Sandpile on uncorrelated site-diluted percolation lattice; from three to two dimensions

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Abstract. The BTW sandpile model is considered on the three dimensional percolation lattice which is tuned by the occupation parameter $p$. Along with the three-dimensional avalanches, we study the avalanches in two-dimensional cross-sections. We use the moment analysis (along with some other methods) to extract the exponents for two separate cases: the lattice at critical percolation ($p = p_c \equiv p_{3D}^c$) and the supercritical one ($p_c < p \leq 1$). Our numerical data is consistent with the conjecture that the three-dimensional avalanches at $p = p_c$ have nearly the same exponents as the regular 2D BTW model. The moment analysis shows that finite size scaling theory is fulfilled, and some hyper-scaling relations hold. The main finding of the paper is the logarithmic dependence of the exponents on $p - p_c$, for which the cut-off exponents $\nu$ change discontinuously from $p = p_c$ to the values for the supercritical case. Moreover we show that there is a singular point $p_0 \approx p_{2D}^c$ ($p_{3D}^c$ and $p_{2D}^c$ being three- and two-dimensional percolation thresholds) for 2D cross-sections, which separate the behaviors to two distinct intervals: $p_{3D}^c \leq p < p_{2D}^c$ which, due to the lack of 2D percolation cluster, has no thermodynamic limit, and $p \geq p_{2D}^c$ which involves the percolated clusters.

Keywords: critical exponents and amplitudes, percolation problems, self-organized criticality
1. Introduction

There is theoretical and experimental interest in the notion of critical phenomena on the fractal lattices in physics. Examples of experimental motivations are the voids of percolating clusters which are filled by (commonly magnetite) nano-particles of a ferromagnetic fluid [1–7]. In the theoretical side the main contribution was made by Gefen, et al [8] in which it was claimed that the critical behavior of the models on the fractal geometries (for which no lower critical dimension can be defined) is tuned by the detail of the topological quantities of the fractal lattice. The cluster fractal dimension, the order of ramification and the connectivity are some examples of these quantities [8]. Examples are the Ising model in a BTW sandpile [9] and the Ising model on the percolation lattices [10–17]. The fluid movement in the porous media is the other important example which can be modeled by invasion percolation [18], or directly by the Darcy’s reservoir model on the percolation lattices [19, 20]. The later application contains the important concept of self-organized criticality on the percolation lattices [19, 21]. Despite of intense research on the topic, there is a little understanding in the literature, specially on the notion of SOC on the site- or bond-diluted systems.

Sandpile model, as a prototype of SOC on the fractal (and other) lattices has some theoretical [21, 22] and empirical [20, 23] attractions. Fortunately the critical behaviors of the sandpile models in three and two dimensions have been vastly studied analytically [24–32] and numerically [20, 21, 33–41]. The exact determination of the exponents of the avalanches of the sandpiles on the imperfect or porous media (which, for example, is expected to have connections with the fluid dynamics in the porous media [20]) is a challenging problem [36], for which a detailed finite-size analysis is required. Along with these issues, the problem of sand grain propagation in a subset of the original system, e.g. the propagation of avalanches in two-dimensional cross-sections of the three-dimensional system, has especial importance [42]. In the theoretical side such an investigation shows which model lives in a $d−1$ dimensional subsystem of a $d$ dimensional system [43].
In this paper we focus on the critical properties of the BTW sandpile model on the three-dimensional site-diluted cubic percolation lattice in terms of the parameter $p$ which tunes the occupation probability of the system. The paper is divided into two distinct parts: In the first part we consider $p$ to be the critical one, i.e. $p = p^{3D}_c$ = percolation threshold for the three-dimensional system. We name this as the critical percolation regime (CPR). In the second part we analyze the case $p_{c} < p \leq 1$ which is named as the supercritical percolation regime (SCPR) (note that it does not imply that the system is not critical). The moment analysis, as a precise tool for extracting the exponents is employed. The resulting exponents are in agreement with the exponents resulting from other methods like the data collapse and direct slope determination. We show that some exponents show logarithmic behaviors with respect to the parameter $p - p_c$ in the case $p > p_c$.

Parallel to the three-dimensional analysis, the avalanches in two-dimensional cross-sections are also investigated and some exponents are obtained which respect to some hyper-scaling relations. The fractal dimension of loops (the exterior perimeter of the connected avalanches) at $p = p_{c}$ is determined to be $D_{F}^{\text{cross-sections}} \approx 1.37$ that is compatible with the fractal dimension of loops of the spin clusters of the Ising model, i.e. $D_{F}^{2D-\text{Ising}} = \frac{11}{8}$ [44]. Also this result along with some other exponents show the similarity of 2D cross-sections in the present paper and the sandpile on two-dimensional site-diluted square percolation lattice [21]. In the SCPR we observe a singular behavior, having its root in the non-percolating character of 2D percolation systems for $p^{3D}_c < p < p^{2D}_c$, in which $p_{c}^{3D}$ and $p^{2D}_c$ stand for the percolation thresholds in 3D and 2D systems.

