Geometrical structure of Weyl invariants for spin three gauge field in general gravitational background in $d = 4$

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Abstract

We construct all possible Weyl invariant actions in $d = 4$ for linearized spin three field in a general gravitational background. The first action is obtained as the square of the generalized Weyl tensor for a spin three gauge field in nonlinear gravitational background. It is, however, not invariant under spin three gauge transformations. We then construct two other nontrivial Weyl but not gauge invariant actions which are linear in the Weyl tensor of the background geometry. We then discuss existence and uniqueness of a possible linear combination of these three actions which is gauge invariant. We do this at the linear order in the background curvature for Ricci flat backgrounds.
1 Introduction

Conformal gravity has attracted considerable attention during the last thirty years [1]-[12], parallel to the development of higher spin gauge field theories [13],[14]. It remained an intriguing task to combine these two developments and to construct an interacting higher spin conformal gauge field theory. These two generalizations and extensions of ordinary gauge and gravity theories share many properties and problems related to the high level and complicated structure of gauge symmetries [15]-[19] and the necessity to include higher derivatives [20]-[22] which raises the issue of the unitarity (see the discussions in [23]-[25]). The interest in these intriguing topics intensified during the last decade with new applications of conformal higher spin theories in the context of the AdS/CFT correspondence. Furthermore, the remarkable trivialization of the partition function in flat space [26]-[31] could be explained by the high level of gauge symmetry. The possibility to obtain the exact partition function in some conformal higher spin field configurations could prove useful for future nontrivial checks of the AdS/CFT conjecture. Studying conformal higher spins is also helpful for the construction of couplings of higher spin gauge fields to conformal currents [32]-[35]. This simple interaction allowed to perform some one loop calculations and to investigate the structure of conformal anomalies of higher spin fields [31], [36]-[38].

In this paper we consider four-dimensional conformal higher spin (spin 3) theory in a general curved background. We use the usual spin 2 Weyl symmetry (with scalar parameter) for the construction of generalized curvature and Weyl tensors from the spin 3 field. We construct the linearized spin 3 Weyl tensor with three covariant derivatives and one spin 3 field. The square of this tensor is our first conformal (Weyl) primary. This is done in a general curved background, thus extending the usual flat space higher spin Weyl tensor of [41] by additional terms containing the background curvature. Then, guided by ideas from spin 2 considerations and using the technology developed in [40] related to the supersymmetric case, we construct all possible primaries with conformal weight -4, which are linear in the background Weyl tensor. For this purpose we have constructed and investigated the conformal properties of the whole hierarchy of generalized spin 3 Christoffel symbols [41] in a curved background.

The main result of our paper is the derivation of a second nontrivial Weyl invariant (3.42). The existence of this additional primary, quadratic in the generalized spin 3

1In this rather technical contribution we do not pretend to cover all relevant references but only mention those which were important for our understanding of the issues involved here.
Christoffel symbols [41] and linear in the gravitational Weyl tensor, opens up the possibility to construct a unique Lagrangian which, besides being invariant under spin 2 and spin 3 Weyl transformations, is also invariant under spin-3 gauge transformations. This will also involve the more trivial Weyl primary (3.24) and the square of the spin 3 Weyl tensor. Unfortunately, for the time being, we were only able to prove invariance in Ricci (and therefore also Bach) flat backgrounds and to linear order in the background Weyl tensor. Even to achieve this required computer assistance. An interesting observation is that the combination obtained from the requirement of spin 2 Weyl symmetry and spin 3 gauge symmetry is automatically invariant under the spin 3 Weyl symmetry, where the spin 3 Weyl transformation is parameterized by a vector field and shifts of the trace of spin 3 gauge field.

More general backgrounds might be possible in the future. In this case one is confronted with the results of [29][30] and also with discussions in [25] and later in [39].

In the next section we give a complete description of the well known spin 2 case: how to construct conformal primaries and the construction of a unique gauge invariant action to all order on background curvature. In Section 3 we apply this technology to the spin three case and obtain all necessary Weyl primaries with conformal weight -4, including the most nontrivial one mentioned above as our main results. Then, in Section 4, we try to find a linear combination of the three conformal primaries to obtain a gauge invariant Lagrangian. This is a formidable task and we succeeded in doing this only for Ricci-flat backgrounds and to linear order in the background Weyl curvature. Details of the construction are relegated to three appendices.

2 Spin Two Example

In this section we demonstrate how the linearized conformal gravity action in $d = 4$ can be obtained from symmetry considerations alone. We show that possible Weyl invariant expressions can be combined into a unique gauge invariant action. To realize this idea we concentrate on the construction of possible primary fields $\mathcal{L}_\Delta(g_{\alpha\beta}, \nabla_\lambda, h_{\mu\nu})$ with weight $\Delta$ with respect to local Weyl transformations, written in terms of the linearized gravitational field $h_{\mu\nu}$ and covariant derivatives $\nabla_\mu$ in a general gravitational background $g_{\mu\nu}$:

$$\delta \mathcal{L}_\Delta(g_{\alpha\beta}, \nabla_\lambda, h_{\mu\nu}) = \Delta \sigma(x) \mathcal{L}_\Delta(g_{\alpha\beta}, \nabla_\lambda, h_{\mu\nu}),$$

$$2$$
The Weyl transformations of background metric and the linearized spin two field are
defined in a similar way
\[ \delta h_{\mu\nu}(x) = 2\sigma(x)h_{\mu\nu}(x), \]
\[ \delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x). \]
i.e. they are both symmetric spin two tensor primaries with conformal dimension 2.

The most interesting primaries are scalars with conformal dimension \(-4\), because they
can be used to construct Weyl invariant actions
\[ S_{\text{Weyl inv}} = \int d^4x \sqrt{g} \mathcal{L}_{-4}(g_{\alpha\beta}, \nabla_{\lambda}, h_{\mu\nu}). \] (2.1)

Details of the notation and definitions can be found in Appendix A. Here we note only
that it is convenient to introduce the notation \( \sigma_{\mu} = \partial_{\mu}\sigma(x) \) for the gradient of the scalar
parameter of the Weyl symmetry. Then e.g.
formula (A.7) for the Weyl transformation
of the Christoffel symbol can be written as
\[ \delta \Gamma^\lambda_{\mu\nu} = \sigma_{(\mu} \delta \Gamma^\lambda_{\nu)} - g_{\mu\nu} g^{\sigma\lambda} \] (2.2)

Using this we can investigate the Weyl transformation of second covariant derivatives of
\( h_{\mu\nu} \), where the four indices have the symmetry of the Young Tableau "window" (curvature). For that we make symmetrization of indices of covariant derivatives \( (\alpha\beta) \) and then
perform antisymmetrization of two pairs of indices \( [\alpha\mu] \) and \( \{\beta\nu\} \). Then correcting this
variation with terms proportional to background Schouten tensor we came to the following
nice transformation of the linearized curvature constructed from second derivatives:
\[ R_{\alpha\mu,\beta\nu} = \frac{1}{4} \left\{ \{\nabla_{[\alpha}, \nabla_{\beta]} h_{\mu]\nu\} - 2K_{[\alpha\beta}\,h_{\mu]\nu]} - K_{[\alpha g_{\mu][\beta h_{\nu]}\tau]} - K_{\beta g_{\nu}[\alpha h_{\tau]}}, \right\} \] (2.3)
\[ \delta R_{\alpha\mu,\beta\nu} = 2\sigma R_{\alpha\mu,\beta\nu} + \frac{1}{2} g_{[\alpha\beta} (\sigma^\tau \nabla_{ adversary} h_{\mu]\nu]} - \nabla_{\nu]} h_{\mu\nu}\tau) \sigma^\tau \right\} - 2\sigma R_{\alpha\mu,\beta\nu} - g_{[\alpha\beta} \sigma^{\tau} G_{\tau,\mu\nu]} \right\}, \]
(2.4)

where
\[ G_{\tau,\mu\nu} = \frac{1}{2} (\nabla_{\mu} h_{\nu}\tau) - \nabla_{\tau} h_{\mu\nu} \] (2.5)

\(^2\)We use in this paper (\(\ldots\)) and \(\langle \ldots \rangle\) brackets for symmetrized sets and \([\ldots]\) and \(\{\ldots\}\) for
antisymmetric pairs of indices. No weight factor is included in the (anti)symmetrization, e.g.
\( v_{(\mu} w_{\nu)} = v_{\mu} w_{\nu} + v_{\nu} w_{\mu}. \)
is the linearized Christoffel symbol. To obtain primary fields we investigate the Weyl
transformations of the linearized Ricci tensor and Ricci scalar, obtained as traces of (2.4).
Then we find that the following combination transforms as
\[
\delta \mathcal{K}_{\mu\nu} = \delta \frac{1}{2}(\mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{6} \mathcal{R}) = -\sigma^\tau \mathcal{G}_{\tau\mu\nu}.
\] (2.6)
This allows us to integrate (2.4) in the form
\[
(\delta - 2\sigma) \mathcal{R}_{\alpha\mu,\beta\nu} = g_{[\alpha[\beta} \delta \mathcal{K}_{\mu\nu]} = (\delta - 2\sigma)(g_{[\alpha[\beta} \mathcal{K}_{\mu\nu]}).
\] (2.7)
We have thus constructed invariant linearized Weyl tensor (a primary field under Weyl
transformations):
\[
\mathcal{W}_{\alpha\mu,\beta\nu} = \mathcal{R}_{\alpha\mu,\beta\nu} - g_{[\alpha[\beta} \mathcal{K}_{\mu\nu]}.
\] (2.8)
Note also that substraction of traces from (2.3) or from the same expression but without
last two terms with background metric leads to the same results. So we can conclude that
linearized Weyl tensor in general background is:
\[
\mathcal{W}_{\alpha\mu,\beta\nu} = \frac{1}{4}\{\nabla_{[\alpha, \nabla_{[\beta} h_{\mu\nu]} - 2K_{[\alpha[\beta} h_{\mu\nu]}\} - \text{traces},
\] (2.9)
and it is a conformal primary:
\[
\delta \mathcal{W}_{\alpha\mu,\beta\nu} = 2\sigma(x)\mathcal{W}_{\alpha\mu,\beta\nu}.
\] (2.10)
The background Weyl tensor is also $\Delta = 2$ primary but without dependence on $h_{\mu\nu}$.
Having these two primaries we can construct:
1) One linear in linearized spin two field relevant ($\Delta = -4$) primary
\[
\mathcal{L}^{lin}_{-4} = W^{\alpha\mu,\beta\nu} \mathcal{W}_{\alpha\mu,\beta\nu} = 2W^{\alpha\mu,\beta\nu}(\nabla_\alpha \nabla_\beta - K_{\alpha\beta})h_{\mu\nu},
\] (2.11)
and corresponding invariant action produces correct equation of motion with Bach tensor
for background metric:
\[
B^{\mu\nu} = (\nabla_\alpha \nabla_\beta - K_{\alpha\beta})W^{\alpha\mu,\beta\nu} = 0.
\] (2.12)
2) One four derivative quadratic primary
\[
\mathcal{L}^{\mathcal{W}^2}_{-4} = \frac{1}{2} \mathcal{W}^{\alpha\mu,\beta\nu} \mathcal{W}_{\alpha\mu,\beta\nu}.
\] (2.13)
3) And several two and zero derivative primaries quadratic in linearized field

\[ \mathcal{L}_{-4}^{WW} = W^\alpha\beta\nu W_{\alpha\beta\rho} h^\rho, \]  
(2.14)

\[ \mathcal{L}_{-4}^{(1)W^2} = W^\alpha\beta\nu W_{\alpha\beta\rho} h^\rho_{\tau}, \]  
(2.15)

\[ \mathcal{L}_{-4}^{(2)W^2} = W_{\alpha\beta\rho} h_{\alpha\beta\rho} h^\tau. \]  
(2.16)

We now turn to the Weyl variation of the linearized Christoffel symbol (2.5)

\[ (\delta - 2\sigma) \mathcal{G}_{\tau;\mu\nu} = -\sigma_\tau h_{\mu\nu} + g_{\mu\nu} h_{\tau\lambda} \sigma^\lambda. \]  
(2.17)

We see that traceless in \( \mu, \nu \) part of Christoffel symbol transforms in a way that quantity \( \sigma_\tau \) arises only with first \( \tau \) index of symbol. Then taking into account transformation low (2.6) we can guess the last nontrivial primary with four derivatives and second order on spin three gauge field:

\[ \mathcal{L}_{-4}^{WG^2} = \frac{1}{2} W^\alpha\beta\nu \left( \mathcal{G}_{\tau;\alpha\beta} \mathcal{G}_{\tau;\mu\nu} - 2 h_{\alpha\beta} \mathcal{K}_{\mu\nu} \right), \]  
(2.18)

with conformal weight -4:

\[ \delta \mathcal{L}_{-4}^{WG^2} = -4\sigma(x) \mathcal{L}_{-4}^{WG^2} \]  
(2.19)

Now we consider the linearized gauge invariance:

\[ \bar{\delta} h_{\mu\nu} = \nabla_{(\mu \epsilon_\nu)}. \]  
(2.20)

The main goal now is to find unique gauge invariant combination of the primaries presented above. To find that we start from gauge variation of the last one and try to cancel at least some part from variation of (2.13). Immediately we see that cancelation can be observed only up to total derivative terms and therefore gauge invariance exists only on the level of Weyl invariant actions (2.1) where corresponding Lagrangians are our \(-4\) weight primaries (2.13)-(2.16) and (2.18). Doing in this direction and hiding long calculation in appendix B, we arrive to the following unique ”gauge invariant” combination of our primaries

\[ \mathcal{L}_{-4}^{GI} = \mathcal{L}_{-4}^{WG^2} + \frac{1}{4} \mathcal{L}_{-4}^{W^2} - \frac{1}{16} \mathcal{L}_{-4}^{WW} + \frac{1}{32} \mathcal{L}_{-4}^{(1)W^2} - \frac{1}{64} \mathcal{L}_{-4}^{(2)W^2} \]  
(2.21)

with the property that corresponding action transforms in respect to gauge transformation (2.20) as follows

\[ \bar{\delta} \int d^4x \sqrt{g} \mathcal{L}_{-4}^{GI} = \int d^4x \sqrt{g} \left\{ -\frac{1}{2} B^{\alpha\beta} \mathcal{L}_{\epsilon_\alpha\beta} \right\}. \]  
(2.22)

\[ \text{Here } \mathcal{L}_{\epsilon} \text{ is Lie derivative in direction of gauge vector parameter } \epsilon^\mu \text{ in background metric } g_{\alpha\beta} \text{ defined by rule } \mathcal{L}_{\epsilon} T^\mu = \epsilon^\nu \nabla_\nu T^\mu - \nabla_\tau \epsilon^\mu T^\tau + \nabla_\nu \epsilon^\tau T^\nu. \]
Therefore we prove that in the background with zero Bach tensor (conformal gravitational background) gauge and Weyl invariant action is:

\[
S_{GI} = \frac{1}{8} \int d^4x \sqrt{g} W^\alpha_{\mu,\beta\nu} W_{\alpha \mu,\beta\nu} + \frac{1}{2} \int d^4x \sqrt{g} W^\alpha_{\mu,\beta\nu} \left( G^\tau_{\gamma\alpha\beta} G^\tau_{\gamma\mu\nu} - 2 h^\tau_{\gamma\alpha\beta} \mathcal{K}_{\mu\nu} \right) - \frac{1}{16} \int d^4x \sqrt{g} \left\{ W^\alpha_{\mu,\beta\nu} W_{\alpha \mu,\beta\nu} h^\rho_\rho - \frac{1}{2} W^\alpha_{\mu,\beta\nu} W_{\alpha \mu,\beta\nu} \left[ h^\rho_\rho h^\tau_\tau - \frac{1}{4} h^\rho_\rho h^\tau_\tau \right] \right\} \quad (2.23)
\]

