We calculate the effect of polarization-dependent scattering by disorder on the degree of polarization-entanglement of two beams of radiation. Multi-mode detection converts an initially pure state into a mixed state with respect to the polarization degrees of freedom. The degree of entanglement decays exponentially with the number of detected modes if the scattering mixes the polarization directions and algebraically if it does not.

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I. INTRODUCTION

A pair of photons in the Bell state \((|HV⟩ + |VH⟩)/√2\) can be transported over long distances with little degradation of the entanglement of their horizontal (H) and vertical (V) polarizations. Polarization-dependent scattering has little effect on the degree of entanglement, as long as it remains linear (hence describable by a scattering matrix) and as long as the photons are detected in a single spatial mode only. This robustness of photon entanglement was demonstrated dramatically in a recent experiment \([1]\) and theory \([2, 3]\) on plasmon-assisted entanglement transfer.

Polarization-dependent scattering may significantly degrade the entanglement in the case of multi-mode detection. Upon summation over \(N\) spatial modes the initially pure state of the Bell pair is reduced to a mixed state with respect to the polarization degrees of freedom. This loss of purity diminishes the entanglement — even if the two polarization directions are not mixed by the scattering.

The transition from pure-state to mixed-state entanglement will in general depend on the detailed form of the scattering matrix. However, a universal regime is entered in the case of randomly located scattering centers. This is the regime of applicability of random-matrix theory \([\text{I, II}]\). As we will show in this paper, the transmission of polarization-entangled radiation through disordered media reduces the degree of entanglement in a way which, on average, depends only on the number \(N\) of detected modes. (The average refers to an ensemble of disordered media with different random positions of the scatterers.) The degree of entanglement (as quantified either by the concurrence \([\text{III}]\) or by the violation of a Bell inequality \([\text{IV, V}]\)) decreases exponentially with \(N\) if the disorder randomly mixes the polarization directions. If the polarization is conserved, then the decrease is a power law \((\propto N^{-1}\) if both photons are scattered and \(\propto N^{-1/2}\) if only one photon is scattered).
similarly for the second medium.

A subset of $N_1$ out of the $M_1$ modes are detected in the first detector. We relabel the modes so that $n = 1, 2, \ldots, N_1$ are the detected modes. This subset is contained in the four vectors $u^+_n$, $u^-_n$, $u^{+*}_n$, $u^{-*}_n$ of length $N_1$ each. We write these vectors in bold face, $u_{\pm}$, omitting the mode index. Similarly, the second detector detects $N_2$ modes, contained in vectors $v_{\pm}$. A single or double dot between two pairs of vectors denotes a single or double contraction over the mode indices: $a \cdot b = \sum_n a_n b_n$, $\mathbf{a} : \mathbf{b} = \sum_{n,m} a_{n,m} c_{m,n} d_n$.

The maximal value $C$ of the concurrence is obtained through the maximal violation of the Bell-inequality, which we may set at 1. The assumption of independent variables statistically distributed as independent random variables ignores the orthonormality constraint of the vectors $u_m$. The normalization is now given simply by $Z = \sum_{\sigma, \tau} a_{\sigma \tau} a_{\sigma \tau}^*$.

The pure state has density matrix $\Psi_{n\sigma, m\tau} = \psi_{n\sigma, m\tau}$.

By tracing over the detected modes the pure state is reduced to a mixed state with respect to the polarization degrees of freedom. The reduced density matrix is $4 \times 4$, with elements

$$
\rho_{\sigma \tau, \sigma' \tau'} = \frac{1}{Z}(u_{+\sigma} v_{-\tau} + u_{-\sigma} v_{+\tau}): (v^*_{-\tau} u^*_{+\sigma} + v^*_{+\tau} u^*_{-\sigma}),
$$

(2.2)

$$
Z = \sum_{\sigma, \tau} (u_{+\sigma} v_{-\tau} + u_{-\sigma} v_{+\tau}): (v^*_{-\tau} u^*_{+\sigma} + v^*_{+\tau} u^*_{-\sigma}).
$$

(2.3)

