Fast and Space-optimal Low-rank Factorization in the Streaming Model With Application in Differential Privacy

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Abstract

In this paper, we consider the problem of computing a low-rank factorization of an \(m \times n\) matrix in the general turnstile update model. We consider both the private and non-private setting.

1. In the non-private setting, we give a space-optimal algorithm that computes a low-rank factorization. Our algorithm maintains three sketches of the matrix instead of five as in Boutsidis et al. (STOC 2016). Our algorithm takes \(O(1)\) time to update the sketch and computes the factorization in time linear in the sparsity and the dimensions of the matrix.

2. In the private setting, we study low-rank factorization in the framework of differential privacy and under turnstile updates. We give two algorithms with respect to two levels of privacy. Both of our privacy levels are stronger than earlier studied privacy levels, namely that of Blocki et al. (FOCS 2012), Dwork et al. (STOC 2014), Hardt and Roth (STOC 2012, STOC 2013), and Hardt and Price (NIPS 2014).

   (a) In our first level of privacy, \(\text{Priv}_1\), we consider two matrices as neighboring if their difference has a form \(uv^T\) for some unit vectors \(u\) and \(v\). Our private algorithm with respect to \(\text{Priv}_1\) matches the optimal space bound up to a logarithmic factor and is optimal in the terms of the additive error incurred. The algorithm is also efficient and takes time linear in the input sparsity of the matrix and quadratic in \(\min\{m, n\}\). Our bound quantitatively improve the result of Hardt and Roth (STOC 2012) by a factor of \(\sqrt{k \log(1/\delta)}\) when \(m \leq n\), a scenario considered by Hardt and Roth (STOC 2012).

   (b) Our second level, \(\text{Priv}_2\), generalizes \(\text{Priv}_1\). In \(\text{Priv}_2\), we consider two matrices as neighboring if their difference has unit Frobenius norm. Our private algorithm with respect to \(\text{Priv}_2\) is computationally more efficient than our first algorithm – it uses \(O(\log(m+n))\) time to update and computes the factorization in time linear in the input sparsity and the dimensions of the matrix. This algorithm incurs optimal additive error and uses optimal space when \(n \ll m\).

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1 Introduction

In the list compiled by Dongarra and Sullivan [18] that “assemble the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century”, the fourth entry is “The Decompositional Approach to Matrix Computation”. Technically, matrix decomposition is not an algorithm, but its inclusion is often argued for by its immense application in many areas of computation [47]. An incomplete list of applications of matrix factorization includes data mining [4], recommendation systems [21], conic fitting problems [67], information retrieval [55, 59], system and control [46], web search [11, 44], chemometrics [60, 65], clustering [14, 19, 48], solving PDE [30, 31], and learning distributions [2, 41]. For example, consider the following two graphs,

Graph 1. In the graph $G_1$, there is an edge between two nodes $(u, v) \in V$ if there is a student who did his undergraduate from a university $u$ and then moved to the company/university $v$. The weight on the edge is the number of all such students who moved from $u$ to $v$ after graduation.

Graph 2. The graph $G_2$ is a generalization of $G_1$. The contribution of a person to the weight of an edge between two nodes $(u, v)$ is proportional to the time spent after moving from $u$ to $v$ since graduation.

The weight on every edge $(i, j)$ depends on the number of undergraduates who moved from $i$ to $j$, this makes such graphs vital for companies and universities to target incoming graduates. One would, therefore, like to build a data mining system or a recommendation system based on the statistics of these graphs or perform efficient search of any entry in these graphs. In order to build such a system, let us look closely at some of the features of these graphs. The adjacency matrices of $G_1$ and $G_2$ have large Frobenius norms and can be dense (depending on the number of graduating students). Therefore, storing the graphs requires a lot of storage. Since students get an offer in an arbitrary order; $G_1$ is updated in a turnstile manner. Likewise people change job in arbitrary order during their career, i.e., $G_2$ is updated in a turnstile manner. This makes storing and updating a sketch, that allows efficient construction of a recommendation system or perform data-mining, challenging.

Low-rank approximation. To assuage these issues, a general technique used in the literature is to compute a low-rank approximation (LRA) of the adjacency matrix in the turnstile update model and use this approximation to perform all the computational tasks. This technique is used by many previous works where matrix factorization is used. As a result, LRA of large matrices (and not just adjacency matrices of graphs) has received a lot of attention in the recent past. Most of the proposed algorithms used either random sketching [11, 40, 58] or random sampling [3, 20, 22, 15, 29, 56, 57]. The run-time of these algorithms were improved substantially in subsequent works [13, 16, 45, 50, 54, 53, 58]. However, these works either assumed that the matrix is static or used a lot of storage when matrices are updated in the general turnstile manner. Recently, Boutsidis et al. [8] gave the first space-optimal and low communication cost algorithm to compute LRA; however, they do not optimize the computational cost of their algorithm.

Low-rank approximation and privacy. There is also a natural privacy concern associated with such graphs and many applications where LRA is used. The privacy issue has been recently exemplified by the deanonymization of Netflix datasets [52] and Hunch and Amazon [9]. In the latter case, no data is directly released (as in the Netflix challenge), but items similar to a given item are shown to the users. This was one of the motivations for the work of McSherry and Mironov [49]. Even training data of learning theoretic algorithms have natural privacy concerns. This motivated the line of research of differentially private learning [37, 61].
Therefore, a more natural requirement would be to compute low-rank approximation of a matrix while providing some robust privacy guarantee, like differential privacy \[24\]. Differentially-private LRA has been studied by many recent works (see, Section 1.3 for more details), but these algorithms are optimized for the additive error incurred, and not for the space used by the algorithm or the computational cost to compute the LRA. Moreover, except for Dwork et al. \[28\] and Upadhyay \[63\], none of the known results work in the streaming model. The algorithms of Dwork et al. \[28\] and Upadhyay \[63\] works only in the row-wise update model and not in the general turnstile update model as in \(G_1\) and \(G_2\). Further, Dwork et al. \[28\] approximates only the right singular vectors.

1.1 Problem Description

In this paper, we study time-efficient and sublinear space algorithms for computing low-rank factorization when the input matrix is updated in a general turnstile manner (see Problem 1). If there were no space or time constraints, then computing a low-rank factorization is equivalent to computing a low-rank approximation and we can use any of the earlier known results. However, under the time and space constraints considered in this paper, it is not clear how can we use a low rank approximation to get a low-rank factorization. In order to keep our results as general as possible, we do not assume any structure in the matrices, like symmetry or incoherence.

In the non-private setting, our goal is to construct a data-structure such that (i) the data-structure uses space sub-linear in the size of the data and its overall update time is linear in the sparsity of the private matrix, (ii) performing the low-rank factorization using these data-structures leads to a small multiplicative error. In the private setting, we further require that the data-structure does not lead to a privacy leak of the entries in the input matrix. We consider differential privacy for the privacy framework. Differential privacy is a well-established notion of privacy with many desirable features such as robustness and modular composition.

Definition 1. A randomized algorithm \(\mathcal{M}\) gives \((\varepsilon, \delta)\)-differential privacy, if for all neighbouring databases \(\mathbf{A}, \mathbf{A}'\) and all subsets \(S\) in the range of \(\mathcal{M}\), \(\Pr[\mathcal{M}(\mathbf{A}) \in S] \leq \exp(\varepsilon) \Pr[\mathcal{M}(\mathbf{A}') \in S] + \delta\), where the probability is over the coin tosses of \(\mathcal{M}\).

Since we consider differential privacy in the streaming setting, we define it in terms of neighboring streams.

Definition 2. A randomized algorithm \(\mathcal{M}\) gives \((\varepsilon, \delta)\)-differential privacy in the streaming model, if for all neighbouring data-streams \(\mathbf{A}, \mathbf{A}'\) and all subsets \(S\) in the range of \(\mathcal{M}\), \(\Pr[\mathcal{M}(\mathbf{A}) \in S] \leq \exp(\varepsilon) \Pr[\mathcal{M}(\mathbf{A}') \in S] + \delta\), where the probability is over the coin tosses of \(\mathcal{M}\).

Equipped with the above definition, we can define our model. We enumerate the key aspects of the (differentially private) turnstile update model of computation as follows.

- There are three entities: a stream generator \(S\), a (database) curator \(C\), and an analyst \(A\). The stream generator \(S\) has a database matrix as input which it streams to the curator.
- \(S\) starts the process at time \(\tau = 0\) and \(C\) initializes its data structure to \(D_0\). The stream generator \(S\) provides \(C\) with the database matrix in a general turnstile manner, i.e., the stream consists of pairs \((q, c_\tau)\), where \(q \in [m] \times [n]\) is the location and \(c_\tau\) is the item to be added or deleted at location \(q\).
- \(C\) update its data structure to \(D_\tau\) using \(D_{\tau-1}\) and the data received during the time \(\tau\). At any time, \(C\) uses space that is at most sublinear in the size of the streamed matrix. and takes total time that is linear in the length of the stream.
Once the stream is over at $\tau$, $\mathcal{A}$ requests $\mathcal{C}$ to perform certain tasks to which $\mathcal{C}$ responds using only $\mathcal{D}_\tau$.

- In the setting of differential privacy, we further require that the responses of the curator should also satisfy Definition 2 for neighboring streams.

We can now define the problem that we consider in this paper.

**Problem 1.** $(\alpha, \beta, \gamma, k)$-LRF. Given parameters $\alpha, \beta, \gamma$, a private $m \times n$ matrix $\mathbf{A}$ (received in a turnstile streaming model) and the target rank $k$, find a rank-$k$ matrix factorization $\tilde{\mathbf{U}}_k, \tilde{\Sigma}_k, \tilde{\mathbf{V}}_k$ such that

$$
\Pr \left[ \| \mathbf{A} - \tilde{\mathbf{U}}_k \tilde{\Sigma}_k \tilde{\mathbf{V}}_k^T \|_F \leq (1 + \alpha) \min_{\text{rank}(\mathbf{A}_k) \leq k} \| \mathbf{A} - \mathbf{A}_k \|_F + \gamma \right] \geq 1 - \beta.
$$

When privacy is also a concern, we require that the algorithm that computes $(\alpha, \beta, \gamma, k)$-LRF also preserves $(\varepsilon, \delta)$-differential privacy. When an algorithm only outputs a rank-$k$ matrix (and not a factorization), then we say that the algorithm computes $(\alpha, \beta, \gamma, k)$-low rank approximation of the input matrix $\mathbf{A}$.

Central to the notion of differential privacy is the concept of neighboring database. In this paper, we consider databases presented in the form of matrices. To motivate our definition of neighboring databases, we refer back to the graphs $G_1$ and $G_2$. Graphs like $G_1$ are well studied in the literature of differential privacy (see, for example, [6, 28, 34, 33, 62, 63]) under the privacy level known as edge-level privacy. In edge-level privacy, the presence or absence of a person corresponds to change in a single edge. On the other hand, graphs like $G_2$ are not very well studied. One of the main problems with such graphs is that a person might have been in many places during his career; thereby, presence or absence of a person would be reflected on many edges. The straightforward applications of earlier results on edge-level privacy [6, 34, 62, 63] to the graphs of form $G_2$ would lead to a large privacy loss. In this paper, we ask the question whether we can achieve comparable privacy loss in the scenario as in graph $G_2$ without compromising on the correctness or privacy loss. This motivates us to consider a stronger privacy level than edge-level privacy. We define our privacy level next.

**Granularity of privacy.** We consider two different levels of privacy. In the first level, $\text{Priv}_1$, two databases are neighboring if the difference of their corresponding matrix representation has the form $uv^T$ for some unit vectors $u$ and $v$. In the second level, $\text{Priv}_2$, we consider two databases as neighboring if the difference of their matrix representations has Frobenius norm at most 1. Note that $\text{Priv}_1$ is a special case of $\text{Priv}_2$. We consider two streams neighboring if they are formed by streaming neighboring matrices.

### 1.2 Our Contributions and Technical Overview

In this paper, we do not assume any upper bound on the Frobenius norm of the input matrix. For example, adjacency matrices corresponding to both $G_1$ and $G_2$ can have arbitrary large Frobenius norm. In fact, none of our results depend on the Frobenius norm, unlike some private algorithms [28, 34]. All of our results are for the turnstile update model. Our contributions in this paper are as follows.

1. **Non-private fast and space-optimal algorithm.** Our first result is a space-optimal and update time efficient algorithm for outputting LRF of a matrix in the turnstile update model. Our algorithm is different from Boutsidis et al. [8]. We show that one only need three sketches (one for approximating row-space, one for approximating column-space and one for approximating the matrix), and not five sketches as used by Boutsidis et al. [8], to get similar bounds. Even if one does count the update-time efficiency, we believe that maintaining three sketches instead of five would be more efficient in actual implementation (and not asymptotic analysis).
2. Space-optimal private streaming algorithm. We also convert our non-private algorithm to give a private algorithm that is space-optimal under a stronger privacy level, \( \text{Priv}_{\text{I}} \) (see, Section 1.2.1) in the general turnstile update model. The following is an informal statement of Theorem 15.

**Theorem (informal statement of Theorem 15).** Let \( \mathbf{A} \) be an \( m \times n \) matrix with \( \Omega(n) \) non-zero entries. Let \( p = \max \{ m, n \} \). Then there is an \((\epsilon, \delta)\)-differentially private algorithm that receives the private matrix in a turnstile streaming model, uses \( O_\beta((m + n)(k + \epsilon^{-1})\alpha^{-1}) \) space, and outputs a rank-\( k \) factorization \( \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^T \) of \( \mathbf{A} \) in time \( O(\Omega(n)) + O_\delta(nk^2\alpha^{-2} + mk^2\alpha^{-3} + k^3\alpha^{-4}) \) such that

\[
\| \mathbf{A} - \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^T \|_F \leq (1 + \alpha)\| \mathbf{A} - \mathbf{A}_k \|_F + O_\delta \left( \epsilon^{-1} \left( \sqrt{kpn(1 + 1/\alpha)} + \sqrt{kmn/(p + p_\alpha)} \right) \right),
\]

where \( O_\delta(\cdot) \) hides a factor of \( \text{poly}(\log(1/\delta)) \) and \( \mathbf{A}_k \) is the best rank-\( k \) approximation.

To prove our result, we give an alternate proof of one of the intermediate theorems of Boutsidis et al. [8]. Our alternate proof uses two optimization problems and uses the solution to these optimization problems. We feel that our proof is simpler. It also gives explicit optimum solutions to the two problems, which makes it easy to extend to (and get a tight bound in) the case of private low-rank factorization. We also prove a lemma (Lemma 13) that gives the best rank-factorization. We feel that our proof is simpler. It also gives explicit optimum solutions to the two problems, which makes it easy to extend to (and get a tight bound in) the case of private low-rank factorization.
Comparison of Non-private Algorithms.

1.2.1 Comparison with Previous Works

One can consider that Priv\_run time of general than Blocki et al. ∑ Comparison of private algorithms (privacy level). work in Table 1. of Clarkson and Woodruff [13] in the protocol of Boutsidis et al. learning of this paper, Woodruff [66] showed that one can obtain the similar bound by using the embedding the first space-optimal algorithm for low-rank approximation; however, they run in time $O$ the data. However, this is not captured by either the edge-level privacy or by the node-level privacy.

Hardt and Roth [34] showed that any differentially private low-rank approximation incurs an $\Omega(\sqrt{kn+k})$ additive error by showing a reduction to the linear reconstruction attack [17]. They showed the lower bound when the differentially private mechanism can access the private matrix as many times as possible. Our bound matches the lower bound, up to a small multiplicative factor, while allowing the private mechanism to access the private matrix only in a turnstile manner.

3. Optimal error private streaming algorithm.

Our last contribution is an algorithm that has constant update time and computes private low-rank factorization efficiently with an optimal additive error under the stronger level of privacy, Priv\_2 (see, Section 1.2.1).

**Theorem** (informal statement of Theorem 28). Let $\mathbf{A}$ be an $m \times n$ matrix with $nn(\mathbf{A})$ non-zero entries. Then there is an $(\varepsilon, \delta)$-differentially private algorithm that receives the private matrix in a turnstile streaming model, updates in $O((\log(m + n))$ time, uses $O_{\varepsilon}(m + n\varepsilon^{-2})k\alpha^{-2}$ space, and outputs a rank-$k$ factorization $\mathbf{U}_k \Sigma_k \mathbf{V}_k^T$ of $\mathbf{A}$ in $O(nn(\mathbf{A})) + O_{\varepsilon}(nk^{2}\alpha^{-3} + mk^{2}\alpha^{-5} + k^{3}\alpha^{-6})$ time such that

$$\|\mathbf{A} - \mathbf{U}_k \Sigma_k \mathbf{V}_k^T\|_F \leq (1 + \alpha)\|\mathbf{A} - \mathbf{A}_k\|_F + \widetilde{O}_\delta \left( (1 + \alpha)\sqrt{km} + \frac{n(1 + \alpha)}{\alpha^3} \left( k + \frac{1}{\alpha} \right) \frac{1}{\varepsilon} \right),$$

where $\widetilde{O}_\delta(\cdot)$ hides a factor of $\text{poly}(\log(1/\delta))$ and $\mathbf{A}_k$ is the best rank-$k$ approximation.

**Optimality of the result.** The above result matches the run-time of the most efficient non-private algorithm up to a factor of $\alpha^{-3}$ [13, 50]. We match the lower bound of $\Omega(\sqrt{kn+k})$ on the additive error given by Hardt and Roth [34], up to a small multiplicative factor (and constant if $\alpha = \sqrt{2} - 1$), while allowing the private mechanism to access the private matrix only once and that, too, in a turnstile manner.

