Stability Criteria via Common Non-strict Lyapunov Matrix for Discrete-time Linear Switched Systems

Xiongping Dai\textsuperscript{a}, Yu Huang\textsuperscript{b}, Mingqing Xiao\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China
\textsuperscript{b}Department of Mathematics, Zhongshan (Sun Yat-Sen) University, Guangzhou 510275, People’s Republic of China
\textsuperscript{c}Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA

Abstract

Let $S = \{S_1, S_2\} \subset \mathbb{R}^{d \times d}$ have a common, but not necessarily strict, Lyapunov matrix (i.e. there exists a symmetric positive-definite matrix $P$ such that $P - S_k^T P S_k \geq 0$ for $k = 1, 2$). Based on a splitting theorem of the state space $\mathbb{R}^d$ (Dai, Huang and Xiao, arXiv:1107.0132v1[math.PR]), we establish several stability criteria for the discrete-time linear switched dynamics

$$x_n = S_{\sigma_n} \cdots S_{\sigma_1}(x_0), \quad x_0 \in \mathbb{R}^d \text{ and } n \geq 1$$

governed by the switching signal $\sigma: \mathbb{N} \to \{1, 2\}$. More specifically, let $\rho(A)$ stand for the spectral radius of a matrix $A \in \mathbb{R}^{d \times d}$, then the outline of results obtained in this paper are: (1) For the case $d = 2$, $S$ is absolutely stable (i.e., $\|S_{\sigma_n} \cdots S_{\sigma_1}\| \to 0$ driven by all switching signals $\sigma$) if and only if $\rho(S_1), \rho(S_2)$ and $\rho(S_1 S_2)$ all are less than 1; (2) For the case $d = 3$, $S$ is absolutely stable if and only if $\rho(A) < 1$ for $A \in \{S_1, S_2\}^\ell$ for $\ell = 1, 2, 3, 4, 5, 6,$ and $8$. This further implies that for any $S = \{S_1, S_2\} \subset \mathbb{R}^{d \times d}$ with the generalized spectral radius $\rho(S) = 1$ where $d = 2$ or $3$, if $S$ has a common, but not strict in general, Lyapunov matrix, then $S$ possesses the spectral finiteness property.

Keywords: Linear switched/inclusion dynamics, non-strict Lyapunov matrix, asymptotic stability, finiteness property

2010 MSC: 93D20, 37N35

1. Introduction

1.1. Motivations

Let $\mathbb{R}^{d \times d}$ be the standard topological space of all $d$-by-$d$ real matrices where $2 \leq d < +\infty$, and for any $A \in \mathbb{R}^{d \times d}$, by $\rho(A)$ we denote the spectral radius of $A$. In addition, we identify $A$ with

\footnote{Project was supported partly by National Natural Science Foundation of China (Grant Nos. 11071112 and 11071263), the NSF of Guangdong Province and in part by NSF 0605181 and 1021203 of the United States.}

Email addresses: xpdai@nju.edu.cn (Xiongping Dai), stshyu@mail.sysu.edu.cn (Yu Huang), mxiao@math.siu.edu (Mingqing Xiao)
1 INTRODUCTION

its induced operator $A(\cdot) : x \mapsto Ax$ for $x \in \mathbb{R}^d$. Let $S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d \times d}$ be a finite set with $2 \leq K < +\infty$. We consider the stability and stabilization of the linear inclusion/control dynamics

$$x_n \in \{S_1, \ldots, S_K\}(x_{n-1}), \quad x_0 \in \mathbb{R}^d \text{ and } n \geq 1. \quad (1.1)$$

As in [12, 10], we denote by $\Sigma^+_n$ the set of all admissible control signals $\sigma : \mathbb{N} \to \{1, \ldots, K\}$, equipped with the standard product topology. Here and in the sequel $\mathbb{N} = \{1, 2, \ldots\}$ and for any $\sigma \in \Sigma^+_n$ we will simply write $\sigma(n) = \sigma_n$ for all $n \geq 1$.

For any input $(x_0, \sigma)$, where $x_0 \in \mathbb{R}^d$ is an initial state and $\sigma = (\sigma_n)_{n=1}^{+\infty} \in \Sigma^+_n$ a control (switching) signal, there is a unique output $(x_n(x_0, \sigma))_{n=1}^{+\infty}$, called an orbit of the system (1.1), which corresponds to the unique solution of the discrete-time linear switched dynamics

$$x_n = S_{\sigma_n} \cdots S_{\sigma_1}(x_0), \quad x_0 \in \mathbb{R}^d \text{ and } n \geq 1 \quad (1.2)$$
driven/governed by the switching signal $\sigma$. Then as usual, $S$ is called (asymptotically) stable driven by $\sigma$ if

$$\lim_{n \to +\infty} \|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\| = 0 \forall x_0 \in \mathbb{R}^d; \text{ or equivalently, } \|S_{\sigma_n} \cdots S_{\sigma_1}\| \to 0 \text{ as } n \to +\infty.$$ 

$S$ is said to be absolutely stable if it is stable driven by all switching signals $\sigma \in \Sigma^+_n$; see, e.g., [16]. We note that the stability of $S$ is independent of the norm $\| \|$ used here.

It is a well-known fact that if each member $S_k$ of $S$ shares a common Lyapunov matrix; i.e., there exists a symmetric positive-definite matrix $Q \in \mathbb{R}^{d \times d}$ such that

$$Q - S_k^T Q S_k > 0 \quad (1 \leq k \leq K),$$

then $S$ is absolutely stable. Here $^T$ stands for the transpose operator of matrices or vectors. An essentially weak condition is that each member $S_k$ of $S$ shares a common, “but not necessarily strict,” Lyapunov matrix; that is, there exists a symmetric positive-definite matrix $P$ such that

$$P - S_k^T P S_k \geq 0, \quad 1 \leq k \leq K. \quad (1.3a)$$

Here “$A \geq 0$” means $x^T A x \geq 0 \forall x \in \mathbb{R}^d$. Associated to the weak Lyapunov matrix $P$ as in (1.3a), we define the vector norm on $\mathbb{R}^d$ as

$$\|x\|_P = \sqrt{x^T P x} \forall x \in \mathbb{R}^d. \quad (1.3b)$$

(We also write its induced operator/matrix norm on $\mathbb{R}^{d \times d}$ as $\| \cdot \|_P$.) Then, $\|S_k\|_P \leq 1$ for all $1 \leq k \leq K$. Condition (1.3a) is both practically important and academically challenging, for example, [20, 1, 18, 2, 25] for the continuous-time case and [16] for discrete case. Indeed, it is desirable in many practical issues and is closely related to periodic solutions and limit cycles, see, e.g., [5, 6] and [22, Proposition 18]; in addition, if $S_k, 1 \leq k \leq K$, are paracontractive (i.e., $x^T S_k^T S_k x \leq x^T x$ for all $x \in \mathbb{R}^d$, and “=’ holds if and only if $S_k(x) = x$, see, e.g., [24]), then condition (1.3a) holds.

In this paper, we will study the stability of $S$ that satisfies condition (1.3a). Even under condition (1.3a), the stability of every subsystems $S_k$ does not implies the absolute stability of $S$, as shown by Example 6.6 constructed in Section 6. So, our stability criteria — Theorems A, B, C, and D — established in this paper, are nontrivial.
1.2. Stability driven by nonchaotic switching signals

Under condition (1.3a), in [3] for the continuous-time case, Balde and Jouan provided a large class of switching signals for which a large class of switched systems are stable, by considering nonchaotic inputs and the geometry of $\omega$-limit sets of the matrix sequences $(S_{\sigma} \cdots S_{\sigma})^{n_{1}}_{n=1}$.

Recall from [3, Definition 1] that a switching signal $\sigma = (\sigma_{n})_{n=1}^{\infty} \in \Sigma_{K}$ is said to be nonchaotic, if to any sequence $(n_{i})_{i=1}^{\infty}$ any $m \geq 1$ there corresponds some integer $\delta$ with $2 \leq \delta \leq m + 1$ such that $\forall \ell_{0} \geq 1, \exists \ell \geq \ell_{0}$ so that $\sigma$ is constant restricted to some subinterval of $[n_{i}, n_{i} + m]$ of length greater than or equal to $\delta$. A switching signal $\sigma \in \Sigma_{K}^{d}$ is said to be generic [16] (or regular in [3]) if each alphabet in $\{1, \ldots, K\}$ appears infinitely many times in the sequence $\sigma = (\sigma_{n})_{n=1}^{\infty}$.

Then our first stability criterion obtained in this paper can be stated as follows:

**Theorem A.** Let $S = \{S_{1}, \ldots, S_{K}\} \subset \mathbb{R}^{d \times d}$ satisfy condition (1.3a) with $\rho(S_{k}) < 1$ for all $1 \leq k \leq K$. Then

$$\|S_{\sigma_{n}} \cdots S_{\sigma_{1}}\| \to 0 \quad \text{as } n \to +\infty$$

for any nonchaotic switching signal $\sigma = (\sigma_{n})_{n=1}^{\infty} \in \Sigma_{K}^{d}$.

We note that in Theorem A, if $\sigma$ is additionally generic (regular), then this statement is a direct consequence of [3, Theorem 3]. However, without the genericity of $\sigma$, here we need to explore an essential property of a nonchaotic switching signal; see Lemma 2.1 below. In the case of $d = 2$ and $K = 2$, an ergodic version of Theorem A will be stated in Corollary 5.3 in Section 5.

As is shown by Example 6.6 mentioned before, under the assumption of Theorem A, one cannot expect the stability of $S$ driven by an arbitrary switching signal.

1.3. A splitting theorem driven by recurrent signals

Next, we consider another type of switching signal — recurrent switching signal, which does not need to be nonchaotic and balanced and which seems more general from the viewpoint of ergodic theory. In fact, all recurrent switching signals form a set of total measure 1.

Corresponding to a switching signal $\sigma = (\sigma_{n})_{n=1}^{\infty} \in \Sigma_{K}^{d}$, for the system $S$ we define two important subspaces of the state space $\mathbb{R}^{d}$:

$$E^{(s)}(\sigma) = \left\{ x_{0} \in \mathbb{R}^{d} : \|S_{\sigma_{n}} \cdots S_{\sigma_{1}}(x_{0})\|_{\rho} \to 0 \text{ as } n \to +\infty \right\}$$

and

$$E^{(c)}(\sigma) = \left\{ x_{0} \in \mathbb{R}^{d} : \exists (n_{i})_{i=1}^{\infty} \not\to +\infty \text{ such that } \lim_{i \to +\infty} S_{\sigma_{n_{i}}} \cdots S_{\sigma_{1}}(x_{0}) = x_{0} \right\} ;$$

called, respectively, the stable and central manifolds of $S$ driven by $\sigma$. Here $E^{(s)}(\sigma)$ and $E^{(c)}(\sigma)$ are indeed independent of the norm $\| \cdot \|_{\rho}$.

A switching signal $\sigma = (\sigma_{n})_{n=1}^{\infty} \in \Sigma_{K}^{d}$ is called recurrent under the classical one-sided Markov shift transformation, $\theta: \sigma(\cdot) \mapsto \sigma(\cdot + 1)$, of $\Sigma_{K}^{d}$, if for any $\ell \geq 1$ there exists some $m$ sufficiently large such that

$$(\sigma_{1}, \ldots, \sigma_{\ell}) = (\sigma_{1+m}, \ldots, \sigma_{\ell+m}).$$

We have then, for $S$, the following important splitting theorem of the state space $\mathbb{R}^{d}$ based on a recurrent switching signal:
1 INTRODUCTION

Splitting Theorem ([13]). Let $S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d \times d}$ satisfy condition (1.3a). Then, for any recurrent switching signal $\sigma \in \Sigma^2$ it holds

$$\mathbb{R}^d = E^i(\sigma) \oplus E^r(\sigma) \quad \text{and} \quad S_{\sigma}(E^{i,c}(\sigma)) = E^{i,c}(\sigma(\cdot + 1)).$$

This theorem is a special case of a more general result [13, Theorem B'']. So in this case, if the central manifold $E^r(\sigma) = \{0\}$ then $S$ is stable driven by the recurrent switching signal $\sigma$. This splitting is in fact unique under the Lyapunov norm $\| \cdot \|_\rho$.

