FELDMAN-KATOK PSEUDOMETRIC AND THE GIKN CONSTRUCTION OF NONHYPERBOLIC ERGODIC MEASURES

DOMINIK KWIECIAK AND MARTHA ŁĄCKA

Abstract. The GIKN construction was introduced by Gorodetski, Ilyashenko, Kleptsyn, and Nalsky in [Functional Analysis and its Applications, 39 (2005), 21–30]. It gives a nonhyperbolic ergodic measure which is a weak* limit of a special sequence of measures supported on periodic orbits. This method was later adapted by numerous authors and provided examples of nonhyperbolic invariant measures in various settings. We prove that the result of the GIKN construction is always a loosely Kronecker measure in the sense of Ornstein, Rudolph, and Weiss (equivalently, standard measure in the sense of Katok, another name is loosely Bernoulli measure with zero entropy). For a proof we introduce and study the Feldman-Katok pseudometric $\bar{F}_K$.

The pseudodistance $\bar{F}_K$ is a topological counterpart of the $f$-bar metric for finite-state stationary stochastic processes introduced by Feldman and, independently, by Katok, later developed by Ornstein, Rudolph, and Weiss. We show that every measure given by the GIKN construction is the $\bar{F}_K$-limit of a sequence of periodic measures. On the other hand we prove that a measure which is the $\bar{F}_K$-limit of a sequence of ergodic measures is ergodic and its entropy is smaller or equal than the lower limit of entropies of measures in the sequence. Furthermore we demonstrate that $\bar{F}_K$-Cauchy sequence of periodic measures tends in the weak* topology either to a periodic measure or to a loosely Kronecker measure.

The opening question of [16], that is, "To what extent is a behaviour of a generic dynamical system hyperbolic?" is one of the oldest questions in dynamics. Abraham and Smale [1] demonstrated that hyperbolic diffeomorphisms are not dense in the space of all diffeomorphisms of a manifold. This motivated Pesin [30] to introduce a weaker notion of hyperbolic invariant measure, that is, an ergodic invariant measure with non-vanishing Lyapunov exponents. If a diffeomorphism has hyperbolic nonwandering set, then all its invariant measures are hyperbolic. The converse is not true: there are nonhyperbolic diffeomorphisms with all invariant measures hyperbolic [2, 9].

As the value of a Lyapunov exponent is very sensitive to perturbations one might expect that hyperbolic measures appear generically. Shub and Wilkinson [33] posed a problem which can be paraphrased as: Does a generic smooth dynamical system on a compact Riemannian manifold has nonzero Lyapunov exponents for every measure obtained through the Krylov-Bogolyubov procedure from the Lebesgue measure? Gorodetski, Ilyashenko, Kleptsyn, and Naksy [19] speculated that if one relaxes the assumption on the measure, then the answer is negative. They conjectured that there exists an open set $U$ in the space of diffeomorphisms of the three dimensional torus $T^3$ such that any diffeomorphism from $U$ has an ergodic invariant measure with at least one zero Lyapunov exponent. The GIKN construction presented in [16] was the first step towards that conjecture, which was later settled by Kleptsyn and Nalsky [22]. The method used in [16] was inspired by the technique of approximating the ergodic systems by periodic ones introduced in [21].

A nonhyperbolic ergodic invariant measure arising from the GIKN construction is the weak* limit of measures supported on a sequence of periodic points with special properties. Each periodic orbit in this sequence can be divided into two parts: the shadowing part and the tail. Their key features are:

1. The shadowing part takes a large proportion (growing to 1 as $n$ goes to $\infty$) of each periodic orbit in the sequence. Furthermore, images of each point on the shadowing part of the $n$-th orbit are $\gamma_n$-close to the $(n-1)$-th orbit for the number of iterates equal to the primary period of the $(n-1)$-th orbit and the series formed by $\gamma_n$’s is summable. This is used to show that the limit measure is ergodic.

2. There is a fixed (center) direction such that the Lyapunov exponent in that direction along the sequence of periodic orbits decreases to zero. To achieve this each periodic orbit spends a small proportion of its primary period (this part of the orbit is called the tail) in a region which is far from the previous orbits and is chosen so that the Lyapunov exponent in the
center direction along the whole orbit has smaller absolute value than the same exponent of previously constructed orbits. This also guarantees that the limit measure is nonatomic. By Proposition 2.7 of \cite{7} the support of the limit measure is the topological limit of periodic orbits carrying measures in the sequence. This approach was subsequently adapted in \cite{2, 3} to find nonhyperbolic measures for partially hyperbolic dynamics. In particular, it is proved in \cite{10} that for a \( C^1 \)-generic diffeomorphism every nonhyperbolic homoclinic class carries a nonhyperbolic invariant measure given by the GIKN construction. Furthermore, under mild assumptions the support of that measure is the whole class. (Note that Bochi, Bonatti and Díaz \cite{5} \cite{6} introduced recently a different construction of nonuniformly hyperbolic measures. These measures are supported on the \( \omega \)-limit set of a point which is \textit{controlled at any scale}, see \cite{5} for more details.)

The GIKN method is tailored to control Lyapunov exponents and ergodicity, and so far little was known about other properties of the resulting measure. In particular, the question whether the GIKN construction always leads to a measure with zero entropy was around for some time. One of the authors heard it during a minicourse presented by Lorenzo J. Díaz at the conference \textit{Global dynamics beyond uniform hyperbolicity} in Olmué, Chile in September 2015. It is noted in \cite{5} that the repetitive nature of a measure obtained through the GIKN construction suggests that its entropy is zero, but this heuristics did not convince all researchers, see \cite{5}. According to Díaz (personal communication) the question was raised by Jérôme Buzzi in Orsay.

We prove that the entropy in question is always zero. For a proof we introduce a pseudometric \( \bar{F}_K \) generated by a continuous map \( T : X \to X \) on a compact metric space \( X \). We call it the Feldman-Katok pseudometric, because it is inspired by the \( f \)-bar metric \( f \) introduced by Feldman (and, independently, by Katok) for finite state stationary stochastic processes. The transition from \( f \) to \( \bar{F}_K \) is done in a similar vain as the extension of Ornstein’s \( d \)-bar metric to a dynamically defined Besicovitch pseudometric on a dynamical system, see \cite{26, 32, 35}.

Before going into details, let us digress to make a brief description of works of Feldman \cite{13}, Katok \cite{19}, and Ornstein, Rudolph, and Weiss \cite{28}. Feldman \cite{13} studied the isomorphism problem in ergodic theory. He introduced a new property (called \textit{loose Bernoullicity}) for finite partitions of a measure-preserving system \((X, B, \mu, T)\) and thus for \( T \) itself. He used it to construct important examples of \( K \)-automorphisms which are loosely Bernoulli, but not Bernoulli. In particular, these measure-preserving systems have positive entropy. The definition of loosely Bernoulli partition follows Ornstein’s definition \cite{27} of very weak Bernoulli partition with the Hamming distance of strings of symbols of length \( n \) replaced by the weaker \textit{edit metric} \( f_n \). The edit metric between two strings (words) of length \( n \) equals \( 1 - k/n \) where \( k \) is the minimum number of symbols which has to be removed from each string so that remaining strings are identical. Feldman’s idea revived the theory of \textit{Kakutani equivalence} and was subsequently extended by Ornstein, Rudolph, and Weiss \cite{28}. Kakutani equivalence is a natural equivalence relation between transformations preserving an ergodic nonatomic measure, which is weaker than the usual notion of isomorphism. Note that entropy is not an invariant for Kakutani equivalence, but it follows from Abramov’s formula that this relation preserves classes of zero, positive and finite, and infinite entropy transformations. Furthermore, in each of these three entropy classes the Kakutani equivalence classes of loosely Bernoulli transformations form the simplest equivalence class, see \cite{28} for more details. Positive and finite (infinite) entropy transformations Kakutani equivalent to a Bernoulli shift coincide with loosely Bernoulli and positive and finite (infinite) entropy transformations. Zero entropy loosely Bernoulli transformations form the Kakutani equivalence class of any ergodic rotation of a compact infinite group (Kronecker system). According to Feldman and Nadler \cite{14} members of the latter equivalence class are called \textit{loosely Kronecker} following a suggestion of Marina Ratner. We also adapt this terminology.

Independently, Katok \cite{19} (partly in collaboration with Sataev \cite{20}, who contributed to the subject as well, see \cite{31}) also studied the problem of classification of measure-preserving transformations up to the Kakutani equivalence, which he called \textit{monotone equivalence}. He introduced the notion of \textit{standard automorphism}, which is defined as any member of the monotone (Kakutani) equivalence class of the ergodic invertible measure preserving system with discrete spectrum consisting of all the roots of unity. The latter system is isomorphic to an ergodic group rotation on the universal odometer. Katok presented, among the other results, a criterion for standardness in terms of coding using as a basis the same edit metric on words \( f_n \) as Feldman. Using this criterion
he proved independently of [22] that the notions: loosely Bernoulli and standard coincide for zero entropy (in the terminology presented above).

The Feldman-Katok pseudometric $F_K$ introduced here provides a link between ideas of Feldman and Katok phrased in terms of words over finite alphabet and topological conditions imposed on a sequence of periodic points in the GIKN construction. (Note that these conditions were presented implicitly in [16]. We use here an explicit formulation which is due to Bonatti, Díaz, and Gorodetski [7].) It turns out that the GIKN construction yields a sequence of periodic points which is $F_K$-Cauchy. As it might be of independent interest we study general $F_K$-Cauchy sequences with every element generic for some ergodic measure. Of course this applies to $F_K$-convergent sequences of periodic points. We prove that the $F_K$-Cauchy sequences of generic points leads naturally to an ergodic measure, which we call $F_K$-limit of the sequence. This generalizes the criterion for ergodicity provided in [16]. It also holds that the entropy function is lower semicontinuous with respect to that $F_K$-convergence. Therefore the $F_K$-limit of zero entropy measures (in particular, the result of the GIKN construction) has zero entropy. Finally, we show using Katok’s criterion [19], that a nonatomic measure which is an $F_K$-limit of a sequence of periodic measures is loosely Kronecker (standard in Katok’s terminology), that is, is Kakutani equivalent to an ergodic aperiodic group rotation. This applies to measures defined in [4] [7] [10] [11] [16] [22] by the GIKN construction. In other words, each such measure is isomorphic to a measure-preserving systems arising by taking an aperiodic ergodic group rotation and the transformation induced by it on appropriately chosen measurable subset of the group.

1. Basic Definitions and Notation

Throughout this paper $\mathbb{N}$ stands for the set of positive integers, $|A|$ is the cardinality of a set $A$, and $\chi_A$ is its characteristic function. Unless otherwise stated $i, j, k, \ell, m, n$ denote integers. Note that in the following we will often define objects depending on previously fixed parameters, but our notation rarely will reflect that dependence.

Let $\bar{d}(A)$ be the upper density of a set $A \subset \mathbb{N} \cup \{0\}$, that is

$$\bar{d}(A) = \limsup_{n \to \infty} \left| \frac{A \cap \{0, \ldots, n - 1\}}{n} \right|.$$  

The definition of lower density is analogous, with the lower limit replacing the upper one. The set $A \subset \mathbb{Z}$ has density $\alpha$ if its lower and upper density are both equal to $\alpha$.

1.1. Dynamical systems. Throughout we assume that $X$ is a compact metric space, $\rho$ is a metric for $X$, and $T: X \to X$ is a continuous map. We denote by $X^\infty$ the family of all $X$-valued sequences indexed by $\mathbb{N} \cup \{0\}$. We always endow $X^\infty$ with the product topology. Typically, we write $x = (x_i)$ or $\underline{x} = (x_i)_{i=0}^\infty$ for elements of $X^\infty$. By $\sigma$ we denote the shift operator acting on $X^\infty$ as $\sigma(x) = (x_{i+1})_{i=0}^\infty$. Note that $\sigma$ is continuous. Given $x \in X$, we distinguish between the orbit of $x$, which is a set $\{T^n(x) : n \geq 0\} \subset X$ and the trajectory of $x$ which is a sequence $(\underline{x}_T = (T^j(x))_{j=0}^\infty) \in X^\infty$.

1.2. Note on invertibility. Observe that by default we work here with noninvertible transformations, but some of the results we invoke from the literature are stated assuming that the transformations at hand are invertible. Fortunately, it is easy to see that in all such instances the theorem we need holds for noninvertible transformations.

