MOND-like acceleration in integrable Weyl geometric gravity

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Abstract

In a Weyl geometric scalar tensor theory of gravity we replace the quadratic kinetic Lagrangian of the scalar field by a cubic term, similar to the one of Bekenstein and Milgrom’s first relativistic MOND theory (AQUAL). In Einstein-scalar field gauge of the Weylian metric, the scale connection expresses an additional acceleration adding to the (Riemannian)metrical component known from Einstein gravity. It becomes MOND-like in the static weak field approximation, while the Riemannian component remains Newtonian. Near mass centers the energy-momentum tensor of the scalar field acquires spatial inhomogeneities containing a considerable amount of energy. These inhomogeneities have consequences comparable to the ones attributed to dark matter, as far as cluster dynamics and gravitational lensing are concerned.

Introduction

Shortly after M. Milgrom’s original proposal of his modified Newtonian dynamics (MOND) as an explanation for the observed anomalies in galaxy rotation curves, Bekenstein and Milgrom showed how a MOND-ian dynamics could be derived from a Lagrangian of a scalar field \( \phi \) involving a kinetic term proportional to \( \tilde{f}(a_o^{-2}(\nabla \phi)^2) \) with a non-linear functional \( \tilde{f} \) (Bekenstein/Milgrom 1984). A case distinction between the Newton regime and a MOND regime had to be inbuilt by hand into the functional \( \tilde{f} \). In the appendix of their paper they indicated how their “a-quadratic” (AQUAL) Lagrangian could be adapted to general relativity in a Jordan-Brans-Dicke (JBD) framework. This approach was the first of a collection of different attempts to cope with MOND phenomenology in general relativistic frameworks (TeVeS, Einstein aether, and others).

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\(^1a_o\) denoted the typical new constant of the MOND hypothesis \( a_o \approx \frac{1}{8}H_1, \) \( H_1 \) Hubble constant in length units.
The relativistic a-quadratic Lagrangian approach itself ("RAQUAL") suffered from several deficiencies noticed by the authors from the outset: gravitational waves appeared to propagate with velocity greater than that of light; cluster dynamics and gravitational lensing could not be accounted for. Moreover, the different conformal aspects in JBD theory, "Jordan frame" and "Einstein frame", entered the analysis in a way typical for JBD-theory at the time.

In the meantime it has become clear that such different, conformally related, "frames" are better analyzed in terms of integrable Weyl geometry (and Omote-Dirac gravity without the curvature (Yang-Mills) term of the scale connection). Here we see that, after writing the original AQUAL Lagrangian in scale invariant form, the non-linear kinematic term of the scalar field can be simplified to a cubic expression in the gradient of the invariant scalar field. A scale covariant quantity proportional to the scalar field can be identified, in Einstein gauge, with a constant \( \tilde{a}_0 \) which plays a role analogous to the MOND constant \( a_0 \), but is not identical with it (section 1).

Already the conceptual clarification achieved by this move is interesting: In the usual weak field static approximation of gravity the metrical representation of the Newton potential is kept intact for the Riemannian component of the Weyl metric; the Weylian scale connection in Einstein gauge induces an additional acceleration for the dynamics of test bodies. It has the scale invariant version of the scalar field (in Riemann gauge) as its potential. As the scale connection is an integral component of the Weylian metric, the additional acceleration is part of an extended metrical theory of gravity; it needs no additional structural element (section 3). Specifying these general considerations to the case of a scalar field with the cubic Lagrangian introduced in section 1 leads, in good approximation, to a MOND-like modified Poisson equation very much like in RAQUAL. But here it governs only the ("anomalous") additional acceleration, while the Riemannian component remains governed by the ordinary Poisson equation (which will acquire an additional source term, as we shall see in a moment). The quality of the approximation is estimated for the deep MOND regime and seems acceptable (section 4).

A new feature arises from the evaluation of the energy-momentum tensor of the scalar field in the Weyl geometric framework. The most important contributions to the energy tensor derive from boundary terms in varying the modified Hilbert action. They give rise to quadratically decreasing inhomogeneities in the energy density near point-like mass centers, which are

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2Still in later presentations Bekenstein conceived the Jordan frame as the "the metric measured by rods and clocks, hence the physical metric", while Einstein frame played the role of a "primitive metric" which governed the Einstein-Hilbert action "in order not ot break violently with GR . . ." (Bekenstein 2004, 5f.).

3(Quiros e.a. 2013) or (Scholz 2014b, sec. 3). For reasons, independent of JBD theory, to restrict Omote-Dirac gravity far below the Planck scale to integrable Weyl geometry see, e.g., (Scholz 2014b, sec. 4.2).
no longer negligible (section 5). First estimates indicate that their amount is at least as large as expected from a dark matter halo. Moreover, the quadratically decaying energy density enters the right hand side (r.h.s.) of the Poisson equation of the Newton approximation. The ensuing modification of the “naked” Poisson equation of the Newton potential leads to another addition to the Newton acceleration, proportional to the MOND acceleration of the scale connection. The effect of both additions together is to be equated with the empirically determined acceleration in the deep MOND regime (section 6). This demands the constant \( \tilde{a}_0 \) to be \( \frac{1}{16} a_o \).

At the end of the article it is discussed, why the Weyl geometric adaptation of the RAQUAL approach seems quite well able to cope with cluster dynamics and gravitational lensing effects, while this was not the case, and was even excluded on seemingly principled grounds, for conformal modifications of Riemannian metric gravity theories in general (section 7). A series of open questions is shortly presented and argued that it is worthwhile to deal with them, because of the unificatory potential of the present approach.

1 Assumptions and Lagrangian

We work in the framework of integrable Weyl geometric gravity, with a gravitational scalar field non-minimally coupled to scalar curvature. Here we deal with a classical, real valued gravitational scalar field \( \phi \), rescaling with (Weyl) weight \( w(\phi) = -1 \) under conformal changes of the Riemannian component of the Weyl metric.

A scale choice (scale gauge) specifies a triple \((g, \varphi, \phi)\) consisting of the Riemannian component of the Weyl metric \( g = g_{\mu\nu} dx^\mu dx^\nu \), its scale connection (“Weyl field”) \( \varphi = \varphi_\mu dx^\mu \) and the scalar field \( \phi \). A scale transformation to \((\bar{g}, \bar{\varphi}, \bar{\phi})\) is given by \( \bar{g} = \Omega^2 g, \bar{\varphi} = \varphi - d \log \Omega, \bar{\phi} = \Omega^{-1} \phi \) with an everywhere positive real valued function \( \Omega \).

