A Review of Symmetry Algebras of Quantum Matrix Models in the Large-$N$ Limit

C.-W. H. Lee and S. G. Rajeev

Department of Physics and Astronomy, P.O. Box 270171, University of Rochester, Rochester, New York 14627

June 8, 1999

Abstract

This is a review article in which we will introduce, in a unifying fashion and with more intermediate steps in some difficult calculations, two infinite-dimensional Lie algebras of quantum matrix models, one for the open string sector and one for the closed string sector. Physical observables of quantum matrix models in the large-$N$ limit can be expressed as elements of these Lie algebras. We will see that both algebras arise as quotient algebras of a larger Lie algebra. We will also discuss some properties of these Lie algebras not published elsewhere yet, and briefly review their relationship with well-known algebras like the Cuntz algebra, the Witt algebra and the Virasoro algebra. We will also review how Yang–Mills theory, various low energy effective models of string theory, quantum gravity, string-bit models, and quantum spin chain models can be formulated as quantum matrix models. Studying these algebras thus help us understand the common symmetry of these physical systems.
1 Introduction

As physicists, there are a number of phenomena in strong interaction, quantum gravity and condensed matter physics which we want to understand.

Quantum chromodynamics (QCD) is the widely accepted theory of strong interaction. It postulates that the basic entities participating strong interaction are quarks, antiquarks and gluons. In the high-energy regime, the theory displays asymptotic freedom. The coupling among these entities becomes so weak that we can use perturbative means to calculate experimentally measurable quantities like differential cross sections in particle reactions. Indeed, the excellent agreement between perturbative QCD and high energy particle phenomena form the experimental basis of the theory.

Nevertheless, strong interaction manifests itself not only in the high-energy regime but also in the low-energy one. Here, the strong coupling constant becomes large. Quarks, anti-quarks and gluons are permanently confined to form bound states called hadrons, like protons and neutrons. Hadrons can be observed in laboratories. We can measure their charges, spins, masses and other physical quantities. One challenging but important problem in physics is to understand the structures of hadrons within the framework of QCD; this serves as an experimental verification of QCD in the low-energy regime. Hadronic structure can be described by something called a structure function which tells us the (fractional) numbers of constituent quarks, antiquarks or gluons carrying a certain fraction of the total momentum of the hadron. The structure function of a proton has been measured carefully [1]. There has been no systematic theoretical attempt to explain the structure function until very recently [2]. In this work, the number of colors $N$ is taken to be infinitely large as an approximation. The resulting model can be treated as a classical mechanics [4, 5, 6]; i.e., the space of observables form a phase space of position and momentum, and the dynamics of a point on this phase space is governed by the Hamiltonian of this classical system and a Poisson bracket. This is because quantum fluctuations abate in the large-$N$ limit — the Green function of a product of color singlets is dominated by the product of the Green functions of these color singlets,

\footnote{See Ref.3 for a more pedagogical and updated account.}

\footnote{Ref.4 provides an excellent discussion for such a geometric formulation of classical mechanics.
and other terms are of subleading order. It is possible to derive a Poisson bracket for Yang–Mills theory in the large-$N$ limit. This Poisson bracket can be incorporated into a commutative algebra of dynamical variables to form something called a Poisson algebra.

As an initial attempt, only quarks and anti-quarks in the QCD model in Ref.[2] are dynamical. A more realistic model should have dynamical gluons in addition to quarks. Gluons carry a sizeable portion of the total momentum of a proton and are thus significant entities. One notable feature of a gluon field is that it is in the adjoint representation of the gauge group and carries two color indices. These can be treated as row and column indices of a matrix. This suggests gluon dynamics can be described by an abstract model of matrices.

Besides strong interaction phenomena, another fundamental question in physics is how one quantizes gravity. The most promising solution to this problem is superstring theory. Here we postulate that the basic dynamical entities are one-dimensional objects called strings. Quarks, gluons, gravitons and photons all arise as excitations of string states. If the theory consists of bosonic strings only, the ground states will be tachyons. To remove tachyons, fermions are introduced into the theory in such a way that there exists a symmetry between bosons and fermions. This boson–fermion symmetry is called supersymmetry. A superstring theory which is free of quantum anomaly must be ten-dimensional. Since we see only four macroscopic dimensions, the extra ones have to be compactified.

A partially non-perturbative treatment of superstring theory is through entities called D$p$-branes. They are extended objects spanning $p$ dimensions. Open strings stretch between D$p$-branes in the remaining dimensions, which are all compactified. There is a non-trivial background gauge field permeating the whole ten-dimensional space-time. The dynamics of the end points of open strings can be regarded as the dynamics of D$p$-branes themselves, each of which behaves like space-time of $p$ dimensions.

Different versions of string theory were put forward in the 80’s. Lately, evidence suggests that there exist duality relationship among these different theories.

---

3For an introductory discussion on this point, see Refs.[8] and [9].

4Supersymmetry could also be viewed as a symmetry which unifies, in a non-trivial manner, the space-time symmetry described by the Poincaré group, and the local gauge symmetry at each point of space-time. The symmetry between bosons and fermions then come as a corollary. See Refs.[12] or [13] for further details.
versions of string theory and so there is actually only one theory for strings. The most fundamental formulation of string theory is called M(atrix)-theory \[14\]. Currently, there is a widely-believed M-theory conjecture which states that in the infinite momentum frame (a frame in which the momentum of a physical entity in one dimension is very large), M-theory \[14\] can be described by the quantum mechanics of an infinite number $N$ of point-like D0-branes, the dynamics of which is in turn described by a matrix model with supersymmetry.

Besides the M-theory conjecture, there are a number of different ways of formulating the low-energy dynamics of superstring theory as supersymmetric matrix models \[11\].

We can use matrix models to describe condensed matter phenomena, too. One major approach condensed matter physicists use to understand high-$T_c$ superconductivity, quantum Hall effect and superfluidity is to mimic them by integrable models like the Hubbard model \[15\]. (This is a model for strongly correlated electron systems. Its Hamiltonian consists of some terms describing electron hopping from site to site, and a term which suppresses the tendency of two electrons to occupy the same site. We will write down the one-dimensional version of this model in a later section.) It turns out that the Hubbard model and many other integrable models can actually be formulated as matrix models with or without supersymmetry.

Thus matrix models provide us a unifying formalism for a vast variety of physical phenomena. Now, we would like to propose an algebraic approach to matrix models. The centerpiece of the classical mechanical model of QCD in Refs.\[6\] and \[2\] is a Poisson algebra. We can write the Hamiltonian as an element of this Poisson algebra, and can describe the dynamics of hadrons and do calculations through it. In string theory, the string is a one-dimensional object and so it sweeps out a two-dimensional surface called a worldsheet as time goes by. Worldsheet dynamics possesses a remarkable symmetry called conformal symmetry — the Lagrangian is invariant under an invertible mapping of worldsheet coordinates $x \rightarrow x'$ which leaves the worldsheet metric tensor $g_{\mu\nu}(x)$ invariant up to a scale, i.e., $g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x)$, where $\Lambda(x)$ is a non-zero function of worldsheet coordinates. This invertible mapping is called the conformal transformation. It turns out that the conformal charges, i.e., the conserved charges associated with conformal symmetry, in particular the Hamiltonian of bosonic string theory, can be written as elements of a Lie algebra called the Virasoro algebra \[16, 17\]. Through the Virasoro algebra,
we learn a lot about string theory like the mass spectrum and the S-matrix elements.

The relationship between conformal symmetry and the Virasoro algebra illustrates one powerful approach to physics — identify the symmetry of a physical system, express the symmetry in terms of an algebra, and use the properties of the algebra to work out the physical behavior of the system. Sometimes, the symmetry of the physical system is so perfect that it completely determines the key properties of the system.

The above argument suggests that we may get fruitful discovery in gluon dynamics, M-theory and superconductivity if there is a Lie algebra for a generic matrix model, and we are able to write its Hamiltonian in terms of this Lie algebra, which expresses a new symmetry in physics.

2 Formalism and Examples of Quantum Matrix Models in the Large-$N$ Limit

Examples of quantum matrix models fall into three broad classes: Yang–Mills theory, string theory and one-dimensional quantum spin systems.

It should be obvious why Yang–Mills theory can be expressed as a quantum matrix model. Identify the trivial vacuum state $|0\rangle$ with respect to the annihilation operators characterized by Eq. (1) below, i.e., the action of any of these annihilation operators on this vacuum state yields 0. A typical physical state is a linear combination of traces of products of creation operators acting on this vacuum state. (In the context of Yang–Mills theory, this is a color-singlet state of a single meson or glueball.) A typical observable is a linear combination of the trace of a product of creation and annihilation operators. Naively, we may think that the ordering of these operators is arbitrary, depending on the details of that physical model only. Actually, the planar property of the large-$N$ limit \cite{18} comes into play here, and only those traces in which all creation operators come to the left of all annihilation operators survive the large-$N$ limit. Moreover, the large-$N$ limit brings about a dramatic simplification of the action of this collective operator on a physical state — the unique trace of a product of creation operators acting on the vacuum will not be broken up into a product of more than one trace of products of creation operators acting on the vacuum, and it is not possible
for a product of more than one trace of products of creation operators to merge into just one trace of products of creation operators. In other words, an observable propagates a color-singlet state into a linear combination of color-singlet states [19].

This explains why we can mimic a string as a collection of string bits. In this context, a physical state is a string, open or closed, and the concluding statement of the previous paragraph implies that an observable replaces a segment of the string with another segment, without breaking it and without joining several strings together. This is a desirable simplifying feature for some string models. An important feature of this string-bit model is that wherever this segment lies on the string, it will be replaced by the new segment at exactly the same location. In other words, the action of an observable involves neighbouring string bits only, and is translationally invariant with respect to the string.

This brings us to quantum spin chain models. Since the two most important properties of a quantum spin chain model is that a spin interacts with neighbouring spins (not necessarily nearest neighbouring spins though) only, and the interaction is translationally invariant, we can write down a typical quantum spin chain model as a quantum matrix model in the large-$N$ limit, so long as the boundary conditions match.

We are going to present these ideas systematically in this section. First of all, we will present an abstract formalism of quantum matrix models. We will then use a Yang–Mills theory with only bosonic adjoint matter fields to serve as a generic example to show how Yang–Mills theories with different matter contents are expressed as quantum matrix models. Next, we will turn to string theory, and see the various ways quantum matrix models are related to it. Finally, we will present a formal way of transcribing a quantum spin chain system into a quantum matrix model in the large-$N$ limit, and give some examples from bosonic spin systems. Since many of them are known to be exactly integrable, and have even been exactly solved, we thus give here some examples of quantum matrix models, with or without supersymmetry, which are exactly integrable or even exactly solved. These models either involve cyclically symmetric spin chains satisfying the periodic boundary condition, or involve open spin chains satisfying open boundary conditions.
2.1 Formulation of Quantum Matrix Models in the Large-$N$ Limit

Let us give a unifying and more comprehensive review of the formulation of quantum matrix models in our previous papers [20, 21]. The multi-index notations introduced in Appendix A of Ref. [20] and that of Ref. [21] will be used extensively. The reader can understand these two appendices without reading the main texts of those two references.

Think of the row and column indices of those annihilation and creation operators as the row and column indices of an element in $U(N)$. We will call them color indices, as this is the case in Yang–Mills theory (Subsection 2.2). Let $a_{\mu_1}^{\mu_2}(k)$ be an annihilation operator of a boson in the adjoint representation for $1 \leq k \leq \Lambda$. (Here $\Lambda$ is a positive integer.) Let $\chi_{\mu}(\lambda)$ be an annihilation operator of a fermion in the fundamental representation for $1 \leq \lambda \leq \Lambda_F$. (Here $\Lambda_F$ is also a positive integer.) Lastly, let $\bar{\chi}_{\mu}(\lambda)$ be an annihilation operator of a fermion in the conjugate representation. We will call $k$ and $\lambda$ quantum states other than color. The corresponding creation operators are $a_{\mu_1}^{\dagger \mu_2}(k)$, $\chi_{\mu}^{\dagger}(\lambda)$ and $\bar{\chi}_{\mu}^{\dagger}(\lambda)$ with appropriate values for $k$ and $\lambda$. We will say that these operators create an adjoint parton, a fundamental parton and a conjugate parton, respectively. Most of these operators commute with one another except the following non-trivial cases:

\[
\left[a_{\mu_1}^{\mu_2}(k_1), a_{\mu_3}^{\mu_4}(k_2)\right] = \delta_{k_1 k_2} \delta_{\mu_3}^{\mu_1} \delta_{\mu_4}^{\mu_2}; \quad (1)
\]

\[
\left[\bar{\chi}_{\mu_1}(\lambda_1), \bar{\chi}_{\mu_2}^{\dagger}(\lambda_2)\right] = \delta_{\lambda_1 \lambda_2} \delta_{\mu_1}^{\mu_2}; \quad (2)
\]

and

\[
\left[\chi_{\mu_1}(\lambda_1), \chi_{\mu_2}^{\dagger}(\lambda_2)\right] = \delta_{\lambda_1 \lambda_2} \delta_{\mu_1}^{\mu_2}. \quad (3)
\]

There are two families of physical states (or color-invariant states in the context of gauge theory). One family consists of linear combinations of states of the form

\[
\bar{\phi}^{\lambda_1} \otimes s^K \otimes \phi^{\lambda_2} \equiv N^{-(c+1)/2} \bar{\chi}_{\mu_1}^{\dagger}(\lambda_1) \alpha_{\nu_1}^{\mu_2}(k_1) \alpha_{\nu_2}^{\mu_3}(k_2) \cdots \alpha_{\nu_{c+1}}^{\mu}(k_c) \chi_{\nu_c}^{\dagger}(\lambda_2) |0\rangle.
\]

Here we use the capital letter $K$ to denote the integer sequence $k_1, k_2, \ldots, k_c$. Unless otherwise specified, the summation convention applies to all repeated
color indices throughout this whole review. We will justify the use of the notation \( \otimes \) later. This term carries a factor of \( N \) raised to a certain power to make its norm finite in the large-\( N \) limit. (The proof that this is the correct power is similar to that given in Appendix A in which we will prove a closely related statement.) We will call these open singlet states. These are open string states in string-bit model. The other family consists of linear combinations of states of the form

\[
\Psi^K \equiv N^{-c/2} \alpha_{v_1}^{+v_2}(k_1) \alpha_{v_2}^{+v_3}(k_2) \cdots \alpha_{v_c}^{+v_1}(k_c) |0\rangle.
\]

These are closed singlet states. These are closed string states in string-bit model. This state is manifestly cyclic:

\[
\Psi^{K_1 K_2} = \Psi^{K_2 K_1}.
\]

Figs. 1(a) and (b) show a typical open singlet state, whereas Figs. 1(c) and (d) show a typical closed singlet state.

Now let us construct physical operators acting on these singlet states. It turns out that there are five families of them. The first family consists of finite linear combinations of operators of the form

\[
\Xi_{\lambda_2} \otimes f^I_1 \otimes \Xi_{\lambda_4} \equiv N^{-(a+b+2)/2} \chi_{\mu_1}^{\lambda_1} (\lambda_1) a_{\mu_1}^{\lambda_2} (i_1) \cdots a_{\mu_{a+1}}^{\lambda_{a+1}} (i_{a}) \chi_{\mu_{a+1}}^{\lambda_3} (\lambda_3) \cdot \\
\bar{\chi}_{\nu_1} (\lambda_2) a_{\nu_1}^{\lambda_4} (j_1) \cdots a_{\nu_{b+1}}^{\lambda_{b+1}} (j_{b}) \chi_{\nu_{b+1}}^{\lambda_5} (\lambda_5),
\]

where \( a = \#(I) \) and \( b = \#(J) \). We say that this is an operator of the first kind. Figs. 2(a) and (b) show the diagrammatic representations of this family of operators. In the planar large-\( N \) limit, a typical term of an operator of the first kind propagates an open singlet state to another open singlet state:

\[
\Xi_{\lambda_2} \otimes f^I_1 \otimes \Xi_{\lambda_4} \left( \bar{\phi}^{\lambda_5} \otimes s^K \otimes \phi^{\lambda_6} \right) = \delta_{\lambda_2}^{\lambda_5} \delta_{\lambda_4}^{\lambda_6} \bar{\phi}^{\lambda_1} \otimes s^I \otimes \phi^{\lambda_3}.
\]

However, it annihilates a closed singlet state:

\[
\Xi_{\lambda_2} \otimes f^I_1 \otimes \Xi_{\lambda_4} \left( \Psi^K \right) = 0.
\]
Figure 1: (a) A typical open singlet state in detail. It consists of the creation operators of a conjugate parton of the quantum state $\lambda_1$, four adjoint partons of the quantum states $k_1$, $k_2$, $k_3$ and $k_4$, and a fundamental parton of the quantum state $\lambda_2$. These creation operators are represented by a solid square at the top, four solid circles in the middle, and a solid square at the bottom, respectively. All these operators carry color indices. If two circles, or a circle and a square, are joined by a solid line, then the two corresponding operators share a color index, all possible values of which we sum over, regardless of how thick the solid line is. The arrow indicates the direction of the integer sequence $\hat{K}$. (b) A simplified diagrammatic representation of an open singlet state. Here conjugate and fundamental partons are ignored. We will ignore them in all future simplified diagrams. The adjoint partons in between are represented by the integer sequence $\tilde{K}$. There is no relationship between the length of the thick line and the number of adjoint partons it carries. (c) A typical closed singlet state $\Psi^K$ in detail. This state consists of a series of adjoint partons of the quantum states $k_1$, $k_2$, $\ldots$, and $k_5$, and is cyclically symmetric. (d) A simplified diagrammatic representation of the closed singlet state. We use the integer sequence $K$ to represent it. Cyclic symmetry is manifest. The size of this big circle does not tell us the number of indices the circle carries.
Figure 2: (a) A typical operator of the first kind. On the left, there is a solid square representing the creation operator of a conjugate parton with the quantum state $\lambda_1$. Below it are two solid circles representing the creation operators of adjoint partons with the quantum states $i_1$ and $i_2$. At the bottom is another solid circle representing the creation operator of a fundamental parton with the quantum state $\lambda_3$; on the right, there is a hollow square representing the annihilation operator of a conjugate parton with the quantum state $\lambda_2$. Below it are four hollow circles representing the annihilation of adjoint partons with the quantum states $j_1$, $j_2$, $j_3$ and $j_4$. At the bottom is another hollow square representing the annihilation operator of a fundamental parton with the quantum state $\lambda_4$. Again two circles, or a circle and a square, are joined by a solid or thin line whenever the corresponding operators share a common color index with all possible values. The arrows indicate the directions of the integer sequences $I$ and $\dot{J}$. Notice that $\dot{J}$ is put in reverse. (b) An operator of the first kind in brief. The sequence of creation operators is simplified to a thick line, and the sequence of annihilation operators is simplified to a thin line. $\dot{J}^*$ is the reverse of $\dot{J}$. The lengths of the two lines have no relationship with the numbers of creation or annihilation operators they carry. (c) The action of an operator of the first kind on the adjoint parton portion of an open singlet state (Eq. (8)). The dotted lines connect the line segments to be ‘annihilated’ together. This action produces an open singlet state as shown on the right of the arrow.
for a long time [13], and is ultimately related to the planarity of the large-$N$
limit [18]. We will provide a non-rigorous diagrammatic proof of this fact in
Appendix [3] where we work on the action of an operator of the second kind
(to be defined below) on an open singlet state. The reader can easily work
out the actions of operators of other kinds by the same reasoning.

