Representation of Integral Dispersion Relations by Local Forms

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Abstract

The representation of the usual integral dispersion relations (IDR) of scattering theory through series of derivatives of the amplitudes is discussed, extended, simplified, and confirmed as mathematical identities. Forms of derivative dispersion relations (DDR) valid for the whole energy interval, recently obtained and presented as double infinite series, are simplified through the use of new sum rules of the incomplete Γ functions, being reduced to single summations, where the usual convergence criteria are easily applied. For the forms of the imaginary amplitude used in phenomenology of hadronic scattering at high energies, we show that expressions for the DDR can represent, with absolute accuracy, the IDR of scattering theory, as true mathematical identities. Besides the fact that the algebraic manipulation can be easily understood, numerical examples show the accuracy of these representations up to the maximum available machine precision.

As consequence of our work, it is concluded that the standard forms sDDR, originally intended for high energy limits, are an inconvenient and incomplete separation of terms of the full expression, leading to wrong evaluations.

Since the correspondence between IDR and the DDR expansions is linear, our results have wide applicability, covering more general functions, built as combinations of well studied basic forms.

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I. INTRODUCTION

We here deal with the one-subtracted integral dispersion relations (IDR) used in high energy scattering, as introduced originally to study the properties of the complex functions that represent the scattering amplitudes [1, 2]. These relations are derived from general principles of causality, analyticity and crossing, and provide connections between the real and imaginary parts of the amplitudes, helping in the disentanglement of the expressions that represent the observable probabilities. The relations are written for amplitudes that have specific behavior under the operation of crossing symmetry, that connects \( a + b \rightarrow a + b \) and \( a + b \rightarrow a + \bar{b} \) reaction channels (\( \bar{b} \) is the antiparticle of \( b \)).

These dispersion relations are derived using the Cauchy theorem, having the form of infinite integrations over the energy of the scattering process. The applications in phenomenology are restricted because experimental information is limited to finite energy values. In spite of this limitation, applications of IDR to the analysis of hadronic scattering were numerous in the years that followed the original work.

The conversion of integrations over infinite energy intervals into infinite series of derivatives of the amplitudes in a single energy leads to the so called derivative dispersion relations (DDR). The program has run through a long and rather slow progress, which started in 1969 [3], when, in a specific dynamical model in Regge theory framework, Gribov and Migdal wrote a relation in which real and imaginary amplitudes are connected by a single derivative with respect to the logarithm of the energy.

More general relations of local form were written for the even and odd (under crossing) amplitudes [4], through algebraic transformations and series expansions that required appropriate analytical properties of the imaginary amplitude. At about the same time, similar derivative dispersion relations were written [5] and applied to the analysis of hadronic scattering within the Regge dynamics approach. The expressions of these DDR are formed by infinite series of derivatives of increasing order, with the form of a symbolic tangent operator in the derivatives with respect
to the logarithm of the energy, as written below in Eqs. (3, 4).

The need of mathematical convergence of the series, which depends on the analytic form of the imaginary amplitudes, has restricted the applicability and credibility of the method, and soon it was analysed in both aspects of mathematical validity and physical usefulness. Examples of physical interest were given [6] for which the series are divergent, as in the presence of resonance poles in general, and in particular for a Breit-Wigner amplitude.

Convergence conditions were discussed by Heidrich and Kazes [7] who proved that the representation of the amplitude as a series in the logarithm of the energy must have an infinite convergence radius; thus the amplitude must be an entire function of the energy logarithm. Again it was here observed that in the presence of singularities the DDR representation is not justified.

The use of DDR with a Regge representation of the amplitudes, consisting only of log, log-squared and power terms of the energy, without resonance poles, first introduced by Kang and Nicolescu [5], was discussed by Bujak and Dumbrajs [8]. The authors warn for the spurious singular behavior of the tangent series for specific values of the power parameters, which is a peculiar feature of the series summation in the original form of DDR obtained in the low energy approximation.

Concerns with the mathematical validity of the DDR lead Fischer and Kolář, in a series of papers [9, 10, 11], to the establishment of conditions under which the series of derivatives have meaning, converging at a point or in an energy range.

The DDR technique was considered of practical usefulness, in spite of the limitations on the form of functions to which it could be applied, and it was actually used in phenomenological investigations in pp and p̅p scattering. Limitations in this use of DDR were due to the lack of knowledge of the importance of the approximations involved in their derivation, of which the so called high energy approximation is the most important, and also the role of a free parameter $\alpha$ that appeared in the algebraic manipulation leading from IDR to DDR [4]. Use of the integration constant $K$ that appears in the one-subtracted IDR, and sometimes of the value of the quantity $\alpha$, as free parameters, filled gaps between phenomenological parameterizations and
A comprehensive account of the progress of DDR, with clear description of limitations and successes of the method in its applications to hadronic (pp and p\(\bar{p}\)) scattering, is found in [12]. As far as the applications to hadronic scattering are concerned, practical uses of DDR only consider a few forms of energy dependence of the imaginary amplitude that are required to represent the data [13].

A main step in the progress came with the extension of the DDR to the low energy region, up to the threshold [14], with a derivation free of the high energy approximation, opening the way to the true mathematical connection between integral and local representations of scattering amplitudes. The expressions given for the DDR are double infinite series, with rather complicated form, but really presenting no practical difficulty for their numerical summation, as the series have quickly decreasing terms. The only remaining difficulty may rest in the characterization of convergence criteria for double series. Applications to the description of pp and p\(\bar{p}\) data have been very successful, showing perfect superposition of evaluations of the real amplitudes through IDR and the full exact DDR.

In the present paper we deal with the establishment of mathematical relations through which local expressions represent infinite dispersion integrals exactly. We are not concerned here with phenomenological fittings or choice of parameters so that we drop the additive constant \(K\) and put the parameter \(\alpha = 1\). Using sum rules of the incomplete gamma functions which have been derived before [15], we are able to reduce the double series to single summations, where convergence properties are of elementary knowledge. The connections thus established between principal value integrals and series summations are true mathematical identities, and even lead to the establishment of some new formulae of mathematical interest. Our results are explicitly presented for all functions used in hadronic scattering phenomenology [12, 13, 14]. Since the connections between integral and local dispersion relations are linear, we can study simple forms separately, without concern on their superposition to construct physical amplitudes. For imaginary amplitudes of the form of functions of the energy logarithms, built as sums of products of the type \(\ln(E/m)^k \times (E/m)^\lambda\),
it seems that no restriction or approximation needs to be introduced to represent by
single series the integral dispersion relations of scattering theory. The convergence
criteria are the same as the conditions for validity of the original integral forms.

The standard forms of DDR, called sDDR, viciously affected by the high energy
approximation, often have simple expressions in terms of elementary functions, but
are frequently inaccurate as representations of the original IDR that they intend
to substitute. They even create spurious singular behavior where the principal
value integrals are regular. Our investigation of the asymptotic behavior of the full
DDR shows also that the sDDR may even be incomplete as a high energy limit
approximation.

Our work leads to the conclusion that the derivation of Ávila and Menon [14]
is correct, and shows that the separation between standard sDDR and correction
terms is unnecessary and misleading, and from now on should be considered only
for historical reference.

The exact mathematical correspondence between integral and local dispersion
relations opens new lines of investigation, not only in hadronic scattering, but also
in other areas of physics where the use of dispersion techniques is important.

In the present work we take as given the description of the historical facts and
critical analysis about derivative dispersion relations [12], and start from the recent
results of Ávila and Menon [14] that obtained expressions for DDR valid for the
whole energy interval above the physical threshold.

The paper develops the analytical treatment of cases of imaginary amplitudes of
forms \((\ln(E/m))^k \times (E/m)^\lambda\), where \(k\) is an integer. First considering imaginary am-
plitudes \(\text{Im } F_+\) and \(\text{Im } F_-\) of form \((E/m)^\lambda\), where series of derivatives become series
of usual functions, we recall a mathematical relation [15] that expresses in closed
form the sums over the incomplete \(\Gamma\) functions that appear in the double series. The
resulting single series are studied in detail. As illustration of the analytical results,
numerical comparison of maximum machine accuracy between the principal value
integral and the DDR representation is successfully performed.

