Envelope of Mid-Hyperplanes of a Hypersurface

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Abstract. Given 2 points of a smooth hypersurface, their mid-hyperplane is the hyperplane passing through their mid-point and the intersection of their tangent spaces. In this paper we study the envelope of these mid-hyperplanes (EMH) at pairs whose tangent spaces are transversal. We prove that this envelope consists of centers of conics having contact of order at least 3 with the hypersurface at both points. Moreover, we describe general conditions for the EMH to be a smooth hypersurface. These results are extensions of the corresponding well-known results for curves. In the case of curves, if the EMH is contained in a straight line, the curve is necessarily affinely symmetric with respect to the line. We show through a counter-example that this property does not hold for hypersurfaces.

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1. Introduction

Given a pair of points in a smooth convex planar curve, its mid-line is the line that passes through its mid-point and the intersection of the corresponding tangent lines. If these tangent lines are parallel, the mid-line is the line through $M$ parallel to both tangents. When both points coincide, the mid-line is just the affine normal at the point. The envelope of mid-lines is an important affine invariant symmetry set associated with the curve. It is important in computer graphics and has been studied by many authors ([2], [3], [4], [5], [7]). The envelope of mid-lines of planar curves can be divided into 3 parts: The Affine Envelope Symmetry Set (AESS), corresponding to pairs with non-parallel tangent lines, the Mid-Points Parallel Tangent Locus (MPTL), corresponding to pairs with parallel tangent lines, and the Affine Evolute, corresponding to coincident points.

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The concept of mid-line has a quite natural generalization to a hypersurface $S$ in the affine $(N+1)$-space: For a pair $(p_1, p_2)$ in $S$, the mid-hyperplane is the affine hyperplane that passes through the mid-point $M$ of $(p_1, p_2)$ and the intersection of the tangent spaces at $p_1$ and $p_2$ (note that this intersection is a co-dimension 2 affine space). It is then natural to ask what is the structure of the envelope of mid-hyperplanes. In this paper, we study this set assuming that the tangent spaces at $p_1$ and $p_2$ are transversal (for $N = 1$, this set is the AESS). We shall call Envelope of Mid-Hyperplanes (EMH) the envelope of mid-hyperspaces of pairs $(p_1, p_2)$ with transversal tangent spaces. The envelope of mid-planes $(N=2)$ with parallel tangent planes is called MPTS and has been studied in [7]. The envelope of mid-planes $(N=2)$ corresponding to coincident points is called Affine Mid-Planes Evolute and has been studied in [1].

The AESS is very well studied and coincides with the locus of center of conics having contact of order $\geq 3$ with the curve at 2 points. Moreover, if both contacts are of order exactly 3, the AESS is regular ([3],[5]). In this article we prove that each point of the EMH is the center of a conic having contact of order $\geq 3$ with the hypersurface at 2 points. Moreover, we describe a general condition for the regularity of the EMH that generalizes the regularity condition for curves. This general condition is algebraic, but we give geometric interpretations in some particular cases.

The reflection property of the AESS is very significant for symmetry recognition: If the AESS is contained in a straight line $l$, then the curve itself is invariant under an affine reflection with axis $l$ ([3],[5]). Unfortunately, the reflection property does not extend to the EMH. We give an example where the EMH is contained in a plane $(N=2)$ but the surface $S$ is not invariant under an affine reflection. An interesting question here is to understand which kind of symmetry is implied by the inclusion of the EMH in a hyperplane.

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2. Review of affine differential geometry of hypersurfaces

In this section we review some basic concepts of affine differential geometry of hypersurfaces in $(N + 1)$-space (for details, see [6]). Denote by $D$ the canonical connection and by $[....]$ the standard volume form in the affine $(N + 1)$-space. Let $S$ be a hypersurface and denote by $X(S)$ the tangent bundle of $S$. Given a transversal vector field $\xi$, write the Gauss equation

$$ D_X Y = \nabla_X Y + h(X, Y)\xi, $$

where $h$ is a symmetric bilinear form and $\nabla$ is a torsion free connection in $S$. We shall assume that $h$ is non-degenerate, which is independent of the choice of $\xi$. The volume form induces a volume form in $S$ by the relation

$$ \theta(X_1, ..., X_N) = [X_1, ..., X_N, \xi]. $$
The metric $h$ also defines a volume form in $S$: Given $X_i \in \mathfrak{X}(S)$, $1 \leq i \leq N$, let

$$\theta_h(X_1, \ldots, X_N) := |\det(h(X_i, X_j))|^{\frac{1}{2}}.$$ 

Next theorem is fundamental in affine differential geometry ([6], ch.II):