1.4. Almost sure stability

Under condition (1.3a), let $K_{\|\cdot\|_\rho}(S_k) = \{x \in \mathbb{R}^d : \|S_k(x)\|_\rho = \|x\|_\rho\}$ for $1 \leq k \leq K$. We note that if $\|S_1\|_\rho < 1$ then $K_{\|\cdot\|_\rho}(S_1) = \{0\}$.

Next, using the above splitting theorem, we can obtain the following almost sure stability criterion:

**Theorem B.** Let $S = \{S_1, S_2\} \subset \mathbb{R}^{d \times d}$ satisfy (1.3a) and $K_{\|\cdot\|_\rho}(S_1) \cap K_{\|\cdot\|_\rho}(S_2) = \{0\}$, where $d = 2$ or $3$. Then, if $\mathbb{P}$ is a non-atomic ergodic probability measure of the one-sided Markov shift transformation $\theta : \Sigma^+ \to \Sigma^+$ defined by $\sigma(\cdot) \mapsto \sigma(\cdot + 1)$, there holds

$$\|S_{\sigma_1} \cdots S_{\sigma_n}\|_\rho \to 0 \quad \text{as} \quad n \to +\infty$$

for $\mathbb{P}$-a.e. $\sigma \in \Sigma^+$.

We consider a simple example. Let $S = \{S_1, S_2\}$ with $S_1 = \text{diag}(\frac{1}{2}, \frac{1}{2})$ and $S_2 = \text{diag}(1, 1)$. Then, $K_{\|\cdot\|_\rho}(S_1) = \{0\}$ and $K_{\|\cdot\|_\rho}(S_2) = \mathbb{R}^2$, where $\| \cdot \|_2$ stands for the usual Euclidean norm. So, $K_{\|\cdot\|_\rho}(S_1) \cap K_{\|\cdot\|_\rho}(S_2) = \{0\}$. Clearly, $S$ is not absolutely stable. This shows that under the situation of Theorem B, it is necessary to consider the almost sure stability.

1.5. Absolute stability and finiteness property

For absolute stability, we can obtain the following two criteria Theorems C and D, which show the stability is decidable in the cases of $d = 2, 3$ under condition (1.3a).

**Theorem C.** Let $S = \{S_1, S_2\} \subset \mathbb{R}^{2 \times 2}$ satisfy condition (1.3a). Then, $S$ is absolutely stable if and only if $\rho(A) < 1$ for all $A \in \{S_1, S_2\}^\ell$ for $\ell = 1, 2$.

**Theorem D.** Let $S = \{S_1, S_2\} \subset \mathbb{R}^{3 \times 3}$ satisfy condition (1.3a). Then, $S$ is absolutely stable if and only if $\rho(A) < 1$ for all $A \in \{S_1, S_2\}^\ell$ for $\ell = 1, 2, 3, 4, 5, 6$, and $8$.

On the other hand, the accurate computation of the generalized spectral radius of $S$, introduced by Daubechies and Lagarias in [15] as

$$\rho(S) = \lim_{n \to +\infty} \max_{\sigma \in \Sigma^2} \sqrt[n]{\rho(S_{\sigma_1} \cdots S_{\sigma_n})} \quad (= \sup_{n \geq 1} \sqrt[n]{\rho(S_{\sigma_1} \cdots S_{\sigma_n})}),$$

is very important for many subjects. If one can find a finite-length word $(w_1, \ldots, w_n) \in \{1, \ldots, K\}^n$ for some $n \geq 1$, which realizes $\rho(S)$, i.e.,

$$\rho(S) = \sqrt[n]{\rho(S_{w_1} \cdots S_{w_n})},$$

then $S$ is said to have the spectral finiteness property. A brief survey for some recent progresses regarding the finiteness property can be found in [14, §1.2].
2 SWITCHED SYSTEMS DRIVEN BY NONCHAOTIC SWITCHING SIGNALS

Under condition (1.3a), we have \( \rho(S) \leq 1 \). If \( \rho(S) < 1 \) then \( S \) is absolutely stable; see, e.g., [16]. If \( \rho(S) = 1 \) then \( \| \cdot \|_p \) is just an extremal norm for \( S \) (see [4, 28, 9] for more details). In [16], Gurvits proved that if \( S \) has a polytope\(^1\) extremal norm on \( \mathbb{R}^d \), then it has the spectral finiteness property. However, the Lyapunov norm \( \| \cdot \|_p \) defined as in (1.3b) does not need to be a polytope norm, for example, \( P = I_d \) the identity matrix which is associated with the usual Euclidean norm \( \| \cdot \|_2 \) on \( \mathbb{R}^d \).

As a consequence of the statements of Theorems C and D, we can easily obtain the following spectral finiteness result.

**Corollary.** Let \( S = [S_1, S_2] \subset \mathbb{R}^{d \times d} \) satisfy condition (1.3a) with \( \rho(S) = 1 \). Then the following two statements hold.

1. For the case \( d = 2 \), there follows \( 1 = \max \{ \rho(S_1), \rho(S_2), \sqrt{\rho(S_1 S_2)} \} \).
2. In the case \( d = 3 \), there holds \( 1 = \max \{ \sqrt[n]{\rho(S_{\sigma_1} \cdots S_{\sigma_n})} | w \in \{1, 2\}^n, n = 1, 2, 3, 4, 5, 6, 8 \} \).

**Proof.** Let \( d = 2 \). Assume \( \{ \rho(S_1), \rho(S_2), \sqrt{\rho(S_1 S_2)} \} < 1 \). Then Theorem C implies that \( S \) is absolutely stable and so \( \rho(S) < 1 \), a contradiction. Similarly, we can prove the statement in the case \( d = 3 \).

It should be pointed out that if \( \rho(S) < 1 \), then \( \rho(S) \) does not need to be attained by these maximum values defined as in the above corollary.

1.6. Outline

The paper is organized as follows. We shall prove Theorem A in Section 2. In fact, we will prove a more general result (Theorem 2.3) than Theorem A there. Since the above Splitting Theorem is very important for the proofs of Theorems B, C, and D, we will give some notes on it in Section 3. Then, Theorem B will be proved in Section 4. Section 5 will be devoted to proving Theorems C and D. We will construct some examples in Section 6 to illustrate applications of our Theorems stated here. Finally, we will end this paper with some concluding remarks in Section 7.

2. Switched systems driven by nonchaotic switching signals

This section is devoted to proving Theorem A stated in Section 1.2 under the guise of a more general result.

For any integer \( 2 \leq K < +\infty \), we recall that a switching signal \( \sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma^3 \) is called nonchaotic, if to any sequence \( (n_i)_{i \geq 1} \nearrow +\infty \) and any \( m \geq 1 \) there corresponds some \( \delta \) with \( 2 \leq \sigma \leq m + 1 \) such that for all \( \ell \geq 1 \), there exists an \( \ell \geq \ell_0 \) so that \( \sigma \) is constant restricted to some subinterval of \([n_\ell, n_\ell + m]\) of length greater than or equal to \( \delta \). Clearly, a constant switching signal \( \sigma \) with \( \sigma(n) \equiv k \) is nonchaotic.

Then from definition, we can obtain the following lemma, which discovers the essential property of a nonchaotic switching signal.

**Lemma 2.1.** Let \( \sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma^3 \) be a nonchaotic switching signal. Then, there exists some alphabet \( k \in \{1, \ldots, K\} \) such that for any \( \ell \geq 1 \) and any \( \ell' \geq 1 \), there exists an \( n_\ell \geq \ell \) so that \( \sigma_{n_\ell + 1} = \cdots = \sigma_{n_\ell + \ell} = k \).

---

\(^1\)A norm \( \| \cdot \| \) on \( \mathbb{R}^d \) is called a (real) polytope norm, if the unit sphere \( S_{\| \cdot \|} = \{ x \in \mathbb{R}^d : \| x \| = 1 \} \) is a polytope in \( \mathbb{R}^d \); see, e.g., [16].
3 \( \omega \)-LIMIT SETS FOR PRODUCT BOUNDED SYSTEMS

Proof. First, we can choose a sequence \( (n_i)_{i \geq 1} \) such that \( n_{i+1} - n_i \to +\infty \) and some \( k \in \{1, \ldots, K\} \), which are such that \( n_i - n_{i+1} \sim +\infty \) and \( \sigma_{n_i} = k \) for all \( i \geq 1 \). Now from the definition of nonchaotic property with \( m = 1 \), it follows that we can choose a subsequence of \( (n_i)_{i \geq 1} \), still write, without loss of generality, as \( (n_i)_{i \geq 1} \), such that \( \sigma_{n_i} = \sigma_{n_i+1} = k \) for all \( i \geq 1 \). Repeating this procedure for \( (n_i + 1)_{i \geq 1} \) proves the statement.

Lemma 2.1 shows that the \( \omega \)-limit set of a nonchaotic switching signal contains at least one constant switching signal, under the sense of the classical Markov shift transformation.

The following fact is a simple consequence of the classical Gel’fand spectral formula, which will be refined in Section 5 for the Lyapunov norm \( \| \cdot \|_p \).

Lemma 2.2. For any \( A \in \mathbb{R}^{d \times d} \) and any matrix norm \( \| \cdot \| \) on \( \mathbb{R}^{d \times d} \), if \( p(A) < 1 \) then there is an integer \( N \geq 1 \) such that \( \| A^N \| < 1 \).

For \( S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d \times d} \), it is said to be product bounded, if there is a universal constant \( \beta \geq 1 \) such that
\[
\| S_{\sigma_k} \cdots S_{\sigma_1} \| \leq \beta \quad \forall \sigma \in \Sigma^+ \quad \text{and} \quad n \geq 1.
\]

This property does not depend upon the norm \( \| \cdot \| \) used here.

If \( S \) is product bounded, then one always can choose a vector norm \( \| \cdot \| \) on \( \mathbb{R}^d \) such that its induced operator norm \( \| \cdot \| \) on \( \mathbb{R}^{d \times d} \) is such that \( \| S_k \| \leq 1 \) for all \( 1 \leq k \leq K \). Then the norm \( \| \cdot \| \) on \( \mathbb{R}^d \) acts as a Lyapunov function for \( S \). However, there does not need to exist a common, not strict in general, “quadratic” Lyapunov function/matrix \( P \) as in (1.3a). So, the following theorem is more general than Theorem A stated in Section 1.2.

Theorem 2.3. Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d \times d} \) be product bounded. If \( p(S_k) < 1 \) for all \( 1 \leq k \leq K \), then \( S \) is stable driven by any nonchaotic switching signals \( \sigma \in \Sigma^+ \).

Proof. Without loss of generality, let \( \| \cdot \| \) be a matrix norm on \( \mathbb{R}^{d \times d} \) such that \( \| S_k \| \leq 1 \) for all \( 1 \leq k \leq K \). Let \( \sigma = (\sigma_{n})_{n=0}^{\infty} \in \Sigma^+ \) be an arbitrary nonchaotic switching signal. Let \( k \) be given by Lemma 2.1. Since \( p(S_k) < 1 \), by Lemma 2.2 we have some \( m \geq 1 \) such that \( \| S_k^m \| < 1 \).

