Harish-Chandra modules over the generalized
$W$-algebra $W(2,2)$
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Abstract: In this paper, using the theory of $A$-cover developed in [2], we completely classify all simple Harish-Chandra modules for any generalized $W$-algebra $W(2,2)$. As a byproduct, we also obtain the classification of simple Harish-Chandra modules over the classical $W$-algebra $W(2,2)$ studied in [7,12,17].

Key words: generalized $W$-algebra $W(2,2)$, Harish-Chandra module, weight module.

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1 Introduction

In the representation theory of infinite-dimensional Lie algebras, there is a very important class of weight modules called Harish-Chandra modules (namely, weight modules with finite-dimensional weight spaces). It is well-known that the classification of simple Harish-Chandra modules over the Virasoro algebra (also called $N=0$ superconformal algebra), conjectured by Kac (see [13]), was given in [20]. Combining [23] with [27], a new method was presented to obtain this classification. After that, a lot of general versions of the Virasoro algebra have been investigated by some authors. Those include, but are not limited to, the generalized Virasoro algebra (see, e.g., [11,19,22,25,28,31]), the generalized Heisenberg-Virasoro algebra (see [10,15,18]), the $W$-algebra $W(2,2)$ (see [7,12,17]), the loop-Virasoro algebra (see [12]), and so on. To classify all simple Harish-Chandra modules over the Lie algebra $W_n$ of vector fields on $n$-dimensional torus, Billig and Futorny developed a very useful technique called $A$-cover theory in [2]. The new result gained here was a generalized version of Mathieu’s classification theorem for the Virasoro algebra. From then on, the $A$-cover theory was used in some other Lie (super)algebras (see, e.g., [3,4,7,32]).

The $W$-algebra $W(2,2)$ was introduced in [33] by Zhang and Dong for the investigated the classification of simple vertex operator algebras generated by two weight 2 vectors. The centerless $W$-algebra $W(2,2)$ $\overline{W}[Z]$ can be obtained from the point of view of non-relativistic analogues of the conformal field theory. By using the “non-relativistic limit” on a pair of commuting algebras vect$(S^1) \oplus$ vect$(S^1)$ (see [26]) via a group contraction, one has the following generators

$$L_q = -t^{q+1} \frac{d}{dt} - (q+1)t^q y \frac{d}{dy} - (q+1)\sigma t^q - q(q+1)\eta t^{q-1} y,$$

$$W_q = -t^{q+1} \frac{d}{dy} - (q+1)\eta t^q,$$

where $q \in \mathbb{Z}$, $\sigma$ and $\eta$ are respectively the scaling dimension and a free parameter. Then $\overline{W}[Z]$ is the Lie algebra with the basis $\{L_q, W_q \mid q \in \mathbb{Z}\}$ and the non-vanishing commutators
as follows
\[ [L_q, L_{q'}] = (q' - q)L_{q+q'}, \quad [L_q, W_{q'}] = (q' - q)W_{q+q'} \]
for \( q, q' \in \mathbb{Z} \). It is easy to see that \( \mathcal{W}[\mathbb{Z}] \) is an infinite-dimensional extension of an algebra called either non-relativistic or conformal Galilei algebra \( \text{CGA}(1) \cong \langle L_{\pm1, 0}, W_{\pm1, 0} \rangle \) (see [9]). As everyone knows, the relationship with conformal algebras makes it widely studied in string theory (see [6]). Furthermore, \( \mathcal{W}[\mathbb{Z}] \) can be realized by the semidirect product of the Witt algebra \( \mathcal{V}[\mathbb{Z}] \) and the \( \mathcal{V}[\mathbb{Z}] \)-module \( A_{0, -1} \) of the intermediate series in [14], that is, \( \mathcal{W}[\mathbb{Z}] \cong \mathcal{V}[\mathbb{Z}] \ltimes A_{0, -1} \). What we concern most is the approach of realizing \( \mathcal{W}[\mathbb{Z}] \) from a truncated loop-Witt algebra (see, e.g., [11, 16]). The detailed description on it will be shown in Section 3.1 which is closely associated with the usage of \( \mathcal{A} \)-cover theory.

The aim of this paper is to present a completely classification of simple Harish-Chandra modules over the generalized \( W \)-algebra \( W(2, 2) \) (see Section 2.1). In particular, we classify the simple Harish-Chandra modules over the high rank \( W \)-algebra \( W(2, 2) \) (i.e., rank \( k \) \( W \)-algebra \( W(2, 2) \)), which reobtains the classification result of classical \( W \)-algebra \( W(2, 2) \) (when \( k = 1 \)) studied in [7, 12, 17]. The method here mainly depends on the theory of \( \mathcal{A} \)-cover given in [2].

The paper is organized as follows. In Section 2, we introduce some notations and definitions related to generalized \( W \)-algebra \( W(2, 2) \) and Harish-Chandra modules. We also recall some known classification theorems over several related Lie algebras for later use. In Section 3, we give a classification of all simple Harish-Chandra modules for any generalized \( W \)-algebras \( W(2, 2) \). We first use the theory of \( \mathcal{A} \)-cover developed in [2] to determine the simple Harish-Chandra modules over the higher rank \( W \)-algebra \( W(2, 2) \), which give the classification result over the classic \( W \)-algebra \( W(2, 2) \) as a byproduct (also see [7, 12, 17]). Note that the method we employed here is conceptional and non-computational. At last, by the similar job in [10, 11], we can generalize this result to the general case.

Throughout the present article, we denote by \( \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N} \) and \( \mathbb{Z}_+ \) the sets of complex numbers, real numbers, integers, nonnegative integers and positive integers, respectively. All vector spaces and Lie algebras are over \( \mathbb{C} \). All simple modules are considered to be non-trivial. For a Lie algebra \( \mathfrak{L} \), we use \( U(\mathfrak{L}) \) to denote the universal enveloping algebra.

2 Preliminaries

2.1 The generalized \( W \)-algebra \( W(2, 2) \) and its cuspidal module

Let \( \Gamma \) be an additive subgroup of \( \mathbb{C} \). We first present the definition of generalized \( W \)-algebra \( W(2, 2) \)
\[ \mathcal{W}[\Gamma] = \bigoplus_{\alpha \in \Gamma} \mathbb{C}L_\alpha \oplus \bigoplus_{\alpha \in \Gamma} \mathbb{C}W_\alpha \oplus \mathbb{C}C, \]
which satisfies the following Lie brackets

\[ [L_\alpha, L_\beta] = (\beta - \alpha) L_{\alpha+\beta} + \delta_{\alpha+\beta,0} \frac{\alpha^3 - \alpha}{12} C, \]
\[ [L_\alpha, W_\beta] = (\beta - \alpha) W_{\alpha+\beta} + \delta_{\alpha+\beta,0} \frac{\alpha^3 - \alpha}{12} C, \tag{2.1} \]
\[ [W_\alpha, W_\beta] = [\mathcal{W}[\Gamma], C] = 0, \]

where \( \alpha, \beta \in \Gamma \) and \( C \) is the central element. Clearly, \( \mathcal{W}[\Gamma] \) has an infinite-dimensional Lie subalgebra \( \mathcal{V}[\Gamma] := \text{span}\{L_\alpha, C \mid \alpha \in \Gamma\} \), which is called generalized Virasoro algebra. The quotient algebras \( \mathcal{V}[\Gamma] = \mathcal{W}[\Gamma] / \mathcal{C} \) and \( \mathcal{V}[\Gamma] = \mathcal{W}[\Gamma] / \mathcal{C} \) are respectively called generalized centerless \( W \)-algebra \( W (2, 2) \) and generalized \( \mathcal{Z} \)-algebra \( W (2, 2) \). When \( \Gamma = \mathbb{Z} \), we say that \( \mathcal{V}[\mathbb{Z}] \) and \( \mathcal{W}[\mathbb{Z}] \) are respectively classical Virasoro algebra and classical \( \mathcal{W} \)-algebra \( W (2, 2) \). When \( \Gamma = \mathbb{Z}^k \) for some \( k \in \mathbb{Z}_+ \), the resulting algebras \( \mathcal{V}[\mathbb{Z}^k] \) and \( \mathcal{W}[\mathbb{Z}^k] \) are respectively called rank \( k \) Virasoro algebra and rank \( k \) \( \mathcal{W} \)-algebra \( W (2, 2) \).