Because we are working in an integrable Weyl geometry, there is a scale gauge \((\bar{g}, 0, \bar{\phi})\) in which the scale connection vanishes, \( \bar{\varphi} = 0 \). By obvious reason it is called Riemann gauge. Writing \( \bar{\phi} \) in exponential form, \( \bar{\phi} = e^{\omega} \), we may just as well use

\[
\omega := \log \bar{\phi}
\]  

as a scale invariant expression for the scalar field.

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4Like in (Omote 1971, Dirac 1973, Romero e.a. 2011, Almeida e.a. 2014, Quiros 2014, Scholz 2014b) and many others. For Weyl geometry and Weyl structures, their derivatives and curvature expressions one may consult, among others, (Gilkey e.a. 2011, Yuan/Huang 2013) and the appendix of (Miritzis 2004).

5The scalar field may be related to a complex valued one \( \Phi \) by \( \phi^2 = \langle \Phi \Phi^* \rangle \).

6In the language of Jordan-Brans-Dicke gravity Riemann gauge corresponds to a “Jordan frame”. Of course, Riemann gauge, or Jordan frame, are specified only up to a global factor.
Another important gauge \((g, \varphi, \phi_c)\) arises from scaling the values of \(\phi\) to a constant, \(\phi(x) = \phi_c\) for all \(x\); we call it scalar field gauge.\(^7\) The components of the scale connection in this gauge are given by \(\varphi_c = -\partial_c \omega\) (appendix 1. eq. (54)). This shows that the scale covariant scalar field \(\phi\), its scale invariant version \(\omega\), and the (integrable) scale connection \(\varphi\) are closely related. They can be considered as different mathematical representations of the same dynamical entity (scalar field \(\leftrightarrow\) scale connection).

In any gauge \((g, \varphi, \phi_c)\) the Weylian affine connection \(\Gamma\) derives from the Levi-Civita connection \(g\Gamma\) of its Riemannian component by

\[
\Gamma_{\mu \nu \lambda} = g \Gamma_{\mu \nu \lambda} + \varphi \Gamma_{\mu \nu \lambda} = \delta^\mu_\nu \varphi_\lambda + \delta^\mu_\lambda \varphi_\nu - g_{\nu \lambda} \varphi^\mu. \tag{2}
\]

We denote the covariant derivative with regard to \(\Gamma\) by \(\nabla\) and the scale covariant derivative of fields \(X\) with weight \(w\) by \(D\):

\[
DX = \nabla X + w \varphi \otimes X \tag{3}
\]

In index notation \((\nabla X)_\mu = \partial_\mu X^\mu + \Gamma^\mu_{\nu \kappa} X^\kappa\), while \(DX\) is \((DX)_\mu = \partial_\mu X^\mu + \Gamma^\mu_{\nu \kappa} X^\kappa + w \varphi_\nu X^\mu\). \(DX\) is again scale covariant of weight \(w\); \(\nabla X\) is not.

To be consistent with both signature choices for \(g\), we introduce

\[
\epsilon_{\text{sig}} = \begin{cases} +1 & \text{if sign}(g) = (3, 1) \sim (-+++) \\ -1 & \text{if sign}(g) = (1, 3) \sim (+---) \end{cases} \tag{4}
\]

Our Lagrangian density will be

\[
L = L \sqrt{|g|} \quad \text{with} \quad |g| := |\det g|,
\]

\[
L = L_{HW} + L_{V4} + L_\phi + L_m,
\]

with a (classical) matter term \(L_m\), given by an expression of scale weight \(w(L_m) = -4\) (like in the case of the matter terms of the standard model fields) under which test particles follow the Weyl geometric path structure. The latter is strongly supported by the analysis of the stream lines of a Klein-Gordon field (in WKB approximation) (Audretsch e.a. 1984), if one assumes a structure conserving transition from the quantum world to classical particle motion after decoherence. It can be understood as a compatibility criterion of the matter Lagrangian with the EPS axioms for a generalized theory of gravity (Ehlers/Pirani/Schild). We introduce it as an additional postulate which deserves further investigation.\(^8\)

\[
L_{HW} = \frac{\epsilon_{\text{sig}}(\xi \phi)^2 R}{2} \quad \text{Hilbert-Weyl term,} \tag{5}
\]

\(^7\)For \(\phi_c = \xi^{-1} E_{pl}\) it is called Einstein gauge (see below).

\(^8\) The postulate could be stated as an action principle for point particles with the scale invariant action: \(S_{\text{pp}} = \int \phi_{\text{comp}} \sqrt{g(\gamma\bar{\gamma})} \, d\tau\) (with \(\gamma\) timelike curves parametrized by \(\tau\), \(\phi_{\text{comp}}\) the ”compensating field” like in appendix 1); but the question of consistency or derivability would still persist. In (Pucheu e.a. 2014) it is derived for a weak extension of Einstein gravity, rewritten scale covariantly using Weyl geometry (by means of the contracted Bianchi identity applied to the energy-momentum of dust-like matter, like in ordinary Einstein gravity). This approach might be generalizable. The condition of EPS compatibility is analyzed in great generality in (Di Mauro e.a. 2010).
\[ L_{V4} = -\frac{\lambda}{4} \phi^4 \quad \text{quartic potential term of } \phi, \quad (6) \]
\[ L_\phi = \frac{2}{3} (\xi \phi^2 (\eta^{-1} \phi)^{-1} f(\epsilon \sigma a^{-2} D^\nu \phi D_\nu \phi) \quad \text{gradient term of } \phi, \quad (7) \]

with \( f(x) = \begin{cases} x^3 & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0 \end{cases} \).

\( \xi, \lambda, \eta \) are constants to be interpreted later; \( R \) is the Weyl geometric scalar curvature (scale covariant of weight \( w(R) = -2 \)). The scale weight of all \( L \)-terms is -4, which implies the scale invariance of the Lagrangian density \( L \).

Let us introduce the abbreviations:

\[ a \cdot b := \epsilon \sigma a_{\nu b^\nu} \text{ for vectors or covectors } a, b, \]
\[ |a| := \sqrt{a_\nu a^\nu} \]
\[ \|a\| := \begin{cases} \sqrt{a \cdot a} & \text{for } a \cdot a \geq 0 \\ 0 & \text{for } a \cdot a \leq 0 \end{cases} \quad (8) \]
\[ \nabla f := (\partial_\nu f)_{\nu=0,...,3} \text{ for a scale invariant function } f \quad (9) \]
\[ \nabla^2 f := \nabla \cdot \nabla f = \epsilon \sigma \partial_\nu \partial^\nu f \quad (10) \]

According to (8) we have in particular

\[ \|\nabla \omega\| = \sqrt{\nabla \omega \cdot \nabla \omega} \text{ for positive radicand, otherwise } = 0. \quad (11) \]

Because of \( D_\nu \phi = \phi \partial_\nu \omega \) (appendix 1, eq. 50), the scalar field Lagrangian can be written as:

\[ L_\phi = \frac{2}{3} (\xi \phi^2 (\eta^{-1} \phi)^{-1} \|\nabla \omega\|^3 \quad (12) \]

It is cubic in the gradient of the scale invariant scalar field and of the correct scale weight \(-4\), because \( w(\|\nabla \omega\|) = w(\|\nabla \omega\|) = -1 \).

