ALTERNATIVE LAGRANGIANS OBTAINED BY SCALAR DEFORMATIONS

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Abstract. We study non-conservative like SODEs admitting explicit Lagrangian descriptions. Such systems are equivalent to the system of Lagrange equations of some Lagrangian \( L \), including a covariant force field which represents non-conservative forces. We find necessary and sufficient conditions for the existence of a differentiable function \( \Phi : \mathbb{R} \to \mathbb{R} \) such that the initial system is equivalent to the system of Euler-Lagrange equations of the deformed Lagrangian \( \Phi(L) \). We give various examples of such deformations.

1. Introduction

Consider a system of second order ordinary differential equations (SODE), in normal form,

\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \quad i \in \mathbb{R}^n.
\]

A geometric approach to the study of the solutions of such a SODE consists in identifying the system (1.1) with a second order vector field or a semi-spray. This means that \( S \) is a vector field on \( TM \), \( M \) an \( n \)-dimensional differentiable manifold, with \( JS = C \), where \( J \) is the vertical endomorphism and \( C \) the Liouville vector field [13].

Locally, \( S \) can be represented in the natural basis of \( TM \) as follows:

\[
S = y^i \frac{\partial}{\partial y^i} - 2G^i(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i}.
\]

The solutions of (1.1) are called the geodesics of the semi-spray.

An approach to the inverse problem of the calculus of variations searches for the existence of a multiplier matrix \( g_{ij}(x, \dot{x}) \) which relates the equations of the geodesics of a semi-spray with the Euler-Lagrange equations derived from a Lagrangian function \( L \):

\[
g_{ij} \left( \frac{d^2 x^i}{dt^2} + 2G^j(x, \dot{x}) \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i}.
\]

The Lagrangian \( L \) is determined from the condition that the multiplier matrix coincides with the Hessian of \( L \). Necessary and sufficient conditions for the existence of such a multiplier matrix are known as Helmholtz conditions [4].

In some particular cases there exist several different Lagrangians, called alternative Lagrangians, such that the system (1.1) is equivalent to the system of Euler-Lagrange equations of each of those Lagrangians. Alternative Lagrangian can be used, for example, to construct constant of motions [7].

One can work with alternative Lagrangians in order to simplify computations. Given a geometric structure on a differentiable manifold, it is possible to consider more general structures on the same manifold, related to the first one, and study the relations between the geometric objects induced by these different structures. For example, in [3], the authors considered a Finsler manifold \( (M, F) \) and using a differential deformation \( \alpha : \mathbb{R}_+ \to \mathbb{R} \), they constructed a Lagrange manifold \( (M, L = \alpha(F^2)) \). In general the computations in the associated Lagrange manifold are simpler and one can obtain useful information about the initial Finsler manifold. The examples presented on this article involve Antonelli’s ecological metric and Synge metric [1, 2], with various applications in biology and physics.

During the last few years non-standard Lagrangians are considered good candidates to explain non-conservative dynamical systems. Non-standard Lagrangians are Lagrangians that cannot be expressed as differences between kinetic energy terms and potential energy terms. They can be of exponential type, of power-law type, or even radical type [9]. They were used, for instance, to study second order Ricati and Abel equations [8] or non-inertial dynamics [11].

The above aspects encouraged us to investigate alternative Lagrangians obtained as deformations of other Lagrangians.
Suppose that the system (1.1) is variational, i.e. it is equivalent with the system \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0 \). Consider a deformation of the initial Lagrangian \( \Psi \circ L \), where \( \Psi : \mathbb{R} \to \mathbb{R} \) is a differentiable function of class \( C^2 \). The equations of motion of \( \Psi(L) \) have the form

\[
\frac{d}{dt} \left( \frac{\partial \Psi(L)}{\partial \dot{x}^i} \right) - \frac{\partial \Psi(L)}{\partial x^i} = \sigma_i(x, \dot{x}),
\]

where \( \sigma_i(x, \dot{x}) \) is a covariant tensor field that represents some external forces. Hence the initial system appears as a non-conservative mechanical system, with respect to the alternative Lagrangian \( \Psi(L) \). The most studied cases are dissipative and gyroscopic systems \([5, 10, 16, 18]\).

In a recent paper, \([6]\), starting with a given Lagrangian \( L \), the authors determined all deformations \( \Psi \) such that \( \Psi(L) \) is dynamically equivalent to \( L \), i.e. \( \Psi(L) \) and \( L \) give the same dynamical vector field. This paper was the starting point for the study developed here. This paper was also inspired by \([1, 3, 5, 9, 11, 17, 19]\), in which the authors searched for alternative or non-standard Lagrangians.

In this article we are interested in the following situation. We start with a non-conservative like system, i.e. a SODE in normal form (1.1), which can be written in an equivalent way as in (1.3), for some Lagrangian starting point for the study developed here. This paper was also inspired by \([1, 3, 5, 9, 11, 17, 19]\), in which the authors searched for alternative or non-standard Lagrangians.

We give various examples of explicit deformations, some of them with possible applications in theoretical physics or applied mathematics. The examples of the article start with regular Lagrangians \( L \) and include both regular and singular Lagrangians \( \Phi(L) \).

