On exact sampling of the first passage event of Lévy process with infinite Lévy measure and bounded variation

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Abstract

We present an exact sampling method for the first passage event of a Lévy process. The idea is to embed the process into another one whose first passage event can be sampled exactly, and then recover the part belonging to the former from the latter. The method is based on several distributional properties that appear to be new. We obtain general procedures to sample the first passage event of a subordinator across a regular non-increasing boundary, and that of a process with infinite Lévy measure, bounded variation, and suitable drift across a constant level or interval. We give examples of application to a rather wide variety of Lévy measures.

Keywords and phrases. First passage; Lévy process; bounded variation; subordinator; creeping; Dirichlet distribution

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1 Introduction

The first passage event of a Lévy process is an important topic in applied probability and has received extensive study [cf. 2–4, 12–14, 21, 24, 36, and references therein]. However, although practically important and conceptually interesting, its exact sampling remains a subtle issue. In particular, when the Lévy measure of a process has infinite integral, except for a few well-known cases, the distribution of the process is not analytically available, which poses a significant hurdle to the exact sampling of its first passage event.

Real-valued Lévy processes with bounded variation form a large class. Since each such process is the difference between two independent subordinators, i.e., non-decreasing Lévy processes, many properties of Lévy processes with bounded variation can be deduced from those of subordinators. By themselves, subordinators not only play a significant role in the theory of Lévy processes 3, 11, 20, but also have important applications 18, 22, 33. As in the general case, for most subordinators with infinite Lévy measure, exact sampling methods for the first passage event are lacking, although differential equations can be used to evaluate some parameters involved 37, 38.

In this paper, we present a method to sample the first passage event for a rather wide range of real-valued Lévy processes with bounded variation. From now on, by sampling we always mean exact sampling. The method can sample 1) the first passage event of a subordinator across a regular non-increasing boundary, where the meaning of “regular” is specified in Section 4, 2) the first passage event of a Lévy process with bounded variation and non-positive drift across a positive constant level, and 3) the first exit event of a Lévy

1
process with bounded variation and no drift out of a closed interval with 0 as an internal point. As a by-product, the method also provides a way to sample infinitely divisible random variables alternative to one in [8].

We first give an overview of the method for subordinators, which is the main issue, and then an overview of its extension to real-valued Lévy processes with bounded variation.

1.1 Overview of method for subordinators

In a nutshell, the idea is to embed a subordinator of interest into a “carrier” subordinator whose first passage event is tractable, and to utilize the latter to sample for the former. To put the idea into perspective, think of a Lévy measure that can be decomposed as

$$\varphi(x) \, dx + \chi(dx), \quad \text{with} \quad \varphi(x) = \mathbf{1} \{0 < x \leq r\} \gamma e^{-qx}x^{-\alpha-1},$$

(1.1)

where $r$, $\gamma > 0$, $q \geq 0$, $\alpha \in (0, 1)$, and $\chi$ is a finite measure on $(0, \infty)$. This type of Lévy measures coincide with those that can be decomposed as $\tilde{\varphi}(x) \, dx + \chi(dx)$ with $\tilde{\varphi}(x) = (\gamma + O(x)) x^{-\alpha-1}$ as $x \downarrow 0$, and give rise to many interesting processes discovered recently [6, 25, 28, 30], such as Lamperti-stable process, whose Lévy density is $\mathbf{1} \{x > 0\} e^{\beta x}(e^x - 1)^{-\alpha-1}$, $\beta < \alpha + 1$. For this particular process, we can set $r = \infty$ in (1.1). However, in general, $r$ has to be finite.
A subordinator with Lévy density \( \mathbf{1} \) can be represented as the sum of two independent subordinators, one with Lévy density \( \varphi \), the other with Lévy measure \( \chi \). Since the latter is compound Poisson and only poses minor problems, we shall ignore it altogether here. Denoting by \( Z \) the subordinator with Lévy density \( \varphi \), it can be embedded into a stable subordinator as follows. Let \( X_2 \) and \( X_3 \) be independent subordinators which are also independent from \( Z \) and have Lévy densities \( \mathbf{1} \{ 0 < x \leq r \} (1 - e^{-q x}) x^{-\alpha - 1} \) and \( \mathbf{1} \{ x > r \} x^{-\alpha - 1} \), respectively. Then \( S = Z + X_2 + X_3 \) is a stable subordinator with index \( \alpha \).

One can expect that the first passage event of \( S \) is tractable, which indeed is the case. Supposing the first passage event of \( S \) is sampled, we then have to recover the part due to \( Z \) from the sample values. This is the main issue we have to address.

In general, let \( Z \) be a subordinator and \( c \) a non-increasing function on \((0, \infty)\). If \( Z \) has positive drift \( d > 0 \), then we can consider \( Z(t) - dt \) and \( c(t) - dt \) instead. Therefore, without loss of generality, let \( Z \) be a pure jump process. Our method requires some regularity of \( c \). For now we ignore the issue and consider how to jointly sample the random variables \( T = \{ t > 0 : Z(t) > c(t) \} \), \( Z(T^-) \), and \( Z(T) \). Suppose the Lévy measure of \( Z \) can be written as \( \mathbf{1} \{ 0 < x \leq r \} e^{-q x} \Lambda(dx) \), where \( 0 < r \leq \infty \) and \( q \geq 0 \), such that the Lévy measure \( \Lambda(dx) \) gives rise to a subordinator \( S \) whose first passage event across any regular non-increasing boundary on \((0, \infty)\) can be sampled. Represent \( S \) as \( S = Z + X_2 + X_3 \), such that \( Z \), \( X_2 \) and \( X_3 \) are independent, with \( X_2 \) and \( X_3 \) having Lévy measures \( \mathbf{1} \{ 0 < x \leq r \} (1 - e^{-q x}) \Lambda(dx) \) and \( \mathbf{1} \{ x > r \} \Lambda(dx) \), respectively. Like \( Z \), assume \( X_2 \) and \( X_3 \) are pure jump processes.

The scheme of the method is shown in Fig. 1. To start with, instead of \( c(t) \), let \( b(t) = c(t) \wedge r \) be the “target boundary” for \( S \) to cross and \( \tau \) the corresponding first passage time. By assumption, we can sample \((\tau, S(\tau^-), S(\tau))\). Observe the following simple but crucial fact: since \( S(\tau^-) \leq b(\tau) \leq r \), \( S \) can only have jumps of size no greater than \( r \) in \((0, \tau)\). Thus \( Z(\tau^-) + X_2(\tau^-) = S(\tau^-) \). Now given \( \tau = t \) and \( S(\tau^-) = s \leq r \), we have to sample \( Z(\tau^-) \), which is possible for two reasons. First, the conditional distribution of \( Z(\tau^-) \) is the same as that of \( Z(t) \) given \( S(t) = s \), if \( t \) is fixed beforehand (cf. Section 3). Second, using the properties of exponential tilting and upper truncation of Lévy measures on \((0, \infty)\), the latter conditional distribution can be sampled (cf. Section 3). In panel (a) of Fig. 1 the sampled \( Z(\tau^-) \) is less than \( S(\tau^-) \). However, since \( X_2 \) is compound Poisson, \( Z(\tau^-) \) can be equal to \( S(\tau^-) \) with positive probability. Having got \( Z(\tau^-) \), we next sample \( Z(\tau) \). The jump of \( S \) at \( \tau \) is \( \Delta_S = S(\tau) - S(\tau^-) \). By independence, only one of \( Z \), \( X_2 \) and \( X_3 \) can have a jump at \( \tau \), so \( \Delta_Z \) is either 0 or \( \Delta_S \). Fig. 1 illustrates two scenarios. If \( \Delta_S > r \), then clearly \( \Delta_Z = 0 \). If \( \Delta_S \leq r \), then by comparing the Lévy measures of \( Z \) and \( X_2 \), with probability \( e^{-q \Delta_S} \) (resp. \( 1 - e^{-q \Delta_S} \)), \( \Delta_S \) belongs to \( Z \) (resp. \( X_2 \)), giving \( \Delta_Z = \Delta_S \) (resp. \( \Delta_Z = 0 \)) (cf. Sections 3 and 4). Thus \( Z(\tau) = Z(\tau^-) + \Delta_Z \) can be sampled. If \( Z(\tau) < c(\tau) \), then by strong Markov property, we can renew the procedure, but now with starting time point at \( \tau \) and starting value of \( S \) equal to \( Z(\tau) \). As can be expected, the procedure eventually stops, giving a sample value of \((T, Z(T^-), Z(T))\).

Note that, if \( c \) is decreasing, then it is possible for \( S \) to creep across \( c \), i.e., \( \Delta_S = 0 \), as marked by \( * \) in panel (c). In this case, although \( \Delta_Z = 0 \), if \( g > 0 \), it is possible that \( Z(\tau) < S(\tau) \) and so the procedure has to be renewed; see the scenario marked by \( B \) in panel (c). The characterization of creeping when \( c \) is linear is known [3, 17]. However, the case where \( c \) is non-linear appears to be still unresolved.
Figure 2: Sampling of the first passage event of $Z = Z^+ - Z^-$ across $a > 0$, where $Z^\pm$ are independent subordinators.

To implement the scheme, one can first sample $\tau$, then $(S(\tau^-), S(\tau))$ conditional on $\tau$, and finally $(Z(\tau^-), Z(\tau))$ conditional on $(\tau, S(\tau^-), S(\tau))$. Among these samplings, the one for $\tau$ typically is the simplest. The other samplings require several theoretical results, which will be obtained in Section 3. Finally, it is easy to introduce a terminal point $K \leq \infty$ and sample $(T', Z(T'-), Z(T'))$, with $T' = T \wedge K$. In particular, if $K = 1$ and $c \equiv \infty$, the method samples an infinitely divisible random variable with upper truncated Lévy measure $\mathbf{1}\{0 < x \leq r\} e^{-qx} \Lambda(dx)$. As a passing remark, the so-called Vervaat perpetuity correspond to $q = 0$ and $\Lambda(dx) = adx$, whose exact sampling is known [10, 15].

1.2 Overview of extensions

The method described in Section 1.1 can be extended to Lévy process with bounded variation, as each such process is the difference of two independent subordinators. Fig. 2 illustrates the sampling of the first passage event across a constant level $a > 0$ by a Lévy process with non-positive drift. Write the process as $Z^+ - Z^-$, where $Z^\pm$ are independent subordinators, with $Z^+$ having no drift. The sampling can be thought of as having $Z^+$ to “catch up” with $a + Z^-$. To start with, let $a$ be the target boundary for $Z^+$ to cross and $\tau_1$ the corresponding first passage time. It is evident that before $\tau_1$, $Z^+$ stays below $a + Z^-$. However, at $\tau_1$, since $Z^+$ has a jump, it is possible for $Z^+$ to pass $a + Z^-$. We can use the method in Section 1.1 to sample $Z^+(\tau_1)$. Meanwhile, since $Z^-$ is independent of $\tau_1$, we can use the modification mentioned at the end of Section 1.1 to sample $Z^-(\tau_1)$. If $a + Z^-(\tau_1) < Z^+(\tau_1)$, then $\tau_1$ is the first passage time of $Z^+ - Z^-$ across $a$. If $a + Z^-(\tau_1) > Z^+(\tau_1)$, then set $a + Z^-(\tau_1)$ as the new target boundary for $Z^+$ to cross, this time with $\tau_1$ as the starting time point and $Z^+(\tau_1)$ the starting value for $Z^+$. As long as $\lim_{t \to \infty} Z(t) = \infty$ w.p. 1, the procedure eventually stops (cf. Section 5). Here the assumption that $Z$ has non-positive drift is critical, since otherwise $Z^+$ can creep across $a + Z^-$ with positive probability. When this happens, the first passage times of $Z^+$ across the target boundaries shown in Fig. 2 will
converge but never be equal to the first passage time of $Z$ across $a$, resulting in the iteration going on forever. It can be seen that if $T = \inf\{t > 0 : Z(t) - Z(t) > a\}$, then the procedure samples $(T, Z^+(T), Z^+(T), Z^-(T))$. As in Section 1.1, a terminal point $K \leq \infty$ can be introduced so that one can sample $(T', Z^+(T'), Z^+(T'), Z^-(T'))$, where $T' = T \wedge K$. If $K < \infty$, the procedure eventually stops w.p. 1, whether or not $\lim_{t \to \infty} Z(t) = \infty$.

It is also possible to sample the first exit event of $Z = Z^+ - Z^-$ out of a fixed interval $[-a^-, a^+]$ with $a^+ > 0$ if it has no drift. To see how the method in Section 1.1 can be extended to this case, it is convenient to consider the “phase plot” of the Lévy process $W = (Z^-, Z^+)$, which shows the trajectory of $W$ on the $x$-$y$ plane without time axis. Then the first exit of $Z$ out of $[-a^-, a^+]$ can be depicted as the first exit of $W$ out of the band $\{(x, y) : -a^\leq y - x \leq a^+\}$. Suppose $Z^\pm$ each satisfies the assumption in Section 1.1 so that, for example, $Z^+$ has Lévy measure $\{0 < x \leq r^+\} \exp\{-q^+x\} \Lambda^+(dx)$, where $\Lambda^+(dx)$ is the Lévy measure of a subordinator $S^+$ whose first passage event across any constant level can be sampled. To start with, let $b^- = a^- \wedge r^-$, $b^+ = a^+ \wedge r^+$, and set the top and right sides of the rectangle $[0, b^-] \times [0, b^+]$ to be the target boundary. In panel (a) of Fig. 3, $r^- > a^-$ and $r^+ < a^+$, resulting in the rectangle as shown. Now sample the first passage event of $S = (S^-, S^+)$ across the target boundary. To do this, we can first independently sample the first passage times of $S^\pm$ across $b^\pm$. If, as shown in panel (a), $S^-$ makes a crossing at time $\tau$ before $S^+$, then sample $(S^-(\tau^-), S^-(\tau))$, and sample $S^+(\tau^-) = S^+(\tau)$ conditional on $S^+(\tau) < b^+$. We next can use the method in Section 1.1 to recover $Z^\pm(\tau)$. In the scenario shown in Fig. 3 since the jump of $S^-$ at $\tau$ is greater than $r^-$, it is not part of $Z^-$, and so we end up with $W(\tau)$ as in panel (b). The procedure is then renewed. As long as $Z \neq 0$, the procedure will stop eventually and we get the first exit event of $Z$ (cf. Sections 3 and 5). It can be seen that if $T = \inf\{t > 0 : Z(t) \notin [-a^-, a^+]\}$, then the procedure samples $(T, Z^+(T), Z^+(T))$. As in Section 1.1, a terminal point $K \leq \infty$ can be introduced so that one can sample $(T', Z^+(T'), Z^+(T'))$ with $T' = T \wedge K$.
1.3 Organization of the paper

Section 2 fixes notation and recalls preliminary results. Section 3 obtains distributional properties needed by the above methods. The basic tools are results on subordinators and fluctuation theory [cf. 3], although some recent developments on the first passage event of a general Lévy process could be exploited [cf. 11, 12, 14, 23]. In addition, we need to get some detail on various conditional distributions of a subordinator as well as on creeping. Sections 4–6 present procedures to implement the methods and show their validity, and identify major sampling issues involved. Sections 7–9 show examples of application of the procedures to several types of Lévy measures. The first type is given in (1.1). The second type consists of finite sums of Lévy measures of the form (1.1), which require additional techniques. The third type consists of $1 \{0 < x \leq r\} x^{-1} e^{-x} \, dx + \chi(dx)$, with $\chi$ a finite measure on $(0, \infty)$. This type gives rise to the aforementioned Vervaat perpetuity and the Beta process in survival analysis, which has Lévy density $1 \{x > 0\} e^{-cx}/(1 - e^{-x})$ with $c > 0$ and belongs to the Beta-class processes [18, 23, 25, 31]. Rejection sampling and the Dirichlet distribution play an important role in these examples [9, 16, 32].