Of course this action can be obtained from expansion of the action for conformal gravity:

\[
S_{W(G)} = \frac{1}{2} \int d^4x \sqrt{G} W^\alpha_{\mu,\beta\nu} (G) W_{\alpha \mu,\beta\nu} (G) \quad (2.24)
\]

up to second order on fluctuation $h_{\mu\nu}$ around general background $g_{\mu\nu}$:

\[
G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \quad (2.25)
\]

### 3 Linearized Weyl Tensor and Other Primaries for Spin 3 Field

Now we construct Weyl tensor for spin 3 field in general gravitational background. To do that first of all we should define gravitational Weyl transformation for spin 3 field setting 4 for conformal dimension of third rank symmetric tensor $h_{\mu\nu\lambda}$:

\[
\delta h_{\mu\nu\lambda}(x) = 4 \sigma(x) h_{\mu\nu\lambda}(x). \quad (3.1)
\]

Using the same as in previous section convenient vector notation $\sigma_\mu = \partial_\mu \sigma(x)$ for gradient of scalar parameter of Weyl symmetry and investigating Weyl variation of the third covariant derivative of $h_{\mu\nu\lambda}$ with two set of three symmetrized indices, we arrive to the following basic object

\[
H_{\alpha\beta\gamma,\mu\nu\lambda} = \frac{1}{6} \nabla_{(\alpha} \nabla_{\beta} \nabla_{\gamma)} h_{\mu\nu\lambda} - \frac{2}{3} \nabla_{(\alpha} K_{\beta\gamma)} h_{\mu\nu\lambda} + \frac{4}{3} K_{\alpha\beta} \nabla_\lambda h_{\mu\nu\gamma} + \frac{4}{3} K_{\beta\gamma} \nabla_\mu h_{\nu\lambda\alpha} + \frac{4}{3} K_{\gamma\alpha} \nabla_\nu h_{\lambda\mu\beta}. \quad (3.2)
\]

Weyl transformation of which include only first derivative of scalar parameter $\sigma_\mu$. Another remarkable properties of this object is that after antisymmetrization over the index pairs.
\{\alpha, \mu\}, \{\beta, \nu\} \text{ and } \{\gamma, \lambda\} \text{ we obtain index properties of Young Tableau with two row and three column and call it Riemann curvature for spin three linearized gauge field:}

\begin{align}
R_{\alpha\mu,\beta\nu,\gamma\lambda} &= H_{\alpha\beta\gamma,\mu\nu\lambda} - H_{\mu\beta\gamma,\alpha\nu\lambda} - H_{\alpha\nu\gamma,\mu\beta\lambda} + H_{\mu\nu\gamma,\alpha\beta\lambda} \\
&- H_{\alpha\beta\lambda,\mu\nu\gamma} + H_{\mu\beta\lambda,\alpha\nu\gamma} + H_{\alpha\nu\lambda,\mu\beta\gamma} - H_{\mu\nu\lambda,\alpha\beta\gamma}, \\
R_{\alpha\mu,\beta\nu,\gamma\lambda} &= R_{\beta\nu,\alpha\mu,\gamma\lambda} = R_{\alpha\mu,\gamma\lambda,\beta\nu} = R_{\gamma\lambda,\beta\nu,\alpha\mu}, \\
R_{\alpha\mu,\beta}[\nu,\gamma\lambda] &= R_{\alpha\mu,\beta\nu,\gamma\lambda} + R_{\alpha\mu,\beta\gamma,\nu\lambda} + R_{\alpha\mu,\beta\lambda,\nu\gamma} = 0.
\end{align}

(3.3)

The last condition is first Bianchi identity for spin 3 curvature. Then we can observe that Weyl variation of curvature is linear on background metric

\[
\delta R_{\alpha\mu,\beta\nu,\gamma\lambda} = O(g; \sigma^7; \nabla^2; h).
\]

(3.6)

So we see that this, proportional to \(g_{\alpha\beta}\), variation of \(R_{\alpha\mu,\beta\nu,\gamma\lambda}\) can be completely removed by subtraction of traces from curvature \(R_{\alpha\mu,\beta\nu,\gamma\lambda}\). To subtract traces we note first that due to symmetry properties we have unique first trace with one pair symmetric and another antisymmetric pair of indices and satisfying Bianchi identity obtaining from trace of (3.3)

\begin{align}
R_{\mu\nu,\gamma\lambda} &= g^{\alpha\beta} R_{\alpha\mu,\beta\nu,\gamma\lambda}, \\
R_{\mu\nu,\gamma\lambda} &= R_{\nu\mu,\gamma\lambda} = -R_{\mu\nu,\lambda\gamma}, \\
R_{\mu[\nu,\gamma\lambda]} &= R_{\mu\nu,\gamma\lambda} + R_{\mu\gamma,\lambda\nu} + R_{\mu\lambda,\nu\gamma} = 0.
\end{align}

(3.7)\quad (3.8)\quad (3.9)

The second trace we can take in two ways

\begin{align}
R_{\gamma\lambda}^{(1)} &= g^{\mu\nu} R_{\mu\nu,\gamma\lambda} = -R_{\gamma\lambda}^{(1)}, \\
R_{\nu\lambda}^{(2)} &= g^{\mu\gamma} R_{\mu\nu,\gamma\lambda} = -R_{\nu\lambda}^{(2)}.
\end{align}

(3.10)\quad (3.11)

Antisymmetrical properties of (3.11) connected with the fact that it is second trace of (3.3) with three antisymmetric pairs of indices.

\[
R_{\nu\lambda}^{(2)} = g^{\alpha\beta} g^{\mu\gamma} R_{\alpha\mu,\beta\nu,\gamma\lambda}.
\]

(3.12)

Moreover the Bianchi identity (3.9) relates this two second traces:

\[
R_{\gamma\lambda}^{(1)} = R_{\gamma\lambda}^{(2)} - R_{\lambda\gamma}^{(2)} = 2R_{\gamma\lambda}^{(2)}.
\]

(3.13)

To substruct traces from spin three curvatre we introduce analog of gravitational Schouten tensor in spin three case (d=4):

\[
K_{\mu\nu,\gamma\lambda} = \frac{1}{4} \left[ R_{\mu\nu,\gamma\lambda} - \frac{1}{10} \left( g_{\mu\nu} R_{\gamma\lambda}^{(1)} + g_{\gamma\lambda} R_{\nu\lambda}^{(2)} \right) \right],
\]

(3.14)
with the same symmetry properties as (3.8), (3.9) and define wanted spin three Weyl tensor in the form

$$W_{\alpha \mu, \beta \nu, \gamma \lambda} = R_{\alpha \mu, \beta \nu, \gamma \lambda} - g_{[\alpha \{ \beta K_{\mu \nu} \} \gamma \lambda] - g_{[\alpha \{ \gamma K_{\mu \lambda} \} ; \beta \nu} - g_{[\gamma \{ \beta K_{\lambda \nu} \} ; \alpha \mu}, \quad (3.15)$$

$$\delta W_{\alpha \mu, \beta \nu, \gamma \lambda} = 4\sigma(x) W_{\alpha \mu, \beta \nu, \gamma \lambda}. \quad (3.16)$$

In the full analogy with spin 2 case our spin 3 Weyl tensor is the same weight primary as the spin 3 gauge field but constructed from third covariant derivatives of it. Squaring this sixth rank and +4 weight primary tensor and contracting with 6 metric tensors $g^{\mu \nu}$ of weight $-2$, we obtain first scalar primary Lagrangian with weight $-4$ relevant for Weyl invariant action:

$$L_{W^2}^{4} = W_{\alpha \mu, \beta \nu, \gamma \lambda} W_{\alpha \mu, \beta \nu, \gamma \lambda}. \quad (3.17)$$

And we know from spin 2 consideration that it is not enough for gauge invariance. This action is not gauge invariant in general background: non invariance here is proportional to the curvature of the background and arose from the commutators of derivatives coming from definition of curvature with covariant derivative from gauge transformation of spin three field:

$$\delta_{\epsilon} h_{\mu \nu \lambda} = \nabla_{\mu} \epsilon_{\nu \lambda} + \nabla_{\nu} \epsilon_{\lambda \mu} + \nabla_{\lambda} \epsilon_{\mu \nu}. \quad (3.18)$$

Therefore the expected solution of this problem is in the possible existence of other Weyl invariant primaries (but not gauge invariant) with four or two covariant derivatives and first or second order on background curvature. Then the resulting combination of Weyl invariant actions can compensate non invariance of (3.17) supplemented with some reasonable restriction on background metric $g^{\mu \nu}$. To make the next step in this direction, we start looking at Weyl noninvariant part of spin three curvature written in terms of spin three Schouten tensor (3.14) and understand that the most interesting property of this tensor is it’s Weyl transformation:

$$\delta K_{\mu \nu, \gamma \lambda} = 2\sigma K_{\mu \nu, \gamma \lambda} + \frac{1}{3} \sigma^{\tau} \Gamma^{(2)}_{\tau \gamma \lambda \mu \nu} - \frac{1}{12} g_{[\gamma (\mu} \Gamma^{(2)}_{\tau \alpha \nu \lambda] \alpha} \sigma^{\tau} + \frac{1}{24} g_{[\gamma (\mu} \Gamma^{(2)}_{\tau \lambda \nu \alpha] \alpha} \sigma^{\tau} + \frac{1}{24} g_{[\gamma (\mu} \Gamma^{(2)}_{\tau \lambda \nu \alpha] \alpha} \sigma^{\tau} \quad (3.19)$$

We see that this object transforms through the so called second spin three Christoffel symbol $\Gamma^{(2)}_{\tau \gamma \lambda \mu \nu}$ and it’s traces.

So we should turn to the spin three analog of Christoffel symbols so called Freedman-deWitt hierarchy of Christoffel symbols for higher spin case [41]. The hierarchy means that in this spin three case we have in addition to spin three curvature two other important
objects: first and second Christoffel symbols with one and two covariant derivatives. First one is defined in unique way in general background:

$$\Gamma^{(1)}_{\gamma;\mu\nu\lambda} = \nabla_\gamma h_{\mu\nu\lambda} - \nabla_{\{\mu} h_{\nu\lambda\}}$$,

(3.20)

with the following Weyl variation:

$$\delta \Gamma^{(1)}_{\gamma;\mu\nu\lambda} = 4\sigma \Gamma^{(1)}_{\gamma;\mu\nu\lambda} + 4\sigma_\gamma h_{\mu\nu\lambda} - 2g_{\{\mu} h_{\nu\lambda\}}\sigma^\tau.$$

(3.21)

The second Christoffel symbol we can define through the first Christoffel symbol and curvature corrections in a way:

$$\Gamma^{(2)}_{\beta\gamma;\mu\nu\lambda} = \nabla_{\{\beta} \Gamma^{(1)}_{\gamma;\mu\nu\lambda} - \frac{1}{2} \nabla_{\mu} \Gamma^{(1)}_{\langle\beta\gamma\rangle;\nu\lambda} - 8K_{\beta\gamma} h_{\mu\nu\lambda} + 2K_{(\mu<\beta} h_{\gamma)\nu\lambda)} + 2g_{\{\mu} h_{\lambda\}} h_{\tau\beta\gamma} - g_{(\langle\mu(K_{\gamma} h_{\lambda)}} h_{\tau\beta\gamma} - 2g_{(\mu<\beta} K_{\gamma)\nu\lambda} + 2g_{(\mu\nu(\langle\tau} h_{\lambda)\beta\gamma}.$$

(3.22)

In this case Weyl transformation of (3.22) could be obtained only after enough long but straightforward calculations So we arrive to the following transformation rule:

$$\delta \Gamma^{(2)}_{\beta\gamma;\mu\nu\lambda} = 4\sigma \Gamma^{(2)}_{\beta\gamma;\mu\nu\lambda} + 3\sigma_\tau \Gamma^{(1)}_{\beta\gamma;\mu\nu\lambda} + 2g_{\{\mu} h_{\lambda\}} h_{\tau\beta\gamma} - g_{(\langle\mu(K_{\gamma} h_{\lambda)}} h_{\tau\beta\gamma} - 2g_{(\mu<\beta} K_{\gamma)\nu\lambda} + 2g_{(\mu\nu(\langle\tau} h_{\lambda)\beta\gamma}.$$

(3.23)

Comparing (3.23) with (2.17) we see that our second Christoffel symbol (last in hierarchy) could play a role of usual gravitational one and participate in construction of spin 3 generalization of the invariant (2.18).

In principal we have now all necessary ingredients to construct a spin three analog of invariant (2.18). The obstacle to construct it is only technical. But before we can easily construct analog of more simple primary (2.14) where linearized Weyl tensor contracted with background one and multiplied by trace of graviton field to get proper conformal weight.

