Quantum radiation reaction force on a one-dimensional cavity with two relativistic moving mirrors

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We consider a real massless scalar field inside a cavity with two moving mirrors in a two-dimensional spacetime, satisfying Dirichlet boundary condition at the instantaneous position of the boundaries, for arbitrary and relativistic laws of motion. Considering vacuum as the initial field state, we obtain formulas for the exact value of the energy density of the field and the quantum force acting on the boundaries, which extend results found in previous papers [1–4]. For the particular cases of a cavity with just one moving boundary, non-relativistic velocities, or in the limit of infinity length of the cavity (a single mirror), our results coincide with those found in the literature.

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The Dynamical Casimir effect has been investigated since the 1970s [5–7], and has attracted growing attention. It is related to problems like particle creation in cosmological models and radiation emitted by collapsing black holes [6, 8], decoherence [9], entanglement [10], the Unruh effect [11], among others. Models of a single mirror have been investigated and also cavities with one moving boundary have been studied in many papers (for a review see Ref. [12]). In contrast, the problem of a cavity with two moving boundaries has been investigated recently and relatively few papers on this subject are found in the literature (for instance, Refs. [13–17]). A cavity with two oscillating mirrors can exhibit situations of constructive and destructive interference in the number of created particles, depending on the relation among the phase difference of each boundary, the amplitudes and frequencies of oscillation [13–15]. Ji, Jung and Soh [14], considering the expansion of the quantizing field over a instantaneous basis and a perturbative approach, investigated the problem of interference in the particle creation for a one-dimensional cavity with two boundaries moving according to prescribed, non-relativistic and oscillatory (small amplitudes) laws of motion. Dalvit and Mazzitelli [15] extended the field solution obtained by Moore [3] for the case of a one-dimensional cavity with two moving boundaries, deriving a set of generalized Moore’s equations, also obtaining the expected energy-momentum tensor for this model, generalizing the corresponding formula obtained by Fulling and Davies [6]. In Ref. [15] the set of generalized Moore’s equations was solved for the case of a resonant oscillatory movement with small amplitude, via renormalization-group procedure. Li and Li [2] applied the geometrical method, proposed by Cole and Schieve [18], to solve exactly the generalized Moore equations obtained by Dalvit and Mazzitelli [15], and also used numerical methods to obtain the behavior of the energy density in a cavity for particular sinusoidal laws of motion, with small amplitude [3]. On the other hand, as far as we know, there is no paper in literature devoted to obtain formulas which enable us to get directly exact values for the quantum force and energy density in a nonstatic cavity for arbitrary laws of motion for the moving boundaries, including non-oscillating movements with large amplitudes, which are out of reach of the perturbative approaches found in the literature.

In the present paper we consider a real massless scalar field satisfying the Klein-Gordon equation (we assume throughout this paper $\hbar = c = 1$): $(\partial_t^2 - \partial_x^2) \phi(t, x) = 0$, and obeying Dirichlet conditions imposed at the left boundary located at $x = L(t)$, and also at the right boundary located at $x = R(t)$, where $L(t)$ and $R(t)$ are arbitrary prescribed laws of motion, with $R(t < 0) = L_0$ and $L(t < 0) = 0$, where $L_0$ is the length of the cavity in the static situation. Considering vacuum as the initial field state, we obtain formulas for the exact value of the energy density of the field and the quantum force acting on the boundaries, showing that the energy density in a given point of the spacetime can be obtained by tracing back a sequence of null lines, connecting the value of the energy density at the given spacetime point to a certain known value of the energy at a point in the “static zone”, where the initial field modes are not affected by the disturbance caused by the movement of the boundaries. Our formulas generalize those found in literature [1], where this problem is approached for a cavity with only one moving mirror. For the particular cases of a cavity with just one moving boundary, non-relativistic velocities, or in the limit of large length of the cavity (a single mirror), our results coincide with those found in the literature [1, 4, 19].