\( L_\phi \) is a scale covariant adaptation of the relativistic "a-quadratic Lagrangian" (AQUAL) \( \frac{a^2}{8\pi G} f(\frac{\|\nabla \phi\|^2}{a^2}) \) introduced in the first relativistic MOND theory (Bekenstein/Milgrom 1984), with the specification \( f(x) \approx 2x^3 \) for \( x \ll 1 \) and \( f(x) \approx 1 \) for \( x \gg 1 \), which distinguishes between the MOND and Newton regimes. In our framework, it seems possible to avoid this differentiation, because the MOND-like acceleration appears here as an additional contribution to the Einstein-Newton acceleration, not as an overall substitute. It remains to be seen whether the consequences of this simplification are empirically acceptable. On the other hand, we do have to introduce a cut off, \( f(x) = 0 \) for \( x \leq 0 \), in order to avoid an unacceptable contribution of this term to the energy-momentum of the scalar field in the cosmological limit.
After all, the scale invariant cubic (SIC) Lagrangian looks a bit surprising, although less artificial than in the original relativistic AQUAL theory. But one should not forget that, different from the quadratic kinetic term of Klein-Gordon type, it has been constructed with a definite (“inductive”) relation to the empirically founded MOND phenomenology in mind. The role of the different factors involving $\xi$ and $\eta$ will become clear in the next paragraph.

The Lagrangian links up to Einstein gravity, with reduced Planck energy $E_{pl}$ respectively Planck length $L_{pl}$ (suppressing here and elsewhere obvious factors $c$ and $\hbar$), if in the scalar field gauge $(g, \varphi, \phi_c)$

$$\xi \phi_c = (8\pi G)^{\frac{3}{2}} = E_{pl} \leftrightarrow L_{pl}^{-1}. \quad (13)$$

Here, as otherwise, $\equiv$ indicates that the equation holds only in a specific gauge made clear by the context (in most cases in Einstein gauge). Moreover it links up to Bekenstein/Milgrom’s RAQUAL approach if

$$\eta^{-1} \phi_c = \tilde{a}_o. \quad (14)$$

Here $\tilde{a}_o$ is analogous to the MOND acceleration $a_o \approx \frac{1}{6}H$, where $H$ denotes the Hubble parameter $H = H_0 \leftrightarrow H_1$. Below we find $\tilde{a}_o \approx \frac{2}{10}$ if we want to derive the classical MOND acceleration by the total effects of the scalar field (see section 6). Einstein gravity is (precisely) contained in our approach as the special case with $\omega = \text{const}$. Then Riemann gauge and Einstein gauge coincide and the scalar field is dynamically inert.\footnote{(Scholz 2014a, sect.3), (Romero e.a. 2011).}

In the following we will understand by Einstein gauge the scalar field gauge with (13) and (14) (more precisely: Einstein-MOND gauge).

$\phi_c^{-1}$ stands between the the largest and smallest physically conceivable length units in the universe $\tilde{a}_o^{-1}$ and $L_{pl}$; or the other way round:

$$\tilde{a}_o \overset{\eta}{\longrightarrow} \phi_c \overset{\xi}{\longrightarrow} E_{pl} \leftrightarrow L_{pl}^{-1}$$

The product of our typical coefficients is (by definition) the proportion of these extremal quantities:

$$\eta \cdot \xi = \frac{E_{pl}}{\tilde{a}_o} = \frac{\tilde{a}_o^{-1}}{E_{pl}} \sim 10^{63} \quad (15)$$

We may assume $\xi$ and $\eta$ to be at roughly comparable orders of magnitude. Then $\phi_c$ lies close to the geometrical mean between the extremes $\tilde{a}_o, E_{pl}$.

2. Dynamical equations

In varying with regard to $\delta g^{\mu\nu}$ one has to be careful with boundary terms from the Hilbert-Weyl term, which do not arise with the constant coefficient
\[
\frac{1}{2\sqrt{|g|}} \frac{\delta L_{\text{HW}}}{\delta g^{\mu\nu}} = \epsilon_{\text{sig}} \frac{\xi^2}{2} \left( \phi^2 (\text{Ric} - \frac{R}{2} g)_{\mu\nu} - D_{(\mu} D_{\nu)} \phi^2 + D^\lambda D_\lambda \phi^2 g_{\mu\nu} \right)
\]

Here \( \text{Ric} \) and \( R \) are the Weyl geometric Ricci tensor and scalar curvature respectively. Remember that \( D_\mu \) denotes the scale covariant derivative of Weyl geometry, depending on the scale weight \( w = w(X) \) of a field \( X \).

The variation of the other terms is straightforward. The variation of \( L_\phi \) gives a peculiar energy-momentum contribution from the scalar field to the r.h.s. (see below, (19), (20)). The energy-momentum tensor of matter is defined as usual:

\[
T^{(m)}_{\mu\nu} := -\epsilon_{\text{sig}} \frac{1}{\sqrt{|g|}} \frac{\delta L_{\text{m}}}{\delta g^{\mu\nu}}
\]

One thus derives the *scale invariant Einstein equation* of Weyl geometric gravity (eq. (18)), where the r.h.s. consists of the energy-momentum of matter \( T^{(m)} \) and \( \Theta \), the energy tensor of the scalar field (up to the constant \( 8\pi G \)). The latter decomposes,

\[
\Theta = \Theta^{(I)} + \Theta^{(II)}
\]

into a term (I) proportional to the Riemannian component of the metric \( g \), and another one (II) which is not:

\[
\Theta^{(I)} = (\epsilon_{\text{sig}} (\xi \phi)^{-2} L_{V_{4}} + \epsilon_{\text{sig}} (\xi \phi)^{-2} L_{\phi} - \phi^{-2} D^\lambda D_\lambda \phi^2) g \tag{19}
\]

\[
\Theta^{(II)}_{\mu\nu} = \phi^{-2} D_{(\mu} D_{\nu)} \phi^2 - 2(\eta^{-1} \phi)^{-1} \| \nabla \omega \| \partial_{\mu} \omega \partial_{\nu} \omega \tag{20}
\]

In (19) the summand \( \epsilon_{\text{sig}} (\xi \phi)^{-2} L_{V_{4}} g = -\epsilon_{\text{sig}} \frac{3}{4} \xi^{-2} \phi^2 g \) is a scale covariant form of the cosmological constant term \( \Lambda g \) with:

\[
\Lambda = \frac{\lambda}{4} \xi^{-2} \phi^2 \quad (\text{variable}) , \quad \Lambda = \frac{\lambda}{4} \xi^{-2} \phi^2_c \quad (\text{constant}) .
\]