1.2.1 Comparison with Previous Works

**Comparison of Non-private Algorithm.** In the non-private setting, our algorithm runs in time $\widetilde{O}_\beta(nn(\mathbf{A}) + (m + n)k^2/\alpha^2 + k^3/\alpha^3)$ and uses $O((n + m)k\log(1/\beta)/\alpha)$ space. Clarkson and Woodruff [13] achieves a run time of $\widetilde{O}_\beta(nn(\mathbf{A}) + mk^2/\alpha^2 + nk^2/\alpha^4)$ and uses $\widetilde{O}_\beta(mk/\alpha + nk/\alpha^2)$ space. Boutsidis et al. [8] gave the first space-optimal algorithm for low-rank approximation; however, they run in time $O(mnk)$. After learning of this paper, Woodruff [66] showed that one can obtain the similar bound by using the embedding of Clarkson and Woodruff [13] in the protocol of Boutsidis et al. [8]. We tabulate our result with previous work in Table 1.

**Comparison of private algorithms (privacy level).** The granularity of privacy used in this paper is more general than Blocki et al. [6], Dwork et al. [28], Hardt and Roth [34, 35], Hardt and Price [33], and Upadhyay [63]. One can consider that Priv\_1 and Priv\_2 lies between the node-level privacy [43] and edge-level privacy. For example, in the adjacency matrix corresponds to $G_2$, every edge $e = (u, v)$ is given a weight $\sum_{i}(t_{i,e}/t_i)$, where $t_{i,e}$ is the time spent by person $i$ after moving from $u$ to $v$ and $t_i$ is the total time lapsed since his/her graduation. In this case, Priv\_2 corresponds to the presence or absence of a single individual in the data. However, this is not captured by either the edge-level privacy or by the node-level privacy.
Hardt and Roth [34] and Upadhyay [63] studied problems in Appendix C. We give a comparison of the results in Table 2. Problem 2 differs from Problem 1 only in the sense that Problem 2 only requires the low-rank approximation while Problem 1 also requires low-rank factorization. Below, we enumerate the key differences between our result and previous results [34, 63].

1. Additive error. Both of our bounds improves on Upadhyay [63] by a factor of $k^{3/2} \alpha \sqrt{\log(1/\delta)}$. To make a reasonable comparison with Hardt and Roth [34], we consider their result with no-coherence.
assumption and our result when $\alpha = \sqrt{2} - 1$. Combining Theorem 4.2 and 4.7 in Hardt and Roth [34] results in an additive error $O((\sqrt{km \log(1/\delta)} + ck \log(k/\delta)\sqrt{n})\varepsilon^{-1})$, where $c$ is the maximum entry in the projection matrix. In other words, Theorem 15 improve the result of Hardt and Roth [34] by a $c\sqrt{k \log(1/\delta)}$ factor when $m \leq n$.

2. Time and space efficiency. The result of Hardt and Roth [34] is applicable when the private matrix is not dense (see the discussion in Hardt and Roth [34, Sec 1.1]). However, their algorithm does not use low-sparsity of the input matrix. On the other hand, the running and update time of our algorithms depends on the number of non-zero entries of the input matrix and uses optimal space. Hardt and Roth [34] and Upadhyay [63] uses $O(kmn)$ time and $O(mn)$ space. Therefore, even if the matrix is not sparse, our algorithms are still more efficient in terms of both the space and time.

3. Streaming constraints. Hardt and Roth [34] is a private version of the two-pass algorithm of Halko et al. [32]. The algorithm of Upadhyay [63] is one-pass, but assumes that the matrix is streamed row-wise. A row-wise update is an easier problem (with respect to the space required) compared to the turnstile update model as illustrated by Clarkson and Woodruff [11].

4. Applicability. The result of Hardt and Roth [34] is applicable when the Frobenius norm is bounded by $O(\sqrt{n})$ and $m \ll n$. We do not put such restrictions on the size of $m$ and $n$ and the Frobenius norm.

1.3 Related Work

There has been a lot of work that used random projection [11, 40, 58] or random sampling [8, 20, 22, 15, 29, 56, 57] to give low-rank approximation. Many of the latter algorithms were improved substantially in the follow-up work [8, 13, 16, 45, 50, 54, 58]. Differential privacy was introduced by Dwork et al. [24]. Since then, many algorithms for preserving differential privacy have been proposed in the literature (see, Dwork and Roth [26]). Dwork et al. [25] first considered streaming algorithms with privacy under the model of pan-privacy, where the internal state is known to the adversary. Subsequently, there have been some works on online private learning [28, 37, 61] for various tasks. There are some recent works on differentially private LRA as well. Blum et al. [7] first studied this problem in the Frobenius norm. This was improved by Hardt and Roth [34] under the low coherence assumption. Upadhyay [63] later showed that one can make the two-pass algorithm of Hardt and Roth [34] single-pass. Differentially-private LRA has been studied in the spectral norm as well by many works [10, 42, 35, 33]. Recently, Dwork et al. [28] gave a tighter analysis of the algorithm of Blum et al. [7] and used it to give a private online algorithm for covariance matrices.

**Organization of the paper.** In Section 3 we give our non-private algorithm for LRF in the general turnstile update model. In Section 4 we transform our non-private algorithm to a private algorithm under $Priv_1$ while incurring optimal additive error. In Section 5 we give an algorithm that has efficient update time under more finer granularity of privacy, $Priv_2$.

2 Notations, Preliminaries, and Known Results

We let $\mathbb{N}$ to denote the set of natural numbers. We use bold-face capital letters to denote matrices and bold-face small letters to denote vectors. We denote by $0_{m \times n}$ the all-zero $m \times n$ matrix and by $I_n$ the $n \times n$ identity matrix. For a matrix $A$, we denote the best $k$-rank approximation of $A$ by $[A]_k$ and its Frobenius norm by $\|A\|_F$. The singular-value decomposition (SVD) of an $m \times n$ rank-$r$ matrix $A$ is a decomposition of $A$ as a product of three matrices, $A = U\Sigma V^T$ such that $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$ has orthonormal
columns and $\Sigma \in \mathbb{R}^{r \times r}$ is a diagonal matrix with singular values of $\mathbf{A}$ on its diagonal. For a matrix $\mathbf{A}$, we use the symbol $r(\mathbf{A})$ to denote its rank and $\Delta(\mathbf{A})$ to denote its determinant. If $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T$ with the diagonal entries of $\Sigma$ being $\{\sigma_1, \ldots, \sigma_r\}$, then $\Delta(\mathbf{A}) = \prod_i \sigma_i$. The Moore-Penrose pseudo-inverse of a matrix $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T$ is denoted by $\mathbf{A}^+$ and has a SVD $\mathbf{A}^+ = \mathbf{V} \Sigma^+ \mathbf{U}^T$, where $\Sigma^+$ consists of inverses of only non-zero singular values of $\mathbf{A}$. A matrix $\mathbf{A}$ has a left-inverse (right-inverse, respectively) if and only if it has full column rank (full row rank, respectively). If $\mathbf{A}$ has a left-inverse, then $\mathbf{A}^+ = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ and if $\mathbf{A}$ has right-inverse, then $\mathbf{A}^+ = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1}$.

**Probability theory.** We use the notation $\text{Rad}(p)$ to denote a distribution with support $\pm 1$ such that $+1$ is sampled with probability $p$ and $-1$ is sampled with probability $1 - p$. An $n \times n$ Walsh-Hadamard matrix $\mathbf{H}_n$ is constructed recursively as follows: $\mathbf{H}_n = \begin{pmatrix} \mathbf{H}_{n/2} & \mathbf{H}_{n/2} \\ \mathbf{H}_{n/2} & -\mathbf{H}_{n/2} \end{pmatrix}$ and $\mathbf{H}_1 := 1$. A randomized Walsh-Hadamard matrix $\mathbf{W}_n$ is formed by multiplying $\mathbf{H}_n$ with a diagonal matrix whose diagonal entries are picked i.i.d. from $\text{Rad}(1/2)$. We drop the subscript $n$ where it is clear from the context.

Given a random variable $x$, we denote by $\mathcal{N}(\mu, \rho^2)$ the fact that $x$ has a normal Gaussian distribution with mean $\mu$ and variance $\rho^2$. The multivariate Gaussian distribution is a generalization of univariate Gaussian distribution. Let $\mu$ be an $n$-dimensional vector. An $N$-dimensional multivariate random variable, $x \sim \mathcal{N}(\mu, \Lambda)$, where $\Lambda = \mathbb{E}[(x - \mu)(x - \mu)^T]$ is the $N \times N$ covariance matrix, has the probability density function given by $\text{PDF}_X(x) := \frac{e^{-\frac{1}{2}x^T \Lambda^{-1} x}}{\sqrt{(2\pi)^N |\Lambda|}}$. If $\Lambda$ has a non-trivial kernel space, then the multivariate distribution is undefined. However, in this paper, all our covariance matrices have only trivial kernel. Multivariate Gaussian distributions is invariant under affine transformation, i.e., if $y = \mathbf{A}x + b$, where $\mathbf{A} \in \mathbb{R}^{M \times N}$ is a rank-$M$ matrix and $b \in \mathbb{R}^M$, then $y \sim \mathcal{N}(\mathbf{A}\mu + b, \mathbf{A}\Lambda\mathbf{A}^T)$.

**Random Projections.** We use various concepts and results from the theory of random projections.

**Definition 3.** Let $\alpha, \beta > 0$. A distribution $\mathcal{D}$ over $t \times n$ random matrices satisfies $(\alpha, \beta)$-Johnson-Lindenstrauss property (JLP) if, for any unit vector $\mathbf{x} \in \mathbb{R}^n$, we have $\|\mathbf{F}\mathbf{x}\|_2^2 = (1 \pm \alpha)$ with probability $1 - \beta$ over $\Phi \sim \mathcal{D}$.

**Definition 4.** A distribution $\mathcal{D}_R$ of $t \times m$ matrices satisfies $(\alpha, \beta)$-subspace embedding for generalized regression if it has the following property: for any matrices $\mathbf{P} \in \mathbb{R}^{m \times n}$ and $\mathbf{Q} \in \mathbb{R}^{m \times n'}$ such that $r(\mathbf{P}) \leq r$, with probability $1 - \beta$ over $\Phi \sim \mathcal{D}_R$, if $\tilde{\mathbf{X}} = \text{argmin}_X \|\Phi(\mathbf{PX} - \mathbf{Q})\|_F$, then $\|\mathbf{PX} - \mathbf{Q}\|_F \leq (1 + \alpha) \min_{\mathbf{X} \in \mathbb{R}^{n \times n'}} \|\mathbf{PX} - \mathbf{Q}\|_F$.

**Definition 5.** A distribution $\mathcal{D}_A$ over $t \times m$ matrices satisfies $(\alpha, \beta)$-affine embedding if it has the following property: for any matrices $\mathbf{P} \in \mathbb{R}^{m \times n}$ and $\mathbf{Q} \in \mathbb{R}^{m \times n'}$ such that $r(\mathbf{P}) \leq r$, with probability $1 - \beta$ over $\mathbf{S} \sim \mathcal{D}_A$, simultaneously for all $\mathbf{X} \in \mathbb{R}^{n \times n'}$, $\|\mathbf{S}(\mathbf{AX} - \mathbf{B})\|_F^2 = (1 \pm \alpha) \|\mathbf{AX} - \mathbf{B}\|_F^2$.

We use the symbol $\mathcal{D}_R$ to denote a distribution that satisfies $(\alpha, \beta)$-subspace embedding for generalized regression. We use the symbol $\mathcal{D}_A$ to denote a distribution that satisfies $(\alpha, \beta)$-affine subspace embedding. One example of a distribution that satisfies $\mathcal{D}_R$ with $t = O(\alpha^{-2} \log(1/\beta))$ is the distribution of random matrices whose entries are sampled i.i.d. from $\mathcal{N}(0, 1/t)$. Sarlos [58] observed that a random matrix sampled from a distribution that satisfies the JLP is also a $(\alpha, \beta)$-subspace embedding for generalized regression for rank-$r$ matrix as long as it approximates matrix multiplication with multiplicative error $\sqrt{\alpha/r}$. The latter follows from the standard result on any transform with JLP with projected dimension $t = O(r + \log(1/\beta)/\alpha^2)$. Using Indyk and Motwani [36], we only need to project to a $t = O(r\alpha^{-1} \log(1/\beta))$-dimensional subspace. Since random Gaussian matrices satisfies the JLP, this implies the following fact.

**Fact 6.** [(39 [58])] Let $\mathbf{P} \in \mathbb{R}^{m \times n}$ be a matrix of rank $r$ and $\mathbf{Q} \in \mathbb{R}^{m \times n'}$ be an $m \times n'$ matrix. Let $\mathcal{D}$ be a distribution of matrices over $\mathbb{R}^{t \times n}$ with entries sampled i.i.d. from $\mathcal{N}(0, 1/t)$. Then there exists a
Initialization. Set \( t = O(k\alpha^{-1} \log k) \), \( v = O(k\alpha^{-2} \log k) \). Sample \( \Phi \in \mathbb{R}^{n \times t}, \Psi \in \mathbb{R}^{t \times m} \) such that \( \Phi^T \sim \mathcal{D}_R \) and \( \Psi \sim \mathcal{D}_R \) satisfies Lemma \[7\] Sample \( S \in \mathbb{R}^{t \times m}, T \in \mathbb{R}^{v \times n} \) such that \( S \sim \mathcal{D}_A \) and \( T^T \sim \mathcal{D}_A \) satisfies Lemma \[8\] Initialize an all zero matrices, \( Y_c := O^{m \times t}, Y_r := O^{k \times v}, \) and \( Z := O^{v \times v}. \)

Update rule. Suppose at time \( t \), the stream is \((i_t, j_t, x_t)\), where \((i_t, j_t) \in [m] \times [n]\). Update the matrices by the following rule: (i) \( (Y_c)_{i_t,j_t} = (Y_c)_{i_t,j_t} + A_s \Phi_j \) for all \( 1 \leq j \leq t \), (ii) \( (Y_r)_{i_t,j_t} = (Y_r)_{i_t,j_t} + A_s \Psi_j \) for all \( 1 \leq j \leq t \), and (iii) \( Z_{i,j} = Z_{i,j} + S_{i,j} \Phi_j \) for all \( 1 \leq i \leq n, 1 \leq j \leq n'. \)

Computing the factorization. Once the matrix is streamed, we follow the following steps.

1. Compute a matrix \( U \in \mathbb{R}^{m \times t} \) whose columns are orthonormal basis for the column space of \( Y_c \) and matrix \( V \in \mathbb{R}^{k \times n} \) whose rows are the orthonormal basis for the row space of \( Y_r \).
2. Compute a SVD of \( U \). Let it be \( U_i, \Sigma_i, V^T \). Compute a SVD of \( V^T \). Let it be \( S_i, \Sigma_i, V_i^T \).
3. Compute a SVD of \( \tilde{V}_s \Sigma_s \tilde{U}_s \). Compute a SVD of \( \tilde{V}_s^T \). Let it be \( U', \Sigma', V' \).
4. Output the matrix \( U' \Sigma' \) comprising of left singular vectors, diagonal matrix \( \Sigma' \), and the matrix \( V' \) with right-singular vectors. Denote by \( M_k = U' \Sigma' \).

Figure 1: Space-efficient Low-rank Factorization (Optimal-Space-LRF)

t = O(r/\alpha \log(r/\beta)) such that \( \mathcal{D} \) is an \((\alpha, \beta))\)-subspace embedding for generalized regression.

We use the following known results shown by Clarkson and Woodruff \[12\].

Lemma 7. (\[12\] Lem 41, Lem 46) There is a distribution \( \mathcal{D}_R \) over \( \mathbb{R}^{t \times m} \) that satisfies (i) \((\alpha, \beta)\)-subspace embedding for generalized regression, and (ii) if \( t = O(\alpha^{-2} \log^2 m) \), then for any \( m \times n \) matrix \( D \), \( \|\Phi D\|_F^2 = (1 + \alpha)\|D\|_F^2 \), and (iii) if \( t = O(r/\alpha \log(r/\alpha) \log(1/\beta)) \), such that, if \( R \sim \mathcal{D}_R \) and \( RQ \) can be computed in \( O(n(nQ) + tn^2 \log t) \) time for rank \( r \) matrices \( P \).

Lemma 8. (\[12\] Thm 39, Thm 42) There exists a distribution \( \mathcal{D}_A \) over \( \mathbb{R}^{v \times m} \) that satisfies \((\alpha, \beta)\)-affine embedding, and (ii) if \( v = \Theta(\alpha^{-2}) \) and an \( m \times d \) matrix \( D \), \( \|SD\|_F^2 = (1 + \alpha)\|D\|_F^2 \), and (iii) if \( v = O(p/\alpha^2 \log(r) \log(1/\beta)) \), then for any rank \( r \) matrix \( D \in \mathbb{R}^{m \times n} \) and a matrix \( E \in \mathbb{R}^{m \times n'} \), \( SE \) and \( SD \) can be computed in \( O(n(nD) + nmSE) + O((n + n'n^2 \log v) \log v) \) time.

Differential privacy. We use the following results about differential privacy in this paper.

Lemma 9. (Post-processing \[23\].) Let \( \mathcal{M}(D) \) be an \((\varepsilon, \delta)\)-differential private mechanism for a database \( D \), and let \( h \) be any function, then any mechanism \( M' := h(\mathcal{M}(D)) \) is also \((\varepsilon, \delta)\)-differentially private.

Theorem 10. (Composition \[27\].) Let \( \varepsilon_0, \delta_0 \in (0, 1) \), and \( \delta' > 0 \). If \( \mathcal{M}_1, \cdots, \mathcal{M}_t \) are each \((\varepsilon, \delta)\)-differential private mechanism, then the mechanism \( \mathcal{M}(D) := (\mathcal{M}_1(D), \cdots, \mathcal{M}_t(D)) \) releasing the concatenation of each algorithm is \((\varepsilon', \ell \delta + \delta')\)-differentially private for \( \varepsilon' < \sqrt{2\ell \ln(1/\delta')}\varepsilon_0 + 2\ell \delta_0^2 \).

