ASYMPTOTIC BEHAVIOR OF GROUND STATES OF
GENERALIZED PSEUDO-RELATIVISTIC HARTREE EQUATION

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Abstract. With appropriate hypotheses on the nonlinearity $f$, we prove the existence of a ground state solution $u$ for the problem

$$\sqrt{-\Delta + m^2} u + V u = (W * F(u)) f(u) \quad \text{in } \mathbb{R}^N,$$

where $V$ is a bounded potential, not necessarily continuous, and $F$ the primitive of $f$. We also show that any of this problem is a classical solution. Furthermore, we prove that the ground state solution has exponential decay.

1. Introduction

In this paper we consider a generalized pseudo-relativistic Hartree equation

$$\sqrt{-\Delta + m^2} u + V u = (W * F(u)) f(u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $N \geq 2$, $F(t) = \int_0^t f(s)ds$, assuming that the nonlinearity $f$ is a $C^1$ function, non-negative in $[0, \infty)$, that satisfies

(f1) $\lim_{t \to 0} \frac{|f(t)|}{t} = 0$;

(f2) $\lim_{t \to \infty} \frac{f(t)}{t^{\theta-1}} = 0$ for some $2 < \theta < 2^\# = \frac{2N}{N-1}$;

(f3) $\frac{f(t)}{t}$ is increasing for all $t > 0$.

We also postulate

(V1) $V$ is continuous and satisfies $V(y) + V_0 \geq 0$ for every $y \in \mathbb{R}^N$ and some constant $V_0 \in (0, m)$;

(V2) $V_\infty = \lim_{|y| \to \infty} V(y) > 0$;

(V3) $V(y) \leq V_\infty$ for all $y \in \mathbb{R}^N$, $V(y) \neq V_\infty$;

(W1) $0 \leq W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ is radial, with $r > \frac{N}{N(2-\theta)+\theta}$.

Therefore, we aim to generalize the results obtained by Coti Zelati and Nolasco [9] and Cingolani and Secchi [6]. In the last paper, the authors have studied the
equation
\[ \sqrt{-\Delta + m^2} u + V u = (W * u^\theta)|u|^{\theta-2}u, \]
supposing, additionally to our hypotheses, that the potential \( V \) is continuous and has a horizontal asymptote for \( N \geq 3 \). If \( k \in \mathbb{N} \), our work covers the case
\[ W(x) = \frac{|x|^k}{1 + |x|^l}, \]
while the hypothesis \( W(y) \to 0 \) when \( |y| \to \infty \) is explicitly assumed in \([6, \text{Section } 7]\). Furthermore, the homogeneity of the equation is a key ingredient in the proofs presented. So, applying different methods, we generalize \([6]\). A careful reading of our paper will also show that it generalizes \([9]\).

The equation
\[
\begin{cases}
i \partial_t u = \sqrt{-\Delta + m^2} + G(u) & \text{in } \mathbb{R}^N, \\
u(x, 0) = \phi(x), & x \in \mathbb{R}^N
\end{cases}
\]  
(1.2)

where \( N \geq 2 \), \( G \) is a nonlinearity of Hartree type, \( m > 0 \) denotes the mass of bosons in units, was used to describe the dynamics of pseudo-relativistic boson stars in astrophysics. See \([4, 13, 8, 20]\) for more details. For the study of semiclassical analysis of the non-relativistic Hartree equations we would like to quote the papers \([5, 15, 23, 26]\) and the recent work \([7]\) as well. For the Hartree equation without external potential \( V \), we cite \([20]\) for radial ground state solution, \([18]\) for uniqueness and nondegeneracy of ground state solutions, and \([9, 10]\) for the existence of positive and radially symmetric solutions. In \([22]\) is treated some Hartree problem imposing that the external potential \( V \) is radial, while in \([6]\) this condition is dropped.

By considering an extension problem from \( \mathbb{R}^N \) to \( \mathbb{R}^{N+1} \), an alternative definition of \( \sqrt{-\Delta + m^2} \) is well-known (see \([9]\) or \([3]\)), so that equation \((1.1)\) can be written as
\[
\begin{cases}
-\Delta u + m^2 u = 0, & \text{in } \mathbb{R}^{N+1}, \\
-\frac{\partial u}{\partial x}(0, y) = -V(y)u(0, y) + (W(y) * F(u(0, y)))f(u(0, y)) & \text{in } \mathbb{R}^N.
\end{cases}
\]  
(1.3)

We summarize our results:

\[\text{Theorem 1.} \quad \text{Suppose that conditions (} f_1)-(f_3)\), \( (V_1) \) and \( (W_h) \) are valid. Then, problem \((1.3)\) has a non-negative ground-state solution \( w \in H^1(\mathbb{R}^{N+1}) \).\]

\[\text{Theorem 2.} \quad \text{Assuming that hypotheses already stated are satisfied by } f, V \text{ and } W, \text{ any solution } v \text{ of problem } \((1.3)\) \text{ satisfies}\]
\[v \in C^{1,\alpha}(\mathbb{R}^{N+1}) \cap C^2(\mathbb{R}^{N+1})\]
\[\text{and therefore is a classical solution of } \((1.3)\).\]

We also prove that the ground station solution has exponential decay:

\[\text{Theorem 3.} \quad \text{Let } w \text{ be the ground state solution obtained in Theorem } 7. \text{ Then } w(x, y) > 0 \text{ in } [0, \infty) \times \mathbb{R}^N \text{ and, for any } \alpha \in (V_0, m) \text{ there exists } C > 0 \text{ such that}\]
\[0 < w(x, y) \leq Ce^{-(m-\alpha)\sqrt{x^2+|y|^2}}e^{\alpha x}\]
for any \((x, y) \in [0, \infty) \times \mathbb{R}^N\). In particular,
\[
0 < w(0, y) \leq Ce^{-\delta|y|}, \quad \forall \ y \in \mathbb{R}^N,
\]
where \(0 < \delta < m - V_0\).

The natural setting for problem (1.3) is the Sobolev space
\[
H^1(\mathbb{R}^N_{++}) = \left\{ u \in L^2(\mathbb{R}^N_{++}) : \int_{\mathbb{R}^N_{++}} |\nabla u|^2 \, dx \, dy < \infty \right\}
\]
endowed with the norm
\[
\|u\|^2 = \int_{\mathbb{R}^N_{++}} (|\nabla u|^2 + u^2) \, dx \, dy.
\]

**Notation.** The norm in the space \(\mathbb{R}^N_{++}\) will be denoted by \(\|\cdot\|\). For all \(q \in [1, \infty]\), we denote by \(|\cdot|_q\) the norm in the space \(L^q(\mathbb{R}^N)\) and by \(\|\cdot\|_q\) the norm in the space \(L^q(\mathbb{R}^N_{++})\).

It is well-known that traces of functions \(H^1(\mathbb{R}^N_{++})\) are in \(H^{1/2}(\mathbb{R}^N)\) and that every function in \(H^{1/2}(\mathbb{R}^N)\) is the trace of a function in \(H^1(\mathbb{R}^N_{++})\), see [25]. Denoting \(\gamma : H^1(\mathbb{R}^N_{++}) \rightarrow H^{1/2}(\mathbb{R}^N)\) the linear function that associates the trace \(\gamma(v) \in H^{1/2}(\mathbb{R}^N)\) of the function \(v \in H^1(\mathbb{R}^N_{++})\), then \(\ker \gamma = H^1_0(\mathbb{R}^N_{++})\).

The immersions
\[
H^1(\mathbb{R}^N_{++}) \hookrightarrow L^q(\mathbb{R}^N_{++}) \quad \text{(1.4)}
\]
\[
H^{1/2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \quad \text{(1.5)}
\]
are continuous for any \(q \in [2, 2^*]\) and \([2, 2#]\) respectively, where
\[
2^* = \frac{2(N + 1)}{N - 1} \quad \text{and} \quad 2# = \frac{2N}{N - 1}. \quad \text{(1.6)}
\]

The space \(H^{1/2}(\mathbb{R}^N)\) is defined by means of Fourier transforms; therefore, we can not change \(\mathbb{R}^N\) to a bounded open set \(\Omega \subset \mathbb{R}^N\). However (see [11]), \(H^{1/2}(\mathbb{R}^N) = W^{1/2, 2}(\mathbb{R}^N)\) and \(W^{1/2, 2}(\Omega)\) is well-defined for an open set \(\Omega \subset \mathbb{R}^N\). We recall its definition. Let \(u : \Omega \rightarrow \mathbb{R}\) a measurable function and \(\Omega\) a bounded open set (that, in the sequel, we suppose to have Lipschitz boundary). Denoting
\[
[u]^2_\Omega = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 1}} \, dx \, dy
\]
and
\[
W^{1/2, 2}(\Omega) = \left\{ u \in L^2(\mathbb{R}^N) : [u]^2_\Omega < \infty \right\},
\]
then \(W^{1/2, 2}(\Omega)\) is a reflexive Banach space (see, e.g., [11] and [12]) endowed with the norm
\[
\|u\|_{W^{1/2, 2}(\Omega)} = |u|_2 + [u]_\Omega.
\]
The proof of the next result can be found in [11, Theorem 4.54].

**Theorem 4.** The immersion $W^{1/2,2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any $q \in [1, 2^#)$.

As usual, the immersion $W^{1/2,2}(\Omega) \hookrightarrow L^{2^#}(\Omega)$ is continuous; see [11, Corollary 4.53]. We denote the norm in the space $L^q(\Omega)$ by $\| \cdot \|_{L^q(\Omega)}$.

