Pre-Lagrangian Submanifolds In Contact Manifolds*

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Abstract

In this article, we prove the non-existence of exact pre-Lagrangian submanifolds in contact manifolds by using the Gromov’s nonlinear Fredholm alternative.

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1 Introduction and Results

Let $U$ be a $(2n - 1)$-dimensional manifolds. A contact structure $\xi$ on $U$ is a completely nonintegrable codimension 1 tangent distribution. It means that $\xi$ can be defined, at least locally, by a 1–form $\lambda$ with $\lambda \wedge (d\lambda)^{n-1} \neq 0$. Note that if $n$ is odd then the contact distribution $\xi$ is automatically orientable. For an even $n$ the existence of a contact structure implies the orientability of the ambient manifold $U$. In both cases, the coorientability of $\xi$ implies that $\xi$ and $U$ are both orientable. We will assume from now on that $\xi$ is

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coorientable and fix its orientation. Then $\xi$ can be globally defined by a 1-form $\lambda$, which is determined up to a multiplication by a positive function. Let $SU = U \times [0, \infty]$. We will still denote by $\lambda$ the pull back of $\lambda$ by the projection of $SU = U \times R_+$ on the first factor and denote by $t$ the projection on the second. Then the form $\omega = d(a\lambda)$ defines a symplectic structure on $SU$ (indeed, $(d(a\lambda))^n = a^{n-1}a \wedge \lambda \wedge d\lambda^{n-1} \neq 0$). The map $(x, a) \to (x, a/f(x))$ induces an isomorphism of forms $d(a\lambda)$ and $d(a\mu)$ for $\mu = f\lambda$. Therefore, the symplectic manifold $(SU, \omega)$ depends, up to a symplectomorphism, only on the contact manifolds $(U, \xi)$ and not on the choice of the 1-form $\lambda$.

For an $n$-dimensional manifolds $M$ let us denote by $P_+T^*(M)$ the oriented projective cotangent bundle of $M$ with the contact structure $\xi$ defined by the form $pdq$. The manifold $P_+T^*(M)$ can also be considered a space of cooriented $(n-1)$-dimensional contact elements of $M$. With this interpretation the plane $\xi_x$ of $\xi$ at a point $x = (p, q)$, $q \in M$, $p \in T^*_q(M)$, consists of infinitesimal deformations of $\xi_x$, which leaves fixed the point of contact $q \in M$. Then the symplectization $\text{Sympl}(P_+T^*(M), \xi)$ is isomorphic to $T^*(M) \setminus M$ with the standard symplectic structure $\omega = d(pdq)$, for more example, see[1-6].

According to [2,4], the following notion was suggested by D. Bennequin.

An $n$-dimensional submanifolds $L$ of the $(2n-1)$-dimensional contact manifold $(U, \xi)$ is called pre-Lagrangian if it satisfies the following two conditions:

a. $L$ is transverse to $\xi$;

b. The distribution $\xi \cap T(L)$ is integrable and can be defined by a closed 1-form.

For any pre-Lagrangian submanifold $L \subset U$ there exists a Lagrangian submanifold $\tilde{L} \subset S_\xi U$ such that $\pi(\tilde{L}) = L$. The cohomology class $\lambda \in H^1(L; R)$, such that $\pi^*\lambda = [\alpha_\xi]\tilde{L}$, is defined uniquely up to multiplication by a non-zero constant. Conversely, if $L \subset U$ is the (embedded) image of a Lagrangian submanifold $\tilde{L} \subset S_\xi U$ under the projection $S_\xi U \to U$ then $L$ is pre-Lagrangian(see[2,4]). Thus with any pre-Lagrangian submanifold $L \subset U$ one can canonically associate a projective class of the form $\lambda$. The main result of this paper is following:

**Theorem 1.1** There does not exist any pre-Lagrangian submanifold $L \subset U$ with the canonical projective class equal to zero, especially any simply connected manifold cannot be embedded in $(U, \xi)$ as a pre-Lagrangian submanifold.