In the following, we shall recall some definitions related to the weight module. Consider a non-trivial module \( M \) over \( \mathcal{V}[\Gamma] \) or \( \mathcal{W}[\Gamma] \). We always assume that the action of the central element \( C \) is a scalar. The module \( M \) is said to be trivial if the action of whole algebra on \( M \) is trivial. Denote \( M_\lambda = \{v \in M \mid L_0 v = \lambda v\} \), which is called a weight space of weight \( \lambda \in \Gamma \). We call that \( M \) is a weight module if \( M = \bigoplus_{\lambda \in \Gamma} M_\lambda \). Set \( \text{Supp}(M) = \{\lambda \mid M_\lambda \neq 0\} \), which is called the support (or called the weight set) of \( M \). The indecomposable weight module \( M \) with all weight spaces one-dimensional is called the intermediate series module.

**Definition 2.1.** Let \( M \) be a weight module over \( \mathcal{W}[\Gamma] \).

1. If \( \dim(M_\lambda) < +\infty \) for all \( \lambda \in \text{Supp}(M) \), then \( M \) is called Harish-Chandra module.

2. If there exists some \( K \in \mathbb{Z}_+ \) such that \( \dim(M_\lambda) < K \) for all \( \lambda \in \text{Supp}(M) \), then \( M \) is called cuspidal (or uniformly bounded).

Now we show a class of cuspidal \( \mathcal{W}[\Gamma] \)-modules, which are exactly intermediate series modules for \( \mathcal{W}[\Gamma] \).

**Definition 2.2.** For any \( g, h \in \mathbb{C} \), the \( \mathcal{W}[\Gamma] \)-module \( M(g, h; \Gamma) \) has a \( \mathbb{C} \)-basis \( \{v_\beta \mid \beta \in \Gamma\} \) and the \( \mathcal{W}[\Gamma] \)-action:

\[ L_\alpha v_\beta = (g + \beta + h\alpha)v_{\alpha+\beta}, \quad W_\alpha v_\beta = Cv_\beta = 0. \]

Clearly, the modules \( M(g, h; \Gamma) \) are isomorphic to the intermediate series modules of \( \mathcal{V}[\Gamma] \). By [31], we see that the modules \( M(g, h; \Gamma) \) are reducible if and only if \( g \in \Gamma \) and \( h \in \mathbb{C} \).
We use \( \overline{M}(g, h; \Gamma) \) to denote the unique non-trivial simple subquotient of \( M(g, h; \Gamma) \). Then \( \text{Supp}(\overline{M}(g, h; \Gamma)) = g + \Gamma \) or \( \text{Supp}(\overline{M}(g, h; \Gamma)) = \Gamma^* \). We also define \( \overline{M}(g, h; \Gamma) \) as intermediate series modules of \( W[\Gamma] \).

### 2.2 Generalized highest weight modules

In this subsection, a general class of Lie algebras are considered. Assume that \( H = \sum_{\alpha \in \Gamma} H_\alpha \) is a \( \Gamma \)-graded Lie algebra such that \( H_0 \) is abelian. And the gradation of \( H \) is the root space decomposition with respect to \( H_0 \).

Let \( g \) be a subgroup of \( \Gamma \) such that \( \Gamma = g \oplus \mathbb{Z} \mu \) for some \( \mu \in \Gamma \). We define the subalgebra of \( H \) as follows

\[
H_g = \bigoplus_{\alpha \in g} H_\alpha, \quad H_g^+ = \bigoplus_{\alpha \in g, m \in \mathbb{Z}_+} H_{\alpha + m\mu}, \quad H_g^- = \bigoplus_{\alpha \in g, m \in \mathbb{Z}_+} H_{\alpha - m\mu}.
\]

Let \( K \) be a simple \( H_g \)-module. Then \( K \) can be extended to an \( (H_g^+ + H_g^-) \)-module by defining \( H_g^+ K = 0 \). Now we can define the generalized Verma module \( V_{g, \mu, K} \) for \( H \) as

\[
V_{g, \mu, K} = \text{Ind}_{H_g^+ + H_g^-}^{H_g} K = U(H) \bigotimes_{U(H_g^+ + H_g^-)} K.
\]

It is easy to know that \( V_{g, \mu, K} \) has a unique simple quotient module for \( H \) and we write it as \( P_{g, \mu, K} \). Then \( P_{g, \mu, K} \) is called a simple highest weight module. As far as we know, the generalized Verma module (or generalized highest weight module) was introduced and investigated in some other references (see, e.g., \([5, 8, 21]\)).

Now we introduce an arbitrary \( H \) with \( \Gamma \cong \mathbb{Z}^k \) for \( k \in \mathbb{Z}_+ \). Fix a basis of \( \Gamma \) and identify \( \Gamma \) with \( \mathbb{Z}^k \). Assume that \( M \) is a weight module for \( H \). Then \( M \) is called dense if \( \text{Supp}(M) = \lambda + \Gamma \) for some \( \lambda \in H_0^\ast \). On the other hand, if there exist \( \lambda \in \text{Supp}(M), \tau \in \mathbb{R}^k \setminus \{0\} \) and \( \beta \in \Gamma \) such that

\[
\text{Supp}(M) \subseteq \lambda + \beta + \Gamma^{(\tau)}_{\leq 0},
\]

where \( \Gamma^{(\tau)}_{\leq 0} = \{ \alpha \in \Gamma \mid (\tau|\alpha) \leq 0 \} \) and \( (\tau|\alpha) \) is the usual inner product in \( \mathbb{R}^k \), then \( M \) is called cut. Obviously, the modules \( P_{g, \mu, K} \) defined above are cut modules. If there exist a \( \mathbb{Z} \)-basis \( \{\epsilon_1, \ldots, \epsilon_k\} \) of \( \Gamma \) and \( K \in \mathbb{Z}_+ \) such that \( H_\alpha v = 0 \) for all \( \alpha = \sum_{i=1}^k \alpha_i \epsilon_i \) with \( \alpha_i > K, i \in \{1, \ldots, k\} \), then the element \( v \in M \) is called a generalized highest weight vector.

The following general result of cut \( H \)-modules appeared in Theorem 4.1 of \([21]\).

**Theorem 2.3.** Let \( [H_\alpha, H_\beta] = H_{\alpha + \beta} \) for all \( \alpha \neq \beta \in \Gamma \cong \mathbb{Z}^k, k \in \mathbb{Z}_+ \). Assume that \( M \) is a simple weight module over \( H \), which is neither dense nor trivial. If \( M \) contains a generalized highest weight vector, then \( M \) is a cut module.
2.3 The know results

The classification theorems of simple Harish-Chandra modules over the classical $W$-algebra $W(2,2)$ and generalized Virasoro algebra will be recalled in this subsection.

It is well known that simple Harish-Chandra modules over the Virasoro algebra and $\mathbb{Q}$-Virasoro algebra were respectively classified in [20] and [22]. These results were extended to arbitrary generalized Virasoro algebras (see [11,19,30]), and we summarize them as follows.

