Logical network implementation for cluster states and graph codes

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In a previous paper a straightforward construction method for quantum error correcting codes, based on graphs, has been presented. These graph codes are directly related to cluster states which have been introduced by Briegel and Raussendorf. We show that the preparation of a cluster state as well as the coding operation for a graph code, can be implemented by a logical network. Concerning the qubit case each vertex corresponds to an Hadamard gate and each edge corresponds to a controlled not gate.

I. INTRODUCTION

Recently, we discussed a new construction method for quantum error correcting codes, based on graphs [1]. The error correcting capabilities of a "graph code" can directly be derived from the structure of the graph which also allows to tailor quantum error correcting codes systematically.

In the present paper we discuss how graph codes can be implemented on a quantum computer by a logical network. This problem has also been analyzed by several authors with respect to quantum error correcting codes being obtained by other construction methods [2,3].

The subsequent results are based on a direct relation between graph codes and the so called "cluster states". These states have been introduced by H.J. Briegel and R. Raussendorf for performing quantum algorithms by means of local von Neumann measurements [4,5]. Besides these aspects, cluster states can also be used for studying concepts multi-particle entanglement [6].

A cluster state is determined by the following objects:

- A finite number of levels,
- A graph consisting of a finite number of vertices and edges, where two vertices are connected by at most one edge,
- Non-zero integral numbers (weights), each of them being attached to an edge.

The number of levels is the dimension of a single elementary quantum system, which we call here a "quantum digit". For example, considering an atom or ion in a trap, the states of a quantum digit are density matrices on a Hilbert space spanned by finitely many energy eigenvectors, each of them belonging to an eigenvalue of multiplicity one. A two level or binary quantum digit is just a qubit.

A cluster state corresponds to a system of multiple quantum digits whose positions are labeled by the vertices of the graph. Each edge of the graph can be viewed as an interaction between the quantum digits corresponding to vertices which are connected by it. The non-zero integral number (weight) attached to the edge can be viewed as the strength of the interaction. As example, one may think of a two dimensional optical lattice structure, where each quantum digit occupy a point within a two dimensional cubic lattice with next neighbor interaction.

As far as operations on an optical lattice are concerned, the preparation of a cluster state needs only one elementary operation applied to an initial product state – each qubit is in a superposition of "0" and "1". This operation acts globally on the lattice, creating entanglement between neighbored qubits. Once a cluster state is prepared, it can be used as a resource for performing any quantum algorithm by local measurements [7].

For a graph with finitely many qubits on which local Hadamard gates and CNOT gates can be performed in a controlled manner, we show in Section 3 that a cluster state for an arbitrary graph can be prepared by applying a logical network to the initial state where each qubit is prepared in the state "0". For a graph with v vertices and l edges, the number of steps which are needed to perform the corresponding network is l + v. For each vertex there is a Hadamard gate and for each edge a CNOT gate. This network can be derived from the graph by a systematic algorithm.

This also opens the discussion for comparing complexity measures for quantum algorithms based on one- and two-qubit gates, on one hand, and globally parallel operations, on the other hand.

Graph codes [1] are directly related to cluster states. Consider the cluster state for a graph, we select for each quantum digit, we wish to encode, a vertex. These selected vertices are called "input vertices". The remaining vertices are called "output vertices". As we shall see in Section 3, the coding operation for a graph code can be split into four main steps. For the qubit case, it works as follows:

1: The input qubits are prepared in the state we wish to protect against errors.
2: Each output qubit is prepared in the state "0".
3: A discrete dynamics is applied which creates entanglement between those qubits sitting at vertices connected by an edge. Here it is given by first performing a Hadamard transform on each input qubit, then applying the network which creates the cluster state and finally one operates again with a Hadamard transform on each input qubit.
4: Each input qubit is measured in the "computational basis", i.e. the basis given by the states "0" and "1".

For a graph with k input vertices, n output vertices and l edges, as we already know, the network for creating a cluster state can be realized by k + n + l elementary gate operations. Implementing the dynamics of step 3 would cost at most 3k + n + l steps since there are two additional Hadamard gates for each input. Provided there are no edges between input
vertices, the number of gates can be reduced to \( k + n + l \) as we shall see in Section III. The corresponding logical network can be derived systematically from the graph analogously to the network for cluster states.

Considering the case where only one input digit is present, we derive logical networks for graph codes which operate on the output digits. In comparison to the networks, described before, it makes use of less resources. We give in Section IV an algorithm associating to a graph with \( n \) output vertices and \( l \) edges a logical network which implements the corresponding code with \( n + l - 1 \) elementary gate operations.

