DELINE–LUSZTIG CONSTRUCTIONS FOR DIVISION ALGEBRAS AND
THE LOCAL LANGLANDS CORRESPONDENCE

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ABSTRACT. In 1979, Lusztig proposed a cohomological construction of supercuspidal representations of reductive $p$-adic groups, analogous to Deligne–Lusztig theory for finite reductive groups. In this paper we establish a new instance of Lusztig’s program. Precisely, let $D$ be the quaternion algebra over a local non-Archimedean field $K$ of positive characteristic, and let $X$ be the $p$-adic Deligne–Lusztig ind-scheme associated to $D^\times$. There is a natural correspondence between quasi-characters of the (multiplicative group of the) unramified quadratic extension of $K$ and representations of $D^\times$ given by $\theta \mapsto H_i(X)[\theta]$. We show that this correspondence is a bijection (after a mild restriction of the domain and target), and matches the bijection given by local Langlands and Jacquet–Langlands.

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1. INTRODUCTION

Deligne–Lusztig [DL76] theory gives a geometric description of the irreducible representations of finite groups $G$ of Lie type. In [L79], Lusztig suggests an analogue of Deligne–Lusztig theory for $p$-adic groups $G$. He introduces a certain infinite-dimensional variety $X$ which has a natural action of $G \times T$, and defines $\ell$-adic homology groups $H_i(X)$ functorial for this action. One thus obtains a correspondence $\theta \mapsto H_i(X)[\theta]$ between characters of $T$ and representations of $G$. In this paper, we study this correspondence when $G$ is the multiplicative group of the non-split quaternion algebra over a local function field of positive characteristic and $T$ is the unramified torus. We prove that Lusztig’s conjectural construction indeed gives rise to supercuspidal representations of division algebras and furthermore gives a geometric realization of the local Langlands and Jacquet–Langlands correspondences (LLC and JLC).

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Consider the following set-up. Let $K$ be a non-Archimedean field of positive characteristic with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q = \mathcal{O}_K/\pi$ for a fixed uniformizer $\pi$, and let $L \supset K$ be the unramified extension of degree $n$ with ring of integers $\mathcal{O}_L$. A smooth character $\theta: L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is said to be of level $h$ if $h$ is the smallest integer such that $\theta$ is trivial on $1 + \pi^h \mathcal{O}_L$. In the situation that $T = L^\times$ and $G = D_{1/n}^\times$, where $D_{1/n}$ is the central division algebra over $K$ with invariant $1/n$, the local Langlands and Jacquet–Langlands correspondences (LLC and JLC) give rise to a correspondence between characters of $T$ and representations of $G$: to a smooth character $\theta: L^\times \to \overline{\mathbb{Q}}_\ell^\times$, one can associate a smooth irreducible $n$-dimensional representation $\sigma_\theta$ of the Weil group $\mathcal{W}_K$ of $K$, which corresponds to an irreducible supercuspidal representation $\pi_\theta$ of $\text{GL}_n(K)$ (via LLC), which finally corresponds to an irreducible representation $\rho_\theta$ of $D_{1/n}^\times$ (via JLC).

The main theorem of this paper is:

**Main Theorem** (Rough Formulation). Let $n = 2$. For a broad class of characters $\theta: L^\times \to \overline{\mathbb{Q}}_\ell^\times$, there exists an $r$ (dependent on the level of $\theta$) such that

$$H_i(X)[\theta] = \begin{cases} \rho_0 & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}$$

In pictorial form, we have

$$\begin{array}{c} \theta \downarrow \sigma_\theta \downarrow \pi_\theta \downarrow \rho_\theta \\ p \text{-adic Deligne–Lusztig} \\ \mathcal{G}_K(n) \downarrow \mathcal{A}_K(n) \downarrow \mathcal{A}'_K(n) \\ \mathcal{X} = \{ \text{characters } L^\times \to \overline{\mathbb{Q}}_\ell^\times \text{ with trivial Gal}(L/K)\text{-stabilizer} \} \\ \mathcal{G}_K(n) = \{ \text{smooth irreducible dimension-$n$ representations of the Weil group } \mathcal{W}_K \} \\ \mathcal{A}_K(n) = \{ \text{supercuspidal irreducible representations of } \text{GL}_n(K) \} \\ \mathcal{A}'_K(n) = \{ \text{smooth irreducible representations of } D_{1/n}^\times \} \end{array}$$

1.1. **What is known.** In [B12], Boyarchenko presents a method for explicitly calculating the representations $H_i(X)[\theta]$ and does so for a special class of characters $\theta$ in the case when $G$ is the multiplicative group of the central division algebra with Hasse invariant $1/n$ over a local field $K$, and $T = L^\times$, where $L$ is the unramified degree-$n$ extension of $K$. The approach is to reduce the computation to a problem involving certain finite unipotent groups and then develop a “Deligne–Lusztig theory” for these groups.

Before we continue, we must introduce some terminology. Let $D_{1/n}$ denote the central division algebra with Hasse invariant $1/n$ over $K = \mathbb{F}_q((\pi))$ for $q$ a $p$-power and let $L = \mathbb{F}_{q^n}((\pi))$. The level of a smooth character $\theta: L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is the smallest integer $h$ such that $\theta$ is trivial on $U_L^h := 1 + \pi^h \mathcal{O}_L \subset \mathcal{O}_L^\times$, where $\mathcal{O}_L$ is the ring of integers of $L$. The set of characters of $L^\times$ has a natural action by $\text{Gal}(L/K)$. We say that $\theta$ is primitive if for any $\gamma \in \text{Gal}(L/K)$, both $\theta$ and $\theta/\theta^\gamma$ have the same level. (Equivalently, $\theta$ is primitive if its restriction to $U_L^{n-1}$ has trivial $\text{Gal}(L/K)$-stabilizer.)
In Section 1 we recall the unipotent situation established by Boyarchenko in [B12]. We describe a unipotent group scheme $U_{h}^{n,q}$ over $\mathbb{F}_{p}$ together with a subscheme $X_{h} \subset U_{h}^{n,q}$ that comes with a left action by $U_{L}^{1}/U_{L}^{h}$ and a right action by $U_{h}^{n,q}(\mathbb{F}_q)$. This unipotent group depends on three parameters that are determined by the set-up in the following way. If $\theta$ is a character of level $h$, then the computation of the eigenspaces $H_{i}(X)[\theta]$ for $D_{1/n}^{\infty}$ over $K = \mathbb{F}_{q}(\pi)$ will reduce to a computation of $H_{i}^{\prime}(X_{h}, \overline{\mathbb{Q}}_{\ell})[\chi]$ for $U_{h}^{n,q}(\mathbb{F}_q)$, where $\chi$ is the character of $U_{L}^{1}/U_{L}^{h}$ induced by $\theta$. To be completely clear, the three parameters $n, q, h$ correspond respectively to the Hasse invariant of the division algebra, the size of the residue field of $K$, and the level of $\theta$.

In [BW14], Boyarchenko and Weinstein give a complete description of the $U_{2}^{n,q}(\mathbb{F}_q)$-representations $H_{c}^{\prime}(X_{2}, \overline{\mathbb{Q}}_{\ell})[\chi]$. They prove the following

**Theorem (Boyarchenko and Weinstein).** Given a character $\chi: U_{L}^{1}/U_{L}^{h} \to \overline{\mathbb{Q}}_{\ell}$, there exists a unique $r$ such that $H_{i}^{\prime}(X_{2}, \overline{\mathbb{Q}}_{\ell})[\chi]$ vanishes when $i \neq r$ and is an irreducible $U_{h}^{n,q}(\mathbb{F}_q)$-representation when $i = r$. Furthermore, every irreducible representation of $U_{h}^{n,q}(\mathbb{F}_q)$ occurs with multiplicity one in $\bigoplus_{i \in \mathbb{Z}} H_{c}^{\prime}(X_{2}, \overline{\mathbb{Q}}_{\ell})$.

It turns out that the scheme $X_{2}$ is very closely related to a certain open affinoid of the Lubin-Tate tower (see [BW14]), and in op. cit. Boyarchenko and Weinstein use the above theorem to give a purely local proof of the local Langlands and Jacquet–Langlands correspondences for a broad class of supercuspidals (those whose Weil parameters are induced from a primitive character of an unramified degree-$n$ extension). In [BW13], Boyarchenko and Weinstein use this result to give a geometric realization of the local Langlands and Jacquet–Langlands correspondences in this class of supercuspidals.

The analogue of the above theorem of Boyarchenko and Weinstein for $U_{h}^{n,q}(\mathbb{F}_q)$ when $h > 2$, however, was almost completely unknown. Indeed, the only higher level situation known was the case $h = 3$ and $n = 2$, which Boyarchenko computed in [B12] (see Theorem 5.20 of [B12]). The following conjecture appears in [B12].

**Conjecture 1.1 (Boyarchenko).** Given a character $\chi: U_{L}^{1}/U_{L}^{h} \to \overline{\mathbb{Q}}_{\ell}$, there exists $r \geq 0$ such that $H_{i}^{\prime}(X_{h}, \overline{\mathbb{Q}}_{\ell})[\chi]$ vanishes when $i \neq r$ and is an irreducible $U_{h}^{n,q}(\mathbb{F}_q)$-representation when $i = r$.

Assuming Conjecture 1.1 holds, Boyarchenko gives a complete description of the $D_{1/n}^{\infty}$-representations $H_{i}(X)[\theta]$ based on the $U_{h}^{n,q}(\mathbb{F}_q)$-representations $H_{i}(X_{h})[\chi]$ (see Proposition 5.19 of [B12]). Thus, the problem of determining the representations $H_{i}(X)[\theta]$ arising from Lusztig’s $p$-adic analogue of Deligne–Lusztig varieties, depends only on the Deligne–Lusztig theory of the finite unipotent group $U_{h}^{n,q}(\mathbb{F}_q)$.

**Remark 1.2.** The varieties constructed in [L79] are not the affine Deligne–Lusztig varieties. In [I13], Ivanov shows that there are no nontrivial morphisms from the cohomology of the affine Deligne–Lusztig to $D_{1/2}^{\infty}$-representations of level $> 1$. This is not true for the varieties in [L79] associated to division algebras (see Theorem 7.1).

### 1.2. Outline of this paper.

In this paper, we prove Conjecture 1.1 when $n = 2$ and $\chi: U_{L}^{1}/U_{L}^{h} \to \overline{\mathbb{Q}}_{\ell}$ has the property that its restriction $\psi := \chi|_{U_{L}^{1}/U_{L}^{h}}$ has trivial $\text{Gal}(\mathbb{F}_q^{\text{sep}}/\mathbb{F}_q)$-stabilizer. (In this situation, we say that $\psi$ has conductor $q^{2}$.) Using Proposition 5.19 of [B12], we can then describe the representations $H_{i}(X)[\theta]$ for primitive $\theta$ of arbitrary level.
Let $A_\psi$ denote the set of such $\chi$ and let $G_\psi$ denote the set of irreducible representations of $U_h^{2,q}(\mathbb{F}_{q^2})$ restricting to a multiple of $\psi$. In Section 3, we recall the following theorem of Corwin (see Theorem 2.3 in [C74]):

**Theorem (Corwin).** There exists a bijection

$$A_\psi \leftrightarrow G_\psi, \quad \chi \mapsto \rho_\chi.$$

Using an explicit description of this bijection, we prove a certain character formula in Section 4 that plays a crucial role in Section 6.

In Section 5, we prove that there are no nontrivial morphisms from $\rho_\chi$ to $H^i_c(X_h, \overline{\mathbb{Q}}_\ell)$ if $i \neq h - 1$. This allows us to apply a variant of a Deligne–Lusztig fixed point formula (see Lemma 2.13 of [B12]) in order to compute subspaces of intertwiners in $H^i_c(X_h, \overline{\mathbb{Q}}_\ell)$. These computations, done in Section 6, allow us to prove

**Theorem (6.3).** The cohomology groups $H^i_c(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ vanish when $i \neq h - 1$ and

$$H^{h-1}_c(X_h, \overline{\mathbb{Q}}_\ell)[\chi] \cong \rho_\chi.$$

Moreover, the Frobenius $Fr_{q^2}$ acts on $H^{h-1}_c(X_h, \overline{\mathbb{Q}}_\ell)$ by multiplication by $(-1)^{h-1}q^{h-1}$.

**Remark 1.3.** Carrying out the arguments of Section 4, 5, and 6 in the special case $h = 3$ gives a different proof of Theorem 5.20 of [B12].

**Remark 1.4.** Following [BW14], we say that a finite-type variety $S$ over $\mathbb{F}_Q$ is a maximal variety if

$$\#S(\mathbb{F}_Q) = \sum_{i \in \mathbb{Z}} Q^{i/2} \dim H^i_c(S, \overline{\mathbb{Q}}_\ell).$$

This bound is achieved if and only if $Fr_{Q^2}$ acts on $H^i_c(S, \overline{\mathbb{Q}}_\ell)$ via the scalar $(-1)^i Q^{i/2}$.

In [BW14], they show that the varieties $X^{2,q}_h$ are maximal by computing the action of $Fr_{q^2}$. Theorem 6.3 shows that $X^{2,q}_h$ behaves like a maximal variety in middle degree; however, at the writing of this paper, we do not know the action of $Fr_{q^2}$ on the cohomology of $X^{2,q}_h$ outside the middle degree. Note also that although $X_h(\mathbb{F}_q)$ has as many points as possible as a subvariety of $A^{2(h-1)}(\mathbb{F}_q)$, but $X_h$ may have extra cohomology that makes the bound $\sum Q^{i/2} \dim H^i_c(S, \overline{\mathbb{Q}}_\ell)$ exceed $\#A^{2(h-1)}(\mathbb{F}_q) = q^{4(h-1)}$.

It is worth noting here that the above theorem (Theorem 6.3) is stronger than Conjecture 1.1; it requires considerably more work to prove $H^{h-1}_c(X_h, \overline{\mathbb{Q}}_\ell)[\chi] \cong \rho_\chi$ than to prove its irreducibility (compare the proofs of Theorems 6.1 and 6.2). Because we have an explicit description of the $U_h^{2,q}(\mathbb{F}_q)$-representations $H^i_c(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$, we can use Proposition 5.19 of [B12] to explicitly describe the $D_{1/2}^{2,q}$-representations $H^i(X, \overline{\mathbb{Q}}_\ell)[\theta]$. The final theorem in this paper, whose rough formulation was stated as the Main Theorem, compares the correspondence

$$\theta \mapsto H^i_c(X, \overline{\mathbb{Q}}_\ell)[\theta]$$

to known correspondences between characters of $L^\times$ and representations of division algebras.

We can now formulate the Main Theorem more precisely.

**Theorem (7.1).** Let $\theta : L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a primitive character of level $h$ and let $\rho_\theta$ be the $D_{1/2}^{2,q}$ representation corresponding to $\theta$ under the local Langlands and Jacquet–Langlands correspondences. Then $H^i_c(X, \overline{\mathbb{Q}}_\ell)[\theta] = 0$ for $i \neq h - 1$, and

$$H^{h-1}_c(X, \overline{\mathbb{Q}}_\ell)[\theta] \cong \rho_\theta.$$
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2. Definitions and notation

The definitions of this section are established in Section 5.2 of [B12]. We include them here in the rank-2 case for the convenience of the reader.

Let \( K = \mathbb{F}_q(\pi) \) and let \( L = \mathbb{F}_q^2(\pi) \) be the degree-2 unramified extension of \( K \). For any \( r \geq 1 \), let \( U_L^r = 1 + \pi^r \mathcal{O}_L \) where \( \mathcal{O}_L = \mathbb{F}_q[\pi] \) is the ring of integers of \( L \). Let \( D \) be the quaternion algebra over \( K \). Explicitly, \( D = L / (\Pi / (\Pi^2 - \pi) \), where \( L / (\Pi) \) is the twisted polynomial ring with the commutation relation \( \Pi \cdot a = \varphi(a) \cdot \Pi \), where \( \varphi \) is the nontrivial element of \( \text{Gal}(L/K) \). Let \( U_D^r = 1 + \Pi^r \mathcal{O}_D \) where \( \mathcal{O}_D = \mathcal{O}_L(\Pi) / (\Pi^2 - \pi) \) is the ring of integers of \( D \).

For any \( \mathbb{F}_q \)-algebra \( A \), consider the twisted polynomial ring \( A[\pi] / (\tau^2 - \pi, \tau^{2(h-1)+1}) \)

Now define

\[
U_h^2(q)(A) := \{ 1 + a_1 \tau + \cdots + a_{2(h-1)} \tau^{2(h-1)} \} \subseteq \mathcal{R}_{h,2,q}(A), \\
H(A) := \{ 1 + a_2 \tau^2 + \cdots + a_{2(h-1)} \tau^{2(h-1)} \} \subseteq U_h^2(q(A).
\]

The functor \( A \mapsto U_h^2(q)(A) \) is representable by \( A^{2(h-1)} \) and the group scheme \( U_h^2(q) \) has a natural filtration

\[
\{ 1 \} \subset H_2(h-1) \subset H_2(h-1) - 1 \subset \cdots \subset H_2 \subset H_1 = U_h^2(q),
\]

where \( H_k := \{ 1 + \sum a_i \tau^i : i \geq k \} \).

Remark 2.1. We have canonical isomorphisms

\[
U_L^1 / U_L^1 \cong H(\mathbb{F}_q), \\
F_q^2 / U_L^1 \cong H_{2}(\mathbb{F}_q), \\
U_h^2(q) \cong U_D^2 / U_D^{2(h-1)+1}.
\]

These isomorphisms, together with the assignment \( \tau \mapsto \Pi \) induces

\[
U_h^2(q)(\mathbb{F}_q) \cong U_D^2 / U_D^{2(h-1)+1}.
\]

2.1. The varieties \( X_h \).

Definition 2.2. For any \( \mathbb{F}_q \)-algebra \( A \), let \( \text{Mat}_2(A) \) denote the ring of all \( n \times n \) matrices \( (b_{ij}) \) with \( b_{ii} \in A[\pi] / (\pi^h) \), \( b_{ij} \in A[\pi] / (\pi^{h-1}) \) for \( i < j \), and \( b_{ij} \in \pi A[\pi] / \pi^h \) for \( i > j \). The determinant induces a multiplicative map \( \det : \text{Mat}_2(A) \rightarrow A[\pi] / (\pi^h) \).

For any \( \mathbb{F}_q \)-algebra \( A \), consider the morphism

\[
\iota_h : \mathcal{R}_{h,2,q}(A) \rightarrow \text{Mat}_2(A)
\]

given by

\[
\iota_h(\sum a_i \tau^i) := \left( \begin{array}{cc} a_0 + a_2 \pi + \cdots + a_{2(h-1)} \pi^{h-1} & a_1 + a_3 \pi + \cdots + a_{2(h-1)-1} \pi^{h-2} \\ a_0 q^2 \pi + a_2 q^2 \pi^2 + \cdots + a_{2(h-1)-1} q^2 \pi^{h-2} & a_1 q^2 + a_2 q^2 \pi + \cdots + a_{2(h-1)-1} q^2 \pi^{h-2} \end{array} \right),
\]

\[
\iota_h(\sum a_i \tau^i) := \left( \begin{array}{cc} a_0 + a_2 \pi + \cdots + a_{2(h-1)} \pi^{h-1} & a_1 + a_3 \pi + \cdots + a_{2(h-1)-1} \pi^{h-2} \\ a_0 q^2 \pi + a_2 q^2 \pi^2 + \cdots + a_{2(h-1)-1} q^2 \pi^{h-2} & a_1 q^2 + a_2 q^2 \pi + \cdots + a_{2(h-1)-1} q^2 \pi^{h-2} \end{array} \right).
\]
By the calculations of Section 4.2 of [B12], the \( p \)-adic Deligne–Lusztig construction \( X \) described in [L79] can be identified with the set

\[
\left\{ \begin{pmatrix} A_0 & A_1 \\ \pi \varphi(A_1) & \varphi(A_0) \end{pmatrix} \in \GL_2(\overline{K}^{nr}) : \det \in K^\times \right\},
\]

which can be realized as the \( \mathbb{F}_q \)-points of an ind-scheme

\[
\bar{X} = \bigsqcup_{m \in \mathbb{Z}_h} \Xi_{m}. \]

Here, \( \Xi_h^{(m+1)} = \Xi_h^{(m)} \cdot \Pi \) and for any \( \mathbb{F}_q \)-algebra \( A \),

\[
\Xi_h^{(0)}(A) = \left\{ \iota_h(\sum a_i \tau^i) : \varphi(\det) = \det(A[\pi]/(\pi^h))^\times \right\},
\]

where \( \varphi \) is induced from the \( q \)-Frobenius \( x \mapsto x^q \) on \( A \).

**Definition 2.3.** For any \( \mathbb{F}_q \)-algebra \( A \), define

\[
X_h(A) := U_{h}^{2,q}(A) \cap \iota_h^{-1}(\Xi_h^{(0)}(A)).
\]

**Remark 2.4.** Note that \( \Xi_h^{(0)}(A) \) is a disjoint union of \( q^2 - 1 \) copies of \( X - h \).

### 2.2. Group actions.

The map \( \iota_h \) has the following property, which we will refer to as Property \( \dagger \). If \( A \) is an \( \mathbb{F}_q^2 \)-algebra, then \( \iota_h(xy) = \iota_h(x)\iota_h(y) \) for all \( x \in U_{h}^{2,q}(A) \) and all \( y \in U_{h}^{2,q}(\mathbb{F}_q^2) \). Moreover, for \( y \in U_{h}^{2,q}(\mathbb{F}_q^2) \), we have \( \det(y) \in \mathbb{F}_q[\pi]/(\pi^h) \). It therefore follows that \( X_h \) is stable under right-multiplication by \( U_{h}^{2,q}(\mathbb{F}_q^2) \). We denote by \( x \cdot g \) the action of \( g \in U_{h}^{2,q}(\mathbb{F}_q^2) \) on \( x \in X_h \).