Special discussion is made of the behavior of DDR in the cases \(k = 0, 1, 2\) in
the proximity of values of $\lambda (\lambda \to 0)$ where spurious singularities appear in the expressions of sDDR. Through numerical tables we show how the correction terms of the full DDR introduce singularities of opposite sign that exactly compensate the sDDR disease, leading to results that regularize the full DDR and are confirmed by the original IDR.

II. GENERAL FORMULAE

We are here concerned with the study of problems in the strong interaction of fundamental particles, such as proton-proton (pp) and proton-antiproton (p\bar{p}) collisions at high energies. All masses being equal, these are processes of the kind $m + m \to m + m$. Protons have antiprotons as their antiparticles, and the pp and p\bar{p} channels are connected by the operation of crossing symmetry, leading to the definition of combined amplitudes, which are even (subscript +) or odd (subscript −) under such operation.

When dispersion relations are constructed, care must be taken of the behavior at large distances in the complex plane, in order to find the proper function for which the relations are valid. The factors introduced as needed for convergence at infinity appear in the form of subtractions, which introduce external constants. In the case of high energy scattering here considered, one subtraction is needed, and the Integral Dispersion Relations (IDR) are written \cite{1,2}

\begin{equation}
\text{Re} \ F_+(E, t) = K + \frac{2E^2}{\pi} P \int_{m}^{+\infty} dE' \frac{\text{Im} \ F_+(E', t)}{E'^2(E'^2 - E^2)} \tag{1}
\end{equation}

and

\begin{equation}
\text{Re} \ F_-(E, t) = \frac{2E}{\pi} P \int_{m}^{+\infty} dE' \frac{\text{Im} \ F_-(E', t)}{(E'^2 - E^2)} , \tag{2}
\end{equation}

respectively for the amplitudes which are even and odd under crossing. $K$ is the subtraction constant of the one-subtracted dispersion relation, $m$ is the proton mass, $E$ is the total energy of the incident particle in the laboratory system, and $t$ is the momentum transfer in the elastic collision ($t$ will be written as equal to zero from now on). The pp and p\bar{p} channels are governed by the combinations

$$F_{pp} = F_+ + F_- , \quad F_{p\bar{p}} = F_+ - F_- .$$
This is a well defined theoretical framework, but it has the practical difficulty that it requires the knowledge of imaginary amplitude in the whole energy interval, from threshold \( m \) to infinity, for the calculation of the real part. Extrapolations to far away energies go beyond safe phenomenological or model-building grounds. This difficulty leads to the development of local relations, in which the real amplitude at a given energy is expressed in terms of derivatives of the imaginary part at the same energy, or in a nearby range.

With the notation and kinematics used in Eqs. (1) and (2), these DDR introduced for the high energy conditions are written

\[
\text{Re } F_+(E, t) = K + E \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \left[ \text{Im } F_+(E, t) \right],
\]

(3)

\[
\text{Re } F_-(E, t) = \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \text{Im } F_-(E, t).
\]

(4)

These forms are derived for high energies (\( m \) goes to zero or \( E \) goes to infinity). They are the traditional DDR, often applied in the description of \( pp \) and \( p\bar{p} \) cross sections [12], called standard derivative dispersion relations, sDDR, by Ávila and Menon [14], to distinguish them from extended forms eDDR that aim at covering also the low energy region, and which are discussed in the present work.

The constant \( K \), which is important in phenomenology, would be transferred without any action from Eq. (1) to the final exact formulae that we write, and will be dropped from now on.

III. CONDITIONS FOR CONVERGENCE

The introduction of DDR raised concerns about its mathematical validity, due to the infinite series of operator derivatives of increasing order, symbolically represented by the tangent operator.

In practice, since we do not know the exact analytical forms of the theoretical amplitudes, in the applications of DDR at high energies one deals with phenomenological representations, obtained from fitting or from models, in terms of simple functions of \( E \) and \( \ln E \), used in the description of \( pp \) and \( p\bar{p} \) at high energies. For
these cases, the questions of convergence and mathematical validity can be treated directly, and may even become trivial, with explicit sums of the resulting series. Actually, in asymptotic limits the whole tangent series can eventually be reduced to its first term.

However, in principle dignified relations should be written and valid for the exact amplitudes, whatever be their analytical forms. Eichmann and Dronkers [6] have shown that the DDR are valid only for certain classes of entire functions of \( \ln(E/m) \). In a series of papers, Kolár and Fischer [9, 10, 11] established theorems fixing the conditions for the validity of the representations of amplitudes in terms of series of derivatives. The main results are expressed as follows.

**Theorem**: Let \( f : \mathbb{R}^1 \to \mathbb{R}^1 \). The series

\[
\tan \left[ \frac{\pi}{2} \frac{d}{dx} \right] f(x)
\]

converges at a point \( x \) in \( \mathbb{R}^1 \) if and only if the series

\[
\sum_{n=0}^{\infty} \frac{d^{2n+1}}{dx^{2n+1}} f(x)
\]

is convergent.

Besides, another theorem implies that the convergence in an energy interval requires that \( f(x) \) be an entire function of complex \( x \) and that the series of positive orders in derivatives

\[
\sum_{n=0}^{\infty} \frac{d^{2n}}{dx^{2n}} f(x)
\]

also converges.

These important theorems allow the examination of the convergence of the tangent series that appear in the applications of DDR to high energy scattering, where simple functions like \( x, x^2, e^{\lambda x}, xe^{\lambda x}, \ldots \) appear. Here \( x \) is written in the place of \( \ln(E/m) \).

**IV. LOW ENERGIES**

The high energy approximations restrict the use of DDR in regions of a few GeV where models and different descriptions of pp and p\( \bar{p} \) scattering are tested and
compared. Ávila and Menon [14] presented a derivation of DDR avoiding the high energy approximations, and showed results representing well the exact IDR, even close to the threshold ($E \approx m$). The results, presented in the form of double infinite sums, have their conditions of convergence difficult to prove. In this section we test these results in some cases of interest.

The results of Ávila and Menon are presented in the following form, called eDDR (extended Derivative Dispersion Relations). For even parity they write

$$\text{Re } F_+(E) = E \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \frac{\text{Im } F_+(E)}{E} + \Delta^+(E, m),$$  \hspace{1cm} (5)

where the correction term $\Delta^+$ is given by

$$\Delta^+(E, m) = \frac{E}{\pi} \ln \left| \frac{m - E}{m + E} \right| \frac{\text{Im } F_+(m)}{m}$$

$$+ \frac{2E}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k + 1, (2p + 1) \ln(E/m))}{(2p + 1)^{k+2}k!} \frac{d^{k+1}}{d(\ln E)^{k+1}} \frac{\text{Im } F_+(E)}{E}.$$  \hspace{1cm} (6)

For the odd relation they obtain

$$\text{Re } F_-(E) = \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) \frac{\text{Im } F_-(E)}{E} + \Delta^-(E, m),$$  \hspace{1cm} (7)

where

$$\Delta^-(E, m) = \frac{1}{\pi} \ln \left| \frac{m - E}{m + E} \right| \frac{\text{Im } F_-(m)}{m}$$

$$+ \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k + 1, (2p + 1) \ln(E/m))}{(2p + 1)^{k+2}k!} \frac{d^{k+1}}{d(\ln E)^{k+1}} \frac{\text{Im } F_-(E)}{E}.$$  \hspace{1cm} (8)