Eq.(8) justifies the use of the direct product symbol $\otimes$. This equation
shows that $\bar{\Xi}_{\lambda_1}^\lambda_{\lambda_2}$ acts as a $\Lambda_F \times \Lambda_F$ matrix on
the vector $\bar{\phi}^\lambda_{\lambda_1}$ in a $\Lambda_F$-dimensional space, $f^I_j$ acts as
an infinite-dimensional matrix on the vector $s^K$ in an infinite-dimensional
space, and $\Xi^\lambda_{\lambda_2}$ acts as another $\Lambda_F \times \Lambda_F$ matrix on
the vector $\phi^\lambda_{\lambda_2}$ in another $\Lambda_F$-dimensional space. Thus an operator of the first
kind is an element of the direct product $gl(\Lambda_F) \otimes F \otimes gl(\Lambda_F)$.
Here, $gl(\Lambda_F)$ is the Lie algebra of the general linear group $GL(2\Lambda_F)$, and the infinite-
dimensional Lie algebra $F$ spanned by $f^I_j$ is isomorphic to the inductive
limit $gl^+\infty$ of the $gl(n)$’s as $n \to \infty$.

Operators of the second kind are finite linear combinations of operators
of the form

$$\bar{\Xi}_{\lambda_1}^\lambda_{\lambda_2} \otimes f^I_j \equiv N^{-\frac{a+b}{2}} \bar{X}^\mu_1 (\lambda_1) a^\mu_2 (i_1) a^\mu_3 (i_2) \cdots a^\mu_{a+b+1} (i_a) \\
\cdot a^\nu_{a+b+1} (j_{b-1}) a^\nu_{b} (j_{b-1}) \cdots a^\nu_{b} (j_1) \bar{X}^\mu_2 (\lambda_2).$$

(10)

A typical operator of this kind is depicted in Fig. 3(a) and (b). An operator
of the second kind acts on the end with a conjugate parton and propagates
an open singlet state to a linear combination of open singlet states:

$$\bar{\Xi}_{\lambda_1}^\lambda_{\lambda_2} \otimes f^I_j \left( \bar{\phi}^\lambda_{\lambda_3} \otimes s^K \otimes \phi^{\lambda_4} \right) = \delta_{\lambda_1}^{\lambda_3} \sum_{K_1, K_2 = K} \delta^K_j \bar{\phi}^\lambda_{\lambda_1} \otimes s^K \otimes \phi^{\lambda_4}.$$ 

(11)

On the R.H.S. of this equation, there will only be a finite number of non-zero
terms in the sum (bounded by the number of ways of splitting $K$ into
subsequence), so there is no problem of convergence. For example, if $\bar{K}$
is shorter than $J$, the right hand side will vanish. We can visualize this equation
in Fig. 3(c). This equation shows why we can treat this operator as a direct
product of the operators $\bar{\Xi}_{\lambda_1}^\lambda_{\lambda_2}$, $f^I_j$ and the identity operator. The first operator
acts as a $\Lambda_F \times \Lambda_F$ matrix on $\bar{\phi}^{\lambda_3}$’s, the second one acts on $s^K$, whereas the
last one acts trivially on $\phi^{\lambda_4}$’s. This operator annihilates a closed singlet state:

$$\bar{\Xi}_{\lambda_1}^\lambda_{\lambda_2} \otimes f^I_j \left( \psi^K \right) = 0.$$ 

(12)
Figure 3: (a) A typical operator of the second kind with five creation and four annihilation operators in detail. The solid and hollow squares represent creation and annihilation operators of conjugate partons, respectively. (b) An operator of the second kind in brief. (c) The action of an operator of the second kind on the adjoint parton portion of an open singlet state (Eq. (11)).
Figure 4: (a) A typical operator of the third kind with four creation and three annihilation operators in detail. This time the solid and hollow squares are creation and annihilation operators of fundamental partons. Notice the difference between the orientations of the arrows here and those in Fig. 3(a). This reflects the difference in the manner the color indices are contracted in Eqs. (10) and (13). (b) An operator of the third kind in brief. (c) The action of an operator of the third kind on the adjoint parton portion of an open singlet state (Eq. (14)).
Operators of the third kind are very similar to those of the second kind. They are linear combinations of operators of the form

\[
\rho^I_j \otimes \Xi^\lambda_1_{\lambda_2} \equiv N^{-(\hat{a} + \hat{b})/2} \chi^\mu_1 (i_\hat{a}) a^\mu_1 a^\mu_2 (i_{\hat{a}-1}) \cdots a^\mu_{\hat{b}-1} (i_{\hat{b}-1}) \cdot a^\mu_{\hat{b}} (i_{\hat{b}}) \chi^\nu (\lambda_2).
\]

(13)

A typical operator of this kind is depicted in Fig. 4(a) and (b). They act on the end with a fundamental parton instead of a conjugate parton as shown below:

\[
\rho^I_j \otimes \Xi^\lambda_1_{\lambda_2} (\bar{\phi}^\lambda \otimes \sigma^I) = \delta^\lambda_{\lambda_2} \sum_{K_1 K_2 = K} \bar{\phi}^\lambda \otimes \sigma^I_{K_1} \otimes \phi^\lambda.
\]

(14)

Fig. 4(c) shows this action diagrammatically. This equation shows that a term of an operator of the third kind is a direct product of the identity operator, the operator \(\rho^I_j\) and the operator \(\Xi^\lambda_1_{\lambda_2}\). Like the previous two kinds of operators, an operator of the third kind also annihilates a closed singlet state:

\[
\rho^I_j \otimes \Xi^\lambda_1_{\lambda_2} (\Psi^K) = 0.
\]

(15)

Operators of the fourth kind are the most non-trivial among all physical operators. They are finite linear combinations of operators of the form

\[
\gamma^I_J \equiv N^{-(a + b - 2)/2} a^\mu_1 (i_1) a^\mu_2 (i_2) \cdots a^\mu_{\hat{b}} (i_{\hat{b}}) \cdot a^\mu_{\hat{b}-1} (j_{\hat{b}-1}) a^\mu_{\hat{b}} (j_{\hat{b}}) \chi^\nu \chi_{\nu} (\lambda_2).
\]

(16)

We may write \(\gamma^I_J\) as \(\sigma^I_J\) in the ensuing paragraphs. Unlike the operators of the first three kinds, both \(I\) and \(J\) in an operator of the fourth kind must be non-empty sequences. Figs. 5(a) and (b) show the diagrammatic representations of such an operator. In the planar large-\(N\) limit, it replaces some sequences of adjoint partons on an open singlet state with some other sequences, producing a linear combination of this kind of states:

\[
\gamma^I_J (\bar{\phi}^\lambda \otimes \sigma^I \otimes \phi^\lambda) = \bar{\phi}^\lambda \otimes \left( \sum_{K_1 K_2 K_3 = K} \delta^K_{\gamma^I_J} \sigma^I_{K_1} \otimes \phi^\lambda \right).
\]

(17)
This action is depicted in Fig. 5(c). When it acts on a closed singlet state, it also replaces some sequences of adjoint partons with others:

\[
\gamma^I_J \Psi^K = \delta^K_J \Psi^I + \sum_{K_1 K_2 = K} \delta^K_{K_2} \Psi^{K_1} + \sum_{K_1 K_2 = K} \delta^K_{K_1} \Psi^{K_2} \\
+ \sum_{K_1 K_2 K_3 = K} \delta^K_{J} \Psi^{IK_1} + \sum_{K_1 K_2 = K} (-1)^{\epsilon(K_1) \epsilon(K_2)} \delta^K_{K_2} \Psi^{IK_1} \\
+ \sum_{J_1 J_2 = J} \sum_{K_1 K_2 K_3 = K} \delta^K_{J_1} \delta^K_{J_2} \Psi^{IK_2};
\]

(18)

This action is depicted in Fig. 5(d).

Operators of the fifth kind form the last family of physical operators. They are finite linear combinations of operators of the form

\[
\tilde{f}^I_J \equiv N^{-(a+b)/2} a_{\mu_1}^{i_1} a_{\mu_2}^{i_2} \cdots a_{\mu_a}^{i_a} \cdot a_{\nu_1}^{j_1} a_{\nu_2}^{j_2} \cdots a_{\nu_b}^{j_b}.
\]

(19)

A typical operator of this kind is shown in Figs. 6(a) and (b). This operator possesses the following interesting cyclic property:

\[
\tilde{f}^I_{J_2 J_1} = \tilde{f}^I_{J_1 J_2}, \quad \text{and}
\]

\[
\tilde{f}^I_{J_2 J_1} = \tilde{f}^I_{J_1 J_2}.
\]

(20)

It annihilates an open singlet state:

\[
\tilde{f}^I_J \left( \phi^{\lambda_1} \otimes s^K \otimes \phi^{\lambda_2} \right) = 0.
\]

(21)

It may replace a closed singlet state with another one, though:

\[
\tilde{f}^I_J \Psi^K = \delta^K_J \Psi^I + \sum_{J_1 J_2 = J} \delta^K_{J_2 J_1} \Psi^I.
\]

(22)

This action is depicted in Fig. 5(c).

We will say more about the mathematical properties of these physical operators in later sections. In the remainder of this section, we will see how these physical states and operators show up in actual physical systems, starting with Yang–Mills theory.
Figure 5: (a) A typical operator of the fourth kind with two creation and two annihilation operators in detail. The conjugate and fundamental partons in an open singlet state are unaffected by the action of this operator. (b) An operator of the fourth kind in brief. (c) The action of an operator of the fourth kind on an open singlet state (Eq. (17)). (d) The action of an operator of the fourth kind on a closed singlet state (Eq. (18)).
Figure 6: (a) A typical operator of the fifth kind with three creation and two annihilation operators in detail. The upper circle, labelled by $I$, represents creation operators, whereas the lower circle, labelled by $J$ represents annihilation operators. Note that the sequence $J$ is put in the clockwise instead of the counterclockwise direction. (b) An operator of the fifth kind in brief. $J^*$ is the reverse of $J$. (c) The action of an operator of the fifth kind on a closed singlet state (Eq. (22)).
2.2 Examples: Yang–Mills Theory

The simplest example for this class of models is 2-dimensional Yang–Mills theory with adjoint matter fields. For the sake of pedagogy, we will give a detailed exposition below. The following presentation is closely parallel to Ref.[22]. As will be shown below, there is no dynamics for the gauge bosons in 2 dimensions. The quantum matrices in the above abstract formalism is realized as canonically quantized adjoint matter fields. Different matrices carry different momenta. The linear momentum and Hamiltonian of the model can be expressed as linear combinations of the five families of operators introduced above.

We need a number of preliminary definitions. Let $g$ be the Yang–Mills coupling constant, $\alpha$ and $\beta$ ordinary space-time indices, $A_\alpha$ a Yang–Mills potential and $\phi$ a scalar field in the adjoint representation of the gauge group $U(N)$, and $m$ the mass of this scalar field. Both $\phi$ and $A_\alpha$ are $N \times N$ Hermitian matrix fields. If we treat the Yang–Mills potential as the Lie-algebraically valued connection form and $\phi$ a fiber on the base manifold of space-time in the language of differential geometry\footnote{An authoritative text on differential geometry which explains the notion of a fiber bundle is Ref.[23]. However, readers who are not interested in differential geometry may well take the covariant derivative to be given shortly for granted.}, then the covariant derivative is

$$D_\alpha \phi = \partial_\alpha \phi + i [A_\alpha, \phi].$$

The Minkowski space action of this model is

$$S = \int d^2 x \text{Tr} \left[ \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4 g^2} F_{\alpha\beta} F^{\alpha\beta} \right]. \quad (23)$$

Introduce the light-cone coordinates

$$x_+ = \frac{1}{\sqrt{2}} (x_0 + x_1)$$

and

$$x_- = \frac{1}{\sqrt{2}} (x_0 - x_1).$$

Take $x^+$ as the time variable. Choose the light-cone gauge

$$A_\alpha = 0.$$
The action is now simplified to

\[ S = \int dx^+ dx^- \text{Tr} \left[ \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2 g^2} (\partial_- A_+)^2 + A_+ J^+ \right], \]

(24)

where

\[ J^{\mu_1}_{\mu_2} = i [\phi, \partial_- \phi]_{\mu_1}^{\mu_2} \]

is the longitudinal momentum current. Note that \( A_+ \) has no time dependence in Eq. (24) and so the gauge field is not dynamical at all in the light-cone gauge. Let us split \( A_+ \) into its zero mode \( A_{+,0} \) (this is a mode which is independent of \( x_- \)) and non-zero mode \( A_{+,n} \), i.e.,

\[ A_+ = A_{+,0} + A_{+,n}. \]

The Lagrange constraints for them are

\[ \int dx^- J^+ = 0, \text{ and} \]

(25)

\[ \partial_-^2 A_{+,n} - g^2 J^+ = 0, \]

(26)

respectively. Now we can use these two equations to eliminate \( A_+ \) in Eq. (24) and get

\[ S = \int dx^+ dx^- \text{Tr} \left( \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2 g^2} J^+ \right). \]

(27)

The light-cone momentum and Hamiltonian are

\[ P^\pm \equiv \int dx^- T^\pm. \]

It follows from Eq. (27) that their expressions in this model are

\[ P^+ = \int dx^- \text{Tr} (\partial_- \phi)^2, \text{ and} \]

(28)

\[ P^- = \int dx^- \text{Tr} \left( \frac{1}{2} m^2 \phi^2 - \frac{1}{2} g^2 J^+ \right). \]

(29)

We now quantize the system. Eq. (27) implies that the canonical quantization condition is

\[ \left[ \phi^{\mu_1}_{\mu_2}(x^-), \partial_- \phi^{\mu_3}_{\mu_4}(\bar{x}^-) \right] = \frac{i}{2} \delta^{\mu_1}_{\mu_4} \delta^{\mu_3}_{\mu_2} \delta(x^- - \bar{x}^-). \]

(30)
Then a convenient field decomposition is
\[
\phi_{\mu_1}(x^+ = 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[ a_{\mu_1}^+(k^+) e^{-ik^+x^-} + a_{\mu_1}^+(k^+) e^{ik^+x^-} \right].
\] (31)

If the annihilation and creation operators here satisfy the canonical commutation relation Eq. (1), except that the Kronecker delta function \( \delta_{k_1 k_2} \) in Eq. (1) is replaced with the Dirac delta function \( \delta(k_1 - k_2) \) here, then the adjoint matter field will satisfy Eq. (30). A state can be built out of a series of creation operators acting on the trivial vacuum, and it takes the form
\[
a_{\nu_1}^+ (k_1^+) a_{\nu_2}^+ (k_2^+) \cdots a_{\nu_c}^+ (k_c^+) |0\rangle.
\]

However, in general this state does not satisfy the Lagrangian constraint Eq. (29). Only those of the form given in Eq. (30) satisfy this constraint. We thus obtain a model whose physical states are the physical states introduced in the previous subsection.

Using Eqs. (28) and (30), we can quantize the light-cone momentum and get
\[
P^+ = \int_0^\infty dk k \gamma^k_k,
\] (32)
where \( \gamma^k_k \) is defined in Eq. (16). We have dropped the superscript + in \( k \) to simplify the notation. This is a concrete example in which a physical observable is expressed as a linear combination of the operators introduced in the previous section. (Here \( k \) can take on an infinite number of values, and the regulator \( \Lambda \to \infty \). Nonetheless, taking the regulator to infinity has an influence on technical details only, since we are talking about a field theory without divergences in this limit.) To obtain a similar formula for the light-cone Hamiltonian, we need to express, in momentum space, the longitudinal momentum current as a quantum operator first. Define the longitudinal momentum current in momentum space to be
\[
\tilde{J}^+ (k) = \frac{1}{\sqrt{2\pi}} \int dx^- J^+(x^-) e^{-ik^+x^-}.
\] (33)

Then, for \( q > 0 \),
\[
\tilde{J}^+_{\mu_2} (\pm q) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty dp \frac{2p + q}{\sqrt{p(p + q)}} \left[ a_{\nu_1}^\dagger (p) a_{\mu_1}^\dagger (p + q) \right] \]

20
\[-a_{\nu}^{\mu_1}(p)a_{\mu_2}^{\nu}(p + q)\]
\[+ \int_0^q dp \frac{q - 2p}{\sqrt{p(q - p)}}a_{\mu_2}^{\nu}(p)a_{\nu}^{\mu_1}(q - p),\]

(34)

and

\[\tilde{J}_{\mu_2}^{+\mu_1}(q) = [\tilde{J}_{\mu_2}^{+\mu_1}(-q)]^\dagger.\]

(35)

We can now use Eqs. (29), (30), (33), (34) and (35) to obtain

\[P^- = \frac{1}{2}m^2 \int_0^\infty \frac{dk}{k} k g^k_k + \frac{g^2N}{4\pi} \int_0^\infty \frac{dk}{k} C g^k_k \]
\[+ \frac{g^2N}{8\pi} \int_0^\infty \frac{dk_1dk_2dk_3dk_4}{\sqrt{k_1k_2k_3k_4}} \{ A\delta(k_1 + k_2 - k_3 - k_4)g^{k_4k_3}_{k_1k_2} \]
\[+ B\delta(k_1 - k_2 - k_3 - k_4)g^{k_2k_3k_4}_{k_1} + g^{k_1k_2k_3k_4}_{k_4}\},\]

(36)

where

\[A = \frac{(k_2 - k_1)(k_4 - k_3)}{(k_1 + k_2)^2} - \frac{(k_3 + k_1)(k_4 + k_2)}{(k_4 - k_2)^2};\]
\[B = \frac{(k_2 + k_1)(k_4 - k_3)}{(k_4 + k_3)^2} - \frac{(k_4 + k_1)(k_3 - k_2)}{(k_3 + k_2)^2};\]

and

\[C = \int_0^k dp \frac{(k + p)^2}{p(k - p)^2}.\]

Eq. (36) shows that the light-cone Hamiltonian is a linear combination of operators of the fourth kind.

