fBLS – a fast-folding BLS algorithm

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ABSTRACT
We present fBLS – a novel fast-folding technique to search for transiting planets, based on the fast-folding algorithm (FFA), which is extensively used in pulsar astronomy. For a given lightcurve with \( N \) data points, fBLS simultaneously produces all the binned phase-folded lightcurves for an array of \( N_p \) trial periods. For each folded lightcurve produced by fBLS, the algorithm generates the standard BLS periodogram and statistics. The number of performed arithmetic operations is \( O(N_p \cdot \log N_p) \), while regular BLS requires \( O(N_p \cdot N) \) operations. fBLS can be used to detect small rocky transiting planets, with periods shorter than one day, a period range for which the computation is extensive. We demonstrate the capabilities of the new algorithm by performing a preliminary fBLS search for planets with ultra-short periods in the Kepler main-sequence lightcurves. In addition, we developed a simplistic signal validation scheme for vetting the planet candidates. This two-stage preliminary search identified all known ultra-short planet candidates and found three new ones.

Key words: planets and satellites: detection – methods: data analysis – techniques: photometric

1 INTRODUCTION
Exoplanets with orbital periods shorter than 1 day are usually called ultra-short-period (USP) planets. This somewhat arbitrary division should serve as a criterion for periodicity search campaigns rather than describing an independent class of planets. However, since most USP planets are terrestrial (e.g., Winn et al. 2018), this term is often used to describe rocky planets that reside within a few stellar radii from their host stars.

It is expected that studies of these extreme systems will shed light on various aspects of planet formation, star-planet interaction, and orbital evolution (Owen & Wu 2013; Lopez 2017; Millholland & Spalding 2020). Constraints on the statistical properties of the USP planet population are of particular interest; for example, several studies targeted the distribution of their orbital inclinations (e.g., Dai et al. 2018) and sizes (e.g., Uzsoy et al. 2021). These studies build upon the known population of ~200 USP planets and planet candidates, most of which were discovered by Kepler, either during the primary mission of the spacecraft or as a part of its second, K2, phase (e.g., Sanchis-Ojeda et al. 2014; Adams et al. 2021).

USP planets should be relatively easy to detect because planets at short orbital periods are more likely to transit their host star and produce more transit signals than planets at longer orbital periods. Therefore, one might expect that photometric surveys would favor their detection. However, since USP planets are mostly terrestrial, ground-based surveys often lack the precision required for this task. As a result, most USP planets were discovered in high-precision data, using designated search algorithms (e.g., Ofir & Dreizler 2013; Sanchis-Ojeda et al. 2014; Caceres et al. 2019; Adams et al. 2021).

A significant challenge of the search for USP planets is the intensive computation involved in discovering signals with relatively short periodicity. Consider, for example, an evenly-sampled lightcurve with a temporal baseline, \( \tau \), of four years. In order for two trial frequencies tested by the search algorithm to be independent, they should be separated by about \( 1/\tau \approx 0.0007 \) cycles per day (e.g., Vanderplas 2018). Therefore, the resulting number of trial periods between \( 1–1000 \) days is \( \approx 1,400 \), while between \( 0.1–1 \) day there are \( \approx 13,000 \). Consequently, the time required to perform a periodicity search in the USP frequency regime can become relatively long.

In practice, the number of required trial periods may be larger than the number of independent frequencies, even if the lightcurve is uniformly sampled. This is because the accumulated inaccuracy over the entire baseline should be smaller than the transit duration, which spans ~10% of the orbital phase in the USP regime. For example, if one allows up to ~25% inaccuracy compared to the USP transit duty-cycle, the frequency grid spacing should be \( \approx 0.025/\tau \), yielding \( \approx 500,000 \) trial periods in the range \( 0.1–1 \) day.

In order to reduce the computation involved, one can make use of efficient periodicity detection algorithms. For example, Ofir & Dreizler (2013) focused their search on a carefully selected grid of trial periods, while Sanchis-Ojeda et al. (2014) harnessed the efficiency of fast Fourier transform, at the cost of losing some power due to the division of the transit signal into several harmonics. Alternatively, the amount of available computational power can be increased, as Caceres et al. (2019) did in their autoregressive planet search campaign. In this work, we present fBLS, a fast algorithm that reduces the computational time required for searching transit-like signals while preserving statistical efficiency. To do so, we combine two long-standing techniques – the fast-folding algorithm (FFA; Staelin 1969; Kondratiev et al. 2009) and box least-squares periodogram (BLS; Kovács, Zucker & Mazeh 2002). Note that a fast-folding-based tran-
sitting search was also deployed by Petigura et al. (2013) in their search for transiting planets in the period range of 5–50 days. Other astronomical applications used similar or closely related algorithms. Some examples include the use of FFA in pulsar astronomy (e.g., Morello et al. 2020), the development of fast dispersion measure transform algorithm (FDMT; Zackay & Ofek 2017) for the detection of radio bursts, and the use of fast Radon transform for streak detection in astronomical images (Nir et al. 2018).

We demonstrate the capabilities of fBLS by applying it to Kepler data in the search for USP planets. Our preliminary analysis demonstrates the algorithm’s ability to detect shallow transits while significantly reducing the required computation time.

The paper is structured as follows: Section 2 presents the fBLS algorithm. Section 3 describes the preliminary analysis – the selected Kepler sample, the search methodology, the verification process, and our results. Section 4 summarizes the results and discusses the possible applications of fBLS.

2 OUTLINE OF THE ALGORITHM

There are two main parts of the fBLS algorithm. First, using FFA, the input lightcurve is transformed into a set of folded and binned lightcurves (or simply, ‘folds’) generated for different trial periods. Then, BLS is applied to each fold in the set to form a standard BLS periodogram.

We introduce FFA using a simple example in Section 2.1; explain how to calculate the fBLS periodogram in Section 2.2; elaborate on the analysis of realistic, irregularly sampled lightcurves in Section 2.3; discuss the computational complexity in Section 2.4, and summarize the capabilities of the algorithm in Section 2.5.

2.1 Fast folding

FFA efficiently transforms a given lightcurve into a matrix in which each row represents a fold generated by a different trial period. In the context of this work, folds are arrays of size $m$ in which each entry represents a segment of the periodic cycle (i.e., a ‘phase bin’) and contains the sum of all measurements that correspond to it. We note that we set the input lightcurve to FFA to be mean-subtracted.

In the following, we assume that the input lightcurve comprises of exactly $2^m$ evenly-spaced measurements and set the temporal unit of measure to be the sampling interval. We use these assumptions only to simplify the discussion; generally, FFA is not restricted by the lightcurve length or photometric sampling scheme. The sampling restrictions are relaxed in Section 2.3, where we discuss the analysis of realistic lightcurves. FFA can produce folds over an arbitrary period range (e.g., Morello et al. 2020), however, in this simple introduction, we restrict the discussion to the period range $m \leq P \leq m + 1$, and neglect the influence of data binning.

