On the strong metric dimension of generalized butterfly graph, starbarbell graph, and $C_m \odot P_n$ graph

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Abstract. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For every pair of vertices $u, v \in V(G)$, the interval $I[u, v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in V(G)$ strongly resolves two vertices $u$ and $v$ if $u$ belongs to a shortest $v - s$ path or $v$ belongs to a shortest $u - s$ path. A vertex set $S$ of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The strong metric basis of $G$ is a strong resolving set with minimal cardinality. The strong metric dimension $sdim(G)$ of a graph $G$ is defined as the cardinality of strong metric basis. In this paper we determine the strong metric dimension of a generalized butterfly graph, starbarbell graph, and $C_m \odot P_n$ graph. We obtain the strong metric dimension of generalized butterfly graph is $sdim(B\Gamma_n) = 2n - 2$. The strong metric dimension of starbarbell graph is $sdim(SB_{m1,m2,...,mn}) = \sum_{i=1}^{n}(m_i - 1) - 1$. The strong metric dimension of $C_m \odot P_n$ graph are $sdim(C_m \odot P_n) = 2m - 1$ for $m \geq 3$ and $n = 2$, and $sdim(C_m \odot P_n) = 2m - 2$ for $m \geq 3$ and $n \geq 2$.

1. Introduction
The concept of strong metric dimension was presented by Sebő and Tannier [6] in 2004. Oellermann and Peters-Fransen [5] defined for two vertices $u$ and $v$ in a connected graph $G$, the interval $I[u, v]$ between $u$ and $v$ to be collection of all vertices that belong to some shortest path. A vertex $s$ strongly resolves two vertices $u$ and $v$ if $u \in I[u, s]$ or $u \in I[v, s]$. A set $S$ of vertices in a connected graph $G$ is a strong resolving set for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of a strong resolving set of $G$ is called its strong metric dimension and is denoted by $sdim(G)$.

Some researchers have investigated the strong metric dimension to some graph classes. In 2004 Sebő and Tannier [6] observed that the strong metric dimension of complete graph $K_n$, cycle graph $C_n$, and tree. In 2012, Kratica et al. [4] observed that the strong metric dimension of corona product graph. In the same year, Kratica et al. [1] determined that the metric dimension of hamming graph $H_{n,k}$. Kratica et al. [2] determined that the metric dimension of convex polytope $D_n$ and $T_n$ in 2012 too. In 2013 Yi [8] determined that $sdim(G) = 1$ if only if $G$ is path graph and $sdim(G) = n - 1$ if only if $G$ is complete graph. Kusmayadi et al. [3] determined the strong metric dimension of some related wheel graph such as sunflower graph, $t$-fold wheel graph, helm graph, and friendship graph. In this paper, we determine the strong metric dimension of a generalized butterfly graph, starbarbell graph, and $C_m \odot P_n$ graph.
2. Main Results

2.1. Strong Metric Dimension

Let \( G \) be a connected graph with vertex set \( V(G) \), edge set \( E(G) \), and \( S = \{s_1, s_2, \ldots, s_k\} \subset V(G) \). Oellermann and Peters-Fransen \([5]\) defined the interval \( I[u,v] \) between \( u \) and \( v \) to be the collection of all vertices that belong to some shortest \( u-v \) path. A vertex \( s \in S \) strongly resolves two vertices \( u \) and \( v \) if \( u \in I[v,s] \) or \( v \in I[u,v] \). A vertex set \( S \) of \( G \) is a strong resolving set of \( G \) if every two distinct vertices of \( G \) are strongly resolved by some vertex of \( S \). The strong metric basis of \( G \) is a strong resolving set with minimal cardinality. The strong metric dimension of a graph \( G \) is defined as the cardinality of strong metric basis denoted by \( sdim(G) \).

We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. \([2]\).

**Lemma 2.1** Let \( u, v \in V(G), u \neq v \),

(i) \( d(w,v) \leq d(u,v) \) for each \( w \) such that \( u \ w \in E(G) \), and
(ii) \( d(u,w) \leq d(u,v) \) for each \( w \) such that \( v \ w \in E(G) \).

