Stable Approximation Algorithms for the Dynamic Broadcast Range-Assignment Problem

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Abstract

Let $P$ be a set of points in $\mathbb{R}^d$ (or some other metric space), where each point $p \in P$ has an associated transmission range, denoted $\rho(p)$. The range assignment $\rho$ induces a directed communication graph $G_\rho(P)$ on $P$, which contains an edge $(p, q)$ iff $|pq| \leq \rho(p)$. In the broadcast range-assignment problem, the goal is to assign the ranges such that $G_\rho(P)$ contains an arborescence rooted at a designated root node and the cost $\sum_{p \in P} \rho(p)^2$ of the assignment is minimized.

We study the dynamic version of this problem. In particular, we study trade-offs between the stability of the solution—the number of ranges that are modified when a point is inserted into or deleted from $P$—and its approximation ratio. To this end we introduce the concept of $k$-stable algorithms, which are algorithms that modify the range of at most $k$ points when they update the solution. We also introduce the concept of a stable approximation scheme, or SAS for short. A SAS is an update algorithm $\text{Alg}$ that, for any given fixed parameter $\varepsilon > 0$, is $k(\varepsilon)$-stable and that maintains a solution with approximation ratio $1 + \varepsilon$, where the stability parameter $k(\varepsilon)$ only depends on $\varepsilon$ and not on the size of $P$. We study such trade-offs in three settings.

- For the problem in $\mathbb{R}^1$, we present a SAS with $k(\varepsilon) = O(1/\varepsilon)$. Furthermore, we prove that this is tight in the worst case: any SAS for the problem must have $k(\varepsilon) = \Omega(1/\varepsilon)$. We also present algorithms with very small stability parameters: a 1-stable $(6 + 2\sqrt{5})$-approximation algorithm—this algorithm can only handle insertions—a (trivial) 2-stable 2-approximation algorithm, and a 3-stable 1.97-approximation algorithm.

- For the problem in $S^1$ (that is, when the underlying space is a circle) we prove that no SAS exists. This is in spite of the fact that, for the static problem in $S^1$, we prove that an optimal solution can always be obtained by cutting the circle at an appropriate point and solving the resulting problem in $\mathbb{R}^1$.

- For the problem in $\mathbb{R}^2$, we also prove that no SAS exists, and we present a $O(1)$-stable $O(1)$-approximation algorithm.

Most results generalize to when the range-assignment cost is $\sum_{p \in P} \rho(p)^\alpha$, for some constant $\alpha > 1$.

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1 Introduction

The broadcast range-assignment problem. Let $P$ be a set of points in $\mathbb{R}^d$, representing transmission devices in a wireless network. By assigning each point $p \in P$ a transmission range $\rho(p)$, we obtain a communication graph $G_\rho(P)$. The nodes in $G_\rho(P)$ are the points
from $P$ and there is a directed edge $(p, q)$ iff $|pq| \leq \rho(p)$, where $|pq|$ denotes the Euclidean distance between $p$ and $q$. The energy consumption of a device depends on its transmission range: the larger the range, the more energy it needs. More precisely, the energy needed to obtain a transmission range $\rho(p)$ is given by $\rho(p)^\alpha$, for some real constant $\alpha > 1$ called the \textit{distance-power gradient}. In practice, $\alpha$ depends on the environment and ranges from 1 to 6 \cite{1}. Thus the overall cost of a range assignment is $\text{cost}_{\alpha}(\rho(P)) := \sum_{p \in P} \rho(p)^\alpha$, where we use $\rho(P)$ to denote the set of ranges given to the points in $P$ by the assignment $\rho$. The goal of the range-assignment problem is to assign the ranges such that $G_\rho(P)$ has certain connectivity properties while minimizing the total cost \cite{5}. Desirable connectivity properties are that $G_\rho(P)$ is $(h$-hop) strongly connected \cite{11, 12, 13, 27} or that $G_\rho(P)$ contains a broadcast tree, that is, an arborescence rooted at a given source $s \in P$. The latter property leads to the \textit{broadcast range-assignment problem}, which is the topic of our paper.

The broadcast range-assignment problem has been studied extensively, sometimes with the extra condition that any point in $P$ is reachable in at most $h$ hops from the source $s$. For $\alpha = 1$ the problem is trivial in any dimension: setting the range of the source $s$ to $\max\{|sp| : p \in P\}$ and all other ranges to zero is optimal; however, for any $\alpha > 1$ the problem is \textsc{NP}-hard in $\mathbb{R}^d$ for $d \geq 2$ \cite{8, 22}. Approximation algorithms and results on hardness of approximation are known as well \cite{19, 22, 10}. Many of our results will be on the 1-dimensional (or: linear) broadcast range-assignment problem. Linear networks are important for modeling road traffic information systems \cite{3, 29} and as such they have received ample attention. In $\mathbb{R}^1$, the broadcast range-assignment problem is no longer \textsc{NP}-hard, and several polynomial-time algorithms have been proposed, for the standard version, the $h$-hop version, as well as the weighted version \cite{10, 7, 19, 17, 2}. The currently fastest algorithms for the (standard and $h$-hop) broadcast range-assignment problem run in $O(n^2)$ time \cite{10}.

All results mentioned so far are for the static version of the problem. Our interest lies in the dynamic version, where points can be inserted into and deleted from $P$ (except the source, which should always be present). This corresponds to new sensors being deployed and existing sensors being removed, or, in a traffic scenario, cars entering and exiting the highway. Recomputing the range assignment from scratch when $P$ is updated may result in all ranges being changed. The question we want to answer is therefore: is it possible to maintain a close-to-optimal range assignment that is relatively stable, that is, an assignment for which only few ranges are modified when a point is inserted into or deleted from $P$? And which trade-offs can be achieved between the quality of the solution and its stability?

To the best of our knowledge, the dynamic problem has not been studied so far. The online problem, where the points from $P$ arrive one by one (there are no deletions) and it is not allowed to decrease ranges, is studied by De Berg et al. \cite{18}. This restriction is arguably unnatural, and it has the consequence that a bounded approximation ratio cannot be achieved. Indeed, let the source $s$ be at $x = 0$, and suppose that first the point $x = 1$ arrives, forcing us to set $\rho(s) := 1$, and then the points $x = i/n$ arrive for $1 \leq i < n$. In the optimal static solution at the end of this scenario all points, except the rightmost one, have range $1/n$; for $\alpha = 2$ this induces a total cost of $n \cdot (1/n)^2 = 1/n$. But if we are not allowed to decrease the range of $s$ after setting $\rho(s) = 1$, the total cost will be (at least) 1, leading to an unbounded approximation ratio. Therefore, \cite{18} analyze the competitive ratio: they compare the cost of their algorithm to the cost of an optimal offline algorithm (which knows the future arrivals, but must still maintain a valid solution at all times without decreasing any range). As we will see, by allowing to also decrease a few ranges, we are able to maintain solutions whose cost is close even to the static optimum.

\textbf{Our contribution.} Before we state our results, we first define the framework we use to
analyze our algorithms. Let $P$ be a dynamic set of points in $\mathbb{R}^d$, which includes a fixed source point $s$ that cannot be deleted.

An update algorithm $\text{alg}$ for the dynamic broadcast range-assignment problem is an algorithm that, given the current solution (the current ranges of the points in the current set $P$) and the location of the new point to be inserted into $P$, or the point to be deleted from $P$, modifies the range assignment so that the updated solution is a valid broadcast range assignment for the updated set $P$. We call such an update algorithm $k$-stable if it modifies at most $k$ ranges when a point is inserted into or deleted from $P$. Here we define the range of a point currently not in $P$ to be zero. Thus, if a newly inserted point receives a positive range it will be counted as receiving a modified range; similarly, if a point with positive range is deleted then it will be counted as receiving a modified range. To get a more detailed view of the stability, we sometimes distinguish between the number of increased ranges and the number of decreased ranges, in the worst case. When these numbers are $k^+$ and $k^-$, respectively, we say that $\text{alg}$ is $(k^+, k^-)$-stable. This is especially useful when we separately report on the stability of insertions and deletions; often, when insertions are $(k_1, k_2)$-stable then deletions will be $(k_2, k_1)$-stable.

We are not only interested in the stability of our update algorithms, but also in the quality of the solutions they provide. We measure this in the usual way, by considering the approximation ratio of the solution. As mentioned, we are interested in trade-offs between the stability of an algorithm and its approximation ratio. Of particular interest are so-called stable approximation schemes, defined as follows.

▶ **Definition 1.** A stable approximation scheme, or SAS for short, is an update algorithm $\text{alg}$ that, for any given yet fixed parameter $\varepsilon > 0$, is $k(\varepsilon)$-stable and that maintains a solution with approximation ratio $1 + \varepsilon$, where the stability parameter $k(\varepsilon)$ only depends on $\varepsilon$ and not on the size of $P$.

Notice that in the definition of a SAS we do not take the computational complexity of the update algorithm into account. We point out that, in the context of dynamic scheduling problems (where jobs arrive and disappear in an online fashion, and it is allowed to re-assign jobs), a related concept has been introduced under the name robust PTAS: a polynomial-time algorithm that, for any given parameter $\varepsilon > 0$, computes a $(1 + \varepsilon)$-approximation with re-assignment costs only depending on $\varepsilon$, see e.g. [33] and [32].

We now present our results. Recall that $\text{cost}_{\alpha}(\rho(P)) := \sum_{p \in P} \rho(p)^{\alpha}$, is the cost of a range assignment $\rho$, where $\alpha > 1$ is a constant. To make the results easier to interpret, we state the results for $\alpha = 2$; the dependencies of the bounds on the parameter $\alpha$ can be found in the theorems presented in later sections.

- In Section 2, we present a SAS for the broadcast range-assignment problem in $\mathbb{R}^1$, with $k(\varepsilon) = O(1/\varepsilon)$. We prove that this is tight in the worst case, by showing that any SAS for the problem must have $k(\varepsilon) = \Omega(1/\varepsilon)$.

- Our SAS (as well as some other algorithms) needs to know an optimal solution after each update. The fastest existing algorithms to compute an optimal solution in $\mathbb{R}^1$ run in $O(n^2)$ time. In Section 2, we show how to recompute an optimal solution in $O(n \log n)$ time after each update, which we believe to be of independent interest. As a result, our SAS also runs in $O(n \log n)$ time per update.

- There is a very simple 2-stable 2-approximation algorithm. We show that a 1-stable algorithm with bounded approximation ratio does not exist when both insertions and deletions must be handled. For the insertion-only case, however, we give a 1-stable ($6 + 2\sqrt{5}$)-approximation algorithm. We have not been able to improve upon the approximation
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Figure 1 The structure of an optimal solution. The non-filled points are zero-range points, the solid black points all have a standard range (for \(\ell_{|L|}\) and \(r_{|R|}\) the standard range is zero), except for the root-crossing point which (in this example) has a long range.

ratio 2 with a 2-stable algorithm, but we show that with a 3-stable we can get a 1.97-approximation. Due to lack of space, these results are delegated to the appendix.

Next we study the problem in \(S^1\), that is, when the underlying 1-dimensional space is circular. This version has, as far as we know, not been studied so far. We first prove that in \(S^1\) an optimal solution for the static problem can always be obtained by cutting the circle at an appropriate point and solving the resulting problem in \(R^1\). This leads to an algorithm to solve the static problem optimally in \(O(n^2 \log n)\) time. We also prove that, in spite of this, a SAS does not exist in \(S^1\).

Finally, we consider the problem in \(R^2\). Based on the no-SAS proof in \(S^1\), we show that the 2-dimensional problem does not admit a SAS either. In addition, we present an 17-stable 12-approximation algorithm for the 2-dimensional version of the problem.

Omitted proofs and results, and some other additional material, are given in the appendix.

2 Maintaining an optimal solution in \(R^1\)

Before we can present our stable algorithms for the broadcast range-assignment problem in \(R^1\), we first introduce some terminology and we discuss the structure of optimal solutions. We also present an efficient subroutine to maintain an optimal solution.

2.1 The structure of an optimal solution

Several papers have characterized the structure of optimal broadcast range assignments in \(R^1\), in a more or less explicit manner. We use the characterization by Caragiannis et al. [7], which is illustrated in Figure 1 and described next.

Let \(P := L \cup \{s\} \cup R\) be a point set in \(R^1\). Here \(s\) is the designated source node, \(L := \{\ell_1, \ldots, \ell_{|L|}\}\) contains all points from \(P\) to the left of \(s\), and \(R := \{r_1, \ldots, r_{|R|}\}\) contains all points to the right of \(s\). The points in \(L\) are numbered in order of increasing distance from \(s\), and the same is true for the points in \(R\). The points \(\ell_{|L|}\) and \(r_{|R|}\) are called extreme points. In the following, and with a slight abuse of notation, we sometimes use \(p\) or \(q\) to refer a generic point from \(P\)—that is, a point that could be \(s\), or a point from \(R\), or a point from \(L\). Furthermore, we will not distinguish between points in \(P\) and the corresponding nodes in the communication graph \(G_\rho(P)\).

For a non-extreme point \(r_i \in R\), we define \(r_{i+1}\) to be its successor; similarly, \(\ell_{i+1}\) is the successor of \(\ell_i\). The source \(s\) has (at most) two successors, namely \(r_1\) and \(\ell_1\). The successor of a point \(p\) is denoted by \(\text{succ}(p)\); for an extreme point \(p\) we define \(\text{succ}(p) = \text{NIL}\).

If \(\text{succ}(p) \neq \text{NIL}\), we call \(p\) the predecessor of \(q\) and we write \(\text{pred}(q) = p\). A chain is a path in the communication graph \(G_\rho(P)\) that only consists of edges connecting a point to its successor. Thus a chain either visits consecutive points from \(\{s\} \cup R\) from left to right, or it visits consecutive points from \(\{s\} \cup L\) from right to left. It will be convenient to consider the empty path from \(s\) to itself to be a chain as well.
Consider a range assignment $\rho$. We say that a point $q \in P$ is *within reach* of a point $p \in P$ if $|pq| \leq \rho(p)$. Let $\mathcal{B}$ a broadcast tree in $\mathcal{G}_\rho(P)$—that is, $\mathcal{B}$ is an arborescence rooted at $s$. A point in $R \cup L$ in $\mathcal{B}$ is called root-crossing in $\mathcal{B}$ if it has a child on the other side of $s$; the source $s$ is root-crossing if it has a child in $L$ and a child in $R$. The following theorem, which holds for any distance-power gradient $\alpha > 1$, is proven in [7].

**Theorem 2 ([7]).** Let $P$ be a point set in $\mathbb{R}^1$. If all points in $P \setminus \{s\}$ lie to the same side of the source $s$, then the optimal solution induces a chain from $s$ to the extreme point in $P$. Otherwise, there is an optimal range assignment $\rho$ such that $\mathcal{G}_\rho(P)$ contains a broadcast tree $\mathcal{B}$ with the following structure:

- $\mathcal{B}$ has a single root-crossing point, $p^*$.
- $\mathcal{B}$ contains a chain from $s$ to $p^*$.
- All points within reach of $p^*$, except those on the chain from $s$ to $p^*$, are children of $p^*$.
- Let $r_i$ and $\ell_j$, be the rightmost and leftmost point within reach of $p^*$, respectively. Then $\mathcal{B}$ contains a chain from $r_i$ to $r_{|R|}$, and a chain from $\ell_j$ to $\ell_{|L|}$.

From now on, whenever we talk about optimal range assignments and their induced broadcast trees, we implicitly assume that the broadcast tree has the structure described in Theorem 2. Note that the communication graph $\mathcal{G}_\rho(P)$ induced by an optimal range assignment $\rho$ can contain more edges than the ones belonging to the broadcast tree $\mathcal{B}$. Obviously, for $\rho$ to be optimal it must be a minimum-cost assignment inducing $\mathcal{B}$.

Define the *standard range* of a non-extreme point $r_i \in R$ to be $|r_ir_{i+1}|$; the standard range of the extreme point $r_{|R|}$ is defined to be zero. The standard ranges of the points in $L$ are defined similarly. The source $s$ has two standard ranges, $|sf_1|$ and $|sr_1|$. A range assignment in which every point has a standard range is called a *standard solution*; a standard solution may or may not be optimal. Note that, in the static problem, it is never useful to give a point a non-zero range that is smaller than its standard range. Hence, we only need to consider three types of points: standard-range points, zero-range points, and long-range points. Here zero-range points are non-extreme points with a zero range, and a point is said to have a long range if its range is greater than its standard range. Theorem 2 implies that an optimal range assignment has the following properties; see also Figure 1.

- There is at most one long-range point.
- The set $Z \subset P$ of zero-range points (which may be empty) can be partitioned into two subsets, $Z_{\text{left}}$ and $Z_{\text{right}}$, such that $Z_{\text{left}}$ consists of consecutive points that lie to the left of the source $s$, and $Z_{\text{right}}$ consists of consecutive points to that lie to the right of $s$.

### 2.2 An efficient update algorithm

Using Theorem 2 an optimal solution for the broadcast range-assignment problem can be computed in $O(n^2)$ time [10]. Below we show that maintaining an optimal solution under insertions and deletions can be done more efficiently than by re-computing it from scratch; using a suitable data structure, we can update the solution in $O(n \log n)$ time. This will also be useful in later sections, when we give algorithms that maintain a stable solution.

Recall that an optimal solution for a given point set $P$ has a single root-crossing point, $p^*$. Once the range $\rho(p^*)$ is fixed, the solution is completely determined. Since $\rho(p^*) = |p^*p|$ for some point $p \neq p^*$, there are $n - 1$ candidate ranges for a given choice of the root-crossing point $p^*$. The idea of our solution is to implicitly store the cost of the range assignment for each candidate range of $p^*$ such that, upon the insertion or deletion of a point in $P$, we can in $O(\log n)$ time find the best range for $p^*$. By maintaining $n$ such data structures $T_p$, one for each choice of the root-crossing point $p^*$, we can then find the overall best solution.
The data structure for a given root-crossing point. Next we explain our data structure for a given candidate root-crossing point $p^*$. We assume without loss of generality that $p^*$ lies to the right of the source point $s$; it is straightforward to adapt the structure to the (symmetric) case where $p^*$ lies to the left of $s$, and to the case where $p^* = s$.

