Stability of the Cauchy horizon in Kerr–de Sitter spacetimes

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ABSTRACT

We begin a program of work aimed at examining the interior of a rotating black hole with a non–zero cosmological constant. The generalisation of Teukolsky’s equation for the radial mode functions is presented. It is shown that the energy fluxes of scalar, electromagnetic and gravity waves are regular at the Cauchy horizon whenever the surface gravity there is less than the surface gravity at the cosmological horizon. This condition is narrowly allowed, even when the cosmological constant is very small, thus permitting an observer to pass through the hole, viewing the naked singularity along the way.
1. Introduction

The fate of an observer who falls into a black hole is an interesting and important issue within the context of general relativity. The situation of interest is when the black hole is charged or rotating, because then the spacetime can be continued beyond the black hole into another universe, with naked singularities and causality violation to contend with.

We know that for a black hole in empty space the observer’s progress is impeded by a singularity at the Cauchy horizon [1,2], but the situation is different if there is a cosmological constant. In the non–rotating case we now know that there is a finite range of parameters for which the Cauchy horizon is fully stable [3–6]. In these cases an observer could pass into the hole and see the naked singularity. Since a cosmological constant does not contradict the laws of physics in any known way, this journey constitutes at the very least a valid thought experiment in which the predictive power of general relativity breaks down. The evolution of the spacetime beyond the Cauchy horizon requires a knowledge of ‘initial’ conditions at the singularity and this lies beyond the limitations of current theory.

In this paper we examine the stability of the Cauchy horizon for a rotating hole when a cosmological constant is present by consideration of the energy fluxes of scalar, electromagnetic and gravity waves propagating on the black hole background. We begin by separating the field equations into equations for radial and angular functions. For this step in the calculation the introduction of a cosmological constant becomes an asset because it makes manifest the symmetry of the metric under the interchange of the radial coordinate $r$ and the coordinate $\mu = a \cos \theta$. It is the strength of this symmetry combined with the algebraic character of the curvature tensor that makes possible the separation of the fields and ultimately makes the problem solvable.

The significance of gravitational wave equations to the stability lies in the fact that they result from perturbing the full set of Einstein’s field equations [1]. Therefore an arbitrarily small perturbation, provided it does not grow in size anywhere in the spacetime region of interest, can be analysed by purely linear equations. This is to be contrasted with the unstable case, familiar from the ordinary Kerr metric, where it is necessary to solve non–linear equations because an initially small perturbation becomes large near to the Cauchy horizon [2].

As a matter of fact, gravitational wave propagation is only part of the full gravitation perturbation analysis because there are other degrees of freedom in the metric, but it ought to contain all of the physical degrees of freedom relevant to the stability question. To be certain, it only remains to be shown that the remaining gravitational perturbation equations separate into similar wave equations to deduce stability in the full gravity–matter system. We hope to establish the complete separation of the equations in a later paper.

We find stability requires that the surface gravity $\kappa$ of the cosmological horizon should be larger than the surface gravity of the Cauchy horizon; the same condition as in the charged case. The parameter range for which this condition holds are found in the following section, in which various properties of the spacetime are explained. The parameter range is small, but finite.

2. The spacetime

The metric of a stationary black hole in de Sitter space was first given by Carter [7]
and takes the form,

\[ ds^2 = \rho^2 (\Delta_r^{-1} dr^2 + \Delta_\mu^{-1} d\mu^2) + \rho^{-2} \Delta_\mu \omega^1 \otimes \omega^1 - \rho^{-2} \Delta_r \omega^2 \otimes \omega^2, \]

(2.1)

with one–forms,

\[ \chi^2 \omega^1 = dt - a^{-1} \sigma_r^2 d\phi, \quad \chi^2 \omega^2 = dt - a^{-1} \sigma_\mu^2 d\phi. \]

(2.2)

The metric is parameterised as follows,

\[ \sigma_r = (a^2 + r^2)^{1/2}, \quad \Delta_r = (a^2 + r^2)(1 - \frac{1}{3} \Lambda r^2) - 2Mr + Q^2 \]

\[ \sigma_\mu = (a^2 - \mu^2)^{1/2}, \quad \Delta_\mu = (a^2 - \mu^2)(1 + \frac{1}{3} \Lambda \mu^2) \]

(2.3)

and

\[ \rho^2(r) = r^2 + \mu^2, \quad \chi^2 = 1 + \frac{1}{3} \Lambda a^2, \quad \Omega(r) = a/(r^2 + a^2). \]

(2.4)

The function \( \Omega \) is explained below. Note that the subscripts are used to denote functions that depend only on the coordinate indicated by the subscript, and a prime will always denote differentiation with respect to this coordinate.

The metric is a solution of the Einstein–Maxwell equations with a non–zero cosmological constant \( \Lambda \) and the electromagnetic vector potential of a charge \( Q \),

\[ A = \frac{Qr}{\rho^2} \omega^2. \]

(2.5)

To see that this metric really does represent a black hole we introduce the angle \( \theta \), where \( \mu = a \cos \theta \). The surfaces of constant radius \( r \) and time \( t \) are distorted spheres, parameterised by spherical polar coordinates \( \theta \) and \( \phi \) in the usual way.

The metric has coordinate singularities at the roots of \( \Delta_r = 0 \). For a range of parameters shown in figure 1 there are four real roots, which shall be labelled in decreasing order by \( r_1, r_2, r_3 \) and \( r_4 \). The roots \( r_2 \) and \( r_3 \) represent the outer and inner horizons respectively of the black hole, whilst \( r_1 \) represents a de Sitter horizon. The fourth root \( r_4 \) is negative and non–physical.

It is possible to set up coordinate systems for which the metric is regular on any particular horizon. Following section 56 in ref. [1], in the region outside the event horizon there exists two simple families of null geodesics with tangent vectors

\[ \dot{t} = \chi^4 \frac{\sigma_r^2}{\Delta_r} \omega, \quad \dot{r} = \pm \chi^2 \omega, \quad \dot{\mu} = 0, \quad \text{and} \quad \dot{\phi} = \chi^4 \frac{a}{\Delta_r} \omega, \]

(2.6)

where a dot denotes a derivative with respect to the geodesic affine parameter and \( \omega \) is a constant. Corresponding 'light–like' coordinates which are fixed along each family can therefore be defined by \( u = t - r^* \) and \( v = t + r^* \), where

\[ dr^* = \chi^2 \Delta_r^{-1} \sigma_r^2 dr. \]

(2.7)
Additionally, for rotating black holes, there is a unique combination of the killing vectors that is null on the (future) event horizon, \( l = \partial_t + \Omega(r_2)\partial_\phi \). This leads us to introduce an azimuthal angle coordinate
\[
\phi_2 = \phi - \Omega(r_2)t
\]
on the horizon, which is constant along the integral curves of \( l \). Physically, this corresponds to the dragging of inertial frames near to the hole.

The null–killing vector on the horizon \( r_i \) can be used also to define the surface gravity, \( \nabla l^2 = \kappa_i l \). The explicit expression for \( \kappa_i \) follows from squaring both sides of the expression and taking the limit as \( r \to r_i \) [8],
\[
\kappa_i = \frac{1}{6}\Lambda\chi^{-2}\sigma^{-2}\prod_{i \neq j}|r_i - r_j|.
\]
(There is a typographical mistake in ref. [9]).

The extension of the metric through the event horizon was given by Carter in ref [7] (with some corrections given in ref. [8]). The regular coordinate system is given in terms of the surface gravity,
\[
U = -\kappa_2^{-1}e^{-\kappa_2 u}, \quad V = \kappa_2^{-1}e^{\kappa_2 v}
\]
The coordinates \( u \) and \( v \) will be defined inside the event horizon as shown in figure 2.