Then there does not exist vertex \( a \in V(G), a \neq u,v \) that strongly resolves vertices \( u \) and \( v \).

**Property 2.1** If \( S \subset V(G) \) is strong resolving set of graph \( G \), then for every two vertices \( u,v \in V(G) \) satisfying conditions 1 and 2 of Lemma 2.1, obtained \( u \in S \) or \( v \in S \).

**Property 2.2** If \( S \subset V(G) \) is strong resolving set of graph \( G \), then for every two vertices \( u,v \in V(G) \) satisfying \( d(u,v) = diam(G) \), obtained \( u \in S \) or \( v \in S \).

2.2. The Strong Metric Dimension of Generalized Butterfly Graph

Weisstein \([7]\) defined the butterfly graph as a planar undirected graph with 5 vertices and 6 edges. In this research, we obtain generalized butterfly graph. We define a generalized butterfly graph as a double shell in which each shell has any \( n \) order and for every shell have the same order. The generalized butterfly graph \( BF_n \) can be depicted as in Figure 1.

![Generalized butterfly graph BF_n](image)

**Figure 1.** Generalized butterfly graph \( BF_n \)

**Lemma 2.2** For every integer \( n \geq 3 \), if \( S \) is a strong resolving set of generalized butterfly graph \( BF_n \) then \( |S| \geq 2n-2 \).

**Proof.** Let us consider a pair of vertices \( (v_i, v_j) \) with \( i, j = 1, 2, \ldots, 2n-1 \) satisfying both of the conditions of Lemma 2.1. According to Property 2.1, we obtain \( v_i \in S \) or \( v_j \in S \). It means that \( S \) contains one vertex from distinct sets \( X_{ij} = \{v_i, v_j\} \) with \( i, j = 1, 2, \ldots, 2n-1 \) and \( i \neq j \). The minimum number of vertices from distinct sets \( X_{ij} \) is \( 2n - 2 \). Therefore, \( |S| \geq 2n - 2 \). \( \square \)
Lemma 2.3 For every integer \( n \geq 3 \), a set \( S = \{v_1, v_2, \ldots, v_{2n-2}\} \) is a strong resolving set of generalized butterfly graph \( BF_n \).

Proof. We prove that every two distinct vertices \( u, v \in (BF_n) \setminus S \), there exists a vertex \( s \in S \) which strongly resolves \( u \) and \( v \). There are two pairs of vertices from \( V(BF_n) \setminus S \).

(i) A pair of vertices \((c, v_i)\).
For every integer \( i = 2n - 1, 2n \) and \( j = n + 1, n + 2, \ldots, 2n - 2 \), \( d(v_i, v_j) = 2 = diam(BF_n) \), we obtain the shortest \( v_i - v_j \) path: \( v_i, c, v_j \). Thus, \( c \in I[v_i, v_j] \).

(ii) A pair of vertices \((v_{2n-1}, v_{2n})\).
For \( i = 2n - 2 \), \( d(v_i, v_j) = 2 = diam(BF_n) \), we obtain the shortest \( v_{2n-2} - v_n \) path: \( v_{2n-2}, v_{2n-1}, v_{2n} \). Thus, \( v_{2n-1} \in I[v_{2n-2}, v_{2n}] \).

For every possible pairs of vertices, there exists a vertex \( s \in S \) which strongly resolves every two distinct vertices \( BF_n \setminus S \). Thus, \( S \) is a strong resolving set of \( BF_n \). \( \square \)

Theorem 2.1 Let \( BF_n \) be the generalized butterfly graph with \( n \geq 3 \). Then \( sdim(BF_n) = 2n - 2 \).

