ON EXISTENCE OF LOG MINIMAL MODELS II

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Abstract. We prove that the existence of log minimal models in dimension $d$ essentially implies the LMMP with scaling in dimension $d$. As a consequence we prove that a weak nonvanishing conjecture in dimension $d$ implies the minimal model conjecture in dimension $d$.

1. Introduction

We work over a fixed algebraically closed field $k$ of characteristic zero. See section 2 for notation and terminology. Remember that a lc pair $(X/Z, B)$ is called pseudo-effective if $K_X + B$ is pseudo-effective/Z, that is, if there is a sequence of $\mathbb{R}$-divisors $M_i \geq 0$ such that $K_X + B \equiv \lim_{i \to \infty} M_i$ in $N^1(X/Z)$. The pair is called effective if $K_X + B \equiv M/Z$ for some $M \geq 0$.

The following two conjectures are, at the moment, the most important open problems in birational geometry and the classification theory of algebraic varieties.

Conjecture 1.1 (Minimal model). Let $(X/Z, B)$ be a lc pair. If it is pseudo-effective then it has a log minimal model, and if it is not pseudo-effective then it has a Mori fibre space.

Conjecture 1.2 (Abundance). Let $(X/Z, B)$ be a lc pair. If $K_X + B$ is nef/Z, then it is semi-ample/Z.

For a brief history of the many results on the minimal model conjecture see the introduction to [1]. On the other hand, there has been little progress regarding the abundance conjecture in higher dimension. The main conceptual obstacle to abundance is the following problem.

Conjecture 1.3 (Weak nonvanishing). Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial dlt pair. If $K_X + B$ is pseudo-effective/Z, then it is effective/Z, that is, $K_X + B \equiv M/Z$ for some $M \geq 0$. 

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This conjecture is not only at the heart of the abundance conjecture but it is also closely related to the minimal model conjecture. In fact, we show that it implies the minimal model conjecture.

**Theorem 1.4.** Assume the weak nonvanishing conjecture (1.3) in dimension $d$. Then, the minimal model conjecture (1.1) holds in dimension $d$; moreover, if $(X/Z, B)$ is a $\mathbb{Q}$-factorial dlt pair of dimension $d$, then there is a sequence of divisorial contractions and log flips starting with $(X/Z, B)$ and ending up with a log minimal model or a Mori fibre space of $(X/Z, B)$.

The proof of this theorem is given via the following results and [1, Proposition 3.4].

**Theorem 1.5.** Assume the minimal model conjecture (1.1) in dimension $d$ for pseudo-effective $\mathbb{Q}$-factorial dlt pairs. Let $(X/Z, B + C)$ be a $\mathbb{Q}$-factorial lc pair of dimension $d$ such that

1. $K_X + B + C$ is nef$/Z$,
2. $B, C \geq 0$, and
3. $(X/Z, B)$ is dlt.

Then, we can run the LMMP$/Z$ on $K_X + B$ with scaling of $C$, and it terminates if either

- $B \geq H \geq 0$ for some ample$/Z \mathbb{R}$-divisor $H$, or
- $C \geq H \geq 0$ for some ample$/Z \mathbb{R}$-divisor $H$, or
- $\lambda \neq \lambda_i$ for any $i$ where $\lambda$ and $\lambda_i$ are as in Definition 2.3.

**Corollary 1.6.** Assume the minimal model conjecture (1.1) in dimension $d$ for pseudo-effective $\mathbb{Q}$-factorial dlt pairs. Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial dlt pair of dimension $d$. Then, there is a sequence of divisorial contractions and log flips starting with $(X/Z, B)$ and ending up with a log minimal model or a Mori fibre space $(Y/Z, B_Y)$. In particular, the corresponding birational map $Y \dasharrow X/Z$ does not contract divisors.

**Corollary 1.7.** Assume the minimal model conjecture (1.1) in dimension $d$ for pseudo-effective $\mathbb{Q}$-factorial dlt pairs. Then, the minimal model conjecture (1.1) holds in dimension $d + 1$ for effective lc pairs.

We sometimes refer to some of the results of [3, 4]. Actually, to prove the main results of this paper we only need two pages of [4], that is [4, Theorem 2.6], the rest that we need can be easily incorporated into the framework of [4] and this paper.
2. Basics

Let $k$ be an algebraically closed field of characteristic zero fixed throughout the paper.

A pair $(X/Z, B)$ consists of normal quasi-projective varieties $X, Z$ over $k$, an $\mathbb{R}$-divisor $B$ on $X$ with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier, and a projective morphism $X \to Z$. For a prime divisor $D$ on some birational model of $X$ with a nonempty centre on $X$, $a(D, X, B)$ denotes the log discrepancy.

A pair $(X/Z, B)$ is called pseudo-effective if $K_X + B$ is pseudo-effective$/Z$, that is, up to numerical equivalence$/Z$ it is the limit of effective $\mathbb{R}$-divisors. The pair is called effective if $K_X + B$ is effective$/Z$, that is, there is an $\mathbb{R}$-divisor $M \geq 0$ such that $K_X + B \equiv M/Z$.

By a log flip$/Z$ we mean the flip of a $K_X + B$-negative extremal flipping contraction$/Z$ for some lc pair $(X/Z, B)$ (cf. [1, Definition 2.3]), and by a pl flip$/Z$ we mean a log flip$/Z$ such that $(X/Z, B)$ is $\mathbb{Q}$-factorial dlt and the log flip is also an $S$-flip for some component $S$ of $\lfloor B \rfloor$.

A sequence of log flips$/Z$ starting with $(X/Z, B)$ is a sequence $X_i \to X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$-flip$/Z$, $B_i$ is the birational transform of $B_1$ on $X_1$, and $(X_1/Z, B_1) = (X/Z, B)$.

In this paper, special termination means termination near $\lfloor B \rfloor$ of any sequence of log flips$/Z$ starting with a pair $(X/Z, B)$, that is, the log flips do not intersect $\lfloor B \rfloor$ after finitely many of them.

**Definition 2.1** A pair $(Y/Z, B_Y)$ is a log birational model of $(X/Z, B)$ if we are given a birational map $\phi: X \dashrightarrow Y/Z$ and $B_Y = B^\sim + E$ where $B^\sim$ is the birational transform of $B$ and $E$ is the reduced exceptional divisor of $\phi^{-1}$, that is, $E = \sum E_j$ where $E_j$ are the exceptional/X prime divisors on $Y$. A log birational model $(Y/Z, B_Y)$ is a nef model of $(X/Z, B)$ if in addition

1. $(Y/Z, B_Y)$ is $\mathbb{Q}$-factorial dlt, and
2. $K_Y + B_Y$ is nef$/Z$.

