The Maximum of an Asymmetric Simple Random Walk with Reflection

STEVEN FINCH

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ABSTRACT. Consider the extreme value of a Bernoulli random walk on the one-dimensional integer lattice, with reflection at 0, over a finite discrete time interval. Only the asymmetric (biased) case is discussed. Asymptotic mean/variance results are given as the time interval length approaches infinity. We similarly solve an elementary traffic light problem from queueing theory.

Let $X_0 = 0$ and $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables satisfying

$$
P(X_i = 1) = p, \quad P(X_i = -1) = q, \quad p + q = 1, \quad p \leq q$$

for each $1 \leq i \leq n$. Define $S_0 = X_0$ and

$$S_j = \begin{cases} |S_{j-1} + X_j| & \text{strong reflection at the origin,} \\
\max\{S_{j-1} + X_j, 0\} & \text{weak reflection at the origin}
\end{cases}$$

for each $1 \leq j \leq n$. The simple reflected random walk $S_0, S_1, S_2, \ldots, S_n$ is symmetric if $p = q$ and asymmetric if $p < q$. Let

$$M_n = \max_{0 \leq j \leq n} S_j$$

denote the maximum value of the walk over the time interval $[0, n]$. We shall focus entirely on the asymmetric case; a survey of related results (including those for a symmetric walk with reflection) appears elsewhere [1].

For the strong scenario, we have

$$\mathbb{E}(M_n) \sim \frac{\ln(n)}{\ln\left(\frac{2}{p}\right)} + \frac{\gamma + \ln\left(\frac{(1-2p)^2}{2q^2}\right)}{\ln\left(\frac{2}{p}\right)} + \frac{1}{2} + \phi(n)$$

as $n \to \infty$ and, for the weak scenario,

$$\mathbb{E}(M_n) \sim \frac{\ln(n)}{\ln\left(\frac{2}{p}\right)} + \frac{\gamma + \ln\left(\frac{p(1-2p)^2}{q^2}\right)}{\ln\left(\frac{2}{p}\right)} + \frac{1}{2} + \psi(n).$$

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The symbol $\gamma$ denotes Euler’s constant \cite{2}; $\varphi$ and $\psi$ are periodic functions of $\log q/p(n)$ with period 1 and of small amplitude. It is not surprising that the strong means are slightly larger than the weak means (because a weakly reflected walk can dwell at the origin indefinitely). This is provably true since $1/2 > p$. For both scenarios, we have

$$\mathbb{V}(M_n) \sim \frac{\pi^2}{24} \frac{1}{\ln\left(\frac{q}{p}\right)^2} + \frac{1}{12} + \omega(n)$$

as $n \to \infty$ and the function $\omega$, like $\varphi$ and $\psi$, is effectively negligible.

Different expressions emerge for a certain traffic light problem \cite{3}. Let $\ell \geq 1$ be an integer. Let $X_0 = 0$ and $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables satisfying

$$\mathbb{P}\{X_i = 1\} = p, \quad \mathbb{P}\{X_i = 0\} = q \quad \text{if } i \equiv 1, 2, \ldots, \ell \mod 2\ell;$$

$$\mathbb{P}\{X_i = 0\} = p, \quad \mathbb{P}\{X_i = -1\} = q \quad \text{if } i \equiv \ell + 1, \ell + 2, \ldots, 2\ell \mod 2\ell$$

for each $1 \leq i \leq n$. Define $S_0 = X_0$ and $S_j = \max\{S_{j-1} + X_j, 0\}$ for all $1 \leq j \leq n$. Thus cars arrive at a one-way intersection according to a Bernoulli($p$) distribution; when the signal is red ($1 \leq i \leq \ell$), no cars may leave; when the signal is green ($\ell + 1 \leq i \leq 2\ell$), a car must leave (if there is one). The quantity $M_n = \max_{0 \leq j \leq n} S_j$ is the worst-case traffic congestion (as opposed to the average-case often cited). Only the circumstance when $\ell = 1$ is amenable to rigorous treatment, as far as is known. We have

$$\mathbb{E}(M_n) \sim \frac{\ln(n)}{2\ln(2/p)} + \frac{\gamma + \ln\left(\frac{p(1-2p)^2}{4q^2}\right)}{2\ln(2/p)} + \frac{1}{2} + \psi(n),$$

$$\mathbb{V}(M_n) \sim \frac{\pi^2}{24} \frac{1}{\ln\left(\frac{q}{p}\right)^2} + \frac{1}{12} + \omega(n)$$

as $n \to \infty$, assuming $p < q$. For $\ell > 1$, only computer simulation-based estimates are available.

