$N = 2$ supersymmetric pseudodifferential symbols and super $W$-algebras

Stéphane Gourmelen

Institut de Physique Nucléaire de Lyon, IN2P3/CNRS
Université Claude Bernard
43, boulevard du 11 novembre 1918
F - 69622 - Villeurbanne Cedex

Abstract

We study the superconformally covariant pseudodifferential symbols defined on $N = 2$ super Riemann surfaces. This allows us to construct a primary basis for $N = 2$ super $W^{(n)}_{KP}$-algebras and, by reduction, for $N = 2$ super $W_n$-algebras.

1 Introduction

$W$-symmetry plays an important rôle in the context of two-dimensional conformal field theories [1, 2] and their applications to critical phenomena and to string theory [3]. Classical $W$-algebras first appeared in the study of integrable systems, namely of generalized KdV hierarchies [4]. More precisely, the $W_n$-algebra arises as the second hamiltonian structure of the $n$-th KdV hierarchy whose Poisson brackets are defined on the manifold of differential operators of order $n \geq 2$. In the simplest case ($n = 2$), this algebra coincides with the Virasoro algebra, as was first noticed in ref. [5]. This fact exhibits the connection between $W_n$-algebras and conformal field theory and suggests to apply conformal symmetry to the formulation of classical $W_n$-algebras. This was done in ref. [6] (see also references therein) where it was shown that the $W_n$-algebra possesses a ‘primary basis’ of generators consisting of a projective connection (the Virasoro generator) and $n - 2$ primary fields which transform like $k$-forms ($k = 3, \ldots, n$) under the Virasoro flow.

Besides $W_n$-algebras there are other $W$-algebras which are said to be ‘infinite’ because they contain an infinity of independent generators. These algebras are related to each other by reductions, truncations or contractions [7]. In particular, every $W_n$-algebra can be obtained by reduction from the infinite $W^{(n)}_{KP}$-algebra [8]. The latter is obtained by applying the second hamiltonian structure of $W_n$ to pseudodifferential symbols rather than
differential operators. It was shown in ref. [9] that such a symbol can be parametrized by a projective connection and an infinity of primary fields, thus providing a primary basis for every $\mathcal{W}_{KP}^{(n)}$-algebra ($n \geq 2$).

$\mathcal{W}$-algebras admit supersymmetric extensions which manifest themselves in the context of superstring or super-Toda field theories. The second hamiltonian structure of $N = 1$ super $\mathcal{W}$-algebras was constructed in superspace in ref. [11]. Their primary basis was determined in ref. [11] for super $\mathcal{W}_n$ and generalized to super $\mathcal{W}_{KP}^{(n)}$ [12]. In this paper, we are interested in $N = 2$ super $\mathcal{W}$-algebras. They have been extensively studied in the $N = 1$ formalism (see [13, 14, 15, 16] and references therein) until their formulation in $N = 2$ superspace was discovered [17, 18]. $N = 2$ super $\mathcal{W}_n$-algebras were shown to admit a primary basis which was constructed in ref. [19]. The aim of this paper is to study the $N = 2$ supersymmetric pseudodifferential symbols and to apply them to the determination of a primary basis for $N = 2$ super $\mathcal{W}_{KP}$-algebras in $N = 2$ superspace.

In section 2, we summarize some concepts and tools of $N = 2$ superconformal symmetry and superconformally covariant operators. These are used in section 3 to study $N = 2$ supersymmetric pseudodifferential symbols. We pay a particular attention to the so-called Bol symbols (parametrized by a superprojective connection) which are studied systematically. A particular class of them is generalized in section 4 and applied to the formulation of $N = 2$ super $\mathcal{W}_{KP}$-algebras in a primary basis.

2 $N = 2$ supersymmetric differential operators

For further details concerning the notions summarized in this section, we refer to ref. [19].

2.1 Geometric framework

$N = 2$ supersymmetry  In order to make $N = 2$ supersymmetry manifest, all considerations will be carried out on a compact two-dimensional $N = 2$ supermanifold $\Sigma$ [20] with local coordinates $z \equiv (z, \theta, \bar{\theta})$ and their complex conjugates (c.c.) $\bar{z} \equiv (\bar{z}, \theta^-, \bar{\theta}^-)$. Here, $z, \bar{z}$ are even and $\theta, \bar{\theta}, \theta^-, \bar{\theta}^-$ are odd Grassmann numbers. The tangent space is spanned by the derivatives $(\partial, D, \bar{D})$ (and their c.c. $(\bar{\partial}, D_-, \bar{D}_-)$) defined by $\partial = \frac{\partial}{\partial z}$, $D = \frac{\partial}{\partial \theta} + \frac{1}{2} \bar{\partial} \theta$, $\bar{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2} \partial \theta$. Their graded Lie brackets $\{D, \bar{D}\} = \bar{\partial}$, $D^2 = 0 = \bar{D}^2$ are those of the $N = 2$ supersymmetry algebra.