2. Preliminaries

For an \( n \)-dimensional smooth manifold \( M \), denote by \( TM \) its tangent bundle. The Local coordinates \( (x^i, y^i) \) on \( M \) induce local coordinates \( (x^i, y^i) \) on \( TM \). We will use the following notations: \( C^\infty(TM) \) for the real algebra of smooth functions on \( TM \), \( \mathcal{X}(TM) \) for the Lie algebra of smooth vector fields on \( TM \) and \( \Lambda^k(TM) \) for the \( C^\infty(TM) \) module of differentiable \( k \)-forms on \( TM \).

The canonical submersion \( \pi : TM \to M \) induces a natural foliation on \( TM \). The tangent spaces to the leaves of this foliation determine a regular \( n \)-dimensional distribution, \( VTM : u \in TM \to V_uTM = \text{Ker} d_u \pi \subset T_uTM \), which is called the vertical distribution. A vector field \( X \in \mathcal{X}(TM) \) is vertical if \( X_u \in V_uTM \), \( \forall u \in TM \).

Consider \( \mathbb{C} \in \mathcal{X}(TM) \) the Liouville (dilation) vector field and \( J \) the tangent structure (vertical endomorphism), locally given by:

\[
\mathbb{C} = y^i \frac{\partial}{\partial y^i}, \quad J = \frac{\partial}{\partial y^i} \otimes dx^i.
\]
A semi-spray, or a second order vector field, is a globally defined vector field on $TM$, $S \in \mathfrak{X}(TM)$, that satisfies $JS = 0$. Locally, a semi-spray can be expressed as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

with $G^i$ smooth functions on $TM$.

Any system of SODE in normal form (1.1) can be identified with a semi-spray on $TM$, with local coefficients $G^i(x, y)$.

A semi-spray is called a spray if it is homogeneous of degree 2 with respect to fiber coordinates $y$, it means that $[\mathbb{C}, S] = S \iff \text{the functions } G^i(x, y) \text{ are homogeneous of degree } 2: \mathbb{C}(G^i) = 2G^i$.

A Lagrangian on $TM$ is a smooth function $L: TM \to \mathbb{R}$ whose Hessian with respect to the fiber coordinates $\theta$

$$(2.1)\quad g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}$$

is nontrivial. We say that $L$ is a regular Lagrangian if the Poincaré-Cartan 2-form $dd_jL$ is a symplectic form on $TM$, where $d_j = i_j \circ d - d \circ i_j$. Locally, the regularity condition of a Lagrangian $L$ is equivalent to the fact that the Hessian (2.1) of $L$ has maximal rank $n$ on $TM$.

For a Lagrangian $L$, we consider $E_L = \mathbb{C}(L) - L$ its Lagrangian energy.

A form on $TM$ is called semi-basic if it vanishes whenever one of its arguments is vertical. A vector valued form on $TM$ is called semi-basic if it takes vertical values and it vanishes whenever one of its arguments is vertical. For example, the tangent structure $J$ is a vector valued semi-basic 1-form.

For an arbitrary semi-spray $S$ and a Lagrangian $L$, the following 1-form (the Lagrange differential [20]) is a semi-basic 1-form:

$$\delta_S L = \mathcal{L}_S d_j L - dL = \left\{ S \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} \right\} dx^i.$$

We introduce next some of the derivations used in this article, within the Frölicher-Nijenhuis formalism, [13].

Consider a vector valued $p$-form $P$ on $TM$. We will denote by $i_P : \Lambda^k(TM) \to \Lambda^{k+p-1}(TM)$ the derivation of degree $(p-1)$, given by

$$i_P \alpha(X_1, ..., X_{k+p-1}) = \frac{1}{p!(k-1)!} \sum_{\sigma \in S_{k+p-1}} \text{sign}(\sigma) \alpha \left( P(X_{\sigma(1)}, ..., X_{\sigma(p)}), X_{\sigma(p+1)}, ..., X_{\sigma(k+p-1)} \right),$$

where $S_{k+p-1}$ is the permutation group of $\{1, ..., k + p - 1\}$. Particularly, for the vertical endomorphism $J$ and $\theta \in \Lambda^1(TM)$, we obtain $(i_J \theta)(X) = \theta(JX), \forall X \in \mathfrak{X}(TM)$. The derivation $i_P$ is trivial on functions and hence it is uniquely determined by its action on $\Lambda^1(TM)$. It follows that $i_P$ is a derivation of $i_w$-type.

We denote by $d_P : \Lambda^k(TM) \to \Lambda^{k+p}(TM)$ the derivation of degree $p$, given by

$$d_P = [i_P, d] = i_P \circ d - (-1)^{p-1} d \circ i_P.$$  

In particular, $(d_J \theta)(X, Y) = (JX)(\theta Y) - (JY)(\theta X) - \theta(JX, Y) + \theta(JY, X) - \theta(JX, Y), \forall X, Y \in \mathfrak{X}(TM), \forall \theta \in \Lambda^1(TM)$. Evidently $d_X = \mathcal{L}_X$ is the Lie derivative if $X \in \mathfrak{X}(TM)$. The derivation $d_P$ commutes with the exterior derivative $d$ and hence it is uniquely determined by its action on $C^\infty(TM)$. It follows that $d_P$ is a $d_w$-type derivation.

For two vector valued forms $K$ and $P$ on $TM$, of degrees $k$ and $p$, we consider the Frölicher-Nijenhuis bracket $[K, P]$, which is the vector valued $(k + p)$-form, uniquely determined by

$$(2.2)\quad d_{[K, P]} = [d_K, d_P] = d_{k \circ d_p} - (-1)^{kp} d_p \circ d_K.$$

For example, $[X, J] = \mathcal{L}_X J = \mathcal{L}_X \circ J - J \circ \mathcal{L}_X, \forall X \in \mathfrak{X}(TM)$.

