Solutions of time-fractional differential equations using homotopy analysis method

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Abstract: We have used the homotopy analysis method to obtain solutions of linear and non-linear fractional partial differential differential equations with initial conditions. We replace the first order time derivative by $\psi$-Caputo fractional derivative, and also we compare the results obtained by the homotopy analysis method with the exact solutions.

Keywords: Homotopy analysis method, time-fractional differential equations, $\psi$-Caputo fractional derivative

1 Introduction

Fractional order differential equations, systems of fractional algebraic-differential equations and fractional integrodifferential equations have been widely studied. Methods for obtaining analytical solutions to these problems, in its nonlinear form, are commonly used, among them the Adomian decomposition method [1, 4, 10], homotopy perturbation method (HPM) [10] and homotopy analysis method (HAM) [7, 8]. Jafari and Seifi [5] have obtained solutions for linear and nonlinear fractional diffusion and wave equations by means of the HAM. Ganjiani [3] has discussed only nonlinear fractional differential equations and Xu et al. [17] have discussed fractional partial differential equations subject to the boundary conditions and initial condition, both by means of homotopy analysis method. Sotlo et al. [15] have applied HAM for solving integrodifferential equations and Zhang et al. [18] have investigate numerical solutions of higher-order fractional integrodifferential equations with boundary conditions. Zurigat et al. [19, 20] have used HAM to solve systems of fractional algebraic-differential equations.

Since, we are interested in the analytical solution of fractional partial differential equations, we must choose a particular fractional derivative. There are several types of fractional derivatives defined in terms of a respective fractional integral [2, 11, 13, 16]. Perhaps the various ways of approaching the fractional derivative reside in the fact that, until now, we don’t have a classic geometric interpretation, as in the case of an ordinary differential where we associate the concept of derivative with the concept of tangency. Here, we choose the $\psi$-Caputo fractional derivative [2] to discuss our applications. The choice of this fractional differentiation operator is due to the fact that when we derive a constant, the result is identically zero and as a particular case recovers the classical Caputo fractional derivative.

The paper is organized in the following way. In Section 2 we present some basic definitions of the fractional calculus. In Section 3 we describe the HAM, and three examples are present in 4. The first approach, we discuss linear time-fractional diffusion equation; the second approach a nonlinear time-fractional gas-dynamic equation is investigated. The third approach, we discuss nonlinear time-fractional KdV equation. At the end of each application numerical solutions to show the efficiency of the method are presented. Concluding remarks close the paper.
2 Fractional calculus

In this section we present some concepts of the fractional calculus that are useful in the remainder of the text.

**Definition 1.** Let $\alpha > 0$, $I = [a,b]$ be a finite or infinite interval, $f$ an integrable function defined on $I$ and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in I$. The left fractional integral of $f$ with respect to another function $\psi$ of order $\alpha$ is defined as

$$I_{a+}^{\alpha,\psi} [f(x,t)] = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(x,\tau)d\tau.$$  \hspace{1cm} (1)

For $\alpha = 0$, we have

$$I_{a+}^{0,\psi} [f(x,t)] = f(x,t).$$

**Definition 2.** Let $\alpha > 0$, $n \in \mathbb{N}$, $I$ is the interval $-\infty \leq a < b \leq \infty$, $f,\psi \in C^n(I)$ two functions such that $\psi$ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left $\psi$-Caputo fractional derivative of $f$ of order $\alpha$ is given by

$$C D_{a+}^{\alpha,\psi} [f(x,t)] = I_{a+}^{n-\alpha,\psi} \left( \frac{1}{\psi'(t)} \frac{\partial}{\partial t} \right)^n f(x,t),$$

where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}, \quad n = \alpha \text{ for } \alpha \in \mathbb{N}.$$  

To simplify notation, we will use the abbreviated notation

$$f^{[n],\psi}(x,t) = \left( \frac{1}{\psi'(t)} \frac{\partial}{\partial t} \right)^n f(x,t).$$

**Property 1.** Let $f \in C^n[a,b]$, $\alpha > 0$ and $\delta > 0$, \cite{2}.

1. $f(t) = (\psi(t) - \psi(a))^{\delta-1}$, then

$$I_{a+}^{\alpha,\psi} f(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha + \delta)} (\psi(t) - \psi(a))^\alpha + \delta - 1.$$  

2. $I_{a+}^{\alpha,\psi} C D_{a+}^{\alpha,\psi} [f(x,t)] = f(x,t) - \sum_{k=0}^{n-1} \frac{f^{[k],\psi}(x,a)}{k!}(\psi(t) - \psi(a))^k$, where $n - 1 < \alpha < n$ with $n \in \mathbb{N}$.

