THE CKM MATRIX AND THE HEAVY QUARK EXPANSION

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These lectures contain an elementary introduction to heavy quark symmetry and the heavy quark expansion. Applications such as the expansion of heavy meson decay constants and the treatment of inclusive and exclusive semileptonic $B$ decays are included. The use of heavy quark methods for the extraction of $|V_{cb}|$ and $|V_{ub}|$ is presented in some detail.

1 Heavy Quark Symmetry

In these lectures I will introduce the ideas of heavy quark symmetry and the heavy quark limit, which exploit the simplification of certain aspects of QCD for infinite quark mass, $m_Q \rightarrow \infty$. We will see that while these ideas are extraordinarily simple from a physical point of view, they are of enormous practical utility in the study of the phenomenology of bottom and charmed hadrons. One reason for this is the existence not just of an interesting new limit of QCD, but of a systematic expansion about this limit. The technology of this expansion is the Heavy Quark Effective Theory (HQET), which allows one to use heavy quark symmetry to make accurate predictions of the properties and behavior of heavy hadrons in which the theoretical errors are under control. While the emphasis in these lectures will be on the physical picture of heavy hadrons which emerges in the heavy quark limit, it will be important to introduce enough of the formalism of the HQET to reveal the structure of the heavy quark expansion as a simultaneous expansion in powers of $\Lambda_{\text{QCD}}/m_Q$ and $\alpha_s(m_Q)$. However, what I hope to leave you with above all is an appreciation for the simplicity, elegance and coherence of the ideas which underlie the technical results which will be presented. The interested reader is also encouraged to consult a number of excellent reviews which typically cover in more detail the material in these lectures.
1.1 Introduction

We begin by recalling the properties of charged current interactions in the Standard Model. They are mediated by the interactions with the $W^\pm$ bosons, which for the quarks take the form.

\[( \bar{u} \bar{c} \bar{t} ) \gamma^\mu (1 - \gamma^5) V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix} W^\mu + \text{h.c.} \]

The $3 \times 3$ unitary matrix $V_{\text{CKM}}$ is

\[V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}.\]

The elements of $V_{\text{CKM}}$ have a hierarchical structure, getting smaller away from the diagonal: $V_{ud}$, $V_{cs}$ and $V_{tb}$ are of order 1, $V_{us}$ and $V_{cd}$ are of order $10^{-1}$, $V_{cb}$ and $V_{ts}$ are of order $10^{-2}$, and $V_{ub}$ and $V_{td}$ are of order $10^{-3}$. By contrast, except for small effects associated with neutrino masses, the interaction of the $W^\pm$ with the leptons is flavor diagonal.

The CKM matrix is of fundamental importance, because it is the low energy manifestation of the higher-energy physics which breaks the global flavor symmetries of the Standard Model. In the absence of the Yukawa couplings, the quark sector of the Standard Model may be characterized by its gauge symmetry $SU(3) \times SU(2) \times U(1)$, and its global symmetry $U(3)_Q \times U(3)_U \times U(3)_D$ which rotates the triplets of colored fields $Q^i_L$, $U^i_R$ and $D^i_R$ among each other. The global symmetry group has a total of $3 \times 3^2 = 27$ generators. With the addition of fields $\phi$ and $\tilde{\phi}$ (possibly composite) carrying Higgs and conjugate-Higgs quantum numbers, one may write Yukawa-type interactions,

\[\lambda^j_u \left( \bar{Q}^i_L \phi U^j_R \right) + \lambda^j_d \left( \bar{Q}^i_L \phi D^j_R \right) + \text{h.c.}\]

which break the flavor symmetries explicitly. The complex matrices $\lambda^j_u$ and $\lambda^j_d$ correspond to $2 \times (2 \times 3^2) = 36$ independent parameters. They break the global flavor symmetries completely, except for a remaining conserved baryon number $U(1)_B$. The $27 - 1 = 26$ broken generators may be used to rotate 26 of the Yukawa couplings to zero, leaving $36 - 26 = 10$ physical parameters.

When $\phi$ and $\tilde{\phi}$ get vacuum expectation values, one may go to the mass eigenstate basis for the quark fields, in which case the ten parameters are six quark masses and four parameters characterizing $V_{\text{CKM}}$. An independent examination of $V_{\text{CKM}}$ confirms this counting. A $3 \times 3$ unitary matrix has 9
parameters, of which \( \binom{3}{2} = 3 \) are angles and the remaining 6 are complex phases. However, one may adjust 5 relative phases of the mass eigenstate quark fields, leaving 6 − 5 = 1 physical phase in \( V_{\text{CKM}} \). Thus \( V_{\text{CKM}} \) is indeed characterized by four parameters, one of which is a CP-violating phase.

It is an instructive exercise, left to the reader, to perform the analogous counting for the general case of \( U(N_f)_Q \times U(N_f)_U \times U(N_f)_D \) global flavor symmetry. One finds that the Yukawa couplings contain \( N_f^2 + 1 \) physical parameters, of which \( 2N_f \) are quark masses. The remaining \( (N_f - 1)^2 \) parameters characterize \( V_{\text{CKM}} \), with \( \frac{1}{2}N_f(N_f - 1) \) being angles and \( \frac{1}{2}(N_f - 1)(N_f - 2) \) being complex phases. In particular, one notes that for \( N_f = 2 \) there is no CP violation in the weak interactions.

Why is an understanding of QCD crucial to the study of the properties of \( V_{\text{CKM}} \)? As an example, consider semileptonic b decay, \( b \to c \ell \bar{\nu} \), from which one would like to extract \( |V_{cb}| \). This process is mediated by a four-fermion operator,

\[
\mathcal{O}_{bc} = \frac{G_F V_{cb}}{\sqrt{2}} \bar{c} \gamma^\mu (1 - \gamma^5) b \bar{\nu} \gamma_\mu (1 - \gamma^5) \ell .
\]

The weak matrix element is easy to calculate at the quark level,

\[
A_{\text{quark}} = \langle c \ell \bar{\nu} | \mathcal{O}_{bc} | b \rangle = \frac{G_F V_{cb}}{\sqrt{2}} \bar{u}_c(p_c) \gamma^\mu (1 - \gamma^5) u_b(p_b) \bar{\nu}_\ell (p_\ell) \gamma_\mu (1 - \gamma^5) v_\nu(p_\nu) .
\]

However, \( A_{\text{quark}} \) is only relevant at very short distances; at longer distances, QCD confinement implies that free \( b \) and \( c \) quarks are not asymptotic states of the theory. Instead, nonperturbative QCD effects “dress” the quark level transition \( b \to c \ell \bar{\nu} \) to a hadronic transition, such as

\[
B \to D \ell \bar{\nu} \quad \text{or} \quad B \to D^* \ell \bar{\nu} \quad \text{or} \quad \ldots
\]

(In these lectures, we will use a convention in which a \( B \) meson contains a \( b \) quark, not a \( \bar{b} \) antiquark.) The hadronic matrix element \( A_{\text{hadron}} \) depends on nonperturbative QCD as well as on \( G_F V_{cb} \), and is difficult to calculate from first principles. To disentangle the weak interaction part of this complicated process requires us to develop some understanding of the strong interaction effects.

There are a variety of methods by which one can do this. Perhaps the most popular, historically, has been use of various quark potential models. While these models are typically very predictive, they are based on uncontrolled assumptions and approximations, and it is virtually impossible to estimate the theoretical errors associated with their use. This is a serious defect if one builds such a model into the experimental extraction of a weak coupling constant such
as $V_{cb}$, because the uncontrolled theoretical errors then infect the experimental result.

These are issues which are important for the extraction of all the elements of $V_{CKM}$. Let us pause now to review our current experimental knowledge of each of the magnitudes. The results are taken from the Particle Data Book[1] (The phases of the matrix elements must be extracted from CP violating asymmetries, as discussed elsewhere at this school.)

We start with the submatrix describing mixing among the first two generations. The parameter $|V_{ud}|$ is measured by studying the rates for neutron and nuclear $\beta$ decay. Here the isospin symmetry of the strong interactions may be used to control the nonperturbative dynamics, since the operator $\bar{d}\gamma^\mu(1-\gamma^5)u$ which mediates the decay is a partially conserved current associated with a generator of chiral $SU(2)_L \times SU(2)_R$. The current data yield

$$|V_{ud}| = 0.9735 \pm 0.0008,$$

so $|V_{ud}|$ is known at the level of 0.1%. The parameter $|V_{us}|$ is measured similarly, via $K \to \pi\ell\nu$ and $\Lambda \to p\ell\bar{\nu}$. Here chiral $SU(3)_L \times SU(3)_R$ must be used in the hadronic matrix elements, since a strange quark is involved; because the $m_s$ corrections are larger, $|V_{us}|$ is only known to 1%:

$$|V_{us}| = 0.2196 \pm 0.0023.$$

The $V_{CKM}$ elements involving the charm quark are not so well measured. One way to extract $|V_{cs}|$ is to study the decay $D \to K\ell^\pm\nu_\ell$. Unfortunately, there is no symmetry by which one can control the matrix element $\langle K|\bar{s}\gamma^\mu(1-\gamma^5)c|D\rangle$, since flavor $SU(4)$ is badly broken. One is forced to resort to models for these matrix elements. The reported value is

$$|V_{cs}| = 1.04 \pm 0.16,$$

but it must be said that this error estimate is not on very firm footing, and should probably be taken to be substantially larger.

An alternative is to measure $V_{cs}$ from inclusive processes at higher energies. For example, one can study the branching fraction for $W^+ \to c\bar{s}$, which can be computed using perturbative QCD. The result of a preliminary analysis is

$$|V_{cs}| = 1.00 \pm 0.13,$$

consistent with the model-dependent measurement. In this case, however, the error is largely experimental, and is unpolluted by hadronic physics. Similarly, one extracts $|V_{ud}|$ from deep inelastic neutrino scattering, using the process
\[ \nu_\mu + d \to e + \mu^- \] This inclusive process may be computed perturbatively in QCD, leading to a result with accuracy at the level of 10%,

\[ |V_{cd}| = 0.224 \pm 0.016. \] (11)

The elements of \( V_{\text{CKM}} \) involving the third generation are, for the most part, harder to measure accurately. The branching ratio for \( t \to b \ell^+ \nu \) can be analyzed perturbatively, but the experimental data are not very good. The present bound on \( |V_{tb}| \)

\[ \frac{|V_{td}|^2}{|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2} = 0.99 \pm 0.29. \] (12)

If one imposed the unitarity constraint \( |V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 1 \), then this would amount to a 15% measurement of \( |V_{tb}| \), but this unitarity constraint is one of the properties of \( V_{\text{CKM}} \) which one is trying to test. More generally, in fact, one should be wary of constraints on \( V_{\text{CKM}} \) which impose unitarity as part of the analysis; while one often obtains tighter constraints in this way, these constraints have a different meaning than do direct measurements. What the comparison of a constrained and unconstrained “measurement” really tells you is whether the direct determination may be used to test the unitarity of \( V_{\text{CKM}} \).

Unfortunately, there are as yet no direct extractions of \( |V_{td}| \) or \( |V_{ts}| \). One often speaks of these elements being measured in \( B_d - \overline{B}_d \) and \( B_s - \overline{B}_s \) mixing, but again, the hypothesis that the Standard Model is responsible for these processes is something that one really wants to check. The correct way to view this part of the experimental program is to say that the Standard Model, including the unitarity of \( V_{\text{CKM}} \), constrains \( V_s \) and \( V_d \) severely enough that testable predictions can be made for the mixing parameters \( \Delta m_d \) and \( \Delta m_s \).

This leaves us with the matrix elements \( V_{ub} \) and \( V_{cb} \), for which we need an understanding of \( B \) meson decay. In these lectures we will discuss an approach to understanding the relevant hadronic physics which exploits the fact that the \( b \) and \( c \) quarks are heavy, by which we mean that \( m_b, m_c \gg \Lambda_{\text{QCD}} \). The scale \( \Lambda_{\text{QCD}} \) is the typical energy at which QCD becomes nonperturbative, and is of the order of hundreds of MeV. The physical quark masses are approximately \( m_b \approx 4.8 \text{ GeV} \) and \( m_c \approx 1.5 \text{ GeV} \). The formalism which we will develop will not make as many predictions as do potential models. However, the compensation will be that we will develop a systematic expansion in powers of \( \Lambda_{\text{QCD}}/m_{b,c} \), within which we will be able to do concrete error analysis. In particular, we will be able to estimate the error associated with the fact that \( m_c \) may not be very close to the asymptotic limit \( m_c \gg \Lambda_{\text{QCD}} \). Even where this error may be substantial, the fact that it is under control allows us to maintain predictive power in the theory.

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1.2 The heavy quark limit

Consider a hadron $H_Q$ composed of a heavy quark $Q$ and “light degrees of freedom”, consisting of light quarks, light antiquarks and gluons, in the limit $m_Q \to \infty$. The Compton wavelength of the heavy quark scales as the inverse of the heavy quark mass, $\lambda_Q \sim 1/m_Q$. The light degrees of freedom, by contrast, are characterized by momenta of order $\Lambda_{\text{QCD}}$, corresponding to wavelengths $\lambda_l \sim 1/\Lambda_{\text{QCD}}$. Since $\lambda_l \gg \lambda_Q$, the light degrees of freedom cannot resolve features of the heavy quark other than its conserved gauge quantum numbers. In particular, they cannot probe the actual value of $\lambda_Q$, that is, the value of $m_Q$.

We may draw the same conclusion in momentum space. The structure of the hadron $H_Q$ is determined by nonperturbative strong interactions. The asymptotic freedom of QCD implies that when quarks and gluons exchange momenta $p$ much larger than $\Lambda_{\text{QCD}}$, the process is perturbative in the strong coupling constant $\alpha_s(p)$. On the other hand, the typical momenta exchanged by the light degrees of freedom with each other and with the heavy quark are of order $\Lambda_{\text{QCD}}$, for which a perturbative expansion is of no use. For these exchanges, however, $p < m_Q$, and the heavy quark $Q$ does not recoil, remaining at rest in the rest frame of the hadron. In this limit, $Q$ acts as a static source of electric and chromoelectric gauge field. The chromoelectric field, which holds $H_Q$ together, is nonperturbative in nature, but it is independent of $m_Q$. The result is that the properties of the light degrees of freedom depend only on the presence of the static gauge field, independent of the flavor and mass of the heavy quark carrying the gauge charge.

There is an immediate implication for the spectroscopy of heavy hadrons. Since the interaction of the light degrees of freedom with the heavy quark is independent of $m_Q$, then so is the spectrum of their excitations. It is these excitations which determine the spectrum of heavy hadrons $H_Q$. Hence the splittings $\Delta_i \sim \Lambda_{\text{QCD}}$ between the various hadrons $H_Q^i$ are independent of $Q$ and, in the limit $m_Q \to \infty$, do not scale with $m_Q$.

For example, the bottom and charmed meson spectra are shown schematically in Fig. [1] in the limit $m_b, m_c \gg \Lambda_{\text{QCD}}$. The light degrees of freedom are in exactly the same state in the mesons $B_i$ and $D_i$, for a given $i$. The offset $B_i - D_i = m_b - m_c$ is just the difference between the heavy quark masses; in no way does the relationship between the spectra rely on an approximation $m_b \approx m_c$.

