SOME WELL-POSEDNESS AND ILL-POSEDNESS RESULTS FOR THE INLS EQUATION

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Abstract. We consider the initial value problem associated to the inhomogeneous nonlinear Schrödinger equation,

\[ iu_t + \Delta u + \mu |x|^{-b}|u|^\alpha u = 0, \quad u_0 \in H^s(\mathbb{R}^N) \text{ or } u_0 \in H^s(\mathbb{R}), \]

with \( \mu = \pm 1, b > 0, s \geq 0 \) and \( 0 < \alpha \leq \frac{4 - 2b}{N - 2s} \). By means of an adapted version of the fractional Leibniz rule, we prove new local well-posedness results in Sobolev spaces for a large range of parameters. We also prove some ill-posedness results for this equation, through a delicate analysis of the associated Duhamel operator.

1. Introduction

In this work, we consider the inhomogeneous nonlinear Schrödinger equation (INLS)

\[ iu_t + \Delta u + \mu |x|^{-b}|u|^\alpha u = 0, \quad (1.1) \]

where \( u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C} \), \( \mu = \pm 1 \), \( 0 < b < \min\{2, N\} \) and

\[ 0 < \alpha \leq \alpha_s, \text{ with } \alpha_s = \begin{cases} \frac{4 - 2b}{N - 2s}, & \text{if } s < N/2, \\ \infty, & \text{if } s \geq N/2. \end{cases} \]

This model has been a topic of intense research in the last few years ([14], [15], [7], [1], [2], [9], [19], [20]). Our aim in this work is to study the well-posedness of the initial value problem (IVP) associated to equation (1.1). First, we prove well-posedness in the usual Strichartz framework for initial data either in \( H^s \) or \( H^s \) and in both subcritical and critical cases. Second, we prove some weak ill-posedness results for large values of \( b + s \), by performing a refined analysis of the Duhamel operator.

We are particularly interested to address the fractional regularity. In this direction, Aloui and Tayachi [1] made use of Lorentz spaces to study the cases \( 0 \leq s < \min\{1, N/2\} \), \( 0 < b < \min\{2, N - 2s\} \) and \( 0 < \alpha \leq \alpha_s \). The same strategy was used by An and Kim [3] to study the critical case \( \alpha = \alpha_s \) with the restrictions \( 0 \leq s < N/2 \) and \( 0 < b < \min\{2, N - s, N/2 + 1 - s\} \). Another method, based on weighted \( L^p \) spaces, was introduced by Kim, Lee and Seo [19] to study the cases \( N \geq 3, 0 < s < 1/3 \) with some technical restrictions on the range of \( b \) and \( \alpha \). The more natural approach, relying on the Strichartz estimates in classical Sobolev spaces, was implemented by Guzmán [15] establishing the local well-posedness in \( H^s \), for \( 0 \leq s \leq \min\{1, N/2\} \), \( 0 < b < \min\{2, N/3\} \) and \( 0 < \alpha < \alpha_s \). An and Kim [2] studied the cases \( 0 \leq s < \min\{N, 1 + N/2\} \), \( 0 < b < \min\{2, N - s, N/2 + 1 - s\} \) and \( 0 < \alpha < \alpha_s \) using similar ideas. In these last two works, the starting point of the analysis is to split the space domain in different regions (around/far from the origin) since \( |x|^{-b} \) and its fractional derivatives fail to be in every \( L^p \) space, \( 1 \leq p \leq \infty \). However, the non-local nature of the fractional derivative requires a careful treatment of the nonlinear estimates and these authors overlooked this step using directly the fractional Leibniz rule locally in space (see for instance inequalities (3.29) and (3.44) in [2]). This difficulty does not appear in the works [1], [3] and [19] mentioned above, since one does not need to decompose the spatial domain when working with Lorentz and weighted \( L^p \) spaces. Here, we develop a modification of the fractional Leibniz rule to overcome this obstacle and successfully give a complete proof of local well-posedness in \( H^s \) based on Strichartz estimates in the classical Sobolev spaces. We
expect this new estimate to be applicable to other problems that involve fractional derivatives locally in space.

Recall that the critical Sobolev index is given by

\[ s_c = \frac{N}{2} - \frac{2 - b}{\alpha}, \]

in the sense that the \( \dot{H}^{s_c} \) norm remains invariant under the scaling \( \lambda \mapsto \lambda^{\frac{4s}{N-2s}} u(\lambda x, \lambda^2 t) \).

The first part of the paper is devoted to the well-posed results. Our approach is based in the fixed point method. More precisely, we want to show that the Duhamel operator

\[ \Phi(u)(x, t) = e^{it\Delta}u_0(x, t) + i\mu \int_0^t e^{i(t-\tau)\Delta}|x|^{-b}|u(x, \tau)|^\alpha u(x, \tau)\,d\tau, \quad t \in [0, T], \]

(1.2)

where \( e^{it\Delta} \) denotes the Schrödinger group, has a fixed point in a suitable complete metric space. This will follow either from a contraction argument or through the application of some stability results.

We start by considering the IVP associated to (1.1) with initial data in inhomogeneous Sobolev spaces. We refer to Section 2 for the precise definition of the auxiliary spaces mentioned in the statements below.

**Theorem 1.1** (Well-posedness in \( H^s \)). Let \( N \geq 1, s \geq 0, \alpha > 0 \) and \( 0 < b < \min \{2, N-s, \frac{N}{2} + 1-s\} \). Moreover, if \( \alpha \) is not an even integer, assume additionally that \( s < \alpha + 1 \). We then have:

(a) (Subcritical well-posedness in \( H^s \)) If

\[ 0 < \alpha < 2 - \frac{2b}{N-2s}, \quad s < \frac{N}{2}, \]

\[ \infty, \quad s \geq \frac{N}{2}, \]

then, for any \( u_0 \in H^s \), there exists \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution \( u \in C([0, T] : H^s) \cap (1-\Delta)^{-s/2}S(L^2, [0, T]) \) to (1.1) with initial datum \( u_0 \).

(b) (Critical well-posedness in \( H^s \)) If \( \alpha = \frac{2-2b}{N-2s} \), then for any \( u_0 \in H^s \), there exists \( T = T(u_0) > 0 \) such that there is a unique solution \( u \in C([0, T] : H^s) \cap (1-\Delta)^{-s/2}S(L^2, [0, T]) \) to (1.1) with initial datum \( u_0 \). Moreover, if \( \|u_0\|_{H^s} \) is sufficiently small, then the solution is global.

In the subcritical case, the above result extends the one obtained by Aloui and Tayachi [1], by relaxing the assumption \( b + 2s < N \) to \( b + s < N \). In the critical case, it extends the result by An and Kim [2], where they require the more restrictive assumption \( \|s| < \alpha + 1 \), if \( \alpha \) is not an even integer.

We also provide a detailed proof of this result using a generalized fractional Leibniz rule suitable for our context (see Lemmas 2.3 and Corollary 2.4).

We continue our well-posedness study considering now the initial data in homogeneous Sobolev spaces with \( 0 \leq s_c \leq s \leq 1 \).

**Theorem 1.2** (Well-posedness in \( \dot{H}^s \)). Let \( N \geq 1, 0 \leq s \leq 1 \) such that \( s < \frac{N}{2} \), and \( 0 < b < \min \{\frac{N}{2} + 1-s, N-s, 2\} \).

(a) (Subcritical well-posedness in \( \dot{H}^s \)) If \( 0 < \alpha < \frac{2-2b}{N-2s} \) and \( \alpha \geq 1 \) for \( N = 1 \), then for any \( u_0 \in \dot{H}^s \), there exists \( T = T(\|u_0\|_{\dot{H}^s}) > 0 \) such that there is a unique solution \( u \in C([0, T] : \dot{H}^s) \cap \dot{D}^{-s}S(L^2, [0, T]) \) to (1.1) with initial datum \( u_0 \).

(b) (Critical well-posedness in \( \dot{H}^s \)) If \( \alpha = \frac{2-2b}{N-2s} \), then for any \( u_0 \in \dot{H}^s \), there exists \( T = T(u_0) > 0 \) such that there is a unique solution \( u \in C([0, T] : \dot{H}^s) \cap \dot{D}^{-s}S(L^2, [0, T]) \) to (1.1) with initial datum \( u_0 \). Moreover, if \( \|u_0\|_{\dot{H}^s} \) is sufficiently small, then the solution is global.

Although the proofs here extend in a similar fashion to the case \( \max\{0, s_c\} \leq s < \frac{N}{2} \), we find that the increased technicality involved, albeit not being considerably deep, would impair the legibility of this manuscript. That said, the case \( s_c < 0, s = 0 \) is already considered standard (it is treated, e.g., in [14] and [15]), and the case \( s > 1 \) can be reduced to the case \( 0 < s \leq 1 \) by including up to \( |s|-1 \) derivatives in the corresponding norms.

\[^1\text{Here } \lceil s \rceil = \min \{m \in \mathbb{N} : m \geq s \} \text{ and } \lfloor s \rfloor = \max \{m \in \mathbb{N} : m \leq s \} \text{ denote the ceiling and floor functions, respectively.}\]
As an immediate consequence of part (a) we deduce a lower bound for the blow-up up rate of finite time solutions (see Corollary 3.1). This rate was used as an additional assumption in [4] to prove the blow up of the critical norm for the intercritical INLS equation. Our result implies that it always holds for this type of solution. The proof of part (b) relies in a different technique based in critical stability results in \( H^s \).

In all of the above results, well-posedness was achieved under the assumptions \( b + s < \min\{N, N/2 + 1\} \). These restrictions on \( b \) and \( s \) stem from the spatial norms used in the Strichartz estimates (in particular, from estimating \( D^s(|x|^{-b})|u|^\alpha u \sim |x|^{-b-s}|u|^\alpha u \) in a \( L^p \) space, with \( p \geq \min\{1, N/2 + 1\} \). It is hence natural to investigate whether these conditions are indeed necessary. We consider separately the cases

\[
b + s > N/2 + 1 \quad \text{or} \quad b + s > N.
\]

In each of the above cases, we are able to show some weak ill-posed results, in the sense that it is not possible to find a fixed point for the operator \( (1.2) \) in the classical Strichartz spaces (see Definition 2.2) using the contraction mapping principle. Our results in this direction read as follows.

**Theorem 1.3** (Weak ill-posedness for \( b + s < N \)). Suppose that

\[
N \geq 3, \quad 0 < b < 2, \quad \frac{N}{2} + 1 < b + s < N, \quad \max\left\{ \frac{N}{2} - 1, 0 \right\} < s < \frac{N}{2}, \quad \alpha < \frac{4 - 2b}{N}.
\]

Given \( \varphi \in C^\infty_c([0, 2] \times \mathbb{R}^N) \),

\[
(1) \quad \text{if} \quad \int_0^\infty y^{\frac{N}{2} - 1} \left( \int_0^1 e^{iyr} r^{\frac{N}{2} - s}(1 - r)^{\frac{N - b - s}{2}} dr \right) dy \neq 0, \quad (1.3)
\]

and \( \varphi(t, 0) = 1, \ t \in [0, 2], \) then

\[
\|D^s \Phi(\varphi)\|_{S(L^2,(0,2))} = \infty.
\]

\[
(2) \quad \text{if} \quad b + s > (N + 3)/2 \quad \text{and, for large} \ k \in \mathbb{R},
\]

\[
\varphi(t, 0) = e^{-ikt}, \ t \in [0, 2],
\]

then there exists a constant \( C = C\left( \|\phi\|_{S(L^2,(0,2))} + \|D^\alpha \phi\|_{S(L^2,(0,2))} \right) \) such that

\[
\|D^s \Phi(\varphi)\|_{L^\infty_t L^2_x((0,2) \times \mathbb{R}^N)} \geq k^{\frac{b + s - 2}{4(s - N/2)}} - C.
\]

**Remark 1.4.** Based on numerical simulations, we believe condition (1.3) holds for almost every possible value of \( b \) and \( s \). In Lemma 4.8 we prove rigorously that (1.3) holds for \( 4 < b + s < 6 \).