Equations (5) and (7) are meant to be valid, for appropriate functional forms of the imaginary amplitude, for any energy above the physical threshold $E = m$. As a previous verification, before making use of our sum rules for the $\Gamma$ functions, we present in this section practical examples comparing, in accurate numerical terms, the original IDR (supposed to be exact) with the improved eDDR. We take as working examples the forms $(E/m)^{\lambda}$ for $\text{Im } F_+$ and $\text{Im } F_-$, and choose a very low energy $E = 2m$ where the corrections are expected to be large. We then evaluate the contribution of each term to the real amplitude, through IDR and through sDDR and eDDR, to check for differences.
To exemplify the use of Eqs. (5) and (7) in calculations in local form of the integral dispersion relations in Eqs. (1) and (2), we evaluate the differences between the RHS of Eqs. (1) and (3) and between the RHS of Eqs. (2) and (4) and compare with the correction terms $\Delta^+(E, m)$ and $\Delta^-(E, m)$ given in Eqs. (5) and (7) respectively. With these forms for the imaginary amplitudes the tangent series of derivatives in sDDR are explicitly summable. We have in the even case

$$\text{Re} F_+ = (E/m) \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) (E/m)^{\lambda - 1} = (E/m)\lambda \tan \left[ \frac{\pi}{2} (\lambda - 1) \right]$$

and in the odd case

$$\text{Re} F_- = \tan \left( \frac{\pi}{2} \frac{d}{d \ln E} \right) (E/m)^{\lambda} = (E/m)\lambda \tan \left[ \frac{\pi}{2} \lambda \right].$$

(9)

(10)

Results for $\lambda$ equal to 0.5 , 0.75 and 1.0 are given in Table I below. The Cauchy Principal Value integrations were made with the Mathematica 6.0 program. The case $\lambda = 1$ is not applicable to the odd case, as the integral in IDR diverges, and also the sDDR gives the infinite value $\text{Re} F_- = E \tan(\pi/2)$. The agreement is absolutely perfect, showing that for this example the expressions for $\Delta^+(E, m)$ and $\Delta^-(E, m)$ [14] are the exact and final corrections to sDDR.

The value $\lambda = 0$ in $\text{Im} F_+ = (E/m)^{\lambda}$ is regular in the IDR integral of Eq. (11), but leads to a singular value for sDDR of Eq. (9). This spurious singularity in sDDR must be compensated by a similar singularity of opposite sign in $\Delta^+$. This is actually the case, as exhibited numerically in Table II and shown graphically in Fig. II.

The same behavior occurs in cases like $\ln(E/m)(E/m)^{\lambda}$ and $(\ln(E/m))^2 (E/m)^{\lambda}$ which we have studied similarly. Illustrative numerical tables are presented in the appendix.

In the next sections we show how to put in closed form one of the sums in $\Delta^+$ and $\Delta^-$, so that the convergence conditions can be studied with the usual elementary methods.
TABLE I: The low energy correction terms, with $\text{Im} F_+ = (E/m)^\lambda$, $\text{Im} F_- = (E/m)^\lambda$, $E = 2m$. The standard DDR gives wrong values for the real amplitudes of Eqs. (1), (2). The correction terms of Eqs. (5) and (7) fill the gaps with all precision of the machine.

| $\lambda$ | $\text{Re} F_+$ (IDR) | $\text{Re} F_+$ (sDDR) | Expected $\lambda$ correction | $\Delta^+(E, m)$
|-----------|----------------------|----------------------|-----------------------------|------------------|
| 0.5       | -0.0665721174        | -1.4142135624        | +1.3476414450               | +1.3476414450    |
| 0.75      | +0.2202873309        | -0.6966213995        | +0.9169087304               | +0.9169087304    |
| 1.0       | +0.6993983051        | 0.0                 | +0.6993983051               | +0.6993983051    |

| $\lambda$ | $\text{Re} F_-$ (IDR) | $\text{Re} F_-$ (sDDR) | Expected $\lambda$ correction | $\Delta^-(E, m)$
|-----------|----------------------|----------------------|-----------------------------|------------------|
| 0.5       | +1.6536021594        | +1.4142135624        | +0.2393885970               | +0.2393885970    |
| 0.75      | +4.2675819488        | +4.0602070605        | +0.2073748883               | +0.2073748883    |

V. REDUCTION OF THE CORRECTION TERMS TO SINGLE SERIES:
FORMS $(E/m)^\lambda$

In terms of the dimensionless variable

$$\xi \equiv \ln(E/m)$$

the Ávila-Menon corrections [14] to the standard derivative dispersion relations (sDDR) read

$$
\Delta^+(E, m) = -\frac{1}{\pi} e^{\xi} \ln \left| \frac{1 - e^{\xi}}{1 + e^{\xi}} \right| \text{Im} F_+(E = m)
+ \frac{2}{\pi} e^{\xi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k+1, (2p+1)\xi)}{(2p+1)^{k+2} k!} \frac{d^{k+1}}{d\xi^{k+1}} \left( e^{-\xi} \text{Im} F_+(E) \right)
$$

(11)

and

$$
\Delta^-(E, m) = -\frac{1}{\pi} \ln \left| \frac{1 - e^{\xi}}{1 + e^{\xi}} \right| \text{Im} F_-(E = m)
+ \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k+1, (2p+1)\xi)}{(2p+1)^{k+2} k!} \frac{d^{k+1}}{d\xi^{k+1}} \left( \text{Im} F_-(E) \right).
$$

(12)

The incomplete gamma function, with the first argument being an integer, appearing in the right hand sides of both corrections, can be written in the form [16].

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FIG. 1: sDDR and correction term $\Delta^+$ evaluated separately for the example $\text{Im} F_+ = (E/m)^{\lambda}$ in the vicinity of $\lambda = 0$. The two parts become singular with opposite signs, while the sum (not shown in figure), giving the full DDR and the IDR is finite. The plot corresponds to the numbers given in Table II which shows that the full DDR gives precise description of the principal value integral. This example shows dramatically that the separation of sDDR and $\Delta$ terms is very inconvenient.

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\[ \Gamma(k + 1, z) = k! e^{-z} \sum_{n=0}^{k} \frac{z^n}{n!}, \]  
\[ \text{(13)} \]

from which it follows immediately the recurrence relation

\[ \Gamma(k + 1, z) = k \Gamma(k, z) + e^{-z} z^k. \]  
\[ \text{(14)} \]

This function admits the representations

\[ \Gamma(k + 1, z) = e^{-z} z^k \binom{-k}{1; 1} (-1/z), \]  
\[ \text{(15)} \]

in terms of generalized Bessel polynomials, and

\[ \Gamma(k + 1, z) = k! - e^{-z} \frac{z^{k+1}}{k+1} \binom{1}{1; k+2; -z}, \]  
\[ \text{(16)} \]

in terms of confluent hypergeometric functions. These representations may be useful, not only for the numerical evaluation of the incomplete gamma function, but also
TABLE II: Real amplitudes from $F_+$ of power form $\text{Im} F_+ = (E/m)^\lambda$ at several energies, with $\lambda$ approaching the critical value $\lambda = 0$. The standard DDR contribution and correction terms $\Delta^+$ added together reproduce perfectly the direct calculation of the IDR principal value integral (calculated using the Mathematica 6 software). For $\lambda \to 0$, sDDR and the corrections $\Delta^+$ have singularities of form $\approx 1/\lambda$ of opposite signs, that cancel in the sum.

