Reply To “Comment on ‘Quantum String Seal Is Insecure’ ”

H. F. Chau

Department of Physics, University of Hong Kong, Pokfulam Road, Hong Kong and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong

(Dated: February 2, 2008)

In Phys. Rev. A 76, 056301 (2007), He claimed that the proof in my earlier paper [Phys. Rev. A 75, 012327 (2007)] is insufficient to conclude the insecurity of all quantum string seals because my measurement strategy cannot obtain non-trivial information on the sealed string and escape detection at the same time. Here, I clarify that our disagreement comes from our adoption of two different criteria on the minimum amount of information a quantum string seal can reveal to members of the public. I also point out that He did not follow my measurement strategy correctly.

PACS numbers: 03.67.Dd, 03.67.Hk, 89.20.Ff, 89.70.+c

In Ref. [1], He said that the key measurement strategy used in my paper [2] to show the insecurity of all quantum string seals was in the form

\[ Q_{ij} = a(\nu)I + b(\nu)|i\rangle\langle j|. \] (1)

(See Eq. (29) in Ref. [2].) He then attempted to find a "loophole" in my conclusion by means of two "counter-example" quantum string seals.

His first scheme (Scheme A) is a family of quantum string seals each with a different sealed string length. In Scheme A, each bit in a string is independently encoded as a publicly accessible state in the form \( \cos \theta_i |b_i\rangle + \sin \theta_i |\bar{b}_i\rangle \), where \( b_i \) is the value of the bit and \( |\theta_i| \leq \Theta/n^\alpha \). Here \( \Theta < \pi/4 \) and \( \alpha < 1/2 \) are two positive constants, and \( n \) is the bit string length. The probabilities of correctly determining a particular bit of the string and the entire string are \( \approx \cos^2(\Theta/n^\alpha) \) and \( \approx \cos^{2n}(\Theta/n^\alpha) \), respectively [1, 3]. In other words, as \( n \to \infty \), the chance of correctly finding out the entire string is negligibly even though the percentage of correctly determined bits approaches 1. Furthermore, the number of incorrectly determined bits increases without bound in the large \( n \) limit.

In his second scheme (Scheme B), the classical message \( i \) is encoded as a publicly accessible quantum state \( \sum_j \lambda_{ij} |j\rangle \). He claimed that (the magnitudes of) \( \lambda_{ij} \)'s should be very small (large) if the contents of the messages \( i \) and \( j \) were irrelevant (close) [1]. In both schemes, He said that to follow the instructions in my paper [2], a member of the public (Bob) had to apply the measurement operators in the form \( Q_{ij} \)'s to each encoded qubit. Nevertheless, this measurement strategy has little chance to obtain non-trivial information on the sealed string and escape detection simultaneously. He further claimed that Bob is extremely likely to be caught when the parameter \( \nu \) used in \( Q_{ij} \) approaches 1 because the maximum probability of correctly determining the entire sealed string can be made arbitrarily small by increasing \( n \) [1].

One can judge the validity of He's claim by answering the following two questions: What is the minimum (classical) information a quantum string seal can reveal to Bob? And is He really using my measurement strategy reported in Ref. [2] in both schemes?

To answer the first question, let us recall that the objective of a quantum seal is to allow detection of Bob's measurement with a high probability without concealing the sealed message [4]. There are at least three possible ways to define what is the meaning of non-concealment in this context; and I list them in the order of decreasing ability to recover the sealed message.

**Criterion A.** There is a measurement for Bob in such a way that the conditional entropy \( H_{\text{cond}} \) of the sealed message given the measurement results is less than a fixed non-negative number \( H_{\text{crit}} \) independent of the string length \( n \). Moreover, the mutual information \( I \) between the sealed message and the measurement result divided by the entropy \( H \) of the sealed message is of the order of 1. Thus the expected number of bits whose values are wrongly determined by Bob is finite even though Bob's chance of correctly determining the entire sealed string can be low.

**Criterion B.** The value \( H_{\text{cond}}/H \to 0 \) (and hence \( I/H \to 1 \)) in the limit of \( n \to \infty \). In other words, the percentage of incorrectly determined bits approaches 0 in the large \( n \) limit although the number of incorrectly determined bits may approach infinity.

**Criterion C.** The value \( H_{\text{cond}}/H < c \) (and hence \( I/H \geq 1 - c \) in the large \( n \) limit, where \( c \) is a fixed positive number of order of 1). That is, the percentage of incorrectly determined bits is bounded by a non-zero value in the large \( n \) limit.

Clearly, Scheme A proposed by He satisfies Criterion B but not Criterion A. Furthermore, independently sealing each bit of a classical string by an imperfect quantum bit seal is an example of a family of quantum seals obeying only Criterion C.