2 Preliminaries

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, denote by $\|x\|$ its $L^1$ norm $\sum_i |x_i|$. The convention $\inf \emptyset = \infty$ will be used all along; “p.d.f.” will stand for “probability density function with respect to Lebesgue measure”. For $a, b > 0$, Beta$(a, b)$ denotes the distribution with p.d.f. $1 \{0 < x < 1\} x^{a-1}(1-x)^{b-1}/B(a, b)$, where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, Gamma$(a, b)$ denotes the one with p.d.f. $1 \{x > 0\} x^{a-1} e^{-x/b}/[b^a\Gamma(a)]$, and Exp$(b)$ denotes the one with p.d.f. $1 \{x > 0\} e^{-x/b}$. Finally, Unif$(0, 1)$ denotes the uniform distribution on $(0, 1)$.

Let $\hat{\nu}$ and $\nu$ be two probability measures on a measurable space $\Omega$ satisfying $d\hat{\nu}/d\nu \propto \varrho \leq C$, where $\varrho \geq 0$ is a known function and $C > 0$ a known constant. Then $\hat{\nu}$ can be sampled as follows: keep sampling $\xi \sim \nu$ and $U \sim \text{Unif}(0, 1)$ until $CU \leq \rho(\xi)$. This procedure is the well-known rejection sampling, which is exact and stops w.p. 1 [3, 16, 32].

Let $\nu$ be an infinitely divisible distribution on $(0, \infty)$ with Lévy measure $\Lambda$. Given $q > 0$, $\Lambda_q(dx) = e^{-qx} \Lambda(dx)$ is known as an exponentially tilted version of $\Lambda$. If $\nu_q$ is the infinitely divisible distribution with Lévy measure $\Lambda_q$, then $\nu_q(dx) \propto e^{-qx} \nu(dx)$ [3, 16, 20]. By setting $g(x) = e^{-qx} 1 \{x > 0\}$, rejection sampling can be used to sample $\nu_q$ based on $\nu$.

The Dirichlet distribution $Di(a_1, \ldots, a_k)$ with parameters $a_1, \ldots, a_k > 0$ is a generalization of the Beta distribution. It can be defined as a distribution on $\mathbb{R}^k$, such that for any measurable function $g \geq 0$ on $\mathbb{R}^k$ and $\omega \sim Di(a_1, \ldots, a_k)$,

$$
\mathbb{E}[g(\omega)] = \frac{\Gamma(a_1 + \cdots + a_k)}{\Gamma(a_1) \cdots \Gamma(a_k)} \int 1 \{\text{all } x_i \geq 0\} g(x) \prod_{i=1}^k x_i^{a_i-1} \, dx_1 \cdots dx_{k-1},
$$

where in the integral $x_k = 1 - x_1 - \cdots - x_{k-1}$ instead of a variate, and $x = (x_1, \ldots, x_k)$. The distribution has p.d.f. $\Gamma(a_1 + \cdots + a_k) \prod_{i=1}^k x_i^{a_i-1} / \prod_{i=1}^k \Gamma(a_i)$ with respect to the degenerate measure $\sigma_k(dx) = 1 \{x_i \geq 0, \|x\| = 1\} \, dx_1 \cdots dx_{k-1} \delta(dx_k + x_1 + \cdots + x_{k-1} - 1)$, where $\delta$ is the Dirac measure at 0. For convenience, we will refer to it as the p.d.f. of $Di(a_1, \ldots, a_k)$. Also, if $k = 1$, then for $a > 0$, define $Di(a)$ to be the Dirac measure at 1.
3 Distributions of the first passage event

We need several distributional properties to implement the method introduced in Section 3.1. The main issue is, provided a subordinator $Z$ can be embedded into another subordinator $S = Z + X_2 + X_3$, how to recover the first passage event of $Z$ from the one sampled for $S$. As noted in Section 1.1, we need to take into account the possibility that $S$ creeps across a boundary. Also, for both the method for subordinators and its extension to Lévy processes, we need to make sure the corresponding sampling procedures eventually stop.

3.1 Results for subordinators

Let $X = (X_1, \ldots, X_d)$ be a Lévy process taking values in $[0, \infty)^d$ with Lévy measure $\Pi$ and Laplace exponent $\int (1 - e^{-\langle \theta, x \rangle}) \Pi(dx)$, $\theta \in [0, \infty)^d$. In the application later, $X_1 = Z$, $d = 3$, and $X_i$ are independent. By definition, $\Pi$ has no mass at $\{0\}$. Let $\Delta_X$ be the jump process of $X$. The process $S = \|X\| = X_1 + \cdots + X_d$ is a subordinator with Lévy measure $\Pi_S(dx) = \int_{[0,\infty)^d} 1 \{\|z\| \in dz\} \Pi(dz)$, $x \in (0, \infty)$, and jump process $\Delta_S = \|\Delta_X\|$. In the rest of this section, we shall always assume

$$\Pi_S(0, \infty) = \infty. \quad (3.1)$$

Denote $\Pi_S(x) = \Pi_S(x, \infty)$, and given $c \in C(0, \infty)$,

$$\tau_c = \tau^S_c = \inf \{t > 0 : S(t) > c(t)\}. \quad (3.2)$$

If $c(t) \equiv a > 0$, the notation $\tau_a$ will be used instead of $\tau_c$. To implement the method introduced in Section 3.1, we will first sample $\tau_c$, which is often easy, then $(S(\tau_c^\cdot), \Delta_S(\tau_c^\cdot))$ conditional on $\tau_c$, and finally $(X(\tau_c^\cdot), \Delta_X(\tau_c^\cdot))$ conditional on $(\tau_c, S(\tau_c^-), \Delta_S(\tau_c^-))$. Among the results below, part 2) of Theorem 3.1 will be used for the conditional sampling of $(X(\tau_c^\cdot), \Delta_X(\tau_c^\cdot))$, while Theorem 3.1 will be used for the conditional sampling of $(S(\tau_c^\cdot), \Delta_S(\tau_c^\cdot))$.

**Theorem 3.1.** Let $c \in C(0, \infty)$ be non-increasing with $c(0+) > 0$ and (3.1) hold.

1) Let $\Omega = (0, \infty) \times [0, \infty)^d \times ([0, \infty)^d \setminus \{0\})$. For $(t, u, v) \in \Omega$, let

$$P \{\tau_c \in dt, X(\tau_c^-) \in du, \Delta_X(\tau_c) \in dv\} = 1 \{0 \leq c(t) - \|u\| < \|v\|\} dt P \{X(t) \in du\} \Pi(dv),$$

$$P \{\tau_c \in dt, X(\tau_c^-) \in du, \Delta_X(\tau_c) = 0\} = P \{\tau_c \in dt, S(\tau_c) = c(t)\} P \{X(t) \in du | S(t) = c(t)\}. \quad (3.3)$$

2) For $t > 0$, $s \in [0, c(t)]$, $z > c(t) - s$, and $u, v \in [0, \infty)^d$,

$$P \{X(s^-) \in du, \Delta_X(s^-) \in dv | \tau_c = t, S(\tau_c^-) = s, \Delta_S(\tau_c^-) = z\} = P \{X(t) \in du | S(t) = s\} \Pi_z(dv), \quad (3.4)$$

with $\Pi_z(dv) = P \{V \in dv | \|V\| = z\}$, where $V$ is a random vector following the distribution

$$\nu_a(dv) = 1 \{\|v\| > a\} \Pi(dv)/\Pi_S(a),$$

$$\nu_a(dv) = 1 \{\|v\| > a\} \Pi(dv)/\Pi_S(a),$$

7
with \( a \in (0, z) \) a fixed number. The conditional probability measure \( \Pi_z(dv) \) is independent of the choice of \( a \in (0, z) \). Furthermore,

\[
P \{ X(\tau_c-) \in du \mid \tau_c = t, \Delta S(\tau_c) = 0 \} = P \{ X(t) \in du \mid S(t) = c(t) \}. \tag{3.6}
\]

**Proof.** 1) It is clear that \( 0 < \tau_c < \infty \) w.p. 1. We first show (3.3). Following the proof for Proposition III.2 in [3], let \( f \geq 0 \) be a Borel function on \( \Omega \) such that \( f(t, u, v) = 0 \) when \( \|v\| = c(t) - \|u\| \). Then

\[
f(\tau_c, X(\tau_c-), \Delta X(\tau_c)) = \sum_t f(t, X(t-), \Delta X(t)) \{ 0 \leq c(t) - S(t-) < \|\Delta X(t)\| \}. \tag{3.7}
\]

For each \( t > 0 \), define function \( H_t(v) = f(t, X(t-), v) \{ 0 \leq c(t) - S(t-) < \|v\| \} \) on \([0, \infty)^d\). Since \( H = (H_t) \) is a predictable process with respect to the filtration generated by \( \Delta X \), by (3.7) and the compensation formula,

\[
E[f(\tau_c, X(\tau_c-), \Delta X(\tau_c))]
= \int_0^\infty dt \int f(t, X(t-), v) \{ 0 \leq c(t) - S(t-) < \|v\| \} \Pi(dv)
= \int_0^\infty dt \int f(t, u, v) \{ 0 \leq c(t) - \|u\| < \|v\| \} \Pi(dv)
= \int_0^\infty \Pi(dv) \int f(t, u, v) \{ 0 \leq c(t) - \|u\| < \|v\| \} dt
\]

where \( (a) \) is due to \( X(t-) \sim X(t) \) for any \( t > 0 \). Since \( f \) is arbitrary, this shows (3.6) for \( (t, u, v) \in \Omega \) with \( \|v\| \neq c(t) - \|u\| \). It remains to consider \( (t, u, v) \in \Omega \) with \( \|v\| = c(t) - \|u\| \). In this case, the right hand side of (3.3) is 0. If we define \( f(t, u, v) = 1 \{ v = c(t) - u > 0 \} \) for \( t > 0, u \geq 0 \) and \( v > 0 \), then by similar argument as above based on the compensation formula, but directly applied to \( S \),

\[
P\{S(\tau_c-) < S(\tau_c) = c(\tau_c)\} = \int_0^\infty dt \int P\{S(t) \in du\} \Pi_S(\{c(t) - u\}).
\]

For each \( t \), there is only a countable set of \( u \) with \( \Pi_S(\{c(t) - u\}) > 0 \). On the other hand, under assumption (3.1), the distribution of \( S(t) \) is continuous, i.e., \( P\{S(t) = u\} = 0 \) for any \( u \); see Theorem 27.4. As a result, \( \int P\{S(t) \in du\} \Pi_S(\{c(t) - u\}) = 0 \) for each \( t \), and so the multiple integral is 0. Finally, the proof of (3.6) is complete by

\[
P\{\Delta X(\tau_c) \neq 0, S(\tau_c) = c(\tau_c)\} = P\{\Delta X(\tau_c) = 0\} = 0. \tag{3.8}
\]

Now consider (3.3). Under (3.1), \( S \) is strictly increasing w.p. 1. Clearly, \( \Delta X(\tau_c) = 0 \) implies \( S(\tau_c) = c(\tau_c) \). On the other hand, from (3.8), on the event \( S(\tau_c) = c(\tau_c), \Delta X(\tau_c) = 0 \) w.p. 1. Define \( \tau_* = \inf\{t \geq 0 : S(t) = c(t)\} \). Then w.p. 1,

\[
\{\tau_* < \infty\} = \{\tau_c = \tau_*\} = \{S(\tau_c) = c(\tau_c)\}. \tag{3.9}
\]
Let \( f \geq 0 \) be a Borel function on \((0, \infty) \times [0, \infty)^d\) with bounded support. Then 
\[
\mathbb{E}[f(\tau_v, X(\tau_v-))1 \{S(\tau_v) = c(\tau_v)\}]
\]
can be expressed in two ways. First, from (3.8), it equals
\[
\int f(t, u)1 \{\|u\| = c(t)\} P \{\tau_c \in dt, X(\tau_c-) \in du, \Delta_X(\tau_c) = 0\}. \tag{3.10}
\]
Second, from (3.8) and (3.9), the expectation also equals
\[
\mathbb{E}[f(\tau_v, X(\tau_v))1 \{S(\tau_v) = c(\tau_v)\}] = \mathbb{E}[f(\tau_v, X(\tau_v))1 \{\tau_v < \infty\}]
\]
\[
= \int \mathbb{E}[f(t, X(t))|\tau_v = t] P\{\tau_c \in dt\} = \int f(t, u) P\{X(t) \in du|\tau_v = t\} P\{\tau_v \in dt\}.
\]
From the definition of \( \tau_v \) and (3.9), the last integral is equal to
\[
\int f(t, u) P \{\tau_c \in dt, S(\tau_c) = c(\tau_c)\} P \{X(t) \in du|S(t) = c(t)\}. \tag{3.11}
\]
Since \( f \) is arbitrary, comparing the integrals in (3.10) and (3.11) then yields
\[
1 \{\|u\| = c(t)\} P \{\tau_c \in dt, X(\tau_c-) \in du, \Delta_X(\tau_c) = 0\}
\]
\[
= P \{\tau_c \in dt, S(\tau_c) = c(\tau_c)\} P \{X(t) \in du|S(t) = c(t)\}.
\]
Since the qualifier \( 1 \{\|u\| = c(t)\} \) can be removed from the identity, (3.14) follows.

