SUPERGRAVITY VACUA AND LORENTZIAN LIE GROUPS

ALI CHAMSEDDINE, JOSÉ FIGUEROA-O’FARRILL, AND WAFIC SABRA

Abstract. We classify maximally supersymmetric backgrounds (vacua) of chiral (1,0) and (2,0) supergravities in six dimensions and, by reduction, also those of the minimal $N=2$ supergravity in five dimensions. Up to R-symmetry, the (2,0) vacua are in one-to-one correspondence with (1,0) vacua, and these in turn are locally isometric to Lie groups admitting a bi-invariant lorentzian metric with anti-selfdual parallelising torsion, which we classify. We then show that the five-dimensional vacua are homogeneous spaces arising canonically as the spaces of right cosets of spacelike one-parameter subgroups.

CONTENTS

1. Introduction 1

2. (1,0) and (2,0) supergravities in six dimensions 2

3. Maximally supersymmetric solutions 4

3.1. Vacua of (1,0) supergravity 4

3.2. Vacua of (2,0) supergravity 4

4. Anti-selfdual lorentzian Lie groups 6

4.1. Lie algebras with an invariant metric 6

4.2. Anti-selfdual lorentzian Lie algebras 7

5. Five-dimensional vacua and Kaluza–Klein reduction 10

5.1. Geometric preliminaries 10

5.2. Possible Kaluza–Klein reductions 12

5.3. Five-dimensional vacua 13

6. Conclusions and summary of results 14

Acknowledgments 14

References 14

1. Introduction

This paper is concerned with maximally supersymmetric backgrounds of minimal supergravity theories in dimensions five and six; that is, chiral six-dimensional supergravity theories of the type (1,0) and (2,0) and the minimal $N=2$ five-dimensional supergravity theories. Much progress has been made recently in dimension five [1] in the determination of supersymmetric backgrounds by rewriting the conditions for the existence of a Killing spinor in terms of bispinors of the Killing spinor, that is, the differential forms in the Fierz decomposition of the square of a Killing spinor. Together with a knowledge of the orbit structure of the spinor representation, in particular the isomorphism type of the stabiliser, this method has proven useful in characterising the supersymmetric solutions in terms of bundle constructions and other standard geometric constructions. Nevertheless and despite vigorous effort [1, Section 5] there does not exist a proven list of the
maximally supersymmetric vacua of the minimal $N=2$ supergravity. The list in [1, Section 5.4] includes flat space, a symmetric plane wave discovered by Meessen [2], the hitherto unknown Gödel solution, and a one-parameter family of solutions interpolating between $\text{AdS}_2 \times S^3$ and $\text{AdS}_3 \times S^2$ and which can be interpreted [3] as near-horizon geometries of supersymmetric rotating black holes [4]. In addition to these five-dimensional vacua, there are other conjecturally maximally supersymmetric vacua mentioned in [1, Section 5.4] which have yet to be identified. One of the aims in the present paper is to clarify this situation.

The vacua in the above list have been studied from a variety of points of view [5, 6] and in particular they have been shown to be related by dimensional reduction from the minimal chiral supergravity in six dimensions [5]. The existence of so many Kaluza–Klein reductions preserving all the supersymmetry is a very unusual phenomenon deserving of a conceptual explanation. In this paper we will provide such an explanation and as a result we will be able to give a list of all possible five-dimensional vacua. Our results will be phrased purely in terms of Lie algebraic data.

The paper is organised as follows. In Section 2 we describe the chiral six-dimensional supergravities of types $(1,0)$ and $(2,0)$. In Section 3 we show that the $(1,0)$ vacua (up to local isometry) are in one-to-one correspondence with six-dimensional Lie groups admitting a lorentzian metric and anti-selfdual parallelising torsion. Moreover we show that up to the action of the R-symmetry group, the $(2,0)$ vacua are in one-to-one correspondence with the $(1,0)$ vacua. In Section 4 we determine the six-dimensional anti-selfdual lorentzian Lie groups. The corresponding vacua are given by flat space, $\text{AdS}_3 \times S^3$ and Meessen’s symmetric plane wave. In Section 5 we discuss the Kaluza–Klein reduction to five dimensions of these vacua. We observe that because the six-dimensional vacua are parallelised Lie groups the space of right cosets of any (spacelike) one-parameter subgroup is a smooth maximally supersymmetric vacuum solution of the minimal $N=2$ supergravity theory in five dimensions. We therefore classify such subgroups and hence such vacua. Indeed we find other reductions in addition to the list in [1, Section 5.4]; although it still remains to identify them with the extra possibilities in [1].

For a discussion of lorentzian Lie groups in the construction of type II supergravity backgrounds see [7, 8].

**Note added**

While this paper rested undisturbed in our computer hard drives, the paper [9] appeared where all supersymmetric backgrounds of the $(1,0)$ supergravity theory are determined and the maximally supersymmetric ones classified. This last result is obtained using the methods of [10, 11], whereas in the present paper we employ a Lie theoretic method which allows in addition to obtain all the five-dimensional vacua by reduction.

2. $(1,0)$ and $(2,0)$ Supergravities in Six Dimensions

We describe the field content and Killing spinor equations of $(1,0)$ [12] and $(2,0)$ [13, 14] chiral supergravities in six dimensions. We start as usual by describing the relevant spinorial representations, which in signature $\text{(1,5)}$ correspond to symplectic Majorana–Weyl spinors. More precisely, the spin group $\text{Spin}(1,5)$ is isomorphic to $\text{SL}(2, \mathbb{H})$, whence the irreducible spinorial representations are quaternionic (i.e., pseudoreal) of complex dimension 4. There are two inequivalent representations $S_{\pm}$

---

1The Gödel solution was not discussed in [5], but the authors of that paper subsequently showed that it can be obtained by reducing the six-dimensional symmetric plane wave of [2] (private communication).
which are distinguished by their chirality. Let $S_1$ denote the fundamental representation of $\text{Sp}(1)$: it is a quaternionic representation of complex dimension 2, and similarly let $S_2$ denote the fundamental representation of $\text{Sp}(2)$, which is a quaternionic representation of complex dimension 4. The tensor products $S_+ \otimes S_1$ and $S_+ \otimes S_2$ are complex representations of $\text{Spin}(1,5) \times \text{Sp}(1)$ and $\text{Spin}(1,5) \times \text{Sp}(2)$, respectively, with a real structure. We will let

$$S = [S_+ \otimes S_1] \quad \text{and} \quad \mathbb{S} = [S_+ \otimes S_2]$$

denote the underlying real representations. Clearly $S$ is a real representation of dimension 8 and $\mathbb{S}$ is a real representation of dimension 16. The reality condition corresponds to the symplectic Majorana condition.

The field content of minimal $(1,0)$ supergravity in six dimensions consists of a metric $g$, an anti-selfdual three-form $H$ and a gravitino which is a one-form with values in the spinor bundle associated to $S$, which we will also denote $S$. As a check, notice that there is a match of physical degrees of freedom, there being 12 bosonic and 12 fermionic.

On the other hand, the field content of minimal $(2,0)$ supergravity consists of a metric $g$, a $V$-valued anti-selfdual three-form $H$, where $V$ is the five-dimensional real representation of the R-symmetry group $\text{Sp}(2) \cong \text{Spin}(5)$, and a gravitino which is a one-form with values in $S$, the spinor bundle associated to $S$. Again we check that there are 24 physical bosonic and 24 physical fermionic degrees of freedom.

Next we discuss the Killing spinor equations. In $(1,0)$ supergravity let $\varepsilon$ be a section of $S$, and the Killing spinor equation is

$$D_\mu \varepsilon = \nabla_\mu \varepsilon + \frac{1}{8} H_{\mu}^{ab} \Gamma_{ab} \varepsilon = 0 \ ,$$

where $\nabla$ is the spin connection. Notice that $D$ is in fact a spin connection with torsion three-form $H$.

