ON SHATTERING, SPLITTING AND REAPING PARTITIONS

Lorenz Halbeisen
Université de Caen
France

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Abstract

In this article we investigate the dual-shattering cardinal \( \mathcal{H} \), the dual-splitting cardinal \( \mathcal{S} \) and the dual-reaping cardinal \( \mathcal{R} \), which are dualizations of the well-known cardinals \( h \) (the shattering cardinal, also known as the distributivity number of \( P(\omega)/\text{fin} \)), \( s \) (the splitting number) and \( r \) (the reaping number). Using some properties of the ideal \( \mathcal{J} \) of nowhere dual-Ramsey sets, which is an ideal over the set of partitions of \( \omega \), we show that \( \text{add}(\mathcal{J}) = \text{cov}(\mathcal{J}) = \mathcal{H} \). With this result we can show that \( \mathcal{H} > \omega_1 \) is consistent with ZFC and as a corollary we get the relative consistency of \( \mathcal{H} > t \), where \( t \) is the tower number. Concerning \( \mathcal{S} \) we show that \( \text{cov}(\mathcal{M}) \leq \mathcal{S} \) (where \( \mathcal{M} \) is the ideal of the meager sets). For the dual-reaping cardinal \( \mathcal{R} \) we get \( p \leq \mathcal{R} \leq r \) (where \( p \) is the pseudo-intersection number) and for a modified dual-reaping number \( \mathcal{R}' \) we get \( \mathcal{R}' \leq d \) (where \( d \) is the dominating number). As a consistency result we get \( \mathcal{R} < \text{cov}(\mathcal{M}) \).

1 The set of partitions

A partial partition \( X \) (of \( \omega \)) consisting of pairwise disjoint, nonempty sets, such that \( \text{dom}(X) := \bigcup X \subseteq \omega \). The elements of a partial partition \( X \) are called the blocks of \( X \) and \( \text{Min}(X) \) denotes the set of the least elements of the blocks of \( X \). If \( \text{dom}(X) = \omega \), then \( X \) is called a partition. \( \{\omega\} \) is the partition such that each block is a singleton and \( \{\{\omega\}\} \) is the partition containing only one block. The set of all partitions containing infinitely (resp. finitely) many blocks is denoted by \( (\omega)^{\omega} \) (resp. \( (\omega)^{<\omega} \)). By \( (\omega)^{\omega} \) we denote the set of all infinite partitions such that at least one block is infinite. The set of all partial partitions with \( \text{dom}(X) \in \omega \) is denoted by \( (\mathbb{N}) \).

Let \( X_1, X_2 \) be two partial partitions. We say that \( X_1 \) is coarser than \( X_2 \), or that \( X_2 \) is finer than \( X_1 \), and write \( X_1 \subseteq X_2 \) if for all blocks \( b \in X_1 \) the set \( b \cap \text{dom}(X_2) \) is the union of some sets \( b_i \cap \text{dom}(X_1) \), where each \( b_i \) is a block of \( X_2 \). (Note that if \( X_1 \) is coarser than \( X_2 \), then \( X_1 \) is in a natural way also contained in \( X_2 \).) Let \( X_1 \cap X_2 \) denotes the finest partial partition which is coarser than \( X_1 \) and \( X_2 \) such that \( \text{dom}(X_1 \cap X_2) = \text{dom}(X_1) \cup \text{dom}(X_2) \). Similarly \( X_1 \sqcup X_2 \) denotes the coarsest partial partition which is finer than \( X_1 \) and \( X_2 \) such that \( \text{dom}(X_1 \sqcup X_2) = \text{dom}(X_1) \cup \text{dom}(X_2) \).

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If $f$ is a finite subset of $\omega$, then $\{f\}$ is a partial partition with $\text{dom}(\{f\}) = f$. For two partial partitions $X_1$ and $X_2$ we write $X_1 \sqsubseteq^* X_2$ if there is a finite set $f \subseteq \text{dom}(X_1)$ such that $X_1 \cap \{f\} \subseteq X_2$ and say that $X_1$ is coarser* than $X_2$. If $X_1 \sqsubseteq^* X_2$ and $X_2 \sqsubseteq^* X_1$ then we write $X_1 \equiv X_2$. If $X \equiv \{\omega\}$, then $X$ is called trivial.

Let $X_1, X_2$ be two partial partitions. If each block of $X_1$ can be written as the intersection of a block of $X_2$ with $\text{dom}(X_1)$, then we write $X_1 \preceq X_2$. Note that $X_1 \preceq X_2$ implies $\text{dom}(X_1) \subseteq \text{dom}(X_2)$.

We define a topology on the set of partitions as follows. Let $X \in (\omega)^\omega$ and $s \in (\mathbb{N})$ such that $s \subseteq X$, then $(s,X)^\omega := \{Y \in (\omega)^\omega : s \preceq Y \land Y \subseteq X\}$ and $(X)^\omega := (\emptyset,X)^\omega$. Now let the basic open sets on $(\omega)^\omega$ be the sets $(s,X)^\omega$ (where $X$ and $s$ as above). These sets are called the dual Ellentuck neighborhoods.

The topology induced by the dual Ellentuck neighborhoods is called the dual Ellentuck topology (cf. [CS]).

2 On the dual-shattering cardinal $\mathfrak{f}$

Four cardinals

We first give the definition of the dual-shattering cardinal $\mathfrak{f}$.