Proof. By using Lemma 2.3 a set \( S = \{v_1, v_2, \ldots, v_{2n-2}\} \) is a strong resolving set of generalized butterfly graph \( BF_n \) with \( n \geq 3 \). According to Lemma 2.2, \( |S| \geq 2n - 2 \), \( S \) is a strong metric basis of generalized butterfly graph \( BF_n \). Hence, \( sdim(BF_n) = 2n - 2 \). \( \square \)

2.3. The Strong Metric Dimension of Starbarbell Graph
Starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \) is a graph obtained from a star graph \( S_n \) and \( n \) complete graph \( K_{m_i} \) by merging one vertex from each \( K_{m_i} \) and the \( i^{th} \)-leaf of \( S_n \), where \( m_i \geq 3 \), \( 1 \leq i \leq n \), and \( n \geq 2 \). In this paper, we obtain the strong metric dimension of starbarbell graph denoted by \( SB_{m_1,m_2,\ldots,m_n} \). The starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \) can be depicted as in Figure 2.

![Figure 2. Starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \)](image)

Lemma 2.4 For every integer \( m \geq 3 \) and \( n \geq 2 \), if \( S \) is a strong resolving set of starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \) then \( |S| \geq \sum_{i=1}^{n} (m_i - 1) - 1 \).

Proof. We prove for every two distinct vertices \((v_{i,j}, v_{k,l}) \in V(SB_{m_1,m_2,\ldots,m_n})\) for every \( i, k = 1, 2, \ldots, n \) with \( i \neq k \) and \( j, l = 2, \ldots, m_i \) so \( d(v_{i,j}, v_{k,l}) = 4 = diam(SB_{m_1,m_2,\ldots,m_n}) \) then by using Property 2.2, \( v_{i,j} \in S \) or \( v_{k,l} \in S \). Therefore, \( S \) contains at least one vertex from distinct set \( X_{i,j,k,l} = \{v_{i,j}, v_{k,l}\} \) for \( i, k = 1, 2, \ldots, n \), \( i \neq k \), and \( j, l = 2, \ldots, m_i \). The minimum number of vertices from distinct set \( X_{i,j,k,l} \) is \( \sum_{i=1}^{n} (m_i - 1) - 1 \). Therefore, \( |S| \geq \sum_{i=1}^{n} (m_i - 1) - 1 \). \( \square \)
Lemma 2.5 For every integer \( m \geq 3 \) and \( n \geq 2 \), a set \( S = \{v_{1,2}, v_{1,3}, \ldots, v_{1,m_1}, v_{2,2}, v_{2,3}, \ldots, v_{2,m_2}, \ldots, v_{n,2}, v_{n,3}, \ldots, v_{n,m_n-1}\} \) is a strong resolving set of starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \).

Proof. We prove that for every two distinct vertices \( u, v \in V(SB_{m_1,m_2,\ldots,m_n}\setminus S) \), \( u \neq v \), there exists a vertex \( s \in S \) which strongly resolves \( u \) and \( v \). There are pairs four possible pairs of vertices.

(i) A pair of vertices \( (u,v_{n,m_n}) \).
For every integer \( i \in \{1,2,\ldots,n-1\} \) and \( j \in \{2,3,\ldots,m_i\} \), \( d(v_{i,j},v_{n,m_n})=4=diam(SB_{m_1,m_2,\ldots,m_n}) \), we obtain the shortest \( v_{n,m_n}-v_{i,j} \) path: \( v_{n,m_n}, v_{n,1}, u, v_{i,1}, v_{i,j} \). Thus, \( u \in I[v_{n,m_n},v_{i,j}] \).

(ii) A pair of vertices \( (u,v_1) \).
For every integer \( i \in \{1,2,\ldots,n-1\} \) and \( j \in \{2,3,\ldots,m_i\} \), \( d(u,v_{i,j})=2 \), we obtain the shortest \( u-v_{i,j} \) path: \( u, v_{i,1}, v_{i,j} \). Thus, \( v_{i,1} \in I[u,v_{i,j}] \). For \( i = n \) and \( l \in \{2,3,\ldots,m_i-1\} \), \( d(u,v_{n,l})=2 \), we obtain the shortest \( u-v_{n,l} \) path: \( u, v_{n,1}, v_{n,l} \). Thus, \( v_{n,1} \in I[u,v_{n,l}] \).

