Zeta functions on tori using contour integration

Emilio Elizalde, Klaus Kirsten, Nicolas Robles and Floyd Williams

Abstract

A new, seemingly useful presentation of zeta functions on complex tori is derived by using contour integration. It is shown to agree with the one obtained by using the Chowla-Selberg series formula, for which an alternative proof is thereby given. In addition, a new proof of the functional determinant on the torus results, which does not use the Kronecker first limit formula nor the functional equation of the non-holomorphic Eisenstein series. As a bonus, several identities involving the Dedekind eta function are obtained as well.

Contents

1 Introduction 1
2 Argument principle technique 2
3 Description of the problem 5
4 Eigenvalue problem set up 8
5 Main result 10
6 Additional results and consequences 15
   6.1 The Chowla-Selberg series formula 15
   6.2 The Nan-Yue Williams formula 16
   6.3 The Kronecker first limit formula 17
   6.4 The Lambert series 18
   6.5 Functional equation for the remainder 19
7 Conclusion and outlook 22

1 Introduction

Zeta regularization and the theory of spectral zeta functions are powerful and elegant techniques that allow one to assign finite values to otherwise manifestly infinite quantities in a unique and well-defined way [7, 8, 12].

Suppose we have a compact smooth manifold $M$ with a Riemannian metric $g$ and a corresponding Laplace-Beltrami operator $\Delta = \Delta(g)$, where $\Delta$ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty.$$ (1.1)
If we denote by \( n_j \) the finite multiplicity of the \( j \)-th eigenvalue \( \lambda_j \) of \( \Delta \) then, by a result of H. Weyl [28], which says that the asymptotic behavior of the eigenvalues as \( j \to \infty \) is \( \lambda_j \sim j^{2/\dim M} \), we can construct the corresponding spectral zeta function as

\[
\zeta_M(s) = \sum_{j=1}^{\infty} \frac{n_j}{\lambda_j^s}, \tag{1.2}
\]

which is well-defined for \( \operatorname{Re}(s) > \frac{1}{2} \dim M \). Minakshisundaram and Pleijel [17] showed that \( \zeta_M(s) \) admits a meromorphic continuation to the whole complex plane and that, in particular, \( \zeta_M(s) \) is holomorphic at \( s = 0 \). This, in turn, means that \( \exp[-\zeta_M'(0)] \) is well-defined and Ray and Singer [22] set the definition

\[
\det(\Delta) = \prod_{k=1}^{\infty} \lambda_k^{n_k} := e^{-\zeta_M'(0)}, \tag{1.3}
\]

where it is understood that the zero eigenvalue of \( \Delta \) is not taken into the product. For the reader’s understanding, the motivation for this definition comes from the formal computation

\[
\exp \left[ -\frac{d}{ds} \bigg|_{s=0} \sum_{k=1}^{\infty} n_k \lambda_k^s \right] = \exp \left[ \sum_{k=1}^{\infty} n_k \log \lambda_k \right] = \prod_{k=1}^{\infty} e^{n_k \log \lambda_k} = \prod_{k=1}^{\infty} \lambda_k^{n_k}. \tag{1.4}
\]

As long as the spectrum is discrete, definition (1.3) is suitable for more general operators on other infinite dimensional spaces. In particular, it is useful for Laplace-type operators on smooth manifolds of a vector bundle over \( M \), see e.g. [19, 29].

### 2 Argument principle technique

Let us introduce the basic ideas used in this article by considering a generic one dimensional second order differential operator \( \mathcal{O} := -d^2/dx^2 + V(x) \) on the interval \([0, 1]\), where \( V(x) \) is a smooth potential. Let its eigenvalue problem be given by

\[
\mathcal{O} \phi_n(x) = \lambda_n \phi_n(x), \tag{2.1}
\]

and choose Dirichlet boundary conditions \( \phi_n(0) = \phi_n(1) = 0 \). This problem can be translated to a unique initial value problem [14, 15, 16]

\[
(\mathcal{O} - \lambda)u_\lambda(x) = 0, \tag{2.2}
\]

with \( u_\lambda(0) = 0 \) and \( u_\lambda'(0) = 1 \). The eigenvalues \( \lambda_n \) then follow as the solutions to the equation

\[
u_\lambda(1) = 0, \tag{2.3}
\]

where \( u_\lambda(1) \) is an analytic function of \( \lambda \). Let us recall the argument principle from complex analysis. It states that if \( f \) is a meromorphic function inside and on some counterclockwise contour \( \gamma \) with \( f \) having neither zeroes nor poles on \( \gamma \), then

\[
\int_\gamma dz \frac{f'(z)}{f(z)} = 2\pi i (N - P), \tag{2.4}
\]
where \( N \) and \( P \) are, respectively, the number of zeros and poles of \( f \) inside the contour \( \gamma \). There is a slightly stronger version of this statement called the generalized argument principle, stating that if \( f \) is meromorphic in a simply connected set \( D \) which has zeroes \( a_j \) and poles \( b_k \), if \( g \) is an analytic function in \( D \), and if we let \( \gamma \) be a closed curve in \( D \) avoiding \( a_j \) and \( b_k \), then

\[
\sum_{j} g(a_j)n(\gamma,a_j) - \sum_{k} g(b_k)n(\gamma,b_k) = \frac{1}{2\pi i} \int_{\gamma} dz \frac{g'(z)f'(z)}{f(z)},
\]

(2.5)

where \( n(\gamma,a) \) is the winding number of the closed curve \( \gamma \) with respect to the point \( a \notin \gamma \), defined as

\[
n(\gamma,a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} \in \mathbb{Z}.
\]

(2.6)

If we let \( f \) be a polynomial with zeros \( z_1, z_2, \ldots \) and \( g(z) := z^s \), then

\[
\frac{1}{2\pi i} \int_{\gamma} dz \ z^s \frac{f'(z)}{f(z)} = z_1^s + z_2^s + \cdots,
\]

(2.7)

or equivalently

\[
\frac{1}{2\pi i} \int_{\gamma} dz \ z^s \frac{dz}{z} \log f(z) = \sum_n z_n^s.
\]

(2.8)

Taking into account the asymptotic properties of \( u_{\lambda}(1) \) and making the substitutions \( z \to \lambda \) and \( s \to -s \), we see that [12, 13, 15]

\[
\frac{1}{2\pi i} \int_{\gamma} d\lambda \lambda^{-s} \frac{d}{d\lambda} \log u_{\lambda}(1) = \sum_n \lambda_n^{-s} := \zeta_O(s),
\]

(2.9)

since the eigenvalues \( \lambda_n \) are solutions of \( u_{\lambda}(1) = 0 \). As before, \( \gamma \) is a counterclockwise contour that encloses all eigenvalues, which we assume to be positive; see Fig. 2.1.

![Integration contour \( \gamma \).](image)

The pertinent remarks for the case when finitely many eigenvalues are non-positive are given in [13]. It is important to note that the asymptotic behavior of \( u_{\lambda}(1) \) as \( |\lambda| \to \infty \) is given by [15, 16]

\[
u_{\lambda}(1) \sim \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}.
\]

(2.10)
This implies that the integral representation for $\zeta_O(s)$ is valid for $\operatorname{Re}(s) > \frac{1}{2}$ and, therefore, we must continue it analytically if we are to take its derivative at $s = 0$ to compute the determinant of $O$.

The next step \cite{14, 15} necessary to evaluate this integral is to deform the contour suitably. These deformations are allowed provided one does not cross over poles or branch cuts of the integrand. By assumption, for our integrand the poles are on the real axis and, as customary, we define the branch cut of $\lambda^{-s}$ to be on the negative real axis. This means that, as long as the behavior at infinity is appropriate, we are allowed to deform the contour to the one given in Fig. 2.2.

