SOME RESULTS ON THE SCATTERING THEORY FOR NONLINEAR SCHröDINGER EQUATIONS IN WEIGHTED $L^2$ SPACE

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ABSTRACT. We investigate the scattering theory for the nonlinear Schrödinger equation $i\partial_t u + \Delta u + \lambda |u|^\alpha u = 0$ in $\Sigma = H^1(\mathbb{R}^d) \cap L^2(|x|^2; dx)$. We show that scattering states $u^\pm$ exist in $\Sigma$ when $\alpha_d < \alpha < \frac{4}{d-2}$, $d \geq 3$, $\lambda \in \mathbb{R}$ with certain smallness assumption on the initial data $u_0$, and when $\alpha(d) \leq \alpha < \frac{4}{d-2}(\alpha \in [\alpha(d), \infty)$, if $d = 1, 2$), $\lambda > 0$ under suitable conditions on $u_0$, where $\alpha_d$, $\alpha(d)$ are the positive root of the polynomial $dx^2 + dx - 4$ and $dx^2 + (d-2)x - 4$ respectively. Specially, when $\lambda > 0$, we obtain the existence of $u^\pm$ in $\Sigma$ for $u_0$ below a mass-energy threshold $M[u_0]_\sigma E[u_0] < \lambda - 2\tau M[Q]_\sigma E[Q]$ and satisfying an mass-gradient bound $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \lambda^{-1} \|Q\|_{L^2} \|\nabla Q\|_{L^2}$ with $\sigma = \frac{4-(d-2)\alpha}{\alpha d-4}$, $\tau = \frac{2}{\alpha d-4}$ and $Q$ is the ground state. We also study the convergence of $u(t)$ to the free solution $e^{it\Delta}u^\pm$ in $\Sigma$, where $u^\pm$ is the scattering state at $\pm \infty$ respectively.

Keywords: Nonlinear Schrödinger equation; Scattering theory; Oscillating data; Weighted spaces; Lorentz space

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1. INTRODUCTION

In this paper we study the scattering theory for the nonlinear Schrödinger equation

$$
\begin{cases}
    i\partial_t u + \Delta u + \lambda |u|^\alpha u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^d \\
    u(0, x) = u_0(x) \in \Sigma, & x \in \mathbb{R}^d
\end{cases}
$$

in weighted space $\Sigma = H^1(\mathbb{R}^d) \cap L^2(|x|^2; dx)$, where $d$ denotes the spatial dimension, $\lambda \in \mathbb{R} \setminus \{0\}$ and $0 < \alpha < \frac{4}{d-2}$ ($0 < \alpha < \infty$ if $d = 1, 2$).

As is well-known, if $\lambda < 0$, or $\lambda > 0$ and $\alpha < 4/d$, the unique solution $u(t)$ to the Cauchy problem (1.1) is global in time and bounded in $H^1(\mathbb{R}^d)$, and $u \in C(\mathbb{R}, \Sigma)$ (see e.g. [3]). If $\lambda > 0$, $\frac{4}{d} \leq \alpha < \frac{4}{d-2}$ ($4/d \leq \alpha < \infty$, if $d = 1, 2$), the local well-posedness in $\Sigma$ has been established by using Kato’s fixed point method to the equivalent integral equation (Duhamel’s formula)

$$
u(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^\alpha u(\tau) d\tau,$$
in an appropriate space (see [3, 8, 10]), where \((e^{it\Delta})_{t \in \mathbb{R}}\) is the one parameter Schrödinger group. More precisely, given \(u_0 \in \Sigma\), there exists \(T > 0\) and a unique solution \(u \in C([-S, T], \Sigma)\) of (1.1), which can be extended to a maximal existence interval \((-T_{\text{min}}, T_{\text{max}})\). This solution either exist globally or blow up in finite time, the global versus blow-up dichotomy is associated inseparably with the mass-energy threshold condition of the initial data \(u_0\) (see [5, 6, 7, 11]). Moreover, for arbitrary \(u_0 \in \Sigma\), the corresponding solution \(u(t)\) satisfies the mass and energy conservation laws:

\[
M[u(t)] = \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M[u_0],
\]

\[
E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{\lambda}{\alpha + 2} \int_{\mathbb{R}^d} |u(t, x)|^{\alpha + 2} dx = E[u_0],
\]

and the pseudo-conformal conservation law

\[
\frac{d}{dt} \left(\|x + 2it\nabla u(t)\|^2_{L^2} - \frac{8\lambda^2}{\alpha + 2} \|u(t)\|_{L^{\alpha + 2}}^{\alpha + 2}\right) = 4\lambda \frac{\alpha d - 4}{\alpha + 2} t \|u(t)\|_{L^{\alpha + 2}}^{\alpha + 2}.
\]

Thus we will denote the mass and energy by \(M[u]\) and \(E[u]\) respectively, with no reference to the time \(t\).

If the solution \(u(t)\) is global in time, we will care about its asymptotic behavior as \(t \to \pm \infty\). To state our results on this topic, we will introduce some basic notions of scattering theory (see [3]) below.

Let \(u_0 \in \Sigma\) be such that the corresponding solution \(u\) of (1.1) is defined for all \(t \geq 0\), i.e., \(T_{\text{max}} = \infty\). If the limit

\[
u^+ = \lim_{t \to +\infty} e^{-it\Delta} u(t)
\]

exists in \(\Sigma\), we say that \(u^+\) is the scattering state of \(u_0\) at \(+\infty\). Also, if \(u_0 \in \Sigma\) is such that the solution of (1.1) is defined for all \(t \leq 0\), i.e., \(T_{\text{min}} = \infty\), and if the limit

\[
u^- = \lim_{t \to -\infty} e^{-it\Delta} u(t)
\]

exists in \(\Sigma\), we say that \(u^-\) is the scattering state of \(u_0\) at \(-\infty\).

We observe that saying that \(u_0\) has a scattering state at \(\pm \infty\) is a way of saying that \(u(t)\) behaves as \(t \to \pm \infty\) like the solution \(e^{it\Delta} u^\pm\) of the linear Schrödinger equation. We set

\[
\mathcal{R}_+ = \{u_0 \in \Sigma : T_{\text{max}} = \infty \text{ and the limit (1.6) exists}\}
\]

and

\[
\mathcal{R}_- = \{u_0 \in \Sigma : T_{\text{min}} = \infty \text{ and the limit (1.7) exists}\},
\]

which denote the set of initial values \(u_0\) that have a scattering state at \(\pm \infty\).

Remark 1.1. We can see that changing \(t\) to \(-t\) in the equation (1.1) corresponds to changing \(u\) to \(\overline{u}\), which means changing \(u_0\) to \(\overline{u}_0\). So we have

\[
\mathcal{R}_- = \overline{\mathcal{R}_+} = \{u_0 \in \Sigma : \overline{u}_0 \in \mathcal{R}_+\}.
\]
The scattering theory for (1.1) in weighted space Σ has been quite extensively studied (see [2, 8, 9, 16, 17, 20, 21]). It is well-known that if $\alpha \leq \frac{2}{d}$, then no scattering theory can be developed for equation (1.1) (see [20]). In the defocusing case $\lambda < 0$, low energy scattering theory holds in Σ provided $4/(d + 2) < \alpha < 4/(d - 2)(2 < \alpha < \infty$, if $d = 1$). Moreover, if $\alpha \geq \alpha(d) = \frac{2 - d + \sqrt{d^2 + 12d + 1}}{2d}$, then scattering theory holds in whole Σ space (see [8, 16, 21]), and we notice that the asymptotic completeness for $\frac{2}{d} < \alpha < \alpha(d)$ was established recently in [18]. For the focusing case $\lambda > 0$, to our best knowledge, there is only a little positive answers, there is no low energy scattering if $\alpha < 4/(d + 2)$, but when $\alpha > 4/(d + 2)$, a low energy scattering theory holds in Σ (see [2]). If $\alpha \geq \frac{4}{d}$, some solutions will blow up in finite time.

In this paper we are mainly concerned with the scattering theory in Σ for focusing NLS, but we also study the convergence of a global solution $u(t)$ of (1.1) to the free solutions generated by its scattering states $u^\pm$ in Σ, our main results is Theorem 3.1, 5.1, 5.3, 5.4, 5.6, 5.7, 6.2 and Corollary 5.5.

The content of our paper can be mainly divided into four parts.

1. Cazenave and Weissler have proved in [2] that if $\alpha > \frac{4}{d+2}$ and $\|u_0\|_\Sigma$ is small, then scattering states $u^\pm$ exist in Σ at $\pm\infty$. They also have shown in [2] that if $\lambda > 0$, $\alpha < \frac{4}{d+2}$, then the scattering theory for small data in Σ fails, there are initial values $u_0 \in \Sigma$ with arbitrary small norm $\|u_0\|_\Sigma$ that do not have a scattering state, even in the sense of $L^2(\mathbb{R}^d)$. However, by applying the pseudo-conformal transformation and studying the resulting nonautonomous NLS, for $\lambda \in \mathbb{R} \setminus \{0\}$, $d \geq 3$, $u_0 \in \Sigma$, let $v_0 = e^{-\frac{|i|x|^2}{4}} u_0$, we show in Theorem 3.1 that if $\alpha > \alpha_d = \frac{d + \sqrt{d^2 + 12d + 1}}{2d}$ and $u_0$ is such that $\|v_0\|_{L^2}$ is small enough, then scattering states $u^\pm$ exist in Σ at $\pm\infty$. Note that $d \geq 3$, max {$\frac{2}{d}, \frac{4}{d+2}$} $< \alpha(d) < \frac{4}{d+2}$, thus Theorem 3.1 extends the scattering theory for small initial values to the range $\alpha_d < \alpha \leq \frac{4}{d+2}$.

2. Next, we consider the scattering theory for the focusing NLS with $\alpha \geq \alpha(d) = \frac{2 - d + \sqrt{d^2 + 12d + 1}}{2d}$, it's obvious that this problem is closely associated with the decay property of the solution $u(t)$. Cazenave and Weissler have proposed in [2] a notion of “positively rapidly decaying solutions” (i.e., a positively global solution $u$ of (1.1) which satisfies $\|u\|_{L^\alpha((0,\infty),L^{\alpha+2})} < \infty$, where $\alpha = \frac{2\alpha(\alpha + 2)}{4 - \alpha(d - 2)}$, $0 < \alpha < \frac{4}{d+2}$, and $0 < \alpha < \infty$, if $d = 1, 2$), and characterized the sets $\mathcal{R}_\pm$ in the case $\lambda > 0$, $\alpha > \alpha(d)$ in terms of rapidly decaying solutions (refer to [2], Theorem 4.12). But we can see that if $\alpha \leq \alpha(d)$, the rapidly decaying solutions should decay faster than the optimal rate $\tau \sim t^{-\frac{\alpha(d)}{2\alpha(d - 2)}}$ as $t \to +\infty$, that’s impossible unless $u \equiv 0$ (see [2], Proposition 3.15). Thus we give a refined definition of Rapidly Decaying Solution below:

**Definition 1.2.** Suppose $\alpha(d) \leq \alpha < \frac{4}{d+2}$ ($\alpha(d) \leq \alpha < \infty$, if $d = 1, 2$). A positively global solution $u$ of (1.1) is rapidly decaying if

\[ \|u\|_{L^\alpha((0,\infty),L^{\alpha+2})} < \infty, \quad for \quad \alpha = \alpha(d); \]
where $a = \frac{2\alpha(\alpha+2)}{4-\alpha(d-2)}$, and $\| \cdot \|_{L^{a,\infty}}$ denotes the weak $L^a$ norm. Correspondingly, we say a negatively global solution $u$ of (1.1) is rapidly decaying if it satisfies (1.10) or (1.11) with the time interval changed into $(-\infty, 0)$ respectively.

In Theorem 5.3 we show that if $\alpha = \alpha(d), d \neq 2$, the positively (resp. negatively) rapidly decaying solutions have scattering states at $+\infty$ (resp. $-\infty$), and we characterized the sets $\mathcal{R}_\pm$ for $\lambda > 0$ in terms of rapidly decaying solutions in Theorem 5.6. Our method is mainly based on the lower case $\alpha$ devoted to the scattering theory for focusing NLS in $\Sigma$.

Moreover, a byproduct is the sets $\mathcal{R}_\pm$ are unbounded subsets of $L^2(\mathbb{R}^d)$ (see Theorem 5.6).

3. We also investigate the scattering theory in $\Sigma$ under certain suitable assumptions on initial data $u_0$ (see Theorem 6.1, Theorem 6.4 and Theorem 6.7). Specially, note that for $\frac{4}{d-2} < \alpha < \frac{4}{d-2}$, there are a lot of literature devoted to the scattering theory for focusing NLS in $H^1(\mathbb{R}^d)$ for initial data $u_0 \in H^1(\mathbb{R}^d)$ below a mass-energy threshold and satisfying an mass-gradient bound (see Kenig and Merle [11], Killip and Visan [13] for the energy-critical case $\alpha = \frac{4}{d-2}$, [5] and [7] for the 3D cubic case, and [6] for the general energy-subcritical case). We will show in Theorem 6.7 that if $\lambda > 0, \frac{4}{d} < \alpha < \frac{4}{d-2} (\alpha \in (\frac{4}{d}, \infty)$, if $d = 1, 2$), assume $u_0 \in \Sigma$ below a mass-energy threshold $\int |u_0|^\sigma E[u_0] < \lambda^{-2\tau} M|Q|^\sigma E[Q]$ and satisfying an mass-gradient bound $\|u_0\|_{L^2}^\sigma \|\nabla u_0\|_{L^2} < \lambda^{-\tau} \|Q\|^\sigma_{L^2} \|\nabla Q\|_{L^2}$, then scattering states $u^\pm$ exist in $\Sigma$ at $\pm\infty$, where $\sigma = \frac{4-(d-2)\alpha}{ad-4}, \tau = \frac{2}{ad-4}$ and $Q$ is the ground state solution to $-\Delta Q + Q = |Q|^\alpha Q$.