The paper has been organized as follows: In the section 2 we introduce the problem and the motivation. Section 3 has been devoted to the statistical properties of the model in the critical regime. The SCPR exponents have been analyzed in section 4 with two subsections: three and two dimensional systems. We end the paper by a conclusion in section 5.

2. Motivation and model definition

Let us first introduce the ordinary BTW model on the regular cubic lattice which defines the dynamics of sand grains. The sand grains are distributed randomly through the lattice, so that we have a local height field $h$ over the lattice, for which the constraint is that no site has a height higher than $2d$ ($d$ = spatial dimension of the system which is three in this paper), i.e. $h(i)$ takes the numbers from the set $\{1, 2, ..., 2d\}$ for each site $i$. The system is open, i.e. adding or losing grains is allowed. The dynamic of the system is as follows: A random site $(i)$ is chosen and a grain is added to this site, i.e. $h(i) \rightarrow h(i) + 1$. If the resulting height is lower than a critical value ($h(i) \leq h_{c} = 2d$), another site is chosen for adding the grain. But if this height exceeds the critical value ($h(i) > h_{c}$), then this site becomes unstable and topples. During this toppling, the height of the original site $i$ is lowered by a number equal to its neighbors ($h(i) \rightarrow h(i) - 2d$) and the content of each of its neighbors is increased by one in such a way that the total number of grains is conserved. The single toppling process can be expressed via the relation $h(i) \rightarrow h(i) - \Delta_{ij}$ in which
\[ \Delta_{i,j} = \begin{cases} 
-1 & \text{if } i \text{ and } j \text{ are neighbors} \\
2d & \text{if } i = j \\
0 & \text{otherwise} 
\end{cases} \] (1)

As a result of this toppling, the neighboring sites may become unstable and topple. This process continues until reaching the state in which all sites of the system become stable. An avalanche is defined as the chain of activity that is triggered by adding a grain to a stable state until another stable state is reached. Now another site is chosen for injection and the process continues. The sand grains can leave the system via the boundaries in which, for a single toppling, one or two sand grains (depending on the position of the boundary site) can dissipate. Generally we have two kinds of configurations: transient and recurrent ones. The transient configurations may happen once in the early evolution steps and shall not happen again and the recurrent configurations take place in the steady state of the system. In this state of the system, the grain input and output rates are statistically equal and the statistical observables are statistically constant in time. All of the configurations in this state occur with the same probability. For a good review see [26].

Now we turn to the main problem of the present paper, i.e. the BTW model on the site-diluted cubic percolation lattice. A percolation lattice is constructed simply by the following rule: each lattice site is occupied (active) by the probability \( p \) and is un-occupied (inactive) by the probability \( 1 - p \). There is a critical occupation probability (percolation threshold) \( p_c \) such that for \( p \geq p_c \) there are some connected clusters (involving the set of sites of the same type) which percolates, namely percolated or spanning percolation clusters. For the cubic lattice the critical threshold is nearly \( p_{c3D} \approx 0.32 \) for site-percolation. We show the \( p = p_c \) case by the critical percolation system. For other occupations \( (p_c < p \leq 1) \) we use the phrase supercritical percolation regime. At each \( p \) a single spanning percolation cluster is chosen as the host for the sandpile model. For the simulation of the grain propagation in the spanning percolation clusters we use the simple rules of the BTW model as stated above, except that the sand grains cannot enter the un-occupied sites, i.e. when an unstable site has \( z \) occupied neighbors, during a local toppling its grain content decreases by \( z \) and each of its activate neighbors increases by one. Also the sand grains have the chance to leave the system via the boundary sites of the spanning percolation cluster. To realize the statistical properties of the metric (percolation) space, we perform the simulations on many spanning percolation clusters, i.e. we change the spanning clusters (with the same \( p \)) after extracting some avalanches. When a new percolation cluster with corresponding \( p \) is constructed, we wait until the new cluster reaches its steady state. Therefore our statistical analysis contains averaging over avalanches in the steady state of the BTW, and different realizations of the spanning cluster.

The problem of two-dimensional propagation of sand grains (energy) in three dimensional systems seems to be very important from the theoretical (and sometimes experimental) side. More precisely the important question in the theoretical physics is that how the information in \( d+1 \) dimensions would be reflected to its \( d \) dimensional subsystem. For this purpose one should map the original \( d+1 \) dimensional model to a \( d \)-dimensional one and measure how some information are lost and how the degrees of freedom of the subtracted dimension affect the \( d \)-dimensional model, i.e. which model...
lives in the lower dimensional system. If the subtracted dimension be temporal, then one is looking at a \textit{frozen} model with no dynamics. The investigation of the contour lines of statistical systems \cite{41} and the ground state of the quantum systems \cite{45} are some examples. A more interesting situation is the case in which the subtracted dimension is spatial one. As an example we have earlier proposed that the cross sections of three-dimensional BTW model share some critical behaviors with the 2D Ising model \cite{42, 43}. In this paper along with the three-dimensional analysis, we study the avalanches in two-dimensional slices, i.e. cross-sections of the three-dimensional system.