Two partitions $X_1, X_2 \in (\omega)^\omega$ are called almost orthogonal ($X_1 \perp_* X_2$) if $X_1 \cap X_2 \notin (\omega)^\omega$, otherwise they are compatible ($X_1 \parallel X_2$). If $X_1 \cap X_2 = \{\omega\}$, then they are called orthogonal ($X_1 \perp X_2$). We say that a family $A \subseteq (\omega)^\omega$ is maximal almost orthogonal (mao) if $A$ is a maximal family of pairwise almost orthogonal partitions. A family $\mathcal{H}$ of mao families of partitions shatters a partition $X \in (\omega)^\omega$, if there are $H \in \mathcal{H}$ and two distinct partitions in $H$ which are both compatible with $X$. A family of mao families of partitions is shattering if it shatters each member of $(\omega)^\omega$. The dual-shattering cardinal $\mathfrak{f}$ is the least cardinal number $\kappa$, for which there exists a shattering family of cardinality $\kappa$.

One can show that $\mathfrak{f} \leq \mathfrak{f}$ and $\mathfrak{f} \leq \mathfrak{D}$ (cf. [CMW]), (where $\mathfrak{D}$ is the dual-splitting cardinal).

Two cardinals related to the ideal of nowhere dual-Ramsey sets

Let $C \subseteq (\omega)^\omega$ be a set of partitions, then we say that $C$ has the dual-Ramsey property or that $C$ is dual-Ramsey, if there is a partition $X \in (\omega)^\omega$ such that $(X)^\omega \subseteq C$ or $(X)^\omega \cap C = \emptyset$. If the latter case holds, we also say that $C$ is dual-Ramsey$^0$. If for each dual Ellentuck neighborhood $(s,Y)^\omega$ there is an $X \in (s,Y)^\omega$ such that $(s,X)^\omega \subseteq C$ or $(s,X)^\omega \cap C = \emptyset$, we call $C$ completely dual-Ramsey. If for each dual Ellentuck neighborhood the latter case holds, we say that $C$ is nowhere dual-Ramsey.

REMARK 1: In [CS] it is proved, that a set is completely dual-Ramsey if and only if it has the Baire property and it is nowhere dual-Ramsey if and only if it is meager with respect to the dual Ellentuck topology. From this it follows, that a set is nowhere dual-Ramsey if and only if the complement contains a dense and open subset (with respect to the dual Ellentuck topology).
Let $\mathcal{J}$ be set of partitions which are completely dual-Ramsey. The set $\mathcal{J} \subseteq \mathcal{P}(\omega^\omega)$ is an ideal which is not prime. The cardinals $\text{add}(\mathcal{J})$ and $\text{cov}(\mathcal{J})$ are two cardinals related to this ideal.

$\text{add}(\mathcal{J})$ is the smallest cardinal $\kappa$ such that there exists a family $\mathcal{F} = \{ J_\alpha \in \mathcal{J} : \alpha < \kappa \}$ with $\bigcup \mathcal{F} \not\in \mathcal{J}$.

$\text{cov}(\mathcal{J})$ is the smallest cardinal $\kappa$ such that there exists a family $\mathcal{F} = \{ J_\alpha \in \mathcal{J} : \alpha < \kappa \}$ with $\bigcup \mathcal{F} = (\omega)^\omega$.

Because $(\omega)^\omega \not\in \mathcal{J}$, it is clear that $\text{add}(\mathcal{J}) \leq \text{cov}(\mathcal{J})$. Further it is easy to see that $\omega_1 \leq \text{add}(\mathcal{J})$. In the next section we will show that $\text{add}(\mathcal{J}) = \text{cov}(\mathcal{J})$.

The distributivity number $d(\mathcal{W})$

A complete Boolean algebra $\langle B, \leq \rangle$ is called $\kappa$-distributive, where $\kappa$ is a cardinal, if and only if for every family $\langle u_{\alpha i} : i \in I_\alpha, \alpha < \kappa \rangle$ of members of $B$ the following holds:

$$\prod_{\alpha < \kappa} \sum_{i \in I_\alpha} u_{\alpha i} = \sum_{f \in \prod_{\alpha < \kappa} I_\alpha} \prod_{\alpha < \kappa} u_{\alpha f(\alpha)}.$$ 

It is well known (cf. [Je2]) that for a forcing notion $\langle P, \leq \rangle$ the following statements are equivalent:

- $\text{r.o.}(P)$ is $\kappa$-distributive.
- The intersection of $\kappa$ open dense sets in $P$ is dense.
- Every family of $\kappa$ maximal anti-chains of $P$ has a common refinement.
- Forcing with $P$ does not add a new subset of $\kappa$.

Let $\mathcal{J}$ be the ideal of all finite sets of $\omega$ and let $\langle (\omega)^\omega / \mathcal{J}, \leq \rangle =: \mathcal{W}$ be the partial order defined as follows:

$$p \in \mathcal{W} \iff p \in (\omega)^\omega,$$

$$p \leq q \iff p \subseteq^* q.$$ 

The distributivity number $d(\mathcal{W})$ is defined as the least cardinal $\kappa$ for which the Boolean algebra $\text{r.o.}(\mathcal{W})$ is not $\kappa$-distributive.

The four cardinals are equal

Now we will show, that the four cardinals defined above are all equal. This is a similar result as in the case when we consider infinite subsets of $\omega$ instead of infinite partitions (cf. [P] and [BPS]).

**FACT 2.1** If $T \subseteq (\omega)^\omega$ is an open and dense set with respect to the dual Ellentuck topology, then it contains a mao family.

**PROOF:** First choose an almost orthogonal family $\mathcal{A} \subseteq T$ which is maximal in $T$. Now for an arbitrary $X \in (\omega)^\omega$, $T \cap (X)^\omega \neq \emptyset$. So, $X$ must be compatible with some $A \in \mathcal{A}$ and therefore $\mathcal{A}$ is mao. -
LEMMA 2.2 $\mathfrak{h} \leq \text{add}(\mathcal{J})$.