It is possible to extend the metric through each of the horizons in turn and obtain the complete Penrose diagram, a part of which is shown in figure 3 [7,8]. The fully extended spacetime stretches arbitrarily far in each direction. It is possible to find world–lines that pass through the black hole from region \( I \), through regions \( II \) and \( III \) and on to the outside.

The exterior geometry of a collapsing star would only take in part of the picture, the remainder being replaced by the geometry of the stellar interior. In order to retain the axial symmetry of the solution it is also necessary to place a second body of equal mass around the point of the universe that is antipodal to the centre of the star. This second body can be thought of as representing the rest of the matter in the universe.

Inside region \( III \) lies the naked singularity and the possibility of causality violation from closed timelike curves. These curves pass beyond the singularity at \( r = 0 \), which only appears in the \( \mu = 0 \) plane, into the next region in the Penrose diagram. The problem for an observer who tries to enter these regions lies in crossing the Cauchy horizon between regions \( II \) and \( III \) at \( r = r_3 \).

As an observer crosses the Cauchy horizon any radiation from outside appears to be either red–shifted or blue–shifted depending on the relative values of the surface gravity at the Cauchy and de Sitter horizons, red–shifted if the Cauchy horizon value is the smaller. In the non–rotating case, this was also the Cauchy horizon stability condition [4,5].

It is possible to split up the parameter space of the black hole metrics into regions depending on the relative sizes of the surface gravity on various horizons. We write the difference between the surface gravities at \( r_i \) and \( r_j \) as
\[
\kappa_i - \kappa_j = \frac{1}{6}\Lambda\chi^{-2}\frac{|r_i - r_j|}{(r_i^2 + a^2)(r_j^2 + a^2)}p_{ij},
\]
where \( p_{ij} \) is a polynomial in the radii.

If two of the radii are equal, then the surface gravities of the respective horizons both vanish. In other cases the relative magnitudes of the surface gravities are determined by the signs of the polynomials. Explicit expressions for these polynomials are,

\[
p_{12} = (r_1 r_2 + r_3 r_4 - 2a^2)(r_1^2 - r_2^2)
\]
\[
p_{13} = (r_1 r_3 + r_2 r_4)(r_1 + r_3)^2 + 2(r_1 r_3 - a^2)(r_1 r_3 - r_2 r_4).
\]

The black-hole parameters for which \( p_{ij} = 0 \) can be found by first writing \( p_{ij} \) in terms of \( x = r_1 r_2 - r_3 r_4 \) or \( x = r_1 r_3 - r_2 r_4 \) in the case of \( p_{12} \) or \( p_{13} \) respectively. From the conditions on roots of \( \Delta_r = 0 \) it is possible to deduce quite independently that \( x \) satisfies

\[
M^2 = \frac{1}{36} \Lambda^2 x^3 + \frac{1}{12} \Lambda (1 - \frac{1}{3} \Lambda a^2) x^2 + \frac{1}{3} \Lambda (a^2 + Q^2) x + (1 - \frac{1}{3} \Lambda a^2)(a^2 + Q^2).
\]

Solving \( p_{ij} = 0 \) for \( x \) gives an expression \( x(a,Q,\Lambda) \) which can be substituted in this equation to give a condition \( M(a,Q,\Lambda) \) for equal surface gravities.

The situation in which the surface gravities on the de Sitter and event horizons are equal gives rise to gravitational instantons and was discussed in reference [8]. From equation (2.12), the solution for this case is \( x = 2a^2 \) and leads to a condition from (2.13),

\[
M^2 = Q^2 \chi^2 + a^2 \chi^4.
\]

The borderline of the stability criterion is that \( \kappa_1 = \kappa_3 \). This can be found by solving \( p_{13} = 0 \) numerically and substituting the result into (2.13), leading to the condition shown in figure 1. The line lies very close to the line where the inner and outer horizons coincide. The region of stability between the two lines OA is small, but finite in a uniform measure on the space of parameters \( M, Q \) and \( a \) for fixed \( \Lambda \).

3. The scalar wave equation

We will take a massless scalar field \( \Phi \), with wave equation

\[
\nabla^2 \Phi = 0,
\]

(3.1)
to demonstrate how the wave–equation separates on the black hole background [10].

In terms of coordinates the Laplacian becomes

\[
\nabla^2 = \frac{1}{\rho^2} \partial_r \Delta_r \partial_r + \frac{1}{\rho^2} \partial_\mu \Delta_\mu \partial_\mu + \frac{\rho^2}{\Delta_\mu} \partial_1^2 - \frac{\rho^2}{\Delta_r} \partial_2^2
\]

(3.2)

where \( \partial_1 \) and \( \partial_2 \) are dual vectors to \( \omega_1 \) and \( \omega_2 \),

\[
\partial_1 = \frac{\chi^2}{\rho^2} (\sigma_\mu \partial_t + a \partial_\phi)
\]
\[
\partial_2 = \frac{\chi^2}{\rho^2} (\sigma_\tau \partial_t + a \partial_\phi).
\]

(3.3)

We look for separable solutions of form

\[
\Phi = R(r)S(\mu)e^{-i\omega t}e^{im\phi}.
\]

(3.4)
The angular and radial terms in the wave equation separate immediately, with angular part

\[-\partial_\mu \Delta_\mu \partial_\mu S + \frac{K_\mu^2}{\Delta_\mu} S = \lambda S, \quad (3.5)\]

where

\[K_\mu = \chi^2 (am - \sigma^2 \omega). \quad (3.6)\]

We require that the solutions be regular at \(\mu = \pm a\). Scaling \(\mu\) by \(a\) shows that the eigenvalues \(\lambda\) depend upon the parameters \(m, \omega a\) and \(\Lambda a^2\). We also add another parameter \(l\) to label the position of the eigenvalue in sequence, starting from the value \(|m|\).

The corresponding eigenfunctions will then be denoted by \(S(lm\omega; \mu)\). Some values of the angular eigenvalues obtained numerically are tabulated in tables 1–3 for various parameter values. The cosmological constant has been re-expressed there as \(\Lambda = 3/\alpha^2\).

For \(\Lambda = 0\) the equation reduces to the equation for spheroidal wave functions,

\[((a^2 - \mu^2) S')' + \left[\lambda + 2am\omega - \omega^2(a^2 - \mu^2) - a^2 m^2 (a^2 - \mu^2)^{-1}\right] S = 0. \quad (3.7)\]

The spheroidal eigenvalues denoted by \(\lambda_l^m\) [11] are then related to \(\lambda\) by a shift of \(2am\omega\). We shall be interested later in the analyticity properties of the angular functions in a strip of the complex \(\omega\) plane below the real axis. Numerical studies for spheroidal wave functions indicate that there are branch cuts away from the real axis and suggest that these functions are regular in the region \(|\omega| < cl^2\) for some constant \(c\) [11].

