Lottery Tickets in Linear Models: An Analysis of Iterative Magnitude Pruning

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Abstract

We analyse the pruning procedure behind the lottery ticket hypothesis Frankle and Carbin [4], iterative magnitude pruning (IMP), when applied to linear models trained by gradient flow. We begin by presenting sufficient conditions on the statistical structure of the features under which IMP prunes those features that have smallest projection onto the data. Following this, we explore IMP as a method for sparse estimation.

1 Introduction

The lottery ticket hypothesis [4] asserts that a randomly initialised, densely connected feed-forward neural network contains a sparse sub-network that, when trained in isolation, attains equal or higher accuracy than the full network. These sub-networks are called lottery tickets and the method used to find them is iterative magnitude pruning (IMP). A network is given a random initialisation, trained by some form of gradient descent for a specified number of iterations and a proportion of its smallest weights (by absolute magnitude) are deleted. The remaining weights are then reset to their initialised values and the network is retrained. This procedure can be performed multiple times, resulting in a sequence of sparse yet trainable sub-networks.2

This simple procedure gives quite surprising results. The sub-networks uncovered by IMP are trainable from their original initialisation and achieve accuracies comparable with, and often better than, the full network. The same sub-networks perform poorly when reinitialised [4]. Moreover, as observed in [18], many of the sub-networks found by IMP have better than random test loss at initialisation, suggesting that IMP has some ability to select good inductive biases for a problem. These results point to avenues for better computational and memory efficiency in neural networks, as well as to properties of deep networks and their training dynamics that we do not fully understand. In particular, it is not yet known why weight magnitude provides a good signal on which to base a pruning heuristic in neural networks. In other words, why is IMP effective in neural networks?

Our paper aims to address this question for linear models, with the hope that this will lay the groundwork for similar study in neural networks. We make a minor adaptation to the version of IMP used in [4], shown in Algorithm 1, the difference being that in Algorithm 1 we consider pruning one weight per iteration, while in [4] a proportion of the weights are pruned. It is straightforward to

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These sub-networks are weight sub-networks, formed by setting to 0 entries of the weight matrices (edges in the graph). This is in contrast to the neuron pruning, which removes entire neurons (nodes in the graph). All mentions of sub-network in this paper refer to weight sub-networks.

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extend our results to the latter case. It is similarly easy to modify them to discrete time gradient descent with appropriate step sizes.

Algorithm 1: Iterative Magnitude Pruning

\begin{algorithm}
\textbf{Input:} Loss function $L: \mathbb{R}^p \rightarrow \mathbb{R}$, training time $T \in \mathbb{R}_+$, initialisation $\mathbf{w}^{\text{init}} \in \mathbb{R}^p$, iterations of pruning $q < p$.
\textbf{Output:} $\mathbf{w}^{(q)}(T)$
\begin{algorithmic}[1]
\State Set $M = \mathbb{I}_p$
\For{$k = 0$ to $q$}
\State Initialise $\mathbf{w}^{(k)}(0) = M \mathbf{w}^{\text{init}}$
\State Train $\mathbf{w}^{(k)}(t) = -M \nabla L(\mathbf{w}^{(k)}(t))$ for $t \in [0, T]$
\State Set $i = \arg \min_{j \in [p]} \left\{ \| \mathbf{w}^{(k)}(t) \| : M_{ij} = 1 \right\}$
\State Set $M_{ii} = 0$
\EndFor
\Return $\mathbf{w}^{(q)}(T)$
\end{algorithmic}
\end{algorithm}

1.1 Related Work

The lottery ticket hypothesis [4] has sparked a lot of interest. Already, there are more follow up works than it is possible to cover in this article. To name a few: more reliable discovery of lottery tickets in deep networks with rewinding [5], generalising lottery tickets across tasks [12], constructing networks that perform well with random weights [18, 6], pruning at initialisation [7] and finding lottery tickets in randomly weighted networks [15]. For an empirical comparison of various pruning methods see [1].

Theoretical work has mainly focussed on lottery tickets at initialisation. Malach et al. [10] showed that any continuous function can be approximated by a sub-network of a sufficiently large, randomly weighted neural network. Following this, there is a body of work on the approximation properties of lottery tickets at initialisation [13, 14]. The theory relating to IMP is relatively undeveloped, which is a key motivation for this paper.

We note some similarities between IMP and methods in compressive sensing, which is a subfield of signal processing that attempts to find sparse reconstructions of noisy signals. For a systematic review see [16] and for a comprehensive discussion of thresholding and similar denoising techniques see [11, Chapter 11]. Most similar to IMP are two well known thresholding methods that we discuss below, each of which attempts to recover a sparse signal $s \in \mathbb{R}^p$ from noise corrupted measurements $y \in \mathbb{R}^n$.