**Theorem 1.2** Let $(U, \lambda)$ be a closed contact manifold and $\varphi : L \to (U, \lambda)$ a closed Pre-Lagrangian embedding, then $[\varphi^*(\lambda)] \neq 0$ in $H^*(L, R)$, especially $H^1(L) \neq 0$.

**Sketch of proofs**: We will work in the framework proposed by Gromov in [5]. In Section 2, we study the linear Cauchy-Riemann operator and
sketch some basic properties. In section 3, we study the space $\mathcal{D}(V, W)$ of contractible disks in manifold $V$ with boundary in Lagrangian submanifold $W$ and construct a Fredholm section of tangent bundle of $\mathcal{D}(V, W)$. In Section 4, we use the Gromov’s trick in [5] to estimate the energy of the solution $s$ of the nonlinear Cauchy-Riemann equations. In the final section, we use Gromov’s nonlinear Fredholm trick to complete our proof as in [5].

2 Linear Fredholm Theory

For $100 < k < \infty$ consider the Hilbert space $V_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$, such that $u(z) \in R^n \subset C^n$ for almost all $z \in \partial D$. $L_{k-1}$ denotes the usual Hilbert $L_{k-1}$-space $H^{k-1}(D, C^n)$. We define an operator $\bar{\partial} : V_p \mapsto L_p$ by

$$\bar{\partial}u = u_s + iu_t \quad (2.1)$$

where the coordinates on $D$ are $(s, t) = s + it$, $D = \{z \mid |z| \leq 1\}$. The following result is well known (see [5]).

**Proposition 2.1** $\bar{\partial} : V_p \mapsto L_p$ is a surjective real linear Fredholm operator of index $n$. The kernel consists of the constant real valued maps.

Let $(C^n, \sigma = -Im(\cdot, \cdot))$ be the standard symplectic space. We consider a real $n$-dimensional plane $R^n \subset C^n$. It is called Lagrangian if the skew-scalar product of any two vectors of $R^n$ equals zero. For example, the plane $p = 0$ and $q = 0$ are Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of $R^{2n}$ is called the Lagrangian-Grassmanian $\Lambda(n)$. One can prove that the fundamental group of $\Lambda(n)$ is free cyclic, i.e. $\pi_1(\Lambda(n)) = Z$. Next assume $(\Gamma(z))_{z \in \partial D}$ is a smooth map associating to a point $z \in \partial D$ a Lagrangian subspace $\Gamma(z)$ of $C^n$, i.e. $(\Gamma(z))_{z \in \partial D}$ defines a smooth curve $\alpha$ in the Lagrangian-Grassmanian manifold $\Lambda(n)$. Since $\pi_1(\Lambda(n)) = Z$, one have $[\alpha] = ke$, we call integer $k$ the Maslov index of curve $\alpha$ and denote it by $m(\alpha)$, see (1).

Now let $z : S^1 \mapsto R^n \subset C^n$ be a smooth curve. Then it defines a constant loop $\alpha$ in Lagrangian-Grassmanian manifold $\Lambda(n)$. This loop defines the Maslov index $m(\alpha)$ of the map $z$ which is easily seen to be zero.

Now let $(V, \omega)$ be a symplectic manifold and $W \subset V$ a closed Lagrangian submanifold. Let $u : D^2 \rightarrow V$ be a smooth map homotopic to constant map with boundary $\partial D \subset W$. Then $u^* TV$ is a symplectic vector bundle and $(u|_{\partial D})^* TW$ be a Lagrangian subbundle in $u^* TV$. Since $u$ is contractible, we can take a trivialization of $u^* TV$ as

$$\Phi(u^* TV) = D \times C^n$$

and

$$\Phi(u|_{\partial D})^* TW) \subset S^1 \times C^n$$
Let
\[ \pi_2 : D \times C^n \to C^n \]
then
\[ \bar{u} : z \in S^1 \to \pi_2(\Phi(u|_{\partial D})) TW(z) \in \Lambda(n). \]
Write \( \bar{u} = u|_{\partial D} \).

