Maxwell, Gravitation, and Hodge

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Abstract

Since Einstein’s fundamental paper of 1915, gravitation has been synonymous with General Relativity, while theories based on Lorentz invariance have been dismissed as formal analogies. Consequently modern cosmology focusses exclusively on Einstein’s geometric theory, which is unnecessarily cumbersome in regions of weak fields, while Maxwell’s equations are far more tractable.

It is shown here that Maxwell’s equations are an artifact of Hodge theory, geometric in nature, independent of any specific physical mechanisms, and valid for any force field, attractive or repulsive, generated by a material density and current. In particular, Maxwell’s equations apply to weak gravitational fields – those in Minkowski space-time.

Classical analysis and linear partial differential equations therefore have a role to play in modern problems in cosmology. The gravitational field generated by the rotation of a spherical body is computed explicitly using a multipole expansion; and the method is compared with the Lense-Thirring theory of General Relativity. Maxwell’s equations for gravitation are a suitable model for the dynamics of galaxies and galaxy clusters, and the problem of dark matter is discussed from the perspective of dynamic anomalies, which are not accounted for by Newtonian mechanics.
1 Introduction

Does the Earth’s rotation produce a gravitational field? That is the prediction of Albert Einstein’s General Theory of Relativity, based on the Lense-Thirring theory [7], [14] and the notion of frame-dragging. Since the 1970’s attempts have been made to detect this component of the gravitational field by the precession of a gyroscope in orbit around the Earth. The precession of the spin of a gyroscope is given in Steven Weinberg’s treatise Gravitation and Cosmology [16]; and the prediction was made there that the effect would be detected. Measurements of such gyroscopic forces have recently been reported by NASA [2] in the Gravity Probe B experiments.

On the other hand, such a component of the gravitational field is easily deduced, and computed explicitly, from classical linear harmonic analysis. The Helmholtz decomposition [5] asserts that any smooth vector field $F$ on $\mathbb{R}^3$ can be decomposed as the sum of an irrotational and a solenoidal field,

$$F = -\nabla \phi + \nabla \times A. \quad (1)$$

Helmholtz was led to his theorem in his studies of fluid flow by attempting to resolve the velocity field into separate components of irrotational (potential) and rotational (solenoidal) flows. In the theory of the electromagnetic field, $\phi$ and $A$ are respectively the scalar and vector potentials. In the classical (Newtonian) theory of gravitation, however, the field is presumed at the outset to be irrotational, despite the fact that there is not a shred of evidence, mathematical or physical, to support that assumption. If, on the contrary, the postulate were dropped, $B = \nabla \times A$ would constitute precisely the solenoidal component of the gravitational field predicted by the Lense-Thirring theory and demonstrated by the NASA program.

Efforts to extend Maxwell’s ideas to gravitation date back to a paper published by Oliver Heaviside in 1893 [4] entitled “A Gravitational and Electromagnetic Analogy”. Heaviside’s attempt at a field theory of gravitation was followed by Lorentz (1900) [8] and Poincaré (1905) [11]. Poincaré believed that Lorentz invariance was a fundamental fact of physics; and at the end of his paper, in a section entitled Hypothèses sur la Gravitation, he proposed a rudimentary Lorentz-covariant form of the gravitational field in which “l’onde gravifique, . . . étant supposée se propager avec la vitesse de la lumière . . . peut se partager en trois composantes, la première une vague analogie avec
The gravitational field due to a rotating Earth rotation is obtained in an explicit, closed form in §6 using a multipole expansion in spherical harmonics in §6. Special relativity enters only tangentially, through the postulate that the speed of light is a fundamental
constant of nature, and General Relativity does not enter at all. The significance of the Lense-Thirring theory, outlined in §7 is not that it demonstrates that the rotation of spherical body produces a solenoidal component of the gravitational field, but that it provides a perturbative framework to calculate the relativistic corrections to the classical solution.

Maxwell’s equations for gravitation are valid in flat space-time, hence everywhere except the immediate vicinity of a massive body, such as a star or black hole; they are therefore applicable to problems in galactic dynamics. Since dark matter hypothetically does not interact with the electromagnetic field, any “evidence” for it is necessarily indirect and takes the form of dynamical anomalies which cannot be explained by Newtonian mechanics [13]. A brief recap is given in §8 of this paper.

2 Hodge Theory

The Helmholtz decomposition leads to an extended problem in potential theory in which the scalar and vector potentials are obtained as solutions to the partial differential equations

\[ \Delta \phi = -\frac{\rho}{\epsilon}, \quad \nabla \times \nabla \times A = \mu J, \]  

(2)

where \( \epsilon \) and \( \mu \) are physical parameters of the theory. In the particular case of the electromagnetic field they are called the electric and magnetic inductances, and are related to the laws of Gauss and Ampère. The electrostatic field \( E = -\nabla \phi \) is the force per unit charge acting on a charged particle; while \( B \), the magnetic induction, exerts a force per unit charge on a particle moving with velocity \( v \) given by \( v \times B \).

The difference in the physical natures of \( E \) and \( B \) points up a fundamental dissonance in the use of vector analysis in (1): \( \nabla \phi \) is a polar vector field, while \( \nabla \times A \) is a field of axial vectors. These transform differently under the group of rigid motions, have different physical dimensions, and act differently physically. How, then, can one “add” them together?

The dissonance is resolved by reformulating the problem in terms of differential forms. The polar vector field \( E = E \cdot dx \) is replaced by the 1-form \( E = E \cdot dx \); while the field of axial vectors \( B \) is replaced by the 2-form \( B = B \cdot dS \), where \( dS \) is an oriented element of surface area. One is then led to the Hodge decomposition of differential forms, and more
generally, to the elegant mathematical theory known as Hodge theory. To fix language and notation, we give here a quick review of the basic ideas. An introductory account of differential forms can be found in a number of sources, e.g. Harley Flanders [3]. An extensive account of their use in electrodynamics and general relativity is given in Misner et.al..

We assume the reader is familiar with the basic operations of wedge product $\wedge$ and exterior derivative $d$ on $p$-forms $\Lambda_p$ and that the theorems of Green, Gauss, and Stokes are collected in a single theorem, known as Stokes’ theorem,

$$\int\int_{\Omega} d\omega = \int_{\partial\Omega} \omega. \quad (3)$$

Here, $\omega \in \Lambda_p$ has differentiable coefficients, and $\Omega$ is a $p + 1$ dimensional, oriented manifold embedded in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$.

