Wear of foundation with a nonuniform coating by rough punch

K E Kazakov

Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia

kazakov-ke@yandex.ru

Abstract. Plane problem of contact interaction for a viscoelastic foundation with multilayer coating and a rigid punch is considered. It is assumed that thin coating consists of several different elastic layers. The rigidity of the each layer depends on longitudinal coordinate and can be described by own rapidly changing function. We obtain basic integral equation and additional conditions for this problem. The analytic solution of the problem is presented.

Introduction

In works [1] and [2] we solved some contact-wear problems. Paper [1] was dedicated to the obtaining of solutions for the plane contact-wear problem for flat punch and foundation with coating which has longitudially nonuniformity. A mixed integral equation with one complex function was obtained for this problem. Paper [2] considered an axisymmetric problem in the case where the punch had a complex shape. A mixed integral equation with two complex functions (punch shape and coating nonuniformity) was obtained for this problem.

In the current work we state and solve plane contact-wear problem for surface nonuniform foundation and rough punch. This plane problem has more degrees of freedom than axisymmetric problem [2]; punch can move and rotate. It has another kernels, operators, basis and etc. Some new effects can be detected in a plane problem.

1. Statement of the linear wear plane problem

Elastic layer of constant thickness lies on an underlying undeformable foundation (figure 1). Another elastic layer of constant thickness lies on the first layer. There are ideal contact between the layers and between lower layer and underlying foundation. We assume that upper layer is nonuniform and it is soft, i.e. its variable rigidity does not exceed the rigidity of lower layer [3]. We assume that rigidity of upper layer depends on longitudinal coordinate, i.e. $R = R(x)$.

A rigid punch indents into such foundation along the axis $Oz$. Moreover the punch move along the axis perpendicular to the plane $Oxz$ with an average velocity $V$. The average modulus of the punch velocity is taken into account when considering the reciprocating motion. The layered foundation is worn due to described punch motion.

Further we will assume that the contact region is constant and bounded by the boundaries $x = -a$ and $x = a$, the characteristic dimension $2a$ of the contact region is much greater than the upper layer thickness $h_1$, and there is a smooth contact between the punch and upper layer.
Numerous experiments have shown that, in the case of linear wear, the velocity of body surface points is proportional to the normal load and inversely proportional to the material hardness [4–8]:

\[ v_w(x,t) = -k_w V q(x,t) \]

where \( q(x,t) \) is normal contact pressure, \( H(x) \) is hardness of coating (upper layer), \( k_w \) is constant and depends on several intensities of vibrations. Then the vertical displacement of the top face of coating due to wear has the form

\[ u_w(x,t) = -\int_{\tau_0}^{t} v_w(x,\tau) d\tau = -k_w V \int_{\tau_0}^{t} q(x,\tau) d\tau. \tag{1} \]

Here \( \tau_0 \) is time of start punching. The vertical displacement of the coating top face due to applied load \(-q(x,t)\) has a form [9]

\[ u_q(x,t) = -\frac{q(x,t)h_1}{R(x)} - \frac{2(1-\nu_2^2)}{\pi E_2} \int_{-a}^{a} k_{pl} \left( \frac{x-\xi}{h_2} \right) q(\xi,t) d\xi, \tag{2} \]

Here \( R(x) = E_1(x)[1-\nu_1(x)]/[1-\nu_1(x)-2\nu_1^2(x)] \) (see, for example, [9]); \( \nu_1(x), E_1(x) \) and \( \nu_2, E_2 \) are where the Poisson’s ratios and the Young modulus of the upper and bottom layers, respectively; \( k_{pl}(s) = \int_{0}^{\infty} L(u) \cos(su)/u du \) and is the plane contact problem kernel [10] \( (L(u) = [2\kappa_2 \sinh(2u) - 4u]/[2\kappa_2 \cosh(2u) + \kappa_2^2 + 1 + 4u^2], \kappa_2 = 3 - 4\nu_2) \). There exist several cases when the hardness of the materials is proportional to their contact rigidity (see, for example, [11] and [12]), i.e.

\[ H(x) = k_H R(x). \tag{3} \]

