OVERCONVERGENT SUBANALYTIC SUBSETS IN THE FRAMEWORK OF BERKOVICH SPACES

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ABSTRACT. We study the class of overconvergent subanalytic subsets of a $k$-affinoid space $X$ when $k$ is a non-archimedean field. These are the images along the projection $X \times \mathbb{B}^n \to X$ of subsets defined with inequalities between functions of $X \times \mathbb{B}^n$ which are overconvergent in the variables of $\mathbb{B}^n$. In particular, we study the local nature, with respect to $X$, of overconvergent subanalytic subsets. We show that they behave well with respect to the Berkovich topology, but not to the $G$-topology. This gives counter-examples to previous results on the subject, and a way to correct them. Moreover, we study the case $\dim(X) = 2$, for which a simpler characterisation of overconvergent subanalytic subsets is proven.

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Motivations. Let us consider a complete non-archimedean field $k$ (assumed to be algebraically closed in this introduction for simplicity). Since non-archimedean fields are totally disconnected, one can not define the notion of analytic spaces over $k$ as easily as in the case of $\mathbb{R}$ or $\mathbb{C}$. Tate [Tat71] developed such a theory, and called his spaces rigid spaces, whose building blocks are affinoid spaces. However, these spaces are not endowed with a classical topology, but with a Grothendieck topology ($G$-topology). Afterwards, V. Berkovich developed another viewpoint for $k$-analytic geometry [Ber90,Ber93]. His spaces, called $k$-analytic spaces, or Berkovich spaces, have more points than the corresponding rigid spaces and are equipped with a topology which is locally arcwise connected. Moreover, in this theory, affinoid spaces are compact. R. Huber also developed another viewpoint, in the setting of adic spaces [Hub96], and there also exists an approach, initiated by M. Raynaud, using formal geometry (see [BL93] for instance).

If $X, Y$ are $k$-analytic spaces, and $\varphi : Y \to X$ is an analytic map, it is natural to wonder what is the shape of $\varphi(Y)$. By analogy with Chevalley’s theorem and Tarski-Seidenberg theorem, one would like to be able to describe such images $\varphi(Y)$ using only functions of $X$.

Without assumption on $\varphi$, this is impossible: one needs some kind of compactness at some point. One reasonable restriction is to consider analytic maps $\varphi : Y \to X$ where $X$ and $Y$ are affinoid spaces.

In this context the first natural approach to define a semianalytic set of a $k$-affinoid space as a finite boolean combination of sets defined by inequalities $\{|f| \leq |g|\}$ between analytic functions. But the class of semianalytic sets is not big enough: there exist some morphisms of affinoid spaces $\varphi : Y \to X$ such that $\varphi(Y)$ is not semianalytic.

To overpass this problem, one has to consider more functions on an affinoid space $X$ than the analytic ones. In the framework of $\mathbb{Z}_p^n$, Jan Denef and Lou Van den Dries have given [DvdD88] a good description of images of analytic maps $\varphi : \mathbb{Z}_p^m \to \mathbb{Z}_p^n$. Their main idea is to allow division of functions. In the framework of rigid geometry, where $\mathbb{Q}_p$ has to be replaced by some non-archimedean algebraically closed field $k$, this idea of allowing divisions has been developed in two ways.

The first one is due to Leonard Lipshitz [Lip93,LR00b,Lip88,LR96] and rests on the introduction of an algebra $S_{m,n}$ of restricted analytic functions on products of closed and open balls. This allows L. Lipshitz to define for each affinoid space $X$ the class of subanalytic sets of $X(k)$ (in terms of analytic functions of $X$, division and composition with $S_{m,n}$), and to prove that subanalytic sets are stable under analytic maps between affinoid spaces.

A second approach has been developed by Hans Schoutens in [Sch94a]. This leads to the definition of overconvergent subanalytic sets of $X(k)$. Namely, overconvergent subanalytic sets of $X(k)$ form a subclass of the subanalytic sets as defined by L. Lipshitz. Overconvergent subanalytic sets are only stable under overconvergent analytic maps between affinoid spaces. For instance, if $\varphi : \mathbb{B}^n \to X$ is an analytic map which can be analytically extended to a polydisc of radius $r > 1$, then $\varphi(\mathbb{B}^n)$ is an overconvergent subanalytic set of $X$.

Overconvergent subanalytic sets. Hans Schoutens used the language of rigid geometry. We now summarize his results. First let $D : k^2 \to k$ be defined by

$$D(x,y) = \begin{cases} \frac{x}{y} & \text{if } |x| \leq |y| \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $A$ be an affinoid algebra and $X$ its affinoid space. The algebra $A(\langle D \rangle)$ is defined as the smallest $k$-algebra of functions $f : X(k) \to k$ such that

- $A(\langle D \rangle)$ contains the functions induced by $A$.
- If $f, g \in A(\langle D \rangle)$, then $D(f,g) \in A(\langle D \rangle)$.
- If $f \in A(Y_1, \ldots, Y_n)$ is overconvergent in the variables $Y_i$, and $g_1, \ldots, g_n \in A(\langle D \rangle)$ satisfy $|g_i|_{\sup} \leq 1$, then $f(g_1, \ldots, g_n) \in A(\langle D \rangle)$.

Stability under overconvergent maps is contained in the following result (we denote by $\mathbb{B}$ the closed unit disc).
Theorem. \cite{Sch94a} For a subset $S \subset X(k)$ the following are equivalent:

- there exists $n \in \mathbb{N}$, a semianalytic set $T$ of $X \times \mathbb{A}^n(k)$ defined by inequalities $\{|f| \leq |g|\}$ where $f$ and $g$ are overconvergent with respect to the variables of $\mathbb{A}^n$ such that $S = \pi(T)$ where $\pi : X \times \mathbb{A}^n(k) \to X(k)$ is the first projection. We call such sets overconvergent subanalytic sets.
- $S$ is a boolean combination of inequalities $\{|f| \leq |g|\}$ where $f,g \in \mathcal{A}(\langle D \rangle)$

For instance, if $\varphi : \mathbb{A}^n \to X$ is an overconvergent map (in the sense that it can be extended to a polydisc of radius greater than one), then $\varphi(\mathbb{A}^n)$ is overconvergent subanalytic (take for $T$ the graph of $\varphi$).

Results of this text. In this article we explain how Berkovich spaces are well suited to study overconvergent subanalytic sets. Indeed, the definitions that we have given above (semianalytic, overconvergent subanalytic) can be given in the framework of Berkovich spaces. For instance if we consider $X = \mathbb{A}^2$ with coordinate functions $T_1, T_2$, the inequality $\{|T_1| \leq |T_2|\}$ naturally defines two sets

$$S_{\text{rig}} = \{(t_1,t_2) \in (k^\circ)^2 \mid |t_1| \leq |t_2|\}$$
$$S_{\text{Berko}} = \{x \in \mathcal{M}(k(T_1,T_2)) \mid |T_1(x)| \leq |T_2(x)|\}$$

Of course $S_{\text{rig}} \subset S_{\text{Berko}}$. More precisely, $S_{\text{rig}}$ is the set of rigid points of $S_{\text{Berko}}$.

This new approach with Berkovich spaces allows us to simplify the proof of the theorem of \cite{Sch94a} mentioned above. The part 2 of \cite{Sch94a} A combinatorial lemma is replaced by a simple compactness argument in Berkovich spaces.

Then we consider the local behaviour of overconvergent subanalytic sets. If $X$ is an affinoid space there are two ways to consider local behaviour on $X$.

1. The $G$-topology, where a covering of $X$ is a finite covering $\{X_i\}$ by affinoid domains.
2. The Berkovich topology \cite{Ber90,Ber93} on $X$ seen as a Berkovich space, which is a real topology.

If $S$ is an overconvergent subanalytic set of $X$ and $U$ an affinoid domain of $X$, it is easy to see that $S \cap U$ is an overconvergent subanalytic set of $U$. It is then natural to wonder if overconvergent subanalytic sets fit well with one of these topologies. We give the following answers.

Proposition. 2.4 There exists an affinoid space $X$, some subset $S \subset X$, and a finite covering $\{X_i\}$ of $X$ by affinoid domains such that for all $i$, $S \cap X_i$ is overconvergent subanalytic in $X_i$, but $S$ is not overconvergent subanalytic in $X$.

In other words, being overconvergent subanalytic is not local with respect to the $G$-topology. This contradicts some results of \cite{Sch94a}, for instance \cite{Sch94a} QE theorem p. 270, Proposition 4.2, Theorem 5.2.

We prove however that the Berkovich topology corrects this.

Theorem. 1.44 A subset $S \subset X$ is overconvergent subanalytic if and only if for every $x \in X$ (seen as a Berkovich space), there exists $V$ an affinoid neighbourhood of $x$ (for the Berkovich topology) such that $S \cap V$ is overconvergent subanalytic.

In other words, being overconvergent subanalytic is a local property, but with respect to the Berkovich topology.

The mistake in \cite{Sch94a} which we point out in proposition 2.4 lead to other mistakes in further work of H. Schoutens \cite{Sch94c,Sch94b}. In particular \cite{Sch94b} which relies on the false results of \cite{Sch94a} claims that if $k$ is algebraically closed of characteristic 0, then a subset of the unit bidisc is overconvergent subanalytic if and only if it is rigid-semianalytic (i.e. semianalytic locally for the $G$-topology). But the counterexample we give in proposition 2.4 proves that this equivalence does not hold. Anyway, the proofs of \cite{Sch94b} rely on some false equivalences of \cite{Sch94a}.

We show that the Berkovich topology allows to correct the results of \cite{Sch94b}.
Theorem. Let us assume that $k$ is algebraically closed. Let $X$ be a good quasi-smooth (equivalently rig-smooth, or regular) strictly $k$-analytic space of dimension 2. Then a subset $S$ of $X$ is overconvergent subanalytic if and only if it is locally semianalytic.

Here, we say that $S$ is locally semianalytic if for every $x \in X$, there is an affinoid neighbourhood $V$ of $x$ such that $S \cap V$ is semianalytic in $V$.

Ideas behind the proofs. We want to point out that the two equivalent characterizations of overconvergent subanalytic sets which were given in [Sch94a] and which we have reminded on page 3 are not very manageable. In particular it is hard to prove that some set is not overconvergent subanalytic using these characterizations, whereas we have much more tools to say that a subset is semianalytic or not. In order to overpass this difficulty, we have introduced a third characterization of overconvergent subanalytic sets which is more geometric. We remark that the quotient of two analytic functions $f$ and $g$ is not analytic any more, but becomes analytic if one blows up $(f,g)$. With this in mind, in order to describe a subset of $X$ defined by inequalities $\{|f| \leq |g|\}$ with $f,g \in A(\mathcal{D})$ we can consider some finite sequences of blow ups $\tilde{X} \to X$ and project some semianalytic sets of $\tilde{X}$ outside the exceptional locus (with some extra condition for the overconvergence condition). We call such subsets overconvergent constructible (see 1.8 for a precise definition). The idea of looking at analytic functions above some blowup of $X$ had already appeared in [LR00a, 2.3 (iv)].

With this in mind we would like to restate more precisely the results of this paper.

First, we prove theorem 1.37 which asserts that if $X$ is an affinoid space, $S \subset X$ is overconvergent subanalytic if and only if it is overconvergent constructible, using at some point the compactness of the Berkovich space $X$.

Then, according to the definition of an overconvergent constructible set, it is easy to prove that overconvergent subanalytic sets are local for the Berkovich topology (proposition 1.44).

To justify our counterexample in proposition 2.4 we use the more geometric approach of overconvergent constructible sets which allows to use results on semianalytic sets. Ultimately, our argument relies on the study of some Gauss point in en embedded curve in the polydisc, which strengthens our feeling that Berkovich spaces are well suited to study overconvergent subanalytic sets between overconvergent subanalytic sets and overconvergent constructible sets.

Finally, we want to mention one more benefit of overconvergent constructible sets. In the author’s thesis it is proved (proposition 2.4.1) that if $k$ is algebraically closed, $S$ a locally closed overconvergent subanalytic set of the compact $k$-analytic space $X$, and if we consider a prime number $\ell \neq \text{char}(k)$, then the étale cohomology groups with compact support of the germ $(S,X)$ (see [Ber93, 3.4,5.1])

$$H^i_c((S,X), \mathbb{Q}_\ell)$$

are finite dimensional $\mathbb{Q}_\ell$-vector spaces. Here again the idea is that (thanks to the presentation of $S$ as an overconvergent constructible set) we can reduce to the case where $S$ is semianalytic, and in that case, the finiteness result is proved in [Mar13, proposition 2.2.3] (which ultimately relies on a finiteness result for affinoid spaces proved by V. Berkovich).

Organisation of the paper.

In section 1 we define constructible data of $X$, in order to define overconvergent constructible subsets. Note that we do not assume that $k$ is algebraically closed contrary to [Sch94a]. In section 2 we introduce overconvergent subanalytic subsets. In section 3 we have tried to carefully treat Weierstrass division, trying to be as general as possible (namely our results hold for an arbitrary ultrametric Banach algebra, and an arbitrary radius of convergence). In section 4 we prove that overconvergent constructible and overconvergent subanalytic subsets are the same. The proof of this result which appears in [Sch94a], is here simplified by the use of Berkovich spaces: in particular, the quite technical section 2 of [Sch94a] A combinatorial lemma is replaced by a simple compactness argument (see theorem 1.37). In 5 we try to handle the following problem: how to pass from a definition that works only for $k$-affinoid spaces to a more local definition, with the hope that in the affinoid case the local and the global definitions would
coinide. As we said earlier, trying to do this with the G-topology will not work. If in contrary we do this with the Berkovich topology, the definitions will be compatible. In section 1.6 we explain how these results can be extended to $k$-affinoid spaces (by opposition to strictly $k$-affinoid spaces). In addition, in that case, we can allow the field $k$ to be trivially valued.

In section 2 we give some counter-examples to erroneous statements of [Sch94a]. Precisely, in [Sch94a] five classes of subsets were defined: globally strongly subanalytic, globally strongly $D$-semianalytic, strongly subanalytic, locally strongly subanalytic and strongly $D$-semianalytic subsets. The three last classes were defined from the first two ones by adding "G-local" at some point. In [Sch94a] it was claimed that these five classes agree. We explain that this is not the case, namely from these five classes, the first two ones indeed agree, but not the last three ones, which are larger (see figure 1, p. 28). The main idea is that if one replaces "G-local" by "locally for the Berkovich topology", the results of [Sch94a], for instance the theorem on p. 270, become true. Let us give one of the counter-examples that we study:

Example 0.1. Let $X = \mathbb{B}^2$ be the the closed bidisc, $0 < r < 1$ with $r \in \sqrt{|k^2|}$, $f \in k\{r^{-1}x\}$ a nalytic function whose radius of convergence is exactly $r$ and such that $\|f\| < 1$. We then define

$$S = \{(x, y) \in \mathbb{B}^2 \mid |x| < r \text{ and } y = f(x)\}.$$ Then (see proposition 2.4) $S$ is rigid-semianalytic, but not overconvergent subanalytic. The Berkovich approach is here helpful since to prove this, we use a point $\eta$ of the Berkovich bidisc which is not a rigid point, and some properties of its local ring $O_{X, \eta}$.

Finally, in section 3 we correct the proof of [Sch94b] (which rested on the erroneous results of [Sch94a] and [Sch94c]) and restrict the hypothesis of it. Namely, we prove that when $k$ is algebraically closed, and $X$ is a good quasi-smooth (equivalently rig-smooth) strictly $k$-analytic space of dimension 2, then overconvergent subanalytic subsets are in fact locally semianalytic. Not only do we give a correct proof of this theorem, but moreover, this result is more general than the result of [Sch94b], where $X$ was the bidisc and where it was assumed that the characteristic of $k$ was 0.

Contribution of this text. We want to stress the fact that section 1 is highly inspired by the work of H. Schoutens. In particular, the definition we give of a constructible datum, and the resulting definition of an overconvergent constructible subset, is a geometric formulation of what is done in [Sch94a] concerning $D$-strongly semianalytic subsets. In particular, the proof of theorem 1.37 is very close to the proof of [Sch94a] Th 5.2. We have however decided to include a proof of theorem 1.37 for three reasons. First, the compactness argument that we use in theorem 1.37 seems to us enlightening, and a way to see that Berkovich spaces are relevant in this context. Secondly, we have the feeling that replacing the strongly $D$-semianalytic subsets of [Sch94a] by our overconvergent constructible subsets is more geometric and gives a better understanding of the situation. Finally, the mistakes in [Sch94a], that we explain in section 2, have the following consequences: most of the statements of [Sch94a] become false. For instance, [Sch94a] Theorem 5.2] is false as we prove in section 2. In this context it seemed to us relevant to write section 1.

The same remarks hold for section 3. A statement analogous to theorem 8.12 was claimed in [Sch94b]. However, in this article, it was assumed, and used in the proofs, that the five classes of subsets introduced in [Sch94a] were the same; but since we prove that this is not the case, the proofs of [Sch94b] are erroneous.

Finally, let us mention that another proof of theorem 1.37 has also been given in [CL11] 4.4.10.

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However, it has to be noted that we could have written this proof in the context of adic spaces, and used a similar argument of quasi-compactness.
1. Overconvergent constructible subsets

With a few exceptions that will be specified, $\mathcal{A}$ will be a strictly $k$-affinoid algebra, and $X$ the strictly $k$-affinoid space $\mathcal{M}(\mathcal{A})$.

1.1. Constructible data.

**Definition 1.1.** Let $X$ be a $k$-affinoid space whose $k$-affinoid algebra is $\mathcal{A}$. A subset $S$ of $X$ is called **semianalytic** if it is a finite boolean combination of sets of the form $\{x \in X \mid |f(x)| \leq |g(x)|\}$ where $f$ and $g \in \mathcal{A}$ (by finite boolean combination, we mean finitely many use of the set-theoretical operators $\cap$, $\cup$ and $\forall$). A subset of the form $\{x \in X \mid |f_i(x)| \delta_i |g_i(x)| \forall i = 1 \ldots n\}$ with $f_i$ and $g_i \in \mathcal{A}$, and $\delta_i \in \{\leq, <\}$ will be called **basic semianalytic**.

**Remark 1.2.** With a repeated use of the rule $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ one can show that any semianalytic subset of $X$ is a finite union of basic semianalytic subsets.

**Definition 1.3.** Let $(X, S)$ be a $k$-germ in the sense of [Ber93, 3.4]; this just means that $S$ is a subset of $X$. An elementary constructible datum of $(X, S)$, is the following datum. Let $f, g \in \mathcal{A}$. Let also $r$ and $s$ be two real numbers such that $r > s > 0$ and $s \in \sqrt{|k^*|}$. Let $T := \varphi^{-1}(S) \cap \{y \in R \mid g(y) \neq 0 \text{ and } |f(y)| \leq s |g(y)|\}$. Then $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum. If $\psi : (Y', T') \simeq (Y, T)$ is an isomorphism of $k$-germs, and $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum, if we set $\varphi' = \varphi \circ \psi$, then we will also say that $(Y', T') \xrightarrow{\varphi'} (X, S)$ is an elementary constructible datum.

**Remark 1.4.** If $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum, then $\varphi(T) \subset S$, and $\varphi$ realizes a homeomorphism between $T$ and its image $\varphi(T)$. Moreover

$$\{y \in Y \mid |f(y)| \leq s |g(y)| \neq 0\}$$

is an analytic domain of $Y$, and can be identified through $\varphi$ with the analytic domain of $X$

$$\{x \in X \mid |f(x)| \leq s |g(x)| \neq 0\}.$$ 

**Definition 1.5.** Let $(X, S)$ be a $k$-germ. A **constructible datum** is a sequence

$$(Y, T) = (X_n, S_n) \xrightarrow{\varphi_n} (X_{n-1}, S_{n-1}) \rightarrow \cdots \rightarrow (X_1, S_1) \xrightarrow{\varphi_1} (X_0, S_0) = (X, S)$$

where for $i = 1 \ldots n$, $(X_i, S_i) \xrightarrow{\varphi_i} (X_{i-1}, S_{i-1})$ is an elementary constructible datum. Let $\varphi = \varphi_1 \circ \cdots \circ \varphi_n$. Then we will denote this constructible datum by

$$(Y, T) \xrightarrow{\varphi} (X, S).$$

We will say that the complexity of $\varphi$ is $n$.

In the particular case $S = X$, i.e. $(X, S) = (X, X)$, we will denote the constructible datum by:

$$(Y, T) \xrightarrow{\varphi} X,$$

and we will call it a constructible datum of $(X, S)$. This is actually the case that will mainly interest us, but partly for technical reasons we have chosen to use $k$-germs.

**Remark 1.6.** If $(Y, T) \xrightarrow{\varphi} X$ is a constructible datum, it follows easily from the above definitions that $T$ is a semianalytic subset of $Y$. 

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*[BER93]* Bernardi, J.-C. *Introduction to $p$-adic Analysis*. Cambridge University Press, 1993.
Remark 1.4 implies that if \((Y,T) \rightarrow (X,S)\) is a constructible datum, \(\varphi_T : T \rightarrow (Y,T)\) induces a homeomorphism between \(T\) and \(\varphi(T)\). It is also clear that if \((Z,U) \rightarrow (Y,T)\) is a constructible datum and \((Y,T) \rightarrow (X,S)\) is another one, then \((Z,U) \rightarrow (Y,T)\) is also a constructible datum.

We want to point out that in the definition of a constructible datum, \(n\) cannot be recovered from \(\varphi\) alone.

**Definition 1.7.** Let \((X_i,S_i) \rightarrow (X,S)\), \(i = 1\ldots m\) be \(m\) constructible data of the \(k\)-germ \((X,S)\). We will say it forms a **constructible cover** of \((X,S)\) if \(\bigcup_{i=1}^{m} \varphi_i(S_i) = S\).

**Definition 1.8.** Let \(X\) be a \(k\)-affinoid space. A subset \(C\) of \(X\) is said to be an **overconvergent constructible subset** of \(X\) if there exist \(m\) constructible data \((X_i,S_i) \rightarrow (X,S)\) for \(i = 1\ldots m\) such that \(\bigcup_{i=1}^{m} \varphi_i(S_i) = C\).

