Finiteness conditions of S-Cohn-Jordan Extensions

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Abstract

Let a monoid $S$ act on a ring $R$ by injective endomorphisms and $A(R; S)$ denote the $S$-Cohn-Jordan extension of $R$. Some results relating finiteness conditions of $R$ and that of $A(R; S)$ are presented. In particular necessary and sufficient conditions for $A(R; S)$ to be left noetherian, to be left Bézout and to be left principal ideal ring are presented. This also offers a solution to Problem 10 from [6].

Throughout the paper $R$ stands for an associative ring with unity and $\phi$ denotes an action of a multiplicative monoid $S$ on a ring $R$ by injective endomorphisms. By this we mean that a homomorphism $\phi: S \rightarrow \text{End}(R)$ is given, such that $\phi(s)$ is an injective endomorphism of $R$, for any $s \in S$. It is assumed that all endomorphisms of $R$ preserve unity.

We say that an over-ring $A(R; S)$ of $R$ is an $S$-Cohn-Jordan extension of $R$ if it is a minimal over-ring of $R$ such that the action of $S$ on $R$ extends to the action of $S$ on $A(R; S)$ by automorphisms (Cf. Definition 1.1). A classical result of Cohn (see Theorem 7.3.4 [1]) says that if the monoid $S$ possesses a group $S^{-1}S$ of left quotients, then $A(R; S)$ exists, moreover it is uniquely determined up to an $R$-isomorphism. The above mentioned theorem of Cohn was originally formulated in much more general context of $\Omega$-algebras, not just rings. The construction of $A(R; S)$ was given as a limit of a suitable directed system. The possibility of enlarging an object and replacing the action of endomorphisms by the action of automorphisms is a powerful tool, similar to a localization. This is indeed the case. One can see [3], [4], [7], [8], [9], [10] for examples of such applications in various algebraic contexts.

Jordan in [2] began systematic studies of relations between various algebraic properties of a ring $R$ and that of $A(R; S)$ in the case $S = \langle \sigma \rangle$ is a monoid generated by an injective endomorphism $\sigma$ of $R$. Then Matczuk in [5] started such investigations in the case $S$ is an arbitrary monoid acting, by injective endomorphisms, on a ring $R$. This paper can be considered as a continuation of works carried out in [5]. One can also consult [5] and

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The aim of this paper is to continue investigation of some finiteness conditions of the S-Cohn-Jordan extension $A(R; S)$ of $R$ in terms of properties of $R$ and the action of $S$. The basic idea, which goes back to Jordan [2], is to compare left ideals $I$ of $A(R; S)$ with its orbits $\{\phi(s)(I) \cap R \mid s \in S\}$ in $R$.

The paper is organised as follows: In a short Section 1 we present technicalities needed in the next section. Those generalize and rework Theorem 2.10 of [5] which gives a correspondence between left ideals of $A(R; S)$ and certain admissible sets of left ideals of $R$. In the same time it simplifies considerations from [5] and makes the paper self contained.

In Section 2 we give necessary and sufficient conditions for $A(R; S)$ to be noetherian (Theorem 2.4), left principal ideal ring (Theorem 2.7) and to be left Bézout ring (Proposition 2.9). Some applications and examples are presented. In particular it appears that $A(R; S)$ is always left Bézout provided $R$ is such. The behaviour of the noetherian property is much more complicated. Even when $S$ is a cyclic monoid, one can find examples of rings $R$ and $A(R; S)$ showing that one of those rings is left noetherian but the other is not.

Theorem 2.4 gives an answer to Problem 10 posed in [6]. The characterization presented in the statement (2) of this theorem is a generalization of the one obtained by Jordan in [2], in the case when $S = \langle \sigma \rangle$, where $\sigma$ is an injective endomorphism of $R$. However the ideas for his proof are different from ours.

1 Preliminaries

Henceforth $R$ stands for an associative unital ring and $S$ denotes a monoid which possesses a group $S^{-1}S$ of left quotients. Recall that this is the case exactly when the monoid $S$ is left and right cancellative and satisfies the left Ore condition. That is, for any $s_1, s_2 \in S$, there exist $t_1, t_2 \in S$ such that $t_1s_1 = t_2s_2$.

Let $\phi: S \to \text{End}(R)$ denote the action of $S$ on $R$ by injective endomorphisms. For any $s \in S$, the endomorphism $\phi(s) \in \text{End}(R)$ will be denoted by $\phi_s$.

**Definition 1.1.** An over-ring $A(R; S)$ of $R$ is called an S-Cohn-Jordan extension (CJ-extension, for short) of $R$ if:

1. the action of $S$ on $R$ extends to an action of $S$ (also denoted by $\phi$) on $A(R; S)$ by automorphisms, i.e. $\phi_s$ is an automorphism of $A(R; S)$, for any $s \in S$.

2. every element $a \in A(R; S)$ is of the form $a = \phi_s^{-1}(b)$, for some suitable $b \in R$ and $s \in S$.

As it was mentioned in the introduction, the CJ-extension $A(R; S)$ exists and is uniquely defined up to an $R$-isomorphism (see also [5]).

Hereafter, as in the above definition, $\phi_s$ will also denote the automorphism $\phi(s)$ of $A(R; S)$ and $\phi_s^{-1}$ will stand for its inverse $(\phi_s)^{-1}$, where $s \in S$. In particular, the preimage in $R$ of a subset $X$ of $R$ under the action of $s \in S$ is equal to $\phi_s^{-1}(X) \cap R$. 


