Some Riemann Hypotheses from Random Walks over Primes

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Abstract

The aim of this article is to investigate how various Riemann Hypotheses would follow only from properties of the prime numbers. To this end, we consider two classes of $L$-functions, namely, non-principal Dirichlet and those based on cusp forms. The simplest example of the latter is based on the Ramanujan tau arithmetic function. For both classes we prove that if a particular series involving sums of multiplicative characters over primes is $O(\sqrt{N})$, then the Euler product converges in the right half of the critical strip. When this result is combined with the functional equation, the non-trivial zeros are constrained to lie on the critical line. We argue that this $\sqrt{N}$ growth is a consequence of the series behaving like a one-dimensional random walk. Based on these results we obtain an equation which relates every individual non-trivial zero of the $L$-function to a sum involving all the primes. Finally, we briefly mention subtle differences for principal Dirichlet $L$-functions due to the existence of the pole at $s = 1$, in which the Riemann $\zeta$-function is a particular case.

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I. INTRODUCTION

Montgomery [1] conjectured that the pair correlation function between the ordinates of the Riemann zeros on the critical line satisfy the GUE statistics of random matrix theory. On the other hand, Riemann [2] obtained an exact formula for the prime number counting function $\pi(x)$ in terms of the non-trivial zeros of $\zeta(s)$. This suggests that if the Riemann Hypothesis is true, then this should imply some kind of randomness of the primes. It has been remarked by many authors that the primes appear random, and this is sometimes referred to as pseudo-randomness of the primes [4].

In this article we address the following question, which is effectively the reverse of the previous paragraph. What kind of specific pseudo-randomness of the primes would imply the Riemann Hypothesis? This requires a concrete characterization. We provide such a characterization by arguing that certain deterministic trigonometric sums over primes, involving multiplicative functions, behave like random walks, namely grow as $\sqrt{N}$. However, we are not able to fully prove this $\sqrt{N}$ growth, and thus take it as a conjecture. It should be emphasized that we do not introduce any new probabilistic aspect to the original problem.

The main result of this paper can be stated as follows. Consider $L$-functions based on non-principal Dirichlet characters and on cusp forms. We prove that, assuming the claim from the previous paragraph concerning the random walk behavior, the Euler product converges to the right of the critical line.

This article is largely based on the ideas in [3] and is intended to clarify it with more precise statements. There is an important difference between the cases mentioned above and principal Dirichlet $L$-functions, where $\zeta(s)$ is a particular case, and this is emphasized more here. We will not consider this latter case in detail, but we briefly mention these subtleties in the last section of the paper.

II. ON THE GROWTH OF SERIES OF MULTIPLICATIVE FUNCTIONS OVER PRIMES

In this section we consider the asymptotic growth of certain trigonometric sums over primes involving multiplicative arithmetic functions. We propose that these sums have the same growth as one-dimensional random walks.
Let \( c(n) \) be a multiplicative function, i.e. \( c(1) = 1 \) and \( c(mn) = c(m)c(n) \) if \( m \) and \( n \) are coprime integers, and let \( p \) denote an arbitrary prime number. We can always write \( c(p) = |c(p)|e^{i\theta_p} \). Now consider the trigonometric sum

\[
C_N = \sum_{n=1}^{N} \cos \theta_{p_n}
\]

where \( p_n \) denotes the \( n \)th prime; \( p_1 = 2, p_2 = 3 \), and so forth. We wish to estimate the size of this sum, specifically how its growth depends on \( N \).

### A. Non-Principal Dirichlet Characters

#### 1. The Main Conjecture

Now let \( c(n) = \chi(n) \) be a Dirichlet character modulo \( k \), where \( k \) is a positive integer. The function \( \chi \) is completely multiplicative, i.e. \( \chi(1) = 1 \), \( \chi(mn) = \chi(m)\chi(n) \) for all \( m,n \in \mathbb{Z}^+ \), and obeys the periodicity \( \chi(n) = \chi(n+k) \). Its values are either \( \chi(n) = 0 \), or \( |\chi(n)| = 1 \) if and only if \( n \) is coprime to \( k \). For a given \( k \) there are \( \varphi(k) \) different characters which can be labeled as \( \{\chi_1, \ldots, \chi_{\varphi(k)}\} \). The arithmetic function \( \varphi(k) \) is the Euler totient. We will omit the index of the character except for \( \chi_1 \) which denotes the principal character, defined as \( \chi_1(n) = 1 \) if \( n \) is coprime to \( k \), and \( \chi_1(n) = 0 \) otherwise. The Riemann \( \zeta \)-function corresponds to the trivial principal character with \( k = 1 \).

For a non-principal character the non-zero elements correspond to \( \varphi(k) \)th roots of unity given by \( \chi(n) \equiv e^{i\theta_n} = e^{2\pi i\nu_n/\varphi(k)} \) for some \( \nu_n \in \mathbb{Z} \). The distinct phases of these roots of unity form a discrete and finite set denoted by

\[
\theta_n \in \Phi \equiv \{\phi_1, \phi_2, \ldots, \phi_r\}, \quad \text{where } r \leq \varphi(k).
\]

Here \( r \) depends on the particular character \( \chi \).

There is an important distinction between principal verses non-principal characters. The principal characters satisfy

\[
\sum_{n=1}^{k-1} \chi_1(n) = \varphi(k) \neq 0,
\]

while non-principal characters satisfy

\[
\sum_{n=1}^{k-1} \chi(n) = 0.
\]
The above relation (4) shows that the angles in $\Phi$ are equally spaced over the unit circle for non-principal characters. On the other hand, this is not the case for principal characters due to (3); in fact the angles $\theta_n$ are all zero.

For the sake of clarity, let us now simply state the main hypothesis that the remainder of this work relies upon. We cannot prove this conjecture, however we will subsequently provide supporting, although heuristic, arguments.

**Conjecture 1.** Let $p_n$ be the $n$th prime and $\chi(p_n) = e^{i\theta_{p_n}} \neq 0$ the value of a non-principal Dirichlet character modulo $k$. Consider the series

$$C_N = \sum_{n=1}^{N} \cos \theta_{p_n}.$$  \hspace{1cm} (5)

Then $C_N = O(\sqrt{N})$ as $N \to \infty$.

