The rigidity of the graphs of homology spheres minus one edge

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Abstract

We prove that for any prime homology \((d - 1)\)-sphere \(\Delta\) of dimension \(d - 1 \geq 3\) and any edge \(e \in \mathcal{S}\), the graph \(G(\Delta) - e\) is generically \(d\)-rigid. This confirms a conjecture of Nevo and Novinsky.

1 Introduction

The main object of this paper is the notion of generic rigidity. We now briefly mention a few relevant definitions, deferring the rest until later sections. Recall that a \(d\)-embedding of a graph \(G = (V, E)\) is a map \(\psi : V \rightarrow \mathbb{R}^d\). This embedding is called rigid if there exists an \(\epsilon > 0\) such that if \(\psi : V \rightarrow \mathbb{R}^d\) satisfies \(\text{dist}(\phi(u), \psi(u)) < \epsilon\) for every \(u \in V\) and \(\text{dist}(\psi(u), \psi(v)) = \text{dist}(\phi(u), \phi(v))\) for every \(\{u, v\} \in E\), then \(\text{dist}(\psi(u), \psi(v)) = \text{dist}(\phi(u), \phi(v))\) for every \(u, v \in V\). A graph \(G\) is called generically \(d\)-rigid if the set of rigid \(d\)-embeddings of \(G\) is open and dense in the set of all \(d\)-embeddings of \(G\).

The first substantial mathematical result concerning rigidity can be dated back to 1813, when Cauchy proved that any bijection between the vertices of two convex 3-polytopes that induces a combinatorial isomorphism and an isometry of the facets, induces an isometry of the two polytopes. Based on Cauchy’s theorem and on later results by Dehn and Alexandrov, in 1975 Gluck [6] gave a complete proof of the fact that the graphs of all simplicial 3-polytopes are generically 3-rigid. Later Whiteley [11] extended this result to the graphs of simplicial \(d\)-polytopes for any \(d \geq 3\). Many other generalizations have been made since, including, for example, the following theorem proved by Fogelsanger.

**Theorem 1.1.** [5] Let \(d \geq 3\). The graph of a minimal \((d - 1)\)-cycle complex is generically \(d\)-rigid. In particular, the graphs of all homology \((d - 1)\)-spheres are generically \(d\)-rigid.

The rigidity theory of frameworks is a very useful tool for tackling the lower bound conjectures. For a \((d - 1)\)-dimensional simplicial complex \(\Delta\), we define \(g_2(\Delta) := f_1(\Delta) - df_0(\Delta) + \binom{d+1}{2}\), where \(f_1\) and \(f_0\) are the numbers of edges and vertices of \(\Delta\), respectively. By interpreting \(g_2(\Delta)\) as the dimension of the left kernel of the rigidity matrix of \(\Delta\), Kalai [7] proved that the \(g_2\)-number of an arbitrary triangulated manifold \(\Delta\) of dimension at least three is nonnegative (thus reproving the Lower Bound Theorem due to Barnette [3], [4]). Furthermore, Kalai showed that \(g_2(\Delta) = 0\) is attained if and only if \(\Delta\) is a stacked sphere. Kalai’s theorem was then extended to the class of
normal pseudomanifolds by Tay [10], where Theorem 1.1 served as a key ingredient in the proof. We refer to [8] for another application of the rigidity theory to the Balanced Lower Bound Theorem.

It might be tempting to conjecture that the graph of a non-stacked homology sphere $\Delta$ minus any edge of $\Delta$ is also generically $d$-rigid. This is not true in general; for example, let $\Delta$ be obtained by stacking over a facet of any $(d - 1)$-sphere $\Gamma$, and let $e$ be any edge not in $\Gamma$. In this case the graph of $\Delta - e$ is not generically $d$-rigid. However, Nevo and Novinsky [9] showed that this statement does hold if, in addition, one requires that $\Delta$ is prime (i.e., $\Delta$ has no missing facets) and $g_2(\Delta) = 1$. They raised the following question.