**Definition 3.** Let $\alpha > 0$ and $a > 0$. The one-parameter Mittag-Leffler function has the power series representation \cite{2} \cite{2}.

$$E_{\alpha}[\psi(t) - \psi(a)] = \sum_{m=1}^{\infty} \frac{(\psi(t) - \psi(a))^m}{\Gamma(m\alpha + 1)}. \hspace{1cm} (2)$$

3 Homotopy analysis method

In this section we introduce the basic ideas of the HAM by means of the description of general nonlinear problems.
We consider the following nonlinear differential equation in a general form

\[ \mathcal{N}[u(x, t)] = 0, \]  
(3)

where \( \mathcal{N} \) is a nonlinear differential operator, \( x \) and \( t \) are independent variables and \( u \) is an unknown function. We then construct the so-called zero-order deformation equation

\[ (1 - p)\mathcal{L}[\phi(x, t; p) - u_0(x, t)] = phH(x, t)\mathcal{N}[\phi(x, t; p)], \]  
(4)

where \( p \in [0, 1] \) is an embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( H(x, t) \) is an auxiliary function and \( \phi(x, t; p) \) is a function of \( x, t \) and \( p \). Let \( u_0(x, t) \) be an initial approximation of Eq.(3) and \( \mathcal{L} = C_{D_{a+}^{\alpha, \psi}} \) denotes an auxiliary linear differential operator with the property

\[ \mathcal{L}[\phi(x, t)] = 0, \quad \text{for} \quad \phi(x, t) = 0. \]

When \( p = 0 \) and \( p = 1 \), we have

\[ \phi(x, t; 0) = u_0(x, t), \quad \text{and} \quad \phi(x, t; 1) = u(x, t), \]

respectively. As the embedding parameter \( p \) increases from 0 to 1, the solution \( \phi(x, t; p) \) depends upon the embedding parameter \( p \) and varies from the initial approximation \( u_0(x, t) \) to the solution \( u(x, t) \).

Expanding \( \phi(x, t; p) \) in a Taylor’s series with respect to \( p \), we have

\[ \phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)p^m, \]  
(5)

where

\[ u_m(x, t) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \phi(x, t; p) \bigg|_{p=0}. \]

Assume that the auxiliary parameter \( h \), the auxiliary function \( H(x, t) \), the initial approximation \( u_0(x, t) \), and the auxiliary linear operator \( \mathcal{L} = C_{D_{a+}^{\alpha, \psi}} \) are so properly chosen that the series, Eq.(5), converges at \( p = 1 \). Then, the series Eq.(5), at \( p = 1 \), becomes

\[ u(x, t) = \phi(x, t; 1) = u_m(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t). \]

Differentiating Eq.(4), \( m \) times with respect to \( p \), then setting \( p = 0 \), and dividing it by \( m! \), we obtain the \( m \)-th order deformation equation

\[ \mathcal{L}[u_m(x, t) - \mathcal{X}_m u_{m-1}(x, t)] = hH(x, t)R_m(\bar{u}_{m-1}, x, t), \]  
(6)

with \( \bar{u}_m = \{u_0(x, t), u_1(x, t), \ldots, u_n(x, t)\} \) and

\[ R_m(\bar{u}_{m-1}, x, t) = \frac{1}{(m - 1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}[\phi(x, t; p)] \bigg|_{p=0} \]

where we have introduced the notation

\[ \mathcal{X}_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]  
(7)

Operating the fractional integral operator \( \int_{a+}^{\alpha, \psi} \), given by Eq.4, on both sides of Eq.(6), we have

\[ u_m(x, t) = \mathcal{X}_m u_{m-1}(x, t) - \mathcal{X}_m \sum_{k=0}^{n-1} \frac{u_{m-1}(x, t)}{k!} (\psi(t) - \psi(a))^k + hH(x, t)\int_{a+}^{\alpha, \psi} [R_m(\bar{u}_{m-1}, x, t)], \quad m \geq 1. \]  
(8)
Thus, we obtain $u_1(x,t), u_2(x,t), \cdots$ by means of Eq.(8). The $M$th-order approximation of $u(x,t)$ is given by
\[ u(x,t) = \sum_{m=0}^{M} u_m(x,t), \]
and for $M \to \infty$, we get an accurate approximation of Eq.(3).

