ON CONNECTIVE $K$-THEORY OF ELEMENTARY ABELIAN 2-GROUPS AND LOCAL DUALITY

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Abstract. The connective $ku$-(co)homology of elementary abelian 2-groups is determined as a functor of the elementary abelian 2-group. The argument requires only the calculation of the rank one case and the Atiyah-Segal theorem for $KU$-cohomology together with an analysis of the functorial structure of the integral group ring. The methods can also be applied to the odd primary case. These results are used to analyse the local cohomology spectral sequence calculating $ku$-homology, via a functorial version of local duality for Koszul complexes. This gives a conceptual explanation of results of Bruner and Greenlees.

1. Introduction

The calculation of the $ku$-(co)homology of finite groups is an interesting and highly non-trivial problem. The case of elementary abelian $p$-groups illustrates important features; these groups were first calculated by Ossa [Oss89] and were studied further by Bruner and Greenlees [BG03], exhibiting a form of duality via local cohomology. Neither of these references exploit the full naturality of the functors $V \mapsto ku^*(BV_+)$ and $V \mapsto ku_*(BV_+)$. This paper shows how studying these as functors of the elementary abelian $p$-group $V$ gives a new and conceptual approach. The methods apply to any prime $p$; the case $p = 2$ is privileged here since this requires an additional filtration argument when studying the local cohomology. Moreover, this is the case of interest when extending the methods to the study of $ko$-(co)homology (cf. [BG10]); this will be developed elsewhere. The main results of the first part of the paper give complete descriptions of the functors $V \mapsto ku^*(BV_+)$ (Theorem 5.19) and $V \mapsto ku_*(BV_+)$ (Theorem 5.22).

For $ku$-cohomology, the only input which is required is the graded abelian group structure of $ku^*(BZ/2_+)$ and the identification of the functor $V \mapsto KU^0(BV_+)$ for periodic complex $K$-theory, which is provided by the Atiyah-Segal completion theorem. In particular, the method gives a conceptual proof of an algebraic form of Ossa’s theorem [Oss89], which gives a reduction to the rank one case. Both Ossa [Oss89] and Bruner and Greenlees [BG03] use Ossa’s theorem as a starting point for their calculations.

To give a full functorial description, the integral group ring functor $V \mapsto Z[V]$ is studied, stressing the functorial viewpoint and extending results of Passi and others [PV77]. Of independent interest is the observation that the quotients which arise from studying the filtration of $Z[V]$ by powers of the augmentation ideal are self-dual under Pontrjagin duality (see Theorems 5.14 and 5.12).

Similarly, for $ku$-homology, only knowledge of $ku_*(BZ/2_+)$ is required, together with an understanding of the functor $V \mapsto KU_1(BV_+)$. The arguments make
explicit the functorial nature of the duality between $ku^*(BV_+)$ and $ku_*(BV_+)$, via Pontrjagin duality.

The second part of the paper applies these results to give an analysis of the local cohomology spectral sequence relating $ku^*(BV_+)$ to $ku_*(BV_+)$ (see Theorem [5.20]), this sheds light upon the description given by Bruner and Greenlees [BG03]: local duality appears as an explicit functor defined in the functorial context. The key observation which explains the origin of the differentials in the local cohomology spectral sequence comes from the analysis of $ku^*(BV_+)$, which shows how the v-torsion $\text{tors}_v ku^*(BV_+)$ and the v-cotorsion of $ku^*(BV_+)$ are related.

The functorial description of $V \mapsto ku^*(BV_+)$ identifies the mod-p cohomology of the spaces of the $\Omega$-spectrum for $ku$ (up to nilpotent unstable modules), via Lannes’ theory [Lan92]. This gives a conceptual framework for understanding the results of [Sto63, Sin68], and can be related to the description of the mod-p homology in terms of Hopf rings [Har91] (which does not a priori retain information on the action of the Steenrod algebra). This will be explained elsewhere.

## Contents

1. Introduction 1
2. Background 2
3. The integral group ring functor 4
4. Milnor derivations 8
5. Connective complex K-cohomology and homology 10
6. Local duality 18
7. Filtering symmetric powers and Koszul complexes 21
8. The local cohomology spectral sequence 24
References 31

## 2. Background

### 2.1. Definitions and notation.

Fix a prime $p$ and let $\mathbb{F}$ denote the prime field $\mathbb{F}_p$; $\mathcal{V}^f$ denotes the full subcategory of finite-dimensional spaces in the category $\mathcal{V}$ of $\mathbb{F}$-vector spaces and vector space duality is denoted by $(-)^{\bigvee} : \mathcal{V}^f \to \mathcal{V}^f$.

**Notation 2.1.** The category of functors from $\mathcal{V}^f$ to abelian groups is denoted $\mathcal{F} \mathcal{A}$ and the full subcategory of functors with values in $\mathcal{V}$ is denoted $\mathcal{F}$.

The categories $\mathcal{F} \mathcal{A}, \mathcal{F}$ are tensor abelian, with structure induced from $\mathcal{A}/b$. (For basic properties of $\mathcal{F}$, see [Kuh94], [Kuh94b], [Kuh95], or [FFSS99].) There is an exact Pontrjagin duality functor which generalizes the duality for $\mathcal{F}$ introduced in [Kuh94].

**Definition 2.2.** Let $D : \mathcal{F} \mathcal{A}^{\text{op}} \to \mathcal{F} \mathcal{A}$ be the functor defined on $F \in \mathcal{F} \mathcal{A}$ by

$$DF(V) := \text{Hom}_{\mathcal{A}/b}(F(V^2), \mathbb{Z}/p^{\infty}).$$

Recall that the socle of an object is its largest semi-simple subobject and the head its largest semi-simple quotient.

**Example 2.3.** The symmetric powers, divided powers and exterior powers are fundamental examples of (polynomial) functors in $\mathcal{F}$. For $n \in \mathbb{N}$, the $n$th symmetric power functor $S^n$ is defined by $S^n(V) := (V^\otimes n)/\mathcal{S}_n$, the $n$th divided power functor by $\Gamma^n(V) := (V^\otimes n)^{\mathcal{S}_n}$ and the $n$th exterior power functor identifies as $\Lambda^n(V) \cong (V^\otimes n \otimes \text{sign})^{\mathcal{S}_n}$, where sign is the sign representation of $\mathcal{S}_n$. By convention, these functors are zero for negative integers $n$. There is a duality relation $S^n \cong D\Gamma^n$,
whereas the functor $\Lambda^n$ is self-dual (there is a canonical isomorphism $DA^n \cong \Lambda^n$).

For $p = 2$, the functor $\Lambda^n$ is the head of $S^n$ and the socle of $\Gamma^n$.

These functors are examples of graded exponential functors; for example, the exponential structure induces natural coproducts $S^n \cong S^i \otimes S^j$ and products $S^i \otimes S^j \cong S^n$, for integers $i + j = n$, and, upon evaluation on $V \in \text{Ob} \mathcal{F}$, these correspond to the primitively-generated Hopf algebra structure on a polynomial algebra.

**Example 2.4.** Yoneda’s lemma provides the standard injective and projective objects of $\mathcal{F}$ (the case of $\mathcal{F}_\text{set}$ is considered in Section 3). The projective functor $P_\mathcal{F}$ is the functor $V \mapsto \mathbb{F}[V]$, which corepresents evaluation at $\mathbb{F}$; the injective functor $I_\mathcal{F}$ is given by $V \mapsto \mathbb{F}^V$ (the vector space of set maps) and represents the dual evaluation functor $F \mapsto DF(\mathbb{F})$. Duality provides the relation $I_\mathcal{F} \cong DF(P_\mathcal{F})$, which relates the canonical decompositions $I_\mathcal{F} \cong \mathbb{F} \oplus T_\mathcal{F}$ and $P_\mathcal{F} \cong F \oplus T_\mathcal{F}$, where $F$ is the constant functor and $T_\mathcal{F}$ (respectively $P_\mathcal{F}$) is the complementary constant-free summand.

The functor $I_\mathcal{F}$ is ungraded exponential and has associated diagonal $\Delta : I_\mathcal{F} \to I_\mathcal{F} \otimes I_\mathcal{F}$ and multiplication $\mu : I_\mathcal{F} \otimes I_\mathcal{F} \to I_\mathcal{F}$; these morphisms induce $T_\mathcal{F} \to T_\mathcal{F} \otimes T_\mathcal{F}$ and $T_\mathcal{F} \otimes T_\mathcal{F} \to T_\mathcal{F}$ respectively and, in both cases, these are the unique non-trivial morphisms of the given form. Dually, there is a product $P_\mathcal{F} \otimes P_\mathcal{F} \to P_\mathcal{F}$ and coproduct $P_\mathcal{F} \to P_\mathcal{F} \otimes P_\mathcal{F}$.

**Notation 2.5.** For $n > 0$ an integer:

1. let $P^n_\mathcal{F}$ be the image of the iterated product $\mu^{(n-1)} : P^{\otimes n}_\mathcal{F} \to P_\mathcal{F}$ and $q_{n-1}P^n_\mathcal{F}$ denote its cokernel, so that there is a short exact sequence

   $$0 \to P^{\otimes n}_\mathcal{F} \to P^n_\mathcal{F} \to q_{n-1}P^n_\mathcal{F} \to 0;$$

2. dually, let $p_{n-1}P^n_\mathcal{F}$ denote the kernel of the iterated diagonal $T^n_\mathcal{F} \cong P^{\otimes n}_\mathcal{F}$.

**Lemma 2.6.** Suppose that $p = 2$. Then

1. $T_\mathcal{F}$ is the injective envelope of $\Lambda^1$, is uniserial, $T_\mathcal{F} \cong \text{colim}_{p \mathcal{F}}$, and there are non-split short exact sequences

   $$0 \to p_n T_\mathcal{F} \to p_{n+1} T_\mathcal{F} \to \Lambda^{n+1} \to 0;$$

2. $P_\mathcal{F}$ is the projective cover of $\Lambda^1$, is uniserial, $P_\mathcal{F} \cong \text{lim}_{p_n P_\mathcal{F}}$, and there are short exact sequences

   $$0 \to \Lambda^{n+1} \to q_{n+1}P_\mathcal{F} \to q_n P_\mathcal{F} \to 0.$$

The functor $q_n P_\mathcal{F}$ is the dual of $p_n T_\mathcal{F}$.

The fact that $p_n T_\mathcal{F}$ has a simple socle (for $n > 0$) implies that it is easy to detect non-triviality of a subobject:

**Lemma 2.7.** Suppose that $p = 2$ and let $n > 0$ be an integer.

1. If $G \subset p_n T_\mathcal{F}$, then $G = 0$ if and only if $G(\mathbb{F}) = 0$.
2. If $H \subset q_n P_\mathcal{F}$, then $H = q_n P_\mathcal{F}$ if and only if $H(\mathbb{F}) \neq 0$.

**Proof.** The two statements are equivalent by duality, hence it suffices to prove the first. The socle of $p_n T_\mathcal{F}$ is $\Lambda^1$; if $G \neq 0$ then $\Lambda^1 \subset G$, hence $G(\mathbb{F}) \neq 0$, since $\Lambda^1(\mathbb{F}) = 0$. The converse is obvious. □

**Remark 2.8.** There are analogous results for odd primes, taking into account the weight splitting of the category $\mathcal{F}$ provided by the action of the units $\mathbb{F}_p$ (cf. [Kuhl94a]).
3. The integral group ring functor

This section provides a functorial analysis of the structure of the integral group ring functor; throughout, the prime is taken to be 2 (there are analogous results for odd primes). These results are necessary to complete the full functorial description of the \( ku \)-cohomology of elementary abelian 2-groups but are not required for the proof of the algebraic version of Ossa’s theorem.

3.1. The functors \( P_Z, P_{Z_2} \).

Notation 3.1. Let

1. \( P_Z \) denote the integral group ring functor \( V \mapsto Z[V] \) and \( P_Z \) the augmentation ideal, so that there is a direct sum decomposition \( P_Z \cong Z \oplus P_Z \) in \( F A \);
2. \( P_{Z_2} \) denote the functor \( Z_2 \otimes P_Z \) and \( P_{Z_2} \) the functor \( Z_2 \otimes P_Z \), where \( Z_2 \) denotes the 2-adic integers.

Yoneda’s lemma implies:

Lemma 3.2. The functor \( P_Z \) is projective in \( F A \) and corepresents evaluation on \( F \).

The ring structure of \( Z[V] \) gives a morphism \( \mu : P_Z \otimes P_Z \to P_Z \), which induces \( \mu : P_{Z_2} \otimes P_{Z_2} \to P_{Z_2} \). There is a reduced diagonal \( \Delta : P_{Z_2} \to P_{Z_2} \otimes P_{Z_2} \); the composition with the canonical projection \( P_Z \to P_{Z_2} \) is determined by the element \( ([1] - [0]) \otimes ([1] - [0]) \in (P_Z \otimes P_Z)(F) \), by Yoneda.

Definition 3.3. For \( n \in \mathbb{N} \), let \( P^n_Z \) denote the image of the iterated product \( \mu^{n-1} : P_Z \otimes P_Z \to P_Z \) (respectively \( P^n_{Z_2} \subset P_{Z_2} \)).

The structure of \( P_{Z_2}(V) \) and the filtration

\[
\ldots \subset P_Z^{n+1}(V) \subset P_Z^n(V) \subset \ldots \subset P_Z^1(V) = P_Z(V)
\]

for a fixed \( V \) has received much attention (see [PV77, BV00], for instance). These references do not exploit functoriality.

Lemma 3.4.

1. There is a natural isomorphism \( (P_{Z_2})/2 \cong P_{Z_2} \).
2. For \( n \in \mathbb{N} \), the canonical surjection \( P_{Z_2} \to P_Z \) induces a commutative diagram in \( F A \):

\[
\begin{array}{ccc}
P_{Z_2} & \xrightarrow{\mu} & P_Z \\
\downarrow & & \downarrow \\
P_Z & \xrightarrow{\mu} & P_{Z_2} \\
\end{array}
\]

3. The head of \( P_{Z_2} \) is the functor \( \Lambda^1 \).

Proof. The commutative diagram follows from the fact that \( Z \to F \) induces a morphism of group rings \( Z[V] \to F[V] \); the remaining statements are clear. \( \square \)

Lemma 3.5. The composite \( P_Z \xrightarrow{\Delta} P_Z \otimes P_Z \xrightarrow{\mu} P_Z \) is the morphism \( P_Z \xrightarrow{\gamma} P_Z \). Hence, for \( n \in \mathbb{N} \), there are inclusions of subobjects of \( P_{Z_2} \):

- \( 2^n P_Z \subset P_{Z_2}^{n+1} \) and \( 2^n P_{Z_2} \subset P_{Z_2}^{n+1} \).

