An approach to F-theory

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Abstract

We consider BPS configurations in theories with two timelike directions from the perspective of the supersymmetry algebra. We show that whereas a BPS state in a theory with one timelike variable must have positive energy, in a theory with two times any BPS state must have positive angular momentum in the timelike plane, in that $Z_{00} > 0$, where 0 and $\tilde{0}$ are the two timelike directions. We consider some generic BPS solutions of theories with two timelike directions, and then specialise to the study of the (10,2) dimensional superalgebra for which the spinor operators generate 2-forms and 6-forms. We argue that the BPS configurations of this algebra relate to F-theory in the same way that the BPS configurations of the eleven dimensional supersymmetry algebra relate to M-theory. We show that the twelve dimensional theory is one of fundamental 3-branes and 7-branes, along with their dual partners. We then formulate the new intersection rules for these objects. Upon reduction of this system we find the algebraic description of the IIB-branes and the M-branes. Given these correspondences we may begin an algebraic study of F-theory.

1 Introduction

A long standing problem in string theory has been the explanation of the self S-duality of the type IIB superstring. F-theory [1] provides a higher dimensional mechanism which suggests a solution to this problem. F-theory is defined such that given a bundle manifold $\mathcal{M}$ which is a $T^2$ fibre over a base space $\mathcal{B}$

$$F \text{ on } \mathcal{M} \equiv \text{IIB on } \mathcal{B}. \quad (1.1)$$
By allowing the internal moduli of the torus to vary over the base, the $SL(2, \mathbb{Z})$ symmetry of the IIB theory is explained. By necessity, F-theory is a twelve dimensional structure and involves in some way two timelike directions. This has created an interest in theories with dimensions and signatures beyond those common to supergravity [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In this paper we adopt the attitude that some local theory exists in (10,2) dimensions, in the same spirit as M-theory. We think of M-theory as being some quantum structure which reduces to the superstring theories and the eleven dimensional supergravity theory in different limits. Many of the properties of M-theory in the extremal limits may be inferred from an investigation of the different types of solution to the low energy supergravity theories. In a supergravity theory, there exist local supersymmetry transformations relating the bosonic and fermionic fields to each other. In such a theory, however, the behaviour of the long range fields is governed by the superalgebra structure. A study of the algebra thus provides us with many constraints on the global properties of the local theory [14]. Given these restrictions we may infer properties about the underlying M-theory. As an example, by studying the superalgebra it was deduced that one could define the Green-Schwarz superstring solution in ten dimensional flat space [15]. It was realised that this ought to imply the existence of a string-like solution to the corresponding supergravity theory, which was later written down explicitly [16]. In a similar way, a membrane-like solution to the eleven dimensional supergravity equations was constructed [17], corresponding to the Green-Schwarz membrane which exists in a flat superspace in eleven dimensional spacetime. This basic process has now become much more elaborate, and the supergravity interpretation of a very general class of solutions to the rigid supersymmetry theories are now well understood [18, 19]. Although simple, this approach is very effective and gives us a handle on the possible types of object, such as 2-branes and 5-branes, which appear in the different limits of M-theory. We now wish to ask the question: Which objects appear in F-theory? Very little is certain because there is no known low energy theory to which we can turn for guidance. We do, however, have at our disposal candidate superalgebras. We therefore make a start by studying the brane structure of superalgebras in spacetimes of signature (10,2), which we shall loosely call F-theory. By analogy with results from supergravity it is to be expected that the objects in a theory of rigid supersymmetry branes will be in 1-1 correspondence with those in the local version of F-theory.
1.1 Two-timing theories

Although it is currently unclear as to whether or not the additional timelike dimension in F-theory is to be treated as physical, in the same sense as the extra dimensions in string theory correspond to extra dimensions in spacetime, or merely as some auxiliary variable, there is potentially much to be learned from the study of theories with two times. Although there are many suggestions as to the nature of such theories, in our opinion the major question which has yet to be addressed is thus: Which theory should we be studying? There are obviously many ways in which some theory with two timelike directions may be constructed, but how may we be sure that it corresponds to the one relevant to physics? One of the more sensible places to begin with is the known supergravities. However, there are many ways in which Minkowski signature supergravity theories may be related to theories with different spacetime signature. The most obvious method is to consider simply the low energy bosonic sectors of the known theories and their proposed generalisations, imposing the requirement that they match up when linked via a dimensional reduction or duality transformation. This approach has been discussed in various contexts [5, 9, 20]. As regards the full underlying supersymmetric theories, however, these procedures are potentially misleading: A reason for this is that the Einstein term in the low energy effective action arises because the anticommutator of two supersymmetry generators produces a translation. In the local theory this creates the diffeomorphisms which allow us to define an Einstein theory of gravity. Unfortunately there is no guarantee in a supersymmetry theory with two times that the translation generators appear in the algebra in a consistent way. The reason for this apparent discrepancy is that the symmetry properties of products of gamma-matrices are related to the signature of the spacetime, whereas the number of spin degrees of freedom grows rapidly with increasing dimension [3]. With few exceptions [21, 10, 22], the entire supergravity structure which we are used to dealing with is based on the understanding that there is one timelike direction, and a certain amount of care must consequently be taken when discussing possible ‘low energy supergravities’ in a space with two times. Indeed, even the meaning of the term ‘low energy’ must be carefully discussed, as an absence of translations leads to the lack of a simple notion of energy itself!

