Permutation groups, pattern involvement, and Galois connections

Erkko Lehtonen and Reinhard Pöschel

Technische Universität Dresden, Institut für Algebra
01062 Dresden, Germany

Abstract

There is a connection between permutation groups and permutation patterns: for any subgroup $G$ of the symmetric group $S_\ell$ and for any $n \geq \ell$, the set of $n$-permutations involving only members of $G$ as $\ell$-patterns is a subgroup of $S_n$. Making use of the monotone Galois connection induced by the pattern avoidance relation, we characterize the permutation groups that arise via pattern avoidance as automorphism groups of relations of a certain special form. We also investigate a related monotone Galois connection for permutation groups and describe its closed sets and kernels as automorphism groups of relations.

1 Introduction

The theory of pattern-avoiding permutations has been an active field of research over the past decades. For any permutations $\tau = \tau_1 \tau_2 \ldots \tau_\ell \in S_\ell$ and $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n \ (\ell \leq n)$, the permutation $\pi$ is said to involve $\tau$, or $\tau$ is called a pattern of $\pi$, if there exists a substring $\pi_{i_1} \pi_{i_2} \ldots \pi_{i_\ell}$ ($i_1 < i_2 < \cdots < i_\ell$) that is order-isomorphic to $\tau$. If $\pi$ does not involve $\tau$, then $\pi$ is said to avoid $\tau$. For more background and a survey on permutation patterns, we refer the reader to the monograph by Kitaev [6].

The theory of permutation groups is a classical field of algebra that needs no special introduction here; see, e.g., Dixon and Mortimer [5]. At first
sight, permutation patterns do not seem to have much to do with permutation groups. However, and perhaps surprisingly, there is a relevant connection. Namely, every pattern of the composition of two permutations equals the composition of some patterns of the two permutations (see Lemma 2.6). This fact seems to have received limited attention, although it was reported in the 1999 paper by Atkinson and Beals [1], which deals with classes of permutations closed under pattern involvement and composition. An important consequence of this simple yet crucial observation is that for any permutation group $G \leq S_n$, every level of the class $\text{Av}(S_n \setminus G)$ of permutations avoiding the complement of $G$ is a permutation group.

This raises the question which permutation groups arise as sets of $n$-permutations avoiding some sets of $\ell$-permutations. We will refer to such groups as $\ell$-pattern subgroups of $S_n$. Our aim in this paper is to address this question by making use of Galois connections. As for any binary relation, the pattern avoidance relation induces a monotone Galois connection – referred to as $(\text{Pat}(\ell), \text{Comp}(n))$ – between the sets $S_\ell$ and $S_n$ of permutations of the $\ell$-element set and the $n$-element set, respectively. As an answer to the question, we would like to describe the closed sets of this monotone Galois connection, as well as those of its modification $(\text{gPat}(\ell), \text{gComp}(n))$ for permutation groups.

This paper is organized as follows. In Section 2, we introduce the necessary basic definitions related to permutations and patterns. In Section 3, we introduce the monotone Galois connections $(\text{Pat}(\ell), \text{Comp}(n))$ between the subsets of $S_\ell$ and $S_n$ and $(\text{gPat}(\ell), \text{gComp}(n))$ between the subgroups of $S_\ell$ and $S_n$, and we establish some of their basic properties. In Section 4, we describe the $\ell$-pattern subgroups of $S_n$ as automorphism groups of relations. We obtain a simpler description for the special class of $\ell$-pattern subgroups of $S_n$ that are of the form $\text{Comp}(n) G$ for some subgroup $G \leq S_\ell$. This is presented in Section 5, in which we also describe the Galois closures and kernels of $(\text{gPat}(\ell), \text{gComp}(n))$ as automorphism groups of relations. Finally, in Section 6, we make some concluding remarks and indicate possible directions for further research.
2 Preliminaries

General notation

The letter $\mathbb{N}$ stands for the set of nonnegative integers, and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. For any $n \in \mathbb{N}_+$, let $[n] := \{1, \ldots, n\}$. The power set of a set $A$ is denoted by $\mathcal{P}(A)$, and the set of all $\ell$-element subsets of $[n]$ is denoted by $\mathcal{P}_\ell(n)$. Let $A_n^\neq$ be the set of all $n$-tuples on a set $A$ with pairwise distinct entries.

We will always compose functions from right to left, and we often denote functional composition simply by juxtaposition. Thus $f \circ g(x) = f(g(x))$.

Since an $n$-tuple $a = (a_1, \ldots, a_n) \in A^n$ is formally a map $[n] \to A$, we can readily consider the image $\text{Im} a = \{a_1, \ldots, a_n\}$ and form compositions of tuples with other functions. In particular, for any $\sigma \in S_n$, we have $a \sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$, and for any $\varphi: A \to B$, we have $\varphi a = (\varphi(a_1), \ldots, \varphi(a_n))$.

We often write an $n$-tuple $(a_1, \ldots, a_n)$ as a string $a_1 a_2 \ldots a_n$, where $a_i = \pi(i)$ for all $i \in [n]$. On the other hand, any $n$-tuple $a = (a_1, \ldots, a_n) \in [n]^n$ with no repeated elements, when viewed as a map $[n] \to [n]$, is a permutation of $[n]$. In this paper, we will utilize these manifestations of permutations (bijective maps, tuples or strings), and we may switch between them at will, depending on which one we find the most convenient in each situation. In Figure 1 we illustrate this with the graphical representation of the permutation $\pi = 31524 \in [5]^5$ (the circles (shaded or not) represent the graph of the mapping $\pi \in S_5$).
We will also use the conventional cycle notation for permutations: if \( a_1, a_2, \ldots, a_\ell \) are distinct elements of \([n]\), then \( (a_1 \ a_2 \cdots \ a_\ell) \) denotes the permutation that maps \( a_\ell \) to \( a_1 \) and \( a_i \) to \( a_{i+1} \) for \( 1 \leq i \leq \ell - 1 \) and keeps the remaining elements fixed. Such a permutation is called a cycle, or an \( \ell \)-cycle. Every permutation is a product of pairwise disjoint cycles. For the permutation in Figure 1 we have \( \pi = (1 \ 3 \ 5 \ 4 \ 2) \).

The following permutations will appear frequently in what follows:

- the identity permutation, or the ascending permutation \( \iota_n := 1 \ 2 \ \ldots \ n \),
- the descending permutation \( \delta_n := n(n-1) \ldots 1 \),
- the natural cycle \( \zeta_n := 23 \ldots n1 = (1 \ 2 \ \cdots \ n) \).

**Permutation patterns and functional composition**

First we recall some standard terminology from the theory of permutation patterns (see, e.g., Bóna \([3]\) or Kitaev \([6]\)). For any string \( u = u_1u_2 \ldots u_m \) of distinct integers, the reduction or reduced form of \( u \), denoted by \( \text{red}(u) \), is the permutation obtained from \( u \) by replacing its \( i \)-th smallest entry with \( i \), for \( 1 \leq i \leq m \). A permutation \( \tau \in S_\ell \) is a pattern (or an \( \ell \)-pattern, if we want to emphasize the number \( \ell \)) of a permutation \( \pi \in S_n \), or \( \pi \) involves \( \tau \), denoted \( \tau \leq \pi \), if there exists a substring \( u = \pi[I] = \pi_i_1 \pi_i_2 \ldots \pi_i_\ell \) of \( \pi = \pi_1 \ldots \pi_n \)
(where $I = \{i_1, i_2, \ldots, i_\ell\}$, $i_1 < i_2 < \cdots < i_\ell$) such that $\text{red}(u) = \tau$. If $\tau \not\leq \pi$, the permutation $\pi$ is said to avoid $\tau$.

**Example 2.1.** Let $\pi = 31524 \in S_5$ and $I = \{1, 3, 4\}$. Then $u = \pi[I] = 352$ and the corresponding pattern is $\text{red}(u) = 231 \in S_3$ (denoted by $\pi_I$ using Definition 2.3, cf. also the shaded circles in Figure 1).

We denote by $\text{Pat}(\ell) \pi$ the set of all $\ell$-patterns of $\pi$, i.e., $\text{Pat}(\ell) \pi := \{\tau \in S_\ell \mid \tau \leq \pi\}$.

**Example 2.2.** For any $\ell, n \in \mathbb{N}_+$ with $\ell < n$,

$$\text{Pat}(\ell) \iota_n = \{\iota_{\ell}\}, \quad \text{Pat}(\ell) \delta_n = \{\delta_{\ell}\}, \quad \text{Pat}(\ell) \zeta_n = \{\iota_{\ell}, \zeta_{\ell}\}.$$

The pattern involvement relation $\leq$ is a partial order on the set $P := \bigcup_{n \geq 1} S_n$ of all finite permutations. Downward closed subsets of $P$ under this order are called permutation classes. For a permutation class $C$ and for $n \in \mathbb{N}_+$, the set $C^{(n)} := C \cap S_n$ is called the $n$-th level of $C$. For any set $B \subseteq P$, let $\text{Av}(B)$ be the set of all permutations that avoid every member of $B$. It is clear that $\text{Av}(B)$ is a permutation class and, conversely, every permutation class is of the form $\text{Av}(B)$ for some $B \subseteq P$.

The notions introduced in the previous paragraphs can be expressed in terms of order-isomorphisms and functional composition as follows.

**Definition 2.3.** For any $I \in \mathcal{P}_\ell(n)$, let $h_I: [\ell] \to I$ be the order-isomorphism $([\ell], \leq) \to (I, \leq)$. For $\pi \in S_n$, define the permutation $\pi_I: [\ell] \to [\ell]$ as $\pi_I = h^{-1}_{\pi(I)} \circ \pi \circ h_I$ (here $\pi(I) := \{\pi(a) \mid a \in I\}$ and to be precise, $\pi_I = h^{-1}_{\pi(I)} \circ \pi_I \circ h_I$).

