In the covariant light-front quark model, we investigate the $B_c \to h_c, \chi_{c0,1,2}$ form factors. The form factors are evaluated in space-like kinematic region and are recasted to the physical region by adopting the exponential parametrization. We also study the semileptonic $B_c$ decays and find that branching fractions for the $B_c \to (h_c, \chi_{c0,1,2})l\bar{\nu}(l = e, \mu)$ decays have the order $10^{-3}$ while branching fractions for $B_c \to (h_c, \chi_{c0,1,2})\tau\bar{\nu}_\tau$ are suppressed by one order. These predictions will be tested on the forthcoming hadron colliders.

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I. INTRODUCTION

$B$ meson weak decays provide a golden place to extract magnitudes and phases of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, which can test the origins for CP violation in and beyond the standard model (SM). Semileptonic and nonleptonic $B$ meson decays have received extensive interests and achieved many great successes. Experimentally, the two $B$ factories have accumulated more than $10^9 B - \bar{B}$ events; measurements are becoming more and more precise. On the theoretical side, apart from contributions proportional to the form factors, the so-called nonfactorizable diagrams and some other radiative corrections are also taken into account. All of these are making $B$ physics suitable for search of new particles and new phenomena (see Ref. [1] for a review).

Compared with $B$ decays, $B_c$ meson decays have received much less experimental considerations. The mass of a $B_c\bar{B}_c$ pair has exceeded the threshold of $\Upsilon(4S)$ thus the $B_c$ meson can not be generated on the two $B$ factories. But $B_c$ meson decays have a promising prospect on the forthcoming hadron colliders. The Large Hadron Collider (LHC) experiment, which is scheduled to run in the very near future, will produce plenty of $B_c$ events. The LHCb collaboration has the desire to perform a comprehensive investigation on $B_c$ meson decays. With more and more data accumulated in the future, the study on $B_c$ mesons will be of great interests: (1) $B_c$ contains two different heavy flavors, the spectroscopy may be different with the light meson or the meson with only one heavy quark. It serves as a different laboratory to study the strong interactions. (2) $B_c$ meson can weakly decay via the $b \to q$ transition like lighter $B_{u,d,s}$ mesons, but the dynamics is dramatically different. (3) Moreover, the charm quark can also decay via weak interactions, where the $b$ quark acts as a spectator. The CKM matrix element $|V_{cs}| \sim 1$ is much larger than $|V_{cb}| \sim 0.04$ in $b$ quark decays. Decays of the charm quark contribute much more to the decay width of the $B_c$ meson. Although the phase space in $c \to d,s$ decays is much smaller than that in $b \to c$ transitions, the former ones provide about 70% contributions to the decay width of $B_c$. This results in a larger decay width and a much smaller lifetime than the $B$ meson: $\tau_{B_c} \ll \frac{1}{4} \tau_B$. (4) The two heavy $b$ and $c$ quarks can annihilate and they provide new kinds of weak decays with sizable partial decay widths. The pure leptonic or radiative leptonic decay can be used to extract the $B_c$ decay constant and the CKM matrix element $V_{cb}$ [2,3].

Semileptonic $B_c$ decays are simpler than nonleptonic $B_c$ decays: the leptonic part can be perturbatively evaluated
leaving only hadronic form factors unknown. In two-body nonleptonic $B_c$ decays, most channels are also dominated by the $B_c$ transition form factors. Thus the study of $B_c$ transition form factors is essentially required. In the present work, we will use the light-front quark model to analyze the form factors $B_c$ decays into p-wave charmonia. This can be viewed as a continuation of our previous work [4]. The light front QCD approach has some unique features which are particularly suitable to describe a hadronic bound state [5]. Based on this approach, a light-front quark model with many advantages is developed [6, 7, 8, 9, 10]. This model can provide a relativistic treatment of the movement of the hadron. It also gives a fully treatment of the hadron spin by using the so-called Melosh rotation. The light front wave functions are expressed in terms of their fundamental quark and gluon degrees of freedom. They are independent of the hadron momentum and thus are explicitly Lorentz invariant. In the covariant light-front quark model, the spurious contribution, which is dependent on the orientation of the light-front, becomes irrelevant in physical observables and that makes the light-front quark model more selfconsistent. This model has been successfully extended to investigate the decay constants and form factors of the $s$-wave and $p$-wave mesons [11, 12, 13].

Our paper is organized as follows. The formalism of the covariant light-front quark model and numerical results for form factors are presented in the next section. The decay rates of semi-leptonic $B_c$ decays are discussed in Section III. Our conclusions are given in Section IV.

II. FORM FACTORS IN THE COVARIANT LIGHT-FRONT QUARK MODEL

$B_c \to S, A$ ($S, A$ denotes a scalar meson or an axial-vector meson, respectively) form factors are defined by

$$\langle S(P')|A_\mu|B_c(P)\rangle = \left( P_\mu - \frac{m_B^2 - m_S^2}{2q^2} q_\mu \right) F_1^{B_c,S}(q^2) + \frac{m_B^2 - m_S^2}{2q^2} q_\mu F_0^{B_c,S}(q^2),$$

$$\langle A(P',\epsilon'\mu)|A_\mu|B_c(P')\rangle = -\frac{1}{m_B - m_A} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{\mu'\nu\lambda} P_\alpha q^\beta A^{\alpha\gamma}(q^2),$$

$$\langle A(P',\epsilon'\mu)|V_\mu|B_c(P')\rangle = -i \left\{ (m_B - m_A) \epsilon^{\mu'\nu} V_1^{B_c,A}(q^2) - \frac{\epsilon^{\mu'\nu} \cdot P}{m_B - m_A} P_\mu V_2^{B_c,A}(q^2) \right\},$$

where $P = P' + P''$, $q = P' - P''$ and the convention $\epsilon_{0123} = 1$ is adopted. To smear the singularity at $q^2 = 0$, the relation $V_3^{B_c,A}(0) = V_0^{B_c,A}(0)$ is required, and

$$V_3^{B_c,A}(q^2) = \frac{1}{m_B - m_A} V_1^{B_c,A}(q^2) - \frac{m_B + m_A}{2m_A} V_2^{B_c,A}(q^2).$$

Form factors of $B_c$ decays into a tensor meson are defined by

$$\langle T(P'',\epsilon''\mu)|V_\mu|B_c(P')\rangle = h(q^2) \epsilon_{\mu\nu\alpha\beta\gamma} \epsilon^{\mu'\nu\lambda} P_\alpha q^\beta,$$

$$\langle T(P'',\epsilon''\mu)|A_\mu|B_c(P')\rangle = -i \left\{ k(q^2) \epsilon^{\mu'\nu\lambda} \epsilon^{\mu'\nu\lambda} + \epsilon^{\mu'\nu\lambda} \epsilon^{P\mu\nu} \epsilon_{\alpha\beta\gamma} [P_\mu b_+(q^2) + q_\mu b_-(q^2)] \right\},$$

where the polarization tensor, which satisfies $\epsilon_{\mu\nu} P^{\mu\nu} = 0$, is symmetric and traceless. The spin-2 polarization tensors can be constructed using spin-1 polarization vectors:

$$\hat{\epsilon}_{\mu\nu}(p, \pm 2) = \epsilon_{\mu}(\pm) \epsilon_{\nu}(\pm), \quad \hat{\epsilon}_{\mu\nu}(p, \pm 1) = \frac{1}{\sqrt{2}} [\epsilon_{\mu}(\pm) \epsilon_{\nu}(0) + \epsilon_{\nu}(\pm) \epsilon_{\mu}(0)],$$

$$\hat{\epsilon}_{\mu\nu}(p, 0) = \frac{1}{\sqrt{6}} [\epsilon_{\mu}(+) \epsilon_{\nu}(-) + \epsilon_{\nu}(+) \epsilon_{\mu}(-)] + \sqrt{\frac{2}{3}} \epsilon_{\mu}(0) \epsilon_{\nu}(0).$$