**Remark 1.9.** Using the notations of definition 1.3, when \((Y,T) \rightarrow (X,S)\) is an elementary constructible datum, with \(Y = \mathcal{M}(A(r^{-1}t)/(f - tg))\), then \(T\) (and hence \(\varphi(T)\)) are defined with the function \(t\) which mimics the function \(\varphi\), when it has a sense and its norm is \(\leq s\). In addition the condition \(r > s\) is here to make sure that the new functions of \(B\) are overconvergent in \(t = \frac{f}{g}\), that we see as a function on the analytic domain \(\{x \in X \mid |f(x)| \leq s|g(x)| \neq 0\}\).

The following three results are formal consequences of the previous definitions.

**Lemma 1.10.** If \((Y,T) \rightarrow (X,S)\) is an elementary constructible datum and \((Z,U) \rightarrow (X,S)\) is a morphism of \(k\)-germs, let \(\psi\) be the cartesian product of \(k\)-germs:

\[(Y,T) \rightarrow (X,S)\]

\[(Y,T) \times (X,S) \rightarrow (Z,U)\]

Then, \((Y,T) \times (X,S) \rightarrow (Z,U)\) is an elementary constructible datum. Moreover if set

\[(Y,T) \times (X,S) (Z,U) = (Y',T')\]

then, \((\varphi \circ \psi')(T') = \varphi(T) \cap \psi(U)\).

**Corollary 1.11.** Let \((Y,T) \rightarrow (X,S)\) be a constructible datum

\[(Y,T) = (X_n,S_n) \rightarrow (X_n,S_n) \cdots \rightarrow (X_0,S_0) = (X,S)\]

and let \((X',S') \rightarrow (X,S)\) be a morphism of \(k\)-germs. Let us consider the cartesian product:

\[(Y',T') \rightarrow (X',S')\]

Then \((Y',T') \rightarrow (X',S')\) is a constructible datum and \((\psi \circ \varphi')(T') = \varphi(T) \cap \psi(S')\).
Corollary 1.12. Let \((X_1, T_1) -\varphi - (X, S)\) and \((X_2, T_2) -\varphi' - (X, S)\) be two constructible data (with the same target). Let us consider the fibered product

\[
\begin{array}{c}
(X_1, T_1) \\ \downarrow \varphi \\
\downarrow \psi' \\
(Z, U) \\ \downarrow \varphi' \\
(X_2, T_2)
\end{array}
\]

Then \((Z, U) -\psi' - (X_1, T_1)\) and \((Z, U) -\psi' - (X_2, T_2)\) are constructible data. Moreover \((\varphi \circ \psi')(U) = (\psi \circ \varphi')(U)\).

Proof. Lemma 1.10 is a direct consequence of definition 1.3. Corollary 1.11 is then proved by induction on the complexity of \(\varphi\) using lemma 1.10. Similarly, corollary 1.12 is proved by induction on the complexity of \(\psi\) using corollary 1.11.

Proposition 1.13. (1) If \(T \subseteq X\) is a semianalytic subset of \(X\) then \(T\) is an overconvergent constructible subset of \(X\).

(2) Let \(C \subseteq T\) be an overconvergent constructible subset of \(Y\) and let \((Y, T) -\varphi - X\) be a constructible datum. Then \(\varphi(C)\) is an overconvergent constructible subset of \(X\).

(3) The class of overconvergent constructible subsets of \(X\) is stable under finite boolean combinations.

Proof.

(1) Consider the elementary constructible datum \((X, T) \overset{id}{\longrightarrow} X\).

(2) By definition, there exist some constructible data \((Y_i, T_i) -\varphi_i - Y\), for\( i = 1 \ldots m\), such that

\[ C = \bigcup_{i=1}^m \varphi_i(T_i). \]

Now if we define \(\psi_i := \varphi \circ \varphi_i\), then \((Y_i, T_i) -\psi_i - X\) are \(m\) constructible data, and \(\varphi(C) = \varphi(\bigcup_{i=1}^m \varphi_i(Y_i)) = \bigcup_{i=1}^m \psi_i(T_i)\), so it is an overconvergent constructible subset of \((X, S)\).

(3) Stability under finite union is a direct consequence of the definition 1.8, as for intersection, it is a consequence of corollary 1.12. And if \(C \subseteq X\) is an overconvergent constructible subset of \(X\), let us show that \(X \setminus C\) is also overconvergent constructible.

By definition, \(C = \bigcup_{i=1}^m \varphi_i(S_i)\) where \((X_i, S_i) -\varphi_i - X\) are some constructible data. We do it by induction on \(c\), the maximum of the complexity of the \(\varphi_i\)'s.

If \(c = 0\), then \(C\) is a semianalytic subset of \(X\) so \(X \setminus C\) is semianalytic, hence overconvergent constructible.

If \(c > 0\) and we assume the result for \(c' < c\), then

\[ X \setminus C = X \setminus ( \bigcup_{i=1}^m \varphi_i(S_i) ) = \bigcap_{i=1}^m (X \setminus \varphi_i(S_i)) \]

so we can assume that \(m = 1\), that is to say, we can assume that \(C = \varphi(T)\) where \((Y, T) -\varphi - X\) is a constructible datum of complexity \(c\). Then

\[ \varphi = \psi \circ \varphi' : (Y, T) -\varphi' - (Y', T') -\psi - X \]

where the complexity of \(\varphi'\) is \(c - 1\) and \(\psi\) is an elementary constructible datum. Now

\[ X \setminus \varphi(T) = \psi(T' \setminus \varphi'(T)) \cup (X \setminus \psi(T)) \]

because \(\varphi'|_T\) and \(\psi|_{T'}\) are injective maps. By induction hypothesis,

\[ T' \setminus \varphi'(T) = T' \cap (Y' \setminus \varphi'(T)) \]
is an overconvergent constructible subset of \( Y' \), thus according to (1), so is \( \psi(T' \setminus \varphi'(T)) \).

Finally, if the elementary constructible datum \( \psi \) is associated with \( f, g, r \) and \( s \), by definition,
\[
T' = \{ y \in R \mid |f(y)| \leq s|g(y)| \neq 0 \}
\]
for some semianalytic subset \( R \) of \( Y' \). And if we define
\[
\hat{T} = \{ y \in Y' \setminus R \mid |f(y)| \leq s|g(y)| \neq 0 \},
\]
then
\[
X \setminus \psi(T') = \psi(\hat{T}) \cup \{ y \in X \mid |f(y)| > s|g(y)| \} \cup \{ y \in X \mid g(y) = 0 \}.
\]
Thus, it is also overconvergent constructible in \( X \).

Let \( x \in X \), and \( U \) be an affinoid neighbourhood of \( x \). Shrinking \( U \) if necessary, we can assume [Ber90, 2.5.15] that \( U \) is a rational domain of the form \( X(\mathbf{L}^{-1} f \frac{1}{g}) \) still contains \( x \). For each \( i \), we pick a real number \( s_i \) such that \( \frac{f}{g} < s_i < r_i \) and \( s_i \in \sqrt{k^*} \). For each \( i \), we consider the elementary constructible datum \( (X_i, S_i, X) \) defined by \( X_i = \mathcal{A}(r_i^{-1} t_i)/(f_i - t_i g_i) \), and \( S_i = \{ p \in X_i \mid |f_i(p)| \leq s_i |g(p)| \} \). One checks that \( \varphi_i(S_i) \) is a neighbourhood of \( x \). Now if we take the fibered product of all these elementary constructible data, we obtain (using corollary 1.12) the following constructible datum:
\[
\left( X \left( \mathbf{L}^{-1} \frac{f}{g} \right), X \left( \mathbf{L}^{-1} \frac{f}{g} \right) \right) \xrightarrow{\varphi} X
\]
Here \( \varphi \) just corresponds to the embedding of the affinoid domain \( X(\mathbf{L}^{-1} f \frac{1}{g}) \). Moreover \( \varphi \left( X(\mathbf{L}^{-1} f \frac{1}{g}) \right) \), that we might identify with \( X(\mathbf{L}^{-1} f \frac{1}{g}) \), is a neighbourhood of \( x \). We can sum up this in the following lemma:

**Lemma 1.14.** Let \( X \) be a strictly \( k \)-affinoid space. Let \( x \in X \) and \( U \) be an affinoid neighbourhood of \( x \). Then there exists a constructible datum \( (Y, T) \) such that \( T \) is an affinoid domain of \( Y \), \( \varphi \) is the embedding of an affinoid domain \( Y \rightarrow X \) such that \( Y \) is in fact an affinoid subdomain of \( U \), and \( \varphi(T) \) is an affinoid neighbourhood of \( x \).

**Corollary 1.15.** Let \( X \) be a strictly \( k \)-affinoid space. Being overconvergent constructible in \( X \) is a local property.

**Proof.** First, if \( S \subset X \) is overconvergent constructible, and \( U \) is an affinoid domain of \( X \), then \( S \cap U \) is overconvergent constructible.

On the other hand, let us assume that locally for the Berkovich topology, \( S \) is overconvergent constructible, that is to say, let us assume that for all \( x \in X \) there exists an affinoid neighborhood \( U \) of \( x \) such that \( S \cap U \) is overconvergent constructible. Then according to lemma 1.14 there exists a constructible datum \( (Y, T) \) such that \( Y \) is the embedding of an affinoid domain, \( Y \subset U \), and \( T \) is an affinoid neighborhood of \( y \) such that \( T \) is also overconvergent constructible in \( T \), and then \( \varphi(T) \cap S \) is overconvergent constructible in \( X \) (see proposition 1.13 (2)). But since \( \varphi(T) \) is an affinoid neighbourhood of \( x \), by compactness of \( X \) we conclude that \( S \) is overconvergent constructible.

**1.2. Overconvergent subanalytic subsets.** We will denote by \( \mathbb{B} \) (resp. \( \mathbb{B}_r \) for \( r > 0 \)) the closed disc of radius 1 (resp. \( r \)), and if \( n \) is an integer, \( \mathbb{B}^n \) and \( \mathbb{B}_r^n \) will denote the corresponding closed polydiscs.

More generally, if \( \mathbf{L} = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \) is a polyradius, we will denote by
\[
\mathbb{B}_{\mathbf{L}} = \mathcal{M}(k\{\mathbf{L}^{-1} t \}) = \mathcal{M}(k\{r_1^{-1} t_1, \ldots, r_n^{-1} t_n\})
\]
the polydisc of radius $r$, and $\mathring{B}(r)$ the corresponding open polydisc. When the number $n$ will be clear from the context, we will write $1$ for $(1,\ldots,1) \in \mathbb{R}^n$, and $\emptyset$ or $0$ for $(0,\ldots,0) \in \mathbb{R}^n$. Finally, $\rho > r$ will mean that $\rho_i > r_i$ for $i = 1, \ldots, n$.

**Definition 1.16.** Let $X$ be a strictly $k$-affinoid space. A subset $S \subset X$ is said to be an overconvergent subanalytic subset of $X$ if there exist $n \in \mathbb{N}$, $r > 1$, and $T \subseteq X \times \mathbb{B}^n_r$ a semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n_r \to X$ is the natural projection.

**Lemma 1.17.** Let $f : Y \to X$ be a morphism of strictly $k$-affinoid spaces and $S$ an overconvergent subanalytic subset of $X$. Then $f^{-1}(S)$ is an overconvergent subanalytic subset of $Y$. In particular, if $V$ is a strictly affinoid domain of $X$ and $S$ an overconvergent subanalytic subset of $X$, then $S \cap V$ is an overconvergent subanalytic subset of $V$.

**Proof.** Let $r > 1$ and $T \subseteq X \times \mathbb{B}^n_r$ be a semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. Let us consider the following cartesian diagram:

\[
\begin{array}{c}
Y \times \mathbb{B}^n_r \xrightarrow{f'} X \times \mathbb{B}^n_r \\
\pi' \downarrow \quad \quad \quad \pi \downarrow \\
Y \xrightarrow{f} X
\end{array}
\]

Then $f^{-1}(S) = f'^{-1}(\pi(T \cap (X \times \mathbb{B}^n))) = \pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n)))$. The last inequality holds because $\mathbf{U}$ is a cartesian diagram. Now $\pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n)) = \pi'(f'^{-1}(T) \cap (Y \times \mathbb{B}^n)) = \pi'(T \cap (Y \times \mathbb{B}^n))$ where $T' = f'^{-1}(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Hence $f^{-1}(S) = \pi'(T' \cap (Y \times \mathbb{B}^n))$ is an overconvergent subanalytic subset of $Y$. $\square$

**Lemma 1.18.** Let $\varphi : X \to Y$ be a closed immersion.

1. If $\varphi$ is a morphism of affinoid spaces, and $S$ is a semianalytic subset of $X$, then $\varphi(S)$ is a semianalytic subset of $Y$.

2. Let $S$ be an overconvergent subanalytic subset of $X$, then $\varphi(S)$ is an overconvergent subanalytic subset of $Y$.

**Proof.**

1. Write $Y = \mathcal{M}(A)$ and $X = \mathcal{M}(A/I)$ where $I = (a_1,\ldots,a_m)$ is an ideal of $A$. Then, if $S = \{x \in X \mid |f_i(x)|_0 |g_i(x)|, i = 1\ldots n\}$ with $f_i, g_i \in A/I$, we can find functions $F_i, G_i \in A$ such that $F_i = f_i$ and $G_i = g_i$. In that case one checks that, $\varphi(S) = \{y \in Y \mid |F_i(y)|_0 |G_i(y)|, i = 1\ldots n\} \cap \{y \in Y \mid a_j(y) = 0, j = 1\ldots m\}$, which is indeed semianalytic.

2. Since the problem is local on $Y$, we can assume that $Y$ is affinoid, and hence, $X$ is also affinoid. By definition, this means that there exists $T \subseteq X \times \mathbb{B}^n_r$ for some $r > 1$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. We then consider the following cartesian diagram:

\[
\begin{array}{c}
X \times \mathbb{B}^n_r \xrightarrow{\varphi'} Y \times \mathbb{B}^n_r \\
\pi' \downarrow \quad \quad \quad \pi \downarrow \\
X \xrightarrow{\varphi} Y
\end{array}
\]

But $\varphi'$ is also a closed immersion, so according to (1), $T' = \varphi'(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Then one checks that $\pi(T' \cap (Y \times \mathbb{B}^n)) = \pi(\varphi'(T) \cap (Y \times \mathbb{B}^n)) = \varphi(\pi'(T \cap (X \times \mathbb{B}^n))) = \varphi(S)$. $\square$

**Lemma 1.19.** Let us assume that $S \in \sqrt{|k|^{\times}}$. Then, $k\{S^{-1}X\}$ is a strictly $k$-affinoid algebra (see [Ber90, 2.1.1] and [BCR87, 6.1.5.4]). For the same reasons, if $\rho > \Sigma$ and $T \subseteq X \times \mathbb{B}_\Sigma$ is a semianalytic subset, then $\pi(T \cap (X \times \mathbb{B}_\Sigma))$ is an overconvergent subanalytic subset of $X$. 

Proof. Indeed let $s \in \sqrt{|k^\times|}$ and $r \in \mathbb{R}^n$ such that $s < r$, and $T \subseteq X \times \mathbb{B}_r$ a semianalytic subset of $X \times \mathbb{B}_r$. Let us show that $\pi(T \cap (X \times \mathbb{B}_s))$ is overconvergent subanalytic in the sense of definition 1.16. To avoid complications, we assume that $n = 1$ (but the proof is similar for an arbitrary $n$). Let then $s \in \sqrt{|k^\times|}$ and $r > s$. Up to multiplication by some $\mu \in k^\times$ small enough, we can assume that $s \leq 1$. Since $s \in \sqrt{|k^\times|}$, there exist $\lambda \in k^\times$ and $m \in \mathbb{N}$ such that $s^m = |\lambda|$. Then in

$$\mathbb{B}_{(r, (\frac{s}{\lambda})^m)} = \mathcal{M}(k\{r^{-1}y, \left(\left(\frac{s}{\lambda}\right)^m\right)^{-1}t\})$$

let us consider the Zariski-closed subset defined by $y^m = \lambda t$, i.e. $V(y^m - \lambda t)$. Then, the map:

$$\begin{align*}
\mathbb{B}_r & \to \mathbb{B}_{(r, (\frac{s}{\lambda})^m)} \\
x & \mapsto (x, \frac{r^m}{\lambda})
\end{align*}$$

identifies $\mathbb{B}_r$ with the Zariski closed subset $V(y^m - \lambda t)$ and moreover, since $s \leq 1$

$$\begin{align*}
\mathbb{B}_s & \to \mathbb{B}^2 \\
x & \mapsto (x, \frac{r^m}{\lambda})
\end{align*}$$

identifies $\mathbb{B}_s$ with the Zariski-closed subset of $\mathbb{B}^2$, $V(y^m - \lambda t)$. Taking the fibre product with $X$ we then obtain:

$$
\begin{array}{ccc}
X \times \mathbb{B}_r & \xrightarrow{\pi} & X \times \mathbb{B}_{(r, (\frac{s}{\lambda})^m)} \\
\downarrow & & \downarrow \\
X \times \mathbb{B}_s & \xrightarrow{\pi} & X \times \mathbb{B}^2
\end{array}
$$

Hence if $T \subseteq X \times \mathbb{B}_r$ is semianalytic, $T' := \alpha(T)$ is also semianalytic in $X \times \mathbb{B}_{(r, (\frac{s}{\lambda})^m)}$ and $\alpha(T) \cap (X \times \mathbb{B}^2) = \beta(T \cap (X \times \mathbb{B}_s))$. So $\pi(T \cap (X \times \mathbb{B}_s)) = \pi(T' \cap (X \times \mathbb{B}^2))$ is well overconvergent subanalytic in the sense of definition 1.16. \hfill \Box

We now take a few lines to explain an alternative definition which emphasizes the relevance of $k$-Germ in this context.

**Definition 1.20.** Let $(X, S)$ be a $k$-Germ [Ber93, 3.4]. A subset $T \subseteq S$ is a semianalytic subset of $(X, S)$ if there exists a representative $(W, S)$ of $(X, S)$, with $W$ a $k$-affinoid space, and $R \subseteq W$ a semianalytic subset of $W$ such that $T = S \cap R$.

**Lemma 1.21.** Let $W$ be a neighbourhood of $X \times \mathbb{B}^n$ in $X \times \mathbb{A}^n_k$. Then there exists $r > 1$ such that $W \supseteq X \times \mathbb{B}^n_r$.

**Proof.** If necessary, we can assume that $W \subseteq X \times \mathbb{B}^n_s$ with $s > 1$ and also assume that $W$ is open. In that case, $Z := (X \times \mathbb{B}^n_s) \setminus W$ is a compact subset of $X \times \mathbb{B}^n_s$, and by assumption $Z \cap (X \times \mathbb{B}^n) = \emptyset$, so

$$Z \subseteq \bigcup_{i=1}^{n} \bigcup_{r>1} \{p \in X \times \mathbb{B}^n_s \mid |T_i(p)| > r\}.$$ 

Hence by compactness, there exists $r > 1$ such that $Z \subseteq \bigcup_{i=1}^{n} \{p \in X \times \mathbb{B}^n_s \mid |T_i(p)| > r\}$, which says that $W \supseteq X \times \mathbb{B}^n_r$. \hfill \Box

Hence $S$ is an overconvergent subanalytic subset of $X$ if and only if there exist an integer $n$, and $T$ a semianalytic subset of the $k$-Germ $(X \times \mathbb{A}^n_k, X \times \mathbb{B}^n)$ such that $S = \pi(T)$. 

\hfill \Box
1.3. **Weierstrass preparation.** In this section, \( A \) will be a (ultrametric) complete normed ring i.e. satisfies the inequality \( \|ab\| \leq \|a\|\|b\| \) and \( \|a+b\| \leq \max(\|a\|,\|b\|) \) \([BGR84] 1.2.1.1\).

If \( r > 0 \), on \( A\{r^{-1}X\} \) we will consider the following norm: if \( g = \sum_{n \in \mathbb{N}} a_n X^n \in A\{r^{-1}X\} \) then
\[
\|g\| = \max_{n \geq 0} \|a_n\| r^n.
\]
If \( m \in \mathbb{N} \), we will denote by \( A_m[X] \) the subset of \( A[X] \) made of the polynomials of degree less or equal to \( m \).

**Definition 1.22.** An element \( u \in A \) is a multiplicative unit if \( u \) is invertible and for all \( a \in A \),
\[
\|ua\| = \|u\|\|a\|.
\]

Note that if \( u \) and \( v \) are multiplicative units, so is \( uv \).

**Lemma 1.23.** An element \( u \in A \) is a multiplicative unit if and only if \( u \in A^* \) and \( \|u^{-1}\| = \|u\|^{-1} \).

**Proof.** If \( u \) is a multiplicative unit, \( 1 = \|uu^{-1}\| = \|u\|\|u^{-1}\| \), so \( \|u^{-1}\| = \|u\|^{-1} \).

Conversely let us assume that \( u \) is invertible and that \( \|u^{-1}\| = \|u\|^{-1} \). Let then \( a \in A \). The following holds:
\[
\|a\| = \|u^{-1}(ua)\| \leq \|u^{-1}\|\|ua\| = \|u\|^{-1}\|ua\|.
\]
So \( \|ua\| \geq \|u\|\|a\| \). Since in any case the reverse inequality \( \|ua\| \leq \|u\|\|a\| \) holds, we conclude that \( \|ua\| = \|u\|\|a\| \).

**Remark 1.24.** As a consequence, if \( u \in A \) and \( \|u\| < 1 \), then \((1 + u)\) is a multiplicative unit because
\[
\|1 + u\| = 1 = \sum_{n \geq 0} (-u)^n = \|1 + u^{-1}\|^{-1}
\]

Let us note also that if \( u \) is a multiplicative unit, for all \( x \in M(A) \), \( |u(x)| = \|u\| \). Indeed, the definition of \( M(A) \) implies that
\[
(2) \quad |u(x)| \leq \|u\|,
\]
hence \( |u(x)|\|u^{-1}(x)\| \leq \|u\|\|u^{-1}\| = 1 \). So the inequality \(2\) could not be strict, thus \( |u(x)| = \|u\| \).