The main supporting argument is an analogy with one dimensional random walks, which are known to grow as $\sqrt{N}$. Although the series $C_N$ is completely deterministic, its random aspect stems from the pseudo-randomness of the primes [4], which is largely a consequence of their multiplicative independence. The event of an integer being divisible by a prime $p$ and also divisible by a different prime $q$ are mutually independent. A simple argument is Kac’s heuristic [5]: let $P_m(n)$ denote the probability that an integer $m$ is divisible by $n$. The probability that $m$ is even, i.e. divisible by 2, is $P_m(2) = 1/2$. Similarly, $P_m(n) = 1/n$. We therefore have $P_m(pq) = (pq)^{-1} = P_m(p)P_m(q)$, and the events are independent. Because of the multiplicative property of $c(n)$ this independence of the primes extends to quantities involving $c(p)$, in that $c(p)$ is independent of $c(q)$ for primes $p \neq q$. Moreover, if $\{\theta_p\}$ are equidistributed over a finite set of possible angles, then the deterministic sum $\Pi$ is expected to behave like a random walk since each term $\cos \theta_p$ mimics an independent and identically distributed (iid) random variable. Analogously, if we build a random model capturing the main features of $\Pi$, it should provide an accurate description of some of its important global properties.

Let us provide a more detailed argument. First a theorem of Dirichlet addresses the identically distributed aspect:

**Theorem 1** (Dirichlet). Let $\chi(n) = e^{i\theta_n} \neq 0$ be a non-principal Dirichlet character modulo $k$ and $\pi(x)$ the number of primes less than $x$. These distinct roots of unity form a finite and
discrete set, \( \theta_n \in \Phi = \{ \phi_1, \phi_2, \ldots, \phi_r \} \) with \( r \leq \varphi(k) \). Then for a prime \( p \) we have

\[
P(\theta_p = \phi_i) = \lim_{x \to \infty} \frac{\# \{ p \leq x : \theta_p = \phi_i \}}{\pi(x)} = \frac{1}{r}
\]

(6)

for all \( i = 1, 2, \ldots, r \), where \( P(\theta_p = \phi_i) \) denotes the probability of the event \( \theta_p = \phi_i \) occurring.

**Proof.** Let \( [a_i] \) denote the residue classes modulo \( k \) for \( a_i \) and \( k \) coprime, namely the set of integers \( [a_i] = \{ a_i \mod k \} \). There are \( \varphi(k) \) independent classes and they form a group. Of these classes let the set of integers \( [a_i] \) denote the particular residue class where \( \chi(a_i \mod k) = e^{i\phi_i} \).

Then

\[
P(\theta_p = \phi_i) = P(p = a_i \mod k).
\]

(7)

Dirichlet’s theorem states that there are an infinite number of primes in arithmetic progressions, and \( P(p = a_i) = 1/r \) independent of \( a_i \). In particular,

\[
\pi(x,a,k) = \# \{ p < x, p \equiv a \mod k \text{ with } (a,k) = 1 \} = \frac{\pi(x)}{\varphi(k)}
\]

(8)

in the limit \( x \to \infty \). (See for instance [6, Chap. 22].)

Next consider the joint probability defined by

\[
P(\theta_p = \phi_i, \theta_q = \phi_j) = \lim_{x \to \infty} \frac{\# \{ p, q \leq x : \theta_p = \phi_i \text{ and } \theta_q = \phi_j \}}{\pi^2(x)}
\]

(9)

for all \( i, j = 1, 2, \ldots, r \). The events \( p = a_i \) and \( q = a_j \) (mod \( k \)) are independent due to the multiplicative independence of the primes. Thus one expects

\[
P(\theta_p = \phi_i, \theta_q = \phi_j) = P(\theta_p = \phi_i|\theta_q = \phi_j)P(\theta_q = \phi_j) = P(\theta_p = \phi_i)P(\theta_q = \phi_j) = \frac{1}{r^2}.
\]

(10)

In other words, for a randomly chosen prime, each angle \( \phi_i \in \Phi \) is equally likely to be the value of \( \theta_p \), i.e., \( \theta_p \) is uniformly distributed over \( \Phi \). Moreover, \( \theta_p \) and \( \theta_q \) are independent. Thus the series (5) should behave like a random walk, and this is the primary motivation for Conjecture 1. The \( \sqrt{N} \) growth is straightforward to show on average. Using the above probabilities and the independence (10), one has the following expectation value in the limit \( N \to \infty \):

\[
E[C_N^2] = \sum_{n=1}^{N} E[\cos^2 \theta_{pn}] = N/2
\]

(11)

where we have used \( E[\cos \theta_p \cos \theta_q] = E[\cos \theta_p]E[\cos \theta_q] = 0 \) for \( p \neq q \). This implies \( E[|C_N|] = O(\sqrt{N}) \).
2. A Probabilistic Model

Let us build a simple statistical analogue of (5) which also supports Conjecture 1. Based on Theorem 1, let $\theta_n$ be an iid random variable with probability distribution $P(\theta_n = \phi_i) = 1/r$, where $\theta_n \in \phi_1, \ldots, \phi_r$. Let $X_n = \cos \theta_n$ for some fixed $n$. We thus have

$$E[X_n] = \frac{1}{r} \sum_{i=1}^{r} \cos \phi_i = 0,$$

since the angles $\phi_i$ are equally spaced over the unit circle, and

$$(\sigma_{X_n})^2 = E[X_n^2] = \frac{1}{r} \sum_{i=1}^{r} \cos^2 \phi_i = \frac{1}{2} + \frac{1}{2r} \sum_{i=1}^{r} \cos (2\phi_i).$$

If $\chi(n)$ is complex the last sum vanishes since $\{2\phi_i\} \subset \Phi$. This implies $\sigma_{X_n} = 1/\sqrt{2}$. On the other hand, if the non-vanishing elements are all real, $\chi(n) \in \{+1, -1\}$, then $2\phi_i = 0$ implying $\sigma_{X_n} = 1$. Thus $\sigma_{X_n} < \infty$ and the central limit theorem applies to the new series $\tilde{C}_N = \sum_{n=1}^{N} X_n$, i.e. $\frac{\tilde{C}_N}{\sigma_X \sqrt{N}} \to \mathcal{N}(0,1)$ or equivalently

$$\lim_{N \to \infty} P \left( \frac{\tilde{C}_N}{\sigma_X \sqrt{N}} < K \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} e^{-x^2/2} dx = 1 - \frac{e^{-K^2/2}}{\sqrt{2\pi}} \left( \frac{1}{K} - O(K^{-3}) \right).$$

Therefore, $\tilde{C}_N = O(\sqrt{N})$ with very high probability $P = 1 - \epsilon$, with $\epsilon \to 0$ very fast according to the second equality in (14); for instance with $K = 5$ we have $P = 0.9999997$.