**Theorem 1.5** (Weak ill-posedness for \( b + s > N \)). Suppose that

\[
N \geq 1, \quad 0 < b < 2, \quad \max\{N, 2\} < b + s < N + 2,
\]

\[
b + s \neq N + 1, \quad \max\left\{ \frac{N}{2} - 1, 0 \right\} < s < \frac{N}{2}, \quad \alpha < \frac{4 - 2b}{N}.
\]

There exists \( \epsilon > 0 \) small such that, if \( \varphi \in L^\infty((0,2), C^\infty_c(\mathbb{R}^N)) \) satisfies \( \phi(t, 0) = \mathbb{1}_{[0,\epsilon]}(t), \ t \in [0, 2], \) then

\[
\|D^s \Phi(\varphi)\|_{L^\infty_t L^2_x((0,2) \times \mathbb{R}^N)} = \infty.
\]

The main difference in the assumptions of the last two theorems relies in the size of \( b + s \). Indeed, when \( b + s < N \), we use a result of Cazenave and Weissler [6] (see Lemma 4.4 below) to understand the asymptotic for the linear evolution \( e^{it\Delta} |x|^{-b-s} \). In the case \( b + s > N \), the weight \( |x|^{-b-s} \) is no longer locally integrable and Lemma 4.4 does not apply. We extend the asymptotic expansion of [6] to the case \( \max\{N, 2\} < b + s < N + 2 \) with the additional technical restriction \( b + s \neq N + 1 \) (see Lemma 4.10).

It should be noted that these ill-posedness results can be extended to other ranges of \( \alpha \), as long as there exists \( 0 < \tilde{s} < s \) such that (see Lemma 4.2)

\[
b + \tilde{s} < \min\left\{ \frac{N}{2} + 1, N \right\}, \quad \alpha < \frac{4 - 2b}{N - 2\tilde{s}}.
\]
Remark 1.6. Our weak ill-posedness results do not exclude the possibility of local well-posedness in general, but only in the usual Strichartz spaces. It may still happen that the introduction of a stronger norm, capable of detecting and excluding the anomalous behavior displayed in Theorems 1.3 and 1.5, gives rise to a fixed-point structure.

This paper is organized as follows. In Section 2, we established some preliminary estimates. The well-posed theory is discussed in Section 3. The last section is devoted to the proofs of the ill-posed results.

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2. Preliminaries

2.1. Notation. Let us start this section by introducing the notation used throughout the paper. We use $C$ to denote various constants that may vary line by line. Given any positive quantities $a$ and $b$, the notation $a \lesssim b$ or $a = O(b)$ means that $a \leq Cb$, with $C$ uniform with respect to the set where $a$ and $b$ vary. If necessary, we use subscript to indicate different parameters the constant may depend on. We also make use of the $o(\lambda)$ notation, which means a quantity that converges to zero as $\lambda$ tends to zero. We denote by $p'$ the Hölder conjugate of $1 \leq p \leq \infty$ and we use $a^+$ and $a^-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for a sufficiently small $\varepsilon > 0$.

2.2. Linear estimates. We make use of some mixed space-time Lebesgue spaces, which are defined in order to satisfy the so-called Strichartz estimates. Such estimates are named after an article of Strichartz [21], based on estimates for Fourier restrictions on a sphere. They were later generalized by Kato [16], Keel-Tao [17] and by Foschi [11].

Definition 2.1. If $N \geq 1$ and $s \in (-1, 1)$, the pair $(q, r)$ is called $\dot{H}^s$-admissible if it satisfies the condition

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s,$$

where

$$2 \leq q, r \leq \infty, \quad \text{and} \quad (q, r, N) \neq (2, \infty, 2).$$

In particular, if $s = 0$, we say that the pair is $L^2$-admissible.

Definition 2.2. Given $N > 2$, consider the set

$$A_0 = \left\{ (q, r) \text{ is } L^2\text{-admissible} \Big| 2^+ \leq r \leq \frac{2N}{N-2} \right\}.$$

For $N > 2$ and $s \in (0, 1)$, consider also

$$A_s = \left\{ (q, r) \text{ is } \dot{H}^s\text{-admissible} \Big| \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \right\}$$

and

$$A_{-s} = \left\{ (q, r) \text{ is } \dot{H}^{-s}\text{-admissible} \Big| \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \right\}.$$

When $N = 1, 2$ we replace the upper bound for the parameter $r$ in the definitions of $A_s$ and $A_{-s}$ by an arbitrarily large number. In the case $s = 1$, which we only consider for $N \geq 3$, we need to employ some values of $r$ which are larger than $2N/(N-2)$. The result which allows us to do so comes from Foschi [11], and one needs to be careful in the definition of the corresponding spaces to ensure the validity of the desired estimate. We then let $0 < \epsilon_0 < \epsilon_1 \ll 1$ be sufficiently small and define...
\[ A_1 = \left\{ (q, r) \text{ is } \dot{H}^1\text{-admissible} \, \bigg| \, \frac{2N}{N - 2 - \epsilon_0/2} \leq r \leq \frac{2N}{N - 2 - \epsilon_0} \right\} \]

and

\[ A_{-1} = \left\{ (q, r) \text{ is } \dot{H}^{-1}\text{-admissible} \, \bigg| \, \frac{2N}{N - 2 + 2\epsilon_1} \leq r \leq \frac{2N}{N - 2 - \epsilon_1} \right\}. \]

Note that, with the above definition of \( A_{\pm 1} \), if \((q, r) \in A_1 \) and \((\tilde{q}, \tilde{r}) \in A_{-1} \), then \( \frac{1}{q} + \frac{1}{\tilde{q}} \in [1 - 2\epsilon_1 - \epsilon_0, 1 - \frac{\epsilon_0 - \epsilon_1}{2}] \subset [0, 1] \), which enables us to use the Kato-Strichartz estimates obtained by Foschi, see [11] Remark 1.11.

Given a time-interval \( I \subset \mathbb{R} \) and \( 0 \leq s \leq 1 \), we define the following Strichartz norm

\[ \|u\|_{S(\dot{H}^s, I)} = \sup_{(q, r) \in A_s} \|u\|_{L^q_t L^r_x(I)}, \]

and the dual Strichartz norm

\[ \|u\|_{S'(\dot{H}^{-s}, I)} = \inf_{(q, r) \in A_{-s}} \|u\|_{L^{q'}_t L^{r'}_x(I)}. \]

If \( s = 0 \), we shall write \( S(\dot{H}^0, I) = S(L^2, I) \) and \( S'(\dot{H}^0, I) = S'(L^2, I) \). If \( I = \mathbb{R} \), we will omit \( I \). We also consider the norm

\[ \left\| (1 - \Delta)^{s/2} u \right\|_{L^2_x} = \left( \|u\|^2_{S(L^2, I)} + \|D^s u\|^2_{S(L^2, I)} \right)^{1/2}. \]

Note that none of these norms allow for the \( L^\infty \) norm on time. The main reason for this is to allow the \( S(\dot{H}^s) \) norm to be small in small time intervals. Since the \( L^\infty_t L^2_x \) norm also plays an important role, we define

\[ \|u\|_{S(L^2, I)} = \|u\|_{S(L^2, I)} + \|u\|_{L^\infty_t L^2_x(I)}. \]

In this work, we consider the following Kato-Strichartz estimates for \( s \geq 0 \) (Cazenave [5], Keel and Tao [17], Kato [16], Foschi [11])

\[ \|e^{it\Delta} f\|_{S(L^2)} \lesssim \|f\|_{L^2}, \]

\[ \|e^{it\Delta} f\|_{S(\dot{H}^s)} \lesssim \|f\|_{\dot{H}^s}, \quad \text{(2.1)} \]

\[ \left\| \int_\mathbb{R} e^{i(t-k)\Delta} g(k, \tau) d\tau \right\|_{S(L^2, I)} + \left\| \int_0^t e^{i(t-k)\Delta} g(k, \tau) d\tau \right\|_{S(L^2, I)} \lesssim \|g\|_{S(L^2, I)}; \]

and

\[ \left\| \int_\mathbb{R} e^{i(t-k)\Delta} g(k, \tau) d\tau \right\|_{S(\dot{H}^s, I)} + \left\| \int_0^t e^{i(t-k)\Delta} g(k, \tau) d\tau \right\|_{S(\dot{H}^s, I)} \lesssim \|g\|_{S(\dot{H}^s, I)}. \quad \text{(2.2)} \]

2.3. Fractional chain and Leibniz rules.

**Lemma 2.3** (Fractional chain rule for Lipschitz-type functions [8] (c.f. [18])). Let \( 0 < s < 1 \) and suppose \( F \in C(\mathbb{C}) \) and \( G \in C(\mathbb{C} : [0, \infty)) \) satisfy, for all \( u, v \),

\[ |F(u) - F(v)| \lesssim (G(u) + G(v))|u - v|. \]

If \( 1 < p, p_1, p_2 < +\infty \) are such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), then

\[ \|D^s F(u)\|_{L^p} \lesssim \|G(u)\|_{L^{p_1}} \|D^s u\|_{L^{p_2}}. \]

If the desired function is not Lipschitz, but Hölder continuous instead, we have the following version of the chain rule:
Lemma 2.4 (Fractional chain rule for Hölder-continuous functions [22]). Let $F$ be a Hölder-continuous function of order $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < \infty$ and $\frac{1}{\alpha} < \sigma < 1$, we have

$$||D^s F(u)||_{L^p} \lesssim ||u||_{L^{(1 - \frac{1}{\alpha})p_1}}^{\alpha - \frac{s}{\sigma}} ||D^\sigma u||_{L^\sigma}^{\frac{s}{\sigma}};$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{1}{\alpha\sigma})p_1 > 1$.

We also prove generalizations of the Leibniz rule suitable for singular weights.

Lemma 2.5 (Generalized Leibniz 1). Let $0 < s < 1$, $f, g \in S(\mathbb{R}^N)$, $A$ be a Lebesgue-measurable set and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p}$, with $p \in (0, \infty)$, $p_i, q_i \in (1, \infty)$ for $i \in \{1, 2\}$. Then

$$||D^s(fg) - (D^s f)g - f(D^s g)||_{L^p} \lesssim ||f||_{L^p(A)} ||D^s g||_{L^q_1} + ||f||_{L^p_2(A^c)} ||D^s g||_{L^q_2}$$

Proof. In [18], the Theorem A.8 proves $T : L^p_i \times L^q_i \to L^p_i$, $i \in \{1, 2\}$ is a bounded bilinear operator. By writing $f = 1_{A}f + 1_{A^c}f$, the result follows.