For small values of \(\Lambda\) it is possible to relate the properties of the angular functions to the spheroidal wave functions by a suitable approximation scheme. The angular functions are the stationary points of an integral,

\[I[S] = \int_{-a}^a d\mu \left\{ \Delta_\mu (S')^2 + \left( K_\mu^2/\Delta_\mu \right) S^2 \right\}. \quad (3.8)\]

We can separate the \(\Lambda = 0\) part and write

\[I = \sum_n I^{(n)}[S, S](\Lambda/3)^n, \quad (3.9)\]

where

\[I^{(0)}[S_1, S_2] = \int_{-a}^a d\mu \left\{ \sigma_\mu^2 S_1'S_2' + \sigma_\mu^{-2} (am - \sigma_\mu^2 \omega)^2 S_1 S_2 \right\} \]

\[I^{(1)}[S_1, S_2] = \int_{-a}^a d\mu \left\{ \mu^2 \sigma_\mu^2 S_1'S_2' + \left( 1 + a^2/\sigma_\mu^2 \right) (am - \sigma_\mu^2 \omega)^2 S_1 S_2 \right\} \quad (3.10)\]

\[I^{(n)}[S_1, S_2] = \int_{-a}^a d\mu \left( -\mu^2 \right)^{n-1} \sigma_\mu^2 (am - \sigma_\mu^2 \omega)^2 S_1 S_2. \]

These give series for small \(\Lambda\),

\[\lambda_l = \sum_n \lambda_l^{(n)} (\Lambda/3)^n, \quad S_l = \sum_n S_l^{(n)} (\Lambda/3)^n. \quad (3.11)\]
where $S_l^{(0)}$ is the spheroidal wave function. Thus

$$\lambda_l^{(1)} = I^{(1)}[S_l^{(0)}, S_l^{(0)}], \quad S_l^{(1)} = \sum_{q \neq l} (\lambda_l^{(0)} - \lambda_q^{(0)})^{-1} I^{(1)}[S_l^{(0)}, S_q^{(0)}] S_q^{(0)}. \quad (3.12)$$

This result was used to check the values in the tables. Continuing further with the series, it is possible to show that the $S_l^{(n)}$ are regular functions of $\omega$ when $S_l^{(0)}$ is regular.

Returning now to the wave equation and taking the radial part,

$$-\partial_r (\Delta_r \partial_r R) + \lambda R - \frac{K^2}{\Delta_r} R = 0, \quad (3.13)$$

where

$$K_r = \chi^2 (am - \sigma_r^2 \omega). \quad (3.14)$$

It is possible to write

$$\sigma_r \partial_r (\Delta_r \partial_r R) = \left( \frac{\chi^4 \sigma_r^4}{\Delta_r} \frac{d^2}{dr^*2} \right) \sigma_r R - \left( \frac{r \Delta_r}{\sigma_r^3} \right) \sigma_r^2 R. \quad (3.15)$$

This allows the radial equation to be written in the form of a scattering problem,

$$\left( \frac{d^2}{dr^*2} + V(r) \right) \sigma_r R = 0 \quad (3.16)$$

where the potential $V(r)$ takes the form

$$V = (\omega - m\Omega)^2 - \frac{\lambda \Delta_r}{\chi^4 \sigma_r^4} - \frac{\Delta_r}{\chi^4 \sigma_r^3} \left( \frac{r \Delta_r}{\sigma_r^3} \right)'. \quad (3.17)$$

This potential has been plotted as a function of $r^*$ between the outer and inner horizons in figure 4. There is a potential well in the region covered by the plot and a potential barrier outside of the event horizon. The main difference from the non–rotating case is that the values of the potential as $r^* \to \infty$ and $r^* \to -\infty$ need no longer be equal.

4. The electromagnetic wave equations

In this section we will examine the Maxwell field equations on a Kerr-de Sitter back- ground using the Newman–Penrose [12] formalism. This development follows the lines of the analysis for a metric with vanishing cosmological constant [13–15]. The main difference lies in exploiting to the full the symmetrical appearance of the metric with respect to $r$ and $\mu$.

We begin with an orthonormal tetrad of vector fields,

$$e_r = \frac{\sqrt{\Delta_r}}{\rho} \partial_r, \quad e_2 = \frac{\chi^2}{\rho \sqrt{\Delta_r}} (\sigma_r^2 \partial_t + a \partial_\phi),$$

$$e_\mu = \frac{\sqrt{\Delta_\mu}}{\rho} \partial_\mu, \quad e_1 = -\frac{\chi^2}{\rho \sqrt{\Delta_\mu}} (\sigma_\mu^2 \partial_t + a \partial_\phi). \quad (4.1)$$
The vectors dual to $\omega^1$ and $\omega^2$ have been used as in section 2. Newman–Penrose formalism consists of writing the connection forms in a complex null–tetrad basis,

\begin{align*}
  l &= \frac{1}{\sqrt{2}}(e_r + e_2), \quad n = -\frac{1}{\sqrt{2}}(e_r - e_2) \\
  m &= \frac{1}{\sqrt{2}}(e_\mu + ie_1), \quad \bar{m} = \frac{1}{\sqrt{2}}(e_\mu - ie_1). \\
\end{align*}

(4.2)

These vectors satisfy $l.n = -1$ and $m.\bar{m} = 1$. (Inside the event horizon we will use the modulus of $\Delta \overline{r}$ for scaling the tetrad vectors and choose $l$ always to be in the direction of $\partial_V$, where $V$ is the Kruskal coordinate.)

The notation for the covariant derivatives along the tetrad vectors consists of

\begin{align*}
  D &= \nabla_l, \quad \Delta = \nabla_n, \quad \delta = \nabla_m, \quad \bar{\delta} = \nabla_{\bar{m}}. \\
\end{align*}

(4.3)

The connection forms are then expressed as in the following table,

\begin{align*}
  Dl &= (\overline{\epsilon + \bar{\epsilon}})l - \bar{\kappa}m - \kappa \bar{m}, \quad Dm = (\epsilon - \bar{\epsilon})m - \kappa n + \pi l, \\
  \Delta l &= (\gamma + \bar{\gamma})l - \tau m - \tau \bar{m}, \quad \Delta m = (\gamma - \bar{\gamma})m - \tau n + \bar{\pi} l, \\
  \delta l &= (\beta + \alpha)l - \sigma m - \sigma \bar{m}, \quad \delta m = (\beta - \alpha)m - \sigma n + \lambda l, \\
  \bar{\delta} l &= (\alpha + \beta)l - \sigma m - \rho \bar{m}, \quad \bar{\delta} m = (\alpha - \beta)m - \rho n + \mu l. \\
\end{align*}

(4.4)

For the Kerr–de Sitter metric some of the connection components vanish,

$$\kappa = \sigma = \lambda = \nu = 0.$$  

(4.5)

The vanishing of these connection components is a feature common to all black hole solutions. The remaining connection components are

\begin{align*}
  \rho &= \mu = \frac{1}{\sqrt{2} \rho^3} \sqrt{\Delta_r}, \quad \epsilon = \gamma = -\frac{1}{2\sqrt{2} \rho^3} \sqrt{\Delta_r} + \frac{1}{2\sqrt{2}} \frac{\sqrt{\Delta_r'}}{\rho}, \\
  \tau &= -\pi = \frac{i}{\sqrt{2} \rho^3} \sqrt{\Delta_\mu}, \quad \beta = -\alpha = \frac{i}{2\sqrt{2} \rho^3} \sqrt{\Delta_\mu} + \frac{1}{2\sqrt{2}} \frac{\sqrt{\Delta_\mu'}}{\rho}, \\
\end{align*}

(4.6)

where

$$c = r + i\mu.$$  

(4.7)

The electromagnetic field is described by three complex scalars $\phi_0$, $\phi_1$ and $\phi_2$, with the radial fields in $\phi_1$ and the axial fields in $\phi_0 + \phi_2$. They satisfy Maxwell’s equations,

\begin{align*}
  D\phi_1 - \bar{\delta}\phi_0 &= (\pi - 2\alpha)\phi_0 + 2\rho \phi_1 - \kappa \phi_2, \\
  \delta\phi_1 - \Delta\phi_0 &= (\mu - 2\gamma)\phi_0 + 2\tau \phi_1 - \sigma \phi_2, \\
  D\phi_2 - \bar{\delta}\phi_1 &= (\rho - 2\epsilon)\phi_2 + 2\pi \phi_1 - \lambda \phi_0, \\
  \delta\phi_2 - \Delta\phi_1 &= (\tau - 2\beta)\phi_2 + 2\mu \phi_1 - \nu \phi_0. \\
\end{align*}

