ON PRESENTATIONS OF INTEGER POLYNOMIAL POINTS OF SIMPLE GROUPS OVER NUMBER FIELDS

AMIR MOHAMMADI & KEVIN WORTMAN

In this paper we prove the following

Theorem 1. Let $K$ be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $G$ be a connected, noncommutative, absolutely almost simple algebraic $K$-group. If the $K$-rank of $G$ equals 2, then $G(\mathcal{O}_K[t])$ is not finitely presented.

Actually, we will prove a slightly stronger version of Theorem 1 by showing that if $G(\mathcal{O}_K[t])$ is as in Theorem 1 then $G(\mathcal{O}_K[t])$ is not of type $FP_2$.

0.1. Related results. Krstić-McCool proved that $GL_3(A)$ is not finitely presented if there is an epimorphism from $A$ to $F[t]$ for some field $F$ [K-M].

Suslin proved that $SL_n(A[t_1,\ldots,t_k])$ is generated by elementary matrices if $n \geq 3$, $A$ is a regular ring, and $K_1(A) \cong A^\times$ [Su]. Grunewald-Mennicke-Vaserstein proved that $Sp_{2n}(A[t_1,\ldots,t_k])$ is generated by elementary matrices if $n \geq 2$ and $A$ is a Euclidean ring or a local principal ideal ring [G-M-V].

In Bux-Mohammadi-Wortman, it’s shown that $SL_n(\mathbb{Z}[t])$ is not of type $FP_{n-1}$ [B-M-W]. The case when $n = 3$ is a special case of Theorem 1.

While most of the results listed above allow for more general rings than $\mathcal{O}_K[t]$, the result of this paper, and the techniques used to prove it, are distinguished by their applicability to a class of semisimple groups that extends beyond special linear and symplectic groups.

1. PRELIMINARY AND NOTATION

Throughout the remainder, we let $G$ be as in Theorem 1 and we let $\Gamma = G(\mathcal{O}_K[t])$.

The writing of this paper was supported in part by NSF Grant No. DMS-0905891.
Let $L$ be an algebraically closed field containing $K((t^{-1}))$ fixed once and for all. In the sequel the Zariski topology is defined with this fixed algebraically closed field in mind.

Let $S$ be a maximal $K$-split torus of $G$. Let \{\alpha, \beta\} be a set of simple $K$-roots for $(G, S)$, and define $T = (\ker(\alpha))^o$, the connected component containing the identity.

Let $P$ be a maximal $K$-parabolic subgroup of $G$ that has $Z_G(T)$ as a Levi subgroup where $Z_G(T)$ denotes the centralizer of $T$ in $G$. Let $U$ be unipotent radical of $P$. We have $P = UZ_G(T)$. We can further write

$$P = UHMT$$

where $H \leq Z_G(T)$ is a simple $K$-group of $K$-rank 1 and $M$ is a $K$-anisotropic torus contained in the center of $Z_G(T)$.

If $x \in K((t^{-1}))$ is algebraic over $K$ then $x \in K$, hence $G$ has $K((t^{-1}))$-rank 2 as well and $P$ is a $K((t^{-1}))$-maximal parabolic of $G$. It also follows that $H$ has $K((t^{-1}))$-rank 1 and that $M$ is $K((t^{-1}))$-anisotropic.

We let $G, S, P, U, M, H$ and $T$ denote the $K((t^{-1}))$-points of $G, S, P, U, M, H$, and $T$, respectively.

Let $X$ denote the Bruhat-Tits building associated to $G$. This is a 2-dimensional simplicial complex, and the apartments (maximal flats) correspond to maximal $K((t^{-1}))$-split tori.

We fix once and for all a $K$-embedding of $G$ in some $\text{SL}_n$. Using this embedding we realize $G(K[t])$ and $\Gamma$ as subgroups of $\text{SL}(K[t])$ and $\text{SL}(O_K[t])$ respectively. This embedding also gives an isometric embedding of $X$ into $\mathbb{A}_{n-1}$, the building of $\text{SL}_n(K((t^{-1})))$; see [La].