Let us start considering the field operator, solution of the wave equation, given by [15]

$$\hat{\phi}(t, x) = \sum_{k=1}^{\infty} \left[ \hat{a}_k \psi_k(t, x) + \hat{a}_k^\dagger \psi_k^\ast(t, x) \right],$$

where the field modes are

$$\psi_k(t, x) = \frac{i}{\sqrt{4\pi k}} \left[ e^{-ik\pi G(v)} - e^{-ik\pi F(u)} \right], \quad (1)$$

$$L_0$$
with \( v = t + x, \ u = t - x, \) and

\[
\begin{align*}
G'[t + L(t)] - F'[t - L(t)] &= 0 \quad (2a) \\
G'[t + R(t)] - F'[t - R(t)] &= 2. \quad (2b)
\end{align*}
\]

The set of equations (2), obtained by Dalvit anduzzi exploiting the conformal invariance of the model [15], is a generalization of the Moore equation (2), which can be recovered doing \( L(t) = 0 \) in these equations. The renormalized energy density in the cavity is given by [15]

\[
\langle T_{00}(t,x) \rangle = -f_G(v) - f_F(u),
\]

where

\[
f_G(z) = \frac{1}{24\pi} \left\{ \frac{G''''(z)}{G'(z)} - \frac{3}{2} \left[ \frac{G'''(z)}{G'(z)} \right]^2 + \frac{\pi^2}{2} [G''(z)]^2 \right\},
\]

\[
f_F(z) = \frac{1}{24\pi} \left\{ \frac{F''''(z)}{F'(z)} - \frac{3}{2} \left[ \frac{F'''(z)}{F'(z)} \right]^2 + \frac{\pi^2}{2} [F''(z)]^2 \right\}.
\]

(3b)

In the present paper we use Eqs. (2a), (2b), (4a) and (4b) to obtain the following set of equations for the functions \( f_G \) and \( f_F \):

\[
\begin{align*}
f_G[t + R(t)] &= f_F[t - R(t)] A_R(t) + B_R(t), \quad (5a) \\
f_G[t + L(t)] &= f_F[t - L(t)] A_L(t) + B_L(t), \quad (5b)
\end{align*}
\]

with

\[
A_q(t) = \left[ \frac{1 - q'(t)}{1 + q'(t)} \right]^2, \quad (6)
\]

\[
B_q(t) = -\frac{1}{12\pi} \frac{q''''(t)}{[1 + q'(t)]^3 [1 - q'(t)]} \\
-\frac{1}{4\pi} \frac{q''(t)q'(t)}{[1 + q'(t)]^2 [1 - q'(t)]^2}, \quad (7)
\]

where, hereafter, \( q \) can represent \( R \) or \( L \). Eqs. (5a) and (5b) are an extension of the corresponding equation for \( f \), valid for a cavity with just one moving mirror, found in Ref. [3]. If we consider the particular case of \( L(t) = 0 \) in Eq. (5), we recover the corresponding result found in Ref. [3]. For \( t < 0 \) we have \( f_G(v) = f_F(u) = f^s \), extending the work done in Ref. [2]. Let us assume that \( \tilde{t}, \tilde{x} \) belongs to region IV, and that the null line \( v = z_1 \) intersects the moving mirror trajectory at the point \( [t_1, R(t_1)] \) (see Fig. 2(a)), so that \( \tilde{t} + \tilde{x} = t_1 + R(t_1) \). We have \( f_G(v)|_{v=z_1} = f_G[t_1 + R(t_1)] \). Using the Eq. (5a), we get \( f_G[t_1 + R(t_1)] = f_F[t_1 - R(t_1)] A_R(t_1) + B_R(t_1) \).

Now, our aim is to solve the Eqs. (5a) and (5b) recursively, using a geometrical point of view. Let us examine the cavity in the nonstatic situation (\( t > 0 \)). The field modes in Eq. (11) are formed by left and right-propagating parts. As causality requires, the field in region I (\( v < L_0 \) and \( u < 0 \)) (see Fig. 1) is not affected by the boundaries motion, so that, in this sense, this region is considered as a “static zone”. In region II (\( v > L_0 \) and \( u < 0 \)), the right-propagating parts of the field modes remain unaffected by the boundaries motion, so that region II is also a static zone for these modes. On the other hand, the left-propagating parts in region II are, in general, affected by the boundary movement. Similarly, in region III (\( u > 0 \) and \( v < L_0 \)), the left-propagating parts of the field modes are not affected by the boundaries motion, but the right-propagating parts are. In region IV (\( v > L_0 \) and \( u > 0 \)), both the left and right-propagating parts are affected. In summary, the functions corresponding to the left and right-propagating parts of the field modes are considered in the static zone if their arguments are, respectively, \( v < L_0 \) and \( u < 0 \). Then, we have \( f_G(v < L_0) = f^s \) and \( f_G(u < 0) = f^s \).
where, for $t = f(t)$ if $f$ can say that the number of reflections $n_G$ of one reflection ($n_G = 1$), whereas in Fig. 2(b) we see the case $n_G = 2$.