Theorem 11. (Gaussian mechanism \[23\].) Let \( x, y \in \mathbb{R}^n \) be any two vectors such that \( \|x - y\|_2 \leq c \). Let \( \rho = c^{-1} \sqrt{\log(1/\delta')} \), and \( g \sim N(0, \rho^2) \) be a vector with each entries sampled i.i.d. Then for any \( s \in \mathbb{R}^n \), \( \Pr[x + g \in s] \leq \varepsilon' \Pr[y + g \in s] + \delta \).

3 Optimal Space Streaming Algorithm for Low-rank Factorization

In this section, we present our non-private algorithm. Our algorithm is different from that of Boutsidis et al. \[8\] and has a nice symmetric form which allows us to extend this algorithm to the private setting. Our algorithm maintains three sketches (namely, \( Y_r = \Psi A \) for approximating the row-space, \( Y_r = A \Phi \) for approximating the column-space and \( Z = SAT^T \) for approximating the matrix), instead of five sketches as maintained by Boutsidis et al. \[8\], to get similar bounds. The main theorem of this section is stated below.
Theorem 12. Let $m, n \in \mathbb{N}$ and $\alpha, \beta$ be the input parameters. Let $k$ be the desired rank of the low-rank factorization. Given an $m \times n$ matrix $A$ in a turnstile model, the factorization $\bar{U}, \bar{\Sigma}, \bar{V}$ outputted by the algorithm Optimal-Space-LRF, presented in Figure 1, is a $k$-rank factorization of the matrix $A$ such that:

1. With probability $1 - O(\beta)$ over the random coins of the algorithm Optimal-Space-LRF,
   \[ \|M_k - A\|_F \leq (1 + \alpha)\|A - [A]_k\|_F, \text{ where } M_k = \bar{U}\bar{\Sigma}\bar{V}^T. \]

2. Optimal-Space-LRF uses at most $O((m + n)k\alpha^{-1} \log(1/\beta) \log k)$ space.

3. The update-time of Optimal-Space-LRF is $O(\log(n + m))$. The total time for the computation of low-rank factorization by Optimal-Space-LRF is $O(nn(A) \log(1/\beta)) + O((m + n) \log(1/\beta)k^2\alpha^{-2})$ time.

Proof. Part 2 of Theorem 12 follows straightforwardly from the choice of $v$ and the dimensions of all the sketches, $Y_v = A\Phi$, $Y_{\bar{v}} = \Psi A$, and $Z = SAT^T$. For the update time in part 3 of Theorem 12 note that $\Phi$ and $\Psi$ operates in input-sparsity time, which is constant, plus an $O(\log(m + n))$ term. The same is true for the update time of $Z$. For the factorization time in part 3 of Theorem 12 we need to analyze the running time of the algorithm presented in Figure 1. We analyze it as follows.

1. Computing $Y_v$ requires $O(nn(A)) + ml^2$ time and computing $Y_{\bar{v}}$ requires $O(nn(A)) + nt^2$ time.
   Therefore, computing $Y_v$ and $Y_{\bar{v}}$ requires $O(nn(A)) + \tilde{O}((m + n)k(k + \alpha^{-1}))$ time. Computing $Z$ requires $\tilde{O}(k^2/\alpha^4 + nn(A)) + O(nk(k + \alpha^{-3}))$

2. Computing the orthonormal basis $U$ and $V$ requires $nt^2 + mt^2 = \tilde{O}((m + n)k^2\alpha^{-2})$ time.

3. Computing the SVD of the matrix $SU$ and $TV^T$ requires $vt^2 + tv^2 = \tilde{O}(k^3\alpha^{-6})$.

4. Computation of $[\bar{U}^T\bar{Z}\bar{V}]_k$ requires $O(nn(A)) + tv^2 = O(nn(A)) + \tilde{O}(k^3\alpha^{-5})$ time.

5. Computation of the SVD in Step 3 requires $(m + n)t^2 = \tilde{O}((m + n)k^2\alpha^{-2})$ time.

To prove part 1, we first invoke Theorem 50. Theorem 50 relates $\min_{X_r(X) \leq k} \|A\Phi X\Psi A - A\|_F$ with $(1 + \alpha)^2\|A - [A]_k\|_F$ as follow.

\[
\min_{X_r(X) \leq k} \|A\Phi X\Psi A - A\|_F \leq (1 + \alpha)^2\|A - [A]_k\|_F. \tag{1}
\]

Since $V$ is a matrix whose rows are an orthonormal basis for the row space of $\Psi A$ and $U$ is a matrix whose columns are an orthonormal basis for the column space of $A\Phi$, we have

\[
\min_{Y_r(Y) \leq k} \|UXV - A\|_F \leq \min_{X_r(X) \leq k} \|A\Phi XV - A\|_F \leq \min_{X_r(X) \leq k} \|A\Phi X\Psi A - A\|_F. \tag{2}
\]

Combining equation (1) and equation (2), we have with probability $1 - 2\beta$ over $\Phi, \Psi \sim D_R$,

\[
\min_{Y_r(Y) \leq k} \|UXV - A\|_F \leq (1 + \alpha)^2\|A - [A]_k\|_F. \tag{3}
\]

Lemma 13. Let $U$ and $V$ be as defined above and $\tilde{X} = \arg\min_{X_r(X) = k} \|S(UXV^T - A)T\|_F$. Let $D_A$ be a distribution that satisfies $(\alpha, \beta)$-subspace embedding. Then with probability $1 - O(\beta)$ over $S, T^T \sim D_A$,

\[
\|(UXV - A)\|_F \leq (1 + \alpha)^2\|A - [A]_k\|_F.
\]

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Proof. Set $p = k/\alpha$, $D = U$ and $E = A$ in the statement of Lemma 8 and $v = O(k/\alpha^3 \log(k/\alpha))$. Let us restrict our attention to only rank $k$ matrices $X$ and denote by $\tilde{X} = \text{argmin}_{X, r(X) = k} \|UXV - A\|_F$ and $\hat{X} = \text{argmin}_{X, r(X) = k} \|S(UXV^T - A)\|_F$. Then we have with probability $1 - O(\beta)$ over $S \sim \mathcal{D}_A$,

$$\min_{X, r(X) = k} \|UXV - A\|_F = \|\tilde{U}XV - A\|_F = (1 + 3\alpha)^{-1}\|S(\tilde{U}XV - A)\|_F.$$  \hspace{1cm} (4)

Substituting $D = V^T$, $X = (SU\tilde{X})^T$ and $E = (SA)^T$ in the statement of Lemma 8, with probability $1 - O(\beta)$ over $S, T^T \sim \mathcal{D}_A$,

$$(1 + 3\alpha)^{-1}\|S(U\tilde{X}V - A)\|_F = (1 + \alpha)^{-1}\|V^T(SU\tilde{X})^T - (SA)^T\|_F
= (1 + \alpha)^{-2}\|T(V^T(SU\tilde{X})^T - (SA)^T)\|_F
= (1 + \alpha)^{-2}\|S(U\tilde{X}V - A)T^T\|_F
\geq (1 + \alpha)^{-2} \min_{X, r(X) \leq k} \|S(UXV - A)T^T\|_F
= (1 + \alpha)^{-2}\|S(U\tilde{X}V - A)T^T\|_F
= (1 + \alpha)^{-3}\|(UXV - A)T^T\|_F = (1 + \alpha)^{-4}\|(U\tilde{X}V - A)\|_F. \hspace{1cm} (5)$$

Combining equation (5) with equation (3) and equation (4), we have with probability $1 - O(\beta)$ over the coins of Optimal-Space-LRF,

$$\|(U\tilde{X}V - A)\|_F \leq (1 + \alpha)^6\|A - [A]_k\|_F.$$ \hspace{1cm} (6)

This completes the proof of Lemma 13. \hfill \square

To finalize the proof of Theorem 12, we need to compute $\hat{X} = \text{argmin}_{X, r(X) \leq k} \|S(UXV - A)T^T\|_F$. We use the following lemma to compute $\hat{X}$. This lemma is a generalization of one of the previous results of Clarkson and Woodruff [11].

Lemma 14. Let $R$ be a matrix with orthonormal rows and $C$ be orthonormal columns. Then

$$\min_{X, r(X) = k} \|CXR - F\|_F = \|C[C^TFR]^kR - F\|_F.$$  \hspace{1cm} (7)

Proof. For any matrix $Y$ of appropriate dimension, we have $\langle F - CC^TF, CC^TF - CYR \rangle = 0$. This is because $F - CC^TF = (I - CC^T)F$ lies in space orthogonal to $C(C^TF - YR)$. By Theorem 45,

$$\|F - CYR\|_F^2 = \|F - CC^TF\|_F^2 + \|CC^TF - CYR\|_F^2
= \|F - CC^TF\|_F^2 + \|C^TF - YR\|_F^2.$$ \hspace{1cm} (8)

Again, for any matrix $Y$ of appropriate dimensions, we have $\langle C^TFR^TR - YR, C^TF - C^TFR^TR \rangle = 0$. This is because $C^TFR^TR - YR = (C^TFR - Y)R$ lies in the space spanned by $R$, and $C^TF - C^TFR^TR = C^TF(I - R^TR)$ lies in the orthogonal space. By Theorem 45, we have

$$\|C^TF - YR\|_F^2 = \|C^TF - C^TFR^TR\|_F^2 + \|C^TFR^TR - YR\|_F^2.$$ \hspace{1cm} (9)

Since $\|C^TF - C^TFR^TR\|_F^2$ is independent of $Y$, we just bound the term $\|C^TFR^TR - YR\|_F^2$. Substituting $Y = [CFR]_k$ and using the fact that multiplying $R$ from the right does not change the Frobenius norm and $[C^TFR^T]_k$ is the best $k$-rank approximation to the matrix $C^TFR^T$, for all rank-$k$ matrices $Z$, we have

$$\|C^TFR^TR - [C^TFR^T]_kR\|_F^2 \leq \|C^TFR^TR - ZR\|_F^2.$$ \hspace{1cm} (9)
Combining equation (9) with equation (8) and Theorem 45, we have
\[ \| C^T F - |CFR|_k R \|_F^2 \leq \| C^T F - C^T FR^T R \|_F^2 + \| C^T FR^T R - ZR \|_F^2. \]
(10)
Combining equation (10) with equation (7), the fact that \( C \) has orthonormal columns, and Theorem 45 we have
\[ \| F - C|CFR|_k R \|_F^2 \leq \| F - CC^T F \|_F^2 + \| CC^T F - ZR \|_F^2. \]
\[ = \| F - CC^T F \|_F^2 + \| CC^T F - CZR \|_F^2. \]
This completes the proof of Lemma 14. \( \square \)

We now complete the proof of Theorem 12. Recall that \( SU = U_s \Sigma_s V_s^T \) and \( TV^T = U_t \Sigma_t V_t^T \). Invoking Lemma 14 with \( C = U_s, R = V_t^T \) and \( F = Z = \text{SAT}^T \), we get
\[ [\tilde{U}_s \tilde{V}_t]_k = [\tilde{U}_s \text{SAT}^T \tilde{V}_t]_k = \arg\min_{X, r(X) \leq k} \| \tilde{U}_s XV - \text{SAT}^T \|_F. \]
\[ \Rightarrow \tilde{X} = \tilde{V}_s \tilde{\Sigma}_t [\tilde{U}_s \tilde{V}_t]_k \tilde{\Sigma}_t \tilde{U}_t^T = \arg\min_{X, r(X) \leq k} \| S(UXV - A)T^T \|_F. \]
Substituting equation (11) in equation (6), and adjusting the value of \( \alpha \) by a constant, we have the final result. \( \square \)

4 Space-Optimal Private Algorithm for Low-rank Factorization Under \( \text{Priv}_1 \)

In this section, we convert the algorithm presented in the last section to a differentially private algorithm. When \( m \leq n \), we maintain sketches \( Y_c = \tilde{A} \Phi, Y_r = \Psi \tilde{A}, \) and \( Z = \text{SAT}^T \) for \( \tilde{A} = (A \quad \sigma_{\min} I_m) \); else we maintain \( Y_r = (\tilde{A} \Phi)^T, Y_c = (\Psi \tilde{A})^T, \) and \( Z = (\text{SAT}^T)^T \) for \( \tilde{A} = (A^T \quad \sigma_{\min} I_n) \). In this section, we consider \( \text{Priv}_1 \) as the granularity of privacy, a generalization of previously studied privacy levels \([6, 28, 34, 33, 63]\). Figure 2 gives the description of our algorithm. Our main result in this section is as follows.

Theorem 15. Let \( m, n, k \in \mathbb{N} \) and \( \alpha, \beta, \varepsilon, \delta \) be the input parameters. Let \( s = \max\{m, n\} \) and \( \sigma_{\min} = 16 \log(1/\delta)/\sqrt{t(1 + \alpha)(1 - \alpha)^{-1}} \log(1/\delta)/\varepsilon \). Given an \( m \times n \) matrix \( A \) in a turnstile update model, \( \text{Priv-Space-Optimal-LRF} \), described in Figure 2, outputs a factorization \( \text{UU}', \Sigma', V' \) such that

1. With probability \( 1 - O(\delta) - 2^{-\varepsilon^2/8} \) over the random coins of \( \text{Priv-Space-Optimal-LRF} \),
\[ \| A - M_k \|_F \leq (1 + \alpha)\| A - [A]_k \|_F + O(\sigma_{\min} \sqrt{m + n} + (\rho_2 v + \rho_1 \sqrt{k(m + n)})(1 \pm \alpha)). \]
2. \( \text{Priv-Space-Optimal-LRF} \) is \((3\varepsilon, 3\delta)\)-differentially private.
3. The space used by \( \text{Priv-Space-Optimal-LRF} \) is \( O((m + n)(k + 1/\alpha)\alpha^{-1} \log k \log(1/\delta)). \)
4. The initialization time of the algorithm is \( O(m + n) \), the update time is \( O(s^2 (k + 1/\alpha) \alpha^{-1}) \), and the total time to perform the factorization is \( O(n m (A)^2 \log(1/\delta)) + O((nk^2 \alpha^{-2} + mk^2 \alpha^{-2} + k^3 \alpha^{-4}) \log(1/\delta)) \).

Part [4] and [3] of Theorem 15 follows as in the proof of Theorem 12 by setting the values of \( t \) and \( v \). Note that here the update time also requires multiplication by a random Gaussian matrix. In Section 4.1, we prove part [2] of Theorem 15. Then in Section 4.2, we prove part [1] of Theorem 15.
Initialization. Let $t = O(\max\{k\alpha^{-1}, \alpha^{-2}\} \log(k) \log(1/\delta)), v = O(\max\{k\alpha^{-2}, \alpha^{-4}\} \log k \log(1/\delta)$, and $u = \min\{m, n\}$. Sample $\Omega \sim N(0, 1)^{n \times t}$. Let $\Phi \in \mathbb{R}^{(m+n) \times u}$, $\Psi \in \mathbb{R}^{t \times u}$ such that $\Phi^T \sim D_R$ and $\Psi \sim D_R$ satisfies Lemma\ref{lem:strong-concentration}. Let $\rho_1 = \sqrt{(1 + \alpha)\ln(1/\delta)/\varepsilon}$ and $\rho_2 = (1 + \alpha)\sqrt{\ln(1/\delta)/\varepsilon}$. Let $S \in \mathbb{R}^{v \times u}, T \in \mathbb{R}^{v \times (m+n)}$ such that $S \sim N(0, \rho_1^2 I_{(m+n)})$ and $N_2 \sim N(0, \rho_2^2 I_{v \times v})$.

Define $\Phi = t^{-1} \Phi \Omega \in \mathbb{R}^{(m+n) \times t}$. Set $\sigma_{\min} = 16 \log(1/\delta) \sqrt{t(1 + \alpha)(1 - \alpha)^{-1}\ln(1/\delta)/\varepsilon}$.

Case 1: When $u = m$, set $\hat{A} = (0^{m \times n} \sigma_{\min} I_m)$. Compute $Y_c = \hat{A} \cdot \Phi^T, Y'_c = \Psi \hat{A}$, and $Z' = S \hat{A}^T T$.

**Update rule.** When $(i_r, j_r, A_r)$, where $(i_r, j_r) \in [m] \times [n]$, is updated, the matrices by the following rule: (i) $(Y_c)_{i_r, j_r} = (Y'_c)_{i_r, j_r} + A_r \hat{\Phi}_{j_r, r}$, for all $1 \leq j \leq t$, (ii) $(Y'_c)_{i_r, j_r} = (Y'_c)_{i_r, j_r} + A_r \hat{\Psi}_{i_r, j}$, for all $1 \leq j \leq t$, and (iii) $Z'_{i_r, j} = Z'_{i_r, j} + S_{i_r, i} T_{j, j}, A_T = S_{i_r, i} T_{j, j}$, for all $1 \leq i, j \leq v$.

**Computing the factorization.** Once the matrix is updated, we follow the following steps.

1. Compute $Y_r = Y'_c + N_1 = \hat{\Psi} \hat{A} + N_1$, and $Z' = Z' + N_2 = S \hat{A}^T T + N_2$.
2. Compute a matrix $U \in \mathbb{R}^{m \times t}$ whose columns are an orthonormal basis for the column space of $Y_c$ and matrix $V \in \mathbb{R}^{t \times (m+n)}$ whose rows are an orthonormal basis for the row space of $Y_r$.
3. Compute a SVD of SU $\in \mathbb{R}^{t \times t}$. Let it be $U_s \Xi_s V_s^T$. Compute a SVD of $V T^T \in \mathbb{R}^{t \times v}$. Let it be $U_s V_s^T$.
4. Compute a SVD of $\nabla_s \hat{\Sigma} \hat{\Sigma}_s^T \nabla_s V_s^T \in \mathbb{R}^{t \times t}$. Let it be $U_s \hat{\Sigma}_s V_s^T$.
5. Output the matrix $U U'$ comprising of left singular vectors, diagonal matrix $\Sigma'$ and the matrix $V^T V'$ comprising of right-singular vectors. Denote by $M_k = U U' \hat{\Sigma} (V^T V')^T$.