Definition 1.2. A set \( \{X_s\}_{s \in S} \) of subsets of \( R \) is called \( S \)-admissible if, for any \( k, s \in S \), we have \( R \cap \phi_{s^{-1}}(X_{sk}) = X_k \). For such a set let \( \Delta(\{X_s\}_{s \in S}) = \bigcup_{s \in S} \phi_{s^{-1}}(X_s) \subseteq A(R; S) \).

Remark 1.3. Let \( \{X_s\}_{s \in S} \) be an \( S \)-admissible set. Then \( \phi_s(X_k) \subseteq X_{sk} \), for any \( k, s \in S \). Indeed \( \phi_s(X_k) = \phi_s(R \cap \phi_{s^{-1}}(X_{sk})) \subseteq \phi_s(R) \cap X_{sk} \subseteq X_{sk} \).

Lemma 1.4. Let \( X \) be a subset of \( A(R; S) \) and \( \Gamma(X) = \{X_s = \phi_s(X) \cap R\}_{s \in S} \). Then \( \{X_s\}_{s \in S} \) is an \( S \)-admissible set of subsets of \( R \) and \( X = \bigcup_{s \in S} \phi_{s^{-1}}(X_s) \), i.e. \( \Delta \Gamma(X) = X \).

Proof. Let \( s, k \in S \). Notice that \( R \cap \phi_{s^{-1}}(X_{sk}) = R \cap \phi_{s^{-1}}(\phi_{sk}(X) \cap R) = R \cap \phi_k(X) \cap \phi_{s^{-1}}(R) \). Then \( \phi_s(X_k) \subseteq X_{sk} \), as \( R \subseteq \phi_{s^{-1}}(R) \). This shows that \( \{X_s\}_{s \in S} \) is an \( S \)-admissible set.

The inclusion \( X \subseteq \bigcup_{s \in S} \phi_{s^{-1}}(X_s) \) is a consequence of the fact that for any \( x \in X \), there is \( s \in S \) such that \( \phi_s(x) \in R \). The reverse inclusion holds, since \( \phi_s \) is monic, for every \( s \in S \).

Notice that the set of all \( S \)-admissible sets has a natural partial ordering given by

\[ \{X_s\}_{s \in S} \preceq \{Y_s\}_{s \in S} \text{ if and only if } X_s \subseteq Y_s, \text{ for all } s \in S. \]

Proposition 1.5. There is an order-preserving one-to-one correspondence between the set \( \mathcal{L} \) of all subsets of \( A(R; S) \) ordered by inclusion and the partially ordered set \( \mathcal{R} \) of all \( S \)-admissible sets of subsets of \( R \). The correspondence is given by maps \( \Delta \) and \( \Gamma \) defined above.

Proof. By Lemma 1.4 the maps \( \Delta \) and \( \Gamma \) are well-defined and satisfy \( \Delta \Gamma = \text{id}_\mathcal{L} \). Clearly both maps preserve the ordering.

Let \( \{X_k\}_{k \in S} \) be an \( S \)-admissible set of subsets of \( R \). Then

\[ (1.1) \quad \{X_k\}_{k \in S} \leq \Gamma \Delta(\{X_k\}_{k \in S}) = \{Y_k\}_{k \in S} \]

where \( Y_k = R \cap \phi_k(\bigcup_{s \in S} \phi_{s^{-1}}(X_s)) = \bigcup_{s \in S} \phi_{ks^{-1}}(X_{sk}) \). Let \( a \in Y_k \). Then there are \( s \in S \) and \( b \in X_s \) such that \( a = \phi_{ks^{-1}}(b) \). Since \( S \) satisfies the left Ore condition, we can pick \( t, l \in S \) such that \( tk = ls \). Hence \( a = \phi_{tl^{-1}}(b) \) and \( \phi_l(a) = \phi_l(b) \in \phi_l(X_s) \subseteq X_{ts} = X_{tk} \), where the last inclusion is given by Remark 1.3. Therefore we obtain \( a \in R \cap \phi_{tl^{-1}}(X_{tk}) = X_k \), as \( \{X_s\}_{s \in S} \) is an \( S \)-admissible set. This shows that \( Y_k \subseteq X_k \), for any \( k \in S \). This together with \((1.1)\) yield that \( \{X_k\}_{k \in S} \leq \Gamma \Delta(\{X_k\}_{k \in S}) \) and complete the proof of the proposition.

Proposition 1.6. Let \( A \) be an over-ring of \( R \) such that the action of \( S \) on \( R \) extends to the action of \( S \) on \( A \) by automorphisms. Then \( B = \bigcup_{s \in S} \phi_{s^{-1}}(R) \) is a CJ-extension of \( R \).

Proof. Let \( a, b \in B \) and \( k, l \in S \) be such that \( \phi_k(a), \phi_l(b) \in R \). Since \( S \) satisfies the left Ore condition, there are \( s, t, w \in S \) such that \( sk = tl = w \). Then \( \phi_w(a) = \phi_{sk}(a), \phi_w(b) = \phi_u(b) \in R \). This implies that \( a - b, ab \in \phi_{w^{-1}}(R) \subseteq B \) and shows that \( B \) is a subring of \( A \).