**Lemma 2.1** Let \( u : (D, \partial D^2) \to (V, W) \) be a \( C^k \)-map (\( k \geq 1 \)) as above. Then,
\[ m(u|_{\partial D}) = 0 \]
Proof. Since \( u \) is contractible in \( V \) relative to \( W \), we have a homotopy \( \Phi_s \) of trivializations such that
\[ \Phi_s(u^*TV) = D \times C^n \]
and
\[ \Phi_s(u|_{\partial D})^*TW \subset S^1 \times C^n \]
Moreover
\[ \Phi_0(u|_{\partial D})^*TW = S^1 \times R^n \]
So, the homotopy induces a homotopy \( \bar{h} \) in Lagrangian-Grassmanian manifold. Note that \( m(\bar{h}(0, \cdot)) = 0 \). By the homotopy invariance of Maslov index, we know that \( m(u|_{\partial D}) = 0 \).

Consider the partial differential equation
\[ \bar{\partial}u + A(z)u = 0 \text{ on } D \quad (2.2) \]
\[ u(z) \in \Gamma(z) R^n \text{ for } z \in \partial D \quad (2.3) \]
\[ \Gamma(z) \in GL(2n, R) \cap Sp(2n) \quad (2.4) \]
\[ m(\Gamma) = 0 \quad (2.5) \]

For \( 100 < k < \infty \) consider the Banach space \( \tilde{V}_k \) consisting of all maps \( u \in H^{k,2}(D, C^n) \) such that \( u(z) \in \Gamma(z) \) for almost all \( z \in \partial D \). Let \( L_{k-1} \) the usual \( L_{k-1} \)-space \( H_{k-1}(D, C^n) \) and
\[ L_{k-1}(S^1) = \{ u \in H^{k-1}(S^1) | u(z) \in \Gamma(z) R^n \text{ for } z \in \partial D \} \]
We define an operator \( P: \tilde{V}_k \to L_{k-1} \times L_{k-1}(S^1) \) by
\[ P(u) = (\bar{\partial}u + Au, u|_{\partial D}) \quad (2.6) \]
where \( D \) as in (2.1).

**Proposition 2.2** \( \bar{\partial} : \tilde{V}_p \to L_p \) is a real linear Fredholm operator of index \( n \).
3 Nonlinear Fredholm Theory

3.1. Adapted metrics in symplectic manifold \((M, \omega)\). A Riemannian metric \(g\) on \(M\) is called adapted (to the symplectic form \(\omega\)) if \(g + \sqrt{-1}\omega\) is a Hermitian metric with respect to some almost complex structure \(J : T(M) \to T(M)\) preserving \(g\) and \(\omega\). This is equivalent to the existence of a \(g\)-orthonormal coframe \(x_i, y_i, i = 1, \ldots, n = \dim M/2\), at each point in \(M\) such that \(\omega\) equals \(\sum x_i \wedge y_i\) at this point. Yet another equivalent definition reads

\[
||dH||_g = ||\text{grad}_\omega H||_g
\]

for all smooth functions \(H\) on \(M\), where, recall, \(\text{grad}_\omega H\) is the (Hamiltonian) vector field which is \(\omega\)-dual to \(dH\).

Let us show that a complete adapted metric always exists.

Lemma 3.1 (Eliashberg-Gromov[3]). Every symplectic manifold \(M = (M, \omega)\) admits a complete adapted metric \(g\).

Proof (due to [3]). The required metric will be constructed starting with arbitrary adapted metric \(g_0\) and applying a certain symplectic automorphism \(A\) of \(T^*(M)\) to it. This \(A\) is constructed with an exhaustion of \(M\) by compact domains with smooth boundaries \(S_i\) expanding \(g_0\) transversally to all \(S_i\). Namely, we take small \(\varepsilon_i\)-neighborhoods \(N_i \subset M\) of \(S_i\), normally (with respect to \(g_0\)) decomposed as \(N_i = S_i \times [-\varepsilon_i, \varepsilon_i]\). We denote by \(\Sigma_i \subset T(N_i)\) and \(\nu_i \in T(N_i)\) the subbundles tangent and normal to the slices \(S_i \times t, t \in [-\varepsilon_i, \varepsilon_i]\), respectively and take some symplectic automorphisms \(A_i : T(N_i) \to T(N_i)\) preserving the decomposition \(T(N_i) = \Sigma_i \oplus \nu_i\) and acting on \(\nu_i\) by \(A_i(\nu) = 2\nu\). Then \(A\) is taken equal to \(Id\) outside all \(N_i\) and \(A|T(N_i) = \text{def} A_i\) where \(\varphi_i(s, t) = \varphi_i(t)\) is a suitable sequence of positive functions on \([-\varepsilon_i, \varepsilon_i]\) such that \(\varphi_i\) vanish at ends \(\pm \varepsilon_i\) and are large and fast growing with \(i\) on the subsegments \([-\varepsilon_i/2, \varepsilon_i/2]\). Clearly \(g = Ag_0\) is complete (as well as adapted) for suitable \(\varphi_i\).

