Gibbs’ theorem for open systems with incomplete statistics

G. B. Bağcı

Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey

Gibbs’ theorem, which is originally intended for canonical ensembles with complete statistics has been generalized to open systems with incomplete statistics. As a result of this generalization, it is shown that the stationary equilibrium distribution of inverse power law form associated with the incomplete statistics has maximum entropy even for open systems with energy or matter influx. The renormalized entropy definition given in this paper can also serve as a measure of self-organization in open systems described by incomplete statistics.

Keywords: incomplete statistics, Gibbs’ theorem, S-theorem, renormalized entropy, self-organization

I. INTRODUCTION

Many different entropy measures such as Rényi [1], Sharma-Mittal [2] and Tsallis measures [3] have been proposed in order to generalize Boltzmann-Gibbs (BG) entropy. Despite their differences though, all these entropy measures share a common feature in that they are all based on the assumption of complete statistics, which is even shared by the entropy measure they claim to generalize i.e., BG entropy. The assumption of complete statistics implies that all states regarding the system is countable and known completely by us so that we have full knowledge of the interactions taking place in the system of interest, thereby implying the ordinary normalization condition \( \sum p_i = 1 \). However, this scenario is challenged in some cases of interest since one cannot obtain all the information regarding the system under investigation in these particular situations. One such possibility is fractal phase spaces [4, 5], since one can then have singularities and inaccessible points, which invalidates the assumption of complete statistics.

Recently, Wang [6] proposed a new nonadditive entropy based on Tsallis entropy by replacing the complete normalization condition with an incomplete one depending on a free positive parameter \( q \). This new framework is called the formalism of incomplete statistics (IS). The parameter \( q \) plays the role of Hausdorff dimension divided by the topological dimension of the phase space when IS formalism is applied to nonequilibrium systems evolving in hierarchically heterogeneous phase space, connecting the concept of information and the topological dimension of some fractal sets as pointed out earlier by El Naschie [7]. However, the general physical interpretation of this parameter is related to the neglected interaction. When no interaction is neglected, the parameter \( q \) becomes equal to unity, thereby reducing the results of IS formalism to those of complete statistics described by BG entropy.

Despite the simplicity of the proposal by Wang and the treatments of some important issues in this framework such as zeroth law of thermodynamics [8] and the IS formalism with different \( q \) indices [9], there are still open problems in IS formalism like the definition of physical temperature [10, 11]. The aim of this paper is to clarify another such issue concerning IS formalism, namely, its applicability to open systems and the ordering of the entropies.

The Gibbs’ theorem states that the canonical equilibrium distribution, of all the normalized distributions having the same mean energy, is the one with maximum entropy. However, the Gibbs’ theorem rests on two assumptions. Firstly, the stationary equilibrium distribution, being canonical, is of exponential form. Secondly, Gibbs assumed that all the compared distributions have the same mean energy values. The former assumption has recently been challenged by Abe and Rajagopal [12]. Although they originally aimed their generalization for Tsallis entropy, their treatment can be considered valid also for IS formalism, since Abe and Rajagopal relied on the use of stationary equilibrium distributions of the q-exponential form, which are common for both Tsallis entropy and IS formalism. However, these authors only considered the generalization of the Gibbs’ theorem, assuming the equality of the mean energies of the distinct distributions.

The generalization of Gibbs’ theorem to open systems was first made by Klimontovich and is called the S-theorem [13-17]. Open systems are those in which the average internal energy is not constant because of the interactions with the environment. It is in this sense that the Gibbs’ theorem needs to be generalized, since Gibbs assumed the equality of the mean energies of the distributions. Unfortunately, Klimontovich’s treatment is based on the use of BG entropy, which in turn is based on the assumption of complete statistics and yields to stationary equilibrium distributions of exponential form.

*Electronic address: baris.bagci@ege.edu.tr
In this paper, we generalize Gibbs’ theorem for open systems described by incomplete formalism. We show that the stationary equilibrium distribution of inverse power law form associated with IS formalism has maximum entropy even in open systems with energy or matter influx. The general derivation here can be used as a criterion for self-organization in open systems with incomplete statistics.

The paper is organized as follows: In Section II, we give a heuristic review of IS formalism. The generalization of Gibbs’ theorem for open systems with incomplete statistics is given in Section III. The conclusions is given in Section IV.

