The non-Urysohn number of a topological space

Ivan S. Gotchev
Department of Mathematical Sciences,
Central Connecticut State University,
New Britain, CT 06050, USA
E-mail: gotchevi@ccsu.edu

Abstract

We call a nonempty subset $A$ of a topological space $X$ finitely non-Urysohn if for every nonempty finite subset $F$ of $A$ and every family $\{U_x : x \in F\}$ of open neighborhoods $U_x$ of $x \in F$, $\cap \{\text{cl}(U_x) : x \in F\} \neq \emptyset$ and we define the non-Urysohn number of $X$ as follows: $\nu(X) := 1 + \sup\{|A| : A \text{ is a finitely non-Urysohn subset of } X\}$.

For a topological space $X$ and any subset $A$ of $X$ we prove the following inequalities: (1) $|\text{cl}_A(A)| \leq |A|^\kappa(X) \cdot \nu(X)$, (2) $|A| \leq (|A| \cdot \nu(X))^{\kappa(X)}$, (3) $|X| \leq \nu(X)^{\kappa(X)}$, and (4) $|X| \leq \nu(X)^{\kappa(X)}$.

In 1979, A. V. Arhangel’skii asked if the inequality $|X| \leq 2^{\kappa(X)}$ was true for every Hausdorff space $X$. It follows from the third inequality that the answer of this question is in the affirmative for all spaces with $\nu(X)$.

We also give a simple example of a Hausdorff space $X$ such that $|\text{cl}_A(A)| > |A|^\kappa(X)$ and $|\text{cl}_A(A)| > (|A| \cdot U(X))^{\kappa(X)}$, where $U(X)$ is the Urysohn number of $X$, recently introduced by Bonanzinga, Cammaroto and Matveev. This example shows that in (1) and (2) above, $\nu(X)$ cannot be replaced by $U(X)$ and answers some questions posed by Bella and Cammaroto (1988), Bonanzinga, Cammaroto and Matveev (2011), and Bonanzinga and Pansera (2012).

1 Introduction

Let $X$ be a topological space and for $x \in X$ let $N_x$ denote the family of all open neighborhoods of $x$ in $X$. For a nonempty subset $A$ of $X$ we denote by $U_A$ the set of all families $U = \{U_a : a \in A, U_a \in N_a\}$ and by $C_A$ the set of all families $C = \{\overline{U_a} : a \in A, U_a \in N_a\}$.

The $\theta$-closure of a set $A$ in a space $X$ is the set $\text{cl}_\theta(A) = \{x \in X : \text{for every } U \in N_x, \overline{U} \cap A \neq \emptyset\}$. A is called $\theta$-closed if $A = \text{cl}_\theta(A)$ and $A$ is $\theta$-dense if $\text{cl}_\theta(A) = X$ (see [14]). The smallest $\theta$-closed set containing $A$, i.e. the intersection of all $\theta$-closed sets containing $A$, is denoted by $|A|_\theta$ and is called the $\theta$-closed hull of $A$ [5]. The $\theta$-density of a space $X$ is $d_\theta(X) := \min\{|A| : A \subset X, \text{cl}_\theta(A) = X\}$.

Recall that a space $X$ is called Urysohn if every two distinct points in $X$ have disjoint closed neighborhoods.

Definition 1.1 ([2]). For a topological space $X$, $\kappa(X)$ is the smallest cardinal number $\kappa$ such that for each point $x \in X$, there is a collection $V_x$ of closed neighborhoods of $x$ such that $|V_x| \leq \kappa$ and if $W \in N_x$ then $\overline{W}$ contains a member of $V_x$.

Remark 1.2. In [2], $\kappa(X)$ is defined only for Hausdorff spaces but clearly $\kappa(X)$ is well-defined for every topological space $X$. Also, an example of a Urysohn space $X$ is constructed in [2] such that $\kappa(X) < \chi(X)$, where $\chi(X)$ is the character of the space $X$.

2010 Mathematics Subject Classification: Primary 54A25; Secondary 54D10

Key words and phrases: Cardinal function, $\theta$-closure, $\theta$-closed set, Urysohn number of a space, non-Urysohn number of a space, (maximal) finitely non-Urysohn subset of a space.
Definition 1.3 ([7]). The Urysohn number of a topological space \( X \), denoted by \( U(X) \), is the smallest cardinal \( \kappa \) such that for every \( A \subset X \) with \( \|A\| \geq \kappa \), there exists a family \( \mathcal{C} \in \mathcal{C}_A \) such that \( \cap \mathcal{C} = \emptyset \).

Spaces \( X \) with \( U(X) = n \) for some integer \( n \geq 2 \) appeared first in [5] and [8] under the name \( n \)-Urysohn and were studied further in [9]. In [11] such spaces were called finitely-Urysohn.

Clearly, \( U(X) \geq 2 \) and \( U(X) \leq |X|^+ \) for every topological space \( X \). If \( X \) is Hausdorff then \( U(X) \leq |X| \) and \( X \) is Urysohn if and only if \( U(X) = 2 \) [7].

2 On some questions related to the cardinality of the \( \theta \)-closed hull

It was shown in [5] Theorem 1] that for every Urysohn space \( X \), \( |\mathcal{C}| \leq |\mathcal{C}|^{\chi(X)} \) and the authors asked if that inequality holds true for every Hausdorff space (see [5] Question). In [7] the authors extended that result to all spaces with finite Urysohn number.

Theorem 2.1 ([7] Proposition 4]). For a set \( A \) in a space \( X \), if \( U(X) \) is finite then \( |\mathcal{C}| \leq |\mathcal{C}|^{\chi(X)} \).