2. Stabilizers of the $\Gamma$-action on its Euclidean building

Lemma 2. If $X$ is the Euclidean building for $G$, then the $\Gamma$ stabilizers of cells in $X$ are $FP_m$ for all $m$.

Proof. We first recall the proof of [B-M-W] Lemma 2. Let $x_0 \in \mathbb{A}_{n-1}$ be the vertex stabilized by $\text{SL}_n(K[[t^{-1}]]))$. We denote a diagonal matrix in $\text{GL}_n(K((t^{-1})))$ with entries $s_1, s_2, \ldots, s_n \in K((t^{-1}))^\times$ by $D(s_1, s_2, \ldots, s_n)$, and we let $\mathcal{G} \subseteq \mathbb{A}_{n-1}$ be the sector based at $x_0$ and containing vertices of the form $D(t^{m_1}, t^{m_2}, \ldots, t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \ldots \geq m_n$.

The sector $\mathcal{G}$ is a fundamental domain for the action of $\text{SL}_n(K[t])$ on $\mathbb{A}_{n-1}$ (see [So]). In particular, for any vertex $z \in \mathbb{A}_{n-1}$, there is some $h_z' \in \text{SL}_n(K[t])$ and some integers $m_1 \geq m_2 \geq \ldots \geq m_n$ with $z = h_z'D_z(t^{m_1}, t^{m_2}, \ldots, t^{m_n})x_0$. We let $h_z = h_z'D_z(t^{m_1}, t^{m_2}, \ldots, t^{m_n})$. 

For any \( N \in \mathbb{N} \), let \( W_N \) be the \((N+1)\)-dimensional vector space
\[
W_N = \{ p(t) \in \mathbb{C}[t] \mid \deg(p(t)) \leq N \}
\]
which is endowed with the obvious \( K \)-structure. If \( N_1, \ldots, N_{n^2} \) in \( \mathbb{N} \) are arbitrary then let
\[
G_{\{N_1, \ldots, N_{n^2}\}} = \{ x \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(x) = 1 \}
\]
where \( \det(x) \) is a polynomial in the coordinates of \( x \). To be more precise this is obtained from the usual determinant function when one considers the usual \( n \times n \) matrix presentation of \( x \), and calculates the determinant in \( \text{Mat}_n(\mathbb{C}[t]) \).

For our choice of vertex \( z \in \tilde{A}_{n-1} \) above, the stabilizer of \( z \) in \( \text{SL}_n(K((t^{-1})) \) equals \( h_z \text{SL}_n(K[[t]]) h_z^{-1} \). And with our fixed choice of \( h_z \), there clearly exist some \( N_i \in \mathbb{N} \) such that the stabilizer of the vertex \( z \) in \( \text{SL}_n(K[t]) \) is \( G_{\{N_i^z, \ldots, N_{n^2}^z\}}(K) \). Furthermore, conditions on \( N_i^z \) force a group structure on \( G_z = G_{\{N_i^z, \ldots, N_{n^2}^z\}} \). Therefore, the stabilizer of \( z \) in \( \text{SL}_n(K[t]) \) is the \( K \)-points of the affine \( K \)-group \( G_z \), and the stabilizer of \( z \) in \( \text{SL}_n(\mathcal{O}_K[t]) \) is \( G_z(\mathcal{O}_K) \).

Let \( \sigma \) be a cell in \( \tilde{A}_{n-1} \). The action of \( \text{SL}_n(K[t]) \) on \( \tilde{A}_{n-1} \) is type preserving, so if \( \sigma \subset \mathfrak{A} \) is a simplex with vertices \( z_1, z_2, \ldots, z_m \), then the stabilizer of \( \sigma \) in \( \text{SL}_n(\mathcal{O}_K[t]) \) is
\[
(G_{z_1} \cap \cdots \cap G_{z_m})(\mathcal{O}_K)
\]
Which implies that the stabilizer of \( \sigma \) in \( \Gamma \) is \( G_\sigma(\mathcal{O}_K) \) where \( G_\sigma = G \cap G_{z_1} \cap \cdots \cap G_{z_m} \).