By definition of \( B \), \( \phi_{k^{-1}}(B) \subseteq B \) and \( B \subseteq \phi_{l}(B) \) follows, for any \( k \in S \). The left Ore condition implies for any \( k, s \in S \) we can find \( l, t \in S \) such that \( ks^{-1} = tl^{-1} \). Then \( \phi_k(\phi_{s^{-1}}(R)) = \phi_{l^{-1}}(\phi_l(R)) \subseteq \phi_{k^{-1}}(R) \). This means that also \( \phi_k(B) \subseteq B \), for \( k \in S \). Now it is easy to complete the proof.
We will say that a subset $X$ of $A(R; S)$ is $S$-invariant if $\phi_s(X) \subseteq X$, for all $s \in S$.

Direct application of Proposition 1.6 gives the following:

**Corollary 1.7.** Let $T$ be an $S$-invariant subring of $R$. Then $\bigcup_{s \in S} \phi^{-1}_s(T) \subseteq A(R; S)$ is a CJ-extension of $T$.

**Proposition 1.8.** Let $T$ be an $S$-invariant subring of $R$ and $B = \bigcup_{s \in S} \phi^{-1}_s(T) \subseteq A(R; S)$. Let $X$ be a subset of $A(R; S)$ and $\{X_s\}_{s \in S} = \Gamma(X)$. Then:

1. $X$ is an additive subgroup (a subring) of $A(R; S)$ iff for any $s \in S$, $X_s$ is an additive subgroup (a subring) of $R$.

2. $X$ is a left (right) $B$-submodule of $A(R; S)$ iff for any $s \in S$, $X_s$ is a left (right) $T$-submodule of $R$.

**Proof.** (1). If $X$ is an additive subgroup (a subring) of $A(R; S)$, then so is $X_s = \phi_s(X) \cap R$, for any $s \in S$.

Suppose now, that $\{X_s\}_{s \in S}$ consists of additive subgroups (subrings) of $R$. Let $a, b \in X$. Then there are $s, t \in S$ such that $\phi_s(a) \in X_s$ and $\phi_t(b) \in X_t$. By the left Ore condition of $S$, we can pick $k, l \in S$ such that $ks = lt = w$. Then, making use of Remark 1.3, we have $\phi_w(a) = \phi_k(\phi_s(a)), \phi_w(b) = \phi_l(\phi_t(b)) \in X_w$. Now it is easy to complete the proof of (1).

(2). We will prove only the left version of the statement (2). Suppose that $X$ is a left $B$-submodule of $A(R; S)$ and let $s \in S$. Then $TX_s \subseteq R \cap B\phi_s(X) = R \cap \phi_s(BX) \subseteq X$, as $B = \phi_s(B)$. This together with (1) show that $X_s$ is a left $T$-submodule of $R$.

Suppose now, that $\{X_s\}_{s \in S}$ consists of left $T$-submodules of $R$. Let $b \in B$ and $x \in X$. Then there exist $s, t \in S$ be such that $\phi_s(b) \in T$, $\phi_t(x) \in X_t$. Since $T$ is $S$-invariant, similarly as in the proof of (1), we can find $w \in S$ such that $\phi_w(b) \in T$ and $\phi_w(x) \in X_w$. Then $\phi_w(bx) \in X_w$ and $bx \in \phi_{w^{-1}}(X_w) \subseteq X$ follows. This together with (1) completes the proof.

Let $T$ be an $S$-invariant subring of $R$. We will say that an $S$-admissible set $\{X_s\}_{s \in S}$ of subsets of $R$ is an $S$-admissible set of left (right) $T$-modules if each $X_s$ is a left (right) $T$-module. Propositions 1.6 and 1.8 imply the following:

**Corollary 1.9.** Let $T$ be an $S$-invariant subring of $R$ and $B = \bigcup_{s \in S} \phi^{-1}_s(T) \subseteq A(R; S)$. There is a one-to-one correspondence between the set of all left (right) $B$-submodules of $A(R; S)$ and the set of all $S$-admissible sets of left (right) $T$-submodules of $R$.

**Remark 1.10.** 1. If we take $T = R$ in the above corollary, then $B = A(R; S)$ and the corollary gives one-to-one correspondence between the set of all, left, right, two-sided ideals of $A(R; S)$ and the set of all $S$ admissible sets of all left, right, two-sided ideals of $R$, respectively.

2. Let $W, T$ be $S$-invariant subrings of $R$ such that $\bigcup_{s \in S} \phi^{-1}_s(W) = \bigcup_{s \in S} \phi^{-1}_s(T) = B \subseteq A(R; S)$ (for example assume $S$ is commutative and take $W = R$ and $T = \phi_t(R)$, for some $t \in S$). Then an $S$-admissible set $\{X_s\}_{s \in S}$ consists of left $W$-submodules as it corresponds to a $B$-submodule of $A(R; S)$. On the other hand, observe that $T$ is a left $T$-module and it does not have to be a left $W$-module.
Lemma 1.11. Let \( T \) be an \( S \)-invariant subring of \( R \), \( B = \bigcup_{s \in S} \phi_s^{-1}(T) \) its \( CJ \)-extension of \( T \) contained in \( A(R; S) \). Then, for any subset \( X \) of \( R \) and \( k \in S \) we have \( B\phi_k(X) \cap R = \bigcup_{s \in S} \phi_s^{-1}(T\phi_{sk}(X)) \cap R \).

