ZETA FUNCTIONS OF TOTALLY RAMIFIED $p$-COVERS OF THE PROJECTIVE LINE

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Abstract. In this paper we prove that there exists a Zariski dense open subset $U$ defined over the rationals $\mathbb{Q}$ in the space of all one-variable rational functions with arbitrary $\ell$ poles of prescribed orders, such that for every geometric point $f$ in $U(\mathbb{Q})$, the $L$-function of the exponential sum of $f$ at a prime $p$ has Newton polygon approaching the Hodge polygon as $p$ approaches infinity. As an application to algebraic geometry, we prove that the $p$-adic Newton polygon of the zeta function of a $p$-cover of the projective line totally ramified at arbitrary $\ell$ points of prescribed orders has an asymptotic generic lower bound.

1. Introduction

This paper investigates the asymptotics of the zeta functions of $p$-covers of the projective line which are totally (wildly) ramified at arbitrary $\ell$ points. Our approach is via Dwork’s method on one-variable exponential sums.

Throughout this paper we fix positive integers $\ell, d_1, \ldots, d_\ell$, and let $d := \sum_{j=1}^\ell d_j + \ell - 2$. For simplicity we assume $d \geq 2$ if $\ell = 1$. Let $P_1 = \infty, P_2 = 0, P_3, \ldots, P_\ell$ be fixed poles in the projective line over $\overline{\mathbb{Q}}$ of orders $d_1, \ldots, d_\ell$, respectively. Let $f$ be a one-variable function over $\overline{\mathbb{Q}}$ with these prescribed $\ell$ poles. It can be written in a unique form of partial fractions (Theorem 13, Introduction):

$$f = \sum_{i=1}^{d_1} a_{1,i}x^i + \sum_{j=2}^{\ell} \sum_{i=1}^{d_j} a_{ji}(x - P_j)^{-i}$$

with $a_{ji} \in \overline{\mathbb{Q}}$. (Remark: we have assumed that $f$ has a vanishing constant term because this does not affect the $p$-adic Newton polygons of $f$.) Let $\mathcal{A}$ be the space of $a_{ji}$'s with $\prod_{j=1}^\ell a_{ji,d_j} \neq 0$. It is an affine ($\sum_{j=1}^\ell d_j$)-space over $\mathbb{Q}$. Let the Hodge polygon of $\mathcal{A}$, denoted by $\text{HP}(\mathcal{A})$, be the lower convex graph of the piecewise-linear function defined on the interval $[0,d]$ passing through the two endpoints $(0,0)$ and $(d,d/2)$ and assuming every slope in the list below of (horizontal) length 1:

$$\frac{\ell-1}{0, \ldots, 0; 1, \ldots, 1}, \frac{d_1-1}{1 \ d_1}, \frac{d_2-1}{1 \ d_2}, \ldots, \frac{d_{\ell-1}}{1 \ d_{\ell-1}}.$$

A non-smooth point on a polygon (as the graph of a piece-wise linear function) is called a vertex. We remark that the classical and geometrical ‘Hodge polygon’ for any curve (including Artin-Schreier curve as a special case) is the one with

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end points (0, 0) and (d, d/2) and one vertex at (d/2, 0). So the Hodge polygon in our paper is different from the classical Hodge polygon. We anticipate a p-adic arithmetic interpretation of our Hodge polygon, but it remains an open question.

In [13] it is shown that in the case ℓ = 1 there is a Zariski dense open subset \( \mathcal{U} \) defined over \( \mathbb{Q} \) such that every geometric closed point \( f \) in \( \mathcal{U}(\mathbb{Q}) \) has p-adic Newton polygon approaching the Hodge polygon as \( p \) approaches infinity. Wan has proposed conjectures regarding multivariable exponential sums, including the above as a special case (see [10, Conjecture 1.15]). This series of study traces back at least to Katz [4, Introduction], where Katz proposed to study exponential sums in families instead of examining one at a time. He systematically studied families of multivariable Kloosterman sum in [4].

Let \( \mathbb{Q}_f \) be the extension field of \( \mathbb{Q} \) generated by coefficients \( a_{ji} \)'s and poles \( P_1, \ldots, P_\ell \) of \( f \). For every prime number \( p \) we fix an embedding \( \mathbb{Q} \rightarrow \mathbb{Q}_p \) once and for all. This fixes a place \( \mathcal{P} \) in \( \mathbb{Q}_f \) lying over \( p \) of residue degree \( a \) for some positive integer \( a \). As usual, we let \( E(x) = \exp(\sum_{i=0}^{\infty} x^i/p^i) \) be the \( p \)-adic Artin-Hasse exponential function. Let \( \gamma \) be a root of the \( p \)-adic \( \log E(x) \) with \( \text{ord}_p(\gamma) = \frac{1}{p} \). Then \( E(\gamma) \) is a primitive \( p \)-th root of unity and we set \( \zeta_p := E(\gamma) \). Let \( \mathbb{F}_p \) be the finite field of \( p^a \) elements. For \( k \geq 1 \), let \( \psi_k : \mathbb{F}_p^x \rightarrow \mathbb{Q}(\zeta_p)^{\times} \) be a nontrivial additive character of \( \mathbb{F}_q \). Henceforth we fix \( \psi_k(\cdot) = \zeta_p^{T_k \cdot \Tr_k^{-1} (\cdot)} \). Let \( \prod_{j=1}^{\ell} d_j \), and all poles and leading coefficients \( a_{j, d_j} \) of \( f \) be \( p \)-adic units. Let all coefficients \( a_{j, d_j} \) of \( f \) are \( p \)-adically integral. (These are satisfied when \( p \) is large enough.) Let \( S_k(f \mod \mathcal{P}) = \sum_x \psi_k(f(x) \mod \mathcal{P}) \) where the sum ranges over all \( x \) in \( \mathbb{F}_p^x \setminus \{ \mathcal{P}_1, \ldots, \mathcal{P}_\ell \} \) (where \( \mathcal{P}_j \) are reductions of \( P_j \mod \mathcal{P} \)). The \( L \)-function of \( f \) at \( p \) is defined as

\[
L(f \mod \mathcal{P}; T) = \exp(\sum_{k=1}^{\infty} S_k(f \mod \mathcal{P}) T^k / k).
\]

This function lies in \( \mathbb{Z}[\zeta_p][T] \) of degree \( d \). It is independent of the choice of \( \mathcal{P} \) (that is, the embedding of \( \mathbb{Q} \rightarrow \mathbb{Q}_p \) for \( p \) large enough, but we remark that its Newton polygon is independent of the choice of \( \mathcal{P} \) for all \( p \) (see [15 Section 1]). One notes immediately that for every prime \( p \) (coprime to the leading coefficients, the poles and their orders) we have a map \( \mathbb{N}_p(\cdot) \) which sends every \( p \)-adic integral point \( f \) of \( \mathbb{A}(\mathbb{Q}_p) \) to the Newton polygon \( \mathbb{N}_p(f) \) of the \( L \)-function of exponential sums of \( f \) at \( p \). Given any \( f \in \mathbb{A}(\mathbb{Q}) \), we have for \( p \) large enough that \( f \in \mathbb{A}(\mathbb{Z}_p) \) and hence we obtain the Newton polygon \( \mathbb{N}_p(f) \) of \( f \) at \( p \). Presently it is known that \( \mathbb{N}_p(f) \) lies over \( \mathbb{H}(\mathbb{A}) \) for every \( p \). These two polygons do not always coincide. (See [15 Introduction].) Some investigation on first slopes suggests the behavior is exceptional if \( p \) is small (see [4 Introduction]). There has been intensive investigation on how the (Archimedean) distance between \( \mathbb{N}_p(f) \) and \( \mathbb{H}(\mathbb{A}) \) on the real plane \( \mathbb{R}^2 \) varies when \( p \) approaches infinity. Inspired by Wan’s conjecture [10 Conjecture 1.15] (proved in [13] for the one-variable polynomial case), we believe that “almost all” points \( f \) in \( \mathbb{A}(\mathbb{Q}) \) satisfies \( \lim_{p \to \infty} \mathbb{N}_p(f) = \mathbb{H}(\mathbb{A}) \). Our main result is the following.

**Theorem 1.1.** Let \( \mathbb{A} \) be the coefficients space \( \{a_{ji}\} \) of the \( f \)'s as in [4]. There is a Zariski dense open subset \( \mathcal{U} \) defined over \( \mathbb{Q} \) in \( \mathbb{A} \) such that for every geometric closed point \( f \) in \( \mathcal{U}(\mathbb{Q}) \) one has \( f \in \mathcal{U}(\mathbb{Z}_p) \) for \( p \) large enough (only depending on
\[
\lim_{p \to \infty} \text{NP}_p(f) = \text{HP}(A).
\]

The two polygons \(\text{NP}_p(f)\) and \(\text{HP}(A)\) coincide if and only if \(p \equiv 1 \mod \text{lcm}(d_j)\) (see [15, Theorem 1.1]). The case \(\ell = 1\) is known from [13, Theorem 1.1]. For \(p \not\equiv 1 \mod \text{lcm}(d_j)\), the point \(f = x^{d_1} + \sum_{j=1}^{\ell} (x - P_j)^{-d_j}\) does not lie in \(U\). This means \(U\) is always a proper subset of \(A\).

For any \(\overline{f} \in A(F_\ell)\) and the (generalized) Artin-Schreier curve \(C_{\overline{f}}: y^p - y = \overline{f}\), let \(\text{NP}(C_{\overline{f}}, F_\ell)\) be the usual \(p\)-adic Newton polygon of the numerator of the zeta function of \(C_{\overline{f}}/F_\ell\). If it is shrunk by a factor of \(1/(p-1)\) vertically and horizontally, we denote it by \(\text{NP}(C_{\overline{f}}, F_\ell)_{p-1}\).