**Problem 1.2.** [9, Problem 2.11] Let $d \geq 4$ and let $\Delta$ be a prime homology $(d - 1)$-sphere. Is it true that for any edge $e$ in $\Delta$, the graph $G(\Delta) - e$ is generically $d$-rigid?

In this paper we give an affirmative answer to the above problem. The proof is based on the rigidity theory of frameworks. Specifically, we first verify the base cases $d = 4$ and $g_2 = 1$, and then prove the result by inducting on both the dimension and the value of $g_2$.

The paper is organized as follows. In Section 2 after reviewing some preliminaries on simplicial complexes, we introduce the rigidity theory of frameworks and summarize several well-known results in this field. We then prove our main result (Theorem 3.4) in Section 3.

## 2 Preliminaries

A *simplicial complex* $\Delta$ on vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \subseteq \Delta$. The *dimension* of a face $\sigma$ is $\dim(\sigma) = |\sigma| - 1$, and the *dimension* of $\Delta$ is $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \subseteq \Delta\}$. The *facets* of $\Delta$ are maximal faces of $\Delta$ under inclusion. We say that a simplicial complex $\Delta$ is *pure* if all of its facets have the same dimension. A *missing* face of $\Delta$ is any subset $\sigma$ of $V(\Delta)$ such that $\sigma$ is not a face of $\Delta$ but every proper subset of $\sigma$ is. A pure simplicial complex $\Delta$ is *prime* if it does not have any missing facets.

For a simplicial complex $\Delta$, we denote the graph of $\Delta$ by $G(\Delta)$. If $G = (V, E)$ is a graph and $U \subseteq V$, then the *restriction* of $G$ to $U$ is the subgraph $G|_U$ whose vertex set is $U$ and whose edge set consists of all of the edges in $E$ that have both endpoints in $U$. We denote by $C(G)$ the graph of the cone over a graph $G$, and by $K(V)$ the complete graph on the vertex set $V$. For brevity of notation, in the following we will use $G + e$ (resp. $G - e$) to denote the graph obtained by adding an edge $e$ to (resp. deleting $e$ from) $G$.

In this paper we focus on the graphs of a certain class of simplicial complexes. Given an edge $e = \{a, b\}$ of a simplicial complex $\Delta$, the contraction of $e$ to a new vertex $v$ in $\Delta$ is the simplicial complex

$$\Delta^{kv} := \{F \in \Delta : a, b \not\in F\} \cup \{F \cup \{v\} : F \cap \{a, b\} = \emptyset\text{ and either } F \cup \{a\} \in \Delta\text{ or } F \cup \{b\} \in \Delta\}.$$  

A simplicial complex $\Delta$ is a *simplicial sphere* if the geometric realization of $\Delta$, denoted as $||\Delta||$, is homeomorphic to a sphere. Let $\tilde{H}_*(\Gamma, k)$ denote the reduced singular homology of $||\Gamma||$ with coefficients in $k$. The *link* of a face $\sigma$ is $\text{lk}_\Delta \sigma := \{\tau - \sigma : \tau \subseteq \Delta, \sigma \subseteq \tau\}$, and the *star* of $\sigma$ is $\text{st}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$. For a pure $(d - 1)$-dimensional simplicial complex $\Delta$ and a field $k$, we say that $\Delta$ is a homology sphere over $k$ if $\tilde{H}_*(\text{lk}_\Delta \sigma; k) \cong \tilde{H}_*(S^{d - 1 - |\sigma|}; k)$ for every face $\sigma \in \Delta$, including the empty face. We have the following inclusion relations:
boundary complexes of simplicial $(d-1)$-polytopes ⊆ simplicial $(d-1)$-spheres
⊆ homology $(d-1)$-spheres.

It follows from Steinitz’s theorem that when $d = 3$, all three classes above coincide. When $d \geq 4$, all three inclusions are strict.