| $\lambda$ | $E/m$ | $\text{Re} F_+(\text{IDR})$ | $\text{Re} F_+(\text{sDDR})$ | $\Delta^+(E,m)$ | $\text{Re} F_+(\text{full DDR})$ |
|-----------|-------|-----------------------------|-----------------------------|-----------------|--------------------------------|
| 0.1       | 2     | -0.313205331434             | -6.766911322477             | 6.453705991043  | -0.313205331434               |
|           | 10    | -1.579297346713             | -7.948542225578             | 6.369244878869  | -1.579297346709               |
|           | 100   | -3.640393754044             | -10.006621794500            | 6.366228040456  | -3.640393754044               |
|           | 1000  | -6.231392436474             | -12.597590463309            | 6.366198026828  | -6.231392436481               |
| 0.01      | 2     | -0.346383461421             | -64.099508812478            | 63.753125348801 | -0.346383463676               |
|           | 10    | -1.474336661559             | -65.139497143396            | 63.665160481840 | -1.474336661556               |
|           | 100   | -2.994782036559             | -66.656790947530            | 63.662008910971 | -2.994782036558               |
|           | 1000  | -4.547449511041             | -68.209427064530            | 63.661977553485 | -4.547449511046               |
| 0.001     | 2     | -0.349370735776             | -637.060672574268           | 636.711301837638| -0.349370736616               |
|           | 10    | -1.463837806700             | -638.086807702120           | 636.622969894093| -1.463837808012               |
|           | 100   | -2.937945523972             | -639.557749708240           | 636.619804184254| -2.937945523972               |
|           | 1000  | -4.412309900131             | -641.032082558884           | 636.619772685732| -4.412309900136               |
| 0         | 2     | -0.349699152566             | $-\infty$                  | $+\infty$       | -0.349699152566               |
|           | 10    | -1.462672076500             | $-\infty$                  | $+\infty$       | -1.462672076497               |
|           | 100   | -2.931710562938             | $-\infty$                  | $+\infty$       | -2.931710562937               |
|           | 1000  | -4.397613274962             | $-\infty$                  | $+\infty$       | -4.397613274967               |

for algebraic simplifications of the summations in Eqs. (11) and (12), as we show below.

Important simplification of the double summations in the right hand sides of Eqs. (11) and (12) may occur for certain forms of $\text{Im} F_+$ and $\text{Im} F_-$. Let us consider the
expression

$$\mathcal{E}_p \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k + 1, (2p + 1)\xi)}{(2p + 1)^{k+2} k!} \frac{d^{k+1}}{d\xi^{k+1}} (e^{\lambda\xi}) , \quad \text{(18)}$$

which appears when $F$ has a power form $e^{\lambda}$. Using the representation (15), it can be written in the form

$$\mathcal{E}_p = -\lambda e^{-(2p+1)\xi} \frac{e^{\lambda\xi}}{(2p + 1)^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} 2F_0 (-k, 1; -1/(2p + 1)\xi) (\lambda\xi)^k . \quad \text{(19)}$$

The sum rule

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2F_0 (-n, \gamma; w) u^n = e^{-u} (1 - w)^{-\gamma}, \quad \text{for } |wu| < 1 , \quad \text{(20)}$$

a particular form of a known relation [17, Sec. 19.10, Eq. (25)] [18, Eq. 6.8.1.2]

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} p+qF_q (-n, (a_p); (b_q); x) = e^{p} \gamma F_q (-n, (a_p); (b_q); -tx) , \quad \text{(21)}$$

that stems from a more general expansion of a hypergeometric function in hypergeometric functions [19, Eq. (1.6)]

$$\left. p+qF_q+s \left( \begin{array}{c} a_1, \ldots, a_p, b_1, \ldots, b_r \\ c_1, \ldots, c_q, d_1, \ldots, d_s \end{array} \right) \right| z w = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n (-z)^n}{(c_1)_n \cdots (c_q)_n \ n!} \frac{z^n}{n!} \quad \text{for } \left. pF_q \left( \begin{array}{c} a_1+n, \ldots, a_p+n \\ c_1+n, \ldots, c_q+n \end{array} \right) \right| \gamma F_s \left( \begin{array}{c} -n, b_1, \ldots, b_r \\ d_1, \ldots, d_s \end{array} \right) \right| w \ , \quad \text{(22)}$$

has been used before [15, Eq. (11)], can be checked by merely expanding its right hand side in powers of $u$, and can be used to reduce to single summations the double sums that appear in Eqs. (11) and (12). It allows us to write

$$\mathcal{E}_p = -\lambda e^{\lambda\xi} \frac{e^{-(2p+1)\xi}}{(2p + 1)^2} \frac{1}{((1 + \lambda/(2p + 1))^{-1} e^{-\lambda\xi}} , \quad \text{(23)}$$

that is,

$$\mathcal{E}_p = -\lambda e^{-(2p+1)\xi} \frac{e^{\lambda\xi}}{(2p + 1)(2p + 1 + \lambda)} , \quad \text{(24)}$$

provided

$$\frac{|\lambda|}{2p + 1} < 1 . \quad \text{(25)}$$

Therefore, if the imaginary parts of the scattering amplitudes have energy dependencies of the form

$$\text{Im } F_\pm = A_\pm \exp(\beta_\pm \xi) , \quad \text{(26)}$$
the corrections given by Eqs. (11) and (12) can be written respectively as

\[
\Delta^+(E, m) = A_+ \left[ -\frac{1}{\pi} e^\xi \ln \left| \frac{1 - e^\xi}{1 + e^\xi} \right| + \sum_{p=0}^{p_+ - 1} \frac{e^{-2p\xi}}{(2p + 1)^2} \sum_{k=0}^{\infty} \frac{(1 - \beta_+)^k \xi^k}{k!} \right]^{2F_0} \left( -k, 1; \frac{-1}{(2p + 1)\xi} \right)
\]

\[
\Delta^-(E, m) = A_- \left[ -\frac{1}{\pi} \ln \left| \frac{1 - e^\xi}{1 + e^\xi} \right| - \sum_{p=p_-}^{p_- - 1} \frac{e^{-2p\xi}}{(2p + 1)^2} \sum_{k=0}^{\infty} \frac{(-\beta_-)^k \xi^k}{k!} \right]^{2F_0} \left( -k, 1; \frac{-1}{(2p + 1)\xi} \right)
\]

and

\[
\Delta^+(E, m) = A_+ \left[ -\frac{1}{\pi} e^\xi \ln \left| \frac{1 - e^\xi}{1 + e^\xi} \right| + \sum_{p=0}^{p_+ - 1} \frac{e^{-2p\xi}}{(2p + 1)^2} \sum_{k=0}^{\infty} \frac{(1 - \beta_+)^k \xi^k}{k!} \right]^{2F_0} \left( -k, 1; \frac{-1}{(2p + 1)\xi} \right)
\]

\[
\Delta^-(E, m) = A_- \left[ -\frac{1}{\pi} \ln \left| \frac{1 - e^\xi}{1 + e^\xi} \right| - \sum_{p=p_-}^{p_- - 1} \frac{e^{-2p\xi}}{(2p + 1)^2} \sum_{k=0}^{\infty} \frac{(-\beta_-)^k \xi^k}{k!} \right]^{2F_0} \left( -k, 1; \frac{-1}{(2p + 1)\xi} \right)
\]

where \( p_+ \) and \( p_- \) represent the lowest integers verifying, respectively,

\[
2p_+ + 1 > |\beta_+ - 1|, \quad 2p_- + 1 > |\beta_-|.
\]

As a practical reference, we mention that a parametrization [14] of the scattering amplitudes to fit the proton-proton and antiproton-proton data is written

\[
\text{Im } F_+(E) = X E^{\alpha_+(0)} + Y_+ E^{\alpha_+(0)},
\]

\[
\text{Im } F_-(E) = Y_- E^{\alpha_-(0)}.
\]

Since the parameters giving the best fit verify the inequalities

\[
1 > |\alpha_+(0) - 1|, \quad 1 > |\alpha_-(0) - 1|, \quad 1 > |\alpha_-(0)|,
\]

Eq. (27) with \( p_+ = 0 \) and Eq. (28) with \( p_- = 0 \) are applicable in this case. Then, replacing \( \ln |(1 - \exp(\xi))/(1 + \exp(\xi))| \) by its series expansion, Eqs. (27) and (28) can be written

\[
\Delta^+(E, m) = A_+ \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{e^{-2p\xi}}{2p + \beta_+}, \quad 0 < \beta_+ < 2,
\]

\[
\Delta^-(E, m) = A_- \frac{2}{\pi} e^{-\xi} \sum_{p=0}^{\infty} \frac{e^{-2p\xi}}{2p + 1 + \beta_-}, \quad -1 < \beta_- < 1.
\]
Obviously, the series in the right-hand sides of these equations converge faster than the geometric one \( \sum_{p=0}^{\infty} (m^2/E^2)^p \) for any value of \( \beta \), although they contain singular terms when \( \beta \) coincides with an even or odd non-positive integer. As it becomes clear from our derivation, the restrictions on the values of \( \beta \) in Eqs. (33) and (34) are imposed to guarantee the applicability of the sum rule Eq. (20) in the replacement of the double sums in (11) and (12) by the single sums in (33) and (34).