As a preliminary step, the lightcurve is divided into $2^m$ consecutive sections of length $m$,

$$f_0^{(0)} \cdot f_1^{(0)} \cdot f_2^{(0)} \cdot \cdots \cdot f_i^{(0)} \cdot \cdots \cdot f_{2^m-1}^{(0)},$$

where the subscript represents their chronological order and the superscript indicates the current, $0^\text{th}$, FFA level. The sections are ordered below the other according to their timing, forming a matrix of dimensions $2^m \times m$, such that the index $i$ and $\phi$ are the matrix row and column numbers, respectively.

A periodic transit signal will appear as a dark line (‘streak’) crossing through the matrix with a slope determined by the deviation of its period, $P$, from the section length, $m$, as Figure 1 depicts. The illustrated ‘transits’ are drawn as dark circles and taken to be shorter than one sampling interval in duration. If the period is close to the matrix width, namely $|P - m| \leq 2^{-n}$, the discretely sampled signal is likely to appear vertical, as demonstrated on the left panel. If $P$ equals exactly $m + 1$, the signal will appear as a diagonal streak, as illustrated on the right panel. The middle panel presents an intermediate case, where $m < P < m + 1$, demonstrating how the slope of the line is related to the relation between the periodicity of the pulse and the dimensions of the matrix.

Having the initial data matrix properly structured, the folding procedure starts from the bottom-up by co-adding consecutive lightcurve sections. To do so, we divide the matrix into pairs,

$$(f_0^{(0)} \cdot f_1^{(0)}), (f_2^{(0)} \cdot f_3^{(0)}), \ldots , (f_{2^{n-2}} \cdot f_{2^{n-1}}),$$

and, for each pair, calculate the integrated flux along all possible streaks. However, considering the assumed period range, there are in fact only two possibilities to combine two consecutive folds: a vertical streak or a shift of one bin. Notably, these options are equivalent to folding the two sections with trial periods of $m$ and $m + 1$, respectively. The transformation of the first two matrix rows, for example, is

$$f_0^{(1)} \phi = f_0^{(0)} \phi + f_1^{(0)} \phi, \quad f_1^{(1)} \phi = f_0^{(0)} \phi + f_1^{(0)} \phi (\phi + 1 \mod m),$$

where the superscripts indicate the transition from the $0^\text{th}$ to the $1^\text{st}$ FFA level. The integrated profiles are arranged as a matrix with dimensions identical to the original one, as Figure 2 demonstrates.

For the $2^\text{nd}$ FFA level, we proceed by co-adding two-section folds. To do so, we add folds separated by a single row, thus combining information from four consecutive sections of the original lightcurve. The first four matrix rows, for example, are transformed according to

$$f_0^{(2)} \phi = f_0^{(1)} \phi + f_2^{(1)} \phi, \quad f_1^{(2)} \phi = f_0^{(1)} \phi + f_2^{(1)} \phi (\phi + 1 \mod m),$$

$$f_2^{(2)} \phi = f_1^{(1)} \phi + f_3^{(1)} \phi (\phi + 1 \mod m), \quad f_3^{(2)} \phi = f_1^{(1)} \phi + f_3^{(1)} \phi (\phi + 2 \mod m).$$

Each matrix row now represents a four-section fold, generated over a refined period grid, $m, m + \frac{1}{2}, m + \frac{3}{2}$ and $m + 1$, listed with respect to the order of equations in the equation array above.

The integration process is repeated $n$ times, wherein each step...
increasingly distant rows are added, generating folded profiles that represent a longer portion of the original lightcurve. On each step, the period resolution is increased until, eventually, the matrix becomes a large set of folds, where each row is a combination of all original lightcurve sections. Each row in the final matrix is associated with a trial period according to

$$P(i) = m + \frac{i}{2^n - 1},$$

where $i$, as before, is the matrix row number.

The phase shifts between additions of different folds encapsulate the fundamental concept of FFA. On each step, two profiles that were sampled on consecutive time intervals and folded with the same trial period are summed twice: first, using the same period with which they were folded, and second, assuming that the period inaccuracy accumulated to a shift of one phase bin between the two time-intervals (Lovelace & Sutton 1969). This approach enables one to extend the FFA to be applied over a broader period range. In this case, the resulting number of trial periods can be different from the number of rows in the $0^{th}$ FFA step.

In the limiting case of an under-sampled signal, where the transit duration is comparable or shorter than the sampling interval, as illustrated in Figures 1 and 2, such a binning scheme might yield ambiguous results. For example, because changes in the exact time-of-transit within the sampling interval generate different folded patterns. However, if the signal is sufficiently sampled, the fast-folded profile will converge to the expected phase-folded shape of the signal in a manner that will not impair periodicity detection efforts. This is demonstrated in Figure 3, showing the results of FFA for Kepler-1604b (Morton et al. 2016), an USP with a reported orbital period of ~0.68 day. We have detected this planet with half of the literature-reported period, which fell outside the search limits and was reported with a double period. However, alternative explanations should also be explored.

This heuristic description of the process is only meant to provide intuition regarding the manner in which the data are folded. Several recent studies presented and discussed FFA and its properties in-depth (e.g., Zackay & Ofek 2017; Nir et al. 2018; Morello et al. 2020), we refer the reader to these publications for further elaboration. Nevertheless, we also provide a short bottom-up prescription for FFA in Appendix A and an online demonstrative Python notebook, describing the different parts of the algorithm.\(^1\)

### 2.2 fBLS

We use FFA to prepare the data for a BLS search. In practice, we use FFA to generate the following data products:

- (i) a matrix of folds, $F$;
- (ii) a matrix of counts, $N$;
- (iii) and, a vector of trial periods, $\tilde{P}$.

As discussed in 2.1, each row in $F$ represents a folded lightcurve in which each phase-bin contains the summed flux of the mean-subtracted lightcurve, namely $\overline{f}(t)$, for measurement time $t$. The counts matrix, $N$, contains the number of measurements summed in the process. Under the lightcurve length and period range assumptions discussed above, the dimensions of the two matrices are identical, $2^n \times m$. The trial period vector is arranged such that its $i$th entry represents the period with which the corresponding matrix row was folded.

In order to detect a periodic signal, each trial period is assigned

\(^1\) A Google Colaboratory notebook is available online via this link.
copies of the original matrices with column shifts cyclically. Define the mean flux and phase-weights. The convolution is done by adding folded profiles.