(iii) A pair of vertices \( (v_{i,1},v_{n,m_n}) \).
For every integer \( i \in \{1,2,\ldots,n-1\} \) and \( j \in \{2,3,\ldots,m_i\} \), \( d(v_{i,j},v_{n,m_n})=4=diam(SB_{m_1,m_2,\ldots,m_n}) \), we obtain the shortest \( v_{i,j}-v_{n,m_n} \) path: \( v_{i,j}, v_{i,1}, u, v_{n,1}, v_{n,m_n} \). Thus, \( v_{i,1} \in I[v_{i,1},v_{n,m_n}] \).

(iv) A pair of vertices \( (v_{i,1},v_{k,1}) \).
For every integer \( i,k \in \{1,2,\ldots,n-1\}, i \neq k \), and \( l \in \{2,3,\ldots,m_i\} \), \( d(v_{i,1},v_{k,l})=3 \), we obtain the shortest \( v_{i,1}-v_{k,l} \) path: \( v_{i,1}, u, v_{k,1}, v_{k,l} \). Thus, \( v_{k,1} \in I[v_{i,1},v_{k,l}] \). For \( k = n \) and \( l \in \{2,3,\ldots,m_i-1\} \), \( d(v_{1,1},v_{k,l})=3 \), we obtain the shortest \( v_{1,1}-v_{k,l} \) path: \( v_{1,1}, u, v_{k,1}, v_{k,l} \). Thus, \( v_{k,1} \in I[v_{1,1},v_{k,l}] \).

From every possible pairs of vertices, there exists a vertex \( s \in S \) which strongly resolves every two distinct vertices \( SB_{m_1,m_2,\ldots,m_n} \setminus S \). Thus \( S \) is a strong resolving set of starbarbell graph \( SB_{m_1,m_2,\ldots,m_n} \).

Theorem 2.2 Let \( SB_{m_1,m_2,\ldots,m_n} \) be the starbarbell graph with \( m \geq 3 \) and \( n \geq 2 \). Then \( sdim(SB_{m_1,m_2,\ldots,m_n}) = \sum_{i=1}^{n} (m_i - 1) - 1 \).

Proof. By using Lemma 2.5, we have a set \( S = \{v_{1,2}, v_{1,3}, \ldots, v_{1,m_1}, v_{2,2}, v_{2,3}, \ldots, v_{2,m_2}, \ldots, v_{n,2}, v_{n,3}, \ldots, v_{n,m_n-1}\} \) is a strong resolving set of \( SB_{m_1,m_2,\ldots,m_n} \) graph with \( m \geq 3 \) and \( n \geq 2 \). According to Lemma 2.4, \( |S| \geq \sum_{i=1}^{n} (m_i - 1) - 1 \), \( S \) is a strong metric basis of \( SB_{m_1,m_2,\ldots,m_n} \). Hence, \( sdim(SB_{m_1,m_2,\ldots,m_n}) = \sum_{i=1}^{n} (m_i - 1) - 1 \).

2.4. The Strong Metric Dimension of \( C_m \odot P_n \) Graph
The corona product \( C_m \odot P_n \) graph is graph obtained from \( C_m \) and \( P_n \) by taking one copy of \( C_m \) and \( n \) copies of \( P_n \) and joining by an edge each vertex from \( i^{th} \)-copy of \( P_n \) with the \( i^{th} \)-vertex of \( C_m \). The \( C_m \odot P_n \) can be depicted as in Figure 3.
Lemma 2.6 For every integer \( m \geq 3 \) and \( n \geq 2 \), if \( S \) is a strong resolving set of \( C_m \odot P_n \) graph then \(| S | \geq 2m - 1\).