In $(2,0)$ supergravity, let $\varepsilon$ be a section of $\mathbb{S}$. The Killing spinor equation is

$$D_\mu \varepsilon = \nabla_\mu \varepsilon + \frac{1}{8} H^i_{\mu}^{ab} \Gamma_{ab} \gamma_i \varepsilon = 0 \ ,$$

where we have chosen an orthonormal basis $e_i$ for $V$, so that $H = H^i e_i$ and $\gamma_i$ are the corresponding generators of $\text{Cl}(V)$.

The equations of motion consist of the Einstein equations relating the Ricci tensor of $g$ to the energy-momentum tensor of the three-forms, and the fact that these three-forms are closed. Notice that in $(2,0)$ supergravity, the anti-selfduality of the $H^i$ imply that $H^i \wedge H^j = 0$ for all $i,j$.

Maximal supersymmetry implies that the connections $D$ acting on $S$ or $\mathbb{S}$ should be flat. In the case of $(1,0)$ supergravity, $D$ is a spin connection with torsion and maximally supersymmetric solutions correspond to six-dimensional lorentzian manifolds admitting a flat metric connection with anti-selfdual closed torsion three-form. We will see that this means that the manifold is locally isometric to a Lie group with a bi-invariant lorentzian metric.\footnote{The observation that Lie groups with bi-invariant lorentzian metrics yield $(1,0)$ vacua was made independently by Meessen and Ortín (private communication).} In the case of $(2,0)$ supergravity, $D$ does not have such an obvious geometrical interpretation, but we will see below that, up to the natural action of the R-symmetry group on maximally supersymmetric solutions, the $(2,0)$ vacua are in one-to-one correspondence with the $(1,0)$ vacua. In more concrete terms, we will show that a $(2,0)$ vacuum can be $R$-transformed to one where at most one $H^i$ is nonzero. The flatness equations then reduce to those in $(1,0)$ supergravity.
3. Maximal Supersymmetric Solutions

In this section we will study the flatness of the spinor connections $\Omega$ and $\Omega^\perp$.

3.1. Vacua of $(1,0)$ Supergravity. The curvature of the $(1,0)$ connection $\Omega$ is given by

$$-[D_\mu, D_\nu] = -[\nabla_\mu + \frac{1}{2} H_{\mu a} \Gamma^{a b}, \nabla_\nu + \frac{1}{2} H_{\nu c} \Gamma^{c d}]$$

$$= \frac{1}{4} \left( R_{\mu \nu a b} - \frac{1}{4} \nabla_\mu H_{\nu c} a b - \frac{1}{8} \left( H_{\mu a c} H_{\nu c} b - H_{\nu a c} H_{\mu c} b \right) \right) \Gamma^{a b},$$

where we have used that $\Gamma^{a b} \Gamma^c_d + \Gamma^{c a} \Gamma^b_d = 2 g^{a b} \delta_d^c$. The quantity

$$R_{\mu \nu a b} - \frac{1}{4} \nabla_\mu H_{\nu c} a b - \frac{1}{8} \left( H_{\mu a c} H_{\nu c} b - H_{\nu a c} H_{\mu c} b \right)$$

is the curvature of a metric connection with torsion three-form $H$, whose flatness implies that the spacetime is locally isometric to a Lie group admitting a bi-invariant metric with parallelising torsion $H$. This is a well-known result due to Wolf [15, 16] based on earlier work of Élie Cartan and Schouten [17, 18].

Let us sketch how this arises. The vanishing of the curvature $\Omega$ actually implies several independent equations corresponding to the decomposition of the curvature tensor into different algebraic types. We first rewrite the flatness condition without reference to the local frame:

$$R_{\mu \nu a b} - \frac{1}{4} \nabla_\mu H_{\nu c} a b - \frac{1}{8} \left( H_{\mu a c} H_{\nu c} b - H_{\nu a c} H_{\mu c} b \right) = 0.$$  

The crucial observation is that whereas $R_{\mu \nu a b}$ satisfies the algebraic Bianchi identity $\tilde{R}_{[\mu \nu a b]} = 0$ the other terms do not in general. Therefore in the first instance, the above equation breaks into two equations: one sets $T_{\mu \nu a b}$ equal to the component of

$$T_{\mu \nu \rho \sigma} := \frac{1}{4} \nabla_\mu H_{\nu \rho \sigma} + \frac{1}{8} \left( H_{\mu \nu \rho \sigma} - H_{\nu \mu \rho \sigma} \right)$$

which obeys the algebraic Bianchi identity, and the other equation says that

$$T_{[\mu \nu \rho \sigma]} = 0.$$  

Decomposing this last equation further into algebraic types, and using the fact that $dH = 0$ we obtain two equations: the first says that $\nabla H = 0$ and the second is the Jacobi identity for $H$:

$$H_{[\mu \nu] \tau \rho} H_{\tau \rho \sigma} = 0.$$  

Indeed, since $H$ is parallel, so is the Riemann tensor $R$, whence the spacetime is locally symmetric. This means that we can work at a point in the spacetime, on whose tangent space the metric $g$ induces a lorentzian scalar product and $H$ induces a Lie bracket compatible with the metric. In other words, we have the structure of a Lie algebra with an invariant metric on the tangent space at any point. This implies that the spacetime is locally isometric to a Lie group admitting a bi-invariant metric and whose parallelising three-form $H$ is anti-selfdual; indeed, the left-invariant vector fields are precisely those vector fields which are covariantly constant with respect to the connection $D$.

Finally, the Riemann curvature is further given in terms of $H$ by

$$R_{\mu \nu a b} = \frac{1}{4} \left( H_{\mu \nu a} H_{\tau c} - H_{\nu \mu c} H_{\tau a} \right).$$  

3.2. Vacua of $(2,0)$ Supergravity. Similarly, the curvature of the $(2,0)$ connection $\Omega^\perp$ is

$$-[D_\mu, D_\nu] = -[\nabla_\mu + \frac{1}{2} H_{\mu i} \Gamma_{i a b}, \nabla_\nu + \frac{1}{2} H_{\nu i} \Gamma_{i c d}]$$

$$= \frac{1}{4} R_{\mu \nu a b} \Gamma_{i a b} - \frac{1}{4} \nabla_\mu H_{\nu i} a b \Gamma_{i a b} \gamma_i - \frac{1}{6} H_{\mu a b} H_{\nu c d} \Gamma_{a b} \gamma_i \Gamma^c d \gamma_j.$$
Flatness implies the vanishing of all the independent components of the curvature. Using that \( \gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij} \), we obtain the following set of equations, which are equivalent to the flatness of the connection \( D \):

\[
R_{\mu\nu\ ab} - \frac{1}{8} \left( H_{\mu \ ac} H_{\nu \ db} - H_{\nu \ ac} H_{\mu \ db} \right) g^{cd} = 0 \tag{6}
\]
\[
\nabla_{\mu} H_i^{\ nu} ab - \nabla_{\nu} H_{\mu \ ab} = 0 \tag{7}
\]
\[
H_{\mu \ ab}^{[i} H^{\ j]}_{\nu \ cd} e^{abdef} = 0 \tag{8}
\]
\[
H_{\mu \ ab}^{[i} H^{\ j]}_{\nu \ cd} e^{abdef} = 0 , \tag{9}
\]

where we have used that

\[
[\Gamma^{\ ab}_{\ i} \Gamma^{\ cd}_{\ j}] = [\Gamma^{\ ab}_{\ i} \Gamma^{\ cd}_{\ j}] \delta_{ij} + 2 \left( \Gamma^{abcd} + g^{ad} g^{bc} - g^{ac} g^{bd} \right) \gamma_{ij}
\]

and that on chiral spinors \( \Gamma^{abcd} = \pm \frac{1}{4} \epsilon^{abcd} \Gamma_{ef} \).