(4.8)
We follow reference [1] and introduce new radial and angular derivatives defined by

\[ D_n = \partial_r + iK_r \Delta_r, \quad D_n^\dagger = \partial_r - iK_r \Delta_r, \]

with \( K_r \) defined in equations (3.14) and (3.6). These are respectively purely radial or angular in character. Useful properties of these derivatives are that

\[ D_n + n \bar{c} = (\bar{c})^{-n} D_n(\bar{c})^n; \quad D_n = \Delta_r^{-n} \Delta'_r \]

and similarly for \( \mathcal{L} \). For scalar functions that have the dependence \( \exp(i(m\phi - \omega t)) \) on \( \phi \) and \( t \) it is possible to write the directional derivatives along the tetrad vectors as

\[ D = \left( \frac{\Delta_r}{2\rho^2} \right)^{1/2} D_0, \quad \Delta = -\left( \frac{\Delta_r}{2\rho^2} \right)^{1/2} D_0^\dagger, \quad \delta = \left( \frac{\Delta_r}{2\rho^2} \right)^{1/2} \mathcal{L}_0. \]

We can now write Maxwell’s equations in terms of \( D \) and \( \mathcal{L} \), using the connection components from equation (4.6). We will rescale the Maxwell fields first, to \( \varphi_i = \bar{c}\phi_i \), and then the equations become

\[ \sqrt{\Delta_r} \left( D_0 + \frac{1}{\bar{c}} \right) \varphi_1 = \sqrt{\Delta_\mu} \left( \mathcal{L}_{1/2}^\dagger + \frac{i}{\bar{c}} \right) \varphi_0 \]

\[ -\sqrt{\Delta_r} \left( D_{1/2}^\dagger - \frac{1}{\bar{c}} \right) \varphi_0 = \sqrt{\Delta_\mu} \left( \mathcal{L}_0 - \frac{i}{\bar{c}} \right) \varphi_1 \]

\[ \sqrt{\Delta_r} \left( D_{1/2} - \frac{1}{\bar{c}} \right) \varphi_2 = \sqrt{\Delta_\mu} \left( \mathcal{L}_{0}^\dagger - \frac{i}{\bar{c}} \right) \varphi_1 \]

\[ -\sqrt{\Delta_r} \left( D_0^\dagger + \frac{1}{\bar{c}} \right) \varphi_1 = \sqrt{\Delta_\mu} \left( \mathcal{L}_{1/2} + \frac{i}{\bar{c}} \right) \varphi_2. \]

The field \( \varphi_1 \) may be eliminated from (4.13),

\[ \sqrt{\Delta_\mu} \left( \mathcal{L}_0 - \frac{i}{\bar{c}} \right) \sqrt{\Delta_\mu} \left( \mathcal{L}_{1/2}^\dagger + \frac{i}{\bar{c}} \right) \varphi_0 = -\sqrt{\Delta_r} \left( D_0 + \frac{1}{\bar{c}} \right) \sqrt{\Delta_r} \left( D_{1/2}^\dagger - \frac{1}{\bar{c}} \right) \varphi_0 \]

From the forms of \( D \) and \( \mathcal{L} \), and of \( K_r \) and \( K_\mu \), this can be simplified to

\[ \left( \mathcal{L}_{-1/2} \Delta_\mu \mathcal{L}_{1/2}^\dagger + D_{-1/2} \Delta_r D_{1/2}^\dagger + 2i \omega \chi^2 c \right) \varphi_0 = 0. \]

Similarly,

\[ \left( \mathcal{L}_{-1/2}^\dagger \Delta_\mu \mathcal{L}_{1/2} + D_{-1/2}^\dagger \Delta_r D_{1/2} - 2i \omega \chi^2 c \right) \varphi_2 = 0. \]
This is the complex conjugate of the equation for $\varphi_0$. The result for $\varphi_1$ can be obtained by subtracting the middle two equations in (4.13),

$$K_\mu \phi_1 = -\frac{1}{2} \sqrt{\Delta_\mu \Delta_r} \left( \mathcal{D}^\dagger_{1/2} \varphi_0 + \mathcal{D}_{1/2} \varphi_2 \right). \quad (4.17)$$

The equations separate in the same way as the scalar field equation, that is we set

$$\varphi_0 = R_0(\mu) S_0(\mu) e^{i m \phi} e^{-i \omega t}. \quad (4.18)$$

Then,

$$\left( \mathcal{L}_{-1/2} \Delta_\mu \mathcal{L}^\dagger_{1/2} - 2 \omega \chi^2 \mu \right) S_0 = -\lambda S_0$$

$$\left( \mathcal{D}_{-1/2} \Delta_r \mathcal{D}^\dagger_{1/2} + 2 i \omega \chi^2 r \right) R_0 = \lambda R_0. \quad (4.19)$$

The second of these equations is the analogue for Kerr–de Sitter of Teukolsky’s equation for Kerr[13]. For spin $s$ we would have,

$$\left( \mathcal{D}_{-s/2} \Delta_r \mathcal{D}^\dagger_{s/2} + 2(2s - 1)i \omega \chi^2 r \right) R = \lambda R. \quad (4.20)$$

We would like to express the radial equation in terms of the $r^*$ coordinate as we did for the scalar field. Before doing this it is advantageous to introduce electromagnetic potentials rather than the fields themselves because these are directly comparable to the scalar field, particularly when we come to evaluate energy fluxes. In order to proceed we need the Teukolsky and Starobinsky identities,

$$\mathcal{D}_{-1/2} \Delta_r \mathcal{D}_{1/2} R_2 = C R_0, \quad \mathcal{D}^\dagger_{-1/2} \Delta_\mu \mathcal{D}^\dagger_{1/2} R_0 = C R_2, \quad (4.21)$$

where

$$C^2 = \lambda^2 + 4 \omega (a m - a^2 \omega) \chi^4. \quad (4.22)$$

There are similar identities for $\mathcal{L}$. The derivation of these identities follows the corresponding results in reference [1].

Define

$$R_0 = \mathcal{D} \Delta_{1/2}^r R_2, \quad R_2 = \mathcal{D}^\dagger \Delta_{1/2}^r R_0. \quad (4.23)$$

It follows from the identities that

$$R_0 = C^{-1} \Delta_{1/2}^r R_0, \quad R_2 = C^{-1} \Delta_{1/2}^r \mathcal{D}^\dagger R_2. \quad (4.24)$$

Combining the two sets of radial equations gives

$$\mathcal{D}^\dagger \Delta_r \mathcal{D} R_0 = C R_2, \quad \mathcal{D} \Delta_r \mathcal{D}^\dagger R_2 = C R_0. \quad (4.25)$$

These expand into a form,

$$\left( \partial_r \Delta_r \partial_r + K^2_r / \Delta_r - 2 i \omega \chi^2 r \right) R_0 = C R_2$$

$$\left( \partial_r \Delta_r \partial_r + K^2_r / \Delta_r + 2 i \omega \chi^2 r \right) R_2 = C R_0 \quad (4.26)$$
This coupled form of the equations seems a step backwards, but it will still be acceptable for our analysis. The equations can be written as a scattering problem,

\[
\left( \frac{d^2}{dr^2} + V_s(r) \right) \sigma_r F = 0 \quad (4.27)
\]

where

\[
F = \begin{pmatrix} R_0 + R_2 \\ R_0 - R_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_+ & U \\ -U & V_- \end{pmatrix},
\]

and

\[
V_\pm = (\omega - m\Omega)^2 \pm \frac{\mathcal{C}\Delta_r}{\chi^4\sigma_r^4} - \frac{\Delta_r}{\chi^4\sigma_r^3} \left( \frac{r\Delta_r}{\sigma_r^3} \right) ', \quad U = -\frac{2i\omega r\Delta_r}{\chi^2\sigma_r^4}. \quad (4.29)
\]

It would be possible to separate these equation using the methods described in [1], but this will not be necessary.