The latter series $\tilde{C}_N$ is of course not identical to the original deterministic series (5), however we expect their growth to be the same if $C_N$ indeed behaves like a random walk as in Conjecture 1. One can think of $C_N$ as one member of the ensemble $\{\tilde{C}_N\}$. If it is a typical member, namely not on the extreme tails of the normal distribution, then $C_N = O(\sqrt{N})$.

3. Numerical Evidence

Let us also provide numerical evidence for the above statements. In Figure 1 (a) we have an example with $k = 7$. Notice that $P(\theta_p = \phi_i) = 1/6 = 0.1666 \cdots$. This table was computed with $x = 10^9$ in (6). One can see the equally spaced angles over the unit circle, and the numerical results verify that $\theta_p$ is uniformly distributed over $\Phi$, as stated in Theorem 1. In Figure 1 (b) we show that (5) is $O(\sqrt{N})$, also for the same character with $k = 7$, in
FIG. 1. (a) Numerical verification of (6) with $k = 7$. Notice the equally spaced angles over the unit circle, and the corresponding probabilities shown in the table. We use $x = 10^9$. (b) Numerical verification of (5) (blue dots) also with $k = 7$, in comparison with $\frac{1}{3}\sqrt{N}$ (solid red line).

agreement with the previous probabilistic argument. Let us also check (9). All the joint probabilities are shown in the following matrix:

\[
P(\phi_i, \phi_j) = 10^{-2} \times \begin{bmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \\
2.7293 & 2.7679 & 2.7454 & 2.7518 & 2.7583 & 2.7647 \\
2.8071 & 2.7842 & 2.7908 & 2.7973 & 2.8038 \\
2.7616 & 2.7680 & 2.7745 & 2.7810 \\
2.7745 & 2.7810 & 2.7875 \\
2.7875 & 2.7940 & \\
2.8006 
\end{bmatrix}
\]

Here we used only $x = 5 \times 10^4$. These values are all close to the predicted theoretical value $P(\phi_i, \phi_j) = (1/6)^2 = 2.7777 \cdots \times 10^{-2}$, and get even closer with higher $x$.

B. Fourier Coefficients of Cusp Forms

Let us extend the above arguments to the Fourier coefficients of cusp forms. We will review the general Hecke theory in Section [IV] where we will explain the significance of being a cusp form. The simplest and best-known example is the weight $k = 12$ modular form, which is the 24th power of the Dedekind $\eta$-function

\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n,
\]

(15)
where \( q = e^{2\pi i z} \) and \( \Im(z) > 0 \). Here the Fourier coefficients \( \tau : \mathbb{N} \to \mathbb{Z} \) are known as the Ramanujan \( \tau \) arithmetic function. More generally, let us refer to the Fourier coefficients of cusp forms as \( c(n) \), where \( c : \mathbb{N} \to \mathbb{R} \) is a multiplicative function. Thus we can write

\[
c(n) = |c(n)| \cos \theta_n \quad \text{with} \quad \theta_n \in \{0, \pi\}. \tag{16}
\]

The series to consider is now

\[
C_N = \sum_{n=1}^N \cos \theta_n \quad \text{with} \quad \cos \theta_n = \pm 1, \tag{17}
\]

and resembles even more closely the original discrete random walk. Let us assume that Theorem 1 holds in the same way but now we have \( \Phi = \{0, \pi\} \), and then \( P(\theta_p = 0) = P(\theta_p = \pi) = 1/2 \). For any two primes \( p \) and \( q \) these two events are independent. As a consequence we have the analog of Conjecture 1:

**Conjecture 2.** The sum (17) obeys the bound \( C_N = O(\sqrt{N}) \) as \( N \to \infty \).

**Remark 1.** Deligne [7] proved that \( |c(p)| \leq 2p^{(k-1)/2} \). This implies that we can write

\[
c(p) = 2p^{(k-1)/2} \cos \alpha_p \tag{18}
\]

where \( \alpha_p \) is called a Frobenius angle. The aspect of a uniform distribution in Theorem 1 can be seen as a weaker form of Sato-Tate conjecture [8]. Whereas (17) only concerns the signs \( \cos \theta_n = \pm 1 \), Sato-Tate is much more specific. It asserts that \( \alpha_p \) in (18) is uniformly distributed over \([0, \pi]\) according to the function \( \frac{2}{\pi} \sin^2 \beta \), with \( \beta \in [0, \pi] \). Thus, Sato-Tate conjecture would imply our assumption that the signs of \( c(p) \) are equally likely to be +1 or −1.

### III. CONVERGENCE OF THE EULER PRODUCT FOR NON-PRINCIPAL DIRICHLET L-FUNCTIONS

Let \( s = \sigma + it \) be a complex variable. Given a Dirichlet character \( \chi \) modulo \( k \) we have the Dirichlet \( L \)-series

\[
L(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}. \tag{19}
\]

The domain of convergence of generalized Dirichlet series are always half-planes. Such series converge absolutely for \( \sigma > \sigma_a \), where \( \sigma_a \) is referred to as the abscissa of absolute
convergence. There is also an abscissa of convergence $\sigma_c \leq \sigma_a$. For all Dirichlet series we have $\sigma_a = 1$. The analytic properties of are as follows. If $\chi$ is non-principal, then $L(s, \chi)$ is analytic in the half-plane $\sigma > 0$ with no poles. If $\chi = \chi_1$ is principal, then $L(s, \chi_1)$ has a simple pole at $s = 1$, but it is analytic everywhere else. In this case $\sigma_c = \sigma_a = 1$. There is a functional equation relating $L(s, \chi)$ to $L(1 - s, \chi)$, thus the critical line is $\sigma = 1/2$; the critical strip is the region $0 \leq \sigma \leq 1$ where all the non-trivial zeros lie. From now on we consider only non-principal characters $\chi \neq \chi_1$.

