Secondary mathematics teachers learning to do and teach mathematical modeling: a trajectory

Rose Mary Zbiek1 · Susan A. Peters2 · Benjamin Galluzzo3 · Stephanie J. White2

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Abstract
This study explores secondary mathematics teachers’ perceptions of the experiences that contributed to their capacities to understand mathematical modeling and to facilitate students’ modeling experiences. The retrospective research methods and transformative learning theory frame used in the study honor teachers as adult learners and value their perspectives while providing a way to study the complexity of learning to model and to teach modeling. Data analysis identified triggers and knowledge dilemmas that challenged and prompted teacher learning as well as opportunities to resolve dilemmas through rational discourse and critical reflection. Patterns in teacher-identified meaningful learning experiences reveal a trajectory with strands that address aspects of doing and teaching mathematical modeling: mathematics, social aspects of learning, real-world contexts, student thinking, and curriculum. Results of this study provide a holistic view of learning to do and teach mathematical modeling, complementing studies of designed professional learning interventions that out of necessity target specific parts of the modeling process. The results both support and challenge common teacher education content and practices. The study illustrates the usefulness of retrospective methods to understand teachers as lifelong learners.

Keywords Mathematical modeling · Professional learning · Transformative learning theory · Retrospective methods

Rose Mary Zbiek
rmz101@psu.edu
Susan A. Peters
s.peters@louisville.edu
Benjamin Galluzzo
bgalluzz@clarkson.edu
Stephanie J. White
stephanie.white.1@louisville.edu
1 The Pennsylvania State University, University Park, PA, USA
2 University of Louisville, Louisville, KY, USA
3 Clarkson University, Potsdam, NY, USA

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Introduction

Mathematical modeling (MM) is challenging to do and to teach. Teachers who teach MM have had various MM learning and teaching experiences in school, university, teacher education, and professional development (PD) experiences. Some of the experiences might have addressed MM explicitly; others might have contributed implicitly to understanding MM learning and teaching. In this study, we explore teachers’ perceptions of the experiences that contributed to their capacities to understand MM and to facilitate students’ MM experiences. This work values teachers’ voice and considers professional learning as long-term, evolving professional work of teachers, complementing studies of the effectiveness of planned PD and teacher preparation MM experiences. By focusing on in-service teachers, the study yields a trajectory of professional learning that transcends current literature, which attends primarily to pre-service teachers.

Literature

We first explicate what MM is and then look to the literature to discuss competence in doing and teaching MM.

What is MM?

Literature in MM includes theoretical discussions of what MM is (e.g., Kaiser & Sriraman, 2006) and offers numerous definitions of MM (Cirillo et al., 2016). Scholars (e.g., Abassian et al., 2020; Kaiser & Sriraman, 2006; Treffert-Thomas et al., 2017) have categorized perspectives on MM, such as educational modeling (related to didactical and conceptual goals and schooling) or socio-critical goals (related to a critical understanding of the world). Common to all views is the idea of movement between a real world and mathematics. To capture this primary feature of MM, we view MM as “a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena” (Garfunkel & Montgomery, 2019, p. 8). Developing insights into complex real-world phenomena requires knowledge of the phenomena and knowledge of mathematics to mathematically model the phenomena. To teach MM is to facilitate the MM activity of students and to help students understand MM as a mathematical practice, a habit of mind, and an inquiry stance on the world.

Competence in MM

To facilitate students in their MM efforts, teachers need to develop competency in MM. Blomhøj and Højgaard Jensen (2007) define competence as “someone’s insightful readiness to act in response to the challenges of a given situation” (p. 47). Maaß (2006) offered a widely accepted description of competency in MM that addresses overall competence in terms of modeling competencies: “Modeling competencies include the skills and abilities to perform modeling processes appropriately and goal-oriented as well as the willingness to put these into action” (p. 117). Maaß and others (e.g., Anhalt et al., 2018; Kaiser & Brand, 2015) categorize MM competencies in terms of phases of the MM cycle.
Simplifying (or structuring) and mathematizing competencies capture the way in which the real-world situation is conceptualized and construed as a problem to solve, respectively. Working mathematically involves using mathematical knowledge and strategies to solve a problem within the mathematical world. Maaß’s other two competency areas are interpreting mathematical results in the real-world situation and validating by verifying a solution or critically reflecting on the assumptions, model, and solution. Teachers might report experiences that address one or more of these areas.

Differentiating among competencies has led to the observation that some competencies are more challenging than others. For example, Galbraith and Stillman (2006) deem understanding the problem context as in simplifying/structuring to be “a major starting hurdle” and mathematizing as “one of the most challenging parts of the modeling cycle” (p. 159). Cai and colleagues. (2015) posed the question of what level of familiarity with disciplines other than mathematics is needed for MM teachers. We anticipated that teachers would find benefit in experiences that enrich their understanding of simplifying and mathematizing, and perhaps experiences that enhance their understanding of other disciplines.

Blomhøj and Højgaard Jensen (2007) described a holistic approach to MM, which engages students in the full modeling process. In contrast, an atomistic approach engages students in only selected aspects of the process at any one time. The question arises: Which approach is better? Kaiser and Brand (2015) concluded “both approaches foster students’ modelling competency in all dimensions and have strengths and weaknesses” (p. 145). We find it useful to consider both approaches and anticipate that the MM-related experiences that teachers deem helpful to their professional growth will foreground one or more MM competencies and perhaps illuminate MM as a holistic process.

Competence in teaching MM

In addition to deep understanding of MM, teachers learn how to successfully facilitate their students’ modeling (e.g., Blum & Borromeo Ferri, 2009; Cai et al., 2014). Decades of studies of PD and research on teacher knowledge exist in several traditional core areas of school mathematics (e.g., algebra, rational number, function, proof) and have been summarized by scholars (e.g., Jacobs & Spangler, 2017; Strutchens et al., 2017; Sztajn et al., 2017). Less attention has been given to how teachers come to know and facilitate MM. Wess and colleagues (2021) offer a summary of professional competence in teaching that builds on the work of Shulman (1986, 1987), including the Teacher Education and Development Study-Mathematics (Tatto et al., 2008) and Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers (Kunter et al., 2013) project. Scholars working from multiple modeling perspectives on learning and teaching modeling stress the value of teachers’ engagement in modeling tasks and attention to student thinking, including Lesh and colleagues (e.g., Koellner-Clark & Lesh, 2003; Schorr & Lesh, 2003) in their work with model eliciting activities and Wubbels and colleagues (1997) in the use of context in Realistic Mathematics Education. Wess and colleagues argue for the uniqueness of professional competence in MM teaching with assumptions that pre-service teachers are trained in understanding of MM competences and student difficulties—a body of literature to which mathematics educators have contributed empirically and theoretically (Borromeo Ferri, 2018).