The input lightcurve to be of zero mean, a property inherited by the convolution method. We note that under the simplifying assumptions defined at the beginning of the section, we have defined the lightcurve to be in transit. We note that the flux levels are analytically determined by the data for a given epoch and duration.

In the following, considering the Kepler data, we assume that the photometric uncertainty is approximately constant throughout the lightcurve. Because the operation is done for each fold (matrix row) separately, we describe the formulae in terms of the operations made for the i-th row of $F$ and $N$. First, we calculate the mean flux, $s_i$ and phase-weights, $r_i$, for the i-th fold,

$$s_i[\phi] = F_i,\phi / n_{\text{tot}},$$
$$r_i[\phi] = N_i,\phi / n_{\text{tot}},$$

where $n_{\text{tot}}$ is the total number of integrated data-points in each fold. We note that under the simplifying assumptions defined at the beginning of this section, $n_{\text{tot}}$ is $2^m$. As stated earlier, we have defined the input lightcurve to be of zero mean, a property inherited by the folded profiles.

A simple convolution procedure calculates the BLS statistic from the mean-flux and phase-weights. The convolution is done by adding copies of the original matrices with columns shifted cyclically. Define the index permutation,

$$\sigma_w : \phi \mapsto (\phi + w - 1) \mod m.$$  \hspace{1cm} (5)

For a given width, $w$, the convolution is done according to the recursion formula

$$s^{(w)}_i[\phi] = s^{(w-1)}_i[\phi] + s_i[\sigma_w(\phi)],$$
$$r^{(w)}_i[\phi] = r^{(w-1)}_i[\phi] + r_i[\sigma_w(\phi)],$$

where $s^{(1)}_i = s_i$ and $r^{(1)}_i = r_i$. In these terms, the SR statistic of each period (fold) is the maximal score of the fold, considering all allowed phases and widths, namely

$$\text{SR}_i = \max_{\phi, w} \left( \frac{|s^{(w)}_i[\phi]|}{\sqrt{r^{(w)}_i[\phi]}(1 - r^{(w)}_i[\phi])} \right).$$  \hspace{1cm} (7)

where the phase, $\phi$, and width, $w$, are calculated in terms of phase bins. The process results in a vector of scores, SR, one for each period in $P$. An example of an fBLS periodogram for the Kepler data is provided in Figure 3.

In practice, the fBLS score is an approximation of the BLS score. This is because by binning the lightcurve we have introduced timing round-off errors. We discuss the expected effects of binning and show that the resulting error can be controlled in Appendix B. We demonstrate the fBLS score calculation in the online notebook.1

### 2.3 Irregular sampling and time-dependent photometric uncertainty

We have described the FFA algorithm under strict length, sampling, and period range assumptions. However, astronomical time series are often irregularly sampled. Even in the case of space missions, such as Kepler, photometric acquisition can be interrupted, for example, by data broadcasting and spacecraft maneuvers.
First, we relax the requirement for the specific length of the input time series. If dividing the lightcurve into $2^m$ sections of size $m$ is impossible, the lightcurve is padded by zeros that are not counted in the folding procedure. We note that this concept can be extended to rectify data sets that are almost regularly sampled: one may divide the lightcurve into evenly sampled temporal bins, such that most bins contain one measurement, and some remain empty. From that point on, FFA operates based on data addition. Hence bins with no information content will not affect the result.

However, this simplistic solution becomes wasteful for highly irregular sampling. Therefore, the first FFA step is performed in a ‘brute-force’ manner. This is done by dividing the lightcurve into short sections of identical durations, folding them in the usual manner, using the measurement time modulo the trial period, and binning the data according to the desired number of bins in each fold. When folding the data explicitly using the modulo function, the irregular sampling rate has no effect.

For example, consider an irregularly sampled lightcurve that spans a time interval of $2^m n$, where $n$ is an integer, but $m$ is some positive real number larger than 1. Now, we divide the lightcurve into $2^m$ sections of length $2^m$, but note that the number of points in each section is not constant due to the irregular sampling. We then replace the first FFA step by folding and binning each section twice; first, with period $m$ and then, with period $m + 1$. After this brute-force step, the folds can be arranged and coadded according to the standard FFA prescription. A demonstration of this procedure is provided in the online notebook.

Another assumption that can be similarly relaxed is the requirement of constant photometric precision. Again, we recall that FFA is essentially a fast summation scheme. In order for fBLS to account for time-varying uncertainties it is possible to fold the inverse-variance-weighted flux, $\tilde{f}$, and inverse-variance weights, $\tilde{n}$, instead of summing over the flux and number of points, as described Section 2.2. This implies that the input time-series to algorithm should be

$$\tilde{f}(t) = \omega(t)f(t), \quad \text{and} \quad \tilde{n}(t) = \omega(t),$$

where the weights, $\omega$, are given by $\sigma^{-2}/\sum\sigma^{-2}$, and $\sigma$ is the uncertainty of the measurement taken at time $t$. According to the requirements in Kovács et al. (2002), the uncertainty of the samples are assumed to be additive and Gaussian, and $\omega f$ are assumed to be of zero arithmetic mean. In this case, the calculation will converge to the standard BLS statistic with time-varying photometric uncertainty.

### 2.4 Computational complexity

Based on the fBLS prescription described above and provided in Appendix A, we can estimate the algorithm’s number of arithmetic operations. We express the number of operations in terms of the number of points in the lightcurve, $N$, number of trial periods, $N_p$, number of phase-bins, $N_b$, and number of trial widths, $N_w$.

FFA operates for $\log_2 N_p$ levels. This is because, as Figure 2 illustrates, upon transition from one level to the next, the number of trial periods is increased by a factor of two, and the same factor decreases the number of blocks. Therefore, the overall volume of data is conserved throughout the process, and the number of required steps scales logarithmically. On each level, pairs of folds are added twice; hence, the number of operations within each level scales as the total number of folds times the number of phase-bins. Overall, the number of operations required to generate the fold is

$$N_{\text{fold}} \sim N_p \cdot N_b \cdot \log_2 N_p + N_{\text{init}}.$$  \hfill (9)

where $N_{\text{init}}$ is an additional additive term that represents the initialization procedure, such as the brute-force folding stage, required to account for irregular sampling. The computational complexity of $N_{\text{init}}$ is usually subdominant. A similar point was recently discussed and demonstrated by Panahi & Zucker (2021), who addressed the detection of transiting planets in highly sparse and irregularly sampled lightcurves.