Let $\mathcal{R}_{p^*}$ be the set of all ranges we need to consider for $p^*$, for the current set $P$. The range of a root-crossing point must extend beyond the source point. Hence,  

$$ \mathcal{R}_{p^*} := \{ |p^*p| : p \in P \text{ and } |p^*p| > |p^*s| \}. $$

Let $\lambda_1, \ldots, \lambda_m$ denote the sequence of ranges in $\mathcal{R}_{p^*}$, ordered from small to large. (If $\mathcal{R}_{p^*} = \emptyset$, there is nothing to do and our data structure is empty.) As mentioned, once we fix a range $\lambda_j$ for the given root-crossing point $p^*$, the solution is fully determined by Theorem 2: there is a chain from $s$ to $p^*$, a chain from the rightmost point within range of $p^*$ to the right-extreme point, and a chain from the leftmost point within range of $p^*$ to the left-extreme point. We denote the resulting range assignment for $P$ by $\Gamma(P, p^*, \lambda_j)$.

Our data structure, which implicitly stores the costs of the range assignments $\Gamma(P, p^*, \lambda_j)$ for all $\lambda_j \in \mathcal{R}_{p^*}$, is an augmented balanced binary search tree $T_{p^*}$. The key to the efficient maintenance of $T_{p^*}$ is that, upon the insertion of a new point $p$ (or the deletion of an existing point), many of the solutions change in the same way. To formalize this, let $\Delta_j$ be the signed difference of the cost of the range assignment $\Gamma(P, p^*, \lambda_j)$ before and after the insertion of $q$, where $\Delta_j$ is positive if the cost increases. Figure 2 shows various possible values for $\Delta_j$, depending on the location of the new point $q$ with respect to the range $\lambda_j$. As can be seen in the figure, there are only four possible values for $\Delta_j$. This allows us to design our data structure $T_{p^*}$ such that it can be updated using $O(1)$ bulk updates of the following form:

Given an interval $I$ of range values and an update value $\Delta$, add $\Delta$ to the cost of $\Gamma(P, p^*, \lambda_j)$ for all $\lambda_j \in I$.

In Appendix A we define the information stored in $T_{p^*}$ and we show how bulk updates can be done in $O(\log n)$ time. We eventually obtain the following theorem.

Theorem 3. An optimal solution to the broadcast range-assignment problem for a point set $P$ in $\mathbb{R}^1$ can be maintained in $O(n \log n)$ per insertion and deletion, where $n$ is the number of points in the current set $P$.

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1 When $P$ lies completely to one side of $s$, then the range assignment is formally not root-crossing. We permit ourselves this slight abuse of terminology because by considering $s$ as root-crossing point, setting $\rho(s) := |ssucc(s)|$ and adding a chain from succ($s$) to the extreme point, we get an optimal solution.
3 A stable approximation scheme in \( \mathbb{R}^1 \)

In this section we use the structure of an optimal solution provided by Theorem 2 to obtain a SAS for the 1-dimensional broadcast range-assignment problem. Our SAS has stability parameter \( k(\varepsilon) = O((1/\varepsilon)^{1/(\alpha-1)}) \), which we will show to be asymptotically optimal.

The optimal range assignment can be very unstable. Indeed, suppose the current point set is \( P := \{s, r_1, \ldots, r_n\} \) with \( s = 0 \) and \( r_i = i \) (\( 1 \leq i \leq n \)), and take any \( \alpha > 1 \). Then the (unique) optimal assignment \( \rho_{\text{opt}} \) has \( \rho_{\text{opt}}(s) = \rho_{\text{opt}}(r_1) = \cdots = \rho_{\text{opt}}(r_{n-1}) = 1 \) and \( \rho_{\text{opt}}(r_n) = 0 \). If now the point \( \ell_1 = -n \) is inserted, then the optimal assignment becomes \( \rho_{\text{opt}}(s) = n \) and \( \rho_{\text{opt}}(r_1) = \cdots = \rho_{\text{opt}}(r_n) = \rho_{\text{opt}}(\ell_1) = 0 \), causing \( n \) ranges to be modified.

Next, we will define a feasible solution, referred to as a canonical range assignment \( \rho_k \) that is more stable than an optimal assignment, while still having a cost close to the cost of an optimal solution. Here \( k \) is a parameter that allows a trade-off between stability and quality of the solution. The assignment \( \rho_k \) for a given point set \( P \) will be uniquely determined by the set \( P \)—it does not depend on the order in which the points have been inserted or deleted. This means that the update algorithm simply works as follows. Let \( \rho_k(P) \) be the canonical range assignment for a point set \( P \), and suppose we update \( P \) by inserting a point \( q \). Then the update algorithm computes \( \rho_k(P \cup \{q\}) \) and it modifies the range of each point \( p \in P \cup \{q\} \) whose canonical range in \( \rho_k(P \cup \{q\}) \) is different from its canonical range in \( \rho_k(P) \). The goal is now to specify \( \rho_k \) such that (i) many ranges in \( \rho_k(P \cup \{q\}) \) are the same as in \( \rho_k(P) \), (ii) the cost of \( \rho_k(P) \) is close to the cost of \( \rho_{\text{opt}}(P) \).

The instance in the example above shows that there can be many points whose range changes from being standard to being zero (or vice versa) when preserving optimality of the consecutive instances. Our idea is therefore to construct solutions where the number of points with zero range is limited, and instead give many points their standard range; if we do this for points whose standard range is relatively small, then the cost of this solution remains bounded compared to the cost of an optimum solution. We now make this idea precise.

Consider a point set \( P \) and let \( \rho_{\text{opt}} \) be an optimal range assignment satisfying the structure described in Theorem 2. Assuming there are points in \( P \) on both sides of the source, \( \rho_{\text{opt}} \) induces a broadcast tree \( B \) with the structure depicted in Figure 3. Let \( \rho_{st}(p) \) be the standard range of a point \( p \). The canonical range assignment \( \rho_k \) is now defined as follows.

- If all points from \( P \) lie to the same side of \( s \), then \( \rho_k(p) := \rho_{\text{opt}}(p) \) for all \( p \in P \). Note that in this case \( \rho_k(p) = \rho_{st}(p) \) for all \( p \in P \).
- Otherwise, let \( Z \) be the set of zero-range points in \( \rho_{\text{opt}}(P) \). If \( |Z| \leq k \) then let \( Z_k := Z \); otherwise let \( Z_k \subseteq Z \) be the \( k \) points from \( Z \) with the largest standard ranges, with ties broken arbitrarily. We define \( \rho_k \) as follows.
  - \( \rho_k(p) := \rho_{\text{opt}}(p) \) for all \( p \in P \setminus Z \). Observe that this means that \( \rho_k(p) = \rho_{st}(p) \) for all \( p \in P \setminus Z \) except (possibly) for the root-crossing point.
  - \( \rho_k(p) := 0 \) for all \( p \in Z_k \).
  - \( \rho_k(p) := \rho_{st}(p) \) for all \( p \in Z \setminus Z_k \).

Notice that \( \rho_k \) is a feasible solution since \( \rho_k(p) \geq \rho_{\text{opt}}(p) \) for each \( p \in P \). The next lemma analyzes the stability of the canonical range assignment \( \rho_k \). Recall that for any range assignment \( \rho \)—hence, also for \( \rho_k \)—and any point \( q \) not in the current set \( P \), we have \( \rho(q) = 0 \) by definition. The proof of the following lemma is in the appendix.

\textbf{Lemma 4.} Consider a point set \( P \) and a point \( q \notin P \). Let \( \rho_{\text{old}}(p) \) be the range of a point \( p \) in \( \rho_k(P) \) and let \( \rho_{\text{new}}(p) \) be the range of \( p \) in \( \rho_k(P \cup \{q\}) \). Then

\[ |\{p \in P \cup \{q\} : \rho_{\text{new}}(p) > \rho_{\text{old}}(p)\}| \leq k + 3 \quad \text{and} \quad |\{p \in P \cup \{q\} : \rho_{\text{new}}(p) < \rho_{\text{old}}(p)\}| \leq k + 3. \]
Next we bound the approximation ratio of $\rho_k$.

Lemma 5. For any set $P$ and any $\alpha > 1$, we have
\[
\text{cost}_\alpha(\rho_k(P)) \leq \left(1 + \frac{2^{\alpha}}{k^{\alpha-1}}\right) \cdot \text{cost}_\alpha(\rho_{\text{opt}}(P)).
\]

Proof. If all points in $P$ lie to the same side of $s$ then $\rho_k(P) = \rho_{\text{opt}}(P)$, and we are done. Otherwise, let $p^*$ be the root-crossing point. The only points receiving a different range in $\rho_k(P)$ when compared to $\rho_{\text{opt}}(P)$ are the points in $Z \setminus Z_k$; these points have $\rho_k(p) = \rho_{\omega_k}(p)$ while $\rho_{\text{opt}}(p) = 0$. This means we are done when $Z \setminus Z_k = \emptyset$. Thus we can assume that $|Z| > k$, so $Z \setminus Z_k \neq \emptyset$. Assume without loss of generality that $\rho_{\text{opt}}(p^*) = 1$. As each $p \in Z$ is within reach of $p^*$, we have $\sum_{p \in Z} \rho_{\omega_k}(p) \leq 2^{\alpha}$. Since $Z_k$ contains the $k$ points with the largest standard ranges among the points in $Z$, we have $\max \{\rho_{\omega_k}(p) : p \in Z \setminus Z_k\} \leq 2/k$. Hence,
\[
\sum_{p \in Z \setminus Z_k} \rho_k(p)^\alpha = \sum_{p \in Z \setminus Z_k} \rho_{\omega_k}(p)^\alpha = \sum_{p \in Z \setminus Z_k} \rho_{\omega_k}(p)^{\alpha-1} \cdot \rho_{\omega_k}(p) \leq \left(\frac{2}{k}\right)^{\alpha-1} \sum_{p \in Z \setminus Z_k} \rho_{\omega_k}(p) \leq \frac{2^{\alpha}}{k^{\alpha-1}}.
\]
(The analysis can be made tighter by using that $\sum_{p \in Z \setminus Z_k} \rho_{\omega_k}(p) \leq 2 - k \max_{p \in Z \setminus Z_k} \rho_{\omega_k}(p)$, but this will not change the approximation ratio asymptotically.) We conclude that
\[
\frac{\text{cost}_\alpha(\rho_k(P))}{\text{cost}_\alpha(\rho_{\text{opt}}(P))} \leq \frac{\sum_{p \in P \setminus (Z \setminus Z_k)} \rho_k(p)^\alpha + \sum_{p \in Z \setminus Z_k} \rho_k(p)^\alpha}{\sum_{p \in P \setminus (Z \setminus Z_k)} \rho_{\text{opt}}(p)^\alpha} \leq 1 + \frac{2^{\alpha}}{k^{\alpha-1}},
\]
where the last inequality follows because we have $\rho_k(p) = \rho_{\text{opt}}(p)$ for all $p \in P \setminus (Z \setminus Z_k)$ and $\sum_{p \in P \setminus (Z \setminus Z_k)} \rho_{\text{opt}}(p)^\alpha \geq 1$.

By maintaining the canonical range assignment $\rho_k$ for $k = (2^{\alpha/\epsilon}/\epsilon)^{1/(\alpha-1)} = O((1/\epsilon)^{1/(\alpha-1)})$ we obtain the following theorem.

Theorem 6. There is a SAS for the dynamic broadcast range-assignment problem in $\mathbb{R}^1$ with stability parameter $k(\epsilon) = O((1/\epsilon)^{1/(\alpha-1)})$, where $\alpha > 1$ is the distance-power gradient. The time needed by the SAS to compute the new range assignment upon the insertion or deletion of a point is $O(n \log n)$, where $n$ is the number of points in the current set.

Next we show that the stability parameter $k(\epsilon)$ in our SAS is asymptotically optimal.

Theorem 7. Any SAS for the dynamic broadcast range-assignment problem in $\mathbb{R}^1$ must have stability parameter $k(\epsilon) = \Omega((1/\epsilon)^{1/(\alpha-1)})$, where $\alpha > 1$ is the distance-power gradient.

Proof. Let $\text{ALG}$ be a $k$-stable algorithm, where $k \geq 4$ and $k^{\alpha-1} \geq \frac{1}{2^{\alpha+1}}\left(\frac{1}{2^{\alpha}}+1\right)$ and $k$ is even, and let $\rho_{\text{alg}}$ be the range assignment it maintains. Note that the condition on $k$ is satisfied for $k$ large enough. We will show that the approximation ratio of $\text{ALG}$ is at least $1 + \frac{1}{\frac{2^{\alpha+1}}{k^{\alpha-1}}}$. Since a SAS has approximation ratio $1 + \epsilon$, this implies that the stability parameter $k(\epsilon)$ of $\text{ALG}$ must satisfy $k(\epsilon) = \Omega((1/\epsilon)^{1/(\alpha-1)})$.

Consider the point set $P := \{s, r_1, r_2, \ldots, r_{2k}\}$, where $s = 0$ and $r_i = i/(2k)$ for $i = 1, 2, \ldots, 2k$. We consider two cases.

Case I: The number of zero-range points in $\rho_{\text{alg}}(P)$ is at least $k/2$, where we assume without loss of generality that all points with range less than $1/(2k)$ actually have range zero. The cheapest possible solution in this case is to have exactly $k/2$ zero-range points, $k$ points with range $1/(2k)$, and $k/2$ points with range $1/k$, for a total cost
\[
\text{cost}_\alpha(\rho_{\text{alg}}(P)) \geq k \cdot \left(\frac{1}{2k}\right)^\alpha + k/2 \cdot \left(\frac{1}{k}\right)^\alpha = \left(1 + \frac{2^{\alpha-1} - 1}{2}\right) \cdot 2k \left(\frac{1}{2k}\right)^\alpha.
\]
An optimal solution has cost $2k \cdot (1/(2k))^\alpha$, and so the approximation ratio of ALG in Case I is at least $1 + 2^{\frac{1}{2}} - 1$, which is at least $1 + \frac{1}{2^{\frac{1}{2}} - 1}$, since $k^{\alpha - 1} \geq 2^{\frac{1}{2}} - 1$.

**Case II:** The number of zero-range points $\rho_{\text{alg}}(P)$ is less than $k/2$. Now suppose the point $\ell_1 = -1$ arrives. Since $\rho_{\text{alg}}(P)$ had less than $k/2$ zero-range points and ALG can modify at most $k$ ranges, $\rho_{\text{alg}}(P \cup \{\ell_1\})$ has less than $3k/2$ zero-range points. Hence, at least $k/2$ points in $P \cup \{\ell_1\}$ have a range that is at least $1/(2k)$, one of which must have a range at least 1. This implies that $\text{cost}_\alpha(\rho_{\text{alg}}(P \cup \{\ell_1\})) \geq 1 + (k/2 - 1) \cdot \left(\frac{1}{2k}\right) \geq 1 + \frac{1}{2^{\frac{1}{2}} - 1}$, where the last inequality holds since $k/2 - 1 \geq k/4$ (because $k \geq 4$). An optimal range assignment on $P \cup \{\ell_1\}$ has $\rho_{\text{opt}}(s) = 1$ and all other ranges equal to zero, for a total cost of 1, and so the approximation ratio of ALG in Case II is at least $1 + \frac{1}{2^{\frac{1}{2}} - 1}$ as well. ◀

# 4 The problem in $S^1$

We now turn to the setting where the underlying space is $S^1$, that is, the points in $P$ lie on a circle and distances are measured along the circle. In Section 4.1, we prove that the structure of an optimal solution in $S^1$ is very similar to the structure of an optimal solution in $\mathbb{R}^1$ as formulated in Theorem 2. In spite of this, and contrary to the problem in $\mathbb{R}^1$, we prove in Section 4.2 that no SAS exists for the problem in $S^1$.

Again, we denote the source point by $s$. The clockwise distance from a point $p \in S^1$ to a point $q \in S^1$ is denoted by $d_{cw}(p, q)$, and the counterclockwise distance by $d_{ccw}(p, q)$. The actual distance is then $d(p, q) := \min(d_{cw}(p, q), d_{ccw}(p, q))$. The closed and open clockwise interval from $p$ to $q$ are denoted by $[p, q]^{cw}$ and $(p, q)^{cw}$, respectively.

## 4.1 The structure of an optimal solution in $S^1$

Here we prove that the structure of an optimal solution in $S^1$ is very similar to the structure of an optimal solution in $\mathbb{R}^1$. The heart of this proof is the following lemma, whose (rather intricate) proof is given in Appendix D.1. Define the covered region of $P$ with respect to a range assignment $\rho$, denoted by $\text{cov}(\rho, P)$, to be the set of all points $r \in S^1$ such that there exists a point $p \in P$ with $\rho(p) \geq d(p, r)$.

**Lemma 8.** Let $P$ be a point set in $S^1$ with $|P| > 2$ and let $\rho_{\text{opt}}$ be an optimal range assignment for $P$. Then there exists a point $r \in S^1$ such that $r \notin \text{cov}(\rho_{\text{opt}}, P)$.

Lemma 8 implies that an optimal solution for an instance in $S^1$ corresponds to an optimal solution for an instance in $\mathbb{R}^1$ derived as follows. For a point $r \in S^1$, define the mapping $\mu_r : P \rightarrow \mathbb{R}^1$ such that $\mu_r(s) := 0$, and $\mu_r(p) := d_{cw}(s, p)$ for all $p \in [s, r]^{cw}$, and $\mu_r(p) := -d_{ccw}(s, p)$ for all $p \in [r, s]^{cw}$. Let $\mu_r(P)$ denote the resulting point set in $\mathbb{R}^1$.

**Theorem 9.** Let $P$ be an instance of the broadcast range-assignment problem in $S^1$. There exists a point $r \in S^1$ such that an optimal range assignment for $\mu_r(P)$ in $\mathbb{R}^1$ induces an optimal range assignment for $P$. Moreover, we can compute an optimal range assignment for $P$ in $O(n^2 \log n)$ time, where $n$ is the number of points in $P$.