II. LOGICAL NETWORKS FOR CLUSTER STATES

To begin with, we briefly describe here how a cluster state is constructed from a finite number of levels \( d \) and a weighted graph \( \Gamma \) with vertices \( V \). The classical configuration space for describing a digit is given by a cyclic group \( \mathbb{Z}_d \). The states of a single quantum digit are density operators on the Hilbert space of "wave functions" \( \ell_2(\mathbb{Z}_d^V) \) on \( \mathbb{Z}_d \). Obviously, the quantum digit for the two elementary group elements \( d \) and \( \ell \) is constructed from a finite number of levels \( d \) and \( \ell \) quantum digits for the two elementary gates.

In our context a quantum register consists of quantum digits labeled by a finite set \( V \) of "positions". Thus the states of the quantum register are density operators on \( \ell_2(\mathbb{Z}_d^V) \) where \( \mathbb{Z}_d^V \) is the group of tuples \( g = (g_i|i \in V) \). It is convenient to introduce a basis in \( \ell_2(\mathbb{Z}_d^V) \), called the computational basis: \( \{|g\rangle\|g\in G^V\} \) where \( |g\rangle \) is the indicator function of the point \( g \).

A weighted graph is given by the symmetric matrix \( \Gamma = (\Gamma(i,j)|i,j \in V) \), where \( \Gamma(i,j) \) is the integral number assigned to the edge \( \{i,j\} \) being zero if there is no edge between \( i \) and \( j \). The cluster state, corresponding to \( \Gamma \), is represented by a vector \( \Psi^\Gamma \) on \( \mathbb{Z}_d^V \). It is given by assigning the value

\[
\Psi^\Gamma(g) := d^{-\frac{V}{2}} \prod_{(i,j) \in \Gamma} \chi(g_i|g_j)^{\Gamma(i,j)}
\]

(1)

to a group element \( g = (g_i|i \in V) \in \mathbb{Z}_d^V \). Here we have introduced for two group elements \( g, h \in \mathbb{Z}_d \) the phase

\[
\chi(g|h) = \exp\left(\frac{2\pi i}{d} gh\right)
\]

(2)

A logical network for preparing a state, represented by a vector \( \Psi \) in \( \ell_2(\mathbb{Z}_d^V) \), is given by a sequence of "quantum gates", applied successively to a "ground" state \( |0\rangle \) (0 is the zero in \( \mathbb{Z}_d \)), such that the resulting vector is \( \Psi \). Quantum gates are elementary unitary operations, each of them acting only on a single or two quantum digits. In some cases operations on three quantum digits are also considered as elementary gates, e.g. the Toffoli gate.

As it turns out, the following elementary gate operations are needed for building a logical network for a cluster state:

- The local Fourier transform \( F_i \) operating on the quantum digit at \( i \)

\[
F_i|g\rangle := \frac{1}{\sqrt{d}} \sum_{h \in \mathbb{Z}_d} \chi(g_i|h) \{h, g^i\}
\]

(3)

with \( g = (g_i|i \in V) \) and \( g^i := (g_i|i \in V \setminus \{i\}) \) and its inverse \( F_i^* \).

- The \( n \)-controlled shift gate \( c(i,j)^n \) with control digit at position \( i \) and target digit at position \( j \) is given by

\[
c(i,j)^n|g\rangle := |g_j + ng_i, g^i\rangle
\]

(4)

and its inverse \( c(i,j)^{-n} \).

In graphical representation of a logical network the local Fourier transform as well as the \( n \)-controlled shift gate is symbolized as shown in Figure 1.

![FIG. 1. From left to right: The local Fourier transform at position \( i \), the \( n \)-controlled shift operation.](image1)

![FIG. 2. The symbol for the \( n \)-controlled phase gate at positions \( i,j \).](image2)

A procedure for preparing a cluster state \( |\rangle \) can directly be expressed in terms of local Fourier transforms and controlled phase gates whose graphical symbols are given by Figure 2. Each \( n \)-controlled phase gate is a composition of two local Fourier transforms and one \( n \)-controlled shift gate: Acting on the positions \( i,j \), this gate is given by

\[
u(i,j)^n := F_j c(i,j) F_j^* = F_j c(i,j) F_j^*.
\]

(5)

According to the definition of the cluster state \( |\rangle \), the vector \( \Psi \) can be obtained by first applying for each vertex a local Fourier transform and second operating with a controlled phase gate on the quantum digits corresponding to a pair of vertices which are connected by an edge. Introducing the cluster state creation operator

\[
u = \prod_{(i,j) \in \Gamma} \prod_{j \in V} F_j
\]

(6)

the cluster state wave function is:

\[
\Psi^\Gamma = \nu|0\rangle.
\]

(7)
In order to obtain a logical network in terms of controlled shift gates and local Fourier transforms, one just have to substitute the controlled phase gates \( e(i, j) \) in Equation (\ref{equation:controlled_phase}) by \( F_j e(i, j) F_j^* \). This procedure is represented by Figure 5 for the cluster state corresponding to the graph in Figure 3. As a consequence, we obtain a network implementation which uses a local Fourier transform for each vertex and a controlled shift gate for each edge.