4. Finally we study the asymptotic behavior of $\|u(t) - e^{it\Delta} u^\pm\|_{\Sigma}$ under the assumption $u^\pm$ exist at $\pm\infty$. In general, since $e^{it\Delta}$ is not an isometry of $\Sigma$, it is not known whether we can deduce $\|u(t) - e^{it\Delta} u^\pm\|_{\Sigma} \to 0$ from the scattering asymptotic property $\|e^{-it\Delta} u(t) - u^\pm\|_{\Sigma} \to 0$. A positive answer has been given by Bégout [11] for $d \leq 2, \alpha > \frac{4}{d}$, and $3 \leq d \leq 5, \alpha > \frac{8}{d+2}$. Our paper extends this result under certain suitable conditions on $u_0$. Theorem 6.2 will show that if we assume $u_0 \in \Sigma \cap W^{1,\rho'}$, then we
can obtain the convergence \( \| u(t) - e^{it\Delta} u^\pm \|_\Sigma \to 0 \) for \( 3 \leq d \leq 9, \alpha > \frac{16}{3d+2} \),

where \( \rho = \frac{4d}{2d-\alpha(d-2)} \).

The rest of the paper is organized as follows. In Section 2, we will give some preliminaries and notations. In Section 3 we will prove Theorem 3.1 for the small initial data scattering and in Section 4 we prove a lower bound estimate for the \( L^{\alpha+2} \) norm of singular solutions to the nonautonomous equation (1.12). Section 5 is devoted to some \( \Sigma \) scattering results for (1.1) in the focusing case with \( \alpha \geq \alpha(d) \) and Section 6 to the study on the asymptotic convergence of the scattering solution to a free solution in \( \Sigma \).

2. Notations and preliminaries

2.1. Some notations. Throughout this paper, we use the following notation. \( \bar{z} \) is the conjugate of the complex number \( z \), \( \Re z \) and \( \Im z \) are respectively the real and imaginary part of the complex number \( z \). All function spaces involved are spaces of complex valued functions. We denote by \( p' \) the conjugate of the exponent \( p \in [1, \infty] \) defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \), and \( L^p = L^p(\mathbb{R}^d) \) with norm \( \| \cdot \|_{L^p} ; H^s = H^s(\mathbb{R}^d) \) with norm \( \| \cdot \|_{H^s} \); and for all \( (f,g) \in L^2 \times L^2 \), the scalar product \( (f,g)_{L^2} = \Re \int f(x)g(x)dx \).

Let \( L^q(\mathbb{R}, L^r_x(\mathbb{R}^d)) \) denote the mixed Banach space with norm defined by

\[
\| u \|_{L^q(\mathbb{R}, L^r_x(\mathbb{R}^d))} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q},
\]

with the usual modifications when \( q \) or \( r \) is infinity, or when the domain \( \mathbb{R} \times \mathbb{R}^d \) is replaced by a smaller region of spacetime such as \( I \times \mathbb{R}^d \). In what follows positive constants will be denoted by \( C \) and will change from line to line. If necessary, by \( C(\ast, \cdots, \ast) \) we denote positive constants depending only on the quantities appearing in parentheses continuously.

We denote by \( (e^{it\Delta})_{t \in \mathbb{R}} \) the Schrödinger group, which is isometric on \( H^s \) and \( \dot{H}^s \) for every \( s \geq 0 \), and satisfies the Dispersive estimate and Strichartz’s estimates (for more details, see Keel and Tao [12]). We will use freely the well-known properties of the Schrödinger group \( (e^{it\Delta})_{t \in \mathbb{R}} \) (see e.g. Chapter 2 of [3] for an account of these properties). In convenience, we will introduce the definition of “admissible pair” below, which plays an important role in space-time estimates.

**Definition 2.1.** We say that a pair \( (q, r) \) is admissible if

\[
\frac{2}{q} = \delta(r) = d \left( \frac{1}{2} - \frac{1}{r} \right)
\]

and \( 2 \leq r \leq \frac{2d}{d-2} \) (\( 2 \leq r \leq \infty \) if \( d = 1 \), \( 2 \leq r < \infty \) if \( d = 2 \)). Note that if \( (q, r) \) is an admissible pair, then \( 2 \leq q \leq \infty \), the pair \((\infty, 2)\) is always admissible, and the pair \((2, \frac{2d}{d-2})\) is admissible if \( d \geq 3 \).
2.2. **Generalized Hölder’s and Young’s inequality in Lorentz spaces.**

Our paper involves estimates in the general Lorentz spaces $L^{p,q}(0 < p < \infty, 0 < q \leq \infty)$ equipped with the norm

$$
\|f\|_{L^{p,q}(X,\mu)} = p^{1/q} \|\lambda \mu(\{|f| \geq \lambda\})^{1/p}\|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}
$$

(refer to [19] for a review). A special case is the weak $L^p$ space $L^{p,\infty}(0 < p < \infty)$ with norm defined by $\|f\|_{L^{p,\infty}(X,\mu)} = \sup_{\lambda > 0} \lambda \mu(\{|f| \geq \lambda\})^{1/p}$. An useful formula is for any $0 < p, r < \infty$ and $0 < q \leq \infty$,

$$
\|f\|_{L^{p,q}(X,\mu)} \leq \|f\|_{L^{p,r}(X,\mu)} \|f\|_{L^{r,q}(X,\mu)}.
$$

If $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a monotone non-increasing function, we have

$$
\|f\|_{L^{p,q}(X,\mu)} = \|f(t)^{1/p}\|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}.
$$

Below we give the refined Hölder’s and Young’s inequality for Lorentz spaces $L^{p,q}$, due to O’Neil (see [19] and [15] for the proof).

**Lemma 2.2.** If $0 < p_1, p_2, p < \infty$ and $0 < q_1, q_2, q \leq \infty$ obey $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then

$$
\|fg\|_{L^{p,q}(X,\mu)} \leq C \|f\|_{L^{p_1,q_1}(X,\mu)} \|g\|_{L^{p_2,q_2}(X,\mu)}.
$$

If $1 < p_1, p_2, p < \infty$ and $1 \leq q_1, q_2, q \leq \infty$ obey $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then

$$
\|f \ast g\|_{L^{p,q}(X,\mu)} \leq C \|f\|_{L^{p_1,q_1}(X,\mu)} \|g\|_{L^{p_2,q_2}(X,\mu)}.
$$

If we use (2.4) with $q = q_2 = 2$ and $q_1 = \infty$ instead of the Hardy-Littlewood-Sobolev inequality, we obtain the following Strichartz’s estimate in Lorentz spaces.

**Lemma 2.3.** we have the following properties:

$$
\|e^{i\Delta} \varphi\|_{L^{q,2}(\mathbb{R}, L^r)} \leq C \|\varphi\|_{L^2} \text{ for every } \varphi \in L^2(\mathbb{R}^d);
$$

$$
\|\int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau\|_{L^{q,2}(\mathbb{R}, L^r)} \cap L^{\infty}(I, L^2) \leq C \|f\|_{L^{q',2}(I, L^{r'}(\mathbb{R}^d))},
$$

for every $f \in L^{q',2}(I, L^{r'}(\mathbb{R}^d))$ and some constant $C$ independent of $I$, where $(q,r)$ is an admissible pair, $2 < q < \infty$, $I$ is an interval of $\mathbb{R}$ such that $0 \in I$.

2.3. **Properties of the operator $P_t = x + 2it\nabla$.** Let $P_t$ be the partial differential operator on $\mathbb{R}^{d+1}$ defined by $P_t u(t,x) = (x + 2it\nabla) u(t,x)$. Operator $P_t$ has the following important commutative properties:

$$
\left[ P_t, i\partial_t + \Delta \right] = 0,
$$

$$
P_t e^{it\Delta} = e^{it\Delta} x, \quad e^{-it\Delta} P_t = xe^{-it\Delta},
$$

Where $[\cdot, \cdot]$ is the commutator bracket. An easy calculation shows that if $t \neq 0$, then

$$
P_t u = (x + 2it\nabla) u = 2it e^{\frac{|x|^2}{4t}} \nabla(e^{-\frac{|x|^2}{4t}} u),
$$

for every $u \in L^2(\mathbb{R}^d)$. The operator $P_t$ is a smoothing operator, i.e., it maps a function $u \in L^2(\mathbb{R}^d)$ to a function $P_t u \in C(\mathbb{R}^d)$, where $C(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$. Moreover, $P_t$ is a contraction in $L^2(\mathbb{R}^d)$, meaning that for every $u \in L^2(\mathbb{R}^d)$,

$$
\|P_t u\|_{L^2} \leq \|u\|_{L^2}.
$$

This property follows from the fact that $P_t$ is a smoothing operator. It can be shown that $P_t$ is also a Fourier multiplier with symbol $e^{it\xi^2}$, where $\xi$ is the Fourier variable. This property is used in the proof of Strichartz’s estimate.
and so
\[ \| (x + 2it\nabla)u \|_{L^2}^2 = 4t^2 \| \nabla (e^{-i|x|^2}u) \|_{L^2}^2. \]

Let \( v(t, x) = e^{-i|x|^2}u(t, x) \), it follows from (2.7) that
\[ |P_t(\|u\|_p^p) v | = 2t \| \nabla (e^{-i|x|^2} |u|^{\alpha} u) \| = 2t \| \nabla (|v|^{\alpha} v) \|. \]

From the above identity, (2.7) and H"older's inequality, it follows that for any \( 1 \leq p, q, r \leq \infty \) such that \( \frac{1}{r} = \frac{p}{b} + \frac{1}{q} \),
\[ (2.8) \quad \| P_t(\|u\|_p^p) v | \|_{L^r} \leq C \|t\| \|v\|_{L^q} \| \nabla v \|_{L^r} \leq C \|u\|_{L^r} \|P_t u\|_{L^q}. \]

2.4. Applications of the Pseudo-conformal Transformation. We will investigate the scattering problem for (1.1) by applying the pseudo-conformal transformation (see Chapter 7 in [3] for a review). By Remark 1.1, we can mainly concern about positively global solution \( u(t) \) defined on \((0, +\infty)\), the scattering problem for \( t \to -\infty \) can be treated similarly.

we consider the variables \((t, x) \in \mathbb{R} \times \mathbb{R}^d\) defined by
\[ (2.9) \quad t = \frac{s}{1-s}, \quad x = \frac{y}{1-s}, \quad \text{or equivalently,} \quad s = \frac{t}{1+t}, \quad y = \frac{x}{1+t}. \]

Given \( 0 \leq a < b \leq \infty \) and \( u \) defined on \((a, b) \times \mathbb{R}^d\), we set
\[ (2.10) \quad v(s, y) = (1-s)^{-\frac{d}{2}} u\left( \frac{s}{1-s}, \frac{y}{1-s} \right) e^{-i|\frac{y}{1-s}|^2} = (1+t)^{\frac{d}{2}} u(t, x) e^{-i|\frac{y}{1+t}|^2} \]

for \( y \in \mathbb{R}^d \) and \( \frac{a}{1+a} < s < \frac{b}{1+b} \). In particular, if \( u \) is defined on \((0, \infty)\), then \( v \) is defined on \((0, 1)\). One easily verifies that \( u \in C([a, b], \Sigma) \) if and only if \( v \in C([\frac{a}{1+a}, \frac{b}{1+b}], \Sigma)(0 \leq a < b < \infty \) are given).

Furthermore, a straightforward calculation shows that \( u \) satisfies (1.1) on \((a, b)\) if and only if \( v \) satisfies the nonautonomous Cauchy problem
\[ (2.11) \quad \left\{ \begin{array}{l} i\partial_s v + \Delta_y v + \lambda(1-s) \frac{a d-4}{2} |v|^{\alpha} v = 0, \quad s > 0, \quad y \in \mathbb{R}^d \\ v(0, y) = v_0(y) = u_0(x) e^{-i|\frac{y}{1+b}|^2} \in \Sigma, \quad y \in \mathbb{R}^d \end{array} \right. \]
on the interval \((\frac{a}{1+a}, \frac{b}{1+b})\). Note that the term \((1-s)^{\frac{ad-4}{2}} \) is regular, except possibly at \( t = 1 \), where it is singular for \( \alpha < \frac{4}{d} \). Moreover, the following identities hold:
\[ (2.12) \quad \| v(s) \|_{L_{d+2}^{\beta+2}}^{\beta+2} = (1+t)^{\frac{d}{2}} \| u(t) \|_{L_{d+2}^{\beta+2}}^{\beta+2}, \quad \text{for} \quad \beta \geq 0, \]
\[ (2.13) \quad \| \nabla v(s) \|_{L^2}^2 = \frac{1}{4} \| (x + 2i(1+t)\nabla)u(t) \|_{L^2}^2, \]
\[ (2.14) \quad \| \nabla u(t) \|_{L^2}^2 = \frac{1}{4} \| (y - 2i(1-s)\nabla)v(s) \|_{L^2}^2. \]

Hence if we set
\[ E_1(s) = \frac{1}{2} \| \nabla v(s) \|_{L^2}^2 - (1-s)^{\frac{ad-4}{2}} \lambda \frac{\lambda}{\alpha+2} \| v(s) \|_{L^{\alpha+2}}, \]

and so
\[ \| (x + 2it\nabla)u \|_{L^2}^2 = 4t^2 \| \nabla (e^{-i|x|^2}u) \|_{L^2}^2. \]
\[ E_2(s) = (1 - s)^{\frac{d-\alpha d}{2}} \frac{1}{2} \| \nabla v(s) \|_{L^2}^2 - \frac{\lambda}{\alpha + 2} \| v(s) \|_{L^{\alpha+2}}^{\alpha+2}, \]

it follows from the the pseudo-conformal conservation law for (1.1) that

\[ \frac{d}{ds} E_1(s) = -(1 - s)^{\frac{\alpha d - 4}{2}} \frac{\lambda}{\alpha + 2} \| v(s) \|_{L^{\alpha+2}}^{\alpha+2}, \]

\[ \frac{d}{ds} E_2(s) = (1 - s)^{\frac{2 - \alpha d - 4}{2}} \| \nabla v(s) \|_{L^2}^2. \]

By applying the fixed point theorem in an appropriate space to the following equivalent integral equation for the nonautonomous equation (2.11)

\[ v(s) = e^{is\Delta} v_0 + i\lambda \int_0^s e^{i(s-\tau)\Delta} (1 - \tau)^{\frac{\alpha d - 4}{2}} \| v(\tau) \|_{L^{\alpha+2}}^{\alpha+2} d\tau, \]

Cazenave and Weissler obtained the following local well-posedness result for (2.11) in [2] (see [3] or [2], Theorem 3.4 for the proof).