The definition of the tangent structure $J$ leads to $[J, J] = 0$ and $d_J^2 = 0$ follows from the formula (2.2). Therefore, any $d_J$-exact form is $d_J$-closed and according to a Poincaré-type Lemma [21], any $d_J$-closed form is locally $d_J$-exact.

**Definition 1.** a) Given a semi-spray $S$, we say that $S$ is Lagrangian if there exists a (locally defined) Lagrangian $L$ such that $\delta_S L = 0$. Locally it means that the solutions of the system (1.1) are among the solutions of the Euler-Lagrange equations of some Lagrangian $L : \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0$. If $L$ is regular, these two systems are equivalent [4].
b) Consider $S$ a semi-spray and $\sigma \in \Lambda^1(TM)$ a semi-basic 1-form. We say that $S$ is of Lagrangian type with covariant force field $\sigma$ if there exists a (locally defined) Lagrangian $L$ such that $\delta_S L = \sigma$. It means that the solutions of the system (1.1) are among the solutions of the system of Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = \sigma(x^i, \dot{x}^i).$$

If the Lagrangian $L$, which we search for, is regular, then the two systems (1.1) and (2.3) are equivalent [4].

### 3. Deformations of Lagrangians

Consider a SODE in normal form (1.1), which can be written in an equivalent form as in (2.3), for some Lagrangian $L$. The system (2.3) is equivalent to $\delta_S L = \sigma$, where $S$ is the semi-spray on $TM$ with local coefficients $G^i(x, y)$ and $\sigma$ is the semi-basic 1-form $\sigma_i(x, y)dx^i$. It means that $S$ is of a Lagrangian type with covariant force field $\sigma$.

In this section we determine necessary and sufficient conditions for the existence of non-constant, differentiable of class $C^2$ deformations $\Phi : \mathbb{R} \to \mathbb{R}$, such that the system (1.1) is equivalent to the system of Euler-Lagrange equations of $\Phi(L)$:

$$\frac{d}{dt} \left( \frac{\partial \Phi(L)}{\partial \dot{x}^i} \right) - \frac{\partial \Phi(L)}{\partial x^i} = 0 \iff \delta_S \Phi(L) = 0,$$

Now $S$ is a Lagrangian second order vector field corresponding to the Lagrangian $\Phi(L)$.

Using Frölicher-Nijenhuis formalism and the theory of derivations, we obtain the main result of the paper.

**Theorem 1.** Let $S \in \mathcal{X}(TM)$ be a semi-spray of Lagrangian type with covariant force field $\sigma$ and $L$ the corresponding (local) Lagrangian on $TM$, with $S(L) \neq 0$ and $C(L) \neq 0$. Then there exists a non-constant, differentiable of class $C^2$ function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $S$ is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$ if and only if the following three conditions are satisfied:

\[
\begin{align*}
\sigma &= \frac{S(E_L)}{C(L)} d_j L, \\
\frac{\Phi''(L)}{\Phi'(L)} &= -\frac{S(E_L)}{S(L)C(L)}, \\
\left( \Phi'' \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j} + \Phi' \frac{\partial^2 L}{\partial y^i \partial y^j} \right)_{i,j} &\neq O_n.
\end{align*}
\]

**Proof.** We investigate the relation between the equations of motion for $L$ and for its deformation $\Phi(L)$.

We compute the Lagrange differential of the deformed Lagrangian $\Phi(L)$.

\[
\delta_S \Phi(L) = \left[ S \left( \frac{\partial \Phi(L)}{\partial y^i} - \frac{\partial \Phi(L)}{\partial x^i} \right) \right] dx^i = \left[ S \left( \frac{\Phi'(L) \frac{\partial L}{\partial y^i}}{\Phi'(L)} - \frac{\partial \Phi(L)}{\partial x^i} \right) \right] dx^i = \left[ \Phi''(L) S(L) \frac{\partial L}{\partial y^i} + \Phi'(L) S \left( \frac{\partial L}{\partial y^i} \right) - \Phi'(L) \frac{\partial L}{\partial x^i} \right] dx^i.
\]

Using $d_j L = \frac{\partial L}{\partial y^i} dx^i$, the above relation is equivalent to

\[
\delta_S \Phi(L) = \Phi''(L) S(L) d_j L + \Phi'(L) \delta_S L.
\]

Suppose that $S$ is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$.

The conditions $\delta_S L = \sigma$ and $\delta_S \Phi(L) = 0$ are simultaneously satisfied if and only if

\[
\Phi''(L) S(L) d_j L + \Phi'(L) \sigma = 0.
\]

Let $X = S$ and $K = J$ in the formula

\[
i_X d_K = -d_K i_X + \mathcal{L}_{KX} + i_{[K,X]},
\]

then

\[
i_S d_J + d_i i_S = \mathcal{L}_{ij} + i_{[i,j]}.
\]

Applying $i_S$ to the equation (3.3) we get

\[
\Phi''(L) = -\frac{S(E_L)}{S(L)C(L)} \Phi'(L).
\]

This formula is obtained using (3.5), noting that

\[
i_S \sigma = i_S (\mathcal{L}_S d_j L - dL) = i_S (dS d_j L + i_S dd_j L - dL) = i_S (dC(L) - dL) = i_S dE_L = (\mathcal{L}_S - dS) (E_L) = S(E_L).
\]
ALTERNATIVE LAGRANGIANS OBTAINED BY SCALAR DEFORMATIONS