Proof. Let us consider a pair of vertices \((v_{i,j},k,l)\) with \( i, k = 1, 2, \ldots, m, i \neq k \), and \( j, l = 1, 2 \), satisfying both of the conditions of Lemma 2.1. According to Property 2.1, we obtain \( v_{i,j} \in S \) or \( v_{k,l} \in S \). It means that \( S \) contains one vertex from distinct sets \( X_{i,j,k,l} = \{v_{i,j}, v_{k,l}\} \) with \( i, k = 1, 2, \ldots, m, i \neq k \), and \( j, l = 1, 2 \). The minimum number of vertices from distinct sets \( X_{i,j,k,l} \) is \( 2m - 1 \). Therefore, \(| S | \geq 2m - 1\). \( \square \)

Lemma 2.7 For every integer \( m \geq 3 \) and \( n \geq 2 \), a set \( S = \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \ldots, v_{m,1}\} \) is a strong resolving set of \( C_m \odot P_n \) graph.

Proof. We prove that every two distinct vertices \( u, v \in (C_m \odot P_n) \setminus S \), there exists a vertex \( s \in S \) which strongly resolves \( u \) and \( v \). There are two pairs of vertices from \( V(C_m \odot P_n) \setminus S \).

(i) A pair of vertices \((v_i, v_m,2)\).
For every integer \( i = 1, 2, \ldots, m \) and \( j = 1, 2 \) with \( 1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor \), \( d(v_{\left\lfloor \frac{m-1}{2} \right\rfloor,j}, v_{m,2}) = \left\lfloor \frac{m}{2} \right\rfloor + 2 \), we obtain the shortest \( v_{\left\lfloor \frac{m-1}{2} \right\rfloor,j} - v_{m,2} \) path : \( v_{\left\lfloor \frac{m-1}{2} \right\rfloor,j}, v_{\left\lfloor \frac{m-1}{2} \right\rfloor,1}, v_{\left\lfloor \frac{m-1}{2} \right\rfloor - 1}, \ldots, v_1, v_m, v_{m,2} \). So that \( v_i \in I[v_{\left\lfloor \frac{m-1}{2} \right\rfloor,j}, v_{m,2}] \).

For every integer \( i = 1, 2, \ldots, m \) and \( j = 1, 2 \) with \( \left\lfloor \frac{m-1}{2} \right\rfloor \leq i \leq n \), \( d(v_{\left\lceil \frac{m+1}{2} \right\rceil,j}, v_{m,2}) = \left\lceil \frac{m}{2} \right\rceil + 2 \), we obtain the shortest \( v_{\left\lceil \frac{m+1}{2} \right\rceil,j} - v_{m,2} \) path : \( v_{\left\lceil \frac{m+1}{2} \right\rceil,j}, v_{\left\lceil \frac{m+1}{2} \right\rceil,1}, v_{\left\lceil \frac{m+1}{2} \right\rceil + 1}, \ldots, v_{m-1}, v_m, v_{m,2} \). So that \( v_i \in I[v_{\left\lceil \frac{m+1}{2} \right\rceil,j}, v_{m,2}] \).

(ii) A pair of vertices \((v_i, v_k)\).
For every integer \( i, k = 1, 2, \ldots, m - 1 \), and \( j, l = 1, 2 \), without loss of generality, \( 1 \leq i < k \leq n \), \( v_i \) and \( v_k \) will be strongly resolved by \( v_{i,j} \), so that \( v_i \in I[v_{i,j}, v_k] \).

For every possible pairs of vertices, there exists a vertex \( s \in S \) which strongly resolves every two distinct vertices \( C_m \odot P_n \setminus S \). Thus \( S \) is a strong resolving set of \( C_m \odot P_n \). \( \square \)

Lemma 2.8 For every integer \( m \geq 3 \) and \( n \geq 3 \), if \( S \) is a strong resolving set of \( C_m \odot P_n \) graph then \(| S | \geq mn - 2\).
Proof. Let us consider a pair of vertices \( (v_{i,j}, k,l) \) with \( i, k = 1, 2, \ldots, m, \) \( i \neq k, \) and \( j, l = 1, 2, \ldots, n - 1, \) satisfying both of the conditions of Lemma 2.1. According to Property 2.1, we obtain \( v_{i,j} \in S \) or \( v_{k,l} \in S. \) It means that \( S \) contains one vertex from distinct sets \( X_{i,j,k,l} = \{ v_{i,j}, v_{k,l} \} \) with \( i, k = 1, 2, \ldots, m, \) \( i \neq k, \) and \( j, l = 1, 2, \ldots, n - 1. \) The minimum number of vertices from distinct sets \( X_{i,j,k,l} \) is \( mn - 2. \) Therefore, \( | S | \geq mn - 2. \) □