(6)
where $\epsilon$ is the polarization vector for a vector meson. If the recoiling meson is moving on the plus direction of the $z$ axis, their explicit structures are chosen as

$$
\epsilon_\mu(0) = \frac{1}{m_T} (|\vec{p}_T|, 0, 0, E_T),
\epsilon_\mu(\pm) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0),
$$

where the $E_T$ and $p_T$ is the energy and the magnitude of the momentum of the tensor meson in the $B_c$ rest frame, respectively. $m_T$ denotes the tensor meson’s mass. In $B_c$ meson decays, it is useful to define a new polarization vector $\epsilon_T$ for the tensor meson

$$
\epsilon_{T\mu}(h) = \epsilon_{\mu\nu}(p, h) p_{B_c}^\nu
$$

which satisfies

$$
\epsilon_{T\mu}(\pm 2) = 0, \quad \epsilon_{T\mu}(\pm 1) = \frac{1}{\sqrt{2}} \epsilon(0) \cdot p_{B_c} \epsilon_\mu(\pm), \quad \epsilon_{T\mu}(0) = \sqrt{\frac{2}{3}} \epsilon(0) \cdot p_{B_c} \epsilon_\mu(0).
$$

The contraction is evaluated as $\epsilon(0) \cdot p_{B_c} = m_{B_c} |p_T| / m_T$. We can see that the new polarization vector plays a similar role with the polarization vector for a vector meson, regardless of the nontrivial factors $\sqrt{2}$ or $\sqrt{2/3}$. In analogy with the $B_c \rightarrow A$ transition, one can define the following form factors for convenience:

$$
A^{B,T} = -(m_{B_c} - m_T) h(q^2), \quad V_1^{B,T} = -\frac{k(q^2)}{m_{B_c} - m_T}, \quad V_2^{B,T} = (m_{B_c} - m_T) b_+(q^2),
V_0^{B,T}(q^2) = \frac{m_{B_c} - m_T}{2m_T} V_1^{B,T}(q^2) - \frac{m_{B_c} + m_T}{2m_T} V_2^{B,T}(q^2) - \frac{q^2}{2m_T} b_-(q^2),
$$

where these form factors $A^{B,T}, V_0^{B,T}$ have nonzero mass dimensions.

We will work in the $q^+ = 0$ frame and employ the light-front decomposition of the momentum $P' = (P'^-, P'^+, P'_\perp)$, where $P'^\pm = P^0 \pm |p_T|$, so that $P'^2 = P'^- P'^+ - P'^2_\perp$. The incoming (outgoing) meson have the momentum of $P' = p'_1 + p_2$ ( $P'' = p''_1 + p_2$ and the mass of $M'$ ($M''$). The quark and antiquark inside the incoming (outgoing) meson have the mass $m_{i''}(m)$ and $m_{q}$ and the momenta are denoted as $p''_{i'}$ and $p_2$ respectively. These momenta can be expressed in terms of the internal variables $(x_1, p'_{\perp})$ as:

$$
p'_{1,2} = x_{1,2} P'^{\perp}, \quad p'_{1,2,\perp} = x_{1,2} P'_\perp \pm p'_{\perp},
$$

with $x_1 + x_2 = 1$. Using these internal variables, one can define some useful quantities for the incoming meson:

$$
M_0^{i''} = (\epsilon' + e_2)^2 = \frac{p'^2_2 + m_{i''}^2}{x_1} + \frac{p'^2_2 + m_2^2}{x_2}, \quad \tilde{M}_0' = \sqrt{M_0'^2 - (m'_{i''} - m_2)^2},
\epsilon'_{i} = \sqrt{m_{i''}^2 + p'^2_\perp + p'^2_2}, \quad p' = \frac{x_2 M_0'}{2} \frac{m_2^2 + p'^2_2}{2x_2 M_0'},
$$

here $e_i$ can be interpreted as the energy of the quark or the antiquark and $M_0'$ can be viewed kinematic invariant mass of the meson system. The definition of the internal quantities for the outgoing meson is similar. To calculate the amplitude for the transition form factor, we require the following Feynman rules for the meson-quark-antiquark vertices ($i\Gamma_M$):

$$
i\Gamma' = H' \gamma_5,
i\Gamma' = H'_{\gamma_5},
$$
where $\Gamma_{A}^{I}$ contribute to physical quantities such as decay constants and form factors in terms of the corresponding current matrix elements. In order to solve this problem, Jaus proposed the covariant light-front approach which permits a systematic way to deal with the zero-mode contributions. In practice, we use the light-front decomposition of the loop momentum and perform the integration over the minus component using the contour method. If the covariant vertex functions are not singular when performing the integration, the transition amplitude will pick up the singularities in the antiquark propagator. The integration then leads to

$$B_{c}^{S} = -i^{3} \frac{N_{c}}{(2\pi)^{4}} \int d^{4}p_{1}' |H_{A}^{S}| N_{1}^{(\nu)} N_{0}^{(\nu)} S_{\mu}^{B_{c}^{S}}.$$  

where $N_{1}^{(\nu)} = p_{1}'^{(\nu)2} - m_{1}'^{(\nu)2}$, $N_{0} = p_{2}^{2} - m_{2}^{2}$. The function $S^{B_{c}^{S}}_{\mu}$ is derived using the Lorentz contraction

$$S^{B_{c}^{S}}_{\mu} = Tr[(p_{1}'' + m_{1}'')^{\nu}\gamma_{\mu}\gamma_{5}(p_{1}' + m_{1}')\gamma_{5}(-p_{2} + m_{2})]$$

$$= 2p_{1}''[M_{2}^{2} + M_{2} \nu - q^{2} - 2N_{2} - (m_{1}' - m_{2})^{2} - (m_{1}'' + m_{2})^{2} + (m_{1}' + m_{1}'')^{2}]$$

$$+ q_{\mu}[q^{2} - 2M_{2}^{2} + N_{1}' - N_{1}'' + 2N_{2} + 2(m_{1}' - m_{2})^{2} - (m_{1}' + m_{1}'')^{2}]$$

$$+ P_{\nu}[q^{2} - N_{1}' - N_{1}'' - (m_{1}' + m_{1}'')^{2}].$$  

In practice, we use the light-front decomposition of the loop momentum and perform the integration over the minus component using the contour method. If the covariant vertex functions are not singular when performing the integration, the transition amplitude will pick up the singularities in the antiquark propagator. The integration then leads to

