Universal models of the constitutive relations for transversely isotropic compressible composites with finite strains

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Abstract. A model of a transversely isotropic compressible elastic medium with finite strains is proposed. The model belongs to the class of the so-called universal models, which are formulated in terms of energy (conjugated) pairs of stress and strain tensors - simultaneously for several types of pairs. An algorithm is proposed for calculating the constants included in the constitutive relations of this model, based on a comparative analysis of the results of calculating deformation diagrams under uniaxial loading according to this model and using the asymptotic averaging method. An example of numerical modeling for a layered composite is given.

1. Introduction
To calculate the effective characteristics of composites with small deformations, there are a large number of different methods, we only note some work in this field [1-10]. Asymptotic homogenization method (AHM, asymptotic averaging method) is the most promising method for solving the problems of calculating the effective characteristics of composites. AHM was proposed by N.S. Bakhvalov, G.P. Panasenko, Bensoussan A., Lions JL, Papanicalaou G [2] Sanchez-Palencia E [3]. The method of asymptotic averaging is quite well developed at the present time and has been successfully numerically implemented for various problems of mechanics, but mainly for linear problems [4,5]. In [11-14] this AHM method was applied to calculate the nonlinear elastic properties of layered composites with finite strains. An algorithm was proposed for constructing strain diagrams of layered composites with finite strains based on the asymptotic theory of averaging of nonlinear-elastic composites with a periodic structure, for the case of finite strains, and also taking into account the incompressibility of the layer materials. In [15,16], a method was developed for constructing defining relations for layered incompressible composites with finite strains. In this method, first, the effective defining relations for the composite are selected as for a homogeneous anisotropic medium, and then the constants included in these relations are calculated from the condition of the best fit to the calculated strain diagrams obtained using the asymptotic homogenization method. To construct a model of an effective anisotropic medium with finite strains, we use the theory of universal models proposed in [15,17], in which several energy (conjugated) pairs of stress-strain tensors are applied at once. The aim of this work is to apply this method to construct defining relations for layered compressible composites with finite strains.

2. Model of transversely isotropic compressible media with finite deformations.
Let us consider a nonlinear elastic layered composite material (NLCM), which in an initial configuration $K_0$ characterized by a Cartesian basis $e_α$, is a system of periodically repeating layers orthogonal...
to the axis \(Oe_3\). We introduce the Cartesian coordinates \(x^k\) as the Lagrangian coordinates \(X^i\) in the basis, i.e. \(X^i = x^i\), and define the law of motion of the composite in the form \(x^k = x^k(X^i)\), where \(x^k\) are the Cartesian coordinates of the material points in the current configuration \(K\). We introduce \(r_a = Q^k_a e_k\) - local basis vectors in \(K\), where 
\[ Q^k_a = \frac{\partial x^k}{\partial X^a} \] is the Jacobian matrix, and also 
\[ F = r_k \otimes e^k \] is the deformation gradient, \(\otimes\) is the sign of the tensor product. For the deformation gradient \(F\), a polar expansion 
\[ \mathbf{F} = \mathbf{U} \cdot \mathbf{O} \] takes place \([15]\), where \(\mathbf{U}\) is the left distortion tensor and \(\mathbf{O}\) is the rotation tensor. We introduce energy measures of deformation \([15]\)
\[ G = \frac{1}{n-III} U^{n-III}, \quad n = I, II, IV, V \] (1) 
Energy stress tensors \(\mathbf{T} = ^4\mathbf{E}^{-1} \cdot \mathbf{T}\) correspond to these measures, where \(^4\mathbf{E}\) are the energy equivalence tensors \([16]\), which depend only on \(\mathbf{F}\), and \(\mathbf{T}\) is the Cauchy true stress tensor. Since the properties of the NLCM in the reference configuration do not change during rotations in the plane orthogonal to \(Oe_3\), the NLCM can be considered as transversally isotropic medium with the symmetry group \(T3\) \([17]\). For constructing analytic constitutive relations of NLCM, we apply universal models of compressible transversely isotropic media, the general theory of which is formulated in \([15]\). The most general form of writing these universal models in tensor form for the so-called models of the class \(A_n\) has the following form \([15]\):
\[ T = \mathbf{D} \frac{\partial \psi}{\partial \mathbf{C}}, \quad n = I, II, IV, V \] (2) 
where \(n\) is the model number in the class \(A_n\), \(\mathbf{D}\) is the averaged density of the composite, \(\mathbf{T}\) is the energy stress tensor, \(\mathbf{C}\) is the deformation energy tensor (rank 2 tensor), \(\psi\) is the Helmholtz free energy, which is a function of the tensor invariants \(I_{1-5}(\mathbf{C})\) with respect to the symmetry group transversally-isotropic medium \(T3\)
\[ \psi = \psi(I_{1}(\mathbf{C}),...,I_{5}(\mathbf{C})) \] (3) 
Then, for a compressible transversely isotropic medium, the following representation of the constitutive relations holds (2)
\[ T = \sum_{r=1}^{5} \mathbf{C} \mathbf{I}_{r, C}, \quad \mathbf{C} = \frac{\partial \psi}{\partial \mathbf{C}}, \quad I_{r, C} = \frac{\partial I_r(\mathbf{C})}{\partial \mathbf{C}}. \] (4) 
For a transversely isotropic medium, symmetric tensors of the second rank have 5 invariants. Let us choose the following invariants as a functional basis for invariants of a transversely isotropic medium \([15-17]\)
\[
I_1 = (\mathbf{E} - c_3^2) \cdot \mathbf{C}, \quad I_2 = c_3^2 \cdot \mathbf{C}, \quad I_3 = \text{det}(\mathbf{C}), \quad I_4 = \mathbf{C}^2 \cdot \mathbf{E} - I_2^2 + 2I_3, \\
I_5 = \left( (\mathbf{E} - c_3^2) \cdot \mathbf{C} \right) \left( c_3^2 \cdot \mathbf{C} \right), \quad I_6 = \mathbf{C}^2 \cdot \mathbf{E} - I_2^2 + 2I_3, 
\] (5)
where $\mathbf{c}_3^2 = \mathbf{c}_3 \otimes \mathbf{c}_3$ is the directing tensor of the 2nd rank, and $\mathbf{c}_I$ is the orthonormal basis of the principal axes of transversal isotropy.