**Remark 1.25.** If \( \varphi : A \to B \) is a contractive morphism of normed rings (i.e. \( \|\varphi(a)\| \leq \|a\| \) for all \( a \in A \)), then \( \varphi \) sends multiplicative units to multiplicative units. Indeed we have the sequence of inequalities:
\[
1 = \|\varphi(u)\varphi(u)\| = \|\varphi(u)\|\|\varphi(u)\| \leq \|\varphi(u)\| = \|u\|^{-1} \leq \|u\|^{-1} = 1.
\]

This remark will apply in the following context: when \( A \) is a strictly \( k \)-affinoid algebra and we look at a morphism \( \varphi : A \to B = A\{r^{-1}X\}/I \) with \( I \) any ideal, and \( B \) is equipped with the quotient norm inherited from \( A\{r^{-1}X\} \). In this situation, \( \varphi \) is contractive. This is the case when we consider \( \varphi \) the morphism of a constructible datum \((Y,T) \to X \).

Note that if \( \varphi \) is not contractive, multiplicative units are not necessarily preserved. For instance consider \( A = k\{t\} \) and \( B = k\{2^{-1}x,y\}/(y-x^2) \) that we equip with the residue norm. These \( k \)-affinoid algebras are isomorphic through \( \varphi : t \mapsto x \), and if we choose \( \pi \in k \) such that \( \frac{1}{2} < |\pi| < 1 \), then \( u := 1 + \pi t \) is a multiplicative unit of \( A \), but not \( \varphi(u) \). Note however that if the field \( k \) is stable (for instance in our situation, where \( k \) is a non-archimedean complete field, \( k \) is stable if \( \text{char}(k) = 0 \), or if it is algebraically closed, or a discrete valuation field \([BGR84] 3.6.2\)), one might say that any morphism of reduced affinoid algebra is contractive. Indeed, if \( k \) is stable, and \( A \) is a reduced affinoid algebra, then it is a distinguished affinoid algebra \([BGR84] 6.4.3\), i.e. the supremum seminorm is a residue norm on \( A \). If \( B \) is also reduced, so that the supremum seminorm is an admissible norm on it, and \( \varphi : A \to B \) is a morphism of affinoid algebras, then \( \varphi \) is contractive.
Definition 1.26. Let $r > 0$ be a real number and $s \in \mathbb{N}$. An element $g = \sum_{n \geq 0} g_n X^n$ of $A\{r^{-1}X\}$ is called $X$-distinguished of order $s$ if $g_s$ is a multiplicative unit, $\|g_s\| r^s = \|g\|$ and for all $n > s$, $\|g_n\| r^n < \|g_s\| r^s$. Note that in that case, $g$ is necessarily a non zero element since $g_s \neq 0$.

Remark 1.27. We can extend the previous remark saying that if $\varphi : A \to B$ is a contractive morphism and $g = \sum_{n \in \mathbb{N}} g_n X^n \in A\{r^{-1}X\}$ is $X$-distinguished of order $s$, then $\varphi(g) = \sum_{n \in \mathbb{N}} \varphi(g_n) X^n \in B\{r^{-1}X\}$ and it is an $X$-distinguished element of $B\{r^{-1}X\}$ of order $s$. This applies in particular when $\varphi$ is the morphism of a constructible datum $(Y,T) \mapsto s$.

Lemma 1.28. Let $g = \sum_{m \in \mathbb{N}} g_m X^m \in A\{r^{-1}X\}$ be $X$-distinguished of order $s$.

1. Then for all $q = \sum_{k \in \mathbb{N}} q_k X^k \in A\{r^{-1}X\}$, $\|g q\| = \|g\| \|q\|$.

2. Let us set $g q = \sum_{l \in \mathbb{N}} c_l X^l$, and let us assume that $q \neq 0$. Let us denote by $k_0$ the greatest rank such that $\|q_{k_0}\| r^{k_0} = \|q\|$. Then $\|g q\| = \|c_{s+k_0}\| r^{s+k_0}$ and $\|c_{s+k_0}\| = \|g_s\| \|q_{k_0}\|$.

Proof. First, without any hypothesis, it is true that

$$\|g q\| \leq \|g\| \|q\|. \quad (3)$$

Conversely, by definition,

$$c_{s+k_0} = \sum_{m+k=s+k_0} g_m q_k. \quad (4)$$

Let then $m$ and $k$ be two integers such that $m + k = s + k_0$.

If $k > k_0$, by definition of $k_0$, $\|q_k\| r^k < \|q_{k_0}\| r^{k_0}$. So, using that $g_s$ is a multiplicative unit, we obtain:

$$\|g_m q_k\| r^{s+k_0} = \|g_m q_k\| r^{m+k} \leq \|g_m\| r^m \|q_k\| r^k < \|g_s\| r^s \|q_{k_0}\| r^{k_0} = \|g_s q_{k_0}\| r^{s+k_0}. \quad (5)$$

Thus,

$$\|g_m q_k\| < \|g_s q_{k_0}\|. \quad (6)$$

Thus, (4), (6) and the ultrametric inequality imply that $\|c_{s+k_0}\| = \|g_s q_{k_0}\|$. And since $g_s$ is a multiplicative unit, $\|g_s q_{k_0}\| = \|g_s\| \|q_{k_0}\|$.

Finally we obtain that $\|g q\| \geq \|g_s\| r^s \|q_{k_0}\| r^{k_0} = \|q\| \|g\|$, which with (3) ends the proof. \qed

Proposition 1.29. Weierstrass Division. Let $g \in A\{r^{-1}X\}$ be $X$-distinguished of order $s$.

If $f = \sum_{n \in \mathbb{N}} f_n X^n \in A\{r^{-1}X\}$ there exists an unique couple $(q,R) \in A\{r^{-1}X\} \times A_{s-1}[X]$ such that

$$f = g q + R. \quad (7)$$

Moreover

$$|f| = \max(\|g\| \|q\|, \|R\|). \quad (8)$$

Proof. First, let us show that if a couple $(q,R)$ satisfies (7), then it must satisfy the equality (8). Because of the ultrametric inequality, $\|f\| \leq \max(\|g\| \|q\|, \|R\|)$. For the reverse inequality, we distinguish two cases.

If $\|g q\| \neq \|R\|$, then $\|f\| = \max(\|g q\|, \|R\|) = \max(\|g\| \|q\|, \|R\|)$ according to lemma 1.28.

Otherwise $\|g q\| = \|g\| \|q\| = \|R\|$, and we use again lemma 1.28 and its notations (so $g q = \sum_{l \in \mathbb{N}} c_l X^l$). We get $\|g q\| = \|c_{s+k_0}\| r^{s+k_0}$. Since $R$ is a polynomial of degree $d$ with $d < s$, and since $f = g q + R$, and $d < s + k_0$, the coefficient $f_{s+k_0}$ of $f$ is $c_{s+k_0}$, hence $\|f\| \geq \|c_{s+k_0}\| r^{s+k_0} = \|g\| \|q\|$. \qed
This finally proves that \( \|f\| = \text{max}(\|g\|, \|R\|) \).

From this we can conclude that the couple \((q, R)\) is unique because if \(f = gq' + R'\) is another decomposition, we have \(0 = g(q - q') + (R - R')\) and since \(\|g\| \neq 0\), \(\|q - q'\| = \|R - R'\| = 0\), i.e. \(R = R'\) and \(q = q'\).

Let us now show the existence of such a decomposition. Let us set

\[
g' := \sum_{m=0}^{s} g_m X^m.
\]

In particular, \(\|g\| = \|g'\|\) because \(g\) is \(X\)-distinguished of degree \(s\). Let us set

\[
\kappa := \frac{\max_m(\|g_m\| r_m)}{\|g\| \|s\|}.
\]

Since \(g\) is \(X\)-distinguished of order \(s\), \(\kappa < 1\). Actually, if \(\kappa = 0\) (which would mean that \(g = g'\)), replace \(\kappa\) by \(\frac{1}{\kappa}\). In any case \(\|g - g'\| \leq \kappa \|g\|\) and \(\kappa \in ]0, 1[\).

Next, let \(N \in \mathbb{N}\) and let us set

\[
f' := \sum_{k=0}^{N} f_k X^k.
\]

Let us assume that \(N\) is big enough to fulfill \(\|f - f'\| \leq \kappa \|f\|\). In particular, \(\|f'\| = \|f\|\).

By definition and hypothesis, \(g' \in A[X]\) is of degree \(s\) and possesses an invertible dominant coefficient, which is \(g_s\). Hence in \(A[X]\), one can carry out euclidean division by \(g'\) [Lan02, 4.1.1], which gives \(f' = g'q + R\), with \(R \in A_{s-1}[X]\) and \(q \in A[X]\). We can then apply the norm equality that we have shown in the first part of the proof, (because \(g'\) is also \(X\)-distinguished of order \(s\)): \(\|f'\| = \text{max}(\|g'\|, \|q\|, \|R\|)\). In particular \(\|q\| \leq \frac{\|f'\|}{\|g'\|} = \frac{\|f\|}{\|g\|}\) so that

\[
\|g\| \|q\| \leq \|f\|.
\]

Moreover \(\|R\| \leq \|f'\| = \|f\|\). Thus the following holds:

\[
f = f' + (f - f') = g'q + R + (f - f') = gq + R + (f - f') + (g' - g)q.
\]

By definition of \(g'\) and of \(\kappa\), \(\|g' - g\| \leq \kappa \|g\|\), so

\[
\|(g' - g)q\| \leq \|g\| \|q\| \kappa \leq \|f\|
\]

In addition, by hypothesis,

\[
\|f - f'\| \leq \kappa \|f\|.
\]

Hence if we set

\[
h := f - f' + (g' - g)q = f - (gq + R),
\]

according to \(\text{(9)}\) and \(\text{(10)}\), we obtain that \(\|h\| \leq \kappa \|f\|\).

To sum up, we have found some \(\kappa \in ]0, 1[\) such that

\[
\forall f \in A\{r^{-1}X\}, \text{ there exist } q' \in A\{r^{-1}X\}, R' \in A_{s-1}[X] \text{ such that } \|f - (gq' + R')\| \leq \kappa \|f\|.
\]

This allows us to define by induction two Cauchy sequences \((q^i, R^i) \in A\{r^{-1}X\}\) and \((R^i) \in A_{s-1}[X]\) such that \(\|f - (gq^i + R^i)\| \leq \kappa^i \|f\|\) in the following way.

We start with \((q^0, R^0) = (0, 0)\).

In order to perform the induction step, let \(i > 0\) be given and let us assume that \((q^i, R^i)\) is defined. We set \(h^i := f - (gq^i + R^i)\), which by induction hypothesis fulfills \(\|h^i\| \leq \kappa^i \|f\|\).

According to \(\text{(10)}\), we can define \(q' \in A\{r^{-1}X\}\) and \(R' \in A_{s-1}[X]\) such that \(h^i = gq' + R' + h^i\) with \(\|q'\| \leq \frac{|h^i|}{\|g\|} \leq \kappa^i \frac{\|f\|}{\|g\|}\), and \(\|R'\| \leq \|h^i\| \leq \kappa^i \|f\|\) and \(\|h^i\| \leq \kappa \|h^i\| \leq \kappa^{i+1} \|f\|\). Then we set

\[
q^{i+1} := q^i + q' \text{ and } R^{i+1} := R' + R^i.
\]

Then \(\|f - (gq^{i+1} + R_{i+1})\| = \|h^i - (gq + R)\| = \|h^i\| \leq \kappa^{i+1} \|f\|\). By construction \(\|q^{i+1} - q^i\| = \|q'\| \leq \kappa^i \frac{\|f\|}{\|g\|}\) and \(\|R^{i+1} - R^i\| = \|R'\| \leq \kappa^i \|f\|\), so these sequences are well Cauchy sequences. This ends our induction.
Now, by completeness of $A\{r^{-1} X\}$ and $A_{s-1}[X]$ the sequences $(q^i)$ and $(R^i)$ have a limit, that we denote by $q \in A\{r^{-1} X\}$ and $R \in A_{s-1}[X]$, which satisfy $f = qq + R$ as we wanted. □

**Corollary 1.30. Weierstrass Preparation.** Let $g \in A\{r^{-1} X\}$ be a $X$-distinguished element of order $s$. There exists an unique couple $(w, e) \in A_s[X] \times A\{r^{-1} X\}$ such that $w$ is a unitary polynomial of degree $s$, $e$ is a multiplicative unit of $A\{r^{-1} X\}$, and $g = ew$.

**Proof.** Using Weierstrass division, we can write $X^s = gq + R$ with $\|X^s\| = \max(\|g\|\|q\|, \|R\|)$, and $R \in A[X]_{s-1}$. Let us set

$$w := X^s - R = gq.$$ 

So $w \in A_s[X]$ is a unitary polynomial. Since $g$ is $X$-distinguished of order $s$, according to lemma 1.28 and if we denote by $k_0$ the greatest index such that $\|q_k\| = \|q\|$, and $w = \sum_{i=0}^{s} w_i X^i$, we obtain

$$\|w\| = \|gq\| = \|(gq)s+k_0\| = \|w_{s+k_0}\| = \|s+k_0\|.$$ 

But since $w \in A_s[X]$, necessarily, $s + k_0 = s$ and $k_0 = 0$. Hence, by definition of $k_0$, for all $k > 0$, $\|q_k\| > \|q_k\|^r k$.

The coefficient of degree $s$ in $gq$ being 1, (because $gq = X^s - R$), we have the equality

$$1 = gq_k + g_1 q_{s+1} + \ldots + g_s q_0$$

and since $k_0 = 0$, and $g$ is $X$-distinguished of order $s$, we obtain, with the same reasoning that we have already used in the course of the proof of lemma 1.28 that $\|g_k\| > \|g_{s-k}\|$ for $i = 1 \ldots s$. So $\|g_0\| = 1$, and $g_0 = 1 - (g_{s-1} q_1 + \ldots + g_0 q_s)$, with $\|g_0 q_1 + \ldots + g_0 q_s\| < 1$. Thus, $g_0$ is a multiplicative unit. Moreover, since $g_s$ is also a multiplicative unit, $g_0$ is also a multiplicative unit, and $\|q_0\| = \|g_0\| - 1$. Hence

$$q = g_0 (1 + \frac{q_1}{q_0} X + \ldots + \frac{q_k}{q_0} X^k + \ldots)$$

and since $k_0 = 0$ (so $\|q_i\|^r i < \|q_0\|$ for $i > 0$) and $g_0$ is a multiplicative unit, $\|g_0\|^r i < 1$ for all $i > 0$. Hence

$$1 + \frac{q_1}{q_0} X + \ldots + \frac{q_k}{q_0} X^k + \ldots$$

is a multiplicative unit of $A\{r^{-1} X\}$, and according to (12), $q$ is also a multiplicative unit. So $g = q^{-1}(X^s - R)$, with $q^{-1}$ a multiplicative unit and $X^s - R$ a unitary polynomial of degree $s$. So if we set $e := q^{-1}$, and $w = X^s - R$ we have the expected result: $g = ew$.

As for the uniqueness of this decomposition, if $g = ew$, $e$ and $w$ being as in the statement of the corollary, then $w = X^s + R$ with $R \in A_{s-1}[X]$, and $X^s = w - R = e^{-1} g + (-R)$ which is the Weierstrass division of $X^s$ by $g$. Hence $e$ and $R$ are unique and $w$ too because $w = X^s + R$. □

Let us assume that $\mathcal{A}$ is a $k$-affinoid algebra, let $(r_1, \ldots, r_n)$ be a polyradius, and let us set $A := A\{r_1^{-1} X_1, \ldots, r_n^{-1} X_n\}$. Then if we set $r = r_n, A\{r_1^{-1} X_1, \ldots, r_n^{-1} X_n\} = A\{r^{-1} X\}$, and we can introduce the notion of an element $X$-distinguished, apply Weierstrass theory to them, which corresponds to the classical one, especially if $\mathcal{A} = k$, where we find the classical Tate algebra $k\{r_1^{-1} X_1, \ldots, r_n^{-1} X_n\}$.

Now we state a result that we will need in the next section.

**Lemma 1.31.** Let $\varepsilon > 0$ be given and $\mathbf{r} > 0$ be a polyradius. Let us assume that $\mathcal{A}$ is noetherian, and let us consider

$$f = \sum_{\nu \in \mathbb{N}^n} f_\nu X^\nu \in A\{r^{-1} X\}.$$ 

Then there exists a finite subset $J \subseteq \mathbb{N}^n$, and for all $\nu \in J$, a series $\phi_\nu \in A\{r^{-1} X\}$ satisfying $\|\phi_\nu\| < \varepsilon$, such that

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu)$$

and such that in the $\phi_\nu$’s, not terms $X^\mu$ with $\mu \in J$ appear. Moreover, if we fix some $\mu \in \mathbb{N}^n$, we can assume that $\mu \in J$. 


Proof. Let us denote by $I$ the ideal generated by the family \( \{ f_\nu \}_{\nu \in \mathbb{N}^n} \). Since $A$ is noetherian, there exists $J$ a finite subset of $\mathbb{N}^n$ such that $I = A( f_\nu )_{\nu \in J}$. So for all $\mu \in \mathbb{N}^n \setminus J$ one can find a decomposition $f_\mu = \sum_{\nu \in J} f_\nu a^\mu_\nu$ with $a^\mu_\nu \in A$. In fact, using [BGR84] 3.7.3.1, we can even assume$^2$ that there exists a real constant $C > 0$ such that
\[
\forall \mu \in \mathbb{N}^n, \forall \nu \in J, \| a^\mu_\nu \| \leq C \| f_\mu \|.
\]
Then, let us define for $\nu \in J$
\[
\phi_\nu = \sum_{\mu \in \mathbb{N}^n \setminus J} a^\mu_\nu X^\mu.
\]
Since $\| a^\mu_\nu \| \leq C \| f_\mu \|$, $\phi_\nu \in A(\mathbb{R}^{-1}X)$. Hence, in $A(\mathbb{R}^{-1}X)$, the following equality is satisfied:
\[
f = \sum_{\nu \in J} f_\nu (X^\nu + \sum_{\mu \in \mathbb{N}^n \setminus J} a^\mu_\nu X^\mu) = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu).
\]
Now, if $\nu_0 \notin J$ we set $J' = J \cup \{ \nu_0 \}$, $\phi'_\nu_0 := 0$, and for $\nu \in J$, $\phi'_\nu := \sum_{\mu \in \mathbb{N}^n \setminus J} a^\mu_\nu X^\mu$. One checks that the properties mentioned above still hold, namely $\| a^\mu_\nu \| \leq C \| f_\mu \|$, where the constant $C$ has not been changed, and
\[
f = \sum_{\nu \in J'} f_\nu (X^\nu + \phi'_\nu).
\]
Moreover,
\[
C \| f_\mu \| \| \mu \| \to +\infty \to 0.
\]
so there exists a finite set $K \subset \mathbb{N}^n$ such that
\[
\forall \nu \in J, \forall \mu \in \mathbb{N}^n \setminus K, \| a^\mu_\nu \| < \varepsilon.
\]
Hence if we increase $J$ adding the elements of $K \setminus J$ to $J$, we will manage to obtain a decomposition
\[
f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu)
\]
such that $\| \phi_\nu \| < \varepsilon$ for all $\nu \in J$.

1.4. Equivalence of the two notions. From now on, $A$ will be a $k$-affinoid algebra, and $\mathbb{R} \subset \mathbb{R}_+^n$ a polyradius such that $\mathbb{R} > \frac{1}{2}$ and we will set $A(\mathbb{R}^{-1}X) = A(\mathbb{R}^{-1}X_1, \ldots , \mathbb{R}^{-1}X_n)$. If $\nu \in \mathbb{N}^n$ we will set
\[
X^\nu := X_1^{\nu_1} X_2^{\nu_2} \cdots X_n^{\nu_n}.
\]
If $\nu = (\nu_1, \ldots , \nu_n) \in \mathbb{N}^n$, we will set
\[
|\nu|_\infty = \max_{i=1 \ldots n} \nu_i.
\]
If $\mathbb{R} \in \mathbb{R}_+^n$ and $\nu \in \mathbb{N}^n$, we will set
\[
\mathbb{R}^\nu = \prod_{i=1}^n \mathbb{R}_{\nu_i}^{\nu_i}.
\]
When $\mu, \nu \in \mathbb{N}^n$, we will say that $\mu <_{\text{lex}} \nu$ when $\mu$ is smaller than $\nu$ with respect to the lexicographic order, that is to say when there exists an index $m$ such that $\mu_m < \nu_m$ and $\mu_{m-1} = \nu_{m-1}, \ldots , \mu_1 = \nu_1$. $^2$

$^2$ Indeed, consider
\[
\psi : A^J \to I \quad (a_\nu)_{\nu \in J} \to \sum_{\nu \in J} a_\nu f_\nu
\]
According to [BGR84] 3.7.3.1, $I$ is a complete normed $A$-module, and $\psi$ is a continuous map of normed $A$-modules. Hence there exists a constant $C$ such that $\| \psi(x) \| \leq C \| x \|$ for all $x \in A^J$. 

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We will use the following notation. If \( A \) is a \( k \)-affinoid algebra, \( f = \sum_{n \in \mathbb{N}} a_n T^n \in A[r^{-1}T] \) and \( x \in \mathcal{M}(A) \), we will denote by \( f_x \) the element of \( \mathcal{H}(x)[r^{-1}T] \) defined by \( f_x = \sum_{n \in \mathbb{N}} a_n(x) T^n \).

Since \( A \) is noetherian, we can apply lemma [1.31](#) to it.