In the second stage, we apply BLS to the matrix of folded lightcurves. For a given fold, the number of operations required to calculate the BLS score scales with the number of trial widths and the number of bins. Overall, the number of operations required to generate the BLS scores is

$$N_{\text{score}} \sim N_p \cdot N_b \cdot N_w.$$  \hfill (10)

The total number of arithmetic operations of fBLS, which is the sum of of operations taken at both stages, therefore scales as

$$N_{\text{fBLS}} \sim N_p \cdot (\log_2 N_p + N_w) \cdot N_b + N_{\text{init}}.$$  \hfill (11)

For comparison, a standard BLS implementation requires

$$N_{\text{BLS}} \sim N_p \cdot N \cdot N_p,$$  \hfill (12)

arithmetic operations, where $N_p$ represents the number of phases searched by BLS. If no binning is used, the number of phases scales with the number of points in the lightcurve.

We compared the performance of fBLS to that of the BLS implementation by AstroPy\textsuperscript{2}. By default, the AstroPy BLS implemen-

![Figure 4. Runtime as a function of the number of points in the lightcurve.](image)

Black squares – fBLS, after removing the 0.96 second Numba compilation overhead (see text). The total fBLS computation time appears as gray squares. Python implementation. Circles – BLS; Cython-based AstroPy implementation. The AstroPy BLS version was applied twice: first, assuming a predefined set of transit durations (yellow); and second, taking the set of trial durations to scale with the number of points in the lightcurve (white). The number of trial periods in all three test cases is identical, and all tests were performed on the same machine, with an identical setup. The thin gray lines in the background represent linear dependence for reference, and the dashed black lines represent an extrapolation of the runtime curves.

\textsuperscript{2} AstroPy BLS implementations is Cython-based (docs.astropy.org).
mentation produces a binned lightcurve for each trial period, then searches it for transits over a predetermined set of transit durations. The Astropy BLS implementation is Cython accelerated, while fBLS is written in Python and is partially accelerated with Numba.\(^3\) The results of the runtime experiment, described below, are given in Figure 4.

We tested Astropy’s BLS implementation twice: first, by setting a grid of ten transit durations, and second, by setting the number of durations to one-fifth of the number of points in the data. The first test optimized the runtime, while the second qualitatively mimicked a standard BLS implementation without data binning. In both cases, the oversampling factor was set to the default value of ten. As of fBLS, we defined the folded lightcurves to have forty phase bins and searched over ten phase-width values for the transit. All three tests were made for the same period grid, defined by fBLS. The runtime ratio between BLS and fBLS is 10–20 for simulated lightcurves with \(10^5–10^6\) points, and grows to \(\sim 100\) for lightcurves of \(\sim 10^6\) points.

Admittedly, fBLS requires \(\sim 0.96\) seconds of computation overhead due to the Numba compilation time. We did not attempt to avoid this overhead, and it can probably be reduced if required. After removing this constant term, the linearity of the algorithm is made clear. We note that even with the current setting, it is evident that for lightcurves longer than \(\sim 10^4\) fBLS approaches linear time as the contribution from the overhead becomes less significant. For comparison, the BLS implementation by Astropy is super-linear even for the case of predetermined durations.

### 2.5 Summary of capabilities

For a periodicity search to be effective, it should sample a sufficiently refined period grid; on the other hand, sampling too many trial periods may be computationally expensive, rendering the search inefficient. As Section 2.1 demonstrates, fBLS produces a periodogram for a set of ‘sufficiently different’ trial periods, determined by the duration of the lightcurve and the required number of bins in each fold. This property, inherited from the FFA, enables the algorithm to avoid redundant computations without impairing the result. Furthermore, as Figure 4 demonstrates, fBLS computes the search statistic in an efficient manner that scales linearly with the number of photometric measurements.

In order to optimize the sensitivity of the search, the trial period themselves are assigned with the matched-filtering statistic (e.g., Mood et al. 1974). Like the classical BLS algorithm, fBLS uses this statistic assuming that the transit shape can be approximated as a box-shaped signal, making it more efficient compared to Fourier-transform-based search campaigns (but also see Hippke & Heller 2019, and our discussion below). As a result, fBLS can identify small transiting planets, for which the transit depth is small compared to the typical noise level of the lightcurve. We discuss how the lightcurve binning affects the statistical power of the search in Appendix B and show that the loss of sensitivity can be controlled.

In summary, fBLS is computationally and statistically efficient. The algorithm becomes advantageous as the number of points and trial period grows. One such example is the search for USP planets in the Kepler data, which we discuss below as a proof-of-concept. Nevertheless, we emphasize that the capabilities of fBLS relate not only to the USP regime and discuss other possible applications in Section 4.

### 3 A DEMONSTRATION OF FBLS: KEPLER USP PLANETS

In this section, we demonstrate the fBLS capabilities by applying it to a large sample of Kepler lightcurves in search for USP planets.

Our aim is to demonstrate the applicability of fBLS to realistic data sets, and particularly to the USP domain, rather than obtaining the astrophysical properties of the population of planets. However, some challenges in performing such a search are related to the validity and robustness of the detected transit signals and not only to the computational and statistical efficiency of the search algorithm. Therefore, apart from the search itself, we developed a simplistic signal validation scheme which is presented here. This two-stage approach will serve as a basis for a follow-up study in which we apply fBLS to investigate the occurrence of USP planets.

#### 3.1 Sample selection

Not all types of stars are suitable for USP planet search campaigns. In particular, giant stars would have engulfed planets in such close orbits, and so will most massive main-sequence (MS) stars. We, therefore, restrict the analysis to the lightcurves of MS stars, which are either Sun-like or of later spectral type. This is done by selecting a region in the color-magnitude diagram (CMD) of Kepler targets, using the gaia-kepler.fun crossmatch of Kepler and Gaia DR2 targets.\(^4\) In addition, since the stellar radius estimate is crucial for determining the nature of the transiting object, we only consider at this demonstration phase systems with relative parallax error below 10%.

A CMD of Kepler systems is presented in Figure 5. The selected region (zone I) contains \(\sim 87,000\) MS target stars that are Sun-like or of later spectral type. After removing systems that were not observed in all four Kepler seasons (see below), we are left with \(\sim 80,600\) targets in our sample. Because in this work we merely intend to introduce the algorithm, we opted to exclude from the sample all targets that have a reported threshold-crossing event (TCE) in the Kepler archive,\(^5\) which will be discussed in a future publication. This step left 74,510 stars in our sample (but see the discussion in Section 4). Typical to Kepler, the average number of light-curve points in our sample was \(\sim 65,000\), where fBLS is more efficient than BLS by a factor of \(\sim 15\) (see Figure 4).

#### 3.2 fBLS search

For the selected targets, we retrieved and detrended the Kepler DR25 long-cadence lightcurves (exposure time of 29.4-min; also see Murphy 2012), using the Lightkurve package,\(^6\) and conducted an fBLS periodicity search in the range of 0.2–1.0 days. Each periodogram consisted of SR scores for \(\sim 250,000\) frequencies, which were calculated based on 80-bin fast-folded profiles. The search targeted transits that span up to one-fifth of the orbital phase because wider signals can be efficiently detected with Fourier analysis (see, for example, Sanchis-Ojeda et al. 2014).