1.3. Symbolic systems. In the case where $X = \mathcal{A}$ is finite, we equip it with the discrete topology and call $\mathcal{A}^\infty$ the shift space over the alphabet $\mathcal{A}$. For $k \in \mathbb{N}$ we define $\Omega_k = \{0, 1, \ldots, k - 1\}^\infty$. We call the elements of $\mathcal{A}^n$ words of length $n$ over $\mathcal{A}$. Let $\mathcal{A}^+ = \bigcup_{n \geq 1} \mathcal{A}^n$ denote the set of all words over $\mathcal{A}$ and $|u|$ stands for the length of $u \in \mathcal{A}^+$. Every word $u \in \mathcal{A}^+$ determines a cylinder set $[u] \subset \mathcal{A}^\infty$ consisting of all sequences in $\mathcal{A}^\infty$, whose first $|u|$ symbols coincide with $u$. Cylinders form a clopen base for the topology of $\mathcal{A}^\infty$ and the family of all finite disjoint unions of cylinders generate the Borel $\sigma$-algebra $\mathcal{B}$ of the compact metrizable space $\mathcal{A}^\infty$. Given two $n$ words $u = u_0 u_1 \ldots u_{n-1}$ and $w = w_0 w_1 \ldots w_{n-1}$ over $\mathcal{A}$ we define the Hamming distance

$$d_n(u, w) = \frac{1}{n} |\{0 \leq j < n : u_j \neq w_j\}|$$ 

and the edit distance
\[ f_n(u, w) = 1 - \frac{k}{n}, \]
where \( k \) is the largest among those integers \( \ell \) such that for some \( 0 \leq i_1 < i_2 < \ldots < i_\ell < n \) and \( 0 \leq j_1 < j_2 < \ldots < j_\ell < n \) we have \( u_{i_s} = w_{j_s} \) for \( s = 1, \ldots, \ell \). For two infinite sequences \( \omega = \omega_1 \omega_2 \omega_3 \ldots \), \( \omega' = \omega'_1 \omega'_2 \omega'_3 \ldots \) in \( \mathcal{A}^\infty \) we set
\[
d(\omega, \omega') = \limsup_{n \to \infty} d_n(\omega, \omega') = \limsup_{n \to \infty} d_n(\omega_1 \omega_2 \ldots \omega_{n-1}, \omega'_1 \omega'_2 \ldots \omega'_{n-1})
\]
\[ = d(\{ j \geq 0 : \omega_j \neq \omega'_j \}), \]
\[
f(\omega, \omega') = \limsup_{n \to \infty} f_n(\omega, \omega') = \limsup_{n \to \infty} f_n(\omega_1 \omega_2 \ldots \omega_{n-1}, \omega'_1 \omega'_2 \ldots \omega'_{n-1}).
\]
This defines useful pseudometrics on \( \mathcal{A}^\infty \). Another important pseudometric on \( \mathcal{A}^\infty \) is given by
\[
(1) \quad \bar{f}(\omega, \omega') = \inf \{ \varepsilon > 0 : \text{there are increasing sequences } (i_s), (i'_s) \in \mathbb{N}^\infty \text{ of lower density}
\]
\[ \text{at least } 1 - \varepsilon \text{ for which } \omega_{i_r} = \omega'_{i'_r} \text{ for all } r \geq 0. \}
\]
It is easy to see that \( f(\omega, \omega') \leq \bar{f}(\omega, \omega') \). Actually, \( \bar{f} \) and \( f \) are uniformly equivalent pseudometrics on \( \mathcal{A}^\infty \), see [28].

1.4. Measure-preserving systems. Most of the standard texts on ergodic theory work with measure-preserving transformations of standard Lebesgue spaces. The latter are measure spaces arising as completions of probability measures on Polish metric spaces endowed with their Borel \( \sigma \)-algebras. In this approach it is hard to consider different measures on the same underlying space, since the \( \sigma \)-algebra depends nontrivially on the measure. This is the primary reason we work in the Borel category.

1.5. Invariant measures, generic sequences. We write \( \mathcal{X}_B \) for the Borel \( \sigma \)-algebra of \( X \) and \( \mathcal{M}(X) \) for the set of all Borel probability measures on \( X \). By \( \mathcal{M}_T(X) \) we denote \( T \)-invariant measures in \( \mathcal{M}(X) \). Then the quadruple \( X = (X, \mathcal{X}_B, \mu, T) \) is a measure-preserving system, which is invertible, whenever \( T \) is a homeomorphism. We give \( \mathcal{M}(X) \) the weak* topology, which is compact and metrizable. In this topology a sequence \( (\mu_n)_{n=1}^\infty \) converges to \( \mu \) in \( \mathcal{M}(X) \) if and only if for every continuous function \( \varphi : X \to \mathbb{R} \) the sequence \( \int \varphi \, d\mu_n \) tends to \( \int \varphi \, d\mu \) in \( \mathbb{R} \). It is well known that the weak* topology on \( \mathcal{M}(X) \) is induced by Prokhorov metric
\[
D_P(\mu, \nu) = \inf \{ \varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ for every Borel set } B \subset X \},
\]
where \( B^\varepsilon \) denotes the \( \varepsilon \)-ball of \( B \), that is the set \( \{ y \in X : \text{dist}(y, B) < \varepsilon \} \). For \( x \in X \), let \( \delta(x) \in \mathcal{M}(X) \) be the Dirac measure supported on \( \{ x \} \). Let \( m(x, n) \) denote the \( n \)-empirical measure of \( x = (x_j)_{j=0}^\infty \), that is
\[
m(x, n) = \frac{1}{n} \sum_{j=0}^{n-1} \delta(x_j).
\]
We put \( m_T(x, n) = m(x_T, n) = m((T^n(x))_{j=0}^\infty, n) \) for \( x \in X \). A measure \( \mu \in \mathcal{M}(X) \) is generated by \( x \in X \) if \( \mu \) is the limit of some subsequence of \( (m(x, n))_{n=1}^\infty \), and the set of all measures generated by \( x \) is denoted by \( \hat{\omega}(x) \). If \( \hat{\omega}(x) = \{ \mu \} \) for some \( \mu \in \mathcal{M}(X) \), then we say that \( x \in X^\infty \) is a generic sequence for \( \mu \), and write \( x \in \text{Gen}(\mu) \). If, in addition, \( \mu \) is a \( T \)-invariant ergodic measure, then we say that \( x \in X^\infty \) is an ergodic sequence.

In case, where \( x = x_T \) for some \( z \in X \) we denote \( \hat{\omega}(x_T) \) by \( \hat{\omega}_T(z) \), and we call \( z \) a generic point (or, ergodic point) if its trajectory \( x_T \) is generic (respectively, ergodic) sequence.

One reason for considering the more general notions of generic sequences in \( X^\infty \) is that every invariant measure has a generic sequence in \( X^\infty \) [26, 34], while a non-ergodic invariant measure may have no generic points. Furthermore, one can choose a generic sequence which is a quasi-orbit. A quasi-orbit is built from longer and longer pieces of orbits in such a way that the set of positions at which a quasi-orbit switches from one piece of genuine orbit to another has zero asymptotic density.

**Definition 1.** We say that \( x = (z_n)_{n=0}^\infty \in X^\infty \) is a quasi-orbit for \( T \) if \( d(\{ n \geq 0 : z_{n+1} \neq T(z_n) \}) = 0 \).
It is easy to see that if \( \underline{\omega} \in X^\infty \) is a quasi-orbit, then \( \dot{\omega}(\underline{\omega}) \subseteq \mathcal{M}_T(X) \). Furthermore, the GIKN construction yields a quasi-orbit generic for the invariant measure it produces. We work with that quasi-orbit to demonstrate the properties of the underlying measure.

1.6. Processes. Given a compact metric space we write \( P^m(X) \) for the set of all Borel measurable partitions of \( X \) into at most \( m \) sets, called atoms. For \( P \in P^k \) we write \( P = \{P_0, \ldots, P_{k-1}\} \) regardless of the actual number of nonempty elements in \( P \). We agree that \( P_j \) is the empty set \( \emptyset \) if \( j \) is strictly greater than \( |P| \). Let \( X = (X, \mathcal{B}, \mu, T) \) be a measure preserving system and let \( P = \{x_0, P_1, \ldots, P_{k-1}\} \in P^k(X) \). We identify \( P \) with a function \( P : X \to \{0, \ldots, k-1\} \) defined by \( P(x) = j \) for \( x \in P_j \). The pair \( (X, P) \) is called a process, see [15, p. 273]. A coding of \( \underline{x} = (x_j)_{j=0}^\infty \subseteq X^\infty \) is \( P(\underline{x}) = (P(x))_{j=0}^\infty \subseteq \Omega_k \). The map \( P : X \to \Omega_k \) given by \( \overline{P}(x) = P(\underline{x}) \) defines a homomorphism of \( X \) and \( (\Omega_k, \mathcal{B}, \mu_P, \sigma) \), where \( \mu_P = \overline{P}_*(\mu) \) is the pushforward of the measure \( \mu \). For \( n \geq 0 \) and \( \underline{x} \in X^\infty \) let \( \phi^n_P(\underline{x}) = \overline{P}(x_0) \overline{P}(x_1) \cdots \overline{P}(x_{n-1}) \in \{0, 1, \ldots, k-1\}^n \). Note that for \( x \in X \) we have \( \phi^n_P(\underline{x}) = P^n(x) \), where \( P^n \) is the \( n \)-th join of \( P \) given by

\[
P^n = \bigvee_{j=0}^{n-1} T^{-j}(P)
\]

and elements of \( P^n \) are identified with \( n \)-words over \( \{0, 1, \ldots, k-1\} \). The measure preserving system \( (\Omega_k, \mathcal{B}, \mu_P, \sigma) \) is called the symbolic representation of \( X \) with respect to the partition \( P \) and \( \mu_P \) is the symbolic representation measure of \( \mu \). We endow \( P^k \) with the distance \( d^n_1 \) given for \( P, Q \in P^k \) by

\[
d^n_1(P, Q) = \frac{1}{2} \sum_{j=0}^{k-1} \mu(P_j \Delta Q_j) = \frac{1}{2} \sum_{j=0}^{k-1} \int_X |\chi_{P_j} - \chi_{Q_j}| \, d\mu = \mu(\{x \in X : P(x) \neq Q(x)\}).
\]

Note that the definition of \( d^n_1 \) takes into account the order of the partition’s elements. We tacitly identify Borel partitions \( P, Q \in P^k \) with \( d^n_1(P, Q) = 0 \). With this identification \( d^n_1 \) is a complete metric for \( P^k \).

1.7. Entropy. For a finite measurable partition \( P \) and \( \mu \in \mathcal{M}_T(X) \) we denote by \( h(\mu, P) \) the entropy of \( P \) with respect to \( \mu \) and \( T \) and by \( h(\mu) \) the entropy of \( \mu \) with respect to \( T \), that is \( h(\mu) = \sup P h(\mu, P) \), where \( h(\mu, P) = \inf_{n \in \mathbb{N}} -\sum_{P \in P^n} \mu(P) \log \mu(P) \). The real-valued function \( P \mapsto h(\mu, P) \) is uniformly continuous on \( P^k \) equipped with \( d^n_1 \) [15, Lemma 15.9(5)].

1.8. Faithful coding. For \( P \in P^k(X) \) we define \( \partial P = \partial P_0 \cup \ldots \cup \partial P_{k-1} \). A partition \( P \in P^k(X) \) with \( \mu(\partial P) = 0 \) is called faithful for \( X \).

Lemma 2. Let \( P \in P^k(X) \) be such that \( \mu(\partial P) = 0 \). If \( \underline{x} \in X^\infty \) is generic for \( \mu \in \mathcal{M}_T(X) \), then \( \omega = P(\underline{x}) \subseteq \Omega_k \) is a generic point for the measure \( \mu_P \).

Proof. Note that the boundary in a topological space has the following two properties \( \partial(Y \cap Z) \subseteq \partial Y \cup \partial Z \) for \( Y, Z \subseteq X \) and \( \partial T^{-1}(U) \subseteq T^{-1}(\partial U) \) for any \( U \subseteq X \) and continuous map \( T : X \to X \). Using this and \( \mu(\partial P_j) = 0 \) for every \( 1 \leq j \leq k \) we see that

\[
\mu(\partial (\bigcap_{i=0}^{m-1} T^{-i}(P_{j_i})) = 0 \quad \text{for every } m \geq 1 \text{ and } 1 \leq j_0, j_1, \ldots, j_{m-1} \leq k.
\]

In other words, \( \mu(\partial P^m) = 0 \) for all \( m \geq 1 \). Then for every \( m \geq 1 \) and \( 1 \leq j_0, j_1, \ldots, j_{m-1} \leq k \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \chi_A(x_j) = \mu(A), \quad \text{for } A = \bigcap_{i=0}^{m-1} T^{-i}(P_{j_i}) \subseteq P^m.
\]

Note that \( \omega = P(\underline{x}) \) is an orbit for \( \sigma \) and observe that \( \omega \) is generic for a \( \sigma \)-invariant measure \( \mu' \) such that

\[
\mu'([j_0j_1 \ldots j_{m-1}]) = \mu(\bigcap_{i=0}^{m-1} T^{-i}(P_{j_i})) \quad \text{for every } m \geq 1 \text{ and } 1 \leq j_0, j_1, \ldots, j_{m-1} \leq k.
\]

Hence \( \mu' \) and \( \mu_P \) agree on cylinders. This implies \( \mu' = \mu_P \).
1.9. Kakutani equivalence. Let $X = (X, \mathcal{F}_B, \mu, T)$ be a measure preserving system. For a set $B \in \mathcal{F}_B$ with $\mu(B) > 0$ and $x \in B$ we define the return time $n(x) = \inf\{k > 0 : T^k(x) \in B\}$. This function is finite for $\mu$-almost every $x \in B$ and we define the induced transformation $T_B : B \to B$ by $T_B(x) = T^{n(x)}(x)$. Measure preserving systems $X = (X, \mathcal{F}_B, \mu, T)$ and $Y = (Y, \mathcal{F}_B, \nu, S)$ are Kakutani equivalent (recall, that Katok calls this relation $\nu$ on $A$ ergodic shift invariant measures on $\mathcal{F}_B$). This equivalence is a direct corollary of Propositions 2.6 and 2.7 in [28]. We rephrase them in a way suitable for our purposes. The first result is a direct corollary of Proposition 2.6 and 2.7 in [28]. We state it for the set of all joinings of $\mu$ and $\nu$. Similarly, $J_n(\mu, \nu)$ denotes the set of all measures $\lambda_n$ on $\mathcal{F}_B^n \times \mathcal{F}_B^n$ whose marginals are $\mu_n$ and $\nu_n$.