The above formalism can be easily generalized to different cases. Studying glueballs in quantum chromodynamics [24] requires the incorporation of more than one adjoint matter fields. Take the large-\(N\) limit as an approximation for pure Yang–Mills theory. If we assume that the gluon field, i.e., the gauge field, is not dependent on the transverse dimensions (in other words, we take dimensional reduction as a further approximation), the gluon field will precisely be the adjoint matter fields. The number of adjoint matter fields is the same as the number of transverse dimensions in the system.

To study mesons, we need to include fundamental and conjugate matter fields [25]. They will serve as quark and antiquark fields, respectively. The reader can find the expressions for the light-cone momentum and Hamiltonian of a dimensionally reduced model of mesons in terms of the first four kinds of operators in Ref. [20].

21
2.3 Examples: Quantum Gravity and String Theory

There are several ways to study quantum gravity and its most promising candidate, string theory, in terms of quantum matrix models. One idea is based on the crucial observation that the dual of a Feynman diagram can be taken as a triangulation of a planar surface, which serves as a lattice approximation to a geometrical surface. The partition function for two-dimensional quantum gravity can then be approximated by a matrix model in the large-$N$ limit [26, 27]. Whether the matrix model is classical or quantum, and the exact form of its action depends on what the conformal matter field coupled to quantum gravity is. This, in turn, is equivalent to a string theory with certain dimensions [28]. This approach has the virtue that it reveals some non-perturbative behavior of string theory. For example, a 3-dimensional non-critical string theory is equivalent to a model of two-dimensional quantum gravity coupled with conformal matter with the conformal charge $c = 2$. Then this model of quantum gravity is further mapped to a one-matrix model in the large-$N$ limit with $\phi^3$ interaction, i.e., the action of this matrix model is

$$S = \int d^2 x \text{Tr} \left( \frac{1}{2} \partial_\alpha \Phi \partial^\alpha \Phi + \frac{1}{2} \mu \Phi^2 - \frac{1}{3\sqrt{N}} \lambda \Phi^3 \right),$$  \hspace{1cm} (37)$$

where $\Phi$ is an $N \times N$ matrix, and $\mu$ and $\lambda$ are constants.

The second way is to consider the low-energy dynamics of string theory [11]. For example, if we exclude all Feynman diagrams with more than one loop in a bosonic open string theory with $N$ Chan–Paton factors, we will retain only a tachyonic matrix field $\varphi$ with three-tachyon coupling, and Yang–Mills gauge bosons minimally coupled to tachyons. The action is

$$S = \frac{1}{g^2} \int d^{26} x \left[ -\frac{1}{2} \text{Tr}(D_\alpha \varphi D^\alpha \varphi) + \frac{1}{2\alpha'} \text{Tr}(\varphi^2) + \frac{1}{3} \sqrt{\frac{2}{\alpha'}} \text{Tr}(\varphi^3) \right. \right. \left. \left. - \frac{1}{4} \text{Tr}(F_{\alpha\beta} F^{\alpha\beta}) \right],$$ \hspace{1cm} (38)$$

where $\alpha'$ is the Regge slope, and $g$ is the gauge boson coupling constant.

The third way is to approximate a string by a collection of string bits [29]. A closed singlet state then represents a closed string, and an open singlet state represents an open string.
The fourth way is to consider the low-energy dynamics of Dp-branes \[11\]. Let \(\xi^1, \xi^2, \ldots, \text{and } \xi^p\) be the coordinates inside the \(p\)-brane. The action turns out to be another matrix model:

\[
S = -T_p \int d^{p+1}\xi e^{-\Phi} \left[ -\det(G_{ab} + B_{ab} + 2\pi\alpha'F_{ab}) \right]^{1/2},
\]

where \(T_p\) is the brane tension,

\[
G_{ab}(\xi) = \frac{\partial X^\alpha}{\partial \xi^a} \frac{\partial X^\beta}{\partial \xi^b} G_{\alpha\beta}(X(\xi))
\]

and

\[
B_{ab}(\xi) = \frac{\partial X^\alpha}{\partial \xi^a} \frac{\partial X^\beta}{\partial \xi^b} B_{\alpha\beta}(X(\xi))
\]

are the induced metric and antisymmetric tensor on the brane, and \(F_{ab}\) is the background Yang–Mills field. If supersymmetry is incorporated into the theory, and if we further assume that space–time is essentially flat, the action shown in Eq. (39) will be turned into a supersymmetric Yang–Mills theory.

Closely related to the D-brane action is the M-theory conjecture and the matrix string corollary. According to this corollary, the action of type-IIA string theory in the infinite momentum frame is given by the matrix form of the Green–Schwarz action \[30\].

The presence of a multitude of methods to transcribe string theory to matrix models shows their intimate relationship.

### 2.4 Examples: Quantum Spin Chains

Quantum spin chain systems form another major class of models which can be expressed as quantum matrix models. There are various reasons to study quantum spin chain systems. The essence of many condensed matter and high energy phenomena are captured by them. For example, we can use the Ising model to study ferromagnetism, lattice gas, order-disorder phase transition \[31\], lattice quantum gravity and even string theory \[32\]. In addition, the integrability of these models provides rich insight in pure mathematics like quantum groups and knot theory \[10\]. The Bethe ansatz \[33\] and the closely related Yang–Baxter equations \[34, 35\] are well known to be powerful tools in studying and exactly solving many of these spin chain systems.
These tools provide us a way to determine the integrability of and solve the associated quantum matrix models. A better knowledge in quantum matrix models should thus help us understand exactly integrable models and condensed matter systems better, and vice versa.

There is a one-to-one correspondence between spin chain systems whose collective spin chain states are cyclically symmetric and satisfy the periodic boundary condition, spin chain systems satisfying open boundary conditions, and quantum matrix systems in the large-$N$ limit. (A connection between spin systems and matrix models was observed previously in Ref. [36].) We will illustrate the idea using the simplest spin chain model, the one-dimensional quantum Ising model satisfying the periodic boundary condition $[37, 38, 39]$. A typical collective state of the whole spin chain can be characterized by a sequence of $c$ 2-dimensional column vectors, where $c$ is the number of sites in the spin chain. Spin-up and spin-down states at the $p$-th site are characterized by the $p$-th column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. A cyclically symmetric spin chain state can be constructed by summing over all cyclic permutations of this sequence of $c$ column vectors. The Hamiltonian $H_{\text{Ising}}^{\text{spin}}$ of this spin chain model is

$$H_{\text{Ising}}^{\text{spin}}(\tau, \lambda) = \sum_{p=1}^{c} \tau_p^z + \lambda \sum_{p=1}^{c} \tau_p^x \tau_{p+1}^x.$$ 

In this equation, $\tau_p^x$, $\tau_p^y$, and $\tau_p^z$ are Pauli matrices at the $p$-th site. They act on the $p$-th column vector. $\lambda$ is a real constant. Two Pauli matrices at different sites (i.e., with different subscripts) commute with each other. Moreover, they satisfy the periodic boundary condition $\tau_{c+1}^{x,y,z} = \tau_1^{x,y,z}$.

We can paraphrase the model in terms of the states and operators of quantum matrix models as follows. A closed singlet state corresponds to a cyclically symmetric spin chain state. We allow the existence of only one. Each adjoint parton corresponds to a spin. There are 2 possible quantum states (i.e., $\Lambda = 2$) other than color for an adjoint parton. $a_1^\dagger(1)$ (color indices are suppressed) corresponds to a spin-up state, whereas $a_1^\dagger(2)$ corresponds to a spin-down state. Therefore a typical collective state of a cyclically
symmetric spin chain can be denoted by $\Psi^K$, where $K$ is an integer sequence of $c$ numbers, each of which is either 1 or 2.

Let us turn to the Hamiltonian in Eq.(12). Consider the operator $\tau^x_p \tau^{x}_{p+1}$. It turns
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{p+1}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{p+1},
\]
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{p+1} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{p+1},
\]
where the subscripts $p$ and $p+1$ tell us that they are the $p$-th and $(p+1)$-th column vectors, respectively, into
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{p+1}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{p+1},
\]
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{p+1} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{p+1},
\]
respectively. Let us rewrite this in the language of matrix model. let $k_p$ and $k_{p+1}$ be the $p$-th and $(p+1)$-th numbers in $K$. If $(k_p, k_{p+1}) = (1, 1), (1, 2), (2, 1)$ or $(2, 2)$, then they will be replaced with $(2, 2), (2, 1), (1, 2)$ or $(1, 1)$, respectively. This implies that we can identify $\sum_{p=1}^c \tau^x_p \tau^{x}_{p+1}$ in the spin chain model to be $\gamma^1_{11} + \gamma^2_{12} + \gamma^1_{21} + \gamma^2_{22}$ in the matrix model. Likewise, $\sum_{p=1}^c \tau^z_p$ can be identified with $\gamma^1_1 - \gamma^2_2$. As a result, the Hamiltonian $H_{\text{matrix}}$ of this integrable matrix model in the large-$N$ limit is
\[
H_{\text{matrix}} = H_0 + \lambda V
\]
where
\[
H_0 = \gamma^1_1 - \gamma^2_2; \quad \text{and} \quad V = \left[ \gamma^2_{11} + \gamma^2_{12} + \gamma^1_{21} + \gamma^2_{22} \right].
\]
We can further rewrite this formula in terms of the creation and annihilation operators $a^\dagger$ and $a$:
\[
H_{\text{Ising}} = \text{Tr} \left[ a^\dagger(1)a(1) - a^\dagger(2)a(2) \right] + \frac{\lambda}{N} \text{Tr} \left[ a^\dagger(2)a^\dagger(2)a(1)a(1) + a^\dagger(2)a^\dagger(1)a(2)a(1) + a^\dagger(1)a^\dagger(2)a(1)a(2) + a^\dagger(1)a^\dagger(1)a(2)a(2) \right].
\]
Let us underscore that the matrix model defined by the Hamiltonian $H_{\text{king}}$ in Eqs. (44) or (45) is an integrable matrix model in the large-$N$ limit.

Notice that the Hamiltonian of the Ising spin chain model possesses translationally invariant terms which describe nearest neighbour interactions only. In general, any spin chain model which possesses the same kind of terms can be transcribed to a quantum matrix model in the large-$N$ limit by the equation

$$\gamma_J^I = \sum_{p=1}^{c} X_{p}^{i_1 j_1} X_{p+1}^{i_2 j_2} \cdots X_{p+a-1}^{i_a j_a}$$

where $X_{p}^{ij}$ is the *Hubbard operator* defined by its action: if the spin chain state at the $p$-th state is $j$, then it is changed to $i$; otherwise, the operator annihilates the spin chain state and we get 0. The reader can take a look at Refs. [40] or [21] for some more examples of exactly integrable quantum matrix models obtained from exactly integrable spin chain models satisfying the periodic boundary condition.

There is also a one-to-one correspondence between open spin chains and quantum matrix models in the large-$N$ limit. The relation between a $\gamma$ and Hubbard operators is almost the same as that in Eq. (46), except that the summation index $p$ runs from 1 to $c - a + 1$ this time. It can be easily seen that the corresponding Hubbard operators for an $l_I^I$ and an $r_I^I$ are

$$l_I^I = X_1^{i_1 j_1} X_2^{i_2 j_2} \cdots X_a^{i_a j_a}$$

and

$$r_I^I = X_{n-a+1}^{i_1 j_1} X_{n-a+2}^{i_2 j_2} \cdots X_n^{i_a j_a} ,$$

respectively.

Let us consider an integrable spin-$\frac{1}{2}$ XXZ model [41] to see how the transcription is put into practice. The Hamiltonian $H_{\text{spin}}^{\text{spin}}$ of this spin chain model is

$$H_{\text{spin}}^{\text{spin}} = \frac{1}{2} \sin \gamma \left[ \sum_{p=1}^{c-1} (\tau_p^x \tau_{p+1}^x + \tau_p^y \tau_{p+1}^y + \cos \gamma \tau_p^z \tau_{p+1}^z) + i \sin \gamma (\coth \xi - \tau_1^z + \coth \xi + \tau_c^z) \right] ,$$

respectively.
where $\gamma \in (0, \pi)$ and both $\xi_-$ and $\xi_+$ are arbitrary constants. The Hamiltonian $H_{\text{matrix}}^{\text{XXZ}}$ of the associated integrable matrix model is

$$H_{\text{matrix}}^{\text{XXZ}} = \frac{1}{2\sin \gamma} \left\{ 2(\sigma_{12}^{11} + \sigma_{21}^{12} + \cos \gamma(\sigma_{11}^{11} - \sigma_{12}^{12} - \sigma_{21}^{21} + \sigma_{22}^{22})) + i \sin \gamma \left[ \coth \xi_-(l_1^1 - l_2^2) + \coth \xi_+(r_1^1 - r_2^2) \right] \right\}. \quad (50)$$

Other examples of exactly integrable spin chain models satisfying open boundary conditions and the corresponding quantum matrix models in the large-$N$ limit can be found in Ref. [20].

We also have a transcription rule between fermionic spin chain models and integrable supersymmetric quantum matrix models [42].

3 A Lie Algebra for Open Strings

As we have just been seen in the previous chapter, quantum matrix models in the large-$N$ limit are of very wide applicability in physics. This leads us naturally to the following question: can we do something more in addition to writing down the physical observables of these models as linear combinations of the operators introduced in Subsection 2.1? One major milestone in understanding the physics of these models will be to obtain the spectra of some of these observables, or at least to obtain their key qualitative properties.

A number of systematic analytic approaches have been developed to solve classical matrix models in the large-$N$ limit, another class of models widely used in physics [43, 44, 45]. On the other hand, the method employed for studying the spectra of the Hamiltonians of various versions of Yang–Mills theory in recent years is numerical in nature (e.g., Ref. [22]). A lot of effort has been spent to develop more accurate and sophisticated numerical analysis, and we have a fairly large body of knowledge in the numerical solutions of these quantum matrix models. It should be nice to see how much this approach can tell us about the various interesting physical systems. Nonetheless, it should also be nice if we have several other approaches to the same problem; some results which may be difficult to obtain in one approach may be pretty trivial in another. A combination of the knowledge gained from different approaches can lead us to much deeper understandings of the physics.
In Section 1, we indicated that understanding the symmetry of a physical system can lead to the elucidation of important features of it. This provides us another approach: what is the symmetry of quantum matrix models? A continuous symmetry in physics is usually expressed as a Lie group, or its infinitesimal form, a Lie algebra. Is there a Lie algebra for quantum matrix models? Or more specifically speaking, are the physical observables of quantum matrix models elements of a Lie algebra?

Yes, such a Lie algebra exists.

Indeed, we introduced two Lie algebras, the grand (open string) algebra and the algebra $\mathcal{C}_{\Lambda}$, for open and closed bosonic strings in Refs. [20] and [21], respectively. The fact that we can give a unifying account of all quantum observables for open and closed singlet states suggests that there may be a way to treat these two Lie algebras on an equal footing. In fact, they are quotient algebras of a bigger algebra which in turn is a subalgebra of a precursor algebra. Some familiarity of the grand open string algebra and $\mathcal{C}_{\Lambda}$, however, is necessary to see how this works. We will therefore review a hierarchy of subalgebras and quotient algebras, from the simplest to the most non-trivial, of a simplified version of the grand open string algebra in which there are only one degree of freedom for the fermion in the fundamental representation, and one degree of freedom for the anti-fermion in the conjugate representation. We will also derive some results about the structure of these algebras not published elsewhere.

Another point worthy mentioning here is that these algebras are related intimately with the Cuntz algebra [46, 47] and the Witt algebra [48, 49, 50], the central extension of which is the Virasoro algebra [16]. Further understandings of these well-known algebras can help us a lot in explaining the physics of quantum matrix models.

After acquiring some expertise in this kind of Lie algebraic arguments, we will study the full algebra, the grand string algebra, for all five kinds of operators, and some more subalgebras of it in subsequent sections.

3.1 Operators of the First Three Kinds

We will review what we have learnt so far about the Lie algebras for the operators of the first three kinds in Ref. [20] which will be useful in constructing the grand string algebra later, and state some other properties of them which have not been published yet.
The adjoint matter portion of the operators of the first kind form a Lie algebra with the following Lie bracket

\[
[f^I_J, f^K_L] = \delta^K_J f^I_L - \delta^I_J f^K_L. 
\] (51)

This Lie algebra \( F_\Lambda \) can be derived from the commutator of the associative algebra of these operators. \( F_\Lambda \) is isomorphic to \( gl_{+\infty} \), the Lie algebra, with the usual bracket, of all complex matrices \((a_{ij})_{i,j} \in \mathbb{Z}_+\) such that the number of nonzero \( a_{ij} \) is finite. The isomorphism is given by a one-one correspondence between the multi-indices \( I \) and the set of natural numbers \( \mathbb{Z}_+ r \), for example by a lexicographic ordering \(^6\).

Operators of the second kind form another Lie algebra. This is again the commutator of an associative algebra with the multiplication law

\[
l^I_J l^K_L = \delta^K_J l^I_L + \sum_{J_1,J_2=J} \delta^K_{J_1} l^{I}_{J_2} + \sum_{K_1,K_2=K} \delta_{J_1} l^{I}_{K_2}.
\] (52)

The associativity is demonstrated in Appendix [B]. The resulting Lie bracket is

\[
[l^I_J, l^K_I] = \delta^K_J l^I_L + \sum_{J_1,J_2=J} \delta^K_{J_1} l^{I}_{J_2} + \sum_{K_1,K_2=K} \delta_{J_1} l^{I}_{K_2} - \delta^K_{L_1} l^{I}_{L_2} - \sum_{I_1,I_2=I} \delta_{I_1} l^{I}_{I_2}.
\] (53)

The first three terms on the R.H.S. of Eq.(53) are diagrammatically represented in Fig.7. We will call the Lie algebra defined by Eq.(53) the leftix algebra or \( L_\Lambda \). (We justify this name as an abbreviation of ‘left multi-matrix algebra’.)

---

\(^6\)This is a total ordering \( \triangleright \) among integer sequences such that \( \hat{I} \triangleright \hat{J} \) for the sequences \( \hat{I} \) and \( \hat{J} \) if either

1. \( \#(\hat{I}) > \#(\hat{J}) \); or
2. \( \#(\hat{I}) = \#(\hat{J}) = a \neq 0 \), and there exists an integer \( r \leq a \) such that \( i_1 = j_1, i_2 = j_2, \ldots, i_{r-1} = j_{r-1} \) and \( i_r > j_r \).