**Theorem 2.4.** Let $\Gamma$ be an arbitrary additive subgroup of $\mathbb{C}$.

(a) If $\Gamma \cong \mathbb{Z}$, then any non-trivial simple Harish-Chandra module for $V[\Gamma]$ is either a module of intermediate series, or a highest/lowest weight module;

(b) If $\text{rank}(\Gamma) = 1$ and $\Gamma \not\cong \mathbb{Z}$, then any non-trivial simple Harish-Chandra module for $V[\Gamma]$ is a module of intermediate series;

(c) If $\text{rank}(\Gamma) > 1$, then any non-trivial simple Harish-Chandra module for $V[\Gamma]$ is either a module of intermediate series or isomorphic to $P^{V[\Gamma]}_{\mathfrak{g},\mu,K}$ for some $\mu \in \Gamma^*$, a subgroup $\mathfrak{g}$ of $\Gamma$ with $\Gamma = \mathfrak{g} \oplus \mathbb{Z}\mu$ and a non-trivial simple intermediate series $V[\mathfrak{g}]$-module $K$.

**Lemma 2.5.** (see [15]) Let $\mathcal{P} = P^{V[\Gamma]}_{\mathfrak{g},\gamma,K}$ be the simple module defined in Theorem 2.4. Then for any subgroup $\tilde{\Gamma}$ of $\Gamma$ and $\lambda \in \text{Supp}(\mathcal{P})$, the $V[\tilde{\Gamma}]$ module $P_{\lambda+\tilde{\Gamma}}$ is cuspidal if and only if $\tilde{\Gamma} \subseteq \mathfrak{g}$.

Using Lemma 2.5 and similar proof in [15], we get the following result for the generalized $W$-algebra $W(2,2)$.

**Lemma 2.6.** Assume that $\mathcal{P} = P^{W[\Gamma]}_{\mathfrak{g},\gamma,K}$ is the simple module defined in Section 2.2 with $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{Z}_+$. Then for any subgroup $\tilde{\Gamma}$ of $\Gamma$ and $\lambda \in \text{Supp}(\mathcal{P})$, the $W[\tilde{\Gamma}]$ module $P_{\lambda+\tilde{\Gamma}}$ is cuspidal if and only if $\tilde{\Gamma} \subseteq \mathfrak{g}$.

The classification of Harish-Chandra modules over the classical $W$-algebra $W(2,2)$ was given in [17], which was reobtained in [7,12] by some new ideas.

**Theorem 2.7.** Any non-trivial simple Harish-Chandra module over $W[\mathbb{Z}]$ is either a module of intermediate series, or a highest/lowest weight module.

Let $V$ be a Harish-Chandra module over $V[\mathbb{Z}]$ or $W[\mathbb{Z}]$. Then $V$ is called *positively truncated* if $\text{Supp}(V) \subseteq a - \mathbb{N}$ for some $a \in \mathbb{C}$ and is called *negatively truncated* if $\text{Supp}(V) \subseteq a + \mathbb{N}$ for some $a \in \mathbb{C}$. Now, let us recall the following result about the indecomposable Harish-Chandra modules over $W[\mathbb{Z}]$ (see [16]).
Theorem 2.8. Assume that $\mathcal{M}$ is a Harish-Chandra module for $\mathcal{W}[\mathbb{Z}]$ without nonzero trivial submodules nor nonzero trivial quotient modules. Then $\mathcal{M}$ has a direct sum decomposition of three submodules, that is to say, $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^0 \oplus \mathcal{M}^-$, where $\mathcal{M}^+$ is positively truncated, $\mathcal{M}^-$ is negatively truncated and $\mathcal{M}^0$ is cuspidal.

3 Harish-Chandra modules over $\mathcal{W}[\Gamma]$

3.1 Cuspidal module

In this subsection, we shall determine the simple cuspidal module for the higher rank $W$-algebra $W(2,2)$, namely, $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{Z}_+$. Let $\Gamma = \bigoplus_{i=1}^{k} \mathbb{Z}\epsilon_i$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ is a $\mathbb{Z}$-basis of $\Gamma \subseteq \mathbb{C}$. Given $\alpha \in \Gamma$, we set $\alpha = \sum_{i=1}^{k} \alpha_i \epsilon_i$ for $\alpha_i \in \mathbb{Z}$. For any $\alpha, \beta \in \Gamma$ with $\alpha_i, \beta_i \in \mathbb{N}, i \in \{1, \ldots, k\}$, we denote

$$\alpha \beta = \alpha_1 \beta_1 \cdots \alpha_k \beta_k \quad \text{and} \quad \beta! = \beta_1! \cdots \beta_k!.$$  

Conveniently, we denote $\partial := \frac{d}{dt}$. The generalized $W$-algebra $W(2,2)$ can be realized from truncated generalized loop-Witt algebra $\overline{\mathcal{W}}[\Gamma] \otimes (\mathbb{C}[x]/(x^2))$ (see, e.g., [12, 17]), namely,

$$L_\alpha = t^{\alpha+1} \partial \otimes 1, \quad W_\alpha = t^{\alpha+1} \partial \otimes x,$$

where $\alpha \in \Gamma$. Denote $\mathcal{A} = \text{span}\{t^\alpha \otimes 1 \mid \alpha \in \Gamma\}$, which is a unital associative algebra with multiplication $(t^\alpha \otimes 1)(t^\beta \otimes 1) = t^{\alpha+\beta} \otimes 1$ for $\alpha, \beta \in \Gamma$. For convenience, we write $t^{\alpha+1} \partial = t^{\alpha+1} \partial \otimes 1$ and $t^\alpha = t^\alpha \otimes 1$ for $\alpha \in \Gamma$.

3.1.1 $\mathcal{A}\overline{\mathcal{W}}[\Gamma]$-module

In the following, we shall describe the structure of cuspidal $\overline{\mathcal{W}}[\Gamma]$-modules that admit a compatible action of the commutative unital algebra $\mathcal{A}$.

Definition 3.1. (see [2]) A module $M$ is called an $\mathcal{A}\overline{\mathcal{W}}[\Gamma]$-module if it is a module for both $\overline{\mathcal{W}}[\Gamma]$ and the commutative unital algebra $\mathcal{A} = \mathbb{C}[t^\pm 1] \otimes 1$ with these two structures being compatible:

$$y(fv) = (yf)v + f(yv) \quad \text{for} \quad f \in \mathcal{A}, y \in \overline{\mathcal{W}}[\Gamma], v \in M. \quad (3.1)$$

Let $M$ be a weight module over $\mathcal{A}\overline{\mathcal{W}}[\Gamma]$. From (3.1), we see that the action of $\mathcal{A}$ is compatible with the weight grading of $M$:

$$\mathcal{A}_\alpha M_\lambda \subset M_{\alpha+\lambda} \quad \text{for} \quad \alpha, \lambda \in \Gamma.$$  

We suppose that $\mathcal{A}\overline{\mathcal{W}}[\Gamma]$-module $M$ has a weight space decomposition, and one of the weight spaces is finite-dimensional. Based on all non-zero homogeneous elements of $\mathcal{A}$ are invertible,
we know that all weight spaces of $M$ have the same dimension. Then $M$ is also a free $\mathcal{A}$-module of a finite rank. It is clear that $\mathcal{A}\mathcal{W}[\Gamma]$-module $M$ is cuspidal (as $\mathcal{W}[\Gamma]$-modules).

Assume that $M$ is a cuspidal $\mathcal{A}\mathcal{W}[\Gamma]$-module. Let $W = M_g$ for $g \in \Gamma$ and $\dim(W) < \infty$. From $M$ is a free $\mathcal{A}$-module, we can write

$$M \cong \mathcal{A} \otimes W.$$ 

Lemma 3.2. Let $M$ defined as above. For any $\alpha, n \in \Gamma$, we have $W_\alpha(t^n v) = t^n(W_\alpha v)$ for $v \in M$.