Considering the graph in Figure 3, the corresponding logical network expressed in terms of controlled phase gates and local Fourier transforms is given by Figure 4 below.

**Proposition II.1** For a weighted graph \( \Gamma \) with \( v \) vertices and \( l \) edges the cluster state creation operator \( u_\Gamma \) can be decomposed into a product of \( v + l \) unitary operations which are local Fourier transforms or controlled shift gates.

We postpone the proof of the proposition to Appendix A and describe an algorithm which associates to a weighted graph \( \Gamma \) a logical network expressing \( u_\Gamma \) in terms of controlled shift gates and local Fourier transforms.

**Algorithm II.2** Choose an enumeration of the vertices by \( V = \{0, \ldots, N\} \) and perform step 1 for the vertex \( j = 0 \):

1. Apply the Fourier transform \( F_j \) and proceed with step 2 for the vertex \( k = j + 1 \).
2. Apply the controlled shift gate \( e(j, k)^{\Gamma(j,k)} \).
3. Repeat step 2 for the vertex \( k' = k + 1 \) until \( k' = N \).
4. Repeat steps 1-3 by starting step 1 for the vertex \( j' = j + 1 \) until \( j' = N \).

It will directly follow from the proof of Proposition II.1 that the Algorithm II.2 yields the correct decomposition of \( u_\Gamma \). In particular, by applying it to the graph in Figure 3, one obtains the logical network implementation presented in Figure 5.

**III. FROM CLUSTER STATES TO GRAPH CODES**

In this section we tackle the question, how graph codes can be realized by preparing a cluster state (a globally parallel operation) and local measurement operations.

To start with, we briefly describe here the concept of graph codes, introduced in \cite{1}. Consider a graph \( \Gamma \) with vertices \( V \) and choose a subset of "input" vertices \( X \subset V \) which reasonably contains less elements than the set of "output" vertices \( Y = V \setminus X \). The quantum code, associated with this choice of input and output vertices, is given by a linear map \( v_{\Gamma} \) mapping the Hilbert space for the input register \( l_2(\mathbb{Z}_4^d) \) into the Hilbert space for the output register \( l_2(\mathbb{Z}_4^d) \) as follows: A basis vector \( |h\rangle \in l_2(\mathbb{Z}_4^d) \) with \( h \in \mathbb{Z}_4^d \) is mapped to

\[
v_{\Gamma} |h\rangle = d^{|X|} \sum_{g \in G^X} \Psi_{\Gamma}(h, g) \ |g\rangle \tag{8}
\]

where \( \Psi_{\Gamma} \in l_2(G^V) \) is the wave function for the cluster state, associated with \( \Gamma \). The range of \( v_{\Gamma} \) is then a candidate for a protected subspace corresponding to a quantum error correcting code. In particular, if \( \Gamma \) is a weighted graph, associated with a quantum error correcting code (See \cite{1}), then \( v_{\Gamma} \) is automatically an isometry, i.e. \( v_{\Gamma}^* v_{\Gamma} = 1 \).

Concerning the Heisenberg picture, the isometry \( v_{\Gamma} \) implements a "coding channel" which is the completely positive unital map, assigning to an operator (observable of the output system) \( a \) on \( l_2(\mathbb{Z}_4^d) \) to an observable of the input system:

\[
a \mapsto v_{\Gamma}^* a v_{\Gamma} \ . \tag{9}
\]

For later purpose, it is convenient to introduce for a finite set \( V \) the algebra \( \mathfrak{A}(V) \) of all bounded operators on \( l_2(\mathbb{Z}_4^d) \). For a subset \( K \subset V \) the algebra \( \mathfrak{A}(K) \) can be identified with the subalgebra of \( \mathfrak{A}(V) \) which consists of those operators acting...
nontrivially only on the tensor factors corresponding to the subset $K$. Moreover, we denote by $\mathcal{C}(V)$ the abelian algebra of function on $Z_d^V$, which can be identified with the algebra of multiplication operators in $\mathfrak{A}(V)$.