![Figure 2.2: Deformed integration contour $\gamma$.](image)

The result of this deformation, after shrinking the contour to the negative real axis, is

$$
\zeta_O(s) = \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \log u_{-\lambda}(1).
$$

To establish the limits of the validity of this integral representation we must examine the behavior of the integrand for $\lambda \to \infty$, namely one has \cite{13, 15}

$$
u_{-\lambda}(1) \sim \frac{\sin(i\sqrt{\lambda})}{(i\sqrt{\lambda})} \sim \frac{e^{\sqrt{\lambda}}}{2\sqrt{\lambda}}.
$$

Thus, to leading order in $\lambda$, the integrand behaves as $\lambda^{-s-\frac{1}{2}}$, which means that convergence at infinity is established for $\operatorname{Re}(s) > \frac{1}{2}$, as we discussed above. On the other hand, when $\lambda \to 0$ the behavior $\lambda^{-s}$ follows. Consequently the integral representation (2.11) is well defined for

$$
\frac{1}{2} < \operatorname{Re}(s) < 1.
$$

The analytic continuation to the left is accomplished by subtracting the leading $\lambda \to \infty$ asymptotic behavior of $u_{-\lambda}(1)$ \cite{12}. Carrying out this procedure results in one part that is finite at $s = 0$ and another part for which the analytic continuation can be constructed relatively easily. The partition of $\zeta_O$ that keeps the $\lambda \to 0$ term unaffected and that should improve the $\lambda \to \infty$ part is accomplished by splitting the integration range as

$$
\zeta_O(s) = \zeta_O,\text{asy}(s) + \zeta_O,\text{f}(s),
$$

(2.14)
where
\[
\zeta_{O,f}(s) = \frac{\sin \pi s}{\pi} \int_0^1 d\lambda \lambda^{-s} \frac{d}{d\lambda} \log u_-(1) + \frac{\sin \pi s}{\pi} \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \log \left[ u_- \left( \frac{2\sqrt{\lambda}}{e^{\sqrt{\lambda}}} \right) \right],
\]
and
\[
\zeta_{O,asy}(s) = \frac{\sin \pi s}{\pi} \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \log \left[ \frac{e^{\sqrt{\lambda}}}{2\sqrt{\lambda}} \right].
\]
Clearly, we have constructed \( \zeta_{O,f}(s) \) in such a way that it is already analytic at \( s = 0 \) and, thus, its derivative at \( s = 0 \) can be computed immediately
\[
\zeta'_{O,f}(0) = -\log[2 e^{-1} u_0(1)].
\]
For this case, the analytic continuation to a meromorphic function on the complex plane now follows from
\[
\int_1^\infty d\lambda \lambda^{-\alpha} = \frac{1}{\alpha - 1}, \quad \text{for} \quad \Re(\alpha) > 1.
\]
Applying the above to \( \zeta_{O,asy}(s) \) yields
\[
\zeta_{O,asy}(s) = \frac{\sin \pi s}{2\pi} \left( \frac{1}{s - 1/2} - \frac{1}{s} \right),
\]
and thus
\[
\zeta'_{O,asy}(0) = -1.
\]
The contribution from both terms then becomes
\[
\zeta'_0(0) = -\log[2 u_0(1)].
\]
Note how we could numerically evaluate the determinant of \( O \) without using a single eigenvalue explicitly [15].

3 Description of the problem

Let us next introduce the notions needed for the investigations of the Eisenstein series. Let \( M \) be a compact smooth manifold with dimension \( d \) and let \( s \in \mathbb{C} \) with \( \Re(s) > d/2 \), furthermore let \( \mathbb{H} \) denote the upper half-plane \( \mathbb{H} = \{ \tau = \tau_1 + i\tau_2, \tau_1 \in \mathbb{R}, \tau_2 > 0 \} \).

**Definition 3.1.** For \( c \in \mathbb{R}_+ \) and \( \vec{r} \in \mathbb{R}_+^d \) the homogeneous Epstein zeta function is defined as [10]
\[
\zeta_e(s,c|\vec{r}) := \sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{(c + r_1 m_1^2 + \cdots + r_d m_d^2)^s}.
\]
If \( c = 0 \) then it is understood that the summation ranges over \( \vec{m} \neq \vec{0} \).
Definition 3.2. Let $\mathbb{Z}_2^* = \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and $\tau \in \mathbb{H}$ with $\text{Re} \, \tau = \tau_1$ and $\text{Im} \, \tau = \tau_2$. For $\text{Re}(s) > 1$ the nonholomorphic Eisenstein series is defined as

$$E^*(s, \tau) := \sum_{(m,n) \in \mathbb{Z}_2^*} \frac{\tau_2^s}{|m + n\tau|^{2s}}. \quad (3.2)$$

Note that for $\tau = i$ the nonholomorphic Eisenstein series is related to the homogeneous Epstein zeta function by

$$E^*(s, i) = \sum_{(m,n) \in \mathbb{Z}_2^*} \frac{1}{(m^2 + n^2)^s} = \zeta_E(s, 0|\vec{1}_2), \quad (3.3)$$

where $\vec{1}_2 = (1,1)$. The non-holomorphic Eisenstein series is not holomorphic in $\tau$ but it can be continued analytically beyond $\text{Re}(s) > 1$ except at $s = 1$, where there is a simple pole with residue equal to $\pi$.

Definition 3.3. For $\tau \in \mathbb{H}$, the Dedekind eta function is defined as

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

While $\eta(\tau)$ is holomorphic on the upper half-plane, it cannot be continued analytically beyond it. The fundamental properties are that it satisfies the following functional equations.

**Proposition 3.1.** One has

$$\eta(\tau + 1) = e^{\pi i /12} \eta(\tau),$$

$$\eta(-\tau^{-1}) = \sqrt{-i\tau} \eta(\tau),$$

for $\tau \in \mathbb{H}$.

The first equation is very easy to show and the proof of the second one can be found in a book of modular forms, see for instance [1]. With this in mind, the constant term in the Laurent expansion of $E^*(s, \tau)$ is given in the following theorem (see, e.g., [29]).

**Theorem 3.1** (Kronecker’s first limit formula).

$$\lim_{s \to 1} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) = 2\pi(\gamma - \log 2 - \log \tau_2^{1/2} |\eta(\tau)|^2).$$

A modular form of weight $k > 0$ and multiplier condition $C$ for the group of substitutions generated by $\tau \to \tau + \lambda$ and $\tau \to -\frac{1}{\tau}$ is a holomorphic function $f(\tau)$ on $\mathbb{H}$ satisfying [11]

(i) $f(\tau + \lambda) = f(\tau),$

(ii) $f(-\frac{1}{\tau}) = C(\tau^k) f(\tau),$

(iii) $f(\tau)$ has a Taylor expansion in $e^{2\pi i \tau/\lambda}$ (cf (i)): $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau/\lambda}$, i.e. "$f$ is holomorphic at $\infty$".
The space of such \( f \) is denoted by \( M(\lambda, k, C) \) and furthermore if \( a_0 = 0 \) then \( f \) is a cusp form. The group of substitutions generated by \( \tau \to \tau + 1 \) and \( \tau \to -\frac{1}{\tau} \) is

\[
\text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\},
\]

therefore modular forms of weight \( k \) satisfy

\[
f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau).
\]

The Dedekind eta function is a modular form of weight \( k = \frac{1}{2} \) and we may assume without loss of generality that either \( c > 0 \) or \( c = 0 \) and \( d = 1 \). Moreover, if \( c = 0 \) and \( d = 1 \), then it satisfies

\[
\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \varepsilon(a, b, c, d)(c\tau + d)^{1/2}\eta(\tau),
\]

where \( \varepsilon(a, b, c, d) = e^{\frac{b\pi i}{12}} \), and if \( c > 0 \) then

\[
\varepsilon(a, b, c, d) = \exp \left( \frac{i\pi a + d}{12} - s(d, c) - \frac{1}{4} \right),
\]

where \( s(h, k) \) is the Dedekind sum

\[
s(h, k) = \sum_{n=1}^{k-1} \frac{n}{k} \left( \frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor - \frac{1}{2} \right).
\]

Finally, the non-holomorphic Eisenstein series can alternatively be defined as

\[
E^*(s, \tau) = \zeta_R(2s) \sum_{\gcd(m,n)=1} \frac{\tau_2^s}{|m\tau + n|^{2s}},
\]

where \( \zeta_R(s) \) is the Riemann zeta function, and it is unchanged by the substitutions

\[
\tau \to \frac{a\tau + b}{c\tau + d}
\]

coming from any matrix of \( \text{SL}(2, \mathbb{Z}) \). Selberg and Chowla were interested in the problem of the analytic continuation of \( E^*(s, \tau) \) as a function of \( s \) and its functional equation. Their idea was to consider the Fourier expansion of \( E^*(s, \tau) \) given by

\[
E^*(s, \tau) = E(s, \tau_1 + i\tau_2) = \sum_{m \in \mathbb{Z}} a_m(s, \tau_2) e^{2\pi im\tau_1},
\]

where \( a_m \) is the Fourier coefficient

\[
a_m(\tau_2, s) = \int_0^1 E(s, \tau_1 + i\tau_2) e^{-2\pi im\tau_1} d\tau_1.
\]

The explicit formulas for these coefficients are given by [1] [11]

\[
a_0 = 2\zeta_R(2s)\tau_2^s + 2\phi(s)\zeta_R(2s - 1)\tau_2^{1-s},
\]

7
where
\[ \phi(s) = \sqrt{\frac{\pi}{s - \frac{1}{2}}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)}, \]
and
\[ a_n = 2^{1/2} \tau_1^{-1/2} K_{s-1/2}(2\pi |n| \tau_2) \frac{|n|^{s-1}}{\pi^s \Gamma(s)} \sigma_{1-2s}(n), \]
where for \( n \geq 1 \) and \( v \in \mathbb{C} \), we let
\[ \sigma_v(n) := \sum_{0 < d, d \mid n} d^v \]
denote the divisor function. In general, \( \phi(s) \) is called the constant or scattering term and \( a_n \) with \( n \geq 1 \) is the non-trivial term.