**Theorem 2.4.** Assume that \( \lambda \in \mathbb{R}, \)

\[ \frac{4}{d + 2} < \alpha < \frac{4}{d - 2} \quad (2 < \alpha < \infty, \text{ if } d = 1). \]

It follows that for every \( s_0 \in \mathbb{R} \) and \( \psi \in \Sigma, \) there exist \( S_m(s_0, \psi) < S_M(s_0, \psi) \) and a unique, maximal solution \( v \in C([S_m, S_M], \Sigma) \) of equation (2.11). The solution \( v \) is maximal in the sense that if \( S_M < \infty \) (respectively, \( S_m > -\infty \)), then \( \| v(s) \|_{H^1} \to \infty \) as \( s \uparrow S_M \) (respectively, \( s \downarrow S_m \)). In addition, if \( S_M = 1 \), then \( \liminf_{s \uparrow 1} \{ (1 - s)^\delta \| v(s) \|_{H^1} \} > 0 \) with \( \delta = \frac{d+2}{4} - \frac{1}{\alpha} \) if \( d \geq 3 \), \( \delta < 1 - \frac{1}{\alpha} \) if \( d = 2 \), and \( \delta = \frac{1}{2} - \frac{1}{\alpha} \) if \( d = 1 \).

The following useful observation indicates the inseparable relationship between the asymptotic behavior of \( u(t) \) as \( t \to +\infty \) and \( v(s) \) as \( s \to 1 \)(refer to [2], Proposition 3.14 for the proof).

**Proposition 2.5.** Let \( u \in C([0, \infty), \Sigma) \) be a solution of equation (1.1) and let \( v \in C([0, 1), \Sigma) \) be the corresponding solution of (2.11) defined by (2.10). It follows that \( e^{-it\Delta} u(t) \) has a strong limit in \( \Sigma \) as \( t \to +\infty \) if and only if \( v(s) \) has a strong limit in \( \Sigma \) as \( s \uparrow 1 \), in which case

\[ \lim_{t \to +\infty} e^{-it\Delta} u(t) = e^{i\frac{\alpha d}{2}} e^{-i\Delta} v(1) \quad \text{in } \Sigma. \]

3. SCATTERING FOR SMALL INITIAL DATA

In this section we consider the scattering theory in \( \Sigma \) for small initial values \( u_0 \) under the assumption that \( \lambda \in \mathbb{R} \setminus \{0\}, \alpha \leq \frac{4}{d+2}. \) Note that if \( \lambda > 0, \alpha < \frac{4}{d+2}, \) there are initial values \( u_0 \in \Sigma \) with arbitrary small norm \( \| u_0 \|_{\Sigma} \) that do not have a scattering state, even in the sense of \( L^2(\mathbb{R}^d) \)(see [2]). We obtain the following result.
Theorem 3.1. Assume $d \geq 3$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\alpha_{d} < \alpha < \frac{4}{d-2}$, where $\alpha_{d} = \frac{-d + \sqrt{d^2 + 16d}}{2d}$. Then there exists $\varepsilon_{0} > 0$ with the following property. Let $u_{0} \in H^{2} \cap \mathcal{F}(H^{2}) \subset \Sigma$, $v_{0} = e^{-i\frac{|x|^{2}}{4}}u_{0}$ and let $u$ be the corresponding maximal solution of (1.1). If $\|v_{0}\|_{H^{2}} \leq \varepsilon_{0}$ (assuming further $\|u_{0}\|_{H^{1}} \leq \varepsilon_{0}$ when $\lambda > 0$, $\alpha \geq \frac{4}{d}$), then the solution $u$ is global and scatters as $t \to \pm \infty$.

Proof. For the prove of the scattering properties, we deal with only the positive time $t \to +\infty$, since $t \to -\infty$ can be treated in the same way. Under the assumptions of Theorem 3.1 it is well known that there exists $\varepsilon_{0} > 0$ such that the solution $u$ of Cauchy problem (1.1) is global and bounded in $H^{1}(\mathbb{R}^{d})$, moreover, $u \in C(\mathbb{R}, \Sigma)$. Thus we have $v(s, y)$ (the Pseudo-conformal Transformation of $u(t, x)$, see Section 2 for a review) defined by (2.10) satisfies the following nonautonomous integral equation

\begin{equation}
(3.1)
v(s) = e^{i\lambda}\Delta v_{0} + i\lambda \int_{0}^{s} e^{i(s-\tau)\Delta} (1 - \tau) \frac{\alpha_{d} - 4}{2 \tau} |v(\tau)|^{\alpha} v(\tau) d\tau
\end{equation}

on the interval $(0, 1)$, and $v \in C([0, 1], \Sigma)$.

Let $(\gamma, \rho)$ be the admissible pair defined by

\begin{equation}
(3.2)\gamma = \frac{4(\alpha + 2)}{\alpha(d - 2)}, \quad \rho = \frac{d(\alpha + 2)}{d + \alpha},
\end{equation}

and index $\rho^{\ast} = \frac{d(\alpha + 2)}{d - 2}$. One easily verifies that $\frac{1}{\rho} = \frac{\alpha}{\rho} + \frac{1}{\rho}$ and $L^{\rho^{\ast}}(\mathbb{R}^{d}) \hookrightarrow W^{1, \rho}(\mathbb{R}^{d})$. Therefore, by applying the dispersive estimates (see [2]) and Hölder’s inequality to the integral equation (3.1), we have

\begin{align*}
\|\nabla v(s)\|_{L^{\rho}} & \leq \|\nabla(e^{i\lambda}\Delta v_{0})\|_{L^{\rho}} + C \int_{0}^{s} (1 - \tau)^{\frac{\alpha_{d} - 4}{2 \tau}} (s - \tau)^{-\frac{\rho}{2}} \|v(\tau)\|_{L^{\rho^{\ast}}}^{\alpha} \|\nabla v(\tau)\|_{L^{\rho}} d\tau \\
& \leq C \|v_{0}\|_{H^{2}} + C \int_{0}^{s} (1 - \tau)^{\frac{\alpha_{d} - 4}{2 \tau}} (s - \tau)^{-\frac{\rho}{2}} \|\nabla v(\tau)\|_{L^{\rho^{\ast}}}^{\alpha + 1} d\tau.
\end{align*}

Note that $\alpha > \alpha_{d}$, we have $\frac{4 - \alpha_{d}}{2} + \frac{2}{\gamma} < 1$, thus we can deduce from Hölder’s inequality that

\begin{equation}
(3.3)\int_{0}^{s} (1 - \tau)^{\frac{\alpha_{d} - 4}{2 \tau}} (s - \tau)^{-\frac{\rho}{2}} d\tau \leq C(\alpha, d).
\end{equation}

Set $\Theta(s) = \sup_{\tau \in [0, s]} \|\nabla v(\tau)\|_{L^{\rho}}$, for $0 < s < 1$. Then we can deduce from the above two estimates immediately that

\begin{equation}
(3.4)\Theta(s) \leq C \|v_{0}\|_{H^{2}} + C \Theta(s)^{\alpha + 1} \text{ for all } 0 < s < 1.
\end{equation}

Note that $u_{0} \in H^{2} \cap \mathcal{F}(H^{2})$, one easily verifies that $v_{0} \in H^{2}$ and $v \in C([0, 1), H^{2})$, thus we have $\Theta \in C([0, 1))$ and

\begin{equation}
(3.5)\lim_{s \to 0} \Theta(s) = \|\nabla v_{0}\|_{L^{\rho}} \leq C \|v_{0}\|_{H^{2}}.
\end{equation}

Applying (3.5), we deduce easily that if $\|v_{0}\|_{H^{2}} \leq \varepsilon_{0}$ where $\varepsilon_{0} > 0$ is sufficiently small so that $(2C\varepsilon_{0})^{\alpha + 1} < \varepsilon_{0}$, then

\begin{equation}
(3.9)\Theta(s) \leq 2C \|v_{0}\|_{H^{2}} \text{ for all } 0 < s < 1.
\end{equation}
Letting $s \uparrow 1$, we deduce in particular that

$$\sup_{s \in (0, 1)} \|v(s)\|_{L^{\rho^*}} \leq C \sup_{s \in (0, 1)} \|\nabla v(s)\|_{L^{\rho}} < \infty. \quad (3.6)$$

Therefore we deduce from identity (2.12) the following decay estimate for $u(t, x)$:

$$\|u(t)\|_{L^{\rho^*}} \leq C (1 + t)^{-d \frac{1}{2} - \frac{1}{\rho^*}} \text{ for all } t \geq 0. \quad (3.7)$$

Therefore, it follows from Strichartz’s estimates that for every $t \geq T \geq 0$,

$$\|u\|_{L^\gamma((0, t), W^{1, \rho})} \leq C\|u_0\|_{H^1} + C \left( \int_0^T \|u(\tau)\|_{L^{\rho^*}}^\gamma \left( \int_0^\tau \|u(\tau')\|_{L^{\rho^*}}^\gamma \, d\tau' \right)^\frac{\gamma}{\gamma - 2} \, d\tau \right) \|u\|_{L^\gamma((0, T), W^{1, \rho})}$$

$$+ C \left( \int_T^t \|u(\tau)\|_{L^{\rho^*}}^\gamma \left( \int_0^\tau \|u(\tau')\|_{L^{\rho^*}}^\gamma \, d\tau' \right)^\frac{\gamma}{\gamma - 2} \, d\tau \right) \|u\|_{L^\gamma((T, t), W^{1, \rho})}. \quad (3.8)$$

Using (3.7), we get

$$\|u(\tau)\|_{L^{\rho^*}}^\gamma \leq C (1 + \tau)^{-\frac{\alpha d - 8}{2(\gamma - 2)}}. \quad (3.9)$$

Note that since $\alpha > \alpha_d = \frac{d + \sqrt{d^2 + 16d}}{2d}$, we have $\alpha d \gamma - 8 > 2(\gamma - 2)$. Therefore for $T$ large enough,

$$C \left( \int_T^t \|u(\tau)\|_{L^{\rho^*}}^\gamma \left( \int_0^\tau \|u(\tau')\|_{L^{\rho^*}}^\gamma \, d\tau' \right)^\frac{\gamma}{\gamma - 2} \, d\tau \right) \leq \frac{1}{2}. \quad (3.10)$$

On the other hand, $u \in L^\infty((0, T), H^1(\mathbb{R}^d)) \cap L^q((0, T), W^{1, \rho'}(\mathbb{R}^d))$. Therefore, it follows from (3.3) and (3.3) that

$$\|u\|_{L^\gamma((0, \infty), W^{1, \rho})} \leq C + \frac{1}{2} \|u\|_{L^\gamma((0, \infty), W^{1, \rho})}. \quad (3.11)$$

Letting $t \uparrow \infty$, we obtain

$$\|u\|_{L^\gamma((0, \infty), W^{1, \rho})} \leq 2C. \quad (3.12)$$

This also implies that $|u|^\alpha u \in L^\gamma'((0, \infty), W^{1, \rho'}(\mathbb{R}^d))$. Applying again Strichartz’s estimates, one obtains the result for every admissible pair. Let $P_t u = (x + 2it \nabla)u$, by applying Strichartz’s estimates, we obtain

$$\|P_t u\|_{L^\gamma((0, t), L^\rho)} \leq C \|x u_0\|_{L^2} + C \left( \int_0^T \|u(\tau)\|_{L^{\rho^*}}^\gamma \left( \int_0^\tau \|u(\tau')\|_{L^{\rho^*}}^\gamma \, d\tau' \right)^\frac{\gamma}{\gamma - 2} \, d\tau \right) \|P_t u\|_{L^\gamma((0, T), L^\rho)}$$

$$+ C \left( \int_T^t \|u(\tau)\|_{L^{\rho^*}}^\gamma \left( \int_0^\tau \|u(\tau')\|_{L^{\rho^*}}^\gamma \, d\tau' \right)^\frac{\gamma}{\gamma - 2} \, d\tau \right) \|P_t u\|_{L^\gamma((T, t), L^\rho)}. \quad (3.13)$$

for every $0 \leq T \leq t$. Then one concludes similarly as above that

$$\|(x + 2it \nabla)u\|_{L^\gamma((0, \infty), L^\rho)} \leq 2C, \quad (3.14)$$

and $P(|u|^\alpha u) \in L^\gamma'((0, \infty), L^{\rho'}(\mathbb{R}^d))$. Therefore for $0 < t < s$, by Strichartz’s estimates, we have

$$\|e^{-it\Delta} u(t) - e^{-is\Delta} u(s)\|_{H^1} \leq C \|u|^\alpha u\|_{L^\gamma'((t, s), W^{1, \rho'})}, \quad (3.15)$$

$$\|x(e^{-it\Delta} u(t) - e^{-is\Delta} u(s))\|_{L^2} \leq C \|(x + 2it \nabla)|u|^\alpha u\|_{L^\gamma'((t, s), L^{\rho'})}. \quad (3.16)$$
Thus, we get immediately
\[ \|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{H^1} \to 0, \]
\[ \|x(e^{-it\Delta}u(t) - e^{-is\Delta}u(s))\|_{L^2} \to 0, \]
as \( t, s \to \infty \). Hence there exists \( u^+ \in \Sigma \) such that \( e^{-it\Delta}u(t) \to u^+ \) in \( \Sigma \) as \( t \to \infty \).

