On a characterization of Arakelian sets

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Abstract

Let $K$ be a compact set in the complex plane $\mathbb{C}$, such that its complement in the Riemann sphere, $(\mathbb{C} \cup \{\infty\}) \setminus K$, is connected. Also, let $U \subseteq \mathbb{C}$ be an open set which contains $K$. Then there exists a simply connected open set $V$ such that $K \subseteq V \subseteq U$. We show that if the set $K$ is replaced by a closed set $F$ in $\mathbb{C}$, then the above lemma is equivalent to the fact that $F$ is an Arakelian set in $\mathbb{C}$. This holds more generally, if $\mathbb{C}$ is replaced by any simply connected open set $\Omega \subseteq \mathbb{C}$. In the case of an arbitrary open set $\Omega \subseteq \mathbb{C}$, the above extends to the one point compactification of $\Omega$. As an application we give a simple proof of the fact that the disjoint union of two Arakelian sets in a simply connected open set $\Omega$ is also Arakelian in $\Omega$.

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1. Introduction

K.-G. Grosse-Erdmann in [13] and G. Costakis in [4] stated and proved, independently, the following.

Lemma 1.1. Let $K \subseteq \mathbb{C}$ be a compact set with $(\mathbb{C} \cup \{\infty\}) \setminus K$ connected. If $U$ is an open set in $\mathbb{C}$ containing $K$, then there exists a simply connected open set $V$ such that $K \subseteq V \subseteq U$. 

The above lemma gave several applications. In particular we used it recently in [7]. I was interested in knowing if the above lemma still holds if the compact set $K$ is replaced by a closed subset $F$ in $\mathbb{C}$. The answer is negative in general. A counterexample is the following:

$$F = \bigcup_{n=1}^{\infty} \left( \left\{ \left( \sum_{i=1}^{n} \frac{1}{2^i} \right) \times [0, n] \right\} \cup \left( \left[ \sum_{i=1}^{n} \frac{1}{2^i}, \sum_{i=1}^{n+1} \frac{1}{2^i} \right] \times \{n\} \right) \right) \cup \left\{ \{1\} \times [0, \infty) \right\}.$$  

This set $F$ relates to the well-known Arakelian’s Approximation Theorem [1].

**Theorem 1.2.** Let $F$ be a closed set in the complex plane $\mathbb{C}$. Then every function $f : F \to \mathbb{C}$ continuous on $F$ and holomorphic in $F^0$ ($f \in A(F)$) can be uniformly approximated on $F$ by entire functions, $g \in H(\mathbb{C})$, if and only if the following hold:

1. $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected
2. $(\mathbb{C} \cup \{\infty\}) \setminus F$ is locally connected at $\infty$.

Yet, in [14] one finds another proof of Theorem 1.2 based on Mergelyan’s theorem, where conditions (i) and (ii) are replaced by the next equivalent condition:

3. $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected and for every closed disk $D$, in $\mathbb{C}$, the union of all bounded components of $\mathbb{C} \setminus (F \cup D)$ is bounded. A set satisfying (iii) (or equivalently (i) and (ii)) is called an Arakelian set in $\mathbb{C}$. In the present article we prove the following.

**Theorem 1.3.** Let $F$ be a closed subset of $\mathbb{C}$. Then the following are equivalent:

1. For every open set $U \subseteq \mathbb{C}$, which contains $F$, there exists a simply connected open set $V$ such that $F \subseteq V \subseteq U$
2. $F$ is an Arakelian set in $\mathbb{C}$.

More generally, Arakelian sets may be defined for the arbitrary open set $\Omega \subseteq \mathbb{C}$. The question is for a relatively closed set $F$ in $\Omega$ whether every function $f \in A(F)$ can be uniformly approximated on $F$ by holomorphic functions $g \in H(\Omega)$. This was completely settled by Arakelian in [2], where he extended Theorem 1.2. In this version one considers the one point compactification of $\Omega$, $\Omega \cup \{\alpha\}$. The relatively closed set $F \subseteq \Omega$ for which the approximation is possible is called Arakelian set in $\Omega$ and again a purely topological description is possible.

We extend Theorem 1.3 replacing the complex plane $\mathbb{C}$ by any simply connected open set $\Omega \subseteq \mathbb{C}$. This means that the open set $V$, in the extended version of Theorem 1.3 (1), is still simply connected; that is $(\mathbb{C} \cup \{\infty\}) \setminus V$ is connected. Also, we can further extend Theorem 1.3 in the general case of any open set $\Omega \subseteq \mathbb{C}$, not necessarily simply
connected. In this case the open set $V$ is not simply connected, but its complement $(\Omega \cup \{\alpha\}) \setminus V$ in the one point compactification of $\Omega$ has to be connected.