We now describe a left action of \( H(\mathbb{F}_q^2) \) on \( X_h \). We can identify \( H(\mathbb{F}_q^2) \) with the set \( \iota_h(H(\mathbb{F}_q^2)) \). Note that by Property \( \dagger \), the map \( \iota_h \) actually preserves the group structure of \( H(\mathbb{F}_q^2) \). Since \( \iota_h \) is injective, then we in fact have a group isomorphism \( H(\mathbb{F}_q^2) \cong \iota_h(H(\mathbb{F}_q^2)) \). Explicitly, this isomorphism is given by

\[
1 + \sum a_{2i} \tau^{2i} \mapsto \begin{pmatrix} 1 + \sum a_{2i} \pi \tau^{2i} \\ 0 \\ 0 + \sum a_{2i} \pi \tau^{2i} \end{pmatrix}.
\]

For any \( \mathbb{F}_q^2 \)-algebra \( A \), the natural left-multiplication action of \( \iota_h(H(\mathbb{F}_q^2)) \) on \( \Mat_3(A) \) stabilizes \( X_h(A) \). This defines a left action of \( H(\mathbb{F}_q^2) \) on \( X_h \). We denote by \( g \ast x \) the action of \( g \in H(\mathbb{F}_q^2) \) on \( x \in X_h \).

**Remark 2.5.** Let \( Z(U_{h}^{2,q}(\mathbb{F}_q^2)) \) denote the center of \( U_{h}^{2,q}(\mathbb{F}_q^2) \). This is a subgroup of \( H(\mathbb{F}_q^2) \). By direct computation, one sees that the left action of \( Z(U_{h}^{2,q}(\mathbb{F}_q^2)) \subset H(\mathbb{F}_q^2) \) and the right action of \( Z(U_{h}^{2,q}(\mathbb{F}_q^2)) \subset U_{h}^{2,q}(\mathbb{F}_q^2) \) coincide.

### 3. The representation theory of \( U_{h}^{2,q}(\mathbb{F}_q^2) \)

In this section, we will describe a class of irreducible representations of \( U_{h}^{2,q}(\mathbb{F}_q^2) \) that are in bijection with certain characters \( \chi : U_{h}^{1}/U_{h}^{2} \to \overline{\mathbb{Q}}_{\ell}^\times \). Recall that we have canonical isomorphisms \( U_{h}^{1}/U_{h}^{2} \cong H(\mathbb{F}_q^2) \) and \( \mathbb{F}_q^2 \cong U_{h}^{2,q}/U_{h}^{1,q} \cong H_{2(q-1)}(\mathbb{F}_q^2) \).

The additive characters of \( \mathbb{F}_q^r \) have a natural action by \( \text{Gal}(\mathbb{F}_q^r/\mathbb{F}_q) \). We say that a character \( \psi : \mathbb{F}_q^r \to \overline{\mathbb{Q}}_{\ell}^\times \) has conductor \( q^m \) if its stabilizer in \( \text{Gal}(\mathbb{F}_q^r/\mathbb{F}_q) \) is \( \text{Gal}(\mathbb{F}_q^r/\mathbb{F}_q^{q^m}) \). In this paper we

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1Warning: This is not the same as the left-multiplication action of \( H(\mathbb{F}_q^2) \subset H(A) \) on \( U_{h}^{2,q}(A) \).
will only work with the case when \( n = 2 \) and only work with characters \( \psi: \mathbb{F}_{q^2} \to \overline{\mathbb{Q}}_l^\times \) that have conductor \( q^2 \). In this case, this just means that there exists some \( x \in \mathbb{F}_{q^2} \) such that \( \psi(x^q) \neq \psi(x) \).

Let \( \mathcal{A}_\psi \) denote the set of all characters \( \chi: U_L^1/U_L^h \to \overline{\mathbb{Q}}_l^\times \) whose restriction to \( U_L^{h-1}/U_L^h \) is equal to \( \psi \). Let \( \mathcal{G}_\psi \) denote the set of irreducible representations of \( U_L^{2,q}(\mathbb{F}_{q^2}) \) wherein \( H_{2(h-1)}(\mathbb{F}_{q^2}) \) acts via \( \psi \). The following is a theorem of Corwin (see Theorem 2.3 in [C74]). In Sections 3.1 and 3.2, we give an explicit description of this correspondence.

**Theorem 3.1** (Corwin). If \( \psi \) has conductor \( q^2 \), then there exists a bijection between \( \mathcal{A}_\psi \) and \( \mathcal{G}_\psi \). Furthermore, every representation in \( \mathcal{G}_\psi \) has dimension \( q^{h-1} \).

Given \( \chi \in \mathcal{A}_\psi \), denote by \( \rho_\chi \) the corresponding representation in \( \mathcal{G}_\psi \). We now recall the construction of \( \rho_\chi \) explained in [C74].

Consider the following subgroups:

\[
H'(\mathbb{F}_{q^2}) := \begin{cases} 
\{1 + \sum a_i \tau^i : i \text{ is even OR } i > h - 1 \text{ is odd} \} & \text{if } h \text{ is odd}, \\
\{1 + \sum a_i \tau^i : i \text{ is even OR } i \geq h - 1 \text{ is odd; } a_{h-1} \in \mathbb{F}_q \} & \text{if } h \text{ is even}.
\end{cases}
\]

Define

\[
H'_0(\mathbb{F}_{q^2}) := \{1 + \sum a_i \tau^i : i = 2(h-1) \text{ OR } i > h - 1 \text{ is odd} \} \subset U_L^{2,q}(\mathbb{F}_{q^2})
\]

and if \( h \) is even, define

\[
H'_1(\mathbb{F}_{q^2}) := \{1 + \sum a_i \tau^i : i = 2(h-1) \text{ OR } i \geq h - 1 \text{ is odd; } a_{h-1} \in \mathbb{F}_q \} \subset U_L^{2,q}(\mathbb{F}_{q^2}).
\]

The dependence of these subgroups on the parity of \( h \) reflects the dependence of the construction of \( \rho_\chi \) on the parity of \( h \).

3.1. **Case: \( h \) odd.** We first construct \( \rho_\chi \) in the case that \( h \) is odd. For \( \chi \in \mathcal{A}_\psi \), define a character \( \chi^h \) of \( H'(\mathbb{F}_{q^2}) \) as

\[
\chi^h(1 + \sum a_i \tau^i) = \chi(1 + a_2 \pi + \cdots + a_{2(h-1)} \pi^{h-1}).
\]

Then Theorem 2.3 of [C74] shows that \( \rho_\chi \cong \text{Ind}_{H'(\mathbb{F}_{q^2})}^{U_L^{2,q}(\mathbb{F}_{q^2})} (\chi^h) \).

3.2. **Case: \( h \) even.** Now we construct \( \rho_\chi \) in the case that \( h \) is even. For \( \chi \in \mathcal{A}_\psi \), we would like to construct an extension \( \chi' \) of \( \chi \) to \( H'(\mathbb{F}_{q^2}) \). Section 3 of [C74] gives one way to do this. We present a different approach to obtaining this extension.

We first recall some general facts about group representations. Suppose that \( G \) is a group and \( H, K, N \subset G \) are subgroups such that \( H = K \cdot N \). Note that if \( \chi \) is a character of \( K \) and \( \theta \) is a character of \( N \) such that \( \chi = \theta \) on the intersection \( K \cap N \), then the function \( f(k \cdot n) := \chi(k) \theta(n) \) is well-defined. Now let \( \chi \) and \( \theta \) be multiplicative. If \( K \) normalizes \( N \) and \( K \) centralizes \( \theta \), then in fact

\[
f(k_1 n_1 k_2 n_2) = f(k_1 k_2 (k_2^{-1} n_1 k_2 n_2)) = \chi(k_1 k_2) \theta(n_1 n_2) = f(k_1 k_2 n_1 n_2),
\]

so \( f \) is multiplicative.

We now apply the above to the situation when \( K = H(\mathbb{F}_{q^2}) \), \( N = H'_1(\mathbb{F}_{q^2}) \) and \( H = H'(\mathbb{F}_{q^2}) \). Recall that we have

\[
H(\mathbb{F}_{q^2}) := \{1 + \sum a_i \tau^i : i \text{ is even} \}
\]

\[
H'_1(\mathbb{F}_{q^2}) := \{1 + \sum a_i \tau^i : i = 2(h-1) \text{ OR } i \geq h - 1 \text{ is odd; } a_{h-1} \in \mathbb{F}_q \}
\]

\[
H'(\mathbb{F}_{q^2}) := \{1 + \sum a_i \tau^i : i \text{ is even OR } i \geq h - 1 \text{ is odd; } a_{h-1} \in \mathbb{F}_q \}
\]
Note that \( H'_1(\mathbb{F}_q) \) is an abelian subgroup of \( U^2_q(\mathbb{F}_q) \) containing \( H'_0(\mathbb{F}_q) \) as an index-\( q \) subgroup. Thus there are \( q \) extensions of \( \bar{\psi} \) to \( H'_1(\mathbb{F}_q) \). Given such an extension \( \theta \) and given \( \chi \in \mathcal{A}_\psi \), we wish to construct a character \( \bar{\chi} \) of \( H'(\mathbb{F}_q) \) that extends both \( \chi \) and \( \bar{\psi} \). (This is the analogue of \( \chi' \) in the case that \( h \) is odd.)

We see that \( H(\mathbb{F}_q^2) \cap H'_1(\mathbb{F}_q^2) = H_{2(h-1)}(\mathbb{F}_q^2) \) and that \( \chi \) and \( \theta \) agree on this intersection. Now define

\[
\bar{\chi}_\theta(kn) = \chi(k)\theta(n) \quad \text{for } k \in H(\mathbb{F}_q^2) \text{ and } n \in H'_1(\mathbb{F}_q^2).
\]

This is well-defined.

**Lemma 3.2.** The map \( \bar{\chi}_\theta : H'(\mathbb{F}_q^2) \to \overline{\mathbb{Q}_q}^\times \) is a group homomorphism.

**Proof.** It is enough to show that \( H(\mathbb{F}_q^2) \) normalizes \( H'_1(\mathbb{F}_q^2) \) and that \( H(\mathbb{F}_q^2) \) centralizes \( \theta \). Write \( k = 1 + \sum a_i\tau^i \in H(\mathbb{F}_q^2) \) and \( n = 1 + \sum b_i\tau^i \in H'_1(\mathbb{F}_q^2) \). Since the only nonzero terms of \( k \) are \( a_2\tau^{2i} \), then \( n \) only differs by \( knk^{-1} \) in the coefficients of \( \tau^i \) for \( i \) odd and \( > h - 1 \). Thus \( H(\mathbb{F}_q^2) \) normalizes \( H'_1(\mathbb{F}_q^2) \). Moreover, the coefficient of \( \tau^{2(h-1)} \) in \( knk^{-1}n^{-1} \) is equal to \( b_{2(h-1)} \). Since \( \theta \) is an extension of \( \bar{\psi} \), then it follows that \( H(\mathbb{F}_q^2) \) centralizes \( \theta \). This completes the proof. \( \square \)

To define an extension of \( \chi \) to a character of \( H'(\mathbb{F}_q^2) \), we needed to pick a character \( \theta \). The next lemma shows that this choice is an auxiliary one of no consequence.

**Lemma 3.3.** Let \( \theta_1 \) and \( \theta_2 \) be extensions of \( \bar{\psi} \) to \( H'_1(\mathbb{F}_q^2) \). Let \( \bar{\chi}_i := \bar{\chi}_{\theta_i} \) for \( i = 1, 2 \). Then

\[
\text{Ind}_{H'(\mathbb{F}_q^2)}^{U^2_q(\mathbb{F}_q^2)}(\bar{\chi}_1) \cong \text{Ind}_{H'(\mathbb{F}_q^2)}^{U^2_q(\mathbb{F}_q^2)}(\bar{\chi}_2).
\]

**Proof.** Suppose that \( \theta_1 \) and \( \theta_2 \) are any extensions of \( \bar{\psi} \) to \( H'_1(\mathbb{F}_q^2) \). Recall that the corresponding characters \( \bar{\chi}_1 \) and \( \bar{\chi}_2 \) of \( H'(\mathbb{F}_q^2) \) are defined as

\[
\bar{\chi}_i(kn) = \chi(k)\theta_i(n),
\]

where \( k \in H(\mathbb{F}_q^2) \) and \( n \in H'_1(\mathbb{F}_q^2) \). Note that for any \( g \in U^2_q(\mathbb{F}_q^2) \), we have

\[
gkn(kn^{-1}) = (gkg^{-1})(gng^{-1}).
\]

Now consider the element \( g = 1 - a\tau^{h-1} \in U^2_q(\mathbb{F}_q^2) \). Since \( h \) is even, \( h - 1 \) is odd, and \( \bar{\chi}_1(gkg^{-1}) = \chi(k) \). Therefore

\[
g\bar{\chi}_1(kn) = \chi(k) \cdot g\theta_1(n).
\]

We thus see that to show that \( \bar{\chi}_1 \) and \( \bar{\chi}_2 \) are \( U^2_q(\mathbb{F}_q^2) \)-conjugate, it suffices to show that there exists a \( g = 1 - a\tau^{h-1} \in U^2_q(\mathbb{F}_q^2) \) such that \( g\theta_1 = \theta_2 \).

Now, for any \( n = 1 + \sum b_i\tau^i \in H'_1(\mathbb{F}_q^2) \),

\[
gng^{-1} = \left( 1 - a\tau^{h-1} \right) \left( 1 + \sum_{h-1 \le i} b_i\tau^i \right) \left( 1 + a\tau^{h-1} + a^{q+1}\tau^{h-1} \right)
\]

\[
= 1 + \left( \sum_{h-1 \le i < 2(h-1)} b_i\tau^i \right) + \left( b_{2(h-1)} + b_{h-1}a^q - ab_{h-1}^q \right) \tau^{2(h-1)}.
\]

Thus

\[
g\theta_1(n) = \theta_1(n)(b_{h-1}a^q - ab_{h-1}^q).
\]

From here, we need only show that \( \# \{ g : g = 1 - a\tau^{h-1} \} = q \), where \( \theta \) is any extension of \( \bar{\psi} \) to \( G \).
Noting that \( b_{h,-1} \in F_q \) since \( b \in G \), the above computation shows that
\[
\theta_1(gng^{-1}) = \theta_1(n)\psi(b_{h,-1}a^q - ab_{h,-1}).
\]
Since \( \psi \) has trivial \( \text{Gal}(F_{q^2}/F_q) \)-stabilizer, then in particular it is nontrivial on \( \ker \text{Tr}_{F_{q^2}/F_q} \). It is not difficult to show that every additive character of \( F_q \) can be written as \( b \mapsto \psi(b(a^q - a)) \) for some \( a \in F_{q^2} \). This completes the proof. \( \square \)

We define
\[
\rho_\chi := \text{Ind}_{H_0(F_{q^2})}^{U_{h}^2(F_{q^2})} (\tilde{\chi} \theta).
\]
Note that Lemma 3.3 justifies the suppression of \( \theta \) in this definition.

### 3.3. Final conclusions

Recall that \( H'_0(F_{q^2}) := \{ 1 + \sum a_i \tau^i : i = 2(h-1) \text{ OR } i > h - 1 \text{ is odd} \} \subset U_{h}^2(F_{q^2}) \)
and for an additive character \( \psi \) of \( F_{q^2} \), define the character \( \tilde{\psi} \) of \( H'_0(F_{q^2}) \) as
\[
\tilde{\psi} : H'_0(F_{q^2}) \to \overline{\mathbb{Q}}_\ell, \quad 1 + \sum a_i \tau^i \mapsto \psi(a_{2(h-1)}).
\]

**Lemma 3.4.** Let \( \psi \) be an additive character of \( F_{q^2} \) with conductor \( q^2 \). If \( \rho \) is an irreducible representation of \( U_{h}^2(F_{q^2}) \) where \( H'_{2(h-1)}(F_{q^2}) \) acts by \( \psi \), then the restriction of \( \rho \) to \( H'_0(F_{q^2}) \) must contain \( \psi \).

**Proof.** We prove this inductively. Let
\[
\begin{align*}
G_1 := \{ 1 + a_2(h-1)\tau^{2(h-1)-1} + a_2(h-1)\tau^{2(h-1)} \}, \\
G_2 := \{ 1 + a_{2(h-1)-3}\tau^{2(h-1)-3} + a_2(h-1)-1\tau^{2(h-1)-1} + a_2(h-1)\tau^{2(h-1)} \}, \\
\vdots \\
G_{\lfloor (h-1)/2 \rfloor} := H'_0.
\end{align*}
\]
Since
\[
1 + a_{2(h-1)-1}\tau^{2(h-1)-1} + a_2(h-1)\tau^{2(h-1)} = (1 + a_2(h-1)-1\tau^{2(h-1)-1})(1 + a_2(h-1)\tau^{2(h-1)}),
\]
then every extension of \( \psi \) to \( G_1(F_{q^2}) \) is of the form
\[
1 + a_{2(h-1)-1}\tau^{2(h-1)-1} + a_2(h-1)\tau^{2(h-1)} \mapsto \nu(a_{2(h-1)-1})\psi(a_2(h-1))
\]
for some additive character \( \nu \) of \( F_{q^2} \). Let \( \psi_1 \) denote the extension of \( \psi \) to \( G_1(F_{q^2}) \) given by
\[
\psi_1(1 + a_{2(h-1)-1}\tau^{2(h-1)-1} + a_2(h-1)\tau^{2(h-1)}) := \psi(a_2(h-1)).
\]
For \( g_1 = 1 - b_1 \tau \) and \( h = 1 + \sum a_i \tau^i \in G_1(F_{q^2}) \), we have
\[
g_1 \psi_1(h) = \psi_1(g_1 h g_1^{-1}) = \psi_1(h)\psi(b_1 a_2^{2(h-1)-1} - b_0 a_2(h-1)) \psi(a_2(h-1)).
\]
Since \( \psi \) has conductor \( q^2 \), every character of \( F_{q^2} \) is of the form \( y \mapsto \psi(xy^q - x^qy) \) for some \( x \in F_{q^2} \). Thus for any additive character \( \nu \) of \( F_{q^2} \), there exists an \( g_1 \) such that \( g_1 \psi_1(h) = \psi_1(h)\nu(a_{2(h-1)-1}) \).

We may therefore conclude that the restriction of \( \rho \) to \( G_1(F_{q^2}) \) contains \( \psi_1 \).

We now work on extending \( \psi_1 \) to \( G_2(F_{q^2}) \). Since
\[
h = h_0(1 + a_{2(h-1)-3}\tau^{2(h-1)-3}),
\]
where \( h \in G_2(\mathbb{F}_{q^2}) \) and \( h_0 \in G_1(\mathbb{F}_{q^2}) \), then every extension of \( \psi_1 \) to \( G_2(\mathbb{F}_{q^2}) \) is of the form

\[
h \mapsto \nu(a_2(h_{-1})\psi_1(h_0),
\]

where as before, \( \nu \) is some additive character of \( \mathbb{F}_{q^2} \). Let \( \psi_2 \) denote the extension of \( \psi_1 \) to \( G_2(\mathbb{F}_{q^2}) \) given by

\[
\psi_2(1 + \sum a_i \tau^i := \psi_1(1 + a_2(h_{-1})\tau^{2(h-1)} + a_2(h_{-1})\tau^{2(h-1)} = \psi(a_2(h_{-1})).
\]

For \( g_2 = 1 - b_3 \tau^3 \) and \( h = 1 + \sum a_i \tau^i \in G_2(\mathbb{F}_{q^2}), \) we have

\[
g_2^2 \psi_2(h) = \psi_2(g_2 h g_2^{-1}) = \psi_2(h)\psi_2(b_3 a_2(h_{-1}) - b_3 a_2(h_{-1}) - 3).
\]

As before, this shows that the restriction of \( \rho \) to \( G_2(\mathbb{F}_{q^2}) \) contains \( \psi_2 \).

Continuing this for each \( G_i \), we see that the conclusion of the Lemma holds.

We can now prove a theorem that will be crucial to the work in Section 5.

**Theorem 3.5.** Let \( \psi \) be a character of \( \mathbb{F}_{q^2} \) with conductor \( q^2 \). Then

\[
V_\psi := \text{Ind}_{H_0^1(\mathbb{F}_{q^2})}^{U_{q^2}^1(\mathbb{F}_{q^2})} (\psi) \cong \bigoplus_{\theta} \bigoplus_{\chi \in A_{\theta}} \bar{\chi}_\theta,
\]

where \( \rho \) ranges over the elements of \( G_\psi \).

**Proof.** In the case that \( h \) is odd, this follows directly from Theorem 3.1 together with the construction in Section 3.1.

In the case that \( h \) is even, we have

\[
\text{Ind}_{H_0^1(\mathbb{F}_{q^2})}^{H^1(\mathbb{F}_{q^2})} (\psi) \cong \bigoplus_{\theta} \bigoplus_{\chi \in A_{\theta}} \bar{\chi}_\theta,
\]

where \( \theta \) varies over the extensions of \( \psi \) to \( H_1^1(\mathbb{F}_{q^2}) \). Indeed, this is true by a counting argument: The maximum number of extensions of \( \psi \) to \( H^1(\mathbb{F}_{q^2}) \) is \([H^1(\mathbb{F}_{q^2}) : H_0^1(\mathbb{F}_{q^2})] = q^{3(h-1)+1}/q^{h+1} = q^{2(h-1)}-1\). On the other hand, it is clear that \( \bar{\chi}_\theta \) is an extension of \( \psi \) and varying \( \chi \) and \( \theta \) give rise to \( q^{2(h-1)+1} \) distinct extension \( \nu \). Therefore in fact every such \( \nu \) is of the form \( \bar{\chi}_\theta \). Combining Equation (1), Lemma 3.3, and Theorem 3.1, the desired result follows.