II. INCOMPLETE STATISTICS

The point of departure of IS formalism is the nonadditive Tsallis entropy [3]

\[ S_{Tsallis}^{q}(p) = \frac{\sum_{i}^{W} p_{i}^{q} - \sum_{i}^{W} p_{i}}{1 - q}, \]  

(1)

where \( p_{i} \) is the probability of the system in the \( i \)th microstate and \( W \) is the total number of the configurations of the system. The Boltzmann constant \( k \) is set to unity throughout the paper. The entropic index \( q \) is a real number, which characterizes the degree of nonadditivity since Tsallis entropy obeys the following pseudo-additivity rule:

\[ S_{q}(A + B)/k = [S_{q}(A)] + [S_{q}(B)] + (1 - q)[S_{q}(A)][S_{q}(B)], \]  

(2)

where \( A \) and \( B \) are two independent systems with \( p_{ij}(A+B) = p_{i}(A)p_{j}(B) \). Obviously, this formalism assumes that we have complete access to all random variables since the summation in Eq. (1) is over the total number of configurations \( W \). This assumption also lies at the center of the normalization of probability distribution, since it considers that all the information relevant to the physical system under consideration is accessible. On the other hand, IS formalism challenges this issue since a complete description of a physical system requires the knowledge of the exact Hamiltonian and the exact solution of its corresponding equations of motion. However, it is possible in practice that we may not know analytically all the interactions, which must appear in the relevant Hamiltonian. This failure in our knowledge of the exact Hamiltonian results in the incompleteness of the countable states. If this is the case, the ordinary normalization of the probability distribution will not hold and must be avoided. This observation lies at the heart of IS formalism, since Wang proposes to replace the usual i.e., complete normalization \( \sum_{i}^{W} p_{i} = 1 \) with the incomplete normalization given by

\[ \sum_{i}^{w} p_{i}^{q} = 1, \]  

(3)

where \( w \) denotes the states accessible to us. Since these states do not now form a complete set, \( w \) can be greater or smaller than the real number of all possible states. The incompleteness parameter \( q \) is positive and can be considered as a measure of neglected interactions. When all the interactions are taken into account, it is equal to 1, recovering the ordinary normalization. As a result of incomplete normalization, the expectation value of an observable \( \hat{O} \) is consistently given by

\[ \langle \hat{O} \rangle = \sum_{i=1}^{w} p_{i}^{q} O_{i}. \]  

(4)

The incomplete normalization given by Eq. (3) enables one to write a new entropy based on incomplete statistics whose point of departure is Tsallis entropy in Eq. (1). This new entropy expression reads

\[ S_{q}(p) = \frac{1 - \sum_{i}^{w} p_{i}}{1 - q}. \]  

(5)

The stationary equilibrium distribution for IS entropy in the canonical case can be found by applying the Lagrange method.
\[ \delta(S_q + \frac{\alpha}{1-q} \sum_{i=1}^{w} p_i^q - \alpha \beta \sum_{i=1}^{w} p_i^q \varepsilon_i) = 0. \] (6)

The resulting canonical equilibrium distribution [6] in IS formalism is then given by

\[ p_{eq}^i = \left[ 1 - (1-q)\beta \varepsilon_i \right]^{1/(1-q)}, \] (7)

apart from normalization. The energy of the \( i \)th microstate is denoted by \( \varepsilon_i \), where the Lagrange multipliers are denoted by \( \alpha \) and \( \beta \). However, it is easy to verify that the canonical equilibrium distribution given by Eq. (7) is not invariant under the uniform translation of the energy parameter \( \varepsilon_i \). Moreover, the Lagrange multiplier \( \beta \) is not identical with the inverse temperature \([10, 11]\).

These difficulties are easily overcome by the maximization of the IS entropy in Eq. (5) as follows

\[ \delta(S_q - \alpha \sum_{i=1}^{w} p_i^q - \beta \sum_{i=1}^{w} p_i^q \varepsilon_i) = 0, \] (8)

which results

\[ p_{eq}^i = \left[ 1 - (1-q)\beta (\varepsilon_i - U_q) / \sum_{j=1}^{w} p_j \right]^{1/(1-q)}, \] (9)

where \( U_q = \sum_{i=1}^{w} p_i^q \varepsilon_i \) is the internal energy and the normalization constant is not explicitly written. This canonical equilibrium distribution is invariant under uniform translation of energy parameter and the Lagrange multiplier \( \beta \) is identical with inverse temperature \([11]\).