Since there is no pole at $s = 1$, for $\chi \neq \chi_1$, it is possible that $\sigma_c < \sigma_a$. In fact $\sigma_c = 0$. This is easy to see from the Dirichlet series convergence test as follows. Set $t = 0$ and write $L(s, \chi) = \sum_n \chi(n) \ell_n$ where $\ell_n = 1/n^\sigma$. One has $\ell_n > \ell_{n+1}$ and $\lim_{n \to \infty} \ell_n = 0$ if $\sigma > 0$. Now, due to (4),

$$\left| \sum_{n=1}^N \chi(n) \right| \leq c$$

for all integers $N$ and for some constant $c$. In fact,

$$c = \max \left\{ j \sum_{n=1}^j \chi(n), \; j = 1, 2, \ldots, k - 2 \right\}. \quad (21)$$

Thus, since convergence of Dirichlet series are always half-planes, the series (19) converges for all complex $s$ with $\Re(s) > 0$.

Due to the completely multiplicative property of $\chi$ one has the Euler product formula

$$L(s, \chi) = \prod_{n=1}^{\infty} \left( 1 - \frac{\chi(p_n)}{p_n^s} \right)^{-1}. \quad (22)$$

Because the left hand side converges for $\Re(s) > 0$ this opens up the possibility that the right hand side converges for $\Re(s) > \sigma_c$ for some $\sigma_c > 0$. We will argue that $\sigma_c = 1/2$ and that the above Euler product formula is valid for $\Re(s) > 1/2$ since both sides of the equation converge there.

Taking the formal logarithm on both sides, and assuming the principal branch, we have

$$\log L(s, \chi) = -\sum_{n=1}^{\infty} \log \left( 1 - \frac{\chi(p_n)}{p_n^s} \right) = X(s, \chi) + R(s, \chi)$$

where

$$X(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(p_n)}{p_n^s}, \quad R(s, \chi) = \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \frac{\chi(p_n)^m}{mp_n^{ms}}. \quad (24)$$
Now $R(s, \chi)$ absolutely converges for $\sigma > 1/2$, therefore
\[
\log L(s, \chi) = X(s, \chi) + O(1).
\] (25)

Thus, convergence of the Euler product to the right of the critical line depends only on $X(s, \chi)$. We proceed by analyzing what are the necessary conditions such that $X(s, \chi)$ also converges for $\sigma > 1/2$.

We will need an upper bound on prime gaps $g_n = p_{n+1} - p_n$. The Cramér-Granville conjecture, namely $g_n < c \log^2 p_n$ with $c \geq 1$, would serve our purposes, however, it remains unproven. Fortunately, the following weaker result will prove to suffice:

**Proposition 1.** Let $g_n = p_{n+1} - p_n$ be the gap between consecutive primes. For $N > 4$ we have
\[
\sum_{n=1}^{N} g_n < \sum_{n=1}^{N} \log^2 p_n.
\] (26)

**Proof.** A proof based on a formula of Selberg was given in [9]. Clearly (26) would follow trivially from the stronger Cramér-Granville conjecture with $c = 1$. Here we provide a simpler proof of (26). Let $a(n) = 1$ if $n$ is a prime and $a(n) = 0$ otherwise, and $f(n) = \log^2 n$. Through Abel’s summation formula we have
\[
\sum_{p \leq x} \log^2 p = \sum_{n \leq x} a(n)f(n) = \pi(x) \log^2 x - 2 \int_{2}^{x} \pi(t) \frac{\log t}{t} dt
\] (27)
where $\sum_{n \leq x} a(n) = \pi(x)$. Here it is enough to use the rough approximation
\[
\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\left( \frac{x}{\log^3 x} \right),
\] (28)
which replaced into (27) yields
\[
\sum_{p \leq x} \log^2 p = x \log x - x + O\left( \frac{x}{\log x} \right).
\] (29)

Now choose $x = p_{N+1} - \delta$ and let $\delta$ be arbitrarily small. Analyzing the right hand side of (29) it is obvious that there exists an $x_0$ such that $x(\log x - 1 + o(1)) \geq x$ for $x \geq x_0$. We can estimate such an $x_0$ by requiring $\log x - 1 > 1$, which implies $x > 7.39 > p_4$. Noticing that $p_{N+1} = 2 + \sum_{n=1}^{N} g_n$ we then obtain (26). \(\square\)

**Theorem 2.** Let $L(s, \chi)$ be a non-principal Dirichlet $L$-function. Assuming Conjecture 1 and using the result from Proposition 1, the generalized Dirichlet series $X(s, \chi)$ defined in (24) has abscissa of convergence $\sigma_c = 1/2$. This implies that the Euler product (22) also has the half-plane of convergence given by $\sigma > 1/2$.  

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Proof. It is enough to consider the real part $S(s,\chi)$ of $X(s,\chi)$ since analogous arguments apply to the imaginary part. Consider its partial sum $S_N(s,\chi) = \sum_{n=1}^{N} a_n b_n$ where
\begin{align*}
a_n &= p_n^{-\sigma}, \quad b_n = \cos(t \log p_n - \theta_{p_n}), \quad \theta_{p_n} = \arg \chi(p_n). \quad (30)
\end{align*}
Through summation by parts this can be rewritten as
\begin{align*}
S_N &= a_N B_N + \sum_{n=1}^{N-1} B_n(a_n - a_{n+1}), \quad B_n = \sum_{k=1}^{n} b_k. \quad (31)
\end{align*}
Taking the absolute value and using
\begin{align*}
a_n - a_{n+1} &= \sigma \frac{g_n}{p_n^{\sigma+1}} + O\left(\frac{g_n^2}{p_n^{\sigma+2}}\right) \quad (32)
\end{align*}
we therefore have
\begin{align*}
|S_N| &\leq \frac{|B_N|}{(p_N)^{\sigma}} + \sum_{n=1}^{N-1} |B_n| \left(\sigma \frac{g_n}{p_n^{\sigma+1}} + O\left(\frac{g_n^2}{p_n^{\sigma+2}}\right)\right). \quad (33)
\end{align*}
Let us first set $t = 0$. Then $B_N$ becomes [5] which is bounded as $B_N = O(\sqrt{N})$ when $N \to \infty$. Therefore, the first term in (33) vanishes for $\sigma > 1/2$ as $N \to \infty$. Moreover, using the unconditional bound proved by Baker, Harman and Pintz [10], namely $g_n < p_n^{\theta}$ with $\theta = 0.525$, we conclude that higher order terms in (33) converge for $\sigma > 1/2$. Thus, all that remains is the first term inside the summand in (33). Performing another summation by parts on this term, and using the result (26), we conclude that
\begin{align*}
\sum_{n=1}^{\infty} \frac{g_n}{p_n^{\sigma+1/2}} < \sum_{n=1}^{\infty} \log^2 p_n < \sum_{n=1}^{\infty} \log^2 n. \quad (34)
\end{align*}
Now we have
\begin{align*}
\int_{1}^{\infty} \frac{\log^2 x}{x^{\sigma+1/2}} \, dx = \frac{2}{(\sigma - 1/2)^3} \quad (35)
\end{align*}
which is finite for $\sigma > 1/2$, implying convergence of the first sum in (34) in this region. Finally, since convergence of generalized Dirichlet series are always half-planes, it follows that $S(s,\chi)$, and thus also $X(s,\chi)$, converges for all $s = \sigma + it$ with $\sigma > \sigma_c = 1/2$.