The new radial functions can be used to define new fields,

\[
\Phi_0 = R_0(r)S_0(\mu)e^{im\phi}e^{-i\omega t}. \quad (4.30)
\]

These are related to the electromagnetic potentials. The result (4.24) shows that the Maxwell fields are given by

\[
\phi_0 = \frac{\sqrt{\Delta_r}}{C\mathcal{C}}\mathcal{D}\Phi_0, \quad \phi_2 = \frac{\sqrt{\Delta_r}}{C\mathcal{C}}\mathcal{D}^\dagger\Phi_2, \quad (4.31)
\]

and from (4.17),

\[
\phi_1 = \frac{\sqrt{\Delta_r}\Delta_\mu}{2K_\mu}(\phi_0 + \phi_1) - \frac{\sqrt{\Delta_\mu}}{2K_\mu}(\Phi_0 + \Phi_2). \quad (4.32)
\]

We have not fully explored the numerous relationships that exist between the mode functions because they are not needed for our stability analysis, but there are many equivalent ways to express the above results.

5. The gravitational wave equations

In the Newman–Penrose formalism, the Weyl tensor is described by five complex scalars \( \Psi_0 \ldots \Psi_4 \). Of these, only \( \Psi_2 \) is non–vanishing in black hole solutions. The remaining scalars can be used to describe gravitational wave perturbations of the metric. In this section we will separate the equations for these perturbations on the Kerr–de Sitter background. The perturbed metric has other degrees of freedom, but we leave those equations for another occasion.

The Bianchi identities for the Weyl tensor components read

\[
\begin{align*}
(\delta - 4\alpha + \pi)\Psi_0 - (D - 2\epsilon - 4\rho)\Psi_1 &= 3\kappa\Psi_2 \\
(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 &= 3\sigma\Psi_2 \\
(\delta + 4\beta - \tau)\Psi_4 - (\Delta + 2\gamma + 4\mu)\Psi_3 &= -3\nu\Psi_2 \\
(D + 4\epsilon - \rho)\Psi_4 - (\delta + 4\pi + 2\alpha)\Psi_3 &= -3\lambda\Psi_2.
\end{align*} \quad (5.1)
\]
The connection components $\kappa$, $\sigma$, $\lambda$ and $\nu$ all vanish for the unperturbed metric and the values appearing here are therefore the perturbed values. We need two Ricci identities to solve for these,

\[
(D - \rho - \tilde{\rho} - 3\epsilon + \varepsilon)\sigma - (\delta - \tau + \tilde{\pi} - \tilde{\alpha} - 3\beta)\kappa = \Psi_0
\]
\[
(\Delta - \mu - \tilde{\mu} + 3\gamma - \gamma)\lambda - (\delta + 3\alpha + \tilde{\beta} + \pi - \tau)\nu = -\Psi_4.
\]  

(5.2)

The angular and radial operators $L$ and $D$ that where used for the Maxwell fields can be used here also. We rescale the Weyl scalars first, $\psi = c^2\Psi$ and the connection components $\tilde{\kappa} = c^{1/2}(\tilde{c})^{-1/2}\kappa$. Then,

\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_1^* + \frac{3i}{\tilde{c}}\right)\psi_0 - \sqrt{\Delta_r}\left(D_{-1/2} + \frac{3}{\tilde{c}}\right)\psi_1 = 3\sqrt{2}(\tilde{c}\psi_2)\kappa,
\]
\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_{-1/2} - \frac{3i}{\tilde{c}}\right)\psi_1 + \sqrt{\Delta_r}\left(D_+^{\dagger} - \frac{3}{\tilde{c}}\right)\psi_0 = -3\sqrt{2}(\tilde{c}\psi_2)\sigma,
\]
\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_1 + \frac{3i}{\tilde{c}}\right)\psi_4 + \sqrt{\Delta_r}\left(D_{-1/2} + \frac{3}{\til{c}}\right)\psi_3 = -3\sqrt{2}(\til{c}\psi_2)\nu,
\]
\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_{-1/2} - \frac{3i}{\til{c}}\right)\psi_3 - \sqrt{\Delta_r}\left(D_1 - \frac{3}{\til{c}}\right)\psi_4 = 3\sqrt{2}(\til{c}\psi_2)\lambda.
\]  

(5.3)

and

\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_{-1/2} - \frac{3i}{\til{c}}\right)\kappa - \sqrt{\Delta_r}\left(D_{-1/2} + \frac{3}{\til{c}}\right)\sigma = -\sqrt{2}\til{c}\til{c}^{-2}\psi_0,
\]
\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_{-1/2}^* - \frac{3i}{\til{c}}\right)\nu + \sqrt{\Delta_r}\left(D_{-1/2}^* + \frac{3}{\til{c}}\right)\lambda = \sqrt{2}\til{c}\til{c}^{-2}\psi_4.
\]  

(5.4)

The Weyl scalar

\[
\Psi_2 = -\frac{M}{(\til{c})^3} - \frac{Q^2\til{c}}{(\til{c})^3}.
\]  

(5.5)

If the charge $Q = 0$, we can eliminate $\psi_1$ from the first two equations by using equation (5.4). We will restrict our attention to this case alone, then

\[
\sqrt{\Delta_\mu}\left(\mathcal{L}_{-1/2} - \frac{3i}{\til{c}}\right)\sqrt{\Delta_\mu}\left(\mathcal{L}_1^* + \frac{3i}{\til{c}}\right)\psi_0 + \sqrt{\Delta_r}\left(D_{-1/2} + \frac{3}{\til{c}}\right)\sqrt{\Delta_r}\left(D_1^* - \frac{3}{\til{c}}\right)\psi_0 = 6\frac{cM}{(\til{c})^2}\psi_0.
\]  

(5.6)

After simplification, this gives

\[
\left(\mathcal{L}_{-1}\Delta_\mu\mathcal{L}_1^* + D_{-1}\Delta_rD_1^* + 6i\omega\chi^2c - 2\Lambda\rho^2\right)\psi_0 = 0.
\]  

(5.7)

Similarly, the Weyl scalar $\psi_4$ satisfies the complex conjugate equation. The remaining field $\psi_3$ is gauge dependent and can be made to satisfy similar equations or made to vanish, depending on the choice of tetrad in the perturbed metric.
We separate the equation as before with a radial function $R_0(r)$ and an angular function $S_0(\mu)$. These functions satisfy

\[
\left( L - 1 \Delta \mu L^\dagger - 6 \omega \chi^2 \mu - 2 \Lambda \mu^2 \right) S_0 = -\lambda S_0
\]
\[
\left( D - 1 \Delta \mu D^\dagger + 6 i \omega \chi^2 r - 2 \Lambda r^2 \right) R_0 = \lambda R_0.
\]

These equations are consistent with the new version of Teukolsky’s equation which was proposed in the previous section, but with the addition of a cosmological constant term. This term plays a similar role to a mass in the case of a massive scalar.