Explicitly, the total ordering can be presented as

\[
\emptyset < 1 < 2 < \cdots < \Lambda < 11 < 12 < \cdots < 1\Lambda \\
< 21 < \cdots < \Lambda 1 < \cdots < \Lambda \Lambda < 111 \cdots.
\]
To understand the properties of this Lie algebra better, let us determine its Cartan subalgebra and the root spaces with respect to this Cartan subalgebra. We will follow the definition of the Cartan subalgebra given by Humphreys \cite{51}. To understand this definition, we need a number of preliminary notions. Let $L$ be a Lie algebra. Define the descending central series by $L_0 = L$, $L_1 = [L, L]$, $L_2 = [L, L_1]$, \ldots, $L_i = [L, L_{i-1}]$. $L$ is called nilpotent if $L^n = 0$ for some $n$. The normalizer of a subalgebra $K$ of $L$ is defined by $N_L(K) = \{x \in L | [x, K] \subset K\}$. A Cartan subalgebra of a Lie algebra $L$ is a nilpotent subalgebra which is equal to its normalizer in $L$. A Cartan subalgebra of a Lie algebra is a nilpotent subalgebra of it such that the normalizer of the nilpotent subalgebra is the nilpotent subalgebra itself. It turns out that all vectors of the form $l^I_\dot{I}$, where $I$ is either empty or an arbitrary finite integer sequence of integers between 1 and $\Lambda$ inclusive, span a Cartan subalgebra of the algebra $\hat{L}_{\Lambda}$. The proof of this proposition can be found in Appendix C.

This also gives a one-one correspondence (counting rule) between the set of values of the indices and the set of natural numbers: $\emptyset \rightarrow 1, 1 \rightarrow 2, \ldots, \Lambda \rightarrow \Lambda + 1, 1\Lambda \rightarrow \Lambda + 2$, etc..
As we have learnt in Ref.[20], the action of the linear combination
\[ f^K_L = l^K_L - \sum_{j=1}^{\Lambda} l^K_j L_j \]  
(54)
is
\[ f^K_L s^M = \delta^M_L s^K, \]  
(55)
so this linear combination satisfies
\[ [l^I_J, f^K_L] = \sum_{K_1 K_2 = K} \delta^K_{K_1} f^K_{L_1} L_2 - \sum_{l_1 l_2 = L} \delta^I_{l_1} f^K_{l_2}. \]  
(56)
This formula is depicted in Fig.8. It follows from this equation that
\[ [l^I_J, f^K_L] = \left( \sum_{K_1 K_2 = K} \delta^I_{K_1} f^K_{L_1} L_2 - \sum_{l_1 l_2 = L} \delta^I_{l_1} f^K_{l_2} \right) f^K_L. \]  
(57)
As a result, every \( f^K_L \) is a root vector of \( L_\Lambda \). Moreover, these are the only root vectors of the algebra \( \hat{L}_\Lambda \). A proof that there are no root vectors other than the \( f^K_L \) will be provided in Appendix D. We therefore conclude that every root space is one-dimensional.

![Figure 8: The Lie bracket between \( l^I_J \) and \( r^K_L \). Only the first term on the R.H.S. of Eq.(56) is shown.](image)

We have also learnt from Ref.[20] that \( F_\Lambda \) is a proper ideal of the algebra \( \hat{L}_\Lambda \). There exists the exact sequence of Lie algebras
\[ 0 \to F_\Lambda \to \hat{L}_\Lambda \to L_\Lambda \to 0. \]
Moreover, $\hat{L}_\Lambda$ and $L_\Lambda$ are the extended Cuntz–Lie algebra and the Cuntz–Lie algebra, respectively.

We can get further insight of $\hat{L}_\Lambda$ and $L_\Lambda$ by considering the special case $\Lambda = 1$. Then all integer sequences are repetitions of the number 1 a number of times. We can then simplify the notations and write $s^K$ as $s_{#(K)}$ and $l^I_j$ as $l_{#(I)}^j$. The action of $l^a_b$, where $\hat{a}$ and $\hat{b}$ are the numbers of integers in the various sequences, on $s^c$, where $\hat{c}$ is also a non-negative integer, is given by

$$l^a_b s^c = \theta(\hat{b} \leq \hat{c}) s^{\hat{a}+\hat{c}-\hat{b}}, \quad (58)$$

where $\theta$(condition) is 1 if the condition holds, and 0 otherwise. The Lie bracket Eq.(53) is simplified to

$$[l^a_b, l^c_d] = \theta(\hat{c} \leq \hat{b}) l^{\hat{a}+\hat{c}-\hat{b}}_{b+d-\hat{c}} + \theta(\hat{b} < \hat{c}) l^{\hat{a}+\hat{c}-\hat{b}}_{d} - \theta(\hat{a} \leq \hat{d}) l^{\hat{a}}_{b+d-\hat{a}} - \theta(\hat{d} < \hat{a}) l^{\hat{a}+\hat{d}-\hat{a}}_{b}.$$

The set of all vectors of the form $l^a_b$ span a Cartan subalgebra. The root vectors are given by $f^a_b = l^a_b - l^{a+1}_{b+1}$. This can be deduced from Eq.(54). The action of $f^a_b$ on $s^c$, which can be derived from Eq.(58), is

$$f^a_b s^c = \delta^c_b s^a. \quad (60)$$

The corresponding eigenequation, which can be deduced from Eq.(57), is

$$[l^a_b, f^c_d] = (\theta(\hat{a} \leq \hat{c}) - \theta(\hat{c} \leq \hat{a})) f^c_d. \quad (61)$$

As in an earlier discussion, the subspace $F_1$ spanned by all the vectors of the form $f^c_d$ form a proper ideal of this $\Lambda = 1$ algebra $\hat{L}_1$. We can deduce from Eq.(56) that

$$[l^a_b, f^c_d] = \theta(\hat{b} \leq \hat{c}) f^{\hat{a}+\hat{c}-\hat{b}}_{d} - \theta(\hat{a} \leq \hat{d}) f^{\hat{a}}_{b+d-\hat{a}}. \quad (62)$$

We can now form the quotient algebra of cosets of the form $v + F_\Lambda$, where $v$ is an arbitrary vector of the algebra $\hat{L}_1$. This quotient algebra is spanned by the cosets $l^0_0 + F_\Lambda$ and $l^0_b + F_1$, where $a$ and $b$ run over all 0, 1, · · · , $\infty$. It is straightforward to show that the following Lie brackets hold:

$$[l^0_0 + F_1, l^0_0 + F_1] = F_1;$$
$$[l^0_0 + F_1, l^0_0 + F_1] = F_1; \quad \text{and}$$
$$[l^0_b + F_1, l^0_d + F_1] = F_1. \quad (63)$$
Therefore, this quotient algebra is an Abelian algebra.

The relationship between $\hat{L}_\Lambda$ and the Cuntz algebra acquires a new meaning in this special case. Let $a^\dagger \equiv l_0^a$ and $a \equiv l_1^a$. $a^\dagger$ and $a$ are the building blocks of $\hat{L}_1$ because

$$l_0^a = (a^\dagger)^a (a)^b.$$  \hfill (64)

$a^\dagger$ and $a$ satisfy

$$aa^\dagger = 1.$$  \hfill (65)

This is the defining relation for the Toeplitz algebra \[2\]. Note that the commutator between $a$ and $a^\dagger$ is a finite-rank operator, the projection operator to the vacuum state. Thus if we quotient the Toeplitz algebra by $F_1$, we will get an Abelian algebra generated by the operators satisfying

$$aa^\dagger = 1, \quad \text{and} \quad a^\dagger a = 1.$$  \hfill (66)

This is just the algebra of functions on a circle, and is consistent with the fact that the quotient algebra characterized by Eq.(63) is Abelian. Thus we can regard the Cuntz algebra as a non-commutative multi-dimensional generalization of the algebra of functions on the circle.

The Lie algebra $\hat{R}_\Lambda$ spanned by the operators of the third kind is similar to $\hat{L}_\Lambda$. Its Lie bracket is

$$[r^i_j, r^K_L] = \delta^K_j r^i_L + \sum_{J_1J_2 = J} \delta^K_{J_2} r^i_{J_1L} + \sum_{K_1K_2 = K} \delta^K_{J_1} r^i_{J_2L} - \delta^K_{J_2} r^i_{J_1L} - \sum_{L_1L_2 = L} \delta^{J_2}_{L_2} r^i_{L_1j} - \sum_{L_1L_2 = L} \delta^{J_1}_{L_2} r^i_{L_1j}.$$  \hfill (67)

A vector in $F_\Lambda$ can be expressed as an element in $\hat{R}_\Lambda$ as well:

$$f^K_L = r^K_L - \sum_{i=1}^{\Lambda} r^i_K r^i_L.$$  \hfill (68)

Furthermore, $F_\Lambda$ is a proper ideal of $\hat{R}_\Lambda$, as is clear from the Lie bracket.

$$[r^i_j, f^K_L] = \sum_{K_1K_2 = K} \delta^K_{J_1} f^K_{J_2L} - \sum_{L_1L_2 = L} \delta^K_{J_2} f^K_{J_1L}.$$  \hfill (69)
All the operators of the first three kinds together form a bigger Lie algebra $M_\Lambda$. This Lie algebra embraces all the ones discussed so far as subalgebras. The additional Lie bracket needed is one between an operator of the second kind and that of the third kind, as shown:

$$\left[ l^i_j, r^K_L \right] = \sum J_1 J_2 = J \quad J_1 J_2 = K \quad I_1 I_2 = I \quad L_1 L_2 = L$$

Verifying that the action of the R.H.S. on an arbitrary open singlet state $s^M$ gives $l^i_j(r^K_L s^M) - r^K_L(l^i_j s^M)$ is not enough to validate this binary operation between these two operators as a Lie bracket because Eqs. (54) and (68) together imply that the set of all finite linear combinations of the operators in $\hat{L}_\Lambda$ and $\hat{R}_\Lambda$ is not linearly independent, which in turn implies that the Jacoby identity may not be satisfied. We will properly justify Eq. (70) in the Section 3, where we will see that we can treat $M_\Lambda$ as a subalgebra of yet another larger Lie algebra. A typical term in the above equation is depicted in Fig. 9.

![Diagram of Lie bracket](image)

Figure 9: A diagrammatic representation of a Lie bracket of $M_\Lambda$, Eq. (70). Only the first term on the R.H.S. of this equation is shown.

### 3.2 A Lie Algebra for the Operators of the Fourth Kind

We now review the most non-trivial of all subalgebras of the grand open string algebra, the centrix algebra $\Sigma_\Lambda$. If we retain only the adjoint matter of the action of an operator of the fourth kind on an open singlet state in
Figure 10: Diagrammatic representations of the first nine terms on the R.H.S. of Eq.(73).
Eq. (17), we will get
\[ \sigma^I_J s^K = \sum_{A,B} \delta^K_A \delta^A_B s^{IB} \] (71)
(c.f. Fig. 3(c)). It is impossible to find a finite linear combination of this kind of operators whose action on an arbitrary singlet state is exactly the same as the composite operator \( \sigma^I_J \sigma^K_L \). (We will put off proving this assertion because we are going to prove a similar but more general statement in the next section in Appendix G.) Nevertheless, a remarkable thing here is that the Lie bracket between two operators of the fourth kind can still be equated with a finite linear combination of this kind of operators by the requirement
\[ ([\sigma^I_J, \sigma^K_L]) s^P \equiv \sigma^I_J (\sigma^K_L s^P) - \sigma^K_L (\sigma^I_J s^P) \] (72)
for any arbitrary sequence \( \dot{P} \). Then
\[
\left[ \sigma^I_J, \sigma^K_L \right] = \delta^K_J \sigma^I_L + \sum_{J_1J_2=J} \delta^K_{J_2} \sigma^{J_1}L + \sum_{K_1K_2=K} \delta^{J_1}K \sigma^K_{L_2} \\
+ \sum_{J_1J_2=J} \delta^{K_1}J \sigma^{K_2}L + \sum_{J_1J_2=J} \delta^{K_1}J \sigma^{K_2}L_2 + \sum_{K_1K_2=K} \delta^{K_1}J \sigma^{K_2}L \\
+ \sum_{J_1J_2=J} \delta^{K_1}J \sigma^K_{L_3} + \sum_{K_1K_2=K} \delta^{K_1}J \sigma^K_{L_3} + \sum_{K_1K_2=K} \delta^{K_1}J \sigma^{K_2}L_3 \\
-(I \leftrightarrow K, J \leftrightarrow L). \] (73)
The proof of this equation is given in detail in Appendix E as a demonstration of how we carry out computations involving multi-indices. Fig. 10 gives diagrammatic representations of the first nine terms. We will call the Lie algebra defined by Eq. (73) the centrix algebra \( \hat{\Sigma}_\Lambda \).

Let us explore the structure of \( \hat{\Sigma}_\Lambda \). All vectors of the form \( \sigma^I_J \), where \( I \) is an arbitrary finite integer sequence of integers between 1 and \( \Lambda \) inclusive, span a Cartan subalgebra \( \hat{\Sigma}^0_\Lambda \) of the centrix algebra. The proof is pretty much the same as the one shown in Appendix E.

Next, the reader can easily verify the following actions, which we have mentioned in Ref. [20]:
\[ \left( \sigma^I_J - \sum_{i=1}^\Lambda \sigma^i_J \right) s^K = \sum_{K_1K_2=K} \delta^K_{J_1} s^{IK_2} \] (74)
Figure 11: Diagrammatic representations of Eq. (79). Only the first six terms on the R.H.S. of this equation are shown here.

and

$$\left( \sigma_j^I - \sum_{j=1}^{\Lambda} \sigma_j^I J_j \right) \sigma^K = \sum_{K_1K_2=K} \delta^{K_2K_1} J_1^I.$$  \hspace{1cm} (75)$$

These are exactly the action of the operators $l_j^I$ and $r_j^I$. Thus,

$$l_j^I = \sigma_j^I - \sum_{i=1}^{\Lambda} \sigma_j^I i_j; \quad \text{and} \quad r_j^I = \sigma_j^I - \sum_{j=1}^{\Lambda} \sigma_j^I J_j.$$  \hspace{1cm} (76)$$

$$r_j^I = \sigma_j^I - \sum_{j=1}^{\Lambda} \sigma_j^I J_j.$$  \hspace{1cm} (77)$$

We remind the reader that unlike Eqs. (54) and (58), here $I$ and $J$ must be non-empty. Combining Eqs. (76) and (54), or Eqs. (77) and (58), we obtain

$$f_j^I = \sigma_j^I - \sum_{j=1}^{\Lambda} \sigma_j^I J_j - \sum_{i=1}^{\Lambda} \sigma_i^I i_j + \sum_{i,j=1}^{\Lambda} \sigma_i^I J_j.$$  \hspace{1cm} (78)$$
(We are going to use a generalized version of these relations in a later section.) Let $F'_\Lambda$, $\hat{L}'_\Lambda$ and $R'_\Lambda$ be vector spaces spanned by $f^I_J$'s, $l^I_J$'s and $r^I_J$'s, respectively, and let $M'_\Lambda$ be the sum of all these vector spaces. All four vector spaces are proper ideals of $\hat{\Sigma}_\Lambda$.

For the sake of completeness, we review the Lie bracket relations between an operator of the fourth kind and an operator of another kind:

$$\left[\sigma^I_J, l^K_L\right] = \delta^K_L^J I + \sum_{J_1J_2=J} \delta^K_{J_1} l^K_{J_2} I + \sum_{K_1K_2=K} \delta^K_{K_1} l^K_{K_2} I$$

$$+ \sum_{K_1K_2=K} \delta^K_{K_1} l^K_{K_2} I + \sum_{J_1J_2=J} \delta^K_{J_1} l^K_{J_2} I + \sum_{K_1K_2K_3=K} \delta^K_{K_2} l^K_{K_3} I$$

$$- \delta^K_{L_1J} - \sum_{I_1I_2=I} \delta^K_{I_1} l^K_{I_2} I - \sum_{L_1L_2=L} \delta^K_{L_1} l^K_{L_2} I$$

$$- \sum_{L_1L_2=L} \delta^K_{L_1} l^K_{L_2} I - \sum_{L_1L_2=L} \delta^K_{L_1} l^K_{L_2} I - \sum_{L_1L_2L_3=L} \delta^K_{L_1} l^K_{L_2} I_3; \quad (79)$$
Figure 13: A diagrammatic representation of Eq. (81). Only the first term on the R.H.S. of this equation is shown here.

\[
\left[ \sigma^I_J, r^K_L \right] = \delta^K_J r^I_L + \sum_{J_1J_2=J} \delta^K_{J_2} r^I_{J_1L} + \sum_{K_1K_2=K} \delta^K_{J_2} r^{K_1I}_{L}
\]

\[
+ \sum_{K_1K_2=K} \delta^K_{J_1} r^{K_2}_{L} + \sum_{J_1J_2=J} \delta^K_{J_2} r^{J_1K_2}_{L} + \sum_{K_1K_2K_3=K} \delta^K_{J_2} r^{J_1K_3}_{L} + \sum_{J_1J_2=J} \delta^K_{J_2} r^{J_1K_3}_{L} + \sum_{K_1K_2K_3=K} \delta^K_{J_2} r^{J_1K_3}_{L}
\]

\[
- \delta^K_{L_1L_2J} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L}
\]

\[
- \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} r^{J_1}_{L}
\]

\[
\left[ \sigma^I_J, f^K_L \right] = \sum_{K_1K_2K_3=K} \delta^K_{J_2} f^{K_1}_{I} r^{K_3}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} f^{K_1}_{I} r^{K_3}_{L}
\]

Eqs. (79), (80), and (81) are illustrated in Figs. 11, 12, and 13, respectively. It follows from these three equations that the quotient algebra \( V_\Lambda \equiv \tilde{\Sigma}_\Lambda / \tilde{M}_\Lambda' \) is the Lie algebra of a set of outer derivations of the Cuntz algebra. As we have shown in Ref. [20], \( V_\Lambda \) simplifies precisely to the Witt algebra in the special case \( \Lambda = 1 \). Thus, the Lie algebra of this set of outer derivations of the Cuntz algebra is a generalization of the algebra of vector fields on the unit circle.

Let us make some remarks on root vectors. We obtain from Eq. (81) that

\[
\left[ \sigma^I_J, f^K_L \right] = \left( \sum_{K_1K_2K_3=K} \delta^K_{J_2} f^{K_1}_{I} r^{K_3}_{L} - \sum_{L_1L_2=I} \delta^{L_2}_{L_1} f^{K_1}_{I} r^{K_3}_{L} \right) \]

As a result, every \( f^K_L \) is a root vector. Moreover, there are no root vectors other than \( f^K_L \)'s, and a proof of this statement will be given in Appendix F. Hence every root space is one-dimensional.
There is a way of splitting $\hat{\Sigma}_\Lambda$ into ‘raising’ operators, ‘diagonal’ operators and ‘lowering’ operators. Let $\hat{\Sigma}^+_\Lambda$ be the vector space spanned by all elements of the form $\sigma^I_J$ such that $I > J$ (See the footnote in the previous subsection). Then it can be proved by checking term by term on the R.H.S. of Eq.(73) that $\hat{\Sigma}^+_\Lambda$ is indeed a subalgebra of the centrix algebra\footnote{Take the fourth term as an example. If both $\sigma^I_J$ and $\sigma^K_L$ $\in$ $\hat{\Sigma}^+_\Lambda$, then $I K_2 > J K_2 = J_1 J_2 K_2 = J_1 K_1 K_2 = J_1 K > J_1 L$. Hence $\sigma^I_{J,L} \in \hat{\Sigma}^+_\Lambda$ also.}. Likewise, let $\hat{\Sigma}^-_\Lambda$ be the vector space spanned by all elements of the form $\sigma^I_J$ such that $I < J$. Then it also follows from Eq.(73) that $\hat{\Sigma}^-_\Lambda$ is also a subalgebra. Moreover, we have

$$\hat{\Sigma}_\Lambda = \hat{\Sigma}^+_\Lambda \oplus \hat{\Sigma}^0_\Lambda \oplus \hat{\Sigma}^-_\Lambda.$$ 

4 Open String and Closed String Algebras

In the previous section, we studied a Lie algebra for bosonic open strings, and gained some expertise in how to study its structure. We will use this skill to fulfill our promise in the preceding section, which is to study the full Lie algebra for open and closed strings, or in other words, the Lie algebra for the five kinds of operators discussed in Section 2.1.