We will now analyse these equations. Equation 7, together with the fact that \( dH = 0 \), implies that \( \nabla H = 0 \); whence it follows, from equation 6, that the Riemann curvature tensor is covariantly constant, whence the spacetime is locally symmetric. Equation 9 is identically satisfied for anti-selfdual \( H^i \). Finally, using the anti-selfduality of \( H^i \) we can rewrite equation 8 as

\[
H_{\ abm}^{[i} H^{\ j]}_{\ cdm} = 0 , \tag{10}
\]

where we have expressed \( H^i \) in terms of a frame \( \{ e_a \} \). In fact, since the spacetime is locally symmetric, it is enough to work at a point on whose tangent space \( g \) induces a lorentzian scalar product and \( H^i \) are constant-coefficient anti-selfdual three-forms.

We claim that equation 10 implies that \( H \) is decomposable; that is, \( H_{\ abc} = H_{\ ab} v^i \), where \( v \) is a unit vector and \( H_{\ abc} \) is anti-selfdual. If all \( H^i = 0 \) this is clear, so let us suppose that at least one \( H^i \) is nonzero. Without loss of generality we can let it be \( H^1 \) by relabelling the frame if necessary. Because of anti-selfduality, we know that \( \epsilon_0 H^1 \) is different from zero. Being a two-form in the five-dimensional space perpendicular to \( e_0 \), we can rotate in that space in such a way that \( \epsilon_0 H^1 = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4 \), where \( \alpha \neq 0 \). Since equation 10 is homogeneous, we can rescale \( H^1 \) such that \( \alpha = 1 \). Finally we perform an R-symmetry transformation on \( H^i \) in such a way that \( H^i_{012} = 0 \) for \( i \neq 1 \). In other words, by suitably changing basis we arrive at \( H_{012}^i = 0 \) and \( H_{012}^1 = 1 \), \( H_{034}^1 = \beta \) and all other \( H_{0ab}^1 = 0 \). Inserting this Ansatz into equation 10 for \( i = 1, a = 0, b = 1 \) and letting \( j > 1, c, d \) vary, we obtain at once that \( H^j = 0 \) for \( j > 1 \). In other words and re-inserting the parameter \( \alpha \), the only nonzero \( H^i \) is

\[
H^1 = \alpha (e_{012} + e_{345}) + \beta (e_{034} + e_{125}) ,
\]

with \( \alpha \neq 0 \), where we have introduced the notation \( e_{012} = e_0 \wedge e_1 \wedge e_2 \), and so on.

Finally we insert this result into equation 6 to obtain

\[
R_{\mu\nu\ ab} = - \frac{1}{8} \left( H_{\mu \ ac} H_{\nu \ db} - H_{\nu \ ac} H_{\mu \ db} \right) g^{cd} , \tag{11}
\]

where we have let \( H^1 = H \). Comparing this equation with 6 we see that they are indeed the same, whence there is a one-to-one correspondence between (2,0) vacua (up to R-symmetry) and (1,0) vacua. This means, in particular, that the (1,0) vacua actually possess 16 supercharges.

More precisely, we see that any (1,0) vacuum \((g, H)\) gives rise to a (2,0) vacuum \((g, H^i)\) simply by letting \( H = H \otimes v \) with \( v \) a unit vector in \( V \); that is, \( H^i = H v^i \), with \( v^i v^i = 1 \). Under the R-symmetry group Sp(2), \( H \) transforms as a vector and hence the Sp(2)-transform of any vacuum is also a vacuum. What we have shown is that every (2,0) vacuum is in the R-symmetry orbit of a (1,0) vacuum.

It thus remains to determine the (1,0) vacua, a task to which we now turn.
4. ANTI-SELFDUAL LORENTZIAN LIE GROUPS

In the previous section we concluded that $(1,0)$ vacua are locally isometric to six-dimensional Lie groups admitting a bi-invariant metric whose parallelising torsion three-form is anti-selfdual. In this section we determine such Lie groups up to local isometry.

A Lie group $G$ admits a bi-invariant metric (that is, a metric invariant under both left and right multiplication) if and only if its Lie algebra $\mathfrak{g}$ admits a scalar product $\langle -, - \rangle$ which is invariant (under the adjoint action):

$$\langle [X,Y], Z \rangle = \langle X, [Y, Z] \rangle,$$

for all $X, Y, Z \in \mathfrak{g}$. Relative to a basis $X_a$ for $\mathfrak{g}$, let $g_{ab} = \langle X_a, X_b \rangle$ and $[X_a, X_b] = f_{abc}^\gamma X_c$. Then the invariance of the metric simply says that $f_{abc} = f_{ab}^\gamma g_{dc}$ is totally skew-symmetric. We say that a Lie group $G$ is lorentzian if it has a bi-invariant lorentzian metric. Simply-connected lorentzian Lie groups are in one-to-one correspondence with Lie algebras admitting an invariant lorentzian scalar product; that is $g_{ab}$ is lorentzian. Given such a Lie algebra there is a canonical invariant three-form $h \in \Lambda^3 \mathfrak{g}^*$ defined by

$$h(X, Y, Z) = \langle X, [Y, Z] \rangle,$$

or equivalently $h_{abc} = f_{abc}$. This form gives rise to a bi-invariant differential three-form $H \in \Omega^3(G)$ on the Lie group by defining it to be $h$ on left-invariant vector fields and extending tensorially to arbitrary vector fields. Being bi-invariant, $H$ is both closed and co-closed. Acting on three-forms in six dimensions (and lorentzian signature) the Hodge $\star$ operator obeys $\star^2 = \text{id}$, whence we can define selfdual and anti-selfdual three-forms. We will say that a lorentzian Lie group is anti-selfdual if $H$ is an anti-selfdual three-form. Similarly, we will say that its Lie algebra is an anti-selfdual Lie algebra.

In this section we will classify six-dimensional anti-selfdual lorentzian Lie algebras.

4.1. Lie algebras with an invariant metric. It is well-known that reductive Lie algebras — that is, direct products of semisimple and abelian Lie algebras — admit invariant scalar products: Cartan’s criterion allows us to use the Killing forms on the simple factors and any scalar product on an abelian Lie algebra is trivially invariant. Another well-known example of Lie algebras admitting an invariant scalar product are the classical doubles. Let $\mathfrak{h}$ be any Lie algebra and let $\mathfrak{h}^*$ denote the dual space on which $\mathfrak{h}$ acts via the coadjoint representation. The definition of the coadjoint representation is such that the dual pairing $\mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{R}$ is an invariant scalar product on the semidirect product $\mathfrak{h} \rtimes \mathfrak{h}^*$ with $\mathfrak{h}^*$ an abelian ideal. The Lie algebra $\mathfrak{h} \rtimes \mathfrak{h}^*$ is called the classical double of $\mathfrak{h}$ and the invariant metric has split signature $(r, r)$ where $\dim \mathfrak{h} = r$.