In this paper, we propose to recover these results by contour integration on the complex plane without Fourier techniques.

Indeed, the existence of two (or more) general methods of obtaining analytic continuations and functional equations of zeta functions has been known since Riemann’s 1859 paper on the distribution of prime numbers. The table below summarizes the dates of some of the most common zeta functions.

| \( \zeta \) function       | Poisson summation | Contour integration |
|---------------------------|-------------------|---------------------|
| Riemann zeta function     | Riemann (1859)    | Riemann (1859)      |
| Lerch zeta function       | Apostol (1951)    | Lerch (1874)        |
| Hurwitz zeta function     | Fine (1951)       | Hurwitz (1882)      |
| Dirichlet \( L \)-function| de la Vallée Poussin (1896) | Berndt (1973) |
| Dedekind zeta function    | Hecke (1917)      |                     |
| Epstein zeta function     | Epstein (1903), Chowla-Selberg (1947) | E. K. R. W. (2013) |

4 Eigenvalue problem set up

For fixed \( \tau = \tau_1 + i\tau_2 \in \mathbb{H} \) the general \( \tau \)-Laplacian is
\[ \Delta_\tau = -\frac{1}{\tau_2^2} \left[ \left( \frac{\partial^2}{\partial x^2} + \tau_1 \frac{\partial}{\partial y} \right)^2 + \left( \tau_2 \frac{\partial}{\partial y} \right)^2 \right] = -\frac{1}{\tau_2^2} \left[ \frac{\partial^2}{\partial x^2} + (\tau_1^2 + \tau_2^2) \frac{\partial^2}{\partial y^2} + 2\tau_1 \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right]. \tag{4.1} \]

Now we let \( M = S^1 \times S^1 \) be a complex torus and the corresponding integral lattice is (see Fig. 3.1)
\[ \mathcal{L}_\tau := \{ a + b\tau \mid a, b \in \mathbb{Z} \}, \quad M := \mathbb{C} \setminus \mathcal{L}_\tau. \tag{4.2} \]

The relevant eigenvalue problem is
\[ \Delta_\tau \phi_\lambda(x, y) = \lambda^2 \phi_\lambda(x, y), \tag{4.3} \]
with periodic boundary conditions
\[ \phi_\lambda(x, y) = \phi_\lambda(x + 1, y), \quad \frac{\partial}{\partial x} \phi_\lambda(x, y) = \frac{\partial}{\partial x} \phi_\lambda(x + 1, y), \tag{4.4} \]
on $x$, as well as
\[ \phi_\lambda(x, y) = \phi_\lambda(x, y + 1), \quad \frac{\partial}{\partial y} \phi_\lambda(x, y) = \frac{\partial}{\partial y} \phi_\lambda(x, y + 1), \] (4.5)
on $y$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.1.png}
\caption{$\tau$ parametrizes the complex structure of this parallelogram \cite{19}.}
\end{figure}

By taking the eigenfunctions to be
\[ \phi_{m,n}(x, y) = e^{-2\pi i m x} e^{-2\pi i n y}, \] (4.6)
with $(m, n) \in \mathbb{Z}_2^\ast$, we see that
\[ \Delta_\tau \phi_{m,n}(x, y) = \frac{(2\pi)^2}{\tau_2^2} [m^2 + 2\tau_1 m n + (\tau_1^2 + \tau_2^2)n^2] \phi_{m,n}(x, y) = \frac{(2\pi)^2}{\tau_2^2} |m + n\tau|^2 \phi_{m,n}(x, y). \] (4.7)

Therefore, the eigenvalues to consider are
\[ \lambda_{m,n}^2 = \frac{(2\pi)^2}{\tau_2^2} |m + n\tau|^2 = \frac{(2\pi)^2}{\tau_2^2} (m + n\tau)(m + n\bar{\tau}), \quad (m, n) \in \mathbb{Z}_2^\ast, \] (4.8)
and we define the following spectral function.

**Definition 4.1.** For $\text{Re}(s) > 1$ and $\tau \in \mathbb{H}$, the associated spectral zeta function of the general $\tau$-Laplacian on the complex torus is defined to be
\[ \zeta_{\Delta_\tau}(s) := \sum_\lambda (\lambda^2)^{-s} = (2\pi)^{-2s} \tau_2^{-2s} \sum_{(m,n) \in \mathbb{Z}_2^\ast} |m + n\tau|^{-2s} \]
\[ = (2\pi)^{-2s} \tau_2^{-s} E^*(s, \tau), \] (4.9)
where $E^*(s, \tau)$ is the non-holomorphic Eisenstein series.
5 Main result

A majority of the cases treated in the literature of spectral zeta functions [7, 8, 15] have eigenvalues which give rise to homogeneous Epstein zeta functions, of the type (3.1), with no mixed terms $mn$ (or equivalently, where there is no mixed partial derivative $\partial^2 / \partial x \partial y$ in the Laplacian). In particular, the inhomogeneous Epstein zeta function $\zeta_E(s) = \sum_{(m,n) \in \mathbb{Z}^2} Q(m,n)^{-s}$ for a general quadratic form $Q(m,n) = am^2 + bmn + cn^2$ with $b \neq 0$ has not been computed with the argument principle, only with Poisson summation. This therefore constitutes a new application of the contour integration method which does not exist in the literature and where different insights are gained (in [24] contour integration is used but the essential step is accomplished through Fourier methods). More general cases with $Q'(m,n) = am^2 + bmn + cn^2 + dm + en + f$, where $f$ is a real positive constant, and in higher dimensions as well, have been treated in [7] by means of Poisson summation.

We split the summation into $n = 0, m \in \mathbb{Z}\setminus\{0\}$ and $n \neq 0, m \in \mathbb{Z}$. Thus we write

$$\zeta_{\Delta_{\tau}}(s) = (2\pi)^{-2s} \tau_2^{2s} \sum_{m=-\infty}^{\infty} m^{-2s} + (2\pi)^{-2s} \tau_2^{2s} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [(m+n\tau)(m+n\bar{\tau})]^{-s}$$

$$:= 2(2\pi)^{-2s} \tau_2^{2s} \zeta_R(2s) + (2\pi)^{-2s} \tau_2^{2s} \zeta_I(s),$$

where $\zeta_R(s)$ denotes the Riemann zeta function and

$$\zeta_I(s) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [(m+n\tau)(m+n\bar{\tau})]^{-s}.$$

We represent the zeta function $\zeta_I(s)$ in terms of a contour integral; the summation over $m$ is expressed using $\sin(\pi k) = 0$. Thus, we write

$$\zeta_I(s) = \sum_{n=-\infty}^{\infty} \int_{\gamma} \frac{dk}{2\pi i} [(k+n\tau)(k+n\bar{\tau})]^{-s} \frac{d}{dk} \log \sin(\pi k),$$

where $\gamma$ is the contour in Fig. 4.1.

![Figure 5.1: Contour $\gamma$ enclosing the eigenvalues.](image)
where $\alpha \in \mathbb{R}$ and $\alpha \geq 0$. Writing $\tau = \tau_1 + i\tau_2$, we have

$$k^2 + n^2|\tau|^2 + kn\bar{\tau} + kn\tau + \alpha = 0,$$

so that

$$k = -n\tau_1 \pm \sqrt{n^2(\tau_1^2 - |\tau|^2)} - \alpha = -n\tau_1 \pm \sqrt{-n^2\tau_2^2 - \alpha}.$$ 

We notice that

$$-n^2\tau_2^2 - \alpha \leq 0,$$

and the branch cuts are something like the ones given in Fig. 4.2.

![Figure 5.2: Location of the branch points](image)

Let us denote the branch points by $z^n_b$ and $\bar{z^n}_b$, thus

$$z^n_b = -n\tau_1 + i\sqrt{n^2\tau_2^2} = -n\tau_1 + i|n|\tau_2.$$ 

The natural deformation is, therefore, as indicated in Fig. 4.3.