It is clear that to get scalar combination from six and four rank traceless tensors we need some antisymmetric second rank tensor constructed from spin three field and one

5Note that the last term of (3.23) can be also written through the first Christoffel symbol in some sophisticated way:

$$6g_{(\mu\nu(\langle\tau} h_{\lambda)\beta\gamma} \sigma^\tau = \frac{3}{2} g_{(\mu\nu(\langle\beta\gamma(\rangle} \langle\tau\text{; }\rangle h_{\lambda)} + \frac{3}{2} g_{(\mu\nu\sigma^\tau(1)} \Gamma^{(1)}_{\langle\beta\gamma\rangle;\nu} - \frac{3}{2} g_{(\mu\nu(\langle\beta\gamma)\nu} \sigma^\tau.$$
derivative. Looking at the traces of transformations of first Christoffel symbol we can define the following primary:

\[
L_{-4}^W = \frac{1}{4} \Gamma_{[\gamma,\lambda] \rho}^{(1)} W_{\alpha\mu,\beta\nu} \gamma^\lambda W^{\alpha\mu,\beta\nu} - h_\rho^\lambda \nabla^\gamma W_{\gamma,\lambda,\alpha\mu,\beta\nu} W^{\alpha\mu,\beta\nu},
\]

(3.24)

where we used relation

\[
\delta \nabla^\gamma W_{\gamma,\alpha\mu,\beta\nu} = 2 \sigma \nabla^\gamma W_{\gamma,\alpha\mu,\beta\nu} + 2 \sigma^\gamma W_{\gamma,\alpha\mu,\beta\nu}.
\]

(3.25)

To discover an analog of (2.18) we should change to more irreducible objects in respect to Weyl transformation. These are traceless parts of our spin three field and Christoffel symbols defined above:

\[
h^{T}_{\mu\nu\lambda} = h_{\mu\nu\lambda} - \frac{1}{6} g_{(\mu\nu} h_{\lambda)}^\alpha,
\]

(3.26)

\[
\Gamma_{\gamma;\mu\nu\lambda}^{(1)} = \Gamma_{\gamma;\mu\nu\lambda}^{(1)} - \frac{1}{6} \Gamma_{\gamma;\alpha\nu}^\lambda (\lambda g_{\mu\nu})
\]

(3.27)

\[
\Gamma_{(1);T}^{\gamma;\mu\nu\lambda} = \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} - \frac{1}{4} g_{[\gamma(\mu} \Gamma^{(1);T}_{\alpha\nu)]\lambda},
\]

(3.28)

\[
\Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)} = \Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)} - \frac{1}{6} g_{(\mu\nu} \Gamma^{(1);T}_{\beta;\alpha\nu)}^\lambda,
\]

(3.29)

\[
\Gamma_{(1);T}^{\gamma;\mu\nu\lambda} = \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} - \frac{1}{4} g_{[\gamma(\mu} \Gamma^{(1);T}_{\beta;\alpha\nu)]\lambda},
\]

(3.30)

where \(\Gamma_{\gamma;\alpha\mu,\beta\nu}^{(1)}\) and \(\Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)}\) are traceless in both pair of symmetric and anti symmetric indices of (3.22). For full list of Weyl transformations of traces of first two Christoffel symbols and corresponding traceless parts of them we refer to Appendix C. Here we present only the most important variations of traceless field and Christoffel symbols widely used for integration of new nontrivial invariant:

\[
\delta h^{T}_{\mu\nu\lambda} = 4 \sigma h^{T}_{\mu\nu\lambda},
\]

(3.31)

\[
\delta \Gamma_{\gamma;\mu\nu\lambda}^{(1)} = 4 \sigma \Gamma_{\gamma;\mu\nu\lambda}^{(1)} + 4 \sigma \gamma h^{T}_{\mu\nu\lambda},
\]

(3.32)

\[
\delta \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} = 4 \sigma \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} + 4 \sigma \gamma h^{T}_{\mu\nu\lambda} - g_{(\gamma \mu} h^{T}_{\nu)]\lambda) \sigma^\tau,
\]

(3.33)

\[
\delta \Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)} = 4 \sigma \Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)} + 3 \sigma \beta \Gamma_{\gamma;\lambda;\mu\nu}^{(1);T} + 3 \sigma_{\gamma} \Gamma_{\beta;\gamma;\lambda;\mu\nu}^{(2)} + 3 \sigma_{\beta} \Gamma_{\gamma;\lambda;\mu\nu}^{(2)} + 3 \sigma_{\tau} \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} + 3 \sigma_{\tau} \Gamma_{(1);T}^{\gamma;\mu\nu\lambda} - \frac{1}{4} g_{[\gamma(\mu} \gamma_{\nu)]\lambda\beta},
\]

(3.34)

But index \(\beta\) of second Christoffel is out of this game.
\[ \delta \Gamma_{\mu \nu \lambda}^T = 2 \sigma \Gamma_{\mu \nu \lambda}^T + 8 \sigma \Gamma_{\tau \mu \nu \lambda}^{(1)T}, \]  

(3.35)

where we introduced notation

\[ \Gamma_{\mu \nu \lambda}^T = g^{\alpha \beta} \Gamma_{\alpha \beta ; \mu \nu \lambda}^{(2)T}. \]  

(3.36)

Then subtracting trace from spin three Schouten tensor \( K_{\mu \nu \gamma \lambda}^{(T)} \) we see from (3.19) that

\[ \delta K_{\mu \nu \gamma \lambda}^{(T)} = \frac{1}{3} \sigma \beta \Gamma_{\beta [\gamma ; \lambda] \mu \nu}^{(2)T}. \]  

(3.37)

So we see that traceless parts of some field, Christoffel symbols and spin three Schouten tensor

\[ h_{\mu \nu \lambda}^{(T)}; \Gamma_{\gamma ; \mu \nu \lambda}^{(1)T}; \Gamma_{\beta [\gamma ; \lambda] \mu \nu}^{(2)T}; \Gamma_{\mu \nu \lambda}^{(T)}; K_{\mu \nu \gamma \lambda}^{(T)}. \]  

(3.38)

transforms in close and more or less simple way in respect to Weyl transformations (3.31)-(3.35) and we can use this property for construction of possible primaries. To illustrate this idea we should separate from (3.38) a subset of two terms with close Weyl variations \( \{ h_{\mu \nu \lambda}^{(T)}; \Gamma_{\gamma ; \mu \nu \lambda}^{(1)T} \} \) obtained from \( h_{\mu \nu \lambda}^{T} \) and one covariant derivative. After that we construct primary combination by coupling it with similar set formed from background Weyl and Cooton tensors \( \{ W_{\alpha \beta , \rho} ; C_{\alpha \beta , \rho} \} \). The most important point here is that the both sets start from primary field and end with nonprimary field constructed from previous one and from one covariant derivative. Then the cross coupled combination of them

\[ W_{\alpha \beta , \rho} \Gamma_{\gamma , \mu \nu \lambda}^{(1)T} = C_{\alpha \beta , \rho} h_{\mu \nu \lambda}^{T} \]  

(3.39)

is the primary tensor with weight four. In the same spirit we can introduce another primary tensor with two set of three symmetrized indices:

\[ T_{\alpha \beta \gamma}^{\mu \nu \lambda} = \Gamma_{\tau , \mu \nu \lambda}^{(1)T} \Gamma_{\alpha \beta \gamma}^{(1)T} - \frac{1}{2} ( h_{\mu \nu \lambda}^{T} \Gamma_{\alpha \beta \gamma}^{(2)T} + h_{\mu \nu \lambda}^{T} \Gamma_{\alpha \beta \gamma}^{(1)T} ), \]  

(3.40)

\[ T_{\alpha \beta}^{\mu \nu} = T_{\alpha \beta \lambda}^{\mu \nu}, \quad T_{\alpha}^{\mu \nu} = T_{\alpha \nu}^{\mu \nu}. \]  

(3.41)

This weight zero primary (3.40) and corresponding first traces (3.41) we incorporate with our whole set (3.38) to construct the most nontrivial primary starting from background weyl tensor and square of second Christoffel tensor. This primary is playing role of analog of (2.18) in spin three case. The ideology of construction we described above and

\[ \text{Note that } \Gamma_{\mu \nu \lambda}^{T} \text{ coincides in flat background with traceless part of Fronsdal equation and therefore it is gauge invariant in zero order on curvature.} \]
illustrated with some simple cases. But the proof is so long and complicated that we put it in Appendix C, presenting here only the final result for this primary:

\[ L_{-4}^{\text{WTT}} = \frac{2}{3} W_{\mu \tau \rho \nu} \Gamma^{(2)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} + \frac{22}{9} W_{\gamma \mu \rho \tau} \Gamma^{(2)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} - \frac{1}{6} W_{\gamma \lambda \rho \tau} \Gamma^{(2)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} \]

\[ - \left[ \nabla_\gamma W_{\mu \nu \rho \tau} - 8 \nabla_\mu W_{\nu \gamma \rho \tau} + 6 C_{\mu \rho}^{\gamma \delta} \Gamma^{(2)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} \right] \left( \frac{4}{3} \Gamma^{(1)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} - \frac{1}{2} \Gamma^{(1)T,T}_{\lambda \tau \rho} T_{\lambda}^{(\gamma,\lambda)\mu \nu} - 16 h^{(T)}_{\lambda \tau \rho} K^{(\gamma,\lambda)}_{\mu \nu} \right) \]

\[ - \left[ 12 W_{\mu \tau \nu \rho} \Gamma^{(1)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} + 44 W_{\gamma \mu \tau \rho} \Gamma^{(1)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} - 3 W_{\gamma \lambda \tau \rho} \Gamma^{(1)T,T}_{\beta(\gamma,\lambda)} T_{\lambda}^{\beta(\gamma,\lambda)\mu \nu} \right] K^{(\mu \nu,\gamma \lambda)}_{\lambda \tau \rho} \]

\[ - 2 \left[ (\nabla^\sigma \nabla_\rho + 4 K^\sigma \nabla_{\alpha \beta}) W_{\alpha \beta}^{\mu \nu} T_{\mu \nu}^{\alpha \beta} + \left[ 4 K^{\mu \tau} W_{\alpha \tau \beta}^{\mu \nu} - 3 (\square + 2 J) W_{\alpha \beta}^{\mu \nu} \right] T_{\mu \nu}^{\alpha \beta} \right] \]

where we introduced new notation:

\[ \tilde{\Gamma}^{\beta(\gamma,\lambda)\mu \nu}_{(2)T,T} = \Gamma^{\beta(\gamma,\lambda)\mu \nu}_{(2)T,T} - \frac{3}{8} \left( g^{\beta \gamma} \Gamma^{\lambda \mu \nu}_{T} - \frac{1}{4} g^{\gamma \lambda} \Gamma^{\mu \nu \beta}_{T} \right) \]

shifting second Christoffel symbol by gauge invariant in zero order on background curvature terms. This modified Christoffel symbol transforms without third line in (3.34).

4 On Gauge Invariant Action for Conformal Spin Three

In this section we address the final issue which is the construction of a Weyl and gauge invariant action for a spin 3 field in a background gravitational field. Following the prescription of [40] we can use approach described here for spin two. In other words we try to construct gauge invariant combination of Weyl invariant actions obtained from primary Lagrangians of the previous section. We should start from gauge variation

\[ \delta \epsilon h_{\mu \nu \lambda} = \nabla_\mu \epsilon_{\nu \lambda} + \nabla_\nu \epsilon_{\lambda \mu} + \nabla_\lambda \epsilon_{\mu \nu} \]

of action

\[ S_{W^2} = \int d^4 x \sqrt{g} L_{-4}^{W^2} \]

It is worth to note that contracting second trace of primary (3.41) with background Bach tensor we quickly construct another Weyl invariant action:

\[ S_{B}^{W} = \frac{1}{2} \int d^4 x \sqrt{g} B^\alpha \beta \left[ g^{\rho \sigma} \Gamma^{(1)T,T}_{\mu \alpha \rho} \Gamma^{(1)T,T}_{\nu \beta \sigma} - h_{\mu \beta}^{T,T} T_{\rho \sigma}^{\alpha \beta} \right] \]
constructed from square of spin three linearized Weyl tensor \((3.17)\) and try to cancel this using partial integration with the variation of action from second nontrivial invariant action \((3.42)\)

\[
S_{WTT} = \int d^4x \sqrt{g} L_{-4}^{W^2},
\]

(4.3)

integrating remaining terms to other possible Weyl invariant part of action constructed from \((3.24)\)

\[
S_{WW} = \int d^4x \sqrt{g} L_{-4}^{WW}.
\]

(4.4)

In contrast to Weyl invariance, this symmetry holds between Weyl primaries only up to total derivatives. We therefore use actions instead of primary lagrangians.

The full check of gauge invariance is very tedious, and an analytical treatment to all orders in the background curvature is out of reach. Instead with some experience we can guess the right combination and then check with the help of the computer. For this we used the Mathematica package xAct. Following the spin two case, presented in the Appendix B, we expect that a particular linear combination of the three Weyl invariant terms found in \((4.2)\), \((4.3)\) and \((4.4)\), might be gauge invariant, at least to first order in background curvature. We showed this, with the help of the computer, with the further assumption of a Ricci flat background. This implies that Schouten and Bach tensors vanish and we therefore work to linear order in the Weyl tensor of the background geometry. Even this turned out to be a formidable task with the result

\[
\delta \epsilon \left[ S_{W^2} - \frac{8}{5} S_{WTT} + \frac{4}{3} S_{WW} \right] = 0 + O(R^2, K_{\alpha \beta}, B)
\]

(4.5)

The Lagrangians \(S_{W^2}, S_{WTT}\) and \(S_{WW}\) were given in \((3.17), (3.42)\) and \((3.24)\). We choose this particular framework for two reasons: first it significantly shortens the computing time and second it gives the possibility to express the terms in a way where the variation problem could be reduced to the problem of solving a system of linear equations. Another long check of relation \((4.5)\) leads us to results that the parameter \(\epsilon_{\mu \nu}\) in transformation \((4.1)\) could be non traceless as it should be in Fronsdal theory. The last important relation is the following: The action we proposed in \((4.5)\)

\[
S_{CGI} = S_{W^2} - \frac{8}{5} S_{WTT} + \frac{4}{3} S_{WW}
\]

(4.6)

is not only conformal and gauge invariant in first order on background Weyl tensor, but invariant also in respect to spin 3 Weyl transformation (shifting of trace):

\[
\delta_{\alpha} h_{\mu \nu \lambda} = g_{\mu \nu} \alpha_{\lambda} + g_{\nu \lambda} \alpha_{\mu} + g_{\lambda \mu} \alpha_{\nu}
\]

(4.7)
5 Conclusion and outlook

In this paper we investigated the structure of Weyl covariant primaries in $d = 4$. This primaries are relevant for using as a Weyl invariant Lagrangians, expressed through the corresponding members of hierarchy of generalized Christoffel symbols [41] and Weyl tensor for linearized spin 3 gauge field in general gravitational background. The main result is that in addition to the linearized spin 3 Weyl tensor corrected with background curvature terms we can construct additional nontrivial Weyl primary in full analogy with spin 2 case. This primary is linear in background Weyl tensor and quadratic in linearized second Christoffel symbol. This Christoffel symbol is the last before curvature in corresponding hierarchy for spin 3 case [41]. A possible combination of these primaries, in principal, can be interpreted as a gauge and Weyl invariant action with corresponding restriction on background geometry. This could be investigated in the future. Here we only briefly discuss the possible combination of these invariants in linear on background Weyl tensor approximation using computer calculations. Also it is reasonable in the future obtain connections with the results of [29] where authors using another methods claim that for gauge invariance in the second order on background curvature we should introduce interaction with additional spin one field. Unfortunately by technical reason, at the moment, we can make some checking of gauge invariance using computer only in first order on background Weyl tensor. Another reasonable task here to obtain better understanding of connections with the results of [39] and [30].

