Records and sequences of records from random variables with a linear trend

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Received 9 August 2010
Accepted 21 September 2010
Published 13 October 2010

Online at stacks.iop.org/JSTAT/2010/P10013
doi:10.1088/1742-5468/2010/10/P10013

Abstract. We consider records and sequences of records drawn from discrete time series of the form \( X_n = Y_n + cn \), where the \( Y_n \) are independent and identically distributed random variables and \( c \) is a constant drift. For very small and very large drift velocities, we investigate the asymptotic behavior of the probability \( p_n(c) \) of a record occurring in the \( n \)th step and the probability \( P_N(c) \) that all \( N \) entries are records, i.e. that \( X_1 < X_2 < \cdots < X_N \). Our work is motivated by the analysis of temperature time series in climatology, and by the study of mutational pathways in evolutionary biology.

Keywords: slow dynamics and ageing (theory), stochastic processes (theory), mutational and evolutionary processes (theory), extreme value statistics
1. Introduction

A record is an entry in a discrete time series that is larger (upper record) or smaller (lower record) than all previous entries. In this sense, a record is an extreme value that is defined relative to all previous values in the time series. Record events are of interest in various areas of life and science such as climatology [1]–[5] and sports [6,7], but also in biology [8]–[10]. A record is usually a rare and remarkable event that will be remembered by observers. Not without good reason the term record originates from the Latin verb recordari—to recall, to remind.

The classic results for records drawn from series of independent and identically distributed (i.i.d.) random variables (RVs) are well established, see [11]–[14] for a review. In this work we concentrate on two important quantities in particular. The first one is the probability for a certain entry in a time series to be a record, and the second one is the probability that the entries of a time series are ordered or, in other words, that all events are records. For i.i.d. RVs both these quantities are completely universal for all continuous probability density functions. This can be shown by the so-called stick-shuffling argument: the last one of n identically distributed entries (sticks) in a time series is equally likely to be a record as all other entries, and therefore the probability $p_n$ for the $n$th event to be a record, henceforth referred to as the record rate, is given by

$$p_n = \frac{1}{n}.$$  

(1)
Accordingly the expected mean number of records $R_n$ up to a time $n$ can be obtained by computing the harmonic sum: $R_n = \sum_{i=1}^{n} 1/k \approx \ln(n) + \gamma + O(1/n)$, where $\gamma \approx 0.577215...$ is the Euler–Mascheroni constant. From similar considerations one obtains the statistics of waiting times between record-breaking events which turn out to be universal as well. It is equally straightforward to compute the probability for all events in a series of length $N$ to be ordered in size. Since this case is only one of $N!$ possible and equally likely permutations of all $N$ events, the ordering probability $P_N$ is given by

$$P_N = \frac{1}{N!}.$$  \hspace{1cm} (2)

We conclude that the two quantities of interest are related by

$$P_N = \prod_{n=1}^{N} p_n,$$  \hspace{1cm} (3)

which reflects the fact that record events are independent in the i.i.d. case [11, 13]. We will return to this point below in section 2. In contrast to the properties of record times, the distributions of record values are not completely universal, but their asymptotic behavior falls into three different universality classes that are analogous to the universality classes of extreme value statistics: the Weibull class of distributions with finite support, the Gumbel class of distributions with exponential-like tails, and the Fréchet class of power law tailed distributions [15]–[17].

Given that the statistics of records for i.i.d. RVs is well understood, it is natural to ask what happens when the underlying time series is correlated, or when the RVs are drawn from a distribution that varies in time. An important example of a correlated random process is the random walk, and the record statistics of this process was recently analyzed by Majumdar and Ziff [18, 19]. The simplest realization of a time-dependent distribution is the linear drift model (LDM) first considered by Ballerini and Resnick [20, 21]. In this model the $n$th entry in the time series is of the form

$$X_n = Y_n + cn,$$  \hspace{1cm} (4)

where $c$ is a constant and the $Y_n$ are i.i.d. RVs. In this simple scenario the probability density $f_n(x)$ of $X_n$ is of the form $f_n(x) = f(x - cn)$ with a fixed probability density $f(y)$ and the corresponding cumulative distribution function $F(y) = \int_{-\infty}^{y} dy' f(y')$, which is the distribution of the i.i.d. part $Y_n$ of $X_n$. We will usually consider upper records and assume $c > 0$.

The LDM was originally introduced as a model for sports records in improving populations [20], and it has recently appeared in the context of the dynamics of elastic manifolds in random media [22]. An important motivation for the present work comes from the interest in the consequences of global warming for the occurrence of temperature records [3, 4]. In [5, 23] the effect of warming on daily temperature measurements was modeled using a Gaussian probability density with a linear trend, and it was shown that this very simple model is capable of quantitatively describing the statistics of record-breaking temperatures at European and American weather stations. In the climate context the drift speed $c$ is typically small compared to the standard deviation of $f$, which suggests considering the behavior of the record rate $p_n(c)$ for small $c$ and finite $n$.  

\textbf{doi:10.1088/1742-5468/2010/10/P10013}
This approach is complementary to previous work on the LDM [20]–[22], [24], which has mostly been concerned with the asymptotic behavior of the record rate for $n \to \infty$.

Another application that motivates our research comes from the study of adaptive paths in evolutionary biology. In this context, a path is a collection of mutations that change the genotype of an organism into another genotype of higher fitness. Given that mutation rates are small, the evolution of a population usually proceeds one mutation at a time. For a given set of $N$ mutations, there are then $N!$ distinct paths which correspond to the different orders in which the mutations can occur. Since a mutation spreads in the population only if it confers a fitness advantage, a given pathway is accessible to adaptive evolution only if the fitness values of the intermediate genotypes increase monotonically along the path, that is, if they are arranged in ascending order [25,26].

In view of the complexity of real fitness landscapes, the intricate interactions between different mutations are often modeled by assigning fitness values at random to genotypes [10]. One such model, which is closely related to the LDM, was introduced by Aita et al in the context of protein evolution [27]. In this model the fitness $X_n$ of a particular intermediate genotype with $n$ mutations is assumed to consist of an i.i.d. RV $Y_n$ and a systematic part $cn$, where $c > 0$ if the mutations move the population closer to the global fitness peak, and the value of $c$ (relative to the standard deviation of the $Y_n$) can be adjusted to tune the ruggedness of the fitness landscape. Taking into account also the initial genotype with no mutations, a total of $N+1$ genotypes with fitness values $X_0, X_1, \ldots, X_N$ are encountered along a path. The probability for a path to be accessible in this model is then just $P_{N+1}(c)$, and the expected number of accessible paths is $N!P_{N+1}(c)$.