A form $\omega \in \Lambda_p$ is said to be closed if $d\omega = 0$ and exact if $\omega = d\chi$, where $\chi \in \Lambda_{p-1}$. Since $d^2 = 0$, a $p$-form is closed if it is exact. In a simply connected region the two conditions are equivalent; and it will be sufficient to restrict ourselves to this case. A necessary and sufficient condition for $\omega \in \Lambda_p$ to be exact in a region $U$ (not necessarily simply connected) is that its integral over every closed $p$ manifold $\Sigma \subset U$ vanish:

$$\int\int_{\Sigma} \omega = 0, \quad \text{whenever} \quad \partial\Sigma = \emptyset.$$

In the special case of a 1-form $E$, $\Sigma$ is a closed path, and the integral above is a line integral called the circulation. If the circulation vanishes for every smooth closed path, then regardless of the topology of the region, $E$ is exact, and there exists a 0-form $\phi$ (that is, a single-valued function) such that $E = -d\phi$. It is standard convention to normalize the potential to vanish at infinity, so that it is explicitly given by

$$\phi(x) = \int_{x}^{\infty} E, \quad x \in \mathbb{E}^3. \quad (4)$$

In the static case the electric and gravitational fields are both conservative, so that their corresponding 1-forms are exact; but the electrostatic potential is positive, while the gravitational potential is negative. This reflects the fact that the gravitational force is attractive,
while the electrostatic force, defined in terms of like charges, is repulsive.

Hodge’s theory has evolved considerably since his book [6] was published, and some of the language and notation has changed. The book written by Flanders, though not complete, nevertheless offers a readable introduction to the subject, especially for those interested in physical applications. The survey article by Raoul Bott [11] is especially worthwhile for a glimpse from the perspective of topology. He explains the topological aspects of Hodge’s work and also discusses Maxwell’s equations and the Yang-Mills equations in connection with Hodge theory. Note, however, that his discussion is restricted to Maxwell’s equations in a vacuum (no charge or current), and to the equations on \( \mathbb{E}^4 \), in which case they are elliptic. Below, when we come to Maxwell’s dynamical equations, we shall work on the space of 2-forms over Minkowski space-time \( \mathfrak{M}^4 \), which leads to a hyperbolic system.

The Hodge star operation \(*\) on \( p \)-forms, which plays a fundamental role in potential theory, is defined as follows: Given an oriented volume element \( dv \) on a smooth manifold \( \mathcal{M} \) and \( \omega \in \Lambda_p(\mathcal{M}) \), \( *\omega \) is defined as the \( n-p \) form for which \( \omega \wedge *\omega = dv \). For example, the standard (right-handed) volume element on \( \mathbb{E}^3 \) is \( dv = dx^1 \wedge dx^2 \wedge dx^3 \). The associated Hodge star operation is

\[
* \text{d}x^i = \text{d}x^j \wedge \text{d}x^k, \quad *1 = dv, \quad ** = \text{id}.
\]

(5)

Here \( i, j, k \) denote the integers 1,2,3 in cyclic order, and \( \text{id} \) denotes the identity mapping. The star operation is extended to \( \mathbb{E}^n \) in the obvious way; the reader may check that \( ** = (-1)^{n+1} \text{id} \) on \( \mathbb{E}^n \).

An inner product, called the Hodge duality, is defined for \( \xi, \eta \in \Lambda_p(\mathbb{E}^n) \) by

\[
(\xi, \eta) = \iiint_{\mathbb{E}^n} \xi \wedge *\eta \, dv,
\]

(6)

where \(*\) is the operation associated with the oriented volume element \( dv \). The Hodge duality implicitly defines a formal adjoint to the exterior derivative \( d \). Called the coderivative, it maps \( \Lambda_{p+1} \) to \( \Lambda_p \), and is defined by the relation

\[
(d \xi, \eta) = (\xi, \delta \eta) \quad \xi \in \Lambda_p, \ \eta \in \Lambda_{p+1}
\]

(7)

where \( \xi, \eta \) have compact support on \( \mathbb{E}^n \). On \( \Lambda_p(\mathbb{E}^n) \) the co-derivative is given by \( \delta = \delta_p = (-1)^{p-1} *^{-1} d * \) (13, Proposition 2.1).
A differential form \( \xi \) is **co-closed** if \( \delta \xi = 0 \) and **co-exact** if \( \xi = \delta \eta \).

The reader should check that \( \delta : \Lambda^p \to \Lambda^{p-1} \), and that on a manifold with trivial topology such as \( \mathbb{E}^n \), \( \delta^2 = 0 \) and a differential form is co-exact if and only if it is co-closed. A differential form is called harmonic if it is both closed and co-closed, i.e. \( d\xi = \delta\xi = 0 \). On \( \mathbb{E}^2 \) these two equations are precisely the Cauchy-Riemann equations; and so this system of equations can be called the **generalized Cauchy-Riemann** equations. If \( \xi \) is harmonic then \( \Delta \xi = 0 \), where \( \Delta = d\delta + \delta d \) is the Laplacian.

The Hodge decomposition on a compact manifold states that every (smooth) \( p \)-form \( \omega \) can be decomposed as \( \omega = d\xi + \delta\eta + \alpha \), where \( \alpha \) is a harmonic form. Since \( \langle d\xi, \delta\eta \rangle = \langle \xi, \delta^2\eta \rangle = 0 \), etc. it is clear that the subspaces of exact, co-exact, and harmonic forms are mutually orthogonal; hence the Hodge decomposition can also be written as

\[
\Lambda^p = [\text{im} \, d] \oplus [\text{im} \, \delta] \oplus [\ker d \cap \ker \delta].
\]

This formulation of Hodge’s theorem leads to a simple proof based on the orthogonal decomposition theorem of Hilbert spaces (see Riesz and Nagy §34):

**Proposition 2.1** Let \( \mathcal{H} \) be a Hilbert Space, \( \mathcal{E} \) be a subspace and \( \mathcal{B} = \mathcal{E}^\perp \) its orthogonal complement. Then every element \( F \in \mathcal{H} \) can be uniquely decomposed as \( F = E + B \), where \( E \in \mathcal{E} \) and \( B \in \mathcal{B} \). We write \( \mathcal{H} = \mathcal{E} \oplus \mathcal{B} \).

We denote by \( \Lambda_p(\mathbb{E}^n) \) the Hilbert space of \( p \)-forms with finite Hodge norms \( \langle F, F \rangle < +\infty \) for \( F \in \Lambda_p \). In the applications here, \( n = 3 \) and the potentials, both scalar and vector, are solutions of \( \Delta \phi = 0 \) where \( \rho \) has compact support. They are thus solutions of \( \Delta \phi = 0 \), etc. in exterior domains; hence they decay as \( r^{-1} \) as \( r \to \infty \). The fields themselves are first order derivatives of the potentials, specifically, \( \nabla \phi \) and \( \nabla \times A \). They therefore decay as \( r^{-2} \) at infinity and are regular at the origin; hence they are square integrable.