2) The process \((X, S)\) is a Lévy process with \(\Pi_{(X,S)}(dv, dz) = \Pi(dv) \delta(dz - \|v\|)\). With similar argument as 1), for \( t > 0, s \geq 0, z > 0, u \in [0, \infty)^d, \) and \( v \in [0, \infty)^d \setminus \{0\}, \)
\[
P \{\tau_c \in dt, X(\tau_c-) \in du, S(\tau_c-) \in ds, \Delta_X(\tau_c) \in dv, \Delta_S(\tau_c) \in dz\}
\]
\[
= 1 \{0 \leq c(t) - s < z\} dt P \{X(t) \in du, S(t) \in ds\} \Pi(dv) \delta(dz - \|v\|).
\]
On the other hand, applying (3.3) directly to \( S \), we get
\[
P \{\tau_c \in dt, S(\tau_c) \in ds, \Delta_S(\tau_c) \in dz\} = 1 \{0 \leq c(t) - s < z\} dt P \{S(t) \in ds\} \Pi_S(dz).
\]
Therefore, in order to get (3.5), it suffices to show \( \Pi(dv) \delta(dz - \|v\|) = \Pi_z(dv) \Pi_S(dz) \), which is equivalent to saying that for any measurable \( A \subset ([0, \infty)^d \setminus \{0\}) \times (0, \infty), \)
\[
\int 1 \{(v, z) \in A\} \Pi(dv) \delta(dz - \|v\|) = \int 1 \{(v, z) \in A\} \Pi_z(dv) \Pi_S(dz),
\]
The left hand side is \( \int 1 \{(v, \|v\|) \in A\} \Pi(dv). \) To evaluate the right hand side, for any \( \varepsilon > 0, \) let \( A_\varepsilon = \{(v, z) \in A : z > \varepsilon\}. \) Observe that \( 1 \{z \geq \varepsilon\} \Pi_S(dz)/\Pi_S(\varepsilon) \) is the distribution of \( \|V\| \) for \( V \sim \nu_\varepsilon. \) Then, from the definition of \( \Pi_z(dv), \)
\[
\int 1 \{(v, z) \in A_\varepsilon\} \Pi_z(dv) \Pi_S(dz)
\]
\[
= \Pi_S(\varepsilon) \int 1 \{(v, z) \in A_\varepsilon\} P\{V \in dv|\|V\| = z\} P\{\|V\| \in dz\}
\]
\[
= \Pi_S(\varepsilon) \int 1 \{(v, \|v\|) \in A_\varepsilon\} P\{V \in dv\} = \int 1 \{(v, \|v\|) \in A_\varepsilon\} 1 \{\|v\| \geq \varepsilon\} \Pi(dv).
\]
Let \( \varepsilon \downarrow 0. \) Since \( A_\varepsilon \uparrow A, \) the last integral converges to \( \int 1 \{(v, \|v\|) \in A\} \Pi(dv). \) This completes the proof of (3.5). Finally, since \( \Delta_X(\tau_v) = 0 \) if and only if \( \Delta_S(\tau_v) = 0, \) and by (3.3), \( P\{\Delta_S(\tau_v) \neq 0, S(\tau_v) = c(\tau_v)\} = 0, \) (3.3) follows from (3.4). \( \square \)
Corollary 3.2. For \( a > 0 \) and \( t > 0 \), define
\[
    \psi_a(t) = \int_0^a \Pi_S(a-u)P\{S(t) \in du\}.
\]
Then, under the same assumption as Theorem 3.1,
\[
P\{\tau_c \in dt, S(\tau_c) > c(\tau_c)\} = \psi_{c(t)}(t) \, dt = dt \int_0^{c(t)} \Pi_S(c(t) - u)P\{S(t) \in du\}.
\]
In particular, for constant \( a \in (0, \infty) \), \( \tau_a \) has p.d.f. \( \psi_a(t) \).

Proof. Apply (3.3) in Theorem 3.1 directly to \( \tau_a \) to get
\[
P\{\tau_c \in dt, S(\tau_c) > c(\tau_c)\} = dt \int 1_{\{0 \leq c(t) - u < v\}}P\{S(t) \in du\} \Pi_S(dv),
\]
which is (3.13). Since \( P\{S(\tau_a) > a\} = 1 \) [3, Theorem III.4], \( P\{\tau_a \in dt\} = \psi_a(t) \, dt \). \( \Box \)

Definition 3.3. \( S \) is said to satisfy the continuous density condition, if \( S(t) \) has a p.d.f. \( g_t \) on \((0, \infty)\) for each \( t > 0 \) and the mapping \((t, x) \to g_t(x)\) is continuous on \((0, \infty) \times (0, \infty)\).

We next obtain the p.d.f. of \( \tau_c \) at the event that \( S \) creeps across a differentiable segment of \( c \). For linear \( c \), the result is shown in [17]. The following lemma is proved in Appendix.

Lemma 3.4. Under the continuous density condition on \( S \), the mapping \((a, t) \to \psi_a(t)\) is continuous on \((0, \infty) \times (0, \infty)\), where \( \psi_a(t) \) is the p.d.f. of \( \tau_a \) in (3.12).

Proposition 3.5. Let \( c \in C(0, \infty) \) be non-increasing with \( c(0^+) > 0 \). If \( c \) is differentiable on an open non-empty \( G \subset (0, \infty) \) and \( S \) satisfies the continuous density condition, then
\[
P\{\tau_c \in dt, S(\tau_c) = c(\tau_c)\} = -c'(t)g_t(c(t)) \, dt, \quad t \in G.
\]

Proof. It suffices to consider \( t \in G \) with \( c(t) > 0 \). Given such \( t \), \( a := c(t) \) is fixed. Letting \( q(\varepsilon) = P\{t - \varepsilon < \tau_a \leq t\} \), for \( \varepsilon > 0 \), we have \( q(\varepsilon) = P\{S(t - \varepsilon) < c(t - \varepsilon), S(t) \geq a\} = q_1(\varepsilon) + q_2(\varepsilon) \), where \( q_1(\varepsilon) = P\{S(t - \varepsilon) < a \leq S(t)\} \), \( q_2(\varepsilon) = P\{a \leq S(t - \varepsilon) < c(t - \varepsilon)\} \).

Let \( \tau_a = \inf \{s > 0 : S(s) > a\} \). Then \( q_1(\varepsilon) = P\{t - \varepsilon < \tau_a \leq t\} \). By Corollary 3.2 and Lemma 3.4 \( \psi_a \) is the continuous p.d.f. of \( \tau_a \), so the function \( P\{\tau_a \leq \cdot\} \) is differentiable with derivative \( \psi_a(t) \) at \( t \). Then \( q_1(\varepsilon)/\varepsilon \to \psi_a(t) \) as \( \varepsilon \downarrow 0 \). On the other hand, \( q_2(\varepsilon) = \int_0^{c(t)-c(t)} g_{t-\varepsilon}(a+x) \, dx \). Since \( (t, x) \to g_t(x) \) is continuous on \((0, \infty) \times (0, \infty)\) and \( c \) is differentiable at \( t \), \( q_2(\varepsilon)/\varepsilon \to -c'(t)g_t(c(t)) \) as \( \varepsilon \downarrow 0 \). We thus get
\[
    \lim_{\varepsilon \downarrow 0} \frac{P\{\tau_c \leq t\} - P\{\tau_c \leq t - \varepsilon\}}{\varepsilon} = -c'(t)g_t(c(t)) + \psi_{c(t)}(t).
\]

Similarly, as \( \varepsilon \downarrow 0 \), \( \varepsilon^{-1}[P\{\tau_c \leq t + \varepsilon\} - P\{\tau_c \leq t\}] \) has the same limit. It follows that \( P\{\tau_c \leq \cdot\} \) is differentiable everywhere in the open set \( \{t \in G : c(t) > 0\} \), and hence its derivative \( -c'(t)g_t(c(t)) + \psi_{c(t)}(t) \) is the p.d.f. of \( \tau_c \) on the set \( [3, \text{Theorem } 7.21] \). Then by Corollary 3.2
\[
    \frac{d}{dt} P\{\tau_c \in dt\} = -c'(t)g_t(c(t)) + \frac{P\{\tau_c \in dt, S(\tau_c) > c(\tau_c)\}}{dt},
\]
which yields (3.14). \( \Box \)
We need one more lemma before getting the second main result of this section.

**Lemma 3.6.** Let $S$ satisfy the continuous density condition. If $c$ is continuous and non-increasing on $(0, \infty)$ with $c(0^+) > 0$, then $P \{ \tau_c \in A \} = 0$ for any $A \subset (0, \infty)$ with $\ell(A) = c(A) = 0$, where $c(A)$ is the Riemann-Stieltjes integral $\int 1 \{ x \in A \cap (0, \infty) \} \, dc(x)$.

**Theorem 3.7.** Let $c$ be an absolutely continuous non-increasing function on $(0, \infty)$ with $c(0^+) > 0$. Suppose $c$ is differentiable on $(0, \infty) \setminus F$ for some closed set $F$ with $\ell(F) = 0$. Then under the continuous density condition on $S$, w.p. 1, $\tau_c \in (0, \infty) \setminus F$ and for $u \in [0, \infty)^d$ and $v \in [0, \infty)^d \setminus \{0\}$,

\[
P \{ X(\tau_c -) \in du, \Delta_X(\tau_c) \in dv \, | \, \tau_c \} = Z(\tau_c)^{-1} \nu_1(du, dv \, | \, \tau_c), \quad (3.15)
\]
\[
P \{ X(\tau_c -) \in du, \Delta_X(\tau_c) = 0 \, | \, \tau_c \} = Z(\tau_c)^{-1} \nu_2(du \, | \, \tau_c), \quad (3.16)
\]

where for $t \in (0, \infty) \setminus F$,

\[
\nu_1(du, dv \, | \, t) = 1 \{ 0 \leq c(t) - \|u\| < \|v\| \} P \{ X(t) \in du \} \Pi(du),
\]
\[
\nu_2(du \, | \, t) = -c'(t) g_t(c(t)) P \{ X(t) \in du \} \Pi(S(t) = c(t)),
\]
\[
Z(t) = -c'(t) g_t(c(t)) + \int_0^{c(t)} \Pi_S(c(t) - s) \Pi \{ S(t) \in ds \}.
\]

**Proof.** Since $c$ is absolutely continuous, $c(F) = 0$, so by Lemma 3.6, $\tau_c \in (0, \infty) \setminus F$ w.p. 1. By Theorem 3.11 and Proposition 3.5 for $t \in (0, \infty) \setminus F$, $u \in [0, \infty)^d$, $v \in [0, \infty)^d \setminus \{0\}$,

\[
P \{ \tau_c \in dt, X(\tau_c -) \in du, \Delta_X(\tau_c) \in dv \}
\]
\[
= 1 \{ 0 \leq c(t) - \|u\| < \|v\| \} dt \Pi \{ X(t) \in du \} \Pi(du) = dt \nu_1(du, dv \, | \, t),
\]
\[
P \{ \tau_c \in dt, X(\tau_c -) \in du, \Delta_X(\tau_c) = 0 \}
\]
\[
= -c'(t) g_t(c(t)) dt \Pi \{ X(t) \in du \} \Pi(S(t) = c(t)) = dt \nu_2(du \, | \, t).
\]

Integrate over $u$ and $v$ to get $P \{ \tau_c \in dt \} = Z(t) dt$. Then (3.15) follows. \qed

### 3.2 Results for general Lévy processes with bounded variation

For a process $X$ taking values in $\mathbb{R}$, denote

\[
\overline{X}(t) = \sup \{ X(s) : s \leq t \}, \quad \underline{X}(t) = \inf \{ X(s) : s \leq t \}.
\]

The following results will be used to validate the extension described in Section 12.

**Proposition 3.8.** Let $X$ be a Lévy process taking values in $\mathbb{R}$ with bounded variation and non-positive drift. Suppose $X$ is not compound Poisson. Then for any $a > 0$,

\[
P \{ \exists t > 0 \text{ such that } \overline{X}(s) < a \text{ all } s < t, X(t-) = a \text{ or } X(t) = a \} = 0.
\]

**Proof.** For each $t > 0$, denote $A_t = \{ \overline{X}(s) < a \text{ all } s < t \}$. We first show

\[
P \{ \exists t > 0 \text{ s.t. } X(t-) = a, \Delta_X(t) \neq 0 \}
\]
\[
= 0 = P \{ \exists t > 0 \text{ s.t. } A_t, X(t) = a, \Delta_X(t) \neq 0 \}. \quad (3.18)
\]
Let \( \Pi \) be the Lévy measure of \( X \). Given \( \varepsilon > 0 \),

\[
\mathbb{1}\{\exists t > 0 \text{ s.t. } X(t-) = a, \ |\Delta_X(t)| \geq \varepsilon\} \leq \sum_{t:|\Delta_X(t)| \geq \varepsilon} \mathbb{1}\{X(t-) = a\}.
\]

Take expectation on both side and apply the compensation formula to get

\[
P\{\exists t > 0 \text{ s.t. } X(t-) = a, \ |\Delta_X(t)| \geq \varepsilon\} \
\leq \Pi(\mathbb{R} \setminus (-\varepsilon, \varepsilon)) \int_0^\infty P\{X(t-) = a\} \, dt \leq \Pi(\mathbb{R} \setminus (-\varepsilon, \varepsilon)) U(\{a\}),
\]

where \( U(\cdot) = \int P\{X(t) \in \cdot\} \, dt \) is the potential measure of \( X \). Since \( U \) is diffuse [3], Proposition I.15 and \( \Pi(\mathbb{R} \setminus (-\varepsilon, \varepsilon)) < \infty \), the left hand side is 0 for any \( \varepsilon > 0 \), showing the first half of (3.18). On the other hand, since \( A_t \) implies \( X(t-) \leq a \),

\[
P\{\exists t > 0 \text{ s.t. } A_t, \ X(t) = a, \ \Delta_X(t) \neq 0\} \leq P\{\exists t > 0 \text{ s.t. } \Delta_X(t) = a - X(t-) > 0\},
\]

which is 0 by the argument for Proposition III.2 in [3]. This shows the second half of (3.18).