2. Preliminaries

Let us suppose that $u \in H^1(\mathbb{R}^{N+1}) \cap C^\infty_0(\mathbb{R}^{N+1})$ and $u(x, y) \geq 0$. Let us proceed heuristically: since

$$|u(0, y)|^t = \int_0^\infty \frac{\partial}{\partial x} |u(x, y)|^t \, dx = \int_\infty^0 t|u(x, y)|^{t-2}u(x, y)\frac{\partial}{\partial x}u(x, y) \, dx,$$

it follows from Hölder’s inequality

$$\int_{\mathbb{R}^N} |\gamma(u)|^t = \int_{\mathbb{R}^N} |u(0, y)|^t \, dy \leq \int_{\mathbb{R}^N} \int_0^\infty t|u(x, y)|^{t-1}|\nabla u(x, y)| \, dx \, dy \leq t \left( \int_{\mathbb{R}^{N+1}^+} |u|^{2(t-1)} \right)^{1/2} \left( \int_{\mathbb{R}^{N+1}^+} |\nabla u|^2 \right)^{1/2} \leq t \|u\|_{(t-1)\frac{N}{2}} \|\nabla u\|_2. \tag{2.1}$$

So, in order to apply the immersion $H^1(\mathbb{R}^{N+1}) \hookrightarrow L^q(\mathbb{R}^{N+1})$ we must have $2 \leq 2(t-1) \leq \frac{2(N+1)}{N-1}$, that is,

$$2 \leq t \leq \frac{2N}{N-1} = 2^#. \tag{2.2}$$

By density of $H^1(\mathbb{R}^{N+1}) \cap C^\infty_0(\mathbb{R}^{N+1})$ in $H^1(\mathbb{R}^{N+1})$, the estimate (2.1) is valid for all $u \in H^1(\mathbb{R}^{N+1})$.

Taking into account (1.4), Young’s inequality applied to (2.1) yields

$$|\gamma(u)|_t \leq \|u\|_{2(t-1)}^{(t-1)/t} (t\|\nabla u\|_2)^{1/t} \leq \frac{t-1}{t} \|u\|_{2(t-1)} + \|\nabla u\|_2 \leq C_t \|u\|, \tag{2.3}$$

where $C_t$ is a constant. We summarize:

$$|\gamma(u)| \in L^t(\mathbb{R}^N), \ \forall \ t \in [2, 2^#]. \tag{2.4}$$

The inequality (2.3) will also be valuable in the special case $t = 2$:

$$|\gamma(u)|_2^2 \leq \|u\|_2 (2\|\nabla u\|_2) \leq \lambda \int_{\mathbb{R}^{N+1}^+} u^2 + \frac{1}{\lambda} \int_{\mathbb{R}^{N+1}^+} |\nabla u|^2 \tag{2.5}$$

where $\lambda > 0$ is a parameter, the last inequality being a consequence of Young’s inequality.
Remark 2.1. It follows from \((f_3)\) that \(f\) satisfies the Ambrosetti-Rabinowitz inequality \(2F(t) \leq f(t)t, \) for all \(t > 0.\) Furthermore, it follows from \((f_1)\) and \((f_2)\) that, for any fixed \(\xi > 0,\) there exists a constant \(C_\xi\) such that
\[
|f(t)| \leq \xi t + C_\xi t^{\theta-1}, \quad \forall \ t \geq 0 \tag{2.6}
\]
and analogously
\[
|F(t)| \leq \xi t^2 + C_\xi t^\theta \leq C(t^2 + t^\theta), \quad \forall \ t \geq 0. \tag{2.7}
\]
Observe that \(\gamma(u) \in L^p(\mathbb{R}^N)\) and \(\gamma(u) \in L^2(\mathbb{R}^N)\) imply \(F(\gamma(u)) \in L^1(\mathbb{R}^N)\).

**Proposition 2.1** (Hausdorff-Young). Assume that, for \(1 \leq p, q, s \leq \infty,\) we have \(f \in L^p(\mathbb{R}^N), g \in L^q(\mathbb{R}^N)\) and
\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}.
\]
Then
\[
|f \ast g|_s \leq |f|_p|g|_q.
\]
We now enhance the result given by \((2.4).\) Observe that \(N \frac{1}{N(2-\theta)+\theta} \geq 1\) and \(N \frac{N}{N(2-\theta)+\theta} = 1\) if, and only if \(N = \theta = 2.\)

The results in the sequel will be useful when addressing the regularity of the solution of problem \((1.3).\)

**Lemma 2.1.** Concerning hypothesis \((W_h)\) we have:

(i) if \(r \in \left(\frac{N}{N(2-\theta)+\theta}, \frac{2N}{N(2-\theta)+\theta}\right],\) there exists \(p \in \left[1, \frac{2N}{(N-1)\theta}\right]\) such that
\[
|\gamma(u)|^\theta \in L^p(\mathbb{R}^N)
\]
and
\[
\frac{1}{p} + \frac{1}{r} = 1 + \frac{N(2-\theta) + \theta}{2N}.
\]
Furthermore, \(F(\gamma(u)) \in L^p(\mathbb{R}^N)\) and
\[
|W_1 \ast F(\gamma(u))| =: g \in L^{2N/[N(2-\theta)+\theta]}(\mathbb{R}^N).
\]

(ii) if \(r'\) denotes the conjugate exponent of \(r\) and \(r > \frac{2N}{N(2-\theta)+\theta},\) then
\[
F(\gamma(u)) \in L^{r'}(\mathbb{R}^N) \text{ and } W_1 \ast F(\gamma(u)) \in L^\infty(\mathbb{R}^N).
\]

**Proof.** (i) We verify the values of \(r\) that satisfy the equality
\[
\frac{1}{p} + \frac{1}{r} = 1 + \frac{N(2-\theta) + \theta}{2N}.
\]
Observe that \(r \in \left(\frac{N}{N(2-\theta)+\theta}, \frac{2N}{N(2-\theta)+\theta}\right]\) if, and only if, \(p \in \left[1, \frac{2N}{(N-1)\theta}\right].\)

As consequence of \((2.4)\) \(|\gamma(u)|^\theta \in L^p(\mathbb{R}^N)\) and thus \(|\gamma(u)|^2 \in L^p(\mathbb{R}^N)\) and \((2.7)\) yields \(F(\gamma(u)) \in L^p(\mathbb{R}^N).\) So, \(|W_1 \ast F(\gamma(u))| = g \in L^{2N/[N(2-\theta)+\theta]}(\mathbb{R}^N)\) follows from the Hausdorff-Young inequality.
Corollary 2.1. We have $|W * F(\gamma(u))| \leq C + g$ with $g \in L^{2N/(N(2-\theta)+\theta)}(\mathbb{R}^N)$. 

Proof. An immediately consequence of Lemma 2.1, since we conclude that $F(\gamma(u)) \in L^{r'}(\mathbb{R}^N)$ and $W_1 * F(\gamma(u)) \in L^{\infty}(\mathbb{R}^N)$ is consequence of Proposition 2.1. $\square$

Theorem 2.2. For all $t \in \left(2, \frac{2N}{N-1}\right)$, we have $|\gamma(u)|^{\theta-2} \leq 1 + g_2$, where $g_2 \in L^N(\mathbb{R}^N)$.

Proof. We have $|\gamma(u)|^{\theta-2} = |\gamma(u)|^{\theta-2}\chi_{\{\gamma(u)\leq 1\}} + |\gamma(u)|^{\theta-2}\chi_{\{\gamma(u)\geq 1\}} \leq 1 + g_2$, with $g_2 = |\gamma(u)|^{\theta-2}\chi_{\{\gamma(u)\geq 1\}}$. If $(\theta - 2)N \leq 2$, then

$$\int_{\mathbb{R}^N} |\gamma(u)|^{(\theta-2)N}\chi_{\{\gamma(u)\geq 1\}} \leq \int_{\mathbb{R}^N} |\gamma(u)|^{2}\chi_{\{\gamma(u)\geq 1\}} \leq \int_{\mathbb{R}^N} |\gamma(u)|^{2} < \infty.$$

When $2 < (\theta - 2)N$, then $(\theta - 2)N \in \left(2, \frac{2N}{N-1}\right)$ and $|\gamma(u)|^{\theta-2} \in L^N(\mathbb{R}^N)$ as outcome of (2.4). $\square$

Lemma 2.3. For all $\theta \in \left(2, \frac{2N}{N-1}\right)$ we have $h = g|\gamma(u)|^{\theta-2} \in L^N(\mathbb{R}^N)$, where $g$ is the function of Lemma 2.2.