**Remark 3.2.** Since \( \|v_0\|_{H^2} \leq C\|u_0\|_{H^2 \cap F(H^2)} \), the smallness assumptions in Theorem 3.1 can be deduced from assuming that \( \|u_0\|_{H^2 \cap F(H^2)} \) is small.

Note that \( \max\{\frac{2}{d}, \frac{4}{d+2}\} < \alpha_d < \frac{4}{d+2} \), we claim that for \( \lambda > 0 \), there exists a constant \( K > 0 \), such that for any \( C_0 \geq K \), there exists an initial data \( u_0 \) satisfying \( \|u_0\|_{H^2 \cap F(H^2)} = C_0 \), which do not have a scattering state, even in the sense of \( L^2(\mathbb{R}^d) \). To see this, let \( \varphi \in \Sigma \) be a nontrivial solution of the equation
\[ -\Delta \varphi + \varphi = \lambda |\varphi|^\alpha \varphi. \]
Given \( \omega > 0 \), set \( \varphi_\omega(x) = \omega^{\frac{1}{2}} \varphi(x/\omega) \). It follows that \( u_\omega(t, x) = e^{i\omega t} \varphi_\omega(x) \) satisfies (1.1) and does not have any strong limit as \( t \to \pm \infty \) in \( L^2(\mathbb{R}^d) \).

One easily verifies that if \( \alpha > \alpha_d \), then \( \|\varphi_\omega\|_{H^2 \cap F(H^2)} \to \infty \) as \( \omega \to 0 \) and \( \omega \to \infty \), which indicates that \( K = \inf_{\omega \in (0, \infty)} \|\varphi_\omega\|_{H^2 \cap F(H^2)} \) is attained. This proves our claim. Furthermore, if \( \alpha < \frac{4}{d+2} \), since \( \|\varphi_\omega\|_{H^2 \cap F(H^2)} \to 0 \) as \( \omega \to 0 \), there are arbitrary small initial values \( u_0 \in H^2 \cap F(H^2) \) that do not have a scattering state.

4. LOWER BOUND ESTIMATES FOR THE SINGULAR SOLUTIONS OF NONAUTONOMOUS EQUATION

In this section we will present a lower bound estimates for the \( L^{\alpha+2} \) norm of the global solutions \( u(t, x) \) to Cauchy problem (1.1) that do not scatter as \( t \to \pm \infty \) in the case \( \lambda > 0 \). Therefore, we could deduce from these results that the global solution to (1.1) with fast decay must scatter at \( \pm \infty \), we will apply the following Propositions in this Section to the investigation on “rapidly decaying solutions” in Section 5.

**Lemma 4.1.** Assume \( \lambda \in \mathbb{R} \setminus \{0\} \), \( \frac{4}{d+2} < \alpha < \frac{4}{d+2} \) (2 < \( \alpha < \infty \), if \( d = 1 \)). Let \( v_0 \in \Sigma \) and \( v \in C((S_m, S_M), \Sigma) \) be the corresponding maximal solution of (2.11) given by Theorem 2.4. Then if \( S_M(0, v_0) = 1 \), we have the following lower estimate of the blowup rate
\[ \|\nabla v(s)\|_{L^2}^2 \geq C(1 - s)^{-2\theta} \]
for some constant \( C > 0 \) and all \( s \in [0, 1) \), where \( \theta = \frac{d+2}{4} - \frac{1}{\alpha} \) if \( d \geq 3 \), \( \theta = 1 - \frac{2}{\alpha} \) if \( d = 2 \), and \( \theta = 1 - \frac{1}{\alpha} \) if \( d = 1 \).

**Proof.** By Theorem 2.4, Lemma 4.1 holds for \( d \geq 2 \), therefore we need only consider \( d = 1 \). Fix \( s_0 \in [0, 1) \). Let \( f(s) = \lambda(1 - s)^{\frac{d}{2}} \), for \( 0 \leq s < 1 \). It follows from equation (2.11) and Strichartz’s estimates that
\[ \|v\|_{L^\infty((0, s), H^1)} \leq C\|v(s_0)\|_{H^1} + C\|f|v|^\alpha v\|_{L^1((0, s), H^1)} \]
for all \( s \in (s_0, 1) \). On the other hand,

\[
\|v\|_{H^1}^2 \leq C \|v\|_{L^\infty}^2 \|v\|_{H^1}
\]

and, by Gagliardo-Nirenberg’s inequality,

\[
\|v\|_{L^\infty} \leq C \|v\|_{H^1}^{1/2} \|v\|_{L^2}^{1/2} \leq C \|v\|_{H^1}^{1/2}.
\]

Therefore, we deduce from the above three inequalities \[(4.2), (4.3)\] and \[(4.4)\] that there exists a constant \( K > 0 \) independent of \( s_0 \) and \( s \) such that

\[
\|v\|_{L^\infty((s_0, s), H^1)} \leq K \|v(s_0)\|_{H^1} + K \|f\|_{L^1(s_0, s)} \|v\|_{L^\infty((s_0, s), H^1)}^{1 + \frac{\alpha}{2}}.
\]

Now, since by Theorem \( 2.4 \) we have

\[
\limsup_{s \uparrow s_1} \|v(s)\|_{H^1} = \infty,
\]

there exists \( s_1 \in (s_0, 1) \) such that \( \|v\|_{L^\infty((s_0, s_1), H^1)} = (K + 1)\|v(s_0)\|_{H^1} \).

Letting \( s = s_1 \) in \( (4.5) \), we obtain

\[
\|v(s_0)\|_{H^1} \leq K((K + 1)\|v(s_0)\|_{H^1})^{1 + \frac{\alpha}{2}} \|f\|_{L^1(s_0, s_1)},
\]

hence

\[
1 \leq CK(K + 1)^{1 + \frac{\alpha}{2}} \|v(s_0)\|_{H^1} \|v(s_0)\|_{H^1}^{\alpha - 2}.
\]

Since \( s_0 \in [0, 1) \) is arbitrary, we obtain for \( d = 1 \) that

\[
\|\nabla v(s)\|_{L^2} \geq C(1 - s)^{-\frac{\alpha - 2}{\alpha}},
\]

for some constant \( C > 0 \) and all \( s \in [0, 1) \). This closes our proof. \( \square \)

**Proposition 4.2.** Assume \( \lambda > 0 \), \( \frac{4}{d+2} < \alpha \leq \frac{4}{d}(2 < \alpha \leq 4 \), if \( d = 1 \). Let \( u_0 \in \Sigma \) and \( u \) be the corresponding maximal solution of \( (1.1) \), then if \( u \) is positively(resp. negatively) global and doesn’t scatter at \( +\infty \)(resp. \( -\infty \)), we have

\[
\|u(t)\|_{L^{\alpha+2}} \geq C(1 + |t|)^{-\frac{2(1-\theta)}{\alpha + 2}},
\]

for all \( t \in (0, +\infty) \)(resp. \( t \in (-\infty, 0) \), where \( \theta \) is defined the same as in Lemma \( 4.4 \). Moreover, for \( d \geq 3 \), \( \alpha = \alpha(d) = \frac{2 - d + \sqrt{d^2 + 12d + 4}}{2d} \), we can derive a better lower estimate

\[
\|u(t)\|_{L^{\alpha+2}} \geq C(1 + |t|)^{-\frac{\alpha d}{2(\alpha + 2)} \log(1 + |t|)}^{\frac{1}{\alpha + 2}},
\]

for all \( t \in (0, +\infty) \)(resp. \( t \in (-\infty, 0) \)).

**Proof.** We will only deal with the positive time, since \( (-\infty, 0) \) can be treated similarly. By Proposition \( 2.5 \) we deduce from the assumption positively global solution \( u \) doesn’t have scattering state \( u^+ \) at \( +\infty \) that the nonautonomous equation \( (2.11) \) blows up at \( s = 1 \)(i.e., \( S_M(0, v_0) = 1 \)). Hence by Theorem \( 2.4 \) one has

\[
\limsup_{s \uparrow s_1} \|v(s)\|_{H^1} = \infty.
\]
Furthermore, by Lemma 4.1, we get
\begin{equation}
\|\nabla v(s)\|_{L^2}^2 \geq C(1-s)^{-2\theta}
\end{equation}
for some constant $C > 0$ and all $s \in [0, 1)$. Note that $\lambda > 0$, $\alpha \leq \frac{4}{d}$, we deduce from (2.15) that \( \frac{d}{ds}E_1(s) \leq 0 \), and so
\begin{equation}
\frac{1}{2}\|\nabla v(s)\|_{L^2}^2 \leq E_1(0) + \frac{\lambda}{\alpha + 2}(1-s)^{\frac{\alpha d - 4}{2}}\|v(s)\|_{L^{\alpha+2}}^{\alpha + 2}.
\end{equation}
Note that $\lambda > 0$, $\frac{4}{d+2} < \alpha \leq \frac{4}{d} (2 < \alpha \leq 4$, if $d = 1)$, from (4.10) and (4.11) we infer that
\begin{equation}
\|v(s)\|_{L^{\alpha+2}}^{\alpha + 2} \geq C(1-s)^{-\frac{4}{d} + \frac{4 - 2d}{2}} \|\nabla v(s)\|_{L^2}^2 \geq C(1-s)^{-\frac{4}{d} + \frac{4 - 2d}{2}} \|\nabla v(s)\|_{L^2}^2 \geq C(1-s)^{-\frac{4}{d} + \frac{4 - 2d}{2}}
\end{equation}
for some constant $C > 0$ and all $s \in [0, 1)$. Thus it follows from identity (2.9) and (2.12) that
\begin{equation}
\|u(t)\|_{L^{\alpha+2}} \geq C(1 + t)^{-\frac{2(1 - \theta)}{\alpha + 2}} \text{ for all } t \in [0, \infty).
\end{equation}
This indicates that the lower estimate (4.7) holds for positive time $(0, +\infty)$. Furthermore, if $d \geq 3$, $\alpha = \alpha(d)$, we deduce from (2.16) and (4.10) that
\begin{equation}
\frac{\lambda}{\alpha + 2}\|v(s)\|_{L^{\alpha+2}}^{\alpha + 2} + E_2(0) = \frac{1}{2}(1-s)^{\frac{4 - 2d}{2}}\|\nabla v(s)\|_{L^2}^2 + \frac{4 - \alpha d}{4}\int_0^s (1-\tau)^{\frac{2 - 2d}{2}}\|\nabla v(\tau)\|_{L^2}^2 d\tau 
\end{equation}
and
\begin{equation}
\geq C + C \int_0^s (1-\tau)^{-1}d\tau \geq C + C \log(\frac{1}{1-s}),
\end{equation}
from which we get immediately
\begin{equation}
\|v(s)\|_{L^{\alpha+2}}^{\alpha + 2} \geq C \log(\frac{1}{1-s})
\end{equation}
for some constant $C > 0$ and all $s \in [0, 1)$. Thus it follows from identity (2.9) and (2.12) that
\begin{equation}
\|u(t)\|_{L^{\alpha+2}} \geq C(1 + t)^{-\frac{2(1 - \theta)}{\alpha + 2}} \log(1 + t)\frac{1}{\alpha + 2}
\end{equation}
for all $t \in [0, \infty)$. This closes the proof of Proposition 4.2 for positive time $(0, +\infty)$, and the arguments for $(-\infty, 0)$ being similar. \(\square\)

When $\alpha > \frac{4}{d}$, there is a lower estimate of integral form.

**Proposition 4.3.** Assume $\lambda > 0$, $\frac{4}{d} < \alpha < \frac{4}{d-2}(\frac{4}{d} < \alpha < \infty$, if $d = 1, 2$). Let $u_0 \in \Sigma$ and $u$ be the corresponding maximal solution of (1.1), then if $u$ is positively(resp. negatively) global and doesn’t scatter at $+\infty$ (resp. $-\infty$), then there exists a $t_0 > 0$ such that
\begin{equation}
\left| \int_0^t (1 + |\tau|)\|u(\tau)\|_{L^{\alpha+2}}^{\alpha + 2} d\tau \right| \geq C(1 + |t|)^{2\theta},
\end{equation}
for all $t \in [t_0, +\infty)$ (resp. $t \in (-\infty, -t_0]$), where $\theta$ is defined the same as in Lemma 4.1.
Moreover, by Lemma 4.1, we get
\[ \limsup_{s \uparrow 1} \|v(s)\|_{H^1} = \infty. \]

Moreover, by Lemma 4.1, we get
\[ \|\nabla v(s)\|^2_{L^2} \geq C(1 - s)^{-2\theta} \]
for some constant \( C > 0 \) and all \( s \in [0, 1) \). From (2.15) we can deduce the following identity
\[ \frac{1}{2} \|\nabla v(s)\|^2_{L^2} = E_1(0) + \frac{\lambda}{\alpha + 2} (1 - s)^{\frac{\alpha + d}{2}} \|v(s)\|_{L^{\alpha + 2}}^{\alpha + 2} \]
\[ + \frac{\lambda(\alpha d - 4)}{2(\alpha + 2)} \int_{0}^{s} (1 - \tau)^{\frac{\alpha d - 2}{2}} \|v(\tau)\|_{L^{\alpha + 2}}^{\alpha + 2} d\tau. \]
Let \( h(s) = \frac{\lambda}{\alpha + 2} \int_{0}^{s} (1 - \tau)^{\frac{\alpha d - 2}{2}} \|v(\tau)\|_{L^{\alpha + 2}}^{\alpha + 2} d\tau, \) using (4.16) and (4.17) we get
\[ (1 - s)^{\frac{4 - \alpha d}{2}} \left( (1 - s)^{4 - \alpha d} h(s) \right)' \geq C(1 - s)^{-2\theta} \]
for some constant \( C > 0 \) and all \( s \in [0, 1) \). From (2.15) we can deduce the following identity
\[ (1 - s)^{\frac{4 - \alpha d}{2}} h(s) \geq C\{(1 - s)^{-2\theta + \frac{4 - \alpha d}{2}} - 1\} \text{ for all } s \in [0, 1). \]
Since \( \alpha > \frac{4}{d} \), one easily verifies that \(-2\theta + \frac{4 - \alpha d}{2} < 0\), thus there exists a \( 0 < s_0 < 1 \) such that
\[ h(s) \geq C(1 - s)^{-2\theta} \]
for all \( s \in [s_0, 1) \). Applying (2.9) and (2.12), (4.20) yields
\[ \int_{0}^{t} (1 + \tau) \|u(\tau)\|_{L^{\alpha + 2}}^{\alpha + 2} d\tau \geq C(1 + t)^{2\theta}, \]
for all \( t \in \left[ \frac{s_0}{1 - s_0}, +\infty \right) \). This closes the proof of Proposition 4.3 for positive time \((0, +\infty)\), and the arguments for \((-\infty, 0)\) being similar. \(\square\)

5. SCATTERING THEORY FOR THE FOCUSING NLS

As is well known, if \( \alpha \geq \alpha(d) \), scattering theory holds in whole \( \Sigma \) space in the defocusing case (see [8, 15, 21]), this section is devoted to the studying on the scattering theory for the focusing NLS.