Next, we give two applications of our results. One of these states that if $\Omega$ is a simply connected open subset of $\mathbb{C}$, then the union of two disjoint Arakelian sets in $\Omega$ is also Arakelian in $\Omega$. We notice that when $\Omega$ is not simply connected the above fails.

N. Tsirivas in [16] proved a variation of Lemma 1.1 without the assumption that $(\mathbb{C} \cup \{\infty\}) \setminus K$ is connected; see also [6]. There are indications that we could obtain similar variations corresponding to our results, but we have not yet managed to do so.

Some partial extensions of Theorem 1.3 can be achieved in the case of Riemann surfaces ([8]). This is another possible direction for further investigation in connection with our results. P. M. Gauthier suggested that some alternative proofs could relate to Runge’s pairs and harmonic approximation [3], [9], [10], [11].

Finally, we mention that it is open to characterize all subsets $F \subseteq \mathbb{C}$, such that the conclusion of Lemma 1.1 holds. We can easily find examples of sets $F$, which are not relatively closed in any simply connected open set $\Omega \subseteq \mathbb{C}$; in some of these examples the conclusion of Lemma 1.1 holds, but in some others it does not.

2. The results

In [2] N. U. Arakelian proved the following theorem.

**Theorem 2.1.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $F$ a relatively closed subset of $\Omega$. Then every function $f \in A(F)$ can be uniformly approximated on $F$ by functions $g \in H(\Omega)$ holomorphic in $\Omega$, if and only if the following hold:

i) $(\Omega \cup \{\alpha\}) \setminus F$ is connected and

ii) $(\Omega \cup \{\alpha\}) \setminus F$ is locally connected at $\alpha$, where $\Omega \cup \{\alpha\}$ is the one point compactification of $\Omega$.

Such a set $F$ is also called an Arakelian set in $\Omega$.

We say that $B$ is a ”hole of $F$" in $\Omega$, iff $B$ is a component of $\Omega \setminus F$, which is contained in a compact subset of $\Omega$. Note that $(\Omega \cup \{\alpha\}) \setminus F$ is connected iff $F$ has no holes in $\Omega$.

**Proposition 2.2.** A closed set $F$ in $\Omega$, without holes, is an Arakelian set in $\Omega$, if and only if for every compact set $K \subseteq \Omega$, the union of all holes of $F \cup K$ in $\Omega$, is contained in a compact subset of $\Omega$. 


For the case $\Omega = \mathbb{C}$ see [14]. We include a proof of the general case, for the sake of completeness.

**Proof.** ($\Rightarrow$) Suppose that there is a compact set $K \subseteq \Omega$, such that the union of all holes in $\Omega$ of $F \cup K$ is either unbounded or has zero distance from the boundary of $\Omega$. The complement $(\Omega \cup \{\alpha\}) \setminus K$ is a neighborhood of $\alpha$, in $\Omega \cup \{\alpha\}$. Hence, there exists a neighborhood $W \subseteq (\Omega \cup \{\alpha\}) \setminus K$ of $\alpha$, in $\Omega \cup \{\alpha\}$, such that $W \cap [(\Omega \cup \{\alpha\}) \setminus F]$ is connected. There exists a hole $B$ of $F \cup K$ such that $B \cap W \neq \emptyset$, because $(\Omega \cup \{\alpha\}) \setminus W$ is contained in a compact subset of $\Omega$. Observe that $bd(B) \subseteq F \cup K$. Since $W \cap [(\Omega \cup \{\alpha\}) \setminus F]$ and $F \cup K$ are disjoint, it follows that $W \cap [(\Omega \cup \{\alpha\}) \setminus F] \subseteq B \cup [(\Omega \cup \{\alpha\}) \setminus F]$, which is a contradiction.