The following result follows immediately from Lemma 2.5 and Hölder inequality.

Corollary 2.6 (Generalized Leibniz 2). Let $0 < s < 1$, $f, g \in S(\mathbb{R}^N)$, $A, B$ be Lebesgue-measurable sets and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p}$, with $p \in (1, \infty)$, $q_i, r_i \in (1, \infty)$ for $i \in \{1, 2, 3, 4\}$. Then

$$||D^s(fg) - (D^s f)g - f(D^s g)||_{L^p} \lesssim ||D^s f||_{L^p_1(A)} ||g||_{L^q_1} + ||D^s f||_{L^p_2(A^c)} ||g||_{L^q_2} + ||f||_{L^p_3(B)} ||D^s g||_{L^q_3} + ||f||_{L^p_4(B^c)} ||D^s g||_{L^q_4}$$

The above result is essential for dealing with the inhomogeneous term in the nonlinearity, since in view of integrability restrictions we need to consider two different Lebesgue norms in order to study the function $|x|^{-b}$. In the next result, we explore this idea.

Lemma 2.7. If $a > 0$, $1 < p < \frac{N}{a}$, $0 \leq s < \frac{N}{p} - a$, and $q_{\pm \eta}$ is such that $\frac{1}{p} = \frac{a + u}{N} + \frac{1}{q_{\pm \eta}}$, for small $\eta > 0$, then

$$||D^s(|x|^{-a}f)||_p + ||D^s(|x|^{-a}f) - (D^s|x|^{-a})f||_p \lesssim \eta \left[||D^s f||_{q_{\pm \eta}} ||D^s f||_{q_{-\eta}}\right]^{\frac{1}{2}}$$

Proof. Define $q_{\pm \eta}$ as

$$\frac{1}{p} = \frac{a + s \pm \eta}{N} \pm \frac{1}{q_{\pm \eta}},$$

For $0 < s < 1$, using Lemma 2.4 Hölder and denoting by $B_R$ the ball of radius $R > 0$ centered at the origin, we write

$$||D^s(|x|^{-a}f)||_p + ||D^s(|x|^{-a}f) - (D^s|x|^{-a})f||_p \lesssim ||D^s|x|^{-a}||_{L^{\frac{N}{a+s+\eta}(B_R^1)}} ||f||_{q_{a,s}} + ||D^s|x|^{-a}||_{L^{\frac{N}{a+s-\eta}(B_R^2)}} ||f||_{q_{a,-s}}$$

$$+ |||x|^{-a}||_{L^{\frac{N}{a+s}(B_R^1)}} ||D^s f||_{q_{a,0}} + |||x|^{-a}||_{L^{\frac{N}{a+s}(B_R^2)}} ||D^s f||_{q_{a,0}} \lesssim \eta R_1^\eta ||f||_{q_{a,s}} + R_1^{-\eta} ||f||_{q_{a,-s}} + \eta R_2^\eta ||D^s f||_{q_{a,0}} + R_2^{-\eta} ||D^s f||_{q_{a,0}}.$$
we find 
\[\|D^s(|x|^{-a}f)\|_p + \|D^s(|x|^{-a}f) - (D^s|x|^{-a})f\|_p \lesssim \|f\|_{q_0, r} \|f\|_{q_0, s, r}^{\frac{1}{2}} + \|D^s f\|_{q_0, 0} \|D^s f\|_{q_0, -s, r}^{\frac{1}{2}}.\]

Finally by Sobolev \(\|f\|_{q_2, s, r} \lesssim \|D^s f\|_{q_2, 0, r} = \|D^s f\|_{q_2, s, r}\) and we finish the proof for \(0 < s < 1\). The case \(s = 0\) is analogous, and the case \(s \geq 1\) follows from induction, by writing \(D^s \sim D^{s-1} \nabla\), iterating the argument and using the classical Leibniz rule \(\partial^\alpha(fg) = \sum_{\alpha_1 + \alpha_2 = \alpha} \partial^{\alpha_1} f \partial^{\alpha_2} g\), for \(\alpha, \alpha_1, \alpha_2 \in \mathbb{Z}_{\geq 0}^N\).

### 2.4. Nonlinear estimates

Now, we use the previous results to deduce some scale-invariant nonlinear estimates.

**Lemma 2.8.** Let \(N \geq 1\), \(s \geq 0\), \(0 < \alpha \leq \frac{4 - 2b}{N - 2s}\), if \(s < N/2\) or \(0 < \alpha < \infty\), if \(s \geq N/2\) and \(0 < b < \min\{N/2 + 1 - s, N - s, 2\}\). Then the following inequalities hold

- If \(\alpha\) is an even integer or \(s < \alpha + 1\), there exists \(0 < \theta \leq 1\) such that:
  \[
  \|x|^{-b}|u|^\alpha v\|_{S'(L^2, I)} \lesssim \left|\frac{\theta x}{\theta x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^\theta \|v\|_{S(L^2, I)}.
  \]
  (2.3)

\[
\|D^s(|x|^{-b}|u|^\alpha u)\|_{S'(L^2, I)} \lesssim \left|\frac{\theta x}{\theta x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^{\alpha + 1} \|D^s u\|_{S(L^2, I)}.
\]  (2.4)

- If \(0 \leq s_c < s \leq 1\), \(s < N/2\), then
  \[
  \|D^s(|x|^{-b}|u|^\alpha u)\|_{S'(L^2, I)} \lesssim \left|\frac{x}{x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^\theta \|D^s u\|_{S(L^2, I)}.
  \]  (2.5)

Moreover, if we additionally assume that \(\alpha \geq 1\) for \(N = 1\), there exists \(0 \leq s_0 \leq \min\{\alpha, s\}\) such that
\[
\|D^{s_0}(|x|^{-b}|u|^\alpha v)\|_{S'(H^{-s_0}, I)} \lesssim \left|\frac{x}{x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(H^{-s_0}, I)}\right)^\theta \|D^{s_0} v\|_{S(H^{-s_0}, I)}.
\]  (2.6)

- If \(0 \leq s_c \leq 1\):
  \[
  \|x|^{-b}|u|^\alpha v\|_{S(H^{-s_c}, I)} \lesssim \|u\|_{S(H^{-s_c}, I)} \|v\|_{S(H^{-s_c}, I)},
  \]  (2.7)

\[
\|D^s(|x|^{-b}|u|^\alpha u)\|_{S'(L^2, I)} \lesssim \|u\|_{S(H^{-s_c}, I)} \|D^s u\|_{S(L^2, I)}.
\]  (2.8)

**Proof.** Throughout the whole proof, we let \(\eta = \eta(N, s, b, \alpha) > 0\) be sufficiently small and define precisely \(r_i^\pm\) and \(q_i^\pm\) as to satisfy \(\frac{1}{r_1} = \frac{1}{r_1} \pm \frac{\eta}{N}\) and \(\frac{1}{q_i} = \frac{1}{q_i} \pm \frac{\eta}{2}\), respectively. We start with the proof of the first inequalities. To prove (2.3), let \((\gamma, \rho) \in A_0\) and \(0 \leq \gamma \leq \rho\) be such that \(\gamma = s < N/2\) and \(s_c < \gamma < N/2\) if \(s \geq N/2\). We define the relations
\[
\frac{1}{\rho} = \frac{b + \eta}{N} + \frac{1}{r_1^+}, \quad \frac{1}{r_1} = \frac{\alpha}{r_2} + \frac{1}{r_3}, \quad \frac{1}{r_2} = \frac{1}{r_3} \frac{1}{N}, \quad \frac{2}{q_3} = \frac{N}{2} - \frac{N}{r_3},
\]  (2.9)

which implies
\[
\frac{1}{\gamma} = \frac{\alpha(s - s_c)}{2} + \frac{\alpha + 1}{q_3}.
\]

Then use Lemma 2.7 Hölder and Sobolev to write
\[
\|x|^{-b}|u|^\alpha v\|_{L_1^{s', s} L_2^{s', s}} \lesssim \left[\|x|^{-b}|u|^\alpha v\|_{L_1^{s', s}} \|x|^{-b}|u|^\alpha v\|_{L_2^{s', s}}\right]^\theta \|v\|_{L_2^{s', s}} \|u\|_{L_1^{s', s}} \|v\|_{L_2^{s', s}}^\theta
\]

\[
\lesssim \left|\frac{\theta x}{\theta x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^\theta \|v\|_{L_2^{s', s}} \|u\|_{L_1^{s', s}} \|v\|_{L_2^{s', s}}^\theta
\]

\[
\lesssim \left|\frac{x}{x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^\theta \|D^s u\|_{S(L^2, I)} \|D^s v\|_{S(L^2, I)}
\]

\[
\lesssim \left|\frac{x}{x + \theta^s b}\right| \left(\|1 - \Delta\|^s/2\|u\|_{S(L^2, I)}\right)^\theta \|D^s u\|_{S(L^2, I)} \|D^s v\|_{S(L^2, I)}
\]
and, invoking also Lemmas 2.3 and 2.4 (see also [12, Proposition 1.3] for a direct proof in the case \( s > 1 \)),

\[
\|D^s(|x|^{-b}|u|^a v)\|_{L^{\gamma}_x L^{r_3}_y} \lesssim \left( \left\| D^s(|u|^a v)\|_{L^{r_3}_x} \right\| D^s(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^s(|u|^a v)\|_{L^{r_3}_x} \right\| D^s(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}}.
\]

We then claim it is possible to have \((q_3^\pm, r_3^\pm), (q_3, r_3) \in A_0\). Indeed, using the relations (2.9) we have

\[
\frac{1}{\rho} = \frac{b}{N} + \frac{1}{r_1} = \frac{b}{N} + \frac{\alpha + 1}{r_3} - \frac{\alpha s}{N} = \frac{b}{N} + \frac{\alpha + 1}{r_2} - \frac{\tilde{s}}{N},
\]

with the restrictions \(1 \leq r_1, r_2 < \infty\) and \(2 < \rho, r_3 < 2^*\), where

\[
2^* = \begin{cases} \frac{\infty}{N}, & N = 1, 2, \\ \frac{2N}{N-2}, & N \geq 3. \end{cases}
\]

Thus, in dimensions \(N = 1, 2\), such choice is possible if \(\rho\) is chosen in the interval

\[
\max \left\{ \frac{N}{N - b + \alpha s}, \frac{N}{N - b - \tilde{s}} \right\} < \rho < \frac{2N}{N - 2b - \alpha(N - 2\tilde{s})}.
\]

On the other hand, if \(N \geq 3\), we have

\[
\max \left\{ \frac{2N}{N - 2b + 2 + \alpha(2 - (N - 2\tilde{s}))}, \frac{N}{N - b - \tilde{s}} \right\} < \rho < \min \left\{ \frac{2N}{N - 2}, \frac{2N}{N - 2b - \alpha(N - 2\tilde{s})} \right\}.
\]

In both cases, by the definition of \(\tilde{s}\), the restriction on \(\alpha\) and the fact that \(b + s < N/2 + 1\), simple calculations imply that we can find such \(\rho\).

Now, by defining \(0 < \theta \leq 1\) as to satisfy

\[
\theta(s - s_c) = \tilde{s} - s_c,
\]

the estimates (2.3) and (2.4) are proved by the embedding \(W^{s, r_3} \hookrightarrow W^{\tilde{s}, r_3}\). Recalling that \(\tilde{s} = s\) if \(s < N/2\), estimate (2.5) follows.