Proof. The first statement is straightforward (cf. [BV00 Lemma 3.2]); this gives rise to the natural inclusion \( 2^n P_Z \subset P_{Z_2}^{n+1} \). The final statement follows by induction. \( \square \)
Lemma 3.6. For $n \in \mathbb{N}$, there is a unique non-trivial morphism $\mathcal{T}_Z^n \rightarrow \mathcal{T}_F$ in $\mathcal{F} \mathcal{A}$ and this induces a surjection
\[
\frac{\mathcal{T}_Z^n}{\mathcal{T}_Z^{n+1}} \twoheadrightarrow p_n \mathcal{T}_F.
\]

Proof. A morphism $\mathcal{T}_Z^n \rightarrow \mathcal{T}_F$ factors naturally across $(\mathcal{T}_Z^n)/2$. It is straightforward to see that $((\mathcal{T}_Z^n)/2)(F) = F$ and $((\mathcal{T}_Z^n)/2)(0) = 0$; this implies that there is a unique non-trivial morphism $(\mathcal{T}_Z^n)/2 \rightarrow \mathcal{T}_F$, by Yoneda.

The composite $\mathcal{T}_Z^{\otimes n} \rightarrow \mathcal{T}_Z^n \rightarrow \mathcal{T}_F$ factorizes across the projection $\mathcal{T}_Z^{\otimes n} \rightarrow \mathcal{T}_F^{\otimes n}$. Again, there is a unique non-trivial morphism from $\mathcal{T}_F^{\otimes n}$ to $\mathcal{T}_F$; for $n = 1$, this is the composite $\mathcal{T}_F \rightarrow \Lambda^1 \hookrightarrow \mathcal{T}_F$ and, for $n > 1$,
\[
\mathcal{T}_F^{\otimes n} \rightarrow \mathcal{T}_F^{\otimes n} \mu^{(n-1)} \rightarrow \mathcal{T}_F,
\]
where the first morphism is the iterated tensor product of the morphism $\mathcal{T}_F \rightarrow \mathcal{T}_F$; it follows easily that the composite has image $p_n \mathcal{T}_F$. Hence the morphism $\mathcal{T}_F \rightarrow \mathcal{T}_F^{\otimes n}$ has image $p_n \mathcal{T}_F$, by Lemma 2.6.

Finally, the fact that $\mathcal{T}_F^2$ is the kernel of $\mathcal{T}_F \rightarrow \Lambda^1$ implies that the composite
\[
\mathcal{T}_Z^{n+1} \rightarrow \mathcal{T}_Z^n \rightarrow \mathcal{T}_F
\]
is trivial, giving the stated factorization. \hfill \Box

A non-functorial version of the following result (expressed using different notation) is proved in [PV77]. Since the functorial version is required here, a direct proof is given.

Proposition 3.7. For $n > 0$ an integer, the following statements hold:

1. there is a short exact sequence:
\[
0 \rightarrow 2\mathcal{T}_Z^n \rightarrow \mathcal{T}_Z^{n+1} \rightarrow \mathcal{T}_F^{n+1} \rightarrow 0;
\]

2. the canonical morphism $\mathcal{T}_Z^n/\mathcal{T}_Z^{n+1} \rightarrow p_n \mathcal{T}_F$ is an isomorphism.

Proof. The statements are proved in parallel by induction upon $n$. For $n = 1$, consider the commutative diagram
\[
\begin{array}{c}
2\mathcal{T}_Z^\varepsilon \rightarrow \mathcal{T}_Z^2 \rightarrow \mathcal{T}_Z^2/2\mathcal{T}_Z \\
\downarrow \downarrow \downarrow \\
2\mathcal{T}_Z^\varepsilon \rightarrow \mathcal{T}_Z \rightarrow \mathcal{T}_F.
\end{array}
\]
The image of $\mathcal{T}_Z^2$ in $\mathcal{T}_F$ is $\mathcal{T}_F^2$, by Lemma 3.4, which proves the first statement. The second statement follows by studying the cokernels of the monomorphisms.

For the inductive step, suppose the result true for $n < N$. Lemma 3.5 provides a natural inclusion $2\mathcal{T}_Z^n \subset \mathcal{T}_Z^{n+1}$, for all natural numbers $i$; for current purposes, it is sufficient to work with the inclusion $2^{i+1}\mathcal{T}_Z^n \subset \mathcal{T}_Z^{n+1}$. The proof proceeds by providing upper and lower bounds for $\mathcal{T}_Z^{n+1}/2^{n+1}\mathcal{T}_Z$. It is sufficient to do this in the Grothendieck group of $\mathcal{F} \mathcal{A}$, since the functors considered below only have finitely many composition factors of a given isomorphism type, which allows comparison arguments. (In fact, evaluation on finite dimensional vector spaces allows the arguments to be reduced to objects which are finite.)

The element of the Grothendieck group associated to an object $F$ is denoted $[F]$; for the functors considered here, this lies in the submonoid $\prod_{\lambda} \mathbb{N}$ indexed by the isomorphism classes of simple objects $S_\lambda$ of $\mathcal{F} \mathcal{A}$; for two objects $M = \Sigma M_\lambda[S_\lambda]$ and $N = \Sigma N_\lambda[S_\lambda]$, write $M \leq N$ if $M_\lambda \leq N_\lambda$, for all $\lambda$. \hfill \Box
The inductive hypothesis implies that:

\[ \left[ \frac{P^N}{2NP_Z} \right] = \sum_{i=1}^{N} \left[ \frac{P}{2^{i+1}P_Z} \right] \]

in the Grothendieck group.

The inclusion \( 2P^N \rightarrow P^{N+1} \) induces a monomorphism

\( (P^N_Z / 2NP_Z) \cong 2P^N_Z / 2NP^{N+1} \rightarrow P^{N+1}_{Z} / 2NP^{N+1}_Z; \)

the composite of this morphism with the projection \( P^{N+1}_Z / 2NP^{N+1}_Z \rightarrow P^{N+1}_Z \) is clearly trivial, hence this gives a lower bound for \( P^{N+1}_Z / 2NP^{N+1}_Z; \)

\[ \sum_{i=1}^{N+1} \left[ \frac{P}{2^{i+1}P_Z} \right] \leq \left[ \frac{P^{N+1}}{2NP^N_Z} \right], \]

with equality if and only if the first statement holds.

Consider the inclusion \( P^{N+1}_Z \hookrightarrow P^N_Z \), which induces the monomorphism

\( P^{N+1}_Z / 2NP^{N+1}_Z \rightarrow P^N_Z / 2NP^N_Z \)

with cokernel which surjects to \( p_N^N \), by Lemma 3.5 this gives the inequality:

\[ \left[ \frac{P^N}{2NP^N_Z} \right] \geq \left[ \frac{P^{N+1}}{2NP^{N+1}_Z} \right] + [p_N^N], \]

with equality if and only if the second statement holds.

The short exact sequence

\[ 0 \rightarrow P^N \cong (2^N P_Z / 2NP^N_Z) \rightarrow \frac{P^N}{2NP^N_Z} \rightarrow \frac{P^N}{2NP^{N+1}_Z} \rightarrow 0 \]

and the inductive hypothesis give:

\[ \left[ \frac{P^N}{2NP^{N+1}_Z} \right] = \left[ \frac{P}{2^{i+1}P_Z} \right] + \sum_{i=1}^{N} \left[ \frac{P}{2^{i+1}P_Z} \right]. \]

Now \( \left[ \frac{P}{2^{i+1}P_Z} \right] = [p^{N+1}_Z] + [p_N^N], \) hence (1) implies that

\[ \sum_{i=1}^{N+1} \left[ \frac{P}{2^{i+1}P_Z} \right] \geq \left[ \frac{P^{N+1}}{2NP^{N+1}_Z} \right], \]

with equality if and only if the second statement holds. Hence, both inequalities are equalities and the inductive step is established.

**Corollary 3.8.** For \( V \in \text{Ob} \mathcal{V}^f \), the topologies on the abelian group \( P_Z(V) \) induced by the \( 2 \)-adic filtration \( 2 \frac{P}{P_Z} \) and by the filtration \( P_Z \) are equivalent.

**Proof.** By Lemma 3.5 \( 2^n P_Z \subset P^{n+1}_Z \); conversely, it is straightforward to show, using Proposition 3.7 that \( P^{n+k}(\mathbb{F}^n) \subset 2^k P_Z(\mathbb{F}^n). \)

**3.2. The structure of the quotients \( \frac{P_Z}{P^{n+1}_Z}. \)**

**Notation 3.9.** For \( n \in \mathbb{N} \), let \( \frac{P}{P^n_Z} \) denote the quotient \( \frac{P_Z}{P^{n+1}_Z}. \)

**Lemma 3.10.** For \( n > 0 \) an integer:

1. \( 2^{n+1} \frac{P}{P^n_Z} = 0 \) and \( \frac{P}{P^n_Z} \cong \mathbb{Z}/2^n; \)
2. there are short exact sequences:

\[ 0 \rightarrow p_n^N \rightarrow \frac{P}{P^n_Z} \rightarrow \frac{P}{P^{n-1}_Z} \rightarrow 0 \]

\[ 0 \rightarrow \frac{P}{P^{n-1}_Z} \rightarrow \frac{P}{P^n_Z} \rightarrow q_n^N \rightarrow 0; \]
3. the largest subfunctor \( 2 \frac{P}{P^n_Z} \) of \( \frac{P}{P^n_Z} \) annihilated by 2 is isomorphic to \( p_n^N \).
Proof. The first statement is clear and the first short exact sequence is provided by Proposition 3.7. The second short exact sequence is induced by the inclusion $2\mathcal{T}_Z \hookrightarrow \mathcal{T}_Z$, since Proposition 3.7 implies that $2\mathcal{T}_Z \cap \mathcal{T}_Z$ is isomorphic to $\mathcal{T}_Z$ under the isomorphism $\mathcal{T}_Z \cong 2\mathcal{T}_Z$.

The proof that the largest subfunctor of $R^n_\mathcal{F}$ annihilated by 2 is $p_n\mathcal{T}_\mathcal{F}$ is by induction on $n$; for $n = 1$, $R^1_\mathcal{F} \cong \Lambda^1$ and the result is immediate. For the inductive step, the short exact sequence

$$0 \to p_n\mathcal{T}_\mathcal{F} \to R^n_\mathcal{F} \to R^{n-1}_\mathcal{F} \to 0,$$

implies that there is an exact sequence

$$0 \to p_n\mathcal{T}_\mathcal{F} \to 2R^n_\mathcal{F} \to p_{n-1}\mathcal{T}_\mathcal{F},$$

where the right hand term is given by the inductive hypothesis. To complete the result, it suffices to show that the image of $2R^n_\mathcal{F}$ in $p_{n-1}\mathcal{T}_\mathcal{F}$ is trivial; hence, by Lemma 2.7 it suffices to show this after evaluation on $\mathcal{F}$. This follows from the fact that $R^n_\mathcal{F}(\mathcal{F}) \cong \mathbb{Z}/2^n$.

Lemma 3.11. For $n > 0$ an integer, the functor $R^n_\mathcal{F}$ has simple head $\Lambda^1$ and simple socle $\Lambda^1$.

Proof. The functor $R^n_\mathcal{F}$ is a non-trivial quotient of $\mathcal{T}_Z$, which has simple head $\Lambda^1$ by Lemma 3.3, hence $R^n_\mathcal{F}$ has simple head $\Lambda^1$.

The proof that the socle is $\Lambda^1$ is by induction on $n$, starting from the case $n = 1$, which is clear, since $R^1_\mathcal{F} \cong \Lambda^1$ is simple. The exact sequence $0 \to p_n\mathcal{T}_\mathcal{F} \to R^n_\mathcal{F} \to R^{n-1}_\mathcal{F} \to 0$ shows that the socle of $R^n_\mathcal{F}$ is either $\Lambda^1$ or $\Lambda^1 \oplus \Lambda^1$. The latter possibility is excluded since $R^n_\mathcal{F}(\mathcal{F}) \cong \mathbb{Z}/2^n$, by Lemma 3.10.

Theorem 3.12. Let $n > 0$ be an integer. The functor $R^n_\mathcal{F}$ is self-dual; more precisely, any surjection $\mathcal{T}_Z \twoheadrightarrow DR^n_\mathcal{F}$ factors canonically across an isomorphism $R^n_\mathcal{F} \cong DR^n_\mathcal{F}$.

Proof. The proof is by induction upon $n$, starting from the case $n = 1$, when $R^1_\mathcal{F}$ is the simple functor $\Lambda^1$, which is self-dual. The factorization statement follows from the fact that $\mathcal{T}_Z$ has simple head.

For the inductive step, $R^n_\mathcal{F}$ has simple socle $\Lambda^1$, hence $DR^n_\mathcal{F}$ has simple head $\Lambda^1$ and there exists a surjection $\mathcal{T}_Z \twoheadrightarrow DR^n_\mathcal{F}$ (by projectivity of $\mathcal{T}_Z$). This gives rise to a morphism of short exact sequences

$$\begin{array}{cccccc}
0 & \to & \mathcal{T}_Z & \to & 2\mathcal{T}_Z & \to & p_n\mathcal{T}_\mathcal{F} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & DR^{n-1}_\mathcal{F} & \to & DR^n_\mathcal{F} & \to & q_n\mathcal{T}_\mathcal{F} & \to & 0,
\end{array}$$

where the lower exact sequence is the dual of the first exact sequence of Lemma 3.10. The commutativity of the right hand square follows from the fact that there is a unique non-trivial morphism $\mathcal{T}_Z \twoheadrightarrow q_n\mathcal{T}_\mathcal{F}$ and the surjectivity of the left hand vertical morphism is seen by evaluating on $\mathcal{F}$, since $DR^{n-1}_\mathcal{F}$ has simple head.

By the inductive hypothesis, the left hand vertical morphism factorizes across an isomorphism $DR^{n-1}_\mathcal{F} \cong DR^n_\mathcal{F}$. In particular, this implies that $\mathcal{T}_Z \twoheadrightarrow DR^n_\mathcal{F}$ lies in the kernel of $\mathcal{T}_Z \twoheadrightarrow DR^n_\mathcal{F}$, hence this induces a surjection

$$R^n_\mathcal{F} \twoheadrightarrow DR^n_\mathcal{F},$$

which is an isomorphism, since the objects have finite composition series with isomorphic associated graded functors.
Definition 3.13. Let $R^\infty_Z$ be the direct limit of the diagram
$$R^1_Z \hookrightarrow R^2_Z \hookrightarrow R^3_Z \hookrightarrow \ldots$$
of monomorphisms provided by Lemma 3.10.

Theorem 3.14. There are Pontrjagin duality isomorphisms:
$$R^\infty_Z \cong D\mathcal{T}_Z$$
$$\mathcal{T}_Z \cong DR^\infty_Z.$$

Proof. Observe that the functor $R^\infty_Z$ takes values in torsion $2$-groups. By construction and Corollary 3.8, the functor $\mathcal{T}_Z$ is isomorphic to the inverse limit of the natural system of quotients
$$\ldots \rightarrow R^n_Z \rightarrow R^{n-1}_Z \rightarrow \ldots \rightarrow R^1_Z \cong \Lambda^1.$$
Applying the Pontrjagin duality functor and using the fact that each $R^n_Z$ is self dual, gives a direct system which is isomorphic to that defining $R^\infty_Z$ (the latter fact follows from the proof of Theorem 3.12). The result follows from Pontrjagin duality for abelian groups.

