Structural Controllability of a Consensus Network With Multiple Leaders

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Abstract—This paper examines the structural controllability for a group of agents, called followers, connected to each other based on the consensus law under commands of multiple leaders, which are agents with superior capabilities, over a fixed communication topology. It is proved that the graph-theoretic sufficient and necessary condition for the set of followers to be structurally controllable under the leaders’ commands is leader–follower connectivity of the associated graph topology. This shrinks to graph connectivity for the case of solo leader. In the approach, we explicitly put into account the dependence among the entries of the system matrices for a consensus network using the linear parameterization technique.

Index Terms—Multiagent systems, structural controllability.

I. INTRODUCTION

In Recent years, due to the importance of analyzing the complex systems, the notion of structural controllability has been retaken into consideration. Defined as controllability of systems for almost every parameter values, structural controllability has a wide range of applications from robotics [2] to biological systems [3]. Lin [4] first introduced the structural controllability for single input linear time-invariant (LTI) dynamical systems. He provided a graph-theoretic representation that guarantees the structural controllability for LTI systems, i.e., controllability for almost every parameter values. The new notion of controllability that Lin introduced, encouraged other researchers to investigate the interaction among systems’ parameters.

The authors of [5] presented an algebraic representation of Lin’s theorem and also extended the theorem to scrutinize the structural controllability for multi-input LTI systems. The aforementioned studies dealt with dynamical systems in which each of the nodes represents a first-order dynamical system. The structural controllability of multi-input/multiputput high-order systems was investigated in [6]–[8]. In most cases, one may need to examine beyond the fact that whether a system is structurally controllable or not. For instance, when dealing with uncontrollable systems, declaring the maximum controllable subspace enables us to know our ability to control the system (see [9]–[12]). Moreover, in applications such as systems biology, the choice of input nodes (driver nodes) is so broad that selecting a proper set of nodes to ensure the controllability becomes a crucial problem in cell reprogramming or in cancer treatment (see [13]–[16]). The minimum number of driver nodes to guarantee the structural controllability of LTI systems was determined in [13]. In there, the authors provided a polynomial algorithm to determine the driver nodes.

These studies have a common assumption that the nonzero entries of the pair \((A, B)\) are independent of each other with free choices. Despite the wide application of this theory, it cannot analyze a system with the same scalar values appearing in more than one place in the pair \((A, B)\), i.e., entries of the pair \((A, B)\) cannot be assigned arbitrarily, which is indeed the case for many dynamical systems. To overcome this dilemma, one can represent the system by a linearly parameterized model. The authors of [1] extended the notion of structural controllability to linearly parameterized systems with binary assumptions. This extension has a great application in the areas such as cooperative control of multiagent systems where there exists an inherent dependence between the entries of the state and input matrices that capture the network topology.

In this paper, we examine the structural controllability for a group of agents equipped with a consensus law. The notion of controllability for such a setup was first proposed by [17]. This reference exploited controllability to examine the possibility that a group of interconnected agents through consensus law can be steered to any desired configuration under the command of a single leader. Several necessary and sufficient algebraic conditions for controllability of multiagent systems based on eigenvectors of the associated Laplacian graph were introduced in [17]. The problem was then developed further in [18]–[21]. For instance, in [19], it was concluded that devoid of eigenvalue sharing between the Laplacian matrix associated with the follower set only and the Laplacian matrix corresponding to the whole topology is both necessary and sufficient for controllability. The only condition necessary for the controllability of followers under multiple leaders was provided in [21]. The graph-theoretic representation of these results was introduced in [22]. Moreover, the structural controllability of multiagent systems with a switching topology and the structural controllability of higher order multiagent systems were studied in [23] and [24], respectively. Another sphere of research relies on the behavior of the system before and after establishing link or agent removal. Robustness of structural controllability against node and link removal was investigated in [25] and [26].

In this paper, we exploit the linear parameterization technique to deal with the dependence among the entries of the pair \((A, B)\) when analyzing the structural controllability of interconnected linear systems. The same problem for the case of solo leader was addressed in [22]. Even though the results reported in [22] are correct, the authors neglected the above-mentioned inherent dependence in the main proof stated there. Moreover, the authors in [23] addressed the structural controllability for a group of interconnected agents with multiple leaders under a switching topology and provided the sufficient and necessary condition. Similar to [22], in this reference there exists an implicit assumption about independence of entries appearing in the \(A\) and \(B\) matrices. There seems no clear clue to fix the flaw of the proofs within the same framework. Therefore, we aim to provide an alternative rigorous proof using the linear parameterization technique recently developed in [1].

The rest of this paper is organized as follows. The terminology and concepts used in this paper are defined in Section II. The problem formulation is given in Section III. We study the case where there...
exists only one leader among agents in Section IV. Then the results of this section are exploited in Section V to examine the multiple leaders case. Finally, Section VI concludes the paper.