We can enrich this picture by recalling that the heavy quarks and light

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aTop quarks decay too quickly for a static chromoelectric field to be established around them, so the simplifications discussed here are not relevant to them.
degrees of freedom also carry spin. The heavy quark has spin quantum number $S_Q = \frac{1}{2}$, which leads to a chromomagnetic moment

$$\mu_Q = \frac{g}{2m_Q}. \quad (13)$$

Note that $\mu_Q \to 0$ as $m_Q \to \infty$, and the interaction between the spin of the heavy quark and the light degrees of freedom is suppressed. Hence the light degrees of freedom are insensitive to $S_Q$; their state is independent of whether $S_Q^z = \frac{1}{2}$ or $S_Q^z = -\frac{1}{2}$. Thus each of the energy levels in Fig. 1 is actually doubled, one state for each possible value of $S_Q^z$.

To summarize, what we see is that the light degrees of freedom are the same when combined with any of the following heavy quark states:

$$Q_1(\uparrow), \quad Q_1(\downarrow); \quad Q_2(\uparrow), \quad Q_2(\downarrow); \quad \ldots \quad Q_{N_h}(\uparrow), \quad Q_{N_h}(\downarrow), \quad (14)$$

where there are $N_h$ heavy quarks (in the real world, $N_h = 2$). The result is an $SU(2N_h)$ symmetry which applies to the light degrees of freedom. A new symmetry means new nonperturbative relations between physical quantities. It is these relations which we wish to understand and exploit.
The light degrees of freedom have total angular momentum $J_\ell$, which is integral for baryons and half-integral for mesons. When combined with the heavy quark spin $S_Q = \frac{1}{2}$, we find physical hadron states with total angular momentum

$$J = |J_\ell \pm \frac{1}{2}|.$$  

(15)

If $S_\ell \neq 0$, then these two states are degenerate. For example, the lightest heavy mesons have $S_\ell = \frac{1}{2}$, leading to a doublet with $J = 0$ and $J = 1$. In the charm system we find that the states of lowest mass are the spin-0 $D$ and the spin-1 $D^*$; the corresponding bottom mesons are the $B$ and $B^*$. The heavy quark spin operator $S_Q$ exchanges these two states. Writing the spin wave function $|m_Q, m_\ell\rangle$, we have

$$|M\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |M^*(J^3 = 0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle).$$  

(16)

Then it is easy to show that

$$S_Q |M\rangle = \frac{1}{2} |M^*(J^3 = 0)\rangle, \quad S_Q |M^*(J^3 = 0)\rangle = \frac{1}{2} |M\rangle.$$  

(17)

When effects of order $1/m_Q$ are included, the chromomagnetic interactions split the states of given $S_\ell$ but different $J$. This “hyperfine” splitting is not calculable perturbatively, but it is proportional to the heavy quark magnetic moment $\mu_Q$. This gives its scaling with $m_Q$: 

$$m_{D^*} - m_D \sim 1/m_c$$

$$m_{B^*} - m_B \sim 1/m_b.$$  

(18)

From this fact we can construct a relation which is a nonperturbative prediction of heavy quark symmetry,

$$m_{B^*}^2 - m_B^2 = m_{D^*}^2 - m_D^2.$$  

(19)

Experimentally, $m_{B^*}^2 - m_B^2 = 0.49 \text{ GeV}^2$ and $m_{D^*}^2 - m_D^2 = 0.55 \text{ GeV}^2$, so this prediction works quite well. Note that this relation involves not just the heavy quark symmetry, but the systematic inclusion of the leading symmetry violating effects.

Generally, the mass of a heavy hadron $H_Q$ may be expanded in inverse powers of $m_Q$

$$m(H_Q) = m_Q + \tilde{\Lambda}_Q + \mathcal{O}(1/m_Q),$$  

(20)

where $\tilde{\Lambda}_Q$ is independent of $Q$ and is associated with the energy of the light degrees of freedom in the hadron $H_Q$. For the lowest lying $J_\ell = \frac{1}{2}$ doublet,
this quantity is usually just referred to as $\bar{\Lambda}$. From dimensional considerations, one expects $\bar{\Lambda}$ to be of order of a few hundred MeV.

So far, we have formulated heavy quark symmetry for hadrons in their rest frame. Of course, we can easily boost to a frame in which the hadrons have arbitrary four-velocity $v^\mu = \gamma(1, \vec{v})$. For heavy quarks $Q_1$ and $Q_2$, the symmetry will then relate hadrons $H_1(v)$ and $H_2(v)$ with the same velocity but with different momenta. This distinguishes heavy quark symmetry from ordinary symmetries of QCD, which relate states of the same momentum. To remind ourselves of this distinction, henceforth we will label heavy hadrons explicitly by their velocity: $D(v), D^*(v), B(v), B^*(v)$, and so on.

1.3 Semileptonic decay of a heavy quark

Now let us return to the semileptonic weak decay $b \to c \ell \bar{\nu}$, but now consider it in the heavy quark limit for the $b$ and $c$ quarks. Suppose the decay occurs at time $t = 0$. For $t < 0$, the $b$ quark is embedded in a hadron $H_b$; for $t > 0$, the $c$ quark is dressed by light degrees of freedom to $H'_c$. Let us consider the lightest hadrons, $H_b = B(v)$ and $H'_c = D(v')$. Note that since the leptons carry away energy and momentum, in general $v \neq v'$.

What happens to the light degrees of freedom when the heavy quark decays? For $t < 0$, they see the chromoelectric field of a point source with velocity $v$. At $t = 0$, this point source recoils instantaneously to velocity $v'$; the color neutral leptons do not interact with the light hadronic degrees of freedom as they fly off. The light quarks and gluons then must reassemble themselves about the recoiling color source. This nonperturbative process will generally involve the production of an excited state or of additional particles; the light degrees of freedom can exchange energy with the heavy quark, so there is no kinematic restriction on the excitations (of energy $\sim \Lambda_{QCD}$) which can be formed. There is also some chance that the light degrees of freedom will reassemble themselves back into a ground state $D$ meson. The amplitude for this to happen is a function only of the inner product $w = v \cdot v'$ of the initial and final velocities of the color sources. This amplitude, $\xi(w)$, is known as the Isgur-Wise function.

Clearly, the kinematic point $v = v'$, or $w = 1$, is a special one. In this corner of phase space, where the leptons are emitted back to back, there is no recoil of the source of color field at $t = 0$. As far as the light degrees of freedom are concerned, nothing happens! Their state is unaffected by the decay of the heavy quark; they don’t even notice it. Hence the amplitude for them to remain in the ground state is exactly unity. This is reflected in a

\[^b\text{The weak decay occurs over a very short time } \delta t \sim 1/M_W \ll 1/\Lambda_{QCD}.\]
nonperturbative normalization of the Isgur-Wise function at zero recoil,

\[ \xi(1) = 1. \]  

As we will see, this normalization condition is of enormous phenomenological use. It will be extremely important to understand the corrections to this result for finite heavy quark masses \( m_b \) and, especially, \( m_c \).

The weak decay \( b \to c \) is mediated by a left-handed current \( \bar{c}\gamma^\mu(1 - \gamma^5)b \). Not only does this operator carry momentum, but it can change the orientation of the spin \( S_Q \) of the heavy quark during the decay. For a fixed light angular momentum \( J_\ell \), the relative orientation of \( S_Q \) determines whether the physical hadron in the final state is a \( D \) or a \( D^* \). However, the light degrees of freedom are insensitive to \( S_Q \), so the nonperturbative part of the transition is the same whether it is a \( D \) or a \( D^* \) which is produced. Hence heavy quark symmetry implies relations between the hadronic matrix elements which describe the semileptonic decays \( B \to D\ell \bar{\nu} \) and \( B \to D^*\ell \bar{\nu} \).

It is conventional to parameterize these matrix elements by a set of scalar form factors. These are defined separately for the vector and axial currents, as follows:

\[
\begin{align*}
\langle D(v')|\bar{c}\gamma^\mu b|B(v)\rangle &= h_+(w)(v + v')^\mu + h_-(w)(v - v')^\mu \\
\langle D^*(v', \epsilon)|\bar{c}\gamma^\mu b|B(v)\rangle &= h_V(w)\epsilon^\mu \alpha^\beta \epsilon'_\alpha v'_\beta \\
\langle D(v')|\bar{c}\gamma^\mu \gamma^5 b|B(v)\rangle &= 0 \\
\langle D^*(v', \epsilon)|\bar{c}\gamma^\mu \gamma^5 b|B(v)\rangle &= h_{A_1}(w)(w + 1)\epsilon^\mu - \epsilon^* \cdot v[h_{A_2}(w)v^\mu + h_{A_3}(w)v'^\mu]. 
\end{align*}
\]

The set of form factors \( h_i(w) \) is the one appropriate to the heavy quark limit. Other linear combinations are also found in the literature. In any case, the form factors are independent nonperturbative functions of the recoil or equivalently, for fixed \( m_b \) and \( m_c \), of the momentum transfer. However, in the heavy quark limit they correspond to a single transition of the light degrees of freedom, being distinguished from each other only by the relative orientation of the spin of the heavy quark. Hence they may all be written in terms of the single function \( \xi(w) \) which describes this nonperturbative transition. As we will derive later, the result is a set of relations

\[
\begin{align*}
h_+(w) &= h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \\
h_-(w) &= h_{A_2}(w) = 0,
\end{align*}
\]

which follow solely from the heavy quark symmetry. Of course, all of the form factors which do not vanish inherit the normalization condition (21) at zero recoil. This result is a powerful constraint on the structure of semileptonic decay in the heavy quark limit.
1.4 Heavy meson decay constant

As a final example of the utility of the heavy quark limit, consider the coupling of the heavy meson field to the axial vector current. This is conventionally parameterized by a decay constant; for example, for the $B^{-}$ meson we define $f_{B}$ via

$$(0|\bar{u}\gamma^{\mu}\gamma^{5}b|B^{-}(p_{B})) = if_{B}p_{B}^{\mu}.$$  \hspace{1cm} (24)

What is the dependence of the nonperturbative quantity $f_{B}$ on $m_{B}$? To address this question, we rewrite Eq. (24) in a form appropriate to taking the heavy quark limit, $m_{B} \to \infty$ (which is equivalent to $m_{b} \to \infty$). This entails making explicit the dependence of all quantities on $m_{B}$. First, we trade the $B^{-}$ momentum for its velocity,

$$p_{B}^{\mu} = m_{B}v_{\mu}.$$  \hspace{1cm} (25)

Second, we replace the usual $B^{-}$ state, whose normalization depends on $m_{B}$,

$$\langle B(p_{1})|B(p_{2})\rangle = 2E_{B}\delta^{(3)}(\vec{p}_{1} - \vec{p}_{2}),$$  \hspace{1cm} (26)

by a mass-independent state,

$$|B(v)\rangle = \frac{1}{\sqrt{m_{B}}}|B(p_{B})\rangle,$$  \hspace{1cm} (27)

satisfying

$$\langle B(v_{1})|B(v_{2})\rangle = 2\gamma\delta^{(3)}(\vec{p}_{1} - \vec{p}_{2}).$$  \hspace{1cm} (28)

Then Eq. (24) becomes

$$\sqrt{m_{B}}\langle 0|\bar{u}\gamma^{\mu}\gamma^{5}b|B^{-}(v)\rangle = if_{B}m_{B}v^{\mu}.$$  \hspace{1cm} (29)

The nonperturbative matrix element $\langle 0|\bar{u}\gamma^{\mu}\gamma^{5}b|B^{-}(v)\rangle$ is independent of $m_{B}$ in the heavy quark limit. Hence, we see that in this limit $f_{B}$ takes the form

$$f_{B} = m_{B}^{-1/2} \times \text{(independent of } m_{B}).$$  \hspace{1cm} (30)

This makes explicit the scaling of $f_{B}$ with $m_{B}$. It is more interesting to write this as a prediction for the ratio of charmed and bottom meson decay constants. We find

$$\frac{f_{B}}{f_{D}} = \sqrt{\frac{m_{D}}{m_{B}}} + O\left(\frac{\Lambda_{QCD}}{m_{D}}, \frac{\Lambda_{QCD}}{m_{B}}\right).$$  \hspace{1cm} (31)

For the physical bottom and charm masses, of course, the correction terms proportional to $\Lambda_{QCD}/m_{Q}$ could be important.
2 Effective Field Theories

2.1 General Considerations

A central observation which underlies much of the theoretical study of $B$ mesons is that physics at a wide variety of distance (or momentum) scales is typically relevant in a given process. At the same time, the physics at different scales must often be analyzed with different theoretical approaches. Hence it is crucial to have a tool which enables one to identify the physics at a given scale and to separate it out explicitly. Such a tool is the operator product expansion, used in conjunction with the renormalization group. Here a general discussion of its application is given.

Consider the Feynman diagram shown in Fig. 2, in which a $b$ quark decays nonleptonically. The virtual quarks and gauge bosons have virtualities $\mu$ which vary widely, from $\Lambda_{\text{QCD}}$ to $M_W$ and higher. Roughly speaking, these virtualities can be classified into a variety of energy regimes: (i) $\mu \gg M_W$; (ii) $M_W \gg \mu \gg m_b$; (iii) $m_b \gg \mu \gg \Lambda_{\text{QCD}}$; (iv) $\mu \approx \Lambda_{\text{QCD}}$. Each of these momenta corresponds to a different distance scale; by the uncertainty principle, a particle of virtuality $\mu$ can propagate a distance $x \approx 1/\mu$ before being reabsorbed. At a given resolution $\Delta x$, only some of these virtual particles can be distinguished, namely those that propagate a distance $x > \Delta x$. For example, if $\Delta x > 1/M_W$, then the virtual $W$ cannot be seen, and the process whereby it is exchanged would appear as a point interaction. By the same token, as $\Delta x$ increases toward $1/\Lambda_{\text{QCD}}$, fewer and fewer of the virtual gluons can be seen explicitly. Finally, for $\mu \approx \Lambda_{\text{QCD}}$, it is inappropriate to speak of virtual gluons at all, because at such low momentum scales QCD becomes strongly interacting and an expansion in terms of individual gluons is inadequate.

It is useful to organize the computation of a diagram such as is shown in
Fig. 2 in terms of the virtuality of the exchanged particles. This is important both conceptually and practically. First, it is often the case that a distinct set of approximations and approaches is useful at each distance scale, and one would like to be able to apply specific theoretical techniques at the scale at which they are appropriate. Second, Feynman diagrams in which two distinct scales $\mu_1 \gg \mu_2$ appear together can lead to logarithmic corrections of the form $\alpha_s \ln^n(\mu_1/\mu_2)$, which for $\ln(\mu_1/\mu_2) \sim 1/\alpha_s$ can spoil the perturbative expansion. A proper separation of scales will include a resummation of such terms.