To complete the proof, it only remains to show

\[
P\{\exists t > 0 \text{ s.t. } A_t \text{ and } X(t-) = X(t) = a\} = 0,
\]

or equivalently, \( P\{\tau^* \leq \infty\} = 0 \), where \( \tau^* = \inf\{t > 0 : A_t \text{ and } X(t-) = X(t) = a\} \). Let \( \tau_a = \inf\{t > 0 : X(t) > a\} \). Clearly, \( \{\tau^* < \tau_a\} \subset \{\tau^* < \infty\} \subset \{\tau^* \leq \tau_a\} \). Since \( \{\tau^* = \tau_a < \infty\} \subset \{X \text{ creeps across } a \text{ at } \tau_a\} \), has 0 probability [cf. 3, Exercise VI.9], \( P\{\tau^* < \infty\} = P\{\tau^* < \tau_a\} \). Let \( \eta \sim \text{Exp}(1) \) be independent of \( X \). If \( P\{\tau^* < \tau_a\} > 0 \), then \( P\{\tau^* < \eta < \tau_a\} > 0 \) and hence \( P\{X(\eta) = a\} > 0 \). However, from the fluctuation identity [3, Theorem VI.5], \( X(\eta) \) is either constant 0 or infinitely divisible with Lévy measure \( \nu(dx) = 1\{x > 0\} \int_0^\infty t^{-1} e^{-t} P\{X(t) \in dx\} \, dt \). In the latter case, since \( U \) is diffuse, \( \nu \) is also diffuse, implying the distribution of \( X(\eta) \) is continuous on \((0, \infty)\) [cf. 30, Remark 27.3]. As a result, \( P\{X(\eta) = a\} = 0 \). The contradiction implies \( P\{\tau^* < \infty\} = 0 \). \( \square \)

Applying the result to \( X \) and \(-X\) respectively and using union-sum inequality, we get

**Corollary 3.9.** Let \( X \) be a Lévy process taking values \( \mathbb{R} \) with bounded variation and no drift. Suppose \( X \) is not compound Poisson. Then for any \( a > 0 \), \( b > 0 \),

\[
P\{\exists t > 0 \text{ such that } -b < \underline{X}(s) \leq \overline{X}(s) < a \text{ all } s < t, X(t-) \text{ or } X(t) = -b \text{ or } a\} = 0.
\]

## 4 Sampling of first passage event for a subordinator

We call a function \( c \) “regular” if it satisfies the conditions in Theorem 3.7 i.e., \( c \) is absolutely continuous and non-increasing on \((0, \infty)\) with \( c(0+) > 0 \), and is differentiable on \((0, \infty) \setminus F\), where \( F \) is a closed set of Lebesgue measure 0. Note that if \( c \) is regular, then for any constant \( a > 0 \), \( c \land a \) is also regular.

Let \( Z \) be a subordinator with Lévy measure \( \Pi \) and no drift, such that 1) \( \Pi(0, \infty) = \infty \) and \( \Pi \) can be decomposed as

\[
\Pi(dx) = e^{-q^2} \mathbb{1}\{x \leq r\} \Lambda(dx) + \chi(dx), \quad q \geq 0, \quad r > 0,
\]

(4.1)
with \( \chi(0, \infty) < \infty \), and 2) letting \( S \) be a subordinator with Lévy measure \( \Lambda \) and no drift, its passage event across any regular function can be sampled.

To utilize \( S \) to sample the first passage event of \( Z \) across a regular boundary \( c \), let \( X_1, X_2, X_3 \), and \( Q \) be independent subordinators with no drift, and with Lévy measures \( e^{-qx} \mathbf{1} \{ x \leq r \} \Lambda(dx) \), \((1 - e^{-qx}) \mathbf{1} \{ x \leq r \} \Lambda(dx) \), \( \mathbf{1} \{ x > r \} \Lambda(dx) \), and \( \chi \), respectively.

Among the four, only \( X_1 \) is not compound Poisson. Represent \( Z \) and \( S \) as

\[
Z = X_1 + Q, \quad S = \|X\| = X_1 + X_2 + X_3 \quad \text{with} \quad X = (X_1, X_2, X_3)'.
\]

Denote \( \tau_c^Z = \inf \{ t > 0 : Z(t) > c(t) \} \), \( \tau_c^S = \inf \{ t > 0 : S(t) > c(t) \} \), and \( \Delta_Q \) the jump process of \( Q \).

Table 1: Sampling of \((\tau, Z(\tau-), \Delta_Z(\tau))\), where \( \tau = \tau_c^Z \wedge K \). For \( c \) is a regular function or \( \infty \), \( 0 < K \leq \infty \) (finite if \( c \equiv \infty \)).

| * | Set \( T = H = D = 0 \), \( A = K \), \( b(\cdot) \equiv c(\cdot) \) |
|---|---|
| 1. If \( D = 0 \), then sample \((D, J) \sim (\tau_Q \wedge A, \Delta_Q(\tau_Q \wedge A))\), where \( \tau_Q = \inf \{ t : \Delta_Q(t) > 0 \} \). |
| 2. Sample \( t_1 \sim \tau_{b\wedge r}^S \) and set \( t = t_1 \wedge D \). |
| 3. If \( t = t_1 < D \), then sample \((s, v) \sim (S(t-), \Delta_S(t))\) conditional on \( \tau_{b\wedge r}^S = t \). |
| 4. If \( t = D < t_1 \), then sample \( s \sim S(t)\) conditional on \( S(t) < b(t) \wedge r \) and set \( v = 0 \). |
| 5. Sample \( x \sim X_1(t)\) conditional on \( X_1(t) + X_2(t) = s \). |
| 6. If \( v > 0 \), then sample \( U \sim \text{Unif}(0, 1) \) and reset \( v \leftarrow v \mathbf{1} \{v \leq r, U \leq e^{-qv}\} \). |
| 7. Update \( T \leftarrow T + t \). Set \( \Delta = v + 1 \{ t = D \} J, z = x + \Delta \), and update \( H \leftarrow H + z \). |
| 8. If \( z < b(t) \) and \( t < A \), then update \( A \leftarrow A - t \), \( D \leftarrow D - t \), \( b(\cdot) \leftarrow b(\cdot + t) - z \), and go back to step 1 to start a new iteration; else output \((T, H - \Delta, \Delta)\) and stop. |

**Theorem 4.1.** If \( c \) is a regular function and \( 0 < K \leq \infty \), then the procedure in Table 1 stops w.p. 1, and its output is a sample value of \((\tau, Z(\tau-), \Delta_Z(\tau))\), where \( \tau = \tau_c^Z \wedge K \). The claim is still true if \( c \equiv \infty \) and \( K < \infty \).

**Proof.** We only prove the claim for the case where \( c \) is a regular function. The case where \( c \equiv \infty \) and \( K < \infty \) can be similarly proved.

Consider the first iteration. With \( A = K \), \( D = t_Q \wedge K \). Note \( Z(t) = X_1(t) \) for \( t < D \).

In step 2, with \( b = c \), \( t_1 \) is a sample value of \( \tau_{b\wedge r}^S \) and \( t_1 \) that of \( \tau^* := \tau_{b\wedge r}^S \wedge \tau_Q \wedge K \). By independence, \( t_1 \neq D \) w.p. 1. If \( t_1 > D \), then w.p. 1, \( S(D-) = S(D) < c(D) \wedge r \). Thus the pair \((s, v)\) generated by steps 3 and 4 is a sample value of \((\tau^* - , \Delta_S(\tau^*))\) conditional on \( \tau^* = t \). Given \((\tau^*, S(\tau^*-), \Delta_S(\tau^*)) = (t, s, v)\), steps 5 and 6 sample \( X_1(\tau^*-) \) and \( \Delta_1(\tau^*) \) from their joint conditional distribution, where \( \Delta_1 \) is the jump process of \( X_1 \). If \( t = t_1 < D \), i.e., \( S \) crosses \( c \wedge r \) before \( D \), then by part 2) of Theorem 4.1, \( X_1(\tau^*-) \) and \( \Delta_1(\tau^*) \) are independent under the conditional distribution, following the distribution of
X_1(t) \text{ conditional on } S(t) = s \text{ and that of } \Delta_1(t) \text{ conditional on } \Delta_S(t) = v, \text{ respectively. This is still true if } t = D < t_1, \text{ as } X(D-) = X(D) \text{ and } \Delta_1(D) = \Delta_S(D) = 0 \text{ w.p. 1. By } s \leq c(t) \wedge r \leq r, \mathbb{P}\{X_1(t) \in s | S(t) = s\} = \mathbb{P}\{X_1(t) \in s | X_1(t) + X_2(t) = s\}, \text{ hence the sampling of } x \text{ in step 5. Clearly, } \Delta_S(t) = 0 \text{ implies } \Delta_1(t) = 0. \text{ Suppose } \Delta_S(t) = v > 0. \text{ The support of } \Pi_X \text{ is within } \{(x_1, x_2, x_3) : x_i \geq 0, \text{ at most one is nonzero}\}, \text{ such that for } y > 0, \Pi_X(dy \times \{0\} \times \{0\}) = e^{-qv}1\{y \leq r\} \Lambda(dy), \Pi_X(\{0\} \times dy \times \{0\}) = (1-e^{-qy})1\{y \leq r\} \Lambda(dy), \text{ and } \Pi_X(\{0\} \times \{0\} \times dy) = 1\{y > r\} \Lambda(dy). \text{ Then by Theorem 3.1} \mathbb{P}\{\Delta_1(t) \in dy | \Delta_S(t) = v\} = \Pi_v(dy \times \{0\} \times \{0\}) = 1\{y = v \leq r\} e^{-qv}, \text{ hence the updating of } v \text{ in step 6. Put together, the triplet } (t, x, v) \text{ generated by the end of step 6 is a sample value of } (\tau^*, X_1(\tau^*), \Delta_1(\tau^*)) \text{ and } \Delta \text{ in step 7 is a sample value of } \Delta_Z(\tau^*) = \Delta_1(\tau^*) + \Delta_Q(\tau^*) \text{ and } z \text{ that of } Z(\tau^*) = X_1(\tau^*) + \Delta_Z(\tau^*). \text{ If the condition of termination is not satisfied, we can renew the sampling by shifting the origin to } (t, Z(t)). \text{ This justifies the updating of } A \text{ and } b \text{ in step 8. Note that } D \text{ is the distance in time to the current jump of } Q. \text{ Once } D \text{ becomes 0, the next jump of } Q \text{ will be sampled.}

Let } T_0 = 0, \text{ and for } n \geq 1, (T_n, H_n, \Delta_n) \text{ the value of } (T, H, \Delta) \text{ obtained by the end of the } n^{th} \text{ iteration. By induction, we can make the following conclusion. For } n \geq 1, \text{ if } Z(T_{n-1}) < c(T_{n-1}) \text{ and } T_{n-1} < K, \text{ then}

\[ T_n = \inf\{t > T_{n-1} : S(t) - S(T_{n-1}) > [c(t) - Z(T_{n-1})] \wedge r \text{ or } \Delta_Q(t) > 0\} \wedge K, \tag{4.2} \]

\[ H_n = Z(T_n) \text{ and } \Delta_n = \Delta_Z(T_n). \text{ Evidently, } Z(T_{n-1}) = H_n - \Delta_n. \]

To show that the procedure stops w.p. 1 and returns } (\tau, Z(\tau), \Delta_Z(\tau)), \text{ it suffices to show } \mathbb{P}\{T_n = \tau \text{ eventually}\} = 1. \text{ It is clear that } T_0 < \tau. \text{ For } n \geq 1, \text{ if } T_{n-1} < \tau, \text{ then, since } Z \text{ is strictly increasing w.p. 1, } Z(T_{n-1}) < Z(\tau-) \leq c(\tau) \leq c(T_{n-1}). \text{ Then by (4.2), } T_n > T_{n-1}. \text{ Observe that in this case, for any } t \in (T_{n-1}, T_n),

\[ Z(t) - Z(T_{n-1}) = X_1(t) - X_1(T_{n-1}) \leq S(t) - S(T_{n-1}) \leq c(t) - Z(T_{n-1}), \]

giving } Z(t) \leq c(t) \text{ and hence } T_n \leq \tau. \text{ Therefore, if } T_n \neq \tau \text{ for all } n \geq 1, \text{ then } T_n \text{ is strictly increasing and strictly less than } \tau. \text{ Let } \theta = \lim T_n. \text{ Then } \theta \leq \tau < \infty. \text{ For } n \gg 1, \text{ the compound Poisson processes } X_2, X_3 \text{ and } Q \text{ make no jumps in } (T_n, \theta), \text{ and so } S(T_{n+1}) - S(T_n) = X_1(T_{n+1}) - X_1(T_n) = Z(T_{n+1}) - Z(T_n). \text{ Meanwhile, since } Z(T_n) < c(T_n) \leq c(T_1) < \infty, \text{ for } n \gg 1, 0 < Z(T_{n+1}) - Z(T_n) < r. \text{ Then we get}

\[ r > Z(T_{n+1}) - Z(T_n) = S(T_{n+1}) - S(T_n) \geq [c(T_{n+1}) - Z(T_n)] \wedge r. \]

It is easy to see that the inequalities imply } Z(T_{n+1}) \geq c(T_{n+1}). \text{ The contradiction shows that w.p. 1, } T_n = \tau \text{ for some } n. \]

\[ \square \]

5 Extensions to Lévy processes with bounded variation

5.1 Non-positive drift, positive constant level

Let } Z \text{ be a Lévy process with non-positive drift, such that its Lévy measure } \Pi \text{ satisfies}

\[ \Pi(\mathbb{R}) = \infty, \int_{-\infty}^{\infty} (|x| \wedge 1) \Pi(dx) < \infty. \tag{5.1} \]
Given constants $a > 0$ and $0 < K < \infty$, let $\tau^Z_a = \inf\{t > 0 : Z(t) > a\}$ and $\tau = \tau^Z_a \wedge K$. Decompose $Z$ as $Z^+ - Z^-$, where $Z^\pm$ are independent subordinators with Lévy measures

$$
\Pi^+(dx) = 1\{x > 0\} \Pi(dx), \quad \Pi^-(dx) = 1\{x > 0\} \Pi(-dx),
$$

(5.2)

respectively, with $Z^+$ having no drift. Since the drift of $Z$ is non-positive, if $\tau^Z_a < \infty$, then w.p. 1, $Z$ makes a positive jump at $\tau^Z_a$ [cf. Exercise VI.9]. Meanwhile, $Z$ makes no jump at $K$ w.p. 1. Therefore the only possible jump that $Z$ can make at $\tau$ is positive, giving $\Delta_Z(\tau) = \Delta_{Z^+}(\tau)$, $Z^+(\tau) = Z^+(\tau-) + \Delta_Z(\tau)$, and $Z^-(\tau) = Z^-(\tau)$.

We thus consider the sampling of $(\tau, Z^+(\tau-), Z^-(\tau), \Delta_Z(\tau))$. Table 2 describes a procedure to do this, which essentially follows the description in Section 4.2 but allows a terminal point $K \leq \infty$ to be included. Note that by assumption, $Z^\pm$ cannot be both compound Poisson. If $\Pi^+$ (resp. $\Pi^-$) can be decomposed as in (4.1), then the procedure in Table 1 can be directly called in step 1 (resp. 2) in Table 2. On the other hand, if one of $Z^\pm$ is compound Poisson, the corresponding step is straightforward.

Table 2: Sampling of $(\tau, Z^+(\tau-), Z^-(\tau), \Delta_Z(\tau))$, where $\tau = \tau^Z_a \wedge K$, $a$ is a positive constant, and $0 < K \leq \infty$

* Set $T = H^+ = H^- = 0$, $A = K$, $b = a$
  1. Sample $(t, z^+, v) \sim (\tau^+, Z^+(\tau^+) - \Delta_Z^+(\tau^+))$, where $\tau^+ = \tau_b^Z + A$. Set $x = z^+ + v$.
  2. Sample $z^- \sim Z^-(t)$.
  3. Update $T \leftarrow T + t$, $H^+ \leftarrow H^+ + x$, $H^- \leftarrow H^- + z^- - x$.
  4. If $x - z^- < b$ and $t < A$, then update $A \leftarrow A - t$, $b \leftarrow b + z^- - x$, go back to step 1 to start a new independent iteration; else output $(T, H^+ - v, H^-, v)$ and stop.