\( \square \)

### 4. A CHARACTER FORMULA

One can show that every irreducible representation of \( D^\times \) can be obtained from the irreducible representations of certain subquotients \( U_{2q}^1/U_{q}^1 \) in a manner similar to that described in Section 7 (see [C74] for more details). Thus computing the characters of representations of division algebras reduces to computing the characters of representations of these subquotients. In [C89], Corwin outlines these computations but only writes down the final character formula (of representations of division algebras). One can also find similar computations in [CH77].

In this section we establish a character formula for certain representations of \( U_{2q}^1/U_{q}^{2(h-1)+1} \cong U_{h}^{2,q}(\mathbb{F}_{q^2}). \) The main consequence of this formula is that we will be able to decompose the irreducible representations \( \rho_\chi \) of \( U_{h}^{2,q}(\mathbb{F}_{q^2}) \) as representations of the subgroup \( H(\mathbb{F}_{q^2}) \subset U_{h}^{2,q}(\mathbb{F}_{q^2}) \). Moreover, we will show that for an additive character \( \psi: \mathbb{F}_{q^2} \to \overline{\mathbb{Q}}_l^\times \) of conductor \( q^r \), the elements of \( G_\psi \) are uniquely determined by their restrictions to \( H(\mathbb{F}_{q^2}) \). The character formula (Theorem 4.1) and its consequences (Corollaries 4.2 and 4.3) play a fundamental role in Section 6.
We establish some notation first. Recall the subgroup \( H \subset U^2_q \) and, abusing notation, define
\[
H := H(F_q) = \{1 + \sum_{i=1}^{h-1} a_{2i} \tau^{2i} \} \subset U^2_q(F_q).
\]
We will also need the subgroups
\[
N_k := \{1 + \sum a_i \tau^i : i \text{ even OR } i > k \} \subset U^2_q(F_q),
\]
\[
M := N_{h-1} \subset U^2_q(F_q).
\]
Given a character \( \chi : H \to \mathbb{Q} \times \ell \), let \( \chi^\# \) be the character of \( M \) defined in Section 3 (note that \( M = H'_0(F_q) \)). Let \( \rho_\chi \) be the irreducible \( U^2_q(F_q) \)-representation constructed in Section 3 and recall that
\[
\text{Ind}^{U^2_q(F_q)}_{M}(\chi^\#) \cong \begin{cases}
\rho_\chi & \text{if } h \text{ is odd}, \\
q \cdot \rho_\chi & \text{if } h \text{ is even}.
\end{cases}
\]
Define
\[
G_k := \{1 + \sum a_i \tau^i \in H : a_{2i} \in \mathbb{F}_q \text{ for } 1 \leq i \leq k \} \subseteq H.
\]
We define \( G_0 := H \). Note that the center of \( U^2_q(F_q) \) is exactly \( G_{h-2} \). We thus have a tower of subgroups
\[
Z(U^2_q(F_q)) = G_{h-2} \subset G_{h-3} \subset \cdots \subset G_1 \subset G_0 = H.
\]
In this section, we will often write \( 1 + \sum a_i \tau^i = \sum a_i \tau^i \), where it is understood that \( \tau^0 = 1 \) and \( a_0 = 1 \).

The main results of this section are the following theorem and its corollaries. All proofs are in Section 4.1.

**Theorem 4.1.** Let \( \chi \) be a character of \( H \) whose restriction to \( H_{2(h-1)} \) has conductor \( q^2 \). Let \( \rho_\chi \) denote the irreducible \( U^2_q(F_q) \)-representation constructed in Section 3. Then as elements of the Grothendieck group of \( H \),
\[
\rho_\chi = (-1)^h \left( q \cdot \chi + \sum_{i=1}^{h-2} (-1)^i(q + 1) \text{Ind}^H_{G_i}(\chi) \right).
\]

Since \( H \) is abelian, Theorem 4.1 allows us to easily read off the decomposition of \( \rho_\chi \) as a representation of \( H \).

**Corollary 4.2.** Let \( \chi \) be as in Theorem 4.1. Let \( \mathcal{A}(\chi) \) be the collection of all characters \( \theta : H \to \mathbb{Q}_\ell^\times \) such that, for some even \( k \), \( \theta \) agrees with \( \chi \) on \( G_{h-2-k} \) but not on \( G_{h-2-k-1} \).

(a) If \( h \) is odd, then the restriction of \( \rho_\chi \) to \( H \) comprises
\[
\begin{cases}
1 \text{ copy of } \chi, \\
q + 1 \text{ copies of } \theta, \text{ for } \theta \in \mathcal{A}(\chi).
\end{cases}
\]

(b) If \( h \) is even, then the restriction of \( \rho_\chi \) to \( H \) comprises
\[
\begin{cases}
q \text{ copies of } \chi, \\
q + 1 \text{ copies of } \theta, \text{ for } \theta \in \mathcal{A}(\chi).
\end{cases}
\]

An immediate consequence of Corollary 4.2 is
Corollary 4.3. Let \( \rho \) be an irreducible representation of \( U^{2,q}_{h}(\mathbb{F}_{q^{2}}) \) wherein \( H_{2(h-1)}(\mathbb{F}_{q^{2}}) \) acts via a character \( \psi \) of conductor \( q^{2} \). Then \( \rho \) is uniquely determined by its restriction to \( H(\mathbb{F}_{q^{2}}) \).

4.1. Proof of Theorem 4.1 and Corollary 4.2. Corollary 4.2 follows easily from Theorem 4.1, which in turn is a corollary of the following proposition.

Proposition 4.4. Let \( s \in H \).
(a) If \( s \in G_{h-2} \), then
\[
\text{Tr} \rho_{\chi}(s) = q^{h-1} \chi(s).
\]
(b) If \( s \in G_{h-2-k} \setminus G_{h-2-k+1} \) for some \( 1 \leq k \leq h-2 \), then
\[
\text{Tr} \rho_{\chi}(s) = (-1)^{k} q^{h-1-k} \chi(s).
\]

The organization of this section is as follows. We begin by presenting a sequence of lemmas (Lemmas 4.7 to 4.16). The first of these (up to Lemma 4.14) have straightforward, computational proofs, and so we omit them. This sequence of lemmas will allow us to prove, in quick succession, Proposition 4.4, Theorem 4.1, and Corollary 4.2.

Remark 4.5. The representation \( \text{Ind}_{\mathbb{M}}^{U^{2,q}_{h}(\mathbb{F}_{q^{2}})}(\chi^{2}) \) is a sum of copies of \( \rho_{\chi} \); it consists of 1 copy when \( h \) is odd and \( q \) copies when \( h \) is even. Thus, to prove Proposition 4.4, it suffices to compute the sum
\[
\text{Tr} \text{Ind}_{\mathbb{M}}^{U^{2,q}_{h}(\mathbb{F}_{q^{2}})}(\chi^{2})(s) = \frac{1}{|\mathbb{M}|} \sum_{t \in U^{2,q}_{h}(\mathbb{F}_{q^{2}})} \chi_{s}^{t}(ts^{-1}) \quad \text{for } s \in H,
\]
where
\[
\chi_{s}^{t}(g) = \begin{cases} \chi^{2}(g) & \text{if } g \in \mathbb{M}, \\ 0 & \text{otherwise}. \end{cases}
\]

Since \( G_{h-2} \) is the center of \( U^{2,q}_{h}(\mathbb{F}_{q^{2}}) \), Proposition 4.4(a) is easy. Lemmas 4.7 through 4.16 build up to the proof of Proposition 4.4(b).

Remark 4.6. Note that
\[
s = 1 + \sum_{i=1}^{h-1} s_{2i} \tau^{2i} \in G_{h-2-k} \setminus G_{h-2-k+1}
\]
is equivalent to the conditions
\[
s_{2i} \in \mathbb{F}_{q} \quad \text{if } i \leq h - 2 - k \quad \text{and} \quad s_{2(h-2-k+1)} \notin \mathbb{F}_{q}.
\]

Lemma 4.7. Every element of \( U^{2,q}_{h}(\mathbb{F}_{q^{2}}) \) can be written in the form
\[
(1 - a_{1} \tau^{1})(1 - a_{3} \tau^{3}) \cdots (1 - a_{2(h-1)} \tau^{2(h-1)-1}) \cdot g \quad \text{for some } g \in H.
\]

Lemma 4.8. Let \( s = \sum s_{i} \tau^{i} \in U^{2,q}_{h}(\mathbb{F}_{q^{2}}) \) and let \( r \) be an odd integer with \( 1 \leq r \leq 2(h-1) \). Then if we let \( s' = \sum s'_{i} \tau^{i} = (1 - ar^{r})s(1 - ar^{r})^{-1} \) for some \( s \in \mathbb{F}_{q^{2}} \), we have
\[
s'_{n} = s_{n} + \sum_{r(l+1)+2m=n, l \geq 0} -a^{q^{l+q^{l-1}+\cdots+q+1}}(s_{2m} - s_{2m}) + \sum_{r(l+1)+2m+1=n, l \geq 0} -a^{q^{l-1+q^{l-2}+\cdots+q+1}}(as_{2m+1} - a^{q}s_{2m+1}).
\]
For the next few lemmas, we need an auxiliary definition.

**Definition 4.9.** If \( s \in 1 + \sum a_i \tau^i \in U_h^{2q}(\mathbb{F}_{q^2}) \) is such that

\[
\begin{cases}
  s_{2i} \in \mathbb{F}_q & \text{if } i \leq h - 2 - k, \\
  s_{2i} \notin \mathbb{F}_q & \text{if } i = h - 1 - k, \\
  s_i = 0 & \text{if } i \text{ is odd and } i \leq 2(h - 1) - k,
\end{cases}
\]

(Property \( \ast \))

then we will say that \( s \) satisfies Property \( \ast \) for \( k \).

**Remark 4.10.** It is implicit in the formulation of Property \( \ast \) that we must have \( k \leq h - 2 \). Thus the second condition (regarding which odd coefficients vanish) implies that \( s \in M \). Note further that if \( s \in H \) satisfies Property \( \ast \), then \( s \in G_{h-2-k} \backslash G_{h-1-k} \). \( \diamond \)

**Lemma 4.11.** Suppose that \( s \in U_h^{2q}(\mathbb{F}_{q^2}) \) satisfies Property \( \ast \) for \( k \). Then for any \( q \in H \), the element \( gsg^{-1} \) also satisfies Property \( \ast \) for \( k \). Furthermore, if we write \( gsg^{-1} = s' = \sum s'_i \tau^i \), then

\[
s'_{2i} = s_{2i} \quad \text{for all } i.
\]

**Lemma 4.12.** Suppose that \( s \in U_h^{2q}(\mathbb{F}_{q^2}) \) satisfies Property \( \ast \) for \( k \). Then for any odd \( r > k \) and \( a \in \mathbb{F}_{q^2} \), the element \((1 - a \tau^{r})s(1 - a \tau^{r})^{-1} \) also satisfies Property \( \ast \) for \( k \). Furthermore, if we write \((1 - a \tau^{r})s(1 - a \tau^{r})^{-1} = s' = \sum s'_i \tau^i \), then

\[
s'_{2i} = s_{2i} \quad \text{for all } i.
\]

**Lemma 4.13.** Suppose \( s = 1 + \sum s_i \tau^i \in U_h^{2q}(\mathbb{F}_{q^2}) \) satisfies Property \( \ast \) for \( k \). Then for any \( t \in N_k \), the element \( tst^{-1} \) also satisfies Property \( \ast \) for \( k \). Furthermore, if \( tst^{-1} = s' = \sum s'_i \tau^i \), then

\[
s'_{2i} = s_{2i} \quad \text{for all } i.
\]

**Lemma 4.14.** Let \( s \in G_{h-2-k} \backslash G_{h-2-k+1} \). Then

\[
\chi^\circ(s) = \chi^\circ(tst^{-1}) = \chi(s) \quad \text{for any } t \in N_k.
\]

**Lemma 4.15.** Suppose \( s \) satisfies Property \( \ast \) for \( k \) where \( k \) is odd. Then

\[
\sum_{a \in \mathbb{F}_{q^2}^\times} \chi^\circ((1 - a \tau^k)s(1 - a \tau^k)^{-1}) = -(q + 1)\chi^\circ(s).
\]

**Proof.** In a manner similar to Lemma 4.12, it is a straightforward computation to show

\[
\chi^\circ((1 - a \tau^k)s(1 - a \tau^k)^{-1}) = \chi^\circ(s \cdot (1 - \alpha^{r+1}(s_{2(h-1)-k}^q - s_{2(h-1)-k})^{r^{h-1}}))
\]

\[
= \chi^\circ(s) \cdot \psi(-\alpha^{r+1}(s_{2(h-1)-k}^q - s_{2(h-1)-k})).
\]

Note that since \( s \) satisfies Property \( \ast \) for \( k \) and \( a \neq 0 \), then \(-\alpha^{r+1}(s_{2(h-1)-k}^q - s_{2(h-1)-k}) \neq 0 \). Furthermore, this element is in ker(Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}). Since \( \psi \) has conductor \( q^2 \), we know that its restriction to ker(Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}) is nontrivial. Notice that ranging \( a \in \mathbb{F}_{q^2}^\times \) allows \(-\alpha^{r+1}(s_{2(h-1)-k}^q - s_{2(h-1)-k}) \) to take each nonzero value of ker(Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}) exactly \( q + 1 \) times. Therefore, for any \( s \) satisfying Property \( \ast \) for \( k \),

\[
\sum_{a \in \mathbb{F}_{q^2}^\times} \chi^\circ((1 - a \tau^k)s(1 - a \tau^k)^{-1}) = \chi^\circ(s) \sum_{a \in \mathbb{F}_{q^2}^\times} \psi(-\alpha^{r+1}(s_{2(h-1)-k}^q - s_{2(h-1)-k}))
\]

\[
= -(q + 1) \cdot \chi^\circ(s).
\]

□
Lemma 4.16. Let \( s \in G_{h-2-k} \setminus G_{h-2-k+1} \). Let \( n_1, \ldots, n_r \) be a decreasing sequence of consecutive odd numbers starting from \( n_1 = 2(h-1) - 1 \) and assume that \( n_r < k \). Then
\[
\sum_{a_1, \ldots, a_r \in \mathbb{F}_q^2, a_r \neq 0} \chi_\sigma((1 - a_r \tau^{n_r}) \cdots (1 - a_1 \tau^{n_1}) s(1 - a_1 \tau^{n_1})^{-1} \cdots (1 - a_r \tau^{n_r})^{-1}) = 0.
\]

Proof. Let \( g = (1 - a_r \tau^{n_r}) \cdots (1 - a_1 \tau^{n_1}) \). If \( n_r + 2(h-1-k) \leq h-1 \), then we see from Lemma 4.8 that the coefficient of \( \tau^{n_r+2(h-1-k)} \) in \( gs^{-1} \) is \(-a_r(s_{2(h-1-k)}^q - s_{2(h-1-k)}) \neq 0 \). Thus \( gs^{-1} \notin M \) and \( \chi_\sigma^i(gs^{-1}) = 0 \). We may therefore assume that \( n_r > 2k - (h-1) \).

Let \( f \) be such that \( n_r + n_f = 2k \) (so that \( n_r + n_f + 2(h-1-k) = 2(h-1) \)). Note that such an \( f \) exists by the assumption \( n_r < k \). To prove the lemma, we will prove the following sum identity. Fix \( a_i \in \mathbb{F}_q^2 \) for \( 1 \leq i \leq r \), and assume that \( a_r \neq 0 \). Then
\[
\sum_{a_f \in \mathbb{F}_q^2} \chi_\sigma((1 - a_r \tau^{n_r}) \cdots (1 - a_1 \tau^{n_1}) s(1 - a_1 \tau^{n_1})^{-1} \cdots (1 - a_r \tau^{n_r})^{-1}) = 0.
\]

It is clear that once this is established, the lemma follows immediately. We thus focus the rest of the proof on proving Equation (2).

Write \( g = (1 - a_r \tau^{n_r}) \cdots (1 - a_1 \tau^{n_1}) \). We must study the contribution of \( a_f \) in \( gs^{-1} \). By construction \( n_f \) is odd and \( s \in H \). Thus \( a_f \) can only contribute to the coefficient of \( \tau^{2i} \) in conjunction with (at least) one of the other \( a_i \)'s for \( 1 \leq i \leq r \).

For convenience, let \( gs^{-1} = 1 + \sum c_i \tau^i \). First observe that since \( s \in G_{h-2-k} \), we have \( s_{2i}^q - s_{2i} = 0 \) for \( 1 \leq i \leq h - 2 - k \), and thus the smallest odd \( i \) such that \( a_f \) has a nonzero contribution to \( c_i \) is when \( i = 2(h-1-k) + n_f > 2(h-1-k) + 2k - (h-1) = h - 1 \). Thus we see that \( gs^{-1} \in M \) for \( a_f = 0 \) if and only if \( gs^{-1} \in M \) for any \( a_f \in \mathbb{F}_q^2 \) (remember that \( g \) depends on \( a_f \)).

If \( gs^{-1} \notin M \), then we are done. Now assume \( gs^{-1} \in M \). By construction, the value of \( \chi_\sigma^i(gs^{-1}) \) depends only on the coefficients \( c_{2i} \) for \( 1 \leq i \leq h - 1 \). If \( a_f \) contributes to some \( c_{2i} \), then we must have
\[
2i \geq n_f + n_r + 2(h - 1 - k) = 2(h-1).
\]

Thus \( a_f \) only contributes to \( c_{2(h-1)} \). Furthermore, its contribution to \( c_{2(h-1)} \) is
\[
a_r a_f^q s_{2(h-1-k)}^q - a_r s_{2(h-1-k)} a_f^q - a_f s_{2(h-1-k)}^q a_f^q + s_{2(h-1-k)} a_f^q a_f^q
\]
\[
= -(a_r a_f^q + a_f^q a_r) (s_{2(h-1-k)}^q - s_{2(h-1-k)})^q.
\]

(One can see this by computing \( (1 - a_r \tau^{n_r})(1 - a_f \tau^{n_f})s(1 - a_f \tau^{n_f})^{-1}(1 - a_r \tau^{n_r})^{-1} \). Thus
\[
\chi_\sigma(g) = \chi_\sigma(g^{-1}) = \chi_\sigma(\gamma) \cdot \psi(-(a_r a_f^q + a_f^q a_r) (s_{2(h-1-k)}^q - s_{2(h-1-k)})),
\]
where \( \gamma \) does not depend on the choice of \( a_f \). Therefore
\[
\sum_{a_f \in \mathbb{F}_q^2} \chi_\sigma(g^{-1}) = \chi_\sigma(\gamma) \cdot \sum_{a_f \in \mathbb{F}_q^2} \psi(-(a_r a_f^q + a_f^q a_r) (s_{2(h-1-k)}^q - s_{2(h-1-k)})).
\]

Note that for any \( c \in \mathbb{F}_q \), any solution \( x \) to \( a_r x^q + a_f^q x = c \) must satisfy \( x^q = x \). Thus varying \( a_f \in \mathbb{F}_q^2 \), the quantity \( a_r a_f^q + a_f^q a_r \) takes the value of each element \( \mathbb{F}_q \) exactly \( q \) times. Since \( s_{2(h-1-k)}^q \neq \mathbb{F}_q \), \( - (a_r a_f^q + a_f^q a_r) (s_{2(h-1-k)}^q - s_{2(h-1-k)}) \) attains each value of \( \ker \text{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q} \).
We will use this at various points in this proof. Thus by Remark 4.5, we have
\[\sum_{a_f \in \mathbb{F}_{q^2}} \psi(-(a_r a_f^2 + a_f^2)(s_2^q - s_2(h-1-k))) = 0.\]

Equation (2) follows. \(\square\)

We are now ready to prove Proposition 4.4, Theorem 4.1, and Corollary 4.2.

**Proof of Proposition 4.4.** It is easy to see that
\[
|N_k| = \begin{cases} 
q^{4(h-1-k)} & \text{if } k \text{ is even,} \\
q^{4(h-1)-(k+1)} & \text{if } k \text{ is odd.}
\end{cases}
\]

We will use this at various points in this proof.

If \(s \in G_{h-2}\), then \(s\) is central in \(U_{h}^{2,q}(F_{q^2})\). Thus for any \(t \in U_{h}^{2,q}(F_{q^2})\), \(tst^{-1} = s\) and
\[
\frac{1}{|M|} \sum_{t \in U_{h}^{2,q}(F_{q^2})} \chi_{\sigma}^{2}(tst^{-1}) = \frac{|U_{h}^{2,q}(F_{q^2})|}{|M|} \chi(s) = \begin{cases} 
q^{h-1} \cdot \chi(s) & \text{if } h \text{ is odd} \\
q^{h} \cdot \chi(s) & \text{if } h \text{ is even.}
\end{cases}
\]

Thus by Remark 4.5, we have
\[
\Tr \rho_\chi(s) = q^{h-1} \cdot \chi(s).
\]

This proves (a).