The values of \( \Delta^+ (E, m) \) and \( \Delta^- (E, m) \) for some values of \( \beta \) are shown as functions of the energy in Figs. 2 and 3 respectively. The corrections for the even amplitude decrease with the energy to reach finite limits \( 2/\pi \beta_+ \) asymptotically. The corrections \( \Delta^- (E, m) \) of the odd amplitude decrease to zero as the energy increases. The difference of behavior in the even and odd cases is mainly determined by the factor \( e^{-\xi} = m/E \). The behaviors of Eqs. (33), (34), in the limits of large \( \xi \) correspond to a mathematical relation

\[
\lim_{x_0 \to -\infty} \left[ \frac{2}{\pi} x_0^2 \int_1^{+\infty} \frac{x^\lambda - 1}{x^2 - x_0^2} \tan \left( \frac{\pi}{2} (\lambda - 1) \right) \right] = \frac{2}{\pi \lambda}, \quad 0 < \lambda < 2. \tag{35}
\]

To show the relative importance of the correction terms compared to the values of expressions given by sDDR, we draw in Figs. 4 and 5 the ratios

FIG. 2: Corrections to the even DDR, given by \( \Delta^+ (E, m) \), for some values of \( \beta_+ \), as function of the energy. The nonzero values of \( \Delta^+ \) at high energies show that the sDDR do not give the correct and complete asymptotic forms.
FIG. 3: Corrections to the odd DDR, given by $\Delta^-(E,m)$, for some values of $\beta_-$, as function of the energy .

$\Delta^+(E,m)/\text{Re}F_+(\text{sDDR})$ and $\Delta^-(E,m)/\text{Re}F_-(\text{sDDR})$ as functions of the energy, for a few values of $\beta_+$ and $\beta_-$. 

FIG. 4: Ratios between the corrections to the even DDR, given by $\Delta^+(E,m)$, and the standard high energy values given by sDDR , as functions of the energy, for a few values of $\beta_+$. 

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VI. FULL DDR FOR $\text{Im} \, F$ WRITTEN AS COMBINATION OF TERMS $(E/m)^{\lambda} (\ln(E/m))^k$

As mentioned before, sDDR suffer from serious deficiencies and are not able to replace the IDR, even at high energies. Addition of the Ávila-Menon corrections, to give what we call full derivative dispersion relations, copes with those deficiencies. We present, in this Section, a procedure to obtain the full DDR in a general case of the imaginary part of the amplitude being given by a sum

$$\text{Im} \, F = \sum_{n} c_n (E/m)^{\lambda_n} (\ln(E/m))^{k_n}, \quad k_n \text{ integer},$$  \hspace{1cm} (36)$$

with known coefficients $c_n$. If the sum is infinite, the series must obey the convergence criteria mentioned before.

The reason why we consider the form (36) for the imaginary part of the amplitude is mainly because it is an entire function of the logarithm of the scattering energy, a condition that, as shown by Eichmann and Dronkers [6], guarantees the convergence
of series like that appearing in the sDDR, by expansion of the tangent function. Besides this, the finite energy correction to the sDDR due to each term in the right hand side of Eq. (36) admits a very simple form: a single series, obviously convergent and easily calculable. We leave aside the crucial problem of the possibility of representing the scattering amplitude in the whole energy interval by sums of the form (36). As pointed out also in Ref. [6], singularities on the real $E$-axis due to inelastic thresholds and resonance poles for complex $E$ make difficult such representation with the sufficient accuracy as to avoid uncontrollable errors in $\text{Re } F$, like those discussed in Ref. [10, Sec. V]. According to the comments of an anonymous Referee of a previous version of this paper, we do not solve the problem of finding a trustworthy representation of the scattering amplitude as a superposition of entire functions of $\ln(E/m)$. The issue remains open.

The procedure to obtain full DDR in the case of $\text{Im } F$ as given in Eq. (36) is based on the linearity of dispersion relations and benefits from the fact that

$$\left(\ln(E/m)\right)^n = \frac{\partial^n}{\partial \lambda^n}(E/m)^\lambda \bigg|_{\lambda=0}. \quad (37)$$

We now focus our attention on the case of even scattering amplitudes. The odd case is treated analogously.

We first consider the particular case

$$\text{Im } F_+(E, m) = (E/m)^\lambda, \quad (38)$$

already discussed in previous Sections. Equation (9) for the even sDDR gives

$$\text{Re } F_{+, sDDR}(E) = (E/m)^\lambda \tan \left[\frac{\pi}{2}(\lambda - 1)\right] = -(E/m)^\lambda \cot \left[\frac{\pi}{2}\lambda\right]$$

$$= (E/m)^\lambda \left[-\frac{2}{\pi\lambda} + \sum_{n=1}^{\infty} \frac{2^{2n} |B_{2n}|}{(2n)!}\left(\frac{\pi\lambda}{2}\right)^{2n-1}\right], \quad 0 < \lambda < 2,$$

$$0 < \lambda < 2,$$

(39)

whereas the Ávila-Menon correction, as obtained before, is written

$$\Delta^+(E) = \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p}}{2p + \lambda}, \quad 0 < \lambda < 2.$$  

(40)
Adding the two contributions we obtain the full DDR for \( \text{Im} F_+(E, m) = (E/m)\lambda \),

\[
\text{Re } F_+(E) = -\frac{2}{\pi \lambda} ((E/m)^\lambda - 1) - (E/m)^\lambda \left[ \cot \left( \frac{\pi \lambda}{2} \right) - \frac{2}{\pi \lambda} \right] \\
+ \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{2p + \lambda}, \quad -2 < \lambda < 2, \tag{41}
\]

or, equivalently,

\[
\text{Re } F_+(E) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\lambda^n (\ln(E/m))^{n+1}}{(n+1)!} + (E/m)^\lambda \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \pi^{2n}}{(2n)!} \left( \frac{\pi \lambda}{2} \right)^{2n-1} \\
+ \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{2p + \lambda}, \quad -2 < \lambda < 2. \tag{42}
\]

Notice that the range of values of \( \lambda \) for which the last two equations are valid is larger than that of Eqs. (39) and (40). This is not surprising in view of the exact cancellation, when summing, of the singularities occurring in the right hand sides of Eqs. (39) and (40) for \( \lambda = 0 \). This fact has already been discussed at the end of Section IV and is illustrated in Tables II to IV. Derivation \((k\text{-fold})\) with respect to \( \lambda \) gives the full DDR for \( \text{Im} F_+(E, m) = (E/m)^\lambda \ (\ln(E/m))^k \) in the form,

\[
\text{Re } F_+(E) = -\frac{2}{\pi} (\ln(E/m))^{k+1} \sum_{n=0}^{\infty} \frac{\lambda^n (\ln(E/m))^{n}}{n! (n+k+1)} \\
+ \frac{2}{\pi} (E/m)^\lambda \sum_{j=0}^{k} \binom{k}{j} (\ln(E/m))^{k-j} \sum_{n=j+1}^{\infty} \frac{|B_{2n}| \pi^{2n}}{(2n)!} (2n-j)_j \lambda^{2n-1-j} \\
+ \frac{2}{\pi} (-1)^k k! \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{(2p + \lambda)^{k+1}}, \quad -2 < \lambda < 2. \tag{43}
\]

Alternatively, but much more tediously, the last equation could be obtained from Eq. (44) by direct computation and making use, to simplify the double series in Eq. (45), of sum rules stemming from Eq. (20) by successive derivations of both sides with respect to \( u \).