Lemma 2.9 For every integer \( m \geq 3 \) and \( n \geq 3, \) a set \( S = \{ v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,(n-2)} \} \) is a strong resolving set of \( C_m \odot P_n \) graph.

Proof. We prove that every two distinct vertices \( u, v \in (C_m \odot P_n) \setminus S, \) there exists a vertex \( s \in S \) which strongly resolves \( u \) and \( v. \) There are three pairs of vertices from \( V(C_m \odot P_n) \setminus S. \)

(i) A pair of vertices \( (v_i, v_{m,l}). \)
For every integer \( i = 1, 2, \ldots, m, \) \( j = 1, 2, \ldots, n, \) and \( l \in \{ n - 1, n \} \) with \( 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor, \) \( d(v_{\lfloor \frac{m-1}{2} \rfloor}, v_{m,l}) = \lfloor \frac{m}{2} \rfloor + 1, \) we obtain the shortest \( v_{\lfloor \frac{m-1}{2} \rfloor}, v_{m,l} \) path: \( v_{\lfloor \frac{m-1}{2} \rfloor}, v_{\lfloor \frac{m-1}{2} \rfloor}, v_{\lfloor \frac{m-1}{2} \rfloor} - 1, \ldots, v_1, v_{m,l}. \) So that \( v_i \in I[v_{\lfloor \frac{m-1}{2} \rfloor}, v_{m,l}]. \)

(ii) A pair of vertices \( (v_i, v_k) \)
For every integer \( i, k = 1, 2, \ldots, m - 1, \) and \( j, l = 1, 2, \) without loss of generality, \( 1 \leq i < k \leq n, \) and \( v_i \) and \( v_k \) will be strongly resolved by \( v_{i,j}, \) so that \( v_i \in I[v_{i,j}, v_k]. \)

(iii) A pair of vertices \( (v_{m,(n-1)}, v_{m,n}). \)
For every integer \( i = n - 2, \) \( d(v_{m,i}, v_{m,n}) = 2, \) we obtain the shortest \( v_{m,i} - v_{m,n} \) path: \( v_{m,i}, v_{m,(n-1)}, v_{m,n}. \) So that, \( v_{m,(n-1)} \in I[v_{m,i}, v_{m,n}]. \)

For every possible pairs of vertices, there exists a vertex \( s \in S \) which strongly resolves every two distinct vertices \( C_m \odot P_n \). Thus \( S \) is a strong resolving set of \( C_m \odot P_n. \) □

Theorem 2.3 Let \( C_m \odot P_n \) be the corona product of cycle graph and path graph with \( m \geq 3 \) and \( n \geq 2, \) then
\[
\text{sdim}(C_m \odot P_n) = \begin{cases} 2m - 1, & m \geq 3 \text{ and } n = 2; \\ mn - 2, & m \geq 3 \text{ and } n \geq 3. \end{cases}
\]

Proof. For \( m \geq 3 \) and \( n \geq 2, \) it is easy to see that \( sd(C_m \odot P_n) = 2m - 1. \) By Lemma 2.8 and Lemma 2.9, we have \( sd(C_m \odot P_n) = mn - 2 \) for \( m \geq 3 \) and \( n \geq 3. \) □

3. Concluding Remark
According to the discussion above it can be concluded that the strong metric dimension of a generalized butterfly graph \( BF_n, \) a starbarbel graph \( SB_{m_1, m_2, \ldots, m_n}, \) and \( C_m \odot P_n \) graph are as stated in Theorem 2.1, Theorem 2.2, and Theorem 2.3 respectively.

Open Problem: Determine the strong metric dimension of Hanoi graph.

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