We are now in a position to review basic definitions of rigidity theory of frameworks. Given a graph $G$ and a $d$-embedding $\phi$ of $G$, we define the matrix $\operatorname{Rig}(G, \phi)$ associated with a graph $G$ as follows: it is an $f_1(G) \times df_0(G)$ matrix with rows labeled by edges of $G$ and columns grouped in blocks of size $d$, with each block labeled by a vertex of $G$; the row corresponding to $\{u,v\} \in E$ contains the vector $\phi(u) - \phi(v)$ in the block of columns corresponding to $u$, the vector $\phi(v) - \phi(u)$ in columns corresponding to $v$, and zeros everywhere else. It is easy to see that for a generic $\phi$ the dimensions of the kernel and image of $\operatorname{Rig}(G, \phi)$ are independent of $\phi$. Hence we define the rigidity matrix of $G$ as $\operatorname{Rig}(G, d) = \operatorname{Rig}(G, \phi)$ for a generic $\phi$. It follows from [2] that $G$ is generically $d$-rigid if and only if $\operatorname{rank}(\operatorname{Rig}(G, d)) = df_0(G) - \binom{d+1}{2}$. The following lemmas summerize a few additional results on framework rigidity.

**Lemma 2.1** (Cone Lemma, [11]). $G$ is generically $(d - 1)$-rigid if and only if $C(G)$ is generically $d$-rigid.

Since the star of any face $\sigma$ in a homology sphere is the join of $\sigma$ with the link of $\sigma$, and since the link of $\sigma$ is a homology sphere, Theorem 1.1 along with the cone lemma implies the following corollary.

**Corollary 2.2.** Let $d \geq 4$ and let $\Delta$ be a homology $(d - 1)$-sphere. Then the graph of $\text{st}_\Delta \sigma$ is generically $d$-rigid for any face $\sigma$ with $|\sigma| \leq d - 3$.

**Lemma 2.3** (Gluing Lemma, [2]). Let $G_1$ and $G_2$ be generically $d$-rigid graphs such that $G_1 \cap G_2$ has at least $d$ vertices. Then $G_1 \cup G_2$ is also generically $d$-rigid.

**Lemma 2.4** (Replacement Lemma, [7]). Let $G$ be a graph and $U$ a subset of $V(G)$. If both $G|_U$ and $G \cup K(U)$ are generically $d$-rigid, then $G$ is generically $d$-rigid.

Finally we state a variation of the gluing lemma.

**Lemma 2.5.** Let $G_1$ and $G_2$ be two graphs, and assume that $a, b \in U = V(G_1 \cap G_2)$. Assume further that $G_1$ and $G_2$ satisfy the following conditions: 1) the set $U$ contains at least $d$ vertices, including $a$ and $b$, 2) both $G_1$ and $G_2 + \{a, b\}$ are generically $d$-rigid, and 3) $G_1|_U = G_2|_U$. Then $G_1 \cup G_2$ is also generically $d$-rigid.

**Proof:** The second condition implies that $G_1 + \{a, b\}$ is generically $d$-rigid. Since $G_i + \{a, b\}$ are generically $d$-rigid for $i = 1, 2$ and their intersection contains at least $d$ vertices, by the gluing lemma, $G := (G_1 \cup G_2) + \{a, b\}$ is generically $d$-rigid. Note that by condition 3), the restriction of $G$ to $V(G_1)$ is $G_1 + \{a, b\}$. Replacing $G_1 + \{a, b\}$ by the generically $d$-rigid graph $G_1$, we obtain the graph $G_1 \cup G_2$, which is also generically $d$-rigid by the replacement lemma.

### 3 Proof of the main theorem

In this section we will prove our main result, Theorem 3.4. We begin with the following lemma that is originally due to Kalai. We give a proof here for the sake of completeness.
Lemma 3.1. Let \( d \geq 4 \) and let \( \Delta \) be a homology \((d - 1)\)-sphere. If \( \sigma \) is a missing \( k \)-face in \( \Delta \) and \( 2 \leq k \leq d - 2 \), then \( G(\Delta) - e \) is generically \( d \)-rigid for any edge \( e \subseteq \sigma \).