$$N_{1}^{(\nu)} \rightarrow \tilde{N}_{1}^{(\nu)} = x_{1}(M_{2}^{(\nu)2} - M_{0}^{(\nu)2}),$$

$$H_{M}^{(\nu)} \rightarrow \tilde{H}_{M}^{(\nu)},$$

$$W_{M}^{(\nu)} \rightarrow \tilde{W}_{M}^{(\nu)}.$$
\[
\int \frac{d^4p_1'}{N_1'N_1'^*N_2} H'_p H''_S S^{B,S} \rightarrow -i\pi \int \frac{dx_2d^2p_1'}{x_2N_1'N_1'^*} h'_p h''_S S^{B,S},
\]
where
\[
M''_0 = \frac{p''_1 + m''_1}{x_1} + \frac{p''_2 + m''_2}{x_2}
\]
with \(p''_1 = p'\perp - x_2 q\perp\). The explicit forms of \(h'_M\) and \(w'_M\) used in this work are given by
\[
\begin{align*}
\hat{h}'_p &= (M'^2 - M_0'^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0'^2}} \varphi', \\
\hat{h}'_S &= \sqrt{\frac{2}{3}} \hat{h}'_A = (M'^2 - M_0'^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0'^2}} \frac{\tilde{M}_0'^2 - \varphi'^p}{\varphi'^p}, \\
\hat{h}'_A &= (M'^2 - M_0'^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0'^2}} \varphi'^p, \\
w'_A &= \frac{\tilde{M}_0'}{m_1 - m_2}, \quad w'_A = 2
\end{align*}
\]
where \(\varphi'\) and \(\varphi'^p\) is the light-front wave function for s-wave and p-pave mesons, respectively. After this integration, the conventional light-front model is recovered but manifestly the covariance is lost as it receives additional spurious contributions proportional to the lightlike four vector \(\hat{\omega} = (0, 2, 0, 0)\). The undesired spurious contributions can be eliminated by the inclusion of the zero mode contribution which amounts to performing the \(p^-\) integration in a proper way in this approach. The specific rules under this \(p^-\) integration are derived in Ref. [10, 11] and are displayed in Appendix A.

Using Eqs. (15)–(20) and taking the advantage of the rules in Ref. [10, 11], the \(B_c \rightarrow S\) form factors are straightforwardly given by
\[
\begin{align*}
F_1^{B,S}(q^2) &= \frac{N_c}{16\pi^3} \int dx_2d^2p_\perp \frac{H'_p H''_S}{x_2N_1'N_1'^*} \left[ x_1(M'^2 - M_0'^2) + x_2q^2 \
&\quad - x_2(m'_1 + m''_1)^2 - x_1(m'_1 - m_2)^2 - x_1(m''_1 + m_2)^2 \right], \\
F_0^{B,S}(q^2) &= \frac{N_c}{q \cdot P 16\pi^3} \int dx_2d^2p_\perp \frac{2H'_p H''_S}{x_2N_1'N_1'^*} \left\{ - x_1x_2M'^2 - p'^2_\perp + (m''_1 + m_2)(x_2m'_1 + x_1m_2) \
&\quad + 2\frac{q \cdot P}{q^2} \left( \frac{p'^2_\perp + (p'_\perp \cdot q\perp)^2}{q^2} \right) + 2\frac{(p'_\perp \cdot q\perp)^2}{q^2} \frac{x_2(q^2 + q \cdot P)}{q^2} \right\},
\end{align*}
\]

Similarly, one can derive the \(B_c \rightarrow A, T\) form factors and we refer to Appendix B for tedious expressions of these form factors.

The light front wave function \(\varphi'\) can be obtained by solving the relativistic Schrödinger equation with a phenomenological potential. But in fact except for some limited cases, the exact solution is not obtainable. In practice, a phenomenological wave function to describe the hadronic structure is preferred. In this work, we will use the simple Gaussian-type wave function which has been extensively examined in the literature [11, 12, 13]
\[
\begin{align*}
\varphi' &= \varphi'(x_2, p'_\perp) = 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{dp'_z}{dx_2}} \exp \left( -\frac{p'^2_\perp + p'^2_z}{2\beta^2} \right), \\
\varphi'^p &= \varphi'^p(x_2, p'^\perp) = \sqrt{\frac{2}{\beta^2}} \varphi', \quad \frac{dp'_z}{dx_2} = \frac{c'_1c_2}{x_1x_2M_0'},
\end{align*}
\]
The parameters $\beta$’s, which describe the momentum distribution, are usually fixed by mesons’ decay constants whose analytic expressions are also given in [11]. The decay constant for the $B_c$ meson is employed by

$$f_{B_c} = (400 \pm 40) \text{MeV},$$

which gives $\beta_{B_c} = (0.89_{-0.074}^{+0.075}) \text{GeV}$. This value is a bit smaller than results provided by Lattice QCD method [14]

$$f_{B_c} = (489 \pm 4 \pm 3) \text{MeV}.$$  \hfill (24)

The other inputs, including masses (in units of GeV) of the constituent quarks and hadrons, $V_{cb}$ and the lifetime of $B_c$, are used as [13]

$$m_c = 1.4, \quad m_b = 4.8, \quad m_{B_c} = 6.286,$$

$$m_{b_c} = 3.52528, \quad m_{\chi_0} = 3.41476, \quad m_{\chi_1} = 3.51066, \quad m_{\chi_2} = 3.5562$$

$$V_{cb} = 41.2 \times 10^{-3}, \quad \tau_{B_c} = (0.46 \pm 0.07) \text{ps}.$$  \hfill (25)

The constituent quark masses are close to those used in the literature [4, 11, 12, 13]. The shape parameter $\beta_c$ is used as:

$$B_{m} = (0.7 \pm 0.1) \text{GeV}$$

which corresponds to $|f_{\chi_1}| = (340_{-101}^{+119}) \text{MeV}$. For the other shape parameters, we will assume the same values and introduce a relatively large uncertainty to compensate the different Lorentz structures:

$$\beta_{\chi_0} = \beta_{\chi_2} = \beta_{b_c} = (0.7 \pm 0.1) \text{GeV}.$$

Unlike the light quark, the heavy bottom and charm quark have large masses. In the heavy quark limit $m_{c,b} \to \infty$, the $B_c$ and charmonium systems obey the heavy quark symmetry which is helpful to simplify the dynamics in transition form factors and decay amplitudes. In particular, the large momentum of the heavy quark can be projected out and the remanent momentum is of the order of the hadronic scale. For $B_c$ and charmonia, the transverse momentum is of the order of the hadronic scale. For $B_c$ and charmonia, the two constituents are both heavy and move non-relativistically. After projecting the large mass scale, the dynamic scale is of the order $m_cv$ and $m_cv^2$, where $v$ is the relative velocity of the quark-anti-quark pair. Then physical quantities can be expanded in terms of $1/m_c, 1/m_b$ in the effective theory. In the present analysis, the expansion in $1/m_c$ and $1/m_b$ is not used and physical quantities contain a tower of contributions with different orders. The $B_c$ and the charmonia are directly made of two heavy quarks and the dynamics is reflected by the light-front wave functions. Despite of the different treatments, the leading power behavior should be the same. In $B_c$ and charmonia, the transverse momentum is of the order of $m_cv$. Inferred from Eq. (22), the parameter $\beta'$ in the light-front wave function is of the order of $m_cv$.