Acknowledgements:

RM is grateful to Stefan Theisen and Sergei Kuzenko for many valuable discussions, useful conversations and support. RM also acknowledges hospitality of Albert Einstein Institute in Potsdam-Golm when this project was initiated. The work of RM was supported in part by the Alexander von Humboldt Foundation and by the Science Committee of the Ministry of Science and Education of the Republic of Armenia under contract 15T-1C233.
6 Appendix A. Notations and Conventions

We work in a $d = 4$ dimensional curved space with general metric $g_{\mu\nu}$ and use the following conventions for covariant derivatives and curvatures:

$$\nabla_{\mu} V^\rho_{\lambda} = \partial_{\mu} V^\rho_{\lambda} + \Gamma^\rho_{\mu\sigma} V^\sigma_{\lambda} - \Gamma^\sigma_{\mu\lambda} V^\rho_{\sigma}, \quad (A.1)$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_{\mu} g_{\nu\lambda} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu}), \quad (A.2)$$

$$[\nabla_{\mu}, \nabla_{\nu}] V^\rho_{\lambda} = R^\rho_{\mu\nu,\sigma} V^\sigma_{\lambda} - R^\sigma_{\mu\nu,\lambda} V^\rho_{\sigma}, \quad (A.3)$$

$$R^\rho_{\mu\nu,\lambda} = \partial_{\mu} \Gamma^\rho_{\nu\lambda} - \partial_{\nu} \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} , \quad (A.4)$$

$$R^\mu_{\lambda} = R^\rho_{\mu\rho\lambda} , \quad R = R^\mu_{\mu} . \quad (A.5)$$

The corresponding local conformal transformations (Weyl rescalings)

$$\delta g_{\mu\nu} = 2\sigma(x) g_{\mu\nu} , \quad \delta g^{\mu\nu} = -2\sigma(x) g^{\mu\nu} , \quad (A.6)$$

$$\delta \Gamma^\lambda_{\mu\nu} = \partial_{\mu} \sigma \delta^\lambda_{\nu} + \partial_{\nu} \sigma \delta^\lambda_{\mu} - g_{\mu\nu} \partial^\lambda \sigma , \quad (A.7)$$

$$\delta R^\rho_{\mu\nu,\lambda} = \nabla_{\mu} \partial_{\nu} \sigma \delta^\rho_{\lambda} - \nabla_{\nu} \partial_{\lambda} \sigma \delta^\rho_{\mu} + g_{\mu\lambda} \nabla_{\nu} \partial^\rho \sigma - g_{\nu\lambda} \nabla_{\mu} \partial^\rho \sigma , \quad (A.8)$$

$$\delta R^\mu_{\lambda} = (d - 2) \nabla_{\mu} \partial_{\lambda} \sigma + g_{\mu\lambda} \Box \sigma , \quad \delta R = -2\sigma R + 2(d - 1) \Box \sigma , \quad (A.9)$$

are first order in the infinitesimal local scaling parameter $\sigma$.

We then introduce the Weyl ($W$) and Schouten ($K$) tensors, as well as the scalar $J$

$$R_{\mu\nu} = (d - 2) K_{\mu\nu} + g_{\mu\nu} J , \quad J = \frac{1}{2(d - 1)} R , \quad (A.11)$$

$$W^\rho_{\mu\nu,\lambda} = R^\rho_{\mu\nu,\lambda} - K_{\mu\lambda} \delta^\rho_{\nu} + K_{\nu\lambda} \delta^\rho_{\mu} - K^\rho_{\nu} g_{\mu\lambda} + K^\rho_{\mu} g_{\nu\lambda} , \quad (A.12)$$

$$\delta K_{\mu\nu} = \nabla_{\mu} \partial_{\nu} \sigma , \quad \delta J = -2\sigma J + \Box \sigma , \quad (A.13)$$

$$\delta W^\rho_{\mu\nu,\lambda} = 0 , \quad (A.14)$$

which are more convenient because their conformal transformations are ”diagonal”.

To describe the Bianchi identity with these tensors, we have to introduce the so called Cotton tensor

$$C_{\mu\nu,\lambda} = \nabla_{\mu} K_{\nu\lambda} - \nabla_{\nu} K_{\mu\lambda} , \quad (A.16)$$

$$\delta C_{\mu\nu,\lambda} = -\partial_\alpha \sigma W_{\mu\nu,\lambda}^\alpha , \quad C_{\mu\nu,\lambda}|_{\alpha=0} = 0 . \quad (A.17)$$
All important properties of these tensors following from the Bianchi identity can then be listed as

\[ \nabla_{[\alpha} W_{\mu\nu],\lambda}^\rho = g_{\lambda[\alpha} C_{\mu\nu]}^\rho - \delta_{\lambda[\alpha}^\rho C_{\mu\nu],\lambda}, \quad \text{(A.18)} \]
\[ \nabla_{\alpha} W_{\mu\nu,\lambda}^\alpha = (3 - d) C_{\mu\nu,\lambda}, \quad \text{(A.19)} \]
\[ \nabla^\mu K_{\mu\nu} = \partial_{\nu} J, \quad \text{(A.20)} \]
\[ C_{\mu\nu,\nu}^\alpha = 0, \quad \nabla^\lambda C_{\mu\nu,\lambda} = 0. \quad \text{(A.21)} \]

Finally we introduce the last important conformal object in the above listed hierarchy, namely the symmetric and traceless Bach tensor

\[ B_{\mu\nu} = \nabla^\lambda C_{\lambda\mu,\nu} + K_{\alpha}^\lambda W_{\lambda\mu,\nu}^\alpha, \quad \text{(A.22)} \]

whose conformal transformation and divergence are expressed in terms of the Cotton and the Schouten tensors as follows

\[ \delta B_{\mu\nu} = -2\sigma B_{\mu\nu} + (d - 4) \nabla^\lambda \sigma (C_{\lambda\mu,\nu} + C_{\lambda\nu,\mu}), \quad \text{(A.23)} \]
\[ \nabla^\mu B_{\mu\nu} = (d - 4) C_{\alpha\nu,\beta} K_{\alpha\beta}. \quad \text{(A.24)} \]

Note that only in four dimensions the Bach tensor is conformal invariant and divergenceless. We use also other dimension dependent \((d=4)\) relation for any traceless tensor with symmetries of Weyl tensor:

\[ W_{[\alpha}^{[\mu \beta \gamma]} = 0. \quad \text{(A.25)} \]

From last relation we can derive the following important identities:

\[ W^{\alpha\mu,\beta\rho} W_{\alpha\mu,\beta\lambda} = \frac{1}{4} \delta^\rho_\lambda W^{\alpha\mu,\beta\nu} W_{\alpha\mu,\beta\nu}, \quad \text{(A.26)} \]
\[ \mathcal{W}_{\alpha\mu,\beta}^{\alpha\mu,\beta}(\rho) = \frac{1}{2} \gamma^\rho_\lambda W_{\alpha\mu,\beta}^{\alpha\mu,\beta} W_{\alpha\mu,\beta\nu}, \quad \text{(A.27)} \]
\[ \frac{1}{4} W^{\alpha\mu,\beta\nu} W_{\alpha\mu,\beta\nu}(h_\rho h_\tau - h_\rho h_\tau) = 2h_\rho h_\lambda h_\nu W_{\rho,\lambda}^{\alpha\beta} W_{\alpha\mu,\beta\nu} + h_\rho h_\lambda h_\nu W_{\rho,\lambda}^{\alpha\mu} W_{\alpha\mu,\beta\nu} - 2W_{\rho,\lambda}^{\alpha\beta} h_\rho h_\lambda W_{\alpha,\nu}^{\alpha\beta} h_\mu h_\nu. \quad \text{(A.28)} \]

## 7 Appendix B. Spin Two Details

To prove (2.22) we start from gauge variation (2.20) of the initial object

\[ R_{\alpha\mu,\beta\nu} = \frac{1}{4} \{ \nabla_{[\alpha}, \nabla_{\beta]} h_{\rho\nu]} \}. \quad \text{(B.1)} \]
This can be rewritten in the following form
\[
\delta R_{\alpha\mu,\beta\nu} = -\frac{1}{4}[R_{\alpha\mu,[\beta} \tau^\delta h_{\nu]\tau} + R_{\beta\nu,[\alpha} \tau^\delta h_{\mu]\tau}] + \mathcal{L}_\epsilon R_{\alpha\mu,\beta\nu}, \tag{B.2}
\]
where \( R_{\alpha\mu,\beta\nu} \) is background curvature and \( \mathcal{L}_\epsilon \) is Lie derivative in direction of gauge parameter vector \( \epsilon^\tau \):
\[
\mathcal{L}_\epsilon R_{\alpha\mu,\beta\nu} = \epsilon^\tau \nabla_\tau R_{\alpha\mu,\beta\nu} + R_{\alpha\mu,\tau[\nu} \nabla_\beta] \epsilon^\tau + R_{\beta\nu,\tau[\mu} \nabla_\alpha] \epsilon^\tau. \tag{B.3}
\]
So we see that the following improved linearized curvature
\[
\bar{\delta} R_{\alpha\mu,\beta\nu} = \frac{1}{4}[\nabla_{[\alpha,} \nabla_{\beta]} h_{\mu]\nu]} + R_{\alpha\mu,\tau[\nu} \tau^\delta h_{\mu]\tau} + R_{\beta\nu,\tau[\mu} \tau^\delta h_{\mu]\tau} \tag{B.4}
\]
transforms covariantly:
\[
\delta \bar{R}_{\alpha\mu,\beta\nu} = \mathcal{L}_\epsilon \bar{R}_{\alpha\mu,\beta\nu}. \tag{B.5}
\]
Then after some calculations we can see that (B.4) can be rewritten using linearized Christoffel symbols (2.5) in the form:
\[
\bar{R}_{\alpha\mu,\beta\nu} = \nabla_{[\alpha} G_{\beta]}^\tau g_{\mu\nu} + R_{\alpha\mu,\tau[\nu} \tau^\delta h_{\mu]\tau} \tag{B.6}
\]
and coincides with the linearized expansion of usual nonlinear curvature for general metric (2.25).

Expanding background curvatures on Weyl and Schouten tensors we obtain the following relations:
\[
\bar{R}_{\alpha\mu,\beta\nu} = R_{\alpha\mu,\beta\nu} + h_{[\alpha,} \tau^\delta K_{\beta]\nu]} + \frac{1}{2}[W_{\alpha\mu,[\beta} \tau^\delta h_{\nu]\tau} + W_{\beta\nu,[\alpha} \tau^\delta h_{\mu]\tau}], \tag{B.7}
\]
\[
\delta \bar{R}_{\alpha\mu,\beta\nu} = \mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} + g_{[\alpha,} \tau^\delta \mathcal{L}_\epsilon K_{\beta]\nu]} - \frac{1}{4}[W_{\alpha\mu,[\beta} \tau^\delta \delta h_{\nu]\tau} + W_{\beta\nu,[\alpha} \tau^\delta \delta h_{\mu]\tau}], \tag{B.8}
\]
where \( \mathcal{R} \) here is the same as in (2.3)
\[
R_{\alpha\mu,\beta\nu} = \frac{1}{4}[\nabla_{[\alpha,} \nabla_{\beta]} h_{\mu]\nu]} - 2K_{[\alpha,} h_{\beta]\nu]} - \tau^\delta K_{[\alpha} g_{\beta]} h_{\mu]\nu]} + \tau^\delta K_{[\alpha} g_{\beta]} h_{\mu]\nu]}, \tag{B.9}
\]
Then taking traces from (B.8) we arrive to the following variations
\[
\bar{\delta} R_{\mu\nu} = g^{\alpha\beta} \bar{\delta} R_{\alpha\mu,\beta\nu} = 2\mathcal{L}_\epsilon K_{\mu\nu} - \frac{1}{2}W_{\alpha\mu,\beta\nu} \delta h^{\alpha\beta} + g_{\mu\nu}(\delta h_{\alpha\beta} K_{\alpha\beta} + \mathcal{L}_\epsilon \mathcal{J}), \tag{B.10}
\]
\[
\bar{\delta} \mathcal{R} = g^{\mu\nu} \bar{\delta} R_{\mu\nu} = 6\bar{\delta} \mathcal{J} = 6(\delta h_{\alpha\beta} K_{\alpha\beta} + \mathcal{L}_\epsilon \mathcal{J}), \tag{B.11}
\]
\[
\bar{\delta} K_{\mu\nu} = \frac{1}{2} \bar{\delta}(R_{\mu\nu} - g_{\mu\nu} \mathcal{J}) = \mathcal{L}_\epsilon K_{\mu\nu} - \frac{1}{4}W_{\alpha\mu,\beta\nu} \delta h_{\alpha\beta}, \tag{B.12}
\]
we obtain for linearized Weyl tensor (2.8)

\[\delta W_{\alpha\mu,\beta\nu} = L_\epsilon W_{\alpha\mu,\beta\nu} - \frac{1}{4} [W_{\alpha\mu,\beta\nu} \tau \delta h_{\rho\tau} + W_{\beta\nu,\alpha\tau} \tau \delta h_{\rho\tau} - g_{[\alpha} W_{\mu,\nu]}^\rho \tau \delta h_{\rho\tau}] \]  

(B.13)

Comparing (2.3), (2.8) and (B.6), (B.7) we see that linearized curvature obtained from gauge consideration connected with Weyl invariant traceless linearized Weyl tensor in the following way:

\[\tilde{\mathcal{R}}_{\alpha\mu,\beta\nu} = W_{\alpha\mu,\beta\nu} + g_{[\alpha} K_{\mu,\nu]}^\beta + h_{[\alpha} K_{\mu,\nu]}^\beta + \frac{1}{2} [W_{\alpha\mu,\beta\tau} \tau h_{\nu\tau} + W_{\beta\nu,\alpha\tau} \tau h_{\mu\tau}] \]  