### III. OPEN SYSTEMS WITH INCOMPLETE STATISTICS

The Gibbs’ theorem states that the canonical equilibrium distribution has the maximum entropy compared to any other distribution with the same mean energy. Therefore, it can be considered as an indication of the importance of the canonical ensemble since it is the state with maximum entropy. However, Gibbs’ theorem relies on two major assumptions. Firstly, the system under consideration is governed by complete statistics, which is described by BG measure. Due to this assumption, Gibbs’ theorem is limited to the cases where the stationary equilibrium distribution is exponential. Secondly, Gibbs’ theorem is limited to the cases where the average internal energy is kept constant. It is our aim in this paper to generalize Gibbs’ theorem for open systems with incomplete statistics. It is highly probable indeed that one cannot write all the interactions governing such a system, since open systems are subject to many interactions. Therefore, their equations of motion are not fully solvable.

This in turn will result in not all states being accessible to us, justifying the use of incomplete statistics. In addition, an open system might have a metastable stationary state described by an inverse power law as IS suggests instead of an exponential as Klimontovich assumed \([13-15]\). These considerations force us to generalize Gibbs’ theorem for open systems with incomplete statistics. In order to do this, we first define a new quantity named renormalized entropy \( R_{IS}^q \) as

\[ R_{IS}^q \equiv S_{neq}^q(r) - \tilde{S}_{eq}^q(\tilde{p}_{eq}). \] (10)

The generalization of Gibbs’ theorem is now equivalent to showing that renormalized entropy expression in Eq. (10) is negative i.e., \( R_{IS}^q < 0 \), since this implies that \( S_{eq}^q > S_{neq}^q \). However, we know that this cannot be proved on the basis of ordinary Gibbs’ theorem, since it assumes that the internal energy is kept constant, which is not the case for open systems. Therefore, Klimontovich generalized Gibbs’ theorem for open systems with complete statistics by equating the mean energies of the equilibrium and nonequilibrium states \([13-15]\). Due to this equalization of the effective mean energy, we denote the equilibrium entropy by a tilde since this is not the original equilibrium entropy but the one obtained after the effective mean energy equalization. Two distinct incomplete probability distributions i.e. \( \tilde{p}_{eq} \) and \( r \) in Eq. (10) denotes the renormalized equilibrium and nonequilibrium probability distributions respectively. The
corresponding IS entropy expressions are denoted by \( \tilde{S}^{eq}(\tilde{p}_{eq}) \) and \( S^{neq}(r) \). From now on, we will drop the subscript from the equilibrium probability distribution so that it should be understood that the probability distributions \( p \) and \( \tilde{p} \) denote the ordinary and renormalized equilibrium distributions, respectively. The renormalized equilibrium probability distribution \( \tilde{p} \) and nonequilibrium probability distribution \( r \) obey the incomplete normalization summarized by Eq. (3) i.e., \( \sum_i^w \tilde{p}^q_i = \sum_i^w r^q_i = 1 \), which is implicitly taken into account in the definition of IS entropy given by Eq. (5).

In order to proceed, we need to define effective mean energy in terms of the equilibrium state associated with the incomplete statistics. This can be achieved by defining

\[
U_{eff} \equiv - \ln_q p_i, \tag{11}
\]

where the \( q \)-logarithm is simply defined as

\[
\ln_q(x) = \frac{x^{1-q} - 1}{1-q}. \tag{12}
\]

This definition of effective mean energy is central to our generalization and therefore requires some explanation. The effective mean energy is defined in terms of the unnormalized equilibrium distribution. In this sense, if we apply this definition to the unnormalized equilibrium distribution given by Eq. (7), we see that \( U_{eff} = \beta \varepsilon_i \). The application of the effective mean energy to the canonical equilibrium distribution in Eq. (9) however results in \( U_{eff} = q \beta (\varepsilon_i - U_q) \). This observation explains why it is called effective mean energy since it is always proportional to the multiplication of the Lagrange multiplier \( \beta \) associated with the internal energy constraint and the energy of the \( i \)th microstate.

The calculations in this paper are general in the sense that both equilibrium distributions can be used although a consistent treatment would be through the adoption of the canonical equilibrium distribution in Eq. (9) due to its explicit dependence on temperature through \( \beta \). The open systems are usually treated by using a control parameter, which controls the matter or energy influx into the system due to its interaction with the environment. In this sense, the state with the zero value of the control parameter is the equilibrium distribution and all the other stationary states with control parameter values different than zero correspond to nonequilibrium distributions.