Most in-service teachers in the United States have not had the benefit of a concentrated program in MM. A shift from familiar problem solving to MM requires “a new set of teaching and learning skills” (Herget & Torres-Skoumal, 2007, p. 385). Some scholars posit...
teaching skills that might be important to teaching in general and especially important in teaching MM. To illustrate how teachers might see general teaching skills as relevant, we next describe pressing on student answers and supporting group work. These teaching skill areas seem especially relevant to in-service teachers faced with teaching MM. For example, in the teacher MM competence test designed by Wess and colleague, teachers need to press on student responses to identify student difficulties and then support student groups to work through those difficulties.

Press on student answers

Some of the practices useful in teaching MM are practices that are difficult in general for teachers to implement. For example, MM teachers need to work with students to evaluate and improve models. As Giordano and Weir (1997) observed, “the objective is not to confirm or deny the model (we already know it is not precisely correct because of the simplifying assumptions we have made), but rather to test its reasonableness [i.e., validity]” (p. 38). Such work requires teachers to press on student answers and engage students in pressing on their own and their classmates’ MM solutions. More specifically, teachers engage students in providing additional information and reasons rather than only the steps needed to solve a problem and in verifying answers or solutions by triangulating strategies that use different representational forms (Jacobs & Spangler, 2017; Kazemi & Stipek, 2001). Pressing on student answers also requires teachers to press on solutions when student reasoning deviates from what teachers or curriculum writers intended, as Czocher (2019) observed.

In doing classroom MM, students and teachers have a special need to press on relatively correct ideas to make them better as well as on ideas rather than answers (e.g., Staples, 2007). Teachers can find it difficult to press on correct answers (Webb et al., 2008). Teachers can also find it difficult to encourage students to address their incorrect answers rather than to immediately pronounce answers wrong or to correct students’ work for them (Schleppenbach et al., 2007). Such observations suggest why engaging students in evaluating and revising models might be a competency area that needs additional support in preparation to teach MM. To help students become competent in assessing and revising models, teachers need to help students press on their assumptions and ideas that underlie the models, which likely include both correct and less correct notions.

Facilitating group work

Teachers support the social context of MM as well as the cognitive side of student MM. Blum (2015) describes group work as especially suitable and effective for modeling due to its social nature and the group as an environment for modelers’ cognitive work. Frejd and Bergsten (2016) underscore how large-scale MM is done in teams. Ikeda and Stephens (2001) provide evidence that group discussion while engaged in MM may help student achievement. MM requires students to take responsibility for their own and others’ processes (Borromeo Ferri, 2018). Borromeo Ferri (2018) provides evidence that groups of students of mixed ages could work together productively despite differences in level of competency. She underscores how MM problems are self-differentiated, which allows a good MM problem to be used with a range of students.

Although students can work together productively on MM problems, some, if not all, students working on MM problems might engage in mathematics that they do not yet understand well enough to use readily in their MM work. Facilitating MM thus requires
teachers to help students develop new mathematical understanding as well as challenge a teacher’s ability to support the social aspect of teamwork. Learning to teach MM seems to require a teacher to develop proficiency in what Baxter and Williams (2010) identify as social scaffolding (supporting students as they learn to work with each other) and analytic scaffolding (supporting students in building personal mathematical knowledge). We anticipated teachers might seek competence in social and analytic scaffolding strategies.

**Professional learning experiences**

Professional learning opportunities in MM for in-service teachers can be found in schools, at professional conferences, and in informal settings. Recent studies suggest that completing a short MM module or a teacher preparation course can offer teachers opportunities to experience and deepen foundational understandings of the MM process (Anhalt & Cortez, 2016; Anhalt et al., 2018; Cetinkaya et al., 2016) and to develop foundational ideas for teaching MM (Cetinkaya et al., 2016).

Teachers’ experiences with MM might occur in less obvious ways than MM courses or workshops. Greer and Verschaffel (2007) note three levels of MM that individuals might experience: explicit (with attention on the MM process), implicit (engagement without awareness that it is MM), and critical (examining role of MM within mathematics, science, and society). Teachers might report experiences that they did not initially perceive as professional learning in doing and teaching MM. Examples might include implicit MM in an applied mathematics, science, or engineering course project or critical MM done as activities to understand and act on equity and inclusion in mathematics.

Teachers’ understandings, practices, and beliefs might be challenged, expanded, or transformed in multiple ways as they continue to better understand how to do and teach MM. Researchers note progress in understanding how teachers develop understanding of MM and teaching MM yet note the potential for additional research to clarify and augment their findings (e.g., Besser et al., 2015; Cetinkaya et al., 2016). We seek to answer the overarching question of how teachers learn to do and teach MM by looking at experiences through the eyes of teachers who have engaged with MM doing and teaching. We want to understand what teachers view as helpful and, from their stories and through their eyes, develop further insights into how teachers learn to do and teach MM. Our theoretical perspective elaborates mechanisms for teacher learning.

**Theoretical perspective**

The purpose of this study is to understand what in-service teachers identify, implicitly as well as explicitly, as important experiences that may have supported their learning. Much of their learning about doing and teaching MM likely occurred during adulthood. We use transformative learning theory (Mezirow, 1985, 1991, 2000) as the theoretical grounding for this work because this theory attends to how adults learn—by critically assessing assumptions formed from a multitude of life experiences.

Transformative learning theory provides insights into the process of learning, including the way in which knowledge schemes and perspectives change during different forms of learning and the mechanisms that influence the process. A growing body of research, including work in teaching and teacher education that is framed by transformative learning theory, provides insights into the process of transformative learning and
the factors that influence the process (Taylor, 2017). In this study, we focus on changes in schemes and perspectives to understand better how the experiences that the teachers identify are supportive for learning about doing and teaching MM.

In general, transformative learning occurs in response to an epochal (or disorienting) dilemma or multiple smaller (or incremental) dilemmas triggered by stimuli that signal “dissatisfaction with current ways of thinking” (Marsick & Watkins, 2001, p. 29). We consider both types of dilemmas. We focus on experiences that trigger dilemmas that teachers seek to resolve regarding insufficient knowledge of MM or the teaching of MM. Although not all triggers lead to learning, triggers can separate learning in general from transformative learning. Transformative learning is more than accumulation of knowledge; it is a profound change in a teacher’s existing knowledge scheme or perspective. If a teacher believes they learned about MM or its teaching from an experience, we identify any trigger(s) and related dilemma(s) as we work to understand how the experience contributed to learning.

Existing research offers insights into different experiences that can trigger knowledge dilemmas for teachers. Peters (2014) found that engagement in workshop or conference activities focused on key problems or concepts, dialogue with colleagues and knowledgeable others about statistics, formulating responses to students’ thought-provoking questions, and planning to teach statistics for the first time were experiences that high school statistics teachers recalled as sources of triggers for dilemmas related to their knowledge of statistics. Similar triggering opportunities for MM teachers seem likely given the open-ended nature of MM tasks and unexpected responses from students, especially for teachers new to teaching MM. PD activities based on challenging content can trigger knowledge dilemmas. For example, McCulloch and colleagues (2019) targeted teachers’ typically limited views of function as algebraic expressions and graphs (e.g., Even, 1993) to trigger knowledge dilemmas for preservice secondary mathematics teachers. Peters and colleagues (Peters, 2018; Peters & Stokes-Levine, 2019; Peters et al., 2014) targeted statistical concepts as triggers for professional learning by middle and high school mathematics teachers. We look for triggers of knowledge dilemmas resulting from all types of experiences.