**Proof.** Let $r \in S^1$ be a point such that $r \notin \text{cov}(\rho_{\text{opt}}, P)$, which exists by Lemma 8. Consider the mapping $\mu_r$. Any feasible range assignment for $\mu_r(P)$ induces a feasible range assignment for $P$ in $S^1$, since $d(p, q) \leq |\mu_r(p) - \mu_r(q)|$ for any two points $p, q \in P$. Conversely, an optimal range assignment for $P$ induces a feasible range assignment for $\mu_r(P)$, since the point $r$ is not covered in the optimal solution. This proves the first part of the theorem.
Now let \( P := \{s, p_1, \ldots, p_n\} \), where the points \( p_i \) are ordered clockwise from \( s \). For \( 0 \leq i \leq n \), let \( r_i \) be a point in \((p_i, p_{i+1})^{cw}\), where \( p_0 = p_{n+1} = s \). Since \( \mu_{r_i} = \mu_r \) for any \( r \in (p_i, p_{i+1})^{cw} \), an optimal solution can be computed by finding the best solution over all mappings \( \mu_{r_i} \). The only difference between \( \mu_{r_i} \) and \( \mu_{r_{i+1}} \) is the location that \( p_{i+1} \) is mapped to, so after computing an optimal solution for \( \mu_1(P) \) in \( O(n^2 \log n) \) time, we can go through the mappings \( \mu_2, \ldots, \mu_n \) and update the optimal solution in \( O(n \log n) \) time using Theorem 3. Hence, an optimal range assignment for \( P \) can be computed in \( O(n^2 \log n) \) time.

4.2 Non-existence of a SAS in \( S^1 \)

We have seen that an optimal solution for a set \( P \) in \( S^1 \) can be obtained from an optimal solution in \( \mathbb{R}^1 \), if we cut \( S^1 \) at an appropriate point \( r \). It is a fact however that the insertion of a new point into \( P \) may cause the location of the cutting point \( r \) to change drastically.

Next we show that this means that the dynamic problem in \( S^1 \) does not admit a SAS.

▶ Theorem 10. The dynamic broadcast range-assignment problem in \( S^1 \) with distance power gradient \( \alpha > 1 \) does not admit a SAS. In particular, there is a constant \( c_\alpha > 1 \) such that the following holds: for any \( n \) large enough, there is a set \( P := \{s, p_1, \ldots, p_{2n+1}\} \) and a point \( q \) in \( S^1 \) such that any update algorithm \( \text{ALG} \) that maintains a \( c_\alpha \)-approximation must modify more than \( 2n/3 - 1 \) ranges upon the insertion of \( q \) into \( P \).

The rest of this section is dedicated to proving Theorem 10. We will prove the theorem for

\[
c_\alpha := \min \left( 1 + 2^\alpha - 4, \frac{1}{8} \cdot \frac{\alpha}{2^\alpha - 1}, 1 + \frac{3^{\alpha} - 3^\alpha - 1}{3 \cdot 2^\alpha + 2}, 1 + \frac{4^{\alpha} - 2^\alpha - 2}{4(2^{\alpha} + 1)} \right).
\]

Note that each term is a constant strictly greater than 1 for any fixed constant \( \alpha > 1 \). In particular, for \( \alpha = 2 \) we have \( c_2 = 1 + \frac{1}{14} \).

Let \( P := \{s, p_1, \ldots, p_{2n+1}\} \), where \( d_{cw}(p_i, p_{i+1}) = 2 \) for odd \( i \) and \( d_{cw}(p_i, p_{i+1}) = 1 \) for even \( i \); see Fig. 3(i). Let \( d_{cw}(s, p_1) = \delta \), where \( \delta = (2^\alpha + 1)n \). Finally, let \( d_{cw}(p_{2n+1}, q) = d_{cw}(q, s) = x\delta \), where \( x\delta = \frac{1}{4} + \left(\frac{1}{2}\right)^{\alpha+1} \). Note that \((1/2)^\alpha < x\delta < 1/2\) for any \( \alpha > 1 \).

Let \( \rho(p) \) denote the range given to a point \( p \) by \( \text{ALG} \). A directed edge \((p, p')\) in the communication graph induced by \( \rho \) is called a clockwise edge if \( \rho(p) \geq d_{cw}(p, p') \), and it is called a counterclockwise edge if \( \rho(p) \geq d_{cw}(p, p') \). Observe that we may assume that no edge \((p, p')\) is both clockwise and counterclockwise, because otherwise \( \rho(p) \geq (\delta + 3n + 2x\delta)/2 \),
which is much too expensive for an approximation ratio of at most $c_\alpha$. Define the range 
$\rho(p)$ of a point in $P$ to be \textit{cw-minimal} if $\rho(p)$ equals the distance from $p$ to its clockwise 
neighbor in $P$. Similarly, $\rho(p)$ is \textit{ccw-minimal} if $\rho(p)$ equals the distance from $p$ to its 
counterclockwise neighbor. The idea of the proof is to show that before the insertion of $q$, 
most of the points $s, p_1, \ldots, p_{2n+1}$ must have a \textit{cw-minimal} range, while after the insertion 
most points must have a \textit{ccw-minimal} range. This will imply that many ranges must be 
modified from being \textit{cw-minimal} to being \textit{ccw-minimal}.

Before the insertion of $q$, giving every point a \textit{cw-minimal} range leads to a feasible 
assignment of total cost $\delta^\alpha + (2^\alpha + 1)n = 2\delta^\alpha$. After the insertion of $q$, giving every point a 
\textit{ccw-minimal} range leads to a feasible assignment of total cost $2(x\delta)^\alpha + (2^\alpha + 1)n = (2x^\alpha + 1)\delta^\alpha$. 
Hence, if $\text{OPT}()$ denotes the cost of an optimal range assignment, then we have:

\textbf{Observation 11.} $\text{OPT}(P) \leq 2\delta^\alpha$ and $\text{OPT}(P \cup \{q\}) \leq (2x^\alpha + 1)\delta^\alpha < 2\delta^\alpha$.

We first prove a lower bound on the total cost of the points $p_1, \ldots, p_{2n+1}$. Intuitively, only $o(n)$ of those points can be reached from $s$ or $q$ (otherwise the range of $s$ or $q$ would be too expensive) and the cheapest way to reach the remaining points will be to use only \textit{cw-minimal} or \textit{ccw-minimal} ranges. A formal proof of the lemma is given in Appendix D.2.

\textbf{Lemma 12.} $\sum_{i=1}^{2n+1} \rho(p_i)^\alpha \geq (2^\alpha + 1)n - o(n)$, both before and after the insertion of $q$.

The following lemma gives a key property of the construction.

\textbf{Lemma 13.} The point $p_{2n+1}$ cannot have an incoming counterclockwise edge before $q$ is inserted, and the point $p_1$ cannot have an incoming clockwise edge after $q$ has been inserted.

\textbf{Proof.} The cheapest incoming counterclockwise edge for $p_{2n+1}$ before the insertion of $q$ is 
from $s$, but this is too expensive for \textsc{alg} to achieve approximation ration $c_\alpha$. Similarly, the 
cheapest incoming clockwise edge for $p_1$ is from $s$, but this is too expensive after the insertion 
of $q$. The computations for both cases can be found Appendix D.2.

We are now ready to prove that many edges must change from being \textit{cw-minimal} to being 
\textit{ccw-minimal} when $q$ is inserted.

\textbf{Lemma 14.} Before the insertion of $q$, at least $4n/3 + 1$ of the points from \{s, p_1, \ldots, p_{2n}\} 
have a \textit{cw-minimal} range and after the insertion of $q$ at least $4n/3 + 1$ of the points from 
\{q, p_1, \ldots, p_{2n}\} have a \textit{ccw-minimal} range.

\textbf{Proof.} We prove the lemma for the situation before $q$ is inserted; the proof for the situation 
after the insertion of $q$ is similar. Observe that before and after the insertion of $q$, the 
distance between any two points is either 1, 2 or at least 3. Hence, in what follows we may 
assume that $\rho(p) \in \{0, 1, 2\} \cup [3, \infty)$ for any point $p \in P \cup \{q\}$.

It will be convenient to define $p_0 := s$ (although we may still use $s$ if we want to 
stress that we are talking about the source). Recall that $p_{2n+1}$ does not have an incoming 
counterclockwise edge in the communication graph $G\rho(P)$ before the insertion of $q$. Let $\pi^*$ 
be a minimum-hop path from $s$ to $p_{2n+1}$ in $G\rho(P)$. Since $p_{2n+1}$ does not have an incoming 
counterclockwise edge and $\pi^*$ is a minimum-hop path, all edges in $\pi$ are clockwise. We assign 
each point $p_j$ with $1 \leq j \leq 2n + 1$ to the edge $(p_i, p_j)$ in $\pi^*$ such that $i + 1 \leq j \leq t$, and we 
define $A(p_i, p_t) := \{p_{i+1}, \ldots, p_t\}$ to be the set of all points assigned to $(p_i, p_t)$. We define the 
\textit{excess} of a point $p_i \in A(p_i, p_t)$ to be

$\text{excess}(p_j) := \frac{1}{|A(p_i, p_j)|} \left( \rho(p_i)^\alpha - \sum_{p \in A(p_i, p_j)} d(p_{c-1}, p_i)^\alpha \right)$. 

We say that an edge \((p_i, p_k)\) in \(\pi^*\) is \(cw\)-minimal if \(p_i\) has a \(cw\)-minimal range. Note that if a point \(p_j\) is assigned to a \(cw\)-minimal edge, then this is the edge \((p_{j-1}, p_j)\) and \(\text{excess}(p_j) = 0\).

Intuitively, \(\text{excess}(p_j)\) denotes the additional cost we pay for reaching \(p_j\) compared to reaching it by a \(cw\)-minimal edge, if we distribute the additional cost of a non-\(cw\)-minimal edge over the points assigned to it. Because each of the points \(p_1, \ldots, p_{2n+1}\) is assigned to exactly one edge on the path \(\pi^*\), we have

\[
\sum_{p_i \in \pi^*} \rho(p_i)^\alpha \geq \sum_{j=1}^{2n+1} d(p_{j-1}, p_j)^\alpha + \sum_{j=1}^{2n+1} \text{excess}(p_j) \geq \text{OPT}(P) + \sum_{j=1}^{2n+1} \text{excess}(p_j) \tag{1}
\]

where the second inequality follows from Observation 11 and because \(p_0 = s\). The following claim is proved in Appendix D.2 (Essentially, the smallest possible excess is obtained when \(|A(p_i, p_k)| \in \{1, 2, 3\}; the three terms in the claim correspond to these cases.)

Claim. If \(p_j\) is not assigned to a \(cw\)-minimal edge then \(\text{excess}(p_j) \geq c'_\alpha \), where \(c'_\alpha = \min\left(2^\alpha - 1, \frac{3^\alpha - 2^\alpha - 1}{2}, \frac{4^\alpha - 3^\alpha - 2}{3}\right)\).

Now suppose for a contradiction that less than \(4n/3 + 1\) points from \(\{s, p_1, \ldots, p_{2n+1}\}\) have a \(cw\)-minimal range. Then at least \(2n/3 + 1\) points \(p_j\) have \(\text{excess}(p_j) \geq c'_\alpha\) by the claim above. By Inequality (1) the total cost incurred by \(\text{ALG}\) is therefore more than

\[
\text{OPT}(P) + c'_\alpha \cdot (2n/3) = \text{OPT}(P) + \frac{c'_\alpha}{3(2^\alpha + 1)} \cdot 2(2^\alpha + 1)n
\]

\[
> \left(1 + \min\left(\frac{3^\alpha - 2^\alpha - 1}{2}, \frac{4^\alpha - 3^\alpha - 2}{3}\right)\right) \cdot \text{OPT}(P)
\]

\[
\geq c_\alpha \cdot \text{OPT}(P)
\]

which contradicts the approximation ratio achieved by \(\text{ALG}\).

Lemma 14 implies that at least \(4n/3\) of the points \(p_1, \ldots, p_{2n+1}\) have a \(cw\)-minimal range before \(q\) is inserted, and at least \(4n/3\) of those points have a \(ccw\)-minimal range after the insertion. Hence, at least \(2n + 1 - 2 \cdot (2n/3 + 1) = 2n/3 - 1\) points must change from being \(cw\)-minimal to being \(ccw\)-minimal, thus finishing the proof of Theorem 10.

### 5 The 2-dimensional problem

The broadcast range-assignment problem is \(\text{NP}\)-hard in \(\mathbb{R}^2\), so we cannot expect a characterization of the structure of an optimal solution similar to Theorem 2. Using a similar construction as in \(\mathbb{S}^1\) we can also show that the problem in \(\mathbb{R}^2\) does not admit a SAS.

**Theorem 15.** The dynamic broadcast range-assignment problem in \(\mathbb{R}^2\) with distance power gradient \(\alpha > 1\) does not admit a SAS. In particular, there is a constant \(c_\alpha > 1\) such that the following holds: for any \(n\) large enough, there is a set \(P := \{s, p_1, \ldots, p_{2n+1}\}\) and a point \(q\) in \(\mathbb{R}^2\) such that any update algorithm \(\text{ALG}\) that maintains a \(c_\alpha\)-approximation must modify at least \(2n/3 - 1\) ranges upon the insertion of \(q\) into \(P\).

**Proof.** We use the same construction as in \(\mathbb{S}^1\), where we embed the points on a square and the distances used to define the instance are measured along the square; see Fig. 2(ii). For any two points on the same edge of the square, their distance along the square is the same as their distance in \(\mathbb{R}^2\). Moreover, we know that no range can be larger than \(3\delta\), otherwise the range assignment is already too expensive. Note that the number of points \(p_i\) within distance
δ from a corner is $o(n)$. Using this fact we can argue that all lemmas from Section 4.2 still hold. (For example, in the proof of Lemma 14 we can simply ignore the excess of $o(n)$ points.) This is argued more extensively in Appendix E.1.

Although the problem in $\mathbb{R}^2$ does not admit a SAS, there is a relatively simple $O(1)$-stable $O(1)$-approximation algorithm for $\alpha \geq 2$. The algorithm is based on a result by Ambühl [1], who showed that a minimum spanning tree (MST) on $P$ gives a 6-approximation for the static broadcast range-assignment problem: turn the MST into a directed tree rooted at the source $s$, and assign as a range to each point $p \in P$ the maximum length of any of its outgoing edges. To make this solution stable, we set the range of each point to the maximum length of any of its incident edges (not just the outgoing ones). Because an MST in $\mathbb{R}^2$ has maximum degree 6, this leads to 17-stable 12-approximation algorithm; see Appendix E.2.

6 Concluding remarks

We studied the dynamic broadcast range-assignment problem from a stability perspective, introducing the notions of $k$-stable algorithms and stable approximation schemes (SASs). Our results provide a fairly complete picture of the problem in $\mathbb{R}^1$, in $S^1$, and in $\mathbb{R}^2$. In particular, we presented a SAS in $\mathbb{R}^1$ that has an asymptotically optimal stability parameter, and showed that the problem does not admit a SAS in $S^1$ and $\mathbb{R}^2$. Future work can focus on improving (the upper and/or lower bounds for) the approximation ratios we have obtained for algorithms with constant stability parameter. In particular, it is open whether there exists a 2-stable algorithm with approximation ratio less than 2 in $\mathbb{R}^1$. It would also be interesting to use develop algorithm with small stability parameter in $S^1$, possibly using the relation we proved between the structure of an optimal structure in in $S^1$ and in $\mathbb{R}^1$.

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A data structure for maintaining an optimal solution

Below we give a detailed description of the data structure to maintain an optimal solution in $\mathbb{R}^1$. For convenience, we repeat somematerial from the main text, so that the description below is self-contained.

Recall that an optimal solution for a given point set $P$ has a single root-crossing point, $p^*$. Once the range of $p^*$ is fixed, the solution is completely determined. The range of $p^*$ is defined by some other point from $P$—we have $\rho(p^*) = |p^*p|$ for some point $p \neq p^*$—and so there are $n - 1$ candidate ranges for a given choice of the root-crossing point $p^*$. The idea of our solution is to implicitly store the cost of the range assignment for each candidate range of $p^*$ such that, upon the insertion or deletion of a point in $P$, we can in $O(\log n)$ time find the best range for $p^*$. By maintaining $n$ such data structures $\mathcal{T}_{p^*}$, one for each choice of the root-crossing point $p^*$, we can then find the overall best solution.

Besides the data structures $\mathcal{T}_{p^*}$ which are described below, we also maintain a global data structure $\mathcal{T}_P$ that supports the following operations.

- Find the predecessor $\text{pred}(q)$ and successor $\text{succ}(q)$ in $P$ of a query point $q$.
- Given two points $p, p' \in P$, report the total cost of the chain from $p$ to $p'$.
- Insert or delete points from $P$.

By implementing $\mathcal{T}_P$ as a suitably augmented binary search tree—see the book by Cormen et al. [15] Chapter 15] for the design and maintenance of such structures—each of these operations can be performed in $O(\log n)$ time.

The data structure for a given root-crossing point. Next we explain our data structure for a given candidate root-crossing point $p^*$. We assume without loss of generality that $p^*$ lies to the right of the source point $s$; it is straightforward to adapt the structure to the (symmetric) case where $p^*$ lies to the left of $s$, and to the case where $p^* = s$.

Let $\mathcal{R}_{p^*}$ be the set of all ranges we need to consider for $p^*$, for the current set $P$. The range of a root-crossing point must extend beyond the source point. Hence,

$$\mathcal{R}_{p^*} := \{|p^*p| : p \in P \text{ and } |p^*p| > |p^*s|\}.$$

Let $\lambda_1, \ldots, \lambda_m$ denote the sequence of ranges in $\mathcal{R}_{p^*}$, ordered from small to large. (If $\mathcal{R}_{p^*} = \emptyset$, there is nothing to do and our data structure is empty.) As mentioned, once we fix a range $\lambda_j$ for the given root-crossing point $p^*$, the solution is fully determined by Theorem 2: there is a chain from $s$ to $p^*$, a chain from the rightmost point within range of $p^*$ to the right-extreme point, and a chain from the leftmost point within range of $p^*$ to the left-extreme point. We denote the resulting range assignment $\Gamma$ for $P$ by $\Gamma(P, p^*, \lambda_j)$.