Define
\[ f_n(\mu, \nu) = \inf_{\lambda_n \in J_n(\mu, \nu)} \int_{\mathcal{F}_B^n \times \mathcal{F}_B^n} f_n(u, v) \lambda_n(u, v). \]

One can prove (see [28]) that the following upper limit defines the $f$-bar distance between measures $\mu$ and $\nu$ on $\mathcal{F}_B^\infty$:
\[ \bar{f}(\mu, \nu) = \limsup_{n \to \infty} f_n(\mu, \nu). \]

Ornstein’s $d$-bar metric $d$ on $\mathcal{M}_\sigma(\mathcal{F}_B^\infty)$ is defined analogously with the Hamming distance $d_n$ on $\mathcal{F}_B^n$ replacing the edit distance $f_n$ in (2), see [32].

1.10. Feldman’s $f$ metric for ergodic shift-invariant measures on $\mathcal{F}_B^\infty$. Let $\mu$ and $\nu$ be ergodic shift invariant measures on $\mathcal{F}_B^\infty$. By $\mu_n$, respectively $\nu_n$, we denote the restriction of $\mu$, respectively $\nu$ to the set of all $n$-cylinders, that is, these are measures that $\mu$, respectively $\nu$, define on $\mathcal{F}_B^n$ via the projections onto first $n$ coordinates. A joining of $\mu$ and $\nu$ is any $\sigma \times \sigma$ invariant measure on $\mathcal{F}_B^\infty \times \mathcal{F}_B^\infty$ whose marginals are $\mu$ and $\nu$. We write $J(\mu, \nu)$ for the set of all joinings of $\mu$ and $\nu$. Similarly, $J_n(\mu, \nu)$ denotes the set of all measures $\lambda_n$ on $\mathcal{F}_B^n \times \mathcal{F}_B^n$ whose marginals are $\mu_n$ and $\nu_n$.

Define
\[ f_n(\mu, \nu) = \inf_{\lambda_n \in J_n(\mu, \nu)} \int_{\mathcal{F}_B^n \times \mathcal{F}_B^n} f_n(u, v) \lambda_n(u, v). \]

One can prove (see [28]) that the following upper limit defines the $f$-bar distance between measures $\mu$ and $\nu$ on $\mathcal{F}_B^\infty$:
\[ \bar{f}(\mu, \nu) = \limsup_{n \to \infty} f_n(\mu, \nu). \]

Ornstein’s $d$-bar metric $d$ on $\mathcal{M}_\sigma(\mathcal{F}_B^\infty)$ is defined analogously with the Hamming distance $d_n$ on $\mathcal{F}_B^n$ replacing the edit distance $f_n$ in (2), see [32].

1.11. Properties of $\bar{f}$. For the readers convenience we include here some statements extracted from [28]. We rephrase them in a way suitable for our purposes. The first result is a direct corollary of Propositions 2.6 and 2.7 in [28].

**Lemma 3.** For every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mu$ and $\nu$ are shift invariant ergodic measures on $\mathcal{F}_B^\infty$ and there are generic points $\omega$ for $\mu$ and $\omega'$ for $\nu$ with $\bar{f}(\omega, \omega') < \delta$, then $\bar{f}(\mu, \nu) < \varepsilon$.

Another consequence of Propositions 2.6 and 2.7 is discussed in [28, p. 12—13]. We state it explicitly. Note that the integrand on the left-hand side is $\bar{f}$ defined by (1).

**Lemma 4.** If $\mu$ and $\nu$ are shift invariant ergodic measures on $\mathcal{F}_B^\infty$, then for every $\varepsilon > 0$ we can find an ergodic joining $\lambda$ of $\mu$ and $\nu$ such that
\[ \int_{\mathcal{F}_B^\infty \times \mathcal{F}_B^\infty} \bar{f}(\omega, \omega') \ d\lambda(\omega, \omega') < \bar{f}(\mu, \nu) + \varepsilon. \]

Finally, we note that [28, Proposition 3.4] says that entropy function $\mu \mapsto h(\mu)$ is uniformly continuous with respect to the $\bar{f}$-metric on the space of ergodic shift-invariant measures on $\mathcal{F}_B^\infty$.

1.12. Loosely Kronecker systems. An ergodic measure preserving system $X = (X, \mathcal{F}_B, \mu, T)$ is loosely Kronecker if it has zero entropy and for every finite Borel partition $\mathcal{P}$ of $X$ and every $\varepsilon > 0$ there are $n > 0$ and a set $A_n$ of atoms of $\mathcal{F}_B^n = \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{P})$ such that $\mu(A_n) > 1 - \varepsilon$ and $\bar{f}_n(u, w) < \varepsilon$ for $u, w \in A_n$ (here, as usual, we identify atoms of the partition $\mathcal{P}_n$ with words of length $n$ over the alphabet $\{0, 1, \ldots, \{|\mathcal{P}| - 1\}\}$).

## 2. GIKN construction

We present the construction of Gorodetski, Ilyashenko, Kleptsyn, and Nalsky from [16] following the exposition provided by Bonatti, Díaz, and Gorodetski in [17]. Recall that $X$ is a compact metric space, $\rho$ is a metric for $X$, and $T : X \to X$ is a continuous map.
2.1. Topological backbone of the GIKN construction. In this paragraph we sketch the main features of the GIKN construction. Our theory applies to any measure defined in this way

Definition 5. We say that a $T$-periodic orbit $\Gamma$ is a $(\gamma, \kappa)$-good approximation of a $T$-periodic orbit $\Lambda$ if there exist a subset $\Delta$ of $\Gamma$ with $|\Delta|/|\Gamma| \geq \kappa$ and a constant-to-one surjection $\psi: \Delta \to \Lambda$ (called $(\gamma, \kappa)$-projection) such that for each $y \in \Delta$ and $0 \leq j < |\Lambda|$ we have

$$\rho(T^j(y), T^j(\psi(y))) < \gamma.$$ 

The following is a slightly reformulated [7] Lemma 2.5. The proof that $\mu$ is ergodic [7] invokes [16] Lemma 2.

Theorem 6. Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of $T$-periodic orbits and assume that $|\Gamma_n|$ increases with $n$. For each $n$ let $\mu_n$ be the ergodic measure supported on $\Gamma_n$. If there exist sequences of positive real numbers $(\gamma_n)_{n=1}^\infty$ and $(\kappa_n)_{n=1}^\infty$ satisfying

(1) for each $n$ the orbit $\Gamma_{n+1}$ is a $(\gamma_n, \kappa_n)$-good approximation of $\Gamma_n$,
(2) $\sum_{n=1}^\infty \gamma_n < \infty$,
(3) $\prod_{n=1}^\infty \kappa_n > 0$,

then $\mu_n \subset \mathbb{N}$ weak* converges to an ergodic measure $\mu$ supported on the topological limit of $(\Gamma_n)_{n \in \mathbb{N}}$, that is,

$$\text{supp} \mu = \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \Gamma_n.$$ 

Definition 7. We call the sequence of $T$-periodic orbits fulfilling the conditions from Theorem 6 the GIKN sequence and a measure provided by Theorem 6 the result of the GIKN construction.

2.2. Lyapunov exponents. We discuss the framework in which the GIKN construction is used to find nonhyperbolic measures. This part is logically independent from the rest of the paper, because our results apply to any measure obtained as weak* limit of periodic measures fulfilling the assumptions of Theorem 6.

Let $M$ be a smooth Riemannian manifold with $\dim M = m$. If $f: M \to M$ is a diffeomorphism and $\mu$ is an ergodic $f$-invariant measure, then there exist a set $\Lambda \subset M$ of full $\mu$-measure and real numbers $\chi^i_\mu \leq \cdots \leq \chi^m_\mu$ such that for every $x \in \Lambda$ and nonzero vector $v \in \mathcal{T}_x M$ one has

$$\lim_{n \to \infty} \frac{1}{n} \log \|Df^n_x(v)\| = \chi^i_\mu$$

for some $i = 1, \ldots, m$.

The number $\chi^i_\mu$ is the $i$th Lyapunov exponent of the measure $\mu$. If there exists a closed $f$-invariant set $\Xi \subset M$ and a $Df$-invariant direction field $\mathcal{E} = (E_x)_{x \in \Xi} \subset T\Xi M$ with $\dim E_x = 1$ for $x \in \Xi$, then for every measure $\nu \in \mathcal{M}_f(M)$ with supp $\nu \subset \Xi$ there is a Lyapunov exponent $\chi^\mathcal{E}(\nu)$ of $\nu$ associated with $\mathcal{E}$ in the following sense: for $\nu$-a.e. $x \in \Xi$ and $v \in E_x$ is nonzero one has

$$\lim_{n \to \infty} \frac{1}{n} \log \|Df^n_x(v)\| = \chi^\mathcal{E}(\nu).$$

Furthermore, if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of ergodic $f$-invariant measures supported on $\Xi$, $\mu \in \mathcal{M}_f^e(M)$ and $\mu_n \to \mu$ as $n \to \infty$ in the weak* topology (it implies that supp $\mu \subset \Xi$ as well), then $\chi^\mathcal{E}(\mu_n) \to \chi^\mathcal{E}(\mu)$ as $n \to \infty$ [10] Lemma 1.

Theorem 8. Assume that $f: M \to M$ is a diffeomorphism of a smooth Riemannian compact manifold with a closed $f$-invariant set $\Xi \subset M$ and a $Df$-invariant direction field $\mathcal{E} = (E_x)_{x \in \Xi} \subset T\Xi M$ with $\dim E_x = 1$ for $x \in \Xi$. Let $(\Gamma_n)_{n \in \mathbb{N}} \subset \Xi$ be a sequence of $f$-periodic orbits and suppose that $|\Gamma_n|$ increases to infinity as $n \to \infty$. For each $n$ let $\mu_n$ be the ergodic measure supported on $\Gamma_n$. Furthermore, assume that the following conditions hold:

(1) There exist sequences of positive real numbers $(\gamma_n)_{n=1}^\infty$ and a constant $C$ such that for each $n$ the orbit $\Gamma_{n+1}$ is a $(\gamma_n, 1 - C|\chi^\mathcal{E}(\mu_n)|)$-good approximation of $\Gamma_n$;
(2) There exists a constant $0 < \alpha < 1$ such that

$$|\chi^\mathcal{E}(\mu_{n+1})| < \alpha|\chi^\mathcal{E}(\mu_n)|;$$
(3) $\gamma_n < \frac{\min_{1 \leq i \leq n} d_i}{3 \cdot 2^n}$, where $d_i$ denotes the minimal distance between different points in $\Gamma_i$. 


Then \((\Gamma_n)_{n \in \mathbb{N}} \subset \Xi\) is a GIKN sequence and \((\mu_n)_{n \in \mathbb{N}}\) weak* converges to an ergodic measure \(\mu\) with uncountable support equal to the topological limit of \((\Gamma_n)_{n \in \mathbb{N}}\), that is,

\[
\text{supp} \mu = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \Gamma_n.
\]

Furthermore, \(\mu\) is nonhyperbolic, since \(\chi^2(\mu) = 0\).

3. Feldman-Katok Pseudometric

The metrics \(\bar{d}\) and \(\bar{f}\) on \(\mathcal{M}_p(\mathcal{A}^\infty)\) are related with identically denoted pseudometrics on \(\mathcal{A}^\infty\). This connection is described in more details in [28, 32, 35]. From the point of view of ergodic theory there is no need to extend \(\bar{d}\) and \(\bar{f}\) from \(\mathcal{A}^\infty\) to more general metric spaces. It turns out, however, that for some geometric applications an extension of \(\bar{d}\), called the Besicovitch pseudometric, is very useful (see [26] and references therein). Here we first recall the definition of Besicovitch pseudometric, then introduce Feldman-Katok pseudometric, which extends \(\bar{f}\) to general metric spaces.