Defining relation (3) for the Helmholtz free energy we will set in an analytical form in the following form

$$
\rho \Psi = \rho \Psi_0 + \frac{1}{2} \left( l_{11} I_1^{2n_1} + 2l_{12} I_1^{n_1} I_2^{n_2} + l_{22} I_2^{2n_2} \right) + l_{33} I_3^{n_3} + l_{44} I_4^{2n_4},
$$

(6)

where $\Psi_0$ is the initial value (constant) of the potential $\Psi$, and $n_1, n_2, n_3, n_4$ and $l_{11}, l_{12}, l_{22}, l_{33}, l_{44}$ are elastic constants.

We calculate the derivatives $\rho \partial \Psi / \partial I_i$:

$$
\phi_1 = \rho \partial \Psi / \partial I_1 = J (n_1 I_1^{2n_1-1} + n_2 I_2^{n_2-1})
$$

$$
\phi_2 = \rho \partial \Psi / \partial I_2 = J (n_2 I_2^{2n_2-1} + n_2 I_1^{n_1-1})
$$

$$
\phi_3 = \rho \partial \Psi / \partial I_3 = J n_3 I_3^{n_3-1}, \quad \phi_4 = \rho \partial \Psi / \partial I_4 = J n_4 I_4^{n_4-1},
$$

(7)

$$
J = \rho / \rho^0
$$

The derivatives $J \partial \Psi / \partial I_i$ are calculated using the rules of differentiation of the tensor invariants [17]. Then, in the component record in the Cartesian coordinate system, the constitutive relations (4) will take the form:

$$
T_{ij}^{(n)} = \phi_1 (\delta_{ij} - \delta_{i3} \delta_{j3}) + \phi_2 \delta_{i3} \delta_{j3} +
$$

$$
+ (\phi_3 - 2\phi_4) \left( C_{13}^{(n)} (\delta_{i3} \delta_{j3} + \delta_{i3} \delta_{j1}) + C_{23}^{(n)} (\delta_{i2} \delta_{j3} + \delta_{i3} \delta_{j2}) \right) + 2\phi_4 \left( C_{13}^{(n)} \delta_{j1} \delta_{j3} \right)
$$

Where $T_{ij}, C_{ij}$ are the components of the tensors $T^{(n)}$ and $C^{(n)}$ in the basis $\mathbf{c}_I$. After finding the energy stress tensor, the Cauchy $T$ and Piola-Kirchhoff stress tensors $P = \mathcal{F}^{-1} \cdot T$ are determined.

3. Method for finding the model parameters of a transversely isotropic medium

The constants $l_{11}, l_{12}, l_{22}, l_{33}, l_{44}$ and $n_1, n_2, n_3, n_4$ entering into relations (6), (7) are found from the condition of the best approximation using the model (8) of the strain curves $P = \mathcal{F} \cdot F$ obtained on the basis of direct numerical solution of problems in the nuclear field that arise in the method of asymptotic averaging (AHM) [12,13]. Comparison of deformation diagrams using the effective anisotropic medium model (EAM model) and the AHM method is carried out for some standard macroscopic deformation problems in which a uniform stress-strain state is realized with $P$ and $F$ independent of coordinates. A numerical method for solving the local problem over periodicity cell (PC) was proposed in [12,13]. This method was used in this work.