A schematic graph is presented in figure 1(a) showing the total set up of a percolated host cluster with a three-dimensional avalanche and its cross-section. Also a real two-dimensional (2D) sample is presented in the figure 1(b), to visualize how the height-field varies over the 2D sample. In figure 1(a), a spanning percolation cluster along the Z axis has been shown with an avalanche which has been shown by a red area. A X–Z plate has been sketched in this figure, showing the mentioned cross-sections which passes from the center of mass of the host cluster. In the figure 1(b) the set of color sites show the cross section of the percolated cluster. The light yellow sites are not contained in the avalanche and the other colors show various connected components of an avalanche cluster in that cross-section.

In the critical state one expects a power-law behavior for the local and geometrical quantities. For example for the ordinary BTW model the distribution functions behave like $P(x) \sim x^{-\tau_x}$ ($x =$ the statistical quantities in three- and two-dimensional systems). The estimation of these exponents is a challenging problem, for which a detailed finite-size analysis is required. For finite systems, the finite-size scaling (FSS) theory predicts that \cite{33}:

$$P_x(x, L) = L^{-\beta_x} g_x(xL^{-\nu_x}),$$

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Figure 1. (a). A schematic set-up of the problem. The 2D cross-section has been shown right at $y = \frac{L}{3}$. The resulting 2D avalanche (in the cross-section) can apparently be non-simply-connected (this is true also for 3D avalanches in the presence of non-active sites, but for the cross-sections it can be due to the distortions of the 3D avalanche). (b) A sample of 2D height configuration in a cross-section.
in which \( g_x \) is a universal function and \( \beta_x \) and \( \nu_x \) are some exponents that are related by \( \tau_x = \frac{\beta_x}{\nu_x} \). In the next sections we test if the avalanche distribution satisfies the above scaling form. The exponent \( \nu_x \) determines the cutoff behavior of the probability distribution function. If FSS works, all distributions \( P_x(x, L) \) for various system sizes have to collapse, including their cutoffs. One can simply show that \( r_{\text{cutoff}} \sim L \), i.e. the cutoff radius should scale linearly with the system size \( L (\nu_x = 1) \), so that for all observables one gets \( \nu_x = \gamma_{x} \) [33].

The relation (2) is only correct for mono-fractal systems. To investigate the monofractality and multi-fractality, we use the method of moment analysis presented in [46]. To this end, we should calculate the \( q \)th moment of the \( x \) variable

\[
\langle x^q \rangle = \int P_x(x, L) x^q dx \sim L^{\sigma_x(q)},
\]

(3)

in which \( \sigma_x(q) = \nu_x (q - \tau_x + 1) \) for mono-fractal systems. It is seen that for monofractal systems \( \sigma_x(q) \) has the linear behavior in terms of \( q \), i.e. \( \sigma_x(q + 1) - \sigma_x(q) = \nu_x \). It is a serious test for monofractality and multi-fractality of the system, which estimates the exponents. In the following sections we use this analysis. Also we note that there is a hyper-scaling relation between the \( \tau \) exponents and the fractal dimensions \( \gamma_{x,y} \), which are defined by the relation \( x \sim y^{\gamma_{x,y}} \), namely:

\[
\gamma_{x,y} = \frac{\tau_y - 1}{\tau_x - 1}.
\]

(4)

This relation is valid only when the conditional probability function \( p(x|y) \) is a function with a very narrow peak for both \( x \) and \( y \) variables. Despite of high precision of the moment analysis method, the fact that the large events dominate the summations in this method, causes some noises in the results. Due to this, we have used the data collapse technique according to the scaling relation (2), and also the exponents have directly been estimated by measuring the slopes of the log–log plots (up to cut-off values).

3. Critical percolation lattice; \( p = p_c \)

In this section we focus on the case \( p = p_c \). In [20, 21] it has been numerically shown that for 2D BTW on the uncorrelated percolation lattice, the fractal dimension of loops in the supercritical case \( p > p_c \) becomes identical to the one for the \( p = 1 \) case, i.e. \( D_F(p) \to \frac{5}{4} = D_2^{\text{BTW}}(p = 1) \) in the limit \( L \to \infty \), and only \( p = p_c \) has different exponents (compatible with the 2D critical Ising model). Based on this result and some other results (the distribution functions) it was speculated that the (quenched) disorder is irrelevant in the 2D BTW model. Therefore the case \( p = p_c \) is of special importance in such an analysis. Note that in the percolation theory, in the supercritical region \( (p > p_c) \) there is, almost surely, a single spanning cluster which is a compact cluster away from criticality with dimension 2 (in the scaling limit).
We have two types of quantities in 3D systems and their 2D cross sections: the fractal dimensions and the distribution functions for the statistical observables. The three-dimensional quantities studied in this paper are as follows:

- The avalanche mass \( M_3 \) which is the total number of sites involved in a three-dimensional avalanche.
- The three-dimensional gyration radius \( R_3 \) which is defined by
  \[
  R_3^2 \equiv \frac{1}{M_3} \sum_{i=1}^{M_3} (\vec{r}_i - \vec{r}_{\text{com}})^2,\]
  i.e. it is the gyration radius of points involved in a three-dimensional avalanche. In this formula \( \vec{r}_i \equiv (x_i, y_i, z_i) \) is the position vector of the \( i \)th point of the avalanche and \( \vec{r}_{\text{com}} \equiv (x_{\text{com}}, y_{\text{com}}, z_{\text{com}}) \) satisfies the relation
  \[
  \frac{1}{M_3} \sum_{i=1}^{M_3} \vec{r}_i = \vec{r}_{\text{com}},
  \]
  i.e. it is the center of mass of the avalanche.
- The number of topplings in a three-dimensional avalanche (avalanche size) \( s_3 \).

Let us first consider the 3D avalanches. The numerical analysis right at \( p = p_c \) is hard and bothersome, since the typical time needed to reach a stable configuration is related inversely to the number of the boundary sites which is very low in \( p = p_c \). On the other hand due to the diluteness of sites the samples are commonly small, the fact which affects the quality of the results. The way out of these deficiencies is to firstly choose the percolated cluster with larger boundary sites, and secondly increase the number of samples, so that some (rare) large avalanches have the chance to appear. In this work we have generated over \( 5 \times 10^8 \) samples for each \( L \) and \( p \). The simulation on each SPC \( p \) (spanning percolation cluster tuned by the occupation parameter \( p \)) has been started with a random \( \tilde{h} \) configuration and the statistical analysis has been carried out in the steady states. The lattice sizes considered in this work are \( L = 64, 128, 256 \) and \( 500 \). After extracting \( 10^5 \) avalanche samples, another percolated cluster is chosen and so on. The SPC’s have been chosen so that the fraction of N.O. boundary sites is not that small. For the distribution functions there are some cut values above which the linear behavior of the log–log graph is destroyed and the graph falls off rapidly. In this scale the finite size plays the dominant role. This cut-off for the observable \( x \) scales with \( L \) by the relation \( x^{\text{cut}} \sim L^{\nu_x} \) in which \( \nu_x \) has been defined in the previous section. This exponent can be obtained by means of moment analysis. We have observed that all of the fractal dimensions at \( p = p_c \) (three and two dimensional systems) scale with \( L \) by the relation:

\[
\gamma_{xy}^{p_c}(L \to \infty) - \gamma_{xy}(L)^{p_c} \propto \frac{1}{L},
\]

in which the quantities \( \gamma_{xy}(L \to \infty)^{p_c} \) are of interest. The \( \tau \) exponents however saturate in some final value \( \tau_x(L \to L_{\text{max}})^{p=p_c} \). This should be compared with the one for two dimensional regular BTW model, i.e. \( \tau_x(L)_{d=2}^{\text{regular BTW}} = \tau_{x,\infty} - \frac{\text{const.}}{\ln(L)} \) [37].

In the figure 2(a) we have shown the distribution function of the three-dimensional gyration radius \( P(R_3) \) for various rates of the lattice sizes \( L \). As can be seen in the inset of figure 2(a), \( \tau_{R_3} \) saturates at final value \( 1.5 \pm 0.1 \). The results of the moment analysis have been shown in figures 2(c) and (d). The linearity of the log–log plot of \( \langle M_3^q \rangle \) in terms of \( L \) is evident in the figure 2(c) whose slopes increase as \( q \) increases. The corresponding exponent \( \sigma_{M_3}(q) \) and \( \sigma_{R_3}(q) \) has been shown in

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The exponents of the critical case \( p = p_c \) for the 3D system. (a) The distribution function of the three-dimensional gyration radius \( R_3 \). Inset: the finite size dependence of \( \tau_{R_3} \). (b) The finite-size dependence of the fractal dimension \( \gamma_{M_3R_3} \) defined as \( M_3 \sim R_3^{\gamma_{M_3R_3}} \). (c) The \( q \)th moment of \( M_3 \) in terms of \( L \) for various rates of \( q \). (d) The exponent \( \sigma_{M_3}(q) \) and \( \sigma_{R_3}(q) \) in terms of \( q \). When the graph is fitted by \( \sigma_x = \nu_x q + b_x \), then the corresponding \( \tau \) exponent is obtained via \( \tau_x = 1 + \frac{b_x}{\nu_x} \).