To achieve this purpose, we need to define a precursor Lie algebra which we will call the heterix algebra among some physical operators. These physical operators are nothing but the ones defined in Subsection 2.1, except that the ranges of values of the quantum states other than color (see the second paragraph in Subection 2.1) are changed. We will derive this algebra in Subsection 4.1. Then the grand string algebra will be seen as a subalgebra of the heterix algebra. Readers who are only interested in the definition of this algebra may skip this section, occasionally returning to it to look up the relevant definitions. In Subsection 4.2, we will give the definition of a algebra of operators acting on open and closed singlet states. We will call this the ‘grand string algebra’. The difference between this Lie algebra and the precursor algebra is that there are more than one degree of freedom at the ends of the open singlet states. At first glance, this algebra is somewhat ‘larger’ than the precursor algebra. Ironically, we will derive this grand string algebra as a subalgebra of the precursor algebra by manipulating the numbers of degrees of freedom in an accompanying appendix.
In Subsection 4.3, we will derive a Lie algebra just for open strings. This will be a quotient algebra of the grand string algebra. In Section 4.4, we will derive a Lie algebra for closed strings as another quotient algebra of the grand string algebra. We will see that the corresponding Lie algebra, which we will call the ‘cyclix algebra’, for bosonic closed strings has a closed relationship with the Witt algebra, too.

To have a glimpse of how to use these algebras in the study of physical systems, we will turn to the Ising model again in Section 4.5. It is well known that there are many ways of solving the quantum Ising model in one dimension. One method which is close to the spirit of the original way Onsager himself solved the model [37] is via the ‘Onsager algebra’ [53]. We will see that actually this Onsager algebra is a subalgebra of the cyclix algebra, and we can use the cyclix algebra directly to obtain some conserved quantities of the Ising matrix model. This example may give us a clue of how to use these Lie algebras more effectively in the future.

### 4.1 Derivation of a Precursor Algebra

We are going to derive a precursor algebra. The grand string algebra will be identified as a subalgebra of this algebra in the next section.

Let $\alpha^\mu_{\nu}(k)$ be an annihilation operator for a boson in the adjoint representation for $1 \leq k \leq \Lambda + 2\Lambda_\varphi$, and let $\overline{\chi}_\mu$ and $\chi^\mu$ be annihilation operators for an antifermion in the conjugate representation and a fermion in the fundamental representation, respectively. Moreover, let $\alpha^{\mu\nu}_{\mu\nu}(k)$, $\overline{\chi}^{\mu}$ and $\chi^{\mu}$ be the corresponding creation operators. The annihilation and creation operators satisfy the usual canonical (anti)-commutation relations, the non-vanishing ones being

\[
\left[\alpha^{\mu_1}_{\mu_2}(k_1), \alpha^{\nu_3}_{\nu_4}(k_2)\right] = \delta_{k_1 k_2} \delta^{\mu_3}_{\mu_4} \delta^{\mu_1}_{\mu_2}, \tag{83}
\]

\[
\left[\overline{\chi}_{\mu_1}, \overline{\chi}^{\mu_2}\right]_+ = \delta^{\mu_2}_{\mu_1}, \tag{84}
\]

and

\[
\left[\chi^{\mu_1}, \chi^{\mu_2}\right]_+ = \delta^{\mu_1}_{\mu_2}. \tag{85}
\]
Again we introduce two families of color-invariant singlet states. A typical open singlet state is a linear combination of the states of the form
\[ s_I^K \equiv N^{-(c+1)/2} \chi_u^{v_1} \alpha_{v_1}^{v_2} (k_1) \alpha_{v_2}^{v_3} (k_2) \cdots \alpha_{v_{c+1}}^{v_c} (k_c) \chi_u^{v_{c+1}} | 0 \rangle . \] (86)
We denote by \( \mathcal{T}_o \) the Hilbert space of all these open singlet states. A typical closed singlet state is a linear combination of the states of the form
\[ \Psi^K \equiv N^{-c/2} \alpha_{v_1}^{v_2} (k_1) \alpha_{v_2}^{v_3} (k_2) \cdots \alpha_{v_c}^{v_1} (k_c) | 0 \rangle . \] (87)
The Hilbert space of all closed singlet states will be denoted by \( \mathcal{T}_c' \).

We need only two families of color-invariant operators acting on \( \mathcal{T}_o \) and \( \mathcal{T}_c' \) to establish our main results. One family consists of operators of the form
\[ \tilde{f}^I_j \equiv \sum_{i_a} \alpha^{v_3}_\mu (i_1) \alpha^{v_2}_\mu (i_2) \cdots \alpha^{v_1}_\mu (i_a, \epsilon (i_a)) \alpha^{v_3}_\nu (j_b) \alpha^{v_2}_\nu (j_{b-1}) \cdots \alpha^{v_1}_\nu (j_1) . \] (88)
In the large-\( N \) limit, the actions of this operator on singlet states read (c.f. Ref. [19] or Appendix A of this article)
\[ \tilde{f}^I_j s_I^K = 0; \quad \text{and} \quad \tilde{f}^I_j \Psi^K = \sum_{j_1, j_2 = j} \delta^K_{j_1} \Psi^K, \] (89)
(90)
It is clear that the sum on the right hand side of Eq.(89) is a finite one. The other family consists of operators of the form
\[ \gamma^I_j \equiv N^{-(a+b-2)/2} \alpha^{v_3}_{\mu_1} (i_1) \alpha^{v_2}_{\mu_2} (i_2) \cdots \alpha^{v_1}_{\mu_a} (i_a) \alpha^{v_3}_{\nu_1} (j_b) \alpha^{v_2}_{\nu_2} (j_{b-1}) \cdots \alpha^{v_1}_{\nu_1} (j_1) . \] (91)
In the large-\( N \) limit, this operator propagates singlet states in the following manner:
\[ \gamma^I_j s_I^K \equiv \sum_{K_1, K_2, K_3 = K} \delta^K_{K_1} s^{K_2 K_3}, \quad \text{and} \quad \gamma^I_j \Psi^K \equiv \delta^K_{j} \Psi^K + \sum_{K_1, K_2 = K} \delta^K_{j} \Psi^{K_2 K_1} + \sum_{K_1, K_2 = K} \delta^K_{j} \Psi^{K_1 K_2} \] (92)
\[ + \sum_{K_1, K_2, K_3 = K} \delta^K_{j} \Psi^{K_2 K_3 K_1} \] (93)
(c.f. Eqs. (2) and (3) in Ref. [21]). The set of all $\tilde{f}_{ij}'$’s and $\gamma_{ij}'$’s is linearly independent. This was proved in Appendix A of Ref. [21]. (Notice that the open singlet states are needed for the linear independence. If we simply considered their actions on closed singlet states only, the set would not be linear independent.)

The next thing we are going to do is to construct a precursor Lie algebra out of the two families of color-invariant operators. If the product of any two of these operators were well defined, i.e., if we could write down the product as a finite linear combination of color-invariant operators, the easiest way to obtain a Lie algebra would certainly be to define the Lie bracket of two operators as a sum or difference of their products in different orders of the operators. However, such a product is actually not well defined; this is shown in Appendix G. Now consider the commutator of two operators. It can be shown that the actions of these commutators on singlet states are identical to the actions of some observables. If we define the commutator to be the color-invariant operator whose action on $T_{o}^{\prime} \oplus T_{c}^{\prime}$ is identical to it, then we have

$$\left[\tilde{\gamma}_{J}^{I}, \tilde{\gamma}_{L}^{K}\right] = \delta_{J}^{K} \gamma_{L}^{I} + \sum_{J_{1}J_{2}=J} \delta_{J_{2}}^{K} \gamma_{J_{1}L}^{I} + \sum_{K_{1}K_{2}=K} \delta_{J}^{K_{1}} \gamma_{J_{1}K_{2}}^{I} $$

$$+ \sum_{J_{1}J_{2}=J} \delta_{J_{2}}^{K_{2}} \gamma_{J_{1}L}^{I} + \sum_{J_{1}J_{2}J_{3}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}J_{2}L}^{I} + \sum_{K_{1}K_{2}=K} \delta_{J}^{K_{2}} \gamma_{J_{1}K_{2}L}^{I}$$

$$+ \sum_{J_{1}J_{2}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}L}^{I} + \sum_{J_{1}J_{2}J_{3}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}J_{2}L}^{I} + \sum_{K_{1}K_{2}K_{3}=K} \delta_{J}^{K_{2}} \gamma_{J_{1}K_{2}K_{3}L}^{I}$$

$$+ \sum_{J_{1}J_{2}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}L}^{I} + \sum_{J_{1}J_{2}J_{3}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}J_{2}L}^{I} + \sum_{K_{1}K_{2}K_{3}=K} \delta_{J}^{K_{2}} \gamma_{J_{1}K_{2}K_{3}L}^{I}$$

$$+ \sum_{J_{1}J_{2}J_{3}=J} \delta_{J_{2}}^{K_{1}} \gamma_{J_{1}J_{2}L}^{I} - (I \leftrightarrow K, J \leftrightarrow L), \quad (94)$$

43
Figure 14: Possible terms in $\gamma^J_I(\gamma^K_L \Psi^M)$. (a) This is the case when $\gamma^J_I$ and $\gamma^K_L$ act on disjoint segments in $\Psi^M$. This term is cancelled in the Lie superbracket. (b) Partial overlap between $J$ and $K$. This is the case when $J_1 = K_2$. This term is equivalent to the corresponding one in $\gamma^K_L J_2^I \Psi^M$. (c) Complete overlap between $J$ and $K$. This time $J = K$. This term is equivalent to a term in $\gamma^K_L \Psi^M$. 
Figure 15: Other possible terms in $\gamma_j^I (\gamma_I^K \Psi^M)$. (a) Another partial overlap between $J$ and $K$. This time $J_2 = K_1$. This term is equivalent to a term in $\gamma_j^I(K_2 \Psi^M)$. (b) This partial overlap corresponds to the case when $J_1 = K_3$ and $J_3 = K_1$. Note that we have turned $\gamma_j^I$ by 180 degrees. This term is equivalent to $\tilde{f}_{j_2K_2}^I \Psi^M$.

That these equations hold true was demonstrated in Appendix D of Ref. [21]. Unfortunately, the rigorous proof is not illuminating. To enlighten the reader, we would like to discuss the following special cases in Eq. (94) to see why it makes sense for the right hand sides of these equations to appear in the above manner.

Consider $\gamma_j^I (\gamma_I^K \Psi^M)$, and consider a term $t_1$ in $\gamma_j^I (\gamma_I^K \Psi^M)$ produced by the operations of $\gamma_j^I$ and $\gamma_I^K$ on disjoint sequences of creation operators in $\Psi^M$ (Fig. 14(a)), i.e., no annihilation operator in $\gamma_j^I$ acts on any creation operator in $\gamma_I^K$. This term is identical to the term $t_2$ in $\gamma_I^K (\gamma_j^I \Psi^M)$ produced by the operations of $\gamma_j^I$ and $\gamma_I^K$ on the same disjoint sequences in $\Psi^M$. Since during the process of producing $t_1$ and $t_2$, there is no contraction at all...
Figure 16: Diagrammatic representations of the tenth to twelfth terms on the R.H.S. of Eq. (94).
between any operator in $\gamma'_{I\bar{J}}$ and any in $\gamma'_{\bar{K}L}$ (any contraction between $\gamma'_{I\bar{J}}$ and $\gamma'_{\bar{K}L}$ will produce terms which we are not considering right now), $t_2$ can be obtained from $\gamma'_{I\bar{J}}(\gamma'_{\bar{K}L}\Psi^M)$ as well by interchanging $\gamma'_{I\bar{J}}$ and $\gamma'_{\bar{K}L}$ first before these two color-invariant operators act on $\Psi^M$. As a result, $t_1$ is cancelled by $t_2$ in the commutator.

Therefore, during the process of producing terms that are not killed by the commutator, there must be contraction(s) among some operators in $\gamma'_{I\bar{J}}$ and some in $\gamma'_{\bar{K}L}$. If we perform such contractions first before contracting the annihilation operators in $\gamma'_{I\bar{J}}$ and $\gamma'_{\bar{K}L}$ with the creation operators in $\Psi^M$, in general we will obtain a color-invariant operator of the form $\text{Tr} (\alpha^\dagger \cdots \alpha^\dagger \alpha \cdots \alpha \cdots \alpha \cdots \alpha \cdots \alpha) \cdots \text{Tr} (\alpha^\dagger \cdots \alpha^\dagger \alpha \cdots \alpha \cdots \alpha \cdots \alpha \cdots \alpha) \cdots$ multiplied by a factor of $N$ raised to some power. However, Figs. 14(b), (c) and Fig. 13 clearly reveals that only color-invariant operators of the forms $\text{Tr} (\alpha^\dagger \cdots \alpha^\dagger \alpha \cdots \alpha)$ and $\text{Tr} (\alpha^\dagger \cdots \alpha^\dagger) \text{Tr} (\alpha \cdots \alpha)$ survive the large-$N$ limit. All possibilities of producing color-invariant operators in these two forms are in Eqs. (74) to (96).

The heterix algebra $\hat{\Gamma}_{A|A}$ is the Lie algebra defined by the Lie brackets in Eqs. (74) to (96). Figs. 13, 17 and 18 shows the diagrammatic representations of these three equations.

The reader can take a look at Ref. [21] to learn the structure of this Lie
Figure 18: Diagrammatic representations of the first two terms on the R.H.S. of Eq. (95), and the first term on the R.H.S. of Eq. (96).
4.2 Definition of the Grand String Algebra

We are going to give a precise definition of a Lie algebra of the operators defined in Section 2.1 acting on closed or open singlet states. Unlike the open singlet states in Section 4.1, the numbers of degrees of freedom of the fundamental and conjugate fields are arbitrary. The proof that this is really a Lie algebra will be given in Appendix H.

Consider a vector space of all finite linear combinations of all five families of operators listed in Table 1.

| operator of which kind | expression                                      |
|------------------------|-------------------------------------------------|
| first                  | $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes f^I_J \otimes \Xi^{\lambda_3}_{\lambda_4}$ |
| second                 | $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes i^I_J$                                           |
| third                  | $r^I_J \otimes \Xi^{\lambda_3}_{\lambda_4}$                                                   |
| fourth                 | $\gamma^I_J$                                                                                   |
| fifth                  | $\tilde{f}^I_J$                                                                                  |

Table 1: Definitions of physical observables.

In this table, $1 \leq \lambda_1, \lambda_2, \lambda_3, \text{ and } \lambda_4 \leq \Lambda_F$, $I$ and $J$ are non-empty sequences of integers between 1 and $\Lambda$ inclusive, and $\hat{I}$ and $\hat{J}$ are empty or non-empty sequences of integers also between 1 and $\Lambda$ inclusive. The reader can regard the five types of vectors as operators defined in Eq.(129), or as axiomatic entities satisfying a set of Lie brackets to be described immediately. We call a vector or an operator an operator of the first, second, third, fourth or fifth kind if it is a finite linear combination of operators, all of which are of the first, second, third, fourth or fifth form enlisted above, respectively.

Let us describe the Lie brackets among different kinds of operators. The Lie bracket between an operator of the fifth kind, and an operator of the first, second or third kind is trivial:

$$\left[ \Xi^{\lambda_1}_{\lambda_2} \otimes f^I_J \otimes \Xi^{\lambda_3}_{\lambda_4}, \tilde{f}^K_L \right] = 0;$$
$$\left[ \Xi^{\lambda_1}_{\lambda_2} \otimes i^I_J, \tilde{f}^K_L \right] = 0; \text{ and}$$
The operators of the first kind form a proper ideal of the Lie algebra:

\[
[\Xi_{\lambda_2}^f \otimes t_j^I, \Xi_{\lambda_1}^3 \otimes t_L^K] = \\
\delta_{\lambda_2}^I \delta_{\lambda_1}^L \Xi_{\lambda_2}^{iL} \Xi_{\lambda_1}^{iK} + \delta_{\lambda_1}^L \delta_{\lambda_2}^I \Xi_{\lambda_2}^{iI} \Xi_{\lambda_1}^{iK} - \delta_{\lambda_1}^I \delta_{\lambda_2}^L \Xi_{\lambda_2}^{iK} \Xi_{\lambda_1}^{iI}.
\]

The operators of the second kind form a subalgebra. So are the operators of the third kind:

\[
[\Xi_{\lambda_2}^f \otimes t_j^I, \Xi_{\lambda_1}^3 \otimes t_L^K] = \\
\delta_{\lambda_2}^I \delta_{\lambda_1}^L \Xi_{\lambda_2}^{iL} \Xi_{\lambda_1}^{iK} + \delta_{\lambda_1}^L \delta_{\lambda_2}^I \Xi_{\lambda_2}^{iI} \Xi_{\lambda_1}^{iK} - \delta_{\lambda_1}^I \delta_{\lambda_2}^L \Xi_{\lambda_2}^{iK} \Xi_{\lambda_1}^{iI}.
\]

Eqs. (97) to (99) together with the following relations show that operators of
the first three kinds as a whole form another ideal:

\[
\left[ \Xi_{\lambda_2} \otimes I_j^L, \gamma^K_L \otimes \Xi_{\lambda_4}^{\lambda_3} \right] =
\left( \begin{array}{c}
\Xi_{\lambda_2}^{\lambda_1} \otimes \\
\sum_{j_1, j_2 = j} \delta_{j_2}^{K_1} f_{j_1 L}^{i K_2} - \sum_{j_1, j_2 = j} \delta_{L_1}^{i_2} f_{j_1 L_2}^{i_1 K_2}
\end{array} \right) \otimes \Xi_{\lambda_4}^{\lambda_3},
\]

\[
\left[ \Xi_{\lambda_2}^{\lambda_1} \otimes I_j^L, \gamma^K_L \right] =
\left( \begin{array}{c}
\Xi_{\lambda_2}^{\lambda_1} \otimes \\
\sum_{j_1, j_2 = j} \delta_{j_1}^{K_1} f_{j_1 L_2}^{i K_2} - \sum_{j_1, j_2 = j} \delta_{L_1}^{i_2} f_{j_1 L_2}^{i_1 K_2}
\end{array} \right) \otimes \Xi_{\lambda_4}^{\lambda_3},
\]

\[
\left[ I_j^L, \gamma^K_L \otimes \Xi_{\lambda_4}^{\lambda_3} \right] =
\left( \begin{array}{c}
\delta_{j}^{K_1} f_{j}^{i K_2} + \sum_{K_1, K_2 = K} \delta_{j_1}^{K_1} f_{j_1 L}^{i K_2} + \sum_{j_1, j_2 = j} \delta_{j_2}^{K_1} f_{j_1 L_2}^{i K_2}
\end{array} \right) \otimes \Xi_{\lambda_4}^{\lambda_3}.
\]
These equations also reveal that operators of the first kind form a proper ideal of the algebra spanned by operators of the first three kinds.

