Fractional-Spin Integrals of Motion for the Boundary Sine-Gordon Model at the Free Fermion Point

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Abstract

We construct integrals of motion (IM) for the sine-Gordon model with boundary at the free Fermion point ($\beta^2 = 4\pi$) which correctly determine the boundary S matrix. The algebra of these IM ("boundary quantum group" at $q = 1$) is a one-parameter family of infinite-dimensional subalgebras of twisted $sl(2)$. We also propose the structure of the fractional-spin IM away from the free Fermion point ($\beta^2 \neq 4\pi$).
1 Introduction

Much is known about trigonometric solutions of the Yang-Baxter equations, primarily because the algebraic structure underlying these equations has been identified and elucidated—namely, quantum groups \([1]\). In contrast, much less is known about corresponding solutions of the boundary Yang-Baxter equation \([2]\), because the relevant algebraic structure (some sort of “boundary quantum group”) has not yet been formulated.

Motivated in part by such considerations, we have undertaken together with A. B. Zamolodchikov a project \([3]\) to construct fractional-spin integrals of motion \([4] - [6]\) of the sine-Gordon model with boundary \([7]\), which should generate precisely such an algebraic structure. A further motivation for this work is to determine the exact relation between the parameters of the action and the parameters of the boundary \(S\) matrix given in \([7]\).

We focus here on the special case of the free Fermion point \((\beta^2 = 4\pi)\) \([8]\). We construct Fermionic integrals of motion (IM) which correctly determine the boundary \(S\) matrix, and which flow to the “topological charge” as the “boundary magnetic field” \(h\) is varied from \(h = 0\) to \(h \to \infty\). The algebra of these IM (“boundary quantum group” at \(q = 1\)) is a one-parameter \((h)\) family of infinite-dimensional subalgebras of twisted \(\widehat{\mathfrak{sl}}(2)\). We also give an Ansatz for the structure of the fractional-spin IM away from the free Fermion point \((\beta^2 \neq 4\pi)\). A preliminary account of this work has appeared in Refs. \([9]\) and \([10]\).

We now outline the contents of the paper. In Sec. 2 we briefly review some basic properties of both bulk and boundary sine-Gordon field theory for general values of \(\beta\). In Sec. 3 we consider the special case of the free Fermion point. We first review the Fermionic description of the fractional-spin IM for the bulk theory. There follows the main part of the paper, in which we generalize these results to the boundary theory. In particular, we describe the Fermionic IM and their algebra, and we verify that these IM correctly determine the boundary \(S\) matrix. In Sec. 4 we propose the structure of the fractional-spin IM away from the free Fermion point. We conclude in Sec. 5 with a brief discussion of some open problems.

2 Sine-Gordon field theory

We briefly review here some basic properties of the sine-Gordon field theory for general values of \(\beta\). We first consider the bulk theory. We describe the soliton/antisoliton \(S\) matrix \([11]\) and the fractional-spin integrals of motion \([6]\). We then turn to the boundary sine-Gordon theory, and describe the boundary \(S\) matrix \([7]\).
2.1 Bulk theory

The Lagrangian density of the bulk sine-Gordon field theory in Minkowski spacetime is given by

\[ \mathcal{L}_0 = -\frac{1}{2} (\partial_\mu \Phi)^2 + \frac{m_0^2}{\beta^2} \cos(\beta \Phi), \]  

(1)

where \( \Phi(x,t) \) is a real scalar field, \( m_0 \) has dimensions of mass, and \( \beta \) is a dimensionless coupling constant. For \( 4\pi \leq \beta^2 \leq 8\pi \), the particle spectrum consists only of solitons and antisolitons, with equal masses \( m \), and with “topological charge”

\[ T = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \Phi(x,t) \]  

(2)

equal to +1 or −1, respectively. The particles’ two-momenta \( p_\mu \) are conveniently parameterized in terms of their rapidities \( \theta \):

\[ p_0 = m \cosh \theta, \quad p_1 = m \sinh \theta. \]  

(3)

This theory is integrable; i.e., it possesses an infinite number of integer-spin integrals of motion. Consequently, the scattering of solitons is factorizable. (See [11] and references therein.) For the case of two particles with rapidities \( \theta_1 \) and \( \theta_2 \), the two-particle \( S \) matrix \( S(\theta) \) (where \( \theta = \theta_1 - \theta_2 \)) is defined by

\[ A_{a_1}(\theta_1)^\dagger A_{a_2}(\theta_2)^\dagger = S_{a_1}^{b_1} b_2(\theta) A_{b_2}(\theta_2)^\dagger A_{b_1}(\theta_1)^\dagger, \]  

(4)

where \( A_{\pm}(\theta)^\dagger \) are “particle-creation operators”. The \( S \) matrix obeys the Yang-Baxter equation

\[ S_{12}(\theta) S_{13}(\theta + \theta') S_{23}(\theta') = S_{23}(\theta') S_{13}(\theta + \theta') S_{12}(\theta), \]  

(5)

and therefore has the form

\[ S(\theta) = \rho(\theta) \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix}, \]  

(6)

where

\[ a(\theta) = \sin[\lambda(\pi - u)] \]

\[ b(\theta) = \sin(\lambda u) \]

\[ c(\theta) = \sin(\lambda \pi) \]  

(7)
and \( u = -i \theta \). The relation between the coupling constant \( \beta \) and the parameter \( \lambda \) of the \( S \) matrix

\[
\lambda = \frac{8\pi}{\beta^2} - 1
\]

(8)
can be inferred \[11\] from semiclassical results for the \( S \) matrix and the mass spectrum. The unitarization factor \( \rho(\theta) \) can be found in Ref \[11\].

In addition to having an infinite number of integrals of motion (IM) of integer spin, the sine-Gordon model also has fractional-spin IM. Within the framework of Euclidean-space perturbed conformal field theory \[12\], the IM \( Q_\pm \) and \( \bar{Q}_\pm \) (with Lorentz spin \( \lambda \) and \( -\lambda \), respectively) are given by \[6\]

\[
Q_\pm = \frac{1}{2\pi i} \left( \int dz J_\pm + \int d\bar{z} H_\pm \right),
\]

\[
\bar{Q}_\pm = \frac{1}{2\pi i} \left( \int d\bar{z} J_\pm + \int dz H_\pm \right),
\]

(9)

where, up to a normalization factor,

\[
J_\pm(x, t) = \exp \left[ \pm \frac{2i}{\beta} \phi(x, t) \right], \quad \bar{J}_\pm(x, t) = \exp \left[ \mp \frac{2i}{\beta} \bar{\phi}(x, t) \right],
\]

\[
H_\pm(x, t) = m_0 \frac{\beta^2}{\beta^2 - 2} \exp \left[ \pm i \left( \frac{2}{\beta} - \hat{\beta} \right) \phi(x, t) \mp i\hat{\beta} \bar{\phi}(x, t) \right],
\]

\[
\bar{H}_\pm(x, t) = m_0 \frac{\beta^2}{\beta^2 - 2} \exp \left[ \mp i \left( \frac{2}{\beta} - \hat{\beta} \right) \bar{\phi}(x, t) \pm i\hat{\beta} \phi(x, t) \right],
\]

(10)

with \( \hat{\beta} = \beta/\sqrt{4\pi} \) and where \( \phi(x, t) \) and \( \bar{\phi}(x, t) \) are given by

\[
\phi(x, t) = \frac{\sqrt{4\pi}}{2} \left( \Phi(x, t) + \int_{-\infty}^{x} dy \ \partial_t \Phi(y, t) \right),
\]

\[
\bar{\phi}(x, t) = \frac{\sqrt{4\pi}}{2} \left( \Phi(x, t) - \int_{-\infty}^{x} dy \ \partial_t \Phi(y, t) \right),
\]

(11)

respectively. An appropriate regularization prescription is implicit in the above expressions for the currents.

