l + 1)$ where in both cases $l$ ranges from $-\infty$ to $\infty$. Using the definition of $\hat{\theta}$ from (7) in (A.3) results in the RHS of (17).

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