4 Applications

In this section we apply the HAM to solving linear and nonlinear fractional partial differential equations.

Application 1. Let $t > 0$, $x > 0$ and $u = u(x,t)$. Consider the linear time-fractional diffusion equation \([4, 5]\)
\[ C D_{a+}^{\alpha,\psi} u = \frac{\partial^2 u}{\partial x^2} + u, \quad 0 < \alpha < 1, \] (9)
whose solution satisfies the initial condition
\[ u(x,a) = \cos(\pi x). \] (10)

In order to solve Eq.(9) by means of HAM, satisfying the initial condition given by Eq.(10), it is convenient to choose the initial approximation
\[ u_0(x,t) = \cos(\pi x) \] (11)
and the linear differential operator
\[ L[\phi(x,t;p)] = C D_{a+}^{\alpha,\psi}[\phi(x,t;p)], \]
satisfying the property
\[ L[c] = 0, \]
where $c$ is an arbitrary constant. We define the nonlinear differential operator
\[ N[\phi(x,t;p)] = C D_{a+}^{\alpha,\psi}[\phi(x,t;p)] - \frac{\partial^2}{\partial x^2}[\phi(x,t;p)] - \phi(x,t;p). \] (12)

Using Eq.(12) and the assumption $H(x,t) = 1$ we construct the zero-order deformation equation
\[ (1 - p)L[\phi(x,t;p) - u_0(x,t)] = p h N[\phi(x,t;p)]. \] (13)

Obviously, when $p = 0$ and $p = 1$, we get
\[ \phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \phi(x,t;1) = u(x,t), \]
respectively. So the $m$th-order deformation equation is
\[ L[u_m(x,t) - \mathcal{X}_m u_{m-1}(x,t)] = h R_m(\bar{u}_{m-1}, x,t), \] (14)
subject to the initial condition $u_m(x,a) = 0$ where $\mathcal{X}_m$ is defined by Eq.(7) and
\[ R_m(\bar{u}_{m-1}, x,t) = C D_{a+}^{\alpha,\psi} u_{m-1}(x,t) - \frac{\partial^2}{\partial x^2} u_{m-1}(x,t) - u_{m-1}(x,t). \]
Now we apply the integral fractional operator $I_{a+}^{\alpha,\psi}$ on both sides of Eq. (14) to get

$$C D_{a+}^{\alpha,\psi}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar I_{a+}^{\alpha,\psi} \left[ C D_{a+}^{\alpha,\psi} u_m(x,t) - \frac{\partial^2}{\partial x^2} u_{m-1}(x,t) - u_{m-1}(x,t) \right],$$

whose solution has the form

$$u_m(x,t) = \sum_{k=0}^{n-1} \frac{[k]!}{k!} (\psi(t) - \psi(a))^k - \chi_m u_{m-1}(x,t)
\begin{equation}
+ \chi_m \sum_{k=0}^{n-1} \frac{u_{m-1}(x,a)}{k!} (\psi(t) - \psi(a))^k
\end{equation}
- I_{a+}^{\alpha,\psi} \left[ \frac{\partial^2}{\partial x^2} u_{m-1}(x,t) + u_{m-1}(x,t) \right], \quad m \geq 1.
$$

For $0 < \alpha < 1$, then $n = 1$, we can rewrite the above equation as

$$u_m(x,t) = (\chi_m + \hbar) u_{m-1}(x,t) - (\chi_m + \hbar) u_{m-1}(x,a)
\begin{equation}
- \hbar I_{a+}^{\alpha,\psi} \left[ \frac{\partial^2}{\partial x^2} u_{m-1}(x,t) + u_{m-1}(x,t) \right], \quad m \geq 1.
\end{equation}
$$