The cluster state wave function occurs explicitly in the formula for the graph code (6). This suggests the following scheme for implementing the code by preparing a cluster state and doing local measurement operations:

1: Prepare the input digits in the state which one wish to protect against errors.
2: Prepare the output digits in the ground state corresponding to the vector $|0_Y\rangle$ (See (12) below).
3: Apply to each input digit an inverse Fourier transform. By acting on all quantum digits, apply the cluster state creation operator $u_C$. Operate with a Fourier transform on each input (See (11)).
4: Perform a local measurement on each input digit in the computational basis (See (14)).

Concerning the scheme above, the coding procedure depends on the outcome of the measurement of the input digits. Hence, besides the quantum output register we obtain an additional classical output. This can be modeled (in the Heisenberg picture) by the unital completely positive map $C_T$ which maps the operator $a \otimes f$ in $\mathfrak{A}(Y) \otimes \mathcal{C}(X)$ to

$$C_T(a \otimes f) := d^{-|X|} \sum_{h \in Z_d^X} \hat{u}(h)^* a v_f \hat{u}(h) f(h) \quad (10)$$

where $h \mapsto \hat{u}(h)$ is the representation of $Z_d^X$ by multiplier operator as given in Appendix B.

The following operations, given in terms of completely positive maps are building blocks for performing $C_T$:

- An automorphism $\alpha_T$ describes a discrete dynamics of the system consisting of the input and output digits. It acts on the corresponding observable algebra by mapping an operator $b$ on $l_2(Z_d^X)$ to

$$\alpha_T(b) := F_X u_C^* F_X^* b F_X u_C F_X^* \quad (11)$$

where $u_C$ is the cluster state creation operator $\mathfrak{A}$. The ground state preparation of the output digits $P_Y$ is a completely positive unital map, sending an operator $b$, acting on all quantum digits, to the operator

$$P_Y(b) = w_Y^* b w_Y \quad (12)$$

acting on the inputs. We introduce the isometry $w_K$ by assigning to $\Psi \in l_2(Z_d^{V\setminus K})$ the vector

$$w_K \Psi := |0_K\rangle \otimes \Psi $$

where $0_K$ is the zero in $Z_d^K$.

- A measurement of the input digits in the computational basis, corresponds to the unital completely positive map $M_X$ mapping the operator $a \otimes f$ in $\mathfrak{A}(Y) \otimes \mathcal{C}(X)$ to

$$M_X(a \otimes f) := \sum_{h \in Z_d^X} u(h) w_X a w_X^* u(h)^* f(h) \quad (14)$$

The following statement, which we prove in Appendix C, justifies the suggested coding scheme, described above:

**Proposition III.1** Let $\Gamma$ be a weighted graph, associated with a quantum error correcting code. By adopting the notations, given above, the isometry $v_r$ fulfills the identity

$$v_r = d^{-|X|} X^{w} F_X u_r F_X^* X_Y^* \quad (15)$$

In particular, the coding operation $C_T$ satisfies

$$C_T = P_Y \circ \alpha_T \circ M_X \quad (16)$$

**IV. LOGICAL NETWORKS FOR GRAPH CODES**

For getting the logical network, which performs step 3 of the previous section, we have to decompose the operator $F_X u_C F_X^*$ into a product of controlled shift gates and local Fourier transforms, which is a straightforward task since we know already how to decompose the cluster state creation operator $u_C$. As a result, we get with help of Proposition II.1:

**Proposition IV.1** Let $\Gamma$ be a weighted graph, associated with a quantum error correcting code, having $v$ vertices, $l$ edges and no edges between input vertices. Then the operator $F_X u_C F_X^*$ is a product of $v + l$ elementary gates being either controlled shift gates or local Fourier transforms.

As it is already mentioned in [1], links between input vertices can always be removed without affecting the error correcting capabilities of the corresponding code. Therefore, the assumptions in Proposition IV.1 (the proof can be found in Appendix C) can be made without loss of generality.

Analogously to the Algorithm II.2, we also obtain an algorithm for decomposing the operator $F_X u_C F_X^*$:

**Algorithm IV.2** Choose an enumeration of the input vertices by $X = \{0, \ldots, k\}$ and the output vertices $Y = \{k + 1, \ldots, v\}$. Perform step 1 for the input vertices $j = 0$ and $i = 1$:

1: Apply the controlled shift gate $c(j, i)^{F(i, j)}$.
2: Repeat step 1 for the vertices $j$ and $i' = i + 1$ until $i' = k$.
3: Apply the Fourier transform $F_{i'}$ and proceed with step 1 for the input vertices $j' = j + 1$ and $i' = j + 2$ if $i' \leq k$. Otherwise, proceed with step 4.
4: Apply the Algorithm II.2 starting with the output vertex $k + 1$.