The operators of the fifth kind form yet another proper ideal of this algebra:

\[
\left[ \tilde{f}_J, \tilde{f}_L^K \right] = \sum_{K_1 = K} \delta^K_{J_2} \tilde{f}_L^K + \sum_{K_1 = K} \delta^K_{J_1} \tilde{f}_L^K - (I \leftrightarrow K, J \leftrightarrow L) \tag{101}
\]

and

\[
\left[ \gamma_J^K, \tilde{f}_L^K \right] = \sum_{K_1 = K} \delta^K_{J_1} \tilde{f}_L^K + \sum_{K_1 = K} \delta^K_{J_2} \tilde{f}_L^K + \sum_{K_1 = K} \delta^K_{J_3} \tilde{f}_L^K
\]

Finally, the Lie bracket between two operators of the fourth kind is a linear combination of operators of the fourth and fifth kinds:

\[
\left[ \gamma_J^K, \gamma_L^K \right] = \sum_{J_1, J_2 = J} \delta^{K_1}_{J_2} \gamma_{J_1 L} + \sum_{J_1, J_2 = J} \delta^{K_2}_{J_1} \gamma_{J_2 L} + \sum_{J_1, J_2 = J} \delta^{K_3}_{J_1} \gamma_{J_2 L}
\]
We call the Lie algebra defined by the Lie brackets from Eqs. (97) to (103) the grand string algebra.

Let $\mathcal{T}_o$ be the vector space of all finite linear combinations of singlet states of the form $\bar{\phi}^\rho_1 \otimes s^K \otimes \phi^\rho_2$ where $1 \leq \rho_1, \rho_2 \leq \Lambda_F$ and all integers are between 1 and $\Lambda$ inclusive in $K$, which may be empty. Also let $\mathcal{T}_c$ be the vector space of all finite linear combinations of singlet states of the form $\Psi^K$ such that all integers are again between 1 and $\Lambda$ inclusive in $K$, which has to be non-empty, and that $\Psi^K$ satisfies Eq. (6). If we treat the operators $\mathcal{T}_o \oplus \mathcal{T}_c$ as the ones defined in the Subsection 4.1, we will find that the actions of an operator of the first kind are precisely given by Eqs. (8) and (9); those of the second kind by Eqs. (11) and (12); those of the third kind by Eqs. (14) and (15); those of the fourth kind by Eqs. (17) and (18); and those of the fifth kind by Eqs. (21) and (22). Alternatively, we can define the actions of the operators of the five kinds on $\mathcal{T}_o \oplus \mathcal{T}_c$ by these ten equations, and show that $\mathcal{T}_o \oplus \mathcal{T}_c$ provides a representation for this Lie algebra.

4.3 Open String Algebra

Now that we have the grand string algebra at hand, we are going to derive the Lie algebra of operators acting on open singlet states only as a quotient algebra of it. We will call this the open string algebra, and identify some subalgebras of it. The Lie algebra of operators acting on closed singlet states only will be considered in the next section.

Since every element of the grand string algebra maps a state in $\mathcal{T}_o$ to a state in $\mathcal{T}_o$, and a state in $\mathcal{T}_c$ to a state in $\mathcal{T}_c$, $\mathcal{T}_o$ and $\mathcal{T}_c$ furnish two representation spaces to the algebra. The representations of the operators in
the algebra, of course, form a Lie algebra for each of these two representation spaces. However, since none of these two representation spaces provide faithful representations to the grand string algebra, some operators vanish.

Consider the Lie algebra generated by the representation $T_0$ first. We are going to call this algebra the open string algebra. Since all operators of the fifth kind sends any state in $T_0$ to zero, there are only 4 kinds of operators in this algebra. The Lie brackets of these operators are the same as those in the previous section, except that we set all $\bar{f}_j$’s to be zero, i.e., only Eq. (103) needs to be modified. Let us write $\gamma$ as $\sigma$ in this open string algebra. Note that the operators in this algebra are not linearly independent; there are many relations among them. For example,

$$\sum_{\lambda=1}^{\Lambda} \bar{\Xi}^\lambda \otimes l^I_j = \sigma^I_j - \sum_{i=1}^{\Lambda} \sigma^{iI}_{ij};$$

$$\sum_{\lambda=1}^{\Lambda} \bar{r}^I_j \otimes \Xi^\lambda = \sigma'_I_j - \sum_{j=1}^{\Lambda} \sigma'^{Ij}_j;$$

$$\sum_{\lambda_2=1}^{\Lambda} \bar{\Xi}^{\lambda_2} \otimes f^I_j \otimes \Xi^{\lambda_3}_{\lambda_3} = \Xi^{\lambda_1}_{\lambda_2} \otimes l^I_j - \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes \sum_{j=1}^{\Lambda} l^I_{jj}; \text{ and}$$

$$\sum_{\lambda_1=1}^{\Lambda} \bar{\Xi}^{\lambda_1} \otimes f^I_j \otimes \Xi^{\lambda_4}_{\lambda_4} = r^I_j \otimes \Xi^{\lambda_3}_{\lambda_4} - \sum_{i=1}^{\Lambda} r^{iI}_{ij} \otimes \Xi^{\lambda_3}_{\lambda_4}. \quad (104)$$

There are many subalgebras and ideals in the open string algebra. They were described in Table 3 in Ref. [20].

### 4.4 Closed String Algebra

Let us turn our attention to the Lie algebra of operators acting on closed singlet states only. It is generated by the representation space $T_c$. We will call this the closed string algebra or the cyclix algebra $\hat{C}_\Lambda$. Since acting any operator of the first three kinds on $T_c$ yields zero, this Lie algebra is obtained by considering the Lie bracket among operators of the fourth and fifth kinds only. Thus the closed string algebra is characterized by Eqs. (101), (102) and (103). We will write $\gamma$ as $g$ in the closed string algebra. Again the operators are not linearly independent; in fact, any operator of the fifth kind can be
written as a linear combination of operators of the fourth kind as

\[ \tilde{f}_j = g_j - \sum_{k=1}^{\Lambda} g_{jk} \]  \hspace{1cm} (105) 

or

\[ \tilde{f}_j = g_j - \sum_{k=1}^{\Lambda} g_{k,j}. \]  \hspace{1cm} (106) 

These two relations together with

\[ \tilde{f}_{I_1 I_2} = \tilde{f}_{I_2 I_1} \]  \hspace{1cm} (107) 

and

\[ \tilde{f}_{J_1 J_2} = \tilde{f}_{J_2 J_1} \]  \hspace{1cm} (108) 

can generate many other relations. From Eq.(102), we see that the set of all \( \tilde{f}_j \)'s span a proper ideal \( \tilde{F}'_\Lambda \) for the cyclix algebra. We conjecture that \( \hat{C}_\Lambda \) is precisely the quotient of the Lie algebra given by Eqs.(101), (102) and (103) by the kernel of Eqs.(105) and (106). We showed in Ref.[21] that this conjecture is true for \( \Lambda = 1 \). Shown in the same reference was the fact that if \( \Lambda = 1 \), then the quotient of the closed string algebra by \( \tilde{F}'_1 \) and \( \tilde{M}'_1 \), viewed as another proper ideal of the closed string algebra, yields the Witt algebra. This fact should be much more transparent in the unified account of both the open string algebra and the closed string algebra in this article: set \( \Lambda = \Lambda_F = 1 \) in the grand string algebra, then quotient out \( \tilde{M}'_1 \) (which is spanned by all operators of the first three kinds and some linear combinations of operators of the fourth kind shown in Eq.(129) with the primes removed) and \( \tilde{F}'_1 \) (which is spanned by all operators of the fifth kind and some linear combinations of operators of the fourth kind shown in Eqs.(105) and (106)), and we will get the Witt algebra. The two apparently different ways of obtaining the Witt algebra in Refs.[20] and [21] are just different orders of quotienting out the same set of operators.

A Cartan subalgebra and the associated root vectors of the \( \Lambda = 1 \) closed string algebra is discussed in Ref.[21].

55
4.5 The Ising Model Revisited

In this and the last section, we have mostly been studying issues in mathematics. Let us see how to use these mathematical results in physics. A good starting point should be a model which is simple enough that we know a great deal of it so that we can see how the algebraic point of view we have been discussing fit in. In Section 2.4, we introduced a number of exactly solvable matrix models. The Ising model is the simplest, and let us look into this model to see what can be learnt.

There are many different ways of solving the Ising model. One which is closer to the original spirit of Onsager’s approach is via the so-called Onsager algebra [37, 53]. Consider a system whose Hamiltonian has the form

\[ H = H_0 + V \]

(109)

with the two terms in the Hamiltonian satisfying the Dolan–Grady conditions [54, 55]:

\[ [H_0, [H_0, [H_0, V]]] = 16[H_0, V], \]

(110)

and

\[ [V, [V, [V, H_0]]] = 16[V, H_0]. \]

(111)

Then we can construct operators satisfying an infinite-dimensional Lie algebra

\[ [A_m, A_n] = 4G_{m-n}, \quad [G_m, A_n] = 2A_{n+m} - 2A_{n-m}, \quad \text{and} \quad [G_m, G_n] = 0 \]

(112)

by the following recursion relations:

\[
A_0 = H_0, \quad A_1 = V, \quad A_{n+1} - A_{n-1} = \frac{1}{2}[G_1, A_n], \\
G_1 = \frac{1}{4}[A_1, A_0], \quad \text{and} \quad G_n = \frac{1}{4}[A_n, A_0].
\]

(113)

This Lie algebra is called the Onsager Lie algebra. It is known to be isomorphic to a fixed-point subalgebra of the \( sl(2) \) loop algebra \( \mathcal{L}(sl(2)) \) with respect to the action of a certain involution [53]. We can now use the Onsager algebra to generate an infinite number of conserved quantities [57]:

\[
Q_m = -\frac{1}{2} (A_m + A_{-m} + \lambda A_{m+1} + \lambda A_{-m+1}).
\]

(114)
Thus any such system should be integrable.

In the case of the Ising model we choose $H_0$ and $V$ as above. The Lie
brackets of the cyclix algebra then allows us to verify the Dolan–Grady con-
ditions. Thus we can construct the Onsager algebra as a proper subalgebra
of $\hat{C}_A$. Because of the relationship between the Onsager algebra and the loop
algebras, this suggests a deep relationship between $\hat{C}_A$ and the loop algebras.

Certainly, if we understand some other features of the algebras we have
described in this article better, we will be in a better position to under-
stand Yang–Mills theory, the low energy behavior of string theory, and many
integrable systems.

Acknowledgments

We thank V. John, S. Okubo, T. D. Palev and O.T. Turgut for discussions.
We were supported in part by funds provided by the U.S. Department of
Energy under grant DE-FG02-91ER40685.

Appendix

A Actions of Operators on Physical States in
the Large-$N$ Limit

We are going to illustrate (though not rigorously prove) why in the planar
large-$N$ limit, an operator representing a term in a dynamical variable sends
singlet states to singlet states. We are going to confine ourselves to the action
of operators of the second kind defined in Section 2.1 only. The reasoning is
similar for operators of other kinds.

Assume that the operator is of the form Eq.(10) and the open singlet
state is of the form Eq.(4). Let $\hat{a} = \#(\hat{I})$, $\hat{b} = \#(\hat{J})$ and $\hat{c} = \#(\hat{K})$. So there
are $\hat{a}$ creation operators and $\hat{b}$ annihilation operators for adjoint partons in
the operator of the second kind, and $\hat{c}$ creation operators for adjoint partons
in the open singlet state. There is a factor of $N^{-(\hat{a}+\hat{b})/2}$ in the operator of the
second kind and a factor of $N^{-(\hat{c}+1)/2}$ in the open singlet state, so initially
there is a total factor of $N^{-(\hat{a}+\hat{b}+\hat{c}+1)/2}$. A term in the final state is either
an open singlet state or a product of an open singlet state and a number of
closed singlet states. No matter how many closed singlet states there are in this term, \( N \) should be raised to the power of \(- (\dot{a} - \dot{b} + \dot{c} + 1)/2\) in order that the term survives the large-\( N \) limit. It is therefore that only the operations which produce a factor of \( N\dot{b} \) survive the large-\( N \) limit.

To clarify the argument, we need to refine the diagrammatic representations of an operator of any kind and a singlet state more carefully. Note that in Figs. 19(a) and (c), each square carries one color index whereas each circle carries two color indices. We can put these indices at the ends of the thick or thin lines attaching to them, as is shown in Figs. 19(a) and (b).

![Figure 19](image-url)

**Figure 19:** (a) A typical open singlet state given by Eq. (4). The color indices are explicitly shown. Note that each square carries 1 color index, whereas each circle carries 2 color indices. Moreover, the color indices at the two ends of a connecting solid line are the same. (b) An operator of the second kind given by Eq. (10). Only the color indices of the annihilation operators are shown. (c) The action of an operator of the second kind with no annihilation operator of an adjoint parton on an open singlet state. To get a non-vanishing final state, we need \( \lambda_2 = \rho_1 \). The creation and annihilation operators removed in the final state are depicted as dotted squares or dotted circles. Algebraically the dotted line joining the operator of the second kind and the open singlet state is \( \delta_{\mu a+1}^{\nu_1} \). Clearly the final state is an open singlet state.

Consider the case when there are no annihilation operators for adjoint partons in the operator of the second kind, i.e., \( \dot{b} = 0 \). Fig. 19(c) shows the action of such an operator on an open singlet state. It destroys the solid square in Fig. 19(a) and the hollow square in Fig. 19(b). The ends
of the lines originally attached to the squares are now joined together by a dotted line with an appearance different from other dotted lines in the diagrams. Algebraically this dotted line is a Kronecker delta function. The factor involving \( N \) in the final state is \( N^{-(\dot{a}+\dot{c}+1)/2} \). This is precisely the factor for an open singlet state with \( \dot{a} + \dot{c} \) partons in the adjoint representation.

\[
\lambda_2 = \lambda_3; j_1 = k_1
\]

(a)

\[
\lambda_2 = \lambda_3; j_1 = k_1
\]

(b) \( \lambda_2 = \lambda_3; j_1 = k_6 \)

Figure 20: The action of an operator of the second kind on an open singlet state. (a) The annihilation operator of an adjoint parton in the operator of the second kind contracts with the first creation operator of an adjoint parton in the sequence of adjoint partons in the initial open singlet state. The final state is also an open singlet state. This action survives the large-N limit. (b) Here this annihilation operator contracts with the creation operator of a later adjoint parton in the sequence. The final state is a product of an open singlet and a closed singlet. However, this term is negligible in the large-N limit.

Now consider the case when there is one annihilation operator of an adjoint parton in the operator of the second kind, i.e., \( \dot{b} = 1 \). The action of this operator needs to produce a factor of \( N \) in order for the final state to survive in the large-N limit. Consider the following two cases illustrated separately in Figs. 20(a) and (b). In the former diagram, the annihilation operator of an adjoint parton in the operator of the second kind contracts with the creation operator of the first adjoint parton in the initial open singlet state. This results in a close loop with no squares or circles but two dotted lines only. Algebraically this loop is the factor \( \delta_{\nu_1}^{\nu_1} \delta_{\nu_1}^{\nu_1} \), which in turn is equal to \( N \). The remaining parts of this diagram form an open singlet state. Thus the singlet
state survives the large-$N$ limit. In the latter diagram, the annihilation operator of the adjoint parton in the operator of the second kind contracts with the creation operator of a later adjoint parton in the adjoint parton sequence of the initial open singlet state. This time the final state is a product of an open singlet together with a closed singlet. However, there is no closed loop with solid and dotted lines only. This implies that no extra factor of $N$ is produced and so this term can be neglected in the large-$N$ limit.

Let us now turn to the case when $b$ is an arbitrary positive integer. The annihilation operator of a conjugate parton in the operator of the second kind will contract with the creation operator of a conjugate parton in the initial open singlet state. This will produce 1 dotted line. The $b$ annihilation operators of adjoint partons in the operator of the second kind will contract with $b$ creation operators of adjoint partons in the initial meson state. This

Figure 21: (a) Contraction of circles. See the text for details. (b) The surviving term in the large-$N$ limit of the action of an operator of the second kind with $b$ annihilation operators of adjoint partons on an open singlet state. The final state is also an open singlet state.
will further produce $2b$ dotted lines. Hence there are $2b + 1$ dotted lines. One of these dotted lines has to be recruited to join the creation and annihilation operators in the final state. In order for the final state to survive in the large-$N$ limit, we need a factor of $N^b$, as explained above. Since the minimum number of dotted lines to produce a closed loop without squares or circles is 2, the factor $N^b$ can be obtained only if there are $b$ closed loops, every closed loop has 2 dotted lines only, and there are no circles or squares in any closed loop. This can be done only if the 2 dotted lines join 2 adjacent pairs of creation and annihilation operators. Moreover, only 1 dotted line can be uninvolved in any closed loops. Consider Fig. 21(a), where we contract 2 circles $h_1$ and $s_1$, producing 2 dotted lines $l_1$ and $l_2$. Since one of these dotted lines has to lie within a closed loop, a pair of circles adjacent to $h_1$ and $s_1$ has to be contracted. In Fig. 21(a), $h_2$ and $s_2$ are contracted and hence we obtain $l_3$ and $l_4$. Then either $l_1$ or $l_4$ (or both) has to lie within a closed loop. If we continue this argument, we will obtain Fig. 21(b). As is clear from the figure, the final state is an open singlet state. Thus we conclude that in the large-$N$ limit, an operator of the second kind propagates open singlet states to open singlet states. The actions of operators of other kinds can be understood similarly.