Notwithstanding the aforementioned complications, the algebraic study of BPS configurations allows us to deduce with confidence some properties of the F-theory spectrum. Once the concrete two-timing supersymmetric BPS solutions are known, it is possible to discuss the relationships to Minkowski signature theories.
2 BPS configurations in general supersymmetric theories

One of the most useful concepts of supergravity, or indeed any low energy limit of a theory which is only understood perturbatively, is that of the BPS state: From the BPS configuration, which is a special solution to the effective theory at weak coupling, we can deduce information about the corresponding quantum theory at strong coupling. The reason for this is that the BPS states lie in short representations of the supersymmetric theory, and as such should remain so even as the coupling is switched on. These states are central to the study of Minkowski signature theories, and if there is a physical link between M-theory and F-theory then it should be provided by the BPS states in each. We therefore discuss BPS conditions in a general $T = 2$ supersymmetric theory and then specialise to F-theory.

2.1 Supersymmetry algebras

The supersymmetry algebra is the graded generalisation of the isometry algebra of the tangent space of a spacetime manifold. A Lorentz supersymmetric theory is one such that the isometries are given by the Lorentz algebra, which is generated by the Lorentz rotations $M_{\mu\nu}$, supplemented by an anticommutator $\{Q^\alpha, Q^\beta\}$ and a commutator $[Q^\alpha, M_{\mu\nu}]$, where $Q$ is a spinorial operator. In addition, other central terms $Z_{\mu_1...\mu_n}$ may appear in the algebra. This leads to the rich brane structure of M-theory.

Generically, a quantum vacuum of a supersymmetric theory is defined to be a state $|0\rangle$ in an Hilbert space which is annihilated by all the fields

$$Q^\alpha|0\rangle = M_{\mu\nu}|0\rangle = Z_{\mu_1...\mu_n}|0\rangle = 0.$$ (2.2)

A general quantum state $|\psi\rangle$ is not projected to zero by the fields, and is labelled by the eigenvalues of the various operators. In addition to $|0\rangle$ and $|\psi\rangle$, due to the anticommuting nature of the fermionic spinor operators, we may define the intermediate class of BPS states. These are states which are annihilated by some linear combination of the spinor generators, which is equivalent to the condition that

$$\det \left( \{Q^\alpha, Q^\beta\} \right) = 0.$$ (2.3)

These solutions preserve some fraction of the supersymmetry equal to the ratio of the number of zero eigenvalues of the anticommutator to the dimensionality of the spin space, and as a result lie in short representations of the general supersymmetry algebra. This result holds quite independently of the particular details of the supersymmetry algebra, and allows us to simply generalise to the situations with two times. We shall say that a
BPS state is given by any background which is a consistent solution to (2.3). By consistent we mean that the generators of the solution must satisfy the appropriate underlying super Lie algebra. The important terms to consider are the anticommutators of the spinor generators with themselves, which most generally take the form

\[ \{ Q^\alpha, Q^\beta \} = \sum_{n=1}^{\lfloor D/2 \rfloor} (C\Gamma^{\mu_1\ldots\mu_n})^{\alpha\beta} Z_{\mu_1\ldots\mu_n}. \] (2.4)

The sum may be taken to terminate at the integer part of half the dimension by employing the Poincaré duals of the higher spin Z fields, if we ignore topological complications. We take the spinor Q to be Majorana, in which case the left hand side of (2.4) is clearly a real symmetric matrix, which is positive definite since we choose to work in a Hilbert space. As such, the matrix is diagonalisable and has zero or positive eigenvalues. The components \( Z_{\mu_1\ldots\mu_n} \) must be chosen so as to make the right hand side of the equation symmetric, with non-negative eigenvalues; in particular this means that the trace of the right hand side must be positive.

2.2 \( T = 1, D > 2 \)

For the signature \((D - 1, 1)\) Clifford algebra, the charge conjugation matrix used to raise and lower spinor indices may be taken to be \( C = \Gamma^0 \), the timelike gamma-matrix. In this case we find that the \( Tr(CT^{\mu_1\ldots\mu_n}) \) are zero except for \( Tr(C\Gamma^0) = Tr(1) = D \). To see this note that \( CT^{\mu_1\ldots\mu_n} \) is an anti-symmetrised product of less than \( D \) gamma matrices. The trace of an even number of anticommuting matrices is always zero, due to the cyclic property of \( Tr \). If we have an odd number of anticommuting matrices, then we insert \( 1 = (\Gamma^{(s)})^2 \) into the trace, where \( \Gamma^{(s)} \) is a spacelike gamma matrix not in the product; we may always do this because the integer \( n \) of the sum in (2.4) only runs up to \( D/2 \). This means that the trace of the right hand side of (2.4) is zero unless \( Z_0 \equiv P_0 \neq 0 \). Since we must sum over symmetric \( (CT^{\mu_1\ldots\mu_n})^{\alpha\beta} \), this implies the existence of negative eigenvalues, which is a contradiction. We thus arrive at the positive energy condition

- A BPS state in a \( T = 1, D > 2 \) theory must have \( P_0 > 0 \).

This is a pleasing result, since it tells us from algebraic considerations that any brane configuration of a general low energy supersymmetric theory must have non-zero energy. In addition, the role of the momentum is uniquely singled out from the set of possible central charges \( Z^{\mu_1\ldots\mu_d} \).

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1In a Pontryagin space, which is an indefinite metric Hilbert space, this restriction is waived and we may define more general supersymmetry theories. This, however, is at the expense of introducing additional complications in the quantisation procedure.
2.3 \( T = 2, D \geq 4 \)

For a theory with two times, the crucial point to notice is that the charge conjugation matrix is given by the product of the two timelike gamma matrices, \( C = \Gamma^0 \Gamma^\tilde{0} \) \[^{[26]}\]. In this case, the only term on the right hand side of \((2.4)\) with non-zero trace is \( C \Gamma^0 \Gamma^\tilde{0} M^\mu_{0\tilde{0}} \).

By a similar argument to that for the \( T = 1 \) case, we find that

- A BPS state in a \( T = 2, D \geq 4 \) theory must have \( Z^0_{\tilde{0}0} > 0 \).