Proof. Let \( x \in B\phi_k(X) \cap R \). Then \( x = \sum_{i=1}^{n} b_i \phi_k(x_i) \in R \), where \( b_i \in B \) and \( x_i \in X \), for \( 1 \leq i \leq n \). Let \( s \in S \) be such that \( \phi_s(b_i) \in T \), for all \( 1 \leq i \leq n \). Then \( \phi_s(x) = \sum_{i=1}^{n} \phi_s(b_i) \phi_{sk}(x_i) \in T\phi_{sk}(X) \). This shows that \( B\phi_k(X) \cap R \subseteq \bigcup_{s \in S} \phi_s^{-1}(T\phi_{sk}(X)) \cap R \). The reverse inclusion is clear as, for any \( s \in S \), we have \( \phi_{s^{-1}}(T) \subseteq B \) and \( \phi_{s^{-1}}(\phi_{sk}(X)) \subseteq \phi_k(X) \). \( \square \)

Definition 1.12. Let \( T, W \) be \( S \)-invariant subrings of \( R \). For any \((T,W)\)-subbimodule \( M \) of \( R \) and \( k \in S \) we define \( c^{(T,W)}_k(M) = \bigcup_{s \in S} \phi_s^{-1}(T\phi_{sk}(M)W) \cap R \).

Proposition 1.13. Let \( M \) be a \((T,W)\)-subbimodule of \( R \), where \( T,W \) are \( S \)-invariant subrings of \( R \) and \( B = A(T; S), C = A(W, S) \subseteq A(R; S) \). Then \( \{c^{(T,W)}_s(M)\}_{s \in S} \) is an admissible set of \((T,W)\)-bimodules associated to the \((B,C)\)-subbimodule \( BMC \) of \( A(R; S) \).

Proof. Let us consider \((B,C)\)-subbimodule \( BMC \) of \( A(R; S) \). Since \( \phi_s(B) = B \) and \( \phi_s(C) \), for all \( s \in S \), we have \( \Gamma(BMC) = \{B\phi_s(M)C \cap R\}_{s \in S} \). Now, the proof is a direct consequence of a bimodule versions of Corollary [L.9] and Lemma [L.11]. \( \square \)

Definition 1.14. Let \( M \) be a \((T,W)\)-subbimodule of \( R \), where \( T,W \) are \( S \)-invariant subrings of \( R \) and \( B = A(T; S), C = A(W, S) \subseteq A(R; S) \). We say that \( M \) is \((T,W)\)-closed if \( M = BMC \cap R \).

The following proposition offers an internal \((in R)\) characterization of \((T,W)\)-closed subbimodules of \( R \).

Proposition 1.15. For a \((T,W)\)-subbimodule \( M \) of \( R \) the following conditions are equivalent:

1. \( M \) is \((T,W)\)-closed.
2. \( c^{(T,W)}_{id}(M) = M \)
3. \( R \cap \phi_{s^{-1}}(T\phi_s(M)W) \subseteq M \), for any \( s \in S \).

Proof. Recall that \( c^{(T,W)}_{id}(M) = \bigcup_{s \in S} \phi_s^{-1}(T\phi_s(M)W) \cap R \). The equivalence \((1) \iff (2)\) is given by Proposition [L.13]. The implication \((2) \Rightarrow (3)\) is a tautology.

The statement \((3)\) yields that \( c^{(T,W)}_{id}(M) \subseteq M \) and clearly \( M \subseteq c^{(T,W)}_{id}(M) \). This shows that \((3) \Rightarrow (2)\) and completes the proof of the proposition. \( \square \)

Let us notice that if \( V \) is a \((B,C)\)-subbimodule of \( A(R; S) \), then \( V \cap R \) is a \((T,W)\)-subbimodule of \( R \) and \( V \cap R \subseteq B(V \cap R)C \cap R \subseteq V \cap R \), i.e., \( V \cap R \) is a \((T,W)\)-closed subbimodule of \( R \).

Proposition 1.16. Let \( T,W \) be \( S \)-invariant subrings of \( R \). Then:
1. If \( \{X_s\}_{s \in S} \) is an \( S \)-admissible set of \((T, W)\)-subbimodules of \( R \), then \( X_s \) is a closed \((T, W)\)-subbimodule of \( R \), for each \( s \in S \).

2. Let \( T_1 \subseteq T \) and \( W_1 \subseteq W \) be \( S \)-invariant subrings. Then any \((T, W)\)-closed subbimodule \( M \) of \( R \) is closed as \((T_1, W_1)\)-subbimodule.

Proof. By Corollary \( \text{[L.9]} \), there is \((B, C)\)-submodule \( V \) of \( A(R; S) \) such that \( X_s = \phi_s(V) \cap R \). This together with the observation made just before the proposition, gives (1).

The statement (2) is an easy exercise if we use the description (3) of closeness from of Proposition \( \text{[L.15]} \).

\[ \square \]

2 Applications

In this section we restrict our attention to left ideals, i.e. we take \( T = R \) and \( W \) is the subring of \( R \) generated by 1. In this case, for \( k \in S \), we will write \( c_k \) instead of \( c_k^{(T, W)} \). That is, by Proposition \( \text{[L.13]} \), \( c_k(M) = A(R; S)\phi_k(M) \cap R \), for any left ideal \( M \) of \( R \).