The complex numbers that enter into the density matrix are conveniently grouped into a pair of Hermitian positive definite matrices $a$ and $b$, with elements $a_{\sigma \tau, \sigma' \tau'} = u_{\sigma \tau} \cdot u^*_{\sigma' \tau'}$, $b_{\sigma \tau, \sigma' \tau'} = v_{\sigma \tau} \cdot v^*_{\sigma' \tau'}$. One has

$$
Z \rho_{\sigma \tau, \sigma' \tau'} = a_{+\sigma, +\tau} b_{-\tau, -\sigma} + a_{-\sigma, -\tau} b_{+\tau, +\sigma} + a_{-\sigma, +\tau} b_{+\tau, -\sigma} + a_{+\sigma, -\tau} b_{-\tau, +\sigma} + a_{+\sigma, +\tau} b_{-\tau, -\sigma} + a_{-\sigma, -\tau} b_{+\tau, +\sigma} + a_{-\sigma, +\tau} b_{+\tau, -\sigma} + a_{+\sigma, -\tau} b_{-\tau, +\sigma}.
$$

(2.4)

Inferred from the Bell inequality violation.

As a special case we will also consider what happens if only one of the two beams is scattered. The other beam reaches the photodetector without changing its mode or polarization, so we may set $v^\pm_n = \delta_{m,n} \delta_{\sigma, \pm}$. This implies $b_{\sigma \tau, \sigma' \tau'} = \delta_{\sigma, \tau} \delta_{\sigma', \tau'}$, hence

$$
Z \rho_{\sigma \tau, \sigma' \tau'} = a_{+\sigma, \tau} a_{\tau, \sigma'},
$$

(2.8)

where we have defined $\bar{\tau} = -\tau$. The normalization is now given simply by $Z = \sum_{\sigma, \tau} a_{\sigma \tau, \sigma \tau}$.

### III. RANDOM-MATRIX THEORY

For a statistical description we use results from the random-matrix theory (RMT) of scattering by disordered media \[.\] According to that theory, the real and imaginary parts of the complex scattering amplitudes $u^\sigma_n$ are statistically distributed as independent random variables with the same Gaussian distribution of zero mean. The variance of the Gaussian drops out of the density matrix; we fix it at 1. The assumption of independent variables ignores the orthonormality constraint of the vectors $u_m$, which is justified if $N_1 \ll M_1$. Similarly, for $N_2 \ll M_2$ the real and imaginary parts of $v^\sigma_n$ have independent Gaussian distributions with zero mean and a variance which we may set at 1.

The reduced density matrix of the mixed state depends on the two independent random matrices $a$ and $b$, according to Eq. (2.4). The matrix elements are not independent. We calculate the joint probability distribution of the matrix elements, using the following result from
RMT [12]: Let $W$ be a rectangular matrix of dimension $p \times (k+p)$, filled with complex numbers with distribution

$$P(\{W_{nm}\}) \sim \exp \left( -c \text{Tr} W W^\dagger \right), \quad c > 0. \quad (3.1)$$

Then the square matrix $H = W W^\dagger$ (of dimension $p \times p$) has the Laguerre distribution

$$P(\{H_{nm}\}) \sim (\text{Det} H)^k \exp(-c \text{Tr} H). \quad (3.2)$$

Note that $H$ is Hermitian and positive definite, so its eigenvalues $h_n$ $(n = 1, 2, \ldots, p)$ are real positive numbers. Their joint distribution is that of the Laguerre unitary ensemble,

$$P(\{h_n\}) \sim \prod_n h_n^k e^{-ch_n} \prod_{i<j} (h_i - h_j)^2. \quad (3.3)$$

The factor $(h_i - h_j)^2$ is the Jacobian of the transformation from complex matrix elements to real eigenvalues. The eigenvectors of $H$ form a unitary matrix $U$ which is uniformly distributed in the unitary group.