4. Milnor derivations

This section establishes the fundamental ingredient, Proposition 4.10, to the proof of an algebraic version of Ossa’s theorem; the prime $p$ is taken to be $2$.

4.1. Milnor derivations on symmetric powers.

Definition 4.1. For $i \in \mathbb{N}$, let $Q_i : S^1 \rightarrow S^{2i+1}$ denote the iterated Frobenius $x \mapsto x^{2i+1}$ and also its extension $S^n \rightarrow S^{n+2i+1}$ to a derivation of $\bigoplus S^*$, defined by the composite
$$S^n \xrightarrow{\Delta} S^{n-1} \otimes S^1 \xrightarrow{1 \otimes Q_i} S^{n-1} \otimes S^{2i+1} \xrightarrow{\mu} S^{n+2i+1}.$$

Lemma 4.2. For $i, j \in \mathbb{N}$, $Q_i \circ Q_j = 0$ and the derivations $Q_i, Q_j$ commute. Hence the graded algebra $\bigoplus S^*$ is a module in $\mathcal{F}$ over the exterior algebra $\Lambda(Q_i | i \geq 0)$; in particular, $(S^*, Q_i)$ has the structure of a commutative differential graded algebra in $\mathcal{F}$.

Proof. Straightforward.

Notation 4.3. For $j > 0$ an integer, let $S^* / (x^{2j})$ denote the truncated symmetric power functor, so that $S^n / (x^{2j})$ is the cokernel of the composite
$$S^{n-2j} \otimes S^1 \xrightarrow{1 \otimes Q_j^{-1}} S^{n-2j} \otimes S^{2j} \xrightarrow{\mu} S^n.$$

In the following statement, the degree corresponds to the grading inherited from that of $S^*$.

Proposition 4.4. For $i \in \mathbb{N}$, the homology $H(S^*, Q_i)$ is the truncated symmetric power functor $S^*/(x^{2j})$, concentrated in even degrees. Explicitly:
$$H(S^*, Q_i)^k \cong \begin{cases} 0 & k \equiv 1 \pmod{2} \\ S^d / (x^{2j}) & k \equiv 2d. \end{cases}$$

Proof. The result follows from the calculation of the homology of the differential graded algebra $(\mathcal{F}[x], dx = x^{2i+1})$, which is the truncated polynomial algebra $\mathcal{F}[y]/y^{2d}$, where $y = x^{2j}$. The homology of the tensor product of such algebras is calculated by using the Künneth theorem. It remains to show that this corresponds to the stated functorial isomorphism.
The above establishes that the homology is concentrated in even degrees; moreover, in degree $k = 2d$, the Frobenius $S^d \to S^{2d}$ maps to the cycles in degree $k$ and induces a surjection onto the homology.

It is straightforward to check that, for integers $i, d \geq 1$, there is a commutative diagram in $\mathcal{F}$:

$$
\begin{array}{ccc}
S^{d-2i} \otimes S^1 & \xrightarrow{\Phi \otimes 1} & S^{2d-2i+1} \otimes S^1 \\
\downarrow 1 \otimes Q_{i-1} & & \downarrow Q_i \\
S^{d-2i} \otimes S^2 & \xrightarrow{\mu} & S^d & \xrightarrow{\Phi} & S^{2d},
\end{array}
$$

where $\Phi$ is the Frobenius. It follows that the morphism $S^d \to H(S^*, Q_i)^{2d}$ factorizes across the canonical projection $S^d \to S^d/(x^2)$. This completes the proof. \hfill $\square$

**Remark 4.5.** For $i = 0$, the homology is $\mathbb{F}$ concentrated in degree zero; in particular, the $Q_0 : S^n \to S^{n+1}$ induce an exact complex

$$0 \to S^1 \to S^2 \to S^3 \to \ldots .$$

For $i = 1$, there is non-trivial homology in even degrees; the homology of the complex $S^{2d-3} \to S^{2d} \to S^{2d+3}$ is $\Lambda^d$. Moreover, the proof of the Proposition gives an exact complex

$$S^d \oplus S^{2d-3} (\Phi, Q_1) S^{2d} Q_3 S^{2d+3}.$$

**4.2. The $Q_0$-kernel complex.**

**Notation 4.6.** For $n \in \mathbb{N}$, let $K_n$ denote the kernel of $Q_0 : S^n \to S^{n+1}$.

By construction, there are short exact sequences $0 \to K_n \to S^n \to K_{n+1} \to 0$, for $n \geq 0$. Initial values of $K_n$ are $K_0 = \mathbb{F} = S^0$, $K_1 = 0$, $K_2 = \Lambda^1 = S^1 \hookrightarrow S^2$, $K_3 = \Lambda^2$.

**Lemma 4.7.** The derivation $Q_1$ on $S^*$ induces a differential $Q_1 : K_n \to K_{n+3}$ and there is a short exact sequence of complexes

$$0 \to (K_{i+3\bullet}, Q_1) \to (S^{i+3\bullet}, Q_1) \to (K_{i+1+3\bullet}, Q_1) \to 0,$$

where $0 \leq i < 3$ and $\bullet \geq 0$.

**Proof.** A consequence of the exactness of the $Q_0$-complex in positive dimensions and the commutation of $Q_0, Q_1$ (Lemma 4.2). \hfill $\square$

**Notation 4.8.** For $n \in \mathbb{N}$, let $L_n$ denote the image of $Q_1 : K_{n-3} \to K_n$ and $\tilde{L}_n$ the kernel of $Q_1 : K_n \to K_{n+3}$.

The following is clear:

**Proposition 4.9.** The graded functors $K_\bullet, \tilde{L}_\bullet$ have unique graded algebra structures such that the canonical inclusions $\tilde{L}_\bullet \hookrightarrow K_\bullet \hookrightarrow S^*$ are morphisms of commutative graded algebras in $\mathcal{F}$.

**Proposition 4.10.** The homology of the complexes $(K_{i+3\bullet}, Q_1)$ is determined by

$$\tilde{L}_n/L_n \cong \begin{cases} 
\mathbb{F} & n = 0 \\
0 & n \equiv 1 \text{ mod } 2 \\
p_d \mathbb{T}_\mathbb{F} & n = 2d > 0.
\end{cases}$$
Proof. The proof is by induction upon $n$; the case $n = 0$ is clear. It is straightforward to show that the odd degree homology is trivial (independently of the calculation of the even degree homology).

For the inductive step, consider the commutative diagram arising from the short exact sequence of complexes given by Lemma 4.7:

\[
\begin{array}{c}
S^{n-6} \to K_{n-5} \\
K_{n-3} \to S^{n-3} \to K_{n-2} \\
K_{n} \to S^{n} \to K_{n+1} \\
K_{n+3} \to S^{n+3} \to K_{n+4};
\end{array}
\]

the homology $H$ at the middle of the left hand column is calculated in terms of the known homologies in the other two columns, via the long exact sequence in homology.

For $n = 2d > 0$, there is a short exact sequence

\[0 \to p_d I_{\overline{F}} \to H \to \Lambda^d \to 0,\]

by the inductive hypothesis for the left hand term and Proposition 4.4 for the exterior power, using the vanishing of homology in odd degrees. It suffices to show that $p_d I_{\overline{F}}$ is a subquotient of $H$.

The Frobenius $\Phi : S^d \to S^{2d}$ maps to $K_{2d}$ and the image lies in the kernel $\tilde{L}_{2d}$ of $K_{2d} \to K_{2d+3}$. There is a unique non-trivial morphism $S^d \to T_{\overline{F}}$ and this has image $p_d I_{\overline{F}}$; since $I_{\overline{F}}$ is injective, this extends to give a commutative diagram

\[
\begin{array}{c}
K_{2d-3} \\
S^{2d} \to \tilde{L}_{2d} \\
p_d I_{\overline{F}} \to T_{\overline{F}}.
\end{array}
\]

The composite morphism $K_{2d-3} \to T_{\overline{F}}$ is trivial, since $K_{2d-3}(\overline{F}) = 0$; it follows that $p_d I_{\overline{F}}$ is a subquotient of the homology $H$, as required. \qed

Example 4.11. It is straightforward to verify that $L_n = 0$ for $n \leq 5$, hence the initial values of $\tilde{L}_i$ are given by

\[
\tilde{L}_n = \begin{cases} 
0 & n \in \{1, 3, 5\} \\
\Lambda^1 & n = 2 \\
S^2 & n = 4.
\end{cases}
\]

5. Connective complex K-cohomology and homology

A complete functorial description of both $V \mapsto ku^*(BV_\ast)$ and $V \mapsto ku_\ast(BV_\ast)$ is given, using the results of Section 3 and Section 4.
5.1. Recollections. The Postnikov towers of $ku$ and $KU$ provide morphisms $ku \to HZ$ and $ku \to KU$ of commutative ring spectra relating connective (resp. periodic) complex $K$-theory $ku$ (resp. $KU$) and the integral Eilenberg-MacLane spectrum $HZ$.

There are cofibre sequences $\Sigma^2ku \overset{v}{\to} ku \to HZ$, $HZ \overset{2}{\to} HZ \overset{\beta}{\to} HF$, where $v$ is multiplication by the Bott element $(ku_* \cong \mathbb{Z}[v], \text{with } |v| = 2)$. These give rise to (generalized) Bocksteins; in particular:

**Notation 5.1.** Let $Q$ denote the first $k$-invariant of $ku$, given by the composite $HZ \to \Sigma^3ku \to \Sigma^3HZ$.

Recall that the Milnor derivation $Q_1 \in HF^3HF$ is the commutator $[Sq^2, Sq^1]$ (the Milnor derivation $Q_0$ is the Bockstein $\beta = Sq^1$).

**Lemma 5.2.** There is a commutative diagram:

\[
\begin{array}{ccc}
HZ & \overset{Q}{\longrightarrow} & \Sigma^3HZ \\
\downarrow & & \downarrow \rho \\
HF & \overset{Q_1}{\longrightarrow} & \Sigma^3HF.
\end{array}
\]

**Proof.** The morphism $Q$ is the image under the integral Bockstein $HF^2HZ \to HZ^3HZ$ of the class of $Sq^2$ (recall that $HF^*HZ \cong \mathcal{A}/(Sq^1)$) [Ada74, proof of III.16.6]. Hence there is a commutative diagram

\[
\begin{array}{ccc}
HZ & \overset{Q}{\longrightarrow} & \Sigma^3ku \longrightarrow \Sigma^3HZ \\
\downarrow & & \downarrow \rho \\
HF & \overset{Q_1}{\longrightarrow} & \Sigma^3HF.
\end{array}
\]

Since the composite $Sq^1 \circ \rho$ is trivial, the result follows.

**Notation 5.3.** The morphisms in (co)homology induced by $Q$ (respectively $Q_1$) will be denoted simply $Q$ (resp. $Q_1$).

**Lemma 5.4.** For $Y$ a spectrum, the following conditions are equivalent:

1. $HZ_*Y \overset{Q}{\to} HF_*Y$ is a monomorphism;
2. the Bockstein complex $(HF_*Y, \beta)$ is exact;
3. $HZ_*Y \overset{\rho}{\to} HF_*Y$ is a monomorphism;
4. the Bockstein complex $(HF_*Y, \beta)$ is exact.

When these conditions are satisfied, the respective morphisms $Q$ are determined by the commutative diagrams:

\[
\begin{array}{ccc}
HZ_*Y & \overset{Q}{\longrightarrow} & HZ_*^{*+3}Y \\
\downarrow \rho & & \downarrow \rho \\
HF_*Y & \overset{Q_1}{\longrightarrow} & HF_*^{*+3}Y.
\end{array}
\]

\[
\begin{array}{ccc}
HZ_*Y & \overset{Q}{\longrightarrow} & HZ_*^{*-3}Y \\
\downarrow \rho & & \downarrow \rho \\
HF_*Y & \overset{Q_1}{\longrightarrow} & HF_*^{*-3}Y.
\end{array}
\]

**Proof.** The equivalence of the conditions is standard and the commutative diagrams follow from Lemma 5.2; these determine $Q$, since the vertical morphisms are injective, by hypothesis.
Example 5.5. The hypotheses of Lemma 5.4 are satisfied for $Y = \Sigma^\infty BV$, where $V$ is an elementary abelian 2-group. In particular, the action of $\Omega$ on $H\mathbb{Z}^*(BV)$ can be understood in terms of the action of $\Lambda(Q_0, Q_1)$ on $H\mathbb{F}^*(BV)$.

5.2. On $v$-torsion and cotorsion. There is a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ on the category of $\mathbb{Z}[v]$-modules where $\mathcal{T}$ is the category of $v$-torsion modules (every element is annihilated by some power of $v$) and $\mathcal{F}$ the category of $v$-cotorsion modules (those $\mathbb{Z}[v]$-modules $M$ which embed in $M[\frac{1}{v}]$). This torsion theory has associated torsion functor $\text{tors}_v$ and cotorsion functor $\text{cotors}_v$ so that, for a $\mathbb{Z}[v]$-module, there is a natural short exact sequence of $\mathbb{Z}[v]$-modules:

$$0 \to \text{tors}_v M \to M \to \text{cotors}_v M \to 0.$$ 

The submodule of $M$ annihilated by $v$ is written $\text{ann}_v M$.

Lemma 5.6. For a $\mathbb{Z}[v]$-module $M$, there is an isomorphism $\text{tors}_v M \cap v M \cong \text{v tors}_M$, hence $\text{tors}_v M \to M$ induces a monomorphism $(\text{tors}_v M)/v \hookrightarrow M/v$.

Moreover, there is a short exact sequence of $\mathbb{Z}[v]$-modules:

$$0 \to \text{cotors}_v M \xrightarrow{\delta} \text{cotors}_v M \to (M/v)/((\text{tors}_v M)/v) \to 0.$$ 

Proof. Straightforward. 

For a $\mathbb{Z}[v]$-module $M$, there is a composite

$$\text{ann}_v M \hookrightarrow \text{tors}_v M \twoheadrightarrow (\text{tors}_v M)/v.$$ 

The following algebraic result is required in the proof of Proposition 5.12.

Lemma 5.7. For $M$ a graded $\mathbb{Z}[v]$-module, the following conditions are equivalent:

\begin{enumerate}
  \item $\text{ann}_v M = \text{tors}_v M$;
  \item $\text{ann}_v M \to (\text{tors}_v M)/v$ is injective.
\end{enumerate}

If, for each degree $d$, there exists $N(d) \in \mathbb{N}$ such that

$$(v^N(d) M \cap \text{tors}_v M)^d = 0$$

then these conditions are equivalent to

\begin{enumerate}
  \setcounter{enumi}{2}
  \item $\text{ann}_v M \to (\text{tors}_v M)/v$ is surjective.
\end{enumerate}

Proof. If $\text{ann}_v M = \text{tors}_v M$, then $\text{tors}_v M \cong (\text{tors}_v M)/v$ and hence condition (1) implies both (2) and (3).