![Figure 5.3: Newly deformed contour enclosing the branch cuts](image)

When shrinking the contours to the branch cuts, the parametrizations will be done according to the following:

1. for the upper contour

$$k = z^n_b + e^{i\pi / 2}u, \quad u \in (\infty, 0],
\quad k = z^n_b + e^{-3i\pi / 2}u, \quad u \in [0, \infty);$$
2. similarly for the lower contour

\[ k = z_b^n + e^{-i\pi/2}u, \quad u \in [0, \infty), \]

\[ k = \bar{z}_b^n + e^{3i\pi/2}u, \quad u \in (\infty, 0]. \]  

For \( \zeta_I(s) \) this gives

\[ \zeta_I(s) = \sum_{n=-\infty}^{\infty} \left\{ \int_0^\infty \frac{du}{2\pi i} [(z_b^n + e^{i\pi/2}u + n\tau)(z_b^n + e^{i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + e^{i\pi/2}u]) \right. \\
+ \left. \int_0^\infty \frac{du}{2\pi i} [(z_b^n + e^{-3i\pi/2}u + n\tau)(z_b^n + e^{-3i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + e^{-3i\pi/2}u]) \right. \\
+ \left. \int_0^\infty \frac{du}{2\pi i} [(z_b^n + e^{-i\pi/2}u + n\tau)(z_b^n + e^{-i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + e^{-i\pi/2}u]) \right. \\
+ \left. \int_0^\infty \frac{du}{2\pi i} [(z_b^n + e^{3i\pi/2}u + n\tau)(z_b^n + e^{3i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + e^{3i\pi/2}u]) \right\}. \]

We next rewrite the integrands using the fact that \( z_b^n \) and \( \bar{z}_b^n \) solve the quadratic equation (5.1) with \( \alpha = 0 \). We then use the notation

\[ \zeta_I(s) = \zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) + \zeta_I^{(3)}(s) + \zeta_I^{(4)}(s) \]

to denote each individual (infinite) sum above. Let us start with \( \zeta_I^{(1)}(s) \): we begin by computing

\[(z_b^n + e^{i\pi/2}u + n\tau)(z_b^n + e^{i\pi/2}u + n\bar{\tau}) = e^{i\pi}(u^2 + 2u|n|\tau_2),\]

so that we have

\[ \zeta_I^{(1)}(s) = \sum_{n=-\infty}^{\infty} (-e^{-i\pi s}) \int_0^\infty \frac{du}{2\pi i} (u^2 + 2u|n|\tau_2)^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + iu]). \]

Redoing the same computation for \( \zeta_I^{(2)}(s) \), but replacing \( e^{i\pi/2} \) with \( e^{-3i\pi/2} \) yields

\[(z_b^n + e^{-3i\pi/2}u + n\tau)(z_b^n + e^{-3i\pi/2}u + n\bar{\tau}) = e^{-i\pi}(u^2 + 2u|n|\tau_2),\]

so that

\[ \zeta_I^{(2)}(s) = \sum_{n=-\infty}^{\infty} e^{i\pi s} \int_0^\infty \frac{du}{2\pi i} (u^2 + 2u|n|\tau_2)^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + iu]) \]

and, combining the two terms,

\[ \zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\infty du (u^2 + 2u|n|\tau_2)^{-s} \frac{d}{du} \log \sin(\pi[z_b^n + iu]). \]

We next consider the log terms in order to perform the analytic continuation. A suitable rewriting is

\[ \sin(\pi[z_b^n + iu]) = \frac{e^{i\pi(\tau_1 + i|n|\tau_2)} - e^{-i\pi(\tau_1 + i|n|\tau_2)}}{2i} = \frac{1}{2i} e^{\pi u - i\pi z_b^n} (1 - e^{-2\pi u + 2i\pi z_b^n}). \]

Note that \( e^{2i\pi z_b^n} = e^{2i\pi(-n\tau_1 + i|n|\tau_2)} \) is exponentially damped for large \(|n|\). We therefore write

\[ \zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\infty du (u^2 + 2u|n|\tau_2)^{-s} \frac{d}{du} \log[e^{\pi u - i\pi z_b^n} (1 - e^{-2\pi u + 2i\pi z_b^n})]. \]
\[ \zeta_I^{(1,1)}(s) = \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} du (u^2 + 2u |n| \tau_2)^{-s} \left\{ \pi + \frac{d}{du} \log(1 - e^{-2\pi u + 2i\pi z_b^n}) \right\} \]

where we define
\[ \zeta_I^{(1,1)}(s) := \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} du (u^2 + 2u |n| \tau_2)^{-s}, \]

and
\[ \zeta_I^{(1,2)}(s) := \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} du (u^2 + 2u |n| \tau_2)^{-s} \frac{d}{du} \log(1 - e^{-2\pi u + 2i\pi z_b^n}). \]

To compute the derivative at \( s = 0 \) we only need to find the analytical continuation of \( \zeta_I^{(1,1)}(s) \) since \( \zeta_I^{(1,2)}(s) \) is already valid for all \( s \in \mathbb{C} \). In order to accomplish this continuation we note that, for \( \frac{1}{2} < \text{Re}(s) < 1 \),

\[ \int_{0}^{\infty} du \ u^{-s}(u + 2x)^{-s} = x^{1-2s} \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}}. \quad (5.2) \]

With this in mind, we write
\[ \zeta_I^{(1,1)}(s) = \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} du (u^2 + 2u |n| \tau_2)^{-s} = 2 \sin(\pi s) \sum_{n=1}^{\infty} \int_{0}^{\infty} du u^{-s}(u + 2n\tau_2)^{-s} \]

\[ = 2 \sin(\pi s) \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}} \tau_2^{1-2s} \sum_{n=1}^{\infty} n^{1-2s} = \frac{\sqrt{\pi} \Gamma(s-\frac{1}{2})}{\Gamma(s)} \tau_2^{1-2s} \zeta_R(2s-1). \]

The branch in the lower half-plane is handled accordingly. In order to simplify the integrand, we note the analogy between the first and third case: \( e^{i\pi/2} \to e^{-i\pi/2}, z_b^n \to \bar{z}_b^n \). Thus
\[ (z_b^n + e^{-i\pi/2} u + n\tau)(\bar{z}_b^n + e^{-i\pi/2} u + n\bar{\tau}) = e^{-i\pi}(u^2 + 2u |n| \tau_2), \]

so that the third function is
\[ \zeta_I^{(3)}(s) = \sum_{n=-\infty}^{\infty} (e^{i\pi s})^n \int_{0}^{\infty} du 2\pi i (u^2 + 2u |n| \tau_2)^{-s} \frac{d}{du} \log \sin(\pi [\bar{z}_b^n - iu]). \]

Similarly, for the last function it follows that
\[ \zeta_I^{(4)}(s) = \sum_{n=-\infty}^{\infty} (-e^{-i\pi s})^n \int_{0}^{\infty} du 2\pi i (u^2 + 2u |n| \tau_2)^{-s} \frac{d}{du} \log \sin(\pi [\bar{z}_b^n - iu]). \]

Adding up these two terms yields
\[ \zeta_I^{(3)}(s) + \zeta_I^{(4)}(s) = \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} du (u^2 + 2u |n| \tau_2)^{-s} \frac{d}{du} \log \sin(\pi [\bar{z}_b^n - iu]). \]

Going through a similar manipulation of the log sin term as above allows us to write
\[ \sin(\pi [\bar{z}_b^n - iu]) = \frac{e^{i\pi (\bar{z}_b^n - iu)} - e^{-i\pi (\bar{z}_b^n - iu)}}{2i} = \frac{1}{2i} e^{\pi u + i\pi \bar{z}_b^n} (1 - e^{-2\pi u - 2i\pi z_b^n}). \]

13
We note that $\pi u + i\pi z^n_0 = \pi u + i\pi (-n\tau_1 - i |n| \tau_2)$ and so $e^{-2\pi u - 2i\pi z^n_0}$ is, like in the previous case, exponentially damped as $|n| \to \infty$. Once more, using (5.2),

$$\zeta_f^{(3)}(s) + \zeta_f^{(4)}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\infty du(u^2 + 2u |n| \tau_2)^{-s} \left[ \pi + \frac{d}{du} \log(1 - e^{-2\pi u - 2i\pi z^n_0}) \right]$$

$$= \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \tau_2^{-2s} \zeta_R(2s - 1)$$

$$+ \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\infty du(u^2 + 2u |n| \tau_2)^{-s} \frac{d}{du} \log((1 - e^{-2\pi u - 2i\pi z^n_0})(1 - e^{-2\pi u + 2i\pi n\tau})).$$

Therefore, the final result is

$$\zeta_{\Delta_r}(s) = 2(2\pi)^{-2s} \tau_2^{-2s} \zeta_R(2s) + (2\pi)^{1-2s} \tau_2^{-2s} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{\pi} \Gamma(s)} \tau_2^{-1-2s} \zeta_R(2s - 1)$$

$$+ (2\pi)^{-2s} \tau_2^{2s} \sin(\pi s) \sum_{n=-\infty}^{\infty} \int_0^\infty du(u^2 + 2un\tau_2)^{-s} \frac{d}{du} \log((1 - e^{-2\pi u - 2i\pi n\tau})(1 - e^{-2\pi u + 2i\pi n\tau})),$$

which is now valid for all $s \in \mathbb{C}\{1\}$. Re-writing the sum so that it goes from $n = 1$ to $n = \infty$ we have thus proved the following result.