The goal of this work is not a study of the properties of the USP planet population but rather to demonstrate the detection capabilities. We, therefore, focus on the challenging cases of transits shallower than 200 ppm. From the set of shallow-depth candidates, we removed targets that do not meet the following conditions:

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3 Numba is a just-in-time compiler for Python (numba.readthedocs.io).

4 gaia-kepler.fun: 1” crossmatch table, accessed at March 21st 2021.

5 See the online Kepler TCE table.

6 See the online Lightkurve documentation.
After applying all four tests, we were left with a sample of 81 targets that yielded significant detections. We further discuss these systems below.

### 3.3 Candidate validation

Despite having a statistically significant response to the box-shaped filter, some detected systems may be false-positive identifications of an astrophysical origin. *Kepler*’s point spread function, of ~6.4" full width at half maximum, may be large compared to the angular separation between stars in its field (e.g., Lillo-Box *et al.* 2014). Therefore, transit-shaped signals may originate from diluted light from eclipsing-binary stars in the background.

One way to identify these diluted binaries, without resorting to data obtained by other instruments, is exploiting *Kepler*’s ‘seasonal cycle’ (see, for example, Wu *et al.* 2010). Every three months *Kepler* was rotated by 90° about its optical axis to maintain the solar array exposure. The level of background contamination consequentially varied between consecutive quarters, as did the target position and orientation on the detector. As a result, diluted eclipses of background binary stars may change in depth between different seasons. The position of the photometric centroid can also vary as the flux contribution from the diluted binary changes. In these cases, the centroid position may be correlated with the phase curve of the binary star and/or its orientation on the detector. We, therefore, performed three additional validation tests:

(i) *Seasonal transit-depth variation*. We separately folded each *Kepler* season with the fBLS period. Like the odd-even test, we compared the depth values obtained from each season to that obtained when analyzing the entire lightcurve using a $\chi^2$ test.

(ii) *Seasonal flux-correlated centroid position*. We searched for a correlation between the photometric centroid position and the target flux level for each season. The combined correlation significance for all four seasons was estimated using Fisher’s method (e.g., Fisher 1992). The entire analysis was done separately for the row and column directions on the detector.

(iii) *Yearly transit-depth variation*. Similarly to the seasonal depth analysis, we compared between the transit depths obtained during each of the four years of the spacecraft. This was done in order to identify spurious signals originating from long-term effects, systematic or otherwise, that may vary with time.

We have set a significance threshold of 1% for each of these tests, rejecting all targets that failed one of these tests by having a p-value below this limit.

The validation process rejected most of the systems and reduced the sample to 9 systems, which we visually inspected. We excluded three additional targets, KIC 3628897, 9824396, and 7810181, because their lightcurves demonstrated a significant out-of-transit modulation, suggesting that the target is a binary star or contaminated by systematic artifacts. In Appendix C we provide a list of the rejected targets and a demonstration of two systems that did not pass these thresholds (see Table C1 and Figure C1).

### 3.4 The fBLS detections

Finally, we were left with six candidates, listed in Table 1, with their folded lightcurves plotted in Figure 6. Three of these targets – KIC 2718885, 6359893, and 11187332 were discovered by Sanchis-Ojeda *et al.* (2014). Because these targets never received an entry in the NASA Exoplanet Archive, they also appear in our sample.
Figure 6. Folded lightcurves of the detected candidates. Kepler ID and folding period are listed in each panel. The top three panels present the three new candidates that passed our validation tests. The bottom three are the three known candidates that were discovered by Sanchis-Ojeda et al. (2014) but also appeared in our sample. Note that KIC 271885, 6293500, and 6359893 were identified with a double period due to the period range boundaries set in our search. The expected orbital period for these candidates is half the listed folding period.

Table 1. Properties of the detected fBLS candidates that passed all validation tests. The top three systems are new identifications, and the bottom three were discovered by Sanchis-Ojeda et al. (2014) but also appear in our sample (see text).

| KIC    | V    | Teff (K) | R (Rsun) | Period (day) | Width (phase) | Depth (ppm) | SR | SNR | Stellar parameters reference |
|--------|------|----------|----------|--------------|---------------|-------------|----|-----|--------------------------------|
| 6293500 | 14.3 | ~5200    | ~0.77    | 0.34863*     | 0.162         | 59          | 22 | 32  | Stassun et al. (2019)         |
| 9217391 | 16.1 | ~5200    | ~0.76    | 0.66976      | 0.0625        | 77          | 19 | 8.7 | Stassun et al. (2019)         |
| 9835433 | 15.8 | ~6000    | ~1.00    | 0.32502      | 0.05          | 72          | 16 | 8.9 | Stassun et al. (2019)         |
| 2718885 | 14.8 | ~5700    | ~1.45    | 0.39467*     | 0.05          | 53          | 12 | 10  | Martinez et al. (2019)        |
| 6359893 | 15.9 | ~5700    | ~1.91    | 0.36087*     | 0.075         | 110         | 30 | 13  | Martinez et al. (2019)        |
| 11187332| 15.4 | ~5600    | ~0.95    | 0.30599      | 0.0875        | 58          | 16 | 13  | Martinez et al. (2019)        |

* This signal was detected by fBLS with twice the orbital period.

In the following, we briefly review the new targets presented in Table 1. A detailed characterization of these targets is beyond the scope of this methodological paper.

**KIC 6293500** is a K-type MS star. As Figure 6 shows, the detected signal was found with twice the suggested orbital period of the planet candidate. It is also evident that the fBLS width is underestimated as the signal deviates from the assumed rectangular shape. The width of the transit-shaped signal, when adopting the 4.2 hours period, covers ~50% of the phase. Given the estimated stellar radius, this transit duration is large even when accounting for Kepler’s 30-minute photometric integration time. The Gaia renormalized unit weight error (RUWE; Lindegren et al. 2018) value for this target is 1.211, which raises the possibility that this target is not a single star (Belokurov et al. 2020). We have searched Gaia for other sources that fall within Kepler’s aperture mask and identified a source with a magnitude difference of ~3.5 in the G band. Therefore, we suspect that the signal is not necessarily caused by a transiting planet but could originate from another, astrophysical or instrumental, source. However, the signal had passed our simplistic quantitative and qualitative validation tests. Further study is beyond the scope of this work.

**KIC 9217391** is a K-type MS star. The detected depth of the signal corresponds to a planet candidate of ~0.75 R\(_\oplus\). This radius, along with the detected 16 hour orbital period, places this candidate in a relatively well-populated region of USP planets parameter-space (e.g., Uzsoy et al. 2021). We searched for other sources that fall within Kepler’s aperture mask and did not find any other possible stars within Gaia’s magnitude limit. The RUWE value for this target is ~1.012, consistent with a single main-sequence primary star.