In theories with two times, therefore, a special role is given to the \( Z_{\mu\nu} \) terms, and the other \( p \)-form charges have no particular constraints (except that the positivity condition must be satisfied). This is an important point: In two time theories we are quite generally allowed configurations which have zero \( P_0 \); the only requirement is that the angular momentum in the timelike plane, \( Z^0_{\tilde{0}0} \), is positive. Hints of this type of behaviour have occurred in the literature, in the context of the Two-Timing Monopole \[^{[21]}\]: This is the analogue of the Kaluza-Klein monopole \[^{[27]}\] in which the internal direction is a timelike circle, and therefore exists in a spacetime of signature \((3,2)\). Interestingly, it seems that there are no constraints on the energy of the solution. In general, for any supersymmetric theory with two times, there is no intrinsic concept of energy or length in the same way as for \( T = 1 \) supersymmetry. One corollary of this is that we do not need concern ourselves with the problems that arise if one tries to define a naive \( T = 2 \) quantum supersymmetric system. The resolution of the quantisation procedure remains mysterious, although it is likely to be formally possible with path integral formulation. This form of quantisation has already been carried out for the \((2,2)\)-string \[^{[28]}\] \[^{2}\].

3 \( T = 1 \) SUSY and \( T = 2 \) SUSY

We now consider the basic supersymmetry theories containing BPS solutions with one or two timelike directions. If \( T = 1 \) then due to the positive energy condition the simplest possible non-trivial anticommutator is

\[
\{ Q^\alpha, Q^\beta \}_{T=1} = (C \Gamma^\mu)^{\alpha\beta} P_\mu.
\]

(3.5)

The simplest superalgebra with a non-trivial bosonic sector which includes the Lorentz rotations consistent with this anticommutator is in fact the standard super Poincaré algebra. This has the non-zero terms additional to \((3.5)\)

\[
[M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\sigma} \eta_{\mu\rho} + M_{\mu\rho} \eta_{\nu\sigma} - M_{\nu\rho} \eta_{\sigma\mu} - M_{\sigma\mu} \eta_{\nu\rho}
\]

\[^{2}\text{It is worth noting that a nice property of path integrals in a } T = 2 \text{ spacetime is that the action is real, as opposed to the imaginary quantity which occurs in a Minkowski signature theory.}\]
\[ [M_{\mu\nu}, P_{\rho}] = P_{\mu}\eta_{\nu\rho} - P_{\nu}\eta_{\mu\rho} \]
\[ [Q^\alpha, M_{\mu\nu}] = \frac{1}{2}(\Gamma_{\mu\nu})^\alpha_\beta Q^\beta . \] (3.6)

In analogy with choosing \( Z^\mu \to P^\mu \), it seems natural to make the identification \( Z_{\mu\nu} \to M_{\mu\nu} \). This was the approach discussed in [2]. If we adopt this identification then the two-time commutator analogous to (3.5) is

\[ \{Q^\alpha, Q^\beta\}_{T=2} = (C\Gamma^{\mu\nu})^{\alpha\beta} M_{\mu\nu} . \] (3.7)

Since only the rotation generators appear in this expression, it has a simpler extension than the Poincaré algebra: The full algebra, called sio\(_2\) is given by (3.7) with the additional terms

\[ [M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\sigma}\eta_{\mu\rho} + M_{\mu\rho}\eta_{\nu\sigma} - M_{\nu\rho}\eta_{\mu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} \]
\[ [Q^\alpha, M_{\mu\nu}] = \frac{1}{2}(\Gamma_{\mu\nu})^\alpha_\beta Q^\beta . \] (3.8)

It is a simple matter to check that a representation of the supersymmetric sio\(_2\) system is given by

\[ M_{\mu\nu} = \tilde{M}_{\mu\nu} + \frac{1}{4}\theta^\alpha (\Gamma_{\mu\nu})^{\alpha\beta} \partial^\beta \]
\[ Q^\alpha = \partial^\alpha - \frac{1}{2}(\Gamma_{\mu\nu})^{\alpha}_\beta M_{\mu\nu} , \] (3.9)

where \( \tilde{M} \) is the orbital part of the angular momentum operator, defined by

\[ \tilde{M}_{\mu\nu} = X_{[\mu|\partial_{\nu]}} . \] (3.10)

Note that for consistency of this algebra we require that

\[ (\Gamma^{\hat{\mu}\hat{\nu}})^{(\alpha\beta}(\Gamma^{\hat{\mu}\hat{\nu}})^{\gamma\delta)} = 0 , \] (3.11)

where the indices \( (\hat{\mu}\hat{\nu}) \) occur whenever \( M_{\hat{\mu}\hat{\nu}} \) is non-zero. This is easily seen to be the case by evaluation of the \( \{QQQ\} \) super Jacobi identity. Of course, if we choose \( \{Q, Q\} \) to be centrally extended with a \( Z^{\mu\nu} \) term, then the algebra is always consistent. For the purpose of the results obtained in this paper, this distinction is of no consequence.

