EQUIVARIANT COMPACTIFICATIONS OF REDUCTIVE GROUPS

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Abstract. We study equivariant projective compactifications of reductive groups obtained by closing the image of a group in the space of operators of a projective representation. We describe the structure and the mutual position of their orbits under the action of the doubled group by left/right multiplications, the local structure in a neighborhood of a closed orbit, and obtain some conditions of normality and smoothness of a compactification. Our methods of research use the theory of equivariant embeddings of spherical homogeneous spaces and of reductive algebraic semigroups.

Bibliography: 36 items.

0. Introduction

Let \( G \) be a connected reductive complex algebraic group. We may regard \( G \) as a symmetric homogeneous space \((G \times G)/\text{diag} \ G\) under the group \( G \times G \) acting by left/right multiplications. Equivariant embeddings and, in particular, compactifications (completions) of \( G \) were constructed and studied in a number of papers. A completion of \( G = \text{PGL}_n(\mathbb{C}) \) via “complete collineations” was constructed by Simple [Sem]. Later, Neretin generalized this construction to other classical groups (see survey [Ner]). Neretin’s compactification may be considered as a particular case of the “wonderful” completion of a symmetric space, and in particular, of a semisimple adjoint group [CP-I] (see [S]).

A more general class of regular completions was considered in [CP-II] with applications to the intersection theory on homogeneous spaces. The geometry and cohomology of equivariant compactifications were studied in [CP], [St], [BCP], [LP], [St2], [Br3], [BP], [AB2]. The theory of equivariant embeddings of reductive groups is a particular case of the theory of spherical varieties [LV], [Kn], [Br3] (see [E]). On the other hand, affine equivariant embeddings of reductive groups are nothing else, but reductive algebraic semigroups, which were intensively studied...
in a series of papers by Putcha and Renner \cite{Pu1}, \cite{Re1}, \cite{PR}, \cite{Pu2}, \cite{Re2}, and also by Vinberg \cite{Vin}, Rittatore \cite{Rit} and Alexeev–Brion \cite{AB1}.

The compactifications of Semple–Neretin and the “wonderful” completion of de Concini and Procesi \cite{CP-I} were based on explicit constructions of embeddings in projective spaces. However, the general theory of equivariant embeddings \cite{LV} followed another way: the description of embeddings and the study of their geometric properties goes on in terms of certain objects of discrete convex geometry (colored cones and fans) and of their combinatorics, see \S4.

Here we come back to a constructive viewpoint and consider the following natural class of compactifications of $G$. Let $G : \mathbb{P}(V)$ be a faithful projective representation. Then $G$ embeds in the space $\mathbb{P}(\text{End}(V))$ of projective linear operators, where $\text{End}(V)$ is the algebra of linear operators on $V$. The projective closure $X = \overline{G} \subseteq \mathbb{P}(\text{End}(V))$ is a $(G \times G)$-equivariant projective compactification of $G$. In fact, $X$ is completely determined by the set of highest weights of $G : V$.

Our goal is to extract from these data some geometric information about $X$ including $(G \times G)$-orbit structure (Theorem \S 5), local structure in a neighborhood of a closed orbit (Propositions \S7), conditions of normality (\S10) and smoothness (\S11). The problem of normality is important, in particular, because we can apply a well-developed theory of spherical varieties to normal $X$. For instance, in this case all orbit closures (including $X$ itself) have rational singularities \cite{Po}, \cite{BI}, there is an explicit description of the Picard group \cite{Br1} and, in certain cases, of the cohomology ring \cite{BCP}, \cite{LP}, \cite{St2}, \cite{Br4}, there are vanishing theorems for higher cohomology of nef line bundles \cite{BI}, etc.

Now we explain the structure of the paper. Basic notation is fixed in \S2. In \S3 we prove a technical lemma on decomposing tensor powers of $G$-modules into irreducibles. In \S3 we consider a particular case of Luna’s fundamental lemma, which is required in examining smoothness of $X$. A brief exposition of the theory of spherical varieties, which is applied to studying projective compactifications of $G$, is given in \S4. As our compactification $X$ is apriori a non-normal variety, it is important to study its normalization $\tilde{X} \to X$. Some properties of the normalization of a spherical variety are discussed in Proposition \S1 and in \S5 for the particular case of toric varieties.

Another important tool for the study of $X$ is the local structure of the projective action described in \cite{BLV}. It provides a way to construct transversal slices to closed orbits and to reduce the study of the geometry of an action to actions of Levi subgroups on affine subvarieties. These results are recalled in \S6.

The group $G$ as a spherical (even symmetric) homogeneous space under $G \times G$ is considered in \S7. Here we compute all combinatorial data required for the theory of spherical embeddings. In \S8 we proceed to the
study of $X$, first by describing the closed orbits and the local structure of $X$ in their neighborhoods in terms of the weight polytope of $G : V$. We observe that transversal slices to closed orbits have the structure of algebraic semigroups, whose unit groups are Levi subgroups in $G$. These results allow to compute combinatorial data describing $\tilde{X}$, which in turn are applied to describing the orbit structure of $X$ in §9.

Normality and smoothness of $X$ is discussed in §§10,11. Due to the local nature of these properties, everything is reduced to the case, where $X$ is replaced by a transversal slice to a closed orbit, which is a reductive algebraic semigroup. While an effective normality criterion for the general case requires information on decomposing tensor products of reductive group representations and on branching to Levi subgroups, one can formulate some necessary or sufficient conditions and in certain cases (e.g. for regular highest weights) even criteria of normality. A criterion of smoothness is given in Theorem 9.

In §12 we illustrate the results obtained above by the study of equivariant compactifications of simple algebraic groups in the spaces of projective linear operators of fundamental and adjoint representations. Some of the results obtained here are related to similar results of Putcha–Renner [PR], [Re2] (orbital decomposition), and of Faltings [Fal], Kannan [Kan] and de Concini [Con] (normality) for reductive algebraic semigroups.

The aim of this paper is twofold. Together with obtaining new results, we gather in this paper and generalize some known results on reductive group embeddings, which are scattered in the literature. Therefore we tried to make the exposition maximally self-contained by including proofs of some known assertions (see e.g. §7).

Remark. For simplicity, we work over the field $\mathbb{C}$ of complex numbers. However our approach is purely algebraic, and all results are valid over any algebraically closed base field of characteristic 0.

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1. Notation

$G$ is a connected reductive complex algebraic group.
$B \supseteq T,U$ are a fixed Borel subgroup, a maximal torus, and a maximal unipotent subgroup in $G$.
$\Delta = \Delta_G$ is the root system of $G$ relative to $T$.
$\Delta^+ \supseteq \Pi = \{\alpha_1, \ldots, \alpha_l\}$ are the subsystems of positive and simple roots relative to $B$.
$\alpha^\vee$ is the coroot dual to $\alpha$ (i.e., a 1-parameter subgroup in $T$).
$e_\alpha$ is a root vector in $\mathfrak{g}$ corresponding to $\alpha$. 

\[ u_\alpha(t) = \exp(te_\alpha) \] is a root unipotent 1-parameter subgroup.

\[ W = W_G = N_G(T)/T \] is the Weyl group.

\[ s_\alpha \in W \] is the reflection corresponding to \( \alpha \in \Delta \) (or its representative in \( N_G(T) \)).

\[ Q = \mathbb{Z}\Delta \] is the root lattice.

\[ \mathfrak{X} = \mathfrak{X}(T) \] is the character lattice of \( T \).

\[ C = C_G \subseteq \mathfrak{X} \otimes \mathbb{Q} \] is the positive Weyl chamber. We use the same notation for the positive Weyl chamber in the dual space \( \text{Hom}(\mathfrak{X}, \mathbb{Q}) \) identified with \( \mathfrak{X} \otimes \mathbb{Q} \) via a \( W \)-invariant inner product.

\[ \langle \cdot, \cdot \rangle \] is the pairing between elements of dual spaces.

\[ X^+ = X_G^+ = C \cap \mathfrak{X} \] is the semigroup of dominant weights.

\( \omega_1, \ldots, \omega_l \) are the fundamental weights.

\[ V(\lambda) = V_G(\lambda) \] is an irreducible representation of highest weight \( \lambda \).

\[ v_\mu \in V(\lambda) \] is (any) eigenvector of \( T \)-weight \( \mu \). In particular, \( v_\lambda \) is a highest weight vector.

\[ M_\lambda \] is the isotypic component of highest weight \( \lambda \) in a \( G \)-module \( M \).

\[ \mathcal{P} = \mathcal{P}(V) \] is the weight polytope of a linear representation \( G : V \).

\[ \langle v \rangle \in \mathbb{P}(V) \] is the point in the projective space corresponding to a vector \( v \in V \).

End\((V)\) is the algebra of linear operators on \( V \).

\( \mathbb{C}(X) \) is the field of rational functions on a variety \( X \).

\( \mathbb{C}[X] \) is the algebra of polynomial functions on an affine variety \( X \).

\( \tilde{X} \) is the normalization of \( X \).

Lie algebras of algebraic groups are denoted by the respective lowercase Gothic letters. The signs “\( \times \), \( \lhd \)” denote almost (semi)direct products of algebraic groups (we allow intersections in finite subgroups).

For any parabolic \( P \subseteq G \), \( P \supseteq T \), we denote by \( P^- \) the opposite parabolic (and we also write \( P = P^+ \)), and by \( P_u \) the unipotent radical.

\section{2. A Result from Representation Theory}

Let \( G : V \) be any rational linear representation.

\textbf{Lemma 1} (cf. [AB2, Lemma 4.9]). For any \( \mu \in C \cap \mathcal{P} \) there is \( n \) such that the decomposition of \( V^\otimes n \) contains a simple submodule \( V(n\mu) \).

\textbf{Proof.} Put \( \mathcal{M} = \{ \mu \in C \mid \exists n \in \mathbb{N} : V^\otimes n \hookrightarrow V(n\mu) \} \). Clearly, \( \mathcal{M} \subseteq \mathcal{P} \).

We have to prove \( \mathcal{M} = C \cap \mathcal{P} \).

Observe that \( \mathcal{M} \) is convex. Indeed, for any \( \mu, \nu \in \mathcal{M} \) consider any convex combination \( r \mu + s \nu, \) \( r = p/m, \) \( s = q/m, \) \( p, q \geq 0, \) \( p+q = m \). For some \( n \) we have \( V^\otimes n \hookrightarrow V(n\mu), V(n\nu) \), whence \( V^\otimes nm \hookrightarrow V(n\mu)^\otimes p \otimes V(n\nu)^\otimes q \hookrightarrow V(nm(r \mu + s \nu)) \). Therefore \( r \mu + s \nu \in \mathcal{M} \).

Thus it suffices to prove \( \mathcal{M} = C \cap \mathcal{P} \) for irreducible \( V = V(\lambda), \) \( \lambda \in \mathfrak{X}^+ \). Furthermore, we may assume \( G \) to be semisimple.

First, \( \lambda \in \mathcal{M} \). Secondly, \( 0 \in \mathcal{M} \), since \( V^\otimes n \supset \bigwedge^n V = V(0) \) for \( n = \dim V \).
Other vertices $\mu$ of $C \cap P$ are the intersection points of the faces of $P$ at $\lambda$, whose direction subspaces are spanned by simple root subsystems, with perpendicular faces of $C$. If $L \subset G$ is the respective Levi subgroup, then the face is of the form $P_L = P \cap (\lambda + \langle \Delta_L \rangle)$, and $\mu$ is the center of $P_L$. Moreover, $V(\lambda)$ contains $V_L(\lambda)$, whose weight polytope is $P_L$. Under restriction to the commutator subgroup of $L$, $\mu$ maps to 0. Hence $V(\lambda) \otimes n \leftarrow V_L(n\mu)$. But an $L$-highest weight vector of weight $n\mu$ in $V(\lambda) \otimes n$ is automatically a highest weight vector for $G$ (adding $\alpha \in \Delta^+ \setminus \Delta_L^+$ moves the weight outside the weight polytope $nP$), whence $\mu \in M$. \hfill \Box

3. Luna’s fundamental lemma

The following result is an easy particular case of the fundamental lemma [Lu, II.2] in the étale slice theory [loc. cit.]. For convenience of a reader, we provide a proof.

Lemma 2. Suppose $X \subseteq V$ is a smooth closed $G$-stable subvariety with a dense orbit, and $0 \in X$. Then the projection $\pi : X \to T_0X \subseteq V$ along a $G$-stable complementary subspace is an isomorphism.