To account for the lack of differentiability, the decomposition must incorporate the notion of weak derivatives. We say that \( d\alpha = \beta \) in the weak sense if \( \langle d\alpha, \eta \rangle = \langle \beta, \delta\eta \rangle \) for all differential forms \( \eta \in \Lambda_{p+1}(\mathbb{E}^n) \) with \( C^1 \) coefficients and compact support. A similar definition applies to the equation \( \delta\alpha = \beta \). Note that if \( d\alpha = \beta \) in the weak sense and \( \alpha \) is itself \( C^1 \), then \( \beta \) is continuous and \( d\alpha = \beta \) in the ordinary (strong) sense.
Theorem 2.2 The Hilbert spaces $\Lambda_p(\mathbb{R}^n)$ decompose into the direct sum of the $L^2$ exact and co-exact forms: $\Lambda_p = [d\Lambda_{p-1}] \oplus [\delta\Lambda_{p+1}]$, where $[d\Lambda_{p-1}]$ denotes the $L^2$ closure of the linear set $\{dA : A \in \Lambda_{p-1}\}$, etc. and by default, $[d\Lambda_n] = [\delta\Lambda_0] = 0$. Thus every differential form $F \in \Lambda_p$ can be written as $F = dA + \delta\Phi$, where $A \in \Lambda_{p-1}$ and $\Phi \in \Lambda_{p+1}$.

Moreover, since $** = (-1)^{n+1}id,$

$$\star [d\Lambda_p] = [\delta\Lambda_{n-p}],$$

(9)

Proof: The harmonic forms satisfy $d\alpha = \delta\alpha = 0$, hence each term must be harmonic in the usual sense. By Liouville’s theorem, each coefficient of $\alpha$ must therefore be a constant; and since the Hodge norm of $\alpha$ must be finite, those constants must vanish. So there are no harmonic components in the Hodge decomposition on Euclidean spaces.

The orthogonal decomposition (8) therefore follows immediately from Proposition 2.1.

The identity (9) follows very simply from the following: $\star [d\Lambda_p] = \{\star d\alpha : \alpha \in \Lambda_p\} = \{\star d \beta : \beta = (-1)^{n+1} \star \alpha \in \Lambda_{n-p}\} = [\delta\Lambda_{n-p}].$ ■

Finally, we shall need the expression for $dS$, introduced above, for the vector element of surface area on a 2 dimensional surface $S$ embedded in $\mathbb{R}^3$. It is given by $dS = (X_u \times X_v) \, du \wedge dv$, where $X = (x^1(u,v), x^2(u,v), x^3(u,v))$ is a parametrization of a neighborhood of $S$ by local coordinates $u, v$. We leave it to the reader to verify the identities

$$dx^i \wedge dx^j = \frac{\partial(x^i, x^j)}{\partial(u, v)} \, du \wedge dv, \quad B = B_j \, dx^k \wedge dx^l.$$

Here and throughout this article, the expression for $B$ signifies a summation over $j, k, l$ from 1 to 3 in cyclical order.

3 Stationary Field Theory

Maxwell’s equations of electrodynamics in vector form are (see Stratton)

$$\nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \text{div} \, B = 0; \quad (10)$$

$$\nabla \times H - \frac{\partial D}{\partial t} = J, \quad \text{div} \, D = \rho. \quad (11)$$
Here, $E$ is the electric field, $B$ the magnetic induction, $H$ the magnetic field, and $D$ the electric displacement introduced by Maxwell. The first pair of equations constitute the differential form of Faraday’s Law, and the second the Maxwell-Ampère Law. The system is closed with the two constitutive relations

$$D = \varepsilon E, \quad B = \mu H,$$  \hspace{1cm} (12)

where $\varepsilon$ and $\mu$ are the electric and magnetic permittivities.

As we noted above, equations (12) ignore the fundamentally different vectorial properties of $E$, $H$ and $D$, $B$. Furthermore, the left side of (11) is an axial vector, while the right side is a polar vector. In the following, we shall prove

**Theorem 3.1** The fields $E, B, H, D$ of electromagnetic theory arise naturally in the Hodge decompositions of $\Lambda_p(\mathbb{R}^3)$ for $p = 1, 2$. The 1-forms $E = E \cdot dx$ and $H = H \cdot dx$ are respectively the exact and co-exact 1-forms; while $B = B \cdot dS$ and $D = D \cdot dS$ are the exact and co-exact 2-forms. The Laws of Gauss and Ampère take the form $D = \varepsilon \ast E$ and $B = \mu \ast H$, where $\varepsilon$ and $\mu$ are the electric and magnetic inductances, and $\ast$ is the Hodge star operation. The derivation carries over to attractive force fields, the only difference being that $\varepsilon$ and $\mu$ are then negative.

With the above conventions Faraday’s Law can be written in the integral form

$$\int \int_S E \wedge dt + B = 0, \quad \partial S = 0,$$  \hspace{1cm} (13)

where $S$ is a closed 2-manifold in 4 dimensional space-time, i.e. $\partial S = 0$. The condition (13) implies that the Faraday 2-form $F = E \wedge dt + B$ is an exact 2-form in Minkowski space-time. The two conditions (10) and (13) are equivalent for smooth $E$ and $B$ (13). For, if (13) holds then $F$ is exact; hence $dF = 0$; and if $E$ and $B$ are smooth, a direct calculation shows that (10) holds. Conversely, if (10) holds $F$ is closed; and since the topology of Minkowski space-time is trivial, $F$ is exact.

Denote the Hodge decompositions of $\Lambda_1$ and $\Lambda_2$ on $\mathbb{R}^3$ by $\Lambda_1 = \mathcal{E} \oplus \mathcal{H}$ and $\Lambda_2 = \mathcal{B} \oplus \mathcal{D}$, where $\mathcal{E}$ and $\mathcal{B}$ are respectively: $\mathcal{E} = \{E : E = -d\phi\}$, $\mathcal{B} = \{B : B = dA, A \in \Lambda_1\}$. By (11),

$$\mathcal{D} = \ast \mathcal{E}, \quad \mathcal{B} = \ast \mathcal{H};$$  \hspace{1cm} (14)
and by the Hodge decomposition any $F \in \Lambda_1(\mathbb{R}^3)$ may be written as

$$F = -d\phi + \ast dA,$$  \hspace{1cm} (15)

where $\phi$ is a 0-form and $A$ a 1-form.

Consider the two integrals

$$\mathcal{C}(\gamma) = \oint_{\gamma} F, \quad \mathcal{F}(S) = \iint_S \ast F$$

where $\gamma$ is a smooth closed curve and $S$ a smooth closed surface. The line integral $\mathcal{C}(\gamma)$ is called the circulation of $F$ around $\gamma$; and $\mathcal{F}(S)$ is called the flux of $F$ through $S$. By (15),

$$\mathcal{F}(S) = -\iint_S \ast d\phi, \quad \mathcal{C} = \oint \ast dA. \hspace{1cm} (16)$$

Gauss’ Law of electrostatics states that the electric flux through a closed surface is proportional to the enclosed electric charge, and is associated with the first integral above. It may be interpreted as a general axiom of conservative force fields, as follows. Let $E = -\nabla \phi$ be any conservative force field, and $E = E \cdot dx = -d\phi$ the associated exact 1-form. Then

$$\mathcal{F}(S) = -\iint_S \ast d\phi = \iint_S \ast E = \iint_S E \cdot dS;$$

hence $\mathcal{F}(S)$ is precisely the flux of the lines of force through the closed surface $S$.