**Proposition 5.1.** Suppose $\overline{\lim}_{t \to \infty} Z(t) = \infty$ w.p. 1 or $K < \infty$. Then the procedure in Table 2 stops w.p. 1, and its output is a sample value of $(\tau, Z^+(\tau-), Z^-(\tau), \Delta_Z(\tau))$.

**Proof.** Let $T_0 = 0$, $H^+_0 = H^- = 0$, and for $n \geq 1$, let $(T_n, H^+_n, H^-_n, v_n)$ be the value of $(T, H^+, H^-, v)$ at the end of the $n$th iteration. By induction, for $n \geq 1$, the procedure has to continue into the $n$th iteration if and only if $Z(T_k) < a$ and $T_k < K$ for all $0 \leq k < n$, and in this case,

$$
T_n = \inf\{t > T_{n-1} : Z^+(t) - Z^+(T_{n-1}) > a - Z(T_{n-1})\} \wedge K > T_{n-1},
$$

(5.3)

and $H^+_n = Z^+(T_n)$, $H^-_n = Z^-(T_n)$, $v_n = \Delta_Z(T_n)$. By $Z^+(T_n-) - Z^+(T_{n-1}) \leq a - Z(T_{n-1})$,

$$
Z^+(T_n-) - Z^-(T_{n-1}) \leq a.
$$

(5.4)

We show that for $n \geq 1$, if the procedure has to continue into the $n$th iteration, then

$$
\sup\{Z(t) : t < T_n\} < a \quad w.p. 1
$$

(5.5)
Consider \( n = 1 \). Since \( Z \) is right-continuous, there is \( \varepsilon > 0 \) such that \( Z(\varepsilon) < a \), where \( Z \) is defined as in (3.17). Since at least one of \( Z^\pm \) is strictly increasing, by (5.4), for \( \varepsilon \leq s < t < T_1 \), \( Z(s) \leq Z^+(t) - Z^-(\varepsilon) < Z^+(T_n -) \leq a \). As a result, \( Z(t) < a \). If (5.5) is not true, then there must be \( Z(T_1 -) = a \). However, by Proposition 3.8, the probability for this to happen 0. We then get (5.5) for \( n = 1 \). For \( n \geq 2 \), by renewal argument, if the procedure has to continue into the \( n \)th iteration, then sup \( \{Z(t) - Z(T_{n-1}) : T_{n-1} \leq t < T_n \} < a - Z(T_{n-1}) \), which together with \( \Pi \) yields (5.5).

By assumption, \( \tau < \infty \) w.p. 1. To finish the proof, it suffices to show w.p. 1, \( T_n = \tau \) eventually. The compliment of the event has two cases. The first one is that the procedure stops at the end of an iteration with \( T_n \neq \tau \). In this case, by (5.5), \( T_n < \tau \leq K \). Now \( T_n < \tau \) implies \( Z(T_n) \leq a \), while \( T_n < K \) together with the stopping rule of the procedure implies \( Z(T_n) \geq a \). Thus \( Z(T_n) = a \). Also by (5.5), \( Z(t) < a \) for all \( t < T_n \). However, by Proposition 3.8, the probability of this case is 0. The second case is that the procedure goes on forever. In this case, by (5.5), \( Z(T_n) < a \) and \( T_n < \tau \) for all \( n \). Then by (5.3), \( T_n \) is strictly increasing with a limit \( \theta \leq \tau \). Now for any \( t < \theta \), \( Z(t) < a \). Meanwhile, by \( Z^+(T_{n+1}) - Z^+(T_n) \geq a - Z(T_n) > 0 \), letting \( n \to \infty \) yields \( Z(\theta-) = a \). By Proposition 3.8, the probability for such \( \theta \) to exist is also 0.

### 5.2 No drift, first exit out of an interval

Now consider the sampling of the first exit from an interval. Let \( Z \) be a Lévy process with no drift, such that its Lévy measure \( \Pi \) satisfies (5.1). Given constants \( a^\pm > 0 \) and \( 0 < K \leq \infty \), denote \( I = [-a^-, a^+] \) and let \( \tau^Z \) = \( \inf \{ t > 0 : Z(t) \notin I \} \). As in Section 5.1, write \( Z = Z^+ - Z^- \), where \( Z^\pm \) are independent subordinators with Lévy measures \( \Pi^\pm \), respectively, and no drift. Then \( Z \) makes a positive jump if it first exits \( I \) at \( a^+ \), and a negative jump if it first exists \( I \) at \( -a^- \). We thus consider the sampling of \( \tau, Z^+(\tau-), Z^-(\tau-), \Delta Z^+(\tau), \Delta Z^-(\tau) \).

Suppose that \( \Pi^\pm \) can be decomposed as in (4.1). To be specific, for \( \sigma \in \{\pm\} \),

\[
\Pi^\sigma(dx) = \exp(-q^\sigma x) 1 \{x \leq r^\sigma\} \Lambda^\sigma(dx) + \chi^\sigma(dx),
\]

where \( q^\sigma \geq 0 \) and \( 0 < r^\sigma \leq \infty \) are constants and \( \chi^\sigma(0, \infty) < \infty \), such that, letting \( S^\sigma \) be a subordinator with Lévy measure \( \Lambda^\sigma \) and no drift, its passage event across any positive constant level can be sampled. In the case where \( Z^\sigma \) is compound Poisson, we simply set \( S^\sigma \equiv 0 \) and the time of the first passage of \( S^\sigma \) across any positive boundary to be \( \infty \). Note that, since \( \Pi(\mathbb{R}) = \infty \), at most one \( Z^\sigma \) is compound Poisson.

To utilize \( S^\pm \) to sample \( (\tau, Z^+(\tau-), Z^-(\tau-), \Delta Z^+(\tau), \Delta Z^-(\tau)) \), the idea is similar to that in Section 4. Table 3 describes a procedure to do this. In each iteration, we have to monitor two passage times, i.e., the times when \( S^\sigma \) cross \( b^\sigma \land r^\sigma \), \( \sigma \in \{\pm\} \), respectively, where \( b^\sigma \) is a constant obtained from \( a^\sigma \). To simplify notation, denote by \( S_{\min}^\sigma(\sigma) \) the first passage time of \( S^\sigma \) across \( b^\sigma \land r^\sigma \). Represent \( S^\sigma = \|X^\sigma\| = X^\sigma_1 + X^\sigma_2 + X^\sigma_3 \), where \( X^\sigma = (X^\sigma_1, X^\sigma_2, X^\sigma_3) \) and \( X^\sigma_i \) are subordinators with Lévy measures \( \exp(-q^\sigma x) 1 \{x \leq r^\sigma\} \Lambda^\sigma(dx), \|1 - \exp(-q^\sigma x)\| 1 \{x \leq r^\sigma\} \Lambda^\sigma(dx) \), and \( 1 \{x > r^\sigma\} \Lambda^\sigma(dx) \), respectively. All of \( X^\sigma_i, i \leq 3, \sigma \in \{\pm\} \) are assumed to be independent with no drift.
Table 3: Sampling of \((\tau, Z^+(\tau-), Z^-(\tau-), \Delta_{Z^+}(\tau), \Delta_{Z^-}(\tau))\), where \(\tau = \tau_f^\sharp \wedge K\), \(I = [-a^-, a^+]\), \(a^-, a^+ > 0\) are constants, and \(0 < K \leq \infty\)

* Set \(T = H^+ = H^- = D = 0\), \(A = K\), \(b^+ = a^+, b^- = a^-\). Let \(Q\) be a compound Poisson process with Lévy measure \(\chi^+ + \chi^-\).

1. If \(D = 0\), then sample \((D, J) \sim (\tau_Q \wedge A, \Delta_Q(\tau_Q \wedge A))\) and set \(J^+ = J \vee 0\), \(J^- = (-J) \vee 0\), \(\tau = \inf\{t : \Delta_Q(t) \neq 0\}\).

2. For \(\sigma \in \{\pm\}\), sample \(t^\sigma \sim \tau_{\nu^\sigma}(\sigma)\). Set \(t = t^+ \wedge t^- \wedge D\). (Note: w.p. 1, \(t^\pm\) and \(D\) are different from each other.)

3. For \(\sigma \in \{\pm\}\), sample \((x^\sigma, v^\sigma) \sim (X_1^\sigma(t^-), \Delta_X^\sigma(t^-))\) conditional on \(\tau^+ \wedge \tau^- \wedge D = t\), by applying steps 3–6 in Table 1 to \(X^\sigma\).

4. Update \(T \leftarrow T + t\). For \(\sigma \in \{\pm\}\), set \(\Delta^\sigma = v^\sigma + 1\{t = D\} J^\sigma\), \(z^\sigma = x^\sigma + \Delta^\sigma\), and update \(H^\sigma \leftarrow H^\sigma + z^\sigma\).

5. If \(z^+ - z^- \in (-b^-, b^+)\) and \(t < A\), then update \(A \leftarrow A - t\), \(D \leftarrow D - t\), \(b^+ \leftarrow b^+ + z^- - z^+, b^- \leftarrow b^- + z^+ - z^-, \) and go back to step 1 to start a new iteration; else output \((T, H^+ - \Delta^+, H^- - \Delta^-, \Delta^+, \Delta^-)\) and stop.

**Proposition 5.2.** Suppose \(Z \not\equiv 0\). Then the procedure in Table 3 stops w.p. 1, and its output is a sample value of \((\tau, Z^+(\tau-), Z^-(\tau-), \Delta_{Z^+}(\tau), \Delta_{Z^-}(\tau))\).

**Proof.** First, \(\tau < \infty\) w.p. 1 as \(\lim_{t \to \infty} |Z(t)| = \infty\) [Theorem VI.12]. Let \(T_0 = 0\), \(H^+_0 = H^-_0 = 0\), and for \(n \geq 1\), let \((T_n, H^+_n, H^-_n, \Delta^+_n, \Delta^-_n)\) be the value of \((T, H^+, H^-, \Delta^+, \Delta^-)\) obtained by the end of the \(n\)th iteration. By induction and the same argument as in the proof of Theorem 4.1 for \(n \geq 1\), the procedure has to continue into the \(n\)th iteration if and only if \(Z(T_k) \in (-a^-, a^+)\) and \(T_k < K\) for \(0 \leq k < n\), and in this case,

\[
T_n = \inf\{t > T_{n-1} : S^+(t) - S^+(T_{n-1}) > a^+ - Z(T_{n-1}), \text{ or } S^-(t) - S^-(T_{n-1}) > a^- + Z(T_{n-1}), \text{ or } \Delta_Q(t) > 0\} \wedge K > T_{n-1},
\]

\[
H^+_n = Z^+(T_n), \quad H^-_n = Z^-(T_n), \quad \Delta^+_n = \Delta_{Z^+}(T_n), \quad \Delta^-_n = \Delta_{Z^-}(T_n),
\]

and, as in the proof of Proposition 5.1, \(Z^+(T_n) - Z^-(T_n) \leq a^+\), \(\sup\{Z(t) : t < T_n\} < a^+\), and likewise, by considering \(-Z(t), Z^-(T_n) - Z^+(T_n) \leq a^-, \inf\{Z(t) : t < T_n\} > -a^-\). The rest of the proof follows that of Proposition 5.1 except Corollary 3.3 is used. \(\square\)

### 6 Sampling issues involved

The procedures in previous sections involve several types of sampling, some of which are standard, while the others have to be dealt with on a case-by-case basis. Consider the procedure in Table 1. The main task of its step 1 is

sample the first jump of a compound Poisson process on \((0, \infty)\). \hspace{1cm} (6.1)
The rest of the procedure requires a subordinator $S$ with infinite Lévy measure $\Lambda$ and analytically tractable properties be available. Under this prerequisite, given regular boundary $c$, $t > 0$, and $0 < s \leq r$, the main tasks of steps 2–5 are to sample

\begin{eqnarray}
\tau^S_c &=& \inf\{t > 0 : S(t) > c(t)\}, \\
(S(t-), \Delta S(t)), & \text{conditional on } & \tau^S_c = t, \\
S(t), & \text{conditional on } & S(t) < c(t), \text{ and} \\
X_1(t), & \text{conditional on } & X_1(t) + X_2(t) = s,
\end{eqnarray}

respectively, where $X_1$, $X_2$ are independent subordinators with no drift and with Lévy measures $1 \{x \leq r\} e^{-qx} \Lambda(dx)$ and $1 \{x \leq r\} (1 - e^{-qx}) \Lambda(dx)$, respectively. The procedures in Tables 2 and 3 also boil down to (6.1)–(6.5).

For (6.1), recall that if a compound Poisson process $Q$ has Lévy measure $\chi \neq 0$, then the time and size of its first jump are independent with p.d.f. $g e^{-qx} \{t > 0\}$ and distribution $\chi/q$, respectively, where $q = \int d\chi$ [cf. 3, 36]. For complicated $\chi$, a rejection sampling method known as “thinning” can be used [cf. 9, 16]. Let $\mu$ be a Lévy measure such that 1) $\alpha = \int d\mu < \infty$ is readily available, 2) $\alpha^{-1}\mu$ is easy to sample, and 3) $d\chi = \rho d\mu$ for some function $\rho \leq 1$ that is easy to compute. Then the thinning based on $\mu$ is as follows.

* Set $t = 0$
  1. Sample $s$ from the distribution with p.d.f. $\alpha e^{-\alpha s} \{s > 0\}$. Update $t \leftarrow t + s$.
  2. Sample $x$ from the distribution $\alpha^{-1}\mu$ and $U \sim \text{Unif}(0,1)$.
  3. If $U \leq \rho(x)$, then stop and output $(t, x)$ as a sample of the time and size of the first jump of $Q$; else go back to step 1.

For (6.2), since $S$ is strictly increasing w.p. 1 and $c$ is non-increasing,

$$
P\{\tau^S_c \leq t\} = P\{S(t) \geq c(t)\},$$

with each side being continuous and strictly increasing in $t > 0$. If the probability distribution of $S(t)$ is analytically available, then $\tau^S_c$ may be sampled by inversion method, i.e., sample $U \sim \text{Unif}(0,1)$ and return the unique value of $t$ satisfying $P\{S(t) \geq c(t)\} = U$ [cf. 2]. Alternatively, if $S$ has scaling property, it can be utilized to sample $\tau^S_c$. Both possibilities are illustrated later. The sampling for (6.3) heavily relies on the results in Section 3 and its detail need be dealt with on a case-by-case basis. The sampling for (6.4) has the following generic solution: keep sampling $x \sim S(t)$ until $x \leq a$. However, by utilizing the structure of $S(t)$, it is possible to make the sampling significantly more efficient.