Proof. The map $\pi$ is étale at 0. The set of points, where $\pi$ is not étale, is closed and $G$-stable. As closed orbits are separated by polynomial invariants, $\{0\}$ is the unique closed orbit in $X$, hence $\pi$ is étale. By Zariski’s Main Theorem, $\pi$ decomposes into an open immersion $X \hookrightarrow \overline{X}$ and a finite morphism $\overline{X} \to T_0X$, so that $\mathbb{C}[\overline{X}]$ is the integral closure of $\mathbb{C}[T_0X]$ in $\mathbb{C}[X]$. But $\overline{X}$ also has a dense orbit, whence a unique closed orbit, which means $\overline{X} \setminus X = \emptyset$. Therefore $\pi$ is a finite étale covering. But $\pi^{-1}(0) = \{0\}$ ($X$ has a unique $G$-fixed point), hence $\deg \pi = 1$, i.e., $\pi$ is an isomorphism. \hfill \Box

4. Spherical varieties

A normal $G$-variety $X$ is called spherical if $B$ has an open orbit on $X$. The more so, $X$ has an open $G$-orbit, which can be identified with a homogeneous space $G/H$ by choosing a base point. Thus $X$ may be considered as an equivariant embedding of $G/H$. All equivariant embeddings of a spherical homogeneous space $G/H$ are described in terms of combinatorics of certain objects from convex geometry (cones, polytopes) related to $G/H$. This theory is due to Luna and Vust [LV]. For a transparent exposition, see [Kn], [Br3].

The algebra of regular functions on a spherical homogeneous space $G/H$ is multiplicity free, i.e., nonzero isotypic components $\mathbb{C}[G/H]_{(\lambda)} \cong V(\lambda)$ are simple $G$-modules.

Let $\mathcal{X}(G/H)$ be the weight lattice of all rational $B$-eigenfunctions on $G/H$ (which are determined by their weights uniquely up to proportionality). Consider the dual space $E = \text{Hom}(\mathcal{X}(G/H), \mathbb{Q})$. Any (discrete $\mathbb{Q}$-valued) valuation of $\mathbb{C}(G/H)$ determines by restriction to
the multiplicative subgroup of rational $B$-eigenfunctions a homomorphism $\mathfrak{x}(G/H) \to \mathbb{Q}$, i.e., a point in $E$.

**Theorem 1** ([Kn, 2.8, 6.3], [Br3, 3.1, 4.2]). The set of $G$-invariant valuations of $\mathbb{C}(G/H)$ maps to $E$ injectively, and its image is a solid polyhedral cone $V$ containing the image of the negative Weyl chamber under the natural projection to $E$.

$B$-invariant divisors on $G/H$ (i.e., irreducible components of the complement to the open $B$-orbit, there are finitely many of them) determine a finite set $D$ of valuations of $\mathbb{C}(G/H)$ (the set of colors), but the map $\rho : D \to E$ is no more injective in general.

**Definition.** A colored cone is a pair $(C, F)$, where $F \subseteq D$, $\rho(F) \not\ni 0$, and $C$ is a strictly convex polyhedral cone generated by $\rho(F)$ and by finitely many vectors from $V$, so that $\text{int} C$ (the relative interior) intersects $V$.

A colored face of the colored cone is a colored cone $(C', F')$, where $C'$ is a face of $C$ and $F' = \rho^{-1}(\rho(F) \cap C')$.

A colored fan is a finite collection of colored cones $(C_i, F_i)$ closed under passing to a colored face and such that $\text{int} C_i \cap \text{int} C_j \cap V = \emptyset$, $\forall i \neq j$ (i.e., the cones intersect in faces inside $V$).

**Theorem 2** ([Kn 4.3], [Br3 3.4]). There is a bijection between all equivariant normal embeddings $X \hookrightarrow G/H$ and all colored fans. Furthermore, $G$-orbits $Y \subseteq X$ are in bijection with colored cones $(C_Y, F_Y)$ of a given colored fan. The set of colors $F_Y$ corresponds to all $B$-stable divisors on $G/H$, whose closures contain $Y$, and $G$-stable divisors containing $Y$ determine the generators of those edges of $C_Y$ which do not intersect $\rho(F_Y)$.

The geometry of a spherical variety is determined by its fan.

**Theorem 3** ([Kn 4.2, 5.2], [Br3 3.4]). Suppose $X \leftrightarrow G/H$ is a spherical variety, and $Y, Y' \subseteq X$ are $G$-orbits. Then $Y \subseteq Y'$ iff $(C_Y, F_Y)$ is a face of $(C_Y, F_Y)$. In particular, $G/H$ corresponds to $(0, \emptyset)$, and $Y$ is projective iff $C_Y$ is a solid cone. $X$ is complete iff its fan covers all $V$.

Every spherical variety is covered by simple open subvarieties, which contain a unique closed orbit. In particular, affine spherical varieties are simple. The colored fan of a simple variety is determined by the colored cone of the closed orbit and consists of all its colored faces.

Affine spherical varieties have a more explicit description.

**Theorem 4** ([Kn 7.7]). $G/H$ admits an affine embedding iff $\rho(D)$ is contained in an open half-space of $E$. There is a bijection between all equivariant normal affine embeddings $X \leftrightarrow G/H$ and all strictly convex polyhedral cones $C \subset E$ generated by $\rho(D)$ and by finitely many vectors
from \( \mathcal{V} \). Further,
\[
\mathbb{C}[X] = \bigoplus_{\lambda \in \mathfrak{x}(G/H) \cap \mathcal{C}^\vee} \mathbb{C}[G/H]_{(\lambda)}
\]
and \( \mathbb{C}[X]^U = \mathbb{C}[\mathfrak{x}(G/H) \cap \mathcal{C}^\vee] \) is the semigroup algebra of the semigroup of lattice points in the dual cone \( \mathcal{C}^\vee \). The colored fan of \( X \) is the set of all colored cones of \( (\mathcal{C}, \mathcal{D}) \).

**Remark.** The pair \( (\mathcal{C}, \mathcal{D}) \) of Theorem 4 might not be a colored cone in the sense of the above definition: \( \text{int} \mathcal{C} \) may have empty intersection with \( \mathcal{V} \). However we may consider its colored faces in accordance with the definition. There exists a largest face \( \mathcal{C}' \subseteq \mathcal{C} \) whose interior intersects \( \mathcal{V} \). The respective colored face \( (\mathcal{C}', \mathcal{F}') \) corresponds to the closed orbit and determines the colored fan of \( X \).

**Proposition 1.** Let \( X \leftarrow G/H \) be a quasiprojective embedding of a spherical homogeneous space. Then the normalization map \( \tilde{X} \rightarrow X \) is bijective on the set of \( G \)-orbits.

**Proof.** If \( X \) is affine, then \( \mathbb{C}[X] = \bigoplus_{S \subseteq \mathfrak{x}^+} \mathbb{C}[G/H]_{(S)} \), where \( S \subseteq \mathfrak{x}^+ \) is a finitely generated semigroup, and \( \mathbb{Z}S = \mathfrak{x}(G/H) \). The following objects are in bijective correspondence: \( G \)-orbits on \( X \), \( G \)-stable closed irreducible subvarieties of \( X \), \( G \)-stable prime ideals in \( \mathbb{C}[X] \), the (some, but generally not all) respective \( T \)-stable prime ideals in \( \mathbb{C}[X]^U = \mathbb{C}[S] \) (the semigroup algebra of \( S \)), (some) sets of weights of the form \( S \cap (\Sigma \setminus \Sigma') \), where \( \Sigma = \mathbb{Q}_+S \), and \( \Sigma' \) is its face.

After normalization, \( S \) is replaced by \( \mathfrak{x}(G/H) \cap \Sigma \), but the cone \( \Sigma \), its faces etc. do not change.

The projective case is reduced to the affine case by passing to the affine cone over \( X \) and extending \( G \) by homotheties. The general case reduces to the projective case by taking the projective closure. \( \square \)

**Example.** Toric varieties are a particular case of spherical varieties. Here \( G = B = T \) is a torus, and we may assume \( H = \{e\} \), if we want, after replacing \( T \) by \( T/H \). The lattice \( \mathfrak{x}(T/H) \) coincides with the character lattice of \( T/H \). Every \( T \)-invariant valuation of \( \mathbb{C}(T/H) \) is proportional to a valuation given by the order in \( t \rightarrow \infty \) of a function restricted to a 1-parameter subgroup \( \gamma(t) \in T/H \). Its value at an eigenfunction of weight \( \lambda \) equals \( \langle \gamma, \lambda \rangle \), whence \( V = E \). There are no colors, and colored cones become usual cones, and a fan is just a finite collection of strictly convex polyhedral cones intersecting along faces.

A simple toric variety \( X \) given by a cone \( \mathcal{C} \) is affine, and its coordinate algebra \( \mathbb{C}[X] = \mathbb{C}[\mathfrak{x}(T/H) \cap \mathcal{C}^\vee] \) is the semigroup algebra of the semigroup of lattice points in the dual cone.

A projective toric variety can be also defined by a polytope in \( E^* = \mathfrak{x}(T/H) \otimes \mathbb{Q} \) which is dual to the fan, i.e., the fan consists of dual cones to the cones at all vertices of the polytope (and of all their faces).
5. **The polytope of the closure of a torus orbit**

Consider a linear representation $T : V$ and a vector $v \in V$. The *support* $\text{supp} \ v$ is the convex hull of all weights in the weight decomposition of $v$.

**Proposition 2.** The polytope of the projective toric variety $\overline{T\langle v \rangle}$ equals $-\text{supp} \ v$.

**Proof.** Consider the weight decomposition $v = v_{\lambda_0} + \cdots + v_{\lambda_n}$, complete the set $v_{\lambda_0}, \ldots, v_{\lambda_n}$ to a weight basis of $V$, and consider the dual coordinates $x_{\lambda}$ (of weights $-\lambda$) as homogeneous coordinates in $\mathbb{P}(V)$. The variety $X = \overline{T\langle v \rangle}$ is covered by affine charts $X_i = \{x_{\lambda_i} \neq 0\}$. To be definite, consider $X_0$. Then $\mathbb{C}[X_0] = \mathbb{C}[x_{\lambda_1}/x_{\lambda_0}, \ldots, x_{\lambda_n}/x_{\lambda_0}]$ is the semigroup algebra of the semigroup $S_0 = \mathbb{Z}_+(\lambda_0 - \lambda_1) + \cdots + \mathbb{Z}_+(\lambda_0 - \lambda_n)$. This semigroup generates the cone $\Sigma_0$ at the vertex $-\lambda_0$ of $-\text{supp} \ v$. Hence $\mathbb{C}[X_0] = \mathbb{C}[\mathbb{Q} S_0 \cap \Sigma_0]$. Thus the fan of $X$ is dual to $-\text{supp} \ v$. □

**Corollary** (of the proof). $\overline{T\langle v \rangle}$ is normal iff $\mathbb{Q} S_i \cap \Sigma_i = S_i$, $i = 0, \ldots, n$

6. **The local structure of a projective action**

Consider a linear representation $G : V$ and fix a lowest weight vector $v_\lambda \in V$. Let $P^- = P(\lambda) = G\langle v_\lambda \rangle$, and $P \supseteq B$ be the opposite parabolic. Consider the Levi decompositions $P = P_u \times L$, $P^- = P_u^- \times L$. We have a decomposition $V = p_u v_\lambda \oplus \mathbb{C} v_\lambda \oplus M$ in a direct sum of $L$-modules. Consider the dual highest weight vector $v_{-\lambda} \in V^*$ and the open subset $\hat{V} = p_u v_\lambda \oplus \mathbb{C}^* v_\lambda \oplus M$, which is the complement to the hyperplane $\{\langle v_{-\lambda}, \cdot \rangle = 0\}$.

**Theorem 5** ([BLV]). The $P$-action on $\mathbb{C} v_\lambda \oplus M$ yields an isomorphism $\mathbb{P}(\hat{V}) \cong P \times_L M(-\lambda) = P_u \times M(-\lambda)$ where $M(-\lambda) = M$ is equipped with the $L$-action twisted by the character $-\lambda$ and embedded in $\mathbb{P}(V)$ as $\{\langle v_\lambda \rangle\} \times M = \mathbb{P}(v_\lambda + M)$.

It follows that any projective $G$-variety $X \subseteq \mathbb{P}(V)$ containing the closed orbit $Y = G\langle v_\lambda \rangle$ contains the affine open $P$-stable subset $\tilde{X} = X \cap \mathbb{P}(\hat{V}) \cong P \times_L Z = P_u \times Z$ intersecting $Y$, where $Z \subseteq M(-\lambda)$ is a closed affine $L$-subvariety.