We also consider how teachers resolve knowledge dilemmas to advance their knowledge about MM or teaching MM. According to transformative learning theory, learning is prompted by dilemmas for which existing knowledge does not provide resolution (Merriam & Caffarella, 1999). Reflection on questions of what one knows and does is a main mechanism for learning in general (e.g., Wheatley, 1992). Transformative learning requires critical reflection, which focuses on previously unexamined assumptions and the reasoning behind what one knows (Cranton, 2006; Mezirow, 1985). Recognizing insufficiencies in their knowledge, learners may take actions to construct the knowledge needed to resolve their dilemmas. The actions might include rational discourse, through which learners encounter alternative views. As learners use alternative views to examine their underlying assumptions, they may either integrate the new view(s) into their schemes and perspectives or transform their schemes and perspectives. For the previously described programs, McCulloch and colleagues and Peters and colleagues incorporated not only planned triggers but also opportunities for critical reflection and rational discourse to facilitate teachers’ dilemma resolution.

Understanding teachers’ perceptions of their experiences means exploring their recollections with respect to triggers and dilemmas. In this study, we sought to identify triggers and dilemmas in experiences that are deemed important by teachers of MM for their learning of how to do and teach MM. We note triggers and the critical reflection, rational discourse, and actions that the triggers prompted and that teachers pursued. We explore
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teachers’ opportunities for critical reflection, rational discourse, and dilemma resolution to understand why the teachers’ experiences could be powerful for their learning.

**Methods**

To understand the nature of experiences that teachers perceived as contributing to their evolving understanding of MM and how to facilitate MM with students, we use a phenomenological methodology (Moustakas, 1994; Vagle, 2018). In this case the phenomenon is learning how to do and teach MM.

**Participants**

To have secondary mathematics teachers who experienced learning and teaching MM, we recruited teachers who (a) had experience in facilitating MM, (b) were committed to teaching MM, (c) participated in PD on MM at the national level, and (d) served as leaders in MM teaching (e.g., led MM PD, produced MM curriculum materials). Sufficient for phenomenological methods (Vagle, 2018), the purposeful sample consisted of five teachers from five different geographic regions across the United States. The teachers varied greatly in their backgrounds related to factors typically considered in phenomenological research, including sex (Moustakas, 1994) and teacher preparation experiences. The three female and two male teachers taught for a mean of 14.8 years (with a standard deviation of 7.56 years). Three taught in public schools, and two taught in private schools. One teacher has an undergraduate degree in teaching; three teachers have graduate degrees in education; one teacher has no teaching degree; and three teachers have undergraduate mathematics degrees. We use pseudonyms and present the details about participants in aggregate form to preclude revealing potentially identifying information about individual teachers.

**Data sources**

The current phenomenological study explores teachers’ perceptions of experiences that contributed to their capacities to understand MM and to facilitate students’ MM experiences. Self-report is one technique appropriate for phenomenological studies (Seidman, 2006). Collection and analysis of self-reported retrospective data typically raises questions about the reliability of participants’ memories and the accuracy of those memories. To value the teachers’ perspectives without imposing our lenses on learning and teaching MM, we employed event history calendars (EHCs) and critical incident descriptions (CIs)—two established instruments that were refined by Peters (2009, 2014). They incorporate retrospective techniques that minimize recall effects (Eisenhower et al., 1991), such as use of landmark events as retrieval cues.

**Event history calendars**

To obtain the collection of experiences that teachers valued, we used EHCs. EHCs incorporate memory retrieval techniques to allow individuals to accurately reconstruct past experiences (Martyn & Belli, 2002; Morselli et al., 2019). Individual completion of EHCs has been shown to produce data as good as interviewer-administered EHCs (Morselli et al.,
An EHC is a table in which columns contain cues for recording behaviors chronologically and rows include significant activities related to the goals of the research (Freedman et al., 1988). The EHC template for this study (Appendix A) was structured by rows of experiences, such as undergraduate, graduate, or professional learning experiences in modeling or teaching modeling. Teachers could add other experiences to the calendar. Columns included landmark events and the years in which they occurred, such as MM-focused competitions, release of MM-related curriculum and policy documents, and MM-intense professional conferences. This orderly review of experiences enables participants to have greater recall than experiences recalled haphazardly (Eisenhower et al., 1991), and recall from completing EHCs compares favorably with perceptions captured in survey data years prior to calendar completion (Freedman et al., 1988). The teachers also recorded background information about people, places, and feelings associated with the experience to provide insights into characteristics of their learning experiences and their perceptions beyond the dates of the experiences.

Critical incident descriptions

Specific experiences can be especially powerful for learning. CIs provide opportunities “to highlight particular, concrete, and contextually specific aspects” (Brookfield, 1990, p. 180) of unique positive and negative experiences (Cuddapah, 2005) that are significant to individuals—experiences that individuals tend to remember well (Eisenhower et al., 1991). For each of one positive and one negative experience, the teachers recorded the time and place, other individuals involved in the experiences, rationale for selecting the experiences, and their thoughts and feelings about the experiences and their effects (Appendix B). The CIs provide a window into participants’ implicit assumptions and beliefs about experiences (Butterfield et al., 2005). From these descriptions, we gained insights into teachers’ triggers and dilemmas and how they resolved (or failed to resolve) those dilemmas.

Interviews

Interviews complement background information from the EHCs and CIs and teacher-provided resumés. The first author established rapport with the teachers and conducted two semi-structured interviews with them, following recommendations for phenomenological studies (Seidman, 2006). Each interview lasted 60 to 120 min.

A first semi-structured interview elicited teachers’ perceptions about and experiences with learning and learning to facilitate MM. The first question in the protocol asked which experience was the most important for learning MM or for teaching MM. Informed by transformative learning theory, follow-up questions focused on uncovering triggers, dilemmas, and conditions related to the experience. The questions addressed such things as the teacher’s feelings about the experience, activities that were part of the experience, how the teacher learned about the experience, how students were involved, and whether support was provided for the teacher to participate in the experience. Questions about the activities, the people involved, and perceived sense of community revealed potential for rational discourse and critical reflection. The set of follow-up questions were repeated for each additional pivotal formal or informal learning or teaching experience that the teacher identified. The researchers also reviewed teachers’ EHCs, CIs, and resumés to gain a sense of the temporal positioning of educational experiences, to become familiar with experiences listed as
pivotal or influential, and to augment the interview protocol with questions unique to each teacher to clarify and elaborate the nature and impact of the experiences.