Proof. Application of the Hölder inequality yields

$$\int_{\mathbb{R}^N} (g|\gamma(u)|^{\theta-2})^N \leq \left(\int_{\mathbb{R}^N} g^{N\alpha}\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^N} (|\gamma(u)|^{(\theta-2)N})^{\alpha'}\right)^{\frac{1}{\alpha'}},$$

if we define $\alpha$ so that $\alpha N = 2N/[N(2-\theta)+\theta]$. Thus, $\alpha' = 2/[N(2-\theta)+\theta]$ and we have $\alpha' N(\theta - 2) = 2N/[N - 1]$. Since both integrals of the right-hand side of the last inequality are integrable, we are done. $\square$

We now handle the existence of the “energy” functional. We denote by $L^p_q(\mathbb{R}^N)$ the weak $L^q(\mathbb{R}^N)$ space and by $|\cdot|_{q_w}$ its usual norm (see [19]). The next result is a generalized version of the Hardy-Littlewood-Sobolev inequality:

Proposition 2.2 (Lieb [19]). Assume that $p, q, r \in (1, \infty)$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Then, for some constant $N_{p,q,t} > 0$ and for any $f \in L^p(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ and $h \in L^q(\mathbb{R}^N)$, we have the inequality

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t)h(t-s)g(s)dt ds \leq N_{p,q,t}|f|_p|g|_q|h|_{q_w}.$$
Lemma 2.4. For a positive constant $C$ holds

$$\left| \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u)) \right| \leq C \left( \|u\|_2^2 + \|u\|_\theta^2 \right).$$

Proof. Let us denote

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} [W * F(\gamma(u))] F(\gamma(u)).$$

Since $W = W_1 + W_2$,

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} [W_1 * F(\gamma(u))] F(\gamma(u)) + \frac{1}{2} \int_{\mathbb{R}^N} [W_2 * F(\gamma(u))] F(\gamma(u))$$

$$=: J_1(u) + J_2(u).$$

Let us suppose that $|\gamma(u)|^\theta \in L^1(\mathbb{R}^N)$ for some $t \geq 1$. Then $|\gamma(u)|^2 \in L^t(\mathbb{R}^N)$ and $F(\gamma(u)) \in L^t(\mathbb{R}^N)$ (as consequence of (2.7)). Application of Proposition 2.2 yields

$$|J_1(u)| = \left| \frac{1}{2} \int_{\mathbb{R}^N} W_1 * F(\gamma(u)) F(\gamma(u)) \right| \leq N |W_1| \|rF(\gamma(u))\|_t.$$  \hspace{1cm} \text{(2.8)}

(Observe that, in order to apply the immersion $H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^q(\mathbb{R}^{N+1}_+)$, we must have $t \theta / 2 < 2N/(N - 1)$, that is, $r > N/[N(2 - \theta) + \theta]$.)

In the case $W_2 \in L^\infty(\mathbb{R}^N)$ we can take $t = 1$, therefore

$$|J_2(u)| = \left| \frac{1}{2} \int_{\mathbb{R}^N} W_2 * F(\gamma(u)) F(\gamma(u)) \right| \leq C \left( |\gamma(u)|_2^2 + |\gamma(u)|_\theta^2 \right)^2 \leq C' \left( \|u\|_2^2 + \|u\|_\theta^2 \right)^2.$$  \hspace{1cm} \text{(2.9)}

From (2.10) and (2.9) results the claim. \hfill \Box

Lemma 2.5. The functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla u|^2 + m^2 u^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(y)|\gamma(u(y))|^2$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u))$$

$$=: I_1(u) + I_2(u) - \Psi(u)$$

is well-defined.

Proof. Of course

$$I_1(u) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla u|^2 + m^2 u^2) \leq \frac{k}{2} \|u\|^2 < \infty,$$
if we take \( k = \max\{1, m^2\} \). Since hypothesis (V1) implies \(|V(y)| < C\), we have

\[
|I_2(u)| = \left| \frac{1}{2} \int_{\mathbb{R}^N} V(y) |\gamma(u(y))|^2 \right| \leq \frac{C}{2} \int_{\mathbb{R}^N} |\gamma(u)|^2 = C' |\gamma(u)|_2^2 \leq C'' \|u\|^2.
\]

Taking into account Lemma 2.3, the proof is complete. \( \square \)

Since the derivative of the energy functional is given by

\[
I'(u) \cdot \varphi = \int_{\mathbb{R}^{N+1}_+} \left[ \nabla u \cdot \nabla \varphi + m^2 u \varphi \right] + \int_{\mathbb{R}^N} V(y) \gamma(u) \gamma(\varphi)
- \int_{\mathbb{R}^N} (W * F(\gamma(u))) f(\gamma(u)) \gamma(\varphi), \quad \forall \, \varphi \in H^1(\mathbb{R}^{N+1}_+),
\]

we see that critical points of \( I \) are weak solutions \( 1.3 \).

Because we are looking for a positive solution, we suppose that \( f(t) = 0 \) for \( t < 0 \).

**Proposition 2.3.** The quadratic form

\[
u \mapsto \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla u|^2 + m^2 u^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(y) |\gamma(u(y))|^2
\]

defines an norm in the space \( H^1(\mathbb{R}^{N+1}_+) \), which is equivalent to the norm \( \| \cdot \| \).

**Proof.** We keep up with the notation already introduced and note that \( I_2(u) \geq -(1/2)V_0 \int_{\mathbb{R}^N} |\gamma(u)|^2 \). Furthermore, as consequence of (2.5), we have

\[
\int_{\mathbb{R}^N} |\gamma(u)|^2 \leq m \int_{\mathbb{R}^{N+1}_+} |u|^2 + \frac{1}{m} \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2.
\]

Therefore,

\[
I_1(u) + I_2(u) \geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla u|^2 + m^2 u^2) - \frac{V_0 m}{2} \int_{\mathbb{R}^{N+1}_+} |u|^2 - \frac{V_0}{2m} \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2
= \frac{1}{2} \left( 1 - \frac{V_0}{m} \right) \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 + \frac{1}{2} m(m - V_0) \int_{\mathbb{R}^{N+1}_+} |u|^2.
\]

Defining \( K = \min \left\{ \frac{1}{2} \left( 1 - \frac{V_0}{m} \right), \frac{1}{2} m(m - V_0) \right\} > 0 \), we conclude that

\[
I_1(u) + I_2(u) \geq K \|u\|^2.
\]

By applying (2.12) it easily follows that

\[
I_1(u) + I_2(u) \leq \frac{1}{2} \left( 1 + \frac{V_0}{m} \right) \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 + \frac{1}{2} (m^2 + V_\infty m) \int_{\mathbb{R}^{N+1}_+} |u|^2
\leq C \|u\|^2
\]

for a constant \( C > 0 \). We are done. \( \square \)
3. Mountain pass geometry and Nehari manifold

Lemma 3.1. \( I \) satisfies the mountain pass theorem geometry. More precisely,

(i) There exist \( \rho, \delta > 0 \) such that \( I|_{S} \geq \delta > 0 \) for all \( u \in S \), where

\[
S = \{ u \in H^1(\mathbb{R}^N+1) : \| u \| = \rho \}.
\]

(ii) For each \( u_0 \in H^1(\mathbb{R}^N+1) \) such that \( (u_0)_+ \neq 0 \), there exists \( \tau \in \mathbb{R} \), satisfying \( \| \tau u_0 \| > \rho \) and \( I(\tau u_0) < 0 \).

Proof. Since we have already showed that

\[
I_1(u) + I_2(u) \geq K \| u \|^2
\]

and so \( I(u) \geq K \| u \|^2 - \Psi(u) \geq K \| u \|^2 - C \left( \| u \|^2 + \| u \|^\theta \right)^2 \), we obtain (i) by choosing \( \rho > 0 \) small enough.

In order to prove (ii), fix \( u_0 \in H^1(\mathbb{R}^N+1) \setminus \{0\} \) such that \( u_0 \geq 0 \). For all \( t > 0 \) consider the function \( g_{u_0} : (0, \infty) \to \mathbb{R} \) defined by

\[
g_{u_0}(t) = \Psi\left( \frac{tu_0}{\| u_0 \|} \right)
\]

where, as before,

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u)).
\]

An easy calculation shows that

\[
g'_{u_0}(t) = \frac{2}{t} \int_{\mathbb{R}^N} \left( W * F\left( \frac{tu_0}{\| u_0 \|} \right) \right) \frac{d}{dt} \left( \frac{tu_0}{\| u_0 \|} \right) \frac{1}{2} (\gamma\left( \frac{tu_0}{\| u_0 \|} \right))^2 \geq \frac{4}{t} g_{u_0}(t),
\]

the last inequality being a consequence of the Ambrosetti-Rabinowitz inequality. Observe that \( g'_{u_0}(t) > 0 \) for \( t > 0 \).

Thus, we obtain

\[
\ln g_{u_0}(t)|_{1}^{\| u_0 \|} \geq 4 \ln t|_{1}^{\| u_0 \|} \Rightarrow \frac{g_{u_0}(\tau\| u_0 \|)}{g_{u_0}(1)} \geq (\tau\| u_0 \|)^4,
\]

proving that

\[
\Psi(\tau u_0) = g_{u_0}(\tau\| u_0 \|) \geq D (\tau\| u_0 \|)^4.
\]

for a constant \( D > 0 \).

It follows from (2.13) that

\[
I(\tau u_0) \leq C\tau^2\| u_0 \|^2 - D\tau^4\| u_0 \|^4.
\]

Thus, it suffices to take \( \tau \) large enough. \( \Box \)

The existence of a Palais-Smale sequence \( (u_n) \subset H^1(\mathbb{R}^N+1) \) such that

\[
I'(u_n) \to 0 \quad \text{and} \quad I(u_n) \to c,
\]

where

\[
c = \inf_{\alpha \in \Gamma} \max_{t \in [0, 1]} I(\alpha(t)),
\]

where

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u)).
\]

and so \( I(u) \geq K \| u \|^2 - \Psi(u) \geq K \| u \|^2 - C \left( \| u \|^2 + \| u \|^\theta \right)^2 \), we obtain (i) by choosing \( \rho > 0 \) small enough.