These IM obey the so-called \( q \)-deformed twisted affine \( sl(2) \) algebra \( (sl_q(2)) \) with zero center, where

\[
q = e^{-i\pi(\lambda-1)}.
\]

(12)
Moreover, these IM have nontrivial commutation relations with the particle-creation operators, which in terms of the matrix notation

\[
A(\theta)^\dagger = \begin{pmatrix} A_+(\theta)^\dagger & A_-(\theta)^\dagger \end{pmatrix}
\]

are of the form

\[
\begin{align*}
Q_\pm A(\theta)^\dagger &= q^{\pm \sigma_3} A(\theta)^\dagger Q_\pm + \alpha e^{i\lambda \theta} \sigma_\mp A(\theta)^\dagger,
\tilde{Q}_\pm A(\theta)^\dagger &= \bar{q}^{\mp \sigma_3} A(\theta)^\dagger \tilde{Q}_\pm + \bar{\alpha} e^{-i\lambda \theta} \sigma_\mp A(\theta)^\dagger,
T A(\theta)^\dagger &= A(\theta)^\dagger T + \sigma_3 A(\theta)^\dagger,
\end{align*}
\]

(13)

where the values of \( \alpha \) and \( \bar{\alpha} \) depend on the normalization and regularization of the currents, with

\[
\alpha \to 1, \quad \bar{\alpha} \to 1 \quad \text{for} \quad q \to 1.
\]

(14)

Associativity of the products in \( QA_{a_1}(\theta_1)^\dagger A_{a_2}(\theta_2)^\dagger \) and invariance of the vacuum \( Q|0\rangle = 0 \) (where \( Q = Q_\pm, T \) or \( Q = \tilde{Q}_\pm, T \)); or, equivalently,

\[
[S, \Delta(Q)] = 0
\]

(15)

(where \( \tilde{S} \) is the “S operator” and \( \Delta \) is the comultiplication) leads to the \( S \) matrix (3), up to the unitarization factor. Note that this derivation of the \( S \) matrix yields the relation (8) between the coupling constant \( \beta \) and the parameter \( \lambda \) of the \( S \) matrix. Also note that only half of the fractional-spin IM are needed to determine the \( S \) matrix. This approach of computing \( S \) matrices is a realization in quantum field theory of the general program [8] of linearizing the problem of finding trigonometric solutions of the Yang-Baxter equations.

2.2 Boundary theory

Following Ghoshal and Zamolodchikov [7], we consider the sine-Gordon field theory on the negative half-line \( x \leq 0 \), with action

\[
S = \int_{-\infty}^{\infty} dt \left\{ \int_{0}^{0} dx \mathcal{L}_0 + M \cos \frac{\beta}{2} (\Phi - \Phi_0) \right\}_{x=0},
\]

(16)

where \( \mathcal{L}_0 \) is given by Eq. (4), and \( M \) and \( \Phi_0 \) are parameters which specify the boundary conditions. There is compelling evidence that this choice of boundary conditions preserves the integrability of the bulk theory, and hence, the scattering of the solitons remains factorizable. The boundary \( S \) matrix \( R(\theta) \), which is defined by [4]

\[
A(\theta)^\dagger B = R(\theta) A(-\theta)^\dagger B,
\]

(17)
where $B$ is the boundary operator, describes the scattering of solitons and antisolitons off the boundary. The boundary $S$ matrix must satisfy the boundary Yang-Baxter equation \[2\]
\[S_{12}(\theta_1 - \theta_2) R_1(\theta_1) S_{21}(\theta_1 + \theta_2) R_2(\theta_2) = R_2(\theta_2) S_{12}(\theta_1 + \theta_2) R_1(\theta_1) S_{21}(\theta_1 - \theta_2),\] and therefore has the form \[3\], \[13\]
\[R(\theta) = r(\theta) \begin{pmatrix} \cosh(\lambda \theta + i \xi) & -\frac{i}{2} k_+ \sinh(2\lambda \theta) \\ -\frac{i}{2} k_- \sinh(2\lambda \theta) & \cosh(\lambda \theta - i \xi) \end{pmatrix}.\] Evidently, this matrix depends on the three boundary parameters $\xi, k_+, k_-; however, by a suitable gauge transformation, the off-diagonal elements can be made to coincide. Thus, the boundary $S$ matrix in fact depends on only two boundary parameters, as does the action \[16\]. The relation between these two sets of parameters is not known for general values of $\beta$.

3 Free Fermion point

As discussed in the Introduction, we would like to construct fractional-spin IM for the sine-Gordon field theory with boundary for general values of $\beta$. We consider in this Section the special case $\beta^2 = 4\pi$ which corresponds to $q = 1$ (i.e., undeformed algebra). We begin by reviewing the Fermionic description of the fractional-spin IM for the bulk theory \[14\]. We then generalize these results to the boundary theory. In particular, we describe the Fermionic IM and their algebra, and we verify that these IM correctly determine the boundary $S$ matrix.

3.1 Bulk theory

As is well known \[8\], the bulk sine-Gordon field theory \[1\] with $\beta^2 = 4\pi$ is equivalent to a free massive Dirac field theory. The corresponding Lagrangian density is\[4\]
\[L_0^{FF} = \frac{i}{2} \bar{\Psi} \overset{T}{\partial} \Psi - im \bar{\Psi} \Psi \\
= -i \left[ \bar{\psi}_- \partial_+ \psi_+ + \bar{\psi}_+ \partial_- \psi_- - \psi_- \partial_- \psi_+ - \psi_+ \partial_- \psi_- - m(\bar{\psi}_- \psi_+ - \psi_- \bar{\psi}_+) \right],\] where $\Psi = \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right)$, $\Psi^\dagger = \left( \begin{array}{c} \psi_- \\ \psi_+ \end{array} \right)$. Evidently, the Lagrangian has a $U(1)$ symmetry; $\psi_+$, $\bar{\psi}_+$ have charge $+1$ and $\psi_-, \bar{\psi}_-$ have charge $-1$.