Finally, some comment on (6.5). Give $t > 0$, for the non-trivial case $g > 0$, if $S(t)$ has a bounded p.d.f. $g_t$, then in principle rejection sampling can be used [3, 10, 13, 32]. Indeed, since the Lévy measure of $X_1(t)$ is $e^{-qx} \nu(dx)$, where $\nu(dx) = t 1 \{x \leq r\} \Lambda(dx)$ is that of $X_1(t) + X_2(t)$, $P\{X_1(t) \in dx\} \propto e^{-qx} P\{X_1(t) + X_2(t) \in dx\}$. Recall $S = X_1 + X_2 + X_3$, where $X_3$ has Lévy measure $1 \{x > r\} \Lambda(dx)$. Since $X_3(t)$ is either 0 or $> r$, $X_1(t) + X_2(t)$ has p.d.f. $g_t(x)/P\{X_3(t) = 0\}$ on $(0, r]$. It follows that $X_1(t)$ has a p.d.f. on $(0, r]$ in proportion to $e^{-qx} g_t(x)$, and given $s \in (0, r]$,

$$
P\{X_1(t) \in dx \mid X_1(t) + X_2(t) = s\} \propto e^{-qx} g_t(x) P\{X_2(t) \in s - dx\}.$$
Thus to sample (6.5), we may keep sampling $v \sim X_2(t)$ and $U \sim \text{Unif}(0, 1)$ until $v \leq s$ and $\sup_x g_t(x) \cdot U \leq e^{-q(s-v)}g_t(s-v)$ and then output $s - v$. Here, since $X_2(t)$ is compound Poisson, its sampling is standard [9, 16]. However, a problem is that $g_t$ can be hard to evaluate. To get around the problem, the structure of $S(t)$ needs to be exploited.

7 Exponentially tilted upper truncated stable Lévy density

Consider the measure $\Pi(dx) = \mathbf{1}\{x \leq r\} e^{-qx} \Lambda(dx) + \chi(dx)$ specified in (1.1), where $\Lambda(dx) = \mathbf{1}\{x > 0\} \gamma x^{-1-\alpha} dx$ with $\alpha \in (0, 1)$. Let $c$ be a regular function as defined in Section 4. As an application of the procedure in Table 1, we next show an algorithm to sample the first passage event of a Lévy process with non-positive or no drift across a constant level or interval, when its Lévy measure is $\mathbf{1}\{0 < x \leq r^+\} \gamma^+ e^{-q^+ x} x^{-1-\alpha^+} dx + \mathbf{1}\{-r^- \leq x < 0\} \gamma^- e^{-q^- x} x^{-1-\alpha^-} dx + \chi(dx)$ with $\alpha^\pm \in (0, 1)$. We omit detail on this.

Let $S$ be a stable subordinator with Lévy measure $\Lambda$, so that for $\lambda, t > 0$, $\mathbb{E}[e^{-\lambda S(t)}] = \exp\{-\gamma \Gamma(1-\alpha)\alpha^{-1} \lambda^\alpha\}$. By scaling of time, assume $\gamma = \alpha/\Gamma(1-\alpha)$ without loss of generality. Then $S(1)$ is a “standard” stable variable with p.d.f.

$$f(x) = \frac{\alpha}{(1-\alpha)x} \int_0^x h(x, \theta) \, d\theta, \quad \text{where}$$

$$h(x, \theta) = \mathbf{1}\{x > 0\} h_0(\theta) x^{-1/(1-\alpha)} \exp\{-h_0(\theta) x^{-\alpha/(1-\alpha)}\}, \quad \text{with}$$

$$h_0(\theta) = \sin[(1-\alpha)\theta] \sin(\alpha \theta) \theta^{\alpha/(1-\alpha)} (\sin \theta)^{-1/(1-\alpha)}. \quad (7.1)$$

The sampling of $S(1)$ is well-known [7, 9, 39]. Define function

$$\psi(x) = \mathbf{1}\{x \neq 0\} x^{-1}(1-e^{-x}) + \mathbf{1}\{x = 0\}. $$

Let $0 < K \leq \infty$ and $\tau = \inf\{t > 0 : Z(t) > c(t)\} \wedge K$. An algorithm to sample $(\tau, Z(\tau-), \Delta Z(\tau))$ is as follows.

* Set $T = H = D = 0$, $A = K, b(\cdot) \equiv c(\cdot)$, $M_\alpha = (1-\alpha)^{1-1/\alpha} \alpha^{-1-1/\alpha} e^{-1/\alpha}$.
* Sample $(D, J)$ as in step 1 in Table 1
* Sample $S(1)$. Set $t_1$ such that $t_1^{1/\alpha} S(1) = b(t_1) \wedge r$, $t = t_1 \wedge D$, and $z = b(t) \wedge r$.
* If $t = t_1 < D$, then set

$$w_0 = - \frac{d(b(u) \wedge r)}{du} \bigg|_{u=t}, \quad w_1 = \frac{\alpha^{-1}}{\alpha(1-\alpha)}$$

and do the following steps. (Note: w.p. 1, $b(u) \wedge r$ is differentiable at $t$ with a non-positive derivative.)

(a) Sample $\iota \in \{0, 1\}$ such that $P\{\iota = i\} = w_i/(w_0 + w_1)$. If $\iota = 0$, then set $s = z, v = 0$; else sample $\beta_1 \sim \text{Beta}(1, 1-\alpha), \beta_2 \sim \text{Beta}(\alpha, 1)$, and set $s = \beta_1 z, v = (z-s)/\beta_2$. 

19
We next justify the algorithm. All its steps correspond 1-to-1 to those in Table 1, and only steps 2, 3 and 5 contain new detail. Denote θ, ζ, V with (1 being the unique solution to S, V = {τ = B(t−1/α, x, θ), if D(t−1/α, x, θ) < 0, set τ = 0, \( \tau = \sup_s \rho(s, v) \)), and set x = sβ, \( \theta = \prod_i \psi(q(s - x) \omega_i) \)).

We next justify the algorithm. All its steps correspond 1-to-1 to those in Table 1, and only steps 2, 3 and 5 contain new detail. Denote a = h(1/α, s, θ)/M, which is regular. With \( t_1 \) being the unique solution to \( t^{1/\alpha} S(1) = a(t) \), from [6,0] and scaling property of S, \( P\{\tau_a^S = t_1 \} = \int \rho(s, v) ds dv \), with \( z = a(t), w_0 = |a'(t)|, w_1 = \gamma z^{-1/\alpha}/[\alpha(1 - \alpha)] \), and \( \rho \) is the following p.d.f.

\[
\rho(s, v) = \begin{cases} 0 & \text{if } s < v < z \alpha(1 - \alpha) z^{-1/\alpha} v^{-1/\alpha}. \\
1 & \text{if } s = v < z \alpha(1 - \alpha) z^{-1/\alpha} v^{-1/\alpha}.
\end{cases}
\]

Define random vector (ς, ζ, V), such that \( P\{\varsigma = 0 \} = 1 - P\{\varsigma = 1 \} = w_0/(w_0 + w_1) \), \( P\{\zeta = 0 \} = 0 \), \( P\{\zeta = 1 \} = 1 \), \( P\{\varsigma \in ds, V \in dv | \varsigma = 1 \} = \rho(s, v) ds dv \). Let \( \vartheta \sim \text{Unif}(0, \pi) \) be independent of (ς, ζ, V). Then by \( g_\vartheta(s) = t^{-1/\alpha} f(t^{1/\alpha} s) \), where f given in (7.1),

\[
P\{S(t−) \in ds, \Delta_S(t−) \in dv | \tau_a^S = t\} \propto g_\vartheta(s) \left[ w_0 \delta(ds - \zeta) \delta(dv) + w_1 \rho(s, v) ds dv \right],
\]

with \( z = a(t), w_0 = |a'(t)|, w_1 = \gamma z^{-1/\alpha}/[\alpha(1 - \alpha)] \), and \( \rho \) is the following p.d.f.

\[
\rho(s, v) = \begin{cases} 0 & \text{if } s < v < z \alpha(1 - \alpha) z^{-1/\alpha} v^{-1/\alpha}. \\
1 & \text{if } s = v < z \alpha(1 - \alpha) z^{-1/\alpha} v^{-1/\alpha}.
\end{cases}
\]

Define random vector (ς, ζ, V), such that \( P\{\varsigma = 0 \} = 1 - P\{\varsigma = 1 \} = w_0/(w_0 + w_1) \), \( P\{\zeta = 0 \} = 0 \), \( P\{\zeta = 1 \} = 1 \), \( P\{\varsigma \in ds, V \in dv | \varsigma = 1 \} = \rho(s, v) ds dv \). Let \( \vartheta \sim \text{Unif}(0, \pi) \) be independent of (ς, ζ, V). Then by \( g_\vartheta(s) = t^{-1/\alpha} f(t^{1/\alpha} s) \), where f given in (7.1),

\[
P\{S(t−) \in ds, \Delta_S(t−) \in dv | \tau_a^S = t\} \propto \int h(t^{1/\alpha} s, \theta) P\{\vartheta \in d\vartheta, t \in dt, \zeta \in ds, V \in dv \}, \]

with the integral only over \( \vartheta \) and i. It is seen that \( \zeta \sim (1 - U_1^{1/(1-\alpha)}) z \) and conditional on \( \zeta, V \sim (z - \zeta) U_2^{1-\alpha} \), with \( U_1, U_2 \text{ i.i.d. } \sim \text{Unif}(0, 1) \). Thus step 3(a) samples (ς, ζ, V). Next, for \( x > 0 \) and \( \theta \in (0, \pi) \), by change of variable \( s = x^{-\alpha/(1-\alpha)} \) in the expression of \( h \),

\[
h(x, \theta) \leq \sup_{\theta \in (0, \pi)} \left[ h_0(\theta) \times \sup_{s > 0} (s^{1/\alpha} e^{-h_0(\theta)s}) \right] = \alpha^{-1/\alpha} e^{-1/\alpha} \sup_{\theta \in (0, \pi)} h_0(\theta)^{1-1/\alpha}.
\]
Since $\sin(t\theta)/\sin(\theta) \geq t$ for $\theta \in (0, \pi)$ and $t \in (0, 1)$, $h_0(\theta) \geq (1 - \alpha)\alpha^{\alpha/(1 - \alpha)}$, giving $h(x, \theta) \leq M_\alpha$. Thus, step 3 is a rejection sampling procedure for the distribution which is proportional to $h(t^{-1/\alpha}s, \theta)P\{\theta \in d\theta, t \in dt, \zeta \in ds, V \in dv\}$. Then by (7.3), $(s, v)$ is a sample of $(S(t-), \Delta S(t))$ conditional on $\tau_a^S = t$. The entire step 3 is now justified.

By step 5 in Table 1, given $(\tau^*, S(\tau^*-)) = (t, s)$ with $s \in (0, r]$, we need to sample $X_1(t)$ conditional on $X_1(t) + X_2(t) = s$, where $X_1$ and $X_2$ are independent subordinators with Lévy measures $1 \{x \leq r\} e^{-qx} \Lambda(dx)$ and $1 \{x \leq r\} (1 - e^{-qx}) \Lambda(dx)$, respectively. By (6.7),

$$P\{X_1(t) \in dx \mid X_1(t) + X_2(t) = s\} \propto e^{-qx} g_t(x)P\{X_2(t) \in s - dx\}, \quad 0 \leq s \leq s.$$

Since $X_2(t)$ is compound Poisson with Lévy density $\lambda(x) = 1 \{0 < x \leq r\} t\gamma(1 - e^{-qx})x^{-1 - \alpha}$,

$$P\{X_2(t) \in s - dx\} \propto 1 \{0 \leq x \leq s\} \left[\delta(s - dx) + \sum_{k=1}^\infty \frac{\lambda^k(s - x) dx}{k!}\right],$$

where $\lambda^k$ is the $k$-fold convolution of $\lambda$. For $w > 0$ and $k > 1$,

$$\lambda^k(w) = w^{k-1} \int \prod_{i=1}^k \lambda(wv_i) \sigma_k(dv),$$

where $\sigma_k$ is the measure specified in Section 2. Since $0 \leq w \leq s \leq r$, by the definition of $\psi$ and Dirichlet distribution, for any $v = (v_1, \ldots, v_k)$ with $v_i \geq 0$ and $\|v\| = 1$,

$$\prod_{i=1}^k \lambda(wv_i) = (t\gamma)^k \prod_{i=1}^k \frac{1 - e^{-qwv_i}}{qwv_i} = (t\gamma)^k q^k w^{-k\alpha} \prod_{i=1}^k \psi(qwv_i) \prod_{i=1}^k \frac{1}{v_i^\alpha},$$

$$= w^{-k\alpha} \frac{(t\gamma q(1 - \alpha))^k}{\Gamma(k(1 - \alpha))} f_k(v) \prod_{i=1}^k \psi(qwv_i),$$

where $f_k$ denotes the p.d.f. of Di$(a_1, \ldots, a_k)$, with all $a_i = 1 - \alpha$. Denote by $\omega_k = (\omega_{k1}, \ldots, \omega_{kk})$ a random variable following the Dirichlet distribution. Then

$$\lambda^k(s - x) dx = (s - x)^{k(1 - \alpha) - 1} dx \times \frac{(t\gamma q(1 - \alpha))^k}{\Gamma(k(1 - \alpha))} \mathbb{E} \left[\prod_{i=1}^k \psi(q(s - x)\omega_{ki})\right].$$

Note that $1 \{0 \leq x \leq s\} k(1 - \alpha)(s - x)^{k(1 - \alpha) - 1}/s^{k(1 - \alpha)}$ is the p.d.f. of $s\beta_k$, where $\beta_k \sim \text{Beta}(1, k(1 - \alpha))$. Then, with $C_k$ the same as in the algorithm, we get

$$\lambda^k(s - x) dx = k! C_k \mathbb{P}\{s\beta_k \in dx\} \mathbb{E} \left[\prod_{i=1}^k \psi(q(s - x)\omega_{ki})\right].$$

The above identity also holds for $k = 1$. Combining with (7.1), we get

$$P\{X_1(t) \in dx \mid X_1(t) + X_2(t) = s\} \propto e^{-q x} g_t(x) \left\{\delta(s - dx) + \sum_{k=1}^\infty C_k \mathbb{P}\{s\beta_k \in dx\} \mathbb{E} \left[\prod_{i=1}^k \psi(q(s - x)\omega_{ki})\right]\right\}$$

$$\propto \int_0^\pi e^{-q x} h(t^{-1/\alpha} x, \theta) d\theta \left\{\delta(s - dx) + \sum_{k=1}^\infty C_k \mathbb{P}\{s\beta_k \in dx\} \mathbb{E} \left[\prod_{i=1}^k \psi(q(s - x)\omega_{ki})\right]\right\}.$$
Starting from the last integral expansion, the treatment is similar to step 3. Define random vector \((\kappa, \zeta, \omega)\), such that \(\kappa \in \{0, 1, 2, \ldots\} \) with \(P\{\kappa = k\} \propto C_k\), conditional on \(\kappa = 0\), \(\zeta = s\), \(\omega = 0\), and conditional on \(\kappa = k \geq 1\), \(\zeta \sim s \beta_k\) and \(\omega \sim \omega_k\) are independent. For any \(x \in [0, s]\) and \(w \in \mathbb{R}^k\), define \(g(x, w) = \prod_{i=1}^k \psi(q(s - x)w_i)\), with \(k\) equal to the dimension of \(w\). Finally, let \(\vartheta \sim \text{Unif}(0, \pi)\) be independent from \((\kappa, \zeta, \omega)\). Then

\[
P\{X_1(t) = dx | X_1(t) + X_2(t) = s\}
\]

\[
\propto \int e^{-qx} h(t^{-1/\alpha} x, \theta) g(x, w) P\{\vartheta \in d\theta, \kappa \in dk, \zeta \in dx, \omega \in dw\},
\]

where the integral is only over \(\theta, k,\) and \(w\). Based on this, it is seen step 5 is a rejection sampling procedure of \(X_1(t)\) conditional on \(X_1(t) + X_2(t) = s\).