In nonrelativistic QCD (NRQCD) [16], the kinematic energy has the order of $m_cv^2$. If it can be identified as the hadronic scale, the $\beta'$ is of the order of $\sqrt{m_c \Lambda_{QCD}}$, which is expected to be larger than the shape parameter in the light meson system. This feature is also confirmed by the numerical result: the shape parameters $\beta$’s for charmonium and $B_c$ meson are larger than those for light mesons such as $\beta''_{c,b} = 0.3102 \text{GeV}$ [11].

In the NRQCD framework, the light cone distribution amplitudes of the s-wave charmonia have been comprehensively investigated in Refs. [17, 18, 19, 20]. Recently, the analysis has been generalized to the p-wave charmonia [21]. In their notation, the distribution is described by the matrix elements of the nonlocal operators, while in the light-front quark model, we use the coupling vertex. Moreover, the distribution amplitudes in these two frameworks are also different. In Ref. [21], the leading twist light-cone distribution amplitude is expanded into Gegenbauer polynomials and the Gegenbauer moments are studied in the QCD sum rules. The authors in Ref. [21] also propose the following model

$$\Phi(\xi, \mu \sim m_c) = c(\beta_p)(1-\xi^2)\exp\left(-\frac{\beta_p}{1-\xi^2}\right),$$

\hfill (26)
TABLE I: Results for the $B_c \to \chi_{c0, c1, c2}, h_c$ form factors and fitted parameters $c_1$ and $c_2$. The first type of uncertainties are from the shape parameters of the $p$-wave charmonia and the second ones are from the $B_c$ meson decay constants.

| $F_{\bar{B}_c X_{c0}}$ | $F_{\bar{B}_c X_{c0}}'$ | $F_{c_{\text{mix}}}$ | $c_1$ | $c_2$ |
|-----------------------|-----------------------|----------------------|------|------|
| $F(0)$                | $c(0)$                | $c_{\text{mix}}$     |      |      |
| $F_{\bar{B}_c X_{c1}}$ | $F_{\bar{B}_c X_{c1}}'$ | $F_{c_{\text{mix}}}$ |      |      |
| $A_{B_c X_{c2}}$      | $A_{B_c X_{c2}}'$     | $A_{c_{\text{mix}}}$ |      |      |
| $V_{B_c X_{c1}}$      | $V_{B_c X_{c1}}'$     | $V_{c_{\text{mix}}}$ |      |      |
| $V_{1} B_{c X_{c1}}$  | $V_{1} B_{c X_{c1}}'$ | $V_{1 c_{\text{mix}}}$ |      |      |
| $V_{2} B_{c X_{c1}}$  | $V_{2} B_{c X_{c1}}'$ | $V_{2 c_{\text{mix}}}$ |      |      |
| $\beta_{B_c X_{c2}}$  | $\beta_{B_c X_{c2}}'$ | $\beta_{c_{\text{mix}}}$ |      |      |
| $\beta_{B_c X_{c2}}$  | $\beta_{B_c X_{c2}}'$ | $\beta_{c_{\text{mix}}}$ |      |      |

where $c(\beta_P)$ is a normalization constant. The parameter $\xi$ is defined as $\xi = 2u - 1$, where $u$ is the momentum fraction of the charm quark. This form is dramatically different with the one in Eq. (22) used in this framework. The main part in Eq. (20) is the exponential term $\exp(1/\beta_P)$ which corresponds to $\exp(1/\beta_P)$ in the distribution amplitude in Eq. (22). For charmonia, the momentum $p_z^2$ is simplified as

$$p_z^2 = \frac{(x_1 - x_2)^2}{x_1x_2}(p_T^2 + m_c^2).$$

Considering the longitudinal part, we can see that there is an additional factor $(x_1 - x_2)^2 = \xi^2$ in the one used in this framework. It indicates that the distribution amplitude used in this work is sharper at the region around $x_2 \sim 0.5$.

At last, the parameter $\beta_P$ is dimensionless and is different with $\beta'$. In the heavy quark limit, $u$ is close to $1/2$ and $\beta_P$ is of the order $1$: $\beta_P = 3.4^{+1.0}_{-0.9}$. Comparing the longitudinal part of the two distribution amplitudes, one can obtain the relation between $\beta$ and $\beta_P$: $\beta_P \sim 2m_c^2((x_1 - x_2)^2)$. The typical value of $((x_1 - x_2)^2)$ is of the order $\Lambda_{QCD}/m_c$ which also indicates $\beta_P \sim 1$. These different distribution amplitudes are expected to induce sizable differences to the resultant form factors. They can be discriminated or constrained by the available data of transition form factors in the future.

Because of the condition $q^+ = 0$ imposed during the course of calculation, form factors can be directly studied only at spacelike momentum transfer $q^2 = -q_A^2 \leq 0$ which are not relevant for the physical decay processes. It has been proposed in [11] to parameterize form factors as explicit functions of $q^2$ in the space-like region and then analytically extend them to the time-like region. To shed light on the momentum dependence, one needs a specific model to parameterize the form factors and we will choose a three-parameter form

$$F(q^2) = F(0)\exp(c_1s + c_2s^2),$$

where $s = q^2/m_{B_c}$ and $F$ denotes any one of the $B_c \to S, A, T$ form factors. In the fitting procedure, form factors in the region $q^2 = [-15] \text{GeV}^2$ are studied and the fitted results for $c_1$ and $c_2$ are collected in table [1]. We should point out that we have adopted a negative decay constant for $x_{c1}$ so that the $B_c \to A$ form factors are both positive.
III. SEMILEPTONIC $B_c$ DECAYS

With the form factors at hand, one can directly perform the analysis of semileptonic $B_c$ decays whose differential decay widths are given by

$$\frac{d\Gamma(B_c \to S l \bar{\nu})}{dq^2} = \left( \frac{q^2 - m_l^2}{q^2} \right)^2 \frac{\sqrt{\lambda(m_{B_c}, m_S, q^2)} G_f^2 V_{cb}^2}{384 m_{B_c}^3 \pi^3} \times \frac{1}{q^2} \times \left\{ (m_l^2 + 2q^2)\lambda(m_{B_c}, m_S, q^2)[F_1^{BcS}(q^2)]^2 + 3m_l^2 \lambda(m_{B_c}, m_S, q^2)[F_0^{BcS}(q^2)]^2 \right\}, \quad (29)$$

$$\frac{d\Gamma(L)(B_c \to A l \bar{\nu})}{dq^2} = \left( \frac{q^2 - m_l^2}{q^2} \right)^2 \frac{\sqrt{\lambda(m_{B_c}, m_A, q^2)} G_f^2 V_{cb}^2}{384 m_{B_c}^3 \pi^3} \times \frac{1}{q^2} \times \left\{ (m_l^2 + 2q^2)\lambda(m_{B_c}, m_A, q^2)[V_{L}^{BcA}(q^2)]^2 + 3m_l^2 \lambda(m_{B_c}, m_A, q^2)[V_{0}^{BcA}(q^2)]^2 \right\}, \quad (30)$$