Since the proof given in [7] does not apply for spaces with infinite Urysohn numbers the authors naturally asked the following question.

Question 2.2 ([7] Question 5]). Is it true that for a set \( A \) in a (assume Hausdorff if necessary) space \( X \), \( |\mathcal{C}| \leq |\mathcal{C}|^{\chi(X)}U(X) \)?

In [9] the authors improved the inequality in Theorem 2.1 as follows and asked if even a stronger inequality than the one in Question 2.2 holds true.

Theorem 2.3 ([9] Proposition 7]). For a set \( A \) in a space \( X \), if \( U(X) \) is finite then \( |\mathcal{C}| \leq |\mathcal{C}|^{\kappa(X)} \).

Question 2.4 ([9] Question 9]). Is it true that for a set \( A \) in a (Hausdorff) space \( X \), \( |\mathcal{C}| \leq |\mathcal{C}|^{\kappa(X)}U(X) \)?

The following example shows that the answer of Question 2.2 (and therefore of the other two questions) is in the negative even for Hausdorff spaces with Urysohn numbers \( U(X) = \omega \). Even more, our example shows that for Hausdorff spaces it is even possible that \( |\mathcal{C}| > |\mathcal{C}|^{\chi(X)}U(X) \) and \( |\mathcal{C}| > |\mathcal{C}|^{\chi(X)}U(X) \).

Example 2.5. Let \( \mathbb{N} \) denote the set of all positive integers, for \( m \in \mathbb{N} \) let \( \mathbb{N}_m := \{n : n \in \mathbb{N}, n \leq m\} \), \( \mathbb{R} \) be the set of all real numbers, and \( \mathcal{C} = |\mathcal{R}| \). Let also \( S := \{1/n : n \in \mathbb{N}\} \cup \{0\} \) and \( \mathbb{N} \times S \) be the subspace of \( \mathbb{R} \times \mathbb{R} \) with the inherited topology from \( \mathbb{R} \times \mathbb{R} \). Let \( \alpha \) be an initial ordinal and \( |B_\beta| \) be a family of \( \alpha \) many pairwise disjoint copies of \( \mathbb{N} \times S \). We will refer to the points in \( B_\beta \) as \( (n, r) \), where \( n \in \mathbb{N} \) and \( r \in S \). For each ordinal number \( \beta < \alpha \), let \( M_\beta := B_\beta \cup \{\beta\} \) be the topological space with a topology such that \( \{\beta\} \) is closed in \( M_\beta \), all points in \( B_\beta \) have the topology inherited from \( \mathbb{R} \times \mathbb{R} \) and the point \( \beta \) has basic neighborhoods all sets of the form \( \{\beta\} \cup \{m \times (S \setminus \{0\})\}_{m \in \mathbb{N}} \). Now, let \( X \) be the topological space obtained from the disjoint union of all spaces \( M_\beta \), \( \beta < \alpha \) after identifying for each \( n \in \mathbb{N} \) all points of the form \( (n, 0) \beta \), \( \beta < \alpha \). We will denote those points by \( (n, 0) \). Then it is not difficult to verify that \( X \) is a Hausdorff space (but not Urysohn) with Urysohn number \( U(X) = \omega \), \( \chi(X) = \kappa(X) = \omega \), and if \( A \) is the subset \( \{(n, 0) : n \in \mathbb{N}\} \) of \( X \) then \(|\mathcal{C}| = |\mathcal{C}| = \kappa \) and \(|\mathcal{C}|^{\chi(X)}U(X) = |\mathcal{C}|^{\chi(X)}U(X) = \omega^\omega \cdot \omega = \mathcal{C} \).

Therefore if \( \alpha > \mathcal{C} \) then we have \( |\mathcal{C}| > |\mathcal{C}|^{\chi(X)}U(X) \) and \( |\mathcal{C}| > (|\mathcal{C}| \cdot U(X))^{\chi(X)} \).

3 Spaces with finite versus spaces with infinite Urysohn numbers

We begin with the following lemma.

Lemma 3.1. Let \( A \) be a nonempty subset of a topological space \( X \) such that \( \cap \mathcal{C} \neq \emptyset \) for every \( \mathcal{C} \in \mathcal{C}_A \). Then \( A \subset \cap\{\mathcal{C} \cap \mathcal{C} : \mathcal{C} \in \mathcal{C}_A\} \).

Proof. Let \( \mathcal{C}_0 \in \mathcal{C}_A \) and let \( G = \cap \mathcal{C}_0 \neq \emptyset \). Suppose that there exists \( a_0 \in A \) such that \( a_0 \notin \mathcal{C}(G) \). Then there is \( W_{a_0} \in \mathcal{N}_{a_0} \) such that \( W_{a_0} \cap G = \emptyset \). Let \( \mathcal{U}_{a_0} := \overline{W}_{a_0} \cap \mathcal{W}_{a_0} \), where \( \overline{W}_{a_0} \in \mathcal{C}_0 \) and \( \mathcal{U}_{a_0} \in \mathcal{N}_{a_0} \). Then the family \( \mathcal{C}_1 := \{\mathcal{U}_{a_0}\} \cup \{\mathcal{U}_a : a \in \mathcal{C}_0, a \in A \} \{\mathcal{U}_{a_0}\} \) has the property that \( \cap \mathcal{C}_1 = \emptyset \), a contradiction. Therefore \( A \subset \mathcal{C}(\cap \mathcal{C}) \) for every \( \mathcal{C} \in \mathcal{C}_A \), hence \( A \subset \cap\{\mathcal{C} \cap \mathcal{C} : \mathcal{C} \in \mathcal{C}_A\} \).
Theorem 3.2. Let $X$ be a topological space and $1 < n < \omega$. Then $U(X) = n$ if and only if there exists a set $A \subset X$ with $|A| = n - 1$ such that $A = \cap \{\text{cl}(\cap C) : C \in C_A\}$ and $\cap \{\text{cl}(\cap C) : C \in C_B\} = \emptyset$ for every set $B$ with $|B| \geq n$.