Suppose condition (2) holds and consider $x \in \text{tors}_v M$, so that there exists $t \in \mathbb{N}$ such that $v^tx \neq 0$ and $v^{t+1}x = 0$. Hence $v^tx \in \text{ann}_v M$; the hypothesis implies that $v^tx$ is not in $vM$, hence $t = 0$ and $x \in \text{ann}_v M$, as required.

Now suppose that condition (3) holds, under the additional hypothesis. Consider $x \in \text{tors}_v M$; by surjectivity of $\text{ann}_v M \twoheadrightarrow (\text{tors}_v M)/v$, $x = a + vy$, for $a \in \text{ann}_v M$ and $y \in \text{tors}_v M$. An induction shows that, for $0 < n \in \mathbb{N}$, $x = a + v^ny_n$, for some $y_n \in \text{tors}_v M$; it suffices to show that $v^ny_n = 0$ for $n \gg 0$. By construction $v^ny_n \in (v^n M \cap \text{tors}_v M)^{|x|}$, hence the hypothesis implies that the element is zero for $n \geq N(|x|)$.

Remark 5.8. The example $\mathbb{Z}[v]/v^\infty$ shows that the conditions (1) and (3) are not equivalent without the additional hypothesis.

The following Lemma applies when considering $ku$-cohomology of a spectrum and allows the cotorsion to be related to periodic $K$-theory.

Lemma 5.9. \cite{BG03} Chapter 1 If $Z$ is a connective spectrum, there is a natural isomorphism $KU^*(Z) \cong ku^*(Z)[\frac{1}{[2]}]$.

For the applications, $Z$ will be the suspension spectrum of a space, hence the connectivity hypothesis is not restrictive.
Lemma 5.10. For $X$ a spectrum:

1. there is a natural exact sequence of $\mathbb{Z}[v]$-modules:
   \[ 0 \to \text{tors}_v ku_*(X) \to ku_*(X) \to \text{cotor}_v ku_*(X) \to 0 \]
   and a natural isomorphism
   \[ \text{cotor}_v ku_*(X) \cong \text{image}\{ku_*(X) \to KU_*(X)\}; \]

2. there is a natural exact sequence of $\mathbb{Z}[v]$-modules:
   \[ 0 \to \text{tors}_v ku^*(X) \to ku^*(X) \to \text{cotor}_v ku^*(X) \to 0 \]
   and, if $X$ is connective, a natural isomorphism
   \[ \text{cotor}_v ku^*(X) \cong \text{image}\{ku^*(X) \to KU^*(X)\}. \]

Lemma 5.11. For $X$ a spectrum, there are natural short exact sequences

\[ 0 \to ku_*(X)/v \to HZ_*(X) \to \text{ann}_v ku_{*-1}(X) \to 0; \]
\[ 0 \to ku^*(X)/v \to HZ^*(X) \to \text{ann}_v ku^{*+1}(X) \to 0. \]

Moreover, there are natural inclusions

\[ \text{Im} \Omega \subset (\text{tors}_v ku_*(X))/v \subset ku_*(X)/v \subset \text{Ker} \Omega; \]
\[ \text{Im} \Omega \subset (\text{tors}_v ku^*(X))/v \subset ku^*(X)/v \subset \text{Ker} \Omega. \]

Proof. The proofs for homology and cohomology are formally the same, hence consider $ku$-homology. The short exact sequence is induced by the cofibre sequence $\Sigma^2 ku \to ku \to H\mathbb{Z}$. The inclusion $(\text{tors}_v ku_*(X))/v \subset ku_*(X)/v$ is provided by Lemma 5.10; the outer inclusions are clear, from the definition of $\Omega$. \hfill \Box

Proposition 5.12. Let $Z$ be a connective spectrum.

1. The following conditions are equivalent:
   a) $\text{ann}_v ku_*(Z) = \text{tors}_v ku_*(Z)$;
   b) $(\text{tors}_v ku_*(Z))/v \cong \text{Im} \Omega$;
   c) $ku_*(Z)/v \cong \text{Ker} \Omega$.
   If these conditions are satisfied, then $(ku_*(Z)/v)/(\text{tors}_v ku_*(Z))/v) \cong \text{Ker} \Omega/\text{Im} \Omega$ and there is a short exact sequence of $\mathbb{Z}[v]$-modules:
   \[ 0 \to \text{cotor}_v ku_{*-2}(Z) \to \text{cotor}_v ku_*(Z) \to \text{Ker} \Omega/\text{Im} \Omega \to 0 \]
   and a pullback:
   \[
   \begin{array}{c}
   0 \\ \cong \\
   0 \\
   0 \\
   \end{array}
   \begin{array}{c}
   \xrightarrow{\text{cotor}_v ku_*(Z)} \\
   \text{cotor}_v ku_*(Z) \\
   \text{Ker} \Omega/\text{Im} \Omega \\
   \text{Ker} \Omega/\text{Im} \Omega \\
   \end{array}
   \]

2. Suppose that, for each degree $d$, there exists an integer $N(d)$ such that $(v^{N(d)} ku^*(Z) \cap \text{tors}_v ku^*(Z))^d = 0$, then the following conditions are equivalent:
   a) $\text{ann}_v ku^*(Z) = \text{tors}_v ku^*(Z)$;
   b) $(\text{tors}_v ku^*(Z))/v \cong \text{Im} \Omega$;
   c) $ku^*(Z)/v \cong \text{Ker} \Omega$.
   If these conditions are satisfied, then $(ku^*(Z)/v)/(\text{tors}_v ku^*(Z))/v) \cong \text{Ker} \Omega/\text{Im} \Omega$ and there is a short exact sequence of $\mathbb{Z}[v]$-modules:
   \[ 0 \to \text{cotor}_v ku^{*+2}(Z) \to \text{cotor}_v ku^*(Z) \to \text{Ker} \Omega/\text{Im} \Omega \to 0 \]
and a pullback:

\[
0 \longrightarrow \text{tors}_{v} \text{ku}^{*}(Z) \longrightarrow \text{ku}^{*}(Z) \longrightarrow \text{cotor}_{v} \text{ku}^{*}(Z) \longrightarrow 0
\]

Proof. The hypotheses imply that, for \( M \in \{ \text{ku}^{*}(Z), \text{ku}^{*}(Z) \} \), the three conditions of Lemma 5.7 are equivalent.

The equivalence of the conditions (a), (b), (c) follows from an analysis of the short exact sequences of Lemma 5.11; consider \( \text{ku}^{*}\)-homology (the argument for \( \text{ku}^{*}\)-cohomology is similar), so that there is a commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{ku}^{*}(Z)/v \\
\to & \downarrow & \downarrow \\
0 & \to & H\mathbb{Z}^{*}(Z) \\
\to \text{Ann}_{v} \text{ku}^{*}(Z) & \to & 0 \\
\end{array}
\]

in which the middle row and column are both short exact.

By the five lemma, \( \text{Im}\Omega \cong \text{Ann}_{v} \text{ku}^{*}(Z) \) if and only if \( \text{Ker}\Omega \cong \text{ku}^{*}(Z)/v \).

Moreover, Lemma 5.7 implies that the following three conditions are equivalent:

1. \( \text{Ann}_{v} \text{ku}^{*}(Z) = \text{tors}_{v} \text{ku}^{*}(Z) \);
2. \( \text{Im}\Omega \cong (\text{tors}_{v} \text{ku}^{*}(Z))/v \);
3. \( \text{Im}\Omega \cong \text{Ann}_{v} \text{ku}^{*}(Z) \).

This shows that conditions (a), (b), (c) are equivalent. The consequences follow, using Lemma 5.6 to provide the short exact sequence which calculates \( \text{cotor}_{v} \text{ku}^{*}(Z)/v \).

□

Remark 5.13.

1. Under the hypotheses of the Proposition, \( \text{ku}^{*}(Z) \) (respectively \( \text{ku}^{*}(Z) \)) is determined by \( (H\mathbb{Z}^{*}(Z), \Omega) \) (resp. \( (H\mathbb{Z}^{*}(Z), \Omega) \)), up to the analysis of the \( v \)-adic filtration of \( \text{cotor}_{v} \text{ku}^{*}(Z) \) (resp. \( \text{cotor}_{v} \text{ku}^{*}(Z) \)).
2. If \( H\mathbb{Z}^{*}(Z) \to H\mathbb{F}^{*}(Z) \) is a monomorphism (so that the hypotheses of Lemma 5.4 are satisfied), then the data is provided by \( H\mathbb{F}^{*}(Z) \), considered as a \( \Lambda(Q_{0}, Q_{1}) \)-module and there is a form of duality between \( \text{ku}^{*}(Z) \) and \( \text{ku}^{*}(Z) \).
3. Related considerations for connected Morava \( K \)-theories occur in [Lel82].

The nilpotency hypothesis of Proposition 5.12 is supplied by the following result when considering the \( \text{ku} \)-cohomology of spaces.

Lemma 5.14. [BG03, Lemma 1.5.8] Let \( Y \) be a space such that \( \text{ku}^{*}(Y) \) is a Noetherian \( \mathbb{Z}[v] \)-algebra. Then there exists a natural number \( N \) such that \( v^{N} \text{tors}_{v} \text{ku}^{*}(Y) = 0 \).

Example 5.15. For \( G \) a finite group, \( \text{ku}^{*}(BG_{+}) \) is a Noetherian \( \mathbb{Z}[v] \)-algebra (see [BG03, Section 1.1] for example).

When considering the unreduced \( \text{ku} \)-cohomology of a space, under the cohomological hypothesis of Proposition 5.12 the following result is clear.
Proposition 5.16. (Cf. [BG03]) Let $Y$ be a space such that $\text{tors}_{v}ku^*(Y_+) \cong \text{ann}_{v}ku^*(Y_+)$, then the algebra structure of $ku^*(Y_+)$ is determined by the induced algebra structures of $ku^*(Y_+)/v$ and $\text{cotor}_{v}ku^*(Y_+)$. In particular, there is a monomorphism of algebras

$$ku^*(Y_+) \to KU^*(Y_+) \prod HZ^*(Y_+)$$

where $\text{cotor}_{v}ku^*(Y_+)$ is considered as a subalgebra of $KU^*(Y_+)$ and $ku^*(Y_+)/v \cong \text{Ker} \Omega$, as a subalgebra of $HZ^*(Y_+)$. 

5.3. The $ku$-cohomology of $BV_s$. The above discussion applies in considering the group $ku$-cohomology $ku^*(BG_s)$. The periodic $K$-theory $KU^*(BG_s)$ of the finite group $G$ is known by the Atiyah-Segal completion theorem to be trivial in odd degrees and isomorphic in even degrees to the completion $R(G)_I$, where $I$ is the augmentation ideal of the complex representation ring $R(G)$.

Here we restrict the case to elementary abelian $2$-groups, and $V \mapsto ku^*(BV_\mathbb{Z})$ is considered as a contravariant functor of $V \in \text{Ob } \mathcal{V}$. The result is proved by applying Proposition 5.17 for which an understanding of $HZ^*(BV_\mathbb{Z})$ and the action of $\Omega$ is required.

Proposition 5.17. There are natural isomorphisms:

$$HZ^n(BV_\mathbb{Z}) \cong \left\{ \begin{array}{ll} \mathbb{Z} & n = 0 \\
K_n(V^\mathbb{Z}) & n > 0 \end{array} \right.$$  

$$(\text{Im} \Omega)^n \cong L_n(V^\mathbb{Z})$$  

$$(\text{Ker} \Omega)^n \cong \left\{ \begin{array}{ll} \mathbb{Z} & n = 0 \\
\hat{L}_n(V^\mathbb{Z}) & n > 0 \end{array} \right.$$  

Proof. The algebra $HF^*(BV_\mathbb{Z})$ is naturally isomorphic to $S^*(V^\mathbb{Z})$ and integral reduced cohomology $HZ^*(BV)$ embeds in $HF^*(BV)$ as the kernel of the Bockstein operator. Hence (paying attention to the behaviour in unreduced cohomology), the result follows from Lemma 5.4 using the definition of the functors $L_n$, $L$ and $\hat{L}_n$ from Section 4. 

Lemma 5.18. There are identities $\text{tors}_{v}ku^*(B0_+) = 0 = \text{tors}_{v}ku^*(BZ/2_+)$.  

Proof. This follows from the identification of $ku^*(BZ/2_+)$ using the Gysin sequence (cf. [BG03 Section 2.2]).

Theorem 5.19. For $V \in \text{Ob } \mathcal{V}$, there are natural isomorphisms

$$\text{tors}_{v}ku^*(BV_\mathbb{Z}) \cong \text{ann}_{v}ku^*(BV_\mathbb{Z}) \cong (\text{Im} \Omega)^v$$  

$$ku^*(BV_\mathbb{Z})/v \cong (\text{Ker} \Omega)^v$$

and $\text{tors}_{v}ku^n(BV_\mathbb{Z}) \to ku^n(BV_\mathbb{Z})/v$ is an isomorphism for $n$ odd and, for $n = 2d > 0$, there is a natural short exact sequence:

$$0 \to \text{tors}_{v}ku^{2d}(BV) \to ku^{2d}(BV)/v \to p_d\mathcal{T}_F(V^\mathbb{Z}) \to 0.$$  

The surjection $\text{cotor}_{v}ku^*(BV_\mathbb{Z}) \to \text{cotor}_{v}ku^*(BV_\mathbb{Z})/v$ induces a pullback diagram of short exact sequences

$$\begin{array}{cccccc}
0 & \longrightarrow & \text{tors}_{v}ku^*(BV_\mathbb{Z}) & \longrightarrow & ku^*(BV_\mathbb{Z}) & \longrightarrow & \text{cotor}_{v}ku^*(BV_\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Im} \Omega & \longrightarrow & \text{Ker} \Omega & \longrightarrow & (\text{cotor}_{v}ku^*(BV_\mathbb{Z})/v) & \longrightarrow & 0.
\end{array}$$

There are natural isomorphisms $\text{cotor}_{v}ku^*(BV_\mathbb{Z}) \cong \mathbb{Z}[v] \oplus \text{cotor}_{v}ku^*(BV)$ and $\text{cotor}_{v}ku^{2d}(BV) \cong \mathcal{P}_Z(V^\mathbb{Z})$.
Proposition 5.21. constructing an isomorphism of short exact sequences (for \( d > 0 \)):

\[
\begin{array}{c}
0 \longrightarrow \text{cotor}_s ku^{2d+2}(BV) \overset{v}{\longrightarrow} \text{cotor}_s ku^{2d}(BV) \longrightarrow p_d T_F(V^2) \longrightarrow 0 \\
\cong \\
0 \longrightarrow p_d T_F(V^2) \longrightarrow 0.
\end{array}
\]

\[\def\fboxrule{0pt}\fbox{0} \overset{\cong}{\longrightarrow} \fbox{0} \overset{\cong}{\longrightarrow} \fbox{0} \]

Proof. The first part of the Theorem follows from Proposition 5.12 since Lemma 2.7 implies that the cohomological hypothesis is satisfied. To apply the Proposition, it is sufficient to show that \( \text{Im} \Omega \cong (\text{cotor}_s ku^*(BV_+))/v \).