Sections 1 and 2 cover generating functions and singularity analyses corresponding to strongly RRWs and weakly RRWs, respectively. Section 5 does likewise for the traffic light problem ($\ell = 1$). We focus on the calculation of moments in Section 3. Sections 4 and 6 provide extensive verification by use of simulation.
1. Strong Scenario

For \( k = 1, 2, \ldots \), define \((k + 1) \times (k + 1)\) matrices

\[
A_1 = \begin{pmatrix} 0 & q & 0 \\ 1 & 0 & q \\ 0 & p & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & q & 0 & 0 \\ 1 & 0 & q & 0 \\ 0 & p & 0 & q \\ 0 & 0 & p & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & q & 0 & 0 & 0 \\ 1 & 0 & q & 0 & 0 \\ 0 & p & 0 & q & 0 \\ 0 & 0 & p & 0 & q \\ 0 & 0 & 0 & p & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & q & 0 & 0 & 0 & 0 \\ 1 & 0 & q & 0 & 0 & 0 \\ 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & p & 0 & q \\ 0 & 0 & 0 & 0 & p & 0 \end{pmatrix}, \quad \ldots
\]

and column vectors \( \varepsilon_{k+1} = (1, 0, 0, \ldots, 0), \quad 1_{k+1} = (1, 1, \ldots, 1) \). It is not difficult (e.g., starting with \( [1] \)) to show that

\[
\mathbb{P} \{ M_n \leq k \} = 1_{k+1} A_n^k \varepsilon_{k+1}
\]

and \( A^n \) denotes \( n^{\text{th}} \) matrix power. This yields a generating function

\[
G_k(z) = \sum_{n=0}^{\infty} \mathbb{P} \{ M_n \leq k \} z^n = 1_{k+1} \sum_{n=0}^{\infty} (A_k z)^n \varepsilon_{k+1} = 1_{k+1} (I - A_k z)^{-1} \varepsilon_{k+1}
\]

\[
= \frac{\det (B_k)}{\det (I - A_k z)} = \frac{Q_k}{R_k}
\]

where \( B_k \) is obtained from \( I - A_k z \) via replacing the first row by \( 1_{k+1}' \). Expanding the determinant with respect to the last row, we find linear recursive formulas

\[
R_k = R_{k-1} - p q z^2 R_{k-2}, \quad R_0 = 1, \quad R_1 = 1 - q z^2;
\]

\[
Q_k = Q_{k-1} - p q z^2 Q_{k-2} + p^{k-1} z^k, \quad Q_{-1} = 0, \quad Q_0 = 1.
\]

Setting \( t = \sqrt{1 - 4pqz^2} \), explicit solutions are as follows:

\[
R_k = \frac{1}{2t^2} \left[ u \left( \frac{1 + t}{2} \right)^k + v \left( \frac{1 - t}{2} \right)^k \right],
\]

\[
Q_k = \frac{1}{2t(1 - z)} \left[ u \left( \frac{1 + t}{2} \right)^k + v \left( \frac{1 - t}{2} \right)^k - 2t z (p z)^k \right]
\]
where

\[
u = 1 + t - 2qz^2 = 1 + t - 2q \left( \frac{1 - t^2}{4pq} \right)
\]

\[
= \left( \frac{1 + t}{2p} \right) [2p - (1 - t)] = \left( \frac{1 + t}{2p} \right) (-1 + 2p + t),
\]

\[
v = -1 + t + 2qz^2 = -1 + t + 2q \left( \frac{1 - t^2}{4pq} \right)
\]

\[
= \left( \frac{1 - t}{2p} \right) [-2p + (1 + t)] = \left( \frac{1 - t}{2p} \right) (1 - 2p + t).
\]