Compelling numerical evidence for Theorem 2 has already been given in [3].

**Corollary 1.** If Conjecture 2 is true unconditionally, then Theorem 2 is also true unconditionally, implying that all non-trivial zeros of a non-principal Dirichlet L-function must be on the critical line $\sigma = 1/2$, which is the (generalized) Riemann Hypothesis.
Proof. The argument is very simple, analogous to showing that there are no zeros with \( \sigma > 1 \). From (25) a zero \( \rho \) of \( L(\rho, \chi) = 0 \) requires \( X(\rho, \chi) \to -\infty \). Thus if \( X(s, \chi) \) converges for \( \sigma > 1/2 \) there are no zeros of \( L(s, \chi) \) in this region. From the functional equation, which is a symmetry between \( L(s, \chi) \) and \( L(1-\overline{s}, \chi) \), it implies no non-trivial zeros with \( \sigma < 1/2 \). Since it is known that there are infinite zeros in the critical strip \( 0 < \sigma < 1 \), they must all be on the line \( \sigma = 1/2 \).

Remark 2. Notice that Theorem 2 is a conditional result based on Conjecture 1. However, if we use instead the weaker result of subsection II.A.2 based on the probabilistic model, which is motivated by Theorem 1, then Theorem 2 can be stated unconditionally by asserting that the Euler Product (22) converges in probability, or converges weakly, for \( \sigma > 1/2 \).

IV. CONVERGENCE OF THE EULER PRODUCT FOR L-FUNCTIONS BASED ON CUSP FORMS

In this section we show that the same reasoning applies to \( L \)-functions based on cusp forms. The modular group \( SL_2(\mathbb{Z}) \) is the group of \( 2 \times 2 \) integer matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with unit determinant \( ad - bc = 1 \). A modular form \( f(z) \) of weight \( k \) by definition satisfies

\[
f \left( \frac{az+b}{cz+d} \right) = (cz+d)^k f(z).
\]

The \( SL_2(\mathbb{Z}) \) transformations imply the periodicity \( f(z+1) = f(z) \), thus it has a Fourier series

\[
f(z) = \sum_{n=0}^{\infty} c(n) q^n, \quad q \equiv e^{2\pi iz}.
\]

If \( c(0) = 0 \) then \( f \) is called a cusp form. We will only consider level-1, entire modular forms, which are defined to be analytic in the upper half plane \( \Im(z) > 0 \), and do not have \( n < 0 \) terms in their Fourier expansion. This requires \( k \) to be an even integer with \( k \geq 4 \). In the classification of such forms an important role is played by the Eisenstein series

\[
G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m+nz)^k}
\]

with \( (m,n) \neq (0,0) \), which is a weight \( k \) modular form. From the Fourier coefficients (37) one can define the Dirichlet series

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.
\]
Based on the analogy with the Dirichlet $L$-functions, one can ask about the analytical properties of $L(s, f)$, such as the existence of a functional equation and Euler product. The answer to these questions was settled by Hecke, who proved a bijection between modular forms and Dirichlet series satisfying many of the properties of Dirichlet $L$-functions. The validity of a Riemann Hypothesis for $L(s, f)$ remains an open question; however, it was conjectured to be true by Ramanujan for the $L$-function based on $c(n) = \tau(n)$.

A. Hecke Theory

We review Hecke’s theory by briefly listing its main results [11].

1. Since the linear combination of two modular forms of the same weight is still a modular form, one can define $\mathcal{M}_k$ as the linear vector space of modular forms of weight $k$. These are finite dimensional spaces: $\dim \mathcal{M}_k = \lceil k/2 \rceil$ if $k \equiv 2 \text{ mod } 12$, and $\dim \mathcal{M}_k = \lceil k/12 \rceil + 1$ if $k \not\equiv 2 \text{ mod } 12$. The functions $G_{k>6}$ can all be expressed as polynomials in $G_4$ and $G_6$, thus a basis for $\mathcal{M}_k$ is

$$f(z) = \sum_{(a,b) \in \mathbb{Z}^2 \atop 4a+6b=k} c_{a,b} G_a^4 G_b^6$$

with $(a, b) > (0, 0)$ and for some complex coefficients $c_{a,b}$.

2. Cusp forms also form a vector space $\mathcal{M}_{k,0}$, and $\dim \mathcal{M}_{k,0} = \dim \mathcal{M}_k - 1$. The lowest cusp form occurs at $k = 12$ and is unique. It is the modular discriminant $\Delta$ in equation (15). In the above basis

$$\Delta = \frac{1}{(2\pi)^{12}} \left[ (60 G_4)^3 - 27(140 G_6)^2 \right].$$

There is a unique cusp form, i.e. $\dim \mathcal{M}_{k,0} = 1$, also for $k \in \{16, 18, 20, 22, 26\}$, equal to $\{\Delta G_4, \Delta G_6, \Delta G_4^2, \Delta G_4 G_6, \Delta G_4^2 G_6\}$, respectively.