So, total displacement is a sum of displacement \( u_w(x,t) \) and displacement \( u_q(x,t) \), i.e. (according to (1)–(3))

\[ u(x,t) = -\frac{k_w V}{H(x)} \int_{\tau_0}^{t} q(x,\tau) d\tau - \frac{k_H q(x,t)h_1}{H(x)} - \frac{2(1-\nu_2^2)}{\pi E_2} \int_{-a}^{a} k_{pl} \left( \frac{x-\xi}{h_2} \right) q(\xi,t) d\xi. \tag{4} \]
But it is easy to show that total displacement is equal to
\[ u(x, t) = -\delta(t) - \alpha(t)x + g(x), \]
where \( \delta(t) \) and \( \alpha \) are the punch settlement and tilt angle, \( g(x) \) is gap between coating and punch in nondeformable state \( (g(x) \geq 0, \exists x_0 \in [-a, a]: g(x_0) = 0) \). Note that if the stamp is rough, the function \( g(x) \) rapidly changes. Using this relation and equation (4) we obtain main integral equation of our problem

\[
\frac{k_w V}{H(x)} \int_{\tau_0}^{t} q(x, \tau) d\tau + \frac{k_H q(x, t) h_1}{H(x)} - \frac{2(1 - \nu_2^2)}{\pi E_2} \int_{-a}^{a} k_{pl} \left( \frac{x - \xi}{h_2} \right) q(\xi, t) d\xi = \delta(t) + \alpha(t)x - g(x),
\]

where \( x \in [-a, a] \), \( t \geq \tau_0 \).

The resulting equation should be supplemented by equilibrium conditions of the punch:

\[
\int_{-a}^{a} q(\xi, t) d\xi = P(t), \quad \int_{-a}^{a} \xi q(\xi, t) d\xi = P(t)e(t), \quad t \geq \tau_0,
\]
where \( P(t) \) is applied force and \( e(t) \) is eccentricity.

We use following change of variables in equations (5) and (6)

\[
x^* = \frac{x}{a}, \quad \xi^* = \frac{\xi}{a}, \quad t^* = \frac{t}{\tau_0}, \quad \tau^* = \frac{\tau}{\tau_0}, \quad \lambda = \frac{H}{a}, \quad V^* = \frac{k_w V \tau_0}{k_H h_1},
\]

\[
\delta^*(t^*) = \frac{\delta(t)}{a}, \quad \alpha^*(t^*) = \alpha(t), \quad g^*(x^*) = \frac{g(x)}{a}, \quad m^*(x^*) = -\frac{k_{pl} h_1 E_2}{2(a(1 - \nu_2^2)H(x))},
\]

\[
q^*(x^*, t^*) = \frac{2(1 - \nu_2^2)q(x, t)}{E_2}, \quad P^*(t^*) = \frac{2(1 - \nu_2^2)P(t)}{E_2 a^2}, \quad M^*(t^*) = \frac{2(1 - \nu_2^2)P(t)e(t)}{E_2 a^2},
\]

\[
F^* f(x^*) = \int_{-1}^{1} k_{pl}^*(x^*, \xi^*) f(\xi^*) d\xi^*, \quad k_{pl}^*(x^*, \xi^*) = \frac{1}{\pi} k_{pl} \left( \frac{x - \xi}{H} \right) = \frac{1}{\pi} k_{pl} \left( \frac{x^* - \xi^*}{\lambda} \right).
\]

Having omitted asterisks in the relations obtained, we arrive at the integral equation and additional condition in the form

\[
m(x) \left[ q(x, t) + V \int_{1}^{t} q(x, \tau) d\tau \right] + \int_{-1}^{1} k_{pl}(x, \xi) q(\xi, t) d\xi = \delta(t) + \alpha(t)x - g(x),
\]

\[
\int_{-1}^{1} q(\xi, t) d\xi = P(t), \quad \int_{-1}^{1} \xi q(\xi, t) d\xi = P(t), \quad x \in [-1, 1], \quad t \geq 1.
\]

Note that functions \( m(x) \) and \( q(x) \) in (8) connect with hardness of upper layer and punch base form, respectively; these parameters can be described by a rapidly changing function. Hardness can be described even by discontinuous function.

So it is required to solve mixed integral equation (i.e. equations with two different integral operators, see [14]) with 2 rapidly changing functions.