figure 2(d), from which one finds \( \nu_{M_3} = 3.32(4) \), \( \nu_{R_3} = 1.12(4) \), \( \tau_{M_3} = 1.18(4) \) and \( \tau_{R_3} = 1.49(4) \). The fact that \( \nu_{R_3} \) is nearly unity is expected for the critical systems (see above comments). It is seen that \( \nu_{M_3} \) is consistent with \( \nu_{M_3}^{p=1} = 3.00(2) \). We have found that \( \nu_{3} = 3.11(3) \simeq \nu_{M_3} \), just like the regular lattice result according to which \( \nu_{p=1}^{M_3} = \nu_{p=1}^{M_3} \), although the values are different [33]. The full information of the exponents \( \tau \) and \( \nu \) have been reported in the table 1 for \( M_3, R_3 \) and \( s_3 \). In the first two rows the analytical and calculated values have been reported for \( p = 1 \) which show a complete agreement. The results of the moment analysis as well as the maximum lattice size have been shown in the last three rows which show agreement. It is seen that the exponents for \( s_3(p = p_c) \) and \( M_3(p = p_c) \) are not the same in contrast to the case \( p = 1 \), having its root in the fact that the number of topplings of a typical site in an avalanche is larger than unity for \( p = p_c \). Note that for the regular lattice in three and four dimensions these two quantities are nearly the same, showing that the probability that a site topples more than one is very
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Table 1. The exponents τ and ν of the distribution functions of $M_3$, $R_3$ and $s_3$ in 3D at the critical regime, i.e. $p = p_c$. The τ exponents saturate at $\tau_x(L = L_{\text{max}})^{p_c}$ which has been reported in the last row. $\tau_x^{p=1,d=3}$ (theoretical) and $\tau_x^{p=1,d=3}$ (calculated) and $\tau_x^{\text{regular BTW}}(d = 2)$ have also been reported for comparison. The results of moment analysis, as well as the maximum lattice size have been shown.

| $x$                      | $M_3$    | $R_3$    | $s_3$    |
|--------------------------|----------|----------|----------|
| $\tau_x^{p=1,d=3}$      | 4/3      | 2        | $\approx 4/3$ |
| $\tau_x^{p=1,d=3}$      | 1.33(3)  | 1.98(3)  | 1.34(3)  |
| $\tau_x^{\text{regular BTW}}(d = 2)$ | 1.25     | 1.59     | 1.25     |
| $\mu^{p=p_c,d=3}$       | 3.32(4)  | 1.12(4)  | 3.11(3)  |
| $\mu^{p=p_c,d=3}$       | 1.18(4)  | 1.49(4)  | 1.02(3)  |
| $\tau(L = 500)^{p=p_c,d=3}$ | 1.23(3)  | 1.50(4)  | 1.05(3)  |

low. Although the resulting exponents differ significantly from the case $p = 1$, there are interestingly some similarities with the regular two-dimensional avalanches of the BTW model which has been listed in the third row of table 1 (from the reference [37]). Note that the difference is seen only for the last one, i.e. $\tau_{s_3}$.

The fractal dimension $\gamma_{M_3 R_3}$ has been shown in figure 2(b) for various lattice sizes. We see that this exponent extrapolates to 2.78(8) as $L \to \infty$. This can be interpreted as the effect of empty (un-occupied) sites which lead this exponent to differ from the result for $p = 1$, i.e. $\gamma_{M_3 R_3}^{p=1} \approx 3$. Observe its similarity to the mass-radius exponent of spanning percolation cluster $\gamma_{M_3 R_3}^{\text{percolation}} \approx 2.52$ [47].

Let us clarify the method of analysis of two-dimensional cross sections. Consider a 3D avalanche starting at a randomly chosen site. We consider the section of avalanche at the plane $z = z_0$ (which passes through the center of mass of the spanning percolation cluster). The avalanche cluster on this plane has one or more disconnected clusters. Let $n(s)$ be the average number of clusters having exactly $s$ sites on this plane. Then, $n(s)$ has the scaling form $n(s) \sim s^{-\nu}$. Also for each connected element of 2D avalanche there is an exterior frontier which is a loop with length $l$ containing the avalanche.

The quantities which are analyzed in the cross-sections are the following:

- The mass of 2D avalanches $M_2$ which is the total number of sites involved in a connected component of 2D cross-section of an avalanche. Note that different 2D clusters are given equal weight in averaging.
- The loop lengths $l$ which is the length of the loop that is the external perimeter of a connected component of 2D cross-section of an avalanche.
- The area inside loops $a$ which is the total area that is contained in the loop which was defined above.
- The gyration radius of loops ($r$) and 2D mass $R_2$ for the cross-sections.
- The number of topplings in the cross-section avalanche $s_2$. $s_2$ is the 2D avalanche size, i.e. the total number of topplings in 2D avalanches and $n_{\text{sites}}^{2D}$ is the number of toppled sites (not necessarily connected component) of an avalanche.