3.1. Besicovitch pseudometric \(D_B\). For \(x = (x_j)_{j=0}^{\infty}, z = (z_j)_{j=0}^{\infty} \in X^\infty\) we define the Besicovitch pseudometric \(D_B\) on \(X^\infty\) as

\[
D_B(x, z) = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(x_j, z_j).
\]

It is known [26] that \(D_B\) is uniformly equivalent to \(D_B'\) given by

\[
D_B'(x, z) = \inf \left\{ \delta > 0 : \bar{d}(x_i, z_i) \leq \delta \text{ for all } i \right\}.
\]

Given \(T : X \to X\) the Besicovitch pseudometric \(D_B\) on \(X\) is defined by \(D_B(x, y) := D_B(x \cdot_T, y \cdot_T)\). If \(X = \mathcal{A}^\infty\) and \(\rho\) is any metric compatible with the topology on \(\mathcal{A}^\infty\), then the Besicovitch pseudometric \(D_B\) and the \(\bar{d}\)-bar pseudometric \(\bar{d}\) are uniformly equivalent on \(\mathcal{A}^\infty\).

3.2. Pseudometric \(F_K\). Let \(x = (x_j)_{j=0}^{\infty}, z = (z_j)_{j=0}^{\infty} \in X^\infty, \delta > 0, \text{ and } n \in \mathbb{N}\).

**Definition 9.** An \((n, \delta)\)-match of \(x\) and \(z\) is an order preserving bijection \(\pi : D(\pi) \to R(\pi)\) such that \(D(\pi), R(\pi) \subset \{0, 1, \ldots, n-1\}\) and for every \(i \in D(\pi)\) we have \(\rho(x_i, z_{\pi(i)}) < \delta\).

**Definition 10.** The fit \(|\pi|\) of an \((n, \delta)\)-match \(\pi\) is the cardinality of \(D(\pi)\).

**Definition 11.** An \((n, \delta)\)-match is maximal if its fit is the largest possible. If there is no \((n, \delta)\)-match \(\pi\) with \(|\pi| \geq 1\), then the empty match \(\pi_0\) with \(|\pi_0| = 0\) is the maximal one. The \((n, \delta)\)-gap between \(x\) and \(z\) is given by

\[
f_{n, \delta}(x, z) = 1 - \frac{\max\{|\pi| : \pi \text{ is an } (n, \delta)\text{-match of } x \text{ with } z\}}{n}.
\]

Note that if \(X = \mathcal{A}^\infty\) for some finite alphabet \(\mathcal{A}\) and we endow \(X = \mathcal{A}^\infty\) with the standard metric given by

\[
\rho(\omega, \omega') = \begin{cases} 0, & \text{if } \omega \neq \omega', \\ 2^{-\min\{j \geq 0 \mid \omega_j \neq \omega'_j\}}, & \text{otherwise}, \end{cases}
\]

then (with a minor abuse of notation) we have \(f_{n, \delta}(x, z) = f_{n, \delta}(x_0x_1 \ldots x_{n-1}, z_0z_1 \ldots z_{n-1})\) for any \(x, z \in \mathcal{A}^\infty\).

We note some properties of the \((n, \delta)\)-gap function \(f_{n, \delta}\) that hold for any \(x, z \in X^\infty, \varepsilon, \delta > 0, \text{ and } n \in \mathbb{N}\).

**Fact 12.** If \(\delta < \delta'\), then \(f_{n, \delta}(x, z) \leq f_{n, \delta'}(x, z)\).

**Fact 13.** If \(q \geq 1\), then \(f_{n, \delta}(x, z) \leq f_{n+q, \delta}(x, z) + q/2\).

**Fact 14.** If \(f_{n, \delta}(x, z) < \varepsilon\), then \(D_P(m(x, n), m(z, n))) < \max\{\delta, \varepsilon\}\).

**Fact 15.** Assume that \(x\) and \(z\) are periodic with a common period \(N\). If \(p^{(n)}\) denotes the fit of a maximal \((nN, \delta)\)-match, then the sequence \((p^{(n)})_{n=1}^{\infty}\) is subadditive, that is \(p^{(\ell)} + p^{(m)} \geq p^{(\ell + m)}\) for every \(\ell, m \in \mathbb{N}\).
Definition 16. The $f_δ$-pseudodistance between $\underline{x}$ and $\bar{\underline{z}}$ is given by
\[ f_δ(\underline{x}, \bar{\underline{z}}) = \limsup_{n \to \infty} f_{n, δ}(\underline{x}, \bar{\underline{z}}). \]

Fact 17. If $i ≥ 0$, then $f_δ(\underline{x}, \sigma^i(\bar{\underline{z}})) = f_δ(\underline{x}, \sigma^i(\bar{\underline{z}}))$. In particular, $f_δ(\underline{x}, \sigma(\bar{\underline{z}})) = 0$.

Fact 18. If $\underline{x}$ and $\bar{\underline{z}}$ are periodic sequences with a common period $N$, then
\[ f_δ(\underline{x}, \bar{\underline{z}}) = \inf_{n \in \mathbb{N}} f_{nN, δ}(\underline{x}, \bar{\underline{z}}) = \lim_{n \to \infty} f_{nN+q, δ}(\underline{x}, \bar{\underline{z}}) \]
for every $0 < q < N$.

Definition 19. The Feldman-Katok pseudometric on $X^\infty$ is given by
\[ F_K(\underline{x}, \underline{z}) = \inf \{ δ > 0 : f_δ(\underline{x}, \underline{z}) < δ \}. \]

The Feldman-Katok pseudometric on $X$, also denoted by $F_K$, is defined for $x, z \in X$ by
\[ F_K(x, z) := F_K(\underline{x}, \underline{z}). \]

The symbols $f_{n, δ}(x, z)$ and $f_δ(x, z)$ have the obvious meaning.

Fact 20. If $\underline{x}, \underline{z} \in X^\infty$ and $f_δ(\underline{x}, \underline{z}) ≤ \varepsilon$, for some $δ, \varepsilon > 0$, then $F_K(\underline{x}, \underline{z}) ≤ δ + \varepsilon$.

Proof. If $f_δ(\underline{x}, \underline{z}) ≤ \varepsilon$, then $f_{δ+\varepsilon}(\underline{x}, \underline{z}) < δ + \varepsilon$ by Fact 19. Thus $F_K(\underline{x}, \underline{z}) ≤ δ + \varepsilon$. □

Remark 21. By Fact 20 the set $\{ δ > 0 : f_δ(\underline{x}, \underline{z}) < δ \}$ is nonempty for every $\underline{x}, \underline{z} \in X^\infty$. Furthermore, $0 ≤ f_δ(\underline{x}, \underline{z}) ≤ 1$ for all $\underline{x}, \underline{z} \in X^\infty$ and $δ > 0$. This together with Fact 20 imply that $0 ≤ F_K(\underline{x}, \underline{z}) ≤ 1$ for every $\underline{x}, \underline{z} \in X^\infty$.

Fact 22. The function $F_K$ is a pseudometric on $X^\infty$, as well as on $X$.

Definition 23. We say that $x, y \in X$ are orbitally related and write $x \sim y$ if $T^i(x) = T^j(y)$ for some $i, j ≥ 0$.

Fact 24. If $x \sim x'$ and $y \sim y'$, then $F_K(x, y) = F_K(x', y')$.

Proof. It is enough to prove that $f_δ(x, y) = f_δ(x', y')$ for every $δ > 0$. Thus, we fix $δ > 0$. If $x \sim x'$ and $y \sim y'$, then there are $i, j, m, n ≥ 0$ such that $T^i(x) = T^j(x')$ and $T^m(y) = T^n(y')$. Using Fact 17 repeatedly we have
\[ f_δ(x, y) = f_δ(T^i(x), y) = f_δ(T^j(x'), y) = f_δ(x', y), \]
and similarly, $f_δ(x', y') = f_δ(x', y')$. □

By Fact 24, the Feldman-Katok pseudometric depends rather on the separation between forward orbits of given points, than of the points alone.

3.3. Comparison with the Besicovitch Pseudometric. We note that our results about Feldman pseudometric generalize those known for Besicovitch pseudometric (see [20] for more details).

Lemma 25. If $\underline{x}, \underline{z} \in X^\infty$, then $F_K(\underline{x}, \underline{z}) ≤ D_B'(\underline{x}, \underline{z})$.

Proof. Define
\[ d_{n, δ}(\underline{x}, \underline{z}) = \frac{1}{n} \cdot \left| \{ 0 ≤ j < n : \rho(x_j, z_j) ≥ δ \} \right|. \]
If $D_B'(\underline{x}, \underline{z}) < δ$ for some $δ > 0$ then for all $n$ large enough $d_{n, δ}(\underline{x}, \underline{z}) < δ$. It follows that there exists an $(n, δ)$-match of $\underline{x}$ with $\underline{z}$. Therefore $F_K(\underline{x}, \underline{z}) ≤ D_B'(\underline{x}, \underline{z})$. □
4. $\hat{F}_K$-CONVERGENCE OF MEASURES AND ITS PROPERTIES

It is not clear whether $\hat{F}_K$ corresponds to a metric on $\mathcal{M}_T^f(X)$ or $\mathcal{M}_T(X)$ as it happens for $\hat{f}$ on $\mathcal{X}^\infty$. Nevertheless, we may use $\hat{F}_K$ to define a certain notion of “convergence” for $\mathcal{M}_T(X)$.

**Definition 26.** We say that a sequence of measures $(\mu_n)_{n=1}^\infty \subset \mathcal{M}_T(X)$ converges in $\hat{F}_K$ or $\hat{F}_K$-converges to $\mu \in \mathcal{M}_T(X)$ if there exists a sequence of quasi-orbits $(x^{(n)})_{n=1}^\infty \subset X^\infty$ with $\hat{\omega}(x^{(n)}) = \{\mu_n\}$ such that for some $\mu$-generic quasi-orbit $\underline{z} \in X^\infty$ we have $\hat{F}_K(\underline{z}, x^{(n)}) \to 0$ as $n \to \infty$.

Keeping in mind that we are interested in the GIKN construction, we will examine properties of the $\hat{F}_K$-convergence in a special case, where the quasi-orbits $(x^{(n)})_{n=1}^\infty \subset X^\infty$ are actually orbits (for each $n \in \mathbb{N}$, $z^{(n)}$ is the orbit of a $\mu_n$-generic point).

4.1. “Completeness” of $\hat{F}_K$-convergence. Since $\hat{F}_K$ is a pseudometric on $X^\infty$, given $T: X \to X$ on $X$ such notions as $\hat{F}_K$-Cauchy sequence or $\hat{F}_K$-limit have obvious meaning. The first important feature of the $\hat{F}_K$-convergence is that it has enough “completeness” for our purposes: an $\hat{F}_K$-Cauchy sequence of orbits defines a quasi-orbit which is its $\hat{F}_K$-limit.

**Definition 27.** We say that a sequence of quasi-orbits $(x^{(n)})_{n=1}^\infty \subset X^\infty$ is $\hat{F}_K$-Cauchy if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\hat{F}_K(\underline{x}, x^{(k)}) < \varepsilon$ for $k, \ell \geq N$.

Repeating the proof of [3, Proposition 2] we get the following fact.

**Lemma 28.** If $(x^{(n)})_{n=1}^\infty \subset X$ is an $\hat{F}_K$-Cauchy sequence on $X$, then there is a quasi-orbit $\underline{z} = (z_j)_{j=0}^\infty \subset X^\infty$ such that $\hat{F}_K(\underline{z}, x^{(n)}) \to 0$ as $n \to \infty$.

In fact, the proof of the above proposition yields that there are $0 = m_0 < m_1 < m_2 < \ldots$ and a subsequence $(x^{(n)})_{n=1}^\infty$ of $(x^{(n)})_{n=1}^\infty$ such that $d(\{m_j : j \geq 0\}) = 0$ and $\underline{z}$ is given by $z_n = T^n(x^{(m_j)})$ for $m_{j-1} \leq n < m_j$.

We do not know whether one can claim in Lemma 28 that there is a point whose orbit is a $\hat{F}_K$-limit of the $\hat{F}_K$-Cauchy sequence. We can prove it only under an additional assumption about $T: X \to X$. It turns out that the asymptotic average shadowing property (it is a generalization of the shadowing property introduced by Gu [17]) is sufficient. The asymptotic average shadowing property follows from most of the version of the specification property considered in the literature, see [23, 25, 26].

**Definition 29.** A sequence $\underline{x} = (x_j)_{j=0}^\infty \subset X^\infty$ is asymptotic average pseudo-orbit for $T: X \to X$ if

$$D_B(T, \underline{x}, \underline{z}) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \rho(T(z_j), z_{j+1}) = 0.$$ 

We say that $\underline{x} = (x_j)_{j=0}^\infty \subset X^\infty$ is asymptotically shadowed in average by $x \in X$, if

$$D_B(\underline{x}, \underline{z}) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \rho(T^j(x), z_j) = 0.$$ 

(Note that then $D_B(\underline{x}, \underline{z}) = \hat{F}_K(\underline{x}, \underline{z}) = 0$ as well.) A system $(X, T)$ has the asymptotic average shadowing property if every asymptotic average pseudo-orbit of $T$ is asymptotically shadowed in average by some point.