Let \(s \in G_{h-2-k} \setminus G_{h-2-k+1}\). We first handle the case when \(k\) is even. We have
\[
\sum_{t \in U_{h}^{2,q}(F_{q^2})} \chi_{\sigma}^{2}(tst^{-1}) = \sum_{t \in N_k} \chi_{\sigma}^{2}(tst^{-1}) + \sum_{t \notin N_k} \chi_{\sigma}^{2}(tst^{-1}).
\]

By Lemma 4.14, we know
\[
(1) = |N_k| \cdot \chi(s).
\]

By Lemma 4.7, we know that every element \(t \in U_{h}^{2,q}(F_{q^2})\) can be written in the form
\[(1 - a_1 \tau^1)(1 - a_3 \tau^3) \cdots (1 - a_{2(h-1)-1} \tau^{2(h-1)-1}) \cdot g\]
for some \(g \in H\). Since \(H\) is abelian, this implies that \(gsg^{-1} = s\), and the assumption \(t \notin N_k\) implies that there exists \(i\) odd with \(i < k\) such that \(a_i \neq 0\). Thus
\[
(2) = |H| \cdot \sum_{a \in \mathbb{F}_{q^2}, a_i \neq 0} \chi_{\sigma}^{2}(asa^{-1}) = 0,
\]
where \(a = (1 - a_1 \tau^1)(1 - a_3 \tau^3) \cdots (1 - a_{2(h-1)-1} \tau^{2(h-1)-1})\), and the last equality holds by Lemma 4.16.

Therefore,
\[
\frac{1}{|M|} \sum_{t \in U_{h}^{2,q}(F_{q^2})} \chi_{\sigma}^{2}(tst^{-1}) = \frac{|N_k|}{|M|} \cdot \chi(s) = \begin{cases} 
q^{h-1-k} \cdot \chi(s) & h \text{ odd,} \\
q^{h-k} \cdot \chi(s) & h \text{ even.}
\end{cases}
\]

Recalling Remark 4.5, this finishes the proof of the proposition in the case \(k\) is even.
Now let $k$ be odd. By Lemma 4.7, we have
\[
\sum_{t \in U_{k} \setminus (F^{2})} \chi_{0}^{\sharp}(tst^{-1}) = \sum_{t \in N_{k}} \chi_{0}^{\sharp}(tst^{-1}) + \sum_{\substack{t \in N_{k} \\ a \in F_{q}^{x}}} \chi_{0}^{\sharp}((1 - a\tau^{k})tst^{-1}(1 - a\tau^{k})^{-1}) \\
+ \sum_{t \in N_{k} \setminus N_{k} \setminus (F^{2})} \chi_{0}^{\sharp}(tst^{-1}).
\]

By Lemma 4.14, we know
\[
(1) = |N_{k}| \cdot \chi(s).
\]

By Lemma 4.13, we know that given $s \in G_{h-2-k} \setminus G_{h-2-k+1}$ and $t \in N_{k}$, we have that $tst^{-1}$ satisfies Property $\star$ for $k$ and that $\chi_{0}^{\sharp}(tst^{-1}) = \chi(s)$. Thus by Lemma 4.15, we have
\[
\sum_{a \in F_{q}^{x}} \chi_{0}^{\sharp}((1 - a\tau^{k})tst^{-1}(1 - a\tau^{k})^{-1}) = -(q + 1)\chi_{0}^{\sharp}(tst^{-1}) = -(q + 1)\chi(s).
\]

Therefore
\[
(2) = -|N_{k}|(q + 1) \cdot \chi(s).
\]

By the same argument as the case when $k$ is even, it follows from Lemma 4.7 and Lemma 4.16 that
\[
(3) = 0.
\]

Therefore,
\[
\frac{1}{|M|} \sum_{t \in U_{k} \setminus (F^{2})} \chi_{0}^{\sharp}(tst^{-1}) = \frac{1}{|M|} \left( |N_{k}| \cdot \chi(s) - |N_{k}|(q + 1) \cdot \chi(s) \right)
= \begin{cases} 
-q^{h-1-k} \cdot \chi(s) & \text{if } h \text{ odd}, \\
-q^{h-k} \cdot \chi(s) & \text{if } h \text{ even}.
\end{cases}
\]

By Remark 4.5, this finishes the proof of the proposition when $k$ is odd. \qed

**Proof of Theorem 4.1.** Consider the (virtual) $H$-representation
\[
\rho = (-1)^{h} \left( q \cdot \chi + \sum_{i=1}^{h-2} (-1)^{i}(q + 1) \text{Ind}_{G_{i}}^{H}(\chi) \right).
\]

Since $H$ is abelian, its trace is very easy to calculate: using $|H|/|G_{i}| = q^{i}$, for any $s \in H$,
\[
\text{Tr} \rho(s) = (-1)^{h} \left( q \cdot \chi(s) + \sum_{i=1}^{h-2} (-1)^{i}(q + 1) \text{Tr} \text{Ind}_{G_{i}}^{H}(\chi)(s) \right)
= (-1)^{h} \left( q \cdot \chi(s) + \sum_{i=1}^{h-2} (-1)^{i}(q + 1)q^{i} \cdot \mathbb{1}_{G_{i}}(s) \cdot \chi(s) \right).
\]

Therefore:
(a) If $s \in G_{h-2}$, then
\[ \text{Tr} \rho(s) = (-1)^h \cdot \chi(s) \cdot \left( q + \sum_{i=1}^{h-2} (-1)^i(q+1)q^i \right) = q^{h-1} \cdot \chi(s). \]

(b) If $s \in G_{h-2-k} \setminus G_{h-2-k+1}$, then
\[ \text{Tr} \rho(s) = (-1)^h \cdot \chi(s) \cdot \left( q + \sum_{i=1}^{h-2-k} (-1)^i(q+1)q^i \right) = (-1)^k q^{h-1-k} \cdot \chi(s). \]

Comparing this with Proposition 4.4, we see that $\rho_X = \rho$ as elements of the Grothendieck group of $H$.

**Proof of Corollary 4.2.** Given a character $\theta : H(\mathbb{F}_{q^2}) \to \overline{\mathbb{Q}}_\ell^\times$, we can read off its multiplicity from the result of Theorem 4.1. Indeed, since $H$ is abelian, then if $\theta$ is an $H$-character that agrees with $\chi$ on some subgroup $G_m$ but not on $G_{m-1}$, then it occurs exactly once in $\text{Ind}_{G_{m-1}}^G(\chi)$ for every $i \geq m$ and does not occur in $\text{Ind}_{G_{m-2}}^G(\chi)$ for $i \leq m-1$. Therefore:

(a) The character $\chi$ occurs in $\rho_X$ with multiplicity equal to
\[ (-1)^h \left( q - (q+1) + (q+1) - \cdots + (-1)^{h-2}(q+1) \right) = \begin{cases} 1 & \text{if } h \text{ is odd,} \\ q & \text{if } h \text{ is even.} \end{cases} \]

(b) Let $\theta$ be a character of $H$ such that, for some odd $k$, $\theta$ agrees with $\chi$ on $G_{h-2-k}$ but not on $G_{h-2-k-1}$. Then $\theta$ occurs in $\rho_X$ with multiplicity equal to
\[ (-1)^h \left( (-1)^{h-2-k}(q+1) + \cdots + (-1)^{h-2}(q+1) \right) = 0. \]

(c) Let $\theta$ be a character of $H$ such that, for some even $k$, $\theta$ agrees with $\chi$ on $G_{h-2-k}$ but not on $G_{h-2-k-1}$. Then $\theta$ occurs in $\rho_X$ with multiplicity equal to
\[ (-1)^h \left( (-1)^{h-2-k}(q+1) + \cdots + (-1)^{h-2}(q+1) \right) = q + 1. \]

(d) Let $\theta$ be a character of $H$ that does not agree with $\chi$ on $G_{h-2}$. Since $G_{h-2}$ is in the center of $U_h^{2,q}(\mathbb{F}_{q^2})$, then the restriction of $\rho_X$ to $G_{h-2}$ must be a sum of $\chi|_{G_{h-2}}$. Therefore the multiplicity of $\theta$ in $\rho_X$ must be 0. \hfill \square

### 5. Morphisms between $H_\mathfrak{a}(X_h)$ and representations of $U_h^{2,q}(\mathbb{F}_{q^2})$

Let $H_h^\bullet(X_h) = \bigoplus_{\mathfrak{a} \in \mathbb{Z}} H_h^\mathfrak{a}(X_h, \overline{\mathbb{Q}}_\ell)$. The aim of this section is to compute the space of homomorphisms $\text{Hom}_{U_h^{2,q}(\mathbb{F}_{q^2})}(V_\psi, H_h^\bullet(X_h))$. Recall that
\[ V_\psi = \text{Ind}_{H_0(\mathbb{F}_{q^2})}^{U_h^{2,q}(\mathbb{F}_{q^2})}(\overline{\psi}), \]

where $\overline{\psi}$ is the extension of $\psi$ to $H_0(\mathbb{F}_{q^2})$ defined in Section 3.

The following theorem is a straightforward calculation, but is a crucial result. We describe a clean way to express the equations cutting out the scheme $X_h \subseteq U_h^{2,q}$. This will be heavily used in this section, as well as in the next section.

**Theorem 5.1.** The scheme $X_h \subseteq U_h^{2,q}$ is defined by the vanishing of the polynomials
\[ f_{2k} := (a_{2k}^q - a_{2k}) + \sum_{i=1}^{2k-1} (-1)^i a_{2k}^q (a_{2k-i}^q - a_{2k-i}) \]
for \( 1 \leq k \leq h - 1 \).

The following theorem is the main result of this section.

**Theorem 5.2.** Let \( \psi \) be an additive character of \( \mathbb{F}_{q^2} \) with conductor \( q^2 \). If \( h \) is odd, then

\[
\dim \text{Hom}_{U^2_h(\mathbb{F}_{q^2})}(V_\psi, H^i_c(X_h, \mathbb{Q}_\ell)) = \begin{cases} 
q^{2(h-2)} & \text{if } i = h - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

If \( h \) is even, then

\[
\dim \text{Hom}_{U^2_h(\mathbb{F}_{q^2})}(V_\psi, H^i_c(X_h, \mathbb{Q}_\ell)) = \begin{cases} 
q^{2(h-2)+1} & \text{if } i = h - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, the Frobenius \( \text{Fr}_{q^2} \) acts on \( \text{Hom}_{U^2_h(\mathbb{F}_{q^2})}(V_\psi, H^{h-1}_c(X_h, \mathbb{Q}_\ell)) \) via multiplication by the scalar \((-1)^{h-1}q^{h-1}\).

This is proven in Section 5.1. As a corollary to Theorem 5.2, we have the following.

**Corollary 5.3.** Let \( \psi \) be an additive character of \( \mathbb{F}_{q^2} \) with conductor \( q^2 \). If \( \chi: U^1_h/U^h_L \rightarrow \mathbb{Q}_\ell^* \) is a character that restricts to \( \psi \) on \( U^h_L/U^h_L \), then \( H^i_c(X_h, \mathbb{Q}_\ell)[\chi] = 0 \) for all \( i \neq h - 1 \).

**Proof.** The left action of \( U^1_h/U^h_L \) and the right action of \( U^2_h(\mathbb{F}_{q^2}) \) on \( X_h \) agree on \( U^h_L/U^h_L \cong H^{2(h-1)}(\mathbb{F}_{q^2}) \). Therefore, since \( U^h_L/U^h_L \) acts by \( \psi \) on \( H^c_c(X_h, \mathbb{Q}_\ell)[\chi] \), then \( H^{2(h-1)}(\mathbb{F}_{q^2}) \) also acts by \( \psi \). We know from our analysis of the representations of \( U^2_h(\mathbb{F}_{q^2}) \) that every irreducible component of \( H^i_c(X_h, \mathbb{Q}_\ell)[\chi] \) appears in \( V_\psi \), so this forces \( H^i_c(X_h, \mathbb{Q}_\ell)[\chi] = 0 \) if \( i \neq h - 1 \). \( \square \)

This will allow us to compute intertwining spaces using Lemma 2.13 of [B12]. This will be exploited in Section 6.

### 5.1. Proof of Theorem 5.2.

The structure of the proof is as follows. We first use Proposition 2.3 of [B12] to reduce the computation of \( \text{Hom}_{U^2_h(\mathbb{F}_{q^2})}(V_\psi, H^i_c(X_h, \mathbb{Q}_\ell)) \) to the computation of the cohomology of a certain scheme \( S \) with coefficients in a certain constructible \( \mathbb{Q}_\ell \) sheaf \( \mathcal{F} \). Then, to compute \( H^i_c(S, \mathcal{F}) \), we apply (a slightly more general version of) Proposition 2.10 of [B12] inductively. This will allow us to reduce the computation of \( H^i_c(X, \mathcal{F}) \) to a computation involving a 0-dimensional scheme in the case that \( h \) is odd, and a computation involving a 1-dimensional scheme in the case that \( h \) is even. Because the computation is identical until this final step, we treat these to cases simultaneously until the very last step.

We start with a slight generalization of Proposition 2.10 of [B12] that has been tailored for our purposes.

**Proposition 5.4.** Let \( q \) be a power of \( p \), let \( n \in \mathbb{N} \), and let \( \psi: \mathbb{F}_{q^n} \rightarrow \mathbb{Q}_\ell^* \) be a character that has conductor \( q^m \). Let \( S_2 \) be a scheme of finite type over \( \mathbb{F}_{q^n} \), put \( S = S_2 \times \mathbb{A}^1 \) and suppose that a morphism \( P: S \rightarrow \mathbb{G}_a \) has the form

\[
P(x, y) = f(x)^q y - f(x)^q y^{q^n} + \alpha(x, y)q^m - \alpha(x, y) + P_2(x).
\]

Here, \( j \) is some integer \( j \) not divisible by \( m \); \( f, P_2: S_2 \rightarrow \mathbb{G}_a \) are two morphisms; and \( \alpha: S_2 \times \mathbb{A}^1 \rightarrow \mathbb{G}_a \) is a morphism. Let \( S_3 \subset S_2 \) be the subscheme defined by \( f = 0 \) and let \( P_3 = P_2|_{S_3}: S_3 \rightarrow \mathbb{G}_a \). Then for all \( i \in \mathbb{Z} \), we have

\[
H^i_c(S, P^* \mathcal{L}_\psi) \cong H^{i-2}_c(S_3, P_3^* \mathcal{L}_\psi)(-1)
\]
as vector spaces equipped with an action of $\Fr_{q^n}$, where the Tate twist $(-1)$ means that the action of $\Fr_{q^n}$ on $H^i_{\text{et}}(S_2, P_3^* \mathcal{L}_\psi)$ is multiplied by $q^n$.

Proof of Proposition 5.4. Let $P'(x, y) = f(x)q^y - f(x)q^{y - 1} + P_2(x)$ and $P''(x, y) = \alpha(x, y)q^m - \alpha(x, y)$. We show that the pullbacks $P^* \mathcal{L}_\psi$ and $(P')^* \mathcal{L}_\psi$ are isomorphic. Since $\psi$ has conductor $q^m$, the pullback of $\mathcal{L}_\psi$ by the map $z \mapsto z^q$ is trivial, and so thus $(P'')^* \mathcal{L}_\psi$ is trivial. Since $\mathcal{L}_\psi$ is additive, then we have shown that $P^* \mathcal{L}_\psi$ and $(P')^* \mathcal{L}_\psi$ are isomorphic and thus by Proposition 2.10 of [B12],

$$H^i_{\text{et}}(S, P^* \mathcal{L}_\psi) \cong H^i_{\text{et}}(S, (P')^* \mathcal{L}_\psi) \cong H^i_{\text{et}}(S_3, P_3^* \mathcal{L}_\psi)(-1)$$

as vector spaces equipped with an action of $\Fr_{q^{2^n}}$. \hfill \Box

We now return to the proof of Theorem 5.2.

Step 0. We first need to establish some notation. I have tried to make this notation reminiscent of that used in the proof of Proposition 6.5 in [BW14].

- Let $I'$ denote the set of integers $j$ such that $h - 1 < j < 2(h - 1)$ and $2 \nmid j$. Put $I = I' \cup \{2(h - 1)\}$.
- Put $J = [2(h - 1)] \setminus I$, where $[n] = \{1, \ldots, n\}$.
- Put $I_0 := I'$ and $J_0 := J$. Then define $I_1 := I_0 \setminus \{2(h - 1) - 1\}$ and $J_1 := J_0 \setminus \{1\}$. This describes a recursive construction of $I_k$ and $J_k$; namely, one obtains $I_k$ from $I_{k-1}$ by removing the largest odd number and one obtains $J_k$ from $J_{k-1}$ by removing the smallest odd number. This defines indexing sets $I_k$ and $J_k$ for $1 \leq k \leq \lfloor (h - 1)/2 \rfloor$.
- Note that if $h$ is odd, then

\[
I_{\lfloor (h - 1)/2 \rfloor} = I_{\lfloor (h - 1)/2 \rfloor} = \emptyset,
\]

\[
J_{\lfloor (h - 1)/2 \rfloor} = J_{\lfloor (h - 1)/2 \rfloor} = \{2, 4, \ldots, 2(h - 2)\}.
\]

If $h$ is even, then

\[
I_{\lfloor (h - 1)/2 \rfloor} = I_{\lfloor (h - 1)/2 \rfloor} = \emptyset,
\]

\[
J_{\lfloor (h - 1)/2 \rfloor} = J_{\lfloor (h - 1)/2 \rfloor} = \{2, 4, \ldots, 2(h - 2)\} \cup \{h - 1\}.
\]

This distinction is exactly why our inductive argument reduces to a 0-dimensional scheme in the case that $h$ is odd and a 1-dimensional scheme in the case that $h$ is even.

- Note that $H_0' = \{1 + \sum a_i \tau^i : i \in I\}$.
- For a finite set $T \subset \mathbb{N}$, we will write $\mathbb{A}[T]$ to denote affine space with coordinates $x_i$ for $i \in T$.

Step 1. We apply Proposition 2.3 of [B12] to the following set-up:

- $G = U^2,q$ and $H = H_0'$, both defined over $\mathbb{F}_{q^2}$
- the morphism $s: U^2,q/H_0' \to U^2,q$ defined by sending the $i$th coordinate to the coefficient of $\tau^i$; that is, identify $U^2,q/H_0'$ with affine space with coordinates indexed by $J$, and set $s: (x_i)_{i \in J} \mapsto 1 + \sum_{i \in J} x_i \tau^i$.
- the algebraic group homomorphism $f: H_0' \to \mathbb{G}_a$ given by projection to the last coordinate. That is, $f: 1 + \sum_{i \in J} a_i \tau^i \mapsto a_{2(h-1)}$. (From the definition of $H_0'$, it is easy to see that this map is a homomorphism.)
- an additive character $\psi: \mathbb{F}_{q^2} \to \overline{\mathbb{Q}}_l^\times$
- a locally closed subvariety $Y_h \subset U^2,q$ which is chosen so that $X_h = L_{q^2}^{-1}(Y_h)$
Since $X_h$ has a right-multiplication action by $U^2_{h,q}(\mathbb{F}_q)$, the cohomology groups $H^i(X_h,\mathbb{Q}_l)$ inherit a $U^2_{h,q}(\mathbb{F}_q)$-action. For each $i \geq 0$, Proposition 2.3 of [B12] implies that we have a vector space isomorphism

$$\text{Hom}_{U^2_{h,q}(\mathbb{F}_q)}(V_\psi, H^i(X_h,\mathbb{Q}_l)) \cong H^i_\beta(\psi, P^*L_\psi)$$

compatible with the action of Fr$_q$. Here, $L_\psi$ is the Artin-Schreier local system on $\mathbb{G}_a$ corresponding to $\psi$, the morphism $\beta: (U^2_{h,q}/H_0) \times H_0 \to U^2_{h,q}$ is given by $\beta(x,g) = s(\text{Fr}_q(x)) \cdot g \cdot s(x)^{-1}$, and the morphism $P: \beta^{-1}(Y_h) \to \mathbb{G}_a$ is the composition $\beta^{-1}(Y_h) \hookrightarrow (U^2_{h,q}/H_0) \times H_0 \xrightarrow{\text{pr}} H_0 \xrightarrow{l} \mathbb{G}_a$.

We now work out an explicit description of $\beta^{-1}(Y_h) \subset \mathbb{A}[J] \times H_0$ (keep in mind that we identified $U^2_{h,q}/H_0$ with $\mathbb{A}[J]$). For $1 \leq l \leq (h - 1)$, recall the polynomial described in Theorem 5.1

$$f_2l := (a_{2l}^2 - a_{2l}) + \sum_{i=1}^{2l-1} (-1)^i a_{2l-i}^2(a_{2l-1}^2 - a_{2l-1}).$$

Write $x = (x_i)_{i \in J} \in \mathbb{A}[J]$ and $g = 1 + \sum_{i \in I} x_i \tau_i \in H_0^1(\mathbb{F}_q)$. (Note that $I \cap J = \emptyset$; the $x_i$ in $x$ and $x_i$ in $g$ are independent of each other.) For each $i \in I$, we can write $x_i = y_i^2 - y_i$ for $y_i \in \mathbb{F}_q$, so that $g = L_{y_i}(y)$, where $y := 1 + \sum_{i \in J} y_i \tau_i$. Therefore

$$\beta(x,g) = \text{Fr}_q(s(x)) \cdot L_{y_i}(y) \cdot s(x)^{-1} = L_{y_i}(s(x) \cdot y).$$

We see that $\beta(x,g) \in Y_h$ if and only if $s(x) \cdot y \in X_h$. Let $s(x) \cdot y = 1 + \sum a_i \tau_i = a$. By Theorem 5.1, we know that $s(x) \cdot y \in X_h$ if and only if $f_2l(a) = 0$ for all $l$ with $1 \leq l \leq h - 1$.