A second interview was held several weeks after the first interview to capture recollections of experiences not mentioned previously and for member checking of our evolving interpretations. The time between interviews provided an opportunity for teachers to reflect on their experiences and the meaning of their experiences (Peters, 1991; Seidman, 2006) and for researchers to determine any remaining questions or areas in need of clarification, including questions about learning experiences that were not identified as pivotal by the teachers. Researchers contacted teachers after the second interview for any additional clarification needed or to obtain documents that teachers identified during the interviews as important for their learning to do or teach MM.

**Data analysis**

Our analysis of teachers’ documents and interviews followed systematic procedures recommended for phenomenological studies (Vagle, 2018) and used by Peters (2009, 2014). Both interviews were videorecorded, transcribed, and annotated prior to analysis.

For each teacher, each researcher viewed the interview videos and did a line-by-line reading of the transcript. They highlighted text and made margin notes about evidence related to potential learning experiences, triggers, rational discourse, critical reflection, conditions for transformative learning, or any other aspects of the experience that raised a question for research group discussion. One example of an experience noted by the research team is the following excerpted passage from Dahane’s interview: “So that whole science—like, like it brought [pause] why we’re doing this. You know, it pulled it in. And, um, that’s when I started realizing that I wasn’t actually being taken care of in terms of professional development.” Dahane’s PD experience in science appeared to trigger a potentially epochal dilemma related to teaching.

We used the interview transcript passages, highlights, and notes along with the text and experiences identified in teachers’ CIs and EHCs to create chronologies of experiences that teachers explicitly or implicitly identified as contributing to their capacities to do or teach MM and their perceptions of characteristics of experiences that helped or hindered their development. The resulting chronologies were useful for synthesizing and condensing the data. We used the chronologies in our group discussions to come to agreement on the important facets of each teacher’s story and to create a phenomenologically reduced storyline that included the dilemmas the teacher faced and their reactions to the dilemmas (Vagle, 2018). For example, we noted Karina’s engagement with a “ridiculously amazing” problem as a triggering event. To capture the conditions of the experience, we noted characteristics of the problem (e.g., relatable context, open-ended questions) and the learning setting (e.g., safe space to explore and take risks; provided scaffolding) that she identified as important for her early learning of MM. The chronologies also informed our understanding of teacher learning over time.

We used constant comparison (Glaser & Strauss, 1967) in repeated readings and discussions of each teacher’s story to identify both commonalities and differences in teachers’ stories and emerging themes across teachers’ experiences in learning to do and teach MM. For example, Karina elaborated on multiple solutions as part of her teacher preparation experience; Phil mentioned them as part of his professional learning; and Viv noted them as things she learned to value in her students’ work. We looked across the five teachers’
stories and confirmed that each teacher noted an experience with learning about multiple solutions for mathematics problems.

We placed the teachers’ chronologies alongside each other to see patterns in the order in which learning happened. Phil noted that his district had stressed the use of four representations: graph, table, verbal, equation. He recounted how he incorporated the four representation forms in his lessons. After a workshop, his notion of four representations expanded to multiple solutions, illustrating the trajectory link: MULTIPLE REPRESENTATIONS → MULTIPLE SOLUTIONS. Karina underscored the learning impact of a college mathematics problem that was presented in three forms: one that allowed for multiple solutions, a more open form, and then a MM problem. Her learning illustrates the links: MULTIPLE SOLUTIONS → OPEN-ENDED MATH PROBLEM → FULL MM EXPERIENCE. By piecing together chains of experiences such as these and triangulating with experiences of all five teachers, we arrived at the trajectory.

Findings

The purpose of this study is to examine MM teachers’ perceptions of experiences that contributed to their capacities to understand MM and to facilitate students’ MM experiences using the framework of transformative learning theory. Our findings are not the teachers’ or our assessment of their specific experiences but rather an articulation of themes regarding what they experienced as triggers and described as valuable characteristics of their experiences. The themes and patterns across teachers’ chronologies fall into a trajectory of growth as a doer and teacher of MM, with connections to critical reflection and rational discourse. We turn to that trajectory and then to examples of the teachers’ stories.

A trajectory

The five teachers recounted similar kinds of experiences (e.g., conversations with colleagues, reading coaching guides for MM competitions, connecting experiences with and without MM, connecting with others at professional conferences, engaging in MM problems that were of particular interest to them). The experiences prompted growth in understanding MM, teamwork, curriculum, context, or student learning, yet were tied to the teacher’s current understandings in one or more of the other areas but not necessarily explicitly connected to MM. The teachers’ experiences suggest their learning falls into a trajectory that intertwines five knowledge schemes that constitute their MM doing and teaching meaning perspectives: MM as Mathematics, Social Aspects of Working Together, Context Awareness, Attention to Students, and Nature of Curriculum. The schemes differ from the structural view of MM teaching competence used by Wess and colleagues (2021) and others to prepare and assess preservice teachers in at least two major ways. The schemes include the importance of context, and they begin with in-service teachers’ mathematics teaching competence in general and evolve empirically rather than arise from theory to structure preservice teachers’ training in MM teaching competence.

Figure 1 represents the Doing and Teaching MM knowledge perspective. It illustrates the five schemes and conveys the complexity of learning to do and teach MM. The five schemes are color-coded and labeled. Each scheme captures a particular aspect of doing and teaching MM. MM as Mathematics is MM as the use of mathematics to answer real-world problems that transcend typical school word problems. Social Aspects of Working
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Fig. 1  Trajectory with five schemes

Together are the structure and function of groups to collaboratively solve problems. Context Awareness is knowing about real-world contexts and leveraging them to pose mathematical problems. Attention to Students is a focus on student affect and thinking and comfort with eliciting and prompting productive struggle. Nature of Curriculum is knowledge of curriculum content and recommendations with a vision of how students engage in MM and how teachers facilitate students’ MM work.

Learning occurs within each scheme as the teacher moves from left to right in the diagram. For the diagram, vertically aligned elements of different schemes seemed to happen relatively concurrently in the teachers’ experiences. To unpack the complexity of learning to do and teach MM, we share examples of triggers and dilemmas experienced by the teachers and connect the triggers to the diagram.

Examples

For each of the examples that follow, we describe a trigger and dilemma, indicate where in the trajectory the learning occurred, note affect as well as critical reflection and rational discourse, and discuss and illustrate how the learning integrates two or more of the five schemes to show how the teachers’ Doing and Teaching MM knowledge perspectives changed. The examples are snapshots illustrative of the teachers’ perceptions related to particular experiences and do not represent the totality of the experiences that the teachers shared. They illustrate both the variety of settings in which triggers arise and how critical reflection and rational discourse influence and support the teachers’ learning related to the dilemma.
Summer workshop

After teaching for 10 years, Dahane attended a summer workshop with science and technology teachers. As he said, “I was indifferent about the experience, thinking that it would be like any other professional development I have previously done. I was wrong!” He noted that too many of these prior professional learning experiences “immediately gave in to the shortcut nature of the modeling using a calculator instead of going deeper into the mathematics study of the model produced.” Dahane noticed that science teachers did wonderful demonstrations to inspire scientific ideas and formulas, although they did not pursue the mathematics behind the formulas. Dahane recalled, “I could see how the math formulas are coming alive in their lessons but when it was time to dive into the math they suddenly stopped and trivialized the approach (all you have to do is plug in!!)” He viewed their demonstrations as “bringing mathematics to life” and important in encouraging students to go into STEM fields. The science experience triggered an affective dilemma for Dahane in that he realized what PD could be and that he “wasn’t actually being taken care of in terms of professional development.”