**Proof:** Let $\langle S_\alpha : \alpha < \lambda < \mathfrak{h} \rangle$ be a sequence of nowhere dual-Ramsey sets and let $T_\alpha \subseteq (\omega) \setminus S_\alpha$ ($\alpha < \lambda$) be such that $T_\alpha$ is open and dense with respect to the dual Ellentuck topology (which is always possible by the Remark 1). For each $\alpha < \lambda$ let

$$T^*_\alpha := \{ X \in (\omega) : \exists Y \in T_\alpha (X \sqsubseteq^* Y \land \neg(X \neq^* Y)) \}.$$  

It is easy to see, that for each $\alpha < \lambda$ the set $T^*_\alpha$ is open and dense with respect to the dual Ellentuck topology.

Let $U_\alpha \subseteq T^*_\alpha$ ($\alpha < \lambda$) be mao. Because $\lambda < \mathfrak{h}$, the set $\langle U_\alpha : \alpha < \lambda \rangle$ can not be shattering. Let for $\alpha < \lambda U^*_\alpha := \{ X \in (\omega) : \exists Z_\alpha \in U_\alpha (X \sqsubseteq^* Z_\alpha) \}$, then $U^*_\alpha \subseteq T_\alpha$ and $\bigcap_{\alpha < \lambda} U^*_\alpha$ is open and dense with respect to the dual Ellentuck topology:

$\bigcap_{\alpha < \lambda} U^*_\alpha$ is open: clear.

$\bigcap_{\alpha < \lambda} U^*_\alpha$ is dense: Let $(s,Z)^\omega$ be arbitrary. Because $\langle U_\alpha : \alpha < \lambda \rangle$ is not shattering,

there is a $Y \in (s,Z)^\omega$ such that $\forall \alpha < \lambda \exists X_\alpha \in U_\alpha (Y \sqsubseteq^* X_\alpha)$. Hence, $Y \in \bigcap_{\alpha < \lambda} U^*_\alpha$.

Further we have by construction

$$\bigcap_{\alpha < \lambda} U^*_\alpha \cap \bigcup_{\alpha < \lambda} S_\alpha = \emptyset,$$

which completes the proof. $\dashv$

LEMMA 2.3 $\mathfrak{h} \leq d(\mathfrak{d})$.

**Proof:** Let $\langle T_\alpha : \alpha < \lambda < \mathfrak{h} \rangle$ be a sequence of open and dense sets with respect to the dual Ellentuck topology. Now the set $\bigcap_{\alpha < \lambda} U^*_\alpha$, constructed as in Lemma 2.2, is dense (and even open) and a subset of $\bigcap_{\alpha < \lambda} T_\alpha$. Therefore $\mathfrak{h} \leq d(\mathfrak{d})$. $\dashv$

LEMMA 2.4 $\text{add}(\mathcal{J}) \leq \mathfrak{h}$.

**Proof:** Let $\langle R_\alpha : \alpha < \mathfrak{h} \rangle$ be a shattering family and $P_\alpha := \{ X : \exists Y \in R_\alpha (X \sqsubseteq^* Y) \}$.

For each $\alpha < \mathfrak{h}$, $P_\alpha$ is dense and open with respect to the dual Ellentuck topology:

$P_\alpha$ is open: clear.

$P_\alpha$ is dense: Let $(s,Z)^\omega$ be arbitrary and $X \in (s,Z)^\omega$. Because $R_\alpha$ is mao, there is a $Y \in R_\alpha$ such that $X' := X \sqcup Y \in (\omega)^\omega$. Now let $X'' := X'$ such that $X'' \in (s,Z)^\omega$, then $X'' \sqsubseteq^* Y$.

Now we show that $\bigcap_{\alpha < \mathfrak{h}} P_\alpha = \emptyset$ and therefore $\bigcup_{\alpha < \mathfrak{h}} (\omega)^\omega \setminus P_\alpha = (\omega)^\omega$. Assume there is an $X \in \bigcap_{\alpha < \mathfrak{h}} P_\alpha$, then $\forall \alpha < \mathfrak{h} \exists Y_\alpha \in R_\alpha (X \sqsubseteq^* Y_\alpha)$. But this contradicts that $\langle R_\alpha : \alpha < \mathfrak{h} \rangle$ is shattering. $\dashv$
LEMMA 2.5 $d(\mathcal{W}) \leq \mathfrak{f}$.  

PROOF: In the proof of Lemma 2.4 we constructed a sequence $\langle P_\alpha : \alpha < \mathfrak{f} \rangle$ of open and dense sets with an empty intersection. Therefore $\bigcap_{\alpha<\mathfrak{f}} P_\alpha$ is not dense. \hfill ⊣

COROLLARY 2.6 $\text{cov}(\mathfrak{f}) \leq \mathfrak{f}$.  

PROOF: In the proof of Lemma 2.4 in fact we proved that $\text{cov}(\mathfrak{f}) \leq \mathfrak{f}$. \hfill ⊣

COROLLARY 2.7 $\text{add}(\mathfrak{f}) = \text{cov}(\mathfrak{f}) = d(\mathcal{W}) = \mathfrak{f}$.  

PROOF: It is clear that $\text{add}(\mathfrak{f}) \leq \text{cov}(\mathfrak{f})$. By the Lemmas 2.3 and 2.5 we know that $\mathfrak{f} = d(\mathcal{W})$. Further by the Lemma 2.2 and the Corollary 2.6 it follows that $\mathfrak{f} \leq \text{add}(\mathfrak{f}) \leq \text{cov}(\mathfrak{f}) \leq \mathfrak{f}$. Hence we have $\text{add}(\mathfrak{f}) = \text{cov}(\mathfrak{f}) = d(\mathcal{W}) = \mathfrak{f}$. \hfill ⊣

COROLLARY 2.8 The union of less than $\mathfrak{f}$ completely dual-Ramsey sets is dual-Ramsey, but the union of $\mathfrak{f}$ completely dual-Ramsey sets can be a set, which does not have the dual-Ramsey property.  