If $X$ is a spherical $G$-variety, then $Z$ is an affine spherical $L$-variety. It is easily seen from the structure of $\tilde{X}$ that $\mathfrak{X}(X) = \mathfrak{X}(\tilde{X}) = \mathfrak{X}(Z)$, and $\mathcal{C}_Y = \mathcal{C}_Y \cap Z$. The only delicate point is that the cone of invariant valuations can increase and the set of colors can decrease, because some colors may become $L$-stable divisors. Combined with Theorem 4, this yields an effective description of the local structure of a spherical variety by its fan.
7. A reductive group as a spherical homogeneous space

The group $G$ is a homogeneous space under $G \times G$ acting by left and right multiplications with stabilizer $\text{diag} \; G$ of $e$. Fix a Borel subgroup $B^- \times B \subseteq G \times G$. The Bruhat decomposition implies that $G = (G \times G)/\text{diag} \; G$ is a spherical homogeneous space. The combinatorial data related to this space in the sense of the [118] were computed by Vust [Vu] (in the more general context of symmetric spaces) and by Rittatore [Rii]. We reproduce their results below. The following result is well known [Kr, II.3.1, Satz 3]:

Proposition 3. $\mathbb{C}[G] = \bigoplus_{\lambda \in \mathfrak{X}^+} \mathbb{C}[G]_{(\lambda)}$, where $\mathbb{C}[G]_{(\lambda)} \cong V(\lambda)^* \otimes V(\lambda)$ is the linear span of the matrix elements of the representation $G : V(\lambda)$.  

Corollary. $\mathfrak{X}(G) \cong \mathfrak{X}$

The eigenfunction $f_\lambda(g) = \langle v_-, gv\lambda \rangle$ has highest weight $(-\lambda, \lambda)$, i.e., $\lambda$ under our identifications.

Proposition 4. $\mathbb{C}[G]_{(\lambda)} \cdot \mathbb{C}[G]_{(\mu)} = \mathbb{C}[G]_{(\lambda+\mu)} \oplus \bigoplus_i \mathbb{C}[G]_{(\lambda+\mu-\beta_i)}$, where $\lambda + \mu - \beta_i$ are highest weights of all “lower” irreducible components in the decomposition $V(\lambda) \otimes V(\mu) = V(\lambda + \mu) \oplus \ldots$, so that $\beta_i \in \mathbb{Z}_+ \Pi$.

Proof. $\mathbb{C}[G]_{(\lambda)} \cdot \mathbb{C}[G]_{(\mu)}$ is generated by products of matrix elements of $G : V(\lambda)$ and $G : V(\mu)$, i.e., by matrix elements of $G : V(\lambda) \otimes V(\mu)$. $\square$

Corollary. $\mathcal{V} = -C$ is the negative Weyl chamber.

Proof. It is easy to see that a $G$-invariant valuation of $\mathbb{C}(G)$ is constant at any isotypic component $\mathbb{C}[G]_{(\lambda)}$, and its restriction is a linear function of $\lambda$, i.e., $\nu \in E = \text{Hom}(\mathfrak{X}, \mathbb{Q})$. The value of the valuation at $\forall f \in \mathbb{C}[G]$ equals $\min_{f(\lambda) \neq 0} \langle \nu, \lambda \rangle$, where $f(\lambda)$ is the projection of $f$ on $\mathbb{C}[G]_{(\lambda)}$. Multiplying functions from $\mathbb{C}[G]_{(\lambda)}$ and $\mathbb{C}[G]_{(\mu)}$, we deduce from Proposition 3 that $\langle \nu, \beta_i \rangle \leq 0$ is necessary for $\nu$ to define a valuation. Since $\beta_i$ are positive combinations of simple roots, and multiples of all simple roots are among them for appropriate $\lambda, \mu \in \mathfrak{X}^+$ (see e.g. [2]), these inequalities define $-C$.

Conversely, each $\nu \in -C$ defines a $\mathbb{Q}$-valued function on $\mathbb{C}[G]$ (denoted by the same letter by abuse of notation) satisfying the additive property of a valuation by the above formula. To verify the multiplicative property, take $p, q \in \mathbb{C}[G]$ and choose $\gamma \in \text{int}(-C)$ such that $\min \langle \gamma, \lambda \rangle$ over all $\lambda$ with $p(\lambda) \neq 0$, $\langle \nu, \lambda \rangle = \min$, and $\min \langle \gamma, \mu \rangle$ over all $\mu$ with $q(\mu) \neq 0$, $\langle \nu, \mu \rangle = \min$, are reached at the unique points $\lambda_0$ and $\mu_0$, respectively. Then $pq = f(\lambda_0 + \mu_0) + \sum f(\chi)$, where $\langle \nu, \chi \rangle \geq \langle \nu, \lambda_0 + \mu_0 \rangle$, $\langle \gamma, \chi \rangle \geq \langle \gamma, \lambda_0 + \mu_0 \rangle$, and at least one of these inequalities is strict. Hence $\nu(pq) = \langle \nu, \lambda_0 + \mu_0 \rangle = \nu(p) + \nu(q)$, and we are done. $\square$

It follows from the Bruhat decomposition, that $(B^- \times B)$-stable divisors on $G$ are of the form $D_i = B^- s_{\alpha_i} B$. If $G$ is of simply connected type (i.e., $G$ is a direct product of a torus and a simply connected
semisimple group), then \( \omega_i \in \mathfrak{X} \), and \( D_i \) is defined by the equation 
\[ f_{\omega_i} = 0. \]
More precisely, consider a curve \( g_j(t) = u_{-\alpha_j}(t)\alpha_j^\vee(t^{-1})u_{\alpha_j}(-t) \)
in the big Bruhat cell. Then \( \lim_{t \to \infty} g_j(t) = s_{\alpha_j} \in D_j \) (everything takes place in an \( SL_2 \)-subgroup, where it is computed explicitly), and
\[
f_{\omega_i}(g_j(t)) = \langle v_{-\omega_i}, \alpha_j^\vee(t^{-1})v_{\omega_i} \rangle = t^{-\langle \omega_i, \alpha_j^\vee \rangle} \langle v_{-\omega_i}, v_{\omega_i} \rangle
\]
Hence \( f_{\omega_i} \) has order 1 along \( D_j \) for \( i = j \) and 0 for \( i \neq j \). Therefore
the colors look like \( \rho(D_j) = \alpha_j^\vee \). This conclusion remains valid for arbitrary \( G \), because \( G \) is covered by a group of simply connected type with the colors being the preimages of the colors of \( G \).

These results, due to Vust [Vu], allow us to apply the theory of \( G \)-equivariant embeddings of \( G \). Affine embeddings have another remarkable property.

**Theorem 6** ([Vu]). An affine \((G \times G)\)-equivariant embedding \( X \hookrightarrow G \) is an algebraic semigroup with unit, and \( G \) is the group of invertibles in \( X \).

**Proof.** The actions of the left and right copy of \( G \times G \) on \( X \) define coactions \( \mathbb{C}[X] \to \mathbb{C}[G] \otimes \mathbb{C}[X] \) and \( \mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[G] \), which are
the restrictions to \( \mathbb{C}[X] \subseteq \mathbb{C}[G] \) of the comultiplication \( \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G] \).
Hence the image of \( \mathbb{C}[X] \) lies in \( \mathbb{C}[G] \otimes \mathbb{C}[X] \cap \mathbb{C}[X] \otimes \mathbb{C}[G] = \mathbb{C}[X] \otimes \mathbb{C}[X] \), and we have a comultiplication in \( \mathbb{C}[X] \). Now \( G \) is open in \( X \) and consists of invertibles. For any invertible \( x \in X \), we have
\[
xG \cap G \neq \emptyset \implies x \in G.
\]

8. **Projective compactification of a reductive group**

Let \( V = \bigoplus_{i=0}^m V(\lambda_i) \) (\( \lambda_i \in \mathfrak{X}^+ \)) be a faithful representation of \( G \). Then
\( G \hookrightarrow \mathbb{P}V = \bigoplus \text{End}(V(\lambda_i)) \subseteq \text{End}(V) \cong V \otimes V^* \). Passing to
the projectivization, we have \( G \hookrightarrow \mathbb{P} = \mathbb{P}(V) \iff \mathfrak{X} = \sum \mathbb{Z}(\lambda_i - \lambda_j) + Q. \) This condition can always be achieved by adding to \( V \) a
trivial representation, and we will assume it or, more generally, we will consider faithful projective representations \( G : \mathbb{P}(V) \) (which can be lifted to a linear representation of rather a finite cover of \( G \), than
\( G \) itself). Moreover, we may assume all \( \lambda_i \) to be distinct.

Our aim is to study \( X = \overline{G} \subseteq \mathbb{P} \).

The image of the identity in \( \mathbb{V} \) is \( e = \sum v_{\mu} \otimes v_{-\mu} \), where \( v_{\pm \mu} \) run
over dual weight bases in \( V, V^* \). Under the above identification \( \mathfrak{X}((T \times T)/\text{diag}T) \cong \mathfrak{X}(T) \), we have \( \text{supp} e = -\mathbb{P} \).

All closed \((G \times G)\)-orbits in \( \mathbb{P} \) are of the form \( Y_i = (G \times G)y_i, \)
\( y_i = \langle v_{\lambda_i} \otimes v_{-\lambda_i} \rangle \).

The local structure of \( \mathbb{P} \) in a neighborhood of \( Y_i \) looks as follows ([10]).

Let \( P = P(\lambda_i) \) be the projective stabilizer of \( v_{\lambda_i} \), and \( P = P_0 \times L \),
\( P^- = P^-_u \times L \) be the Levi decompositions. Then \( P \times P^- = (G \times G)y_i \).
Consider the open subset
\[
\mathbb{V} = \mathbb{C}(v_{\lambda_i} \otimes v_{-\lambda_i}) \oplus (p^-_u v_{\lambda_i} \otimes v_{-\lambda_i}) \oplus (v_{\lambda_i} \otimes p^-_u v_{-\lambda_i}) \oplus \mathbb{M}
\]
in \( V \), where \( M \subset V \) is an \((L \times L)\)-stable subspace and \( C(v_{\lambda_0} \otimes v_{-\lambda_0}) \oplus M \ni e \). The affine \((P^- \times P)\)-stable chart \( \hat{P} = \mathbb{P}(\hat{V}) \) intersects \( Y_i \), so that \( \hat{P} \cong (P^- \times P) \times (L \times L) M(-\lambda_i, \lambda_i) = P_u^- \times P_u \times M(-\lambda_i, \lambda_i) \).

**Proposition 5.** \( X \ni Y_i \iff \lambda_i \) is a vertex of \( P \)

**Proof.** If \( \lambda_i \) is a vertex of \( P \), then there exists a 1-parameter subgroup \( \gamma(t) \in T \) such that \( \langle \gamma, \lambda_i \rangle > \langle \gamma, \mu \rangle \), Then

\[
\gamma(t) = \left( \sum \gamma(t) v_\mu \otimes v_{-\mu} \right) \rightarrow y_i \quad \text{as} \ t \to \infty
\]

Conversely, assume \( Y_i \subset X \). Consider the local structure of \( X \) in a neighborhood of \( Y_i \). \( X \) contains the \((P^- \times P)\)-stable affine open subset \( X = P_u^- \times P_u \times Z \), where \( Z = X \cap M(-\lambda_i, \lambda_i) = (L \times L) e = T \) is an affine \((L \times L)\)-equivariant embedding of \( L \) with the unique fixed point \( y_i \). Then by Theorem 7, \( C[Z] = \bigoplus_{\lambda \in S} C[L]_{(\lambda)} \), where \( C \) is a strictly convex solid cone generated by \( \Delta_L^+ \) and by finitely many vectors from \( V_L = -C_L \), so that \( \text{int} C \cap V_L \neq \emptyset \). Hence \( C[Z] = \bigoplus_{\lambda \in S} C[L]_{(\lambda)} \), where \( S \subset X_L^+ \cap C^\vee \) is a finitely generated semigroup such that \( \mathbb{Z} S \cap C^\vee = \mathfrak{X} \cap C^\vee \).

There exists a 1-parameter subgroup \( \gamma \in \text{int} C \cap -C_L \), \( \gamma \perp \Delta_L \). It defines a non-negative grading of \( C[Z] \): \( \deg C[Z]_{(\lambda)} = \langle \gamma, \lambda \rangle \), and functions of zero degree are constant. Consequently the action of \( \gamma(t) \) contracts the whole \( Z \) (and in particular, \( e \)) to \( y_i \) as \( t \to \infty \), whence \( -\lambda_i \) is a vertex of \( \text{supp} e = -P \). \( \Box \)

Let us give a more detailed study of the local structure of \( X \) in a neighborhood of a closed orbit, say \( Y_0 \). We keep the previous notation. Put \( \hat{X} = X \cap \hat{P} = \{ f_{\lambda_0} \neq 0 \} \). (Here \( f_{\lambda_0} = v_{-\lambda_0} \otimes v_{\lambda_0} \) is regarded as a linear function on \( V \).) Then \( \hat{X} \cong (P^- \times P) \times (L \times L) \hat{Z} = P_u^- \times P_u \times Z \), where \( Z = X \cap \{ y_0 \} \times M \) = \( T \) is an equivariant affine embedding.

More precisely, if \( V = C v_{\lambda_0} \oplus V_0 \) is an \( L \)-stable decomposition, then \( Z \subset \text{End}(V_0(-\lambda_0)) \subset M(-\lambda_0, \lambda_0) \), whence

**Proposition 6.** Let \( \mu_1, \ldots, \mu_s \in X_L^+ \) be all highest weights of \( L : V \), except \( \lambda_0 \). The representations \( L : V_L^\dagger(\mu_j - \lambda_0) \) extend to \( Z \) and induce a closed embedding \( Z \hookrightarrow V_0 = \bigoplus \text{End}(V_L(\mu_j - \lambda_0)) \).