**Corollary 30.** If $(X, T)$ satisfies the asymptotic average shadowing property, then $\hat{F}_K$ is a complete pseudometric on $X$.

**Proof.** Fix an $\hat{F}_K$-Cauchy sequence $(x^{(n)})_{n=1}^\infty \subset X$. Note that the quasi-orbit $\underline{z} \in X^\infty$ provided by Lemma 28 is an asymptotic average pseudo-orbit and satisfies $\hat{F}_K(\underline{x}, \underline{z}) = 0$. Pick $x \in X$ which asymptotically shadows in average $\underline{z}$. Then

$$\hat{F}_K(x, x^{(n)}) \leq \hat{F}_K(\underline{x}, \underline{z}) + \hat{F}_K(\underline{z}, x^{(n)}) \to 0 \quad \text{as } n \to \infty.$$ 

Therefore $x$ is an $\hat{F}_K$-limit of $(x^{(n)})_{n=1}^\infty$ in $X$.

It turns out that the set of generic quasi-orbits is $\hat{F}_K$-closed. More is true: $\hat{\omega}(x)$ depends $\hat{F}_K$-continuously on $\underline{x}$ when we consider $\underline{x}$ as a point in the hyperspace of $\mathcal{M}(X)$.
Definition 31. The hyperspace of a compact metric space $Z$ endowed with metric $\rho_Z$ is the set of nonempty closed subsets of $Z$ endowed with the Hausdorff metric induced by $\rho_Z$.

Fact 32. If $\hat{F}_K(x, z) < \varepsilon$ for some $\varepsilon > 0$, then $D_H(\hat{\omega}(\hat{x}), \hat{\omega}(\hat{z})) < \varepsilon$, where $D_H$ is the Hausdorff metric on the hyperspace of $\mathcal{M}(X)$ endowed with $D_P$.

Proof. If $\hat{F}_K(x, z) < \varepsilon$, then for some $\tilde{F}_K(x, z) \leq \delta < \varepsilon$ we have $\tilde{f}_n(x, z) < \delta$ for all $n$ large enough and $D_P(m(\hat{x}, n), m(\hat{z}, n)) < \varepsilon$ by Fact 14. This implies that $D_H(\hat{\omega}(\hat{x}), \hat{\omega}(\hat{z})) < \varepsilon$.

Although the proof of the following fact is short, its importance justifies calling it a theorem.

Theorem 33. Let $(\hat{x}^{(n)})_{n=1}^{\infty} \in X^\infty$ be such that for each $n \in \mathbb{N}$ there is $\mu_n \in \mathcal{M}(X)$ with $\hat{x}^{(n)} \in \text{Gen}(\mu_n)$. If $x \in X^\infty$ and $\tilde{F}_K(x, z^{(n)}) \to 0$ as $n \to \infty$, then there exists $\mu \in \mathcal{M}(X)$ such that $\mu_n \to \mu$ as $n \to \infty$ in $\mathcal{M}(X)$.

Proof. By Fact 32, $\tilde{F}_K(x, z^{(n)}) \to 0$ as $n \to \infty$ implies that $\hat{\omega}(z^{(n)}) \to \hat{\omega}(x)$ as $n \to \infty$ in the hyperspace of $\mathcal{M}(X)$. Since $z^{(n)}$ is generic for $\mu_n$, we have $\hat{\omega}(z^{(n)}) = \{\mu_n\}$ for $n \in \mathbb{N}$. The family of all singletons is closed in the hyperspace and homeomorphic with $\mathcal{M}(X)$. Thus $\hat{\omega}(x)$ must also be a singleton, that is $\hat{\omega}(x) = \{\mu\}$ for some $\mu \in \mathcal{M}(X)$ and $\mu_n \to \mu$ as $n \to \infty$ in $\mathcal{M}(X)$.

An immediate consequence of Lemma 28 and Theorem 33 is the following.

Corollary 34. An $\hat{F}_K$-Cauchy sequence of generic points uniquely determine an invariant measure.

Definition 35. An invariant measure determined by a $\hat{F}_K$-Cauchy sequence of generic points is called the $\hat{F}_K$-limit of the corresponding sequence of measures.

5. GIKN sequence is $\hat{F}_K$-Cauchy

Careful inspection of concrete examples of $(\gamma, \kappa)$-good approximations presented in the literature shows that it is always the case that the two periodic orbits considered are $(\gamma + (1 - \kappa))$-close with respect to $\hat{F}_K$. Yet the abstract definition of $(\gamma, \kappa)$-good approximation does not immediately imply that $(\gamma, \kappa)$-projection $\psi$ defines a match. The reason is that a match is order preserving and the definition above does not require that. Hence we need the following technical lemma.

Lemma 36. If a $T$-periodic orbit $\Gamma$ is a $(\gamma, \kappa)$-good approximation of a $T$-periodic orbit $\Lambda$, then for any choice of $x \in \Gamma$ and $z \in \Lambda$ for which the requested property holds. Let $\psi: \Delta \to \Lambda$ be the $(\gamma, \kappa)$-projection from $\Gamma$ to $\Lambda$.

Proof. It follows from Fact 24 that it is enough to show that there exists $x_0 \in \Gamma$ and $z_0 \in \Lambda$ for which the requested property holds. Let $\psi: \Delta \to \Lambda$ be the $(\gamma, \kappa)$-projection from $\Gamma$ to $\Lambda$. Pick any $x_0 \in \Delta$ and let $z_0 = \psi(x_0) \in \Lambda$. By Fact 20 it is enough to show that $f_s(x_0, z_0) < 1 - \kappa$.

Define $q := |\Delta|$. Furthermore, since $x_0$ and $z_0$ are periodic, we conclude from Fact 13 and Fact 14 that it is sufficient to find for any multiple $p$ of $|\Gamma|$ and $|\Delta|$ a $(p + q, \gamma)$-match $\pi$ of $x_0$ with $z_0$ such that $|\pi| \geq kp$. Therefore from now on we fix $p$ which is a multiple of $|\Gamma|$ and $|\Delta|$. For $j \geq 1$ define $x_j = T^j(x_0)$ and $z_j = T^j(z_0)$. Note that $(z_j)_{j=0}^{\infty}$ is a $q$-periodic sequence. We will abuse the notation and treat $\{x_0, \ldots, x_{p-1}\}, \{z_0, \ldots, z_{p-1}\}$ as sets with $p$ elements, still denoted by $\Gamma$ and $\Delta$. Furthermore we extend $\psi$ to a function from $\Delta \subset \{x_0, \ldots, x_{p-1}\}$ to $\{z_0, \ldots, z_{p-1}\}$ where $|\Delta| = r$ and $r/p \geq \kappa$.

We are going to define the match $\pi$ by performing an inductive construction with at most $|\Delta|$ steps and at each step we will extend the domain of $\pi$ by at least one element. Enumerate elements of $\Delta$ as $y_0, \ldots, y_{r-1}$ in such a way that the order induced by this indexing of $\Delta$ coincides with the one induced by enumerating the elements of $\Gamma$ as $x_0, \ldots, x_{p-1}$. By definition $x_0 = y_0$. Let $\theta: \{0, \ldots, r-1\} \to \{0, 1, \ldots, p-1\}$ be a function such that $\theta(s)$ for $0 \leq s < r$ is the position of $y_s$ in the sequence $x_0, \ldots, x_{p-1}$, that is, $\theta(s) = j$ if and only if $y_s = x_j$. We begin with $D(\pi) = \emptyset$ and $\pi(0) = 0$. By definition, $z_{\theta(0)} = \psi(x_0) = z_0$, hence $\rho(x_0, z_{\theta(0)}) < \gamma$, and $\pi$ is a $(p + q, \gamma)$-match.

Now assume that we have already performed some number of steps of our construction and we have obtained a $(p + q, \gamma)$-match $\pi$ such that $0 \leq s < r$ is the largest integer satisfying: $|\pi| \geq s + 1$, $\theta(s) \in D(\pi)$, $\pi(\theta(s)) \leq \theta(s)$ and $z_{\pi(\theta(s))} = \psi(\theta(s))$. It follows from our construction that $s + 1$ is always greater or equal than the number of steps performed. We can extend $\pi$ by setting $\pi(\theta(s) + i) = \theta(s) + i$ for $0 < i < q$. Now, there are two cases to consider: either the domain of...
\( \pi \) contains \( \{\theta(0), \ldots, \theta(r-1)\} \) and we are done or there is \( s < t < r \) which is the smallest integer such that \( \pi \) is not defined at \( \theta(t) \). Clearly, in the latter case \( \theta(t) - \theta(s) \geq q \). Since \( (z_j)_{j=0}^\infty \) is a \( q \)-periodic sequence there exists \( 1 \leq \ell < q \) such that \( z_{\theta(s)+\ell} = \psi(y_1) \) and \( \pi(\theta(s)) < \theta(s) + \ell \leq \theta(t) \). We want to set \( \pi(\theta(t)) = \theta(s) + \ell \), but to keep \( \pi \) increasing we need to remove first at most \( q - 1 \) elements from \( D(\pi) \) as defined so far, namely those in \( D(\pi) \cap \{\theta(s) + \ell, \ldots, \theta(s) + q - 1\} \). But then we can set \( \pi(\theta(t) + i) = \theta(t) + i = \theta(s) + \ell + i \) for \( 1 \leq i < q \), because
\[
z_{\pi(\theta(t) + i)} = T^i(z_{\theta(t)}), \quad \rho(T^i(z_{\theta(t)}), T^i(\psi(y_1))) < \gamma.
\]
For our new \( \pi \) we see that the largest integer \( s \) in \( \{0, 1, \ldots, r-1\} \) satisfying \( |\pi| \geq s+1 \), \( \theta(s) \in D(\pi) \), \( \pi(\theta(s)) \leq \theta(s) \) and \( z_{\pi(\theta(s))} = \psi(y_s) \) is larger or equal \( t \). Thus in a finite number of steps our procedure will produce a \((p + q, \gamma)\)-match \( \pi \) with \( |D(\pi)| \geq sp \).

As a consequence we see that any measure obtained through the GIKN construction is the \( \hat{F}_K \)-limit of periodic measures.

**Theorem 37.** Let \( (\Gamma_n)_{n \in \mathbb{N}} \) be a sequence of \( T \)-periodic orbits and assume that \( |\Gamma_n| \) increases with \( n \). For each \( n \) let \( \mu_n \) be the ergodic measure supported on \( \Gamma_n \). If there exist sequences of positive real numbers \( (\gamma_n)_{n=1}^\infty \) and \( (\kappa_n)_{n=1}^\infty \) such that
(1) for each \( n \) the orbit \( \Gamma_{n+1} \) is a \((\gamma_n, \kappa_n)\)-good approximation of \( \Gamma_n \),
(2) \( \sum_{n=1}^\infty \gamma_n < \infty \),
(3) \( \prod_{n=1}^\infty \kappa_n > 0 \),
then for any choice \( x_n \in \Gamma_n \) the sequence \( (x_n)_{n=1}^\infty \) is \( \hat{F}_K \)-Cauchy.

6. \( \hat{F}_K \)-LIMIT OF ERGODIC MEASURES IS ERGODIC

We are going to show that an \( \hat{F}_K \)-limit of a sequence of ergodic measures must be ergodic. To this end we first present a criterion for ergodicity of a measure generated by a quasi-orbit. We obtain it by an easy adaptation of an analogous criterion for orbits given by Oxtoby [29].

6.1. Auxiliary terminology and results. For \( k \in \mathbb{N} \), \( \tilde{\varphi} = (x_j)_{j=0}^\infty \in X^\infty \) and \( \varphi \in C(X) \) let \( A_k(\varphi, \tilde{\varphi}) \) denote the Birkhoff average along \( x_0, x_1, \ldots, x_{k-1} \), that is,
\[
A_k(\varphi, \tilde{\varphi}) = \frac{1}{k} \sum_{j=0}^{k-1} \varphi(x_j) = \int_X \varphi \, d\mu(\tilde{\varphi}, k).
\]
For \( x \in X \) we write \( A_k(\varphi, x) \) for the Birkhoff average along orbit segment of length \( k \), that is \( A_k(\varphi, x) := A_k(\varphi, \tilde{\varphi}_T) \). Recall that a sequence \( \tilde{\varphi} \in X^\infty \) (a point \( x \in X \), respectively) is generic for some measure \( \mu \) if and only if for every \( \varphi \in C(X) \) the sequence \( A_k(\varphi, \tilde{\varphi}) \) (respectively, \( A_k(\varphi, x) \)) converges as \( k \to \infty \). We denote the corresponding limit by \( \varphi^*(\tilde{\varphi}) \) (respectively, \( \varphi^*(x) \)). It is easy to see that for any \( t \in \mathbb{N} \) we have \( \varphi^*(\tilde{\varphi}) = \varphi^*(\sigma^t(\tilde{\varphi})) \), respectively \( \varphi^*(x) = \varphi^*(T^t(x)) \). For a generic sequence \( \tilde{\varphi} \) we put
\[
A^*_k(\varphi, x_T) := |A_k(\varphi, (x_j)_{j=0}^\infty) - \varphi^*((x_j)_{j=0}^\infty)| = |A_k(\varphi, \sigma^t(\tilde{\varphi})) - \varphi^*(\tilde{\varphi})|.
\]
Furthermore, for a generic point \( x \) and \( t \in \mathbb{N} \) we define
\[
A^*_k(\varphi, T^t(x)) := |A_k(\varphi, T^t(x)) - \varphi^*(x)|.
\]

We say that \( t \geq 0 \) initiates an \((\alpha, \varphi)\)-bad segment in a generic sequence \( \tilde{\varphi} = (x_j)_{j=0}^\infty \), where \( \alpha > 0 \) and \( \varphi \in C(X) \) if \( A^*_k(\varphi, \sigma^t(\tilde{\varphi})) > \alpha \).

