Two-loop vacuum diagrams and tensor decomposition

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Abstract

General algorithms for tensor reduction of two-loop massive vacuum diagrams are discussed. Some explicit useful formulae are presented.

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1. General Remarks

In the talk given by one of the authors (A. D.) at the AIHENP-95 Conference in Pisa, some recent (i.e. since AIHENP-93) developments in two-loop calculations were reviewed. The main results discussed in the talk were published [1, 2, 3] and were obtained in collaboration with N.I. Ussyukina [4], F.A. Berends and V.A. Smirnov [2], and the other author of this paper, J.B. T. [2, 3].

In ref. [1] (see also in ref. [4]), some new results for two-loop three-point diagrams with massless internal particles were obtained, including the diagrams with irreducible numerators. The paper [2] continues the project started in refs. [5, 6] and provides explicit results for the small momentum expansion of two-loop self-energy diagrams for all cases when the smallest physical threshold is zero (“zero-threshold expansion”). To do this, general results on asymptotic expansions of Feynman diagrams were applied (see e.g. in the reviews [7] and references therein). In the paper [3], we have established a useful connection between two-loop massive vacuum diagrams and off-shell massless triangle diagrams. This connection is valid for all values of the space-time dimension \( n \) and masses \( m_i \) (which are related to the external momenta squared of the triangle diagram via \( m_i^2 = p_i^2, i = 1, 2, 3 \)). As a corollary, the analytic result for the \( \varepsilon \)-part (\( \varepsilon \equiv (4 - n)/2 \)) of two-loop vacuum diagram was obtained in terms of trilogarithms. For the case of equal masses, a new transcendental constant (related to log-sine integral \( \text{Ls}_3(\frac{2}{3}\pi) \)) is shown to appear.

Since it is impossible to give a detailed description of all these results in a short contribution, and also because the main results have been already published, we decided to concentrate ourselves on an important particular problem related to two-loop vacuum diagrams, namely the problem of tensor decomposition.

Tensor reduction of Feynman integrals containing loop momenta with uncontracted Lorentz indices in the numerator is very important for various realistic calculations in the Standard Model (and beyond). For one-loop diagrams with different numbers of external lines, several approaches and algorithms were developed [8]. For two-loop vacuum and self-energy diagrams, the problem was considered e.g. in refs. [9, 10], whilst the three-point two-loop case is more complicated and requires results for the integrals with irreducible numerators [4].

The problem of finding general algorithms for the tensor decomposition of two-loop vacuum diagrams is interesting because it is connected with the calculation of coefficients of the small momentum expansion [4], also for the three-point case [11]. In the three-point case, the problem is more tricky (even for scalar integrals), because we have two independent external momenta to contract with. Some relevant formulae for cases when the numerator is contracted with one or two external vectors can be found in refs. [6, 12]. We also note that the general result for a special case (when two masses are equal and the third is zero) was presented in ref. [10].
2. Tensor Decomposition

In this paper, we shall use the following notation:

\[ I \{\text{something}\} \equiv \int \int d^n p \ d^n q \ \{\text{something}\} \ F \left( p^2, q^2, (pq) \right), \tag{1} \]

and we are interested in expressing the integrals

\[ I \left[ p_{\mu_1} \ldots p_{\mu_{N_1}} \ q_{\sigma_1} \ldots q_{\sigma_{N_2}} \right] \tag{2} \]

in terms of scalar integrals. In eq. (1), \( F (p^2, q^2, (pq)) \) is an arbitrary scalar function depending on Lorentz invariants of the loop momenta \( p \) and \( q \). In the special case when this function does not depend on \((pq)\) we shall write it as \( F (p^2, q^2) \). Usually, this function is a product of propagators,

\[ (p^2-m_1^2)^{-\nu_1} (q^2-m_2^2)^{-\nu_2} \left((p-q)^2-m_3^2\right)^{-\nu_3} \text{ or } (p^2-m_1^2)^{-\nu_1} (q^2-m_2^2)^{-\nu_2}, \quad \tag{3} \]

but all formulae we are going to discuss are valid for arbitrary scalar functions.