**Proposition 7.** \( C[Z] = \bigoplus_{\lambda \in S} C[L]_{(\lambda)} \), where \( S \subset X_L^+ \) is a semigroup generating the cone \( \Sigma_0 \) of \( C \cap P \) at \( \lambda_0 \), and \( \mathbb{Z} S = \mathfrak{X} \).

**Proof.** The homogeneous coordinate algebra of \( P \) is \( C[V] = S^* V^* \). The set of weights of homogeneous polynomials of degree \( n \) is contained in \( -nP \oplus nP \). As \( C[P] = \bigoplus S^n V^*/f_{\lambda_0}^n \), the weights of homogeneous polynomials of degree \( n \) on \( \hat{P} \) lie in \( n(-P + \lambda_0) \oplus n(P - \lambda_0) \). Hence the highest weights of the \((L \times L)\)-module \( C[Z] \subset C[L] \) lie in \( C_L \cup \bigcup n(P - \lambda_0) = \Sigma_0 \) (under the antidiagonal embedding \( \mathfrak{X} \subset \mathfrak{X}(T \times T) \)).
On the other hand, for each $\mu \in C_L \cap \mathcal{P}$, Lemma 4 yields $V_{\mu} \leftrightarrow V_L(n\mu)$ for some $n$. Using Proposition 1 for $L$, it is easy to derive $f_{n\mu}/f_{\lambda_0} \in \mathbb{C}[Z]$. Therefore $S$ generates $\Sigma_0$.

Corollary. $\mathbb{C}[\tilde{Z}] = \bigoplus_{\lambda \in \mathfrak{X} \cap \Sigma_0} \mathbb{C}[L]_{(\lambda)}$

Remark. If $\lambda_0$ is a regular weight (i.e., $\lambda_0 \in \text{int } C$), then $L = T$, $\Sigma_0$ is the cone of $\mathcal{P}$ at $\lambda_0$, and $\mathbb{C}[Z] = \mathbb{C}[S]$ is the semigroup algebra of the semigroup $S$ generated by $\mu - \lambda_0$, where $\mu$ runs over all weights of $V$ (cf. §5). In the general case, the description of $S$ requires decomposing the $G$-module $V$ into simple $L$-modules and also tensor products of simple $L$-modules (see §10). Normality of $Z$ means $S = \mathfrak{X} \cap \Sigma_0$.

Consider the spherical variety $\tilde{X}$. By Proposition 1 the normalization map $\tilde{X} \rightarrow X$ is bijective on the set of orbits. In particular, $\tilde{X} \leftarrow G/H, Y_i$.

Theorem 7. The colored fan of $\tilde{X}$ is generated by colored cones of closed orbits $Y_i \subset \tilde{X}$ over all vertices $\lambda_i \in \mathcal{P}$: $\mathcal{C}_{Y_i} = \Sigma_0^{\gamma_i}$, where $\Sigma_i$ is the cone of $C \cap \mathcal{P}$ at $\lambda_i$, and $\mathcal{F}_{Y_i} = \Pi_{Y_i}$ is the subsystem of simple coroots of the respective Levi subgroup $L_i = Z_G(\lambda_i)$. (Recall that colors are identified with simple coroots.)

Proof. The assertion on $\mathcal{C}_{Y_i}$ stems from Proposition 7 and from arguments in §6. Let us deal with colors.

Note that $D_j \in \mathcal{F}_{Y_i} \iff y_i \in \overline{D_j}$. As $f_{\gamma_i}(y_i) \neq 0$ and $f_{\lambda_i}$ has order $\langle \lambda_i, \alpha_j \rangle$ along $D_j$, it follows that $D_j \notin \mathcal{F}_{Y_i}$ whenever $\lambda_i \not\perp \alpha_j \iff \alpha_j \not\in \Pi_{L_i}$. Conversely, assume $\alpha_j \in \Pi_{L_i}$, and let $\gamma(t) \in T$ be a 1-parameter subgroup such that $\langle \gamma, \lambda_i \rangle > \langle \gamma, \mu \rangle$, $\forall \mu \in \mathcal{P}$, $\mu \neq \lambda_i$. Then as $t \rightarrow \infty$,

$$
\gamma(t)s_{\alpha_j} = \left\langle \sum \gamma(t)v_{s_{\alpha_j}^\mu} \otimes v_{-\mu} \right\rangle = \left\langle \sum t^{\langle \gamma, \alpha_j \rangle}v_{s_{\alpha_j}^\mu} \otimes v_{-\mu} \right\rangle \rightarrow y_i,
$$

i.e., $\overline{D_j} \ni y_i$. □

Remark. In case, where all vertices of $\mathcal{P}$ are regular weights, the fan of $X$ has no colors and is a subdivision of $-C$. Transversal slices $Z_i$ to closed orbits $Y_i$ at $y_i$ are affine toric varieties, and their cones are dual to the cones of $\mathcal{P}$ at $\lambda_i$. These slices are contained in a projective toric variety $\overline{T} \subset X$ (or $\tilde{X}$). Clearly, $\overline{T}$ is $W$-stable ($W$ or, more precisely, $N_G(T)$ acts by conjugation). It follows from the local structure that $\overline{T}$ intersects all $Y_i$ transversally at the points $w(y_i), w \in W$, which are all $T$-fixed points of $\overline{T}$. The fan of $\overline{T}$ is the $W$-span of the fan of $X$, and its polytope equals $\mathcal{P}$ by Proposition 5.

Left cosets $v\overline{T}$, $v \in W$, intersect all $Y_i$ transversally at the points $(vw, w)y_i$, $w \in W$, which are all $(T \times T)$-fixed points of $X$. The closure of the normalizer of the torus $\overline{N_G(T)} = \bigcup_{w \in W} v\overline{T}$ is a disjoint union of cosets. (Otherwise distinct $v\overline{T}$ would share fixed points.)
In case, where $V = V(\lambda_0)$ is irreducible, $X$ is normal (even smooth, cf. §§10,11 and [CP-I]) and does not depend on the choice of (regular) $\lambda_0$. This compactification is called wonderful [CP-I]. The general case of regular weights was considered in [CP-II] and in a number of other papers.

9. Orbits

According to the general theory of spherical varieties and Theorem 7, $(G \times G)$-orbits in $X$ whose closures contain a given closed orbit $Y_i$ are in bijection with colored faces of the colored cone $(\Sigma_i^\vee, \Pi_i^\vee)$ or, equivalently, with faces $C \subseteq \Sigma_i^\vee$ such that $(\text{int} C) \cap (-C) \neq \emptyset$. We reformulate this description in “dual” terms of $P$.

For any face $\Gamma \subseteq P$, denote by $|\Gamma|$ its direction subspace, by $\langle \Gamma \rangle$ its linear span, and put $\|\Gamma\| = |\Gamma| \oplus (\langle \Gamma \rangle^\perp \cap \langle Q \rangle)$. (If $G$ is semisimple and $\Gamma$ is a proper face, then $\|\Gamma\|$ is its supporting hyperplane shifted to 0.) We say that $\gamma \in \text{Hom}(X, Q)$ is a supporting function for $\Gamma$ in $P$ if $\langle \gamma, \Gamma \rangle = \text{const} < \langle \gamma, P \setminus \Gamma \rangle$, i.e., $\gamma$ is the l.h.s. of the equation of a supporting hyperplane for $\Gamma$.

Proposition 8. There is a bijection between all $(G \times G)$-orbits $Y \subset X$ and all faces $\Gamma \subseteq P$ such that $(\text{int} \Gamma) \cap C \neq \emptyset$. Here $C_Y$ is dual to the cone of $C \cap P$ at the face $C \cap \Gamma$, and $F_Y = \{D_j \mid \alpha_j \perp (\langle \Gamma \rangle^\perp \cap \langle Q \rangle)\}$.

“Colored” orbits (i.e., those having $F_Y \neq \emptyset$) correspond to faces lying at the boundary of $C$. The adherence of orbits (i.e., inclusion of orbit closures) corresponds to the inclusion of the respective faces.

Proof. Assume $Y \supseteq Y_i$, i.e., $C_Y$ is a face of $\Sigma_i^\vee$. We have $\gamma \in \text{int} C_Y \iff \gamma$ is a supporting function for the dual face $\Sigma_Y = \Sigma_i \cap C^\vee_Y$ of $\Sigma_i$, i.e., $\langle \gamma, \Sigma_i \rangle \geq 0$ and $\Sigma_Y$ is distinguished by the equation $\langle \gamma, \cdot \rangle = 0$. Thus colored faces $\Sigma_Y^\vee$ correspond to faces of $\Sigma_i$, or of $C \cap P$ at $\lambda_i$, cut out by supporting functions $\gamma \in -C$. But such faces of $C \cap P$ are obtained by intersecting $C$ with faces $\Gamma \subseteq P$ of the same dimension cut out by $\gamma$ in $P$.

The description of colors stems from $\Pi_i^\vee = \Pi^\vee \cap \lambda_i^\perp$ and $F_Y = \Pi_i^\vee \cap |\Gamma|^\perp$. The assertion on adherence follows from Theorem 3 because duality reverts inclusion of faces. 

Let us make the above description of orbits more explicit. We introduce the following notation.

For any subspace $N \subseteq X \otimes \mathbb{Q}$ orthogonal to some dominant weight, denote by $P_N$ the parabolic in $G$ generated by $B$ and by the roots $\alpha \in N$. Let $P^\pm_N = (P^\pm)_u \times L_N$ be the Levi decompositions, $L_N \supseteq T$, $L'_N$ be the commutator subgroup of $L_N$. For any sublattice $\Lambda \subseteq X$, denote by $T^\Lambda \subseteq T$ the diagonalizable group which is the common kernel of all characters $\lambda \in \Lambda$.

For any face $\Gamma \subseteq P$, denote by $V_\Gamma$ the sum of weight subspaces of $V$ with weights in $\Gamma$, and by $V^\perp_\Gamma$ its $T$-stable complement. Let $e_\Gamma =$
Let $\sum_{\mu \in \Gamma} v_\mu \otimes v_{-\mu}$ be the projector on $V_\Gamma$ w.r.t. the decomposition $V = V_\Gamma \oplus V'_\Gamma$.

Observe that $V_\Gamma, V'_\Gamma$ are $L_{|\Gamma|}$- and even $L'_{|\Gamma|}$-stable, $L'_{|\Gamma|} \cap L'_{|\Gamma|}$, and the action $L'_{|\Gamma|}$ of $V_\Gamma$ is trivial. Indeed, adding roots $\alpha \in |\Gamma| \setminus |\Gamma|$ moves $\Gamma$ outside $\mathcal{P}$, i.e., the respective root vectors act on $V_\Gamma$ trivially. This means that $\Delta \cap |\Gamma| = (\Delta \cap |\Gamma|) \cup (\Delta \cap (\Gamma) \perp)$ is a disjoint orthogonal union.

In the subspaces $|\Gamma|, \langle \Gamma \rangle$, consider generating sublattices

$$|\Gamma|_Z = \sum_{\lambda_i, \lambda_j \in \Gamma} \mathbb{Z}(\lambda_i - \lambda_j) + \mathbb{Q} \cap |\Gamma|$$

$$\langle \Gamma \rangle_Z = \sum_{\lambda_i \in \Gamma} \mathbb{Z} \lambda_i + \mathbb{Q} \cap |\Gamma|$$

(To be rigorous, these lattices depend not only on $\mathcal{P}$ and $\Gamma$, but on the initial set of highest weights $\lambda_0, \ldots, \lambda_m$.) Note that $\langle \Gamma \rangle_Z$ is the weight lattice of $T : V_\Gamma$.

**Theorem 8.** The orbit $Y \subset X$ corresponding to the face $\Gamma \subseteq \mathcal{P}$ is represented by $y = \langle e_\Gamma \rangle$. Stabilizers look like

$$(G \times X)_y = \left( (P_{|\Gamma|} u \times (P_{-|\Gamma|}) u \right) \times \left( (L'_{|\Gamma|} \perp T|\Gamma| \times L'_{|\Gamma|} \perp T|\Gamma| \perp) \cdot \text{diag} L_{|\Gamma|} \right)$$

$$\dim Y = \dim G - \dim L_{(\Gamma)\perp} + \dim \Gamma$$

**Proof.** Take $\gamma \in (\text{int} \mathcal{C}_\Gamma) \cap \mathcal{N}$. Then

$$\gamma(t) = \left\langle \sum \gamma(t)v_\mu \otimes v_{-\mu} \right\rangle = \left\langle \sum t^{(\gamma, \mu)}v_\mu \otimes v_{-\mu} \right\rangle \rightarrow y \quad \text{as } t \rightarrow 0$$

whence $y \in \overline{T}$. Moreover, $f_\lambda(\gamma(t)) = t^{(\lambda, \gamma)}c_\lambda, c_\lambda = f_\lambda(e) \neq 0, \forall \lambda \in \mathcal{X}$.