The graph in Figure 6 yields an example for a quantum error detecting code, encoding two quantum digits $\{0, 1\}$, into four $\{2, \ldots, 5\}$ and detecting all errors which affect one quantum digit.
Performing the steps 1-3 of the Algorithm IV.2 for the graph in Figure 6, one obtains the part "Steps 1-3" of the logical network depicted in Figure 7. The part "Step 4" is obtained from an application of the Algorithm II.2 by starting with vertex \{2\}. This is nothing else but the logical network implementation for the cluster state associated with the subgraph which is obtained by removing the input vertices \{0,1\}.

In the previous section, we have given an algorithm which associates a logical network to a weighted graph implementing the corresponding quantum error correcting code. Here the network operates on the input digits as well as on the output digits. After applying the network, a local measurement of the inputs is performed which completes the coding procedure.

In this section, we discuss the construction of local networks for graph codes, which only operate on the output digits and which do not require a measurement procedure after applying the network.

We give here a practicable solution for the case that there is one input vertex. Furthermore we require that there exists an output vertex being connected with the input by an edge with weight 1.

**Proposition V.1** Let \( \Gamma \) be a weighted graph with \( l \) edges, input vertex \( \{0\} \) and output vertices \( \{1, \cdots, n\} \). If \( \Gamma(0,1) = 1 \), then there exists a unitary operator \( z_\Gamma \), acting on the output digits, such that

\[
v_\Gamma = z_\Gamma \cdot w_{\{2, \cdots, n\}} \quad .
\]

Furthermore, \( z_\Gamma \) can be decomposed into a product of \( l + n - 1 \) elementary gate operations which are either controlled shifts or local Fourier transforms.

We prove the proposition in Appendix D. The coding operation is performed by the following two steps:

- Prepare the output digit \( \{1\} \) in the state one wishes to protect. The remaining output digits are prepared in the state, corresponding to the vector \( \{0_{\{2, \cdots, n\}}\} \).
- Apply the logical network which implements the unitary operator \( z_\Gamma \).

The Algorithm V.2, given below, associates to each graph, which satisfies the assumptions of Proposition V.1, a decomposition of \( z_\Gamma \) into elementary gates.

**Algorithm V.2** Perform step 1 for the output vertex \( i = 2 \):

1. Apply the controlled shift gate \( c(1,i)\Gamma(0,i) \).
2. Repeat step 1 for the vertex \( i' = i + 1 \) until \( i' = n \).
3. Apply the Algorithm II.2 starting with the output vertex \( j = 1 \).

We illustrate the statement of Proposition V.1 by considering the quantum error correcting code which corresponds to the graph in Figure 3 with input vertex \( \{0\} \).

By Proposition III.1, the network in Figure 8 represents the coding operation in the following way: Each output digit \( y = 1, \cdots, 5 \) is prepared in the state \( \{|0\} \), the input digit \( x = 0 \) is prepared in the state \( |h\rangle \). After applying the logical network, which implements \( F^*_0 u_1 F^*_0 \), a selective measurement is performed on the input by collecting those measurement outcomes for which each input digit is in the state \( |0\rangle \). This selection procedure yields a factor \( d^{-1/2} \) and Figure 8 represents the operator \( d^{-1/2}v_\Gamma \).

\[
d^{1/2} |b\rangle \quad \text{after} \quad \text{Applying logical network} \quad = \quad |h\rangle
\]

**FIG. 6.** The graph for a quantum error detecting code of length 4 encoding two quantum digits (inputs \{0,1\}) and detecting one error.

**FIG. 7.** Logical network for implementing the quantum error correcting code corresponding to the graph of Figure 6 with inputs \{0,1\}. The horizontal dashed lines correspond to classical wires.

**FIG. 8.** Network representation of the operator \( d^{-1/2}v_\Gamma \) for the graph in Figure 3 with input vertex \( \{0\} \).

**FIG. 9.** Useful identity.
Now we make use of the identity represented in Figure 9. If we replace in Figure 8 the part within the dashed frame by the network on the right hand side of Figure 9, we obtain the network depicted in Figure 10 below. Due to the identity in Figure 9, we gain a factor $d^{l/2}$ and Figure 10 represents the full isometry $\varphi_F$ operating on the basis vector $|h\rangle$, $h \in \mathbb{Z}_d$.

By applying the Algorithm [V.2] directly to the graph in Figure 3 (with input vertex $\{0\}$) one indeed obtains the network in Figure 10.

VI. CONCLUSION AND OUTLOOK

The ability of preparing cluster states yields a resource for performing quantum algorithms [4], on one hand, and it provides a natural mechanism for protecting quantum information against errors, on the other hand. In the present paper we have developed the following features:

- Logical networks for preparing general cluster states can be derived from their defining graphs in a systematic manner. The number of elementary operations which is used by the network is the number of vertices plus the number of edges of the graph.