Some explicit results for the cases when the integral (2) was contracted with the external vector (momentum) \( k \) were given in ref. [6] (eqs. (B.10) and (B.9)). They are also valid when the integrands (2) are replaced by \( F (p^2, q^2, (pq)) \) (in eq. (B.10)) or \( F (p^2, q^2) \) (in eq. (B.9)), respectively. Moreover, we note that eq. (B.9) can be generalized to the case with two external momenta \( k_1 \) and \( k_2 \),

\[
\int \int d^n p \ d^n q \ F \left( p^2, q^2 \right) \left[2(k_1 p)\right]^{N_1} \left[2(k_2 q)\right]^{N_2} \left[2(pq)\right]^{N_3}\right|_{N_1+N_3-\text{even}}_{N_2+N_3-\text{even}} = \frac{N_1! \ N_2! \ N_3!}{(n/2)_{(N_1+N_2)/2} \ (n/2)_{(N_1+N_3)/2} \ (n/2)_{(N_2+N_3)/2}} \times \sum_{2j_1+j_3=N_1} \sum_{2j_2+j_3=N_2} \frac{(k_1^{j_1} k_2^{j_2} k_3^{j_3})^{N_3} (n/2)_{(N_1+N_3-j_3)/2} (N_3-j_3)!}{j_1! \ j_2! \ j_3! ((N_3-j_3)/2)!} \times \int \int d^n p \ d^n q \ F \left( p^2, q^2 \right) \left(p^2\right)^{(N_1+N_3)/2} \left(q^2\right)^{(N_2+N_3)/2}, \tag{4} \]

where \((a)_{j} = \Gamma(a+j)/\Gamma(a)\) denotes the Pochhammer symbol. If \((N_1 + N_3)\) or \((N_2 + N_3)\) is odd, the integral on the l.h.s. is equal to zero.

Since we have no external momenta in (3), only tensor structures constructed of metric tensors may be involved in the tensor decomposition. Therefore, (3) vanishes if \( N \equiv N_1 + N_2 \) is odd. For even \( N \), the suitable tensor structures should be symmetric with respect to two subsets of indices, \((\mu_1, \ldots, \mu_{N_1})\) and \((\sigma_1, \ldots, \sigma_{N_2})\). All independent tensor structures can be constructed in the following way. Let us take a product of \( j_1 \) metric tensors \( g_{\mu_1 \mu_2} \), \( j_2 \) tensors \( g_{\sigma_1 \sigma_2} \) and \( j_3 \) tensors \( g_{\mu_3 \sigma_3} \) (we should remember that the conditions \(2j_1 + j_3 = N_1\) and \(2j_2 + j_3 = N_2\) should be satisfied). Now, let us consider all possible permutations of \( \mu \)'s and all possible permutations of \( \sigma \)'s producing distinct products of
metric tensors, and take the sum of all these terms (with the coefficients equal to one). So, the obtained tensor structure is symmetrized in μ’s and σ’s, and it is

\[ \{ j_1, j_2, j_3 \} = g_{\mu_1 \mu_2} \cdots g_{\mu_{2j_1-1} \mu_{2j_1}} g_{\sigma_1 \sigma_2} \cdots g_{\sigma_{2j_2-1} \sigma_{2j_3}} g_{\mu_{2j_1+1} \sigma_{2j_2+1}} \cdots g_{\mu_{2j_1+j_3} \sigma_{2j_2+j_3}} + \text{permutations.} \] (5)

For given \( N_1 \) and \( N_2 \), the number of independent tensor structures (3) is

\[ T(N_1, N_2) = \min \left( \left\lceil \frac{N_1}{2} \right\rceil, \left\lceil \frac{N_2}{2} \right\rceil \right) + 1, \] (6)

where \( \left\lceil \frac{N_i}{2} \right\rceil \) is the integer part of \( N_i/2 \), and the number of terms on the r.h.s. of (3) is

\[ c_{j_1 j_2 j_3} = \frac{N_1! N_2!}{2^{j_1+j_2} j_1! j_2! j_3!}. \] (7)

Note that due to the conditions \( 2j_1 + j_3 = N_1 \) and \( 2j_2 + j_3 = N_2 \), at given \( N_1 \) and \( N_2 \) the tensor structures (3) are completely defined by one index, \( j_3 \).