And we call a nef model $(Y/Z, B_Y)$ a log minimal model of $(X/Z, B)$ if in addition

3. for any prime divisor $D$ on $X$ which is exceptional$/Y$, we have

$$a(D, X, B) < a(D, Y, B_Y)$$
Definition 2.2 (Mori fibre space) A log birational model \((Y/Z, B_Y)\) of a lc pair \((X/Z, B)\) is called a Mori fibre space if \((Y/Z, B_Y)\) is \(\mathbb{Q}\)-factorial dlt, there is a \(K_Y + B_Y\)-negative extremal contraction \(Y \to T/Z\) with \(\dim Y > \dim T\), and
\[
a(D, X, B) \leq a(D, Y, B_Y)
\]
for any prime divisor \(D\) (on birational models of \(X\)) and the strict inequality holds if \(D\) is on \(X\) and contracted/\(Y\).

Our definitions of log minimal models and Mori fibre spaces are slightly different from the traditional ones, the difference being that we do not assume that \(\phi^{-1}\) does not contract divisors. Even though we allow \(\phi^{-1}\) to have exceptional divisors but these divisors are very special; if \(D\) is any such prime divisor, then \(a(D, X, B) = a(D, Y, B_Y) = 0\). Actually, in the plt case, our definition of log minimal models and the traditional one coincide (see [1, Remark 2.6]).

Definition 2.3 (LMMP with scaling) Let \((X_1/Z, B_1 + C_1)\) be a lc pair such that \(K_{X_1} + B_1 + C_1\) is nef/\(Z\), \(B_1 \geq 0\), and \(C_1 \geq 0\) is \(\mathbb{R}\)-Cartier. Suppose that either \(K_{X_1} + B_1\) is nef/\(Z\) or there is an extremal ray \(R_1/Z\) such that \((K_{X_1} + B_1) \cdot R_1 < 0\) and \((K_{X_1} + B_1 + \lambda_1 C_1) \cdot R_1 = 0\) where
\[
\lambda_1 := \inf\{ t > 0 \mid K_{X_1} + B_1 + tC_1 \text{ is nef/}Z\}
\]
When \((X_1/Z, B_1)\) is \(\mathbb{Q}\)-factorial dlt, the last sentence follows from [1] 3.1. If \(R_1\) defines a Mori fibre structure, we stop. Otherwise assume that \(R_1\) gives a divisorial contraction or a log flip \(X_1 \dashrightarrow X_2\). We can now consider \((X_2/Z, B_2 + \lambda_1 C_2)\) where \(B_2 + \lambda_1 C_2\) is the birational transform of \(B_1 + \lambda_1 C_1\) and continue. That is, suppose that either \(K_{X_2} + B_2\) is nef/\(Z\) or there is an extremal ray \(R_2/Z\) such that \((K_{X_2} + B_2) \cdot R_2 < 0\) and \((K_{X_2} + B_2 + \lambda_2 C_2) \cdot R_2 = 0\) where
\[
\lambda_2 := \inf\{ t > 0 \mid K_{X_2} + B_2 + tC_2 \text{ is nef/}Z\}
\]
By continuing this process, we obtain a sequence of numbers \(\lambda_i\) and a special kind of LMMP/\(Z\) which is called the LMMP/\(Z\) on \(K_{X_1} + B_1\) with scaling of \(C_1\); note that it is not unique. This kind of LMMP was first used by Shokurov [7]. When we refer to termination with scaling we mean termination of such an LMMP. We usually put \(\lambda = \lim \lambda_i\).

Special termination with scaling means termination near \([B_1]\) of any sequence of log flips/\(Z\) with scaling of \(C_1\), i.e. after finitely many steps, the locus of the extremal rays in the process do not intersect \([B_1]\).

When we have a lc pair \((X/Z, B)\), we can always find an ample/\(Z\) \(\mathbb{R}\)-Cartier divisor \(C \geq 0\) such that \(K_X + B + C\) is lc and nef/\(Z\), so we can run the LMMP/\(Z\) with scaling assuming that all the necessary ingredients exist, eg extremal rays, log flips.
3. Extremal rays

We need a result of Shokurov on extremal rays \cite{9}. Since we need stronger statements than those stated in \cite{9}, we give detailed proofs here (see also \cite{2}). Some parts of our proof are quite different from the originals. As a corollary, we give a short proof of a result of Kawamata on flops connecting minimal models.

Let \(X \rightarrow Z\) be a projective morphism of normal quasi-projective varieties. A curve \(\Gamma\) on \(X\) is called extremal if it generates an extremal ray \(R/Z\) which defines a contraction \(X \rightarrow S/Z\) and if for some ample divisor \(H\) we have \(H \cdot \Gamma = \min\{H \cdot \Sigma\}\) where \(\Sigma\) ranges over curves generating \(R\). If \((X/Z, B)\) is dlt and \((K_X + B) \cdot R < 0\), then by \cite[Theorem]{8} there is a curve \(\Sigma\) generating \(R\) such that \((K_X + B) \cdot \Sigma \geq -2 \dim X\).

On the other hand, since \(\Gamma\) and \(\Sigma\) both generate \(R\) we have

\[
(K_X + B) \cdot \Gamma = (K_X + B) \cdot (H \cdot \Sigma)
\]

hence

\[
(K_X + B) \cdot \Gamma = ((K_X + B) \cdot \Sigma)(H \cdot \Sigma) \geq -2 \dim X
\]

(3.0.1)

Remark 3.1 Let \(X/Z\) be a \(\mathbb{Q}\)-factorial dlt variety, \(F\) be a reduced divisor on \(X\), and \(V\) be a rational affine subspace of the \(\mathbb{R}\)-vector space of divisors generated by the components of \(F\). By \cite[1.3.2]{7}, the set

\[
\mathcal{L} = \{\Delta \in V \mid (X/Z, \Delta) \text{ is lc}\}
\]

is a rational polytope, that is, it is the convex hull of finitely many rational points in \(V\). For any \(\Delta \in \mathcal{L}\) and any extremal curve \(\Gamma/Z\) the boundedness \((K_X + \Delta) \cdot \Gamma \geq -2 \dim X\) holds as in (3.0.1). Even though \((X/Z, \Delta)\) may not be dlt but we can use the fact that \((X/Z, a\Delta)\) is dlt for any \(a \in [0, 1)\).

Let \(B_1, \ldots, B_r\) be the vertices of \(\mathcal{L}\), and let \(m \in \mathbb{N}\) such that \(m(K_X + B_j)\) are Cartier. For any \(B \in \mathcal{L}\), there are nonnegative real numbers \(a_1, \ldots, a_r\) such that \(B = \sum a_j B_j\), \(\sum a_j = 1\), and each \((X/Z, B_j)\) is lc. Moreover, for any curve \(\Gamma\) on \(X\) the intersection number \((K_X + B) \cdot \Gamma\) can be written as \(\sum a_j n_j/m\) for certain \(n_1, \ldots, n_r \in \mathbb{Z}\). If \(\Gamma\) is extremal, then the \(n_j\) satisfy \(n_j \geq -2m \dim X\).