3.2. Construction of Lagrangian submanifolds. Let \((V', \omega')(\omega = d\alpha')\) be an exact symplectic manifold and \(W' \subset V'\) a closed submanifolds, we call \(W'\) an exact Lagrangian submanifold if \(\alpha'|W'\) an exact form, i.e., \(\alpha'|W' = df\). Consider an isotopy of Lagrange submanifolds in \(V'\) given by a \(C^\infty\)-map \(F' : W' \times [0, 1] \to V'\) and let \(\tilde{\omega}'\) be the pull-back of the form \(\omega'\) to \(W' \times [0, 1]\). The form \(\tilde{\omega}'\) clearly is exact since \(\omega' = d\alpha', \tilde{\omega}' = d\tilde{\alpha}'\), where the \(1\)-form \(\tilde{\alpha}'\) is closed on \(W' \times t\) for \(t \in [0, 1]\). Recall that \(F'\) is called an exact isotopy if the class \([\tilde{\omega}'|W' \times t]\) \(\in H^1(W' = W' \times t; R)\) is constant in \(t \in [0, 1]\), for more detail see[5, 2.3B]

Let \(U\) a contact manifold and \(L \subset U\) an exact pre-Lagrangian submanifold, as proved in [4], that one can choose a contact form \(\lambda\) on \(U\) such that \((V', \omega') = (U \times R_+, d(\alpha))\) and \(d(\lambda)|L \times \{1\} = 0\) and \(\lambda|L\) is exact. So \(W' = \{1\} \times L \subset V'\) an exact Lagrangian submanifold in \(V'\) and the manifold \((SU, d(\alpha))\) has a canonical diffeotopy \(v' \to sv'\) for \(s \in [0, \infty)\). The induced
isotopy on $L$ clearly is Lagrange; it is exact if and only if the form $l'|W'$ is exact. The isotoped manifolds $W'_s = s(W')$ are disjointed from $W'$ for any $s$. We choose a positive number $s_0$ small enough which will be determined in section 5 and define

$$F' : W' \times [0, 1] \to SU$$

as

$$F'((w, 1), t) = (w, 1 + ts_0)$$

Then one can easily check that $F'$ is an exact Lagrangian isotopy of $W'$ in $SU$.

Let $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$. As in [5], we use figure eight trick to construct a Lagrangian submanifold in $V$ through the Lagrange isotopy $F'$ in $V'$. Fix a positive $\delta < 1$ and take a $C^\infty$-map $\rho : S^1 \to [0, 1]$, where the circle $S^1$ ia parametrized by $\Theta \in [-1, 1]$, such that the $\delta$–neighborhood $I_0$ of $0 \in S^1$ goes to $0 \in [0, 1]$ and $\delta$–neighbourhood $I_1$ of $\pm 1 \in S^1$ goes $1 \in [0, 1]$. Let

$$\tilde{l} = -\psi(w', \rho(\Theta))\rho'(\Theta)d\Theta$$

be the pull-back of the form $\tilde{l}' = -\psi(w', t)dt$ to $W' \times S^1$ under the map $(w', \Theta) \to (w', \rho(\Theta))$ and assume without loss of generality $\Phi$ vanishes on $W' \times (I_0 \cup I_1)$.