Proof:  Let \( \tau = \sigma \setminus e \). The dimension of \( \text{lk}_\Delta \tau \) is

\[
\dim \text{lk}_\Delta \tau = d - 1 - |\tau| = (d - 1) - (|\sigma| - 2) \geq d + 1 - (d - 1) = 2,
\]

so \( \text{lk}_\Delta \tau \) is generically \((d - |\tau|)\)-rigid. By Corollary 2.2, \( \text{st}_\Delta \tau \) is generically \( d \)-rigid. Note that \( e \notin \text{st}_\Delta \tau \), and the induced subgraph of \( G(\Delta) \) on \( W = V(\text{st}_\Delta \tau) \) contains a generically \( d \)-rigid subgraph \( G(\text{st}_\Delta \tau) \). Applying the replacement lemma on \( W \) (that is, replacing \( G(\Delta)|_W \) with \( G(\Delta)|_{W - e} \)), we conclude that the resulting graph \( G(\Delta) - e \) is also generically \( d \)-rigid.

The following proposition was mentioned in [9] without a proof.

Proposition 3.2. Let \( \Delta \) be a prime homology \((d - 1)\)-sphere with \( g_2(\Delta) = 1 \), where \( d \geq 4 \). Then for any edge \( e \in \Delta \), the graph \( G(\Delta) - e \) is generically \( d \)-rigid.

Proof:  By Theorem 1.3 in [9], \( \partial \Delta = \sigma_1 \ast \sigma_2 \), where \( \sigma_1 \) is the boundary complex of an \( i \)-simplex for some \( i \geq \frac{d+1}{2} \), and \( \sigma_2 \) is either the boundary complex of a \((d + 1 - i)\)-simplex, or a cycle graph \((c_1, \ldots, c_k)\) when \( i = d - 2 \). If \( e \in \sigma_1 \), then \( G(\Delta) - e \) is generically \( d \)-rigid by Lemma 3.1. Now assume that \( e \) contains a vertex \( v \) in \( \sigma_2 \). Note that \( \sigma_2 \setminus \{v\} \) is either a simplex or a path graph. In the former case, the graph of \( \Delta \setminus \{v\} \) is the complete graph on \( d + 1 \) vertices, and hence it is generically \( d \)-rigid. In the latter case, since the graph of \( \sigma_2 \ast \{c_i, c_{i+1}\} \) is also the complete graph on \( d + 1 \) vertices, by the gluing lemma, \( G(\Delta \setminus \{v\}) \) is generically \( d \)-rigid. Finally, the graph \( G(\Delta) - e \) is obtained by adding to \( G(\Delta \setminus \{v\}) \) the vertex \( v \) and \( \deg v - 1 \geq d \) edges containing \( v \). Hence \( G(\Delta) - e \) is generically \( d \)-rigid.

Proposition 3.3. Let \( \Delta \) be a prime homology 3-sphere. For any edge \( e \in \Delta \), the graph \( G(\Delta) - e \) is generically 4-rigid.