To assess the asymptotics for the coefficients of \(G_k(z)\), it suffices to examine the zero \(z_k\) of its denominator that is closest to the origin. The equation \(R_k = 0\) can be rewritten as

\[
\left( \frac{1 + t}{1 - t} \right)^k = \frac{v}{-u} = \frac{1 - t}{1 + t} \cdot \frac{1 - 2p + t}{1 - 2p - t},
\]

that is,

\[
\left( \frac{1 + t}{1 - t} \right)^{k+1} = \frac{1 - 2p + t}{1 - 2p - t}.
\]

For suitably large \(k\), there is exactly one solution \(t_k\) of the preceding equation with positive real part; further, \(t_k\) is real and satisfies \(0 < t_k < 1\). The details (involving Rouché’s theorem) are omitted. It follows that \(z_k\) is real; as both \(z_k\) and \(-z_k\) are zeroes of \(R_k\), we choose \(z_k\) to be positive (without loss of generality). Further,

\[
z_k - \frac{1}{(p/q)^k} \to \frac{(1 - 2p)^2}{2q^2}
\]

for the strong scenario as \(k \to \infty\), which implies that \([4, 5]\)

\[
\mathbb{P}\{M_n \leq \log_{q/p}(n) + h\} \sim \exp \left[ -\frac{(1 - 2p)^2}{2q^2} \left( \frac{q}{p} \right)^{-h} \right]
\]

as \(n \to \infty\). Consequences of such a discrete Gumbel distributional limit will be explored shortly.
2. Weak Scenario

The analog of the $A_k$ matrix here is

$$
\begin{pmatrix}
q & q & 0 & 0 & \cdots & 0 & 0 \\
p & 0 & q & 0 & \cdots & 0 & 0 \\
0 & p & 0 & q & \cdots & 0 & 0 \\
0 & 0 & p & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & q & 0 \\
0 & 0 & 0 & 0 & \cdots & p & 0 \\
\end{pmatrix}
$$

which gives rise as before to recursions

$$
R_k = R_{k-1} - pqz^2 R_{k-2}, \quad R_{-1} = 1, \quad R_0 = 1 - qz;
$$

$$
Q_k = Q_{k-1} - pqz^2 Q_{k-2} + p^k z^k, \quad Q_{-1} = 0, \quad Q_0 = 1.
$$

Solving these, we have

$$
R_k = \frac{1}{2t} \left[ u \left( \frac{1+t}{2} \right)^k + v \left( \frac{1-t}{2} \right)^k \right],
$$

$$
Q_k = \frac{1}{2t(1-z)} \left[ u \left( \frac{1+t}{2} \right)^k + v \left( \frac{1-t}{2} \right)^k - 2t(pz)^{k+1} \right]
$$

where

$$
u = 1 + t - q(1+t)z - 2pqz^2 = 1 + t - q(1+t)\sqrt{\frac{1-t^2}{4pq}} - 2pq \left( \frac{1-t^2}{4pq} \right)
$$

$$
= \left( \frac{1+t}{2} \right) \left[ 2 - q \sqrt{\frac{1-t^2}{pq}} - (1-t) \right] = \left( \frac{1+t}{2} \right) \left( 1 + t - \sqrt{\frac{q}{p}} \sqrt{1-t^2} \right),
$$

$$
v = -1 + t + q(1-t)z + 2pqz^2 = -1 + t + q(1-t)\sqrt{\frac{1-t^2}{4pq}} + 2pq \left( \frac{1-t^2}{4pq} \right)
$$

$$
= \left( \frac{1-t}{2} \right) \left[ -2 + q \sqrt{\frac{1-t^2}{pq}} + (1+t) \right] = \left( \frac{1-t}{2} \right) \left( -1 + t + \sqrt{\frac{q}{p}} \sqrt{1-t^2} \right).
The equation $R_k = 0$ can be rewritten as
\[
\left(\frac{1 + t}{1 - t}\right)^k = \frac{-v}{u} = \frac{1 - t}{1 + t} \cdot \frac{1 - t - \sqrt{\frac{2}{p} \sqrt{1 - t^2}}}{1 + t - \sqrt{\frac{2}{p} \sqrt{1 - t^2}}},
\]
that is,
\[
\left(\frac{1 + t}{1 - t}\right)^{k+1} = \frac{1 - t - \sqrt{\frac{2}{p} \sqrt{1 - t^2}}}{1 + t - \sqrt{\frac{2}{p} \sqrt{1 - t^2}}}.
\]
By reasoning akin to earlier,
\[
z_k - 1 \left(\frac{p}{q}\right)^k \to \frac{p(1 - 2p)^2}{q^2}
\]
for the weak scenario as $k \to \infty$, which implies that
\[
P\{M_n \leq \log_{q/p}(n) + h\} \sim \exp \left[ -\frac{p(1 - 2p)^2}{q^2} \left(\frac{q}{p}\right)^{-h}\right]
\]
as $n \to \infty$.