For an \(\mathbb{R}\)-divisor \(D = \sum d_i D_i\) where the \(D_i\) are the irreducible components of \(D\), define \(\|D\| := \max\{|d_i|\}\).
Proposition 3.2. Let $X/Z$, $F$, $V$, and $\mathcal{L}$ be as in Remark [3.1] and fix $B \in \mathcal{L}$. Then, there are real numbers $\alpha, \delta > 0$, depending on $(X/Z, B)$ and $F$, such that

1. if $\Gamma$ is any extremal curve/Z and if $(K_X + B) \cdot \Gamma > 0$, then $(K_X + B) \cdot \Gamma > \alpha$;
2. if $\Delta \in \mathcal{L}$, $||\Delta - B|| < \delta$ and $(K_X + \Delta) \cdot R \leq 0$ for an extremal ray $R/Z$, then $(K_X + B) \cdot R \leq 0$;
3. let $\{R_t\}_{t \in \mathcal{T}}$ be a family of extremal rays of $\overline{NE}(X/Z)$. Then, the set
   $$\mathcal{N}_T = \{\Delta \in \mathcal{L} \mid (K_X + \Delta) \cdot R_t \geq 0 \text{ for any } t \in T\}$$
   is a rational polytope;
4. if $K_X + B$ is nef/Z, then for any $\Delta \in \mathcal{L}$ and for any sequence $X_1 \to X_{i+1}/\mathbb{Z}$ of $K_X + \Delta$-flips/Z which are flops with respect to $(X/Z, B)$ and any extremal curve $\Gamma/Z$ on $X_i$, if $(K_{X_i} + B_i) \cdot \Gamma > 0$, then $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ where $B_i$ is the birational transform of $B$;
5. assumptions as in (5). In addition suppose that $||\Delta - B|| < \delta$. If $(K_X + \Delta_i) \cdot R \leq 0$ for an extremal ray $R/Z$ on some $X_i$, then $(K_X + B_i) \cdot R = 0$ where $\Delta_i$ is the birational transform of $\Delta$.

Proof. (1) If $B$ is a $\mathbb{Q}$-divisor, then the statement is trivially true even if $\Gamma$ is not extremal. If $B$ is not a $\mathbb{Q}$-divisor, let $B_1, \ldots, B_r$, $a_1, \ldots, a_r$, and $m$ be as in Remark [3.1]. Then,

$$(K_X + B) \cdot \Gamma = \sum a_j (K_X + B_j) \cdot \Gamma$$

and if $(K_X + B) \cdot \Gamma < 1$, then there are only finitely many possibilities for the intersection numbers $(K_X + B_j) \cdot \Gamma$ because $(K_X + B_j) \cdot \Gamma \geq -2 \dim X$. So, the existence of $\alpha$ is clear for (1).

(2) If the statement is not true then there is an infinite sequence of $\Delta_t \in \mathcal{L}$ and extremal rays $R_t/Z$ such that for each $t$ we have

$$(K_X + \Delta_t) \cdot R_t \leq 0 \Rightarrow (K_X + B) \cdot R_t > 0,$$

and $||\Delta_t - B||$ converges to 0. Let $B_1, \ldots, B_r$ be the vertices of $\mathcal{L}$ which are rational divisors as $\mathcal{L}$ is a rational polytope. Then, there are nonnegative real numbers $a_1, \ldots, a_r$ and $a_{1,t}, \ldots, a_{r,t}$ such that $B = \sum a_j B_j$, $\sum a_j = 1$ and $\Delta_t = \sum a_{j,t} B_j$, $\sum a_{j,t} = 1$. Since $||\Delta_t - B||$ converges to 0, $a_j = \lim_{t \to \infty} a_{j,t}$. Perhaps after replacing the sequence with an infinite subsequence we can assume that the sign of $(K_X + B_j) \cdot R_t$ is independent of $t$, and that for each $t$ we have an extremal curve $\Gamma_t$ for $R_t$. Now, if $(K_X + B_j) \cdot \Gamma_t \leq 0$, then it is bounded from below
hence there are only finitely many possibilities for this number and we could assume that it is independent of \( t \). On the other hand, if \( a_j \neq 0 \), then \((K_X + B_j) \cdot \Gamma_t\) is bounded from below and above because

\[
(K_X + \Delta_t) \cdot \Gamma_t = \sum a_{j,t}(K_X + B_j) \cdot \Gamma_t \leq 0
\]

hence there are only finitely many possibilities for \((K_X + B_j) \cdot \Gamma_t\) and we could assume that it is independent of \( t \).

Assume that \( a_j \neq 0 \) for \( 1 \leq j \leq l \) but \( a_j = 0 \) for \( j > l \). Then, it is clear that

\[
(K_X + \Delta_t) \cdot \Gamma_t = (K_X + B) \cdot \Gamma_t + \sum_{j \leq t} (a_{j,t} - a_j)(K_X + B_j) \cdot \Gamma_t + \sum_{j > t} a_{j,t}(K_X + B_j) \cdot \Gamma_t
\]

would be positive by (1) if \( t \gg 0 \), which gives a contradiction.

(3) We may assume that for each \( t \in T \) there is some \( \Delta \in \mathcal{L} \) such that \((K_X + \Delta) \cdot R_t < 0\), in particular, \((K_X + B_j) \cdot R_t < 0\) for a vertex \( B_j \) of \( \mathcal{L} \). Since the set of such extremal rays is discrete, we may assume that \( T \subseteq \mathbb{N} \).

Obviously, \( \mathcal{N}_T \) is a convex compact subset of \( \mathcal{L} \). If \( T \) is finite, the claim is trivial. So we may assume that \( T = \mathbb{N} \). By (2) and by the compactness of \( \mathcal{N}_T \), there are \( \Delta_1, \ldots, \Delta_n \in \mathcal{N}_T \) and \( \delta_1, \ldots, \delta_n > 0 \) such that \( \mathcal{N}_T \) is covered by \( \mathcal{B}_i = \{ \Delta \in \mathcal{L} \mid ||\Delta - \Delta_i|| < \delta_i \} \) and such that if \( \Delta \in \mathcal{B}_i \) with \((K_X + \Delta) \cdot R_t < 0\) for some \( t \), then \((K_X + \Delta) \cdot R_t = 0\). If

\[
T_i = \{ t \in T \mid (K_X + \Delta) \cdot R_t < 0 \text{ for some } \Delta \in \mathcal{B}_i \}
\]

then by construction \((K_X + \Delta_i) \cdot R_t = 0\) for any \( t \in T_i \). Then, since the \( \mathcal{B}_i \) give an open cover of \( \mathcal{N}_T \), we have \( \mathcal{N}_T = \bigcap_{1 \leq i \leq n} \mathcal{N}_{T_i} \). So, it is enough to prove that each \( \mathcal{N}_{T_i} \) is a rational polytope and by replacing \( T \) with \( T_i \), we could assume from the beginning that there is some \( \Delta \in \mathcal{N}_T \) such that \((K_X + \Delta) \cdot R_t = 0\) for every \( t \in T \). If \( \dim \mathcal{L} = 1 \), this already proves the claim. If \( \dim \mathcal{L} > 1 \), let \( \mathcal{L}^1, \ldots, \mathcal{L}^p \) be the proper faces of \( \mathcal{L} \). Then, each \( \mathcal{N}^{j}_{T} = \mathcal{N}_{T} \cap \mathcal{L}^{j} \) is a rational polytope by induction. Moreover, for each \( \Delta'' \in \mathcal{N}_T \) which is not \( \Delta \), there is \( \Delta' \) on some proper face of \( \mathcal{L} \) such that \( \Delta'' \) is on the line segment determined by \( \Delta \) and \( \Delta' \). Since \((K_X + \Delta) \cdot R_t = 0\) for every \( t \in T \), if \( \Delta' \in \mathcal{L}^j \), then \( \Delta' \in \mathcal{N}^j_T \). Hence \( \mathcal{N}^j_T \) is the convex hull of \( \Delta \) and all the \( \mathcal{N}^j_T \). Now, there is a finite subset \( T' \subset T \) such that

\[
\cup \mathcal{N}^j_T = \mathcal{N}^j_{T'} \cap (\cup \mathcal{L}^j)
\]