Let the material density be $\rho \geq 0$; since $\rho \, dv$ is a closed 3-form, there is a 2-form $D$ such that $dD = \rho \, dv$. By Stokes’ theorem,

$$Q(S) = \iint_S D = \iiint_{U_S} \rho \, dv$$  \hspace{1cm} (17)

is the total amount of material contained within $S$. Gauss’ Law asserts that $Q(S) = \varepsilon \mathcal{F}(S)$ for all closed surfaces $S$; hence

$$D = \varepsilon \ast E, \quad \text{Gauss’ Law.} \hspace{1cm} (18)$$

It follows that

$$\delta E = \ast d \ast E = \frac{1}{\varepsilon} \ast dD = \frac{\rho}{\varepsilon}, \hspace{1cm} (19)$$
Theorem 3.2  In the case of an inverse square law on $\mathbb{E}^3$, $\epsilon$ is given by

$$\epsilon = \frac{1}{4\pi G},$$  \hspace{1cm} (20)

where $G$ is the physical constant which determines the strength of the field produced by $q$. The parameter $\epsilon$ is positive or negative, according as the field is repulsive or attractive.

Proof: For a point source $q$ at the origin the displacement $D$ is given by

$$D = \frac{q}{4\pi r^3} x^k \wedge dx^l.$$  \hspace{1cm} (21)

The reader may verify by direct calculation that $dD = 0$ for $r > 0$; hence the integral of $D$ over any closed surface $S$ enclosing the origin is equal to the integral over the sphere $S_R$ of radius $R$ centered at the origin. By Stokes’ theorem

$$\int_{S_R} D = \frac{q}{4\pi R^3} \int_{S_R} x_j dx^k \wedge dx^l = \frac{q}{4\pi R^3} \int_{B_R} 3 \, dv = q,$$

where $B_R$ denotes the interior of the sphere.

Let $E = E \cdot dx$ be the 1-form associated with the field $E$ produced by a point source $q$ at the origin. Then

$$E = Gq \hat{r} \cdot dx = Gq \frac{x_j dx^j}{r^3}; \quad *E = Gq \frac{x_j dx^k \wedge dx^l}{r^3}.$$  \hspace{1cm} (22)

Equation (20) follows by comparing (21) and the second equation in (22).

The lines of force exit or enter the region bounded by $S$, and $E \cdot dS$ is respectively positive or negative, according as the force is repulsive or attractive. Since $\rho \geq 0$ in either case, $\epsilon$ is positive or negative according as the force is repulsive or attractive.  

For the electrostatic field $G$ is called the Coulomb constant, after Coulomb, who first measured it in 1785 using a torsion balance he had developed. In the case of gravitation, $G$ is called the Cavendish constant, after Henry Cavendish, who was the first to determine the constant accurately in 1798, also using a torsion balance. The Coulomb constant is positive while the Cavendish constant is negative.

Now we turn to Ampère’s Law. In the stationary case, conservation of the material requires that $\nabla \cdot J = 0$, equivalently, that $d \ast J = 0$.
Consequently $\ast J = J \cdot dS$ is a closed 2-form, and there is a 1-form $H$ such that

$$dH = \ast J.$$  \hspace{1cm} (23)

In the language of vector analysis, (23) takes the form $\nabla \times H = J$, and is known as Ampère’s Law. The result, however, is an immediate mathematical consequence of the conservation of charge ($d \ast J = 0$) and involves no physical assertion. For our purposes, it is more appropriate to link Ampère’s Law with the parameter $\mu$ and to frame it as a constitutive law, analogous to Gauss’ law of electrostatics: the circulation of $F$ around a closed path $\gamma$ is proportional to the current passing through any surface spanning $\gamma$:

$$C(\gamma) = \mu \int\int_{S_\gamma} \ast J \quad \text{Ampère’s Law.} \hspace{1cm} (24)$$

**Theorem 3.3** Putting $B = dA$ in the Hodge decomposition (15) we have

$$dB = 0, \quad B = \mu \ast H \quad \delta B = \mu J, \quad \delta dA = \mu J \hspace{1cm} (25)$$

**Proof:** By (16) and (23) we have

$$\int\int_{S_\gamma} \ast J = \int\int_{S_\gamma} dH = \oint_{\gamma} H, \quad C(\gamma) = \oint_{\gamma} \ast dA.$$

Since (24) holds for all closed loops $\gamma$, it follows that $\ast dA = \mu H$, and the relations (25) follow immediately. $$

For convenience we derive the equations for $\phi$ and $A$ in vector notation. For the irrotational component $E$ we have $E = -\nabla \phi$, $D = \varepsilon E$, $\nabla \cdot D = \rho$, hence

$$\Delta \phi = -\frac{\rho}{\varepsilon}. \hspace{1cm} (26)$$

This is Poisson’s equation for the electrostatic potential. For the vector potential we have $B = \nabla \times A$, $\nabla \times B = \mu J$, hence

$$\nabla \times \nabla \times A = \mu J. \hspace{1cm} (27)$$

Without loss of generality, we can assume that $\text{div} \cdot A = 0$; hence, by the vector identity $\nabla \times \nabla \times a = \nabla (\nabla \cdot a) - \Delta a$, equation (27) reduces to

$$\Delta A = -\mu J. \hspace{1cm} (28)$$
4 Units and Dimensions

The derivations of the static equations in the previous section were purely mathematical, but their application to physical cases requires that we attach physical dimensions to the quantities of interest. In particular, Maxwell’s equations of electrodynamics require for their validity that

\[ \frac{1}{\varepsilon \mu} = c^2, \]  

(29)

where \( c \) is the speed of light. The proof of (29) does not follow from the static equations alone; it requires the dynamical equations, as well as basic results from special relativity, including Einstein’s axiom of special relativity that the speed of light is a fundamental constant of nature. In this and the next section we shall establish (29) for general force fields, both attractive and repulsive.

The relationship (29) was already known to Maxwell for the electromagnetic field, and he cited it specifically in his paper. For the case of gravitation, only the units of mass, length, and time, denoted by \( m, \ell, \) and \( \tau \), enter the discussion; but when the field is generated by a material other than mass, an additional unit is needed. For electromagnetism, that unit is charge. In general, we shall assume that the force field is generated by a material density \( \rho \) called charge and measured in units denoted by \( q \). Thus, the basic physical units of measurement of a general force field are mass \( m \), length \( \ell \), time \( \tau \), and charge \( q \). In the special case of gravitation, the material is mass, and \( q = m \).

The primary variables of dynamics are velocity \( v \), acceleration \( a \), force \( f \), momentum \( p \), and energy \( \varepsilon \) with physical dimensions

\[ v = \frac{\ell}{\tau}, \quad a = \frac{\ell}{\tau^2}, \quad f = ma, \quad p = mv, \quad \varepsilon = mv^2. \]

The material density \( \rho \) has dimensions \( q/\ell^3 \).