As explained in the beginning of this section, we can consider the mapping $h_I$ also as an $\ell$-tuple in $I^\ell \subseteq [n]^\ell$, namely, as the tuple consisting of the elements of $I$ in increasing order, or, using the notation for substrings, $h_I = t_n[I]$.

**Lemma 2.4.** Let $n, \ell \in \mathbb{N}_+$ with $\ell \leq n$.

(i) For any $a \in [n]^n$ and $I \in \mathcal{P}_\ell(n)$, it holds that $a[I] = a \circ h_I$.

(ii) For any $u \in \mathbb{N}_+^\ell$, it holds that $\text{red}(u) = h^{-1}_{\text{Im}u} \circ u$.

(iii) For all $\pi \in S_n$, $\tau \in S_\ell$, we have $\tau \leq \pi$ if and only if $\tau = \pi_I$ for some $I \in \mathcal{P}_\ell(n)$.
Proof. (i) Consider the composite function \( a \circ h_I \). The inner function \( h_I \) provides the string \( i_1 i_2 \ldots i_\ell \) of indices, where \( \{i_1, i_2, \ldots, i_\ell\} = I \) and \( i_1 < i_2 < \cdots < i_\ell \). Subsequent application of \( a \) gives the substring \( a_{i_1} a_{i_2} \ldots a_{i_\ell} = a[I] \) of \( a \).

(ii) Follows immediately from the definition of \( \text{red}(u) \) and \( h_{\text{im}(u)} \).

(iii) Being a composition of bijective maps, \( \pi_I = h_{\pi[I]}^{-1} \circ \pi \circ h_I \) is clearly a permutation of \( [\ell] \). By parts (i) and (ii), we have \( h_{\pi[I]}^{-1} \circ \pi \circ h_I = \text{red}(\pi[I]) \), so \( \pi_I \) is indeed a pattern of \( \pi \). Every \( \ell \)-pattern of \( \pi \) obviously arises in this way for some \( I \in \mathcal{P}_\ell(n) \).

In order to provide examples of the notions defined above and to help the reader feel comfortable with the formalism, we provide a simple, almost mechanical proof of the well-known fact that pattern involvement is preserved under reverses, complements, and inverses of permutations. Recall that the reverse of \( \pi \) is \( \pi^r = \pi \circ \delta_n \), and the complement of \( \pi \) is \( \pi^c = \delta_n \circ \pi \). For example, \( \pi = 31524 \) we have \( \pi^r = 42513 \) (the tuple \( \pi \) in reverse order) and \( \pi^c = 35142 \) (in each place we have the “complement” to \( n + 1 \): \( 6 = 3 + 3 = 1 + 5 = 5 + 1 = 2 + 4 = 4 + 2 \)).

**Lemma 2.5.** The following identities hold for any \( \pi \in S_n \) and \( I \in \mathcal{P}_\ell(n) \):

(i) \( (\pi^r)_I = (\pi_{\delta_n(I)})^r \),

(ii) \( (\pi^c)_I = (\pi_I)^c \),

(iii) \( (\pi^{-1})_I = (\pi_{\pi^{-1}(I)})^{-1} \).

Proof. The identities are verified by straightforward calculations:

\[
(\pi^r)_I = h_{\pi[I]}^{-1} \circ \pi^r \circ h_I = h_{\pi[I]}^{-1} \circ \pi \circ \delta_n \circ h_I \\
= h_{\pi[I]}^{-1} \circ \pi \circ h_{\delta_n(I)} \circ h_{\pi[I]}^{-1} \circ \delta_n \circ h_I \\
= \pi_{\delta_n(I)} \circ (\delta_n)_I = \pi_{\delta_n(I)} \circ \delta_1 = (\pi_{\delta_n(I)})^r,
\]

\[
(\pi^c)_I = h_{\pi[I]}^{-1} \circ \pi^c \circ h_I = h_{\delta_n(\pi[I])}^{-1} \circ \delta_n \circ \pi \circ h_I \\
= h_{\delta_n(\pi[I])}^{-1} \circ \delta_n \circ h_{\pi[I]} \circ h_{\pi[I]}^{-1} \circ \pi \circ h_I \\
= (\delta_n)_{\pi[I]} \circ \pi_I = \delta_1 \circ \pi_I = (\pi_I)^c,
\]

\[
(\pi^{-1})_I = h_{\pi^{-1}[I]}^{-1} \circ \pi^{-1} \circ h_I = (h_I^{-1} \circ \pi \circ h_{\pi^{-1}[I]})^{-1} \\
= (h_{\pi[I]}^{-1} \circ \pi \circ h_{\pi[I]})^{-1} = (\pi_{\pi^{-1}(I)})^{-1}. \qed
\]
More interestingly, our formalism reveals that every $\ell$-pattern of a composition of permutations is a composition of $\ell$-patterns of the respective permutations. (Lemma 2.6(i) rephrases \[1, Lemma 3\] in a slightly generalized way.)

**Lemma 2.6.** Let $\pi, \tau \in S_n$, let $\ell \in [n]$, and let $I \in P_\ell(n)$. Then the following statements hold.

(i) $(\pi \tau)_I = \pi_{\tau(I)} \circ \tau_I$.

(ii) $\text{Pat}^{(\ell)} \pi \tau \subseteq (\text{Pat}^{(\ell)} \pi)(\text{Pat}^{(\ell)} \tau)$.

**Proof.** (i) The identity is verified by straightforward calculation:

$$(\pi \tau)_I = h^{-1}_{(\pi \circ \tau)(I)} \circ \pi \circ \tau \circ h_I = h^{-1}_{\pi(I)} \circ \pi \circ h^{-1}_{\tau(I)} \circ \tau \circ h_I = \pi_{\tau(I)} \circ \tau_I.$$ (ii) Let $\sigma \in \text{Pat}^{(\ell)} \pi \tau$. Then there exists $I \in P_\ell(n)$ such that $\sigma = (\pi \tau)_I$. By part (i), we have $(\pi \tau)_I = \pi_{\tau(I)} \circ \tau_I$. Since $\pi_{\tau(I)} \in \text{Pat}^{(\ell)} \pi$ and $\tau_I \in \text{Pat}^{(\ell)} \tau$, we conclude that $\sigma \in (\text{Pat}^{(\ell)} \pi)(\text{Pat}^{(\ell)} \tau)$.

**Remark 2.7.** The converse inclusion $(\text{Pat}^{(\ell)} \pi)(\text{Pat}^{(\ell)} \tau) \subseteq \text{Pat}^{(\ell)} \pi \tau$ does not hold in general, and it is easy to find examples where we have a strict inclusion $\text{Pat}^{(\ell)} \pi \tau \subsetneq (\text{Pat}^{(\ell)} \pi)(\text{Pat}^{(\ell)} \tau)$. For example, let $\pi = \tau = 132$ and $\ell = 2$. Then $\pi \tau = 123$ and $\text{Pat}^{(2)} \pi = \text{Pat}^{(2)} \tau = \{12, 21\}$, $\text{Pat}^{(2)} \pi \tau = \{12\}$, $(\text{Pat}^{(2)} \pi)(\text{Pat}^{(2)} \tau) = \{12, 21\}$.

Using our formalism, it is also easy to prove the well-known fact that the pattern involvement relation is a partial order. Furthermore, every covering relationship in this order links permutations of two consecutive lengths.

**Lemma 2.8.** Assume that $\ell \leq m \leq n$.

(i) If $I \in P_\ell(m)$ and $J \in P_m(n)$, then $h_J \circ h_I = h_{h_J(I)}$.

(ii) For all $\sigma \in S_\ell$, $\pi \in S_m$, $\tau \in S_n$, it holds that if $\sigma \leq \pi$ and $\pi \leq \tau$, then $\sigma \leq \tau$.

(iii) For all $\sigma \in S_\ell$, $\tau \in S_n$, it holds that if $\sigma \leq \tau$, then there exists $\pi \in S_m$ such that $\sigma \leq \pi \leq \tau$. 

Proof. (i) For any \( i \in [\ell] \), the \( i \)-th smallest element of the set \( I \) is \( h_I(i) \). The order-isomorphism \( h_J : [m] \to J \subseteq [n] \) maps the \( i \)-th smallest element of \( I \) to the \( i \)-th smallest element of \( h_J(I) \), which is \( h_{h_J(I)}(i) \). Thus, \((h_J \circ h_I)(i) = h_J(h_I(i)) = h_{h_J(I)}(i) \).

(ii) Our hypotheses assert that there exist \( I \in \mathcal{P}_\ell(m) \) and \( J \in \mathcal{P}_m(n) \) such that \( \sigma = \pi_I = h^{-1}_{\pi(I)} \circ \pi \circ h_I \) and \( \pi = \tau_J = h^{-1}_{\tau(J)} \circ \tau \circ h_J \), where \( h_I \) and \( h_{\pi(I)} \) map \([\ell]\) into \([m]\), and \( h_J \) and \( h_{\tau(J)} \) map \([m]\) into \([n]\). Observe that \( h_{\tau(J)} \circ \pi = \tau \circ h_J \). Then, by part (i), we have

\[
 h_{\pi(I)}^{-1} \circ h_{\tau(J)}^{-1} = (h_{\tau(J)} \circ h_{\pi(I)})^{-1} = (h_{h_{\tau(J)}(\pi(I))})^{-1} = (h_{\tau(h_J(I))})^{-1}.
\]

Consequently,

\[
 \sigma = h_{\pi(I)}^{-1} \circ h_{\tau(J)}^{-1} \circ \tau \circ h_J \circ h_I = (h_{\tau(h_J(I))})^{-1} \circ \tau \circ h_J = \tau_{h_J(I)},
\]

which shows that \( \sigma \leq \tau \).