To prove (2.6), we first consider the case \( \alpha < 1 \) and \( N \geq 2 \). Define \( \sigma = (1 - \eta)s \) and \( s_0 = (1 - \eta)^2 \alpha s \), so that \( 0 < s_0 < \alpha \) and \( \frac{\alpha}{2} < \sigma < s \leq 1 \). Let \((\gamma, \rho) \in A_{(s - s_0)}, (q_3^\pm, r_3^\pm) \in A_{s - s_0}\), and \((q_8, r_8) \in A_0\). From Lemmas 2.1, 2.6 and 2.4 and Sobolev, we have

\[
\|D^{s_0}(|x|^{-b}|u|^a v)\|_{L^{\gamma}_x L^{r_3}_y} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \times \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}} \lesssim \left( \left\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right\| D^{s_0}(|u|^a v)\|_{L^{r_3}_x} \right)^{\frac{1}{2}}.
\]
where
\[
\frac{1}{\rho'} = \frac{b \pm \eta}{N} + \frac{1}{r_1^2}, \quad \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_4} + \frac{1}{r_5}, \quad \frac{1}{r_2} = \frac{1}{r_6} + \frac{1}{r_7}, \quad \left(\frac{\alpha - s_0}{\sigma}\right)r_6 > 1, \quad \frac{1}{r_3} = \frac{1}{r_5} - \frac{s_0}{N}, \quad \frac{1}{r_8} = \frac{1}{\alpha r_4} + \frac{s}{N} = \frac{1}{\alpha - \frac{s_0}{\sigma} r_6} + \frac{s}{N} = \frac{1}{\alpha - \frac{s_0}{\sigma} r_7} + \frac{s - \sigma}{N}.
\]

We choose \(\frac{1}{r_8} = \frac{1}{2} - \eta\), so that \(0 < 1/ \left(\frac{\alpha - s_0}{\sigma}\right) r_6 = 1/2 - s/N - \eta < 1\), and \(r_5 = \rho\). With this choice and recalling that \(\alpha = \frac{4 - 2b}{N - 2s_c}\), the above relations imply
\[
\frac{1}{\rho} = \frac{N - 2}{2N} + \frac{\alpha(s - s_c)}{2N} + \frac{\alpha \eta}{2} \quad \text{and} \quad \frac{1}{\gamma} = \frac{\alpha(s - s_c)}{2} + \frac{\alpha}{q_8} + \frac{1}{q_5}.
\]

In order to have \((\gamma, \rho) \in A_{-(s-s_0)}\), we need to satisfy the condition
\[
\frac{N - 2}{2N} < \frac{1}{\rho} < \frac{N - 2(s - s_0)}{2N},
\]
which is true, for \(\eta > 0\) sufficiently small, since \(\alpha(1 + s_c) > 0\). By then calculating the \(L^{\gamma'}_1\)-norm and using H"older and the corresponding scaling relations, one obtains (2.6) in this case.

The case \(\alpha \geq 1\) follows by setting \(s_0 = s\) and using the same spaces as in (2.10) to write
\[
\|D^s(|x|^{-b}|u|^{\alpha}v)\|_{L^{\gamma'}_1 L^{\gamma''}_2} \lesssim \left[I \frac{\|D^s u\|_{L^{q_1}_1 L^{q_2}_2}}{t}\right] \left[\|D^s u\|_{L^{q_1}_1 L^{q_2}_2} \|D^s v\|_{L^{q_3}_1 L^{q_4}_2}\right].
\]

Now, we turn to the proof of (2.7). Let \((\gamma, \rho) \in A_{-s_c}\) and write
\[
\|x|^{-b}|u|^{\alpha}v\|_{L^{\gamma'}_1 L^{\gamma''}_2} \lesssim \left[I \frac{\|u^{\alpha} v\|_{L^{q_1}_1 L^{q_2}_2}}{t}\right] \left[\|u^{\alpha} v\|_{L^{q_1}_1 L^{q_2}_2} \|v\|_{L^{q_3}_1 L^{q_4}_2}\right],
\]
where
\[
\frac{1}{\rho'} = \frac{b \pm \eta}{N} + \frac{1}{r_1^2}, \quad \frac{1}{r_1} = \frac{\alpha}{r_2} + \frac{1}{r_3}, \quad (q_j, r_j) \in A_{s_c}, \quad j = 2, 3.
\]

We start considering \(s_c < 1\). If \(N \geq 2\), we set \(\frac{1}{r_2} = \frac{N - 2s_c}{2N} - \eta\) and \(r_3 = \rho\) to get
\[
\frac{1}{\rho} = \frac{N - 2}{2N} + \frac{\alpha \eta}{2}.
\]

On the other hand, if \(N = 1\), we set \(\frac{1}{\rho} = \frac{N - 2}{2N} - \eta\) and \(r_3 = \rho\) to deduce
\[
\frac{1}{r_2} = \frac{1 - b - 2\eta}{\alpha} < \frac{2 - b}{\alpha} = \frac{1 - 2s_c}{2}.
\]

In both cases, we satisfy the required conditions to ensure \((\eta, \rho) \in A_{-s_c}\) and \((q_j, r_j) \in A_{s_c}, \quad j = 2, 3\).

It remains to consider the case \(s_c = 1\), which requires \(N \geq 3\). We now choose fix \(0 < \epsilon \ll 1\), impose \(0 < \eta \ll 1\) and set \(r_2 = r_3 = \frac{2N - 2s_c}{2}\) to obtain
\[
\frac{1}{\rho} = \frac{N - 2 + (1 + (4 - 2b)/(N - 2))\epsilon}{2N}.
\]

Therefore, choosing \(\epsilon_0, \epsilon_1\) in the definition of \(A_{\pm 1}\) (c.f. Definition 2.2) as
\[
\epsilon_0 = \left[1 + \frac{2 - b}{2(N - 2)}\right] \epsilon, \quad \epsilon_1 = \left(1 + \frac{2 - b}{N - 2}\right) \epsilon,
\]
we ensure \((\gamma, \rho) \in A_{-1}\) and \((q_2, r_2), (q_3, r_3) \in A_1\).

It remains to prove (2.3). In the same fashion as (2.10), for \((\gamma, \rho) \in A_0\), we write
\[
\|D^s(|x|^{-b}|u|^{\alpha}u)\|_{L^{\gamma'}_1 L^{\gamma''}_2} \lesssim \|u^{\alpha} u\|_{L^{q_1}_1 L^{q_2}_2} \left[\|D^s u\|_{L^{q_1}_1 L^{q_2}_2} \|D^s v\|_{L^{q_3}_1 L^{q_4}_2}\right].
\]
with the relations
\[
\frac{1}{\rho^2} = \frac{b + \eta}{N} + \frac{1}{r_1^2}, \quad \frac{1}{r_1} = \frac{\alpha}{\rho} + \frac{1}{r_3}, \quad (q_2, r_2) \in \mathcal{A}_c, \quad (q_3^\pm, r_3^\pm) \in \mathcal{A}_0.
\]
As in the proof of (2.7), when \( s_c < 1 \) we choose \( \frac{1}{r_2} = \frac{N-2s_c}{2N} - \eta \) and \( r_3 = \rho \) if \( N \geq 2 \), and \( \frac{1}{\rho} = \eta \) and \( r_3 = \rho \) if \( N = 1 \). For the case \( s_c = 1 \), which requires \( N \geq 3 \), we set \( r_2 = \frac{2N}{N-2} \), \( r_3 = \rho \) and obtain
\[
\frac{1}{\rho} = \frac{N - 2 + \epsilon(2 - b)/(N - 2)}{2N} > \frac{N - 2}{2N}.
\]
For all these choices we ensure \((\gamma, \rho), (q_3^\pm, r_3^\pm) \in \mathcal{A}_0 \) and \((q_2, r_2) \in \mathcal{A}_1 \). \( \square \)

3. **Well-posedness**

In this section we show the existence results stated in Theorems (1.1)-(1.2).

### 3.1. Well-posedness in inhomogeneous Sobolev spaces.

**Proof of Theorem (1.1)** We consider each case separately.

(a) Fixed \( u_0 \in H^s \), let
\[
E = \left\{ u \in L_t^\infty H_x^s([0, T]) : (1 - \Delta)^{s/2} u \in \tilde{S}(L^2, [0, T]), \right\}
\]
be a (complete) metric space with the metric
\[
\rho(u, v) = \| u - v \|_{S(L^2, [0, T])}.
\]

Then, applying the linear estimates (2.1)-(2.2) and the nonlinear estimates (2.3)-(2.4) to the operator (1.2), we obtain
\[
\| (1 - \Delta)^{s/2} \Phi(u) \|_{S(L^2, [0, T])} \leq C \| u_0 \|_{H^s} + C T^{-\alpha/(s+1)} \| (1 - \Delta)^{s/2} u \|_{S(L^2, [0, T])}^{\alpha+1}
\]
\[
\leq C \left[ 1 + O(T^{-\alpha/(s+1)} \| u_0 \|_{H^s}) \right] \| u_0 \|_{H^s},
\]
\[
\| \Phi(u) - \Phi(v) \|_{S(L^2, [0, T])} \leq O(T^{-\alpha/(s+1)} \| u_0 \|_{H^s}) \| u - v \|_{S(L^2, [0, T])}.
\]

Thus, choosing \( T = \delta \| u_0 \|_{H^s}^{-\alpha/(s+1)} \) for \( \delta > 0 \) small enough (depending only on universal constants) we deduce the result by standards arguments.

(b) In this case, for \( \delta_0 > 0 \) be chosen later, we let \( T > 0 \) be such that
\[
\| (1 - \Delta)^{s/2} e^{it \Delta} u_0 \|_{S(L^2, [0, T])} < \delta_0.
\]

We then define the complete metric space \((E, \rho)\) by
\[
E = \left\{ u \in L_t^\infty H_x^s([0, T]) : (1 - \Delta)^{s/2} u \in \tilde{S}(L^2, [0, T]), \right\}
\]
\[
\| u \|_{L_t^\infty H_x^s([0, T])} \leq 2C \| u_0 \|_{H^s},
\]
\[
\| (1 - \Delta)^{s/2} u \|_{S(L^2, [0, T])} \leq 2 \| (1 - \Delta)^{s/2} e^{it \Delta} u_0 \|_{S(L^2, [0, T])},
\]
\[
\rho(u, v) = \| u - v \|_{S(L^2, [0, T])}.
\]

Thus, using the same ideas as in part (a) and (3.1), we have
\[
\| \Phi(u) \|_{L_t^\infty H_x^s([0, T])} \leq C \| u_0 \|_{H^s} + C \| (1 - \Delta)^{s/2} u \|_{S(L^2, [0, T])}^{\alpha+1}
\]
\[
\leq \left[ 1 + O(\delta_0)^\alpha \right] \| u_0 \|_{H^s},
\]
\[
\| (1 - \Delta)^{s/2} \Phi(u) \|_{S(L^2, [0, T])} \leq \| (1 - \Delta)^{s/2} e^{it \Delta} u_0 \|_{S(L^2, [0, T])} + C \| (1 - \Delta)^{s/2} u \|_{S(L^2, [0, T])}^{\alpha+1}
\]
\[
\leq \left[ 1 + O(\delta_0)^\alpha \right] \| (1 - \Delta)^{s/2} e^{it \Delta} u_0 \|_{S(L^2, [0, T])},
\]
\[
\| \Phi(u) - \Phi(v) \|_{S(L^2, [0, T])} \leq \delta_0 \| u - v \|_{S(L^2, [0, T])}.
\]
The result follows by choosing \( \delta_0 > 0 \) small enough, depending only on universal constants. Note that, if
\[
\|(1 - \Delta)^{s/2} e^{it\Delta} u_0\|_{S(L^2, \mathbb{R})} < \delta_0,
\]
then the solution exists for all \( t \in \mathbb{R} \).

\[ \square \]

3.2. Well-posedness in homogeneous Sobolev spaces.

**Proof of Theorem 1.2 (a).** Fixed \( u_0 \in \dot{H}^s \), let
\[
E = \left\{ u \in L^\infty_t \dot{H}_x^s([0, T] \times \mathbb{R}^N) : u \in L^\infty_t L^{2N}([0, T] \times \mathbb{R}^N), \right. 
\]
\[
D^s u \in \dot{S}(L^2, [0, T]), 
\]
\[
\|D^s u\|_{\dot{S}(L^2, [0, T])} \leq 2C\|u_0\|_{H^{s\alpha}} \}
\]
be a metric space with the metric
\[
\rho(u, v) = \|D^{s_0}(u - v)\|_{S(H^{s - s_0}, [0, T])},
\]
where \( s_0 \) is given in Lemma 2.8.