**Proposition 1.32.** Let \( f = \sum_{\nu \in \mathbb{N}^n} f_\nu X^\nu \in A[\mathbb{L}^{-1}X] \). There exists a constructible covering of \( X, (X_i, S_i) \xrightarrow{\varphi_i} X, i = 1..N \), such that, if we consider the following cartesian diagrams:

\[
\begin{align*}
(X_i, S_i) &\xrightarrow{\varphi_i} X \\
\pi_i &\xrightarrow{} \pi \\
X_i \times \mathbb{B}_\mathbb{L} &\xrightarrow{(\phi_i)} X \times \mathbb{B}_\mathbb{L}
\end{align*}
\]

and if we denote by \( A_i \) the \( k \)-affinoid algebra of \( X_i \), for all \( i = 1..N \), there exist \( a_i \in A_i \) and a function \( g_i = \sum_{\nu \in \mathbb{N}^n} g_{i,\nu} X^\nu \in A_i[\mathbb{L}^{-1}X] \)

such that

- For all \( i \), the family \( \{ g_{i,\nu} \}_{\nu \in \mathbb{N}^n} \) generates the unit ideal in \( A_i \).
- For all \( i \), \( \varphi_i^*(f)|_{\pi_i} = (a_i g_i)|_{\pi_i} \).

**Proof.** According to lemma [1.31](#) (here we will not use the extra condition \( \|\varphi\| < \varepsilon \) of this lemma), we can find a finite subset \( J \subseteq \mathbb{N}^n \), and for \( \nu \in J \) some \( \phi_\nu \in A[\mathbb{L}^{-1}X] \) such that:

\[
f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu).
\]

Let us fix any \( r > 1 \), and for each \( \nu \in J \), let us consider the constructible datum \( (X_\nu, T_\nu) \xrightarrow{\varphi_\nu} X \) where the affinoid algebra of \( X_\nu \) is \( A[\mathbb{L}^{-1}T_\nu] \mid_{\mu \in J \setminus \{\nu\}} / (f_\mu - t_\mu f_\nu) \), and

\[
T_\nu := \{ x \in X_\nu \mid |f_\nu(x)| \leq |f_\nu(x)| \forall \kappa \in J \setminus \{\nu\} \text{ and } f_\nu(x) \neq 0 \}.
\]

This gives rise to the following cartesian diagrams:

\[
\begin{align*}
(X_\nu, T_\nu) &\xrightarrow{\varphi_\nu} X \\
\pi' &\xrightarrow{} \pi \\
X_\nu \times \mathbb{B}_\mathbb{L} &\xrightarrow{\phi_\nu} X \times \mathbb{B}_\mathbb{L}
\end{align*}
\]

Now,

\[
\varphi_\nu^*(f) = f_\nu (X^\nu + \phi_\nu + \sum_{\mu \in J \setminus \{\nu\}} t_\mu (X^\mu + \phi_\mu)).
\]

For \( \nu \in J \), we set

\[
g_\nu = X^\nu + \phi_\nu + \sum_{\mu \in J \setminus \{\nu\}} t_\mu (X^\mu + \phi_\mu).
\]

Hence,

\[
\varphi_\nu^*(f) = f_\nu g_\nu.
\]

Moreover, if we set \( g_\nu = \sum_{\nu \in \mathbb{N}^n} g_{\nu,\mu} X^\mu \), according lemma [1.31](#) the coefficient of index \( \nu \), \( g_{\nu,\nu} \), is 1, so the coefficients of \( g_\nu \) generate the unit ideal. Finally, let us denote by \( I \) the ideal of \( A \).
generated by the family \((a_\nu)_{\nu \in J}\). By construction, \(I\) also equals the ideal generated by \((a_\nu)_{\nu \in \mathbb{N}^n}\). Then, according to the definition of the \(T'_\nu\) that:

\[
\bigcup_{\nu \in J} \phi_\nu(T'_\nu) = \{x \in X \mid \exists \nu \in J \text{ such that } f_\nu(x) \neq 0\} = X \setminus V(I).
\]

Thus, if we set \(T = V(I)\), then \((X, T) \xrightarrow{id} X\) is an elementary constructible datum and \(id^*(f)|_T = f|_T = 0\).

Now if we regroup the constructible data \((X_\nu, T_\nu) \xrightarrow{\varphi_\nu} X\), for \(\nu \in J\), with \((X, T) \xrightarrow{\varphi} X\), we obtain the desired constructible cover. \(\square\)

**Definition 1.33.** Let \(r \in \mathbb{R}^*_+\) be a polyradius and \(d_1, \ldots, d_{n-1}\) some integers such that

\[
\forall i = 1 \ldots n - 1, \ r_i^{d_i} \leq r_i.
\]

Then

\[
\sigma : \begin{cases} 
X_i \mapsto X_i + X_n^{d_n} \quad \text{for } 1 \leq i \leq n - 1 \\
X_n \mapsto X_n
\end{cases}
\]

is an automorphism of \(A\{\underleftarrow{r}^{-1}X\}\). We will call such an automorphism (as well as the automorphism it induces on the \(k\)-analytic space \(B_\rho\)) a Weierstrass automorphism.

**Remark 1.34.** If \(r > \frac{1}{2}\), we will use that \(\sigma\) induces a 'classical' Weierstrass automorphism of \(A\{X_1, \ldots, X_n\}\), hence of \(X \times B^n\).

Remind the following classical result. If \(f \in k\{X_1, \ldots, X_n\}\), then there exists a Weierstrass automorphism \(\sigma\) of \(k\{X_1, \ldots, X_n\}\) such that \(\sigma(f)\) is \(X_n\)-distinguished. Roughly speaking, the next lemma says that if \(A\) is a \(k\)-affinoid algebra, \(f \in A\{X_1, \ldots, X_n\}\) is overconvergent, then locally on \(X = M(A)\), we can obtain an analogous result.

**Proposition 1.35.** Let \(A\) be a \(k\)-affinoid algebra. Let \(X = M(A)\) and let \(x \in X\). Let \(r \in \mathbb{R}^n\) be a polyradius such that \(r > 1\).

1. Let \(f \in A\{\underleftarrow{r}^{-1}X\}\) such that \(f_x \neq 0\). Then there exist an affinoid neighbourhood \(V = M(B)\) of \(x\), a polyradius \(\rho\) such that \(1 < \rho \leq \underleftarrow{r}\) and \(\sigma\) a Weierstrass automorphism of \(B\{\rho^{-1}X\}\) such that in \(B\{\rho^{-1}X\}\)

\[
\sigma(f) = ag
\]

where \(a \in B\) and \(g \in B\{\rho^{-1}X\}\) is \(X_n\)-distinguished.

2. More generally, let us consider \(m\) functions \(f_1, \ldots, f_m \in A\{\underleftarrow{r}^{-1}X\}\) such that for all \(i\) \((f_i)_x \neq 0\). Then there exist an affinoid neighbourhood \(V = M(B)\) of \(x\), a polyradius \(\rho\) such that \(1 < \rho \leq \underleftarrow{r}\) and \(\sigma\) a Weierstrass automorphism of \(B\{\rho^{-1}X\}\) such that for all \(i\)

\[
\sigma(f_i) = a_i g_i
\]

where \(a_i \in B\) and \(g_i \in B\{\rho^{-1}X\}\) is \(X_n\)-distinguished.

**Proof.** We first prove (1).

**Step 1.** Let us write

\[
f = \sum_{\nu \in \mathbb{N}^n} f_\nu x^\nu \in A\{\underleftarrow{r}^{-1}X\}.
\]

Let us consider \(\mu \in \mathbb{N}^n\) the greatest index with respect to the lexicographic order such that

\[
\max_{\nu \in \mathbb{N}^n} |f_\nu(x)| = |f_\mu(x)|.
\]

Since by assumption \(f_x \neq 0\), it is true that \(f_\mu(x) \neq 0\). According to lemma [181] there exists a finite set \(J \subset \mathbb{N}^n\) such that \(\mu \in J\), and for each \(\nu \in J\) a series \(\phi_\nu \in A\{\underleftarrow{r}^{-1}X\}\) which satisfies

\[
\|\phi_\nu\|_{A\{\underleftarrow{r}^{-1}X\}} < 1
\]

such that

\[
f = \sum_{\nu \in J} f_\nu (x^\nu + \phi_\nu).
\]
Step 2. Let us consider some $\nu \in J$ and let us assume that 
\[ |f_\nu(x)| < |f_\mu(x)|. \]
Then we pick some $a, b \in \mathbb{R}$ such that 
\[ |f_\nu(x)| < a < b < |f_\mu(x)|. \]
Next, let us introduce the affinoid domain of $X$:
\[ W := \{ z \in X \mid |f_\nu(z)| \leq a \text{ and } b \leq |f_\mu(z)| \} = M(B). \]
By construction, $W$ is an affinoid neighbourhood of $x$, $f_\mu$ is invertible in $B$ and 
\[ \| f_\nu \|_B \leq \frac{a}{b} < 1. \]
So we can write:
\[ f_\nu(X^\nu + \phi_\nu) = f_\mu \left( \frac{f_\nu}{f_\mu} (X^\nu + \phi_\nu) \right). \]
Next we consider some polyradius $1 < \rho \leq r$. Clearly 
\[ \rho \rightarrow 1. \]
So we can choose some $\rho$ close enough to 1 such that 
\[ \| f_\nu (X^\nu + \phi_\nu) \|_{B_{\{\rho^{-1}X\}}} < 1. \]
But since 
\[ f_\nu(X^\nu + \phi_\nu) = f_\mu \left( \frac{f_\nu}{f_\mu} (X^\nu + \phi_\nu) \right), \]
if we set 
\[ \phi'_\mu := \phi_\mu + \frac{f_\nu}{f_\mu} (X^\nu + \phi_\nu) \]
we still have that $\| \phi'_\mu \|_{B_{\{\rho^{-1}X\}}} < 1$ and 
\[ f_\mu(X^\mu + \phi_\mu) + f_\nu(X^\nu + \phi_\nu) = f_\mu(X^\mu + \phi'_\mu). \]
Hence we can remove $\nu$ from $J$ and replace $\phi_\mu$ by $\phi'_\mu$. The equality (16) will still be satisfied.
If we repeat this process for each $\nu \in J$ such that $|f_\nu(x)| < |f_\mu(x)|$, we can assume that 
\[ \forall \nu \in J, \ |f_\nu(x)| = |f_\mu(x)|. \]
Thus, according to the definition of $\mu$, this implies that $\mu$ is the greatest index in $J$ with respect to the lexicographic order.
Step 3. Then we set 
\[ d := 1 + \max_{\nu \in J} |\nu|. \]
Since by assumption $1 < r$, if we take $s > 1$ which is close enough to 1, we can assert that 
\[ 1 < (s^{d-1}, s^{d-2}, \ldots, s^d, s) \leq r. \]
We fix a number $s > 1$, which satisfies (17), and we set 
\[ \rho := (s^{d-1}, s^{d-2}, \ldots, s^d, s). \]
In these conditions, it is easy to check that $\rho$ satisfies condition (15) of definition 1.33 (actually, $\rho$ has been defined in (18) to further this goal), so

$$
\sigma(f_\nu(X^\nu + \phi_\nu)) = f_\mu(\sigma(X^\nu) + \sigma(\phi_\nu)) = f_\mu \left( \frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu)) \right).
$$

Since $\|\sigma(\phi_\nu)\|_{B(B^{-1}X)} = \|\phi_\nu\|_{B(B^{-1}X)} < 1$, in fact we can chose $s$ close enough to 1, so that

$$
\|\sigma(X^\nu)\|_{B(B^{-1}X)} = \|X^\nu\|_{B(B^{-1}X)} = \prod_{k=1}^{n} \left( s^{d^{n-k}} \right)^{\nu_k} = s \left( \sum_{k=1}^{n} \nu_k d^{n-k} \right).
$$

Moreover, we remark that $\sum_{k=1}^{n} \nu_k d^{n-k}$ is nothing else but the integer encoded by $\nu$ in base $d$.

Then by construction, for all $\nu \in J \setminus \{ \mu \}$ we have $\nu < \text{lex} \mu$, it follows that for $\nu \in J \setminus \{ \mu \}$

$$
\sum_{k=1}^{n} \nu_k d^{n-k} + 1 \leq \sum_{k=1}^{n} \mu_k d^{n-k}.
$$

As a corollary,

$$
\|\sigma(X^\nu)\|_{B(B^{-1}X)} \leq \|\sigma(X^\mu)\|_{B(B^{-1}X)}.
$$

Let us now consider some $s' \in \mathbb{R}$, such that $1 < s' < s$ and let us consider

$$
V := \{ z \in X \mid \forall \nu \in J \setminus \{ \mu \}, \ |f_\nu(z)| \leq s'|f_\mu(z)| \}.
$$

Then by construction, $V$ is an affinoid neighbourhood of $x$. Let us then replace then $B$ by the affinoid algebra of $V$. Then by construction still, for all $\nu \in J \setminus \{ \mu \}$,

$$
\|\frac{f_\nu}{f_\mu}\|_B \leq s' < s.
$$

So according to (21)

$$
\|\frac{f_\nu}{f_\mu}(X^\nu)\|_{B(B^{-1}X)} < s\|\sigma(X^\nu)\|_{B(B^{-1}X)} \leq \|\sigma(X^\mu)\|_{B(B^{-1}X)}.
$$

So, according to (19), we can assume that

$$
\|\frac{f_\nu}{f_\mu}(\sigma(\phi_\nu))\|_{B(B^{-1}X)} \leq s\|\sigma(\phi_\nu)\|_{B(B^{-1}X)} = s\|\phi_\nu\|_{B(B^{-1}X)} < 1 \leq \|\sigma(X^\nu)\|.
$$

Thus

$$
\sigma(f_\nu(X^\nu + \phi_\nu)) = f_\mu \left( \frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu)) \right)
$$

where

$$
\|\frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu))\|_{B(B^{-1}X)} < \|\sigma(X^\nu)\|_{B(B^{-1}X)}.
$$
Step 4. So

\[ \sigma(f) = f_\mu \left( \sigma(X^\mu) + \sigma(\phi_\mu) + \sum_{\nu \in J \setminus \{\mu\}} \frac{f_\nu}{f_\mu} (\sigma(X^\nu) + \sigma(\phi_\nu)) \right) \]

Hence if we set

\[ \phi = \sigma(\phi_\mu) + \sum_{\nu \in J \setminus \{\mu\}} \frac{f_\nu}{f_\mu} (\sigma(X^\mu) + \sigma(\phi_\nu)) \]

the preceding inequalities imply that \( \|\phi\|_{B(T^{-1}X)} < \|\sigma(X^\mu)\|_{B(T^{-1}X)} \), and by construction

\[ \sigma(f) = f_\mu (\sigma(X^\mu) + \phi) \]

It follows that \( \sigma(X^\mu) + \phi \) is \( X_n \)-distinguished of order \( \sum_{k=1}^{n} \mu_k d^{n-k} \), which ends the proof of (1).

For the proof of (2), it suffices to remark that we could have handled the proof of (1) simultaneously for all the \( f_i \)'s. The main point being that in step 3, we have to take some \( d \) big enough that works for all \( f_i \)'s simultaneously. \( \square \)

**Lemma 1.36.** If \( S \) is an overconvergent constructible subset of \( X \), then \( S \) is overconvergent subanalytic subset of \( X \).

**Proof.** It is sufficient to prove that if \( (Y,T) \to X \) is a constructible datum, then \( \varphi(T) \) is overconvergent subanalytic in \( X \).

We claim that if \( \varphi \) is a constructible datum of complexity \( n \), there exist some polyradii \( s, r \in \mathbb{R}^n \) such that \( s \in \sqrt{|k|}^{-n} \) and \( 0 < s < r \), and some closed immersion \( \iota \):

\[
\begin{array}{c}
Y \xleftarrow{\varphi} X \times \mathbb{B}_r \\
\downarrow{\iota} \\
X
\end{array}
\]

such that \( \iota(T) \subset X \times \mathbb{B}_s \). Indeed this follows from the definition of a constructible datum, and is proved easily by induction on the complexity of the constructible datum \( \varphi \).

Hence \( \varphi(T) = \pi(\iota(T)) \), and since \( \iota(T) \) is a semianalytic subset of \( X \times \mathbb{B}_r \) such that

\[ \iota(T) \subset X \times \mathbb{B}_s \]

it follows that \( \pi(\iota(T)) \) is an overconvergent subanalytic subset of \( X \). \( \square \)

**Theorem 1.37.** Let \( S \subset X \). If \( S \) is overconvergent subanalytic, \( S \) is also overconvergent constructible.

**Proof.** Let \( S \) be an overconvergent subanalytic subset of \( X \). By definition, there exist \( r > 1 \), \( T \) a semianalytic subset of \( X \times \mathbb{B}_r \) such that \( S = \pi(T \cap (X \times \mathbb{B}^n)) \). We then show by induction on \( n \) that \( S \) is overconvergent subanalytic.

If \( n = 0 \), there is nothing to prove since in that case, \( X \) is then a semianalytic subset of \( X \), in particular it is an overconvergent constructible subset.

Let then \( n > 0 \) be given and let us assume that the theorem holds for integers \( < n \). In order to prove the theorem, we can actually assume that \( T \) is a basic semianalytic subset (see remark [1.2]), i.e. that there are \( 2m \) functions \( f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{A}(r^{-1}X) \) and \( \diamond_j \in \{\leq, <\} \) for \( j = 1 \ldots m \) such that

\[
(22) \quad T = \{ x \in X \times \mathbb{B}^n_r \mid |f_j(x)|^{\diamond_j} |g_j(x)| \mid j = 1 \ldots m \}.
\]
Step 1. According to proposition 1.32 we can find a constructible covering $(X_i, S_i) \rightarrow X$ where $X_i = \mathcal{M}(B_i)$ which induces the following cartesian diagram

$$
\begin{array}{ccc}
X_i \times \mathbb{B}_r^n & \xrightarrow{\varphi'_i} & X \times \mathbb{B}_r^n \\
\pi_i \downarrow & & \downarrow \pi \\
X_i \rightarrow & & X
\end{array}
$$

such that for all $j = 1 \ldots m$,

(23) \quad \varphi'_i(F_j) |_{\pi_i^{-1}(S_i)} = (a_j^iF_j^i)|_{\pi_i^{-1}(S_i)}

(24) \quad \varphi'_i(G_j) |_{\pi_i^{-1}(S_i)} = (b_j^iG_j^i)|_{\pi_i^{-1}(S_i)}

where $a_j^i, b_j^i \in B_i$, $F_j^i, G_j^i \in B_i\{\mathbb{A}^{-1}X\}$, and the coefficients of $F_j^i$ (resp. of $G_j^i$) generate the unit ideal in $B_i$. Then for each $i$ we set

$$
T_i := \{ x \in X_i \times \mathbb{B}_r^n \mid |a_j^iF_j^i(x)| \Delta_j |b_j^iG_j^i(x)| \ j = 1 \ldots m \}.
$$

So (23) and (24) imply precisely that

$$
T_i \cap \pi_i^{-1}(S_i) = \varphi'_i(T) \cap \pi_i^{-1}(S_i).
$$

So if we set

$$
U_i := \pi_i(T_i \cap (X_i \times \mathbb{B}_r^n))
$$

then,

$$
\varphi_i(S_i \cap U_i) = \varphi_i(S_i) \cap S
$$

hence since the $\varphi_i(S_i)$ form a covering of $X$,

$$
S = \bigcup_{i=1}^n \varphi(S_i \cap U_i).
$$

So if we prove that $\varphi_i(S_i \cap U_i)$ is overconvergent constructible, we are done.

But actually, since each $S_i$ is overconvergent constructible in $X_i$ (it is even semianalytic, see remark 1.6) if we prove that $U_i$ is an overconvergent constructible subset of $X_i$, then it will follow that $S_i \cap U_i$ is an overconvergent constructible subset of $X_i$, and then according to proposition 1.13 (2), $\varphi_i(S_i \cap U_i)$ will be overconvergent constructible in $X$. Thus, we restrict to prove that $U_i$ is overconvergent constructible in $X_i$.