There is one subtlety in the definition of $\Gamma(P, p^*, \lambda_j)$, namely when there are no points within reach of $p^*$ to, say, the right of $p^*$; see Figure 1. Such a solution can never be optimal, but we must maintain it nevertheless, because the range $\lambda_j$ may become relevant later. To deal with this situation, we will insert a dummy point whose location coincides with $p^*$ and that is defined to be the predecessor of $\text{succ}(p^*)$. Note that the dummy will become a zero-range point as soon as an actual point is inserted that is within the range of $p^*$ and lies to the same side of $p^*$ as the dummy.

---

2 When all points in $P$ lies to the same side of $s$, then the range assignment is formally not root-crossing, but we will permit ourselves this slight abuse of terminology. Notice that in this case the range assignment induced by considering $s$ as root-crossing point and setting $\rho(s) := |s \text{succ}(s)|$ gives a chain from $s$ to the extreme point as solution, which is optimal.
Our data structure, which implicitly stores the costs of the range assignments \( \Gamma(P, p^*, \lambda_j) \) for all \( \lambda_j \in \mathcal{R}_{p^*} \), is an augmented balanced binary search tree \( T_{p^*} \), defined as follows.

- The leaves of \( T_{p^*} \) are in one-to-one correspondence with the candidate ranges in \( \mathcal{R}_{p^*} \): the leftmost leaf corresponds to \( \lambda_1 \), the next left to \( \lambda_2 \), and so on. From now on, with a slight abuse of notation, we use \( \lambda_j \) to refer to a range in \( \mathcal{R}_{p^*} \) as well as to the corresponding leaf.
- Each leaf stores, besides the corresponding range \( \lambda_j \), a value \( f(\lambda_j) \). Initially \( f(\lambda_j) \) will be equal to the cost of \( \Gamma(P, p^*, \lambda_j) \). Later this may no longer be the case, however.
- The internal nodes of \( T_{p^*} \) are augmented with extra information, as follows. For an internal node \( v \), let \( \mathcal{R}_{p^*}(v) \subseteq \mathcal{R}_{p^*} \) be the set of all ranges stored in the leaves of the subtree rooted at \( v \). The node \( v \) stores the following additional information, besides the splitting values that we have because \( T_{p^*} \) is a search tree on the ranges in \( \mathcal{R}_{p^*} \):
  - A correction value \( \Delta(v) \in \mathbb{R} \).
  - A value \( \text{min-cost}(v) \) defined as follows. For a range \( \lambda_j \in \mathcal{R}_{p^*}(v) \) define the local cost of \( \lambda_j \) at \( v \) to be \( f(\lambda_j) + \sum u \Delta(u) \), where the sum is over all nodes \( u \) on the path from \( v \) (and including \( v \)) to \( \lambda_j \). Then \( \text{min-cost}(v) \) is defined to be the minimum local cost over all ranges in \( \mathcal{R}_{p^*}(v) \).
  - A range \( \lambda_j \in \mathcal{R}_{p^*}(v) \) whose local cost at \( v \) is \( \text{min-cost}(v) \). This range is denoted by \( \text{best-range}(v) \).

Our update algorithm will ensure the following invariant:

For any range \( \lambda_j \in \mathcal{R}_{p^*} \), the total cost of \( \Gamma(P, p^*, \lambda_j) \) is equal to \( f(\lambda_j) + \sum u \Delta(u) \), where the sum is over all nodes on the search path from root \( T_{p^*} \) to \( \lambda_j \).

In other words, Invariance \( [5] \) states that, for any range \( \lambda_j \), the local cost of \( \lambda_j \) at the root of \( T_{p^*} \) is equal to the actual cost of \( \Gamma(P, p^*, \lambda_j) \). Since \( \mathcal{R}_{p^*}(\text{root}(T_{p^*})) = \mathcal{R}_{p^*} \), this implies that \( \text{min-cost} \) equals the minimum cost that can be obtained by a solution that uses \( p^* \) as a root-crossing point.

**Updating the data structure.** We now describe how to update the structure upon the insertion of a new point. Deletions can be handled in a symmetrical manner. To simplify the presentation, we assume that no two points in \( P \) coincide; the solution is easily adapted to the case where \( P \) can be a multi-set. Let \( \Delta_j \) be the (signed) difference of the cost of the range assignment \( \Gamma(P, p^*, \lambda_j) \) before and after the insertion of \( q \), where \( \Delta_j \) is positive if the cost increases. Figure \( 5 \) shows various possible values for \( \Delta_j \), depending on the location of the new point \( q \). The figure is for the generic case, when \( Z_{\text{left}}, Z_{\text{right}} \neq \emptyset \) and there are points to the right as well as to the left of the interval that is within reach of the root-crossing point \( p^* \). Lemma \( [10] \) which is easy to verify, gives the values for \( \Delta_j \) for all cases, where we write \( p < q \) when a point \( p \) is to the left of a point \( q \).
Lemma 16. Let \( \Delta_j := \text{cost}(\Gamma(P \cup \{q\}, p^*, \lambda_j)) - (\Gamma(P, p^*, \lambda_j)) \). If \( s < q < p^* \) or \( p^* < s < q \) we have

\[
\Delta_j = |\text{pred}(q) q|^{\alpha} + |q \text{ succ}(q)|^{\alpha} - |\text{pred}(q) \text{ succ}(q)|^{\alpha}
\]

Otherwise we have

\[
\Delta_j =
\begin{cases}
  |q \text{ succ}(q)|^{\alpha} - |\text{pred}(q) \text{ succ}(q)|^{\alpha} & \text{if } |p^* q| \leq \rho_j < |p^* \text{ succ}(q)| \\
  0 & \text{if } \rho_j \geq |p^* \text{ succ}(q)| \\
  |\text{pred}(q) q|^{\alpha} + |q \text{ succ}(q)|^{\alpha} - |\text{pred}(q) \text{ succ}(q)|^{\alpha} & \text{if } \lambda_j < |p^* q| \text{ and } \text{succ}(q) \neq \text{NIL} \\
  |\text{pred}(q) q|^{\alpha} & \text{if } \lambda_j < |p^* q| \text{ and } \text{succ}(q) = \text{NIL}
\end{cases}
\]

Lemma 16 implies that, after computing \( \text{pred}(q) \) and \( \text{succ}(q) \), we can update our data structure using \( O(1) \) bulk updates of the following form:

Given an interval \( I \) of range values and an update value \( \Delta \), add \( \Delta \) to the cost of \( \Gamma(P, p^*, \lambda_j) \) for all \( \lambda_j \in I \).

We cannot afford to do this explicitly, so we implement bulk updates by updating the auxiliary information stored in \( T_{\log n} \) nodes in \( T_{p^*} \), as follows.

1. Let \( \lambda_{\min} \) and \( \lambda_{\max} \) be the two endpoints of the interval \( I \) (possibly \( \lambda_{\max} = \infty \)). By searching with \( \lambda_{\min} \) and \( \lambda_{\max} \) in \( T_{p^*} \), identify a collection \( C(I) \) of \( O(\log n) \) nodes in \( T_{p^*} \), such that \( \lambda_i \in I \) if and only if the leaf storing \( \lambda_i \) is a descendant of a node in \( C(I) \).
2. Add \( \Delta \) to the correction values \( \Delta(v) \) of all nodes \( v \in C(I) \) and to the value \( \text{min-cost}(v) \).
3. Update the values \( \Delta(v) \), \( \text{min-cost}(v) \), and \( \text{best-range}(v) \) of the \( O(\log n) \) ancestors of the nodes in \( C(v) \) in a bottom-up manner.

Since algorithms for updating this type of auxiliary information are rather standard we omit further details.

Besides updating the costs of the assignments \( \Gamma(P, p^*, \lambda_j) \), we may also need to introduce another candidate range for \( p^* \). In particular, we need to introduce the range \( |p^* q| > |p^* s| \). To this end we need to compute the cost of the range assignment \( \Gamma(P, p^*, |p^* q|) \). After computing \( \text{pred}(q) \), \( \text{succ}(q) \), and the cost of \( O(1) \) chains—this can all be done in \( O(\log n) \) time using the global tree \( T_p \)—we can compute the cost of \( \Gamma(P, p^*, |p^* q|) \) in \( O(1) \) time. We then insert a leaf \( w \) for the range \( |p^* q| \) into \( T_p \), with \( f(|p^* q|) \) equal to the just computed cost. Finally, we update the values \( \text{min-cost}(v) \) and \( \text{best-range}(v) \) of the ancestors...
v of w whose the current value of \( \text{min-cost}(v) \) is larger than \( f([p^* q]) \). (Re-balancing \( T_{p^*} \), when necessary, can be done in a standard manner \cite[Chapter 15]{14}.)

The following lemma summarizes the discussion above.

**Lemma 17.** \( T_{p^*} \) can be updated in \( O(\log n) \) time per insertion and deletion.

**Putting it all together.** To summarize, upon the insertion of a new point \( q \) into \( P \), we first update each tree \( T_{p^*} \), as described above. This takes in \( O(\log n) \) time per tree, so \( O(n \log n) \) time in total. Then we update the global tree \( T_P \) in \( O(\log n) \) time. Finally, we create a tree \( T_q \) with \( q \) being the root-crossing point. This can be done in \( O(n \log n) \) time, by inserting the points from \( P \) one by one as described above. Thus inserting a new point \( q \) can be done in \( O(n \log n) \) time in total, after which we know the cost of the optimal solution for \( P \cup \{q\} \). Deletions can be handled in a similar manner, so we obtain the following theorem.

**Theorem 3.** An optimal solution to the broadcast range-assignment problem for a point set \( P \) in \( \mathbb{R}^1 \) can be maintained in \( O(n \log n) \) per insertion and deletion, where \( n \) is the number of points in the current set \( P \).

### B Missing proofs for Section 3

#### B.1 Proof of Lemma 4

**Lemma 4.** Consider a point set \( P \) and a point \( q \notin P \). Let \( \rho_{\text{old}}(p) \) be the range of a point \( p \) in \( \rho_k(P) \) and let \( \rho_{\text{new}}(p) \) be the range of \( p \) in \( \rho_k(P \cup \{q\}) \). Then

\[
|\{p \in P \cup \{q\} : \rho_{\text{new}}(p) > \rho_{\text{old}}(p)\}| \leq k + 3 \quad \text{and} \quad |\{p \in P \cup \{q\} : \rho_{\text{new}}(p) < \rho_{\text{old}}(p)\}| \leq k + 3.
\]

**Proof.** The range of a point \( p \in P \cup \{q\} \) can increase due to the insertion of \( q \) only if

(i) \( p = q \) and \( \rho_{\text{new}}(q) > 0 \), or
(ii) \( p \) is a zero-range point in \( \rho_k(P) \), or
(iii) \( p \) is the root-crossing point in \( \rho_k(P \cup \{q\}) \), or
(iv) the standard range of \( p \) increases due to the insertion of \( q \), or
(v) \( p = s \) and, out of the two standard ranges it has, \( s \) gets assigned a larger one in \( \rho_k(P \cup \{q\}) \) than in \( \rho_k(P) \).

Recall that we defined \( \rho_k \) such that the number of zero-range points is at most \( k \). Furthermore, at most one standard range can increase due to the insertion of \( q \), namely, the standard range of a point that is extreme in \( P \) but not in \( P \cup \{q\} \). When this happens, however, \( q \) is extreme in \( P \cup \{q\} \) and so \( \rho_{\text{new}}(q) = 0 \); this implies that cases (i) and (iv) cannot both happen. Hence, \( |\{p \in P \cup \{q\} : \rho_{\text{new}}(p) > \rho_{\text{old}}(p)\}| \leq k + 3 \).

The range of a point \( p \) can decrease only if

(i) \( p \) is a zero-range point in \( \rho_k(P \cup \{q\}) \), or
(ii) \( p \) is the root-crossing point in \( \rho_k(P) \), or
(iii) the standard range of \( p \) decreases due to the insertion of \( q \), or
(iv) \( p = s \) and, out of the two standard ranges it has, \( p \) gets assigned a smaller one in \( \rho_k(P \cup \{q\}) \) than in \( \rho_k(P) \).

Since the only point whose standard range decreases is the predecessor of \( q \) in \( P \), we conclude that \( |\{p \in P \cup \{q\} : \rho_{\text{new}}(p) < \rho_{\text{old}}(p)\}| \leq k + 3 \).

\[\blacksquare\]
B.2 Proof of Theorem 6

Theorem 6. There is a SAS for the dynamic broadcast range-assignment problem in $\mathbb{R}^1$ with stability parameter $k(\varepsilon) = O((1/\varepsilon)^{1/(\alpha-1)})$, where $\alpha > 1$ is the distance-power gradient. The time needed by the SAS to compute the new range assignment upon the insertion or deletion of a point is $O(n \log n)$, where $n$ is the number of points in the current set.

Proof. Our SAS maintains the canonical range assignment $\rho_k$ for $k = (2^{\alpha}/\varepsilon)^{1/(\alpha-1)} = O((1/\varepsilon)^{1/(\alpha-1)})$. We then have $\text{cost}_\alpha(\rho_k(P)) \leq (1 + \varepsilon) \cdot \rho_{\text{opt}}(P)$ by Lemma 5. Furthermore, the number of modified ranges when $P$ is updated is $2k + 6$ by Lemma 4. To determine the assignment $\rho_k$, we need to know an optimal assignment $\rho_{\text{opt}}$ with the structure from Theorem 2. Such an optimal assignment can be maintained in $O(n \log n)$ time per update, by Theorem 3. Once we have the new optimal assignment, the new optimal assignment can be determined in $O(n)$ time.

C 1-Stable, 2-Stable, and 3-Stable Algorithms in $\mathbb{R}^1$

In Section 3 we presented a $(2k + 6)$-stable algorithm whose approximation ratio is $1 + 2^{\alpha}/k^{\alpha-1}$, which provided us with a SAS. For small $k$ the algorithm is not very good: the most stable algorithm we can get is 6-stable, by setting $k = 0$. A more careful analysis shows that the approximation ratio of this 6-stable algorithm is 3, for $\alpha = 2$. In this section we study more stable algorithms. We first present a 1-stable $O(1)$-approximation algorithm; obviously, this is the best we can do in terms of stability. This algorithm can only handle insertions. (Deletions would be 3-stable.) We then present a straightforward 2-stable 2-approximation algorithm. Finally, we show that it is possible to get an approximation ratio strictly below 2 using a 3-stable algorithm; see Table 1 for an overview.

| $\ell$-stable Algorithm | Approximation Ratio | Remarks |
|-------------------------|---------------------|---------|
| $\ell = 1$              | $6 + 2\sqrt{5} \approx 10.47$ | $\alpha = 2$, insertions only |
| $\ell = 2$              | 2                   | for any $\alpha > 1$ |
| $\ell = 3$              | 1.97                | $\alpha = 2$ |

Table 1 An overview of the approximation ratio of 1-stable, 2-stable and 3-stable algorithms

C.1 A 1-stable insertion-only algorithm

We first describe our algorithm for the one-sided version of the problem, where all points in $P$ lie to the same side of the source. Let $P = \{s, p_1, \ldots, p_n\}$, where the points are numbered in order of increasing distance to the source. It will be convenient to define $p_0 := s$. Our algorithm maintains a range assignment $\rho$ that satisfies the following invariant.

- There is a path $\pi^*$ in $G_\rho(P)$ from $p_0$ to $p_n$ such that for each edge $(p_i, p_j)$ on the path we have $\rho(p_i) = |p_ip_j|$ and $i < j \leq i + 4$. For an edge $(p_i, p_j)$ on $\pi^*$, we call the subsequence $p_i, \ldots, p_j$ a block, and we denote it by $B[i, j]$.

- A point $p_k$ in a block $B[i, j]$ is a zero-range point, unless $B[i, j]$ consists of five points (including $p_i$ and $p_j$) of which $p_k$ is the middle one. In the latter case $\rho(p_k) = |p_kp_j|$.

Algorithm 1-STABLE-INSERT, presented below, shows how to insert a point $q$ into $P$. Figure 6 shows the life cycle of a block in the solution maintained by the algorithm.
Algorithm 1 1-Stable-Insert\((P, q)\)
1:▷ By default \(ρ(q) = 0\), so we only set \(ρ(q)\) when it receives a non-zero range.
2: if \(q\) is extreme then
3: Set \(ρ(\text{pred}(q)) := |\text{pred}(q)|\), thus creating a new block.
4: else
5: Let \(B[i, j]\) be the block containing \(q\), after the insertion of \(q\).
6: if \(B[i, j]\) consists of at most four points then
7: Do nothing.
8: else if \(B[i, j]\) consists of five points then
9: Set \(ρ(p_{\text{mid}}) := |p_{\text{mid}}|\), where \(p_{\text{mid}}\) is the middle point from \(B[i, j]\).
10: else
11: Let \(p_{\text{mid}} \not\in \{p_i, p_j\}\) be the point in \(B[i, j]\) with non-zero range.
12: Split the block \(B[i, j]\) by decreasing the range of \(p_i\) to \(|p_ip_{\text{mid}}|\).

It is readily verified that 1-Stable-Insert maintains the invariant—hence, the solution remains valid—and that it is 1-stable. We now analyze its approximation ratio.

Lemma 18. Algorithm 1-Stable-Insert maintains a \(c_α'\)-approximation of an optimal solution for the one-sided range-assignment problem in \(\mathbb{R}^1\), where the approximation ratio \(c_α'\) depends on the distance-power gradient \(α\). For \(α = 2\) the approximation ratio is \(c_2 = 3 + \sqrt{5}\).