Varying \( \delta \omega \), i.e. with regard to the scale invariant version of the scalar field, one uses (51) (valid in any gauge) and (11) to find:

\[
\frac{\partial}{\partial \omega} \phi = \frac{\partial}{\partial \omega} e^{\omega} + f \varphi = e^{\omega} + f \varphi = \phi ,
\]

\[
\frac{\partial}{\partial (\partial_{\nu} \omega)} \| \nabla \omega \| = \frac{\partial}{\partial (\partial_{\nu} \omega)} \sqrt{\epsilon_{\text{sig}} \partial_{\mu} \omega \partial_{\mu} \omega} = \frac{\epsilon_{\text{sig}}}{\| \nabla \omega \|} \partial_{\nu} \omega
\]

The Euler-Lagrange condition leads to a raw form of the scalar field equation. It can be simplified by subtracting the trace (\( tr \ldots \)) of the Einstein equation (see appendix 2). In Einstein(-MOND) gauge the arising final
version of the scalar field equation is:
\[ \tilde{a}_0^{-1} \nabla \cdot (\| \nabla \omega \| \nabla \omega) + B_1 + B_2 = -\epsilon_{\text{sig}} 4\pi G \text{ tr} T^{(m)} \] (22)

with a MOND-typical l.h.s term plus two “nuisance” terms
\[ B_1 = -6(\nabla^2 \omega + \| \nabla \omega \|^2) \] (23)
\[ B_2 = \frac{\tilde{a}_0^{-1}}{2} \| \nabla \omega \| \nabla \omega \cdot \nabla \log |g| \] (24)

Of course, the Einstein equation (18) and the scalar field equation (22) constitute an interdependent system of differential equations. \( B_1 \) and \( B_2 \) distract from the basic simplicity of the scalar field equation. We shall study it here under simplifying conditions only: a static weak field case under constraints which make the \( B \)-terms negligible (deep MOND case), and a cosmological limit in which \( B_2 \) vanishes and \( B_1 \) reduces to \(-6\nabla^2 \omega\).

3. The scale connection induces an additional acceleration

Free fall of test particles in Weyl geometric gravity follows scale covariant geodesics \( \gamma(\tau) \) of weight \( w(\dot{\gamma}) = -1 \). Their path structure is given by the scale invariant affine connection \( \Gamma \) of eq. (2),
\[ \ddot{x}^\mu + \Gamma^\mu_{\nu\kappa} \dot{x}^\nu \dot{x}^\kappa = 0 \iff \nabla_{\dot{x}} \dot{x} = 0. \] (25)

But their parametrization has to be adapted to the metric \( g \) of any gauge \((g, \varphi, \phi)\) such that always \( g(\dot{\gamma}, \dot{\gamma}) = -\epsilon_{\text{sig}} \). Then the geodesic equation has to be rewritten using the scale covariant derivative
\[ D_{\dot{x}} \dot{\gamma} = 0 \iff \ddot{\gamma}^\mu + \Gamma^\mu_{\nu\kappa} \dot{\gamma}^\nu \dot{\gamma}^\kappa - \varphi_{\nu} \dot{\gamma}^\nu \dot{\gamma}^\mu = 0, \] (26)
(the scale covariance term has been underlined).

Slow (non-relativistic) motions are described by a differential equation formally identical to the one in Einstein gravity, but with scale covariant derivatives of the Weyl geometric affine connection rather than that of the (Riemannian) Levi-Civita one.

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Footnotes:

11 Geodesics are said to be parametrized scale covariantly if to any scale choice \((g, \varphi, \phi)\) there is a parametrization \( \gamma : \mathbb{R} \to M \) with the same image (trace of the curve) such that the norm condition \( g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) = -1 \) holds independently of the gauge; in indices \( g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu = -1 \). More generally, a path \( \gamma \) in a Weylian spacetime manifold \( M \) is given a scale covariant parametrization of weight \(-1\), if to any scale choice \((g, \varphi, \phi)\) a parametrization \( \gamma : \mathbb{R} \to M \) is given, which changes under rescaling of the metric in such a way that \( g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) \) is independent of the gauge.

12 That (25) and (26) characterize the same path structure can be verified by the criterion of projective equivalence for two connections \( \Gamma, \tilde{\Gamma} \), which is \( (\tilde{\Gamma} - \Gamma)_{\nu\kappa} X^\nu X^\kappa \sim X^\mu \) for any vector field \( X \). (26) is the geodesic equation with regard to \( \tilde{\Gamma}^\mu_{\nu\kappa} = \Gamma^\mu_{\nu\kappa} - \frac{1}{2} (\delta^\mu_\nu \varphi_\kappa + \delta^\mu_\kappa \varphi_\nu) \).
Coordinate acceleration $a$ with regard to proper time $t$ for a slow motion parametrized by $x(t)$ is given (analogous to Einstein gravity) by

$$a^j = \frac{d^2x^\mu}{dt^2} \approx -\Gamma^j_{\infty},$$

(27)

Because of (2) the total acceleration decomposes into

$$a^j = -g^j_{\infty} - \varphi \Gamma^\mu_{\nu\lambda} = a^j_R + a^j_W,$$

(28)

with $a^j_R = -g^j_{\infty}$ the Riemann-Einstein component and an additional acceleration $a^j_W = -\Gamma^j_{\infty}$ due to the Weylian scale connection.

For a (diagonalized) weak field approximation in Einstein gauge,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| \ll 1,$$

(29)

with $\eta = \epsilon_{\text{sig}} \text{diag}(-1, +1, +1, +1)$,

$$a^j_R = -g^j_{\infty} \approx \frac{1}{2} \eta^{ij} \partial_j h_{\infty}$$

is the well-known Riemann-Einstein component of the acceleration. In the light of (2) and (54) the Weylian component becomes

$$a^j_W = g^j_{\infty} \varphi^j = g^j_{\infty} \partial_j \omega \approx -\partial_j \omega \approx \varphi_j,$$



4. MOND approximation and a cosmological limit case

Assuming constraints under which the nuisance terms $B_1, B_2$ can be neglected, the scalar field equation (22) reduces to

$$\ddot{a}_o^{-1} \nabla \cdot (\|\nabla \omega\| \nabla \omega) = -\epsilon_{\text{sig}} 4\pi G \text{tr} T^{(m)} \approx -\epsilon_{\text{sig}} 4\pi G \text{tr} T^{(m)},$$