Substituting $\Phi''(L)$ in (3.4) we obtain the required expression for $\sigma$ in (3.1).
If we want $\Phi(L)$ to be also a Lagrangian, we investigate its Hessian with respect to the fiber coordinates $(y^i)$.
We denote $\hat{g}_{ij} = \frac{\partial^2 \Phi(L)}{\partial y^i \partial y^j}$. It results $\hat{g}_{ij} = \Phi'' \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j}$. Hence we are searching for deformations $\Phi$ such that
\[
\left( \Phi'' \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j} + \Phi' \hat{g}_{ij} \right) = \text{a non-trivial matrix. Therefore the non-constant function } \Phi \text{ has to satisfy the condition}
\]
\[
\exists i, j \in \mathbb{N}, \; \hat{g}_{ij} \neq -\frac{\Phi'' \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j}}{\Phi'}.
\]
Evidently $\Phi(L)$ is a regular Lagrangian if and only if $\rank \left( \Phi'' \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j} + \Phi' \hat{g}_{ij} \right) = n$ on $TM$.
The converse is trivial.

Remark. For $\sigma = 0$, i.e. studying the deformations of conservative mechanical system, we reobtain the result of [6]: $\Phi(L)$ is dynamically equivalent to $L$ if and only if $\Phi(t) = at + b, a, b \in \mathbb{R}, a \neq 0$ or $L$ is a constant of motion for $S$ i.e. $S(L) = 0$. In the second case, $\Phi$ can be any differentiable function.

We remark that having a semi-spray such that its equations of motions can be written as $\delta_S L = \sigma$, we are able to apply a simple test to check if $S$ is Lagrangian with respect to an alternative Lagrangian $\Phi(L)$. For this we verify if condition (3.1) is satisfied. In the affirmative case, we have to integrate equation (3.2) and obtain the deformation $\Phi$. But this is possible only if the expression $\frac{S(E_L)}{S(L)C(L)}$ can be written as an integrable function of the initial Lagrangian $L$. Also, the matrix (3.3) has to be non-trivial, respectively of maximal rank $n$, depending on what kind of Lagrangian $\Phi(L)$ we search for, a singular or a regular one.

In the last section we will consider some classes of Lagrangian’s deformations, inspired by the references, that satisfy the conditions
\[
\delta_S L = \frac{S(E_L)}{C(L)}d_jL, \; \exists f : \mathbb{R} \rightarrow \mathbb{R} : -\frac{S(E_L)}{S(L)C(L)} = f(L)
\]
and condition (3.3), where $f : \mathbb{R} \rightarrow \mathbb{R}$ is an integrable function.

If we suppose that $\Phi$ is a strictly increasing function, then
\[
\Phi(L) = \int \exp \left( \int f(L)dL \right) dL, \; \delta_S \Phi(L) = 0.
\]

Homogeneous Lagrangians. Suppose that $S$ is a spray of Lagrangian type with covariant force field $\sigma$ and the corresponding Lagrangian $L$. In this subsection, we assume that $\sigma$ and $L$ are both homogeneous of order $p$. It means that $[C, L] = C(L) = pL$ and $[C, \sigma] = L_C\sigma = p\sigma$. We will work on $T_0M$, the tangent bundle with the zero section removed.

If $p = 1 \Leftrightarrow C(L) = L$, then $E_L = 0$ and formula (3.1) implies $\sigma = 0$. This case was treated in [6]. Therefore we assume $p > 1$.

Theorem 2. Let $S \in \mathfrak{X}(T_0M)$ be a spray of Lagrangian type with covariant force field $\sigma$, with $\sigma$ a semi-basic 1-form on $T_0M$, homogeneous of order $p > 1$. Let $L$ be the corresponding (local) Lagrangian on $TM$, also homogeneous of order $p$. Suppose that $L$ has positive values. Then there exists a differentiable of class $C^2$, strictly increasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$, such that $S$ is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$, if and only if
\[
(3.6) \quad \left( \frac{1}{p} \frac{\partial L}{\partial y^i} \frac{\partial L}{\partial y^j} + L \frac{\partial^2 L}{\partial y^i \partial y^j} \right) \neq 0_n
\]
and
\[
d_jL \wedge \sigma = 0.
\]
Moreover, the deformation $\Phi$ is given by
\[
\Phi(L) = aL \dot{\hat{r}} + b, \; a, b \in \mathbb{R}.
\]
Proof. Suppose that $S$ is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$: $\delta_S \Phi(L) = 0$. From the above theorem, using $\mathbb{C}(L) = pL$ and $E_L = (p - 1)L$, we deduce

$$\Phi'' = \left( \frac{1}{p} - 1 \right) \frac{1}{L}$$

and

$$\sigma = \left( 1 - \frac{1}{p} \right) S \left( \ln L \right) d_j L.$$ 

It follows that $d_j L \wedge \sigma = 0$.

Conversely, suppose that $\sigma$ is homogeneous of order $p > 1$ and $d_j L \wedge \sigma = 0$. Applying $i_S$ to this relation, it yields

$$\mathbb{C}(L) \sigma - i_S \sigma d_j L = 0 \iff pL \sigma - (p - 1)S(L)d_j L = 0 \iff \sigma = \left( 1 - \frac{1}{p} \right) \frac{S(L)}{L} d_j L.$$ 

Hence $\sigma = \frac{S(E_L)}{\mathbb{C}(L)} d_j L$. Using again the last theorem, it results that $\delta_S \Phi(L) = 0$.