As standard problems of macroscopic deformation, we consider the class of problems of 3 axis stretching — compression of a plate in the form of a parallel-facet which is parallel to the coordinate axes. The law of movement of the plate with such movements is given in the form $x^\alpha = k_\alpha X^\alpha$, where
\[ k_\alpha(t) \] are unknown functions of time. The strain gradient for this motion has the form
\[ F = \sum_{\alpha=1}^{3} k_\alpha e_\alpha^2. \]
The energy strain tensors have the form
\[ C = \frac{1}{n - III} (U^{n-III} - E) = \frac{1}{n - III} \sum_{\alpha=1}^{3} (k_\alpha^{n-III} - 1) e_\alpha^2. \] (9)

Substituting (9) into (5), we find the expressions for the invariants
\[ I_1 = C_{11}^{(n)} + C_{22}^{(n)} = \frac{1}{n - III} (k_1^{n-III} + k_2^{n-III} - 2), \quad I_2 = C_{33}^{(n)} = \frac{1}{n - III} (k_3^{n-III} - 1) \]
\[ I_3 = 0, \quad I_4 = C_{11}^{(n)} + C_{22}^{(n)} = \frac{1}{n - III} ((k_1^{n-III} - 1)^2 + (k_2^{n-III} - 1)^2). \] (10)

Substituting (9) and (10) into (8), we find that the energy stress tensors, as well as the Cauchy stress tensor, will also have a diagonal form:
\[ T = \sum_{\alpha=1}^{3} T_{\alpha\alpha} e_\alpha^2, \quad P = \sum_{\alpha=1}^{3} Jk_\alpha T_{\alpha\alpha} e_\alpha^2, \quad T_{\alpha\alpha} = k_\alpha^{n-III} T_{\alpha\alpha}^{(n)}, \] (11)

\[ T_{\alpha\alpha} = k_\alpha^{n-III} \left( \phi_1 + \frac{\phi_2 (k_3^{n-III} - 1)}{n - III} \right) \delta_{\alpha\alpha} + \left( \phi_1 + \frac{\phi_2 (k_2^{n-III} - 1)}{n - III} \right) \delta_{2\alpha} + \varphi_2 \delta_{3\alpha}. \]

4. Uniaxial tension of a composite

Let us consider the case of stretching a plate in the plane of transversal isotropy in the direction of the axis \( e_1 \). In this case, from the system of equations (15) we obtain the relationship between stress \( P_{11} \) and \( k_1 \), i.e., the strain diagram under uniaxial tension (compression)
\[ P_{11} = J n_4 l_{44} l_{12} k_1^{n-III+1} \frac{k_1^{n-III} - f(k_1^{n-III} - 1) - 1}{n - III} I_4^{n_1-1} \]
where
\[ I_4 = \frac{1}{n - III} ((k_1^{n-III} - 1)^2 + (f(k_1^{n-III} - 1))^2) \]
and the function \( y = f(x) \) is a solution to the equation
\[ (l_{11} - l_{12}) \left( \frac{x + y}{n - III} \right)^{2n_1-1} = - \frac{l_{44}}{n - III} n_4 \left( \frac{x^2 + y^2}{n - III} \right)^{n_1-1} \]

From the solution of this problem, we also find the dependence \( k_2 = k_2(k_1) \) of transverse compression on longitudinal tension

Similarly, we find the strain diagram \( P_{33} = P_{33}(k_3) \) and the diagram \( k_3 = k_3(k_1) \) of the transverse compression of the composite under tension in the direction of the axis \( e_3 \).

5. Calculation of model parameters for NLCM

Calculations were carried out according to the developed technique for a layered composite, the PC of which consisted of 2 materials: duotan and rubber SKN. Figures 1 and 2 show graphs of defor-
The approximation diagrams $P_{11} = P^{(\gamma)}_{11}(k_1)$ and $k_2 = k^{(\gamma)}_2(k_1)$ obtained using a numerical “experiment” (based on the direct solution of local problems in nuclear materials) for the AI and AV models and using approximations based on the AHM model and using the EAM model.

The approximation error for the considered “experimental” data using the EAM model was 10-12%.

**Figure 1.** Deformation diagrams $P_{11} = P^{(\gamma)}_{11}(k_1)$ for uniaxial deformation of a plate using the AV model with the help of AHM and EAM methods for a 2-layer SKN duotan rubber composite

**Figure 2.** Deformation diagrams $k_2 = k^{(\gamma)}_2(k_1)$ for uniaxial plate tension using the AV model according to AHM and EAM methods for a 2-layer SKN duotan rubber composite

### 6. Conclusions

A model of an effective transversely isotropic nonlinear elastic compressible medium with finite strains is proposed, which belongs to the class of universal models formulated in terms of simultaneously 4 classes of energy pairs of stress – strain tensors. The model is used to model the deformation diagrams of layered compressible elastic composites with finite deformations and a periodic structure, using the universal representation of the defining relations for composite layers.

A method is proposed for finding the elastic effective constants of a composite model by solving the problem of minimizing the deviation of the composite strain diagrams obtained for a set of standard problems using direct numerical solution of problems on a periodicity cell using the AHM method and using an approximation diagram using the EAM method.
The numerical simulation of the strain diagrams of compressible layered composites with finite strains was carried out using the AHM and EAM methods. It is shown that the EAM method allows one to obtain defragmentation diagrams with acceptable engineering accuracy.

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