Stable propagation of a modulated particle beam in a crystal channel

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The propagation of a modulated beam of charged particles in a planar crystal channel is investigated. It is demonstrated that the beam preserves its modulation at sufficiently large penetration depths to ensure the feasibility of using a crystalline undulator as a coherent source of hard X rays. This finding is a crucial milestone in developing a new type of lasers radiating in the hard X ray and gamma ray range.

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In this Letter we study for the first time the evolution of a modulated particle beam in a planar crystal channel and demonstrate that it preserves its modulation at sufficiently large penetration depths to ensure the feasibility of using a crystalline undulator as a coherent source of hard X rays. Solving this problem is of crucial importance in the theory of the Crystal Undulator based Laser (CUL) [1, 2, 3] — a new electromagnetic radiation source in hard X and gamma ray range.

Channeling takes place if charged particles enter a single crystal at small angle with respect to crystallographic planes or axes [4]. The particles get confined by the interplanar or axial potential and follow the shape of the corresponding planes and axes.

A single crystal with periodically bent crystallographic planes can force channeling particles to move along nearly sinusoidal trajectories and radiate in hard X and gamma ray frequency range (see Fig. 1). The feasibility of such a device, known as the 'crystalline undulator', was demonstrated theoretically a decade ago [1] (further developments as well as historical references are reviewed in [5]). Its experimental study is on the way within the PECU project [6].

FIG. 1: Schematic representation of the crystalline undulator.

The advantage of the crystalline undulator is in extremely strong electrostatic fields inside a crystal which are able to steer the particles much more effectively than even the most advanced superconductive magnets. Due to this fact, such an undulator can radiate powerful electromagnetic waves in the hard X and soft gamma ray range, where conventional sources with comparable intensity are unavailable [7].

Even more powerful and coherent radiation will be emitted if the probability density of the particles in the beam is modulated in the longitudinal direction with the period \( \lambda \), equal to the wavelength of the emitted radiation. In this case, the electromagnetic waves emitted in the forward direction by different particles have approximately the same phase [8]. Therefore, the intensity of the radiation becomes proportional to the beam density squared (in contrast to the linear proportionality for an unmodulated beam). This increases the photon flux by orders of magnitude relative to the radiation of unmodulated beam of the same density. The radiation of a modulated beam in an undulator is a keystone of the physics of free-electron lasers [9]. It can be considered as a classical counterpart of the stimulated emission in quantum physics. Therefore, if similar phenomenon takes place in a crystalline undulator, it can be referred to as the lasing regime of the crystalline undulator.

The feasibility of CUL radiating in hard X ray and gamma ray range was considered for the first time in [1, 2]. Recently, a two-crystal scheme, the gamma klystron, has been proposed [3].

A simplified model used in the cited papers assumed that all particle trajectories follow exactly the shape of the bent channel. In reality, however, the particle moving along the channel also oscillates in the transverse direction with respect to the channel axis (see the shape of the trajectory in Fig. 1). Different particles have different amplitudes of the oscillations inside the channel (Fig. 2, upper panel). Similarly, the directions of particle momenta in \((xz)\) plane are slightly different (Fig. 2, lower panel). Even if the speed of the particles along their trajectories is the same, the particles oscillating with different amplitudes or the particles with different trajectory slopes with respect to \(z\) axis have slightly different \(z\) components of their velocities. As a result, the beam gets demodulated. An additional contribution to the beam demodulation comes from incoherent collisions.
of the channeling particles with the crystal constituents.

In the case of an unmodulated beam, the length of the crystalline undulator and, consequently, the maximum accessible intensity of the radiation is limited by the dechanneling process. The channeling particle gradually gains the energy of transverse oscillation due to collisions with crystal constituents. At some point this energy exceeds the maximum value of the interplanar potential and the particle leaves the channel. The average penetration length at which this happens is known as the dechanneling length. The dechannelled particle does not follow the sinusoidal shape of the channel any more and, therefore, does not contribute to the undulator radiation. Hence, the reasonable length of the crystalline undulator is limited to a few dechanneling lengths. A longer crystal would attenuate rather than produce the radiation. Since the intensity of the undulator radiation is proportional to the undulator length squared, the dechanneling length is the main restricting factor that has to be taken into account when the radiation output is calculated.