![Diagram](image)

**FIG. 2:** Sequence of null lines (dotted lines) connecting a point $(t, x)$ to a static zone. The dashed lines are null lines separating region I from II and III, and these ones from region IV, as presented in Fig. 1. In Fig. 2(a), we see the case of one reflection ($n_G = 1$), whereas in Fig. 2(b) we see the case $n_G = 2$.

Can say that the number of reflections $n_G$ to get into the static zone is, in this case, $n_G = 1$. On the other hand, if $t_1 - R(t_1) > 0$ (case shown in Fig. 2(b)), we can draw another null line $v = t_2 + L(t_2)$ intersecting the world line of the left boundary at the point $[t_2, L(t_2)]$, with $t_1 - R(t_1) = t_2 - L(t_2)$. In this case we have, using [5b], $f_G|t_1 + R(t_1) = \{f_G|t_2 + L(t_2)| - B_L(t_2)\} A_R(t_1)/A_L(t_2) + B_R(t_1)$. If $t_2 + L(t_2) < L_0$ (see Fig. 2(b)), then

$$f_G(z) = f^{(s)}A_G(z) + \tilde{B}_G(z),$$

where, for $n_G(z)$ even, we have

$$A_G(z) = \prod_{k=0}^{n_G(z)-1} \left(1 - \delta_{k,0}\right) \frac{A_R[t_{2k+1}(z)]}{A_L[t_{2k}(z)]} + \delta_{k,0},$$

(9a)

$$\tilde{B}_G(z) = - \sum_{k=0}^{n_G(z)-1} \left(1 - \delta_{k,0}\right) \frac{B_R[t_{2k+1}(z)] A_L[t_{2k}(z)]}{A_R[t_{2k+1}(z)]} + \delta_{k,0},$$

(9b)

with $\delta$ symbolizing Kronecker’s delta function. For $n_G(z)$ odd we have

$$A_G(z) = \prod_{k=0}^{n_G(z)-1} \left(1 - \delta_{k,0}\right) \frac{A_R[t_{2k+1}(z)]}{A_L[t_{2k}(z)]} + \delta_{k,0},$$

(10a)

$$\tilde{B}_G(z) = - \sum_{k=0}^{n_G(z)-1} \left(1 - \delta_{k,0}\right) \frac{B_R[t_{2k+1}(z)] A_L[t_{2k}(z)]}{A_R[t_{2k+1}(z)]} + \delta_{k,0}.$$  

(10b)

Note that the number $n_{G}$ of reflections and the sequence of instants $t_1, \ldots, t_{n_G}$ depend on the argument $z$. The set of instants mentioned in Eqs. (9) and (10) are calculated via [2]:

$$z = t_1 + R(t_1),$$

$$t_{2l+1} - R(t_{2l+1}) = t_{2l+2} - L(t_{2l+2}),$$

$$t_{2l+2} + L(t_{2l+2}) = t_{2l+3} + R(t_{2l+3}),$$

$l = 0, 1, 2, \ldots$.

To solve recursively the set of equations [5] for $f_R$, we start assuming that the null line $u = t - \tilde{x}$ intersects the world line of the left mirror at the point $[\tilde{t}_1, L(\tilde{t}_1)]$, so that $\tilde{t} - \tilde{x} = \tilde{t}_1 - L(\tilde{t}_1)$. Thus we have $f_R(u)_{u=\tilde{t}_1-z} = f_R[\tilde{t}_1 - L(\tilde{t}_1)]$. Using the Eq. (9), we get

$$f_R[\tilde{t}_1 - L(\tilde{t}_1)] = \{f_G[\tilde{t}_1 + L(\tilde{t}_1)] - B_L(\tilde{t}_1)\} / A_L(\tilde{t}_1).$$

If $\tilde{t}_1 + L(\tilde{t}_1) < L_0$, then the null line $\tilde{t} = \tilde{t}_1 + L(\tilde{t}_1)$ is already in the static zone, so that we can write

$$f_G[\tilde{t}_1 + L(\tilde{t}_1)] = f^{(s)},$$

and also $f_R[\tilde{t}_1 - L(\tilde{t}_1)] = \{f^{(s)} - B_L(\tilde{t}_1)\} / A_L(\tilde{t}_1), and we can say that the number of reflections $n_R$ to get into the static zone is, in this case, $n_R = 1$ (see Fig. 3(a)). On the other hand, if $\tilde{t}_1 + L(\tilde{t}_1) > L_0$ (as shown in Fig. 3(b)), we need to find $f_G[\tilde{t}_1 + L(\tilde{t}_1)]$ recursively via Eq. (8). In general, we get

$$f_R(z) = f^{(s)}A_F(z) + \tilde{B}_F(z),$$

(12)

where

$$\tilde{A}_F(z) = \frac{\tilde{A}_G[\tilde{t}_1(z) + L[\tilde{t}_1(z)]]}{A_L[\tilde{t}_1(z)],}$$