Hard thresholding, studied in the wavelet basis by Donoho and Johnstone [3], performs least-squares estimation $\hat{s}$ of $s$ and then applies elementwise to $\hat{s}$ the thresholding operator $H_\tau(z) = 1 \{ |z| > \tau \} z$. Iterative hard thresholding [2] consists of linear projection of $s$ onto a feature matrix $\Psi \in \mathbb{R}^{n \times p}$ and then iteratively solving the linear system $y = \Psi s$, thresholding the solution at each stage. Specifically, iterative hard thresholding estimates $s$ by $\hat{s}$ using the update rule $\hat{s}^n_i = H_\tau(\hat{s}^n_i + \eta \Psi^\top (y - \Psi \hat{s}^n_i))$ with initialisation chosen by the user and where $\eta$ is a step size. The argument of $H_\tau$ is exactly the update from gradient flow on the squared error loss $\frac{1}{2} \| y - \Psi \hat{s} \|^2_2$. IMP can therefore be viewed as a variation of iterative hard thresholding, where the threshold operator is replaced by the restriction to a subset of indices that is chosen at time $T$ of each training run.

1.2 Setup and Notation

We write $[n]$ for the set $\{1, \ldots, n\}$. Let the training inputs be $x_1, \ldots, x_n$ and targets be $y_1, \ldots, y_n \in \mathbb{R}$. For the parameters we write $\mathbf{w} \in \mathbb{R}^p$ and the features are written $\phi_i(x)$ for $i \in [p]$. In this work we consider the training data and features to be deterministic, but it is possible to extend our main result to a random design setting. Let $X \in \mathbb{R}^{n \times d}$ and $y \in \mathbb{R}^p$ be the usual row stacking of the training examples and write $\Phi \in \mathbb{R}^{n \times p}$ for the row stacking of the features. For each $i \in [p]$, write $\phi_i(X) \in \mathbb{R}^n$ for the vector with components $(\phi_i(x_1), \ldots, \phi_i(x_n))^\top$. A linear model is any predictor $f_\Theta(x) = \Theta^\top \phi(x)$ that is linear in the features, where $\Theta \in \mathbb{R}^p$ are the learned parameters. The features can be arbitrary non-linear functions, for instance, the outputs of
For any matrix the diagonal
the weight then pruned is
will prune according to the alignment
3
We conclude that, if Algorithm 1 the diagonal matrix \( M \in \mathbb{R}^{p \times p} \) records the weights pruned so far, that is \( M_{ij} = 1 \) \( \{ i = j \land w_{ij} \text{ not yet pruned} \} \).
Pruning can be seen as either removing features entirely (so reducing the dimension of \( \Phi \)) or setting to 0 the corresponding weights or column of \( \Phi \) (but preserving all the dimensions). For the work in this paper these perspectives are equivalent and we will use each description interchangeably.

2 Warm Up: Pruning Heuristic of IMP

In this section we give an analysis to demonstrate that, given some statistical assumptions on the features, IMP preferentially prunes the features that explain the data the least in terms of linear projections. We call this pruning heuristic the alignment heuristic. Specifically, the alignment heuristic prunes \( w_{ij} \) where \( i = \arg\min_j \{ |\phi_j(X)^\top y| : M_{ij} = 1 \} \). We consider examples of various \( \Sigma \) and examine the pruning heuristic that arises. For now we assume that \( \Sigma \) is full rank, so training to convergence with \( L(w) = \frac{1}{2m} \| \Phi w - y \|_2^2 \) gives \( w^{(i)}(\infty) = \frac{1}{2} \Upsilon_i^{-1} P_i^\top y \) where \( \Upsilon_i \) and \( P_i \) are, respectively, the restrictions of \( \Sigma \) and \( \Phi \) to the parameters not yet pruned at iteration \( i \) of Algorithm 1. \( \Upsilon_i \) will always be invertible as long as \( \Sigma \) is invertible, which follows from the Cauchy interlace theorem and \( \Sigma \) being positive definite. We will relax our notation back to \( \Omega \) and \( \Phi \) for the following examples, with the understanding that each of these results apply for each iteration of Algorithm 1, restricting the matrices accordingly. In particular, it should be noted that in the following examples the conditions on \( \Sigma \) transfer to conditions on \( \Upsilon_i \) and \( P_i \). Finally, for the rest of this section we normalise the features so that \( \Sigma_{ii} = 1 \) \( \forall i \in [p] \).