(32)

well known from the AQUAL approach.\footnote{Weinberg 1972, 213ff.} or, for Weyl geometry, (Scholz 2005b, eq.(60)).\footnote{Bekenstein/Milgrom 1984, Bekenstein 2004.}

For a point-like mass source $M$ the r.h.s becomes $-\epsilon_{\text{sig}} 4\pi G \text{tr} T^{(m)} = 4\pi G M \delta(p)$ (delta distribution at $p$).
Considering an Euclidean approximation (with coordinates \(y = (y_1, y_2, y_3)\)) for \(g_{\mu\nu} \approx \eta_{\mu\nu}\), the corresponding (fundamental) solution is:

\[
\omega \approx \sqrt{GM\tilde{a}_o} \log |y| \quad (33)
\]
\[
\nabla \omega \approx \sqrt{GM\tilde{a}_o} \frac{y}{|y|^2}, \quad \nabla^2 \omega \approx \frac{\sqrt{GM\tilde{a}_o}}{|y|^2}
\]

For the MOND approximation, (33), we get from (23)

\[
B_1 \approx -6 \left( \frac{\sqrt{GM\tilde{a}_o}}{|y|^2} + \frac{GM\tilde{a}_o}{|y|^2} \right).
\]

As \(0 \ll GM\tilde{a}_o \ll \sqrt{GM\tilde{a}_o} \ll 1\), the first term dominates. In the deep MOND case (\(dM\), appendix 3), with \(|y| \geq \sqrt{\frac{GM}{a_o}}\) it is bounded by

\[
\frac{\sqrt{GM\tilde{a}_o}}{|y|^2} \leq_{dM} a_o \sqrt{\frac{\tilde{a}_o}{GM}}.
\]

For stars and for galaxies\(^{16}\) we find

\[
B_1 \leq \begin{cases} 
10^{-17}H_o & \text{for stars}, \\
10^{-22}H_o & \text{for galaxies}.
\end{cases} \quad (34)
\]

\(B_2\) vanishes in the Euclidean approximation (\(\nabla \log |g| = 0\)), but not, e.g., in the Schwarzschild metric. There we get with \(\omega\) like in (33)

\[
\nabla \omega \cdot \nabla \log |g| \approx \frac{4\sqrt{GM\tilde{a}_o}}{|y|^2}
\]

and

\[
B_2 \approx \frac{2GM}{|y|^2} \leq_{dM} 2a_o \sqrt{\frac{\tilde{a}_o}{GM}},
\]

comparable to \(B_1\). Both can safely be neglected in the deep MOND regime, as is shown by (34).

Finally we want to make a short observation with regard to the cosmological limit. For this limit we use the idealizing assumption of a homogeneous matter distribution. Then the invariant scalar field does not depend on spacelike coordinates,

\[
\omega(x) = \omega(t), \quad \nabla \omega = (\partial_t \omega, 0, 0, 0).
\]

\(\|\nabla \omega\|\) vanishes and with it the MOND-typical term and \(B_2\) in (22). The scalar field equation reduces to

\[
\nabla^2 \omega = \partial_t^2 \omega = \epsilon_{\text{sig}} \frac{2}{3} \pi G |T| T^{(m)}.
\]

\(^{16}\tilde{a}_o \approx \frac{1}{2} H \sim 10^{-28} \text{ cm}^{-1}, \tilde{a}_o\) of the same order of magnitude (see below, section 6), \(GM \sim 10^5 \text{ cm}\) for typical stars and \(GM \sim 10^{16} \text{ cm}\) for typical galaxies.
In the vacuum case we get:

\[ \omega(t) = \text{const}, \quad \varphi = (\text{const}, 0, 0, 0) \]

This condition is satisfied for a simple type of time-homogeneous stationary solutions of the vacuum Einstein equation (18). In Einstein gauge it has the (Riemannian) geometry of an Einstein universe and a non-vanishing Weylian scale connection \( \varphi = (H, 0, 0, 0) \) which encodes the cosmological redshift.\(^1\)

5. Inhomogeneities of the scalar field energy tensor

We now want to address the inhomogeneities in the distribution of the scalar field, induced by matter. We use the static weak field approximation (29), in Einstein gauge, near a mass center. Then \( \omega(x) \) depends only on the spacelike coordinates of \( x = (x_0, \ldots, x_3) \), which we denote separately by \( y := (y_1, y_2, y_3) = (x_1, x_2, x_3) \). The energy-momentum tensor of the scalar field \( T(\phi) = (8\pi G)^{-1} \Theta \) is given by (19), (20). The second term of the energy density of \( \Theta^{(I)} \) vanishes. It only remains

\[ \Theta^{(II)}_{oo} = \phi^{-2} D_o D_o \phi^2, \]

which cancels with the index= 0 summand of the last term in \( \Theta^{(I)}_{oo} \). In the (static) weak field case, the cosmological constant contribution \( L_{V4} g_{oo} \) does not contribute to the inhomogeneity of the energy density. The inhomogeneity contribution of \( \Theta_{oo} \) is (here \( g_{oo} \approx \eta_{oo} \approx -\epsilon_{sig} \))

\[ \Theta_{oo}^{(in)} \approx (-\epsilon_{sig}(\xi \phi)^{-2} L_o + \phi^{-2} D_j D^j \phi^2) \epsilon_{sig}, \quad j = 1, 2, 3 \]

\[ \approx -\frac{2}{3} (\eta^{-1} \phi)^{-1} \| \nabla \omega \|^3 + \phi^{-2} D^2 \phi^2, \]

where we have used the following abbreviation analogous to (10):

\[ D^2 := \epsilon_{sig} D_j D^j \quad (j = 1, \ldots, 3) \]

With (53) (appendix 1) we get:

\[ \Theta_{oo}^{(in)} \approx -\frac{2}{3} (\eta^{-1} \phi)^{-1} \| \nabla \omega \|^3 + 2 \left( 2 \| \nabla \omega \|^2 + \nabla^2 \omega + \epsilon_{sig} \Gamma^j_{jk} \partial^k \omega \right) \]

In the MOND approximation, \( a_o^{-1} \| \nabla \omega \|^3 \approx a_o^{-1} G M a_o \sqrt{G M a_o} |y|^{-2} \) and \( \| \nabla \omega \|^2 \approx G M a_o |y|^{-2} \) are much smaller than the other terms (0 \( \ll \) GM \( \ll \) \( \sqrt{G M a_o} \ll 1 \):

\[ \nabla^2 \omega \approx \frac{\sqrt{G M a_o}}{|y|^2} \]

\(^{17}\) (Scholz 2005a, Scholz 2009).
and, for the Schwarzschild metric,

$$\epsilon_{stg} \Gamma^j_{jk} \vartheta^k \omega \approx \frac{2 \sqrt{GM \tilde{a}_o}}{|y|^2}$$