If \( \psi \subset X \) is a cell, then we let \( \sigma_1, \ldots, \sigma_k \) be simplices of \( \tilde{A}_{n-1} \) such that their union contains \( \psi \), and such that their union is contained in the union of any other set of simplices of \( \tilde{A}_{n-1} \) that contains \( \psi \).

The group \( \Gamma \) may not act on \( X \) type-preservingly, but the stabilizer of \( \psi \) in \( \Gamma \) will contain a finite index subgroup that fixes \( \psi \) pointwise. Because \( \Gamma \) does act type-preservingly on \( \tilde{A}_{n-1} \), we have that the stabilizer of \( \psi \) in \( \Gamma \) contains
\[
(G_{\sigma_1} \cap \cdots \cap G_{\sigma_k})(\mathcal{O}_K)
\]
as a finite index subgroup. This is an arithmetic group, and Borel-Serre [B-S] proved that any such group is \( FP_m \) for all \( m \).

\[\square\]

3. An unbounded ray in \( \Gamma \backslash X \)

The group \( \Gamma \) does not act cocompactly on \( X \). Our next lemma is a generalization of Mahler’s compactness criterion, and it will help us
identify a ray in $X$ whose projection to $\Gamma \setminus X$ is proper. Our proof is similar to [B-M-W, Lemma 11].

**Lemma 3.** If $e \in X$, $a \in G$, $u \in \Gamma$ is nontrivial, and $a^{-n}ua^n \to 1$ as $n \to \infty$, then $\{\Gamma a^n : n \geq 0\} \subset \Gamma \setminus X$ is unbounded.

**Proof.** Since $G$ acts on $X$ with bounded point stabilizers, it suffices to show that $\{\Gamma a^n : n \geq 0\} \subset \Gamma \setminus G$ is unbounded.

If $\{\Gamma a^n : n \geq 0\}$ is bounded, then it is contained in a set $\Gamma B$ where $B \subset G$ is a bounded set. Thus, for any $a^n$, we have $a^n = \gamma b$ for some $\gamma \in \Gamma$ and $b \in B$. Hence $a^{-n}ua^n = b^{-1}\gamma^{-1}u\gamma b$.

Because $u$ is nontrivial, $\gamma^{-1}u\gamma \in \Gamma - 1$ is bounded away from 1, and thus $b^{-1}\gamma^{-1}u\gamma b$ is bounded away from 1. That’s a contradiction. \qed

4. **An unbounded semisimple element in $H(\mathcal{O}_K[t])$**

Recall that $H$ has $K((t^{-1}))$-rank 1 (and $K$-rank 1), hence the Bruhat-Tits building of $H$, which will be denoted by $X_H$, is a tree. Let $S'$ be a maximal $K$-split, thus $K((t^{-1}))$-split, torus of $H$ and let $Q^+$ and $Q^-$ be opposite $K$-parabolic subgroups of $H$ with Levi subgroup $Z_H(S')$.

We denote the unpotent radical of $Q^\pm$ as $R_u(Q^\pm)$, and we let $Q^\pm = Q^\pm(K((t^{-1})))$, $R_u(Q^\pm) = R_u(Q^\pm)(K((t^{-1})))$, and $S' = S'(K((t^{-1})))$.

See [Se, Proposition 25] for the next lemma.

**Lemma 4.** Let $u^+ \in R_u(Q^+)$ and $u^- \in R_u(Q^-)$ and let $F^\pm = \text{Fix}_{X_H}(u^\pm)$. Assume that $F^+ \cap F^- = \emptyset$. Then $u^+u^-$ is a hyperbolic isometry of $X_H$.