Thus, for an arbitrary \( \varepsilon > 0 \) there is an \( l \geq 1 \) such that \( \| S_k^m \| < \varepsilon \). From Lemma 2.1, it follows that as \( n \to +\infty \),
\[
\| S_{\sigma_n} \cdots S_{\sigma_1} \| \leq \| S_{\sigma_{n-l+1}} \cdots S_{\sigma_{n-l+1}} \| < \varepsilon.
\]

So, \( \| S_{\sigma_n} \cdots S_{\sigma_1} \| \to 0 \) as \( n \to +\infty \), since \( \varepsilon > 0 \) is arbitrary. This completes the proof of Theorem 2.3.

Under condition (1.3a), the statement of Theorem 2.3 will be strengthened by Corollary 5.3 in Section 5.

3. \( \omega \)-limit sets for product bounded systems

In this section, we will introduce \( \omega \)-limit sets and give some notes on our splitting theorem stated in Section 1.3 that is very important for our arguments in the next sections.
3. \( \omega \)-LIMIT SETS FOR PRODUCT BOUNDED SYSTEMS

3.1. \( \omega \)-limit sets of a trajectory

We now consider the linear inclusion (1.1) generated by \( S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d \times d} \) where \( 2 \leq K < +\infty \), as in Section 1. The classical one-sided Markov shift transformation

\[
\theta: \Sigma^+_K \rightarrow \Sigma^+_K
\]

is defined as

\[
\sigma = (\sigma_n)_{n=1}^{\infty} \mapsto \theta(\sigma) = (\sigma_{n+1})_{n=1}^{\infty} \quad \forall \sigma \in \Sigma^+_K.
\]

**Definition 3.1** ([23, 24, 3]). Let \( x_0 \in \mathbb{R}^d \) be an initial state and \( \sigma = (\sigma_n)_{n=0}^{\infty} \in \Sigma^+_K \) a switching signal. The set of all limit points of the sequence \( (S_{\sigma_1} \cdots S_{\sigma_n}(x_0))_{n=0}^{\infty} \) in \( \mathbb{R}^d \) is called the \( \omega \)-limit set of \( S \) at the input \((x_0, \sigma)\). We denote it by \( \omega(x_0, \sigma) \) here.

It is easy to see that for any switching signal \( \sigma \), the corresponding switched system is asymptotically stable if and only if \( \omega(x_0, \sigma) = \emptyset \forall x_0 \in \mathbb{R}^d \). Thus we need to consider the structure of \( \omega(x_0, \sigma) \) in order to study the stability of the switched dynamics induced by \( S \).

**Lemma 3.2.** Assume \( S \) is product bounded; that is, there is a matrix norm \( \| \cdot \| \) on \( \mathbb{R}^{d \times d} \) such that \( \| S_n \| \leq 1 \) for all \( 1 \leq k \leq K \). Then, for any initial data \( x_0 \in \mathbb{R}^d \) and any switching signal \( \sigma \), the following two statements hold.

1. The \( \omega \)-limit set \( \omega(x_0, \sigma) \) is a compact subset contained in a sphere \( \{x \in \mathbb{R}^d; \|x\| = r\} \), for some \( r \geq 0 \).

2. The trajectory \( \langle x_n(x_0, \sigma) \rangle_{n=1}^{\infty} \) in \( \mathbb{R}^d \) tends to 0 as \( n \rightarrow \infty \) if and only if there exists a subsequence of it which tends to 0.

**Proof.** Since the sequence \( \langle \|S_{\sigma_1} \cdots S_{\sigma_n}(x_0)\| \rangle_{n=0}^{\infty} \) is nonincreasing in \( \mathbb{R} \) for any \( \sigma \in \Sigma^+_K \), it is convergent as \( n \rightarrow +\infty \). Denoted by \( r \) its limit, we have the statement (1). The statement (2) follows immediately from the statement (1). This proves Lemma 3.2.

In the case (2) of this lemma, we call the orbit \( \langle x_n(x_0, \sigma) \rangle_{n=1}^{\infty} \) with initial value \( x_0 \) asymptotically stable.

We note here that Lemma 3.2 is actually proved in [24, 3] for the continuous-time case, but [3] is under the condition that each member of \( S \) shares a common, not strict in general, quadratic Lyapunov function and [24] under an additional assumption of “paracontraction” except the Lyapunov function. In Section 3.3, we will consider the \( \omega \)-limit set of a matrix trajectory \( \langle S_{\sigma_1} \cdots S_{\sigma_n}(x_0) \rangle_{n=1}^{\infty} \). In addition, in the continuous-time case, \( \omega(x_0, \sigma) \) is a connected set. This is an important property needed in [24, 3].

For a given switching signal, to consider the stability of the corresponding switched system, we need to classify which kind of initial values in \( \mathbb{R}^d \) makes the corresponding orbits asymptotically stable. It is difficult to have such classification for a general switching signal. In the following, for the recurrent switching signal, we have a classification result.

3.2. Decomposition for general extremal norm

In this subsection, we will introduce a preliminary splitting theorem of the state space \( \mathbb{R}^d \) which plays the key in our classification.

First, we recall from [21, 27] that for a topological dynamical system \( T: \Omega \rightarrow \Omega \) on a separable metrizable space \( \Omega \), a point \( w \in \Omega \) is called “recurrent”, provided that one can find a
positive integer sequence \( n_i \to +\infty \) such that \( T^n(w) \to w \) as \( i \to +\infty \). And \( w \in \Omega \) is said to be "weakly Birkhoff recurrent" [29] (also see [10]), provided that for any \( \varepsilon > 0 \), there exists an integer \( N_\varepsilon > 1 \) such that

\[
\sum_{i=0}^{N_\varepsilon - 1} I_{B(w, \varepsilon)}(T^i(w)) \geq j \quad \forall j \in \mathbb{N},
\]

where \( I_{B(w, \varepsilon)} : \Omega \to [0, 1] \) is the characteristic function of the open ball \( B(w, \varepsilon) \) of radius \( \varepsilon \) centered at \( w \) in \( \Omega \). We denote by \( R(T) \) and \( W(T) \), respectively, the set of all recurrent points and weakly Birkhoff recurrent points of \( T \). It is easy to see that \( R(T) \) and \( W(T) \) both are invariant under \( T \) and \( W(T) \subset R(T) \).

In the qualitative theory of ordinary differential equation, this type of recurrent point is also called a "Poisson stable" motion, for instance, in [21].

For the one-sided Markov shift \( (\Sigma_k, \theta) \), it is easily checked that every periodically switched signal is recurrent. And \( \sigma = (\sigma_i)_{i=0}^{\infty} \in R(\theta) \) means that there exists a subsequence \( n_i \to +\infty \) such that \( \theta^{n_i}(\sigma) \to \sigma \) as \( i \to +\infty \). This implies that

\[
S_{\sigma_i} \cdots S_{\sigma_1} \to S_{\sigma_i} \cdots S_{\sigma_1} \quad \text{as } i \to +\infty
\]

for any \( n \geq 1 \). We should note that for any two finite-length words \( w \neq w' \), the switching signal \( \sigma = (w', w, w, w, \ldots) \) is not recurrent.

For any function \( A : \Omega \to \mathbb{R}^{d \times d} \), the cocycle \( A_T : \mathbb{N} \times \Omega \to \mathbb{R}^{d \times d} \) driven by \( T \) is defined as

\[
A_T(n, w) = A(T^{n-1}w) \cdots A(w)
\]

for any \( n \geq 1 \) and all \( w \in \Omega \). Now, our basic decomposition theorem can be stated as follows:

**Theorem 3.3** ([13, Theorem B’]). Let \( T : \Omega \to \Omega \) be a continuous transformation of a separable metrizable space \( \Omega \). Let \( A : \Omega \to \mathbb{R}^{d \times d} \) be a continuous family of matrices with the property that there exists a norm \( \| \cdot \| \) such that

\[
\|A_T(n, w)\| \leq 1 \quad \forall n \geq 1 \text{ and } w \in \Omega.
\]

Then for any recurrent point \( w \) of \( T \), there corresponds a splitting of \( \mathbb{R}^d \) into subspaces

\[
\mathbb{R}^d = E'(w) \oplus E''(w),
\]

such that

\[
\lim_{n \to +\infty} \|A_T(n, w)(x)\| = 0 \quad \forall x \in E'(w)
\]

and

\[
\|A_T(n, w)(x)\| = \|x\| \quad \forall n \geq 1 \quad \forall x \in E''(w).
\]

Here \( \| \cdot \| \) does not need to be a Lyapunov norm \( \| \cdot \|_p \) as in (1.3b) and further the central manifold \( E''(\sigma) \) is not necessarily unique and invariant. Although \( \|A_T(n, w)E'(w)\| \) converges to 0, yet \( \|A_T(n, w)E'(w)\| \) does not need to converge exponentially fast, as is shown by [13, Example 4.6].

However, under the assumptions of Theorem 3.3, if \( w \) is a weakly Birkhoff recurrent point of \( T \), we have the following alternative results:
\textbf{Theorem 3.4.} Let $T: \Omega \to \Omega$ be a continuous transformation of a separable metrizable space $\Omega$. Let $A: \Omega \to \mathbb{R}^{d \times d}$ be a continuous family of matrices with the property that there exists a norm $\| \cdot \|$ such that $\|A_T(n, w)\| \leq 1$ for all $n \geq 1$ and $w \in \Omega$. If $w \in \Omega$ is a weakly Birkhoff recurrent point of $T$, then either

\[
\|A_T(n, w)\| \xrightarrow{\text{exponentially fast}} 0 \quad \text{as } n \to +\infty,
\]

or

\[
\|A_T(n, T^i(w))\| = 1 \quad \forall i \geq 0 \quad \text{for } n \geq 1.
\]

\textit{Proof.} If there exist $i \geq 0$ and $n \geq 1$ such that $\|A_T(n, T^i(w))\| < 1$ then from $T^i(w) \in W(T)$ and [10, Theorem 2.4], it follows that

\[
\|A_T(n, T^i(w))\| \xrightarrow{\text{exponentially fast}} 0 \quad \text{as } m \to +\infty.
\]

This completes the proof of Theorem 3.4. \qed

3.3. Decomposition under a weak Lyapunov matrix

For a recurrent switching signal $\sigma = (\sigma_i)_{i=1}^{\infty}$ of $S$, to consider its stability, it is essential to compute the stable manifold $E^s(\sigma)$. From the proof of Theorem 3.3 presented in [13], we know that $E^s(\sigma)$ is the kernel of an idempotent matrix that is a limit point of $S_{\sigma_i} \cdots S_{\sigma_1}$ with $\theta^i(\sigma) \to \sigma$ as $i \to +\infty$.

However, in applications, it is not easy to identify which subsequence $(n_i)_{i=1}^{\infty}$ with this property. In this subsection, instead of the product boundedness, we assume the more strong condition (1.3a) with induced norm $\| \cdot \|_p$ on $\mathbb{R}^d$.

In this case, we can calculate the stable manifold $E^s(\sigma)$ for any switching signal $\sigma$ (not necessarily recurrent) of $S$. To do this end, we first consider the geometry of the limit sets $\omega(x_0, \sigma)$ of $S$ driven by $\sigma$. For the similar results in continuous-time switched linear systems, see [3].

For any switching signal $\sigma = (\sigma_i)_{i=1}^{n_{\text{max}}} \in \Sigma^*_K$, on the other hand, we will consider the sequence $(S_{\sigma_i} \cdots S_{\sigma_1})_{i=1}^{n_{\text{max}}}$ of matrices and let $\omega(\sigma)$ denote the set of all limit points of this sequence in $\mathbb{R}^{d \times d}$.