Thus, from inequalities (2.6) and (2.5), we deduce
\[
\|D^s \Phi(u)\|_{\dot{S}(L^2, [0, T])} \leq C\|u_0\|_{H^s} + CT^{\frac{s-1}{2}} \|D^s u\|_{S(L^2, [0, T])}^{\alpha+1} \|D^s u\|_{H^s},
\]
\[
\leq C[1 + O(T^{-\frac{s-1}{2}} \|u_0\|_{H^s})^\alpha] \|u_0\|_{H^s},
\]
\[
\|D^{s_0} \Phi(u) - \Phi(v)\|_{S(H^{s - s_0}, [0, T])} \leq O(T^{-\frac{s-1}{2}} \|u_0\|_{H^s})^\alpha \|D^{s_0} (u - v)\|_{S(H^{s - s_0}, [0, T])}.
\]

We then choose \( T = \delta \|u_0\|_{H^s}^{-\frac{2}{\alpha}} \) for \( \delta > 0 \) small enough (depending only on universal constants).

From the nonlinear estimates, one can also prove a blow-up rate for subcritical finite-time blow up.

**Corollary 3.1 (Subcritical blow-up rate).** Under the conditions of Theorem 1.2 (a), if the corresponding solution \( u \) blows up in finite positive time \( T > 0 \), then
\[
\|u(t)\|_{H^s} \gtrsim \frac{1}{(T - t)^{\frac{1}{\alpha}}}
\]
for any \( t \in [0, T) \).

**Proof.** We first observe that the local theory (in the subcritical case) implies
\[
\|u(t)\|_{H^s} \to \infty, \text{ as } t \nearrow T.
\]

Now, assume by contradiction that there exists a sequence \( t_n \not\to T \) such that, for all \( n \),
\[
(T - t_n)^{\frac{1}{\alpha}} \|u(t_n)\|_{H^s} < \frac{1}{n}.
\]

From the Duhamel formula and arguing as in the proof of Theorem 1.2 (a), for \( t \in [t_n, T) \), there exists \( C_0 > 0 \) such that
\[
\|D^s u\|_{\dot{S}(L^2, [t_n, t])} \leq C_0 \|u(t_n)\|_{H^s} + \frac{C_0}{n^{\alpha} \|u(t_n)\|_{H^s}^{\alpha+1}} \|D^s u\|_{S(L^2, [t_n, t])}^{\alpha+1}.
\]

Consider the function \( f(x) = x - C_0 \|u(t_n)\|_{H^s} + \frac{C_0}{n^{\alpha} \|u(t_n)\|_{H^s}^{\alpha+1}} x^{\alpha+1} \). A simple computation reveals that it has a global maximum at \( x^*_n = \frac{n^{\alpha} \|u(t_n)\|_{H^s}}{\alpha C_0^{\alpha+1} (\alpha + 1)} \), \( f(x^*_n) = \left( \frac{\alpha n}{\alpha + 1} \right)^{\alpha+1} C_0 \|u(t_n)\|_{H^s} \) and \( f(0) < 0 \). Thus, for \( n_0 > \frac{|C_0 (\alpha + 1)|^{\alpha+1}}{\alpha n^{\alpha} \|u(t_n)\|_{H^s}^{\alpha+1}} \) we have that \( f(x^*_n) > 0 \).

Setting \( x(t) = \|D^s u\|_{S(L^2, [t_n, t])} \), from (3.3) we have that \( f(x(t)) \leq 0 \), for all \( t \in [t_n, T) \). A continuity argument then implies that \( x(t) \leq x^*_n \) and therefore
\[
\|D^s u\|_{S(L^2, [t_n, t])} \lesssim \|u(t_n)\|_{H^s}.
\]
So, estimate (3.3) implies \( u \in L_t^\infty \dot{H}^s_x([0,T]) \), which is a contradiction with (3.2).

To prove Theorem 1.2(b), we rely on the stability theory.

**Lemma 3.2** (Critical stability). Let \( 0 \leq s \leq 1 \), \( s < N/2 \), \( \alpha = \frac{4 - 2b}{N - 4} \), \( u_0 \in H^s \) and \( \tilde{u} \) be a \( C^0_t H^s_x \) solution to

\[
i \partial_t \tilde{u} + \Delta \tilde{u} + \mu |x|^{-b} |\tilde{u}|^\alpha \tilde{u} = e,
\]

where \( \mu = \pm 1 \). Assume that the boundedness conditions

\[
\|D^s \tilde{u}\|_{\hat{S}(L^2, t)} \leq E, \quad \|D^s e\|_{\hat{S}(L^2, t)} \leq E
\]

and the smallness conditions

\[
\|e^{it \Delta} (u_0 - \tilde{u}_0)\|_{\hat{S}(\dot{H}^s_x, t)} \leq \epsilon, \quad \|e\|_{\hat{S}(\dot{H}^{-s}_x, t)} \leq \epsilon,
\]

hold for \( 0 < \epsilon < \epsilon_0 = \epsilon_0(E) \). Then there is a solution \( u \) to (1.1) with initial datum \( u_0 \) such that

\[
\|u - \tilde{u}\|_{\hat{S}(\dot{H}^s_x, t)} \lesssim E \epsilon
\]

and

\[
\|D^s u\|_{\hat{S}(L^2, t)} \lesssim E 1.
\]

**Proof.** Write \( w = u - \tilde{u} \) and split \( I \) in a finite number \( N \sim E/\delta \) of intervals \( I_j = [t_j, t_{j+1}] \) such that

\[
\|\tilde{u}\|_{\hat{S}(\dot{H}^s_x, I_j)} \sim \delta \quad \text{for all} \quad j.
\]

Let \( F_j = \{ w : \|w\|_{\hat{S}(\dot{H}^s_x, I_j)} \leq 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{\hat{S}(\dot{H}^s_x, I_j)} + 2CE, \|D^s w\|_{\hat{S}(L^2, I_j)} \leq 2\|w(t_j)\|_{\dot{H}^s_x} + 2CE \} \)

be the (complete) metric space with the metric

\[
\rho_j(w_1, w_2) = \|w_1 - w_2\|_{\hat{S}(\dot{H}^s_x, I_j)}
\]

and define

\[
\Phi_j(w) = e^{i(t-t_j)\Delta} w(t_j) + i\mu \int_{t_j}^t e^{i(t-\tau)\Delta} \left[ |x|^{-b} |w + \tilde{u}|^\alpha (w(\tau) + \tilde{u}(\tau)) - |x|^{-b} |\tilde{u}|^\alpha (\tilde{u}(\tau)) \right] d\tau
\]

\[
- i\mu \int_{t_j}^t e^{i(t-\tau)\Delta} e(\tau) d\tau.
\]

Then,

\[
\|\Phi_j(w)\|_{\hat{S}(\dot{H}^s_x, I_j)} \leq \|e^{i(t-t_j)\Delta} w(t_j)\|_{\hat{S}(\dot{H}^s_x, I_j)}
\]

\[
+ C \left[ \|w\|_{\hat{S}(\dot{H}^s_x, I_j)} + \|\tilde{u}\|_{\hat{S}(\dot{H}^s_x, I_j)} \right] \|w\|_{\hat{S}(\dot{H}^s_x, I_j)} + C\epsilon
\]

\[
\leq [1 + o_{\delta,\epsilon}(1)]\|e^{it\Delta} w(t_j)\|_{\hat{S}(\dot{H}^s_x, I_j)} + [1 + o_{\delta,\epsilon}(1)]C\epsilon,
\]

\[
\|D^s \Phi_j(w)\|_{\hat{S}(L^2, I_j)} \leq C\|w(t_j)\|_{\dot{H}^s_x} + C\|w + \tilde{u}\|_{\hat{S}(\dot{H}^s_x, I_j)} (\|D^s w\|_{\hat{S}(L^2, I_j)} + \|D^s \tilde{u}\|_{\hat{S}(L^2, I_j)})
\]

\[
+ C\|\tilde{u}\|_{\hat{S}(\dot{H}^s_x, I_j)} \|D^s \tilde{u}\|_{\hat{S}(L^2, I_j)} + CE
\]

\[
\leq [1 + o_{\delta,\epsilon}(1)]C\|w(t_j)\|_{\dot{H}^s_x} + [1 + o_{\delta,\epsilon}(1)]CE,
\]

\[
\|\Phi_j(w_2) - \Phi_j(w_1)\|_{\hat{S}(\dot{H}^s_x, I_j)} \leq C \left[ \|w_2\|_{\hat{S}(\dot{H}^s_x, I_j)} + \|w_1\|_{\hat{S}(\dot{H}^s_x, I_j)} + \|\tilde{u}\|_{\hat{S}(\dot{H}^s_x, I_j)} \right] \|w_2 - w_1\|_{\hat{S}(\dot{H}^s_x, I_j)}
\]

\[
\leq o_{\delta,\epsilon}(1)\|w_2 - w_1\|_{\hat{S}(\dot{H}^s_x, I_j)}.
\]

It is clear that we can impose \( \delta \) and \( \epsilon \) small enough, in such a way that both depend only on universal constants, concluding that each \( \Phi_j \) is a contraction on \( F_j \). By inductively constructing the solution on \( I_0, I_1, \cdots, I_{N-1} \), we have a solution \( w \) defined on \( I \) such that, for all \( j > 0 \),

\[
\|D^s w\|_{\hat{S}(L^2, I_j)} \leq 2C\|w(t_j)\|_{\dot{H}^s_x} + 2CE,
\]

\[
\|w\|_{\hat{S}(\dot{H}^s_x, I_j)} \leq 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{\hat{S}(\dot{H}^s_x, I_j)} + 2CE.
\]
Noting that \( \|w(t_j)\|_{H^s} \leq \|D^\alpha w\|_{S(L^2, I_{j-1})} \), we get, by (3.5), \( \|D^\alpha w\|_{S(L^2, I)} \lesssim C^N E \sim C^2 E \). Now, observing, by Duhamel, that

\[
e^{i(t-t_j)\Delta} w(t_j) = e^{i(t-t_{j-1})\Delta} w(t_{j-1}) + i\mu \int_{t_{j-1}}^{t_j} e^{i(t-\tau)\Delta} \left[ |x|^{-\beta} w + \tilde{u}^\alpha (w(\tau) + \tilde{u}(\tau)) - |x|^{-\beta} |\tilde{u}|^\alpha \tilde{u}(\tau) \right] d\tau
\]

we get

\[
\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(H^s)} \leq 2\|e^{i(t-t_{j-1})\Delta} w(t_{j-1})\|_{S(H^s)} + 2C\epsilon.
\]