**Proof.** Let $x$ be the midpoint between $F^+$ and $F^-$. Let $p_1$ be the path between $x$ and $F^+$ and let $p_2$ be the path between $x$ and $F^-$, and let $\psi$ be an edge containing $x$, contained in $p_1 \cup p_2$, not contained in $p_2$, and oriented towards $F^+$.

Notice that $u^-p_2 \cup p_2$ is an embedded path between $x$ and $u^-x$ and that $p_1 \cup u^+p_1 \cup u^+p_2 \cup u^+u^-p_2$ is an embedded path between $x$ and $u^+u^-x$. The edge $u^+u^-\psi$ is a continuation of the latter path that is oriented away from both $u^+u^-x$ and $x$.

If $u^+u^-$ is elliptic, then it fixes the midpoint of the path between $x$ and $u^+u^-x$ and maps $\psi$ to an oriented edge pointed towards $x$. Therefore, $u^+u^-$ is hyperbolic. \qed

**Lemma 5.** There exists elements $u^\pm \in R_u(Q^\pm)(\mathcal{O}_K[t])$ of arbitrarily large norm.

**Proof.** After perhaps replacing $\alpha$ with $2\alpha$, there is a root group $U_\alpha \leq R_u(Q^\pm)$ and a $K$-isomorphism of algebraic groups $f : \mathbb{A}^k \to U_\alpha$ for some affine space $\mathbb{A}^k$. 

The regular function \( f \) is defined by polynomials \( f_i \in K[x_1, \ldots, x_k] \). Because \( f \) maps the identity element to the identity element, each \( f_i \) has a constant term of 0.

The field of fractions of \( \mathcal{O}_K \) is \( K \). We let \( N \) be the product of the denominators of the coefficients of the \( f_i \). Then the image under \( f \) of the points \((Nt^1, \ldots, Nt^j)\) forms an unbounded sequence in \( j \) of points in \( U_\alpha(\mathcal{O}_K[t]) \).

\[ \square \]

**Lemma 6.** There exists a hyperbolic isometry \( b \in \text{H}(\mathcal{O}_K[t]) \) of the tree \( X_H \).

**Proof.** Let \( \ell' \subseteq X_H \) be the geodesic corresponding to \( S' \), and choose \( u\pm \in R_u(Q)\mathcal{O}_K[t] \) of sufficient norm such that \( \ell' \cap F^+ \) is disjoint from \( \ell' \cap F^- \). Since \( F^+ \) and \( F^- \) are convex, and \( \ell' - (F^+ \cup F^-) \) is the geodesic between them, it follows that \( F^+ \cap F^- = \emptyset \). Now apply Lemma 4. \( \square \)

5. **Construction of cycles in \( X \) near \( \Gamma \)**

Let \( b \in \text{H}(\mathcal{O}_K[t]) \) be as in Lemma 6, and let \( S'' \) be the \( K((t^{-1})) \)-split one dimensional torus corresponding to the axis of \( b \) in \( X_H \). Define the \( K((t^{-1})) \)-split torus \( A = \langle S'', T \rangle \leq P \) and let \( A = A(K((t^{-1}))) \). Let \( A \) denote the apartment in \( X \) corresponding to \( A \).

Recall that any unbounded element \( a \in T \) translates \( A \), and that the axis for the translation is any geodesic in \( A \) that joins \( P \) with its opposite parabolic \( P^{\text{op}} \), as usual \( P^{\text{op}} = P^{\text{op}}(K((t^{-1}))) \) where \( P^{\text{op}} \) is the opposite parabolic containing \( Z_G(T) \).