**Step 2.** This is a necessary preparation step before we apply Proposition 5.4. As in Step 1, let $x = (x_i)_{i \in J} \in \mathbb{A}[J]$ and $s(x) = 1 + \sum_{i \in J} x_i \tau_i \in U^2_{h,q}(\mathbb{F}_q)$. Let $g = 1 + \sum_{i \in I} x_i \tau_i \in H_0^1(\mathbb{F}_q)$ and let $y_i$ be such that $x_i = y_i^2 - y_i$ for $i \in I$ so that $g = L_{y_i}(y)$, where $y = 1 + \sum_{i \in I} y_i \tau_i$. Recall that we wrote $s(x) \cdot y = 1 + \sum a_i \tau_i = a$.

From direct computation, we can write down an explicit description of each coefficient $a_i$ in terms of $x$ and $y$. For convenience, let $r = 2[\lfloor h/2 \rfloor]$. Then

$$a_i = \begin{cases} 
  x_i & \text{if } i \leq r, \\
  y_i + x_2 y_{i-2} + x_4 y_{i-4} + \cdots + x_{i-(r+1)} y_{r+1}^{-1} y_{i-1}^{(r+1)} & \text{if } r < i \text{ and } i \text{ is odd}, \\
  x_1 y_i^{-1} + x_3 y_{i-3}^{-1} + \cdots + x_{i-(r+1)} y_{r+1}^{-1}^{-1} + x_i & \text{if } r < i < 2(h - 1) \text{ and } i \text{ is even}, \\
  y_{2(h-1)} + x_1 y_{2(h-1)-1} + x_3 y_{2(h-1)-3} + \cdots + x_{2(h-1)-(r+1)} y_{r+1}^{-1} + x_{2(h-1)-(r+1)} y_{r+1}^{-1} & \text{if } i = 2(h - 1).
\end{cases}$$

Fix $1 \leq l \leq h - 1$. The polynomial $f_2l(a)$ is a priori a polynomial in $x_i$ for $i \in I$ and $y_i$ for $i \in I$. In this step, we show that, after setting $x_i = y_i^2 - y_i$ for $i \in I$, the expression $f_2l(a)$ is actually a polynomial in $x_i$ for $i \in I \cup J$.

First observe that the monomials occurring in $f_2l(s(x) \cdot y)$ can involve $y_i$ for at most one $i$. More precisely, a monomial occurring $f_2l(a)$ takes one of the following forms:

(i) It is a product of powers of $x_i$’s.

(ii) It involves $y_{2(h-1)}$.

(iii) It is of the form $x_i^j x_j^\beta y_i^\gamma$, where $i \geq 0$ is even, $j \leq 2[\lfloor h/2 \rfloor]$ is odd, and $k \geq 2[\lfloor h/2 \rfloor] + 1$ is odd.

(As usual, we set $x_0 = 1$.)
We need to show that setting \( x_i = y_i^2 - y \) for \( i \in I \) allows us to write the monomials in (ii) and (iii) as expressions involving only \( x_i \)'s for \( i \in I \cup J \).

The term \( y_{2(h-1)} \) only occurs in the polynomial \( f_{2l}(a) \) for \( l = h - 1 \). Its contribution to \( f_{2(h-1)}(a) \) is

\[
y_{2(h-1)}^2 - y_{2(h-1)} = x_{2(h-1)}^2,
\]

so this takes care of (ii).

Now pick \( i, j, k \) with \( i + j + k = 2l \) so that \( i \geq 0 \) is even, \( j \leq 2\lfloor h/2 \rfloor \) is odd, and \( k > 2\lfloor h/2 \rfloor \) is odd. Then \( y_k, x_i \), and \( x_j \) occur in \( f_{2l}(a) \) in the terms

\[
a_i^q(a_j^{q^2} - a_{j+k}) + a_j^q(a_i^{q^2} - a_i) - a_i^q(a_{i+k}^{q^2} - a_{i+k}) - a_{i+k}^q(a_j^{q^2} - a_j),
\]

and are exactly

\[
x_i^q((x_jy_k^q)^q - x_jy_k^q) + (x_jy_k^q)^q(x_i^q - x_i) - x_j^q((x_iy_k^q)^q - x_iy_k^q) - (x_iy_k^q)^q(x_j^q - x_j).
\]

Note that monomials of the form \( x_i^q x_j^q y_k^q \) do not occur in \( a_i^q(a_{i+j} - a_{i+j}) \) or \( a_i^q(a_i^q - a_k) \) (see Equation (7)). The above simplifies to

\[
x_i^q x_j^q(y_k^{q+2} - y_k^{q+1}) - x_i x_j^q(y_k^{q+1} - y_k^q) - x_j^q x_j(y_k^q - y_k^{q+1}) + x_j^q x_j^q(y_k^{q+1} - y_k^{q+2}).
\]

By assumption, \( i \) is even and \( j \) is odd, which means that each expression involving \( y_k \)'s is of the form \( y_k^{m+2n} - y_k^{m} \) for some \( n \). Since \( y_k^2 - y_k = x_k \), we then have

\[
y_k^{m+2n} - y_k^m = (y_k^{2n} - y_k)^m = (x_k^{2n-2} + x_k^{2n-4} + \cdots + x_k^2 + x_k)^m.
\]

This takes care of (iii) and thus we have shown that for any \( 1 \leq l \leq h - 1 \), \( f_{2l}(a) \) is a polynomial in terms of \( x_i \) for \( i \in I \cup J \). We will write \( F_{2l} \) to mean the polynomial \( f_{2l}(a) \) viewed as a polynomial in \( x_i \) for \( i \in I \cup J \).

**Step 3.** Let \( P^{(0)} = x_{2(h-1)} - F_{2(h-1)} \). By Step 2, \( P^{(0)} \) is a polynomial in terms of \( x_i \) for \( i \in I_0 \cup J = I_0 \cup J_0 \). Recall from Step 1 that \( \beta(x, y) \in Y_h \) if and only if \( s(x) \cdot y \in X_h \). If \( s(x) \cdot y \in X_h \), then we must have \( F_{2(h-1)} = F_{2(h-1)}(s(x) \cdot y) = 0 \), so \( P^{(0)} = x_{2(h-1)} \). Thus we see that the \( 2(h-1) \)th coordinate of \( \beta^{-1}(Y_h) \subset \mathbb{A}[I \cup J] \) is uniquely determined by the other coordinates. We can therefore rewrite this scheme as a subscheme \( S^{(0)} \) of \( \mathbb{A}[I \cup J] \). Furthermore the morphism \( P^{(0)} : S^{(0)} \to \mathbb{G}_a \) is exactly the restriction of the morphism \( P : \beta^{-1}(Y_h) \to \mathbb{G}_a \) introduced in Step 1. Thus

\[
H^*_c(\beta^{-1}(Y_h), P^* L_\psi) \cong H^*_c(S^{(0)}, (P^{(0)})^* L_\psi).
\]

**Step 4.** In the next two steps, we describe an inductive application of Proposition 5.4.

We apply Proposition 5.4 to the following set-up:

- Let \( S^{(0)} \) be as in Step 3. Explicitly, it is the subscheme of \( \mathbb{A}[I \cup J_0] \) defined by the equations \( F_{2l} = 0 \) for \( l < h - 1 \), where \( \mathbb{A}[I \cup J_0] \) is the affine space \( \mathbb{A}^{2(h-1)-1} \) with coordinates labelled by \( x_i \) for \( i \in I \cup J_0 \).
- Let \( S_2^{(0)} \) denote the subscheme of \( \mathbb{A}[I_1 \cup J_0] \) defined by the same equations.
- Note that \( S^{(0)} = S_2^{(0)} \times \mathbb{A}[\{2(h-1) - 1\}] \), since \( x_{2(h-1)-1} \) has no contribution to \( F_{2l} \) for \( l < h - 1 \).
- Let \( f : S_2^{(0)} \to \mathbb{G}_a \) be defined as the projection to \( x_1 \).
Thus the only terms involving $x$ exist. Moreover, the remaining terms in $H_c^{1-2}(S_3^{(0)}, (P_3^{(0)})^* \mathcal{L}_\psi (-1))$ as vector spaces equipped with an action of $\text{Fr}_{q^2}$, where the Tate twist $(-1)$ means that the action of $\text{Fr}_{q^2}$ on $H_c^{1-2}(S_3^{(0)}, (P_3^{(0)})^* \mathcal{L}_\psi )$ is multiplied by $q^2$.

Before we proceed, we must show that one can indeed decompose $P^{(0)}$ into the form described in Equation (8). Using Theorem 5.1 together with the explicit equations for the coordinates of the product $s(x \cdot y) := a$ described in Equation (7), we see that the only terms in $x_{2(h-1)} - f_{2(h-1)}(a)$ involving $y_{2(h-1)-1}$ occur in the expression

$$-(a_2^{(0)} + a_1^{(0)}(a_2^{(0)} - a_2^{(0)})) + a_2^{(0)}(a_2^{(0)} - a_2^{(0)}) + a_2^{(0)}(a_2^{(0)} - a_1^{(0)})$$

and are exactly

$$-(x_1 y_2^{(0)} - x_1^{(0)} y_2^{(0)}) + x_1^{(0)}(y_2^{(0)} - y_2^{(0)}) + y_2^{(0)}(x_1^{(0)} - x_1^{(0)}).$$

Thus the only terms involving $x_{2(h-1)-1}$ in $P^{(0)}$ are

$$x_1^{(0)} x_{2(h-1)-1} - x_1^{(0)} x_{2(h-1)-1}.$$

Moreover, the remaining terms in $P^{(0)}$ only involve indices in $I_1 \cup J_0$. This proves that the decomposition in (8) exists.

Remark 5.5. Note that since $S_3^{(0)}$ was defined to be the subscheme of $S_2^{(0)} \subset \mathbb{A}[I_1 \cup J_0]$ cut out by $x_1$, we can actually view $S_3^{(0)}$ as a subscheme of $\mathbb{A}[I_1 \cup J_1]$. Thus what we have done in this step is reduce a computation about a subscheme of $\mathbb{A}[I_0 \cup J_0]$ to a computation about a subscheme of $\mathbb{A}[I_1 \cup J_1]$.

Step 5. We now apply Proposition 5.4 again. We apply it to the following set up.

- Let $S^{(1)} := S_3^{(0)} \subset \mathbb{A}[I_1 \cup J_1].$
- Let $S_2^{(1)}$ be the subscheme of $S^{(1)}$ cut out by $x_{2(h-1)-3}$ so that we can in fact view $S_2^{(1)}$ as a subscheme of $\mathbb{A}[I_2 \cup J_1].$
- Note that $S^{(1)} = S_2^{(1)} \times \mathbb{A}[[2(h-1) - 3]]$ since $x_1 = x_{2(h-1)-1} = 0$ implies that $x_{2(h-1)-3}$ does not contribute to $F_{2l}$ for $l < h - 1$.
- Let $f: S_2^{(1)} \to \mathbb{G}_a$ be defined as the projection to $x_3$.
- For $v \in S_2^{(1)}$ and $w = x_{2(h-1)-3}$, we may write

$$P^{(1)}(v, w) := P_3^{(0)}(v, w) = f(v)^q w + f(v)^2 w^q + (f(v)w^q)(f(v)w^q)^q + P_2^{(1)}.$$
Let \( S_3^{(1)} \subset S_2^{(1)} \subset \mathbb{A}[I_2 \cup J_1] \) be the subscheme defined by \( f = x_3 = 0 \) and let \( P_3^{(1)} := P_2^{(1)}|_{S_3^{(1)}} : S_3^{(1)} \to \mathbb{G}_a \).

Then by Proposition 5.4, for all \( i \in \mathbb{Z} \),

\[
H_i^c(S^{(1)}, (P^{(1)})^*\mathcal{L}_\psi) \cong H_{i-2}^c(S_3^{(1)}, (P_3^{(1)})^*\mathcal{L}_\psi)(-1)
\]

as vector spaces equipped with an action of \( \text{Fr}_{q^2} \).

As before, we must verify that one can indeed decompose \( P^{(1)} \) into the form described in Equation (9). This computation will turn out to be very similar to the computation in Step 4. Again using Theorem 5.1 together with Equation (7), we see that once we set \( x_1 = 0 \) and \( x_{2(h-1)-1} = 0 \), the only terms in \( x_{2(h-1)} - f_2(h-1)(s(x) \cdot y) \) involving \( y_{2(h-1)-3} \) occur in the expression

\[-(a_{2(h-1)}^2 - a_{2(h-1)}) + a_k^2(a_{2(h-1)-3}^2 - a_{2(h-1)-3}) + a_{2(h-1)-3}^2(a_{2(h-1)-3}^2 - a_{2(h-1)-3})\]

and are

\[-((x_3 y_{2(h-1)-3})^q - (x_3 y_{2(h-1)-3}^q)) + x_3^q(y_{2(h-1)-3}^q - y_{2(h-1)-3}) + y_{2(h-1)-3}^q(x_3^q - x_3).\]

Thus the only terms involving \( x_{2(h-1)-3} \) in \( P^{(0)} \) are

\[x_3^q x_{2(h-1)-3} - x_3^q x_{2(h-1)-3}^q + x_3 x_{2(h-1)-3}^q - x_3^q x_{2(h-1)-3}^q.\]

Moreover, the remaining terms in \( P^{(1)} \) only involve indices in \( I_1 \cup J_0 \). This verifies (9).

**Remark 5.6.** Each time we iterate Step 5, it will be of the following form. Let \( k \) be a positive odd integer < \( h - 1 \). We will have \( S = S_2 \times \mathbb{A}[[2(h-1)-k]] \) with \( f : S_2 \to \mathbb{G}_a \) defined as the projection to \( x_k \). For \( v \in S_2 \) and \( w = x_{2(h-1)-k} \), we may write

(10) \[
P(v, w) = f(v)^q w - f(v)^q w^q + (f(v) g(w) - (f(v) g(w))^q) + P_2,
\]

where \( g(w) = w^{q^{k-2}} + w^{q^{k-4}} + \cdots + w \). (In the notation of Proposition 5.4, \( \alpha(v, w) = -f(v) g(w) \).)

Let \( S_3 \subset S_2 \) be the subscheme defined by \( f = x_k = 0 \) and let \( P_3 = P_2|_{S_3} : S_3 \to \mathbb{G}_a \). Then by Proposition 5.4,

\[
H_i^c(S, P^*\mathcal{L}_\psi) \cong H_{i-2}^c(S_3, P_3^*\mathcal{L}_\psi)(-1)
\]

as vector spaces equipped with an action of \( \text{Fr}_{q^2} \). To see (10), observe that once we set \( x_l = x_{2(h-1)-l} = 0 \) for \( l \) odd and \( l < k \), the only terms in \( x_{2(h-1)} - f_2(h-1)(s(x) \cdot y) \) involving \( y_{2(h-1)-k} \) occur in the expression

\[-(a_{2(h-1)}^2 - a_{2(h-1)}) + a_k^2(a_{2(h-1)-k}^2 - a_{2(h-1)-k}) + a_{2(h-1)-k}^2(a_{2(h-1)-k}^2 - a_{2(h-1)-k}).\]

Thus we see that the only terms involving \( y_{2(h-1)-k} \) are

\[-((x_k y_{2(h-1)-k})^q - (x_k y_{2(h-1)-k}^q)) + x_k^q(y_{2(h-1)-k}^q - y_{2(h-1)-k}) + y_{2(h-1)-k}^q(x_k^q - x_k),\]
which simplifies to
\[
- (x_k^2 y_{2(h-1) - k} - x_k y_{2(h-1) - k}^2) + x_k^2 x_{2(h-1)-k} + y_{2(h-1)-k}^2 (x_k^2 - x_k)
\]
\[
= - x_k^2 (y_{2(h-1) - k} - y_{2(h-1) - k}^2) + x_k (y_{2(h-1) - k} - y_{2(h-1) - k}) + x_k^2 x_{2(h-1) - k}
\]
\[
= x_k^2 x_{2(h-1) - k} - x_k^2 x_{2(h-1) - k} + x_k (x_{2(h-1) - k}^2 + x_{2(h-1) - k}^2 + \cdots + x_{2(h-1) - k})
\]
\[
- x_k^2 (x_{2(h-1) - k}^2 + x_{2(h-1) - k}^2 + \cdots + x_{2(h-1) - k})
\].

This verifies (10) and allows us to use Proposition 5.4 to iterate the induction.

\(\diamondsuit\)

**Step 6, odd case.** Iterating Step 5, we reduce the computation about the cohomology of \(S^{(0)}\) to a computation about the cohomology of \(S^{((h-1)/2)} := S_3^{((h-3)/2)}\), which is the subscheme of \(\mathbb{A}[I_{(h-1)/2} \cup J_{(h-1)/2}]\) defined by the equations
\[
x_2^2 - x_2 = 0, \quad x_4^2 - x_4 = 0, \quad \ldots, \quad x_{2(h-2)}^2 - x_{2(h-2)} = 0.
\]
These equations come from the equations given in Theorem 5.1 together with setting \(x_i = 0\) for all odd \(i\). Recalling that \(I_{(h-1)/2} \cup J_{(h-1)/2} = \{2, 4, \ldots, 2(h-2)\}\), we see that \(S^{((h-1)/2)}\) is a 0-dimensional scheme with \(q^{2(h-2)}\) points and \(\text{Fr}_{q^2}\) acts trivially on the cohomology. Therefore
\[
\dim H^i_c(S^{((h-1)/2)}, (P^{((h-1)/2)})^* \mathcal{L}_\psi) = \begin{cases} q^{2(h-2)} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

**Step 6, even case.** Iterating Step 5, we reduce the computation about the cohomology of \(S^{(0)}\) to a computation about the cohomology of \(S^{((h-2)/2)} := S_3^{((h-4)/2)}\), which is the subscheme of \(\mathbb{A}[I_{(h-2)/2} \cup J_{(h-2)/2}]\) defined by the equations
\[
x_2^2 - x_2 = 0, \quad x_4^2 - x_4 = 0, \quad \ldots, \quad x_{2(h-2)}^2 - x_{2(h-2)} = 0.
\]
Recalling that \(I_{(h-2)/2} \cup J_{(h-2)/2} = \{2, 4, \ldots, 2(h-2)\} \cup \{h-1\}\), we see that \(S^{((h-2)/2)}\) is a one-dimensional scheme. Moreover \(P^{((h-2)/2)}\) is the morphism
\[
P^{((h-2)/2)} : S^{((h-2)/2)} \rightarrow \mathbb{G}_a, \quad (x_i)_{i \in I_{(h-2)/2} \cup J_{(h-2)/2}} \mapsto x_{h-1}^q (x_{h-1}^{q^2} - x_{h-1}).
\]
The above shows that
\[
H^i_c(S^{((h-2)/2)}, (P^{((h-2)/2)})^* \mathcal{L}_\psi) \cong H^i_c(\mathbb{G}_a, P^* \mathcal{L}_\psi)^{\otimes q^{2(h-2)}},
\]
where the morphism \(P\) is defined as
\[
P : \mathbb{G}_a \rightarrow \mathbb{G}_a, \quad x \mapsto x^q (x^{q^2} - x).
\]
We now compute the right-hand-side cohomology groups in the same way as in Sections 6.5 and 6.6 in [BW14]. We may write \(P = p_1 \circ p_2\) where \(p_1(x) = x^q - x\) and \(p_2(x) = x^{q^2+1}\). Since \(p_1\) is a group homomorphism, then \(p_1^* \mathcal{L}_\psi \cong \mathcal{L}_\psi \circ p_1\), where \(\mathcal{L}_\psi \circ p_1\) is the multiplicative local system on \(\mathbb{G}_a\) corresponding to the additive character \(\psi \circ p_1 : \mathbb{F}_{q^2} \rightarrow \mathbb{Q}_{q^2}^\times\). By assumption, \(\psi\) has trivial
Gal($\mathbb{F}_{q^2}/\mathbb{F}_q$)-stabilizer, and so $\psi \circ p_1$ is nontrivial. Furthermore, $\psi \circ p_1$ is trivial on $\mathbb{F}_q$. Thus the character $\psi \circ p_1: \mathbb{F}_{q^2} \to \mathbb{T}_q^\times$ satisfies the hypotheses of Proposition 6.12 in [BW14], and thus

$$\dim H^i_c(\mathbb{G}_a, \mathbb{P}^n \mathcal{L}_\psi) = \dim H^i_c(\mathbb{G}_a, p_2^2 \mathcal{L}_{\psi \circ p_1}) = \begin{cases} q & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the Frobenius $\text{Fr}_{q^2}$ acts on $H^1_c(\mathbb{G}_a, \mathbb{P}^n \mathcal{L}_\psi)$ via multiplication by $-q$.

Putting this together, we have

$$\dim H^i_c(S^{((h-2)/2)}), (\mathbb{P}^{((h-2)/2)})^* \mathcal{L}_\psi) = \begin{cases} q^{2(h-2)+1} & \text{if } i = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and the Frobenius $\text{Fr}_{q^2}$ acts on $H^1_c(\mathbb{G}_a, \mathbb{P}^n \mathcal{L}_\psi)$ via multiplication by $-q$.

Step 7. We now put together all of the boxed equations. We have

$$\text{Hom}_{U^2_q(\mathbb{F}_{q^2})}(V_{\psi}, H^i_c(X_h, \mathbb{T}_q^\times)) \cong H^{i-2((h-1)/2)}(S^{((h-1)/2)}), (\mathbb{P}^{((h-1)/2)})^* \mathcal{L}_\psi)(-[(h-1)/2]).$$

This proves Theorem 5.2.