$N = 2$ superconformal symmetry  Since we will be interested in $N = 2$ superconformal symmetry, we require the supermanifold $\Sigma$ to be a super Riemann Surface (SRS). This means that local coordinate systems on $\Sigma$ are related by superconformal transformations. By definition [21], a superconformal transformation of local coordinates of $\Sigma$ is a superdiffeomorphism $(z, \bar{z}) \mapsto (z', \bar{z}')$ satisfying the following three properties (as well as the c.c.
relations):

(i) \[ z' = z'(z) \iff D_z' = 0 = \bar{D}_z' \]

(ii) \[ D\theta' = 0 = \bar{D}\theta' \]

(iii) \[ Dz' = \frac{1}{2} \theta' D\theta' \quad , \quad \bar{D}z' = \frac{1}{2} \bar{\theta}' \bar{D}\bar{\theta}' . \]

These relations imply that \( D \) and \( \bar{D} \) transform homogeneously,

\[ D' = e^w D \quad , \quad \bar{D}' = e^{\bar{w}} \bar{D} , \]

where \( e^{-w} \equiv D\theta' , Dw = 0 \) and \( e^{-\bar{w}} \equiv \bar{D}\bar{\theta}' , \bar{D}\bar{w} = 0 \).

**Superprojective connection** The super Schwarzian derivative \( S(z', z) \) associated to a superconformal change of coordinates \( z \mapsto z'(z) \) is defined by

\[
-S(z', z) = \frac{\partial \bar{D}\theta'}{\partial \theta'} - \frac{\partial D\theta'}{\partial \theta'} + \frac{\partial \theta'}{\partial \bar{D}\theta'} \frac{\partial \theta'}{\partial D\theta'} = 2 e^{-\frac{1}{2}(w+\bar{w})} [D, \bar{D}] e^{\frac{1}{2}(w+\bar{w})} .
\]

The following study will make an extensive use of a superprojective connection. This is a superfield \( R \equiv R_{\theta\bar{\theta}}(z) \) which is locally superanalytic (i.e. \( D_R = 0 = \bar{D}_R \)) and which transforms under a superconformal transformation of coordinates according to

\[ R'(z') = e^{w+\bar{w}} [R(z) - S(z', z)] . \]

Such a field can be globally defined on compact SRS’s of arbitrary genus \([22, 21]\).

### 2.2 Superconformally covariant differential operators

A superconformal (or primary) field of superconformal weight \( (p, q) \) is a function \( C_{p,q} \in C^\infty(\Sigma) \) (i.e. the space of supersmooth functions on \( \Sigma \)) which transforms under a superconformal change of local coordinates according to

\[ C'_{p,q}(z'; \bar{z}') = e^{pw+q\bar{w}} C_{p,q}(z; \bar{z}) \quad (p, q \in \mathbb{Z} / 2 , \quad p + q \in \mathbb{Z}) . \]

The space of these fields will be denoted by \( F_{p,q} \).

**Definition** 1 A generic superdifferential operator on \( \Sigma \) is locally defined by

\[
\mathcal{L} = \sum_{n=0}^{n_{\max}} \left( a_n + \alpha_n D + \beta_n \bar{D} + b_n [D, \bar{D}] \right) \partial^n ,
\]

where \( a_n, b_n \) and \( \alpha_n, \beta_n \) are, respectively, even and odd superfields belonging to \( C^\infty(\Sigma) \). Such an operator is called superconformally covariant (or covariant for short) if it maps primary fields of some weight \( (p, q) \) to primary fields of some weight \( (p', q') \):

\[
\mathcal{L} \equiv \mathcal{L}_{p,q} : F_{p,q} \rightarrow F_{p',q'} .
\]
Note that the correspondence (3) is equivalent to the following transformation law under superconformal changes of local coordinates:

\[
(L_{p,q})' = e^{\rho w + q \bar{\rho} \bar{w}} L_{p,q} e^{-p w - q \bar{w}}.
\]  

(4)

In order to construct such covariant operators, it is convenient to use a superaffine connection. This is a collection of superfields \(B \equiv B_{\theta}, \bar{B} \equiv \bar{B}_{\bar{\theta}}\) which are locally defined on \(\Sigma\) and which satisfy the following three conditions: they are locally superanalytic, they satisfy the chirality conditions \(DB = 0 = \bar{D} \bar{B}\) and they transform under a superconformal change of local coordinates according to \(B'(z') = e^w [B(z) + D \bar{w}]\), \(\bar{B}'(\bar{z}') = e^{\bar{w}} [\bar{B}(\bar{z}) + \bar{D} w]\).