Proof. For any $\beta, n \in \Gamma$, by (3.1), we have

$$W_\beta(t^n v) = (W_\beta t^n)v + t^n(W_\beta v).$$

(3.2)

Note that $W_\beta t^n = nt^{n+\beta} \otimes x = n(t^{n+\beta} \otimes 1)(1 \otimes x)$. It is clear that $[1 \otimes x, \mathcal{W}[\Gamma] \oplus \mathcal{A}] = 0$. Then there is a homomorphism of algebras $\chi : 1 \otimes x \to \mathbb{C}$ such that $1 \otimes x$ acts on $M$ as $\chi(1 \otimes x) \in \mathbb{C}$. So the action of $W_\beta t^n$ on $M$ can be written as $n\mu t^{n+\beta}$, where $\mu \in \mathbb{C}$. Now from

$$0 = [W_\alpha, W_\beta](t^n v) = n\mu^2(\beta - \alpha)t^{n+\beta+n}v,$$

we get $\mu = 0$ by taking $n \neq 0, \alpha \neq \beta$, namely, $(W_\beta t^n)M = 0$. Putting this into (3.2), one has $W_\alpha(t^n v) = t^n(W_\alpha v)$ for $v \in M$. The lemma has been proved.

Remark 3.3. We note that the $\mathcal{A}\mathcal{W}[\Gamma]$-module is a module for the semidirect product Lie algebras $\mathcal{W}[\Gamma] \rtimes \mathcal{A}$ (the action of $\mathcal{A}$ as a unital commutative associative algebra). The Lie brackets between $\mathcal{W}[\Gamma]$ and $\mathcal{A}$ are given by $[L_m, t^n] = nt^n, [W_m, t^n] = 0$ for $m, n \in \Gamma$.

For $m \in \Gamma$, we consider the following operator

$$\mathfrak{D}(m) : W \to W.$$ 

It can be defined as the restriction to $W$ of the composition $t^{-m} \circ (t^{m+1}\partial)$ regarded also as an operator on $M$. Note that $\mathfrak{D}(0) = g\operatorname{Id}$.

According to (3.1), Lemma 3.2 and the finite-dimensional operator $\mathfrak{D}(m)$, we get the action on $M$ as follows

$$L_m(t^n v) = (t^{m+1}\partial)(t^n v) = nt^{n+m}v + t^{n+m}\mathfrak{D}(m)v, \quad W_m(t^n v) = t^n(W_m v),$$

(3.3)

where $m, n \in \Gamma, v \in W$. Based on (2.1) and (3.3), it is easy to derive the Lie bracket (also see Lemma 3.2 in [1]):

$$[\mathfrak{D}(s), \mathfrak{D}(m)] = (m - s)\mathfrak{D}(s + m) - m\mathfrak{D}(m) + s\mathfrak{D}(s).$$

(3.4)

Next, we show that $\mathfrak{D}(m)$ can be expressed as a polynomial in $m = (m_1, \ldots, m_k)$. 

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Theorem 3.4. Assume that $M$ is a cuspidal $\mathcal{A}\mathcal{W}[\Gamma]$-module, $M = \mathcal{A} \otimes W$, where $W = M_g, g \in \Gamma$. Then the action of $\mathcal{W}[\Gamma]$ on $M$ is presented as

$$L_m(t^nv) = nt^{m+n}v + t^{m+n}D(m)v, \quad W_m(t^nv) = t^n(W_mv),$$

$m, n \in \Gamma, v \in W$, where the family of operators $D(m) : W \to W$ can be shown as an $\text{End}(W)$-valued polynomial in $m = (m_1, \ldots, m_k)$ with the constant term $D(0) = g\text{Id}$, and $\text{Id}$ is the identification endomorphism of $W$.

Proof. By $m \in \Gamma$, we can write $m = \sum_{i=1}^{k} m_\epsilon_i$, where $m_\epsilon_i \in \mathbb{Z}$, $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ is a $\mathbb{Z}$-basis of $\Gamma$. According to Theorem 2.2 in [1], we obtain that $D(m_\epsilon_i)$ is a polynomial in $m_\epsilon_i$ with coefficients in $\text{End}(W)$ for all $i \in \{1, \ldots, k\}$. Now suppose that $D(\sum_{i=1}^{j-1} m_\epsilon_i)$ is a polynomial in $\alpha_1, \ldots, \alpha_{j-1}$ for some $1 < j \leq k$. For $m_j \in \mathbb{Z}$, it follows from (3.4) that

$$\left(\sum_{i=1}^{j-1} m_\epsilon_i - m_j \epsilon_j\right)D(\sum_{i=1}^{j} m_\epsilon_i) = [D(m_j \epsilon_j), D(\sum_{i=1}^{j-1} m_\epsilon_i)] + \sum_{i=1}^{j-1} (m_\epsilon_i)D(\sum_{i=1}^{j-1} m_\epsilon_i) - (m_j \epsilon_j)D(\alpha_j \epsilon_j).$$

Consider $m_j \neq 0$. Then from the linearly independence of $\epsilon_1, \ldots, \epsilon_j$, one has $\sum_{i=1}^{j-1} m_\epsilon_i - m_j \epsilon_j \neq 0$. By the induction assumption, we conclude that $D(\sum_{i=1}^{j} m_\epsilon_i)$ is a polynomial in $m_1, \ldots, m_j$, where $1 < j \leq k$. Choosing $j = k$, one can see that $D(m)$ is a polynomial in $m_1, \ldots, m_k$. By the definition of operator $D(m)$, one has $D(0) = g\text{Id}$ for $g \in \mathbb{C}$. We complete the proof.

Now we can write $D(m)$ in the form (also see [18, 15])

$$\sum_{\tilde{i} \in \mathbb{N}^k} \frac{m_{\tilde{i}}}{\tilde{i}!} D^{(\tilde{i})}, \quad (3.5)$$

where $\tilde{i}! = \prod_{j=1}^{k} \tilde{i}_j!$ and only has a finite number of the nonzero operators $D^{(\tilde{i})} \in \text{End}(W)$. 

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Note that $\mathcal{D}(0) = D^{(0)}$. For $m, s \in \Gamma$, by (3.4), we have

$$
\sum_{i,j \in \mathbb{N}^k} \frac{s_i m_j}{i! j!} [D^{(i)}, D^{(j)}] = (\sum_{i \in \mathbb{N}^k} \frac{s_i}{i!} D^{(i)})(\sum_{j \in \mathbb{N}^k} \frac{m_j}{j!} D^{(j)}) - (\sum_{i \in \mathbb{N}^k} \frac{s_i}{i!} D^{(i)})(\sum_{j \in \mathbb{N}^k} \frac{m_j}{j!} D^{(j)}) = [\mathcal{D}(s), \mathcal{D}(m)] = (s - m)\mathcal{D}(s + m) - s\mathcal{D}(s) + m\mathcal{D}(m) = \sum_{l=1}^{k-1} \left( \sum_{i,j \in \mathbb{N}^k} \frac{s_i m_j}{i! (j - l)! (i - l)!} D^{(i+j-l)} - \sum_{i,j \in \mathbb{N}^k} \frac{s_i m_j}{i! (i - l)! j!} D^{(i+j-l)} \right). \tag{3.6}
$$

Comparing the coefficients of $\frac{s_i m_j}{i! j!}$ in (3.6), we check that

$$
[D^{(i)}, D^{(j)}] = \begin{cases} 
\sum_{l=1}^{k} (\sum_{i,j \in \mathbb{N}^k} \frac{s_i m_j}{i! (j - l)! (i - l)!} D^{(i+j-l)}) & \text{if } \tilde{i}, \tilde{j} \in \mathbb{N}^k \setminus \{0\}, \\
0 & \text{if } \tilde{i} = 0 \text{ or } \tilde{j} = 0. 
\end{cases} \tag{3.7}
$$