The inhomogeneity part of the energy density is thus (in Einstein gauge, MOND approximation, Schwarzschild metric as Riemannian component of the Weylian metric)

$$T_{oo}^{(in)} = (8\pi G)^{-1} \Theta_{oo}^{(in)} \approx (8\pi G)^{-1} \frac{6 \sqrt{GM \tilde{a}_o}}{|y|^2}$$

(41)

The inhomogeneity part of the scalar field energy appears on the r.h.s of the Einstein equation in addition to the inhomogeneities of the matter term, as some kind of transparent matter/energy contribution. In order to estimate its amount about a spherical symmetric astrophysical structure, we have to integrate $\Theta_{oo}^{(in)}$ from one center of inhomogeneity to roughly the next one,

$$E^{(in)} = \int T_{oo}^{(in)} dv = (8\pi G)^{-1} \int \Theta_{oo}^{(in)} dv = E_{pl} \cdot L_{pl}^{-1} \int \Theta_{oo}^{(in)} dv,$$

with the 3-dimensional volume form $dv$, and omitting the $c, \hbar$ factors. If $r_1$ denotes half the distance to the next center of inhomogeneity and $r_o$ the distance from which the inhomogeneity energy is considered, we get

$$E^{(in)} \approx E_{pl} \cdot L_{pl}^{-1} \int_{r_o}^{r_1} \Theta_{oo}^{(in)} 4\pi r^2 dr$$

$$\approx 24\pi \sqrt{GM \tilde{a}_o} \frac{r_1 - r_o}{L_{pl}} E_{pl}$$

(42)

Let us make a rough estimate for stars as constitutive elements of a galaxy, taking a sun-sized star $M_\odot \sim 10^{57} GeV \sim 10^{38} E_{pl}$, $GM_\odot \sim 10^9 cm$, a typical distance to the next star $r_1 \sim 1 pc \sim 10^{18} cm$, and beginning the integration with the onset of the deep MOND regime, $r_o \sim 10^{-1} pc$ (appendix 3). Assuming $\tilde{a}_o$ at one order of magnitude below $a_o \approx \frac{1}{6} H \sim 10^{-29} cm^{-1}$ (cf. section 6), we find

$$E^{(in)}_\odot \sim 10 \cdot 10^{5-30} \cdot 10^{18+33} \sim 10^{39} E_{pl} \sim 10 M_\odot.$$  

(43)

That is a surprisingly high contribution of the scalar field energy to the total gravitating star mass in the distant field. If our model is realistic, this would mean that the gross contribution of stars to the gravitational mass of a galaxy may be up to an order of magnitude larger than the net contribution of their baryonic mass. It has to be investigated whether such a high amount is compatible with mass to light ratios inferred for modelling galaxy dynamics.
For galaxies the estimation is more complicated, because spherical symmetry is broken. Even so, the inhomogeneities of $\Theta$ arising from a scalar field solution with the r.h.s. of (22) the baryonic mass of a galactic disk will establish a considerable amount of additional transparent scalar field energy (in addition to the scalar field energy about the single stars, estimated above).\footnote{If we perform a toy computation for galaxies analogous to that for stars, we find for $M_{\text{gal}} \sim 10^{11} M_\odot \sim 10^{45} E_{\text{pl}}$, $GM_{\text{gal}} \sim 10^{18} \text{cm}$ and a mean distance to the next galaxy in the cluster $r_1 \sim 1 \text{ Mpc} \sim 10^{24} \text{cm}$: $E_{\text{gal}}^{(\text{in})} \sim 10 \cdot 10^{16-36} \cdot 10^{24+33} \sim 10^{51} E_{\text{pl}} \sim 100 M_{\text{gal}}$. The non-spherical shape of galaxies reduces, of course, the value considerably.}

Therefore we have to expect considerable influences on cluster dynamics and on lensing effects by $E^{(\text{in})}$ of galaxies.

6. Additional Newton acceleration and determination of $\tilde{a}_o$

In the Newtonian limiting case the inhomogeneity energy of the scalar field about a point-like mass center (41) enters also the right hand side of the Poisson equation. In the MOND approximation we have:

$$\nabla^2 \phi_N(y) = 4\pi G \left( M \delta(p) + \frac{6\sqrt{GM\tilde{a}_o}}{8\pi G |y|^2} \right)$$

(44)

The total Newton potential consists now of two components, $\phi_N = \phi_{N_1} + \phi_{N_2}$, with $\phi_{N_1}(y) = -\frac{GM}{|y|}$ from the point-like mass itself and a contribution from the quadratically decreasing continuous source

$$\nabla^2 \phi_{N_2}(y) \approx 3\sqrt{GM\tilde{a}_o} \frac{|y|}{|y|^2}.$$

(45)

The solution for the latter is (cf. (33))

$$\phi_{N_2}(y) \approx 3\sqrt{GM\tilde{a}_o} \log |y|.$$

(46)

Accordingly the total Newton acceleration consists of two contributions $a_N = a_{N_1} + a_{N_2}$ with

$$a_{N_1} = -GM \frac{y}{|y|^3} \quad \text{and} \quad a_{N_2} \approx -3\sqrt{GM\tilde{a}_o} \frac{y}{|y|^2}.$$

(47)

The second term is three times a MOND-like acceleration with the modified constant $\tilde{a}_o$. The total correction of the original Newton dynamics of a point-like source in our approach is ((31), (33), (47))

$$a_{\text{add}} = a_{N_2} + a_W \approx -4\sqrt{GM\tilde{a}_o} \frac{y}{|y|^2}.$$

(48)

Finally we can specify the value of our $\tilde{a}_o$ for which our model gives a total additional acceleration which agrees with the acceleration of the empirical MOND model:

$$\tilde{a}_o = \frac{a_o}{16} \approx \frac{H}{100} \approx 8 \cdot 10^{-31} \text{cm} \leftrightarrow 2 \cdot 10^{-20} \text{s}^{-1}.$$

(49)

Then $a_{\text{add}} \approx -\sqrt{GM\tilde{a}_o} \frac{y}{|y|^2}$, with $a_o$ the usual MOND acceleration.
7. Discussion

Our assimilation of the original (R)AQUAL Lagrangian to Weyl geometric gravity has shown quite convincing properties. The Weyl geometric approach with its consequent scale invariance, respectively its scale covariant expressions, is conceptually clearer than the “2 metric approach” of the Jordan-Brans-Dicke framework in the AQUAL theory. Einstein gauge and Riemann gauge (or any other gauge) are here logically equivalent. Which one is used, may be made dependent on the specific problem context one is dealing with. It should, perhaps, be added that *Einstein gauge gives the most immediate expression* to measured quantities; in this sense it may be considered as the *appearing gauge* or, a little misleading, as a “physical gauge”. The additional degree of freedom (in comparison to Einstein gravity) is regulated by the scalar field equation (22). This equation can just as well be read as a condition for the Weylian scale connection (in Einstein gauge).