\[1\] Our conventions are $\eta^{00} = -1 = -\eta^{11}$; $\bar{\Psi} = \Psi^\dagger \gamma^0$; $\gamma^0 = -i \sigma_2$; $\gamma^1 = \sigma_1$, where $\sigma_i$ are the Pauli matrices; $\partial_\pm = \frac{1}{2} (\partial_0 \pm \partial_1)$
By solving the field equations
\[
\partial_- \psi_\pm = \frac{m}{2} \tilde{\psi}_\pm, \quad \partial_+ \tilde{\psi}_\pm = \frac{m}{2} \psi_\pm, \quad (21)
\]
we are led (following the conventions of \cite{7}) to the mode expansions
\[
\psi_+(x, t) = \sqrt{\frac{m}{4\pi}} \int_{-\infty}^{\infty} d\theta \, e^{\frac{\theta}{2}} \left[ \omega A_-(\theta) e^{ip \cdot x} + \omega^* A_+(\theta)^\dagger e^{-ip \cdot x} \right],
\]
\[
\tilde{\psi}_+(x, t) = \sqrt{\frac{m}{4\pi}} \int_{-\infty}^{\infty} d\theta \, e^{-\frac{\theta}{2}} \left[ \omega^* A_-(\theta) e^{ip \cdot x} + \omega A_+(\theta)^\dagger e^{-ip \cdot x} \right], \quad (22)
\]
where \( p \cdot x = p_0 t + p_1 x \) with \( p_\mu \) given by Eq. (3), and \( \omega = e^{i\pi/4}, \omega^* = e^{-i\pi/4} \). Canonical quantization implies that the only nonvanishing anticommutation relations for the modes are
\[
\{ A_\pm(\theta), A_\pm(\theta') \} = \delta(\theta - \theta'). \quad (23)
\]
We regard \( A_\pm(\theta)^\dagger \) as creation operators.

The fractional-spin integrals of motion (9) and the topological charge (2) have at the free Fermion point the form \cite{14}
\[
Q_\pm = -\frac{i}{m} \int_{-\infty}^{\infty} dx \, \psi_\pm \dot{\psi}_\pm = \int_{-\infty}^{\infty} d\theta \, \dot{\theta} A_\pm(\theta)^\dagger A_\mp(\theta),
\]
\[
\bar{Q}_\pm = -\frac{i}{m} \int_{-\infty}^{\infty} dx \, \tilde{\psi}_\pm \dot{\tilde{\psi}}_\pm = \int_{-\infty}^{\infty} d\theta \, e^{-\dot{\theta}} A_\pm(\theta)^\dagger A_\mp(\theta),
\]
\[
T = -\int_{-\infty}^{\infty} dx \left( \psi_\pm \dot{\psi}_\mp + \tilde{\psi}_\pm \dot{\tilde{\psi}}_\mp \right) = \int_{-\infty}^{\infty} d\theta \left[ A_+(\theta)^\dagger A_+(\theta) - A_-(\theta)^\dagger A_-(\theta) \right], \quad (24)
\]
where the dot (\( \dot{\cdot} \)) denotes differentiation with respect to time. Here we normalize the IM so that their commutation relations with the particle-creation operators are given by Eq. (13) with \( \alpha = \bar{\alpha} = 1 \).

Let us set \( Q_{\pm 1} \equiv Q_\pm, Q_{\mp 1} \equiv \bar{Q}_\pm, T_0 \equiv T \). By closing the algebra of these IM, we obtain the infinite-dimensional twisted affine Lie algebra \( \hat{sl}(2) \) with zero center \cite{14}:
\[
[Q_n, Q_m] = T_{n+m}, \quad [T_n, Q_m] = \pm 2Q_{n+m}, \quad (25)
\]
(all other commutators vanish), where
\[
\begin{align*}
Q_n^+ &= \int_{-\infty}^{\infty} d\theta \, e^{-n\theta} A_+(\theta)^\dagger A_-(\theta), \\
Q_n^- &= \int_{-\infty}^{\infty} d\theta \, e^{-n\theta} A_-(\theta)^\dagger A_+(\theta), \\
\end{align*}
\quad n \text{ odd}, \quad (26)
\]

\[ T_n = \int_{-\infty}^{\infty} d\theta \, e^{-n\theta} \left[ A_+(\theta)^\dagger A_+(\theta) - A_-^{\dagger}(\theta) A_-(\theta) \right], \quad n \text{ even}. \]  

Note that

\[ Q^\dagger_n = Q_n^-; \quad T^\dagger_n = T_n. \]  

The expressions in coordinate space for \( Q^\pm_n \) and \( T_n \) have derivatives of highest order \(|n|\).

### 3.2 Boundary theory

The sine-Gordon field theory with boundary (16) for \( \beta^2 = 4\pi \) is equivalent to [9], [15]

\[ S = \int_{-\infty}^{\infty} dt \left\{ \int_{-\infty}^{0} dx \, L^0_{FF} + L_{\text{boundary}} \right\}, \tag{29} \]

where \( L^0_{FF} \) is given by Eq. (20), and \( L_{\text{boundary}} \) is given by

\[ L_{\text{boundary}} = \left. \frac{i}{2} \left[ e^{i(\phi+\varphi)} \bar{\psi}_+ \psi_+ + e^{-i(\phi+\varphi)} \bar{\psi}_- \psi_- + a \dot{a} ight] - h \left( e^{i\phi} \bar{\psi}_+ + e^{-i\phi} \bar{\psi}_- + e^{i\phi} \psi_+ + e^{-i\phi} \psi_- \right) \right|_{x=0}, \tag{30} \]

where \( a(t) \) is a Fermionic boundary degree of freedom, and \( h, \phi, \varphi \) are real parameters which specify the boundary conditions. Varying the action gives (upon eliminating \( a(t) \) through its equations of motion) the following boundary conditions

\[ \left( e^{i\phi} \bar{\psi}_+ - e^{-i\phi} \bar{\psi}_- + e^{i\phi} \psi_+ - e^{-i\phi} \psi_- \right) \big|_{x=0} = 0, \]  

\[ \left. \frac{d}{dt} \left( e^{i\phi} \bar{\psi}_+ + e^{-i\phi} \bar{\psi}_- - e^{i\phi} \psi_+ - e^{-i\phi} \psi_- \right) \right|_{x=0} - h^2 \left( e^{i\phi} \bar{\psi}_+ + e^{-i\phi} \bar{\psi}_- + e^{i\phi} \psi_+ + e^{-i\phi} \psi_- \right) \big|_{x=0} = 0. \]  

This action is a generalization of the one for the off-critical Ising field theory (free massive Majorana field) with a boundary magnetic field \( h \). The action given in [15] differs slightly from (30); specifically, it involves two Fermionic boundary degrees of freedom instead of one. However, it gives the same boundary conditions (31), (32), apart from minor differences in notation. The relation between the parameters of the Bosonic and Fermionic actions is discussed in [15].
Computing the boundary $S$ matrix $R(\theta)$ according to the definition (17) and the mode expansions (22) along the lines of Ref. [7], we obtain the result (19), with $\lambda = 1$ and

\[ e^{2i\xi} = \frac{e^{-i(\phi-\varphi)} - \frac{m}{\hbar^2}}{e^{i(\phi-\varphi)} - \frac{m}{\hbar^2}}, \quad (33) \]

\[ k_{\pm} = -\frac{me^{\mp i(\phi+\varphi)}}{\hbar \sqrt{1 - \frac{2m}{\hbar^2} \cos(\phi - \varphi) + \frac{m^2}{\hbar^4}}}, \quad (34) \]

\[ r(\theta) = -\frac{\sqrt{1 - \frac{2m}{\hbar^2} \cos(\phi - \varphi) + \frac{m^2}{\hbar^4}} \cosh(\phi - \varphi) - i \sinh \theta - \frac{m}{\hbar^2} \cosh^2 \theta}{\cosh(\phi - \varphi) - i \sinh \theta - \frac{m}{\hbar^2} \cosh^2 \theta}. \quad (35) \]

Indeed, the action (29), (30) was formulated specifically to reproduce this result of [7].