Proof:  The proof has a similar flavor to the proof of Proposition 1 in [12]. If \( e \) is an edge in a missing 2-face of \( \Delta \), then by Lemma 3.1, \( G(\Delta) - e \) is generically 4-rigid. Now assume that \( e = \{a, b\} \) does not belong to any missing 2-face of \( \Delta \). We claim that \( \text{lk}_\Delta \ e = \text{lk}_\Delta a \cap \text{lk}_\Delta b \). If \( v \in \text{lk}_\Delta a \cap \text{lk}_\Delta b \), then \( e = \{a, b\}, \{a, v\} \) and \( \{b, v\} \) are edges of \( \Delta \). Hence, by our assumption, \( \{a, b, v\} \subseteq \Delta \), and so \( v \in \text{lk}_\Delta e \). Also if \( e' = \{c, d\} \in \text{lk}_\Delta a \cap \text{lk}_\Delta b \), then \( e' \ast \partial e \subseteq \Delta \). Since \( e \) does not belong to any missing 2-face of \( \Delta \), it follows that \( c, d \in \text{lk}_\Delta e \). Hence \( (e' \ast \partial e) \cup (e \ast \partial e') \subseteq \Delta \), which by the primeness of \( \Delta \) implies that \( e \ast e' \subseteq \Delta \), i.e., \( e' \in \text{lk}_\Delta e \). Finally, if \( \text{lk}_\Delta a \cap \text{lk}_\Delta b \) contains a 2-dimensional face \( \tau \) whose boundary edges are \( e_1, e_2 \) and \( e_3 \), then the above argument implies that \( e_i \cup \{b\} \subseteq \text{lk}_\Delta a \) for \( i = 1, 2, 3 \). Hence \( \partial (\tau \cup \{b\}) \subseteq \text{lk}_\Delta a \), and so \( \text{lk}_\Delta a \) is the boundary complex of a 3-simplex. This contradicts the fact that \( \Delta \) is prime. We conclude that both \( \text{lk}_\Delta e \) and \( \text{lk}_\Delta a \cap \text{lk}_\Delta b \) are 1-dimensional. Furthermore, \( \text{lk}_\Delta a \cap \text{lk}_\Delta b \subseteq \text{lk}_\Delta e \). However, it is obvious that the reverse inclusion also holds. This proves the claim.

If \( \text{lk}_\Delta e \) is a 3-cycle, then the filled-in triangle \( \tau \) determined by \( \text{lk}_\Delta e \) is not a face of \( \Delta \). Otherwise, by the fact that \( \tau \cup (\text{lk}_\Delta e \ast \{a\}) \) and \( \tau \cup (\text{lk}_\Delta e \ast \{b\}) \) are subcomplexes of \( \Delta \) and by the primeness of \( \Delta \), we obtain that \( \tau \cup \{a\}, \tau \cup \{b\} \subseteq \Delta \). Then since \( \text{lk}_\Delta e = \text{lk}_\Delta a \cap \text{lk}_\Delta b \), we conclude that \( \tau \in \text{lk}_\Delta e \), contradicting that \( \text{lk}_\Delta e \) is 1-dimensional. Hence we are able to construct a new sphere \( \Gamma \) from \( \Delta \) by replacing \( \text{st}_\Delta e \) with the suspension of \( \tau \) (indeed, \( \Gamma \) and \( \Delta \) differ in a bistellar flip),
and therefore \( G(\Delta) - e = G(\Gamma) \) is generically 4-rigid. Next we assume that \( \text{lk}_\Delta e \) has at least 4 vertices. By [9, Proposition 2.3], the edge contraction \( \Delta^{ie} \) of \( \Delta \) is also a homology sphere. Assume that in a \( d \)-embedding \( \psi \) of \( G(\Delta) \), both \( a \) and \( b \) are placed at the origin, \( V(\text{lk}_\Delta e) = \{u_1, \ldots, u_t\} \), \( V(\text{lk}_\Delta a) - V(\text{lk}_\Delta e) = \{v_1, \ldots, v_m\} \) and \( V(\text{lk}_\Delta b) - V(\text{lk}_\Delta e) = \{w_1, \ldots, w_n\} \). The rigidity matrix of \( G(\Delta) - e \) can be written as a block matrix

\[
M := \text{Rig}(G(\Delta) - e, \psi) = \begin{pmatrix} A & B \\ 0 & R \end{pmatrix},
\]

where the columns of \( B \) and \( R \) correspond to the vertices in \( \text{st}_\Delta a \cup \text{st}_\Delta b \), and the rows of \( R \) correspond to the edges containing either \( a \) or \( b \) but not both. For convenience, we write \( v_i \) (resp. \( u_i, w_i \)) to represent \( \psi(v_i) \) (resp. \( \psi(u_i) \) and \( \psi(w_i) \)). Then