From Eq. (11) and Eq. (15), we obtain

$$u_0(x,t) = \cos(\pi x),
\begin{equation}
u_1(x,t) = -\hbar I_{a+}^{\alpha,\psi} \left[ \frac{\partial^2}{\partial x^2} u_0(x,t) + u_0(x,t) \right] = -\hbar (1 - \pi^2) \cos(\pi x) \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)},
\end{equation}
\begin{equation}
u_2(x,t) = -(1 + \hbar)(1 - \pi^2) \cos(\pi x) \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} + \hbar^2 (1 - \pi^2)^2 \cos(\pi x) \frac{(\psi(t) - \psi(a))^{2\alpha}}{\Gamma(2\alpha + 1)},
\end{equation}
\begin{equation}
\vdots
\end{equation}
$$

An accurate approximation of Eq. (9) is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots,$$

and, when $h = -1$, we have

$$u(x,t) = \cos(\pi x) \left[ 1 + \frac{(1 - \pi^2)}{\Gamma(\alpha + 1)} (\psi(t) - \psi(a))^\alpha + \frac{(1 - \pi^2)^2 (\psi(t) - \psi(a))^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots \right]
\begin{equation}
= \cos(\pi x) \left[ 1 + \sum_{m=1}^{\infty} \frac{(1 - \pi^2)^m (\psi(t) - \psi(a))^{m\alpha}}{\Gamma(m\alpha + 1)} \right],
\end{equation}
$$

or in terms of the one-parameter Mittag-Leffler function Eq. (2),

$$u(x,t) = \cos(\pi x) E_\alpha [(1 - \pi^2)(\psi(t) - \psi(a))^\alpha].
\begin{equation}
(16)
\end{equation}
$$

We note two important special cases of Eq. (16). First taking $\psi(t) = t$ and $a = 0$. In this case the solution Eq. (16) takes the form

$$u(x,t) = \cos(\pi x) E_\alpha [(1 - \pi^2)t^\alpha].
\begin{equation}
(17)
\end{equation}
$$

Eq. (17) recovers the solutions found by Jafari and Seifi [3], obtained by means of the HAM and Jafari and Daftardar-Gejji [4] using the Adomian decomposition method.
On the other hand, if $\psi(t) = \ln t$ and $a > 0$, the solution Eq. (16) becomes

$$u(x, t) = \cos(\pi x) E_\alpha \left[ (1 - \pi^2) \left( \frac{\ln t}{a} \right)^\alpha \right].$$

(18)
Figura 3: Approximate solutions $u(0.1,t)$ using 3-terms, $\psi(t) = \ln t$, $a = 1$ and $\alpha \to 1$. Solid line (exact solution using Eq.(13)): $h = -1$, dashdotted: $h = -0.7$, and dotted: $h = -1.2$.

**Application 2.** Let $t > 0$, $x > 0$ and $u = u(x,t)$. Consider the nonlinear time-fractional gas-dynamic equation [14]

$$ C \partial_{a+}^\alpha \psi u + u \cdot \frac{\partial u}{\partial x} - u + u^2 = 0, \quad 0 < \alpha < 1 \tag{19} $$

whose solution satisfies the initial condition

$$ u(x,a) = e^{-x}. \tag{20} $$

In order to solve Eq.(19), we choose the initial approximation

$$ u_0(x,t) = e^{-x} $$

and the linear operator

$$ \mathcal{L}[\phi(x,t;p)] = C \partial_{a+}^\alpha [\phi(x,t;p)], $$

with the property $\mathcal{L}[c] = 0$, where $c$ is a constant. From Eq.(19), we define the nonlinear differential operator

$$ \mathcal{N}[\phi(x,t;p)] = C \partial_{a+}^\alpha [\phi(x,t;p)] + \phi(x,t;p) \cdot \frac{\partial}{\partial x} [\phi(x,t;p)] - \phi(x,t;p) + [\phi(x,t;p)]^2. $$

Taking $H(x,t) = 1$, we construct the zero-order deformation equation

$$ (1-p)\mathcal{L}[\phi(x,t;p) - u_0(x,t)] = p\mathcal{N}[\phi(x,t;p)]. \tag{21} $$