The force field \( \mathbf{E} \), has dimensions of force per unit charge, hence

\[ [\mathbf{E}] = \frac{m \ell}{q \tau^2}, \quad [\ast E] = [\mathbf{E} \cdot d\mathbf{S}] = \frac{m \ell^3}{q \tau^2}. \]  

(30)

The displacement 2-form \( D = \mathbf{D} \cdot d\mathbf{S} \) must have the dimension of \( q \), since its integral over a 2 dimensional surface \( S \) produces the total charge contained within; hence it follows by Gauss’ Law (18) that

\[ [\varepsilon] = \frac{q^2 \tau^2}{m \ell^4}. \]  

(31)
The magnetic field $H$ is generated by the current $J$. We have

$$[J] = [\rho v] = \frac{q}{\ell^3} \frac{\ell}{\tau} = \frac{q}{\ell^2 \tau}, \quad [\star J] = [J \cdot dS] = \frac{q}{\tau}.$$ 

Now note that the exterior derivative $d$ is homogeneous of degree 0 with respect to dimension; that is $[d\omega] = [\omega]$ for any p-form $\Omega$. It therefore follows from (23) that $[H] = [dH] = [\star J] = q/\tau$; and, since $[H] = [H] \cdot \ell$, that

$$[H] = \frac{q}{\ell \tau}. \quad (32)$$

The dimensions obtained in (30), (31), and (32) coincide with those in Stratton, §1.8.

The units of $B = B \cdot dS$ in the case of the electromagnetic field are defined to be webers; but to relate this new unit to the standard units one turns to Faraday’s Law of electromagnetic induction to obtain

$$[B] = 1 \text{ weber} = 1 \frac{\text{kilogram} \cdot \text{meter}^2}{\text{coulomb} \cdot \text{second}}. \quad (33)$$

Since $[B] = [B]\ell^2$, two immediate corollaries of (33) are

$$[B] = \frac{m}{q\tau} \quad \text{and} \quad [B][v] = [E], \quad (34)$$

where $v$ is a velocity. Note that these two statements are equivalent.

In the theory of electromagnetism the force on a particle of charge $q$ moving with velocity $v$ is given by the Lorentz force $f = q(E + v \times B)$. The following theorem shows that the same result holds for any field, including the gravitational field.

**Theorem 4.1** The two statements in (34) are equivalent to the statement that $[\mu e] = \tau^2 \ell^{-2}$. The relation $[B]c = [E]$ is a consequence of Lorentz invariance,

**Proof:** If $[\mu e] = \tau^2 \ell^{-2}$, then by (31) we find that $[\mu] = m\ell/q^2$. By (23) and (32) we have

$$[B] = [\mu \star H] = [\mu][H \cdot dS] = \frac{m\ell^3}{q^2} \frac{q}{\ell \tau} = \frac{m\ell^2}{q \tau},$$

and (34) follows. The argument is reversible. The proof of the second statement requires notions from Special Relativity and will be proved in the next section.
5 Dynamic field theory

Hodge theory for manifolds embedded in $\mathbb{R}^n$ leads to elliptic systems of partial differential equations; whereas Maxwell’s dynamical equations for the electromagnetic field are hyperbolic. They are the field theory of special relativity, and are naturally formulated on Minkowski space-time $\mathbb{M}^4$. This space is obtained from $\mathbb{R}^4$ by setting $x_4 = ict$. This is the point of view taken in Stratton, and we shall follow it here.

Substantive differences arise when one attempts to extend Hodge’s formalism to $\mathbb{M}^4$. The equation $F = dA$ where $F$ is a 2-form, for example, is now a hyperbolic system. This fact arose in our mathematical proof of Faraday’s Law in [13]. The Hodge duality on $\Lambda_2(\mathbb{M}^4)$ is indefinite, so that the space is no longer a Hilbert space; and the proof of the Hodge decomposition given in §2 is not valid. The discussion of Maxwell’s equations and Yang Mills theory in Bott’s article is restricted to the Euclidean space $\mathbb{R}^4$, since the Hodge decomposition of 2-forms is explicitly limited to the elliptic case. The situation can be finessed to a certain degree, however, as we shall see; and the Hodge decomposition on $\mathbb{M}^4$ is not needed. Nevertheless, Maxwell’s dynamical equations arise naturally out of this extended formalism, and are obtained for both repulsive and attractive forces.

The Hodge star operation on $\mathbb{R}^4$ associated with the oriented volume element $dv \wedge dx^4$ is

$$\begin{align*}
* dx^j &= dx^k \wedge dx^l \wedge dx^4 & * dx^4 &= -dv \quad (35) \\
* dx^j \wedge dx^k &= dx^l \wedge dx^4 & * dx^j \wedge dx^4 &= dx^k \wedge dx^l \quad (36) \\
* dv &= dx^4, & * dx^j \wedge dx^k \wedge dx^4 &= -dx^l. \quad (37)
\end{align*}$$

Note that $** = -id$ in 4 dimensions.

The star operation on $\mathbb{M}^4$ is obtained by simply setting $x_4 = ict$ in these equations. The Hodge duality on $\mathbb{M}^4$ is given by

$$\langle A, B \rangle = \frac{1}{ic} \int_{\mathbb{M}^4} A \wedge * B. \quad (38)$$

Using this, the reader may verify that the Hodge duality is positive definite on $\Lambda_p(\mathbb{M}^4)$ for $p \neq 2$. 

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Lemma 5.1 Every exact form in $\Lambda_2(\mathbb{M}^4)$ can be written as

$$F = E \wedge dt + B,$$  \hspace{1cm} (39)

where $E = E_j dx^j$, and $B = B_i dx^j \wedge dx^k$, the sums running over $1 \leq i,j,k \leq 3$, is an exact 2-form.

Proof: The exact forms in $\Lambda_2$ are given by $F = dA$, where $A = A_j dx^j$, and

$$dA = \sum_{j<k\leq3} \left( \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) dx^j \wedge dx^k$$

$$+ \sum_{j=1}^3 \left( \frac{\partial A_4}{\partial x^j} - \frac{\partial A_j}{\partial x^4} \right) dx^j \wedge dx^4.$$  

Putting

$$B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \hspace{0.5cm} E_j = \frac{(\partial A_1)}{ic} - \frac{(\partial A_j)}{ic}.$$  \hspace{1cm} (40)

we obtain (39). $\blacksquare$

$F$ is called the Faraday 2-form. Note that if we put $A_4 = -\phi/ic$ we obtain

$$E_j = -\frac{\partial \phi}{\partial x^j} - \frac{\partial A_j}{\partial t}, \hspace{0.5cm} \text{or} \hspace{0.5cm} E = -\nabla \phi - \frac{\partial A}{\partial t},$$

which is the standard representation of the electric field in terms of the scalar and vector potentials in electromagnetic theory (see Stratton, §1.21).

We can now complete the proof of Theorem 4.1. Since the components of the Lorentz transformations are dimensionless variables, all coefficients $A_j$ of the 4-potential $A = A_j dx^j$ have the same dimension. It follows from (40) that $[E] = |B|ic$.

Lemma 5.2 The Hodge star operation interchanges exact $p$ forms with co-exact $4-p$ forms; and $\lambda$ is a real form if and only if $*\lambda$ is imaginary.