**Proposition 5.1.** The spectral zeta function of $\Delta_r$ on $S^1 \times S^1$ can be written as

$$\zeta_{\Delta_r}(s) = 2(2\pi)^{-2s} \tau_2^{-2s} \zeta_R(2s) + (2\pi)^{1-2s} \tau_2^{-2s} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{\pi} \Gamma(s)} \tau_2^{-1-2s} \zeta_R(2s - 1)$$

$$+ \frac{2 \sin(\pi s)}{\pi} \left( \frac{2\pi}{\tau_2} \right)^{-2s} \sum_{n=1}^{\infty} \int_0^\infty du(u^2 + 2un\tau_2)^{-s} \frac{d}{du} \log((1 - e^{-2\pi u - 2i\pi n\tau})(1 - e^{-2\pi u + 2i\pi n\tau})).$$

(5.3)

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}\{1\}$.

Before proceeding to explain the term on the second line of (5.3), we first compute the functional determinant we were interested in. The derivative of the last expression at $s = 0$ is obtained in terms of the Dedekind eta function:

$$\zeta_{\Delta_r}(0) = -\log \tau_2^2 + \frac{\pi \tau_2}{3} + 2 \sum_{n=1}^{\infty} \int_0^\infty du \frac{d}{du} \log((1 - e^{-2\pi u - 2i\pi n\tau})(1 - e^{-2\pi u + 2i\pi n\tau}))$$

$$= -\log \tau_2^2 + \frac{\pi \tau_2}{3} - 2 \sum_{n=1}^{\infty} \left[ \log(1 - e^{-2i\pi n\tau}) + \log(1 - e^{-2\pi u + 2i\pi n\tau}) \right]$$

$$= -\log \tau_2^2 + \frac{\pi \tau_2}{3} - 2 \left( \log \eta(\tau) - \frac{\pi i \tau}{12} + \log \eta(-\tau) + \frac{\pi i \tau}{12} \right)$$

$$= -\log \tau_2^2 + \frac{\pi \tau_2}{3} - 2 \left( \log[\eta(\tau)\eta(-\tau)] + \frac{\pi i}{12}(\tau - \tau) \right)$$

$$= -\log(\tau_2^2 |\eta(\tau)|^4),$$

since $\eta(-\tau) = \overline{\eta(\tau)}$. Therefore, we have the following result [3, 4, 21, 29].
Theorem 5.1. The functional determinant of the \( \tau \)-Laplacian on the complex torus is
\[
\det(\Delta_\tau) = \tau_2^2 |\eta(\tau)|^4. \tag{5.4}
\]

Proof. By definition
\[
\det(\Delta_\tau) = \exp(-\zeta'_\Delta(0)) = \tau_2^2 |\eta(\tau)|^4,
\]
as claimed. \( \square \)

We note that in order to obtain the value of \( \zeta'_\Delta(0) \) we have not used Kronecker’s first limit formula, which in turn, depends on the functional equation of \( E^*(s, \tau) \). Thus, viewed under this optic, the method of contour integration is cheaper in the sense that it requires less resources to provide the functional determinant. For a derivation of this theorem using the Kronecker formula, see, for instance, [29].

6 Additional results and consequences

6.1 The Chowla-Selberg series formula

In fact, the term on the second line of (5.3) can be shown to be the term from the Chowla-Selberg series formula obtained through Poisson summation methods. To see this, let us compute the log-terms further. Expanding the logarithm,
\[
\log(1 - e^{-2\pi u - 2i\pi n\bar{\tau}}) = -\sum_{k=1}^{\infty} \frac{1}{k} e^{-2\pi ku - 2i\pi nk\bar{\tau}},
\]
and for \( \text{Re}(s) < 1 \) the integral becomes
\[
\Upsilon_{\tau}(s, n) := \int_0^\infty du u^{-s} (u + 2n\tau_2)^{-s} \frac{d}{du} \log[(1 - e^{-2\pi u - 2i\pi n\bar{\tau}})]
\]
\[
= -\int_0^\infty du u^{-s} (u + 2n\tau_2)^{-s} \frac{d}{du} \sum_{k=1}^{\infty} \frac{1}{k} e^{-2\pi ku - 2i\pi nk\bar{\tau}}
\]
\[
= 2\pi \sum_{k=1}^{\infty} \int_0^\infty du u^{-s} (u + 2n\tau_2)^{-s} e^{-2\pi ku - 2i\pi nk\bar{\tau}}
\]
\[
= 2\pi \sum_{k=1}^{\infty} e^{-2i\pi nk\bar{\tau}} \int_0^\infty du u^{-s} (u + 2n\tau_2)^{-s} e^{-2\pi ku}
\]
\[
= 2\pi \sum_{k=1}^{\infty} e^{-2i\pi nk\bar{\tau}} K_{1/2-s} (2\pi kn\tau_2) \frac{1}{\sin(\pi s)\Gamma(s)} \frac{2\pi kn\tau_2 k^{s-1/2} \pi^s}{\sqrt{n\tau_2}} (n\tau_2)^{1-s}
\]
\[
= 2\pi^{1+s} \tau_2^{1/2-s} \sum_{k=1}^{\infty} e^{-2\pi nk\tau_1} K_{s-1/2} (2\pi kn\tau_2) \quad \tau_1 = \sqrt{-\frac{1}{\pi}} \tau
\]
\[
\text{where } K_n(z) \text{ is the Bessel function of the second kind, defined by }
\]
\[
K_\nu(z) := \frac{1}{2} \int_0^\infty dt \exp \left[ -\frac{z}{2} \left( t + \frac{1}{t} \right) \right] t^{\nu-1}.
\]
Moreover, the integral of the other log-term only differs in that $e^{-2\pi ink\tau}$ is replaced by $e^{2\pi ink\tau}$. Thus, noting \( \tau_2 + i\tau = \tau_2 + i(\tau_1 + i\tau_2) = i\tau_1 \), we have

\[
\Upsilon(s, n) := \int_0^\infty du u^{-s} (u + 2n\tau_2)^{-s} \frac{d}{du} \log \left[ (1 - e^{-2\pi u + 2i\pi\tau}) \right] = \frac{2\pi^{1+s}\tau_2^{-1/2-s}}{\sin(\pi s)\Gamma(s)} \sum_{k=1}^\infty e^{2\pi ink\tau_1} k^{s-1/2} n^{1/2-s} K_{1/2-s}(2\pi kn\tau_2).
\]

If we set the third term in (5.3) to be \( Q(s, \tau) \), then we see that

\[
Q(s) = \frac{2\sin(\pi s)}{\pi} \left( \frac{2\pi}{\tau_2} \right)^{-2s} \sum_{n=1}^\infty \sum_{k=1}^\infty \left[ \Upsilon(s, n) + \Upsilon(s, n) \right] = \frac{2\sin(\pi s)}{\pi} \left( \frac{2\pi}{\tau_2} \right)^{-2s} \sum_{n=1}^\infty \sum_{k=1}^\infty e^{2\pi ink\tau_1} k^{s-1/2} n^{1/2-s} K_{1/2-s}(2\pi kn\tau_2)
\]

\[
+ \frac{2\sin(\pi s)}{\pi} \left( \frac{2\pi}{\tau_2} \right)^{-2s} \sum_{n=1}^\infty \sum_{k=1}^\infty e^{-2\pi ink\tau_1} k^{s-1/2} n^{1/2-s} K_{1/2-s}(2\pi kn\tau_2)
\]

\[
= \frac{2^{3-2s} \pi^{-s} \tau_2^{-1/2+s}}{\Gamma(s)} \sum_{n=1}^\infty \sum_{k=1}^\infty \cos(2\pi nk\tau_1) \left( \frac{k}{n} \right)^{s-1/2} K_{1/2-s}(2\pi kn\tau_2).
\]

The key is now to relate the expression inside the double sum to the divisor function. The main property we are interested in is the following way of changing the double sum for a single sum while bringing in the divisor function

\[
\sum_{n=1}^\infty \left[ \sum_{k=1}^\infty e^{\pm 2\pi ink\tau_1} K_{1/2-s}(2\pi kn\tau_2) \left( \frac{k}{n} \right)^{s-1/2} \right] = \sum_{n=1}^\infty \sigma_{1-2s}(n)e^{\pm 2\pi in\tau_1} K_{1/2-s}(2\pi n\tau_2)n^{s-1/2}.
\]

For a proof of a more general result of this type, see for instance [29]. Thus, switching back to \( E^*(s, \tau) \) instead of \( \zeta_{\Delta_1}(s) \), we have arrived at the following theorem, which now holds for all \( s \neq 1 \) by analytic continuation [2].

**Theorem 6.1** (Chowla-Selberg series formula). One has

\[
E^*(s, \tau) = 2\tau_2^2 \zeta_R(2s) + 2\pi \tau_2^{-s} \Gamma(s - \frac{1}{2}) \zeta_R(2s - 1) + \frac{8\pi^s \tau_2^{-1/2}}{\Gamma(s)} \sum_{n=1}^\infty \sigma_{1-2s}(n) \cos(2\pi n\tau_1) K_{1/2-s}(2\pi n\tau_2)n^{s-1/2},
\]

for \( \tau \in \mathbb{H} \) and \( s \in \mathbb{C} \setminus \{1\} \).