(iii) By the assumption that \( \sigma \leq \tau \), there exists \( I \in \mathcal{P}_\ell(n) \) such that \( \sigma = \pi_I = h^{-1}_{\pi(I)} \circ \tau \circ h_I \). Let \( J \) be any \( m \)-element subset of \([n]\) satisfying \( I \subseteq J \), and let \( \pi := \tau_J \). We have \( \pi \leq \tau \) by definition, and it remains to show that \( \sigma \leq \pi \). Consider

\[
 \pi_{h_J^{-1}(I)} = h^{-1}_{\pi(h_J^{-1}(I))} \circ \pi \circ h_{h_J^{-1}(I)} = h^{-1}_{\tau_J(h_J^{-1}(I))} \circ \tau_J \circ h_{h_J^{-1}(I)}
\]

\[
 = h^{-1}_{\tau_J(h_J^{-1}(I))} \circ h_{\tau(J)}^{-1} \circ \tau \circ h_J \circ h_{h_J^{-1}(I)}.
\]

By part (i), we have \( h_J \circ h_{h_J^{-1}(I)} = h_{h_J(h_J^{-1}(I))} = h_I \), and

\[
 h_{\tau_J(h_J^{-1}(I))}^{-1} \circ h_{\tau(J)}^{-1} = (h_{\tau(J)} \circ h_{\tau_J(h_J^{-1}(I))} \circ \tau \circ h_J) \circ h_{h_J^{-1}(I)})^{-1} = (h_{h_{\tau(J)}(h_J^{-1}(I))} \circ \tau \circ h_J) \circ h_{h_J^{-1}(I)})^{-1} = h_{\tau(I)}^{-1}.
\]

Thus, \( \pi_{h_J^{-1}(I)} = h_{\tau(I)}^{-1} \circ \tau \circ h_I = \tau_I = \sigma \), which shows that \( \sigma \leq \pi \).

\[\square\]

3 Pattern involvement and Galois connections

Galois connections

A pair \((f, g)\) of maps \( f : \mathcal{P}(A) \to \mathcal{P}(B), g : \mathcal{P}(B) \to \mathcal{P}(A)\) between the power sets of sets \( A \) and \( B \) is called an \((\text{antitone})\) Galois connection or a
monotone Galois connection, respectively, if for all \(X \subseteq A, Y \subseteq B\) we have
\[
X \subseteq g(Y) \iff f(X) \supseteq Y \quad \text{or} \quad X \subseteq g(Y) \iff f(X) \subseteq Y,
\]
respectively.

We collect some well-known facts. For a Galois connection the mappings \(f\) and \(g\) are antitone (order-reversing) and both compositions \(f \circ g\) and \(g \circ f\) are closure operators. For a monotone Galois connection the mappings \(f\) and \(g\) are monotone, \(f\) and \(g\) are called a lower and an upper adjoint, resp., and the composition \(g \circ f\) is a closure operator while \(f \circ g\) is a kernel operator.

Each binary relation \(R \subseteq A \times B\) induces an antitone Galois connection \((f, g)\) as well as a monotone Galois connection \((f^*, g^*)\) between the power set lattices \(\mathcal{P}(A)\) and \(\mathcal{P}(B)\) via
\[
\begin{align*}
f(X) &:= \{ b \in B \mid \forall a \in X : (a, b) \in R \}, \\
g(Y) &:= \{ a \in A \mid \forall b \in Y : (a, b) \in R \}, \\
f^*(X) &:= A \setminus \{ b \in B \mid \forall a \in X : (a, b) \in R \}, \\
g^*(Y) &:= \{ a \in A \mid \forall b \in B \setminus Y : (a, b) \in R \},
\end{align*}
\]
where \(X \subseteq A\) and \(Y \subseteq B\). Moreover, each Galois connection between \(\mathcal{P}(A)\) and \(\mathcal{P}(B)\) is induced by a suitable relation \(R \subseteq A \times B\) (note \((a, b) \in R \iff b \in f(\{a\}) \iff b \notin f^*(\{a\})\)).

Some further properties are mentioned in the following where we consider a particular monotone Galois connection induced by the pattern avoidance relation. For general background and further information on Galois connections, we refer the reader to the book [1].

**The operators** \(\text{Pat}^{(\ell)}\) and \(\text{Comp}^{(n)}\)

Let \(\ell, n \in \mathbb{N}_+\) with \(\ell \leq n\). We say that a permutation \(\tau \in S_n\) is compatible with a set \(S \subseteq S_\ell\) of \(\ell\)-permutations if \(\text{Pat}^{(\ell)}\) \(\tau \subseteq S\). For \(S \subseteq S_\ell, T \subseteq S_n\), we write
\[
\begin{align*}
\text{Comp}^{(n)} S &:= \{ \tau \in S_n \mid \text{Pat}^{(\ell)}\tau \subseteq S \}, \\
\text{Pat}^{(\ell)} T &:= \bigcup_{\tau \in T} \text{Pat}^{(\ell)}\tau.
\end{align*}
\]

Thus, \(\text{Comp}^{(n)} S\) is the set of all \(n\)-permutations compatible with \(S\), and \(\text{Pat}^{(\ell)} T\) is the set of all \(\ell\)-patterns of permutations in \(T\). It is not difficult to
verify that

\[ \text{Comp}^{(n)} S = \{ \tau \in S_n \mid \forall \sigma \in S_\ell \setminus S : \sigma \not\leq \tau \}, \]
\[ \text{Pat}^{(t)} T = S_\ell \setminus \{ \sigma \in S_\ell \mid \forall \tau \in T : \sigma \not\leq \tau \}. \]

Consequently, \( \text{Pat}^{(t)} \) and \( \text{Comp}^{(n)} \) are precisely the lower and upper adjoints of the monotone Galois connection between \( \mathcal{P}(S_\ell) \) and \( \mathcal{P}(S_n) \) induced by the pattern avoidance relation \( \not\leq \). Therefore,

\[ \forall S \subseteq S_\ell \forall T \subseteq S_n : \text{Pat}^{(t)} T \subseteq S \iff T \subseteq \text{Comp}^{(n)} S. \quad (3.1) \]

Furthermore, \( \text{Pat}^{(t)} \) \( \text{Comp}^{(n)} \) and \( \text{Comp}^{(n)} \) \( \text{Pat}^{(t)} \) are kernel and closure operators, respectively. The kernels and closures are just all the sets of the form \( \text{Pat}^{(t)} T \) and \( \text{Comp}^{(n)} S \), respectively. In particular we have

\[ \text{Pat}^{(t)} \text{Comp}^{(n)} S \subseteq S, \quad \text{Comp}^{(n)} \text{Pat}^{(t)} \text{Comp}^{(n)} S = \text{Comp}^{(n)} S, \]
\[ T \subseteq \text{Comp}^{(n)} \text{Pat}^{(t)} T, \quad \text{Pat}^{(t)} \text{Comp}^{(n)} \text{Pat}^{(t)} T = \text{Pat}^{(t)} T. \]

The upper adjoint \( \text{Comp}^{(n)} \) has the following remarkable property, on which the current work builds.

**Proposition 3.1.** If \( S \) is a subgroup of \( S_\ell \), then \( \text{Comp}^{(n)} S \) is a subgroup of \( S_n \).

**Proof.** Assume that \( S \leq S_\ell \). Note that \( \text{Comp}^{(n)} S \) is nonempty, because \( S \) is a group and hence contains the identity, from which it follows, by Example 2.2, that \( \text{Comp}^{(n)} S \) contains the identity. Let \( \pi, \tau \in \text{Comp}^{(n)} S \). Thus \( \text{Pat}^{(t)} \pi, \text{Pat}^{(t)} \tau \subseteq S \). According to Lemmas 2.5 and 2.6 we have

\[ \text{Pat}^{(t)} \pi^{-1} = (\text{Pat}^{(t)} \pi)^{-1} := \{ \sigma^{-1} \mid \sigma \in \text{Pat}^{(t)} \pi \}, \]
\[ \text{Pat}^{(t)} \pi \tau \subseteq (\text{Pat}^{(t)} \pi)(\text{Pat}^{(t)} \tau) = \{ \sigma \sigma' \mid \sigma \in \text{Pat}^{(t)} \pi, \sigma' \in \text{Pat}^{(t)} \tau \}. \]

Since \( S \) is a group, it contains all products and inverses of its elements, so we have \( \text{Pat}^{(t)} \pi^{-1} \subseteq S \) and \( (\text{Pat}^{(t)} \pi)(\text{Pat}^{(t)} \tau) \subseteq S \). Consequently, \( \pi^{-1}, \pi \tau \in \text{Comp}^{(n)} S \). This implies that \( \text{Comp}^{(n)} S \) is a subgroup of \( S_n \). \( \square \)

Using the standard terminology of the theory of permutation patterns, Proposition 3.1 can be rephrased as follows.