In the *first step* of our analysis (sections 3 and 4), the *scale connection* gives – in MOND approximation – an *additional MOND-like acceleration* with the invariant scalar field $\omega$ as its potential (31). So far our analysis is quite close to RAQUAL, the main difference being that the Newton approximation of Einstein gravity remains a partial contribution of our MOND approximation ((28), (30)). It is, in the first step, only amended by an additional component of the acceleration due to the scale connection of the Weylian metric.

In a *second step* we have analyzed the inhomogeneities in the energy density of the scalar field/scale connection (sections 5 and 6) and found that they *modify the total Newton potential* of the static weak field approximation considerably (41), (44). This comes about without any additional stipulation, just by analyzing the r.h.s of the scale invariant Einstein equation. This consequence of our approach changes the situation for *cluster dynamics and gravitational lensing* considerably, compared with other relativistic MOND approaches. Of course we could here lay down only the fundamentals of the modification. Detailed investigation of the consequences have to follow and are very welcome.

Given that the problems of *cluster dynamics and gravitational lensing* seem to have been even more important for giving up the original RAQUAL approach than the appearance of unphysical faster-than-light perturbations, it is interesting to see why a similar derivation of a changed Poisson equation has not been made there. Apparently it has become a conviction, even a kind of “folk theorem”, that strictly scale covariant (or conformal) metrical

\[\text{Whoever thinks of free fall as being governed by a Levi-Civita connection in the Riemannian sense, may just as well argue for Riemann gauge as “physical”. The unification of such different aspects necessitates a consequently Weyl geometric perspective.}\]

\[\text{\textcopyright Bekenstein 2004, 6}\]
approaches can never lead to a derivation of gravitational lensing effects.\textsuperscript{21}

But the argument given for this conviction relies crucially on the conditional clause made explicit in the following statement

“so long as the $\psi$ field [corresponding to our $\omega$, E.S.] contributes comparably little to the energy-momentum tensor, it cannot affect light deflection . . .” (Bekenstein 2004, 6).

Why does this condition not apply to our Weyl geometric extension of essentially the same Lagrangian like in RAQUAL? The answer can be read off from (38), digging back to (19) and (16). The crucial difference in our energy-momentum tensor to the one used in most JBD-approaches (not in all\textsuperscript{22}) comes from the consideration of the \textit{boundary terms in varying the Hilbert action} to which the scalar field is non-minimally coupled\textsuperscript{23}. It is these terms, roughly $D_\mu D_\nu \phi^2$ and $D_\nu D^\nu \phi^2$, which contribute essentially to the energy-momentum of Weyl geometric gravity. The successful adaptation of RAQUAL to Weyl geometric gravity seems a strong support for this perspective.

Of course, a series of questions remains:

- Is the scalar field energy contribution to the total disk mass (43) consistent with observational estimates of the mass-to-light ratio of galaxies?

- Is the additional scalar field mass/energy in fact able to contribute essentially to solving the cluster dynamics and gravitational lensing problems?

- Is the “unphysical” propagation of scalar field perturbations suppressed in our approach (by the inertial effects from the enhanced energy density), or does it persist?

- Our considerations have been concentrated on the deep MOND regime. What happens in the Newton regime, or even the Schwarzschild regime? What are (approximate) solutions of the scalar field equation (22) in these domains and how do they affect the dynamics? Is the re-introduction of the case distinction in the “classical” MOND approach between deep MOND and Newton regimes necessary, or is it justified to avoid it?

- What can we say about the transition between deep MOND regimes and the cosmological limit?

\textsuperscript{21}See, among others, (Sanders 2010, 146f.)

\textsuperscript{22}(Fujii/Maeda 2003, 40ff.)

\textsuperscript{23}See the literature in fn. 10
Finally, is the geodesic principle (EPS compatibility) derivable for a sufficiently large class of classical matter Lagrangians or, at least, consistent with it?

Whatever the answers to these questions will be, it seems striking that in our model the effects usually ascribed to “dark matter” or to “dark energy” appear here as features of the energy-momentum of the scalar field/scale connection. The empirically most directly accessible dark matter effects are due to the inhomogeneity of the energy density of the scalar field (41), dark energy to the metric proportional component of the energy-momentum tensor (19). The scalar field $\phi$, its scale invariant version $\omega$, and the scale connection $\varphi$ are mutually interdependent expressions of the same dynamical entity which enhances the Riemannian component of the Weylian metric to its full expression $(g, \varphi, \phi)$. In this way, the Weyl geometric AQUAL model leads to a unification of the “dark sector”. In the end, both features can be attributed to the gravitational field itself, if one takes the Weyl geometric extension of the Riemannian metric seriously.

Even if one or more of the above questions cannot be answered soon, or not with satisfying results, such considerations show that the change of perspective opened by the transition from Riemannian geometry and Einstein gravity to Weyl geometry and the according generalization of gravitation theory is worthwhile to explore. In the light of present disillusions with regard to the perspective of explaining the dark sector by modified gravity theories, the Weyl geometric approach with the SIC Lagrangian may change the profile of some of the outstanding problems. Perhaps they become answerable in a surprising way.

Appendix 1: Scale invariant version of scalar field

In Riemann gauge $(\tilde{g}, 0, \tilde{\phi})$ we write $\tilde{\phi} = e^{\omega}$. By definition $\omega$ is not affected by regauging, therefore

$$D_\nu \omega = \partial_\nu \omega.$$  \hspace{1cm} (50)

It is a scale invariant version of the scalar field.

Any scale gauge $(g, \varphi, \phi)$ arises from Riemann gauge, $g = \Omega^2 \tilde{g}$, for some $\Omega$. Then

$$\varphi = -d \log \Omega \leftrightarrow \Omega = e^{-\int \varphi}.$$  

\hspace{1cm} (Starkman 2011)

\hspace{1cm} (Kroupa e.a. 2012)
here $\int \varphi$ is an abbreviated notation for integrating the 1-form $\varphi$ along any curve from a fixed initial point to the point $x$ of spacetime considered (underdetermination only up to a point independent constant). We thus get

$$\tilde{\phi} = \Omega \phi,$$

$$\omega = \log \tilde{\phi} = \log \phi - \int \varphi,$$

$$\phi = \Omega^{-1} e^{\omega} = e^{\omega + \int \varphi}.$$ (51)

In some of the recent literature $\varphi_{\text{comp}} := e^{\int \varphi}$ is considered on its own (with $\omega = 0$) (Almeida e.a. 2014, Pucheu e.a. 2014). Because of the gauge transformation for the scale connection it transforms with weight $w(\varphi_{\text{comp}}) = -1$ like $\phi$. But it does not essentially contribute to the dynamics besides giving it a scale covariant expression ("compensating field"). Restricting to $\varphi_{\text{comp}}$ boils down to considering Einstein gravity in scale covariant form. The result is a dynamically trivial Weyl geometric extension of Einstein gravity (and Riemannian geometry).