By Proposition 5.17 and Proposition 4.10,

\[ (\text{Ker} \Omega/\text{Im} \Omega)^n \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
0 & n \text{ odd} \\
p_d T_F(V^2) & n = 2d, \; d > 0.
\end{cases} \]

In odd degrees \( \text{Im} \Omega = \text{Ker} \Omega \), and the result follows from the inclusions given in Lemma 5.10.

It remains to show that the inclusion \((\text{Im} \Omega)^{2d} \hookrightarrow \text{cotor}_s ku^{2d}(BV_+)/v\) is an isomorphism for \( d \in \mathbb{N} \). For \( d = 0 \), both terms are zero; for \( d > 0 \), the cokernel is a subfunctor of \( V \mapsto p_d T_F(V^2) \) by the inclusions given in Lemma 5.10 and the above identification of \( \text{Ker} \Omega/\text{Im} \Omega \), hence it suffices to show that the cokernel is trivial when evaluated on \( V = F \), by Lemma 2.7. This follows from the fact that \( \text{cotor}_s ku^*(BZ/2) = 0 \), by Lemma 5.18.

Finally, consider \( \text{cotor}_s ku^*(BV_+) \). For \( d = 0 \), the result is the Atiyah-Segal completion theorem; the structure in higher degree follows by induction on \( d \) from the results of Section 5.1 in particular Proposition 5.7 using the identification of the functors

\[ \text{cotor}_s ku^*(BV_+)/v \cong \text{Ker} \Omega/\text{Im} \Omega \]
given above. \( \square \)

Remark 5.20. Proposition 5.18 applies to \( ku^*(BV_+) \) to give a description of the algebra structure (cf. [BG03]).

5.4. The ku-homology of elementary abelian 2-groups. The \( ku \)-homology of elementary abelian 2-groups can be determined as for \( ku \)-cohomology, by applying Proposition 5.12 for which an understanding of \( HZ_*(BV_+) \) and the action of \( \Omega \) is required.

Proposition 5.21. There are natural isomorphisms:

\[ HZ_n(BV_+) \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
DK_{n+1}(V) & n > 0
\end{cases} \]

\[ (\text{Im} \Omega)_n \cong DL_{n+4} \]

\[ (\text{Ker} \Omega)_n \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
D(K_{n+1}/L_{n+1})(V) & n > 0
\end{cases} \]

Moreover:

\[ (\text{Ker} \Omega/\text{Im} \Omega)_n \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
0 & n \equiv 0 \mod 2 \\
p_d T_F(V) & n = 2d - 1 > 0
\end{cases} \]

Proof. There is a natural isomorphism \( HF_n(BV_+) \cong \Gamma^0(V) \) and \( HZ \overset{\rho}{\to} HF \) induces a monomorphism in reduced homology, so Lemma 5.4 applies. It remains to identify the functors upon dualizing, with the attendant shift in indexing.
There is a natural isomorphism
\[ HZ_n(BV_+) \cong DK_{n+1}(V), \]
for \( n > 0 \) and the natural transformation \( \Omega \) is induced by \( DQ_1 \).

Consider the \( Q_1 \) complex \( K_{n-3} \to K_n \to K_{n+3} \) (for \( n > 3 \)); this gives rise to short exact sequences
\[
0 \to L_n \to \tilde{L}_n \to H_n \to 0 \\
0 \to \tilde{L}_n \to K_n \to L_{n+3} \to 0 \\
0 \to H_n \to K_n/L_n \to L_{n+3} \to 0,
\]
where \( H_n \) denotes the homology.

In the dual complex, \( DK_{n+3} \overset{DQ_1}{\to} DK_n \overset{DQ_1}{\to} DK_{n-3} \):
\[
\text{Im}DQ_1 \cong DL_{n+3} \\
\text{Ker}DQ_1 \cong D(K_n/L_n),
\]
the inclusion \( \text{Im}DQ_1 \to \text{Ker}DQ_1 \) is dual to \( K_n/L_n \to L_{n+3} \) and the cokernel is \( DH_n \). This proves the first statement (taking into account the unreduced homology in degree zero and the degree shift); the calculations of Example 4.11 show that the expressions are correct in low degrees.

The final statement follows from Proposition 4.10, by dualizing. \( \square \)

**Theorem 5.22.** For \( V \in \text{Ob} Y^\ell \), there are natural isomorphisms:
\[
\text{tors}_v ku_n(BV_+)^{\cong} \text{ann}_v ku_n(BV_+) \cong (\text{Im} \Omega)_n \\
ku_n(BV_+)/v \cong (\text{Ker} \Omega)_n
\]
and the inclusion \( \text{tors}_v ku_n(BV_+) \hookrightarrow ku_n(BV_+) \) induces a natural short exact sequence
\[
0 \to \text{tors}_v ku_n(BV_+) \to ku_n(BV_+)/v \to \text{cotor}_v ku_n(BV_+)/v \to 0,
\]
where
\[
\text{cotor}_v ku_n(BV_+)/v \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
0 & 0 < n \equiv 0 \mod 2 \\
qdP_{v}(V) & n = 2d - 1 > 0.
\end{cases}
\]
There is a pullback diagram of short exact sequences
\[
\begin{array}{cccccc}
0 & \to & \text{tors}_v ku_n(BV_+) & \to & ku_n(BV_+) & \to & \text{cotor}_v ku_n(BV_+) & \to & 0 \\
\sim & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Im} \Omega & \to & \text{Ker} \Omega & \to & (\text{cotor}_v ku_n(BV_+))/v & \to & 0.
\end{array}
\]
For \( d > 0 \) an integer, there is a natural isomorphism
\[
\text{cotor}_v ku_{2d-1}(BV) \cong R^d_{Z}(V)
\]
inducing an isomorphism of short exact sequences
\[
\begin{array}{cccccc}
0 & \to & \text{cotor}_v ku_{2d-1}(BV) & \xrightarrow{\times v} & \text{cotor}_v ku_{2d+1}(BV) & \to & q_{d+1}P_{v}(V) & \to & 0 \\
\sim & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R^d_{Z}(V) & \to & R^{d+1}_{Z}(V) & \to & q_{d+1}P_{v}(V) & \to & 0,
\end{array}
\]
where \( R^d_{Z} \hookrightarrow R^{d+1}_{Z} \) is the dual of the natural projection \( R^{d+1}_{Z} \to R^d_{Z} \).
Proof. There are natural monomorphisms

\[ \text{Im}(\Omega(V)) \hookrightarrow ku_*(BV_+)/v \hookrightarrow \text{Ker}(V); \]

by Proposition 5.21 in positive even degree, these are isomorphisms; in degree \( n = 2d - 1 > 0 \), the quotient \( (\text{Ker}/\text{Im}(\Omega))(V) \) is \( q_d\mathcal{P}_F(V) \), hence there is a natural inclusion

\[ ((ku_*(BV_+)/v)/\text{Im}(\Omega(V)))_{2d-1} \hookrightarrow q_d\mathcal{P}_F(V). \]

To prove the result, by Proposition 5.12 it suffices to show that this is an isomorphism; hence, by Lemma 2.7 it suffices to show that the left hand side is non-trivial when evaluated on \( F \), for all \( d > 0 \). It is straightforward to verify that \( \text{Im}\Omega_{od}(\mathcal{F}) = 0 \), thus it suffices to show that \( ku_*(B\mathbb{Z}/2+) / v \) is non-trivial in all odd degrees, which follows from the structure of \( ku_*(B\mathbb{Z}/2+) \) as a graded abelian group (see [BG03, Section 3.4], for example).

Finally, the identification of \( \text{cotor}_ku_*(BV_+) \) follows from the results of Section 3.2 in particular the short exact sequences of Lemma 3.10, the self-duality of the functors \( R^0_\mathcal{F} \) (Theorem 3.12) and the Pontryagin duality between \( R^\infty_\mathcal{F} \) and \( \mathcal{P}_\mathbb{Z}/2z \) (Theorem 3.14). (Compare the proof of Theorem 5.19) \( \square \)

Remark 5.23. By [BG03, Proposition 3.2.1], the universal coefficient spectral sequence calculating \( ku^*(BV) \) from \( ku_*(BV) \) collapses to the short exact sequence

\[ 0 \to \text{Ext}^1_{ku_*}(\Sigma^2\text{tors}_ku_*(BV), ku_*) \to ku^*(BV) \to \text{Ext}^1_{ku_*}(\Sigma^3\text{cotor}_ku_*(BV), ku_*) \to 0. \]

This is isomorphic to

\[ 0 \to \text{tors}_ku^*(BV) \to ku^*(BV) \to \text{cotor}_ku^*(BV) \to 0 \]

and explains the duality between \( ku \)-homology and \( ku \)-cohomology of \( BV \). The analysis of [BG03, Section 4.12] can be made functorial to give the identification of \( \text{cotor}_ku_*(BV_+) \).

6. Local duality

An equivariant version of local duality (with respect to the action of the general linear groups \( \text{Aut}(V) \)) is given as it arises in the current context; this gives a refinement of the results of [BG03, Section 4.7].

6.1. Categories of \( S^\bullet \)-modules. Throughout this section, the prime \( p \) is arbitrary. The fact that the functor \( S^\bullet \) takes values in graded vector spaces of finite type will be used without further comment.

Definition 6.1.

1. Let \( S^\bullet-\text{mod}_\mathcal{F} \) denote the category of graded right \( S^\bullet \)-modules in \( \mathcal{F} \) and \( S^\bullet \)-module morphisms.
2. For \( V \in \text{Ob} \mathcal{F} \), let \( S^\bullet(V) - \text{mod}_{\text{Aut}(V)} \) denote the category of graded right \( S^\bullet(V) \)-modules in \( \text{Aut}(V) \)-modules and \( S^\bullet(V) \)-module morphisms.

Remark 6.2. The choice to work with right modules is dictated by the notation adopted for Koszul complexes.

An object of \( S^\bullet-\text{mod}_\mathcal{F} \) is a graded functor \( M^\bullet \), equipped with a structure morphism \( M^\bullet \otimes S^\bullet \to M^\bullet \) which is unital and associative; this can be expressed in terms of the components \( M^b \otimes S^a \to M^{a+b} \). A similar description holds for \( S^\bullet(V) \)-modules.

Lemma 6.3. For \( V \in \text{Ob} \mathcal{F} \), the categories \( S^\bullet-\text{mod}_\mathcal{F} \) and \( S^\bullet(V) - \text{mod}_{\text{Aut}(V)} \) are tensor abelian. Moreover:
(1) the forgetful functor $S^\bullet - \text{mod}_\mathcal{F} \to \mathcal{F}$ is exact and admits an exact left adjoint $- \otimes S^\bullet : \mathcal{F} \to S^\bullet - \text{mod}_\mathcal{F}$ which is monoidal;
(2) the forgetful functor $S^\bullet(V) - \text{mod}_{\text{Aut}(V)} \to \text{Aut}(V) - \text{mod}$ is exact and admits an exact left adjoint $- \otimes S^\bullet(V) : \text{Aut}(V) - \text{mod} \to S^\bullet(V) - \text{mod}_{\text{Aut}(V)}$ which is monoidal;
(3) evaluation at $V$, $\mathcal{F} \to \text{Aut}(V) - \text{mod}$, induces an exact tensor functor $S^\bullet - \text{mod}_\mathcal{F} \to S^\bullet(V) - \text{mod}_{\text{Aut}(V)}$.

Proof. Clear.

Definition 6.4. For $N \in \text{Ob} S^\bullet - \text{mod}_\mathcal{F}$, let $\text{Hom}_V^\bullet (\cdot, N)$ be the functor $(S^\bullet - \text{mod}_\mathcal{F})^{\text{op}} \to S^\bullet(V) - \text{mod}_{\text{Aut}(V)}$ defined by

$$\text{Hom}_V^\bullet(M, N) := \text{Hom}_{S^\bullet(V)}(M(V), N(V))$$

where the right hand side is equipped with the usual grading and $\text{Aut}(V)$ acts via conjugation.

Remark 6.5. The above definition can be refined to give a coefficient system for the general linear groups over $\mathbb{F}$, associated to the pair $M, N$ of graded $S^\bullet$-modules in $\mathcal{F}$.

Lemma 6.6. For $F \in \text{Ob} \mathcal{F}$ and $V \in \text{Ob} \mathcal{F}^f$, there is a natural isomorphism:

$$\text{Hom}_V^\bullet(F \otimes S^\bullet, S^\bullet) \cong F(V)^2 \otimes S^\bullet(V)$$
in $\text{Aut}(V) - \text{mod}$, where $F(V)^2$ is equipped with the contragredient action.

Proof. Straightforward.

The following is clear:

Lemma 6.7. Let $F, G \in \text{Ob} \mathcal{F}$ and $\alpha : F \otimes S^\bullet \to G \otimes S^\bullet$ be a morphism of $S^\bullet - \text{mod}_\mathcal{F}$, induced by $\tilde{\alpha} : F \to G \otimes S^\bullet$ in $\mathcal{F}$.

Then $\text{Hom}_V^\bullet(\alpha, S^\bullet)$ identifies with the morphism $G(V)^2 \otimes S^\bullet(V) \to F(V)^2 \otimes S^\bullet(V)$ of $S^\bullet(V) - \text{mod}_{\text{Aut}(V)}$ induced by $\gamma : G(V)^2 \to F(V)^2 \otimes S^\bullet(V)$ in $\text{Aut}(V) - \text{mod}$, where $\gamma$ is adjoint (in the category $\text{Aut}(V) - \text{mod}$) to the morphism $G(V)^2 \otimes S^\bullet(V)^2 \to F(V)^2$ dual to the evaluation of $\tilde{\alpha}$ on $V$.

6.2. The dualizing functor. Recall that the exterior power functors are self-dual, so that:

Lemma 6.8. For $n \in \mathbb{N}$, there is a natural isomorphism of contravariant functors of $V$: $\Lambda^n(V^2) \cong \Lambda^n(V)^2$.

This is combined with the following duality result, when restricting to the consideration of the $\text{Aut}(V)$-action:

Lemma 6.9. Let $0 \leq j \leq r$ be integers and $V \in \text{Ob} \mathcal{F}^f$ have rank $r$. The composite

$$\Lambda^r(V) \otimes \Lambda^j(V)^2 \xrightarrow{\Lambda^j \otimes \Lambda^r} \Lambda^{r-j}(V) \otimes \Lambda^j(V) \otimes \Lambda^j(V)^2 \to \Lambda^{r-j}(V),$$

where the second morphism is induced by evaluation $\Lambda^j(V) \otimes \Lambda^j(V)^2 \to F$, induces an isomorphism $\Lambda^r(V) \otimes \Lambda^j(V)^2 \cong \Lambda^{r-j}(V)$ in $\text{Aut}(V) - \text{mod}$, where $\Lambda^j(V)^2$ is equipped with the contragredient $\text{Aut}(V)$-module structure.