\textbf{Definition 3.5 ([28, 3])}. The set $\omega(\sigma)$ is called the $\omega$-limit set of $S$ driven by $\sigma$, for any $\sigma \in \Sigma^*_K$.

From condition (1.3a), it follows immediately that $\omega(\sigma)$ is non-empty and compact. But it may not be a semigroup in the sense of matrix multiplication when $\sigma$ is not a recurrent switching signal. We note that if $\sigma \in \mathbb{R}(\theta)$ then from the proof of [13, Theorem 4.2], $\omega(\sigma)$ contains a nonempty compact semigroup and so there is an idempotent element in $\omega(\sigma)$.

Parallel to Lemma 3.2, we can obtain the following result.

\textbf{Lemma 3.6.} Under condition (1.3a), there follows the following statements.

(a) For any switching signal $\sigma \in \Sigma^*_K$ of $S$, it holds that

\[
\omega(\sigma) \subset \{M \in \mathbb{R}^{d \times d}: \|M\|_p = r\},
\]

for some constant $0 \leq r \leq 1$; if $\sigma$ is further recurrent, then either $r = 0$ or 1.
Proof. We first note that from (3.1a) and (3.1b), it follows immediately that $\|S_i\|_p \leq 1$ for all indices $1 \leq k \leq K$.

For the statement (b), we let $(x_0, \sigma) \in \mathbb{R}^d \times \Sigma_k^+$ be arbitrary. If $M \in \omega(\sigma)$, it is clear that $M(x_0) \in \omega(\sigma)$. Conversely, let $y \in \omega(\sigma)$ be arbitrary. By the definition of $\omega(x_0, \sigma)$ there exists an increasing sequence $(n_i)$ such that

$$y = \lim_{i \to \infty} S_{\sigma_{n_i}} \cdots S_{\sigma_1}(x_0).$$

The product boundedness condition implies that the sequence $(S_{\sigma_{n_i}} \cdots S_{\sigma_1})_{i=1}^{\infty}$ has a convergent subsequence, whose limit element is denoted by $M$. Thus $y = M(x_0)$.

For the statement (c) of Lemma 3.6, let $M, N \in \omega(\sigma)$ be arbitrary. As $\|S_i\|_p \leq 1$ for all $1 \leq k \leq K$, from Lemma 3.2 we have

$$\|M(x)\|_p = \|N(x)\|_p \quad \forall x \in \mathbb{R}^d.$$

That is

$$x^T (M^T PM - N^T PN)x = 0 \quad \forall x \in \mathbb{R}^d.$$ It follows, from the symmetry of the matrix $M^T PM - N^T PN$, that

$$M^T PM = N^T PN.$$ This proves the statement (c) of Lemma 3.6.

Finally, the statement (a) of Lemma 3.6 comes from the statement (c) and Theorem 3.3. In fact, let $M, N \in \omega(\sigma)$ be arbitrary. Then there are vectors $x, y \in \mathbb{R}^d$ such that

$$\|x\|_p = \|y\|_p = 1, \quad \|M\|_p = \|M(x)\|_p, \quad \text{and} \quad \|N\|_p = \|N(y)\|_p.$$ So, from (c) it follows that

$$\|M\|_p = \sqrt{x^T M^T P M x} = \sqrt{x^T N^T P N x} \leq \|N\|_p = \sqrt{y^T N^T P N y} = \sqrt{y^T M^T P M y} \leq \|M\|_p.$$ This together with Theorem 3.3 proves the statement (a) of Lemma 3.6.

Thus the proof of Lemma 3.6 is completed. \qed

Let $M \in \omega(\sigma)$. Then $\sqrt{M^T PM}$ is a nonnegative-definite matrix which does not depend on the choice of the matrix $M \in \omega(\sigma)$ by the statement (c) of Lemma 3.6 and is uniquely decided by the switching signal $\sigma$. So, we write

$$Q_\sigma = \sqrt{M^T PM} \quad \forall M \in \omega(\sigma). \quad (3.1)$$

The continuous-time case of the following statement (1) of Proposition 3.7 has already been proved by Balde and Jouan [3, Theorem 1] using the polar decomposition of matrices.
Proposition 3.7. Under condition (1.3a), for any switching signal \( \sigma = (\sigma_n)_{n=1}^{\infty} \) of \( S \), there hold the following two statements.

1. The switching signal \( \sigma \) is asymptotically stable for \( S \); that is,
   \[
   \lim_{n \to \infty} S_{\sigma_n} \cdots S_{\sigma_1}(x_0) = 0 \quad \forall x_0 \in \mathbb{R}^d,
   \]
   if and only if \( Q_\sigma = 0 \);

2. If \( Q_\sigma \neq 0 \), then
   \[
   \lim_{n \to \infty} \|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\|_p = \|Q_\sigma(x_0)\|_2 \quad \forall x_0 \in \mathbb{R}^d.
   \]

So, the stable manifold of \( S \) at \( \sigma \) is such that \( E^s(\sigma) = \text{kernel of } Q_\sigma \); that is

\[
\lim_{n \to \infty} \|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\|_p = 0 \quad \forall x_0 \in E^s(\sigma).
\]

Here \( \| \cdot \|_2 \) denotes the Euclidean vector norm on \( \mathbb{R}^d \).

Proof. The statement (1) holds trivially from the statement (a) of Lemma 3.6 or from the statement (2) to be proved soon. We next will prove the statement (2). For that, let \( Q_\sigma \neq 0 \). For an arbitrary \( x_0 \in \mathbb{R}^d \), by the definition of \( Q_\sigma \) as in (3.1) there exists a subsequence \( (n_i)_{i \geq 1} \) and some \( M \in \omega(\sigma) \) such that

\[
\lim_{i \to \infty} \|S_{\sigma_{n_i}} \cdots S_{\sigma_1}(x_0)\|_p = \|M(x_0)\|_p = \sqrt{x_0^T Q_\sigma^2 x_0} = \sqrt{x_0^T Q_\sigma^2 Q_\sigma x_0} = \|Q_\sigma(x_0)\|_2.
\]

Therefore, by (1.3a) we have

\[
\lim_{n \to \infty} \|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\|_p = \|Q_\sigma(x_0)\|_2.
\]

This thus completes the proof of Proposition 3.7.

We note here that if \( Q_\sigma \) is idempotent, then from Proposition 3.7 we have \( E^s(\sigma) = \text{Im}(Q_\sigma) \) and \( \mathbb{R}^d = E^s(\sigma) \oplus E^s(\sigma) \). Because in general there lacks the recurrence of \( \sigma \), one cannot define a central manifold \( E^s(\sigma) \) satisfying \( \mathbb{R}^d = E^s(\sigma) \oplus E^s(\sigma) \) as done in Theorem 3.3. However, we will establish another type of splitting theorem in the case \( d = 2 \) for \( S \) driven by a general switching signal, not necessarily recurrent.

For that, we first introduce several notations for the sake of our convenience. For any given \( A \in \mathbb{R}^{d \times d} \) and any vector norm \( \| \cdot \| \) on \( \mathbb{R}^d \), write

\[
\|A\|_{co} = \min \left\{ \|A(x)\| : x \in \mathbb{R}^d \text{ with } \|x\| = 1 \right\}, \tag{3.2}
\]

called the co-norm (also minimum norm in some literature) of \( A \) under \( \| \cdot \| \).

Definition 3.8. Under condition (1.3a), for any switching signal \( \sigma \in \Sigma^\tau_+ \) the numbers

\[
r_E(\sigma) := |M|_p \quad \text{and} \quad r_I(\sigma) := |M|_{p,co},
\]

for \( M \in \omega(\sigma) \), are called the \( \omega \)-exterior and \( \omega \)-interior radii of \( S \) driven by \( \sigma \), respectively.
Lemma 3.9. Under the Lyapunov norm \(\|\cdot\|_p\) as in (1.3b), there \(K_{\|\cdot\|_{p\infty}}(A)\) and \(K_{\|\cdot\|_{p\sigma}}(A)\) both are linear subspaces of \(R^d\) for any \(A \in \mathbb{R}^{d \times d}\).

**Proof.** Let \(A \in \mathbb{R}^{d \times d}\) be arbitrarily given. By definitions, we have

\[
x \in K_{\|\cdot\|_{p\sigma}}(A) \iff x^T |A|_p Px - x^T A^T PAx = 0
\]

\[
\iff x^T (|A|_p P - A^T PA)x = 0
\]

\[
\iff |G(x)|_2 = 0
\]

\[
\iff x \in \ker(G).
\]

Here \(G^2 = |A|_p P - A^T PA \geq 0\) is symmetric. Since \(\ker(G)\), the kernel of \(x \mapsto Gx\), is a linear subspace of \(R^d\), \(K_{\|\cdot\|_{p\sigma}}\) is also a linear subspace of \(R^d\).

On the other hand, for any \(x \in \mathbb{R}^d\) we have \(|A(x)|_p \geq |A|_{p\co} \cdot \|x\|_p\). So,

\[
x^T (A^T PA - |A|_{p\co} P)x \geq 0 \quad \forall x \in \mathbb{R}^d.
\]

Let \(H^2 = A^T PA - |A|_{p\co} P\), which is symmetric and nonnegative-definite. Then it holds that \(K_{\|\cdot\|_{p\co}}(A) = \ker(H)\), a linear subspace.

Thus, the proof of Lemma 3.9 is completed.

Now, the improved splitting theorem can be stated as follows:

**Theorem 3.10.** Let \(S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{2 \times 2}\) satisfy condition (1.3a). Then, for any switching signal \(\sigma \in \Sigma_K\), not necessarily recurrent, there exists a splitting of \(\mathbb{R}^2\) into subspaces

\[
\mathbb{R}^2 = K_{\|\cdot\|_{p\sigma}}(\sigma) \oplus K_{\|\cdot\|_p}(\sigma)
\]

such that

\[
\lim_{n \to +\infty} \|S_{\sigma_n} \cdots S_{\sigma_1} x_0\|_p = r_f \|x_0\|_p \quad \forall x_0 \in K_{\|\cdot\|_{p\sigma}}(\sigma),
\]

\[
\lim_{n \to +\infty} \|S_{\sigma_n} \cdots S_{\sigma_1} x_0\|_p = r_E \|x_0\|_p \quad \forall x_0 \in K_{\|\cdot\|_p}(\sigma),
\]

and

\[
r_f \|x_0\|_p < \lim_{n \to +\infty} \|S_{\sigma_n} \cdots S_{\sigma_1} x_0\|_p < r_E \|x_0\|_p \quad \forall x_0 \in \mathbb{R}^2 - K_{\|\cdot\|_{p\sigma}}(\sigma) \cup K_{\|\cdot\|_p}(\sigma).
\]
4. ASYMPTOTICAL STABILITY UNDER A WEAK LYAPUNOV MATRIX

Proof. Let \( r_l < r_E \) and \( M \in \omega(\sigma) \). Define \( \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) = \mathcal{K}_{\|\cdot\|_{L^p}}(M) \) and \( \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) = \mathcal{K}_{\|\cdot\|_{L^p}}(M) \). From the statement (2) of Proposition 3.7, it follows that \( \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) \) and \( \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) \) both are independent of the choice of \( M \). So, \( \mathbb{R}^2 = \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) \oplus \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) \) from Lemma 3.9. We note that if \( r_l = r_E \), then \( \|\|_{\|\cdot\|_{L^p}}(\sigma) = \mathbb{R}^2 \). This completes the proof of Theorem 3.10. \( \square \)

In the case where \( \sigma \) is recurrent, one can easily see that

\[
E'(\sigma) = \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma) \quad \text{and} \quad E'(\sigma) = \mathcal{K}_{\|\cdot\|_{L^p}}(\sigma).
\]

4. Asymptotical stability under a weak Lyapunov matrix

In this section, we will discuss the stability of switched linear system with a common, but not necessarily strict, quadratic Lyapunov function. In this case, a criteria for stability is derived without computing the limit matrix \( Q_\sigma \) as in (3.1). We still assume \( S \) is composed of finitely many subsystems. That is, \( S = \{S_1, \ldots, S_K\} \) with \( 2 \leq K < +\infty \).