In contrast, not only the shape of the trajectory but also the particles positions with respect to each other along z axis are important for the lasing regime. If this positions become random because of the beam demodulation, the intensity of the radiation drops even if the particles are still in the channeling mode. Hence, it is the beam demodulation rather than dechanneling that restricts the intensity of the radiation of CUL. Understanding this process and estimating the characteristic length at which this phenomenon takes place is, therefore, a cornerstone of the theory of this new radiation source.

Let us consider the distribution \( f(t, z; \xi, E_y) \) of the beam particles with respect to the angle between the particle trajectory and axis \( z \) in the \((xz)\) plane \( \xi = \arcsin p_x/p \approx p_x/p \) and the energy of the channeling oscillation \( E_y = p_x^2/2E + U(y) \) [11]. Here \( p, p_x \) and \( p_y \) are, respectively, the particle momentum and its \( x \) and \( y \) components, \( U(y) \) is the interplanar potential, and \( E \) is the particle energy (we will consider only ultrarelativistic particles, therefore \( E \approx p \)). It can be shown that the evolution of this distribution in the crystal channel with the time \( t \) and the longitudinal coordinate (penetration depth into the crystal) \( z \) can be described by the following differential equation of Fokker-Planck type:

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z} v_z = D_0 \left[ \frac{\partial}{\partial E_y} \left( E_y \frac{\partial f}{\partial E_y} \right) + \frac{1}{E} \frac{\partial^2 f}{\partial \xi^2} \right].
\]

Here \( D_0 \) is the diffusion coefficient that is dominated by the scattering of the beam particles by lattice electrons. The particle longitudinal velocity averaged over the undulator period, \( v_z \), is given by

\[
v_z = \left( 1 - \frac{1}{2\gamma^2} - \frac{\xi^2}{2} - \frac{E_y}{2E} \right)
\]

If the beam is periodically modulated (bunched) the distribution can be represented as a Fourier series:

\[
f(t, z; \xi, E_y) = \sum_{j=-\infty}^{\infty} g_j(z; \xi, E_y) \exp(i\omega t).
\]

with \( g_j(z; \xi, E_y) = g_{-j}(z; \xi, E_y) \) to ensure the real value of the particle distribution. Since Eq. (1) is linear, it is sufficient to consider only one harmonic. Substituting \( f(t, z; \xi, E_y) = g(z; \xi, E_y) \exp(i\omega t) \) one obtains

\[
i\omega g(z; \xi, E_y) + \frac{\partial g}{\partial z} v_z = D_0 \left[ \frac{\partial}{\partial E_y} \left( E_y \frac{\partial g}{\partial E_y} \right) + \frac{1}{E} \frac{\partial^2 g}{\partial \xi^2} \right].
\]

To simplify this equation, we make the substitution \( g(z; \xi, E_y) = \exp(-i\omega z) \tilde{g}(z; \xi, E_y) \) and assume that the variation of \( \tilde{g}(z; \xi, E_y) \) within the modulation period is small: \( \partial g/\partial z \ll \omega \tilde{g}(z; \xi, E_y) \). This allows us to neglect the terms \((1 - v_z)\partial \tilde{g}/\partial z \) while keeping the terms \((1 - v_z)i\omega \tilde{g}(z; \xi, E_y) \). The resultant partial differential equation for \( \tilde{g}(z; \xi, E_y) \) can be solved by the method of separation of variables. Putting \( \tilde{g}(z; \xi, E_y) = \mathcal{Z}(z)\Xi(\xi)\mathcal{E}(E_y) \), we obtain a set of ordinary differential equations:

\[
\frac{D_0}{E} \frac{d}{dE_y} \left( \frac{d\mathcal{E}}{dE_y} \right) - i\omega E_y = C_z,
\]

\[
\frac{D_0}{\mathcal{E}(E_y)} \frac{d}{dE_y} \left( E_y \frac{d\mathcal{E}(E_y)}{dE_y} \right) = C_y,
\]

\[
1 - \frac{\omega^2}{2E^2} = C_z.
\]

where \( C_z, C_\xi \) and \( C_y \) do not depend on any of the variables \( z, \xi \) and \( E_y \) and satisfy the condition

\[
C_z = C_\xi + C_y.
\]
Eq. (5) has the form of the Schrödinger equation for the harmonic oscillator. Its eigenvalues and eigenfunctions are, respectively,

$$C_{n} = - (1 + i) \sqrt{\frac{\omega D_0}{E}} \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots$$

and

$$\Xi_n(\xi) = H_n \left( e^{i\pi/4} \frac{\sqrt{\omega E}}{2D_0} \right) \exp \left( - \frac{1 + i}{4} \sqrt{\frac{\omega E}{D_0} \xi^2} \right).$$

Here $H_n(\ldots)$ are Hermite polynomials.