Step 2. We can then replace $X$ by one of the $X_i$’s and assume that $T$ is defined by

(25) \quad $T = \{ x \in X \times \mathbb{B}_r^n \mid |a_jf_j(x)| \Delta_j |b_jg_j(x)| \ j = 1 \ldots m \}$

with $a_j, b_j \in A$, $f_j, g_j \in A\{\mathbb{A}^{-1}X\}$ such that for all $j$, the coefficients of $f_j$ (resp. of $g_j$) generate the unit ideal of $A$. In this situation we must show that $S$ is overconvergent constructible in $X$ where

$$
S = \pi(T \cap (X \times \mathbb{B}_r^n)).
$$

Let then $x \in X$. The above property of the $f_j$’s and $g_j$’s implies that $(f_j)_x \neq 0$ and $(g_j)_x \neq 0$. So we can apply proposition 1.35 to them. Thus there exist an affinoid neighbourhood $V = \mathcal{M}(B)$ of $x$, some polyradius $\frac{1}{2} < \rho \leq \frac{3}{2}$ and some Weierstrass automorphism $\sigma$ of $B\{\rho^{-1}X\}$ such that for each $j$,

(26) \quad $\sigma(f_j) = \alpha_jF_j$

(27) \quad $\sigma(g_j) = \beta_jG_j$
where $\alpha_j, \beta_j \in B$ and $F_j, G_j$ are $X_n$-distinguished elements of $B\{\rho^{-1}X\}$. Let us then consider the following commutative diagram:

$$
\begin{array}{ccc}
V \times B & \xrightarrow{\pi} & X \times B \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X & & X
\end{array}
$$

where $\iota$ is the embedding of the affinoid domain $V \times B$ in $X \times B$. Then let us set 

$$T' := \iota^{-1}(T) \text{ and } T'' := \sigma^{-1}(\iota^{-1}(T)).$$

First it is clear that

$$S \cap V = \pi(T \cap (X \times B^n))$$

and

$$= \pi(T \cap (V \times B^n))$$

Step 3. So replacing $X$ by $V$, $T$ by $T''$, $\alpha_j\alpha_j$ by $\alpha_j$, $b_j\beta_j$ by $b_j$, $F_j$ by $f_j$ and $G_j$ by $g_j$, we can assume that

$$T = \{ x \in X \times B^n \mid |a_j f_j(x)| |b_j g_j(x)| \} = 1 \ldots m \}.$$

where $a_j, b_j \in A$ and $F_j, G_j \in A\{r^{-1}X\}$ are $X_n$-distinguished in $A\{r^{-1}X\}$. Then, we apply the Weierstrass preparation theorem to $f_j$ and $g_j$. So there exist $e_j, e'_j \in A\{r^{-1}X\}$ some multiplicative units, and $w_j, w'_j$ some unitary polynomials of $A\{r_1^{-1}X_1, \ldots, (r_{n-1})^{-1}X_{n-1}\}[X_n]$ such that

$$f_j = e_j w_j$$

and

$$g_j = e'_j w'_j.$$

So if we set

$$P_j := a_j w_j$$

and

$$Q_j := b_j w'_j,$$

we have that $P_j, Q_j \in A\{r_1^{-1}X_1, \ldots, (r_{n-1})^{-1}X_{n-1}\}[X_n]$. In addition, since $e_j, e'_j$ are multiplicative unit, for all $x \in X \times B^n$, $|e_j(x)| = ||e_j|| \in \sqrt{|k^\times|}$. So we finally obtain that

$$T = \{ x \in X \times B^n \mid |a_j f_j(x)| |b_j g_j(x)| \} = 1 \ldots m \}$$

and

$$= \{ x \in X \times B^n \mid ||e_j|||P_j(x)| |e'_j||Q_j(x)| \} = 1 \ldots m \}.$$

Let us consider the projection along the last coordinate of $B$:

$$X \times B \xrightarrow{\pi_1} X \times B_{(r_1, \ldots, r_{n-1})} \xrightarrow{\pi_2} X$$

according to [Duc03, 2.5] $\pi_1(T \cap (X \times B^n)$ is a semianalytic subset of $X \times B_{(r_1, \ldots, r_{n-1})}$. So by induction hypothesis,

$$\pi_2(\pi_1(T \cap (X \times B^n))$$
is overconvergent constructible in \( X \). Since \( \pi_2 \circ \pi_1 = \pi \), this proves that \( S \) is overconvergent constructible and ends the proof. \( \square \)

We have then proved

**Theorem 1.38.** Let \( X \) be a strictly \( k \)-affinoid space, and \( S \subset X \). Then \( S \) is overconvergent subanalytic if and only if \( S \) is overconvergent constructible.

Hence thanks to this theorem we can use some obvious properties of overconvergent subanalytic (resp. constructible) subsets to prove less obvious results about overconvergent constructible (resp. subanalytic) subsets. For instance we can obtain the non-trivial result concerning overconvergent subanalytic subsets:

**Proposition 1.39.** Let \( X \) be a strictly \( k \)-affinoid space. The class of overconvergent subanalytic subsets of \( X \) is stable under finite boolean combination.\(^3\)

**Proof.** This was proven for overconvergent constructible subsets in proposition 1.13. \( \square \)

In the same way, we obtain a non-obvious stability property for overconvergent constructible subsets:

**Corollary 1.40.** Let \( \underline{r} \in \mathbb{R}^n \) be a polyradius such that \( r > \frac{1}{2} \), and \( S \subset X \times \mathbb{B}_r \) be an overconvergent subanalytic (or constructible) subset of \( X \times \mathbb{B}_r \). Then \( \pi(S \cap (X \times \mathbb{B}_r)) \) is an overconvergent subanalytic (or constructible) subset of \( X \).

**Proof.** If \( S \) is an overconvergent subanalytic subset of \( X \times \mathbb{B}_r \), by definition, there exists \( s > 1 \), an integer \( m \) and \( T \) a semianalytic subset of \( X \times \mathbb{B}_r \times \mathbb{B}_s^m \) such that \( S = \pi_2(T \cap ((X \times \mathbb{B}_r) \times \mathbb{B}_s^m)) \) where \( \pi_2 : (X \times \mathbb{B}_r) \times \mathbb{B}_s^m \rightarrow X \times \mathbb{B}_r \) is the natural projection. Hence \( \pi(S \cap (X \times \mathbb{B}_r)) = \pi_2(T \cap ((X \times \mathbb{B}_r) \times \mathbb{B}_s^m)) = \pi_2(T \cap (X \times \mathbb{B}_s^{m+n})) \) where \( \pi_2 : X \times \mathbb{B}_s^m \rightarrow X \) is the natural projection (so \( \pi_2 = \pi \circ \pi_1 \)). Hence \( S \) is an overconvergent subanalytic subset of \( X \). \( \square \)

1.5. **From a global to a local definition.**

**Definition 1.41.** Let \( \mathcal{P} \) be the data, for each \( k \)-affinoid space \( X \), of a family \( \mathcal{P}_X \) of subsets of \( X \). If \( S \) is a subset of a \( k \)-affinoid space \( X \), we will say that \( S \) satisfies the property \( \mathcal{P} \) if \( S \in \mathcal{P}_X \).

We will say that

- The property \( \mathcal{P} \) is a \( G \)-local property if for all \( k \)-affinoid spaces \( X \) and any subset \( S \) of \( X \), \( S \cap X_i \) satisfies the property \( \mathcal{P} \) if and only if for all finite affinoid coverings \( \{X_i\} \) of \( X \), \( S \cap X_i \) satisfies the property \( \mathcal{P} \) relatively to \( X_i \) (i.e. \( S \cap X_i \in \mathcal{P}_{X_i} \)).

- The property \( \mathcal{P} \) is a local property if for all affinoid spaces \( X \) and any subset \( S \) of \( X \), \( S \in \mathcal{P}_X \) if and only if for all \( x \in X \), there exists an affinoid neighbourhood \( U \) of \( x \) such that \( S \cap U \in \mathcal{P}_U \).

Note that using the compactness of affinoid spaces, saying that \( \mathcal{P} \) is a local property is equivalent to requiring that for all \( k \)-affinoid spaces \( X \) and any \( S \subset X \), \( S \) satisfies \( \mathcal{P} \) if and only if for any finite affinoid covering \( \{X_i\} \) of \( X \) such that \( \{X_i\} \) is also a covering of \( X \), then \( S \cap X_i \in \mathcal{P}_{X_i} \). As a consequence, if \( \mathcal{P} \) is a \( G \)-local property, then it is also a local property.

**Example 1.42.** A consequence of Kiehl’s theorem [BGR83] 9.4.3 is that the class of Zariski-closed subsets of affinoid spaces defines a class which is \( G \)-local.

**Definition 1.43.** Let \( X \) be a good \( k \)-analytic space. A wide covering of \( X \) is a covering \( \{X_i\} \) such that the \( X_i \)'s are affinoid domains of \( X \) and \( \{X_i\} \) is also a covering of \( X \).

**Proposition 1.44.** Let \( X \) be a strictly \( k \)-affinoid space, and \( S \) a subset of \( X \). The following assertions are equivalent:

1. \( S \) is an overconvergent subanalytic subset of \( X \).

\(^3\)In fact, the only non-trivial result is that overconvergent subanalytic subsets are stable under taking complementary.
(2) For all wide covering \( \{X_i\} \) of \( X \), \( X_i \cap S \) is an overconvergent subanalytic subset of \( X_i \).

(3) There exists a wide covering \( \{X_i\} \) of \( X \) such that \( X_i \cap S \) is overconvergent subanalytic in \( X_i \) for all \( i \).

(4) For all \( x \in X \) there exists an affinoid neighbourhood \( V \) of \( x \) such that \( V \cap S \) is overconvergent subanalytic in \( V \).

(5) For all \( x \in X \) there exist \( V_1, \ldots, V_n \) some affinoid domains of \( X \) such that \( V_1 \cup \ldots \cup V_n \) is a neighbourhood of \( x \) and such that \( V_i \cap S \) is overconvergent subanalytic in \( V_i \) for all \( i \).

The property (4) implies that the class of overconvergent subanalytic subsets is local in the sense of definition 1.41.

**Proof.** (1) \( \Rightarrow \) (2) is obvious and is a consequence of lemma 1.17.

(2) \( \Rightarrow \) (3) is clear.

(3) \( \Leftrightarrow \) (4) follows from the compactness of \( X \).

(4) \( \Rightarrow \) (1) follows from the analogous statement for overconvergent constructible subsets (corollary 1.15) and theorem 1.38.

\[ \square \]

**Definition 1.45.** Let \( X \) be a strictly \( k \)-analytic space. A subset \( S \subset X \) is called overconvergent subanalytic if for all \( x \in X \) there exists \( V \) a strictly affinoid neighbourhood of \( x \) such that \( S \cap V \) is overconvergent subanalytic in \( V \) (according to definition 1.10).

According to the last proposition, when \( X \) is a \( k \)-affinoid space, this definition is compatible with the previous one (definition 1.10).

**Definition 1.46.** Let \( X \) be a good strictly \( k \)-analytic space. A subset \( S \) of \( X \) is called locally semianalytic if for all \( x \in X \) there exists \( V \) some strictly affinoid neighbourhood of \( x \) such that \( V \cap S \) is semianalytic in \( V \).

**Corollary 1.47.** Let \( X \) be a good strictly \( k \)-analytic space. The class of locally semianalytic subsets of \( X \) is contained in the class of overconvergent constructible subsets of \( X \).

**Corollary 1.48.** Let \( X \) be a strictly \( k \)-affinoid space and let \( S \subset X \) be a subset of \( X \). Then \( S \) is an overconvergent subanalytic subset of \( X \) if and only if there exist \( r > 1 \), an integer \( n \), and \( T \subset X \times \mathbb{B}_r^n \) a locally semianalytic subset, such that \( S = \pi(T \cap (X \times \mathbb{B}_r^n)) \).

**Proof.** The first implication is true because a semianalytic subset of \( X \times \mathbb{B}_r^n \) is in particular a locally semianalytic subset of \( X \times \mathbb{B}_r^n \).

Conversely, if \( S = \pi(T \cap (X \times \mathbb{B}_r^n)) \) where \( T \) is a locally semianalytic subset of \( X \times \mathbb{B}_r^n \), then according to corollary 1.47 \( T \) is overconvergent subanalytic in \( X \times \mathbb{B}_r^n \), so according to corollary 1.40 \( \pi(T \cap (X \times \mathbb{B}_r^n)) \) is also overconvergent subanalytic.

\[ \square \]

**Lemma 1.49.** Let \( \varphi : X \to Y \) be a morphism of good strictly \( k \)-analytic spaces, and \( S \subset Y \) be a locally semianalytic subset. Then \( \varphi^{-1}(S) \) is locally semianalytic.

**Proof.** Let \( x \in X \), \( y = \varphi(x) \). There exists an affinoid neighbourhood \( V \) of \( y \) such that \( V \cap S \) is semianalytic in \( V \). Let \( W \) be an affinoid neighbourhood of \( x \) in \( \varphi^{-1}(V) \). Then \( W \cap \varphi^{-1}(S) \) is semianalytic in \( W \).

\[ \square \]

**Theorem 1.50.** Let \( \varphi : Y \to X \) be a morphism of strictly \( k \)-affinoid spaces, and \( U \) an affinoid domain of \( Y \) such that \( U \subset \text{Int}(Y/X) \). If \( S \) is an overconvergent subanalytic subset of \( Y \) then \( \varphi(U \cap S) \) is an overconvergent subanalytic subset of \( X \).

**Proof.** According to [Ber90, Prop 2.5.9] there exist \( t > 0 \) and \( A(\mathbb{Z}^{-1}t) \to B \) an admissible epimorphism which hence identifies \( Y \) with a Zariski closed subset of \( X \times \mathbb{B}_r \), such that under this identification, \( U \subset X \times \mathbb{B}_r \). We can assume that \( t \in \sqrt{|k|^n} \). If we denote by \( \Gamma(\varphi) \) the graph of \( \varphi \), this induces a Zariski closed embedding of \( Y \simeq \Gamma(\varphi) \supset X \times \mathbb{B}_r \). Now since \( S \) is an overconvergent subanalytic subset of \( Y \), according to lemma 1.18 \( i(S) \) is an overconvergent subanalytic subset of \( X \times \mathbb{B}_r \). Finally, \( U \) is a semianalytic subset of \( Y \) (because of Gerritzen-Grauert theorem), so \( i(U) \) is also semianalytic in \( X \times \mathbb{B}_r \), and by assumption, \( i(U) \subset X \times \mathbb{B}_r \).
so \( i(U \cap S) \subseteq X \times \mathbb{B}_k \) is an overconvergent constructible subset of \( X \times \mathbb{B}_k \), and according to corollary 1.40 \( \pi(i(U \cap S)) \) is an overconvergent subanalytic subset of \( X \). But this set is precisely \( \varphi(U \cap S) \). \( \square \)

**Proposition 1.51.** Let \( \varphi : Y \to X \) be a morphism of good strictly \( k \)-analytic spaces, \( S \) an overconvergent subanalytic subset of \( Y \) such that \( \overline{S} \to |X| \) is proper and \( |S| \subseteq \text{Int}(Y/X) \). Then \( \varphi(S) \) is an overconvergent subanalytic subset of \( X \).

**Proof.** If \( X' \) is an affinoid domain of \( X \) and if we consider the cartesian diagram :

\[
\begin{array}{ccc}
S & \subseteq & Y \\
\psi' & \phi & \to \\
Y' & \subseteq & \psi' \\
\end{array}
\]

then \( \psi'^{-1}(S) \) is closed in \( Y' \) and contains \( \psi'^{-1}(S) = S' \) so \( S' \subseteq \overline{S} \subseteq \psi'^{-1}(S) \), and since properness is stable under base change, \( \psi'^{-1}(\overline{S}) \to |X'| \) is proper, and since \( \overline{S} \) is closed, \( \overline{S} \to |X'| \) is proper. Moreover, \( \psi'^{-1}(\text{Int}(Y/X)) \subseteq \text{Int}(Y'/X') \) (\( \text{[Ber90] 3.1.3 (iii)} \)) so \( \overline{S} \subseteq \psi'^{-1}(S) \subseteq \text{Int}(Y'/X') \). So \( S' \) and \( \varphi' \) fulfil the hypotheses of the proposition. Hence, since the property we want to check is local on \( X \), we can assume that \( X \) is a \( k \)-affinoid space, hence that \( \overline{S} \) is compact.

Now for every \( y \in \overline{S} \) we can find an affinoid neighbourhood \( U \) such that \( U \subseteq \text{Int}(Y/X) \), because \( \text{Int}(Y/X) \) is open. Then, \( \varphi(U \cap S) \) is an overconvergent subanalytic subset of \( X \) according to theorem 1.50. Since \( \overline{S} \) is compact we can extract from this a finite covering of \( \overline{S} \), which finishes to prove that \( \varphi(S) \) is overconvergent subanalytic. \( \square \)

**Corollary 1.52.** Let \( \varphi : Y \to X \) be a proper morphism of good strictly \( k \)-analytic spaces. \( \text{[Ber90] p.50} \) that it means that \( |\varphi| : |Y| \to |X| \) is proper and that \( \partial(Y/X) = \emptyset \). Let \( S \) be an overconvergent subanalytic subset of \( Y \). Then \( \varphi(S) \) is an overconvergent subanalytic subset of \( X \).

**Definition 1.53.** A morphism \( \varphi : Y \to X \) of good \( k \)-analytic spaces is locally extendible without boundary if, for all \( y \in Y \), there exists an affinoid neighbourhood \( U \) of \( y \), \( Y' \) a \( k \)-affinoid space that contains \( U \) as an affinoid domain, and \( \psi : Y' \to X \) that extends \( \varphi|_U \), such that \( U \subseteq \text{Int}(Y'/X) \).

Remark that using again \( \text{[Ber90] 3.1.3 (iii)} \), this property is stable under base change.

**Proposition 1.54.** Let \( \varphi : Y \to X \) be a compact morphism of good strictly \( k \)-analytic spaces (i.e. \( |\varphi| : |Y| \to |X| \) is proper \( \text{[Ber90] p.50} \) which is locally extendible without boundary. Then \( \varphi(Y) \) is an overconvergent subanalytic subset of \( X \).

**Proof.** We can assume that \( X \) is a \( k \)-affinoid space, so \( Y \) is compact. Then for all \( y \in Y \) we can find an affinoid neighbourhood \( U \) of \( y \) and \( Y' \) a \( k \)-affinoid space that contains \( U \), and \( \psi : Y' \to X \) that extends \( \varphi|_U \), such that \( U \subseteq \text{Int}(Y'/X) \). Then, according to theorem 1.50 \( \varphi(U) \) is an overconvergent subanalytic subset of \( X \) (take \( S = Y' \)). Hence by compactness of \( Y \), \( \varphi(Y) \) is overconvergent subanalytic. \( \square \)

1.6. The non strict case. In this section, \( k \) will be an arbitrary non-Archimedean field (possibly trivially valued).

One of the advantages of Berkovich’s approach is the possibility to use arbitrary \( \lambda \in \mathbb{R}_+ \) to define inequalities. It is then natural to give the following definitions:

**Definition 1.55.** Let \( \mathcal{A} \) be a \( k \)-affinoid algebra, and let us set \( X = \mathcal{M}(\mathcal{A}) \).

1. A subset \( S \subseteq X \) is called non-strictly semianalytic if it is a boolean combination of subsets

\[
\{ x \in X \mid |f(x)| \leq \lambda |g(x)| \}
\]

where \( f, g \in \mathcal{A} \) and \( \lambda \in \mathbb{R}_+ \).
(2) A subset $S \subseteq X$ is called non-strictly overconvergent subanalytic if there exist an integer $n \in \mathbb{N}$, a real number $r > 1$, a non-strictly semianalytic set $T \subseteq X \times \mathbb{B}^n$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$, where $\pi : X \times \mathbb{B}^n \to X$ is the first projection.

**Remark 1.56.** Let $X$ be a strictly $k$-affinoid space and let $S \subseteq X$. The following implication holds:

$$S \text{ is semianalytic } \Rightarrow S \text{ is non-strictly semianalytic.}$$

However, if $\sqrt{|k|} \subseteq \mathbb{R}^*_+$, the converse implication is false. Indeed, let $r \in ]0,1[$ such that $r \notin \sqrt{|k|}$, let $X = \mathbb{B}^1 = \mathcal{M}(k[t])$ and let $S = \{ x \in \mathbb{B} \mid |t(x)| = r \}$. By definition, $S$ is a non-strictly semianalytic set of $\mathbb{B}^1$, but we claim that it is not semianalytic. Indeed, we will see in [2.13] that semianalytic sets are entirely determined by their rigid points, that is to say, if $S_1$ and $S_2$ are semianalytic subsets of $X$, then, $S_1 = S_2$ if and only if $S_1 \cap X_{rig} = S_2 \cap X_{rig}$. Since in our example, $S \cap X_{rig} = \emptyset$, if $S$ was semianalytic, it would then be empty, but $S$ is not empty. Actually $S = \{ \eta \}$.

**Definition 1.57.** Let $X$ be a $k$-affinoid space. Let $(X, S)$ be a $k$-germ, $f, g \in \mathcal{A}$, $0 < s < r$ where $r, s \in \mathbb{R}$, and

$$Y = \mathcal{M}(\mathcal{A}(r^{-1}t)/(f - tg)) \rightarrow X$$

and $T = \varphi^{-1}(S) \cap R \cap \{ y \in Y \mid |f(y)| \leq s|g(y)| \neq 0 \}$ where $R$ is a non-strictly semianalytic subset of $Y$. Then we say that $(Y, T) \rightarrow (X, R)$ is a non-strictly elementary constructible datum.

The only difference with definition [1.8] is that we do not assume any more that $s \in \sqrt{|k|}$, and that $R$ is allowed to be non-strictly semianalytic, that is to say defined with inequalities involving some arbitrary $\lambda \in \mathbb{R}$.

Then we mimic definition [1.5] and say that a non-strictly constructible datum $(Y, T) \rightarrow (X, S)$ is a composite $\varphi = \varphi_1 \circ \ldots \circ \varphi_n$ where each $\varphi_i$ is a non-strictly elementary constructible datum. Finally, if $(X, S_i) \rightarrow (X, i = 1 \ldots n)$ are non-strictly constructible data, we say that $S := \sqcup_{i=1}^n \varphi_i(S_i)$ is a non-strictly overconvergent constructible set.

We claim that all results we have proven in this section for overconvergent subanalytic (resp. constructible) sets remain valid for non-strictly overconvergent subanalytic (resp. constructible) sets. For instance:

**Theorem 1.58.** Let $X$ be a $k$-affinoid space. $S \subseteq X$ is non-strictly overconvergent subanalytic if and only if it is non-strictly overconvergent constructible.

In this context, we want to stress out that for instance propositions [1.44] [1.51] also remain true.

2. Study of various classes

2.1. Many families. In this section $X = \mathcal{M}(\mathcal{A})$ will be a strictly $k$-affinoid space. The aim of this section is to first recall the definitions of the various classes of rigid/locally/strongly/D-semianalytic/subanalytic subsets of $X$ that are defined in [Sch94a].

We now give the following definitions. A subset $S \subseteq X$ is called:

(a) semianalytic if it is a boolean combination of subsets of the form $\{ x \in X \mid |f(x)| \leq \epsilon \}$, with $f, g \in \mathcal{A}$.

(b) Locally semianalytic, if for all $x \in X$ there exists an affinoid neighbourhood $V$ of $x$ such that $S \cap V$ is semianalytic in $V$.

(c) rigid-semianalytic if there is a finite affinoid covering $\{ X_i \}_{i=1}^n$ such that $S \cap X_i$ is semianalytic in $X_i$ for all $i$.

(d) Overconvergent subanalytic has been defined in definition [1.8]. As we proved in the previous section, this corresponds also to overconvergent constructible subsets. Moreover, our definition of overconvergent subanalytic subset is the same as the definition of globally strongly subanalytic of [Sch94a] 1.3.8.1. In [Sch94a] it is proven and it is correct that this is equivalent to the class of globally strongly $D$-semianalytic subsets [Sch94a] 1.3.2].
(e) $G$-overconvergent subanalytic if there exists a finite affinoid covering $\{X_i\}$ of $X$ such that $S \cap X_i$ is overconvergent constructible in $X_i$ for all $i$. This corresponds to the notion of strongly $D$-semianalytic subset in [Sch94a, 1.3.7.1].