Choose a closed orbit $Y_0 \subseteq Y$ corresponding to a certain vertex $\lambda_0 \in C$ of $\Gamma$. In the notation of $\mathcal{X}$ $y \in X$, whence $y \in Z$. We have to prove that $y \in Y \cap Z$.

It follows from the local structure of $X$ (6) that $Y \cap Z$ is an $(L \times L)$-orbit in $Z$ and $C_{Y \cap Z} = C_Y$ is a face of $\Sigma_0$. The ideal of $\overline{Y \cap Z}$ (the closure in $Z$) as an $(L \times L)$-submodule in $\mathbb{C}[Z]$ is given by its highest weight vectors $f_\lambda, \lambda \in \Sigma_0 \setminus C^+_Y = \Sigma_0 \setminus |\Gamma|$. Thence $f_\lambda(y) = 0 \iff f_\lambda|_{Y \cap Z} = 0$.

Furthermore, for all $(b, b^-) \in B \times B^-$ we have

$$(b, b^-)f_\lambda(y) = f_\lambda\left(\langle b^{-1} c_T b^- \rangle\right) = f_\lambda(b_T^{-1} e_T b_T^-)$$

(Here $b^T = u_T^\pm b_T^\pm$ are the decompositions in $B^\pm = (P_{|\Gamma|}^\pm)_u \times B_{|\Gamma|}^\pm, B_{|\Gamma|}^\pm = B^\pm \cap L_{|\Gamma|}$.) The latter expression may be regarded as the value of a function of weight $(-\lambda, \lambda)$ at $b_T^{-1} b_T^- \in L_{|\Gamma|}$, since the identity maps to $e_T$ under the representation $L_{|\Gamma|} : V_\Gamma$. As the two “big cells” $B_{|\Gamma|} B_{|\Gamma|}^- B_{|\Gamma|} \subseteq \overline{L_{|\Gamma|}}$ intersect in a dense subset, for almost all $(b, b^-)$ there exist $(c_T, c_T^-) \in B_{|\Gamma|}^- \times B_{|\Gamma|}$ such that $f_\lambda(b_T^{-1} e_T b_T^-) = f_\lambda(c_T^- e_T c_T) = d_\lambda f_\lambda(y), d_\lambda \neq 0$. (Recall that $f_\lambda$ is an eigenfunction for $B_{|\Gamma|}^- \times B_{|\Gamma|}$.)
It follows that \( f_\lambda(y) = 0 \) iff \( C[Z](\lambda) \) vanishes at \( y \), whence \( y \in Y \cap Z \) and is not contained in smaller orbits. This implies \( y \in Y \cap Z \).

Now we pass to stabilizers. For \( (g, h) \in G \times G \) we have: \( (g, h)^y = y \) iff \( ge_Gh^{-1} \) is proportional to \( e_G \) iff

1. \( gV_G = V_G \),
2. \( hV'_G = V'_G \),
3. the actions of \( g, h \) on \( V_G \cong V/V'_G \) differ by a scalar multiple.

The condition [1] means that \( g \in P_{||\Gamma||} \), [2] \( \iff h \in P_{||\Gamma||}^- \), and the kernels of the actions \( P_{||\Gamma||}^\pm : V_G \cong V/V'_G \) are \( (P_{||\Gamma||})_y \times (L'_{||\Gamma||})_y \), i.e., we may assume below that \( g, h \in L_{||\Gamma||} \). But now [3] just means that, up to multiplying by one and the same element of \( L_{||\Gamma||} \), \( g, h \in T_{||\Gamma||} \).

The formula for \( \dim Y \) easily follows from the structure of \( (G \times G)_y \) and of \( L'_{||\Gamma||} \).

\( \square \)

**Remark.** The theorem generalizes the results of de Concini–Procesi for wonderful and regular completions (see [CP-I], [CP-II], [Br4]) and those of Putcha–Renner [PR], [Pu2], [Re2], and Vinberg [Vin, Thm.7] for algebraic semigroups. A direct link to algebraic semigroups is provided by considering the cone over \( X \) in \( V \). It is an algebraic semigroup, whose group of invertibles is the extension of \( G \) by homotheties (cf. [Re2]). The idea of computing stabilizers is taken from [Vin, §7].

The results close to Proposition 8 and Theorem 8 were obtained for normal affine and projective embeddings of \( G \) (and even in a more general context) by Alexeev and Brion [AB1], [AB2]. When the preliminary text of this paper was written, the author knew about the paper of Kapranov [Kap] in which all assertions of Proposition 8 and Theorem 8 except for stabilizers and dimensions, where proved, see [Kap, 2.4.2]. However the proof therein seems to be incomplete. The author is indebted to M. Brion for this reference.

**Corollary.** \( \overline{T} \) intersects all \((G \times G)\)-orbits in \( X \). \( T \)-orbits in the intersection of \( \overline{T} \) with a \((G \times G)\)-orbit are permuted by \( W \) transitively.

**Remark.** An assertion similar to the first one holds for any spherical variety [Br3, 2.4]. A simplest “transcendental” proof of the corollary is obtained by closing in \( X \) the Cartan decomposition \( G = KTK \), where \( K \subset G \) is a maximal compact subgroup. One can give an “algebraic” proof in the same way by considering the Iwahori decomposition of \( G(\mathbb{C}((t))) \) instead of the Cartan decomposition, cf. [Br3, 2.4, Exemple 2].

On the other hand, this corollary can in used to obtain a simple proof of Theorem 8 as follows. It is an easy exercise in toric geometry (cf. §5) that \( y = \langle e_\Gamma \rangle \) over all faces \( \Gamma \subset P \) form a complete set of \( T \)-orbit representatives in \( \overline{T} \). Thus these \( y \) represent all \((G \times G)\)-orbits in \( X \). Now it is easy to deduce from the structure of the respective orbit \( Y \) as a homogeneous space given by \((G \times G)_y \) that \( Y_{\text{diag}T} \) is a
union of $T$-orbits permuted by $W$ transitively. Therefore $y = \langle e_\Gamma \rangle$ over those $\Gamma \subseteq \mathcal{P}$ with $(\text{int } \Gamma) \cap C \neq \emptyset$ form a complete set of $(G \times G)$-orbit representatives in $X$.

However we are also interested in the combinatorial data of the embedding theory (colored cones) related to these orbits, so our proof is different.

10. Normality

The questions of normality and smoothness are of local nature. Thus it suffices to examine them at points of some closed orbit $Y_0 \subset X$. A general normality criterion is essentially contained in Propositions 6,7 and the subsequent remarks.

We say that weights $\mu_1, \ldots, \mu_s \in \mathfrak{X}^+$ $G$-generate a semigroup $S$ if $S$ consists of all highest weights $\mu_1 + \cdots + k_s \mu_s - \beta$ ($\beta \in \mathbb{Z}_+ \Pi$) of $G$-modules $(\mu_1) \otimes k_1 \otimes \cdots \otimes (\mu_s) \otimes k_s$, $k_1, \ldots, k_s \in \mathbb{Z}_+$.

Proposition 9. In the notation of §3 let $\mu_1, \ldots, \mu_s \in \mathfrak{X}_L^+$ be all highest weights of $L : V$, except $\lambda_0$. Then $X$ is normal at points of $Y_0$ iff $\mu_1 - \lambda_0, \ldots, \mu_s - \lambda_0$ $L$-generate $\mathfrak{X} \cap \Sigma_0$.

Proof. Note that $X$ is normal at points of $Y_0$ iff $Z$ is normal. By Proposition 4 $Z \subset V_0 = \bigoplus \text{End}(V_L(\mu_j - \lambda_0))$, hence $\mathbb{C}[Z] \subset \mathbb{C}[L]$ is generated by the components $\mathbb{C}[L]_{(\mu_j - \lambda_0)}$. By Proposition 4 $\mathbb{C}[Z] = \bigoplus_{\lambda \in S} \mathbb{C}[L]_{(\lambda)}$, where $S \subset \mathfrak{X}_L^+$ is the semigroup $L$-generated by $\mu_j - \lambda_0$, $j = 1, \ldots, s$. Now the assertion follows from the corollary of Proposition 7.

In order to apply this criterion, one requires the information on decomposing $V$ into simple $L$-modules and on decomposing tensor products of $L$-modules. The first problem is eliminated by the following lemma:

Lemma 3. The weights $\lambda_1 - \lambda_0, \ldots, \lambda_m - \lambda_0, -\alpha_j \in -(\Pi \setminus \Pi_L)$ are highest weights of $L : V(\lambda_0)$ and they $L$-generate all highest weights of $L : V(\lambda_0)$.

Proof. Observe that $v_{\lambda_i}, e_{-\alpha_j} v_{\lambda_0}$ are highest weight vectors of $L : V$, whence the first assertion. Furthermore, for every $i = 0, \ldots, m$ there exists a unique $L$-submodule $V_L(\lambda_i) \hookrightarrow V$ (generated by $v_{\alpha_i}$) and

$$V = \sum_{k,i} g_{k,i} v_{\lambda_i} = \sum_{k,i} p^{-} \cdots p^{-} \cdot V_L(\lambda_i)$$

The algebra $p^{-}$ is generated by $I$ and $e_{-\alpha_j}, \alpha_j \notin \Pi_L$, and it contains simple $L$-submodules $g_L(\lambda_0) \cong V_L(\lambda_0)$ with highest weight vectors $e_{-\alpha_j}$. Therefore

$$V = \sum_{n,i,j_1,\ldots,j_n} g_L(-\alpha_{j_1}) \cdots g_L(-\alpha_{j_n}) \cdot V_L(\lambda_i)$$
But \( g_L(-\alpha_{j_1}) \cdots g_L(-\alpha_{j_n}) \cdot V_L(\lambda_i) \) is a quotient module of \( V_L(-\alpha_{j_1}) \otimes \cdots \otimes V_L(-\alpha_{j_n}) \otimes V_L(\lambda_i) \). This yields the second assertion of the lemma.

Thus in Proposition 9 one may replace \( \mu_1 - \lambda_0, \ldots, \mu_s - \lambda_0 \) by \( \lambda_1 - \lambda_0, \ldots, \lambda_m - \lambda_0, -\alpha_j \in -(\Pi \setminus \Pi_L) \). (Although sometimes it is more convenient to operate with all \( L \)-highest weights if they are known.) However the problem of decomposing tensor products of modules with \( L \)-generating highest weights into simple \( L \)-modules remains. (For normality one has to obtain all generators of \( \mathfrak{X} \cap \Sigma_0 \) among highest weights of all occurring simple \( L \)-submodules.) That is why this criterion is not really effective in the general case. However it implies simple sufficient conditions.

**Corollary.** \( X \) is normal (at points of \( Y_0 \)) if \( \lambda_1 - \lambda_0, \ldots, \lambda_m - \lambda_0 \) generate \( \mathfrak{X} \cap \Sigma_0 \).

The necessary condition for normality obtained by Renner for reductive algebraic semigroups (see [Re1, 6.4]) extends to projective compactifications.

**Proposition 10.** If \( X \) is normal, then \( \overline{T} \) is normal.

**Proof.** Replacing \( G \) by \( L \), \( X \) by \( Z \), \( \overline{T} \) by the closure of \( T \) in \( L \), we reduce the problem to the case of an affine embedding. We have \( Z \subseteq V_0 = \bigoplus \text{End}(V_L(\nu_j)) \), where \( \nu_j \in \mathfrak{X}_L^+ \) \( L \)-generate \( \Sigma_0 \cap \mathfrak{X} \). We can increase \( V_0 \) by adding new \( L \)-highest weights in such a way that \( \nu_j \) will generate \( \Sigma_0 \cap \mathfrak{X} \). Then \( Z = \overline{T} \subseteq V_0 \) will not change (the highest weights of \( \mathbb{C}[Z] \) are the same), but now the weights of \( T : V_L(\nu_j) \) will generate the semigroup \( W_L \Sigma_0 \cap \mathfrak{X} \) of all lattice points of \( W_L \Sigma_0 \), the cone of \( \mathcal{P} \) at \( \lambda_0 \). The respective semigroup algebra is the coordinate algebra of \( \overline{T} \subseteq Z \), whence \( \overline{T} \) is normal.

**Remark.** This condition can be effectively verified (§5), and in the case, where all vertices of \( \mathcal{P} \) are regular weights, the condition coincides with the general normality criterion. However this condition is not sufficient in the general case.