For \( \lambda \in \mathbb{R}, \alpha(d) < \alpha < \frac{4}{d-2}(\alpha(d) < \alpha < \infty, \text{ if } d = 1, 2) \), let
\[ \|u\|_{X_\infty} = \sup_{0 < t < \infty} t^\beta \|u(t)\|_{L^{\alpha + 2}}, \|\varphi\|_{W_\infty} = \sup_{0 < t < \infty} t^\beta \|e^{it\Delta} \varphi\|_{L^{\alpha + 2}}, \]
where \( \beta = \frac{4-(d-2)\alpha}{2\alpha(d+2)} \). Cazenave and Weissler proved in [4] that there exists \( \varepsilon_0 > 0 \) with the following property. Let \( u_0 \in H^1(\mathbb{R}^d) \) and let \( u \) be the

\[ \text{Proof.} \]
corresponding unique, maximal strong $H^1$ solution of \((1.1)\), if \(\|u_0\|_{W_\infty} \leq \varepsilon \leq \varepsilon_0\), then the solution \(u\) is positively global, and \(\|u\|_{X_\infty} \leq 2\varepsilon\). The same conclusion also holds for \((-\infty,0)\) with the usual modification.

So it’s natural to ask if \(u_0 \in \Sigma\) satisfying \(\|u_0\|_{W_\infty} \leq \varepsilon_0\), does the corresponding global solution \(u\) scatter in \(\Sigma\)? In the defocusing case, it’s well known that the answer is yes; as to the focusing case, we will give a positive answer below.

**Theorem 5.1.** Assume \(\lambda > 0\), \(\alpha(d) < \alpha < \frac{4}{d-2}\) if \(d \geq 3\), and \(4 \leq \alpha < \infty\), if \(d = 1\). Let \(u_0 \in \Sigma\) and let \(u\) be the corresponding unique, maximal solution of \((1.1)\). Then there exists \(\varepsilon_0 > 0\) such that, if \(\|u_0\|_{W_\infty} \leq \varepsilon_0\), then the solution \(u\) is positively global and scatters at \(+\infty\) in \(\Sigma\). The same conclusion also holds for \((-\infty,0)\) with the usual modification on assumptions.

**Proof.** We have known that(see [3, 4]) there exists a \(\varepsilon_0 > 0\), such that if \(\|u_0\|_{W_\infty} \leq \varepsilon_0\), then the solution \(u\) is positively global, and

\[
(5.2) \quad \|u\|_{X_\infty} \leq 2\varepsilon_0.
\]

This property also holds for any \(0 < \varepsilon \leq \varepsilon_0\). We will prove the scattering properties by contradiction arguments below.

If \(d \geq 3\), \(\alpha(d) < \alpha < \frac{4}{d-2}\), or \(d = 1\), \(\alpha = 4\), we deduce from \((5.2)\), \((5.1)\) and Proposition \(\ref{prop:4.2}\) that, if \(u\) doesn’t scatter at \(+\infty\), then

\[
(5.3) \quad C_1(\alpha, d) \varepsilon_0 (1 + t)^{-\frac{4-2\alpha}{2\alpha}} \geq \|u(t)\|_{L^{\alpha+2}} \geq C(1 + t)^{-\frac{4-2\alpha}{2\alpha}}
\]

for some constants \(C_1, C > 0\), and all \(t \in [1, +\infty)\). It is absurd if we take \(\varepsilon_0\) sufficiently small such that \(C/C_1 > \varepsilon_0\).

If \(d \geq 3\), \(\frac{4}{d} < \alpha < \frac{4}{d-2}\), or \(d = 1\), \(4 < \alpha < \infty\), it follows from \((5.2)\), \((5.1)\) and Proposition \(\ref{prop:4.3}\) that, if \(u\) doesn’t scatter at \(+\infty\), then there exists a \(t_0 > 0\) such that

\[
(5.4) \quad C_2(\alpha, d) \varepsilon_0 (1 + t)^{2\theta_2} \geq \int_0^{t_0} (1 + \tau)^2 \|u(\tau)\|_{L^{\alpha+2}} d\tau \geq C(1 + t)^{2\theta_2}
\]

for some constants \(C_2, C > 0\), and all \(t \in [t_0, +\infty)\), where \(\theta_2\) is defined the same as in Lemma \(\ref{lem:4.1}\). Therefore it is absurd if we take \(\varepsilon_0\) sufficiently small such that \(C/C_2 > \varepsilon_0\). This closes the proof of Theorem \(\ref{thm:5.1}\) for positive time \((0, +\infty)\), and the arguments for \((-\infty,0)\) being similar.

**Remark 5.2.** Note that we can deduce from the isometric properties of \((e^{it\Delta})_{t \in \mathbb{R}}\) and dispersive estimate(refer to [3]) that

\[
(1 + |t|)^{\frac{\alpha d}{(\alpha + 2)}} \|e^{it\Delta} u_0\|_{L^{\alpha+2}} \leq C \|u_0\|_{H^1 \cap L^{\frac{\alpha+2}{\alpha+4}}}
\]

Since \(\beta < \frac{\alpha d}{2(\alpha + 2)}\), we infer that \(\|u_0\|_{W_\infty} \leq C \|u_0\|_{H^1 \cap L^{\frac{\alpha+2}{\alpha+4}}}\). Thus by Theorem \(\ref{thm:5.1}\) for \(\alpha > \alpha(d)\), there exists \(\varepsilon_0 > 0\) such that if

\[
\|u_0\|_{H^1 \cap L^{\frac{\alpha+2}{\alpha+4}}} \leq \varepsilon_0,
\]

then the corresponding solution \(u\) is global and scattering states \(u^\pm\) exist in \(\Sigma\) at \(\pm\infty\).
The concept “rapidly decaying solutions” (see Definition 1.2) plays an important role in the scattering theory for focusing NLS. When \( \lambda > 0 \), \( \alpha > \alpha(d) \), Cazenave and Weissler obtained the scattering results for “rapidly decaying solutions”, in terms of which they characterized the sets \( \mathcal{R}_\pm \) (refer to [2]). We will extend this work to the critical power \( \alpha = \alpha(d) \) below.

**Theorem 5.3.** Assume \( \lambda > 0 \), \( \alpha = \alpha(d) = \frac{2-d+\sqrt{d^2+12d+4}}{2d} \) and \( d \geq 1 \), \( d \neq 2 \). Let \( u_0 \in \Sigma \) be such that the corresponding solution \( u \) is positively (resp. negatively) global with rapid decay (see Definition 1.2, i.e.,

\[
\|u\|_{L^a,\infty((0,\infty),L^{\alpha+2})} < \infty \quad (\text{resp.} \quad \|u\|_{L^a,\infty((\infty,0),L^{\alpha+2})} < \infty),
\]

where \( a = \frac{2\alpha+2}{4-\alpha(d-2)} \), then \( u_0 \) has scattering state at \( +\infty \) (resp. \( -\infty \)).

**Proof.** By Remark 1.1, we need only deal with the positive time \( t \to +\infty \).

We consider separately the cases \( d \geq 3 \) and \( d = 1 \).

First we consider the simpler case \( d \geq 3 \). We argue by contradiction and assume that \( u \) doesn’t scatter at \( +\infty \). Then by (5.5), Proposition 4.2 and (2.1), we get immediately

\[
\infty > \|u\|_{L^a,\infty((0,\infty),L^{\alpha+2})} \geq C\|\left(1+t\right)^{-\frac{a(d)}{2\alpha+\alpha+1}} \log(1+t)\|^\frac{1}{\alpha+2}\|u\|_{L^a,\infty((0,\infty)},
\]

which is absurd. This closes our proof for \( d \geq 3 \).

Now we proceed to the case \( d = 1 \). We argue by contradiction and assume that \( u \) doesn’t scatter at \( +\infty \). Then by Proposition 2.5, the nonautonomous equation (2.11) blows up at \( s = 1 \) (i.e., \( S_M(0,\nu_0) = 1 \)). Hence by Theorem 2.4, one has

\[
\lim_{s \uparrow 1} \|v(s)\|_{H^1} = \infty.
\]

Since \( \alpha = \alpha(d) < \frac{4}{d} \), one can deduce from the energy estimates that the positively global \( u \) is bounded in \( H^1(\mathbb{R}^d) \), so by using (5.6) and (2.13), we get

\[
\lim_{t \to +\infty} \|P_t u(t)\|_{L^2} = \lim_{t \to +\infty} \|(x+2it\nabla)u(t)\|_{L^2} = \infty.
\]

On the other hand, it follows from (2.6) and Strichartz’s estimates that

\[
\|P_t u(\tau)\|_{L^\infty((t,T),L^2)} \leq C\|P_t u(t)\|_{L^2} + C\|P_t(\vert u\vert^\alpha u(t))\|_{L^1((t,T),L^2)}
\]

for any \( 0 < t < T < \infty \). We define a multiplier \( M_t \) by \( M_t = e^{\frac{i\alpha t^2}{4t}} \). Then by Gagliardo-Nirenberg’s inequality, combined with (2.7) and (2.8) we have
thus we deduce from (5.7) that there exists $T$ and hence $u
onumber$

$$K$$ is constant and hence by assumption (5.5) and (5.8) we obtain that there exists a con-

$$\|\|_{L^\infty((t,T),L^2)}$$ such that

and hence by assumption (5.5) and (5.8) we obtain that there exists a constant $K$ independent of $t$ and $T$ such that 

$$\|P_\tau u\|_{L^\infty((t,T),L^2)} \leq K\|P_\tau u(t)\|_{L^2} + K\|P_\tau u\|_{L^\infty((t,T),L^2)}\].$$

Note that $u \in C((0,\infty),\Sigma)$, so $\|P_\tau u\|_{L^\infty((t,T),L^2)}$ is continuous and nondecreasing about $T$ on $(t,\infty)$ and note also that

$$\|P_\tau u\|_{L^\infty((t,T),L^2)} \rightarrow \|P_\tau u(t)\|_{L^2}, \text{ as } T \downarrow t,$$

thus we deduce from (5.7) that there exists $T_0 \in (t,\infty)$ such that $\|P_\tau u\|_{L^\infty((t,T_0),L^2)} = (K+1)\|P_\tau u(t)\|_{L^2}$. Letting $T = T_0$ in (5.10), we obtain

$$\|P_\tau u(t)\|_{L^2} \leq K((K+1)\|P_\tau u(t)\|_{L^2})^{\frac{1}{\alpha+4}}\|P_\tau u\|_{L^2}^{\frac{\alpha+1}{\alpha+4}}\|P_\tau u\|^2_{L^\infty((t,T_0),L^2)}.$$

Since by (2.2) we have $\|P_\tau u\|^2_{L^\infty((t,T_0),L^2)} \leq \|P_\tau u\|^2_{L^\infty((t,T_0),L^2)} \leq C t^{-\frac{\alpha+4}{\alpha+2}}$, we deduce from (5.11) that

$$1 \leq C K(K+1)\|P_\tau u(t)\|_{L^2}^{\frac{2\alpha}{\alpha+4}} t^{-\frac{\alpha+4}{2(\alpha+3)}},$$

and hence

$$\|P_\tau u(t)\|_{L^2} = \|(x + 2it\nabla)u(t)\|_{L^2} \geq C t^{-\frac{1}{\alpha}}$$

for some constant $C > 0$ and arbitrary $t \in [0,\infty)$. Now taking the $L^2_x$ coupling of the equation (1.1) and $it^{\frac{\alpha d}{2}} \Delta u + t^{\frac{\alpha d}{2}-1} x \cdot \nabla u - i\frac{1}{2} t^{-\frac{\alpha d}{2}} - 2 u$ and taking the real part, we obtain

$$\partial_t N(t) = -2 - \frac{\alpha d}{2} t^{\frac{\alpha d}{2}-3} \|(x + 2it\nabla)u(t)\|_{L^2}^2,$$

where

$$N(t) = t^{\frac{\alpha d}{2}-2} \|(x + 2it\nabla)u(t)\|_{L^2}^2 - \frac{8\lambda}{\alpha + 2} t^{\frac{\alpha d}{2}} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2}.$$
Integrating the identity (5.13), we get (5.15)
\[ t^{\frac{\alpha-d}{2}} \| P_t u \|_{L^2}^2 - \frac{8\lambda}{\alpha + 2} t^{\frac{\alpha-d}{2}} \| u(t) \|_{L^{\alpha+2}}^2 = N(1) - (2 - \frac{\alpha}{2}) \int_1^t \tau^{\frac{\alpha-d-6}{2}} \| P_\tau u(\tau) \|_{L^2}^2 d\tau. \]