Proof. Suppose first that there exists a subset $A$ of $X$ with $|A| = n - 1 \geq 1$, such that $A = \cap \{\text{cl}(\cap C) : C \in C_A\}$ and $\cap \{\text{cl}(\cap C) : C \in C_B\} = \emptyset$ for every set $B$ with $|B| \geq n$. Then $A \subseteq \text{cl}(\cap C)$ for every $C \in C_A$. Hence $\cap C \neq \emptyset$ for every $C \in C_A$ and therefore $U(X) > |A| = n - 1$. Thus $U(X) > n$. Then there exists a set $B \subset X$ with $|B| = n$ such that $\cap C \neq \emptyset$ for every $C \in C_B$. Then it follows from Lemma 3.3 that $B \subset \cap \{\text{cl}(\cap C) : C \in C_B\} = \emptyset$, a contradiction. Therefore $U(X) = n$.

Now let $U(X) = n > 1$. Then for every set $B \subset X$ such that $|B| = n$ there exists $C \in C_B$ such that $\cap C = \emptyset$ and therefore $\cap \{\text{cl}(\cap C) : C \in C_B\} = \emptyset$. Also, there exists a set $A$ with $|A| = n - 1$ such that for every $C \in C_A$ we have $\cap C \neq \emptyset$. Then it follows from Lemma 3.3 that $A \subseteq \cap \{\text{cl}(\cap C) : C \in C_A\}$. To show that $A = \cap \{\text{cl}(\cap C) : C \in C_A\}$, suppose that there is $x \in \cap \{\text{cl}(\cap C) : C \in C_A\}$. Therefore $U \cap (\cap C) \neq \emptyset$ for every $U \in \mathcal{N}_x$ and every $C \in C_A$. Then for the set $B := A \cup \{x\}$ we have that if $C' \in C_B$ then $\cap C' \neq \emptyset$. Thus $U(X) > |B| = n$, a contradiction.

Remark 3.3. Consider the sets $A := \{(n, 0) : n < \omega\} \subset X$ and $\alpha \subset X$ in Example 2.2 and let $B_1$ and $B_1$ be a nonempty finite subset and an infinite subset of $\alpha$, respectively. If $C \subset C_B$ then $\cap C \subset A$ and $B_1 \subset \cap \{\text{cl}(\cap C) : C \in C_B\} = \text{cl}(A) = \alpha$, while there is $C \in C_B$ such that $\cap C = \emptyset$, hence $\cap \{\text{cl}(\cap C) : C \in C_B\} = \emptyset$. Therefore the space $X$ in Example 2.2 shows that Theorem 3.2 is not always valid when $U(X)$ is infinite even for Hausdorff spaces $X$, or in other words, the subsets in spaces with finite and infinite Urysohn numbers that determine the Urysohn number have different properties. Therefore we should not be surprised that theorems which are valid for spaces with finite Urysohn numbers are not necessarily valid for spaces with infinite Urysohn numbers (see Section 2).

The following two observations are valid for topological spaces with finite or infinite Urysohn numbers.

Lemma 3.4. Let $X$ be a topological space and $A$ be a nonempty subset of $X$. If $\cap C \neq \emptyset$ for every $C \in C_F$ and every finite nonempty subset $F$ of $A$ then $A \subseteq \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A, |F| < \omega\}$. 

Proof. Let $F$ be a nonempty subset of $A$, $C_0 \in C_F$, and $G = \cap C_0$. Suppose that there exist $a_0 \in A$ such that $a_0 \notin \text{cl}(G)$. Then there is $W_{a_0} \in \mathcal{N}_{a_0}$ such that $W_{a_0} \cap G = \emptyset$. Let $W_{a_0} := \cap W_{a_0}$ if $a_0 \notin F$ and $W_{a_0} := U_{a_0} \cap W_{a_0}$ if $a_0 \in F$, where $U_{a_0} \in \mathcal{N}_{a_0}$ and $U_{a_0} \in \mathcal{N}_{a_0}$. Then the family $C_1 := \{W_{a_0} \cup \{U_{a_0} : U_{a_0} \in C_{a_0}, a \in F, \emptyset \neq U_{a_0}\}\}$ has the property that $\cap C_1 = \emptyset$, a contradiction. Therefore $A \subseteq \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A, |F| < \omega\}$. 

Theorem 3.5. Let $X$ be a topological space and $A$ be a nonempty subset of $X$. If $\cap C \neq \emptyset$ for every $C \in C_F$ and every nonempty finite subset $F$ of $A$ then there exists a subset $M$ of $X$ such that $A \subseteq M$ and $M = \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset M, |F| < \omega\}$. 