**Corollary 1.2.** Let notations be as in Theorem 1.1 and the above. For any \(\overline{f} \in A(F_\ell)\) we have \(\text{NP}(C_{\overline{f}}, F_\ell)_{p-1}\) lies over \(\text{HP}(A)\) with the same endpoints, and they coincide if and only if \(p \equiv 1 \mod \text{lcm}(d_j)\). Moreover, there is a Zariski dense open subset \(U\) defined over \(\mathbb{Q}\) in \(A\) such that for every geometric closed point \(f\) in \(U(\mathbb{Q})\) one has \(f \in U(Z_p)\) for \(p\) large enough (only depending on \(f\)), and \(\lim_{p \to \infty} \text{NP}(C_{\overline{f}}, F_\ell)_{p-1} = \text{HP}(A)\).

**Proof.** This follows from the theorem above and a similar argument as the proof of Corollary 1.3 in [15], which we shall omit here. \(\square\)

**Remark 1.3.** (1) The result in Theorem 1.1 and Corollary 1.2 does not depend on where those \(\ell\) poles are (as long as they are distinct).

(2) By Deuring-Shafarevic formula (see for instance [3, Corollary 1.5]), one knows that \(\text{NP}_p(f)\) always has slope-0 segment precisely of horizontal length \(\ell - 1\). By symmetry it also has slope-1 segment of the same length. See Remark 1.4 of [15].

Plan of the paper is as follows: section 2 introduces sheaves of (infinite dimensional) \(\varphi\)-modules over some affinoid algebra arising from one-variable exponential sums. We consider two Frobenius maps \(\alpha_1\) and \(\alpha_a\). Section 4 is the main technical part, where major combinatorics of this paper is done. After working out several combinatorial observations we are able to reduce our problem to an analog of the one-variable polynomial case as that in [13]. Now back to Section 3 we improve the key lemma 3.5 of [13] to make the generic Fredholm polynomial straightforward to compute. Section 5 uses \(p\)-adic Banach theory to give a new transformation theorem from \(\alpha_1\) to \(\alpha_a\) for any \(a \geq 1\). This approach is very different from [9] or [14]. It sheds some new light on \(p\)-adic approximations of \(L\)-functions of exponential sums and we believe that it will find more application in the future. Finally at the end of section 5 we prove our main result Theorem 1.1.

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2. Sheaves of \( \varphi \)-modules over affinoid algebra

The purpose of this section is to generalize the trace formula (see [14, Section 2]) for an exponential sum to that for families of exponential sums. See [2] for fundamentals in rigid geometry and see [1] for an excellent setup for rigid cohomology related to \( p \)-adic Dwork theory.

Let \( \mathcal{O}_1 := \mathbb{Z}_p[\zeta] \) and \( \Omega_1 := \mathbb{Q}_p(\zeta) \). Fix a positive integer \( a \). Let \( \Omega_a \) be the unramified extension of \( \Omega_1 \) of degree \( a \) and \( \mathcal{O}_a \) its ring of integers. Let \( \tilde{P}_1, \ldots, \tilde{P}_\ell \) in \( \mathcal{O}_a^\times \) be Teichmüller lifts of \( \mathcal{T}, \ldots, \mathcal{T}_\ell \) in \( \mathbb{F}_{p^{\ell}} \). Similarly let \( A_{j,i} \) be that of \( \mathcal{T}_{j,i} \) and let \( \tilde{A} \) denote the sequence of \( A_{j,i} \) (we remark that for most part of the paper \( \tilde{A} \) will be treated as a variable). Let \( \tau \) be the lift of Frobenius to \( \Omega_a \) which fixes \( \Omega_1 \). Then \( \tau(A_{j,i}) = A_{j,i}^p \). Let \( 1 \leq j \leq \ell \). Pick a root \( \gamma^{1/d_j} \) of \( \gamma \) in \( \mathbb{Q}_p \) (or in \( \mathbb{Z}_p \) all the same) for the rest of the paper, and denote \( \Omega'_1 := \Omega_1(\gamma^{1/d_1}, \ldots, \gamma^{1/d_\ell}) \). Let \( \Omega'_1 \) be its ring of integers. Let \( \Omega'_a := \Omega_a \Omega'_1 \) and let \( \Omega'_a \) be its ring of integers. Then the affinoid algebra \( \Omega'_a(\tilde{A}) \) (with \( \tilde{A} \) as variables) forms a Banach algebra over \( \Omega'_a \) under the supremum norm.

Let \( 0 < r < 1 \) and \( r \in |\Omega'_a|_p \). Let \( \mathbb{A}_r \) be the affinoid with \( \ell \) deleted discs centering at \( \tilde{P}_1, \ldots, \tilde{P}_\ell \) each of radius \( r \) on the rigid projective line \( \mathbb{P}^1 \) over \( \Omega'_a \) (as defined in [15]). The topology on \( \mathbb{A}_r \) is given by the fundamental system of strict neighborhood \( \mathbb{A}_{r'} \) with \( r < r' < 1 \) and \( r' \in |\Omega'_a|_p \). Let \( \mathbb{A} \) be \( \mathbb{A}_r \) for some unspecified \( r \) sufficiently close to \( 1^- \) (the precise bound on the size of \( r \) was discussed in [15, Section 2]). Let \( \mathcal{H}(\Omega'_a) \) be the ring of rigid analytic functions on \( \mathbb{A} \) over \( \Omega'_a \). Then it is a \( p \)-adic Banach space over \( \Omega'_a \).

It consists of functions in one variable \( X \) of the form \( \xi = \sum_{i=0}^{\infty} c_{i,j} X^i + \sum_{j=2}^{\ell} \sum_{i=1}^{\infty} c_{j,i} (X - \tilde{P}_j)^{-i} \) where \( c_{j,i} \in \Omega'_a \) and \( \forall j \geq 1, \lim_{i \to \infty} \frac{|c_{i,j}|_{p^{\ell_j}}}{r^i} = 0 \). Its norm is defined as \( ||\xi|| = \max_j (\sup_i |c_{i,j}|_{p^{\ell_j}}) \). (See [15, Section 2.1].) Let \( \mathcal{H}(\Omega'_a(\tilde{A})) := \mathcal{H}(\Omega'_a) \otimes_{\Omega'_a} \Omega'_a(\tilde{A}) \) where \( \otimes \) means \( p \)-adic completion after the tensor product. It is a \( p \)-adic Banach modules over \( \Omega'_a(\tilde{A}) \) with the natural norm on the tensor product of two Banach modules defined by the followings. For any \( \sum v \otimes w \in \mathcal{H}(\Omega'_a(\tilde{A})) \) let \( ||\sum v \otimes w|| = \inf(\max_i (||v_i|| \cdot ||w_i||)) \), where the inf ranges over all representatives \( \sum v_i \otimes w_i \) with \( \sum v \otimes w = \sum_i v_i \otimes w_i \). From the \( p \)-adic Mittag-Leffler decomposition theorem derived in [15, Section 2.1], we can generalize it to the decomposition of \( \Omega'_a(\tilde{A}) \) as a Banach \( \Omega'_a(\tilde{A}) \)-module. Write \( X_1 = X \) or \( X_j = (X - \tilde{P}_j)^{-1} \) for \( 2 \leq j \leq \ell \). Let \( Z_j = \gamma^{1/d_j} X_j \). Note that \( \tilde{b}_w = \{1, Z_1, \ldots, Z_{\ell_j}\}_{i \geq 1} \) is a formal basis of the Banach \( \Omega'_a(\tilde{A}) \)-module \( \mathcal{H}(\Omega'_a(\tilde{A})) \), that is, every \( v \in \mathcal{H}(\Omega'_a(\tilde{A})) \) can be written uniquely as an infinite sum of \( \tilde{b}_w \)'s with \( \tilde{c}_{i,j} \in \Omega'_a(\tilde{A}) \) and \( \frac{\tilde{c}_{i,j}|_{p^{\ell_j}}}{r^i} \to 0 \) as \( i \to \infty \), where \( r' = r p^{-\frac{1}{m-1}} \). The Banach module \( \mathcal{H}(\Omega'_a(\tilde{A})) \) is orthonormalizable (even though \( \tilde{b}_w \) is not its orthonormal basis).

In this paper we extend the \( \tau \)-action so that it acts on \( \gamma^{1/d_j} \) trivially for any \( J \). Below we begin to construct the Frobenius operator \( \alpha_1 \) on \( \mathcal{H}(\Omega'_a(\tilde{A})) \). Recall the \( p \)-adic Artin-Hasse exponential function \( E(X) \). Take expansion of \( E(\gamma X) \) at \( X \) one gets \( E(\gamma X) = \sum_{m=0}^{\infty} \lambda_m X^m \) for some \( \lambda_m \in \mathcal{O}_1 \). Clearly \( \text{ord}_p \lambda_m \geq \frac{m}{p^{m-1}} \) for all \( m \geq 0 \). In particular, for \( 0 \leq m \leq p - 1 \) the equality holds and \( \lambda_m = \frac{m}{p^{m-1}} \). Let
F_j(X_j) := \prod_{i=1}^{d_j} E(\gamma A_{j,i} X_j^i). Then

F_j(X_j) = \sum_{n=0}^{\infty} F_{j,n}(A_{j,1}, \ldots, A_{j,d_j}) X_j^n,

where \( F_{j,n} := 0 \) for \( n < 0 \) and for \( n \geq 0 \)

\[ F_{j,n} := \sum \lambda_{m_1} \cdots \lambda_{m_{d_j}} A_{j,1}^{m_1} \cdots A_{j,d_j}^{m_{d_j}}, \]

where the sum ranges over all \( m_1, \ldots, m_{d_j} \geq 0 \) and \( \sum_{k=1}^{d_j} km_k = n \). It is clear that \( F_{j,n} \) lies in \( \mathcal{O}_1[A_{j,1}, \ldots, A_{j,d_j}] \). One observes that \( F_j(X_j) \in \mathcal{O}_1(A_{j,1}, \ldots, A_{j,d_j}) \langle X_j \rangle \), the affinoid algebra in one variable \( X_j \) (actually it lies in \( \mathcal{O}_1[A_{j,1}, \ldots, A_{j,d_j}] \langle X_j \rangle \)).