As before, we will introduce potentials to facilitate the stability analysis. This makes use of the identities,

\[
\mathcal{D}_{-1} \Delta \mathcal{D} \Delta \mathcal{D}_1 R_4 = \mathcal{C} R_0, \quad \mathcal{D}_{-1}^\dagger \Delta \mathcal{D}^\dagger \mathcal{D}^\dagger \Delta \mathcal{D}_1^\dagger R_0 = \bar{\mathcal{C}} R_4.
\]

The modulus of the constant $\mathcal{C}$ is given by

\[
|\mathcal{C}|^2 = \lambda^2 \lambda_1^2 - 20 K_0 K_2 \lambda \lambda_1 + 24 K_2^2 a^2 \lambda + K_2 (K_0^2 + M^2)
\] \[+ 8 \lambda (a^2 \lambda \lambda_1 - 4 K_0^2 \lambda + 6 a^2 K_0 K_2) + 16 \Lambda^2 a^4
\]

where

\[
K_0 = (a m - a^2 \omega) \chi^2, \quad K_2 = -2 \omega \chi^2, \quad \lambda_1 = \lambda + 2 - \frac{2}{3} \Lambda a^2.
\]

Define

\[
R_0 = \mathcal{D} \Delta \mathcal{D}_1 R_4, \quad R_4 = \mathcal{D}^\dagger \Delta \mathcal{D}^\dagger \Delta \mathcal{D}_1^\dagger R_0,
\]

and also

\[
\Phi_i = R_i(r) S_i(\theta) e^{im\phi} e^{-i\omega t}.
\]

The equations for these radial functions form a coupled set of second order equations. (This can be shown by repeated use of the original equation to reduce the order, but the resulting equations are quite complicated). The Weyl curvature scalars are recoverable from,

\[
\Psi_4 = \frac{\Delta}{\mathcal{C} c^2} \mathcal{D}^\dagger \Phi_4, \quad \Psi_0 = \frac{\Delta}{\mathcal{C} c^2} \mathcal{D} \Phi_0.
\]

The remaining scalars $\Psi_1$ and $\Psi_3$ can be chosen to be zero by adopting a suitable gauge.
6. Regularity at the Cauchy horizon

We shall now examine the behaviour of an incoming flux of energy as it reaches the Cauchy horizon. The energy flux will be set up initially in the form of waves that enter the black hole from the outside. These waves are affected by the geometry of spacetime in two ways, their wavelengths are stretched or compressed and they get scattered. It is the compression of the waves that leads to instability, but it is necessary to consider the scattering problem in some detail. We begin therefore with some general remarks about scattering problems. The following treatment is based very closely on Chandrasekhar and Hartle [16].

Consider the following scattering problem

\[
\frac{d^2 f}{dx^2} + V(x)f = 0 \tag{6.1}
\]

where the potential has an exponential expansion as \(x \to \pm \infty\),

\[
V(x) = \begin{cases} 
\sum_{n=0}^{\infty} c_n^+ e^{-2n\kappa_+ x} & x \to \infty \\
\sum_{n=0}^{\infty} c_n^- e^{2n\kappa_- x} & x \to -\infty \end{cases} \tag{6.2}
\]

Suppose also that \(c_0^\pm\) is parameterised by a frequency \(\omega\),

\[
c_0^\pm = (\omega - m\Omega_\pm)^2. \tag{6.3}
\]

A potential of this form is called Yukawian [17] and leads to special analyticity properties of the scattering coefficients which are derived in the appendix.

In the case of waves entering the black hole there are two scattering regions, the first outside the hole in region I and the second inside region II. The radial modes will be separated into \(\leftarrow\) modes which enter from the cosmological horizon and \(\rightarrow\) modes that are scattered from the past event horizon. These are shown in figure 5. Near to the Cauchy horizon, the asymptotic limits of the modes are

\[
\sigma_r R \sim Ae^{ik_3 r^*} + Be^{-ik_3 r^*} \tag{6.4}
\]

in both cases.

The scattering amplitudes can be found by combining two scattering problems with coordinates \(x = r^*\) in both regions. This gives a potential of the required form. The scattering problems are related to those in the appendix by the substitutions

\[
\begin{align*}
\text{I} & : r_- = r_2, \quad r_+ = r_1, \quad f_2 = \sigma_r \frac{R}{r}, \quad f_1 = \sigma_r \frac{R}{r}; \\
\text{II} & : r_- = r_2, \quad r_+ = r_3, \quad f_2 = (T_{II}/T_I)\sigma_r \frac{R}{r}, \quad f_1 = (T_{II}/R_I)\sigma_r \frac{R}{r}. \tag{6.5}
\end{align*}
\]

Consequently,

\[
\frac{\mathcal{A}}{\frac{T_I}{T_{II}}} = \frac{\mathcal{A}}{\frac{R_I}{T_{II}}}. \tag{6.6}
\]
The full expansion of the field $\Phi$ will be of the form $\Phi = \tilde{\Phi} + \bar{\Phi}$, with

$$\tilde{\Phi} = \sum_{l,m} \int \frac{d\omega}{2\pi} \tilde{W}(lm\omega) \tilde{R}(\omega; r) S(lm\omega; \theta) e^{im\phi - i\omega t}.$$  \hfill (6.7)

The functions $W(lm\omega)$ determine the initial conditions on an initial surface in region I. We will take $W$ to be a regular function of $\omega$ with isolated poles.

Close to the Cauchy horizon, the asymptotic form of the mode functions can be used, and then

$$\tilde{\Phi} = \sum_{l,m} \int \frac{d\omega}{2\pi r} \tilde{W}(lm\omega) S(lm\omega; \theta)(\tilde{A} e^{-ik_3 v} + \tilde{B} e^{ik_3 u}) e^{im(\phi - \Omega t)}.$$  \hfill (6.8)

Recall that $\phi - \Omega(r_3) t$ is the regular coordinate $\phi_3$. Furthermore, $k_3$ is taken to be real or in the lower half plane and therefore the exponential terms converge at the Cauchy horizon, where $v \to \infty$. In fact, the only possible divergences in the field $\Phi$ at the horizon would arise from poles in $A(\omega)$, and these can be removed by taking the principal value of the integral.

The energy flux in the fields would be measured by the components of the stress–energy tensor. A coordinate frame that is regular on the Cauchy horizon, such as the Kruskal coordinate frame, should be used. A typical component of the stress–energy for a scalar field would be

$$T_{VV} = -\partial_V \Phi \partial_V \Phi.$$  \hfill (6.9)

The part of the stress–energy tensor that is able to diverge at the Cauchy horizon is the derivative with respect to $V$. This is because

$$\partial_V \Phi = e^{\kappa_3 v} \partial_v \Phi.$$  \hfill (6.10)

At the Cauchy horizon, $v \to \infty$, and the exponential diverges.

For the electromagnetic field, the stress tensor components in the null tetrad frame are functions of the Maxwell fields $\phi_0$, $\phi_1$ and $\phi_2$, for example

$$T_{abl}^{a'b} = \phi_0 \bar{\phi}_0, \quad T_{ab}^{n'a'n'b} = \phi_2 \bar{\phi}_2, \quad T_{ab}^{l'a'n'b} = \phi_1 \bar{\phi}_1.$$  \hfill (6.11)

This frame has to be rescaled to obtain the Kruskal frame,

$$\partial_V = e^{\kappa_3(u+v)/2} l, \quad \partial_U = e^{-\kappa_3(u+v)/2} n.$$  \hfill (6.12)

The potentially diverging stress–energy tensor component is then

$$T_{VV} = \frac{2\rho^2}{C^2 e^2} (\partial_V \Phi_0)^*(\partial_V \Phi_0),$$  \hfill (6.13)

after using expressions for the Maxwell fields given in equations (4.31).
In the case of the gravitational waves we are interested in whether components of the curvature tensor remain finite at the Cauchy horizon. We have

\[ C_{abcd}^{\mu \lambda m b c m d} = -\Psi_0, \quad C_{abcd}^{\nu \lambda \bar{m} b c \bar{m} d} = -\Psi_4. \] (6.14)

Again, the important terms are the ones depending upon the \( V \) derivatives of \( \Phi \). (The basis vectors \( m \) and \( \bar{m} \) have to be Lorentz boosted to a regular frame.)