6. The Representations $H^*_c(X_h)[\chi]$

Let $K := H'(\mathbb{F}_{q^2})$, where $H'$ is defined as in Section 3. Let $\psi: \mathbb{F}_{q^2} \to \mathbb{T}_q^\times$ be a character of conductor $q^2$ and let $\chi \in \mathcal{A}_\psi$. In this section, we will compute the representation $\sigma_\chi := H^{h-1}_c(X_h, \mathbb{T}_q^\times)[\chi]$ by computing its restriction to $H := H(\mathbb{F}_{q^2})$. It will turn out that $\sigma_\chi$ is irreducible and therefore by Corollary 4.3, determining $\sigma_\chi$ as a representation of $H$ will be enough to determine $\sigma_\chi$ as a representation of $U^2_q(\mathbb{F}_{q^2})$.

Recall that the left action of $U^1_c/U^h_c$ and right action of $U^2_q(\mathbb{F}_{q^2})$ on $X_h$ induce a $(U^1_c/U^h_c \times U^2_q(\mathbb{F}_{q^2}))$-module structure on $H^{h-1}_c(X_h, \mathbb{T}_q^\times)$. The primary object of interest in this section is the subspace $H^{h-1}_c(X_h, \mathbb{T}_q^\times)\chi_1, \chi_2 \subset H^{h-1}_c(X_h, \mathbb{T}_q^\times)$ wherein $U^1_c/U^h_c \times H(\mathbb{F}_{q^2})$ acts by $\chi_1 \otimes \chi_2$. Here, $\chi_1$ and $\chi_2$ are characters of $U^1_c/U^h_c \cong H(\mathbb{F}_{q^2})$.

We first present the main theorems of this section.

**Theorem 6.1.** Let $\psi: \mathbb{F}_{q^2} \to \mathbb{T}_q^\times$ be a character of $q^2$ and let $\chi \in \mathcal{A}_\psi$. Then $H^{h-1}_c(X_h, \mathbb{T}_q^\times)[\chi]$ is an irreducible representation of $U^2_q(\mathbb{F}_{q^2})$.

Theorem 6.1 proves Conjecture 5.18 of [B12] (this was restated in Section 1 of this paper). We prove this in Section 6.2. However, it will be important to know exactly which representation $H^{h-1}_c(X_h, \mathbb{T}_q^\times)[\chi]$ is. Thus we need the following finer statement.

**Theorem 6.2.** Let $\psi: \mathbb{F}_{q^2} \to \mathbb{T}_q^\times$ be a character of conductor $q^2$ and let $\chi_1 \in \mathcal{A}_\psi$. Then for any character $\chi_2: U^1_c/U^h_c \to \mathbb{T}_q^\times$,

$$\dim H^{h-1}_c(X_h, \mathbb{T}_q^\times)\chi_1, \chi_2 = (-1)^h \left( q \cdot \langle \chi_1, \chi_2 \rangle + \sum_{i=1}^{h-2} (-1)^i(q+1) \cdot \langle \chi_1, \chi_2 \rangle \mathcal{G}_i \right).$$

We prove this in Section 6.1. Note that Theorem 6.1 is a consequence of Theorem 6.2. However, because the proof of Theorem 6.2 is complicated, we hope that proving Theorem 6.1 independently (in Section 6.2) will illustrate the flavor of the computation in a simpler situation.

As a consequence of Theorem 6.2, we have
Theorem 6.3. Let \( \psi : \mathbb{F}_{q^2} \to \mathbb{Q}_l^\times \) be a character of conductor \( q^2 \) and let \( \chi \in A_\psi \). Then

\[
H^i_c(X_h, \mathbb{Q}_l)[\chi] = \begin{cases} 
\rho_\chi & \text{if } i = h - 1, \\
0 & \text{otherwise.} 
\end{cases}
\]

Proof. We know from Theorem 5.2 that \( H^i_c(X_h, \mathbb{Q}_l)[\chi] = 0 \) if \( i \neq h - 1 \). Let \( \sigma_\chi \) is a representation of \( U_{2, q}^h(\mathbb{F}_{q^2}) \) wherein \( H^2_h(\mathbb{F}_{q^2}) \) acts by some character \( \psi \) with conductor \( q^2 \).

Theorem 6.2 implies that

\[
\sigma_\chi = (-1)^h(q \cdot \chi + \sum_{i=1}^{h-2} (-1)^i(q + 1) \cdot \text{Ind}_{G_i}^H(\chi)),
\]

which implies that \( \dim \sigma_\chi = q^{h-1} \). By Theorem 3.1, we know that if \( \rho \) is an irreducible representation of \( U_{2, q}^h(\mathbb{F}_{q^2}) \) such that \( H^2_h(\mathbb{F}_{q^2}) \) acts by \( \psi \), then \( \dim \rho = q^{h-1} \). Therefore \( \sigma_\chi \) is irreducible.

Thus Corollary 4.3 implies that the \( U_{2, q}^h(\mathbb{F}_{q^2}) \)-representation \( \sigma_\chi \) is determined by its restriction to the subgroup \( H \). Finally, Equation (11) and Theorem 4.1 allow us to conclude that \( \sigma_\chi \cong \rho_\chi \) as \( U_{2, q}^h(\mathbb{F}_{q^2}) \)-representations. \( \square \)

6.1. Proof of Theorem 6.2. By Theorem 5.2, we can apply Lemma 2.13 of [B12] to our situation and get

\[
\dim H^{h-1}_c(X_h, \mathbb{Q}_l)[\chi_1, \chi_2] = \frac{1}{q^{\delta(h-1)}} \sum_{g, \gamma \in H} \chi_1(g)^{-1} \chi_2(\gamma) \cdot N(g, \gamma)
\]

(12)

\[
= \frac{1}{q^{\delta(h-1)}} \sum_{g, \gamma \in H \atop g_2i = h_2i \text{ for } 1 \leq i \leq h - 2} \chi_1(g)^{-1} \chi_2(\gamma) \cdot N(g, \gamma)
+ \frac{1}{q^{\delta(h-1)}} \sum_{g, \gamma \in H \atop \exists k < h - 2 \text{ s.t. } h_{2k} \neq g_{2k}} \chi_1(g)^{-1} \chi_2(\gamma) \cdot N(g, \gamma),
\]

(13) and (14)

where

\[
g = 1 + \sum g_i \tau^i
\]

\[
\gamma = 1 + \sum h_i \tau^i
\]

\[
N(g, \gamma) = \# \{ x \in X_h(\mathbb{F}_q) : g \cdot \text{Fr}_{q^2}(x) = x \cdot \gamma \}.
\]

We compute line (13) in Proposition 6.4 and line (14) in Proposition 6.11.
In both of these situations, we need to analyze the set of solutions to a large system of equations. These equations are:

\[ x_{2k}^q - x_{2k} = \sum_{i=1}^{2k-1} (-1)^{(i+1)} x_{i}^q (x_{2k-i}^q - x_{2k-i}) \quad \text{for } 1 \leq k \leq h - 1 \]

\[ x_{2k}^q - x_{2k} = \sum_{i=1}^{k} [(h_{2i} - g_{2i})x_{2k-2i}] \quad \text{for } 1 \leq k \leq h - 1 \]

\[ x_{2k+1+1}^q - x_{2k+1} = \sum_{i=1}^{k} [(h_{2i}^q - g_{2i})x_{2k+1-2i}] \quad \text{for } 1 \leq k \leq h - 2 \]

The equations of Type (*) are equivalent to the condition that \( x \in X_h(\mathbb{F}_q) \) (this was proved in Theorem 5.1). The equations of Type (**) and Type (***) are equivalent to the condition that \( g \star \text{Fr}_q(x) = x \cdot h \), where we write \( g = 1 + \sum_{i=1}^{h-1} g_{2i} \tau_{2i} \) and similarly for \( h \). (These two actions were defined in Section 2.) We will call the above equations the Type (*) equation for \( 2k \), the Type (**) equation for \( 2k \), and the Type(***) for \( 2k + 1 \), respectively. Furthermore, when we refer to these equations as polynomials, we view them as multivariate polynomials in the \( x_i \)’s.

6.1.1. Computation of line (13). We prove a sequence of lemmas that build up to the following

**Proposition 6.4.** Let \( \psi: \mathbb{F}_{q^2} \rightarrow \mathbb{F}_k^* \) be a character with conductor \( q^2 \) and let \( \chi_1 \in A_\psi \). Then

\[ \text{Line (13)} = (-1)^h \left( \chi_1, \chi_2 \right) \cdot q + \sum_{i=1}^{h-2} (-1)^i \left( \chi_1, \chi_1 \right) g_i \cdot (q + 1) \right). \]

**Lemma 6.5.** Assume that \( h_{2i} = g_{2i} \) for \( 1 \leq i \leq h - 2 \). Then \( x_{2k}^q - x_{2k} = 0 \) for \( 1 \leq k \leq h - 2 \).

**Proof.** This is just a simple execution of induction. It is clear that this is true for \( k = 1 \). Now assume that it is true for \( k < h - 2 \), and we can show that it is true for \( k + 1 \). Indeed, by assumption, \( h_{2i} = g_{2i} \) for \( 1 \leq i \leq h - 2 \), so the induction hypothesis implies that the Type (**) equation for \( 2(k + 1) \) simplifies to

\[ x_{2k}^q - x_{2k} = \sum_{i=1}^{k+1} h_{2i}x_{2k-2i} - g_{2i}x_{2k-2i} = \sum_{i=1}^{k+1} (h_{2i} - g_{2i})x_{2k-2i} = 0. \]

**Important Remark 6.6.** The key observation that we will capitalize on in the next few lemmas is the following. The Type (*) equations "intertwine" the equations of Type (**) and Type (***)

Using Lemma 6.5 and substituting Type (**) and (***) equations into Type (*) equations, we
Thus:

\[ h_{2k} - g_{2k} = \sum_{i \text{ odd}} \sum_{1 \leq i \leq 2k-3} x_i^q \left( \sum_{j \text{ odd}} (g_{2k-i-j} - g_{2k-i-j})x_j - g_{2k-i-j} (x_j^q - x_j) \right) \]

\[ = \sum_{i \text{ odd}} \sum_{1 \leq i \leq 2k-3} x_i^q \left( \sum_{j \text{ odd}} (g_{2k-i-j} - g_{2k-i-j})x_j \right) - \sum_{2 \leq j \leq 2k-2} g_{2k-i} (x_i^q - x_i). \]

Thus:

\[ (\dagger) \quad h_{2k} - g_{2k} = \sum_{i \text{ odd}} \sum_{1 \leq i \leq 2k-3} x_i^q \left( \sum_{j \text{ odd}} (g_{2k-i-j} - g_{2k-i-j})x_j \right) \quad \text{for } 1 \leq k \leq h-1 \]

Recall that \( h_{2k} - g_{2k} = 0 \) for \( 1 \leq k \leq h-2 \) by assumption.

**Lemma 6.7.** Assume that \( h_{2i} = g_{2i} \) for \( 1 \leq i \leq h-2 \). If \( g_{2k} \in \mathbb{F}_q \) for \( 1 \leq k \leq h-2 \), then

\[ N(g, \gamma) = \begin{cases} q^{4(h-1)} & \text{if } g_{2(h-1)} = h_{2(h-1)} \\ 0 & \text{otherwise} \end{cases} \]

**Proof.** If \( g_{2k} \in \mathbb{F}_q \) for \( 1 \leq k \leq h-2 \), then all the coefficients in the Type (\( \dagger \)) equations vanish, imposing no further conditions on the \( x_i \)'s but forcing \( h_{2(h-1)} - g_{2(h-1)} = 0 \). Therefore the number of solutions to the equations of Types (\( \ast \)) through (\( \ast \ast \ast \)) are the solutions to \( x_i^q - x_i = 0 \) together with the the solutions to (\( \ast \ast \)) and (\( \ast \ast \ast \)). Thus we have \( q^2 \) choices for each \( x_k \), and the desired conclusion follows.

**Lemma 6.8.** Assume that \( h_{2i} = g_{2i} \) for \( 1 \leq i \leq h-2 \). Pick \( k \geq 1 \) and suppose that \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i \leq h-(2k+1) \) and \( g_{2(h-2k)} \notin \mathbb{F}_q \). Then

\[ N(g, \gamma) = \begin{cases} q^{2(2(h-1)-k)} & \text{if } h_{2(h-1)} = g_{2(h-1)} \\ (q+1)q^{2(2(h-1)-k)} & \text{if } 0 \neq h_{2(h-1)} - g_{2(h-1)} \in \ker \text{Tr}_{q^2/\mathbb{F}_q}, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** All the coefficients in the Type (\( \dagger \)) equations for \( 2 \leq 2j \leq 2(h-2k) \) vanish so we get the empty conditions \( h_{2j} - g_{2j} = 0 \), yielding no additional restrictions on any of the \( x_i \)'s. The first nontrivial restriction comes from the Type (\( \dagger \)) equation for \( 2(h-2k+1) \):

\[ h_{2(h-2k+1)} - g_{2(h-2k+1)} = x_1^q (g_{2(h-2k)}^2 - g_{2(h-2k)})x_1. \]

By assumption, the left-hand side vanishes and the coefficient of \( x_1 \) on the right-hand side is nonvanishing, which forces \( x_1 = 0 \). This extra condition implies that the subsequent Type (\( \dagger \)) equation (i.e. the Type (\( \dagger \)) equation for \( 2(h-2k+2) \)) simplifies to

\[ h_{2(h-2k+2)} - g_{2(h-2k+2)} = 0, \]

imposing no additional constraints on the \( x_i \)'s. The subsequence Type (\( \dagger \)) equation simplifies to

\[ h_{2(h-2k+3)} - g_{2(h-2k+3)} = x_3^q (g_{2(h-2k)}^3 - g_{2(h-2k)})x_3, \]
which forces $x_3 = 0$ since the left-hand side vanishes. This continues until the equation
\[ h_2(h-1) - g_2(h-1) = x^q_{2k-1}(g^q_{2(h-2k)} - g_2(h-2k))x_{2k-1}. \]

Thus we see that regardless of whether $h_2(h-1)$ and $g_2(h-1)$ agree, Equations (i) force $x_1 = 0, x_3 = 0, \ldots, x_{2k-3} = 0$. Note that equation (**) for $2k-1$ implies that $x_{2k-1} \in \mathbb{F}_q$. Thus we see that if $h_2(h-1) = g_2(h-1)$, then this last displayed equations implies that we have the additional constraint that $x_{2k-1} = 0$. Furthermore, this gives $q+1$ choices for $x_{2k-1}$ if $h_2(h-1) - g_2(h-1) \in \ker \text{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q}$ and no choices for $x_{2k-1}$ otherwise. We see from equations of Type (***) and (*** *) that regardless of whether $h_2(h-1)$ and $g_2(h-1)$ agree, we have $q^2$ choices for the remaining $x_i$'s (i.e. for even $i \leq 2k-2$ and all $i > 2k-1$). The desired conclusion follows.

**Lemma 6.9.** Assume that $h_{2i} = g_{2i}$ for $1 \leq i \leq h-2$. Pick $k \geq 1$ and suppose that $g_{2i} \in \mathbb{F}_q$ for $1 \leq i \leq h - (2k + 2)$ and $g_{2i} \notin \mathbb{F}_q$. Then

\[ N(g, \gamma) = \begin{cases} 
q^{2(h-1) - k} & \text{if } h_2(h-1) = g_2(h-1), \\
0 & \text{otherwise.}
\end{cases} \]

**Proof.** We argue as in the proof of Lemma 6.8. All the coefficients in the Type (i) equations for $2 \leq 2j \leq 2(h - 2k - 1)$ vanish so we get the empty conditions $h_{2j} - g_2j = 0$, yielding no additional restrictions on any of the $x_i$'s. The first nontrivial restriction comes from the Type (i) equation for $2(h - 2k)$:

\[ h_2(h-2k) - g_2(h-2k) = x^q_2(g^q_2(h-(2k+1)) - g_2(h-(2k+1)))x_1. \]

Thus, for odd $l$ with $1 \leq l \leq 2k-1$, the Type (i) equation for $2(h - (2k + 1) + l)$ reduces to

\[ h_2(h-(2k+1)+l) - g_2(h-(2k+1)+l) = x^q_2(g^q_2(h-(2k+1)) - g_2(h-(2k+1)))x_1, \]

which forces $x_1 = 0$. But then this implies that the Type (i) equation for $2(h - 1)$ simplifies to

\[ h_2(h-1) - g_2(h-1) = 0. \]

Thus there are no solutions if $h_2(h-1) \neq g_2(h-1)$. If $h_2(h-1) = g_2(h-1)$, then we get $q^2$ solutions for each of the $x_i$'s other than the odd $i, 1 \leq i \leq 2k-1$. The desired conclusion follows.

**Lemma 6.10.** Assume that $h_{2i} = g_{2i}$ for $1 \leq i \leq h-2$ and let $\psi$ be the restriction of $\chi_1$ to $U^{h-1}_L/U^h_L$. By assumption, we can write $\gamma = g \cdot (1 + \epsilon q^{2(h-1)})$. Then

\[ \sum_{\epsilon \in \mathbb{F}_q^2} \psi(\epsilon) \cdot N(g, \gamma) = \begin{cases} 
q^{4(h-1)} & \text{if } g \in G_{h-2}, \\
q \cdot q^{2(h-1)-k} & \text{if } g \in G_{2(h-(2k+1))} \supset G_{2(h-(2k+2))}, k \geq 1, \\
q^{2(h-1)-k} & \text{if } g \in G_{2(h-(2k+2))} \supset G_{2(h-(2k+3))}, k \geq 1.
\end{cases} \]

**Proof.** By assumption $\psi$ is a nontrivial additive character $\mathbb{F}_q^2 \to \mathbb{Q}_q^\times$. Recalling the definition of $G_i$ from Section 4, it is easy to see that the first and third cases of the lemma follow from Lemmas 6.7 and 6.9 respectively. To see the second case, Lemma 6.8 implies that

\[ \sum_{\epsilon \in \mathbb{F}_q^2} \psi(\epsilon) \cdot N(g, \gamma) = -q \cdot q^{2(h-1)-k} + \sum_{\epsilon \in \ker \text{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q}} \psi(\epsilon) \cdot (q + 1)q^{2(h-1)-k} \]

\[ = -q \cdot q^{2(h-1)-k}. \]

We are now ready to prove Proposition 6.4.
Proof of Proposition 6.4. First notice that by the assumption \( h_{2i} = g_{2i} \) for \( 1 \leq i \leq h - 2 \), we can write Line (13) as

\[
\frac{1}{q^{h(h-1)}} \sum_{g \in H} \chi_2(g) / \chi_1(g) \cdot \psi(e) \cdot N(g, \gamma),
\]

where \( \gamma = g \cdot (1 + \epsilon t^{2(h-1)}) \). Also notice that

\[
\sum_{g \in H} = \sum_{g \in G_{h-2}} + \sum_{g \in G_{h-3} \setminus G_{h-2}} + \cdots + \sum_{g \in G_1 \setminus G_2} + \sum_{g \in H \setminus G_1}.
\]

We first analyze each of the summands. Pick \( k \geq 1 \). Then:

\[
\sum_{g \in G_{h-2}} \frac{\chi_2(g)}{\chi_1(g)} \sum_{e \in \mathcal{F}_{g,2}} \psi(e) \cdot N(g, \gamma)
= \langle \chi_1, \chi_2 \rangle_{G_{h-2}} \cdot |G_{h-2}| \cdot q^{2(2(h-1))} = q^{5(h-1)} \cdot \langle \chi_1, \chi_2 \rangle_{G_{h-2}} \cdot q
\]

\[
\sum_{g \in G_{h-(2k+1)}} \frac{\chi_2(g)}{\chi_1(g)} \sum_{e \in \mathcal{F}_{g,2}} \psi(e) \cdot N(g, \gamma)
= \left( \langle \chi_1, \chi_2 \rangle_{G_{h-(2k+1)}} \cdot |G_{h-(2k+1)}| - \langle \chi_1, \chi_2 \rangle_{G_{h-2k}} \cdot |G_{h-2k}| \right) \cdot -q \cdot q^{2(2(h-1)-k)}
\]

\[
\sum_{g \in G_{h-(2k+2)}} \frac{\chi_2(g)}{\chi_1(g)} \sum_{e \in \mathcal{F}_{g,2}} \psi(e) \cdot N(g, \gamma)
= \left( \langle \chi_1, \chi_2 \rangle_{G_{h-(2k+2)}} \cdot |G_{h-(2k+2)}| - \langle \chi_1, \chi_2 \rangle_{G_{h-(2k+1)}} \cdot |G_{h-(2k+1)}| \right) \cdot q^{2(2(h-1)-k)}
\]

Now we put this together. From the definition, it is easy to see that

\[
|G_{h-n}| = q^{h-2+n}.
\]

We now analyze the coefficient of \( \langle \chi_1, \chi_2 \rangle_{G_{h-n}} \).