Taking product over \( j = 1, \ldots, \ell \), we have that \( F(X) := \prod_{j=1}^{\ell} F_j(X_j) \) lies in \( \mathcal{H}(\mathcal{O}_a(\bar{A})) \). Let \( \tau_*^{a-1} \) be the push-forward map of \( \tau^{a-1} \), that is, for any function \( \xi \), \( \tau_*^{a-1}(\xi) = \tau^{a-1} \circ \xi \circ \tau \). For example, \( \tau_*^{a-1}(B/(X - P^p)) = \tau^{a-1}(B)/(X - P) \) for any \( B \in \mathcal{O}_1(\bar{A}) \) and \( P \) a Teichmüller lift of some \( \mathfrak{F} \). Let \( U_p \) be the Dwork operator and let \( F(X) \) denote the multiplication map by \( F(X) \), as defined in [15, Section 2]. Let \( \alpha_1 := \tau_*^{a-1} \circ U_p \circ F(X) \) denote the composition map.

Let \( S \) be the affinoid over \( \Omega'_a \) with affinoid algebra \( \Omega'_a(\bar{A}) \). If \( L \) is a sheaf of \( p \)-adic Banach \( \Omega'_a(\bar{A}) \)-module (with formal basis) and \( \alpha_1 \) is the Frobenius map which is \( \tau^{a-1} \)-linear with respect to \( \Omega'_a(\bar{A}) \), then we call the pair \((L, \alpha_1)\) a sheaf of \( \varphi \)-module of infinite rank.

Note that the pair \((\mathcal{H}(\Omega'_a(\bar{A})), \alpha_1)\) can be considered as sections of a sheaf of \( \Omega'_a(\bar{A}) \)-module of infinite rank on \( A \). This is intimately related to Wan’s nuclear \( \sigma \)-module of infinite rank (see [11]) if replacing his \( \sigma \) by our \( \tau^{a-1} \).

Wan has defined \( L \)-functions of nuclear \( \sigma \)-modules and he also showed that it is \( p \)-adic meromorphic on the closed unit disc (see Wan’s papers [11][12][15][16] which proved Dwork’s conjecture). Finally we define \( \alpha_a := \alpha_1^a \).

Recall that \( \alpha_1 \) is a \( \tau^{a-1} \)-linear (with respect to \( \Omega'_a(\bar{A}) \)) completely continuous endomorphism on the \( p \)-adic Banach module \( \mathcal{H}(\Omega'_a(\bar{A})) \) over \( \Omega'_a(\bar{A}) \). Let \( 1 \leq J_1, J \leq \ell \). Write \( (\alpha Z_j)_{\mathfrak{F}_j} = \sum_{i=0}^{\infty} (\tau^{a-1} C_{J_{i+1},i})_{\mathfrak{F}_j} \) for some \( C_{J_{i+1},i} \) in \( \Omega'_a(\bar{A}) \). The matrix of \( \alpha_1 \), consisting of all these \( \tau^{a-1} C_{J_{i+1},i} \)’s, is a nuclear matrix (see section 5). This matrix is the subject of the next section. Below we extend Dwork, Monsky and Reich’s trace formula to families of one-variable exponential sums.

**Theorem 2.1.** Let \( \mathcal{T} = \sum_{i=1}^{d_1} \mathfrak{p}_i, i x + \sum_{j=2}^{\ell} \sum_{i=1}^{d_j} \mathfrak{p}_j, i (x - P_j)^{-i} \in \mathbb{A}(\mathbb{F}_q) \) and let \( \mathcal{T} \) be its Teichmüller lift with coefficient \( \mathfrak{p}_j \) being lifted to \( A_{j,i} \). Let \( \mathcal{H}(\Omega'_a(\bar{A})) \), be the Banach module \( \mathcal{H}(\Omega'_a(\bar{A})) \) for some suitably chosen \( 0 < r < 1 \) with \( r \in \Omega'_a \) close enough to \( 1^{-1} \). Then

\[
L(\mathcal{T}/\mathbb{F}_q, T) = \frac{\det(1 - T\alpha_a | \mathcal{H}(\Omega'_a(\bar{A})))}{\det(1 - Tq\alpha_a | \mathcal{H}(\Omega'_a(\bar{A})))}
\]

lies in \( \mathcal{O}_a(\bar{A})[T] \) as a polynomial of degree \( d \) in \( T \). Its Teichmüller specialization of \( \bar{A} \) in \( \mathcal{O}_a \) lies in \( \mathbb{Z} \mathcal{F}_q[T] \).

**Proof.** The proof is similar to that of [15] Lemma 2.7]. Let \( \mathcal{H}^I := \bigcup_{0 < r < 1} \mathcal{H}(\Omega'_a(\bar{A}))_r \). Then it is the Monsky-Washnitzer dagger space. Then \( \alpha_a \) is a completely continuous endomorphism on \( \mathcal{H}^I \) and the determinant \( \det(1 - T\alpha_a | \mathcal{H}^I) = \det(1 -
\[ T \alpha_0 | \mathcal{H}(\Omega'_n(\overline{A}))_r \} \text{ for any } r \text{ within suitable range in } (0, 1) \text{ is independent of } r. \] Finally one knows that the coefficients are all integral so lies in \( \mathcal{O}_n \) and coefficient of \( T^m \) vanishes for all \( m > d \). We omit details of the proof. \[ \square \]

3. Explicit approximation of the Frobenius matrix

This section uses some standard techniques in \( p \)-adic approximation and it is very technical. The readers are recommended to skip it at first reading and continue at the next section.

3.1. The nuclear matrix. Let notation be as in the previous section. Assign \( \phi(1) = 0 \). Let \( \phi(Z^j) = \frac{n^{-1}}{j} \) for \( j \leq 2 \) or \( \frac{n^{-1}}{j^2} \) for \( j \geq 3 \). Order the elements in \( \hat{b}_w \) as \( e_1, e_2, \cdots \) such that \( \phi(e_1) \leq \phi(e_2) \leq \cdots \). Consider the infinite matrix representing the endomorphism \( \alpha_0 \) of the \( \Omega'_n(\overline{A}) \)-module \( \mathcal{H}(\Omega'_n(\overline{A})) \) with respect to the basis \( \hat{b}_w \). This matrix can be written as \( \tau^{m-1} \mathcal{M} \), where each entry is \( \tau^{m-1} C^{n,i}_{J_1,J} \) for \( 1 \leq J_1, J \leq \ell \).

Our goal of this section is to collect delicate information about entries of the matrix \( \mathcal{M} \). Recall the is polynomial \( F_{J,n,J} \) in \( \mathcal{O}_n[\overline{A}] \) as in \( 3 \), which we have already built up some satisfying knowledge. Below we will express \( C^{n,i}_{J_1,J} \) as a polynomial expression in these \( F_{J,n,J} \)’s. In this paper the formal expansion of \( C^{n,i}_{J_1,J} \) will always mean the formal sum in \( \mathcal{O}_n[\overline{A}] \) by the composition of \( 3 \) and the formula in Lemma \( 3 \).

For \( n, i \geq 1 \), and if \( J_1 = 1 \) or \( J_1 = 1 \) then for \( i \geq 0 \) or for \( n \geq 0 \) respectively one has

\[ C^{n,i}_{J_1,J} = \begin{cases} \gamma^\frac{1}{2} - \frac{a}{J_1} H_{J_1,J}^{n,p,i} & J_1 = 1, 2 \\ \gamma^\frac{1}{2} - \frac{a}{J_1} \sum_{m=n}^{np} C^{n,m}_{J_1,J} H_{J_1,J}^{m,i} & J_1 \geq 3 \end{cases} \]

where \( C^{n,i}_{J_1,J} \in \mathbb{Z}_p \) is defined in \( 15 \) Lemma 3.1 and \( H_{J_1,J}^{n,i} \) in \( \mathcal{O}_n[\overline{A}] \) is formulated in Lemma \( 3 \) below. Indeed, we recall that \( C^{n,m} \) is actually a rational integer and it only depends on \( n, m \) and \( p \).

Lemma 3.1. Let \( \vec{i} := (n_1, \ldots, n_\ell) \in \mathbb{Z}_\geq 0^\ell \).

(1) For \( i, n \geq 0 \), then \( H_{n,i}^{n,i} \) is equal to

\[ \sum \left( F_{1,n_1} \left( \sum_{j \neq 1} F_{J,m_j} \left( \frac{n_j + i - 1}{m_j + i - 1} \right) \hat{P}_{J,m_j}^{n_j - m_j} \right) \right) \prod_{j \neq 1, J} \left( \sum_{m_j = 0}^{n_j} F_{J,m_j} \left( \frac{n_j - 1}{m_j - 1} \right) \hat{P}_{J,m_j}^{n_j - m_j} \right), \]

where the sum ranges over all \( n_1 \in \mathbb{Z}_\geq 0^\ell \) such that \( n = n_1 + i - \sum_{j=2}^{\ell} n_j \) and the + or − depends on \( J = 1 \) or \( J \neq 1 \), respectively.
(2) For \( J_1, J \neq 1 \), one has that \( H_{n_1}^{n_i} \) is equal to

\[
\sum \left( F_{J_1,n_1}, \left( \sum_{m_1 = n_1}^{\infty} F_{1,m_1} \left( \frac{m_1 + i}{n_1} \right) \hat{P}_{J_1}^{m_1 + i - n_1} \right) \right) \cdot \prod_{j \neq 1, J_1, J} \left( \sum_{m_j = 0}^{\infty} F_{J,m_j} (-1)^{m_j} \left( \frac{n_j + m_j - 1}{m_j - 1} \right) (\hat{P}_j - \hat{P}_{J_1})^{-(n_j + m_j)} \right)
\]

where the sum ranges over all \( n \in \mathbb{Z}_{\geq 0} \) such that \( n = n_{J_1} + i - \sum_{j \neq J_1, J} n_j \) if \( J = J_1 \) and \( n = n_{J_1} - \sum_{j \neq J_1} n_j \) if \( J \neq J_1 \).