We turn now to the problem of constructing IM. In analogy with the bulk IM (24), we define the following charges on the half-line:

\[ Q_\pm = -\frac{i}{m} \int_{-\infty}^{0} dx \, \psi_\pm^{*} \psi_\pm, \]

\[ \bar{Q}_\pm = -\frac{i}{m} \int_{0}^{\infty} dx \, \bar{\psi}_\pm^{*} \bar{\psi}_\pm, \]

\[ T = -\int_{-\infty}^{0} dx \left( \psi_\pm \psi_\pm^{*} + \bar{\psi}_\pm \bar{\psi}_\pm^{*} \right). \quad (36) \]

Evidently, these charges are of the form $\int_{-\infty}^{\infty} dx \, J_0$, where $J_0$ is the time component of a two-component current $J^\mu$ which is conserved ($\partial_\mu J^\mu = 0$). Nevertheless, in general these charges (unlike $\int_{-\infty}^{\infty} dx \, J^0$) are not separately conserved, since $J^1|_{x=0} \neq 0$.

For future convenience, we also define (in analogy with (23), (27)) the quantities

\[ Q_n^+ = \int_{0}^{\infty} d\theta \, e^{-n\theta} A_+(\theta)^{\dagger} A_-(\theta) \]

\[ Q_n^- = \int_{0}^{\infty} d\theta \, e^{-n\theta} A_-(\theta)^{\dagger} A_+(\theta) \]

\[ T_n = \int_{0}^{\infty} d\theta \, e^{-n\theta} \left[ A_+(\theta)^{\dagger} A_+(\theta) - A_-(\theta)^{\dagger} A_-(\theta) \right], \quad n \text{ even}, \quad (38) \]

which also obey the twisted $\widehat{sl}(2)$ algebra

\[ [Q_n^+, Q_m^-] = T_{n+m}, \]

\[ [T_n, Q_m^\pm] = \pm 2 Q_{n+m}^\pm. \quad (39) \]
We shall see that for the boundary theory (29), (30), the integrals of motion are linear combinations of (37) - (38), and the algebra of these IM is a subalgebra of (39). Before treating the general case, it is useful to first consider the special cases of “fixed” \((h \to \infty)\) and “free” \((h = 0)\) boundary conditions.

“Fixed” boundary conditions

For “fixed” boundary conditions \((h \to \infty)\), the Fermion fields satisfy

\[
\begin{align*}
\left. \left( \psi_+ + e^{-i(\phi-\varphi)} \bar{\psi}_+ \right) \right|_{x=0} &= 0, \\
\left. \left( \psi_- + e^{i(\phi-\varphi)} \bar{\psi}_- \right) \right|_{x=0} &= 0.
\end{align*}
\]

These boundary conditions preserve the \(U(1)\) symmetry of the bulk Lagrangian.

One can show using the field equations that for “fixed” boundary conditions there are three linearly independent combinations of the charges (36) which are integrals of motion:

\[
\begin{align*}
\hat{Q}_1^+ \text{fixed} &= 2 \left( \bar{Q}_+ e^{2i(\phi-\varphi)} Q_+ \right), \\
\hat{Q}_1^- \text{fixed} &= 2 \left( \bar{Q}_- e^{-2i(\phi-\varphi)} Q_- \right), \\
\hat{T}_0 \text{fixed} &= 2 T.
\end{align*}
\]

In analogy with the bulk analysis, we wish to rewrite these IM in terms of Fourier modes. A direct – but laborious – approach is to substitute into the above expressions the mode expansions (22), to use the boundary \(S\) matrix to express modes with negative rapidity in terms of modes with positive rapidity (see Eq. (17)), and then to perform the \(x\) integrals and use certain identities which are obeyed by the boundary \(S\) matrix. The result is

\[
\begin{align*}
\hat{Q}_1^+ \text{fixed} &= 2 \left( Q_1^+ e^{\pm 2i(\phi-\varphi)} Q_{-1}^\pm \right), \\
\hat{T}_0 \text{fixed} &= 2 T_0,
\end{align*}
\]

where \(Q_n^\pm\) and \(T_n\) are given by Eqs. (37), (38). We now observe that the same result can be obtained much more readily by assuming that (for the purpose of writing a conserved charge in terms of Fourier modes) we can make the replacement

\[
\int_{-\infty}^{0} dx \to \frac{1}{2} \int_{-\infty}^{\infty} dx
\]

9
in the coordinate-space expression for the charges (36); and then make in the final result the replacement
\[ \int_{-\infty}^{\infty} d\theta \rightarrow 2 \int_{0}^{\infty} d\theta. \] (44)

We shall use this device extensively in the work that follows.

Moreover, we find higher-derivative integrals of motion which in terms of Fourier modes are given by
\[
\hat{Q}^\pm_{n \text{ fixed}} = Q^\pm_n + Q^\pm_{2-n} + e^{\pm i 2(\phi - \varphi)} (Q^\pm_{n-2} + Q^\pm_{-n}) , \quad n \text{ odd} \geq 1 , \\
\hat{T}^{}_{n \text{ fixed}} = T_n + T_{-n} , \quad n \text{ even} \geq 0 ,
\] (45)

where \( Q^\pm_n \) and \( T_n \) are given by Eqs. (37), (38). (See Appendix A for details.) Notice that the expressions for \( \hat{Q}^\pm_{n \text{ fixed}} \) and \( \hat{T}^{}_{n \text{ fixed}} \) are invariant under \( n \rightarrow 2-n \) and \( n \rightarrow -n \), respectively.

The IM for “fixed” boundary conditions obey the algebra (we now drop the label “fixed”)
\[
[\hat{Q}^+_n, \hat{Q}^-_m] = \hat{T}^{}_{n+m} + \hat{T}^{}_{n+m-4} + \hat{T}^{}_{2+n-m} + \hat{T}^{}_{2-n+m} + 2 \cos(2(\phi - \varphi)) (\hat{T}^{}_{n-m} + \hat{T}^{}_{n+m-2}) , \\
[\hat{T}^{}_{n}, \hat{Q}^\pm_m] = \pm 2 (\hat{Q}^\pm_{m+n} + \hat{Q}^\pm_{m-n}) ,
\] (46)

and all other commutators vanish.