An immediate corollary of (2) is that the expected number of accessible paths of length $N$ in a completely random fitness landscape without any average uphill slope ($c = 0$) is $1/(N+1)$ [28,29]. When the endpoint of the path is the global fitness maximum of the landscape, the number of fitness values that are required to be ordered reduces to $N$ and the expected number of paths is $N!P_N = 1$ in the completely random case.

Here we consider both the record rate $p_n$ and the ordering probability $P_N$ for the linear drift model. We distinguish between a small drift $c$ that is much smaller than the characteristic width of the distribution (in most cases the standard deviation), and a large drift that is much larger than this width. Both cases are of practical relevance. In section 2 we discuss the general properties of record statistics for systems with linear drift, with particular emphasis on the correlations between record events. In the subsequent section 3 we will present new results for small $c$. We examine the record statistics for members of the three extreme value classes individually and find the corresponding asymptotic behaviors. In section 4 we analyze the case of large $c$. Throughout Monte Carlo simulations are used to confirm the analytical results. Finally, in section 5 we present a brief summary, discuss related issues and give an outlook on further possible research. Some of the calculational details are relegated to appendices.

2. General theory and an exactly solvable example

The values taken by the $\{X_i\}_{i \in \{1,\ldots,n\}}$ are stochastically independent. The probability that all $n$ values are less than a given value $x$ factorizes to $\prod_{i=1}^n \int_{-\infty}^x dx_i f_i(x_i) = \prod_{i=1}^n F_i(x)$. Here $f_i$ and $F_i$ are, as stated in the introduction, the probability densities and cumulative distribution functions of the $X_i$. Thus, given the value $y_n$ of the i.i.d. part $Y_n$ of $X_n$, the probability that all previous RVs $\{X_i\}_{i \in \{1,\ldots,n-1\}}$ are smaller than $X_n$ is
\[
\prod_{i=1}^{n-1} f_{y_n+i}^\infty \, dy_{n-i} \, f(y_{n-i}).
\]
The probability that a RV \( X_n \) drawn from a general time-dependent distribution \( F_n(x) \) is a record is therefore given by [30]

\[
P[X_n = \max_{i \in \{1, \ldots, n\}} \{X_i\}] = p_n = \int_{-\infty}^\infty dx \, f_n(x) \prod_{i=1}^{n-1} F_i(x),
\]
which reduces to

\[
p_n(c) = \int_{-\infty}^\infty dx \, f(x) \prod_{i=1}^{n-1} F(x + ci)
\]
for the LDM. It was shown in [20] that the limiting record rate

\[
p(c) \equiv \lim_{n \to \infty} p_n(c) = \int_{-\infty}^\infty dx \, f(x) \prod_{i=1}^{n-1} F(x + ci)
\]
exists and is nonzero for \( c > 0 \) provided the distribution \( f(y) \) of the i.i.d. part in (4) has a finite first moment. For \( c = 0 \), (6) can be evaluated directly and, with the substitution \( u = F(x) \), one obtains

\[
p_n(c = 0) = \int_{-\infty}^\infty dx \, f(x) F(x)^{n-1} = \int_{F(-\infty) = 0}^{F(\infty) = 1} du \, u^{n-1} = \frac{1}{n},
\]
independent of \( F \), as already shown in (1).

The other quantity under consideration in this article, the ordering probability \( P_N \), can be expressed as

\[
P[X_1 < X_2 < \cdots < X_N] = P_N(c) = \int_{-\infty}^\infty dx_N \, f_N(x_N) \cdots \int_{-\infty}^\infty dx_1 \, f_1(x_1) \mathbb{1}_{x_1 < x_2 \cdots < x_N}.
\]
Inserting \( f_n(x) = f(x - cn) \) the indicator function \( \mathbb{1}_{x_1 < x_2 < \cdots < x_N} \) can be absorbed in the integral boundaries to yield

\[
P_N(c) = \int_{-\infty}^\infty dy_N \, f(y_N) \int_{-\infty}^{y_N} dy_{N-1} \cdots \int_{-\infty}^{y_2} dy_1 \, f(y_1).
\]
As for \( p_n(c) \), this equation can be solved for arbitrary \( F \) only in the case \( c = 0 \). Using again the substitution \( u = F(x) \) in turn in all the \( N \) integrals, starting from the inside, one obtains the result already derived in (2),

\[
P_N(c = 0) = \int_{-\infty}^\infty dy_N \, f(y_N) \int_{-\infty}^{y_N} dy_{N-1} \cdots \int_{-\infty}^{y_2} dy_2 \, f(y_2) \, F(y_2)
\]

\[
= \int_{-\infty}^\infty dy_N \, f(y_N) \int_{-\infty}^{y} dy_{N-1} \cdots \int_{-\infty}^{F(y_1)} du \, u
\]

\[
= \frac{1}{2} \int_{-\infty}^\infty dy_N \, f(y_N) \int_{-\infty}^{y_N} dy_{N-1} \cdots \int_{-\infty}^{F(y_1)} du \, u^2 = \cdots = \frac{1}{N!}.
\]
The reason for re-deriving the two previous results is that here this is done in a way that in principle generalizes to arbitrary \( c \).

For \( c > 0 \), the exact evaluation of equations (6) and (10) has proven difficult, but in the case where the \( Y_n \) are Gumbel distributed, i.e. \( F(y) = \exp(-e^{-y}) \), one can use the fact

doi:10.1088/1742-5468/2010/10/P10013
that this distribution obeys the relation \( F(y + a) = F(y) \exp(-a) \) to explicitly perform the integration in (6) [20,21]. With the abbreviation \( \alpha \equiv e^{-c} \) and the substitution \( u = F(y) \) one obtains

\[
p_n(c) = \int_{-\infty}^{\infty} dy \, f(y) F(y)^{\sum_{i=1}^{n-1} \alpha^i} = \frac{1 - e^{-c}}{1 - e^{-nc}}
\]

by use of the incomplete geometric series. Keeping \( c \) fixed, one obtains in the limit \( n \to \infty \) the asymptotic record rate \( p(c) = 1 - e^{-c} \), while for \( c \to 0 \) one recovers the i.i.d. result \( p_n = 1/n \). For \( c < 0 \) the record rate is seen to decay exponentially in \( n \), which implies that the expected number of records \( R_n \) remains finite for \( n \to \infty \). We suspect this to be a general feature of the LDM with \( c < 0 \), but are not aware of a proof of this fact.