$$\frac{d\Gamma(L)(B_c \to A l \bar{\nu})}{dq^2} = \left( \frac{q^2 - m_l^2}{q^2} \right)^2 \frac{\sqrt{\lambda(m_{B_c}, m_A, q^2)} G_f^2 V_{cb}^2}{384 m_{B_c}^3 \pi^3} \times \frac{1}{q^2} \times \left\{ (m_l^2 + 2q^2)\lambda(m_{B_c}, m_A, q^2) \left| A^{BcA}(q^2) \right|^2 \right\}, \quad (31)$$

where the superscript +(-) denotes the right-handed (left-handed) states of axial-vector mesons, while the subscript $L$ denotes that the axial-vector in the final state is longitudinally polarized. $m_l$ is the lepton’s mass and $\lambda(m_{B_c}, m_l^2, q^2) = (m_{B_c}^2 + m_l^2 - q^2)^2 - 4m_{B_c}m_l^2$ with $i = S, A$. The combined transverse and total differential decay widths are given by:

$$\frac{d\Gamma_T}{dq^2} = \frac{d\Gamma^+}{dq^2} + \frac{d\Gamma^-}{dq^2}, \quad \frac{d\Gamma_L}{dq^2} = \frac{d\Gamma^-}{dq^2} + \frac{d\Gamma_T}{dq^2}. \quad (32)$$

Expressions for the decay width of $B_c \to T l \bar{\nu}$ can be obtained by the decay width of $B_c \to A l \bar{\nu}$ decays:

$$\frac{d\Gamma_L(B \to T l \bar{\nu})}{dq^2} = \frac{1}{2} \frac{\lambda(m_{B_c}, m_T^2, q^2)}{4m_T^2} \times \frac{d\Gamma_L(B \to A l \bar{\nu})}{dq^2} \bigg|_{V_{0,1,2}^{BcA} \to V_{0,1,2}^{BcT}},$$

$$\frac{d\Gamma_T(B \to T l \bar{\nu})}{dq^2} = \frac{2}{3} \frac{\lambda(m_{B_c}, m_T^2, q^2)}{4m_T^2} \times \frac{d\Gamma_T(B \to A l \bar{\nu})}{dq^2} \bigg|_{(V_{1}^{BcA, A B_cA}) \to (V_{1}^{BcT, A B_cT})}, \quad (33)$$

where the form factors $V_{0,1,2}, A$ of $B_c \to T$ decays have nontrivial dimensions and thus the two functions $\frac{d\Gamma_L(B \to A l \bar{\nu})}{dq^2} \bigg|_{V_{0,1,2}^{BcA} \to V_{0,1,2}^{BcT}}$ and $\frac{d\Gamma_L(B \to A l \bar{\nu})}{dq^2} \bigg|_{(V_{1}^{BcA, A B_cA}) \to (V_{1}^{BcT, A B_cT})}$ do not have the correct dimensions with the conventional $\frac{d\Gamma}{dq^2}$. It is compensated by the prefactor $\frac{\lambda(m_{B_c}^2, m_T^2, q^2)}{4m_T^2}$ which also have nonzero mass dimensions.

Our results for these semileptonic $B_c$ decays are collected in table III. Since electrons and muons are very light compared with the charm quark, we can safely neglect the masses of these two kinds of leptons. The uncertainties are from those in the form factors; the first kind of uncertainties are from the shape parameters of the charmonia and the second ones are from the $B_c$ decay constant. The third uncertainties are from that in the $B_c$ lifetime. Several remarks on the results are given in order. First of all, branching fractions for the $B_c \to (h_c, \chi_{c0,1,2}) l \bar{\nu}(l = e, \mu)$ decays have the order $10^{-3}$ while branching fractions of $B_c \to (h_c, \chi_{c0,1,2}) \tau \bar{\nu}_\tau$ are suppressed by one order. In the covariant light-front quark model, branching fractions of the $B_c \to \eta_c l \bar{\nu}$ and $B_c \to J/\psi l \bar{\nu}$ decays are about one percent. From these results, the branching ratios of $B_c \to (h_c, \chi_{c0,1,2}) l \bar{\nu}$ are smaller by a factor of 2 – 10. There are two main reasons for these differences: the form factors and phase space. For example, if we set the mass of $\chi_{c0}$ equal to that of $\eta_c$, the branching ratio of $B_c \to \chi_{c0} l \bar{\nu}$ becomes 0.40%. The larger form factors of $B_c \to \eta_c$ will enhance the branching fraction by a factor of 1.68. Secondly, polarizations $\delta_T$ of $B_c \to h_c l \bar{\nu}$ and $B_c \to \chi_{c1} l \bar{\nu}$

$$BR(B_c \to \eta_c e \bar{\nu}_e) = (0.67^{+0.09+0.04+0.04-0.10}_{-0.09-0.07-0.04-0.10})\%,$$

$$BR(B_c \to J/\psi e \bar{\nu}_e) = (1.40^{+0.03+0.01+0.15+0.23}_{-0.03-0.14-0.23})\%. \quad (34)$$

Compared with these decays, the branching ratios of $B_c \to (h_c, \chi_{c0,1,2}) l \bar{\nu}$ are smaller by a factor of 2 – 10. There are two main reasons for these differences: the form factors and phase space. For example, if we set the mass of $\chi_{c0}$ equal to that of $\eta_c$, the branching ratio of $B_c \to \chi_{c0} l \bar{\nu}$ becomes 0.40%. The larger form factors of $B_c \to \eta_c$ will enhance the branching fraction by a factor of 1.68. Secondly, polarizations $\delta_T$ of $B_c \to h_c l \bar{\nu}$ and $B_c \to \chi_{c1} l \bar{\nu}$
TABLE II: Branching ratios (in units of %) of semileptonic $B_c \rightarrow (h_c, \chi_{c0,1,2}) \bar{l} \nu (l = e, \mu)$ and $B_c \rightarrow (h_c, \chi_{c0,1,2}) \tau \bar{\nu}_\tau$ decays.

|                | $B_c \rightarrow \chi_{c0} \bar{l} \nu$ | $B_c \rightarrow \chi_{c1} \bar{l} \nu$ | $B_c \rightarrow h_c \bar{l} \nu$ | $B_c \rightarrow \chi_{c2} \bar{l} \nu$ |
|----------------|----------------------------------------|----------------------------------------|---------------------------------|---------------------------------|
| This work      | 0.21$^{+0.02+0.01+0.00+0.00}_{-0.04-0.01-0.01}$ | 0.14$^{+0.00+0.01+0.00}_{-0.01-0.01-0.02}$ | 0.31$^{+0.05+0.01+0.05}_{-0.08-0.01-0.05}$ | 0.11$^{+0.04+0.02+0.00}_{-0.06-0.02-0.03}$ |
| CCWZ [22]      | 0.12                                   | 0.15                                   | 0.18                            | 0.19                            |
| IKS [23]       | 0.17                                   | 0.092                                  | 0.27                            | 0.17                            |
| IKS [24]       | 0.18                                   | 0.098                                  | 0.31                            | 0.20                            |
| HNV [25]       | 0.11                                   | 0.066                                  | 0.17                            | 0.13                            |

are dramatically different. As indicated from table II the form factors $V_1$ and $V_2$ for $B_c \rightarrow h_c$ have different signs.