II. PRELIMINARIES

A. Structural Controllability

Roughly speaking, the concept of controllability, as a paramount property of control systems, examines the capability of a system to steer from any initial state to some desired final value within its entire configuration space under a proper control law. The answer to this examination is given by controllability tests such as the Kalman’s rank condition [27] or Popov–Belevitch–Hautus (PBH) controllability test [28]. However, for many dynamical systems, the system’s parameters are not precisely known and in some cases, the existence or absence of system parameters is the only accessible information. In addition to this, some systems have time-variant parameters, and it is computationally hard to determine the controllability of these systems during the whole process. In order to overcome these challenges, Lin in his seminal paper [4] introduced the notion of structural controllability. The Lin’s theorem provided a test for checking the controllability of structured LTI systems, which are LTI systems whose entries of $A$ and $B$, i.e., the state and input matrices are either zero or independently free parameters. As defined in [4], the pair of matrices $(A, B)$ with each entry either being a zero value or an arbitrarily chosen scalar not depending on other entries, is structurally controllable if there exists a real controllable pair, say $(\hat{A}, \hat{B})$, with the same structure of zero entries as $(A, B)$. Consequently, the system is concluded to be controllable for almost every parameter values.

The result in [4] is insightful but the definition of structural controllability introduced in [4] needs to be slightly modified to be applicable to systems with some inherent dependencies among the entries of $A$ and $B$. One should note that it is ubiquitous in many practical scenarios such as in biological systems, where one explores gene–gene interactions, or links, between the entries of $A$ and $B$. The interconnecting links are related to each other. To accommodate these systems, one needs to modify the structural controllability definition in [4] and propose new test tools. One way is to represent the system in the linearly parameterized form [29]. This approach is a convenient method to analyze LTI systems with parameter repetition in their state and/or input matrices. We briefly review linear parameterization of structured systems in the next subsection.

B. Linear Parameterization

Consider the LTI system given as

$$\dot{x} = A(w)x + B(w)u$$

(1)

where $A(w) \in \mathbb{R}^{n \times n}$ and $B(w) \in \mathbb{R}^{n \times m}$ are functions of an arbitrarily selected vector $w = [w_1\ w_2\ \ldots\ w_n]$. Suppose the matrices $(A, B)$ have $p$ nonzero entries.

For the more general scenario with $\sigma \leq p$, the matrix pair $(A, B)$ can be linearly parameterized as

$$A_{n \times n}(w) = \sum_{k \in \mathbb{Q}} c_k w_k r_{k1}, \quad B_{n \times m}(w) = \sum_{k \in \mathbb{Q}} c_k w_k r_{k2}$$

(2)

where $\mathbb{Q} = \{1, \ldots, \sigma\}$, $c_k \in \mathbb{R}^n$, $r_{k1} \in \mathbb{R}^{1 \times n}$, and $r_{k2} \in \mathbb{R}^{1 \times m}$. We provide the following example for further explanation of linear parameterization.

Example 2.1: Consider the following equation

$$\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_3
\end{bmatrix} =
\begin{bmatrix}
-w_1 - w_2 - w_3 & w_2 & w_3 \\
w_2 & -w_2 & 0 \\
w_3 & 0 & -w_3
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
+ \begin{bmatrix} w_1 \\ 0 \\ 0 \end{bmatrix} u.$$  

(3)

The above LTI system attains the pair $(A, B)$ which is a function of $[w_1 w_2 w_3]^T$, and its associated linear parameterization can be represented as

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_{11} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad r_{12} = 1$$

$$c_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad r_{21} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad r_{22} = 0$$

$$c_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r_{31} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad r_{32} = 0.$$  

(4)

It is obvious that the vectors $c_1$, $c_2$, and $c_3$ are linearly independent of each other and $\sigma = 3$.

The pair $(A(w), B(w))$ is called structurally controllable if there exists a parameter vector $w \in \mathbb{R}^{\sigma}$ for which the pair $(A(w), B(w))$ is controllable [1]. We adapt the same definition in this paper.

It is worthwhile noting that, for $\sigma = \sigma$, the definitions of structural controllability in [1] and [4] are identical and the structural controllability of the system in (1) can be studied by the results in [4]. The structural controllability of the system (1) for $\sigma = \sigma$ is also explored in [29] from an algebraic point of view. Despite the well-approved algebraic structural controllability conditions in [29], in most cases, the graph-theoretic perspective provides more insights regarding hidden relations that undergo among system’s parameters. In the next subsection, we give a short review on some graph theory concepts exploited in this paper.