4.1. Stability of generic recurrent switching signals

Now for \( \sigma = (\sigma_n)_{n \in \mathbb{N}} \in \Sigma^\omega_\sigma \), if \( \text{Card}(\{\sigma_n = k\}) = \infty \) for all \( 1 \leq k \leq K \) then \( \sigma \) is called “generic.” Recall that a switching signal \( \sigma = (\sigma_n)_{n \in \mathbb{N}} \in \Sigma^\omega_\sigma \) is said to be stable for \( S \) if

\[
\|S_{\sigma_1} \cdots S_{\sigma_n}\| \to 0 \quad \text{as} \quad n \to +\infty.
\]

(Note that the stability is independent of the chosen norm \( \|\cdot\| \).) As is known, a switching system which is asymptotically stable for all periodically switching signals does not need to be asymptotically stable for all switching signals in general [8, 7, 19, 17]. However we can obtain the following result.

Lemma 4.1. If all recurrent switching signals are stable for \( S \), then it is asymptotically stable driven by all switching signals in \( \Sigma^\omega_\sigma \).

Proof. Since the set \( R(\theta) \) of all recurrent switching signals has full measure 1 for all ergodic measures with respect to \( (\Sigma^\omega_\sigma, \theta) \), the result follows from [11, Lemma 2.3]. \( \square \)

By Lemma 4.1, to obtain the asymptotic stability of \( S \), it suffices to prove that it is only asymptotically stable driven by all recurrent switching signals.

In addition, we need the following lemma.

Lemma 4.2. Under condition (1.3a), if \( \|S_k\|_p = 1 \) and \( \mathcal{K}_{\|\cdot\|_{L^p}}(S_k) \) is \( S_k \)-invariant, then \( \rho(S_k) = 1 \).

Here \( \mathcal{K}_{\|\cdot\|_{L^p}}(S_k) \) is defined as in (3.3).

Proof. The statement comes obviously from Lemma 3.9. \( \square \)

In the following, for simplicity, we just consider a switched system which is composed of two subsystems. That is, \( K = 2 \).

Lemma 4.3. Under condition (1.3a) with \( K = 2 \) (i.e., \( S = \{S_1, S_2\} \)), if \( \|S_1\|_p = \|S_2\|_p = 1 \) and

\[
\mathcal{K}_{\|\cdot\|_{L^p}}(S_1) \cap \mathcal{K}_{\|\cdot\|_{L^p}}(S_2) = \{0\},
\]

and at least one of them is invariant (i.e., \( S_1(\mathcal{K}_{\|\cdot\|_{L^p}}(S_1)) = \mathcal{K}_{\|\cdot\|_{L^p}}(S_1) \) or \( S_2(\mathcal{K}_{\|\cdot\|_{L^p}}(S_2)) = \mathcal{K}_{\|\cdot\|_{L^p}}(S_2) \)), then every generic switching signal is stable for \( S \).
4 ASYMPTOTICAL STABILITY UNDER A WEAK LYAPUNOV MATRIX

Proof. Assume that $\mathbb{K}_{\|\cdot\|_P}(S_1)$ is $S_1$-invariant. (Otherwise, if $\mathbb{K}_{\|\cdot\|_P}(S_2)$ is $S_2$-invariant, the proof is the same.) Let $\sigma = (\sigma_n)_{n=1}^{\infty}$ be a generic switching signal; that is, in $(\sigma_n)_{n=1}^{\infty}$, both 1 and 2 appear infinitely many times. Then there exists a subsequence $(\sigma_{n_i})$ such that

$$\sigma_{n_i} = 1 \quad \text{and} \quad \sigma_{n_i+1} = 2 \quad \forall i \geq 1.$$  

For a given initial value $x_0 \in \mathbb{R}^d$, consider the subsequence $\{S_{\sigma_{n_i}-1} \cdots S_{\sigma_1}(x_0)\}_{i=1}^{\infty}$. By the assumption (1.3a), it has a convergent subsequence in $\mathbb{R}^d$. Without loss of generality, we assume that

$$S_{\sigma_{n_i}-1} \cdots S_{\sigma_1}(x_0) \to y \in \mathbb{R}^d \quad \text{as} \quad i \to +\infty.$$  

Thus

$$S_{\sigma_{n_i}+1}S_{\sigma_{n_i}}S_{\sigma_{n_i}-1} \cdots S_{\sigma_1}(x_0) \to S_2S_1(y),$$

as $i \to +\infty$. By the statement (1) of Lemma 3.2, we have

$$\|S_2S_1(y)\|_P = \|S_1(y)\|_P = \|y\|_P.$$  

Thus $y \in \mathbb{K}_{\|\cdot\|_P}(S_1)$ and $S_1(y) \in \mathbb{K}_{\|\cdot\|_P}(S_1)$. From the $S_1$-invariance of $\mathbb{K}_{\|\cdot\|_P}(S_1)$ it follows that

$$S_1(y) \in \mathbb{K}_{\|\cdot\|_P}(S_1) \cap \mathbb{K}_{\|\cdot\|_P}(S_2).$$

So $S_1(y) = 0$ and so is $y$. From the statement (2) of Lemma 3.2, we have

$$S_{\sigma_n} \cdots S_{\sigma_1}(x_0) \to 0 \quad \text{as} \quad n \to +\infty.$$  

That is, $\sigma$ is a stable switching signal for $S$. This proves Lemma 4.3.  

Both $E(\sigma)$ in [10, §5.2.2] and $V_1$ in [3, Lemma 1] are invariant. Unfortunately, here our subspace $\mathbb{K}_{\|\cdot\|_P}(S_k)$ does not need to be $S_k$-invariant in general. See Example 6.2 in Section 6. If this is the case, we still have, however, the following criterion.

**Theorem 4.4.** Under conditions (1.3a) and (4.1) with $S = \{S_1, S_2\} \subset \mathbb{R}^{d \times d}$, the following two statements hold.

1. If $d = 2$, then all generic recurrent switching signals $\sigma \in \Sigma_2^+$, which satisfy

$$\sigma \neq (1, 2, 1, 2, \ldots),$$

are stable for $S$.

2. If $d = 3$, then all generic recurrent switching signals $\sigma \in \Sigma_3^+$ such that

$$\sigma \neq (w, w, w, \ldots), \quad \text{where} \ w \in \{(1, 2), (2, 1), (1, 2, 2), (2, 1, 1)\},$$

are stable for $S$.  

4 ASYMPTOTICAL STABILITY UNDER A WEAK LYAPUNOV MATRIX

Proof. First, if \( \|S_1\|_p < 1 \) or \( \|S_2\|_p < 1 \), then every generic switching signal is stable for \( S \) and hence the statements (1) and (2) trivially hold. So, we next assume \( \|S_1\|_p = \|S_2\|_p = 1 \). This implies that \( \dim \mathbb{R}^{k+1} = 1 \) for \( k = 1, 2 \).

For the statement (1) of Theorem 4.4, from (4.1) it follows that \( \dim \mathbb{R}^{k+1} = 1 \) for \( k = 1, 2 \). Let \( \sigma = (\sigma_n)_{n=1}^{\infty} \) be a given generic recurrent switching signal such that

\[
\sigma(\cdot + n) \neq (1, 2, 1, 2, 1, 2, \ldots) \quad \forall n \geq 1.
\]

(4.2)

From Theorem 3.3, there corresponds a splitting of \( \mathbb{R}^2 \) into subspaces

\[
\mathbb{R}^2 = E'(\sigma) \oplus E''(\sigma),
\]

such that

\[
\lim_{n \to +\infty} \|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\|_p = 0 \quad \forall x_0 \in E'(\sigma)
\]

and

\[
\|S_{\sigma_n} \cdots S_{\sigma_1}(x_0)\|_p = \|x_0\|_p \quad \forall n \geq 1 \quad \forall x \in E''(\sigma).
\]

To prove that \( \sigma \) is a stable switching signal for \( S \), we need to prove that \( E'(\sigma) = \{0\} \). By the genericity of \( \sigma \) and (4.2), \( \sigma \) must contain the word (1, 1, 2) or (2, 2, 1). Without loss of generality, we assume that

\[
(\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2).
\]

Thus we have

\[
\|S_2S_1(x_0)\|_p = \|S_1(x_0)\|_p = \|x_0\|_p \quad \forall x_0 \in E''(\sigma)
\]

These imply that

\[
\{x_0, S_1(x_0) \in \mathbb{R}^{k+1}, S_1(x_0) \in \mathbb{R}^{k+1}, S_2 \}
\]

Suppose that \( x_0 \neq 0 \). It follows from \( \dim \mathbb{R}^{k+1} = 1 \) that there exists a real number \( \lambda \) with \( |\lambda| = 1 \) such that

\[
S_1(x_0) = \lambda x_0.
\]

This means that \( x_0 \) is an eigenvector of \( S_1 \) with eigenvalue \( \lambda \). So

\[
S_1S_1(x_0) = \lambda^2 x_0 \in \mathbb{R}^{k+1}.
\]

Therefore

\[
S_1(x_0) \in \mathbb{R}^{k+1} \cap \mathbb{R}^{k+1} = \{0\}.
\]

Thus we have \( S_1S_1(x_0) = 0 \), which implies \( x_0 = 0 \), a contradiction.

Next, for proving the statement (2) of Theorem 4.4 that \( d = 3 \), by (4.1), we have that one of \( \mathbb{R}^{k+1} \), \( \mathbb{R}^{k+1} \) has dimension 1 and the other has dimension at least 1 and at most 2.

If both \( \mathbb{R}^{k+1} \) and \( \mathbb{R}^{k+1} \) have dimension 1, then by the same argument as in the statement (1), all generic recurrent switching signals satisfying (4.2) are stable for \( S \).

Next, we assume that, for example,

\[
\dim \mathbb{R}^{k+1} = 1 \quad \text{and} \quad \dim \mathbb{R}^{k+1} = 2.
\]
We claim that for any generic recurrent switching signal \( \sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma^+, \) if
\[
\sigma(\cdot + n) \notin \{ (0, 1, 2, \ldots, 0.1, \ldots), (0, 1, 2, 0, 1, 2, \ldots, 0, 1, 2, \ldots) \} \quad \forall n \geq 1. \tag{4.3}
\]
then \( \sigma \) is stable for \( S \). There is no loss of generality in assuming \( \sigma_1 = 1 \); otherwise replacing \( \sigma \) by \( \sigma(\cdot + n) \) for some \( n \geq 1 \). Then,
\[
K_{\| \cdot \|_p}(S_1) = E'(\sigma) \quad \text{if} \ E'(\sigma) \neq \{0\},
\]
where \( E'(\sigma) \) is given by Theorem 3.3.

Whenever the word 11 appears in the sequence \( (\sigma_n)_{n=1}^{\infty} \), \( K_{\| \cdot \|_p}(S_1) \) is \( S \)-invariant. Then, Lemma 4.3 follows that \( \sigma \) is stable for \( S \). Next, we assume 11 does not appear in \( (\sigma_n)_{n=1}^{\infty} \). If 121 appears in \( (\sigma_n)_{n=1}^{\infty} \) then \( 121212 \cdots \) must appear too, a contradiction. So, 121 cannot appear in \( (\sigma_n)_{n=1}^{\infty} \). Then 122 must appear. If 1221 appears in \( (\sigma_n)_{n=1}^{\infty} \) then \( 12212212 \cdots \) must appear too, a contradiction. Thus, the word 1222 must appear in \( (\sigma_n)_{n=1}^{\infty} \).