### 6.2 The Nan-Yue Williams formula

As a byproduct, we may now equate the two ‘remainders’ in the above expressions of \( E^*(s, \tau) \), recalling that the one in (5.3) has to be multiplied by \( (2\pi)^2 \tau_2^{-s} \), to get

\[
Q(s, \tau) = \frac{8\pi^s \tau_2^{-1/2}}{\Gamma(s)} \sum_{n=1}^\infty \sigma_{1-2s}(n) \cos(2\pi n\tau_1) K_{1/2-s}(2\pi n\tau_2)n^{s-1/2}
\]
\[ = 2τ_s^2 \sin(\pi s) \sum_{n=1}^{\infty} \int_0^{\infty} \frac{du}{u^2 + 2unτ_2} - s \frac{d}{du} \log[(1 - e^{-2πu - 2iπnτ})(1 - e^{-2πu + 2iπnτ})]. \]

Using the Euler reflection formula for the \(\Gamma\)-function we obtain

\[
\sum_{n=1}^{\infty} \sigma_1(n) \cos(2πnτ_1)K_{1/2}(2πnτ_2)n^{s-1/2} = \frac{τ_s^{-1/2}}{4} \left( \frac{τ_2}{π} \right)^s \frac{1}{Γ(1 - s)} \sum_{n=1}^{∞} \int_0^{∞} \frac{du}{u^2 + 2unτ_2} - s \frac{d}{du} \log[(1 - e^{-2πu - 2iπnτ})(1 - e^{-2πu + 2iπnτ})]. \]

(6.1)

At \(s = 0\) the integral is easily evaluated by a similar computation to the one done previously for \(ζ_τ' \Delta_τ(0)\), so that we are left with the following consequence.

**Corollary 6.1.** For \(τ ∈ \mathbb{H}\), one has

\[
\sum_{n=1}^{∞} \sigma_1(n) \cos(2πnτ_1)K_{1/2}(2πnτ_2)n^{-1/2} = - \frac{τ_s^{-1/2}}{2} \log|η(τ)| - \frac{τ_2^{1/2}π}{24}. \]

(6.3)

In [20] this is given in a slightly different context and

\[
\sum_{n=1}^{∞} \sigma_1(n)K_{1/2}(2πn)n^{-1/2} = - \frac{1}{2} \log η(i) - \frac{π}{24} ≈ 0.000936341.
\]

for the particular case \(τ = i\).

### 6.3 The Kronecker first limit formula

Let us now go back to the Chowla-Selberg formula. Using the functional equation of the divisor function and the Bessel function, respectively,

\[ σ_ν(n) = n^νσ_{-ν}(n), \quad K_ν(x) = K_{-ν}(x), \]

it is not difficult to show that the following functional equation for \(E^*(s, \tau)\) holds

\[ π^{1-2s}Γ(s)E^*(s, \tau) = Γ(1 - s)E^*(1 - s, \tau). \]

(6.2)

Using this functional equation and the result of the determinant of the Laplacian (5.4) we can now reverse the steps of the proof of Kronecker’s first limit formula [29]. All known proofs (e.g. [18, 20, 24, 25, 29]) use a variation of some kind of Poisson summation. The present one is a new proof in the sense that no techniques from Fourier analysis are ever used.

Interesting steps in the direction of special functions and complex integration were worked out in [25, 27] using the theory of multiple Gamma functions and Barnes zeta functions. Furthermore, using the Barnes double gamma function and the Selberg zeta function, the determinants of the \(n\)-sphere and spinor fields on a Riemann surface are found in [23]; see also [5, 6].

**Proof of Kronecker’s first limit formula.** First we define

\[ f(s) := Γ(s)π^{2-2s}/Γ(2 - s), \]

(6.3)
so that
\[ f(1) = 1, \quad \lim_{s \to 1} \frac{f(s) - f(1)}{s - 1} = f'(1) = -2 \log \pi - 2\gamma, \] (6.4)
where \( \gamma \) is Euler’s constant. The key idea is to use the result we already got
\[ \zeta'_{\Delta_r}(0) = -\log \tau_2^2 |\eta(\tau)|^4, \] (6.5)
from where
\[
\frac{\partial}{\partial s} \bigg|_{s=0} E^*(s, \tau) := \lim_{s \to 1} \frac{E^*(s, \tau) - E^*(0, \tau)}{s - 0} = \lim_{s \to 1} \frac{E^*(1 - s, \tau) + 1}{1 - s}
\]
\[
= \lim_{s \to 1} \left( \frac{E^*(1 - s, \tau) \Gamma(s)}{\Gamma(s)(1 - s)} - \frac{1}{s - 1} \right) = \lim_{s \to 1} \left( \frac{\Gamma(s)}{\Gamma(1 - s)} \frac{\pi^{1-2s}}{\tau_2^2} E^*(s, \tau) - \frac{1}{s - 1} \right)
\]
\[
= \lim_{s \to 1} \left[ \frac{\Gamma(s)}{\Gamma(2 - s)} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) + \frac{\Gamma(s)}{\Gamma(2 - s)} \frac{\pi}{s - 1} - \frac{1}{s - 1} \right]
\]
\[
= \lim_{s \to 1} \frac{\Gamma(s)}{\Gamma(2 - s)} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) + \lim_{s \to 1} \frac{f(s) - f(1)}{s - 1} 
\]
\[
= \frac{1}{\pi} \lim_{s \to 1} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) - 2 \log \pi - 2\gamma. \] (6.7)
Comparing (6.6) and (6.7) yields
\[
\frac{1}{\pi} \lim_{s \to 1} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) - 2 \log \pi - 2\gamma = -\log(4\pi^2 \tau_2 |\eta(\tau)|^4) \] (6.8)
and re-arranging
\[
\lim_{s \to 1} \left( E^*(s, \tau) - \frac{\pi}{s - 1} \right) = 2\pi(\gamma - \log 2 - \log \tau_2^{1/2} |\eta(\tau)|^2), \] (6.9)
the desired result follows.

### 6.4 The Lambert series

We may re-write (6.1) as
\[
Q(s, \tau) := \sum_{n=1}^{\infty} \sigma_{1-2s}(n) \cos(2\pi n \tau_1) K_{1/2-s}(2\pi n \tau_2)n^{s-1/2}
\]
\[
= \frac{\tau_2^{-1/2}}{4} \left( \frac{\tau_2}{\pi} \right)^s \frac{1}{\Gamma(2 - s)}
\]

\[ \times \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{dx}{2x+2n\tau_{2}} \left[ \frac{d}{dx} \left( x^{2}+2xn\tau_{2} \right)^{-s+1} \right] \frac{dx}{dx} \log \left[ (1-e^{-2\pi x-2\pi i n\tau})(1-e^{-2\pi x+2\pi i n\tau}) \right], \]

so that, after integrating by parts, we have

\[ Q(s, \tau) = \tau^{-1/2} \left( \frac{\tau}{\pi} \right)^{s} \frac{1}{\Gamma(2-s)} \sum_{n=1}^{\infty} \left[ \frac{2\pi (1-2e^{2\pi i n\tau}+e^{2\pi i n(\tau+\bar{\tau})})}{(e^{2\pi i n\tau}-1)(e^{2\pi i n\tau}-1)} \right] \Gamma(2-s)\Gamma(s-\frac{1}{2}) (n\tau_{2})^{1-2s} \]

\[ + \int_{0}^{\infty} dx \, g(s, x, n, \tau_{2}) \frac{d^{2}}{dx^{2}} \log \left[ (1-e^{-2\pi x-2\pi i n\tau})(1-e^{-2\pi x+2\pi i n\tau}) \right], \]

where

\[ g(s, x, n, \tau_{2}) := x \, {}_{2}F_{1} \left( 1-s, s, 2-s, -\frac{x}{2n\tau_{2}} \right) \left( 1 + \frac{x}{2n\tau_{2}} \right)^{s} (x(x+2n\tau_{2}))^{-s}, \]

with \( {}_{2}F_{1} \) the usual hypergeometric function. Setting \( s = 1 \), we have

\[ g(1, x, n, \tau_{2}) = \frac{1}{2n\tau_{2}} \]

and the integral can, once again, be easily evaluated

\[ Q(1, \tau) = -\frac{\tau_{2}^{1/2}}{4} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-2e^{2\pi i n\tau}+e^{2\pi i n(\tau+\bar{\tau})}}{(e^{2\pi i n\tau}-1)(e^{2\pi i n\tau}-1)}. \]

The case \( \tau = i \) yields

\[ Q(1, i) = \sum_{n=1}^{\infty} \sigma_{-1}(n) K_{-1/2}(2\pi n) n^{1/2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n e^{2\pi n} - 1} \approx 0.00936341. \]

Using the fact that

\[ K_{-1/2}(2\pi n) = \frac{1}{2\sqrt{n}} e^{-2\pi n}, \]

we obtain

\[ \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2\pi n} = \sum_{n=1}^{\infty} \frac{1}{n e^{2\pi n} - 1}, \]

which is a special case of the Lambert series [24]

\[ \sum_{n=1}^{\infty} \sigma_{\alpha}(n) q^{n} = \sum_{n=1}^{\infty} \frac{n^{\alpha} q^{n}}{1-q^{n}}; \]

with \( \alpha = -1 \) and \( q = e^{-2\pi} \). To obtain a different \( \sigma_{\alpha}(n) \) term, with \( \alpha \in -\mathbb{Z}_{+} \) we have to perform additional processes of integration by parts.