**Theorem 2.1.** There exists, up to sign, a unique transversal vector field $\xi$ such that $\nabla \theta = 0$ and $\theta = \theta_h$. The vector field $\xi$ is called the affine normal vector field and the corresponding metric $h$ the Blaschke metric of the surface.

For $p \in S$, let $\nu(p)$ be the linear functional in $(N + 1)$-space such that

$$\nu(p)(\xi) = 1 \quad \text{and} \quad \nu(p)(X) = 0 \quad \forall \ X \in T_pS.$$ 

The differentiable map $\nu$ is called the conormal map. It satisfies the following property ([6], ch.II):

**Proposition 2.2.** Let $S$ be a non-degenerate hypersurface and $\nu$ the conormal map. Then

$$D_Y \nu(X) = 0 \quad \text{and} \quad D_Y \nu(X) = -h(Y, X), \quad \forall \ X, Y \in \mathfrak{X}(S).$$

**Corollary 2.3.** If $X \in \mathfrak{X}(\mathbb{R}^{N+1})$ is any vector field, then

$$D_Y \nu(X) = -h(Y, X^T), \quad Y \in \mathfrak{X}(S),$$

where $X = X^T + \lambda \xi, \lambda \in \mathbb{R}$ and $X^T$ is the tangent component of $X$.

**Proof.** We have

$$D_Y \nu(X) = D_Y \nu(X^T + \lambda \xi) = D_Y \nu(X^T) + \lambda D_Y \nu(\xi) = -h(Y, X^T),$$

thus proving the corollary. \qed

**Lemma 2.4.** Assume that $S$ is the graph of $f(x, y), y = (y_1, \ldots, y_{N-1})$, i.e.,

$$\psi(x, y) = (x, y, f(x, y))$$

is a parameterization of $S$. Then, at any point $(x, y)$, $h(\psi_x, \psi_{y_j}) = 0$ if and only if $f_{x y_j} = 0$.

**Proof.** Observe first that, from equation (1), we obtain

$$[\psi_x, \psi_y, \psi_{x y_j}] = h(\psi_x, \psi_{y_j}) [\psi_x, \psi_y, \xi].$$

On the other hand, a direct calculation shows that

$$[\psi_x, \psi_y, \psi_{x y_j}] = f_{x y_j}.$$ 

Since $[\psi_x, \psi_y, \xi] \neq 0$, we conclude the lemma. \qed

3. **Envelope of Mid-Hyperplanes**

Let $S$ be a non-degenerate convex hypersurface. Take points $p_1, p_2 \in S$ and let $S_1 \subset S$ and $S_2 \subset S$ be open subsets around $p_1$ and $p_2$, respectively. Denote $h_1$ and $h_2$ the Blaschke metrics of $S_1$ and $S_2$, respectively. We shall assume that the tangent spaces at points of $S_1$ are transversal to tangent spaces at points of $S_2$. 
3.1. Basic definitions
Denote by $M(p_1, p_2)$ the mid-point and by $C(p_1, p_2)$ the mid-chord of $p_1$ and $p_2$, i.e.,
\[ M(p_1, p_2) = \frac{p_1 + p_2}{2}, \quad C(p_1, p_2) = \frac{p_1 - p_2}{2}. \]
The mid-hyperplane of $(p_1, p_2)$ is the affine hyperplane that contains $M(p_1, p_2)$ and the intersection $Z$ of the tangent spaces at $p_1$ and $p_2$. Let $F : S_1 \times S_2 \times \mathbb{R}^{N+1} \to \mathbb{R}$ be given by
\[ F(p_1, p_2, X) = (\nu_2(C)\nu_1 + \nu_1(C)\nu_2)(X - M). \]
where $\nu_i$ is the co-normal map of $S_i$. It is not difficult to verify that, for $(p_1, p_2)$ fixed, $F(p_1, p_2, X) = 0$ is the equation of the mid-hyperplane.