Therefore, by (3.5),

\[
\|w\|_{S(H^s, I)} \lesssim 2^N C\epsilon \sim 2^2 C\epsilon.
\]

\[\square\]

**Proof of Theorem 1.2(b).** Let \( \{u_{0,n}\}_n \) be a sequence of functions in \( H^s \) such that \( \|u_{n,0} - u_0\|_{H^s} \to 0 \) as \( n \to +\infty \). Let \( T_0 > 0 \) be such that

\[
\|e^{it\Delta} u_0\|_{S(H^s, [0,T_0])} < \delta_0,
\]

for a small \( \delta_0 > 0 \) to be chosen later. Note that, if \( n \) is large enough, then

\[
\|e^{it\Delta} u_{n,0}\|_{S(H^s, [0,T_0])} \lesssim \delta_0.
\]

By Duhamel, Strichartz and Lemma 2.8 one then has, for all \( t \in [0,T_0] \):

\[
\|u_n\|_{S(H^s, [0,t])} \lesssim \delta_0 + \|u_n\|_{S(H^s, [0,t])}^{\alpha+1}.
\]

This bootstraps to \( \|u_n\|_{S(H^s, [0,T_0])} \lesssim \delta_0 \), if \( \delta_0 \) is small (depending only on universal constants). We then have

\[
\|D^\alpha u_n\|_{S(L^2, [0,T_0])} \lesssim \|u_0\|_{H^s} + \|u_n\|_{S(H^s, [0,T_0])}^\alpha \|D^\alpha u_n\|_{S(L^2, [0,T_0])},
\]

which finally implies

\[
\|D^\alpha u_n\|_{S(L^2, [0,T_0])} \lesssim \|u_0\|_{H^s}.
\]

Thus, we can employ Lemma 3.2 (with \( \epsilon \equiv 0 \)) to guarantee that the sequence of solutions \( \{u_n\}_n \) is Cauchy in the norm \( \|\cdot\|_{S(H^s, [0,T_0])} \), therefore converging to, say, \( u \). By reflexivity, uniqueness of strong and weak limits and lower semicontinuity of the norm, passing to a subsequence, we can assume that \( u \in L^\infty_T H^s((0,T_0]) \cap D^{-\alpha} S(L^2, [0,T_0]) \). Strichartz estimates again let us write, for \( [t_1, t_2] \subset [0,T_0] \):

\[
\|u(t_2) - u(t_1)\|_{H^s} \lesssim \|e^{i(t_2-t_1)\Delta} - I\| u_0\|_{H^s} + \|u\|_{S(H^s, [t_1, t_2])}^\alpha \|D^\alpha u\|_{S(L^2, [0,T_0])},
\]

which shows that \( u \in C^0_T H^s((0,T_0]) \). Uniqueness also follows from Strichartz: if \( u \) and \( v \) satisfy the Duhamel formula with initial datum \( u_0 \), then (3.6), Strichartz and Lemma 2.8 imply

\[
\|u\|_{S(H^s, [0,T_0])} + \|v\|_{S(H^s, [0,T_0])} \lesssim \delta_0.
\]

Thus, estimating the difference, we get

\[
\|u - v\|_{S(H^s, [0,T_0])} \lesssim a_{\delta_0}(1) \|u - v\|_{S(H^s, [0,T_0])},
\]

which implies \( u \equiv v \). Similarly to the well-posedness result in \( H^s \), we remark that if

\[
\|e^{it\Delta} u_0\|_{S(H^s, \mathbb{R})} < \delta_0,
\]

then the solution is defined for all \( t \in \mathbb{R} \). \[\square\]
4. Ill-Posedness

We now turn to the proof of the ill-posedness results (Theorems 1.3 and 1.5). The main idea is to use Hölder regularity in $x$ to replace $|u(t, x)|^{\alpha}u(t, x)$ by $|u(t, 0)|^{\alpha}u(t, 0)$. The resulting Duhamel term then involves the linear evolution of the weight $|x|^{-b-s}$, which can be determined (almost) explicitly.

More precisely, set $f = |u|^{\alpha}u$. We decompose

$$
D^{s} \int_{0}^{t} e^{i(t-\tau)\Delta} |x|^{-b} f(\tau, x) d\tau = \int_{0}^{t} e^{i(t-\tau)\Delta} \left[ D^{s} (|x|^{-b} f(\tau, x)) - (D^{s} |x|^{-b}) f(\tau, x) \right] d\tau \\
+ \int_{0}^{t} e^{i(t-\tau)\Delta} \left[ (D^{s} |x|^{-b}) f(\tau, x) - f(\tau, 0) \right] d\tau \\
+ \int_{0}^{t} e^{i(t-\tau)\Delta} (D^{s} |x|^{-b}) f(\tau, 0) d\tau =: I + II + III.
$$

The following two lemmas state that $I$ and $II$ are well-behaved in the Strichartz space:

**Lemma 4.1.** If $s > N/2 - 1$ and $b + s > N/2$,

$$
\|I\|_{S(L^2, I)} \lesssim \|u\|_{S(L^2, I)} + \|D^{s} u\|_{S(L^2, I)}^{\alpha + 1}
$$

**Proof.** We split the estimate in the regions $B = B_{1}(0)$ and $B^{c}$.

**Step 1. Control near the origin.** In $B$, we make use of the Hölder estimate

$$
|f(t, x) - f(t, 0)| \lesssim |x|^{\theta} \|f(t)\|_{W^{s, \tilde{r}}}, \quad \theta = s - N_{r} \leq 1.
$$

$$
\left\| \int_{0}^{t} e^{i(t-\tau)\Delta} \left[ (D^{s} |x|^{-b}) (f(\tau, x) - f(\tau, 0)) \right] d\tau \right\|_{S(L^2, I)} \lesssim \|D^{s} |x|^{-b} (f(t, x) - f(t, 0))\|_{L^q_t L^{q'}_x(B)} \\
\lesssim \|x|^{-b-s+\theta} \|f(t)\|_{W^{s, \tilde{r}}(B)} \lesssim \|x|^{-b-s+\theta} \|f(t)\|_{W^{s, \tilde{r}}(B)}
$$

Here we require

$$
b + s - \theta < \frac{N}{r}, \quad \text{that is,} \quad \frac{1}{r} + \frac{1}{\tilde{r}} < 1 - \frac{b}{N}.
$$

Using the fractional chain rule (Lemma 2.3) and Sobolev embedding,

$$
\|f(t)\|_{W^{s, \tilde{r}}} \lesssim \|u(t)\|_{L^{q'}_{t, \tilde{r}}}^{\alpha q' \tilde{r} / \tilde{q} - s} \|u(t)\|_{W^{s, r}}^{\alpha q' \tilde{r} / \tilde{q} - s} \lesssim \|u(t)\|_{W^{s, N_{r} / s}}^{\alpha q' \tilde{r} / \tilde{q} - s} \|u(t)\|_{W^{s, r}}^{\alpha q' \tilde{r} / \tilde{q} - s}.
$$

Let $(\tilde{q}, \tilde{r})$ and $(q_{0}, N_{r}/s)$ be Strichartz admissible. Integrating in time,

$$
\|f\|_{L^q_t W^{s, \tilde{r}} < q_{0}, \tilde{r}} \lesssim \|u\|_{L^{q'}_{t, \tilde{r}}}^{\alpha q' \tilde{r} / \tilde{q} - s} \|u\|_{L^q_t W^{s, \tilde{r}}}^{\alpha q' \tilde{r} / \tilde{q} - s} \lesssim \|u\|_{L^{q'}_{t, \tilde{r}}}^{\alpha q' \tilde{r} / \tilde{q} - s} \|u\|_{L^q_t W^{s, \tilde{r}}}
$$

In the last step, we need

$$
\frac{\alpha q' \tilde{r}}{\tilde{q} - q'} < q_{0}, \quad \text{that is,} \quad \frac{\alpha q' \tilde{r}}{\tilde{q} - q'} < q_{0} < 1 - \frac{N_{r}}{2} + \frac{N_{r}}{2} \left( \frac{1}{r} + \frac{1}{\tilde{r}} \right).
$$

Imposing

$$
\frac{1}{r} + \frac{1}{\tilde{r}} = \left( 1 - \frac{b}{N} \right)^{-1},
$$

the condition on $\alpha$ simplifies to $\alpha < \frac{4 - 2b}{N_{r} - 2}$, which holds.

In conclusion, the choices of $r, \tilde{r}$ are restricted by

$$
\tilde{r}, r, \frac{N_{r}}{s} \in [2, 2N/(N - 2)], \quad s - \frac{N}{r} > 0.
$$

This turns out to be equivalent to $s > N/2 - 1$ and

$$
\max \left\{ \frac{1}{2} - \frac{b}{N}, \frac{N - 2}{2N}, \frac{s - 1}{N} \right\} < \frac{1}{r} \leq \min \left\{ \frac{1}{2} - \frac{b}{N}, \frac{1}{N}, \frac{1}{2}, \frac{s}{N} \right\}
$$
Integrating in time,

\[ b + s > \frac{N}{2}, \quad b < 2, \quad s > \frac{N}{2} - 1. \]

**Step 2. Control away from the origin.**

\[
\left\| \int_0^t e^{(t-\tau)\Delta} \left[ \left( D^s |x|^{-b} \right) (f(\tau, x) - f(\tau, 0)) \right] d\tau \right\|_{S(L^2, I)} \lesssim \| |x|^{-b-s} \|_{L^\infty_t L^1_x(B^c)} \lesssim \| |x|^{-b-s} \|_{L^2(B^c)} \| f \|_{L^1_t L^\infty_x} \lesssim \| u \|^\alpha_{L^2_t L^\infty_x} \lesssim \| u \|^\alpha_{L^2_t W^{s,N+}/s},
\]

as long as \( \alpha + 1 < q_0 \), that is

\[ \alpha < \frac{4 - N + 2s}{N - 2s}. \]

Since \( \alpha < (4 - 2b)/(N - 2s) \), the above condition is verified for \( b + s > N/2 \).