Obviously, when $p = 0$ and $p = 1$, we get

$$ \phi(x,t;0) = u_0(x,t) = e^{-x} \quad \text{and} \quad \phi(x,t;1) = u(x,t), $$

respectively. The $m$th-order deformation equation is given by

$$ \mathcal{L}[u_m(x,t) - X_m u_{m-1}(x,t)] = hR_m(\bar{u}_{m-1},x,t), \tag{22} $$

subject to the initial condition $u_m(x,a) = 0$, where

$$ R_m(\bar{u}_{m-1},x,t) = C \partial_{a+}^\alpha u_{m-1}(x,t) + \sum_{i=0}^{m-1} u_i(x,t) \cdot \frac{\partial}{\partial x} u_{m-1-i}(x,t) - u_{m-1}(x,t) $$

$$ + \sum_{i=0}^{m-1} u_i(x,t) u_{m-1-i}(x,t). \tag{23} $$
Operating the fractional integral operator $I_{a+}^{α,ψ}$ on both sides of Eq.\([22]\), we have

$$u_m(x,t) = (X_m + h)u_{m-1}(x,t) - (X_m + h)u_{m-1}(x,a) + hI_{a+}^{α,ψ} \left[ \sum_{i=0}^{m-1} u_i(x,t) \cdot \frac{∂}{∂x} u_{m-1-i}(x,t) - u_{m-1}(x,t) + \sum_{i=0}^{m-1} u_i(x,t) u_{m-1-i}(x,t) \right].$$

(24)

In this way, we obtain

$$u_0(x,t) = e^{-x},$$

$$u_1(x,t) = hI_{a+}^{α,ψ} \left\{ u_0(x,t) \cdot \frac{∂}{∂x} u_0(x,t) - u_0(x,t) + [u_0(x,t)]^2 \right\} = -he^{-x} \frac{(\psi(t) - \psi(a))^α}{Γ(α + 1)},$$

$$u_2(x,t) = (1+h)u_1(x,t) - (1+h)u_1(x,a) + hI_{a+}^{α,ψ} \left\{ u_0(x,t) \cdot \frac{∂}{∂x} u_1(x,t) + u_1(x,t) \cdot \frac{∂}{∂x} u_0(x,t) - u_1(x,t) - 2u_1(x,t) \cdot u_0(x,t) \right\} = -(1+h)he^{-x} \frac{(\psi(t) - \psi(a))^α}{Γ(α + 1)} + h^2 e^{-x} \frac{(\psi(t) - \psi(a))^{2α}}{Γ(2α + 1)},$$

$$u_3(x,t) = (1+h)u_2(x,t) - (1+h)u_2(x,a) + hI_{a+}^{α,ψ} \left\{ u_0(x,t) \cdot \frac{∂}{∂x} u_2(x,t) + u_1(x,t) \cdot \frac{∂}{∂x} u_2(x,t) - u_1(x,t) \cdot u_2(x,t) + 2u_0(x,t) \cdot u_2(x,t) + [u_1(x,t)]^2 \right\} = -(1+h)^2he^{-x} \frac{(\psi(t) - \psi(a))^α}{Γ(α + 1)} + 2h^2(1+h)e^{-x} \frac{(\psi(t) - \psi(a))^{2α}}{Γ(2α + 1)} - h^3 e^{-x} \frac{(\psi(t) - \psi(a))^{3α}}{Γ(3α + 1)},$$

$$\vdots$$

The solution $u(x,t)$ is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

and if $h = -1$, we have

$$u(x,t) = e^{-x} \left[ 1 + \frac{(\psi(t) - \psi(a))^α}{Γ(α + 1)} + \frac{(\psi(t) - \psi(a))^{2α}}{Γ(2α + 1)} + \frac{(\psi(t) - \psi(a))^{3α}}{Γ(3α + 1)} + \cdots \right]$$

$$= e^{-x} E_α[(\psi(t) - \psi(a))^α].$$

(25)

In particular, if $ψ(t) = t$ and $a = 0$, we can write the solution as

$$u(x,t) = e^{-x} E_α(t^α)$$

(26)

and if $α \to 1$, we obtain the solution found by Shone and Patra \([14]\) using the fractional complex transform and a new iterative method, this is, $u(x,t) = e^{t-x}$. On the other hand, if $ψ(t) = \ln t$ and $a > 0$, we obtain

$$u(x,t) = e^{-x} E_α \left[ \left( \frac{\ln \frac{t}{a} }{a} \right)^α \right].$$

(27)
Figura 4: Approximate solutions $u(0.2, t)$ using 4-terms and exact solution of Eq. (19) subject the initial condition Eq. (20) with $\psi(t) = t$, $a = 0$ and $\alpha \to 1$. Solid line (exact solution): $h = -1$, dashdotted: $h = -0.6$, and dashed: $h = -1.4$.