3. There exists a basis of $\mathcal{M}_k$ which are eigenvectors (eigenforms) of certain Hecke operators $T_n$ with multiplicative properties. The Fourier coefficients for these eigenforms have the following multiplicative property:

$$c(m)c(n) = \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$
where \( d \mid (m,n) \) are the divisors of \((m,n)\). This reduces to \( c(m)c(n) = c(mn) \) when we have \((m,n) = 1\), and it can also be shown that

\[
c(p^{r+1}) = c(p)c(p^r) - p^{k-1}c(p^{r-1}).
\] (43)

This multiplicative property changes the form of the Euler product slightly in comparison to the completely multiplicative case of Dirichlet \( \chi \). Now we have

\[
L(s, f) = \prod_{n=1}^{\infty} \left( 1 - \frac{c(p_n)}{p_n^s} + \frac{1}{p_n^{2s-k+1}} \right)^{-1}
\] (44)

which converges absolutely for \( \sigma > k/2 + 1 \).

4. The Dirichlet series \( L(s, f) \) can be analytically continued to the whole complex \( s \)-plane, with the possible exception of a single pole. For cusp forms \( L(s, f) \) is entire, i.e. has no poles. Non-cusp forms have a pole at \( s = k \). We have the functional equation

\[
\Lambda(s, f) \equiv (2\pi)^{-s}\Gamma(s)L(s, f) = (-1)^{k/2}\Lambda(k-s, f).
\] (45)

5. For cusp forms we have

\[
c(n) = O(n^k),
\] (46)

whereas for non-cusp forms \( c(n) = O(n^{2k-1}) \).

**B. Convergence of the Euler Product**

Henceforth we consider only cusp forms. The analysis of Section III extends straightforwardly if one uses a non-trivial result of Deligne. For the arithmetic function \( \tau(n) \), Ramanujan conjectured that it actually grows more slowly than \(46\), namely \( \tau(p) = O(p^{11/2}) \). This was only proved in 1974 by Deligne [7] as a consequence of his proof of the Weil conjectures.

**Theorem 3** (Deligne). For cusp forms we have

\[
|c(p)| \leq 2p^{(k-1)/2}.
\] (47)

This theorem implies that \(39\) converges absolutely for \( \sigma > (k+1)/2 \). Based on the functional equation \(45\) the critical line is \( \sigma = k/2 \), and the critical strip is the region \( (k-1)/2 \leq \sigma \leq (k+1)/2 \).
Theorem 4. Let \( L(s, f) \) be an \( L \)-function based on a cusp form \( f \) with Euler product given by (44). Assuming Conjecture 2 and using Proposition 1, the Euler product converges to the right of the critical line \( \sigma > k/2 \).

Proof. The arguments are nearly the same as in Theorem 2. Taking the logarithm of (44) one has \( \log L(s, f) = X(s, f) + O(1) \), where \( O(1) \) denotes absolutely convergent terms for \( \sigma > k/2 \) which is a consequence of (47). Here we have

\[
X(s, f) = \sum_{n=1}^{\infty} \frac{c(p_n)}{p_n^s}.
\]

(48)

Therefore, all relies on the region of convergence of (48). Without loss of generality let us set \( t = 0 \), thus \( S = X(\sigma, f) = \sum_n |c(p_n)| \cos \theta_{p_n} p_n^{-\sigma} \). We then have exactly the inequality (33) but now with

\[
B_N = \sum_{n=1}^{N} |c(p_n)| \cos \theta_{p_n}.
\]

(49)

Applying summation by parts once more to (49), together with (47) and Conjecture 2, we conclude that \( B_N = O(p^{k/2}_N) \). Therefore, the lowest order term of interest here is

\[
\sum_{n=1}^{\infty} \frac{g_n}{p_n^{\sigma+k/2+1}} < \sum_{n=1}^{\infty} \frac{\log^2 p_n}{p_n^{\sigma+k/2+1}},
\]

(50)

where we used Proposition 1. This is the analog of (34) and by the same argument converges for \( \sigma > k/2 \). Since (48) is a Dirichlet series we conclude that it converges for all \( s = \sigma + it \) with \( \sigma > k/2 \).

Corollary 2. If Conjecture 2 is true unconditionally then Theorem 4 follows and the Riemann Hypothesis is true for \( L \)-functions based on cusp forms.

Proof. This simple argument is the same as in Corollary 1. Convergence of \( X(s, f) \) on \( \sigma > k/2 \) does not allow zeros of \( L(s, f) \) in this region. The functional equation (45) then forces the zeros to lie on the critical line \( \sigma = k/2 \), since it excludes them from the left half of the critical strip \( \sigma < k/2 \).

Example 1. It is interesting to see how the above arguments fail for non-cusp forms. For simplicity, let us consider the Eisenstein series \( G_4 \) which is a modular form of weight \( k = 4 \). It has the Fourier series

\[
G_4(z) = \frac{\pi^4}{45} \left( 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right).
\]

(51)
where $\sigma_m(n) = \sum_{d|n}d^m$ is the divisor sum function. From the Fourier coefficients one defines the Dirichlet $L$-series

$$L(s, G_4) = \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^s}. \quad (52)$$

The arithmetic function $\sigma_3(n)$ is multiplicative, so $L(s, G_4)$ has an Euler product. It also satisfies a functional equation relating $L(s, G_4)$ to $L(4-s, G_4)$. Now if $G_4$ were a cusp form, then Corollary 2 would assert that the non-trivial zeros have real part $\sigma = 2$. But this is evidently false since $L(s, G_4)$ is related to the product of two Riemann $\zeta$ functions

$$L(s, G_4) = \zeta(s)\zeta(s-3). \quad (53)$$

From the above identity, $L(s, G_4)$ actually does not have zeros on the region $1 \leq \sigma \leq 3$. The zeros of $\zeta(s)$ on the critical line imply that $L(s, G_4)$ has zeros on $\sigma = 1/2$ and also on $\sigma = 7/2$. The reason Theorem 4 fails is due to a higher growth of $|c(p)|$. The bound (47) is simply not valid for a non-cusp forms, and the Euler product for $L(s, G_4)$ does not converge for every point with $\sigma > 2$. For instance, if we just use the bound $c(n) = O(n^{2k-1})$ for non-cusp forms we can only conclude convergence for $\sigma > 15/2$.