C. Graph Notation

Weighted digraphs to represent dynamical systems were exploited in [4]. This graph representation not only shows the existence of directed interactions, or links, between the entries of $A$ and $B$, but also reveals the strength of those links. This way of demonstrating dynamical systems enabled the author of [4] to introduce graph-theoretic descriptions for structural controllability of single input LTI systems. In this paper, we deploy the flow graph representation to study the dynamical system (1) from graph-theoretic point of view.

Consider the weighted graph $G$ with its node set $V = \{v_1, v_2, \ldots, v_{N_V}\}$, edge set $E = \{(e_1, e_2), (e_2, e_3), \ldots, (e_{N_E}, e_{N_E})\}$, and weight set $W$ correspondence to each link $W = \{w_1, w_2, \ldots, w_{N_W}\}$. Let $N_V$ and $N_E$ be the number of the nodes and the edges, respectively. Then graph representation of the dynamical system (1), which is called the flow graph denoted by $F_G$, is a digraph. It includes $n + m$ vertices $V = \{v_1, v_2, \ldots, v_{n+m}\}$, where the input nodes take the last $m$ indices, i.e., $v_{n+1}, \ldots, v_{n+m}$. Moreover, if the $\{i, j\}$ entry of the matrix $[A, B]$ is nonzero, there exists a link from $v_j$ to $v_i$. Two nodes are called neighbors if there exists an edge that corresponds to these two nodes. A path is a set of edges that connect a set of distinct nodes. The
digraph is called connected provided that there exists a bidirectional path between every two different vertices.

The flow graph $F_n$ has a spanning forest rooted at vertices $v_{n+1}, v_{n+2}, \ldots, v_{n+m}$ if for any other node of the graph, say $v_j \in \{v_1, v_2, \ldots, v_n\}$, there exists a path from one of the root nodes $v_i \in \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\}$ to $v_j$.

Finally, the Laplacian matrix associated with the graph $G$ is defined as

$$L_{(i,j)} = \begin{cases} \sum_{i \neq j} a_{ij} & i = j \\ -a_{ij} & i \in \text{neighborhood of } j \\ 0 & \text{otherwise} \end{cases}$$  \quad (5)$$

For the sake of clear explanation, we represent the flow graph of the system (3) in Fig. 1.

Remark 2.1: It is noteworthy that the vectors $c_1$ and $r_{1i}$ have graph-theoretic implications in the system’s associated flow graph. On one hand, a nonzero entry of $c_1$, say $j$, captures an outgoing edge to node $v_j$ with weight $w_1$ in its associated flow graph. On the other hand, in $r_{1i}$ a nonzero entry expresses an outgoing edge from node $v_i$ in the associated flow graph, and $j$ is the index of that nonzero entry. This is further demonstrated in the following example.

Example 2.2: Consider the following system:

$$\dot{x} = \begin{bmatrix} w_2 & 0 & w_2 \\ w_1 & 0 & 0 \\ w_1 & 0 & 0 \end{bmatrix} x.$$  \quad (6)$$

One can verify that this system can be linearly parameterized with vectors $r_{11}, r_{12}$, and $c_1$ related to weight $w_1$ and $w_2$ as

$$c_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad r_{11} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad r_{12} = 0$$

and

$$c_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_{21} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad r_{22} = 0.$$  \quad (7)$$

The flow graph of this system is represented in Fig. 2.

As stated in Remark 2.1, each of the nodes $v_2$ and $v_3$ in the corresponding flow graph (the indices of nonzero entries of $c_1$) has an ingoing edge with weight $w_1$. Moreover, the nonzero entry of $r_{11}$ (the first entry) determines the outgoing edge from node $v_1$ within the associated flow graph. This system has two outgoing edges from $v_1$ (one to node $v_2$ and the other one to node $v_3$) with weight $w_1$. It is worth noting that the vectors $c_2$ and $r_{22}$ both have a nonzero first entry. This means that the node $v_1$ has a self loop with weight $w_2$.

**D. Matrix-Algebraic Terminology**

In this subsection, we state some notions and results that help us in establishing the main result of this paper.

The generic rank denoted by $\text{g-rank}[]$ of linearly parameterized matrix

$$M(w) = \begin{bmatrix} A(w) & B(w) \end{bmatrix} = \sum_{k \in \mathbb{Q}} c_k w_k r_k$$

where $r_k = [r_{k1} \ r_{k2}]$, is the maximum rank of $M(w)$ for all possible values of $w$. Furthermore, the pair $(A, B)$ is irreducible if there exists no permutation matrix $Q$ such that

$$QAQ^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} \quad QB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$  \quad (8)$$

where $A_{11} \in \mathbb{R}^{h \times h}, \ A_{12} \in \mathbb{R}^{(n-h) \times h}, \ A_{22} \in \mathbb{R}^{(n-h) \times (n-h)}, \ B_2 \in \mathbb{R}^{(n-h) \times m}$, and $1 \leq h \leq n$. It then becomes evident that the system is structurally controllable if the pair $(A, B)$ is irreducible and its associated g-rank is equal to the number of states, i.e., $n$.