In order to prove (ii), fix \( u_0 \in H^1(\mathbb{R}^N+1) \setminus \{0\} \) such that \( u_0 \geq 0 \). For all \( t > 0 \) consider the function \( g_{u_0} : (0, \infty) \to \mathbb{R} \) defined by

\[
g_{u_0}(t) = \Psi\left( \frac{tu_0}{\| u_0 \|} \right)
\]

where, as before,

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u)).
\]

An easy calculation shows that

\[
g'_{u_0}(t) = \frac{2}{t} \int_{\mathbb{R}^N} \left( W * F\left( \frac{tu_0}{\| u_0 \|} \right) \right) \frac{d}{dt} \left( \frac{tu_0}{\| u_0 \|} \right) \frac{1}{2} (\gamma\left( \frac{tu_0}{\| u_0 \|} \right))^2 \geq \frac{4}{t} g_{u_0}(t),
\]

the last inequality being a consequence of the Ambrosetti-Rabinowitz inequality. Observe that \( g'_{u_0}(t) > 0 \) for \( t > 0 \).

Thus, we obtain

\[
\ln g_{u_0}(t)|_{1}^{\| u_0 \|} \geq 4 \ln t|_{1}^{\| u_0 \|} \Rightarrow \frac{g_{u_0}(\tau\| u_0 \|)}{g_{u_0}(1)} \geq (\tau\| u_0 \|)^4,
\]

proving that

\[
\Psi(\tau u_0) = g_{u_0}(\tau\| u_0 \|) \geq D (\tau\| u_0 \|)^4.
\]

for a constant \( D > 0 \).

It follows from (2.13) that

\[
I(\tau u_0) \leq C\tau^2\| u_0 \|^2 - D\tau^4\| u_0 \|^4.
\]

Thus, it suffices to take \( \tau \) large enough. \( \Box \)

The existence of a Palais-Smale sequence \( (u_n) \subset H^1(\mathbb{R}^N+1) \) such that

\[
I'(u_n) \to 0 \quad \text{and} \quad I(u_n) \to c,
\]

where

\[
c = \inf_{\alpha \in \Gamma} \max_{t \in [0, 1]} I(\alpha(t)),
\]

where

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (W * F(\gamma(u))) F(\gamma(u)).
\]
Lemma 3.2. There exists $\beta > 0$ such that $\|u\| \geq \beta$ for all $u \in \mathcal{N}$.

An alternative characterization of $c$ is obtained by a standard method: for $u_+ \neq 0$, consider the function $\Phi(t) = I_1(tu) + \int_0^t (\frac{\partial}{\partial s} \Psi(su)) ds$, preserving the notation of Lemma 3.1. The proof of Lemma 3.1 assures that $\Psi(tu) > 0$ for $t$ small enough, $\Psi(tu) < 0$ for $t$ large enough and $g_u(t) > 0$ if $t > 0$. Therefore, $\max_{t \geq 0} \Psi(t)$ is achieved at a unique $t_u = t(u) > 0$ and $\Psi(tu) > 0$ for $t < t_u$ and $\Psi(tu) < 0$ for $t > t_u$. Furthermore, $\Psi(t_u u) = 0$ implies that $t_u u \in \mathcal{N}$.

The map $u \mapsto t_u (u \neq 0)$ is continuous and $c = c^*$, where

$$c^* = \inf_{u \in H^1(\mathbb{R}_+^N) \setminus \{0\}} \max_{t \geq 0} I(tu).$$

For details, see [24] Section 3 or [14].

Standard arguments prove the next affirmative:

Lemma 3.3. Let $(u_n) \subset H^1(\mathbb{R}_+^N)$ be a sequence such that $I(u_n) \to c$ and $I'(u_n) \to 0$, where

$$c = \inf_{u \in H^1(\mathbb{R}_+^N) \setminus \{0\}} \max_{t \geq 0} I(tu).$$

Then $(u_n)$ is bounded and (for a subsequence) $u_n \rightharpoonup u$ in $H^1(\mathbb{R}_+^N)$.

Lemma 3.4. Let $U \subseteq \mathbb{R}^N$ be any open set. For $1 < p < \infty$, let $(f_n)$ be a bounded sequence in $L^p(U)$ such that $f_n(x) \to f(x)$ a.e. Then $f_n \to f$.

The proof of Lemma 3.4 can be found, e.g., in [17] Lemma 4.8, Chapitre 1.

4. The limit problem

In this section we consider a variant of problem (1.3), changing the potential $V(y)$ for $V_\infty$.

Theorem 5. Assuming $(f_1)$, $(f_2)$, $(f_3)$ and $(W_h)$, problem

$$\begin{cases}
-\Delta u + m^2 u = 0 & \text{in } \mathbb{R}_+^{N+1} \\
-\frac{\partial u}{\partial x} = -U_\infty u + [W \ast F(u)] f(u), & (x, y) \in \{0\} \times \mathbb{R}^N \cong \mathbb{R}^N,
\end{cases} \quad (P_{\infty})$$

has a non-negative ground state solution.
Proof. Let \((u_n)\) be the minimizing sequence given by Lemma 3.1. Then, there exist \(R, \delta > 0\) and a sequence \((z_n) \subset \mathbb{R}^N\) such that
\[
\liminf_{n \to \infty} \int_{B_R(z_n)} |\gamma(u_n)|^2 \geq \delta. \tag{4.1} \]

If false, a result of Lions (see [21]) guarantees that \(\gamma(u_n) \to 0\) in \(L^q(\mathbb{R}^N)\) for \(2 < q < 2^*\), thus implying that
\[
\int_{\mathbb{R}^N} (W * F(\gamma(u_n))) f(\gamma(u_n)) \gamma(u_n) \to 0,
\]
contradicting Lemma 3.2.

We define
\[
v_n(x) = u_n(x - z_n).
\]

From (4.1) we derive that
\[
\int_{B_R(0)} |\gamma(v_n)|^2 \geq \frac{\delta}{2}.
\]

We observe that the energy functional
\[
I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + m^2 u^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty |u|^2
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^N} [W * F(\gamma(u))] F(\gamma(u))
\]
and its derivative as well are translation invariant. Therefore, it also holds that
\[
I'_\infty(v_n) \to 0 \quad \text{and} \quad I_\infty(v_n) \to c_\infty,
\]
where
\[
c_\infty = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} I_\infty(tu).
\]

(Observe that all reasoning in Section 3 is valid for \(I_\infty\) and its minimizing sequence.)

Since \((v_n)\) is bounded (see Lemma 3.3) it follows that \(v_n \to v\). A standard argument shows that we can suppose \(v_n(x) \to v(x)\) a.e. in \((\mathbb{R}^{N+1}_+)\), \(v_n \to v\) in \(L^s_{loc}(\mathbb{R}^{N+1}_+)\) for all \(s \in [2, 2^*)\), \(\gamma(v_n(x)) \to \gamma(v(x))\) a.e. in \(\mathbb{R}^N\) and \(\gamma(v_n) \to \gamma(v)\) in \(L^q_{loc}(\mathbb{R}^N)\) for all \(q \in [p, p^*]\).

We will show that \(v \in H_\infty = \{u \in H^1(\mathbb{R}^{N+1}_+) \setminus \{0\} : I'_\infty(u) \cdot u = 0\}\).

For all \(\varphi \in C_0^\infty(\mathbb{R}^{N+1}_+)\), let us consider \(\psi_n = (v_n - v)\varphi \in H^1(\mathbb{R}^{N+1}_+)\). We have
\[
\langle I'_\infty(v_n), \psi_n \rangle = \int_{\mathbb{R}^{N+1}_+} \nabla v_n \cdot \nabla \psi_n + \int_{\mathbb{R}^{N+1}_+} m^2 v_n \psi_n + \int_{\mathbb{R}^N} V_\infty(\gamma(v_n)) \gamma(\psi_n)
\]
\[
- \int_{\mathbb{R}^N} (W * F(\gamma(v_n))) f(\gamma(v_n)) \gamma(\psi_n)
\]
\[
= J_1 + J_2 + J_3 - J_4. \tag{4.2}
\]

We start considering
\[
J_4 = \int_{\mathbb{R}^N} (W * F(\gamma(v_n))) f(\gamma(v_n)) \gamma(\psi_n).
\]
Because \( \lim_{n \to \infty} (I'_\infty(v_n), (v_n - v)\varphi) = 0 \), it follows from [1, Lemma 3.5] that \( J_4 \to 0 \) when \( n \to \infty \) and thus is easily verified that \( J_2 + J_3 - J_4 \to 0 \) when \( n \to \infty \). We now consider \( J_1 \):

\[
J_1 = \iint_{\mathbb{R}^{N+1}_+} \nabla v_n \cdot \nabla ((v_n - v)\varphi) \\
= \iint_{\mathbb{R}^{N+1}_+} \nabla v_n \cdot \varphi \nabla (v_n - v) + \iint_{\mathbb{R}^{N+1}_+} \nabla v_n \cdot (v_n - v) \nabla \varphi \\
= \iint_{\mathbb{R}^{N+1}_+} |\nabla (v_n - v)|^2 \varphi + \varphi \nabla v \cdot \nabla (v_n - v) + \nabla v_n \cdot (v_n - v) \nabla \varphi.
\]

We infer that

\[
\lim_{n \to \infty} \iint_{\mathbb{R}^{N+1}_+} |\nabla (v_n - v)|^2 \varphi = - \lim_{n \to \infty} \iint_{\mathbb{R}^{N+1}_+} \varphi \nabla v \cdot \nabla (v_n - v) \\
- \lim_{n \to \infty} \iint_{\mathbb{R}^{N+1}_+} (v_n - v) \nabla v_n \cdot \nabla \varphi.
\]

Since

\[
\lim_{n \to \infty} \iint_{\mathbb{R}^{N+1}_+} \varphi \nabla v \cdot \nabla (v_n - v) = 0 \quad \text{and} \quad \lim_{n \to \infty} \iint_{\mathbb{R}^{N+1}_+} (v_n - v) \nabla v_n \cdot \nabla \varphi = 0
\]

(because \( \nabla v_n \) is bounded), we deduce that

\[ \nabla v_n \to \nabla v \quad \text{a.e. in} \quad \mathbb{R}^{N+1}_+. \]

Thus

\[ I'_\infty(v) = 0 \]

and \( v \in \mathcal{N}_\infty \).