2.2 Example I: Weak $b$ Decays

As an example, consider the weak decay of a $b$ quark, $b \to c\bar{u}d$, which is mediated by the decay of a virtual $W$ boson. Viewed with resolution $\Delta x < 1/M_W$, the decay amplitude involves an explicit $W$ propagator and is proportional to

$$\bar{c}\gamma^\mu(1 - \gamma^5)b\bar{d}\gamma_\mu(1 - \gamma^5)u \times \frac{(ig_2)^2}{p^2 - M_W^2},$$

where $p^\mu$ is the momentum of the virtual $W$. Since $m_b \ll M_W$, the kinematics constrains $p^2 \ll M_W^2$, so the virtuality of the $W$ is of order $M_W$, and it travels a distance of order $1/M_W$ before decaying. Viewed with a lower resolution, $\Delta x > 1/M_W$, the process $b \to c\bar{u}d$ appears to be a local interaction, with four fermions interacting via a potential which is a $\delta$ function where the four particles coincide. This can be seen by making a Taylor expansion of the amplitude in powers of $p^2/M_W^2$,

$$\bar{c}\gamma^\mu(1 - \gamma^5)b\bar{d}\gamma_\mu(1 - \gamma^5)u \times \frac{(i\partial)^2}{M_W^2} \left[ 1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots \right].$$

The coefficient of the first term is just the usual Fermi decay constant, $G_F/\sqrt{2}$. The higher order terms correspond to local operators of higher mass dimension. In the sense of a Taylor expansion, the momentum-dependent matrix element (32), which involves the propagation of a $W$ boson between two spacetime points, is identical to the matrix element of the following infinite sum of local operators:

$$\frac{G_F}{\sqrt{2}}\bar{c}\gamma^\mu(1 - \gamma^5)b \left[ 1 + \frac{(i\partial)^2}{M_W^2} + \frac{(i\partial)^4}{M_W^4} + \ldots \right] \bar{d}\gamma_\mu(1 - \gamma^5)u,$$

where the derivatives act on the entire current on the right. This expansion of the nonlocal product of currents in terms of local operators, sometimes known
as an operator product expansion, is valid so long as \( p^2 \ll M_W^2 \). For \( B \) decays, the external kinematics requires \( p^2 \leq m_b^2 \), so this condition is well satisfied. In this regime, one may consider a nonrenormalizable effective field theory, with interactions of dimension six and above. The construction of such a low energy effective theory is also known as matching. As it is nonrenormalizable, the effective theory is defined (by construction) only up to a cutoff, in this case \( M_W \). The cutoff is explicitly the mass of a particle which has been removed from the theory, or integrated out. If one considers processes in which one is restricted kinematically to momenta well below the cutoff, the nonrenormalizability of the theory poses no technical problems. Although the coefficients of operators of dimension greater than six require counterterms in the effective theory (which may be unknown in strongly interacting theories), their matrix elements are suppressed by powers of \( p^2/M_W^2 \). To a given order in \( p^2/M_W^2 \), the theory is well-defined and predictive.

From a modern point of view, in fact, such nonrenormalizable effective theories are actually preferable to renormalizable theories, because the nonrenormalizable terms contain information about the energy scale at which the theory ceases to apply. By contrast, renormalizable theories contain no such explicit clues about their region of validity.

In principle, it is possible to include effects beyond leading order in \( p^2/M_W^2 \) in the effective theory, but in practice, this is usually quite complicated and rarely worth the effort. Almost always, the operator product expansion is truncated at dimension six, leaving only the four-fermion contact term. Corrections to this approximation are of order \( m_b^2/M_W^2 \sim 10^{-3} \).

### 2.3 Radiative Corrections

At tree level, the effective theory is constructed simply by integrating out the \( W \) boson, because this is the only particle in a tree level diagram which is off-shell by order \( M_W^2 \). When radiative corrections are included, gluons and light quarks can also be off-shell by this order. Consider the one-loop diagram shown in Fig.\( \text{[Diagram]}. \) The components of the loop momentum \( k^\mu \) are allowed to take all values in the loop integral. However, the integrand is cut off both in the ultraviolet and in the infrared. For \( k > M_W \), it scales as \( d^4k/k^6 \), which is convergent as \( k \to \infty \). For \( k < m_b \), it scales as \( d^4k/k^3m_bM_W^2 \), which is convergent as \( k \to 0 \). In between, all momenta in the range \( m_b < k < M_W \) contribute to the integral with roughly equivalent weight.

As a consequence, there is potentially a radiative correction proportional to \( \alpha_s \ln(M_W/m_b) \). Even if \( \alpha_s(\mu) \) is evaluated at the high scale \( \mu = M_W \), such a term is not small in the limit \( M_W \to \infty \). At \( n \) loops, there is potentially a
Figure 3: The nonleptonic decay of a $b$ quark at one loop.

term of order $\alpha_s^n \ln^n(M_W/m_b)$. For $\alpha_s \ln(M_W/m_b) \sim 1$, these terms need to be resummed for the perturbation series to be predictive. The technique for performing such a resummation is the renormalization group.

The renormalization group exploits the fact that in the effective theory, operators such as

$$O_I = \bar{c}_i \gamma^\mu (1 - \gamma^5) b^i \bar{d}_j \gamma_\mu (1 - \gamma^5) u^j$$

receive radiative corrections and must be subtracted and renormalized. (Here the color indices $i$ and $j$ are explicit.) In dimensional regularization, this means that they acquire, in general, a dependence on the renormalization scale $\mu$. Because physical predictions are independent of $\mu$, in the renormalized effective theory the operators must be multiplied by coefficients with a compensating dependence on $\mu$. It is also possible for operators to mix under renormalization, so the set of operators induced at tree level may be enlarged once radiative corrections are included. In the present example, a second operator with different color structure,

$$O_{II} = \bar{c}_i \gamma^\mu (1 - \gamma^5) b^i \bar{d}_j \gamma_\mu (1 - \gamma^5) u^j,$$

is induced at one loop. The interaction Hamiltonian of the effective theory is then

$$\mathcal{H}_{\text{eff}} = C_I(\mu)O_I(\mu) + C_{II}(\mu)O_{II}(\mu),$$

and it satisfies the differential equation

$$\mu \frac{d}{d\mu} \mathcal{H}_{\text{eff}} = 0.$$
By computing the dependence on $\mu$ of the operators $O_i(\mu)$, one can deduce the $\mu$-dependence of the Wilson coefficients $C_i(\mu)$. In this case, a simple calculation yields

$$C_{I,II}(\mu) = \frac{1}{2} \left[ \left( \frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{6/23} \pm \left( \frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{-12/23} \right].$$

(39)

For $\mu = m_b$, these expressions resum all large logarithms proportional to $\alpha_s^n \ln^n(M_W/m_b)$.

The decays which are observed involve physical hadrons, not asymptotic quark states. For example, this nonleptonic $b$ decay can be realized in the channels $B \to D\pi$, $B \to D^*\pi\pi$, and so on. The computation of partial decay rates for such processes requires the analysis of hadronic matrix elements such as

$$\langle D\pi | \bar{c}\gamma^\mu (1 - \gamma^5) b \bar{u}\gamma^\nu (1 - \gamma^5) d | \bar{b} \rangle.$$  

(40)

Such matrix elements involve nonperturbative QCD and are extremely difficult to compute from first principles. However, they have no intrinsic dependence on large mass scales such as $M_W$. Because of this, they should naturally be evaluated at a renormalization scale $\mu \ll M_W$, in which case large logarithms $\ln(M_W/m_b)$ will not arise in the matrix elements. By choosing such a low scale in the effective theory (37), all such terms are resummed into the coefficient functions $C_i(m_b)$. As promised, the physics at scales near $M_W$ has been separated from the physics at scales near $m_b$, with the renormalization group used to resum the large logarithms which connect them. In fact, as we will see in the next section, nonperturbative hadronic matrix elements are usually evaluated at an even lower scale $\mu \approx \Lambda_{QCD} \ll m_b$, explicitly resumming all perturbative QCD corrections.

3 Heavy Quark Effective Theory

We have already extracted quite a bit of nontrivial information from the heavy quark limit. We have found the scaling of various quantities with $m_Q$, we have studied the implications for heavy hadron spectroscopy, and we have found nonperturbative relations among the hadronic form factors which describe semileptonic $b \to c$ decay. However, all of these results have been obtained in the strict limit $m_Q \to \infty$. If the heavy quark limit is to be of more than academic interest, and is to provide the basis for quantitative phenomenology, we have to understand how to include corrections systematically. There are actually two types of corrections which we would like to include. Power corrections are subleading terms in the expansion in $\Lambda_{QCD}/m_Q$; those proportional to
\(\Lambda_{QCD}/m_c\) are the most worrisome, because of the relatively small charm quark mass. Logarithmic corrections arise from the implicit dependence of quantities on \(m_Q\) through the strong coupling constant \(\alpha_s(m_Q) \sim 1/\ln(m_Q/\Lambda_{QCD})\). For the physical values of \(m_b\) and \(m_c\), either of these could be important. What we need is a formalism which can accommodate them both.

In short, we need to go from a set of heavy quark symmetry predictions in the \(m_Q \to \infty\) limit, to a reformulation of QCD which provides a controlled expansion about this limit. The formalism which does the job is the Heavy Quark Effective Theory, or the HQET. The purpose of the HQET is to allow us to extract, explicitly and systematically, all dependence of physical quantities on \(m_Q\), in the limit \(m_Q \gg \Lambda_{QCD}\). In these lectures, we will develop only enough of the technology to treat the dominant leading effects, providing indications along the way of how one would carry the expansion further.

The HQET, as formulated here, was developed in a series of papers going back to the late 1980’s, \(4, 5, 7, 9, 10, 11, 12, 13, 14, 15, 16\) which the reader who is interested in tracking its historical development may consult.

### 3.1 The effective Lagrangian

Consider the kinematics of a heavy quark \(Q\), bound in a hadron with light degrees of freedom to make a color singlet state. The small momenta that \(Q\) typically exchanges with the rest of the hadron are of order \(\Lambda_{QCD} \ll m_Q\), and they never take \(Q\) far from its mass shell, \(p_Q^2 = m_Q^2\). Hence the momentum \(p_Q^{\mu}\) can be decomposed into two parts,

\[
p_Q^{\mu} = m_Q v^{\mu} + k^{\mu},
\]

where \(m_Q v^{\mu}\) is the constant on-shell part of \(p_Q^{\mu}\), and \(k^{\mu} \sim \Lambda_{QCD}\) is the small, fluctuating “residual momentum”. The on-shell condition for the heavy quark then becomes

\[
m_Q^2 = (m_Q v^{\mu} + k^{\mu})^2 = m_Q^2 + 2m_Q v \cdot k + k^2.
\]

In the heavy quark limit we may neglect the last term compared to the second, and we have the simple condition

\[
v \cdot k = 0
\]

for an on-shell heavy quark. Here the velocity \(v^{\mu}\) functions as a label; since soft interactions cannot change \(v^{\mu}\), there is a velocity superselection rule in the heavy quark limit, and \(v^{\mu}\) is a good quantum number of the QCD Hamiltonian.
We find the same result by taking the \( m_Q \to \infty \) limit of the heavy quark propagator,
\[
\frac{i}{\not{p} - m_Q + i\epsilon} \to \frac{1 + \not{v}}{2} \frac{i}{v \cdot \not{k} + i\epsilon}.
\] (44)
In this limit the propagator is independent of \( m_Q \). The projection operators
\[
P_{\pm} = \frac{1 \pm \not{v}}{2}
\] (45)
project onto the positive \( (P_+ \) and negative \( (P_- \) frequency parts of the Dirac field \( Q \). This is clear in the Dirac representation in the rest frame, in which \( P_+ \) and \( P_- \) project, respectively, onto the upper two and lower two components of the heavy quark spinor. In the limit \( m_Q \to \infty \), in which \( Q \) remains almost on shell, only the “large” upper components of the field \( Q \) propagate; mixing via zitterbewegung with the “small” lower components is suppressed by \( 1/2m_Q \).

Hence the action of the projectors on \( Q \) is
\[
P_+ Q(x) = Q(x) + O(1/m_Q), \quad P_- Q(x) = 0 + O(1/m_Q).
\] (46)
More precisely, these relations should be understood as pertaining to those modes of the field \( Q(x) \) which annihilate heavy quarks and antiquarks in a heavy meson.

The momentum dependence of the field \( Q \) is given by its action on a heavy quark state,
\[
Q(x) |Q(p)\rangle = e^{-ip \cdot x} |0\rangle.
\] (47)
If we now multiply both sides by a phase corresponding to the on-shell momentum,
\[
e^{im_Qv \cdot x} Q(x) |Q(p)\rangle = e^{-ik \cdot x} |0\rangle,
\] (48)
the right side of this equation is independent of \( m_Q \). Hence the left side must be, as well. Combining this observation with the argument of the previous paragraph, we are motivated to define a \( m_Q \)-independent effective heavy quark field \( h_v(x) \),
\[
h_v(x) = e^{im_Qv \cdot x} P_+ Q(x).
\] (49)
Note that the effective field carries a velocity label \( v \) and is a two-component object. The modifications to the ordinary field \( Q(x) \) project out the positive frequency part and ensure that states annihilated by \( h_v(x) \) have no dependence on \( m_Q \). Hence, these are reasonable candidate fields to carry representations of the heavy quark symmetry. Of course, the small components cannot be
neglected when effects of order $1/m_Q$ are included. In the HQET they are represented by a field

$$H_v(x) = e^{im_Qv \cdot x} P_- Q(x). \quad (50)$$

The field $H_v(x)$ vanishes in the $m_Q \to \infty$ limit.

The ordinary QCD Lagrange density for a field $Q(x)$ is given by

$$\mathcal{L}_{\text{QCD}} = \bar{Q}(x) (i\gamma \cdot D - m_Q) Q(x), \quad (51)$$

where $D_\mu = \partial_\mu - ig A_\mu^a T^a$ is the gauge covariant derivative. To find the Lagrangian of the HQET, we substitute

$$Q(x) = e^{-im_Qv \cdot x} h_v(x) + \ldots \quad (52)$$

into $\mathcal{L}_{\text{QCD}}$ and expand. With the aid of the projection identity $P_+ \gamma^\mu P_+ = v^\mu$, we find

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v(x) i\gamma \cdot D h(x). \quad (53)$$

This simple Lagrangian leads to the propagator we derived earlier,

$$\frac{i}{v \cdot k + ie}, \quad (54)$$

and to an equally simple quark-gluon vertex,

$$ig T^a v^\mu A^a_\mu. \quad (55)$$

Note that both the propagator and the vertex are independent of $m_Q$, reflecting the heavy quark flavor symmetry. They also have no Dirac structure, reflecting the heavy quark spin symmetry. Our intuitive statements about the structure of heavy hadrons have been promoted to explicit symmetries of the QCD Lagrangian in the limit $m_Q \to \infty$.