To apply this to the matrix $a$ we set $c = 1/2$, $p = 4$, $k = N_1 - 4$. We first assume that $N_1 \geq 4$, to ensure that $k \geq 0$. Then

$$P(\{a_{\sigma \tau, \sigma' \tau'}\}) \sim (\text{Det} a)^{N_1-4} \exp \left( -\frac{1}{2} \text{Tr} a \right), \quad (3.4)$$

$$P(\{a_n\}) \sim \prod_n a_n^{N_1-4} e^{-a_n/2} \prod_{i<j} (a_i - a_j)^2, \quad (3.5)$$

where $a_1, a_2, a_3, a_4$ are the real positive eigenvalues of $a$. The $4 \times 4$ matrix $U$ of eigenvectors of $a$ is uniformly distributed in the unitary group. If $N_1 = 1, 2, 3$ we set $c = 1/2$, $p = N_1$, $k = 4 - N_1$. The matrix $a$ has $4 - N_1$ eigenvalues equal to 0. The $N_1$ non-zero eigenvalues have distribution

$$P(\{a_n\}) \sim \prod_n a_n^{N_1-4} e^{-a_n/2} \prod_{i<j} (a_i - a_j)^2. \quad (3.6)$$

The distribution of the matrix elements $b_{\sigma \tau, \sigma' \tau'}$ and of the eigenvalues $b_n$ is obtained upon replacement of $N_1$ by $N_2$ in Eqs. (3.3), (3.5), and (3.6).

### IV. ASYMPTOTIC ANALYSIS

We wish to average the concurrence (2.5) and pseudo-concurrence (2.7) with the RMT distribution of Sec. III. The result depends only on the number of detected modes $N_1, N_2$ in the two photodetectors. Microscopic details of the scattering media become irrelevant once we assume random scattering. The averages $\langle C \rangle$, $\langle C' \rangle$ can be calculated by numerical integration [13]. Before presenting these results, we analyze the asymptotic behavior for $N_1 \gg 1$ analytically. We assume for simplicity that $N_1 = N_2 \equiv N$. It is convenient to scale the eigenvalues as

$$a_n = 2N(1 + \alpha_n), \quad b_n = 2N(1 + \beta_n). \quad (4.1)$$

The distribution of the $\alpha_n$'s and $\beta_n$'s takes the same form

$$P(\{\alpha_n\}) \sim \exp \left( -N \sum_{n=1}^4 [\alpha_n - \ln(1 + \alpha_n)] + O(1) \right), \quad (4.2)$$

where $O(1)$ denotes $N$-independent terms. The bulk of the distribution lies in the region $\sum_n \alpha_n^2 \lesssim 1/N \ll 1$, localized at the origin. Outside of this region the distribution decays exponentially $\propto \exp[-N f(\{\alpha_n\})]$, with

$$f(\{\alpha_n\}) = \sum_{n=1}^4 [\alpha_n - \ln(1 + \alpha_n)]. \quad (4.3)$$

The concurrence $C$ and pseudo-concurrence $C'$ depend on the rescaled eigenvalues $\alpha_n, \beta_n$ and also on the pair of $4 \times 4$ unitary matrices $U, V$ of eigenvectors of $a$ and $b$. Both quantities are independent of $N$, because the scale factor $N$ in Eq. (4.2) drops out of the density matrix upon normalization.

The two quantities $C$ and $C'$ are identically zero when the $\alpha_n$'s and $\beta_n$'s are all $\ll 1$ in absolute value. For a nonzero value one has to go deep into the tail of the eigenvalue distribution. The average of $C$ is dominated by the “optimal fluctuation” $\alpha_n^{\text{opt}}, \beta_n^{\text{opt}}, C_n^{\text{opt}}, V^{\text{opt}}$ of eigenvalues and eigenvectors, which minimizes $f(\{\alpha_n\}) + f(\{\beta_n\})$ in the region $C > 0$. The decay

$$\langle C \rangle \approx \exp \left( -N [f(\{\alpha_n^{\text{opt}}\}) + f(\{\beta_n^{\text{opt}}\})] \right) \equiv e^{-AN} \quad (4.4)$$

of the average concurrence is exponential in $N$, with a coefficient $A$ of order unity determined by the optimal fluctuation. The average $\langle C' \rangle \approx e^{-BN}$ also decays exponentially with $N$, but with a different coefficient $B$ in the exponent. The numbers $A$ and $B$ can be calculated analytically for the case that only one of the two beams is scattered.