Example 2. Consider the Euler product based on the Ramanujan $\tau$ function:

$$L(s, \tau) = \lim_{N \to \infty} P_N(s, \tau), \quad P_N(s, \tau) = \prod_{n=1}^{N} \left(1 - \frac{\tau(p_n)}{p_n^s} + \frac{1}{p_n^{2s-11}} \right)^{-1}. \quad (54)$$

Here $L(s, \tau)$ is the $L$-function, the analytic continuation of the $L$-series. It is straightforward to numerically verify the above results as shown in Figure 2. We expect the results to be more accurate as we increase $N$. We choose $t = 0$ in the rightmost column to explicitly show that there are no divergences on the real line.

V. AN EQUATION RELATING NON-TRIVIAL ZEROS AND PRIMES

Riemann proved that the non-trivial zeros of $\zeta(s)$ dictate the distribution of the primes: by summing over the infinite number of zeros one can reconstruct the prime number counting function $\pi(x)$ exactly. Assuming the Euler product converges to the right of the critical line we can actually establish a converse. For instance, consider non-principal and primitive Dirichlet $L$-functions and the counting formula $N^+(T, \chi)$ for the number of zeros in the region $0 < \sigma < 1$ and $0 < t < T$. From Corollary all zeros have $\sigma = 1/2,$
FIG. 2. Numerical results for the Euler product based on Ramanujan \( \tau \). The table shows some values for two different points inside the critical strip as we increase \( N \). Note that there are no divergences when \( t = 0 \). We also have two plots for \( s = 6.25 + it \). In the first case \( t \in [-15, 20] \), and in the second case \( t \in [50, 80] \). We choose \( N = 10^2 \) since for higher \( N \) the curves are indistinguishable. The (shaded) black line is \(|L(s, \tau)|\) and the blue line \(|P_N(s, \tau)|\), against \( t \).

and replacing (23) into the well-known function \( S_\chi(T) = \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, \chi \right) \) implies that \( N^+(T, \chi) \) depends only on the primes.

One can obtain an exact equation relating zeros to primes as follows. Consider zeros \( \rho_n = 1/2 + it_n \) of non-principal and primitive Dirichlet \( L \)-functions. In [12] we derived the following transcendental equation on the critical line without assuming the Riemann Hypothesis:

\[
\vartheta_{k,a}(t_n) + \lim_{\delta \to 0^+} \Im \log \left( \frac{L \left( \frac{1}{2} + \delta + it_n, \chi \right)}{L \left( \frac{1}{2} + \delta, \chi \right)} \right) = (n - \frac{1}{2}) \pi
\]  

(55)

where \( n = 1, 2, \ldots \) and \( t_n > 0 \). Above \( \vartheta_{k,a}(t) = \Im \log \Gamma \left( \frac{1}{4} + \frac{a}{2} + it \right) - \frac{1}{2} \log \frac{\pi}{k} \), where \( a = 1 \) if \( \chi(-1) = -1 \) and \( a = 0 \) if \( \chi(-1) = 1 \). The above equation is actually also valid for \( \zeta(s) \) and other principal \( L \)-functions. We argued in [12] that if the above equation has a solution for every \( n \), then it saturates the counting formula on the entire strip \( N^+ (T, \chi) \), implying that all zeros must be on the critical line. However, we were not able to justify that this equation does have a solution for every \( n \). The issue is whether the \( \delta \) limit in (55) is well-defined. Assuming Theorem 2 this is indeed the case since the Euler product converges to the right
of the critical line. We therefore have

\[
S_\chi(t) = \frac{1}{\pi} \text{arg } L \left( \frac{1}{2} + it, \chi \right) = \frac{1}{\pi} \Im \log L \left( \frac{1}{2} + it, \chi \right) = \frac{-1}{\pi} \lim_{\delta \to 0^+} \Im \sum_p \log \left( 1 - \frac{\chi(p)}{p^{1/2 + \delta + it}} \right).
\]

(56)

This series converges with \( \delta > 0 \) thus the above limit is well-defined. Note that it is well-defined even on a zero \( t = t_n \), since we get arbitrarily close but we never actually touch the critical line. Moreover, this shows that \( S_\chi(t) = O(1) \) as long as one stays to the right of the critical line. This justifies that (55) has a solution for every \( n \).

More interestingly, equation (55) together with (56) no longer makes any reference to the \( L \)-function itself. Remarkably, this shows that every single zero \( \rho_n = 1/2 + it_n \) is determined by all of the primes, which is a converse of Riemann’s result for \( \pi(x) \) as a sum over zeros. One can actually in practice solve for zeros using only primes from equation (55) with the replacement (56). An interesting question which remains unanswered by the above analysis is how large an \( N \) is required in order to obtain a particular desired accuracy of a zero. In Table I we provide some numerical data concerning this question.

| \( N \) | \( t_1 \) | error (%) |
|--------|--------|----------|
| 1      | 5.57869| 7.3      |
| 10     | 5.24273| 0.85     |
| \( 10^2 \) | 5.20071| 0.05     |
| \( 10^3 \) | 5.19936| 0.02     |
| \( 10^4 \) | 5.19596| 0.04     |
| \( 10^5 \) | 5.19946| 0.02     |
| \( 10^6 \) | 5.19947| 0.02     |

TABLE I. The first zero on the upper critical line for the Dirichlet character \( \chi \) modulo 7 used in Example II A 3, calculated only from knowledge of the first \( N \) primes from equations (55) and (56). The actual value is \( t_1 = 5.198116 \cdots \). One can see how the accuracy increases with \( N \), but rather slowly.

The same argument holds for cusp forms through (44); see [12, eq. (57)]. Although not discussed in [12], it is now clear that this only applies to cusp forms. For non-cusp forms the transcendental equation will not have a solution. This is evident from the previous Example I considering \( G_4 \).
VI. THE CASE OF RIEMANN ζ AND PRINCIPAL DIRICHLET

In this last section we briefly remark on the case of principal Dirichlet characters. The Riemann ζ-function corresponds to the principal character with modulus $k = 1$. Much of the same reasoning of the previous cases apply, however with some subtle new issues that complicate the analysis. In these cases $L(s, \chi)$ has a simple pole at $s = 1$, therefore the abscissas of convergence of the Euler product are $\sigma_a = \sigma_c = 1$. Thus, the Euler product formally diverges for $\sigma \leq 1$. Nevertheless, let us revisit the arguments of Theorem 2. For principal characters we have $\theta_{p_n} = 0$ in (30) hence

$$B_N(t) = \sum_{n=1}^{N} \cos(t \log p_n). \quad (57)$$

Setting $t = 0$, then $B_N = N$, and the same arguments imply convergence of the Euler product for $\sigma > 1$, which is consistent with what was just stated above.