It turns out that all Lie algebras admitting an invariant scalar product can be obtained by a mixture of these constructions. Let $\mathfrak{g}$ be a Lie algebra with an invariant scalar product $\langle -, - \rangle$. We will let $X_\alpha$ be a basis for $\mathfrak{g}$ such that $[X_\alpha, X_\beta] = f_{\alpha\beta}^\gamma X_\gamma$ and such that $\langle X_\alpha, X_\beta \rangle = g_{\alpha\beta}$. Now let $\mathfrak{h}$ act on $\mathfrak{g}$ as skew-symmetric derivations; that is, preserving both the Lie bracket and the scalar product. We will let $H_i$ be a basis for $\mathfrak{h}$. The action of $\mathfrak{h}$ on $\mathfrak{g}$ is given by $f_{i\alpha\beta}$, such that

$$H_i \cdot X_\alpha = f_{i\alpha\beta} X_\beta.$$

For future use we define $f_{i\alpha\beta} = f_{\alpha\gamma} g_{\gamma\beta}$. First of all, since $\mathfrak{h}$ acts on $\mathfrak{g}$ preserving the scalar product, we have a linear map

$$\mathfrak{h} \to \Lambda^2 \mathfrak{g}^*,$$
with dual map

\[ c : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{h}^* , \]

where we have used the invariant scalar product to identity \( \mathfrak{g} \) and \( \mathfrak{g}^* \) equivariantly. Explicitly, \( c(X_\alpha \wedge X_\beta) = f_{\alpha \beta} H^i \), where \( H^i \) is the canonical dual basis of \( \mathfrak{h}^* \). Since \( \mathfrak{h} \) preserves the Lie bracket in \( \mathfrak{g} \), this map is a cocycle, whence it defines a class \( [c] \in H^2(\mathfrak{g}; \mathfrak{h}^*) \) in the second Lie algebra cohomology of \( \mathfrak{g} \) with coefficients in the trivial module \( \mathfrak{h}^* \). Let \( \mathfrak{g} \times_c \mathfrak{h}^* \) denote the corresponding central extension. The Lie bracket of \( \mathfrak{g} \times_c \mathfrak{h}^* \) is such that \( \mathfrak{h}^* \) is central and if \( X, Y \in \mathfrak{g} \), then

\[ [X, Y] = [X, Y]_\mathfrak{g} + c(X, Y) , \]

where \([-, -]_\mathfrak{g}\) is the original Lie bracket on \( \mathfrak{g} \). In terms of the basis chosen above,

\[ [X_\alpha, X_\beta] = f_{\alpha \beta}^\gamma X_\gamma + f_{\alpha \beta} H^i . \]

Now \( \mathfrak{h} \) acts naturally on \( \mathfrak{g} \times_c \mathfrak{h}^* \) preserving the Lie bracket; the action on \( \mathfrak{h}^* \) being given by the coadjoint representation. This then allows us to define the double extension of \( \mathfrak{g} \) by \( \mathfrak{h} \),

\[ \mathfrak{d}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{h} \ltimes (\mathfrak{g} \times_c \mathfrak{h}^*) \]

as a semidirect product. Details of this construction can be found in \[19, 20\]. The remarkable fact is that \( \mathfrak{d}(\mathfrak{g}, \mathfrak{h}) \) admits an invariant scalar product:

\[
\begin{pmatrix}
X_\beta & H_j & H^i \\
X_\alpha & g_{\alpha \beta} & 0 & 0 \\
H_i & 0 & B_{ij} & \delta_j^i \\
H^i & 0 & \delta_j^i & 0
\end{pmatrix}
\]

(12)

where \( B \) is any invariant symmetric bilinear form on \( \mathfrak{h} \).

We say that a Lie algebra with an invariant scalar product is indecomposable if it cannot be written as the direct product of two orthogonal ideals. A theorem of Medina and Revoy \[19\] (see also \[21\] for a refinement) says that an indecomposable (finite-dimensional) Lie algebra with an invariant scalar product is one of the following:

1. one-dimensional,
2. simple, or
3. a double extension \( \mathfrak{d}(\mathfrak{g}, \mathfrak{h}) \) where \( \mathfrak{h} \) is either simple or one-dimensional and \( \mathfrak{g} \) is a (possibly trivial) Lie algebra with an invariant scalar product.

Any (finite-dimensional) Lie algebra with an invariant scalar product is then a direct sum of indecomposables.

4.2. Anti-selfdual lorentzian Lie algebras. Notice that if the scalar product on \( \mathfrak{g} \) has signature \((p, q)\) and if \( \dim \mathfrak{h} = r \), then the scalar product on \( \mathfrak{d}(\mathfrak{g}, \mathfrak{h}) \) has signature \((p + r, q + r)\). Therefore indecomposable lorentzian Lie algebras are either reductive or double extensions \( \mathfrak{d}(\mathfrak{g}, \mathfrak{h}) \) where \( \mathfrak{g} \) has a positive-definite invariant scalar product and \( \mathfrak{h} \) is one-dimensional. In the reductive case, indecomposability means that it has to be simple, whereas in the latter case, since the scalar product on \( \mathfrak{g} \) is positive-definite, \( \mathfrak{g} \) must be reductive. A result of \[20\] (see also \[21\]) then says that any semisimple factor in \( \mathfrak{g} \) splits off resulting in a decomposable Lie algebra. Thus if the double extension is to be indecomposable, then \( \mathfrak{g} \) must be abelian. In summary, an indecomposable lorentzian Lie algebra is either simple or a double extension of an abelian Lie algebra by a one-dimensional Lie algebra and hence solvable (see, e.g., \[19\]).

These considerations make possible the following enumeration of six-dimensional lorentzian Lie algebras:

1. \( \mathbb{E}^{1,5} \)
(2) $E^{1,2} \oplus so(3)$
(3) $E^3 \oplus so(1,2)$
(4) $so(1,2) \oplus so(3)$
(5) $\mathfrak{d}(E^4, \mathbb{R})$

where the last case actually corresponds to a family of Lie algebras, depending on the action of $\mathbb{R}$ on $E^4$, which is given by a homomorphism $\mathbb{R} \to so(4)$.

Imposing the condition of anti-selfduality trivially discards cases (2) and (3) above. Case (1) is the abelian Lie algebra with Minkowski metric. We now investigate in more detail the remaining two cases, starting with case (5) which requires more attention.

### 4.2.1. A six-dimensional Nappi–Witten vacuum

Let $e_i$, $i = 1, 2, 3, 4$, be an orthonormal basis for $E^4$, and let $e_- \in \mathbb{R}$ and $e_+ \in \mathbb{R}^*$, so that together they span $\mathfrak{d}(E^4, \mathbb{R})$. The action of $\mathbb{R}$ on $E^4$ defines a map $\rho : \mathbb{R} \to \Lambda^2 E^4$, which can be brought to the form $\rho(e_-) = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$ via an orthogonal change of basis in $E^4$ which moreover preserves the orientation. The Lie brackets of $\mathfrak{d}(E^4, \mathbb{R})$ are given by

$$
\begin{align*}
[e_-, e_1] &= \alpha e_2 & [e_-, e_3] &= \beta e_4 \\
[e_-, e_2] &= -\alpha e_1 & [e_-, e_4] &= -\beta e_3 \\
[e_1, e_2] &= \alpha e_+ & [e_3, e_4] &= \beta e_+
\end{align*}
$$

and the scalar product is given (up to scale) by

$$
\langle e_-, e_- \rangle = b \quad \langle e_+, e_- \rangle = 1 \quad \langle e_+, e_j \rangle = \delta_{ij}.
$$

The first thing we notice is that we can set $b = 0$ without loss of generality by the automorphism fixing all $e_1, e_+$ and mapping $e_- \mapsto e_- - \frac{\beta}{2} e_+$. We will assume that this has been done and that $\langle e_-, e_- \rangle = 0$. A straightforward calculation shows that the three-form $f_{abc}$ is anti-selfdual if and only if $\beta = \alpha$. Let us put $\beta = \alpha$ from now on. We must distinguish between two cases: if $\alpha = 0$, then the resulting algebra is abelian and is precisely $E^{1,5}$. On the other hand if $\alpha \neq 0$, then rescaling $e_\pm \mapsto \alpha^{\pm 1} e_\pm$ we can effectively set $\alpha = 1$ without changing the scalar product. Finally we notice that a constant rescaling of the scalar product can be undone by an automorphism of the algebra. As a result we have two cases: $E^{1,5}$ (obtained from $\alpha = 0$) and the algebra

$$
\begin{align*}
[e_-, e_1] &= e_2 & [e_-, e_3] &= e_4 \\
[e_-, e_2] &= -e_1 & [e_-, e_4] &= -e_3 \\
[e_1, e_2] &= e_+ & [e_3, e_4] &= e_+
\end{align*}
$$

with scalar product given by

$$
\langle e_+, e_- \rangle = 1 \quad \text{and} \quad \langle e_+, e_j \rangle = \delta_{ij}.
$$