Proof. Let $\alpha$ be an initial ordinal such that $\alpha = |X|^+$ and let $A$ satisfies the hypotheses of our claim. Then it follows from Lemma 3.3 that $A \subseteq \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A, |F| < \omega\}$. Suppose that there is $x_0 \in \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A, |F| < \omega\} \setminus A$. Then $U \cap (\cap C) \neq \emptyset$ for every $U \in \mathcal{N}_{a_0}$, every $C \in C_F$ and every nonempty finite subset of $A$. Then for the set $A_1 := A \cup \{x_0\}$ we have that if $F$ is a nonempty finite subset of $A_1$ and $C \in C_F$ then $\cap C \neq \emptyset$. Then, according to Lemma 3.3, $A_1 \subset \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A_1, |F| < \omega\}. \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A_1, |F| < \omega\} \setminus A_1$. Define $A_\beta := A_\gamma \cup \{x_\gamma\}$. If $\beta = \gamma + 1$ for some $\gamma < \beta$ and $A_\gamma \subseteq \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A_\gamma, |F| < \omega\}$ then we choose $x_\gamma \in \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A_\gamma, |F| < \omega\} \setminus A_\gamma$ and define $A_\beta := A_\gamma \cup \{x_\gamma\}$. If $\beta$ is a limit ordinal then $A_\beta := \bigcup \{A_\gamma : \gamma < \beta\}$. In that case it is clear that $\cap C \neq \emptyset$ for every $C \in C_F$ and every nonempty finite subset of $A_\beta$ since every such $F$ is a subset of $A_\beta$ and $\alpha \neq |X|$. Therefore, according to Lemma 3.3, we have $A_\beta \subset \cap \{\text{cl}(\cap C) : C \in C_F, \emptyset \neq F \subset A_\beta, |F| < \omega\}$. Then we stop and take $M$ to be $A_\beta$. This process will eventually stop since for each $\gamma < \beta < \alpha$ we have $A_\gamma \subseteq A_\beta \subseteq X$ and $\alpha \neq |X|$. \qed
4 The non-Urysohn number of a space

Motivated by the observations in the previous section we give the following definition.

Definition 4.1. A nonempty subset $A$ of a topological space $X$ is called finitely non-Urysohn if for every nonempty finite subset $F$ of $A$ and every $C \in \mathcal{C}_F$, $\cap C \neq \emptyset$. $A$ is called maximal finitely non-Urysohn subset of $X$ if $A$ is a finitely non-Urysohn subset of $X$ and if $B$ is a finitely non-Urysohn subset of $X$ such that $A \subset B$ then $A = B$.

Remark 4.2. (a) Using Lemma 3.3 and Definition 4.1 one can easily verify that a nonempty subset $M$ of a topological space is maximal finitely non-Urysohn if and only if $M = \cap \{\text{cl}_x(\cap C) : C \in \mathcal{C}_F, \emptyset \neq F \subset M, |F| < \omega\}$.

(b) It follows from Theorem 3.1 and Remark 4.2(a) that every finitely non-Urysohn subset of a topological space is contained in a maximal one.

(c) Using disjoint union of spaces as those constructed in Example 2.5 one can construct a Hausdorff topological space with (disjoint) maximal finitely non-Urysohn subsets with different cardinality.

(d) In a Urysohn space $X$ the only (maximal) finitely non-Urysohn subsets of $X$ are the singletons.

Now we are ready to introduce the concept of a non-Urysohn number of a topological space $X$.

Definition 4.3. Let $X$ be a topological space. We define the non-Urysohn number $nu(X)$ of $X$ as follows: $nu(X) := 1 + \sup\{|M| : M$ is a (maximal) finitely non-Urysohn subset of $X\}$.

Remark 4.4. It follows from Theorem 3.2 and Definition 4.3 that if $X$ is a topological space then $nu(X) = U(X)$. Also, $nu(X) \geq 2$ and $nu(X) \geq U(X)$ for every topological space $X$.

5 On the cardinality of the $\theta$-closed hull

In Theorem 4.1 using the cardinal invariant non-Urysohn number of a space, we give an upper bound for $|\text{cl}_\theta(A)|$ and $|A|$ of a subset $A$ in a topological space $X$. That theorem generalizes simultaneously all the results included in Theorem 3.1. The proof of Theorem 4.2 follows proofs given in [5], [2], [8], [7] or [9].

Theorem 5.1. Let $X$ be a space and $A \subset X$.

(a) If $X$ is Urysohn then $|A|_\theta \leq |A|^{\kappa(X)}$ [2];

(b) If $X$ is Urysohn then $|\text{cl}_\theta(A)| \leq |A|^{\kappa(X)}$ [2];

(c) If $U(X)$ is finite then $|A|_\theta \leq |A|^{\kappa(X)}$ [8], [7];

(d) If $U(X)$ is finite then $|A|_\theta \leq |A|^{\kappa(X)}$ [2].

Theorem 5.2. Let $A$ be a subset of a topological space $X$. Then $|\text{cl}_\theta(A)| \leq |A|^{\kappa(X)} \cdot nu(X)$ and $|A|_\theta \leq (|A| \cdot nu(X))^{\kappa(X)}$.

Proof. Let $\kappa(X) = m$, $nu(X) = u$, and $|A| = r$. For each $x \in X$ let $\mathcal{V}_x$ be a collection of closed neighborhoods of $x$ with $|\mathcal{V}_x| \leq m$ and such that if $W$ is a closed neighborhood of $x$ then $W$ contains a member of $\mathcal{V}_x$. For every $x \in \text{cl}_\theta(A)$ and every $V \in \mathcal{V}_x$, fix a point $a_{x,V} \in V \cap A$, and let $A_x := \{a_{x,V} : V \in \mathcal{V}_x\}$. Let also $\Gamma_x := \{V \cap A_x : V \in \mathcal{V}_x\}$. Then $\Gamma_x$ is a centered family (the intersection of any finitely many elements of $\Gamma_x$ is nonempty). It is not difficult to see that there are at most $\tau^n$ such centered families. Indeed $A_x \in |A|^{\leq m}$ and $V \cap A_x \in |A|^{\leq m}$, for every $V \in \mathcal{V}_x$. Since each centered family $\Gamma_x$ is a subset of $|A|^{\leq m}$ and $|\Gamma_x| \leq m$, the cardinality of the set of all such families is at most $|A|^{(\leq m)} = |A| = \tau^m$.