### 6.5 Functional equation for the remainder

By using the functional equation of the Riemann zeta function as well as the functional equation (6.2) of the non-holomorphic Eisenstein series we can isolate another functional equation for the remainder, namely

\[ \pi^{1-2s} \Gamma(s) Q(s, \tau) = \Gamma(1-s) Q(1-s, \tau). \]
We now ask the question of whether a functional equation can also be derived by contour integration. Let us first look at a simpler example. In [14], the operator

\[ P := -\frac{d^2}{dt^2} \]

is considered. Under Dirichlet boundary conditions, the eigenvalues are given by

\[ \lambda_n = n^2, \]

where \( n \) is a non-negative integer. The spectral zeta function associated to this operator is

\[ \zeta_P(s) := \sum_{n} (\lambda_n)^{-s} = \sum_{n=1}^{\infty} n^{-2s}. \]  

(6.11)

Let us for a moment pretend that we are not aware that this series represents the Riemann zeta function. The above series is convergent for \( \text{Re}(s) > \frac{1}{2} \). With

\[ F(\lambda) := \frac{1}{2i\sqrt{\lambda}} (e^{i\sqrt{\lambda} \pi} - e^{-i\sqrt{\lambda} \pi}), \]

the contour integral method yields

\[ \zeta_P(s) = \frac{1}{2\pi i} \int_{\gamma} d\lambda \lambda^{-s} \frac{d}{d\lambda} \log F(\lambda) = \frac{\sin(\pi s)}{\pi} \int_{0}^{\infty} dx x^{-s} \frac{d}{dx} \log \frac{1}{2\sqrt{x}} (e^{\sqrt{x} \pi} - e^{-\sqrt{x} \pi}), \]

where \( \gamma \) is the Hankel contour enclosing the eigenvalues \( \lambda_n \) depicted in Fig. 2.1. To obtain the second integral, the deformation to the negative real axis depicted in Fig. 2.2 is performed. From the behaviour of the integrand, this integral representation is seen to be valid for \( \frac{1}{2} < \text{Re}(s) < 1 \).

Furthermore, splitting the integral as \( \int_{0}^{1} dx + \int_{1}^{\infty} dx \) yields

\[ \zeta_P(s) = \frac{\sin(\pi s)}{2s - 1} \int_{0}^{\infty} dx x^{-s} \frac{d}{dx} \log \left[ e^{\sqrt{x} \pi} (1 - e^{-2\sqrt{x} \pi}) \right] + \frac{\sin(\pi s)}{\pi} \int_{1}^{\infty} dx x^{-s} \frac{d}{dx} \log \left[ e^{\sqrt{x} \pi} (1 - e^{-2\sqrt{x} \pi}) \right]. \]  

(6.12)

This is now valid for \( -\infty < \text{Re}(s) < 1 \) and it allows us to find \( \zeta_P(0) = -\frac{1}{2} \) as well as \( \zeta'_P(0) = -\log(2\pi), \) which was the goal in [14] (more specifically, the functional determinant of the operator \( P \)). In fact, the above expression also tells us that \( \zeta_P(-k) = 0 \) for \( k = 1, 2, 3, \ldots \). To construct the analytic continuation over \( \mathbb{C} \) we focus on \( \text{Re}(s) < 0 \) so that, splitting the second integral as a sum of two logarithms, we obtain

\[ \zeta_P(s) = \frac{\sin(\pi s)}{2s - 1} \int_{0}^{\infty} dx x^{-s-1/2} e^{2\pi \sqrt{x} - 1}. \]

Performing the change \( s \to \frac{s}{2} \) and \( y = 2\pi \sqrt{x}, \) we can write the above as

\[ \zeta_P\left(\frac{s}{2}\right) = (2\pi)^{1-u} \frac{\sin\left(\frac{\pi s}{2}\right)}{\pi} \int_{0}^{\infty} dy \frac{y^{-s}}{e^{y} - 1}. \]

Next, we set \( s = 1 - u \) so that \( \text{Re}(u) = 1 - \text{Re}(s) > 1, \) thus

\[ \zeta_P\left(\frac{1-u}{2}\right) = (2\pi)^{1-u} \frac{\cos\left(\frac{\pi u}{2}\right)}{\pi} \int_{0}^{\infty} du \frac{y^{u-1}}{e^{y} - 1}, \]  

(6.13)
By real variable methods \cite{26} this last integral, which represents the Mellin transform of \(1/(e^y-1)\), is seen to yield
\[
\int_0^\infty \frac{y^{u-1}}{e^y-1} \, dy = \Gamma(u)\zeta_R(u), \quad \text{Re}(u) > 1.
\]
Finally, by the use of the Euler reflection formula we have
\[
\zeta_P\left(\frac{1}{2} \right) = (2\pi)^{1/2} \Gamma\left(1 - \frac{1}{2} \right) \zeta_P\left(1 - \frac{1}{2} \right) = 2\pi^{1/2} \Gamma\left(1 - \frac{1}{2} \right) \sin\left(\frac{\pi}{2} \right).
\]
This is the functional equation of the spectral zeta function \(\zeta_P(u)\). It shows that \(\zeta_P(u)\) has a meromorphic continuation to \(\mathbb{C}\) with a simple pole at \(u = \frac{1}{2}\) with residue equal to \(\frac{1}{2}\).

The question is now whether a similar argument can be reproduced for the non-holomorphic Eisenstein series for which we know there exists a functional equation.

To make matters easier, we take the case \(\tau = i\). Equation (4-5) of \cite{15} shows that the remainder \(Q(s, i)\) can be written as
\[
Q(s, i) = 4\frac{\sin(\pi s)}{\pi} \tau^s \sum_{n=1}^{\infty} \int_0^\infty \frac{dz}{(z-n^2)^s} \frac{d}{dz} \log(1 - e^{-2\pi\sqrt{z}}).
\]
We can re-write this as
\[
Q(1-s, i) = 4\frac{\sin(\pi s)}{\pi} \tau^s \sum_{n=1}^{\infty} \int_0^\infty du u^{s-1} \left( \frac{1}{e^{2\pi\sqrt{n^2 + u}} - 1} - \frac{\pi}{\sqrt{n^2 + u}} \right),
\]
so that in fact we are interested in computing the Mellin transform of
\[
h(x) := \frac{1}{e^{2\pi\sqrt{n^2 + u}} - 1} - \frac{\pi}{\sqrt{n^2 + u}}.
\]
This is to be compared with \((6.13)\) but in this case it is more difficult to evaluate the integral, except for the case \(n = 0\) for which we would go back to the Riemann zeta function. Thus, instead of obtaining the functional equation \((6.10)\) we can use \((6.10)\) to conclude that the infinite sum is
\[
\sum_{n=1}^{\infty} \int_0^\infty du u^{s-1} \left( \frac{1}{e^{2\pi\sqrt{n^2 + u}} - 1} - \frac{\pi}{\sqrt{n^2 + u}} \right) = \frac{\pi^{2s}}{4} Q(s, i).
\]
An alternative way to produce the functional equation of the non-holomorphic Eisenstein series is as follows, see \cite{31}.