When \( \sigma \) contains the word \( (2, 2, 2, 1) \), assume that, for example,
\[
(\sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}, \sigma_{n+4}) = (2, 2, 2, 1).
\]
Then we have
\[
\|S_1S_2S_2S_2(x_0)\|_p = \|S_2S_2S_2(x_0)\|_p = \|S_2S_2(x_0)\|_p = \|S_2(x_0)\|_p = \|x_0\|_p \quad \forall x_0 \in E'(\sigma(\cdot + n)),
\]
which show that for all \( x_0 \in E'(\sigma(\cdot + n)) \),
\[
\{ x_0, S_2(x_0), S_2S_2(x_0) \} \subset K_{\| \cdot \|_p}(S_2), \quad S_2S_2S_2(x_0) \in K_{\| \cdot \|_p}(S_1).
\]
If \( x_0 \) and \( S_2(x_0) \) are linear dependent, that is,
\[
S_2(x_0) = \lambda x_0,
\]
for some \( \lambda \) with \( |\lambda| = 1 \), then \( S_2S_2S_2(x_0) = \lambda x_0 \in K_{\| \cdot \|_p}(S_2) \). So
\[
S_2S_2S_2(x_0) \in K_{\| \cdot \|_p}(S_1) \cap K_{\| \cdot \|_p}(S_2) = \{0\},
\]
which implies that \( x_0 = 0 \). On the other hand, if \( x_0 \) and \( S_2(x_0) \) are linear independent, then
\[
S_2S_2(x_0) = \lambda x_0 + \alpha S_2(x_0),
\]
for some \( \lambda \) and \( \alpha \), since \( \dim K_{\| \cdot \|_p}(S_2) = 2 \). Thus \( S_2S_2S_2(x_0) \) is a linear combination of \( S_2(x_0) \) and \( S_2S_2(x_0) \). So it is also in \( K_{\| \cdot \|_p}(S_2) \). Therefore
\[
S_2S_2S_2(x_0) \in K_{\| \cdot \|_p}(S_1) \cap K_{\| \cdot \|_p}(S_2) = \{0\},
\]
which shows \( x_0 = 0 \). Thus \( E'(\sigma(\cdot + n)) = \{0\} \) and then \( E'(\sigma) = \{0\} \).

Similarly, when \( \dim K_{\| \cdot \|_p}(S_1) = 2 \) and \( \dim K_{\| \cdot \|_p}(S_2) = 1 \), we can prove that all generic recurrent switching signals, but the following four periodic switching signals
\[
(1, 1, 1, \ldots), (2, 2, 2, \ldots), (\widehat{2, 1, 2, 1, \ldots}), (2, 1, 1, 2, 1, 1, \ldots),
\]
are stable for \( S \).

This completes the proof of Theorem 4.4. \( \square \)
5 Absolute Stability of a Pair of Matrices

We have the following remarks on Theorem 4.4.

**Remark 1.** Similarly, we can consider a switched linear system composed of two subsystems on \( \mathbb{R}^d \) with \( d \geq 4 \). In this case, under the assumptions (1.3a) and (4.1), if either \( \|K\|_r(S_1) \) or \( \|K\|_r(S_2) \) has dimension 1, then all generic recurrent switching signals but finitely many periodic signals are stable for \( S \).

**Remark 2.** Under the assumptions on Theorem 4.4, in order to obtain the stability for all recurrent switching signals, we just need to check finitely many periodic signals to see whether they are stable for \( S \).

**Remark 3.** Theorem 4.4 suggests an easy computable sufficient condition of asymptotically stable for switched linear systems which are composed of two subsystems. In fact, Remark 2 provides a direct way to check the stability of all recurrent signals, which implies the asymptotically stable of the systems by Lemma 4.1.

We can also discuss the stability of switched linear systems composed of finite many subsystems similarly. But it is troublesome to formulate the corresponding assumptions. Here we will give an example to illustrate such conditions in Section 6.

4.2 Almost sure stability

Let \((\Sigma^+_K, \mathcal{B})\) be the Borel \( \sigma \)-field of the space \( \Sigma^+_K \) and then the one-sided Markov shift map \( \theta: \sigma(\cdot) \mapsto \sigma(\cdot + 1) \) is measurable. A Borel probability measure \( P \) on \( \Sigma^+_K \) is said to be \( \theta \)-invariant, if \( P = P \circ \theta^{-1} \), i.e. \( P(B) = P(\theta^{-1}(B)) \) for all \( B \in \mathcal{B} \). A \( \theta \)-invariant probability measure \( P \) is called \( \theta \)-ergodic, provided that for \( B \in \mathcal{B} \), \( P((B \setminus \theta^{-1}(B)) \cup (\theta^{-1}(B) \setminus B)) = 0 \) implies \( P(B) = 1 \) or 0.

An ergodic measure \( P \) is called non-atomic, if every singleton set \( \{\sigma\} \) has \( P \)-measure 0.

Using Theorem 4.4, we can easily prove Theorem B stated in Section 1.4.

**Proof of Theorem B.** Let \( P \) be an arbitrary non-atomic \( \theta \)-ergodic measure on \( \Sigma^+_2 \). Then from the Poincaré recurrence theorem (see, e.g., [27, Theorem 1.4]), it follows that \( P \)-a.e. \( \sigma \in \Sigma^+_2 \) are recurrent. In addition, since \( P \) is non-atomic, we obtain that \( P \)-a.e. \( \sigma \in \Sigma^+_2 \) are non-periodic and generic. This completes the proof of Theorem B from Theorem 4.4. \( \square \)

We note that in the proof of Theorem B presented above, the deduction of the genericity of \( \sigma \) needs the assumption \( K = 2 \).

5. Absolute stability of a pair of matrices with a weak Lyapunov matrix

We now deal with the case \( S = \{S_1, S_2\} \subset \mathbb{R}^{2 \times d} \), where \( S_1 \) and \( S_2 \) both are stable and share a common, but not necessarily strict, quadratic Lyapunov function. For any \( A \in \mathbb{R}^{d \times d} \), we denote by \( \rho(A) \) the spectral radius of \( A \).

Our first absolute stability result Theorem C is restated as follows:

**Theorem 5.1.** Let \( S = \{S_1, S_2\} \subset \mathbb{R}^{2 \times 2} \) satisfy condition (1.3a). Then, \( S \) is absolutely stable (i.e., \( \|S_{\sigma_n} \cdots S_{\sigma_1}\| \to 0 \) as \( n \to +\infty \), for all switching signals \( \sigma \in \Sigma^+_2 \)) if and only if there holds that \( \rho(S_1) < 1, \rho(S_2) < 1, \) and \( \rho(S_1S_2) < 1. \)
5 ARBITRARY STABILITY OF A PAIR OF MATRICES

Proof. We only need to prove the sufficiency. Let \( \rho(S_1) < 1, \rho(S_2) < 1 \), and \( \rho(S_1 S_2) < 1 \). Let \( \sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma_2^1 \) be an arbitrary recurrent switching signal. Clearly, if \( \sigma \) is not generic, then it is stable for \( S \). So we assume \( \sigma \) is generic and recurrent. Then, from Theorem 3.3 there exists a splitting of \( \mathbb{R}^2 \) into subspaces:

\[
\mathbb{R}^2 = E'(\sigma) \oplus E''(\sigma).
\]

If \( \dim E'(\sigma) = 0 \), then \( \sigma \) is stable for \( S \); and if \( \dim E'(\sigma) = 2 \) then either \( \rho(S_1) = 1 \) or \( \rho(S_2) = 1 \), a contradiction. We now assume \( \dim E'(\sigma) = 1 \).

Then, \( \dim K_{||\rho|}(S_1) = 1 \) and \( \dim K_{||\rho|}(S_2) = 1 \). It might be assumed, without loss of generality, that \( \sigma_1 = 1 \) and then we have \( K_{||\rho|}(S_1) = E'(\sigma) \). From this, we see

\[
\sigma_2 = 2, \sigma_3 = 1, ..., \sigma_{2n} = 2, \sigma_{2n+1} = 1, ... .
\]

This contradicts \( \rho(S_1 S_2) = \rho(S_2 S_1) < 1 \).

Therefore, \( E'(\sigma) = [0] \) and \( S \) is absolutely stable from Lemma 4.1.

So, Theorem C is proved.

Next, we need a simple fact for considering higher dimensional cases.

Lemma 5.2 ([26, Corollary]). Let \( A \in \mathbb{R}^{d \times d} \) be a stable matrix (i.e., \( \rho(A) < 1 \)) such that

\[
D - A^T D A \geq 0
\]

for some symmetric, positive-definite matrix \( D \). Then \( D - (A^d)^T D A^d > 0 \).

This lemma refines Lemma 3.2. From it, we can obtain a simple result which improves the statement of Theorem A in the case of \( d = 2 \) and \( K = 2 \).

Corollary 5.3. Let \( S = \{ S_1, S_2 \} \subset \mathbb{R}^{2 \times 2} \) satisfy condition (1.3a). If \( \rho(S_1) < 1 \) and \( \rho(S_2) < 1 \), then for any \( \theta \)-ergodic probability measure \( \mathbb{P} \) on \( \Sigma_2^1 \), \( S \) is stable driven by \( \mathbb{P} \)-a.e. \( \sigma \in \Sigma_2^1 \) as long as \( \mathbb{P} \) satisfies \( \mathbb{P}((12, 12, 12, \ldots), (21, 21, 21, \ldots)) = 0 \).

Proof. Since \( \mathbb{P} \) is ergodic and \( \mathbb{P}((12, 12, 12, \ldots), (21, 21, 21, \ldots)) = 0 \), we have

\[
\mathbb{P}(\{\sigma \in \Sigma_2^1 | \sigma(\cdot + n) = (12, 12, 12, \ldots) \text{ or } (21, 21, 21, \ldots) \text{ for some } n \geq 1\}) = 0.
\]

Now, let \( \sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma_2^1 \) be arbitrary. Then, \( \sigma \) can consist of the following 2-length words:

\[
11, 22, 12, 21.
\]

If 11 (or 22) appears infinitely many times in \( (\sigma_n)_{n=1}^{\infty} \), then from Lemma 5.2 it follows that \( S \) is stable driven by \( \sigma \). Next, assume 11 and 22 both only appear finitely many times in \( (\sigma_n)_{n=1}^{\infty} \) and let \( a = 12 \) and \( b = 21 \). Then, one can find some \( N \geq 1 \) such that

\[
\sigma(\cdot + N) = (a, a, a, \ldots).
\]

Note here that if \( ab \) appears \( m \) times in \( (\sigma_n)_{n=1}^{\infty} \), then 22 must appear \( m \) times; if \( ba \) appears \( m \) times in \( (\sigma_n)_{n=1}^{\infty} \), then 11 must appear \( m \) times. So, \( S \) is stable driven by \( \mathbb{P} \)-a.e. \( \sigma \in \Sigma_2^1 \).

This completes the proof of Corollary 5.3.
The condition $\mathbb{P}(\{(12, 12, 12, \ldots), (21, 21, 21, \ldots)\}) = 0$ means that $\mathbb{P}$ is not distributed on the periodic orbit of the one-sided Markov shift $(\Sigma^+_{R^2}, \theta)$:

$\{(12, 12, \ldots), (21, 21, \ldots)\}.$

This corollary shows that $S$ is “completely” almost sure stable up to only one ergodic measure supported on a periodic orbit generated by the word 12.