**KIC 9835433** is a G-type MS star. Several studies have found the target star to be photometrically variable, probably due to magnetic activity, with a rotation period of ~13 days and variability amplitude of ~0.16% (Mehrabi et al. 2017; Reinhold et al. 2017). A careful inspection of the data folded with twice the orbital period revealed some phase modulation and the possibility of an odd-even transit...
depth difference. However, our simplistic box-shaped depth comparison did not detect that. *Gaia* reveals two nearby sources, fainter by $\sim$4.5 magnitudes in the G band, but the RUWE value for this target, $\sim$1.015, is consistent with a single main-sequence primary star. Further analysis of this system, which is beyond the scope of this study, is needed to determine the validity of this signal.

3.5 Performance of the algorithm

As noted above, most of the systems found by fBLS are false discoveries, probably originated from background sources or instrumental artifacts (e.g., Coughlin et al. 2014; Van Cleve & Caldwell 2016). However, the falsifiability of these systems has nothing to do with the fBLS search since any search campaign is susceptible to these phenomena due to the mission characteristics. We, therefore, opted to discuss the entire bulk of targets, true- and false-discoveries alike.

To assess the performance of the algorithm, we present the transit depth and SNR of the 81 targets as a function of the detected period in Figure 7, together with the planets, planet-candidates and false positives from the NASA exoplanet catalogue. The top panel shows the fBLS reported depth, calculated by assuming a rectangular transit shape, and the depths of *Kepler*’s targets as reported in the NASA exoplanet catalogue. By definition, all transits in our sample are shallower than 200 ppm. As seen in the figure, most of them have periods in the range of 0.2–0.4 day, a period range for which only very few targets were known before.

The bottom panel presents the detection SNR, as reported by fBLS and the NASA exoplanet catalogue. By definition, the fBLS sample presented here is limited to be of SNR larger than 8.5. Unlike the transit depth and orbital period, which are proxies of the physical dimensions of the planet, star, and orbit, the transit SNR is also governed by the observed noise of the target star. The figure suggests, again, that the fBLS detections are concentrated in a scarcely populated parameter region.

Figure 8 presents a period histogram of our false and true candidates, together with the reported objects in the NASA catalogue. The top panel suggests that the period distribution of false-positive identifications tends to prefer a few distinct frequencies, probably caused by contamination from bright variable stars (Coughlin et al. 2014; Thompson et al. 2018). Apart from these frequencies, the period distribution appears uniform between 0.5 and 1.0 days. Below 0.5 day the rate of false discoveries sharply drops, as most search campaigns did not target this region. A KS-test comparing the period distribution of false-positives between 0.6 and 0.9 day to a uniform yields a p-value of 15%.

The bottom panel of Figure 8 shows a decline in the occurrence rate of *Kepler*’s USP planets and planet-candidates at orbital periods below $\sim$0.6 day (also see Winn et al. 2018; Uzsoy et al. 2021). As opposed the the top panel, in this panel, the detected candidates are provided according to their physical period, rather than the double period detected in the search. A few systems stand out at the very margin of this distribution, with orbital periods at $\sim$0.3 and shorter than 0.2 day. The fBLS algorithm is tailored for detecting these transiting planets.

It is also evident from Figures 7 and 8 that most false-positive identifications of fBLS populate the period range $\sim$0.25–0.4 day, whereas most false-positives reported in the NASA exoplanet catalogue are of orbital periods larger than $\sim$0.5 day. A noticeable gap in the false-positive population separates these two groups. While the cause of this gap is unclear, it is unlikely to originate from an inherent property of fBLS but rather from the sampling, noise or lightcurve detrending procedure. An excess of false positives below 0.4 day could originate from the detection of false-positive signals with periods of $\sim$0.1–0.2 day, which are detected with twice their actual period, due to the search predefined period range. Another plausible interpretation is that these false detections are related to spurious signals, either because of the inherent properties of the spacecraft and its instruments or imperfections in the data whitening procedure. A detailed study of the noise properties in the USP regime is beyond the scope of this work and deferred to a future study of the USP population.

4 SUMMARY AND DISCUSSION

We introduced fBLS – a fast-folding efficient BLS algorithm. fBLS naturally defines a non-redundant array of trial periods, produces a set of folded lightcurves, and derives the BLS statistics, enabling the detection of shallow transit signals. fBLS is well suited for searching short-period small rocky planets in lightcurves obtained over long temporal baselines at high sampling rates.

We demonstrated the capabilities of fBLS by analyzing a large sample of $\sim$75,000 *Kepler* lightcurves in search for transiting planets with orbital periods shorter than one day. The selected sample targeted MS stars with no reported transit-like signal in the NASA exoplanet catalogue. The analysis yielded 81 detections with SNR larger than 8.5, out of which our vetting process identified 75 as false discoveries.

Six candidates passed our simplistic validation scheme, out of which three are new discoveries. KIC 9217391 is particularly promising, with a period of $\sim$0.67 day and planet radius of $\sim$0.75 $R_\oplus$. The other two candidates require further validation, which is beyond the scope of this work. The remaining three detections were found to be known planet-candidates, discovered by Sanchis-Ojeda et al. (2014), but do not have an entry in the NASA exoplanet catalogue. These three known candidates were found with fBLS SNR larger than 10. The results of the preliminary fBLS search performed here, together with the planet candidates reported by NASA, suggest a small group of planets with periods $\lesssim$0.4 day. These candidates are quite interesting, as their composition, origin, and evolutionary paths are still unclear.

The choice of limiting the detection at SNR of 8.5 was quite arbitrary. fBLS can find USP planets of either smaller dimensions compared to their host or around fainter host stars. This can be done by bootstrap validation, which can assure the significance of the detection (e.g., Thompson et al. 2018). This and other quality assurance measures are important to validate signals with low SNR (Burke et al. 2019). The search for those USP planets, some of which should be with a smaller radius, is deferred to the next paper.

The efficiency of fBLS also makes statistical injection-recovery tests easily feasible. These tests will help place tighter constraints on the margins of the USP planet mass, radius, and composition distributions (e.g., Winn et al. 2018; Dai et al. 2019; Uzsoy et al. 2021). fBLS also makes computationally intensive signal validation processes, such as time slides (e.g., Zackay et al. 2021), easily accessible.

Since planets at orbits of a few hours are scarce, the reliability of individual candidates is vital to study their population as a whole. For instance, the estimated radii of KIC 27188835 and 6359893 indicate that these stars could be slightly evolved or be members of binary systems (see Table 1 and Figure 5), and as a result, it is difficult to constrain the planetary nature of these candidates based on *Kepler*’s photometry alone (but see Judkovsky et al. 2021, for example).