Consider frames \(\{Z_1, \ldots, Z_{N-1}\}\) of $Z$, each $Z_j$ being smooth functions of $(p_1, p_2) \in S_1 \times S_2$. Consider also vector fields $Y_1, Y_2$ such that $Y_i(p_1, p_2)$ is tangent to $S_i$ and $h_i$-orthogonal to $Z$.

We want to find $X$ satisfying $F = F_{p_1} = F_{p_2} = 0$, for some $p_1 \in S_1$, $p_2 \in S_2$. Since $Y_1$ and $Z_j$ are $h_i$-orthogonals, \(\{Y_1, Z_1, \ldots, Z_{N-1}\}\) is a basis of $T_{p_i}S_i, i = 1, 2$. Thus we have to find $X$ in the following system:
\[
\begin{align*}
F(p_1, p_2, X) &= 0 \\
F_{p_1}(p_1, p_2, X)(Y_1) &= 0 \\
F_{p_2}(p_1, p_2, X)(Y_2) &= 0 \\
F_{p_1}(p_1, p_2, X)(Z_j) &= 0 \\
F_{p_2}(p_1, p_2, X)(Z_j) &= 0.
\end{align*}
\]
The notation $F_{p_i}(p_1, p_2, X)(W)$ corresponds to the partial derivative of $F$ with respect $p_i$ in the direction $W \in T_{p_i}S_i$, thus keeping $p_j, j \neq i$ and $X$ fixed.

3.2. Solutions of the system

We begin with the following simple lemma:

**Lemma 3.1.** We have that
\[ D_{Y_1}\nu_1 = a\nu_1 + b\nu_2 \quad \text{and} \quad D_{Y_2}\nu_2 = \bar{a}\nu_1 + \bar{b}\nu_2, \]
where $a, b, \bar{a}, \bar{b}$ are given by
\[ a = -\frac{h_1(Y_1, X_2)}{\nu_1(X_2)}, \quad b = -\frac{h_1(Y_1, X_1)}{\nu_2(X_1)}, \quad \bar{a} = -\frac{h_2(Y_2, X_2)}{\nu_1(X_2)}, \quad \bar{b} = -\frac{h_2(Y_2, X_1)}{\nu_2(X_1)}, \]
for any $X_1 \in T_{p_1}S_1, X_2 \in T_{p_2}S_2$.

**Proof.** Take a basis \(\{\nu_1, \nu_2, \zeta_1, \ldots, \zeta_{N-1}\}\) of the dual space $\mathbb{R}^{N+1}$. Thus we can write the linear functional $D_{Y_1}\nu_1$ as a linear combination of the basis vector, i.e., $D_{Y_1}\nu_1 = a\nu_1 + b\nu_2 + \sum_{j=1}^{N-1} c_j \zeta_j$. Since $D_{Y_1}\nu_1(Z_j) = -h_1(Y_1, Z_j) = 0$ we obtain $c_j = 0$ and so $D_{Y_1}\nu_1 = a\nu_1 + b\nu_2$. Applying $D_{Y_1}\nu_1$ to any tangent vector field $X_1$ on $S_1$ we get $D_{Y_1}\nu_1(X_1) = b\nu_2(X_1)$, thus proving the formula for $b$. The other formulas are proved similarly. \(\square\)
Proposition 3.2. The first three equations of the system \(\text{(4)}\) admit a solution if and only if
\[
\nu_1(C) = -\lambda \nu_2(C), \tag{5}
\]
where
\[
\lambda = \left( \frac{\nu_1^2(Y_2) h_1(Y_1, Y_1)}{\nu_2^2(Y_1) h_2(Y_2, Y_2)} \right)^{1/3}. \tag{6}
\]