The obtained exponents for two-dimensional cross-sections of avalanches are more interesting and have been reported in table 2. The exponents have been obtained, using
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Table 2. The exponents of the distribution functions of $s_2$, $n_{\text{sites}}$, $M_2$, $r$, $R_2$ and $a$ in two-dimensional cross-sections at the critical regime, i.e. $p = p_c$. The exponents have been calculated via the moment analysis. For $\tau^\text{Ising}$ the [44] has been used.

| $x$   | $s_2$  | $n_{\text{sites}}$ | $M_2$  | $r$   | $R_2$  | $a$  |
|-------|--------|--------------------|--------|-------|--------|------|
| $\tau_{\text{moment anal.}}$ | 1.46(9) | 1.32(9)            | 3.00(9) | 1.4(8) | 3.00(9) | 2.39(9) |
| $\tau^\text{Ising}$       | —      | —                  | 2.31   | —     | 3.4    | 2.75 |

Table 3. The fractal dimensions $\gamma_{M_3 R_3}$, $\gamma_{M_2 R_2}$, $\gamma_{l_r}$ and $\gamma_{l_a}$ at the critical regime, i.e. $p = p_c$. The observed finite-size relation is $\gamma(L \to \infty)_{\text{xy}}^p - \gamma(L)_{\text{xy}}^p \propto \frac{\beta_{\text{xy}}}{p_c^\text{xy}} \frac{L^\delta - 1}{L^\delta - 1}$ has also been reported for testing the hyper-scaling relation. The same exponents for the Ising model have been shown for comparison, after [44].

| $(x, y)$ | $(M_3, R_3)_{d=3}$ | $(M_2, R_2)_{\text{cross-section}}$ | $(l, r)_{\text{cross-section}}$ | $(l, a)_{\text{cross-section}}$ |
|----------|-------------------|------------------------------------|-------------------------------|-------------------------------|
| $\tau_{L \to \infty}^{M_3 R_3}$ | 2.5(2)            | 1.1(2)                             | 0.9(1)                        | 0.7(1)                        |
| $\gamma_{L \to \infty}^{p_{M_3}}$ | 2.8(1)            | 1.2(1)                             | 1.37(5)                       | 0.87(3)                       |
| $\gamma_{2D \text{ Ising}}^{M_2 R_2}$ | —                 | —                                  | 1.375(5)                      | —                             |

the moment analysis. The exponent $n_{\text{sites}}$ is compatible with the same exponent of the regular two-dimensional avalanches of the BTW model, i.e. $\tau_{n_{\text{sites}}} (p = p_c) \approx 1.32$. For comparison, the exponents of the Ising model have been shown in the last row of the table 2. The corresponding fractal dimensions have also been calculated which have been shown in the table 3. $D_F \equiv \gamma_{l_r}$ is interpreted as the most important exponent from which the universality classes of the 2D critical models can be obtained. The fractal dimension of loops for the 2D cross-section avalanches (2DCSA) is $D_F^{\text{2DCSA}} (p = p_c) = 1.37 \pm 0.05$ which is compatible with the fractal dimension of the external perimeter of the spin clusters of the 2D critical Ising model, i.e. $D_F^{\text{Ising}} = \frac{11}{8}$ [10]. It is seen that the fractal dimension $\gamma_{M_3 R_3}$ is different substantially from the one for $p = 1$ case ($\gamma_{M_3 R_3} \approx 1.2$, whereas $\gamma_{p=1 M_3 R_3} \approx 2$).

To be more precise about identification of the (3D and 2D) model, we should study its properties out of the percolation threshold, i.e. $p_c < p \leq 1$ which is the subject of the next section.

4. Super-critical percolation region

In this section we observe how things change in the SCPR, i.e. $p_c < p \leq 1$. The quantities to be investigated are the same as the previous section. In the $p_c < p \leq 1$ regime, just like the case $p = p_c$, the critical behaviors are seen with varying exponents to be reported in the next two sub-sections. The results of this part supports the hypothesis that the exponents change logarithmically with respect to $x \equiv p - p_c$ in the SCPR.

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4.1. Three dimensions

For calculating the exponents of the distribution functions of the three-dimensional quantities, along with the direct determining the slopes, we have used the moment analysis for all amounts of $p$. The full information of the graphs have been gathered in figures 3 and 4. In the figure 3(a) we have shown the distribution function of the 3D mass for $L = 500$. It is seen that the slopes change smoothly from $\tau_{M_3}(p_c)$ to $\tau_{M_3}(p = 1)$. By the moment analysis, in addition to extracting the exponents, we can calculate the $\nu$ exponent which controls the cut value (e.g. $R_3^{\text{cut}}$ for $R_3$). In figure 3(b) we have shown $\sigma_{M_3}(q)$ and $\tau_{M_3}$ in its inset. Our results reveal that all of the $\nu$ exponents fall off rapidly from its value in $p = p_c$ to that of $p > p_c$ which is nearly constant (see figures 3(b), (d) and (f)).