PROOF: Follows from Remark 1 and Corollary 2.7. \hfill ⊣

On the consistency of $\mathfrak{f} > \omega_1$

First we give some facts concerning the dual-Mathias forcing. The conditions of dual-Mathias forcing are pairs $\langle s, X \rangle$ such that $s \in (\mathbb{N})$, $X \subseteq (\omega)^\omega$ and $s \subseteq X$, stipulating $\langle s, X \rangle \leq \langle t, Y \rangle$ if and only if $(s,X)^\omega \subseteq (t,Y)^\omega$. It is not hard to see that similar to Mathias forcing, the dual-Mathias forcing can be decomposed as $\mathcal{W} \ast P_\mathcal{D}$, where $\mathcal{W}$ is defined as above and $P_\mathcal{D}$ denotes dual-Mathias forcing with conditions only with second coordinate in $\mathcal{D}$, where $\mathcal{D}$ is an $\mathcal{W}$-generic object.

Further, because dual-Mathias forcing has pure decision (cf. [CS]), it is proper and has the Laver property and therefore adds no Cohen reals.

If we make an $\omega_2$-iteration of dual-Mathias forcing with countable support, starting from a model in which the continuum hypothesis holds, we get a model in which the dual-shattering cardinal $\mathfrak{f}$ is equal to $\omega_2$.

Let $V$ be a model of CH and let $P_{\omega_2} := \langle P_\alpha, \hat{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of dual-Mathias forcing, i.e. $\forall \alpha < \omega_2 : \models_{P_\alpha} \text{"Q}_\alpha \text{ is dual-Mathias forcing"}$. In the sequel we will not distinguish between a member of $\mathcal{W}$ and its representative. In the proof of the following theorem, a set $C \subseteq \omega_2$ is called $\omega_1$-club if $C$ is unbounded in $\omega_2$ and closed under increasing sequences of length $\omega_1$.

THEOREM 2.9 If $G$ is $P_{\omega_2}$-generic over $V$, where $V \models \text{CH}$, then $V[G] \models \mathfrak{f} = \omega_2$.  

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PROOF: In $V[G]$ let $\langle D_\nu : \nu < \omega_1 \rangle$ be a family of open dense subsets of $\mathcal{W}$. Because dual-Mathias forcing is proper and by a standard Löwenheim-Skolem argument, we find a $\omega_1$-club $C \subseteq \omega_2$ such that for each $\alpha \in C$ and every $\nu < \omega_1$ the set $D_\nu \cap V[G_\alpha]$ belongs to $V[G_\alpha]$ and is open dense in $\mathcal{W}^{\mathcal{M}[G_\alpha]}$. Let $A \in \mathcal{W}^{\mathcal{M}[G_\alpha]}$ be arbitrary. By properness and genericity and because $P_{\omega_2}$ has countable support, we may assume that $A \in G(\alpha)'$ for an $\alpha \in C$, where $G(\alpha)'$ is the first component according to the decomposition of Mathias forcing of the $Q_\alpha[G_\alpha]$-generic object determined by $G$. As $\alpha \in C$, $G(\alpha)'$ clearly meets every $D_\nu$ ($\nu < \omega_1$). But now $X_\alpha$, the $Q_\alpha$-generic partition (determined by $G(\alpha)''$) is below each member of $G(\alpha)'$, hence below $A$ and in $\bigcap_{\nu < \omega_1} D_\nu$. Because $A$ was arbitrary, this proves that $\bigcap_{\nu < \omega_1} D_\nu$ is dense in $\mathcal{W}$ and therefore $d(\mathcal{W}) > \omega_1$.

Again by properness of dual-Mathias forcing $V[G] \models 2^{\omega_0} = \omega_2$ and we finally have $V[G] \models \mathfrak{f} = \omega_1$. 

In the model constructed in the proof of Theorem 2.9 we have $\mathfrak{f} > t$, where $t$ is the well-known tower number (for a definition of $t$ cf. [vDo]). Moreover, we can show

COROLLARY 2.10 The statement $\mathfrak{f} > \text{cov}(\mathcal{M})$ is relatively consistent with ZFC, (where $\mathcal{M}$ denotes the ideal of meager sets).

PROOF: Because dual-Mathias forcing is proper and does not add Cohen reals, also forcing with $P_{\omega_2}$ does not add Cohen reals. Further it is known that $t \leq \text{cov}(\mathcal{M})$ (cf. [PV] or [BJ]). Now because forcing with $P_{\omega_2}$ does not add Cohen reals, in $V[G]$ the covering number $\text{cov}(\mathcal{M})$ is still $\omega_1$ (because each real in $V[G]$ is in a meager set with code in $V$). This completes the proof. 

REMARK 2: In [vDo] Theorem 3.1.(c) it is shown that $\omega \leq \kappa < t$ implies that $2^\kappa = 2^{\omega_0}$. We do not have a similar result for the dual-shattering cardinal $\mathfrak{f}$. If we start our forcing construction $P_{\omega_2}$ with a model $V \models \text{CH} + 2^{\omega_1} = \omega_3$, then (again by properness of dual-Mathias forcing) $V[G] \models \mathfrak{f} = \omega_2 = 2^{\omega_0} < 2^{\omega_1} = \omega_3$, so that $G$ is $P_{\omega_2}$-generic over $V$.