(3) For \( J_1 \neq 1 \) and \( J = 1 \) we have that \( H_{n_1}^{n_i} \) is equal to

\[
\sum \left( F_{J_1,n_1}, \left( \sum_{m_1 = n_1 - i}^{\infty} F_{1,m_1} \left( \frac{m_1 + i}{n_1} \right) \hat{P}_{J_1}^{m_1 + i - n_1} \right) \right) \cdot \prod_{j \neq 1, J_1, J} \left( \sum_{m_j = 0}^{\infty} F_{J,m_j} (-1)^{m_j} \left( \frac{n_j + m_j - 1}{m_j - 1} \right) (\hat{P}_j - \hat{P}_{J_1})^{-(n_j + m_j)} \right)
\]

where the sum ranges over all \( n \in \mathbb{Z}_{\geq 0} \) such that \( n = n_{J_1} - \sum_{j \neq J_1} n_j \).

Proof. We shall use \( \hat{P}_{1, n}^{m} \) to mean expansion at \( \hat{P}_j \). Clearly for any \( J_1 \) one has

\[
F_{J_1}(X_{J_1})X_{J_1}^{i} \overset{\hat{P}_{J_1}}{=} \sum_{n=0}^{\infty} F_{J_1,n}X^{n+i}.
\]

For \( J \geq 2 \) one has the expansion at \( \hat{P}_1 = \infty \):

\[
F_{J}(X_{J})X_{J}^{i} = \sum_{m=0}^{\infty} F_{J,m}(X^{-1}(1 - \hat{P}_{J}X^{-1})^{-1})^{m+i} = \sum_{m=0}^{\infty} F_{J,m} \left( \sum_{k=m+i}^{\infty} \frac{1}{1+m+i} X^{-k} \right) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} F_{J,m} \left( \frac{n+i-1}{m+i-1} \right) \hat{P}_{J}^{n-m}X^{-n-i} \right).
\]

For \( J_1 \neq 1 \) and \( J \neq 1, J_1 \), its expansion at \( \hat{P}_{J_1} \) is:

\[
F_{J}(X_{J})X_{J}^{i} = \sum_{m=0}^{\infty} F_{J,m}(X_{J_1}^{-1} - (\hat{P}_{J} - \hat{P}_{J_1}))^{-(m+i)} = \sum_{m=0}^{\infty} F_{J,m} \left( \sum_{n=0}^{\infty} \frac{1}{1+m+i} X^{-n} \right) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} F_{J,m} \left( \frac{n+i-1}{m+i-1} \right) \hat{P}_{J_1}^{n-m}X_{J_1}^{-n} \right).
\]
LEMMA 3.3.

For $J_1 \neq 1$ and $J = 1$ then one has

$$F_j(X)X_j^m \equiv \sum_{n=0}^{\infty} \left( \sum_{m=-n}^{\infty} F_{j,m} \left( \frac{m+i}{n} \right) \hat{p}_{J_1}^{m+i-n} \right) X_j^{-n}.$$ 

By $F(X)X_j^m = (F_j(X)X_j^m) \cdot \prod_{j \neq J} F_j(X_j)$, and Key Computational Lemma of [15], one can compute and obtain $(F(X)X_j^m)\hat{p}_{J_1}$ for the case $J_1 = 1$ or $J_1 \neq 1$ presented respectively in the two formulas in our assertion. This proves the lemma.

**Remark 3.2.** If we are dealing with the case of unique pole at $\infty$ then one sees easily that $C_{1,1}^\ast$ lies in $O'(\hat{A})$. This greatly reduces the complexity of situation.

The following results were presented in [15]. See Section 3 and in particular, Theorem 3.7 of [15] for a proof. We shall use $t_{J_1}$ to denote the lower bound in Lemma 3.3 (c).

**Lemma 3.3.** Let notation be as above.

(a) For all $J$ and $n,J$ we have $\text{ord}_p(F_{J,n,J}) \geq \frac{\lfloor n + t_{J_1} \rfloor}{p-1} \geq \frac{n_{J}}{J(p-1)}$.

(b) For all $J,J$, and all $n,i$ we have $\text{ord}_p(H_{J,i,J}^n) \geq \frac{n-i}{J(p-1)}$.

(c) For any $J_1$ and any $n$ we have $\text{ord}_p(C_{1,J_1,J}^{*}) \geq \frac{n}{J} \text{ or } \frac{n-1}{J} \text{ depending on } J_1 = 1, 2 \text{ or } 3 \leq J_1 \leq \ell$. Moreover, $\text{ord}_p(C_{1,J_1,J}^{*}) \geq \frac{\lfloor np/2 \rfloor}{J(p-1)}$ or $\frac{(n-1)(p-i-1)}{J(p-1)}$ depending on $J_1 = 1, 2$ or $3 \leq J_1 \leq \ell$, respectively.

**3.2. Approximation by truncation.** From the previous subsection one has noticed an unpleasant feature of $C_{1,J_1,J}^{*}$ for the purpose of approximation by $F_{J_1,n,J}$’s. First, in the sum for $C_{1,J_1,J}^{*}$ when $J_1 \geq 3$, the range of $m$ is too ‘large’. Second, $H_{J_i,J}^n$ of Lemma 3.1 is generally an infinite sum of $F_{J_1,n,J}$’s. In this subsection we will define an approximation in terms of truncated finite sum of $F_{J_1,n,J}$’s. Below we prove two lemmas which will be used for approximation in Lemma 3.3 (c).

For any integer $0 < t < p$, let $C_{1,J_1,J}^{t}$ be the same as $C_{1,J_1,J}^{*}$ except for $J_1 \geq 3$ its sum ranges over all $m$ in the sub-interval $[(n-1)p+1, (n-1)p+t]$.

**Lemma 3.4.** Let $3 \leq J_1 \leq \ell$, $1 \leq J \leq \ell$. Let $n \leq d_{J_1}$ and $i \leq d_{J}$.

1. For $p$ large enough, one has

$$\text{ord}_p(C_{1,J_1,J}^{n,i} - tC_{1,J_1,J}^{n,i}) \geq \frac{n-1}{d_{J_1}} + \frac{d}{p-1}.$$  

2. There is a constant $\beta > 0$ depending only on $d$ such that for $t \geq \beta$ one has

$$\text{ord}_p(C_{1,J_1,J}^{n,i} - tC_{1,J_1,J}^{n,i}) \geq \frac{n-1}{d_{J_1}} + \frac{d}{p-1}.$$  

**Proof.** (1) By [15] Lemma 3.1, one knows that for any $m \leq (n-1)p$ one has $\text{ord}_p(C_{1,J_1,J}^{n,m}) \geq 1$ and hence $\text{ord}_p(C_{1,J_1,J}^{n,i}) \geq 1 + \left( \frac{t}{d_{J_1} - \frac{n}{p-1}} \right) 1$. For $n \leq d_{J_1}$ and $p$ large enough one has $1 + \left( \frac{t}{d_{J_1} - \frac{n}{p-1}} \right) 1 > \frac{n-1}{d_{J_1}} + \frac{d}{p-1}$. Combining these two inequalities, one concludes.

(2) We may assume $J_1 \geq 3$. Then for any $1 \leq v \leq p$, by Lemma 3.3

$$\text{ord}_p(H_{1,J_1,J}^{n-1,p+v,i}) \geq \frac{(n-1)p+v-i}{d_{J_1}(p-1)} > \left( \frac{n}{d_{J_1}} - \frac{i}{d_{J}} \right) 1 + \frac{n-1}{d_{J_1}} + \frac{d}{p-1},$$
if \( v \geq \beta \) for some \( \beta > 0 \) only depending on \( d \). Therefore,

\[
\text{ord}_p (pC_{n,i}^{p,i} - \ell C_{n,i}^{n,i}) > \frac{n-1}{d_{J_i}} + \frac{d}{p-1}.
\]

This finishes our proof. \( \square \)

Fix \( \beta \) for the rest of the paper. We will truncate the infinite expansion of \( H_{j_1,i,J}^{n,i} \).

Let \( w > 0 \) be any integer. For \( J_1 = 1,2 \) let \( wH_{j_1,i,J}^{n,i} \) be the sub-sum in \( H_{j_1,i,J}^{n,i} \) where \( \vec{\alpha} = (n_1, \ldots, n_\ell) \) such that \( n_{J_1} - np \) and \( n_j \) lie the interval \([-w,w]\) for \( j \neq J_1 \). Similarly, for \( J_1 \geq 3 \) let \( wH_{j_1,i,J}^{n,i} \) be the sub-sum of \( H_{j_1,i,J}^{n,i} \) where \( \vec{\alpha} \) ranges over the finite set of vectors \((n_1, \ldots, n_\ell)\) such that \( n_{J_1} - (n-1)p \) and \( n_j \) lie in the interval \([-w,w]\) for \( j \neq J_1 \). Consider \( \beta C_{n,i,J}^{n,i} \) as a polynomial expression in \( H_{j_1,i,J}^{n,i} \)'s, then we set \( wK_{n,i,J}^{n,i} := \beta C_{n,i,J}^{n,i} (wH_{j_1,i,J}^{n,i}) \).

**Lemma 3.5.** There is a constant \( \alpha \) depending only on \( d \) such that

\[
\text{ord}_p (\beta C_{n,i,J}^{n,i} - \alpha K_{n,i,J}^{n,i}) > \frac{n-1}{d_{J_i}} + \frac{d}{p-1}.
\]

**Proof.** This part is similar to Lemma 3.4(2), so we omit its proof. \( \square \)

### 3.3. Minimal weight terms

The *weight* of a monomial (with nonzero coefficient) \((\prod_{j=1}^{J} \prod_{i=1}^{d_j} A_{i,j}^{k_{i,j}})\) in \( O_\alpha [\vec{A}] \) is defined as \( \sum_{j=1}^{J} \sum_{i=1}^{d_j} ik_{i,j} \). For example, the weight of \( A_{1,2}^a A_{1,3}^b \) is equal to \( 2a + 3b \). We will later utilize the simple observation that every monomial in \( F_{j,n,i} \) is of weight \( n_j \).