Case 2: When $u = n$, set $\hat{A} = (0^{n \times m} \sigma_{\min} I_n)$. Compute $Y_r = (\hat{A} \Phi^T, Y'_c = (\hat{A} \Psi)^T$, and $Z' = (S \hat{A}^T T)^T$.

**Update rule.** When $(i_r, j_r, A_r)$, where $(i_r, j_r) \in [m] \times [n]$, is updated, the matrices by the following rule: (i) $(Y'_c)_{j_r, r} = (Y'_{r})_{j_r, r} + A_r \hat{\Phi}_{r, j_r}$, for all $1 \leq j \leq t$, (ii) $(Y'_{r})_{j_r, r} = (Y'_{r})_{j_r, r} + A_r \hat{\Psi}_{i_r, j_r}$, for all $1 \leq j \leq t$, and (iii) $Z'_{i_r, j} = Z'_{i_r, j} + T_{i_r, i} A_r S_{j, j}$, for all $1 \leq i, j \leq v$.

**Computing the factorization.** Once the matrix is updated, we follow the following steps.

1. Compute $Y_c = (Y'_c + N_1)^T = (\hat{A} \Phi + N_1)^T$, and $Z' = (Z' + N_2)^T = (S \hat{A}^T T + N_2)^T$.
2. Compute a matrix $U \in \mathbb{R}^{(m+n) \times t}$ whose columns are an orthonormal basis for the column space of $Y_c$ and matrix $V \in \mathbb{R}^{t \times n}$ whose rows are an orthonormal basis for the row space of $Y_r$.
3. Compute a SVD of $U T^T \in \mathbb{R}^{t \times t}$. Let it be $U_s \Xi_s V_s^T$. Compute a SVD of $V S^T \in \mathbb{R}^{t \times v}$. Let it be $U_s V_s^T$.
4. Compute a SVD of $\nabla_s \hat{\Sigma}_s^T \nabla_s V_s^T \in \mathbb{R}^{t \times t}$. Let it be $U_s \hat{\Sigma}_s V_s^T$.
5. Output the matrix $U U'$ comprising of left singular vectors, diagonal matrix $\Sigma'$ and the matrix $V^T V'$ comprising of right-singular vectors. Denote by $M_k = U U' \hat{\Sigma} (V^T V')^T$.

Figure 2: Space-efficient Private Low-rank Approximation Under Priv$_1$ (Private-Space-Optimal-LRF)

### 4.1 Proof of Part 2 of Theorem 15

**Lemma 16.** If $\sigma_{\min}$ and $\rho$ be as in Theorem 15, then the algorithm presented in Figure 2, Private-Space-Optimal-LRF, is $(3\varepsilon, 3\delta)$-differentially private.

**Proof.** We prove the lemma when $m \leq n$. The case for $m \geq n$ is analogous after inverting the roles of $\hat{\Phi}$ and $\hat{\Psi}$. Let $A$ and $A'$ be two neighboring matrices, i.e., $E = A - A' = uv^T$. Then $\hat{A}$ and $\hat{A'}$, constructed by Optimal-Space-Private-LRF, has the following property: $A' = \hat{A} + (E \ 0)$.

**Claim 17.** If $\rho_1 = \sqrt{(1 + \alpha)\ln(1/\delta)/\varepsilon}$ and $\rho_2 = (1 + \alpha)\sqrt{\ln(1/\delta)/\varepsilon}$, then publishing $Y_r$ and $Z$ preserves $(2\varepsilon, 2\delta)$-differential privacy.
Proof. We use the second claims of Lemma 7 and Lemma 8, i.e., \(|SD|^2 = (1 \pm \alpha)||D||^2\) and \(|ΨD|^2 = (1 \pm \alpha)||D||^2\) for all D, where \(S \sim D_A\) and \(Ψ \sim D_R\). Let \(Λ\) and \(Λ'\) be two neighboring matrices such that \(E = Λ - Λ' = uv^T\). Then \(||S(0 Σ) T||^2 \leq (1 + \alpha)||S(0 Σ) T||^2 = (1 + \alpha)^2\). Publishing \(Z\) preserves \((ε, δ)\)-differential privacy follows from considering the vector form of the matrix \(SΛT^T\) and \(N_2\) and applying Theorem 11. Similarly, we use Theorem 11 and the fact that, for any matrix \(C\) of appropriate dimension, \(||ΨC|^2 = (1 + \alpha)||C||^2\) to prove that publishing \(ΨΛ + N_1\) preserves differential privacy.

We next prove that \(Y_e\) is \((ε, δ)\)-differentially private. This would complete the proof of Lemma 10 as the lemma would follow by combining Lemma 9 and Theorem 10. One of the key ideas behind the proof is the result of Sarlos [58]. Sarlos [58] observed that since any distribution satisfying the lemma would follow by combining Lemma 9 and Theorem 10. One of the key ideas behind the proof is the proof of privacy follows the same line as the proof of Blocki et al. [6]; however, we need to take care of two things. First, the definition of neighboring matrices considered in this paper is different from that of Blocki et al. [6]. To recall, Blocki et al. [6] considered two matrices neighboring if they differ in at most one row by a unit norm. In our case, we consider two matrices are neighboring if they have the form \(uv^T\) for unit vectors \(u\) and \(v\). Secondly, we multiply the Gaussian matrix to \(Ψ\) and not to \(Ψ\), as in Blocki et al. [6]. However, this is not a problem because \(Ψ\) satisfies \((α, δ)\)-JLP because of the choice of \(t\). Since the rank of \(Λ\) and \(Φ\) are the same with probability \(1 - δ\), the singular values of \(Ψ\) are within a multiplicative factor \((1 \pm α)^{1/2}\) of \(Ψ\). Therefore, \(Ψ = (1 + α)^{1/2}Ψ\) for some unit vector \(Ψ\). That is, \(uv^TΨ = (1 + α)^{1/2}uv^T\) for some unit vectors \(u\) and \(v\). We now show that \(ΨΛ + N_1\) preserves privacy. We prove that each row of the published matrix preserves \((ε_0, δ_0)\)-differential privacy for some appropriate \(ε_0, δ_0\), and then invoke Theorem 10 to prove that the published matrix preserves \((ε, δ)\)-differential privacy.

The proof of privacy follows the same line as the proof of Blocki et al. [6]; however, we need to take care of two things. First, the definition of neighboring matrices considered in this paper is different from that of Blocki et al. [6]. To recall, Blocki et al. [6] considered two matrices neighboring if they differ in at most one row by a unit norm. In our case, we consider two matrices are neighboring if they have the form \(uv^T\) for unit vectors \(u\) and \(v\). Secondly, we multiply the Gaussian matrix to \(Ψ\) and not to \(Ψ\), as in Blocki et al. [6]. However, this is not a problem because \(Ψ\) satisfies \((α, δ)\)-JLP because of the choice of \(t\). Since the rank of \(Λ\) and \(Φ\) are the same with probability \(1 - δ\), the singular values of \(Ψ\) are within a multiplicative factor \((1 + α)^{1/2}\) of the singular values of \(Φ\) due to Sarlos [58]. Therefore, our proof goes through if we scale the singular values of \(Λ\) appropriately. In this sense, we generalize the privacy result of Blocki et al. [6].

Denote by \(Λ = (A \ σ_{min}e_m)\) and by \(Λ' = (A' \ σ_{min}e_m)\), where \(A - A' = uv^T\). Then \(Λ' - Λ = (uv^T 0)\). Let \(U_CΣ_CV^T_C\) be the SVD of \(C = ΛΦ\) and \(U_CΣ_CV^T_C\) be the SVD of \(C = Λ'Φ\). From above discussion, we know that if \(A - A' = uv^T\), then \(C - C' = (1 + α)^{1/2}uv^T\) for some unit vectors \(u\) and \(v\). For notational brevity, in what follows we write \(u\) for \(u\) and \(v\) for \(v\).

Note that both \(C\) and \(C\) is a full rank matrix because of the construction; therefore \(CC^T\) is a full dimensional \(m \times m\) matrix. This implies that the affine transformation of the multi-variate Gaussian is well-defined (both the covariance \((CC^T)^{-1}\) has full rank and \(Δ(CC^T)\) is non-zero). That is, the PDF of the distributions of the rows, corresponding to \(C\) and \(C\), is just a linear transformation of \(N(0, I_{m \times m})\). Let \(y \sim N(0, I^T)\).

\[
PDF_{C}(x) = \frac{1}{\sqrt{(2π)^d|Δ(ΔC^T)|}} e^{-\frac{1}{2}x(ΔC^T)^{-1}x^T} \\
PDF_{C}(x) = \frac{1}{\sqrt{(2π)^d|Δ(ΔC^T)|}} e^{-\frac{1}{2}x(ΔC^T)^{-1}x^T}
\]

We prove the result for a row of the published matrix; the theorem follows from Theorem 10. Let \(ε_0 = \frac{ε}{\sqrt{4r \ln(1/δ) \log(1/δ)}}\) and \(δ_0 = δ/2r\). Let \(x\) be sampled either from \(N(0, CC^T)\) or \(N(0, ΠCC^T)\). It is
straightforward to see that the combination of Claim 18 and 19 proves differential privacy for a row of published matrix. The lemma then follows by an application of Theorem 10 and our choice of \( \varepsilon_0 \) and \( \delta_0 \).

Claim 18. Let \( C \) and \( \varepsilon_0 \) be as defined above. Then

\[
e^{-\varepsilon_0} \leq \sqrt{\frac{\Delta(CC^T)}{\Delta(C^T)}} \leq e^{\varepsilon_0}.
\]

**Proof.** The claim follows simply as in [6] after a slight modification. More concretely, we have \( \Delta(CC^T) = \prod_i \sigma_i^2 \), where \( \sigma_1 \geq \cdots \geq \sigma_m \geq \sigma_{\min}(C) \) are the singular values of \( C \). Let \( \tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_m \geq \sigma_{\min}(\tilde{C}) \) be its singular value for \( \tilde{C} \). There is only one singular value of \( E \) and it is \( \sqrt{1+\alpha} \). This is because \( EE^T = (1+\alpha)vv^T \). To finish the proof of this claim, we use Theorem 46.

Since the singular values of \( C - \tilde{C} \) and \( C - C \) are the same, Lidskii’s theorem (Theorem 46) gives \( \sum_i(\sigma_i - \tilde{\sigma}_i) \leq \sqrt{1+\alpha} \). Therefore, with probability \( 1 - \delta \),

\[
\prod_{i: \tilde{\sigma}_i \geq \sigma_i} \frac{\tilde{\sigma}_i}{\sigma_i} = \prod_{i: \tilde{\sigma}_i \geq \sigma_i} \left( 1 + \frac{\tilde{\sigma}_i - \sigma_i}{\sigma_i} \right) \leq \exp \left( \frac{\varepsilon}{32(1+\alpha)\sqrt{t\log(2/\delta)} \log(t/\delta)} \sum_i(\tilde{\sigma}_i - \sigma_i) \right) \leq e^{\varepsilon_0/2}.
\]

The last inequality holds because \( \Phi \sim \mathcal{D}_R \) satisfies \( (\alpha, \delta) \)-JLP due to the choice of \( t \) (second claim of Lemma 7). Since \( C \) and \( \Lambda \) have same rank, this implies, due to a result by Sarlos [58], that all the singular values of \( C \) are within \( (1 \pm \alpha)^{1/2} \) multiplicative factor of \( \tilde{A} \) (Lemma 48). In other words, \( \sigma_{\min}(C) \geq (1 - \alpha)^{1/2}\sigma_{\min} \). The case for all \( i \in [m] \) when \( \tilde{\sigma}_i \leq \sigma_i \) follows similarly as the singular values of \( E \) and \( -E \) are the same. This completes the proof of Claim 18. \( \square \)

Claim 19. Let \( C, \varepsilon_0, \) and \( \delta_0 \) be as defined earlier. Let \( y \sim \mathcal{N}(0, 1) \). If \( x \) is sampled either from \( Cy \) or \( \tilde{C}y \), then we have

\[
\Pr \left[ |x^T(CC^T)^{-1}x - x^T(\tilde{C}\tilde{C}^T)^{-1}x| \leq \varepsilon_0 \right] \geq 1 - \delta_0.
\]

**Proof.** Without any loss of generality, we can assume \( x = Cy \). The case for \( x = \tilde{C}y \) is analogous. Let \( C - \tilde{C} = vu^T \). Note that \( E((\Omega)_{i,j}) = 0^\alpha \) and \( \text{COV}((\Omega)_{i,j}) = 1 \) if and only if \( i = j \). Then

\[
x^T(CC^T)^{-1}x - x^T(\tilde{C}\tilde{C}^T)^{-1}x = x^T(CC^T)^{-1}(vv^T)(CC^T)^{-1}x - x^T(\tilde{C}\tilde{C}^T)^{-1}x
\]

\[
= x^T(CC^T)^{-1}(C + vu^T)(C + vu^T)^T(\tilde{C}\tilde{C}^T)^{-1}x - x^T(\tilde{C}\tilde{C}^T)^{-1}x
\]

\[
= x^T \left( (CC^T)^{-1}(Cvu^T + vu^TCC^T)^{-1} \right) x
\]

Using the singular value decomposition of \( C = U_C\Sigma_CV_C^T \) and \( \tilde{C} = \tilde{U}_C\tilde{\Sigma}_C\tilde{V}_C^T \), we have

\[
\left( x^T(U_C\Sigma_C^{-1}V_C^T)u \right) \left( v^T(\tilde{U}_C\tilde{\Sigma}_C^{-2}\tilde{U}_C^T)x \right) + \left( x^T(U_C\Sigma_C^{-2}U_C^Tv) \right) \left( u^T(\tilde{V}_C\tilde{\Sigma}_C^{-1}\tilde{U}_C)x \right)
\]

Since \( x \sim Cy \), where \( y \sim \mathcal{N}(0, 1)^t \), we can write the above expression as \( \tau_1 \tau_2 + \tau_3 \tau_4 \), where

\[
\tau_1 = \left( y^TC^T(U_C\Sigma_C^{-1}V_C^T)u \right) \quad \tau_2 = \left( v^T(\tilde{U}_C\tilde{\Sigma}_C^{-2}\tilde{U}_C^T)Cy \right)
\]

\[
\tau_3 = \left( y^TC^T(U_C\Sigma_C^{-2}U_C^Tv) \right) \quad \tau_4 = \left( u^T(\tilde{V}_C\tilde{\Sigma}_C^{-1}\tilde{U}_C)x \right).
\]

Now since \( \|\tilde{\Sigma}\|_2, \|\Sigma\|_2 \geq \sigma_{\min}(C) \), plugging in the SVD of \( C \) and \( C - \tilde{C} = vu^T \), and that every term \( \tau_i \) in the above expression is a linear combination of a Gaussian, i.e., each term is distributed as per
Performing step (i). We start by proving a bound on Part 1 of Theorem 15 follows by fitting together the above three bounds.

(i) Lower bound

\[ \| (V C \Sigma C U_1^T)(U C \Sigma C^{-1} V_1^T) u \|_2 \leq \| u \|_2 \leq 1, \]

\[ \| (V C \Sigma C U_1^T)(U C \Sigma C^{-2} U_2^T) v \|_2 \leq \| \Sigma C^{-2} \|_2 \leq \frac{1}{\sigma_{\min}(C)}, \]

\[ \| v^T (U C \Sigma C^{-2} \tilde{U}_2^T)(U C \Sigma C \tilde{V}_C^T - v u^T) \|_2 \]

\[ \leq \| v^T (U C \Sigma C^{-2} \tilde{U}_2^T) \tilde{U}_C \Sigma C \tilde{U}_C^T \|_2 + \| v^T (U C \Sigma C^{-2} \tilde{U}_2^T) v u^T \|_2 
\leq \frac{1}{\sigma_{\min}(C)} + \frac{1}{\sigma_{\min}(C)}; \]

\[ \| u^T (V C \Sigma C^{-1} \tilde{U}_1^T)(U C \Sigma C \tilde{V}_C^T) \|_2 \]

\[ \leq \| u^T (V C \Sigma C^{-1} \tilde{U}_1^T) (\tilde{U}_C \Sigma C \tilde{V}_C^T) v \|_2 \]

\[ \leq \| u^T (V C \Sigma C^{-1} \tilde{U}_1^T) v \|_2 \leq 1 + \frac{1}{\sigma_{\min}(C)}. \]

Using the concentration bound on the Gaussian distribution, each term, \( \tau_1, \tau_2, \tau_3, \) and \( \tau_4, \) is less than \( \| \tau_i \| \ln(4/\delta_0) \) with probability \( 1 - \delta_0/2. \) The second claim follows from the following inequality:

\[ \Pr \left[ \| x^T (C C^T)^{-1} x - x^T (\overline{C} \overline{C}^T)^{-1} x \| \leq 2 \left( \frac{1}{\sigma_{\min}(C)} + \frac{1}{\sigma_{\min}(C)} \right) \ln(4/\delta_0) \leq \varepsilon_0 \right] \geq 1 - \delta_0, \]

where the second inequality follows from the choice of \( \sigma_{\min} \) and the fact that \( \sigma_{\min}(C) \geq (1-\alpha)^{1/2}\sigma_{\min}. \]

Combining Claims 18 and 19, Lemma 16 follows.