**Example.** Let \( G = \text{Sp}_4 \) and the highest weights of \( V \) be \( \{\lambda_0, \lambda_1\} = \{3\omega_1, 2\omega_2\} \). The weight polytope \( \mathcal{P} \) is given at Fig. (a), the highest weights are indicated by bold dots.
Here \( L = \text{SL}_2 \times \mathbb{C}^* \), \( \Delta_L = \{ \pm \alpha_2 \} \). The weight semigroup of \( \mathbb{C}[\mathcal{T}] \) (the closure is taken in \( \mathbb{Z} \)) is indicated by dots at Fig. (b). Bold dots indicate the subsemigroup of highest weights of \( \mathbb{C}[\mathcal{T}] \) (which is easy to compute using the Clebsch–Gordan formula). Now we can see that \( \mathcal{T} \) is normal and \( \mathbb{Z} \) is not.

11. Smoothness

In the theory of spherical varieties, smoothness is usually a much more subtle property than normality. The general smoothness criterion [Br2, 4.2] is rather intricate. Surprisingly, for projective compactifications of reductive groups, it is easier to verify smoothness, than normality. We retain the notation of §§ 8, 10.

Theorem 9. \( X \) is smooth at points of \( Y_0 \) iff \( L \cong \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_p} \) is a direct product, and only polynomial representations of \( L \) occur in the decomposition of the \( L \)-module \( V_0(-\lambda_0) \) (here \( V = \mathbb{C} v_{\lambda_0} \oplus V_0 \) as above), and all minimal representations of factors occur among them. In the language of roots and weights, this amounts to the following conditions:

1. All simple components of \( L \) are of type \( A \), and there are no more than \( \dim Z(L) \) of them.
2. \( \Sigma_0 \) is simplicial, and moreover, is generated by a basis of \( \mathfrak{X} \).
3. One can enumerate the simple roots in order of their positions at Dynkin diagrams of connected components \( \{\alpha_1^{(k)}, \ldots, \alpha_{n_k-1}^{(k)}\} \) of \( \Pi_L \), \( k = 1, \ldots, q \), and partition the basis of the free semigroup \( \mathfrak{X} \cap \Sigma_0 \) into subsets \( \{\pi_1^{(k)}, \ldots, \pi_{n_k}^{(k)}\} \), \( k = 1, \ldots, p \), \( p \geq q \), in such a way that \( \langle \pi_j^{(k)}, \alpha_j^{(k)} \rangle = 1 \), \( \pi_j^{(k)} \perp \Pi_L \setminus \{\alpha_j^{(k)}\} \), \( n_k \pi_j^{(k)} - j \pi_n^{(k)} \perp \pi_{n_1}^{(1)}, \ldots, \pi_{n_p}^{(p)}, \forall j, k \).
4. Among the weights \( \lambda_1 - \lambda_0, \ldots, \lambda_m - \lambda_0, -\alpha_j \in - (\Pi \setminus \Pi_L) \) there occur all \( \pi_1^{(k)}, 1 \leq k \leq p \).

Proof. Observe that \( X \) is smooth (at points of \( Y_0 \)) iff \( Z \) is smooth, i.e., the problem is reduced to affine embeddings.

As above, let \( \mu_1, \ldots, \mu_s \) be the highest weights of \( L : V \). We have an embedding \( Z \hookrightarrow V_0 = \bigoplus \text{End}(V_L(\mu_j - \lambda_0)), y_0 \mapsto 0 \).
If $Z$ is smooth, then by Luna’s fundamental lemma (38), $Z$ projects onto $T_0Z$ isomorphically under an $(L \times L)$-equivariant projection $\mathcal{V}_0 \to T_0Z$. Renumbering of $\mu_j$ yields

$$Z \cong \text{End}(V_L(\mu_1 - \lambda_0)) \oplus \cdots \oplus \text{End}(V_L(\mu_p - \lambda_0)), \quad p \leq s$$

Let $e \mapsto (e_1, \ldots, e_p)$ under this isomorphism. The projection $Z \to \text{End}(V_L(\mu_k - \lambda_0))$ maps the dense orbit $L \subset Z$ onto the dense orbit $L e_k$, whence $e_k$ is a nonzero scalar operator. After rescaling the above isomorphism, we may assume $e_k$ to be the identity operator. Then the projection maps $L$ homomorphically onto $\text{GL}(V_L(\mu_k - \lambda_0))$. By a dimension argument, $L \cong \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_p}$, $n_k = \dim V_L(\mu_k - \lambda_0)$, and $\mu_k - \lambda_0$ are the highest weights of minimal representations of $\text{GL}_{n_k}$, $k = 1, \ldots, p$.

The semigroup $\mathfrak{X} \cap \Sigma_0$ is $L$-generated by all $\mu_k - \lambda_0$ and consists of highest weights of all polynomial representations of $L$ (i.e., those extendible to $Z$). The description of polynomial representations of $\text{GL}_n$ implies that $\mathfrak{X} \cap \Sigma_0$ is freely generated by the weights $\pi_j^{(k)}$, where $\pi_1^{(k)}$ is the highest weight of a minimal representation of $\text{GL}_{n_k}$ and $V_L(\pi_j^{(k)}) = \bigwedge^j V_L(\pi_1^{(k)})$. Conditions (1)–(4) are easily deduced in view of Lemma 3.

Conversely, if conditions (1)–(4) are satisfied, then $L \cong \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_p}$, and highest weights in $\mathbb{C}[Z]$ correspond to all polynomial representations of $L$. Therefore $Z \cong \text{Mat}_{n_1} \times \cdots \times \text{Mat}_{n_p}$, i.e., it is smooth. \hfill $\square$

Remark. In the case of a regular weight $\lambda_0$, $L = T$, and the smoothness criterion is reduced to conditions (2) and (4). $X$ is smooth (at points of $Y_0$) $\iff \Sigma_0$ is generated by the basis $\pi^{(1)}, \ldots, \pi^{(p)}$ of $\mathfrak{X}$ and $\lambda_0 + \pi^{(k)}$ are in the weight system of $T : V$.

Example. Let $G = \text{SO}_{2l+1}$ and $V = V(\lambda_0)$ be an irreducible representation of the fundamental highest weight $\lambda_0 = \omega_i$. For $i < l$, $X$ will be singular, because it violates condition (1) (or (3) for $l = 2$).

In case of the spinor representation, we have $\lambda_0 = (\varepsilon_1 + \cdots + \varepsilon_l)/2$, where $\pm \varepsilon_1, \ldots, \pm \varepsilon_l$ are the nonzero weights of the tautological representation of $\text{SO}_{2l+1}$. The weights of the spinor representation are $(\pm \varepsilon_1 \pm \cdots \pm \varepsilon_l)/2$, $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$, $\Pi_L = \Pi \setminus \{\varepsilon_l\}$. The highest weights of $L : V$ are $\mu_1 = (\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)/2$, $\mu_2 = (\varepsilon_1 + \cdots - \varepsilon_{l-1} - \varepsilon_l)/2$, $\ldots$, $\mu_l = (-\varepsilon_1 - \cdots - \varepsilon_{l-1} - \varepsilon_l)/2$. The vectors $\pi_k = \mu_k - \lambda_0$ generate $\Sigma_0$ and form a basis of $\mathfrak{X} = \langle \varepsilon_1, \ldots, \varepsilon_l \rangle$. Indeed, $\pi_1 = -\varepsilon_1, \pi_2 = -\varepsilon_1 - \varepsilon_2, \ldots, \pi_l = -\varepsilon_1 - \cdots - \varepsilon_l$. It is also easy to see that condition (3) is verified. Thus $X$ is smooth.

Remark. Using a more subtle version of the local structure in the neighborhood of a non-closed orbit, one can obtain conditions of normality and smoothness at any point of $X$. 

12. Examples

Here we illustrate the general theorems proven above by describing geometric properties of equivariant compactifications of simple algebraic groups in the spaces of projective linear operators of fundamental and adjoint representations. In each case, we describe the structure of the orbit set and examine normality and smoothness of the compactification.

Our results are presented in Tables 1–2. We consider a representation of a simple algebraic group $G$, indicated up to isomorphism in the column “Group”, in a module $V = V(\lambda_0)$, where $\lambda_0$ is a fundamental weight or the highest root. Fundamental representations are denoted in the column “Module” by indicating the highest weight $\lambda$ or the highest root. Fundamental representations are denoted by the symbol Ad. The numeration of simple roots and of fundamental weights, respectively, is taken from [VO] (so that $V(\omega_i)$ always has the minimal dimension).

We consider a $(G \times G)$-variety $X = \text{Ad}G \subseteq \mathbb{P}(\text{End}(V))$. In the column “Orbits” we indicate the dimensions and the Hasse graph of $(G \times G)$-orbits in $X$. For classical groups, we also indicate orbit representatives (given by projectors noted in the column “Module” by indicating the highest weight or the highest root. Fundamental representations are denoted in the column “Module” by indicating the highest weight $\lambda_0$ and those adjoint representations which are not fundamental are denoted by the symbol Ad. The numeration of simple roots and of fundamental weights, respectively, is taken from [VO] (so that $V(\omega_i)$ always has the minimal dimension).

We consider a $(G \times G)$-variety $X = \text{Ad}G \subseteq \mathbb{P}(\text{End}(V))$. In the column “Orbits” we indicate the dimensions and the Hasse graph of $(G \times G)$-orbits in $X$. For classical groups, we also indicate orbit representatives (given by projectors $V \rightarrow V_\Gamma$; if $V = g$, then $V_\Gamma$ is the center of $(p_1|G|)_u$ or $(p_2|G|)_u$). These data can be easily derived from Theorem 8 and the following lemma (cf. [PR, Thm. 2]):

**Lemma 4.** Let $V = V(\lambda_0)$, $\lambda_0 \in \mathfrak{h}^+$, and $L \subseteq P = P(\lambda_0)$ be the standard Levi subgroup. The faces of $\mathcal{P} = \mathcal{P}(V)$ intersecting $C$ are of the form $\Gamma = \mathcal{P} \cap (\lambda_0 + \langle \Pi_\Gamma \rangle)$, where $\langle \Pi_\Gamma \rangle \subseteq \Pi$ is a subsystem of simple roots such that no connected component of $\Pi_\Gamma$ is contained in $\Pi_L$, or $\Pi_\Gamma = \emptyset$. Furthermore, $|\Gamma| = \langle \Pi_\Gamma \rangle$, the systems of simple roots of $L_{|\Gamma|}$ and of $L_{|\Gamma|}^\perp$ are $\Pi_\Gamma$ and $\Pi_{\Gamma^\perp} = \Pi_L \cap \langle \Pi_\Gamma \rangle^\perp$, respectively, and $e_\Gamma$ is the $L_{|\Gamma|}$-equivariant projector onto $V_{L_{|\Gamma|}}(\lambda_0) \subseteq V_G(\lambda_0)$.

In particular, for $\lambda_0 = \omega_i$, $\Pi_\Gamma$ is a connected subsystem of $\Pi$ containing $\alpha_i$, or $\Pi_\Gamma = \emptyset$, and $\Pi_{\Gamma^\perp} = \langle \Pi_\Gamma \rangle^\perp \cap \Pi \setminus \{\alpha_i\}$.

**Proof.** The faces $\Gamma \subseteq \mathcal{P}$, $\text{int} \Gamma \cap C \neq \emptyset$, are cut out by supporting functions $\gamma \in -C$. Consider the root subsystem $\Delta_\gamma = \Delta \cap \gamma^\perp$ with simple roots in $\Pi_\gamma = \Delta_\gamma \cap \Pi$. The space $|\Gamma|$ is generated by those $\alpha \in \Delta_\gamma^+$ such that $e_{-\alpha}v_{\lambda_0} \neq 0$, i.e., $\alpha \in \Delta_\gamma^+ \setminus \Delta_L^+$. In other words, $\alpha$ has a strictly positive linear expression in a connected subsystem $\Pi' \subseteq \Pi_\gamma$, $\Pi' \not\subseteq \Pi_L$. All such $\Pi'$ are linearly expressed in such $\alpha$. Hence $|\Gamma| = \langle \Pi_\Gamma \rangle$, where $\Pi_\Gamma$ is the union of connected components of $\Pi_\gamma$ that are not contained in $\Pi_L$. This yields the description of $\Gamma$. The description of $e_\Gamma$ stems from $V_\Gamma = V_{L_{|\Gamma|}}(\lambda_0)$, and other assertions of the lemma are evident. □
The Hasse diagrams of orbit sets coincide with those for irreducible simple algebraic semigroups (which are nothing else, but the cones over our projective compactifications). The latter are computed in [PR]

We indicate the Levi subgroup of \( P = P(\lambda_0) \) in the column “L” and the nonzero highest weights of \( L : V(\lambda_0) \) in the column “L-weights”. If there are too many of them, then we indicate only \(-\alpha_j \in -(\Pi \ \Pi_\lambda)\) (which \(L\)-generate all other highest weights by Lemma 3). We use the following notation: \( \varepsilon \) is a fixed basic weight of the central 1-torus; \( \varepsilon_1, \ldots, \varepsilon_n \) are the weights of the tautological representation in \( \mathbb{C}^n \) of a classical subgroup of \( GL_n \) \( (\varepsilon_{n-i} = -\varepsilon_i \) for the orthogonal and the symplectic group); \( \pi_i = \varepsilon_1 + \cdots + \varepsilon_i \) is the highest weight of \( \bigwedge^i \mathbb{C}^n \); if \( L \) is represented as a quotient of a direct product of several groups, then the weights of the factors are distinguished by superscripts \( \prime, \prime', \ldots \).