As mentioned above, the important feature of the results for 3D is that no cross-over between two regimes, i.e. $UV$ (small-scale) regime and $IR$ (large-scale) regime, is seen. Instead $\tau_x$ varies linearly with the logarithm of $p - p_c$, i.e. $\tau_x(p) = \zeta_x \ln(p - p_c) + \xi_x$ (note that apparently this relation is not valid for $p$ very close to $p_c$). The same is true for $s_3$ in the figure 3(c). This behavior is seen for all lattice sizes considered in this paper with $L$-dependent $\zeta_x$ and $\xi_x$. The resulting coefficients although do not have a clean scaling behavior in terms of $1/L$ or $1/\ln L$, but saturate properly for $L \gtrsim 256$. Therefore in the table 4 we have reported the results for $L = 500$. The same behavior is seen for $\gamma_{M_3 R_3}$ for which a logarithmic behavior in terms of $p - p_c$ is seen (the inset of figure 4(a)). The finite size dependence of the $\gamma_{M_3 R_3}$ for various occupation numbers ($p$) has been shown in figure 4(b) from which we see that the exponents become nearly saturated for large sizes. It is worth mentioning that the hyper-scaling relation $\gamma_{M_3, R_3} \equiv \frac{\tau_{R_3}^{-1}}{\tau_{M_3}^{-1}}$ is violated for $p_c < p < 1$ and is restored right at $p = p_c$ and $p = 1$.

The sole result of this subsection is the hypothesis that the change of exponents in the supercritical region is logarithmic.

4.2. Two dimensions; cross-section statistics

Now let us turn to the two-dimensional problem in the $p_c < p \leq 1$ regime. The two dimensional problem is interesting since a singular behavior is seen in some occupation number. The BTW model on the two-dimensional site-diluted square lattice has been studied in some other works [20, 21]. This model was shown to have relations with the Darcy model of fluid propagation in porous media [20]. An interesting feature of this study [20] is the point that there is a special amount of occupation probability $p_0$ for which the spanning avalanche probability is maximum. This quantity is in the vicinity of $p = 0.6$. Interestingly we see something like this in the 2D induced model, i.e. the model living in the 2D cross-sections. For most considered quantities, we observed a singular behavior (or a change of behavior) around $p_0 \in (0.5, 0.6)$. To show this effect we have sketched the log–log plot of the distribution function of the 2D avalanche mass $P(M_2)$ in the figure 5(a) for various rates of $p$. In its inset we have shown the quantity $\tau_{M_2}$ in terms of $p$. For small value of $p - p_c$ the exponent is nearly constant up to $p = 0.4$ at which the exponent falls rapidly to another constant which finally saturates at $p = 1$. This behavior occurs also for the most observables considered in this work. Consider for example the $\nu$ exponent in the inset of figure 5(b) in which a cross-over is
seen from small to large values in terms of $p$. As another example let us consider the fractal dimension $\gamma_{lr}$ which has been presented in figure 5(c), whose inset represents the exponent for various lattice sizes. The mentioned singular behavior is seen in a sharp peak at $p = 0.6$.

Figure 3. The histogram plot of (a) $M_3$, (c) $s_3$, and (e) $R_3$ in terms of $p$ for $L = 500$, along with their moment analysis (b) $\sigma_{M_3}$, (d) $\sigma_{s_3}$, and (f) $\sigma_{R_3}$. The logarithmically-$p$-dependent $\tau$ exponents have been shown in the insets of (a), (c) and (e) graphs ($x \equiv p - p_c$), whereas the $p$-dependent $\nu$ exponents have been shown in the insets of (b), (d) and (f) graphs.
An explanation of this observation is the fact that the 2D cross section of a 3D uncorrelated percolation lattice is a real 2D percolation lattice. For a 2D square percolation lattice we know that $p_{\text{2D}}^c = 0.5927$ in which a percolation transition occurs. For the cross sections in the interval $p_{\text{3D}}^c \leq p < p_{\text{2D}}^c$ there exist, almost surely, finite disjoint clusters which cannot be spanning. So, any model on such a lattice can be simply decomposed onto a number (=the number of disjoint clusters) isolated systems on finite clusters. We note that all clusters in this sub-critical regime are finite and the concept of the criticality does not make sense at all, remembering that the critical behavior is always investigated in the thermodynamic limit where the system size goes to infinity. However for the scales much lower than $L$ (≡ the linear size of the finite (non-spanning) cluster) the power-law behaviors of the models are restored, i.e. if we restrict ourselves to small scales (compared to $L$) the power-law behaviors are seen. It defines a phase with non-extended power-law behaviors which occurs in $p_{\text{3D}}^c \leq p < p_{\text{2D}}^c$. In the other phase $p \geq p_{\text{2D}}^c$, the existence of 2D spanning clusters change the thermodynamic description of the system, i.e. there exists the $L \to \infty$ limit which we consider as the other phase. We have numerically shown that the behaviors of these two phases are different. This is responsible for the observed singularity at $p = p_{\text{2D}}^c$. This behavior can also be seen in the inset figure 5(d), in which $\nu_1$ crosses over from small $p$’s to the large ones, and also the inset of figure 5(f) in which $\gamma_{M_2 R_3}$ do such a cross over.