**Corollary 3.2.** The set of \( n \)-permutations avoiding the complement of a subgroup of \( S_\ell \) is a subgroup of \( S_n \).
The operators $\text{gPat}^{(f)}$ and $\text{gComp}^{(n)}$

In view of Proposition 3.1, it makes sense to modify the monotone Galois connection $(\text{Pat}^{(f)}, \text{Comp}^{(n)})$ into one between the subgroup lattices $\text{Sub}(S_\ell)$ and $\text{Sub}(S_n)$ of the symmetric groups $S_\ell$ and $S_n$. Thus we define $\text{gComp}^{(n)}$: $\text{Sub}(S_\ell) \rightarrow \text{Sub}(S_n)$, $\text{gPat}^{(f)}$: $\text{Sub}(S_n) \rightarrow \text{Sub}(S_\ell)$ just by applying the corresponding operators $\text{Comp}^{(n)}$ and $\text{Pat}^{(f)}$ and then taking the generated subgroups. According to Proposition 3.1, $\text{Comp}^{(n)} G \subseteq \text{Sub}(S_n)$ whenever $G \subseteq \text{Sub}(S_\ell)$, and we get:

$$\text{gComp}^{(n)} G := \langle \text{Comp}^{(n)} G \rangle = \text{Comp}^{(n)} G = \{ \tau \in S_n \mid \text{Pat}^{(f)} \tau \subseteq G \},$$

$$\text{gPat}^{(f)} H := \langle \text{Pat}^{(f)} H \rangle = \bigcup_{\tau \in H} \text{Pat}^{(f)} \tau,$$

where $G \subseteq \text{Sub}(S_\ell)$ and $H \subseteq \text{Sub}(S_n)$.

Let us verify that $(\text{gPat}^{(f)}, \text{gComp}^{(n)})$ (or, equivalently, $(\text{gPat}^{(f)}, \text{Comp}^{(n)})$) is indeed an adjoint pair of a monotone Galois connection.

**Lemma 3.3.** For all $G \subseteq \text{Sub}(S_\ell)$ and $H \subseteq \text{Sub}(S_n)$, it holds that $\text{gPat}^{(f)} H \subseteq G$ if and only if $H \subseteq \text{gComp}^{(n)} G$.

**Proof.** First assume that $\text{gPat}^{(f)} H \subseteq G$. Then it holds that $\text{Pat}^{(f)} H \subseteq G$ and we get $H \subseteq \text{Comp}^{(n)} G = \text{gComp}^{(n)} G$ from equivalence (3.1) and Proposition 3.1. Assume then that $H \subseteq \text{gComp}^{(n)} G = \text{Comp}^{(n)} G$. Thus $\text{Pat}^{(f)} H \subseteq G$ by (3.1) and therefore $\text{gPat}^{(f)} H = \langle \text{Pat}^{(f)} H \rangle \subseteq \langle G \rangle = G$. □

The proof of Lemma 3.3 remains valid if $H$ is an arbitrary subset of $S_n$. Thus $(\text{gPat}^{(f)}, \text{gComp}^{(n)})$ can be considered also as a monotone Galois connection between $\mathcal{P}(S_n)$ and $\text{Sub}(S_\ell)$. In particular, the following lemma holds.

**Lemma 3.4.** $\text{gPat}^{(f)} H = \langle \text{gPat}^{(f)} H \rangle$ for $H \subseteq S_n$.

**Proof.** Indeed, we have $\text{gPat}^{(f)} H \subseteq \langle \text{gPat}^{(f)} H \rangle$ by the monotonicity of $\text{gPat}^{(f)}$. Furthermore, because $\text{gComp}^{(n)} \text{gPat}^{(f)}$ is the closure operator associated with the monotone Galois connection $(\text{gPat}^{(f)}, \text{gComp}^{(n)})$ between $\mathcal{P}(S_n)$ and $\text{Sub}(S_\ell)$, we have $H \subseteq \text{gComp}^{(n)} \text{gPat}^{(f)} H$, which implies $\langle H \rangle \subseteq \langle \text{gComp}^{(n)} \text{gPat}^{(f)} H \rangle = \text{gComp}^{(n)} \text{gPat}^{(f)} H$, and therefore $\text{gPat}^{(f)} \langle H \rangle \subseteq \text{gPat}^{(f)} \text{gComp}^{(n)} \text{gPat}^{(f)} H = \text{gPat}^{(f)} H$. □
Subgroups of $S_n$ of the form $\text{Comp}^{(n)} S$ for some subset $S \subseteq S_{\ell}$ are called $\ell$-pattern subgroups of $S_n$. As we have seen in Proposition 3.1, $\text{Comp}^{(n)} S$ is a subgroup of $S_n$ whenever $S$ is a subgroup of $S_{\ell}$. On the other hand, it is well possible that $\text{Comp}^{(n)} S$ is a group even if $S$ is not, and there are subgroups of $S_n$ that are not $\ell$-pattern subgroups for any $\ell < n$.

**Example 3.5.** For all $\ell, n \in \mathbb{N}_+$ with $\ell < n$ and $n \geq 3$, the group $\langle (1 \ n) \rangle \subseteq S_n$ is not an $\ell$-pattern subgroup of $S_n$. The claim is obvious when $\ell \leq 2$, and it is easy to verify for $\ell = 3$ with the help of Proposition 3.6 below. Assume that $\ell \geq 4$, and suppose, to the contrary, that $\langle (1 \ n) \rangle = \text{Comp}^{(n)} S$ for some $S \subseteq S_{\ell}$. Then

$$\text{Comp}^{(n)} \text{Pat}^{(\ell)} \langle (1 \ n) \rangle = \text{Comp}^{(n)} \text{Pat}^{(\ell)} \text{Comp}^{(n)} S = \text{Comp}^{(n)} S = \langle (1 \ n) \rangle,$$

that is, $\langle (1 \ n) \rangle$ is a closed set. It is easy to verify that

$$\text{Pat}^{(\ell)} \langle (1 \ n) \rangle = \{\iota_\ell, (1 \ \ell), \zeta_\ell, \zeta_\ell^{-1}\},$$

$$\text{Comp}^{(n)} \{\iota_\ell, (1 \ \ell), \zeta_\ell, \zeta_\ell^{-1}\} = \{\iota_n, (1 \ n), \zeta_n, \zeta_n^{-1}\}.$$

Thus $\langle (1 \ n) \rangle \not\subseteq \text{Comp}^{(n)} \text{Pat}^{(\ell)} \langle (1 \ n) \rangle$, a contradiction.

These observations raise the questions which subgroups of $S_n$ are $\ell$-pattern subgroups and for which subsets $S \subseteq S_{\ell}$, the set $\text{Comp}^{(n)} S$ is a group. For small values of $\ell$, we can provide a conclusive answer.

**Proposition 3.6.** Let $n, \ell \in \mathbb{N}_+$ with $n \geq \ell$ and $\ell \leq 3$, and let $S$ be a subset of $S_{\ell}$. Then $\text{Comp}^{(n)} S$ is a subgroup of $S_n$ if and only if $S$ is a subgroup of $S_{\ell}$.

**Proof.** The claim clearly holds for $S = \emptyset$, and we may assume that $S$ is nonempty. Sufficiency follows from Proposition 3.1, and we only need to show necessity. The case when $\ell = 1$ is trivial: all nonempty subsets of $S_1$ are subgroups of $S_1$. It is easy to see that the claim holds when $\ell = 2$. The only nonempty subset of $S_2$ that is not a subgroup is $\{21\}$, and $\text{Comp}^{(n)} \{21\} = \{\delta_n\}$ is not a subgroup of $S_n$. It remains to consider the case when $\ell = 3$. Assume that $S \subseteq S_{\ell}$ and $H := \text{Comp}^{(n)} S$ is a subgroup of $S_n$. Since $H$ is a group, it contains the identity permutation $\iota_n$; therefore $S$ must contain all 3-patterns of $\iota_n$, of which there is only one, namely $\iota_3 = 123$. If
descending permutation $\delta$ contains the reverses and complements of its members. Observe that the permutations

\[(1 2) = 213 \ldots n, \quad (1 2)^t = n(n-1) \ldots 312,\]
\[(1 2)^c = (n-1)n(n-2)(n-3) \ldots 1, \quad (1 2)^{rc} = 12 \ldots (n-2)n(n-1)\]

can be obtained from each other by taking reverses and complements, and

\[
\text{Pat}^{(3)}(1 2) = \{123, 213\}, \quad \text{Pat}^{(3)}(1 2)^t = \{321, 312\},
\]
\[
\text{Pat}^{(3)}(1 2)^c = \{321, 231\}, \quad \text{Pat}^{(3)}(1 2)^{rc} = \{123, 132\}.
\]

Since $\{123, 321\} \subseteq S$ and $\{132, 213, 231, 312\} \cap S \neq \emptyset$, one of $\text{Pat}^{(3)}(1 2)$, $\text{Pat}^{(3)}(1 2)^t$, $\text{Pat}^{(3)}(1 2)^c$, $\text{Pat}^{(3)}(1 2)^{rc}$ is included in $S$. Therefore $H$ contains one of $(1 2)$, $(1 2)^t$, $(1 2)^c$, $(1 2)^{rc}$ and hence it contains all of them. It follows that $S_3 \subseteq \text{Pat}^{(3)}H \subseteq S \subseteq S_3$, that is, $S = S_3$, so $S$ is a subgroup of $S_3$.

Case 2. Assume that $\{231, 312\} \cap S \neq \emptyset$. Since

\[
\text{Pat}^{(3)}23 \ldots n1 = \{123, 231\}, \quad \text{Pat}^{(3)}n2 \ldots (n-1) = \{123, 312\},
\]

we have that $\zeta_n = 23 \ldots n1 \in H$ or $\zeta_n^{-1} = n2 \ldots (n-1) \in H$. Since $H$ is a group, it follows that $\langle \zeta_n \rangle \subseteq H$; hence, in fact, both $\zeta_n$ and $\zeta_n^{-1}$ are in $H$, so $\{123, 231, 312\} \subseteq \text{Pat}^{(i)}H \subseteq S$. If $S = \{123, 231, 312\}$, then $S$ is a subgroup of $S_3$. Let us assume that $|S| \geq 4$; then $\{132, 213, 321\} \cap S \neq \emptyset$. We have already dealt with the case when $321 \in S$ (Case 1 above), so we may assume that $\{132, 213\} \cap S \neq \emptyset$. Since

\[
\text{Pat}^{(3)}(1 2) = \{123, 213\}, \quad \text{Pat}^{(3)}(n-1 \ n) = \{123, 132\},
\]

$H$ contains $(1 2)$ or $(n-1 \ n)$. Therefore $H$ includes $\{\zeta_n, (1 2)\}$ or $\{\zeta_n, (n-1 \ n)\}$; these are generating sets of $S_3$, so $H = S_3$. Consequently, $S = S_3$.