Using an affine connection one can introduce supercovariant derivatives

\[
\nabla \equiv \nabla_{p,q} = D - q B : \mathcal{F}_{p,q} \rightarrow \mathcal{F}_{p+1,q}
\]

\[
\bar{\nabla} \equiv \bar{\nabla}_{p,q} = \bar{D} - p \bar{B} : \mathcal{F}_{p,q} \rightarrow \mathcal{F}_{p,q+1}.
\]

By construction, these are nilpotent: \(\nabla^2 = 0 = \bar{\nabla}^2\).

The most general covariant operator that is locally defined on \(\Sigma\) is simply obtained by replacing the two fermionic derivatives \(D\) and \(\bar{D}\) in expression (2) by supercovariant ones, \(\nabla\) and \(\bar{\nabla}\):

\[
L_{p,q} = \sum_{n=0}^{n_{max}} \left( a_n + \alpha_n \nabla + \beta_n \bar{\nabla} + b_n [\nabla, \bar{\nabla}] \right) \{\nabla, \bar{\nabla}\}^n.
\]  

(5)

### 2.3 Bol differential operators

The only compact SRS’s which admit a globally defined affine connection are those of genus one \([22, 23]\). On the other hand, affine and projective connections are locally related by the super Miura transformation\(^1\)

\[
\mathcal{R} = (DB) - (\bar{D} \bar{B}) - BB
\]  

(6)

The fact that a projective connection can always be defined globally on a SRS motivates the following definition.

**Definition 2** A Bol operator is a covariant differential operator on \(\Sigma\) which depends on an unique superfield, namely a projective connection.

It follows directly from this definition that a Bol operator is globally defined on the SRS. In order to construct a Bol operator, we require that the operator \(L_{p,q}\) of eq. (3) only depends on the affine connections \(B, \bar{B}\) through the combination \(\mathcal{R} = DB - \bar{D} \bar{B} - BB\) given by the Miura transformation. By using a variational argument (one imposes that \(\delta L_{p,q} = 0\) while varying \(B\) and \(\bar{B}\) subject to the condition that \(\mathcal{R}\) is fixed), one is led to the following result \([19]\):

\[^1\text{In order to avoid ambiguities, we adopt from now on the following notation for the action of derivatives on a field } C: (\partial C), (D C), (\bar{D} C), ([D, D] C), \ldots \text{ denote derivatives of the field } C \text{ while a derivative acts operatorially otherwise, e.g. } \partial C = (\partial C) + C \partial.\]
Theorem 1  For each superconformal weight \((p, q) \in (\mathbb{Z}/2, \mathbb{Z}/2)\) such that \(- (p + q) \in \mathbb{N}^*\), there exists a Bol operator defined on the whole space \(\mathcal{F}_{p,q}\). This operator is unique up to a global factor and is of order \(n = - (p + q)\). It reads

\[
L_{p,q}(\mathcal{R}) = q(\nabla \bar{\nabla})^n - p(\bar{\nabla} \nabla)^n : \mathcal{F}_{p,q} \longrightarrow \mathcal{F}_{p+n,q+n} .
\]  

(7)

Note that there are other Bol operators which are only defined on appropriate subspaces of \(\mathcal{F}_{p,q}\). 

3  \(N = 2\) supersymmetric pseudodifferential symbols

3.1 Basic definitions and relations

We aim to extend the analysis of sections 2.2 and 2.3 to the pseudodifferential case. A generic pseudodifferential symbol (or symbol for short) is locally defined on \(\Sigma\) by

\[
\mathcal{L} = \sum_{n=-\infty}^{n_{\text{max}}} \left( a_n + \alpha_n D + \beta_n \bar{D} + b_n[D, \bar{D}] \right) \partial^n ,
\]  

(8)

where we have introduced the inverse \(\partial^{-1}\) of the usual derivative:

\[
\partial \partial^{-1} = \partial^{-1} \partial = 1 .
\]  

(9)

The symbol \(\mathcal{L}\) can be divided into its differential part (the summation going from \(n = 0\) to \(n_{\text{max}}\)) and its integral part (the summation going from \(n = -\infty\) to \(-1\)) which will be denoted by \((\mathcal{L})_+\) and \((\mathcal{L})_-\), respectively.

By using the identity

\[
[D, \bar{D}]^2 = \partial^2 ,
\]  

(10)

one immediately verifies that

\[
[D, \bar{D}]^{-1} = \partial^{-2} [D, \bar{D}] .
\]  

(11)

From (8) and (11), it then follows that for all \(\alpha, \beta \in \mathbb{R}\),

\[
\left( \alpha \partial - \beta [D, \bar{D}] \right)^{-1} = \frac{1}{\alpha^2 - \beta^2} \left( \alpha \partial^{-1} + \beta [D, \bar{D}]^{-1} \right) \quad \text{if} \quad \alpha \neq \pm \beta .
\]  

(12)

Since \(\partial = \{D, D\}\), the condition \(\alpha \neq \pm \beta\) reflects the fact that the operators \(DD\) and \(\bar{D} \bar{D}\) are not invertible.