Eq. (6) can be reduced to the Laguerre differential equation, so that its solution can be represented as

$$\mathcal{E}_k(E_y) = \exp \left( - \frac{1 + i}{2} \sqrt{\frac{\omega}{D_0 E}} E_y \right) L_{\nu_k} \left( 1 + i \right) \sqrt{\frac{\omega}{D_0 E}} E_y$$

where $L_{\nu}(\ldots)$ is the Laguerre function[12] and $\nu_k$ is related to the eigenvalue $C_{y,k}$ via

$$C_{y,k} = - \frac{(1 + i)}{2} \sqrt{\frac{D_0 \omega}{E}} (2\nu_k + 1), \quad k = 1, 2, 3, \ldots$$

The eigenvalues can be found by imposing the boundary conditions: the density of the channeling particles becomes zero if the energy of channeling oscillations $E_y$ equals to the interplanar barrier $U_{\text{max}}$:

$$L_{\nu_k} \left( 1 + i \right) \sqrt{\frac{\omega}{D_0 E}} U_{\text{max}} = 0.$$  \hspace{1cm} (13)

Equation (13) has to be solved for $\nu_k$. (the subscript $k$ enumerates different roots of the equation) and the result has to be substituted into (12).

It is convenient to represent the eigenvalues in the form

$$C_{y,k} = \frac{\alpha_k(\kappa)}{L_d} + i \omega \theta_L^2 \beta_k(\kappa).$$ \hspace{1cm} (14)

Here $L_d = \frac{4U_{\text{max}}/j_0^2 D_0}$ is the dechanneling length [10] ($j_{0,k}$ is k-th zero of the Bessel function $J_0(\cdot)$), $\theta_L = \sqrt{2U_{\text{max}}/E}$ is Lindhard’s angle. We introduced the parameter

$$\kappa = \pi \frac{L_d}{\lambda} \theta_L^2,$$ \hspace{1cm} (15)

where $\lambda = 2\pi/\omega$ is the spatial period of the modulation. The functions $\alpha_k(\kappa)$ and $\beta_k(\kappa)$ are to be found by solving numerically Eq. (13) combined with (12).

Using (8), one finds the solution of Eq. (7):

$$Z_{n,k}(z) = \exp \left\{ - \frac{z}{L_d} \left[ \alpha_k(\kappa) + (2n + 1) \frac{\sqrt{\kappa}}{j_{0,1}} \right] - i \omega z \left[ \frac{1}{2\gamma^2} + \theta_L^2 \beta_k(\kappa) + \frac{\theta_L^2 (2n + 1)}{2j_{0,1} \sqrt{\kappa}} \right] \right\}.$$ \hspace{1cm} (16)

Hence, the solution of Eq. (4) is represented as

$$g(z; \xi, E_y) = \exp \left( -i \omega z \right) \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} c_{n,k} \Xi_n(\xi) \mathcal{E}_k(E_y) Z_{n,k}(z),$$ \hspace{1cm} (17)

where the coefficients $c_{n,k}$ are found from the particle distribution at the entrance of the crystal channel. Due to the exponential decrease of $Z_{n,k}(z)$ with $z$ (see (16)), the asymptotic behaviour of $g(z; \xi, E_y)$ at large $z$ is dominated by the term with $n = 0$ and $k = 1$ having the smallest value of the factor $|\alpha_1(\kappa) + (2n + 1) \sqrt{\kappa} |$ in the exponential. Therefore, at sufficiently large penetration depths, the particle distribution depends on $z$ as

$$g(z; \xi, E_y) \propto \exp \left( -z/L_{\text{dm}} - i \omega / u_z \right) z$$

where $L_{\text{dm}}$ is the newly introduced parameter — the dechanneling length:

$$L_{\text{dm}} = \frac{L_d}{\alpha_1(\kappa) + \sqrt{\kappa} / j_{0,1}}$$ \hspace{1cm} (18)

and $u_z$ is the phase velocity of the modulated beam along the crystal channel

$$u_z = \left[ 1 + \frac{1}{2\gamma^2} + \theta_L^2 \left( \beta_1(\kappa) + \frac{1}{2j_{0,1} \sqrt{\kappa}} \right) \right]^{-1}.$$ \hspace{1cm} (19)

This parameter is important for establishing the resonance conditions between the undulator parameters and the radiation wavelength. It will be analyzed elsewhere.