- Recall that \( H = G_0 \). If \( h \) is odd, then from the above we see that the coefficient of \( \langle \chi_1, \chi_2 \rangle_{G_0} \) is
  \[
  |G_0| \cdot -q \cdot q^{4(h-1)-(h-1)} = q^{5(h-1)} \cdot -q.
  \]
  If \( h \) is even, then from the above we see that the coefficient is
  \[
  |G_0| \cdot q^{4(h-1)-(h-2)} = q^{5(h-1)} \cdot q.
  \]

- Let \( k \geq 1 \). The coefficient of \( \langle \chi_1, \chi_2 \rangle_{G_{h-(2k+1)}} \) is
  \[
  |G_{h-(2k+1)}| \cdot (-q \cdot q^{2(2(h-1)-k)} - q^{2(2(h-1)-k)}) = q^{5(h-1)} \cdot (-q - 1).
  \]
  The coefficient of \( \langle \chi_1, \chi_2 \rangle_{G_{h-(2k+2)}} \) is
  \[
  |G_{h-(2k+2)}| \cdot (q^{2(2(h-1)-k)} + q \cdot q^{2(2(h-1)-(k+1))}) = q^{5(h-1)} \cdot (q + 1).
  \]

The desired result follows. \( \square \)
6.1.2. Computation of line (14). In this subsection, we prove the following

**Proposition 6.11.** Let $\psi : \mathbb{F}_q^2 \to \overline{\mathbb{Q}}_\ell^\times$ be a character with conductor $q^2$ and let $\chi_1 \in A_\psi$. Then

$\text{Line (14)} = 0.$

Here is the idea of the proof. First let

$$A_{g, \gamma} := \{ x \in X_h(\mathbb{F}_p) : g \ast \text{Fr}_{q^2}(x) = x \cdot \gamma \}.$$

We will show that a partial solution $(x_1, \ldots, x_{2(h-1)})$ extends to a full solution $(x_1, \ldots, x_{2(h-1)})$ if and only if the partial solution satisfies an equation of the form $ax^q - a^q x + a_0 = 0$ for some nonzero

$a \in \mathbb{F}_q$. The $x$ in this equation will be one of the $x_k$'s. The main work is in giving a nonvanishing condition for coefficients of certain $x_k$'s which will allow us to find such an $a$. This will give us a bijection between $A_{g, h}$ and $A_{g, h + \delta r 2(h-1)}$. Once we have established this, we will be able to prove Proposition 6.11.

**Lemma 6.12.** Let $(x_1, \ldots, x_{2(h-1)})$ be a solution to the equations of type (** and (**). Then

$(x_1, \ldots, x_{2(h-1)})$ also satisfies the equations of type (*) if and only if, for every $k$, the tuple satisfies the equation

$$h_{2k} - g_{2k} = 1 \leq i \leq 2k-1 \quad (2k-i-1)/2 \quad \sum_{j=1}^{(2k-i)/2} (h_{2j} - g_{2j}) x_{2k-i-2j}$$

$$(\dagger) \quad \sum_{1 \leq i \leq 2k-1, \, i \, \text{odd}} x_i^q \left[ \sum_{j=1}^{(2k-i)/2} (h_{2j} - g_{2j}) x_{2k-i-2j} \right] - \sum_{i=1}^{k-1} (h_{2i} - g_{2i}) x_{2k-2i}.$$

**Proof.** First note that a tuple $(x_1, \ldots, x_{2(h-1)})$ satisfying (** and (** can be constructed as follows: pick any $x_1, x_2 \in \mathbb{F}_q$ and then notice that the equations of type (** and (** allow us to choose $x_k$ given $x_1, \ldots, x_{k-1}$.

Now we substitute (** and (** into (*).

$$x_{2k}^q - x_{2k} = \sum_{i=1}^{2k-1} (-1)^{i+1} x_i^q (x_{2k-i} - x_{2k-i})$$

$$= \sum_{i \, \text{odd}} x_i^q \left[ \sum_{j=1}^{(2k-i)/2} (h_{2j} - g_{2j}) x_{2k-i-2j} - g_{2j} (x_{2k-i-2j} - x_{2k-i-2j}) \right]$$

$$- \sum_{i \, \text{even}} x_i^q \left[ \sum_{j=1}^{(2k-i)/2} (h_{2j} - g_{2j}) x_{2k-i-2j} - g_{2j} (x_{2k-i-2j} - x_{2k-i-2j}) \right]$$

$$= \sum_{i \, \text{odd}} x_i^q \left[ \sum_j (h_{2j}^q - g_{2j}) x_{2k-i-2j} \right] - \sum_{i \, \text{even}} x_i^q \left[ \sum_j (h_{2j} - g_{2j}) x_{2k-i-2j} \right]$$

$$- \sum_{j=1}^{k-1} g_{2j} (x_{2k-2j}^q - x_{2k-2j}).
On the other hand,

\[ x_{2k}^2 - x_{2k} = \sum_{i=1}^{k} \left( (h_{2i} - g_{2i})x_{2k-2i} - g_{2i}(x_{2k-2i}^2 - x_{2k-2i}) \right) \]

\[ = (h_{2k} - g_{2k}) + \sum_{i=1}^{k-1} (h_{2i} - g_{2i})x_{2k-2i} - \sum_{i=1}^{k-1} g_{2i}(x_{2k-2i}^2 - x_{2k-2i}). \]

Therefore

\[ h_{2k} - g_{2k} = \sum_{1 \leq i \leq 2k-1 \atop i \text{ odd}} x_i^q \left[ \sum_{j=1}^{(2k-i)/2} (h_{2j}^q - g_{2j})x_{2k-i-j} \right] \]

\[ - \sum_{1 \leq i \leq 2k-1 \atop i \text{ even}} x_i^q \left[ \sum_{j=1}^{(2k-i)/2} (h_{2j} - g_{2j})x_{2k-i-j} \right] - \sum_{i=1}^{k-1} (h_{2i} - g_{2i})x_{2k-2i}. \]

This shows that the above collection of equations imposes the same conditions as the equations of type (\*). \( \square \)

**Lemma 6.13.** Let \( k \) be the smallest \( k \) such that \( h_{2k} \neq g_{2k} \) and assume that \( k \leq h - 2 \). If \((x_1, \ldots, x_{2(h-1)})\) is a solution to the equations (\*) through (**), then:

(a) If \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i \leq k - 3 \), then \( g_{2(k-2)} \in \mathbb{F}_q \) and \( x_1(g_{2(k-2)}^q - g_{2k-2}) \neq 0 \).

(b) If \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i < k - 3 \) and \( g_{2(k-3)} \notin \mathbb{F}_q \), then \( x_3(g_{2(k-3)}^q - g_{2(k-3)}) \neq 0 \).

(c) If \( n \) is odd and \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i < k - n \) and \( g_{2(k-n)} \notin \mathbb{F}_q \), then \( x_n(g_{2(k-n)}^q - g_{2(k-n)}) \neq 0 \).

(d) If \( n > 2 \) is even and \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i < k - n \) and \( g_{2(k-n)} \notin \mathbb{F}_q \), then \( \# = 0 \).

**Proof of (a).** If \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i \leq k - 3 \), then by Lemma 6.12, the tuple \((x_1, \ldots, x_{2(h-1)})\) must satisfy

\[ 0 = h_{2(k-1)} - g_{2(k-1)} = x_1^q((g_{2(k-2)}^q - g_{2(k-2)})x_1) \]

\[ 0 \neq h_{2k} - g_{2k} = x_1^q((g_{2(k-1)}^q - g_{2(k-1)})x_1 + (g_{2(k-2)}^q - g_{2(k-2)})x_3) \]

\[ + x_3^q((g_{2(k-2)}^q - g_{2(k-2)})x_1). \]

If \( x_1 = 0 \), then this automatically implies that \( h_{2k} - g_{2k} = 0 \), which contradicts the assumption that \( h_{2k} \neq g_{2k} \). Therefore \( x_1 \neq 0 \). The first equation above then forces \( g_{2(k-2)} \in \mathbb{F}_q \), and so the second equation simplifies to

\[ 0 \neq h_{2k} - g_{2k} = x_1^q(g_{2(k-1)}^q - g_{2(k-1)})x_1. \]
Proof of (b). If \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i < k - 3 \) and \( g_{2(k-3)} \notin \mathbb{F}_q \), then by Lemma 6.12, we necessarily have
\[
0 = h_{2(k-2)} - g_{2(k-2)} = x_1^q (g_{2(k-3)}^q - g_{2(k-3)})x_1
\]
\[
0 = h_{2(k-1)} - g_{2(k-1)} = x_1^q (g_{2(k-2)}^q - g_{2(k-2)})x_1 + \left( g_{2(k-3)}^q - g_{2(k-3)} \right)x_3
\]
\[
0 \neq h_{2k} - g_{2k} = x_1^q \left( g_{2(k-1)}^q - g_{2(k-1)} \right)x_1 + \left( g_{2(k-2)}^q - g_{2(k-2)} \right)x_1 + \left( g_{2(k-3)}^q - g_{2(k-3)} \right)x_3
\]
By assumption \( g_{2(k-3)} \notin \mathbb{F}_q \), so the first equation forces \( x_1 = 0 \). Then the second equation simplifies to 0 = 0 and the third equation simplifies to
\[
0 \neq h_{2k} - g_{2k} = x_1^q (g_{2(k-2)}^q - g_{2(k-2)})x_3.
\]

Proof of (c) and (d). Now suppose that there is some \( n > 3 \) such that \( g_{2i} \in \mathbb{F}_q \) for \( 1 \leq i < k - n \) and \( g_{2(k-n)} \notin \mathbb{F}_q \). Then since \( h_{2i} = g_{2i} \) for \( 1 \leq i < k \), the equations \((\dagger\dagger)\) simplify to
\[
h_{2m} - g_{2m} = \sum_{1 \leq i \leq 2m-1} x_i^q \left( \sum_{j=k-n}^{(2m-i-1)/2} (g_{2j}^q - g_{2j})x_{2m-2j} \right)
\]
for \( k - n + 1 \leq m \leq k \).

So Equation \((\dagger\dagger)\) for \( 2(k-n+1) \) is
\[
0 = x_1^q (g_{2(k-n)}^q - g_{2(k-n)})x_1,
\]
which forces \( x_1 = 0 \) since by assumption \( g_{2(k-n)} \notin \mathbb{F}_q \). This implies that Equation \((\dagger\dagger)\) for \( 2(k-n+2) \) gives the empty condition 0 = 0. Setting \( x_1 = 0 \), Equation \((\dagger\dagger)\) for \( 2(k - n + 3) \) simplifies to
\[
0 = x_3^q (g_{2(k-n)}^q - g_{2(k-n)})x_3,
\]
which forces \( x_3 = 0 \). Continuing this, we see that:

- For \( 2l + 1 \leq n \), Equation \((\dagger\dagger)\) for \( 2(k-n+2l+1) \) yields
  \[
h_{2(k-n+2l+1)} - g_{2(k-n+2l+1)} = x_{2l+1}^q (g_{2(k-n)}^q - g_{2(k-n)})x_{2l+1},
  \]
  which forces \( x_{2l+1} = 0 \) if \( 2l + 1 < n \), and \( x_3^q (g_{2(k-n)}^q - g_{2(k-n)})x_n \neq 0 \) when \( 2l + 1 = n \). This proves (c).

- For \( 2l \leq n \), the test equation for \( m = k - n + 2l \) gives the condition equation \( h_{2(k-n+2l)} - g_{2(k-n+2l)} = 0 \). In particular, if \( 2l = n \), then we have \( h_{2k} - g_{2k} = 0 \), which is a contradiction. This proves (d).

\[
\square
\]

Lemma 6.14. Let \( k \) be as in Lemma 6.13. Then
\[
(a) \text{ If } g_{2i} \in \mathbb{F}_q \text{ for } 1 \leq i \leq k-3, \text{ then Equation } (\dagger\dagger) \text{ for } 2(h-1) \text{ is of the form } ax_1^q = 0, \text{ where } a = x_1 (g_{2k-2}^q - g_{2k-2}). \text{ Moreover, } x_2(2(h-1)-2(k-1)-1 = a_0. \text{ where } a = x_1 (g_{2k-2}^q - g_{2k-2}). \text{ Moreover, } x_2(2(h-1)-2(k-1)-1 has no contribution to } a \text{ or } a_0.
\]
(b) If $n \geq 3$ is odd and $g_{2i} \in \mathbb{F}_q$ for $1 \leq i < k - n$ and $g_{2(k-n)} \notin \mathbb{F}_q$, then Equation \((\dagger\dagger)\) for $2(h - 1)$ is of the form $ax^q_{2(h-1)-2(k-n)-n} - a^q x_{2(h-1)-2(k-n)-n} + a_0 = 0$, where $a = x_n (g^q_{2(k-n)} - g_{2(k-n)})$. Moreover, $x_2(h-1)-2(k-n)-n$ has no contribution to $a$ or $a_0$.

Proof. First note that since $k \leq h - 2$, then necessarily $2(h - 1) - 2(k - n) - n \neq n$, which automatically implies that $x_2(h-1)-2(k-n)-n$ has no contribution to $a$.

Recall Equation \((\dagger\dagger)\) for $2(h - 1)$:

$$h_{2(h-1)} - g_{2(h-1)} = \sum_{1 \leq i \leq 2(h-1)-1} x^q_i \left[\sum_{j=1}^{(2(h-1)-i-1)/2} (h^q_{2j} - g_{2j}) x_{2(h-1)-i-2j}\right]$$

\[ \begin{align*}
&- \sum_{1 \leq i \leq 2(h-1)-1} x^q_i \left[\sum_{j=1}^{(2(h-1)-i)/2} (h_{2j} - g_{2j}) x_{2(h-1)-i-2j}\right] \\
&- \sum_{i=1}^{(h-1)-1} (h_{2i} - g_{2i}) x_{2(h-1)-2i}.
\end{align*} \]

We prove (a). We need only show that the only terms in Equation (15) involving $x_2(h-1)-2(k-n)-1$ are exactly the terms

$$ax^q_{2(h-1)-2(k-n)-1} - a^q x_{2(h-1)-2(k-n)-1}, \quad \text{where } a = x_1 (g_{2k-2}^q - g_{2k-2}).$$

Clearly any term involving $x_2(h-1)-2(k-n)-1$ must come from the first sum in the equation. These terms are

$$\sum x^q_i (h^q_{2j} - g_{2j}) x_{2(h-1)-2(k-n)-1} + x^q_{2(h-1)-2(k-n)-1} (h^q_{2j} - g_{2j}) x_i,$$

where the sum ranges over $i$ and $j$ such that $i + 2j + 2(h - 1) - 2(k - 1) - 1 = 2(h - 1)$. In particular, if $i \geq 3$, then $2j \leq 2(k - 2)$. We know by assumption that $h_{2j} = g_{2j} \in \mathbb{F}_q$ for $2j \leq 2(k - 3)$ and Lemma 6.13(a) implies $g_{2(k-2)} \in \mathbb{F}_q$, so the coefficient $h^q_{2j} - g_{2j}$ vanishes for $2j \leq 2(k - 2)$. Therefore the sum above simplifies to

$$x^q_i (g^q_{2(k-1)} - g_{2(k-1)}) x_{2(h-1)-2(k-n)-1} + x^q_{2(h-1)-2(k-n)-1} (g^q_{2(k-1)} - g_{2(k-1)}) x_1.$$

Set $a = x_1 (g^q_{2(k-1)} - g_{2(k-1)})$ and notice that since $x_1 \in \mathbb{F}_q^*$, the above expression simplifies to

$$-a^q x_{2(h-1)-2(k-n)-1} + ax^q_{2(h-1)-2(k-n)-1},$$

which is exactly what we wanted to show in (a).

We now prove (b). We need to establish the following statements:

(i) $x_n \in \mathbb{F}_q^*$

(ii) The only term in the equation for $2(h - 1)$ in Lemma 6.12 that contains $x_{2(h-1)-2(k-n)-n}$ are the terms $ax^q_{2(h-1)-2(k-n)-n}$ and $a^q x_{2(h-1)-2(k-n)-n}$.

In the proof of Lemma 6.13(c), we showed that for odd $m$ with $m < n$, we have $x_m = 0$. Then by the equation for $n$ of type (**), we see that we must have

$$x^q_n - x_n = 0,$$

so this shows (i).
To see (ii), we proceed as in the proof of part (a) of this lemma. Clearly any term involving $x_{2(h-1)-2(k-n)-n}$ must come from the first sum in Equation (15). In this sum, the terms involving $x_{2(h-1)-2(k-n)-n}$ are

$$\sum x_i^q(h_i^q - g_{2j})x_{2(h-1)-2(k-n)-n} + x_{2(h-1)-2(k-n)-n}^q(h_i^q - g_{2j})x_i,$$

where the sum ranges over $i$ and $j$ such that $i + 2j + 2(h - 1) - 2(k - n) - n = 2(h - 1)$. Equivalently, $i + 2j = 2(k - n) + n$. Note that this forces $i$ to be odd since $n$ is odd by assumption. If $i < n$, then $x_i = 0$, and thus any terms involving $(h_i^q - g_{2j})$ for $j > 2(k - n)$ vanish. If $j < 2(k - n)$, then by assumption $h_{2j} = g_{2j} \in \mathbb{F}_q$, and so $h_i^q - g_{2j} = 0$ when $j < 2(k - n)$. Therefore the above sum simplifies to

$$x_n^q(g_{2(k-n)}^q - g_{2(k-n)})x_{2(h-1)-2(k-n)-n} + x_{2(h-1)-2(k-n)-n}^q(g_{2(k-n)}^q - g_{2(k-n)})x_n.$$

Set $a = x_n(g_{2(k-n)}^q - g_{2(k-n)})$. By (i), we know that $x_n \in \mathbb{F}_{q^2}$, and thus the above expression simplifies to

$$-a^q x_{2(h-1)-2(k-n)-n} + ax_{2(h-1)-2(k-n)-n},$$

which is exactly what we wanted to show in (b). This completes the proof. \qed

**Definition 6.15.** Let $\delta \in \ker \Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Given a tuple $(x_1, \ldots, x_{2(h-1)}) \in \mathbb{F}_q^{2(h-1)}$ together with $g, \gamma \in H(\mathbb{F}_q^2)$ satisfying the conditions of Line (14), define a tuple $(x_1', \ldots, x_{2(h-1)}')$ in the following way:

- Pick $z$ so that $z^q - z = \delta$
- Pick $y$ such that $ay^q - a^q y + \delta = 0$, where $a$ is as in Lemma 6.14.
- Set $y_{2(h-1)-2(k-n)-n} := y$ and $y_{i} = 0$ for odd such that $i < 2(h-1) - n$ and $i \neq 2(h-1) - 2(k-n) - n$. Here, $k$ is as in Lemma 6.14.
- For each odd $i$ with $i > 2(h-1) - n$, pick $y_{i}$ so that

$$y_i^q - y_i = \sum_{2m \leq i} (h_{2m}^q - g_{2m})y_{i-2m}.$$

Finally, define

$$x_i' = \begin{cases} x_i + y_i & \text{if } i \text{ is odd,} \\ x_i + z & \text{if } i = 2(h-1), \\ x_i & \text{otherwise.} \end{cases}$$

**Lemma 6.16.** Let $k$ be the smallest integer such that $h_{2k} \neq g_{2k}$ and assume that $k \leq h - 2$. Let $(x_1, \ldots, x_{2(h-1)}) \in A_{g, \gamma}$ and $\delta \in \ker \Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Then the tuple $(x_1', \ldots, x_{2(h-1)}')$ defined in Definition 6.15 is an element of $A_{g, \gamma + \delta, \tau_{2(h-1)}}$.

**Proof.** It is easy to see that $x'$ satisfies each Type $(\ast)$ equation. To see that $x'$ satisfies the Type $(\ast \ast)$ equations for the all odd $i$, amounts to checking that

$$(x_i')^q - (x_i') = x_i^q - x_i + \sum_{2m \leq i} (h_{2m}^q - g_{2m})y_{i-2m},$$

where the sum ranges over $i$ and $j$ such that $i + 2j + 2(h - 1) - 2(k - n) - n = 2(h - 1)$. Equivalently, $i + 2j = 2(k - n) + n$. Note that this forces $i$ to be odd since $n$ is odd by assumption. If $i < n$, then $x_i = 0$, and thus any terms involving $(h_i^q - g_{2j})$ for $j > 2(k - n)$ vanish. If $j < 2(k - n)$, then by assumption $h_{2j} = g_{2j} \in \mathbb{F}_q$, and so $h_i^q - g_{2j} = 0$ when $j < 2(k - n)$. Therefore the above sum simplifies to

$$x_n^q(g_{2(k-n)}^q - g_{2(k-n)})x_{2(h-1)-2(k-n)-n} + x_{2(h-1)-2(k-n)-n}^q(g_{2(k-n)}^q - g_{2(k-n)})x_n.$$

Set $a = x_n(g_{2(k-n)}^q - g_{2(k-n)})$. By (i), we know that $x_n \in \mathbb{F}_{q^2}$, and thus the above expression simplifies to

$$-a^q x_{2(h-1)-2(k-n)-n} + ax_{2(h-1)-2(k-n)-n},$$

which is exactly what we wanted to show in (b). This completes the proof. \qed
which certainly holds by the construction of \( y_i \). By Lemma 6.12, it remains only to show that \( x' \) satisfies each Type (\( \dagger \dagger \)) equation. This amounts to showing that for each \( i \),

\[
\sum_{1 \leq j \leq 2i - 1 \atop j \text{ odd}} x_j^2 \left[ \sum_{m=1}^{(2i-j-1)/2} (h_m^2 - g_{2m})x_{2i-j-2m} \right] = \sum_{1 \leq j \leq 2i - 1 \atop j \text{ odd}} (x_j')^q \left[ \sum_{m=1}^{(2i-j-1)/2} (h_m^2 - g_{2m})x_{2i-j-2m} \right]
\]

Note that by construction, if \( j \) is odd and \( j < n \), then \( x_j' = x_j = 0 \) (using the proof of Lemma 6.13(c) here). Furthermore, since we have \( h_m^2 - g_{2m} = 0 \) if \( m < k - n \), then the only potentially nonzero terms on the right-hand side are of the form \((x_j')^q (h_m^2 - g_{2m})x_{2i-j-2m}\) where \( j \geq n \), \( m \geq k - n \), and \( 2(h-1) - j - 2m \geq n \). First assume \( i < h - 1 \). Then it follows that \( j, 2i-j-2m < 2(h-1) - 2(k-n) - n \), and thus \( x_j' = x_j \) and \( x_{2i-j-2m}' = x_{2i-j-2m} \). Therefore equality holds when \( i < h - 1 \).