Proof: If $\lambda$ is exact, then $\lambda = d\alpha$, and $\gamma = *\lambda = *d\alpha = \delta \beta$, where $\beta = *\alpha$. Conversely, if $\gamma$ is a co-exact, then $\gamma = \delta \beta = *d\alpha$, where $\alpha = *\beta$. Hence $*\gamma$ is exact.

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We shall say that a basis form on $\mathbb{R}^4$ is time-like if it contains $dx^4$ and space-like if it does not. A $p$-form $\lambda$ is real if and only if $\overline{\lambda} = \lambda$ and imaginary if $\overline{\lambda} = -\lambda$. Therefore a differential form is real if and only if all its time-like coefficients are imaginary while all its space-like coefficients are real; and it is imaginary if the opposite holds. Since the Hodge star operation interchanges space-like and time-like basis forms, it interchanges real and imaginary differential forms.

In the static case we have $*B = \mu H$, and $*D = \epsilon E$ where $*$ is the Hodge operation on $\mathbb{R}^3$. The forms $D$ and $H$ and the parameters $\epsilon$ and $\mu$ were defined in terms of the Hodge decompositions on $\Lambda_1$ and $\Lambda_2$ over $\mathbb{R}^3$, together with the Laws of Gauss and Ampère. When $*$ is the Hodge operation on $\mathbb{R}^4$ we formally obtain

$$*B = *B_j dx^k \wedge dx^l = \mu H_j dx^j \wedge dx^4 = ic\mu H \wedge dt; \quad (41)$$

$$*E \wedge dt = *\left(\frac{E_j}{ic} dx^j \wedge dx^4\right)$$

$$= \frac{E_j}{ic} dx^k \wedge dx^l = \frac{1}{ic\epsilon} D_j dx^k \wedge dx^l = \frac{1}{ic\epsilon} D \quad (42)$$

These two relations formally define $H$ and $D$ in the dynamic case and provide the dynamic version of the laws of Ampère and Gauss.

In Special Relativity, the material density $\rho$ and current $J = \rho v$ are combined into a single source term, the 1-form $J = J_j dx^j + ic\rho dx^4$. The reader may verify that $(J, J) > 0$. Conservation of material (charge or mass) is expressed by the condition $\delta J = 0$. The 4-potential is given by the 1-form $A = A_j dx^j$. By the Hodge decomposition of $\Lambda_1(\mathbb{R}^4)$, we may write $A = d\phi + \delta \Phi$, but since the field is obtained from $dA$, the term $d\phi$ has no effect; hence we may assume that $\delta A = 0$.

**Theorem 5.3** Every co-exact form $G \in \Lambda^2(\mathbb{R}^4)$ can be written as $G = ic(H \wedge dt - D)$, where $H$ and $D$ are defined by equations (41) and (42). $G$ satisfies the system

$$\delta G = 0, \quad dG = *J, \quad (43)$$

where $J$ is the current. If $F$ is a Faraday 2-form, then $*F = \mu G$; and $F$ satisfies the system of equations

$$dF = 0, \quad \delta F = \mu J. \quad (44)$$
Putting $F = dA$, we have $\delta dA = -\Box A = \mu J$, where
\[
\Box A = \sum_{j=1}^{4} \Box A_j dx^j, \quad \Box = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \tag{45}
\]

**Proof:** We begin by noting that $** = (-1)^p id$ and $\delta = (-1)^{p-1}$ on $\mathbb{R}^4$; hence $\delta = -*d*$ on $\mathbb{R}^4$.

Since $\delta J = d*J = 0$, $*J$ is a closed 3-form, and the second equation in (43) is solvable. Observe that $G$, with $H = H_j dx^j$ and $D = D_j dx^k \wedge dx^l$, represents a general 2-form on $\mathbb{R}^4$. By direct calculation
\[
dG = \left( \frac{\partial H_l}{\partial x^k} - \frac{\partial H_k}{\partial x^l} - ic \frac{\partial D_j}{\partial x^l} \right) dx^k \wedge dx^l \wedge dx^4 - ic \sum_{j=1}^{3} \frac{\partial D_j}{\partial x^j} dv \tag{46}
\]

Since $*J = (J_j dx^k \wedge dx^l \wedge dx^4 - ic \rho dv)$, (43) and (46) give the Maxwell-Ampère equations (11):
\[
\left( \frac{\partial H_l}{\partial x^k} - \frac{\partial H_k}{\partial x^l} - ic \frac{\partial D_j}{\partial x^l} \right) = J_j, \quad \sum_{j=1}^{3} \frac{\partial D_j}{\partial x^j} = \rho.
\]

By (44) and (42), Lemma 5.1, and (29) we obtain
\[
*F = * \left( \frac{E}{ic} \wedge dx^4 + B \right) = \frac{1}{ic} D + \mu H \wedge dx^4
\]
\[
= \mu \left( H \wedge dx^4 + \frac{1}{ic\mu} D \right) = ic\mu (H \wedge dt - D) \tag{47}
\]

We call the 2-forms $G = ic(H \wedge dt - D)$ the Maxwell-Ampère forms.

Since $J$ is a 1-form and $F, G$ are 2-forms,
\[
\delta F = -*d*F = -\mu *dG = -\mu * *J = \mu J,
\]
which is the second equation in (44). It follows that $\delta dA = \delta F = \mu J$.

Finally, let us show prove (45) when $A$ satisfies the Lorentz gauge. Denoting $\partial_k A_j$ by $A_{j,k}$, etc. we have
\[
dA = \sum_{i,j=1}^{4} A_{i,j} dx^j \wedge dx^i = \sum_{i,j=1}^{4} A_{j,i} dx^i \wedge dx^j,
\]
\[
* dA = \sum_{i,j=1}^{4} A_{j,i} dx^k \wedge dx^l, \quad k, l \neq i, j
\]
where \( dx^i \wedge dx^j \wedge dx^k \wedge dx^l \) is equal to the standard volume element \( dv \wedge dx^4 \). Then

\[
\star d \star dA = \sum_{i,j,m=1}^{4} A_{j,im} dx^m \wedge dx^k \wedge dx^l.
\]

Since \( dx^m \wedge dx^k \wedge dx^l = 0 \) unless \( m = i \) or \( j \), this sum reduces to

\[
\sum_{i,j=1}^{4} A_{j,ii} dx^i \wedge dx^k \wedge dx^l + \sum_{i,j=1}^{4} A_{j,ij} dx^j \wedge dx^k \wedge dx^l,
\]

and

\[
\star d \star dA = -\sum_{i,j=1}^{4} A_{j,ii} dx^j + \sum_{i,j=1}^{4} A_{j,ij} dx^i.
\]

Now the first sum is \(-\sum_j \Delta A_j dx^j\), while the second sum is \(\sum_i \partial_i(\sum_j \partial_j A_j)dx^i = 0\) by the Lorentz condition. The Laplacian is \(\Delta = \sum_j \partial_j^2\) on \(\mathbb{E}^4\); but on \(\mathbb{M}^4\) it becomes the wave operator with the substitution \(x_4 = ict\).

This completes the proof of Theorem 5.3.