Note that \( d = 1 \) and \( \alpha = \alpha(d) \) now, applying estimate (5.12), we deduce from (5.15) that
\[ \frac{8\lambda}{\alpha + 2} t^{\frac{\alpha}{2}} \| u(t) \|_{L^{\alpha+2}}^2 + N(1) = t^{\frac{\alpha-d}{2}} \| P_t u \|_{L^2}^2 + (2 - \frac{\alpha}{2}) \int_1^t \tau^{\frac{\alpha-d-6}{2}} \| P_\tau u(\tau) \|_{L^2}^2 d\tau \]
\[ \geq C + C \int_1^t \tau^{-1} d\tau \geq C + C \log t, \]
from which we get immediately (5.16)
\[ \| u(t) \|_{L^{\alpha+2}} \geq Ct^{-\frac{\alpha}{2(\alpha+2)}} (\log t) \frac{1}{\| u \|_{L^{\alpha+2}}} \]
for some constant \( C > 0 \) and all \( t \in [1, \infty) \). Thus by (5.5), (5.16) and (2.1) we get immediately
\[ \infty > \| u \|^2_{L^{\alpha,\infty}(0,\infty), L^{\alpha+2}} \geq C \| (1 + t)^{-\frac{\alpha}{2(\alpha+2)}} \| \log(1 + t) \|_{L^{\alpha,\infty}(0,\infty)} \]
\[ \geq C \| (1 + t)^{-1} \| \log(1 + t) \|_{L^{\alpha,\infty}(0,\infty)} \]
which is absurd. This closes our proof for \( d = 1 \).

The next Theorem indicates a sufficient condition on the initial data \( u_0 \in \Sigma \) at \( \alpha = \alpha(d) \), which guarantees the corresponding global solutions have rapid decay as \( t \to \pm \infty \). Thus by Theorem 5.3 these initial values \( u_0 \) have scattering states at \( \pm \infty \) when \( d \geq 1, \ d \neq 2 \).

**Theorem 5.4.** Let \( \lambda \in \mathbb{R}, \ \alpha = \alpha(d) = \frac{2 - \sqrt{d^2 + 12d + 4}}{2d}, \ d \geq 1, \) and let
\[ a = \frac{2\alpha(\alpha + 2)}{4 - \alpha(d - 2)}. \]
There exists \( \varepsilon_0 > 0 \) such that, if \( u_0 \in H^2 \cap \mathcal{F}(H^2) \subset \Sigma \) and there exists an admissible pair \( (\mu, \nu) \) with \( \max \{ \frac{d(\alpha+2)}{d+a}, 2 \} < \nu \leq \alpha + 2 \) satisfying
\[ (5.17) \sup_{0 \leq t < \infty} (t + 1)^{\frac{\nu}{2}} \| e^{it\Delta} [(x + 2i\nabla)u_0] \|_{L^\nu} + \| e^{it\Delta} u_0 \|_{L^\nu} \leq \varepsilon_0, \]
or respectively,
\[ (5.18) \sup_{-\infty < t \leq 0} (|t| + 1)^{\frac{\nu}{2}} \| e^{it\Delta} [(x + 2i\nabla)u_0] \|_{L^\nu} + \| e^{it\Delta} u_0 \|_{L^\nu} \leq \varepsilon_0, \]
then the corresponding maximal solution \( u \) of (1.1) is a positively (resp. negatively) rapidly decaying solution (see Definition 1.2). Moreover,
\[ u \in L^{q,2}((0, \infty), W^{1,r}(\mathbb{R}^d)) \quad (\text{resp. } u \in L^{q,2}((-\infty, 0), W^{1,r}(\mathbb{R}^d))), \]
\[ Pu \in L^{q,2}((0, \infty), L^r(\mathbb{R}^d)) \quad (\text{resp. } Pu \in L^{q,2}((-\infty, 0), L^r(\mathbb{R}^d))), \]
where operator \( P = x + 2it\nabla \) and \((q,r)\) is an admissible pair with \( r = \alpha + 2 \), therefore scattering state exists in \( \Sigma \) at \(+\infty\) (resp. \(-\infty\)).

**Proof.** By Remark 1.1, we need only to deal with the positive time \((0, +\infty)\). Since \( \alpha = \alpha(d) \), by energy estimates, it’s well known that the solution \( u \) is global and bounded in \( H^1(\mathbb{R}^d) \), moreover, \( u \in C(\mathbb{R}, \Sigma) \). Thus we have \( v(s, y) \) (the Pseudo-conformal Transformation of \( u(t, x) \), see Section 2) defined by \((2.10)\) satisfies the nonautonomous integral equation \((3.1)\) on the interval \((0, 1)\), and \( v \in C([0, 1], \Sigma) \).

Therefore, by applying the dispersive estimates (see [3]) and Hölder’s inequality to the integral equation \((3.1)\), we have

\[
\|v(s)\|_{W^{1, \nu}} \leq \|e^{i\Delta} v_0\|_{W^{1, \nu}} + C \int_0^s (1 - \tau)^{\frac{d+4}{4} - \frac{\nu}{2}} \|v\|^\alpha_{L^{\infty, \nu}} \|v\|_{W^{1, \nu}} d\tau.
\]

Note that by definition of the Pseudo-conformal Transformation (see \((2.9), (2.10)\) and \((2.12)\)), one easily verifies that the condition \((5.17)\) is equivalent to

\[
\sup_{s \in [0, 1]} \|e^{i\Delta} v_0\|_{W^{1, \nu}} \leq \varepsilon_0.
\]

Note also that \( \alpha = \alpha(d) \), \( \max\{\frac{d(\alpha+2)}{d+\alpha}, 2\} < \nu < \alpha+2 \), so we have \( L^{\frac{\alpha+2}{\nu-2}}(\mathbb{R}^d) \hookrightarrow W^{1, \nu}(\mathbb{R}^d) \) and \( \frac{4-ad}{2} + \frac{2}{\nu} < 1 \), hence by Hölder’s inequality we get

\[
\int_0^s (1 - \tau)^{\frac{d+4}{4} - \frac{\nu}{2}} (s - \tau)^{-\frac{\nu}{2}} d\tau \leq C(\nu, d).
\]

Set \( \Phi(s) = \sup_{r \in [0, s]} \|v(r)\|_{W^{1, \nu}} \), for \( 0 < s < 1 \). Then we can deduce from \((5.19)\), \((5.20)\) and \((5.21)\) immediately that

\[
\Phi(s) \leq \varepsilon_0 + C\Phi(s)^{\alpha+1} \text{ for all } 0 < s < 1.
\]

Since \( u_0 \in H^2 \cap C(\mathbb{R}^2) \), it follows from a trivial continuation arguments (similar to the proof of Theorem 3.1) we omit the details here) that if \( \varepsilon_0 \) is small enough such that \( C(2\varepsilon_0)^{\alpha+1} < \varepsilon_0 \), then

\[
\sup_{s \in [0, 1]} \|v(s)\|_{W^{1, \nu}} \leq 2\varepsilon_0.
\]

Since \( \max\{\frac{d(\alpha+2)}{d+\alpha}, 2\} < \nu < \alpha+2 \), which implies \( L^{\alpha+2} \hookrightarrow W^{1, \nu} \), we have

\[
\sup_{s \in [0, 1]} \|v(s)\|_{L^{\alpha+2}} \leq C \sup_{s \in [0, 1]} \|v(s)\|_{W^{1, \nu}} \leq 2C(\nu, d)\varepsilon_0,
\]

combined with identity \((2.12)\), we infer that

\[
\|u(t)\|_{L^{\alpha+2}} \leq 2C\varepsilon_0(t+1)^{-1/\alpha}.
\]

Therefore, we have

\[
\|u\|_{L^{\nu}((0, \infty), L^{\alpha+2})} \leq 2C(\nu, d)\varepsilon_0,
\]

which implies that \( u \) is a positively rapidly decaying solution. Thus by Theorem 5.3 if \( \lambda > 0, d \neq 2 \), scattering state \( u^+ \) exists at \(+\infty\).
Furthermore, by using the Strichartz’s and Hölder’s estimates in Lorentz spaces (see Lemma 2.3, Lemma 2.2), we get there exists K independent of T and $u_0$ such that

$$\|u\|_{L^{q,r}(0,T),W^{1,r}} \leq K\|u_0\|_{H^1} + K\|\mu\|_{L^{q,\infty}(0,T),L^r}\|u\|_{L^{q,r}(0,T),W^{1,r}}$$

for every $0 < T < \infty$, where $(q,r)$ is an admissible pair with $r = \alpha + 2$. Thus by continuation arguments, one can easily deduce from (5.25) and (5.26) that if $\varepsilon_0$ is sufficiently small such that $2^{\alpha+1}K(C\varepsilon_0)^{\alpha} < 1$, then

$$\|u\|_{L^{q,2}((0,\infty),W^{1,r})} \leq 2K\|u_0\|_{H^1}. \tag{5.27}$$

Similarly, one can easily obtain from Strichartz’s estimates and (5.25) that

$$\|(x + 2it\nabla)u\|_{L^{q,2}((0,\infty),L^r)} \leq 2K\|xu_0\|_{L^2}. \tag{5.28}$$

Applying Strichartz’s estimate in Lorentz spaces (Lemma 2.3), combined with (5.25), (5.27) and (5.28) we get for any $0 < t < s$,

$$\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{H^1} \leq C\|u\|_{L^{q,2}((t,s),W^{1,r})} \rightarrow 0,$$

$$\|(x + 2it\nabla)u(t) - e^{-is\Delta}u(s)\|_{L^2} \leq C\|(x + 2it\nabla)u\|_{L^{q,2}((t,s),L^r)} \rightarrow 0,$$

as $t, s \rightarrow \infty$. Therefore, the above two estimates implies that, for general $d \geq 1$, the scattering state $u^+$ exists in $\Sigma$ at $+\infty$. \qed

We will apply Theorem 5.4 to certain type of initial data below. In Corollary 5.5, we will show that when $\alpha = \alpha(d)$, if the initial data is “sufficiently oscillating”, then the corresponding solution will scatter at $\pm \infty$, note that the $L^2$ norm of these initial values is unbounded.

**Corollary 5.5.** Assume $\lambda \in \mathbb{R}$, $\alpha = \alpha(d) \geq 1$. For arbitrary $\varphi \in H^2 \cap \mathcal{F}(H^2) \subset \Sigma$, given $b \in \mathbb{R}$, and let $\tilde{u}_b$ be the corresponding maximal solutions of (1.7) with the initial values $\tilde{u}_{b,0} = e^{i\Delta}(e^{\frac{b|x|^2}{4}}\varphi)$, then there exists $0 < b_0 < \infty$ such that if $b \geq b_0$, the solutions $\tilde{u}_b$ are rapidly decaying as $t \rightarrow \infty$, therefore scattering states $u_b^+$ exist at $+\infty$ and $\tilde{u}_{b,0} \in \mathcal{R}_+$. The same conclusions also hold for $(-\infty,0)$ provided $b \leq -b_0$.

**Proof.** By Remark 1.1, we need only to deal with the positive time $(0, +\infty)$. Applying Theorem 5.4, the key point is to show that condition (5.17) is satisfied for certain special choice of admissible pair $(\mu, \nu)$.

Here we fixed $\mu_0 = \frac{4(d+2)(\alpha+2)}{\alpha + d + 2}$ and $\nu_0 = \frac{(d+2)(\alpha+2)}{\alpha + d + 2}$ to prove (5.17). It’s obvious that such choice of $(\mu_0, \nu_0)$ satisfies the conditions in Theorem 5.4, and $\tilde{u}_{b,0} = e^{i\Delta}(e^{\frac{b|x|^2}{4}}\varphi) \in H^2 \cap \mathcal{F}(H^2)$ for any $b \in \mathbb{R}$ and $\varphi \in H^2 \cap \mathcal{F}(H^2)$. A direct calculation, based on the explicit kernel of the Schrödinger group (refer to [2] for a review) shows that

$$[e^{it}\Delta(e^{\frac{b|x|^2}{4}}\varphi)](x) = e^{\frac{b|x|^2}{4}\frac{1}{1+it}}[D^{\frac{1}{1+it}}e^{\frac{1}{1+it}\Delta}\varphi](x), \tag{5.29}$$
where the dilation operator $D_\beta$, $\beta > 0$, is defined by $D_\beta \omega(x) = \beta^d \omega(\beta x)$.

Note that by (5.29) and simple calculations, we get

\begin{align}
\sup_{0 \leq t < \infty} (t + 1)^{\frac{d}{2}} \{ \| e^{it\Delta} [(x + 2i\nabla)\tilde{u}_{b,0}] \|_{L^p} + \| e^{it\Delta} \tilde{u}_{b,0} \|_{L^p} \}
& = \sup_{1 \leq t < \infty} \frac{2}{t^{\frac{d}{2}}} \{ \| e^{it\Delta} [e^{\frac{bt^2}{t + \frac{1}{2}}} (x\varphi)] \|_{L^p} + \| e^{it\Delta} (e^{\frac{bt}{t + \frac{1}{2}}} \varphi) \|_{L^p} \}
& = \sup_{1 \leq t < \infty} \frac{2}{t^{\frac{d}{2}}} \left\{ \| e^{it\Delta} (x\varphi) \|_{L^p} + \| e^{it\Delta} \varphi \|_{L^p} \right\}
& \leq \sup_{1 \leq t < \infty} \frac{2}{t^{\frac{d}{2}}} C \| \varphi \|_{H^2 \cap \mathcal{F}(H^2)} \leq C b^{-\frac{2}{d}} \to 0,
\end{align}

as $b \to \infty$. Therefore, there exists a $0 < b_0 < \infty$ such that, if $b \geq b_0$, then initial data $\tilde{u}_{b,0}$ satisfies the condition (5.17). The result now follows from Theorem 5.4.