Remark: Recently Spinas showed in [Sp], that $\mathfrak{f} < \mathfrak{b}$ is consistent with ZFC. But it is still open if MA+$\neg$CH implies that $\omega_1 < \mathfrak{b}$.

3 On the dual-splitting cardinals $\mathcal{G}$ and $\mathcal{G}'$

Let $X_1, X_2$ be two partitions. We say $X_1$ splits $X_2$ if $X_1 \parallel X_2$ and it exists a partition $Y \subseteq X_2$, such that $X_1 \perp Y$. A family $\mathcal{S} \subseteq (\omega)^\omega$ is called splitting if for each non-trivial $X \in (\omega)^\omega$ there exists an $S \in \mathcal{S}$ such that $S$ splits $X$. The dual-splitting cardinal $\mathcal{G}$ (resp. $\mathcal{G}'$) is the least cardinal number $\kappa$, for which there exists a splitting family $\mathcal{S} \subseteq (\omega)^\omega$ (resp. $\mathcal{S} \subseteq (\omega)^{\mathcal{M}}$) of cardinality $\kappa$.

It is obvious that $\mathcal{G} \leq \mathcal{G}'$.

First we compare the dual-splitting number $\mathcal{G}'$ with the well-known bounding number $\mathfrak{b}$ (a definition of $\mathfrak{b}$ can be found in [vDo]).

THEOREM 3.1 $\mathfrak{b} \leq \mathcal{G}'$. 

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The page contains a mathematical proof and corollaries about set theory. The proof involves constructing a function and partition to show consistency with ZFC. The corollaries provide additional consistency results. The text is presented in a readable format with proper symbols and notation.
if and only if \( t \preceq s \) and \( B \sqsubseteq A \). (\( s \) is called the stem of the condition.) If \( \langle s, A_1 \rangle, \langle s, A_2 \rangle \) are two \( \mathcal{Q} \)-conditions, then \( \langle s, A_1 \sqcup A_2 \rangle \preceq \langle s, A_1 \rangle, \langle s, A_2 \rangle \). Hence, two \( \mathcal{Q} \)-conditions with the same stem are compatible and because there are only countably many stems, the forcing notion \( \mathcal{Q} \) is \( \sigma \)-centered.

Now we will see, that forcing with \( \mathcal{Q} \) adds an infinite partition which is compatible with all old infinite partitions but is not contained in any old partition. (So, the forcing notion \( \mathcal{Q} \) is in a sense like the dualization of Cohen forcing.)

**Lemma 3.5** If \( G \) is \( \mathcal{Q} \)-generic over \( V \), then \( G \in (\omega)\forall \) and \( V[G] \models \forall X \in (\omega)\forall \cap V(G \parallel X \wedge \neg (X \sqsubseteq^* G)) \).

**Proof:** Let \( X \in V \) be an arbitrary, infinite partition. The set \( D_n \) of \( \mathcal{Q} \)-conditions \( \langle s, A \rangle \), such that

(i) at least one block of \( s \) has more than \( n \) elements,

(ii) at least \( n \) blocks of \( X \) are each the union of blocks of \( A \),

(iii) there are at least \( n \) different blocks \( b_i \in X \), such that \( \bigcup b_i \in s \cap X \),

is dense in \( \mathcal{Q} \) for each \( n \in \omega \). Therefore, at least one block of \( G \) is infinite (because of (i)), \( G \) is compatible with \( X \) (because of (ii)) and \( X \) is not coarser* than \( G \) (because of (iii)). Now, because \( X \) was arbitrary, the \( \mathcal{Q} \)-generic partition \( G \) has the desired properties.

Because the forcing notion \( \mathcal{Q} \) is \( \sigma \)-centered and each \( \mathcal{Q} \)-condition can be encoded by a real number, forcing with \( \mathcal{Q} \) does neither collapse any cardinals nor change the cardinality of the continuum and we can prove the following

**Lemma 3.6** It is consistent with ZFC that \( \mathcal{G}' < \mathfrak{c} \).

**Proof:** [CMW] If make an \( \omega_1 \)-iteration of \( \mathcal{Q} \) with finite support, starting from a model in which we have \( \mathfrak{c} = \omega_2 \), then the \( \omega_1 \) generic objects form a splitting family.

Even if a partition does not have a complement, for each non-trivial partition \( X \) we can define a non-trivial partition \( Y \), such that \( X \perp Y \).

Let \( X = \{ b_i : i \in \omega \} \in (\omega)^\omega \) and assume that the blocks \( b_i \) are ordered by their least element and that each block is ordered by the natural order. A block is called trivial, if it is a singleton. With respect to this ordering define for each non-trivial partition \( X \) the partition \( X^\prec \) as follows.

If \( X \in (\omega)\exists \) then

\[
n \text{is in the } i \text{th block of } X^\prec \iff n \text{ is the } i \text{th element of a block of } X,
\]

otherwise
\(n, m\) are in the same block of \(X^\perp\)

\(n, m\) are both least elements of blocks of \(X\).

It is not hard to see that for each non-trivial \(X \in (\omega)^\omega\), \(X \perp X^\perp\).

A family \(\mathcal{W} \subseteq (\omega)^\omega\) is called \textit{weak splitting}, if for each partition \(X \in (\omega)^\omega\), there is a \(W \in \mathcal{W}\) such that \(W\) splits \(X\) or \(W\) splits \(X^\perp\). The cardinal number \(w\mathfrak{S}\) is the least cardinal number \(\kappa\), for which there exists a weak splitting family of cardinality \(\kappa\). (It is obvious that \(w\mathfrak{S} \leq \mathfrak{S}'\).)