The irreducibility of the system has a graph-theoretic implication which is stated in the following proposition.

Proposition 1: [30] The pair $(A, B)$ is irreducible if and only if the associated flow graph $F_n$ has a spanning forest rooted at $v_{n+1}, \ldots, v_{n+m}$.

Remark 2.2: Proposition 1 was initially developed to address the matrix pairs satisfying the unitary assumption which means that each weight appears only in one entry of the matrix pair $(A(w), B(w))$, i.e., $\sigma = p$, however, the same proof applies to the case in which $\sigma \leq p$ without any modification [1].

**III. PROBLEM FORMULATION**

Our goal in this paper is to investigate structural controllability for a group of interconnected systems with fixed topology of no self-loops. We consider a group of $N$ interconnected agents and focus on the leader–follower framework, where there exist $l$ agents with superior capabilities and access to external commands which we refer to as leaders, while the remainder of agents take the follower role. Without loss of generality, the last $l$ agents are considered as leaders manipulated
by some external input, and the remaining $N - l$ agents are controlled by the consensus law.

Each follower can be modeled as a point mass exerted by an external load as

$$
\dot{x}_i = - \sum_{j \in N_i} a_{ij}(x_i - x_j)
$$

(9)

where $N_i$ is the set that captures the neighbors of the agent $i$, and $a_{ij} \neq 0$ is weight of the edge from $j$ to $i$. For the sake of simplicity in the notation, we suppose that there exists a bijective mapping between two sets $\{a_{ij} \mid i < j \}$ and $\{w_1, w_2, \ldots, w_n\}$. In the rest of this paper, we exploit $w_k$ instead of $a_{ij}$.

Example 3.1: The system (3) can be rewritten as

$$
\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_3
\end{bmatrix} =
\begin{bmatrix}
w_1(v_1 - u) + w_2(v_2 - v_1) + w_3(v_1 - v_3) \\
w_2(v_2 - v_1) \\
w_3(v_3 - v_1)
\end{bmatrix}
$$

so, the graph topology for the consensus network, called the communication topology, is shown in Fig. 3, while the flow graph is in Fig. 1. The leaders do not follow the law in (9), and are controlled exclusively by some external input expressed as

$$
\dot{x}_j = u_j^*
$$

(11)

where $j$ defines the index number of leaders’ vertices $j \in \{N - l + 1, \ldots, N\}$. The aggregated dynamical model of the whole interconnected system can be obtained as [22]

$$
\dot{x} = \begin{bmatrix} A_{(N - l) \times (N - l)} & B_{(N - l) \times 1} \end{bmatrix} x + \begin{bmatrix} 0_{(N - l) \times 1} \end{bmatrix} u_j^*.
$$

(12)

In the set of equations in (12) the leaders’ positions can be seen as inputs to autonomous dynamics captured by followers only. Then the part of above dynamics only associated with followers can be simplified as

$$
\dot{x} = Ax + Bu
$$

(13)

where $A = -L_{ff}$, where $L_{ff}$ is the part of Laplacian matrix associated with followers only. And, the matrix $B$ only captures the interactions between followers and leaders. Our task in paper is to explore the controllability of the system (13) under the commands of multiple leaders and establish a graph-theoretic condition which is both sufficient and necessary for guaranteeing structural controllability.

To explore the controllability of multiagent systems, in the following sections, we first consider the case of single leader and derive the necessary and sufficient condition for this setup. We then extend the theorem to the case with more than one leader.

IV. STRUCTURAL CONTROLLABILITY OF MULTIAGENT SYSTEMS WITH SINGLE LEADER

In this section, a sufficient and necessary condition for structural controllability of a group of agents under a solo leader with fixed communication topology is introduced. To this end, let us first consider an edge with the weight $w_k$ that connects two vertices $v_i$ and $v_j$ within the flow graph $\mathcal{F}_G$ that captures the interactions between entries of $A$ and $B$ matrices in (13). Without loss of generality, we assume that $i < j$. Then these two vectors $c_i \in \mathbb{R}^{n \times 1}$, and $r_k \in \mathbb{R}^{1 \times n}$ have zero entries except their $i$ and $j$ entries i.e.,

$$
c_k^{(i)} = -1 \quad c_k^{(j)} = 1
$$

and

$$
r_k^{(i)} = 1 \quad r_k^{(j)} = -1
$$

(14)