Proof. The result follows from the fact that the product $\Lambda^{r-j}(V) \otimes \Lambda^j(V) \to \Lambda^r(V) \cong F$ defines a perfect pairing and the equivariance of the evaluation map. □
Lemma 6.10. Let \( 1 \leq j \leq r \) be integers, \( V \in \text{Ob} \mathcal{V}^I \) have rank \( r \) and write \( \mu : \Lambda^{r-j+1}(V)^j \otimes \Lambda^1(V)^j \to \Lambda^1(V)^j \) for the product morphism (dual to the evaluation of the coproduct \( \Lambda^j \to \Lambda^{j-1} \otimes \Lambda^1 \)).

Then, under the isomorphism of Lemma 6.9, \( \Lambda^r(V) \otimes \mu \) is \( \text{Aut}(V) \)-equivariantly isomorphic to the morphism

\[
\Lambda^{r-j+1}(V) \otimes \Lambda^1(V)^j \to \Lambda^{r-j}(V)
\]

which is adjoint to the evaluation of the coproduct \( \Lambda^{r-j+1} \to \Lambda^{r-j} \otimes \Lambda^1 \) on \( V \).

Proof. A consequence of the coassociativity of the comultiplication on the exterior power functors and the fact that the multiplication \( \mu \) is dual to the coproduct \( \Lambda^j \to \Lambda^{j-1} \otimes \Lambda^1 \).

\[\square\]

Notation 6.11. For \( i \in \mathbb{N} \) and \( V \in \text{Ob} \mathcal{V}^I \) of rank \( r \), let

(1) \( \tau_i : \Lambda^j \to S^i \) denote the composite of the isomorphism \( \Lambda^1 \cong S^1 \) with the iterated Frobenius \( S^1 \to S^p \) and also the induced Koszul differential

\[
\tau_i : \Lambda^j \otimes S^\bullet \to \Lambda^{j-1} \otimes S^{i+p}
\]

in the category \( S^\bullet - \text{mod} \mathcal{X} \), induced by the composite morphism:

\[\Lambda^j \xrightarrow{\tau} \Lambda^{j-1} \otimes \Lambda^1 \otimes S^p ;\]

(2) \( Kz_i \) denote the Koszul complex in \( S^\bullet - \text{mod} \mathcal{X} \):

\[
\cdots \to \Lambda^j \otimes S^\bullet \xrightarrow{\tau_i} \Lambda^{j-1} \otimes S^{i+p} \to \cdots \to S^{i+rp} \to 0;
\]

(3) \( Kz_i(V) \) denote the Koszul complex in \( S^\bullet(V) - \text{mod} \text{Aut}(V) \):

\[
0 \to \Lambda^r(V) \otimes S^\bullet(V) \to \cdots \to \Lambda^j(V) \otimes S^{i+(r-j)p}(V) \xrightarrow{\tau_i} \Lambda^{j-1}(V) \otimes S^{i+(r-j+1)p}(V) \to \cdots \to S^{i+rp}(V) \to 0.
\]

Proposition 6.12. For integers \( 1 \leq j \leq r \), \( V \in \text{Ob} \mathcal{V}^I \) of rank \( r \) and \( i \in \mathbb{N} \), the morphism

\[
\text{Hom}_{S^\bullet}(\tau_i, S^\bullet) : \text{Hom}_{S^\bullet}(\Lambda^{j-1} \otimes S^\bullet, S^\bullet) \to \text{Hom}_{S^\bullet}(\Lambda^j \otimes S^\bullet, S^\bullet)
\]

is induced by the morphism \( \gamma : \Lambda^{j-1}(V)^j \to \Lambda^j(V)^j \otimes S^p(V) \) such that, under the isomorphism of Lemma 6.9, \( \Lambda^r(V) \otimes \gamma : \Lambda^{r-j}(V) \to \Lambda^{r-j}(V) \otimes S^p(V) \) is \( \text{Aut}(V) \)-equivariantly isomorphic to the evaluation on \( V \) of the Koszul differential.

Proof. Combine Lemma 6.7 with Lemma 6.10. \[\square\]

Corollary 6.13. For \( V \in \text{Ob} \mathcal{V}^I \) of rank \( r \) and \( i \in \mathbb{N} \), there is a natural isomorphism of complexes:

\[
\text{Hom}_{S^\bullet}(Kz_i, \Lambda^r \otimes S^\bullet) \cong Kz_i(V)
\]

in \( S^\bullet(V) - \text{mod} \text{Aut}(V) \).

Proof. After addition of the twisting functor \( \Lambda^r \), the result is an immediate consequence of Proposition 6.12. \[\square\]

Remark 6.14. Taking \( i = 0 \), so that \( Kz_0 \) is the usual Koszul complex, which has homology \( F \) concentrated in homological degree zero, this shows that \( \Lambda^r \otimes S^\bullet \) plays the role of the dualizing object, corresponding to the fact that \( S^\bullet \) is graded Gorenstein (cf. [BH93, Section 3.7]).
6.3. Local cohomology in $S^\bullet(V)\mod_{\text{Aut}(V)}$. In this section, local cohomology is considered with respect to the augmentation ideal $I$ of $S^\bullet(V)$, where $V \in \text{Ob}\mathcal{F}$. It has rank $r$; from the results of the previous section, it follows that the local duality isomorphism (cf. [BH93]) should be interpreted as stating that the local cohomology of the $S^\bullet(V)$-free object of $S^\bullet(V)\mod_{\text{Aut}(V)}$, $F(V) \otimes S^\bullet(V)$, for $F \in \text{Ob}\mathcal{F}$, is concentrated in cohomological degree $r$, where it is isomorphic to

$$\text{Hom}_{\mathcal{F}}(\text{Hom}_{\mathcal{F}}^r(F \otimes S^\bullet, \Lambda^r \otimes S^\bullet), \mathbb{F}).$$

**Remark 6.15.** In the cases of interest here, $F$ takes finite-dimensional values, so that $\text{Hom}_{\mathcal{F}}^r(F \otimes S^\bullet, \Lambda^r \otimes S^\bullet)$ is a graded vector space of finite type.

**Proposition 6.16.** (Cf. [BG03] Lemma 4.7.1.) Let $V \in \text{Ob}\mathcal{F}$ have rank $r$ and $G \in \text{Ob}\mathcal{F}$. Suppose that there exists a complex

$$0 \to F_r \otimes S^\bullet \to F_{r-1} \otimes S^\bullet \to \ldots \to F_0 \otimes S^\bullet \to G \to 0$$

which induces an exact sequence in $S^\bullet(V)\mod_{\text{Aut}(V)}$, after evaluation on $V$. Then the local cohomology of $G(V) \in \text{Ob}\mathcal{F}$ is

$$H^r(V(G)) \cong H_{r-1}(\text{Hom}_{\mathcal{F}}(\text{Hom}_{\mathcal{F}}^r(F) \otimes S^\bullet, \Lambda^r \otimes S^\bullet), \mathbb{F}),$$

up to shift in grading.

7. Filtering symmetric powers and Koszul complexes

To calculate the local cohomology of tors $ku^\bullet(BV_+)$ at the prime two by using a form of local duality, it is necessary to filter and study the associated Koszul complexes (for odd primes, this filtration step is unnecessary). The results of this section refine those of Section 6.3.

7.1. Filtering the symmetric powers. Let $\Phi S^\bullet$ denote the image of the Frobenius, so that $\Phi S^\bullet$ takes values in graded commutative algebras and the canonical morphisms $\Phi S^\bullet \rightarrow \Phi S^\bullet \otimes S^\bullet$ are morphisms of algebras.

**Definition 7.1.** For $t \in \mathbb{N}$, let $f_tS^\bullet \subset S^\bullet$ denote the image of multiplication

$$S^t \otimes \Phi S^\bullet \rightarrow S^\bullet.$$

**Lemma 7.2.** For $t \in \mathbb{N}$,

1. the functors $f_tS^\bullet \subset S^\bullet$ define an increasing filtration of $S^\bullet$:

$$f_0S^\bullet = \Phi S^\bullet \subset f_1S^\bullet \subset f_2S^\bullet \subset \ldots \subset f_tS^\bullet \subset \ldots \subset S^\bullet$$

in $\Phi S^\bullet$-modules;
2. there is an isomorphism of $S^\bullet$-modules:

$$f_tS^\bullet/f_{t-1}S^\bullet \cong \Lambda^t \otimes S^\bullet,$$

where $S^\bullet$ acts on the left hand side by restriction along $S^\bullet \rightarrow \Phi S^\bullet$ and by multiplication on the right hand factor of $\Lambda^t \otimes S^\bullet$; for $V \in \text{Ob}\mathcal{F}$, this restricts to an isomorphism of $S^\bullet(V)$-modules:

$$\bigoplus_{t \geq 0} (f_tS^\bullet/f_{t-1}S^\bullet)(V) \cong S^\bullet(V),$$

where $S^\bullet(V)$ acts on the right hand side via $\Phi$;
3. for $V \in \text{Ob}\mathcal{F}$, the inclusion $f_tS^\bullet(V) \hookrightarrow S^\bullet(V)$ is an isomorphism for $t \geq \dim V$.

**Proof.** Straightforward; to prove that $S^\bullet(V)$ is isomorphic to $\bigoplus_{t \geq 0} (f_tS^\bullet/f_{t-1}S^\bullet)(V)$ as $S^\bullet(V)$-modules, it is sufficient to consider monomial bases. (Note that this statement is not true $\text{Aut}(V)$-equivariantly.)
For notational clarity, shifts in gradings are omitted from the following statement.

**Proposition 7.3.** For $i, t \in \mathbb{N}$, the Milnor derivation $Q_i : S^\bullet \to S^\bullet$ is a morphism of $\Phi S^\bullet$-modules and restricts to $f_i S^\bullet \stackrel{f_i Q_i}{\to} f_{i-1} S^\bullet$; the induced morphism on the filtration quotients

$$\Lambda^i \otimes S^\bullet \cong f_i S^\bullet / f_{i-1} S^\bullet \to f_{i-1} S^\bullet / f_{i-2} S^\bullet \cong \Lambda^{t-1} \otimes S^\bullet$$

is the Koszul differential $\tau_i$.

**Proof.** Straightforward. $\square$

The differentials $\tau_0$ and $\tau_1$ define a bicomplex structure on $\Lambda^\bullet \otimes S^\bullet$, which can be displayed as:

$$
\begin{array}{c}
\Lambda^1 \to S^1 \\
\Lambda^2 \to \Lambda^1 \otimes S^1 \to S^2 \\
\Lambda^3 \to \Lambda^2 \otimes S^1 \to \Lambda^1 \otimes S^2 \to S^3 \\
\Lambda^4 \to \Lambda^3 \otimes S^1 \to \Lambda^2 \otimes S^2 \to \Lambda^1 \otimes S^3 \to S^4 \\
\end{array}
$$

The portion of the bicomplex displayed indicates the essential features:

1. the rows (respectively columns) are Koszul complexes, with differential $\tau_0$ (resp. $\tau_1$);
2. the bicomplex is concentrated in a single quadrant and there are vanishing lines of slope 1/2 and 1.

### 7.2. Filtering the functors $K_n$.

This section establishes the filtered version of Proposition 4.10; the starting point is the functorial homology of the Koszul complexes, which is a standard calculation, related to Proposition 4.3.

**Proposition 7.4.** For $i \in \mathbb{N}$, the homology of $(\Lambda^\bullet \otimes S^\bullet, \tau_i)$ is $S^\bullet / \langle x^2 \rangle$, concentrated in homological degree zero.

The following are analogous to the functors $K_n$ introduced in Notation 4.6:

**Notation 7.5.** For an integer $a \geq 1$ and $b \in \mathbb{N}$, let $K_{a,b}$ denote the image of $\tau_0 : \Lambda^a \otimes S^b \to \Lambda^{a-1} \otimes S^{b+1}$ and, by convention:

$$K_{0,b} := \begin{cases} F & b = 0 \\ 0 & b > 0. \end{cases}$$

**Remark 7.6.**

1. The functor $K_{1,b}$ identifies with the symmetric power $S^{b+1}$.
2. Taking the image of $\tau_0$ rather than the kernel, is best suited for the current application, where the homology of the Koszul complex intervenes.

**Lemma 7.7.** For $0 < n \in \mathbb{Z}$, the filtration $f_i S^\bullet$ induces a filtration of $K_n$ with associated graded:

$$\text{gr} K_n \cong \bigoplus_{a+2b+1=n} K_{a,b}.$$
Proof. For $n > 0$, $K_n$ is the image of $Q_0 : S^{n-1} \to S^n$. Passing to the associated graded, the morphism $Q_0$ induces

$$
\bigoplus_{a+2b+1=n} \tau_0 : \bigoplus_{a+2b+1=n} \Lambda^a \otimes S^b \to \bigoplus_{a+2b+1=n} \Lambda^{a-1} \otimes S^{b+1},
$$

by Proposition 7.3. Moreover, evaluated on $V \in \text{Ob} \mathcal{T}'$, as a morphism of vector spaces, $Q_0$ identifies with $\bigoplus \tau_0$, using the splitting of the filtration in $S^*(V)$-modules given in Lemma 7.2. □

Lemma 7.8. The derivation $\tau_1$ induces a differential $\tau_1 : \mathfrak{R}_{a+1,b-2} \to \mathfrak{R}_{a,b}$. For $a > 0$ and $b \geq 0$, the short exact sequences from the Koszul complexes:

$$
0 \to \mathfrak{R}_{a,b} \to \Lambda^{a-1} \otimes S^{b+1} \to \mathfrak{R}_{a-1,b+1} \to 0
$$

induce a short exact sequence of complexes:

$$
\begin{array}{cccccc}
0 & \to & \mathfrak{R}_{3,b-4} & \to & \Lambda^2 \otimes S^{b-3} & \to & \mathfrak{R}_{2,b-3} & \to & 0 \\
\tau_1 & & \tau_1 & & \tau_1 & & \tau_1 \\
0 & \to & \mathfrak{R}_{2,b-2} & \to & \Lambda^1 \otimes S^{b-1} & \to & \mathfrak{R}_{1,b-1} & \to & 0 \\
\tau_1 & & \tau_1 & & \tau_1 & & \tau_1 \\
0 & \to & \mathfrak{R}_{1,b} & \to & S^{b+1} & \to & 0 & \to & 0.
\end{array}
$$

Proof. The horizontal $\tau_0$-Koszul complexes are acyclic. □

Proposition 7.9. For $b \in \mathbb{N}$, the complex

$$
\ldots \to \mathfrak{R}_{a+1,b-2n} \xrightarrow{\tau_1} \mathfrak{R}_{a,b-2n+2} \to \ldots \to \mathfrak{R}_{1,b} \cong S^{b+1}
$$

has homology $p_{b+1} \mathcal{T}'$ concentrated in homological degree zero.