Taking \( \lambda = 0 \) in (44) we obtain obviously the full DDR for \( \text{Im} F_+(E, m) = (\ln(E/m))^k \),

\[
\text{Re } F_+(E) = -\frac{2}{\pi (k+1)} (\ln(E/m))^{k+1} \\
+ \frac{2}{\pi} k! \sum_{l=1}^{[(k+1)/2]} \frac{(\ln(E/m))^{k+1-2l}}{(k+1-2l)!} \frac{|B_{2l}| \pi^{2l}}{(2l)!} \\
+ \frac{2}{\pi} (-1)^k k! \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{(2p)^{k+1}}. \tag{44}
\]
It is now trivial to obtain the full DDR for $\text{Im} F_+(E, m)$ as in Eq. (36).

In the case of odd amplitudes, proceeding as above, we obtain, for $\text{Im} F_-(E, m) = (E/m)^\lambda (\ln(E/m))^k$, the full DDR expression

$$\text{Re} F_-(E) = \frac{2}{\pi} (E/m)^\lambda \sum_{j=0}^{k} \binom{k}{j} (\ln(E/m))^{k-j} \sum_{n=[j/2]+1}^{\infty} \frac{(2n-1)!|B_{2n}|\pi^{2n}}{(2n)!} (2n-j) \lambda^{2n-1-j}$$

$$+ \frac{2}{\pi} (-1)^k k! \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p+1+\lambda)^{k+1}}, \quad -1 < \lambda < 1,$$

and, taking in this expression $\lambda = 0$, the full DDR for $\text{Im} F_-(E, m) = (\ln(E/m))^k$

$$\text{Re} F_-(E) = \frac{2}{\pi} k! \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \frac{(\ln(E/m))^{k+1-2l}}{(k+1-2l)!} \frac{(2^{2l} - 1)!|B_{2l}|\pi^{2l}}{(2l)!}$$

$$+ \frac{2}{\pi} (-1)^k k! \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p+1)^{k+1}}.$$  

(45)

The exact DDR for $\text{Im} F_-(E, m)$ of the form given by Eq. (36) follows immediately.

For convenience of the reader, we present here the full DDR for the forms most frequently assumed for the imaginary part of the amplitude in the phenomenology of elementary particles, namely

$$\text{Im} F^{(a)}(E) = \ln(E/m),$$  

(47)

$$\text{Im} F^{(b)}(E) = (\ln(E/m))^2,$$  

(48)

$$\text{Im} F^{(c)}(E) = (E/m)^\lambda,$$  

(49)

$$\text{Im} F^{(d)}(E) = (E/m)^\lambda \ln(E/m),$$  

(50)

$$\text{Im} F^{(e)}(E) = (E/m)^\lambda (\ln(E/m))^2.$$  

(51)

Full DDR for cases (a) and (b) can be obtained from Eqs. (44) and (46). The case (c) has been thoroughly discussed in the preceding Sections. Formulae for cases (d) and (e) stem from Eqs. (43) and (45). The resulting expressions are

$$\text{Re} F^{(a)}_+(E) = -\frac{1}{\pi} (\ln(E/m))^2 + \frac{\pi}{6} - \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{(2p)^2},$$

(52)

$$\text{Re} F^{(a)}_-(E) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p+1)^2},$$

(53)

$$\text{Re} F^{(b)}_+(E) = -\frac{2}{3\pi} (\ln(E/m))^3 + \frac{\pi}{3} \ln(E/m) + \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{(2p)^3},$$

(54)
\[
\text{Re } F^{(b)}(E) = \pi \ln(E/m) + \frac{4}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p+1)^3}
\]  
(55)

\[
\text{Re } F^{(c)}(E) = (E/m)^\lambda \tan \left(\frac{\pi}{2} (\lambda - 1)\right) + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p}}{2p + \lambda}, \quad |\lambda - 1| < 1,
\]  
(56)

\[
\text{Re } F^{(d)}(E) = (E/m)^\lambda \tan \left(\frac{\pi}{2}\lambda\right) + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{2p + 1 + \lambda}, \quad |\lambda| < 1,
\]  
(57)

\[
\text{Re } F^{(d)}(E) = -\frac{2}{\pi} (\ln(E/m))^3 \sum_{n=0}^{\infty} \frac{\lambda^n (\ln(E/m))^n}{n! (n + 2)}
\]

\[
+ \frac{2}{\pi} (E/m)^\lambda \sum_{n=1}^{\infty} \frac{\lambda \ln(E/m) + 2n - 1}{2n!} \frac{|B_{2n}| \pi^{2n}}{(2n)!} \lambda^{2n-2}
\]

\[
- \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p-1}}{(2p + 1 + \lambda)^2}, \quad |\lambda - 1| < 1,
\]  
(58)

\[
\text{Re } F^{(d)}(E) = \frac{2}{\pi} (E/m)^\lambda \sum_{n=1}^{\infty} \frac{\lambda \ln(E/m) + 2n - 1}{2n!} \frac{(2^{2n-1}) B_{2n} |\pi^{2n}}{(2n)!} \lambda^{2n-2}
\]

\[
- \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p + 1 + \lambda)^2}, \quad |\lambda| < 1,
\]  
(59)

\[
\text{Re } F^{(c)}(E) = -\frac{2}{\pi} (\ln(E/m))^3 \sum_{n=0}^{\infty} \frac{\lambda^n (\ln(E/m))^n}{n! (n + 3)}
\]

\[
+ \frac{2}{\pi} (E/m)^\lambda \sum_{n=1}^{\infty} \left[ (\lambda \ln(E/m))^2 + 2(2n-1)\lambda \ln(E/m) + (2n-1)(2n-2) \right] \frac{|B_{2n}| \pi^{2n}}{(2n)!} \lambda^{2n-3}
\]

\[
+ \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(E/m)^{-2p}}{(2p + \lambda)^3}, \quad |\lambda - 1| < 1,
\]  
(60)

\[
\text{Re } F^{(c)}(E) = \frac{2}{\pi} (E/m)^\lambda \sum_{n=1}^{\infty} \left[ (\lambda \ln(E/m))^2 + 2(2n-1)\lambda \ln(E/m) \right. \]

\[
+ (2n-1)(2n-2) \frac{(2^{2n-1}) B_{2n} |\pi^{2n}}{(2n)!} \lambda^{2n-3}
\]

\[
+ \frac{4}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{(2p + 1 + \lambda)^3}, \quad |\lambda| < 1.
\]  
(61)

Notice that the right hand side of Eq. (56) should be replaced by that of Eq. (42) for values of \(\lambda\) in the interval \((-2, 0]\).
VII. CONTRIBUTIONS TO THE DDR FOR ARBITRARY FORMS OF IM F

We now present alternative forms to the quantities $\Delta^+$ and $\Delta^-$ of Eqs. (11) and (12), for arbitrary functional forms of the scattering amplitudes, reduced through the use of the sum rules of the incomplete gamma functions. We first consider the even amplitude. The treatment of corrections to the odd case is similar.