But then the convex hull of \( \Delta \) and \( \cup \mathcal{N}^j_T \) is just \( \mathcal{N}^j_{T'} \) and we are done.
(4) Since $K_X + B$ is nef/$Z$, $B \in \mathcal{N}_T$ where we take $\{R_t\}_{t \in \mathcal{T}}$ to be the family of all the extremal rays of $\mathcal{N}(X/Z)$. Since $\mathcal{N}_T$ is a rational polytope by (3), there are nonnegative real numbers $a_1', \ldots, a_r'$, and $m' \in \mathbb{N}$ so that $\sum a_j' = 1$, $B = \sum a_j' B_j'$, and each $m'(K_X + B_j')$ is Cartier where $B_j'$ are the vertices of $\mathcal{N}_T$. Therefore, by the property $K_X + B = \sum a_j' (K_X + B_j')$, the sequence $X_i \rightarrow X_{i+1}/Z_i$ is also a sequence of flops with respect to each $(X/Z, B_j')$. Moreover, $(X_i/Z, B_j')$ is lc and $m'(K_X + B_j')$ is Cartier for any $j, i$ where $B_j'$ is the birational transform of $B_j'$. The rest is as in (1).

(5) Take $L$ to be the line in $V$ which goes through $B$ and $\Delta$ and let $\Delta'$ be the intersection point of $L$ and the boundary of $\mathcal{L}$, in the direction of $\Delta$. So, there are nonnegative real numbers $r, s$ such that $r + s = 1$ and $\Delta = rB + s\Delta'$. In particular, the sequence $X_i \rightarrow X_{i+1}/Z_i$ is also a sequence of $K_X + \Delta'$-flips and $(X_i/Z, \Delta_i')$ is lc where $\Delta_i'$ is the birational transform of $\Delta'$. Suppose that there is an extremal ray $R/\mathcal{L}$ on some $X_i$ such that $(K_{X_i} + \Delta_i) \cdot R \leq 0$ but $(K_{X_i} + B_i) \cdot R > 0$. Let $\Gamma$ be an extremal curve for $R$. By (4), $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ and by (3.0.1) $(K_{X_i} + \Delta_i') \cdot \Gamma \geq -2 \dim X$. Now

$$(K_{X_i} + \Delta_i) \cdot \Gamma = r(K_{X_i} + B_i) \cdot \Gamma + s(K_{X_i} + \Delta_i') \cdot \Gamma > r\alpha - 2s \dim X$$

and it is obvious that this is positive if $r > \frac{2s \dim X}{\alpha}$. In other words, if $\Delta$ is sufficiently close to $B$, then we get a contradiction. Therefore, it is enough to replace the $\delta$ of (2) by one sufficiently smaller. Note that we could also prove (2) in a similar way. $\Box$

In section 4, we will apply the proposition in a way similar to [9]. Proposition 3.2 easily implies the following result of Kawamata [6] on flops connecting log minimal models.

**Corollary 3.3.** Let $(Y_1/Z, B_1)$ and $(Y_2/Z, B_2)$ be two klt pairs such that $K_{Y_1} + B_1$ and $K_{Y_2} + B_2$ are nef/$Z$, and $Y_1$ and $Y_2$ are isomorphic in codimension one. Then, $Y_1$ and $Y_2$ are connected by a sequence of flops$/Z$ with respect to $(Y_1/Z, B_1)$.

**Proof.** Let $H_2$ be a general ample$/Z$ divisor on $Y_2$ and let $H_1$ be its birational transform on $Y_1$. There is $\delta > 0$ such that $(Y_1/Z, B_1 + \delta H_1)$ is klt. Now there is a general ample$/Z$ divisor $H_1'$ on $Y_1$ such that $(Y_2/Z, B_2 + \delta H_2 + \delta' H_1')$ is klt for some $\delta' > 0$ where $H_2$ is the birational transform of $H_1'$. If $\delta$ is sufficiently small, then $K_{Y_1} + B_1 + \delta H_1 + \delta' H_1'$ is nef$/Z$. By [3, 4], we can run the LMMP$/Z$ on $K_{Y_1} + B_1 + \delta H_1$ with scaling of $\delta' H_1'$. After a finite sequence of log flips$/Z$, we end up with $Y_2$. On the other hand, we can lift the sequence to the $\mathbb{Q}$-factorial.
situation and by applying Proposition 3.2 we see that the sequence is a sequence of flops with respect to \((Y_1/Z, B_1)\) if \(\delta\) is sufficiently small. 

Note that if \((Y_1/Z, B_1)\) and \((Y_2/Z, B_2)\) are log minimal models of a klt pair \((X/Z, B)\), then \(Y_1\) and \(Y_2\) are automatically isomorphic in codimension one.

4. Log minimal models and termination with scaling

Proof of Theorem 1.5 Step 1. The fact that we can run the LMMP/Z on \(K_X + B\) with scaling of \(C\) follows from [1, Lemma 3.1]. Note that the log flips required exist by the assumptions since existence of log flips is a special case of existence of log minimal models. Alternatively one can use [3] [4]. We will deal with the termination statement. We may assume that the sequence corresponding to the \(\lambda_i\) is a sequence \(X_i \rightarrow X_{i+1}/Z_i\) of log flips/Z starting with \((X/Z, B)\) where the \(\lambda_i\) are obtained as in Definition 2.3. Remember that \(\lambda = \lim_{i \to \infty} \lambda_i\).

If \(B \geq H \geq 0\) for some ample/Z \(\mathbb{R}\)-divisor \(H\), then the LMMP terminates by [4, Theorem 2.7]. Note that since \(H\) is ample/Z, we can perturb the coefficients of \(B\) and \(C\) to reduce to the situation in which \((X/Z, B + C)\) is klt (cf. [4, Remark 2.4]). If \(C \geq H \geq 0\) where \(H\) is an ample/Z \(\mathbb{R}\)-divisor and if we have \(\lambda > 0\), then the termination follows again from [4, Theorem 2.7].