We now turn our attention to the positivity of \( v \). Seeing that

\[
\iint_{\mathbb{R}^{N+1}_+} (\nabla v \cdot \nabla \varphi + m^2 v \varphi) + \int_{\mathbb{R}^N} V_\infty \gamma(v) \gamma(\varphi) = \int_{\mathbb{R}^N} [W * F(\gamma(v))] f(\gamma(v)) \gamma(\varphi)
\]

and choosing \( \varphi = v^- \), the left-hand side of the equality is positive (by the definition of \( I_\infty \) and equation (3.1) applied to \( I_\infty \)), since \( J_1 + J_2 + J_3 = I_1 + I_2 \geq K\|v\|^2 \), while \( \Psi(v) = J_4 \leq 0 \). We are done.

\[ \square \]

5. Proof of Theorem \[ \square \]

In order to consider the general case of the potential \( V(y) \), we state a well-known result due to M. Struwe:

**Struwe** Lemma 5.1 (Splitting Lemma). Let \( (v_n) \subset H^1(\mathbb{R}^{N+1}_+) \) be such that

\[ I(u_n) \to c, \quad I'(u_n) \to 0 \]

and \( u_n \rightharpoonup u \) weakly on \( X \). Then \( I'(u_0) = 0 \) and we have either

(i) \( u_n \to u \) strongly on \( X \);
there exist \( k \in \mathbb{N}, (y_j^n) \in \mathbb{R}^N \) such that \( |y_j^n| \to \infty \) for \( j \in \{1, \ldots, k\} \) and nontrivial solutions \( u^1, \ldots, u^k \) of problem \( (\mathcal{P}_{\infty}) \) so that

\[
I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(u_j)
\]

and

\[
\left\| u_n - u_0 - \sum_{j=1}^k u^j (\cdot - y_j^n) \right\| \to 0.
\]

**Lemma 5.2.** The functional \( I \) satisfies \((PS)_c\) for any \( 0 \leq c < c_\infty \).

**Proof.** Let us suppose that \( (u_n) \) satisfies

\[
I(u_n) \to c < c_\infty \quad \text{and} \quad I'(u_n) \to 0.
\]

We can suppose that the sequence \( (u_n) \) is bounded, according to Lemma 3.3. Therefore, for a subsequence, we have \( u_n \rightharpoonup u_0 \) in \( H^1(\mathbb{R}^{N+1}) \). It follows from the Splitting Lemma (Lemma 5.1) that \( I'(u_0) = 0 \). Since

\[
I'(u_0) \cdot u_0 = \int_{\mathbb{R}^{N+1}} (|\nabla u_0|^2 + m^2 u_0^2) + \int_{\mathbb{R}^N} V(y)|\gamma(u_0)|^2 - \int_{\mathbb{R}^N} [W \ast F(\gamma(u_0))] f(\gamma(u_0)) \gamma(u_0)
\]

and

\[
I(u_0) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla u_0|^2 + m^2 u_0^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(y)|\gamma(u_0)|^2 - \frac{1}{2} \int_{\mathbb{R}^N} [W \ast F(\gamma(u_0))] F(\gamma(u_0)),
\]

we conclude that

\[
I(u_0) = \int_{\mathbb{R}^N} [W \ast F(\gamma(u_0))] \left( \frac{1}{2} f(\gamma(u_0)) \gamma(u_0) - F(\gamma(u_0)) \right) > 0, \tag{5.1}
\]

as consequence of the Ambrosetti-Rabinowitz condition.

If \( u_n \not\to u \) in \( H^1(\mathbb{R}^{N+1}) \), by applying again the Splitting Lemma we guarantee the existence of \( k \in \mathbb{N} \) and nontrivial solutions \( u^1, \ldots, u^k \) of problem \( (\mathcal{P}_{\infty}) \) satisfying

\[
\lim_{n \to \infty} I(u_n) = c = I(u_0) + \sum_{j=1}^k I_\infty(u^j) \geq kc_\infty \geq c_\infty
\]

contradicting our hypothesis. We are done. \( \square \)

We prove the next result by adapting the proof given in Furtado, Maia e Medeiros [16]:

\[
(i) \quad \text{there exist } k \in \mathbb{N}, (y_j^n) \in \mathbb{R}^N \text{ such that } |y_j^n| \to \infty \text{ for } j \in \{1, \ldots, k\} \text{ and nontrivial solutions } u^1, \ldots, u^k \text{ of problem } (\mathcal{P}_{\infty}) \text{ so that}
\]

\[
I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(u_j)
\]

and

\[
\left\| u_n - u_0 - \sum_{j=1}^k u^j (\cdot - y_j^n) \right\| \to 0.
\]
Lemma 5.3. Suppose that $V(y)$ satisfies (V3). Then
\[ 0 < c < c_\infty, \]
where $c$ is characterized in Lemma 3.3.

Proof. Let $\bar{u} \in N_\infty$ be the weak solution of \( P_{\infty} \) given by Theorem 5 and $t_{\bar{u}} > 0$ be the unique number such that $t_{\bar{u}} \in N$. We claim that $t_{\bar{u}} < 1$. Indeed,
\[
\int_{\mathbb{R}^N} [W * F(\gamma(t_{\bar{u}}))] f(\gamma(t_{\bar{u}})) \gamma(t_{\bar{u}})
\]
\[
= t_{\bar{u}}^2 \int_{\mathbb{R}^{N+1}} (|\nabla \bar{u}|^2 + m^2 \bar{u}^2) + \int_{\mathbb{R}^N} V(y)|\gamma(\bar{u})|^2
\]
\[
< t_{\bar{u}}^2 \int_{\mathbb{R}^{N+1}} (|\nabla \bar{u}|^2 + m^2 \bar{u}^2) + \int_{\mathbb{R}^N} V_\infty|\gamma(\bar{u})|^2
\]
\[
= t_{\bar{u}}^2 \int_{\mathbb{R}^N} |W * F(\gamma(\bar{u}))| f(\gamma(\bar{u})) \gamma(\bar{u})
\]
\[
= t_{\bar{u}}^2 \left( \int_{\mathbb{R}^N} [W * F(\gamma(\bar{u}))] f(\gamma(\bar{u})) \gamma(\bar{u}) + \int_{\mathbb{R}^N} [W * F(\gamma(t_{\bar{u}}))] f(\gamma(\bar{u})) \gamma(\bar{u})
\]
\[
- \int_{\mathbb{R}^N} [W * F(\gamma(t_{\bar{u}}))] f(\gamma(\bar{u})) \gamma(\bar{u}) \right)
\]
thus yielding
\[
0 > \int_{\mathbb{R}^N} [W * F(\gamma(t_{\bar{u}}))] \left( \frac{f(\gamma(t_{\bar{u}}))}{\gamma(t_{\bar{u}})} - \frac{f(\gamma(\bar{u}))}{\gamma(\bar{u})} \right)
\[
+ t_{\bar{u}}^2 \int_{\mathbb{R}^N} [W * (F(\gamma(t_{\bar{u}})) - F(\gamma(\bar{u}))) f(\gamma(u)) \gamma(u).
\]

If $t_{\bar{u}} \geq 1$, since $f(s)/s$ is increasing, the first integral is non-negative and, since $F$ is increasing, the second integral as well. We conclude that $t_{\bar{u}} < 1$.

Lemma 5.3 and its previous comments show that
\[
c \leq \max_{t \geq 0} I(t \bar{u}) = I(t_{\bar{u}} \bar{u}) = \int_{\mathbb{R}^N} [W * F(\gamma(t_{\bar{u}}))] \left( \frac{1}{2} f(\gamma(t_{\bar{u}})) \gamma(t_{\bar{u}}) - F(\gamma(t_{\bar{u}})) \right)
\]
Since
\[
g(t) = \int_{\mathbb{R}^N} [W * F(\gamma(t \bar{u}))] \left( \frac{1}{2} f(\gamma(t \bar{u})) \gamma(t \bar{u}) - F(\gamma(t \bar{u})) \right)
\]
is a strictly increasing function, we conclude that
\[
c = g(t_{\bar{u}}) < g(1) = \int_{\mathbb{R}^N} [W * F(\gamma(\bar{u}))] \left( \frac{1}{2} f(\gamma(\bar{u})) \gamma(\bar{u}) - F(\gamma(\bar{u})) \right) = c_\infty,
\]
proving our result. \( \square \)

Proof of Theorem 1. Let $(u_n)$ be the minimizing sequence given by Lemma 3.1. It follows from Lemmas 5.2 and 5.3 that $u_n \to u$ such that $I(u) = c$ and $I'(u) = 0$. 

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Proposition 6.1. For all \( \beta > 0 \) it holds

\[
|\gamma(v_+)^{1+\beta}|^2_2 \leq 2C_2^2 C_\beta \left( |(V|_\infty + CC_1(2 + M)) |\gamma(v_+)^{1+\beta}|^2_2 
+ C_1 |\beta|_{2N/|N(2-\theta)+\theta|} |\gamma(v_+)^{1+\beta}|^2_{2(2+\theta)} \right),
\]

where \( C_\beta = \max\{m^{-2}, \left(1 + \frac{\beta}{2} \right)\} \), \( C, C_1, \tilde{C} \) and \( M = M(\beta) \) are positive constants and \( g = |W_1 * F(\gamma(v))| \) is the function given by Lemma 2.4.