Proof. The unique optimal solution for the one-sided problem on the current point set \(P = \{p_0, \ldots, p_n\}\) is the chain from \(p_0\) to \(p_n\), which has cost \(\sum_{i=0}^{n-1} |p_ip_{i+1}|^α\). This implies that the approximation ratio of the current range assignment \(ρ\) is bounded by the maximum, over all blocks \(B[i, j]\) in the current assignment, of the quantity \(\sum_{i=0}^{n-1} |p_ip_{i+1}|^α / \sum_{i=0}^{n-2} |p_ip_{i+2}|^α\), where the numerator gives the cost incurred by the algorithm on \(B[i, j]\) and the denominator gives the cost of the optimal solution on \(B[i, j]\).

To analyze this maximum, consider a block \(B[i, j]\) and assume without loss of generality that \(p_i = 0\) and \(p_j = 1\). Clearly, the maximum approximation ratio that can be achieved is when \(B[i, j]\) consists of five points; see the fourth block in Figure 6. Let \(p_{(i+j)/2}\), the middle point in \(B[i, j]\), be located at position \(x\), for some \(0 < x < 1\). Then the cost of the algorithm incurred on \(B[i, j]\) is \(1 + (1-x)^α\). The cost of the optimal solution on \(B[i, j]\) is minimized when the second point in the block is located at position \(x/2\) (this is true because \(α > 1\) and so for \(0 < y < x\) the function \(y^α + (x-y)^α\) is minimized at \(y = x/2\) and, similarly, when the fourth point is located at \((x+1)/2\). Hence,

\[
\frac{\sum_{t=i}^{j-1} |p_{t+1}|^α}{\sum_{t=i}^{j-1} |p_{t+1}|^α} = \frac{1 + (1-x)^α}{2(x/2)^α + 2((1-x)/2)^α} = \frac{2^{α-1} \cdot (1 + (1-x)^α)}{x^α + (1-x)^α}.
\]

Thus the approximation ratio is \(c_α' = \max_{0 ≤ x ≤ 1} \frac{2^{α-1} \cdot (1 + (1-x)^α)}{x^α + (1-x)^α}\). For \(α = 2\) this is maximized at \(x = (3 - \sqrt{5})/2\), giving an approximation ratio \(c_2 = 3 + \sqrt{5}\). ▶

The approximation ratio of Algorithm 1-Stable-Insert for the one-sided range assignment problem in \(\mathbb{R}^1\) is actually tight for \(α = 2\), as the next observation shows.

Figure 6 Life cycle of a block. At the last step, the block is split into two smaller blocks, which start in the middle of their life cycle: one block consists of three points, the other of four points.
Proposition 19. For the one-sided range-assignment problem, the approximation ratio of $3 + \sqrt{5}$ is tight for Algorithm 1-STABLE-INSERT.

Proof. Let $c = (3 - \sqrt{5})/2$, and consider the instance $P = \{s, p_1, p_2, p_3, p_4\}$ with $s = 0, p_1 = 1, p_2 = \frac{1}{2}, p_3 = c, p_4 = \frac{1+c}{2}$. The insertion order of the points is $p_1, p_2, p_3, p_4$.

Clearly, by setting $\rho(s) = \rho(p_2) = \frac{1}{2}$, and $\rho(p_3) = \rho(p_4) = \frac{1+c}{2}$, an optimum solution with $\text{cost}_2(\rho(P)) = \frac{5-2\sqrt{5}}{2}$ is found. Algorithm 1-STABLE-INSERT sets $\rho(s) := 1$ and $\rho(p_3) := 1-c$, and all other ranges to 0. The resulting cost equals $\frac{5-2\sqrt{5}}{2}$, and the ratio follows.

To handle the case with points to both sides of $s$, we proceed as follows. Let $P = L \cup \{s\} \cup R$, where $L$ and $R$ contain the points to the left and to the right of $s$, respectively. We simply run the above algorithm separately on $L \cup \{s\}$ and $\{s\} \cup R$. This way the points in $R \cup L$ are assigned one range, while $s$ gets assigned two ranges; the actual range of $s$ is the largest of these two ranges.

Theorem 20. There exists a 1-stable $c_\alpha$-approximation algorithm for the broadcast range-assignment problem in $\mathbb{R}^1$, where the approximation ratio $c_\alpha$ depends on the distance-power gradient $\alpha$. For $\alpha = 2$ the approximation ratio is $c_2 = 2(3 + \sqrt{5}) \approx 10.47$.

Proof. Recall that our algorithm simply runs the one-sided algorithm separately on $L \cup \{s\}$ and $\{s\} \cup R$, where the actual range of $s$ is defined to be the largest of the two ranges it receives.

To analyze the approximation ratio of this algorithm we use that for any $\alpha > 1$ we have $\text{OPT}(L \cup \{s\} \cup R) \geq \text{max}(\text{OPT}(L \cup \{s\}), \text{OPT}(\{s\} \cup R))$, where $\text{OPT}(\cdot)$ denotes the cost of an optimal range assignment $[13]$. Hence, the cost of the range assignment $\rho$ that we maintain is

$$\text{cost}_\alpha(\rho(L \cup \{s\} \cup R)) \leq \text{cost}_\alpha(\rho(L \cup \{s\})) + \text{cost}_\alpha(\rho(\{s\} \cup R))$$

$$\leq c'_\alpha \cdot (\text{OPT}(L \cup \{s\}) + \text{OPT}(\{s\} \cup R))$$

$$\leq 2c'_\alpha \cdot \text{max}(\text{OPT}(L \cup \{s\}), \text{OPT}(\{s\} \cup R))$$

$$\leq 2c'_\alpha \cdot \text{OPT}(L \cup \{s\} \cup R).$$

Lemma $[13]$ thus implies the theorem.

We note that a 1-stable algorithm $\text{ALG}$ that handles deletions cannot have a bounded approximation ratio, as we show next for $\alpha = 2$. Suppose for a contradiction that $\text{ALG}$ has approximation ratio $c$, where we assume for simplicity that $c$ is an integer. Let $P := \{s, r_1, \ldots, r_{c+1}\}$ where $s = 0$ and $r_i = i/(c+1)$. Then $\text{OPT} = (c+1) \cdot (1/(c+1))^2 = 1/(c+1)$, so $\text{ALG}$ cannot give the source a range of 1. But if we then delete all non-zero points in $P \setminus \{s\}$, the algorithm is stuck: the deletion of a non-zero point already causes a modification, so the algorithm is not allowed to increase any range; hence, the solution is invalid after all non-zero-range points from $P \setminus \{s\}$ have been deleted. One may argue that it is unfair that the algorithm has to pay for the deletion of a non-zero point. We defined it like this to keep the definition symmetric for deletions and insertions, where the algorithm also has to pay for assigning a non-zero range to a new point. It is then actually surprising that for insertions it is possible to obtain a 1-stable algorithm with $O(1)$-approximation ratio.

A lower bound for 1-stable algorithms

The next theorem shows that any 1-stable algorithm in $\mathbb{R}^1$ has an approximation ratio greater than 2.61 for $\alpha = 2$. 

Theorem 21. Any 1-stable algorithm for the dynamic broadcast range-assignment problem in \( \mathbb{R}^1 \) has an approximation ratio greater than or equal to \( \frac{1}{2} \cdot (3 + \sqrt{5}) \approx 2.61 \) for \( \alpha = 2 \), and any 1-stable algorithm has an approximation ratio greater than 2 for \( \alpha > 2 \).

Proof. Let \( \text{ALG} \) be a 1-stable algorithm, and let \( \rho_{\text{alg}} \) be the range assignment it maintains.

Consider the point set \( P := \{s, r_1, r_2, p\} \), where \( s = 0 \), and \( r_i = x_i \) for \( i = 1, 2 \), and \( p = 1 \). Assume \( 0 < x_1 < 1 \) and let \( x_2 = x_1 + \frac{1}{2} \). Also assume after the source, the point \( p \) arrives first, then the point \( r_1 \) and finally \( r_2 \) arrives. Let \( P' := \{s, r_1\} \). Trivially, after the arrival of the point \( p \), we must have \( \rho_{\text{alg}} \geq 1 \) in order to have a feasible solution. After the arrival of \( r_1 \), \( \text{ALG} \) is forced to keep \( \rho_{\text{alg}}(s) \geq 1 \) since \( \text{ALG} \) is 1-stable.

We consider two cases.

Case I: After the arrival of \( r_1 \), \( \text{ALG} \) gives a range of at least \( 1 - x_1 \) to \( r_1 \)

In this case \( \text{ALG} \) cannot decrease any range. So,

\[
\text{cost}_\alpha(\rho_{\text{alg}}(P')) \geq 1 + (1 - x_1)\alpha
\]

An optimal solution on \( P' \) has cost \( x_1^\alpha + (1 - x_1)\alpha \), and so the approximation ratio of \( \text{ALG} \) in Case I is at least \( \frac{1 + (1 - x_1)\alpha}{x_1^\alpha + (1 - x_1)\alpha} \).

Case II: After the arrival of \( r_1 \), \( \text{ALG} \) gives a range less than \( 1 - x_1 \) to \( r_1 \)

Now the point \( r_2 \) arrives. Since \( \text{ALG} \) is 1-stable it cannot decrease the range of the source, so,

\[
\text{cost}_\alpha(\rho_{\text{alg}}(P')) \geq 1
\]

An optimal solution on \( P \) has cost \( x_1^\alpha + 2 \cdot \left(\frac{1 - x_1}{2}\right)\alpha \), and so the approximation ratio of \( \text{ALG} \) in Case II is at least \( x_1^\alpha + 2 \cdot \left(\frac{1 - x_1}{2}\right)^\alpha \).

We conclude that the approximation ratio of any 1-stable algorithm is greater than or equal to at least

\[
\min\left(\frac{1 + (1 - x_1)\alpha}{x_1^\alpha + (1 - x_1)\alpha}, \frac{1}{x_1^\alpha + 2 \cdot \left(\frac{1 - x_1}{2}\right)^\alpha}\right).
\]

For \( \alpha = 2 \) we see that by substituting \( x_1 = \frac{1}{2} - \frac{\sqrt{5}}{2} \) we get the approximation ratio is at least \( \frac{1}{2} \cdot (3 + \sqrt{5}) \). Moreover, for any \( \alpha > 2 \) we get an approximation ratio greater then 2, for instance by substituting \( x_1 = \frac{1}{2} \).

\[\square\]

C.2 A 2-stable algorithm

Obtaining a 2-stable 2-approximation algorithm is straightforward: simply give every point in \( P \) its standard range, where the source \( s \) receives the largest of its (at most) two standard ranges. This induces a broadcast tree consisting of (at most) two chains: a chain from \( s \) to the rightmost point and a chain from \( s \) to the leftmost point. This algorithm is 2-stable: if we insert an extreme point then we increase the range of at most one point, and if we insert a non-extreme point \( q \) we increase the range of \( q \) and decrease the range of its predecessor. (Deletions are symmetrical.) We call this algorithm the standard-range algorithm. It is easy show that the standard-range algorithm gives a 2-approximation \cite{18}.

Observation 22. The standard-range algorithm is a 2-stable 2-approximation algorithm for the dynamic broadcast range-assignment problem in \( \mathbb{R}^1 \), for any power-distance gradient \( \alpha > 1 \). Moreover, the approximation ratio of 2 is tight for this algorithm.
Proof. The fact that the approximation ratio is at most 2 was observed in [18]. For completeness, we give an instance showing this bound is tight.

Define \( P := P(\varepsilon) \cup \{s\} \), where \( s = 0 \) is the source, and \( P(\varepsilon) := \{p_1, p_2, p_3, p_4\} \), where \( p_1 = \varepsilon, p_2 = -\varepsilon, p_3 = 1, \) and \( p_4 = -1 \) for some small \( \varepsilon > 0 \). The insertion order of the points in \( P(\varepsilon) \) is \( p_1, p_2, p_3, p_4 \). Clearly, by setting \( \rho(s) = 1 \) and \( \rho(p_i) = 0 \) for \( i = 1, \ldots, 4 \), we obtain an optimal solution with cost \( \alpha(P) = 1 \). However, the standard-range algorithm will set \( \rho(s) = \varepsilon, \rho(p_1) = \rho(p_2) = 1 - \varepsilon \), and \( \rho(p_3) = \rho(p_4) = 0 \), leading to a solution with cost \( f(\varepsilon) := 2(1 - \varepsilon)^n + \varepsilon^n \). Since \( \lim_{\varepsilon \to 0} f(\varepsilon) = 2 \), this proves we have tightness for any \( \alpha > 1 \).

C.3 A 3-stable algorithm for \( \alpha = 2 \) with approximation ratio less than 2

Given the simplicity of our 2-stable 2-approximation algorithm, it is surprisingly difficult to obtain an approximation ratio strictly smaller than 2. In fact, we have not been able to do this with a 2-stable algorithm. Below we show this is possible with a 3-stable algorithm, at least for the case \( \alpha = 2 \), which we assume from now on.

Recall that for any set \( P \) with points on both sides of the source point \( s \), there is an optimal range assignment inducing a broadcast tree with a single root-crossing point; see Figure [1]. Unfortunately the root-crossing point may change when \( P \) is updated. This may cause many changes if we maintain a solution with a good approximation ratio and the same root-crossing point as the optimal solution. We therefore restrict ourselves to source-based range assignments, where \( s \) is the root-crossing point. The main question is then how large the range of \( s \) should be, and which points within range of \( s \) should be zero-range points.

We now define our source-based range assignment, which we denote by \( \rho_{sb} \), more precisely. It will be uniquely defined by the set \( P \); it does not depend on the order in which points have been inserted or deleted. Let \( \delta \) be a parameter with \( 1/2 < \delta < 1 \); later we will choose \( \delta \) such that the approximation ratio of our algorithm is optimized. We call a point \( p \in P \setminus \{s\} \) expensive if \( \text{succ}(p) \neq \text{NIL} \) and \( |p \cdot \text{succ}(p)| > \delta \cdot |s \cdot \text{succ}(p)| \), and we call it cheap otherwise. The source \( s \) is defined to be always expensive. (This is consistent in the sense that for \( p = s \), the condition \( |p \cdot \text{succ}(p)| > \delta \cdot |s \cdot \text{succ}(p)| \) holds for both successors, since \( \delta < 1 \).) We denote the set of all expensive points in \( P \) by \( P_{\text{exp}} \) and the set of all cheap points by \( P_{\text{cheap}} \). Define \( d_{\text{max}} := \max\{|s \cdot \text{succ}(p)| : p \in P_{\text{exp}}\} \), that is, \( d_{\text{max}} \) is the maximum distance from \( s \) to the successor of any expensive point. We say that a point \( p \in P_{\text{exp}} \) is crucial if \( |s \cdot \text{succ}(p)| = d_{\text{max}} \). Typically there is a single crucial point, but there can also be two: one on the left and one on the right of \( s \). Our source-based range assignment \( \rho_{sb} \) is now defined as follows.

\[
\begin{align*}
\rho_{sb}(s) &:= d_{\text{max}}, \\
\rho_{sb}(p) &:= 0 \text{ for all } p \in P_{\text{exp}} \setminus \{s\}, \text{ and} \\
\rho_{sb}(p) &:= \rho_{st}(p) \text{ for all } p \in P_{\text{cheap}} \text{, where } \rho_{st}(p) \text{ denotes the standard range of a point.}
\end{align*}
\]

The next lemma, whose proof is in the appendix, analyzes the stability of \( \rho_{sb} \). The lemma implies that insertions are \((2,1)\)-stable and deletions are \((1,2)\)-stable.

\[\textbf{Lemma 23.} \text{ Consider a point set } P \text{ and a point } q \notin P. \text{ Let } \rho_{old}(p) \text{ be the range of a point } p \text{ in } \rho_{sb}(P) \text{ and let } \rho_{new}(p) \text{ be the range of } p \text{ in } \rho_{sb}(P \cup \{q\}). \text{ Then} \]

\[\left| \{p \in P \cup \{q\} : \rho_{old}(p) < \rho_{new}(p)\} \right| \leq 2 \text{ and } \left| \{p \in P \cup \{q\} : \rho_{old}(p) > \rho_{new}(p)\} \right| \leq 1.\]

\[\textbf{Proof.} \text{ Due to the insertion of } q, \text{ five types of range modifications can happen.}
\]

\[\begin{align*}
&\text{(i) The point } q \text{ may get a non-zero range because it is cheap and non-extreme.}
\end{align*}\]
(ii) A point \( p \) may move from \( P_{\text{cheap}} \) to \( P_{\text{exp}} \) and become a zero-range point. This can only happen when \( p = \text{pred}(q) \) and \( p \) was extreme before the insertion of \( q \). Hence, \( q \) will be extreme after its insertion, so this cannot occur together with type (i).

(iii) A point \( p \in P_{\text{cheap}} \) may get a smaller range because its standard range decreases. This can only happen when \( p = \text{pred}(q) \), and so it cannot happen together with type (ii).

(iv) A point \( p \) may move from \( P_{\text{exp}} \) to \( P_{\text{cheap}} \) and get a non-zero range. Again, this can only happen when \( p = \text{pred}(q) \), so this cannot happen together with types (ii) or (iii).

(v) The source \( s \) may get a different range because \( d_{\text{max}} \) changes. If \( d_{\text{max}} \) decreases, then \( \text{pred}(q) \) must have been crucial, and so this cannot occur together with types (ii) or (iii). If \( d_{\text{max}} \) increases, then a type (ii) modification must have occurred, which means that types (i), (iii), and (iv) did not occur.

Overall, we have at most one range increase of type (i), at most one range change from any of the types (ii), (iii), (iv), and at most one change of type (v). There can be at most one decrease among these three changes, because if type (v) is a decrease then types (ii) and (iii) did not occur. Finally, there can be at most two increases, because if type (v) is an increase then types (i), (iii), and (iv) did not occur.