Recall (Cf. Remark \( \text{[L.10]}(1) \)) that there is one-to-one correspondence between left ideals of \( A(R; S) \) and \( S \)-admissible sets of left ideals of \( R \). If a left ideal \( L \) of \( A(R; S) \) corresponds to the \( S \)-admissible set \( \{L_s\}_{s \in S} \), we will say that \( L \) is associated to \( \{L_s\}_{s \in S} \) or that \( \{L_s\}_{s \in S} \) is associated to \( L \).

Definition 2.1. We say that an \( S \)-admissible set \( \{L_s\}_{s \in S} \) of left ideals of \( R \) is stable if there exists \( k \in S \) such that \( c_s(L_k) = L_{sk} \), for all \( s \in S \).

The following proposition offers some other characterizations of stability of \( S \)-admissible sets of left ideals.

Proposition 2.2. Let \( \{L_s\}_{s \in S} \) be an \( S \)-admissible set of left ideals of \( R \) and \( L \) be its associated left ideal of \( A(R; S) \). The following conditions are equivalent:

1. \( \{L_s\}_{s \in S} \) is stable.

2. There exists \( k \in S \) such that \( \phi_{sk}(L) = A(R; S)(\phi_{sk}(L) \cap R) \), for any \( s \in S \).

3. There exists \( k \in S \) such that \( \phi_k(L) = A(R; S)(\phi_k(L) \cap R) \).

4. There exist \( k \in S \) and a left ideal \( W \) of \( R \) such that \( \phi_k(L) = A(R; S)W \).

Proof. (1) \( \Rightarrow \) (2). Suppose \( \{L_s\}_{s \in S} \) is stable, that is we can pick \( k \in S \) such that \( c_s(L_k) = L_{sk} \), for all \( s \in S \). Recall that \( L_s = \phi_s(L) \cap R \). This means that \( \{L_{sk}\}_{s \in S} = \{c_s(L_k)\}_{s \in S} \) is an \( S \)-admissible set of left ideals of \( R \) associated to \( \phi_k(L) \). Now, Proposition \( \text{[L.13]} \) applied to \( M = L_k \), yields that the left ideals \( \phi_k(L) \) and \( A(R; S)L_k \) of \( A(R; S) \) have the same associated \( S \)-admissible sets. Hence, by Proposition \( \text{[L.15]} \), \( \phi_k(L) = A(R; S)L_k \). Then, for any \( s \in S \), we have

\[
A(R; S)(\phi_s(\phi_k(L)) \cap R) \subseteq \phi_s(\phi_k(L)) = \phi_s(A(R; S)(\phi_k(L) \cap R)) \subseteq \phi_s(A(R; S)(\phi_k(L) \cap \phi_s(R))) \subseteq A(R; S)(\phi_s(\phi_k(L)) \cap R).
\]
This shows that \( \phi_{sk}(L) = A(R; S)(\phi_{sk}(L) \cap R) \), i.e. (2) holds.

The implications (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are tautologies.

(4) \( \Rightarrow \) (1). Let \( k \in S \) and the left ideal \( W \) of \( R \) be such that \( \phi_k(L) = A(R; S)W \). Eventually replacing \( W \) by \( \phi_k(L) \cap R \), we may additionally assume that \( W = \phi_k(L) \cap R = L_k \). Therefore, by Proposition 1.13, the left ideal \( \phi_k(L) \) of \( A(R; S) \) is associated to the \( S \)-admissible set \( \{c_s(L_k)\}_{s \in S} \). Also, by definition, \( \phi_k(L) \) is associated to \( \{\phi_s(\phi_k(L)) \cap R\}_{s \in S} = \{L_{sk}\}_{s \in S} \). This shows that \( c_s(L_k) = L_{sk} \), for any \( s \in S \) and completes the proof of the implication.

**Corollary 2.3.** Suppose that the \( S \)-admissible set \( \{L_s\}_{s \in S} \) of left ideals of \( R \) is associated to a finitely generated left ideal of \( A(R; S) \). Then \( \{L_s\}_{s \in S} \) is stable.

**Proof.** Let \( L = A(R; S)a_1 + \ldots + A(R; S)a_n \) be a left ideal of \( A(R; S) \) associated to \( \{L_s\}_{s \in S} \) and \( k \in S \) be such that \( \phi_k(a_i) = b_j \in R \), for \( 1 \leq i \leq n \). Then \( \phi_k(L) = A(R; S)W \), where \( W = \sum_{i=1}^{n} Rb_i \). Thus the condition (4) of Proposition 2.2 holds, i.e. \( \{L_s\}_{s \in S} \) is stable.

Recall (Cf. Definition 1.14) that a left ideal \( X \) of \( R \) is closed if \( X = A(R; S)X \cap R \) and that \( A(R; S)X \cap R \) is always a closed left ideal of \( R \). This implies that \( A(R; S)X \cap R \) is the smallest closed left ideal of \( R \) containing \( X \). We will call it the closure of \( X \) and denote by \( \overline{X} \). Proposition 1.13 offers an internal characterization of the closure of \( X \), namely \( \overline{X} = \bigcup_{s \in S} \phi_{s^{-1}}(R\phi_s(X)) \cap R \).

With all the above preparation we are ready to prove the following theorem.