Proof. Since \(\nu_1(Y_1) = \nu_2(Y_2) = 0\), it follows that the derivative \(F_{p_1}(Y_1)\) is given by
\[
D_Y \nu_1(Y_1) = \nu_2(Y_2) = 0, \tag{8}
\]
and
\[
F_{p_1}(Y_1) = \left( 2b \nu_2(C) \nu_2 + \frac{1}{2} \nu_2(Y_1) \nu_1 \right) (X - M) - \frac{1}{2} \nu_1(C) \nu_2(Y_1) + aF. \tag{9}
\]
Similarly
\[
F_{p_2}(Y_2) = \left( 2\bar{a} \nu_1(C) \nu_1 - \frac{1}{2} \nu_1(Y_2) \nu_2 \right) (X - M) - \frac{1}{2} \nu_2(C) \nu_1(Y_2) + \bar{b}F. \tag{10}
\]
Using that \(F = 0\), the equations \(F_{p_1}(Y_1) = 0\) and \(F_{p_2}(Y_2) = 0\) can be simplified to
\[
\left( -\frac{\nu_2(Y_1) \nu_1(C) \nu_2}{\nu_2(C)} + 2b \nu_2(C) \nu_2 \right) (X - M) = \frac{1}{2} \nu_1(C) \nu_2(Y_1), \tag{11}
\]
and
\[
\left( -\frac{2\bar{a} \nu_1^2(C) \nu_2}{\nu_2(C)} - \nu_1(Y_2) \nu_2 \right) (X - M) = \frac{1}{2} \nu_2(C) \nu_1(Y_2). \tag{12}
\]
These equations, after some simple calculations, leads to
\[
b \nu_2^3(C) \nu_1(Y_1) = -\bar{a} \nu_2^3(C) \nu_2(Y_1), \tag{13}
\]
which, together with lemma 3.1, proves the proposition. \(\square\)

Next lemma is a consequence of the first three equations of system \(\text{(4)}\):

Lemma 3.3. From equation \(\text{(5)}\), we can write
\[
C = A \left( Y_1 - \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \sum_{j=1}^{N-1} \alpha_j Z_j, \tag{14}
\]
for some \(A \in \mathbb{R}, \alpha_j \in \mathbb{R}\). Then
\[
X - M = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \sum_{j=1}^{N-1} \beta_j Z_j, \tag{15}
\]
where \(\beta_j \in \mathbb{R}\) and
\[
B = -\frac{\lambda A}{\lambda + 4Ab}. \tag{16}
\]
Proof. It follows from $F = 0$ and equation (5) that $\nu_1(X - M) = \lambda \nu_2(X - M)$. Then equation (9) holds, for some $B \in \mathbb{R}$. From equation (7) we have

$$\nu_2(X - M) = -\frac{\lambda \nu_2(C)}{\lambda \nu_2(Y_1) + 4b \nu_2(C)}.$$

We conclude that

$$B = -\frac{\lambda \nu_2(C)}{\lambda \nu_2(Y_1) + 4b \nu_2(C)} = -\frac{\lambda A}{\lambda + 4Ab},$$

thus proving the lemma. \qed

Next theorem is the main result of the section and says that the geometry of the EMH occurs in the plane generated by $Y_1$ and $Y_2$ (see figure 1).

**Theorem 3.4.** The system (4) admits a solution if and only if $C, Y_1$, and $Y_2$ are co-planar and equation (5) holds. Moreover, the solution of the system is given by

$$X - M = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right),$$

where $\lambda$ and $B$ are given by equations (6) and (10), respectively.