Finally, let \( i = h - 1 \). Then by the above analysis together with Lemma 6.14, showing \( x' \) satisfies the Type (\( \dagger \dagger \)) equation for \( 2(h - 1) \) is equivalent to showing the equality

\[
a(x_{2(h-1) - 2(k-n) - n})^q - a^q (x_{2(h-1) - 2(k-n) - n}) + a_0 + \delta = 0.
\]

But since \((x_1, \ldots, x_{2(h-1)})\) satisfies Equation (\( \dagger \dagger \)) for \( 2(h - 1) \), and since \( x'_{2(h-1) - 2(k-n) - n} = x_{2(h-1) - 2(k-n) - n} + y \) where \( a^q - a^q y + \delta = 0 \), then the above equality holds. This finishes the proof that \((x_1', \ldots, x_{2(h-1)}') \in A_{g,\gamma + \delta^2} \).

**Lemma 6.17.** There is a bijection between \( A_{g,\gamma} \) and \( A_{g,\gamma + \delta^2} \), where \( \delta \in \ker Tr_{\bar{g}^q}\).

**Proof.** Pick \( \delta \in \ker Tr_{\bar{g}^q} \). First notice is that if \( g \) and \( h \) satisfy the hypotheses of Lemma 6.13(d), then \( g \) and \( \gamma + \delta^2 \) also satisfy the hypotheses of Lemma 6.13(d). Thus, by Lemma 6.13(d), \( A_{g,\gamma} \) and \( A_{g,\gamma + \delta} \) are both empty.

From now on, we assume that \( g \) and \( h \) either satisfy the hypotheses in Lemma 6.14(a) or 6.14(b). By Lemma 6.14, the Type (\( \dagger \dagger \)) equation for \( 2(h - 1) \) is of the form \( a(x_{2(h-1) - 2(k-n) - n} - a^q x_{2(h-1) - 2(k-n) - n} + a_0 = 0 \) where \( a = x_n (g_{2(k-n)} - g_{2(k-n)}) \).

Let \((x_1, \ldots, x_{2(h-1)}) \in A_{g,\gamma} \) and let \((x_1', \ldots, x_{2(h-1)}') \) be the element of \( A_{g,\gamma + \delta^2} \) constructed in Definition 6.15. Then we have a map

\[
\varphi_{\delta} : A_{g,\gamma} \rightarrow A_{g,\gamma + \delta^2}, \quad x \mapsto x'.
\]

Now we check that \( \varphi_{\delta} \) is invertible. Using the same notation as in Definition 6.15, it is easy to see that by Lemma 6.16, setting

\[
x_i'' = \begin{cases} 
  x_i - y_i & \text{if } i \text{ is odd} \\
  x_i - z & \text{if } i = 2(h - 1) \\
  x_i & \text{otherwise}
\end{cases}
\]

defines a map

\[
\varphi_{-\delta} : A_{g,\gamma + \delta^2} \rightarrow A_{g,\gamma}, \quad x \mapsto x''
\]

wherein \( \varphi_{\delta} \) and \( \varphi_{-\delta} \) are mutual inverses. Therefore \( \varphi_{\delta} \) must be a bijection. \( \square \)

We are now ready to prove Proposition 6.11.

**Proof of Proposition 6.11.** From Lemma 6.17, \(|A_{g,\gamma}| = |A_{g,\gamma + \delta^2}|\), so

\[
\sum \chi_2(g^{-1}) \cdot \# = \sum' \chi_2(g^{-1}) \sum_{\delta \in \ker Tr} \psi(\delta) \cdot N_{g,\gamma + \delta^2} = 0,
\]

wherein \( \chi_2 \) is a nontrivial character.
where \( \sum \) and \( \sum' \) range over \( g \in H \) and \( \gamma \in H \) satisfying the condition that there exists \( k \leq h - 2 \) such that \( h_{2k} \neq g_{2k} \), and \( \sum' \) has the additional restriction that \( h_{2(h-1)} \in \mathbb{F}_q \). This completes the proof of Proposition 6.11.

Proof of Theorem 6.2. This follows directly from Proposition 6.4 and 6.11.

6.2. Proof of Theorem 6.1. Let \( \psi \) and \( \chi \in \mathcal{A}_q \) be as in the statement of the theorem. Let \( \theta \) be an arbitrary character of \( G_{h-2} \subset H \). To prove Theorem 6.1, we will compute

\[
\dim H_c^{h-1}(X_h, \overline{\mathbb{Q}_t})_{\chi, \theta} = \frac{1}{q^{h-1} \cdot q^{2(h-1)} \cdot q^h} \sum_{g \in H} \chi(g) \cdot N(g, \gamma),
\]

where the above equation follows by Lemma 2.13 of [B12] and \( N(g, \gamma) = \# \{ x \in X_h(\overline{\mathbb{F}_q}) : g \ast \text{Fr}_q^* \} = \# \mathcal{A}_{g, \gamma} \), as in Section 6.1. Since \( G_{h-2} \) is a subgroup of \( H \), then in fact \( x \in \mathcal{A}_{g, \gamma} \) if and only if \( x = (x_1, \ldots, x_{2(h-1)}) \) satisfies Equations (**) through (***) and \( \ell + 1 = 1 + \sum h_i \tau_i \).

Now comes the simplification. It is not difficult to see inductively that since \( \gamma \in G_{h-2} \), Equations (**) through (***) are equivalent to the following:

(i) For \( 1 \leq n \leq 2(h-1) \), we have \( x_n^2 - x_n = 0 \).

(ii) For \( 1 \leq k \leq h-1 \), we have \( h_{2k} = g_{2k} \).

Thus,

\[
N(g, \gamma) = \begin{cases} q^{h-1} & \text{if } g = \gamma, \\ 0 & \text{otherwise}. \end{cases}
\]

Therefore,

\[
\dim H_c^{h-1}(X_h, \overline{\mathbb{Q}_t})_{\chi, \theta} = q^{h-1} \langle \chi, \theta \rangle_{G_{h-2}}.
\]

It follows that \( H_c^{h-1}(X_h, \overline{\mathbb{Q}_t})[\chi] \) has dimension \( q^{h-1} \). Now, \( H_c^{h-1}(X_h, \overline{\mathbb{Q}_t})[\chi] \) is a representation of \( U_h^{2,2}(\mathbb{Q}_q^2) \) wherein \( H_{2(h-1)}(\mathbb{F}_q^2) \) acts by \( \psi \). Therefore, by Theorem 3.1, \( H_c^{h-1}(X_h, \overline{\mathbb{Q}_t})[\chi] \) is irreducible. This completes the proof.

7. Representations of division algebras

Throughout this section, \( \theta : L^{\times} \to \mathbb{Q}_l^{\times} \) will be a primitive character of level \( h \). Recall that \( \theta \) is primitive of level \( h \) if for each \( \gamma \in \text{Gal}(L/K) \), both \( \theta \) and \( \theta / \theta^\gamma \) have level \( h \). This induces a character \( \chi : U_L^{1} / U_L^{h} \to \mathbb{Q}_l^{\times} \) whose restriction to \( U_L^{h-1} / U_L^{h} \cong \mathbb{F}_q^2 \) has conductor \( q^2 \) and will be denoted by \( \psi \).

In this section, we use Theorem 6.3 in order to describe the representations of the division algebra \( D^{\times} := D_1^{\times} / 2 \) arising from Lusztig’s conjectural \( p \)-adic Deligne–Lusztig variety \( X \) (see [L79] and [B12]). We can write \( D = L(\Pi) / (\Pi^2 - \pi) \), where \( L(\Pi) \) is the twisted polynomial ring defined by the commutation relation \( \Pi \cdot a = \phi(a) \cdot \Pi \) (\( \phi \) is the nontrivial element of \( \text{Gal}(L/K) \)), and \( \pi \) is the uniformizer of \( L \). Write \( \mathcal{O}_D = \mathcal{O}_L(\Pi) / (\Pi^2 - \pi) \) for the ring of integers of \( D \). Define \( P_D^0 = \Pi \mathcal{O}_D \) and \( U_D^0 = 1 + P_D^0 \).

There exists a connected reductive group \( G \) over \( K \) such that \( G(K) \) is isomorphic to \( D^{\times} \), and a \( K \)-rational maximal torus \( T \subset G \) such that \( T(K) \) is isomorphic to \( L^{\times} \). We describe \( G \) more explicitly here. Let \( \hat{K}^{nr} \) be the completion of the maximal unramified extension of \( K \) and let \( \varphi \) denote the Frobenius automorphism of \( \hat{K}^{nr} \) (inducing \( x \mapsto x^q \) on the residue field). Letting \( \varpi = (0, \pi) \), the
homomorphism \( F : \text{GL}_2(\bar{K}^\nr) \to \text{GL}_2(\bar{K}^\nr) \) given by \( F(A) = \omega^{-1} A \psi \omega \) is a Frobenius relative to a \( K \)-rational structure whose corresponding algebraic group over \( K \) is \( \mathbb{G} \).

Let \( \bar{G} := \mathbb{G}(\bar{K}^\nr) = \text{GL}_2(\bar{K}^\nr) \) and \( \bar{T} := \mathbb{T}(\bar{K}^\nr) \). Let \( \mathbb{B} \subset \mathbb{G} \otimes_K \bar{K}^\nr \) be the Borel subgroup consisting of upper triangular matrices and let \( \mathbb{U} \) be its unipotent radical. Note that \( \bar{T} \) consists of all diagonal matrices and \( \bar{U} := \mathbb{U}(\bar{K}^\nr) \) consists of unipotent upper triangular matrices. Let \( \bar{U}^- \subset \text{GL}_2(\bar{K}^\nr) \) denote the subgroup consisting of unipotent lower triangular matrices.

The \( p \)-adic Deligne–Lusztig construction \( X \) for \( D^\times \) described in \([L79]\) is the quotient

\[
X := (\bar{U} \cap F^{-1}(\bar{U})) \backslash \{ A \in \text{GL}_2(\bar{K}^\nr) : F(A) A^{-1} \in \bar{U} \}.
\]

In \([B12]\) (see Section 4.2 of \textit{op. cit.}), Boyarchenko proves that \( X \) can be identified\(^2\) with the set

\[
\tilde{X} := \{ A \in \text{GL}_2(\bar{K}^\nr) : F(A) A^{-1} \in \bar{U} \cap F(\bar{U}^-) \}
\]

and describes how to define the homology groups \( H_i(\tilde{X}, \mathbb{Q}_\ell) \) (see Section 4.4 of \textit{op. cit.}). For each \( i \geq 0 \), \( H_i(\tilde{X}, \mathbb{Q}_\ell) \) inherits commuting smooth actions of \( \mathbb{G}(K) \cong D^\times \) and \( \mathbb{T}(K) \cong L^\times \). Given a smooth character \( \theta : L^\times \to \mathbb{Q}^\times_\ell \), we may consider the subspace \( H_i(\tilde{X}, \mathbb{Q}_\ell)[\theta] \subset H_i(\tilde{X}, \mathbb{Q}_\ell) \) wherein \( L^\times \) acts by \( \theta \).

Using Proposition 5.19 of \textit{op. cit.}, we can now describe the cohomology groups \( H_i(\tilde{X}, \mathbb{Q}_\ell)[\theta] \) as representations of the division algebra \( D^\times := D_{1/2}^\times \). For convenience, we restate the description given in this proposition.

- Let \( \rho_\chi \) denote the representation \( H^{h-1}_i(X_h, \mathbb{Q}_\ell)[\chi] \). (Note that by Theorem 6.3, this notation is consistent with the representation \( \rho_\chi \) introduced in Section 3.) This is a representation of \( U^{2d}_{2h}(\mathbb{F}_q) \cong U^1_D/U^2_D \).

- This extends to a representation \( \eta_\theta^\prime \) of \( O^\times_D/U^2_D \) with the property that \( \text{Tr}(\eta_\theta^\prime(\zeta)) = (-1)^{h-1} \theta(\zeta) \).

- This inflates to a representation \( \tilde{\eta}_\theta^\prime \) of \( O_D^\times \).

- This extends to a representation \( \eta_\theta^\prime \) of \( \pi^\times : O_D^\times \) via setting \( \eta_\theta^\prime(\pi) := \eta(\pi) \).

- Set \( \eta_\theta := \text{Ind}_{O_D^\times}^{D^\times}(\eta_\theta^\prime) \) and Proposition 5.19 of \([B12]\) asserts that \( H_i(\tilde{X}, \mathbb{Q}_\ell)[\theta] \cong \eta_\theta \) for \( i = h - 1 \).

Via the local Langlands and Jacquet–Langlands correspondences, there is a bijection between smooth characters of \( L^\times \) and irreducible representations of \( D^\times \). For a character \( \theta : L^\times \to \mathbb{Q}^\times_\ell \), let \( \rho_\theta \) denote the corresponding \( D^\times \)-representation. Theorem 2.6 of \([BW14]\) gives an explicit construction of \( \rho_\theta \) in the case that \( \theta \) is primitive using a geometric ingredient given by the representation \( H^1_i(X_2, \mathbb{Q}_\ell)[\psi] \) of \( U^2_{2h}(\mathbb{F}_q) \). Note that in \([BW14]\), \( X_2 \) is denoted by \( X \) and \( U^2_{2h}(\mathbb{F}_q) \) is denoted by \( U^2_{2h}(\mathbb{F}_q) \).

Our work describes a correspondence between \( L^\times \)-representations and \( D^\times \)-representations arising in Lusztig’s conjectural construction of a local analogue of Deligne–Lusztig theory. A natural question to ask is whether the map

\[
\{ \text{primitive characters of } L^\times \} \quad \xrightarrow{\theta} \quad \{ \text{irreducible representations of } D^\times \}
\]

\[\begin{array}{c}
\theta \\
\downarrow \\
H_4(\tilde{X}, \mathbb{Q}_\ell)[\theta]
\end{array}\]

\(^2\)Since we are in the situation \( n = 2 \), the subgroup \( \bar{U} \cap F^{-1}(\bar{U}) \) is actually trivial. For arbitrary \( n \), the analogous subgroup is not trivial, but then there is more substance to the identification of \( X \) with \( \tilde{X} \).
matches the correspondence given by the local Langlands and Jacquet–Langlands correspondences. It in fact does!

**Theorem 7.1.** Let \( \theta : L^\times \to \mathcal{O}_\ell^\times \) be a primitive character of level \( h \) and let \( \rho_\theta \) be the \( D^\times \)-representation corresponding to \( \theta \) under the local Langlands and Jacquet–Langlands correspondences. Then \( \operatorname{Ind}_{\ell} H \) and if follows that, viewing \( H \) to a representation

\[
\rho_\theta \cong H_{h-1}(\mathcal{X}, \mathcal{O}_\ell)[\theta].
\]

**Proof.** The first assertion is clear from Theorem 6.3 and Proposition 5.19 of [B12]. As the description in Theorem 2.6 of [BW14] depends on the parity of \( h \), we will handle the even-\( h \) and odd-\( h \) cases separately. In the case that \( h \) is odd, the heart of the proof is really in the observation that the image of \( L^\times \cdot U^h_2 \cap U_D^h \) in \( U^{2,q}_h(\mathbb{F}_q^2) \) under the surjection \( U^h_2 \to U^2_h(\mathbb{F}_q^2) \) is exactly the group \( H(\mathbb{F}_q^2) \). The case when \( h \) is even requires a bit more work as we must unravel the connection between the \( U^{2,q}_h(\mathbb{F}_q^2) \)-representations \( H \) and \( L^\times \cdot U_D^\times \cap \mathcal{O}_\ell^\times \).

Pulling back these representations to \( O^\times_D \), we see that

\[
\operatorname{Ind}_{(L^\times \cdot U^h_D) \cap U_D^h} (\eta_\theta) \cong \eta_\theta.
\]

It is sufficient to show that the traces agree on \( \zeta \). But this is easy: The representation \( \operatorname{Ind}_{(L^\times \cdot U^h_D) \cap U_D^h} (\bar{\theta}) \) is the pullback of the representation \( \operatorname{Ind}_{\zeta}(\zeta \times H^2(\mathbb{F}_q^2)) \bar{\theta} \), whose trace on \( \zeta \) is exactly \( \theta(\zeta) \) since any element \( g \in \langle \zeta \rangle \times U^2_h(\mathbb{F}_q^2) \) conjugates \( \zeta \) out of \( \langle \zeta \rangle \times H^2(\mathbb{F}_q^2) \).
It is clear that
\[ \text{Ind}_{L^\infty \cdot \mathcal{O}_D^\infty}^L(\tilde{\theta}) \cong \eta_h'. \]
Noting that \( L^\infty \cdot \mathcal{O}_D^\infty = \pi^\infty \cdot \mathcal{O}_D^\infty \), we may now conclude that
\[ \rho_\theta = \text{Ind}_{L^\infty \cdot U_D^h}^L(\tilde{\theta}) \cong \text{Ind}_{L^\infty \cdot U_D^h}^\pi \cdot \mathcal{O}_D^\infty(\eta_h') \cong H_{h-1}(\tilde{X}, \underline{\mathcal{Q}}_\ell)[\theta]. \]

Now let \( h \) be even. By Theorem 2.6 of [BW14], there is an irreducible representation \( \sigma \) of \( L^\infty \cdot U_D^{h-1} \) such that \( \text{Tr} \sigma(x) = (-1) \cdot \theta(x) \) for each very regular element \( x \in \mathcal{O}_D^\infty \) and the restriction of \( \sigma \) to \( K^\infty \cdot U_2^h : U_2^h \) is a direct sum of copies of a character that equals \( \theta \) on \( K^\infty \cdot U_1^h \) and is trivial on \( 1 + (C' \cap P_D^h) \). Then \( \rho_\theta = \text{Ind}_{L^\infty \cdot U_D^{h-1}}(\sigma) \). Just as in the odd-\( h \) case, we would like to compare \( \rho_\theta \) to the representation \( \eta_h = H_{h-1}(\tilde{X}, \underline{\mathcal{Q}}_\ell)[\theta] \).

The image of \( (L^\infty \cdot U_D^{h-1}) \cap U_2^h \) under the surjection \( U_1^h \to U_2^h \) is equal to \( H''(\mathcal{F}, \mathcal{Q}) \), where
\[ H'' := \{ 1 + \sum a_i \tau^i : i \text{ is even; or } i \geq h - 1 \} \subset U_2^h. \]
Note that \( H''(\mathcal{F}, \mathcal{Q}) \) contains \( H''(\mathcal{F}, \mathcal{Q}) \) as a degree-\( q \) subgroup.

By the proof of Theorem 2.6 of [BW14], \( \sigma \) is constructed as follows. Consider the group \( J = 1 + P_L^{h-1} + (C' \cap P_D^{h-1}) \) and \( J' = 1 + P_L^h + (C' \cap P_D^{h-1}) \). Then we have an isomorphism \( J/J' \cong U_2^h(\mathcal{F}, \mathcal{Q}) \) coming from the natural surjection \( L^\infty \to K^\infty \) and tensor this representation with \( \theta \) to obtain a representation that descends to a representation \( \sigma \) of \( L^\infty \cdot U_D^{h-1} \). The representation \( H_1^c(X_2, \underline{\mathcal{Q}}_\ell)|\psi \) is constructed as follows. Let \( \tilde{\psi} \) be any extension of \( \psi \) to \( \{ 1 + a \tau + b \tau^2 : a \in \mathcal{F}_q \} \subset U_2^h(\mathcal{F}, \mathcal{Q}) \). Then \( H_1^c(X_2, \underline{\mathcal{Q}}_\ell)|\tilde{\psi} \cong \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(\mathcal{F}, \mathcal{Q}) \) as representations of \( U_2^h(\mathcal{F}, \mathcal{Q}) \).

We can realize \( U_2^h(\mathcal{F}, \mathcal{Q}) \) as a subgroup of \( U_2^h(\mathcal{F}, \mathcal{Q}) \) via the inclusion
\[ 1 + a_{h-1} \tau + a_{2(h-1)} \tau^2 \to 1 + a_{h-1} \tau + a_{2(h-1)} \tau^2. \]
Thus \( H_1^c(X_2, \underline{\mathcal{Q}}_\ell)|\tilde{\psi} \cong \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(\mathcal{F}, \mathcal{Q}) \) and as representations of \( (L^\infty \cdot U_D^{h-1}) \cap U_2^h \),
\[ \sigma \cong \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(\mathcal{F}, \mathcal{Q}) \] and as representations of \( (L^\infty \cdot U_D^{h-1}) \cap U_2^h \),
\[ \sigma \cong \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(\mathcal{F}, \mathcal{Q}) \] Therefore,
\[ \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(U_2^h(\mathcal{F}, \mathcal{Q})) \cong U_2^h(\mathcal{F}, \mathcal{Q}) \]
By Proposition 5.19 of [B12], there exists a unique extension of \( H_1^c(X_2, \underline{\mathcal{Q}}_\ell)|\chi \) to a representation of \( \mathcal{O}_D^\infty \) characterized by \( \text{Tr}(\zeta', H_0^c(X_2, \underline{\mathcal{Q}}_\ell)|\chi) = (-1)^{h-1} \theta(\zeta) = -\theta(\zeta) \). This therefore implies that as representations of \( \mathcal{O}_D^\infty \),
\[ \text{Ind}_{U_2^h(\mathcal{F}, \mathcal{Q})}^2(U_2^h(\mathcal{F}, \mathcal{Q})) \cong \tilde{\eta}_h \]
The final conclusion is exactly the same as the argument in the \( h \)-odd case, and this completes the proof of Theorem 7.1. \( \square \)
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