8 Finite mixture of exponentially tilted upper truncated stable Lévy densities

Given \(I \geq 2, r, \gamma_1, \ldots, \gamma_I > 0, q \geq 0,\) and \(\alpha_1, \ldots, \alpha_I \in (0, 1)\), let

\[
\Pi(dx) = \sum_{i=1}^I \varphi_i(x) dx + \chi(dx) \quad \text{with} \quad \varphi_i(x) = 1 \{0 < x \leq r\} \gamma_i e^{-qx} x^{-1-\alpha_i}. \tag{8.1}
\]

The seemingly more general case where different \(\varphi_i(x)\) have different \(r_i\) and \(q_i\) is actually covered by (8.1), once we set \(r = \min r_i, q = \max q_i, \tilde{\varphi}_i(x) = 1 \{0 < x \leq r\} \gamma_i e^{-q_i x} x^{-1-\alpha_i}\), and \(\tilde{\chi}(dx) = \chi(dx) + \sum_i [\varphi_i(x) - \tilde{\varphi}_i(x)] dx\). As in last section, we shall focus on the sampling for subordinators. The extension to Lévy measures \(\sum_{i=1}^I \varphi_i(\pm x) dx + \chi(dx)\) is rather straightforward.

To apply the procedure in Table 1 let \(X_{ij}, i = 1, \ldots, I, j = 1, 2, 3,\) and \(Q\) be independent subordinators with no drift, such that \(X_{i1}, X_{i2},\) and \(X_{i3}\) have Lévy densities \(\varphi_i(x)\), \(1 \{0 < x \leq r\} \gamma_i (1 - e^{-q_i x}) x^{-1-\alpha_i}, 1 \{x > r\} \gamma_i x^{-1-\alpha_i}\), respectively, and \(Q\) has Lévy measure \(\chi\). Let \(S_i = X_{i1} + X_{i2} + X_{i3}\) and \(Z = \sum_i X_{i1} + Q\). Then \(S_1, \ldots, S_I\) are independent stable processes with Lévy densities \(\gamma_i x^{-1-\alpha_i}\), respectively, and no drift, and \(Z\) a subordinator with Lévy measure \(\Pi\) and no drift. Let \(S = (S_1, \ldots, S_I)\). Unlike previous sections, \(S\) is multidimensional. Let \(\Sigma = \|S\| = S_1 + \cdots + S_I\). Then \(\Sigma(t) \sim \sum_{i=1}^I t^{1/\alpha_i} S_i(1)\). Let \(d_i = [\alpha_i / \gamma_i \Gamma(1 - \alpha_i)]^{1/\alpha_i}\). Then each \(d_i S_i(1)\) has Laplace transform \(E[e^{-\lambda d_i S_i(1)}] = \exp(-\lambda^{\alpha_i}), \lambda > 0\), whose p.d.f. \(f_i\) is given in (7.1). Let be \(h_i\) the function in the integral representation of \(f_i\) in (7.1).

Let \(0 < K \leq \infty\) and \(\tau = \inf\{t > 0 : Z(t) > c(t)\} \wedge K\). An algorithm to sample \((\tau, Z(\tau), \Delta_z(\tau))\) is as follows.

---

* Set \(T = H = D = 0, A = K, b() \equiv c(), M_i = (1 - \alpha_i)^{1-1/\alpha_i} \alpha_i^{-1-1/\alpha_i} e^{-1/\alpha_i}, i \leq I\)
1. Sample \((D, J)\) as in step 1 in Table 1
2. Sample \(S(1)\). Set \(t_1\) such that \(\sum_{i=1}^I t_1^{1/\alpha_i} S_i(1) = b(t_1) \wedge r, t = t_1 \wedge D,\) and \(z = b(t) \wedge r.\)
3. If \(t = t_1 < D,\) then set

\[
w_0 = -\frac{1}{\Gamma(I)} \left. \frac{d(b(u) \wedge r)}{du} \right|_{u=t}, \quad w_i = \frac{\gamma_i z^{1-\alpha_i} \Gamma(1 - \alpha_i)}{\alpha_i \Gamma(I + 1 - \alpha_i)}, \quad 1 \leq i \leq I
\]

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22
and do the following steps.

(a) Sample \( t \in \{0, 1, \ldots, I\} \), such that \( P\{t = j\} = w_j/(w_0 + w_1 + \cdots + w_I) \). Sample 
\( \omega = (\omega_1, \ldots, \omega_I) \sim \text{Di}(1, \ldots, 1) \). If \( t = 0 \), then set \( s_i = z\omega_i, i \leq I \), and \( v = 0 \).
If \( t \geq 1 \), then sample \( \beta \sim \text{Beta}(I, 1 - \alpha_i) \), \( \beta' \sim \text{Beta}(\alpha_i, 1) \), and set \( s_i = z\beta\omega_i, i \leq I \), and \( v = (z - s_1 - \cdots - s_I)/\beta' \).

(b) Sample \( \vartheta_1, \ldots, \vartheta_I \sim \text{Unif}(0, \pi) \) and \( U \sim \text{Unif}(0, 1) \) independently. If \( U \geq \prod_{i=1}^{I} [h_i(d_i, t^{-1}/\alpha_i, s_i, \vartheta_i)/M_i] \), then go back to step 3(a).

4. If \( t = D < t_1 \), then sample \( S_1(1), \ldots, S_I(1) \) conditional on \( \sum_{i=1}^{I} D^{1/\alpha_i} S_i(1) \leq z \). Set 
\( s = \sum_{i=1}^{I} D^{1/\alpha_i} S_i(1) \) and \( v = 0 \).

5. For \( i \leq I \), sample \( x_{i1} \) from the distribution of \( X_{i1}(t^-) \) conditional on \( X_{i1}(t^-) + X_{i2}(t^-) = s_i \), by executing step 5 in the algorithm in Section 7 with the corresponding parameters. Set \( x = x_{i1} + x_{i2} + \cdots + x_{iI} \).

6. The rest is the same as steps 6–8 in Table 1.

To justify the algorithm, denote \( a(t) = b(t) \land r \). From (6.6) and scaling property of \( S_i \),
\( P\{r_{\alpha}^{\Sigma} \leq t\} = P\{\Sigma(t) \geq a(t)\} = P\{\sum_{i=1}^{I} t^{1/\alpha_i} S_i(1) \geq a(t)\} \), which leads to step 2.

Given \( r_{\alpha}^{\Sigma} = t_1 \) and \( t' = t \), where \( t' = r_{\Sigma}^{\alpha} \land D \), denote \( z = a(t) \). Denote by \( g_t \) the p.d.f. of \( \Sigma(t) \). By Theorem 3.7, for \( s = (s_1, \ldots, s_I) \) with \( s_i \geq 0 \), and \( u, v \geq 0 \),
\[ P\{S(t^-) \in ds, \Sigma(t^-) \in du, \Delta_{\Sigma}(t^-) \in dv \mid r_{\alpha}^{\Sigma} = t\} \]
\[ \propto g_t(u)P\{S(t) \in ds \mid \Sigma(t) = u\} \]
\[ \times \left[ |a'(t)| \delta(du - z) \delta(dv) + 1 \{0 \leq z - u < v\} \sum_{i=1}^{I} \frac{\gamma_i du dv}{\nu^{1+\alpha_i}} \right]. \]

Since each \( S_i(t) \) has p.d.f. proportional to \( f_i(d_i, t^{-1/\alpha_i} x) \),
\[ g_t(u)P\{S(t) \in ds \mid \Sigma(t) = u\} \propto \prod_{i=1}^{I} f_i(d_i, t^{-1/\alpha_i} s_i) \times P\{u\omega \in ds\} \times \frac{u^{I-1}}{\Gamma(I)}, \]
with \( \omega = (\omega_1, \ldots, \omega_I) \sim \text{Di}(1, \ldots, 1) \). Let \( w_0, w_1, \ldots, w_I \) be as in the algorithm. Then 
\[ \frac{u^{I-1}}{\Gamma(I)} \times |a'(t)| \delta(du - z) \delta(dv) = z^{I-1} w_0 \delta(du - z) \delta(dv), \]
\[ \frac{u^{I-1}}{\Gamma(I)} \times 1 \{0 \leq z - u < v\} \frac{\gamma_i du dv}{\nu^{1+\alpha_i}} = z^{I-1} w_i P\{z\beta_i \in du, (z - u)/\beta'_i \in dv\}, \quad i \geq 1, \]
where \( \beta_i \sim \text{Beta}(I, 1 - \alpha_i) \) and \( \beta'_i \sim \text{Beta}(\alpha_i, 1) \) are independent. The above identities together with (7.1) yield
\[ P\{S(t^-) \in ds, \Sigma(t^-) \in du, \Delta_{\Sigma}(t^-) \in dv \mid r_{\alpha}^{\Sigma} = t\} \]
\[ \propto \int_{\theta_i \in [0, \pi]} \prod_{i=1}^{I} h_i(d_i, t^{-1/\alpha_i} s_i, \vartheta_i) d\theta_1 \cdots d\theta_I P\{u\omega \in ds\} \]
\[ \times \left[ w_0 \delta(du - z) \delta(dv) + \sum_{i=1}^{I} w_i P\{z\beta_i \in du, (z - u)/\beta'_i \in dv\} \right], \]
yielding $1$ identity still holds if $X \sim \chi$ and measure $\Pi$ and no drift. Let $0 < K < t < D$. From step 4 in Table 1 and the scaling property of $S_i$, it is easy to see step 4 samples $\Sigma(D)$ conditional on $D < \tau^*_a$. Finally, note the ultimate goal of step 5 in Table 1 is to sample $\sum_i X_{i1}(t)$ conditional on $\tau^* = t$. Since we have now sampled $(s,v) \sim (S(t), \Delta(t))$ conditional on $\tau^* = t$, it suffices to sample $(X_{i1}(t), \ldots, X_{i1}(t))$ conditional on $(S(t), \Delta(t), \tau^*) = (s,v,t)$. Since in this case, $X_{i1}(t) + X_{i2}(t) = s_i$ and $X_{i3}(t) = 0$, the conditional sampling is equivalent to that of $(x(t), \Phi)$, where $X = (X_{ij}, i = 1, \ldots, I, j = 1, 2, 3)$. By Theorem 3.1 if $t = t_1 < D$, then

$$
P\{X^{(t)} \in dx| \tau^* = t, \Sigma(t) = \|s\|, \Delta(t) = v\} = P\{X(t) \in dx| \Sigma(t) = \|s\|\},$$

$$P\{S^{(t)} \in ds| \tau^* = t, \Sigma(t) = \|s\|, \Delta(t) = v\} = P\{S(t) \in ds| \Sigma(t) = \|s\|\},$$

yielding $P\{X^{(t)} \in dx| \tau^* = t, S^{(t)} = s, \Delta(t) = v\} = P\{X(t) \in dx| S(t) = s\}$. The identity still holds if $t = D < t_1$. By independence, the right hand side is

$$\prod_{i=1}^{I} P\{X_{i1}(t) \in dx_{i1}, X_{i2}(t) \in dx_{i2}, X_{i3}(t) \in dx_{i3}| S_{i1}(t) = s_i\},$$

so to sample $(X_{i1}(t), \ldots, X_{i1}(t))$ conditionally, it suffices to sample $X_{i1}(t)$ independently, each conditional on $S_{i1}(t) = s_i$. Therefore, step 5 in the algorithm in Section 7 can be used. Once $X_{i1}(t)$ are sampled, their sum is a sample value of $\sum_i X_{i1}(t)$.

### 9 Upper truncated Gamma Lévy density

Let $\Pi(dx) = \varphi(x)dx + \chi(dx)$, where

$$\varphi(x) = 1 \{0 < x \leq r\} e^{-r}x^{-1}, \quad r > 0,$$

and $\chi$ is a finite measure on $(0, \infty)$. For a Lévy measure of the more general form $\tilde{\Pi}(dx) = \tilde{\phi}(x)dx + \tilde{\chi}(dx)$, where $\tilde{\phi}(x) = 1 \{0 < x \leq \tilde{r}\} \gamma e^{-q\tilde{r}}x^{-1}$, $\gamma, q > 0$, the sampling of the first passage event can be reduced to that for $\Pi$. Indeed, if $Z$ has Lévy measure $\Pi$, then $Z(t) = q\tilde{Z}(t/\gamma)$ has Lévy measure $\tilde{\Pi}$, with $r = q\tilde{r}$, $\chi(dx) = \tilde{\chi}(dx/q)$ therein. As a result, the first passage event of $\tilde{Z}$ across $\tilde{c}$ can be deduced from that of $Z$ across $qc(t/\gamma)$.

The sampling of the first passage event for $\Pi$ is somewhat simpler than those in previous sections, as exponential tilting is “built in” the Gamma process. Let $X_1, X_2,$ and $Q$ be independent subordinators with no drift and with Lévy measures $\varphi(x)dx, 1 \{x \geq r\} e^{-r}x^{-1}dx$, and $\gamma$, respectively. Then $S = X_1 + X_2$ is a Gamma process and $Z = X_1 + Q$ has Lévy measure $\Pi$ and no drift. Let $0 < K \leq \infty$ and $\tau = \inf\{t > 0: Z(t) > c(t)\} \wedge K$. An algorithm to sample $(\tau, Z(\tau), \Delta_Z(\tau))$ is as follows.

---

* Set $T = H = D = 0$, $A = K$, $b(\cdot) \equiv c(\cdot)$

1. Sample $(D,J)$ as in step 1 in Table 1

2. Sample $U \sim \text{Unif}(0,1)$. Set $t_1$ such that $\int_{b(t_1) \wedge r}^{\infty} x^{t_1-1}e^{-x}dx/\Gamma(t_1) = U$, $t = t_1 \wedge D$, and $z = b(t) \wedge r$.  