($\Leftarrow$) Let $U \subseteq \Omega \cup \{\alpha\}$ be an open neighborhood of $\alpha$ in $\Omega \cup \{\alpha\}$. The set $K = (\Omega \cup \{\alpha\}) \setminus U$ is compact. Therefore, the union of all holes in $\Omega$ of $F \cup K$ is contained in a compact subset of $\Omega$. Let $B_1, B_2, \ldots$ be those holes. Also, let $W = (\Omega \cup \{\alpha\}) \setminus (K \cup B_1 \cup B_2 \cup \cdots)$. Obviously, $W$ is a neighborhood of $\alpha$ and $W \subseteq U$. We notice that $W \cap [(\Omega \cup \{\alpha\}) \setminus F]$ is the union of $\{\alpha\}$ and all the components of $\Omega \setminus (F \cup K)$, which are either unbounded or have zero distance from $bd(\Omega)$. Thus, $W \cap [(\Omega \cup \{\alpha\}) \setminus F]$ is connected and the proof is complete. ■

**Remark 2.3.** In order to determine whether a relatively closed set $F$, without holes, is Arakelian in $\Omega$, it suffices to check the condition of Proposition 2.2 only for an exhausting sequence, $(K_n)_{n \in \mathbb{N}}$, of compact subsets of $\Omega$. Such a sequence can be chosen so that $K_n \subseteq K_{n+1}^0$, $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} K_n = \Omega$ and $K_n$ has no holes in $\Omega$, for all $n \in \mathbb{N}$ ([15]). Also, we can assume that $K_n$ is a finite union of squares in a grid, whose sides are parallel to the coordinate axes and of length $\delta_n > 0$, $n \in \mathbb{N}$.

**Theorem 2.4.** If $F$ is an Arakelian set in $\Omega$, then for every open set $U \subseteq \Omega$, which contains $F$, there exists an open set $V \subseteq \Omega$ such that $F \subseteq V \subseteq U$ and $(\Omega \cup \{\alpha\}) \setminus V$ is connected.

**Proof.** Let $F$ be an Arakelian set in $\Omega$ and $U \subseteq \Omega$ an open set such that $F \subseteq U$. We define $d_x = \min\{\frac{dist(x,F)}{2}, dist(x, \Omega \setminus \Omega), 1\} > 0$, $x \in \Omega \setminus U$. Also, let $(K_n)_{n \in \mathbb{N}}$ be an exhausting sequence of compact subsets of $\Omega$.

- The relatively closed set $\Omega \setminus U$ has a locally finite cover in $\Omega$, $\{D(x_i, d_{x_i})\}_{i=1}^{\infty}$. It suffices to choose a finite cover of disks $D(x, d_x)$, $x \in \Omega \setminus U$, for each of the compact sets $(K_n \setminus K_{n-1}^0) \cap (\Omega \setminus U)$, $n \in \mathbb{N}$, $K_0 = \emptyset$. 

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This implies that every compact subset of $\Omega$ intersects a finite number of disks from $\{D(x_i, d_{x_i})\}_{i=1}^{\infty}$. Observe that $\bigcup_{i=1}^{\infty} D(x_i, d_{x_i})$ is closed in $\Omega$ and the set $\{x_i \mid i = 1, 2, \ldots\}$ has no accumulation points in $\Omega$. Hence, $U_1 = U \setminus \bigcup_{i=1}^{\infty} D(x_i, d_{x_i})$ is open and $F \subseteq U_1 \subseteq U$.

- We say that a point $x \in \Omega$ is joined with $\alpha$ by a curve $\Gamma$ in $E \subseteq \Omega$, if $\Gamma : [0, +\infty) \to E$ is continuous and $\Gamma(0) = x$, $\lim_{t \to +\infty} \Gamma(t) = \alpha$. The image of such a curve, $\Gamma([0, +\infty))$, is relatively closed in $\Omega$.

Each $x_i$ can be joined with $\alpha$ by a curve $\Gamma_i$, $i = 1, 2, \ldots$, such that for every $n \in \mathbb{N}$ only finite curves intersect the compact set $K_n$. Indeed, if $n \in \mathbb{N}$, then there are finitely many $x_i$ contained in the union of $K_n$ and all the holes of $F \cup K_n$ in $\Omega$. The points that we have not already joined with $\alpha$ (induction), are contained in components of $\Omega \setminus (F \cup K_{n-1})$, which are either unbounded or have zero distance from the boundary of $\Omega$. Let $x_i$ be such a point and $E$ the component of $\Omega \setminus (F \cup K_{n-1})$, which contains it. We can construct a curve $\Gamma_i$ in $E$, which joins $x_i$ with $\alpha$.