3. Mean and Variance

Fix $c > 0$ and $r > 1$. Forget temporarily the discrete nature of our distributional limits. To evaluate moments associated with a continuous Gumbel CDF
\[
\exp[-c r^{-y}] = \exp[-e^{-(\ln(r)y - \ln(c))}], \quad -\infty < y < \infty
\]
merely set
\[
\frac{y - \alpha}{\beta} = \ln(r)y - \ln(c)
\]
i.e.,
\[
\frac{1}{\beta} = \ln(r), \quad \alpha = \frac{\ln(c)}{\ln(r)}
\]
i.e., \[6\]
\[
\mathbb{E}(Y) = \alpha + \beta \gamma = \frac{\ln(c) + \gamma}{\ln(r)}, \quad \mathbb{V}(Y) = \frac{\pi^2}{6} \beta^2 = \frac{\pi^2}{6} \frac{1}{\ln(r)^2}.
\]
Return to the discrete domain is achieved by addition of correction terms:
\[
\mathbb{E}(M_n - \log_r(n)) \sim \frac{\ln(c) + \gamma}{\ln(r)} + \frac{1}{2} + \varphi(n),
\]
\[
\mathbb{V}(M_n) \sim \frac{\pi^2}{6} \frac{1}{\ln(r)^2} + \frac{1}{12} + \omega(n)
\]
as $n \to \infty$, via appropriate generalization of \[7, 8\] (who unnecessarily restrict $r$ to be exactly 2). The calculation of higher order moments is also possible.
4. Walk Data

Let $n = 10^{10}$. For each $p \in \{1/5, 1/3, 1/4, 3/7, 1/8\}$, we generated 40000 strongly RRWs and produced an empirical histogram for the maximum $M_n$. Figures 1–5 contain these histograms (in blue) along with our theoretical predictions (in red). The fit is excellent.

Similarly, we generated 40000 weakly RRWs and produced a histogram for the maximum $M_n$. Figures 6–10 contain these histograms along with our theoretical predictions. The fit, again, is excellent.

Each histogram is accompanied by an experimental mean, mean square and standard deviation, as well as our theoretical values.

5. Traffic Light

The analog of the $A_k$ matrix here is

$$
\begin{pmatrix}
1 - p^2 & q^2 & 0 & 0 & \cdots & 0 & 0 \\
p^2 & 2pq & q^2 & 0 & \cdots & 0 & 0 \\
0 & p^2 & 2pq & q^2 & \cdots & 0 & 0 \\
0 & 0 & p^2 & 2pq & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2pq & q^2 \\
0 & 0 & 0 & 0 & \cdots & p^2 & pq
\end{pmatrix}
$$

but recursions for $R_k$ and $Q_k$ are more complicated than those for walks. We have

$$R_k = \tilde{R}_k + p q z \tilde{R}_{k-1},$$

$$Q_k = \tilde{Q}_k + p q z \tilde{Q}_{k-1}$$

and, in turn,

$$\tilde{R}_k = (1 - 2pqz)\tilde{R}_{k-1} - p^2q^2z^2\tilde{R}_{k-2}, \quad \tilde{R}_{-1} = 1, \quad \tilde{R}_0 = 1 - (1 - p^2) z;$$

$$\tilde{Q}_k = (1 - 2pqz)\tilde{Q}_{k-1} - p^2q^2z^2\tilde{Q}_{k-2} + p^{2k}z^k, \quad \tilde{Q}_{-1} = 0, \quad \tilde{Q}_0 = 1.$$
The Maximum of an Asymmetric Simple Random Walk with Reflection