Next, consider a map $\alpha$ of the annulus $S^1 \times [\Phi_-, \Phi_+]$ into $R^2$, where $\Phi_-$ and $\Phi_+$ are the lower and the upper bound of the fuction $\Phi$ correspondingly, such that

(i) The pull-back under $\alpha$ of the form $dx \wedge dy$ on $R^2$ equals $-d\Phi \wedge d\Theta$.

(ii) The map $\alpha$ is bijective on $I \times [\Phi_-, \Phi_+]$ where $I \subset S^1$ is some closed subset, such that $I \cup I_0 \cup I_1 = S^1$; furthermore, the origin $0 \in R^2$ is a unique double point of the map $\alpha$ on $S^1 \times 0$, that is

$$0 = \alpha(0, 0) = \alpha(\pm 1, 0),$$

and $\alpha$ is injective on $S^1 = S^1 \times 0$ minus $\{0, \pm 1\}$.

(iii) The curve $S_0^1 = \alpha(S^1 \times 0) \subset R^2$ “bounds” zero area in $R^2$, that is $\int_{S_0^1} xdy = 0$, for the 1–form $xdy$ on $R^2$.

**Proposition 3.1** Let $V'$, $W'$ and $F'$ as above. Then there exists an exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ given by $F(w', \Theta) = (F'(w', \rho(\Theta)), \alpha(\Theta, \Phi))$.

**Proof.** Similar to [5,2.3B_3].

**3.3. Formulation of Hilbert manifolds.** Now let $(U, \lambda)$ be a contact manifold with contact form $\lambda$. Let $SU = (U \times [0, \infty), d(\alpha \lambda))$ be its symplectization. By Lemma 3.1, one has

**Proposition 3.2** There exists an adapted complete metric on the symplectization $SU = (U \times [0, \infty), d(\alpha \lambda))$ of contact manifolds $(U, \lambda)$. 
In the following we denote by $(V,\omega) = (SU \times R^2, d(a\lambda) \oplus dx \wedge dy))$ with the adapted metric $g \oplus g_0$ and $W \subset V(W = F(W' \times S^1))$ the Lagrangian submanifold constructed in section 3.2.

Let $k \geq 100$ and

\[ D^k(V,W,p) = \{ u \in H^k(D,V) | u(\partial D) \subset W, \ u \text{ homotopic to } u_0 = p, \ u(1) = p \} \]

**Lemma 3.2** Let $W$ be a closed Lagrangian submanifold in $V$. Then, $D^k(V,W,p)$ is a Hilbert manifold with the tangent bundle

\[ T D^k(V,W,p) = \bigcup_{u \in D^k(V,W,p)} \Lambda^{k-1}(u^*TV, u|_{\partial D}^*TL, p) \]  \hspace{1cm} (3.2)

here

\[ \Lambda^{k-1}(u^*TV, u|_{\partial D}^*TL, p) = \{ H^{k-1} - \text{sections of } (u^*TV, (u|_{\partial D})^*TL) \text{ which vanishes at 1} \} \]  \hspace{1cm} (3.3)

\[ \{ H^{k-1} - \text{sections of } (u^*TV, (u|_{\partial D})^*TL) \text{ which vanishes at 1} \} \]  \hspace{1cm} (3.4)

Proof: See [5].

Now we construct a nonlinear Fredholm operator from $D^k(V,W,p)$ to $T D^k(V,W,p)$ follows in [5]. Let $\bar{\partial} : D^k(V,W,p) \to T D^k(V,W,p)$ be the Cauchy-Riemann Section induced by the Cauchy-Riemann operator, locally,

\[ \bar{\partial} u = \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} \]  \hspace{1cm} (3.5)

for $u \in D^k(V,W,p)$.