However, through further analysis of (57) it can be shown that if we draw $t$ at random from the interval $t \in [T, 2T]$ then (57) obeys a central limit theorem when $T \to \infty$ and $N \to \infty$, implying $B_N(t) = O(\sqrt{N})$ in distribution. In the case of (57) the central limit theorem again relies on the multiplicative independence of the primes. Furthermore, it is possible to estimate (57) directly through the prime number theorem

$$B_N(t) = \int_{2}^{p_N} \cos(t \log x) \frac{d\pi(x)}{dx} \, dx \sim \int_{2}^{p_N} \cos(t \log x) \frac{dx}{\log x}. \quad (58)$$

The last integral is expressed in terms of the Ei($z$) function and asymptotically yields

$$B_N(t) \sim \frac{p_N}{\log p_N} \frac{t}{1 + t^2} \sin(t \log p_N). \quad (59)$$

The growth of $B_N(t)$ is then given roughly by the ratio $N/t$. This shows that $B_N = O(\sqrt{N})$ for $N \leq N_c$ with $N_c = O(t^2)$. As we will explain, the need for the cut-off $N_c$ is ultimately attributed to the existence of the pole.

Therefore, for a fixed $t \gg 0$ the analysis in Theorem 2 is still valid as long as we stay below the cutoff $N_c$, i.e. we cannot take the limit $N \to \infty$. This means that in the region $1/2 < \sigma \leq 1$ a truncated product $\prod_{p \leq p_N} (1 - \chi(p)p^{-s})^{-1}$, with $N \leq N_c$, is well-behaved and meaningful despite the fact that the Euler product itself is formally divergent. More specifically, the equality (25) should now read

$$\log L(s, \chi) = X_N(s, \chi) + O(1) + R_N(s, \chi) \quad (60)$$
where again $O(1)$ denotes higher order terms which are absolutely convergent for $\sigma > 1/2$, and $R_N(s, \chi)$ is an error due to truncating $X(s, \chi)$. To claim something more precise it is necessary to compute $R_N(s, \chi)$ without further assumptions, which is beyond the scope of this paper. Yet, if one can show that $R_N(s, \chi)$ becomes small for large $N$ and $t$, then it may be possible to exclude zeros in the region $\sigma > 1/2$ from this argument.

Based on a result of Titchmarsh [13] which extends the partial sum of $\partial_s \log \zeta(s)$ into the critical strip, at the cost of introducing a sum over non-trivial zeros of $\zeta(s)$, Gonek, Keating, and Hughes [14, 15] proposed a truncated Euler product into the critical strip. Assuming the Riemann Hypothesis, Gonek [15] estimated the truncation error for $\sigma \geq \frac{1}{2} + \frac{1}{\log N}$. Simply borrowing this result we thus expect something not larger than

$$R_N(s, \chi) \ll N^{\frac{1}{2}-\sigma} \left( \log t + \frac{\log t}{\log N} \right) + \frac{N}{t^2 \log^2 N} \sim N^{\frac{1}{2}-\sigma} \log t, \quad \text{(61)}$$

where in the last step we assumed $t < N \leq t^2$ for both $N$ and $t$ large. We see that for $\sigma > 1/2$ the error vanishes in the limit of large $t$. Thus the cutoff $N \leq N_c = O(t^2)$ is merely a consequence of the pole at $s = 1$. Although formally divergent, if we stay away from the pole, which requires $t \to \infty$ and $N \to \infty$, even in the case of principal Dirichlet, for all practical purposes the Euler product should still be valid on the right-half part of the critical strip. If this can be shown to be correct in a rigorous way without assuming the Riemann Hypothesis, then it suggests the following approach to the Riemann Hypothesis in this case. It is already known that the non-trivial zeros are on the critical line up to at least $t \sim 10^{10}$ based on numerical work. One can then use the asymptotic validity of the Euler product formula to rule out zeros to the right of the critical line at higher $t$.

VII. CONCLUDING REMARKS

The main goal of this paper was to identify precisely what properties of the prime numbers are responsible for the validity of certain generalized Riemann Hypotheses. We concluded that there are two fundamental properties:

1. Their pseudo-random behaviour which is a consequence of their multiplicative independence. This strongly suggests that trigonometric sums over primes of multiplicative functions behave like random walks, and thus are bounded by the typical $\sqrt{N}$ growth, which led us to the Conjectures 1 and 2. These conjectures were the only unproven
assumptions of this paper, although we provided strong motivating arguments for their validity.

2. The fact that large gaps between consecutive primes do not grow too fast. More specifically, the Riemann Hypothesis is not sensitive to the growth of individual gaps, but only on their overall contribution; see Proposition [1].

From these properties, we proved in Theorem [2] that the Euler product for non-principal Dirichlet $L$-functions converges to the right of the critical line. In Theorem [4] we proved that the same holds for $L$-functions based on cusp forms, where Deligne’s result [47] also plays a major role. This indicates there is some universality to our approach. The original Riemann Hypothesis for $\zeta(s)$ corresponds to the trivial principal Dirichlet character and is thus not subsumed. However, in the last section we proposed how to extend these arguments to such a case, which is more subtle due to the simple pole at $s = 1$. Therefore, the Riemann Hypothesis arises from a delicate balance in the prime numbers. They should behave randomly enough, but at the same time there must be some regularity in the gaps preventing them from growing too fast.

A vast generalization of Hecke’s theory of $L$-functions based on modular forms is the Langlands program [16]. There, the $L$-functions are those of Artin, which are based on Galois number field extensions of the rational numbers. They have Euler products and satisfy functional equations, like the cases studied in this paper. Langlands automorphic forms play the role of modular forms. There also exists the notion of a cuspidal form. It would be very interesting to try and extend the ideas in this paper to study which of these $L$-functions, if any, satisfy a Riemann Hypothesis.

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