(B.14)

This linearized curvature cares all symmetry properties of nonlinear Riemann curvature and satisfies also to usual first Bianchi identity. The second (differential) Bianchi identity for linearized curvature looks like

\[\nabla_\tau \tilde{\mathcal{R}}_{\alpha\mu,\beta\nu} - \mathcal{G}_\beta^\rho \mathcal{R}_{\alpha\mu,\beta\nu} - \mathcal{G}_\mu^\rho \mathcal{R}_{\alpha\mu,\beta\nu} = 0.\]  

(B.15)

Using this and partial integration we can derive another important relation

\[\frac{1}{2} \int d^4 x \sqrt{g} W^{\alpha\mu,\beta\nu} \tilde{\mathcal{R}}_{\alpha\mu,\beta\nu} \nabla_\nu \epsilon_\tau = \frac{1}{4} \int d^4 x \sqrt{g} [W^{\alpha\mu,\beta\nu} \mathcal{L}_\epsilon \tilde{\mathcal{R}}_{\alpha\mu,\beta\nu} + W^{\alpha\mu,\beta\nu} \mathcal{G}_\beta^\rho \mathcal{R}_{\rho\nu,\alpha\tau} \epsilon_\tau - \frac{1}{2} W^{\alpha\mu,\beta\nu} \mathcal{R}_{\rho\nu,\alpha\tau} \mathcal{G}_\beta^\rho \epsilon_\tau].\]  

(B.16)

Now we are good prepared to construct gauge and Weyl invariant action for spin 2 case. For that we should calculate gauge variation of Weyl invariant Lagrangian (2.18). Using (B.12) and

\[\tilde{\delta} \mathcal{G}_{\tau,\mu\nu} = \nabla_\mu \nabla_\nu \epsilon_\tau - \mathcal{R}_{\nu\tau,\rho\mu} \epsilon_\rho,\]  

(B.17)

we obtain

\[\tilde{\delta} S_G = \int d^4 x \sqrt{g} W^{\alpha\mu,\beta\nu} \left( \mathcal{G}_\alpha^\tau (\nabla_\mu \nabla_\nu \epsilon_\tau - \mathcal{R}_{\nu\tau,\rho\mu} \epsilon_\rho) - \delta h_{\alpha\beta} K_{\mu\nu} - h_{\alpha\beta} \mathcal{L}_\epsilon K_{\mu\nu} - \frac{1}{4} h_{\alpha\beta} W_{\tau\mu,\rho\nu} h^{\rho\tau} \right),\]  

(B.18)

where

\[S_G = \int d^4 x \sqrt{g} \mathcal{L}_{\epsilon} W^{\alpha\mu,\beta\nu} \sqrt{g}.\]  

(B.19)

Then doing partial integration and using relations (B.6) and (B.16) we arrive to the following intermediate formula:

\[\tilde{\delta} S_G = \int d^4 x \sqrt{g} \left( -\frac{1}{2} B^\alpha_{\beta\gamma} \mathcal{L}_\epsilon h_{\alpha\beta} - \frac{1}{16} W^{\alpha\mu,\beta\nu} W_{\alpha\mu,\beta\nu} (\mathcal{L}_\epsilon h_{\tau} + \frac{1}{2} \delta [h_{\rho\tau} h^{\rho\tau}]) \right)\]
\[-\frac{1}{8}\delta(W^{\alpha\beta\nu}\epsilon_{\mu
u}W_{\alpha\beta\tau\rho}h^{\rho\tau}) + \frac{1}{4}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} + \frac{1}{4}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon [W_{\alpha\mu,\beta\tau}h^\tau_{\nu}]\). \quad (B.20)

To proceed more we should widely use relations:

\[\bar{\delta}h_{\alpha\beta} = \mathcal{L}_\epsilon g_{\alpha\beta}, \quad (B.21)\]
\[\bar{\delta}h^{\alpha\beta} = -\mathcal{L}_\epsilon g^{\alpha\beta}, \quad (B.22)\]

and (A.26)-(A.28). Using these we arrive to the following variation:

\[\bar{\delta}\left(S_G + \frac{1}{32}\int d^4x\sqrt{g}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}[h^{\rho\tau} - \frac{1}{4}h^\rho_{\rho}h^\tau_{\tau}]\right) = \int d^4x\sqrt{g}\left\{-\frac{1}{2}B^{\alpha\beta}\mathcal{L}_\epsilon h_{\alpha\beta} - \frac{1}{32}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon h^\rho_{\rho}\right\}. \quad (B.23)\]

Then investigating gauge variation of another Weyl invariant action (2.13) we derive:

\[\frac{1}{4}\bar{\delta}\left(S_W + \frac{1}{4}\int d^4x\sqrt{g}\left\{W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^\rho_{\rho} + \frac{1}{8}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^{\rho\tau}h^\tau_{\tau}\right\}\right) = \int d^4x\sqrt{g}\left\{\frac{1}{4}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} + \frac{1}{16}h^\rho_{\rho}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu}\right\}, \quad (B.24)\]

where

\[S_W = \int d^4x\sqrt{g}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}. \quad \text{(B.25)}\]

Summing (B.23) and (B.24) we obtain

\[\bar{\delta}\left(S_G + \frac{1}{4}S_W + \frac{1}{16}\int d^4x\sqrt{g}\left\{W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^\rho_{\rho} + \frac{1}{2}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon h^{\rho\tau}h^\tau_{\tau}\right\}\right) = \int d^4x\sqrt{g}\left\{\frac{1}{4}(W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} + W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu}) + \frac{1}{16}h^\rho_{\rho}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} - \frac{1}{2}B^{\alpha\beta}\mathcal{L}_\epsilon h_{\alpha\beta}\right\}. \quad (B.26)\]

Finally investigating first three terms of r.h.s. of (B.26) we see due to (B.21)-(A.27) that

\[\int d^4x\sqrt{g}\left\{\frac{1}{4}(W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu} + W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu}) + \frac{1}{16}h^\rho_{\rho}W^{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon W_{\alpha\mu,\beta\nu}\right\} = \frac{1}{8}\bar{\delta}\int d^4x\sqrt{g}\left\{W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^\rho_{\rho} + \frac{1}{16}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^\rho_{\rho}h^\tau_{\tau}\right\} + \frac{1}{32}\int d^4x\sqrt{g}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}\mathcal{L}_\epsilon h^\rho_{\rho}. \quad \text{(B.27)}\]

Combining with (B.26) we see that nonintegrable terms with \(\mathcal{L}_\epsilon h^\rho_{\rho}\) cancel together and we arrive to the following final formula:

\[\bar{\delta}\left(S_G + \frac{1}{4}S_W - \frac{1}{16}\int d^4x\sqrt{g}\left\{W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}h^\rho_{\rho} + \frac{1}{2}W^{\alpha\mu,\beta\nu}W_{\alpha\mu,\beta\nu}[h^{\rho\tau}h^\tau_{\tau} - \frac{1}{4}h^\rho_{\rho}h^\tau_{\tau}]\right\}\right) \]
\[ = \int d^4x \sqrt{g} \left\{ - \frac{1}{2} B^{\alpha\beta} \mathcal{L}_{\alpha\beta} \right\}. \] (B.28)

Therefore we prove that in the background with zero Bach tensor gauge and Weyl invariant action is \( (2.23) \).

## 8 Appendix C. Spin Three Details

Here we present the Weyl transformations of first two Christoffel symbols and corresponding traceless parts of them. For first Christoffel symbol transformation looks like:

\[
(\delta - 4\sigma) \Gamma^{(1)}_{\gamma;\mu\nu} = 4\sigma \gamma_{\mu\nu\lambda} - 2g_{(\mu\nu} h_{\lambda)\gamma\tau} \sigma^\tau, \tag{C.1}
\]

\[
(\delta - 6\sigma) \Gamma^{(1)}_{\gamma;\mu} = 4\sigma \gamma_{\mu\lambda} - 12h_{\lambda\gamma\tau} \sigma^\tau, \tag{C.2}
\]

\[
(\delta - 6\sigma) \Gamma^{(1)}_{\gamma;\nu\lambda} = -2g_{\nu\lambda} h_{\mu\tau}^\mu \sigma^\tau, \tag{C.3}
\]

\[
(\delta - 4\sigma) \Gamma^{(1);T}_{\gamma;\mu\nu\lambda} = 4\sigma_{\mu} h_{\nu\lambda}^T, \tag{C.4}
\]

\[
(\delta - 6\sigma) \Gamma^{(1);T}_{\gamma;\nu\lambda} = 4h_{\nu\lambda}^T \sigma^\tau, \tag{C.5}
\]

then introducing notations

\[
t_{\mu\nu\lambda} = \sigma^\tau \Gamma^{(1)}_{\tau;\mu\nu\lambda}, \tag{C.6}
\]

\[
t_{\lambda} = t_{\mu\lambda}^\mu, \tag{C.7}
\]

\[
t_{\mu\nu\lambda}^T = t_{\mu\nu\lambda} - \frac{1}{6}g_{(\mu\nu} t_{\lambda)}, \tag{C.8}
\]

\[
\Gamma_{\mu\nu\lambda}^T = \Gamma_{\alpha;T^{(2)}\alpha\mu\nu\lambda}, \tag{C.9}
\]

we collect all formulas for second Christoffel symbol:

\[
(\delta - 4\sigma) \Gamma^{(2)}_{\beta;\gamma;\mu\nu\lambda} = 3\sigma_{(\beta} \Gamma^{(1)}_{\gamma;\mu\nu\lambda)}
\]

\[
+ 2g_{\beta\gamma} t_{\mu\nu\lambda} - g_{(\mu} h_{\nu\lambda)\gamma\tau} + 2g_{(\mu\nu} t_{\lambda)}, \tag{C.10}
\]

\[
(\delta - 2\sigma) \Gamma^{(2)}_{\beta;\gamma;\lambda\alpha} = 3\sigma_{(\beta} \Gamma^{(1)}_{\gamma;\lambda\alpha)}
\]

\[
+ 8t_{\gamma} h_{\beta\gamma\tau} + 2g_{(\mu\nu} t_{\lambda)} - g_{(\beta} t_{\lambda)} = 3\sigma_{(\beta} \Gamma^{(1)}_{\gamma;\lambda\alpha)}
\]

\[
+ \frac{36}{\lambda} h_{\beta\gamma\tau} + 3g_{(\mu\nu} t_{\lambda)} - 18\Gamma^{(1)}_{\gamma;\beta\gamma\tau} \sigma^\tau, \tag{C.11}
\]

\[
(\delta - 2\sigma) \Gamma^{(2)}_{\beta;\alpha;\nu\lambda} = 3\sigma_{(\beta} \Gamma^{(1)}_{\gamma;\alpha;\nu\lambda)}
\]

\[
- 4t_{\beta\nu} \lambda
\]

\[
- \frac{3}{2} g_{\nu\lambda} h_{\beta;\alpha\tau} + \frac{1}{2} g_{\nu\lambda} t_{\beta} - g_{(\beta} t_{\lambda)}, \tag{C.13}
\]
\[
(\delta - 4\sigma)\Gamma^{(2);T}_{\beta[\gamma;\mu\nu]} = 3\sigma_\beta \Gamma^{(1);T}_{\gamma[\mu\nu]} + 2g_{\beta\gamma}T^T_{\mu\nu} - g_{(\mu;\beta}t_{\gamma\nu;\gamma)} + \frac{2}{3}g_{(\mu\nu}t_{\lambda)\beta\gamma} + \frac{1}{6}g_{(\mu\nu}g_{\lambda)(\beta t_{\gamma})}
\]
\[
(\delta - 2\sigma)\Gamma^T_{\mu\nu} = 8t^T_{\mu\nu},
\]
\[
(\delta - 2\sigma)\Gamma^{(2);T}_{\beta[\gamma;\mu\nu]} \alpha = 3\sigma_\beta \Gamma^{(1);T}_{\gamma[\mu\nu]} + 3\Gamma^{(1);T}_{\beta[\gamma;\mu\nu]} \sigma^\tau - \frac{11}{3}t^T_{\beta\mu\nu}
\]
\[
+ \frac{8}{9}(g_{\nu\lambda}t_\beta - 2g_{\beta(\nu}t_{\lambda)}) = (\delta - 6\sigma)\gamma^{(2);T}_{\beta[\gamma;\mu\nu]},
\]
\[
(\delta - 4\sigma)\Gamma^{(2);T:T}_{\beta[\gamma;\mu\nu]} = (\delta - 4\sigma)[\Gamma^{(2);T}_{\beta[\gamma;\mu\nu} - \frac{1}{4}g_{[\gamma(\mu} \Gamma^{(1);T}_{\beta[\nu)]\lambda]}^\alpha]
\]
\[
+ 3(\sigma_\gamma \Gamma^{(1);T}_{[\beta;\lambda]} + \frac{1}{4}g_{[\gamma(\mu} \Gamma^{(1);T}_{[\beta;\nu)]\lambda}]^\tau \sigma^\tau)
\]
\[
+ 3(g_{\beta[\gamma}T^T_{\lambda]\mu\nu} - \frac{1}{4}g_{[\gamma(\mu} \nu)]^\sigma)
\]
\[
(C.17)
\]

We see that r.h.s of latter consists of three brackets and each of them is just traceless part of first term. To see cancelation of odd terms in this variation we present for completeness two more useful variations:

\[
(\delta - 4\sigma)\Gamma^{(2);T}_{\beta[\gamma;\mu\nu]} = 3\sigma_\beta \Gamma^{(1);T}_{\gamma[\mu\nu]} + 3\sigma_\gamma \Gamma^{(1);T}_{[\beta;\lambda]} + 3g_{\beta[\gamma}T^T_{\lambda]\mu\nu} - \frac{5}{3}g_{[\gamma(\mu}T^T_{\nu)]\beta}
\]
\[
+ \frac{4}{9}(g_{\beta[\gamma}g_{[\mu}t_{\nu]} - g_{\beta(\mu}g_{\nu)}[\gamma t_{\lambda})]
\]
\[
(\delta - 4\sigma)\left[-\frac{1}{4}g_{[\gamma(\mu} \gamma^{(2);T}_{\nu)]}\right] = \frac{3}{4}\sigma_\beta g_{[\gamma(\mu} \Gamma^{(1);T}_{\nu)} + \frac{3}{4}g_{[\gamma(\mu} \Gamma^{(1);T}_{[\nu)]\lambda}]^\tau \sigma^\tau + \frac{11}{12}g_{[\gamma(\mu} \nu)]}
\]
\[\frac{1}{4}g_{\beta(\mu}g_{\nu)}[\gamma t_{\lambda})]
\]

Summing last two variations we restore (C.17).