In addition, Theorem C can be directly deduced from Corollary 5.3 and Lemma 4.1.

For the sake of our convenience, we now restate our second absolute stability result Theorem D as follows:

**Theorem 5.4.** Let $S = \{S_1, S_2\} \subset \mathbb{R}^{3 \times 3}$ satisfy condition (1.3a). Then, $S$ is absolutely stable if and only if there holds the following conditions:

\[
\begin{align*}
\rho(S_1) &< 1, & \rho(S_2) &< 1, & (C1) \\
\rho(S_1), S_2) &< 1, & (C2) \\
\rho(S_1, S_2) &< 1, & (C3) \\
\rho(S_{w_1}, S_{w_2}, S_{w_3}) &< 1 & \forall (w_1, w_2, w_3) &\in [1, 2]^3, (C4) \\
\rho(S_{w_1}, \cdots S_{w_4}) &< 1 & \forall (w_1, \ldots, w_4) &\in [1, 2]^4, (C5) \\
\rho(S_{w_1}, \cdots S_{w_5}) &< 1 & \forall (w_1, \ldots, w_6) &\in [1, 2]^5, (C6) \\
\rho(S_{w_1}, \cdots S_{w_6}) &< 1 & \forall (w_1, \ldots, w_8) &\in [1, 2]^8. (C8)
\end{align*}
\]

We note here that it is somewhat surprising that we do not need to consider the words of length 7.

**Proof.** We need to consider only the sufficiency. Let conditions (C1) – (C8) all hold. According to Lemma 4.1, we let $\sigma = (\sigma_n)_{n=1}^{\infty} \in \Sigma^+_{\mathbb{N}}$ be an arbitrary recurrent switching signal. There is no loss of generality in assuming $\sigma_1 = 1$.

It is easily seen that $0 \leq \dim K_{\|\cdot\|_p}(S_1) \leq 2$ and $0 \leq \dim K_{\|\cdot\|_p}(S_2) \leq 2$ by condition (C1). Then from Theorem 3.3 with $\|\cdot\| = \|\cdot\|_p$, there exists a splitting of $\mathbb{R}^3$ into subspaces:

\[
\mathbb{R}^3 = E'(\sigma) \oplus E'(\sigma) \quad \text{such that } \dim E'(\sigma) \leq \dim K_{\|\cdot\|_p}(S_k) \text{ for } k = 1, 2.
\]

There is only one of the following three cases occurs.

- $\dim E'(\sigma) = 2$;
- $\dim E'(\sigma) = 1$;
- $\dim E'(\sigma) = 0$.

Clearly, if $\sigma$ is not generic, then it is stable for $S$. So we let $\sigma$ be generic in what follows. We also note that $E'(\sigma) \subseteq K_{\|\cdot\|_p}(S_1)$.

Case (a): Let $\dim E'(\sigma) = 2$. Then $\dim K_{\|\cdot\|_p}(S_1) = \dim K_{\|\cdot\|_p}(S_2) = 2$ and further we have $K_{\|\cdot\|_p}(S_1) = E'(\sigma)$. If $\sigma_2 = 1$ then it follows that $K_{\|\cdot\|_p}(S_1)$ is $S_1$-invariant and so $\rho(S_1) = 1$ by Lemma 4.2, a contradiction. Thus, $\sigma_2 = 2$. If $\sigma_3 = 2$ it follows that $K_{\|\cdot\|_p}(S_2)$ is $S_2$-invariant...
and so $\rho(S_2) = 1$ by Lemma 4.2, also a contradiction. So, $\sigma_3 = 1$. Repeating this, we can see $\sigma = (1, 2, 1, 2, 1, 2, \ldots)$, a contradiction to condition (C2). Thus, the case (a) cannot occur.

Case (b): Let $\dim E'(\sigma) = 1$. (This is the most complex case needed to discussion.) We first claim that $\sigma$ does not contain any one of the following two words:

$$(1, 1, 1), (2, 2, 2).$$

In fact, without loss of generality, we let $(\sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}) = (2, 2, 2)$. Choose a vector $x \in E'(\sigma)$ with $\|x\|_P = 1$. Then, $v := \sigma_1 \cdots \sigma_n x \in K_{\|\cdot\|_P}(S_2)$ with $\|v\|_P = 1$. Moreover, $S_2(v)$ and $S_2(S_2(v))$ both belong to $K_{\|\cdot\|_P}(S_2)$ such that with $\|S_2(v)\|_P = \|S_2(S_2(v))\|_P = 1$. Since $S_2(v) \neq \pm v$ (otherwise $\rho(S_2) = 1$), we see $K_{\|\cdot\|_P}(S_2)$ is $S_2$-invariant. So, $\rho(S_2) = 1$ by Lemma 4.2, a contradiction to condition (C1).

Secondly, we claim that if $\sigma$ contains the word of the form $(1, 1, w_1, \ldots, w_m, 1, 1)$ then

$$\rho(S_{w_1}, \ldots, S_{w_m}, S_1 S_1) = 1;$$

and if $\sigma$ contains the word of the form $(2, 2, w_1, \ldots, w_m, 2, 2)$ then

$$\rho(S_{w_1}, \ldots, S_{w_m}, S_2 S_2) = 1.$$

In fact, without loss of generality, we assume that

$$\sigma = (1, \sigma_2, \ldots, \sigma_n, 2, 2, w_1, \ldots, w_m, 2, 2, \ldots).$$

Then, take arbitrarily a vector $x \in E'(\sigma)$ with $\|x\|_P = 1$ and write $v_n := \sigma_1 \cdots \sigma_n x$. So, $v_n$ and $S_2(v_n)$ both belong to $K_{\|\cdot\|_P}(S_2)$ such that $\|v_n\|_P = \|S_2(v_n)\|_P = 1$. On the other hand, $v' := S_{w_1} \cdots S_{w_m} S_2 S_2(v_n)$ and $S_2(v')$ both belong to $K_{\|\cdot\|_P}(S_2)$ with $\|v'\|_P = \|S_2(v')\|_P = 1$. If $v_n \neq \pm v'$ then $K_{\|\cdot\|_P}(S_2)$ is $S_2$-invariant and so $\rho(S_2) = 1$ by Lemma 4.2, a contradiction to condition (C1). Thus, we have $v_n = \pm v'$ and then $\rho(S_{w_1}, \ldots, S_{w_m}, S_2 S_2) = 1$.

Thirdly, we show the case (b), i.e., $\dim E'(\sigma) = 1$, does not occur too. In fact, from the above claims, it follows that $\sigma = (\sigma_n)_{n=1}^{\infty}$ only possesses the following forms:

$$
\begin{align*}
1 & \to \begin{cases} 12 \to \cdots \text{(case (A))} \\
2 & \to \begin{cases} 1 \to \cdots \text{(case (B))} \\
21 \to \cdots \text{(case (C))} \end{cases}
\end{cases}
\end{align*}
$$

(5.1)

Here and in the sequel, “$a \to b$” means that $b$ follows $a$; i.e., $\sigma_n = a$ and $\sigma_{n+1} = b$ for some $n$. For example, in the above figure, “$1 \to 2 \to 21$” means $\sigma_1 = 1, \sigma_2 = 2$ and $(\sigma_3, \sigma_4) = (2, 1)$. In addition, in the following three figures, the symbol “$\times$” means “This case does not happen.”
Thus, \((\sigma_1, \sigma_2, \sigma_3) \neq (1, 1, 2)\) and then
\[(\sigma_1, \sigma_2) \neq (1, 1). \tag{5.2} \]

For the case (C) in the figure (5.1), we have
\[
\begin{align*}
12 & \rightarrow \\
& \left\{ \begin{array}{l}
1 \ (\times \text{by (C3)}) \\
2 \ (\times \text{by (C4)}) \\
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
1221 & \rightarrow \\
& \left\{ \begin{array}{l}
12 \rightarrow \\
2 \rightarrow \\
\end{array} \right.
\end{align*}
\]
Thus \((\sigma_1, \sigma_2, \sigma_3, \sigma_4) \neq (1, 2, 2, 1)\) and then \((\sigma_1, \sigma_2, \sigma_3) \neq (1, 2, 2). \tag{5.3} \]
Finally, for the case (B) in the figure (5.1),
Thus, \((\sigma_1, \sigma_2, \sigma_3) \neq (1, 2, 1)\). Further, from (5.3) it follows \((\sigma_1, \sigma_2) \neq (1, 2)\). This together with (5.2) implies that \((\sigma_1, \sigma_2) \notin [(1, 1), (1, 2)]\), a contradiction.

So, \(\dim E'(\sigma) \neq 1\) and hence case (b) does not occur. Therefore, \(\dim E'(\sigma) = 0\). This implies that \(\sigma\) is stable for \(S\). Therefore \(S\) is absolutely stable from Lemma 4.1.

This completes the proof of Theorem 5.4. \(\square\)

6. Examples

We in this section shall give several examples to illustrate applications of our results. In what follows, let \(\|\cdot\|\) be the usual Euclidean norm on \(\mathbb{R}^d\); that is, \(P = I_d\) in (1.3b).

First, a very simple example is the following.

Example 6.1. Let \(S = \{S_1, S_2\}\) with

\[
S_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad S_2 = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(|\alpha| < 1, \ |\beta| < 1\). It is easy to see that

\[\|S_1\|_2 = \|S_2\|_2 = 1,\]

and that \(\mathcal{K}_{\|\cdot\|}(S_1) = \{(x_1, 0)^T \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}, \mathcal{K}_{\|\cdot\|}(S_2) = \{(0, x_2)^T \in \mathbb{R}^2 \mid x_2 \in \mathbb{R}\}\). So, we can obtain that \(\mathcal{K}_{\|\cdot\|}(S_1) \cap \mathcal{K}_{\|\cdot\|}(S_2) = \{0\}\) and \(\mathcal{K}_{\|\cdot\|}(S_1)\) is \(S_1\)-invariant. Thus the switched linear system \(S\) is asymptotically stable for all switching signals in which each \(k\) in \(\{1, 2\}\) is stable by Lemma 4.3. Also, from Theorem 4.4, it follows that all recurrent signals but the fixed signals \((1, 1, 1, \ldots)\) and \((2, 2, 2, \ldots)\) are stable for \(S\). We note here that the periodic switching signal \((1, 2, 1, 2, \ldots)\) is stable for \(S\).

A more interesting example is the following.

Example 6.2. Let \(S = \{S_1, S_2\}\) with

\[
S_1 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S_2 = \beta \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},
\]

where

\[
\alpha = \sqrt{\frac{3 - \sqrt{5}}{2}}, \quad \beta = \frac{1}{2}.
\]

Then, \(\|S_1\|_2 = \|S_2\|_2 = 1\). A direct computation shows that

\[
\mathcal{K}_{\|\cdot\|}(S_1) = \left\{(x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 = \frac{\sqrt{5} + 1}{2}x_2\right\}
\]

\[
\mathcal{K}_{\|\cdot\|}(S_2) = \left\{(x_1, x_2)^T \in \mathbb{R}^2 \mid x_2 = 2x_1\right\}.
\]

Thus \(\mathcal{K}_{\|\cdot\|}(S_1) \cap \mathcal{K}_{\|\cdot\|}(S_2) = \{0\}\). But they are not invariant. Thus \(S\) is asymptotically stable for all generic recurrent switching signals but the periodic signal \((1, 2, 1, 2, \ldots)\) by Theorem 4.4. Note that the two subsystems themselves are asymptotically stable.
6. Examples

Next, we give an example which is the discretization of the switched linear continuous system borrowed from [3].