- Quantum error correcting codes can be realized by applying preparation procedure for a cluster state to a suitably prepared input state followed by a local measurement operation on the input digits. These coding schemes operates on the input and output digits and they can be expressed in terms of logical networks which uses the same amount of elementary gates as the preparation procedure of the corresponding cluster state.

In order to save resources, one wishes to construct logical networks, implementing quantum error correcting codes, by operating only on a number of digits corresponding to the length of the code (number of outputs). In fact we have given an algorithm which handle the following situation:

- Provided there is one input vertex, a logical network can be associated to a given graph which implements the corresponding graph code. This network operates directly on the output digits and its number of elementary gate operations is the number of outputs plus the number of edges minus one.

Concerning future investigations, it would be desirable to develop similar network representations also for the decoding operations. Here, one possible strategy is to look for a decoding procedure which starts from preparing a cluster state, and then performing local measurements on a suitable set of digits. This problem can be tackled by searching for a graph representation for decoding channels similar to those for the coding operations.

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APPENDIX A: PROOF OF PROPOSITION II.1

Let $\Gamma$ be a weighted graph with vertices $V = \{1, \cdots, v\}$. By introducing the operators

$$u_{ij}^{(v)} = \prod_{k=1}^{v} u(j, k)^{\Gamma(j, k)}$$

the cluster state creation operator $|\varphi\rangle$ can be written as

$$u_F = u_{F_{v+1}}^{(v)} u_{F_{v-2}}^{(v-2)} \cdots u_{F_1}^{(1)} F_{(1, \cdots, v)}$$

We introduce for each vertex $j$ the block of controlled shift operations

$$c_j = \prod_{i=j+1}^{v} c(j, i)^{\Gamma(j, i)}$$

Note that $c(i, j)$ commutes with $c(i, k)$ which implies that the definition of $c_j$ is independent of the order of factors on the right hand side of (A3).

The local Fourier transform $F_j$ commutes with the controlled phase operation $u(j, k)$ (as well as the controlled shift $c(j, k)$) for $i \neq j, k$. Thus we obtain from the definition of the controlled phase gate $[\hat{F}]$ and from [13]:

$$u_{ij}^{(v)} = F_{j} \cdot c(j, v)^{\Gamma(j, v)} F_{v}^*$$

Inserting the identity [A4] into (A2) yields

$$u_F = F_{v} c_{v+1}^F F_{v-1} \cdots c_2^F c_1^F F_{(1, \cdots, v)}$$

APPENDIX B: PROOF OF PROPOSITION II.2

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, \mathcal{H}\}$ which assigns a basis vector $|v\rangle$ to each vertex $v \in \mathcal{V}$.

Let $H$ be a Hamiltonian acting on $\mathcal{H}$ and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, H\}$ which assigns an operator $\hat{H}_v$ to each vertex $v \in \mathcal{V}$.

APPENDIX C: PROOF OF PROPOSITION II.3

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, \mathcal{H}\}$ which assigns a basis vector $|v\rangle$ to each vertex $v \in \mathcal{V}$.

Let $H$ be a Hamiltonian acting on $\mathcal{H}$ and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, H\}$ which assigns an operator $\hat{H}_v$ to each vertex $v \in \mathcal{V}$.

APPENDIX D: PROOF OF PROPOSITION II.4

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, \mathcal{H}\}$ which assigns a basis vector $|v\rangle$ to each vertex $v \in \mathcal{V}$.

Let $H$ be a Hamiltonian acting on $\mathcal{H}$ and $\mathcal{V}$ be a set of vertices. In order to couple the Hilbert space with the vertices, we introduce the mapping $\{\mathcal{V}, H\}$ which assigns an operator $\hat{H}_v$ to each vertex $v \in \mathcal{V}$.
Looking at (A3) each block $c^r_l$ contains $\sum_{k\neq l+1}^{n} 1^{(j,k)}$ controlled shift operations. Therefore, the total number of controlled shift operations in the last line of (A3) is just the number of edges $l$ of the graph. In addition to that, for each vertex there is a elementary Fourier transform and $u_F$ can indeed be decomposed into a product of $v + l$ elementary gate operations being either local Fourier transforms or controlled shift operations. $\square$

APPENDIX B: PROOF OF PROPOSITION $\{\text{B.1}\}$

1. Shift and multiplier

For a subset $K \subset V$, the group $Z^K_d$ can naturally be identified with a subgroup in $Z^V_d$. The group $Z^K_d$ is represented on $l_2(Z^K_d)$ by shift operators according to

$$u(h)|h\rangle = |h + h\rangle$$  \hspace{1cm} (B1)

with $h \in Z^K_d$ and $h' \in Z^V_d$.