4.2 Proof of Part 1 of Theorem 15

In what follows, we analyze the case when \( m \leq n. \) The case when \( n \leq m \) follows analogously due to the symmetry of Private-Space-Optimal-LRF. First note that appending \( A \) with an all zero matrix \( 0^{m \times m} \) has no effect on its \( k \)-rank approximation, i.e., we can analyze \( (A \ 0) \). We break our proof in three main steps.

(i) Lower bound \( \| M_k - \tilde{A} \|_F \) by \( \| M_k - (A \ 0) \|_F \) up to an additive term.

(ii) Relate \( \| M_k - \tilde{A} \|_F \) and \( \| \tilde{A} - [A]_k \|_F \).

(iii) Upper bound \( \| \tilde{A} - [A]_k \|_F \) by a term linear in \( \| A - [A]_k \|_F \) up to an additive factor.

Part 1 of Theorem 15 follows by fitting together the above three bounds.

Performing step (i). We start by proving a bound on \( \| M_k - \tilde{A} \|_F \) by \( \| M_k - A \|_F \) and a small additive term. The following lemma provides such a bound.

**Lemma 20.** Let \( A \) be an \( m \times n \) input matrix, and let \( \tilde{A} = (A \ \sigma_{\min} \overline{I}_m) \) for \( \sigma_{\min} \) defined in Theorem 15. Denote by \( M_k := U U' \Sigma'(\overline{VV'})^T \) the output of Private-Optimal-Space-LRF. Then \( \| M_k - (A \ 0) \|_F \leq \| M_k - \tilde{A} \|_F + \sigma_{\min} \sqrt{m}. \)

**Proof.** The lemma is immediate from the following.

\[ \| M_k - (A \ 0) \|_F - \sigma_{\min} \| \bar{I}_m \|_F \leq \| M_k - (A \ 0) - (0 \ \sigma_{\min} \bar{I}) \|_F = \| M_k - \tilde{A} \|_F, \]

where the first inequality follows from the sub-additivity of norms.


Performing step (ii). This is the most involved part of the proof and uses multiple lemmas as follows.

Lemma 21. Let $\hat{A} = (A, \sigma_{\min} I)$ and denote by $M_k := UU' \Sigma' (VV')^T$. Let $\hat{\Phi} = \Phi \Omega$. Then with probability $1 - O(\delta)$ over the random coins of the algorithm Private-Space-Optimal-LRF,

$$\|M_k - \hat{A}\|_F \leq (1 + \alpha)\|\hat{A} - [\hat{A}]_k\|_F + \|\hat{A} \hat{\Phi}([\hat{A}]_k \hat{\Phi})^\dagger (\Psi \hat{A} \hat{\Phi}([\hat{A}]_k \hat{\Phi})^\dagger)^\dagger N_1\|_F + 2\|S^\dagger N(T^\dagger)^\dagger\|_F.$$

Proof. Let $B = \hat{A} + S^\dagger N(T^\dagger)^T$ and $\hat{\Phi} = t^{-1} \Omega \Phi$. We first use the relation between $\min_{X, T(X) \leq k} \|\hat{A} \hat{\Phi} X \Psi \hat{A} - \hat{A}\|$ and $(1 + \alpha)\|\hat{A} - [\hat{A}]_k\|_F$ shown by Boutsidis et al. [8]. Using the Fact 6 and Lemma 7 if we set $A = \hat{A}$ in equation (39), we have with probability $1 - 2\delta$ over $\hat{\Phi}^T, \Psi \sim D_R$,

$$\|\hat{A} \hat{\Phi}([\hat{A}]_k \hat{\Phi})^\dagger (\Psi \hat{A} \hat{\Phi}([\hat{A}]_k \hat{\Phi})^\dagger)^\dagger \hat{\Psi} - \hat{\Phi}\|_F \leq (1 + \alpha)^2\|\hat{A} - [\hat{A}]_k\|_F.$$

Consider the optimization problem: $\min_{X, r(X) \leq k} \|Y_c X Y - B\|_F$. Define $P_k := ([\hat{A}]_k \hat{\Phi})^\dagger (\Psi \hat{A} \hat{\Phi}([\hat{A}]_k \hat{\Phi})^\dagger)^\dagger$. Since $P_k$ is a rank-$k$ matrix, we have

$$\min_{X, r(X) \leq k} \|Y_c X Y - B\|_F \leq \|Y_c P_k Y - B\|_F = \|Y_c P_k Y - (\hat{A} + S^\dagger N(T^\dagger)^T)\|_F \leq \|Y_c P_k Y - \hat{A}\|_F + \|S^\dagger N(T^\dagger)^T\|_F.$$

Let $S_2 = \hat{A} \hat{\Phi} P_k N_1$. We have an upper bound on $\min_{X, r(X) \leq k} \|Y_c X Y - B\|_F$ in the terms of $(1 + \alpha)\|\hat{A} - [\hat{A}]_k\|_F$ and some additive terms. Since $V$ is a matrix whose rows are an orthonormal basis for the row space of $Y_r$ and $U$ is a matrix whose columns are an orthonormal basis for the column space of $Y_c$, we have

$$\min_{Y, r(Y) \leq k} \|UXV - B\|_F \leq \min_{X, r(X) \leq k} \|Y_c X V - B\|_F \leq \min_{X, r(X) \leq k} \|Y_c X Y - B\|_F. \quad (13)$$

Combining equation (12) and equation (13), we have

$$\min_{Y, r(Y) \leq k} \|UXV - B\|_F \leq (1 + \alpha)^2\|\hat{A} - [\hat{A}]_k\|_F + \|S^\dagger N(T^\dagger)^T\|_F + \|S_2\|_F. \quad (14)$$

Claim 22. Let $U, V, B, A, S, T$ and $N_2$ be as above. Let $D_A$ be a distribution that satisfies $(\alpha, \beta)$-subspace embedding. Let $\hat{X} = \arg\min_{X, r(X) = k} \|S(UX - B)\|_F$, then with probability $1 - O(\delta)$ over $S, T^\dagger \sim D_A$,

$$\|(UXV - B)\|_F \leq (1 + \alpha)^6\|\hat{A} - [\hat{A}]_k\|_F + \|S^\dagger N(T^\dagger)^T\|_F + \|S_2\|_F.$$

Proof. Set $p = t, D = U$ and $E = B$ in the statement of Lemma 8. Let us restrict our attention to only rank $k$ matrices $X$ and denote by $\hat{X} = \arg\min_{X, r(X) = k} \|UXV - B\|_F$ and $\hat{X} = \arg\min_{X, r(X) = k} \|S(UXV - B)/T^\dagger\|_F$. Then we have with probability $1 - 3\delta$ over $S \sim D_A$,

$$\min_{X, r(X) = k} \|UXV - B\|_F = \|UXV - B\|_F = (1 + \alpha)^{-1}\|S(UXV - B)\|_F. \quad (15)$$
Substituting \( D = V^T, X = (SU\tilde{X})^T \) and \( E = (SB)^T \) in the statement of Lemma 8, with probability \( 1 - 4\delta \),
\[
(1 + \alpha)^{-1} \|S(U\tilde{X}V - B)\|_F = (1 + \alpha)^{-1} \|V^T(SU\tilde{X}) - (SB)^T\|_F
\]
\[
= (1 + \alpha)^{-2} \|T(V^T(SU\tilde{X}) - (SB)^T)\|_F
\]
\[
= (1 + \alpha)^{-2} \|S(U\tilde{X}V - B)^T\|_F
\]
\[
\geq (1 + \alpha)^{-2} \min_{x_r(x)} \|S(UXV - B)^T\|_F
\]
\[
= (1 + \alpha)^{-2} \|S(U\tilde{X}V - B)^T\|_F = (1 + \alpha)^{-4} \|S(U\tilde{X}V - B)\|_F. \tag{16}
\]

Combining equation (16) with equation (14) and equation (15), we have with probability \( 1 - O(\delta) \) over the random coins of Private-Optimal-Space-LRF,
\[
\|(S(U\tilde{X}V - B)T^T\|_F \leq (1 + \alpha)^6 \|\bar{A} - [\bar{A}]_k\|_F + \|S^1N_2(T^T)\|_F + \|S_2\|_F. \tag{17}
\]

This completes the proof of Claim 22.

To finalize the proof, we need to compute \( \tilde{X} = \arg\min_{x_r(x) \leq k} \|S(UXV - B)T^T\|_F \). Recall \( S = U_s\Sigma_s\Sigma_s^T \) and \( TV^T = U_t\Sigma_tV_t^T \). Invoking Lemma 14 with \( C = \tilde{U}_s, R = \tilde{V}_t \) and \( \tilde{F} = \tilde{Z} = S\hat{A}T^T + N \), we get
\[
[U_s^T\Sigma V_t^T]_k = [\tilde{U}_s^T SBT^T \tilde{V}_t^T]_k = \arg\min_{x_r(x) \leq k} \|\tilde{U}_s^T\Sigma V_t^T - SBT^T\|_F
\]
\[
\Rightarrow \tilde{X} = \tilde{V}_s\Sigma_t[\tilde{U}_s^T\Sigma V_t^T]_k \tilde{U}_t^T = \arg\min_{x_r(x) \leq k} \|S(UXV - B)T^T\|_F. \tag{18}
\]

Recall, \( \tilde{X} = \tilde{V}_s\Sigma_t[\tilde{U}_s^T\Sigma V_t^T]_k \tilde{U}_t^T = U^t\Sigma^tV^T \). Substituting equation (18) in equation (17) and the fact that \( B = \tilde{A} + S^1N_2(T^T)^T \), we have
\[
\|UU^t\Sigma^t(V^T\Sigma^tV')^T - \tilde{A}\|_F - \|S^1N_2(T^T)^T\|_F \leq \|UU^t\Sigma^t(V^T\Sigma^tV')^T - B\|_F
\]
\[
\leq (1 + \alpha)^6 \|\tilde{A} - [\tilde{A}]_k\|_F + \|S^1N_2(T^T)^T\|_F + \|S_2\|_F
\]
\[
\Rightarrow \|UU^t\Sigma^t(V^T\Sigma^tV')^T - \tilde{A}\|_F \leq (1 + \alpha)^6 \|\tilde{A} - [\tilde{A}]_k\|_F + 2\|S^1N_2(T^T)^T\|_F + \|S_2\|_F.
\]

Scaling the value of \( \alpha \) by a constant completes the proof of Lemma 21.

To compute the additive error, we need to bound \( \|S^1N_2(T^T)^T\|_F \) and \( \|S_2\|_F \). This is done by the following two lemmas.

Claim 23. Let \( D_R \) be a distribution that satisfies \((\alpha, \beta)\)-subspace embedding for generalized regression. Let \( S_2 \) be as defined above. Then with probability \( 1 - \delta \) over \( \tilde{\Phi} \sim D_R \), \( \|S_2\|_F = \rho(1 \pm \alpha)^{-1}\sqrt{kn} \).

Proof. Let \( G = \Psi A \Phi ([\tilde{A}]_k \Phi)^T \). \( G \) is an \( t \times k \) matrix. When \( \alpha \leq 1 \), \( G \) is a full column rank. This implies that there exist a \( t \times k \) matrix \( \tilde{U} \) with orthonormal columns such that \( GG^T = \tilde{U}\tilde{U}^T \). Therefore, \( \Psi S_2 = G \Phi^T N_2 = \tilde{U}\tilde{U}^T N_2 \). From the second claim of Lemma 1 and the choice of the parameter \( t \), \( \|S_2\|_F^2 = (1 \pm \alpha)^{-1} \|\tilde{U}\tilde{U}^T N_2\|_F^2 \). Since every entries of \( N_2 \) are picked i.i.d. and \( \tilde{U}\tilde{U}^T \) is an orthonormal projection onto a \( k \)-dimensional subspace, we have \( \|S_2\|_F = \rho \sqrt{kn}/(1 \pm \alpha) \).
Initialization. Set \( t = O(\max\{k\alpha^{-1}, \alpha^{-2}\} \log(1/\delta)) \), \( v = O(\max\{k\alpha^{-2}, \alpha^{-4}\} \log(1/\delta)) \). Sample \( \Phi \in \mathbb{R}^{m \times t} \) from \( D_R \) as in Lemma 7 and \( S \in \mathbb{R}^{v \times n} \) from \( D_R \) as in Lemma 8. Sample \( N_1 \sim \mathcal{N}(0, \rho^2 I) \), \( N_2 \sim \mathcal{N}(0, \rho^2 I) \). Initialize an all zero \( m \times t \) matrix \( Y' \) and an all zero \( v \times n \) matrix \( Z' \).

Update rule. Suppose at time \( \tau \), the stream is \((i_\tau, j_\tau, A_\tau)\), where \((i_\tau, j_\tau) \in [m] \times [n]\). Update the matrices by the following rule: \( Y'_{i_\tau,j} = Y'_{i_\tau,j} + A_\tau \Phi_{j_\tau,j} \) for all \( 1 \leq j \leq t \) and \( (Z')_{i_\tau,j} = (Z')_{i_\tau,j} + A_\tau S_{i,j} \) for all \( 1 \leq j \leq v \).

Computing the factorization. Once the matrix is streamed, we follow the following steps.

1. Compute \( Y = Y' + N_1 \) and \( Z = Z' + N_2 \).
2. Compute a matrix \( U \in \mathbb{R}^{m \times t} \) whose columns are an orthonormal basis for the column space of \( Y \).
3. Compute the singular value decomposition of \( SU \in \mathbb{R}^{m \times t} \). Let it be \( \tilde{U} \Sigma \tilde{V}^T \).
4. Compute the singular value decomposition of \( \tilde{V}\Sigma^1 |\tilde{U}^T Z|_k \). Let it be \( U^* \Sigma^* V'^T \).
5. Output \( UU^*, \Sigma^* \) and \( V'^* \). Let \( M_k = UU^* \Sigma^* V'^T \).

Figure 3: Private Low-rank Approximation Under \textit{Priv}_2 (Private-Frobenius-LRF)

Using Lemma 47 we know that \( \|N_2\|_F = O(\rho v) \) with probability 99/100. If we instead use the concentration bound of \( \chi^2\)-distribution, then \( \|N_2\|_F = O(\rho v \ell) \) with probability \( 1 - 2^{-\ell^2 \rho^2/8} \).

Performing step (iii). In order to complete the proof, we compute an upper bound on \( \|\hat{A} - [\hat{A}]_k\|_F \). For this, we need the Weyl’s perturbation theorem:

**Theorem 25.** For any \( m \times n \) matrices \( P, Q \), we have \( |\sigma_i(P + Q) - \sigma_i(P)| \leq \|Q\|_2 \), where \( \sigma_i(\cdot) \) denotes the \( i\)-th singular value and \( \|Q\|_2 \) is the spectral norm of the matrix \( Q \).

**Lemma 26.** Let \( d \) be the maximum of the rank of \( A \) and \( \hat{A} \). Let \( \sigma_1, \ldots, \sigma_d \) be the singular values of \( A \) and \( \sigma'_1, \ldots, \sigma'_d \) be the singular values of \( \hat{A} \). Then \( |\sigma_i - \sigma'_i| \leq \sigma \) for all \( 1 \leq i \leq d \).

**Proof.** The lemma follows from the basic application of **Theorem 25**. We can write \( \hat{A} = (A \ 0) + (0 \ \sigma_{\text{min}} \mathbb{I}_m) \). The lemma follows since, by construction, all the singular values of \((0 \ \sigma_{\text{min}} \mathbb{I}_m)\) are \( \sigma_{\text{min}} \).

Since \( \|\hat{A} - [\hat{A}]_k\|_F^2 \leq \sum_{i>k} \sigma_i^2 \) and \( \|A - A_k\|_F^2 \leq \sum_{i>k} \sigma_i^2 \), combining **Lemma 20** and **Lemma 21**, we have the final utility bound.

**Lemma 27.** Let \( \rho \) and \( \sigma_{\text{min}} \) be as defined in **Theorem 15**. With probability 99/100 over the coins of the algorithm **Private-Optimal-Space-LRF**, the output of **Private-Optimal-Space-LRF** satisfies

\[
\| (A \ 0) - M_k \|_F \leq (1 + \alpha)\|A - [A]_k\|_F + O(\sigma_{\text{min}} \sqrt{m} + \rho_1 (1 + \alpha)^{-1} \sqrt{k n} + \rho_2 v). 
\]

The result follows by setting the values of \( \rho \) and \( \sigma_{\text{min}} \).

### 5 Efficient-update and Optimal-error Differentially-private LRF Under \textit{Priv}_2

In the previous section, we gave an space-optimal (when \( k \geq 1/\alpha \)) and optimal additive error algorithm for LRF under \textit{Priv}_1. In this section, we show a differentially private algorithm that outputs a \( k \)-rank factorization of an \( m \times n \) matrix under an even stronger level of privacy, \textit{Priv}_2. Our idea is to show that the algorithm of Clarkson and Woodruff [13] can be made to work in the turnstile update model while incurring optimal additive error. The main result of this section is the following theorem.

**Theorem 28.** Let \( m, n \in \mathbb{N} \) and \( \alpha, \epsilon, \delta \) be the input parameters. Let \( k \) be the desired rank of the factorization and let \( s = \max\{m, n\} \). Given a private input matrix \( A \in \mathbb{R}^{m \times n} \) streamed in a turnstile model, the
factorization $\tilde{U}, \tilde{\Sigma}, \tilde{V}$ outputted by the algorithm, Private-Frobenius-LRF, presented in Figure 3 satisfies the following properties:

1. **Private-Frobenius-LRF** outputs a low rank factorization such that, with probability 99/100 over the coins of the algorithm,

$$\|A - M_k\|_F \leq (1+\alpha)\|A - [A]_k\|_F + O\left((1+\alpha)\sqrt{km} + \sqrt{\frac{n(1+\alpha)}{\alpha^3}}\left(k + \frac{1}{\alpha}\right)\sqrt{\log(1/\delta)}\right),$$

where $M_k := \tilde{U}\tilde{\Sigma}\tilde{V}^T$.