Let us define the operator
\[
V(p) \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k+1,(2p+1)\xi)}{(2p+1)^{k+2}k!} \frac{d^k}{d\xi^k},
\]
that, in view of Eq. (15), can be written
\[
V(p) = -e^{-(2p+1)\xi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} 2^F_0(-k,1;-1/(2p+1)\xi) \xi^k \frac{d^k}{d\xi^k}.
\]

Since we are now dealing with derivative operators, the sum rule in Eq. (20) must be used carefully. For clarity, it should preferably be written as
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2^F_0(-n,\gamma;;w) u^n = e^{-u} 1^F_0(\gamma;;wu).
\]

To abbreviate, let us define the operators
\[
D_\xi \equiv \frac{d}{d\xi}, \quad D_\xi^n \equiv \xi^n \frac{d^n}{d\xi^n},
\]
and their powers
\[
D^n_\xi \equiv \frac{d^n}{d\xi^n}, \quad D^n_\xi \equiv \xi^n \frac{d^n}{d\xi^n}.
\]

Notice that the last definition is an unnatural one. Then, from Eq. (63) written, with this notation, in the form
\[
V(p) = -e^{-(2p+1)\xi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} 2^F_0(-k,1;-1/(2p+1)\xi) D_\xi^k,
\]
one obtains, by using Eq. (64),
\[
V(p) = -e^{-(2p+1)\xi} \exp(-D_\xi) 1^F_0(1;;-\frac{D_\xi}{2p+1}),
\]
that, with the abbreviations

\[
\mathcal{W}_1 \equiv \exp(-D\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n D^n \xi^n}{n!},
\]

(69)

\[
\mathcal{W}_2(p) \equiv (2p + 1 + D\xi)^{-1} = \frac{1}{2p+1} \operatorname{$_2F_0$}(\frac{1}{2}; -D\xi\frac{2p+1}{2p+1})
\]

(70)

can be written as

\[
\mathcal{V}(p) = -\frac{e^{-(2p+1)\xi}}{2p+1} \mathcal{W}_1 \mathcal{W}_2(p).
\]

(71)

Equations (62) and (71) allow to write for the correction to the sDDR for the even amplitude

\[
\Delta^+(E, m) = -\frac{1}{\pi} e^\xi \ln \left| \frac{1 - e^\xi}{1 + e^\xi} \right| \operatorname{Im} F_+(E = m)
\]

\[
-\frac{2}{\pi} e^\xi \sum_{p=0}^{\infty} \frac{e^{-(2p+1)\xi}}{2p+1} \mathcal{W}_1 \mathcal{W}_2(p) \frac{d}{d\xi} \left( e^{-\xi} \operatorname{Im} F_+(E) \right),
\]

(72)

that is, replacing the logarithm by its series expansion,

\[
\Delta^+(E, m) = 2\frac{\pi}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p}}{2p+1} \left[ \operatorname{Im} F_+(E = m) - \mathcal{W}_1 \mathcal{W}_2(p) \frac{d}{d\xi} \left( e^{-\xi} \operatorname{Im} F_+(E) \right) \right],
\]

(73)

or, equivalently,

\[
\Delta^+(E, m) = 2\frac{\pi}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p}}{2p+1} \left[ \operatorname{Im} F_+(E = m) - \mathcal{W}_1 \mathcal{W}_2(p) E \frac{d}{dE} \left( (E/m)^{-1} \operatorname{Im} F_+(E) \right) \right].
\]

(74)

Following an analogous procedure, it can be obtained from Eq. (12)

\[
\Delta^-(E, m) = 2\frac{\pi}{\pi} \sum_{p=0}^{\infty} \frac{(E/m)^{-2p-1}}{2p+1} \left[ \operatorname{Im} F_-(E = m) - \mathcal{W}_1 \mathcal{W}_2(p) E \frac{d}{dE} \left( \operatorname{Im} F_-(E) \right) \right].
\]

(75)

An expression similar to Eq. (74) has been given by Cudell, Martynov and Selyugin [20, 21] to take account of the corrections to the sDDR. Unfortunately, they
do not give details of the derivation of their final formula, that, besides containing an error, turns out to be confusing. They write, for the corrections given in Eq. (74), the expression

\[ -2 \pi \sum_{p=0}^{\infty} \frac{C_+(p)}{2p+1} \left( \frac{m}{E} \right)^{2p}, \]

(76)

where

\[ C_+(p) = \frac{e^{\xi D_\xi}}{2p+1+D_\xi} \left[ \text{Im} F_+(E) - E \frac{d}{dE} \text{Im} F_+(E) \right]. \]

(77)

Firstly, the absence in Eq. (76) of a term like the first one inside the bracket in the right hand side of Eq. (74) suggests that those authors take for granted that \( \text{Im} F_+(E = m) = 0 \). This is true for amplitudes of the form \( \text{Im} F = (\ln(E/m))^{k(E/m)} \), with \( k \) positive integer, but not for amplitudes like \( (E/m)^{\lambda} \).

Secondly, to avoid ambiguities, the factor \( (m/E)^{2p} \) in Eq. (76) should be moved to the left of \( C_+(p) \), as this operator implies derivations with respect to \( E \). Thirdly, and more important, in the expression (77) of \( C_+(p) \) one can find our operators \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), defined in Eqs. (69) and (70), but the order in which they act on the functions at their right is not specified. Of course, the order is not irrelevant, as it can be easily checked in a very simple particular case:

\[ \mathcal{W}_1 \mathcal{W}_2(p) e^{\lambda \xi} = \mathcal{W}_1 \frac{1}{2p+1+\lambda} e^{\lambda \xi} = \frac{1}{2p+1+\lambda} \]

(78)

\[ \mathcal{W}_2(p) \mathcal{W}_1 e^{\lambda \xi} = \mathcal{W}_2(p) e^{\xi} = \frac{1}{2p+1}. \]

(79)

Finally, a factor \( E^{-1} \) is lacking in front of the bracket in the right hand side of Eq. (77).

The usefulness of Eqs. (74) and (75) is conditioned by the convergence of the series involved. As far as we limit ourselves to forms of the type (36), such convergence is guaranteed, provided the parameters \( \lambda_n \) satisfy the restrictions mentioned in the above equations. The problem of the convergence for more general forms of \( \text{Im} F(E) \), is beyond the scope of this paper. Nevertheless, the similitude of the terms constituting the sDDR and those of the low energy corrections, evident in the paper by Ávila and Menon [14, Sec. II, Subsec. D], together with Theorem 4 in Ref. [10], allow us to conjecture that the converge conditions of the series in \( \Delta^+ \) and \( \Delta^- \) are the same as that of the tangent series in sDDR, mentioned in our Section III.

26
It is interesting to write the expression of the sDDR in terms of the operators introduced above. By expanding the tangent in a double sum, one obtains

$$\text{Re } F_{+, sDDR} = \frac{4}{\pi} (E/m) \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(2p+1)^{2k+2}} \frac{d^{2k+1}}{d^2 k+1} (e^{-\xi} \text{Im } F_+), \quad (80)$$

that can be written in the form

$$\text{Re } F_{+, sDDR} = \frac{4}{\pi} (E/m) \sum_{p=0}^{\infty} \mathcal{W}_2(p) \mathcal{W}_3(p) E \frac{d}{dE} ((E/m)^{-1} \text{Im } F_+(E)), \quad (81)$$

with the operator $\mathcal{W}_3(p)$ given by

$$\mathcal{W}_3(p) \equiv (2p + 1 - D_\xi)^{-1} = \frac{1}{2p+1} \pFq{1}{2p+1}{1}{D_\xi}{2p+1} = \frac{1}{2p+1} \sum_{n=0}^{\infty} \frac{1}{(2p+1)^n} D_\xi^n. \quad (82)$$

Analogously, one obtains for the odd case

$$\text{Re } F_{-, sDDR} = \frac{4}{\pi} \sum_{p=0}^{\infty} \mathcal{W}_2(p) \mathcal{W}_3(p) E \frac{d}{dE} (\text{Im } F_-(E)). \quad (83)$$

Of course, the product of operators $\mathcal{W}_2(p) \mathcal{W}_3(p)$ in Eqs. (81) and (83) can be replaced by the operator

$$\mathcal{W}_4(p) \equiv ((2p+1)^2 - D_\xi^2)^{-1} = \frac{1}{(2p+1)^2} \pFq{1}{1}{(2p+1)^2}{D_\xi^2}{(2p+1)^2} = \frac{1}{(2p+1)^2} \sum_{n=0}^{\infty} \frac{1}{(2p+1)^{2n}} D_\xi^{2n}. \quad (84)$$

**VIII. CONCLUSIONS**

Local forms of dispersion relations have potential to become precious tools in theoretical physics. Avoiding the need of the knowledge of amplitudes and data at energies higher than experiments can provide, which limit the use of the integral dispersion relations, they become a useful technique in the interpretation of the experimental information and in the formulation and test of theoretical models.
Since its origins \cite{4}, the procedure to pass from integral to local dispersion relations is based on mathematical manipulations using series expansions and integrations, whose validity are restricted by severe convergence criteria \cite{6, 7, 9, 10, 11}. These requirements limit the forms of the imaginary amplitude for which the procedure is mathematically allowed. This limitation remains imperative, but it is not an objection in the practice of hadronic phenomenology, since the forms of amplitudes tested in the description of data are simple and treatable functions of $E$ and $\ln E$. However, the predictions obtained with the so called standard DDR, based on the high energy approximation, can be absurdly wrong, as it was already pointed out and discussed by Bujak and Dumbrajs \cite{8} and by Kolár and Fischer \cite{10}.