For $L$ with positive values and $\Phi$ a strictly increasing function, we can easily integrate the equation (3.7) and obtain

$$\Phi(L) = aL^\frac{1}{p} + b, \ a, b \in \mathbb{R}.$$ 

The fact that the Hessian matrix of $\Phi(L)$ is non-trivial is equivalent to the condition (3.6).

Remark that we started with a Lagrangian homogeneous $L$ of order $p > 1$, and for $b = 0$, the deformed Lagrangian $\Phi(L)$ is homogeneous of order 1, $\mathbb{C}(\Phi(L)) = \Phi(L)$.

\[\square\]

4. Examples

All the examples of this section satisfy the conditions

$$\delta_S L = \frac{S(E_L)}{\mathbb{C}(L)} d_j L, \ \exists f : \mathbb{R} \to \mathbb{R} : -\frac{S(E_L)}{S(L)\mathbb{C}(L)} = f(L),$$

with $f$ an integrable function. The examples are subordinated to the following cases:

1. If $f$ is a constant function, $f(L) = \gamma = \text{constant}$, it follows $\Phi(L) = \frac{1}{\gamma} \exp(\gamma L) + a, \ a \in \mathbb{R}$. A similar deformation of Lagrangian was presented in [9] [11].

2. If $f(L) = \frac{1}{L+a}, \gamma \in \mathbb{R}^* \setminus \{-1\}, \ a \in \mathbb{R}$, then $\Phi(L) = \frac{1}{1+\gamma} (L + a)^{1+\gamma} + b, \ b \in \mathbb{R}$. It is a deformation used in [3] [9] [11] [15].

   a. As a particular subcase, $f(L) = \left( \frac{1}{p} - 1 \right) \frac{1}{L}, \ p \in \mathbb{N}, \ p > 1$. It follows $\Phi(L) = L^\frac{1}{p} + a, \ a \in \mathbb{R}$. This deformation is obtained when we search for Lagrangians that are homogeneous of order $p > 1$, as the Lagrangians studied in Finsler Geometry [2].

   b. The deformation $\Phi(L) = \frac{1}{p} L^2 + aL + c$ was used by Kawaguchi, T., Miron, R. [15]. They studied the geometry of a generalized Lagrange space obtained from a Riemannian space, using a deformation of the Riemannian metric.

3. If $f(L) = \frac{-1}{L+a}, \ a \in \mathbb{R}$, then it follows $\Phi(L) = b \ln(L + a) + c, \ b, c \in \mathbb{R}$. Such a deformation appeared also in [3].

4. If $f(L) = \frac{-2c}{cL+d}, \ c, d \in \mathbb{R}$, then it follows $\Phi(L) = \frac{aL^2 + b}{cL+d}, \ a, b \in \mathbb{R}, \ ad - bc \neq 0$. This deformation is used before in [3].

Most of the following examples produce regular Lagrangians $\Phi(L)$.

We start with an example corresponding to the first class of Lagrangian’s deformations enumerated above.
Example 1. Let $M$ be a real, 3-dimensional, differentiable manifold. Consider the SODE

\[ \frac{d^2x}{dt^2} + g(x) \frac{dx}{dt} + h(x) = 0, \]

(4.2)

The associated semi-spray is $S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + x^3 y^3 + \left( y^1 \right)^2 + \left( y^2 \right)^2$. We consider the regular Lagrangian

\[ L(x, y) := C_2 + \frac{1}{\gamma} \ln \left[ \gamma \left( x^1 y^1 + x^2 y^2 + x^3 y^3 + \left( y^1 \right)^2 + \left( y^2 \right)^2 \right) - C_1 \right], \]

with $C_1, C_2, \gamma \in \mathbb{R}^*$, defined on an open set such that $x^1 y^1 + x^2 y^2 + x^3 y^3 + \left( y^1 \right)^2 + \left( y^2 \right)^2 > \frac{C_1}{\gamma}$ and the semi-basic 1-form

\[ \sigma = \frac{\gamma \left[ x^1 x^3 y^1 - \left( y^1 \right)^2 - \left( y^2 \right)^2 - \left( y^3 \right)^2 \right]}{\left[ \gamma \left( x^1 y^1 + x^2 y^2 + x^3 y^3 + \left( y^1 \right)^2 + \left( y^2 \right)^2 \right) - C_1 \right]^2} \left[ (x^1 + 2y^1) dx^1 + (x^2 + 2y^2) dx^2 + x^3 dx^3 \right]. \]

By direct computations, we obtain that the system (4.2) is equivalent to $\delta S L = \sigma$ with $L$ and $\sigma$ defined above. We use

\[ S(L) = \exp \left( \gamma (C_2 - L) \right) \left[ (y^1)^2 + (y^2)^2 + (y^3)^2 - x^1 x^3 y^1 \right], \]

\[ d_J L = \exp \left( \gamma (C_2 - L) \right) \left[ (x^1 + 2y^1) dx^1 + (x^2 + 2y^2) dx^2 + x^3 dx^3 \right], \]

\[ C(L) = \exp \left( \gamma (C_2 - L) \right) \left[ x^1 y^1 + x^2 y^2 + x^3 y^3 + 2(y^1)^2 + 2(y^2)^2 \right], \]

\[ S(E_L) = -\gamma C(L) S(L), \quad \delta S L = -\gamma S(L) d_J L. \]

We verify that both conditions (4.1) are satisfied, with $f(L) = \gamma$. Hence, the deformed Lagrangian is given by

\[ \Phi(L)(x, y) = \frac{1}{\gamma} \exp \left( \gamma C_2 \right) \left[ \gamma \left( x^1 y^1 + x^2 y^2 + x^3 y^3 + (y^1)^2 + (y^2)^2 \right) - C_1 \right]. \]

One can easily see that $\Phi(L)$ is a regular Lagrangian.