Scattering of a single beam corresponds to a density matrix $\rho$ which is directly given by the matrix $a$, cf. Eq. (2.5). To find $A$, we therefore need to minimize $f(\{\alpha_n\})$ over the eigenvalues and eigenvectors of $a$ with the constraint $C > 0$,

$$A = \min_{\{\alpha_n\}, U} \left\{ f(\{\alpha_n\}) \right\} \mathcal{C}(\rho(\{\alpha_n\}, U)) > 0. \quad (4.5)$$

The minimum can be found with the help of the following result [13]: The concurrence $C(\rho)$ of the two-qubit density matrix $\rho$, with fixed eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \Lambda_4$ but arbitrary eigenvectors, is maximized upon unitary transformation by

$$\max_\Omega \mathcal{C}(\Omega \rho \Omega^\dagger) = \max \left\{ 0, \Lambda_1 - \Lambda_3 - 2\sqrt{\Lambda_2 \Lambda_4} \right\}. \quad (4.6)$$

(The matrix $\Omega$ varies over all $4 \times 4$ unitary matrices.) With this knowledge, Eq. (4.5) reduces to

$$A = \min_{\{\alpha_n\}} \left\{ f(\{\alpha_n\}) | \alpha_1 - \alpha_3 - 2\sqrt{(1 + \alpha_2)(1 + \alpha_4)} > 0 \right\}, \quad (4.7)$$
where we have ordered $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. This yields for the optimal fluctuation $\alpha_1^{\text{opt}} = 1$, $\alpha_2^{\text{opt}} = \alpha_3^{\text{opt}} = \alpha_4^{\text{opt}} = -1/3$ and

$$A = 3 \ln 3 - 4 \ln 2 = 0.523.$$  

The asymptotic decay $\langle C \rangle \propto e^{-AN}$ is in good agreement with a numerical calculation for finite $N$, see Fig. 2.

The asymptotic decay of the average pseudo-concurrence $\langle C' \rangle$ for a single scattered beam can be found in a similar way, using the result [10].

V. COMPARISON WITH THE CASE OF POLARIZATION-CONSERVING SCATTERING

If the scatterers are translationally invariant in one direction, then the two polarizations are not mixed by the scattering. Such scatterers have been realized as parallel glass fibers [15]. One polarization corresponds to the electric field parallel to the scatterers (TE polarization), the other to parallel magnetic field (TM polarization). The boundary condition differs for the two polarizations (Dirichlet for TE and Neumann for TM), so the scattering amplitudes $u_{++}, v_{++}, u_{--}, v_{--}$ that conserve the polarization can still be considered to be independent random numbers. The amplitudes that couple different polarizations vanish: $u_{+-}, v_{+-}, u_{-+}, v_{-+}$ are all zero.

The reduced density matrix [24] simplifies to

$$Z_{\rho_{\sigma',\sigma}} = \delta_{\sigma,\sigma'} a_{\sigma,\sigma'} b_{\tau,\tau'}.$$  \hspace{1cm} (5.1)

with $\bar{\tau} = -\tau$, $\bar{\tau}' = -\tau'$. We will abbreviate $A_{\sigma\tau} \equiv a_{\sigma,\sigma',\tau}$, $B_{\sigma\tau} \equiv b_{\sigma,\sigma',\tau}$. The concurrence $C$ and pseudo-
concurrence $C'$ are calculated from Eqs. (2.5) and (2.7), with the result

$$C = C' = \frac{2|A_{+-}||B_{+-}|}{A_{++}B_{--} + A_{--}B_{++}}. \tag{5.2}$$

It is again our objective to calculate $\langle C \rangle$ for the case $N_1 = N_2 = N$. The distribution of the matrices $A$ and $B$ follows by substituting $N_1 - 4 \to N - 2$ in Eq. (3.4):

$$P(\{A_{\sigma\tau}\}) \propto (\text{Det} A)^{N-2} \exp \left( -\frac{1}{2} \text{Tr} A \right). \tag{5.3}$$

The average over this distribution was done numerically, see Fig. 3. For large $N$ we may perform the following asymptotic analysis.