Recently, Ávila and Menon \cite{14} reformulated the transformation from integral to derivative dispersion relations avoiding the high energy approximation. Using some examples, simple and frequent in applications, we show in the present paper that their new extended forms are complete representations. The double infinite summations in the expressions are correct, although apparently cumbersome, and in this form it is not trivial to prove their convergence.

In the present work we went a step further in the development of these extended forms of DDR, reducing the double to single series summations for functions of general form $\text{Im } F = (E/m)^\lambda \times (\ln(E/m))^k$, with $k$ an integer, which may satisfy the mathematical conditions in the passage from integral to local dispersion relations. The resulting single summations are easy to analyse in terms of usual convergence criteria, and easy to apply, as the terms decrease rapidly.

We give explicit results for a number of basic forms and indicate the way to more general possibilities. Through analytical and numerical examples we show the practicability of the use of the exact local forms for the principal value integrals of IDR.

Our work shows that the so called standard derivative dispersion relation, sDDR, cannot be trusted, and should not be used in applications before specific tests, with comparison to the exact DDR values. It is convenient, from now on, to work only with the exact forms.
Since the connections from IDR to DDR are linear, results can be combined linearly, and the treatment of the basic cases is expected to be useful in many cases. Our basic concern was with hadronic phenomenology, but we may hope that the exact forms of DDR here discussed can be applied as a tool and serve of inspiration in several other areas of physics. From this development, we believe that a new era of applications of derivative dispersion relations can start.

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APPENDIX A: NUMERICAL ILLUSTRATIONS OF USE OF DDR COMPARED TO IDR

We present in Tables III and IV numerical values of even IDR and DDR calculations of $\text{Re} F_+$, for imaginary amplitudes of forms $\text{Im} F_+ = (E/m)^\lambda \ln(E/m)$ and $\text{Im} F_+ = (E/m)^\lambda \ln^2(E/m)$, in the neighborhood of $\lambda = 0$, where the spurious singularities that appear in the separate parts sDDR and $\Delta_+$ are shown to cancel out, leading to the regular behavior of the full DDR, with accurate matching of the numerical principal value integrals.
TABLE III: Real amplitudes from $F_+$ of form $\text{Im} F_+ = (E/m)^\lambda \ln(E/m)$ at several energies, with $\lambda$ approaching the critical value $\lambda = 0$. The standard DDR contribution and correction terms $\Delta^+$ added together reproduce perfectly the direct calculation of the IDR principal value integral (calculated using the Mathematica 6 software program). For $\lambda \to 0$, sDDR and the corrections $\Delta^+$ have singularities of form $\approx 1/\lambda$ of opposite signs, that cancel in the sum.

| $\lambda$ | $E/m$ | $\text{Re} F_+(IDR)$ | $\text{Re} F_+(sDDR)$ | $\Delta^+(E, m)$ | $\text{Re} F_+(\text{full DDR})$ |
|---|---|---|---|---|---|
| 0.1 | 2 | 0.403965347917 | 64.104711735696 | -63.700746387780 | 0.403965347917 |
| | 10 | -1.157501083855 | 62.505923539517 | -63.663424623364 | -1.157501083847 |
| | 100 | -8.012794316442 | 55.649197356317 | -63.661991672959 | -8.012794316643 |
| | 1000 | -22.610812691731 | 41.051164689357 | -63.661977381116 | -22.610812691759 |
| 0.01 | 2 | 0.335089941935 | 6366.575003517582 | -6366.239913575625 | 0.335089941935 |
| | 10 | -1.167198982181 | 6365.032104422951 | -6366.199303405104 | -1.167198982174 |
| | 100 | -6.387924518333 | 6359.80814915427 | -6366.197739433735 | -6.387924518332 |
| | 1000 | -15.306917952512 | 6350.89080588063 | -6366.197723833389 | -15.306917952549 |
| 0.001 | 2 | 0.328764505695 | 636620.1436892521 | -636619.8149247168 | 0.328764505695 |
| | 10 | -1.165818944785 | 636618.6081426198 | -636619.7739615351 | -1.165818944778 |
| | 100 | -6.2429244888218 | 636613.5294586228 | -636619.7723834814 | -6.2429244888217 |
| | 1000 | -14.728077626309 | 636605.0442901437 | -636619.7723677403 | -14.728077626321 |
| 0 | 2 | 0.328067590880 | $-\infty$ | $+\infty$ | 0.328067590880 |
| | 10 | -1.165643354601 | $-\infty$ | $+\infty$ | -1.165643354595 |
| | 100 | -6.227003476713 | $-\infty$ | $+\infty$ | -6.227003476712 |
| | 1000 | -14.665220640485 | $-\infty$ | $+\infty$ | -14.665220640497 |

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TABLE IV: Real amplitudes from $F_+$ of form $\text{Im} F_+ = (E/m)^\lambda \ln^2(E/m)$ at several energies, with $\lambda$ approaching the critical value $\lambda = 0$. The standard DDR contribution and correction terms $\Delta^+$ added together reproduce perfectly the direct calculation of the IDR principal value integral (calculated using the Mathematica 6 software program). For $\lambda \to 0$, sDDR and the corrections $\Delta^+$ have singularities of form $\approx 1/\lambda$ of opposite signs, that cancel in the sum.

| $\lambda$ | $E/m$ | $\text{Re} F_+(IDR)$ | $\text{Re} F_+(sDDR)$ | $\Delta^+(E,m)$ | $\text{Re} F_+(\text{full DDR})$ |
|-----------|-------|------------------------|------------------------|-----------------|-----------------------------|
| 0.1       | 2     | 0.825818895910         | -1.272.49349885946     | 1273.275168781875 | 0.825818895910             |
|           | 10    | 0.385649697828         | -1.272.855271730595     | 1273.240921428424 | 0.385649697828             |
|           | 100   | -19.862174910070       | -1.293.107333393810     | 1273.239558483750 | -19.862174910061           |
|           | 1000  | -98.840346530164       | -1.372.079891403041     | 1273.239544872647 | -98.840346530396           |
| 0.01      | 2     | 0.708180406300         | -1.273238.877089482     | 1273239.585269881 | 0.708180406300             |
|           | 10    | -0.132971707507        | -1.273239.679276770     | 1273239.546305056 | -0.132971707508           |
|           | 100   | -16.282484687019       | -1.273255.827235536     | 1273239.544750842 | -16.282484687013           |
|           | 1000  | -65.648292253245       | -1.273305.192964580     | 1273239.544735319 | -65.648292253369           |
| 0.001     | 2     | 0.697502589569         | -1.273239544078824E+09  | 1.273239544776237E+09 | 0.697502589569          |
|           | 10    | -0.173398449022        | -1.273239544910241E+09  | 1.273239544736754E+09 | -0.173398449022         |
|           | 100   | -15.940304163616       | -1.273239560675572E+09  | 1.273239544735179E+09 | -15.940304163611        |
|           | 1000  | -63.000875389386       | -1.273239607736127E+09  | 1.273239544735163E+09 | -63.000875389502        |
| 0         | 2     | 0.696327412793         | $-\infty$               | $+\infty$        | 0.696327412793           |
|           | 10    | -0.177778228941        | $-\infty$               | $+\infty$        | -0.177778228942         |
|           | 100   | -15.902527098975       | $-\infty$               | $+\infty$        | -15.902527098969        |
|           | 1000  | -62.713313030736       | $-\infty$               | $+\infty$        | -62.713313030851        |

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