The next example corresponds to the second class of Lagrangian’s deformations.

Example 2. [12] [13] Consider the Liénard-type second-order nonlinear differential equation of the form

\[ \frac{d^2x}{dt^2} + g(x) \frac{dx}{dt} + h(x) = 0, \]

(4.3)

where $g$ and $h$ are real, smooth functions of $x$, defined on an interval. It plays an important role in many areas of applied sciences, cardiology, neurology, biology, mechanics, seismology, chemistry, physics, and cosmology.

If the functions $g, h$ satisfy the Chiellini integrability condition

\[ \frac{d}{dx} \left( \frac{h}{g} \right) = kg, \]

with $k \in \mathbb{R}^*$, then it is possible to construct exact solutions for a first-order Abel equation of the first kind associated to (4.3).

We consider the Liénard equation (4.3) with $g, h$ satisfying Chiellini’s condition with $k$ of the form $-\alpha (\alpha + 1)$ and $\alpha \in \mathbb{R}^*$.

Hence, equation (4.3) can be written in the form $\delta S L = \sigma$, with $S = y \frac{\partial}{\partial x} - (g(x) y + h(x)) \frac{\partial}{\partial y}$ a semi-spray on an 1-dimensional, real, differentiable manifold, $\sigma = 2 \left( \frac{1}{\alpha} h(x) - g(x)y \right) dx$ a semi-basic 1-form and the Lagrangian

\[ L(x, y) = \left( y - \frac{1}{\alpha} \frac{h(x)}{g(x)} \right)^2. \]

We can show that $\sigma$ satisfies the conditions (4.1), by the following computations:

\[ C(L) = 2y \left( y - \frac{1}{\alpha} \frac{h(x)}{g(x)} \right), \quad E_L = y^2 - \frac{1}{\alpha^2} \left( \frac{h(x)}{g(x)} \right)^2 \]

\[ S(L) = 2 \alpha g(x)L, \quad S(E_L) = -g(x)C(L), \quad d_J L = -\frac{1}{g(x)} \sigma. \]
Moreover, \( f(L) = -\frac{1}{2xL} \), hence the deformed Lagrangian is given by
\[
\Phi(L) = \frac{2\alpha a}{2a+1} L^{\frac{2\alpha+1}{2a+1}} + b, \ a, b \in \mathbb{R}.
\]
We know from theorem 1 that for any \( a, b \in \mathbb{R} \), \( \Phi(L) \) verifies \( \delta_S \Phi(L) = 0 \). For \( b = 0 \) and \( a = \frac{\alpha}{2(\alpha+1)} \) we get
\[
\hat{L}(x, y) = \frac{\alpha^2}{(\alpha + 1)(2\alpha + 1)} \left( y - \frac{1}{\alpha} \frac{g(x)}{f(x)} \right)^{2\alpha+1}.
\]
This Lagrangian was obtained in \[12\] using the Jacobi Last Multiplier method.

The next two examples are subordinated to the third class of Lagrangian’s deformations.

**Example 3.** Let \( M \) be a real, 3-dimensional, differentiable manifold. Consider the SODE
\[
(4.4)
\]
and the associated semi-spray is \( S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - 2x^2 \frac{\partial}{\partial y^2} \). The SODE (4.4) is equivalent to \( \delta_S L = \sigma \), for the regular Lagrangian
\[
L(x^2, y^1, y^2, y^3) := F_1(y^1) F_2(y^2) \exp \left( C_2 \left[ \frac{1}{2} (y^2)^2 - (x^2)^2 \right] + C_1 y^2 \right),
\]
where \( F_1(y^1), F_2(y^2) \) are smooth real functions depending only on \( y^1, y^3 \), respectively, with positive values, \( C_1, C_2 \in \mathbb{R}^+ \) and the semi-basic 1-form on TM defined by
\[
\sigma = -2x^2 (2C_2 y^2 + C_1) \exp \left( C_2 \left[ \frac{1}{2} (y^2)^2 - (x^2)^2 \right] + C_1 y^2 \right) \left( F_1 F_2 dx^1 + (C_1 + C_2 y^2) F_1 F_2 dx^2 + F_1 (2F_2) dx^3 \right).
\]
Using some computations we check that \( \sigma \) satisfies the first condition \[11\]:
\[
d_J L = L \left( \frac{F'_1}{F_1} (y^1) dx^1 + (C_1 + C_2 y^2) dx^2 + \frac{F'_2}{F_2} (y^2) dx^3 \right), \quad S(L) = -2x^2 (2C_2 y^2 + C_1) L,
\]
\[
C(L) = \left[ y^1 \frac{F'_1}{F_1} (y^1) + y^2 (C_2 y^2 + C_1) + y^3 \frac{F'_2}{F_2} (y^2) \right] L, \quad S(E_L) = \frac{C(L) S(L)}{L}, \quad \delta_S L = \frac{S(E_L)}{C(L)} d_J L.
\]
Moreover, \( f(L) = -\frac{1}{L} \). Then the deformed Lagrangian is given by
\[
\Phi(L)(x, y) = a \ln \left( F_1(y^1) F_2(y^3) \right) + a \left( C_2 \left[ \frac{1}{2} (y^2)^2 - (x^2)^2 \right] + C_1 y^2 \right) + b, \ a, b \in \mathbb{R}, \quad a \neq 0.
\]
If both functions \( F_i'' F_i - F_i'^2, \ i \in \{1, 2\} \) are non identically zero then \( \Phi(L) \) is a regular Lagrangian.