We call those entries with \( J_1 = J \) the diagonal one (or blocks). As we have seen in Lemma 3.1, the off-diagonal entries are less manageable while the diagonal entries behave well in principle. For any integer \( 0 < t \leq p \), let \( \text{\textsuperscript{t}M} := (tC_{n,i}^{n,i}) \) with respect to the basis arranged in the same order as that for \( M \). Consider the diagonal blocks, consisting of \( pC_{n,i,J}^{n,i} \)'s. Despite \( pC_{n,i,J}^{n,i} \) lives in \( O_\alpha [\vec{A}] \), its minimal weight terms live in \( \hat{P}_j^{\frac{i-n}{\gamma}} F_{j,n,i} \). For \( J \geq 3 \) and \( n \geq 2 \), the minimal weight monomials of \( pC_{n,i,J}^{n,i} \) live in the term

\[
\gamma \frac{i-n}{\gamma} C_{n,(n-1)p+1}^{n,(n-1)p+1} \hat{P}_j^{p-1} F_{j,(n-1)p-1} \]

where \( d_j > n, i \geq 1 \).

**Lemma 3.6.** Let \( p > d_j \) for all \( j \). The minimal weight monomials of \( pC_{n,i,J}^{n,i} \) (with \( J = 1,2 \)) live in the term \( \gamma \frac{i-n}{\gamma} F_{1,np-i} \) where \( d_1 > n, i \geq 0 \) unless \( n = 0 \) and \( i > 0 \).

For \( J \geq 3 \) and \( n \geq 2 \), the minimal weight monomials of \( pC_{n,i,J}^{n,i} \) live in the term

\[
\gamma \frac{i-n}{\gamma} C_{n,(n-1)p+1}^{n,(n-1)p+1} \hat{P}_j^{p-1} F_{j,(n-1)p-1} \]

where \( d_j > n, i \geq 1 \).

**Proof.** This follows from Lemma 3.1. We omit its proof. \( \square \)

Given a \( k \times k \) matrix \( M := (m_{ij})_{1 \leq i,j \leq k} \) with a given formal expansion of \( m_{ij} \in O_\alpha [\vec{A}] \), the formal expansion of \( \det M \) means the formal expansion as \( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^{k} m_{i\sigma(i)} \) where the product is expanded according to the given formal expansion \( m_{ij} \). For example, if \( m_{ij} = C_{n,i,J}^{n,i} \), then its formal expansion is given by composition of \( \text{\textsuperscript{b}} \) and formulas in Lemma 3.1.

**Lemma 3.7.** Let notation be as above and let \( p > d_j \) for all \( j \). Then in the formal expansion of \( \det (pM)^{[k]} \) in \( O_\alpha [\vec{A}] \), all minimal weight terms are from \( \prod_{j=1}^{J} \text{det} pC_{n,i,J}^{n,i} \) (with \( n, i \geq 1 \) in a suitable range for \( J = 1,2 \) and with \( n, i \geq 2 \) for \( J \geq 3 \) ) of the diagonal blocks.
Proof. We will show that picking an arbitrary entry on the diagonal block, every off-diagonal entry on the same row has strictly higher minimal weight among its monomials.

Let $\tilde{A}_J$ stand for the vector $(A_{J,1}, \ldots, A_{J,d_J})$. As we have noticed earlier the polynomial $F_{J,n,i}$ in $O_1[\tilde{A}_J]$ has every monomial of equal weight $n_i$ for any $J$. For simplicity we assume $n_i \geq 1$ here. Using data from Lemma 3.1 we find all minimal weight monomials in $H_{J,-1}^{*}$'s illustrated below by an arrow: $H_{1,j}^{n_{p+1},i} \rightarrow F_{1,n_{p+1},i}, H_{2,j}^{n_{p+1},i} \rightarrow F_{2,n_{p+1},i}, H_{D,j}^{n_{p+1},i} \rightarrow F_{2,n_{p+1},i}$. One also notes that for $J_i \geq 3$ one has that $H_{J_i,1}^{(n_{i-1})p+1,i} \rightarrow F_{J_i,(n_{i-1})p+1,i}$ if $J_i = J$, and $H_{J_i,1}^{(n_{i-1})p+1,i} \rightarrow F_{J_i,(n_{i-1})p+1,i}$ if $J_i \neq J$. One notices from (8) and the above that the minimal weight monomials of $pC_{J_i,j}^{n_{i+1},i}$ live in $H_{J_i,j}^{n_{p+1},i}$ if $J_i = 1, 2$ and in $H_{J_i,j}^{(n_{i-1})p+1,i}$ if $3 \leq J_i \leq \ell$.

Recall that for $J = 1$ the range for $i$ is $i \geq 0$. In all other cases the range is $i \geq 1$. From the above we conclude our claim in the beginning of the proof. Consequently, all minimal weight monomials in the formal expansion of the determinant $M^{[k]}$ come from the diagonal blocks. By Lemma 3.7 $C_{1,1}^{n,i}$ and $C_{J,1}^{n,i}$ (with $J > 3$) both have their minimal weight equal to 0 if $i = 0$ and $> 0$ if $i > 0$. Then it is not hard to conclude that the minimal weight monomials of $\det(C_{1,1}^{n,i} n_{i \geq 0})$ (resp. $\det(C_{J,1}^{n,i} n_{i \geq 2})$).

For $1 \leq J \leq \ell$, let $D_J^{[k]} := \det(F_{J,i,p-1})_{1 \leq i,j \leq k} \in O_1[A_1, \ldots, A_d]$.

Proposition 3.8. Let $p > d_J$ for all $J$. The minimal weight monomials of $\det((pC_{J,i,j}^{n})_{1 \leq i,j \leq k}$ for $J = 1, 2$ (resp. $\det((pC_{J,i,j}^{n})_{2 \leq i,j \leq k}$ for $J \geq 3$) lie in $D_J^{[k]}$ (resp. $D_J^{[k-1]}$). Every monomial of $D_J^{[k]}$ (resp. $D_J^{[k-1]}$) corresponds to a monomial in the formal expansion of $\det((pC_{J,i,j}^{n})_{1 \leq i,j \leq k})$ for $J = 1, 2$ (resp. $\det((pC_{J,i,j}^{n})_{2 \leq i,j \leq k}$ for $J \geq 3$) by the same permutation $\sigma \in S_k$ in the natural way.

Proof. It follows from Lemmas 3.10 and 3.11 above.}

3.4. Local at each pole. For ease of notation, we drop the subindex $J$ for the rest of this subsection. One should understand that $d_J, A_J, F_{i,p-1}, D_n$ stand for $d_J, A_{J,i}, F_{i,p-1,j}, D_n^{[i]}$, respectively. Let $1 \leq n \leq d - 1$ and let $S_n$ be the permutation group. Let $D_n := \det(F_{i,p-1})_{1 \leq i,j \leq n} \in O_1[A_1, \ldots, A_d]$. Then we have the formal expansion of $D_n^{[n]}$:

$$D_n^{[n]} = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{i=1}^{n} g_{\sigma,i},$$

where the second $\sum$ runs over all terms $g_{\sigma,i}$ of the polynomial $F_{i,p-\sigma(1)}$ in $O_1[A_1, \ldots, A_d]$.

Proposition 3.9. Let $1 \leq n \leq d$. Then there is a unique monomial in the above formal expansion of $D_n^{[n]}$ with highest lexicographic order (according to $A_d, \ldots, A_1$). Moreover, the $p$-adic order of this monomial (with coefficient) is minimal among the $p$-adic orders of all monomials in the above formal expansion.

Remark 3.10. We shall fix the unique $\sigma_0$ found in the proposition for the rest of the paper. The minimal $p$-adic order of this monomial (with coefficient) is equal to $[8]$ while every row achieve its minimal order in Lemma 3.10. We shall use this fact later.
Proof. Denote by $r$ the least non-negative residue of $p \mod d$. Recall the $n$ by $n$ matrix $r_n := \{r_{ij}\}_{1 \leq i,j \leq n}$ where $r_{ij} := df \left( \frac{ik}{d} \right) - (ri - j)$. The properties of this matrix can be found in [13, Lemma 3.1]. Let $\prod_{i=1}^{n} h_{\sigma,i}$ be a highest-lexicographic-order-monomial in the formal expansion of $D^{[n]}$. Then $h_{\sigma,i}$ must be the highest-lexicographic-order-monomial in $F_{ip-\sigma(i)}$, which is easily seen to be $c_{ip-\sigma(i)} A_{d}$ or $c_{ip-\sigma(i)} A_{d}^{|\frac{ip-\sigma(i)}{d}|} A_{d-\tau_{i,\sigma(i)}}$ depending on $\tau_{i,\sigma(i)} = 0$ or not, where $c_{ip-\sigma(i)} \in \Omega_1$. We show first that

\[ (7) \]

\[ \sigma(i) = k \text{ for any } 1 \leq i,k \leq n \text{ with } r_{ik} = 0. \]

Suppose that (7) does not hold. Pick a pair $(i,k)$ among the pairs failing (7) such that $|\sigma(i) - k|$ is minimal. Say $\sigma(j) = k$. Define another permutation $\sigma' \in S_n$ by $\sigma'(i) = k$ and $\sigma'(j) = \sigma(i)$ while $\sigma'(s) = \sigma(s)$ for all other $s$. Denote by $h_{\sigma',t}$ the highest-lexicographic-order-monomial in $F_{ip-\sigma'(t)}$. Then it is easy to see that the lexicographic order of $\prod_{t=1}^{n} h_{\sigma',t}$ is strictly higher than that of $\prod_{t=1}^{n} h_{\sigma,t}$, which is a contradiction. Therefore (7) holds.

Notice that for permutations $\sigma'' \in S_n$ satisfying (7), the degree of $A_d$ in $\prod_{t=1}^{n} h_{\sigma'',t}$ does not depend on the choice of $\sigma''$, where $h_{\sigma'',t}$ is the highest-lexicographic-order-monomial in $F_{ip-\sigma''(t)}$. Then the proof of [13, Lemma 3.2] shows that there exists a unique $\sigma_0 \in S_n$ such that $\prod_{t=1}^{n} h_{\sigma_0,t}$ has highest lexicographic order among the corresponding monomials for all $\sigma'' \in S_n$ satisfying (7). In fact, $\sigma_0$ is exactly the permutation in [13, Lemma 3.2]. By the above discussion, this monomial $\prod_{t=1}^{n} h_{\sigma_0,t}$ also has the unique highest lexicographic order in the formal expansion of $D^{[n]}$.