From now on we assume without loss of generality that the source \( s \) is located at \( x = 0 \). We will need the following lemma before we can proceed to prove the performance guarantee.

\[ \text{Lemma 24.} \text{ Let } I \subset \mathbb{R}^1 \text{ be an interval of length } \Delta_1 \text{ at distance } \Delta_2 \text{ from the source, that is, } I = [\Delta_2, \Delta_2 + \Delta_1] \text{ or } I = [-\Delta_1 - \Delta_2, -\Delta_2], \text{ for some } \Delta_2 > 0. \text{ Let } P_{\text{cheap}}(I) \text{ be the set of all cheap points that are in } I \text{ and whose successor lies in } I \text{ as well. Then } \sum_{p \in P_{\text{cheap}}(I)} \rho_{st}(p)^2 \leq \delta(\Delta_1 + \Delta_2)\Delta_1. \]

\[ \text{Proof.} \text{ Let } p \in P_{\text{cheap}}(I). \text{ If } p \text{ is extreme we can ignore } p \text{ because } \rho_{st}(p) = 0, \text{ so assume } p \text{ is not extreme. Since } p \text{ is cheap we then have } \]
\[ \rho_{st}(p) = |p \text{ succ}(p)| \leq \delta \cdot |s \text{ succ}(p)|. \]

Since \( \text{succ}(p) \in I \) we have \( |s \text{ succ}(p)| \leq \Delta_1 + \Delta_2 \), and so \( \rho_{st}(p) \leq \delta(\Delta_2 + \Delta_1) \). Hence,
\[ \sum_{p \in P_{\text{cheap}}(I)} \rho_{st}(p)^2 \leq \delta(\Delta_2 + \Delta_1) \cdot \sum_{p \in P_{\text{cheap}}(I)} \rho_{st}(p) \leq \delta(\Delta_2 + \Delta_1)\Delta_1. \]

We now prove the approximation ratio of our source-based range assignment.

\[ \text{Lemma 25.} \text{ For any point set } P \text{ in } \mathbb{R}^1 \text{ and any } 1/2 < \delta < 1 \text{ we have } \]
\[ \text{cost}_2(\rho_{sb}(P)) \leq c_5 \cdot \text{OPT}, \text{ where } c_5 := \max \left(1 + \delta + \frac{(1 + 5\delta)(1 - \delta)^2}{\delta^2}, \frac{1}{2} + \frac{1}{2}\right) \]
and \( \text{OPT} = \text{cost}_2(\rho_{opt}(P)) \) is the cost of the optimal range assignment on \( P \).

\[ \text{Proof.} \text{ The worst approximation ratio is achieved by a set } P \text{ with points to both sides of the source—indeed, if we have only points to the right of } s, \text{ say, then adding an additional point slightly to the left of } s \text{ will change neither the cost of an optimal solution nor the cost of } \rho_{sb}. \text{ So from now on we assume that } P \text{ has points to both sides of } s. \text{ In the following, with a slight abuse of notation, we will for a point } p \in P \text{ use the notation } p \text{ for the “object” } p \text{ and to its value (that is, its } x\text{-coordinate, if we identify } \mathbb{R}^1 \text{ with the } x\text{-axis). For example, to indicate that a point } p \text{ lies to the left of another point } p' \text{ we may write } p < p'. \text{ We will assume without loss of generality that } s = 0. \]
Case 1: We now bound the cost of the points without loss of generality that in the region left of and including \( q \), that is, |q\ell| = \( \rho_{\text{opt}}(q) \) and/or |qr| = \( \rho_{\text{opt}}(q) \)—in fact, one of these cases must happen.

Let \( \rho_{\text{opt}}(P) \) be an optimal range assignment that induces a broadcast tree \( B \) with the structure of Theorem 2 and let \( q \) denote the root-crossing point in \( B \). With a slight abuse of notation, let \( p_i^* \) denote a crucial point in \( P \), and let \( p_{i+1}^* = \text{succ}(p_i^*) \). If \( p_i^* = s \) then we define \( p_{i+1}^* \) to be a successor of \( s \) at maximum distance from \( s \), so that also in this case we have \( d_{\text{max}} = |sp_{i+1}^*| \). Thus \( \rho_{sb}(s) = d_{\text{max}} = |sp_{i+1}^*| \).

Let \( \ell \) and \( r \) be the leftmost and the rightmost points that are within range of \( q \) in the optimal solution, respectively. Let \( P_{\text{left}} := \{ p \in P : p \leq \ell \} \) be the set of all points to the left of \( \ell \) plus \( \ell \) itself, and let \( P_{\text{right}} := \{ p \in P : p \geq r \} \). Finally, let \( P_{\text{mid}} \) be the set of points in between \( s \) and \( q \), excluding both \( s \) and \( q \); see Figure 7. We now define

\[
C_{sb} := \sum_{p \in P_{\text{left}} \cup P_{\text{mid}} \cup P_{\text{right}}} \rho_{sb}(p)^2 \quad \text{and} \quad C_{\text{opt}} := \sum_{p \in P_{\text{left}} \cup P_{\text{mid}} \cup P_{\text{right}}} \rho_{\text{opt}}(p)^2
\]

as the costs incurred by \( \rho_{sb} \) and \( \rho_{\text{opt}} \) on the sets just defined. Observe that \( C_{\text{opt}} \geq C_{sb} \), because \( \rho_{\text{opt}}(p) = \rho_{sb}(p) \) for all \( p \in P_{\text{left}} \cup P_{\text{mid}} \cup P_{\text{right}} \), and \( \rho_{sb}(p) \leq \rho_{\text{opt}}(p) \) for all \( p \in P \setminus \{s\} \).

We now analyze the costs incurred by \( \rho_{sb} \) and \( \rho_{\text{opt}} \) on the remaining points. We assume without loss of generality that \( q \leq s \), and we let \( x := |qs| \) denote the distance from \( q \) to \( s \). Furthermore, we define \( z := \rho_{\text{opt}}(q) \). Note that \( z \geq x \). We divide the analysis into several cases, depending on the relative position of \( s, q, p_i^*, p_{i+1}^* \).

Case 1: \( p_i^* \) and \( p_{i+1}^* \) lie to the right of \( s \) (possibly \( p_i^* = s \)) and inside the range of \( q \) in the optimal solution.

See Figure 8 for an illustration. Since \( p_i^* \) is crucial we have \( |p_i^*p_{i+1}^*| > \delta \cdot |sp_{i+1}^*| \) and so

\[
|sp_i^*| = |sp_{i+1}^*| - |p_i^*p_{i+1}^*| < \left(1 - \frac{1}{\delta} - 1\right) |p_i^*p_{i+1}^*| = \left(1 - \frac{1 - \delta}{\delta}\right) |p_i^*p_{i+1}^*| \leq \left(1 - \frac{1 - \delta}{\delta}\right)(z - x).
\]

We now bound the cost of the points \( p \notin P_{\text{left}} \cup P_{\text{mid}} \cup P_{\text{right}} \). These are the points \( s, q, p_i^*, p_{i+1}^* \) plus the points in the red regions in Figure 8. Note that \( \rho_{sb}(p_i^*) = 0 \).

By applying Lemma 24 with \( \Delta_1 = \left(1 - \frac{1 - \delta}{\delta}\right)(z - x) \) and \( \Delta_2 = 0 \), we see that the cost incurred by \( \rho_{sb} \) due to the points strictly in between \( s \) and \( p_i^* \) is less than or equal to \( \frac{(1 - \delta)^2}{\delta}(z - x)^2 \). By applying Lemma 24 with \( \Delta_1 = z \) and \( \Delta_2 = x \), the cost incurred by \( \rho_{sb} \) due to the points in the region left of and including \( q \) and within the range of \( q \) is at most \( \delta z(z + x) \). Finally, the cost incurred by \( \rho_{sb} \) due to the points to the right of and including \( p_{i+1}^* \) and within the
range of $q$ is at most $(z - x - |sp^*_i|)^2$. Since $\rho_{sb}(s) = |sp^*_i|$ we obtain

$$\text{cost}_2(\rho_{sb}(P)) \leq |sp^*_i|^2 + \delta z(z + x) + \frac{(1 - \delta)^2}{\delta} (z - x)^2 + (z - x - |sp^*_i|)^2 + C_{sb}.$$

Obviously, $\text{cost}_2(\rho_{opt}(P)) > z^2 + C_{opt}$. Since $C_{sb} \leq C_{opt}$ and $x \leq z$ we conclude

$$\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{(z - x - |sp^*_i|)^2 + |sp^*_i|^2 + \delta z + (1 - \delta)^2}{z^2 + \delta z + (1 - \delta)^2}$$

$$\leq 1 + \frac{1}{\delta} + \frac{(1 - \delta)^2}{\delta} < c_\delta.$$

**Case 2:** $p^*_i$ lies to the left of $q$ but within the range of $q$ and $p^*_{i+1}$ lies outside the range of $q$.

As before, since $p^*_i$ is crucial we have $|p^*_i p^*_{i+1}| > \delta \cdot |sp^*_i| + 1$ and so

$$|sp^*_i| = |sp^*_i| - |p^*_i p^*_{i+1}| < \left(\frac{1 - \delta}{\delta}\right)|p^*_i p^*_{i+1}|.$$

Applying Lemma 24 with $\Delta_1 + \Delta_2 = |sp^*_i|$ and $\Delta_1 = |qp^*_i| \leq z$, we see that the cost incurred by $\rho_{sb}$ due to the points in the region between and including $q$ and $p^*_i$ is at most $(1 - \delta)|p^*_i p^*_{i+1}|z$. Applying Lemma 24 with $\Delta_1 = z - x$ and $\Delta_2 = 0$, we see that the cost incurred by $\rho_{sb}$ due to the points in the region to the right of $s$ and within the range of $q$ is at most $z(z - x)^2$. Furthermore, the cost incurred by $\rho_{sb}$ due to the range of $s$ is equal to $|sp^*_i|^2$.

Let $C_{opt} := C_{opt} - |p^*_i p^*_{i+1}|^2$. Note that in Case 2, $p^*_i$ is the leftmost point in the range of $q$ (the point $\ell$ in Figure 7) and so it is included in $P_{left}$. Since $\rho_{sb}(p^*_i) = 0$ this implies $C_{opt}^* \geq C_{sb}$. Hence,

$$\text{cost}_2(\rho_{opt}(P)) > z^2 + |p^*_i p^*_{i+1}|^2 + C_{opt}^*.$$

Moreover,

$$\text{cost}_2(\rho_{sb}(P)) = |sp^*_i|^2 + (1 - \delta)|p^*_i p^*_{i+1}|z + \delta(z - x)^2 + C_{sb}.$$

Since $C_{sb} \leq C_{opt}^*$ we thus get

$$\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|sp^*_i|^2 + (1 - \delta)|p^*_i p^*_{i+1}|z + \delta(z - x)^2}{z^2 + |p^*_i p^*_{i+1}|^2}.$$

Now if $|p^*_i p^*_{i+1}| \geq z$, and using that $x \leq z$, we find:

$$\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|sp^*_i|^2}{|p^*_i p^*_{i+1}|^2} + \frac{(1 - \delta)|p^*_i p^*_{i+1}|z}{z^2 + |p^*_i p^*_{i+1}|^2} + \frac{\delta(z - x)^2}{2z^2}$$

$$\leq \frac{1}{z^2} + \frac{1}{2} + \frac{\delta}{2}$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{1}{2z^2}.$$
On the other hand, if \(|p_i^*| < z\) we have \(|sp_{i+1}^*| < |p_{i+1}^*|/\delta < z/\delta\) and so we get

\[
\frac{\text{cost}_2(\rho_{ab}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|p_{i+1}^*|^2}{z^2} + \frac{(1-\delta)|p_{i+1}^*|z}{z^2} + \frac{\delta(z-x)^2}{z^2} \\
< \frac{|p_{i+1}^*|^2}{(z^2+|p_{i+1}^*|^2)\delta z} + \frac{(1-\delta)|p_{i+1}^*|z}{z^2} + \frac{\delta(z-x)^2}{z^2} \\
\leq \frac{1}{2\delta^2} + \frac{1-\delta}{2\delta} + \delta.
\]

So we have,

\[
\frac{\text{cost}_2(\rho_{ab}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \max \left( \frac{1}{2} + \frac{1}{\delta^2}, \frac{1}{2\delta^2} + \frac{1-\delta}{2\delta} + \delta \right) = \frac{1}{\delta^2} + \frac{1}{2} \leq c_5.
\]

Case 3: \(p_i^*, p_{i+1}^*\) lie to the left of \(q\) and inside the range of \(q\).

As before, since \(p_i^*\) is crucial we have \(|sp_i^*| < (\frac{1}{\delta^2})|p_i^*|p_{i+1}^*|\), which in Case 3 implies

\[
\text{Figure 10} \text{ Relative position of the points } s, q, p_i^*, p_{i+1}^* \text{ in Case 3. Costs of the points in the green regions are included in } C_{ab} \text{ and } C_{opt}. \text{ The cost for the other regions is analyzed in the text.}
\]

\[x < \frac{(1-\delta)}{\delta} z \text{ and } |qp_i^*| < \frac{(1-\delta)}{\delta} z. \text{ Applying Lemma 24 with } \Delta_1 = |qp_i^*| \text{ and } \Delta_2 = x, \text{ we see that the cost incurred by } \rho_{ab} \text{ due to the points in the region between including } q \text{ and } p_i^* \text{ is at most } \delta \cdot z(x + \frac{(1-\delta)}{\delta} z) \cdot \frac{(1-\delta)}{\delta}. \text{ Applying Lemma 24 with } \Delta_1 = z - x \text{ and } \Delta_2 = 0, \text{ we see that the cost incurred by } \rho_{ab} \text{ due to the points in the region right of } s \text{ and within the range of } q \text{ is at most } \delta(z - x)^2. \text{ Finally, the cost incurred by } \rho_{ab} \text{ due to the points in the region left of } p_{i+1}^* \text{ and within the range of } q \text{ is at most } (z + x - |sp_{i+1}^*|)^2. \text{ Hence,}
\]

\[
\text{cost}_2(\rho_{ab}(P)) > z^2 + C_{opt}
\]

and

\[
\text{cost}_2(\rho_{ab}(P)) \leq |sp_{i+1}^*|^2 + \delta(z-x)^2 + \delta \cdot \frac{(1-\delta)}{\delta} z(x + \frac{(1-\delta)}{\delta} z) + (z + x - |sp_{i+1}^*|)^2 + C_{ab},
\]

Since \(C_{ab} \leq C_{opt}\) this implies

\[
\frac{\text{cost}_2(\rho_{ab}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|sp_{i+1}^*|^2 + (z+x-|sp_{i+1}^*|)^2 + \delta(z-x)^2 + \delta \cdot \frac{(1-\delta)}{\delta} z(x + \frac{(1-\delta)}{\delta} z)}{z^2} \\
= \frac{(z+x)^2 + \delta(z-x)^2 + (1-\delta)zx + (\frac{(1-\delta)}{\delta})^2 z^2}{z^2} \quad \text{(note that } |sp_{i+1}^*|^2 + (z + x - |sp_{i+1}^*|)^2 < (z + x)^2) \\
= \frac{(1+\delta)(\frac{(1-\delta)}{\delta})^2 z^2 + (3-3\delta)x + (1+\delta)x^2}{z^2} \\
= (1+\delta) \frac{(1-\delta)^2}{\delta^2} + (3-3\delta)\frac{x}{z} + 1 + \delta + \frac{(1-\delta)^2}{\delta}.
\]

Now define \(f(y) := (1+\delta)y^2 + (3-3\delta)y + 1 + \delta + \frac{(1-\delta)^2}{\delta}\), then the last term is equal to \(f(x/z)\). Observe that \(f\) is quadratic in \(y\) and that \(0 \leq x/z \leq (1-\delta)/\delta\). Hence,

\[
\frac{\text{cost}_2(\rho_{ab}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \max (f(0), f((1-\delta)/\delta)) \\
= \max (1+\delta + \frac{(1-\delta)^2}{\delta}, 1+\delta + (1+5\delta)\frac{(1-\delta)^2}{\delta}) \\
= 1+\delta + (1+5\delta)\frac{(1-\delta)^2}{\delta} \\
\leq c_6.
\]
Case 4: $p_i^*, p_{i+1}^*$ lie to the left of $q$ but outside the range of $q$; or $p_i^*, p_{i+1}^*$ lie in the region $[s, q]$.

Since $p_i^*$ is crucial we have $|sp_{i+1}^*| < |p_i^*p_{i+1}^*|/\delta$. Clearly the cost incurred by $\rho_{sb}$ due to

$$\text{cost}_2(\rho_{opt}(P)) > z^2 + |p_i^*p_{i+1}^*| + C_{opt}^*$$

and

$$\text{cost}_2(\rho_{sb}(P)) \leq |sp_{i+1}^*|^2 + z^2 + \delta(z - x)^2 + C_{sb}$$

with $C_{sb} \leq C_{opt}^*$. Since $x \leq z$ we thus obtain

$$\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|sp_{i+1}^*|^2 + z^2 + \delta(z - x)^2}{z^2 + |p_i^*p_{i+1}^*|^2} \leq \frac{1}{\delta^2} $$

Case 5: $p_i^*$ lies to the right of $s$ and within the range of $q$ and $p_{i+1}^*$ lies to the right of $s$ and outside the range of $q$; or $p_i^*, p_{i+1}^*$ lie to the right of $s$ and outside the range of $q$.