**Example 1** (\( \Sigma = I \)). Gradient flow converges to \( w_i(\infty) = \Sigma^{-1} \Phi^\top y = \phi_i(X)^\top y \). According to Algorithm 1 the weight then pruned is \( w_i \) where \( i = \arg\min_j \{ |w_j^{(i)}(\infty)| : M_{ij} = 1 \} \). We see immediately that this is equivalent to the alignment heuristic.

**Example 2** (Uniform correlations). We express
\[
\Sigma = I + \alpha(11^\top - I)
\]
for \( \alpha \in (0, 1) \), where \( 1 = (1, 1, \ldots, 1)^\top \in \mathbb{R}^p \). We can use the Sherman-Morrison formula to calculate
\[
\Sigma^{-1} = \frac{1}{1-\alpha} I - \frac{\alpha(1-\alpha)}{1+\alpha(p-1)} 11^\top
\]
and we see that
\[
n w_i(\infty) = \frac{1}{1-\alpha} \phi_i(X)^\top y - \frac{\alpha(1-\alpha)}{1+\alpha(p-1)} \sum_j \phi_j(X)^\top y.
\]
We conclude that, if \( \alpha \ll 1 \) is small enough, then Algorithm 1 will prune according to the alignment heuristic. The case \( \alpha \approx 1 \) gives this outcome too, but in this case \( \Sigma \) is barely invertible. Intuitively, in the case \( \alpha \approx 1 \) the features are similar and pruning one is as good as pruning another.

**Example 3** (Pairwise incoherence). We say that \( \Phi \) satisfies the pairwise incoherence assumption with parameter \( \delta_{pw} \) if
\[
\delta_{pw} = \max_{i,j} \left| \frac{1}{n} (\Phi^\top \Phi)_{ij} - I \{ i = j \} \right|.
\]
Assume that this is the case. Write \( \Sigma = I - A \), then formally we have
\[
\Sigma^{-1} = (I - A)^{-1} = \sum_{j=0}^{\infty} A^j.
\]
The alignment heuristic will be followed if the magnitude of the first term dominates that of the second, which is $O(\delta_{pw})$. Therefore, given a suitable pairwise incoherence assumption, Algorithm 1 prunes according to the alignment heuristic.

3 Support Recovery with IMP

Empirically, IMP has been found to result in sparse yet performant sub-networks. In this section we explore this phenomenon analytically in the context of linear models, where the natural application is to sparse estimation. In our analysis we will make use of a strengthened form of the restricted nullspace property [17].

Definition 4 (Orthogonal nullspace property). Let $S \subset [p]$ be a subset of indices. Define the cone

$$C(S) = \{ x \in \mathbb{R}^p : \| x_S \|_1 \leq \| x \|_1 \}$$

where $x_Q$ are the components of $x$ with indices in $Q$. A matrix $A$ acting on $\mathbb{R}^p$ satisfies the orthogonal nullspace property with respect to $S$ if all elements of the nullspace of $A$ are orthogonal to all elements of $C(S)$

$$\text{null}(A) \perp C(S).$$

Notice that the cone $C(S)$ is the set of vectors whose 1-norm on the index set $S$ dominates that on the other indices $S^c$. In particular, a sparse signal supported on $S$ will belong to $C(S)$. In the context of Theorem 5, the above condition ensures that the signal $s$ is recoverable. We now present our sparse estimation result for Algorithm 1.

Theorem 5 (Sparse Estimation with IMP). Assume that $y = \Phi s + \xi$ for $s \in \mathbb{R}^p$ and that $\{\xi_i : i = 1, \ldots, p\}$ are independent, zero mean sub-Gaussian random variables with variance proxy $\sigma^2$. Let $\Sigma := \frac{1}{n} \Phi^\top \Phi$ and let $\lambda_{\min} > 0$ be the smallest non-zero eigenvalue of $\Sigma$, which we assume to exist. Suppose that $s$ is $k$-sparse and supported on a set $S \subset [p]$ with $|S| = k$, and that $\Sigma$ satisfies the orthogonal nullspace property with respect to $S$. Let $L(w) = \frac{1}{n} \| \Phi w - y \|_2^2$ be the mean squared error loss. Consider running Algorithm 1 with $L$, $w^{\text{init}} = 0$, $T = \infty$ and $q \leq p - k$, and denote the output by $v$. Let $\gamma > 0$, if

$$n \geq \frac{8 \sigma^2}{\gamma^2 \lambda_{\min}} \log(2p/\delta)$$

then, with probability at least $1 - \delta$ we have

(i) $v$ is at least $(p - q)$-sparse.

(ii) No false exclusion above $\gamma$: $v_i \neq 0$ for any $i$ with $|s_i| \geq \gamma$.