Next we show that the $p$-adic order of $\prod_{t=1}^{n} h_{\sigma_0,t}$ is minimal (among the $p$-adic orders of the monomials in the formal expansion of $D^{[n]}$). Let $\prod_{i=1}^{n} g_{\sigma,i}$ be an arbitrary monomial in the formal expansion of $D^{[n]}$. Then clearly $\text{ord}_p(g_{\sigma,i}) \geq \lceil \frac{ip-\sigma(i)}{d} \rceil = \frac{ip-\sigma(i)+r_{i,\sigma(i)}}{d}$ for all $1 \leq i \leq n$. Since $h_{\sigma_0,i}$ is the highest-lexicographic-order-monomial in $F_{ip-\sigma_0(i)}$, one sees easily that $\text{ord}_p(h_{\sigma_0,i}) = \frac{ip-\sigma_0(i)+r_{i,\sigma_0(i)}}{d}$ for all $1 \leq i \leq n$. From (7) it is easy to see that $r_{i,j} - r_{i,\sigma_0(i)} \geq j - \sigma_0(i)$ for all $1 \leq i,j \leq n$. It follows that $\text{ord}_p(\prod_{i=1}^{n} g_{\sigma,i}) \geq \text{ord}_p(\prod_{i=1}^{n} h_{\sigma_0,i})$. □

Remark 3.11. In the proof of Proposition 3.9 we have noticed that the $\sigma_0$ is exactly the permutation in [13, Lemma 3.2]. Therefore, one can always take $t_0 = 0$, that is, $f_n^0(A) \neq 0$ in [13, Lemma 3.5].

4. Newton polygon of $\alpha_1$

Recall that $\text{HP}(A)$ lives on the real plane over the interval $[0,d]$. Because of Remark 1.3 one only has to consider the part of $\text{NP}_p(f)$ with slope $< 1$, that is, to consider the part of $\text{NP}_p(f)$ over the interval $[0,d - \ell]$. This part is our focus of this section. Suppose for some $1 \leq k \leq d - \ell$, the point $(k,c_0)$ is a vertex on $\text{HP}(A)$. Then one notices that

\[ c_0 = \sum_{J=1}^{2} \sum_{i=1}^{k_J} i/d_J + \ell \sum_{J=3}^{k_J} (i-1)/d_J \]

for a sequence of nonnegative integers $k_1, \ldots, k_t$ such that $k_1 + \ldots + k_t = k$. This sequence is unique because $(k,c_0)$ is a vertex. From now on we fix such a $k$.

For our purpose we also fix the residue classes of $p \mod d_J$ for all $J$. Let $r_{J,i,j}$ be the least nonnegative residue of $-(ip - j) \mod d_J$. Let $\sigma_0$ be the permutation
in $S_k$ which is the union of those permutations found in Proposition 3.4 locally at each pole $P_j$. Let $s_{J_i}$ be the rational number defined by

$$s_{J_i} := \frac{(p-1)k_{J,1}(k_{J,1}+1)}{d_{J,1}} + \sum_{i=1}^{k_{J,1}} r_{J,i-1,\sigma_0(i)}$$

where $+$ and $-$ is taken according to $J_1 = 1, 2$ or $J_1 \geq 3$. Let $s_0 := s_1 + \cdots + s_\ell$. Clearly $s_0 - c_0(p-1) < k \leq d - \ell$.

Let $\alpha$ and $\beta$ be the integers chosen in Lemmas 3.4 and 3.5 (they depend only on $d$). Let $Q' := \mathbb{Q}(\gamma^{1/d_1}, \ldots, \gamma^{1/d_{J_1}})$.

**Lemma 4.1.** For any $J_1 = 1, 2, 1 \leq J \leq \ell$ and for any $n, i$ in their range, for $p$ large enough, there is a polynomial $G_{J_1,i}^{n,i}$ in $\mathbb{Q}(\bar{P})[\bar{A}]$ such that

$$\alpha K_{J_1,i}^{n,i} = \gamma^{(p-1)n/d_{J_1}} U_{J_1,n} G_{J_1,i}^{n,i} \mod \gamma^{(p-1)n/d_{J_1}+d+1}.$$  

For the case $J_1 \geq 3$, one has a similar $G_{J_1,i}^{n,i}$ such that

$$\alpha K_{J_1,i}^{n,i} = \gamma^{(p-1)(n-1)/d_{J_1}} U_{J_1,n} G_{J_1,i}^{n,i} \mod \gamma^{(p-1)(n-1)/d_{J_1}+d+1},$$

where $U_{J_1,n}$ is a $p$-adic unit depending only on the row index $(J_1, n)$.

**Proof.** We use the same technique as 13, so we only outline our proof here for the case $J_1 = J = 1$. Let $n_j \in [-\alpha, \alpha]$ for $j \neq J_1$. Let $n_{J_1} = np + \sum_{j \neq J_1} n_j - i$. For any $\bar{n} = (n_1, \ldots, n_\ell)$ in this range, we have

$$F_j, n_j \equiv \gamma^{\frac{n_j}{\sigma_j}} Q_j \mod \gamma^{\frac{n_j \alpha_j}{\alpha_j}+1},$$

and

$$F_{J_1, n_{J_1}} = \gamma^{\frac{n_{J_1}}{\alpha_{J_1}}} V_{J_1} Q_{J_1} \mod \gamma^{\frac{n_{J_1} \alpha_{J_1}}{\alpha_{J_1}}+1},$$

where $Q_j$‘s and $Q_{J_1}$ are in $\mathbb{Q}'[\bar{A}]$ independent of $p$ and $V_{J_1}$ is some $p$-adic unit depending only on the row index $J_1$. Now let $\bar{n}$ be in the range for $\alpha K_{J_1,i}^{n,i}$ such that $n_j$‘s vary in $[-\alpha, \alpha]$ and

$$\frac{n}{d_{J_1}} - \frac{n_{J_1} \alpha_{J_1}}{\alpha_{J_1}} \geq (p-1)n/d_{J_1}.$$  

Then by the formula of Lemma 3.1 (1), and for $p$ large enough,

$$\alpha K_{J_1,i}^{n,i} = \gamma^{\frac{n}{\sigma_j} - \frac{n_{J_1}}{\alpha_{J_1}}} \gamma^{\frac{n_{J_1} \alpha_{J_1}}{\alpha_{J_1}}} W G_{J_1,i}^{n,i} \mod \gamma^{(p-1)n/d_{J_1}+d+1},$$

where $W$ is a suitable $p$-adic unit. The rest of the cases are similar. \qed

**Proposition 4.2.** Let notation be as in Lemma 4.1. Let $K := (\alpha K_{J_1,i}^{n,i})$. For $p$ large enough, there are a polynomial $Y_k$ in $\mathbb{Q}'(\bar{P})[\bar{A}]$ and some $p$-adic unit $U$ such that

$$\det K^{[k]} \equiv \gamma^{c_0(p-1)} U Y_k \mod \gamma^{c_0(p-1)+d+1},$$

where $Y_k$ is the product of the $U_{J_1,n}$‘s for the pairs $(J_1, n)$ whose corresponding row appears in $M^{[k]}$. Now just set $Y_k = \det G^{[k]}$. \qed
Lemma 4.3. Let $1 \leq k \leq d - \ell$. (1) For $p$ large enough one has

$$\text{ord}_p(\det M^{[k]} - \det p^k M^{[k]}) > \frac{s_0}{p - 1}.$$  

(2) Let $\alpha$ and $\beta$ be the integers chosen in Lemmas 4.2 and 4.3 (they depend only on $d$). Then

$$\text{ord}_p(\det p^k M^{[k]} - \det K^{[k]}) > \frac{s_0}{p - 1}.$$  

Proof. (1) Note that $d \geq k > s_0 - c_0(p - 1)$. Note that in $p^k M^{[k]}$ the row minimal $p$-adic order is the same as that for $C^{n,i}_{J_1,J}$. Let $M$ denote specialization of $M$ at variables $\hat{A}$ by assigning $\hat{A}$ as the Teichmüller lifts of coefficients of $f \mod P$ (see [14, Section 1] for more details).

Proposition 4.4. Let $1 \leq k \leq d - \ell$. Let $(k, c_0) \in \mathbb{R}^2$ be a vertex of the slope $< 1$ part of $\text{HP}(\hat{A})$, where $1 \leq k \leq d - \ell$. There is a Zariski dense open subset $\mathcal{U}_k$ defined over $\overline{Q}$ in $\mathbb{A}$ such that if $f \in \mathcal{U}_k(\overline{Q})$ and if $P$ is a prime ideal in the ring of integers of $\mathbb{Q}(f)$ lying over $p$, one has $\lim_{p \to \infty} \text{ord}_p(\det(M(\hat{f})^{[k]})) = c_0$.

Proof. Without loss of generality, we fix the residues of $p$ as above. Consider the $\gamma$-expansions of $\det M^{[k]}$, $\det p^k M^{[k]}$, and $\det K^{[k]}$. By Lemma 4.3 their $\gamma^{s_0}$-coefficients are the same. Proposition 4.2 implies that for $p$ large enough there is a polynomial $G$ in $\mathbb{Q}(\hat{P})[\hat{A}]$ such that the $\gamma^{s_0}$-coefficient is congruent to $U \bmod \gamma$ for some $p$-adic unit $U$. Moreover, from the proofs of Lemmas 4.1 and Proposition 4.2 one observes easily that the monomials of $G$ are a subset of all monomials in the formal expansion of $\det p^k M^{[k]}$ (with all $\gamma^{0}$-factors squeezed out from its coefficients at appropriate places).