In the columns “Normality” and “Smoothness” we indicate whether \( X \) has the respective property. If normality fails, we give a reason for it (see Propositions 7, 9): if \( S \) is the semigroup of highest weights of \( L : \mathbb{C}[Z] \) and \( \mu \not\in S \), but \( k\mu \in S \), then we write “\( \exists \mu, \exists k\mu \)”.
The following two lemmas are helpful in verifying normality:

**Lemma 5.** Suppose \( G_i : V_i \) \( (i = 1, \ldots, s) \) are faithful representations of connected reductive groups such that \( Z(G_i) \) act by homotheties. Let \( Z_i, Z \) be the closures of the images of \( G_i, G_1 \times \cdots \times G_s \) in \( \text{End}(V_i) \), \( \text{End}(V_1 \otimes \cdots \otimes V_s) \), respectively. Then \( Z \) is normal iff \( Z_1, \ldots, Z_s \) are normal.

**Proof.** The algebra \( \mathbb{C}[Z_1 \times \cdots \times Z_s] \) is multi-graded by the action of the \( s \)-dimensional torus \( Z(G_1) \times \cdots \times Z(G_s) \), and \( \mathbb{C}[Z] \) is the invariant algebra of the subtorus \( T_0 = \{ (t_1, \ldots, t_s) \mid t_1 \cdots t_s = 1 \} \). Thus normality of \( Z \) is implied by normality of \( Z_1, \ldots, Z_s \), of direct products and of quotients of normal varieties [Kr] App. I, 4.4, II.3.3, Satz 1.

On the other hand, the action \( Z(G_i) : Z_i \) by homotheties lifts to \( \tilde{Z}_i \). Let \( f_i \in \mathbb{C}[\tilde{Z}_i] \) be an arbitrary homogeneous function. For all \( j \neq i \) choose homogeneous functions \( f_j \in \mathbb{C}[\tilde{Z}_j] \) of the same degree and consider \( f = f_1 \cdots f_s \in \mathbb{C}[Z_1 \times \cdots \times Z_s]^{T_0} \). It is a rational function on \( Z_1 \times \cdots \times Z_s \) which is constant on \( T_0 \)-orbits. It is easy to see that all \( T_0 \)-orbits in \( (Z_1 \setminus \{0\}) \times \cdots \times (Z_s \setminus \{0\}) \) are closed in \( Z_1 \times \cdots \times Z_s \), i.e., generic \( T_0 \)-orbits are closed. Since closed orbits are separated by invariant polynomials, \( f \) is pulled back from a rational function on \( Z \) which is integral over \( \mathbb{C}[Z] \) [Kr] App. I, 3.7, Satz 2, III.3.3, Satz 1.

Thus if \( Z \) is normal, then \( f \in \mathbb{C}[Z] \), whence \( f_i \in \mathbb{C}[Z_i] \). Therefore \( \tilde{Z}_i \) is normal.

**Lemma 6.** Let \( G \) be one of the groups \( GL_n, \mathbb{C}^* \times \text{Sp}_{2l}, \mathbb{C}^* \times \text{SO}_{2l}, \mathbb{C}^* \times \text{Spin}_n, \mathbb{C}^* \times E_l \) \( (l = 6, 7) \), and \( V = V(\mu_0), \mu_0 = \pi_k, \varepsilon + \omega_1, \varepsilon + \omega_1, \varepsilon + \omega_{(l/2)}, \varepsilon + \omega_1 \), respectively. Then the closure \( Z \subseteq \text{End}(V) \) of the image of \( G \) is normal.
from the Pieri formula for the decomposition of \( \mu \) of all polynomial weights is any polynomial weight, that \( S \) is the saturated semigroup consisting of all polynomial weights \( \mu \) such that \( m_2 + \cdots + m_n \geq (k - 1)m_1 \) and \( m_1 + \cdots + m_n \) is divisible by \( k \).

In the case \( G = \text{GL}_n \), all weights in \( S \) are polynomial, i.e., are of the form \( \mu = m_1 \varepsilon_1 + \cdots + m_n \varepsilon_n \), \( m_1 \geq \cdots \geq m_n \geq 0 \). It easily follows from the Pieri formula for the decomposition of \( V(\mu) \otimes V(\pi_k) \), where \( \mu \) is any polynomial weight, that \( S \) is saturated semigroup over \( C \cap \mathcal{P} \).

In the case \( G = \mathbb{C}^x \times \text{Sp}_{2l} \), the dominant weights of \( V(\mu_0)^{\otimes m} \) are of the form \( \mu = m_\varepsilon + m_1 \varepsilon_1 + \cdots + m_l \varepsilon_l \), \( m_1 \geq \cdots \geq m_l \geq 0 \), \( m - m_1 - \cdots - m_l \in 2\mathbb{Z}_+ \). All of them are in \( S \). Indeed, \( k \varepsilon + \varepsilon_1 + \cdots + \varepsilon_k \) are highest weights of \( \mathbf{A}^k V(\mu_0) \), \( 2\varepsilon \) is a highest weight of \( V(\mu_0)^{\otimes 2} \), and all other \( \mu \) are their \( \mathbb{Z}_+ \)-linear combinations. Thus \( S \) is saturated.

The situation in the case \( G = \mathbb{C}^x \times \text{SO}_{2l} \) is similar, but dominant weights are of the form \( \mu = m_\varepsilon + m_1 \varepsilon_1 + \cdots + m_l \varepsilon_l \), \( m_1 \geq \cdots \geq m_l \geq |m_l| \), \( (l - 2)m \geq m_1 + \cdots + m_l - m_1 \), \( m \equiv m_1 \equiv \cdots \equiv m_l \) (mod 2). These conditions determine a saturated semigroup, which is generated by \( \mu_0 = \varepsilon + \omega_l , 2\varepsilon + \omega_{l-2k} \) (\( 1 \leq k \leq l/2 \)), \( 4\varepsilon + \omega_l + \omega_j \) (\( 0 \leq i \leq j \leq l - 2 \), \( i \equiv j \) (mod 2)), \( k \varepsilon + \omega_{l-i} + (k - 2)\omega_{l-i} \) (\( 2 \leq k \leq i \leq l \), \( k \equiv i \) (mod 2)), \( (k + 2)\varepsilon + \omega_{l-i} + \omega_{l-j} + (k - 2)\omega_{l-j} \) (\( 2 \leq k \leq l \), \( j \equiv i \) (mod 2)). (Here \( \omega_0 = 0 \).) It is easy to verify that all of them are highest weights in the respective \( V(\mu_0)^{\otimes m} \) (where \( m \) is the coefficient in \( \varepsilon \)). Thus our semigroup coincides with \( S \).

A similar reasoning applies to \( G = \mathbb{C}^x \times E_7 \). One considers a semigroup \( \mathfrak{X} \cap \Sigma_0 \), where \( \Sigma_0 = \mathbb{Q}_+(C \cap \mathcal{P}) \). It is generated by the following weights: \( k \varepsilon + \omega_k \), \( (k + 9 - l)\varepsilon + \omega_k \) (\( 0 \leq k \leq l - 3 \), \( \omega_0 = 0 \)), \( (l - 2)\varepsilon + \omega_{l-2} + \omega_l \), \( (l-1)\varepsilon + 2\omega_{l-2} , 4\varepsilon + \omega_{l-2} , 2\varepsilon + \omega_{l-1} \), \( (l-1)\varepsilon + \omega_{l-1} + 2\omega_l \), \( l\varepsilon + 3\omega_l , 3\varepsilon + \omega_l \). One verifies that all of them occur as highest weights in \( V(\mu_0)^{\otimes m} \), whence \( S = \mathfrak{X} \cap \Sigma_0 \). We omit routine computations. \( \square \)

Remark. Observe that the list of \( \mu_0 \) restricted to the maximal torus of \( G' \) in Lemma [Fal] is nothing else but the list of minuscule weights of simple algebraic groups, up to diagram automorphisms. It is obvious that if (the restriction of) \( \mu_0 \) is not minuscule, then the semigroup \( S \) is not saturated, whence \( Z \) is not normal. This was first observed by Faltings [Fal], and he also proved the normality of \( Z \) for some minuscule weights. The uniform proof of normality for any minuscule weight was recently
obtained by de Concini [Con] using a representation-theoretic lemma from [Kan]. When the preliminary text of this paper was written, the author learned about the papers [Fal], [Con], [Kan] from M. Brion, to whom he is grateful for these references.

| Group       | Module | Orbits                                      |
|-------------|--------|---------------------------------------------|
| $\mathbb{A}_l = \text{SL}_n$ \( n = l + 1 \) | $\omega_1, \omega_l$ | $Y_r \ni \langle \phi \rangle$, $\text{rk} \phi = r$, $\dim Y_r = 2nr - r^2 - 1$ |
|             |        | $Y_0 \twoheadrightarrow \cdots \twoheadrightarrow Y_l$ |
| $\omega_k$  | $1 < k < l$ | $Y_{sr} \ni \langle \bigwedge^k \mathbb{C}^n \rightarrow \bigwedge^s \mathbb{C}^s \otimes \bigwedge^{k-s} \mathbb{C}^{r-s} \rangle$ |
|             |        | $\dim Y_{sr} = 2nr - r^2 - s^2 - 1$, $\dim Y_{kk} = 2nk - 2k^2$ |
|             |        | $Y_0 \twoheadrightarrow \cdots \twoheadrightarrow Y_{0,k+1}$ |
|             |        | $Y_{k-1,0} \twoheadrightarrow \cdots \twoheadrightarrow Y_{k-1,k+1} \twoheadrightarrow Y_{kk}$ |
| $\text{Ad}$ |        | $\text{dim} Y_0 = n^2 - 1$, $\text{dim} Y_{sr} = n^2 - (r-s)^2 - 2$ |
|             |        | $Y_0 \twoheadrightarrow Y_{11} \twoheadrightarrow \cdots \twoheadrightarrow Y_{n,n-1}$ |

| Group       | Module | Orbits                                      |
|-------------|--------|---------------------------------------------|
| $\mathbb{B}_l = \text{SO}_n$ \( n = 2l + 1 \) | $\omega_k$  | $Y_{sr} \ni \langle \bigwedge^k \mathbb{C}^n \rightarrow \bigwedge^s \mathbb{C}^s \otimes \bigwedge^{k-s} \mathbb{C}^{r-s} \rangle$ |
|             | $k < l$ | $Y_s \ni \langle \bigwedge^k \mathbb{C}^n \rightarrow \bigwedge^s \mathbb{C}^s \otimes \bigwedge^{k-s} \mathbb{C}^{n-2s} \rangle$ |
|             |        | $\mathbb{C}^s \perp \mathbb{C}^{n-2s} \subseteq \mathbb{C}^{r-s}$ are subspaces in $\mathbb{C}^n$ |
|             |        | $\mathbb{C}^s, \mathbb{C}^{r-s}$ are isotropic and $\mathbb{C}^{n-2s}$ is nondegenerate |
|             |        | $\dim Y_{sr} = (2n - 2r - 1)r - s^2 - 1$ |
|             |        | $\dim Y_{kk} = (2n - 3k - 1)k$, $\dim Y_s = n(n-1)/2 - s^2$ |
|             |        | $Y_0 \twoheadrightarrow Y_{0,l} \twoheadrightarrow \cdots \twoheadrightarrow Y_{0,k+1}$ |
|             |        | $Y_{k-1,0} \twoheadrightarrow \cdots \twoheadrightarrow Y_{k-1,k+1} \twoheadrightarrow Y_{kk}$ |
| $\omega_l$  |        | $Y_s \ni \langle S^n \rightarrow \bigwedge^s \mathbb{C}^s \otimes S^{n-2s} \rangle$ |
|             |        | $S^n$ is the spinor space over $\mathbb{C}^n$ |
|             |        | $\mathbb{C}^s \perp \mathbb{C}^{n-2s}$ are subspaces in $\mathbb{C}^n$ |
|             |        | $\mathbb{C}^s$ is isotropic and $\mathbb{C}^{n-2s}$ is nondegenerate |
|             |        | $\dim Y_s = n(n-1)/2 - s^2$ |
|             |        | $Y_0 \twoheadrightarrow \cdots \twoheadrightarrow Y_l$ |
Table 1: (continued)