Proof. Choosing \( \varphi = \varphi_{\beta,T} = v_{\beta,T}^2 \) in (2.11), where \( v_T = \min\{v_+, T\} \) and \( \beta > 0 \), we have \( 0 \leq \varphi_{\beta,T} \in H^1(\mathbb{R}^{N+1}_+) \) and

\[
\int\int_{\mathbb{R}^{N+1}_+} \nabla v \cdot \nabla \varphi_{\beta,T} + m^2 v \varphi_{\beta,T} \nabla \varphi_{\beta,T} = - \int_{\mathbb{R}^N} V(y) \gamma(v) \gamma(\varphi_{\beta,T}) + \int_{\mathbb{R}^N} (W * F(\gamma(v))) f(\gamma(v)) \gamma(\varphi_{\beta,T}), \tag{6.1} \]

Since \( \nabla \varphi_{\beta,T} = v_T^{2\beta} \nabla v + 2\beta v_T^{2\beta-1} \nabla v_T \), the left-hand side of (6.1) is given by

\[
\int\int_{\mathbb{R}^{N+1}_+} \nabla v \cdot \left(v_T^{2\beta} \nabla v + 2\beta v_T^{2\beta-1} \nabla v_T\right) + m^2 v \left(v_T^{2\beta}\right) 
= \int\int_{\mathbb{R}^{N+1}_+} v_T^{2\beta} \left[|\nabla v|^2 + m^2 v^2\right] + 2\beta \int_{D_T} v_T^{2\beta} |\nabla v|^2, \tag{6.2} \]

where \( D_T = \{(x, y) \in (0, \infty) \times \mathbb{R}^N : v_T(x, y) \leq T\} \).

Now we express (6.2) in terms of \( |v_T^{2\beta}|^2 \). For this, we note that \( \nabla(v_T^{2\beta}) = v_T^{2\beta} \nabla v + \beta v_T^{2\beta-1} \nabla v_T \). Therefore,

\[
\int\int_{\mathbb{R}^{N+1}_+} |\nabla(v_T^{2\beta})|^2 = \int\int_{\mathbb{R}^{N+1}_+} v_T^{2\beta} |\nabla v|^2 + (2\beta + \beta^2) \int_{D_T} v_T^{2\beta} |\nabla v|^2,
\]
thus yielding
\[
\|v v_T^2\| = \left( \int_{\mathbb{R}^N} v_T^{2\beta} |\nabla v|^2 + \left( 2\beta + \beta^2 \right) \int_{D_T} v_T^{2\beta} |\nabla v|^2 \right) + \int_{\mathbb{R}^N} (v v_T^2)^2
\]
\[
= \int_{\mathbb{R}^N} v_T^{2\beta} \left( |\nabla v|^2 + |v|^2 \right) + 2\beta \left( 1 + \frac{\beta}{2} \right) \int_{D_T} v_T^{2\beta} |\nabla v|^2
\]
\[
\leq C_\beta \left[ \int_{\mathbb{R}^N} v_T^{2\beta} \left( |\nabla v|^2 + m^2 |v|^2 \right) + 2\beta \int_{D_T} v_T^{2\beta} |\nabla v|^2 \right], \quad (6.3)
\]

where \( C_\beta = \max \{ m^{-2}, \left( 1 + \frac{\beta}{2} \right) \} \). Gathering (6.1), (6.2) and (6.3), we obtain
\[
\|v v_T^2\| \leq C_\beta \left[ - \int_{\mathbb{R}^N} V \gamma(v)^2 \gamma(v_T)^2 \beta \right.
\]
\[
+ \left. \int_{\mathbb{R}^N} (W + F(\gamma(v))) f(\gamma(v)) \gamma(v) \gamma(v_T)^2 \beta \right]. \quad (6.4)
\]

We now start to consider the right-hand side of (6.3). Since \(|f(t)| \leq C_1 (|t| + |t|^\theta - 1)\), Corollary (2.1) shows that it can be written as
\[
\leq C_\beta \left[ |V| \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + \int_{\mathbb{R}^N} (C + g) |f(\gamma(v))| |\gamma(v)| \gamma(v_T)^2 \beta \right]
\]
\[
\leq C_\beta \left[ |V| \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + C \int_{\mathbb{R}^N} C_1 (|\gamma(v)| + |\gamma(v)|^\theta - 1) |\gamma(v)| \gamma(v_T)^2 \beta \right.
\]
\[
+ \left. C_1 \int_{\mathbb{R}^N} g \left( |\gamma(v)| + |\gamma(v)|^\theta - 1 \right) |\gamma(v)| \gamma(v_T)^2 \beta \right]
\]
\[
\leq C_\beta \left[ (|V| + C C_1) \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + C C_1 \int_{\mathbb{R}^N} |\gamma(v)|^\theta - 1 \gamma(v) \gamma(v T)^2 \beta \right.
\]
\[
+ \left. C_1 \int_{\mathbb{R}^N} g \gamma(v v_T^2)^2 + C_1 \int_{\mathbb{R}^N} g |\gamma(v)|^\theta - 2 \gamma(v) \gamma(v T)^2 \beta \right]. \quad (6.5)
\]

Applying Lemmas (2.2) and (2.3) inequality (6.5) becomes
\[
\leq C_\beta \left[ (|V| + C C_1) \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + C C_1 \int_{\mathbb{R}^N} (1 + g_2) \gamma(v v_T^2)^2 \right.
\]
\[
+ \left. C_1 \int_{\mathbb{R}^N} g \gamma(v v_T^2)^2 + C_1 \int_{\mathbb{R}^N} h \gamma(v v_T^2)^2 \right]
\]\[
\leq C_\beta \left[ (|V| + 2 CC_1) \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + C C_1 \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 + C C_1 \int_{\mathbb{R}^N} \gamma(v v_T^2)^2 \left], \quad (6.6)\right.
\]

where \( G = g_2 + h \in L^N(\mathbb{R}^N) \), admitting that \( C C_1 \geq C_1 \).
Because \( |\psi(u)|_{2^\#} \leq C_{2^\#} \|u\| \) for all \( u \in H^1(\mathbb{R}^{N+1}) \), the last inequality is equivalent to
\[
|\gamma(vv_0^\beta)|_{2^\#} \leq C_{2^\#} C_{\beta} \left[ |V|_{\infty} + 2C_1 \int_{\mathbb{R}^N} \gamma(vv_0^\beta)^2 + C_1 \int_{\mathbb{R}^N} g(\gamma(vv_0^\beta))^2 \right. \\
+ C_1 \int_{\mathbb{R}^N} G(\gamma(vv_0^\beta))^2] .
\]

(6.6) \( \text{rhs3} \)

Let us consider the last integral in the right-hand side of (6.6). For all \( M > 0 \), define \( A_1 = \{ G \leq M \} \) and \( A_2 = \{ G > M \} \). Then, whereas \( G \in L^N(\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^N} G\gamma(vv_0^\beta)^2 \leq M \int_{A_1} \gamma(vv_0^\beta)^2 + \left( \int_{A_2} G^N \right)^{1/N} \left( \int_{A_2} \gamma(vv_0^\beta)^{2^\#} \right)^{N-1} \right)^{N/N-1} \\
\leq M \int_{\mathbb{R}^N} \gamma(vv_0^\beta)^2 + \epsilon(M) \left( \int_{\mathbb{R}^N} \gamma(vv_0^\beta)^{2^\#} \right)^{N-1} ,
\]
and \( \epsilon(M) = \left( \int_{A_2} G^N \right)^{1/N} \to 0 \) when \( M \to \infty \).

If \( M \) is taken so that \( \epsilon(M) C_{2^\#} C_{\beta} C_{1} < 1/2 \), we have
\[
|\gamma(vv_0^\beta)|_{2^\#} \leq 2C_{2^\#} C_{\beta} \left[ |V|_{\infty} + C_{1}(2 + M) \right] \int_{\mathbb{R}^N} \gamma(vv_0^\beta)^2 \\
+ C_1 \int_{\mathbb{R}^N} g(\gamma(vv_0^\beta))^2] .
\]

(6.7) \( \text{rhs4} \)

The Hölder inequality guarantees that
\[
\int_{\mathbb{R}^N} g\gamma(vv_0^\beta)^2 \leq |g|_{2N/[N(2-\theta)+\theta]} \left( \int_{\mathbb{R}^N} \gamma(vv_0^\beta)^{2\alpha'} \right)^{1/\alpha'} ,
\]
where
\[
\alpha' = \frac{2N}{N(2-\theta)+\theta} = \frac{2N}{(N-1)\theta} = \frac{2^\#}{\theta} .
\]
Thus,
\[
\int_{\mathbb{R}^N} g\gamma(vv_0^\beta)^2 \leq |g|_{2N/[N(2-\theta)+\theta]} \left| \gamma(vv_0^\beta)^{2^\#} \right|_{2^\#(2/\theta)} \]
and substitution on the right-hand side of (6.7) yields
\[
|\gamma(vv_0^\beta)|_{2^\#} \leq 2C_{2^\#} C_{\beta} \left[ |V|_{\infty} + C_{1}(2 + M) \right] \left| \gamma(vv_0^\beta)^{2} \right| \\
+ C_1 |g|_{2N/[N(2-\theta)+\theta]} \left| \gamma(vv_0^\beta)^{2} \right|_{(2/\theta)} .
\]

(6.8) \( \text{rhs5} \)

Since \( vv_0^\beta \to v_+^{1+\beta} \), it follows from (6.8) that
\[
|\gamma(v_+)^{1+\beta}|_{2^\#} \leq 2C_{2^\#} C_{\beta} \left[ |V|_{\infty} + C_{1}(2 + M) \right] \left| \gamma(v_+)^{1+\beta} \right| \\
+ C_1 |g|_{2N/[N(2-\theta)+\theta]} \left| \gamma(v_+)^{1+\beta} \right|_{(2/\theta)} ,
\]
and we are done. (Observe, however, that \( M \) depends on \( \beta \).) \( \Box \)
**Proposition 6.2.** For all \( p \in [2, \infty) \) we have \( \gamma(v) \in L^p(\mathbb{R}^N) \).