So, the result for the integral (4) (for even \( N_1 + N_2 \)) should look like

\[ I \left[ p_{\mu_1} \cdots p_{\mu_{N_1}} q_{\sigma_1} \cdots q_{\sigma_{N_2}} \right] = \sum_{2j_1+j_3 = N_1} \sum_{2j_2+j_3 = N_2} \{ j_1, j_2, j_3 \} I_{j_1 j_2 j_3}, \] (8)

where \( I_{j_1 j_2 j_3} \) are some scalar integrals which are to be found. A standard way to define \( I_{j_1 j_2 j_3} \) is to consider all independent contractions of (8) (e.g. to contract it with each of the structures (3)). In such a way, one needs to solve a system of \( T(N_1, N_2) \) (see eq. (6)) linear equations.

Due to power-counting reasons, the scalar integrals \( I_{j_1 j_2 j_3} \) should be linear combinations of the integrals (4) with scalar numerators, carrying the same total powers of the loop momenta \( p \) and \( q \). Therefore, we can re-write eq. (8) as

\[ I \left[ p_{\mu_1} \cdots p_{\mu_{N_1}} q_{\sigma_1} \cdots q_{\sigma_{N_2}} \right] = \sum_{2j_1+j_3 = N_1} \sum_{2j_2+j_3 = N_2} \{ j_1, j_2, j_3 \} \sum_{2j'_1+j'_3 = N_1} \sum_{2j'_2+j'_3 = N_2} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} c_{j'_1 j'_2 j'_3} I \left[ (p^2)^{j'_1} (q^2)^{j'_2} (pq)^{j'_3} \right], \] (9)

where \( c_{j_1 j_2 j_3} \) are defined in (8), while \( \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} \) can be considered as the elements of the “decomposition” matrix \( (T \times T, \text{see eq. (3)}) \). Contracting eq. (8) with all possible tensors (3), we get a “column” of \( c_{j_1 j_2 j_3} I \left[ (p^2)^{j_1} (q^2)^{j_2} (pq)^{j_3} \right] \) on the l.h.s. On the r.h.s., we get the “contraction” matrix \( \chi \) with the elements defined as

\[ \chi_{j_1, j_2, j_3; j'_1, j'_2, j'_3} = \text{contraction} \left( \{ j_1, j_2, j_3 \}, \{ j'_1, j'_2, j'_3 \} \right). \] (10)

When speaking about matrices, we understand that only one of \( j \)-indices is relevant in each set \( \{ j_1, j_2, j_3 \} \) (e.g., \( j_3 \)). For given \( N_1 \) and \( N_2 \), the values of \( j_3 \) can be either even (if
$N_1$ and $N_2$ are even) or odd (if $N_1$ and $N_2$ are odd). In order to use the matrix notation in ordinary form, we introduce the generalized index $j \equiv (j_1, j_2, j_3)$ (and $j' \equiv (j'_1, j'_2, j'_3)$, etc.) which takes values from 1 to $T(N_1, N_2)$. In all “non-matrix” formulae, however, we would prefer to keep all the indices $j_1, j_2, j_3$ explicitly. So, in the matrix notation, a corollary of eqs. (9)–(11) is

$$
\sum_{j''} \chi_{jj''} \phi_{j''j'} = \delta_{jj'} .
$$

(11)

In other words, the decomposition matrix $\phi$ is nothing but the inverse contraction matrix, $\chi^{-1}$. Since $\chi_{jj''}$ is symmetric (see (10)), $\phi_{j''j'}$ is symmetric as well.