**Proof.** We must show that $\alpha_j = \beta_j = 0$ at equations (8) and (9), respectively. For this, we shall consider the last two equations of the system (4). The derivative $F_{p_1}(Z_j)$ is given by

$$F_{p_1}(Z_j) = D_{Z_j} \nu_1(C) \nu_2(X - M) + \nu_2(C) D_{Z_j} \nu_1(X - M).$$

Thus, from Corollary 2.3

$$F_{p_1}(Z_j) = -h_1(Z_j, C^T) \nu_2(X - M) - \nu_2(C) h_1(Z_j, (X - M)^T),$$

**Figure 1.** The geometry of the EMH.
where $V^T$ denotes the projection of $V$ in $T_{p_1}S_1$ along the direction of the affine normal of $S_1$ at $p_1$. From equations (8) and (9) we obtain

$$C^T = A \left( Y_1 - \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_1^T \right) + \sum_{k=1}^{N-1} \alpha_k Z_k$$

and

$$(X - M)^T = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_1^T \right) + \sum_{k=1}^{N-1} \beta_k Z_k.$$ 

Substituting these equations in equation (12) we obtain

$$F_{p_1}(Z_j) = -\nu_2(Y_1) \sum_{k=1}^{N-1} (B\alpha_k + A\beta_k) h_1(Z_j, Z_k).$$

Similarly we obtain

$$F_{p_2}(Z_j) = \lambda \nu_2(Y_1) \sum_{k=1}^{N-1} (-\alpha_k B + \beta_k A) h_2(Z_j, Z_k).$$

Since $S_1$ and $S_2$ are convex, $(h_i(Z_j, Z_k))$, $i = 1, 2$, are definite matrices, hence non-degenerate. Moreover, non-parallel tangent planes imply that $\nu_2(Y_1)$ and $\lambda$ are non-zero. Thus equations $F_{p_1}(Z_j) = F_{p_2}(Z_j) = 0$ are equivalent to $\alpha_j B + \beta_j A = -\alpha_j B + \beta_j A = 0$, which implies that $\alpha_j = \beta_j = 0$. □

4. Conics with $3 + 3$ contact with the surface

Given two non-degenerate locally convex hypersurfaces $S_1$ and $S_2$, consider conics that makes contact of order $\geq 3$ with $S_i$ at points $p_i$, $i = 1, 2$, in directions $Y_i$ which are $h_i$-orthogonals to the intersection $Z$ of $T_{p_1}S_1$ and $T_{p_2}S_2$. We shall prove in this section that the set of centers of these $3 + 3$ conics coincides with the set EMH.

Along this section, we shall assume that $S_i$ is the graph of a function $f_i(u_i, v_i)$, $i = 1, 2$, $v_i = (v_{i,1}, ..., v_{i,N-1})$, and consider the normal vectors

$$N_i = (-(f_i)_{u_i}, ..., -(f_i)_{v_{i,j}}, ..., 1) \quad (13)$$

to $S_i$. Let $F : S_1 \times S_2 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(p_1, p_2, X) = ((N_1 \cdot C)N_2 + (N_2 \cdot C)N_1) \cdot (X - M), \quad (14)$$

where $\cdot$ denotes the canonical inner product, $N_i$ is given by equation (13), $M$ is the mid-point of $(p_1, p_2)$ and $C$ is the mid-chord of $(p_1, p_2)$. Then $F = 0$ is the equation of the mid-plane of $(p_1, p_2)$.

**Lemma 4.1.** Assume that the pair $(p_1, p_2)$ generates a point of EMH. Then by an affine change of coordinates, we may assume that $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and $S_1$ and $S_2$ are graphs of

$$f_1(u_1, v_1) = 1 + \epsilon p u_1 - \frac{1}{2}(p^2 + \epsilon) u_1^2 + \sum_{j_1, j_2} a_{j_1, j_2} v_{1, j_1} v_{1, j_2} + O(3) \quad (15)$$
and
\[ f_2(u_2, v_2) = -1 - \epsilon pu_2 + \frac{1}{2}(p^2 + \epsilon)u_2^2 + \sum_{j_1,j_2} b_{j_1,j_2}v_{j_1}u_{j_2} + O(3), \] (16)

where \((p, 0, 0)\) is the corresponding point in EMH, \(\epsilon = \pm 1\), \(\epsilon p < 0\), \(A = (a_{j_1,j_2})\) and \(B = (b_{j_1,j_2})\) are positive or negative definite. As a consequence, \((p, 0, 0)\) is the center of a conic making contact of order \(\geq 3\) with \(S_i\) at \(p_i\) in the direction \(Y_i\) \(h_i\)-orthogonal to the affine space \(Z = \text{span}\{\frac{\partial}{\partial\nu}\}\). If \(\epsilon = 1\), the conic is an ellipse, while if \(\epsilon = -1\), the conic is a hyperbola.