Figura 5: Exact solutions $u(0.2, t)$ using Eq. (26). Solid line: $\alpha \to 1$, dashdotted: $\alpha = 0.75$, and dotted: $\alpha = 0.4$. 
Figura 6: Approximate solutions $u(0.2, t)$ using 4-terms, $\psi(t) = \ln t$, $a = 1$ and $\alpha \rightarrow 1$. Solid line (exact solution using Eq. (27)): $h = -1$, dashdotted: $h = -2$, and dashed: $h = -0.5$.

**Application 3.** Let $t > 0$, $x > 0$, $u = u(x, t)$ and $0 < \alpha < 1$. Consider the following nonlinear time-fractional KdV equation [10, 12]

$$C D_{a+}^{\alpha, \psi} u(x, t) - \frac{\partial}{\partial x}[u(x, t)]^2 + \frac{\partial}{\partial x} \left[ u(x, t) \frac{\partial^2}{\partial x^2} u(x, t) \right] = 0,$$

(28)

whose solution satisfies the initial condition

$$u(x, a) = \sinh^2 \left( \frac{x}{2} \right).$$

(29)

Let $u_0(x, t)$ denote an initial approximation of $u(x, t)$, this is,

$$u_0(x, t) = \sinh^2 \left( \frac{x}{2} \right)$$

(30)

and we choose the linear differential operator $L = C D_{a+}^{\alpha, \psi}$, with the condition $L[c] = 0$ where $c$ is a constant. From Eq. (28), we define the nonlinear differential operator

$$N[\phi(x, t; p)] = C D_{a+}^{\alpha, \psi}[\phi(x, t; p)] + \phi(x, t; p) \cdot \frac{\partial}{\partial x} [\phi(x, t; p)] - \phi(x, t; p) + [\phi(x, t; p)]^2.$$

With the choice $H(x, t) = 1$ we have the zero-order deformation equation

$$(1 - p)L[\phi(x, t; p) - u_0(x, t)] = p h N[\phi(x, t; p)].$$

(31)

Obviously, when $p = 0$ and $p = 1$, we get

$$\phi(x, t; 0) = u_0(x, t) = \sinh^2 \left( \frac{x}{2} \right) \quad \text{and} \quad \phi(x, t; 1) = u(x, t),$$

respectively. The $m$th-order deformation equation can be expressed by

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h R_m(\bar{u}_{m-1}, x, t),$$

(32)

where

$$R_m(\bar{u}_{m-1}, x, t) = C D_{a+}^{\alpha, \psi} u_{m-1}(x, t) - \frac{\partial}{\partial x} \left[ \sum_{i=0}^{m-1} u_i(x, t) \cdot u_{m-1-i}(x, t) \right]$$

$$- \sum_{i=0}^{m-1} u_i(x, t) \frac{\partial^2}{\partial x^2} u_{m-1-i}(x, t).$$
\[
\text{Applying the fractional operator } I_{a+}^{\alpha,\psi} \text{ to this equation we find}
\]
\[
\begin{align*}
\left( X_m + \hbar \right) u_{m-1}(x, t) - (X_m + \hbar) u_{m-1}(x, a) \\
- \hbar I_{a+}^{\alpha,\psi} \left[ \frac{\partial}{\partial x} \left( \sum_{i=0}^{m-1} u_i(x, t) \cdot u_{m-1-i}(x, t) - \sum_{i=0}^{m-1} u_i(x, t) \frac{\partial^2}{\partial x^2} u_{m-1-i}(x, t) \right) \right].
\end{align*}
\]