Let us now introduce the set $s = \{i_1, \ldots, i_s\} \subset q$ where $q$ is the cardinality of set $s$. Then the matrices $C_s$, $R_s$, and $W_s$ can be defined as

$$
C_s \triangleq \begin{bmatrix} c_{i_1} & c_{i_2} & \cdots & c_{i_s} \end{bmatrix}
$$

$$
R_s \triangleq \begin{bmatrix} r_{i_1}^\top & r_{i_2}^\top & \cdots & r_{i_s}^\top \end{bmatrix}
$$

$$
W_s \triangleq \text{diag} \begin{bmatrix} w_{i_1} & w_{i_2} & \cdots & w_{i_s} \end{bmatrix}.
$$

(15)

Example 4.1: Consider the graph topology of system represented in (3) under the leadership of node $v_1$. This system has, as discussed in Example 2.1, three independent parameters i.e., $\sigma = 3$. Thus, for this case, we have

$$
c_1 = c_{v_1 v_4} = \begin{bmatrix} 1 \\
0 \\
0
\end{bmatrix} \quad r_1 = r_{v_1 v_4} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}
$$

$$
c_2 = c_{v_1 v_2} = \begin{bmatrix} -1 \\
1 \\
0
\end{bmatrix} \quad r_2 = r_{v_1 v_2} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}
$$

$$
c_3 = c_{v_1 v_3} = \begin{bmatrix} -1 \\
0 \\
1
\end{bmatrix} \quad r_3 = r_{v_1 v_3} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}
$$

Hence, if we consider $s = q$, the $C_s$, $R_s$, and $W_s$ can be written as follows:

$$
C_s = \begin{bmatrix} 1 & -1 & -1 \\
0 & 1 & 0 \end{bmatrix}
\quad R_s = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & -1 \end{bmatrix}
\quad W_s = \begin{bmatrix} w_1 \\
w_2 \\
w_3
\end{bmatrix}.
$$

(16)

(17)

We need to introduce the notion of transfer matrix for establishing the main result of this paper. The transfer matrix of $\{(c_i, r_1, r_2) | i \in q\}$, denoted by $T$, is a block matrix defined as

$$
T_{i,j} = \begin{cases} r_1 c_i & i, j \in q \\
r_2 c_i & i \in q, j = \sigma + 1 \\
r_3 c_i & i, j \in q \\
r_1 c_i & i \in q, j = \sigma + 1 \\
r_2 c_i & i \in q, j = \sigma + 1
\end{cases}
$$

(18)

We refer to the graph associated with the transfer matrix $T$ as transfer graph denoted by $\mathcal{T}$. This is a directed graph with $\sigma + 1$ vertices.
The transfer graph of the above transfer matrix is shown in Fig. 4.

Fig. 4. Transfer graph of the system defined in (3).

\[ \gamma_1, \ldots, \gamma_s, \gamma_{s+1} \] and an edge from node \( \gamma_j \) to \( \gamma_i \) whenever \( T_{i,j} \) is nonzero.

**Example 4.2:** Consider the system (3), if \( s = \{1\} \subset q, C_s, R_s, \) and \( W_s \) are

\[
\begin{align*}
C_s & = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
R_s & = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \\
W_s & = \begin{bmatrix} w_1 \end{bmatrix}.
\end{align*}
\]

(19)

Similarly, \( C_{q-s}, R_{q-s}, \) and \( W_{q-s} \) are

\[
\begin{align*}
C_{q-s} & = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \\
R_{q-s} & = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\
W_{q-s} & = \begin{bmatrix} w_2 & 0 & w_3 \end{bmatrix}.
\end{align*}
\]

(20)

For the system (3), the transfer matrix \( T \) can be represented as

\[
T = \begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -2 & -1 & 0 \\
1 & -1 & -2 & 0
\end{bmatrix}.
\]

(21)

The transfer graph of the above transfer matrix is shown in Fig. 4.

**Remark 4.1:** As defined in (18), the entry \( T_{i,j} \) with \( i,j \in q \) is obtained by inner product of two vectors \( r_i \) and \( c_j \), and this means \( r_i \cdot c_j = \sum_{k=1}^{n} r_{i}^{(k)} c_{j}^{(k)} = r_{i}^{(1)} c_{j}^{(1)} + \cdots + r_{i}^{(n)} c_{j}^{(n)} \) where \( r_{i}^{(k)} \) and \( c_{j}^{(k)} \) represent the \( k \)th entry of \( r_i \) and \( c_j \), accordingly. Each of these terms has a graph representation. The terms \( r_{i}^{(k)} \) and \( c_{j}^{(k)} \) seek for an outgoing edge from node \( v_k \) with weight \( w_k \), and another ingoing edge to node \( v_k \) with weight \( w_k \), respectively.