We treat the third case. From now on suppose that \(\lambda \neq \lambda_i\) for any \(i\). Pick \(i\) so that \(\lambda_i > \lambda_{i+1}\). Thus, \(\text{Supp} \, C_{i+1}\) does not contain any lc centre of \((X_{i+1}/Z, B_{i+1} + \lambda_i C_{i+1})\) because \((X_{i+1}/Z, B_{i+1} + \lambda_i C_{i+1})\) is lc. Then, by replacing \((X/Z, B)\) with \((X_{i+1}/Z, B_{i+1})\) and \(C\) with \(\lambda_{i+1} C_{i+1}\) we may assume that no lc centre of \((X/Z, B + C)\) is inside \(\text{Supp} \, C\). Furthermore, using induction and the special termination (cf. [1, Lemma 3.6]) we can assume that the log flips do not intersect \(\lfloor B \rfloor\). Since in each step \(K_{X_i} + B_i + \lambda_i C_i\) is anti-ample/Z, the sequence is also a sequence of \(K_X + B + \lambda C\)-flips. By replacing \(B\) with \(B + \lambda C\), \(C\) with \(1 - \lambda)C\), and \(\lambda_i\) with \(\frac{\lambda - \lambda_1}{1 - \lambda}\), we may assume that \(\lambda = 0\).

Step 2. By assumptions there is a log minimal model \((Y/Z, B_Y)\) for \((X/Z, B)\). Let \(\phi: X \rightarrow Y/Z\) be the corresponding birational map. Since \(K_{X_i} + B_i + \lambda_i C_i\) is nef/Z, we may add an ample/Z \(\mathbb{R}\)-divisor \(G^i\) so that \(K_{X_i} + B_i + \lambda_i C_i + G^i\) becomes ample/Z, in particular, it is movable/Z. We can choose the \(G^i\) so that \(\lim_{i \to \infty} G^i_1 = 0\) in \(N^1(X_1/Z)\) where \(G^i_1\) is the birational transform of \(G^i\) on \(X_1 = X\). Therefore,

\[
K_X + B \equiv \lim_{i \to \infty} (K_{X_i} + B_i + \lambda_i C_i + G^i)/Z
\]
which implies that $K_X + B$ is a limit of movable $\mathbb{R}$-divisors.

Let $f: W \to X$ and $g: W \to Y$ be a common log resolution of $(X/Z, B + C)$ and $(Y/Z, B_Y + C_Y)$ where $C_Y$ is the birational transform of $C$. By applying the negativity lemma to $f$, we see that

$$E := f^*(K_X + B) - g^*(K_Y + B_Y) = \sum_D a(D, Y, B_Y)D - a(D, X, B)D$$

is effective (cf. [1, Remark 2.6]) where $D$ runs over the prime divisors on $W$. If $E \neq 0$, let $D$ be a component of $E$. If $D$ is not exceptional$/Y$, then it must be exceptional$/X$ otherwise $a(D, X, B) = a(D, Y, B_Y)$ and $D$ cannot be a component of $E$. By definition of log minimal models, $a(D, Y, B_Y) = 0$ hence $a(D, X, B) = 0$ which again shows that $D$ cannot be a component of $E$. Therefore, $E$ is exceptional$/Y$.

Step 3. Let $B_W$ be the birational transform of $B$ plus the reduced exceptional divisor of $f$, and let $C_W$ be the birational transform of $C$ on $W$. Pick a sufficiently small $\delta \geq 0$. Take a general ample$/Z$ divisor $L$ so that $K_W + B_W + \delta C_W + L$ is dlt and nef$/Z$. Since $(X/Z, B)$ is lc,

$$E' := K_W + B_W - f^*(K_X + B) = \sum_D a(D, X, B)D \geq 0$$

where $D$ runs over the prime exceptional$/X$ divisors on $W$. So,

$$K_W + B_W + \delta C_W = f^*(K_X + B) + E' + \delta C_W = g^*(K_Y + B_Y) + E + E' + \delta C_W$$

Moreover, $E'$ is also exceptional$/Y$ because for any prime divisor $D$ on $Y$ which is exceptional$/X$, $a(D, Y, B_Y) = a(D, X, B) = 0$ hence $D$ cannot be a component of $E'$.

On the other hand, since $Y$ is $\mathbb{Q}$-factorial, there are exceptional$/Y$ $\mathbb{R}$-divisors $F, F'$ on $W$ such that $C_W + F \equiv 0/Y$ and $L + F' \equiv 0/Y$. Now run the LMMP$/Y'$ on $K_W + B_W + \delta C_W$ with scaling of $L$ which is the same as the LMMP$/Y'$ on $E + E' + \delta C_W$ with scaling of $L$. Let $\lambda'_i$ and $\lambda' = \lim_{i \to \infty} \lambda'_i$ be the corresponding numbers. If $\lambda' > 0$, then by step 1 the LMMP terminates since $L$ is ample$/Z$. Since $W \to Y$ is birational, the LMMP terminates only when $\lambda'_i = 0$ for some $i$ which implies that $\lambda' = 0$, a contradiction. Thus, $\lambda' = 0$. On some model $V$ in the process of the LMMP, the pushdown of $K_W + B_W + \delta C_W + \lambda'_i L_W$,

$$K_V + B_V + \delta C_V + \lambda'_i L_V$$

$$\equiv E_V + E'_V + \delta C_V + \lambda'_i L_V$$

$$\equiv E_V + E'_V - \delta F_V - \lambda'_i F'_V/Y$$
is nef$/Y$. Applying the negativity lemma over $Y$ shows that $E_V + E'_V - \delta F_V - \lambda'_i F'_i \leq 0$. But if $i > 0$, then $E_V + E'_V \leq 0$ because $\lambda'_i$ and $\delta$ are sufficiently small. Therefore, $E_V = E'_V = 0$ as $E$ and $E'$ are effective.

**Step 4.** We prove that $\phi: X \to Y$ does not contract any divisors. Assume otherwise and let $D$ be a prime divisor on $X$ contracted by $\phi$. Then $D^\sim$ the birational transform of $D$ on $W$ is a component of $E$ because by definition of log minimal models $a(D, X, B) < a(D, Y, B_Y)$. Now, in step 3 take $\delta = 0$. The LMMP contracts $D^\sim$ since $D^\sim$ is a component of $E$ and $E$ is contracted. But this is not possible because $K_X + B$ is a limit of movable$/Z \mathbb{R}$-divisors and $D^\sim$ is not a component of $E'$ so the pushdown of $K_W + B_W = f^*(K_X + B) + E'$ cannot negatively intersect a general curve on $D^\sim/Y$. Thus $\phi$ does not contract divisors, in particular, any prime divisor on $W$ which is exceptional$/Y$ is also exceptional$/X$. Though $\phi$ does not contract divisors but $\phi^{-1}$ might contract divisors. The prime divisors contracted by $\phi^{-1}$ appear on $W$.

**Step 5.** Now take $\delta > 0$ in step 3 which is sufficiently small by assumptions. By induction and the special termination, when we run the LMMP$/Y$ on $K_W + B_W + \delta C_W$ with scaling of $L$, the extremal rays contracted in the process do not intersect $[B_W]$, after finitely many steps. On the other hand, since $\phi$ does not contract divisors, every exceptional$/Y$ prime divisor on $W$ is a component of $[B_W]$. Therefore, the LMMP terminates because it is an LMMP on the exceptional$/Y \mathbb{R}$-divisor $E + E' - \delta F$. So, we get a model $Y'$ on which the pushdown of $K_W + B_W + \delta C_W$, say $K_{Y'} + B_{Y'} + \delta C_{Y'}$, is nef$/Y$. By step 3, $K_{Y'} + B_{Y'} \equiv E_{Y'} + E'_{Y'} = 0/Y$ where $E_{Y'}$ and $E'_{Y'}$ are the birational transforms of $E$ and $E'$ on $Y'$, respectively. Therefore, $(Y'/Z, B_{Y'})$ is a dlt crepant model of $(Y/Z, B_Y)$.