Case 3. Assume that $\{132, 213\} \cap S \neq \emptyset$. If $|S| = 2$, then $S = \{123, 132\}$ or $S = \{123, 213\}$, so $S$ is a subgroup of $S_3$. Let us assume that $|S| \geq 3$. We have already dealt with the cases when $\{321, 231, 312\} \cap S \neq \emptyset$ (Cases 1, 2 above), so we may assume that $\{132, 213\} \subseteq S$. Then all adjacent transpositions $(i \ i + 1)$ ($1 \leq i \leq n - 1$) are in $H$, because $\text{Pat}^{(3)}(i \ i + 1) \subseteq$
\begin{center}
\begin{tabular}{|c|c|}
\hline
$G$ & $\text{Comp}^{(n)}G$ \\
\hline
$\emptyset$ & $\emptyset$ \\
\{123\} & \{12...n\} \\
\{123, 132\} & $\langle (n-1\ n) \rangle$ \\
\{123, 213\} & $\langle (1\ 2) \rangle$ \\
\{123, 321\} & $\langle \delta_n \rangle$ \\
\{123, 231, 312\} & $\langle \zeta_n \rangle$ \\
$S_3$ & $S_n$ \\
\hline
\end{tabular}
\caption{Permutation groups compatible with subgroups of $S_3$.}
\end{center}

{\{123, 132, 213\} $\subseteq S$. These transpositions generate the symmetric group $S_n$, so $H = S_n$. Consequently, $S = S_3$.}

The cases considered above exhaust all possibilities. This concludes the proof. 

It is easy to determine the groups $\text{Comp}^{(n)}G$ for each subgroup $G$ of $S_3$; these are summarized in Table 1.

In Proposition 3.6, the hypothesis $\ell \leq 3$ is indispensable. In fact, as the following lemma and examples illustrate, counterexamples can be found when $\ell \geq 4$.

**Lemma 3.7.** Assume that $\ell \geq 4$ and $n = \ell + 1$. Then there exists a group $H \leq S_n$ such that $H = \text{Comp}^{(n)}S$ for some subset $S \subseteq S_\ell$, but there is no subgroup $G \leq S_\ell$ such that $H = \text{Comp}^{(n)}G$.

**Proof.** Let $H := \langle (1\ n)^r \rangle = \{12...n, 1(n-1)(n-2)...2n\}$ and $S := \text{Pat}^{(\ell)}H$. Then $S$ comprises the following four permutations:

- $12... (n-1) = \iota_{n-1}$,
- $1(n-1)(n-2)...2 = (1\ 2\ \cdots\ n-1)^r = [(1\ 2\ \cdots\ n-1)^{-1}]^c$,
- $(n-2)(n-3)...1(n-1) = [(1\ 2\ \cdots\ n-1)^{-1}]^r = (1\ 2\ \cdots\ n-1)^c$,
- $1(n-2)(n-3)...2(n-1) = (1\ n-1)^r = (1\ n-1)^c$. 


First, let us verify that $\text{Comp}^{(n)} S = H$. Let $\pi \in \text{Comp}^{(n)} S$. Then $\text{Pat}^{(n-1)} \pi \subseteq S$, by definition. Consider first the case when $12\ldots(n-1) \leq \pi$. Then $321 \not\leq \pi$, so $1(n-1)(n-2)\ldots2$ and $(n-2)(n-3)\ldots1(n-1)$ cannot be $(n-1)$-patterns of $\pi$, because $321$ is involved in both and pattern involvement is a transitive relation (see Lemma 2.8). Similarly, if $n \geq 6$, then $1(n-2)(n-3)\ldots2(n-1)$ cannot be an $(n-1)$-pattern of $\pi$. If $n = 5$, then it may be possible that both 1234 and 1324 are patterns of $\pi$; in this case $\pi$ is either 12435 or 13245. But we have $1243 \leq 12435$ and $2134 \leq 13245$, yet $1243 \not\leq S$ and $2134 \not\leq S$. We conclude that the only $(n-1)$-pattern of $\pi$ is $12\ldots(n-1)$, which implies that $\pi = 12\ldots n$.

Consider then the case when $12\ldots(n-1) \not\leq \pi$. Observe first that every permutation $\tau = \tau_1\tau_2\ldots\tau_{n-1} \in S \setminus \{12\ldots(n-1)\}$ satisfies $\tau_2 > \tau_3$. Thus, from the fact that $\pi_{\{1,\ldots,n-1\}} \in S \setminus \{12\ldots(n-1)\}$, we get $\pi_2 > \pi_3$. At the same time, the only permutation $\tau \in S$ satisfying $\tau_1 > \tau_2$ is $(n-2)(n-3)\ldots1(n-1)$. This implies that $\pi_{\{2,\ldots,n\}} = (n-2)(n-3)\ldots1(n-1)$, so $\pi_n > \pi_2 > \pi_3 > \ldots > \pi_{n-1}$. The only permutation $\tau \in S$ satisfying $\tau_2 > \tau_{n-1}$ is $1(n-1)(n-2)\ldots2$, which implies that $\pi_{\{1,\ldots,n\}} = 1(n-1)(n-2)\ldots2$, so $\pi_{n-1} > \pi_1$. Putting together the above inequalities involving $\pi_i$’s, we get $\pi = 1(n-1)(n-2)\ldots2n$. We conclude that indeed $\text{Comp}^{(n)} S = H$.

We still need to show that there is no subgroup $G \leq S_\ell$ such that $\text{Comp}^{(n)} G = H$. Suppose, to the contrary, that $\text{Comp}^{(n)} G = H$ for a subgroup $G \leq S_\ell$. Then $S = \text{Pat}^{(\ell)} H \subset G$, so $\langle S \rangle \subset \langle G \rangle = G$.

The group $\langle S \rangle$ contains

- $(1\ n-1)c \circ (1\ 2\ \ldots\ n-1)c = (1\ n-1) \circ (1\ 2\ \ldots\ n-1) = (1\ 2\ \ldots\ n-2)$,
- $(1\ 2\ \ldots\ n-1)c \circ (1\ n-1)c = (1\ 2\ \ldots\ n-1) \circ (1\ n-1) = (2\ 3\ \ldots\ n-1)$,
- $(1\ n-1) \circ (1\ n-1)c = (1\ n-1) \circ (1\ n-1) = \delta_{n-1}$,
- $(1\ n-1)c \circ (1\ n-1) = (1\ 2\ \ldots\ n-1) \circ (1\ n-1) = (1\ n-1)$,
- $(1\ 2\ \ldots\ n-1)c \circ \delta_{n-1} = (1\ 2\ \ldots\ n-1) \circ \delta_{n-1} = (1\ 2\ \ldots\ n-1)$.

Since $\langle S \rangle$ contains $\{(1\ n-1), (1\ 2\ \ldots\ n-1)\}$, a generating set of $S_{n-1}$, we conclude that $\langle S \rangle = S_{n-1}$, so $G = S_{n-1}$. But $\text{Comp}^{(n)} S_{n-1} = S_n \neq H$, a contradiction. \qed
Example 3.8. For $n = 6$, $\ell = 4$, the authors have verified with computer (using the GAP algebra system) that a subgroup $H \leq S_n$ is of the form \(\text{Comp}(n)S\) for some subset $S \subseteq S_\ell$ but not of the form \(\text{Comp}(n)G\) for any subgroup $G \leq S_\ell$ if and only if $H$ is one of the following:

\[
\{(2 \, 5)\}, \quad \{(2 \, 4) (3 \, 5)\}, \quad \{(2 \, 5) (3 \, 6)\}, \quad \{(1 \, 3) (4 \, 6)\}, \quad \{(1 \, 4) (2 \, 5)\}, \\
\{(1 \, 5) (2 \, 6)\}, \quad \{(1 \, 4 \, 5) (2 \, 3 \, 6)\}, \quad \{(2 \, 4) (3 \, 5), (1 \, 6) (2 \, 3) (4 \, 5)\}.
\]

Example 3.9. For $n = 7$, $\ell = 4$, examples of subgroups of $S_n$ that are of the form \(\text{Comp}(n)S\) for some subset $S \subseteq S_\ell$ but not of the form \(\text{Comp}(n)G\) for any subgroup $G \leq S_\ell$ include the following:

\[
\{(2 \, 5) (3 \, 6)\}, \quad \{(2 \, 6) (3 \, 7)\}, \quad \{(1 \, 5) (2 \, 6)\}.
\]

For $n = 8$, $\ell = 4$, examples of subgroups of $S_n$ that are of the form \(\text{Comp}(n)S\) for some subset $S \subseteq S_\ell$ but not of the form \(\text{Comp}(n)G\) for any subgroup $G \leq S_\ell$ include the following:

\[
\{(2 \, 6) (3 \, 7)\}.
\]

Note that the above lists may not be exhaustive.

4 On the monotone Galois connection \(\text{(Pat, Comp)}\)

The $\ell$-pattern subgroups of $S_n$ are those subgroups that are of the form \(\text{Comp}(n)S\) for some subset $S \subseteq S_\ell$. As a way of describing such subgroups, we make use of another, classical Galois connection, namely the Galois connection \((\text{Aut}, \text{Inv})\) between permutations and relations. Recall that a $k$-ary relation on a set $A$ is simply a subset of $A^k$. Denote by $\text{Rel}_n^{(k)}$ the set of all $k$-ary relations on $[n]$, and let $\text{Rel}_n := \bigcup_{k \geq 1} \text{Rel}_n^{(k)}$.