Applying Theorem 5.3 and Corollary 5.5, we now characterize the sets $\mathcal{R}^\pm$ in the case $\lambda > 0$, $\alpha = \alpha(d)$ in terms of rapidly decaying solutions. Our result extends the work of Cazenave and Weissler in [2] (see Theorem 4.12 therein), which is devoted to the case $\alpha > \alpha(d)$.

**Theorem 5.6.** Assume $\lambda > 0$ and $\alpha = \alpha(d) = \frac{2 - d + \sqrt{d^2 - 12d + 4}}{2d}$, and let $u_\varphi$ be the corresponding solution of (1.1) with initial data $u(0) = \varphi$. If $d \geq 1$, $d \neq 2$, $\mathcal{R}_+ = \{ \varphi \in \Sigma; u_\varphi$ has rapid decay as $t \to \infty \}$ and $\mathcal{R}_- = \{ \varphi \in \Sigma; u_\varphi$ has rapid decay as $t \to -\infty \}$. Moreover, for $d \geq 1$, $\mathcal{R}_\pm$ are unbounded subsets of $L^2(\mathbb{R}^d)$.

**Proof.** By Remark 1.1, we need only show the result for $\mathcal{R}_+$. First, if $\varphi \in \mathcal{R}_+$, then

\[ \|(x + 2it\nabla)u_\varphi\|_{L^2} \leq C, \]

and hence by Gagliardo-Nirenberg’s inequality, we have

\[ \|u_\varphi(t)\|_{L^{d+2}} \leq C(t^{-1} \|(x + 2it\nabla)u_\varphi\|_{L^2})^{-\frac{d(d-2)}{2(d+2)}} \leq C t^{-\frac{d(d-2)}{2(d+2)}}. \]

Therefore $\|u_\varphi\|_{L^{d+2}(0, \infty), L^{d+2}} < \infty$, that is, $u$ has rapid decay as $t \to \infty$ (see Definition 1.2). Conversely, if $\varphi \in \Sigma$ be such that $u_\varphi$ has rapid decay as $t \to \infty$, then by Theorem 5.3, for $d \geq 1$, $d \neq 2$, the scattering state exists in $\Sigma$ at $\pm \infty$, i.e., $\varphi \in \mathcal{R}_+$. This closes the proof for the first assertion.

Now consider arbitrary $\psi \in H^2 \cap \mathcal{F}(H^2)$, and let $\tilde{u}_{b,0} = e^{it\Delta} (e^{\frac{bt}{t + \frac{1}{2}}} \psi)$, it follows from Corollary 5.5 that $\tilde{u}_{b,0} \in \mathcal{R}_+$, for $b$ large enough. Note that $\|\tilde{u}_{b,0}\|_{L^2} = \|\psi\|_{L^2}$, it follows that $\mathcal{R}_+$ is unbounded subsets of $L^2(\mathbb{R}^d)$.

Next, we will investigate the scattering theory in $\Sigma$ for focusing NLS with $\alpha \geq \frac{4}{3}$ and initial data $u_0 \in \Sigma$ below a mass-energy threshold and satisfying an mass-gradient bound. For the $H^1$ scattering results, see [5, 6, 7].
Theorem 5.7. Assume \( \lambda > 0, d \geq 1, \frac{4}{d} \leq \alpha < \frac{4}{d-2} (\alpha \in [\frac{d}{2}, \infty), \) if \( d = 1, 2). \) If initial data \( u_0 \in \Sigma \) satisfies the following assumptions:

\[ (5.31) \quad \|u_0\|_{L^2} < \lambda^{-\alpha} \|Q\|_{L^2}, \quad \text{for } \alpha = \frac{4}{d}, \]

or

\[ (5.32) \quad M[u_0]^{\sigma} E[u_0] < \lambda^{-2\sigma} M[Q]^{\sigma} E[Q], \]

\[ (5.33) \quad \|u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2} < \lambda^{-\tau} \|Q\|_{L^2}^\sigma \|\nabla Q\|_{L^2}, \quad \text{for } \alpha > \frac{4}{d}, \]

then scattering states \( u^\pm \) exist in \( \Sigma \) at \( \pm \infty, \) where \( \sigma = \frac{4-(d-2)\alpha}{2\alpha d-4}, \quad \tau = \frac{2}{\alpha d-4} \)

and \( Q \) is the ground state solution to \(-\Delta Q + Q = |Q|^\alpha Q.\)

Proof. Note that the conditions in Theorem 5.7 is invariant by taking complex conjugation, so by Remark 1.1 we need only prove the results for positive time \( t \to +\infty. \) Throughout our proof, we define

\[ v(t, x) = e^{-\frac{|x|^2}{4t}} u(t, x), \]

where \( u(t, x) \) is the corresponding maximal \( H^1 \) solution of (1.1). We consider separately the cases \( \alpha = \frac{4}{d} \) and \( \alpha > \frac{4}{d}. \)

Case \( \alpha = \frac{4}{d}. \) It has been proved by M. Weinstein in [22] that the maximal \( H^1 \) solution of (1.1) is global and bounded in \( H^1(\mathbb{R}^d). \) To prove the scattering results, we argue by contradiction and assume that \( u \) doesn’t scatter at \( +\infty. \) Then by Proposition 4.2, we have for all \( t \in (0, +\infty), \)

\[ (5.34) \quad \|u(t)\|_{L^{\alpha+2}} \geq C (1 + |t|)^{\frac{2(1-\theta)}{2(\alpha+2)}}, \]

where \( \theta > 0 \) is defined the same as in Lemma 4.1. Note that the conservation of mass and Gagliardo-Nirenberg’s inequality imply that

\[ (5.35) \quad \frac{1}{2} \|\nabla v(t)\|_{L^2}^2 \leq E[v(t)] + \frac{\lambda}{\alpha + 2} C_{GN} \|\nabla v(t)\|_{L^2}^\alpha \|u_0\|_{L^2}^\sigma, \]

and the best constant \( C_{GN} = \frac{\alpha + 2}{2} \|Q\|_{L^2}^{-\alpha} \) (see Corollary 2.1 in [22]), where \( Q \) is the ground state solution to \(-\Delta Q + Q = |Q|^\alpha Q. \) Thus we can deduce from (5.31) and (5.35) that

\[ (5.36) \quad \|\nabla v(t)\|_{L^2}^2 \leq CE[v(t)]. \]

Since \( \alpha = \frac{4}{d}, \) the pseudo-conformal conservation law (1.5) becomes a precise conservation law

\[ (5.37) \quad 8t^2 E[v(t)] = \|x u_0\|_{L^2}^2, \]

combined with (5.36), we get

\[ \|u(t)\|_{L^{\alpha+2}}^\alpha \|v(t)\|_{L^{\alpha+2}}^\alpha \leq CE[v(t)] \|u_0\|_{L^2}^{\alpha} \leq Ct^{-2} \quad \text{for all } t \in \mathbb{R}, \]

which contradicts (5.34). This closes our proof for \( \alpha = \frac{4}{d}. \)

Case \( \alpha > \frac{4}{d}. \) By scaling, we may assume \( \lambda = 1. \) We first recall some basic
properties of $Q$, the ground state of $-\Delta Q + Q = |Q|^\alpha Q$, by Pohozaev’s identity (see e.g. Corollary 8.1.3 in [3]),
\begin{equation}
\|Q\|_{L^2}^2 = \frac{4 - (d - 2)\alpha}{\alpha d} \|\nabla Q\|_{L^2}^2 = \frac{4 - (d - 2)\alpha}{2(\alpha + 2)} \|Q\|_{L^{\alpha+2}}^{\alpha+2}.
\end{equation}
Using (5.38), the best constant in the Gagliardo-Nirenberg’s inequality
\begin{equation}
\|\omega\|_{L^{\alpha+2}}^{\alpha+2} \leq C_{GN} \|\nabla \omega\|_{L^2}^{\frac{4-(d-2)\alpha}{\alpha d}} \|\nabla Q\|_{L^2}^{\frac{\alpha d}{\alpha+2}}
\end{equation}
can be given by
\begin{equation}
C_{GN} = \frac{2(\alpha + 2)}{\alpha d} \left( \|Q\|_{L^2} \|\nabla Q\|_{L^2} \right)^{-\frac{\alpha d - 4}{2}}.
\end{equation}
Define $f(x) = \frac{1}{4} x^2 - \frac{C_{GN}}{\alpha d} x^\frac{\alpha d}{2}$ for $x > 0$. It follows from Gagliardo-Nirenberg’s inequality (5.39), energy conservation and (5.32) that
\begin{equation}
f(\|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2}) \leq M[u_0]^\alpha E[u_0] < M[Q]^\alpha E[Q],
\end{equation}
and note that if we take $\omega = Q$, then we get equality in (5.39), we infer
\begin{equation}
f(\|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2}) < f(\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}).
\end{equation}
Since $\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}$ is a local maximum point of $f$, we deduce from the continuity of $\|\nabla u(t)\|_{L^2}$ in $t$, the initial mass-gradient bound (5.33) and (5.32) that
\begin{equation}
\|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2} < \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}
\end{equation}
for all $t \in [0, T_{\text{max}})$, which implies the maximal solution $u$ is positively global in time and bounded in $H^1(\mathbb{R}^d)$. Thus we deduce from the Gagliardo-Nirenberg’s inequality that
\begin{equation}
\|\nabla u(t)\|_{L^2}^2 < \frac{2\alpha d}{\alpha d - 4} E[u]
\end{equation}
for all $t \in [0, +\infty)$.
Define a constant $\eta_0 > 0$ by $\eta_0 = \eta_0(u_0) = \frac{M[u_0]^\alpha E[u_0]}{M[Q]^\alpha E[Q]}$, it follows form (5.32) that $\eta_0 < 1$. Using Pohozaev’s identity (5.38) and (5.44), we get
\begin{equation}
\|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2} < \sqrt{\frac{2\alpha d}{\alpha d - 4} (M[u]^\alpha E[u])^\frac{1}{2}} < \eta_0^\frac{1}{2} \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}.
\end{equation}
Note that by the pseudo-conformal conservation law (1.5), energy conservation and (5.34) we have
\begin{equation}
\|\nabla v(t)\|_{L^2}^2 = \frac{\|ux_0\|_{L^2}^2}{4t^2} + \|\nabla u(t)\|_{L^2}^2 - 2E[u] + \frac{\alpha d - 4}{\alpha + 2} t^2 \int_0^t \|u(\tau)\|_{L^{\alpha+2}}^{\alpha+2} d\tau
\end{equation}
\begin{equation}
< \frac{\|ux_0\|_{L^2}^2}{4t^2} + \|\nabla u(t)\|_{L^2}^2 - 2E[u] + \frac{4}{t^2} \int_0^t \tau E[u] d\tau
\end{equation}
\begin{equation}
= \frac{\|ux_0\|_{L^2}^2}{4t^2} + \|\nabla u(t)\|_{L^2}^2.
\end{equation}
Therefore, we deduce easily from (5.46) that
\begin{equation}
\|\nabla v(t)\|_{L^2}^2 < \eta_0^\frac{1}{2} \|\nabla u\|_{L^2}^2
\end{equation}
for all \( t \geq t_0 = \frac{\|u_0\|_2^2}{\sqrt{8[E[u_0] - M[Q] E[Q]]^{1/2}}} - 1 \)^{-\frac{1}{2}}. \) In view of (5.45) and (5.37), we get immediately

\[
\|u_0\|_{L^2}^2 \| \nabla v(t) \|_{L^2}^2 < \eta_0^\frac{1}{4} \| Q \|_{L^2} \| \nabla Q \|_{L^2}
\]

for all \( t \geq t_0 \). Applying Gagliardo-Nirenberg inequality (5.39) and (5.48), we get

\[
\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} < \frac{2(\alpha + 2)}{\alpha d} \eta_0^{\frac{\alpha d - 4}{4}} \| \nabla v(t) \|_{L^2}^2 < \frac{4(\alpha + 2)}{\alpha d - 4} \eta_0^{\frac{\alpha d - 4}{4}} E[v(t)]
\]

for all \( t \geq t_0 \). Note that \( 0 < \eta_0 = \frac{M[u_0] E[u_0]}{M[Q] E[Q]} < 1 \), by the pseudo-conformal conservation law (15), we have

\[
t^2 \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} < \frac{4(\alpha + 2)}{\alpha d - 4} \eta_0^{\frac{\alpha d - 4}{4}} t^2 E[v(t)]
\]

\[
\leq \frac{4(\alpha + 2)}{\alpha d - 4} \eta_0^{\frac{\alpha d - 4}{4}} \int_{t_0}^t \tau \|u(\tau)\|_{L^{\alpha+2}}^{\alpha+2} d\tau
\]

for all \( t \geq t_0 \), therefore we deduce from Gronwall’s inequality that

\[
\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq C t^{-2(1-\eta_0^{\frac{\alpha d - 4}{4}})}
\]

for all \( t \geq t_0 \). Note that \( 0 < \eta_0 < 1 \), (5.51) implies that

\[
\|u(t)\|_{L^{\alpha+2}} \to 0,
\]

as \( t \to +\infty \). Since \( \alpha > \frac{2}{d} \), by Strichartz’s estimates and continuation arguments, one can easily obtain that

\[
u \in L^q((0, \infty), W^{1,r}(-\mathbb{R}^d)), \quad P_t u = (x+2it\nabla)u \in L^q((0, \infty), L^r(-\mathbb{R}^d))
\]

for every admissible pair \((q, r)\), we omit the details here (see e.g. Theorem 7.7.3 in [3], and following the proof of this theorem with \( P_t u \) instead of \( u \)). In particular, by identity (2.13), \( u \in L^\infty((0, \infty), H^1) \) and \( P_t u \in L^\infty((0, \infty), L^2) \) implies that

\[
\sup_{s \in [0, 1]} \|v(s)\|_{H^1} < \infty,
\]

where \( v(s) \) is the pseudo-conformal transformation of \( u(t) \). Thus by Theorem 2.4 and Proposition 2.5, scattering state \( u^+ \) exists in \( \Sigma \) at \( +\infty \). \( \square \)