“Free” boundary conditions

For “free” boundary conditions (\( h = 0 \)), the Fermion fields satisfy
\[
\left( \bar{\psi}^+ - e^{-i(\phi + \varphi)} \psi^- \right) \bigg|_{x=0} = 0 , \\
\left( \bar{\psi}^- - e^{i(\phi + \varphi)} \psi^+ \right) \bigg|_{x=0} = 0 .
\] (47)

These boundary conditions break the \( U(1) \) symmetry of the bulk Lagrangian.

For “free” boundary conditions, there are only two linearly independent combinations of the charges (36) which are integrals of motion: \( \hat{Q}^{}_{1 \text{ free}} \) and \( \hat{Q}^\dagger_{1 \text{ free}} \), where
\[
\hat{Q}^{}_{1 \text{ free}} = 2 \left( \bar{Q}^+ + e^{-2i(\phi + \varphi)} Q^- - ie^{-i(\phi + \varphi)} T \right) ,
\] (48)

Going to Fourier modes, we have
\[
\hat{Q}^{}_{1 \text{ free}} = 2 \left( Q^+_1 + e^{-2i(\phi + \varphi)} Q^-_{-1} - ie^{-i(\phi + \varphi)} T_0 \right) ,
\] (49)
where $Q_n^\pm$ and $T_n$ are given by Eqs. (37), (38).

Moreover, we find higher-derivative integrals of motion which in terms of Fourier modes are given by

$$
\hat{Q}_{n \text{ free}} = Q_n^+ + Q_{2-n}^+ + e^{-2i(\phi+\varphi)} (Q_{n-2}^- + Q_{n}^-),
$$

$$
\hat{T}_{n \text{ free}} = T_n - T_{-n} + T_{-n+4} - T_{n-4},
$$

$$
-2ie^{i(\phi+\varphi)} (Q_{n-1}^+ + Q_{n-3}^+ + Q_{1-n}^+ + Q_{3-n}^+) + 2ie^{-i(\phi+\varphi)} (Q_{n-1}^- + Q_{n-3}^- + Q_{1-n}^- + Q_{3-n}^-),
$$

$n$ odd $\geq 1$, $n$ even $\geq 2$, (50)

together with $\hat{Q}_{n \text{ free}}$. (Here $Q_n^\pm$ and $T_n$ are given by Eqs. (37), (38). See Appendix A for details.) Notice that the expressions for $\hat{Q}_{n \text{ free}}$ and $\hat{T}_{n \text{ free}}$ are invariant under $n \to 2-n$ and $n \to 4-n$, respectively; and $\hat{T}_{n \text{ free}} = \hat{T}_{n \text{ free}}$. The algebra of the IM for “free” boundary conditions is given by (dropping the label “free”)

$$
\begin{align*}
[\hat{Q}_n, \hat{Q}_m^\dagger] &= \hat{T}_{n+m} + \hat{T}_{n-m+2}, \\
[\hat{T}_n, \hat{Q}_m] &= 2(\hat{Q}_{m+n} + 2\hat{Q}_{m+n-2} + \hat{Q}_{m+n-4} + \hat{Q}_{m-n} + 2\hat{Q}_{m-n+2} + \hat{Q}_{m-n+4}), \\
[\hat{T}_n, \hat{Q}_m^\dagger] &= -2(\hat{Q}_{m+n} + 2\hat{Q}_{m+n-2} + \hat{Q}_{m+n-4} + \hat{Q}_{m-n} + 2\hat{Q}_{m-n+2} + \hat{Q}_{m-n+4}),
\end{align*}
$$

and all other commutators vanish.

**General boundary conditions**

A principal result [10] is that for general values of $h$, the quantities

$$
\hat{Q}_1 = 2 \left[ \bar{Q}_+ + e^{-2i(\phi+\varphi)} Q_- - ie^{-i(\phi+\varphi)} \left( 1 - \frac{h^2}{2m} e^{i(\phi-\varphi)} \frac{1}{2} \right) T + \frac{i}{m} e^{-i(\phi+\varphi)} \psi_0 \frac{\partial}{\partial x} \right]_{x=0} \tag{52}
$$

and $\hat{Q}_1^\dagger$ are integrals of motion. A proof is outlined in Appendix B. As $h$ varies from 0 to $\infty$, the conserved charge $\hat{Q}_1$ interpolates between $\hat{Q}_1 \text{ free}$ and $\hat{T}_0 \text{ fixed}$. \(^2\)

\[\hat{Q}_1 \xrightarrow{h \to 0} \hat{Q}_1 \text{ free}\]

\(^2\)In [4], we exhibited a different set of charges $\hat{Q}_1 \text{ old}, \hat{Q}_1^\dagger \text{ old}$ which interpolate instead between $\hat{Q}_1 \text{ free}$.
\[ h \to \infty \quad \frac{\hbar^2}{m} e^{-2i\varphi} \hat{T}_0 \text{ fixed}. \] (53)

Going to Fourier modes, we have

\[ \hat{Q}_1 = \hat{Q}_1 \text{ free} + 2i \frac{\hbar^2}{m} e^{-2i\varphi} T_0, \] (54)

where \( \hat{Q}_1 \text{ free} \) is given by Eq. (50).

We also find higher-derivative IM which in terms of Fourier modes are given by

\[ \hat{Q}_n = \hat{Q}_n \text{ free} + 4 \frac{\hbar^2}{m} e^{-i(3\varphi + \phi)} \sum_{k=1}^{n-1} (-)^k \left( Q_{n-2k}^- + Q_{2k-n}^- \right) + 2i \frac{\hbar^2}{m} e^{-2i\varphi} \left( - \frac{n+1}{4} T_2 + \sum_{k=1}^{n-3} (-)^{k+1} (T_{n-2k+1} + T_{2k+1-n}) \right) + 4 \frac{\hbar^4}{m^2} e^{-4i\varphi} \left( - \frac{n+1}{4} \hat{Q}_1 \text{ fixed} + \sum_{k=1}^{n-3} (-)^{k+1} k \hat{Q}_{n-2k} \text{ fixed} \right), \quad n \text{ odd } \geq 3, \] (55)

\[ \hat{T}_n = \hat{T}_n \text{ free} + \left[ 2 \frac{\hbar^2}{m} \left( e^{i(\phi - \varphi)} + e^{-i(\phi - \varphi)} \right) - 4 \frac{\hbar^4}{m^2} \right] \hat{T}_{n-2} \text{ fixed} + 4i \frac{\hbar^2}{m} \left[ e^{2i\varphi} \left( Q_{n+1}^+ + Q_{n+3}^+ \right) - e^{-2i\varphi} \left( Q_{n+1}^- + Q_{n+3}^- \right) \right], \quad n \text{ even } \geq 2, \] (55)

where \( Q_n^\pm \) and \( T_n \) are given by Eqs. (37), (38), \( Q_n^\pm \text{ fixed} \) and \( T_n \text{ fixed} \) are given by Eq. (45), and \( Q_n^\pm \text{ free} \) and \( T_n \text{ free} \) are given by Eq. (50). (See Appendix B for details.) Note that these IM also interpolate between “free” and “fixed” IM as \( \hbar \) varies from 0 to \( \infty \).