(13a)

$$\tilde{B}_F(z) = \frac{\tilde{B}_G[\tilde{t}_1(z) + L[\tilde{t}_1(z)]]}{A_L[\tilde{t}_1(z)],}$$

(13b)

with the function $\tilde{t}_1(z)$ calculated via

$$z = \tilde{t}_1 - L(\tilde{t}_1).$$

(14)

The formulas [4], [10] and [13] generalize those for $\tilde{A}$ and $\tilde{B}$ found in Ref. [1], which are valid for a cavity with just the right boundary in movement.
From Eqs. (3), (8) and (12), we get the exact formula for the renormalized energy density as

$$
\langle T_{00} (t, x) \rangle = - f(s) \left[ \dot{A}_G (v) + \dot{A}_F (u) - \dot{B}_G (v) - \dot{B}_F (u) \right].
$$

Eq. (15) gives directly the exact values for the energy density in a nonstatic cavity for arbitrary laws of motion $R(t)$ and $L(t)$. Since $T_{00} = T_{11}$ in this model, we have the following exact formulas for the renormalized quantum forces $F_R = \langle T_{00} [t, R(t)] \rangle$ and $F_L = - \langle T_{00} [t, L(t)] \rangle$ (see Refs. [19], [20]) acting, respectively, on the right and left boundaries:

$$
F_R(t) = - f(s) \left\{ A_G [t + R(t)] + \dot{A}_F [t - R(t)] \right\} +
- B_G [t + R(t)] - \dot{B}_F [t - R(t)],
$$

$$
F_L(t) = f(s) \left\{ A_G [t + L(t)] + \dot{A}_F [t - L(t)] \right\} +
+ B_G [t + L(t)] + \dot{B}_F [t - L(t)].
$$

Next we examine the behavior of these forces in each region pointed in Fig. 4.

In region I (Fig. 1), we have $n_G = n_F = 0$. Then, Eqs. (9) and (10) give: $A_G (z) = A_F (z) = 1$ and $B_G (z) = B_F (z) = 0$. This results, as expected, in the static Casimir force

$$
F_R^{(Cas)} = - F_L^{(Cas)} = - \pi/(24L_0^2),
$$

acting on the boundaries.

In region II, we have $n_G = 1$ and $n_F = 0$. For this case, Eq. (12) gives $A_F (u) = 1$ and $B_F (u) = 0$, whereas from Eq. (10) we have $A_G (v) = A_R [t_1 (v)]$ and $B_G (v) = B_R [t_1 (v)]$. To calculate the force $F_R(t)$ in Eq. (16) we do $v \rightarrow t + R(t)$, and obtain $t_1 (v)$ as already discussed: $t + R(t) = t_1 + R(t_1) \Rightarrow t_1 = t$. Then we get $A_G [t + R(t)] = A_R (t)$ and $B_G (t + R(t)) = B_R (t)$. The force $F_R(t)$ on the right boundary in region II, now relabeled as $F_R^{(II)}(t)$ is

$$
F_R^{(II)}(t) = - f(s) \left[ 1 + A_R (t) \right] - B_R (t).
$$

From this formula, we can obtain an analytical result for an arbitrary law of motion $R(t)$. Note that in Eq. (18) the subscript $L$ is not found, since the quantum force for the worldline in region II has no influence of the movement of the left boundary. Considering the limit $L_0 \rightarrow \infty$ we recover the quantum radiation force $F_R^{(u)}$ corresponding to the unbounded field, acting on the left side of a single mirror: $\lim_{L_0 \rightarrow \infty} F_R^{(II)}(t) = F_R^{(u)}(t)$, where

$$
F_R^{(u)}(t) = - B_R (t).
$$

In the non-relativistic limit, from (18) we get $F_R^{(II)} (t) \approx F_R^{(Cas)} + \tilde{R}/(12\pi)$, and adding the limit $L_0 \rightarrow \infty$ we recover the approximate quantum radiation force $F_R^{(II)} (t) \approx \tilde{R}/(12\pi)$, which acts on the left side of a single mirror [21].