### 3.1 BPS solutions

We now search for BPS solutions of the basic \( T = 1 \) and \( T = 2 \) systems with anticommutators (3.5) and (3.7). For \( T = 1 \) we note that in any dimension the determinant of the anticommutator (3.7) is zero if and only if \( P^\mu P_{\mu} = 0 \). The positive energy condition requires that \( P_0 > 0 \), hence the solution is the massless supersymmetric Brinkmann
wave [29]. Although there is a vast literature on the BPS solutions to (3.5) with central extensions in dimensions ten and eleven, many may be derived from this plane wave solution by using $S$, $T$-dualities and dimensional reductions [19]. We may attempt to find analogies when $T = 2$. It is, unfortunately, more difficult to analyse the solution to

$$\det\{Q^\alpha, Q^\beta\}_{T=2} = 0.$$ 

We proceed by considering canonical forms for the components of the rotation generator $M_{\mu\nu}$, which may be obtained from more general cases by the action of Lorentz transformations. To begin a partial classification we first note that $S_{\mu\nu} \equiv M_{\mu\rho}M_{\nu\sigma}\eta^{\rho\sigma}$ is a real symmetric matrix, whose eigenvalues must consist of repeated pairs, since the eigenvalues of an antisymmetric matrix occur in pairs of opposite sign. The eigenvectors of the symmetric matrix may be spacelike, timelike or null. A generalisation of the classification of symmetric matrices in signature (3,1) [30] to signature (2,2) shows us that the only cases consistent with the specific form of $S_{\mu\nu}$ as the square of the rotation matrix are those for which $S_{\mu\nu}$ is diagonalisable in a orthonormal basis of eigenvectors. The corresponding canonical forms of $M_{\mu\nu}$ are as follows

$$\begin{pmatrix}
1 & 2 & 0 & \bar{0} \\
0 & a & b & c \\
-a & 0 & \mp c & \pm b \\
-c & \mp b & \mp a & 0
\end{pmatrix}, \begin{pmatrix}
0 & a & 0 & 0 \\
-a & 0 & 0 & 0 \\
0 & 0 & d & 0 \\
-c & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & b & 0 \\
0 & 0 & 0 & c \\
0 & 0 & d & 0 \\
0 & 0 & 0 & e
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

(3.12)

To compare this result with the standard scenarios, we note that in signature (3,1) we only obtain matrices of the form of the last three matrices of (3.12), whereas in a positive definite space antisymmetric matrices are block diagonal, with off diagonal blocks. Using the basic blocks (3.12) allows us to make some start at the analysis in (10,2).

Suppose first that the angular momentum lies in a (2+2) dimensional subspace. The last two forms of (3.12) are unsuitable for two-timing BPS states, since the 0 $\bar{0}$ component of angular momentum is zero. We thus have two inequivalent choices to investigate, for which we may calculate the determinant explicitly. In any theory with both a timelike and spacelike direction we may represent the gamma matrices as the real matrices

$$\begin{pmatrix}
\Delta_p & 0 \\
0 & -\Delta_p
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},$$

(3.13)

where the $\Delta_p$ are the gamma matrices of the $(S - 1, T - 1)$ Clifford algebra $\mathbb{C}(S,T)$.

\footnote{Since the reality properties of $C(S,T)$ and $C(S - 1, T - 1)$ are the same, we may choose the $\Delta_p$ to be real for a Majorana representation of $C(S,T)$.}
Recall that a BPS configuration is a set of fields such that the determinant of the anticommutator $\{Q^\alpha, Q^\beta\}$ is zero. Equivalently we may search for zero eigenvalues of the matrix. We now do this for the possible canonical forms (3.12):

1. Employing the spin-space reduction (3.13) twice, we see that the characteristic equation of the first matrix of (3.12) is given by

$$\lambda^2 \left( \lambda^2 - 2a \lambda + (a^2 - b^2 - c^2) \right) = 0. \quad (3.14)$$

We therefore have a supersymmetric solution with two zero eigenvalues for any values of $a, b$ and $c$. In the special case for which we have $a^2 = b^2 + c^2$ there is extra supersymmetry due to an additional zero eigenvalue. This additional condition on $a, b$ and $c$ is satisfied if and only if $M^2 \equiv M_{\mu\nu}M^{\mu\nu} = 0$. By squaring the RHS of the expression

$$\{Q, Q\} = \Gamma^{00} \left( a(\Gamma^{00} \pm \Gamma^{12}) + b(\Gamma^{01} \pm \Gamma^{02}) + c(\Gamma^{02} \mp \Gamma^{10}) \right) , \quad (3.15)$$

we see that the $M^2 = 0$ cases are precisely those for which the right hand side of the anticommutator is proportional to a projection operator. This is a pleasing result, and these solutions are naturally to be interpreted as the two-time analogues of the Brinkmann waves. The relation may be seen thus

$$(T = 1) \quad P_0 > 0 , \quad P_\mu P^\mu = 0 \longleftrightarrow M_{\mu\nu}M^{\mu\nu} = 0 , \quad M_{00} > 0 \quad (T = 2) \quad (3.16)$$

2. For the second possible canonical form of the matrix we find that

$$\{Q, Q\} = 2f1 + 2af^{012} , \quad (3.17)$$

which corresponds to a BPS state if and only if $a = |f|$, in which case $M^2 = 4a^2$. This solution naturally represents a $(2+2)$-brane in the 1-2 direction. A Green-Schwarz formulation of the $(2+2)$-brane is provided in [2].

In the next section we use these ideas to study the BPS configurations to the algebra of relevance to F-theory.

4 F-branes

It is well known that properties of a locally supersymmetric theory may be deduced from a study of the corresponding rigid supersymmetry theory: Many properties of M-theory may be deduced from the study of the eleven dimensional supersymmetry algebra which has the anticommutator

$$\{Q^\alpha, Q^\beta\} = (C^\mu)_{\alpha\beta} P_\mu + (C^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (C^{\mu\nu\rho\sigma\lambda})_{\alpha\beta} Z_{\mu\nu\rho\sigma\lambda} . \quad (4.18)$$
From the viewpoint of supersymmetric $T = 1$ physics, this algebra is often considered to be fundamental, since it reduces to all the supersymmetry algebras of the different types of superstring theories in *nine* dimensions, either directly or by appealing to T-duality [14]; consequently all the information about supergravity BPS states may be obtained by studying the solutions to this algebra. However, we need not stop at eleven dimensions: It is well known that the superalgebra (4.18) arises as a Wigner-Inonou contraction of an ortho-symplectic algebra in twelve dimensions [2, 25, 31], which is given by

$$
[M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\sigma} \eta_{\mu\rho} + M_{\mu\rho} \eta_{\nu\sigma} - M_{\mu\sigma} \eta_{\nu\rho} - M_{\sigma\mu} \eta_{\nu\rho},
$$

$$(Q^\alpha, Q^\beta)_{(10,2)} = (CT^{\mu\nu})^{\alpha\beta} Z_{\mu\nu} + (CT^{\mu\nu\rho\sigma\delta})^{\alpha\beta} Z_{\mu\nu\rho\sigma\delta}^+. \quad (4.19)
$$