(f) Strongly subanalytic if there exist an integer $n$, $r > 1$, a subset $T \subseteq X \times \mathbb{B}^r$ which is rigid-semianalytic, such that $S = \pi(T \cap (X \times \mathbb{B}^r))$. This is definition [Sch94a, 1.3.8.1], and we will give an equivalent definition in proposition 2.8.

(g) Locally strongly subanalytic if there exists a finite affinoid covering $\{X_i\}$ of $X$ such that $S \cap X_i$ is strongly subanalytic in $X_i$ for all $i$. This is definition [Sch94a, 1.3.8.2].

In [Sch94a] it is stated that (d),(e),(f) and (g) are equivalent (equivalence of (e), (f), (g) is stated in [Sch94a, Prop 4.2], and the equivalence of (d) and (f) is stated in [Sch94a, Th 5.2]). These results rest on [Sch94a, lemma 4.1] which is false, and we will show indeed that (d), (e) and (f) correspond in general to three different classes. More precisely the aim of this section is to show that these classes satisfy the following relations:

**Figure 1. The hierarchy**

```
Locally strongly subanalytic ⊇ strongly subanalytic ⊇ subanalytic
```

In this figure, Class A ⊇ Class B, means that the class A properly contains the class B, and Class A ⊈ Class B means that the Class A does not contain the class B.

In this diagram, all the inclusions are clear from the definitions, except the inclusion 7 which states that the class of overconvergent subanalytic subsets contains the class of locally semianalytic subsets. But this is precisely the content of corollary 1.44 In comparison with what was stated in [Sch94a], the most striking inequality is probably ⊈ 5 which asserts that rigid-semianalytic subsets are not necessarily overconvergent subanalytic subsets whereas according to [Sch94a, Th 5.2], they should be overconvergent subanalytic. In other words, when you project overconvergent semianalytic subsets, you obtain a class which is not $G$-local (but however local for the Berkovich topology).

In this section we will show that the inclusions (1)-(8) in figure 1 are all proper in general (in the next section we will explain that if $X$ is regular of dimension 2, overconvergent subanalytic subsets correspond to locally semianalytic subsets). We do not know if the inclusion on the left

```
locally strongly subanalytic ⊇ strongly subanalytic
```

is proper.

2.2. Rigid-semianalytic subsets are not necessarily overconvergent subanalytic. Here we prove the inequality ⊈ 5.

**Lemma 2.1.** Let $\eta \in X$ such that $\mathcal{O}_{X,\eta}$ is a field, $S \subset X$ a semianalytic subset. If $\eta \in \overline{S}$, then $\check{S}$ is non empty.

**Proof.** Since

```
\bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{n} S_i
```

we can assume that $S$ is a basic semianalytic subset, i.e. is of the form:

$$S = \left( \bigcap_{i=1}^{m} \{ x \in X \mid |f_i(x)| \leq |g_i(x)| \} \right) \cap \left( \bigcap_{j=1}^{n} \{ x \in X \mid |F_j(x)| < |G_j(x)| \} \right).$$

We use the following decomposition

$$\{ x \in X \mid |f_i(x)| \leq |g_i(x)| \} = \{ x \in X \mid f_i(x) = g_i(x) = 0 \} \cup \{ x \in X \mid |f_i(x)| \leq |g_i(x)| \neq 0 \}$$

and using again that the adherence is stable under finite union, we can assume that $\eta \in \overline{S}$ and that $S$ is of the form:

$$S = \bigcap_{i=1}^{l} \{ x \in X \mid h_i(x) = 0 \} \cap \bigcap_{j=1}^{m} \{ x \in X \mid |f_j(x)| \leq |g_j(x)| \neq 0 \} \cap \bigcap_{k=1}^{n} \{ x \in X \mid |F_k(x)| < |G_k(x)| \}.$$  

Since the subsets $\{ x \in X \mid h_i(x) = 0 \}$ are closed, contain $S$ and $\eta \in \overline{S}$, it follows that $h_i(\eta) = 0$.

Lemma 2.2. Let $\eta \in X$ and let us assume that $\mathcal{O}_{X,\eta}$ is a field. Let $(Y,T) \xrightarrow{\varphi} (X,S)$ be an elementary constructible datum with $Y = \mathcal{M}(\mathcal{A}(r^{-1}t)/(f-tg))$ where $T = \varphi^{-1}(S) \cap \{ y \in R \mid |f(y)| \leq s|g(y)| \neq 0 \}$ with $0 < s < r$, $s \in \sqrt{|k^e|}$ and $R$ a semianalytic subset of $Y$. Let us assume that $\eta \in \varphi(T)$. Then

(a) $g(\eta) \neq 0$.
(b) $|f(\eta)| \leq s|g(\eta)|$.
(c) There exists a neighbourhood $U$ of $\eta$ such that $\varphi^{-1}(U) \xrightarrow{\varphi|_{\varphi^{-1}(U)}} U$ is an isomorphism. If we denote by $\eta'$ the only point of $\varphi^{-1}(U)$ such that $\varphi(\eta') = \eta$, then $\eta' \in \overline{T}$ and $\mathcal{O}_{Y,\eta'}$ is a field.

Proof.

(a) If we had $g(\eta) = 0$, since $\mathcal{O}_{X,\eta}$ is a field, there would exist an affinoid neighbourhood of $\eta, V$, such that $g_{|V} = 0$. Since for $p \in T$, $g(\varphi(p)) \neq 0$ we should have $\varphi(T) \cap V = \emptyset$ which is impossible since $\eta \in \varphi(T)$.

(b) The subset $\{ x \in X \mid |f(x)| \leq s|g(x)| \}$ is a closed subset of $X$ which contains $\varphi(T)$, hence by assumption also $\eta$.

(c) If we set $U = \{ y \in Y \mid g(y) \neq 0 \}$, $\varphi|_{U}$ identifies through an isomorphism $U$ with $\varphi(U) = \{ x \in X \mid |f(x)| \leq r|g(x)| \neq 0 \}$ which is an analytic domain of $X$, and a neighbourhood of $\eta$ according to the two preceding points. So $\eta \in \varphi(U)$, let us say $\eta = \varphi(\eta')$ with $\eta' \in U$. Now, $\mathcal{O}_{Y,\eta'} \simeq \mathcal{O}_{X,\eta}$ is a field and $\eta' \in \overline{T}$.

Corollary 2.3. Let $\eta \in X$ such that $\mathcal{O}_{X,\eta}$ is a field, and let $U$ be an overconvergent subanalytic subset of $X$. If $\eta \in U^\circ$, then $U \neq \emptyset$.

Proof. First, according to theorem 1.37 we can assume that $U$ is an overconvergent constructible subset. Then, using similar arguments as in the beginning of lemma 2.1, we can assume that $U = \varphi(T)$ where $(Y,T) \xrightarrow{\varphi} X$ is a constructible datum. Hence $T$ is a semianalytic subset of $Y$. A repeated use of lemma 2.2 allows us to say that there exists an open neighbourhood $U$ of $\eta$, such that $\varphi^{-1}(U) \xrightarrow{\varphi|_{\varphi^{-1}(U)}} U$ is an isomorphism. Thanks to lemma 2.2 again, we can introduce
\( \eta' \), the only point of \( \varphi^{-1}(U) \) such that \( \varphi(\eta') = \eta \), and assert that \( O_{Y, \eta'} \) is a field and that \( \eta' \in T \).

Now if we consider \( V \) a strictly affinoid neighbourhood of \( \eta' \) contained in \( \varphi^{-1}(U) \), it is true that \( \eta' \in T \cap V \) (the adherence is here taken in \( V \)). Now, \( T \cap V \) is a semianalytic subset of \( V \), so according to lemma \([\text{2.1]}\) \( T \cap V \) has non empty interior in \( V \). We can then deduce that \( T \) has non empty interior in \( X \) whence \( \varphi(T) \) has also non-empty interior. \( \Box \)

Let \( f = \sum_{n \in \mathbb{N}} a_n T^n \) be a series and \( r \in \mathbb{R}_+^* \). We will say that the radius of convergence of \( f \) is exactly \( r \) when \( |a_n|r^n \to 0 \) and \( r \) is maximum for this property.

**Proposition 2.4.** Let \( X = \mathbb{B}^2 = \mathcal{M}(k\{T_1, T_2\}) \) be the closed bidisc, and let \( 0 < r < 1 \) with \( r \in |k^+| \), say \( r = |e| \) for some \( e \in k \), and let \( f \in k\{r^{-1}u\} \) be some function whose radius of convergence is exactly \( r \), and \( ||f|| < 1 \). We then define
\[
S = \{ x \in X \mid |T_1(x)| < r \text{ and } T_2(x) = f(T_1(x)) \}.
\]

Then \( S \) is rigid-semianalytic but not overconvergent subanalytic. As a consequence, the class of overconvergent subanalytic subsets is not \( G \)-local.

**Proof.** In more concrete terms, \( S \) is the set of points of the curve whose equation is \( T_2 = f(T_1) \), restricted to the subset \( \{|T_1| < r\} \). Let us consider
\[
\mathbb{B} \xrightarrow{\psi} X \quad u \mapsto (eu, f(\varepsilon u))
\]
and let us set \( \eta = \psi(g) \) where \( g \) is the Gauss point of \( \mathbb{B} \). Then \( S \subseteq \psi(\mathbb{B}) \) and \( \eta \in T \). According to [\text{Duc}] lemma 2.21, \( O_{X, \eta} \) is a field. Furthermore \( S = \emptyset \) because \( S \subseteq Z := \{ x \in B_{(1,1)} \mid T_2(x) = f(T_1(x)) \} \), which is a Zariski closed subset of dimension 1 of \( B_{(1,1)} \), which itself is of pure dimension 2, so \( Z \) is nowhere dense in \( B_{(1,1)} \) [\text{Ber90} 2.3.7]. Hence according to corollary [\text{2.3}] \( S \) is not overconvergent subanalytic. However, if we consider the covering of \( X \) given by \( X_1 = \{ x \in X \mid |T_1(x)| \leq r \} \), \( X_2 = \{ x \in X \mid |T_1(x)| \geq r \} \), then \( S \cap X_1 \) is indeed semianalytic in \( X_1 \) and \( S \cap X_2 = \emptyset \), so \( S \) is rigid-semianalytic.

Now since the class of overconvergent subanalytic subsets contains the class of semianalytic subsets, if the class of overconvergent subanalytic subsets was \( G \)-local, it should contain the class of rigid-semianalytic subsets, but we have shown that this is false. Hence the class of overconvergent subanalytic subsets is not \( G \)-local. \( \Box \)

**Remark 2.5.** Actually, this example gives directly a counterexample to [\text{Sch94a} lemma 4.1] which in our feeling is the source of mistakes in [\text{Sch94a}].

As a corollary of this we obtain:

**Proposition 2.6.** Let \( 0 < s < r < 1 \) with \( s \in \sqrt{|k^+|} \), \( f \in k\{r^{-1}u\} \) whose radius of convergence is exactly \( r \) such that \( ||f|| < 1 \), and let us set \( B^2 = \mathcal{M}(k\{T_1, T_2\}) \). Define :
\[
S = \{ x \in B^2 \mid |T_1(x)| \leq s \text{ and } T_2(x) = f(T_1(x)) \}.
\]

Then \( S \) is a locally semianalytic subset of \( B^2 \) which is not a semianalytic subset of \( B^2 \).

**Proof.** If \( S \) was a semianalytic subset of \( B^2 \), we could find \( T \subseteq S \) which contains infinitely many points of \( S \) such that \( T \) is a basic semianalytic subset, and even, a finite intersection of sets of the form \( \{ x \in B^2 \mid |g_1(x)| < |g_2(x)| \} \), \( \{ x \in B^2 \mid |g_1(x)| \leq |g_2(x)| \neq 0 \} \) and \( \{ x \in B^2 \mid h(x) = 0 \} \). Since an intersection of the two first kind of sets is a strictly analytic domain, and \( T \subseteq S \), and \( S = \emptyset \), in this intersection, there must be a non-trivial set of the form \( \{ x \in B^2 \mid h(x) = 0 \} \). Now, let us consider in \( B_{(r,1)} = \mathcal{M}(k\{r^{-1}T_1, T_2\}) \) the Zariski-closed subset \( Z = V(T_2 - f(T_1), h) \). By assumption, it is infinite. Moreover, since \( ||f|| < 1 \), \( T_2 - f(T_1) \) is irreducible (see the lemma above) in \( \mathcal{M}(k\{r^{-1}T_1, T_2\}) \), so for reasons of dimension, in \( \mathcal{M}(k\{r^{-1}T_1, T_2\}), V(T_2 - f(T_1)) \subseteq V(h) \). But now if we introduce (as in the preceding proof)
\[
B_r \xrightarrow{\psi} B^2 \quad u \mapsto (u, f(u))
\]
and $\eta = \psi(g)$ where $g$ is the Gauss point of $\mathbb{B}_r$, then $\eta \in V(h)$ (where we see now $V(h)$ as a Zariski closed subset of $\mathbb{B}^3$), $O_{\mathbb{B}^2, \eta}$ is a field, but $V(h) = \emptyset$, and since $V(h)$ is a semianalytic (so overconvergent subanalytic) subset of $\mathbb{B}(r,1)$, this contradicts lemma 2.7.

Let us now show that $S$ is a locally semianalytic subset of $\mathbb{B}^2$. Indeed, take $0 < s < t < r$ with $t, r \in \sqrt{|k^n|}$, and consider $X_1 = \{x \in \mathbb{B}^2 \mid |T_1(x)| \leq r\}$ and $X_2 = \{x \in \mathbb{B}^2 \mid |T_1(x)| \geq t\}$. They define a wide covering of $\mathbb{B}^2$ and $X_1 \cap S$ (resp. $X_2 \cap S$) is semianalytic in $X_1$ (resp. $X_2$), so $S$ is well locally semianalytic in $\mathbb{B}^2$. \hfill $\Box$

We have implicitly used:

**Lemma 2.7.** If $f \in k\{r^{-1}x\}$ and $\|f\| \leq 1$, then $F(x, y) := y - f(x)$ is irreducible in $k\{r^{-1}x, y\}$.\hfill $\Box$

**Proof.** As we have already seen, $V(f)$ is isomorphic to $\mathbb{B}_r$, so is irreducible. \hfill $\Box$

### 2.3. The other inequalities

We will now explain the other inequalities appearing in figure 1.

The following proposition will be implicitly used in the rest of this section. In addition, it illustrates that the mixture of overconvergence and rigid-semianalytic subsets (which is a $G$-local property), is somehow too strong, in the sense that in proposition 2.8 above, the overconvergence condition seems to have disappeared.

**Proposition 2.8.** Let $S \subseteq X$. The following properties are equivalent:

1. $S$ is strongly subanalytic.
2. There exist $\aleph \in \mathbb{N}$ and $T \subseteq X \times \mathbb{B}^\aleph$ a rigid-semianalytic subset such that $S = \pi(T \cap (X \times (\mathbb{B}^\aleph)^n))$ where $\pi : X \times \mathbb{B}^\aleph \to X$ is the natural projection.

**Proof.** Let us show that (1) $\Rightarrow$ (2). Let $S$ be a strongly subanalytic subset of $X$, so there exists $r > 1$, $T \subseteq X \times \mathbb{B}_r$ a rigid-semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^\aleph))$. Decreasing $r$ if necessary, we can assume that $|r| \in \sqrt{|k^n|}$. In fact, using similar arguments as the one given in remark 1.4 we can even assume that $r \in |k|$. Then if we consider the homothety, which is an isomorphism: $h : X \times \mathbb{B}_r \to X \times \mathbb{B}^\aleph$, which can be defined as multiplication of each coordinate of $\mathbb{B}_r$ by $\frac{1}{r}$, this gives the following commutative diagram:

$$
\begin{array}{ccc}
X \times \mathbb{B}_r & \xrightarrow{h} & X \times \mathbb{B}^\aleph \\
\pi \downarrow & & \pi' \downarrow \\
X & & X
\end{array}
$$

and $S = \pi(T \cap (X \times \mathbb{B}^\aleph)) = \pi' \left(h(T) \cap (X \times \mathbb{B}^\aleph)^n\right)$. Now $T' := h(T) \cap (X \times \mathbb{B}^\aleph)$ is a rigid-semianalytic subset of $X \times \mathbb{B}^\aleph$ such that $T' \subseteq X \times (\mathbb{B}^\aleph)^n$ and $S = \pi'(T') = \pi'(T' \cap (X \times (\mathbb{B}^\aleph)^n))$.

Conversely, let $T \subseteq X \times (\mathbb{B}^\aleph)^n$ be a rigid-semianalytic subset of $X \times \mathbb{B}^\aleph$ and $S = \pi(T)$. For any $r > 1$, we can define $X_0 = X \times \mathbb{B}$, and for $i = 1, \ldots, n$, let $X_i = \{(x, t_1, \ldots, t_n) \in X \times \mathbb{B}_r^n \mid |t_i| \geq 1\}$. So $\{X_i\}_{i \in \{0, \ldots, n\}}$ is an admissible covering of $X \times \mathbb{B}_r^n$. By assumption, $T \cap X_0$ is rigid-semianalytic, and $T \cap X_i = \emptyset$ for $i = 1, \ldots, n$. So $T$ is rigid semianalytic in $X \times \mathbb{B}_r^n$, and if we note $\pi : X \times \mathbb{B}_r^n \to X$, $S = \pi(T)$, so $S$ is strongly subanalytic. \hfill $\Box$

**Proposition 2.9.** There exist strongly subanalytic subsets which are not $G$-overconvergent subanalytic.

**Proof.** Let $r > 1$, $X = \mathcal{M}(k\{x, y, z\}) = \mathbb{B}^3$, $Y = \mathcal{M}(k\{x, y, z, t\})$ and $\pi : Y = \mathcal{M}(k\{x, y, z, t\}) \to X = \mathcal{M}(k\{x, y, z\})$, the natural projection. We now choose $f \in k\{t\}$ whose radius of convergence is exactly 1, and such that $\|f\| \leq 1$, and $T = \{(x, y, z, t) \in Y \mid |t| < 1, x = yt, z = yf(t)\}$. It is a rigid-semianalytic subset of $Y$, and $S = \pi(T)$ is a strongly subanalytic subset of $X$ according to the previous proposition. Since the family of closed balls with center the origin is a fundamental system of neighborhoods of the origin, if $S$ was $G$-overconvergent subanalytic, for
some $1 \geq |\mu| = \varepsilon > 0$ small enough, $S' := S \cap B^3$ would be overconvergent subanalytic in $B^3$. We then fix a $y_0 \in k^*$ such that $0 < |y_0| < \varepsilon$, i.e., $\frac{|\mu|}{|\mu|} < 1$ and define $X' := \{(x, y, z) \in B^3 \mid y = y_0\}$. Now $X'$ is isomorphic to the bidisc $B^2 = \{(x, y) \mid |x| \leq \varepsilon \text{ and } |y| \leq \varepsilon\}$, and $S'' := S \cap X'$ should be overconvergent constructible in $X'$ (somehow we use here lemma [1.18](2)). If we make a dilatation of $X'$ by $\frac{1}{\varepsilon}$ it becomes the bidisc of radius 1: the new coordinates are $x'$, $z'$ defined by $x = \mu x'$ and $z = \mu z'$. Now, in these new coordinates:

$$S'' = \{(x', z') \in B^2 \mid |x'| < \frac{|y_0|}{|\mu|} \text{ and } z' = \frac{y_0}{\mu} f(\frac{x' \mu}{y_0})\}$$

should be overconvergent subanalytic in $B^2$. If we put $r := \frac{|y_0|}{|\mu|} < 1$ and $g(x') = \frac{y_0}{\mu} f(\frac{x' \mu}{y_0})$, then the radius of convergence of $g$ is precisely $r$, $||g|| < 1$ so $S'' = \{(x', z') \in B^2 \mid |x'| < r \text{ and } z' = g(x')\}$, $S''$ should be overconvergent subanalytic in $B^2$, but we proved the converse in proposition [2.4].

**Proposition 2.10.** There exist overconvergent subanalytic subsets which are not rigid-semianalytic.

**Proof.** Let $1 < r = |\lambda|$, and $f \in k(r^{-1}X)$ whose radius of convergence is exactly $r$, and such that $||f|| < 1$. We set $X = B^3 = M(k\{x, y, z\})$, $Y = M(k\{x, y, z, r^{-1}t\})$, and

$$T = \{(x, y, z, t) \in Y \mid x = yt, \; z = yf(t), \; |t| \leq 1\}$$

and $S = \pi(T)$, where $\pi : M(k\{x, y, z, r^{-1}t\}) \rightarrow M(k\{x, y, z\})$ is the natural projection. Then $S$ is overconvergent subanalytic. If $S$ was rigid-semianalytic, there would exist $\mu \in k^t$ with $0 < \varepsilon := |\mu| < 1$ such that $S' = S \cap B^3$ is semianalytic in $B^3$ (we again use that if $V$ is an affinoid domain of $B^3$ that contains the origin, then there exists $\varepsilon > 0$ such that $B^3_{|\varepsilon} \subseteq V$).