The relation used to evaluate (12) for the Gumbel case can also be used in (10), as before starting from the innermost integral, which yields

\[
P_N(c) = \int_{-\infty}^{\infty} dy_N \, f(y_N) \int_{-\infty}^{y_{N-1} + c} dy_{N-1} \cdots \int_{-\infty}^{y_2 + c} dy_2 \, f(y_2 + c) = \frac{1}{\alpha + 1} \int_{-\infty}^{\infty} dy_N \, f(y_N) \int_{-\infty}^{x_{N-1} + c} dy_{N-1} \cdots \int_{-\infty}^{x_2 + c} dy_2 \, f(x_2 + c) du \, u^\alpha
\]

\[
= \cdots = \prod_{i=1}^{N-1} \frac{1}{\sum_{k=0}^{i} \alpha^k}.
\]

Summing the geometric series as in (12), one obtains

\[
P_N(c) = (1 - e^{-c})^N \frac{1}{\prod_{i=1}^{N} (1 - e^{-cn})} \equiv (1 - e^{-c})^N Z_N,
\]

where \( Z_N \) is the grand canonical partition function of a system of bosonic particles with energy levels \( n = 1, \ldots, N \) at inverse temperature \( c \). This partition function also occurs as one limit in the integer partition problem (see [31,32] and references therein).

The product \( \prod_{n=1}^{N} (1 - \exp(-cn)) \) in the denominator is the so-called \( q \)-Pochhammer symbol \( (q; q)_N \) with \( q = e^{-c} \). In the limit \( N \to \infty \) with fixed \( c \), one has the asymptotic expression [33]

\[
\lim_{N \to \infty} \prod_{n=1}^{N} (1 - e^{-cn}) \equiv (e^{-c})_\infty \approx \sqrt{\frac{2\pi}{c}} \exp\left(-\frac{\pi}{6c} + \frac{c}{24}\right),
\]

and thus, by inserting this into (14),

\[
P_N(c) \approx \sqrt{\frac{c}{2\pi}} \exp \left( N \ln(1 - e^{-c}) + \frac{\pi}{6c} - \frac{c}{24} \right), \quad N \gg 1.
\]
Figure 1. Comparison of the exact expression (13) and the asymptotic expression (17) to numerical simulation. The exact expression is confirmed, and while there is a clear difference between the simulations and the asymptotic expression for small values of \(c\) in (a), the approximation holds with good accuracy for large \(c\) (inset of (b), lines are the asymptotic expressions). The main plot of (b) demonstrates the scaling between \(N\) and \(c\) according to (17).

On the other hand, taking \(c \gg 1\) at fixed \(N\), one has \(\alpha = \exp(-c) \ll 1\) and thus the geometric series in the denominator of (13) can be approximated to first order in \(\alpha \equiv \exp(-c)\), as \(1/(\sum_{k=0}^{l} \alpha^k) \approx 1 - \alpha + \mathcal{O}(\alpha^2)\). Then (13) becomes

\[
P_N(c) \approx \exp(-(N-1)\alpha) = \exp(-(N-1)e^{-c}), \quad c \gg 1.
\]

This expression is distinguishable from numerical data only in the region of \(c \sim \mathcal{O}(1)\), see figure 1.

Comparing the exact expressions (13) and (12), one sees that the relation (3) obtained in the i.i.d. case remains valid here. This is a consequence of the mutual stochastic independence of record events in the LDM with Gumbel-distributed i.i.d. part \[14, 12, 24\]. In fact the Gumbel distribution is uniquely characterized by the mutual independence of record values and record indicator variables (which indicate whether or not a record occurs at time \(n\)) \[24, 34\].

For \(c > 0\) and arbitrary distribution \(F\), however, the record events in the LDM are not independent. Numerical studies for several different distributions presented in figure 2 show that the records are negatively correlated and seem to repel each other. A more thorough examination of the structure of correlations between record events in this model is currently ongoing research \[35\]. For the purpose of the present discussion we merely note that record events appear to become asymptotically uncorrelated for large \(c\). This fact will be used to derive some asymptotic results for \(P_N(c)\) in section 4. First, however, we consider the case \(c \ll 1\).

3. Record statistics for small drift

3.1. Record rate

In section 2 we gave a general expression for the record rate \(p_n(c)\) of the LDM. Here, we derive the first order term in a series expansion for \(c \ll 1\). If \(c\) is very small (6) can be
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Figure 2. Joint probability of two consecutive record events at times $n = 6$ and $7$, divided by the product of the corresponding record rates. This ratio is unity if record events are stochastically independent. For $c = 0$, this is the case, just as for Gumbel-distributed i.i.d. parts (crosses). Note that for other probability densities, the record events also seem to become increasingly independent as $c$ grows.

simplified as follows:

$$p_n(c) = \int_{-\infty}^{\infty} dy f(y) \prod_{i=1}^{n-1} F(y + ci)$$

$$\approx \int_{-\infty}^{\infty} dy f(y) \prod_{i=1}^{n-1} [F(y) + ci f(y)]$$

$$\approx \int_{-\infty}^{\infty} dy f(y) F^{n-1}(y) + c \frac{n(n-1)}{2} \int_{-\infty}^{\infty} dy f^2(y) F^{n-2}(y)$$

$$= \frac{1}{n} + c I_n$$

(18)

with

$$I_n \equiv \frac{n(n-1)}{2} \int_{-\infty}^{\infty} dy f^2(y) F^{n-2}(y).$$

(19)

This expansion is valid provided $f(y)$ is slowly varying between $y$ and $y + ci$, which strictly speaking requires $nc$ to be small compared to the width of the distribution. In the following we will evaluate the first order correction coefficient $I_n$ for several elementary distributions.