The $Q^\alpha$ in this expression are taken to be 32 component Majorana-Weyl spinors, and the central six-form field is self-dual. This is a maximal supersymmetry algebra in the sense that it has $(528+528)$ degrees of freedom, as does (4.18); yet it has the advantage that it reproduces both the IIA and IIB theories directly in ten dimensions [2]. We call this algebra the F-algebra, and there are clearly good reasons to believe that it could be considered to be Fundamental.

When the F-algebra is reduced, the momentum operator $P_\mu$ is obtained from the Kaluza-Klein reduction of the operator $Z_{\mu\nu}$. If we compactify on the $\tilde{0}$-direction then the momentum is simply $P_\mu = Z_{0\mu}$ and the energy of the $T = 1$ system is given by the component $J = Z_{0\tilde{0}} \rightarrow P_0 = E$. Hence the two positivity conditions $J \geq 0$ and $E \geq 0$ are consistent with each other.

**4.1 Fundamental branes in (10,2)**

We now turn to the question of the branes which appear in F-theory. In analogy with results from M-theory, we anticipate that an algebraic understanding of the solutions of (4.19) will enlighten us as to the nature of F-theory. In what follows we define the two timelike dimensions to be labelled by $0, \tilde{0}$. It is well worth noting that since the F-algebra is of the form

$$(Q, Q) = (CT^{\mu_1\mu_2}) Z_{\mu\nu} + (CT^{\mu_1\cdots\mu_6}) Z_{\mu_1\cdots\mu_6}, \quad (4.20)$$

we would perhaps expect to see 2-branes and 6-branes [32], in accordance with the $T = 1$ theory. As we shall see, in this picture we obtain 3-branes and 7-branes because the charge conjugation matrix is now a product of *two* gamma matrices: In general for a two-timing theory the appearance of a $p$-form charge in the algebra implies the existence of a $(p+1)$-brane.
4.1.1 Case 1: $Z^{\mu\nu\rho\sigma\lambda\delta} = 0$

We begin by setting all of the $Z^{\mu\nu\rho\sigma\lambda\delta}$ terms to zero, and search for vanishing determinant configurations of

$$\{Q^\alpha, Q^\beta\}_{(10,2)} = \left(\Gamma^{00}\Gamma^{\mu\nu}\right)^{\alpha\beta} Z^{\mu\nu}.$$  \hspace{1cm} (4.21)

We find, of course, all the (2+2)-dimensional solutions given in the previous sections, but there also exist solutions such as

$$Z^{0\tilde{0}} = J, Z^{0p} = J_p$$

with

$$\sum_{p=1}^{10} (J_p)^2 = J^2.$$ \hspace{1cm} (4.22)

All of the solutions we have been able to find, excepting the (2+2)-brane, have $Z_{\mu\nu} M^{\mu\nu} = 0$.

4.1.2 Case 2: $Z^{\mu\nu\rho\sigma\lambda\delta} \neq 0$

We now include the central six-form charge terms into the algebra and study the cases with only the necessary $Z^{0\tilde{0}} = J$ part of the rotation non-zero

$$\{Q^\alpha, Q^\beta\} = J(1)^{\alpha\beta} + \left(\Gamma^{00}\Gamma^{\mu\nu\rho\sigma\lambda\delta}\right)^{\alpha\beta} Z^{\mu\nu\rho\sigma\lambda\delta}.$$ \hspace{1cm} (4.23)

In order to find some determinant zero solutions, we look for cases for which the right hand side is a projection operator. In theories with one timelike variable, we are interested in configurations which have compact transverse group. From the algebraic point of view this requires that all the indices on the form $Z_{\mu_1...\mu_p}$ corresponding to the $p$-brane are spacelike variables. We find that there are two possible BPS configurations in (10,2) consistent with this requirement after reduction to a $T = 1$ theory

$$Z_{pqrstu} = J, \quad \text{all other terms zero},$$

$$Z_{\tilde{0}pqqrst} = J, \quad \text{all other terms zero}.$$ \hspace{1cm} (4.24)

The first of these is easily interpreted as being a (6+2)-brane in the $pqrstu$ plane, and the second corresponds to a (6+2)-brane in the $pqrst$ plane, with the $\tilde{0}$ direction in the brane worldvolume. Notice that algebraically this second configuration is equivalent to that of a BPS 5-brane solution in a $T = 1$ theory as follows

$$\{Q, Q\} = J \left(1 + \Gamma^{00}\Gamma^{\tilde{0}pqrst}\right) = J \left(1 - \Gamma^{00}\Gamma^{pqrst}\right).$$ \hspace{1cm} (4.25)

In general, the factor in front of the $Z$ terms need not be equal to $J$ for a general, non-BPS state; for positivity we just require that $J \geq M$. The BPS states are obtained when the bound is saturated. Note that in order to conform to standard notation, we shall
refer to a brane which couples to a six-form charge as a seven brane, since $7+1=6+2$. In this way we interpret the $(5+1)$ brane as a seven brane for which one of the indices of the six-form charge is timelike.