We claim that the mapping $x \mapsto \Gamma_x$ is $(\leq u)$-to-one. Assume the contrary. Then there is a subset $K \subset \text{cl}_\theta(A)$ such that $|K| = u^+$ and every $x \in K$ corresponds to the same centered family $\Gamma$. Since $\nu(X) = u$, there exists a nonempty finite subset $F$ of $K$ and $C \in \mathcal{C}_F$ such that $\cap C = \emptyset$. Then for every $x \in F$ and $U_x \in C$ we have $U_x \cap A_x \in \Gamma$; hence $\Gamma$ is not centered, a contradiction.
Therefore the mapping \( x \to \Gamma_x \) from \( \text{cl}_\theta(A) \) to \( |A|^{\leq m}_{\leq m} \) is \((u \leq u)\)-to-one, and thus
\[
|\text{cl}_\theta(A)| \leq u \cdot (\tau^m)^m = u \cdot \tau^m
\]   
(1)

(Note that the proof that the mapping \( x \to \Gamma_x \) is \((u \leq u)\)-to-one does not depend upon the cardinality of the set \( A \)).

It is not difficult to see (e.g., as in the proof of Theorem 1 in \[3\]) that if we set \( A_0 = A \) and \( A_\alpha = \text{cl}_\theta(\bigcup_{\beta < \alpha} A_\beta) \) for all \( 0 < \alpha \leq m^+ \), then \( |A| = A_{m^+} \). Let \( \kappa = u \cdot \tau \). It follows from (1) that \( |A_2| \leq u \cdot (u \cdot \tau^m)^m = u \cdot \tau^m = (u \cdot \tau)^m = \kappa^m \).

To finish the proof we will show that if \( \alpha \) is such that \( 2 \leq \alpha < m^+ \) then \( |A_\alpha| \leq \kappa^m \), and therefore \( |A_\alpha| \leq \kappa^m \).

Suppose that \( \alpha_0 \leq m^+ \) is the smallest ordinal such that \( |A_\alpha_0| > \kappa^m \). Then we have \( |A_\beta| \leq \kappa^m \) for each \( \beta < \alpha_0 \) and therefore \( |\bigcup_{\beta < \alpha_0} A_\beta| \leq \kappa^m \cdot m^+ = \kappa^m \). Now using (1) again we get \( |A_\alpha_0| \leq u \cdot (\kappa^m)^m = \kappa^m \), a contradiction.

**Corollary 5.3.** If \( X \) is a topological space then \( |X| \leq (d_\theta(X))^\kappa(X) \cdot nu(X) \).

**Remark 5.4.** Example 2.5 shows that the inequality \( |\text{cl}_\theta(A)| \leq |A|^\kappa(X) \cdot \nu(X) \) in Theorem 5.2 is exact. To see that, let \( \alpha \geq \kappa \) and take \( A = \{(n,0) : n \in \mathbb{N}\} \subset X \) and \( M = \{\beta : \beta < \alpha\} \subset X \). Then \( \text{cl}_\theta(A) = A \cup M \) and therefore \( |\text{cl}_\theta(A)| = \alpha \leq |A|^{\kappa(X)} \cdot \nu(X) = \omega^\omega \cdot \alpha \).

To see that the inequality \( |A| \leq (|A| \cdot \nu(X))^\kappa(X) \) in Theorem 5.2 is also exact, one can construct a Hausdorff (non-Urysohn) space \( Y \) and a set \( A \subset Y \) with \( |A| = (|A| \cdot \nu(Y))^\kappa(Y) \) as follows. Take the space \( X_1 := X \) from Example 2.5 and the set \( M = \{\beta : \beta < \alpha\} \subset X_1 \). Represent the set \( M \) as a disjoint union \( \bigcup_{\beta < \alpha} M_\beta \) of countable infinite subsets of \( M \). Take \( \alpha \) many disjoint copies \( X_\beta : \beta < \alpha \) of the space \( X \) and for each \( \beta < \alpha \) identify the points of \( A_\beta \) with the points \( \{(n,0) : n < \omega\} \subset X_\beta \). Call the resulting space \( X_2 \).

Now \( X_1 \subset X_2 \) and in \( X_2 \) we have \( \alpha \) many new copies of the set \( M \). For each such set repeat the previous procedure to obtain the space \( X_3 \) and continue this procedure for each \( n < \omega \). Call the resulting space \( Y \). It is not difficult to see that \( U(Y) = \omega \), \( \chi(Y) = \kappa(Y) = \omega \), \( \nu(Y) = \alpha \), and if \( A \) is the subset \( \{(n,0) : n \in \mathbb{N}\} \) of \( X_1 \) then \( |A| = \alpha^\omega = (\omega \cdot \alpha)^\omega = (|A| \cdot \nu(Y))^\kappa(Y) \). Notice that if \( \alpha > \kappa \) is chosen to be a cardinal with a countable cofinality then \( |A| = \alpha^\omega > \alpha = \omega^\omega \cdot \alpha = |A|^\kappa(Y) \cdot \nu(Y) \) and therefore the right-hand side of the second inequality cannot be replaced by the right-hand side of the first inequality.