In this Letter we concentrate our attention on the demodulation length. This parameter represents the characteristic scale of the penetration depth at which a beam of channeling particles gets demodulated. Fig. 3 presents the dependence of the ratio $L_{\text{dm}}/L_d$ on the parameter $\kappa$. It is seen that the demodulation length approaches the dechanneling length at $\kappa \lesssim 1$. This is an important result, since it means that the demodulation process does not put additional restrictions on the undulator length and, therefore, a strong lasing effect can be expected in the crystalline undulator fed by a modulated beam. On the contrary, the ratio noticeably drops for $\kappa \gtrsim 10$. Therefore, further investigations are needed to clarify whether the lasing regime is feasible for these $\kappa$. 
The dependence of the parameter $\kappa$ on the energy of the emitted photons, $\hbar \omega = 2\pi\hbar/\lambda$, is shown in Fig. 4. The calculation was done for 1 GeV[13] positrons using the formula for the dechanneling length from [5, 10].

As one sees from the figure, there exist such crystal channels that $\kappa = 1$ corresponds to $\hbar \omega = 100 – 300$ keV. Therefore, CUL is most suitable for the application in this photon energy range. Using crystalline undulator for the emission of softer photons $\hbar \omega \lesssim 10$ keV is prevented by very strong absorption of the radiation in the crystal, while the upper limit on the photon energy is set by the decreasing value of $L_{\text{dm}}$.

It is instructive to study the influence of the particle motion in $x$ and $y$ direction on the demodulation length separately. Replacing $\alpha_1(\kappa)$ in (18) with unity means neglecting the motion in the $y$ direction, while omitting the second term in the denominator ignores the motion in $x$ direction. One sees from Fig. 3 that it is mostly the motion in $x$ direction that suppresses the demodulation length at $\kappa \lesssim 10$, while the influence of channeling oscillations is negligible. This suggests the idea that axial channeling, i.e. when motion in both $x$ and $y$ directions has the nature of channeling oscillations, might be more suitable for the lasing regime in the range $\hbar \omega \sim 1$ MeV.

In conclusion, we have studied the propagation of a modulated particle beam in a planar crystal channel. It has been demonstrated that the beam preserves its modulation at the penetration depths which are sufficient for using the crystalline undulator as a source of coherent radiation with the photon energy of hundreds keV.

We restricted our analysis to the beam demodulation in a straight crystal. The influence of the periodic bending on the demodulation length has to be studied at the next step. Another important milestone in the theory of CUL would be developing suitable methods of beam modulation.

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[1] A.V. Korol, A.V. Solov’yov, W. Greiner, J. Phys. G 24, L45 (1998);
[2] Int. J. Mod. Phys. E 8 49 (1999);
[3] A. Kostyuk, A. Korol, A. Solov’yov and W. Greiner, arXiv:0710.4772 [physics.acc-ph].
[4] J. Lindhard, Kong. Danske Vid. Selsk. Mat.-Fys. Medd. 34, 14 (1965).
[5] A.V. Korol, A.V. Solov’yov, W. Greiner, Int. J. Mod. Phys. E 13, 867 (2004).
[6] http://ec.europa.eu/research/fp6/nest/pdf
[7] A.V. Korol, A.V. Solov’yov, W. Greiner, Topics in Heavy Ion Phys., 73 (2005).
[8] V.L. Ginzburg, Izv. Akad. Nauk. SSSR, Ser. Fiz. 11, 165 (1947).
[9] J.M.J. Madey, J. Appl. Phys. 42, 1906 (1971).
[10] V.M. Biruykov, Yu.A. Chesnokov, V.I. Kotov, Crystal Channelling and its Application at High-Energy Accelerators (Springer, Berlin, 1996).
[11] We chose the system of units in such a way that the speed of light is equal to unity. Therefore, mass, energy and momentum have the same dimensionality. This is also true for length and time.
[12] At nonnegative integer values of $\nu$, the Laguerre function is reduced to the well known Laguerre polynomials. In the general case that is relevant to our consideration, it can be represented by an infinite series: $L_\nu(\varepsilon) = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} (m + \nu) \varepsilon^j / (j!)^2$.
[13] Note that $\kappa$ depends weakly (logarithmically) on the particle energy. Therefore, changing the beam energy by an order of magnitude would leave Fig. 4 practically unchanged.