- If $s \geq n$, then by Proposition 2.2 $E$ contains a component $E_s$ of $\Omega \setminus (F \cup K_n)$, which is either unbounded or has zero distance from $\text{bd}(\Omega)$. Further, we can assume that $E_{s-1} \supseteq E_s$, $s \geq n$, where $E_{n-1} = E$. Let $x_{is} \in E_s$ and $\Gamma_{is}$ a curve in $E_{s-1}$, which joins $x_{is} - 1$ with $x_{is}$ (such a curve exists, since $E_{s-1}$ is open and connected), $s \geq n$, where $x_{in-1} = x_i$. The desired curve, $\Gamma_i$, consists of all $\Gamma_{is}$, $s \geq n$.

Thus, the union $\bigcup_{i=1}^{\infty} \Gamma_i$ is closed in $\Omega$ and the open set $V = U_1 \setminus (\bigcup_{i=1}^{\infty} \Gamma_i)$ has the desired properties. Obviously, $F \subseteq V \subseteq U_1 \subseteq U$ and $(\Omega \cup \{\alpha\}) \setminus V = \bigcup_{i=0}^{\infty} (D(x_i, d_{x_i}) \cup \Gamma_i) \cup \{\alpha\}$ is connected, which completes the proof. □

We note that the previous theorem, in the case $\Omega = \mathbb{C}$, is known; see [8], [12].

**Theorem 2.5.** If $F$ is a closed set in $\Omega$, such that for every open set $U \subseteq \Omega$, which contains $F$, there exists an open set $V \subseteq \Omega$ with $F \subseteq V \subseteq \Omega$ and $(\Omega \cup \{\alpha\}) \setminus V$ connected, then $F$ is an Arakelian set in $\Omega$.

**Proof.** First, we notice that $F$ has no holes in $\Omega$. If $B$ is a hole of $F$ in $\Omega$ and $x \in B$, then the open set $U = \Omega \setminus \{x\}$ contains $F$. Hence, there exists an open set $V \subseteq \Omega$ such
that \( F \subseteq V \subseteq U \) and \((\Omega \cup \{\alpha\}) \setminus V \) is connected. It holds \((\Omega \cup \{\alpha\}) \setminus F \supseteq (\Omega \cup \{\alpha\}) \setminus V\). Therefore, the latter is contained in the component of \((\Omega \cup \{\alpha\}) \setminus F\) that contains \(\alpha\). However, \([((\Omega \cup \{\alpha\}) \setminus V) \cup B \neq \emptyset\), which is a contradiction, because \(B\) is a component of \((\Omega \cup \{\alpha\}) \setminus F\) not containing \(\alpha\).

Suppose that \(F\) is not an Arakelian set in \(\Omega\). By Proposition \ref{prop:arakelian-characterization} there exists a compact set \(K \subseteq \Omega\), such that the union of all holes of \(F \cup K\) in \(\Omega\), is either unbounded or has zero distance from \(bd(\Omega)\). Moreover, Remark \ref{rem:finite-union-closed} enables us to assume that \(K\) is a finite union of closed squares in a grid, whose sides are parallel to the coordinate axes and of length \(\delta > 0\). Let \(B_1, B_2, \ldots\) be a sequence of holes of \(F \cup K\) in \(\Omega\) and \(x_n \in B_n, n \in \mathbb{N}\), such that \(x_n \to \alpha, \) as \(n \to +\infty\).

The open set \(U = \Omega \setminus \{x_1, x_2, \ldots\}\) contains \(F\). Thus, there exists an open set \(V\) with \(F \subseteq V \subseteq U\) and \((\Omega \cup \{\alpha\}) \setminus V\) connected. Observe that \((\Omega \cup \{\alpha\}) \setminus V\) intersects \(B_n\) and \((\Omega \cup \{\alpha\}) \setminus \overline{B_n}\), for all \(n \in \mathbb{N}\). This implies that there exists \(y_n \in (\Omega \setminus V) \cap bd(B_n) \cap bd(K), n \in \mathbb{N}\). Since \(bd(K)\) is compact, \((y_n)_{n \in \mathbb{N}}\) has a limit point \(y \in bd(K)\). Also, \(y \in \Omega \setminus V\), because \(\Omega \setminus V\) is closed in \(\Omega\) and \(y \in \Omega\).