2. **Private-Frobenius-LRF** is $(\varepsilon, \delta)$-differentially private.

3. The space used by **Private-Frobenius-LRF** is $O((m + n\alpha^{-2})k\alpha^{-2}\log k\log(1/\delta))$.

4. The update time of **Private-Frobenius-LRF** is $O(\log s)$. It requires $O(nn(A)) + \tilde{O}(nk^2\alpha^{-3} + mk^2\alpha^{-5} + k^3\alpha^{-6})$ to compute LRF, where $nn(A)$ denotes the sparsity of the matrix $A$.

The space required by the algorithm is the space required to store $Y$ and $Z$, which is $mt^2 + nv^2 = O((m + n\alpha^{-2})k\alpha^{-2}\log k\log(1/\delta))$. This proves part 3 of Theorem 28. For the update time in part 3 of Theorem 12, note that $\Phi$ updates in input-sparsity time, which is constant, plus $O(\log(m + n))$. The same is true for the update time of $Z$. For the running time of part 4 of Theorem 28 we have the following:

1. Computing the sketch $Y$ requires $O(nn(A)) + mt^2$ time and computing the sketch $Z$ requires $O(nn(A)) + nv^2$. Therefore, computing $Y$ and $Z$ requires $O(nn(A)) + \tilde{O}(nk^2\alpha^{-3} + (k + \alpha^{-2})^2)$ and $O(nn(A)) + \tilde{O}(nk^2\alpha^{-3} + (k + \alpha^{-2})^2)$, respectively.

2. Computing the orthonormal basis $U$ requires $mt^2 = \tilde{O}(mk^2\alpha^{-4})$ time.

3. Computing the SVD of the matrix $SU$ requires $vt^2 = \tilde{O}(k^3\alpha^7)$.

4. Computation of $[\tilde{U}^T Z]_k$ requires $nv^2 = \tilde{O}(nk^2\alpha^{-6})$ time.

5. Computation of the SVD in Step 4 requires $nt^2 = \tilde{O}(nk^2\alpha^{-4})$ time.

The following lemma proves Part 2 of Theorem 28.

**Lemma 29.** If $\rho = \frac{\sqrt{\varepsilon} - \varepsilon\ln(1/\delta)}{1+\alpha}$, then **Private-Frobenius-LRF** is $(\varepsilon, \delta)$-differentially private.

**Proof.** To prove this lemma, we use the second claims of Lemma 7 and Lemma 8 i.e., for all $D$, $\|SD\|_F^2 = (1 + \alpha)\|D\|_F^2$ for $S \sim D_A$ and $\|\Phi D\|_F^2 = (1 + \alpha)\|D\|_F^2$ for $\Phi \sim D_R$. Let $A$ and $A'$ be two neighboring matrices such that $E = A - A'$ has Frobenius norm 1. Then $\|SE\|_F^2 \leq (1 + \alpha)\|E\|_F^2 = 1 + \alpha$. Publishing $Z$ preserves $(\varepsilon, \delta)$-differential privacy follows from considering the vector form of the matrix $SA$ and $N_2$ and Theorem 11. Similarly, we use the fact that, for any matrix $C$ of appropriate dimension, $\|\Phi C\|_F^2 = (1 + \alpha)\|C\|_F^2$, to prove that publishing $A\Phi + N_1$ preserves differential privacy. The lemma follows by applying Lemma 9 and Theorem 10.

We prove part 1 in two stages by decoupling the effect of the noise matrices $N_1$ and $N_2$. In the first step, we assume that the noise $N_1$ is not introduced and then analyze the algorithm. Later, in Section 5.2 we perform the analysis of the algorithm as presented in Figure 3.
5.1 Simplication Step

We first prove the result without the effect of using $N_1$. That is, in the following analysis, we consider $U_0$, a matrix whose columns are formed by an orthonormal basis of the column-space of $A\Phi$ instead of $U$, a matrix whose columns are formed by an orthonormal basis of the column-space of $A\Phi + N_1$. In this case, we show that $U_0\Sigma^\dagger\{U_0^\dagger Z\}$ is a good approximation of $A$. We show the following

**Theorem 30.** Let $M'_k := U_0\Sigma^\dagger\{U_0^\dagger Z\}$, where $U_0$ is as defined above and rest of the terms as defined in Figure 3. Then with probability $1 - O(\delta)$ over $\Phi \sim D_{R}$ and $S \sim D_A$, 

$$||M'_k - A||_F \leq (1 + \alpha)||A - [A]_k||_F + 2||S^\dagger N_2||_F.$$ 

**Proof.** We prove this lemma using few claims.

**Lemma 31.** Let $A$ be the input matrix. Let $\Phi \sim D_R$, $S \sim D_A$ be as in Figure 3. Let $B = A + S^\dagger N_2$ for $N_2$ defined as in Figure 3. Then

$$\min_{x,X \in S \cap D} \|A\Phi X - B\|_F \leq (1 + \alpha)||A - [A]_k||_F + ||S^\dagger N_2||_F.$$ 

**Proof.** Set $r = k$, $P = [A]_k^\dagger$, and $Q = A^\dagger$ in the statement of Lemma 7. Then using Lemma 7 with $t = O(k/\alpha \log(r/\alpha))$, we have with probability $1 - \beta$ over $\Phi \sim D_R$, 

$$||[A]_k^\dagger X' - A^\dagger||_F \leq (1 + \alpha)\min_{X \in S \cap D} \|[A]_k^\dagger - A\|_F ,$$

where $X' = \arg\min_{X \in S \cap D} \|\Phi^\dagger ([A]_k^\dagger X - A^\dagger)\|_F$. 

Taking the transpose and the fact that the Frobenius norm is preserved under transpose and $X' = ([A]_k^\dagger \Phi)^\dagger (A\Phi)^T$, we have with probability $1 - \delta$ over $\Phi \sim D_R$,

$$\|A\Phi ([A]_k^\dagger \Phi)^\dagger [A]_k - A\|_F \leq (1 + \alpha)||A - [A]_k||_F .$$

Since $([A]_k^\dagger \Phi)^\dagger [A]_k$ has rank at most $k$, we have with probability $1 - \delta$ over $\Phi \sim D_R$,

$$\min_{x,X \in S \cap D} \|A\Phi X - B\|_F \leq \|A\Phi ([A]_k^\dagger \Phi)^\dagger [A]_k - B\|_F \leq \|A\Phi ([A]_k^\dagger \Phi)^\dagger [A]_k - A\|_F + ||S^\dagger N_2||_F$$

(20)

Combining equation (19) and equation (20), with probability $1 - \delta$ over $\Phi \sim D_R$,

$$\min_{x,X \in S \cap D} \|A\Phi X - B\|_F \leq (1 + \alpha)||A - [A]_k||_F + ||S^\dagger N_2||_F$$

(21)

which is what was required. 

Till now, we have related $\min_{x,X \in S \cap D} \|A\Phi X - B\|$ with $(1 + \alpha)||A - [A]_k||_F$ and $||S^\dagger N_2||_F$. Since $U_0$ is a matrix whose columns are an orthonormal basis for the column space of $A\Phi$, we have 

$$\min_{x,X \in S \cap D} \|U_0 X - B\|_F \leq \min_{x,X \in S \cap D} \|A\Phi X - B\|_F.$$ 

(22)

Using equation (22) in Lemma 31 we have

$$\min_{x,X \in S \cap D} \|U_0 X - B\|_F \leq (1 + \alpha)||A - [A]_k||_F + ||S^\dagger N_2||_F.$$ 

(23)

**Lemma 32.** Let $U_0$, $B$, $A$, $S$, and $N_2$ be as in Lemma 31. Let $D_A$ be a distribution that satisfies $(\alpha, \beta)$-subspace embedding. Let $\tilde{X} = \arg\min_{x,X \in S \cap D} \|S(U_0 X - B)\|_F$, then with probability $1 - 2\delta$ over $S \sim D_A$,

$$\|(U_0 \tilde{X} - B)\|_F \leq (1 + \alpha)^3||A - [A]_k||_F + ||S^\dagger N_2||_F.$$
Proof. Set \( p = k/\alpha \), \( D = U_0 \), \( E = B \), and \( v = O(k/\alpha^3 \log(1/\delta)) \) in the statement of Lemma 8. Let us restrict our attention to rank \( k \) matrices \( X \) and denote by \( \tilde{X} = \text{argmin}_{X,r(X) = k} \| U_0 X - B \|_F \) and \( \hat{X} = \text{argmin}_{X,r(X) = k} \| S(UX - B) \|_F \). Then we have with probability \( 1 - 2\delta \) over \( S \sim D_A \),

\[
\min_{X,r(X) = k} \| U_0 X - B \|_F = \| U_0 \tilde{X} - B \|_F = (1 + \alpha)^{-1} \| S(U_0 \tilde{X} - B) \|_F \\
\geq (1 + \alpha)^{-1} \min_{X,r(X)} \| S(U_0 X - B) \|_F \\
= (1 + \alpha)^{-1} \| S(U_0 \hat{X} - B) \|_F = (1 + \alpha)^{-2} \| (U_0 \tilde{X} - B) \|_F. \tag{24}
\]

Combining equation (24) with equation (23), we have

\[
\| (U_0 \tilde{X} - B) \|_F \leq (1 + \alpha)^3 \| A - [A]_k \|_F + \| S^\dagger N_2 \|_F. \tag{25}
\]

This completes the proof. \( \square \)

To finalize the proof of Theorem 30, we need to compute \( \tilde{X} = \text{argmin}_{X,r(X) \leq k} \| S(U_0 X - B) \|_F \) and lower bound \( \| (U_0 \tilde{X} - B) \|_F \). For this, we use the following result by Clarkson and Woodruff [11].

Lemma 33. ([11] Theorem 4.3) Let \( O \) be an orthonormal matrix and \( Z \in \mathbb{R}^{m \times v} \). Then

\[
|O^T Z|_k = \text{argmin}_{X,r(X) \leq k} \| OX - Z \|_F.
\]

Invoking Lemma 33 with \( O = \tilde{U} \) and \( Z = SB \). We get \( \| \tilde{U}^T SB \|_k = \text{argmin}_{X,r(X) \leq k} \| \tilde{U} X - SB \|_F \). This implies that

\[
\tilde{X} = \tilde{V} \Sigma^\dagger [\tilde{U}^T SB]_k = \text{argmin}_{X,r(X) \leq k} \| S(U_0 X - B) \|_F. \tag{26}
\]

Using equation (26) in equation (25) and the fact that \( B = A + S^\dagger N_2 \), by appropriate scaling of \( \alpha \), we have the final result,

\[
\| U_0 \tilde{V} \Sigma^\dagger [\tilde{U}^T Z]_k - A \|_F - \| S^\dagger N_2 \|_F \leq \| U_0 \tilde{V} \Sigma^\dagger [\tilde{U}^T Z]_k - B \|_F \\
\leq (1 + \alpha)^3 \| A - [A]_k \|_F + \| S^\dagger N_2 \|_F \\
\Rightarrow \| U_0 \tilde{V} \Sigma^\dagger [\tilde{U}^T SA]_k - A \|_F \leq (1 + \alpha)^3 \| A - [A]_k \|_F + 2\| S^\dagger N_2 \|_F
\]
as required. \( \square \)

There are various choices of distribution \( D_A \) from which \( S \) is sampled. The first choice is that \( S \) is a subsampled randomized Hadamard transform. We consider another distribution from which \( S \) can be sampled in Section B and show that if we use the alternative distribution discussed in Section B, then it does not affect the bounds presented here. The following lemma bounds the additive error of the simplified (non-private) algorithm when \( S \) is a subsampled Hadamard transform.

Lemma 34. Let \( N_2 \sim \mathcal{N}(0, \rho^2)^{p \times n} \). Then \( \| S^\dagger N_2 \|_F = O(\rho \sqrt{vn}) \) with probability 99/100 over \( S \sim D_A \).

Proof. Using Lemma 47, we have \( E[\| N_2 \|_F^2] = vnp^2 \). The result follows by invoking Lemma 49 and Markov’s inequality. \( \square \)
5.2 Complete Analysis

In this section, we prove the utility guarantee of Theorem\cite{28}. We first show the following result.

**Theorem 35.** Let $M_k = UV\sum_{i}[\bar{U}^TZ]_k$ be the product of the factorization outputted by the algorithm in Figure\cite{3}. Then with probability $1 - O(\delta)$ over $\Phi \sim \mathcal{D}_R$ and $S \sim \mathcal{D}_A$,

$$\|M_k - A\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + 2\|S^TN_2\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F.$$  

**Proof.** We prove the result by proving a series of results. We provide an upper and a lower bound on $\min_{X, r(X) \leq k}\|YX - B\|_F$ in terms of $\|A - [A]_k\|_F$ and the output of the algorithm.

**Lemma 36.** Let $A$ be the input matrix. Let $\Phi \sim \mathcal{D}_R, S \sim \mathcal{D}_A$ be as in Figure\cite{3}. Let $Y = \Phi A + N_1$ and $B = A + S^TN_2$ for $N_1, N_2$ as defined in Figure\cite{3}. Then with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,

$$\min_{X, r(X) \leq k}\|YX - B\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F + \|S^TN_2\|_F.$$  

**Proof.** Set $r = k$, $P = [A]_k^T$, and $Q = A^T$ in Lemma\cite{7}. Then using Lemma\cite{7}, we have

$$\|[A]_k^TX' - A^T\| \leq (1 + \alpha)\min_X\|[A]_k^TX - A^T\|_F,$$  

where $X' = \arg\min_X\||[A]_k^T X - A^T|\|_F$. Taking the transpose and the fact that the Frobenius norm is preserved under transpose and $X' = (([A]_k\Phi)^\dagger(A\Phi)^T$, we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$.

$$\|A\Phi([A]_k\Phi)^\dagger[A]_k - A\|_F \leq (1 + \alpha)\|A - [A]_k\|_F. \quad (27)$$

Moreover, since $([A]_k\Phi)^\dagger[A]_k$ has rank at most $k$ and $Y = \Phi A + N_1$, with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,

$$\begin{align*}
\min_{X, r(X) \leq k}\|YX - B\|_F & \leq \|Y\Phi([A]_k\Phi)^\dagger[A]_k - B\|_F \\
& = \|A\Phi([A]_k\Phi)^\dagger[A]_k + N_1([A]_k\Phi)^\dagger[A]_k - B\|_F \\
& = \|A\Phi([A]_k\Phi)^\dagger[A]_k + N_1([A]_k\Phi)^\dagger[A]_k - A - S^TN_2\|_F \\
& \leq \|A\Phi([A]_k\Phi)^\dagger[A]_k - A\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F + \|S^TN_2\|_F \\
& \leq \|Y\Phi([A]_k\Phi)^\dagger[A]_k - B\|_F. \quad (28)
\end{align*}$$

Combining equation (27) and equation (28), we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$.

$$\min_{X, r(X) \leq k}\|YX - B\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F + \|S^TN_2\|_F. \quad (29)$$

Lemma 36 relates $\min_{X, r(X) \leq k}\|YX - B\|$ with $(1 + \alpha)\|A - [A]_k\|_F, \|N_1([A]_k\Phi)^\dagger[A]_k\|_F$, and $\|S^TN_2\|_F$. Since $U$ is the orthonormal basis for the column space of $Y$, we further have

$$\begin{align*}
\min_{X, r(X) \leq k}\|UX - B\|_F & \leq \min_{X, r(X) \leq k}\|YX - B\|_F. \quad (30)
\end{align*}$$

Combining equation (29) and equation (30), we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$.

$$\min_{X, r(X) \leq k}\|UX - B\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F + \|S^TN_2\|_F. \quad (31)$$

**Lemma 37.** Let $U, B, A, S, N_1$, and $N_2$ be as above, and let $X = \arg\min_{X, r(X) = k}\|S(UX - B)\|_F$. Let $\mathcal{D}_A$ be a distribution that satisfies $(\alpha, \beta)$-subspace embedding. Then with probability $1 - 4\delta$ over $S \sim \mathcal{D}_A$, 

$$\|UX - B\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + \|S^TN_2\|_F + \|N_1([A]_k\Phi)^\dagger[A]_k\|_F.$$ 

\[\]
Proof. Set \( p = k/\alpha \), \( D = U \) and \( E = B \) in the statement of Lemma 8. Let us restrict our attention to rank \( k \) matrices \( X \) and denote by \( \tilde{X} = \arg\min_{X, r(X) = k} \|UX - E\|_F \) and \( \hat{X} = \arg\min_{X, r(X) = k} \|S(UX - B)\|_F \). Then we have with probability \( 1 - \delta \) over \( S \sim D_A \),
\[
\min_{X, r(X) = k} \|UX - B\|_F = \|UX - B\|_F = (1 + \alpha)^{-1} \|S(U\tilde{X} - B)\|_F \\
\geq (1 + \alpha)^{-1} \min_{X, r(X)} \|S(UX - B)\|_F \\
= (1 + \alpha)^{-1} \|S(U\tilde{X} - B)\|_F = (1 + \alpha)^{-2} \|(U\tilde{X} - B)\|_F. 
\]
(32)
Combining equation (32) with equation (31), we have with probability \( 1 - 2\delta \) over \( \Phi \sim D_R \) and \( S \sim D_A \),
\[
\|(UX - B)\|_F \leq (1 + \alpha)^3 \|A - [A]_k\|_F + \|S^T N_2\|_F + \|N_1([A]_k \Phi)^\dagger [A]_k\|_F. 
\]
(33)
This completes the proof by scaling \( \alpha \) by a constant factor. \( \square \)

To finalize the proof of Theorem 35, we need to compute \( \tilde{X} = \arg\min_{X, r(X) \leq k} \|S(UX - B)\|_F \) and lower bound \( \|(UX - B)\|_F \). Invoking Lemma 33 with \( O = \tilde{U} \) and \( Z = SB \), we get \( [\tilde{U}^T Z]_k = [\tilde{U}^T SB]_k = \arg\min_{X, r(X) \leq k} \|UX - SB\|_F \). This implies that
\[
\tilde{X} = \tilde{V} \tilde{\Sigma}^\dagger [\tilde{U}^T Z]_k = \arg\min_{X, r(X) \leq k} \|S(UX - B)\|_F. 
\]
(34)
Using equation (34) in equation (33), and the fact that \( B = A + S^T N_2 \), we have the final result.
\[
\|U\tilde{V} \tilde{\Sigma}^\dagger [U^T Z]_k - A\|_F - \|S^T N_2\|_F \leq \|U\tilde{V} \tilde{\Sigma}^\dagger [U^T S A]_k - B\|_F \\
\leq (1 + \alpha) \|A - [A]_k\|_F + \|S^T N_2\|_F + \|N_1([A]_k \Phi)^\dagger [A]_k\|_F. \\
\Rightarrow \|U\tilde{V} \tilde{\Sigma}^\dagger [U^T Z]_k - A\|_F \leq (1 + \alpha) \|A - [A]_k\|_F + 2\|S^T N_2\|_F + \|N_1([A]_k \Phi)^\dagger [A]_k\|_F. 
\]
This completes the proof of Theorem 35. \( \square \)

Lemma 38. Let \( N_1 \sim \mathcal{N}(0, \rho^2)^{m \times t} \). Then \( \|N_1([A]_k \Phi)^\dagger [A]_k\|_F = O(\rho \sqrt{km(1 + \alpha)}) \) with probability 99/100 over \( \Phi \sim D_R \).