Let $\pi \in S_n$, and let $\varrho \in \text{Rel}_n$. We say that the permutation $\pi$ preserves the relation $\varrho$, or that $\varrho$ is an invariant of $\pi$, or that $\pi$ is an automorphism of $\varrho$, and we write $\pi \triangleright \varrho$, if $\pi(r) \in \varrho$ for every $r \in \varrho$. The preservation relation induces the Galois connection \((\text{Aut}, \text{Inv})\), where

\[
\text{Aut } R := \{\pi \in S_n \mid \pi \triangleright \varrho \text{ for all } \varrho \in R\}, \\
\text{Inv } S := \{\varrho \in \text{Rel}_n \mid \pi \triangleright \varrho \text{ for all } \pi \in S\},
\]
for every $S \subseteq S_n$ and $R \subseteq \text{Rel}_n$. Thus, $\text{Aut } R$ is the automorphism group of $R$. Write $\text{Inv}^{(\ell)} S := \text{Inv } S \cap \text{Rel}^{(\ell)}$.

It is well known that finite permutation groups are precisely the Galois closures of the Galois connection $(\text{Aut}, \text{Inv})$ (see, e.g., Chapter 8 in [8]). A permutation group $H \leq S_n$ is $\ell$-closed, if it is the automorphism group of its $\ell$-ary invariant relations, i.e., $H = \text{Aut } \text{Inv}^{(\ell)} H$.

Let $H \leq S_n$, and let $a = (a_1, \ldots, a_\ell) \in [n]^\ell$. Let

$$a^H := \{\sigma(a) \mid \sigma \in H\} = \{(\sigma(a_1), \ldots, \sigma(a_\ell)) \mid \sigma \in H\}.$$ 

A set of the form $a^H$ for some $a \in [n]_{\neq}^\ell$ is called an $\ell$-orbit of $H$. For $I \in \mathcal{P}_\ell(n)$, recall the map $h_I : [\ell] \to I$ from Definition 2.3 and view it as a tuple $h_I \in [n]_{\neq}^\ell$. Therefore it makes sense to consider the $\ell$-orbit $(h_I)^H$.

For $m \in \mathbb{N}_+$, any group $G \leq S_m$ can be viewed as an $m$-ary irreflexive relation on the set $[m]$ (i.e., a subset of $[m]^m_{\neq}$) whose members are the permutations of $G$ viewed as tuples. We denote this relation by $\gamma_G$. (Formally $\gamma_G = G$, but we prefer to introduce the notation $\gamma_G$ in order to avoid the expression $\text{Aut } G$, because automorphisms of a group $G$ have another fixed meaning.) It holds that $\gamma_G = (1, \ldots, m)^G = (h_{[m]}^G)^G$, where $h_{[m]} : [m] \to [m]$ is just the identity map on $[m]$. Furthermore, the equality $G = \text{Aut } \gamma_G$ holds.

**Proposition 4.1.** Let $H \leq S_n$, and assume that $H = \text{Comp}^{(n)} S$ for some subset $S \subseteq S_l$ (not necessarily a subgroup). Then $H$ is $\ell$-closed, i.e., $H$ is determined by its $\ell$-ary invariant relations:

$$H = \text{Aut } \text{Inv } H = \text{Aut } \text{Inv}^{(\ell)} H.$$ 

In particular, the $\ell$-orbits $(h_I)^H$ are enough to characterize the group:

$$H = \text{Aut } \{(h_I)^H \mid I \in \mathcal{P}_\ell(n)\}.$$ 

**Proof.** Clearly, $H \subseteq \text{Aut } (h_I)^H$ for each $I \in \mathcal{P}_\ell(n)$. Thus, $H \subseteq H'$, where $H' := \text{Aut } \{(h_I)^H \mid I \in \mathcal{P}_\ell(n)\}$. In order to prove the converse inclusion $H' \subseteq H$, it suffices to show that $\text{Pat}^{(\ell)} \tau \subseteq S$ for every $\tau \in H'$. Let $\tau \in H'$ and $I \in \mathcal{P}_\ell(n)$, and consider the pattern $\tau_I$. Since $h_I \in (h_I)^H$ and $\tau \in \text{Aut } (h_I)^H$, we have $\tau \circ h_I \in (h_I)^H$, i.e., $\tau \circ h_I = \pi \circ h_I$ for some $\pi \in H$. Therefore $\tau(I) = \pi(I)$, and since all $\ell$-patterns of $\pi$ are in $S$, we have $\tau_I = h^{-1}_{\pi(I)} \circ \pi \circ h_I = h^{-1}_{\pi(I)} \circ \pi \circ h_I = \pi_I \in S$. 

\[\square\]
In the following theorem, we describe the ℓ-pattern subgroups of $S_n$ as automorphism groups of relations of a certain prescribed form. This theorem provides a description of all groups of the form \( \text{Comp}^{(n)} S \), where $S$ is an arbitrary subset of $S_\ell$. Those ℓ-pattern subgroups that are of the form \( \text{Comp}^{(n)} G \) for some subgroup $G$ of $S_\ell$ have an even simpler description, which we will discuss in the next section (Theorem 5.3).

**Theorem 4.2.** Let $H \leq S_n$, and consider the ℓ-orbits $\varrho_I := (h_I)^H$ for all $I \in \mathcal{P}_\ell(n)$. Then $H$ is of the form $H = \text{Comp}^{(n)} S$ for some $S \subseteq S_\ell$ if and only if

(a) $H = \text{Aut}\{\varrho_I \mid I \in \mathcal{P}_\ell(n)\}$,

(b) the $\varrho_I$ satisfy the following property: for every $x \in [n]_\neq$ we have

$$
(\forall I \in \mathcal{P}_\ell(n) \exists J \in \mathcal{P}_\ell(n) : \text{red}(x[I]) \in \text{red}(\varrho_J)) \implies \forall I \in \mathcal{P}_\ell(n) : x[I] \in \varrho_I.
$$

**Proof.** Concerning the implication in condition (b), observe that the antecedent $\forall I \in \mathcal{P}_\ell(n) \exists J \in \mathcal{P}_\ell(n) : \text{red}(x[I]) \in \text{red}(\varrho_J)$ expresses the fact that each ℓ-pattern of $x$, now considered as a permutation $x \in [n]_\neq = S_n$, coincides with an ℓ-pattern of some permutation from $H$, i.e., $x \in \text{Comp}^{(n)} \text{Pat(ℓ)} H$. Moreover, the consequent $\forall I \in \mathcal{P}_\ell(n) : x[I] \in \varrho_I$ expresses the fact that $x$ coincides with some $\pi \in H$ on each ℓ-element subset; thus $x \triangleright \varrho_I$ for each $I$, i.e., $x \in \text{Aut}\{\varrho_I \mid I \in \mathcal{P}_\ell(n)\}$. Thus the implication states that \( \text{Comp}^{(n)} \text{Pat(ℓ)} H \subseteq \text{Aut}\{\varrho_I \mid I \in \mathcal{P}_\ell(n)\} \).

Assume now that $H = \text{Comp}^{(n)} S$ for some $S \subseteq S_\ell$. Condition (a) is necessary by Proposition 4.1. With the properties of Galois connections (see Section 3), we have $H = \text{Comp}^{(n)} S = \text{Comp}^{(n)} \text{Pat(ℓ)} \text{Comp}^{(n)} S = \text{Comp}^{(n)} \text{Pat(ℓ)} H$, so condition (b) is also necessary.

Assume then that conditions (a) and (b) hold. Since $\text{Comp}^{(n)} \text{Pat(ℓ)}$ is a closure operator, we have $H \subseteq \text{Comp}^{(n)} \text{Pat(ℓ)} H$. By conditions (a) and (b), we also have $\text{Comp}^{(n)} \text{Pat(ℓ)} H \subseteq \text{Aut}\{\varrho_I \mid I \in \mathcal{P}_\ell(n)\} = H$. Thus $H = \text{Comp}^{(n)} \text{Pat(ℓ)} H$. $\square$
5 On the monotone Galois connection (gPat, gComp)

The monotone Galois connection (gPat, gComp) is perhaps more interesting than (Pat, Comp), because it makes a correspondence between well-understood algebraic objects: permutation groups of two different degrees. It may also be the more useful of the two when one wishes to investigate permutation classes in which every level is a group in greater detail than Atkinson and Beals did in [1, 2].

As briefly mentioned in the previous section, the ℓ-pattern subgroups of $S_n$ that are of the form $\text{Comp}^{(n)}G$ for some subgroup $G$ of $S_\ell$ can be described as automorphism groups of relations in a way that is much simpler than the one presented in Theorem 4.2. In fact, as we will see in Theorem 5.3, such subgroups are automorphism groups of a single relation of arity at most $\ell$.

Another goal of this section is to describe the Galois closures $g\text{Comp}^{(n)}g\text{Pat}^{(\ell)}H$ and kernels $g\text{Pat}^{(\ell)}g\text{Comp}^{(n)}G$ as automorphism groups of relations that can be constructed from the group $H$ or $G$. This is done in Theorems 5.3 and 5.10.

In what follows, $\ell$ and $n$ are fixed integers satisfying $\ell \leq n$, and we will make use of the following constructions. Let $k \leq \ell \leq n$, and let $\varrho \subseteq [n]^k$ and $\sigma \subseteq [\ell]^k$. Define the relations $\varrho^\lor \subseteq [\ell]^k$ and $\sigma^\land \subseteq [n]^k$ as

$$\varrho^\lor := \{ h^{-1}_I(r) | r \in \varrho, \text{Im} r \subseteq I \in P_\ell(n) \},$$
$$\sigma^\land := \{ h_J(s) | s \in \sigma, J \in P_\ell(n) \}.$$

For $\sigma \subseteq [\ell]^{\ell}$, we have $\sigma^\land = \{ u \in [n]^{\ell} | \text{red}(u) \in \sigma \}$.