Plane contact problem has four versions of mathematical statements. We construct only one solution for the case with known force \( P(t) \) and moment \( M(t) \), unknown settlement \( \delta(t) \), tilt angle \( \alpha(t) \), and contact pressures \( q(x, t) \). In this case integral equation (8) will have unknown terms both on the left and on the right.
2. Special representation and basis

In main integral equation and additional condition (8), we make new change of variables

\[ q(x, t) = \frac{Q(x, t)}{\sqrt{m(x)}} - (I + V) \frac{g(x)}{m(x)}, \quad k_{pl}(x, \xi) = k(x, \xi) \sqrt{m(x)} \sqrt{m(\xi)}, \]

\[ Vf(t) = -V \int_1^t e^{-V(t-\tau)} f(\tau) d\tau, \]

where \( Q(x, t) \) is new unknown function, \( I \) is identity operator, \( V \) is Volterra operator with kernel \( -Ve^{-V(t-\tau)} \) (note that \( -Ve^{-V(t-\tau)} \) is kernel of resolvent for Volterra integral equation with kernel \( -V \), see [15]). Then the (8) become

\[ Q(x, t) + V \int_1^t Q(x, \tau) d\tau + FQ(x, t) = \frac{\delta(t)}{\sqrt{m(x)}} + \frac{\alpha(t)x}{\sqrt{m(x)}} + \frac{\tilde{\alpha}(t)\tilde{g}(x)}{\sqrt{m(x)}} \equiv f(x, t), \]

\[ \int_{-1}^1 \frac{Q(\xi, t)}{\sqrt{m(\xi)}} d\xi = \tilde{P}(t), \quad \int_{-1}^1 \frac{\xi Q(\xi, t)}{\sqrt{m(\xi)}} d\xi = \tilde{M}(t), \quad x \in [-1, 1], \quad t \geq 1, \]

where

\[ \tilde{F}(f) = \int_{-1}^1 k(x, \xi) f(\xi) d\xi, \quad \tilde{g}(x) = \int_{-1}^1 \frac{k_{pl}(x, \xi) g(\xi)}{m(\xi)} d\xi, \quad \tilde{\alpha}(t) = (I + V)1 = e^{-V(t-1)}, \]

\[ \tilde{P}(t) = P(t) + \tilde{\alpha}(t) \int_{-1}^1 \frac{g(\xi)}{m(\xi)} d\xi, \quad \tilde{M}(t) = M(t) + \tilde{\alpha}(t) \int_{-1}^1 \frac{Q(\xi, t)}{\sqrt{m(\xi)}} d\xi. \]

Note, there is no function \( m(x) \) in the left side of integral equation (10).

We will solve the solution of our problem in the class of time-continuous functions in a Hilbert space \( L_2[-1, 1] \) (e.g., see [14]). First of all we should construct special basis. To this end we will orthonormal on \([-1, 1]\) the linearly independent system of functions

\[ \left\{ \frac{1}{\sqrt{m(x)}}, \frac{x}{\sqrt{m(x)}}, \frac{x^2}{\sqrt{m(x)}}, \ldots \right\} \]

by the formulas (we will use scalar product \((f_1(x), f_2(x)) = \int_{-1}^1 f_1(\xi) f_2(\xi) d\xi\))

\[ J_m = \int_{-1}^1 \frac{\xi^d}{m(\xi)} d\xi, \quad d_{-1} = 1, \quad d_m = \begin{vmatrix} J_0 & \cdots & J_m \\ \vdots & \ddots & \vdots \\ J_m & \cdots & J_{2m} \end{vmatrix}, \]

\[ p^*_m(x) = \frac{1}{d_{m+1}d_m} \begin{vmatrix} J_0 & J_1 & \cdots & J_m \\ \vdots & \ddots & \vdots & \vdots \\ J_{m-1} & J_m & \cdots & J_{2m-1} \\ 1 & x & \cdots & x^m \end{vmatrix}, \quad p_m(x) = \frac{p^*_m(x)}{\sqrt{m(x)}}, \quad m = 0, 1, 2, \ldots, \]

3. Projection method and solution of contact-wear problem

Divide the space \( L_2[-1, 1] \) into two parts (two subspaces): \( L_2^{(1)}[-1, 1] \) and \( L_2^{(2)}[-1, 1] \) whose bases are the functions \( \{p_0(x), p_1(x)\} \) and \( \{p_k(x)\}_{k=2,3,4,\ldots} \), respectively.