From (3.7), the operators $\mathcal{G} = \text{span}\{D^{(i)} \mid \tilde{i} \in \mathbb{N}^k \setminus \{0\}\}$ yield a Lie algebra. For $\tilde{i} \in \mathbb{N}^k$, let $|\tilde{i}| = \tilde{i}_1 + \tilde{i}_2 + \cdots + \tilde{i}_k$. Denote

$$\mathcal{G}_p = \text{span}\{D^{(i)} \mid \tilde{i} \in \mathbb{N}^k, |\tilde{i}| = p - 1\} \quad \text{for } p \in \mathbb{N}.
$$

Then $\mathcal{G} = \bigoplus_{p \in \mathbb{N}} \mathcal{G}_p$ is a $\mathbb{Z}$-graded Lie algebra (also see [1,3,15]). According to (3.7), we know that $\mathcal{G}_0 = \text{span}\{D^{(i)} \mid i = 1, \ldots, k\}$ is a subalgebra of $\mathcal{G}$, whose Lie algebra structure is presented as

$$[D^{(i)}, D^{(j)}] = \epsilon_j D^{(i)} - \epsilon_i D^{(j)},
$$

where $i, j \in \{1, 2, \ldots, k\}$. It is easy to get that $[\mathcal{G}_0, \mathcal{G}_0]$ is nilpotent. Hence, $\mathcal{G}_0$ is a solvable Lie algebra. Now we define the following one-dimensional $\mathcal{G}$-module $V(h) = \mathbb{C}v \neq 0$ for any $h \in \mathbb{C}$:

$$D^{(\epsilon_i)} v = h\epsilon_i v \quad \text{for } i = 1, \ldots, k. \tag{3.8}
$$

The following lemma was proved in [15].

Lemma 3.5. Assume that $T$ and $W$ are finite-dimensional simple modules over $\mathcal{G}_0$ and $\mathcal{G}$, respectively. Then

(a) $T \cong V(h)$ for $h \in \mathbb{C}$.

(b) $D^{(\tilde{i})} W = 0$ for any $\tilde{i} \in \mathbb{N}^k$ with $|\tilde{i}|$ sufficiently large;
Theorem 3.6. Any simple cuspidal $\mathcal{A}\mathcal{W}[\Gamma]$-module is isomorphic to a module of intermediate series $\overline{M}(g, h; \Gamma)$ for some $g, h \in \mathbb{C}$.

Proof. Let $M$ be a simple cuspidal $\mathcal{A}\mathcal{W}[\Gamma]$-module, $M = \mathcal{A} \otimes W$, where $W = M_g, g \in \Gamma$. For $n, \alpha \in \Gamma, v \in W$, based on Theorem 3.4, Lemma 3.5 and (3.5), we check that

$$L_\alpha(t^n v) = nt^{\alpha+n}v + t^{\alpha+n}(\mathcal{D}(\alpha)v)$$

$$= t^{\alpha+n}((n + \mathcal{D}(0) + \sum_{i \in \mathbb{N}\setminus\{0\}} \frac{\alpha^i}{i!} D^{(i)}v)$$

$$= t^{\alpha+n}((n + \mathcal{D}(0) + \sum_{i=1}^{k} \alpha_i D^{(\epsilon_i)}v)$$

$$= t^{\alpha+n}((n + g\text{Id} + \sum_{i=1}^{k} h\alpha_i \epsilon_i)v)$$

$$= (n + g + h\alpha)(t^{\alpha+n}v),$$  (3.9)

where $g, h \in \mathbb{C}$. Using (3.9) and the relations $(\beta - \alpha)W_{\alpha+\beta}(t^n v) = (L_\alpha W_\beta - W_\beta L_\alpha)(t^n v)$, we obtain

$$(\beta - \alpha)t^n(W_{\alpha+\beta}v) = \beta t^{n+\alpha}(W_\beta v).$$  (3.10)

Taking $\beta = 0$ in (3.10), we immediately get $t^n(W_\alpha v) = 0$ for $\alpha \neq 0$. Considering $\alpha = -\beta \neq 0$ in (3.10) again, one checks $t^n(W_0 v) = 0$. Then we conclude $t^n(W_\alpha v) = 0$ for $\alpha, n \in \Gamma$, that is to say, $W_\alpha(t^n v) = 0$. This completes the proof. \qed

3.1.2 $\mathcal{A}$-cover of a cuspidal $\mathcal{W}[\Gamma]$-module

We first recall the definitions of coinduced module and $\mathcal{A}$-cover (see [2]).

Definition 3.7. A module coinduced from a $\mathcal{W}[\Gamma]$-module $M$ is the space $\text{Hom}(\mathcal{A}, M)$ with the actions of $\mathcal{W}[\Gamma]$ and $\mathcal{A}$ as follows

$$(a \varphi)(f) = a(\varphi(f)) - \varphi(a(f)), \quad (y \varphi)(f) = \varphi(yf),$$

where $\varphi \in \text{Hom}(\mathcal{A}, M), a \in \mathcal{W}[\Gamma], f, y \in \mathcal{A}$.

Definition 3.8. An $\mathcal{A}$-cover of a cuspidal module $M$ over $\mathcal{W}[\Gamma]$ is an $\mathcal{A}\mathcal{W}[\Gamma]$-submodule

$$\widehat{M} = \text{span}\{\phi(a, w) \mid a \in \mathcal{W}[\Gamma], w \in M\} \subset \text{Hom}(\mathcal{A}, M),$$

where $\phi(a, w) : \mathcal{A} \to M$ is defined as

$$\phi(a, w)(f) = (fa)(w).$$
The action of $A\mathbb{W}[\Gamma]$ on $\hat{M}$ is given by
\[
b\phi(a, w) = \phi([b, a], w) + \phi(a, bw),
f\phi(a, w) = \phi(fa, w) \quad \text{for } a, b \in \mathbb{W}[\Gamma], w \in M, f \in A.
\]

Let
\[
\mathfrak{K}(M) = \left\{ \sum_{\alpha \in \Gamma} a_{\alpha} \otimes w_{\alpha} \in \mathbb{W}[\Gamma] \otimes M \mid \sum_{\alpha \in \Gamma} (fa_{\alpha}) w_{\alpha} = 0, \forall f \in A \right\}.
\]
Then $\mathfrak{K}(M)$ is an $A\mathbb{W}[\Gamma]$-submodule of $\mathbb{W}[\Gamma] \otimes M$. The $A$-cover $\hat{M}$ can also be constructed as a quotient $A\mathbb{W}[\Gamma]$-module
\[
(\mathbb{W}[\Gamma] \otimes M)/\mathfrak{K}(M),
\]
where $\mathbb{W}[\Gamma]M = M$. Clearly, the following linear map
\[
\Theta : \quad \hat{M} \longrightarrow \mathbb{W}[\Gamma]M
\]
\[
a \otimes w + \mathfrak{K}(M) \longmapsto aw
\]
is a $\mathbb{W}[\Gamma]$-module epimorphism.

**Lemma 3.9.** (see [2]) Let $M$ be a cuspidal module for $\mathbb{W}[\Gamma]$. Then there exists $l \in \mathbb{Z}_+$ such that for all $\alpha, \beta, \gamma \in \Gamma$ the operator $\Omega^{(l, \gamma)}_{\alpha, \beta} = \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma}$ annihilates $M$.

**Lemma 3.10.** Let $M$ be a cuspidal $\mathbb{W}[\Gamma]$-module. Then there exists $r \in \mathbb{Z}_+$ such that for all $\alpha, \beta, \gamma \in \Gamma$ the following two operators
\[
\Omega^{(r, \gamma)}_{\alpha, \beta} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma} \quad \text{and} \quad \tilde{\Omega}^{(r, \gamma)}_{\alpha, \beta} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha-i\gamma} L_{\beta+i\gamma}
\]
annihilate $M$.