**Proof.** Consider \(p_1 \in S_1\) and \(p_2 \in S_2\) with non-parallel tangent planes. By an adequate affine change of variables, we may assume that \(p_1 = (0, 0, 1)\), \(p_2 = (0, 0, -1)\) and the mid-plane of \((p_1, p_2)\) is \(z = 0\). Since, by theorem 3.3, \(Y_1, Y_2\) and \((0, 0, 1)\) are co-planar, we may also assume that \(Y_1\) and \(Y_2\) are in the \(xz\)-plane. We may also assume that the affine space \(Z\), intersection of the tangent planes \(T_{p_1}S_1\) and \(T_{p_2}S_2\) is the \(y\)-space. By lemma 2.6, these conditions implies that the coefficients of \(u_1v_{1,j}\) and \(u_2v_{2,j}\) are zero. Since the tangent plane at \(p_1\) contains \(Z\), \(S_1\) is the graph of a function \(f_1\) of the form
\[ f_1(u_1, v_1) = 1 + \epsilon pu_1 - \frac{1}{2}(p^2 + \epsilon)u_1^2 + \sum_{j_1,j_2} a_{j_1,j_2}v_{j_1}u_{j_2} + O(3), \]
for \(\epsilon = \pm 1\), \(\epsilon p < 0\). The tangent plane to \(S_2\) at \((0, 0, -1)\) is the reflection of the tangent plane to \(S_1\) at \((0, 0, 1)\), so \(S_2\) is the graph of a function \(f_2\) of the form
\[ f_2(u_2, v_2) = -1 - \epsilon pu_2 + \frac{\delta}{2}(p^2 + \epsilon)u_2^2 + \sum_{j_1,j_2} b_{j_1,j_2}v_{j_1}u_{j_2} + O(3), \]
for some \(\delta \in \mathbb{R}\). From these formulas we obtain
\[ N_1 = (-\epsilon p + (p^2 + \epsilon)u_1, -2Av_1, 1), \quad N_2 = (\epsilon p - (p^2 + \epsilon)u_2, -2Bv_2, 1). \]
Using
\[ C = (u_1 - u_2, v_1 - v_2, f_1 - f_2), \quad M = \frac{1}{2}(u_1 + u_2, v_1 + v_2, f_1 + f_2), \]
equation (14) leads to \(F = 2z\) at \(u_1 = v_1 = u_2 = v_2 = 0\). Straightforward but long calculations also leads to \(F_{u_1} = x + pz - p, \quad F_{u_2} = (p^2 - p^2\delta - \delta)x + pz + p, \quad F_{v_1} = Ay, \quad F_{v_2} = By\) at \(u_1 = v_1 = u_2 = v_2 = 0\). Thus the system \(F = F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2}\) at the origin becomes
\[ \begin{cases} -2z = 0 \\ x + pz - p = 0 \\ Ay = 0 \\ (p^2 - p^2\delta - \delta)x + pz + p = 0 \\ By = 0 \end{cases}. \] (17)

Since, by hypothesis, this system admit a solution, this solution must be \((p, 0, 0)\). Thus we conclude that \(\delta = 1\). It is not difficult to verify now that there exists a conic centered at \((p, 0, 0)\) contained in the plane \(xz\) and making
contact of order $\geq 3$ with $S_1$ at $p_1$ and $S_2$ at $p_2$. Moreover, this conic is an ellipse if $p < 0$ and a hyperbola if $p > 0$. □

**Lemma 4.2.** Consider a conic that makes contact of order $\geq 3$ with $S_i$ at points $p_i$, $i = 1, 2$, in directions $Y_i$ which are $h_i$-orthogonals to the intersection $Z$ of $T_{p_1}S_1$ and $T_{p_2}S_2$. Then, by an affine change of coordinates, we may assume that $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by equations (15) and (16), where $(p, 0, 0)$ is the center of the conic and $\epsilon = 1$ if the conic is an ellipse and $\epsilon = -1$ if the conic is a hyperbola. As a consequence, $(p, 0, 0)$ belongs to EMH.