Note that \( b \) acts by translation on \( A \). In fact, \( b \) translates orthogonal to any geodesic in \( A \) that joins \( P \) with \( P^{\text{op}} \). Indeed, choose an element \( w \) of the Weyl group with respect to \( A \) that reflects through a geodesic joining \( P \) and \( P^{\text{op}} \). Thus \( w \) fixes both parabolic groups, and their common Levi subgroup, and hence \( H \). Since \( S' = A \cap H \), \( w \) fixes \( S' \) and thus fixes any axis for \( b \) in \( A \). Therefore, either \( b \) translates orthogonal to any geodesic in \( A \) that joins \( P \) with \( P^{\text{op}} \), or else \( b \) translates along a geodesic in \( A \) that joins \( P \) with \( P^{\text{op}} \). The latter option would contradict Lemma 3 since for any \( e \in A \), we have \( \Gamma b^ae = \Gamma a^e \in \Gamma \setminus X \) and yet there is an unbounded \( a \in T \) such that the ray determined by \( a^e \) is parallel to the ray determined by \( b^ae \) and yet \( a^{-n}ua^n \to 1 \) either for any \( u \in U(\mathcal{O}_K[t]) \) or for any \( u \) in the \( \mathcal{O}_K[t] \)-points of the unipotent radical of \( P^{\text{op}} \).

The spherical Tits building for \( G \) and \( X \) is a graph, and the apartment \( A \) corresponds to a circle in the spherical Tits building. Suppose
this circle has vertices $P_1, \ldots, P_n$ and edges $Q_1, \ldots, Q_n$ where each $P_i$ is a maximal proper $K((t^{-1}))$-parabolic subgroup of $G$ containing $A$, each $Q_i$ is a minimal $K((t^{-1}))$-parabolic subgroup of $G$ containing $A$, and $P_1 = P$. We further assume that mod $n$, the edge $Q_i$ has vertices $P_i$ and $P_i + 1$.

Notice that $U \leq Q_1 \cap Q_n$ since $P = P_1$ contains both $Q_1$ and $Q_n$. That is, any element of $U(O_K)$ fixes the edges $Q_1$ and $Q_n$.

Let $U_1$ be the root group corresponding to the half circle that contains $Q_1$ but not $Q_2$, so that $U_1 \leq U$ but $U_1 \cap Q_2 = 1$. Let $U_n$ be the root group corresponding to the half circle that contains $Q_n$ but not $Q_{n-1}$, so that $U_n \leq U$ but $U_n \cap Q_{n-1} = 1$.

It follows that $U - Q_i$ has codimension in $U$ at least 1 for $i = 2, n-1$. Since $U(O_K)$ is Zariski dense in $U$, there is some $u \in U(O_K) - (Q_2 \cup Q_{n-1})$. It follows that $u$ fixes the edges $Q_n$ and $Q_1$, but no other edges in the circle corresponding to $A$.

Since $u$ is a bounded element of $G$, it fixes a point in $X$. Therefore, $u$ fixes a geodesic ray in $X$ that limits to an interior point of the edge corresponding to $Q_1$ in the spherical building. Any such geodesic ray must contain a point in $A$, which is to say that $u$ fixes a point in $A$.

Define a height function $q : A \to \mathbb{R}$ such that the pre-image of any point is an axis of translation for $b$, such that $s \leq t$ if and only if any geodesic ray in $A$ that eminates from $q^{-1}(s)$ and limits to $P$ contains a point from $q^{-1}(t)$.

Let $F = \{ x \in A \mid ux = x \}$, let $I = \inf_{f \in F} \{ q(f) \}$, and let $E = \{ f \in F \mid q(f) = I \}$. Since the fixed set of $u$ in the circle at infinity of $A$ equals the union of the two edges $Q_1$ and $Q_n$, and since $F$ is convex, $I$ exists and $E$ is either a point of, a subray of, a line segment of, or an entire axis of translation for $b$.

Notice that $E$ is bounded, otherwise $u$ would fix the point at infinity that a subray of $E$ limited to. This point at infinity would have distance $\pi/2$ from the vertex $P$ in the spherical metric, but this is not possible as the previously identified fixed set of $u$ in the boundary circle is centered at $P$ and has radius at most $\pi/3$. (The bound $\pi/3$ is realized exactly when the root system for $G$ is of type $A_2$.) Thus $E$ is either a point or a compact interval.