Thus the longitudinally polarized decay width receives constructive contributions as we can see in Eq. (30). The form factor $A_{B_c,h_c}$ is small which suppresses the transversely polarized decay width. Accordingly, a large $\Gamma_T$ is expected: $\frac{\Gamma_T}{\Gamma_\tau} \simeq 11.1$ for $B_c \rightarrow h_c \bar{l} \nu$ decays and $\frac{\Gamma_T}{\Gamma_\tau} \simeq 4.7$ for $B_c \rightarrow h_c \tau \bar{\nu}_\tau$. The situation is dramatically different for $B_c \rightarrow \chi_{c1}$ decays.

The form factors $V_1^{B_c,\chi_{c1}}$ and $V_2^{B_c,\chi_{c1}}$ have the same sign, which gives destructive contributions to the longitudinally polarized decay width. Form factors $A_{B_c,\chi_{c1}}^{V}$ and $V_1^{B_c,\chi_{c1}}$ are large and thus the minus polarized decay width is large. The polarization fraction $\frac{\Gamma_T}{\Gamma_\tau}$ is reduced and predicted as: $\frac{\Gamma_T}{\Gamma_\tau} \simeq 0.24$ for $B_c \rightarrow \chi_{c1} \bar{l} \nu$ decays and $\frac{\Gamma_T}{\Gamma_\tau} \simeq 0.27$ for $B_c \rightarrow \chi_{c1} \tau \bar{\nu}_\tau$. Compared with results in Refs. [22, 23, 24, 25] for the $B_c \rightarrow (h_c, \chi_{c0,1,2}) \bar{l} \nu$ decays which are collected in table II, we can see that most of our predictions on the semileptonic $B_c$ decays are comparable with their predictions. These results will be tested at the ongoing and forthcoming hadron colliders.

IV. CONCLUSION

Due to its unique properties, $B$ physics has attracted abundant attentions. Measurements on the CKM matrix elements are becoming more and more accurate, which makes the goal to test the CP origins in and beyond SM much more practicable. $B_c$ meson decays provide another promising place to continue the errand in $B$ meson decays, and offer a new window to explore strong interactions. Although the $B_c$ meson can not be generated on the two $B$ factories, it has a promising prospect on the ongoing and forthcoming hadron colliders. The high statistics of $B_c$ meson at the forthcoming hadron colliders can compensate for the hadronic pollution and make it suitable for the precise determination of many standard model parameters. Because of these interesting features, we have studied the $B_c$ transition form factors in the covariant light-front quark model, which are relevant for the semileptonic $B_c$ decays.

Branching fractions of $B_c \rightarrow (h_c, \chi_{c0,1,2}) \bar{l} \nu (l = e, \mu)$ decays have the order $10^{-3}$ while branching fractions of $B_c \rightarrow (h_c, \chi_{c0,1,2}) \tau \bar{\nu}_\tau$ are suppressed by one order. Compared with branching fractions of the $B_c \rightarrow \eta_c \bar{l} \nu$ and $B_c \rightarrow J/\psi \bar{l} \nu$ decays, the branching ratios of $B_c \rightarrow (h_c, \chi_{c0,1,2}) \bar{l} \nu$ are smaller by a factor of $2 - 10$. The polarizations $\Gamma_T$ of $B_c \rightarrow h_c \bar{l} \nu$ and $B_c \rightarrow \chi_{c1} \bar{l} \nu$ are dramatically different: it is very large for $B_c \rightarrow h_c \bar{l} \nu$ but very small for $B_c \rightarrow \chi_{c0} \bar{l} \nu$. Most of our predictions on the semileptonic $B_c$ decays are comparable with results in the literature for the $B_c \rightarrow (h_c, \chi_{c0,1,2}) \bar{l} \nu$ decays. These results will be tested at the ongoing and forthcoming hadron colliders.
APPENDIX A: SOME SPECIFIC RULES IN THE $p^-$ INTEGRATION

When performing the $p^-$ integration, we need to include the zero-mode contribution. This amounts to performing the integration in a proper way in this approach. To be more specific, for $\hat{p}_1'$ under integration we use the following rules [10,11]

\[
\hat{p}'_{1\mu} \equiv P_\mu A_1^{(1)} + q_\mu A_2^{(1)}, \hat{N}_2 \rightarrow Z_2,
\]
\[
\hat{p}'_{1\mu} \hat{p}'_{1\nu} \equiv g_{\mu\nu} A_1^{(2)} + P_\mu P_\nu A_2^{(2)} + (P_\mu q_\nu + q_\mu P_\nu) A_3^{(2)} + q_\mu q_\nu A_4^{(2)},
\]
\[
\hat{p}'_{1\mu} \hat{p}'_{1\nu} \hat{p}'_{1\alpha} \equiv (g_{\mu\nu} P_\alpha + g_{\mu\alpha} P_\nu + g_{\nu\alpha} P_\mu) A_1^{(3)} + (g_{\mu\nu} q_\alpha + g_{\mu\alpha} q_\nu + g_{\nu\alpha} q_\mu) A_2^{(3)} + (P_\mu p_\nu q_\alpha + P_\mu q_\nu P_\alpha + q_\mu P_\nu P_\alpha) A_3^{(3)} + (q_\mu q_\nu P_\alpha + q_\mu P_\nu q_\alpha + P_\mu q_\nu q_\alpha) A_4^{(3)} + q_\mu q_\nu q_\alpha A_5^{(3)},
\]
\[
\hat{p}'_{1\mu} \hat{N}_2 \equiv q_\mu \left[ A_2^{(1)} Z_2 + \frac{q_\perp P}{q^2} A_1^{(2)} \right],
\]
\[
\hat{p}'_{1\mu} \hat{p}'_{1\nu} \hat{N}_2 \equiv g_{\mu\nu} A_1^{(2)} Z_2 + q_\mu q_\nu \left[ A_2^{(2)} Z_2 + 2 \frac{q_\perp P}{q^2} A_1^{(1)} A_2^{(1)} \right],
\]
where the symbol $\doteq$ reminds us that the above equations are true only after integration. In the above equation, $A_j^{(i)}$ are functions of $x_{1,2}$, $p_\perp^2$, $p_\perp \cdot q_\perp$ and $q^2$. Their explicit expressions have been studied in Ref. [10,11]:

\[
Z_2 = \hat{N}_1^t + m_1^2 - m_2^2 + (1 - 2x_1)M^2 + (q^2 + q \cdot P) \frac{p_\perp^2 \cdot q_\perp}{q^2},
\]
\[
A_1^{(1)} = \frac{x_1}{2}, \quad A_2^{(1)} = A_1^{(1)} - \frac{p_\perp \cdot q_\perp}{q^2},
\]
\[
A_1^{(2)} = -p_\perp^2 - \frac{p_\perp \cdot q_\perp}{q^2}, \quad A_2^{(2)} = (A_1^{(1)})^2, \quad A_3^{(2)} = A_1^{(1)} A_2^{(1)},
\]
\[
A_4^{(2)} = (A_2^{(1)})^2 - \frac{1}{q^2} A_1^{(1)}, \quad A_1^{(3)} = A_1^{(1)} A_2^{(1)}, \quad A_2^{(3)} = A_2^{(1)} A_3^{(1)},
\]
\[
A_3^{(3)} = A_1^{(2)} A_2^{(1)}, \quad A_6^{(3)} = A_2^{(3)} A_4^{(1)}, \quad A_5^{(3)} = A_1^{(1)} A_4^{(2)}.
\]

We do not show the spurious contributions in Eq. (A2) since they are numerically vanishing.