It is straightforward to include power corrections to $\mathcal{L}_{\text{HQET}}$. Write $Q(x)$ in terms of the effective fields,

$$Q(x) = e^{-im_Qv \cdot x} [h_v(x) + H_v(x)], \quad (56)$$

and apply the classical equation of motion $(i\gamma \cdot D - m_Q) Q(x) = 0$:

$$i\gamma \cdot D h_v(x) + (i\gamma \cdot (2m_Q) H_v(x) = 0. \quad (57)$$

Multiplying by $P_-$ and commuting $\gamma \cdot \partial$ to the right, we find

$$(iv \cdot D + 2m_Q) H_v(x) = i\gamma_\mu h_v(x), \quad (58)$$
where $D_\mu = D - v^\mu v \cdot D$. We then substitute $Q(x)$ into $\mathcal{L}_{\text{QCD}}$ as before, eliminate $H_v(x)$ and expand in $1/m_Q$ to obtain

$$
\mathcal{L}_{\text{HQET}} = \bar{h}_v i v \cdot D h_v + \frac{1}{i v \cdot D + 2m_Q} i \tilde{D}_\perp h_v
$$

$$
= \bar{h}_v i v \cdot D h_v + \frac{1}{2m_Q} \left[ h_v (i D_\perp)^2 h_v + \frac{g}{2} h_v \sigma^{\alpha\beta} G_{\alpha\beta} h_v \right] + \ldots. \quad (59)
$$

The leading corrections have a simple interpretation, which becomes clear in the rest frame, $v^\mu = (1, 0, 0, 0)$. The spin-independent term is

$$
\frac{1}{2m_Q} \mathcal{O}_K = \frac{1}{2m_Q} \bar{h}_v (i D_\perp)^2 h_v \rightarrow -\frac{1}{2m_Q} \bar{h}_v (i \tilde{D})^2 h_v, \quad (60)
$$

which is the negative of the nonrelativistic kinetic energy of the heavy quark. Because of the explicit factor of $1/2m_Q$, this term violates the heavy flavor symmetry. The spin-dependent part is

$$
\frac{1}{2m_Q} \mathcal{O}_G = \frac{1}{2m_Q} \frac{g}{2} h_v \sigma^{\alpha\beta} G_{\alpha\beta} h_v \rightarrow \frac{1}{4m_Q} \bar{h}_v \sigma^{ij} T^a h_v \times g G_{ij}^a = g \bar{Q}^a \cdot \tilde{B}^a, \quad (61)
$$

which is the coupling of the spin of the heavy quark to the chromomagnetic field. Because it has a nontrivial Dirac structure, this term violates both the heavy flavor symmetry and the heavy spin symmetry. For example, $\mathcal{O}_G$ is responsible for the $D - D^*$ and $B - B^*$ mass splittings. These correction terms will be treated as part of the interaction Lagrangian, even though $\mathcal{O}_K$ has a piece which is a pure bilinear in the heavy quark field.

### 3.2 Effective currents and states

The expansion of the weak interaction current $\bar{c} \gamma^\mu (1 - \gamma^5) b$ is analogous. However, here we must introduce separate effective fields for the charm and bottom quarks, each with its own velocity:

$$
b \rightarrow h_v^b, \quad c \rightarrow h_v^c. \quad (62)
$$

Then a general flavor-changing current becomes, to leading order,

$$
\bar{c} \Gamma b \rightarrow \bar{h}_v^c \Gamma h_v^b, \quad (63)
$$

where $\Gamma$ is a fixed Dirac structure. With the leading power corrections, this is

$$
\bar{c} \Gamma b \rightarrow \bar{h}_v^c \Gamma h_v^b + \frac{1}{2m_b} \bar{h}_v^c \Gamma (i \tilde{D}_\perp) h_v^b + \frac{1}{2m_c} \bar{h}_v^c (-i \tilde{D}_\perp) \Gamma h_v^b + \ldots. \quad (64)
$$
The effective currents, and other operators which appear in the HQET, may often be simplified by use of the classical equation of motion,

\[ iv \cdot D h_v (x) = 0. \]  

However, it is only safe to apply these equations naively at order $1/m_Q$; at higher order the application of the equations of motion involves additional subtleties.\cite{17,18,19}

To complete the effective theory, we need $m_Q$-independent hadron states which are created and annihilated by currents containing the effective fields. For example, there is an effective pseudoscalar meson state $|M(v)\rangle$ which couples to the effective axial current $\bar{q} \gamma^\mu \gamma^5 h_v$, with a coupling $F_M$ which is independent of $m_Q$,

\[ \langle 0 | \bar{q} \gamma^\mu \gamma^5 h_v | M(v) \rangle = i F_M v^\mu . \]  

At lowest order, $F_M$ is related to a conventional decay constant such as $f_B$ by

\[ f_B = F_M / \sqrt{m_B} , \]  

from which we immediately find the relationship \cite{11} between $f_D$ and $f_B$.

3.3 Radiative corrections

We can use the effective Lagrangian $\mathcal{L}_{\text{HQET}}$ to compute the radiative corrections to the matrix element \cite{66}. In particular, we would like to extract the dependence of $F_M$ on $\ln m_Q$. This dependence comes through the one-loop renormalization of the current $\bar{q} \gamma^\mu \gamma^5 Q$. At lowest order, of course, the renormalization is straightforward: we simply compute the set of graphs found in Fig. 4. The result is finite, because the current is (partially) conserved, and we extract a result of the form

\[ \bar{q} \gamma^\mu \gamma^5 Q \left( 1 + \frac{\alpha_s}{4\pi} \ln(m_Q/m_\mu) + \ldots \right) . \]  

21
Note that there is no explicit dependence on the renormalization scale \( \mu \), since there is no divergence to be subtracted.

The same result may be obtained in the effective theory. In this case we must match the currents in full QCD onto HQET currents of the form \( \bar{q} \gamma^\mu \gamma^5 h_v \). This step will induce a matching coefficient containing the explicit dependence on \( m_Q \), which is absent, by construction, from the operators and Lagrangian of the HQET. In addition, the effective current will not necessarily be conserved, since the ultraviolet properties of QCD and the HQET differ. Hence the form of the matching, once radiative corrections are included, is

\[
\bar{q} \gamma^\mu \gamma^5 Q \to C(m_Q, \mu) \times \bar{q} \gamma^\mu \gamma^5 h_v(\mu) .
\]  
(69)

We can deduce the form of \( C(m_Q, \mu) \) by considering the renormalization of the effective current \( \bar{q} \gamma^\mu \gamma^5 h_v \), shown in the last three terms of Fig. 5. These diagrams, computed in the effective theory, are independent of \( m_Q \). However, in general they are divergent, so they depend on the renormalization scale \( \mu \); the renormalization takes the form

\[
\bar{q} \gamma^\mu \gamma^5 h_v(\mu) = \bar{q} \gamma^\mu \gamma^5 h_v(m_q) \times \left( 1 + \gamma_0 \frac{\alpha_s}{4\pi} \ln(m_q/\mu) + \ldots \right) ,
\]  
(70)

where here \( m_q \) acts as an infrared cutoff. The \( \mu \) dependence in the second term must be canceled by \( C(m_Q, \mu) \). Since the logarithm depends on a dimensionless ratio, \( C(m_Q, \mu) \) must be of the form

\[
C(m_Q, \mu) = 1 + \gamma_0 \frac{\alpha_s}{4\pi} \ln(m_Q/\mu) + \ldots
\]  
(71)

Comparing the dependence on \( m_Q \) of \( C(m_Q, \mu) \) and the expansion (68), we see that \( \gamma_0 = \gamma_0 \).

\[\text{In general, the matching procedure at order } \alpha_s \text{ can also induce new Dirac structures } \bar{q} \Gamma h_v. \text{ They do not affect the leading logarithms discussed here.}\]
However, the effective theory allows us to go beyond leading order, and to resum all corrections of the form $\alpha_s^n \ln^m mQ$. We do this with the renormalization group equations, which express the independence of physical observables on the renormalization scale $\mu$. In this case, they require that the $\mu$ dependence of $C(mQ, \mu)$ cancel that of the one loop diagrams in Fig. 5, under small changes in $\mu$:

$$
\mu \frac{d}{d\mu} C(mQ, \mu) = -\gamma_0 \frac{\alpha_s(\mu)}{4\pi}.
$$  (72)

The logarithms are resummed because the partial derivative is promoted to a total derivative with respect to $\mu$, including the implicit dependence on $\mu$ of the coupling constant $\alpha_s(\mu)$:

$$
\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}
$$

$$
\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} + \ldots
$$

$$
\beta_0 = 11 - \frac{2}{3} N_f = \frac{25}{3} \text{ for } N_f = 4,
$$  (73)

where $N_f$ is the number of light flavors. We compute the anomalous dimension $\gamma_0$ from the ultraviolet divergent parts of the one loop diagrams shown in Fig. 5.

It is instructive to perform the radiative correction to the current in detail, since this is different from the diagrams one is used to in ordinary QCD. With the HQET Feynman rules, the diagram may be written in Feynman gauge as

$$
C_f (ig)^2 \mu^\epsilon \int \frac{d^4-q}{(2\pi)^{4-\epsilon}} \bar{v} \gamma^\mu \gamma^5 u_h \frac{i}{q} \gamma^\nu \gamma^5 v \cdot q \frac{i}{v \cdot q} u_h \times \frac{-i}{q^2},
$$  (74)

where $C_f = \frac{4}{3}$ is the color factor. This expression may be simplified to

$$
- C_f (ig)^2 \bar{v} \gamma^\mu \gamma^5 u_h \mu^\epsilon \int \frac{d^4-q}{(2\pi)^{4-\epsilon}} \frac{q^\beta}{q^2 v \cdot q},
$$  (75)

which by Lorentz invariance is simply

$$
- C_f (ig)^2 \bar{v} \gamma^\mu \gamma^5 u_h \mu^\epsilon \int \frac{d^4-q}{(2\pi)^{4-\epsilon}} \frac{1}{q^2}.
$$  (76)

Rotating to Euclidean space, performing the integral and extracting the pole in $\epsilon$, we find

$$
\bar{v} \gamma^\mu \gamma^5 u_h \times (2C_f) \frac{g^2 \mu^\epsilon}{16\pi^2\epsilon} + \text{finite}.
$$  (77)
Since $\mu' = 1 + \epsilon \ln \mu + \ldots$, the one loop contribution to the matrix element then depends on $\ln \mu$ as

$$\mathcal{M}^\mu \gamma^5 u_h \times (2C_f) \frac{\alpha_s}{4\pi} \ln \mu.$$  \hspace{1cm} (78)

The contribution to the anomalous dimension is then $2C_f = \frac{8}{3}$.

The calculation of the wavefunction renormalization of the heavy quark (the third diagram in Fig. 5) is similar, but requires a novel version of the Feynman trick,

$$\frac{1}{ab} = \int_0^\infty \frac{d\lambda}{(a + \lambda b)^2}.$$  \hspace{1cm} (79)

One also has to pick out the term which cancels the $1/v \cdot k$ pole in the heavy quark propagator. With the factor of $\frac{1}{2}$ which accompanies the contributions from wavefunction renormalization, the result is

$$\frac{1}{2} \times \mathcal{M}^\mu \gamma^5 u_h \times (4C_f) \frac{\alpha_s}{4\pi} \ln \mu.$$  \hspace{1cm} (80)

Including the usual QCD renormalization of the light quark field, we find from the three terms in Fig. 5, respectively,

$$\gamma_0 = \frac{8}{3} + \frac{1}{2} \left( -\frac{8}{3} \right) + \frac{1}{2} \left( \frac{16}{3} \right) = 4.$$  \hspace{1cm} (81)

The solution of the renormalization group equation is

$$C(m_Q, \mu) = \left( \frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{-\gamma_0/2\beta_0} = \left( \frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{-6/25}.$$  \hspace{1cm} (82)

This then yields the leading logarithmic correction to the ratio $f_B/f_D$:

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}} \left( \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right)^{6/25}.$$  \hspace{1cm} (83)

The radiative correction is approximately a ten percent effect. In fact, it has a simple physical interpretation. For virtual gluons of “intermediate” energy, $m_c < E_g < m_b$, the bottom quark is heavy but the charm quark is light. Such gluons contribute to the difference between $f_B$ and $f_D$ even in the heavy quark limit.

In summary, then, the purpose of the HQET is to make explicit all dependence of observable quantities on $m_Q$. The logarithmic dependence, through $\alpha_s(m_Q)$, arises from intermediate virtual gluons with $m_c < E_g < m_b$. We
obtain these corrections by computing perturbatively with the HQET Lagrangian, then using the renormalization group to resum the logarithms to all orders. The power dependence, $1/m_Q$, is extracted systematically in the heavy quark expansion. We have seen how to expand the Lagrangian and the states to subleading order; the application of the expansion to a physical decay rate will be presented in the next section.

These lectures are meant to be pedagogical, so we will only treat the leading corrections to a few processes. However, the state of the art goes significantly beyond what will be presented here. For many quantities, not only the leading logarithms, $\alpha_s \ln m_Q$, but the subleading (two loop) logarithms, of order $\alpha_s^{n+1} \ln m_Q$, have been resummed. Similarly, many power corrections are known to relative order $1/m_Q^2$. It is particularly important phenomenologically to take into account the corrections of order $1/m_Q^2$.

4 Exclusive $B$ Decays

We now have the tools we need for an HQET treatment of the exclusive semileptonic transitions $B \to D \ell \bar{\nu}$ and $B \to D^* \ell \bar{\nu}$. Earlier, we argued on physical grounds that in the heavy quark limit all of the hadronic matrix elements which appear in these decays are related to a single nonperturbative function $\xi(w)$. Now we will sharpen this analysis to actually derive these relations, and to include radiative and power corrections. In fact, almost all of our effort will go into the power corrections, since the radiative corrections to the transition currents are computed just as in the previous section.

4.1 Matrix element relations at leading order

The transitions in question require the nonperturbative matrix elements

\begin{align*}
\langle D(v') | \bar{c}\gamma^\mu b | B(v) \rangle, & \quad \langle D^*(v', \epsilon) | \bar{c}\gamma^\mu b | B(v) \rangle, & \quad \langle D^*(v', \epsilon) | \bar{c}\gamma^\mu \gamma^5 b | B(v) \rangle, \quad (84)
\end{align*}

parameterized in terms of form factors as in Eq. (22). Our first task is to derive the relations between these form factors, as promised earlier. These relations depend on the heavy quark symmetry, that is, on the fact that the spin quantum numbers of $Q$ and of the light degrees of freedom are separately conserved by the soft physics. Hence we need a representation of the heavy meson states in which they have well defined transformations separately under the angular momentum operators $S_Q$ and $J_\ell$. In particular, the representation must reflect the fact that a rotation by $S_Q$ can exchange the pseudoscalar meson $M(v)$ with the vector meson $M^*(v, \epsilon)$. 

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The solution is to introduce a “superfield” \( \mathcal{M}(v) \), defined as the 4 × 4 Dirac matrix,

\[
\mathcal{M}(v) = \frac{1 + \slashed{v}}{2} \left[ \gamma^\mu M^\mu_\nu(v, \epsilon) - \gamma^5 M(v) \right] \equiv V(v, \epsilon) + P(v). \tag{85}
\]

Under heavy quark spin rotations \( S_Q \), \( \mathcal{M}(v) \) transforms as

\[
\mathcal{M}(v) \to D(S_Q) \mathcal{M}(v), \tag{86}
\]

and under Lorentz rotations \( \Lambda \), as

\[
\mathcal{M}(v) \to D(\Lambda) \mathcal{M}(\Lambda^{-1} v) D^{-1}(\Lambda). \tag{87}
\]

Here \( D(\cdots) \) is the spinor representation of \( SO(3,1) \). The superfield satisfies the matrix identity

\[
P_+ \mathcal{M}(v) P_- = \mathcal{M}(v), \tag{88}
\]

so it transforms the same way as the product of spinors \( h_v \bar{q} \), representing a heavy quark and a light antiquark moving together at velocity \( v^\mu \).