Consequently, we introduce the operators of orthogonal projection \( P_1 \) and \( P_2 \) mapping the space \( L_2[-1, 1] \) onto \( L_2^{(1)}[-1, 1] \) and \( L_2^{(2)}[-1, 1] \), respectively, \( P_1f(x) = \int_{-1}^1 [p_0(x)p_0(\xi) + \ldots + \ldots + p_{2m}(x)p_{2m}(\xi)] f(\xi) d\xi \) and \( P_2f(x) = \int_{-1}^1 [p_{2m+1}(x)p_{2m+1}(\xi) + \ldots + \ldots + p_0(x)p_0(\xi)] f(\xi) d\xi \)
\[ p_1(x)p_1(\xi)|f(\xi)|\,d\xi, \quad P_2 = I - P_1, \] where \( I \) is the identity operator. Then \( P_i f(x,t) = f_i(x,t) \) and \( P_i Q(x,t) = Q_i(x,t) \) \((i = 1, 2)\).

Then functions \( Q(x,t) \) and \( f(x,t) \) can be represented as sums of functions in the spaces \( L_2^1[-1,1] \) and \( L_2^2[-1,1] \); i.e.,

\[ Q(x,t) = Q_1(x,t) + Q_2(x,t), \quad f(x,t) = f_1(x,t) + f_2(x,t), \]

\[ f_1(x,t) = \left[ \sqrt{J_0}\delta(t) + \frac{J_1}{\sqrt{J_0}}\alpha(t) + g_0\tilde{c}(t) \right] p_0(x) + \left[ \sqrt{\frac{J_0J_2 - J_1^2}{J_0}}\alpha(t) + g_1\tilde{c}(t) \right] p_1(x), \]

\[ f_2(x,t) = \frac{\tilde{c}(t)g_2(x)}{\sqrt{m(x)}}, \]

\[ Q_1(x,t), f_1(x,t) \in L_2^1[-1,1], \quad Q_2(x,t), f_2(x,t) \in L_2^2[-1,1], \]

where, as follows from (12) and (13), the coefficient \( g_0 \) and the function \( g_2(x) \) are determined by the expressions

\[ g_0 = \sum_{l=0}^{\infty} K_{0l} \int_{0}^{1} \frac{p_l(\xi)g(\xi)}{\sqrt{m(\xi)}} \,d\xi, \quad g_1 = \sum_{l=0}^{\infty} K_{1l} \int_{0}^{1} \frac{p_l(\xi)g(\xi)}{\sqrt{m(\xi)}} \,d\xi, \quad \frac{g_2(x)}{\sqrt{m(x)}} = \frac{\tilde{g}(x)}{\sqrt{m(x)}} - g_0p_0(x). \] (14)

Here \( K_{ml} \) are the coefficients of the expansion of the kernel \( k(x,\xi) \) into a double series in the functions \( p_k(x) \) \((k = 0, 1, 2, \ldots)\),

\[ K_{ml} = \int_{0}^{1} \int_{0}^{1} k(x,\xi)p_m(x)p_l(\xi) \,d\xi \,dx. \] (15)

Note that \( Q_1(x,t) \in L_2^1[-1,1] \) can be obtained from the supplementary condition (10),

\[ Q_0(x,t) = z_0(t)p_0(x) + z_1(t)p_1(x), \quad z_0(t) = \frac{\dot{P}(t)}{\sqrt{J_0}}, \quad z_1(t) = \frac{J_0\dot{M}(t) - J_1\dot{P}(t)}{\sqrt{J_0(J_0J_2 - J_1^2)}}, \] (16)

and the right-hand side of the integral equation contains the known term \( f_2(x,t) \in L_2^2[-1,1] \).

In this case, the terms \( Q_2(x,t) \in L_2^2[-1,1] \) and \( f_1(x,t) \in L_2^1[-1,1] \) are unknown. These characteristic features permit classifying the problem thus obtained as a special case of the generalized projection problem, which was posed and solved in [14].