Based on the results, one may claim that there are two separate behaviors at least for the model on the cross-sections, i.e. $(p_c, p_0)$ and $(p_0, 1)$ ($p_0 \approx p_{\text{2D}}^c$) each of which has its own critical behaviors. The fact that $p_0$ is $L$-dependent or not cannot be deduced from the lattice sizes considered in this work.

**Table 4.** The coefficients of the relation $\tau_x(p) = \zeta_x \ln(p - p_c) + \xi_x$ for the exponents of the distribution functions in three dimensions.

| $x$   | $M_3$  | $s_3$  | $R_3$  |
|-------|--------|--------|--------|
| $\zeta_x$ | 0.16(4) | 0.15(4) | 0.54(5) |
| $\xi_x$  | 1.43(5) | 1.41(5) | 2.27(5) |

Figure 4. (a) The log-log plot of $M_3$ in terms of $R_3$ with the corresponding $\gamma_{M_3 R_3}$ in its inset, which changes logarithmically with $x = p - p_c$. (b) Finite-size dependence of $\gamma_{M_3 R_3}$ for various rates of $p$. 

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Figure 5. (a) The histogram plot of $M_2$ (for cross-sections) in terms of $p$ with the exponents (insets) for $L = 500$. (b) $\sigma_{M_2}$ in terms of $p$ with its corresponding $\nu$ exponent in the inset. (c) The fractal dimension $\gamma_{l_r}$ in terms of $p$. Inset shows the behavior of $\gamma_{l_r}$ in terms of $p$ which shows a maximum at $p = p_0 \approx 0.6$. (d) $\sigma_{l}$ in terms of $p$ with its corresponding $\nu$ exponent in the inset. (e) Finite-size dependence of $\gamma_{M_2R_2}$ for various rates of $p$. (f) The log–log plot of $M_2$ in terms of $R_2$ for various amounts of $p$ for $L = 500$. Inset shows the $p$ dependence of $\gamma_{M_2R_2}$.
In the figure 5(f) it is revealed that by increasing $L$ the $\gamma_{M_2 R_2}$ exponent becomes more similar to $\gamma_{M_1 R_2}^{-1}$. This shows that the $p = 1$ properties of the system becomes dominant for $L \to \infty$ which is expected from the general properties of the percolation model. The finite size dependence of this exponent has been shown in figure 5(e), which monitors the approaching of the exponents to the scaling limit. The figure shows that all exponents are nearly saturated for $L \gtrsim 256$ as stated above.

We conclude that although the $p$-dependence of the exponents in 3D is logarithmic, in 2D cross-sections the dependence is not that simple and has some features, e.g. there is a singular behavior at $p = p_0$.

5. Conclusion

In this paper we have considered the three-dimensional BTW model on the uncorrelated site-diluted cubic percolation lattice which is tuned by the occupation probability $p$. Along with the original lattice, we have also considered the two-dimensional cross-sections of the system which crosses the center of mass of the spanning percolation cluster. Our motivation for this has been to investigate the effect of the subtracted dimension in the lower dimensional ($d - 1 = 2$) system, which we name the 2D induced model. We have studied two separate cases: the critical percolation ($p = p_c$) and supercritical ($p_c < p \lesssim 1$) regimes. The fractal dimensions and the distribution functions of various statistical observables have been studied via the moment analysis. For the critical case the finite-size scaling hypothesis were shown to be held, from which some exponents were estimated that have similarities with 2D BTW model, whereas the exponents of the quantities in 2D cross-sections (which satisfy some hyper-scaling relations) have similarities with the 2D Ising universality class. For the supercritical case in three dimensions we have observed that the exponents change logarithmically with $p - p_c$ violating the hyper-scaling relations obtained for the critical case. For the 2D induced model in the supercritical regime we showed that there is a $p$ value ($p_0 \in (0.5, 0.6)$) at which the behavior of the system changes. This is reminiscent of the previously observed occupation number at which the percolation probability becomes maximum in the BTW model on the 2D site-diluted percolation lattice [21]. We explained this observation by the fact that for $p_0^{\text{3D}} < p < p_c^{\text{2D}}$ due to lack of spanning percolation clusters, the cross-sections do not have the thermodynamic limit, whereas for $p \geq p_c^{\text{2D}}$ the 2D spanning clusters exist. This affects the properties of the system in the two mentioned intervals.

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