Proof. Point (i) is obvious. We establish point (ii). Consider the first iteration of Algorithm 1. Let $V$ be the column space of $\Sigma$. Since $\Sigma$ satisfies the orthogonal nullspace condition with respect to the support of $s$, we know that the projection of $s$ on to $V$ is $s$. By diagonalising, we can see that $\Sigma^+ \Sigma$ is exactly this projection. Training until convergence therefore gives

$$w_i(\infty) = s_i + \frac{1}{n} (\Sigma^+ \Phi^\top \xi)_i.$$

Then on this iteration we can be sure not to prune any $i$ with $|s_i| \geq \gamma$ if

$$\frac{1}{n} \| (\Sigma^+ \Phi^\top \xi)_i \| < \gamma / 2 \quad \forall i.$$
By Lemma 6, this happens with probability at least $1 - \delta$ if $n \geq \frac{8\sigma^2}{\gamma^2} \log(2p/\delta)$. To demonstrate (ii), we show that this is sufficient to guarantee (with high probability) that no $i$ with $|s_i| \geq \gamma$ is pruned on any iteration of Algorithm 1. We conclude the proof with this argument.

Let $\Upsilon$ be the sub-matrix of $\Sigma$ formed by removing the rows and columns with indices $i_1, \ldots, i_k$ $k < p$, let $P$ be the sub-matrix of $\Phi$ by removing the same columns. The orthogonal nullspace condition on $\Sigma$ means that the restriction of $s$ has no component in the null of $\Upsilon$ at any iteration. Hence, we need only show that

$$\frac{1}{n} |(\Upsilon + P^\top)\xi_i| < \gamma/2.$$  

Again by Lemma 6, this happens with probability at least $1 - \delta$ if $n \geq \frac{8\sigma^2}{\gamma^2\lambda_{\min}(\Upsilon)} \log(2p/\delta)$. $\Sigma$ is symmetric, so we may apply Cauchy’s interlace theorem (see [8], Theorem 4.3.17) to obtain $\lambda_{\min}(\Upsilon) \geq \lambda_{\min}(\Sigma)$. The proof is complete.

This result tells us that if we have $n = O(\gamma^{-2} \log(2p/\delta))$ samples then, with high probability, IMP can recover the support of a sparse signal wherever it has magnitude at least twice the noise level $\gamma/2$. It is not possible to place guarantees on recovery of components with magnitude smaller than $\gamma$ without assumptions on the relative sizes of the components of $s$. It should be straightforward to modify Theorem 5 to account for random features by a bound on $\lambda_{\min}^{0,0}$.

4 Discussion

We have shown that IMP can recover the support of a sparse signal under mild assumptions on the design matrix and we gave bounds for the estimation error in this setting. Further work may seek to extend the results of this paper to neural networks, a tractable route for which might be found using linearised networks [9]. Alternatively, it may be interesting to consider random features.

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A Sub-Gaussian Concentration

Lemma 6. Define $\Phi \in \mathbb{R}^{n \times p}$ and $\Sigma \in \mathbb{R}^{p \times p}$ with $\Sigma^+$ the Moore-Penrose pseudo-inverse of $\Sigma$ and $\lambda_{\min}^{0,0}$ the non-zero eigenvalue of $\Sigma$ that is smallest in absolute magnitude (which we assume exists). Let $\xi \in \mathbb{R}^n$ be a vector with elements $\xi_i$ that are iid sub-Gaussian with zero mean and variance proxy $\sigma^2$. If $n \geq \frac{28\sigma^2}{\gamma^2\lambda_{\min}(\Sigma)} \log(2p/\delta)$ then with probability at least $1 - \delta$ we have $\max_j |\frac{1}{n}(\Sigma^+\Phi^\top)\xi_j| < \epsilon$.

Proof. Define $\alpha = \frac{1}{n}\Sigma^+\Phi^\top \xi$. Let $A = \frac{1}{n}\Sigma^+\Phi^\top$, then it is straightforward to check that, for any $i$, $\alpha_i = \sum_j A_{ij}\xi_j$ is sub-Gaussian with variance proxy $\sigma^2 \sum_j A_{ij}^2$. In addition to this, we have

$$\sum_j A_{ij}^2 = (AA^\top)_{ii} = \frac{1}{n} \Sigma_{ii}^+ \leq \frac{1}{n} \max_{ij} |\Sigma_{ij}^+| \leq \frac{1}{n} \|\Sigma^+\|_2 = \frac{1}{n\lambda_{\min}^{0,0}},$$

So the standard tail bound gives, for any $i$, $P(|\alpha_i| > \epsilon) \leq 2 \exp \left(-\frac{n\lambda_{\min}^{0,0} \epsilon^2}{2\sigma^2} \right)$ and the conclusion follows from a union bound.

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