We claim that \(y \in F \subseteq V\), which is obviously a contradiction. Indeed, if \(y \notin F\), then there exists \(\varepsilon > 0\) such that \(D(y, \varepsilon) \subseteq \Omega\) does not intersect \(F\). In addition, we can choose \(\varepsilon > 0\), depending on the place of \(y\) in the grid, so that \(D(y, \varepsilon) \setminus K\) has at most two components. This is a contradiction, since \(D(y, \varepsilon) \setminus K \subseteq \Omega \setminus (F \cup K)\) intersects infinite holes from \(\{B_n\}_{n=1}^{\infty}\). The proof is complete. ■

According to Theorems \ref{thm:characterization-arakelian} and \ref{thm:arakelian-connectedness} we have the following characterization of Arakelian sets.

**Theorem 2.6.** Let \(\Omega \subseteq \mathbb{C}\) be an open set and \(\Omega \cup \{\alpha\}\) its one point compactification. A relatively closed set \(F\) is Arakelian in \(\Omega\), if and only if for every open set \(U \subseteq \Omega\), which contains \(F\), there exists an open set \(V \subseteq \Omega\) such that \(F \subseteq V \subseteq U\) and \((\Omega \cup \{\alpha\}) \setminus V\) is connected.

**Lemma 2.7.** Let \(\Omega \subseteq \mathbb{C}\) be a simply connected open set. A set \(G \subseteq \Omega\) has connected complement in \(\Omega \cup \{\alpha\}\), if and only if its complement in the Riemann sphere, \((\mathbb{C} \cup \{\infty\}) \setminus G\), is connected.

**Proof.** \((\Rightarrow)\) Let \(G \subseteq \Omega\) with \((\Omega \cup \{\alpha\}) \setminus G\) connected. Assume that \((\mathbb{C} \cup \{\infty\}) \setminus G\) is not connected. Thus, there are two open sets \(U_1, U_2\) in \(\mathbb{C} \cup \{\infty\}\), such that \(U_i \cap [(\mathbb{C} \cup \{\infty\}) \setminus G] \neq \emptyset, i = 1, 2\). \((\mathbb{C} \cup \{\infty\}) \setminus G \subseteq U_1 \cup U_2\) and \(U_1 \cap U_2 \cap [(\mathbb{C} \cup \{\infty\}) \setminus G] = \emptyset\). Since \((\mathbb{C} \cup \{\infty\}) \setminus \Omega\) is connected and it is contained in \((\mathbb{C} \cup \{\infty\}) \setminus G\), it follows that \((\mathbb{C} \cup \{\infty\}) \setminus \Omega\) is contained in exactly one of the sets \(U_1, U_2).\) Without loss of
generality, we assume that \((\mathbb{C} \cup \{\infty\}) \setminus \Omega \subseteq U_1\). Observe that \((\mathbb{C} \cup \{\infty\}) \setminus U_1\) is a compact subset of \(\Omega\). This implies that \(V_1 = (U_1 \cap \Omega \setminus G) \cup \{\alpha\}\) and the set \(V_2 = U_2 \cap [(\Omega \cup \{\alpha\}) \setminus G] \subseteq (\mathbb{C} \cup \{\infty\}) \setminus U_1 \subseteq \Omega\) are two nonempty disjoint open sets in \((\Omega \cup \{\alpha\}) \setminus G\). Furthermore, it is easy to see that \((\Omega \cup \{\alpha\}) \setminus G \subseteq V_1 \cup V_2\) and thus we obtain a contradiction.

\((\Leftarrow)\) Let \(G \subseteq \Omega\) with \((\mathbb{C} \cup \{\infty\}) \setminus G\) connected. We define \(\phi: \mathbb{C} \cup \{\infty\} \rightarrow \Omega \cup \{\alpha\}\),
\[
\phi(x) = \begin{cases} x, & x \in \Omega \\ \alpha, & x \notin \Omega. \end{cases}
\]
Obviously, \(\phi\) is continuous and so \(\phi((\mathbb{C} \cup \{\infty\}) \setminus G) = (\Omega \cup \{\alpha\}) \setminus G\) is connected. \(\blacksquare\)

Combining Theorem 2.6 with Lemma 2.7, we obtain the following.

**Theorem 2.8.** Let \(\Omega \subseteq \mathbb{C}\) be a simply connected open set and \(F \subseteq \Omega\) a relatively closed set. Then the following are equivalent:

\(i)\) \(F\) is an Arakelian set in \(\Omega\)

\(ii)\) For every open set \(U \subseteq \mathbb{C}\), which contains \(F\), there exists a simply connected open set \(V \subseteq \mathbb{C}\) such that \(F \subseteq V \subseteq U\).

The next corollary is an immediate application of Theorem 2.8.