There is a unique simply-connected Lie group with the above Lie algebra which inherits a bi-invariant lorentzian metric. This Lie group is a six-dimensional analogue of the Nappi–Witten group [22], which is based on the double extension $\mathfrak{d}(R^2, \mathbb{R})$ [20]. We will denote it NW$_6$ as in [23], where one can find a derivation of the metric on this six-dimensional group. It can be seen to be a symmetric plane wave (Hpp-wave in the terminology of [24]) corresponding to a symmetric space of the type discovered by Cahen and Wallach [25]. The supergravity solution was discovered by Meessen [2] who called it KG6 by analogy with the maximally supersymmetric plane wave of eleven-dimensional supergravity discovered by Kowalski-Glikman [26] (see also [24]).
The metric is easy to write down once we choose a parametrisation for the group. The calculation is routine (see, for example, [23]) and the result is
\[ g = 2dx^+dx^- - \frac{1}{4} \sum_i (x^i)^2(dx^-)^2 + \sum_i (dx^i)^2. \] (13)

In these coordinates the three-form \( H \) is given by
\[ H = \frac{2}{3}dx^- \wedge (dx^1 \wedge dx^2 + dx^3 + dx^4). \]

The metric (13) corresponds to a lorentzian symmetric space of the type introduced by Cahen and Wallach [25] and discussed more recently in the context of plane wave solutions of supergravity theories in [24]. However in the present context it appears as a bi-invariant metric on a solvable group. This is not an isolated incident. In fact, it is not hard to characterise those Cahen–Wallach plane wave metrics which are isometric to a bi-invariant metric on a solvable Lie group. Recall that an indecomposable Cahen–Wallach metric takes the form
\[ g = 2dx^+dx^- + \sum_{i,j=1}^n A_{ij} x^i (dx^-)^2 + \sum_{i=1}^n (dx^i)^2, \]
where the symmetric matrix \( A \) is nondegenerate. It is proven in [23] that \( g \) is isometric to a bi-invariant metric on a solvable Lie group if and only if \( A \) is negative-definite and all its eigenvalues have even multiplicity. This means that \( A \) admits a decomposition \( A = J^2 \), where \( J \) is skew-symmetric and nondegenerate, and the Lie algebra is a double extension of \( \mathbb{E}^n \) by \( \mathbb{R} \), where the generator of \( \mathbb{R} \) acts on \( \mathbb{E}^n \) by \( J \). Notice that, as observed in [23], this implies that the IIB maximally supersymmetric wave [27] is isometric to a solvable Lie group with a bi-invariant metric, whereas the maximally supersymmetric M-wave [26] is not.

4.2.2. The Freund–Rubin vacuum. Finally we discuss case (4), with Lie algebra \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \). Let \( e_0, e_1, e_2 \) be a pseudo-orthonormal basis for \( \mathfrak{so}(1,2) \). The Lie brackets are given by
\[ [e_0, e_1] = -e_2 \quad [e_0, e_2] = e_1 \quad [e_1, e_2] = e_0. \]
Similarly let \( e_3, e_4, e_5 \) denote an orthonormal basis for \( \mathfrak{so}(3) \), with Lie brackets
\[ [e_5, e_3] = -e_4 \quad [e_5, e_4] = e_3 \quad [e_3, e_4] = -e_5. \]
The most general invariant lorentzian scalar product on \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \) is labelled by two positive numbers \( \alpha \) and \( \beta \) and is given by
\[ \begin{pmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 \\ -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \end{pmatrix}. \]

Anti-selfduality of the canonical three-form implies that \( \beta = \alpha \). There is a unique simply-connected Lie group with Lie algebra \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \), namely \( \text{SL}(2, \mathbb{R}) \times \text{SU}(2) \), where \( \widetilde{\text{SL}}(2, \mathbb{R}) \) denotes the universal covering group of \( \text{SL}(2, \mathbb{R}) \). This group inherits a one-parameter family of bi-invariant metrics. This solution is none other than the standard Freund–Rubin solution \( \text{AdS}_3 \times S^3 \), with equal radii of curvature, where strictly speaking we should take the universal covering space of \( \text{AdS}_3 \).

\[ ^3 \text{Every Lie group} \ G \text{ with a bi-invariant metric is isometric to the symmetric space} \ (G \times G)/\Delta G, \text{where} \ \Delta G \text{ is the diagonal} \ G \text{ subgroup of} \ G \times G. \text{ However this is not the same symmetric space in the description of Cahen and Wallach.} \]
In summary, the following are the possible vacua of (1, 0) and (up to R-symmetry) (2, 0) supergravity. First of all we have a one-parameter family of Freund–Rubin vacua locally isometric to AdS$_3 \times S^3$, with equal radii of curvature. The anti-selfdual three-form $H$ is then proportional to the difference of the volume forms of the two spaces.

Then we have a six-dimensional analogue NW$_6$ of the Nappi–Witten group, locally isometric to a Cahen–Wallach symmetric space. Finally there is the flat vacuum $\mathbb{R}^{1,5}$. These vacua are related by Penrose limits which can be interpreted as group contractions. The details appear in [29].

5. Five-dimensional vacua and Kaluza–Klein reduction

In this section we will examine the dimensional reductions of the six-dimensional vacua found above. Dimensional reduction usually breaks some supersymmetry: in the ten- and eleven-dimensional supergravity theories, only the flat vacuum remains maximally supersymmetric after dimensional reduction and only by a translation. However for the six-dimensional vacua the situation is different. Indeed, in [5] it was shown that the hitherto known supergravity vacua with eight supercharges in six, five and four dimensions are related by dimensional reduction and oxidation. As we will see presently, this perhaps surprising phenomenon follows from the fact that the six-dimensional vacua are parallelised Lie groups. Our results will also give an a priori explanation to the empirical fact that these vacua are homogeneous [4].

5.1. Geometric preliminaries. We start this section with a technical result which underlies the rest of the section. Let $D$ be a metric connection with torsion $T$. We observe that if a vector field $\xi$ is $D$-parallel then it is Killing. Indeed, $D\xi = 0$ implies that $\nabla_\mu \xi_\nu = \frac{1}{2} \xi^\rho H_{\mu\nu\rho}$, where $H$ is the associated torsion three-form. Therefore we see that $\nabla_\mu \xi_\nu = -\nabla_\nu \xi_\mu$, whence $\xi$ is Killing. Now let $\psi$ be a Killing spinor; that is, $D\psi = 0$. Then the Lie derivative of $\psi$ along $\xi$ is well-defined (see, for example, [28]). Furthermore, it vanishes identically. Indeed, by definition,

$$\mathcal{L}_\xi \psi = \nabla_\xi \psi + \frac{1}{2} [\nabla_\mu \xi_\nu] \Gamma^{\mu\nu} \psi$$

$$= \frac{1}{2} (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu - \xi^\rho H_{\mu\nu\rho}) \Gamma^{\mu\nu} \psi,$$

where we have used that $D\xi = 0$. Since $\xi$ is Killing, $\nabla_\nu \xi_\mu = -\nabla_\mu \xi_\nu$, whence

$$\mathcal{L}_\xi \psi = \frac{1}{4} (\nabla_\mu \xi_\nu - \frac{1}{2} \xi^\rho H_{\mu\nu\rho}) \Gamma^{\mu\nu} \psi,$$

which vanishes when $D\xi = 0$. Moreover it follows from the above calculation that if $\mathcal{L}_\xi \psi = 0$ for all Killing spinors then $\nabla_\mu \xi_\nu = \frac{1}{2} \xi^\rho H_{\mu\nu\rho}$, so that $D\xi = 0$.