As before, we have $|sp_i^*| < \frac{(1-\delta)}{\delta}|p_i^*p_{i+1}^*|$ and $|sp_{i+1}^*| < \frac{1}{\delta}|p_i^*p_{i+1}^*|$. Clearly the cost incurred

$$\text{cost}_2(\rho_{opt}(P)) > z^2 + |p_i^*p_{i+1}^*| + C_{opt}^*$$

and

$$\text{cost}_2(\rho_{sb}(P)) \leq |sp_{i+1}^*|^2 + z^2 + \delta(z - x)^2 + C_{sb}$$

with $C_{sb} \leq C_{opt}^*$. Since $x \leq z$ we thus obtain

$$\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{opt}(P))} \leq \frac{|sp_{i+1}^*|^2 + z^2 + \delta(z - x)^2}{z^2 + |p_i^*p_{i+1}^*|^2} \leq \max \left(1 + \delta, \frac{1}{\delta^2}\right) < c\delta.$$
which is most \( \frac{1}{2} |p_i^* p_{i+1}^*|^2 \). Define \( C_{\text{opt}}^* := C_{\text{opt}} - |p_i^* p_{i+1}^*|^2 \) and observe that \( C_{\text{opt}}^* \geq C_{\text{sb}} \). Then

\[
\text{cost}_2(\rho_{\text{opt}}(P)) > z^2 + |p_i^* p_{i+1}^*|^2 + C_{\text{opt}}^*
\]

and

\[
\text{cost}_2(\rho_{sb}(P)) \leq |s p_{i+1}^*|^2 + z^2 + \frac{(1-\delta)^2}{\delta} |p_i^* p_{i+1}^*|^2 + C_{\text{sb}}
\]

Since \( C_{\text{sb}} \leq C_{\text{opt}}^* \) and \( \delta < 1 \) we conclude

\[
\frac{\text{cost}_2(\rho_{sb}(P))}{\text{cost}_2(\rho_{\text{opt}}(P))} \leq \frac{|s p_{i+1}^*|^2 + z^2 + \frac{(1-\delta)^2}{\delta} |p_i^* p_{i+1}^*|^2}{z^2 + |p_i^* p_{i+1}^*|^2}
\]

\[
\leq \frac{(1+\delta)^2 |p_i^* p_{i+1}^*|^2 + z^2}{z^2 + |p_i^* p_{i+1}^*|^2}
\]

\[
\leq \frac{(1-\delta)^2}{\delta} + \frac{1}{\Delta^2}
\]

\[
< c_\delta.
\]

We conclude that the approximation ratio is bounded by \( c_\delta \) in all cases. ▶

We now want to choose \( \delta \) so as to minimize \( c_\delta = \max \left( 1 + \delta + \frac{(1+5\delta)(1-\delta)^2}{\delta}, \frac{1}{\Delta} + \frac{1}{\Delta} \right) \). The first term is minimized at the real root of the polynomial \( 6\delta^3 - 3\delta - 2 \), whose approximate value is 0.92711; this gives a value that is approximately 1.97. For this value of \( \delta \) the first term dominates the second one, leading to the following theorem.

▶ Theorem 26. There exists a 3-stable 1.97-approximation algorithm for the dynamic broadcast range-assignment problem in \( \mathbb{R}^1 \) for \( \alpha = 2 \).

D Missing proofs for Section 4

D.1 The structure of an optimal solution in \( S^1 \)

In this section we will prove Lemma 8 which states that for any instance in \( S^1 \), there exists a point \( r \in S^1 \) that is not within range of any point. Hence, \( S^1 \) can be cut at \( r \) to obtain an equivalent instance in \( \mathbb{R}^1 \).

Without loss of generality we identify \( S^1 \) with a circle of perimeter 1. Let \( \rho_{\text{opt}} \) be a fixed optimal range-assignment on \( P \). We will need the following simple lemma.

▶ Lemma 27. If \( |P| > 2 \) then \( \rho_{\text{opt}}(p) < \frac{1}{2} \) for all \( p \in P \).

**Proof.** Note that setting \( \rho(s) = \frac{1}{2} \) and \( \rho(p) = 0 \) for all \( p \in P \setminus \{ s \} \) gives a feasible solution. Since \( \rho(s) > 0 \) in any feasible solution, this means that \( \rho_{\text{opt}}(p) < \frac{1}{2} \) for all \( p \neq s \). Hence, it suffices to show that \( \rho_{\text{opt}}(s) < \frac{1}{2} \). If there is no point \( p \in P \) which is diametrically opposite \( s \) then clearly \( \rho_{\text{opt}}(s) < \frac{1}{2} \). Now suppose there is a point \( p \in P \) that lies diametrically opposite \( s \). Let \( q \in P \setminus \{ s, p \} \) be a point that maximizes the distance from \( s \) among all points in \( P \setminus \{ s, p \} \). The point \( q \) exists since \( |P| > 2 \). Note that \( d(s, q) + d(q, p) = \frac{1}{2} \). Hence, setting \( \rho(s) = d(s, q) \) and \( \rho(q) = d(q, p) \) (and keeping all other ranges zero) gives a solution of cost \( d(s, q)^\alpha + d(q, p)^\alpha \), which is less than \( \left( \frac{1}{2} \right)^\alpha \) since \( \alpha > 1 \). Thus \( \rho_{\text{opt}}(s) < \frac{1}{2} \), which finishes the proof. ▶
Before we proceed, we introduce some more notation.

For two points \( p, q \in S^1 \), we let \([p, q]^{cw} \subset S^1\) denote the closed clockwise interval from \( p \) to \( q \). In other words, \([p, q]^{cw}\) is the clockwise arc along \( S^1 \) from \( p \) to \( q \), including its endpoints. Furthermore, we define \((p, q)^{cw}\) to be the open clockwise interval from \( p \) to \( q \). The intervals \([p, q]^{cw}\) and \((p, q)^{cw}\) are defined similarly, but for the counterclockwise direction.

Now consider a directed edge \((p, q)\) in a communication graph \( G_\rho(P) \). We say that \((p, q)\) is a clockwise edge if \( \rho(p) \geq d_{cw}(p, q) \), and we say that it is a counterclockwise edge if \( \rho(p) \geq d_{ccw}(p, q) \). Observe that Lemma 27 implies that an edge cannot be both clockwise and counterclockwise in an optimal range assignment, assuming \(|P| > 2\). Finally, we define the \textit{covered region} of a subset \( Q \subseteq P \) with respect to a range assignment \( \rho \) to be the set of all points \( r \in S^1 \) such that there exists a point \( p \in Q \) such that \( \rho(p) \geq d(p, r) \). We denote this region by \( \text{cov}(\rho, Q) \). Furthermore, the \textit{counterclockwise covered region} of \( Q \), denoted by \( \text{cov}_{ccw}(\rho, Q) \), is the set of all points \( r \in S^1 \) such that there exists a point \( p \in Q \) such that \( \rho(p) \geq d_{ccw}(p, r) \). The \textit{clockwise covered region} of \( Q \), denoted by \( \text{cov}_{cw}(\rho, Q) \), is defined similarly.

We can now state the main lemma of this section.

\textbf{Lemma 8.} Let \( P \) be a point set in \( S^1 \) with \(|P| > 2\) and let \( \rho_{\text{opt}} \) be an optimal range assignment for \( P \). Then there exists a point \( r \in S^1 \) such that \( r \notin \text{cov}(\rho_{\text{opt}}, P) \).

\textbf{Proof.} Let \( d_{\text{hop}}(p, q) \) denote the hop distance from \( p \) to \( q \) in the communication graph \( G_{\rho_{\text{opt}}}(P) \). Let \( B \) be a shortest-path tree rooted at \( s \) in \( G_{\rho_{\text{opt}}}(P) \) with the following properties.

- \( B \) is a shortest-path tree in terms of hop distance, that is, the hop-distance from \( s \) to any point \( p \) in \( B \) is equal to \( d_{\text{hop}}(s, p) \).
- Among all such shortest-path trees, \( B \) maximizes the number of clockwise edges.

For two points \( p, q \in P \), let \( \pi(p, q) \) denote the path from \( p \) to \( q \) in \( B \), and let \(|\pi(p, q)|\) be its length, that is, the number of edges on the path. Note that \(|\pi(s, p)| = d_{\text{hop}}(s, p)\) for any \( p \in P \). Let \( \text{pa}(p) \) denote the parent of a point \( p \) in \( B \) and define

\[
S_{cw} = \{p \in P \setminus \{s\} : (\text{pa}(p), p) \text{ is a clockwise edge}\}
\]

and

\[
S_{ccw} = \{p \in P \setminus \{s\} : (\text{pa}(p), p) \text{ is a counterclockwise edge}\}
\]

Note that \( S_{cw} \cup S_{ccw} = P \setminus \{s\} \). Now define

\[
q_{cw} = \text{the point from } S_{cw} \text{ that maximizes } d_{cw}(s, p),
\]

where \( q_{cw} = s \) if \( S_{cw} = \emptyset \). Similarly, define

\[
q_{ccw} = \text{the point from } S_{ccw} \text{ that maximizes } d_{ccw}(s, p),
\]

where \( q_{ccw} = s \) if \( S_{ccw} = \emptyset \). Let \( \text{anc}(p) \) be the set of ancestors in \( B \) of a point \( p \in P \), that is, \( \text{anc}(p) \) contains the points of \( \pi(s, p) \) excluding the point \( p \). The following observation will be used repeatedly in the proof.

\textbf{Observation.} If \((\text{pa}(p), p)\) is a clockwise edge, then \([s, p]^{cw} \subset \text{cov}(\rho_{\text{opt}}, \text{anc}(p)) \). Similarly, if \((\text{pa}(p), p)\) is a counterclockwise edge, then \([s, p]^{ccw} \subset \text{cov}(\rho_{\text{opt}}, \text{anc}(p)) \).

\textbf{Proof.} Assume \((\text{pa}(p), p)\) is a clockwise edge; the proof for when \((\text{pa}(p), p)\) is a counterclockwise edge is similar. If \( s \in [\text{pa}(p), p]^{cw} \)—this includes the case where
pa(p) = s—then the statement obviously holds, so assume pa(p) ∈ [s, p]\text{cw}. Since (pa(p), p) is a clockwise edge, it then suffices to prove that [s, pa(p)]\text{cw} ⊂ \text{cov}(\rho_{opt}, \text{anc}(p)).

Note that \text{cov}(\rho_{opt}, \text{anc}(p)) is connected, because the points in \text{anc}(p) form a path, namely π(s, pa(p)). Since π(s, p) is shortest path, p \not\in \text{cov}(\rho_{opt}, \text{anc}(pa(p)), which implies that [s, pa(p)]\text{cw} ⊂ \text{cov}(\rho_{opt}, \text{anc}(pa(p))) \subset \text{cov}(\rho_{opt}, \text{anc}(p)).

We now proceed to show that q_{ccw} must lie clockwise from q_{cw}, as seen from s, that is, the situation shown in Fig. 13(i) cannot happen.

\textbf{Claim.} \(d_{cw}(s, q_{cw}) + d_{ccw}(s, q_{ccw}) < 1\).

\textbf{Proof.} Note that \(d_{cw}(s, q_{cw}) + d_{ccw}(s, q_{ccw}) \neq 1\), since otherwise \(q_{cw} = q_{ccw}\) which cannot happen since \(S_{cw} \cap S_{ccw} = \emptyset\).

Now assume for a contradiction that \(d_{cw}(s, q_{cw}) + d_{ccw}(s, q_{ccw}) > 1\), which means that \(q_{ccw} \in [s, q_{cw}]\text{cw}\). Since \(q_{cw}\) is reached from its parent by a clockwise edge, this implies that \(q_{ccw} \in \text{cov}(\rho_{opt}, \text{anc}(q_{cw}))\) by the observation above. Hence, \(d_{hop}(s, q_{cw}) \geq d_{hop}(s, q_{ccw})\). An analogous argument shows that \(d_{hop}(s, q_{cw}) \geq d_{hop}(s, q_{ccw})\). Hence, \(d_{hop}(s, q_{ccw}) = d_{hop}(s, q_{cw})\). This implies that the edge \((pa(q_{cw}), q_{cw})\) passes over \(q_{ccw}\), otherwise some other edge of \(π(s, q_{cw})\) would pass over \(q_{ccw}\) and we would have \(d_{hop}(s, q_{ccw}) < d_{hop}(s, q_{cw})\). But then we also have a shortest path from \(s\) to \(q_{cw}\) whose last edge is a clockwise edge, contradicting the definition of \(\mathcal{B}\).

So we can assume that \(d_{cw}(s, q_{cw}) + d_{ccw}(s, q_{ccw}) < 1\) or, in other words, that \(q_{ccw}\) lies clockwise from \(q_{cw}\), as seen from \(s\). Clearly no point from \(P\) lies in \((q_{cw}, q_{ccw})\text{cw}\). If we have \((q_{cw}, q_{ccw})\text{cw} \not\subset \text{cov}(\rho_{opt}, P)\) then we are done, so assume for a contradiction that \((q_{cw}, q_{ccw})\text{cw} \subset \text{cov}(\rho_{opt}, P)\). This can happen in three ways, each of which will lead to a contradiction.

\textbf{Case I: There exists a point }p^* \in \mathcal{B} \text{ such that } q_{cw} \in \text{cov}_{cw}(\rho_{opt}, \{p^*\})\).

See Fig. 13(ii) for an illustration of the situation. If \(p^* = s\) then \(d_{hop}(s, q_{cw}) = 1\). Since \(q_{cw} \in S_{cw}\) this means that \(q_{cw}\) must also have an incoming clockwise edge from \(s\). But then \(\rho_{opt}(s) \geq \frac{1}{2}\), which contradicts Lemma 27. So \(p^* \neq s\). Now note that \(p^*\) must have an outgoing clockwise edge in \(\mathcal{B}\), else we can reduce the range of \(p^*\) to \(d_{ccw}(p^*, q_{ccw})\), which is smaller than \(d_{ccw}(p^*, q_{cw})\), and still get a feasible solution. Observe that \(p^* \not\in π(s, q_{cw});\)
otherwise we must have \( p^* = pa(q_{cw}) \) (since \( q_{cw} \) lies in the range of \( p^* \)) which contradicts that \( q_{cw} \in S_{cw} \). So for any point from \( P \) in the region \([s, q_{cw}]^{cw}\) there exists a path from \( s \) in the communication graph induced by \( \rho_{opt} \) that does not use \( p^* \). We now have two subcases.

If \( p^* \in [s, q_{cw}]^{cw} \) then clearly \( p^* \in S_{cw} \) (otherwise the definition of \( q_{cw} \) is contradicted). Hence, each point from \( P \) in the region \([s, p^*]^{cw}\) has a path from \( s \) that does not use \( p^* \).

This implies that can reduce the range of \( p^* \) to \( d_{cw}(p^*, q_{cw}) \) and still get a feasible solution.

If \( p^* \in [s, q_{cw}]^{cw} \) then obviously we can also reduce the range of \( p^* \) to \( d_{cw}(p^*, q_{cw}) \) and still get a feasible solution.

So both subcases lead to the desired contradiction.

**Case II:** There exists a point \( p^* \in B \) such that \( q_{cw} \in cov_{cw}(\rho_{opt}, \{p^*\}) \)

In the proof of Case I we never used that \( B \) maximizes the number of clockwise edges. Hence, a symmetric argument shows that Case II also leads to a contradiction.

**Case III:** There are two points \( p_1^*, p_2^* \in P \) such that \([q_{cw}, q_{cw}]^{cw} \subseteq cov_{cw}(\rho_{opt}, \{p_1^*\}) \cup cov_{cw}(\rho_{opt}, \{p_2^*\})\).

See Fig. 13(iii) for an illustration of the situation. We can assume that \( q_{cw} \notin cov_{cw}(\rho_{opt}, \{p_1^*\}) \) and \( q_{cw} \notin cov_{cw}(\rho_{opt}, \{p_2^*\}) \), otherwise we are in Case I or Case II. Now either \( p_2^* \notin \pi(s, p_1^*) \) or \( p_2^* \notin \pi(s, p_1^*) \) or both. Without loss of generality, assume \( p_2^* \notin \pi(s, p_1^*) \). Then \( p_2^* \neq s \) and all points from \( P \) in the region \([s, q_{cw}]^{cw}\) have a path from \( s \) in the communication graph \( G_{\rho_{opt}}(P) \) that does not use \( p_2^* \). The point \( p_2^* \) must have an outgoing counterclockwise edge, else we can reduce the range of \( p_2^* \) to \( d_{cw}(p_2^*, q_{cw}) \) and still get a feasible solution. We have two subcases.

If \( p_2^* \in [s, q_{cw}]^{cw} \) then by reducing the range of \( p_2^* \) to \( d_{cw}(p_2^*, q_{cw}) \) we still get a feasible solution.

If \( p_2^* \in [s, q_{cw}]^{cw} \) then \( p_2^* \) must be reached by a clockwise edge from its parent in \( B \), otherwise the definition of \( q_{cw} \) would be contradicted. Hence, for each point from \( P \) in the region \([s, p_2^*]^{cw}\) there is a path from \( s \) that does not use \( p_2^* \). So again we can reduce the range of \( p_2^* \) to \( d_{cw}(p_2^*, q_{cw}) \) and still get a feasible solution.

Thus both subcases lead to a contradiction.

This finishes the proof of the lemma.

\[ \sum_{i=1}^{2n+1} \rho(p_i)^\alpha \geq (2^n + 1)n - o(n), \] both before and after the insertion of \( q \).

**Proof.** By Observation 11 we have \( \rho(p)^\alpha \leq c_\alpha \cdot 2\delta^\alpha \) and, hence, \( \rho(p) \leq (2c_\alpha)^{1/\alpha} \cdot \delta < 3\delta \), for any point \( p \). Consider the interval \( I = [y_1, y_2]^{cw} \) where \( d_{cw}(s, y_1) = 3\delta \) and \( d_{cw}(q, y_2) = 3\delta \). All the points in \( I \cap P \) are at distance more than \( 3\delta \) from \( s \) or \( q \) and hence \( I \cap P \subseteq \text{cov}(\rho_{opt}, P \setminus \{s, q\}) \). Let \( p_i \in I \cap P \) be the point whose clockwise distance from \( s \) is minimum, and let \( p_j \in I \cap P \) be the point whose counterclockwise distance from \( q \) is minimum. Then the cost of covering all the points in \( I \cap P \) using the points in \( P \setminus \{s, q\} \) is at least \( \sum_{i=1}^{2n-1} d_{cw}(p_i, p_{i+1})^\alpha - 2^\alpha \), where the term \(-2^\alpha \) is because the covered region may leave one interval \([p_i, p_{i+1}]^{cw}\) uncovered. Recall that the cost of assigning all the points in \( P \setminus \{s, q\} \) a cw-minimal range is \((2^n + 1)n \). Note that \( i = O(\delta) \) since \( d_{cw}(s, p_i) \leq 3\delta + 2 \) and \( (2n + 1) - j = O(\delta) \) since \( d_{cw}(p_j, q) \leq 3\delta + 2 \). Hence,

\[ \sum_{i=1}^{2n+1} \rho(p_i)^\alpha \geq (2^n + 1)n - O(\delta) \cdot 2^\alpha \geq (2^n + 1)n - o(n), \]
since $\delta = ((2^\alpha + 1)n)^{1/\alpha} = o(n)$.