The following characterization of ergodic sequences slightly generalizes the one presented by Oxtoby in [29] Section 4. It states that a generic sequence \( \tilde{\varphi} = (x_j)_{j=0}^\infty \) generates an ergodic measure if for every \( \varphi \in C(X) \) and \( \alpha > 0 \) the upper density of the set of integers \( t \) initiating an \((\alpha, \varphi)\)-bad \( k \)-segments converges to zero as \( k \) goes to \( \infty \). Replacing \( \tilde{\varphi} \) by \( T^t(\tilde{\varphi}) \) we obtain a criterion for ergodicity of a measure given by an generic point due to Oxtoby. We omit the proof as it follows the same lines as in [29].

**Theorem 38** (Oxtoby’s Criterion). Let a quasi-orbit \( \tilde{\varphi} = (z_j)_{j=0}^\infty \in X^\infty \) be generic for some \( \mu \in M_T(X) \). Then \( \mu \) is ergodic if and only if for every \( \varphi \in C(X) \) and \( \alpha > 0 \) we have
\[
d(\{t \geq 0 : |A_k(\varphi, \sigma^t(\tilde{\varphi})) - \varphi^*(\tilde{\varphi})| > \alpha\}) \to 0 \text{ as } k \to \infty.
\]
Let \( n \in \mathbb{N}, x \in X \) and \( \bar{z} \in X^{\infty} \). For an \((n, \delta)\)-match \( \pi \) of \( \bar{z}_T \) with \( \bar{z} \) and \( k \leq n \) we define for every \( \ell \in \mathcal{D}(\pi) \cap \{0, 1, \ldots, n-k\} \) a set
\[
\mathcal{D}'_\ell = \{ i \in \mathcal{D}(\pi) : \ell \leq i < \ell + k \text{ and } \pi(\ell) \leq \pi(i) < \pi(\ell + k) \}
\]
and the \((k, \delta)\)-match \( \pi'_k : \mathcal{D}'_\ell \to \pi(\mathcal{D}'_\ell) \) induced by \( \pi \) at \( \ell \) setting \( \pi'_k(\ell) = \pi(i) \) for \( i \in \mathcal{D}(\pi'_k) \). It is easy to see that \( \pi'_k \) is indeed an \((k, \delta)\)-match of \( T^k(x) \) with \( z_{\pi(\ell)} \) satisfying \( \mathcal{D}(\pi'_k) = \mathcal{D}'_\ell \) and \( \mathcal{R}(\pi'_k) = \pi(\mathcal{D}'_\ell) \).

We also need the following technical lemma, which asserts that if an orbit \( \bar{z}_T \) and a quasi-orbit \( \bar{x} \) are sufficiently \( F_K \)-close, then for any \( \varphi \in \mathcal{C}(X) \) and \( k \in \mathbb{N} \) one can find a match \( \pi \) which allows one to find a match \( \pi \) of \( \bar{z}_T \) and \( \bar{z} \) so that the averages of \( \varphi \) over \( k \) segments in \( \bar{z}_T \) and \( \bar{z} \) are also small for most of \( k \)-segments.

**Lemma 39.** Fix \( \varphi \in \mathcal{C}(X) \) and \( \varepsilon > 0 \). Let \( \delta > 0 \) be such that \( y, y' \in X \) and \( \rho(y, y') < \delta \) imply \( |\varphi(y) - \varphi(y')| < \varepsilon \). If \( \bar{x} = (z_j)_{j=0}^\infty \in X^{\infty} \) is a quasi-orbit and \( x \in X \) satisfies \( F_K(z_j, \bar{x}) < \delta \), then for every \( k \in \mathbb{N} \) there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \) there are an \((n, \delta)\)-match \( \pi \) of \( \bar{z}_T \) with \( \bar{z} \) and a set \( A \subset \mathcal{D}(\pi) \) with \( |A| > n(1 - 2\sqrt{\delta} - 2\delta) \) and \( k \) satisfying
\[
|A_k(\varphi, T^k(x)) - A_k(\varphi, \sigma^{\pi(\ell)}(\bar{x}))| \leq \varepsilon + 4\sqrt{\delta}||\varphi||_\infty \quad \text{for every } \ell \in A.
\]

**Proof.** Fix \( k \in \mathbb{N}, \varphi \in \mathcal{C}(X) \) and \( \varepsilon > 0 \). Choose \( N \in \mathbb{N} \) such that for every \( n \geq N \) there exists an \((n, \delta)\)-match \( \pi \) of \( \bar{z}_T \) with \( \bar{z} \) satisfying \(|\pi| > n(1 - \delta)\) and
\[
|\{0 \leq j < n : T(z_{j+1}) \neq z_{j+1+1} \text{ for some } 0 \leq i < k\}| < n\delta.
\]

Fix \( n \geq N \). Define
\[
A_Z = \{0 \leq j < n : T(z_{\pi(j+1)}) \neq z_{\pi(j)+1+1} \text{ for some } 0 \leq i < k\},
\]
\[
A_R = \{0 \leq j < n - k : j \in \mathcal{D}(\pi) \text{ and } |\{0 \leq i < k : \pi(j) + i \notin \mathcal{R}(\pi)\}| \geq \sqrt{\delta}k\},
\]
\[
A_D = \{0 \leq j < n - k : j \in \mathcal{D}(\pi) \text{ and } |\{0 \leq i < k : j + i \notin \mathcal{D}(\pi)\}| \geq \sqrt{\delta}k\}.
\]

Note that \(|A_Z| \leq n\delta\). To estimate \(|A_R|\), define \( \mathcal{R}^c(\pi) = \{0, 1, \ldots, n-1\} \setminus \mathcal{R}(\pi) \), and note that
\[
|\mathcal{R}^c(\pi)| \geq |\{m \in \mathcal{R}^c(\pi) : \pi(j) \leq m < \pi(j) + k \text{ for some } j \in A_R\}|.
\]

On the other hand, for each \( m \in \mathcal{R}^c(\pi) \) the set \( \{j \in A_R : \pi(j) \leq m < \pi(j) + k\} \) has at most \( k \) elements, hence each \( j \in A_R \) implies that there are at least \( \sqrt{\delta} \) members of \( \mathcal{R}^c(\pi) \). Thus \( |\mathcal{R}^c(\pi)| \geq |A_R|\sqrt{\delta} \). This together with \( |\mathcal{R}^c(\pi)| < n\delta \) gives us \(|A_R| < n\sqrt{\delta}\). An analogous reasoning leads to \(|A_D| < n\sqrt{\delta}|. Define
\[
A = ((0, \ldots, n-k-1) \cap \mathcal{D}(\pi)) \setminus (A_Z \cup A_D \cup A_R).
\]

Then \(|A| > n(1 - 2\sqrt{\delta} - 2\delta) - k \) and for every \( j \in A \) we have
(i) the \((k, \delta)\)-match \( \pi'_j \) induced by \( \pi \) at \( j \) satisfies \(|\pi'_j| > (1 - 2\sqrt{\delta})k\),
(ii) \( A_k(\varphi, z_{\pi(j)}) = A_k(\varphi, \sigma^{\pi(\ell)}(\bar{x})) \).

Therefore
\[
|A_k(\varphi, T^j(x)) - A_k(\varphi, \sigma^{\pi(\ell)}(\bar{x}))| = |A_k(\varphi, T^j(x)) - A_k(\varphi, z_{\pi(j)})| \leq \frac{1}{k} \sum_{i \in \mathcal{D}(\pi'_j)} |\varphi(T^j(x)) - \varphi(z_{\pi(j)})| + \frac{1}{k} \sum_{i \in \mathcal{R}(\pi'_j)} |\varphi(T^j(x))| + \frac{1}{k} \sum_{i \in \mathcal{R}(\pi'_j)} |\varphi(z_i)| \leq \varepsilon + 2\sqrt{\delta}||\varphi||_\infty + (1 - 2\sqrt{\delta})k||\varphi||_\infty = \varepsilon + 2\sqrt{\delta}||\varphi||_\infty
\]
and the lemma follows. \( \square \)

6.2. \( F_K \)-limits of ergodic measures are ergodic. We prove the main theorem of this section. As the GIKN sequence of periodic orbits is \( F_K \)-Cauchy by Theorem 37 we see that our Theorem 40 generalizes [16, Lemma 2]. Furthermore, since \( F_K \leq D_B \) this result extends also [26, Theorem].

**Theorem 40.** If \((x^{(p)})_{p=1}^\infty \subset X\) is an \( F_K \)-Cauchy sequence of ergodic points then it determines an ergodic measure.
Proof. By Fact \ref{14} there exists a measure \( \mu \in \mathcal{M}_T(X) \) such that \( \mu_p \to \mu \) as \( p \to \infty \) in the weak* topology on \( \mathcal{M}_T(X) \). We will apply Oxtoby’s criterion (Theorem \ref{18}) to show that \( \mu \) is ergodic.

Let \( \bar{z} = (z_j)_{j=0}^{\infty} \) be a quasi-orbit such that \( f(z_j^{(p)}) \to 0 \) as \( p \to \infty \) provided by Lemma \ref{22}. Clearly, \( \bar{z} \) is generic for \( \mu \). Fix \( \varphi \in \mathcal{C}(X) \) and \( \alpha > 0 \). We need to show that for every \( \eta > 0 \) and all sufficiently large \( k \) the set of \( j \)'s with initiating an \((\alpha, \varphi)\)-bad \( k \)-segment in \( \bar{z} \) has upper density smaller than \( \eta \).

Note that for every \( i, j, k, p \in \mathbb{N} \) we have

\[
|A_k(\varphi, \sigma^j(\bar{z}) - \varphi^*(\bar{z}))| \leq |A_k(\varphi, \sigma^j(\bar{z})) - A_k(\varphi, T^i(x^{(p)}))| + |A_k(\varphi, T^i(x^{(p)})) - \varphi^*(x^{(p)})| + \left| \varphi^*(x^{(p)}) - \varphi^*(\bar{z}) \right|.
\]

By Remark \ref{23} and Fact \ref{14} we can choose \( P(\alpha) \in \mathbb{N} \) such that for every \( p \geq P(\alpha) \) one has \( |\varphi^*(x^{(p)}) - \varphi^*(\bar{z})| \leq \alpha/3 \).

Let \( \delta_0 > 0 \) be such that \( 2\sqrt{\delta_0} + 2\delta_0 < \eta/4 \) and \( 4\sqrt{\delta_0}||\varphi||_{\infty} < \alpha/6 \). Pick \( \delta < \delta_0 \) such that for all \( y, y' \in X \) with \( \rho(y, y') < \delta \) one has \( |\varphi(y) - \varphi(y')| < \alpha/6 \). Note that our choice of constants is motivated by Lemma \ref{22}. Choose \( p \geq P(\alpha) \) such that \( f(z_{\bar{z}}^{(p)^{\delta}}) < \delta \). Pick \( K > 0 \) such that for all \( k \geq K \) the upper density of the set of \( i \)'s initiating an \((\alpha/3, \varphi)\)-bad \( k \)-segment in \( x_i^{(p)} \) is smaller than \( \eta/2 \). We are going to prove that each \( k \geq K \) is “sufficiently large” to imply that the upper density of the set of \( j \)'s initiating an \((\alpha, \varphi)\)-bad \( k \)-segment in \( \bar{z} \) is smaller than \( \eta \). To this end fix any \( k \geq K \).

Let \( N \) be sufficiently large to guarantee that \( k/N < \eta/4 \) and for every \( n \geq N \) we have

(i) there exists an \((n, \delta)\)-match \( \pi \) of \( \bar{z} \) with \( x^{(p)} \) such that \( |\pi| > (1-\delta)n \),

(ii) \( |0 \leq j < n : T(z_{j+1}) \neq z_{j+1+1} \) for some \( 0 \leq i < k | < n\delta \),

(iii) \(|0 \leq i < n : \pi \) initiates an \((\alpha/3, \varphi)\)-bad \( k \)-segment in \( x_i^{(p)} \) \( || \eta/2 \).