**Proof.** Since \( \frac{2N}{N-1} \frac{2}{p} \leq 2 \) never occurs, we have \( 2 < \frac{2N}{N-1} \frac{2}{p} = \frac{2N}{N-1} \frac{2}{p} < 2 \).

According to the Proposition 6.1, we have

\[
|\gamma(v^0 + \beta)|_{2p}^2 \leq \left[ D_1|\gamma(v^0 + \beta)|_{2}^2 + E_1|\gamma(v^0 + \beta)|_{2p}^2 \right],
\]

where \( D_1 \) and \( E_1 \) are positive constants.

Choosing \( \beta_1 + 1 := (\theta/2) > 1 \), it follows from (6.4) that

\[
|\gamma(v^0 + \beta)|_{2p}^2 = |\gamma(v^0)|_{\frac{2p}{\theta}}^2 < \infty,
\]

from what follows that the right-hand side of (6.9) is finite. We conclude that \( |\gamma(v^0)| \in L^{\frac{2N}{N-1} \frac{2}{p}}(\mathbb{R}^N) < \infty \). Now, we choose \( \beta_2 \) so that \( \beta_2 + 1 = (\theta/2)^2 \) and conclude that

\[
|\gamma(v^0)| \in L^{\frac{2N}{N-1} \frac{2}{p}}(\mathbb{R}^N).
\]

After \( k \) iterations we obtain that

\[
|\gamma(v^k)| \in L^{\frac{2N}{N-1} \frac{2}{p}}(\mathbb{R}^N),
\]

from what follows that \( \gamma(v^k) \in L^p(\mathbb{R}^N) \) for all \( p \in [2, \infty) \). Since the same arguments are valid for \( v^- \), we have \( \gamma(v) \in L^p(\mathbb{R}^N) \) for all \( p \in [2, \infty) \).

By simply adapting the proof given in [9], we present, for the convenience of the reader, the proof of our next result:

**Proposition 6.3.** Let \( v \in H^1(\mathbb{R}^{N+1}_+) \) be a weak solution of (1.3). Then \( \gamma(v) \in L^p(\mathbb{R}^N) \) for all \( p \in [2, \infty] \) and \( v \in L^\infty(\mathbb{R}^{N+1}_+) \).

**Proof.** We recall equation (6.3): $\|uv_T^\beta\|^2 \leq C_\beta \left[ -\int_{\mathbb{R}^N} V\gamma(v_T^\beta) + \int_{\mathbb{R}^N} (W*F(\gamma(v)))f(\gamma(v))\gamma(v)\gamma(v_T)^{2\beta} \right]$, where \( C_\beta = \max\{m^{-2}, (1 + \beta^2)\} \).

It follows that \( W*F(\gamma(v)) \in L^\infty(\mathbb{R}^N) \), since \( \gamma(v) \in L^p(\mathbb{R}^N) \) for all \( p \geq 2 \), by Proposition 6.2. We also know that \( |f(t)| \leq C_1 |t| + |t|^{\theta-1} \) and \( V \) is bounded. Therefore, if \( C = \max\{|V|\infty, C_1 W*F(\gamma)|\infty\} \), we have

\[
\|uv_T^\beta\|^2 \leq C_\beta C \left[ \int_{\mathbb{R}^N} \gamma(v_T^\beta) + \int_{\mathbb{R}^N} (\gamma(v)|v_T|^{\theta-1} \gamma(v)\gamma(v_T)^{2\beta} \right]
\]

\[
\leq C_\beta \left[ 2C \int_{\mathbb{R}^N} \gamma(v_T^\beta) + C \int_{\mathbb{R}^N} |\gamma(v)|^{\theta-2} \gamma(v_T^\beta) \right].
\]

Since \( |\gamma(v)|^{\theta-2} = |\gamma(v)|^{\theta-2} \chi_{(|\gamma(v)| \leq 1)} + |\gamma(v)|^{\theta-2} \chi_{(|\gamma(v)| > 1)} \), the fact that

\[
|\gamma(v)|^{\theta-2} \chi_{(|\gamma(v)| > 1)} =: g_3 \in L^{2N}(\mathbb{R}^N)
\]

allows us to conclude that

\[
2C\gamma(v_T^\beta) + C|\gamma(v)|^{\theta-2} \gamma(v_T^\beta) \leq (C_3 + g_3)\gamma(v_T^\beta)^2
\]
for a positive constant $C_3$ and a positive function $g_3 \in L^2(\mathbb{R}^N)$ that depends neither on $T$ nor on $\beta$.

Therefore,

$$\|vv_+^\beta\|^2 \leq \int_{\mathbb{R}^N} (C_3 + g_3) \gamma(vv_+^\beta)^2.$$ \hspace{1cm}

and

$$\|v_+^{\beta+1}\|^2 \leq C_\beta \int_{\mathbb{R}^N} (C_3 + g_3) \gamma(v_+^{\beta+1})^2.$$ \hspace{1cm}

Since

$$\int_{\mathbb{R}^N} g_3 \gamma(v_+^{\beta+1})^2 \leq |g_3|_{2N} \left| \gamma(v_+)^{1+\beta} \right|_2 \left| \gamma(v_+)^{1+\beta} \right|_{2^\beta} \leq |g_3|_{2N} \left( \lambda \left| \gamma(v_+)^{1+\beta} \right|_2^2 + \frac{1}{\lambda} \left| \gamma(v_+)^{1+\beta} \right|_{2^\beta}^2 \right),$$

we conclude that

$$\left| \gamma(v_+)^{1+\beta} \right|_{2^\beta}^2 \leq C_{2^\beta} \|v_+^{\beta+1}\|^2 \leq C_{2^\beta} C_\beta (C_3 + \lambda |g_3|_{2N}) \left| \gamma(v_+)^{1+\beta} \right|_2^2 + \frac{C_{2^\beta} C_\beta |g_3|_{2N}}{\lambda} \left| \gamma(v_+)^{1+\beta} \right|_{2^\beta}^2$$

and, by taking $\lambda > 0$ so that

$$\frac{C_{2^\beta} C_\beta |g_3|_{2N}}{\lambda} < \frac{1}{2},$$

we obtain

$$\left| \gamma(v_+)^{1+\beta} \right|_{2^\beta}^2 \leq C_\beta \left( 2C_{2^\beta} C_3 + 2C_{2^\beta} C_\beta \lambda |g_3|_{2N} \right) \left| \gamma(v_+)^{1+\beta} \right|_2^2.$$ \hspace{1cm}

(6.10)

Since

$$C_4 C_\beta \leq C_4 (m^{-2} + 1 + \beta) \leq M^2 e^{2\sqrt{1+\beta}}$$

for a positive constant $M$, it follows from (6.10) that

$$\left| \gamma(v_+)^{2\beta} \right|_{(1+\beta)} \leq M^{1/(1+\beta)} e^{1/\sqrt{1+\beta}} \left| \gamma(v_+)^{2\beta} \right|_{(1+\beta)}.$$

We now apply an iteration argument, taking $2(1 + \beta_{n+1}) = 2^\beta (1 + \beta_n)$ and starting with $\beta_0 = 0$. This produces

$$\left| \gamma(v_+)^{2\beta} \right|_{(1+\beta_n)} \leq M^{1/(1+\beta_n)} e^{1/\sqrt{1+\beta_n}} \left| \gamma(v_+)^{2\beta} \right|_{(1+\beta_n)}.$$

Because $(1 + \beta_n) = \left( \frac{2^\beta}{2} \right)^n = \left( \frac{N}{N-1} \right)^n$, we have

$$\sum_{i=0}^{\infty} \frac{1}{1 + \beta_n} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{\sqrt{1 + \beta_n}} < \infty.$$ \hspace{1cm}

Thus,

$$\left| \gamma(v_+) \right|_{\infty} = \lim_{n \to \infty} \left| \gamma(v_+)^{2\beta} \right|_{(1+\beta_n)} < \infty,$$

from what follows $\left| \gamma(v_+) \right|_p < \infty$ for all $p \in [2, \infty]$. The same argument applies to $\gamma(v_-)$, proving that $\gamma(v) \in L^p(\mathbb{R}^N)$ for all $p \in [2, \infty]$.\hspace{1cm}
By taking $\lambda = 1$ and $|\gamma(v_+)^{1+\beta}| < C_5$ for all $p$, we obtain for any $\beta > 0$,
\[
\|v_+^{1+\beta}\|_2^2 \leq C_5 (C_\beta + |g_3|_{2N}) C_5^2 + C_\beta |g_3|_{2N} C_5^2.
\] (6.11)

But $\|v_+\|_{2^*(1+\beta)}^{1+\beta} = \|v_+^{1+\beta}\|_{2^*} \leq C_2 \|v_+^{1+\beta}\|$ and for a positive constant $\tilde{c}$ results from (6.11) that
\[
\|v_+\|_{2^*(1+\beta)}^{2(1+\beta)} \leq \tilde{c} C_\beta C_5^{2(1+\beta)}.
\]

Thus,
\[
\|v_+\|_{2^*(1+\beta)} \leq \tilde{c}^{1/2(1+\beta)} C_\beta^{1/2(1+\beta)} C_5
\]
and the right-hand side of the last inequality is uniformly bounded for all $\beta > 0$. We are done.