A family \(U\) is called a \(\pi\)-base for a free ultra-filter \(\mathcal{F}\) over \(\omega\) provided for every \(x \in \mathcal{F}\) there exists \(u \in U\) such that \(u \subseteq x\). Define

\[\pi u := \min\{|U| : U \subseteq [\omega]^\omega\ \text{is a \(\pi\)-base for a free ultra-filter over} \ \omega\}\].

In \([\text{BS}]\) it is proved, that \(\pi u = r\) (see also \([\text{Va}]\) for more results concerning \(r\)).

Now we can give an upper and a lower bound for the size of \(w\mathfrak{S}\).

**Theorem 3.7** \(w\mathfrak{S} \leq r\).

**Proof:** We will show that \(w\mathfrak{S} \leq \pi u\). Let \(U := \{u_i \in [\omega]^\omega : i < \pi u\}\) be a \(\pi\)-basis for a free ultra-filter \(\mathcal{F}\) over \(\omega\). W.l.o.g. we may assume, that all the \(u_i \in U\) are co-infinite. Let \(U = \{Y_u \in (\omega)^\omega : u \in U \land Y_u = \{u_i : u_i = u \lor (u_i = \{n\} \land n \not\in u)\}\}\). Now we take an arbitrary \(X = \{b_i : i \in \omega\} \in (\omega)^\omega\) and define for every \(u \in U\) the sets \(I_u := \{i : b_i \cap u \neq \emptyset\}\) and \(J_u := \{j : b_j \cap u = \emptyset\}\). It is clear that \(I_u \cup J_u = \omega\) for every \(u\).

If we find a \(u \in U\) such that \(|I_u| = |J_u| = \omega\), then \(Y_u\) splits \(X\). To see this, define the two infinite partitions

\[Z_1 := \{a_k : a_k = \bigcup_{i \in I_u} b_i \lor \exists j \in J_u a_k = b_j\}\]

and

\[Z_2 := \{a_k : a_k = \bigcup_{j \in J_u} b_j \lor \exists i \in I_u a_k = b_i\}\.\]

Now we have \(X \cap Y_u = Z_1\) (therefore \(Z_1 \subseteq X, Y_u\)) and \(Z_2 \subseteq X\) but \(Z_2 \perp Y_u\).

(If each block of \(b_i\) is finite, then we are always in this case.)

If we find an \(x \in \mathcal{F}\) such that \(|I_x| < \omega\) (and therefore \(|J_x| = \omega\)), then we find an \(x' \subseteq x\), such that \(|I_x| = 1\) and for this \(i \in I_x\), \(|b_i \setminus x'| = \omega\). (This is because \(\mathcal{F}\) is a free ultra-filter.) Now take a \(u \in U\) such that \(u \subseteq x'\) and we are in the former case for \(X^\perp\). Therefore, \(Y_u\) splits \(X^\perp\).

If we find an \(x \in \mathcal{F}\) such that \(|J_x| < \omega\) (and therefore \(|I_x| = \omega\)), let \(I(n)\) be an enumeration of \(I_x\) and define \(y := x \cap \bigcup_{k \in \omega} b_{I(2k)}\). Then \(y \subseteq x\) and \(|x \setminus y| = \omega\).

Hence, either \(y\) or \(\omega \setminus y\) is a superset of some \(u \in U\). But now \(|J_u| = \omega\) and we are in a former case.

A lower bound for \(w\mathfrak{S}\) is \(\text{cov}(\mathcal{M})\).

**Theorem 3.8** \(\text{cov}(\mathcal{M}) \leq w\mathfrak{S}\).
PROOF: Let $\kappa < \text{cov}(M)$ and $W = \{W_i : i < \kappa\} \subseteq (\omega)^{\omega}$. Assume for each $W_i \in W$ the blocks are ordered by their least element and each block is ordered by the natural order. Further assume that $b_{i(i)}$ is the first block of $W_i$ which is infinite. Now for each $i < \kappa$ the set $D_i$ of functions $f \in \omega^{\omega}$ such that

$$\forall n, m, k \in \omega \, \exists h \in \omega t_1 \in b_n, t_2 \in b_m, t_3, t_4 \in b_k \exists s \in b_{i(i)}$$

$$f(t_1) = f(t_3) \wedge f(t_2) = f(t_4) \wedge \{s' \leq s : f(s') = f(s)\} = k + 1.$$ 

is the intersection of countably many open dense sets and therefore the complement of a meager set. Because $\kappa < \text{cov}(M)$, we find an unbounded function $g \in \omega^{\omega}$ such that $g \in \bigcap_{i < \kappa} D_i$. The partition $G = \{g^{-1}(n) : n \in \omega\} \subseteq (\omega)^{\omega}$ is orthogonal with each member of $W$ and for each $W_i \in W$ and each $k \in \omega$, there exists an $s \in b_{i(i)}$, such that $s$ is the $k$th element of a block of $G$. Hence, $W$ can not be a weak splitting family. \hfill \dashv

4 On the dual-reaping cardinals $R$ and $R'$

A family $R \subseteq (\omega)^{\omega}$ is called reaping (resp. reaping'), if for each partition $X \in (\omega)^{\omega}$ (resp. $X \in (\omega)^{\omega}$) there exists a partition $R \in R$ such that $R \perp X$ or $R \subseteq X$. The dual-reaping cardinal $R$ (resp. $R'$) is the least cardinal number $\kappa$, for which there exists a reaping (resp. reaping') family of cardinality $\kappa$.

It is clear that $R' \leq R$. Further by finite modifications of the elements of a reaping family, we may replace $\subseteq^*$ by $\subseteq$ in the definition above.

If we cancel in the definition of the reaping number the expression “$R \subseteq^* X$”, we get the definition of an orthogonal family.