6. CONVERGENCE OF SCATTERING SOLUTION TO A FREE SOLUTION

Finally we study the asymptotic behavior of \( \|u(t) - e^{it\Delta} u^+\|_{\Sigma} \) under the assumption \( u^\pm \) exist at \( \pm \infty \). In general, since \( e^{it\Delta} \) is not an isometry of \( \Sigma \), it is not known whether we can deduce \( \|u(t) - e^{it\Delta} u^\pm\|_{\Sigma} \to 0 \) from the scattering asymptotic property \( \|e^{-it\Delta} u(t) - u^\pm\|_{\Sigma} \to 0 \). A positive answer has been given by Bézaut [1] for \( d \leq 2, \alpha > \frac{4}{3} \), and \( 3 \leq d \leq 5, \alpha > \frac{4}{d-2} \). Our results in this section extend this work to spatial dimension \( d \leq 9 \) and wider admissible range of \( \alpha \) under certain suitable assumption on initial data \( u_0 \).
Lemma 6.1. Assume $\lambda \in \mathbb{R} \setminus \{0\}$, $\frac{2}{3} < \alpha < \frac{4}{d-2}$, $\frac{3}{2} < \alpha < \infty$, if $d = 1, 2$, and $u_0 \in \mathcal{R}_\pm$, $u \in C(\mathbb{R}, \Sigma)$ is the corresponding global solution of (1.1). Then for any admissible pair $(q, r)$, we have if furthermore $\alpha > \frac{4}{d+2}$, then

$$u \in L^q(\mathbb{R}, W^{1, r}(\mathbb{R}^d)).$$

Proof. For the proof, refer to Bégout [1], Proposition 3.1.

Theorem 6.2. Assume $\lambda \in \mathbb{R} \setminus \{0\}$, $d \geq 3$, $\frac{2}{3} < \alpha < \frac{4}{d-2}$ and $u_0 \in \mathcal{R}_\pm$, $u \in C(\mathbb{R}, \Sigma)$ is the corresponding global solution of (1.1) with scattering states $u_\pm$ at $\pm \infty$. We assume further that $u_0 \in \Sigma \cap W^{1, \rho'}$, where $\rho = \frac{4d}{2d - \alpha(d - 2)}$, then if $3 \leq d \leq 9$, $\alpha > \frac{16}{3d + 2}$, we have

$$\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{\Sigma} \to 0.$$

Remark 6.3. Note that for $3 \leq d \leq 9$, we have $\frac{4}{d} < \frac{16}{3d + 2} < \min\{\frac{8}{d - 2}, \frac{4}{d - 2}\}$, however, when $d \geq 10$, we have $\frac{16}{3d + 2} \geq \frac{4}{d - 2}$.

Proof. By Remark 6.1 we need only prove the results for positive time $t \to +\infty$, for the negative time $t \to -\infty$, we can change $u_0$ to $\tilde{u}_0$ and correspondingly change $u(t)$ to $\tilde{u}(-t)$ to reach the conclusion. Note also that the Schrödinger group is isometric on $H^1$, it’s obvious we have $\|u(t) - e^{it\Delta} u_+\|_{H^1} \to 0$, as $t \to \infty$, we need only prove

$$\lim_{t \to \infty} \|x(u(t) - e^{it\Delta} u_+)\|_{L^2} = 0.$$

Let $(\gamma, \rho)$ be an admissible pair with $\rho(\alpha) = \frac{4d}{2d - \alpha(d - 2)}$, and let $p = p(\chi) = \gamma \gamma$ and $\tilde{p}$ be such that $\frac{1}{p} + \frac{1}{\tilde{p}} = \frac{2}{\gamma}$, where $\chi$ is any number satisfying $\frac{1}{2} < \chi < 1$. Since $u_0$ has scattering state $u_+$ at $+\infty$, by Proposition 2.5 and Theorem 2.4, we have

$$\sup_{s \in [0, 1)} \|v(s)\|_{H^1} < \infty,$$

where $v(s)$ is the pseudo-conformal transformation of $u(t)$. Then by Sobolev embedding and (2.12), we infer

$$\sup_{t \in [0, \infty)} (1 + t)\|u(t)\|_{L^\infty} < \infty.$$

By applying the dispersive estimate and Hardy-Littlewood-Sobolev inequality (see [17]) to the Integral equation (1.1), combined with (6.2), we get

$$\|u\|_{L^p((1, \infty), W^{1, \rho})} \leq C\left(\int_1^\infty t^{-\frac{2p}{p+1-3\alpha}} dt\right)^\frac{1}{p} \|u_0\|_{W^{1, \rho'}} + C\int_1^\infty (t - \tau)^{-\frac{2d}{d-2}} \|u(\tau)\|_{W^{1, \rho'}} d\tau \|\int_1^\infty (t - \tau)^{-\frac{2}{\gamma}} \|u(\tau)\|_{W^{1, \rho}} d\tau \|_{L^p((1, \infty))} \leq C + C\|u\|_{L^\infty} \|u\|_{W^{1, \rho'}} \|L^{\rho'}((1, \infty)) \leq C + C\left(\int_1^\infty (1 + \tau)^{-\frac{pp}{p+1-3\alpha}} d\tau\right)^{1+\frac{1}{p} - \frac{3d}{\gamma}} \|u\|_{L^\infty((1, \infty), W^{1, \rho}).}$$
Since $3 \leq d \leq 9$, $\alpha > \frac{16}{3d+2} > \frac{8}{d+6}$, one easy verifies $\frac{\alpha p}{p+1-3\alpha} > 1$ for all $\chi \in (\frac{1}{2}, 1)$, thus we have

\begin{equation}
(\int_1^\infty (1+\tau)^{-\frac{\alpha p}{p+1-3\alpha}} d\tau)^{\frac{1}{p+1-3\alpha}} < \infty.
\end{equation}

Note that $\alpha > \frac{16}{3d+2} > \frac{4}{d+2}$, by Lemma 6.1 we have

\begin{equation}
\|u\|_{L^\gamma((0,\infty), W^{1,\rho}(\mathbb{R}^d))} < \infty.
\end{equation}

Therefore, we deduce from the above three estimates that

\begin{equation}
u \in L^p(\mathbb{R}^d)
\end{equation}

for all $\chi \in (\frac{1}{2}, 1)$, $\alpha > \frac{16}{3d+2}$.

Now let $H(\chi) = \frac{16\alpha}{(d+6)\chi+2d-2}$, for $\frac{1}{2} < \chi < 1$. One easily verifies that $H(\chi)$ is monotone increasing on the interval $(\frac{1}{2}, 1)$, and we have

\begin{equation}
\lim_{\chi \to \frac{1}{2}^+} H(\chi) = \frac{16}{3d+2}.
\end{equation}

Therefore, for arbitrary fixed $\frac{16}{3d+2} < \alpha_0 < \frac{4}{d+2}$, there exists a $\chi_0 \in (\frac{1}{2}, 1)$ such that

\begin{equation}
\alpha_0 > H(\chi_0).
\end{equation}

Let the corresponding index $p_0 = p(\chi_0) = \chi_0 \gamma_0$ and $\gamma_0 = \gamma(\alpha_0) = \frac{8}{(d-2)\alpha_0}$, $\rho_0 = \rho(\alpha_0)$, from (6.2), (6.5) and (6.6) we deduce that

\begin{equation}
\|u(t)\|_{L^{\gamma_0}(t, \infty), W^{1,\rho_0}}
\leq C \left( \int_t^\infty (1+\tau)^{-\frac{\alpha_0}{\rho_0-1}\gamma_0} d\tau \right)^{\frac{1}{\gamma_0}}
\leq C(1+t)^{-\frac{2\alpha_0}{(d-2)\rho_0(\chi_0)^{-1}}}
\end{equation}

for any $t > 0$. Thus by Strichartz’s estimates, we have for all $t > 0$,

\begin{equation}
\|e^{-it\Delta}u(t) - u^+\|_{H^1} \leq C\|u(t)\|_{L^{\gamma_0}(t, \infty), W^{1,\rho_0}} \leq C(1+t)^{-\frac{2\alpha_0}{(d-2)\rho_0(\chi_0)^{-1}}}.
\end{equation}

Applying the commutative properties of the operator $P_t = x + 2it\nabla$ (see (2.6)), a simple calculation shows that

\begin{equation}
x(u(t) - e^{it\Delta}u^+) = e^{it\Delta}[x(e^{-it\Delta}u(t) - u^+) + 2it\nabla(u^+ - e^{-it\Delta}u(t))].
\end{equation}

Using (6.8) and (6.9), we have

\begin{equation}
\|x(u(t) - e^{it\Delta}u^+)\|_{L^2} \leq \|x(e^{-it\Delta}u(t) - u^+)\|_{L^2} + 2t\|\nabla(e^{-it\Delta}u(t) - u^+)\|_{L^2}
\leq \|x(e^{-it\Delta}u(t) - u^+)\|_{L^2} + C(1+t)^{-\frac{2\alpha_0}{(d-2)\rho_0(\chi_0)^{-1}}}.
\end{equation}

(Note that $u_0 \in \mathcal{R}_+$, so we get

\begin{equation}
\|x(e^{-it\Delta}u(t) - u^+)\|_{L^2} \to 0,
\end{equation}

\begin{equation}
\|\nabla(e^{-it\Delta}u(t) - u^+)\|_{L^2} \to 0.
\end{equation}

\begin{equation}
\|x(e^{-it\Delta}u(t) - u^+)\|_{L^2} \to 0,
\end{equation}

\begin{equation}
\|\nabla(e^{-it\Delta}u(t) - u^+)\|_{L^2} \to 0.
\end{equation}
as \( t \to \infty \), note also by (6.6), we have \( \frac{\alpha_0}{H(x_0)} - 1 > 0 \), thus we deduce from (6.10) that

\[
\| x(u(t) - e^{it\Delta} u^+) \|_{L^2} \to 0,
\]

as \( t \to \infty \), it’s the convergence that we need. Note that \( \alpha_0 \) is an arbitrary number satisfying \( \alpha_0 \in \left( \frac{16}{3d+2}, \frac{1}{d-2} \right) \), thus we have proved the conclusion for all \( \frac{16}{3d+2} < \alpha < \frac{4}{d-2} \), \( 3 \leq d \leq 9 \). \( \square \)

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References

[1] P. Bégout, Convergence to scattering states in the nonlinear Schrödinger equation, Commun. Contemp. Math. 3(2001), no.3, 403-418.
[2] T. Cazenave and F.B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. 147(1992), no.1, 75-100.
[3] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10, New York University, Courant Institute of Mathematical Sciences, New York, American Mathematical Society, Providence, RI, 2003.
[4] T. Cazenave and F.B. Weissler, Scattering theory and self-similar solutions for the nonlinear Schrödinger equation, SIAM J. Math. Anal. 31(2000), 625-650.
[5] T. Duyckaerts, J. Holmer, and S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equations, Math. Res. Lett. 15(2008), 1233-1250.
[6] D. Fang, J. Xie and T. Cazenave, Scattering for the focusing, energy-subcritical NLS, to appear in Sci. China Math.
[7] J. Holmer and S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equations, Commun. Math. Phys. 282(2008), 435-467.
[8] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case, J. Funct. Anal. 32(1979), no. 1, 1-32.
[9] J. Ginibre, T. Ozawa and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations, Ann.Inst.H.Poincaré Phys. Théor., 60(1994),211-239.
[10] T. Kato, On nonlinear Schrödinger equations, II. \( H^s \)-solutions and unconditional well-posedness, J. Anal. Math., 67(1995), 281-306.
[11] C.E. Kenig and F. Merle, Global well-posedness, scattering and blow up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case, Invent. Math. 166(2006), 645-675.
[12] M. Keel and T. Tao, Endpoint Strichartz inequalities, Amer. J. Math., 120(1998), 955-980.
[13] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, Amer. J. Math. 132(2010), no.2, 361424.
[14] K. Nakanishi, Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, J. Funct. Anal., 169(1999), 201-225.
[15] K. Nakanishi, Asymptotically-free solutions for the short-range nonlinear Schrödinger equation, SIAM J. Math. Anal., 32(2001), no.6, 1265-1271.
[16] K. Nakanishi and T. Ozawa, Remarks on scattering for nonlinear Schrödinger equations, NoDEA Nonlinear Differential Equations Appl., 9(2002), no.1, 45-68.
[17] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, New Jersey,1993.
[18] X. Song, *Some results on the scattering theory for a Schrödinger equation with combined power-type nonlinearities*, preprint, arxiv: 1104.2684.

[19] T. Tao, *Lecture Notes on Harmonic Analysis*, UCLA, Department of Mathematics, Los Angeles.

[20] Y. Tsutsumi and K. Yajima, *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Am. Math. Soc., 11 (1984), 186-188.

[21] Y. Tsutsumi, *Scattering problem for nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré Physique Théorique, 43 (1985), 321-347.

[22] M. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Commun. Math. Phys, 87 (1982), no.4, 567-576.

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