3.2 Superconformal covariance of a pseudodifferential symbol

Since we are now dealing with (pseudodifferential) symbols rather than (differential) operators, we have to generalize definition [1]. In fact, the correspondence (3) does not make sense in the present case (because a symbol does not transform a field into another field). However, we can postulate the transformation law (H).
**Definition 3** A pseudodifferential symbol $L_{p,q}$, locally defined on $\Sigma$ by (8), is superconformally covariant (or covariant for short) if it transforms (under a superconformal change of local coordinates) according to

$$(L_{p,q})' = e^{p'w + q'\bar{w}} L_{p,q} e^{-pw - q\bar{w}} ,$$

where $p, q, p', q' \in \mathbb{Z}/2$ and $p + q, p' + q' \in \mathbb{Z}$.

If $L_{p,q}$ is a covariant symbol, then its differential part $(L_{p,q})_+$ and its integral part $(L_{p,q})_-$ are separately covariant. In particular, for the differential part, definition 3 reduces to definition 1 and the analysis of sections 2.2 and 2.3 applies.

Once again superconformal covariance can be ensured locally by introducing supercovariant derivatives in expression (8):

$$L_{p,q} = \sum_{n=-\infty}^{n_{max}} \left( a_n + \alpha_n \nabla + \beta_n \bar{\nabla} + b_n[\nabla, \bar{\nabla}] \right) \{\nabla, \bar{\nabla}\}^n .$$

For later reference, we note that the nilpotency of the supercovariant derivatives allows to obtain the covariant analogon of relations (10)-(12):

$$[\nabla, \bar{\nabla}]^2 = \{\nabla, \bar{\nabla}\}^2$$

$$[\nabla, \bar{\nabla}]^{-1} = \{\nabla, \bar{\nabla}\}^{-2}[\nabla, \bar{\nabla}]$$

$$(\alpha \{\nabla, \bar{\nabla}\} - \beta [\nabla, \bar{\nabla}])^{-1} = \frac{1}{\alpha^2 - \beta^2} (\alpha \{\nabla, \bar{\nabla}\}^{-1} + \beta [\nabla, \bar{\nabla}]^{-1}) \quad \text{if} \quad \alpha \neq \pm \beta .$$

### 3.3 Bol pseudodifferential symbols

As in the differential case, we require a covariant pseudodifferential symbol to be globally defined on any compact SRS:

**Definition 4** A Bol symbol is a superconformally covariant pseudodifferential symbol which depends on an unique superfield, namely a projective connection.

Bol symbols can be determined by using the same variational method as the one used for Bol operators. This leads to the following result :

**Theorem 2** For each superconformal weight $(p, q) \in (\mathbb{Z}/2, \mathbb{Z}/2)$ such that $-(p + q) \in \mathbb{Z}^*$, there exists a Bol symbol which is unique up to a global factor. The latter is of order $n = -(p + q)$ and reads

$$L_{p,q}(\mathcal{R}) = \{\nabla, \bar{\nabla}\}^{n-1}(q\nabla \bar{\nabla}_{p,q} - p\bar{\nabla} \nabla_{p,q}) .$$

$^2$Note that $\{\nabla, \bar{\nabla}\}$ locally reads $\nabla \bar{\nabla}_{p,q} + \bar{\nabla} \nabla_{p,q} = \partial - B\bar{D} - \bar{B}D - p(DB) - q(ar{D}B) + (p - q)B\bar{B}$ and that it is invertible because its leading term $\partial$ is invertible.
For \( n > 0 \), this expression reduces to the Bol operator (7).

The inverse of a Bol symbol (which exists if and only if \( p \neq 0 \) and \( q \neq 0 \)) is also a Bol symbol. In fact, it follows from expression (15) that Bol operators and Bol symbols are related by the following inversion property:

\[
L_{p,q}^{-1} = -\frac{1}{pq} L_{-q,-p} \quad \text{if} \quad p \neq 0 \quad \text{and} \quad q \neq 0.
\]

This relation can be used to determine explicitly purely integral Bol symbols in terms of the projective connection \( \mathcal{R} \) by inverting Bol differential operators.

Before discussing the singular cases \( p = 0 \) or \( q = 0 \), we briefly consider the symmetric case (\( p = q \)).