6 Some cardinal inequalities involving the non-Urysohn number

We recall some definitions.

**Definition 6.1 (\[15, 13\]).** The almost Lindelöf degree of a space \( X \) with respect to closed sets is \( aL_e(X) := \sup\{aL_e(F, X) : F \text{ is a closed subset of } X\} \), where \( aL_e(F, X) \) is the minimal cardinal number \( \tau \) such that for every open (in \( X \)) cover \( U \) of \( F \) there is a subfamily \( U_0 \subset U \) such that \( |U_0| \leq \tau \) and \( F \subset \bigcup\{U : U \in U_0\} \). \( aL_e(X, X) \) is called almost Lindelöf degree of \( X \) and is denoted by \( aL(X) \).

**Remark 6.2.** The cardinal function \( aL_e(X) \) was introduced in \[15\] under the name Almost Lindelöf Degree and was denoted by \( aL(X) \). Here we follow the notation and terminology used in \[13\] and suggested in \[3\].

**Definition 6.3 (\[1\]).** The cardinal function \( wL_e(X) \) is the smallest cardinal \( \tau \) such that if \( A \subset X \) is a closed subset of \( X \) and \( U \) is an open (in \( X \)) cover of \( A \), then there exists \( V \subset U \) with \( |V| \leq \tau \) such that \( A \subset \overline{V} \).

**Definition 6.4 (\[4\]).** The cardinal function \( sL(X) \) is the smallest cardinal \( \tau \) such that if \( A \subset X \) and \( U \) is an open (in \( X \)) cover of \( A \), then there exists \( V \subset U \) with \( |V| \leq \tau \) such that \( A \subset \overline{V} \).

**Definition 6.5 (\[2\]).** The cardinal function \( sL_0(X) \) is the smallest cardinal \( \tau \) such that if \( A \subset X \) and \( U \) is an open (in \( X \)) cover of \( \text{cl}_\theta(A) \), then there exists \( V \subset U \) with \( |V| \leq \tau \) such that \( A \subset \overline{V} \).

Clearly \( sL_0(X) \leq sL(X) \leq wL_e(X) \leq L(X) \) and \( aL_e(X) \leq aL(X) \leq L(X) \), where \( L(X) \) is the Lindelöf degree of \( X \). In \[2\] an example of a Urysohn space \( X \) is constructed such that \( sL_0(X) < sL(X) \). For examples of Urysohn spaces such that \( aL(X) < wL_e(X) \) or \( wL_e(X) < aL_e(X) \) see \[13\] and for an example of a Urysohn space for which \( aL(X) < aL_e(X) < L(X) \) see \[15\] or \[13\].

Here are some cardinal inequalities that involve the cardinal functions defined above. For more related results see the survey paper \[13\].
Theorem 6.6. 
(a) If $X$ is a Hausdorff space, then $|X| \leq 2^{\chi(X)L(X)}$. 
(b) If $X$ is a Hausdorff space, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(c) If $X$ is a Urysohn space, then $|X| \leq 2^{\chi(X)aL(X)}$. 
(d) If $X$ is a Hausdorff space with $U(X) < \omega$, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(e) If $X$ is a Urysohn space, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(f) If $X$ is a topological space with $U(X) < \omega$, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(g) If $X$ is a Hausdorff space, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(h) If $X$ is a Urysohn space, then $|X| \leq 2^{\chi(X)aL_c(X)}$. 
(i) If $X$ is a topological space with $U(X) < \omega$, then $|X| \leq 2^{\chi(X)aL_c(X)}$.

Recently in [9], after proving the inequality given in Theorem 6.6(i), the authors asked the following question.

Question 6.7 ([9 Question 11]). Can one conclude that the inequality 

$$|X| \leq U(X)^{\kappa(X)sL_\theta(X)}$$

is true for every Hausdorff space $X$?

We show below that the answer of the above question is in the affirmative if the Urysohn number $U(X)$ is replaced by the non-Urysohn number $nu(X)$.

Theorem 6.8. For every topological space $X$, $|X| \leq nu(X)^{\kappa(X)sL_\theta(X)}$.

Proof. Let $\kappa(X)sL_\theta(X) = m$ and $nu(X) = u$. For each $x \in X$ let $V_x$ be a collection of closed neighborhoods of $x$ with $|V_x| \leq m$ and such that if $W$ is a closed neighborhood of $x$ then $W$ contains a member of $V_x$. Let $x_0$ be an arbitrary point in $X$. Recursively we construct a family $\{F_\alpha : \alpha < m^+\}$ of subsets of $X$ with the following properties:

(i) $F_0 = \{x_0\}$ and $cl_\theta(\cup_{\beta<\alpha}F_\beta) \subset F_\alpha$ for every $0 < \alpha < m^+$;
(ii) $|F_\alpha| \leq u^m$ for every $\alpha < m^+$;
(iii) for every $\alpha < m^+$, and every $F \subset cl_\theta(\cup_{\beta<\alpha}F_\beta)$ with $|F| \leq m$ if $X \setminus \bigcup \mathcal{U} \neq \emptyset$ for some $\mathcal{U} \subset \mathcal{U}_F$, then $F_\alpha \setminus \bigcup \mathcal{U} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define $F_\alpha$. Since $|F_{\beta}| \leq u^m$ for each $\beta < \alpha$, we have $|\cup_{\beta<\alpha}F_{\beta}| \leq u^m \cdot m^+ = u^m$. Then it follows from Theorem 6.2.2 that $|\cup_{\beta<\alpha}F_{\beta}| \leq u^m$. Therefore there are at most $u^m$ subsets $F$ of $cl_\theta(\cup_{\beta<\alpha}F_{\beta})$ with $|F| \leq m$ and for each such set $F$ we have $|\cup_{U \subset \mathcal{U}}| \leq m^m = 2m \leq u^m$. For each $F \subset cl_\theta(\cup_{\beta<\alpha}F_{\beta})$ with $|F| \leq m$ and each $U \subset \mathcal{U}_F$ for which $X \setminus \bigcup \mathcal{U} \neq \emptyset$ we choose a point in $X \setminus \bigcup \mathcal{U}$ and let $E_\alpha$ be the set of all these points. Clearly $|E_\alpha| \leq u^m$. Let $F_{\beta} = cl_\theta(E_{\alpha} \cup (\cup_{\beta<\alpha}F_{\beta}))$. Then it follows from our construction that $F_\alpha$ satisfies (i) and (ii) while (iii) follows from Theorem 6.2.2.

Now let $G = \cup_{\alpha < m^+}F_\alpha$. Clearly $|G| \leq u^m \cdot m^+ = u^m$. We will show that $G$ is $\theta$-closed. Suppose the contrary and let $x \in cl_\theta(G) \setminus G$. Then for each $U \subset \mathcal{V}_x$ we have $U \cap G \neq \emptyset$ and therefore there is $\alpha_U < m^+$ such that $U \cap F_{\alpha_U} \neq \emptyset$. Since $|\{\alpha_U : U \subset \mathcal{V}_x\}| \leq m$, there is $\beta < m^+$ such that $\beta > \alpha_U$ for every $U \subset \mathcal{V}_x$ and therefore $x \in cl_\theta(F_{\beta}) \subset G$, a contradiction.

To finish the proof it remains to check that $G = X$. Suppose that there is $x \in X \setminus G$. Then there is $V \subset \mathcal{V}_x$ such that $V \cap G = \emptyset$. Hence, for every $y \in G$ there is $V_y \subset \mathcal{V}_y$ such that $V_y \cap Int(V) = \emptyset$. Since $\{Int(V_y) : y \in G\}$ is an open cover of $G$ and $G$ is $\theta$-closed, there is $F \subset G$ with $|F| \leq m$ such that $G \subset \cup_{y \in F}Int(V_y)$. Since $|F| \leq m$, there is $\beta < m^+$ such that $F \subset F_\beta$. Then for $\mathcal{U} := \{Int(V_y) : y \in F\}$ we have $\mathcal{U} \subset \mathcal{U}_F$ and $x \in X \setminus \bigcup \mathcal{U}$. Then it follows from our construction that $F_{\beta+1} \setminus \bigcup \mathcal{U} \neq \emptyset$, a contradiction since $F_{\beta+1} \subset G \subset \overline{\mathcal{U}}$. \qed
Corollary 6.9. For every topological space $X$, $|X| \leq \nu(X)^{2\omega L(X)}$.

Remark 6.10. In 1979, A. V. Arhangel’skiı̆ asked if the inequality $|X| \leq 2^{\omega L(X)}$ was true for every Hausdorff topological space $X$ (see [13, Question 2]). It follows immediately from Corollary 6.9 that the answer of his question is in the affirmative for all spaces with $\nu(X) \leq 2^\omega$. But as Example 3.10 in [12] shows, there are $T_0$-topological spaces for which that inequality is not true (in that example $\nu(X) > 2^\omega$).

Modifying slightly the proof of Theorem 6.8 one can prove the following result.

Theorem 6.11. For every topological space $X$, $|X| \leq \nu(X)^{\kappa(X)aL(X)}$.

Proof. Let $\kappa(X)aL(X) = m$ and $\nu(X) = u$. For each $x \in X$ let $V_x$ be a collection of closed neighborhoods of $x$ with $|V_x| \leq m$ and such that if $W$ is a closed neighborhood of $x$ then $W \cap V_x$ contains a member of $V_x$. Let $x_0$ be an arbitrary point in $X$. Recursively we construct a family $\{F_\alpha : \alpha < m^+\}$ of subsets of $X$ with the following properties:

(i) $F_0 = \{x_0\}$ and $\text{cl}_0(\cup_{\beta < \alpha} F_\beta) \subset F_\alpha$ for every $0 < \alpha < m^+$;

(ii) $|F_\alpha| \leq u^m$ for every $\alpha < m^+$;

(iii) for every $\alpha < m^+$, and every $F \subset \text{cl}_0(\cup_{\beta < \alpha} F_\beta)$ with $|F| \leq m$ if $X \setminus \cup\mathcal{C} \neq \emptyset$ for some $\mathcal{C} \subset \mathcal{C}_F$, then $F_\alpha \setminus \cup\mathcal{C} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define $F_\alpha$. Since $|F_\beta| \leq u^m$ for each $\beta < \alpha$, we have $|\cup_{\beta < \alpha} F_\beta| \leq u^m \cdot m^+ = u^m$. Then it follows from Theorem 5.2 that $|\cup_{\beta < \alpha} F_\beta| \leq u^m$. Therefore there are at most $u^m$ subsets $F$ of $\text{cl}_0(\cup_{\beta < \alpha} F_\beta)$ with $|F| \leq m$ and for each such set $F$ we have $|F| \leq m^m = 2^m \leq u^m$. For each $F \subset \text{cl}_0(\cup_{\beta < \alpha} F_\beta)$ with $|F| \leq m$ and each $\mathcal{C} \subset \mathcal{C}_F$ for which $X \setminus \cup\mathcal{C} \neq \emptyset$ we choose a point $t \in X \setminus \cup\mathcal{C} \neq \emptyset$ and let $E_\alpha$ be the set of all these points. Clearly $|E_\alpha| \leq u^m$. Let $F_\alpha = \text{cl}_0(E_\alpha \cup (\cup_{\beta < \alpha} F_\beta))$. Then it follows from our construction that $F_\alpha$ satisfies (i) and (iii) while (ii) follows from Theorem 5.2.