**Lemma 4.2.** Suppose that, for some \( 0 < \tilde{s} < s < N/2 \),

\[ b + \tilde{s} < \min\{N, 1 + N/2\}, \quad \alpha < (4 - 2b)/(N - 2\tilde{s}). \]

Then

\[ \| I \|_{S(L^2, I)} \lesssim \| u \|_{S(L^2, I)} + \| D^s u \|_{S(L^2, I)}^{\alpha+1} \]

**Proof.** Applying Lemma 2.7

\[ \| D^s (|x|^{-b} f) - (D^s |x|^{-b}) f \|_{L^2_x} \lesssim \left( \| D^s f \|_{L^2_x} + \| D^s f \|_{L^2_x} \right)^{1/2}, \quad \frac{1}{r^2} = \frac{b}{N} + \frac{1}{r_2} \]

We focus on the estimate for \( L^2_x \), the other is treated similarly. We take

\[ \frac{1}{r_2^2} = \frac{1}{r_5} + \frac{1}{r_6} \quad \frac{1}{\alpha r_6} = \frac{1}{r_5} = \frac{\tilde{s}}{N}. \]

Through the fractional chain rule (Lemma 2.3),

\[ \| D^s u \|_{L^2_x}^{\alpha} \lesssim \| u \|_{L^2_x}^{\alpha} \| D^s u \|_{L^2_x} \lesssim \| u \|_{L^2_x}^{\alpha+1} \]

Integrating in time,

\[ \| u \|_{L^2_x} \| D^s u \|_{L^2_x} \lesssim \| u \|_{L^2_x}^{\alpha+1} \| D^s u \|_{L^2_x} \lesssim \| u \|_{L^2_x}^{\alpha+1} \]

as long as \( q'(\alpha + 1) < q_5 \), which is equivalent to \( \alpha < (4 - 2b)/(N - 2\tilde{s}) \).

In conclusion, we are able to bound \( I \) if

\[ r, r_5 \in [2, 2^*], \quad r_2, r_6 \in [1, \infty], \]

where \( 2^* \) is given in (2.11).

For \( N = 1, 2 \), the existence of such exponents is equivalent to

\[ \max \left\{ \frac{(\alpha + 1)\tilde{s}}{N}, \frac{1}{2} - \frac{b}{N} + \frac{\alpha\tilde{s}}{N} \right\} < \min \left\{ 1 - \frac{b}{N} + \frac{\alpha\tilde{s}}{N}, \frac{\alpha + 1}{2} \right\}, \]

while for \( N \geq 3 \), it reduces to

\[ \max \left\{ \frac{(\alpha + 1)\tilde{s}}{N}, (\alpha + 1) \tilde{s} + 1 - \frac{b}{N} + \frac{\tilde{s}\alpha}{N} \right\} < \min \left\{ \frac{N + 2}{2N} - \frac{b}{N} + \frac{\tilde{s}\alpha}{N}, \frac{\alpha + 1}{2} \right\}. \]

In any case, a simple computation shows that the above inequalities hold for \( b + \tilde{s} < \min\{N, 1 + N/2\} \), \( \tilde{s} < N/2 \) and \( \alpha < (4 - 2b)/(N - 2\tilde{s}) \). \(\square\)
Now that we have reduced the analysis to the term \( III \), define
\[
(Tf)(t, x) = \int_0^t f(t - \tau) e^{i\tau \Delta} D^s|x|^{-b} d\tau,
\]
so that
\[
III = T\left( |u|^{\alpha}u \right|_{x=0}^{x=\infty}.
\]
If \( b + s \neq N + k \), \( k \in \mathbb{N}_0 \), \( D^s|x|^{-b} = c_{b,s,N}|x|^{-b-s} \) (see Lemma 4.10 below) and thus
\[
(Tf)(t, x) = c_{b,s,N} \int_0^t f(t - \tau) e^{i\tau \Delta} |x|^{-b-s} d\tau.
\]

**Remark 4.3.** Fix \( u \in \dot{S}(L^2, I) \) with \( D^s u \in \dot{S}(L^2, I) \). Notice that, over any bounded time interval,
\[
\|(|u|^{\alpha}u)(\cdot, 0)\|_{L^q_t} \lesssim \|u\|_{L_t^{\alpha+1}L_x^\infty}^{\alpha+1} \lesssim \|u\|_{L_t^{\alpha+1}W_x^{N+/s}}^{\alpha+1},
\]
which means that \( (|u|^{\alpha}u)(\cdot, 0) \in L^q_t \) if
\[
(\alpha + 1)q \leq q_0, \quad \text{for } (q_0, N^{+}/s) \text{ admissible.}
\]
This holds iff
\[
q < \frac{4}{(N - 2s)(\alpha + 1)}.
\] (4.1)

To derive the precise asymptotics for \( T \), we need an accurate description of \( e^{it\Delta}|x|^{-b-s} \). For the rest of this section, we set
\[
\theta = \frac{b + s}{2}, \quad \beta = \frac{N - b - s}{2}.
\]

**Lemma 4.4 ([11] Proposition 3.3 and Lemma 3.5).** For \( 0 < b + s < N \),
\[
e^{i\tau \Delta}|x|^{-b-s} = \frac{1}{(4it)^\frac{b+s}{2}} \frac{1}{\Gamma((b+s)/2)} \frac{\Delta \left( \frac{|x|^2}{4t} \right)}{t^{b+s}H(y; \theta, \beta)} = \frac{1}{(4it)^\frac{b+s}{2}} \frac{1}{\Gamma((b+s)/2)} \frac{\Delta \left( \frac{|x|^2}{4t} \right)}{t^{b+s}H(y; \theta, \beta)},
\]
where
\[
H(y; \theta, \beta) = \int_0^1 e^{iyr^{\theta-1}(1-r)^{\beta-1}} dr.
\]
Furthermore, one has the following asymptotics for \( H \):
\[
H(y; \theta, \beta) = c_1 y^{-\theta} + c_2 e^{iy} y^{-\beta} + O(y^{\theta-1}) + O(y^{-\beta-1}) \quad \text{as } y \to \infty, \quad c_1, c_2 \neq 0.
\] (4.2)

From the above description, we can already see that the boundedness of \( Tf \) in the Strichartz space \( \dot{S}(L^2, I) \) imposes conditions on \( b + s \):

**Lemma 4.5.** Assume that (1.3) holds. For \( N \geq 3 \), \( \min \{N, 2 + N/2\} > b + s > 2 \) and \( f \equiv 1 \),
\[
\|Tf\|_{\dot{S}(L^2, (1,2))} < +\infty \Rightarrow b + s \leq N/2 + 1.
\]

**Proof.** We have
\[
(T1)(t, x) = \int_0^t \frac{1}{(4it)^\frac{b+s}{2}} \frac{1}{\Gamma((b+s)/2)} \frac{\Delta \left( \frac{|x|^2}{4t} \right)}{t^{b+s}} d\tau \sim \int_{|x|^2/4t}^{\infty} \left| x \right|^{2-2\theta} y^{-2} H(y) dy =: \left| x \right|^{2-2\theta} K \left( \frac{|x|^2}{4t} \right).
\]
Since \( \theta > 1 \), the asymptotic expansion (1.2) of \( H \) implies that \( |K(0)| < \infty \), since \( \theta > 1 \). By (1.3), \( K(0) \neq 0 \). Observe that
\[
\left| K \left( \frac{|x|^2}{4t} \right) - K(0) \right| \lesssim \int_{|x|^2/4t}^{\infty} y^{\theta-2} dy \lesssim \left( \frac{|x|^2}{4t} \right)^{\theta-1}.
\]
Therefore, in the region \( |x| \ll 1, t \in (1,2) \),
\[
(T1)(t, x) \sim |x|^{2-2\theta} K(0).
\]
Taking the \( L^p_x \) norm, the integrability at \( x = 0 \) is equivalent to
\[
2 - 2\theta > -\frac{N}{p}.
\]
Taking $\rho \to 2N/(N-2)$, we obtain $b+s \leq N/2 + 1$. \hfill \Box

**Lemma 4.6.** Under the conditions of Lemma 4.5, if $4 < b + s < 6$, then \& holds.

**Proof.** Consider the region 

$$Q_R = \{ z \in \mathbb{C} : \text{Re } z > 0, \text{ Im } z > 0, 1/R < |z| < R \}, \quad R \gg 1,$$

and, for $z \in Q_R$,

$$g(z, r) = e^{(\theta-2)\ln z} e^{irz} r^{\theta-1}(1-r)^{\beta-1}, \quad G(z) = \int_0^1 g(z, r) dr.$$

Since $G$ is analytic on $Q_R$, $\int_{\partial Q_R} G(z) dz = 0$. Therefore

$$-\int_{1/R}^R y^{\theta-2} H(y) dy = - \int_{1/R}^R \int_0^1 y^{\theta-2e^{iyr}} r^{\theta-1}(1-r)^{\beta-1} dr dz$$

$$= \int_{1/R}^R \int_0^1 z^{\theta-2} e^{(\theta-2)\frac{iy}{r}} e^{-rz} r^{\theta-1}(1-r)^{\beta-1} dr dz$$

$$+ \int_{z \in Q_R, |z|=R} G(z) dz + \int_{z \in Q_R, |z|=1/R} G(z) dz$$

We claim that the last two integrals converge to zero as $R \to \infty$ by dominated convergence. For the first, we use the exponential decay and $\theta < 3$:

$$|g(Re^{i\gamma}, r)| = R^{\theta-2} r^{\theta-1} e^{R \sin \gamma (1-r)^{\beta-1}} \to 0, \quad (\gamma, r) \in (0, \pi/2) \times (0, 1)$$

and

$$|g(Re^{i\gamma}, r)| = R^{\theta-2} r^{\theta-1} e^{-R r \sin \gamma (1-r)^{\beta-1}} \lesssim (R r \sin \gamma)^{\theta-2} e^{-R r \sin \gamma} \frac{r}{(\sin \gamma)^{\theta-2}} (1-r)^{\beta-1}$$

$$\lesssim \frac{r}{(\sin \gamma)^{\theta-2}} (1-r)^{\beta-1} \in L^1((0, \pi/2) \times (0, 1)).$$

For the second, we need $\theta > 2$:

$$\left| g \left( \frac{e^{i\gamma}}{R}, r \right) \right| \lesssim R^{2-\theta} r^{\theta-1} (1-r)^{\beta-1} \to 0$$

and

$$\left| g \left( \frac{e^{i\gamma}}{R} \right) \right| \lesssim \int_0^1 R^{2-\theta} r^{\theta-1} (1-r)^{\beta-1} dr \lesssim \int_0^1 r^{\theta-1} (1-r)^{\beta-1} dr \in L^1((0, \pi/2) \times (0, 1)).$$

Therefore

$$K(0) = \lim_{R \to \infty} \int_{1/R}^R y^{\theta-2} H(y) dy = - \lim_{R \to \infty} \int_{1/R}^R \int_0^1 z^{\theta-2} e^{(\theta-2)\frac{iy}{r}} e^{-rz} r^{\theta-1}(1-r)^{\beta-1} dr dz$$

$$= e^{(\theta-2)\frac{iy}{r}} \int_0^{+\infty} \int_0^1 z^{\theta-2} e^{-rz} r^{\theta-1}(1-r)^{\beta-1} dr dz \neq 0. \hfill \Box$$

In the next three lemmas, we aim to provide an example where $Tf$ is not controllable in $L^\infty_t L^2_x$. Write

$$\text{III} = \int_0^t f(t-\tau, 0) \mathbb{1}_{|x|^2 < \sigma} e^{ix\Delta} (D^s |x|^{-b}) \, d\tau + \int_0^t f(t-\tau, 0) \mathbb{1}_{|x|^2 > \sigma} e^{ix\Delta} (D^s |x|^{-b}) \, d\tau = \text{III}_1 + \text{III}_2.$$