APPENDIX B: EXPRESSIONS OF $B_c \rightarrow A, T$ FORM FACTORS

In this appendix, we collect the analytic expressions of $B_c \rightarrow h_c, \chi_{1,2}$ form factors in the covariant light-front quark model.

\[
A^{B_c \rightarrow h_c}(q^2) = (M' - M'') \frac{N_c}{16\pi^3} \int dx_2 dq_2^2 \frac{2h'_{\mu}h''_{\mu}A}{x_2 N_1^t N_1^t} \frac{1}{w^t_A} p_\perp^2,
\]
\[
V^{B_c \rightarrow h_c}(q^2) = \frac{1}{M' - M''} \frac{N_c}{16\pi^3} \int dx_2 dq_2^2 \frac{h'_{\mu}h''_{\mu}A}{x_2 N_1^t N_1^t} \left[ 4q^2 p_\perp^2 + (p_\perp \cdot q_\perp)^2 \right] \left[ 2x_1(M'^2 + M''^2) \right].
\]
\[-q^2 - q \cdot P - 2(q^2 + q \cdot P) \frac{p_1^2 \cdot q_1}{q^2} - 2(m_1' + m''_1)(m_1' - m_2) \}\), (B2)

\[V_{2}^{B, h_c}(q^2) = -(M' - M'') \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{2h'_p h''_A}{x_2 N_1' N_1''} \]

\[2x_2 q^2 + p_1^2 \cdot q_1 \left[ p_1^2 \cdot p_1'' + (x_1 m_2 + x_2 m_1')(x_1 m_2 + x_2 m_1') \right], \] (B3)

\[V_{0}^{B, h_c}(q^2) = \frac{M' - M''}{2M''} V_{2}^{B, h_c}(q^2) - \frac{M' + M''}{2M''} V_{2}^{B, h_c}(q^2) - \frac{q^2}{2M''} \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{h'_p h''_A}{x_2 N_1' N_1''} \]

\[\frac{4}{w' A} \left( |M|^2 + M'^{\prime 2} - q^2 + 2(m_1' - m_2)(m_2 - m''_1)|A^2_3 + A^2_4 - A^2_2 \right) \]

\[+ Z_2(3A^2_2 - 2A^2_4 - 1) + \frac{1}{2} [x_1 (q^2 + q \cdot P) - 2M'^2 - 2p_1^2 \cdot q_1 - 2m_1'(m_2 - m''_1) \]

\[-2m_2(m_1' - m_2)|A^2_1 + A^2_2| - 1) + q \cdot P \left[ p_1^2 + \frac{(p_1' \cdot q_1)^2}{q^2} \right] \left( 4A^2_2 \right) \right). \] (B4)

\[A_{B, X_1}(q^2) = (M' - M'') \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{2h'_p h''_A}{x_2 N_1' N_1''} \left\{ x_1 m_2 + x_1 m_2 + (m_1' + m''_1) \frac{p_1^2 \cdot q_1}{q^2} \right\}, \] (B5)

\[V_{1}^{B, X_1}(q^2) = -\frac{1}{M' - M''} \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{h'_p h''_A}{x_2 N_1' N_1''} \left\{ 2x_1 (m_2 - m_1')(M''_0^2 + M''_1^2) + 4x_1 m_1'M''_0 \right.

\[+2x_2 m_1' q \cdot P + 2m_2 q^2 - 2x_1 m_2 (M'^2 + M''^2) + 2(m_1' - m_2)(m_1' - m''_1)^2 \]

\[+8(m_1' - m_2) \left[ p_1^2 + \frac{(p_1' \cdot q_1)^2}{q^2} \right] + 2(m_1'' - m''_1)(q^2 + q \cdot P) \frac{p_1^2 \cdot q_1}{q^2} \right\}, \] (B6)

\[V_{2}^{B, X_1}(q^2) = (M' - M'') \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{2h'_p h''_A}{x_2 N_1' N_1''} \left\{ \left( x_1 - x_2 \right)(x_2 m_1' + x_1 m_2) - 2x_1 m_2 - m''_1 + (x_2 - x_1) m_1' \frac{p_1^2 \cdot q_1}{q^2} \right\}, \] (B7)

\[V_{0}^{B, X_1}(q^2) = \frac{M' - M''}{2M''} V_{1}^{B, X_1}(q^2) - \frac{M' + M''}{2M''} V_{2}^{B, X_1}(q^2) - \frac{q^2}{2M''} \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{h'_p h''_A}{x_2 N_1' N_1''} \]

\[\left\{ 2(2x_1 - 3)(x_2 m_1' + x_1 m_2) - 8(m_1' - m_2) \frac{p_1^2}{q^2} + 2 \frac{(p_1' \cdot q_1)^2}{q^2} \right\} \]

\[-\left( 14 - 12x_1 \right) m_1' + 2m''_1 - 8(12x_1) m_2 \frac{p_1^2 \cdot q_1}{q^2} \right\}. \] (B8)

Notice that \(m''_1 = m_2 = m_c\) in the case of the outgoing p-wave charmonia, so \(\frac{1}{w''_A}\) is zero, leading to the vanishing term of \(\frac{1}{w''_A}\) in the form factors.

\[h_{B, X_1}(q^2) = \frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{2h'_p h''_A}{x_2 N_1' N_1''} \left\{ x_1 m_1' + x_1 m_2 + (m_1' - m''_1) \frac{p_1^2 \cdot q_1}{q^2} \right. \]

\[+ \frac{2}{w''_T} \left[ p_1^2 + \frac{(p_1' \cdot q_1)^2}{q^2} \right] + \left[ (m_1' - m''_1)(A^2_3 + A^2_4) + (m_1'' + m_1' - m_2)(A^2_2 + A^2_3) \right. \]

\[\left. - m_1'(A^2_1 + A^2_2) + \frac{2}{w''_T} (2A^3_1 + 2A^3_2 - A^3_2) \right\}, \] (B9)