### Symmetric Bol symbols

For \( p = q \), theorem 2 states that

\[
L_{n}^{\text{sym}}(\mathcal{R}) \equiv \{\nabla, \bar{\nabla}\}^{n-1} \{\nabla, \bar{\nabla}\}_{-\frac{n}{2},-\frac{n}{2}} \quad (n \neq 0)
\]

is a Bol symbol with inverse \((L_{n}^{\text{sym}})^{-1} = L_{-n}^{\text{sym}}\). Interestingly enough, these properties can be generalized to the case \( n = 0 \). The corresponding symbol as given by eq.(17) reads \( L_{0}^{\text{sym}} = \partial^{-1} [D, \bar{D}] \) so that it is a Bol symbol which coincides with its own inverse according to eq.(18) : \((L_{0}^{\text{sym}})^{2} = 1\). For \( n > 0 \), the first symmetric Bol operators have been calculated in [19] and have been shown to appear in the commutation relations of \( N = 2 \) super \( W \)-algebras (see also equations (31) below).

### (Anti-)chiral Bol symbols

Among the Bol symbols given by expression (15), the non-invertible ones correspond to the so-called chiral \((p = 0)\) and anti-chiral \((q = 0)\) solutions :

\[
L_{n}^{\text{chir}}(\mathcal{R}) \equiv \nabla \{\nabla, \bar{\nabla}\}^{n-1} \nabla_{0,-n} = D\{\nabla, \bar{\nabla}\}^{n-1} \bar{D}
\]

\[
L_{n}^{\text{anti}}(\mathcal{R}) \equiv \nabla \{\nabla, \bar{\nabla}\}^{n-1} \nabla_{-n,0} = \bar{D}\{\nabla, \bar{\nabla}\}^{n-1} D
\]

Although they are not invertible (as stated above), they nevertheless are related by a kind of inversion relation, which we will now discuss. We first note that expressions (18) allow us to define the following equivalence classes of symbols,

\[
DJ_{n} \equiv D\{\nabla, \bar{\nabla}\}^{n} \quad \text{modulo} \quad D\lambda \bar{D}
\]

\[
\bar{D}K_{n} \equiv \bar{D}\{\nabla, \bar{\nabla}\}^{n} \quad \text{modulo} \quad \bar{D}\mu D \quad (n \in \mathbb{Z}),
\]

where \( \lambda \) and \( \mu \) are arbitrary pseudodifferential symbols. In the differential case \((n \geq 0)\), this amounts to restricting the domains of definition of the operators \( DJ_{n} \) and \( \bar{D}K_{n} \) to antichiral and chiral superfields, respectively [19]. Obviously the representatives of the equivalence classes (19) are related to the Bol symbols (18) by

\[
L_{n}^{\text{chir}} = DJ_{n-1} \bar{D}
\]

\[
L_{n}^{\text{anti}} = \bar{D}K_{n-1} D
\]
and their interest consists of the fact that they satisfy the two (equivalent) inversion relations

\[ (DJ_n)^{-1} = \bar{D}K_{n-1} \]
\[ (DK_n)^{-1} = DJ_{n-1} \]  

(20)

The latter can be used to determine in a simple way the purely pseudodifferential chiral or antichiral Bol symbols by starting from the differential ones. The simplest examples of symbols given by eqs. (19) read

\[ \bar{D}K_{-1} = \bar{D} \partial^{-1} \]
\[ \bar{D}K_0 = \bar{D} \]
\[ \bar{D}K_1 = \bar{D}[\partial + \mathcal{R}] \]
\[ \bar{D}K_2 = \bar{D}[\partial^2 + 3\partial \partial + (\bar{D}D\mathcal{R}) + 2(D\bar{D}\mathcal{R}) + 2\mathcal{R}^2] \]  

and one can explicitly verify that they satisfy the relations (20). For later reference, we note that the leading terms of the generic antichiral Bol symbol read \((n \in \mathbb{Z})\)

\[ \bar{D}K_n D = \bar{D} \left( \partial^n + c^{(n)} \mathcal{R} \partial^{n-1} + \ldots \right) D \quad \text{with} \quad c^{(n)} = \frac{n(n+1)}{2} . \]  

(22)

4 Covariant symbols and their applications to \(N = 2\) super \(W\)-algebras

The antichiral Bol symbol \(L_n^{\text{anti}}(\mathcal{R})\) given by (18) is a symbol of the generic form

\[ \bar{D}L^{(n)} D = \bar{D} \left[ \partial^n + \sum_{k=1}^{\infty} a_k^{(n)} \partial^{n-k} \right] D \quad (n \in \mathbb{Z}) . \]  

(23)

It is covariant and has the property that it only depends on a projective connection \(\mathcal{R}\). This suggests that, more generally, by requiring a generic symbol of the form (23) to be covariant and globally defined on any SRS, one should obtain a reparametrization of this symbol in terms of a projective connection and some superconformal fields of appropriate weight. This has been worked out in ref. \[19\] for differential operators and the extension to the pseudodifferential case is the following.