We claim that the $\gamma^{s_0}$-coefficient in $\det p^k M^{[k]}$ is nonzero because it has a unique monomial (in variable $\hat{A}$) among all monomials of minimal weight in its formal expansion. We first look locally at an arbitrary pole $P_J$ where $1 \leq J \leq \ell$. By Proposition 4.4 there is a unique local monomial among all terms in $\det D^{[k]}_J$ for $J = 1, 2$ and $\det D^{[k-J^{-1}]}_{1,J}$ for $J \geq 3$. This local monomial corresponds to a permutation $\sigma_{J,0} \in S_k$. Note that the composition of these $\sigma_{J,k}$’s for all $J$ is equal to $\sigma_0$ defined in the beginning of the section. Then the unique monomial we desire is precisely the product of these local monomials (see Lemma 3.7 and Proposition 3.8). By the remark in last paragraph, it is not hard to see that $G \neq 0$.  

Let $\gamma^{>s_0}$ denote all those terms with $p$-adic order $> \frac{m}{p-1}$. Recall from Lemma 4.3 and Proposition 5.2 that one has the $p$-adic unit $U$ (as in the above paragraphs) and some polynomials $G_m'$ and $G'$ (in $\mathbb{Q}(\hat{P})[\hat{A}]$) such that

$$\det(M[k]) = \sum_{c_0 \leq m < s_0} \gamma^m U G'_m + \gamma^{s_0} U G' + \gamma^{>s_0}$$

and $G' \equiv G \mod \gamma$ for the polynomial $G$ (same $G$ as in above paragraphs) in $\mathbb{Q}(\hat{P})[\hat{A}]$ independent of $p$. If $G(f) \neq 0 \mod \mathcal{P}$ (the specialization of $G$ at $f$ over $\mathbb{Q}(\hat{P})$) then $\text{ord}_p(G'(f)) = 0$. For $m < s_0$ one has $\text{ord}_p(G'_m(f)) = 0$ or $\geq 1$. Thus if $G(f) \neq 0$ then for $p$ large enough one has $c_0 \leq \text{ord}_p(\det M[\hat{f}]^{[k]}) \leq \frac{m}{p-1}$. But we already know from the beginning of this section that $0 \leq \frac{m}{p-1} - c_0 \leq \frac{m}{p-1}$ and hence by simple calculus one has that $\lim_{p \to \infty} \text{ord}_p(\det M[\hat{f}]^{[k]}) = c_0$.

Last, taking the norm of $G$ from $\mathbb{Q}(\hat{P})[\hat{A}]$ to $\mathbb{Q}[\hat{A}]$ with the automorphism acting on $\hat{A}$ trivially, one gets a polynomial $g$ in $\mathbb{Q}[\hat{A}]$. Let $\mathcal{V}$ be the complement of the variety defined by $g = 0$ in $\mathcal{A}$. It is Zariski dense in $\mathcal{A}$ because $g \neq 0$. $\square$

5. A transformation theorem from Newton polygons of $\alpha_1$ to $\alpha_a$

We refer the reader to [3] for basic facts about Serre’s theory of completely continuous maps and Fredholm determinants. Let $\mathbb{C}_p$ be the $p$-adic completion of $\mathbb{Q}_p$. For any $\mathbb{C}_p$-Banach spaces $E$ and $F$ that admit orthonormal bases, denote by $\mathcal{C}(E, F)$ the set of completely continuous $\mathbb{C}_p$-linear maps from $E$ to $F$. We say that a matrix $M$ over $\mathbb{C}_p$ is nuclear if there exist a $\mathbb{C}_p$-Banach space $E$ and a $u \in \mathcal{C}(E, E)$ such that $M$ is the matrix of $u$ with respect to some orthonormal basis of $E$. If $M = (m_{ij})_{i,j \geq 1}$ is a matrix over $\mathbb{C}_p$, then $M$ is nuclear if and only if $\lim_{i \to \infty} (\inf_{j \geq 1} \text{ord}_p m_{i,j}) = +\infty$. Recall $\text{ord}_q(\cdot) = \text{ord}_p(\cdot)/a$ for $q = p^a$.

**Lemma 5.1.** Let $\tilde{M} = (M_0, M_1, \cdots, M_{a-1})$ be an $a$-tuple of nuclear matrices over $\mathbb{C}_p$. Set

$$\tilde{M}_{[a]} := \begin{pmatrix} 0 & \cdots & 0 & M_{a-1} \\ M_0 & 0 & & \\ & M_1 & 0 & \\ & & \ddots & 0 \\ & & & M_{a-2} \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $\det(1 - (M_{a-1} \cdots M_1 M_0)T^a) = \det(1 - \tilde{M}_{[a]}T)$.

Lemma 5.1 follows directly from

**Lemma 5.2.** Let $\{E_i\}_{i \in \mathbb{Z}/a\mathbb{Z}}$ be a family of Banach spaces over $\mathbb{C}_p$ that admit orthonormal basis. Set $E = E_0 \oplus E_1 \oplus \cdots \oplus E_{a-1}$ equipped with the supremum norm, that is for $v = (v_0, \ldots, v_{a-1})$ in $E$ one has $||v|| = \max_{i=0}^{a-1} ||v_i||$, where $|| \cdot ||$ are the norms on $E$ and $E_i$’s, respectively. Let $u_i \in \mathcal{C}(E_i, E_{i+1})$ and set $u \in \mathcal{C}(E, E)$ such that $u|_{E_i} = u_i$. Then

$$\det(1 - (u_{a-1} \cdots u_1 u_0)T^a) = \det(1 - uT).$$
Proof. By [8, page 77, Corollaire 3] we have \(\det(1-uT) = \exp(-\sum_{s=1}^{\infty} \text{Tr}(u^s)T^s/s)\). Notice that for any \(s \in \mathbb{Z}_{\geq 1}\), the trace \(\text{Tr}(u_{i+a-1} \cdots u_{i+1} u_i)\) is independent of \(i \in \mathbb{Z}/a\mathbb{Z}\). Clearly \(\text{Tr}(u^s) = 0\) unless \(a|s\). Thus

\[
\det(1-uT) = \exp(-\sum_{s=1}^{\infty} \text{Tr}(u^s)T^s/s) = \exp(-\sum_{s=1}^{\infty} \text{Tr}(u^a s T^{as}/(as))
\]

\[
= \exp(-\sum_{s=1}^{\infty} \sum_{i \in \mathbb{Z}/a\mathbb{Z}} \text{Tr}((u_{i+a-1} \cdots u_{i+1} u_i)^{as}/(as))
\]

\[
= \exp(-\sum_{s=1}^{\infty} \text{Tr}((u_{a-1} \cdots u_{1} u_0)^{as}/s)
\]

\[
= \det(1 - (u_{a-1} \cdots u_{1} u_0)T^a).
\]

This concludes our proof. \(\square\)

Remark 5.3. Lemmas 5.1 and 5.2 still hold when \(\mathbb{C}_p\) is replaced by any field \(K\) equipped with a nontrivial complete non-Archimedean valuation. But we shall not need this more general fact in the present paper.

For any nuclear matrix \(M = (m_{ij})_{i,j \geq 1}\) and \(k \in \mathbb{Z}_{\geq 1}\), denote by \(M^{[k]}\) the \(k \times k\) submatrix of \(M\) consisting of its first \(k\) rows and columns.

Proposition 5.4. Let \(M = (m_{ij})_{i,j \geq 1}\) be a nuclear matrix over \(\mathbb{C}_p\) and let \(g \in \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)\). Fix \(k \in \mathbb{Z}_{\geq 1}\) and denote by \(C_k\) the coefficient of \(T^k\) in \(\det(1 - (M^g^{a-1} \cdots M^g M)T)\). Denote by \(\mathcal{A}\) the set of \(k \times k\) submatrices of \(M\) contained in the first \(k\) rows of \(M\), and denote by \(\mathcal{B}\) the set of all other \(k \times k\) submatrices of \(M\). Set \(t_{\mathcal{A}} = \inf_{W \in \mathcal{A}} \text{ord}_p \det W\) and \(t_{\mathcal{B}} = \inf_{W \in \mathcal{B}} \text{ord}_p \det W\). Consider the following conditions:

(i) \(2\text{ord}_p \det M^{[k]} < t_{\mathcal{A}} + t_{\mathcal{B}}\);
(ii) \(2\text{ord}_p C_k < t_{\mathcal{A}} + t_{\mathcal{B}}\) and \(t_{\mathcal{A}} < t_{\mathcal{B}}\);
(iii) \(\text{ord}_p C_k = \text{ord}_p \det M^{[k]}\).

Then (i) \(\iff\) (ii) \(\iff\) (iii).