Thus, the problem is how to find all $\phi_{j_1j_2j_3\sigma_1\sigma_2\sigma_3}$ for given $N_1$ and $N_2$. One way is to use recurrence relations. We note that the following simple formulae of contracting tensor structures (9) with respect to two indices can be derived:

$$
g_{\mu N_1 \sigma N_2} \{j_1, j_2, j_3\} = (n+2j_1+2j_2+j_3-1) \{j_1, j_2, j_3-1\} + (j_3+1) \{j_1-1, j_2-1, j_3+1\} (12)
$$

$$
g_{\mu N_1 \mu N_1} \{j_1, j_2, j_3\} = (n+2j_1+2j_2-2) \{j_1-1, j_2, j_3\} + 2(j_2+1) \{j_1, j_2+1, j_3-2\} (13)
$$

(and also an analogous formula for contraction with $g_{\sigma N_2 \mu \sigma N_2}$). Using these contractions, the following recurrence relations for $\phi_{j_1j_2j_3\sigma_1\sigma_2\sigma_3}$ can be obtained:

$$
N_1N_2 \left\{ (n+N_1+N_2-j'_3-1) \phi_{j_1j_2j_3; j'_1j'_2j'_3} + (j'_3-1) \phi_{j_1j_2j_3; j'_1j'_2j'_3-2} \right\} = j'_3 \phi_{j_1j_2j_3-1; j'_1j'_2j'_3-1} ,
$$

(14)

$$
N_1 (N_1-1) \left\{ (n+N_1+j'_3-2) \phi_{j_1j_2j_3; j'_1j'_2j'_3} + 2j'_2 \phi_{j_1j_2j_3; j'_1-1j'_2+1j'_3} \right\} = 2 j_1 \phi_{j_1-1; j_2j_3; j'_1-1j'_2j'_3} ,
$$

(15)

and also a relation similar to (13) but with interchanged indices, $1 \leftrightarrow 2$. In eqs. (12)–(15), it is understood that, whenever any of $j$’s becomes negative, the corresponding tensors $\{j_1, j_2, j_3\}$ and $\phi$’s should be taken equal to zero.

If we increase $N_1$ and/or $N_2$ without changing the number of tensor structures (9), it is enough to have only relations (14)–(15) to express “higher” $\phi$’s in terms of “lower” $\phi$’s. If on this step one extra tensor structure appears (for example, when we go from odd $N_1$ and $N_2$ to $(N_1+1)$ and $(N_2+1)$), we need one more relation between $\phi$’s. This extra relation can be obtained by contracting (9) with $k^{\mu_1} \ldots k^{\mu_{N_1}} k^{\sigma_1} \ldots k^{\sigma_{N_2}}$ and using eq. (B.10) of ref. [1],

$$
\sum_{2j'_1+j'_3=N_1 \atop 2j'_2+j'_3=N_2} \phi_{j_1j_2j_3; j'_1j'_2j'_3} c_{j'_1j'_2j'_3} = \frac{1}{2(N_1+N_2)! (n/2)_{(N_1+N_2)/2}} .
$$

(16)

The values of the lowest $\phi$’s can be taken from the results at $N_2 = 0$ and $N_2 = 1$,

$$
I \left[ p_{\mu_1} \ldots p_{\mu_{N_1}} \right] = \frac{1}{2n_{1/2} (n/2)_{N_1/2}} \left\{ \frac{N_1}{2} , 0 , 0 \right\} I \left[ (p^2)^{N_1/2} \right] ,
$$

(17)
Using the recurrence procedure, it is also possible to obtain some less trivial formulae, for $N_2 = 2$ and $N_2 = 3$ and arbitrary $N_1$ (such that $N_1 \geq N_2$):

\[
I \left[ p_\mu \ldots p_{\mu N_1}, q_{\sigma_1} \right] = \frac{1}{2^{(N_1+1)/2} (n/2)^{(N_1+1)/2}} \left\{ \frac{N_1 - 1}{2}, 0, 1 \right\} \left\{ \frac{N_1 - 1}{2}, 0, 1 \right\} I \left[ (p^2)^{(N_1-1)/2}(pq) \right].
\] (18)

Thus, a recursive procedure of calculating $\phi_{j_1,j_2,j_3}^{k_1,k_2,k_3}$ is constructed which provides a general algorithm to calculate the tensor integrals $I$. It is difficult, however, to generalize eqs. (17)–(20) to the case of arbitrary $N_2$.

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