Since the fix set of $u$ in the boundary circle is exactly the union of $Q_1$ and $Q_n$, and since $F$ is convex, $F$ is precisely the union of all geodesic rays eminating from points in $E$ and limiting to points in the arc $Q_1 \cup Q_n$. That is $F$ is a polyhedral region in $A$ that is symmetric with respect to a reflection of $A$ through a geodesic that limits to $P$. 
and the opposite point of $P$. If $E$ is a point, then $F$ has two geodesic rays as its boundary: one ray that limits to $P_2$, and the other that limits to $P_n$. If $E$ is a nontrivial interval, then the boundary of $F$ is the union of $E$, a ray from an endpoint of $E$ that limits to $P_2$, and a ray from the other endpoint of $E$ that limits to $P_n$.

If $E$ is an interval, we label its endpoints $e^+$ and $e^-$ such that $E$ is both oriented in the direction of translation of $b$, and in the direction towards $e^+$, and away from $e^-$. Let $e_0$ be the midpoint of $E$. If $E$ is a point, then $e_0 = e^+ = e^-$ is that point.

For $n_0$ sufficiently large and for any $n \geq n_0$, we define $\sigma_n \subseteq A$ as the geodesic segment between $b^{-n}e^+$ and $b^n e^-$. Notice that $b^{-n}e^+$ is the only point in $\sigma_n$ that is fixed by $g_n = b^{-n}ub^n$, and that $b^n e^-$ is the only point in $\sigma_n$ that is fixed by $h_n = b^n ub^{-n}$.

Recall that $A$ is the apartment corresponding to $A$ and $T \subset A$ is a $K$-split one dimensional torus of $G$. Recall also that $P = UZ_G(T)$. Let $a \in T$ be such that $a^{-n}ua^n \to 1$ as $n \to \infty$ so that $a^n e_0$ converges to the cell at infinity corresponding to $P$ as $n \to \infty$.

Let $\Delta_n$ be the triangle with one face equal to $\sigma_n$, a second face contained in the boundary of $b^{-n}\text{Fix}_A(u) = \text{Fix}_A(g_n)$, a third face contained in the boundary of $b^n\text{Fix}_A(u) = \text{Fix}_A(h_n)$, and vertices $b^n e^-$, $b^{-n} e^+$, and a uniquely determined point $y_n \in \partial\text{Fix}_A(g_n) \cap \partial\text{Fix}_A(h_n)$. Thus $y_n$ converges to the cell at infinity corresponding to $P$ as $n \to \infty$.

Note that

1. $U$ is a unipotent group so $\text{[[[g_n, h_n], \cdots], h_n], h_n]} = 1$ for some fixed number of nested commutators that’s independent of $n$.
2. If $w$ is a word in $\{g_n, h_n, g_n^{-1}, h_n^{-1}\}$ and $d \in \{g_n, h_n, g_n^{-1}, h_n^{-1}\}$, then $w \sigma_n$ and $w d \sigma_n$ are incident.

(1) and (2) imply that the word $\text{[[[g_n, h_n], \cdots], h_n], h_n]}$ (or possibly a subword) describes a 1-cycle that is the union of translates of $\sigma_n$ by subwords of $\text{[[[g_n, h_n], \cdots], h_n], h_n]}$. We name this 1-cycle $c_n$.

The cone of $c_n$ at the point $y_n$ is the topological image of a 2-disk $\phi_n : D^2 \to X$ such that $\phi_n(\partial D^2) = c_n$.

If we let

$$X_0 = \Gamma \sigma_{n_0}$$

then clearly $c_n \in X_0$ for all $n$ since $b, g_n, h_n \in \Gamma$ and $\sigma_n \subseteq (b) \sigma_{n_0}$.