By using the Fourier transform $F_K = \prod_{k \in K} F_k$ on the digits labeled by $K$, one obtains a further representation $\hat{u}$ of $Z^K_d$ by multiplier operators. Namely, the multiplier $u(h)$, $h \in Z^K_d$, is related to the corresponding shift $u(h)$ according to

$$\hat{u}(h)|h\rangle = F_K u(h) F_K^* |h\rangle = \chi(h|h') |h\rangle.$$  \hspace{1cm} (B2)

The multiplication phase is just given by $\chi(h|h') = \prod_{k \in K} \chi(h_k | h'_k)$.

2. Proof of the proposition

Let $|h\rangle$, $h \in Z^X_d$ be a vector of the computational basis of $l_2(Z^X_d)$. The cluster state creation operator $u_F$ is the composition of the Fourier transform on $F_V$ on the full register of all quantum digits and the unitary multiplication operation $\Phi_F$, given by

$$\Phi_F |g\rangle := d^{\frac{|X|}{2}} \Psi_F (|g\rangle |g\rangle)$$  \hspace{1cm} (B3)

where $\Psi_F$ is the cluster state wave function (1) associated with the weighted graph $\Gamma$. Furthermore, we have $w|h\rangle = u(h)w_X$ for each $h \in Z^X_d$. Thus, the right hand side of (15) can be written as

$$w^*_X F_X u_F F^*_X w_Y = w^*_X F_X \Phi_F F_Y w_Y.$$  \hspace{1cm} (B4)

which implies for each $h \in Z^X_d$.

$$w^*_X u(h)^* F_X u_F F^*_X w_Y = w^*_X u(h)^* F_X \Phi_F F_Y w_Y = \chi(h| h') \Phi_F F_Y w_Y$$  \hspace{1cm} (B5)

where we have used the fact that both $\hat{u}(h)$ and $\Phi_F$ are multiplication operators. Furthermore, $\hat{u}(h)$ acts only on the input digits and therefore it commutes with the operator $F_Y w_Y$, which only affects on the output digits.

Now the Fourier transform $F_Y$ on the output digits maps the vector $w_Y|h\rangle = |h, 0_Y\rangle$ to

$$F_Y w_Y |h\rangle = d^{\frac{|Y|}{2}} \sum_{g \in Z^Y_d} |h, g\rangle$$  \hspace{1cm} (B6)

and an application of the operator $\Phi_F$ to (B6) yields the expression

$$\Phi_F F_Y w_Y |h\rangle = d^{\frac{|X|}{2}} \sum_{g \in Z^X_d} \Psi_F (|h, g\rangle |g\rangle).$$  \hspace{1cm} (B7)

Acting with the Fourier transform $F_X$ on (B7) and applying the co-isometry $w^*_X$ afterwards gives

$$w^*_X F_X \Phi_F F_Y w_Y |h\rangle = \sum_{(h', g) \in Z^Y_d} \chi(h|h') \Psi_F (|h, g\rangle |g\rangle) \delta(h'|g)$$

$$= \sum_{(h', g) \in Z^Y_d} \Psi_F (|h, g\rangle |g\rangle) = d^{\frac{|X|}{2}} v_Y |h\rangle.$$  \hspace{1cm} (B8)

Here $\delta$ is the indicator function on $Z^X_d$ of the zero element $0_X$. Note that the co-isometry $w^*_X$ maps the vector $|h, g\rangle$ to $|\delta(h)|g\rangle$ for each $h \in Z^X_d$ and $g \in Z^Y_d$.

Finally, the identity (16) follows directly from (B3), (B5), (B8), the definition of the discrete dynamics $\alpha_F$ (1), the preparation of the outputs $P_Y$ (12) and the measurement of the inputs $M_X$ (14):

$$P_Y \circ \alpha_F \circ M_X (a \otimes f)$$

$$= d^{-|X|} \sum_{h \in Z^X_d} \hat{u}(h)^* v^*_F \hat{u}(h) f(h)$$

$$= C_F (a \otimes f)$$  \hspace{1cm} (B9)

$\square$

APPENDIX C: PROOF OF PROPOSITION $\{\text{C.1}\}$

Let $\Gamma$ be a weighted graph with input vertices $X = \{1, \ldots, k\}$ and output vertices $\{k + 1, \ldots, v\}$. By (C1), the cluster state creation operator $u_F$ can be written as

$$u_F = F_v c^r_{v-1} F_{v-2} \cdots c^r_2 F_2 c^r_1 F_1$$  \hspace{1cm} (C1)

where the blocks of controlled shift operations $c^r_l$ are given by (A3). Since we assume that there are no edges between input vertices, for each input vertex $x \in X$, the operator $c^r_x$ is of the form

$$c^r_x = \prod_{y \in Y} c(x,y)^{\Gamma(x,y)}.$$  \hspace{1cm} (C2)