We scale the matrices $A$ and $B$ as

$$A = 2N(\mathbb{1} + A), \quad B = 2N(\mathbb{1} + B). \tag{5.4}$$

In the limit $N \to \infty$ the Hermitian matrices $A$ and $B$ have the Gaussian distribution

$$P(\{A_{\sigma\tau}\}) \propto e^{-\frac{1}{2} N \text{Tr} AA^\dagger}. \tag{5.5}$$

(The same distribution holds for $B$.) In contrast to the analysis in Sec. IV the concurrence does not vanish in the bulk of the distribution. The average of Eq. (5.2) with distribution (5.5) yields the algebraic decay

$$\langle C \rangle = \frac{\pi}{4N}, \quad N \gg 1, \tag{5.6}$$

in good agreement with the numerical calculation for finite $N$ (Fig. 3).

A completely analytical calculation for any $N$ can be done in the case that only one of the beams is scattered. In that case $B_{\sigma\tau} = 1$ and the concurrence reduces to

$$C = \frac{2|A_{+-}|}{A_{++} + A_{--}}. \tag{5.7}$$

Averaging Eq. (5.7) over the Laguerre distribution (5.3) gives

$$\langle C \rangle = \frac{\sqrt{\pi} \Gamma(N+1/2)}{2 \Gamma(N+1)}. \tag{5.8}$$

For large $N$, the average concurrence (5.8) falls off as

$$\langle C \rangle = \frac{\sqrt{\pi}}{2 \sqrt{N}}, \quad N \gg 1. \tag{5.9}$$

This case is also included in Fig. 3.

VI. CONCLUSION

In summary, we have applied the method of random-matrix theory (RMT) to the problem of entanglement transfer through a random medium. RMT has been used before to study production of entanglement [10, 11, 12]. Here we have studied the loss of entanglement in the transition from a pure state to a mixed state.

A common feature of all these theories is that the results are universal, independent of microscopic details. In our problem the decay of the degree of entanglement depends on the number of detected modes but not on microscopic parameters such as the scattering mean free path.

The origin of this universality is the central limit theorem: The complex scattering amplitude from one mode in the source to one mode in the detector is the sum over a large number of complex partial amplitudes, corresponding to different sequences of multiple scattering. The probability distribution of the sum becomes a Gaussian with zero mean (because the random phases of the partial amplitudes average out to zero). The variance of the Gaussian will depend on the mean free path, but it drops out upon normalization of the reduced density matrix. The applicability of the central limit theorem only requires that the separation of source and detector is large compared to the scattering mean free path, to ensure a large number of terms in the sum over partial amplitudes.

The degree of entanglement (as quantified by the concurrence or violation of the Bell inequality) then depends only on the number $N$ of detected modes. We have identified two qualitatively different types of de-
The decay is exponential $\propto e^{-cN}$ if the scattering mixes spatial modes as well as polarization directions. The coefficient $c$ depends on which measure of entanglement one uses (concurrence or violation of Bell inequality) and it also depends on whether both photons in the Bell pair are scattered or only one of them is. For this latter case of single-beam scattering, the coefficients $c$ are $3 \ln 3 - 4 \ln 2$ (concurrence) and $\ln(11 + 5\sqrt{5}) - \ln 2$ (pseudo-concurrence). The decay is algebraic $\propto N^{-p}$ if the scattering preserves the polarization. The power $p$ is 1 if both photons are scattered and $1/2$ if only one of them is. Polarization-conserving scattering is special; it would require translational invariance of the scatterers in one direction. The generic decay is therefore exponential.

Finally, we remark that the results presented here apply not only to scattering by disorder, but also to scattering by a cavity with a chaotic phase space. An experimental search for entanglement loss by chaotic scattering has been reported by Woerdman et al. [24].

Acknowledgments

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