6. PROOF OF THEOREM [1]

We choose a $\Gamma$-invariant and cocompact space $X_i \subseteq X$ to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq \cup_{i=1}^\infty X_i = X$$
In our present context, Brown’s criterion takes on the following form

**Brown’s Filtration Criterion 7.** By Lemma 2, the group $\Gamma$ is not of type $FP_2$ (and hence not finitely presented) if for any $i \in \mathbb{N}$, there exists some class in the homology group $\tilde{H}_1(X_0, \mathbb{Z})$ which is nonzero in $\tilde{H}_1(X_i, \mathbb{Z})$.

Since $\Gamma \setminus X_i$ is compact it follows from Lemma 3 that for any $i$ there exists some $j_i$ such that $a_j e_0 \not\in X_i$. Choose $n$ sufficiently large so that $a_j e_0 \in \Delta_n \subseteq \phi_n$. Recall that $c_n \subseteq X_0$. Since $X$ is contractible and 2-dimensional, any filling disk for $c_n$ must contain $a_j e_0$. That is, $c_n$ represents a nontrivial class in the homology of $X - \{a_j e_0\}$, and hence is nontrivial in the homology of $X_i$.

7. Other ranks

The proof of Proposition 4.1 in [B-W] gives a short proof that $\text{SL}_2(\mathbb{Z}[t])$ is not finitely generated by examining the action of $\text{SL}_2(\mathbb{Z}[t])$ on the tree for $\text{SL}_2(\mathbb{Q}((t^{-1})))$. Replacing some of the remarks for $\text{SL}_2(\mathbb{Z}[t])$ in that paper with straightforward analogues from lemmas in this paper, it is easy to see that the proof in [B-W] applies to show that if $H$ is a connected, noncommutative, absolutely almost simple algebraic $K$-group of $K$-rank 1, then $H(\mathcal{O}_K[t])$ is not finitely generated.

It seems natural to state the following

**Conjecture 1.** Suppose $H$ is a connected, noncommutative, absolutely almost simple algebraic $K$-group whose $K$-rank equals $k$. Then $H(\mathcal{O}_K[t])$ is not of type $F_k$ or $FP_k$.

The conjecture has been verified when $K = \mathbb{Q}$ and $H = \text{SL}_n$ [B-M-W].

**References**

[B-S] A. Borel, J-P. Serre, *Corners and arithmetic groups*. Comm. Math. Helv. **48** (1973) p. 436-491.

[Br] Brown, K., *Finiteness properties of groups*. J. Pure Appl. Algebra **44** (1987), 45-75.

[B-M-W] Bux, K.-U., Mohammadi, A., Wortman, K. *SL_n(\mathbb{Z}[t]) is not FP_{n-1}*. Comm. Math. Helv. **85** (2010), 151-164.

[B-W] Bux, K.-U., and Wortman, K., *Geometric proof that SL_2(\mathbb{Z}[t^{-1}]) is not finitely presented*. Algebr. Geom. Topol. **6** (2006), 839-852.

[G-M-V] Grunewald, F., Mennicke, J., and Vaserstein, L., *On symplectic groups over polynomial rings*. Math. Z. **206** (1991), 35-56.

[K-M] Krstić, S., and McCool, J., *Presenting GL_n(k(T)).* J. Pure Appl. Algebra **141** (1999), 175-183.
| Ref | Author | Title | Journal/Book Details |
|-----|--------|-------|----------------------|
| La  | Landvogt, E. | Some functorial properties of the Bruhat-Tits building. | J. Reine Angew. Math. 518 (2000) p.213-241. |
| Se  | Serre, J.-P. | Trees. | Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. |
| So  | Soulé, C. | Chevalley groups over polynomial rings. | Homological Group Theory, LMS 36 (1977), 359-367. |
| Su  | Suslin, A. A. | The structure of the special linear group over rings of polynomials. | Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), 235-252. |