A note on coloring line arrangements

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Abstract

We show that the lines of every arrangement of \( n \) lines in the plane can be colored with \( O(\sqrt{n}/\log n) \) colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. [1] by a \( \Theta(\sqrt{\log n}) \) factor. Any further improvement on this bound will improve the best known lower bound on the following problem of Erdős: Estimate the maximum number of points in general position within a set of \( n \) points containing no four collinear points.

1 Introduction

Let \( \mathcal{A} \) be an arrangement of lines in the plane. Denote by \( \chi(\mathcal{A}) \) the minimum number of colors required for coloring the lines of \( \mathcal{A} \) such that there is no monochromatic face in \( \mathcal{A} \), that is, the boundary of every face is colored with at least two colors. Bose et al. [1] proved that \( \chi(\mathcal{A}) \leq O(\sqrt{n}) \) for every simple arrangement \( \mathcal{A} \) of \( n \) lines, and that there are arrangements that require \( \Omega(\log n/\log \log n) \) colors. We improve their upper bound by a \( \Theta(\sqrt{\log n}) \) factor, and extend it to not necessarily simple arrangements.

Theorem 1. The lines of every arrangement of \( n \) lines in the plane can be colored with \( O(\sqrt{n}/\log n) \) colors such that no face of the arrangement is monochromatic.

A set of points in the plane is in general position if it does not contain three collinear points. Let \( \alpha(S) \) denote the maximum number of points in general position in a set \( S \) of points in the plane, and let \( \alpha_4(n) \) be the minimum of \( \alpha(S) \) taken over all sets \( S \) of \( n \) points in the plane with no four point on a line. Erdős pointed out that \( \alpha_4(n) \leq n/3 \) and suggested the problem of determining or estimating \( \alpha_4(n) \). Füredi [3] proved that \( \Omega(\sqrt{n}\log n) \leq \alpha_4(n) \leq o(n) \).

We observe that any improvement of the bound in Theorem 1 would immediately imply a better lower bound for \( \alpha_4(n) \). Indeed, suppose that \( \chi(\mathcal{A}) \leq k(n) \) for any arrangement of \( n \) lines, and let \( P \) be a set of \( n \) points, no four on a line. Let \( P^* \) be the dual arrangement of a slightly perturbed \( P \) (according to the usual point-line duality, see, e.g., [2, §8.2]). Color \( P^* \) with \( k(n) \) colors such that no face is monochromatic, let \( S^* \subseteq P^* \) be the largest color class, and let \( S \) be its dual point set. Observe that the size of \( S \) is at least \( n/k(n) \) and it does not contain three collinear points.

2 Proof of Theorem 1

Let \( \mathcal{A} \) be an arrangement of \( n \) lines. We show that \( O(\sqrt{n}/\log n) \) colors suffice for coloring the lines of \( \mathcal{A} \) such that no face in \( \mathcal{A} \) is monochromatic. Call a subset of lines independent if there is no face that is bounded just by lines from this subset. The proof is based on the following fact.

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**Theorem 2.** There is an absolute constant $c > 0$ such that every arrangement of $n$ lines contains at least $c\sqrt{n \log n}$ independent lines.

We color the lines in $\mathcal{A}$ such that no face is monochromatic by following the same method as in [1]. That is, we iteratively find a large subset of independent lines (whose existence is guaranteed by Theorem 2), color them with the same (new) color, and remove them from $\mathcal{A}$.

Clearly, this algorithm produces a valid coloring. We verify, by induction $n$, that at most $\frac{6c}{c\sqrt{\log n}}$ colors are used in this coloring. We assume the bound is valid for all $n \leq 256$ (by taking sufficiently small $c > 0$). For $n > 256$, we have $\log 4 < \frac{1}{4} \log n$. Let $i$ be the smallest integer such that after $i$ iterations the number of remaining lines is at most $n/4$. Since in each of these iterations at least $c\sqrt{\log n} \geq c\sqrt{\log n}$ vertices are removed, $i \leq \frac{3n/4}{c\sqrt{\log n}} \leq \frac{3}{2c}\sqrt{n/\log n}$. Therefore, by the induction hypothesis the number of colors that the algorithm uses is at most

$$i + \frac{6}{c\sqrt{\log n}} < \frac{3}{c\sqrt{2c}} \sqrt{n/\log n} + \frac{3}{c\sqrt{2c}} \sqrt{n/\log n - \frac{1}{2} \log n} < \frac{3}{c\sqrt{2c}} \sqrt{n/\log n} + \frac{2\sqrt{3}}{c} \sqrt{n/\log n} < \frac{6}{c\sqrt{\log n}}$$

The proof of Theorem 2 is based on a result about independent sets in sparse hypergraphs. A hypergraph $H = (V, E)$ consists of a set of vertices $V$ and a set of edges $E \subseteq 2^V$. If the size of every edge is $k$, then $H$ is $k$-uniform. For a set $Z \subseteq V$ the induced hypergraph $H[Z]$ consists of $Z$ and the edges of $H$ that are contained in $Z$. A set $I \subseteq V$ is an independent set of $H$, if $I$ contains no edge of $H$.

Kostochka et al. [4] proved that every $n$-vertex $(k+1)$-uniform hypergraph in which every $k$ vertices are contained in at most $d < n/(\log n)^{3k+5}$ edges contains an independent set of size $\Omega\left(\left(\frac{n}{\log n}\right)^{1/k}\right)$. In fact, a careful look at their proof reveals the following result, that we stated for $3$-uniform hypergraphs, since this is the case that we need.

**Theorem 3 (1).** Let $H = (V, E)$ be an $n$-vertex $3$-uniform hypergraph, such that every pair of vertices appear in at most $d < n/(\log n)^{12}$ edges. Let $X \subseteq V$ be a random set of vertices chosen independently with probability $p = n^{-2/5}/(\log \log \log n)^{3/5}$, and let $Z$ be an independent set chosen uniformly at random from the independent sets in $H[X]$. Then with high probability $\mathbb{E}[|Z|] \geq \Omega(\sqrt{n \log n})$.

With Theorem 3 in hand we can now prove Theorem 2.

**Proof of Theorem 2.** Let $L$ be a set of $n$ lines and let $\mathcal{A}$ be the corresponding arrangement. For a subset $L' \subseteq L$ we say that a face $f$ of $\mathcal{A}$ is bad with respect to $L'$ if all the lines bounding $f$ are in $L'$.

Define a $3$-uniform hypergraph $H$ whose vertex set are the lines in $\mathcal{A}$ and whose edge set consists of sets of three vertices such that the corresponding lines bound a face of size three in $\mathcal{A}$. Observe that every pair of lines can bound at most four faces of size three, therefore every pair of vertices in $H$ appears in at most four edges. Let $X$ be a random set of vertices (lines) chosen independently with probability $p = n^{-2/5}/(\log \log \log n)^{3/5}$. There are $O(n^2)$ faces in $\mathcal{A}$ and $O(n)$ of them are of size two (since every line can bound at most four such faces). Therefore, the expected number of bad faces with respect to $X$ of size different than three is $O(p^2 n + p^4 n^2) = o(\sqrt{n/\log n})$.

It follows from Theorem 3 and Markov’s inequality that with high probability the set of lines that correspond to $X$ contains a subset $Z$ of $\Omega(\sqrt{n \log n})$ lines such that there are no faces of size three that are bad with respect to $Z$, and the number of bad faces of size different than three is $o(\sqrt{n/\log n})$. By removing from $Z$ a vertex from each bad face we obtain an independent set of lines of size $\Omega(\sqrt{n \log n})$. 

\[\Box\]
References

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