**Corollary 2.9.** If \(\Omega \subseteq \mathbb{C}\) is a simply connected open set, then the disjoint union of two Arakelian sets in \(\Omega\) is also Arakelian in \(\Omega\).

**Proof.** Let \(F_1, F_2\) be two disjoint Arakelian sets in \(\Omega\). Also, let \(U \subseteq \Omega\) an open set, which contains the union \(F_1 \cup F_2\). Since \(F_1\) and \(F_2\) are two disjoint closed sets in \(\Omega\), there exist two disjoint open sets \(G_1, G_2 \subseteq \Omega\) such that \(F_i \subseteq G_i\) and \(F_i \subseteq G_i\). By Theorem 2.8 there two simply connected open sets \(V_1, V_2\) with \(F_i \subseteq V_i \subseteq G_i \cap U, i = 1, 2\). Obviously, it holds \(F_1 \cup F_2 \subseteq V_1 \cup V_2 \subseteq U\) and since \(V_1 \cap V_2 = \emptyset\), every component of \(V_1 \cup V_2\) is simply connected. This implies that \(V_1 \cup V_2\) is a simply connected open set. Thus, according to theorem Theorem 2.8 the closed set \(F_1 \cup F_2\) is Arakelian in \(\Omega\) and the proof is complete. \(\blacksquare\)

In the case \(\Omega = \mathbb{C}\), an alternative proof of the previous result, using Proposition 2.2 can be found in [5]. The following example shows that Corollary 2.9 does not hold when \(\Omega\) is not simply connected.

**Example 2.10.** Let \(\Omega = D(0, 1) \setminus \{0\}\). Also, let \(F_1 = C(0, r_1)\) and \(F_2 = C(0, r_2)\), where \(0 < r_1 < r_2 < 1\). Observe that \(F_1, F_2\) are two disjoint compact subsets of \(\Omega\) with connected complements in the one point compactification of \(\Omega\). Hence, both sets are Arakelian in \(\Omega\). Nonetheless, the union \(F_1 \cup F_2\) is not Arakelian in \(\Omega\), since \((\Omega \cup \{\alpha\}) \setminus (F_1 \cup F_2)\) is not connected. \(\blacksquare\)
Even if $\Omega \subseteq \mathbb{C}$ is a simply connected open set, it is not true that the infinite denumerable union of pairwise disjoint Arakelian sets in $\Omega$ is also Arakelian in $\Omega$.

**Example 2.11.** Let $\Omega = \mathbb{C}$ and let $F_0 = \{2\} \times \mathbb{R}$, $F_n = \left(\left(\sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n} \sum_{i=0}^{n-1} \frac{1}{2^i}\right) \times [0, n]\right) \cup \left(\left(\sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n} \sum_{i=0}^{n-1} \frac{1}{2^i}\right) \times \{n\}\right)$, $n \geq 1$. It is easy to see that each $F_n$, $n = 0, 1, \ldots$, is an Arakelian set in $\mathbb{C}$. However, $F = \bigcup_{n=0}^{\infty} F_n$ is not Arakelian in $\mathbb{C}$, because despite the fact that $F$ is closed and $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected, the union of all holes in $\mathbb{C}$ of $D(0, r) \cup F$, $r \geq 2$, is unbounded. ■

Finally, we present another application of our characterization.

**Corollary 2.12.** Let $\Omega \subseteq \mathbb{C}$ be a simply connected open set and $F \subseteq \Omega$ a relatively closed set. Also, let $f \in A(F)$ satisfying $f(z) \neq 0$, for all $z \in F$. Then there exists a function $g \in A(F)$ such that $f = e^g$.

**Proof.** According to Tietze’s extension theorem, there exists a continuous extension of $f$ on $\Omega$, which we denote by $\tilde{f} : \Omega \to \mathbb{C}$. The open set $U = \Omega \setminus \tilde{f}^{-1}(0)$ contains $F$. By Theorem 2.8, there is a simply connected open set $V$ with $F \subseteq V \subseteq U$. This implies that there exists a continuous function $\tilde{g} : V \to \mathbb{C}$ such that $\tilde{f}\big|_V = e^{\tilde{g}}$. The function $g = \tilde{g}\big|_F$, is obviously continuous on $F$ and $f = e^g$. Since $f\big|_{F^0}$ is holomorphic, $g$ is also holomorphic in $F^0$. Thus, $g \in A(F)$ and the proof is complete. ■

We notice that for $\Omega = \mathbb{C}$ Corollary 2.12 is known; see [12].

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