Before doing this, we show that our formula for $p_n(c)$ can be generalized with respect to the position of the record in the time series. Specifically, we consider the probability that the $k$th event in a time series of length $n$ with linear drift $c$ is a record. For this
purpose we have to consider the following integral instead of (5):
\[
P[X_k = \max(X_1, \ldots, X_n)] = \int_{-\infty}^{\infty} dy \, f_{n}(y) \prod_{i=1, i \neq k}^{n-1} \int_{-\infty}^{y + c(i-k)} dy_i \, f_{i}(y_i). \tag{20}
\]
Evaluating this integral in the same way as shown above, we obtain the following expression:
\[
P[X_k = \max(X_1, \ldots, X_n)] \approx \frac{1}{n} + \frac{c}{2} \left( k^2 - k - (n - k)(n - k - 1) \right) \times \int_{-\infty}^{\infty} dy \, f^2(y) E^{n-2}(y). \tag{21}
\]
Note that for \( k = n \) this expression reduces to our approximation (18) for \( p_n(c) \). Apparently for \( c > 0 \) this expression assumes its maximum for \( k = n \) and its minimum for \( k = 1 \). The last entry has the largest, and the first entry the smallest chance to be the maximum of the series.

3.1.1. Weibull class. Let us start by considering the Weibull class of extreme value statistics, which contains distributions with finite support. A simple example for a member of the Weibull class is a uniform distribution, which takes the value \( 1/2a \) between \( -a \) and \( a \) and 0 outside of this interval. For this case the first order expansion of \( p_n(c) \) is given by
\[
p_{\text{uniform}}^{\text{c}}(c) = \frac{1}{n} + \frac{c}{2} \left( \frac{n(n-1)}{2} \right) \int_{-\infty}^{\infty} dy \left( \frac{1}{2a} \right)^2 \left( \frac{y}{2a} + \frac{1}{2} \right)^{n-1} + O(c^2), \tag{22}
\]
which can be evaluated to yield
\[
p_{\text{uniform}}^{\text{c}}(c) \approx \frac{1}{n} + \frac{c - 1}{4a}. \tag{23}
\]
In this case the correction coefficient \( I_n \) increases linearly with the number of events \( n \).

More generally, we consider distributions of the form
\[
f(y) = \xi(1 - y)^{\xi-1}
\]
with \( \xi > 0 \) and \( 0 < y \leq 1 \). For these distributions we have
\[
p_n(c) \approx \frac{1}{n} + c \frac{n(n-1)}{2} \int_0^1 dy \xi^2(1-y)^{2\xi-2}(1-(1-y)^\xi)^{n-2}. \tag{25}
\]
The integral is divergent for \( \xi < 1/2 \), which indicates that \( p_n(c) \) is a non-analytic function of \( c \); this case will be considered elsewhere. For \( \xi > 1/2 \) we use the substitution \( (1-y) = z^{1/\xi} \) to express the integral in terms of a Beta-function,
\[
p_n(c) \approx \frac{1}{n} + c \xi \frac{n(n-1)}{2} \frac{\Gamma(2-1/\xi)\Gamma(n-1)}{\Gamma(n + 1 - 1/\xi)}. \tag{26}
\]
Using the Stirling approximation for large \( n \) one finally arrives at
\[
p_n(c) \approx \frac{1}{n} + \frac{c\xi}{2} \Gamma \left( 2 - \frac{1}{\xi} \right) n^{1/\xi}, \tag{27}
\]
which shows that \( I_n \) generally increases as a power law in the Weibull class.

DOI:10.1088/1742-5468/2010/10/P10013
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Figure 3. Results of Monte Carlo simulations of the LDM for power law tailed distributions of the Fréchet class. The figure shows the difference between the record rate in the time-independent case for $c = 0$ and the drifting case with drift $c = 0.01$. This difference is given by $(1/c)(p_n(c) - p_n(0))$. The dots correspond to simulations with different tail coefficients $\mu = 1, 2, 3, 5$ averaged over $10^6$ runs, and the lines show the analytic predictions. The first order approximation is very good for $\mu = 1$ and 2 but becomes less accurate for larger $\mu$.

3.1.2. Fréchet class. As a representative of the Fréchet class of extreme value statistics we consider a general power law distribution of the form $f(x) = (1/\mu)x^{-\mu-1}$ for $x > 1$ and $\mu > 0$. For distributions of this kind $p_n(c)$ in the small $c$ expansion is given by

$$p_n(c) \approx \frac{1}{n} + c \frac{n(n-1)}{2} \int_1^{\infty} dy \mu^2 y^{-2-2\mu}(1-y^{-\mu})^{n-2}.$$  \hfill (28)

Again, the integral is very similar to a Beta-function and it can be transformed into one by elementary means. Doing this we find

$$p_n(c) \approx \frac{1}{n} + c\mu \frac{n(n-1) \Gamma(2 + 1/\mu) \Gamma(n-1)}{\Gamma(n + 1/\mu + 1)},$$  \hfill (29)

and using again the Stirling approximation we obtain

$$p_n(c) \approx \frac{1}{n} + \frac{c\mu}{2} \left(2 + \frac{1}{\mu}\right) \frac{1}{n^{1/\mu}}.$$  \hfill (30)

In figure 3 we compare this prediction to simulation results.

While in the case of the Weibull class the correction term $I_n$ increases with $n$, here it decays as a power law $n^{-1/\mu}$. For $\mu > 1$ the decay is slower than the $1/n$-decay of the record rate in the absence of a drift, which implies that the drift will nevertheless dominate the behavior for long times. This is consistent with the fact that the record rate reaches a nonzero asymptotic limit, as given by (7), because $\mu > 1$ implies a finite first moment for the $Y_n$. On the other hand, for $\mu < 1$ the decay of $I_n$ is faster than $1/n$ and

doi:10.1088/1742-5468/2010/10/P10013
the limit on the right-hand side of (7) vanishes for any \( c \), which implies that the drift is asymptotically irrelevant. The borderline situation \( \mu = 1 \) has been studied by De Haan and Verkade [36], who find that the asymptotics depends nontrivially on the value of \( c \) in this case.

In general the results presented so far show that the effect of the drift on a broad distribution is smaller than on a narrower distribution. A similar qualitative trend was found in [30] for probability densities with increasing variance.