\[ \hat{Q}_1 \text{ old} \underset{\hbar \to 0}{\longrightarrow} \hat{Q}_1 \text{ free}, \quad \hat{Q}_1 \text{ fixed} \underset{\hbar \to \infty}{\longrightarrow} e^{-4i\varphi} \hat{Q}_1 \text{ fixed}. \]

The old charges have the drawback of being “nonlocal” in time; indeed, they are constructed in terms of the quantities \( C(\psi^+) \) and \( C(\bar{\psi}^+) \), where \( C(\psi) \) is defined in Eq. (73).

\[ \text{Note that according to our prescription } (43), (44) \text{ for going to Fourier modes, the boundary term in Eq. (52) gives no contribution.} \]
These IM obey the following commutation relations:

\[
[\hat{Q}_1, \hat{Q}_1^\dagger] = 2\hat{T}_2 + 4i\frac{\hbar^2}{m} \left(-e^{2i\varphi}\hat{Q}_1 + e^{-2i\varphi}\hat{Q}_1^\dagger\right),
\]

\[
[\hat{Q}_3, \hat{Q}_1] = 4i\frac{\hbar^2}{m} e^{-2i\varphi}\hat{Q}_3 + 2\frac{\hbar^2}{m} e^{-i(\phi+3\varphi)}\hat{T}_2 + 4i \left(\frac{\hbar^2}{m} e^{-2i\varphi} - 2\frac{\hbar^4}{m^2} e^{-i(\phi+\varphi)}\right)\hat{Q}_1,
\]

\[
[\hat{Q}_3, \hat{Q}_1^\dagger] = 2\hat{T}_2 - 4i\frac{\hbar^2}{m} e^{-2i\varphi}\hat{Q}_3 + 2\frac{\hbar^2}{m} e^{i(\phi-3\varphi)}\hat{T}_2 - 4i \frac{\hbar^2}{m} e^{-2i\varphi}\hat{Q}_1^\dagger + 8i \frac{\hbar^4}{m^2} e^{i(\phi+\varphi)}\hat{Q}_1,
\]

\[
[\hat{T}_2, \hat{Q}_1] = 8\hat{Q}_3 + 8 \left(1 - \frac{\hbar^2}{m} e^{-i(\phi+\varphi)}\right)\hat{Q}_1 - 8\frac{\hbar^2}{m} e^{-i(\phi+3\varphi)}\hat{Q}_1^\dagger,
\]

\[
[\hat{T}_2, \hat{Q}_3] = 4\hat{Q}_5 + 8 \left[1 - \frac{\hbar^2}{m} \left(e^{i(\phi+\varphi)} + e^{-i(\phi+\varphi)}\right) + 2\frac{\hbar^4}{m^2}\right]\hat{Q}_3
\]

\[+ 4 \left(1 - 2\frac{\hbar^2}{m} e^{i(\phi-\varphi)}\right)\hat{Q}_1 + 8 \left(\frac{\hbar^2}{m} e^{-i(\phi+3\varphi)} - 2\frac{\hbar^4}{m^2} e^{-2i(\phi+\varphi)}\right)\hat{Q}_1^\dagger, \tag{56}\]

etc.

**Boundary S matrix revisited**

The IM \(\hat{Q}_1\) and \(\hat{Q}_1^\dagger\) (as well as the higher IM) correctly determine the boundary S matrix. Indeed, the commutation relations of \(\hat{Q}_1\) and \(\hat{Q}_1^\dagger\) with \(A(\theta)^\dagger\) are given by (see Eqs. (54), (59), (57), (58))

\[
[\hat{Q}_1, A(\theta)^\dagger] = Z(\theta) A(\theta)^\dagger, \quad [\hat{Q}_1^\dagger, A(\theta)^\dagger] = Z'(\theta) A(\theta)^\dagger, \tag{57}\]

where the 2 × 2 matrices \(Z(\theta)\) and \(Z'(\theta)\) are given by

\[
Z(\theta) = 2 \begin{pmatrix} -c & b(\theta) \\ a(\theta) & c \end{pmatrix}, \quad Z'(\theta) = 2 \begin{pmatrix} -c^* & a(\theta)^* \\ b(\theta)^* & c^* \end{pmatrix}, \tag{58}\]

and

\[
a(\theta) = e^{-\theta}, \quad b(\theta) = e^{-2i(\phi+\varphi)+\theta}, \quad c = ie^{-i(\phi+\varphi)} - i\frac{\hbar^2}{m} e^{-2i\varphi}. \tag{59}\]

We next observe that

\[
\hat{Q}_1 A(\theta)^\dagger |0\rangle_B = R(\theta) \hat{Q}_1 A(-\theta)^\dagger |0\rangle_B = R(\theta) Z(-\theta) A(-\theta)^\dagger |0\rangle_B \tag{60}\]

\[
= Z(\theta) A(\theta)^\dagger |0\rangle_B = Z(\theta) R(\theta) A(-\theta)^\dagger |0\rangle_B. \tag{61}\]
In the first line, we first use the definition (17) of the boundary $S$ matrix, and we then use the commutation relations together with the fact $\hat{Q}_1|0\rangle_B = 0$; and in the second line we reverse the order of operations. We conclude that
\[ Z(\theta) \ R(\theta) = R(\theta) \ Z(-\theta). \] (62)
An analogous analysis with $\hat{Q}^\dagger_1$ yields
\[ Z'(\theta) \ R(\theta) = R(\theta) \ Z'(-\theta). \] (63)
Solving these two relations for $R(\theta)$ leads directly to the boundary $S$ matrix given by Eqs. (19), (33), (34), up to the unitarization factor.

4 Away from the free Fermion point ($\beta^2 \neq 4\pi$)

We would like to explicitly construct generalizations of the integrals of motion $\hat{Q}_1$ and $\hat{Q}^\dagger_1$ for the sine-Gordon model with boundary for $\beta^2 \neq 4\pi$.

As a preliminary step in this direction, we make the following observation. Define the quantities $\hat{Q}$ and $\hat{Q}'$ by
\[ \hat{Q} = 2 \left[ \alpha^{-1} \bar{Q}_+ + \alpha^{-1} \frac{k_+}{k_-} Q_- - 2 i \frac{e^{-i\xi}}{k_-} \left( \frac{1 - q^{-T}}{q - q^{-1}} \right) \right], \]
\[ \hat{Q}' = 2 \left[ \alpha^{-1} \bar{Q}_- - \alpha^{-1} \frac{k_-}{k_+} Q_+ + 2 i \frac{e^{i\xi}}{k_+} \left( \frac{q^T - 1}{q - q^{-1}} \right) \right], \] (64)
where $Q_\pm$, $\bar{Q}_\pm$, and $T$ have commutation relations with $A(\theta)^\dagger$ given by Eq. (13). A generalization of the argument given in Eqs. (60) - (63) using $\hat{Q}$ and $\hat{Q}'$ instead of $\hat{Q}_1$ and $\hat{Q}^\dagger_1$, respectively, leads to the boundary $S$ matrix $R(\theta)$ given by Eq. (19). For $q \to 1$, one can verify with the help of Eqs. (33) and (34) that the charges $\hat{Q}$ and $\hat{Q}'$ reduce to the expression (54) for $\hat{Q}_1$ in terms of Fourier modes and its Hermitian conjugate, respectively.