6 Gravity of a rotating sphere

The static fields are the solutions of the pair of elliptic equations on \(\mathbb{E}^3\)

\[
\nabla \times \mathbf{E} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}.
\]

The first equation implies that the vector field \(\mathbf{E}\) is irrotational, hence that there is a scalar potential \(\phi\) such that \(\mathbf{E} = -\nabla \phi\). A necessary and sufficient condition for the existence of a solution to the second equation is that \(\text{div} \mathbf{J} = 0\); and the solution is unique if we specify that \(\text{div} \mathbf{H} = 0\).

The equation \(\text{div} \mathbf{B} = 0\) in Faraday’s law [10], is the necessary and sufficient condition that there be a vector field \(\mathbf{A}\) for which \(\mathbf{B} = \nabla \times \mathbf{A}\). Setting \(\mathbf{B} = \mu \mathbf{H}\) and substituting this into the equation above for \(\mathbf{H}\) we obtain the equation

\[
\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}.
\]

The vector potential is uniquely determined by the additional condition that \(\text{div} \mathbf{A} = 0\). In that case the above equation reduces to
\[ \Delta A = -\mu J, \text{ and the vector potential is given by the convolution} \]
\[ A(x) = \mu \iiint \chi(x, y) J(y) dv_y, \quad \chi(x, y) = \frac{1}{4\pi |x - y|}. \quad (48) \]

An interesting special case is that in which \( J \) is generated by a solid sphere such as the Earth rotating with constant angular velocity \( \omega \) about the \( z \) axis. The computation of the exact field provides an interesting application of the theory of spherical harmonics.

Let \( B \) be the sphere of radius \( r_0 \) centered at the origin, and choose spherical coordinates \( r, \theta, \varphi \). In (48) \( |y| = r < r_0 < R = |x| \). Let
\[ \hat{x}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \]
The unit vectors
\[ \hat{e}_r = \hat{x}(\theta, \varphi), \quad \hat{e}_\theta = \frac{\partial \hat{e}_r}{\partial \theta}, \quad \hat{e}_\varphi = \hat{e}_r \times \hat{e}_\theta = \frac{1}{\sin \theta} \frac{\partial \hat{e}_r}{\partial \varphi} \]
form an orthonormal basis of \( \mathbb{R}^3 \) based at \( \hat{x} \). We shall also write \( \hat{r} = \hat{x}, \hat{\theta} = \hat{\theta}, \hat{\varphi} = \hat{\varphi} \), the unit vectors in the direction of increasing \( r, \theta, \varphi \) respectively. We use primed coordinates for a vector \( y \) inside the sphere, and unprimed coordinates for \( x \), which lies outside the sphere. Let \( \gamma \) denote the angle between \( \hat{x}(\theta, \varphi) \) and \( \hat{x}(\theta', \varphi') \); thus
\[ \cos \gamma = \hat{x}(\theta, \varphi) \cdot \hat{x}(\theta', \varphi') = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'). \quad (49) \]

For \( n \geq 0 \) the Spherical Harmonics are
\[ Y_n^m(\theta, \varphi) = \sqrt{\frac{2n + 1}{4\pi} \frac{(n - m)!}{(n + m)!}} P_n^m(\cos \theta) e^{im\varphi}, \quad -n \leq m \leq n, \quad (50) \]
where \( P_n^m(\cos \theta) \) are Ferrer’s associated Legendre polynomials, Whittaker and Watson [17], §15.5. The spherical harmonics for negative \( n \) are obtained by setting \( Y_{-n}^{-m} = Y_n^m \). Observe that with this definition, \( Y_{-1} = Y_0 \).

The spherical harmonics (50) form a complete orthonormal basis for \( L^2(S^2) \) where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \) with the rotationally invariant measure \( \sin \theta \, d\theta \, d\varphi \). They satisfy
\[ \iiint_{S^2} Y_n^m(\theta, \varphi) Y_{n'}^{-m'}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi = \delta_{n,n'} \delta_{m,m'}. \quad (51) \]

The following theorem is the basis for the multipole expansion in spherical harmonics. It is well known in the physics literature and can be found, for example, in Landau and Lifshitz, §41.
Theorem 6.1 Put \( y = r \hat{x}(\theta', \varphi') \) and \( x = R \hat{x}(\theta, \varphi) \). Then for \( r < R \) the fundamental solution \( \chi \) can be expanded

\[
\frac{1}{4\pi|x-y|} = \frac{1}{4\pi\rho} \sum_{n=0}^{\infty} \left( \frac{r}{\rho} \right)^n \sum_{m=-n}^{n} Y_n^m(\theta, \varphi)Y_n^m(\theta', \varphi').
\]

(52)

where \( P_n \) and \( P_n^m \) are respectively the Legendre and associated Legendre polynomials.

Proof: The proof rests on the Legendre addition theorem for the associated Legendre polynomials:

\[
P_n(\cos \gamma) = \sum_{m=-n}^{n} a_{nm}(\theta', \varphi') Y_n^m(\theta, \varphi). \tag{53}
\]

Since the spherical harmonics span the eigenspace of the spherical Laplacian with eigenvalue \( n(n+1) \),

\[
P_n(\cos \gamma) = \sum_{m=-n}^{n} a_{nm}(\theta', \varphi') Y_n^m(\theta, \varphi).
\]

The left side is real and symmetric in \((\theta, \varphi)\) and \((\theta', \varphi')\), and this implies that \( a_{nm}(\theta', \varphi') = \overline{Y_n^m(\theta', \varphi')} \).

Hence \( \chi \) depends only on \( r, R \) and \( \cos \gamma \). The Legendre polynomials \( P_n(z) \) are given by the generating function \((1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(z)\). Thus

\[
\chi(x, y) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos \gamma), \tag{54}
\]

where \( P_n(\cos \gamma) \) are the Legendre polynomials. Inserting (53) into (54) we obtain (52). ■

The velocity field due to a uniform rotation of angular velocity \( \omega \) is \( \mathbf{v}(y) = \omega r \sin \theta' \hat{\varphi}' \). Assuming a uniform material density \( \rho \), the current in spherical coordinates is then \( \mathbf{J} = \kappa \hat{\varphi}' \), \( \kappa = \rho \omega r \sin \theta' \). Expanding \( \hat{\varphi}' \) in the basis at \( \hat{x}, \hat{\varphi}' = \sum_j a_j \hat{e}_j \), where \( a_j = \hat{\varphi}' \cdot e_j(\theta, \varphi) \), we obtain

\[
\hat{\varphi}' = \sin \theta \sin(\varphi - \varphi') \hat{e}_r + \cos \theta \sin(\varphi - \varphi') \hat{e}_\theta + \cos(\varphi - \varphi') \hat{e}_\varphi.
\]

The \( r \) and \( \varphi' \) integrations in (48) can be carried out explicitly. By the orthogonality properties of the sine and cosine over \([0, 2\pi]\), we get

non-zero terms only for \( m = 1 \), and \( \mathbf{A} \) is given by

\[
\frac{\mu \omega \rho}{4\pi R} \sum_{n=0}^{\infty} \frac{r_0^{n+4}}{(n+4)R^n} \left[ \frac{2^{n}(n-1)!}{(n+1)!} P_n^1(\cos \theta) \int_0^\pi \sin^2 \theta' P_n^1(\cos \theta') d\theta' \right] \hat{e}_\varphi.
\]