**Proof.** Consider $p_1 \in S_1$ and $p_2 \in S_2$ with non-parallel tangent planes. By an adequate affine change of variables, we may assume that $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and the mid-plane is $z = 0$. We may also assume that the conic is contained in the $xz$-plane and that the intersection of the tangent planes $T_{p_1}S_1$ and $T_{p_2}S_2$ is the $y$-space. By lemma 2.4, these conditions imply that the coefficients of $u_1v_{1,j}$ and $u_2v_{2,j}$ are zero.

Now assuming that the center of the conic is the point $(p, 0, 0)$, $S_1$ and $S_2$ are graphs of functions given by equations (15) and (16), for some $\epsilon = \pm 1$. As a consequence, $(p, 0, 0)$ satisfies the system (17), which implies that $(p, 0, 0)$ belongs to EMH. □

From the above two lemmas we can conclude the main result of this section.

**Proposition 4.3.** The set of centers of conics which make contact of order $\geq 3$ at points $p_i \in S_i$ at directions $Y_i$ which are $h_i$-orthogonals to the intersection of $T_{p_i}S_i$, $i = 1, 2$, coincides with the set EMH.

5. Regularity of the EMH

In this section, we shall study the regularity of the Envelope of Mid-Hyperplanes.

5.1. A general condition for regularity

Let $F$ be given by equation (14) and consider the map

$$G : \mathbb{R}^{2N} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{2N+1}$$

$$(u_1, v_1, u_2, v_2, X) \mapsto (F, F_{u_1}, F_{v_1}, F_{u_2}, F_{v_2}).$$

Then the set $EMH$ is the projection in $\mathbb{R}^{N+1}$ of the set $G = 0$. If $0 \in \mathbb{R}^{2N+1}$ is a regular value of $G$, then $G^{-1}(0)$ is a $N$-dimensional submanifold of $\mathbb{R}^{3N+1}$. We want to find conditions under which $\pi_2(G^{-1}(0))$ becomes smooth, where $\pi_2(u_1, v_1, u_2, v_2, X) = X$. 

In the case of surfaces (N=2), we can understand better the meaning of the isomorphism. We conclude that the EMH is smooth at this point.

**Theorem 5.1.** If $\Delta \neq 0$, then the EMH is smooth at the point $X$.

**Proof.** Since the mid-plane is non-degenerate, the equalities $F_x = F_y = F_z = 0$ cannot occur simultaneously. Moreover, at points of the envelope, $F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2} = 0$. Thus the hypothesis implies that $JG$ has rank $2N + 1$ and so $G^{-1}(0)$ is a regular $N$-submanifold of $\mathbb{R}^{3N+1}$. Moreover, the hypothesis $\Delta \neq 0$ implies that the differential of $\pi$ restricted to $G^{-1}(0)$ is an isomorphism. We conclude that the EMH is smooth at this point.

### 5.2. The case of surfaces

In the case of surfaces (N=2), we can understand better the meaning of the condition $\Delta \neq 0$. By lemma [4.1], we may assume that $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by

$$f_1(u_1, v_1) = 1 + \epsilon p u_1 - \frac{1}{2}(p^2 + \epsilon)u_1^2 + a_0 u_1^4 + a_1 u_1^3 v_1 + a_2 u_1^2 v_1^2 + a_3 v_1^3 + O(4),$$

$$f_2(u_2, v_2) = -1 - \epsilon p u_2 + \frac{1}{2}(p^2 + \epsilon)u_2^2 + b_0 u_2^3 + b_1 u_2^2 v_2 + b_2 u_2 v_2^2 + b_3 v_2^3 + O(4).$$