**Proof.** Note that $M$ is also a cuspidal module for $V[\Gamma]$. According to Lemma 3.9 there exists $l \in \mathbb{Z}_+$ such that $\Omega^{(l, \gamma)}_{\alpha, \beta} M = 0$ for all $\alpha, \beta, \gamma \in \Gamma$. It follows from this that
\[
0 = \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} (L_{\alpha-(i-1)\gamma} L_{\beta+(i-1)\gamma} - 2L_{\alpha-i\gamma} L_{\beta+i\gamma} + L_{\alpha-(i+1)\gamma} L_{\beta+(i+1)\gamma}) \right) M
\]
\[
= \left( \sum_{i=0}^{l+2} (-1)^i \binom{l+2}{i} L_{\alpha-(i-1)\gamma} L_{\beta+(i-1)\gamma} \right) M.
\]
(3.11)
Setting \( r = l + 2 \) in (3.11), one gets \( \Omega_{\alpha,\beta}^{(r,\gamma)} M = 0 \) for all \( \alpha, \beta, \gamma \in \Gamma \). For any \( s \in \Gamma \), from Lemma 3.9, we immediately get \( [\Omega_{\alpha,\beta}^{(l,\gamma)}, W_s] M = 0 \) for all \( \alpha, \beta, \gamma \in \Gamma \). Now for any \( \alpha, \beta, s \in \Gamma \) and \( \gamma \in \Gamma^* \), we can compute that

\[
0 = \left( \left[ \Omega_{\alpha,\beta}^{(l,\gamma)}, W_{s+\gamma} \right] - 2 \left[ \Omega_{\alpha,\beta}^{(l,\gamma)}, W_s \right] + \left[ \Omega_{\alpha,\beta}^{(l,\gamma)}, W_{s-\gamma} \right] \right) M
\]

\[
= \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta-\gamma+i\gamma}, W_{s+\gamma} \right) - 2 \left[ \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha-i\gamma} L_{\beta+i\gamma}, W_s \right] + \left[ \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha+i\gamma} L_{\beta-\gamma+i\gamma}, W_s \right] \\
+ 2 \left[ \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha+i\gamma} L_{\beta+i\gamma}, W_{s-\gamma} \right] - \left[ \sum_{i=0}^{l} (-1)^i \binom{l}{i} L_{\alpha+i\gamma} L_{\beta+i\gamma}, W_{s-2\gamma} \right] \right) M
\]

\[
= \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( (s + (2 - i)\gamma - \beta) L_{\alpha-i\gamma} W_{\beta+s+i\gamma} \right. \right.
\]

\[
+ (s + (i + 1)\gamma - \alpha) W_{\alpha+s+(i-1)\gamma} L_{\beta+(i-1)\gamma} \n+ 2((s - \beta - i\gamma) L_{\alpha-i\gamma} W_{\beta+s+i\gamma} + (s - \alpha + i\gamma) W_{\alpha+s-i\gamma} L_{\beta+i\gamma}) \n+ (s - \beta - (i + 2)\gamma) L_{\alpha-i\gamma} W_{\beta+s+i\gamma} + (s - \alpha + (i - 1)\gamma) W_{\alpha+s-(i+1)\gamma} L_{\beta+(i+1)\gamma} \n- (s - \beta - (i - 1)\gamma) L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma} - (s - \alpha + (i - 1)\gamma) W_{\alpha+s+(i-1)\gamma} L_{\beta+(i-1)\gamma} \n+ 2((s - \beta - (i + 1)\gamma) L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma} + (s - \alpha + (i - 2)\gamma) W_{\alpha+s-i\gamma} L_{\beta+i\gamma}) \n- (s - \beta - (i + 3)\gamma) L_{\alpha+(i-1)\gamma} W_{\beta+s+(i-1)\gamma} \n- (s - \alpha + (i - 3)\gamma) W_{\alpha+s-(i+1)\gamma} L_{\beta+(i+1)\gamma} \n\left. \right) M
\]

\[
= \left( 2\gamma \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( W_{\alpha+s-(i-1)\gamma} L_{\beta+(i-1)\gamma} - 2 W_{\alpha+s-i\gamma} L_{\beta+i\gamma} + W_{\alpha+s-(i+1)\gamma} L_{\beta+(i+1)\gamma} \right) \right) M
\]

\[
= \left( 2\gamma \sum_{i=0}^{l+2} (-1)^i \binom{l+2}{i} W_{\alpha+s-(i-1)\gamma} L_{\beta+(i-1)\gamma} \right) M.
\]

Moreover, the module \( M \) can be annihilated by the operator \( \tilde{\Omega}_{\alpha,\beta}^{(l+2,0)} \). Then we conclude that \( \tilde{\Omega}_{\alpha,\beta}^{(r,\gamma)} M = \left( \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha-i\gamma} L_{\beta+i\gamma} \right) M = 0 \), where \( r = l + 2 \) and all \( \alpha, \beta, \gamma \in \Gamma \). The lemma holds.

\[\square\]

**Proposition 3.11.** Let \( M \) be a cuspidal module over \( \overline{W}[\Gamma] \). Then the \( \mathcal{A}\)-cover \( \hat{M} \) of \( M \) is also a cuspidal \( \mathcal{A}\overline{W}[\Gamma]\)-module.
Proof. Let $M_\lambda$ be a weight space with weight $\lambda \in \Gamma$. For $\alpha \in \Gamma$, we denote

$$\phi(L_\alpha \cup W_\alpha, M_\lambda) = \left\{ \phi(L_\alpha, w), \phi(W_\alpha, w) \mid w \in M_\lambda \right\} \subset \widehat{M}.$$ 

By considering the weight spaces of $M$, the spaces $\phi(L_\alpha \cup W_\alpha, M_\lambda)$ are finite-dimensional.

Since $\widehat{M}$ is an $A$-module, we see that one of its weight spaces is finite-dimensional. For a fixed weight $\beta \in \Gamma$, we shall show that $\widehat{M}_\beta$ is finite-dimensional. Obviously, the space $\widehat{M}_\beta$ is spanned by the set

$$\left( \bigcup_{\gamma \in \Gamma} \phi(L_{\beta-\gamma}, M_\gamma) \right) \cup \left( \bigcup_{\gamma \in \Gamma} \phi(W_{\beta-\gamma}, M_\gamma) \right).$$

Define a norm on $\Gamma$ as follows

$$\|\alpha\| = \sum_{i=1}^k |\alpha_i|,$$

where $\alpha = \sum_{i=1}^k \alpha_i \varepsilon_i$. According to Lemma 3.10 there exists $r \in \mathbb{N}$ such that for all $\alpha, \beta, \gamma \in \Gamma$ the following two operators

$$\Omega_{\alpha, \beta}^{(r, \gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_{\alpha - i \gamma} L_{\beta + i \gamma} \quad \text{and} \quad \tilde{\Omega}_{\alpha, \beta}^{(r, \gamma)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{\alpha - i \gamma} L_{\beta + i \gamma}$$

annihilate $M$, namely, $\Omega_{\alpha, \beta}^{(r, \gamma)} v = \tilde{\Omega}_{\alpha, \beta}^{(r, \gamma)} v = 0$ for all $\alpha, \beta, \gamma \in \Gamma, v \in M$. Hence, $\Omega_{\alpha, \beta}^{(r, \gamma)} v$ and $\tilde{\Omega}_{\alpha, \beta}^{(r, \gamma)} v$ are both in $\mathcal{R}(M)$.