Having clarified this important issue, we can rewrite the equalization of effective mean energies of the two states as

\[
\langle U_{eff} \rangle^{(req)} = \langle U_{eff} \rangle^{(neq)}, \tag{13}
\]

where superscripts \( (req) \) and \( (neq) \) denote the renormalized equilibrium and ordinary nonequilibrium states, respectively. The corresponding averages must be taken in terms of \( \tilde{p}^q_i \) and \( r^q_i \). The Eq. (13) can be explicitly written as

\[
\sum_{i=1}^w \tilde{p}^q_i U_{eff} = \sum_{i=1}^w r^q_i U_{eff}. \tag{14}
\]

The substitution of the effective mean energy defined in Eq. (11) into Eq. (14) yields

\[
\sum_{i=1}^w \tilde{p}^q_i (p^{1-q}_i - \frac{1}{q-1}) = \sum_{i=1}^w r^q_i (p^{1-q}_i - \frac{1}{q-1}). \tag{15}
\]

Due to the normalization i.e., \( \sum_i^w \tilde{p}^q_i = \sum_i^w r^q_i = 1 \), the above equation can be rewritten as

\[
\sum_{i=1}^w \tilde{p}^q_i p^{1-q}_i = \sum_{i=1}^w r^q_i p^{1-q}_i. \tag{16}
\]

The probability distribution \( \tilde{p} \) can be considered as the normalized and renormalized (i.e., effective mean energy equalization) counterpart of the ordinary equilibrium distribution \( p \). Therefore, we can substitute \( \tilde{p}^q_i = \frac{p^q_i}{\sum_i p^q_i} \) into Eq. (16) to obtain
\[
\sum_{i=1}^{w} \tilde{p}_i = \sum_{i=1}^{w} r_i^{q-1} p_i^{1-q}.
\] (17)

On the other hand, we can also obtain an explicit form of renormalized entropy defined by Eq. (10) by substituting the IS entropy in Eq. (5) explicitly, which gives

\[
R_{IS}^q = \frac{1}{(q-1)} \left( \sum_{i=1}^{w} r_i - \sum_{i=1}^{w} \tilde{p}_i \right).
\] (18)

Making use of the equalization of the effective mean energies of equilibrium and nonequilibrium states given by Eq. (17), we obtain

\[
R_{IS}^q = \frac{1}{(q-1)} \left( \sum_{i=1}^{w} r_i - \sum_{i=1}^{w} r_i^{q-1} \tilde{p}_i^{1-q} \right).
\] (19)

The final step in our treatment is to show that the renormalized entropy \( R_{IS}^q \) is negative for all positive values of the incompleteness parameter \( q \). This can be achieved by rewriting the above renormalized entropy expression as

\[
R_{IS}^q = - \sum_{i=1}^{w} r_i \left( \frac{(r_i/\tilde{p}_i)^{q-1} - 1}{q - 1} \right).
\] (20)

Since \( r_i/p_i \geq 0 \), the following mathematical inequality can be used [18]

\[
\frac{(r_i/\tilde{p}_i)^{q-1} - 1}{q - 1} \geq 1 - \tilde{p}_i/r_i, \quad q > 0.
\] (21)

Multiplying both sides of the above inequality with \( r_i \) and summing over \( i \), we obtain

\[
\sum_{i=1}^{w} r_i \left( \frac{(r_i/\tilde{p}_i)^{q-1} - 1}{q - 1} \right) \geq \sum_{i=1}^{w} (r_i - \tilde{p}_i), \quad q > 0.
\] (22)

Comparing the inequality above with the expressions given by Eqs. (18) and (20), we see that the above inequality takes the form

\[
-R_{IS}^q \geq (q-1)R_{IS}^q, \quad q > 0.
\] (23)

Since the above inequality is valid only for \( q \) values greater than zero, it implies

\[
R_{IS}^q \leq 0.
\] (24)

The equality holds only if the two distributions are the same. Since we assume that the states under question are two different states, one being renormalized equilibrium state and the other being nonequilibrium state, we can drop the equality sign above. Moreover, remembering the original definition of renormalized entropy in Eq. (10), we see that

\[
R_{IS}^q = S_{eq}^{neq} - \tilde{S}_q^{eq} < 0 \Rightarrow \tilde{S}_q^{eq} > S_q^{neq},
\] (25)

thus the (renormalized) equilibrium entropy is greater than the nonequilibrium entropy for open systems with incomplete statistics. Naturally, we recover the result based on complete statistics of BG entropy i.e., \( S^{eq} > S^{neq} \) by taking the \( q \to 1 \) limit in Eq. (25).