To resolve his dilemma, Dahane took charge of his PD as he sought venues to learn about more application possibilities through alternative perspectives. He sought opportunities to coordinate with science teachers in his school because he believed those teachers had the knowledge and resources to carry out the demonstrations. He also met with a local engineer, who told Dahane, “what you guys are teaching is not, not the real thing.” The engineer pushed for teachers to have conversations with students about the kinds of questions they should ask in real-world situations. Dahane incorporated conversations into lessons that had students look at a real-world situation, ask questions about it, and identify variables and assumptions that helped them to make sense of mathematical formulas through simplifying and mathematizing.

Dahane’s experiences with science teachers were ripe for rational discourse. However, according to Dahane, the science and technology teachers at the workshop and in his school were not interested in diving into mathematics. He resolved his dilemma by taking charge of his own PD and picking up ideas for demonstrations and began using demonstrations to make math come to life through “conversations” that he had with students. Dahane may not have seen a need to push harder for rational discourse with science colleagues, and he did not offer evidence of critical reflection in his conversations about MM and teaching MM. The absence of discourse and reflection make sense. Dahane’s dilemma was an affective issue about the quality of provided PD experiences and not a cognitive issue about his own understanding of MM doing and teaching. The resolution of this dilemma resulted in his take-over of his own PD and a change in his practice to enhance students’ engagement in simplifying and mathematizing. We represent Dahane’s learning in the trajectory as a move forward in the Context Awareness scheme. In Fig. 2, the Summer Science diamond and black lines indicate his learning of how to use Physics to Explain Math. The resolution of this dilemma also shows his greater openness to Solve Authentic Word Problems in the MM as Mathematics scheme, as indicated by a black node for Solve Authentic Word Problems in Fig. 2.

Undergraduate course

Karina walked into a college MM class and saw “a gorgeous display of five different kinds of cookies.” The instructor posed the question, “What makes a cookie uncommonly good?” Karina described her experience as a student.

We weren’t expected to know some complex mathematical formula to solve the situation. Instead, we took something we all do (eat), used an object most people love.
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(cookies), applied our prior knowledge to (which cookies do we like to eat), and then looked at how it applied to our familiar math world.

The professor encouraged small groups to “observe, measure, smell, taste, and take in” samples of each kind of cookie. As a class, they considered “all of the different ways we could explore each thought” and “put math [that we were familiar with] to all of our questions.” Karina remembered “having fun that entire class.” Up to that point, mathematics teaching and learning for Karina “always followed a predictable and familiar pattern of I do, we do, you do with a set of math concepts and procedures.” Her prior experiences with MM were limited to two MM competitions as a high school student. The experiences were negative ones because she “couldn’t be helpful as a member of my team, which made me feel a disequilibrium in my identity at the time” and she “didn’t have the guidance or knowledge that I needed.” The experience “stopped my growth as a math learner instead of expanding my definition of what math is.” She encountered MM in parts of other courses, but those courses presented a limited view of MM as “you’re using technology…Or…using a graph with your problem.” Karina’s positive experience with the cookie task served as a trigger for a dilemma in which, “my world view of what math is was in constant disequilibrium.” The cookie task opened her to doing MM and “turned the scary monster [MM] into something I can’t believe I was scared of.”

Karina began resolving her dilemma by continuing to engage in the complete MM cycle with other MM problems and by reflecting on other coursework and assignments. She and her classmates experienced MM in “a safe and warm classroom environment where taking risks and persisting, even when you were unsure of something, was encouraged.” She noted, “I felt challenged during all of these courses, but the

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**Fig. 2** Science workshop trigger as it affects Dahane’s Context scheme and draws on his understandings in MM as Mathematics scheme
productive struggle kept me engaged and thinking about math in different ways each day” and provided a vision of “what it could look like in the classroom.” Previous coursework indicated that Karina could see Open-Ended Math Problems as amenable to multiple approaches, representations, and solutions. For example, an assignment in a class prior to her MM course required Karina to solve a logic problem in two different ways. She used a visual or pictorial approach, a physical manipulative approach, a verbal approach, a tabular approach, and a function or equation approach.

As evidenced in her description of her solutions to the logic problem and the changes in her view of mathematics, Karina is naturally reflective. Karina particularly enjoys rational discourse for the multiple perspectives it offers—perspectives such as those shared by classmates for the cookie task and the “different way of thinking” other educators offer at conferences. We represent Karina’s learning from her dilemma with the cookie problem diamond and black lines in Fig. 3. In the MM as Mathematics scheme, she moves from Open-ended Math Problems towards Full MM Experience. Her openness to engaging in teamwork and rational discourse, particularly when the environment is safe and supportive, are indicative of Productive Teamwork in the Social scheme (as indicated by a black node in Fig. 3). The cookie task, and the MM course in general, contributed to her understanding of MM as an Expected Math Practice and offered her Images of Student MM and Images of MM Facilitation as she simultaneously moved forward in the Curriculum scheme. Black nodes in Fig. 3 denote this growth.
Teresa taught in an urban public school setting before moving and teaching at a private school the year before our study began. Shortly after arriving at her new school, Teresa’s department chair encouraged department members to attend the Teaching Contemporary Mathematics (TCM) conference—a conference that focuses heavily on mathematical modeling. Teresa saw the conference as an opportunity to get good activities while getting to know her new colleagues. While at the conference, she encountered multiple open-ended problems that presenters discussed implementing with students in open ways. Teresa recalled,

One thing that really stood out to me is how much you can do with one simple question, like, how many different ways you can go with students depending on what level math students are in or what students are interested in.

She further noted, “the presenter stated…you’re not telling students anything. They have to look up different options. They have to decide, um, what are different options available, like in the real world.” This “open view,” as Teresa called it, triggered a dilemma for Teresa in that it contrasted with her practices and views at the time: “before that, I always thought of, okay, make it very structured for students…like my fear is always, if I give students too much freedom, they will fall apart.” She began to realize, “Students can handle open-ended problems in mathematics. I should not be afraid to provide students with challenging questions and let them figure out the possible solutions.”

To address her dilemma, Teresa challenged herself “to incorporate more mathematical modeling in my classes.” Shortly after the TCM conference and at the end of a polynomial unit, she engaged students with a “Math and Music” project she learned about at the conference. She described students’ work as:

Students would come up with their own melody ... changed the notes into coordinate points ... used Desmos to graph the points ... thinking about which function, which type of function would best model their graphs ... is it cubic, is it to the fourth power, is it quartic, is it to the root 5, so which model would best represent ... this was done all electronically so they didn’t have to calculate it, they were just looking at the graphs ... they had to do three different transformations.