The next task is to construct analog of (2.18) in spin three case. For this we need to reconsider second Weyl invariant for spin three with Weyl tensor, one derivative and linearized spin 3 field. In this case we start from the same formula (4.1) and replace spin three field with corresponding traceless part:

\[
(\delta - 4\sigma)[\nabla_{(\alpha}^{(\beta} h^T_{\gamma)}_{\mu\nu}\lambda] - C_{(\mu(\alpha,\beta} h^T_{\gamma)}_{\nu\lambda)]} + C_{(\alpha}^{(\beta} g_{\gamma)(\mu} h^T_{\nu\lambda)\rho} + C_{(\alpha}^{(\beta} g_{\gamma)(\mu} h^T_{\nu\lambda)\rho}
\]
\[\frac{1}{2}g_{(\alpha(\beta} C_{\gamma)(\mu} h^T_{\nu\lambda)\rho] - 2g_{(\alpha}^{(\beta} C_{\gamma)(\mu} h^T_{\nu\lambda)\rho]} = -4\sigma (\alpha W_{(\beta}^{\rho} g_{(\gamma)(\mu} h^T_{\nu\lambda)\rho]}
\]
\[
(C.18)
\]

Then using again (C.4) we obtain simply:

\[
(\delta - 4\sigma)[W_{(\alpha}^{\rho} g_{(\gamma)(\mu} h^T_{\nu\lambda)\rho]} = 4\sigma (\alpha W_{(\beta}^{\rho} g_{(\gamma)(\mu} h^T_{\nu\lambda)\rho]}
\]
\[
(C.19)
\]
Then after antisymmetrization of pairs \([\alpha\mu],[\beta\nu],[\gamma\lambda]\) we have

\[
T_{\alpha\gamma,\mu\lambda} = \nabla_{(\alpha} W_{\beta \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T - C_{\alpha\beta\gamma} h_{\nu\lambda}^T + C_{\alpha\beta\gamma} h_{\nu\lambda}^T
- 2g(\alpha\beta C_{\gamma} h_{\nu\lambda}^T) + W_{(\mu} \Gamma_{\nu\lambda)}^{(1)} T_{\rho\tau}
\] (C.20)

So we see that summing \((C.21)\) and \((C.19)\) we obtain exact Weyl invariant tensor with two set of symmetrized indices \((\alpha\beta\gamma)\) and \((\mu\nu\lambda)\)

\[
T_{\alpha\beta;\gamma\rho;\nu\lambda} = 4 \nabla_{(\alpha} W_{\beta \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\alpha} W_{\beta \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\beta} W_{\alpha \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\gamma} W_{\alpha \beta^\rho \nu \lambda)} h_{\nu\lambda}^T
- 8 C_{\alpha\beta\gamma} h_{\nu\lambda}^T - 8 g(\alpha\beta C_{\gamma} h_{\nu\lambda}^T)
- 8 C_{\alpha\beta\gamma} h_{\nu\lambda}^T - 8 g(\alpha\beta C_{\gamma} h_{\nu\lambda}^T)
- W_{(\mu} \Gamma_{\nu\lambda)}^{(1)} T_{\rho\tau} + 3 W_{\beta \gamma^\rho \nu \lambda)} T_{\rho\tau}
- W_{\gamma \lambda}^{(1)} T_{\mu\nu\lambda} + 3 W_{\beta \gamma^\rho \nu \lambda)} T_{\rho\tau} - 12 C_{\gamma} h_{\nu\lambda}^T
\] (C.21)

This is Weyl invariant combination of Weyl tensor, spin 3 field and one covariant derivative with indices organized in a way of Riemann tensor, therefore it is spin 3 version of

\[
\frac{1}{2} [W_{\alpha\beta\gamma} W_{\lambda\nu\mu} W_{\nu\rho} h_{\nu\lambda}^T + W_{\beta\gamma\nu} W_{\nu\mu\lambda} h_{\nu\lambda}^T]
\] (C.22)

coming from linearization \((B.14)\) of gravitational Riemann tensor. So we can couple \((C.21)\) with spin three Weyl tensor and obtain primary:

\[
L_{W\Gamma} = W_{\alpha\beta;\gamma} T_{\alpha\beta;\gamma}
\] (C.23)

Trace of \((C.22)\) is traceless symmetric tensor \(W_{\alpha\mu\beta\nu\gamma\lambda} h_{\mu\nu}\) contracted with Schouten tensor \(K_{\alpha\beta}\) in the last term of Weyl invariant \((2.18)\). Therefore we can try to construct something similar in spin three case. For this purpose we can calculate trace of invariant \((C.21)\)

\[
T_{\mu\nu;\gamma\lambda} = g^{\alpha\beta} T_{\alpha\beta;\gamma\mu;\nu\lambda} = 4 \nabla_{(\alpha} W_{\beta \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\alpha} W_{\beta \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\beta} W_{\alpha \gamma^\rho \nu \lambda)} h_{\nu\lambda}^T + 4 \nabla_{\gamma} W_{\alpha \beta^\rho \nu \lambda)} h_{\nu\lambda}^T
- 8 C_{\alpha\beta\gamma} h_{\nu\lambda}^T - 8 g(\alpha\beta C_{\gamma} h_{\nu\lambda}^T)
- 8 C_{\alpha\beta\gamma} h_{\nu\lambda}^T - 8 g(\alpha\beta C_{\gamma} h_{\nu\lambda}^T)
- W_{(\mu} \Gamma_{\nu\lambda)}^{(1)} T_{\rho\tau} + 3 W_{\beta \gamma^\rho \nu \lambda)} T_{\rho\tau} - 12 C_{\gamma} h_{\nu\lambda}^T
\]
\[ -8g_{\mu\nu}C_{[\gamma}^{\tau} \rho h_{\lambda]\rho}^{T} + 4g_{[\gamma(\mu} C_{\lambda]}^{\tau} \rho h_{\lambda]\rho}^{T} - 12g_{[\gamma(\mu} C_{\nu)}^{\tau} \rho h_{\lambda]\rho}^{T} \]

(C.24)

Now we see that the last expression can be split up to the six primary fields:

\[
I_{\mu\nu;\gamma\lambda}^{(1)} = 4\nabla_{[\gamma} W_{(\mu : \nu)}^{\tau} \rho h_{\lambda]\rho}^{T} + 2W_{(\mu : \nu)}^{\tau} \rho \Gamma_{[\gamma\lambda]\tau\rho}^{(1)T} + 2W_{[\gamma : (\mu} \rho \Gamma_{\nu)\lambda]\tau\rho}^{(1)T} - 2W_{[\gamma(\mu\nu) T \tau\lambda]\rho}^{(1)T} - 8C_{(\mu : \nu)}^{\tau} \rho g_{[\gamma h_{\lambda]\rho}\tau\rho}^{T},
\]

(C.25)

\[
I_{\mu\nu;\gamma\lambda}^{(2)} = 4\nabla_{(\mu} W_{\nu)}^{\tau} \rho h_{\lambda]\rho}^{T} + W_{(\mu : \nu)}^{\tau} \rho \Gamma_{[\gamma\lambda]\tau\rho}^{(1)T} + 3W_{[\gamma : (\mu} \rho \Gamma_{\nu)\lambda]\tau\rho}^{(1)T} + W_{[\gamma(\mu\nu) T \tau\lambda]\rho}^{(1)T} + 4(C_{[\gamma}^{\rho} \mu + C_{(\mu : [\gamma]}^{\rho} h_{\lambda]\rho}) - 4C_{(\mu : \nu)}^{\tau} \rho h_{\lambda]\rho}^{T} - 8g_{\mu\nu} C_{[\gamma}^{\tau} \rho h_{\lambda]\rho}. \]

(C.26)

\[
I_{\mu\nu;\gamma\lambda}^{(3)} = 4\nabla_{[\gamma} W_{\lambda] ; (\mu}^{\tau} \rho h_{\nu]\rho}^{T} - W_{[\gamma : (\mu} \Gamma_{\lambda]\tau\nu\rho}^{(1)T} - 3W_{[\gamma\lambda;\nu) T \tau\rho}^{(1)T} - 4(C_{[\gamma}^{\rho} \mu + C_{(\mu ; [\gamma]}^{\rho} h_{\lambda]\rho}) - 4C_{[\gamma}^{\tau} \rho g_{[\gamma h_{\lambda]\rho}^{T} \tau\rho},
\]

(C.27)

\[
I_{\mu\nu;\gamma\lambda}^{(4)} = 6W_{\gamma\lambda}^{\tau} \rho \Gamma_{[\gamma\lambda]\tau\rho}^{(1)T} - 24C_{\gamma;\lambda}^{\rho} h_{\mu\nu},
\]

(C.28)

\[
I_{\mu\nu;\gamma\lambda}^{(5)} = 3W_{[\gamma(\mu\nu) T \tau\lambda]\rho}^{(1)T} - 12C_{[\gamma;\lambda}^{\rho} h_{\mu\nu]}^{T},
\]

(C.29)

\[
I_{\mu\nu;\gamma\lambda}^{(6)} = -2W_{[\gamma \lambda;\nu) T \tau\rho}^{(1)T} - 8(C_{[\gamma}^{\rho} \mu + C_{(\mu : [\gamma]}^{\rho} h_{\lambda]\rho}) - 8g_{\mu\nu} C_{[\gamma}^{\tau} \rho h_{\lambda]\rho}^{T}.
\]

(C.30)

Then first of all we see that

\[
\sum_{i=1}^{6} I_{\mu\nu;\gamma\lambda}^{(i)} = T_{\mu\nu;\gamma\lambda}.
\]

(C.31)

Moreover contracting with spin three Schouten tensor \( K_{\mu\nu;\gamma\lambda} \) and using Bianchi identity we arrive to the idea that under Bianchi projection the sum of second and third invariant is equal to first:

\[
I_{\mu\nu;\gamma\lambda}^{(1)} K_{\mu\nu;\gamma\lambda} = (I_{\mu\nu;\gamma\lambda}^{(2)} + I_{\mu\nu;\gamma\lambda}^{(3)}) K_{\mu\nu;\gamma\lambda}.
\]

(C.32)

Then fifth invariant is also not independent and connected with fourth.

\[
I_{\mu\nu;\gamma\lambda} = -\frac{1}{2} I_{(\mu;[\gamma\lambda]}^{(4)}.
\]

(C.33)

Therefore we have only 4 independent invariants \( I_{\mu\nu;\gamma\lambda}^{(1)}, I_{\mu\nu;\gamma\lambda}^{(2)}, I_{\mu\nu;\gamma\lambda}^{(4)} \) and \( I_{\mu\nu;\gamma\lambda}^{(6)} \). The last one can be combined with first two to obtain invariant expression with Christoffel symbols with one symmetric and one antisymmetric pair of indices. This is necessary for integration to the square of second Christoffel symbols with the similar organization of indices in the future construction.

\[
J_{\mu\nu;\gamma\lambda}^{(1)} = I_{\mu\nu;\gamma\lambda}^{(1)} + \frac{1}{2} I_{\mu\nu;\gamma\lambda}^{(6)}
\]

\[
= 4\nabla_{[\gamma} W_{(\mu : \nu)}^{\tau} \rho h_{\lambda]\rho}^{T} + 2W_{(\mu : \nu)}^{\tau} \rho \Gamma_{[\gamma\lambda]\tau\rho}^{(1)T} + W_{[\gamma : (\mu} \rho \Gamma_{\nu)\lambda]\tau\rho}^{(1)T} - 2W_{[\gamma(\mu\nu) T \tau\lambda]\rho}^{(1)T} - 4(C_{[\gamma}^{\rho} \mu + C_{(\mu : [\gamma]}^{\rho} h_{\lambda]\rho}) - 8C_{(\mu : \nu)}^{\tau} \rho h_{\lambda]\rho}^{T}.
\]

(C.34)
\[ J^{(2)}_{\mu\nu;\gamma\lambda} = J^{(2)}_{\mu\nu;\gamma\lambda} + \frac{3}{4} f^{(6)}_{\mu\nu;\gamma\lambda} \]
\[ = 4
\n\n\int (\mu \nu) \cdot [\tau_{\rho} h_{\lambda} \gamma T]_{\tau \rho} + W_{(\mu \nu)} \cdot \rho T_{(\gamma \lambda)}^{(1) T} + \frac{3}{2} W_{[\gamma \lambda]} \cdot \rho [\gamma_{\mu}, \lambda_{\nu}]_{\tau \rho} - \Gamma^{(1) T}_{\mu \nu, \gamma \lambda} \]
\[ + W_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} - 2 C_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} - 8 g_{\mu \nu} C_{[\gamma \lambda]} \cdot \rho T_{(\mu \nu)}^{(1) T}. \]
\[ (C.35) \]

This two primaries we can contract with traceless part of Schouten tensor \( K^{(T)}_{\mu\nu;\gamma\lambda} \) with the Weyl transformation \( (3.37) \). For first invariant tensor \( J^{(1)} \) we have:
\[ J^{(1)}_{\mu\nu;\gamma\lambda} K^{(T)}_{(\mu\nu;\gamma\lambda)} = \left[ 16 \nabla_{\gamma} W_{(\mu \nu)} \cdot \tau_{\rho} h_{\lambda T}^{(1) T} - 16 C_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} - 4 W_{(\mu \nu)} \cdot \rho T_{(\gamma \lambda)}^{(1) T} \cdot \tau_{\rho} \right] K^{(T)}_{(\mu\nu;\gamma\lambda)}, \]
\[ (C.36) \]