For a parallelised Lie group $G$, the $D$-parallel vectors are either the left- or right-invariant vector fields, depending on the choice of parallelising connection. For definiteness, we will choose the connection whose parallel sections are the left-invariant vector fields. Left-invariant vector fields generate right translations and are in one-to-one correspondence with elements of the Lie algebra $\mathfrak{g}$. Therefore every left-invariant vector field $\xi$ determines a one-parameter subgroup $K$, say, of $G$ and the orbits of such a vector field in $G$ are the right $K$-cosets. The dimensional reduction along this vector field is smooth and diffeomorphic to the space of cosets $G/K$. We will be interested in subgroups $K$ such that $G/K$ is a five-dimensional lorentzian spacetime, which requires that the right $K$-cosets are spacelike. In other words, we require that the Killing vector $\xi$ be spacelike. Bi-invariance of the metric guarantees that this is the case provided that the Lie algebra element $\xi(\epsilon) \in \mathfrak{g}$ is spacelike relative to the ad-invariant inner product. Further notice that a constant rescaling of $\xi$ does not change its causal property nor the subgroup $K$ it generates: it is simply reparameterised. Therefore, in order to classify all possible reductions (and hence all possible five-dimensional vacua with 8 supercharges) we need to
classify all spacelike elements of \( \mathfrak{g} \) up to scale. Moreover elements of \( \mathfrak{g} \) which are related by isometric automorphisms (e.g., which are in the same adjoint orbit of \( G \)) give rise to isometric quotients. Thus, to summarise, we want to classify spacelike elements of \( \mathfrak{g} \) up to scale and up to automorphisms. For a more detailed explanation of this reasoning, the reader is referred to the papers [29, 30], which also contain the description of the geometric set-up for Kaluza–Klein reduction which we now briefly review.

Let \( X \in \mathfrak{g} \) be spacelike and let \( \xi_X \) denote the corresponding left-invariant vector field. By rescaling \( X \) if necessary we can always take \( X \) (and hence \( \xi_X \)) to have unit norm. Let \( K \) be the one-parameter subgroup of \( G \) generated by \( X \). The natural map \( \pi : G \to G/K \), taking a group element to the right \( K \)-coset it belongs to, is actually a principal fibration with group \( K \). The tangent space \( T_g G \) has a canonical subspace \( \mathcal{V}_g = \ker \pi_* \) corresponding to the span of \( \xi_X(g) \). Because \( \xi_X(g) \) has non-vanishing norm, there is a decomposition

\[
T_g G = \mathcal{V}_g \oplus \mathcal{H}_g
\]

where \( \mathcal{H}_g = \mathcal{V}_g^\perp \) consists of all those tangent vectors at \( g \) which are perpendicular to \( \xi_X(g) \). Because the metric in \( G \) is bi-invariant, we can identify \( \mathcal{H}_g \) with the left-translate (by \( (L_g)_* \)) of those vectors in \( \mathfrak{g} \) which are perpendicular to \( X \). The distribution \( \mathcal{H} \) is a connection in the sense of Ehresmann and has an associated connection one-form \( \alpha \) in \( G \), defined by \( \text{ker} \alpha = \mathcal{H} \) and normalised to \( \alpha(\xi_X) = 1 \). Explicitly, \( \alpha = \xi_X / \|\xi_X\|^2 \), where \( \xi_X \) is the one-form dual to \( \xi_X \). With our choice of normalisation for the Killing vector, \( \|\xi_X\| = \|X\| = 1 \), whence \( \alpha = \xi_X \). We can give an even more explicit expression for \( \alpha \) in terms of the left-invariant Maurer–Cartan one-form \( \theta \). Indeed, \( \alpha = \langle X, \theta \rangle \), where \( \langle -, - \rangle \) is the metric in \( \mathfrak{g} \). To see this, notice that if \( \zeta \) is a vector field in \( G \), then \( \theta(\zeta)(g) = (L_{g^{-1}})_* \zeta(g) \in T_e G = \mathfrak{g} \). Therefore,

\[
\alpha(\zeta)(g) = \langle X, \theta(\zeta)(g) \rangle \\
= \langle X, (L_{g^{-1}})_* \zeta(g) \rangle \\
= \langle (L_g)_* X, \zeta(g) \rangle_g \\
= \langle \xi_X(g), \zeta(g) \rangle_g \\
= \xi_X(\zeta)(g),
\]

where in the third line we use the left-invariance of the metric, in the fourth we have used the left-invariance of \( \xi_X \), and we have introduced the notation \( \langle -, - \rangle_g \) to be the metric at \( g \in G \) with \( \langle -, - \rangle_g := \langle -, - \rangle \) the one in the Lie algebra.

The reduction of the six-dimensional metric to five dimensions gives rise to several geometric structures (see, for example, [29, 30]): a metric \( h \), a dilaton \( \phi \) and a 2-form field strength \( F \). The metric \( h \) is the induced metric on the horizontal distribution \( \mathcal{H} \), the dilaton \( \phi \) is a logarithmic measure of the fibre metric \( \|\xi_X\| \) which in our case is constant, and \( F = d\alpha \). We can give an explicit formula for \( F \) using the Maurer–Cartan structure equations. Indeed,

\[
F = d\alpha = \langle X, d\theta \rangle = -\frac{1}{2} \langle X, [\theta, \theta] \rangle . \tag{14}
\]

In terms of this data, the metric on the \( G \) is given by the usual Kaluza–Klein ansatz

\[
ds^2 = h + \alpha^2 ,
\]

where we have set the dilaton to zero in agreement with the choice of normalisation for \( \xi_X \). More explicitly the metric on the five-dimensional quotient is given by

\[
h = \langle \theta, \theta \rangle - \langle X, \theta \rangle^2 .
\]
In other words, if \( e_\mu \) is a basis for the perpendicular complement of \( X \) in \( g \), with \( \langle e_\mu, e_\nu \rangle = \eta_{\mu\nu} \), then the metric on the five-dimensional quotient is
\[
h = \eta^{\mu\nu} \langle e_\mu, \theta \rangle \langle e_\nu, \theta \rangle.
\]

To reduce the anti-selfdual three-form \( H \) we first decompose it as
\[
H = G_3 + \alpha \wedge G_2,
\]
where \( G_2 = \iota_\xi H \) and \( G_3 \) are horizontal; that is, \( \iota_\xi G_3 = \iota_\xi G_2 = 0 \). Because the Killing vector \( \xi_X \) leaves \( H \) invariant, it follows that also \( G_2 \) and \( G_3 \) are invariant; that is, \( \mathcal{L}_{\xi_X} G_2 = \mathcal{L}_{\xi_X} G_3 = 0 \). In other words, \( G_2 \) and \( G_3 \) are basic; that is, they are pullbacks of forms in the base \( G/K \), which we will denote by the same letters. Because \( dH = 0 \) it follows that \( dG_2 = 0 \) and that \( dG_3 + F \wedge G_2 = 0 \) where \( F = d\alpha \) was defined above. Finally because \( H \) is anti-selfdual, it follows that \( G_3 \) and \( G_2 \) are related by Hodge duality in five dimensions: \( G_3 = \ast_5 G_2 \). In other words, we have that
\[
H = \ast_5 G_2 + \alpha \wedge G_2,
\]
where \( dG_2 = 0 \) and \( d \ast_5 G_2 = -F \wedge G_2 \).