\[
R = \begin{pmatrix}
\begin{array}{cccccccc}
v_1 & \ldots & v_m & u_1 & \ldots & u_\ell & w_1 & \ldots & w_n & a & b \\
\vdots & & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
v_m & & u_1 & & u_\ell & & w_1 & & w_n \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
u_1 & & \ldots & & \ldots & & \ldots & & \ldots \\
\end{array}
\end{pmatrix}
\]

where the rest of the entries not indicated above are 0. We apply the following row and column operations to matrix \( M \): first add the last four columns, i.e. columns corresponding to \( b \) to the corresponding columns of \( a \), then subtract row \((*i)\) from the row \((**i)\) for \( i = 1, \ldots, \ell \). This gives

\[
M'(\psi') = \begin{pmatrix} \text{Rig}(G(\Delta^{ie}), \psi') \begin{array}{cc} \ast \\ 0 \end{array} \begin{array}{c} -u_1 \\ \vdots \\ 0 \end{array} \\ \end{pmatrix},
\]

where \( \psi' \) is the 4-embedding of \( G(\Delta^{ie}) \) induced by \( \psi \), where \( \psi'(v) = \psi(a) = \psi(b) \) for the new vertex \( v \), and \( \psi'(x) = \psi(x) \) for all other vertices \( x \neq a, b \). Since \( \ell = |V(\text{lk}_\Delta e)| \geq 4 \), it follows that the last four columns of \( M'(\psi') \) are linearly independent. Hence for a generic \( \psi' \),

\[
\text{rank}(M) = \text{rank}(M'(\psi')) = \text{rank}(\text{Rig}(G(\Delta^{ie}), 4)) + 4 = (4f_0(\Delta^{ie}) - 10) + 4 = 4f_0(\Delta) - 10.
\]

Since \( 4f_0(\Delta) - 10 \) is the maximal rank that the rigidity matrix of a 4-dimensional framework with \( f_0(\Delta) \) vertices can have, and a small generic perturbation of \( a \) and \( b \) preserves the rank of the rigidity matrix, we conclude that \( \text{rank}(\text{Rig}(G(\Delta) - e, 4)) = \text{rank}(M) = 4f_0(\Delta) - 10 \). Hence \( G(\Delta) - e \) is generically 4-rigid. \( \square \)
In the following we generalize the previous proposition to the case of \( d > 4 \) by inducting on the dimension and the value of \( g_2 \). We fix some notation here. If a homology \((d-1)\)-sphere \( \Delta \) is the connected sum of \( n \) prime homology spheres \( S_1, \ldots, S_n \), then each \( S_i \) is called a prime factor of \( \Delta \). In particular, \( \Delta \) is called stacked if each \( S_i \) is the boundary complex of a \( d \)-simplex. For every stacked \((d-1)\)-sphere \( \Delta \) with \( d \geq 3 \), there exists a unique simplicial \( d \)-ball with the same vertex set as \( \Delta \) and whose boundary complex is \( \Delta \); we denote it by \( \Delta(1) \). We refer to such a ball as a stacked ball.

**Theorem 3.4.** Let \( d \geq 4 \) and let \( \Delta \) be a prime homology \((d-1)\)-sphere with \( g_2(\Delta) > 0 \). Then for any edge \( e \in G(\Delta) \), the graph \( G(\Delta) - e \) is generically \( d \)-rigid.

**Proof:** The two base cases \( g_2(\Delta) = 1, d \geq 4 \) and \( d = 4, g_2(\Delta) \geq 1 \) are proved in Proposition 3.2 and 3.3 respectively. Now we assume that the statement is true for every prime homology \((d_0-1)\)-sphere \( S \) with \( 5 \leq d_0 \leq d \) and \( 1 \leq g_2(S) < g_2(\Delta) \) and every edge \( e \in S \). The result follows from the following two claims. \( \square \)

**Claim 3.5.** Under the above assumptions, if, furthermore, \( g_2(\text{lk}_\Delta u) = 0 \) for some vertex \( u \in V(\Delta) \), then \( G(\Delta) - e \) is generically \( d \)-rigid for any edge \( e \in \Delta \).