Because of these validation challenges, we expect that the study
Figure 7. Top panel: transit depth vs. detected period. Black points (gray circles) represent planets and planet-candidates (false-positives) as reported in the NASA exoplanet catalogue. Black (white) squares are candidates (false-positives) detected in our fBLS search. Bottom panel: signal-to-noise ratio vs. orbital period.

Figure 8. Top panel: Period histogram of false positives. White and gray bars represent false positives reported in the NASA exoplanet catalogue and in this work, respectively. Bottom panel: Period histogram of planets and planet-candidates. The leftmost bin includes KIC 2178885, 5080636, 6293500 and 6359893.
of USP planets will benefit from Gaia’s upcoming third data release (DR3; Gaia Collaboration et al. 2016). This release will include an unprecedented number of binaries and triple systems, and enable the identification of background eclipsing binaries (e.g., Panahi et al. 2021); eclipsing pairs in astrometric triple systems (e.g., Shahaf et al. 2019); and, identify grazing eclipses by unresolved stellar companions (e.g., Belokurov et al. 2020). This additional information can be incorporated into the vetting procedure.

As discussed in Section 2.3, fBLS can be applied to irregularly sampled lightcurves. Compared to Fourier-based periodograms, fBLS is less affected by gaps in the data. This feature is useful when searching for transiting planets in the lightcurves of eclipsing binary stars or multi-planetary systems. This capability of fBLS, together with the Gaia information will help to find new planets in stellar multiple systems, an intriguing emerging planet sub-population (e.g., Schwarz et al. 2016; Martin 2018), and detect new small planets in multiple planetary systems (e.g., Kostov et al. 2019; Kane et al. 2022).

fBLS assumes that a rectangular pulse can approximate the transit shape. However, the efficiency of FFA can be harnessed to extend the analysis to a more extensive set of realistic transit models, increasing the search sensitivity (e.g., Hipke & Heller 2019). In the case of USP planets, for example, accounting for high impact parameters and non-negligible photometric integration time can be used to improve the sensitivity.

Another assumption that fBLS is built upon is that the noise is uncorrelated and Gaussian. For USP planets, transit times are usually shorter than the correlation time of the noise, and therefore correlations are assumed to be negligible. For longer periods, the effect of correlated noise may significantly suppress the detection of small planets (see Pont et al. 2006; Hartman & Bakos 2016; Cubillos et al. 2017), and affect the reliability of our validation scheme. Robnik & Seljak (2020) recently demonstrated that correlated noise induced by stellar variability becomes significant for frequencies \( \lesssim 0.25 \text{ d}^{-1} \), therefore expected to affect the detection of long-duration transit signals. In a follow-up study, Robnik & Seljak (2021) also pointed out that the noise properties may vary significantly between different Kepler stars. We are expanding the capabilities of the algorithm, in terms of its matched-filtering statistic, to account for correlated noise (Ivashenko et al., in preparation; also see Appendix B). This should also be applicable for ground-based lightcurves, like NGTS (Wheatley et al. 2018), for which atmospheric red noise is unavoidable.

We can expect USP planets to be more frequent around small stars, M-type stars in particular. In this sense, the Kepler data, with most lightcurves of G and K stars, is not ideal for the application of fBLS. We therefore plan to apply this new algorithm to the TESS (Ricker et al. 2015) data, for which we anticipate many new candidates. fBLS should also be applicable to the lightcurves of PLATO mission (Shahaf et al. 2021a) expected to be launched by 2026 (Rauer et al. 2014).

In a broader sense, fBLS is conceptually similar to other astronomical applications, such as the identification of streaks in astronomical images or the detection of dispersed radio signals (Zackay & Ofek 2017; Nir et al. 2018). Another course of action in which FFA capabilities can be used is the detection of planets that undergo transit timing, duration, and depth variations (e.g., Shahaf et al. 2021b; Millholland et al. 2021). These applications will be addressed in future studies.

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DATA AVAILABILITY

The data underlying the analysis is publicly available via the Kepler and Gaia online archives.

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APPENDIX A: FAST FOLDING ALGORITHM

A fundamental concept of FFA is the addition of two folded profiles, generated from consecutive sections of the original lightcurve, in order to refine the grid of trial periods. When considering each section separately, the period resolution is limited by the length and sampling rate of a single section. However, by combining the data from two sections, one can refine the period resolution, as small deviations between the trial period and true underlying periodicity, previously undetectable, may now accumulate to a small shift between these two sections. In practice, in order to refine the grid of trial periods, FFA adds folded profiles, generated based on data from consecutive time intervals and folded with the same trial period.

FFA is essentially a bookkeeping procedure. A conceptually simple manner in which the technique can be implemented, is to consider on each step by how many phase bins the two folds should be shifted in order to generate the new, more refined, set of folds. For a detailed description of this approach, see Morello et al. (2020) and references therein. Here, we present an alternative, bottom-up description of the algorithm, which we provide for completeness and clarity. For this purpose, we will use five indices: two matrix indices, \( i \) and \( \phi \), indicating the row and column coordinates, respectively (see Figure 1); two FFA indices, \( f \) and \( p \), which will be defined below; and the FFA level number, \( l \). Two additional auxiliary indices, \( \mu \) and \( \nu \), will be defined and used for ease of notation. A list of all indices, along with their range and role, is given in Table A1.

Groups of folds that originate from the same set of lightcurve sections (‘blocks’; see an illustration in Figure 2) are identified using the \( i' \) index. For example, the first two matrix rows calculated in equation (1) form the first block of the level 1 FFA matrix. The folds within each block correspond to different trial periods and are identified using an additional index, \( p \).

The matrix size is constant, hence its associated indices always span the same range, \( i \in \{0, \ldots, 2^n - 1\} \) and \( \phi \in \{0, \ldots, m - 1\} \). On the contrary, the range of the auxiliary indices changes on each step, as the number of blocks is decreasing and the number of trial periods is increasing by a factor of 2. The FFA indices are therefore identified according to:

\[
 i' \in I_l \equiv \{0, \ldots, 2^n - l - 1\}, \quad p \in P_l \equiv \{0, \ldots, 2^l - 1\}, \quad (A1)
\]

and FFA level is \( l \in \{0, \ldots, n\} \). We note that in the 0th level we obtain the indexing of the initial data matrix, \( i \equiv i' \), and in the final level each matrix row represents a fold of all data points, and \( i \equiv p \) as expected. It is also evident that the column index, \( \phi \), represents the phase of the folded signal.