3.1.3. Gumbel class. The Gumbel class comprises unbounded distributions that decay faster than any power law. A very simple representative of the Gumbel class is the exponential distribution
\[
f(y) = \nu^{-1} e^{-(y/\nu)}.
\]
In this case the first order expansion (18) assumes the following form:
\[
p_n^{\exp}(c) \approx \frac{1}{n} + c \frac{n(n-1)}{2} \int_0^\infty dy \frac{1}{\nu^2} e^{-(2y/\nu)} (1 - e^{-(y/\nu)})^{n-2}.
\]
(31)
The integral can be solved by two partial integrations and one finds
\[
p_n^{\exp}(c) \approx \frac{1}{n} + \frac{c}{2\nu},
\]
(32)
that is, the correction term is independent of \( n \).

The calculation for the Gaussian distribution, arguably the most important member of the Gumbel class, is more complicated. For convenience we consider a Gaussian distribution of unit variance,
\[
f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.
\]
(33)
The integral of interest reads
\[
I_n^{\text{gauss}} = \frac{n(n-1)}{2\sqrt{2\pi}} \int_{-\infty}^\infty dy e^{-y^2} \left( \int_{-\infty}^y dy' e^{-(y'^2/2)} \right)^{n-2},
\]
(34)
which will be evaluated for large \( n \) using the saddle point approximation. With the definition
\[
g(y) := -y^2 + (n-2) \ln \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y dy' e^{-(y'^2/2)} \right)
\]
(35)
we have
\[
I_n^{\text{gauss}} \approx \frac{n(n-1)}{4\pi} \sqrt{-2\pi} \frac{d^2}{d\tilde{y}^2} g(\tilde{y})
\]
(36)
where \( \tilde{y} \) denotes the saddle point of the integral. It turns out that the computation of a practicable series expansion of \( g(y) \) can only be done under some approximations and by using the non-elementary Lambert–W function [37, 38]. In terms of the W function \( W(z) \) defined by the relation \( W(z)e^{W(z)} = z \), we find
\[
\tilde{y} = \sqrt{W \left( \frac{(n-2)^2}{8\pi} \right)}.
\]
(37)
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Figure 4. Results of Monte Carlo simulations from $10^9$ realizations of the LDM with RVs drawn from a normal distribution with standard deviation $\sigma = 1$. The figure shows the normalized difference between the record rate in the time-independent case and the drifting case, $(1/c)(p_n(c) - p_n(0))$. The dots correspond to a simulation with drift velocity $c = 10^{-4}$.

For large $z$ the Lambert-W function can be approximated by $W(z) \approx \ln(z) - \ln(\ln(z))$, which eventually yields

$$p_n^{\text{gauss}}(c) \approx \frac{1}{n} + c \frac{2\sqrt{\pi}}{e^2} \sqrt{\ln \left( \frac{n^2}{8\pi} \right)}.$$  (38)

For a detailed derivation of this result see appendix A. In figure 4 the asymptotic prediction is compared to numerical simulations. The systematic deviations that are visible in this figure can be attributed to strong sub-leading corrections to (38), see appendix A.

As a more general subset of the Gumbel class we also considered distributions of the form $f(y) = C_{\beta} e^{-|y|^\beta}$ with $\beta > 0$ and normalization constant $C_{\beta} = [2\Gamma(1 + 1/\beta)]^{-1}$. The integral of interest then reads

$$I_n = \frac{n(n-1)}{2} C_{\beta} \int_{-\infty}^{\infty} dy e^{-2|y|^\beta} \left( \int_{-\infty}^{y} dy' e^{-|y'|^\beta} \right)^{n-2},$$  (39)

which can again be treated using a saddle point approximation. Ignoring constant prefactors we find that

$$I_n \propto \ln(n)^{1-1/\beta}.$$  (40)

for large $n$, which includes the results for the exponential distribution ($\beta = 1$) and the Gaussian ($\beta = 2$) as special cases. For a detailed derivation of this result see appendix B.

We conclude that the behavior of the correction coefficient $I_n$ in the Gumbel class is generally intermediate between the power law growth for distributions in the Weibull class, and the power law decay for Fréchet-type distributions. Again, the effect of the drift is stronger for distributions that fall off more rapidly (large $\beta$).
3.1.4. Relation to the asymptotic record rate \( p(c) \). It is instructive to compare the asymptotics of the correction term \( I_n \) derived in the preceding subsections to the behavior of the limiting record rate \( p(c) \) for small \( c \), which was studied by Le Doussal and Wiese [22]. Heuristically, the two quantities can be related as follows. We have seen above that, for any choice of \( f(y) \) with a finite first moment, the correction term \( I_n \) becomes large compared to \( 1/n \) for large \( n \). This implies that, for any \( c > 0 \), the first order correction will eventually become comparable to the zeroth order record rate \( 1/n \). The corresponding timescale \( n^* \) can be estimated from

\[
n^* I_{n^*} \sim c.
\] (41)

For times \( n > n^* \) the first order expansion breaks down and the record rate saturates at a nonzero limiting value \( p(c) \). Thus we expect that, in order of magnitude, \( p(c) \sim 1/n^* \). Using the asymptotic results (27), (30), and (40) together with (41) we may then determine the behavior of \( p(c) \) for small \( c \).

\[
 p(c) \sim \begin{cases} 
 c^{\xi/(1+\xi)} & \text{Weibull} \\
 c^{\mu/(\mu-1)} & \text{Fréchet with } \mu > 1 \\
 c|\ln(c)|^{1-1/\beta} & \text{Gumbel}
\end{cases}
\] (42)

agrees with the analysis of [22] in all cases.

3.2. Ordering probability

In this subsection, we derive a first order expansion for the ordering probability \( P_N(c) \). Our main result reads

\[
P_N(c) = \frac{1}{N!} + c \frac{1}{(N-2)!} \int_{-\infty}^{\infty} dx f^2(x) + \mathcal{O}(c^2).
\] (43)

In contrast to the expansion (18) for the record rate, one sees that for \( P_N(c) \) the distribution \( f(x) \) only enters in the form of a non-universal constant but has no influence on the \( N \)-dependence of the correction term. Note, however, that similar to the expansion for \( p_n(c) \), the correction term diverges when \( f^2(x) \) becomes too singular, as is the case for the Weibull-type distribution (24) with \( \xi < 1/2 \).