Let us introduce $y_0 \in k^*$ such that $0 < |y_0| < \frac{\varepsilon}{2}$. In particular $\frac{|\mu|}{|\mu|} = \frac{|\mu|}{|\mu|} < \frac{1}{2}$. Then $X' = \{(x, y, z) \in B^3 \mid y = y_0\}$ is a Zariski-closed subset of $B^3$, isomorphic to a bidisc $B^2$. Now, $S'' := S \cap X'$ is defined by

$$S'' = \{(x, z) \in B^2 \mid \frac{x}{y_0} \leq 1 \text{ and } z = y_0 f\left(\frac{x}{y_0}\right)\}.$$ 

As we said, $X'$ is isomorphic to $B^2$ with coordinates $(x', z')$ where $x = \mu x'$ and $z = \mu z'$. In these new coordinates, $S'' = \{(x', z') \in B^2 \mid \frac{x' \mu}{y_0} \leq 1 \text{ and } z' \mu = y_0 f(\frac{x' \mu}{y_0})\}$. If we define $g(x') = \frac{y_0}{\mu} f(\frac{x' \mu}{y_0})$ and $s = \frac{y_0}{\mu} = \frac{|\mu|}{|\mu|} < \frac{1}{2}$, then $g$ has a radius of convergence which is exactly $\rho$ where $s < \rho = \frac{|\mu|}{|\mu|} < 1$, and $||g|| < ||f|| < 1$, so $S'' = \{(x', z') \in B^2 \mid |x'| \leq s \text{ and } z' = g(x')\}$ should be semianalytic, but is not (see proposition [2.4]).

From this one can deduce:

**Corollary 2.11.** Let $X$ be a strictly $k$-analytic space which contains a closed ball of dimension $\geq 3$. Then there are overconvergent subanalytic subsets of $X$ which are not rigid-semianalytic. In particular, the class of overconvergent subanalytic subsets of $X$ properly contains the class of locally semianalytic subsets of $X$.

In conclusion, in figure [1] we have shown inequalities 1, 4, 5 and 8. Now 2, 3, 6, 7 are set-theoretical consequences of 4, 5 and of the inclusions from the left to the right.

### 2.4. Berkovich points versus rigid points

Let $X = M(\mathcal{A})$ be a strictly $k$-affinoid space. We denote by $X_{\text{rig}}$ the set of rigid points of $X$. When one deals with semianalytic or overconvergent subanalytic subsets $S$ of $X$, one can wonder if things change if we restrict to $S_{\text{rig}} = S \cap X_{\text{rig}}$. Actually the following two propositions show that there is no difference if one works with Berkovich spaces or rigid spaces.

To be precise, let us denote by $\mathcal{B}$ be the free boolean algebra whose set of variables consists in the set of formal inequalities $\{|f| \leq |g|\}$, $\{|f| < |g|\}$ and $\{f = 0\}$, for $f, g \in \mathcal{A}$. We denote by $SA_{\text{rig}}$ the class of semianalytic subsets of $X_{\text{rig}}$ and by $SA_{\text{Ber}}$ the class of semianalytic subsets of $X$.
the Berkovich space $X$. Then we define natural applications $\alpha : B \to SA_{\text{Ber}}$ and $\beta : B \to SA_{\text{rig}}$ where for instance $\alpha(\{f \leq g\}) = \{x \in X \mid |f(x)| \leq |g(x)|\}$ and $\beta(\{f \leq g\}) = \{x \in X_{\text{rig}} \mid |f(x)| \leq |g(x)|\}$. In addition we consider the forgetful map $\iota : SA_{\text{Ber}} \to SA_{\text{rig}}$: if $S \in SA_{\text{Ber}}$ is a semianalytic set, $\iota(S) = S \cap X_{\text{rig}}$. We then obtain the commutative diagram:

$$
\begin{array}{ccc}
B & \xrightarrow{\alpha} & SA_{\text{Ber}} \\
\downarrow{\beta} & & \downarrow{\iota} \\
& SA_{\text{rig}} & 
\end{array}
$$

**Proposition 2.12.** The map $\iota$ is bijective.

**Proof.** First, $\iota$ is surjective by definition.

Now if $\iota(S_1) = \iota(S_2)$, we must show that $S_1 = S_2$. Considering $S_1 \setminus S_2$ and $S_2 \setminus S_1$, we can restrict to show that if $S \in SA_{\text{Ber}}$ and $\iota(S) = \emptyset$, then $S = \emptyset$. According to what has been previously done, we can assume that $S \in SA_{\text{Ber}}$ is a finite intersection of subsets of the form $\{x \in X \mid |f(x)| \leq |g(x)| \neq 0\}$, $\{x \in X \mid |f(x)| < |g(x)|\}$ and $\{x \in X \mid h(x) = 0\}$, and that $\iota(S) = S \cap X_{\text{rig}} = \emptyset$. Passing to $Y = M(A/I)$ where $I$ is the ideal generated by the functions $h$ appearing in the third case ($h(x) = 0)$, we can assume that $S$ is a finite intersection of subsets of the form: $\{x \in X \mid |f(x)| \leq |g(x)| \neq 0\}$ and $\{x \in X \mid |f(x)| < |g(x)|\}$. But then it forms a non empty strictly analytic domain of $X$ so $S \cap X_{\text{rig}} \neq \emptyset$. 

If we denote by $CD$ the family of finite subsets of constructible data of $X$, by $OC$ the family of overconvergent constructible subsets of $X$, and $OC_{\text{rig}}$ the family of subsets of $X_{\text{rig}}$ which are the intersection of an element of $OC$ with $X_{\text{rig}}$, then we can define as above the following commutative diagram:

$$
\begin{array}{ccc}
CD & \xrightarrow{\alpha} & OC \\
\downarrow{\beta} & & \downarrow{\iota} \\
& OC_{\text{rig}} & 
\end{array}
$$

To be precise, if $D \in CD$ is the set of the constructible data $(X_i, T_i) \xrightarrow{\varphi_i} X$, then

$$\alpha(D) = \bigcup_{i=1}^{n} \varphi_i(T_i).$$

**Proposition 2.13.** In the above diagram, $\iota$ is a bijection.

**Proof.** Since we showed that $OC$ (and $OC_{\text{rig}}$) is stable under complementary, we can restrict to show that if $S \in OC$ is such that $\iota(S) = S \cap X_{\text{rig}} = \emptyset$, then $S = \emptyset$. To show this we can even assume that $S = \varphi(T)$, where $(Y, T) \xrightarrow{\varphi} X$ is a constructible datum. But, if $T$ is a non empty semianalytic subset of $Y$, according to proposition $2.12$ $T_{\text{rig}} \neq \emptyset$, so since $\varphi$ preserves the rigid points, $\varphi(T)_{\text{rig}} = S_{\text{rig}}$ is non empty.

3. **Overconvergent subanalytic subsets when $\dim(X) = 2$**

In this section, $k$ will be non-archimedean algebraically closed field. In this context, if $X$ is a $k$-analytic space, it is equivalent to say that $X$ is regular, or that $X$ is quasi-smooth [Duc, section 3]. We will rather use the second terminology.

3.1. **Algebraization of functions.**
Proposition 3.1. Let $X, Y$ be two $k$-affinoid spaces, so that we can consider the cartesian diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_1} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\pi_2} & Y
\end{array}
\]

Let $z \in X \times Y$, and let us denote by $z_1 = \pi_1(z)$, $z_2 = \pi_2(z)$. Let us assume that $z_2 \in Y(k) = Y_{\text{reg}}$.

(a) Let $V$ be an affinoid domain of $X \times Y$ such that $z \in V$. There exists an affinoid domain $U$ of $X$ (which contains $z_1$) such that if $W$ is an affinoid neighbourhood of $z_2$ small enough, $V \cap (X \times W) = U \times W$.

(b) Let $V$ be a neighbourhood of $z$. There exists $U$ (resp. $W$) an affinoid neighbourhood of $z_1$ (resp. $z_2$) such that $V \supset U \times W$

Proof. (a) \cite{Sch94} 2.2] Let us set $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$. First, using the Gerritzen-Grauert theorem, we can assume that $V$ is a rational domain of $X \times Y$ defined by:

\[ V = \{ x \in X \times Y \mid |f_i(x)| \leq |g(x)|, \ i = 1 \ldots n \text{ and } |g(x)| \geq r \} \]

where $f_i, g \in \mathcal{A} \hat{\otimes}_k \mathcal{B}$, and $r > 0$. Since we assume that $z_2 \in Y(k)$, it has a sense to evaluate the functions $f_i, g$ in $z_2$, and we will denote by $f_{i_{z_2}}, g_{z_2}$ the corresponding functions, that we see as elements of $\mathcal{A}$ and of $\mathcal{A} \hat{\otimes}_k \mathcal{B}$. In addition, since $z_2$ is a rigid point of $Y$, there exists an affinoid neighbourhood $T$ of $z_2$ in $Y$ such that

\begin{equation}
\forall i \sup_{x \in X \times T} |(f_i - f_{i_{z_2}})(x)| < r \tag{31}
\end{equation}

\begin{equation}
\sup_{x \in X \times T} |(g - g_{z_2})(x)| < r. \tag{32}
\end{equation}

Since $g = g_{z_2} + (g - g_{z_2})$, we conclude from (32) that if $x \in X \times T$,

\begin{equation}
|g(x)| \geq r \Leftrightarrow |g_{z_2}(x)| \geq r. \tag{33}
\end{equation}

Then since also $g_i = g_{i_{z_2}} + (g_i - g_{i_{z_2}})$, from (31), (32) and (33), we conclude that if $x \in X \times T$

\[ \left( |g(x)| \geq r \text{ and } |f_i(x) - |g(x)|| \Leftrightarrow \left( |g_{z_2}(x)| \geq r \text{ and } |f_{i_{z_2}}(x)| \leq |g_{z_2}(x)| \right) \right. \]

Hence, if we set

\[ U = \{ x \in X \mid |(f_{i_{z_2}})(x)| \leq |g_{z_2}(x)| \text{ and } |g_{z_2}(x)| \geq r \}, \]

then $V \cap (X \times T) = U \times T$. It then follows that if $W$ is an affinoid domain of $Y$ such that $W \subset T$, then $V \cap (X \times W) = U \times W$.

(b) We can assume that $V = V$ is an affinoid neighbourhood of $z$. In (a), $V \cap (X \times W)$ is still a neighbourhood of $z$, since $W$ is an affinoid neighbourhood of $z_2$ (because $z_2$ is a rigid point). If we denote by $s_{z_2} : X \rightarrow X$ the section of $\pi_1$ defined by $s_{z_2}(t) = (t, z_2)$, then

\[ s_{z_2}^{-1}((V \cap (X \times W)) = s_{z_2}^{-1}(U \times W) = U \]

is an affinoid neighbourhood of $x$ (since $s_{z_2}(x) = z$). Thus $U$ is also an affinoid neighbourhood of $z_1$.

\[ \square \]

Remark 3.2. Without the assumption that $z_2 \in Y(k)$ the previous corollary would be false. Indeed assume that $k$ is algebraically closed. Take for instance $X = \mathcal{M}(k\{x\})$ and $Y = \mathcal{M}(k\{y\})$, and let $\varphi : \mathcal{M}(k\{t\}) \rightarrow X \times Y$ be defined by $\varphi(t) = (t, -t)$. Let $\eta$ be the Gauss point of $\mathcal{M}(k\{t\})$ and $z := \varphi(\eta)$. Let $V = \{ p \in \mathcal{M}(k\{x, y\}) \mid |(x+y)(p)| \leq \frac{1}{2} \}$. It is a neighbourhood of $z$. However, $\pi_1(z)$ (resp $\pi_2(z)$) is the Gauss point $z_1 = \eta_X$ of $\mathcal{M}(k\{x\})$ (resp. $z_2 = \eta_Y$ the Gauss point of $\mathcal{M}(k\{y\})$). It is then easy to see, according to the description of an affinoid domain of the unit disc as a Swizz cheese, that there does not exist an affinoid neighbourhood $U$ (resp. $W$) of $\eta_X$ (resp. $\eta_Y$) such that $V \supset U \times W$. For instance for the reason that in $U$ there would necessarily exist a rigid point $x_0 \in \{ x \in k \mid |x| \leq 1 \}$ such that $\overline{x_0} = \hat{0}$ and in $W$ a rigid point $y_0$ such that $\overline{y_0} = \hat{T}$ but $(x_0, y_0) \notin V$ (where $\overline{\pi}$ corresponds to the reduction of $x$ in $\hat{k}$).
Lemma 3.3. Let $x \in X = \mathcal{M}(A)$, and let $f = \sum_{n \in \mathbb{N}} a_n T^n \in \mathcal{A}\{r^{-1}T\}$. Let us assume that $f_x \neq 0$. Then there exist $V = \mathcal{M}(B)$ an affinoid domain of $X$ which contains $x$, $P \in B[T]$, and $u \in B[r^{-1}T]$ a multiplicative unit such that $f|_{V \times \mathbb{B}_r} = uP$.

Proof. Since $f_x = \sum_{n \in \mathbb{N}} a_n(x) T^n \neq 0$, this series is distinguished of some order $s \geq 0$ for some $s > 0$. We remind that this means that $|a_s(x)| r^s = \|f_x\|$ and that $s$ is the greatest rank for this property.

We now use lemma 1.31 in our specific situation where the polyradius $x$ is in fact the real number $r$ and denote by $u \in \mathcal{A}\{r^{-1}T\}$ for $n \in J$ satisfying $\|\phi_n\| < 1$ such that $f = \sum_{n \in J} a_n(X^n + \phi_n)$.

We then define $V$ as the rational domain:

$V = \{ z \in X \mid |a_s(z)| = |a_s(x)| \text{ and } |a_i(z)| r^i \leq |a_s(x)| r^s \text{ for } i \in J \setminus \{s\} \}$

and denote by $B$ the affinoid algebra of $V$. It is then true that $x \in V$. Moreover, on $V = \mathcal{M}(B)$, one checks that $a_s$ is a multiplicative unit, and that on $B[r^{-1}T]$, $f$ is distinguished of order $s$. One can then apply Weierstrass preparation (corollary 1.30) to conclude. □

Remark 3.4. The previous result (lemma 3.3) is false if we remove the assumption $f_x \neq 0$.

Indeed, let us consider a real number $r$ satisfying $0 < r < 1$, and let $f \in k\{r^{-1}x\}$ be a function whose radius of convergence is exactly $r$ and let us assume that $\|f\| < 1$. Let then $A = k[y,t]$, $X = \mathcal{M}(A)$ the unit bidisc, $p$ the rigid point of $X$ corresponding to the origin, and let us consider

$F(y, t, x) = y - tf(x) \in k\{y, t\}\{r^{-1}x\} = \mathcal{A}\{r^{-1}X\}$.

Then we claim that there does not exist $V = \mathcal{M}(B)$ an affinoid domain of $X$ containing $p$ such that $F|_{V \times \mathbb{B}_r} = uP$ where $u$ is a multiplicative unit of $B[r^{-1}T]$ and $P \in B[t]$.

Proof. Indeed otherwise, there would exist some closed bidisc $V$ of radius $s = |\lambda| \in |k^*|$ where $\lambda \in k^*$, and some $P \in k\{s^{-1}y, s^{-1}t\}[x]$ and a multiplicative unit $u \in k\{s^{-1}y, s^{-1}t\}\{r^{-1}x\}$ such that

$$F|_{V \times \mathbb{B}_r} = uP.$$  

Let us fix $t = \lambda$. Then we consider

$G(y, x) = F(y, \lambda, x) = y - \lambda f(x) \in k\{y, r^{-1}x\}$.

According to (34), $G|_{\mathbb{B}_r \times \mathbb{B}_r} = u(y, \lambda, t)P(y, \lambda, t)$. Replacing $y$ by $\frac{y}{\lambda}$ and $f$ by $\lambda f$, we then obtain that

$G(y, x) = y - f(x) \in k\{y, r^{-1}x\}$

$G = uP$

where $u \in k\{y, r^{-1}x\}$ is a multiplicative unit, $P \in k\{y\}[x]$ and $\|f\| < 1$ has a radius of convergence exactly $r < 1$. This implies that if we set

$S := \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } y = f(x)\}$

then

$S = \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } P(x, y) = 0\}$

so $S$ would be semianalytic in $\mathbb{B}^2$, but in section 3 we exploited many times that this is not the case. □

Lemma 3.5 (Local algebraization of a function in a family of rings). Let $n$ be an integer and let us consider $a_0, \ldots, a_n$ some elements of $\{ x \in k \mid |x| \leq 1 \}$ and $r_0, \ldots, r_n$ some positive real numbers. Let $Y \subseteq \mathcal{M}(k\{T\}) = \mathbb{B}$ be the Laurent domain defined by

$Y = \{ y \in \mathcal{M}(k\{T\}) \mid \|(T - a_0)(y)\| \leq r_0 \text{ and } \|(T - a_i)(y)\| \geq r_i, \ i = 1 \ldots n\}$,
and let $X = \mathcal{M}(\mathcal{A})$ be a $k$-affinoid space. Let

$$f \in \mathcal{O}(X \times Y)$$

and let

$$z \in X \times Y$$

such that $\pi_1(z) = x \in X(k)$ and let us set $y := \pi_2(z)$. Assume that $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$ is non-zero\footnote{Here $\mathcal{H}(x) \simeq k$ because $x \in X(k)$.}. Then there exists $V = \mathcal{M}(\mathcal{B})$ an affinoid neighbourhood of $x$, and $Y' \subset Y$ defined by

$$Y' = \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \text{ and } |(T - b_i)(y)| \geq s_i, \ i = 1 \ldots m\}$$

an affinoid neighbourhood of $y$ such that

$$f_{|V \times Y'} = (uP)_{|V \times Y'},$$

where the $s_i$’s are positive real numbers, $b_i \in k^n$, $u$ is a multiplicative unit of $V \times Y'$ and $P \in \mathcal{B}[T, (T - b_1)^{-1}, \ldots, (T - b_m)^{-1}]$.

Remark 3.6. let us mention that in the proof we distinguish two very different cases.

1. If $y$ is a rigid point then $Y'$ can in fact be chosen to be a closed ball, i.e. $m = 0$.
2. Otherwise, if $y$ is not a rigid point, then in fact $s_0 = r_0$, that is to say, we do not have to decrease the radius of the ambient closed ball, but in counterpart, we possibly have to remove some open balls.

Proof. If $y$ is a rigid point, we can indeed find a closed disc $Y'$ which contains $y$ and the result follows from lemma 3.3

If $y$ is not a rigid point, $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$. Then according to classical results on factorization of functions on rational domains of the closed disc (cf [VAPI04, 2.2.9]), there exist $\alpha_1, \ldots, \alpha_N \in k$, $d_1, \ldots, d_N \in \mathbb{N}$, $g$ an invertible function of $\mathcal{O}(Y)$ such that

$$f_x = \prod_{i=1}^{N} (T - \alpha_i)^{d_i} g.$$  \hspace{1cm} (35)

We then set $m = n + N$, $b_i = a_i$ and $s_i = r_i$ for $i = 0 \ldots n$, and $b_{n+j} = \alpha_j$ for $j = 1 \ldots N$ and we take $s_{n+j}$ small enough so that $\{z \in Y \mid |T - \alpha_j(z)| \geq s_{n+j}\}$ is a neighbourhood of $y$ (this is possible because $y$ is not a rigid point). Then we define

$$Y' := \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \text{ and } |(T - b_i)(y)| \geq s_i, \ i = 1 \ldots m\}.$$

Next, we set

$$G = f \prod_{i=1}^{N} (T - \alpha_i)^{-d_i} \in \mathcal{O}(X \times Y').$$

Then, according to (35) $G_x = g$ which does not vanish on $Y'_x$. So there exists an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of $x$ such that $G$ is invertible on $V \times Y'$ because the locus of points $x$ where $G_x$ is invertible is open. Now using the explicit description of $\mathcal{O}(V \times Y')$, we can write

$$G = \sum_{\nu=(\nu_0, \ldots, \nu_m) \in \mathbb{N}^{m+1}} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \ldots (T - b_m)^{-\nu_m}.$$  \hspace{1cm} (36)

Now for $M \geq 0$ set

$$G_M = \sum_{|\nu| \leq M} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \ldots (T - b_m)^{-\nu_m}.$$ 

By definition, $G_M \in \mathcal{B}[T, (T - b_1)^{-1}, \ldots, (T - b_m)^{-1}]$. In addition,

$$G_M \xrightarrow[M \to \infty]{} G,$$

so $G_M$ is invertible for $M$ big enough. For such a $M$,

$$G = G_M + (G - G_M) = G_M (1 + G^{-1}_M (G - G_M)).$$
Moreover, if we take $M$ again larger, we can assume that $\|G^{-1}_M\| = \|G^{-1}\|$, and as a consequence
$$\|G^{-1}_M(G - G_M)\| \xrightarrow[M \to \infty]{} 0.$$ 

Thus, for $M$ large enough, if we set
$$u_M = 1 + G^{-1}_M(G - G_M)$$
then $u_M$ is a multiplicative unit, and according to (36)
$$f = G_M u_M \prod_{i=1}^N (T - \alpha_i)^{d_i}.$$

We then set $u := u_M$ and $P := G_M \prod_{i=1}^N (T - \alpha_i)^{d_i}$ to conclude. $\square$

3.2. **Blowing up.** From now on, $X$ will be a quasi-smooth $k$-analytic space of dimension 2.

We now make two simple remarks that we will use in the proof of theorem 3.12.

**Lemma 3.7.** Let $A$ be a $k$-affinoid algebra, $X = \mathcal{M}(A)$, $0 < r < s$ some real numbers and $h \in A$.

1. Consider the Weierstrass domain of $X$:
$$V = \{ x \in X \mid |h(x)| \leq s \}$$
and let $S$ be a locally semi-analytic subset of $V$ such that
$$S \subseteq \{ x \in X \mid |h(x)| \leq r \}.$$

Then $S$ is also a locally semi-analytic subset of $X$.

2. Consider the Laurent domain of $X$:
$$V = \{ x \in X \mid |h(x)| \geq r \}$$
and let $S$ be a locally semi-analytic subset of $V$ such that
$$S \subseteq \{ x \in X \mid |h(x)| \geq s \}.$$

Then $S$ is also a locally semi-analytic subset of $X$.

**Proof.** Choose a real number $t$ such that $r < t < s$.

1. Let us set $W = \{ x \in X \mid t \leq |h(x)| \}$. Then $\{ V, W \}$ is a wide covering of $X$, and $S \cap V$ is by hypothesis locally semianalytic in $V$, and by assumption, $S \cap W = \emptyset$ so is also locally semianalytic in $W$, hence $S$ is locally semianalytic in $X$.