We are also going to consider the following condition for $\varrho$:

$$\forall r \in [n]^k \forall I, J \in P_\ell(n) : \text{Im} r \subseteq I \land r \in \varrho \implies h_J h_I^{-1}(r) \in \varrho. \quad (5.1)$$

We remark that for $\varrho \subseteq [n]^{\ell}$ ($k = \ell$) and with the notation from Section 2 (cf. Lemma 2.4(ii)) this condition can be written as follows:

$$\forall r, s \in [n]^{\ell} : \{ r \in \varrho \land \text{red}(r) = \text{red}(s) \} \implies s \in \varrho, \quad (5.2)$$

i.e., all tuples with a particular reduced form belong to $\varrho$ whenever one such tuple belongs to $\varrho$. A $k$-ary ($k \leq \ell$) relation satisfying condition (5.1) is called a pattern closed relation, for short pc-relation. For $H \subseteq S_n$ let

$$\text{pcInv} H := \{ \varrho \in \text{Inv} H | \varrho \text{ is a pc-relation} \}$$

denote the set of all invariant pc-relations of $H$. 

Lemma 5.1. Let \( k \leq \ell \leq n \) and \( \varrho \subseteq [n]_{k}^{\ell} \). Then \( \varrho \subseteq \varrho^{\vee \wedge} \). Furthermore, \( \varrho = \varrho^{\vee \wedge} \) if and only if \( \varrho \) is a pc-relation.

Proof. In order to prove \( \varrho \subseteq \varrho^{\vee \wedge} \), let \( r \in \varrho \). Let \( I \in \mathcal{P}_{\ell}(n) \) be any set satisfying \( \text{Im} r \subseteq I \). Then \( h_{I}^{-1}(r) \in \varrho^{\vee} \) by the definition of \( \varrho^{\vee} \), and consequently \( r = h_{I} \circ h_{I}^{-1}(r) \in \varrho^{\vee \wedge} \).

Assume now that \( \varrho \) satisfies condition (5.1). In order to show that \( \varrho^{\vee \wedge} \subseteq \varrho \), let \( s \in \varrho^{\vee \wedge} \). Then \( s = h_{J} \circ h_{J}^{-1}(r) \) for some \( I,J \in \mathcal{P}_{\ell}(n) \) and \( r \in \varrho \) satisfying \( \text{Im} r \subseteq I \). By condition (5.1), we have \( s \in \varrho \).

Assume then that \( \varrho = \varrho^{\vee \wedge} \). Let \( r \in \varrho, I,J \in \mathcal{P}_{\ell}(n) \) such that \( \text{Im} r \subseteq I \). Then \( h_{J}h_{I}^{-1}(r) \in \varrho^{\vee \wedge} \subseteq \varrho \), and we conclude that \( \varrho \) satisfies condition (5.1).

It turns out that every \( \ell \)-ary relation of the form \( \sigma^{\wedge} \) is a pc-relation. This follows immediately from the following lemma.

Lemma 5.2. Let \( \sigma \subseteq [\ell]_{\ell}^{\ell} \). Then \( \sigma^{\wedge \vee} = \sigma \).

Proof. Recall \( \sigma^{\wedge} = \{ h_{I} \circ s \mid s \in \sigma, J \in \mathcal{P}_{\ell}(n) \} \). Thus

\[
\sigma^{\wedge \vee} = \{ h_{I}^{-1} \circ r \mid r \in \sigma^{\wedge}, \text{Im} r \subseteq I \in \mathcal{P}_{\ell}(n) \} = \{ h_{I}^{-1} \circ r \mid r \in \sigma^{\wedge} \} = \{ h_{I}^{-1} \circ h_{J} \circ s \mid s \in \sigma, J \in \mathcal{P}_{\ell}(n) \} = \sigma.
\]

Concerning the second line in the equalities displayed above, note that the inclusion \( \text{Im} r \subseteq I \) holds as an equality, because \( r \) is an \( \ell \)-tuple without repeated entries, i.e., \( |\text{Im} r| = \ell = |I| \); moreover, \( \text{Im}(h_{I} \circ s) = J \).

The following theorem characterizes the closure operator \( \text{gComp}(n) \text{gPat}(\ell) \) of the monotone Galois connection \( \text{gPat}, \text{gComp} \).

Theorem 5.3. (A) \( \text{gComp}(n) \text{gPat}(\ell) H = \text{Aut pcInv} H \) for \( H \leq S_{n} \).

(B) Let \( H \) be a subgroup of \( S_{n} \). Then the following are equivalent:

(a) \( H \) is Galois closed, i.e., \( H = \text{gComp}(n) \text{gPat}(\ell) H \),

(a)' \( \exists G \leq S_{\ell} : H = \text{gComp}(n) G \),

(b) \( H = \text{Aut pcInv} H \),

(c) \( \exists k \leq \ell \exists \varrho \subseteq [n]_{k}^{\ell} : \varrho = \varrho^{\vee \wedge} \wedge H = \text{Aut} \varrho \).
Proof. (B): Clearly, (a) $\Leftrightarrow (a)'$, which follows from the properties of a (monotone) Galois connection. Also (c) $\Rightarrow (b)$ is obvious, since $H \subseteq \text{Aut pcInv } H \subseteq \text{Aut } \varrho = H$. In order to prove (b) $\Rightarrow (a)$, let $H = \text{Aut pcInv } H = \bigcap \{ \text{Aut } \varrho \mid \varrho \in \text{pcInv } H \}$. In Proposition 5.6 below we shall see that Aut $\varrho$ is Galois closed for each pc-relation $\varrho$ (i.e., if $\varrho = \varrho^\gamma$). Thus $H$ is the intersection of Galois closed groups and therefore also Galois closed. It remains to prove (a)$' \Rightarrow (c)$: Assume $H = \text{gComp}^{(n)} G$ for $G \leq S_\ell$. Then $\gamma_G$ is a pc-relation because $\gamma_G^\land \gamma = \gamma_G^\gamma$ by Lemma 5.2 and we have $H = \text{Aut } \gamma_G^\cdot$ by Proposition 5.6 (Recall the notation $\gamma_G$ from Section 4).

Finally, (A) follows from (B). In order to see this, let $H' := \text{Aut pcInv } H$ and $H'' := \text{gComp}^{(n)} \text{gPat}^{(\ell)} H$. Then $H'$ is Galois closed by (b) (since $\text{pcInv } \text{Aut pcInv } H = \text{pcInv } H$, thus $\text{Aut pcInv } H' = H'$) and contains $H$, consequently $H'' \subseteq H'$ (because $\text{gComp}^{(n)} \text{gPat}^{(\ell)}$ is a closure operator). Moreover, by (c) there exists a pc-relation $\varrho$ for the Galois closure $H''$ such that $H'' = \text{Aut } \varrho$. We have $\varrho \in \text{pcInv } H$ since $H \leq H''$. Therefore $H'' = \text{Aut } \varrho \supseteq \text{Aut pcInv } H = H'$, consequently $H' = H''$. \qed

Remark 5.4. Theorem 5.3(A) holds even for an arbitrary subset $H \subseteq S_n$ (since $\text{gPat}^{(\ell)} H = \text{gPat}^{(\ell)} \langle H \rangle$ by Lemma 3.4 and $\text{pcInv } H = \text{pcInv } \langle H \rangle$).

Lemma 5.5. Let $k \leq \ell \leq n$, and let $\sigma \subseteq [\ell]^k_\neq$ be a $k$-ary relation on $[\ell]$. Then we have $\text{gComp}^{(n)} \text{Aut } \sigma \subseteq \text{Aut } \sigma^\cdot$.

Proof. Let $\pi \in \text{Comp}^{(n)} \text{Aut } \sigma$ and let $u \in \sigma^\cdot$. Then $u = h_J \circ s$ for some $s \in \sigma$ and $J \in \mathcal{P}_\ell(n)$. Since $\pi$ is compatible with Aut $\sigma$, we have $\pi_J \in \text{Aut } \sigma$, that is, $\pi_J(s) \in \sigma$. Consequently,

$$
\pi(u) = \pi \circ h_J(s) = h_{\pi(J)} \circ h^{-1}_{\pi(J)} \circ \pi \circ h_J(s) = h_{\pi(J)} \circ \pi_J(s) \in \sigma^\cdot,
$$

and we conclude that $\pi \in \text{Aut } \sigma^\cdot$. \qed

Proposition 5.6. Let $k \leq \ell \leq n$, and let $\varrho \subseteq [n]^k_\neq$ be a $k$-ary pc-relation on $[n]$. Then Aut $\varrho$ is Galois closed, more precisely Aut $\varrho = \text{gComp}^{(n)}$ Aut $\varrho^\cdot$. In particular we have $\text{gComp}^{(n)} G = \text{Aut } \gamma_G^\cdot$ for any $G \leq S_\ell$.