**Example 4.** Let \( M \) be a real, 3-dimensional, differentiable manifold, the SODE
\[
(4.5)
\]
and its associated semi-spray \( S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - 2y^2 \frac{\partial}{\partial y^2} \). The SODE (4.5) is equivalent to \( \delta_S L = \sigma \), with the Lagrangian
\[
L(x^2, y^1, y^2) := \sqrt{2C_2 y^1 + C_4} \exp \left( C_1 \left( x^2 - \frac{1}{4} (y^2)^2 \right) - C_3 y^2 \right),
\]
defined on an open set such that \( 2C_2 y^1 + C_4 > 0, \ C_1, C_2, C_3, C_4 \in \mathbb{R} \), and the semi-basic 1-form
\[
\sigma = 2(C_1 y^2 + C_3) \sqrt{2C_2 y^1 + C_4} \exp \left( C_1 \left( x^2 - \frac{1}{4} (y^2)^2 \right) - C_3 y^2 \right) \left[ \frac{C_2}{2C_2 y^1 + C_4} dx^1 - \frac{1}{2} (C_1 y^2 + 2C_3) dx^2 \right].
\]
We check that $\sigma$ satisfies the first condition (4.1) by computing:

$$S(L) = 2(C_1y^2 + C_3)L, \quad d_JL = L \left[ \frac{C_2}{2C_2y^2 + C_4} dx^1 - \frac{1}{2}(C_1y^2 + 2C_3)dx^2 \right],$$

$$\delta_SL = S(L) \left[ \frac{C_2y^1}{2C_2y^1 + C_4} dx^1 - \frac{1}{2}(C_1y^2 + 2C_3)y^2 \right] = \frac{S(L)}{L} d_JL,$$

$$C(L) = \left[ \frac{C_2y^1}{2C_2y^1 + C_4} - \frac{1}{2}(C_1y^2 + 2C_3)y^2 \right] L,$$

$$S(E_L) = S(L) \left[ \frac{C_2y^1}{2C_2y^1 + C_4} - \frac{1}{2}(C_1y^2 + 2C_3)y^2 \right] = \frac{C(L)S(L)}{L}.$$

It follows $f(L) = -\frac{1}{L}$. Hence, the deformed Lagrangian is

$$\Phi(L)(x, y) = a (C_1[x^2 - \frac{1}{4}(y')^2] - C_3y^2) + \frac{a}{2} \ln(2C_2y^1 + C_4) + b, \ a, b \in \mathbb{R}, \ a \neq 0.$$

Note that $\Phi(L)$ is a singular Lagrangian.

The following examples correspond to the forth class of Lagrangian’s deformations.

**Example 5.** Let $M$ be a real, 3-dimensional, differentiable manifold. Consider the SODE

(4.6)

$$\begin{cases}
\frac{d^2x^1}{dt^2} & = 0, \\
\frac{d^2x^2}{dt^2} & = 0, \\
\frac{d^2x^3}{dt^2} + 2G^3(x, y) & = 0,
\end{cases}$$

and its associated semi-spray $S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} - 2G^3(x, y)\frac{\partial}{\partial y^3}$. The SODE (4.6) is equivalent to $\delta_SL = \sigma$, with the Lagrangian

$$L(x, y) := -\frac{d}{c} - \frac{1}{c^2} C_1 \left( x^1y^1 + y^2[2x^2 + (y^1)^2] + x^3y^1 + C_2 \right)$$

and the semi-basic 1-form

$$\sigma = -\frac{2}{c^2} C_1 \left( x^1y^1 + x^2y^2 + (y^1)^2 + C_2 \right) \left( (x^1 + 2y^1(y^2)^2)dx^1 + (x^2 + 2y^2(y^1)^2)dx^2 + x^3dx^3 \right),$$

where $c, d, C_1 \in \mathbb{R}^*$ and $C_2 \in \mathbb{R}$. We choose the domain of $L$ such that $x^1y^1 + y^2(x^2 + (y^1)^2) + x^3y^3 + C_2 \neq 0$. We verify that $\sigma$ satisfies the first condition (4.1) as follows:

$$d_JL = C_1(cL + d)^2 \left[ (x^1 + 2y^1(y^2)^2)dx^1 + (x^2 + 2y^2(y^1)^2)dx^2 + x^3dx^3 \right],$$

$$C(L) = (cL + d) \left[ -\frac{1}{c} + 3C_1(y^2y^1)^2(cL + d) \right], \quad S(L) = C_1(cL + d)^2 \left[ (y^1)^2 + (y^3)^2 + (y^3)^2 - 2x^3G^3 \right],$$

$$S(E_L) = \frac{2c}{(cL + d)} \left( C(s)S(L) \right), \quad \delta_SL = \frac{2c}{(cL + d)} S(L)d_JL.$$

Therefore both conditions (4.1) are satisfied, with $f(L) = -\frac{2c}{cL + d}$. Hence, the deformed Lagrangian is given by

$$\Phi(L)(x, y) = -\frac{c}{cL} C_1 \left[ x^1y' + x^2y^2 + x^3y^3 + (y^1y^2)^2 + C_2 \right] + \frac{a}{c} \left[ 1 + c d C_1 \left( x^1y^1 + x^2y^2 + x^3y^3 + (y^1y^2)^2 + C_2 \right) \right].$$

In this case $L$ is a regular Lagrangian and $\Phi(L)$ is singular.