*Electronic address: hfchau@hkusua.hku.hk*
A major source of the disagreement between He and myself comes from the fact that I have adopted Criterion A while He used the more lenient Criterion B. I believe that it is more natural to adopt the non-concealment Criterion A as the expected Hamming distance between the Bob’s measurement result and the sealed message is bounded. Since most of the discussion in Ref. [2] was focused on scaling a fixed finite number of possible messages $N$, the distinction between the above three possible non-concealment criteria was not clearly made there.

Now I answer the second question: Is He using my measurement strategy reported in Ref. [2] in his analysis? As I have already pointed out in the second page of Ref. [2] that the maximum probability for Bob to correctly determine the entire sealed string can be small. Thus the first step in constructing the measurement strategy for a quantum string seal is to find a partition $\mathcal{P}$ of the set of all possible sealed messages so that the maximum probability for Bob to correctly determine which element in the partition does the sealed message belong to $p_{\text{max}}$ is of the order of 1. In the case of Scheme A, a possible choice is to partition the $N = 2^n$ possible values of the sealed bit string according to the values of its first $n^{2n}$ bits. In the large $n$ limit, $p_{\text{max}}$ for this choice equals $\exp(-\Theta^2) > 0.5$. In addition, such a probability can be attained by measuring the first $n^{2n}$ qubits in the standard basis and keeping the remaining qubits untouched; and contrary to He’s claim of using $Q_{\text{av}}$’s, my measurement strategy reported in Ref. [2] is to apply the measurement operators $M_i(\nu)$’s defined by Eqs. (11) and (12) in Ref. [2]. It is straightforward to check that using these measurement operators, Bob can obtain $n^{2n}$ bits of information on the sealed string and escape detection with at least $0.5^2 = 0.25$ chance simultaneously for any $1/2 \leq \nu \leq 1$. That is, as $n \to \infty$, the amount of information on the sealed message obtained is infinite although the percentage of information obtained is 0. A comparison of my measurement strategy on families of quantum seals satisfying the three different non-concealment criteria are tabulated in Table I. In particular, only by adopting Criterion A is it always possible to find a partition $\mathcal{P}$ satisfying $\log |\mathcal{P}| \lesssim \log N$ and $p_{\text{max}}$ is of the order of 1.

Let me further clarify. Measurement operators $M_i$’s can be applied to any quantum string seal to obtain information with a high chance of escaping detection. In contrast, $Q_{\text{av}}$’s in the form of Eq. (1) are the measurement operators that maximize the average fidelity between the measured state and the sealed state for a special quantum string seal designed to prove Theorem 1 in Ref. [2] only. In fact, for a general quantum string seal, the measurement operators that maximize the average fidelity between the measured state and the sealed state need not be in the form $Q_{\text{av}}$’s. One example is the (perfect) quantum string seal that encodes the classical message $i$ as $|\phi_i\rangle \equiv \sum_{j=0}^{N-1} \omega_i^{ij} |j\rangle / \sqrt{N}$ for all $i = 0, 1, \ldots, N - 1$, where $\omega_N$ is a primitive $N$th root of unity. Clearly, one can correctly obtain the entire sealed message without being caught using the projective measurement operators $|\phi_i\rangle \langle \phi_i|$’s [3, 4, 5]. Besides, these operators are not in the form of Eq. (1). This example also shows that, contrary to He’s claim in the security analysis of Scheme B [1], it is possible that the magnitudes of $\lambda_{ij}$’s are all equal for all $i, j$.

In conclusion, it is the adoption of different non-concealment criteria that causes the disagreement between He and myself on the security of quantum string seals. Moreover, the issue is further complicated by He’s misuse of $Q_{\text{av}}$’s as the measurement operator in his security analysis. As shown in Table I all quantum string seals are insecure if the non-concealment Criterion A is used. However, some quantum string seals are secure if non-concealment Criteria B or C is used in the large $n$ limit. Note further that in the case of adopting Criterion C, even an honest Bob can obtain only a fraction of the sealed message in the large $n$ limit.

Finally, I remark on passing that He’s claim in Ref. [1] that by linearity, the operator $a|\psi\rangle + \beta |\psi\rangle\langle \psi| |i\rangle$ actually meant applying the identity operator with a certain probability is incorrect. This is because the state $a|\psi\rangle + \beta |\psi\rangle\langle \psi| |i\rangle$ does not equal a mixture of pure states $|\psi\rangle$ and $|i\rangle$.

Acknowledgments

This work was supported by the RGC grant No. HKU 7010/04P of the HKSAR Government.

[1] G. P. He, Phys. Rev. A 76, 056301 (2007).
[2] H. F. Chau, Phys. Rev. A 75, 012327 (2007).
[3] G. P. He, Int. J. Quantum Inf. 4, 677 (2006).
[4] H. Bechmann-Pasquinucci, Int. J. Quantum Inf. 1, 217 (2003).
[5] H. Bechmann-Pasquinucci, G. M. D’Ariano, and C. Mac-
chiavello, Int. J. Quantum Inf. 3, 435 (2005).
[6] H. F. Chau, Phys. Lett. A 353, 31 (2006).