Proof. Let \( C = N_1([A]_k \Phi)^\dagger [A]_k \). Then \( C \Phi = N_1([A]_k \Phi)^\dagger [A]_k \Phi \). Now \( ([A]_k \Phi)^\dagger [A]_k \Phi \) is a projection onto a random subspace of dimension \( k \). Since every entry of \( N_1 \) is picked i.i.d. from \( \mathcal{N}(0, \rho^2) \), \( C \Phi = N_1([A]_k \Phi)^\dagger [A]_k \Phi = N_1 \), where \( N_1 \) is an \( m \times k \) matrix with every entries picked i.i.d. from \( \mathcal{N}(0, \rho^2) \). This is because we can write \( ([A]_k \Phi)^\dagger [A]_k \Phi \) is a projection onto a random subspace of dimension \( k \). Using Lemma 47, this implies that
\[
\mathbb{E}[\|C \Phi\|_F^2] = \mathbb{E}[\|\tilde{N}_1\|_F^2] = \sum_{i,j} \mathbb{E}[(\tilde{N}_1)_{ij}^2] = km \rho^2. 
\]
The result follows using Markov’s inequality and the fact that \( \|C \Phi\|_F^2 = (1 + \alpha) \|C\|_F^2 \). \( \square \)

Theorem 28 now follows from Lemma 34, Lemma 38, Theorem 35, and the choice of \( \rho \) in Lemma 29.

6 Lower-bound for Low-rank Factorization When \( \gamma \neq 0 \)

We give a reduction to the augmented indexing problem, AIND. It is defined as follows.

**Definition 39.** (AIND problem). Alice is given an \( N \)-bit string \( x \) and Bob is given an index \( \text{ind} \in [N] \) together with \( x_{\text{ind} + 1}, \cdots, x_N \). The goal of Bob is to output \( x_{\text{ind}} \).
The communication complexity for solving $A_{\text{IND}}$ is well known due to the result of Miltersen et al. [51].

**Theorem 40.** The minimum bits of communication required to solve $A_{\text{IND}}$ with probability $2/3$, when the message is sent only in one direction, i.e., either from Alice to Bob or from Bob to Alice, is $\Omega(n)$. This lower bound holds even if the index, ind, and the string, $x$, is chosen uniformly at random.

The space lower bound in streaming model is shown by showing that a one-pass streaming algorithm $\text{Alg}$ yields a single round communication protocol for some function $f$. The idea is as follows. On input $x$, Alice invokes $\text{Alg}$ on its input to compute $\text{Alg}(x)$. She then sends the state to Bob, who computes $\text{Alg}(x|y)$ using his input $y$, and uses this to compute the function $f$. The communication is therefore the same as the space required by the algorithm. In what follows, we use the notation $C_{i,j}$ to denote the $i$-th column of the matrix $C$.

**Theorem 41.** Let $m, n, k \in \mathbb{N}$ and $\alpha > 0$. Then any randomized single-pass algorithm for $(\alpha, 5/6, O(m+n), k)$-LRA in the general turnstile model uses $\Omega((m+n)k/\alpha)$ words.

**Proof.** We adapt the proof of Clarkson and Woodruff [11] for the case when $\gamma \neq 0$. Suppose $m \geq n$ and let $a = k/20\alpha$. Without loss of generality, $a$ can be assumed to be at most $n/2$. Let $\ell$ be the word size. We assume Alice has a string $x \in \{-1, +1\}^{(m-a)a}$ and Bob has an index $\text{ind} \in [(m-a)a]$. The idea is to define the matrix $A$ with high Frobenius norm. The matrix $\tilde{A}$ is the summation of the matrix $A$ constructed by Alice and $\tilde{A}$ constructed by Bob.

Alice constructs its matrix $\tilde{A}$ as follows. We partition the set $[a]$ in $\ell$ sets $I_1, \ldots, I_{\ell}$ such that $I_j := \{(j-1)a/\ell + 1, \ldots, ja/\ell\}$. Let $M_{I_i}$ be an $(m-a) \times a/\ell$ matrix for all $1 \leq i \leq \ell$. We form a bijection between entries of $x$ and the entries of $M$ in the following manner. Every entry of $M_{I_i}$ is defined by a unique bit of $x$, i.e., $(M_{I_i})_{j,k} = (-1)^{x_j}10^d$ for $d = (i-1)(m-a)a + (k-1)(m-a) + j$.

$$\tilde{A} = \begin{pmatrix} 0^{a \times a} & 0^{a \times (n-a)} \\ M & 0^{(m-a) \times (n-a)} \end{pmatrix},$$

where $M = (M_{I_{I_1}} \cdots M_{I_{I_{\ell}}})$.

Suppose Bob is given an index $\text{ind} \in [(m-a)a]$ such that $x_{\text{ind}}$ corresponds to the sub-matrix $M_{I_\theta}$. Then we can assume that Bob also knows every entry in the sub-matrix $M_{I_\theta'}$, for $\theta' > \theta$. Bob forms a second level partition of the columns of $M_{I_\theta}$ into $k$ size groups $G_1, \ldots, G_{a/\ell}$ of size $a/\ell$. Due to our construction, there exists a unique $r$ such that $x_{\text{ind}}$ maps to an entry in the sub-matrix formed by columns indexed by one of the second level partition $G_r$. Let $G_r = \{c, c+1, \ldots, c+k-1\}$ be the columns corresponding to the $k$-size group of $I_\theta$ in which ind is present. As its input, Bob streams a matrix $\tilde{A}$ which is an all-zero matrix, except for entries $\tilde{A}_{c+i,c+i} = \zeta$ for $0 \leq i \leq k-1$. In other words, Bob inserts a scaled identity matrix in the stream, where the scaling parameter $\zeta$ is large enough to make sure that most of the error of any randomized algorithm is due to other columns of $\tilde{A}$. We set the value of $\zeta$ as a large polynomial in the error of the algorithm.

Alice runs its randomized single-pass algorithm for $k$-rank approximation on $\tilde{A}$ and send the state to Bob. Bob completes the streaming algorithm using the state received by Alice and by streaming its own matrix $A$. Therefore, the one-pass randomized algorithm gets as input a matrix $A = \tilde{A} + \tilde{A}$ and it is required to output a rank-$k$ matrix $B$ with additive error $\gamma = O(m+n)$. We will show that any such an output allows us to solve $A_{\text{IND}}$. Denote by $A_{G_r}$ the sub-matrix formed by the columns $c, c+1, \ldots, c+k-1$.

Let us first understand the properties of the constructed matrix $A$. To compute the Frobenius norm of this matrix, we need to consider two cases: the case for sub-matrices in which ind belongs, i.e., $M_{I_i}$, and the rest of the matrix. For the sub-matrix corresponding to the columns indexed by $G_r$, the columns of $A_{I_\theta}$ have Euclidean length $(\zeta^2 + (m-a)100^\theta)^{1/2}$. For $\theta' < \theta$, every columns have Euclidean norm
(a(m − a))^{1/2}10^\theta$. Therefore, we have the following:

\[
\|A - [A]_k\|^2_F \leq \frac{(a - k)(m - a)10^\theta}{\ell} + \sum_{\theta' < \theta} \frac{a(m - a)10^\theta}{\ell} \\
\leq \frac{(a - k)(m - a)10^\theta}{\ell} + \frac{a(m - a)10^\theta}{99\ell} \leq 2 \cdot (100)^\theta m^2/\ell = \Gamma
\]

In order to solve $(\alpha, \beta, \gamma, k)$-LRF, the algorithm needs to output $B$ of rank at most $k$ such that, with probability $1 - \beta$ over its random coins,

\[
\|A - B\|_F^2 \leq \left[(1 + \alpha)\sqrt{\Gamma + \gamma}\right]^2 \leq 2(1 + \alpha)\Gamma + 2\gamma^2 \leq 2\Gamma + 100^\theta k(m - a) \left(\frac{1}{10} + \frac{1}{99}\right) + 2\gamma^2 \\
= 4 \cdot (100)^\theta m^2/\ell + 100^\theta k(m - a) \left(\frac{1}{10} + \frac{1}{99}\right) + 2\gamma^2
\]

Let us denote by $Y := 4 \cdot (100)^\theta m^2/\ell + 100^\theta k(m - a) \left(\frac{1}{10} + \frac{1}{99}\right) + 2\gamma^2$. The proof idea is now the following:

(i) Columns of $B$ corresponding to index set in $G_\ell$ are linearly independent.

(ii) Bound the error incurred by $\|A - B\|_F$ in terms of the columns indexed by $G_\ell$.

The idea is to show that most of the error is due to the other columns in $B$; and therefore, sign in the columns in $A$ indexed by $G_\ell$ agrees with that of the signs of those columns in $B$. This allows Bob to solve the AIND problem as Bob can just output the sign of the corresponding position.

Let $R := \{ra/k + 1, \cdots, (r + 1)a/k\}$ and $C := \{c, \cdots, c + k - 1\}$.

The following lemma proves that when $\zeta$ is large enough, then the columns of $B$ corresponding to index set $I_r$ are linearly independent. This proves part (i) of our proof idea.

**Lemma 42.** Let $B(C) := [B_{c1} \cdots B_{c+k-1}]$ be the columns corresponding to the sub-matrix formed by columns $c, \cdots, c + k - 1$ of $B$. If $\zeta \geq 2Y^2$, then the columns of $B(C)$ spans the column space of $[A]_k$.

**Proof.** We will prove the lemma by considering the $k \times k$ sub-matrix, say $Y$, of $B$ formed by the rows indexed by $R$ and the columns indexed by $C$. For the sake of brevity and abuse of notation, let us denote the restriction of $B$ to this sub-matrix $Y := [Y_{11}, \cdots, Y_{kk}]$. Suppose, for the sake of contradiction that the vectors $\{Y_{11}, \cdots, Y_{kk}\}$ are linearly dependent. In other words, there exists a vector $Y_{\tilde{s}}$ and real numbers $a_1, \cdots, a_k$, not all of which are identically zero, such that

\[
Y_{\tilde{s}} = \sum_{j=1, j \neq i}^k a_j Y_{ij}.
\]

From the construction, since Bob inserts a sub-matrix $\zeta I_k$, we know that

\[
\sum_{j=1}^k (Y_{ij} - \zeta)^2 \leq \|A - B\|_F^2 \leq Y. \tag{35}
\]

\[
\sum_{j=1}^k \sum_{p \neq j} Y_{p,j}^2 \leq \|A - B\|_F^2 \leq Y. \tag{36}
\]

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From equation \(35\) and choice of \(\zeta\), for all \(j\), we have \(Y_{j,j} \geq \Upsilon^2\). Further, equation \(36\) implies that \(Y_{p,j} \leq \sqrt{\Upsilon}\). Let \(i\) be the index in \(\{1, \cdots, k\} \setminus \{i\}\) for which \(|a_i|\) attains the maximum value. Then

\[
Y_{i,i} = \sum_{j=1, j \neq i}^{k} a_j Y_{i,j} \geq \Upsilon^2 \text{ implies that there is an } \tilde{i} \in \{1, \cdots, k\} \setminus \{i\} \text{ such that } |a_{\tilde{i}}| \geq \frac{\Upsilon^2}{k \sqrt{\Upsilon}}.
\]

Now consider the \(\tilde{i}\)-entry of \(Y_{:,j}\). Note that \(\tilde{i} \neq i\). Since \(\Upsilon\) depends quadratically on \(m\) and \(\gamma\), we have

\[
\left| \sum_{j=1, j \neq i}^{k} a_j Y_{i,j} \right| \geq |a| (\Upsilon^2 - k \sqrt{\Upsilon}) \geq (\Upsilon^2 - k \sqrt{\Upsilon}) \frac{\Upsilon^2}{k \sqrt{\Upsilon}} > \sqrt{\Upsilon}.
\]

This is a contradiction because \(Y_{p,j} \leq \sqrt{\Upsilon}\) due to equation \(36\) for \(p \neq j\). This completes the proof. \(\square\)

The following lemma proves part (ii) of our proof idea.

**Lemma 43.** Let \(V_{1,1}, \cdots, V_{k,k}\) be the columns of \(B(C)\) and \(\tilde{V}_{1,1}, \cdots, \tilde{V}_{k,k}\) be the restriction of these vectors to the rows \(a + 1, \cdots, n\). Then column \(i\) of \(B\) can be written as linear combination of real numbers \(a_{i,1}, \cdots, a_{i,k}\) of the vectors \(V_{1,1}, \cdots, V_{k,k}\) such that, for all \(j\) and \(i \in R\), \(\eta^2_{i,j} \leq 4/\Upsilon^3\).

**Proof.** Let \(M_{1,1}, \cdots, M_{a,a}\) be the columns of \(M\). The \(i\)-th row of \(B\) can be represented as \(\sum \eta_{i,j} V_{j,i}\), for real numbers \(\eta_{i,j}\), not all of which are identically zero. Then

\[
\Upsilon \geq \|A - B\|_F^2 \geq \sum_{i=1}^{k} (\zeta - V_{r(a/k) + i,j})^2 + \sum_{i=1}^{k} \sum_{j \neq i} V_{r(a/k) + i,j}^2 + \sum_{i=1}^{k} \|M_{r(a/k) + i} - \tilde{V}_{i:i}\|^2
\]

\[
+ \sum_{i \notin G_r} \sum_{j=1}^{k} \left( \eta_{i,j} V_{r(a/k) + j,j} + \sum_{j' \neq j} \eta_{i,j'} V_{r(a/k) + j,j'} \right)^2 + \sum_{i \notin G_r} \left\| M_{i} - \sum_{j=1}^{k} \eta_{i,j} \tilde{V}_{j:j} \right\|^2.
\]

As in the proof of Lemma 42 we have \(|V_{r(a/k) + i,j}| \leq \sqrt{\Upsilon}\) and \(|V_{r(a/k) + i,i}| \geq \Upsilon^2\). Let \(j_i\) be the index such that \(|\eta_{i,j_i}|\) is the maximum. Then the above expression is at least \(|\eta_{i,j_i}|^2 (\Upsilon^2 - k \sqrt{\Upsilon})^2 \geq |\eta_{i,j_i}|^2 \Upsilon^4 / 4\). Since this is less than \(\Upsilon\), the result follows. \(\square\)

We can now complete the proof. First note that since \(M\) is a signed matrix, each \(\tilde{V}_{i}\) in the third term of the above expression is at least \(\sqrt{\Upsilon}\). Therefore, for all \(i \notin S\) and all \(j\)

\[
\left| \sum_{j=1}^{k} \eta_{i,j} \tilde{V}_{j:j} \right| \leq \frac{4k \Upsilon^{1/2}}{\Upsilon^{3/2}} = \frac{4k}{\Upsilon}.
\]

As \(M_i\) is a sign vector and if \(\gamma = O(m + n) = O(m)\), this implies that

\[
\sum_{i \notin G_r} \left\| M_i - \sum_{j=1}^{k} \eta_{i,j} \tilde{V}_{j:j} \right\|^2 \geq \sum_{i \notin G_r} \|M_i\|^2 \left(1 - \frac{4k}{\Upsilon}\right)
\]

\[
\geq O((100)^{9} m^2 / \ell) - O(100^9 a)
\]

\[
\sum_{i=1}^{k} \|M_{r(a/k) + i} - \tilde{V}_{i:i}\|^2 = \sum_{i=1}^{k} \sum_{j=1}^{m-a} (M_{j,r(a/k) + i} - (\tilde{V}_{i})_j)^2 \leq 100^9 k(m - a) \left(\frac{1}{10} + \frac{1}{99}\right) + O(100^9 a)
\]

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Now, since there are in total \( k(m-a) \) entries in the submatrix formed by the columns indexed by \( G_r \), at least \( 1 - \left( \frac{1}{10} + \frac{1}{99} + o(1) \right) \) fraction of the entries have the property that the sign of \( M_{j,ra/k+i} \) matches the sign of \( \tilde{V}_{j,i} \). Since ind is in one of the columns of \( M_{ra/k+1, \cdots M_{ra/k+k}} \), with probability at least \( 1 - \left( \frac{5}{10} + \frac{1}{99} + o(1) \right) \), if Bob outputs the sign of the corresponding entry in \( B \), then Bob succeeds in solving \( \text{AIND} \). This gives a lower bound of \( \Omega((m-a)a) = \Omega(mk\ell/\alpha) \) space. The case when \( m \leq n \) is analogous and gives a lower bound of \( \Omega((m+n)k\ell/\alpha) \). Thus, there is a lower bound of \( \Omega((m+n)k\ell/\alpha) \).