Given a superconformal field \(\mathcal{W}_k\) of weight \((k, k)\) (with \(k \in \mathbb{N}^*\)), the covariant symbol \(\bar{D}M_{\mathcal{W}_k}^{(n)} D\) is defined for \(n \in \mathbb{Z}\) by

\[ \bar{D}M_{\mathcal{W}_k}^{(n)} D = \nabla \sum_{l=0}^{\infty} \left\{ A_{k,l}^{(n)} \left[ (\nabla \nabla)^l \mathcal{W}_k \right] + B_{k,l}^{(n)} \left[ (\nabla \nabla)^l \mathcal{W}_k \right] \right\} \{ \nabla, \nabla \}^{n-k-l} \nabla_{-1,0} . \]  

(24)

It depends linearly on \(\mathcal{W}_k\) and it depends on the projective connection \(\mathcal{R}\) given by the
Miura transformation (3) provided the coefficients are chosen to be

\[ A_{k,l}^{(n)} = \binom{n-k}{l} \binom{k+l}{l} \left( \frac{1}{2k+l} \right), \quad B_{k,l}^{(n)} = \binom{n-k}{l} \binom{k+l-1}{l} \left( \frac{1}{2k+l} \right) \]  

(25)

for \( l = 1, \ldots, n-k \) and \( A_{k0}^{(n)} + B_{k0}^{(n)} = 1. \)

Instead of using the ‘natural’ coefficients \( a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots \) of expression (23) to span the phase space of superfields, we can use symbols of the form (24) to obtain a parametrization which relies on a projective connection \( \mathcal{R} \) and of superconformal fields.

**Theorem 3** The most general symbol of the form (23) which is covariant and globally defined on any SRS is parametrized by a projective connection \( \mathcal{R} \) and an infinity of superconformal fields \( \mathcal{W}_k \) of weight \((k,k)\) – one for each value of \( k \in \{2, \ldots, \infty\} \) – according to

\[ \bar{D}K^{(n)}D = \bar{D} \left[ K_n + \sum_{k=2}^{\infty} M_{\mathcal{W}_k}^{(n)} \right] D \quad (n \in \mathbb{Z}). \]  

(26)

Here \( \bar{D}K_nD \) is the antichiral Bol symbol (13) and \( \bar{D}M_{\mathcal{W}_k}^{(n)}D \) is given by eq. (24).

The superfields \( \mathcal{R} \) and \( \mathcal{W}_2, \mathcal{W}_3, \ldots \) are related to the former by invertible differential polynomials which can be explicitly determined by identifying the expressions (23) and (26):

\[ a_1^{(n)} = c^{(n)} \mathcal{R} \]  

(27)

\[ a_i^{(n)} = \sum_{k=1}^{i} \left[ A_{k,i-k}^{(n)} (\bar{D}D)^{i-k} \mathcal{W}_k \right] + \left[ B_{k,i-k}^{(n)} (\bar{D}D)^{i-k} \mathcal{W}_k \right] + \text{nonlinear terms}. \]  

(28)

The factor \( c^{(n)} \) was given in eq. (22) and we have used the notation \( \mathcal{W}_1 \equiv a_1^{(n)} \) in the last relation.

Thus we have achieved a superconformal (or primary) parametrization of the symbol (23), reflecting its superconformal covariance property. Of course, by starting from the chiral rather than the anti-chiral Bol symbols, one can repeat the whole analysis in order to achieve an analogous parametrization for the symbols of the form \( D\mathcal{J}^{(n)}D \).

**N = 2** super \( \mathcal{W}_{KP}^{(n)} \)-algebras We denote by \( \mathcal{M}_n \) the manifold of covariant symbols of the form (23) and, from now on, we restrict our study to the case \( n \geq 1 \) for which the Bol

\[ \binom{n}{p} = \frac{n(n-1)\ldots(n-p+1)}{p!} \text{ if } p \in \mathbb{N}^* \]

and \( \binom{n}{p} = 1 \) if \( p = 0. \)
symbol $\bar{D}K_nD$ is purely differential and depends on the projective connection $\mathcal{R}$. Following ref. [18], one introduces the residue and the trace of a symbol $L \in \mathcal{M}_n$ by

$$\text{res } L = a^{(n)}_{n+1}, \quad \text{Tr } L = \oint d^3z \text{ res } L \quad (29)$$

where $d^3z = dzd\theta d\bar{\theta}$. This trace can be used to define the pairing of two symbols by $\langle A, B \rangle = \text{Tr}(AB)$. Let $T_L(\mathcal{M}_n)$ and $T^*_L(\mathcal{M}_n)$ denote the tangent and cotangent spaces of $\mathcal{M}_n$ at the point $L$ and $\Phi_U(L)$ and $\bar{\Phi}_U(L)$ denote, respectively, the chiral and antichiral parts of the trace $\text{Tr}(\text{res}[L, U])$. Then the map