\[k_{B, X_1}(q^2) = -\frac{N_c}{16\pi^3} \int dx_2 d^2p_1' \frac{h'_p h''_A}{x_2 N_1' N_1''} \left\{ 2x_1 (m_2 - m_1')(M''_0^2 + M''_1^2) - 4x_1 m_1''M''_0^2 + 2x_2 m_1' q \cdot P \right\} \]
\[\begin{align*}
+ 2m_2 q^2 - 2x_1 m_2 (M'^2 + M''^2) + 2(m'_1 - m_2)(m''_1 + m''_2)^2 \\
+ 8(m'_1 - m_2) \left[ p'^2_\perp + \frac{(p'^1_\perp \cdot q^\perp_\perp)^2}{q^2} \right] + 2(m'_1 + m''_1)(q^2 + q \cdot P) p'^1_\perp \cdot q^\perp_\perp \\
- \frac{4}{q^2 w_T'} \frac{q'^2 + (p'^1_\perp \cdot q^\perp_\perp)^2}{q^2 w_T'} \left[ 2x_1 (M'^2 + M''^2) - q^2 - q \cdot P - 2(q^2 + q \cdot P) p'^1_\perp \cdot q^\perp_\perp \right] \\
- 2(m'_1 - m''_1)(m'_1 - m_2) \right] \\
+ \frac{N_c}{16 \pi^2} \int \frac{d^2 p'_\perp}{x_2 N_1' N_1''} \left\{ 2(A^{(1)}_1 + A^{(2)}_2)|m_2 (q^2 - \hat{N}'_1 - \hat{N}_1'' - m'^2_1 - m''_1 - m'^2_2) \right. \\
- m'_1 (M'^2 - \hat{N}'_1 - m'^2_1 - m'_2) - m''_1 (M'^2 - \hat{N}'_1 - m'^2_1 - m''_1 - m''_2) - 2m'_1 m''_1 m''_2 \right. \\
+ (m'_1 + m''_1)(A^{(1)}_2 Z_2 + \frac{P \cdot q}{q^2} A^{(2)}_1) + 16(m'_1 - m'_2)(A^{(3)}_1 + A^{(3)}_2) + 4(2m'_1 - m''_1 - m_2)A^{(2)}_1 \\
+ \frac{4}{w_T'} \left[ (M'^2 + M''^2 - q^2 + 2m'_1 - m_2)(m''_1 + m''_2)(2A^{(3)}_1 + 2A^{(2)}_2 - A^{(2)}_1) \right] \\
- 4[A^{(2)}_3 Z_2 + \frac{P \cdot q}{q^2} A^{(2)}_1(2)]^2 + 2A^{(2)}_2 \left] \right\}, \\
B^{x = 2}(q^2) = - \frac{N_c}{16 \pi^2} \int \frac{d^2 p'_\perp}{x_2 N_1' N_1''} \left\{ (x_1 - x_2)(x_2 m'_1 + x_1 m_2) - 2[x_2 m_2 + m''_1 + (x_2 - x_1)m'_1] p'^1_\perp \cdot q^\perp_\perp \\
- \frac{2}{q^2 w_T'} \frac{q'^2 + (p'^1_\perp \cdot q^\perp_\perp)^2}{q^2 w_T'} \left[ p'^1_\perp \cdot q^\perp_\perp + (x_1 m_2 + x_2 m'_1)(x_1 m_2 - x_2 m''_1) \right] \\
+ \frac{N_c}{16 \pi^2} \int \frac{d^2 p'_\perp}{x_2 N_1' N_1''} \left\{ 8(m'_1 - m_2)(A^{(3)}_1 + 2A^{(3)}_2 + A^{(5)}_3) - 2m'_1(A^{(1)}_1 + A^{(2)}_2) \right. \\
+ 4(2m'_1 - m''_1 - m_2)(A^{(2)}_2 + A^{(3)}_2 + 2(m'_1 + m''_1)(A^{(2)}_2 + 2A^{(2)}_3 + A^{(4)}_2) \\
+ \frac{2}{w_T'} \left[ (M'^2 + M''^2 - q^2 + 2m'_1 - m_2)(m''_1 + m''_2)(A^{(3)}_3 + 2A^{(3)}_4 + A^{(5)}_5 - A^{(2)}_2 - A^{(3)}_2) \right] \\
+ [q^2 - \hat{N}'_1 - \hat{N}_1'' - (m'_1 + m''_1)(A^{(2)}_2 + 2A^{(3)}_3 + A^{(4)}_2 - A^{(1)}_1 - A^{(2)}_2)] \right\}, \\
B^{x = 2}(q^2) = - \frac{N_c}{16 \pi^2} \int \frac{d^2 p'_\perp}{x_2 N_1' N_1''} \left\{ 2(2x_1 - 3)(x_2 m'_1 + x_1 m_2) - 8(m'_1 - m_2) \right. \\
\frac{p'^2_\perp + 2(p'^1_\perp \cdot q^\perp_\perp)^2}{q^2} \\
- [(14 - 12x_1)m'_1 - 2m''_1 - (8 - 12x_1)m_2] p'^1_\perp \cdot q^\perp_\perp \\
+ \frac{4}{w_T'} \left[ (M'^2 + M''^2 - q^2 + 2m'_1 - m_2)(m''_1 + m''_2)(A^{(2)}_2 + 2A^{(3)}_2 - A^{(1)}_2) \\
+ Z_2(3A^{(1)}_2 - 2A^{(2)}_4 - 1) + \frac{1}{2} [x_1 q^2 + q \cdot P - 2M'^2 - 2p'^1_\perp \cdot q^\perp_\perp - 2m'_1(m''_1 + m''_2) \\
- 2m_2(m'_1 - m_2)](A^{(1)}_1 + A^{(2)}_2 - 1) + q \cdot P \left[ \frac{p'^2_\perp + (p'^1_\perp \cdot q^\perp_\perp)^2}{q^2} \right] (A^{(2)}_2 - 3) \right. \\
+ \frac{N_c}{16 \pi^2} \int \frac{d^2 p'_\perp}{x_2 N_1' N_1''} \left\{ 8(m'_1 - m_2)(A^{(3)}_4 + 2A^{(3)}_5 + A^{(6)}_6) - 6m'_1(A^{(1)}_1 + A^{(2)}_2) \right. \\
+ 4(2m'_1 - m''_1 - m_2)(A^{(2)}_3 + 2A^{(3)}_2 + 2A^{(3)}_2 + A^{(4)}_2) \\
+ \frac{2}{w_T'} \left[ (M'^2 + M''^2 - q^2 + 2m'_1 - m_2)(m''_1 + m''_2)(A^{(2)}_4 + 2A^{(3)}_5 + A^{(6)}_6 - A^{(2)} - A^{(3)}_2) \\
2Z_2(3A^{(2)}_2 - 2A^{(3)}_6 - A^{(2)}_2) + 2 \frac{q \cdot P}{q^2} (6A^{(1)}_2 A^{(2)}_2 - 6A^{(2)}_2 A^{(3)}_2 + \frac{2}{q^2} (A^{(2)}_2)^2 - A^{(2)}_2) \right. \\
\end{align*}\]
\[ q^2 - 2M'^2 + N_1' - N_1'' \left( m_1' + m_2 \right)^2 + 2(m_1' - m_2)^2 \left( A_2^{(2)} + 2A_3^{(2)} + A_4^{(2)} - A_1^{(1)} - A_2^{(1)} \right). \]