The scale covariant derivative of the scalar field in any gauge can be expressed in terms of the latter:

$$D_\nu \varphi = (\partial_\nu - \varphi_\nu) \phi = \partial_\nu e^{\omega + \int \varphi} - \varphi_\nu \phi = (\partial_\nu \omega + \varphi_\nu) \phi - \varphi_\nu \phi = \phi \partial_\nu \omega = \phi D_\nu \omega$$ (52)

Similarly one derives for $||\omega|| > 0$

$$D^j \phi^2 = \partial^j (e^{2(\omega + \int \varphi)}) - 2 \varphi^j \phi^2 = 2 \phi^2 \partial^j \omega,$$

$$D_j D^j \phi^2 = 2 \phi^2 \left( \partial_j \partial^j \omega + 2 \partial_j \omega \partial^j \omega + \Gamma^j_{jk} \partial^k \omega \right);$$

and thus:

$$\phi^{-2} D^2 \phi^2 = \phi^{-2} \epsilon_{\text{sig}} D^h D_h \phi^2 = 4 \|\nabla \omega\|^2 + 2 \nabla^2 \omega + 2 \epsilon_{\text{sig}} \Gamma^j_{jk} \partial^k \omega$$ (53)

If $(g, \varphi, \phi_c)$ denotes a scalar field gauge, in particular Einstein gauge $\phi_c = \xi^{-1} E_{pl}$, we have $\phi_c = \Omega^{-1} \tilde{\phi}$ with $\Omega = \phi_c^{-1} e^{\omega} = \xi E_{pl}^{-1} e^{\omega}$; thus $\varphi \doteq -d \log \Omega \doteq -d \omega$ and

$$\varphi_\nu \doteq -\partial_\nu \omega.$$ (54)

Thus $\omega$ has the formal properties of a potential for the scale connection $\varphi$ in scalar field gauge (and only in this gauge).

Appendix 2: Derivation of the scalar field equation

We calculate the variation in Riemann gauge (then $R$ contains no $\varphi$-terms). Because of scale invariance of the Lagrangian, the result translates straight
forward to any gauge. Using (22) we get:

\[
\frac{\delta L_{HW}}{\delta \omega} = \frac{\delta L_{V4}}{\delta \omega} = \frac{\partial L_{HW}}{\delta \omega} = \epsilon_{sig} \phi^2 R \sqrt{|g|} = 2 L_{HW}
\]

\[
\frac{\delta L_{V4}}{\delta \omega} = \frac{\partial L_{V1}}{\delta \omega} = 4 L_{V4}
\]

\[
\frac{\partial L_{\phi}}{\delta \omega} = L_{\phi}
\]

With (22)

\[
\frac{\partial L_{\phi}}{\partial (\partial \nu \omega)} = 2 \xi \eta \phi \parallel \nabla \omega \parallel \epsilon_{sig} \partial \nu \omega;
\]

thus:

\[
\frac{\partial \nu}{\delta L_{\phi}} = 2 (\xi \phi)^2 \eta (\phi) \phi \parallel \nabla \omega \parallel \epsilon_{sig} \partial \nu \omega
\]

The resulting “raw” scalar field equation is

\[
0 = 2 L_{HW} + 4 L_{V4} + L_{\phi} + (\xi \phi)^2 2 B_2 - 2 (\xi \phi)^2 (\eta^{-1} \phi)^{-1} \nabla \cdot (\parallel \nabla \omega \parallel \nabla \omega)
\]

with \(B_2 = \frac{1}{2} \epsilon_{sig} (\xi \phi)^2 \parallel \nabla \omega \parallel \nabla \omega \cdot \nabla \log |g|\) [24]. The trace of the Einstein equation [18], multiplied by \(\epsilon_{sig} (\xi \phi)^2\), is:

\[
\epsilon_{sig} (\xi \phi)^2 R - L_{V4} + 4 L_{\phi} - 3 L_{\phi} - 3 \epsilon_{sig} \xi^2 D \lambda D \phi^2 + \epsilon_{sig} tr T^{(m)} = 0
\]

Subtracting both and dividing by \(2 (\xi \phi)^2\) leads, in Einstein gauge, to the equation [22],

\[
\tilde{a}_o^{-1} \nabla \cdot (\parallel \nabla \omega \parallel \nabla \omega) + B_1 + B_2 = -\epsilon_{sig} 4 \pi G tr T^{(m)},
\]

with \(B_1 = -6 (\nabla^2 \omega + \parallel \nabla \omega \parallel^2)\) [23] and \(B_2 = \frac{\tilde{a}_o^{-1}}{2} \parallel \nabla \omega \parallel \nabla \omega \cdot \nabla \log |g|\) [24].

**Appendix 3: Deep MOND regime**

The MOND phenomenology works with an anomalous acceleration derived from a potential, \(a_M = -\nabla \phi_M\). For a point mass \(M\) it is \(\phi_M = \sqrt{GMa_o} \log |y|\) (in Euclidean metric), with \(a_o \approx \frac{1}{6} H_o [c]\) (here shorter \(a_o \approx \frac{1}{6} H\) to avoid the factors \(c\) distinguishing \(H_o\) and \(H_1\)). Then

\[
a_M = -\sqrt{GMa_o} \frac{y}{|y|^2}, \quad |a_M| = \frac{\sqrt{GMa_o}}{|y|}.
\]

We consider a point to belong to the deep MOND regime, if the Newton acceleration of the point mass is smaller than the anomalous acceleration:

\[
\frac{GM}{|y|^2} \leq \frac{\sqrt{GMa_o}}{|y|} \quad \iff \quad GM \frac{y}{|y|^2} \leq a_o \quad \iff \quad |y| \geq \sqrt{\frac{GM}{a_o}}
\]

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For stars at the order of our sun the Schwarzschild radius is $2GM_\odot \sim GM_\odot \sim 10^5 \text{cm}$, while $\frac{H}{H} \sim 10^{-29} \text{cm}^{-1}$. The deep MOND regime is reached for $|y| \geq 10^7 \text{cm} \sim 10^4 \text{AU} \sim 10^{-1} \text{pc}$. For the mass of a galaxy with $M_{gal} \sim 10^{11}M_\odot$, idealized to spherical symmetry, the deep MOND regime starts 5 to 6 orders of magnitude higher, $|y| > 10 \text{kpc}$.

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