It is straightforward to verify the transformation properties of the superfield in the rest frame, in which \( v^\mu = (1, 0, 0, 0) \). In this frame, the spinor representation of the angular momentum operator \( \mathbf{J} \) has components \( S^i = \frac{1}{2} \gamma^5 \gamma^0 \gamma^i \). It acts on the superfield by \( \mathbf{J}^i \mathcal{M} = [S^i, \mathcal{M}] \). Defining the polarization vectors \( \epsilon^\pm_\mu = (0, 1/\sqrt{2}, \pm i/\sqrt{2}, 0) \) and \( \epsilon^3_\mu = (0, 0, 0, 1) \), it is easy to check that

\[
\mathbf{J}^2 P = J^3 P = 0, \quad \mathbf{J}^2 V(\epsilon) = 2V(\epsilon), \quad \mathbf{J}^3 V(\epsilon_\pm) = \pm V(\epsilon_\pm), \quad \mathbf{J}^3 V(\epsilon) = 0, \tag{89}
\]

so \( P \) has spin zero and \( V(\epsilon) \) spin one. On the other hand the heavy quark spin operator \( S_Q \) has the same component representation but acts only on the left, \( (S_Q)^i \mathcal{M} = S^i \mathcal{M} \). One may then check that

\[
(S_Q)^3 P = \frac{1}{2} V(\epsilon_3), \quad (S_Q)^3 V(\epsilon_3) = \frac{1}{2} P. \tag{90}
\]

As promised, heavy quark spin transformations exchange the pseudoscalar and vector mesons.

A current which mediates the decay of one heavy quark \( Q \) into another \( Q' \) is of the form \( \bar{h}_v \Gamma h_v \). Under a rotation by \( S_Q \), the effective field \( h_v \) transforms as

\[
h_v \to D(S_Q) h_v, \tag{91}
\]

while \( h_v \) is unchanged. The current would remain invariant if we took \( \Gamma \) to transform as

\[
\Gamma \to \Gamma D^{-1}(S_Q). \tag{92}
\]
we find that the matrix element is restricted to the general form

\[ M(Q') \Gamma h \mathcal{M}(v) \]

On the other hand, the matrix element of superfields

\[ \mathcal{M}'(v') \tilde{h}(v') \Gamma h \mathcal{M}(v) \]

is invariant if we rotate both \( h \) and \( \mathcal{M}(v) \) by the same \( S_Q \). With the transformation law \( \mathcal{M}(v) \) for \( \mathcal{M}(v) \), it follows that the \( S_Q \)-invariant matrix element must be proportional to \( \Gamma \mathcal{M}(v) \). When we also consider rotations under \( S_Q' \), we find that the matrix element is restricted to the general form

\[ \langle \mathcal{M}'(v') | \tilde{h}(v') \Gamma h | \mathcal{M}(v) \rangle = -\sqrt{M_M M_{M'}} \text{Tr} \left[ \mathcal{M}(v') \Gamma \mathcal{M}(v) \right] . \]

The product of masses in front is a convention which restores the relativistic normalization of the states. Note that the heavy quark symmetry allows an arbitrary 4 \( \times \) 4 Dirac matrix \( \tilde{F}(v, v') \) to act on the “light quark” part of the superfields. Its presence reflects the fact that only Lorentz symmetry constrains the spin of the light degrees of freedom during the decay.

A general expansion of \( \tilde{F}(v, v') \) in terms of scalar functions \( F_i(w = v \cdot v') \) takes the form

\[ \tilde{F}(v, v') = F_1(w) + F_2(w)\Phi + F_3(w)\Phi' + F_4(w)\Phi' . \]

However, the identity \( \text{(88)} \) applied to the matrix element \( \text{(94)} \) yields

\[ \tilde{F}(v, v') = P_- \tilde{F}(v, v') P'_- = [F_1(w) - F_2(w) - F_3(w) + F_4(w)] P_- P'_- . \]

In other words, \( \tilde{F}(v, v') \) actually may be taken to be a scalar, which we identify with the Isgur-Wise function,

\[ \tilde{F}(v, v') = \xi(w) . \]

As an exercise, let us apply this formalism to the matrix elements for \( B \to (D, D^*) \ell\nu \). For a given matrix element, we pick out the part of the superfield \( \mathcal{M}(v) \) which is relevant. Hence we find

\[ \langle D(v') | \bar{c} \gamma^\mu b | B(v) \rangle = \langle M_D(v') | \bar{c}_\mu \gamma^\mu h | M_B(v) \rangle \]

\[ = -\sqrt{m_D m_B} \text{Tr} \left[ \gamma^5 P_+ \gamma^\mu P_+ (-\gamma^5) \right] \xi(w) \]

\[ = \sqrt{m_D m_B} \xi(w) (v + v')^\mu \]

\[ \langle D^*(v', \epsilon) | \bar{c} \gamma^\mu \gamma^5 b | B(v) \rangle = \langle M_D^*(v', \epsilon) | \bar{c}_\mu \gamma^\mu \gamma^5 h | M_B(v) \rangle \]

\[ = -\sqrt{m_D^* m_B} \text{Tr} \left[ \gamma^5 P_+ \gamma^\mu P_+ (-\gamma^5) \right] \xi(w) \]

\[ = \sqrt{m_D^* m_B} \xi(w) [(w + 1) \epsilon^\mu - \epsilon^\ast \cdot (v + v')^\mu] \]
\[
(D^* (v', \epsilon) | \bar{c} \gamma^\mu b | B(v)) = \langle M_D^* (v', \epsilon) | \bar{h}_c^\alpha \gamma^\mu h_v^b | M_B (v) \rangle \\
= - \sqrt{m_D \cdot m_B} \mathrm{Tr} \left[ P_+^* \gamma^\mu P_+ (-\gamma^5) \right] \xi (w) \\
= \sqrt{m_D \cdot m_B} \xi (w) i \varepsilon^{\mu \nu \alpha \beta} \xi_\alpha v_\beta v_\beta ,
\]

reproducing explicitly the relations (23) between the independent form factors \(h_j (w)\). We can also derive the normalization condition at \(w = 1\). Consider the matrix element of the \(b\) number current \(\bar{b} \gamma^\mu b\) between \(B\) meson states. In QCD, the matrix element of this current is exactly normalized,

\[
\langle B(v) | \bar{b} \gamma^\mu b | B(v) \rangle = 2 \mu_B^a = 2 m_B v^\mu ,
\]

But in HQET, we have

\[
\langle B(v) | \bar{b} \gamma^\mu b | B(v) \rangle = \langle M_B (v) | \bar{h}_c^\alpha \gamma^\mu h_v^b | M_B (v) \rangle \\
= m_B \xi (v \cdot v) (v + v)^\mu \\
= 2 m_B v^\mu \xi (1) .
\]

Hence the normalization condition at zero recoil,

\[
\xi (1) = 1 ,
\]

follows directly from the conservation of the heavy quark number current.

### 4.2 Power corrections to the matrix elements

The matrix elements we have derived are computed in the strict limit \(m_{b,c} \to \infty\). How are they affected by corrections of order \(1/m_b\) and \(1/m_c\)? There are two sources of \(1/m_Q\) corrections in the effective theory: the corrections (64) to the heavy quark currents, and the corrections (59) to the Lagrangian.

When radiative corrections are included, the expansion of the heavy quark current \(\bar{c} \Gamma b\) in terms of HQET operators has a form which is somewhat more general than Eq. (64),

\[
\bar{c} \Gamma b \to a_0 (\alpha_s) \bar{h}_c^\epsilon \Gamma h_v^b + \frac{a_1 (\alpha_s)}{2 m_b} \bar{h}_c^\epsilon \Gamma^\alpha_i D_\alpha h_v^b + \frac{a'_1 (\alpha_s)}{2 m_c} \bar{h}_c^\epsilon (-i \overleftrightarrow{D}_\alpha) \Gamma^\alpha_i h_v^b + \ldots .
\]

The matrix elements of the power corrections are constrained by heavy quark symmetry in a manner completely analogous to the leading current. In terms of traces over the superfields, we have

\[
\langle M'(v') | \bar{h}_c^\epsilon \Gamma^\alpha_i D_\alpha h_v^b | M(v) \rangle = - \sqrt{M_M M_{M'}} \mathrm{Tr} \left[ \overleftrightarrow{\gamma} (v') \Gamma^\alpha_i M(v) \tilde{G}_\alpha (v, v') \right] ,
\]
where $\hat{G}_\alpha(v, v')$ is another arbitrary $4 \times 4$ Dirac matrix. The matrix element

$$\langle M'(v') | \bar{h}_{v'} (-i\slashed{D}_\alpha) \Gamma^\alpha h_v | M(v) \rangle$$

may also be written in terms of $\hat{G}_\alpha(v, v')$, using charge conjugation.

The $1/m_Q$ corrections $O_K$ and $O_G$ to the Lagrangian contribute somewhat differently. In order to apply heavy quark symmetry, the matrix elements of the local currents, both leading and subleading, must be written in terms of the effective states $|M(v)\rangle$. However, these states are not eigenstates of the Hamiltonian, once $O_K$ and $O_G$ are included in the Lagrangian. Hence, for example, we must allow for the possibility that if an effective state $|M(v)\rangle$ is created at time $t = -\infty$, then $O_K$ or $O_G$ could act on the state before its decay at $t = 0$. This possibility is accounted for by including time-ordered products in which $O_K$ or $O_G$ is inserted along the incoming or outgoing heavy quark line. If we are keeping terms of order $1/m_Q$, only one insertion of $O_K$ or $O_G$ needs to be included. The time-ordered products are of the form

$$\langle D^{(s)}(v) | \bar{\epsilon} \slashed{t} h | B(v) \rangle = \ldots + \frac{1}{2m_c} \langle M'(v') | i \int dy T \left\{ \bar{h}_c^v \Gamma \hat{h}_c^b, O_K^{v'} + O_G^{v'} \right\} | M(v) \rangle$$

$$+ \frac{1}{2m_b} \langle M'(v') | i \int dy T \left\{ \bar{h}_b^v \Gamma \hat{h}_b^c, O_K^{v} + O_G^{v} \right\} | M(v) \rangle,$$

where the ellipses include the current corrections computed earlier. The evaluation of the matrix elements of the time-ordered products will lead to still more nonperturbative functions like $\hat{F}(v, v')$ and $\hat{G}_\alpha(v, v')$.

### 4.3 Corrections at zero recoil

It is straightforward, but not very illuminating, to expand all of the new nonperturbative functions which arise at order $1/m_Q$ in terms of scalar form factors. In the end, the corrections may be parameterized in terms of four functions of the velocity transfer $w$, and a single nonperturbative parameter $\bar{\Lambda}$, all proportional to the mass scale $\Lambda_{QCD}$. The new parameter has a simple interpretation as the “energy” of the light degrees of freedom, and is given by

$$\bar{\Lambda} = \lim_{m_b \to \infty} (m_B - m_b).$$

Instead of a general treatment, however, we will consider the $1/m_Q$ corrections at the zero recoil point $w = 1$. This is clearly the most important case, because it is at this point that the nonperturbative matrix elements are
absolutely normalized in the heavy quark limit. What happens to this normalization condition when $1/m_Q$ corrections are included?

Let us study the corrections to the current in detail. They are described by the nonperturbative function $\hat{G}_\alpha(v, v')$. At $v = v'$, $\hat{G}_\alpha(v, v)$ may be expanded as

$$\hat{G}_\alpha(v, v) = G_1 v_\alpha + G_2 \gamma_\alpha + G_3 v_\alpha \not{\!P} + G_4 \gamma_\alpha \not{\!P}.$$  \hspace{1cm} (109)

But $\hat{G}_\alpha(v, v)$ is subject to the same constraint as $\hat{F}(v, v')$,

$$\hat{G}_\alpha(v, v) = P^- \hat{G}_\alpha(v, v') P^- = (G_1 - G_2 - G_3 + G_4) v_\alpha P^- \equiv G v_\alpha P^-,$$  \hspace{1cm} (110)

and it, too, is equivalent to a Dirac scalar (the same is not true of the general function $\hat{G}_\alpha(v, v')$). Now consider the matrix element where we take $\Gamma^\alpha = v^\alpha$.

Then we have

$$\langle M'(v) | \bar{h} v \cdot D h_v | M(v) \rangle = -\sqrt{M'_M M_M} \text{Tr} \left[ \overline{M}(v) v^\alpha M(v) \right] G v_\alpha = -G \sqrt{M'_M M_M} \text{Tr} \left[ \overline{M}(v) M(v) \right].$$  \hspace{1cm} (111)

But this matrix element vanishes by the classical equation of motion in the effective theory,

$$v \cdot D h_v(x) = 0.$$  \hspace{1cm} (112)

Hence $G = 0 = \hat{G}_\alpha(v, v)$. There are no $1/m_Q$ corrections from the current to the normalization condition at zero recoil.

The same is true of insertions of the corrections $\mathcal{O}_K$ and $\mathcal{O}_G$ to the Lagrangian: their contribution vanishes at $w = 1$. To show this requires the imposition of the conservation of the $b$ number current at order $1/m_b$, much as we derived the normalization of the Isgur-Wise function at leading order. This part of the argument is analogous to the classic nonrenormalization theorem of Ademollo and Gatto. \hspace{1cm} [22]

In the end, we have the result known as Luke’s Theorem. \hspace{1cm} [14] There are no $1/m_c$ corrections from the current to the normalization condition at zero recoil.

The same is true of insertions of the corrections $\mathcal{O}_K$ and $\mathcal{O}_G$ to the Lagrangian: their contribution vanishes at $w = 1$. To show this requires the imposition of the conservation of the $b$ number current at order $1/m_b$, much as we derived the normalization of the Isgur-Wise function at leading order. This part of the argument is analogous to the classic nonrenormalization theorem of Ademollo and Gatto. \hspace{1cm} [22]

In the end, we have the result known as Luke’s Theorem. \hspace{1cm} [14] There are no corrections at zero recoil to the hadronic matrix elements responsible for the semileptonic decays $B \rightarrow D \ell \bar{\nu}$ and $B \rightarrow D^* \ell \bar{\nu}$. The leading power corrections to the normalization of zero recoil matrix elements are only of order $1/m_b^2$. Given that $\Lambda_{QCD}/m_c \sim 30\%$ and $\Lambda_{QCD}^2/m_c^2 \sim 10\%$, the implication is that the leading order predictions at $w = 1$ are considerably more accurate than one might have expected. In addition, away from zero recoil the $1/m_c$ corrections must be suppressed at least by $(w - 1)$.