**Step 6.** As in step 3,

$$E'' := K_W + B_W + C_W - f^*(K_X + B + C) = \sum_D a(D, X, B + C)D \geq 0$$

is exceptional$/X$ where $D$ runs over the prime exceptional$/X$ divisors on $W$. So, by induction and the special termination, the LMMP$/X$ on $K_W + B_W + C_W \equiv E''/X$ with scaling of suitable ample$/Z$ divisors terminate because every component of $E''$ is also a component of $[B_W]$. So, we get a crepant dlt model $(X'/Z, B' + C')$ of $(X/Z, B + C)$ where $K_{X'} + B'$ is the pullback of $K_X + B$ and $C'$ is the pullback of $C$. In fact, $X'$ and $X$ are isomorphic outside the lc centres of $(X/Z, B + C)$ because
the prime exceptional/$X$ divisors on $X'$ are exactly the pushdown of
the prime exceptional/$X$ divisors $D$ on $W$ with $a(D, X, B + C) = 0$,
that is, those which are not components of $E''$. Since $\text{Supp}C'$ does not
contain any lc centre of $(X/Z, B + C)$ by step 1, $(X'/Z, B')$ is a crepant
dlt model of $(X/Z, B)$ and $C'$ is just the birational transform of $C$. Note
that the prime exceptional divisors of $\phi^{-1}$ are not contracted/$X'$ since
their log discrepancy with respect to $(X/Z, B)$ are all 0, and so their
birational transforms are not components of $E''$.

\textit{Step 7.} Remember that $X_1 = X$, $B_1 = B$, and $C_1 = C$. Similarly,
put $X'_1 := X'$, $B'_1 := B'$, and $C'_1 := C'$. Since $K_{X_1} + B_1 + \lambda_1 C_1 \equiv 0/Z_1$,
$K_{X'_1} + B'_1 + \lambda_1 C'_1 \equiv 0/Z_1$. Run the LMMP$/Z_1$ on $K_{X'_1} + B'_1$ with scaling
of $\lambda_1 C'_1$. Since the exceptional locus of $X_1 \rightarrow Z_1$ does not intersect any
lc centre of $(X_1/Z, B_1)$ by step 1, and since $X'_1$ and $X_1$ are isomorphic
outside the lc centres of $(X_1/Z, B_1)$, the LMMP consists of just one
log flip $X'_1 \rightarrow X'_2/Z'_1$ which is the lifting of the log flip $X_1 \rightarrow X_2/Z_1$.
Moreover, $(X'_2/Z, B'_2)$ is a crepant dlt model of $(X_2/Z, B_2)$ where $B'_2$
is the birational transform of $B'_1$. We can continue this process to lift
the original sequence to a sequence $X'_i \rightarrow X'_{i+1}/Z'_i$.

Note that $Y' \rightarrow X'$ does not contract divisors: if $D$ is a prime
divisor on $Y'$ which is exceptional/$X'$, then it is exceptional/$X$ and so
it is exceptional/$Y$ by step 6; but then $a(D, Y, B_Y) = 0 = a(D, X, B)$
and again by step 6 such divisors are not contracted/$X'$, a cotradiction.
Thus, $(Y'/Z, B_{Y'})$ of step 5 is a log birational model of $(X'/Z, B')$
because $B_{Y'}$ is the birational transform of $B'$. On the other hand, assume
that $D$ is a prime divisor on $X'$ which is exceptional/$Y'$. Since
$X \rightarrow Y$ does not contract divisors by step 4, $D$ is exceptional/$X$. In
particular, $a(D, X', B') = a(D, X, B) = 0$; in this case $a(D, Y, B_Y) =
a(D, Y', B_{Y'}) > 0$ otherwise $D$ could not be contracted/$Y'$ by the
LMMP of step 5 which started on $W$ because the birational transform
of $D$ would not be a component of $E + E' + \delta C_W$. So, $(Y'/Z, B_{Y'})$
is actually a log minimal model of $(X'/Z, B')$. Therefore, as in step 4,
$X' \rightarrow Y'$ does not contract divisors which implies that $X'$ and $Y'$
are isomorphic in codimension one. Now replace the old sequence $X_i \rightarrow
X'_i/Z_i$ with the new one $X'_i \rightarrow X'_{i+1}/Z'_i$ and replace $(Y/Z, B_Y)$ with
$(Y'/Z, B_{Y'})$. So, from now on we can assume that $X, X_i$ and $Y$ are
all isomorphic in codimension one. In addition, by step 5, we can also
assume that $(Y/Z, B_{Y} + \delta C_Y)$ is dlt for some $\delta > 0$.

\textit{Step 8.} Let $A \geq 0$ be a reduced divisor on $W$ whose components
are general ample/$Z$ divisors such that they generate $N^1(W/Z)$. By
step 6, \((X_1/Z, B_1 + C_1')\) is obtained by running a specific LMMP on \(K_W + B_W + C_W\). Every step of this LMMP is also a step of an LMMP on \(K_W + B_W + C_W + \varepsilon A\) for any sufficiently small \(\varepsilon > 0\), in particular, \((X_1/Z, B_1 + C_1 + \varepsilon A_1)\) is dlt where \(A_1\) is the birational transform of \(A\). For similar reasons, we can choose \(\varepsilon\) so that \((Y/Z, B_Y + \delta C_Y + \varepsilon A_Y)\) is also dlt. On the other hand, by Proposition 3.2, perhaps after replacing \(\delta\) and \(\varepsilon\) with smaller positive numbers, we may assume that if \(0 \leq \delta' \leq \delta\) and \(0 \leq A_Y' \leq A_Y\), then any LMMP on \(K_Y + B_Y + \delta' C_Y + A_Y'\), consists of only a sequence of log flips which are flops with respect to \((Y/Z, B_Y)\). Note that since \(K_Y + B_Y + \delta' C_Y + A_Y'\) is a limit of movable/Z \(\mathbb{R}\)-divisors, no divisor is contracted by such an LMMP.