Now we start to analyze last expression. Considering variation of the proposed part of invariant
\[ \delta [J^{(1)}_{\mu\nu;\gamma\lambda} K^{(T)}_{(\mu\nu;\gamma\lambda)}] = J^{(1)}_{\mu\nu;\gamma\lambda} \delta K^{(T)}_{(\mu\nu;\gamma\lambda)}, \]
\[ (C.37) \]

and using \( (3.37) \) we should try to integrate variation \( (C.37) \) to terms second order on \( \Gamma^{(2) T;T} \) and linear in gravitational Weyl tensor. To integrate we should use variation \( (C.17) \) where unpleasant trace terms appear again. Doing that with help of the following relation
\[ (\delta - 2 \sigma) \nabla_{\gamma} W_{(\mu \nu)} \cdot \tau_{\rho} = - 2 \sigma_{(\mu \nu)} W_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} + g_{(\mu \nu)} \delta C_{(\lambda \lambda)} + g_{(\mu \nu)} \delta C_{(\lambda \lambda)}, \]

and after integration of some second order on Christoffel symbols terms we arrive to the reminder:
\[ - \frac{7}{3} W_{(\gamma \lambda)} \cdot \rho \tilde{\gamma}_{(\mu \nu)} \Gamma^{(2) T;T}_{[\gamma \lambda] \rho} + 4 W_{[\gamma \lambda]} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} + 8 W_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \rho T_{(\gamma \lambda)}^{(1) T} - \frac{2}{3} W_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} + \frac{4}{3} C_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} + \frac{4}{3} \nabla_{\gamma} W_{(\mu \nu)} \cdot \rho T_{(\gamma \lambda)}^{(1) T} \cdot \tau_{\rho}, \]
\[ (C.38) \]

where
\[ \tilde{\gamma}_{(\mu \nu)} = \sigma_{\alpha} \Gamma^{(1) T}_{\alpha \mu \nu} \cdot \alpha + \Gamma^{(1) T}_{\beta \mu \nu \tau} \cdot \sigma_{\tau} + \tilde{\tau}_{\nu \lambda \beta}. \]
\[ (C.39) \]

To cancel first term of \( (C.38) \) we should use the following general relation for remaining invariant \( (C.28) \)
\[ \frac{A}{3} \delta [J^{(4)}_{(\mu\nu;\gamma\lambda)} K^{(T)}_{(\mu\nu;\gamma\lambda)}] = A \left[ W_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} - 8 C_{(\gamma \lambda)} \cdot \rho T_{(\mu \nu)}^{(1) T} \cdot \tau_{\rho} \right] \delta K^{(T)}_{(\mu\nu;\gamma\lambda)}. \]

24
$$= \delta \frac{A}{18} W_{\gamma \lambda} \frac{\tau \rho}{\beta \gamma \lambda \mu \nu} \Gamma^{(2)T,T}_{\beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} = \delta \frac{2A}{3} C_{\gamma \lambda} \frac{\tau \rho}{\beta \gamma \lambda \mu \nu} \Gamma^{(1)T}_{\beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} + \frac{2A}{3} C_{\gamma \lambda} \frac{\tau \rho}{\beta \gamma \lambda \mu \nu} \delta \Gamma^{[\gamma,\lambda]_{\mu \nu}}$$

$$- \frac{2A}{3} W_{\gamma \lambda \beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} + \frac{A}{3} W_{\gamma \lambda \beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} . \quad \text{(C.40)}$$

So we see that taking $A = 7$ and adding (C.40) to the (C.38) we arrive to the some integrated terms (will collect later in general formula for integrated terms) and the following reminder:

$$\frac{8}{3} W_{\gamma \lambda \mu} \frac{\tau \rho}{\beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} - \frac{16}{3} W_{\gamma \lambda \beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} + 4W_{\mu \tau \rho T,\beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} + \frac{4}{3} W_{\gamma \lambda \beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}}$$

$$+ \frac{4}{3} C_{\gamma \lambda \beta \gamma \lambda \mu \nu} \Gamma^{(1)T}_{\beta \gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} . \quad \text{(C.41)}$$

To continue we should introduce some definitions

$$T^{\mu \nu \lambda}_{\alpha \beta \gamma} = \Gamma^{(1)T}_{\alpha \mu \nu \lambda} \Gamma^{(1)T}_{\tau \alpha \beta \gamma} - \frac{1}{2} (h^{\mu \nu \lambda}_{\alpha \beta \gamma} + h^{\mu \nu \lambda}_{\tau \alpha \beta \gamma}), \quad \text{(C.42)}$$

$$T^{\mu \nu}_{\alpha \beta} = T^{\mu \nu \lambda}_{\alpha \beta \gamma}, \quad T^{\mu}_{\alpha} = T^{\mu \nu}_{\alpha \nu} . \quad \text{(C.43)}$$

Note that (C.42)-(C.43) are Weyl invariant tensors. Then we see that after transformation and integration of some terms in first line of (C.41) using (3.36) and after applying the following important formula

$$\Gamma^{(1)T}_{\alpha \beta \gamma} \delta^{[\gamma,\lambda]_{\mu \nu}} = 3\sigma^{\gamma}_{\mu} T^{\alpha \beta \gamma} - \frac{3}{4} g^{\gamma}_{\mu \nu} \Gamma^{(1)T}_{\alpha \beta \gamma} = \frac{3}{8} \delta^{\gamma}_{\mu \nu} + \frac{3}{8} \delta^{\gamma}_{\lambda \mu \nu} + \frac{1}{2} \delta^{\gamma}_{\mu \nu \lambda} , \quad \text{(C.44)}$$

to the second line of (C.41) we obtain miraculous cancelation of the all overall terms and we arrive to the following reminder expressed through the invariant tensors (C.42)-(C.43):

$$4\sigma^{\gamma}_{\mu} W^{T}_{\gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} - 4\sigma^{\gamma}_{\mu} W^{T}_{\gamma \lambda \mu \nu} \Gamma^{[\gamma,\lambda]_{\mu \nu}} + 24\sigma_{\mu \nu \rho} C^{\alpha \beta \gamma}_{\mu \nu \rho} T^{\mu \nu}_{\alpha \beta \gamma} + 32\sigma_{\mu \nu \rho} C_{\gamma \lambda \mu \nu} T^{\mu \rho}_{\alpha \beta \gamma} . \quad \text{(C.45)}$$

Before continue with this reminder we present the integrated terms during the whole procedure presented from formula (C.37)
\[-3C_\mu^{\tau,\rho} (\Gamma^{\mu\nu}_T \Gamma^{(1) \alpha \mu \nu}_T + \Gamma^{(1) \alpha \nu}_T \Gamma^{\mu \nu}_T). \]

(C.46)

Then to proceed with (C.45) we should use Weyl variations of the following two terms:

\[(\delta - 2 \sigma) B = (\delta - 2 \sigma) (\nabla \sigma \nabla \mu W_{\alpha \mu \beta \nu} + 2K_{\sigma \mu} W_{\alpha \mu \beta \nu} - K_{\sigma \alpha} W_{\mu \beta \nu} - K_{\sigma \beta} W_{\mu \rho \alpha})
+ g_{\rho \alpha} K_{\theta \alpha} W_{\rho \mu \tau \nu} + g_{\rho \beta} K_{\theta \beta} W_{\rho \mu \tau \nu}, \]
\[\nabla \sigma \nabla \mu W_{\alpha \mu \beta \nu} T^{\alpha \beta \mu \nu} \sigma = -10 \sigma \nabla \sigma \nabla \mu W_{\alpha \mu \beta \nu} T^{\alpha \beta \mu \nu} \sigma
+ 3 \sigma \nabla \sigma \nabla \mu \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} \sigma - 4 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} \sigma - 4 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} \sigma - 8 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} \sigma\]
\[\delta C = \delta (\nabla W_{\alpha \mu \beta \nu} + 2JW_{\alpha \mu \beta \nu} T^{\alpha \beta \mu \nu} - 16 \sigma \nabla \sigma \nabla \mu \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} - 8 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu}).
\]

(C.47)

Combining with (C.45) we see that

\[4 \sigma \nabla \mu W_{\nu \sigma \gamma} T^{\gamma \mu \nu} = 4 \sigma \nabla \mu W_{\nu \sigma \gamma} T^{\gamma \mu \nu} + 24 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} + 2 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} + 2 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} + 7 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} + 7 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu}.
\]

(C.48)

It means that after all possible cancelation we arrive to the last term in r.h.s of (C.49). To cancel that we need to consider similar construction for another invariant \( J^{2(\mu \nu \gamma \lambda)}_\mu \nu \gamma \lambda \) (C.39).

Starting now from

\[J^{2(\mu \nu \gamma \lambda)}_\mu \nu \gamma \lambda K^{(\mu \nu \gamma \lambda)}_T \]
\[= \left[ 16 \nabla \mu W_{\nu \gamma} \nabla \mu W_{\gamma \nu} T^{\gamma \mu \nu} - 8C_{(\gamma \lambda \mu \nu)} h_{\nu \lambda \rho} + 2W_{\gamma \lambda \mu} T^{\gamma \mu \nu} + 2 \sigma \nabla \sigma \nabla \mu T^{\alpha \beta \mu \nu} \right] K^{(\mu \nu \gamma \lambda)}_T \]
\[+ \left[ 6W_{\gamma \lambda \mu} T^{\gamma \mu \nu} + 2W_{\gamma \lambda \mu} T^{\gamma \mu \nu} \right] K^{(\mu \nu \gamma \lambda)}_T \]

(C.50)

instead of (C.38) we have

\[-\frac{1}{6} W_{\gamma \lambda \mu} T^{\mu \nu} \nabla \sigma \nabla \mu W_{\nu \gamma} + \frac{1}{3} W_{\gamma \lambda \mu} T^{\mu \nu} \delta_{(2)}^{(T)} T^{(T)} = -\frac{1}{6} W_{\gamma \lambda \mu} T^{\mu \nu} \nabla \sigma \nabla \mu W_{\nu \gamma} + \frac{1}{3} W_{\gamma \lambda \mu} T^{\mu \nu} \delta_{(2)}^{(T)} T^{(T)} 
+ W_{\mu \nu} \nabla \sigma \nabla \mu W_{\nu \gamma} + \frac{1}{3} W_{\gamma \lambda \mu} T^{\mu \nu} \delta_{(2)}^{(T)} T^{(T)}.
\]

(C.51)

Then using again (C.40) with \( A = \frac{1}{2} \) we obtain instead of (C.41)

\[\frac{2}{3} C_{(\gamma \lambda \mu \nu)} T^{(\mu \nu)} \Gamma^{(1)} T^{(T)} T^{(T)} = -\frac{4}{3} W_{\gamma \lambda \mu} \nabla \sigma \nabla \mu W_{\nu \gamma} T^{(1)} T^{(T)} T^{(T)} + \frac{1}{3} C_{\gamma \lambda \mu} T^{(1)} T^{(T)} T^{(T)} 
- \frac{4}{3} W_{\gamma \lambda \mu} T^{(2)} T^{\mu \nu} \nabla \sigma \nabla \mu W_{\nu \gamma} + \frac{1}{3} W_{\gamma \lambda \mu} T^{(2)} T^{\mu \nu} \nabla \sigma \nabla \mu W_{\nu \gamma}.
\]

(C.52)

Then the same miraculous cancelation leads instead of (C.45) to the following reminder:

\[3 \sigma \nabla \mu W_{\nu \gamma} T^{(1)} T^{(T)} T^{(T)} + 2 \sigma \nabla \sigma \nabla \mu W_{\nu \gamma} T^{(1)} T^{(T)} T^{(T)} + 10 \sigma \nabla \sigma \nabla \mu W_{\nu \gamma} T^{(1)} T^{(T)} T^{(T)} + 8 \sigma \nabla \sigma \nabla \mu T^{(1)} T^{(T)} T^{(T)}.
\]

(C.53)
and corresponding integrated terms instead of (C.46) is

\[-L^{(2)} = +\frac{1}{9} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} + \frac{1}{3} W_{\gamma,\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} + \frac{1}{36} W_{\gamma,\lambda,\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} \]

\[-\frac{1}{96} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} \]

\[+ \frac{1}{6} W_{\gamma,\lambda,\mu} \rho \Gamma^{(2)T,T}_{\mu\nu\beta} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - \frac{1}{6} W_{\gamma,\lambda,\beta} \rho \Gamma^{(2)T,T}_{\mu\nu\beta} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{1}{8} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\tau\rho\beta} \left( \frac{10}{3} \Gamma^{(2)T,T}_{\alpha T^{(1)}} - \frac{4}{3} \Gamma^{(2)T,T}_{\alpha T^{(1)}} \right) \]

\[+ \frac{4}{3} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \Gamma^{(1)T}_{\tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - \frac{2}{3} \left( C_{\gamma,\mu} \rho \Gamma^{(1)T}_{\tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - \frac{1}{3} \Gamma^{(1)T}_{\beta, \tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} \right) \]

\[-\frac{1}{2} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \Gamma^{(1)T}_{\tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - \frac{1}{8} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \Gamma^{(1)T}_{\tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} \]

\[+ \frac{1}{4} C_{\gamma,\lambda} \rho \Gamma^{(1)T}_{\tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{3}{4} \Gamma^{(1)T}_{\gamma \tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{3}{4} \Gamma^{(1)T}_{\gamma \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} \right). \]

(C.54)

Then in the similar way (see (C.50)) we can write integration rules for reminder (C.54)

\[3 \sigma^{\lambda} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - 2 \sigma^{\gamma} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + 10 \sigma^{\mu} C_{\gamma,\lambda} \rho \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + 8 \sigma^{\gamma} C_{\gamma,\lambda} \rho \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{3}{10} (B) + \frac{11}{20} (C) \]

\[= \frac{12}{5} C_{\gamma,\lambda} \rho \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}}. \]

(C.55)

Comparing last equation with the (C.50) we see that the following expression will be Weyl Invariant

\[I_{W} = \left[ J^{(1)}_{\mu \nu \gamma \lambda} - 8 J^{(2)}_{\mu \nu \gamma \lambda} + I^{(4)}_{\mu \nu \gamma \lambda} \right] K^{\mu \nu \gamma \lambda}_{(T)} + L^{(1)} - 8 L^{(2)} - 2 B - 3 C. \]

(C.56)

Then multiplying (C.54) by 8 and summing with (3.22) we get

\[L^{(1)} - 8 L^{(2)} = \frac{2}{3} W_{\gamma,\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} + \frac{22}{9} W_{\gamma,\mu,\nu} \rho \Gamma^{(2)T,T}_{\gamma,\nu,\tau} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} - \frac{1}{6} W_{\gamma,\lambda,\mu,\nu} \rho \Gamma^{(2)T,T}_{\gamma,\nu,\tau} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} \]

\[+ \frac{1}{8} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\beta[\nu,\tau] T^{(2)T}} \]

\[+ W_{\gamma,\lambda,\mu} \rho \Gamma^{(2)T,T}_{\mu\nu\beta} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{17}{6} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\tau\rho\beta} \Gamma^{(2)T,T}_{\alpha T^{(1)}} - \frac{4}{3} W_{\mu,\nu} \rho \Gamma^{(2)T,T}_{\tau\rho\beta} \Gamma^{(2)T,T}_{\alpha T^{(1)}} \]

\[- \left( \frac{4}{3} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \rho - \frac{32}{3} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \rho \right) \Gamma^{(1)T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} - 4 C_{\gamma,\mu} \rho \Gamma^{(1)T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + 2 C_{\gamma,\lambda} \rho \Gamma^{(1)T}_{\beta[\gamma,\lambda]} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} \]

\[+ \left( \frac{1}{2} \nabla_{\gamma} W_{\mu,\nu} \rho - \frac{4}{3} \nabla_{\mu} W_{\nu,\gamma} \nabla_{\tau} \rho \right) \Gamma^{(1)T}_{\gamma,\lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{3}{4} C_{\tau,\rho} \rho \Gamma^{(1)T}_{\alpha} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} + \frac{3}{4} \Gamma^{(1)T}_{\beta, \tau \rho \lambda} \Gamma^{\beta[\gamma,\lambda]T\mu\nu}_{\gamma,\lambda T^{(1)T}} \right). \]