**Definition 6.1.** Let \(\lambda_n\) be the eigenvalues of an operator \(\Delta\) on a compact smooth manifold \(M\) with Riemannian metric \(g\). For \(t > 0\) the heat kernel is defined by
\[
K(t) := \sum_n e^{-\lambda_n t}.
\]
Taking the eigenvalues to be \(|m + n\tau| \pi \tau^{-1}\) we get the following result.
Lemma 6.2 (Jacobi-inversion). If

$$K(x, \tau) := \sum_{(m,n) \in \mathbb{Z}^2} \exp(-|m+n\tau| \pi \tau_2^{-1} x)$$

with $x > 0$ then the following functional equation holds

$$K(x, \tau) = x^{-1} K(x^{-1}, \tau). \quad (6.15)$$

This is proved by using Poisson summation in two variables with $f(x, y) = \exp(-(u + v\tau)(u + v\tau_2^{-1} x^{-1})$. By making the change $t = |m + n\tau|^2 \pi x\tau_2^{-1}$ in the integral representation of the $\Gamma$ function, summing over $(m, n) \in \mathbb{Z}^2$ and integrating term by term we see that for $\text{Re}(s) > 1$

$$\tilde{E}(s, \tau) := \frac{1}{2} \pi^{-s} \Gamma(s) E^s(s, \tau) = \frac{1}{2} \pi^{-s} \Gamma(s) \sum_{(m,n) \in \mathbb{Z}^2} Q(m,n,\tau)^{-s} = \frac{1}{2} \int_0^\infty (K(t, \tau) - 1) t^{s-1} dt,$$

where $Q(m, n, \tau) := |m + n\tau|^2 \tau_2$. Using (6.15) leads to

$$\tilde{E}(s, \tau) = \tilde{E}(1-s, \tau).$$

7 Conclusion and outlook

We have shown in this paper that the method of contour integration applied to the eigenvalues of the Laplacian in the torus yields an alternative expression of the Chowla-Selberg series formula which is very useful to immediately and effortlessly obtain the functional determinant of the Laplacian operator as opposed to having to use sophisticated results such as the Kronecker first limit formula. In fact, once the equivalence with the Chowla-Selberg formula is established, the proof of the Kronecker limit formula follows. Moreover, as special cases of the comparison we obtain other interesting formulae, some of which are new and some already well-known, involving the Dedekind eta function as well as the divisor function.

One may, for instance, wish to apply this method to the case of algebraic fields and non-Euclidean space $\mathcal{H}^3$. The points of $\mathcal{H}^3$ are denoted by $z = (x, y)$ with $x = x_1 + x_2i$ with $x_1, x_2 \in \mathbb{R}$ and $y > 0$. The space $\mathcal{H}^3$ is embedded in the Hamiltonian algebra of quaternions so that $z = x + yi \in \mathcal{H}^3$ with $j^2 = -1$ and $ij = -ji$. If we denote by $y(z)$ the third coordinate of $z$ and by $\mathbb{Z}[i]$ the Gaussian number field, then the Eisenstein series is defined by

$$E^s_{\mathbb{Z}[i]}(s, \tau) := \frac{1}{4} \sum_{l,h \in \mathbb{Z}[i]} \frac{y^s}{(|lx + hi|^2 + |y|^2)^{\frac{s}{2}}}.$$

This series converges absolutely for $\text{Re}(s) > 2$ and its Fourier expansion is given by

$$E^s_{\mathbb{Z}[i]}(s, \tau) = y^s + \frac{\pi}{s-1} \frac{\zeta_{K}(s-1)}{\zeta_{K}(s)} y^{s-2} + \frac{2\pi^s y}{\Gamma(s) \zeta_{K}(s)} \sum_{n \in \mathbb{Z}[i]} \frac{|n|^{s-1} \sigma_{1-s}(n) K_{s-1}(2\pi |n| y) e^{2\pi i \text{Re}(n\tau)}}{n \neq 0},$$

where $\zeta_{K}(s)$ is the Dedekind zeta function of the Gaussian number field and $\sigma_{\nu}(n) = \frac{1}{4} \sum_{d|n} |d|^{2\nu}$ with $d \in \mathbb{Z}[i]$. The techniques described here should allow for an alternative representation and derivation of this result.
Acknowledgments

NR wishes to acknowledge partial support of SNF Grant No. 200020-131813/1 as well as Prof. Mike Cranston for the hospitality of the Department of Mathematics of the University of California, Irvine. EE was supported in part by MICINN (Spain), grant PR2011-0128 and project FIS2010-15640, by the CPAN Consolider Ingenio Project, and by AGAUR (Generalitat de Catalunya), contract 2009SGR-994, and his research was partly carried out while on leave at the Department of Physics and Astronomy, Dartmouth College, NH.

References

[1] D. Bump, Automorphic Forms and Representations, Cambridge University Press, 1998.

[2] S. Chowla and A. Selberg, On Epstein’s Zeta Function I, Proc. Natl. Acad. Sci. USA 35:371-374, 1949. A. Selberg and S. Chowla, On Epstein’s Zeta-function, Journal für die reine und angewandte Mathematik 227:86-110, 1967.

[3] P. Deligne et al, Quantum Fields and Strings: A Course for Mathematicians, Volume 2, AMS, Institute of Advanced Study, 2000.

[4] P. Di Francesco, P. Mathieu and D. Senechal, Conformal Field Theory, Springer Verlag, 1997.

[5] J. S. Dowker, Effective action in spherical domains, Commun. Math. Phys. 162:633-648, 1994.

[6] J. S. Dowker, Functional determinants on spheres and sectors, J. Math. Phys. 35:4989-4999, 1994.

[7] E. Elizalde, Ten Physical Applications of Spectral Zeta Functions, Second Edition, Lecture Notes in Physics 855, Springer-Verlag, 2012.

[8] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, Zeta Regularization Techniques with Applications, World Scientific, Singapore, 1994.

[9] P. Epstein, Zur Theorie allgemeiner Zetafunktionen, Math. Ann., 56:615-644, 1903.

[10] P. Epstein, Zur Theorie allgemeiner Zetafunktionen II, Math. Ann., 63:205-216, 1907.

[11] S. S. Gelbart and S. D. Miller, Riemann’s zeta function and beyond, Bulletin (New Series) of the American Mathematical Society, 41:59-112, 2003.

[12] K. Kirsten, Spectral functions in mathematics and physics, Chapman&Hall/CRC, Boca Raton, FL, 2002.

[13] K. Kirsten and A. J. McKane, Functional determinants for general Sturm-Liouville problems, J. Phys. A, 37:4649-4670, 2004.

[14] K. Kirsten and P Loya, Computation of determinants using contour integrals, Am. J. Phys., 76:60-64, 2008.

[15] K. Kirsten, A Window into Zeta and Modular Physics - Functional Determinants in Higher Dimensions via Contour Integrals, MRSI Publications 57:307-328, 2010, Cambridge University Press.
[16] B. M. Levitan and I. S. Sargsjan, *Introduction to spectral theory: Selfadjoint ordinary differential operators*, Translations of Mathematical Monographs 39. AMS, Providence, R. I. 1975.

[17] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian J. Math, 1:242-256, 1949.

[18] Y. Motohashi, *A New Proof of the Limit Formula of Kronecker*, Proc. Japan Acad. 44:614-616, 1968.

[19] M. Nakahara, *Geometry, Topology and Physics*, Second Edition, Graduate Student Series in Physics, Taylor & Francis, 2003.

[20] Z. Nan-Yue and K. S. Williams, *On the Epstein Zeta Function*, Tamkang Journal of Mathematics, 26:165-176, 1995.

[21] J. Polchinski, *String theory, Volume 1: An introduction to the bosonic string*, Cambridge University Press, 1998.

[22] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. 7:145-210, 1971.

[23] P. Sarnack, *Determinants of Laplacians*, Comm. Math. Phys. 110:113-120, 1987.

[24] C. L. Siegel, *On Advanced Analytic Number Theory*, Tata Institute of Fundamental Research, 1959.

[25] T. Shintani, *A Proof of the Classical Kronecker Limit Formula*, Tokyo J. Math. 3:191-199, 1980.

[26] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-function*, Second Edition, Oxford Science Publications, 1986.

[27] I. Vardi, *Determinants of Laplacians and Multiple Gamma Functions*, SIAM J. Math. Anal, 19:493-507, 1988.

[28] H. Weyl, *Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung*, J. Angew. Math. 141:1-11, 1912.

[29] F. L. Williams, *A Window into Zeta and Modular Physics - Lectures on zeta functions, L-functions and modular forms with some physical applications*, MRSI Publications 57:7-100, 2010, Cambridge University Press.

[30] F. L. Williams, *Topics in quantum mechanics*, Progress in Mathematical Physics, no. 27, Birkhäuser Boston Inc., Boston, MA, 2003.

[31] D. Zagier, *Eisenstein series and the Riemann zeta function*, in Automorphic Forms, Representation Theory and Arithmetic, Springer-Verlag, Berlin-Heidelberg-New York (1981) 275-301.

**Institute of Space Science, National Higher Research Council, ICE-CSIC, Facultat de Ciencies, Campus UAB, Torre C5-Par-2A, 08193 Bellaterra (Barcelona), Spain**

**Email:** elizalde@ieec.uab.es; elizalde@math.mit.edu
GCAP-CASPER, Department of Mathematics, Baylor University, One Bear Place, Waco, TX 76798, USA
email: klaus.kirsten@baylor.edu

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
email: nicolas.robles@math.uzh.ch

Department of Mathematics and Statistics, Lederle Graduate Research Tower, 710 North Pleasant Street, University of Massachusetts, Amherst, MA 01003-9305, USA
email: williams@math.umass.edu