**Step 9.** Fix some \(i \gg 0\) so that \(\lambda_i < \delta\). Then, by Proposition 3.2, there is \(0 < \tau \ll \varepsilon\) such that \((X_i/Z, B_i + \lambda_i C_i + \tau A_i)\) is dlt and such that if we run the LMMP on \(K_X + B_i + \lambda_i C_i + \tau A_i\) with scaling of some ample/Z divisor, then it will be a sequence of log flips which would be a sequence of flops with respect to \((X_i/Z, B_i + \lambda_i C_i)\). Moreover, since the components of \(A_i\) generate \(N^1(X_i/Z)\), we can assume that there is an ample/Z \(\mathbb{R}\)-divisor \(H \geq 0\) such that \(\tau A \equiv H + H'/Z\) where \(H' \geq 0\) and \((X_i/Z, B_i + \lambda_i C_i + H + H')\) is dlt. Hence the LMMP terminates by step 1 and we get a model \(T\) on which both \(K_T + B_T + \lambda_i C_T\) and \(K_T + B_T + \lambda_i C_T + \tau A_T\) are nef/Z. Again since the components of \(A_T\) generate \(N^1(T/Z)\), there is \(0 \leq A_T' \leq \tau A_T\) so that \(K_T + B_T + \lambda_i C_T + A_T'\) is ample/Z and \(\text{Supp } A_T' = \text{Supp } A_T\). Now run the LMMP on \(K_Y + B_Y + \lambda_i C_Y + A_T'\) with scaling of some ample/Z divisor where \(A_T'\) is the birational transform of \(A_T'\). The LMMP terminates for reasons similar to the above and we end up with \(T\) since \(K_T + B_T + \lambda_i C_T + A_T'\) is ample/Z. Moreover, the LMMP consists of only log flips which are flops with respect to \((Y/Z, B_Y)\) by Proposition 3.2 hence \(K_T + B_T\) will also be nef/Z. So, by replacing \(Y\) with \(T\) we could assume that \(K_Y + B_Y + \lambda_i C_Y\) is nef/Z. In particular, \(K_Y + B_Y + \lambda_j C_Y\) is nef/Z for any \(j \geq i\) since \(\lambda_j \leq \lambda_i\).

**Step 10.** Pick \(j > i\) so that \(\lambda_j < \lambda_{j-1} \leq \lambda_i\) and let \(r: U \to X_j\) and \(s: U \to Y\) be a common resolution. Then, we have

\[
\begin{align*}
    r^*(K_{X_j} + B_j + \lambda_j C_j) &= s^*(K_Y + B_Y + \lambda_j C_Y) \\
    r^*(K_{X_j} + B_j) &\geq s^*(K_Y + B_Y) \\
    r^*C_j &\leq s^*C_Y
\end{align*}
\]
where the first equality holds because both $K_{X_j} + B_j + \lambda_j C_j$ and $K_Y + B_Y + \lambda_j C_Y$ are nef/Z and $X_j$ and $Y$ are isomorphic in codimension one, the second inequality holds because $K_Y + B_Y$ is nef/Z but $K_{X_j} + B_j$ is not nef/Z, and the third follows from the other two. Now

$$r^*(K_{X_j} + B_j + \lambda_{j-1} C_j)$$

$$= r^*(K_{X_j} + B_j + \lambda_j C_j) + r^*(\lambda_{j-1} - \lambda_j) C_j$$

$$\leq s^*(K_Y + B_Y + \lambda_j C_Y) + s^*(\lambda_{j-1} - \lambda_j) C_Y$$

$$= s^*(K_Y + B_Y + \lambda_{j-1} C_Y)$$

However, since $K_{X_j} + B_j + \lambda_{j-1} C_j$ and $K_Y + B_Y + \lambda_{j-1} C_Y$ are both nef/Z, we have

$$r^*(K_{X_j} + B_j + \lambda_{j-1} C_j) = s^*(K_Y + B_Y + \lambda_{j-1} C_Y)$$

This is a contradiction and the sequence of log flips terminates as claimed. □

**Proof of Corollary 1.6.** Let $H \geq 0$ be an ample/Z divisor such that $K_X + B + H$ is dlt and ample/Z. Now run the LMMP/Z on $K_X + B$ with scaling of $H$. By Theorem 1.5, the LMMP terminates with a log minimal model or a Mori fibre space $(Y/Z, B_Y)$. The claim that $Y \dashrightarrow X$ does not contract divisors is obvious. □

**Lemma 4.1.** Assume the minimal model conjecture \cite{fujino} in dimension $d$ for pseudo-effective Q-factorial dlt pairs. Let $(X/Z, B + C)$ be a Q-factorial lc pair of dimension $d + 1$ such that

1. $K_X + B + C$ is nef/Z,
2. $B, C \geq 0$,
3. $(X/Z, B)$ is dlt,
4. $K_X + B \equiv_Z M \geq 0$ where $\alpha M = M' + C$ for some $\alpha > 0$ and $M' \geq 0$ supported in $\text{Supp} \ [B]$.

Then, we can run an LMMP/Z on $K_X + B + C$ with scaling of $C$ which terminates.

**Proof.** By Theorem \cite{birkar} \cite{fujino} Assumption 5.2.3] is satisfied in dimension $d$ which implies that pl flips exist in dimension $d + 1$ by the main result of \cite{fujino} (cf. \cite[Theorem 2.9]{birkar}). Alternatively, we can simply borrow the existence of log flips from \cite{birkar, fujino}. So, in any case we can run the LMMP/Z on $K_X + B$ with scaling of $C$ by \cite[Lemma 3.1]{birkar} because we only need pl flips. We may assume that any LMMP/Z on $K_X + B$ with scaling of $C$ consists of only log flips.
If $M' = 0$, then $K_X + B + C ≡ \frac{1}{\alpha}C + C/Z$ which implies that $C$ and $K_X + B$ are nef/Z hence we are done. So, from now on we assume that $M' \neq 0$.

By the assumptions, $\text{Supp} M \subseteq \text{Supp}(B + C)$ hence there is a sufficiently small $\tau > 0$ such that

$$\text{Supp}(B + C - \tau M - \tau C) = \text{Supp}(B + C)$$

Put $B' = B - \frac{\tau}{\alpha}M'$ and $C' = C - \tau(\frac{1}{\alpha} + 1)C$ so that

$$K_X + B' + C' ≡ M + C - \frac{\tau}{\alpha}M' - (\frac{\tau}{\alpha} + \tau)C = M + C - \tau(M + C)/Z$$

In particular, $K_X + B' + C'$ is nef/Z. Let $\delta$ be as in Proposition 3.2 chosen for the pair $(X/Z, B' + C')$ where we take $V$ to be the space $V = \{rM' \mid r \in \mathbb{R}\}$. Take $a > 0$ so that $a\alpha \ll \tau$, $||aM'|| < \delta$. $B'' := B - aM' \geq 0$ has the same support as $B$, and $C'' = C - (a + a\alpha)C \geq 0$ has the same support as $C$. Now

$$K_X + B'' + C'' ≡ M + C - aM' - (a + a\alpha)C = M + C - a\alpha(M + C)/Z$$

and $\text{Supp} M' \subseteq \text{Supp} \{B = B'' + aM'\}$. In particular, $K_X + B'' + C''$ is nef/Z.