In region III, we have $n_G = 0$ and $n_F = 1$. For this case, Eqs. (9) and (10) give $A_F (u) = 1/A_L [t_1 (u)]$; $B_F (u) = - B_L [t_1 (u)]/A_L [t_1 (u)]$; $A_G (v) = 1$; $B_G (v) = 0$. Considering $u \rightarrow t - L(t)$ and $t \rightarrow L(t)$, we get $F_L(t)$ on the left boundary in this region, now relabeled as $F_L^{(III)}(t)$ is

$$
F_L^{(III)}(t) = f(s) \left\{ 1 + \frac{1}{A_L (t)} \right\} - \frac{B_L (t)}{A_L (t)}).
$$

Considering the limit $L_0 \rightarrow \infty$ we recover the quantum radiation force $F_L^{(u)}$ corresponding to the unbounded field, acting on the right side of a single mirror: $\lim_{L_0 \rightarrow \infty} F_L^{(III)}(t) = F_L^{(u)}(t)$, where

$$
F_L^{(u)}(t) = - \frac{B_L (t)}{A_L (t)}.
$$

From Eqs. (19) and (21) we recover the total quantum force $F_q^{(u)}(t)$ acting on a single mirror at vacuum, with a prescribed trajectory $x = q(t)$:

$$
F_q^{(u)}(t) = F_q^{(u)}(t) + F_q^{(u)}(t)
$$

$$
= (1 + \dot{q}^2) \left\{ \left[ \dot{q}^2 q/(2\pi) \right]^2 - \dot{q}^2 \right\}^3
+ [\dot{q}^2 / (6\pi)]/(1 - \dot{q}^2)^3,
$$

which is in agreement with that found in literature (see Ref. [19]). In the non-relativistic limit, we reobtain the approximate quantum radiation force $F_q^{(u)}(t) \approx \dot{q}/(6\pi)$ [21].

To compute the total forces $F_{R}^{(tot)}$ and $F_{L}^{(tot)}$ acting on, respectively, the right and left boundaries, for any of
we have an expanding cavity with large amplitude and
motion for the moving boundaries, for vacuum as the
ingital state of the field. Eqs. (5a) and (5b) are an exten-
tion of the corresponding equation for a cavity with just
the boundaries go to an asymptotic behavior of infinity
it goes to zero in the case showed in Fig. (4a), and
κ_R = κ_L = 0.1 (Fig. 4(b)), we have the mirrors in movement with relativistic velocities, but keeping constant the cavity length.

In Figs. 6 and 7, using our formulas (9)-(13) and (22), we plot the time evolution of the quantum force \( F_R^{(tot)}(t) \) and \( F_R(t) \), for, respectively, the cases \( κ_R = -κ_L = 0.1 \) (see Fig. 4(a)), and \( κ_R = κ_L = 0.1 \) (see Fig. 4(b)). We can see discontinuities of the derivatives for \( F_R^{(tot)} \) and \( F_R(t) \). These discontinuities always occur when the front of the wave in the energy density meets the right boundary. In the case, for instance, showed in Fig. 6 when \( t = 0 \) the left boundary starts to move and generate a wave in the energy density, propagating rightward and meeting the right boundary at the instant \( t = \tau_1 \approx 1.05 \), calculated via equation \( \tau_1 - R(\tau_1) = 0 \), and which corresponds to the first discontinuity of the derivative showed in Fig. 5. At \( t = 0 \), another front of wave is generated by the right boundary, propagating leftward and meeting the left boundary at the instant \( \tau_1 \approx 1.05 \), and then reflected back and meeting the right boundary at the instant \( \tau_2 \approx 2.25 \), calculated from the equation \( \tau_2 - R(\tau_2) = \tau_1 - L(\tau_1) \). This instant corresponds to the second discontinuity of the derivative showed in Fig. 6.

Since the length of the cavity remains the same in the case showed in Fig. 4(b), the quantum force \( F_R^{(tot)}(t) \) oscillates around the static Casimir force (Fig. 7), whereas it goes to zero in the case showed in Fig. 6 where the boundaries go to an asymptotic behavior of infinity length and constant velocity.

Summarizing our results, the formulas obtained in the present paper enable us to get directly exact values of the energy density of the field and the quantum force acting on the boundaries of a nonstatic cavity for arbitrary laws of motion for the moving boundaries, for vacuum as the initial state of the field. Eqs. (5a) and (5b) are an extension of the corresponding equation for a cavity with just

to Eq. (24). Fig. 4(a) describes the case \( κ_R = -κ_L = 0.1 \), whereas 4(b) describes the case \( κ_R = κ_L = 0.1 \).
FIG. 6: The solid line shows the total force $F_{R}^{(tot)}(t)$, the dashed line shows the force $F_{R}(t)$, both for the law of movement (24), with $\kappa_R = \kappa_L = 0.1$ and $L_0 = 1$. The dotted line shows the static Casimir force.

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