Theorem 1 represents a graph-theoretic sufficient and necessary condition that guarantees the structural controllability among interconnected agents with fixed topology under single leader. Before we state this result, we first need to introduce some results which enable us to establish the main theorem of this section. Proposition 1 and Lemma 4.1 provide results in order to link the irreducibility property of a linear parameterized representation and the structure of its associated transfer graph when just 1 and 0 values appear within the corresponding \( c_i, r_{i1}, \) and \( r_{i2} \) vectors. This assumption does not hold in our case; thus, we need to extend the proof initially stated in [1].

**Lemma 4.1:** If the pair \( (A, B) \) for the system (13) is irreducible, then the associated transfer graph \( T \) has a spanning tree rooted at \( \gamma_{s+1} \).

In order to prove the above lemma, we follow the approach of [1]. To this end, we first introduce the concept of line graph. The line graph associated with the directed graph \( G \) is also a directed graph \( L(G) \) that represents the adjacencies between the edges of \( G \). Each edge of the original graph \( G \) is presented by a node in its associated line graph \( L(G) \). Thus, the number of edges in \( L(G) \) is equal to the number of vertices in the corresponding \( L(G) \). It is worth mentioning that two edges with the same start and end nodes but different weights in \( G \) are captured as different nodes in \( L(G) \). In order to construct \( L(G) \), one needs parameters associated with each edge in the flow graph \( F \)—namely, start node, end node, and the corresponding weight. Then the node \( i \in k' \) in the line graph if there exist two edges in the flow graph; one from the node \( v_i \) to the node \( v_j \) with weight \( w_k \) and the other one from the node \( v_j \) to \( v_{i'} \) with weight \( w_{k'} \).

**A. Proof of Lemma 4.1**

Suppose that the relation between entries of \( (A, B) \) is captured by the flow graph \( F \) and includes \( \sigma \) independent parameters. Let us consider the equivalent class \( H = \{ i,j \in V_G | k = 0 \} \) with \( V_G \) being the node set of \( L(G) \) and \( \sigma \in \{1, \ldots, \sigma \} \). Hence, a quotient graph like \( 
\] can be introduced in such way that it has \( \sigma \) nodes corresponding to its independent weights. This quotient graph has an edge between two nodes, if there exists at least one edge between the two sets of nodes in the line graph corresponding to these two nodes. In other words, the quotient graph has an edge from node \( k \) to node \( k' \), if there exists a node \( v_j \) such that two edges exist in the flow graph—one from an arbitrary node \( v_i \) to node \( v_j \) with weight \( w_k \) and the other one from \( v_j \) to an arbitrary node \( v_{i'} \) with weight \( w_{k'} \).

Now, we can focus on the transfer matrix. Based on (18), the transfer graph \( T \) has an edge from node \( \gamma_k \) to \( \gamma_k' \) if \( r_{k1} c_{k'} \neq 0 \). Based on (14), we know that for all \( k \)’s at most two entries of \( c_k \) and \( r_{k1} \) are nonzero. In particular, the entry 1 of \( c_k \) has a higher index than that of \( -1 \). The reverse holds for \( r_{k1} \). Therefore, if one calculates \( r_{k1} c_{k'} = \sum_{j=1}^{n} r_{k1}^{(j)} c_{k'}^{(j)} \), there exists at most two nonzero summands say \( r_{k1}^{(j)} c_{k'}^{(j)} + r_{k1}^{(j')} c_{k'}^{(j')} \) which also have the same sign. Hence, one nonzero summands guarantees the existence of an edge from \( \gamma_k \) to \( \gamma_k' \) in the corresponding transfer graph. This is analogous to having an edge from an arbitrary node \( v_i \) to \( v_j \) (\( j \in \{j_1, j_2\} \)) with weight \( w_k \) and another edge from \( v_j \) to an arbitrary node \( v_{i'} \) with weight \( w_{k'} \). Let us now introduce the induced subgraph \( T \) which is obtained from \( T \) by deleting the node \( \gamma_{s+1} \) and its associated edges.
Now based on above-mentioned definition \( s \), we can conclude that \( \hat{\mathcal{G}} \) and \( \hat{T} \) are isomorphic with the bijection that maps vertex \( i \) in \( \hat{\mathcal{G}} \) to vertex \( \gamma_i \) in \( \hat{T} \). If the pair \( (A,B) \) is irreducible, by Proposition 1, \( \hat{\mathcal{G}} \) has a spanning forest rooted at \( v_{n+1}, v_{n+2}, \ldots \), and \( v_{n+m} \). Thus, the associated \( \hat{\mathcal{G}} \) has a spanning forest rooted at \( \{ i \in \mathcal{Q}, r_{i,2} \neq 0 \} \) that captures the weights of edges corresponding to nodes \( v_{n+1}, v_{n+2}, \ldots \), \( v_{n+m} \) in the original flow graph. Consequently, \( \hat{T} \) has a spanning forest rooted at \( \gamma_i \) where is are the indices of roots of spanning forest for the quotient graph. Since there exist \( s \) an edge from \( \gamma_{s+1} \) to each of such \( \gamma_i \) in the transfer graph \( T \), it has a spanning tree rooted at \( \gamma_{s+1} \).