Long but straightforward calculations using formula [14] show that the jacobian matrix $JG1$ at point $(0,0,0,0,p,0,0)$ is given by

$$
\begin{pmatrix}
3p^2 + 3p^4 - 6pa_0 & -2pa_1 & 0 & 0 \\
-2pa_1 & -2ap^2 - 2a_2p & 0 & (a + b)(p^2 + 1) \\
0 & 0 & -3p^4 - 3p^2 - 6pb_0 & -2pb_1 \\
0 & (a + b)(p^2 + 1) & -2pb_1 & -2bp^2 - 2b_2p
\end{pmatrix}.
$$

Thus the condition $\Delta \neq 0$ means that the determinant of this $4 \times 4$ matrix is not zero. We give below geometric interpretations of this condition in some particular cases, but a geometric interpretation in the general case remains to be given.
5.2.1. 3 + 3 contact with quadrics. The condition $a + b = 0$ is equivalent to the existence of a quadric $Q$ with contact of order $\geq 3$ with $S_1$ at $p_1$ and $S_2$ at $p_2$. In this case, the condition $\Delta \neq 0$ can be written as $\delta_1 \delta_2 \neq 0$, where

$$\delta_1 = \det \begin{bmatrix} 3p^2 + 3p^4 - 6pa_0 & -2pa_1 \\ -2pa_1 & -2ap^2 - 2a_3p \end{bmatrix},$$

$$\delta_2 = \det \begin{bmatrix} -3p^2 - 3p^4 - 6pb_0 & -2pb_1 \\ -2pb_1 & -2bp^2 - 2b_3p \end{bmatrix}.$$

The condition $\delta_1 = 0$ means that the quadric $Q$ has in fact a higher order contact with $S_1$ at $p_1$. In fact taking

$$h_1(x, y, z) = (x - p)^2 + z^2 + ay^2,$$

consider the contact function $\bar{h}_1(u_1, v_1) = h_1(\psi_i(u_1, v_1))$. Then $\bar{h}_1(0, 0) = 0$ and $(0, 0)$ is a critical point of $\bar{h}_1$. One can verify that $\delta_1 = 0$ if and only if $(0, 0)$ is a degenerate critical point of $\bar{h}_1$. A similar geometric interpretation holds for $\delta_2$.

5.2.2. The case $a_1 = a_2 = b_1 = b_2 = 0$. In this case we have

$$\Delta = (3p^2 + 3p^4 - 6pa_0) \cdot (-3p^4 - 3p^2 - 6pb_0) \delta,$$

where $\delta = -((a - b)^2p^4 + (a + b)^2 + 2p^2(a + b)^2) < 0$. Then the condition $\Delta \neq 0$ says that the contacts of the conic with $S_1$ at $p_1$ and $S_2$ at $p_2$ are both exactly 3.

6. A counter-example for the reflection property

In [2], it is proved that if the AESS of a pair of planar curves is contained in a line, then there exists an affine reflection taking one curve into the other. This fact is not true for the EMH of a pair of hypersurfaces as the following example shows us.

Consider $\gamma_1(t) = (t, 0, f(t))$ a smooth convex curve and let $\gamma_2(t) = (t - \lambda f(t), 0, -f(t)), \lambda \in \mathbb{R}$, be obtained from $\gamma_1$ by an affine reflection. Let $S_1$ and $S_2$ be rotational surfaces obtained by rotating $\gamma_1$ and $\gamma_2$ around the $z$-axis. $S_1$ and $S_2$ can be parameterized by

$$\phi_1(t, \theta) = (t \cos(\theta), t \sin(\theta), f(t))$$

and

$$\phi_2(t, \theta) = (t - \lambda f(t)) \cos(\theta), (t - \lambda f(t)) \sin(\theta), -f(t)).$$

The intersection $Z$ of the tangent planes at $\phi_1(t, \theta)$ and $\phi_2(t, \theta)$ has direction $(\sin(\theta), -\cos(\theta), 0)$.

Observe that the vectors $Y_1 = (\phi_1)_t$ and $Y_2 = (\phi_2)_t$ are orthogonal to $Z$ in the Blaschke metric. This implies that the EMH of this pair of surfaces is contained in the plane $z = 0$. But it is clear that $S_2$ is not an affine reflection of $S_1$. 

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