Lemma 13. The point $p_{2n+1}$ cannot have an incoming counterclockwise edge before $q$ is inserted, and the point $p_1$ cannot have an incoming clockwise edge after $q$ has been inserted.

Proof. Suppose before insertion of $q$ the point $p_{2n+1}$ has an incoming counterclockwise edge. The cheapest incoming counterclockwise edge would be from $s$ and this is already too expensive. Indeed, if $\rho(s) \geq 2x\delta$ then by Lemma 12 the total cost of the range assignment by ALG is at least

$$
(2x\delta)^\alpha + (2^\alpha + 1)n - o(n) = \left(2^\alpha \cdot \left(\frac{1}{4} + \left(\frac{1}{2}\right)^{\alpha+1}\right) + 1\right) \cdot \delta^\alpha - o(n)
$$

$$
= \left(2^{\alpha-2} + \frac{3}{2}\right) \cdot \delta^\alpha - o(n)
$$

$$
= \left(2^{\alpha-3} + \frac{3}{4}\right) \cdot 2\delta^\alpha - o(n)
$$

$$
= \left(1 + \left(\frac{2^{\alpha-3} - \frac{1}{4}}{4}\right)\right) \cdot 2\delta^\alpha - o(n)
$$

$$
> \left(1 + \frac{1}{2} \cdot \left(\frac{2^{\alpha-3} - \frac{1}{4}}{4}\right)\right) \cdot 2\delta^\alpha \quad \text{for } n \text{ sufficiently large}
$$

$$
\geq c_\alpha \cdot \text{OPT}(P) \quad \text{by definition of } c_\alpha \text{ and Observation 11}
$$

This contradicts the approximation ratio of ALG, proving the first part of the lemma.

Now suppose after the insertion of $q$ the point $p_1$ has an incoming clockwise edge. The cheapest way to achieve this is with $\rho(s) = \delta$, which is too expensive. Indeed, by Lemma 12 the total cost of the range assignment is then at least

$$
\delta^\alpha + (2^\alpha + 1)n - o(n) = \frac{2\delta^\alpha}{(2x^\alpha + 1)\delta^\alpha} \cdot (2x^\alpha + 1)\delta^\alpha - o(n)
$$

$$
\geq \left(1 + \frac{1}{2} \cdot \left(\frac{2\delta^\alpha}{(2x^\alpha + 1)\delta^\alpha} - 1\right)\right) \cdot \text{OPT}(P \cup \{q\}) \quad \text{for } n \text{ sufficiently large}
$$

$$
= \left(1 + \left(\frac{1}{2x^\alpha + 1} - \frac{1}{2}\right)\right) \cdot \text{OPT}(P \cup \{q\})
$$

$$
= \left(1 + \frac{2 - (2x^\alpha + 1)}{2(2x^\alpha + 1)}\right) \cdot \text{OPT}(P \cup \{q\})
$$

$$
= \left(1 + \frac{1 - \left(\frac{1}{2} + \frac{1}{2x^\alpha + 1}\right)}{\frac{1}{2} + \frac{1}{2x^\alpha + 1}}\right) \cdot \text{OPT}(P \cup \{q\}) \quad \text{since } 2x^\alpha = \frac{1}{2} + \frac{1}{2x^\alpha}
$$

$$
\geq c_\alpha \cdot \text{OPT}(P \cup \{q\}) \quad \text{by definition of } c_\alpha \text{ and Observation 11}
$$

This contradicts the approximation ratio of ALG, proving the second part of the lemma.

The claim in the proof of Lemma 14

Claim. If $p_j$ is not assigned to a cw-minimal edge then $\text{excess}(p_j) \geq c'_\alpha$, where $c'_\alpha = \min\left(2^\alpha - 1, \frac{3^\alpha - 2^\alpha - 1}{2}, \frac{4^\alpha - 2^\alpha - 2}{3}\right)$. 
Proof. Consider a non-cw-minimal edge \((p_i, p_t)\). First suppose only a single point \(p_j\) is assigned to \((p_i, p_t)\). Then \(t = i+1\) and \(p_j = p_t\). Hence, \(\rho(p_j) \geq d(p_{j-1}, p_j) + 1\) because we assumed \(\rho(p_j) \in \{0, 1, 2\} \cup [3, \infty)\). Thus when \(|A(p_i, p_j)| = 1\) then
\[\text{excess}(p_j) \geq (d(p_{j-1}, p_j) + 1)^\alpha - d(p_{j-1}, p_j)^\alpha \geq 2^\alpha - 1 \geq c^\alpha_\alpha.\]

Now suppose \(|A(p_i, p_j)| \geq 1\). Let \(z_1\) be the number of points \(p_j \in A(p_i, p_t)\) with \(|A(p_{j-1}, p_j)| = 1\), and let \(z_2\) be the number of points \(p_j \in A(p_i, p_t)\) with \(|A(p_{j-1}, p_j)| = 2\). Since \(|A(p_i, p_j)| > 1\) we have \(z_1 \geq 1\) and \(z_2 \geq 1\) and \(|z_1 - z_2| \leq 1\). When \(|A(p_i, p_j)| = 2\) then \(z_1 = z_2 = 1\), and we are distributing the cost of an edge of length at least 3, minus the costs of edges of length 2 and 1, over two points. Thus in this case we have
\[\text{excess}(p_j) \geq \frac{3^\alpha - 2^\alpha - 1}{2}.\]

Similarly, when \(|A(p_i, p_j)| = 3\) then \(z_1 = 2\) and \(z_2 = 1\) (or vice versa, but that will only lead to a larger excess), and we have
\[\text{excess}(p_j) \geq \frac{4^\alpha - 2^\alpha - 2}{3}.\]

It remains to argue that we do not get a smaller excess when \(|A(p_i, p_j)| \geq 4\). To see this, we compare the excess we get when \((p_i, p_t)\) is an edge of \(\pi\) with the excesses we would get when, instead of \((p_i, p_t)\), the edges \((p_i, p_{i+2})\) and \((p_{i+2}, p_t)\) would be in \(\pi^+\). Note that
\[d(p_i, p_t)^\alpha = (d(p_i, p_{i+2}) + d(p_{i+2}, p_t))^\alpha > d(p_i, p_{i+2})^\alpha + d(p_{i+2}, p_t)^\alpha\]
since \(\alpha > 1\). Hence,
\[
\begin{align*}
d(p_i, p_t)^\alpha - \sum_{t=1+1}^{i} d(p_{i-1}, p_t)^\alpha & \geq \frac{d(p_i, p_{i+2})^\alpha - \sum_{t=1+1}^{i+2} d(p_{i-1}, p_t)^\alpha + d(p_{i-2}, p_{i-2})^\alpha - \sum_{t=1+1}^{2} d(p_{i-1}, p_t)^\alpha}{2} \\
& \geq \frac{d(p_i, p_{i+2})^\alpha - \sum_{t=1+1}^{i+2} d(p_{i-1}, p_t)^\alpha}{2} + d(p_{i-2}, p_{i-2})^\alpha - \sum_{t=1+1}^{2} d(p_{i-1}, p_t)^\alpha
\end{align*}
\]
where the last inequality uses that \(\frac{a1 + a2}{b1 + b2} \geq \min\left(\frac{a1}{b1}, \frac{a2}{b2}\right)\) for any \(a1, a2, b1, b2 > 0\). Thus the excess we get for \((p_i, p_t)\) is at least the minimum of the excesses we would get for \((p_i, p_{i+2})\) and \((p_{i+2}, p_t)\). More generally, when \(|A(p_i, p_j)| \geq 4\) then we can compare the excess for \((p_i, p_t)\) with the excesses we get when we would replace \((p_i, p_t)\) with a path of smaller edges, each being assigned two or three points. The excess for \((p_i, p_{i+2})\) is at least the minimum of the excesses for these shorter edges. (Reducing to edges that are assigned a single point is not useful, since these may be cw-minimal and have zero excess.) This finishes the proof of the claim.

\[\triangleright\]

### E Missing details for Section 5

#### E.1 Proof of Theorem 15

**Theorem 15.** The dynamic broadcast range-assignment problem in \(\mathbb{R}^2\) with distance power gradient \(\alpha > 1\) does not admit a SAS. In particular, there is a constant \(c_0 > 1\) such that the following holds: for any \(n\) large enough, there is a set \(P := \{s, p_1, \ldots, p_{2n+1}\}\) and a point \(q\) in \(\mathbb{R}^2\) such that any update algorithm ALG that maintains a \(c_0\)-approximation must modify at least \(2n/3 - 1\) ranges upon the insertion of \(q\) into \(P\).
Proof. We use the same construction as in $S^1$, where we embed the points on a square and the distances used to define the instance are measured along the square; see Fig. 3(ii). We now discuss the changes needed in the proof to deal with the fact that distances in $\mathbb{R}^2$ between points from $P \cup \{q\}$ may be smaller than when measured along the square. With a slight abuse of terminology, we will still refer to an edge $(p, p')$ that was clockwise in $S^1$ as a clockwise edge, and similarly for counterclockwise edges.

Note that Observation 11 still holds. Now consider Lemma 12. The proof used that the points $p_i$ at distance more than $3\delta$ from $s$ or $q$ must be covered by the ranges of the points $p_1, \ldots, p_{2n+1}$. We now restrict our attention to the points that are also at distance more than $3\delta$ from a corner of the square. Each such point $p_i$ must be covered by the range of some point $p_j$ on the same edge of the square. Hence, the distance in $\mathbb{R}^2$ of from $p_j$ to $p_i$ is the same as the distance in $S^1$, so we can use the same reasoning as before. Thus the exclusion of the points that are at distance at most $3\delta$ from a corner of the square only influences the constant in the $o(n)$ term in the lemma. Hence, Lemma 12 still holds.

The proof of Lemma 13 still holds, since the cheapest counterclockwise edge to $p_{2n+1}$ before the insertion of $q$ is still from $s$ (and the distance from $s$ to $p_{2n+1}$ did not change), and the cheapest clockwise edge to $p_1$ after the insertion of $q$ is still from $s$ (and the distance from $s$ to $p_1$ did not change).

It remains to check Lemma 14. The proof still holds, except that the claim that $\text{excess}(p_j) \geq c'_\alpha$ may not be true for the given value of $c'_\alpha$ when $p_j$ is near a corner of the square, because the distances between points on different edges of the square do not correspond to the distances in $S^1$. To deal with this, we simply ignore the excess of any point within distance $3\delta$ from a corner. This reduces the total excess by $o(n)$. It is easily verified that this does not invalidate the rest of the proof: we have to subtract $o(n)$ from the formulae in Equality (2), but this is still larger than $c_{\alpha} \cdot \text{OPT}(P)$.

We conclude that all lemmas still hold, which proves Theorem 15.

E.2 An $O(1)$-stable $O(1)$-approximation algorithm in $\mathbb{R}^2$

Next we show that there is a relatively simple $O(1)$-stable $O(1)$-approximation algorithm for distance-power gradient $\alpha \geq 2$. Our algorithm is based on a result by Ambühl [1], who showed that a minimum-spanning tree (MST) on $P$ can be used to get a constant-factor approximation to the broadcast range-assignment problem on $P$. The key lemma underlying the result is the following.

\textbf{Lemma 28} ([1]). Let $P$ be any point set in the plane. Let $T_P$ be an MST on $P$, and let $E(T_P)$ be the set of edges of $T_P$. Then, for any distance-power gradient $\alpha \geq 2$, we have $\sum_{e \in T_P} |e|^\alpha \leq 6 \cdot \text{OPT}$, where $\text{OPT} = \text{cost}_\alpha(\text{r}_{\text{opt}}(P))$ is the cost of an optimal range assignment.

In the static problem this immediately gives a $6$-approximation algorithm: turn the MST into a directed tree rooted at the source $s$, and assign as a range to each point $p \in P$ the maximum length of any of its outgoing edges. To apply this in the dynamic setting, we need the following lemma, which implies that for any point set $P$ and any additional point $q$, any MST $T$ on $P$ can be converted to an MST $T'$ on $P \cup \{q\}$ that is very similar to $T$. The result is folklore [12].

\textbf{Lemma 29}. Let $P$ be a set of points in a metric space $X$, and $P' := P \cup \{q\}$ for some point $q \in X$. Let $T$ be any MST on $P$. Then there is an MST $T'$ on $P'$ such that all edges in $T'$ that are not incident to $q$ also occur in $T$. Conversely, let $T$ be any MST on a set
Then there exists an MST $T'$ on $P$ such that all edges in $T$ that are not incident to $q$ also occur in $T'$.

We use this lemma in combination with the following well-known lemma.

Lemma 30. Let $T$ be an MST of a point set in $\mathbb{R}^d$. Then the maximum vertex degree of $T$ is bounded by the Hadwiger number of the corresponding unit ball in $\mathbb{R}^d$. In particular, the maximum vertex degree of an MST in $\mathbb{R}^2$ is at most 6.

We can now prove the following theorem.

Theorem 31. There is a 17-stable 12-approximation algorithm for the dynamic broadcast range-assignment problem in $\mathbb{R}^2$, for power-distance gradient $\alpha \geq 2$.

Proof. Our algorithm will maintain an MST $T$ on the current point set $P$, using Lemma 29. We set the range of each point to be the maximum length of any of its incident edges. Clearly, this defines a feasible solution. We denote the resulting range assignment by $\rho_{mst}$.

We now analyze the stability of $\rho_{mst}$. Consider the insertion of a point $q$, and let $T'$ be the new MST after the insertion has been handled. Observe that, part from the point $q$ itself, only the ranges of the neighbors of $q$ in $T'$ can increase. By Lemma 30 we have $\deg(q) \leq 6$, where $\deg(q)$ denotes the degree of $q$. Hence, the number of ranges that need to be increased is at most 7. Also observe that only the ranges of those points can decrease that had an edge belonging to the edge set $T \setminus T'$ incident to it. Since $\deg(q) \leq 6$, and $T$ and $T'$ have $|P| - 1$ and $|P|$ edges, respectively, we have $|T \setminus T'| \leq 5$. Hence the ranges of at most ten points can decrease. So insertions are (7,10)-stable, and deletions are (10,7)-stable.

It remains to prove the approximation ratio. Using Lemma 28 and noting that every edge in $T$ is adjacent to at most two vertices, we conclude that $\alpha \left( \rho_{mst}(P) \right) \leq 2 \cdot \sum_{e \in T} |e| \alpha \leq 12 \cdot \text{opt}$ for any set $P$, thus finishing the proof.

Some additional previous work on stable algorithms

For an arbitrary dynamic optimization problem, stability of a solution depends on how to measure the difference between two solutions [35]. For instance, in our dynamic broadcast range-assignment problem, we focus on the number of points whose range changes after an insertion or a deletion. For many problems, there is a natural choice for the difference between two solutions. In fact, for some specific problems, trade-offs between the quality of a solution and its stability have been studied before, albeit with varying terminology. We now describe a number of such cases (without claiming to give a complete overview).

Lattanzi and Vassilvitskii [28] consider an online clustering problem: points arrive iteratively, and the goal is to maintain a set of $k$ centers (where $k$ is given) such that each point is assigned to a center while minimizing some given cost function; this setting encompasses the $k$-center, the $k$-median and the $k$-means problem. Lattanzi and Vassilvitskii explicitly focus on the trade-off between the cost of a solution and its stability—they use the term “consistency”—which is defined as the sum, over all iterations, of the quantity $|C_{i+1} \setminus C_i|$, where $C_i$ denotes the set of centers used in the solution after the $i$-th iteration. They show how to maintain a solution that is a constant-factor approximation while the consistency is $O(k \log n)$, where $n$ stands for the number of arriving points. Fichtenberger et al. [21] build on this, and show how to maintain a constant-factor approximation for the $k$-median problem by performing an optimal (up to polylogarithmic factors) number of center swaps. Results of this nature for the uncapacitated facility-location problem can be found in Cohen-Addad et al. [14].
In the online Steiner Tree problem, new points arrive iteratively, and one has to connect each new point to the current tree, thereby maintaining a tree on all points that have arrived. This problem was studied from a stability viewpoint by Imase and Waxman [26]. They show that a simple greedy algorithm that connects the new point to a closest previous point, is \( \log n \) competitive, where \( n \) is the number of arriving points; they also show that it is possible to maintain a 2-competitive tree while making \( O(n^{3/2}) \) swaps. Subsequent contributions (see [30, 24]) have improved those results; in particular, for the metric case, Gu et al. [23] provide an algorithm that, given any fixed \( \varepsilon > 0 \), maintains a tree that is \((1 + \varepsilon)\)-competitive by making at most \( 2n/\varepsilon \) swaps over the course of \( n \) arrivals. In our terminology (see Definition 1) this would be a SAS for metric Steiner Tree, albeit only in an amortized sense and for the insertion-only setting.

In dynamic scheduling problems, jobs arrive and disappear in an online fashion; the goal is to maintain an assignment of jobs to machines that is close to optimal. Clearly, the possibility of re-assigning jobs will help in maintaining high-quality solutions. The cost of re-assigning jobs can be measured by the number of jobs one is allowed to re-assign (the recourse model), or by the total size of the jobs that can be re-assigned (the migration model). When the objective is to minimize the makespan, the trade-off between the cost of re-assinging jobs and quality of the solution has been studied under the name robust PTAS: a polynomial-time algorithm that, for any given parameter \( \varepsilon > 0 \), computes a \((1 + \varepsilon)\)-approximation to the optimal solution with re-assignment costs only depending on \( \varepsilon \); we mention Skutella and Verschae [33], Sanders et al. [32] for such results. Similarly, in dynamic bin packing, there is a trade-off between maintaining a high-quality solution and the amount of repacking that is allowed. We mention work by Feldkord et al. [20], Berndt et al. [5], Epstein and Levin [19].

For the dynamic matching problem, an impressive amount of work has been devoted trade-offs between the quality of a matching and the time needed to update it; see Behnezhad et al. [4] and the references contained therein. In the context of bipartite graphs, Bernstein et al. [6] consider the problem of maintaining a maximum matching, while minimizing the number of replacements, see also Gupta et al. [25].