It remains to construct $\hat{Q}$ and $\hat{Q}'$ explicitly in terms of the local sine-Gordon field $\Phi(x, t)$, to demonstrate their conservation, and to identify their algebra.

5 Discussion

For the case $q = 1$, we have seen that the “boundary quantum group” is a one-parameter ($\hbar$) family of infinite-dimensional subalgebras of twisted $\hat{sl}(2)$. For “fixed” ($\hbar \to \infty$) and “free”
boundary conditions, the subalgebras can be described very explicitly. (See Eqs. (46) and (51), respectively.) However, for general values of \( h \), the subalgebra is evidently more complicated (see Eq. (56)), and we have not yet succeeded in giving a complete characterization of its structure. We remark that, to our knowledge, the general subject of subalgebras of affine Lie algebras remains largely unexplored.

We emphasize that for the general case \( q \neq 1 \), the most pressing problems are to explicitly construct the fractional-spin IM in terms of the local sine-Gordon field, and to identify the algebra of these IM. The boundary sine-Gordon field theory (16) can be regarded as a free scalar conformal field theory with both bulk and boundary integrable perturbations \([12],[7]\). Within this framework, it should be easier to treat the special case of only a boundary perturbation (i.e., conformal bulk). It would also be interesting to identify analogues of fractional-spin IM for integrable quantum spin chains, which can be solved by the Bethe Ansatz. A related question is whether a systematic construction of the fractional-spin IM can be found, in analogy with the so-called quantum inverse scattering (QISM) construction of local IM. Finally, we remark that it should be possible to construct fractional-spin IM for other integrable field theories with boundaries.

6 Acknowledgments

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7 Appendix A: Higher-derivative IM for “fixed” and “free” boundary conditions

We explain here our strategy for constructing higher-derivative integrals of motion. We begin with the case of “fixed” boundary conditions (40). The first step is to observe (using the
field equations and boundary conditions) that the following charges are conserved:

\[ q_{2n+1}^{\text{fixed}} = -\frac{2i}{m^{2n+1}} \int_{-\infty}^{0} dx \left[ \bar{\psi}_+^{(n)} \psi_+^{(n+1)} + e^{2i(\phi-\varphi)} \psi_+^{(n)} \bar{\psi}_+^{(n+1)} \right], \]
\[ t_{2n}^{\text{fixed}} = \frac{2}{m^{2n}} \int_{-\infty}^{0} dx \left[ \psi_+^{(n)} \bar{\psi}_+^{(n)} + \bar{\psi}_+^{(n)} \psi_+^{(n)} \right], \]  \hfill (65)

where \( n \) is a nonnegative integer, and \( ^{(n)} \) denotes \( n^{\text{th}} \)-order time derivative. For \( n = 0 \), these IM coincide with \( \hat{Q}_1^{\pm} \) and \( \hat{T}_0 \) fixed, respectively. (See Eq. \((42)\).) The second step is to express these IM in terms of Fourier modes. Using the prescription \((43), (44)\), we find that

\[ q_{2n+1}^{\text{fixed}} = 2 \int_{0}^{\infty} d\theta \cosh^{2n} \theta \left[ e^{-\theta} + e^{2i(\phi-\varphi)+\theta} \right] A_+^{\dagger}(\theta)A_-(\theta), \]
\[ t_{2n}^{\text{fixed}} = 2 \int_{0}^{\infty} d\theta \cosh^{2n} \theta \left[ A_+^{\dagger}(\theta)A_+(\theta) - A_-(\theta)^{\dagger}A_-(\theta) \right]. \]  \hfill (66)

The third step is to express these IM in terms of the basis \((37), (38)\) by writing \( \cosh \theta \) in terms of exponentials \( e^{\pm \theta} \):

\[ q_1^{\text{fixed}} = 2 \left[ Q_1^+ + e^{2i(\phi-\varphi)} Q_{-1}^+ \right], \]
\[ q_3^{\text{fixed}} = \frac{1}{2} \left[ Q_3^+ + Q_{-1}^+ + e^{2i(\phi-\varphi)} \left( Q_1^+ + Q_{-3}^+ \right) \right] + \frac{1}{2} q_1^{\text{fixed}}, \]
\[ \vdots \]
\[ t_0^{\text{fixed}} = 2T_0, \]
\[ t_2^{\text{fixed}} = \frac{1}{2} (T_2 + T_{-2}) + \frac{1}{2} t_0^{\text{fixed}}, \]
\[ \vdots \]  \hfill (67)

Finally, it is important to note that the IM \( q_{2n+1}^{\text{fixed}} \) and \( t_{2n}^{\text{fixed}} \) for \( n > 0 \) are not “irreducible”; i.e., they are sums of terms which are separately conserved. The “irreducible” conserved quantities evidently are

\[ \hat{Q}_n^{\pm} \] \( n \) odd \( \geq 1 \),
\[ \hat{T}_n \] \( n \) even \( \geq 0 \),

which is the result stated in Eq. \((45)\).
We follow a similar procedure for constructing higher-derivative IM for the case of “free” boundary conditions \( \text{[17]} \). For this case, we have instead of \( \text{[18]} \) the following “reducible” conserved charges:

\[
q_{2n+1 \, \text{free}} = \frac{2i}{m^{2n+1}} \int_{-\infty}^{0} dx \left\{ \tilde{\psi}_+^{(n)} \tilde{\psi}_+^{(n+1)} + e^{-2i(\phi+\varphi)} \psi_+^{(n)} \psi_-^{(n+1)} \right. \\
\left. -me^{-i(\phi+\varphi)} \left[ \psi_-^{(n)} \psi_+^{(n)} + \tilde{\psi}_-^{(n)} \tilde{\psi}_+^{(n)} \right] \right\}, \\
t_{2n+2 \, \text{free}} = \frac{2}{m^{2n+2}} \int_{-\infty}^{0} dx \left\{ \tilde{\psi}_-^{(n)} \tilde{\psi}_-^{(n+1)} + \psi_-^{(n)} \psi_-^{(n+1)} + m \left[ e^{i(\phi+\varphi)} \left( \tilde{\psi}_+^{(n)} \tilde{\psi}_+^{(n+1)} + \psi_+^{(n)} \psi_+^{(n+1)} \right) \right. \\
\left. -e^{-i(\phi+\varphi)} \left( \tilde{\psi}_-^{(n)} \tilde{\psi}_-^{(n+1)} + \psi_-^{(n)} \psi_-^{(n+1)} \right) \right\},
\]

(69)

where \( n \) is a nonnegative integer, and the prime (‘) denotes differentiation with respect to \( x \). Going to Fourier modes and expressing the result in terms of the basis \( \text{[37]}, \text{[38]} \), we eventually arrive at the “irreducible” IM given in Eq. \( \text{[50]} \).