Since the space $D^k(V,W,p)$ is Hilbert manifold, the tangent space $T D^k(V,W,p)$ is trivial, i.e. there exists a bundle isomorphism

\[ \Phi : T D^k(V,W,p) \to D^k(V,W,p) \times E \]

where $E$ is a Hilbert Space. Then the Cauchy-Riemann section $\bar{\partial}$ on $T D^k(V,W,p)$ induces a nonlinear map

\[ \Phi \circ \bar{\partial} : D^k(V,W,p) \mapsto E \]

In the following, we still denote $\Phi \circ \bar{\partial}$ by $\tilde{\partial}$ for convenience. Now we define

\[ F : D^k(V,W,p) \to E \]

\[ F(u) = \Phi(\tilde{\partial} u) \]  \hspace{1cm} (3.7)

**Theorem 3.1** The nonlinear operator $F$ defined in (3.6-3.7) is a nonlinear Fredholm operator of Index zero.
Proof. According to the definition of the nonlinear Fredholm operator, we need to prove that \( u \in \mathcal{D}^k(V, W, p) \), the linearization \( DF(u) \) of \( F \) at \( u \) is a linear Fredholm operator. Note that

\[
DF(u) = D\bar{\partial}_{[u]} \tag{3.8}
\]

where

\[
(D\bar{\partial}_{[u]})v = \frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v \tag{3.9}
\]

with

\[ v|_{\partial D} \in (u|_{\partial D})^*TW \]

here \( A(u) \) is \( 2n \times 2n \) matrix induced by the torsion of almost complex structure, see \([5]\) for the computation.

Observe that the linearization \( DF(u) \) of \( F \) at \( u \) is equivalent to the following Lagrangian boundary value problem

\[
\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v = f, \quad v \in \Lambda^k(u^*TV) \tag{3.10}
\]

\[ v(t) \in T_{u(t)}W, \quad t \in \partial D \tag{3.11} \]

One can check that (3.10-11) defines a linear Fredholm operator. In fact, by proposition 2.2 and Lemma 2.1, since the operator \( A(u) \) is a compact, we know that the operator \( F \) is a nonlinear Fredholm operator of the index zero.

**Definition 3.1** A nonlinear Fredholm \( F : X \to Y \) operator is proper if any \( y \in Y \), \( F^{-1}(y) \) is finite or for any compact set \( K \subset Y \), \( F^{-1}(K) \) is compact in \( X \).

**Definition 3.2** \( \deg(F, y) = \sharp\{F^{-1}(y)\} \mod 2 \) is called the Fredholm degree of a nonlinear proper Fredholm operator (see \([5,11]\)).

**Theorem 3.2** Assume that the nonlinear Fredholm operator \( F : \mathcal{D}^k(V, W, p) \to E \) constructed in (3.6-7) is proper. Then,

\[ \deg(F, 0) = 1 \]

Proof: We assume that \( u : D \to V \) be a \( J \)–holomorphic disk with boundary \( u(\partial D) \subset W \). Since almost complex structure \( J \) tamed by the symplectic form \( \omega \), by stokes formula, we conclude \( u : D^2 \to W \) is a constant map. Because \( u(1) = p \), We know that \( F^{-1}(0) = p \). Next we show that the linearization \( DF(p) \) of \( F \) at \( p \) is an isomorphism from \( T^p\mathcal{D}(V, W, p) \) to \( E \). This is equivalent to solve the equations

\[
\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} = f \tag{3.12}
\]

\[ v|_{\partial D} \subset T_pW \tag{3.13} \]

By Lemma 3.1, we know that \( DF(p) \) is an isomorphism. Therefore \( \deg(F, 0) = 1 \).

**Corollary 3.1** \( \deg(F, w) = 1 \) for any \( w \in E \).

Proof. Using the connectedness of \( E \) and the homotopy invariance of \( \deg \).
4 Non-properness of Fredholm Operator

We shall prove in this section that the operator \( F : D \to E \) constructed in the above section is non proper along the line in [5].

4.1. Anti-holomorphic section. Let \( C = \mathbb{R}^2 \) and \( (V', \omega'), (V, \omega) = (V' \times C, \omega' \oplus \omega_0) \), and \( W \) as in section 3 and \( J = J' \oplus i, g = g' \oplus g_0, g_0 \) the standard metric on \( C \).

Now let \( c \in C \) (here \( C \) the complex plane) be a non-zero vector. We consider the equations

\[
v = (v', f) : D \to V' \times C \\
\partial_J' v' = 0, \bar{\partial} f = c \\
v|_{\partial D} : \partial D \to W
\]

(4.1)

here \( v \) homotopic to constant map \( \{p\} \) relative to \( W \). Note that \( W \subset V \times B_R(0) \) (here \( R \) depends on the \( s_0 \) in section 3.2).