2. Likewise, let us set $W = \{ x \in X \mid |h(x)| \leq t \}$. Then $\{ V, W \}$ is a wide covering of $X$, $S \cap V$ is locally semianalytic in $V$ and $S \cap W = \emptyset$, so $S$ is locally semianalytic in $X$. $\square$

This lemma will be used jointly with the following remark:

**Remark 3.8.** Let us consider a $k$-affinoid space $X = \mathcal{M}(A)$, $f, g \in A$, $0 < s < r$ and
$$(Z, S) \xrightarrow{\varphi} X$$
the elementary constructible datum given by $Z = \mathcal{M}(B)$ where $B = A\{r^{-1}t\}/(f - tg)$ and
$$S = \{ z \in Z \mid |f(z)| \leq s|g(z)| \neq 0 \}.$$ 

Moreover, let
$$(Y, U) \xrightarrow{\psi} (Z, S)$$
be a constructible datum.
A. Let us assume that $g|f$. In other words, there exists $h \in \mathcal{A}$ such that $f = gh$. Let us then consider $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(h - t)$ and $V = \mathcal{M}(\mathcal{C})$. Note that $V$ is the Weierstrass domain of $X$ defined by

$$V = \{x \in X \mid |h(x)| \leq r\}.$$

Let us denote by $\beta$ the map of the immersion of the affinoid domain $V$ inside $X$, and let

$$T = \{x \in V \mid |h(x)| \leq s \text{ and } g(x) \neq 0\}.$$

Since $f - tg = g(h - t)$, $(h - t)|(f - tg)$, and there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

Indeed, $\alpha(T) \subseteq S$, follows from their respective definitions. Conversely, if $z \in S$, $(f - tg)(z) = 0 = g(z)(h - t)(z)$ but since $g(z) \neq 0$, $(h - t)(z) = 0$ which implies that $z \in V$, and by the definition of $S$, it follows that $z \in \alpha(T)$.

Let us then consider the following cartesian diagram of $k$-germs:

$$
\begin{array}{ccc}
(Y, U) & \xrightarrow{\psi} & (Z, S) \xrightarrow{\varphi} X \\
\alpha' \downarrow & & \alpha \downarrow \\
(Y', U') & \xrightarrow{\psi'} & (V, T)
\end{array}
$$

Here, $(Y', U') \xrightarrow{\psi'} (V, T)$ is still a constructible datum according to corollary. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U')) = \psi(U)$, so

$$(37) \quad \varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')).$$

Roughly speaking, we were starting with the constructible datum

$$
(Y, U) \xrightarrow{\psi} (Z, S) \xrightarrow{\varphi} X
$$

such that the elementary constructible datum of $\varphi$ was defined with functions $f$ and $g$ such that $g|f$. And we have been able to replace $\varphi$ by the constructible datum

$$
(Y', U') \xrightarrow{\psi'} (V, T) \xrightarrow{\beta} X
$$

where $V$ is a Weierstrass domain. Note moreover that $T$ and so also $\psi'(U')$ satisfy the hypothesis of lemma 3.7 (1).

B. If $f|g$, there exists $h \in \mathcal{A}$ such that $g = fh$. Let then $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(1 - th)$, $V = \mathcal{M}(\mathcal{C})$. Note that $V$ is the Laurent domain of $X$ defined by

$$V = \{x \in X \mid |h(x)| \geq \frac{1}{r}\}.$$

Let us denote by $\beta$ the map of the immersion of the Laurent domain $V$ inside $X$, and let

$$T = \{x \in V \mid |h(x)| \geq \frac{1}{s} \text{ and } g(x) \neq 0\}.$$

Since $(1 - th)|(f - tg)$, there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

We then consider the following cartesian diagram of $k$-germs:

$$
\begin{array}{ccc}
(Y, U) & \xrightarrow{\psi} & (Z, S) \xrightarrow{\varphi} X \\
\alpha' \downarrow & & \alpha \downarrow \\
(Y', U') & \xrightarrow{\psi'} & (V, T)
\end{array}
$$

Here, $(Y', U') \xrightarrow{\psi'} (V, T)$ is still a constructible datum. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U')) = \psi(U)$, so

$$(38) \quad \varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')).$$
In that case, we were starting with the constructible datum \((Y, U) \xrightarrow{\psi} (Z, S) \xrightarrow{\pi} X\) such that \(f|g\), and we have been able to replace it by the following constructible datum \((Y', U') \xrightarrow{\psi'} (V, T) \xrightarrow{\pi'} X\) where \(V\) is a Laurent domain of \(X\). Note moreover that \(T\) and so also \(\psi'(U')\) satisfies the hypothesis of lemma 3.7 (2).

**Remark 3.9.** We are going to use some blowing up of \(k\)-analytic spaces in the following context: \(X\) will be a quasi-smooth \(k\)-analytic space of dimension 2, and we will blow up a rigid point \(p\) of \(X\). In particular, the resulting blowing up \(\widetilde{X}\) will be still quasi-smooth. To give a precise description of the situation, since \(k\) is algebraically closed, we can assume that \(X = \mathbb{B}^2\) and \(p\) is the origin. The blowing up can then be described with two charts as follows. We consider

\[
X_1 = \mathcal{M}(k\{x, t_1\}) \xrightarrow{\pi_1} \mathbb{B}^2 = \mathcal{M}(k\{x, y\}) \quad \text{and} \quad X_2 = \mathcal{M}(k\{y, t_2\}) \xrightarrow{\pi_2} \mathbb{B}^2 = \mathcal{M}(k\{x, y\})
\]

Then \(\mathbb{B}^2\) is obtained by gluing \(X_1\) and \(X_2\) along the domains \(U_1 = \{z \in X_1 \mid t_1(z) \neq 0\}\) and \(U_2 = \{z \in X_2 \mid t_2(z) \neq 0\}\) via the isomorphism

\[
\begin{align*}
U_1 & \rightarrow U_2 \\
(x, t_1) & \mapsto (xt_1, t_1^{-1}).
\end{align*}
\]

**Proposition 3.10.** Let \(X = \mathcal{M}(A)\) be a quasi-smooth \(k\)-affinoid space of dimension 2 and let \(f, g \in A\). Then there exists a sequence of blowing up of rigid points \(\pi : \widetilde{X} \rightarrow X\) such that for all \(x \in \widetilde{X}\), \(f_x|g_x\) or \(g_x|f_x\). Remark that \(\widetilde{X}\) is still quasi-smooth.

**Proof.** We may assume that \(X\) is irreducible. If \(f = 0\) or \(g = 0\), there is nothing to prove, so we may assume that \(f \neq 0\) and \(g \neq 0\). Likewise, if \(f = g\), there is nothing to do, so we may also assume that \(f - g \neq 0\).

Let \(h = fg(f - g)\). Hence, \(h \neq 0\). We can find a sequence of blowing up of rigid points \(\pi : \widetilde{X} \rightarrow X\) such that \(\pi^*(h)\) is a normal crossing divisor. Indeed, the classical proof (see [Kol07, 1.8]) that works in the algebraic case, or the complex analytic case, can be translated verbatim in our context, and since we are dealing with a compact space, the local procedure of [Kol07, 1.8] needs only to be applied to a finite number of points. Let then \(x \in \widetilde{X}\).

If \(x\) is not a rigid point, \(\mathcal{O}_{\widetilde{X}, x}\) is a field or a discrete valuation ring and the result is clear.

Otherwise, if \(x\) is a rigid point, its local ring is a regular local ring of dimension 2. By assumption, \(h = fg(f - g)\) is a normal crossing divisor, thus can be written in \(\mathcal{O}_{\widetilde{X}, x}\) as

\[
(fg(f - g))_x = u\xi_1^a\xi_2^n
\]

where \(\xi_1, \xi_2\) is a system of local parameters around \(x\) and \(u\) is a unit in \(\mathcal{O}_{\widetilde{X}, x}\). Dividing by the common divisor of \(f_x\) and \(g_x\) in \(\mathcal{O}_{\widetilde{X}, x}\), we can assume for instance that \(f_x = v\xi_1^a\) and \(g_x = w\xi_2^n\) and \(f_x - g_x = z\xi_1^a\xi_2^n\) where \(v, w\) and \(z\) are units of \(\mathcal{O}_{\widetilde{X}, x}\).

If \(p > 0\) then modulo \(\xi_1\) we obtain \(f = 0\), so \(f_x - g_x = w\xi_2^n\) modulo \(\xi_1\). This implies that \(a = 0\) and that \(b = q\). So \(f_x = (f_x - g_x) + g_x\) is divisible by \(\xi_2^n\), and this implies that \(q = 0\). So \(g_x\) is invertible and, \(g_x|f_x\).

And if \(p = 0\), then \(f_x\) is invertible, so \(f_x|g_x\). 

**Lemma 3.11.** Let \(X\) be a good quasi-smooth strictly \(k\)-analytic space of dimension 2.

1. Let \(q \in X_{\text{rig}}\) and \(\pi : \widetilde{X} \rightarrow X\) the blowing-up of \(X\) at \(q\), and let \(S \subseteq \widetilde{X}\) be a locally semianalytic subset. Then \(\pi(S)\) is locally semianalytic.

2. If \(\pi : \widetilde{X} \rightarrow X\) is a succession of blowing-up of rigid points, and \(S \subseteq \widetilde{X}\) is locally semianalytic, then \(\pi(S)\) is also locally semianalytic.

**Proof.** (2) is a consequence of (1) so we only have to show (1).

The problem is local on \(X\), and since outside \(q\), \(\pi\) is a local isomorphism, we can restrict to an affinoid neighbourhood of \(q\), and since \(X\) is regular at \(q\), we can assume that \(X = \mathbb{B}^2\) and \(q\) is the origin.
Then $\pi : \tilde{X} \to X$ can be described with two charts, one of them being
\[
\pi_1 : \quad X_1 = \mathcal{M}(k\{x,t\}) \to X = \mathcal{M}(k\{x,y\}) \\
(x,t) \mapsto (x,tx)
\]
The other chart being analogous we only consider $\pi_1$. Now, changing $S$ in $S \cap X_1$, we can restrict to show that if $S$ is locally semianalytic in $X_1$, so is $\pi_1(S)$. Since $\pi_1$ induces an isomorphism between $X_1 \setminus V(x)$ and $\{p \in \mathbb{B}^2 \mid |y(p)| \leq |x(p)| \neq 0\}$, we only have to show that $\pi_1(S)$ is semianalytic around $q$, the origin of $\mathbb{B}^2$.

Now if for each $p \in E := V(x) \subseteq X_1$ we can find $V_p$ an affinoid neighbourhood of $p$, and $\varepsilon_p > 0$ such that $\pi_1(V_p \cap S) \cap \mathbb{B}_{\varepsilon_p}^2$ is semianalytic in $\mathbb{B}_{\varepsilon}^2 \cap X$, then by compactness of $E$, we can extract $V_1, \ldots, V_n$ a finite covering of $E$ and $\varepsilon > 0$ such that
\[
\bigcup_{i=1}^n (\pi_1(V_i \cap S)) \cap \mathbb{B}_{\varepsilon}^2 = \pi_1(S) \cap \mathbb{B}_{\varepsilon}^2
\]
is semianalytic in $\mathbb{B}_{\varepsilon}^2$. So we fix $p \in E = V(x)$ and try to find $V$ an affinoid neighbourhood of $p$, and $\varepsilon > 0$ such that $\pi_1(V \cap S) \cap \mathbb{B}_{\varepsilon}^2$ is semianalytic in $\mathbb{B}_{\varepsilon}^2$.

Since $S$ is locally semianalytic in $X_1$, we can find $V$ an affinoid neighbourhood of $p$ such that $V \cap S$ is semianalytic in $V$. According to corollary \[41\] we can assume that $\mathbb{B}^2 = \mathbb{B} \times W$ where
\[
W = \{w \in \mathcal{M}(k(t)) \mid |(t-a_0)(w)| \leq r_0 \text{ and }|(t-a_i)(w)| \geq r_i, \ i = 1 \ldots n\}
\]
for some $a_0, \ldots, a_n \in k^\circ$ and $r_0, \ldots, r_n \in \mathbb{R}_+^+$.

To simplify the notations, we can also assume that the semianalytic subset $S$ of $V$ we are dealing with is of the following form:
\[
S = \bigcap_{j=1}^m \{v \in V \mid |f_j(v)| \lambda_j g_j(v)|\}.
\]
Now remind that $V = \mathbb{B} \times W$ with $\mathbb{B} = \mathcal{M}(k\{\varepsilon^{-1}x\})$. So we can factor each $f_j$ and $g_j$ by the greatest power of $x$ which is a factor, hence introduce some integers $b_j, c_j$ such that
\[
S = \bigcap_{j=1}^m \{v \in V \mid |x^{b_j} \tilde{f}_j(v)| \lambda_j |x^{c_j} \tilde{g}_j(v)|\}
\]
where the series $\tilde{f}_j(0,t)$ and $\tilde{g}_j(0,t)$ are non zero, and $f_j = x^{b_j} \tilde{f}_j, g_j = x^{c_j} \tilde{g}_j$. But to simplify the notations, we will use $f_j$ (resp. $g_j$) instead of $\tilde{f}_j$ (resp. $\tilde{g}_j$), so that
\[
S = \bigcap_{j=1}^m \{v \in V \mid |x^{b_j} f_j(v)| \lambda_j |x^{c_j} g_j(v)|\}
\]
where the series $f_j(0,t)$ and $g_j(0,t)$ are non zero.

Then according to lemma \[42\] we can decrease $\varepsilon$ and $W$ so that for each $f_j, g_j \in \{f_1 \ldots f_m, g_1, \ldots, g_m\}$,
\[
f_j = u_j P_j \quad \text{(resp. } g_j = v_j Q_j) \quad \text{where } u_j \quad \text{(resp. } v_j) \quad \text{is a multiplicative unit, and } P_j \quad \text{(resp. } Q_j) \quad \in k\{\varepsilon^{-1}x\}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}]\).

Said differently, and with different notations, there exists an integer $N$ such that $f_j = u_j \frac{P_j}{((t-a_1) \ldots (t-a_n))^N}$ where $u_j$ is a multiplicative unit and $P_j \in k\{\varepsilon^{-1}x\}[t]$ (and resp. $g_j = v_j \frac{Q_j}{((t-a_1) \ldots (t-a_n))^N}$). Hence
\[
|f_j(v)\lambda_j g_j(v)| \Leftrightarrow \bigg|\frac{P_j(v)}{((t-a_1) \ldots (t-a_n))^N(v)}\bigg| \lambda_j \bigg|\frac{Q_j(v)}{((t-a_1) \ldots (t-a_n))^N(v)}\bigg| \Leftrightarrow |P_j(v)|\lambda_j |Q_j(v)|
\]
where
\[
\lambda_j = \frac{\|v_j\|}{\|u_j\|} \in |k^\circ|.
\]
Moreover,
\[
S \cap V = (S \cap \{v \in V \mid x(v) = 0\}) \cup (S \cap \{v \in V \mid x(v) \neq 0\})
\]
\footnote{Here we use the explicit description of affinoid domains of $\mathbb{B}$.}
and \( \pi_1(\{v \in V \mid x(v) = 0\}) = q \), the origin of \( \mathbb{B}^2 \).

So, adding if necessary the origin to \( \pi_1(S \cap \{v \in V \mid v(x) \neq 0\}) \) (which will not change the fact that it is semianalytic), we can restrict to show that \( \pi_1(S \cap \{v \in V \mid v(x) \neq 0\}) \) is semianalytic around the origin. Moreover since on \( \{v \in V \mid v(x) \neq 0\} \), \( \pi_1 \) is bijective, the following holds:

\[
\pi_1 \left( \bigcap_{j=1}^{m} \{v \in V \mid |x^{b_j}f_j(v)|x^{c_j}g_j(v)\} \right) \cap \{v \in V \mid x(v) \neq 0\} = \bigcap_{j=1}^{m} \pi_1 \left( \{v \in V \mid |x^{b_j}f_j(v)|x^{c_j}g_j(v)\} \right) \cap \{v \in V \mid x(v) \neq 0\}.
\]

Now since \( y = tx \) and \( P_j \in k\{x^{-1}\}[t] \) there exists an integer \( M \geq 0 \) such that \( x^MP_j(x,t) \in k\{x^{-1}\}[t] = k\{x^{-1}\}[y] \), i.e. \( x^MP_j(x,t) = \pi^*(P_j(x,y)) \) for some \( \tilde{P}_j(x,y) \in k\{x^{-1}\}[y] \) and such that \( x^M\tilde{Q}_j(x,t) \in k\{x^{-1}\}[y] \), i.e. \( x^M\tilde{Q}_j(x,t) = \pi^*(\tilde{Q}_j(x,y)) \) for some \( \tilde{Q}_j(x,y) \in k\{x^{-1}\}[y] \).

Now on \( \{v \in V \mid v(x) \neq 0\} \),

\[
|x^{b_j}f_j(v)|x^{c_j}g_j(v) \leftrightarrow |x^{M+b_j}f_j(v)|x^{M+c_j}g_j(v) \leftrightarrow |x^{b_j}\tilde{P}_j(\pi_1(v))|x^{c_j}\tilde{Q}_j(\pi_1(v)).
\]

From that we conclude that

\[
z \in \pi_1 \left( \bigcap_{j=1}^{m} \{v \in V \mid |x^{b_j}f_j(v)|x^{c_j}g_j(v)\} \right) \cap \{v \in V \mid x(v) \neq 0\} \]

\[
\leftrightarrow z \in \bigcap_{j=1}^{m} \{z \in \pi_1(V) \mid |x^{b_j}\tilde{P}_j(z)|x^{c_j}\tilde{Q}_j(z)\} \cap \{z \in X \mid x(z) \neq 0\}.
\]

Since \( \pi_1(\mathbb{B}_2) \subseteq \mathbb{B}_2 \) and is semianalytic in \( \mathbb{B}_2 \), we conclude that \( \pi_1(S \cap \{v \in V \mid v(x) \neq 0\}) \) is semianalytic in \( \mathbb{B}_2 \), which ends the proof. \( \Box \)

**Theorem 3.12.** Let \( X \) be a good quasi-smooth strictly \( k \)-analytic space of dimension 2 with \( k \) algebraically closed, and \( S \subseteq X \). Then \( S \) is overconvergent subanalytic subset if and only if \( S \) is locally semianalytic.

**Proof.** Since the problem is local, we can assume that \( X \) is affinoid and that \( S = \phi(U) \) where \( (Y,U) \xrightarrow{\phi} X \) is a constructible datum, and just check that \( S \) is locally semianalytic. We do it by induction on the complexity of \( \phi \). So let \( (Y,U) \xrightarrow{\phi} X \) be a constructible datum, that we decompose as

\[
\phi = (Y,U) \xrightarrow{\psi} Z \xrightarrow{\Delta} X
\]

where \( \chi \) is an elementary constructible datum, and \( \psi \) a constructible datum whose complexity is one less than \( \phi \). So we can introduce \( f, g \in \mathcal{A}, 0 < s < r \) such that \( Z = \mathcal{M}(\mathcal{A}(r^{-1}t))/(f - tg) \).

According to proposition 3.10 we can find a succession of blowing-up of rigid points \( \pi : \tilde{X} \rightarrow X \) such that for all \( x \in \tilde{X}, f_x|g_x \) or \( g_x|f_x \). According to remark 3.9 \( \tilde{X} \) is still quasi-smooth. This gives us the following cartesian diagram:

\[
\begin{array}{ccc}
(Y,U) & \xrightarrow{\phi} & X \\
\pi' \downarrow & & \downarrow \pi \\
(Y',U') & \xrightarrow{\phi'} & \tilde{X}
\end{array}
\]

Then \( \phi(U) = \pi(\phi'(U')) \). Moreover, since \( \tilde{X} \) is compact, we can then find a finite wide covering \( \{X_i\}_{i=1}^{n} \) of \( \tilde{X} \) by affinoid domains such that for all \( i, f|_{X_i}|g|_{X_i}, \) or \( g|_{X_i}|f|_{X_i} \). We denote by
\( \pi_i : X_i \to X \) the composition of the embedding of the affinoid domain \( X_i \to \tilde{X} \) with \( \pi : \tilde{X} \to X \). This gives the following cartesian diagrams:

\[
\begin{array}{ccc}
(Y, U) & \xrightarrow{\psi} & Z \\
\downarrow \pi'_i & \downarrow \pi_i & \downarrow \pi_i \\
(Y_i, U_i) & \xrightarrow{\psi_i} & Z_i \\
\end{array}
\]

Then

\[
\varphi(U) = \pi(\varphi'(U')) = \pi \left( \bigcup_{i=1}^{n} \chi_i(\psi_i(U_i)) \right)
\]

But \( (Y_i, U_i) \xrightarrow{\psi_i} Z_i \) is a constructible datum of lower complexity than \( \varphi \), so that we would like to use our induction hypothesis, and claim that \( \psi_i(U_i) \) is locally semianalytic. However, \( Z_i \) is not necessarily still quasi-smooth so we cannot do that. However, since \( f_i|_{X_i} \mid g_i|_{X_i} \) or \( g_i|_{X_i} \mid f_i|_{X_i} \), according to remark [3.8] we can in fact replace \( Z_i \) by a Weierstrass (or a Laurent) domain of \( X_i \), and hence assume that \( Z_i \) is quasi-smooth. Thus by induction hypothesis \( \psi_i(U_i) \) is locally semianalytic in \( Z_i \).

Next we use lemma 3.7 to assert that \( \chi_i(\psi_i(U_i)) \) is locally semianalytic in \( X_i \). So

\[
\varphi'(U') = \bigcup_{i=1}^{n} (\chi_i(\psi_i(U_i))
\]

is locally semianalytic in \( \tilde{X} \), since \{\( X_i \)\} was a wide covering of \( \tilde{X} \). Finally, according to lemma 3.11 \( \pi(\varphi'(U')) = S \) is also locally semianalytic.

\[
\square
\]

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