**Proof:** Since \( \text{lk}_\Delta u \) is at least 3-dimensional and since \( g_2(\text{lk}_\Delta u) = 0 \), it follows that \( \text{lk}_\Delta u \) is a stacked sphere. Also since \( \Delta \) is prime, the interior faces of the stacked ball \( (\text{lk}_\Delta u)(1) \) are not faces of \( \Delta \) (or otherwise such a face together with \( u \) will form a missing facet of \( \Delta \)). Let

\[
\Gamma := (\Delta \setminus \{u\}) \cup (\text{lk}_\Delta u)(1).
\]

Then \( \Gamma \) is a homology \((d-1)\)-sphere but not necessarily prime. (For more details on this and similar operations, see [13].) Also by the primeness of \( \Delta \), every missing facet \( \sigma \) of \( \Gamma \) must contain a missing facet of \( \text{lk}_\Delta u \). Pick a missing facet \( \tau \) of \( \text{lk}_\Delta u \) and assume that there are \( k \) prime factors of \( \Gamma \) that contain \( \tau \). We first find two facets of \( (\text{lk}_\Delta u)(1) \) that contain \( \tau \) and say they are \( \{v_0\} \cup \tau \) and \( \{v_k\} \cup \tau \). Now assume that the \( k \) prime factors of \( \Gamma \) are \( S_1, S_2, \ldots, S_k \), and each of them satisfies \( S_i \cap S_{i+1} = \tau \cup \{v_i\} \) for some other vertices \( v_1, \ldots, v_{k-1} \in \Delta \) and \( 1 \leq i \leq k-1 \). Furthermore, \( S_j \cap (\text{lk}_\Delta u)(1) = \tau \cup \{v_j\} \) for \( j = 0, k \). Let \( G_\tau := E \cup (\bigcup_{i=1}^{k} G(S_i)) \), where \( E \) is the set of edges connecting \( u \) and the vertices in \( \tau \cup \{v_0, v_k\} \). Since an arbitrary edge \( e \) of \( G(\Delta) \) either contains \( u \) or belongs to one of \( S_i \)'s, it follows that \( G(\Delta) - e = \cup(G_\tau - e) \), where the union is taken over all missing facets \( \tau \) of \( \text{lk}_\Delta u \). By the gluing lemma, it suffices to show that \( G_\tau - e \) is generically \( d \)-rigid for any \( \tau \) and edge \( e \in G_\tau \). We consider the following two cases:

**Case 1:** \( e \in S_i \) for some \( i \), \( S_i \) is not the boundary complex of the \( d \)-simplex, and \( e \notin S_j \) for any other \( j \neq i \). Since \( G(S_i) \) is a generically \( d \)-rigid subgraph of \( \Gamma \), it follows that

\[
g_2(S_i) \leq g_2(\Gamma) = g_2(\Delta) - f_0(\text{lk}_\Delta u) + d < g_2(\Delta).
\]

Furthermore, by the inductive hypothesis on \( g_2 \), \( G(S_i) - e \) is generically \( d \)-rigid for any edge \( e \in \Delta \). Also since \( G(S_i) \) is the induced subgraph of \( G_\tau \) on \( V(S_i) \), by the replacement lemma, \( G_\tau - e \) is generically \( d \)-rigid.