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The theta integrals are easily computed. Setting $z = \cos \theta'$ and using the fact that $P_n^1(z) = \sqrt{1 - z^2}P_n'(z)$ by Ferrer’s formula, we obtain

$$
\int_0^{\pi/2} \sin^2 \theta' P_n^1(\cos \theta') d\theta' = \int_{-1}^1 (1 - z^2)P_n'(z) dz
$$

$$
= (1 - z^2)P_n(z)^1_{-1} + 2 \int_{-1}^1 zP_n(z) dz = \begin{cases} 0 & n \neq 1; \\ \frac{4}{3} & n = 1. \end{cases} \quad (55)
$$

The infinite series thus reduces to the single term for $n = 1$. Noting that $P_1^1(\cos \theta') = \sin \theta'$ we obtain

$$
A(x) = \frac{Ar_0^2}{R^2} \sin \theta \hat{e}_\varphi, \quad A = \frac{\mu \omega r_0^3}{15}
$$

The curl of $A$ in spherical coordinates is

$$
B = \nabla \times A(x) = \frac{Ar_0^2}{R^3}(2 \cos \theta \hat{e}_r + \sin \theta \hat{e}_\theta) \quad (56)
$$

The scalar potential for the $E$ field is given by

$$
\phi(x) = \frac{\rho}{\epsilon} \int \int \int \chi(x, y) dv_y,
$$

where $\epsilon$ is the gravitational induction constant and $\rho$ the mass density. We assume that $\rho$ is constant, though the calculation is easily extended to any spherically symmetric mass distribution. The integrations over $r$ and $\varphi'$ are easily carried out. Since the problem is axially symmetric when $\rho$ is constant, we may use the simplified expansion (54); and it is clear that only the $n = 0$ term survives integration:

$$
\phi(x) = \frac{G\rho}{R} \sum_{n=0}^{\infty} \frac{r_0^{n+3}}{(n + 3)R^n} P_n(\cos \theta) \int_0^{\pi} \int_0^{2\pi} P_n(\cos \theta') \sin \theta' d\theta' d\varphi'
$$

$$
= \frac{G\rho}{R} \frac{4\pi r_0^3}{3r_0^3} = \frac{GM}{R}, \quad (57)
$$

where $M$ is the total mass, and $G = (4\pi\epsilon)^{-1}$. Note that $P_0(\cos \theta) = 1$, and the integrals of $P_n$ for $n \geq 1$ vanish because of the orthogonality of the Legendre polynomials. This formula holds for both repulsive...
and attractive fields provided one takes the constant $G$ to be positive for repulsive fields and negative for attractive fields such as gravity. The field is then given by

$$E = -\nabla \phi = \frac{GM}{R^2} \hat{e}_r. \quad (58)$$

Equations (56) and (58) constitute the Maxwell gravitational field generated by a rotating spherical solid in closed form. The rotational component of the gravitational field is thus a consequence of classical, linear, analysis, not General Relativity.

7 Relativistic corrections

One of the first successes of General Relativity was Einstein’s calculation of the precession of the perihelion of Mercury by solving his equations for a point mass in a vacuum. Kip Thorne ([15], p. 95) points out that the total advance of the perihelion is 1.38 seconds of arc per revolution, 1.28 of which are accounted for by Newtonian theory. Thus the solution based on classical, Newtonian mechanics is highly accurate, and Einstein’s calculation should be regarded as the “relativistic correction”.

In general, however, we should not expect to be able explicitly to calculate the metric tensor associated with a solution of Maxwell’s equations generated by a mass and current density. The theory developed by Hans Thirring [14] and Josef Lense [7] attacks the problem of computing the relativistic corrections by perturbation theory, building on basic ideas of Einstein. Einstein’s General Theory of Relativity involves two principle tensors, the metric tensor and the energy-momentum tensor, coupled by the Einstein Field Equations

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{8\pi G}{c^4} T_{\mu \nu} \quad (59)$$

where $R_{\mu \nu}$ is the Ricci curvature tensor, $R$ the scalar Ricci curvature, $g_{\mu \nu}$ the metric tensor, $G$ the gravitational constant, $c$ the speed of light, and $T_{\mu \nu}$ the energy-momentum tensor. Newton’s laws of motion in a gravitational field are replaced by the geodesic flow of the metric tensor $g$.

Equations (59) are highly nonlinear; but Thirring, and then Lense and Thirring together, developed a perturbation scheme for solving them in the vicinity of a rotating body. In their method, commonly
referred to as “frame-dragging”, the coefficients in the metric tensor are expanded in a series and computed term by term, along with successively higher terms in the energy-momentum tensor. Thus the rotation of the body is incorporated into the metric tensor, resulting in a distortion in the fabric of space-time itself.

The approximation scheme developed by Lense and Thirring has evolved into the Post-Newtonian approximation, and has been developed extensively; see, for example, Weinberg, Chapter 9, and Misner, Thorne, and Wheeler, Chapter 39. Weinberg treats not only the application of the Lense-Thirring method to the precession of gyroscopes orbiting the rotating Earth, but also the precession of the perihelia of elliptical orbits of the planets. Maxwell’s field theory, inasmuch as it lives in flat space-time, constitutes the $0^{th}$ order approximation in the Post-Newtonian scheme. The computation of higher order corrections, as may be expected, is extremely complicated; so that in general only the first order correction may be tractable.

8 Galactic Dynamics

Newton’s theory of gravity models the dynamics of the Solar System with great accuracy, but fails to model the dynamics of spiral galaxies. The most striking anomaly is the “rotation curve,” discovered and documented extensively by Rubin, Ford, and Thonnard [12], in which the velocity of charged particles increases going out from the core of the galaxy. In a central force field, the speeds should drop off as $r^{-1/2}$, where $r$ is the mean distance from the central mass, according to Kepler’s third law of planetary motion, which is fundamental to Newton’s inverse square law of gravity.

The rotation curve, as well as other anomalies in galactic dynamics, have led to the hypothesis of “dark matter” – the existence of elementary particles which do not interact with the electromagnetic field – to account for the observations. The search for dark matter involves theory and experiments in high-energy physics; yet the energy density outside the galactic core is extremely small. Particles created in a high-energy accelerator are unstable and have extremely short lifetimes. To resolve the dynamical issues in galaxies, and in particular to explain the rotation curve, one would have to know the density and distribution of such particles. Thus, the discovery of dark matter in the laboratory cannot by itself resolve the dynamical issues
involved in galaxies.

The most obvious reason for the failure of Newtonian mechanics on a galactic scale is that matter in a galaxy is distributed over tens of thousands of light years, whereas 99% of the mass of the Solar System is in the Sun itself; but a second defect is that disturbances travel infinitely fast. In that case, Maxwell’s equations of gravitation should play a pivotal role in understanding the dynamics of galaxies.

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