We are going to show that for each \( n \geq N \) the number of \( 0 \leq j < n \) such that

\[
A_k(\varphi, \sigma^j(\bar{z}) - \varphi^*(\bar{z})) \leq \alpha
\]

is larger than \( (1-\eta)n \). To this end let \( \bar{\pi} \) be any extension of \( \pi \) to a bijection from \( \{0, \ldots, n-1\} \) onto \( \{0, \ldots, n-1\} \). It follows from the proof of Lemma \ref{22} that conditions \ref{24} and \ref{26} guarantee that the number of integers \( 0 \leq j < n \) for which

\[
|A_k(\varphi, \sigma^j(\bar{z})) - A_k(\varphi, T^{\bar{\pi}(j)}(x^{(p)}))| \leq \alpha/6 + 4\sqrt{\delta_0}||\varphi||_{\infty} < \alpha/3
\]

is larger than \( n(1-2\sqrt{\delta} - 2\delta) - k \) and hence larger than \( n(1-\eta/2) \). Some of these \( j \)'s may initiate an \((\alpha/3, \varphi)\)-bad \( k \)-segment in \( x_i^{(p)} \), but \ref{26} bounds from above the number of such \( j \)'s by \( \eta n/2 \). Therefore setting \( i = \bar{\pi}(j) \) in \ref{28} we see that the number of integers \( 0 \leq j < n \) for which all summands on the right hand side of \ref{26} are bounded above by \( \alpha/3 \) is larger than \( n(1-\eta) \). Since this holds true for any \( n \geq N \) we see that upper density of the set of all \( j \)'s initiating an \((\alpha, \varphi)\)-bad \( k \)-segment in \( \bar{z} \) is smaller than \( \eta \), hence the Oxtoby criterion yields that \( \bar{z} \) is an ergodic sequence.

\( \square \)

7. Lower semicontinuity of entropy in \( \bar{F}_K \)

We are going to show that the function assigning to an ergodic point the entropy of the associated measure is lower semicontinuous with respect to \( \bar{F}_K \).

**Theorem 41.** If a sequence of ergodic measures \( (\mu_n)_{n=1}^{\infty} \) converges in \( \bar{F}_K \) to \( \mu_0 \in \mathcal{M}_T(X) \), then

\[
h(\mu_0) \leq \liminf_{n \to \infty} h(\mu_n).
\]

Before proceeding with the proof, we first note the following technical result allowing us, given an \( \bar{F}_K \)-converging sequence of invariant measures to replace any partition with a faithful partition without much change in \( f \)-distance between codings of generic points of the measures in the sequence.

**Lemma 42.** If \( \delta > 0 \), \( P \in \mathcal{P}^k(X) \) and \( (\mu_j)_{j=0}^{\infty} \subset \mathcal{M}_T(X) \), then there are \( \gamma > 0 \) and \( \mathcal{R} \in \mathcal{P}^{k+1}(X) \) with \( d^*(P, \mathcal{R}) < \delta \) satisfying: if \( z \in X^\infty \) is generic for \( \mu_0 \), \( j \in \mathbb{N} \), and \( z \in X^\infty \) is generic for \( \mu_j \), then \( f(\mathcal{R}(z), \mathcal{R}(z)) < \delta + F_K(z, \mathcal{R}) \) and \( \mu_j(\partial \mathcal{R}) = \mu_0(\partial \mathcal{R}) = 0 \). The same holds if we replace \( f \) by \( \bar{d} \) and \( F_K \) by \( D_B \).
Proof. Fix $\delta > 0$. Using regularity of $\mu_0$ for each $1 \leq j \leq k$ we can find a compact set $R_j \subset P_j$ such that $\mu_0(P_j \setminus R_j) < \delta/(8k^2)$. For $1 \leq i < j \leq k$ the sets $R_i$ and $R_j$ are compact and disjoint, hence
\[
\Delta = \min\{\rho(x,y) : x \in R_i, y \in R_j, 1 \leq i < j \leq k\} > 0.
\]
Set $\tilde{\gamma} = \Delta/2$. For each $1 \leq j \leq k$ and $0 \leq c < \tilde{\gamma}$ let $R_j^c$ be the closed c-hull around $Q_j$, that is, $R_j^c = \{x \in X : \text{dist}(x, Q_j) \leq c\}$ and set $R_j^0 = X \setminus (R_j^c \cup \ldots \cup R_k^c)$. For each $0 \leq c < \tilde{\gamma}$ define the partition $\mathcal{R}^c = \{R_0^c, R_1^c, \ldots, R_k^c\}$. It is easy to see that $d_{1}^{\mu_0}(P, \mathcal{R}^0) < \delta/2$ and $d_{1}^{\mu_0}(\mathcal{R}^{\alpha}, \mathcal{R}^{\beta}) < \delta/2$ for any $0 \leq \alpha, \beta < \tilde{\gamma}$.

For $1 \leq j \leq k$ and $0 < c < \tilde{\gamma}$ define a set $\partial_j Q_j = \{x \in X : \text{dist}(x, Q_j) = c\}$. Note that $\partial_j Q_j$ contains (but need not to be equal) the topological boundary of the set $R_j^c = \{x \in X : \text{dist}(x, Q_j) \leq c\}$. Consider a family of closed sets $\mathcal{C} = \{\partial_j Q_1 \cup \ldots \cup \partial_j Q_k : 0 < c < \tilde{\gamma}\}$. Since elements of $\mathcal{C}$ are pairwise disjoint, only countably many of them can have positive $\mu_j$ measure for some $j \in \mathbb{N} \cup \{0\}$. Therefore the set $E$ of all parameters $0 < c < \tilde{\gamma}$ such that for each $c \in E$ the set $\partial_j Q_1 \cup \ldots \cup \partial_j Q_k \in \mathcal{C}$ is a $\mu_j$-null set for $j \in \mathbb{N} \cup \{0\}$ has at most countable complement in $(0, \tilde{\gamma})$. Thus we can pick $\alpha, \beta \in E$ with $\alpha < \beta$ and $\beta - \alpha > \tilde{\gamma}/2$. Set $\gamma = \tilde{\gamma}/2$. Let $z \in X^{\infty}$ be a generic sequence for $\mu_j$ for some $j \in \mathbb{N}$ and $f(x, z) < \gamma$. Note that $\beta - \alpha > f(x, z)$.

Define $x' = R^{\alpha}(x)$, $x'' = R^{\beta}(x)$, $x' = R^{\alpha}(x)$, and $x'' = R^{\beta}(x)$. (all three points are considered as elements of the alphabet of the shift space over the set $\{0, 1, \ldots, k\}$.

We claim that $f(x', z') < f(x, z) + \delta/2$. Indeed, if for some $m, n \geq 0$ and $1 \leq j \leq k$ we have $x_n \in R_j^\alpha$ and $\rho(x_n, z_m) < \rho(x, z) < \beta - \alpha$, then $R_j^\beta(x_m) = j$. Furthermore, genericity of $z$ for $\mu_0$ and $\partial_j Q_j \in \mathcal{C}$ implies that $d((n \geq 0 : x_n \in R_j^\alpha)) = \mu_0(R_j^\alpha) > \delta/2$. This proves the claim.

Using again that $\partial_j Q_j$ is generic for $\mu_j$ and $\partial_j Q_j$ are $\mu_j$-null we have easily that $d(x', x''') \leq d(x', x''') < d(x', x'') < \delta/2$. This together with the claim above and $f(x', x'') \leq d(x', x'')$ complete the proof for the case. The $d$-part follows the same way.

Proof of Theorem [7]. It follows from Theorem [40] that $\mu_0$ is ergodic and there exists a quasi-orbit $z$ which is generic for $\mu$. Fix $\varepsilon > 0$. Let $\mathcal{P} = \{P_0, \ldots, P_k\}$ be a measurable partition of $X$ with $h(\mu_0, \mathcal{P}) \geq h(\mu_0) - \varepsilon/3$. Let $\zeta > 0$ be so small that $f(y, y') < \zeta$ for two shift ergodic points $y, y' \in \{0, 1, \ldots, k\}^{\infty}$ implies that the entropies of the corresponding ergodic measures differ by at most $\varepsilon/3$ (the existence of such a $\zeta$ is guaranteed by [28 Proposition 3.4]). Since the function $\mathcal{P} \mapsto h(\mu, \mathcal{P})$ is uniformly continuous on $\mathbb{P}^{k+1}$ [15 Lemma 15.9(5)], we may pick $0 < \delta < \varepsilon/2$ such that for any partition $S = \{S_0, S_1, \ldots, S_k\}$ with $d_{1}^{\mu_0}(\mathcal{P}, S) < \delta$ we have $|h(\mu_0, P) - h(\mu_0, S)| < \varepsilon/3$.

Use Lemma to find $\gamma > 0$ for $\mu_0$, $\delta/2$ and $\mathcal{P}$. Let $N \in \mathbb{N}$ be such that $f(x, x^{(N)}(x)) < \min\{\gamma, \delta/2\}$. Take the partition $\mathcal{R}$ provided by Lemma for $x^{(N)}(x)$ and $\mu_N$. Let $x' = \mathcal{R}(x)$ and $x'' = \mathcal{R}(x^{(N)}(x))$. Clearly It follows from Lemma that $\mu'_{\mathcal{R}}$ is a generic point for some measure $\mu'_{\mathcal{R}}$ with $h(\mu'_{\mathcal{R}}) = h(\mu_N, \mathcal{R}) \leq h(\mu_N)$. By Lemma we see that $f(x', x''') < \delta$. Therefore $|h(\mu_0) - h(\mu_N)| < \varepsilon/3$, hence
\[
h(\mu_N) \geq h(\mu_N, S) = h(\mu'_{\mathcal{R}}) \geq h(\mu_0) - \varepsilon.
\]
This finishes the proof.

Note that entropy is continuous if $X = \mathcal{A}^{\infty}$ and we equip $\mathcal{M}(X)$ in $f$ metric (which, as we noted above, is uniformly equivalent with $F_{K}$ induced by a standard metric on $\mathcal{A}^{\infty}$). This is because in that case the entropy function $\mathcal{M}_{T}(X) \ni \mu \mapsto h(\mu) \in \mathbb{R}$ is upper semicontinuous with respect to the weak$^*$ topology on $\mathcal{M}_{T}(X)$. It also holds if $X$ is a manifold and $T$ is of class $C^{\infty}$ or, more generally, if $T$ is asymptotically $h$-expansive, see [12]. But the entropy function need not to be continuous with respect to the $F_{K}$-convergence.

Example 43. Let $\mathcal{A} = \{0\} \cup \{1/k : k \in \mathbb{N}\}$ with the topology inherited from $[0, 1]$. Consider $X = \mathcal{A}^{\infty}$ with any metric compatible with the product topology. Let $\xi^{(n)}$ be a measure on $\mathcal{A}$ uniformly distributed on $\{1/\ell : 2^{n} \leq \ell < 2^{n+1}\}$ and $\mu^{(n)}$ denote the product measure on $X$. It is easy to see that $\mu^{(n)}$ is a shift invariant measure on $\mathcal{A}^{\infty}$ and the sequence $(\mu^{(n)})_{n \in \mathbb{N}}$ converges in $\bar{F}_{K}$ to the measure concentrated on a $\sigma$-fixed point $(0, 0, 0, \ldots) \in \mathcal{A}^{\infty}$. Furthermore, $h(\mu^{(n)}) = n \log 2$, which means that the entropy function cannot be upper semicontinuous.
8. GIKN construction leads to loosely Kronecker measure

We are going to show that $\tilde{F}_K$-limit of periodic measures is either a periodic measure or a loosely Kronecker measure. Since the GIKN construction yields a measure with an uncountable support it must be loosely Kronecker.

**Theorem 44.** An aperiodic $\tilde{F}_K$-limit of periodic measures is loosely Kronecker.

We present the proof at the end of this section. Before that we recall Katok’s criterion for standardness (loosely Kronecker) and formulate two technical lemmas we will need for the proof.

8.1. Katok’s criterion. Let $X = (X, M, \mu, T)$ be a measure preserving system and let $P = \{P_0, P_1, \ldots, P_k-1\} \in \mathbb{P}^k(X)$. Following Katok [19] Definition 9.1 we say that the process $(X, P)$ is $(n, \varepsilon)$-trivial if there exists a word $\omega \in \{0,1,\ldots,k-1\}^n$ such that $\mu(P(\omega)) \geq 1 - \varepsilon$, where $B_\varepsilon(\omega) = \{\omega' \in \{0,1,\ldots,k-1\}^n : f_n(\omega,\omega') < \varepsilon\}$. By [19] Lemma 9.1 if $\omega \in \{0,1,\ldots,k-1\}^n$ and $\beta > 0$ are such that

$$\int_X f_n(P^n(x), \omega) \, d\mu(x) = \int_{\Omega_k} f_n(u_0u_1\ldots u_{n-1}, \omega) \, d\mu_p(u) < \beta,$$

then the process $(X, P)$ is $(n, \sqrt{\beta})$-trivial.

A process $(X, P)$ is $M$-trivial [19] Definition 9.2 if for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for every $n \geq N$ the process $(X, P)$ is $(n, \varepsilon)$-trivial.

**Theorem 45** ([19], Theorem 4, (1)$\Leftrightarrow$(2)). An aperiodic measure preserving system $X$ is loosely Kronecker if and only if for every finite partition $P$ of $X$ the process $(X, P)$ is $M$-trivial.