$$J_L : T^*_L(\mathcal{M}_n) \rightarrow T_L(\mathcal{M}_n) \quad U \mapsto (LU)_+ L - L (UL)_+ + L \Phi_U(L) + \bar{\Phi}_U(L) L \quad (30)$$

determines a hamiltonian structure on the manifold $\mathcal{M}_n$. This hamiltonian map is an $N = 2$ extension of the usual Adler map [23, 4]; it defines a Poisson algebra, namely the $N = 2$ super $\mathcal{W}_{KP}^{(n+1)}$-algebra.

The superfields $a^{(n)}_1$ and $\mathcal{W}_i (i \geq 2)$ constitute a primary basis for this superalgebra. In this basis, the Poisson brackets take a particular form which reflects the superconformal covariance property of these superfields:

$$\begin{align*}
\{a^{(n)}_1(z_2), a^{(n)}_1(z_1)\} &= c^{(n)}_1 \mathcal{L}^{\text{sym}}_2(\mathcal{R}) \delta^{(3)}(z_2, z_1) \\
\{a^{(n)}_1(z_2), \mathcal{W}_k(z_1)\} &= (k \nabla \partial - (\bar{D}V)D - (DV)\bar{D} + (\partial V)) \delta^{(3)}(z_2, z_1) \\
\{\mathcal{W}_k(z_2), \mathcal{W}_l(z_1)\} &= (c^{(n)}_{kl} \mathcal{L}^{\text{sym}}_{k+l}(\mathcal{R}) + \ldots) \delta^{(3)}(z_2, z_1) \quad (31)
\end{align*}$$

The first relation represents the $N = 2$ Virasoro superalgebra. The differential operator $L^{\text{sym}}_2(\mathcal{R})$ is the symmetric Bolt operator given by (17) and explicitly reads $L^{\text{sym}}_2(\mathcal{R}) = \partial D + \mathcal{R} \partial - (D\mathcal{R})\bar{D} - (D\bar{D})\mathcal{R} + (\partial \mathcal{R})$; hence the Virasoro central term reads $\frac{1}{2}n(n + 1) \partial D, \bar{D} \delta^{(3)}(z_2, z_1)$. The second relation reflects the fact that $\mathcal{W}_k$ is a superconformal field of weight $(k, k)$. In the last relation, we have only written the terms which do not depend on the primary fields $\mathcal{W}_i (i \geq 2)$. Since the leading term of the operator $\mathcal{L}^{\text{sym}}_{k+l}(\mathcal{R}) = \partial^{k+l-1}[D, \bar{D}] + \ldots$ does not depends on any field, the central term reads $c^{(n)}_{kl} \partial^{k+l-1}[D, \bar{D}] \delta^{(3)}(z_2, z_1)$.

**N = 2 super $\mathcal{W}_n$-algebras** Interestingly enough, the primary parametrization (24) allows us to split automatically the differential and integral parts of the symbol $\bar{D}K_nD$. In fact, if $k > n$, the symbol (24) is purely integral: $\bar{D}M_{\mathcal{W}_k}^{(n)} D = (\bar{D}M_{\mathcal{W}_k}^{(n)} D)_+$. On the contrary, if $k \leq n$, it is a differential operator (the summation over $l$ is going from 0 to $n - k$ since the coefficients (24) vanish for larger values of $l$): $\bar{D}M_{\mathcal{W}_k}^{(n)} D = (\bar{D}M_{\mathcal{W}_k}^{(n)} D)_+$.

$^4$The map (23) has been determined in ref. [18] for the case of chiral operators of the form $D\mathcal{L}\bar{D}$ and is easily transposed to the antichiral case. As pointed out in this reference, there exists a second quadratic hamiltonian map in the $N = 2$ supersymmetric case.
Thus a symbol $L \in \mathcal{M}_n$ can be easily divided into its differential and integral parts:

$$L_+ = \hat{D} \left[ K_n + \sum_{k=2}^{n} M_{\mathcal{W}_k}^{(n)} \right] D , \quad L_- = \hat{D} \sum_{k=n+1}^{\infty} M_{\mathcal{W}_k}^{(n)} D . \quad (32)$$

According to eqs. (30), $L_- = 0$ implies $J_L(U)_- = 0$ for all $U$. Thus, one can impose the constraint $L_- = 0$. Hence the superfields $\mathcal{W}_k$ with $k > n$ which parametrize $L_-$ generate an ideal $\mathcal{I}_n$ of $\mathcal{W}_{kP}^{(n+1)}$ [4]. This ideal is centerless so that the central charges $c_{kl}^{(n)}$ in (31) vanish if both $k > n$ and $l > n$.