Using the mode decomposition above and the asymptotic forms of the mode functions we can express the potentially divergent term as

\[ \partial \Phi = i e^{\kappa_3 v} \sum_{l,m} \frac{d\omega}{2\pi \sigma_r} \bar{W}(l m \omega) k_3 \bar{A}(\omega) S(l m \omega; \theta) e^{im\phi_3} e^{-ik_3 v}. \] (6.15)

and a similar term with the \( \rightarrow \) amplitude. The integrand is regular with isolated poles which allows us to move the contour of integration away from the real axis. The behaviour of the whole term as \( v \to \infty \) depends upon the poles in the integrand in the lower half of the complex \( \omega \) plane.

The first such pole in \( k_3 A \) can be found from equation (8.11) and figure 6. The crucial feature is that the pole at \( \omega = m\Omega_2 - i\kappa_2 \) in \( 1/T_I \) is matched by an identical pole in \( 1/T_I \) or \( R_I/T_I \). The ratio (6.6) therefore has no pole there. The pole at \( \omega = m\Omega_3 \) also cancels with \( k_3 \), leaving the first pole at \( \omega = m\Omega_3 - i\kappa_3 \).

In order to discuss the behavior of \( W \) we can consider in a similar way the stress–energy tensor component \( T_{VV} \) at the cosmological horizon. Here \( \kappa_3 \) is replaced by \( \kappa_1 \), and we see that the value of \( T_{VV} \) on the horizon is determined by the residue of \( W \) at \( \omega = m\Omega_1 - i\kappa_1 \). Therefore, if the stress energy tensor is non–zero at the cosmological horizon, there should be a pole in \( W \) with imaginary part \(-i\kappa_1 \).

We have stated already our assumption that the angular functions are analytic in a strip bordering the real axis. We conclude that it is possible to distort the contour of integration to pass parallel to the real axis through \(-i\kappa_1 \) but no further, and that \( T_{VV} \) at the Cauchy horizon diverges as \( \exp((\kappa_3 - \kappa_1)v) \). In particular, the stress–energy tensor of the waves is regular at the Cauchy horizon if the surface gravity there is less than the surface gravity at the cosmological horizon.

7. Conclusion

We have seen that there are situations in which the energy fluxes of scalar, electromagnetic and gravity waves remain finite at the Cauchy horizon and allow an observer to pass safely through into the interior of the black hole. The angular velocity of the hole has to be finely tuned to allow this to happen, and of course we assumed the existence of a cosmological constant.

The separation of the field equations that enabled us to make progress was closely associated with the algebraic character of the curvature tensor of the black hole and with the symmetrical appearance of the metric with respect to the angular and radial coordinates. This also explains why the angular and radial equations have such similar properties [1]. We have been able to predict the form of the radial field equations with a cosmological constant for arbitrary spin, thus generalising Teukolsky’s results [13]. We have also found
it necessary to assume some properties of the angular mode functions that merit further study.

The gravitational wave equations form a major part of the full stability analysis for this spacetime. We have left the complete separation of the gravitational perturbations in the metric for another time. The remaining metric components are mostly gauge parameters of little physical interest in the absence of other fields. The more interesting case would include electromagnetic fields as well, but separation is then problematic, even in the standard analysis, particularly when the hole carries an electric charge.

Even in the case that the cosmological constant is very small, there is still a tiny range of values for the rotation rate of the hole leading to a stable Cauchy horizon. This range of values is very close to the extreme limit in which the surface gravity of the event horizon vanishes, a situation that we would expect only to be attainable asymptotically. It therefore seems possible that attempts to spin up a black hole could be used in principle to stabilise the Cauchy horizon.

The stable situation raises the question of how to describe a naked singularity. The classical theory needs to be modified to do this, but we might make some progress by treating the interior geometry as a central force problem with an undetermined set of phase shifts at the singularity. The next question would be what are the consequences of closed timelike curves in the interior of the hole. There is a possibility that linear wave propagation and therefore the stability analysis could still be consistently defined, as they are in asymptotically flat spacetimes [18]. Finally, we would be left with the continuation of the observer’s journey through the hole and into another universe.

8. Appendix

In this appendix we describe the procedure by which the poles of the scattering amplitudes for equation (6.1) can be found, following ref [16] but allowing the potential to approach different constant values at the horizons. Define solutions of the scattering problem $f_1$ and $f_2$ with the asymptotic behaviour

$$
\begin{align*}
    f_1(x, \omega) &\to e^{+ik_+x} & \bar{f}_1(x, \omega) &\to e^{-ik_+x} & x &\to \infty \\
    f_2(x, \omega) &\to e^{-ik_-x} & \bar{f}_2(x, \omega) &\to e^{+ik_-x} & x &\to -\infty \\
\end{align*}
$$

(8.1)

where

$$
    k_\pm = \omega - m\Omega_\pm.
$$

(8.2)

(Note that \(\pm\) refer to \(\pm\infty\), the reverse of ref. [16].)

The solutions $f_1$ and $\bar{f}_1$ form a complete set of solutions to the scattering problem and it is therefore possible to express $f_2$ as a linear combination,

$$
    f_2(x, \omega) = \frac{R(\omega)}{T(\omega)} f_1(x, \omega) + \frac{1}{T(\omega)} \bar{f}_1(x, \omega).
$$

(8.3)

This defines the reflection and transmission coefficients $R(\omega)$ and $T(\omega)$. They can be extracted from $f_1$ and $f_2$ for any value of $x$ by using the Wronskians,

$$
\begin{align*}
    \frac{1}{T(\omega)} &= -\frac{1}{2ik_+} [f_1(x, \omega), f_2(x, \omega)] \\
    R(\omega) &= \frac{1}{2ik_+} [\bar{f}_1(x, \omega), f_2(x, \omega)]
\end{align*}
$$

(8.4)
where \([f, g] = fg' - gf'\).

The scattering problem can be reduced to an integral equation by using the Green function. We first write the differential equation as

\[-\frac{d^2 f}{dx^2} + k_-^2 f = (V(x) - c_0^-) f_2.\]  \hspace{1cm} (8.5)

The solution obtained by regarding the right hand side as a source term is

\[f_2(x, \omega) = e^{-ik_- x} + \int_{-\infty}^x \frac{1}{k_-} \sin k_- (x - x') (V(x') - c_0^-) f_2(x', \omega) \, dx'.\]  \hspace{1cm} (8.6)

The integral equation can be iterated to obtain a series for \(f_2\). set

\[f_2(x, \omega) = e^{-ik_- x} + \sum_{s=1}^{\infty} f_2^{(s)}(x, \omega)\]  \hspace{1cm} (8.7)

then,

\[f_2^{(s)}(x, \omega) = \int_{-\infty}^x \frac{1}{k_-} \sin k_- (x - x_s) (V(x_s) - c_0^-) f_2^{(s-1)}(x_s, \omega) \, dx_s.\]  \hspace{1cm} (8.8)