To prove (43), we set up a Taylor expansion of (10) in \( c \) to first order. With \( P_N(0) = 1/N! \) we have

\[
P_N(c) = \frac{1}{N!} + c \left. \frac{d}{dc} P_N(c) \right|_{c=0} + \mathcal{O}(c^2) \approx \frac{1}{N!} + \int_{-\infty}^{\infty} dx f(x_N) \left. \frac{d}{dc} P(N-1, c, x_N) \right|_{c=0},
\]

where terms of \( \mathcal{O}(c^2) \) and higher have been omitted and

\[
P(N-1, c, x_N) \equiv \int_{-\infty}^{x_{N-1}+c} dy_{N-1} f(y_{N-1}) \int_{-\infty}^{y_{N-1}+c} dy_{N-2} \cdots \int_{-\infty}^{y_2+c} dy_1 f(y_1).
\]

\[
= \int_{-\infty}^{x_{N-1}+c} dy_{N-1} f(y_{N-1}) P(N-2, c, y_{N-1}).
\]

\[\text{doi:10.1088/1742-5468/2010/10/P10013}\]
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Figure 5. Simulations comparing the ordering probability $P_N(c)$ to the first order expansion $P_N(c) = 1/N! + c I_f/(N-2)!$, where $I_f = \int f(y)^2 \, dy$, for (a) Gaussian distribution and $N = 7$, (b) uniform distribution and $N = 5$.

Clearly, the derivative of $P(N-1, c, x_N)$ obeys the recursion relation

$$
\frac{d}{dc} P(N-1, c, x_N) \bigg|_{c=0} = f(x_N) P(N-2, 0, x_N) \\
+ \int_{-\infty}^{x_N+c} dy_{N-1} f(y_{N-1}) \frac{d}{dc} P(N-2, c, y_{N-1}) \bigg|_{c=0}.
$$

Using the same substitutions as in (11), one obtains

$$
P(N-2, c = 0, x_N) = \frac{1}{(N-2)!} F_{N-2}^2(x_N)
$$

and thus

$$
\frac{d}{dc} P(N-1, c, x_N) \bigg|_{c=0} = \frac{f(x_N)}{(N-2)!} F_{N-2}^2(x_N) \\
+ \int_{-\infty}^{x_N+c} dy_{N-1} f(y_{N-1}) \frac{d}{dc} P(N-2, c, y_{N-1}) \bigg|_{c=0}.
$$

Now $P(1, c, x_2) = \int_{-\infty}^{x_2+c} dy_1 f(y_1)$ and thus $(d/dc)P(1, c, x_2)|_{c=0} = f(x_2)$. Putting this into the recursion relation above and integrating over all $y_N$ weighted by $f(y_N)$, we obtain

$$
\frac{d}{dc} P_N(c) \bigg|_{c=0} = \frac{1}{(N-2)!} \int_{-\infty}^{\infty} dy_N f(y_N)^2 F_{N-2}^2(y_N) \\
+ \frac{1}{(N-3)!} \int_{-\infty}^{\infty} dy_N f(y_N) \int_{-\infty}^{y_N} dy_{N-1} f(y_{N-1}) F_{N-3}^2(y_{N-1}) + \cdots \\
+ \frac{1}{0!} \int_{-\infty}^{\infty} dy_N f(y_N) \int_{-\infty}^{y_N} dy_{N-1} f(y_{N-1}) \cdots \int_{-\infty}^{y_1} dy_1 f^2(y_1),
$$

(44)
a sum with $N$ terms, the last of which comprises $N-1$ nested integrals. Somewhat miraculously, as shown in appendix C, this chain of integrals can be collapsed into the simple closed form advertised in (43). Figure 5 compares the asymptotic expression for $P_N(c)$ derived here with numerical simulations.

doi:10.1088/1742-5468/2010/10/P10013
4. Record statistics for large drift

In section 2 we saw that, although record events in the LDM are generally correlated for \( c > 0 \), the correlations tend to diminish for large \( c \) (figure 2). This is in some sense expected, as for \( c \to \infty \) both \( p_n(c) \) and \( P_N(c) \) tend to unity, such that the stochastic independence relation (3) becomes trivially satisfied. Moreover, numerical studies [23] suggest that the rate of convergence of the record rate to its limiting value \( p(c) \) increases with \( c \) and for sufficiently large values is to a good accuracy attained from the very beginning. Thus for large \( c \) (3) can be approximated by

\[
P_N(c) \approx p(c)^N = (1 - \epsilon(c))^N \approx e^{-(N-1)\epsilon(c)},
\]

(45)

where \( \epsilon(c) \) is the probability that \( X_n \) is not a record. For large \( c \), only \( X_{n-1} \) has an appreciable chance of keeping \( X_n \) from being a record. Thus

\[
\epsilon(c) \approx P[X_{n-1} > X_n] = \int_c^\infty dx f^{*2}(x).
\]

(46)

Here \( f^{*2}(x) \) denotes the twofold convolution of the probability density \( f(x) \) of the i.i.d. part of \( X_n \). To quote a few examples:

\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Rightarrow \epsilon(c) = \frac{1}{2} \text{erfc}(c/2) \approx \frac{1}{c\sqrt{\pi}} e^{-c^2/4}
\]

(47)

\[
f(x) = \frac{1}{2} e^{-|x|} \Rightarrow \epsilon(c) = \frac{1}{2} e^{-c} + \frac{c}{4} e^{-c}
\]

(48)

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk e^{-ik|x|} \Rightarrow \epsilon(c) = \frac{1}{2\pi} \int_c^\infty dx \int_{-\infty}^\infty dk e^{-ikx - 2|k|\mu} \approx \gamma_\mu e^{-\mu},
\]

(49)

with

\[
\gamma_\mu = \frac{2\Gamma(1 + \mu) \sin((1/2)\pi\mu)}{\pi\mu}.
\]

(50)

The first two of these examples are from the Gumbel class of extreme value statistics, whereas the third example is from the Fréchet class [15,16]. The asymptotic expression in (47) is from [39], while the one in (49) can straightforwardly be derived from the known expression for the large-\( x \) asymptotics for \( f(x) \), see e.g. [40]. Note that the large-\( c \) asymptotics for the Weibull class is trivial, because both \( p_n \) and \( P_N \) become identically equal to unity once \( c \) exceeds the range of support of \( f(y) \). Inserting the expressions (47)–(49) into (45) and also considering the asymptotics of the exact expression for \( P_N(c) \) derived in (17), we see that in the limit of large \( N \) and \( c \) the behavior of the ordering probability is generally of the approximate form

\[
P_N(c) \approx \exp[-N/N^*(c)],
\]

(51)

where \( N^*(c) \sim e^c \) for the Gumbel and exponential distributions, \( N^*(c) \sim e^{c^2} \) for the Gaussian, and \( N^*(c) \sim e^{c^4} \) for the Lévy distribution.