Example 6.3. Let $S = \{S_1, S_2, S_3\}$ with

\[
S_1 = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad S_3 = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \alpha
\end{pmatrix},
\]

where $|\alpha| < 1$. It is easy to see that $\|S_1\|_2 = \|S_2\|_2 = \|S_3\|_2 = 1$ and

\[
K_{\|\cdot\|_F}(S_1) = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = 0\}, \\
K_{\|\cdot\|_F}(S_2) = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = x_2 = 0\}, \\
K_{\|\cdot\|_F}(S_3) = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = x_3 = 0\}.
\]

Since $K_{\|\cdot\|_F}(S_2) \cap K_{\|\cdot\|_F}(S_3) = \{0\}$ and they are invariant respect to $S_2$ and $S_3$, respectively, we have that any generic switching signal in which either the word $(2, 3)$ or the $(3, 2)$ appears infinitely many times are stable by Lemma 4.3. For the any other generic switching signals $\sigma = (\sigma_1, \sigma_2, \ldots)$, that is, in which both the word $(2, 3)$ and the $(3, 2)$ appear at most finite many times, the matrix $Q_{\sigma}$ defined in (3.1) is

\[
Q_{\sigma} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & a^{k_0} \\
0 & 0 & 0
\end{pmatrix},
\]

for some nonnegative integers $k_0$ and $j_0$ which depend on the times of appearance of $(2, 3)$ and $(3, 2)$ in $\sigma$. Thus by Proposition 3.7, we have

\[
\lim_{n \to \infty} \|S_{\sigma_n} \cdots S_{\sigma_1} x\|_2 = 0, \quad \forall x \in \{(x_1, 0, 0)^T \mid x_1 \in \mathbb{R}\} = \ker(Q_{\sigma}),
\]

\[
\lim_{n \to \infty} \|S_{\sigma_n} \cdots S_{\sigma_1} x\|_2 = \|Q_{\sigma}(x)\|_2, \quad \forall x \in \{(0, x_2, x_3)^T \mid x_2, x_3 \in \mathbb{R}\} = \text{Im}(Q_{\sigma}),
\]

for such kind of generic switching signals.

The following Example 6.4 is associated to Theorem C.

Example 6.4. Let $S = \{S_1, S_2\}$ with

\[
S_1 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
\frac{1}{2} & -1
\end{pmatrix}, \quad S_2 = \sqrt{\frac{3 - \sqrt{5}}{2}} \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

Then, using $\sqrt{\rho(A^TA)} = \|A\|_2$ we have

\[
\rho(S_1) = \frac{1}{2} < 1, \quad \|S_1\|_2 = 1 \quad \text{and} \quad \rho(S_2) = \sqrt{\frac{3 - \sqrt{5}}{2}} < 1, \quad \|S_2\|_2 = 1.
\]

In addition,

\[
\rho(S_1S_2) = \sqrt{\frac{3 - \sqrt{5}}{2}} = \rho(S_2) < 1.
\]

Therefore, $S$ is absolutely stable by Theorem C.
7 CONCLUDING REMARKS

The interesting [22, Proposition 18] implies that if $S = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{d\times d}$ is symmetric (i.e. $A^T \in S$ whenever $A \in S$), then $S$ has the spectral finiteness property; in fact, it holds that $\rho(S) = \sqrt{\rho(A^T A)}$ for some $A \in S$. This naturally motivates us to extend an arbitrary $S$ into a symmetric set $S = S \cup S^T$. Let us see a simple example.

Example 6.5. Let $S = \left\{ A = \sqrt{\frac{3-\sqrt{5}}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$. Then, $S$ satisfies (1.3a) with $\|A\|_2 = 1$ such that $\rho(S) = \sqrt{\frac{3-\sqrt{5}}{2}} < 1$. But for $S = \{A, A^T\}$, $\rho(S) = \sqrt{\rho(A^T A)} = 1 \neq \rho(S)$.

This example shows that the extension $S$ does not work for the original system $S$ needed to be considered here.

Finally, the following Example 6.6 is simple. Yet it is very interesting to the stability analysis of switched systems.

Example 6.6. Let $S = \{S_1, S_2\} \subset \mathbb{R}^{2\times 2}$ with

$$S_1 = \sqrt{\frac{3-\sqrt{5}}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S_2 = \sqrt{\frac{3-\sqrt{5}}{2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Then, using $\sqrt{\rho(A^T A)} = \|A\|_2$ we have

$$\rho(S_1) = \rho(S_2) = \sqrt{\frac{3-\sqrt{5}}{2}} < 1, \quad \|S_1\|_2 = \|S_2\|_2 = 1, \quad \text{and} \quad \rho(S_1S_2) = 1.$$ 

So, $S$ is not absolutely stable. Yet from Corollary 5.3, $S$ is stable driven by $\mathbb{P}$-a.e. $\sigma \in \Sigma_T^\infty$, for any $\theta$-ergodic probability measure $\mathbb{P}$ on $\Sigma_T^\infty$, as long as $\mathbb{P}$ is not the ergodic measure distributed on the periodic orbit

$$((12, 12, 12, \ldots), (21, 21, 21, \ldots)).$$

7. Concluding remarks

In this paper, we have considered the asymptotic stability of a discrete-time linear switched system, which is induced by a set $S = \{S_1, \ldots, S_K\} \subset \mathbb{R}^{d\times d}$ such that each $S_k$ shares a common, but not necessarily strict, Lyapunov matrix $P$ as in (1.3a).

We have shown that if every subsystem $S_k$ is stable then $S$ is stable driven by a nonchaotic switching signal. Particularly, in the cases $K = 2$ and $d = 2, 3$, we have proven that $S$ has the spectral finiteness property and so the stability is decidable.

Recall that $S$ is called periodically switched stable, if $\rho(S_{w_n} \cdots S_{w_1}) < 1$ for all finite-length words $(w_1, \ldots, w_n) \in [1, \ldots, K]^n$ for $n \geq 1$; see, e.g., [16, 12, 10].

Finally, we end this paper with a problem for further study.

Conjecture. Let $S = \{S_1, S_2\} \subset \mathbb{R}^{d\times d}, d \geq 4$, be an arbitrary pair such that condition (1.3a). If $S$ is periodically switched stable, then it is absolutely stable. Equivalently, if $\rho(S) = 1$ there exists at least one word $(w_1, \ldots, w_n) \in [1, 2]^n$ for some $n \geq 1$ such that $\sqrt{\rho(S_{w_n} \cdots S_{w_1})} = 1$.

Since there exist uncountable many pairs $(\alpha, \gamma) \in (0, 1) \times (0, 1)$, for which

$$S_{\alpha, \gamma} = \left\{ S_1 = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \gamma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$
is periodically switched stable such that $\|S_1\| = \|S_2\| = 1$ under some extremal norm $\| \cdot \|$ on $\mathbb{R}^2$; but $S_{\alpha,\gamma}$ is not absolutely stable with $\rho(S_{\alpha,\gamma}) = 1$. See, for example, [8, 7, 19, 17]. So, condition (1.3a) is very important for our Theorems B, C and D and for the above conjecture. In fact, the essential good of $\| \cdot \|_F$ is to guarantee that $K_1\|F\|_F(S_1)$ and $K_1\|F\|_F(S_2)$ are linear subspaces of $\mathbb{R}^d$ in our arguments.

References

[1] A. Bacciotti and R. Lionello, Regularity of Liapunov functions for stable systems, System & Control Letters, 41 (2000), 265–270.
[2] A. Bacciotti and R. Lionello, Liapunov Functions and Stability in Control Theory, 2nd ed., Comm. Control Engng. Ser., Springer-Verlag, Berlin 2005.
[3] M. Balde and P. Jogan, Geometry of the limit sets of linear switched systems, SIAM J. Control Optim., 49 (2011), 1048–1063.
[4] N. Barabanov, Liapunov indicators of discrete inclusions I–III, Autom. Remote Control, 49 (1988), 152–157, 283–287, 558–565.
[5] N. Barabanov, Absolute characteristic exponent of a class of linear non-stationary systems of differential equations, Siberian Math. J., 29 (1988), 521–530.
[6] N. Barabanov, On the Aizerman problem for 3rd-order nonstationary systems, Differ. Equ., 29 (1993), 1439–448.
[7] V. D. Blondel, J. Tsitsiklis and A. A. Vladimirov, An elementary counterexample to the finiteness conjecture, SIAM J. Matrix Anal. Appl., 24 (2003), 963–970.
[8] T. Bouch and J. Mairesse, Asymptotic height optimization for topological IFS, Tetris heaps, and the finiteness conjecture, J. Amer. Math. Soc., 15 (2002), 77–111.
[9] X. Dai, Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices, J. Math. Anal. Appl., 379 (2011), 827–833.
[10] X. Dai, Weakly recurrent switching signals, almost sure and partial stability of linear switched systems, J. Differential Equations, 250 (2011), 3584–3629.
[11] X. Dai, Y. Huang and M. Xiao, Realization of joint spectral radius via ergodic theory, Electron. Res. Announc. Math. Sci., 18 (2011), 22-30.
[12] X. Dai, Y. Huang and M. Xiao, Periodically switched stability induces exponential stability of discrete-time linear switched systems in the sense of Markovian probabilities, Automatica, 47 (2011), 1512–1519.
[13] X. Dai, Y. Huang and M. Xiao, Pointwise stabilization of discrete-time matrix-valued stationary Markov chains, Preprint 2011, arXiv:1107.0132v1 [math.PR].
[14] X. Dai and V. Kozlak, Finiteness property of a bounded set of matrices with uniformly sub-peripheral spectrum, Information processes, 11 (2011), 253–261, arXiv:1106.2298v2 [math.FA].
[15] I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge, Linear Algebra Appl., 161 (1992), 227–263. Corrigendum/addendum, 327 (2001), 69–83.
[16] L. Gurvits, Stability of discrete linear inclusions, Linear Algebra Appl., 341 (1995), 45–85.
[17] K. G. Hare, I. D. Morris, N. Sidorov and J. Tsitsiklis, An explicit counterexample to the Lagarias-Wang finiteness conjecture, Adv. Math., 226 (2011), 4667–4701.
[18] D. J. Hartfiel, Nonhomogeneous Matrix Products, World Scientific, New Jersey London, 2002.
[19] V. S. Kozlak, Structure of extremal trajectories of discrete linear systems and the finiteness conjecture, Autom. Remote Control, 68 (2007), 174–209.
[20] S. Mendenhall and G. L. Slater, A model for helicopter guidance on spiral trajectories, in AIAA Guid. Control Conf., 1980, 62–71.
[21] V. V. Nemytskii and V. V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, New Jersey 1960.
[22] E. Pfeiffer and F. Wirth, Minimax results for the joint spectral radius and transient behavior, Linear Algebra Appl., 428 (2008), 2368–2384.
[23] P. Riedinger, M. Sigalotti and J. Daafouz, On the algebraic characterization of invariant sets of switched linear systems, Automatica, 46 (2010), 1047–1052.
[24] U. Sierks, J.-C. Vivaldi and P. Riedinger, On the convergence of linear switched systems, IEEE Trans. Automat. Control, 56 (2011), 320–332.
[25] Z. Sun, A note on marginal stability of switched systems, IEEE Trans. Automat. Control, 53 (2008), 625–631.
[26] P. P. Vaidyanathan and V. Lii, An improved sufficient condition for absence of limit cycles in digital filters, IEEE Trans. Circuits Systems, VOL. CAS-34 (1987), 319–322.
[27] P. Walters, An Introduction to Ergodic Theory, GTM 79, Springer-Verlag, New York, 1982.
REFERENCES

[28] F. Wirth, The generalized spectral radius and extremal norms, Linear Algebra Appl., 342 (2002), 17–40.
[29] Z. Zuo, Weakly almost periodic point and measure center, Sci. China Ser. A: Math., 36 (1992), 3019–3024.