In summary, we have a more ordered state as the control parameter increases causing the system to recede away from equilibrium. This decrease of entropy on ordering is called self-organization by Haken [19]. Therefore, the renormalized entropy \( R_{IS}^q \) can also be taken as a measure of self-organization for open systems described by IS.
IV. CONCLUSIONS

By generalizing Gibbs’ theorem for open systems described by IS formalism, we have shown that the stationary equilibrium state obtained from IS is the state of maximum entropy even in the presence of energy or matter influx. The treatment here can be considered as a generalization of ordinary S-theorem (and also of Gibbs’ theorem) developed by Klimontovich, since the latter is a particular case of the former in the $q \to 1$ limit. In this sense, the incompleteness seems a more general framework since the results assuming complete statistics can be obtained from IS even in the case of open systems. This generalization of Gibbs’ theorem to IS is necessary, since it is less likely that we will have complete knowledge of the system as the interaction terms governing the system increase as it would generally be the case with open systems. Finally, we have shown that one obtains a more ordered state i.e., a state of lesser entropy as the control parameter increases. In this sense, the renormalized entropy expression obtained in this paper can serve as a criterion of self-organization [19] in open systems described by IS.

V. ACKNOWLEDGEMENTS

I thank Professors Q. A. Wang and Donald H. Kobe for valuable suggestions. I am also grateful to University of North Texas in Denton/Texas for their hospitality during my stay.

[1] Rényi A. On measures of entropy and information. In: Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, vol. 1. University California Press, Berkeley; 1961.
[2] Sharma BD, Mittal DP. New non-additive measures of relative information. J. Math. Sci. 1975;10:28.
[3] Tsallis C. Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys 1988;52:479.
[4] El Naschie MS. On the Electroweak Missing Angle in $E^{(\infty)}$. Chaos, Solitons & Fractals 2000;11(11):1803.
[5] Wang QA. Incomplete information and fractal phase space. Chaos, Solitons & Fractals 2004;19:639.
[6] Wang QA. Incomplete statistics: nonextensive generalizations of statistical mechanics. Chaos, Solitons & Fractals 2001;12:1431.
[7] El Naschie MS. Cantorian distance, statistical mechanics and universal behaviour of multi-dimensional triadic sets. Math Comput Model 1993;17:47.
[8] Wang QA, Le Méhauté A. Unnormalized nonextensive expectation value and zeroth law of thermodynamics. Chaos, Solitons & Fractals 2003;15:537.
[9] Nivanen L, Pezeril M, Wang QA, Le Méhauté A. Applying incomplete statistics to nonextensive systems with different $q$ indices. Chaos, Solitons & Fractals 2005;24:1337.
[10] Ou C, Chen J, Wang QA. Temperature definition and fundamental thermodynamic relations in incomplete statistics. Chaos, Solitons & Fractals 2006;28:518; Ou C, Chen J, Wang QA. Erratum to "Temperature definition and fundamental thermodynamic relations in incomplete statistics". Chaos, Solitons & Fractals 2008;35:1016.
[11] Huang Z, Lin B, Chen J. A new expression of the probability distribution in Incomplete Statistics and fundamental thermodynamic relations. Chaos, Solitons & Fractals (2007). doi:10.1016/j.chaos.2007.09.002.
[12] Abe S, Rajagopal AK. Reexamination of Gibbs’ theorem and nonuniqueness of canonical ensemble theory. Physica A 2001;295:172.
[13] Klimontovich YL. S-theorem. Z Phys B 1987;66:125.
[14] Klimontovich YL. Entropy evolution in self-organization process H-theorem and S-theorem. Physica A 1987;142:390.
[15] Klimontovich YL. Criteria of self-organization. Chaos, Solitons & Fractals 1995;5:1985.
[16] Engel-Herbert H, Ebeling W. The behaviour of the entropy during transitions far from thermodynamic equilibrium : I. Sustained oscillations. Physica A 1988;149:182.
[17] Engel-Herbert H, Ebeling W. The behaviour of the entropy during transitions far from thermodynamic equilibrium : II. Hydrodynamic flows. Physica A 1988;149:195.
[18] Tsallis C. Generalized entropy-based criterion for consistent testing. Phys Rev E 1998;58:1442.
[19] Haken H. Information and Self-organization: A Macroscopic Approach to Complex Systems. Berlin: Springer Verlag; 2000.