As in [14], we act by the operator \( P_2 \) on the integral equation (10) to obtain the equation

\[ Q_2(x,t) + V \int_{1}^{t} Q_2(x,\tau) \,d\tau + P_2 F Q_2(x,t) = -P_2 F Q_1(x,t) + \frac{\tilde{c}(t)g_2(x)}{\sqrt{m(x)}} \] (17)

with known right-hand side for the unknown function \( Q_2(x,t) \). We seek the solution in the form

\[ Q_2(x,t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k(x), \] (18)

where \( z_k(t) \) is the desired function and the eigenfunctions \( \varphi_k(x) \) of the operator \( P_2 F \) are determined by solving the spectral problem for the operator \( P_2 F \),

\[ P_2 F \varphi_k(x) = \gamma_k \varphi_k(x). \] (19)
The eigenfunction system \( \{ \varphi_k(x) \}_{k=2,3,4,...} \) is related to the initial system \( \{ p_k(x) \}_{k=2,3,4,...} \) of basic functions in \( L_2^{(2)}[-1,1] \) as
\[
\varphi_k(x) = \sum_{l=2}^{\infty} \psi_{kl} p_l(x).
\] (20)
Therefore, solving the spectral problem is reduced to solving the system of algebraic equations
\[
\sum_{l=2}^{\infty} K_{ml} \psi_{kl} = \gamma_k \psi_{km}, \quad k, m = 2, 3, \ldots,
\] (21)
where \( K_{ml} \) are the coefficients of the expansion of the kernel \( k(x, \xi) \) in a double series in the functions \( p_k(x) \) \( (k = 0, 1, 2, \ldots) \) determined by formula (15).
We substitute the representation (18) into the integral equation (17) and use relations (11), (12), (19), and (20) to obtain the expression for the function \( z_k(t) \),
\[
z_k(t) = (I + W_k) \left( g_k \tilde{c}(t) - z_0(t) K_k^{(0)} - z_1(t) K_k^{(1)} \right) / (1 + \gamma_k),
\]
where coefficients \( g_k \) and \( K_k^{(0)} \) are calculated by the formulas
\[
g_k = \sum_{m=2}^{\infty} \psi_{km} \sum_{l=2}^{\infty} K_{ml} \int_{-1}^{1} \frac{p_l(\xi) g(\xi)}{m(\xi)} d\xi, \quad K_k^{(0)} = \sum_{m=2}^{\infty} K_{0m} \psi_{km}, \quad K_k^{(1)} = \sum_{m=2}^{\infty} K_{1m} \psi_{km},
\] (22)
and \( W_k \) is the Volterra operator whose kernel \( R_k(t, \tau) \) is the resolvent of the kernel \(-V/(1 + \gamma_k)\):
\[
R_k(t, \tau) = -\frac{V}{1 + \gamma_k} \exp \left[ -\frac{V(t-\tau)}{1 + \gamma_k} \right].
\] (23)
Then the function \( z_k(t) \) can be represented as
\[
z_k(t) = \frac{g_k}{\gamma_k} \left\{ e^{-V(t-1)} - \frac{1}{1 + \gamma_k} \exp \left[ -\frac{V(t-1)}{1 + \gamma_k} \right] \right\} - \frac{K_k^{(0)}}{1 + \gamma_k} \left\{ z_0(t) - \frac{V}{1 + \gamma_k} \int_{1}^{t} \exp \left[ -\frac{V(t-\tau)}{1 + \gamma_k} \right] z_0(\tau) d\tau \right\}
- \frac{K_k^{(1)}}{1 + \gamma_k} \left\{ z_1(t) - \frac{V}{1 + \gamma_k} \int_{1}^{t} \exp \left[ -\frac{V(t-\tau)}{1 + \gamma_k} \right] z_1(\tau) d\tau \right\}.
\] (24)
As a result, by using (9), (13), (16), (18), and (20), we finally obtain the expression for the contact stresses
\[
g(x, t) = \frac{1}{m(x)} \left[ z_0(t) p_0^*(x) + z_1(t) p_1^*(x) + \sum_{k=1}^{\infty} z_k(t) \sum_{m=2}^{\infty} \psi_{km} p_m^*(x) - e^{-V(t-1)} g(x) \right],
\]
where the functions \( z_0(t), z_1(t), \) and \( z_k(t) \) are determined by relations (16) and (24), the coefficients \( \psi_{kl} \) can be obtained from (21), and \( p_0^*(x), p_1^*(x), p_l^*(x) \) can be obtained from (12).
By applying the operator \( P_1 \) to Eq. (10), we obtain the formulas for settlement and tilt angle
\[
\delta(t) = \frac{1}{\sqrt{J_0}} \left\{ z_0(t) + V \int_{0}^{t} z_0(\tau) d\tau - g_0 e^{-V(t-1)} + K_{00} z_0(t) + K_{01} z_1(t) + \sum_{k=2}^{\infty} K_k^{(0)} z_k(t) \right\} - \alpha(t) \frac{J_1}{J_0},
\]
\[
\alpha(t) = \sqrt{\frac{J_0}{J_0 J_2 - J_1^2}} \left\{ z_1(t) + V \int_{0}^{t} z_1(\tau) d\tau - g_1 e^{-V(t-1)} + K_{10} z_0(t) + K_{11} z_1(t) + \sum_{k=2}^{\infty} K_k^{(1)} z_k(t) \right\},
\]
Here the functions $z_0(t)$, $z_1(t)$, and $z_k(t)$ are determined from (16) and (24), the coefficients $g_0$, $g_1$, $K_{00}$, $K_{01}$, $K_{10}$, $K_{11}$, $K_{k}^{(0)}$, and $K_{k}^{(1)}$ are determined from (14), (15), and (22), and the integrals $J_0$, $J_1$, $J_2$ is calculated by formulas (12) ($k = 2, 3, 4, \ldots$).