**Theorem 2.4.** For the CJ-extension \( A(R; S) \) of \( R \) the following conditions are equivalent:

1. \( A(R; S) \) is left noetherian;
2. The ring \( R \) has ACC on closed left ideals and every \( S \)-admissible set of left ideals is stable;
3. Every closed left ideal of \( R \) is the closure of a finitely generated left ideal of \( R \) and every \( S \)-admissible set of left ideals is stable.

**Proof.** (1) \( \Rightarrow \) (2). Suppose \( A(R; S) \) is left noetherian. Let \( X_1 \subseteq X_2 \subseteq \ldots \) be a chain of closed left ideals of \( R \). Since \( A(R; S) \) is left noetherian, there exists \( n \geq 1 \) such that \( A(R; S)X_n = A(R; S)X_{n+m} \), for all \( m \geq 0 \). By assumption, every \( X_i \)'s is closed, so \( X_n = A(R; S)X_n \cap R = A(R; S)X_{n+m} \cap R = X_{n+m} \), for all \( m \geq 0 \). This shows that \( R \) has ACC on closed left ideals.

Since \( A(R; S) \) is left noetherian, every \( S \)-admissible set \( \{L_s\}_{s \in S} \) of left ideals is associated to a finitely generated left ideal of \( A(R; S) \). Hence, by Corollary 2.3 \( \{L_s\}_{s \in S} \) is stable.

(2) \( \Rightarrow \) (3). The proof is a version of a standard argument. Let \( W \) be a closed left ideal of \( R \). Consider the set \( \mathcal{W} \) of all closures \( \overline{I} \), where \( I \) ranges over all finitely generated left ideals \( I \) of \( R \) contained in \( W \). Notice that if \( \overline{I} \in \mathcal{W} \) and \( b \in W \), then \( \overline{I + Rb} \subseteq \overline{W} = W \).

Since \( R \) satisfies ACC on closed left ideals, we can pick a maximal element \( \overline{M} \) in \( \mathcal{W} \) and the remark above yields \( W = \overline{M} \).
(3) $\Rightarrow$ (1). Let $L$ be a left ideal of $A(R;S)$ and \{ $L_s$ $\}_{s \in S}$ its $S$-admissible set of left ideals of $R$. By assumption, \{ $L_s$ $\}_{s \in S}$ is stable. Thus, by Proposition 4.2 there exist $k \in S$ and a left ideal $W$ of $R$ such that $\phi_k(L) = A(R;S)W$. Replacing $W$ by $W$ we may additionally suppose that $W$ is closed. Then, by assumption, there exist $b_1, \ldots, b_n \in R$ such that $W = Rb_1 + \ldots + Rb_n$. Notice that $A(R;S)b_1 + \ldots + A(R;S)b_n \subseteq \phi_k(L) = A(R;S)(R \cap (A(R;S)b_1 + \ldots + A(R;S)b_n)) \subseteq A(R;S)b_1 + \ldots + A(R;S)b_n$. This shows that $\phi_k(L)$ is a finitely generated left ideal of $A(R;S)$. Since $\phi_k$ is an automorphism of $A(R;S)$, $L$ is also finitely generated. \qed

The above theorem gives immediately:

**Corollary 2.5.** Suppose that $R$ left noetherian. Then $A(R;S)$ is left noetherian iff every $S$-admissible set of left ideals of $R$ is stable.

The equivalence (1) $\iff$ (2) in Theorem 2.4 is a generalization of Theorem 5.6 [2] from the case when the monoid $S$ is cyclic to the case when $S$ is a cancellative monoid satisfying the left Ore condition. The idea of the presented proof is completely different from the one used in [2].

It is known that there exist rings $R$ such that only one of $R$ and $A(R;S)$ is left noetherian. The following example, which offers such rings, is a variation of examples from [2].

**Example 2.6.** 1. Let $\sigma$ be the endomorphism of the polynomial ring $\mathbb{Z}[x]$ given by $\sigma(x) = 2x$. One can check that $A(\mathbb{Z}[x]; \langle \sigma \rangle) = \mathbb{Z} + \mathbb{Z}\left[\frac{1}{2}\right][x]x$ is not noetherian.

2. Let $A$ denote the field of rational functions in the set \{ $x_i$ $\}_{i \in \mathbb{Z}}$ of indeterminates over a field $F$ and $\sigma$ be the $F$-endomorphism of $R = F(x_i \mid i \leq 0)|x_i \mid i > 0|$ given by $\sigma(x_i) = x_{i-1}$, for $i \in \mathbb{Z}$. Then $R$ is not noetherian and $A = A(R; \langle \sigma \rangle)$ is a field.

The following theorem offers necessary and sufficient conditions for $A(R;S)$ to be left principal ideal ring.

**Theorem 2.7.** For the CJ-extension $A(R;S)$ of $R$ the following conditions are equivalent:

1. Every left ideal of $A(R;S)$ is principal;

2. Every $S$-admissible set \{ $L_s$ $\}_{s \in S}$ of left ideals of $R$ satisfies the following conditions:
   \( a \) \{ $L_s$ $\}_{s \in S}$ is stable,
   \( b \) There exist $t \in S$ and $b \in R$ such that $L_t = Rb$.

**Proof.** (1) $\Rightarrow$ (2). Let \{ $L_s$ $\}_{s \in S}$ be an $S$-admissible set of left ideals of $R$ and $L$ be its associated left ideal of $A(R;S)$. Since every left ideal of $A(R;S)$ is principal, Corollary 2.3 implies that the property (a) holds.