Claim 1. For any $\alpha, \beta \in \Gamma$, $\widehat{M}_{\alpha + \beta}$ is equal to

$$\text{span} \left\{ \phi(L_{\alpha + \beta} \cup W_{\alpha + \beta}, M_0), \phi(L_{\alpha - \gamma}, M_{\beta + \gamma}), \phi(W_{\alpha - \gamma}, M_{\beta + \gamma}) \mid \gamma \neq -\beta, \|\gamma\| \leq \frac{kr}{2} \right\}.$$ 

For all $q \in \Gamma$ and $w \in M_{\beta + q}$, we have $\phi(L_{\alpha - q}, w)$ and $\phi(W_{\alpha - q}, w)$ in $\widehat{M}_{\alpha + \beta}$. Now we prove this claim by induction on $\|q\|$. If $|q_i| \leq \frac{kr}{2}$ for all $i \in \{1, \ldots, k\}$, the result clears. On the other hand, suppose $|q_i| > \frac{kr}{2}$ for some $i \in \{1, \ldots, k\}$. We may assume $q_i < -\frac{kr}{2}$, and the other case $q_i > -\frac{kr}{2}$ can be proved by the similar method. It is easy to see that $\|q + j \varepsilon_i\| < \|q\|$ for all $j \in \{1, \ldots, r\}$. We only need to give the proof for $\beta + q \neq 0$. From the action of $L_0$ on $M_{\beta + q}$ is a nonzero scalar, we can write $w = L_0 v$, where $v \in M_{\beta + q}$. In the following, we shall verify

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(L_{\alpha - q - j \varepsilon_i}, L_{j \varepsilon_i} v) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(W_{\alpha - q - j \varepsilon_i}, L_{j \varepsilon_i} v) = 0$$
in \( \hat{M} \). Based on Definition 3.8 and Lemma 3.10, for any \( m \in \Gamma \) we deduce

\[
\sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(L_{\alpha - q - j\epsilon_i}, L_{j\epsilon_i} v)(t^m)
\]

\[
= \sum_{j=0}^{r} (-1)^j \binom{r}{j} L_{\alpha + m - q - j\epsilon_i} L_{j\epsilon_i} v = \Omega_{\alpha + m - q, 0}^{(r, \epsilon)} v = 0
\]

and

\[
\sum_{j=0}^{r} (-1)^j \binom{r}{j} \phi(W_{\alpha - q - j\epsilon_i}, L_{j\epsilon_i} v)(t^m)
\]

\[
= \sum_{j=0}^{r} (-1)^j \binom{r}{j} W_{\alpha + m - q - j\epsilon_i} L_{j\epsilon_i} v = \tilde{\Omega}_{\alpha + m - q, 0}^{(r, \epsilon)} v = 0.
\]

Thus, one has

\[
\phi(L_{\alpha - q}, w) = - \sum_{j=1}^{r} (-1)^j \binom{r}{j} \phi(L_{\alpha - q - j\epsilon_i}, L_{j\epsilon_i} v),
\]

(3.12)

\[
\phi(W_{\alpha - q}, w) = - \sum_{j=1}^{r} (-1)^j \binom{r}{j} \phi(W_{\alpha - q - j\epsilon_i}, L_{j\epsilon_i} v).
\]

(3.13)

By induction assumption the right hand sides of (3.12) and (3.13) belong to \( \hat{M}_{\alpha + \beta} \), and so do \( \phi(L_{\alpha - q}, w) \), \( \phi(W_{\alpha - q}, w) \). This proves the claim. Therefore, \( \hat{M}_{\alpha + \beta} \) is finite-dimensional. The proposition follows.

\[
\square
\]

The Claim \( \square \) can also be described as follows.

**Remark 3.12.** For \( \alpha, \beta, q \in \Gamma \) and \( \beta + q \neq 0 \), \( w \in M_{\beta + q} \), we get

\[
\phi(L_{\alpha - q}, w) = \sum_{\|\gamma\| \leq \frac{k_r}{2}} \phi(L_{\alpha - \gamma}, M_{\beta + \gamma}) + \mathfrak{R}(M),
\]

\[
\phi(W_{\alpha - q}, w) = \sum_{\|\gamma\| \leq \frac{k_r}{2}} \phi(W_{\alpha - \gamma}, M_{\beta + \gamma}) + \mathfrak{R}(M).
\]

Now we give a classification for all simple cuspidal \( \overline{W}[\Gamma] \)-modules.

**Theorem 3.13.** Any simple cuspidal \( \overline{W}[\Gamma] \)-module is isomorphic to a module of intermediate series \( \overline{M}(g, h; \Gamma) \) for some \( g, h \in \mathbb{C} \).
Proof. Assume that $M$ is a simple cuspidal $\text{W}[\Gamma]$-module. It is clear that $\text{W}[\Gamma]M = M$. Then there exist an $\mathcal{A}$-cover $\widehat{M}$ of $M$ with a surjective homomorphism $\Theta : \widehat{M} \to M$. It follows from Proposition 3.11 that $\widehat{M}$ is a cuspidal $\mathcal{AW}[\Gamma]$-module. Hence, we can consider the composition series

$$0 = \widehat{M}_0 \subset \widehat{M}_1 \subset \cdots \subset \widehat{M}_c = \widehat{M}$$

with the quotients $\widehat{M}_{i+1}/\widehat{M}_i$ being simple $\mathcal{AW}[\Gamma]$-modules. Let $d$ be the smallest integer such that $\Theta(\widehat{M}_d) \neq 0$. Then by the simplicity of $M$, we obtain $\Theta(\widehat{M}_d) = M$ and $\Theta(\widehat{M}_{d-1}) = 0$. So we get an $\mathcal{AW}[\Gamma]$-epimorphism

$$\overline{\Theta} : \widehat{M}_d/\widehat{M}_{d-1} \to M.$$ 

Now from Theorem 3.10, $\widehat{M}_d/\widehat{M}_{d-1}$ is isomorphic to a module of intermediate series $\overline{M}(g,h;\Gamma)$ for some $g, h \in \mathbb{C}$. We complete the proof. 

Based on the representation theory of $\mathcal{V}[\Gamma]$ studied in [28,29], we see that the action of $C$ on any simple cuspidal $\text{W}[\Gamma]$-modules is trivial. Therefore, the category of simple cuspidal modules over $\text{W}[\Gamma]$ is equivalent to the category of simple cuspidal modules over $\text{W}[\Gamma]$. Then Theorem 3.13 can be described as follows.

**Theorem 3.14.** Let $M$ be a simple cuspidal module over $\text{W}[\Gamma]$. Then $M$ is isomorphic to a simple quotient of intermediate series module $M(g,h;\Gamma)$ for some $g, h \in \mathbb{C}$.

### 3.2 Non-cuspidal modules

In this subsection, we will determine the simple weight modules with finite-dimensional weight spaces which are not cuspidal for the higher rank $W$-algebras $W(2, 2)$. These modules are called generalized highest weight modules exactly and defined in Section 2.2. At last, the results in Section 3.1 and this subsection will be extended to the arbitrary generalized $W$-algebras $W(2, 2)$. Now we let $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{Z}_+$. The results of generalized Virasoro algebras appeared in Theorem 2.4 play a key role in the following proof.

**Theorem 3.15.** Let $M$ be a simple Harish-Chandra module over $\text{W}[\Gamma]$. Then $M$ is either a cuspidal module, or isomorphic to some $P^{\text{W}[\Gamma]}_{g,\mu,\mathcal{K}}$, where $\mu \in \Gamma^*$, $g$ is a subgroup of $\Gamma$ such that $\Gamma = g \oplus \mathbb{Z} \mu$ and $\mathcal{K}$ is a non-trivial simple cuspidal $\text{W}[\Gamma]_g$-module.