(C.57)

In the same way we can collect our starting terms (C.36), (C.50) and (C.41) and obtain

\[\left[ J^{(1)}_{\mu \nu \gamma \lambda} - 8 J^{(2)}_{\mu \nu \gamma \lambda} + I^{(4)}_{\mu \nu \gamma \lambda} \right] K^{\mu \nu \gamma \lambda}_{(T)} \]
Then we can do the following simplifications: First of all note that the following modified by gauge invariant in zero order on background curvature Christoffel symbol

\[ \Gamma^\beta_{\gamma,\lambda}^{[\gamma,\lambda]}(\mu,\nu) = \Gamma^\beta_{\gamma,\lambda}^{(1)}(\mu,\nu) - \frac{3}{4} g^{\beta\gamma}(\mu,\nu) \Gamma^\lambda_{T T} \]  

(C.59)

transforms without third line in (C.17) but with the same in zero order on curvature gauge variation. Shifting all \( \Gamma^\beta_{\gamma,\lambda}^{[\gamma,\lambda]}(\mu,\nu) \) to the \( \Gamma^{\beta\gamma}_{\lambda,\gamma}(\mu,\nu) \) in (C.57) we cancel second and third line there and modify other terms bilinear on first and second Christoffel symbols. Then last modification could be done after the change to the completely traceless object

\[ \Gamma^\gamma_{\gamma,\lambda}^{[\gamma,\lambda]}(\mu,\nu) = \Gamma^\gamma_{\gamma,\lambda}^{(1)}(\mu,\nu) - \frac{1}{4} g^{\gamma\mu}(\mu,\nu) \Gamma^\lambda_{T T,\alpha} \]  

(C.60)

in the first term of fifth line of (C.57). After this transformation we obtain:

\[ L^{(1)} - 8L^{(2)} = \frac{2}{3} W^\gamma_{\mu,\nu} \rho T^{[\gamma,\lambda]}_{\lambda,\mu} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) + 2 \left( \frac{22}{9} W^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) - \frac{1}{6} W^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) \right) \]

(C.61)

Then using the following relation:

\[ 4C^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) - 2 \left( \frac{1}{3} \nabla^\gamma W^\gamma_{\mu,\nu} \rho T^{[\gamma,\lambda]}_{\mu,\nu} \right) \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) + 3 C^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) = 8 C^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) \]  

(C.62)

we arrive to the following compact two line formula:

\[ L^{(1)} - 8L^{(2)} = \frac{2}{3} W^\gamma_{\mu,\nu} \rho T^{[\gamma,\lambda]}_{\lambda,\mu} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) + 2 \left( \frac{22}{9} W^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) - \frac{1}{6} W^\gamma_{\gamma,\lambda} \rho T^{[\gamma,\lambda]}_{\gamma,\lambda} \Gamma^\gamma_{\gamma,\lambda}^{(2)}(T, T) \right) \]

(C.63)

Then applying (C.60) and (C.62) to the first and second line of (C.58) we obtain corresponding cancelation of the last trace term of first line and reduction of terms in second line with the same combination of derivatives of Weyl tensors in brackets:

\[ \left[ J_{\mu,\nu,\gamma,\lambda}^{(1)} + 8 J_{\mu,\nu,\gamma,\lambda}^{(2)} + I_{\mu,\nu,\gamma,\lambda}^{(4)} \right] K_{\mu,\nu,\gamma,\lambda}^{(T)} \]  

28
Combining (C.64) and (C.63) we arrive to the result that the following final expression

$$L_{-4}^{WT} = \left[ (J_{\mu \nu \gamma \lambda}^{(1)} - 8 J_{\mu \nu \gamma \lambda}^{(2)}) K_{(T)}^{\mu \nu \gamma \lambda} + L^{(1)} - 8 L^{(2)} - 2 B - 3 C \right]$$

$$= \frac{2}{3} W_{\mu \tau} \rho T_{\beta \gamma \lambda \tau \rho} + \frac{2}{9} W_{\gamma} (\tau \mu) \rho T_{\beta \gamma \lambda \tau \rho} - \frac{1}{6} W_{\gamma} (\tau \mu) \rho T_{\beta \gamma \lambda \tau \rho} - \frac{1}{2} T_{\beta \gamma \lambda \tau \rho} T_{(1) T} - 16 h_{\tau \rho} T_{(T)}^{K^{\mu \nu \gamma \lambda}}$$

is weight -4 primary field and can be used as a Weyl invariant Lagrangian.

References

[1] E. S. Fradkin and A. A. Tseytlin, “Conformal supergravity,” Phys. Rept. 119, 233 (1985).
[2] E. S. Fradkin and V. Y. Linetsky, “Superconformal Higher Spin Theory in the Cubic Approximation,” Nucl. Phys. B 350 (1991) 274. doi:10.1016/0550-3213(91)90262-V
[3] A. Y. Segal, “Conformal higher spin theory,” Nucl. Phys. B 664 (2003) 59 doi:10.1016/S0550-3213(03)00368-7 [hep-th/0207212].
[4] X. Bekaert, E. Joung and J. Mourad, “Effective action in a higher-spin background,” JHEP 1102 (2011) 048 doi:10.1007/JHEP02(2011)048 [arXiv:1012.2103 [hep-th]]
[5] M. A. Vasiliev, “Bosonic conformal higher-spin fields of any symmetry,” Nucl. Phys. B 829 (2010) 176 doi:10.1016/j.nuclphysb.2009.12.010 [arXiv:0909.5226 [hep-th]].
[6] S. E. Konstein, M. A. Vasiliev and V. N. Zaikin, “Conformal higher spin currents in any dimension and AdS / CFT correspondence,” JHEP 0012 (2000) 018 doi:10.1088/1126-6708/2000/12/018 [hep-th/0010239].
[7] R. R. Metsaev, “Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields,” JHEP 1206 (2012) 062 doi:10.1007/JHEP06(2012)062 [arXiv:0709.4392 [hep-th]].
[8] S. Deser, E. Joung and A. Waldron, “Partial Masslessness and Conformal Gravity,” J. Phys. A 46 (2013) 214019 doi:10.1088/1751-8113/46/21/214019 [arXiv:1208.1307 [hep-th]].
[9] R. R. Metsaev, “Arbitrary spin conformal fields in (A)dS,” Nucl. Phys. B 885 (2014) 734 doi:10.1016/j.nuclphysb.2014.06.013 [arXiv:1404.3712 [hep-th]].

29
[10] R. R. Metsaev, “The BRST-BV approach to conformal fields,” J. Phys. A 49 (2016) no.17, 175401 doi:10.1088/1751-8113/49/17/175401 [arXiv:1511.01836 [hep-th]].

[11] R. R. Metsaev, “Long, partial-short, and special conformal fields,” JHEP 1605 (2016) 096 doi:10.1007/JHEP05(2016)096 [arXiv:1604.02091 [hep-th]].

[12] R. R. Metsaev, “Interacting light-cone gauge conformal fields,” [arXiv:1612.06348 [hep-th]].

[13] C. Fronsdal, “Massless fields with integer spin,” Phys. Rev. D 18, 3624 (1978).

[14] M. A. Vasiliev, “Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions,” Phys. Lett. B 243 (1990) 378. M. A. Vasiliev, “Nonlinear equations for symmetric massless higher spin fields in (A)dS(d),” Phys. Lett. B 567 (2003) 139 [hep-th/0304049].

[15] E. S. Fradkin and V. Y. Linetsky, “Conformal superalgebras of higher spins,” Annals Phys. 198 (1990) 252. doi:10.1016/0003-4916(90)90252-J

[16] M. A. Vasiliev, “Conformal higher spin symmetries of 4-d massless supermultiplets and osp(L,2M) invariant equations in generalized (super)space,” Phys. Rev. D 66 (2002) 066006 doi:10.1103/PhysRevD.66.066006 [hep-th/0106149].

[17] R. Manvelyan, K. Mkrtchyan, R. Mkrtchyan and S. Theisen, JHEP 1310 (2013) 185 doi:10.1007/JHEP10(2013)185 [arXiv:1304.7988 [hep-th]].

[18] E. Joung and K. Mkrtchyan, “Notes on higher-spin algebras: minimal representations and structure constants,” JHEP 1405 (2014) 103 doi:10.1007/JHEP05(2014)103 [arXiv:1401.7977 [hep-th]].

[19] E. Joung and K. Mkrtchyan, “Partially-massless higher-spin algebras and their finite-dimensional truncations,” JHEP 1601 (2016) 003 doi:10.1007/JHEP01(2016)003 [arXiv:1508.07332 [hep-th]].

[20] D. S. Kaparulin, S. L. Lyakhovich and A. A. Sharapov, “Classical and quantum stability of higher-derivative dynamics,” Eur. Phys. J. C 74 (2014) no.10, 3072 doi:10.1140/epjc/s10052-014-3072-3 [arXiv:1407.8481 [hep-th]].

[21] E. Joung and K. Mkrtchyan, “Higher-derivative massive actions from dimensional reduction,” JHEP 1302 (2013) 134 doi:10.1007/JHEP02(2013)134 [arXiv:1212.5919 [hep-th]].

[22] E. Joung and K. Mkrtchyan, “Weyl Action of Two-Column Mixed-Symmetry Field and Its Factorization Around (A)dS Space,” JHEP 1606 (2016) 135 doi:10.1007/JHEP06(2016)135 [arXiv:1604.05330 [hep-th]].

[23] J. Maldacena, “Einstein Gravity from Conformal Gravity,” [arXiv:1105.5632 [hep-th]].

[24] R. R. Metsaev, “Ordinary-derivative formulation of conformal low spin fields,” JHEP 1201 (2012) 064 doi:10.1007/JHEP01(2012)064 [arXiv:0707.4437 [hep-th]].

[25] E. Joung and K. Mkrtchyan, “A note on higher-derivative actions for free higher-spin fields,” JHEP 1211 (2012) 153 doi:10.1007/JHEP11(2012)153 [arXiv:1209.4864 [hep-th]].

[26] E. Joung, S. Nakach and A. A. Tseytlin, “Scalar scattering via conformal higher spin exchange,” JHEP 1602 (2016) 125 doi:10.1007/JHEP02(2016)125 [arXiv:1512.08896 [hep-th]].

[27] M. Beccaria, S. Nakach and A. A. Tseytlin, “On triviality of S-matrix in conformal higher spin theory,” JHEP 1609 (2016) 034 doi:10.1007/JHEP09(2016)034 [arXiv:1607.06379 [hep-th]].
[28] M. Beccaria, X. Bekaert and A. A. Tseytlin, “Partition function of free conformal higher spin theory,” JHEP 1408 (2014) 113 doi:10.1007/JHEP08(2014)113 [arXiv:1406.3542 [hep-th]].

[29] M. Beccaria and A. A. Tseytlin, “On induced action for conformal higher spins in curved background,” Nucl. Phys. B 919 (2017) 359 doi:10.1016/j.nuclphysb.2017.03.022 [arXiv:1702.00222 [hep-th]].

[30] M. Grigoriev and A. A. Tseytlin, “On conformal higher spins in curved background,” J. Phys. A 50 (2017) no.12, 125401 doi:10.1088/1751-8121/aa5e5f [arXiv:1609.09381 [hep-th]].

[31] A. A. Tseytlin, “On partition function and Weyl anomaly of conformal higher spin fields,” Nucl. Phys. B 877 (2013) 598 doi:10.1016/j.nuclphysb.2013.10.009 [arXiv:1309.0785 [hep-th]].

[32] D. Anselmi, “Theory of higher spin tensor currents and central charges,” Nucl. Phys. B 541 (1999) 323 doi:10.1016/S0550-3213(98)00783-4 [hep-th/9808004].

[33] D. Anselmi, “Higher spin current multiplets in operator product expansions,” Class. Quant. Grav. 17 (2000) 1383 doi:10.1088/0264-9381/17/6/305 [hep-th/9906167].

[34] R. Manvelyan and K. Mkrtchyan, “Conformal invariant interaction of a scalar field with the higher spin field in AdS(D),” Mod. Phys. Lett. A 25 (2010) 1333 doi:10.1142/S0217732310033116 [arXiv:0903.0058 [hep-th]].

[35] R. Manvelyan and W. Ruhl, “Conformal coupling of higher spin gauge fields to a scalar field in AdS(4) and generalized Weyl invariance,” Phys. Lett. B 593 (2004) 253 doi:10.1016/j.physletb.2004.04.052 [hep-th/0403241].

[36] R. Manvelyan and W. Ruhl, “The Quantum one loop trace anomaly of the higher spin conformal conserved currents in the bulk of AdS(4),” Nucl. Phys. B 733 (2006) 104 doi:10.1016/j.nuclphysb.2005.10.034 [hep-th/0506185].

[37] R. Manvelyan and W. Ruhl, “The Structure of the trace anomaly of higher spin conformal currents in the bulk of AdS(4),” Nucl. Phys. B 751 (2006) 285 doi:10.1016/j.nuclphysb.2006.06.012 [hep-th/0602067].

[38] S. Acevedo, R. Aros, F. Bugini and D. E. Daz, “On the Weyl anomaly of 4D Conformal Higher Spins: a holographic approach,” JHEP 1711 (2017) 082 doi:10.1007/JHEP11(2017)082 [arXiv:1710.03779 [hep-th]].

[39] T. Nutma and M. Taronna, “On conformal higher spin wave operators,” JHEP 1406 (2014) 066 doi:10.1007/JHEP06(2014)066 [arXiv:1404.7452 [hep-th]].

[40] S. M. Kuzenko, R. Manvelyan and S. Theisen, “Off-shell superconformal higher spin multiplets in four dimensions,” JHEP 1707 (2017) 034 doi:10.1007/JHEP07(2017)034 [arXiv:1701.00682 [hep-th]].

[41] B. de Wit and D. Z. Freedman, “Systematics of Higher Spin Gauge Fields,” Phys. Rev. D 21 (1980) 358. doi:10.1103/PhysRevD.21.358