8.2. Two auxiliary lemmas. In the proof of our main theorem we will need two lesser known properties of $f$.

**Lemma 46.** If $P, R \in \mathbb{P}^k(X)$ and $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ and a measurable set $G \subseteq \text{Gen}(\mu)$ with $\mu(G) > 1 - \varepsilon$ such that for every $x \in G$ and $n \geq N$ we have

$$\bar{f}_n(P(\omega), R(\omega)) < d^R_n(P, R) + \varepsilon.$$

**Proof.** Let $P = \{P_0, P_1, \ldots, P_k-1\}$, $R = \{R_0, R_1, \ldots, R_k-1\}$ and $\Delta = \{P_0 \cap R_0 \cup (P_1 \cap R_1) \cup \ldots \cup (P_{k-1} \cap R_{k-1})\}$. Observe that ergodicity of $\mu$ implies that for $\mu$-a.e. $x \in X$ we have

$$d_n(P(\omega), R(\omega)) = \frac{1}{n} \sum_{j=0}^{n-1} \chi_{\Delta}(T^j(x)) \frac{(n \to 0)}{(n \to 0)} \mu(\Delta) = \mu(\{x \in X : P(x) \neq R(x)\}) = d^R_n(P, R).$$

Given $x \in X$ and $N \in \mathbb{N}$ we define measurable functions $f_{P,R}^n(x) = \bar{f}_n(P(\omega), R(\omega))$ and $\bar{f}_R^P(x) = \sup_{n \geq N} \bar{f}_n(P(\omega), R(\omega))$. The function $f_{P,R}^n$ is the pointwise limit of $f_{P,R}^n$ because

$$\bar{f}_{P,R}^n(x) = \bar{f}(P(\omega), R(\omega)) = \limsup_{n \to \infty} \bar{f}_n(P(\omega), R(\omega)) = \inf_{K \in \mathbb{N}} \sup_{k \geq K} \bar{f}_k(P(\omega), R(\omega)) = \lim_{K \to \infty} \sup_{k \geq K} \bar{f}_k(P(\omega), R(\omega)) = \lim_{K \to \infty} \bar{f}_{P,R}^n(x).$$

Finally, note that $\bar{f}_n(P(\omega), R(\omega)) \leq d_n(P(\omega), R(\omega))$ for all $n \in \mathbb{N}$. Now we finish the proof by applying twice the Egorov Theorem.

**Lemma 47.** If $\mu$ and $\nu$ are ergodic shift invariant measures and $\bar{f}(\mu, \nu) < \varepsilon$, then for $\mu$-a.e. $\xi$ there exists $\xi' \in \text{Gen}(\nu)$ with $\bar{f}(\mu, \nu) \leq \bar{f}(\xi, \xi') < \varepsilon$.

**Proof.** It is enough to show that for every $m \in \mathbb{N}$ there is a set of pairs $(\omega, \omega')$ projecting onto sets of full $\mu$ and $\nu$ measure and such that $\bar{f}(\mu, \nu) \leq \bar{f}(\omega, \omega') < \bar{f}(\mu, \nu) + 1/m$. Therefore we fix $m \in \mathbb{N}$ and use Lemma 46 to find an ergodic joining $\hat{\lambda}$ of $\mu$ and $\nu$ such that

$$\bar{f}(\mu, \nu) \leq \int \bar{f}(u, u') \, d\hat{\lambda}(u, u') < \bar{f}(\mu, \nu) + 1/m.$$
Write \( \hat{\lambda}_n \) for the projection of \( \hat{\lambda} \) onto \( \mathcal{A}^n \times \mathcal{A}^n \). Clearly, \( \hat{\lambda}_n \in J_n(\mu_n, \nu_n) \). Hence, using Fatou’s lemma for bounded functions, we have

\[
\tilde{f}(\mu, \nu) = \lim_{n \to \infty} \tilde{f}_n(\mu, \nu) \leq \limsup_{n \to \infty} \int_{\mathcal{A}^n \times \mathcal{A}^n} \tilde{f}_n(u, u') \, d\hat{\lambda}_n(u, u')
\]

\[
= \limsup_{n \to \infty} \int_{\mathcal{A}^n \times \mathcal{A}^n} \tilde{f}_n(\omega_0, \ldots, \omega_{n-1}, \omega'_0, \ldots, \omega'_{n-1}) \, d\hat{\lambda}(\omega, \omega')
\]

\[
\leq \int_{\mathcal{A}^n \times \mathcal{A}^n} \limsup_{n \to \infty} \tilde{f}_n(\omega_0, \ldots, \omega_{n-1}, \omega'_0, \ldots, \omega'_{n-1}) \, d\hat{\lambda}(\omega, \omega')
\]

\[
= \int_{\mathcal{A}^n \times \mathcal{A}^n} \tilde{f}(\omega, \omega') \, d\hat{\lambda}(\omega, \omega') \leq \int_{\mathcal{A}^n \times \mathcal{A}^n} \tilde{f}(\omega, \omega') \, d\hat{\lambda}(\omega, \omega') \leq \tilde{f}(\mu, \nu) + 1/m.
\]

Since \( \tilde{f} \) is a bounded Borel measurable function on \( \mathcal{A}^\omega \times \mathcal{A}^\omega \) we may use ergodicity of the joining \( \hat{\lambda} \) to conclude that \( \hat{\lambda} \)-a.e. pair \( (\zeta, \zeta') \) in \( \mathcal{A}^\omega \times \mathcal{A}^\omega \) is generic for \( \hat{\lambda} \), satisfies \( \zeta \in \text{Gen}(\mu), \zeta' \in \text{Gen}(\nu) \) and we have

\[
\tilde{f}(\zeta, \zeta') = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(\sigma^j(\zeta), \sigma^j(\zeta')) = \int_{\mathcal{A}^\omega \times \mathcal{A}^\omega} \tilde{f}(\omega, \omega') \, d\hat{\lambda}(\omega, \omega').
\]

Note that the first equality above holds because \( \tilde{f} \) is a \( \sigma \times \sigma \)-invariant function, the second follows from the ergodic theorem. This completes the proof.

The rest of this section is devoted to the proof of Theorem 44.

**Proof of Theorem 44** Let \( \mu = \mu^{(0)} \) be an \( F_K \)-limit of a sequence of periodic measures \( \mu^{(1)}, \mu^{(2)}, \ldots \). For \( n \in \mathbb{N} \) write \( s^{(n)} \) for a periodic point generating \( \mu^{(n)} \). Then \( \mu^{(0)} \) is ergodic and has a generic quasi-orbit \( \xi \) with \( F_K(\xi^{(n)}_i) \to 0 \) as \( n \to \infty \). To apply Katok’s criterion (Theorem 45) we need to show that for every finite partition \( P \) of \( X \) the process \( (X, P) \), where \( X = (X, \mathcal{B}, \mu^{(0)}, T) \) is M-trivial. To this end we choose a partition \( P \in \mathcal{P}^k(X) \) and fix \( \varepsilon > 0 \).

We use Lemma 3 to find \( 0 < \alpha < \varepsilon/6 \) such that \( \tilde{f}(\omega, \omega') < \alpha \) for some \( \omega \in \text{Gen}(\xi), \omega' \in \text{Gen}(\xi) \), where \( \xi \) and \( \zeta \) are shift invariant ergodic measures on \( \Omega_{k+1} \) implies that \( \tilde{f}(\xi, \xi) < \varepsilon/3 \).

Apply Lemma 12 to \( (\mu^{(n)})_{n=0}^{\infty} \) to find \( \gamma = \gamma(\mu, \alpha/2, \mathcal{P}) \) and a partition \( R \in \mathcal{P}^{k+1}(X) \) with \( \mu^{(n)}(\partial R) = 0 \) for \( n = 0, 1, \ldots \). Let \( N' \) be such that \( F_K(\xi^{(n)}_i, x^{(n)}_T) < \min\{\gamma, \alpha/2\} \) for \( n \geq N' \). It follows from Lemma 12 that for \( n \geq N' \) we have

\[
\tilde{f}(R(\xi), R(\xi^{(n)})) < \alpha.
\]

By Lemma 2 for \( n \geq 1 \) the point \( R(\xi^{(n)}) \) is generic for \( R \)-representation measure of \( \mu^{(n)} \) denoted \( \mu^{(n)}_R \) and \( R(\xi) \) is generic for the \( R \)-representation \( \mu^{(0)}_R \) of \( \mu^{(0)} \). The inequality 6 and our choice of \( \alpha \) imply that if \( n \geq N' \), then \( \tilde{f}(\mu^{(0)}_R, \mu^{(n)}_R) < \varepsilon/3 \). It follows from Lemma 17 that given \( n \geq N' \) for \( \mu^{(0)}_R \)-almost every point \( \omega \) in \( \Omega_{k+1} \) there exists a point \( \omega' \) in the orbit of \( R(\xi^{(n)}) \) such that \( \tilde{f}(\omega, \omega') = \tilde{f}(\omega, R(\xi^{(n)})) < \varepsilon/3 \). Fix \( n \geq N' \). Then we have

\[
\limsup_{m \to \infty} \int_X \tilde{f}_m(R(\xi^{(n)}), R(\xi^{(n)})) \, d\mu^{(0)}(z) \leq \int_X \limsup_{m \to \infty} \tilde{f}_m(R(\xi^{(n)}), R(\xi^{(n)})) \, d\mu^{(0)}(z)
\]

\[
= \int_{\mathcal{A}^\omega} \tilde{f}(\omega, R(\xi^{(n)})) \, d\mu^{(0)}(\omega) < \varepsilon/3.
\]

It follows that there is \( N'' \in \mathbb{N} \) such that for \( m \geq N'' \) we have

\[
\int_X \tilde{f}_m(R(\xi^{(n)}), R(\xi^{(n)})) \, d\mu^{(0)}(z) < \varepsilon/2.
\]

By Lemma 18 there exist \( N''' \in \mathbb{N} \) and a set \( G = G(\mathcal{P}, Q) \subset \text{Gen}(\mu) \subset X \) with \( \mu(G) > 1 - \varepsilon/6 \) such that for every \( z \in G \) and \( m \geq N''' \) we have

\[
\tilde{f}_m(\mathcal{P}(\xi^{(n)}), R(\xi^{(n)})) < d^{(0)}_1(\mathcal{P}, \mathcal{R}) + \varepsilon/6 < \varepsilon/3.
\]
If $m \geq N = \max\{N', N''\}$, then
\[ \int_X \bar{j}_m(P(z^T), R(z^T)) d\mu(0)(z) \leq \int_X \bar{j}_m(P(z^T), R(z^T)) d\mu(0)(z) + \int_X \bar{j}_m(R(x^T), R(z^T)) d\mu(0)(z) \]
\[ \leq \varepsilon/6 + \int_G \bar{j}_m(P(z^T), R(z^T)) d\mu(0)(z) + \varepsilon/2 < \varepsilon. \]

Since $\varepsilon > 0$ was arbitrary, we have proved that $(X, P)$ is $M$-trivial. Since $P$ was also arbitrary and $\mu(0)$ is aperiodic we conclude using Theorem 45 that $\mu(0)$ is standard.

As a consequence of Theorem 37, Theorem 40, and Theorem 41, we get the following corollary.

**Theorem 48.** If the invariant measure $\mu$ resulting from the GIKN construction based on a GIKN sequence of periodic orbits $(\Gamma_n)_{n \in \mathbb{N}}$ is aperiodic then $\mu$ is a loosely Kronecker measure (hence, ergodic and with zero entropy) supported on
\[ \text{supp } \mu = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \Gamma_n. \]

Combining Theorem 48 with Theorem 8 we get a strengthening of the latter.

**Theorem 49.** Under the assumptions of Theorem 8 the resulting measure is a nonhyperbolic loosely Kronecker measure (in particular, it is ergodic and has zero entropy) supported on
\[ \text{supp } \mu = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \Gamma_n. \]

Theorems 48 and 49 may replace [7, Lemma 2.5] and they reveal more information about the resulting measure. This applies for example to [10, Theorem B] or [10, Proposition 1.1]. Similar strengthenings are possible for results from [4, 7, 8, 11, 16, 22]. We leave the details to the reader, since presenting them here would require repeating a lot of material from these papers without introducing anything new.

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(D. Kwietniak) Faculty of Mathematics and Computer Science, Jagiellonian University in Kraków, ul. Łojasiewicza 6, 30–348 Kraków, Poland and Institute of Mathematics, Federal University of Rio de Janeiro, Cidade Universitaria - Ilha do Fundão, Rio de Janeiro 21945-909, Brazil

E-mail address: dominik.kwietniak@uj.edu.pl
URL: www.im.uj.edu.pl/DominikKwietniak/

(M. Łącka) Faculty of Mathematics and Computer Science, Jagiellonian University in Kraków, ul. Łojasiewicza 6, 30–348 Kraków, Poland

E-mail address: martha.lacka@doctoral.uj.edu.pl
URL: www.im.uj.edu.pl/MarthaLacka/