**B  Associativity of the Algebra for the Operators of the Second Kind**

The reader can prove the associativity of this algebra by verifying the following identity:

$$l^i_J (l^I_K l^M_N) = (l^I_K l^M_N) l^i_J = \delta^K_J \delta^M_L l^i_N + \sum_{M_1 M_2 = M} \delta^K_J \delta^{M_1} l^i_N \delta^L_L \delta^{M_2}$$

$$+ \sum_{L_1 L_2 = L} \delta^K_J \delta^{M_1} l^i_N \delta^L_L \delta^{M_2} + \sum_{K_1 K_2 = K} \delta^K_J \delta^{M_1} l^i_N \delta^{K_2}_L \delta^{K_2}_N + \sum_{J_1 J_2 = J} \delta^K_J \delta^{M_1} l^i_N \delta^{J_2}_L \delta^{J_2}_N \delta^{J_2}_N \delta^{J_2}_N$$

$$+ \sum_{K_1 K_2 = K} \delta^K_J \delta^{M_1} l^i_N \delta^{K_2}_L \delta^{K_2}_N + \sum_{L_1 L_2 = L} \delta^K_J \delta^{M_1} l^i_N \delta^{L_2}_L \delta^{L_2}_N \delta^{L_2}_N \delta^{L_2}_N$$

$$\hat{K}_1 K_2 = \hat{K} \quad \hat{M}_1 M_2 = \hat{M} \quad \hat{L}_1 L_2 = \hat{L}$$
\[ \sum \delta^{\hat{K}}_{\hat{J}_1} \delta^{\hat{M}_1 \hat{I}_2}_{\hat{L}_2} \delta^{\hat{M}_2 \hat{I}_1}_{\hat{N}_3} \delta^{\hat{M}_3 \hat{I}_1}_{\hat{N}_3} + \sum \delta^{\hat{K}}_{\hat{J}_1} \delta^{\hat{M}_1 \hat{I}_2}_{\hat{L}_2} \delta^{\hat{M}_3 \hat{I}_1}_{\hat{N}_3} \]

\[ \hat{M}_1 \hat{M}_2 = \hat{M} \]

\[ \hat{M}_1 \hat{M}_2 \hat{M}_3 = \hat{M} \]

Q.E.D.

C Cartan Subalgebra of \( \hat{L}_A \)

This can be seen by the following argument. Let us call the subspace spanned by all the \( l_i^j \)'s \( \mathcal{M} \). From Eq.(53), \( [l_i^j, l_i^j] = 0 \) for any integer sequences \( \hat{I} \) and \( \hat{J} \). Thus \( \mathcal{M} \) is an Abelian subalgebra. In particular, \( \mathcal{M} \) is nilpotent. To proceed on, we need the following two lemmas:

Lemma 1 Let

\[ [l_i^j, l_k^k] = \sum_{k=1}^{n} \alpha_{\hat{M}_k \hat{N}_k} l_i^{\hat{M}_k} \]

where \( n \) is a finite positive integer, \( \hat{M}_k \) and \( \hat{N}_k \) are positive integer sequences such that \( l_i^{\hat{M}_k} \neq l_i^{\hat{M}_{k'}} \) for \( k \neq k' \), and \( \alpha_{\hat{M}_k \hat{N}_k} \) are non-zero numerical coefficients. Then

\[ \#(\hat{M}_k) - \#(\hat{N}_k) = \#(\hat{K}) - \#(\hat{L}) \]

for every \( k = 1, 2, \ldots, n \).

This lemma can be proved by using Eq.(53) with \( \hat{J} = \hat{I} \).

Lemma 2 With the same assumptions as in the previous lemma, we have

\[ \#(\hat{M}_k) + \#(\hat{N}_k) \geq \#(\hat{K}) + \#(\hat{L}) \]

for every \( k = 1, 2, \ldots, n \).

This lemma can also be proved by using Eq.(53) with \( \hat{J} = \hat{I} \). Let \( m \) be a positive integer. Now we are ready to show for arbitrary non-zero complex numbers \( \beta_{\hat{L}_i}^{\hat{K}_i} \), where \( i = 1, 2, \ldots, m \) and arbitrary integer sequences \( \hat{L}_i \)'s and
\( \hat{K}_i \)'s such that \( l_{K_i}^j \neq l_{K_i}^{j'} \) for \( i \neq i' \), and \( K_i \neq L_i \) for at least one \( i \), that there exists a sequence \( I \) such that

\[
[l_{I_i}^j, \sum_{i=1}^{m} \beta_{K_i}^j l_{K_i}^j]
\]
does not belong to \( M \). Indeed, let \( j \) be an integer such that

1. \( \#(\hat{K}_j) - \#(\hat{L}_j) \geq \#(\hat{K}_i) - \#(\hat{L}_i) \) for all \( i = 1, 2, \ldots, m \); and
2. \( \#(\hat{K}_j) + \#(\hat{L}_j) \leq \#(\hat{K}_i) + \#(\hat{L}_i) \) for any \( i = 1, 2, \ldots, m \) such that \( \#(\hat{K}_j) - \#(\hat{L}_j) = \#(\hat{K}_i) - \#(\hat{L}_i) \).

If \( \#(\hat{K}_j) \geq \#(\hat{L}_j) \), then consider

\[
[l_{K_j}^j, \sum_{i=1}^{m} \beta_{K_i}^j l_{K_i}^j] = \beta_{K_j}^j l_{K_j}^j - \beta_{K_j}^j \sum_{K_j, K_j = K_j} \delta_{L_j}^{K_j} l_{K_j}^{K_j} + \Gamma,
\]

where

\[
\Gamma = [l_{K_j}^j, \sum_{i=1}^{m} \beta_{K_i}^j l_{K_i}^j]
\]

\[
= \sum_{i=1}^{m} \sum_{k=1}^{n_i} \beta_{K_i}^j l_{K_i}^j l_{N_i}^k + \Gamma_{11} l_{M_i}^k + \Gamma_{12}
\]

(116)

where each \( n_i \) for \( i = 1, 2, \ldots, m \) but \( i \neq j \) is dependent on \( i \). Let us assume that

\[
l_{M_i}^k = l_{L_i}^j
\]

(117)

for some \( i \in \{1, 2, \ldots, m\} \) but \( i \neq j \) and \( k \in \{1, 2, \ldots, n_i\} \). Then \( \#(M_{i_k}) = \#(K_j) \) and \( \#(N_{i_k}) = \#(L_j) \). By Lemmas 17 and 18 we get \( \#(K_i) = \#(K_j) \) and \( \#(L_i) = \#(L_j) \). However, we also know that \( K_i \neq K_j \) or \( L_i \neq L_j \) and so there is no \( k \in \{1, 2, \ldots, n_i\} \) such that Eq.(117) holds. This leads to a contradiction and so we conclude that

\[
l_{M_i}^k \neq l_{L_i}^j
\]

63
for all \( i \in \{1, 2, \ldots, m\} \) but \( i \neq j \) and \( k \in \{1, 2, \ldots, n_i\} \). From Eqs. (113) and (116), we deduce that \([l^j_{K_j}, \sum_{i=1}^m \beta_{K_i}^L L_i]\) does not belong to \( \mathcal{M} \). Similarly, if \#(\( \hat{K}_j \)) \leq \#(\( \hat{L}_j \)), then \([l^i_{L_i}, \sum_{i=1}^m \beta_{K_i}^L L_i]\) does not belong to \( \mathcal{M} \). Hence, the normalizer of \( \mathcal{M} \) is \( \mathcal{M} \) itself. We therefore conclude that \( \mathcal{M} \) is a Cartan subalgebra of the algebra \( \hat{L}_\Lambda \). Q.E.D.

D Root Vectors of \( \hat{L}_\Lambda \)

All we need to do is to show that any root vector has to be of the form given by Eq. (56). Let \( f \equiv \sum_{\hat{P}, \hat{Q}} a_{\hat{P}}^{\hat{Q}} \hat{P}\hat{Q} \), where only a finite number of the numerical coefficients \( a_{\hat{P}}^{\hat{Q}} \neq 0 \), be a root vector. In addition, we can assume without loss of generality that \( \hat{P} \neq \hat{Q} \) if \( a_{\hat{P}}^{\hat{Q}} = 0 \). We can deduce from Eq. (11) that

\[
[l_{j}^{I} s_{K}] = \sum_{K_1 K_2 = K} \delta_{j}^{K_1} s_{K_2}.
\]

Hence,

\[
[l_{P}^{I} s_{K}] = \sum_{I} \sum_{K_1 K_2 = K} \delta_{I}^{K_1} \delta_{P}^{P_{K_2}} s_{I}.
\]

Therefore,

\[
[l_{M}^{I} f] s_{K} = \sum_{I, P, Q} a_{\hat{P}}^{\hat{Q}} \hat{P}\hat{Q} \sum_{I_{1} I_{2} = I} \delta_{I_{1}}^{M} \delta_{I_{2}}^{K_{1}} \delta_{P}^{P_{K_{2}}} - \delta_{I_{1}}^{M} \delta_{I_{2}}^{K_{1}} \delta_{Q}^{Q_{K_{2}}} s_{I_{1}} \delta_{Q}^{Q_{I_{2}}},
\] (118)

Since \( f \) is a root vector, we have

\[
[l_{M}^{I} f] = \lambda_{M} \sum_{P, Q} a_{\hat{P}}^{\hat{Q}} \hat{P}\hat{Q}
\] (119)

where \( \lambda_{M} \) is a root. As a result, we can combine Eqs. (118) and (119) to obtain

\[
\sum_{P, Q} a_{\hat{P}}^{\hat{Q}} \sum_{I_{1} I_{2} = I} \delta_{I_{1}}^{M} \delta_{I_{2}}^{K_{1}} \delta_{P}^{P_{K_{2}}} - \delta_{I_{1}}^{M} \delta_{I_{2}}^{K_{1}} \delta_{Q}^{Q_{K_{2}}} - \sum_{P, Q} a_{\hat{P}}^{\hat{Q}} \lambda_{M} \sum_{K_{1} K_{2} = K} \delta_{I_{1}}^{K_{1}} \delta_{P}^{P_{K_{2}}} = 0
\] (120)
for any integer sequences \( \hat{\imath}, \hat{\kappa} \) and \( \hat{M} \).

Let us find an \( a^S_R \) in \( f \) such that \( \hat{R} \neq \hat{S}, a^S_R \neq 0 \), and \( a^S_{R_1} = 0 \) for all \( R_1 \)’s and \( S_1 \)’s such that \( R_1 \hat{R}_2 = \hat{R} \) and \( S_1 \hat{S}_2 = \hat{S} \) for some \( \hat{R}_2 \) and \( \hat{S}_2 \). The reader can easily convince himself or herself that such an \( a^S_R \) always exists. Let us choose \( \hat{\imath} = \hat{R} \) and \( \hat{\kappa} = \hat{S} \) in Eq.(120). Then we obtain from this equation that

\[
\lambda_M = \sum_{R_1, R_2 = \hat{R}} \delta^R_{R_1} - \sum_{S_1, S_2 = \hat{S}} \delta^S_{S_1},
\]

(121)

Therefore,

\[
\lambda_M - \sum_{i=1}^\Lambda \lambda_{M_i} = \delta^R_M - \delta^S_M.
\]

Thence

\[
\begin{bmatrix}
\delta^R_M - \sum_{i=1}^\Lambda l^R_{\hat{R}_i}, f \\
\delta^S_M - \sum_{i=1}^\Lambda l^S_{\hat{S}_i}, f
\end{bmatrix}
\neq 0; \text{ and}
\]

\[
\begin{bmatrix}
l^S_{\hat{S}_i}, f
\end{bmatrix}
\neq 0.
\]

However, both \( l^R_{\hat{R}} - \sum_{i=1}^\Lambda l^R_{\hat{R}_i} \) and \( l^S_{\hat{S}} - \sum_{i=1}^\Lambda l^S_{\hat{S}_i} \in F_\Lambda \). Thus \( f \in F_\Lambda \). Now Eq.(\text{57}) shows clearly that \( f = f^R_{\hat{S}} \). The same equation also shows that each root vector space must be one-dimensional. Q.E.D.

### E Lie Bracket of \( \hat{\Sigma}_\Lambda \)

We are going to show that the commutator between two operators of the fourth kind, Eq.(\text{73}), defines a Lie bracket between them. This can be done by showing that Eq.(\text{73}) satisfies Eq.(\text{72}). This involves a tedious computation involving a delta function defined in Appendix A of Ref.[20]. The properties of this delta function will be extensively used. Let us consider the action of the commutator of two \( \sigma \)’s on \( s^P \). If \( \hat{P} \) is empty, then Eq.(\text{73}) certainly satisfies Eq.(\text{72}). Therefore we only need to consider the case when \( \hat{P} \) is not empty. In this case we can simply write \( \hat{P} \) as \( P \). Then the action of the commutator on \( s^P \) is

\[
\left[ \sigma_I^L, \sigma^K_J \right] s^P = \sum_Q \left( \delta^I_Q \delta^K_J \delta_L^P + \sum_C \delta^I_Q \delta^C_J \delta^K_L \delta^P_C + \sum_D \delta^I_Q \delta^K_J \delta^D_L \delta^P_D \right)
\]

65
Each of the terms on the R.H.S. of the above equations can be rewritten as follows:

\begin{align*}
\delta^I_j \delta^K_j \delta^P_L &= \delta^K_j \delta^P_L; \\
\sum_C \delta^I_C \delta^K_C \delta^P_L &= \sum_{J_1 J_2 = J} \delta^K_{J_2} \delta^I_{J_1} \delta^P_L; \\
\sum_D \delta^I_D \delta^K_D \delta^P_L &= \sum_{J_1 J_2 = J} \delta^K_{J_2} \delta^I_{J_1} \delta^P_L; \\
\sum_{C,D} \delta^I_C \delta^K_D \delta^P_L &= \sum_{J_1 J_2 J_3 = J} \delta^K_{J_2} \delta^I_{J_1} \delta^P_{J_3}; \\
\sum_A \delta^I_A \delta^K_A \delta^P_L &= \sum_{K_1 K_2 = K} \delta^K_{J_2} \delta^I_{J_1} \delta^P_L; \\
\sum_{A,C} \delta^I_A \delta^K_C \delta^P_L &= \sum_{K_1 K_2 = K} \sum_{E} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I} + \sum_{E} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I} + \sum_{J_1 J_2 = J} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I}; \\
\sum_{A,D} \delta^I_A \delta^K_D \delta^P_L &= \sum_{F} \delta^K_{J_2} \delta_{Q}^{K F I} \delta_{LF}^{P} + \sum_{J_1 J_2 = J} \delta^K_{J_2} \delta_{Q}^{K I} \delta_{LJ}^{P} + \sum_{K_1 K_2 = K} \delta^K_{J_2} \delta_{Q}^{K I} \delta_{LJ}^{P}; \\
\sum_{A,C,D} \delta^I_A \delta^K_C \delta^P_L &= \sum_{J_1 J_2 J_3 = J} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I} + \sum_{J_1 J_2 = J} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I} + \sum_{E,F} \delta^K_{J_2} \delta_{Q}^{K E I} \delta_{EL}^{I};
\end{align*}

(122)
\[
\sum_{B} \delta_{Q}^{IB} \delta_{JB}^K \delta_{L}^{p} = \sum_{K_{1}K_{2}=K} \delta_{j}^{K_{1}} \delta_{Q}^{IK_{2}} \delta_{L}^{p};
\]
\[
\sum_{B,C} \delta_{Q}^{IB} \delta_{JB}^K \delta_{CL}^{p} = \sum_{J_{1}J_{2}=J} \delta_{j_{2}}^{K_{1}} \delta_{Q}^{IK_{2}} \delta_{j_{1}L}^{p} + \sum_{F} \delta_{j}^{K} \delta_{Q}^{L} \delta_{j_{2}L_{F}}^{p};
\]
\[
\sum_{B,D} \delta_{Q}^{IB} \delta_{JB}^K \delta_{LD}^{p} = \sum_{J_{1}J_{2}=J} \sum_{F} \delta_{j_{1}}^{K_{1}} \delta_{Q}^{IF} \delta_{j_{2}F}^{L} + \sum_{F} \delta_{j}^{K} \delta_{Q}^{L_{F}} \delta_{j_{1}L_{F}}^{p};
\]
\[
\sum_{B,C,D} \delta_{Q}^{IB} \delta_{JB}^K \delta_{CLD}^{p} = \sum_{J_{1}J_{2}=J} \sum_{F,G} \delta_{j_{1}}^{K_{1}} \delta_{Q}^{IF} \delta_{j_{2}F}^{L_{G}} + \sum_{G} \delta_{j}^{K} \delta_{Q}^{L_{G}} \delta_{j_{2}L_{G}}^{p};
\]
\[
\sum_{A,B} \delta_{Q}^{AB} \delta_{AJB}^K \delta_{L}^{p} = \sum_{K_{1}K_{2}K_{3}=K} \delta_{j}^{K_{2}} \delta_{Q}^{IK_{3}} \delta_{L}^{p};
\]
\[
\sum_{A,B,C} \delta_{Q}^{AB} \delta_{AJB}^K \delta_{CL}^{p} = \sum_{K_{1}K_{2}K_{3}=K} \sum_{E} \delta_{j_{1}}^{K_{1}} \delta_{Q}^{IK_{2}} \delta_{j_{2}E}^{L_{L}} + \sum_{E} \sum_{J_{1}J_{2}=J} \delta_{j_{2}}^{K_{1}} \delta_{Q}^{E_{Q}} \delta_{j_{1}E_{Q}}^{L_{J_{1}L}};
\]
\[
\sum_{A,B,D} \delta_{Q}^{AB} \delta_{AJB}^K \delta_{LD}^{p} = \sum_{F,G} \delta_{j_{1}}^{K_{1}} \delta_{Q}^{IF} \delta_{j_{2}F}^{L_{G}} + \sum_{G} \sum_{J_{1}J_{2}=J} \delta_{j_{2}}^{K_{1}} \delta_{Q}^{E_{Q}} \delta_{j_{1}E_{Q}}^{L_{J_{1}L}};
\]
\[
\sum_{B,C,D} \delta_{Q}^{IB} \delta_{JB}^K \delta_{CLD}^{p} = \sum_{E,F,G} \delta_{Q}^{E_{Q}} \delta_{Q}^{E_{Q}} + \sum_{E,G} \delta_{Q}^{E_{Q}} \delta_{Q}^{E_{Q}}.
\]
We can now substitute the expressions in Eq. (123) into Eq. (122) to get

\[ [\sigma^I_j, \sigma^K_L] s^P = \]

\[ \sum_Q \left\{ \delta^K_Q (\delta^I_Q \delta^P_L + \sum_E \delta^I_Q \delta^P_E + \sum_F \delta^I_F \delta^P_L + \sum_E \delta^E^I_F \delta^P_E) \right\} + \sum_{J_1 J_2 = J} \sum_{K_1 K_2 = K} \delta^K_{J_2} \delta^I_{J_1 L} + \sum_E \delta^E^I_{K_2} \delta^P_{EJL} + \sum_{J_1 J_2 = J} \sum_{E,F} \delta^K_{J_2} \delta^I_{J_1} \delta^P_{ELF} + \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \delta^P_{ELF} \]

\[ + \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \sum_{E,F} \delta^E^I_{K_2} \delta^P_{ELF} + \sum_{J_1 J_2 = J} \sum_{E,F} \delta^K_{J_2} \delta^I_{J_1} \delta^P_{ELF} + \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \sum_{E,F} \delta^E^I_{K_2} \delta^P_{ELF} \]

\[ + \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \sum_{E,F} \delta^E^I_{K_2} \delta^P_{ELF} + \sum_{J_1 J_2 = J} \sum_{E,F} \delta^K_{J_2} \delta^I_{J_1} \delta^P_{ELF} + \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \sum_{E,F} \delta^E^I_{K_2} \delta^P_{ELF} \]

\[ + \sum_{J_1 J_2 = J} \sum_{K_1 K_2 = K} \delta^K_{J_1} \delta^I_{J_2} \sum_{E,F} \delta^E^I_{K_2} \delta^P_{ELF} \]
where

\[
\Gamma = \delta_{Q}^{IK}\delta_{JL} + \sum_{E} E^{IK}\delta_{EJL} + \sum_{F} F^{IK}\delta_{JFL} + \sum_{G} G^{IK}\delta_{JLG} \\
+ \sum_{E,F} E^{IKF}\delta_{EJFL} + \sum_{F,G} F^{IKG}\delta_{JFLG} + \sum_{E,G} E^{IKG}\delta_{EJLG} \\
+ \sum_{E,F,G} E^{IKFG}\delta_{EJFLG} + (I \leftrightarrow K, J \leftrightarrow L)
\]  

(125)

If we substitute Eq.(124) without \(\Gamma\) into Eq.(122), we will obtain exactly the action of operators on the R.H.S. of Eq.(73) on \(s^{P}\). \(\Gamma\) is reproduced when \(I\) and \(K\) are interchanged with \(J\) and \(L\), respectively, in Eq.(124) and so it is cancelled. Consequently, Eq.(73) is true. Q.E.D.