On closer inspection, this result is more interesting for $B \rightarrow D^* \ell \bar{\nu}$ than for $B \rightarrow D \ell \bar{\nu}$. This is because the leading order matrix element for $B \rightarrow D \ell \bar{\nu}$ vanishes kinematically at zero recoil for a massless lepton in the final state. Hence, in this case the $1/m_c$ corrections are not suppressed as a fractional correction to the lowest order term. \hspace{1cm} [23]
4.4 Extraction of $|V_{cb}|$ from $B \to D^* \ell \bar{\nu}$

An immediate application of these results is the extraction of $|V_{cb}|$ from the exclusive decay $B \to D^* \ell \bar{\nu}$. This process is mediated by the weak operator $O_{bc}$, whose matrix element factorizes as

$$\langle D^* \ell \bar{\nu} | O_{bc} | B \rangle = \frac{G_F V_{cb}}{\sqrt{2}} \langle D^* | \bar{c} \gamma^\mu (1 - \gamma^5) b | B \rangle \langle \ell \bar{\nu} | \bar{\ell} \gamma^\mu (1 - \gamma^5) \nu | 0 \rangle .$$  \hspace{1cm} (113)

The leptonic matrix element may be computed perturbatively, while we treat the hadronic matrix element in the heavy quark expansion. The result is a differential decay rate of the form

$$\frac{d\Gamma}{dw} = \frac{G_F^2}{48\pi^3} |V_{cb}|^2 (m_B - m_{D^*})^2 m_{D^*}^3 (w + 1)^3 \sqrt{w^2 - 1} \left[ 1 + \frac{4w}{w + 1} \frac{m_B^2 - 2w m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2} \right] F^2(w).$$  \hspace{1cm} (114)

All of the HQET analysis goes into the factor $F(w)$, which has an expansion

$$F(w) = \xi(w) + (\text{radiative corrections}) + (\text{power corrections}).$$  \hspace{1cm} (115)

We extract $|V_{cb}|$ by studying the differential decay rate near $w = 1$, where the hadronic matrix elements are known. Of course, this requires extrapolation of the experimental data, since the rate vanishes kinematically at $w = 1$. For massless leptons, only the matrix element $\langle D^* | \bar{c} \gamma^\mu \gamma^5 b | B \rangle$ of the axial current contributes at this point. The analysis of this quantity in the HQET yields an expansion of the form

$$F(1) = \eta_A \left[ 1 + \frac{0}{m_c} + \frac{0}{m_b} + \delta_{1/m^2} + \ldots \right].$$  \hspace{1cm} (116)

The correction $\delta_{1/m^2}$, which contains terms proportional to $1/m_c^2$, $1/m_b^2$, and $1/m_c m_{b}$, is intrinsically nonperturbative. It has been estimated from a variety of models to be small and negative.

$$\delta_{1/m^2} \approx -0.055 \pm 0.035.$$  \hspace{1cm} (117)

Note that the model dependence in the result has been relegated to the estimation of the sub-subleading terms. The radiative correction $\eta_A$ has now been computed to two loops.

$$\eta_A = 0.960 \pm 0.007.$$  \hspace{1cm} (118)
The result is a value for $F(1)$ with errors at the level of 5%,

$$F(1) = 0.91 \pm 0.04.$$  \hspace{1cm} (119)

This is the theory error which the experimental determination of $|V_{cb}|$ will inherit. It is dominated by the uncertainty in the nonperturbative corrections, and it is difficult to see how this can be improved much in the future.

All that is left experimentally is to extrapolate the data to $w = 1$ and extract

$$\lim_{w\to 1} \frac{1}{\sqrt{w-1}} \frac{d\Gamma}{dw}.$$ \hspace{1cm} (120)

Once the kinematic factors in Eq. (114) have been included, this amounts to a direct measurement of the combination $|V_{cb}|F(1)$. Both CLEO and LEP have reported results for this quantity. \cite{19,20} They have taken the slightly different value $F(1) = 0.88 \pm 0.05$, and I have scaled up the theory error of the CLEO result to make it consistent with LEP. Then we have quite consistent results for $V_{cb}$:

- CLEO: $(39.4 \pm 2.1 \pm 2.0 \pm 2.2) \times 10^{-3}$
- LEP average: $(38.4 \pm 1.1 \pm 2.2 \pm 2.2) \times 10^{-3}$. \hspace{1cm} (121)

This value of $|V_{cb}|$ has almost no dependence on hadronic models. In contrast to model-based “measurements”, here the theoretical error is meaningful, in that it is based on a systematic expansion in small quantities.

5 \textbf{Inclusive $B$ Decays}

An exclusive semileptonic $B$ decay, such as $B \to D \ell \bar{\nu}$, is one in which the final hadronic state is fully reconstructed. An inclusive decay, by contrast, is one in which only certain kinematic features, and perhaps the flavor, of the hadron are known. In this case, we need a theoretical analysis in which we sum over all possible hadronic final states allowed by the kinematics. Fortunately, this is possible within the structure of the HQET.

As in the case of exclusive decays, the key theme is the separation of short distance physics, associated with the heavy quark, from long distance physics, associated with the light degrees of freedom. We will also rely on heavy quark spin and flavor symmetry. However, the new ingredient will be the idea of “parton-hadron duality”, which, as we will see, also relies on the heavy quark limit $m_b \gg \Lambda_{QCD}$. 

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5.1 The inclusive decay $B \to X_c \ell\bar{\nu}$

Let us consider the inclusive decay

$$B(p_B) \to X_c(p_X) \ell(p_\ell)\bar{\nu}(p_\nu),$$

(122)

where all that is known about the state $X_c$ are its energy and momentum, and the fact that it contains a charm quark. This decay is mediated by the weak operator $O_{bc}$. It is easy to generalize our discussion to inclusive decays of other heavy quarks, such as $b \to u \ell\nu$ and $c \to s \bar{\ell}\nu$, by replacing $O_{bc}$ with the appropriate weak operator.

The treatment of exclusive decays required both the $b$ and $c$ quarks to be heavy. For inclusive decays we can relax this condition on the $c$ quark, requiring only $m_b \gg \Lambda_{QCD}$. What does the weak decay of the $b$, at time $t = 0$, look like to the light degrees of freedom? For $t < 0$, there is a heavy hadron composed of a point-like color source and light quarks and gluons. At $t = 0$, the point source disappears, releasing both its color and a large amount of energy into the hadronic environment. Eventually, for $t > 0$, this new collection of strongly interacting particles will materialize as a set of physical hadrons. The probability of this hadronization is unity; there is no interference between the hadronization process and the heavy quark decay. There are subleading effects in powers of $\Lambda_{QCD}/m_b$, but they do not alter the probability of hadronization. Rather, they reflect the fact that the $b$ quark is not exactly a static source of color: it has a small nonrelativistic kinetic energy and it carries a spin, both of which affect the kinematic properties of its decay.

As in the case of exclusive decays, we will compute the inclusive semileptonic width $\Gamma(B \to X_c \ell\bar{\nu})$ as a double expansion in powers of $\alpha_s(m_b)$ and $\Lambda_{QCD}/m_b$. The expansion in $\alpha_s(m_b)$ reflects the applicability of perturbative QCD to the short distance part of the process. The heavy quark expansion will be continued to relative order $1/m_b^2$, as there is an analogue of Luke’s Theorem which eliminates power corrections to the rate of order $1/m_b$. These corrections will be written in terms of three nonperturbative parameters. The first, $\bar{\Lambda}$, is defined in Eq. (108). It is essentially the mass of the light degrees of freedom in the heavy hadron, but we will see that it is plagued by an ambiguity of order $\Lambda_{QCD}$ in the definition of the $b$ quark mass. The other two parameters are the expectation values in the $B$ meson of the leading corrections $O_K$ and $O_G$ to $\mathcal{L}_{HQET}$. They are defined as

$$\lambda_1 = \frac{1}{2m_B} \langle B | O_K | B \rangle,$$

$$\lambda_2 = -\frac{1}{6m_B} \langle B | O_G | B \rangle,$$

(123)
where we take the usual relativistic normalization of the states. Hence, $\lambda_1$ may be thought of roughly as the negative of the $b$ quark kinetic energy, and $\lambda_2$ as the energy of its hyperfine interaction with the light degrees of freedom.

Now let us outline the computation. The inclusive decay involves a sum over all possible final states, which is actually a sum over exclusive modes (such as $D, D^*, D\pi, \ldots$), followed by a phase space integral for each mode. We write

$$\Gamma(B \rightarrow X_c \ell \bar{\nu}) = \sum_{X_c} \int \mathcal{D}[P.S.] \left| \langle X_c \ell \bar{\nu} | O_{bc} | B \rangle \right|^2. \quad (124)$$

There is an Optical Theorem for QCD, which follows from the analyticity of the scattering matrix as a function of the momenta of the asymptotic states. Its content is that a transition rate is proportional to the imaginary part of the forward scattering amplitude with two insertions of the transition operator,

$$\Gamma(B \rightarrow X_c \ell \bar{\nu}) = -2 \text{Im} i \int dx \, e^{ik \cdot x} \langle B | T \{ O_{bc}^\dagger(x), O_{bc}(0) \} | B \rangle \equiv 2 \text{Im} T. \quad (125)$$

In what follows, we will write the time-ordered product $T\{O_{bc}^\dagger, O_{bc}\}$ as a series of local operators, using the Operator Product Expansion. As we will see, the applicability of this expansion, and its computation in perturbation theory, will rest on the limit $m_b \gg \Lambda_{\text{QCD}}$. We will then use this limit again to expand the matrix elements of these local operators in the HQET.

The first step is to factorize the integration over the lepton momenta, which can be performed explicitly. Written as a product of currents, $O_{bc}$ takes the form

$$O_{bc} = \frac{G_F V_{cb}}{\sqrt{2}} J_{bc}^\mu J_{\ell \mu}, \quad (126)$$

where

$$J_{bc}^\mu = \bar{c} \gamma^\mu (1 - \gamma^5) b$$
$$J_{\ell \nu} = \bar{\ell} \gamma^\nu (1 - \gamma^5) \nu. \quad (127)$$

Then $T$ can be decomposed as an integral over the total momentum $q^\mu = p_\ell^\mu + p_{\bar{\nu}}^\mu$ transferred to the leptons,

$$T = \frac{1}{2} G_F^2 |V_{cb}|^2 \int dq \, T^{\mu\nu}(q) L_{\mu\nu}(q). \quad (128)$$

Here the lepton tensor is

$$L_{\mu\nu}(q) = \int \mathcal{D}[P.S.] \langle 0 | J_{\mu \nu}^\dagger (\ell \bar{\nu}) \langle \ell \bar{\nu} | J_{\ell \nu} | 0 \rangle = \frac{1}{3\pi} \left( q_\mu q_\nu - q^2 g_{\mu\nu} \right), \quad (129)$$
and the hadron tensor is

\[ T^{\mu\nu}(q) = -i \int dx e^{i q \cdot x} \langle B | T \left\{ J^{\mu+}_{bc}(x), J^{\nu-}_{bc}(0) \right\} | B \rangle. \]  

(130)

We will need the imaginary part, Im \( T^{\mu\nu} \). Where is it nonvanishing? In quantum field theory, a scattering amplitude develops an imaginary part when there can be a real intermediate state, that is, the intermediate particles can all go on their mass shell. Whether this is possible, of course, depends on the kinematics of the external states.

In this case, there are two avenues for creating a physical intermediate state. The first is to act on the external state \( |B\rangle \) with the transition current \( J_{bc}^\mu \). The state which is created has no net \( b \) number and a single charm quark; the simplest possibility is the decay process \( b \to c \). The momentum of the intermediate state is \( p_X = p_B - q \); the condition that it could be on mass shell is simply

\[ p^2_X = (p_B - q)^2 \geq m_D^2. \]  

(131)

If we define scaled variables

\[ p_B^\mu = m_B v^\mu, \quad \hat{q}^\mu = q^\mu / m_B, \quad \hat{m}_D = m_D / m_B, \]  

(132)

this condition becomes

\[ v \cdot \hat{q} \leq \frac{1}{2} \left( 1 + \hat{q}^2 - \hat{m}_D^2 \right). \]  

(133)

Another possibility is to act on \( |B\rangle \) with the conjugate operator \( J_{bc}^{\mu+} \). This operation would produce an intermediate state with two \( b \) quarks and one \( \bar{c} \). For this state to be on shell, the momentum transfer has to satisfy

\[ p^2_X = (p_B + q)^2 \geq (2m_B + m_D)^2, \]  

(134)

that is,

\[ v \cdot \hat{q} \geq \frac{1}{2} \left( 3 - \hat{q}^2 + 4\hat{m}_D + \hat{m}_D^2 \right). \]  

(135)

The physical intermediate states are shown as cuts in the \( v \cdot \hat{q} \) plane in Fig. 6. Also shown is the contour corresponding to the phase space integration over the lepton momentum \( q \). For physical (massless) leptons which are the product of a heavy quark decay, this integral runs over the top of the lower cut, for the range

\[ \sqrt{\hat{q}^2} + i\epsilon \leq v \cdot \hat{q} \leq \frac{1}{2} \left( 1 + \hat{q}^2 - \hat{m}_D^2 \right) + i\epsilon. \]  

(136)
Figure 6: Analytic structure of $T^{\mu\nu}$ in the complex $v \cdot \hat{q}$ plane, for fixed, real $\hat{q}^2$. The integration contour is over the “physical cut”, corresponding to real decay into leptons. The unphysical cuts correspond to other processes.

As indicated by the dotted line, we can continue this contour around the end of the cut and back along the bottom, to $v \cdot \hat{q} = \sqrt{\hat{q}^2} - i\epsilon$. Since $T^{\mu\nu}(v \cdot \hat{q}^*) = -T^{\mu\nu}(v \cdot \hat{q})$ for real $\hat{q}^2$, we compensate for extending the contour by dividing the new integral by two.

We now encounter our central problem. The integral over $v \cdot \hat{q}$ runs over physical intermediate hadron states, which are color neutral bound states of quarks and gluons. Hence the integrand depends intimately on the details of QCD at long distances, which is intrinsically nonperturbative. A perturbative calculation of $T^{\mu\nu}$, which is all we have at our disposal, would appear to be of no use.

The solution is to deform the contour away from the cut, into the complex $v \cdot \hat{q}$ plane, as shown in Fig. 6. Since the scale of momenta is set by $m_b$, the contour is now a distance of order $m_b$ away from the resonances. Since $m_b \gg \Lambda_{\text{QCD}}$, it is reasonable to hope that a perturbative treatment in this region is valid. Essentially, we are saved because we do not need to know $T^{\mu\nu}(q)$ for every value of $q$, just suitable integrals of $T^{\mu\nu}$. That we can use such arguments to compute perturbatively the average value of a hadronic quantity, where at each point the quantity depends on nonperturbative physics, is known as (global) parton-hadron duality.

Parton-hadron duality has the status of being somewhat more than an assumption, since it is known to hold in QCD in the limit $m_b \to \infty$, but somewhat less than an approximation, since it is not known how to compute systematically the leading corrections to it. In any case, the limit $m_b \gg \Lambda_{\text{QCD}}$
plays a crucial role here. By deforming the integration contour a distance of order $m_b$ away from the resonance regime, we find the correspondence in QCD of our earlier intuitive statement: the probability of the decay products materializing as physical hadrons is unity, independent of the kinematics of the short distance process. The local redistribution of probability in phase space due to the presence of hadronic resonances is irrelevant to the total decay. Finally, we should note that since we do not have control over the corrections to local duality, it might work better in some processes than in others, for reasons that need not be apparent from within the calculation. Hence one must be particularly wary of drawing dramatic conclusions from any surprising results of these inclusive calculations.

Let us perform the operator product expansion at tree level, and for decay kinematics. The Feynman diagram is given in Fig. 8, which yields the expression

$$T^{\mu\nu} = \bar{b} \gamma^\mu (1 - \gamma^5) \frac{p_b - \hat{q} + m_c}{(p_b - q)^2 - m_c^2 + i\epsilon} \gamma^\nu (1 - \gamma^5) b. \quad (137)$$

We now write

$$p_\mu^b = m_b v^\mu + k^\mu = m_b (v^\mu + \hat{k}^\mu)$$
$$q^\mu = q^\mu / m_b$$
$$m_c = m_c / m_b$$
$$b(x) = e^{-im_b v \cdot x} h_v(x) + O(1/m_b), \quad (138)$$

and expand in powers of $1/m_b$. Since the operator product expansion is in terms of the effective field $h_v$, a factor of $k^\mu$ corresponds to an insertion of the covariant derivative $iD^\mu$. Operator ordering ambiguities are to be resolved by considering graphs with external gluon fields.