**Example 6.** Let $M$ be a real, 3-dimensional, differentiable manifold. Consider the SODE

(4.7)

$$\begin{cases}
\frac{d^2x^1}{dt^2} & = 0, \\
\frac{d^2x^2}{dt^2} & = 0, \\
\frac{d^2x^3}{dt^2} & = 0.
\end{cases}$$

The associated flat spray is $S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3}$. The SODE (4.7) is equivalent to $\delta_SL = \sigma$, with the Lagrangian
conformal transformation of the ecological metric \([2, 3]\) which is given by

\[
L(x, y) := -\frac{d}{c} \frac{1}{C_1 c^2 [x^1 y^4 + y^2 x^2 + x^3 y^3 + (y^1 y^2 y^3)^2 + C_2]},
\]

and the semi-basic 1-form on TM

\[
\sigma = \frac{2 [\{y^1\}^2 + \{y^2\}^2 + \{y^3\}^2]}{C_1 c^2 [x^1 y^1 + y^2 x^2 + x^3 y^3 + (y^1 y^2 y^3)^2 + C_2]} \left( [x^1 + 2y^1 (y^2 y^3)^2] \, dx^1 + [x^2 + 2y^2 (y^1 y^3)^2] \, dx^2 + [x^3 + 2y^3 (y^1 y^2)^2] \, dx^3 \right),
\]

where \(c, C_1 \in \mathbb{R}^+\) and \(d, C_2 \in \mathbb{R}\). We choose the domain of \(L\) such that \(x^1 y^1 + y^2 x^2 + x^3 y^3 + (y^1 y^2 y^3)^2 + C_2 \neq 0\).

By direct computations, we get

\[
dJ_L = C_1 (cL + d)^2 \left( [x^1 + 2y^1 (y^2 y^3)^2] \, dx^1 + [x^2 + 2y^2 (y^1 y^3)^2] \, dx^2 + [x^3 + 2y^3 (y^1 y^2)^2] \, dx^3 \right),
\]

\[
C(L) = C_1 (cL + d)^2 \left( 5 (y^1 y^2 y^3)^2 - C_2 - \frac{1}{C_1 (cL + d)} \right),
\]

\[
S(E_L) = \frac{2c}{(cL + d)} C(L) S(L),
\]

\[
\delta S_L = \frac{2c}{(cL + d)} S(L) dJ_L.
\]

Hence \(\sigma\) satisfies the conditions \([11]\) and \(f(L) = -\frac{2c}{cL + d}\). Therefore, the deformed Lagrangian is

\[
\Phi(L)(x, y) = -bcC_1 \left[ x^1 y^1 + x^2 y^2 + x^3 y^3 + (y^1 y^2 y^3)^2 + C_2 \right] + \frac{a}{c} \left[ 1 + c \, d \, C_1 \left( x^1 y^1 + x^2 y^2 + x^3 y^3 + (y^1 y^2 y^3)^2 + C_2 \right) \right].
\]

In this case \(L\) and \(\Phi(L)\) are both regular Lagrangians.

The last example is for the case of homogeneous Lagrangians.

**Example 7.** Let \(M\) be a real, 3-dimensional, differentiable manifold. Consider the SODE

\[
\begin{aligned}
\frac{d^2 x^1}{dt^2} - \left[ (y^1)^2 + (y^2)^2 + (y^3)^2 \right] &= 0, \\
\frac{d^3 x^1}{dt^3} &= 0,
\end{aligned}
\]

and its associated spray \(S = y^1 \frac{\partial}{\partial z^1} + y^2 \frac{\partial}{\partial z^2} + y^3 \frac{\partial}{\partial z^3} + [(y^1)^2 + (y^2)^2 + (y^3)^2] \frac{\partial}{\partial z^1} \).

We consider the Lagrangian

\[
L(x, y) = \frac{1}{2} \exp(2x^1) \left[ (y^1)^2 + (y^2)^2 + (y^3)^2 \right].
\]

It is clear that \([13]\) is equivalent to \(\delta S_L = \sigma\), with \(\sigma = 2y^1 \exp(2x^1) \left( y^1 dx^1 + y^2 dx^2 + y^3 dx^3 \right) = 2y^1 dJ_L\).

Both \(L\) and \(\sigma\) are homogeneous of order 2 and \(\sigma \cap dJ_L = 0\). Applying theorem \([2]\) we obtain the deformed Lagrangian

\[
\Phi(L) = a \sqrt{L} + b = \frac{a}{\sqrt{2}} \exp(x^1) \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + b, \quad a, b \in \mathbb{R}.
\]

One can easily show that \(\Phi(L)\) is a singular Lagrangian. Actually, this example represents a special case of the conformal transformation of the ecological metric \([2, 3]\) which is given by \([L(x, y) = \exp(p \phi(x))((y^1)^p + \ldots + (y^n)^p)]\), where \(n = \text{dim}(M)\), by taking \(n = 3, p = 2, \phi(x) = x^1\).

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