A family $O \subseteq (\omega)^{\omega}$ is called orthogonal (resp. orthogonal'), if for each non-trivial partition $X \in (\omega)^{\omega}$ (resp. for each partition $X \in (\omega)^{\omega}$) there exists a partition $O \in O$ such that $O \perp X$. The dual-orthogonal cardinal $O$ (resp. $O'$) is the least cardinal number $\kappa$, for which there exists an orthogonal (resp. orthogonal') family of cardinality $\kappa$. (It is obvious that $O' \leq O$.) Note, that $\sigma = c$, where $c$ is the cardinality of $P(\omega)$ and $c$ is defined like $\mathfrak{D}$ but for infinite subsets of $\omega$ instead of infinite partitions. (Take the complements of a maximal antichain in $[\omega]^\omega$ of cardinality $c$. Because an orthogonal family must avoid all this complements, it has at least the cardinality of this maximal antichain.) It is also clear that each orthogonal family is also a reaping family and therefore $\mathfrak{R}(\sigma) \leq \mathfrak{O}(\sigma)$. Further one can show that $\mathfrak{R}$ is uncountable (cf. [CMW]).

Now we show that $\mathfrak{D}' \leq \mathfrak{D}$, where $\mathfrak{D}$ is the well-known dominating number (for a definition cf. [Dd]), and that $\text{cov}(M) \leq \mathfrak{D}'$.

**Lemma 4.1** $\mathfrak{D}' \leq \mathfrak{D}$.

**Proof:** Let $\{d_i : i < \omega\}$ be a dominating family. Then it is not hard to see that the family $\{D_i : i < \kappa\} \subseteq (\omega)^{\omega}$, where each $D_i$ is constructed from $d_i$ like $D$ from $d$ in the proof of Theorem 1, is an orthogonal family. \hfill \dashv

Let $i$ be the least cardinality of an independent family (a definition and some results can be found in [K]), then
Lemma 4.2 \( D \leq i \).

**Proof:** Let \( I \subseteq [\omega]^\omega \) be an independent family of cardinality \( i \). Let \( I' := \{ r \in [\omega]^\omega : r \trianglerighteq \cap A \setminus \bigcup B \} \), where \( A, B \in [I]^\omega \), \( A \neq \emptyset \), \( A \cap B = \emptyset \) and \( r \trianglerighteq x \) means \( |(r \setminus x) \cup (x \setminus r)| < \omega \). It is not hard to see that \( |I'| = |I| = i \). Now let \( I = I_1 \cup I_2 \) where \( I_1 := \{ X_r \in (\omega)^\omega : r \in I' \land X_r = \{ b_i : b_i = r \lor (b_i = \{ n \} \land n \notin r) \} \} \) and \( I_2 := \{ Y_r : \exists X_r \in I_1 (Y_r = X_r^c) \} \). We see, that \( I \subseteq (\omega)^\omega \) and \( |I| = i \). It leave to show that \( I \) is an orthogonal family.

Let \( Z \in (\omega)^\omega \) be arbitrary and let \( r := \text{Min}(Z) \). If \( r \in I' \), then \( X_r \perp Z \) (where \( X_r \in I_1 \)). And if \( r \notin I' \), then there exists an \( r' \in I' \) such that \( r \cap r' = \emptyset \). But then \( Y_{r'} \perp Z \) (where \( Y_{r'} \in I_2 \)).

Because \( \mathcal{R} \leq D \), the cardinal number \( i \) is also an upper bound for \( \mathcal{R} \). But for \( \mathcal{R} \), we also find another upper bound.

Lemma 4.3 \( \mathcal{R} \leq r \).

**Proof:** Like in Theorem 3.7 we show that \( \mathcal{R} \leq \pi u \). Let \( U := \{ u_i \in [\omega]^\omega : i < \pi u \} \) be as in the proof of Theorem 3.7 and let \( \mathcal{U} = \{ Y_u \in (\omega)^\omega : u \in U \land Y_u = \{ u_i : u_i = \omega \setminus u \lor (u_i = \{ n \} \land n \notin u) \} \} \). Take an arbitrary partition \( X \in (\omega)^\omega \).

Let \( r := \text{Min}(X) \) and \( r_1 := \{ n \in r : \{ n \} \in X \} \). If we find a \( u \in U \) such that \( u \subset r_1 \), then \( Y_u \subset X \). Otherwise, we find a \( u \in U \) such that either \( u \subset \omega \setminus r \) or \( u \subset r \setminus r_1 \) and in both cases \( Y_u \perp X \).

Now we will show, that it is consistent with ZFC that \( D \) can be small. For this we first show, that a Cohen real encode an infinite partition which is orthogonal to each old non-trivial infinite partition. (This result is in fact a corollary of Lemma 5 of [CMW].)

Lemma 4.4 If \( c \in \omega^\omega \) is a Cohen real over \( V \), then \( C := \{ c^{-1}(n) : n \in \omega \} \in (\omega)^{\omega} \cap V[c] \) and \( \forall X \in (\omega)^\omega \cap V (\neg (X \trianglerighteq \{ \omega \}) \rightarrow C \perp X) \).

**Proof:** We will consider the Cohen-conditions as finite sequences of natural numbers, \( s = \{ s(i) : i < n < \omega \} \). Let \( X = \{ b_i : i \in \omega \} \in \mathcal{V} \) be an arbitrary, non-trivial infinite partition. The set \( D_{n,m} \) of Cohen-conditions \( s \), such that

1. \( |\{ i : s(i) = 0 \}| \geq n \),
2. \( \exists k > n \exists i (s(i) = k) \),
3. \( \exists a_n \in b_n \exists a_m \in b_m \exists \exists b_1, a_2 \in b_i (s(a_n) = s(a_1) \land s(a_m) = s(a_2)) \),

is a dense set for each \( n, m \in \omega \). Now, because \( X \) was arbitrary, the infinite partition \( C \) is orthogonal to each infinite partition which is in \( \mathcal{V} \). (Note that because of (i), \( C \in (\omega)^{\omega} \) \)

Now we can show, that \( D \) can be small.