**Lemma 4.7.** If $N/2 < b + s < N$,

$$\left\| \text{III}_1 \right\|_{L^\infty_t L^2_x} \lesssim \|u\|_{\mathcal{S}(L^2_t, 1)} \left( \|u\|_{\mathcal{S}(L^2_t, 1)} + \|D^s u\|_{\mathcal{S}(L^2_t, 1)} \right)^{\alpha+1}.$$
Proof. By duality,
\[
\left| \int_0^T \int_0^t f(t - \tau, 0) \frac{1}{\tau^\theta} H \left( \frac{\tau}{\tau}, \theta, \beta \right) \mathbb{1}_{|x|^2 < \tau} \phi(x, t) d\tau dx dt \right| \leq \int_0^T \int_0^t |f(t - \tau, 0)| \frac{1}{\tau^\theta} \left( \int_{|x|^2 < \tau} |\phi(x, t)| dx \right) d\tau dt \leq \int_0^T \int_0^t |f(t - \tau, 0)| \tau^{\frac{N}{2} - \theta} \|\phi(t)\|_{L^2_x} d\tau dt
\]
Fix
\[
q = \frac{4^-}{(N - 2\beta)(\alpha + 1)} > 1,
\]
which satisfies (4.1). From Hölder and Young’s inequality,
\[
\int_0^T \int_0^t |f(t - \tau, 0)| \tau^{\frac{N}{2} - \theta} \|\phi(t)\|_{L^2_x} d\tau dt \lesssim \|f(\cdot, 0) * (\tau^{\frac{N}{2} - \theta} \mathbb{1}_{[0,T]}))\|_{L^q_x} \|\phi\|_{L^1_t L^2_x}
\]
The boundedness of this last term is equivalent to
\[
\frac{N}{4} - \theta \geq - \frac{1}{q'}, \text{ that is, } \alpha < \frac{4 - 2b}{N - 2s}.
\]

Set \( \eta = \min\{\theta, \beta + 1\} \). Using the asymptotic expansion for \( H \), we write
\[
\text{III}_2 = \int_0^t f(t - \tau, 0) \mathbb{1}_{|x|^2 > \tau} e^{ix\Delta} (D^\tau |x|^{-b}) d\tau = c \int_0^t f(t - \tau, 0) \frac{1}{\tau^\theta} \left( \frac{|x|^2}{\tau} \right)^{-\beta} e^{i|x|^2/\tau} \mathbb{1}_{|x|^2 > \tau} d\tau + \int_0^t f(t - \tau, 0) \frac{1}{\tau^\theta} O \left( \left( \frac{|x|^2}{\tau} \right)^{-\eta} \right) \mathbb{1}_{|x|^2 > \tau} d\tau =: \text{III}_{21} + \text{III}_{22}.
\]

Lemma 4.8. If \( N/2 < b + s < \min\{N, 2 + N/2\} \),
\[
\|\text{III}_{22}\|_{L^\infty \cap L^2_x} \lesssim_{|I|} (\|u\|_{S(L^2_x)} + \|D^\tau u\|_{S(\dot{L}^{1,1})})^{\alpha + 1}.
\]

Proof. The proof follows from similar arguments to those in the previous lemma. Taking \( \phi \in L^1_t L^2_x \),
\[
\int_0^T \int_0^t \left| \frac{f(t - \tau, 0)}{\tau^\theta} \left( \frac{|x|^2}{\tau} \right)^{-\eta} \mathbb{1}_{|x|^2 > \tau} \phi(x, t) d\tau dx dt \right| \lesssim \int_0^T \int_0^t \left| \frac{f(t - \tau, 0)}{\tau^\theta} \right| \left( \frac{|x|^2}{\tau} \right)^{-\eta} \|\phi(t)\|_{L^2_x} d\tau dt
\]
Since \( 4\eta > N \),
\[
\int_{\tau < |x|^2} \frac{\tau^{2\eta}}{|x|^{4\eta}} dx \lesssim \tau^{N/2}
\]
and the estimate follows as in (4.3).

\]

Lemma 4.9. If \( f(t, 0) = e^{-ikt} \), \( k > 0 \) large, then
\[
\|\text{III}_{21}\|_{L^\infty \cap L^2_x([1,2] \times R^N)} \gtrsim k^{\frac{b}{2} + \frac{N}{2} + \frac{3}{2}}.
\]
Proof. To prove the lemma, it suffices to consider the region $|x| \in D = [100/k^{1/2}, 1/2]$. In particular, $|x|^2 < t$. Thus

$$\mathbf{I}_{21} = e^{-ikt} \int_0^{|x|^2} e^{ikr} \left( \frac{|x|^2}{r} \right)^{-\beta} e^{ix|x|^2/2} r e^{-|x|^2/2} dr = e^{-ikt} |x|^2 e^{-2\beta t} e^{i(y+k|x|^2)/y} dy$$

The crucial observation is that the phase has a (unique) nondegenerate critical point at $y = k^{1/2}|x|$. Define

$$P(z) = z + \frac{1}{z}, \quad \lambda = k^{1/2}|x| > 100.$$  

Thus

$$\mathbf{I}_{21} = |x|^2 e^{-2\beta t} e^{i\lambda P(z)} dz$$

Take a bump function $\psi \in C_c^\infty$ localized near $z = 1$. A standard stationary phase argument (see [10 Section 3.1]) shows that

$$\int_{1/\lambda}^\infty z^{-\beta - 2} e^{i\lambda P(z)} dz \sim \lambda^{-1/2}, \quad \int_{1/\lambda}^\infty z^{-\beta - 2} e^{i\lambda P(z)}(1 - \psi(z)) dz \lesssim \lambda^{-1}.$$  

We conclude that

$$\mathbf{I}_{21} \sim |x|^2 e^{-2\beta t} e^{i\lambda P(z)} \sim |x|^{-2(N-1)/2} k^{1/2}(b+s-N+1)$$

Since $|x|^{-2(N-1)}$ is integrable at the origin,

$$\|\mathbf{I}_{21}\|_{L^2(D)} \sim k^{1/2}(b+s-N+1)$$

and the result follows.

Finally, we consider the case $b + s > N$. In this situation, we cannot apply Lemma 4.4 directly. Fortunately, we have

**Lemma 4.10.** Given $\max\{N, 2\} < b + s < N + 2$ such that $b + s \neq N + 1$, one has

$$e^{it\Delta} |x|^{-b-s} = \frac{1}{\beta(4it)^{1/2} \Gamma((b+s)/2)} \left[ (\theta - 1) H \left( \frac{|x|^2}{t}, \theta - 1, \beta + 1 \right) + i|\frac{x|^2}{t} H \left( \frac{|x|^2}{t}, \theta, \beta + 1 \right) \right].$$

As a consequence, for $|x|^2 \gg t$,

$$e^{it\Delta} |x|^{-b-s} \sim \frac{1}{|t|^b} \left[ e^{i\|x|^2/4t} \left( \frac{|x|^2}{t} \right)^{-\beta} + O \left( \left( \frac{|x|^2}{t} \right)^{-\theta + 1} \right) + O \left( \left( \frac{|x|^2}{t} \right)^{-\beta - 1} \right) \right].$$

**Proof.** Fix a test function $\varphi \in C_c^\infty(\mathbb{R}^N)$. From Lemma 4.4 for $0 < \lambda < N$,

$$\langle e^{it\Delta} |x|^{-\lambda}, \varphi \rangle = \left\langle \frac{1}{(4it)^{1/2} \Gamma(\lambda/2)} \frac{1}{H \left( \frac{|x|^2}{4t}, \lambda - N - \lambda \right), \varphi} \right\rangle.$$  

For $2 < \lambda < N$,

$$\frac{N - \lambda}{2} H \left( y; N - \lambda \right) = \frac{N - \lambda}{2} \int_0^1 e^{i\sqrt{\pi}r((\lambda - 2)/2)(1 - r)(N-\lambda)/2} dr = \int_0^1 \partial_r \left( e^{i\sqrt{\pi}r((\lambda - 2)/2)} (1 - r)^{(N-\lambda)/2} dr \right)$$

$$= i\sqrt{\pi} \left( \frac{\lambda - 2}{2}, y; \frac{N + 2 - \lambda}{2} \right) + \frac{\lambda - 2}{2} H \left( y; \frac{\lambda - 2}{2}, \frac{N + 2 - \lambda}{2} \right)$$

Therefore

$$\frac{N - \lambda}{2} \langle e^{it\Delta} |x|^{-\lambda}, \varphi \rangle = \left\langle \frac{1}{(4it)^{1/2} \Gamma(\lambda/2)} \left[ \frac{\lambda - 2}{2} H \left( \frac{|x|^2}{t}, \frac{\lambda - 2}{2}, \frac{N + 2 - \lambda}{2} \right) + i|\frac{x|^2}{t} H \left( \frac{|x|^2}{t}, \frac{\lambda - 2}{2}, \frac{N + 2 - \lambda}{2} \right) \right], \varphi \right\rangle.$$
By [13, Section III.3.2], the l.h.s. is meromorphic in \( \lambda \), with poles at the integers greater or equal than \( N \). On the other hand, the r.h.s. is analytic for \( 2 < \lambda < N + 2 \). By unique continuation, (4.4) follows. The asymptotic development is a direct application of (4.2). □

The high oscillations present in (4.5) can be exploited to give yet another example of unboundedness of \( \|Tf\|_{L^p_tL^q_x} \):

**Lemma 4.11.** In the conditions of Lemma 4.10, if \( f = 1_{[0,\varepsilon]} \), \( \varepsilon \) small,

\[
\|Tf\|_{L^p_tL^q_x((0,2)\times\mathbb{R}^N)} = +\infty.
\]

**Proof.** For \( k \in \mathbb{N} \) large, take a bump function \( \phi \) localized around \( t = 1, x = 0 \), and set \( \phi_k(t,x) = \phi(t,x - \sqrt{8\pi k^2}(1 \uparrow)) \). Then, by the previous lemma,

\[
\int_0^2 \int_{\mathbb{R}^N} Tf(t,x)\phi_k(t,x)dxdt = \int_0^2 \int_{\mathbb{R}^N} \frac{1}{\tau^\beta} \left( \frac{|x|^2}{\tau} \right)^{-\beta} e^{i\frac{|x|^2}{\tau}} \phi_k(t,x)d\tau dxdt
\]

\[
+ \int_0^2 \int_{\mathbb{R}^N} \frac{1}{\tau^\beta} O \left( \left( \frac{|x|^2}{\tau} \right)^{-\min\{\beta+1,\theta-1\}} \right) \phi_k(t,x)d\tau dxdt
\]

Due to the restriction on the support of \( \phi_k \), the second integral can be bounded by \( \|\phi\|_{L^1_tL^\infty_x} \), while the first is comparable to \( k^{-\beta} \), which diverges as \( k \to \infty \) (recall that \( \beta = (N - b - s)/2 < 0 \)). □

**Proof of Theorem 1.5.** The proof is a direct application of Lemmas 4.1, 4.2 and 4.5 (for part (1)) and of Lemmas 4.1, 4.2, 4.7, 4.8 and 4.9 (for part (2)). □

**Proof of Theorem 1.3.** The result is an immediate application of Lemmas 4.1, 4.2 and 4.11. □

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