The following theorem states the necessary and sufficient condition for structural controllability of linear parameterized systems.

**Proposition 2:** [1] A linearly parameterized matrix pair \( (A,B) \) is structurally controllable if and only if

\[
\min_{s \in \mathcal{Q}} (\text{rank} C_s + \text{rank} R_{q,s}) = n \tag{22}
\]

and \( T \) has a spanning tree rooted at \( \gamma_{s+1} \).

We can now present the main result of this section. This theorem states that the connectivity of the system is both the necessary and sufficient condition for guaranteeing the structural controllability of the system (13) when there exists only one leader in the network.

**Theorem 1:** Consider the system (13) under the fixed communication topology \( \mathcal{G} \) and a single leader, \( l = 1 \). This system is structurally controllable if and only if \( \mathcal{G} \) is connected.

**B. Proof of Sufficiency**

Given the topology is connected, there exists at least \( N-1 \) edges among these nodes. We first assume that the connected graph has exactly \( N-1 \) edges. In this case, the parameters \( r_{i,s} \) corresponding to these \( N-1 \) edges are independent of each other, because each one of them establishes a link between two different vertices. The same holds for \( r_{1,1} \) vectors. In the worst case scenario, there exists only one link from the leader to one of the followers. Thus, a connected topology consists of at least \( N-1 = \sigma \) independent \( \mathcal{E}_q \) vectors. A similar argument can be applied for counting the number of independent \( r_{1,1} \) vectors. Thus, as the columns of the matrix \( C_q \) are linearly independent of each other, one can observe that \( \text{rank} C_q = f \). By exploiting the same argument, we can conclude that \( \text{rank} R_{q,s} = \sigma - f \), where \( \sigma \) and \( f \) are the cardinality of the sets \( \mathcal{Q} \) and \( \mathcal{s} \), respectively. We now invoke the Proposition 2. It is easy to see that its first part is satisfied i.e., \( \min_{s \in \mathcal{Q}} (\text{rank} C_s + \text{rank} R_{q,s}) = N-1 \). Now suppose that the number of weights exceeds the number of states, i.e., we have more than \( N-1 \) edges, it is obvious that the \( c_i \) vectors are not independent of each other anymore. The same holds for \( r_{1,1} \) vectors. Let us introduce a subgraph of this topology which is connected, i.e., it has \( N-1 \) edges and contains no simple cycles. Such a subgraph, as already established, satisfies the rank condition in (22) and one can easily conclude that the same should hold for the original graph.

On the other hand, if we have a connected topology with single leader, it is easy to see that there exists a spanning tree rooted at leader’s node. Moreover, Proposition 1 declares that the irreducibility is equivalent to existence of a spanning forest rooted at \( v_{n+1}, v_{n+2}, \ldots, v_{n+m} \). Due to the fact that the system has only one leader, the notion of spanning forest can be considered analogous to the notion of spanning tree. Hence, the system is irreducible. Furthermore, based on Lemma 4.1, we can conclude that \( T \) has a spanning tree rooted at \( \gamma_{s+1} \). Hence, the connectivity of the topology guarantees the existence of a spanning tree for transfer graph.

**C. Proof of Necessity**

We use proof by contradiction to establish this part. Suppose that the system was structurally controllable, while it was not connected. Then the system could be represented as

\[
\dot{x} = \begin{bmatrix}
L_{d_1 \times d_1} & 0_{d_1 \times d_2} \\
0_{d_2 \times d_1} & L_{d_2 \times d_2}
\end{bmatrix} x + \begin{bmatrix} b \\
0_{d_1 + d_2 - 1}
\end{bmatrix} u. \tag{23}
\]

The above system can be considered as two separated subsystems—the connected topology which includes the leader and the rest of the topology. Based on this definition, \( d_1 \) represents the number of nodes in the connected topology that includes the leader and the remaining \( d_2 \) nodes are considered as a different subsystem. According to Kalman’s theorem the controllability matrix for the system (23) can be obtained as

\[
[ \hat{B} \hat{A} \hat{B} \ldots \hat{A} \hat{B} ] = \begin{bmatrix}
b & \ast & \ldots & \ast \\
0 & \ast & \ast & \ast \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ast & \ast & \ast \\
0_{d_2 \times 1} & 0_{d_2 \times 1} & \ldots & 0_{d_2 \times 1}
\end{bmatrix}
\]

where the \( \ast \) captures a zero or a nonzero entry. Consequently, the rank of the controllability matrix is equal or less than \( d_1 \). Also note that the controllability matrix includes a zero matrix of dimension \( d_2 \) by \( n \). This contradicts with the earlier assumption.