Lemma 4.5 It is consistent with ZFC that \( D < \text{cov}(\mathcal{M}) \).
PROOF: If make an \(\omega_1\)-iteration of Cohen forcing with finite support, starting from a model in which we have \(c = \omega_2 = \text{cov}(\mathcal{M})\), then the \(\omega_1\) generic objects form an orthogonal family. Now because this \(\omega_1\)-iteration of Cohen forcing does not change the cardinality of \(\text{cov}(\mathcal{M})\), we have a model in \(\omega_1 = \mathcal{D} < \text{cov}(\mathcal{M}) = \omega_2\) holds. 

Because \(R \leq \mathcal{D}\) we also get the relative consistency of \(R < \text{cov}(\mathcal{M})\). Note that this is not true for \(r\).

As a lower bound for \(R'\) we find \(p\), where \(p\) is the pseudo-intersection number (a definition of \(p\) can be found in [vDo]).

**LEMMA 4.6** \(p \leq \mathcal{R}'\).

**PROOF:** In [Be] it is proved that \(p = m\)-centered, where

\[
m_{\text{\sigma\text{-centered}}} = \min\{\kappa : \text{"MA(\kappa) for \sigma\text{-centered posets" fails}\}.
\]

Let \(\mathcal{R} = \{R_\iota : \iota < \kappa < p\}\) be a set of infinite partitions. Now remember that the forcing notion \(Q\) (defined in section 3) is \(\sigma\)-centered and because \(\kappa < p\) we find an \(X \in (\omega)^{<\omega}\) such that \(\mathcal{R}\) does not reap \(X\). 

As a corollary we get 

**COROLLARY 4.7** If we assume MA, then \(\mathcal{R}' = c\).

**PROOF:** If we assume MA, then \(p = c\).

5 What’s about towers and maximal (almost) orthogonal families?

Let \(\kappa_{\text{mao}}\) be the least cardinal number \(\kappa\), for which there exists an infinite \(\text{mao}\) family of cardinality \(\kappa\). And let \(\kappa_{\text{tower}}\) be the least cardinal number \(\kappa\), for which there exists a family \(F \subseteq (\omega)^{<\omega}\) of cardinality \(\kappa\), such that \(F\) is well-ordered by \(\sqsupset^*\) and \(\neg \exists Y \in (\omega)^{<\omega} \forall X \in F(Y \sqsupset^* X)\).

Now Krawczyk proved that \(\kappa_{\text{mao}} = c\) (cf. [CMW]) and Carlson proved that \(\kappa_{\text{tower}} = \omega_1\) (cf. [Ma]). So, these cardinals do not look interesting. But what happens if we cancel the word “almost” in the definition of \(\kappa_{\text{mao}}\)?

A family \(F \subseteq (\omega)^{<\omega}\) (resp. \(F \subseteq (\omega)_{\omega}\)) is a maximal anti-chain in \((\omega)^{<\omega}\) (resp. \((\omega)_{\omega}\)), if \(F\) is a maximal infinite family of pairwise orthogonal partitions. Let \(\kappa_A\) (resp. \(\kappa_{A'}\)) be the least cardinality of a maximal anti-chain in \((\omega)^{<\omega}\) (resp. \((\omega)_{\omega}\)).

Note that the corresponding cardinal for infinite subsets of \(\omega\) would be equal to \(\omega\).

First we know that \(\text{cov}(\mathcal{M}) \leq \kappa_A, \kappa_{A'}\) (which is proved in [CMW]) and \(b \leq \kappa_{\mathcal{R}'}\) (which one can prove like Theorem [3.1]). Further it is not hard to see that \(\kappa_A \leq \kappa_{A'}\).

But these results concerning \(\kappa_A\) and \(\kappa_{A'}\) are also not interesting, because Spinas showed in [Sp] that \(\kappa_A = \kappa_{A'} = c\).
6 The diagram of the results

Now we summarize the results proved in this article together with other known results.

splitting:

\[
\begin{array}{cccccc}
\text{b} & \mathcal{S}' & \text{c} \\
\text{h} & & \\
\text{h'} & \mathcal{S} & \\
\omega_1 & \text{cov}(\mathcal{M}) & \omega \mathcal{S} & \tau \\
\end{array}
\]

reaping:

\[
\begin{array}{cccccc}
\text{d} & \text{i} & \text{c} \\
\mathcal{O}' & \mathcal{O} & \\
\text{p} & \mathcal{R}' & \mathcal{R} & \tau \\
\end{array}
\]

(In the diagrams, the invariants grow larger, as one moves up or to the right.)

Consistency results:

- \(\text{cov}(\mathcal{M}) < \text{h} ; \text{h} < \text{h} ; \text{h} < \text{cov}(\mathcal{M})\) (this is because \(\text{h} < \text{cov}(\mathcal{M})\) is consistent with ZFC)
- \(s < \mathcal{S} ; \mathcal{S}' < c\)
- \(\mathcal{O} < \text{cov}(\mathcal{M})\)

Note added in proof: Recently, Jörg Brendle informed me that he has proved, that \(\text{MA} + \mathcal{S} < 2^{\aleph_0}\) is consistent with ZFC.

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Lorenz Halbeisen
Departement Mathematik
ETIH-Zentrum
8092 Zürich
Switzerland
E-mail: halbeis@math.ethz.ch