Proof. Notice that \(\text{ord}_p \det M^{[k]} \geq t_{\mathcal{A}}\). So (i) is equivalent to

\[
(13) \quad \min(t_{\mathcal{A}} + t_{\mathcal{B}} - \frac{t_{\mathcal{A}} + t_{\mathcal{B}}}{2}, t_{\mathcal{A}}) > \text{ord}_p \det M^{[k]}.
\]

It suffices to show that (ii) \(\Rightarrow\) (iii). Let \(\tilde{M} := (M, M^g, \cdots, M^g^{a-1})\). Then we have \(\tilde{M}_{[a]}\) in Lemma 5.1 and \(\det(1 - \tilde{M}_{[a]} T) = \det(1 - (M^g^{a-1} \cdots M^g M)T^a)\). Thus \(C_k\) is the coefficient of \(T^a\) in \(\det(1 - \tilde{M}_{[a]} T)\), which is the infinite sum of \((-1)^{ak}\text{det }N\) for \(N\) running over all principal \(ak \times ak\) submatrices of \(\tilde{M}_{[a]}\). Let \(N\) be such a matrix, and let \(N_s\) be the intersection of \(N\) and \(M^g\) as submatrices of \(\tilde{M}_{[a]}\) for all \(0 \leq s \leq a - 1\). It is easy to see that \(\text{det }N = (-1)^{(a-1)k} \prod_{0 \leq s \leq a-1} \text{det }N_s\) or 0 depending on whether every \(N_s\) is a \(k \times k\) matrix or not. So we may assume that every \(N_s\) is a \(k \times k\) matrix. Think of \(N_s\) as a submatrix of \(M^g\) from now on. Let \(X = \{s : 0 \leq s \leq a - 1\}\) and \((N_s)^{g_{s+a-1}} \in \mathcal{A} \setminus \{M^{[k]}\}\) and \(Y = \{s : 0 \leq s \leq a - 1\}\). We shall think of the families \(\{M^g\}_{0 \leq s \leq a-1}\) and \(\{N_s\}_{0 \leq s \leq a-1}\) as parameterized by \(\mathbb{Z}/a\mathbb{Z}\). Then \(X\) and \(Y\) are disjoint subsets of \(\mathbb{Z}/a\mathbb{Z}\). Since \(N\) is principal, the set of the columns of \(N_s\) as a subset in \(\mathbb{Z}_{\geq 1}\) is exactly
the same as the set of the rows of $N_{s-1}$. Consequently, if $s \in X$, then $s-1 \in Y$. Let $Y' = \{s-1 : s \in X\}$ and $Z = (Z/aZ) \setminus (X \cup Y)$. Then $\mathbb{Z}/a\mathbb{Z}$ is the disjoint union of $X \cup Y', Y \setminus Y'$ and $Z$. If $s \in X$, then ord$_p(\det N_s \cdot \det N_{s-1}) \geq t_{s'} + t_{s'}$. If $s \in Y \setminus Y'$, then ord$_p \det N_s \geq t_{s'}$. If $s \in Z$, then ord$_p \det N_s = \text{ord}_p \det M^k$. Therefore
\[\text{ord}_p \det N \geq \min\left(\frac{t_{s'} + t_{s'}}{2}, t_{s'}, \text{ord}_p \det M^k\right),\]
and hence
\[\text{ord}_p C_k \geq \min\left(\frac{t_{s'} + t_{s'}}{2}, t_{s'}, \text{ord}_p \det M^k\right).\]

\[\text{(13)} \Rightarrow (iii): \] Clearly there is a unique $N$ with $X = Y = \emptyset$, i.e. $(N_a)^{-\tau} = M^k$ for all $0 \leq s \leq a - 1$. Denote it by $N$. We have ord$_p \det N = \text{ord}_p \det M^k$. If $N \neq N'$, then $X$ or $Y \setminus Y'$ is nonempty and hence from (13) and the derivation of (14) we see that ord$_p \det N > \text{ord}_p \det M^k$. Now (iii) follows immediately.

\[(ii) \Rightarrow (13): \] (13) follows directly from (ii) and (15). Thus Theorem 5.5 follows from Proposition 5.4.

**Theorem 5.5.** Let $M, g, k$ and $C_k$ be as in Proposition 5.4. Let $h_1 \leq h_2 \leq \cdots$ be a non-decreasing sequence in $\mathbb{R}$ satisfying $h_i \leq \inf_{j \geq 1} \text{ord}_p m_{ij}$ for all $i \geq 1$. Consider the following conditions:

(i) ord$_p \det M^k < \sum_{1 \leq i \leq k} h_i + \frac{h_{k+1} - h_k}{2};$

(ii) ord$_p C_k < \sum_{1 \leq i \leq k} h_i + \frac{h_{k+1} - h_k}{2};$ item [(iii)] ord$_p C_k = \text{ord}_p \det M^k.$

Then (i) $\iff$ (ii) $\iff$ (iii).

**Proof.** Let $t_{s'}$ and $t_{s''}$ be as in Proposition 5.4. Then $\sum_{1 \leq i \leq k} h_i + \frac{h_{k+1} - h_k}{2} \leq \min\left(\frac{t_{s'} + t_{s''}}{2}, t_{s''}\right)$. So (i) follows from (ii) and (15). Thus Theorem 5.5 follows from Proposition 5.4. \hfill $\square$

**Remark 5.6.** Theorem 5.5 is a Wan-type theorem in relating the Newton polygon to its tight lower bound Hodge polygon: In [5, Theorem 8], Wan showed that the Newton polygon for $\alpha_1$ (more precisely, the Fredholm determinant of the nuclear matrix representing $\alpha_1$ with respect to the specific basis) coincides with the Hodge one if and only if the Newton polygon for $\alpha_a$ does. Our result in Theorem 5.5 generalizes it and says that the Newton polygon for $\alpha_1$ is close to the Hodge one if and only if the Newton polygon for $\alpha_a$ is.

**Proof of Theorem 5.5.** For any vertex $(k, c_0) \in \mathbb{R}^2$ (but not the right end point) of the slope $< 1$ part of HP($\mathbb{A}$), where $1 \leq k \leq d - \ell$, let $U_k$ be the Zariski dense open subset in Proposition 4.1. Let $f \in \mathbb{A}_k(\mathbb{Q})$. Then $\lim_{p \to \infty} \text{ord}_p \det(M(\hat{f})^{[k]}) = c_0$. Recall $\phi(\cdot)$ from the beginning of section 3.1. Say the coefficients of $f$ mod $\mathcal{P}$ lie in $\mathbb{F}_p$. Set $M := M(\hat{f})$ and $h_i := \phi(e_i)$ for all $i \geq 1$ in Theorem 5.5. Notice that $\sum_{1 \leq i \leq k} h_i = c_0$. Since $(k, c_0)$ is a vertex of HP($\mathbb{A}$), we have $h_{k+1} > h_k$. In particular, when $p$ is large enough, we have $\text{ord}_p \det M^k < c_0 + \frac{h_{k+1} - h_k}{2}$. Combining this with Lemma 5.3(c), one observes that the hypotheses of Theorem 5.5 are satisfied. Recall the maps $\alpha_1$ and $\alpha_a$ defined in Lemma 2.9 and section 2.5 of [15]. These maps are not the same as the maps defined in section 2 of this article, but are the specialization of those maps in section 2 at the Teichmüller lifts of coefficients of $f$ mod $\mathcal{P}$. Then $M^a$ and $M^a \cdots M^a$ are the matrices of $\alpha_1$ and $\alpha_a$ (over $\Omega_\mathbb{A}^a$) with respect to the formal basis $\tilde{b}_w = \{1, Z^i_1, \cdots, Z^i_\ell\}_{i \geq 1}$ of
H respectively. Notice that $M^r = M$. By Theorem 5.5 one has $\lim_{p \to \infty} \text{ord}_p C_k = c_0$, where $C_k$ is the coefficient of $T^k$ in $\det (1 - (M^{r-1} \cdots M^{r-a})M^{r-a})T) = \det_{\Omega^e}(1 - \alpha_a T)$. Set $\mathcal{U}$ to be the intersection of $\mathcal{U}_k$ for all such vertices $(k, c_0)$. Then for any $f \in \mathcal{U}$, we have $\lim_{p \to \infty} \text{NP}_q(\det_{\Omega^e}(1 - \alpha_a T) \mod T^{d-\ell+1}) = H_P(\mathbb{A})$. Now Theorem 1.1 follows from Remark 1.3 and the fact that the slope < 1 part of $\text{NP}_p(f)$ coincides with $\text{NP}_q(\det_{\Omega^e}(1 - \alpha_a T) \mod T^{d-\ell+1})$ (see [15, Proposition 2.10]).

□

Remark 5.7. (1) Our main result Theorem 1.1 is related but not included in a conjecture of Daqing Wan (see [10, Conjectures 1.12 and 1.14]).

(2) This paper is concerned with the space of all one-variable rational function with fixed poles on the projective line. One naturally wonders if there is a multi-variable generalization of Theorem 1.1. We do not know the answer.

References

[1] Pierre Berthelot: Cohomologie rigide et théorie de Dwork: le ca des sommes exponentielles, in Cohomologie p-adique, Société Mathématique de France, Astérisque 119–120 (1984), 17–49.
[2] S. Bosch; U. Guntzer; R. Remmert: Non-Archimedean analysis, Grundlehren der Mathematischen Wissenschaften Vol. 261, Springer-Verlag, Berlin, 1984.
[3] Richard Crew: Étale $p$-covers in characteristic $p$, Compositio Math., 52 (1984), 31–45.
[4] Nicholas M. Katz: Gauss sums, Kloosterman sums, and monodromy groups, Annals of mathematics studies vol 116, Princeton University Press, 1988.
[5] Paul Monsky: $p$-adic analysis and zeta functions, Lectures in Mathematics, Department of Mathematics, Kyoto University, Kinokuniya Book-Store Co., Ltd., Tokyo, 1970.
[6] Philippe Robba: Index of $p$-adic differential operators III. Application to twisted exponential sums, in Cohomologie p-adique, Société Mathématique de France, Astérisque 119–120 (1984), 191–266.
[7] Jasper Scholten; Hui June Zhu: Hyperelliptic curves in characteristic 2, Math. Research Letters 17 (2002), 905–917.
[8] Jean-Pierre Serre: Endomorphismes complètement continus des espaces de Banach $p$-adiques, Inst. Hautes Études Sci. Publ. Math. 12 (1962), 69–85.
[9] Daqing Wan: Newton polygons of zeta functions and $L$-functions Ann. Math. 137 (1993), 247–293.
[10] Daqing Wan: Variation of Newton polygons for $L$-functions of exponential sums. Asian J. Math. 8 (2004), 427–474.
[11] Daqing Wan: Rank one case of Dwork’s conjecture. J. of Amer. Math. Soc. 13 (2000), 853–908.
[12] Daqing Wan: Higher rank case of Dwork’s conjecture. J. of Amer. Math. Soc. 13 (2000), 807–852.
[13] Hui June Zhu: $p$-adic variation of $L$ functions of one variable exponential sums, I. Amer. J. Math. 125 (2003), 669–690.
[14] Hui June Zhu: Asymptotic variation of $L$ functions of one-variable exponential sums. J. Reine Angew. Math. 572 (2004), 219–233.
[15] Hui June Zhu: $L$ functions of exponential sums over one dimensional affinoids: Newton over Hodge. Inter. Math. Res. Notices., vol 2004, no. 30 (2004), 1529–1550.

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