**Lemma 4.1** Let \( v \) be the solutions of (4.1), then one has the following estimates

\[
E(v) = \int_D (g'(\partial_J' v', J' \partial_J' v') + g'(\partial_J' v', J' \partial_J' v')) \\
+ g_0(\partial_J f, i \partial_J f) + g_0(\partial_J f, i \partial_J f))d\sigma \leq 4\pi R^2.
\]

(4.2)

Proof: Since \( v(z) = (v'(z), f(z)) \) satisfy (4.1) and \( v(z) = (v'(z), f(z)) \in V' \times C \) is homotopic to constant map \( v_0 : D \to \{p\} \subset W \) in \( (V, W) \), by the Stokes formula

\[
\int_D v^*(\omega' \oplus \omega_0) = 0
\]

(4.3)

Note that the metric \( g \) is adapted to the symplectic form \( \omega \) and \( J \), i.e.,

\[
g = \omega(\cdot, J\cdot)
\]

(4.4)

By the simple algebraic computation, we have

\[
\int_D v^* \omega = \frac{1}{4} \int_{D^2} (|\partial v|^2 - |\bar{\partial} v|^2) = 0
\]

(4.5)

and

\[
|\nabla v| = \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2)
\]

(4.6)

Then

\[
E(v) = \int_D |\nabla v| \\
= \int_D \left\{ \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2) \right\} d\sigma \\
= \pi |c|^2_{g_0}
\]

(4.7)
By the equations (4.1), one get

\[ \bar{\partial} f = c \text{ on } D \]  

(4.8)

We have

\[ f(z) = \frac{1}{2}cz + h(z) \]  

(4.9)

here \( h(z) \) is a holomorphic function on \( D \). Note that \( f(z) \) is smooth up to the boundary \( \partial D \), then, by Cauchy integral formula

\[
\int_{\partial D} f(z) \, dz = \frac{1}{2} c \int_{\partial D} \bar{z} \, dz + \int_{\partial D} h(z) \, dz = \pi i c
\]

(4.10)

So, we have

\[ |c| = \frac{1}{\pi} \left| \int_{\partial D} f(z) \, dz \right| \]

(4.11)

Therefore,

\[
E(v) \leq \pi |c|^2 \leq \frac{1}{\pi} \left| \int_{\partial D} f(z) \, dz \right|^2 \\
\leq \frac{1}{\pi} \left| \int_{\partial D} |f(z)||dz||^2 \\
\leq 4\pi |\text{diam}(pr_2(W))|^2 \\
\leq 4\pi R^2.
\]

(4.12)

This finishes the proof of Lemma.

**Proposition 4.1** For \( |c| \geq 3R \), then the equations (4.1) has no solutions.

Proof. By (4.11), we have

\[
|c| \leq \frac{1}{\pi} \left| \int_{\partial D} f(z) \, dz \right| \\
\leq \frac{1}{\pi} \left| \int_{\partial D} \text{diam}(pr_2(W)) ||dz|| \\
\leq \frac{1}{\pi} \left| \int_{\partial D} \text{diam}(pr_2(W)) ||dz|| \\
\leq 2R
\]

(4.13)

It follows that \( c = 3R \) can not be obtained by any solutions.

**4.2. Modification of section** \( c \). Note that the section \( c \) is not a section of the Hilbert bundle in section 3 since \( c \) is not tangent to the Lagrangian submanifold \( W \), we must modify it as follows:

Let \( c \) as in section 4.1, we define

\[
c_{\chi, \delta}(z, v) = \begin{cases} 
  c & \text{if } |z| \leq 1 - 2\delta, \\
  0 & \text{otherwise}
\end{cases}
\]

(4.14)

Then by using the cut off function \( \varphi_h(z) \) and its convolution with section \( c_{\chi, \delta} \), we obtain a smooth section \( c_{\delta} \) satisfying...
\( c_\delta(z, v) = \begin{cases} 
  c & \text{if } |z| \leq 1 - 3\delta, \\
  0 & \text{if } |z| \geq 1 - \delta.
\end{cases} \quad (4.15) \)
for \( h \) small enough by well-known convolution theory.