Moreover, the constraint $L_- = 0$ allows for a hamiltonian reduction: the quotient of $\mathcal{W}_{kP}^{(n+1)}$ by its ideal $\mathcal{I}_n$ is isomorphic to the $N = 2$ super $\mathcal{W}_{n+1}$-algebra generated by the differential operator $L_+$. According to eq.(26), the latter is parametrized by the $n$ superfields $\mathcal{R}$ and $\mathcal{W}_2, .., \mathcal{W}_n$ which thus span a primary basis of the $N = 2$ super $\mathcal{W}_{n+1}$-algebra as shown in ref. [13].

5 Conclusion

By studying $N = 2$ covariant symbols, we have achieved a classification of the Bol symbols which are characterized by their dependence on a superprojective connection. Among them, the antichiral Bol symbols $\hat{D}K_nD$ are of particular interest since they are a special case of symbols $\hat{D}\mathcal{L}^{(n)}D$ of the form (23) whose manifold can be endowed with a hamiltonian structure leading to the $N = 2$ super $\mathcal{W}_{kP}^{(n+1)}$-algebra. We have parametrized such symbols by using a superprojective connection and an infinity of primary superfields $\mathcal{W}_k$ of weights $k = 2, .., \infty$. This provides us with a primary basis of generators for the $N = 2$ super $\mathcal{W}_{kP}^{(n+1)}$-algebra and, by reduction, also one for the super $\mathcal{W}_{n+1}$-algebra. If expressed in this basis, the Poisson brackets take a form which reflects their $N = 2$ superconformal symmetry. This allowed us to study the central terms: the Virasoro central charge $c^{(n)}$ of $\mathcal{W}_{kP}^{(n+1)}$ can be explicitly determined. In principle, the other charges $c_{kl}^{(n)}$ could also be computed by inverting relation (25) and proceeding along the lines of ref. [6]. We conjecture that, due to the choice of the primary basis, these central charges turn out to be diagonal ($c_{kl}^{(n)} = c_{kl}^{(n)} \delta_{kl}$) as in the nonsupersymmetric case.

Acknowledgements The author would like to thank François Gieres for his careful reading of the manuscript and his suggestions.

References

[1] P. Bouwknegt and K. Schoutens. Phys. Rep., 223:183, 1993.

[2] S.V. Ketov. Conformal Field Theory. (World Scientific, 1995).

[3] P.Di Francesco, P. Ginsparg and J. Zinn-Justin. Phys. Rep., 254:1, 1995.
[4] L.A. Dickey. *Soliton Equations and Hamiltonian Systems*, volume 12 of *Advanced Series in Mathematical Physics*. (World Scientific, 1991).

[5] J.-L. Gervais and A. Neveu. *Nucl. Phys.*, B209:125, 1982.

[6] P. Di Francesco, C. Itzykson and J.-B. Zuber. *Commun. Math. Phys.*, 140:543, 1991.

[7] J.M. Figueroa-O’Farrill, J. Mas and E. Ramos. *Phys. Lett.*, B299:41, 1993.

[8] J.M. Figueroa-O’Farrill, J. Mas and E. Ramos. *Commun. Math. Phys.*, 158:17, 1993.

[9] W.-J. Huang. *J. Math. Phys.*, 35:993, 1994.

[10] J.M. Figueroa-O’Farrill and E. Ramos. *Phys. Lett.*, B262:265, 1991.

[11] F.Gieres and S.Theisen. *J.Math.Phys.*, 34:5964, 1993.

[12] W.-J. Huang. *J. Math. Phys.*, 35:2570, 1994.

[13] J.M. Figueroa-O’Farrill and E. Ramos. *Nucl. Phys.*, B368:361, 1992.

[14] K. Huitu and D. Nemeschansky. *Mod. Phys. Lett.*, A6:3179, 1991.

[15] F. Gieres and S. Theisen. *Int. J. Mod. Phys.*, A9:383, 1994.

[16] W.-J. Huang, J.C. Shaw and H.C. Yen. *J. Phys.*, A28:165, 1995.

[17] Z. Popowicz. *Phys.Lett.*, B319:478, 1993.

[18] F. Delduc and L. Gallot. *Commun. Math. Phys.*, 190:395, 1997.

[19] F. Gieres and S. Gourmelen. *J. Math. Phys.*, 39:3453, 1998.

[20] B. DeWitt . *Supermanifolds*. 2nd edition (Cambridge University Press, 1992).

[21] F. Delduc, F. Gieres and S. Gourmelen. *Class. Quantum Grav.*, 14:1623, 1997.

[22] F. Gieres. *Int. J. Mod. Phys.*, A8:1, 1993.

[23] M. Adler. *Invent. Math.*, 50:403, 1979.