Now let $G = \cup_{\alpha < m^+} F_\alpha$. Clearly $|G| \leq u^m \cdot m^+ = u^m$. We will show that $G$ is $\theta$-closed. Suppose the contrary and let $x \in \text{cl}_0(G) \setminus G$. Then for each $U \subseteq V_x$ we have $U \cap G \neq \emptyset$ and therefore there is an $U \cap F_\alpha \neq \emptyset$. Since $|\{U \cap V_x\}| \leq m$, there is $\beta < m^+$ such that $\beta > \alpha_U$ for every $U \subseteq V_x$ and therefore $x \in \text{cl}_0(F_\beta) \subset G$, a contradiction.

To finish the proof it remains to check that $G = X$. Suppose that there is $x \in X \setminus G$. Then there is $V \in \mathcal{V}_x$ such that $V \cap G = \emptyset$. Hence for every $y \in G$ there is $V_y \subseteq V_x$ such that $V_y \cap \text{Int}(V) = \emptyset$ and for every $z \in (X \setminus \{x\}) \setminus V_x$ there is $V_z \subseteq V_y$ such that $V_z \cap G = \emptyset$. Since $\{\text{Int}(V_y) : y \in G\} \cup \{\text{Int}(V_z) : z \in (X \setminus \{x\}) \setminus G\} \cup \{\text{Int}(V)\}$ is an open cover of $X$, there is $F' \subset X$ with $|F'| \leq m$ such that $X \subseteq \cup_{u \in F'} V_U$. Let $F := F' \cap G \neq \emptyset$. Then $G \subset \cup \{V_y : y \in F\}$. Since $|F| \leq m$, there is $\beta < m^+$ such that $F \subset F_\beta$. Then for $C := \{V_y : y \in F\}$ we have $C \subset \mathcal{C}_F$ and $x \in X \setminus \cup\mathcal{C}$. Then it follows from our construction that $F_{\beta + 1} \setminus \cup\mathcal{C} \neq \emptyset$, a contradiction since $F_{\beta + 1} \subset G \subset \cup\mathcal{C}$.

Corollary 6.12. For every topological space $X$ with $\nu(X) < \omega$ (or, equivalently, $U(X) < \omega$), $|X| \leq 2^{\kappa(X)aL(X)}$.

Corollary 6.13. For every Urysohn space $X$, $|X| \leq 2^{\kappa(X)aL(X)}$.

Remark 6.14. In parallel with Definition 6.8 one can introduce the notion of a non-Hausdorff number of a topological space. For results related to that notion see [12].

References

[1] Alas O. T., More topological cardinal inequalities, Colloq. Math. 65 (1993), no. 2, 165–168
[2] Alas O. T., Kočinac Lj. D., More cardinal inequalities on Urysohn spaces, Math. Balkanica (N.S.) 14 (2000), no. 3-4, 247–251
[3] Arhangel’skiı̆ A. V., The power of bicompacta with first axiom of countability, Dokl. Akad. Nauk SSSR 187 (1969), 967–970
[4] Arhangel’škiĭ, A. V., *A generic theorem in the theory of cardinal invariants of topological spaces*, Comment. Math. Univ. Carolin. 36 (1995), no. 2, 303–325

[5] Bella A., Cammaroto F., *On the cardinality of Urysohn spaces*, Canad. Math. Bull. 31 (1988), no. 2, 153–158

[6] Bonanzinga M., Cammaroto F., Matveev M. V., *On a weaker form of countable compactness*, Quaest. Math. 30 (2007), no. 4, 407–415

[7] Bonanzinga M., Cammaroto F., Matveev M. V., *On the Urysohn number of a topological space*, Quaest. Math. 34 (2011), no. 4, 441–446

[8] Bonanzinga M., Cammaroto F., Matveev M. V., Pansera B., *On weaker forms of separability*, Quaest. Math. 31 (2008), no. 4, 387–395

[9] Bonanzinga M., Pansera B., *On the Urysohn number of a topological space II*, Quaest. Math., in press

[10] Cammaroto, F., Catalioto A., Pansera B., Porter J., *On the cardinality of the $\theta$-closed hull of sets II*, preprint available at [http://arxiv.org/abs/1206.6554](http://arxiv.org/abs/1206.6554)

[11] Cammaroto F., Catalito A., Pansera B., Tsaban B., *On the cardinality of the $\theta$-closed hull of sets*, preprint available at [http://arxiv.org/abs/1203.5824](http://arxiv.org/abs/1203.5824)

[12] Gotchev I. S., *The non-Hausdorff number of a topological space*, Topology Proc. 44 (2014), 249–256

[13] Hodel R. E., *Arhangel’skiĭ’s solution to Alexandroff’s problem: a survey*, Topology Appl. 153 (2006), no. 13, 2199–2217

[14] Veličko N. V., *$H$-closed topological spaces*, Mat. Sb. (N.S.) 70 (112) (1966), 98–112

[15] Willard S., Dissanayake U. N. B., *The almost Lindelöf degree*, Canad. Math. Bull. 27 (1984), no. 4, 452–455