Solving these, we have

$$\tilde{R}_k = \frac{1}{2t} \left[ u \left( \frac{1 - 2pqz + t}{2} \right)^k + v \left( \frac{1 - 2pqz - t}{2} \right)^k \right],$$

$$\tilde{Q}_k = \frac{1}{2t(1 - z)} \left[ u \left( \frac{1 - 2pqz + t}{2} \right)^k + v \left( \frac{1 - 2pqz - t}{2} \right)^k - 2t (p^2 z)^{k+1} \right]$$

where

$$u = 1 + t - (1 + 3p + t + pt) q z + 2pq^2 z^2,$$

$$v = -1 + t + (1 + 3p - t - pt) q z - 2pq^2 z^2.$$

Skipping over details, we have

$$\frac{z_k - 1}{(p/q)^{2k}} \rightarrow \frac{p(1 - 2p)^2}{q^3}$$

as \( k \to \infty \) (note the exponent \( 2k \) in the denominator), which implies that

$$P \{ M_n \leq \log_{q^2/p^2}(n) + h \} \sim \exp \left[ -\frac{p(1 - 2p)^2}{2q^3} \left( \frac{q^2}{p^2} \right)^{-h} \right]$$

as \( n \to \infty \) (note the coefficient \( 2 \) in the denominator). Finally, the discussion in Section 3 applies with \( r \) replaced by \( r^2 \).
Figure 4:

Figure 5:
The Maximum of an Asymmetric Simple Random Walk with Reflection

Figure 6:

Figure 7:
The Maximum of an Asymmetric Simple Random Walk with Reflection

Figure 8:

RRW maximums: N=1E10, p=1/4, weak

- Experiment Mean: 19.9821
- Theoretical Mean: 19.9844
- Experiment Mean Squared: 400.7416
- Theoretical Mean Squared: 400.8239
- Experiment Standard Deviation: 1207.6
- Theoretical Standard Deviation: 1202.6

Figure 9:

RRW maximums: N=1E10, p=3/7, weak

- Experiment Mean: 69.9539
- Theoretical Mean: 69.9627
- Experiment Mean Squared: 4913.0321
- Theoretical Mean Squared: 4914.7416
- Experiment Standard Deviation: 4413.7
- Theoretical Standard Deviation: 4467.6
Let $n = 10^{10}$. For each $p \in \{1/5, 1/3\}$, we generated 40000 traffic light queues ($\ell = 1$) and produced an empirical histogram for the maximum $M_n$. Figures 11–12 contain these histograms (in blue) along with our theoretical predictions (in red). The fit is excellent.

Similarly, we generated 40000 TLQs ($\ell = 2$ and $\ell = 3$) and produced a histogram for the maximum $M_n$. Figures 13–16 contain these histograms. A conjecture in [3] – that such distributions do not depend on the value of $\ell$ – is evidently false. The word “theoretical” here refers to the ill-informed predictions emerging from $\ell = 1$. It would be good someday to understand the true distributional limits occurring for $\ell \geq 2$, even if only approximately.

7. Acknowledgements

I am indebted to Stephan Wagner [4, 5] for his expertise in obtaining the discrete Gumbel asymptotics in Section 1. Guy Louchard assured me that his mean/variance formulas [7, 8] indeed apply not just to $q/p = 2$ (i.e., $p = 1/3$), but to all $q/p > 1$ (i.e., $0 < p < 1/2$); he also reminded me that, in Section 5, it’s best to imagine a $2n$-sequence parsed into blocks $\{S_1, S_2\}, \ldots, \{S_{2n-1}, S_{2n}\}$. The creators of Julia, Mathematica and Matlab, as well as administrators of the MIT Engaging Cluster, earn my gratitude every day.
Figure 11: $\ell = 1$

Figure 12: $\ell = 1$
Figure 13: Theory for $\ell = 1$ does not carry over to $\ell = 2$

Figure 14: Theory for $\ell = 1$ does not carry over to $\ell = 2$
Figure 15: Theory for $\ell = 1$ does not carry over to $\ell = 3$

Figure 16: Theory for $\ell = 1$ does not carry over to $\ell = 3$
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Steven Finch
MIT Sloan School of Management
Cambridge, MA, USA
steven_finch@harvard.edu