## 4.2 Fundamental branes and magnetic branes

In traditional supersymmetry, a fundamental brane is an electrically charged object, in the sense that there is a point singularity corresponding to the source. The total charge is then defined to be

$$Z^{\mu_1...\mu_p} = \int_{S^{D-d-1}} \star F_{d+1},$$

(4.26)

where $F$ is the field strength to which the brane with $d = p + t$ extended dimensions couples to. We may thus suppose that for each charge in the supersymmetry algebra there exists an associated fundamental brane. In our case these are the BPS 3- and 7-branes. By writing the action for the coupling to the brane of the field strength $F$ in terms of the Poincaré dual of a $(D-d-1)$ index dual field strength, we may define the magnetic partner of the original electrically charged brane. Such an argument is signature independent and thus we may define magnetic duals in a $T = 2$ theory. This gives us the pairings

$$(2+2) \leftrightarrow (4+2) \quad \text{and} \quad (6+2) \leftrightarrow (0+2).$$

(4.27)

The $(4+2)$-brane is self explanatory; the $(0+2)$-brane corresponds to the two-timing version of a particle, for which the worldsheet is a timelike plane.

## 4.3 Worldvolume theories

We now turn to the question of the worldvolume content of the F-branes. The BPS condition is a question purely about the rigid spacetime supersymmetry of the theory, whereas we presumably require that the fields on the worldsheet form a local supersymmetry theory. A simple way to deduce the contents of the multiplet is to note that for a supersymmetry theory on a worldsheet of dimension greater than one the bosonic and fermionic degrees of freedom must match up. This is true even if the commutator $\{Q, Q\}$ does not generate momentum. To see that this is indeed the case, we note that the representation space of a supersymmetry theory may be considered to be the disjoint union of a bosonic subspace $\mathcal{B}$ and a fermionic subspace $\mathcal{F}$. The supersymmetry generator $Q$ by definition maps $\mathcal{F}$ onto a proper subspace of $\mathcal{B}$ and vice versa. However, the product of two supersymmetry transformations is a bosonic operator, the particular operator being dependent on the theory in question: For the Poincaré supersymmetry it is a momentum generator $P$, whereas more generally it is some other bosonic operator. Each of these possible operators clearly
map the representation space onto itself. For this reason the action of the supersymmetry generator must provide a one to one mapping from $\mathcal{B}$ to $\mathcal{F}$, and we therefore have the result that the worldsheet supermultiplet must have \textit{equal numbers of bosons and fermions}, for both the eleven dimensional superalgebra and the F-algebra.

So, in order to investigate the matching of the bosons with the fermions we need to determine the the on-shell degrees of freedom of tensorial operators in spaces with two timelike directions. To evaluate the answer we see that in a quantum theory each ghost field absorbs one spacelike and one timelike mode, leading to a $T = 1$ vector index possessing $D - 2$ degrees of freedom. A theory with two times ought to have twice as many ghosts as a Minkowskian theory $\text{[33]}$. Therefore in a $T = 2$ theory a vector index has $D - 4$ degrees of freedom in the analogue of the light cone gauge. Such a statement leads to very interesting results in the context of $N = 2$ strings, which have worldsheet theories which are effectively 2-complex dimensional $\text{[34]}$.

We may now apply the matching conditions to the F-branes to find the following worldsheet field contents

\[
\begin{array}{c|c|c}
(s + t) & n_f - n_b & \text{Multiplet} \\
\hline
(0 + 2) & -2 & \{X^\mu_1 \ldots X^\mu_8; \theta^a\} \\
(2 + 2) & 0 & \{X^\mu_1 \ldots X^\mu_6, A^\mu; \theta^a\} \\
(4 + 2) & 2 & \{X^\mu_1 \ldots X^\mu_4, H^{\mu\nu} \sim A^\mu; \theta^a\} \\
(6 + 2) & 4 & \{X^\mu_1 \ldots X^\mu_4, H^{\mu\nu} \sim A^\mu; \theta^a\}
\end{array}
\]

Notice that for a worldsheet of dimension $(6+2)$ a three-form field and a one-form field have the same degrees of freedom. For the $(4+2)$ and the $(6+2)$ branes we may also consider the dual worldsheets fields, to which other F-branes may couple. The field strengths associated with the worldsheet potentials are denoted by $F$.

\[
\begin{array}{c|c|c|c}
\text{Brane} & F & \tilde{F} & \text{Coupling} \\
\hline
(4 + 2) & F^{\mu\nu} & \tilde{F}^{\mu\nu} & (2 + 2) - \text{brane} \\
(6 + 2) & F^{\mu\nu}, F^{\mu\nu}, F^{\mu\nu\rho} \sim (2 + 2)/(4 + 2) - \text{branes}
\end{array}
\]

### 4.4 Intersection rules

Now that we have defined the basic fundamental and magnetic branes associated with the twelve dimensional F-superalgebra we may try to mimic the theory of intersecting branes in Minkowski spacetimes $\text{[19]}$. This approach has proven to be very useful in understanding the restrictions that supersymmetry places on supergravity brane configurations. We therefore wish to investigate briefly the intersection rules for the fundamental $(2+2)$- and $(6+2)$-branes. There are several possibilities, which we detail as follows:
1. For the algebra (4.21) the projectors corresponding to two (2+2)-branes are given by \( \frac{1}{2} (1 + \Gamma^{00p}q) \) and \( \frac{1}{2} (1 + \Gamma^{00r}s) \). For an intersecting configuration to be possible we must have that these projectors commute with each other, so that their product is also a projector. Assuming that \( pq \) and \( rs \) are not the same, the projectors commute provided that the \( p, q, r, s \) are all distinct. Thus two (2+2)-branes may intersect on a (0+2)-brane. This may be viewed schematically as

\[
\begin{array}{c|c|c}
0\tilde{0} & pq & rs \\
0\tilde{0} & pq & rs \\
\end{array}
\]  