Let $a \in A(R;S)$ and $t \in S$ be such that $L = A(R;Sa)$ and $b = \phi_t(a) \in R$. Then $L_t = \phi_t(L) \cap R = A(R;S)b \cap R = Rb$, i.e. the property (b) is satisfied.

(2) $\Rightarrow$ (1). Let $L$ be a left ideal of $A(R;S)$ and \{ $L_s$ $\}_{s \in S}$ be its associated $S$-admissible set of left ideals of $R$. By assumption, \{ $L_s$ $\}_{s \in S}$ is stable. Thus, applying Proposition 2.2, we can pick $k \in S$ such that $\phi_{sk}(L) = A(R;SL_{sk})$, for any $s \in S$. Observe that
\[ \{L_{sk}\}_{s \in S} = \{\phi_s(\phi_k(L)) \cap R\}_{s \in S} \] is an S-admissible set of left ideals associated to \( \phi_k(L) \).

Therefore we can apply (2)(b) to \( \{L_{sk}\}_{s \in S} \) and pick \( t \in S \) and \( b \in R \) such that \( L_{tk} = \overline{Rb} \).

Let us set \( t = lk \). Using the above we have \( \phi_t(L) = A(R; S)L_t \) and \( A(R; S)b \subseteq A(R; S)L_t = A(R; S)\overline{Rb} \subseteq A(R; S)b \). This shows that \( \phi_t(L) = A(R; S)b \) and proves that the left ideal \( L = A(R; S)\phi_{s^{-1}}(b) \) is principal.

\[\text{Remark 2.8.}\]
1. It is not difficult to prove that the condition (2)(b) of the above theorem is equivalent to the condition that every closed left ideal \( X \) of \( R \) is of the form \( X = \phi_{t^{-1}}(\overline{Rb}) \cap R \), for suitable \( t \in S \) and \( b \in R \).
2. Let us remark that the condition (2)(b) always holds, provided every closed left ideal is principal.

Recall that a ring \( R \) is left Bézout if every finitely generated left ideal of \( R \) is principal.

**Proposition 2.9.** For the CJ-extension \( A(R; S) \) of \( R \) the following conditions are equivalent:

1. \( A(R; S) \) is a left Bézout ring;

2. for every \( S \)-admissible set \( \{L_s\}_{s \in S} \) associated to a finitely generated left ideal \( L \) of \( A(R; S) \), there exist \( t \in S \) and \( b \in R \) such that \( L_t = \overline{Rb} \).

**Proof.** Let \( L \) be a finitely generated left ideal of \( A(R; S) \) and \( \{L_s\}_{s \in S} \) its associated \( S \)-admissible set.

If \( A(R; S) \) is left Bézout, then \( L \) is principal. Thus there is \( t \in S \) and \( b \in R \) such that \( \phi_t(L) = A(R; S)b \) and \( L_t = \phi_t(L) \cap R = \overline{Rb} \). This shows that (1) implies (2).

Suppose (2) holds. Then, by Corollary 2.3, \( \{L_s\}_{s \in S} \) is stable.

Now one can complete the proof as in the proof of implication (3) \( \Rightarrow \) (1) of Theorem 2.7.

Notice that the characterization obtained in the above proposition is not nice in the sense that the statement (2) is not expressed in terms of properties of \( R \) but \( A(R; S) \) is involved. Anyway it has the following direct application:

**Corollary 2.10.** Suppose that one of the following conditions is satisfied:

1. Every closed left ideal of \( R \) is principal.
2. \( R \) is left Bézout.

Then \( A(R; S) \) is a left Bézout ring.

**Proof.** Proposition 2.9 and Remark 2.8(2) give the thesis when (1) holds.

Suppose (2) holds. Let \( L = A(R; S)a_1 + \ldots A(R; S)a_n \) and \( t \in S \) be such that \( b_i = \phi_t(a_i) \in R \), \( 1 \leq i \leq n \). By assumption, there exists \( b \in R \) such that \( Rb_1 + \ldots Rb_n = \overline{Rb} \).

Then \( L_t = \phi_t(L) \cap R = A(R; S)b \cap R = \overline{Rb} \) and the thesis is a consequence of Proposition 2.9.

The following example offers a principal ideal domain \( R \) such that \( A(R; S) \) is not noetherian. Of course, by Corollary 2.10, \( A(R; S) \) is left Bézout.
Example 2.11. Let \( A = K[x^{\frac{1}{n}} \mid n \in \mathbb{N}] \), where \( K \) is a field, and \( \sigma \) be a \( K \)-linear automorphism of \( A \) defined by \( \sigma(x) = x^2 \). Then the restriction of \( \sigma \) to \( R = K[x] \) is an endomorphism of \( R \) and it is easy to check that \( A \) is a CJ-extension of \( R \) with respect to the action of \( \sigma \). Notice that \( A \) is not noetherian but it is Bézout, by the above corollary.

In view of Theorem 2.7 and Proposition 2.9 it seems interesting to know when all principal left ideals of \( R \) are closed. We will concentrate on this problem till the end of the paper. It is known (Cf. Lemma 1.16 and Theorem 2.24 of [5]) that if \( R \) is a semiprime left Goldie ring, then:

(i) every regular element \( c \) of \( R \) is regular in \( A(R; S) \);

(ii) \( A(R; S) \) is a semiprime left Goldie ring and \( Q(A(R; S)) = A(Q(R); S) \), where \( Q(B) \) denotes the classical left quotient ring of a left Goldie ring \( B \).