Proof. The proof is by induction upon $b$, using the short exact sequence of complexes provided by Lemma 7.8. The initial case $b = 0$ is by inspection; for the inductive step, use the fact that the $\tau_1$ Koszul complex:

$$
\ldots \to \Lambda^n \otimes S^{b-2n+1} \xrightarrow{\tau_1} \Lambda^{n-1} \otimes S^{b-2n+3} \to \ldots \to S^{b+1}
$$

has homology $\Lambda^{b+1}$ concentrated in homological degree zero, by Proposition 7.4 (with $i = 1$). The proof is completed by the argument employed in the proof of Proposition 4.10. □

7.3. Filtering the functors $L_n$ and $\tilde{L}_n$.

Notation 7.10. For integers $a > 0$, $b \geq 0$, let $\mathfrak{L}_{a,b}$ denote the cokernel of $\tau_1 : \mathfrak{R}_{a+1,b-2} \to \mathfrak{R}_{a,b}$.

Proposition 7.9 implies the following identification:

Lemma 7.11.

1. If $a > 1$, $\mathfrak{L}_{a,b} \cong \text{Ker}\{ \mathfrak{R}_{a,b} \xrightarrow{\tau_1} \mathfrak{R}_{a-1,b+2} \};$

2. $\mathfrak{L}_{1,b} \cong p_{b+1} \mathcal{T}'$.

Recall from Section 4.2 that $Q_1$ induces a morphism $Q_1 : K_{n-3} \to K_n$ with image $L_n$ and kernel $\tilde{L}_{n-3}$. It follows that the cokernel of $Q_1$ occurs in an extension

$$
0 \to \tilde{L}_n/L_n \to \text{Coker}Q_1 \to L_{n+3} \to 0.
$$
Lemma 7.12. For $n > 0$ and $\text{Coker} Q_1$, as above, the filtration $f_* S^\bullet$ induces a finite filtration of $\text{Coker} Q_1$ with associated graded

$$\text{grCoker} Q_1 \cong \bigoplus_{a+2b+1=n, a \geq 1} \mathfrak{L}_{a,b},$$

where, for $2(b+1) = n$, the subobject $\mathfrak{L}_{1,b}$ is isomorphic to $\tilde{L}(2^{b+1})/L_2(2^{b+1}) \cong p_{b+1}T_F$ and there is an induced isomorphism:

$$\text{gr} L_{n+3} \cong \bigoplus_{a+2b+1=n, a \geq 2} \mathfrak{L}_{a,b}.$$

Proof. The result follows as for the proof of Lemma 7.7.

Remark 7.13. For the calculations of local cohomology, it is important that the isomorphisms of Lemma 7.7 and of Lemma 7.12 upon evaluation on $V \in \text{Ob} \mathcal{V}$ correspond to isomorphisms in the category of $S^\bullet(V)$-modules (as in Lemma 7.2).

8. The local cohomology spectral sequence

The local cohomology theorem for $ku$ implies that there is a spectral sequence:

$$E^2 := H^0_f(ku^*(BV_+)) \Rightarrow ku_*(BV_+),$$

where the $E^2$-term is the local cohomology with respect to the augmentation ideal (see [BG03] for generalities on the spectral sequence, for arbitrary finite groups). Here $V$ is taken to be a fixed elementary abelian $2$-group of rank $r$. The aim of this section is to indicate how the spectral sequence can be understood conceptually, by using the functorial calculations introduced in Section 4.

There are two key ingredients: the functorial description of local duality and of local cohomology given in Section 4 and an explanation of the relationship between the local cohomology of $\text{tors}_i ku^*(BV_+)$ and that of $\text{cotor}_s ku^*(BV_+)$. The local cohomology spectral sequence can be made $\text{Aut}(V)$-equivariant but it is clearly not functorial as it stands with respect to arbitrary vector space morphisms; the techniques of this section do however show that the behaviour of the spectral sequence is largely determined by functorial structure.

Remark 8.1. Throughout, the grading shifts resulting from working with graded modules are suppressed. The gradings are not essential for the presentation of the arguments; the reader is encouraged to supply them.

8.1. The case of integral cohomology. The local cohomology spectral sequence for $HZ_*(BV_+)$ already illustrates some of the salient features of the local cohomology spectral sequence. It can also be used in the analysis of the local cohomology spectral sequence for $ku_*(BV_+)$ via the morphism induced by $ku \to HZ$.

Let $V \in \text{Ob} \mathcal{V}$ have rank $r$ and consider the short exact sequence

$$0 \to HZ^*(BV) \to HZ^*(BV_+) \to \mathbb{Z} \to 0$$

relating reduced and unreduced cohomology of $BV$, in the category of $HZ^*(BV_+)$-modules, so that $HZ^*(BV)$ corresponds to the augmentation ideal $I$. There is an induced exact sequence of local cohomology groups:

$$0 \to H_I^0(HZ^*(BV)) \to H_I^0(HZ^*(BV_+)) \to \mathbb{Z} \to H_I^1(HZ^*(BV)) \to H_I^1(HZ^*(BV_+)) \to 0$$

and, for $j > 1$, a natural isomorphism $H_I^1(HZ^*(BV_+)) \cong H_I^1(HZ^*(BV))$.

Hence, up to calculating the connecting morphism $\mathbb{Z} \to H_I^1(HZ^*(BV))$, the local cohomology of $HZ^*(BV_+)$ is determined by that of $HZ^*(BV)$. Moreover, it is clear that $H_I^1(HZ^*(BV))$ is annihilated by $2$, hence it suffices to consider behaviour after reducing mod $2$. 

Notation 8.2. For \( a \in \mathbb{N} \), let \( \sigma_{\geq a} Kz_0 \) denote the brutal truncation to the right of the Koszul complex:

\[
\ldots \to \Lambda^{a+1} \otimes S^{*+1} \to \Lambda^a \otimes S^* \to \Lambda^{a-1} \otimes S^{*+1} \to \ldots \to S^{*+a}.
\]

and let \( \sigma_{\leq a} Kz_0 \) denote the brutal truncation to the left:

\[
\Lambda^a \otimes S^* \to \Lambda^{a+1} \otimes S^{*+1} \to \Lambda^{a+2} \otimes S^{*+2} \to \ldots
\]

Lemma 8.3. For \( a \in \mathbb{N} \),

1. \( \sigma_{\geq a} Kz_0 \) is an \( S^* \)-free resolution of \( \mathfrak{R}_{a, \bullet} \);
2. for \( a \geq 1 \), the complex \( \sigma_{\leq a} Kz_0 \) has homology \( F \) in homological degree 0 and \( \mathfrak{R}_{a+1, \bullet} \) in homological degree \( a \).

Moreover, there are morphisms of complexes

\[
\sigma_{\leq 0} Kz_0 \to \sigma_{\geq 1} Kz_0 \to \sigma_{\geq 2} Kz_0 \to \ldots
\]

\[
\sigma_{\leq 0} Kz_0 \cong S^* \hookrightarrow \sigma_{\leq 1} Kz_0 \hookrightarrow \sigma_{\leq 2} Kz_0 \hookrightarrow \ldots \hookrightarrow Kz_0.
\]

Proof. Clear. \[ \square \]

Proposition 8.4. For \( V \in \text{Ob} \mathcal{F} \) of rank \( r \) and integers \( 0 \leq a \leq b \leq r \), there is a natural isomorphism of complexes:

\[
\text{Hom}_{\mathcal{S}^*}(\sigma_{\geq a} Kz_0, S^* \otimes \Lambda^r) \cong \sigma_{\leq r-a} Kz_0(V)
\]

in \( S^*(V) - \text{mod}_{\text{Aut}(V)} \) and, with respect to these isomorphisms, the surjection \( \sigma_{\geq a} Kz_0 \to \sigma_{\geq b} Kz_0 \) induces the inclusion \( \sigma_{\leq r-b} Kz_0(V) \hookrightarrow \sigma_{\leq r-a} Kz_0(V) \).

In particular, the surjection \( Kz_0 \to \sigma_{\geq 1} Kz_0 \) induces the inclusion:

\[
\sigma_{\leq r-1} Kz_0(V) \hookrightarrow Kz_0(V),
\]

which induces an isomorphism in degree zero homology if \( b < r \).

Proof. The result follows from Corollary 6.13. \[ \square \]

Remark 8.5. It is useful to think of the surjection \( \sigma_{\geq a} Kz_0 \to \sigma_{\geq b} Kz_0 \) as a morphism in an appropriate derived category

\[
\mathfrak{R}_{a, \bullet}[a] \to \mathfrak{R}_{b, \bullet}[b],
\]

where \([a], [b]\) correspond to the shift in homological degree. In particular, for \( a = 0 \), this corresponds to \( F \to \mathfrak{R}_{b, \bullet}[b] \).

The morphism \( \sigma_{\geq a} Kz_0 \to \sigma_{\geq b} Kz_0 \) induces a morphism between local cohomology groups (a generalized connecting morphism), via the identification of local cohomology given in Proposition 6.16 and Proposition 8.4.

Using the above observation, one deduces:

Lemma 8.6. For \( V \in \text{Ob} \mathcal{F} \) of rank \( r > 1 \), the connecting morphism \( H^0_1(|\mathbb{Z}/2) \cong \mathbb{Z}/2 \to H^1_1(\mathbb{H}^* BV) \) induced by the short exact sequence of \( \mathbb{H}^* (BV^+)_- \)-modules

\[
0 \to H^*(BV) \to H^*(BV^+)/2 \to \mathbb{Z}/2 \to 0
\]

is non-trivial.

Remark 8.7. The connecting morphism in the long exact sequence for local cohomology, \( d^1 \), is therefore non-trivial. For a conceptual presentation of the results, it is useful to define an associated \( E^1 \)-page, so that this connecting morphism appears as the \( d^1 \) differential.
The local cohomology \((r > 1)\) is as follows, using Lemma 8.3:

\[
H^j_I(H\mathbb{Z}^*(BV^+)) \cong \begin{cases} 
\mathbb{Z} & j = 0 \\
0 & j = 1 \\
F & 2 \leq j \leq r - 1 \\
H^j_I(H\mathbb{Z}^*(BV)) & j = r.
\end{cases}
\]

Moreover, \(H^r_I(H\mathbb{Z}^*(BV))\) has a finite filtration such that

\[
\text{gr}H^r_I(H\mathbb{Z}^*(BV)) \cong F \oplus \text{gr}H\mathbb{Z}_*(BV),
\]

up to shift in degree, where the filtration on homology \(H\mathbb{Z}_*(BV)\) is induced by the filtration \(f_tS^\bullet\).

The analysis of the local cohomology spectral sequence is straightforward; the local cohomology corresponds to the \(E^2\)-page of the spectral sequence. The permanent cycles in the zero column are given by the subgroup \(2^{r-1}\mathbb{Z}\); the differentials \(d^i\), for \(2 \leq i \leq r\) are all non-trivial, starting from the zero column, and serve to eliminate the extraneous factors of \(F\) which occur above.

Remark 8.8. Heuristically it is useful to consider that the differential \(d^i\) is induced by the surjection of complexes \(Kz_0 \twoheadrightarrow \sigma_{\geq i}Kz_0\) for \(i \geq 1\), using Remark 8.7 to interpret the connecting morphism as \(d^1\).

8.2. Bicomplexes of \(S^\bullet\)-modules. The purpose of this section is to explain the calculation of the local cohomology of \(\text{tors}_{ku^*}(BV^\perp)\); a fundamental point is that the method also calculates the local cohomology of \(\text{cotors}_{ku^*}(BV)/v\) and explains all the differentials in the local cohomology spectral sequence. This relies on the following result, in which grading shifts have been suppressed and, for variance reasons, the cohomology of \(V^\perp\) is considered.

**Proposition 8.9.** (Cf. [BG03 Section 4.6].) For \(V \in \text{Ob}\mathcal{F}\) of rank \(r\), \(ku^*(BV^\perp)\) admits a finite natural filtration with associated graded

\[
\text{gr}\text{tors}_{ku^*}(BV^\perp) \cong \bigoplus_{i=2}^r \Sigma_{i,V}(V)
\]

in \(S^\bullet(V)\mod\text{Aut}(V)\). Moreover, as a module over \(S^\bullet(V)\):

\[
\text{tors}_{ku^*}(BV^\perp) \cong \bigoplus_{i=2}^r \Sigma_{i,V}(V).
\]

**Proof.** The result follows from Lemma 7.12 and Theorem 5.19.

For the consideration of local duality, it is necessary to consider the \(S^\bullet\)-action on the \(\Lambda^a \otimes S^b\)-bicomplex introduced in Section 7.4. This gives a half plane bicomplex with differentials of the form

\[
\begin{array}{cccccccc}
\cdots & \Lambda^{-1} \otimes S^0 \otimes S^3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \tau_0 & \tau_0 & \cdots & \sigma_{\geq 1} \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \Lambda^1 \otimes S^{-1} \otimes S^0 \otimes S^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \tau_0 & \tau_0 & \cdots & \sigma_{\geq 1} \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \Lambda^{r-1} \otimes S^0 \otimes S^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \tau_0 & \tau_0 & \cdots & \sigma_{\geq 1} \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \Lambda^r \otimes S^0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \tau_0 & \tau_0 & \cdots & \sigma_{\geq 1} \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

in the \((s,t)\)-plane.

Consider the following brutal truncations, which are analogues of the truncated Koszul complexes of Notation 8.2.
Definition 8.10. (Cf. [BG03 Section 4.6]) For \( i \in \mathbb{N} \), let

1. \( B(i) \) be the bicomplex in \( S^\bullet - \text{mod}_F \):

\[
B(i)_{s,t} := \begin{cases} 
0 & \text{if } i < s < 0; \\
\Lambda^{s+t} \otimes S^\bullet & \text{if } i \geq s,
\end{cases}
\]

considered as a quotient bicomplex, where the term of lowest total degree is \( \Lambda^i \otimes S^\bullet \), in bidegree \((0, i)\).

2. \( D(i) \) be the bicomplex in \( S^\bullet - \text{mod}_F \):

\[
D(i)_{s,t} := \begin{cases} 
0 & \text{if } t > i; \text{ or } s > 0; \\
\Lambda^{s+t} \otimes S^\bullet & \text{if } t \leq i,
\end{cases}
\]

considered as a sub-bicomplex, where the term of greatest total degree is \( \Lambda^i \otimes S^\bullet \), in bidegree \((0, i)\).

Remark 8.11. When evaluated on \( V \in \text{Ob } V_f \) of rank \( r \), the only non-trivial terms are those with \( s + t \leq r \) and \( t \geq i \). In particular, \( B(i)(V) \) is trivial if \( i > r \).