\[
\text{Thereafter, we successively obtain}
\]
\[
\begin{align*}
u_0(x, t) &= \sinh^2 \left( \frac{x}{2} \right), \\
u_1(x, t) &= -\hbar I_{a+}^{\alpha,\psi} \left[ \frac{\partial}{\partial x} \left( u_0(x, t)^2 - u_0(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) \right) \right] = \hbar \sinh(x) \frac{(\psi(t) - \psi(a))^\alpha}{4\Gamma(\alpha + 1)}, \\
u_2(x, t) &= (1 + \hbar) u_1(x, t) - \hbar I_{a+}^{\alpha,\psi} \left[ \frac{\partial}{\partial x} \left( 2u_0(x, t) \cdot u_1(x, t) - u_0(x, t) \frac{\partial^2}{\partial x^2} u_1(x, t) - u_1(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) \right) \right] \\
&= (1 + \hbar) \hbar \sinh(x) \frac{(\psi(t) - \psi(a))^\alpha}{4\Gamma(\alpha + 1)} + \hbar^2 \cosh(x) \frac{(\psi(t) - \psi(a))^{2\alpha}}{8\Gamma(2\alpha + 1)}, \\
&\vdots
\end{align*}
\]

\[
The \text{second-order approximation of } u(x, t) \text{ is}
\]
\[
\begin{align*}
u(x, t) &= \sinh^2 \left( \frac{x}{2} \right) + \hbar \sinh(x) \frac{(\psi(t) - \psi(a))^\alpha}{4\Gamma(\alpha + 1)} + (1 + \hbar) \hbar \sinh(x) \frac{(\psi(t) - \psi(a))^\alpha}{4\Gamma(\alpha + 1)} \\
&+ \hbar^2 \cosh(x) \frac{(\psi(t) - \psi(a))^{2\alpha}}{8\Gamma(2\alpha + 1)}.
\end{align*}
\]

\[
\text{Taking } \hbar = -1, \text{ we have}
\]
\[
\begin{align*}
u(x, t) &= \sinh^2 \left( \frac{x}{2} \right) - \sinh(x) \frac{(\psi(t) - \psi(a))^\alpha}{4\Gamma(\alpha + 1)} + \cosh(x) \frac{(\psi(t) - \psi(a))^{2\alpha}}{8\Gamma(2\alpha + 1)}.
\end{align*}
\]

\[
\text{If } \psi(t) = t \text{ and } a = 0, \text{ the second-order approximation of } u(x, t) \text{ Eq.} (33) \text{ becomes}
\]
\[
\begin{align*}
u(x, t) &= \sinh^2 \left( \frac{x}{2} \right) - \frac{t^\alpha}{4\Gamma(\alpha + 1)} \sinh(x) + \frac{t^{2\alpha}}{8\Gamma(2\alpha + 1)} \cosh(x).
\end{align*}
\]

\[
\text{This solution is identical to the solution obtained using Rehman et al. [12] the combination of the double Sumudu transform and homotopy perturbation method and also obtained by Momani et al. [10] by homotopy perturbation method.}
\]

\[
\text{On the other hand, if } \psi(t) = \ln t \text{ and } a > 0, \text{ we obtain}
\]
\[
\begin{align*}
u(x, t) &= \sinh^2 \left( \frac{x}{2} \right) - \frac{\sinh(x)}{4\Gamma(\alpha + 1)} \left( \ln \frac{t}{a} \right)^\alpha + \frac{\cosh(x)}{8\Gamma(2\alpha + 1)} \left( \ln \frac{t}{a} \right)^{2\alpha}.
\end{align*}
\]
Figura 7: Approximate solutions $u(1,t)$ using 3-terms, $\psi(t) = t$, $a = 0$ and $\alpha \to 1$. Solid line: $h = -1$, dashdotted: $h = -2$, and dashed: $h = -0.8$.

Figura 8: Approximate solutions $u(1,t)$ using Eq. (35). Solid line: $\alpha \to 1$, dashdotted: $\alpha = 0.8$, and dashed: $\alpha = 0.6$. 
Figura 9: Approximate solutions $u(1,t)$ using Eq. (36). Solid line: $\alpha \to 1$, dashdotted: $\alpha = 0.7$, and dashed: $\alpha = 0.6$.

5 Concluding remarks

In this paper we have presented the HAM to obtain approximate solutions for linear and nonlinear fractional partial differential equations replacing the first order time derivative by the $\psi$-Caputo fractional derivative. We solve fractional partial differential equations, and to obtain explicit series solutions, we have presented numerical solutions. Therefore, we considered different values for $\alpha$ and for auxiliary parameter $h$. It is possible to control the convergence region of the solution series, obtained by means of HAM, adjusting the auxiliary parameter $h$. Mathematica has been used for draw graphs.

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