At this point, we are able to express the FFA recurrence relation...
in terms of the FFA indices, $i'$ and $p$, by
\[
f_i^{(i')} = f_\mu^{(i')} + f_\nu^{(i')} [(\phi + \tilde{S}_i(p) \mod m)],
\]
where
\[
\begin{align*}
  i' &= 2^i \cdot i' + p, \\
  \mu &= 2^i \cdot i' + [p/2], \\
  \nu &= 2^i \cdot i' + [p/2] + 2^{i-1},
\end{align*}
\]
and $\tilde{S}_i$ is the phase-shift vector, given by
\[
\tilde{S}_{i+1} = \tilde{S}_i \odot (\tilde{S}_i + 2^{i-1}).
\]
Here, $\odot$ represents vector concatenation and the first phase-shift vectors are
\[
\begin{align*}
  \tilde{S}_1 &= (0, 1), \\
  \tilde{S}_2 &= (0, 1, 1, 2), \\
  \tilde{S}_3 &= (0, 1, 1, 2, 2, 3, 3, 4).
\end{align*}
\]
Once the FFA procedure is completed, each matrix row corresponds to a trial period as given in equation (3). The phase-shift vector encapsulates a fundamental concept of FFA, as it determines the required shift between added folds, when considering the period range between $m$ and $m + 1$. A similar prescription This prescription can be written in order to account for a broader period range. A Python notebook, demonstrating the fast-folding procedure, is available online.\(^1\)

**APPENDIX B: MATHEMATICAL FRAMEWORK AND APPROXIMATIONS**

This appendix describes the mathematical basis for the BLS statistic and elaborates the approximations made by choosing to calculate it based on the folded and binned, rather than the full data. It also presents some of the limitations of this statistic, which will be addressed in a future publication (Ivashtenko et al., in preparation).

Consider a lightcurve, $f$, which is a function of time, $t$. The photometric uncertainty is assumed to be uncorrelated and Gaussian, with a time-dependent standard deviation $\sigma$. The data are assumed to be sampled from some known underlying periodic model, $h$, such that for a given time, $t$, the photometric measurement is given by
\[
f(t) = A \cdot h(t) + n(t),
\]
where $h$ is $l_2$-normalized, $A$ is the amplitude, and $n(t)$ is a noise term,
\[
n(t) \sim N(0, \sigma^2(t)).
\]

Additionally, since the signal is strictly periodic, we note that
\[
h(t) = h(t \mod P),
\]
where $P$ is the orbital period.

The signal given in equation (B1) has a known functional shape and is characterized by additive Gaussian noise. In this case, the strategy to maximize the detection SNR is the matched-filtering statistic with the template $h$ (see, for example, Mood et al. 1974), given by
\[
S = \sum_t h(t \mod P) f(t) / \sigma^2(t).
\]
As implied by the equation above, the matched-filtering score requires each measurement to be multiplied by the expected response at the exact same moment in time. However, if we allow for round-off errors affecting the search sensitivity, the binning procedure can be used in order to make the search faster. We split the period $P$ into $N$ evenly-spaced bins, as illustrated in Figure B1. We note that the bin step can be smaller or bigger than the time step of data $\Delta t$. A measurement obtained at some time $t$ is multiplied by the response of the model taken on in the closest bin, $I_t$, indexed by
\[
I_t = \left[ \frac{t \mod P}{P} \cdot N \right].
\]
where the brackets denote rounding towards the closest integer. In these terms, the matched-filtering score, given in equation (B4), becomes
\[
S \approx \frac{N}{b} \sum_{b=0}^{b=N} h \left( \frac{b}{N} \right) f(t) / \sigma^2(t).
\]

The right term on the bottom equation presents a sum over $f/\sigma^2$, which in practice describes the aggregation of phase-folded lightcurve data points that fall into one specific bin, i.e., all points such that $I_t$ equals to $b$.

As was mentioned, due to binning, flux is multiplied by template corresponding to different time points. The biggest binning-induced time error, $\delta t_b$, is given by
\[
\delta t_b = \frac{P}{2N}.
\]
If the ingress time is larger than one bin, the linear term of the
uncertainty introduced in the response is proportional to
\[ \delta h_b \propto \frac{dh}{dt} \delta t_b = \frac{dh}{dt} \frac{P}{N \cdot 2N}. \] (B8)

In the case \( \frac{dh}{dt} \) is not varying significantly inside one bin, the error is small, and the statistic remains efficient. The shape of the template is fixed, but the number of bins, \( N \), can be chosen depending on the required precision and complexity. As we will elaborate in the future publication (Ivashtenko et al., in preparation), the loss in SNR behaves as \( \frac{1}{N^2} \) or \( \frac{1}{N} \) depending on whether the transit ingress is resolved or not. The transition between the two regimes for USP planets happens at \( N \sim 50 \).

We note that as follows from equation (B4), the folding providing the highest signal to noise ratio is the one done with the data weighted by inverse variance. In the simple case of the white noise with time-independent photometric uncertainty, the inverse variance may be taken outside, as it was done in the BLS statistic, equation (7). In order to get statistic in the units of SNR, the score should be normalized by its expected standard deviation under assumption of absence of transit. The result yields
\[ \frac{S}{N} = \sum_t \frac{h(t) f(t)}{\sigma^2(t)} \sqrt{\sum_t \frac{h^2(t)}{\sigma^2(t)}}. \] (B9)

If binning is applied, the standard deviation in the denominator also has to be calculated on the binned model.

So far, the noise was assumed to have no correlations. Otherwise, in the case of red noise, the data should be weighted with the inverse covariance matrix. This, together with other details, will be discussed in the follow-up papers.

APPENDIX C: EXAMPLES FOR SEASONAL REJECTIONS

To demonstrate the seasonal validation process, we present two systems that we have rejected based on flux-correlated variations in the centroid position.

**KIC 3323808** is an early type M-dwarf (Gaidos et al. 2016; Claytor et al. 2020), for which fBLS yielded a significant detection of a \( \sim 100 \) ppm transit-like signal in a 4.2 hour period. As Figure C1 shows, the detector column-axis centroid position appears to be correlated with the photometric flux level. A pixel-by-pixel lightcurve analysis of Kepler’s target pixel files suggests that the origin of the signal is indeed flux contamination, probably by the faint background star Gaia EDR3 2099937952015618816.

**KIC 8625236** is a late G-dwarf (e.g., Burke et al. 2015) in which fBLS discovered a \( \sim 50 \) ppm transit-like signal in period 3.97 hours. As Figure C1 shows, the centroid position and photometric flux levels are strongly correlated. In this case, the signal is induced by a nearby eclipsing cataclysmic variable (KIC 8625249; Scaringi et al. 2013) which has an identical orbital period.

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Table C1. Targets vetted as false positive detections. The Vetting column indicates the validation tests in which the target failed, according to their numbering in Section 3.3.