Example 8.12. Taking \( r = 5 \) and \( i = 2 \), \( B(2)(F^5) \) is given by evaluating the following bicomplex on \( F^5 \):

\[
\begin{array}{c}
\Lambda^5 \otimes S^{*-6} \\
\Lambda^5 \otimes S^{*-5} \xrightarrow{\tau_0} \Lambda^4 \otimes S^{*-4} \\
\Lambda^5 \otimes S^{*-4} \xrightarrow{\tau_0} \Lambda^4 \otimes S^{*-3} \xrightarrow{\tau_0} \Lambda^3 \otimes S^{*-2} \\
\Lambda^5 \otimes S^{*-3} \xrightarrow{\tau_0} \Lambda^4 \otimes S^{*-2} \xrightarrow{\tau_0} \Lambda^3 \otimes S^{*-1} \xrightarrow{\tau_0} \Lambda^2 \otimes S^* \\
\end{array}
\]

where \( \Lambda^2 \otimes S^* \) is in \((s, t)\)-degree \((0, 2)\).

Similarly, \( D(2)(F^5) \) is obtained by evaluating the following on \( F^5 \):

\[
\begin{array}{c}
\Lambda^2 \otimes S^* \xrightarrow{\tau_0} \Lambda^1 \otimes S^{*-1} \xrightarrow{\tau_0} S^{*-2} \\
\Lambda^1 \otimes S^{*-1} \xrightarrow{\tau_0} \Lambda^2 \otimes S^{*-2} \\
\end{array}
\]

Remark 8.13.

1. The grading on \( B(i) \) used in [BG03 Chapter 4] (respectively on \( D(i) \)) can be recovered by considering the grading of the ‘generators’ in the lowest (resp. greatest) total degree, since the morphism \( \tau_0 \) raises the degree by 2 and \( \tau_1 \) raises the degree by 4 (the grading is calculated relative to the \( S^\bullet \)-grading, so that the usual gradings of the odd degree generators do not contribute).

2. The homology of the bicomplexes is calculated pointwise, by first evaluating on \( V \in \text{Ob } V_f \); for any \( i \) and \( V \), the bicomplex \( B(i)(V) \) has only finitely many non-zero terms.

The following is clear from the definition.

Lemma 8.14.
(1) There are surjections of bicomplexes in $S^\bullet - \text{mod}_F$:

$$B(0) \twoheadrightarrow B(1) \twoheadrightarrow \ldots \twoheadrightarrow B(i) \twoheadrightarrow B(i+1) \twoheadrightarrow \ldots$$

and, the kernel of $B(i) \twoheadrightarrow B(i+1)$ is the truncated Koszul complex $\sigma_{\leq i} \text{K}_{a_0}$:

$$\ldots \rightarrow \Lambda^{i+s} \otimes S^{*-s} \rightarrow \Lambda^{i+s-1} \otimes S^{*-s+1} \rightarrow \ldots \rightarrow \Lambda \otimes S^*$$

concentrated in $t$-degree $i$.

(2) There are inclusions of bicomplexes:

$$D(0) = S^\bullet \hookrightarrow D(1) \hookrightarrow D(2) \hookrightarrow \ldots \hookrightarrow D(j-1) \hookrightarrow D(j) \hookrightarrow \ldots$$

and the cokernel of $D(j-1) \hookrightarrow D(j)$ is the truncated Koszul complex $\sigma_{\leq j} \text{K}_{a_0}$ shifted so the term of maximal total degree is in bidegree $(0, j)$.

The following result follows from the definition of the objects $\mathfrak{a}_{a,b}$ and $\mathfrak{L}_{a,b}$ given in Section 4.

**Lemma 8.15.** For $0 < a \in \mathbb{Z}$, $\mathfrak{a}_{a,\bullet}$ and $\mathfrak{L}_{a,\bullet}$ are objects of $S^\bullet - \text{mod}_F$ such that the surjections $\Lambda^a \otimes S^\bullet \twoheadrightarrow \mathfrak{a}_{a,\bullet} \twoheadrightarrow \mathfrak{L}_{a,\bullet}$ are morphisms of $S^\bullet - \text{mod}_F$.

To establish the behaviour of the indices, consider the following commutative diagram (for $a \geq 2$):

$$\begin{array}{ccc}
\Lambda^{a+1} \otimes S^{*-2} & \longrightarrow & \mathfrak{a}_{a+1,\bullet}-2 \\
\tau_1 \downarrow & & \downarrow \\
\Lambda^a \otimes S^\bullet & \longrightarrow & \mathfrak{a}_{a,\bullet} \\
\tau_1 \downarrow & & \downarrow \\
\Lambda^{a-1} \otimes S^{*-3} & \longrightarrow & \mathfrak{a}_{a-1,\bullet}+2 \\
\tau_0 \downarrow & & \downarrow \\
\Lambda^a \otimes S^\bullet & \longrightarrow & \mathfrak{L}_{a,\bullet} \\
\end{array}$$

In particular, $\mathfrak{L}_{a,\bullet}$ is a quotient of $\Lambda^a \otimes S^\bullet$ and a subobject of $\Lambda^{a-2} \otimes S^{b+3}$.

In the following Proposition, recall the indexing convention of $B(i)$, which means that the term of lowest total degree is in $(s, t)$-degree $(0, i)$, hence has total degree $i$.

**Proposition 8.16.** For $0 < i \in \mathbb{Z}$, the homology of $\text{Tot}(B(i))$ is concentrated in degree $i$ and $H_i(\text{Tot}(B(i))) \cong \mathfrak{L}_{i,\bullet}$.

**Proof.** The result follows by filtering the bicomplex $B(i)$ using Lemma 8.14 and Proposition 7.9 (cf. [BG03, Proposition 4.6.3]).

**Remark 8.17.** Although the bicomplex $B(0)$ is defined, it does not give a resolution of $F$, since the homology is not concentrated in a single homological degree - which is a consequence of the homology of $(\Lambda^\bullet \otimes S^\bullet, \tau_0)$.

**Proposition 8.18.** For $i \in \mathbb{Z}$, the homology of $\text{Tot}(D(i))$ is as follows.

For $i = 0$:

$$H_m(\text{Tot}(D(0))) \cong \begin{cases} 
S^\bullet & m = 0 \\
0 & \text{otherwise}.
\end{cases}$$

For $i > 0$:

$$H_m(\text{Tot}(D(i))) \cong \begin{cases} 
F^{\oplus i+1} \oplus \bigoplus_{d \geq 1} p_d T_\mathbb{Z} & m = 0 \\
\mathfrak{L}_{i+2,\bullet} & m = i \\
0 & \text{otherwise}.
\end{cases}$$
Moreover, for \( i > 1 \), the short exact sequence of bicomplexes \( 0 \to D(i - 1) \to D(i) \to \sigma_{\leq i} \mathcal{K} \to 0 \) given by Lemma \[8.17\] induces the short exact sequences

\[
\cdots \to H_0(\text{Tot}(D(i))) \to H_0(\text{Tot}(D(i - 1))) \to H_0(\sigma_{\leq i} \mathcal{K}z_0) \to 0 \to \cdots
\]

\[
\cdots \to \mathcal{F}^{\leq i} \oplus \bigoplus_{d \geq 1} p_d \mathcal{F} \to \mathcal{F}^{\leq i + 1} \oplus \bigoplus_{d \geq 1} p_d \mathcal{F} \to \mathcal{F} \to 0 \to \cdots
\]

and

\[
\cdots \to H_i(\text{Tot}(D(i))) \to H_i(\text{Tot}(\sigma_{\leq i} \mathcal{K}z_0)) \to H_{i - 1}(\text{Tot}(D(i - 1))) \to 0 \to \cdots
\]

\[
\cdots \to \mathcal{L}_{i + 2,*} \to \mathcal{K}_{i + 1,*} \to \mathcal{\Sigma}_{i + 1,*} \to 0 \to \cdots
\]

in homology.

**Proof.** The calculation of the homology follows from Proposition \[7.9\] together with the fact that each row of \( D(i) \), considered as a truncated Koszul complex, contributes a factor \( \mathcal{F} \) in homological degree 0.

For \( i > 1 \), the given short exact sequences follow immediately; the exactness of the second sequence is again a consequence of Proposition \[7.9\]. \( \square \)

**Remark 8.19.**

1. The factors \( \mathcal{F} \) (resp. the different factors \( p_d \mathcal{T}_F \)) lie in distinct gradings.
2. The degree \( r - 1 \) homology of \( D(r - 1) \) is the functor \( \mathcal{L}_{r-1,*} \), which is a quotient of \( \Lambda^{r+1} \otimes S^* \). In particular, when evaluated on the rank \( r \) space \( \mathcal{V} \), this is trivial. Thus, the homology of \( D(r - 1) \) evaluated on \( \mathcal{V} \) is concentrated in degree zero.

The following result is proved by using the identification of local cohomology given by Proposition \[9.10\]. Here \( D \) denotes the (graded) duality functor and the identification \( Dp_d \mathcal{T}_F \cong q_d \mathcal{T}_F \) is used to give the duality statement \( D\mathcal{L}_{1,*} \cong \bigoplus_{d > 0} q_d \mathcal{T}_F \); all functors should be understood as being evaluated on \( \mathcal{V} \).

**Corollary 8.20.** (Cf. \[BG03\] Theorem 4.7.3.) For \( \mathcal{V} \in \text{Ob} \mathcal{F}^{\text{f}} \) of rank \( r > 1 \), the local cohomology of \( \mathcal{L}_{i,*} \):

1. for \( i = 1 \), is concentrated in cohomological degree one and \( H_1^i(\mathcal{L}_{1,*}) \cong \mathcal{F}^{\text{gr}} \oplus D\mathcal{L}_{1,*} \);
2. for \( 2 \leq i \leq r - 1 \):

\[
H_i^m(\mathcal{L}_{i,*}) \cong \begin{cases} 
\mathcal{F}^{\text{gr}} \oplus D\mathcal{L}_{1,*} & m = i \\
D\mathcal{L}_{r-i+2,*} & m = r \\
0 & \text{otherwise};
\end{cases}
\]

3. for \( i = r \), is concentrated in cohomological degree \( r \) and \( H_i^r(\mathcal{L}_{i,*}) \cong D\mathcal{S}^* \).

Moreover, the surjection of complexes \( \mathcal{B}(1) \to \mathcal{B}(i) \), for \( 1 < i < r \), induces a surjection: \( H_1^i(\mathcal{L}_{1,*}) \to H_i^i(\mathcal{L}_{i,*}) \) with kernel \( \mathcal{F}^{\text{gr}} \).

**Remark 8.21.**

1. The restriction \( r > 1 \) on the rank is imposed so as to give a unified statement.
2. The grading is again suppressed; the reader is encouraged to calculate the appropriate gradings and to verify that the above yields the Hilbert series specified in \[BG03\] Section 4.7.
8.3. The local cohomology of cotors$_v ku^*(BV_+)$.

Throughout this section, $V \in \text{Ob } \mathcal{V}_+$ has rank $r > 1$. To simplify notation, $Q$ will be written for the (contravariant) functor $V \mapsto \text{cotors}_v ku^*(BV_+)$. Multiplication by $v$ induces a short exact sequence

$$0 \to Q \xrightarrow{\cdot v} Q \to Q/v \to 0$$

(omitting the suspension associated with the grading). Moreover, the augmentation $Q \to \mathbb{Z}[v]$ induces a short exact sequence

$$0 \to \mathbb{Q}/v \to Q/v \to \mathbb{Z} \to 0.$$ 

Theorem 5.19 implies that there is a natural isomorphism $\mathbb{Q}/v \cong \mathfrak{S}_1^*(V^\vee)$. Hence, by using the change of rings isomorphism associated to $ku^*(BV_+) \to H\mathbb{Z}^*(BV_+)$, Corollary 8.20 gives the local cohomology of $\mathbb{Q}/v$; in particular, the ideal $I$ refers here to the augmentation ideal of $ku^*(BV_+)$. 

**Proposition 8.22.** The local cohomology of $Q/v$ is concentrated in cohomological degrees zero and one:

$$H^i_1(Q/v) \cong \mathbb{Z}$$

$$H^1_1(Q/v) \cong \mathbb{F}^{\mathfrak{Br}^{-1}} \oplus D\mathfrak{S}_1^*.$$ 

In particular, $2H^1_1(Q/v) = 0$.

**Proof.** The result follows from the exact sequence

$$0 \to H^0_1(Q/v) \to \mathbb{Z} \to H^1_1(Q/v) \to H^1_1(Q/v) \to 0.$$ 

The connecting morphism is non-trivial (this can be seen by considering the behaviour modulo 2), whence the result. (This also explains the notation $2\mathbb{Z}$).

The local cohomology of $Q$ can now be analysed by using the exact sequence associated to $Q \to Q/v$, which has the form:

$$0 \to H^0_1(Q) \xrightarrow{\cdot v} H^0_1(Q) \to H^1_1(Q/v) \cong 2\mathbb{Z} \to H^1_1(Q) \xrightarrow{i} H^1_1(Q) \to H^1_1(Q/v) \to 0.$$ 

The calculation of $H^i_1(Q)$ is straightforward (cf. [BG03, Section 4.4]) and this implies that the image of the connecting morphism is $\mathbb{Z}/2^{r-1}$. This is sufficient to calculate the local cohomology. A direct approach is taken in [BG03]; the above is preferred here since it stresses the relationship between $H^i_1(Q/v)$ and the $v$-adic filtration of $H^i_1(Q)$.

**Proposition 8.23.** (Cf. [BG03 Lemma 4.5.1]) The associated graded of the $v$-adic filtration of $H^i_1(Q)$ is given by

$$v^i H^i_1(Q)/v^{i+1} H^i_1 Q \cong \begin{cases} \mathbb{F}^{\mathfrak{Br}^{-i-1}} \oplus D\mathfrak{S}_1^* & 0 \leq i \leq r - 3 \\ \mathbb{F} \oplus D\mathfrak{S}_1^* & i = r - 2; \\ D\mathfrak{S}_1^* & i \geq r - 1. \end{cases}$$

**Proof.** To prove that the naturality with respect to $V$ is correct, use the description $Q[\frac{1}{v^r}]/Q$ which is given in [BG03 Proposition 4.4.7].

**Remark 8.24.**

(1) Grading shifts are suppressed.

(2) [BG03 Lemma 4.5.1] is a statement about the 2-adic filtration; this coincides with the $v$-adic filtration (compare the final statement of Lemma 8.10).
Remark 8.25. As in [BG03 Section 4.11], it is more transparent to consider an $E^1$-page of the local cohomology spectral sequence which serves to calculate the local cohomology at the $E^2$-page.

The local cohomology spectral sequence for $ku_*(BV_+)$, The local cohomology of $ku_*(BV_+)$ is determined by using the short exact sequence

$$0 \to \text{tors}_*ku_*(BV_+) \to ku_*(BV_+) \to \text{cotors}_*ku_*(BV_+) \to 0,$$

and the calculation of the local cohomology of $\text{tors}_*ku_*(BV_+)$, which follows from the results of Section 8.2 and of $\text{cotors}_*ku_*(BV_+)$, given in Section 8.3

The connective $K$-theory

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