We now state [9, Proposition 3.9]:

**Proposition 6.4.** Suppose that $v \in H^1(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ is a weak solution of
\[
\begin{align*}
-\Delta v + m^2 v &= 0, & \text{in } \mathbb{R}_+^{N+1}, \\
-\frac{\partial v}{\partial x}(0, y) &= h(y) & \text{for all } y \in \mathbb{R}^N,
\end{align*}
\] (6.12)

where $h \in L^p(\mathbb{R}^N)$ for all $p \in [2, \infty]$. Then $v \in C^\alpha([0, \infty) \times \mathbb{R}^N) \cap W^{1,q}((0, R) \times \mathbb{R}^N)$ for all $q \in [2, \infty)$ and $R > 0$.

In addition, if $h \in C^\alpha(\mathbb{R}^N)$, then $v \in C^{1,\alpha}([0, \infty) \times \mathbb{R}^N) \cap C^2(\mathbb{R}_+^{N+1})$ is a classical solution of (6.12).

**Proof of Theorem**

In the proof of Proposition 6.4 (see [9, Proposition 3.9]), defining
\[
\rho(x, y) = \int_0^x v(t, y) dt,
\]
taking the odd extension of $h$ and $\rho$ to the whole $\mathbb{R}_+^{N+1}$ (which we still denote simply by $h$ and $\rho$), in [9] is obtained that $\rho$ satisfies the equation
\[
-\Delta \rho + m^2 \rho = h & \text{ in } \mathbb{R}_+^{N+1}
\] (6.13)
and $\rho \in C^{1,\alpha}(\mathbb{R}_+^{N+1})$ for all $\alpha \in (0, 1)$ by applying Sobolev’s embedding. Therefore, $v(x, y) = \frac{\partial \rho}{\partial x}(x, y) \in C^\alpha(\mathbb{R}^N)$.

In our case
\[
h(y) = -V(y)v(0, y) + (W \ast F(v(0, y)))f(v(0, y)).
\]

We now rewrite equation (6.13) as
\[
-\Delta \rho + V(y)\frac{\partial \rho}{\partial x}(0, y) + m^2 \rho = \left(W \ast F\left(\frac{\partial \rho}{\partial x}(0, y)\right)\right)f\left(\frac{\partial \rho}{\partial x}(0, y)\right).
\]

Since $f \in C^1$ and $\frac{\partial \rho}{\partial x}(x, y)$ is bounded, the right-hand side of the last equality belongs to $C^\alpha(\mathbb{R}_+^{N+1})$. Thus, classical elliptic boundary regularity yields
\[
\rho \in C^2(\mathbb{R}_+^{N+1}) \Rightarrow v \in C^{1,\alpha}(\mathbb{R}_+^N).
\]

Hence, by applying classical interior elliptic regularity directly to $v$, we deduce that $v \in C^{1,\alpha}(\mathbb{R}_+^N) \cap C^2(\mathbb{R}_+^N)$ is a classical solution of problem (1.3). \qed
7. Proof of Theorem 3

We now adjust [9, Theorem 3.14] to our needs. The original statement guarantees that \( v \in C^\infty((0, \infty) \times \mathbb{R}^N) \), a result that depends on the function \( h \) (of Proposition 6.4) considered in that paper. For the convenience of the reader, we present the proof of the next result:

**Theorem 6.** Suppose that \( v \in H^1(\mathbb{R}^{N+1}_+) \) is a critical point of the energy functional \( I \), then
\[
|v(x, y)| e^{\lambda x} \to 0
\]
as \( x + |y| \to \infty \), for any \( \lambda < m \).

**Proof.** Let us consider a solution \( v \) of the problem
\[
\begin{align*}
-\Delta v + m^2 v &= 0 \\
v(0, y) &= v_0(y) \in L^2(\mathbb{R}^N), \quad y \in \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+.
\end{align*}
\]
By applying the Fourier transform with respect to variable \( y \in \mathbb{R}^N \) we obtain
\[
\mathcal{F}v(x, k) = e^{-\sqrt{(2\pi k)^2 + m^2} x} F v_0(y),
\]
from what follows
\[
\sup_{y \in \mathbb{R}^N} |v(x, y)| \leq C v_0 \leq e^{-m x}.
\]
Since Proposition 6.4 shows that \( v \in W^{1,q}((0, R) \times \mathbb{R}^N) \) for all \( q \in [2, \infty) \) and \( R > 0 \), we conclude that \( |v(x, y)| \to 0 \) when \( |y| \to \infty \) for any \( x \) and \( |v(x, y)| e^{\lambda x} \to 0 \) as \( x + |y| \to \infty \) for any \( \lambda < m \).

We now adapt the proof of [9, Theorem 5.1]. In that paper is assumed that \( W(y) \to 0 \) as \( |y| \to \infty \), a condition that is not necessary.

**Proof of Theorem 3.** We denote
\[
K(y) = W \ast \nabla \left( \frac{\partial w}{\partial x}(0, y) \right).
\]
It follows easily that \( K \) is bounded.

By Theorem 4 we have \( w(x, y) \geq 0 \). Applying Harnack’s inequality we conclude that \( w \) is strictly positive.

Following [9], for any \( R > 0 \) we denote
\[
\begin{align*}
B_R^+ &= \{(x, y) \in \mathbb{R}^{N+1}_+ : \sqrt{x^2 + |y|^2} < R \} \\
\Omega_R^+ &= \{(x, y) \in \mathbb{R}^{N+1}_+ : \sqrt{x^2 + |y|^2} > R \} \\
\Gamma_R^+ &= \{(0, y) \in \mathbb{R}^{N+1}_+ : |y| > R \}
\end{align*}
\]
and define
\[
f_R(x, y) = CR e^{-\alpha x} e^{-(m-\alpha) \sqrt{x^2 + |y|^2}},
\]
where the positive constants $C_R$ and $\alpha \in (V_0, m)$ will be chosen later on. A simple computation shows that

$$\Delta f_R = \left( \alpha^2 + (m-\alpha)^2 + \frac{2\alpha(m-\alpha)x}{\sqrt{x^2+y^2}} - \frac{N(m-\alpha)}{\sqrt{x^2+y^2}} \right) f_R.$$ 

Thus, for $R$ large enough, we have

$$\{ -\Delta f_R + m^2 f_R \geq 0 \quad \text{in } \Omega_R^+ \ight.$$ 

$$\left. -\frac{\partial f_R}{\partial x} = \partial f_R \quad \text{on } \Gamma_R^+. \right.$$ 

We now define

$$\rho(x, y) = f_R(x, y) - w(x, y).$$ 

We clearly have $-\Delta \rho(x, y) - m^2 \rho(x, y) \geq 0$ in $\Omega_R^+$. Choosing

$$C_R = e^{mR} \max_{\partial B_R^+} v,$$

we also have $\rho(x, y) \geq 0$ on $\partial B_R^+$ and $\rho(x, y) \to 0$ when $x + |y| \to \infty$.

We claim that $\rho(x, y) \geq 0$ in $\Omega_R^+$. Defining $z(x, y) = \rho(x, y) e^{\lambda x}$ for some $\lambda \in (V_0, m)$, a straightforward calculation shows that

$$- \Delta \rho + m^2 \rho = e^{-\lambda x} \left( \Delta z + 2\lambda \partial_x z + (m^2 - \lambda^2) z \right).$$ 

Since $- \Delta \rho + m^2 \rho \geq 0$, we conclude that $\Delta z + 2\lambda \partial_x z + (m^2 - \lambda^2) z \geq 0$. Another application of the strong maximum principle yields

$$z(0, y_0) = \inf_{\Gamma_R^+} z = \inf_{\Gamma_R^+} \rho = \rho(0, y_0) < 0.$$

An application of Hopf’s lemma produces $\frac{\partial z}{\partial e}(0, y_0) < 0$, that is,

$$\frac{\partial z}{\partial e}(0, y_0) < 0. \quad \text{(7.1)}$$

Since $\frac{\partial z}{\partial x} = \frac{\partial \rho}{\partial x} e^{\lambda x} + \lambda \rho e^{\lambda x}$, we conclude that

$$\frac{\partial z}{\partial x}(0, y_0) = \frac{\partial \rho}{\partial x}(0, y_0) + \lambda \rho(0, y_0)$$

and so

$$-\frac{\partial z}{\partial x}(0, y_0) = -\frac{\partial f_R}{\partial x}(0, y_0) + \frac{\partial w}{\partial x}(0, y_0) - \lambda f_R(0, y_0) + \lambda w(0, y_0)$$

$$= (\alpha - \lambda) f_R(0, y_0) + V(y_0) w(0, y_0) - K(y_0) f(w(0, y_0))$$

$$+ \lambda w(0, y_0)$$

$$= (\alpha - \lambda) f_R(0, y_0) + (V(y_0) + V_0) w(0, y_0) - K(y_0) f(w(0, y_0))$$

$$+ (\lambda - V_0) w(0, y_0).$$
Now, choosing $\alpha = \lambda$, since $\lambda > V_0$ (so that the last term in the above inequality is non-negative), the positiveness of $(V(y_0) + V_0)w(0, y_0)$ and hypothesis $(f_1)$ guarantees that $-\frac{\partial}{\partial x}(0, y_0) > 0$, thus reaching a contradiction with (7.1). □

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