As an example of the procedure, let us expand the propagator to order $1/m_b^2$. (There are also corrections to the currents at this order, which are
Figure 8: The operator product expansion at tree level.

included in a full calculation.) It is convenient to define the scaled hadronic invariant mass,

\[ \hat{s} = \left(m_b v^\mu - q^\mu\right)^2 / m_b^2 = 1 - 2v \cdot \hat{q} + \hat{q}^2. \]  \hfill (139)

Then we find a contribution to \( T^{\mu\nu} \) of the form

\[ T^{\mu\nu} = \frac{1}{m_b} \bar{h}_v \gamma^\mu (\not{p} - \not{q}) \gamma^\nu (1 - \gamma^5) \left[ \frac{1}{\hat{s} - \hat{m}_c^2 + i\epsilon} + \frac{2k \cdot \hat{q} - \hat{k}^2}{(\hat{s} - \hat{m}_c^2 + i\epsilon)^2 + \ldots} \right] h_v. \]  \hfill (140)

From this expression we can read off the operators which appear in the operator product expansion. Since

\[ \text{Im} \left( \frac{1}{\hat{s} - \hat{m}_c^2 + i\epsilon} \right)^n = \pi (n-1)! (-1)^n \delta^{(n-1)}(\hat{s} - \hat{m}_c^2) \], \hfill (141)

we see that the effect of taking the imaginary part in each term is to put the charm quark on its mass shell. The leading term is a quark bilinear,

\[ \frac{1}{m_b} \bar{h}_v \gamma^\mu (\not{p} - \not{q}) \gamma^\nu (1 - \gamma^5) h_v. \]  \hfill (142)

It is straightforward to compute its matrix element in the HQET using the trace formalism,

\[ \langle B | \bar{h}_v \gamma^\mu (\not{p} - \not{q}) \gamma^\nu (1 - \gamma^5) h_v | B \rangle = 2m_B \left( 2v^\mu v^{\nu} - g^{\mu\nu} - v^\mu \hat{q}^{\nu} - v^{\nu} \hat{q}^\mu + g^{\mu\nu} v \cdot \hat{q} \right). \] \hfill (143)

The leading corrections to this expression are of order \( 1/m_b^2 \); there are no corrections of order \( 1/m_b \), by Luke’s Theorem. Finally, we contract the tensor \( T^{\mu\nu} \) with \( L_{\mu\nu} \) and perform the phase space integration (128). In the end, the result is the same as we would have gotten directly by computing free quark decay,

\[ \Gamma = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} \left( 1 - 8\hat{m}_c^2 + 8\hat{m}_c^6 - \hat{m}_c^8 - 12\hat{m}_c^4 \ln \hat{m}_c^2 \right). \]  \hfill (144)
Of course, if we only intended to reproduce the free quark decay result, we
would never have introduced so much new formalism. The value of the HQET
framework is that it allows us to go beyond leading order and compute the
next terms in the series in $1/m_b^2$. For example, consider the operators induced
by the expansion of the propagator (140). The correction of order $1/m_b$ comes
from the operator
\[ \frac{1}{m_b^2} \frac{2}{(s - m_c^2 + i\epsilon)^2} \bar{h}_v \gamma^\mu (\not{q} - \not{q}) \gamma^\nu (1 - \gamma^5) \hat{q} \cdot iD h_v. \] (145)
However, the matrix element of this operator is of the form
\[ \langle B | \bar{h}_v \Gamma^a (v, q) iD_a h_v | B \rangle, \] (146)
which, as we have seen, vanishes by the classical equation of motion. (In
writing Eq. (140), we have already dropped terms explicitly proportional to
$v \cdot k$, for the same reason.) In fact, since all $1/m_b$ corrections, from any source,
have a single covariant derivative, they all vanish in the same way. This is
the analogue of Luke’s Theorem for inclusive decays.\(^3\) The correction of order $1/m_b^2$ in Eq. (140) is
\[ -\frac{1}{m_b^2} \frac{1}{(s - m_c^2 + i\epsilon)^2} \bar{h}_v \gamma^\mu (\not{q} - \not{q}) \gamma^\nu (1 - \gamma^5) (iD)^2 h_v. \] (147)
The matrix element of this operator is related by the heavy quark symmetry
to $\lambda_1$, the expectation value of $O_K$. The full expansion of $T^{\mu\nu}$ also induces
operators with explicit factors of the gluon field, whose matrix elements are
related to $\lambda_2$.

We now present the result for the inclusive semileptonic decay rate, up
to order $1/m_b^2$ in the heavy quark expansion, and with the complete radiative
correction of order $\alpha_s$. We also include that part of the two loop correction
which is proportional to $\beta_0 \alpha_s^2$. Since $\beta_0 \approx 9$, perhaps this term dominates the
two loop result. In any case, it is interesting for other reasons, as we will see
below.

Let us first consider the decay $B \rightarrow X_u \ell \bar{\nu}$, for which the decay rate simplifies since $m_u = 0$. We find\(^3\)
\[ \Gamma(B \rightarrow X_u \ell \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_b^5 \left[ 1 + \left( \frac{25}{6} - \frac{2\pi^2}{3} \right) \frac{\alpha_s(m_b)}{\pi} \right] \]
\[ - \left( 2.98 \beta_0 + C_u \right) \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2} + \ldots \] (148)

39
When we include the charm mass, it is convenient to write the unknown quark masses in terms of the measured meson masses and the parameters of the HQET. In terms of the spin averaged mass $m_B = (m_B + 3m_{B^*})/4$, we have

$$m_b = m_B - \bar{\Lambda} + \frac{\lambda_1}{2m_B} + \ldots,$$  \hspace{1cm} (149)

and analogously for $m_c$. We then find

$$\Gamma(B \to X_c \ell \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} m_B^5 \times 0.369 \left[ 1 - 1.54\frac{\alpha_s(m_b)}{\pi} - (1.43\beta_0 + C_c) \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 - 1.65 \frac{\bar{\Lambda}}{m_B} \left( 1 - 0.87\frac{\alpha_s(m_b)}{\pi} \right) - 0.95 \frac{\lambda_1^2}{m_B} - 3.18 \frac{\lambda_1}{m_B} + 0.02 \frac{\lambda_2}{m_B} + \ldots \right].$$  \hspace{1cm} (150)

All the coefficients which appear in this expression are known functions of $m_D/m_B$, and are evaluated at the physical point $m_D/m_B = 0.372$. In both $B \to X_u \ell \bar{\nu}$ and $B \to X_c \ell \bar{\nu}$, the power corrections proportional to $\lambda_1$ and $\lambda_2$ are numerically small, at the level of a few percent.

5.2 Renormalons and the pole mass

The inclusive decay rate depends on the heavy quark mass $m_b$, either explicitly, as in Eq. (148), or implicitly through $\bar{\Lambda}$, as in Eq. (150). At tree level, $m_b$ is just the coefficient of the $\bar{b}b$ term in the QCD Lagrangian, but beyond that we are faced with the question of what exactly we mean by $m_b$. Should we take an $\overline{\text{MS}}$ mass, such as $\overline{m}_b(m_b)$? Or should we take the pole mass $m_b^{\text{pole}}$, or maybe some other quantity? The various prescriptions for $m_b$ can vary by hundreds of MeV, and, since the total rate is proportional to $m_b^5$, the question is of practical importance if we hope to make accurate phenomenological predictions.

At a fixed order in QCD perturbation theory, the answer is clear. The heavy quark masses which appear come from poles in quark propagators, so we should take $m_b^{\text{pole}}$ (and $m_c^{\text{pole}}$). This is also the prescription for the mass which cancels out the on-shell part of the heavy quark field in the construction of $L_{\text{HQET}}$. Hence the difference of heavy quark pole masses is known quite well,

$$m_b^{\text{pole}} - m_c^{\text{pole}} = \left( \overline{m}_B - \bar{\Lambda} + \frac{\lambda_1}{2m_B} + \ldots \right) - \left( \overline{m}_D - \bar{\Lambda} + \frac{\lambda_1}{2m_D} + \ldots \right) = 3.34 \text{ GeV} + O(\Lambda_{\text{QCD}}^2/m_Q^2).$$  \hspace{1cm} (151)
Since $\Gamma(B \to X_c \ell \bar{\nu})$ depends approximately as $m_b^2 (m_b - m_c)^3$, the uncertainty due to quark mass dependence is reduced.

The problem, of course, is that there is no sensible nonperturbative definition of $m_b^{\text{pole}}$, since due to confinement there is no actual pole in the quark propagator. Hence a direct experimental determination of a value for $m_b^{\text{pole}}$ to insert into the theoretical expressions (148) and (150) is not possible. How, then, can we do phenomenology?

One approach would be to define $m_b^{\text{pole}}$ to be the pole mass as computed in perturbation theory, truncate at some order, and then estimate the theoretical error from the uncomputed higher order terms. However, it turns out that even within perturbation theory the concept of a quark pole mass is ambiguous. Consider a particular class of diagrams which contribute to $m_b^{\text{pole}}$, shown in Fig. 9. The perturbation theory is developed as an expansion in the small parameter $\alpha_s(m_b)$, so we hope that it will be well behaved. Each of the bubbles represents an insertion of the gluon self-energy, which is proportional at lowest order to $\alpha_s(m_b) \beta_0$. Of course, the infinite sum of the graphs in Fig. 9 can be absorbed into the one loop graph, with a compensating change in the coupling from $\alpha_s(m_b)$ to $\alpha_s(q)$, where $q$ is the loop momentum. The result is an expansion for $m_b^{\text{pole}}$ of the form

$$m_b^{\text{pole}} = m_b(m_b) \left[ 1 + a_1 \alpha_s + (a_2 \beta_0 + b_2) \alpha_s^2 + (a_3 \beta_0^2 + b_3 \beta_0 + c_3) \alpha_s^3 + \ldots \right],$$

where $\alpha_s = \alpha_s(m_b)$. The graphs in Fig. 9 contribute the terms proportional to $\alpha_s^{n+1} \beta_0^n$. Since $\beta_0 \approx 9$ these terms are "intrinsically" larger than ones with fewer powers of $\beta_0$, and we might hope that their sum approximates the full series. However, it is important to realize that the only limit of QCD in which
such terms actually dominate is that of large number of light quark flavors, in which case the sign of $\beta_0$ is opposite to that of QCD. Although this is a physical limit of an abelian theory, we are certainly not close to that limit here. The ansatz of keeping only the terms proportional to $\alpha_s^{n+1}(m_b)\beta_0^n$ is known as “naive nonabelianization” (NNA).

What is most interesting about the series of terms shown in Fig. 9, which takes the form $\sum a_n\alpha_s^n\beta_0^{n-1}$, is that it does not converge. Already in the graphs kept in the NNA ansatz, we are sensitive to the fact that QCD is an asymptotic, rather than a convergent expansion. For large $n$ the coefficients $a_n$ diverge as $n!$, much stronger than any convergence due to the powers $\alpha_s^n$. The series can only be made meaningful if this divergence is subtracted. As with many subtraction prescriptions, there is a residual finite ambiguity.\footnote{This ambiguity, known as an “infrared renormalon”, leads to an ambiguity in the pole mass of order $\delta m_{\text{pole}}^b \sim 100\text{ MeV}$.}

By the definition \textbf{(108)}, $\bar{\Lambda}$ also inherits this ambiguity.

The expressions \textbf{(148)} and \textbf{(150)} are plagued by two problems. The first is the renormalon ambiguity in $m_{\text{pole}}^b$ and $\bar{\Lambda}$. The second is that the perturbative expansion for the rate $\Gamma$ is itself divergent, and also has an infrared renormalon. In the expansion

$$\Gamma = \Gamma_0 \left[ \sum a_n' \alpha_s^n(m_b)\beta_0^{n-1} + \text{(power corrections)} \right],$$

the coefficients $a_n'$ also diverge as $n!$. However, it turns out that these two problems actually cure each other, because the infrared renormalons in $m_{\text{pole}}^b$ and in the perturbation series for $\Gamma$ cancel\footnote{We can exploit this cancelation to improve the predictive power of the theoretical computation of the rate. Without this improvement, the infrared renormalons render the expressions \textbf{(148)} and \textbf{(150)} of dubious phenomenological utility.}

The most reliable approach, theoretically, is to eliminate $m_{\text{pole}}^b$ or $\bar{\Lambda}$ explicitly from the rate by computing and measuring another quantity which also depends on it. For example, let us consider the charmless decay rate $\Gamma(B \to X_u \ell \bar{\nu})$ and the average invariant mass $\langle s_H \rangle$ of the hadrons produced in the decay. Each of these expressions suffers from a poorly behaved perturbation series in the NNA approximation. Ignoring terms of relative order $1/m_b^2$ and writing the rate in terms of $\bar{\Lambda}$ instead of $m_{\text{pole}}^b$, we find to five loop...
order.

\[
\Gamma = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_B^5 \left[ 1 - 2.41 \frac{\alpha_s}{\pi} - 2.98 \left( \frac{\alpha_s}{\pi} \right)^2 \beta_0 - 4.43 \left( \frac{\alpha_s}{\pi} \right)^3 \beta_0^2 \\
- 7.67 \left( \frac{\alpha_s}{\pi} \right)^4 \beta_0^3 - 15.7 \left( \frac{\alpha_s}{\pi} \right)^5 \beta_0^4 + \ldots - 5 \frac{\bar{\Lambda}}{m_B} + \ldots \right] 
\]

\[
= \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_B^5 \left[ 1 - 0.061 - 0.120 - 0.107 - 0.111 - 0.136 + \ldots \\
- 5\bar{\Lambda}/m_B + \ldots \right], 
\] (155)

for \(\alpha_s(m_b) = 0.21\) and \(\beta_0 = 9\). As we see, not only does the perturbation series fail to converge, it does not even have an apparent smallest term, where one should truncate to minimize the error of the asymptotic series. The series for \(\langle s_H \rangle\) exhibits a similar behavior.

\[
\langle s_H \rangle = m_B^2 \left[ 0.20 \frac{\alpha_s}{\pi} + 0.35 \left( \frac{\alpha_s}{\pi} \right)^2 \beta_0 + 0.64 \left( \frac{\alpha_s}{\pi} \right)^3 \beta_0^2 + 1.29 \left( \frac{\alpha_s}{\pi} \right)^4 \beta_0^3 \\
+ 2.95 \left( \frac{\alpha_s}{\pi} \right)^5 \beta_0^4 + \ldots + \frac{7}{10} \frac{\bar{\Lambda}}{m_B} + \ldots \right] 
\]

\[
= m_B^2 \left[ 0.0135 + 0.0141 + 0.0156 + 0.0189 + 0.0261 + \ldots \\
- 7\bar{\Lambda}/10m_B + \ldots \right]. 
\] (156)

However, the situation improves dramatically if we eliminate \(\bar{\Lambda}\) and write \(\Gamma\) directly in terms of \(\langle s_H \rangle\),

\[
\Gamma = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_B^5 \left[ 1 - 7.14 \frac{\langle s_H \rangle}{m_B^2} \\
- 0.064 - 0.020 - 0.0002 - 0.022 - 0.047 + \ldots \right]. 
\] (157)

By truncating this series at its smallest term, 0.0002, we obtain a new expression in which the theoretical errors are under control. The price is that we must now measure a second quantity, \(\langle s_H \rangle\), in the same decay.