Therefore both \( Q(R) \) and \( A(R; S) \) are over-rings of \( R \) included in \( A(Q(R); S) \). Keeping the above notation we have:

**Proposition 2.12.** For a semiprime left Goldie ring \( R \), the following conditions are equivalent:

1. \( Q(R) \cap A(R; S) = R \);
2. \( Rc = \overline{Rc} \) and \( cR = \overline{cR} \), for every regular element \( c \in R \);
3. \( cR = \overline{cR} \), for every regular element \( c \in R \);
4. If \( ca \in R \), then \( a \in R \), provided \( a \in A(R; S) \) and \( c \in R \) is regular.

**Proof.** Let \( c \in R \) be a regular element.

(1) \( \Rightarrow \) (2) Let \( a \in A(R; S) \) be such that \( ac = r \in R \). Then \( a = rc^{-1} \in Q(R) \cap A(R; S) = R \). This shows that \( A(R; S)c \cap R \subseteq Rc \) and implies that \( Rc = \overline{Rc} \). A similar argument works for showing that \( cR = \overline{cR} \).

The implication (2) \( \Rightarrow \) (3) is a tautology.

(3) \( \Rightarrow \) (4) Suppose \( ca \in R \), where \( a \in A(R; S) \). By (3) we have \( cR = \overline{cR} = cA(R; S) \cap R \). Thus there exists \( r \in R \) such that \( ca = cr \) and \( a = r \in R \) follows, as \( c \) is regular in \( A(R; S) \).

(4) \( \Rightarrow \) (1) Let \( r \in R \) be such that \( c^{-1}r = a \in Q(R) \cap A(R; S) \). The condition (4) gives \( a \in R \) and shows that \( Q(R) \cap A(R; S) = R \).

The statement (2) in the above proposition is left-right symmetric thus, additionally assuming that the semiprime ring \( R \) is also right Goldie, we can add to the proposition left versions of statements (3) and (4). However, as the following example shows, we can not do this when \( R \) is not right Goldie.

**Example 2.13.** Let \( D \) denote the field of fractions of the ring \( K[x^{\frac{1}{n}} \mid n \in \mathbb{N}] \) from Example 2.11 and \( \sigma \) be a \( K \)-linear automorphism of \( D \) defined by \( \sigma(x) = x^2 \). Let us consider the skew polynomial ring of endomorphism type (with coefficients written on the left) \( A = D[t; \sigma] \). Then \( \sigma \) can be extended to an automorphism of \( A \) by setting \( \sigma(t) = t \). Let \( R = K(x)[t; \sigma] \subseteq A \). Then the restriction of \( \sigma \) to \( R \) is an endomorphism of \( R \) and
for any $w \in A$, there exists $n \geq 1$ such that $\sigma^n(w) \in R$. This means that $A = A(R; \langle \sigma \rangle)$, where $\langle \sigma \rangle$ denotes the monoid generated by $\sigma$.

It is well known that $R$ is a left Ore domain which is not right Ore. Observe that $t\sqrt{x} = xt \in R$, but $\sqrt{x} \notin R$. Thus, by Proposition 2.12, $Q(R) \cap A \neq R$. In fact, the left localization of $R$ with respect the left Ore set consisting of all powers of $t$ is equal to $D[t, t^{-1}, \sigma]$. Thus $A \subseteq D[t, t^{-1}, \sigma] \subseteq Q(R)$.

We claim that $R$ satisfies the left version of statement (4) from Proposition 2.12. Let $0 \neq c \in R$ and $a \in A$ be such that $ac \in R$. If $a \notin R$, then we can choose such $a = \sum_{i=0}^{n} a_i t^i$ of minimal possible degree, say $n$. Then, by the choice of $n$, $a_n \notin K(x)$. Then also $a_n \sigma^n(c_m) \notin K(x)$, where $c_m$ denotes the leading coefficient of $c \in R = K(x)[t; \sigma]$ and then $ac \notin R$, which is impossible. Thus $a$ has to belong to $R$.

Observe that the ring from Example 2.11 satisfies the assumption of the following proposition.

**Proposition 2.14.** Suppose $R$ is a left Ore domain such that $Q(R) \cap A(R; S) = R$. For a left ideal $L$ of $A$ the following conditions are equivalent:

1. $L$ is principal;
2. $\exists s \in S \exists a \in R$ such that, for all $t \in S$, $\phi_{ts}(L) \cap R = L_{ts} = R\phi_t(a)$.

**Proof.** (1) $\Rightarrow$ (2). Suppose $L = Ab$ and let $s \in S$ be such that $\phi_s(a) \in R$. Then $\phi_s(L) = A\phi_s(b)$. Set $a = \phi_s(b)$. Now the implication is a direct consequence of Proposition 2.12.

(2) $\Rightarrow$ (1). Let $s \in S$ and $a \in R$ be as in (2). Then $\phi_s(L) = L_{ts}R\phi_t(a)$. This means that $\phi_s(L) = \bigcup_{t \in S} \phi_t^{-1}(R)a = A(R; S)a$. Thus $L = A(R; S)\phi_s^{-1}(a)$. $\square$

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