(4.30)

2. The projectors corresponding to two (6+2)-branes may be taken to be given by \( \frac{1}{2} (1 + \Gamma^{00pqrstuv}) \) and \( \frac{1}{2} (1 + \Gamma^{00pqwxyz}) \). For these two projectors to commute, an even number of the spatial indices must match up. Clearly in twelve dimensions this number must be at least two. The results are that the two (6+2)-branes may intersect on (2+2)- and (4+2)-branes, as follows

\[
\begin{array}{c|c|c|c}
0\tilde{0} & pq & rstu & vwxy \\
0\tilde{0} & pq & rstu & vwxy \\
\end{array}
\]

(4.31)

\[
\begin{array}{c|c|c|c}
0\tilde{0} & pqr & tu & vw \\
0\tilde{0} & pqr & tu & vw \\
\end{array}
\]

(4.32)

These intersections are in agreement with the possible worldsheet couplings of the branes given in (4.29).

3. If we choose one of the indices on the six-form to be timelike then it transpires that the the solution is still supersymmetric. In this case the intersection relations ‘contract’ and we obtain (5+1)-brane intersections

\[
\begin{array}{c|c|c|c}
0 & q & rstu & vwxy \\
0 & q & rstu & vwxy \\
\end{array}
\]

(4.33)

\[
\begin{array}{c|c|c|c}
0 & qrs & tu & vw \\
0 & qrs & tu & vw \\
\end{array}
\]

(4.34)

As we shall see in the following section, these 5-brane solutions may be considered to be the trivial lifts to twelve dimensions of M-theory brane configurations.
4.5 Reduction to lower dimensions

We now consider the reduction of the F-branes in a purely super-algebraic way. To proceed we need to consider carefully the effects of the reduction on the supersymmetry theory. We may suppose that the BPS F-brane configurations may be written as

$$\{Q, Q\} = \mathcal{P}, \quad (4.35)$$

for some projection operator $\mathcal{P}$. To obtain the lower dimensional form of this anticommutator in ten and eleven dimensions we must act on the spinors with the projectors $\mathcal{P}_{10} = \frac{1}{2}(1 + \Gamma^0 \ldots \Gamma^9)$ and $\mathcal{P}_{11} = \frac{1}{2}(1 + \Gamma^0 \ldots \Gamma^9 \Gamma^{11})$ respectively. There are two distinct types of supersymmetry theory which may arise after the compactification. Firstly, $\mathcal{P}$ could be mapped onto another projector in the lower dimension. In this case the surviving terms would correspond to a BPS configuration in either type IIB or M-theory. Secondly, there is the possibility that the RHS of the algebra would become zero in the reduced theory. This would correspond to a brane propagating in a trivially realised supersymmetry theory, for which the superspace is flat with no torsion. The role of these types of superspaces (which have been investigated previously [2, 3]) in M-theory has not yet been determined. We shall therefore concentrate on the solutions which are non-trivially realised in lower dimensions, and investigate the various possibilities in turn.

4.5.1 Brinkmann wave analogues

We first look at solutions which have only $Z_{\mu\nu}$ charges, with $Z_{\mu\nu}Z^{\mu\nu} = 0$; the Brinkmann wave analogues. An example which qualitatively covers the features of many higher dimensional solutions is the four dimensional example (3.15), given by

$$\{Q, Q\} = \Gamma^{0\bar{0}} \left( a(\Gamma^{0\bar{0}} \pm \Gamma^{1\bar{s}}) + b(\Gamma^{0\bar{1}} \pm \Gamma^{0\bar{1}\bar{s}}) + c(\Gamma^{0\bar{s}} \pm \Gamma^{1\bar{1}\bar{s}}) \right), \quad (4.36)$$

where the index $s$ may take any spacelike value. We thus have $Z_{\bar{0}0} = \pm Z_{1s}$ etc. For the BPS states we impose the condition that $a^2 = b^2 + c^2$. To obtain a ten dimensional theory we reduce on the torus with coordinates $\bar{0}$ and $\bar{1}$. There are two distinct cases

- If $s = 1, \ldots, 9$ then the surviving terms are $Z_{\bar{0}0}$, $Z_{\bar{0}s}$ and $Z_{\bar{0}1}$. In the reduced theory the anticommutator then becomes

$$\{Q, Q\} = P_0 + (C\gamma^s)P_s \pm (C\gamma^1)P_1, \quad (4.37)$$

where all the quantities are now ten dimensional. Squaring this operator, we find that $(P_0)^2 = (P_1)^2 + (P_s)^2$, which therefore corresponds to a Brinkmann wave with $P^2 = 0$. 

15
If $s = \tilde{1}$ then the quantities which remain after the projection are $Z_{\tilde{0}0} = \pm Z_{11}$ and $Z_{\tilde{0}1} = Z_{10}$

$$\{Q, Q\} = P_0(1 \pm (C\gamma^1)),$$  \hspace{1cm} \hspace{1cm} (4.38)

which is again a massless plane wave.

The reduction of the solutions (4.36) to eleven dimensions on the timelike coordinate $\tilde{0}$ provides us with only one massless plane wave

$$\{Q, Q\} = P_0 \pm (P_1(C\Gamma^1) + P_2(C\Gamma^3)).$$ \hspace{1cm} \hspace{1cm} (4.39)

An example not covered by these cases is (4.22), for which $Z_{\tilde{0}0} = J$ and $Z_{0s} = J_s : \sum_{s=1}^1 (J_s)^2 = J^2$. If we reduce this solution to eleven dimensions then the only surviving generator in the algebra is the $Z_{\tilde{0}0}$ component, and we therefore find a massive particle.