Proof. Let $\pi \in \text{Aut } \varrho$. We show first that $\pi_J \triangleright \varrho^\cdot$ for all $J \in \mathcal{P}_\ell(n)$. Let $s \in \varrho^\cdot$. Then $s = h^{-1}_I(r)$ for some $r \in \varrho$ and Im $r \subseteq I \in \mathcal{P}_\ell(n)$. It follows from condition (5.1) that $h_J h^{-1}_I(r) \in \varrho$. Since $\pi \triangleright \varrho$, we also have $\pi h_J h^{-1}_I(r) \in \varrho$. Consequently,

$$
\pi_J(s) = h^{-1}_{\pi(J)} \circ \pi \circ h_J \circ h^{-1}_I(r) \in \varrho^\cdot,
$$
and we conclude that $\pi \in \text{Comp}^{(n)} \text{Aut} \varrho^\vee = \text{gComp}^{(n)} \text{Aut} \varrho^\vee$. Therefore $\text{Aut} \varrho \subseteq \text{gComp}^{(n)} \text{Aut} \varrho^\vee$. The converse inclusion follows immediately from Lemma 5.5: $\text{gComp}^{(n)} \text{Aut} \varrho^\vee \subseteq \text{Aut} \varrho^\vee$ because $\varrho = \varrho^\vee$ by Lemma 5.1.

Finally, recall that any $G \leq S_\ell$ can be considered as an $\ell$-ary relation $\gamma_G \subseteq [\ell]$. Thus $\varrho := \gamma_G^\wedge$ is a pc-relation and $\varrho^\vee = \gamma_G$. It holds that $\text{Aut} \gamma_G = G$. Thus we get $\text{Aut} \gamma_G^\wedge = \text{Aut} \varrho = \text{gComp}^{(n)} \text{Aut} \varrho^\vee = \text{gComp}^{(n)} \text{Aut} \gamma_G = \text{gComp}^{(n)} G$. 

Now we take a look at the other side of the monotone Galois connection $(\text{gPat}^{(\ell)}, \text{gComp}^{(n)})$, namely at the kernel operator $\text{gPat}^{(\ell)} \text{gComp}^{(n)}$. We want to describe $G' := \text{gPat}^{(\ell)} \text{gComp}^{(n)} G$ as the automorphism group of some relations. Clearly, because of $\text{Aut} \text{Inv} G' = G' \subseteq G = \text{Aut} \text{Inv} G$ we have to extend the set $\text{Inv} G$ in order to get the set $\text{Inv} G'$. We shall see that the following definition will fit our purposes.

**Definition 5.7.** Let $G \leq S_\ell$. A relation $\sigma \subseteq [\ell]^k (k \leq \ell)$ is called a *pattern closed extended invariant* (pc-extended invariant) of $G$ if $\sigma^\wedge = \sigma$ and $\sigma^\wedge \in \text{Inv} \text{Aut} \gamma_G^\wedge$. The set of all pc-extended invariants of $G$ is denoted by $\text{pcExt} G$.

**Remark 5.8.** Note that $\sigma \in \text{pcExt} G$ implies $\sigma^\wedge \in \text{pcInv} \text{Aut} \gamma_G^\wedge$. Analogously to Lemma 5.1 we have $\sigma \subseteq \sigma^\wedge$, and, furthermore, $\sigma^\wedge \vee = \sigma$ is equivalent to the following condition:

$$\forall s \in [\ell]^k \forall I, J \in \mathcal{P}_\ell(n): s \in \sigma \wedge \text{Im} h_J(s) \subseteq J \implies h_I^{-1} h_J(s) \in \sigma.$$  

**Lemma 5.9.** Let $H = \text{Aut} \varrho \leq S_n$ for a pc-relation $\varrho \subseteq [n]^k$ where $k \leq \ell \leq n$. Then we have $\text{gPat}^{(\ell)} H \subseteq \text{Aut} \varrho^\vee$.

**Proof.** Let $\sigma \in \text{Pat}^{(\ell)} H$. Then there exist $\tau \in H$ and $J \in \mathcal{P}_\ell(n)$ such that $\sigma = \tau_J = h_{\tau_J}^{-1} \circ \tau \circ h_J$. Let $u \in \varrho^\vee$. Then $u = h_I^{-1}(r)$ for some $r \in \varrho$ and $I \in \mathcal{P}_\ell(n)$ satisfying $\text{Im} r \subseteq I$. Since $\varrho$ satisfies condition (5.1), we have $h_J h_I^{-1}(r) \in \varrho$. Since $\tau \triangleright \varrho$, we have $\tau h_J h_I^{-1}(r) \in \varrho$. Consequently,

$$\sigma(u) = h_{\tau_J}^{-1} \circ \tau \circ h_J \circ h_I^{-1}(r) \in \varrho^\vee.$$  

This shows that $\text{Pat}^{(\ell)} H \subseteq \text{Aut} \varrho^\vee$. Consequently, $\text{gPat}^{(\ell)} H \subseteq \text{Aut} \varrho^\vee$.

**Theorem 5.10.** $\text{gPat}^{(\ell)} \text{gComp}^{(n)} G = \text{Aut pcExt} G$ for $G \leq S_\ell$.  

Proof. Let \( G_1 := \text{gPat}^{(\ell)} g\text{Comp}^{(n)} G, \) \( G_2 := \text{Aut pcExt} G. \) Then \( H := g\text{Comp}^{(n)} G = \text{Aut} \gamma_{G^\wedge} = \text{Aut pcInv} H \) by Theorem 5.3 and Proposition 5.6.

We prove \( G_1 \subseteq G_2. \) Let \( \sigma \in \text{pcExt} G. \) By Remark 5.8, we have \( \sigma^\wedge \in \text{pcInv} \text{Aut} \gamma_{G^\wedge} = \text{pcInv} H, \) which implies \( \text{Aut} \sigma^\wedge \supseteq \text{Aut pcInv} H = H. \) Using Lemma 5.9, we get \( G_1 = \text{gPat}^{(\ell)} H \subseteq \text{gPat}^{(\ell)} \text{Aut} \sigma^\wedge \subseteq \text{Aut} \sigma^\wedge \wedge \text{Aut} \sigma = \text{Aut} \sigma. \) Consequently, \( G_1 \subseteq \bigcap \{ \text{Aut } \sigma \mid \sigma \in \text{pcExt} G \} = G_2. \)

For the converse implication \( G_2 \subseteq G_1, \) consider \( G_1 \) as the \( \ell\)-ary relation \( \gamma_G = G_1 \subseteq [\ell]^\ell. \) Then \( \gamma_G \) is a pc-extended invariant of \( G. \) Indeed, \( \gamma_G = \gamma_{G^\wedge} \wedge \text{by Lemma 5.2 and } \gamma_{G^\wedge} \) is a pc-relation. Moreover, from Proposition 5.6 and the properties of monotone Galois connections, we get \( \text{Aut} \gamma_{G^\wedge} = g\text{Comp}^{(n)} G_1 = g\text{Comp}^{(n)} G = H. \) Then it follows that \( \gamma_{G^\wedge} \in \text{pcInv} H = \text{pcInv} \text{Aut} \gamma_{G^\wedge}. \) Consequently, \( G_2 = \text{Aut pcExt} G \subseteq \text{Aut} \gamma_{G^\wedge} = G_1. \)

6 Concluding remarks

With the monotone Galois connection \( \text{(Pat}^{(\ell)}, \text{Comp}^{(n)} \text{)} \) at hand, a natural question to ask is what its closed sets and kernels are. In the current paper, we focused on those closed sets that are subgroups of \( S_n. \) A reasonable general description of all closed sets eludes us.

We would like to point out another direction to which our study inevitably leads. Atkinson and Beals [1, 2] studied group classes, i.e., permutation classes in which all levels are groups. In particular, they determined the possible asymptotical behaviours of level sequences \( C^{(1)}, C^{(2)}, \ldots \) of group classes \( C. \) Furthermore, they fully described the group classes in which all levels are transitive groups. As an attempt of refining Atkinson and Beals’s results and looking deeper into the local behaviour of the level sequences of group classes, in [7], one of the current authors set about describing, for an arbitrary group \( G \leq S_n, \) what the sequence

\[
\ldots, \text{gPat}^{(n-2)} G, \text{gPat}^{(n-1)} G, G, \text{gComp}^{(n+1)} G, \text{gComp}^{(n+2)} G, \ldots
\]

looks like. The Galois connections \( (\text{gPat}, \text{gComp}) \) and \( (\text{Pat}, \text{Comp}) \) that were obtained in the current paper might be useful tools in further analysis of group classes.

We would also like to point out that the questions we are considering in this paper can be asked for other definitions of pattern involvement, e.g.,
consecutive, vincular, bivincular, mesh, etc. (see Kitaev [6]). This remains a topic of further investigation.

Acknowledgments

The authors would like to thank Nik Ruškuc for insightful and inspiring discussions.

References

[1] M. D. Atkinson, R. Beals, Permuting mechanisms and closed classes of permutations, in: C. S. Calude, M. J. Dinneen (eds.), Combinatorics, Computation & Logic, Proc. DMTCS ’99 and CATS ’99 (Auckland), Aust. Comput. Sci. Commun., 21, No. 3, Springer, Singapore, 1999, pp. 117–127.

[2] M. D. Atkinson, R. Beals, Permutation involvement and groups, Q. J. Math. 52 (2001) 415–421.

[3] M. Bóna, Combinatorics of Permutations, Discrete Math. Appl. (Boca Raton), Chapman & Hall/CRC, Boca Raton, 2004.

[4] K. Denecke, M. Erné, S. L. Wismath (eds.), Galois Connections and Applications, Math. Appl., vol. 565, Kluwer Academic Publishers, Dordrecht, 2004.

[5] J. D. Dixon, B. Mortimer, Permutation Groups, Grad. Texts in Math., vol. 163, Springer, New York, 1996.

[6] S. Kitaev, Patterns in Permutations and Words, Monogr. Theoret. Comput. Sci. EATCS Ser., Springer, Heidelberg, 2011.

[7] E. Lehtonen, Permutation groups arising from pattern involvement, arXiv:1605.05571.

[8] R. Pöschel, L. A. Kalužnin, Funktionen- und Relationenalgebren, Ein Kapitel der diskreten Mathematik, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979.