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24
3. If \( t = t_1 < D \), then set
\[
\begin{align*}
  w_0 &= - \frac{d(b(u) \land r)}{du} \bigg|_{u=t}, \\
  w_1 &= \frac{2B(t, 1/2)z}{e}, \\
  w_2 &= \frac{1}{t}
\end{align*}
\]
\[
\begin{align*}
  h_1(x, v) &= 1 \left\{ 0 \leq z - x < v \leq z \right\} e^{z-x-v}(1-x/z)^{1/2} \ln[(1-x/z)^{-1}] \\
  h_2(x, v) &= 1 \left\{ 0 \leq z - x < z < v \right\} \frac{ze^{-x}}{v}.
\end{align*}
\]
and do the following steps.
(a) Sample \( i \in \{0, 1\} \), such that \( P\{i = i\} = w_i/(w_0 + w_1 + w_2) \). If \( i = 0 \), then set \( x = z, v = 0, \eta = 1; \) if \( i = 1 \), then sample \( \beta \sim \text{Beta}(t, 1/2), \xi \sim \text{Unif}(0, 1) \), and set \( x = z\beta, v = z(1-\beta)\xi, \eta = h_1(x, v) \); if \( i = 2 \), then sample \( \beta \sim \text{Beta}(t, 1), \xi \sim \text{Exp}(1) \), and set \( x = z\beta, v = z + \xi, \eta = h_2(x, v) \).
(b) Sample \( U \sim \text{Unif}(0, 1) \). If \( U > \eta \), then go back to step 3(a).
4. If \( t = D < t_1 \), then sample \( \gamma \sim \text{Gamma}(D, 1) \) conditional on \( \gamma \leq z \). Set \( x = \gamma, v = 0 \).
5. The rest is the same as steps 6–8 in Table II.

To justify the algorithm, denote \( a(t) = b(t) \land r \). From (6.6) and \( S(t) \sim \text{Gamma}(t, 1) \),
\[
P\{\tau_a^S \leq t\} = P\{S(t) \geq a(t)\} = \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1}e^{-x} \, dx,
\]
which is a continuous and strictly increasing function. The sampling in step 2 is then the standard inversion [cf. 9]. Given \( \tau_0^S = t_1 \) and \( \tau^* = t \), where \( \tau^* = \tau_0^S \wedge D \), by step 3 in Table II if \( t = t_1 < D \), then we need to sample \( (S(t-), \Delta_S(t)) \) conditional on \( \tau_0^S = t \). Let \( g_t \) denote the p.d.f. of \( \text{Gamma}(t, 1) \). Let \( z = a(t) \). From Theorem 3.7 for \( x, v > 0 \),
\[
P\{S(t-) \in dx, \Delta_S(t) \in dv \mid \tau_0^S = t\} \\
\propto |a'(t)|g_t(z)\delta(dx - z)\delta(dv) + 1 \{ 0 \leq z - x < v \} g_t(x)\frac{e^{-v}}{v} dx dv \\
\propto |a'(t)|\delta(dx - z)\delta(dv) + q_1(x, v) dx dv + q_2(x, v) dx dv,
\]
where, letting \( q(x, v) = g_t(x)e^{-v}/[vg_t(z)], q_1(x, v) = 1 \{ 0 \leq z - x < v \leq z \} q(x, v) \) and \( q_2(x, v) = 1 \{ 0 \leq z - x < v \leq \xi \} q(x, v) \). Now \( q(x, v) = (x/z)^{t-1}e^{z-x-v}/v \). Let
\[
\rho_1(x, v) = 1 \{ 0 \leq z - x < v \leq z \} \frac{(x/z)^{t-1}(1-x/z)^{-1/2}}{B(t, 1/2)z} \frac{1}{v \ln[(1-x/z)^{-1}]} \\
\rho_2(x, v) = 1 \{ 0 \leq z - x < v \leq \xi \} \frac{t(x/z)^{t-1}e^{z-x}}{z}.
\]
For \( i = 1, 2, \rho_i(x, v) \) is a p.d.f. and \( q_i(x, v) = w_i h_i(x, v) \rho_i(x, v) \), where \( w_i \) and \( h_i \) are defined in the algorithm. It is easy to check \( h_i(x, v) \leq 1 \) for \( i = 1, 2 \). Define \( h_0(x, v) \equiv 1 \). Define \( (\iota, \zeta, V) \) such that \( \iota \in \{0, 1, 2\} \) with \( P\{\iota = i\} = w_i/(w_0 + w_1 + w_2) \), conditional on \( \iota = 0, \zeta = z \) and \( V = 0 \), and conditional on \( \iota = i \in \{1, 2\}, (\zeta, V) \) has p.d.f. \( \rho_i \). Then
\[
P\{S(t-) \in dx, \Delta_S(t) \in dv \mid \tau_0^S = t\} = \int h_i(x, v)P\{\iota \in di, \zeta \in dx, V \in dv\},
\]
where the integral is only over $i$. It is easy to check that when $(\zeta, \nu)$ has p.d.f. $\rho_1$, then $(\zeta, \nu) \sim (z\beta_1, z(1-\beta_2)\nu)$, where $\beta_1 \sim \text{Beta}(t, 1/2)$ and $U \sim \text{Unif}(0, 1)$, and when $(\zeta, \nu)$ has p.d.f. $\rho_2$, then $(\zeta, \nu) \sim (z\beta'_1, z + \xi)$, where $\beta'_1 \sim \text{Beta}(t, 1)$ and $\xi \sim \text{Exp}(1)$. Then step 3 in the algorithm is a rejection sampling procedure of $(S(t), \Delta_S(t))$ conditional on $\tau_a^S = t$. Since $S(t-)$ sampled by step 3 or 4 is exactly $X_1(t-)$, there is no need for a counterpart of step 5 in Table I. We can directly go to steps 6–8 in Table I.

**Appendix**

To prove Lemmas 3.4 and 3.6, we start with two more lemmas.

**Lemma A.1.** For $t > 0$ and $0 < a < b < \infty$, let

$$L_1(t, a, b) = \int_a^0 \Pi_S(a-u) - \Pi_S(b-u) |g_t(u)| \, du, \quad L_2(t, a, b) = \int_a^b \Pi_S(b-u) g_t(u) \, du.$$  

Then for any $E = [t_0, t_1] \subset (0, \infty)$ and $I = [\alpha, \beta] \subset (0, \infty)$,

$$\lim_{i \to \infty} \sup \{L_i(t, a, b) : t \in E, a, b \in I, 0 \leq b - a \leq r\} = 0, \quad i = 1, 2.$$  \hspace{1cm} (A.1)

**Proof.** Since $\Pi_S$ is non-increasing, for $0 \leq b - a \leq r$, $L_1(t, a, b) \leq L_1(t, a, a + r)$. Fix $\epsilon \in (0, \alpha/2)$ and let $h(u) = |\Pi_S(a-u) - \Pi_S(a+r-u)| g_t(u)$. Then $L_1(t, a, a + r) = J_1 + J_2$, where $J_1 = J_1(t, a, r) = \int_0^\epsilon h$, $J_2 = J_2(t, a, r) = \int_0^\epsilon h$. We have

$$J_1 \leq \int_0^\epsilon \Pi_S(a-u) g_t(u) \, du \leq \Pi_S(a-\epsilon) \int_0^\epsilon g_t(u) \, du = \Pi_S(\alpha-\epsilon) \mathbb{P}\{S(t) \leq \epsilon\} \leq \Pi_S(\alpha-\epsilon) \mathbb{P}\{S(t_0) \leq \epsilon\}$$

and letting $M = \sup \{g_t(u) : t \in E, \epsilon \leq u \leq \beta\}$,

$$J_2 \leq M \int_\epsilon^a [\Pi_S(a-u) - \Pi_S(a-u+r)] \, du \leq M \int_0^\beta [\Pi_S(u) - \Pi_S(u+r)] \, du.$$  

By the assumption on $g_t(x)$, $M < \infty$. Also, $\int_0^\beta \Pi_S(u) \, du = \int_0^\infty (v \wedge \beta) \Pi_S(\nu) < \infty$. Then by monotone convergence, as $r \downarrow 0$, $J_2 \to 0$ uniformly for $(t, a) \in E \times I$. As a result, for $L_1$, the limit in (A.1) is no greater than $\Pi_S(\alpha-\epsilon) \mathbb{P}\{S(t_0) \leq \epsilon\}$. Since $\mathbb{P}\{S(t_0) > 0\} = 1$ and $\epsilon$ is arbitrary, the limit is equal to 0. Thus (A.1) holds for $L_1$.

Next, fixing $\epsilon \in (0, \alpha)$, by change of variable, for $t \in E$, $a, b \in I$ with $0 \leq b - a \leq r$,

$$L_2(t, a, b) = \int_0^\infty \Pi_S(dx) \int_a^{b \wedge (b-x)} g_t \leq \int_0^\epsilon \Pi_S(dx) \int_{b-x}^b g_t + \int_\epsilon^\infty \Pi_S(dx) \int_a^b g_t \leq M' \left[ \int_0^\epsilon x \Pi_S(dx) + r \Pi_S(\epsilon) \right],$$

where $M = \sup \{g_t(u) : t \in E, \alpha - \epsilon \leq u \leq \beta\}$. Therefore, as $r \downarrow 0$, the limit for $L_2$ in (A.1) is no greater than $M' \int_0^\epsilon x \Pi_S(dx)$. Since $\epsilon$ is arbitrary, the limit is 0. 

□
Lemma A.2. Let $h$ be a bounded function on $(0, \infty) \times (0, \infty)$. For $a, t \in (0, \infty)$, define

$$H(a, t) = \int \left\{ u \leq a < x \right\} h(u, x) \mathbb{P}\{S(t) \in du\} \Pi_S(dx - u).$$

Then under the continuous density condition, $H$ is continuous on $(0, \infty) \times (0, \infty)$.

Proof. Without loss of generality, suppose $|h(u, x)| \leq 1$. It suffices to show $H$ is continuous on any $R = [\alpha, \beta] \times [t_0, t_1] \subset (0, \infty) \times (0, \infty)$. Let $(a, s), (b, t) \in R$. Then

$$\left| H(b, t) - H(a, s) \right| \leq \left| H(b, t) - H(a, t) \right| + \left| H(a, t) - H(a, s) \right|$$

Let $L_1$ and $L_2$ be as in Lemma A.1. Let $a' = a \land b$ and $b' = a \lor b$. Then

$$\left| H(b, t) - H(a, s) \right| \leq \int \left\{ 1 \{ u \leq b < x \} - 1 \{ u \leq a < x \} \right\} \mathbb{P}\{S(t) \in du\} \Pi_S(dx - u)$$

$$\leq \int \left( 1 \left\{ u \leq a' < x \leq b' \right\} + 1 \{ a' < u \leq b' < x \} \right) \mathbb{P}\{S(t) \in du\} \Pi_S(dx - u).$$

The right hand side is $L_1(t, a', b') + L_2(t, a', b')$. Then by Lemma A.1 as $(b, t) \to (a, s)$, $L_1(t, a', b') + L_2(t, a', b') \to 0$, giving $H(b, t) - H(a, s) \to 0$.

It only remains to show $H(a, t) - H(a, s) \to 0$. Given $\varepsilon \in (0, \alpha)$, let $M = \sup\{g_t(u) : u \in [\alpha - \varepsilon, \beta], t \in [t_0, t_1]\}$. Then

$$\left| H(a, t) - H(a, s) \right| \leq \int \left\{ u \leq a < x \right\} |g_t(u) - g_s(u)| \Pi_S(dx - u) du$$

$$= \int \left\{ u \leq a \right\} |g_t(u) - g_s(u)| \Pi_S(a - u) du.$$

Bounding the integral on $[a - \varepsilon, a]$ and $[0, a - \varepsilon]$ separately, we obtain

$$\left| H(a, t) - H(a, s) \right| \leq 2M \int_{\varepsilon}^{a} \Pi_S(u) du + \Pi_S(\varepsilon) \int |g_t(u) - g_s(u)| du.$$

Let $t \to s$. Since point-wise convergence of $g_t$ to $g_s$ implies convergence in total variation, $\lim_{t \to s} \left| H(a, t) - H(a, s) \right| \leq 2M \int_{0}^{\varepsilon} \Pi_S(u) du < \infty$. Letting $\varepsilon \to 0$ gets $H(a, t) - H(a, s) \to 0$. $\blacksquare$

Proof of Lemma 3.4. Apply (3.12) and Lemma A.2 with $h(a, t) \equiv 1$ therein. $\blacksquare$

Proof of Lemma 3.6. Let $G_c = \{ t > 0 : c(t) > 0 \}$. Then $G_c$ is a non-empty open interval and $\mathbb{P}\{\tau_c \in G_c\} = 1$. To prove the lemma, it suffices to show that for any $0 < t_0 < t_1 < \infty$ with $[t_0, t_1] \subset G_c$ and $A \subset (t_0, t_1)$ with $\ell(A) = c(A) = 0$, $\mathbb{P}\{\tau_c \in A\} = 0$. Let $\alpha = c(t_1)$ and $\beta = c(t_0)$. Given $\varepsilon > 0$, $A$ can be covered by at most countably many disjoint intervals $(a_i, b_i) \subset (t_0, t_1)$ such that $\sum (b_i - a_i) < \varepsilon$ and $\sum c(a_i) - c(b_i) < \varepsilon$. For each $i$,

$$\mathbb{P}\{\tau_c \in (a_i, b_i)\} \leq \mathbb{P}\{S(a_i) \leq c(a_i), S(b_i) > c(b_i)\}$$

$$\leq \mathbb{P}\{c(b_i) < S(a_i) \leq c(a_i)\} + \mathbb{P}\{S(a_i) \leq c(b_i) < S(b_i)\}$$

$$= \mathbb{P}\{c(b_i) < S(a_i) \leq c(a_i)\} + \mathbb{P}\{\tau_c(b_i) \in (a_i, b_i)\}.$$

Since $a_i, b_i \in [t_0, t_1]$ and $c(a_i), c(b_i) \in [\alpha, \beta]$, by the continuous density condition and Lemma 3.4, the right hand side is no greater than $M_1[c(a_i) - c(b_i)] + M_2(b_i - a_i)$, where $M_1 = \sup g_t(x)$ and $M_2 = \sup \psi_x(t) \text{ over } (t, x) \in [t_0, t_1] \times [\alpha, \beta]$. Therefore, $\mathbb{P}\{\tau_c \in A\} \leq \sum \mathbb{P}\{\tau_c \in (a_i, b_i)\} \leq (M_1 + M_2)\varepsilon$. Since $\varepsilon$ is arbitrary, this yields the proof. $\blacksquare$
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