In fact, in this case we have \( F = G_2 \), whence we are dealing with reductions which truncate consistently to the minimal five-dimensional supergravity theory. Recall that the dimensional reduction of \((1,0)\) supergravity to five dimensions is \( N=2 \) supergravity coupled to a vector multiplet. The minimal \( N=2 \) supergravity is obtained by setting the fields in the vector multiplet to zero. These fields are the “dilaton” whose exponential is the fibre metric and a two-form field strength which is the difference \( F - G_2 \). To show that \( F - G_2 \) vanishes in these reductions, we simply use that \( H = -\frac{1}{\theta} \langle \theta, [\theta, \theta] \rangle \) and compute
\[
G_2 = \iota_\xi H = -\frac{1}{\theta} \langle X, [\theta, \theta] \rangle,
\]
which agrees with the expression for \( F \) derived in (14). This truncation is consistent with supersymmetry since by construction the supersymmetry variations of the fields in the vector multiplet also vanish.

In summary, for the reductions under consideration, we obtain a vacuum of the minimal \( N=2 \) supergravity with bosonic fields \((h,F)\) given by the reduction of \((g,H)\) where \( F = d\alpha \), \( h = g - \alpha^2 \) and \( H = \ast_5 F + \alpha \wedge F \).

5.2. Possible Kaluza–Klein reductions. We now classify the possible Kaluza–Klein reductions to five dimensions, by classifying the spacelike one-parameter subgroups of the Lie groups in question. As outlined above this is achieved by classifying the normal forms of elements of the Lie algebra \( g \) under rescalings and metric-preserving automorphisms.

5.2.1. Spacelike subgroups of \( \mathbb{R}^{1,5} \). The Lie algebra is abelian and hence all automorphisms are outer. The metric-preserving automorphism group is \( O(1,5) \) acting in the obvious way. Up to \( O(1,5) \) any spacelike element in \( \mathbb{R}^{1,5} \) can be rotated so that it generates translation along the fifth spatial coordinate \( x^5 \). The resulting quotient is \( \mathbb{R}^{1,4} \) with vanishing fluxes and constant dilaton.

5.2.2. Spacelike subgroups of \( \widetilde{\text{SL}}(2,\mathbb{R}) \times \text{SU}(2) \). The Lie algebra is now \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(2,3) \). Every nonzero element of \( \mathfrak{so}(3) \) is conjugate under \( \text{SO}(3) \) to any other nonzero element of the same norm, whereas nonzero elements of \( \mathfrak{so}(1,2) \) come in three flavours under \( \text{SO}(1,2) \): elliptic, parabolic or hyperbolic, depending on whether it has positive, zero or negative norm, respectively. Let \( \sigma, \nu \) and \( \tau \) denote respectively a spacelike, null or timelike element in \( \mathfrak{so}(1,2) \). Let us normalise \( \sigma \) and \( \tau \) such that \( \|\sigma\|^2 = 1 \) and \( \|\tau\|^2 = -1 \). Let \( \kappa \) denote any unit-norm element in \( \mathfrak{so}(3) \). Then we have three types of spacelike elements in \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \):

\[
\sigma \oplus \kappa, \quad \nu \oplus \kappa, \quad \tau \oplus \kappa.
\]
(1) \( \xi = a\sigma + bk \), where \( a, b \) are not both zero;
(2) \( \xi = \nu + bk \), where \( b \) is not zero; and
(3) \( \xi = a\sigma + bk \), where \( b^2 > a^2 \).

Rescaling and using the fact that in case (2) we can renormalise the coefficient of \( \nu \) to unit via an SO(1, 2) transformation, we obtain

(1) \( \xi = \frac{a\sigma + \nu}{\sqrt{1 + a^2}} \), where \( a \) is arbitrary;
(1') \( \xi = \sigma \);
(2) \( \xi = \nu + \kappa \); and
(3) \( \xi = \frac{a\sigma + \nu}{\sqrt{1 + a^2}} \), where \(-1 < a < 1\).

Notice that we have split the previous case (1) into two and we have normalised the Killing vector so that it has unit norm.

5.2.3. Spacelike subgroups of \( \text{NW}_6 \). The metric-preserving automorphisms of the Lie algebra \( \mathfrak{n} \) of \( \text{NW}_6 \) have been determined in [23] and they define a group isomorphic to \( \mathbb{R}^4 \rtimes (U(2) \rtimes \mathbb{Z}_2) \), where \( U(2) \rtimes \mathbb{Z}_2 \) is the principal extension of \( U(2) \) by the outer automorphism consisting of complex conjugation, and acts on \( \mathbb{R}^3 \cong \mathbb{C}^2 \) as follows:

\[
(U, 1) \cdot z = Uz \quad \text{and} \quad (U, -1) \cdot z = -U\bar{z},
\]

where \( U \in U(2) \) and \( z \in \mathbb{C}^2 \). Let us consider a spacelike vector in the Lie algebra \( \mathfrak{n} \):

\[
v + v^- e_- + v^+ e_+ \quad \text{with} \quad ||v||^2 + 2v^+v^- > 0,
\]

where \( v = v^e_i \) but we think of it as a vector in \( \mathbb{C}^2 \) by taking explicit complex linear combinations \( e_1 + ie_2 \) and \( e_3 + ie_4 \). The action of \( (z, U, 1) \in \mathbb{C}^2 \rtimes (U(2) \rtimes \mathbb{Z}_2) \) on such a vector is given by

\[
(z, U, 1) \cdot \begin{pmatrix} v \\ v^- \\ v^+ \end{pmatrix} = \begin{pmatrix} Uv - v^-z \\ v^- \\ \bar{z}^t Uv - \frac{i}{2}||z||^2 v^- + v^+ \end{pmatrix}.
\]

We claim that we can always put \( v^+ \) and all components of \( v \) except for \( v^1 \), say, to zero via such an automorphism. There are three cases to consider:

(1) \( v \neq 0 \) and \( v^- \neq 0 \): In this case we act with \((0, U, 1)\) to set all components of \( v \) to zero except for \( v^1 \), say. Then we act with \((z, \bar{1}, 1)\) to set \( v^+ = 0 \) with an appropriate \( z \). Such a \( z \) exists precisely because the vector is spacelike, and moreover it can be chosen so that \( v \) remains with all components zero except for \( v^1 \).

(2) \( v^- = 0 \) and \( v \neq 0 \): Here we act with \((z, \bar{1}, 1)\), for an appropriate \( z \), to set \( v^+ = 0 \) and such that the resulting \( v \) has all components zero except for \( v^1 \). This is clearly possible.

(3) \( v = 0 \) and \( v^- \neq 0 \): Again we act with \((z, \bar{1}, 1)\), for an appropriate \( z \), to set \( v^+ = 0 \) and such that the resulting \( v \) has all components zero except for \( v^1 \), which is again clearly possible.

We summarise this by observing that after a possible rescaling we can always bring a spacelike vector in \( \mathfrak{n} \) to the form \( e_1 + ae_- \), where \( a \) is some arbitrary real parameter.

5.3. Five-dimensional vacua. Each of the Kaluza–Klein reductions of the previous section gives rise to a vacuum solution of the minimal \( N=2 \) supergravity theory. In [1, Section 5] the authors studied the maximally supersymmetric backgrounds of this supergravity theory. Their results include a list of vacua: flat space, a hitherto unknown Gödel-like universe, Meessen’s five-dimensional wave, and a one-parameter family of vacua interpolating between \( \text{AdS}_2 \times S^3 \) and \( \text{AdS}_3 \times S^2 \), which can be understood as the near-horizon geometries of the supersymmetric rotating black holes of [4]. In addition to this list there are three other solutions
which were not yet identified or indeed shown to be maximally supersymmetric. A detailed comparison between our results is hindered by the intrinsic difficulty in comparing metrics which are written in terms of local coordinates. Nevertheless one can hazard a sort of correspondence between our results and those of \[1\] Section 5.