To verify the approximations made in this section, we performed numerical simulations, see figures 6 and 7. The results indicate that our approach, although quite rough and not necessarily well controlled, does indeed capture the interesting regime rather well for sufficiently large \( N \) and \( c \).
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Figure 6. Scaling collapse of $P_N(c)$ as suggested by the asymptotic expression (45) for (a) Laplace density $f(x) = e^{-|x|}/2$ and (b) Lévy density with $\mu = 1.3$. The ordinate is the corresponding expression from equations (48) and (49) respectively. Note that the asymptotic expressions get more accurate for larger $N$. The inset shows direct plots of simulation results (points) versus the asymptotic expression (lines).

Figure 7. Check of the expression for $\gamma_\mu$ from (49). For $N = 1024$, the range $0 \leq c \leq 400$ was numerically explored as for the data shown in figure 6. The curves obtained in this way were then fitted to the form $\exp(-N\gamma_\mu c^{-\mu})$. The value of $\gamma_\mu$ obtained in this way is shown here for various values of $\mu$ and compared to the analytic expression (50).

5. Conclusions

In this paper we considered the statistics of records and sequences of records of random variables with a linear trend as described by (4). We numerically explored the correlations between record events (cf figure 2) and analytically investigated the record rate $p_n(c)$ and

doi:10.1088/1742-5468/2010/10/P10013

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the ordering probability $P_N(c)$ in the limiting regimes of small and large drift velocities, $c \ll 1$ and $c \gg 1$ respectively. For the regime of $c \sim O(1)$, we have not found a generally applicable method. Thus the behavior of $p_n(c)$ and $P_N(c)$ in this regime remains an open problem.

Specifically, we considered the effect of a small linear drift on distributions of the three extreme value classes. While this effect is varying even within the individual classes we still found systematic differences between them. For the Fréchet class of distributions with power law tails we found that the coefficient of the leading order correction to the record rate decays as $I_n \sim n^{-1/\mu}$ for large $n$. This implies a distinction between distributions with and without a finite first moment: for $\mu > 1$ the correction decays more slowly than the unperturbed record rate $1/n$, which implies that the drift dominates asymptotically and $p_n(c)$ attains a nonzero limit for $n \to \infty$; on the other hand, for $\mu < 1$ the drift is asymptotically irrelevant.

For the considered distributions of the Gumbel class the situation was a bit more complicated. For the exponential distribution we found a constant additive correction to the record rate, while for generalized Gaussian probability densities $f \propto e^{-|x|^{\beta}}$ the correction term was shown to be of order $\ln(n)^{1-1/\beta}$, which increases (decreases) with $n$ when $\beta > 1$ ($\beta < 1$). For the distributions of the Weibull class, the effect of the drift is the strongest, and the correction term generally increases as a power law in $n$. Moreover, for highly singular distributions with $\xi < 1/2$ in (24), we found indications for a non-analytic behavior of $p_n(c)$ which will be investigated elsewhere. Generally speaking, narrow distributions are very sensitive to drift, while for broad distributions with heavy tails the effect is much weaker. We have also pointed out that the behavior of the first order correction term $I_n$ obtained in this paper precisely matches earlier results for the asymptotic record rate $p(c)$ [22].

For the probability of a sequence of $N$ consecutive records, we find the following: for $c \ll 1$, the distribution $f(y)$ of the i.i.d. part of $X_n$ enters to leading order in $c$ only as a numerical constant $\int_{-\infty}^{\infty} dx f^2(x)$, see (43), but the $N$-dependence is completely universal for all distributions for which the integral exists. On the other hand, for $c \gg 1$ and $N \gg 1$, the combination in which $c$ and $N$ enter $P_N(c)$ depends explicitly on the tail of the underlying distribution $F$. This indicates that somewhere in the regime of intermediate $c$, there is a crossover in the $c$-dependence of $P_N(c)$ from a highly universal to a less universal form.

The result (43) has important implications in the context of adaptive paths of evolutionary biology: recalling that the expected number of accessible paths between two genotypes which are $N$ mutations apart is given by $N!P_{N+1}$, we see that in the presence of an arbitrarily small drift this quantity increases with $N$ as $cN$. Thus even a weak systematic fitness gradient dramatically increases the accessibility of mutational pathways in the direction of increasing fitness.

Acknowledgments

This work was supported by DFG within SFB 680 Molecular basis of evolutionary innovations and the Bonn–Cologne Graduate School for Physics and Astronomy. We thank Simon Gravel and Damien Simon for useful discussions in the early stages of the project.
Appendix A. Computation of $I_n$ for the Gaussian distribution

We begin by computing the saddle point $\tilde{y}$ defined by $d_y g(\tilde{y}) = 0$, where the function $g(y)$ is given in (35). The saddle point satisfies

$$-2\tilde{y} + (n-2) \frac{e^{-\tilde{y}^2/2}}{\int_{-\infty}^y dy' e^{-y'^2/2}} = 0.$$  \hspace{1cm} (A.1)

For large $n$ this can only be solved by $\tilde{y} \gg 1$, which implies that $\int_{-\infty}^\tilde{y} dy' e^{-y'^2/2} \approx \sqrt{2\pi}$ and reduces (A.1) to

$$\frac{\sqrt{8\pi \tilde{y}}}{n-2} = e^{-\tilde{y}^2/2}. \hspace{1cm} (A.2)$$

By taking the square on both sides of (A.2) one finds that the solution is given in terms of the Lambert-W function $W(z)$ \cite{37,38} as

$$\tilde{y} = \sqrt{W\left(\frac{(n-2)^2}{8\pi}\right)} \hspace{1cm} (A.3)$$

(recall that $W(z)$ is defined implicitly through $W(z)e^{W(z)} = z$). Using (A.2) the function $g$ and its second derivative at the saddle point take the form

$$g(\tilde{y}) \approx -\tilde{y}^2 - 2 \hspace{1cm} (A.4)$$

and

$$d_y^2 g(\tilde{y}) \approx -2(1 + \tilde{y}^2). \hspace{1cm} (A.5)$$

It follows that

$$I_n \approx \frac{n(n-1)}{4\pi} \sqrt{\frac{-2\pi}{d_y^2 g(\tilde{y}) e^{g(\tilde{y})}}} \approx \frac{n(n-1)}{4\pi e^2} \sqrt{\frac{\pi}{1 + \tilde{y}^2}} \approx \frac{2\sqrt{\pi}}{e^2} \tilde{y} \approx \frac{2\sqrt{\pi}}{e^2} \sqrt{\ln\left(\frac{n^2}{8\pi}\right)}.$$  \hspace{1cm} (A.6)