**V. STRUCTURAL CONTROLLABILITY OF MULTIAGENT SYSTEMS UNDER MULTIPLE LEADERS**

The previous section introduced the necessary and sufficient conditions for the structural controllability of interconnected agents under a solo leader. This result enables us to investigate the structural controllability of multiagent systems with more than one leader. To this end, we first need to present the notion of leader–follower connectivity.

**Definition 5.1:** The graph representation of a set of connected agents is called leader–follower connected if there exists at least a leader in each of the associated subgraphs which are totally separated from each other.

**Remark 5.1:** There exists an analogy between the two notions of leader–follower connectivity and having a spanning forest rooted at leaders’ vertices. Based on the definition, if the graph has a spanning forest rooted at some special vertices, e.g., leaders, there exist at least a path between each node of the graph, except leaders, to one of the leaders’ nodes. This property guarantees the existence of at least one leader in every totally separated subgraphs which coincides with Definition 5.1. It is also worthwhile mentioning that for the case of solo leader connectivity is equivalent to having a spanning tree rooted at the leader’s vertex.

Before presenting the main results, it is beneficial to review some information about the whole interconnected system. Provided the multiagent system has \( l \) leaders, the system matrices \( A \) and \( B \) are of dimensions \( (N-l) \times (N-l) \) and \( (N-l) \times l \), accordingly. Moreover, as it was mentioned earlier, there exist \( s \) at least one path to each node from one of the leaders in a leader–follower connected topology. Thus, one can easily conclude that there exist at least \( N-l \) path between leaders and followers. The following theorem establishes that the leader–follower connectivity of the topology associated with the graph is the necessary and sufficient condition for the whole interconnected system to be structurally controllable under multiple leaders.

**Theorem 2:** Consider the system (13) under the communication topology \( \mathcal{G} \) with multiple leaders, i.e., \( l > 1 \). This system is structurally controllable if and only if the system is leader–follower connected.
A. Proof of Sufficiency

The goal is to prove the system is structurally controllable provided that it is leader–follower connected. As mentioned earlier, a leader–follower connected system with $N$ nodes and $l$ leader $s$ has at least $N - l$ edges. Based on the results derived in single leader case, if we have exactly $N - l$ edges, the parameters $c_i$s are linearly independent of each other for every $i \in q$, where $q$ is the set of algebraic independent parameters. The vectors $[r_{i_1}, r_{i_2}]$ are independent of each other as well. Now, if the system is leader–follower connected, again all $c_i$s are linearly independent of each other. Hence, it is easy to see that for every $s \in q$ the matrices $C_i$ and $R_{q,s}$ are full rank—namely, $rank C_i = f$ and $rank R_{q,s} = \sigma - f$. Given that we have $N - l$ independent $w_i$ to assign, we can conclude that $\min_{s \in q} (rank C_i + rank R_{q,s}) = \sigma = N - l$. The latter is equal to number of system states. Besides, as we stated before for the solo leader case, if we have more than $N - l$ edges, it is possible to introduce a subgraph with $N - l$ edges which satisfies the rank condition in (22). On the other hand, the term leader–follower connectivity suggests that the system has a spanning forest rooted at the leaders’ vertices. Due to Proposition 1, this means that the corresponding graph of the system is irreducible. Therefore, based on Lemma 4.1, we can conclude that the transfer graph $T$ has a spanning tree rooted at $\gamma_{s+1}$. Based on these two results, the system is structurally controllable.

B. Proof of Necessity

We use the proof by contradiction to establish the sufficiency part. We assume that the system was not leader–follower connected while it was structurally controllable. Without the loss of generality, we consider that the system consists of two subsystems which are completely separated from each other. One of the subsystems is leader–follower connected and includes all the leaders. This subsystem has $N_1$ nodes. The remaining $N_2$ nodes can be seen as a second subsystem. If we compute the Kalman’s controllability matrix for this system, it is easy to show that the controllability rank is equal or less than $N_1$, and the system is not controllable. This contradicts with the initial assumption, and the proof is finished.

VI. Conclusion

In this paper, the structural controllability of multiagent systems under multiple leaders with fixed topology was scrutinized. The necessary and sufficient condition of structural controllability of multiagent systems for both cases of single and multiple leaders was developed with the help of the linear parameterization technique. We established that the connectivity of graph topology, in the single leader situation, and the leader–follower connectivity of the associated graph, in the multi leader case, stand not only as the necessary condition but also as the sufficient condition. Some possible future research directions include investigation of structural controllability condition for switching and linear time-variant topologies.

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