First of all, flat space is of course the unique maximally supersymmetric reduction of flat space. The near-horizon geometries of the supersymmetric rotating black holes coincide with the reductions (1) and (1') of the $\text{AdS}_3 \times S^3$ vacuum. Reductions (2) and (3) in Section 5.2.2 probably correspond to “near-horizon” limits of the supersymmetric over-rotating black holes; that is, in the regimes where the angular momentum exceeds the physical bound. Such solutions have closed time-like curves and this is consistent with observed phenomena in the similar reductions in \[30\] where the Killing vector generating the reduction, though being spacelike, is a linear combination of a spacelike and a causal Killing vector.\(^4\) In particular, this would mean that our reduction (2) agrees with the solution given by equation (5.102) in \[1\]. Finally both the Gödel and plane wave backgrounds arise as reductions of the six-dimensional wave\(^5\); and the parameter in the reductions of Meessen’s plane wave is probably related to the parameter in the Gödel solution.

Although we have proven that all our reductions are maximally supersymmetric, it remains to identify them. This is work in progress and will be presented elsewhere.

6. Conclusions and summary of results

In this paper we have classified (up to local isometry) the maximally supersymmetric solutions (vacua) of (1,0) and (2,0) supergravities in six dimensions and of the minimal $N=2$ supergravity in five dimensions. The (1,0) vacua are in one-to-one correspondence with six-dimensional Lie groups with a bi-invariant lorentzian metric and with anti-selfdual parallelising torsion. These are easily classified using the known results on lorentzian Lie groups, and we have seen that all vacua are locally isometric to the known vacua: flat space, $\text{AdS}_3 \times S^3$ or Meessen’s symmetric plane wave. Moreover we have also proven that the (2,0) vacua are—up to the action of the R-symmetry group $\text{Sp}(2)$—in one-to-one correspondence with the (1,0) vacua. Finally we have shown that the $N=2$ five-dimensional vacua are spaces of (right) cosets of the above Lie groups by one-parameter subgroups generated by left-invariant vector fields. We have therefore classified the possible reductions by classifying the inequivalent such one-parameter subgroups.

Acknowledgments

It is a pleasure to thank Tomás Ortín and Joan Simón for useful conversations. This work was started while JF was visiting CAMS and it is his pleasure to thank his two co-authors for the invitation, and the Royal Society and CAMS for the support which made the visit possible. JF would also like to thank the Rutgers NHETC and the School of Natural Science of the IAS for hospitality and support during the final stages of this work and for the opportunity to give seminars on these results. The research of JF is partially funded by the EPSRC grant GR/R62694/01. JF is a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme.

References

\[1\] J. Gauntlett, J. Gutowski, C. Hull, S. Pakis, and H. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions.” \texttt{arXiv:hep-th/0209114}.

\(^4\)Joan Simón, private communication.

\(^5\)Originally this is an unpublished result of the authors of \[5\].
[2] P. Meessen, “A small note on pp-wave vacua in 6 and 5 dimensions.” arXiv:hep-th/0111031.

[3] J. Gauntlett, R. Meyers, and P. Townsend, “Supersymmetry of rotating branes,” Phys. Rev. D59 (1999) 025001. arXiv:hep-th/9809085.

[4] R. Kallosh, A. Rajaraman, and W. Wong, “Supersymmetric rotating black holes and attractors,” Phys. Rev. D55 (1997) 3246. arXiv:hep-th/9611094.

[5] E. Lozano-Tellechea, P. Meessen, and T. Ortín, “On $d=4,5,6$ vacua with 8 supercharges,” Class. Quant. Grav. 19 (2002) 5921–5934. arXiv:hep-th/0206200.

[6] N. Alonso-Alberca, E. Lozano-Tellechea, and T. Ortín, “The near-horizon limit of the extreme rotating $d=5$ black hole as a homogeneous spacetime,” Class. Quant. Grav. 19 (2002) 423–430. arXiv:hep-th/0209069.

[7] J. Figueroa-O’Farrill, “On parallelisable NS-NS backgrounds,” Class. Quant. Grav. 20 (2003) 317–342. arXiv:hep-th/0211089.

[8] T. Kawano and S. Yamaguchi, “Dilatonic parallelisable NS-NS backgrounds.” arXiv:hep-th/0306235.

[9] J. Gutowski, D. Martelli, and H. Reall, “All the supersymmetric solutions of minimal six-dimensional supergravity.” arXiv:hep-th/0306235.

[10] J. Figueroa-O’Farrill and G. Papadopoulos, “Maximal supersymmetric solutions of ten- and eleven-dimensional supergravity,” J. High Energy Phys. 03 (2003) 048. arXiv:hep-th/0211089.

[11] J. Figueroa-O’Farrill and G. Papadopoulos, “Plücker-type relations for orthogonal planes.” arXiv:math.AG/0211170.

[12] H. Nishino and E. Sezgin, “Matter and gauge couplings of $N=2$ supergravity in six dimensions,” Phys. Lett. B144 (1984) 187.

[13] J. A. Wolf, “On the geometry and classification of absolute parallelisms. I,” J. Differential Geometry 6 (1971/72) 317–342.

[14] É. Cartan and J. Schouten, “On the geometry of the group manifold of simple and semisimple groups,” Nederl. Akad. Wetensch. Proc. Ser. A 29 (1926) 803–815.

[15] A. Medina and P. Revoy, “Algèbres de Lie et produit scalaire invariant,” Ann. scient. Éc. Norm. Sup. 18 (1985) 553.

[16] J. Figueroa-O’Farrill and S. Stanciu, “Nonsemisimple Sugawara constructions,” Phys. Lett. B327 (1994) 40–46. arXiv:hep-th/9402035.

[17] J. Figueroa-O’Farrill and S. Stanciu, “On the structure of symmetric selfdual Lie algebras,” J. Math. Phys. 37 (1996) 4121–4134. arXiv:hep-th/9506152.

[18] C. Nappi and E. Witten, “A WZW model based on a non-semi-simple group,” Phys. Rev. Lett. 71 (1993) 3751–3753. arXiv:hep-th/9310112.

[19] S. Stanciu and J. Figueroa-O’Farrill, “Penrose limits of Lie branes and a Nappi–Witten braneworld,” J. High Energy Phys. 06 (2003) 025. arXiv:hep-th/0303212.

[20] J. Figueroa-O’Farrill and G. Papadopoulos, “Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory,” J. High Energy Phys. 06 (2001) 036. arXiv:hep-th/0105308.

[21] M. Cahen and N. Wallach, “Lorentzian symmetric spaces,” Bull. Am. Math. Soc. 76 (1970) 585–591.

[22] J. Kowalski-Glikman, “Vacuum states in supersymmetric Kaluza-Klein theory,” Phys. Lett. 134B (1984) 194–196.

[23] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, “A new maximally supersymmetric background of type IIB superstring theory,” J. High Energy Phys. 01 (2002) 047. arXiv:hep-th/0110242.

[24] J. Figueroa-O’Farrill, “On the supersymmetries of Anti-de Sitter vacua,” Class. Quant. Grav. 16 (1999) 2043–2055. arXiv:hep-th/9902066.

[25] J. Figueroa-O’Farrill and J. Simón, “Generalised supersymmetric fluxbranes,” J. High Energy Phys. 12 (2001) 011. arXiv:hep-th/0110170.

[26] J. Figueroa-O’Farrill and J. Simón, “Supersymmetric Kaluza–Klein reductions of M2 and M5-branes,” Adv. Theor. Math. Phys. 6 (2002) 793–793. arXiv:hep-th/0208107.
(AC,WS)  

 CENTER FOR ADVANCED MATHEMATICAL SCIENCES  
 AMERICAN UNIVERSITY OF BEIRUT, LEBANON  

E-mail address: chams@aub.edu.lb, ws00@aub.edu.lb  

(JF)  

 SCHOOL OF MATHEMATICS  
 UNIVERSITY OF EDINBURGH, SCOTLAND  

E-mail address: j.m.figueroa@ed.ac.uk