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A Auxiliary Results

We need the following result about product of pseudo-inverse in the proof of Lemma 49 and Claim 51.

**Theorem 44.** Let A and B be conforming matrices and either,

1. A has orthonormal columns (i.e., \(A^\top A\) is an identity matrix) or,
2. B has orthonormal rows (i.e., \(BB^\top\) is an identity matrix),

then \((AB)^\dagger = B^\dagger A^\dagger\).

We use the following variant of Pythagorean theorem in the proof of Lemma 14.

**Theorem 45.** (Pythagorean theorem). Let A and B be two matrices such that \(A^\top B\) is an zero matrix. Then for any \(C = A + B\), we have \(\|C\|_F^2 = \|A\|_F^2 + \|B\|_F^2\).

We also need the following results for the privacy proof.
Theorem 46. (Lidskii Theorem \cite{5}). Let $A, B$ be $n \times n$ Hermitian matrices. Then for any choice of indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$,
\[ \sum_{j=1}^{k} \lambda_{ij} (A + B) \leq \sum_{j=1}^{k} \lambda_{ij} (A) + \sum_{j=1}^{k} \lambda_{ij} (B), \]
where $\{\lambda_i (A)\}_{i=1}^{n}$ are the eigen-values of $A$ in decreasing order.

Lemma 47. Let $N \sim \mathcal{N}(0, \rho^2)^{m \times n}$ Gaussian matrix. Then with probability $99/100$, $\|N\|_F = O(\rho \sqrt{mn})$.

Proof. The lemma follows from the following computation.
\[ \mathbb{E}[\|C \Phi \|_F^2] = \mathbb{E} \left[ \sum_{i,j} (\tilde{N}_1)_{ij}^2 \right] = \sum_{i,j} \mathbb{E}[ (\tilde{N}_1)_{ij}^2 ] = km \rho^2. \]
The result follows using Markov’s inequality. \hfill \Box

Lemma 48. (Sarlos \cite{59}). Let $\sigma_{\text{max}} = \sigma_1 \geq \cdots \geq \sigma_m = \sigma_{\text{min}}$ be the singular values of an $m \times n$ matrix $A$.

Let $\Phi \in \mathbb{R}^{n \times m}$ be sampled from a distribution that satisfies $(\alpha, \beta)$-JLP. Let $\sigma_{\text{max}} = \sigma_1 \geq \cdots \geq \sigma_m = \sigma_{\text{min}}$ be the singular values of $C := A \Phi$. Then $\sigma_{\text{max}} \leq (1 + \alpha)^{1/2} \sigma_{\text{max}}$ and $\sigma_{\text{min}} \geq (1 - \alpha)^{1/2} \sigma_{\text{min}}$.

Proof. Let $y = \Phi x$. From the definition, we have
\[ \sigma_{\text{max}} = \max_{x \in \mathbb{R}^n, \|x\|_2 = 1} \| \hat{A} \Phi x \|_2. \]

Let $y = \Phi x$. Since $C$ and $\hat{A}$ have the same rank and $\Phi \sim \mathcal{D}_R$, Lemma \cite{7} and the choice of $t$ gives, with probability $1 - \delta$, $(1 + \alpha) \|x\|_2^2 \geq \|y\|_2^2 \geq (1 - \alpha) \|x\|_2^2$. This implies
\[ \sigma_{\text{max}} \leq (1 + \alpha)^{1/2} \max_{y \in \mathbb{R}^m, \|y\|_2 = 1} \| \hat{A} y \|_2 = (1 + \alpha)^{1/2} \sigma_{\text{max}}. \]

The second part of the lemma follows similarly using the fact that $\sigma_{\text{min}} = \min_{x \in \mathbb{R}^n, \|x\|_2 = 1} \| \hat{A} \Phi x \|_2$ and $\|y\|_2^2 \geq (1 - \alpha)$. \hfill \Box

Lemma 49. Let $S$ be a $v \times m$ subsampled randomized Hadamard matrix, where $v \leq m$ and $N \in \mathbb{R}^{v \times n}$. Then we have,
\[ \|S^\dagger N_2\|_F = \|N_2\|_F. \]

Proof. One way to look at the action of $S$ when it is a subsampled Hadamard transform is that it is a product of matrices $W$ and $\Pi_{1:r}$, where $\Pi_{1:r}$ is the matrix formed by the first $r$ rows of a random permutation matrix and $W$ is a randomized Walsh-Hadamard matrix formed by multiplying a Walsh-Hadamard matrix with a diagonal matrix whose non-zero entries are picked i.i.d. from Rad(1/2).

Since $WD$ has orthonormal rows, $S^\dagger = (\Pi_{1:v} WD)^\dagger = (WD)^T (\Pi_{1:v})^\dagger$.
\[ \|S^\dagger N_2\|_F = \| (\Pi_{1:v} WD)^\dagger N_2 \|_F = \| (WD)^T \Pi_{1:v} N_2 \|_F = \| \Pi_{1:v} N_2 \|_F. \]

Using the fact that $\Pi_{1:v}$ is a full row rank matrix and $\hat{\Pi}_{1:v} \hat{\Pi}^T_{1:v}$ is an identity matrix, we have $\hat{\Pi}_{1:v}^\dagger = \hat{\Pi}_{1:v}^T (\hat{\Pi}_{1:v} \hat{\Pi}^T_{1:v})^{-1} = \hat{\Pi}^T_{1:v}$. The result follows. \hfill \Box

Theorem 50. Let $\mathcal{D}_R$ be an $(\alpha, \beta)$-subspace embedding for generalized regression (Definition 4). Then with probability $1 - 2\beta$ over $\Phi^T \sim \mathcal{D}_R$ and $\Psi \sim \mathcal{D}_R$, for any arbitrary $m \times n$ matrix $A$,
\[ \min_{x, r(\mathbf{x}) \leq k} \| A \Phi X \Phi^T A - A \|_F \leq (1 + \alpha)^2 \| A - [A]_k \|_F. \] (37)
We give an alternate proof of Theorem 50. Our proof uses two optimization problems and uses the solution to those optimization problem in a clever way. We feel that our proof is simpler. It also has explicit solutions to the two optimization problems, which makes it easy to extend to the case of private low-rank factorization and get a tight bound.

Proof. We will use Lemma 7 to prove the theorem. Set \( \Phi = \Phi^T, P = [A]_k^T, Q = A^T \). Then for \( \tilde{X} = \arg\min_{X, r(X) \leq k} \| \Phi^T([A]_k^T X - A^T) \| \), we have with probability \( 1 - \beta \) over \( \Phi^T \sim D_R \),

\[
\| [A]_k^T \tilde{X} - A^T \|_F \leq (1 + \alpha) \min_{X, r(X) \leq k} \| [A]_k^T X - A^T \|_F \\
\leq (1 + \alpha) \| [A]_k^T - A^T \|_F .
\]

Here \( \tilde{X} = (\Phi^T [A]_k^T \Phi)^\dagger (A \Phi)^T \). Since Frobenius norm is preserved under transpose, we have

\[
\| A \Phi ([A]_k \Phi)^\dagger [A]_k - A \|_F \leq (1 + \alpha) \| A - [A]_k \|_F .
\]  

(38)

We now use Lemma 7 on the following regression problem:

\[
\min_{X, r(X) \leq k} \| T X - A \|_F, \quad \text{where} \quad T = A \Phi ([A]_k \Phi)^\dagger .
\]

Let \( \hat{X} = \arg\min_{X, r(X) \leq k} \| \Psi (T X - A) \|_F \). Since \([A]_k\) has rank \( k \), Lemma 7 and equation (38) gives with probability \( 1 - \beta \) over \( \Psi \sim D_R \)

\[
\| T \hat{X} - A \|_F = \| A \Phi ([A]_k \Phi)^\dagger \hat{X} - A \|_F \\
\leq (1 + \alpha) \min_{X, r(X) \leq k} \| (A \Phi) ([A]_k \Phi)^\dagger X - A \|_F \\
\leq (1 + \alpha) \| A \Phi ([A]_k \Phi)^\dagger [A]_k - A \|_F \\
\leq (1 + \alpha)^2 \| A - [A]_k \|_F .
\]

Substituting the value of \( \hat{X} = (\Psi T)^\dagger \Psi A \), with probability \( 1 - 2\beta \) over \( \Phi^T, \Psi \sim D_R \), we have

\[
\| A \Phi ([A]_k \Phi)^\dagger (\Psi A \Phi ([A]_k \Phi)^\dagger)^\dagger \Psi A - A \|_F \leq (1 + \alpha)^2 \| A - [A]_k \|_F .
\]  

(39)

Since \(([A]_k \Phi)^\dagger (\Psi A \Phi ([A]_k \Phi)^\dagger)^\dagger \) has rank at most \( k \), this completes the proof because

\[
\min_{X, r(X) \leq k} \| A \Phi X \Psi A - A \|_F \leq \| A \Phi ([A]_k \Phi)^\dagger (\Psi A \Phi ([A]_k \Phi)^\dagger)^\dagger \Psi A - A \|_F \leq (1 + \alpha)^2 \| A - [A]_k \|_F .
\]

\[\square\]

B Improving the Run-time Efficiency of Factorization

If we want a more efficient algorithm, then we can replace \( S \) by the product of a \( v \times v' \) subsampled randomized Hadamard matrix and a \( v' \times m \) sparse subspace embedding matrix, also known as count-sketch matrix \([13, 50]\) (here, \( v = O(k \alpha^{-3} \log(k/\alpha)) \) and \( v' = O(k^2 \alpha^{-4} \log^6(k/\alpha)) \)). Both of these matrices satisfies Definition \([5, 12, \text{Lemma 46}]\). In that case, we have the following claim, which is analogous to the result presented in Lemma \([49] \). In other words, we can use the following claim everywhere where Lemma \([49] \) is used.

Claim 51. Let \( S \) be the product formed by a \( v \times v' \) subsampled randomized Hadamard matrix and a \( v' \times m \) sparse subspace embedding matrix. Then with high probability, for any \( v' \times n \) matrix \( N \), we have

\[
\| S^T N \|_F = \| N \|_F .
\]

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Proof. First note that $S = (\Pi_{1,v'} WD_1)(HD_2)$, where $D_1$ is a $v \times v$ diagonal matrix with non-zero entries sampled from Rad$(1/2)$, $D_2$ is an $m \times m$ diagonal matrix with non-zero entries sampled from Rad$(1/2)$, $W$ is a normalized Hadamard matrix, and $H$ is a matrix formed using a random hash function $h : [m] \to [v]$ such that entry $H_{ij} = 1$ if and only if $h(j) = i$ and 0, otherwise. Therefore, $S^\dagger = ((\Pi_{1,v'} WD_1)(HD_2))^\dagger$.

We claim that $\sqrt{(v/m)HD_2}$ is a matrix with orthonormal rows. First note that $D_2$ is a diagonal matrix with only $\pm 1$ non-zero entries; therefore, if we prove that $\sqrt{v/mH}$ is a matrix with orthonormal rows, then we are done. Now, consider an entry $(HH^T)_{ij}$. Let $H_i$ denote the $i$-th row of the matrix $H$. Note that $H$ is a full row-rank matrix. Therefore, by the construction of the matrix $H$, we have the following:

$$(HH^T)_{ij} = \begin{cases} 0 & i \neq j \\ \|H_i\|_0 & i = j, \end{cases}$$

where $\|H_i\|_0$ denotes the number of non-zero entries in the row $H_i$. Now, since $h(\cdot)$ is a random function that maps to every entry in $[v]$ uniformly, $\mathbb{E}[\|H_i\|_0] = m/v$. Chernoff bound now gives that, with high probability, $\|H\|_0 = m/v$. Therefore, $v/m(HH^T)_{ij} = 1$ when $i = j$. In other words, with high probability, $(HH^T)^{-1} = (v/m)I$.

Now returning to the proof, conditioned on the event that $\sqrt{v/mHD_2}$ has orthonormal rows, we can write $S^\dagger = (HD_2)^\dagger (\Pi_{1,v'} WD_1)^\dagger$. Let $\tilde{S} = (\Pi_{1,v'} WD_1)$. Then we can invoke the proof of Lemma 49 to say that $SN$ gives a matrix $\tilde{N}$ with entries of $N$ permuted according to the permutation $\Pi_{1,v'}$. Therefore, we have $\|S^\dagger N\|_F = ||(HD_2)^\dagger \tilde{N}||_F$ for some random $v \times n$ Gaussian matrix $\tilde{N}$ whose entries are sampled i.i.d. from $N(0, 1)$.

Since $D_2$ is matrix with orthonormal columns, we have $(HD_2)^\dagger \tilde{N} = D_2^\dagger H^\dagger \tilde{N}$. Therefore, with high probability,

$$\|S^\dagger N\|_F = ||H^\dagger \tilde{N}||_F = ||H^\dagger (HH^T)^{-1} \tilde{N}||_F = \frac{v}{m}||H^\dagger \tilde{N}||_F.$$ 

Now, we can write $\frac{v}{m}||H^\dagger \tilde{N}||^2_F$ as

$$\frac{v}{m}||H^\dagger \tilde{N}||^2_F = \frac{v}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \sum_{k=1}^{v} H_{k,i} \tilde{N}_{k,j} \right)^2 = \frac{v}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{N}^2_{h(i),j} = ||\tilde{N}||^2_F.$$ 

\[\square\]

C Problems Studied in Previous Works

Table 3 gives all the previous results under the assumptions made and the problem studied. Below, we define each of these problems and their difference from the problem studied in this paper.

Hardt and Roth [34] and Upadhyay [63] studied the following problem.

**Problem 2.** (Approximation with respect to the Frobenius norm). Given parameters $\alpha, \beta, \tau$, a private $m \times n$ matrix $A$ (where $m \ll n$) and the target rank $k$, find a rank-$k$ matrix $\tilde{A}_k$ such that

$$\Pr \left[ \|A - \tilde{A}_k\|_F \leq (1 + \alpha)\|A - [A]_k\|_F + \tau \right] \geq 1 - \beta.$$ 

Here, two matrices are neighbouring if they differ by single row of unit norm.
We can also study low-rank approximation when the approximation metric is spectral norm, which was the focus of Kapralov and Talwar [42], Hardt and Roth [34], and Hardt and Price [33].

**Problem 3.** (Approximation with respect to the spectral norm). Given parameters $\alpha, \beta, \tau$, a private $m \times n$ matrix $A$ (where $m \ll n$) and the target rank $k$, find a rank-$k$ matrix $\tilde{A}_k$ such that

$$\Pr \left[ \| A - \tilde{A}_k \|_2 \leq (1 + \alpha)\| A - [A]_k \|_2 + \tau \right] \geq 1 - \beta.$$  

Hardt and Price [33] and Jiang et al. [38] consider two matrices as neighboring if they differ in exactly one entry by at most 1. Kapralov and Talwar [42] considered two matrices as neighboring if the difference of their spectral norm is at most 1.

**Difference from this paper.** We consider low-rank factorization with respect to the Frobenius norm while Problem 2 studied only low-rank approximation with respect to the spectral norm. Moreover, granularity of privacy we consider is more general than theirs.

**Approximating the right singular vectors.** Dwork et al. [28] studied the following problem.

**Problem 4.** Given parameters $\alpha, \beta, \tau$ and an $m \times n$ private matrix $A$ (where $m \gg n$), compute a
rank-$k$ matrix $\widetilde{B}_k$ such that
\[ \Pr \left[ \| A^T A - \widetilde{B}_k \| \leq \min_{\text{rank}(B_k) \leq k} \| A^T A - B_k \| + \tau \right] \geq 1 - \beta, \]
where $\| \cdot \|$ denotes either the spectral or the Frobenius norm.

Dwork et al. [28] consider two matrices neighbouring if they differ by at most one row. They further assume that the rows are normalized; therefore, their definition of neighbouring matrices is the same as Hardt and Roth [34].

**Difference from this paper.** We consider low-rank factorization of both the right and the left singular vectors while Problem 4 studied low-rank “approximation” of the right singular vectors. Moreover, granularity of privacy we consider is more general than theirs.

**Lemma 52.** Let $B$ and $C$ be two conforming matrices. Let $\sigma_{\min}(B)$ be the least non-zero singular value of $B$ and $\sigma_{\max}(B)$ be the largest singular value of $B$. Then
\[ \sigma_{\min}(C) \| B \|_F \leq \| CB \|_F \leq \sigma_{\max}(C) \| B \|_F. \]

**Proof.** Let $B = [B_1, \ldots, B_n]$ be the column partitioning of $B$, then the definition of the Frobenius norm, we have
\[ \| CB \|_F^2 = \sum_{i=1}^{n} \| CB_i \|_2^2. \]

Using the fact that
\[ \sigma_{\min}(C) = \min_x \left\{ x^T C^T C x : \| x \|_2 = 1 \right\} \leq \| CB_i \|_2 / \| B_i \|_i, \]
\[ \sigma_{\max}(C) = \max_x \left\{ x^T C^T C x : \| x \|_2 = 1 \right\} \geq \| CB_i \|_2 / \| B_i \|_i, \]
for all $i$, we have Therefore, we have
\[ \sigma_{\min}(C)^2 \sum_{i=1}^{n} \| B_i \|_i^2 \leq \sum_{i=1}^{n} \| CB_i \|_2^2 \leq \sigma_{\max}(C)^2 \sum_{i=1}^{n} \| B_i \|_i^2 \]

or \[ \sigma_{\min}(C) \| B \|_F \leq \| CB \|_F \leq \sigma_{\max}(C) \| B \|_F. \]