In Teresa’s description of the project and its implementation, we saw no evidence of critical reflection about learning opportunities afforded by the project; she saw it as “fun” for students. We also see little evidence of rational discourse in her description of this or other activities. When Teresa attended conferences, she looked for “what can I use with my students tomorrow,” and particularly valued sessions in which “presenters shared all documents that I could use with my students (PowerPoint presentation, information about the problem, sample student’s handout, teacher’s handout, rubric, possible questions, etc.).” During the first interview, Teresa described herself as a listener at conferences, which provided her with opportunities for new ideas but happened in the absence of rational discourse or critical reflection and arguably limited her opportunities for learning.

We noticed what may be a willingness to engage in rational discourse or critical reflection in our second interview with Teresa. Teresa talked with one of her colleagues about planning to use activities from TCM, suggesting the possibility of rational discourse. Additionally, when Teresa engaged in a task sort to place tasks in terms of how
good they are as MM tasks, one of her considerations was the level of struggle for students, suggesting the possibility of critical reflection related to her dilemma. For example, the tasks she considered to be weak were those that “are straight-forward questions that [do] not necessarily require students’ thinking.” The stronger modeling tasks “would be very open-ended so probably for my students they would need to know something” and likely would struggle. She noted that she struggles with identifying good modeling tasks but seeks understanding.

You know, it’s still hard for me ... I want to know which one is a good modeling [problem], which one isn’t a good modeling [problem] ... It’s hard for me to kind of, do it without even knowing, you know what I need ... Like I wanted to understand, this is modeling, this is not.

We see Teresa learning to Trust Students as she progresses from Attend to Student Affect/Community towards engaging students in Productive Struggle in the Attention to Students scheme, as indicated by the diamond in Fig. 4. The other black nodes indicate understandings in other schemes related to this learning. We see, in the MM as Mathematics scheme, Teresa’s openness to more open and Authentic Word Problems. In her Nature of Curriculum scheme, we see Teresa’s awareness of MM as an Expected Math Practice. She enacted real-world problems in her classes and is looking to learn how to implement and provide proper scaffolding for open-ended problems. Teresa’s emphasis on students having fun is consistent with her understanding of Motivating Math using context in the Context Awareness scheme.

![Fig. 4](image)

Teresa’s learning to Trust Students as growth in the Attention to Students scheme with associated understandings in three other schemes
Classroom activity

Viv engaged students in MM in her classroom and through MM competitions. She attended conferences, developed MM problems, and shared ideas and materials with her colleagues as she supported other teachers in their efforts to incorporate MM into their classes. Viv articulated how the importance of context in MM problems fits well with her school’s focus on social justice. Mathematics problems could be “very much tied to the, to the questions of, like how to run a society, like a positive fabric society in a way, or climate change.” Viv also read voraciously and was familiar with MM curriculum and advocacy documents. For example, she used Getting Started, Getting Solutions (Bliss et al., 2014) as “kind of a framework” for MM that gave her “a structure for um, talking about an iterative process” and for “thinking a little bit more about how to teach this as an iterative process.” Viv also noted that she and her school valued the education of the whole student and attended to student interactions. For example, in talking about MM competitions, she described the way in which she coached teams in their ability to work together:

Concrete team building skills, um, I felt like have been helpful to kids, like just to talk about how, how are you going to handle those moments where you feel like you’re not being heard? Or you feel like somebody else is taking over the conversation? Or where, you know, like so, how can we be purposeful?

Viv was similarly reflective about her students’ MM activity and competency. She described her dissatisfaction with her own practice when she realized students were offering only weak references to measurement error when they were asked to critique their models—there were no rich analyses or productive struggle. The observation triggered a dilemma. Viv knew validating and analyzing a model was a MM competency; yet, she believed she did not know well enough how one could analyze a model, and therefore also did not have an image of what more her students might do in their analyses:

I was trying to figure out what it meant when, like the handbook [Getting Started, Getting Solutions], and when people talked about, like I think they talk about error—like in analyzing your answer. Where you’re sort of doing a little bit of error analysis. Um. But that, like analyze your solution. Uh, end part. I was trying to figure out what on earth that was.

To address her dilemma, Viv sought the insights of a well-known teacher of MM she knew from conference presentations and published works. She asked the expert to talk with her. They then had a 35-min conversation at a conference, during which she engaged the expert in her nagging dilemma:

How do you talk about … error analysis, uh, you know, like, is this a good fit? Because that’s the piece that I feel like, you know, if you’re forever in introduction to math modeling workshops, you never get as far as the back half.

Their conversation was rich in rational discourse. The expert teacher described what he did in his classroom and Viv asked questions to probe further into what analyzing a model looks like, why it looks that way, and how she can press on student answers during this phase of the MM cycle. They were able to take “a specific problem and talk through like which pieces, um, he has groups look at first, or that students often look
at first, or the mathematicians that he had worked with had done first.” The conversation, to Viv, was “really useful.” After talking about the experience in our interview, she noted the few places in which students study error and imperfection in school mathematics. She then added, “Maybe that’s the next example I have to figure out is like how do we—how do we write a problem that really gets in at that, that error analysis piece?”.

Viv’s experience with seeking insights into how to analyze models and then facilitate the validation phase of modeling is an example of an expert teacher who had already delved deeply into MM and was refining her MM as Mathematics scheme. In Fig. 5, the Error as Model Evaluation diamond places her key learning based on this trigger and dilemma as a move from Multiple MM Competencies to Full MM Experience, which in turn moved her towards Facilitate Full MM Experience. The other black nodes represent elements of other schemes on which she drew. As her careful preparation of MM teams suggests, Viv already was acutely aware of Productive Teamwork in the Social Aspects of Working Together scheme. Her discomfort with her students’ shallow use of error is indicative of how she understood Productive Struggle in the Attention to Students scheme. In her classroom and in support of her colleagues, Viv illustrated Create & Share MM Materials/Experiences in the Nature of Curriculum scheme. With her deep commitment to Social Justice, she owned Generate Questions as part of the Context Awareness scheme. Viv’s dilemma and its resolution illustrate how experienced MM teachers continue to learn and refine their Facilitate Full MM Experience perspectives. The example also illustrates the power of critical reflection and rational discourse to identify and address areas of improvement regardless of the robustness of teachers’ understanding of doing and teaching MM.

Fig. 5 Viv’s experience with students’ errors as learning in the MM as Mathematics scheme that is supported by understandings in each of the other four schemes
Discussion and implications

Developing a robust understanding of teaching and doing MM is a challenging process. We sought to understand this process as it unfolds over time through the experiences that teachers find meaningful. Our study yields research findings, spurs a new framework, provides research opportunities, and informs teacher education.