In principle, the same procedure works for decays to charm. In practice, it is best to combine a number of determinations of \(\bar{\Lambda}\). This has been done by CLEO, which has performed a comparison of measurements of the moments \(\langle E_\ell \rangle, \langle E_\ell^2 \rangle, \langle s_H - m_D \rangle\) and \(\langle (s_H - m_D^2)^2 \rangle\). The results, reproduced in Fig. [10].
are somewhat disappointing, in that one does not obtain a very consistent determination of $\bar{\Lambda}$ and $\lambda_1$. Perhaps this is just a fluctuation, or perhaps this is a sign that parton-hadron duality is failing in these days.

An alternative approach is to express the width $\Gamma$ in terms of the running mass $m_b(m_b)$ instead of another inclusive observable. Since the $\overline{\text{MS}}$ mass is a short distance quantity, this also eliminates the infrared renormalon, which is associated with long distance physics. However, from a phenomenological point of view, it raises the question of how the running mass is to be determined from experiment. Possibilities include quarkonium spectroscopy, QCD sum rules, and lattice calculations, but in all of these cases it is important to determine reliably the accuracy of the method, and how to deal with renormalon ambiguities in a manner that is consistent with their treatment in the calculation of $\Gamma$. Nevertheless, such an approach, particularly one based on
analyzing quarkonium and production near threshold, should eventually prove fruitful.

5.3 Phenomenology of $V_{cb}$ and $V_{ub}$

Despite this ambiguous situation, groups have presented extractions of $V_{cb}$ based on inclusive semileptonic $b$ decays by simply inserting “reasonable” values for $\bar{\Lambda}$ and $\lambda_1$. Such an approach clearly has its dangers! At any rate, the quoted result is

$$|V_{cb}| = (40.0 \pm 0.4 \pm 2.4) \times 10^{-3}.$$  \hfill (158)

The lack of controversy about this procedure is no doubt due in part to the fact that this number is quite consistent with that determined from the analysis of exclusive decays.

There are additional, much more interesting, problems with extracting $|V_{ub}|$ from the inclusive decay $B \rightarrow X_u\ell\bar{\nu}$. They arise from the problem that the process $B \rightarrow X_c\ell\bar{\nu}$ presents an overwhelming background to $B \rightarrow X_u\ell\bar{\nu}$. The only way to avoid this background is to restrict oneself to a corner of phase space in which charmed final states are kinematically inaccessible. Existing experimental analyses isolate the charmless decays by imposing the requirement $E_\ell > (m_B^2 - m_D^2)/2m_B$ or $s_H < m_D^2$. Unfortunately, the OPE can be shown to break down in these restricted corners of the phase space. 45, 46

I do not have space here to explore this issue in much detail, but it is easy to appreciate the essence of the problem. For massless final states, the OPE is an expansion in powers of the light quark propagator, $1/m_b(1 - v \cdot \hat{q} + \hat{q}^2)$. Over most of the final state phase space, the denominator is of order $m_b$ and the OPE is well behaved. But there exist configurations for which both the denominator vanishes and the operator matrix elements which appear in the OPE are nonzero. It turns out that the dangerous region is when $v \cdot \hat{q} \rightarrow \frac{1}{2}$ and $\hat{q}^2 \rightarrow 0$. The precise form of the divergence depends on the kinematic distribution being studied. The general form, however, is universal. In this “endpoint region”, let $y$ be a scaled variable such that $y \rightarrow 1$ at the kinematic endpoint. For example, for $d\Gamma/dE_\ell$, we take $y = 2E_\ell/m_b$, and for $d\Gamma/ds_H$ we take $y = s_H/\bar{\Lambda}m_b$. Then near $y = 1$, the OPE takes the general form

$$\frac{d\Gamma}{dy} \propto \sum_{n=0}^{\infty} c_n \frac{A_n}{m_B^2(1 - y)^n},$$  \hfill (159)

where $c_n$ are coefficients of order one, and the $A_n$ are moments defined by

$$\langle B(v)|\bar{h}_v iD_{\mu_1} \ldots iD_{\mu_n} h_v |B(v)\rangle/2m_B = A_n v_{\mu_1} \ldots v_{\mu_n} + \ldots.$$  \hfill (160)
The ellipses represent terms involving factors of the metric tensor $g_{\mu\nu}$, which are subleading. Since the $A_n$ are associated with totally symmetric combinations of the covariant derivatives, they may be interpreted roughly as the moments of the heavy quark momentum. Note that as defined, $A_0 = 1$, $A_1 = 0$ and $A_2 = \lambda_1$.

The divergences in Eq. (159) can be controlled only if one integrates over a large enough region near the endpoint, $1 - \delta \leq y \leq 1$. For $\delta \sim \bar{\Lambda}/m_b$, one finds a series for $\Gamma$ which does not converge, since the individual terms are of order $A_n/\bar{\Lambda}^n \sim 1$. This situation reflects a dependence of the shape of $d\Gamma/dE_\ell$ on the entire $b$ quark momentum distribution in the $B$ meson. Since, for example, the window $2.3 \text{ GeV} < E_\ell < 2.6 \text{ GeV}$ corresponds to $\delta \simeq 0.1$, this problem pollutes the extraction of $|V_{ub}|$ from $d\Gamma/dE_\ell$. One is forced to introduce a model for the $b$ wavefunction, with the attendant uncontrolled theoretical uncertainties. The same is true, it turns out, for $d\Gamma/ds_H$. The current “best” measurement of $V_{ub}$ from LEP, based on an inclusive analysis, is

$$|V_{ub}| = [4.05^{+0.39}_{-0.46} \text{(stat.)} + 0.43^{+0.23}_{-0.27} \text{(sys.)} \pm 0.02(\tau_b) \pm 0.16(\text{HQE})] \times 10^{-3},$$

or approximately $|V_{ub}/V_{cb}| = 0.104^{+0.015}_{-0.018}$. While these analyses are experimentally very sophisticated, they rely intensively on a two-parameter model of the $b$ quark wavefunction. Essentially, in such a parameterization all moments of the $b$ momentum distribution are correlated with the first two nonzero ones, a constraint which is unphysical. Even if the two parameters are varied within “reasonable” ranges, it is doubtful that such a restrictive choice of model captures reliably the true uncertainty in $|V_{ub}|$ from our ignorance of the structure of the $B$ meson. While the central value which is obtained in these analyses is reasonable, the realistic theoretical error which should be assigned is not yet well understood.

A recent analysis by CLEO of the exclusive decay $\bar{B} \to \rho \ell \bar{\nu}$ yields

$$|V_{ub}| = [3.25 \pm 0.14(\text{stat.}) + 0.21(\text{syst.}) \pm 0.55(\text{theory})] \times 10^{-3},$$

or approximately $|V_{ub}/V_{cb}| = 0.083^{+0.015}_{-0.016}$, essentially consistent with the LEP result. In this case the reliance on models is quite explicit, since one needs the hadronic form factor $\langle \rho| \bar{u}\gamma^\mu(1 - \gamma^5)b|\bar{B} \rangle$ over the range of momentum transfer to the leptons. The CLEO measurement relies on models based on QCD sum rules, which have uncertainties which are hard to quantify. Hence, just as in the case of the LEP measurement, the quoted errors should not be taken terribly seriously. All of the current constraints are consistent with $|V_{ub}/V_{cb}| = 0.090 \pm 0.025$, where I strongly prefer this more conservative estimate of the theoretical errors. The problem lies not in the experimental analyses, but in our insufficient understanding of hadron dynamics.
Recently it has been pointed out[3] that the one may alternatively reject the charm background by studying the distribution $d\Gamma/dq^2$ and restricting oneself to $q^2 > (m_B - m_D)^2$. Not only does this cut eliminate charmed final states, but it also avoids the troublesome region near $q^2 = 0$. Hence such a determination would not be polluted by the divergence described above. Whether the neutrino reconstruction algorithms of the $B$ Factories will be up to the task of this measurement yet remains to be seen.

6 Concluding Remarks

Unfortunately, we have had time in these lectures only to introduce a very few of the many applications of heavy quark symmetry and the HQET to the physics of heavy hadrons. Since its development less than ten years ago, it has become one of the basic tools of QCD phenomenology. Much of the popularity and utility of the HQET certainly come from its essential simplicity. The elementary observation that the physics of heavy hadrons can be divided into interactions characterized by short and long distances gives us immediately a clear and compelling intuition for the properties of heavy-light systems. The straightforward manipulations which lead to the HQET then allow this intuition to form the basis for a new systematic expansion of QCD. The deeper understanding of heavy hadrons which we thereby obtain is increasingly important as the $B$ Factory Era begins.

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References

1. There are many excellent reviews of heavy quark symmetry and its applications. In particular, see A. V. Manohar and M. B. Wise, *Heavy Quark Physics* (Cambridge University Press, Cambridge, 2000); M. Neubert, *Phys. Rep.* 245, 259 (1994); M. Shifman, preprint TPI-MINN-95-31-T, to appear in *QCD and Beyond, Proceedings of TASI-95*. Much of what appears in these lectures is similar to my 1996 SLAC Summer Institute lecture notes, A. F. Falk, hep-ph/9610363. Material has also been taken
2. For example, see M. Wirbel, B. Stech and M. Bauer, Z. Phys. C29, 637 (1985); J.G. Köhner and G.A. Schuler, Z. Phys. C38, 511 (1988); N. Isgur et al., Phys. Rev. D39, 799 (1989).
3. C. Caso et al., Eur. Phys. J. C3, 1 (1998); update at http://pdg.lbl.gov.
4. N. Isgur and M. B. Wise, Phys. Lett. B232, 113 (1989); Phys. Lett. B237, 527 (1990).
5. M.B. Voloshin and M.A. Shifman, Yad. Fiz. 45, 463 (1987); Yad. Fiz. 47, 511 (1988).
6. S. Nussinov and W. Wetzel, Phys. Rev. D36, 130 (1987).
7. H.D. Politzer and M.B. Wise, Phys. Lett. B206, 681 (1988); Phys. Lett. B208, 504 (1988).
8. E.V. Shuryak, Phys. Lett. B93, 134 (1980).
9. E. Eichten and B. Hill, Phys. Lett. B234, 511 (1990); Phys. Lett. B243, 427 (1990).
10. H. Georgi, Phys. Lett. B240, 447 (1990).
11. B. Grinstein, Nucl. Phys. B339, 253 (1990).
12. A.F. Falk et al., Nucl. Phys. B343, 1 (1990).
13. A.F. Falk and B. Grinstein, Phys. Lett. B247, 406 (1990).
14. M. Luke, Phys. Lett. B252, 247 (1990).
15. A.F. Falk, B. Grinstein and M. Luke, Nucl. Phys. B357, 185 (1991).
16. T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. B355, 38 (1991).
17. H.D. Politzer, Nucl. Phys. B172, 349 (1980).
18. A.F. Falk and M. Neubert, Phys. Rev. D47, 2965 (1993).
19. A.F. Falk, M. Luke and M.J. Savage, Phys. Rev. D49, 3367 (1994).
20. A.F. Falk, Nucl. Phys. B378, 79 (1992).
21. P. Cho and B. Grinstein, Phys. Lett. B285, 153 (1992).
22. M. Ademollo and R. Gatto, Phys. Rev. Lett. 13, 264 (1964).
23. M. Neubert and V. Rieckert, Nucl. Phys. B382, 97 (1992).
24. M. Neubert, Phys. Lett. B264, 455 (1991).
25. M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. D51, 2217 (1995); Erratum, D52, 3149 (1995).
26. M. Neubert, Phys. Lett. B338, 84 (1994).
27. A. Czarnecki, Phys. Rev. Lett. 76, 4124 (1996).
28. For earlier one-loop and one-loop improved calculations, see A.F. Falk and B. Grinstein, Phys. Lett. B247, 406 (1990); Phys. Lett. B249, 314 (1990); M. Neubert, Phys. Rev. D46, 2212 (1992); Phys. Rev. D51, 5924 (1995); Phys. Lett. B341, 367 (1995).
29. B. Barish et al. (CLEO Collaboration), Phys. Rev. D51, 1041 (1995).
30. The LEP results are combined by the LEP $V_{ub}$ Working Group. Details can be found at the web address http://lepvcb.web.cern.ch.
31. M. Neubert, Int. J. Mod. Phys. A11, 4173 (1996).
32. J. Chay, H. Georgi and B. Grinstein, Phys. Lett. B247, 399 (1990).
33. I.I. Bigi, N.G. Uraltsev and A.I. Vainshtein, Phys. Lett. B293, 430 (1992); I.I. Bigi et al., Phys. Rev. Lett. 71, 496 (1993).
34. A.V. Manohar and M.B. Wise, Phys. Rev. D49, 1310 (1994).
35. A. Falk, I. Dunietz and M. B. Wise, Phys. Rev. D51, 1183 (1995).
36. B. Guberina, R.D. Peccei and R. Ruckl, Nucl. Phys. B171, 333 (1980).
37. M. Luke, M.J. Savage and M.B. Wise, Phys. Lett. B343, 329 (1995).
38. Y. Nir, Phys. Lett. B221, 184 (1989).
39. M. Luke, M.J. Savage and M.B. Wise, Phys. Lett. B345, 301 (1995).
40. A.F. Falk, M. Luke and M.J. Savage, Phys. Rev. D53, 2491 (1996); Phys. Rev. D53, 6316 (1996).
41. J. Bartelt et al. (CLEO Collaboration), CLEO-CONF-98-21.
42. R. Poling, proceedings of Lepton-Photon 1999, hep-ex/0003025.
43. M. Beneke and V.M. Braun, Nucl. Phys. B426, 301 (1994); M. Beneke, V.M. Braun and V.I. Zakharov, Phys. Rev. Lett. 73, 3058 (1994).
44. I.I. Bigi et al., Phys. Rev. D50, 2234 (1994).
45. M. Neubert and C.T. Sachrajda, Nucl. Phys. B438, 235 (1995).
46. M. Luke, A.V. Manohar and M.J. Savage, Phys. Rev. D51, 4924 (1995).
47. P. Ball, M. Beneke and V.M. Braun, Phys. Rev. D52, 3929 (1995).
48. For the state of the art, see M. Beneke and A. Signer, Phys.Lett. B471, 233 (1999).
49. M. Neubert, Phys. Rev. D49 (1994) 3392; D49 (1994) 4623; I.I. Bigi et al., Int. J. Mod. Phys. A9 (1994) 2467.
50. A.F. Falk, Z. Ligeti and M.B. Wise, Phys. Lett. B406, 225 (1997); R.D. Dikeman and N.G. Uraltsev, Nucl. Phys. B509, 378 (1998).
51. A.F. Falk et al., Phys. Rev. D49, 4553 (1994).
52. D. Abbaneo et al. (LEP $V_{ub}$ Working Group), LEPVUB-99/01, 1999.
53. B.H. Behrens et al. (CLEO Collaboration), hep-ex/9905056.
54. C. Bauer, Z. Ligeti and M. Luke, Phys. Lett. B479, 395 (2000).