**Case 2:** either \( e \in S_i \) for some \( i \) and \( S_i \) is the boundary complex of the \( d \)-simplex (in this case the edge \( \{v_{i-1}, v_i\} \in S_i \)), or \( e \in \text{lk}_\Delta u \), or \( u \in e \). Hence \( e \in G_\tau' := G(\tau + C) \), where \( C \) is the cycle graph \((u, v_0, \ldots, v_k)\). By Lemma 3.2 \( G_\tau' - e \) is generically \( d \)-rigid for any edge \( e \). The graph \( G_\tau - e \)
(a) The subgraph $G_\tau$: fixing a missing facet $\tau$ of $\text{lk}_\Delta u$, there are three corresponding prime factors $S_1, S_2, S_3$. Here $S_2$ is the boundary complex of the 3-simplex.

(b) $G'_\tau$, obtained from $G_\tau$ by replacing all the blue edges in $G_\tau$ with the red edges $\{v_0, v_1\}$ and $\{v_2, v_3\}$.

Figure 1: The corresponding graphs $G_\tau$ and $G'_\tau$, given the graph $G = G(\Delta)$ and a missing facet in a vertex link.

can be recovered from $G'_\tau - e$ by replacing each edge $\{v_{i-1}, v_i\}$ with the edges in $S_i \setminus G'_\tau$ whenever $S_i$ is not the boundary complex of the $d$-simplex. Note that nothing needs to be done when $S_i$ is the boundary complex of a simplex, since $S_i$ is already a subcomplex of $G'_\tau$. (See Figure 1 for an illustration in a lower dimension case.) Repeatedly applying Lemma 2.5 with $\{a, b\} = \{v_{i-1}, v_i\}$, $G_1 = G(S_i) - e$ and $G_2 + \{a, b\} = G'_\tau - e$, we conclude that $G_\tau - e$ is also generically $d$-rigid. □

Claim 3.6. Under the above assumption, if, furthermore, every vertex link of $\Delta$ has $g_2 \geq 1$, then $G(\Delta) - e$ is generically $d$-rigid for any edge $e \in \Delta$.

Proof: Assume that there is a vertex $u \in \Delta$ such that $\text{lk}_\Delta u$ is the connected sum of prime factors $S_1, \ldots, S_k$ and $e = \{v, w\} \in S_i$ for some $i$. If $e$ is an edge in a missing facet of $\text{lk}_\Delta u$ (which is also a missing $(d-2)$-face of $\Delta$), then by Lemma 3.1 $G(\Delta) - e$ is generically $d$-rigid.

Otherwise, assume first that $g_2(S_i) \neq 0$. Then $G(S_i) - e$ is generically $(d-1)$-rigid by the inductive hypothesis on the dimension. Hence by the gluing lemma and cone lemma, we obtain that $G(\text{st}_\Delta u) - e$ is generically $d$-rigid. By the replacement lemma, $G(\Delta) - e$ is generically $d$-rigid.

Finally, assume that $S_i$ is the boundary complex of a $(d-1)$-simplex, or equivalently, $\text{lk}_\Delta\{u, v, w\}$ is the boundary complex of a $(d-3)$-simplex. If furthermore for any vertex $x \in \text{lk}_\Delta e$, the link of $\{x, v, w\}$ in $\Delta$ is the boundary complex of $(d-3)$-simplex, then $\text{lk}_\Delta e$ must be the boundary complex of a $(d-2)$-simplex. Hence $\text{lk}_\Delta v$ is obtained by adding a pyramid over a facet $\sigma$ of some $(d-2)$-sphere, and $w$ is the apex of the pyramid. Now we construct a new homology $(d-1)$-sphere $\Delta'$ as follows: first delete the edge $e$ from $\Delta$, then add the faces $\sigma, \sigma \cup \{v\}$ and $\sigma \cup \{w\}$ to $\Delta$. It follows that $G(\Delta') = G(\Delta) - e$, which implies that $G(\Delta) - e$ is generically $d$-rigid.

Otherwise, there exists a vertex $x$ such that $\text{lk}_\Delta\{x, v, w\}$ is not the boundary complex of $(d-3)$-simplex. Then we may show that $G(\Delta) - e$ is generically $d$-rigid by applying the same argument as above on $\text{lk}_\Delta x$. This proves the claim. □
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