Empirical and theoretical outcomes

The five teachers who participated in this study experienced the phenomenon of learning about MM and teaching MM, though their individual experiences differed. Their sequences of learning experiences suggested a trajectory with five interconnected knowledge schemes—Mathematical Modeling as Mathematics, Social Aspects of Working Together, Context Awareness, Attention to Students, and Nature of Curriculum—comprising Doing and Teaching Mathematical Modeling perspectives. As they elaborated their experiences, the teachers described triggers and dilemmas that happened when they were at various places in their learning, as illustrated in the four examples.

Each meaningful learning experience recounted by the five teachers highlights the importance of triggers for learning and the variety of settings in which triggers arise (e.g., Addleman et al., 2014; Christie et al., 2015) in contrast to shared settings experienced by educators in other studies framed by transformative learning theory (e.g., preservice teachers using an applet about functions in McCulloch et al., 2019; college of education faculty forced to move their courses online in Terras, 2017). As illustrated in the examples from Dahane, Karina, and Teresa, the teachers encountered triggers while participating in formal professional learning settings, including mathematics and education courses, conferences and workshops, and faculty meetings. Other triggers occurred in the daily work of teaching, such as when teachers planned activities and discussions for mathematics lessons, as illustrated by Viv’s example. Triggers also arose while they interacted with colleagues (e.g., Dahane with science teachers, Teresa with TCM presenters), worked on modeling problems (e.g., Karina with the Cookie Problem), or interacted with students (e.g., Viv with measurement error discussions). The experiences included all three levels of MM that Greer and Verschaffel (2007) identified: explicit (Karina with the Cookie Problem), implicit (Dahane with science teachers), and critical (Phil with a workshop problem focused on food waste).

Reflective of the types of triggers experienced by statistics teachers in Peters (2014) and by university faculty in Terras (2017), triggers in this study prompted dilemmas that challenged teachers’ sense of what mathematics is, how one does mathematics, or what teaching and being a teacher are. Teachers’ resolutions to their dilemmas led to advances in their knowledge schemes and in their doing and teaching MM perspectives (examples of which appear in Figs. 2, 3, 4 and 5). The stories of their dilemma resolutions underscore the importance of teachers engaging in critical reflection to question their premises (e.g., Liu, 2015; Peters, 2014; Taylor, 2016) and rational discourse to consider multiple perspectives (Peters & Stokes-Levine, 2019; Peters et al., 2014). In this study, the teachers’ resolutions to dilemmas reflected their commitment to students and teaching and their perceived need of a safe and supportive environment for learning to teach mathematics in general and MM in particular.
The teachers’ examples illustrate how the five schemes are interrelated. Learning within each scheme is important for learning to facilitate MM. In particular, growth in one scheme can be supported by understandings in other schemes. Viv, for example, experienced a dilemma in the MM as Mathematics scheme and advanced to Full MM Experience. Her learning was supported by her understanding of Productive Struggle (Attention to Students). Viv wanted her students to struggle with challenging ideas. When students provided simplistic responses and were not struggling productively with how to validate a model, she noticed and reflected on why they were not struggling with this topic. Viv’s understanding of Productive Teamwork (Social Aspects of Working Together) was developed to the point of proactively creating groups that included students with different strengths and engaged students in activities to develop their ease in challenging their groupmates’ ideas. When all students in each group similarly referred only to error in measurement, she could not attribute it to differences in ability or to students being unwilling to challenge ideas. The dilemma of her as well as of her students not understanding how to evaluate and test models (MM as Mathematics) became apparent to her. As she identified and resolved her dilemma, Viv blended atomistic and holistic approaches to MM. She targeted her need to learn more about evaluating and testing models from an atomistic perspective while teaching MM from a holistic approach. The blend perhaps is reasonable; Brand (2014, in Kaiser & Brand, 2015) found that neither of the two approaches was superior to the other for student learning.

In some cases, trigger and growth in one scheme pair with growth in another scheme, such as Karina’s dilemma in the MM as Mathematics scheme and growth towards a Full MM Experience. Her work with the Cookie Problem helped her to see MM problems as a specific kind of open-ended mathematics problem. She simultaneously experienced growth in developing an Image of Student MM and an Image of MM Facilitation as she noticed both her experience as a student while engaged in the problem and the efforts of her instructor in presenting the problems and providing a safe intellectual space in which she and her classmates could undertake the challenge of a MM problem.

Insights and implications

The study yields several implications for research. Our analysis of the experiences shared by the five teachers suggested a potential trajectory along which teachers develop understandings of doing and teaching MM. Ongoing research should seek to challenge, refine, and elaborate the nature of the trajectory. Given the affect in the teachers’ stories, from passion about social justice issues to feelings of inadequacy about their understandings of mathematics, future work might also consider the role(s) that affect or attitude (e.g., Asem-papa & Brooks, 2020) plays within the five schemes. Such research might connect to the affective and motivational aspects of professional competence articulated by Wess and colleagues (2021) in their consideration of preservice teachers.

The trajectory integrates five meaning schemes. Existing bodies of research in teacher education and professional learning address aspects of these schemes. For example, teacher noticing (Sherin et al., 2011), use of student thinking (Czocher, 2019; Franke et al., 2015; Staples, 2007; Webb et al., 2008), and facilitation of productive struggle (Warshauer, 2015) are parts of Attention to Students. How teachers learn to model (Anhalt & Cortez, 2016; Anhalt et al., 2018; Besser et al., 2015; Centinkaya et al., 2016) constitutes MM as Mathematics. Social Aspects of Working Together includes facilitation of group work (Borromeo Ferri, 2018), including both social and analytic
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scaffolding (Baxter & Williams, 2010). Some work in mathematical modeling education considers blends of the five strands. For example, scholars (e.g., Koellner & Lesh, 2003; Schorr & Lesh, 2003; Wubbels et al., 1997) integrate the importance of how teachers solve problems (MM as Mathematics) in real-world contexts and how they understand student thinking (Awareness of Students). This study also challenges researchers both to flesh out further how teacher learning occurs within each scheme and to consider how these schemes interact as teachers learn to do and facilitate MM.

In looking at how teacher competency develops along the five schemes in the trajectory, we draw a parallel to the previously mentioned debate between holistic and atomistic approaches that arose in the literature regarding assessment of students’ MM competency (e.g., Blomhøj & Jensen, 2007). Studies that investigate teachers learning about specific nodes or separate schemes take an atomistic approach to understanding how teachers learn to do and teach MM. Retrospective methods, like those used in this study, allow us to see how teachers develop a holistic view of MM doing and teaching. The methods allow for integrating a multitude of learning experiences that unfold over years in a holistic view of teacher professional learning of doing and teaching MM as complex activity with connections among the individual nodes and schemes.