As a consequence $c^r_x$ commutes with all local Fourier transforms $F_{x'}$ on those inputs $x' \in X$ for which $x' \neq x$. Furthermore, the Fourier transform on the inputs $F_X$ commutes with
\[ F_\Gamma, c_{i-1}^\Gamma F_{i-1} \cdots c_k^\Gamma F_{k+1} \cdot (C3) \]

Hence we obtain

\[ F_\Gamma u_\Gamma F^*_\Gamma = F_{(1, \ldots, k)} F_i c_{i-1}^\Gamma F_{i-1} \cdots c_k^\Gamma F_1 F_{(1, \ldots, k)} \]
\[ = F_i c_{i-1}^\Gamma F_{i-1} \cdots c_2^\Gamma F_1 c_1^\Gamma \]
\[ \times F_{(1, \ldots, k)} c_k^\Gamma \cdots c_2^\Gamma \]
\[ = F_i c_{i-1}^\Gamma F_{i-1} \cdots c_k^\Gamma F_1 c_1^\Gamma \]
\[ \times F_{(1, \ldots, k)} c_k^\Gamma \cdots F_2 c_2^\Gamma F_1 c_1^\Gamma \]
\[ = F_i c_{i-1}^\Gamma F_{i-1} \cdots c_k^\Gamma F_1 + \cdots + F_i c_{i-1}^\Gamma F_{i-1} \cdots c_{k+1}^\Gamma F_{k+1} \]
\[ \times F_{(1, \ldots, k)} c_k^\Gamma \cdots F_2 c_2^\Gamma F_1 c_1^\Gamma \cdot (C4) \]

Now we see from (D4) that the operator \( F_\Gamma u_\Gamma F^*_\Gamma \) is a product of \( v + l \) elementary gates, namely a local Fourier transform for each vertex, and a controlled shift gate for each edge. \( \square \)

**APPENDIX D: PROOF OF PROPOSITION V.1**

Given a weighted graph \( \Gamma \) with input vertex by \{0\} and output vertices \{1, \ldots, n\}. According to Proposition III.1 the isometry \( v_\Gamma \) acts on a basis vector \( |h \rangle \), \( h \in \mathbb{Z}_2 \), according to

\[ v_\Gamma |h \rangle = d^2 w_0^\Gamma F_0^\Gamma |h, 0_{(1, \ldots, n)} \rangle \cdot (D1) \]

Making use of the identity \( |A3| \) we find

\[ v_\Gamma |h \rangle = d^2 w_0^\Gamma \prod_{y=k+1}^{k+n} c(0, y)^{\Gamma(0, y)} |h, 0_{(1, \ldots, n)} \rangle \cdot (D2) \]

Here we have used the fact that for any operator \( a \), acting on the outputs \{1, \ldots, n\}, we have \( w_0^\Gamma (1 \otimes a) = aw_0^\Gamma \). Now we compute

\[ d^2 w_0^\Gamma F_0^\Gamma |h, 0_{(1, \ldots, n)} \rangle \]
\[ = \prod_{y=k+1}^{k+n} c(0, y)^{\Gamma(0, y)} |h, 0_{(1, \ldots, n)} \rangle \]
\[ = \sum_{y \in \mathbb{Z}_2} \chi(h|y) \delta(y) |\Gamma(0, 1)h, \ldots, \Gamma(0, n)h \rangle \]
\[ = |\Gamma(0, 1)h, \ldots, \Gamma(0, n)h \rangle \]

which implies by assuming \( \Gamma(0, 1) = 1 \):

\[ d^2 w_0^\Gamma F_0^\Gamma |h, 0_{(1, \ldots, n)} \rangle \]
\[ = \prod_{y=2}^{n} c(1, y)^{\Gamma(0, y)} |h, 0_{(2, \ldots, n)} \rangle \]
\[ = b_0^\Gamma w_{(2, \ldots, n)}^1 |h \rangle \cdot (D4) \]

Here we have introduced the operator

\[ b_0^\Gamma := \prod_{y=2}^{n} c(1, y)^{\Gamma(0, y)} \cdot (D5) \]

As a consequence, the operator

\[ z_\Gamma := F_n c_{(n-1)}^\Gamma F_{n-1} \cdots c_1^\Gamma F_1 b_0^\Gamma \]

fulfills (F7). In particular, \( z_\Gamma \) is a product of \( l + n - 1 \) elementary gates. \( \square \)