Using once more (A.2) to replace $e^{-\tilde{y}^2}$ we obtain

$$I_n \approx \frac{2\sqrt{\pi} n(n-1)}{e^2} \frac{\tilde{y}^2}{(n-2)^2} \sqrt{\frac{1}{1 + \tilde{y}^2}} \approx \frac{2\sqrt{\pi}}{e^2} \tilde{y} \approx \frac{2\sqrt{\pi}}{e^2} \sqrt{\ln\left(\frac{n^2}{8\pi}\right)}.$$  \hspace{1cm} (A.7)

for large $n$, where we have used the expansion \cite{37} $W(z) \approx \ln(z) - \ln(\ln(z))$ to evaluate (A.3). This expansion also shows that the leading corrections to the asymptotic expression (A.7) are of order $\ln(\ln(n^2/8\pi))/\ln(n^2/8\pi)$, which accounts for the relatively large deviations from the numerical results seen in figure 4.
Appendix B. Generalized Gaussian distributions

Here we consider probability densities of the form

$$f(y) = C_\beta e^{-|y|^\beta}$$

with $\beta > 0$ and $C_\beta = [2 \Gamma(1 + 1/\beta)]^{-1}$. We want to evaluate the integral (39) in the saddle point approximation. Introducing the function

$$g(y) = -2y^\beta + (n - 2) \ln \left( C_\beta \int_{-\infty}^{y} dy' e^{-|y'|^\beta} \right),$$

the saddle point equation $d_y g(\tilde{y}) = 0$ reads, for large $n$,

$$\frac{2\beta C_\beta}{n-2} \tilde{y}^{\beta-1} = e^{-\tilde{y}^\beta}.$$ (B.3)

The solution can again be expressed in terms of the Lambert-W function. Defining $\eta := 1 - 1/\beta$ we find

$$\tilde{y} \approx \left( \eta W\left( \eta^{-1} \left( \frac{n-2}{2\beta C_\beta} \right)^{\eta^{-1}} \right) \right)^{1/\beta}.$$ (B.4)

Note that this expression is valid both for $\beta > 1$ ($\eta > 0$) and for $\beta < 1$ ($\eta < 0$), but in the latter case the second real branch of $W(z)$ has to be used [37]. Using the asymptotics of $W(z)$ we obtain

$$\tilde{y} \approx (\ln n - \eta \ln(\eta^{-1} \ln n))^{1/\beta} \approx (\ln n)^{1/\beta}$$ (B.5)

for large $n$.

With the help of (B.3) the function $g$ and its second derivative at the saddle point become

$$g(\tilde{y}) \approx -2\tilde{y}^\beta - 2 \approx -2\tilde{y}^\beta$$ (B.6)

and

$$d_y^2 g(\tilde{y}) \approx -2\beta(\beta - 1)\tilde{y}^{\beta-2} - 2\beta^2\tilde{y}^{2\beta-2} \approx -2\beta^2\tilde{y}^{2\beta-2}$$ (B.7)

for large $\tilde{y}$. Thus, using (B.3), we see that $e^{g(\tilde{y})} \approx e^{-2\tilde{y}^\beta} \sim \tilde{y}^{2\beta-2}/n^2$, and therefore (ignoring all constant prefactors)

$$I_n \sim n^2 \sqrt{\frac{-2\pi}{d_y^2 g(\tilde{y})}} e^{g(\tilde{y})} \sim \tilde{y}^{-\beta-1} \sim (\ln n)^{1-1/\beta}.$$ (B.8)
Appendix C. Proof of an expansion

In this appendix, we will provide the details of the expansion of \( \int_{-\infty}^{\infty} dx f^2(x) \) into the terms on the right-hand side of (44). The starting point is the relation

\[
F^n(x) \int_{-\infty}^{x} dy f^2(y) = n \int_{-\infty}^{x} dy f(y) F^{n-1}(y) \int_{-\infty}^{y} dz f^2(z) + \int_{-\infty}^{x} dz f^2(z) F^n(z), \tag{C.1}
\]

which can be proved by applying integration by parts to the first term on the right-hand side. With the identities \( F^n(\infty) = 1 \) and \( F^n(-\infty) = 0 \), one obtains from (C.1)

\[
\frac{1}{n!} \int_{-\infty}^{\infty} dz f^2(z) = \frac{F^n(x)}{n!} \int_{-\infty}^{x} dz f^2(z) \bigg|_{x=-\infty}^{x=\infty}
= \frac{F^n(\infty)}{n!} \int_{-\infty}^{\infty} dz f^2(z) - \frac{F^n(-\infty)}{n!} \int_{-\infty}^{-\infty} dx f^2(x)
= \frac{1}{n!} \int_{-\infty}^{\infty} dz f^2(z) F^n(z) + \frac{1}{(n-1)!} \int_{-\infty}^{\infty} dz f(z) F^{n-1}(z) \int_{-\infty}^{z} dz' f'(z'). \tag{C.2}
\]

The first term of the sum is already identical to the first term in (44) for \( n = N - 2 \). Using (C.1) on the inner of the two integrals of the second term of the sum above, one obtains

\[
\int_{-\infty}^{\infty} dz f(z) F^{n-1}(z) \int_{-\infty}^{z} dz' f'(z') = \int_{-\infty}^{\infty} dz f(z) \int_{-\infty}^{z} dz' f'(z') F^{n-1}(z')
+ (n-1) \int_{-\infty}^{\infty} dz f(z) \int_{-\infty}^{z} dz' f'(z') F^{n-2}(z') \int_{-\infty}^{z} dz'' f^2(z'').
\]

Dividing by \( (n-1)! \) and putting this back into (C.2) with \( n = N - 2 \), one sees that now the first two terms of the sum agree with (44). By repeating this procedure on the terms that do not yet match and noting that finally \( F^n(z) = 1 \), one has expanded \( \int_{-\infty}^{\infty} dx f^2(x) \) into the RHS of (44), which concludes the proof of (43).

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