Retrospective research methods also honor teachers’ perspectives and voice in studies that explore how teacher learning unfolds. For example, recollection of past experiences underscored the depth of the connections that teachers made between MM and such things as multiple representations, alternative strategies, and productive struggle. The largely untapped empirical usefulness of retrospective methods to probe past experiences could also be helpful in understanding such phenomena as the experience of the COVID-19 pandemic and its early impact on teaching and schools. Retrospective methods might be paired with the transformative learning theory lens used by Terras (2017), who explored personal transformations as a group of college faculty experienced a pre-pandemic move to online courses.

Practically speaking, the current study provides teacher educators with guidance by which to assess the opportunities that our programs present to current and future teachers. Teachers should have opportunities to encounter triggers and time to resolve dilemmas (e.g., McCulloch et al., 2019; Peters, 2014; Terras, 2017) in each of the five schemes. MM can be nurtured in many parts of teacher preparation, including applied mathematics, science, or engineering course projects or critical MM done as activities to understand and act on equity and inclusion in mathematics. Our knowledge of mathematics teacher education programs, especially in the United States, suggests that several elements of the schemes are found in many programs. The gray region in Fig. 6 indicates topics common to syllabi, conferences, and practitioner articles. The ideas resonate with expectations for teacher preparation. For example, in the United States, the Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of Mathematics (2017), which draws on the National Council of Teachers of Mathematics (NCTM) Effective Teaching Practices (2014), include the need for teachers to use and value multiple representations and various approaches (strategies and solutions), to communicate mathematically, to engage students in productive struggle, and to attend to student thinking. The Mathematical Education of Teachers II (Conference Board of the Mathematical Sciences, 2012) includes curves and curve sketching, data collection and analysis, and using mathematics to solve problem. Teachers learn about MM through activities in which ideas needed in MM are implicitly and explicitly present (Greer & Verschaffel, 2007). We need to intentionally draw teachers’ attention to these things and to connections among them as well as to parts of the meaning schemes.
that are not common in teacher education and PD if we want to support teachers in developing robust meaning perspectives for MM and teaching MM.

The comments of some of the teachers identified specific topics that were seemingly underemphasized in their professional learning experiences. For example, Viv pointed out that she had many experiences in how to begin a MM cycle but she never had an experience in learning how to do or facilitate assessment of models. The atomistic approaches to teacher learning that Viv experienced did not engage her in the full MM cycle. In particular, they omitted the key element of validating models. We see a parallel to what Peters (2014) reported about statistics teachers: the teachers noted that design was a key element of statistics but it was often missing in teacher education. The collective teachers’ voices suggest the value of teachers having opportunities to engage in all aspects of MM and to develop holistic competence.

**Conclusion**

The goal of this study was to understand secondary teachers’ perceptions of the experiences that contributed to their understandings of how to do MM and how to facilitate students’ MM experiences. Retrospective methods provided insights into each teachers’ experiences that
occurred over the course of many years. The five teachers in this study valued a variety of experiences that provided triggers and dilemmas related to five schemes: MM as Mathematics, Social Aspects of Working Together, Context Awareness, Attention to Students, and Nature of Curriculum. Teachers’ resolutions to the dilemmas and their learning over time evolved along a trajectory that blended these schemes to form their perspectives for MM and teaching MM.

Several ideas commonly developed in teacher education and PD programs, such as the potential of multiple representations and the value of listening to student thinking, are important to teachers’ early learning about MM doing and teaching. However, additional experiences with explicit connections to MM are necessary for teachers to develop robust understandings needed to do and facilitate MM. The implications of this study support the need for professional learning experiences with embedded opportunities for teachers both to encounter triggers that challenge their current understandings of doing and teaching MM and to resolve their dilemmas. Teachers arguably benefit from safe spaces in which to engage in rational discourse with others and to critically reflect on their experiences. Our ongoing empirical work seeks to better understand teachers’ experiences in learning and teaching MM and the trajectories along which productive sequences of experiences and opportunities for reflection and discourse unfold over time.

**Appendix A**

**Blank event history calendar excerpt**

| Year | 2011 | 2012 | 2013 | 2014 | 2015 | 2016 |
|------|------|------|------|------|------|------|
| Landmark Events → Types of Experiences | COMAP’s Mathematical Modeling Handbook published | NCTM PiA published | GA/ME report released | Blank event calendar excerpt | Where possible, provide a brief description of the experience and the people, places, and feelings associated with the experience. Also briefly describe the salient characteristics of the experience. |
| 1. MS or HS learning experience in modeling | | | | | |
| 2. Undergraduate learning | | | | | |
| 3. Graduate learning | | | | | |
| 4. Experience related to teaching | | | | | |
| 5. Graduate learning related to teaching | | | | | |
| 6. Attending PD* with learning | | | | | |
| 7. Attending PD* with learning | | | | | |
| 8. Teaching a course in which | | | | | |
| 9. Doing a research project | | | | | |
| 10. Conducting PD* related to | | | | | |
| 11. Conducting PD* related to | | | | | |
| 12. Developing lessons or | | | | | |
| 13. Meeting with others to discuss | | | | | |
| 14. Sharing modeling problems with others | | | | | |
| 15. Coaching modeling competitions for students | | | | | |
| 16. Other (Specify) | | | | | |
Appendix B

Critical incident description directions

Think about experiences in which you developed or deepened your understandings of mathematical modeling or how to facilitate students’ mathematical modeling experiences. From your experiences, identify one particularly positive experience and one particularly negative experience related to your informal or formal study of mathematical modeling or how to facilitate students’ mathematical modeling. Please provide written responses to the information requested below. In general, your response to each experience should be approximately one single-spaced page long. You may be asked to expand upon your responses when we meet to discuss your experiences.

Describe one positive experience related to your informal or formal study of mathematical modeling or how to facilitate students’ mathematical modeling— an experience that you recall as being particularly good or that you feel resulted in significant learning on your part. Elaborate on this experience and the timing of the experience. To the extent possible, please address all of the questions listed below in your written response.

Describe one negative experience related to your informal or formal study of mathematical modeling or how to facilitate students’ mathematical modeling—a n experience that you recall as being particularly bad or that you feel affected your perception of your understanding or knowledge of variation or statistics in a negative way. Elaborate on this experience and the timing of the experience. To the extent possible, please address all of the questions listed below in your written response.

List of Questions for each Incident

Details of the experience:

• When, where, and for how long did the experience occur?
• What events or circumstances precipitated the experience or caused the experience to occur in the way in which it did?
• What other people or circumstances played an influential role in the experience?
• How did the experience end?

Reflections on the experience:

• As you reflect on the experience, why do you believe you viewed the events surrounding this experience positively or negatively?
• What emotions did you recall feeling during the experience?
• In response to the experience, what actions did you take?
• What do you believe you learned from the experience?

Effects of the experience:

• How has the experience affected your understanding of mathematical modeling or how to facilitate students’ mathematical modeling?
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- How has the experience affected your teaching of mathematical modeling or how to facilitate students’ mathematical modeling?

Beyond the experience:

- If you could change past events surrounding the experience, what would you change and why?
- If you were to encounter the experience under identical circumstances to those surrounding the original experience, what effect do you believe the experience would have on you today?

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