A toroidal compactification provides us with another such object in ten dimensions.

### 4.5.2 3-branes and 7-branes

The next solution to consider is the three-brane. The reduction of the Green-Schwarz (2+2)-brane was discussed in [2], and was shown to produce the type IIB string and the M-2-brane. We are consequently left to analyse the seven-branes, which have non-vanishing six-form charges. We look at the simplest examples, of the form

$$\{Q^\alpha, Q^\beta\} = J(1)^{\alpha\beta} + (\Gamma^0_{\tilde{0}}\Gamma^\mu\nu\rho\sigma\lambda\delta)^{\alpha\beta} Z_{\mu\nu\rho\sigma\lambda\delta},$$ \hspace{1cm} \hspace{1cm} (4.40)

with a single non-zero six-form component. We are interested in the cases for which $\tilde{P}_{10/11}(C\Gamma^{\mu\nu\rho\sigma\lambda\delta})$ is non-zero. If we reduce to eleven dimensions then the $Z_{pqrsts}$ case is projected to zero, whereas for $Z_{\mu\nu\rho\sigma\lambda\delta} \equiv Z_{\tilde{0}pqrst} = J$ we obtain a BPS saturated five brane

$$\{Q, Q\}_{11} = J(1 + (C\Gamma^{pqrst})Z_{pqrst}).$$ \hspace{1cm} \hspace{1cm} (4.41)

If we perform the double dimensional reduction of the theory on a torus down to the IIB theory then the converse situation holds: It is simple to see that the $Z_{\tilde{0}\tilde{1}pqrst}$ term vanishes, whereas the term corresponding to a (6+2)-brane, $Z_{\tilde{1}pqrst}$, takes us to a five-brane in the IIB theory. This is rather interesting: Although we have seven branes in the twelve dimensional theory, we can only produce five-branes under dimensional reduction! This is pleasing because there are no fundamental or solitonic six-branes in the spectrum of M-theory.

---

4. By altering the parameters of the torus we reduce upon is seems possible in principle to generate the complete $SL(2,\mathbb{Z})$ 5-brane multiplet, in the same manner that the $SL(2,\mathbb{Z})$ invariance of the type IIB string is explained by F-theory.
We should now inquire as to the effect of the reduction on the worldvolume fields of the seven-branes. It is sufficient to note that the supersymmetry theory on the brane will be reduced onto another supersymmetry theory on the new brane in a lower dimension. Since the worldvolume content of the five-branes in ten and eleven dimensions may be deduced from the considerations of a six-dimensional supersymmetry theory, we may be sure that the worldvolume content of the 12-dimensional seven branes will be mapped onto the correct supersymmetry theories for the well known Minkowski signature five-branes. To summarise, we have the correspondences

\[
\begin{array}{c|c|c}
F & M & IIB \\
7 & 5 & 5 \\
3 & 2 & 1 \\
\end{array}
\]  

(4.42)

### 4.5.3 Intersection rules reduction

We now finally wish to reduce the intersection rules we obtained previously. It is important to consider the reduction to M-theory, since the M-branes may be used to reproduce all of the branes in string theory via dimensional reduction and the use of dualities [19]. As a corollary, all of the intersection rules in the type II theories descend from those in eleven dimensions. We find that the reduction of the intersecting F-7-brane configurations on a timelike circle provides us with two M-5-branes intersecting on either 3-branes or strings. Reduction of the intersecting F-3-branes gives two M-2-branes intersecting on a 0-brane. These are precisely the basic M-brane configurations from which all the supergravity brane configurations may be deduced, hence we have complete consistency between the two scenarios.

### 5 Conclusion

We have discussed the basic algebraic consequences of BPS states in supersymmetry theories with two timelike directions. In theories with a single time variable, every low energy BPS configuration must have a positive energy. Quite independently of any one-time considerations, we showed that a supersymmetric BPS state in any four or greater dimensional theory with two times must have a ‘positive angular momentum in the timelike plane’: \( Z_{\tilde{0}0} = J > 0 \). We then studied the algebraic restrictions on the \( p \)-branes arising from the supersymmetry algebra in such theories. This type of procedure has proven to be very useful in M-theory, since each BPS solution of the supersymmetry algebra corresponds to some supergravity configuration which is the low energy limit of an excitation of the underlying quantum theory. Clearly, there is a possibility that the M-theory structure
may derive from some theory with ten spacelike and two timelike directions. If it does, then this theory is sure to be one of supersymmetric brane configurations; otherwise, there is very little which may be said with certainty. For example, it is even unclear as to whether or not the twelve dimensional theory would reduce to the Einstein-Hilbert action in the low energy limit: As we have shown in this paper, there is no fundamental notion of energy in a theory with two timelike directions. The philosophy of the work presented here is that quite generally any supersymmetric theory must in a fundamental way have behaviour governed by the tangent space superalgebra structure. For this reason we studied the F-algebra, which is the algebra most likely to be relevant to M-theory. The basic BPS configurations of the F-algebra are 3-branes and 7-branes, along with magnetic dual partners. This resulting brane structure when reduced to eleven dimension reproduces precisely the brane structure of eleven dimensional superalgebra; when reduced on a null-torus we reproduce the type IIB string and fivebrane solutions. In addition to these standard backgrounds there exists the possibility of a compactification down to a simple supersymmetry sector of the ten and eleven dimensional theories, in which the anticommutator of the supersymmetry generators vanish.

Although the discussion we have presented here is completely classical, it is reassuring to note that if we allow ourselves to use the known duality relations in lower dimensions, there is a direct correspondence between the F-branes and those of the Minkowski signature theories. From this structure, it should now be possible to start to lift the M-branes to the known BPS F-branes. Hence, by using this algebraic correspondence we may begin to understand more of the meaning of F-theory.

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