**F Root Vectors of \(\hat{\Sigma}_{\Lambda}\)**

We need to show that any root vector has to be of the form given by Eq.(82). Let \(f \equiv \sum_{P,Q} a_{P}^{Q}\sigma_{P}^{Q}\), where only a finite number of the numerical coefficients \(a_{P}^{Q} \neq 0\), be a root vector. In addition, we can assume without loss of generality that \(P \neq Q\) if \(a_{P}^{Q} = 0\). Recall from Eq.(71) that

\[
\sigma_{J}^{I} s^{K} = \delta_{J}^{I} s^{I} + \sum_{K_{1}K_{2}=K} \delta_{J}^{K_{1}} s^{I} K_{1} + \sum_{K_{1}K_{2}=K} \delta_{J}^{K_{2}} s^{I} K_{2} + \sum_{K_{1}K_{2}K_{3}=K} \delta_{J}^{K_{2}} s^{I} K_{1} K_{3}
\]

69
Hence,
\[
\sigma_Q^P s^J = \sum_{I} \left( \delta_Q^{J_I} \delta_P^I + \sum_{J_1, J_2=J} \delta_Q^{J_1 J_2} \delta_P^{I_{J_1}} + \sum_{J_1} \delta_Q^{J_1 J_P} \delta_J^I \delta_P^{J_2} \right)
\]
\[
\sum_{J_1, J_2, J_3} \delta_Q^{J_2 J_1 J_3} s^J.
\]

Therefore,
\[
\left[ \sigma_M^M, f \right] s^K = \sum_{I} \left( \delta_M^{I J} \delta_P^J + \sum_{K_1, K_2 = K} \delta_M^{I J} \delta_Q^{K_1 P} + \sum_{K_1, K_2 = K} \delta_M^{I J} \delta_Q^{K_2 P} \right)
\]
\[
+ \sum_{I_1 I_2 = I} \delta_M^{I_1 J_1} \delta_Q^{I_2 J_1} + \sum_{I_1 I_2 = I} \delta_M^{I_1 J_1} \delta_Q^{I_2 J_1}
\]
\[
+ \sum_{I_1 I_2 = I} \delta_M^{I_1 J_1} \delta_Q^{I_2 J_1} \delta_K \delta_P^I
\]
\[
- (P \leftrightarrow M \text{ in the superscripts, } M \leftrightarrow Q \text{ in the subscripts}) \quad (126)
\]
Since $f$ is a root vector, we have

\[
[\sigma^M_M, f] = \lambda_M \sum_{P,Q} a_P^Q a_Q^P. \tag{127}
\]

where $\lambda_M$ is a root. As a result, we can combine Eqs. (126) and (127) together to obtain an equation which is too long to be written down here for any integer sequences $I$, $K$ and $M$.

Let us find an $a^S_R$ in $f$ such that $R \neq S$, $a^S_R \neq 0$, $a^S_{R_1} = a^S_{R_2} = 0$ for all $R_1$’s, $S_1$’s, $R_2$’s and $S_2$’s such that $R_1 R_2 = R$ and $S_1 S_2 = S$, and $a^S_{R_2} = 0$ for all $R_2$’s and $S_2$’s such that $R_1 R_2 R_3 = R$ and $S_1 S_2 S_3 = S$ for some $R_1, R_3, S_1$ and $S_3$. The reader can easily convince himself or herself that such an $a^S_R$ always exists. Let us choose $I = R$ and $K = S$ in Eq. (126). Then when we combine Eqs. (126) and (127), we get

\[
\lambda_M = \delta^R_M + \sum_{R_1 R_2 = R} \delta^R_{R_1} + \sum_{R_1 R_2 = R} \delta^R_{R_2} - \sum_{S_1 S_2 = S} \delta^M_S - \sum_{S_1 S_2 = S} \delta^M_{S_1} - \sum_{S_1 S_2 S_3 = S} \delta^M_{S_2}. \tag{128}
\]

Therefore, we obtain after some manipulation that

\[
\lambda_M - \sum_{j=1}^\Lambda \lambda_{Mj} - \sum_{i=1}^\Lambda \lambda_{iM} - \sum_{i,j=1}^\Lambda \lambda_{iMj} = \delta^R_M - \delta^M_S.
\]

This means

\[
[f^{R_R}_R, f] \neq 0; \text{ and } [f^{S_S}_S, f] \neq 0.
\]

Since $F_{\Lambda}$ is a proper ideal, so $f \in F_{\Lambda}$. Now Eq. (82) shows clearly that $f = f^{R_R}_R$. The same equation also shows that each root vector space must be one-dimensional. Q.E.D.

### G Product of Two Color-Invariant Operators

We will show by contradiction that the product of two color-invariant operators is in general not well defined. Consider the case when $\Lambda = 1$, and
assume that the operators $\alpha^\mu_\nu(1)$ and $\alpha^\nu_\mu(1)$ are bosonic. Let $\gamma^a_\nu = \gamma^1_{11\ldots1}$ and $\tilde{f}^{(a)} = f^{11\ldots1}_{11\ldots1}$, where the number 1 shows up $a$ times in the superscript and $b$ times in the subscript of $\gamma$. Moreover, let $\Psi^{(c)} = \Psi^{(11\ldots1)}$ and $s^c = s^{11\ldots1}$, where the number 1 shows up $c$ times.

Assume that $\gamma^1_1\gamma^1_1 = \sum_{p=1}^r \alpha_p \gamma^p + \sum_{q=1}^s \beta_q \tilde{f}^{(q)}$, where $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_s$ are non-zero complex numbers for some positive integers $r$ and $s$. Then from the equations $\gamma^1_1\gamma^1_1(s^1) = \gamma^1_1(\gamma^1_1 s^1) = 1^2 s^1$, $\gamma^1_1\gamma^1_1(s^2) = \gamma^1_1(\gamma^1_1 s^2) = 2^2 s^1$, and $\gamma^1_1\gamma^1_1(s^3) = \gamma^1_1(\gamma^1_1 s^3) = r^2 s^1$, we deduce that $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_r = 2$. Hence $\gamma^1_1\gamma^1_1 = \gamma^1_1 + 2 \sum_{p=2}^r \gamma^p + \sum_{q=1}^s \beta_q \tilde{f}^{(q)}$. However, $\gamma^1_1\gamma^1_1(s^{r+1}) = (r + 1)^2 s^{r+1}$ and $\gamma^1_1 + 2 \sum_{p=2}^r \gamma^p + \sum_{q=1}^s \beta_q \tilde{f}^{(q)}(s^{r+1}) = (r^2 + 2r - 1)s^{r+1}$, leading to a contradiction.

Thus we assume instead that $\gamma^1_1\gamma^1_1 = \sum_{q=1}^s \beta_q \tilde{f}^{(q)}$, where the $\beta_q$’s are non-zero complex numbers. However, $\gamma^1_1\gamma^1_1(\Psi^{(s+1)}) = (s+1)^2 \Psi^{(s+1)}$ whereas $\sum_{q=1}^s \beta_q \tilde{f}^{(q)} \Psi^{(s+1)} = 0$, leading to a contradiction, too. Consequently, it is impossible to write $\gamma^1_1\gamma^1_1$ as a finite linear combination of $\gamma$’s and $\tilde{f}$’s.

This proof can be easily generalized to include fermions and to the case $\Lambda > 1$. Q.E.D.

## H Grand String Algebra

We would like to show that the binary operations given in Section 4.2 are Lie superbrackets. Thus they constitute a Lie superalgebra.

Define the following operators:

\[
l^{(1)}_{j} \equiv \gamma^{(1)}_{j} - \sum_{i=1}^{\Lambda+2\Lambda} \gamma^{(1)}_{i,j} - \tilde{f}^{(1)}_{j};
\]

\[
\gamma^{(1)}_{j} \equiv \gamma^{(1)}_{j} - \sum_{j=1}^{\Lambda+2\Lambda} \gamma^{(1)}_{j,j} - \tilde{f}^{(1)}_{j} \quad \text{and}
\]

\[
f^{(1)}_{j} \equiv l^{(1)}_{j} - \sum_{j=1}^{\Lambda+2\Lambda} l^{(1)}_{j};
\]

\footnote{This proves our assertion at the beginning of Section 3.2, namely that the product of two operators of the fourth kind cannot be written as a finite linear combination of operators of this kind.}

72
\[ I = r_I J - \sum_{i=1}^{\Lambda+2\Lambda_F} r_{iJ}^{nI} \]
\[ J = \gamma_{iJ}^n - \sum_{i=1}^{\Lambda+2\Lambda_F} \sum_{j=1}^{\Lambda+2\Lambda_F} \gamma_{ij}^{nI} + \sum_{i,j=1}^{\Lambda+2\Lambda_F} \gamma_{ij}^{nI} - \sum_{i=1}^{\Lambda+2\Lambda_F} \sum_{j=1}^{\Lambda+2\Lambda_F} \gamma_{ij}^{nI} + \sum_{i,j=1}^{\Lambda+2\Lambda_F} \gamma_{ij}^{nI} \tag{129} \]

The reader can verify that

\[ l_J s^{IK} = \sum_{K_1,K_2=K} \delta_{K_1}^{K_2} s^{IK_1K_2}; \]
\[ r_J s^{IK} = \sum_{K_1,K_2=K} \delta_{K_2}^{K_1} s^{IK_1K_1}; \]
\[ f_J s^{IK} = \delta_{K}^{K} s^{IK}; \tag{130} \]

and

\[ l_J \Psi^{ik} = r_J \Psi^{ik} = f_J \Psi^{ik} = 0. \tag{131} \]

Consider the subspace of the singlet states spanned by all states of the form

\[ \bar{\phi}^{\rho_1} \otimes s^{K} \otimes \phi^{\rho_2} \equiv s^{\rho_1+\Lambda,K,\rho_2+\Lambda+\Lambda_F} \text{ and } \Psi^K \equiv \Psi^{ik}, \tag{132} \]

where any integer in \( K \) and \( \bar{K} \) is between 1 and \( \Lambda \) inclusive, and \( 1 \leq \rho_1, \rho_2 \leq \Lambda_F \). (The justification of the use of the direct products \( \otimes \) will be obvious shortly.) Eq.(87) tells us that this definition of \( \Psi^K \) satisfies Eq.(6):

\[ \Psi^K = \Psi^{K_2K_1}. \tag{133} \]

Next, consider a subset of color-invariant operators in the heterix algebra consisting of all finite linear combinations of

\[ \gamma_{ij}^l \equiv \gamma_{ij}^l; \]
\[ \tilde{f}_J^l \equiv \tilde{f}_J^l; \]
\[ \Xi_{\lambda_2}^\lambda \otimes l_{\bar{j}}^l \equiv l_{\lambda_1+\Lambda,\lambda_2+\Lambda}; \]
\[ r_{ij}^l \otimes \Xi_{\lambda_4}^\lambda \equiv r_{\lambda_3+\Lambda+\Lambda_F,\lambda_4+\Lambda}; \text{ and} \]
\[ \Xi_{\lambda_2}^\lambda \otimes f_{ij}^l \otimes \Xi_{\lambda_4}^\lambda \equiv f_{\lambda_1+\Lambda,\lambda_2+\Lambda+\Lambda_F}^{\lambda_3+\Lambda+\Lambda_F}. \tag{134} \]
where any integer in $I$, $J$, $\hat{I}$ or $\hat{J}$ is between 1 and $\Lambda$ inclusive, and $1 \leq \lambda_1, \lambda_2, \lambda_3$ and $\lambda_4 \leq \Lambda_F$. (Again it will be obvious shortly why the direct products are appropriate.) It can be shown that this subset of color-invariant operators form a subalgebra of the heterix algebra. *A fortiori*, this subset forms a Lie algebra. Moreover, the subspace of the states defined above is a representation space for this Lie algebra, albeit a *reducible* one according to Eq. (104).

References

[1] R. Brock *et al.*, Rev. Mod. Phys. 67, 157 (1995).

[2] G. S. Krishnaswami and S. G. Rajeev, Phys. Lett. B 441 429 (1998).

[3] S. G. Rajeev, e-print hep-th/9905072.

[4] F. A. Berezin, Commun. Math. Phys. 63, 131 (1978).

[5] L. Yaffe, Rev. Mod. Phys. 54, 407 (1982).

[6] S. G. Rajeev, Int. J. Mod. Phys. A 9, 5583 (1994).

[7] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989).

[8] E. Witten, Nucl. Phys. B 160, 57 (1978).

[9] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985).

[10] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, 1994).

[11] J. Polchinski, *String Theory*, Vols. 1 and 2 (Cambridge University Press, Cambridge, 1998).

[12] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd edition (Princeton University Press, Princeton, N.J., 1992).
[13] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods in Supersymmetry and Supergravity, or a Walk through Superspace* (Institute of Physics Publishing, Bristol, Philadelphia, 1995).

[14] T. Banks, W. Fischler, S. H. Shenkar and L. Susskind, Phys. Rev. D **55**, 5112 (1997).

[15] J. Hubbard, Proc. Royal Soc. London, Ser. A **276**, 238 (1963).

[16] M. A. Virasoro, Phys. Rev. D **1**, 2933 (1970).

[17] P. Goddard and D. Olive, Int. J. Mod. Phys. A **1**, 303 (1986).

[18] G. t’ Hooft, Nucl. Phys. B **72**, 461 (1974).

[19] C. B. Thorn, Phys. Rev. D **20**, 1435 (1979).

[20] C.-W. H. Lee and S. G. Rajeev, Nucl. Phys. B **529**, 656 (1998).

[21] C.-W. H. Lee and S. G. Rajeev, J. Math. Phys. **39**, 5199 (1998).

[22] S. Dalley and I. R. Klebanov, Phys. Rev. D **47**, 2517 (1993).

[23] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Wiley Classics Library Edition (Wiley Interscience, 1996).

[24] F. Antonuccio and S. Dalley, Nucl. Phys. B **461**, 275 (1996).

[25] F. Antonuccio and S. Dalley, Phys. Lett. B **376**, 154 (1996).

[26] F. David, Nucl. Phys. B **257**, 45 (1985).

[27] V. A. Kazakov, Phys. Lett. B **150**, 28 (1985).

[28] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. **254**, 1 (1995).

[29] O. Bergman and C. B. Thorn, Phys. Rev. D **52**, 5980 (1995).

[30] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B **500**, 43 (1997).
[31] C. Domb, in *Ising Model*, edited by C. Domb and M. S. Green, Phase Transitions and Critical Phenomena, Vol. 3 (Academic Press, London, New York, 1974).

[32] J. Ambjorn, B. Durhuus and T. Jonsson, *Quantum Geometry — A Statistical Field Theory Approach*, (Cambridge University Press, Cambridge, 1997).

[33] H. A. Bethe, Z. Phys. 71, 205 (1931).

[34] C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).

[35] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[36] I. R. Klebanov and L. Susskind, Nucl. Phys. B 309, 175 (1988).

[37] L. Onsager, Phys. Rev. 65, 117 (1944).

[38] E. Fradkin and L. Susskind, Phys. Rev. D 17, 2637 (1978).

[39] J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).

[40] C.-W. H. Lee and S. G. Rajeev, Phys. Rev. Lett. 80, 2285 (1998).

[41] F. C. Alcaraz, M. N. Barber and M. T. Batchelor, Phys. Rev. Lett. 58, 771 (1987).

[42] C.-W. H. Lee and S. G. Rajeev, Phys. Lett. B 436, 91 (1998).

[43] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, Comm. Math. Phys. 59, 35 (1978).

[44] M. L. Mehta, Comm. Math. Phys. 79, 327 (1981).

[45] M. R. Douglas, Phys. Lett. B 238, 176 (1990).

[46] J. Cuntz, Commun. Math. Phys., 57, 173 (1977).

[47] D. E. Evans, Publ. RIMS, Kyoto Univ. 14, 383 (1980).

[48] H. Zassenhaus, Hamb. Abh. 13, 1 (1939).
[49] H. J. Chang, Hamb. Abh. 14, 151 (1941).

[50] G. B. Seligman, Modular Lie Algebras, Ergebnisse der Mathematik und Ihrer Grenzgebibete Band 40 (Springer-Verlag, Berlin Heidelberg, 1967).

[51] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory (Springer-Verlag, New York, 1972).

[52] G. J. Murphy, $C^*$-Algebras and Operator Theory (Academic Press, San Diego, 1990).

[53] B. Davies, J. Phys. A: Math. Gen. 23, 2245 (1990).

[54] L. Dolan and M. Grady, Phys. Rev. D 25, 1587 (1982).

[55] B. Davies, J. Math. Phys. 32, 2945 (1991).

[56] S.-S. Roan, Max Planck Institute, Bonn, Report No. MPI/91-70 (1991).

[57] A. Honecker, Ph. D. thesis, hep-th/9503104.