### 8 Appendix B: IM for general boundary conditions

In this Appendix we outline a proof that, for general boundary conditions, the quantity \( \dot{Q}_1 \) given by Eq. \( \text{[22]} \) is an integral of motion. We also indicate how to construct the higher-derivative IM.

We consider first the expression \( \text{[22]} \) for \( \dot{Q}_1 \) except without the boundary term:

\[
P \equiv \int_{-\infty}^{0} dx \left\{ \tilde{\psi}_+ \tilde{\psi}_+ + e^{-2i(\phi+\varphi)} \psi_- \psi_- + \left( -me^{-i(\phi+\varphi)} + h^2 e^{-2i\varphi} \right) (\psi_- \dot{\psi}_+ + \tilde{\psi}_- \tilde{\psi}_+ + \psi_+ \dot{\psi}_- + \tilde{\psi}_+ \tilde{\psi}_-) \right\}.
\]

(70)

Differentiating with respect to time, we obtain using the field equations \( \text{[21]} \) a sum of boundary terms:

\[
\dot{P} = \int_{-\infty}^{0} dx \, \partial_1 \left[ \tilde{\psi}_+ \tilde{\psi}_+ + e^{-2i(\phi+\varphi)} \psi_- \psi_- + \left( -me^{-i(\phi+\varphi)} + h^2 e^{-2i\varphi} \right) \left( \psi_- \dot{\psi}_+ + \tilde{\psi}_- \tilde{\psi}_+ + \psi_+ \dot{\psi}_- + \tilde{\psi}_+ \tilde{\psi}_- \right) \right] \\
= \left\{ -\tilde{\psi}_+ \tilde{\psi}_+ + e^{-2i(\phi+\varphi)} \psi_- \psi_- + h^2 e^{-2i\varphi} \left( \psi_- \psi_+ + \tilde{\psi}_- \tilde{\psi}_+ \right) + m \left[ \tilde{\psi}_+ \psi_+ + e^{-2i(\phi+\varphi)} \psi_- \tilde{\psi}_- - e^{-i(\phi+\varphi)} \left( \psi_- \psi_+ + \tilde{\psi}_- \tilde{\psi}_+ \right) \right] \right\}\bigg|_{x=0},
\]

(71)

where the prime (‘) denotes differentiation with respect to \( x \). Our objective is to express the result as a time derivative of a local boundary term. We proceed by eliminating \( \psi_\pm \) in favor...
of $\tilde{\psi}_\pm$ using the following identities \[ \text{[9]} \] which can be derived from the boundary conditions (31), (32):

\[
\begin{align*}
\left[ \psi_+ - e^{-i(\phi+\varphi)}\tilde{\psi}_- + e^{-i\phi}C(\tilde{\psi}_+) \right]_{x=0} &= 0, \\
\left[ \psi_- - e^{i(\phi+\varphi)}\tilde{\psi}_+ + e^{i\phi}C(\tilde{\psi}_-) \right]_{x=0} &= 0, \quad (72)
\end{align*}
\]

where the quantity $C(\psi)$ is defined by

\[
C(\psi) = \frac{1}{1 + \frac{2\hbar}{m}} \left( e^{i\varphi} \psi + e^{-i\varphi} \psi^{-1} \right). \quad (73)
\]

We find in this way that

\[
\dot{P} = -\partial_0 \left( e^{-i\varphi}C(\tilde{\psi}_+\tilde{\psi}_+) \right)_{x=0} = \partial_0 \left( e^{-i(\phi+\varphi)}/\psi_-\tilde{\psi}_+ \right)_{x=0}. \quad (74)
\]

Therefore, the quantity

\[
\dot{Q}_1 \propto P - e^{-i(\phi+\varphi)}/\psi_-\tilde{\psi}_+ \text{ } |_{x=0} \quad (75)
\]

is an integral of motion.

In a similar manner, we find the following “reducible” higher-derivative IM:

\[
\begin{align*}
q_{2n+1} &= -\frac{2i}{m^{2n+1}} \int_{-\infty}^{0} dx \left\{ \tilde{\psi}_+^{(n)} - h^2 e^{-2i\varphi} \tilde{\psi}_-^{(n-1)} \right\} \left( \tilde{\psi}_+^{(n+1)} - h^2 e^{-2i\varphi} \tilde{\psi}_-^{(n)} \right) \\
& \quad + e^{-i(\phi+\varphi)} \left( \psi_+^{(n)} + h^2 \psi_-^{(n-1)} \right) \left( \psi_+^{(n+1)} + h^2 \psi_-^{(n)} \right) \\
& \quad - me^{-i(\phi+\varphi)} \left[ \left( \psi_+^{(n)} + h^2 \psi_-^{(n-1)} \right) \left( \psi_+^{(n+1)} + h^2 e^{-2i\varphi} \psi_-^{(n-1)} \right) + \left( \psi_-^{(n)} + h^2 \psi_-^{(n-1)} \right) \left( \psi_+^{(n)} + h^2 e^{-2i\varphi} \psi_-^{(n)} \right) \right] \}, \\

\end{align*}
\]

\[
\begin{align*}
t_{2n} &= -\frac{2}{m^{2n}} \int_{-\infty}^{0} dx \left\{ \tilde{\psi}_+^{(n)} - h^2 e^{-2i\varphi} \tilde{\psi}_-^{(n-1)} \right\} \left( \tilde{\psi}_+^{(n)} - h^2 e^{-2i\varphi} \psi_+^{(n-1)} \right) \\
& \quad + \left( \psi_-^{(n)} + h^2 \psi_-^{(n-1)} \right) \left( \psi_+^{(n)} + h^2 \psi_-^{(n+1)} \right) - \frac{m}{2} \left( \psi_+^{(n)} \psi_-^{(n-1)} + \psi_-^{(n)} \psi_+^{(n-1)} \right) \\
& \quad - \frac{m}{2} \left[ e^{i(\phi+\varphi)} \left( \tilde{\psi}_+^{(n)} \tilde{\psi}_-^{(n+1)} + \psi_+^{(n)} \psi_-^{(n)} \right) - e^{-i(\phi+\varphi)} \left( \tilde{\psi}_-^{(n)} \tilde{\psi}_+^{(n-1)} + \psi_-^{(n)} \psi_+^{(n-1)} \right) \right] \\
& \quad - m \hbar^2 \left[ \tilde{\psi}_+^{(n-1)} \psi_-^{(n-1)} + \psi_-^{(n-1)} \tilde{\psi}_+^{(n-1)} \right] \}, \quad (76)
\end{align*}
\]

where $n \geq 1$, and $^{(n)}$ denotes $n^{th}$-order time derivative. For $h = 0$, these charges reduce to those given in Eq. (69). Following the procedure described in Appendix A, we obtain the “irreducible” IM given in Eq. (52).
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