Conclusions

The contact-wear plane problem for regular system of rigid punches with complex base forms and viscoelastic aging layer with nonuniform elastic coating are formulated and solved. The solution for one problem version is constructed in analytical form. Solutions for other versions can be constructed in a similar way. Expressions for contact pressures have a structure in which the functions are associated with the coating contact rigidity and the punch base forms are distinguished by separate terms and factors. This analytical solution allows one to carefully analyze the influence of contact characteristics on the stress-strain state of the foundation and positions of bodies. Moreover, such solution representation allows one to perform efficient calculations of linear wear of foundations in cases when the coating elastic properties and punch base form described by complex rapidly changing functions.

Acknowledgments

The paper is financially supported by the Ministry of Science and Higher Education (State Registration Number AAAA-A17-117021310381-8) and, in part, by the Russian Foundation for Basic Research (under grant No. 18-51-05012-Arm.a).

References

[1] Manzhirov A V and Kazakov K E 2017 Contact problem with wear for a foundation with a surface nonuniform coating Doklady Physics 62 (7) 344–9

[2] Manzhirov A V and Kazakov K E 2018 Axisymmetric problem of fretting wear for a foundation with a nonuniform coating and rough punch AIP Conference Proceedings 1959 070023

[3] Alexandrov V M and Mkitaryan S M 1979 Contact Problems for Bodies with Thin Coatings and Interlayers (Moscow: Nauka) [in Russian]

[4] Pronikov A S 1957 Wear and Durability of Machines (Moscow: Mashgiz) [in Russian]

[5] Khrushchev M M and Babichev M A 1970 Abrasive Wear (Moscow: Nauka) [in Russian]

[6] Collins J 1993 Failure of Materials in Mechanical Design: Analysis, Prediction, Prevention (New York: Wiley)

[7] Goryacheva I G and Dobychin M N 1988 Contact Problems in Tribology (Moscow: Mashinostroenie) [in Russian]

[8] Soldatenkov I A 2010 Wear Contact Problem with Applications to Engineering Calculation of Wear (Moscow: Fizmatkniga) [in Russian]

[9] Alexandrov V M and Kovalenko E V 1986 Problems in Mechanics of Continuous Media with Mixed Boundary Conditions (Moscow: Nauka) [in Russian]

[10] Vorovich I I, Alexandrov V M, and Babeshko V A 1974 Nonclassical Mixed Problems in the Theory of Elasticity (Moscow: Nauka) [in Russian]

[11] Archard J F 1953 Contact and rubbing of flat surfaces Journal of Applied Physics 24 (8) 981–8

[12] Schalliamach A 1954 On the abrasion of rubber Proceedings of the Physical Society, Section B 67 (12) 883–91

[13] Manzhirov A V and Kazakov K E 2018 Modeling the contact interaction between a nonuniform foundation and a rough punch Mathematical Models and Computer Simulations 10 (3) 314–321

[14] Manzhirov A V 2016 A mixed integral equation of mechanics and a generalized projection method of its solution Doklady Physics 61 (10) 489–93

[15] Polyanin A D and Manzhirov A V 2008 Handbook of Integral Equations, 2nd edition (Boca Raton: Chapman & Hall/ CRC)