Repeatedly iterating this equation gives

\[f_2^{(s)}(x, \omega) = \frac{e^{-ik_- x}}{(2ik_-)^s} \int_{-\infty}^{x_0} dx_0 \cdots \int_{-\infty}^{x_s} dx_s \prod_{i=1}^{s} \left\{ \left( e^{2ik_- (x_{i-1} - x_i)} - 1 \right) \sum_{n=1}^{\infty} c_n e^{2n\kappa_- x_i} \right\} \]  \hspace{1cm} (8.9)

where the exponential expansion of \(V\) has been inserted. If we define \(y_i = x_{i-1} - x_i\), then following [16], we can write

\[f_2^{(s)}(x, \omega) = \frac{e^{-ik_- x}}{(2ik_-)^s} \int_0^{\infty} dy_1 \cdots \int_0^{\infty} dy_s \prod_{i=1}^{s} \left\{ (1 - e^{2ik_- y_i}) \sum_{n=1}^{\infty} c_n e^{2n\kappa_- x_i} \right\}.\]  \hspace{1cm} (8.10)

The integrand is a sum of exponentials and the result of integrating over the \(y_i\) introduces poles at \(k_- = -in\kappa_-\), where \(n = 1, 2, \ldots\). Therefore, \(f_2(x, \omega)\) has poles at

\[\omega = m\Omega_- - in\kappa_-.\]  \hspace{1cm} (8.11)

A similar treatment of \(f_1(x, \omega)\) shows that this function has its poles at

\[\omega = m\Omega_+ + in\kappa_+.\]  \hspace{1cm} (8.12)

From equation (8.4), these determine where the poles of \(R(\omega)/T(\omega)\) and \(1/T(\omega)\) lie, as shown in figure 6.
References

1. S. Chandrasekhar, The Mathematical Theory of Black Holes, (New York, Oxford University Press, 1983).
2. W. Israel, in Black Hole Physics, ed V. De Sabbata and Z. Zhang (Amsterdam: Kluwer Academic Publishers 1992).
3. F. Mellor and I. G. Moss, Phys. Rev. D41, 403 (1990).
4. F. Mellor and I. G. Moss, Class. Quantum Grav. 9, L43 (1992).
5. P. R. Brady and E. Poisson, Class. Quantum Grav. 9, 121 (1992).
6. P. R. Brady, D. Nunez and S. Sinha, Phys. Rev. D47, 4239 (1993).
7. B. Carter, in Black Holes ed C. DeWitt and B. S. DeWitt (New York, Gordon and Breach 1973).
8. F. Mellor and I. G. Moss, Class. Quantum Grav. 6, 1379 (1989).
9. G. W. Gibbons and S. W. Hawking, Phys. Rev. D15, 2738 (1977).
10. B. Carter, Comm. Math. Phys. 10, 280 (1968).
11. J. Meixner, F. W. Schäfke and G. Wolf, Mathieu Functions, Spheroidal Functions and their Mathematical Foundations (Heidelberg, Springer–Verlag 1980).
12. E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).
13. S. A. Teukolsky, Phys. Rev. Lett 29, 1114 (1972).
14. A. A. Starobinsky and S. M. Churilov, Soviet Phys. JETP 38, 1 (1973).
15. R. Geroch, A. Held and R. Penrose, J. Math. Phys. 14, 874 (1973).
16. S. Chandrasekhar and J. B. Hartle, Proc. R. Soc. A384, 301 (1982).
17. V. de Alfaro and T. Regge, Potential Scattering (Amsterdam, North Holland 1965).
18. J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, K. S. Thorne and U. Yurtsever, Phys. Rev. D42 1915 (1990).
Figures

1. The parameter space of the black hole for (a) $Q = 0$ and (b) $Q = 0.2$. The mass $M$ and rotation rate $a$ have both been scaled by $\alpha$, where $\alpha^2 = 3/\Lambda$. Coincident horizons occur along the outer borders of the triangular region, $r_2 = r_3$ on OA and $r_1 = r_2$ on AC. Corresponding surface gravities also vanish there. Interior lines denote the cases where $\kappa_1 = \kappa_3$ (the lower line OA) and $\kappa_1 = \kappa_2$ (along OB). The narrow region extending from O to A is the stability region.

2. A Penrose diagram showing the event horizon $r_2$ between regions I (outside the hole) and II (inside the hole).

3. Part of the Penrose diagram of the fully extended spacetime. Horizon 1 is a cosmological de Sitter horizon, horizon 2 a black hole event horizon and horizon 3 an inner Cauchy horizon. Asymptotic regions exist and are labelled as $\mathcal{J}$. Singularities at $r = 0$ are represented by zigzag lines. The fully extended diagram covers the whole plane, including the (blank) regions that can only be reached by passing through $r = 0$.

4. The radial potential (reversed sign) in region II for various values of $l$. The upper three curves have $m = 1$ and the lower three $m = 0$.

5. The inward and outward radial mode functions $\overrightarrow{R}$ and $\overleftarrow{R}$ showing how their asymptotic amplitudes $A$, $B$ and $C$ are associated with the respective horizons.

6. The location of the poles in $1/T$ and $R/T$ for $s = 0$ are shown in the complex $\omega$–plane.
Tables

| $\omega a$ | $(l, m) = (0, 0)$ | $(1, 0)$ | $(1, 1)$ | $(1, -1)$ | $(2, 0)$ |
|-----------|------------------|----------|----------|-----------|----------|
| 0.0       | 0.000            | 2.000    | 2.000    | 2.000     | 6.000    |
| 0.1       | 0.007            | 2.004    | 1.809    | 2.208     | 6.005    |
| 0.2       | 0.027            | 2.016    | 1.631    | 2.432     | 6.019    |
| 0.3       | 0.060            | 2.036    | 1.471    | 2.672     | 6.043    |
| 0.4       | 0.107            | 2.064    | 1.327    | 2.982     | 6.076    |
| 1.0       | 0.667            | 2.400    | 0.800    | 4.800     | 6.476    |

Table 1. Angular eigenvalues for $a/\alpha = 0$.

| $\omega a$ | $(l, m) = (0, 0)$ | $(1, 0)$ | $(1, 1)$ | $(1, -1)$ | $(2, 0)$ |
|-----------|------------------|----------|----------|-----------|----------|
| 0.0       | 0.000            | 2.003    | 2.021    | 2.097     | 6.208    |
| 0.1       | 0.007            | 2.008    | 1.825    | 2.310     | 6.212    |
| 0.2       | 0.028            | 2.020    | 1.645    | 2.538     | 6.228    |
| 0.3       | 0.061            | 2.040    | 1.482    | 2.783     | 6.252    |
| 0.4       | 0.109            | 2.069    | 1.335    | 3.044     | 6.249    |
| 1.0       | 0.680            | 2.473    | 0.857    | 4.954     | 6.694    |

Table 2. Angular eigenvalues for $a/\alpha = 0.1$.

| $\omega a$ | $(l, m) = (0, 0)$ | $(1, 0)$ | $(1, 1)$ | $(1, -1)$ | $(2, 0)$ |
|-----------|------------------|----------|----------|-----------|----------|
| 0.0       | 0.000            | 2.007    | 2.079    | 2.403     | 6.861    |
| 0.1       | 0.008            | 2.012    | 1.871    | 2.628     | 6.867    |
| 0.2       | 0.029            | 2.025    | 1.670    | 2.871     | 6.881    |
| 0.3       | 0.065            | 2.046    | 1.508    | 3.130     | 6.906    |
| 0.4       | 0.116            | 2.077    | 1.352    | 3.407     | 6.943    |
| 1.0       | 0.721            | 2.700    | 1.040    | 5.432     | 7.375    |

Table 3. Angular eigenvalues for $a/\alpha = 0.2$. 