Now let \( c \in C \) be a non-zero vector and \( c_\delta \) the induced anti-holomorphic section. We consider the equations

\[
\begin{align*}
  v &= (v', f) : D \to V' \times C \\
  \bar{\partial}_v v' &= 0, \bar{\partial} f = c_\delta \\
  v|_{\partial D} : \partial D &\to W
\end{align*}
\]
(4.16)
Note that \( W \subset V \times B_R(0) \) for \( 2\pi R^2 \). Then by repeating the same argument as section 4.1., we obtain

**Lemma 4.2** Let \( v \) be the solutions of (4.16) and \( \delta \) small enough, then one has the following estimates

\[
E(v) \leq 4\pi R^2.
\]
(4.17)

and

**Proposition 4.2** For \( |c| \geq 3R \), then the equations (4.16) has no solutions.

**Theorem 4.1** The Fredholm operator \( F : \mathcal{D}^k(V, W, p) \to E \) is not proper.

Proof. If \( F \) is proper, taking a path \( \gamma(\mu) \) connecting 0 and \( c \), then \( F^{-1}(\gamma(\cdot)) \) is a compact set in \( \mathcal{D}^k(V, W, p) \) for \( k \geq 100 \), then the gradients of map \( v \) have a uniform bounds, i.e.,

\[
|\nabla v| \leq c_1 \text{ for } v \in F^{-1}(\gamma(\cdot))
\]
(4.18)
Note that \( v(1) = p \), the above bounds imply

\[
v|_{\partial D} \subset W'(0) \times (-K, K)
\]
(4.19)
for \( K \) large enough which only depends on \( c_1 \). Since \( W'(0) \times (-k, +k) \) is regular embedding in \( V \), we know that \( v|_{\partial D} \) is compact in the submanifold \( W \). Then Theorem 3.1 and 3.2, i.e., that the index of \( F \) is zero and \( \text{deg}(F) = 1 \) implies \( F \) can take the value \( c \) for \( c \geq 3R \), This contradicts Proposition 4.1. So, \( F \) is not proper.
5 Nonlinear Fredholm Alternative

In this section, we use the Sacks-Uhlenbeck-Gromov’s trick and Gromov’s nonlinear Fredholm alternative to prove the existence of $J$-holomorphic disk with boundary in $W$ if $W \subset SU \times C$ is Lagrangian submanifold.

**Proof of Theorem 1.1.** If Theorem 1.1 does not hold, i.e., there exists a exact Pre-Lagrangian submanifold $L$ in a contact manifold $U$, we use the canonical isotopy in the symplectization to construct an very small Lagrangian isotopy of $L$ then by the Gromov’s figure eight construction in section 3.2 we obtain the exact Lagrangian submanifold $W$ in $SU \times C$. By choosing $s_0$ in section 3.2 small enough such that $4\pi R^2$ small enough we conclude that the solutions of (4.16) is bounded by using the monotone inequality of minimal surface since the boundary of solutions of (4.16) remain in the compact manifold $W$. Then for large vector $c \in C$ in equations (4.16) we know that the nonlinear Cauchy-Riemann equations has no solution, this implies that the operator $F$ constructed in section 3.3 is not proper or the solutions of equations (4.16) is non-compact. The non-properness of the operator implies

a. The existence of $J$-holomorphic plane $v : C \to V$ with bounded energy $E(v) \leq E_0$. Since $v$ has a bounded image then by Gromov’s removal singularity theorem we get a non constant map $w : \mathbb{S}^2 \to V$ which contradict the exactness of $V$.

b. The existence of $J$-holomorphic half plane $v : H \to V$ with boundary $\partial H$ in $W$. Since $v$ has a bounded image, then by the Gromov’s removal boundary singularity we get a $J$-holomorphic disks $w : \mathbb{D} \to V$ with boundary in $W$, this contradicts that $W$ is an exact Lagrangian submanifold.

This implies Theorem 1.1 holds.

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