Let $H \geq 0$ be an ample/Z divisor such that $K_X + B + C'' + H$ is dlt and ample/Z. Now run the LMMP/Z on $K_X + B + C''$ with scaling of $H$ and assume that we get a sequence $X_i \dashrightarrow X_{i+1}$ of log flips and divisorial contractions corresponding to extremal rays $R_i$. For each $i$, we have

$$0 > (K_{X_i} + B_i + C''_i) \cdot R_i = (1 - a\alpha)(M_i + C_i) \cdot R_i + aM'_i \cdot R_i$$

where as usual the subscript $i$ for divisors stands for birational transform on $X_i$. By induction on $i$, we may assume that $K_{X_i} + B''_i + C''_i$ is nef/Z which also means that $K_{X_i} + B'_i + C'_i$ is nef/Z. So $M'_i \cdot R_i < 0$ and

$$(K_{X_i} + B'_i + C'_i + aM'_i) \cdot R_i = (1 - \tau)(M_i + C_i) \cdot R_i + aM'_i \cdot R_i$$

$$< (1 - a\alpha)(M_i + C_i) \cdot R_i + aM'_i \cdot R_i < 0$$

which implies that $(K_{X_i} + B'_i + C'_i) \cdot R_i = 0$, by construction, and in turn we get $(K_{X_i} + B''_i + C''_i) \cdot R_i = 0$. Thus, $C_i \cdot R_i > 0$ and $(K_{X_i} + B_i + C_i) \cdot R_i = 0$. So, the above LMMP is an LMMP/Z on $K_X + B$ with scaling of $C$. Since $H$ is ample/Z, the LMMP terminates by the special termination and Theorem 1.5 because the LMMP is a $(-M')$-LMMP and $\text{Supp} M' \subseteq \text{Supp} \{B\}$. Thus, for some $i$, $K_{X_i} + B_i + C''_i = K_{X_i} + B_i + (1 - a - a\alpha)C_i$ is nef/Z.

Now replace $(X/Z, B)$ with $(X_i/Z, B_i)$, $C$ with $C''_i = (1 - a - a\alpha)C_i$, $M$ with $M_i$, $M'$ with $(1 - a - a\alpha)M'_i$, $\alpha$ with $\alpha(1 - a - a\alpha)$, and continue.
the process by starting from the beginning. This process stops again by the special termination and Theorem 1.5.

The underlying idea is that there is an LMMP/Z on $K_X + B$ with scaling of $C$ such that the corresponding numbers $\lambda_i$ and $\lambda$ satisfy the property $\lambda \neq \lambda_i$ for any $i$ and this allows us to use the special termination and apply Theorem 1.5 in lower dimension. □

**Proof of Corollary 1.7.** Let $(X/Z, B)$ be an effective lc pair of dimension $d + 1$. By [1, Proposition 3.4], existence of pl flips in dimension $d + 1$ and the special termination with scaling in dimension $d + 1$ for $\mathbb{Q}$-factorial dlt pairs implies the existence of a log minimal model for $(X/Z, B)$. As mentioned in the proof of Lemma 4.1, existence of pl flips in dimension $d + 1$ follows from the assumptions. However, we have not derived termination with scaling in dimension $d$ from our assumptions when $\lambda = \lambda_i$ for some $i$. But this is not a problem since we can use Lemma 4.1. We analyse the various places in the proof of [1, Proposition 3.4] where the special termination is needed.

In step 1 of the proof of [1, Proposition 3.4] we need to have special termination with scaling of an ample/Z $\mathbb{R}$-divisor for a certain sequence of log flips. This follows from our assumptions by Theorem 1.5. In steps 3, 4, and 5 we need the special termination for some LMMP with scaling in a situation as follows: $(X/Z, B + C)$ is log smooth, $B, C \geq 0$, $K_X + B \equiv \mathbb{Z} M \geq 0$, $\alpha M = M' + C$ for some $\alpha > 0$, $M' \geq 0$ is supported in $\text{Supp}[B]$, and $(Y/Z, B_Y + C_Y)$ is a log minimal model of $(X/Z, B + C)$ where $B_Y$ is the birational transform of $B$ plus the reduced exceptional divisor of $Y \to X$ and $C_Y$ is just the birational transform of $C$. Here we want to run an LMMP/Z on $K_Y + B_Y$ with scaling of $C_Y$ which terminates. Let $f: W \to X$ and $g: W \to Y$ be a common log resolution. By the arguments in step 2 of the proof of Theorem 1.5 we can write $f^*(K_X + B + C) = g^*(K_Y + B_Y + C_Y) + E$ where $E$ is effective, and exceptional/Y. So,

$$f^*(M + C) = f^*\left(\frac{1}{\alpha} M' + \frac{1}{\alpha} C + C\right) \equiv \mathbb{Z} g^*(K_Y + B_Y + C_Y) + E$$

and

$$g_* f^*\left(\frac{1}{\alpha} M' + \frac{1}{\alpha} C + C\right) \equiv \mathbb{Z} K_Y + B_Y + C_Y$$

Now put $M_Y := g_* f^*\left(\frac{1}{\alpha} M' + \frac{1}{\alpha} C + C\right) - C_Y$ and $M'_Y := \alpha M_Y - C_Y$ so that $K_Y + B_Y \equiv M_Y/Z$ and $\alpha M_Y = M'_Y + C_Y$. By construction, every component of $M'_Y$ is either the birational transform of a component of $M'$ or it is an exceptional divisor of $Y \to X$ which in any case would
be a component of \([B_Y]\). Now simply apply Lemma 4.1 to the data: \((Y/Z, B_Y + C_Y), M_Y, \alpha, \) and \(M'_Y\).

In step 6 of the proof of [1, Proposition 3.4] we need special termination to be able to apply [1, Lemma 3.3]. However, the proof of [1, Lemma 3.3] only needs the special termination with scaling of an ample/\( \mathbb{R} \)-divisor applied to a certain sequence of log flips which again follows from our assumptions by Theorem 1.5. \(\square\)

**Proof of Theorem 1.4.** We use induction on \(d\) so assume that the theorem holds in dimension \(d - 1\). In particular, we may assume that the minimal model conjecture (1.1) holds in dimension \(d - 1\). Let \((X/Z, B)\) be a lc pair of dimension \(d\). We may assume that \((X/Z, B)\) is \(\mathbb{Q}\)-factorial dlt by replacing it with a \(\mathbb{Q}\)-factorial dlt crepant model. To construct such a model (cf. step 6 of the proof of Theorem 1.5) we only need the special termination with scaling of an ample/\( \mathbb{R} \)-divisor applied to a certain sequence of log flips which follows from the minimal model conjecture in dimension \(d - 1\) and Theorem 1.5. If \(K_X + B\) is not pseudo-effective/\( \mathbb{Z} \), then by [3][4] there is a Mori fibre space for \((X/Z, B)\). If \(K_X + B\) is pseudo-effective/\( \mathbb{Z} \), then by Conjecture 1.3, it is effective, that is, there is \(M \geq 0\) such that \(K_X + B \equiv M/\mathbb{Z}\). Now the result follows from Corollary 1.7.

The statement concerning \(\mathbb{Q}\)-factorial dlt \((X/Z, B)\) follows from Corollary 1.6, that is, we can run the LMMP/\( \mathbb{Z} \) on \(K_X + B\) with scaling of some ample/\( \mathbb{R} \)-divisor which will end up with a log minimal model or a Mori fibre space. \(\square\)

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