**Proof.** Suppose that $M$ is not a cuspidal module over $\text{W}[\Gamma]$. Let us recall that $M = \bigoplus_{\alpha \in \Gamma} M_\alpha$ where $M_\alpha = \{ w \in M \mid L_0 w = (g + \alpha)w \}$. By Theorem 2.7 we see that the statement holds for any $\Gamma$ of $k = 1$. 

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Now suppose $k \geq 2$. View $M$ as a $\mathcal{V}[\Gamma]$-module. Then based on Theorem 2.4(c), we obtain that the action of the central element $C$ on $M$ is trivial. Hence $M$ can be seen as a $\overline{\mathcal{W}}[\Gamma]$-module. We fix a $\mathbb{Z}$-basis of $\Gamma$, which is also suitable for $\mathbb{Z}^k$. For any $\sigma \in \mathbb{R}^k$ and $g \in \Gamma$, we have the inner product $(\sigma|g)$. It follows from $[\overline{\mathcal{W}}[\Gamma], \overline{\mathcal{W}}[\Gamma]] = \overline{\mathcal{W}}[\Gamma]_{\alpha + \beta}$ that Theorem 2.3 can be applied to $\Gamma$.

Since $M$ is not cuspidal, we can find some rank $k - 1$ direct summand $\bar{\Gamma}$ of $\Gamma$ such that $M_{\bar{\Gamma}}$ is not cuspidal. Without loss of generality, we may assume that $\bar{\Gamma}$ is spanned by $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\} \setminus \{\epsilon_j\}$, where $j \in \{1, 2, \ldots, k\}$. Then there exists some $\bar{\alpha} \in \bar{\Gamma}$ such that

$$\dim(M_{\bar{\alpha}}) > 2k(\dim(M_{\epsilon_j}) + \sum_{i=1, i \neq j}^k \dim(M_{\epsilon_j + \epsilon_i})).$$

(3.14)

For simplicity, we denote $\xi_j = \bar{\alpha} + \epsilon_j$ and $\xi_i = \bar{\alpha} + \epsilon_j + \epsilon_i$ for any $i \in \{1, 2, \ldots, k\} \setminus \{j\}$. Then it is easy to check that the linear transformation sending each $\epsilon_i$ to $\xi_i$ for any $i \in \{1, \ldots, k\}$, has determinant 1 and therefore $\{\xi_i | i = 1, \ldots, k\}$ is also a $\mathbb{Z}$-basis of $\Gamma$. According to (3.14), there exists some nonzero element $w \in M_{\bar{\alpha}}$ such that $L_{\xi_i}w = W_{\xi_i}w = 0$ for all $i \in \{1, \ldots, k\}$. Thus, $w$ is a generalized highest weight vector associated with the $\mathbb{Z}$-basis $\{\xi_i | i = 1, \ldots, k\}$.

It is clear that $M$ is neither dense nor trivial. From Theorem 2.3 there exist some $\beta \in \Gamma$ and $\tau \in \mathbb{R}^k \setminus \{0\}$ such that $\text{Supp}(M) \subset g + \beta + \Gamma^{(r)} \leq 0$. Consider $M$ as a $\mathcal{V}[\Gamma]$-module. Then $M$ has a simple non-trivial $\mathcal{V}[\Gamma]$-subquotient, and we denote it by $\overline{\mathcal{V}}^\lambda$, which is not cuspidal. By Theorem 2.4(c), we know that $\overline{\mathcal{V}}^\lambda \cong P^{\mathcal{V}[\Gamma]}_{\beta, \mu, K}$ for some nonzero $\mu \in \Gamma$, subgroup $g$ of $\Gamma$ with $\Gamma = g \oplus \mathbb{Z}\mu$ and $K$ being a simple intermediate series module over $\mathcal{V}[g]$. Write $\Gamma_\tau = \{\alpha \in \Gamma | (\tau|\alpha) = 0\}$. In particular, we have

$$g - \bar{c}\mu + g \subseteq \text{Supp}(\overline{\mathcal{V}}^\lambda) \subseteq \text{Supp}(M) \subseteq g + \beta + \Gamma^{(r)} \leq 0$$

for sufficiently large $\bar{c} \in \mathbb{Z}_+$. This gives $g = \Gamma_\tau$ and $(\tau|\mu) > 0$.

We let that $\bar{c}_0 \in \mathbb{Z}$ is the maximal number such that $K = M_{g + \bar{c}_0 \mu + g} \neq 0$. Hence, $\mathcal{W}[\Gamma]^+K = 0$. Then it follows from the simplicity of $\mathcal{W}[\Gamma]$-module $M$ that the simple $\mathcal{W}[g]$-module $K$ and $M = P^{\mathcal{V}[\Gamma]}_{\beta, \mu, K}$. At last, note that $K$ is non-trivial and cuspidal. This proves the theorem.

Based on Theorems 3.14 and 3.15 we give a classification of simple Harish-Chandra modules over the higher rank $W$-algebra $W(2, 2)$.

**Theorem 3.16.** Assume that $M$ is a non-trivial simple Harish-Chandra module over $\mathcal{W}[\Gamma]$, where $\Gamma$ is an additive subgroup of $\mathbb{C}$ and $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{Z}_+$. 

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(1) If $M$ is cuspidal, then $M$ is isomorphic to some $\overline{M}(g, h; \Gamma)$ for some $g, h \in \mathbb{C}$;

(2) If $M$ is not cuspidal, then $M$ is isomorphic to $P^{W[\Gamma]}_{g, \mu, K}$ for some $\mu \in \Gamma^*$, a subgroup $g$ of $\Gamma$ with $\Gamma = g \oplus \mathbb{Z}\mu$ and a non-trivial simple intermediate series $W[g]$-module $K$.

Note that Theorem 2.4(a) is a special case of Theorem 3.16 for $k = 1$. By using Theorem 2.4(a), (b) and the same idea in [10], we obtain the following lemma.

**Lemma 3.17.** Assume that $\Gamma$ is an infinitely generated additive subgroup of $\mathbb{C}$ with $\text{rank}(\Gamma) = 1$. Then any non-trivial simple Harish-Chandra module over $W[\Gamma]$ is isomorphic to $\overline{M}(g, h; \Gamma)$ for some $g, h \in \mathbb{C}$.

Now we can generalize the result of Theorem 3.16 to an arbitrary generalized $W$-algebra $W(2,2)$, which is the main results of this paper.

**Theorem 3.18.** Let $\Gamma$ be an arbitrary additive subgroup of $\mathbb{C}$. Let $M$ be a non-trivial simple Harish-Chandra module over $W[\Gamma]$.

(1) If $\Gamma \cong \mathbb{Z}$, then $M$ is either a module of the intermediate series, or a highest/lowest weight module.

(2) If $\text{rank}(\Gamma) = 1$ and $\Gamma \not\cong \mathbb{Z}$, then $M$ is a module of the intermediate series.

(3) If $\text{rank}(\Gamma) > 1$, then $M$ is either a module of the intermediate series or isomorphic to $P^{W[\Gamma]}_{g, \mu, K}$ where $\mu \in \Gamma^*$, $g$ is a subgroup of $\Gamma$ such that $\Gamma = g \oplus \mathbb{Z}\mu$ and $K$ is a non-trivial simple intermediate series module for $W[g]$.

**Remark 3.19.** It is clear that Theorem 3.18(1) is just Theorem 3.16 for $k = 1$ and Theorem 3.18(2) is just Lemma 3.17. Moreover, we remark that the method of the proof of Theorem 3.18(3) is exactly the same as the proof of Theorem 2.4(3). Hence, according to Lemma 2.6, Theorem 2.8 and Theorem 3.16, we transplant the proof of [11] here by replacing $V[\Gamma]$ with $W[\Gamma]$. We omit the detailed process.

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