| Group | Module | Orbits |
|-------|--------|--------|
| \( C_l = \text{Sp}_n \) \( n = 2l \) | \( \omega_k \) | Orbits and their adherence are similar to the case \( B_l \), only \( \bigwedge^{k-s} \mathbb{C}^{n-2s} \) \( s = 0, \ldots, k \) is replaced with its highest component; \( \dim Y_{sr} = (2n-2r+1)r-s^2-1 \), \( \dim Y_{kk} = (2n-3k+1)k \), \( \dim Y_s = n(n+1)/2-s^2 \) |
| \( \text{Ad} \) | \( \dim Y = n(n+1)/2 \), \( \dim Y_r = (2n-2r+1)r-1 \) |
| \( D_l = \text{SO}_n \) \( n = 2l \) | \( \omega_k \) \( k \leq l - 2 \) | \( Y_s, Y_{sr} \) are the same as for \( B_l \), but \( Y_{sl} \) splits in 2 orbits \( Y_{sl}^+ \) related to the choice of \( \mathbb{C}^l = \mathbb{C}^s \oplus \mathbb{C}^{l-s} \) in one of the two connected components of the isotropic Grassmannian. |
| | \( \omega_{l-1}, \omega_l \) | \( Y_s \ni (S^n \rightarrow \bigwedge^l \mathbb{C}^s \otimes S^{n-l}) \), \( Y'_s \ni (S^n \rightarrow \bigwedge^l \mathbb{C}^l \otimes \mathbb{C}^{-s}) \) \( S^m_{\pm} \subset S^m \) are the semispinor spaces \( \mathbb{C}^l = \mathbb{C}^s \oplus \mathbb{C}^{l-s} \subset \mathbb{C}^n \) is isotropic \( \dim Y_s = n(n-1)/2-s^2 \), \( \dim Y'_s = n(n-1)/2-s^2-1 \) \( (0 \leq s \leq l-3) \) \( \dim Y_{l-2} = l^2 + 3l - 7 \), \( \dim Y_{l-1} = l(l-1) \) |
| | \( \omega_1, \omega_2 \) | 14 \( \rightarrow \) 13 \( \rightarrow \) 10 |
| \( F_4 \) | \( \omega_1, \omega_4 \) | 52 \( \rightarrow \) 51 \( \rightarrow \) 48 \( \rightarrow \) 43 \( \rightarrow \) 30 |
| | \( \omega_2, \omega_3 \) | 51 \( \rightarrow \) 50 \( \rightarrow \) 47 \( \rightarrow \) 40 |
| \( E_6 \) | \( \omega_1, \omega_5 \) | 78 \( \rightarrow \) 77 \( \rightarrow \) 74 \( \rightarrow \) 77 \( \rightarrow \) 76 \( \rightarrow \) 73 \( \rightarrow \) 66 \( \rightarrow \) 53 \( \rightarrow \) 32 |
| | \( \omega_2, \omega_4 \) | 77 \( \rightarrow \) 74 \( \rightarrow \) 73 \( \rightarrow \) 77 \( \rightarrow \) 76 \( \rightarrow \) 73 \( \rightarrow \) 72 \( \rightarrow \) 65 \( \rightarrow \) 50 |
| | \( \omega_3 \) | 77 \( \rightarrow \) 76 \( \rightarrow \) 73 \( \rightarrow \) 77 \( \rightarrow \) 76 \( \rightarrow \) 73 \( \rightarrow \) 72 \( \rightarrow \) 74 \( \rightarrow \) 73 |
Table 1: (continued)

| Group | Module | Orbits |
|-------|--------|--------|
| $\omega_6$ | | ![Diagram](image1.png) |
| $E_7$ | $\omega_1$ | $\rightarrow 133 \rightarrow 132 \rightarrow 129$ |
| | | $\rightarrow 132 \rightarrow 131 \rightarrow 128 \rightarrow 121 \rightarrow 108 \rightarrow 87 \rightarrow 54$ |
| | $\omega_2$ | $\rightarrow 129 \rightarrow 128 \rightarrow 127$ |
| | | $\rightarrow 133 \rightarrow 132 \rightarrow 131 \rightarrow 130$ |
| | $\omega_3$ | $\rightarrow 129 \rightarrow 128 \rightarrow 127$ |
| | | $\rightarrow 133 \rightarrow 132 \rightarrow 131 \rightarrow 130 \rightarrow 125 \rightarrow 124$ |
| | $\omega_4$ | $\rightarrow 129 \rightarrow 128 \rightarrow 127$ |
| | | $\rightarrow 132 \rightarrow 131 \rightarrow 130 \rightarrow 125 \rightarrow 124$ |
| | $\omega_5$ | $\rightarrow 129 \rightarrow 128$ |
| | | $\rightarrow 133 \rightarrow 132 \rightarrow 131 \rightarrow 130 \rightarrow 125 \rightarrow 124$ |
| | $\omega_6$ | $\rightarrow 129 \rightarrow 128$ |
| | | $\rightarrow 133 \rightarrow 132 \rightarrow 131 \rightarrow 130 \rightarrow 125 \rightarrow 124$ |
| | $\omega_7$ | $\rightarrow 129 \rightarrow 128$ |
| Group | Module | Orbits |
|-------|--------|--------|
| $E_8$ | $\omega_1$ | $\begin{align*} 248 \rightarrow 247 \rightarrow 244 \\ 247 \rightarrow 246 \rightarrow 243 \rightarrow 236 \rightarrow 223 \rightarrow 202 \rightarrow 169 \rightarrow 14 \end{align*}$ |
| | $\omega_2$ | $\begin{align*} 248 \rightarrow 247 \rightarrow 246 \rightarrow 223 \rightarrow 202 \rightarrow 201 \rightarrow 166 \\ 247 \rightarrow 246 \rightarrow 243 \rightarrow 236 \rightarrow 223 \rightarrow 222 \\ 246 \rightarrow 243 \rightarrow 242 \rightarrow 239 \rightarrow 232 \rightarrow 219 \rightarrow 194 \\ 246 \rightarrow 243 \rightarrow 242 \rightarrow 239 \rightarrow 232 \rightarrow 231 \rightarrow 208 \end{align*}$ |
| | $\omega_3$ | $\begin{align*} 248 \rightarrow 247 \rightarrow 246 \rightarrow 236 \rightarrow 223 \rightarrow 222 \\ 247 \rightarrow 246 \rightarrow 243 \rightarrow 235 \rightarrow 232 \rightarrow 219 \rightarrow 194 \\ 246 \rightarrow 243 \rightarrow 242 \rightarrow 239 \rightarrow 232 \rightarrow 231 \rightarrow 208 \\ 245 \rightarrow 239 \rightarrow 238 \rightarrow 228 \rightarrow 227 \rightarrow 220 \rightarrow 212 \end{align*}$ |
| $\omega_4$ | $\omega_5$ | $\omega_6$ | $\omega_7$ |
### Table 1: (continued)

| Group | Module | Orbits |
|-------|--------|--------|
| \( A_l = SL_n \) \( n = l + 1 \) | \( \omega_1, \omega_2 \) | \( GL_1 \) | \( \pi_1 \) | Yes | Yes |
| \( \omega_k \) | \( 1 < k < l \) | \( PG(\text{End} \times \text{End}_{n-k}) \) | \( \pi'_i + \pi''_i \) | \( i \leq k, n-k \) | Yes | No |
| \( \text{Ad} \) | \( \mathbb{C} \times GL_{n-2} \) | \( \pi'_i, 2\pi'_i, \pi''_i, \pi''_i + \pi'_i \) | \( 2\pi'_i + \pi''_{n-3} - \pi''_{n-2} \) | Yes | \( n \leq 3 \) |
| \( B_l = SO_n \) \( n = 2l + 1 \) | \( \omega_k \) | \( GL_k \times SO_{n-2k} \) | \( \pi'_i + \pi''_m \) | \( m \leq i-j \) | No | No |
| \( \omega_l \) | | | \( m \equiv i-j \) (mod 2) | | |
| \( C_l = Sp_n \) \( n = 2l \) | \( \omega_k \) | \( GL_k \times Sp_{n-2k} \) | \( \pi'_1 + \pi''_1 \) | Yes | Yes |
| \( \omega_l \) | | | \( 2\pi'_1, \ldots, 2\pi'_l \) | \( n = 2 \) | \( \neq \pi'_1, \exists 2\pi'_1 \) |
| \( \text{Ad} \) | \( P(\mathbb{C} \times Sp_{n-2}) \) | \( \pi'_1, 2\pi'_1, 4\pi'_1 \) | \( 2\pi'_1, 3\pi'_1 + \pi''_1 \) | Yes | \( n \leq 4 \) |
| \( D_l = SO_n \) \( n = 2l \) | \( \omega_k \) | \( P(GL_k \times SO_{n-2k}) \) | Same as for \( B_l \) \( \text{and obtained by } \epsilon'_i \leftrightarrow \epsilon''_{i+1} \) | Yes | No |
| \( \omega_l-1, \omega_l \) | \( GL_l/\mathbb{Z}_2 \) | \( \pi'_2, \pi'_4, \ldots \) | Yes | \( n = 6 \) |
| \( G_2 \) | \( \omega_1 \) | \( GL_2 \) | \( \pi'_1, \pi'_2, \pi'_1 + \pi'_2, 2\pi'_2 \) | Yes | Yes |
| \( \omega_2 \) | \( GL_2 \) | \( 2\pi'_1, 3\pi'_1, 2\pi'_2 \) | \( 3\pi'_1 - \pi'_3 \) | No | No |
| \( F_4 \) | \( \omega_1 \) | \( \mathbb{C} \times \text{Spin}_7 \) | \( \epsilon'_1, 2\epsilon'_1, 4\epsilon'_1 \) | \( 2\epsilon'_1 + \omega''_2, 3\epsilon'_1 + \omega''_3 \) | Yes | No |
| \( \omega_2 \) | \( \text{GL}_2 \times \text{SL}_3 \) | \( \pi'_1 + \pi'_1 \) | Yes | No |
| \( \omega_3 \) | \( \text{SL}_3 \times GL_2 \) | \( 2\pi'_1 + \pi'_1, \ldots \) | No | \( \neq \pi'_2 + \pi''_1 + 2\pi''_1 \) |
| \( \omega_4 \) | \( \mathbb{C} \times Sp_6 \) | \( \epsilon'_1 + \omega'_5, 2\epsilon'_1, 4\epsilon'_1 \) | \( 2\epsilon'_1 + 2\omega''_2, 3\epsilon'_1 + \omega''_3 \) | No | \( \neq \epsilon'_1 + \omega''_1, \exists 2\epsilon'_1 + 2\omega''_1 \) |
| \( E_6 \) | \( \omega_1, \omega_5 \) | \( \mathbb{C} \times \text{Spin}_{10} \) | \( \epsilon'_1 + \omega'_5, 2\epsilon'_1 + \omega''_1 \) | Yes | No |
| \( \omega_2, \omega_4 \) | \( \text{GL}_2 \times \text{SL}_5 \) | \( \pi'_1 + \pi'_1 \) | Yes | No |
| \( \omega_3 \) | \( \text{GL}_2 \times \text{SL}_3 \times \text{SL}_3 \) | \( \pi'_1 + \pi'_1 + \pi'_1 \) | Yes | No |

### Table 2: Normality and smoothness

| Group | Module | \( L \) | \( L \)-weights | Normality | Smoothness |
|-------|--------|--------|----------------|-----------|-----------|
| \( A_l = SL_n \) \( n = l + 1 \) | \( \omega_1, \omega_2 \) | \( GL_1 \) | \( \pi_1 \) | Yes | Yes |
| \( \omega_k \) | \( 1 < k < l \) | \( PG(\text{End} \times \text{End}_{n-k}) \) | \( \pi'_i + \pi''_i \) | \( i \leq k, n-k \) | Yes | No |
| \( \text{Ad} \) | \( \mathbb{C} \times GL_{n-2} \) | \( \pi'_i, 2\pi'_i, \pi''_i, \pi''_i + \pi'_i \) | | | | |
| \( B_l = SO_n \) \( n = 2l + 1 \) | \( \omega_k \) | \( GL_k \times SO_{n-2k} \) | \( \pi'_i + \pi''_m \) | \( m \leq i-j \) | No | No |
| \( \omega_l \) | | | \( m \equiv i-j \) (mod 2) | | |
| \( C_l = Sp_n \) \( n = 2l \) | \( \omega_k \) | \( GL_k \times Sp_{n-2k} \) | \( \pi'_1 + \pi''_1 \) | Yes | Yes |
| \( \omega_l \) | | | \( 2\pi'_1, \ldots, 2\pi'_l \) | \( n = 2 \) | \( \neq \pi'_1, \exists 2\pi'_1 \) |
| \( \text{Ad} \) | \( P(\mathbb{C} \times Sp_{n-2}) \) | \( \pi'_1, 2\pi'_1, 4\pi'_1 \) | \( 2\pi'_1, 3\pi'_1 + \pi''_1 \) | Yes | \( n \leq 4 \) |
| \( D_l = SO_n \) \( n = 2l \) | \( \omega_k \) | \( P(GL_k \times SO_{n-2k}) \) |Same as for \( B_l \) \( \text{and obtained by } \epsilon'_i \leftrightarrow \epsilon''_{i+1} \) | Yes | No |
| \( \omega_l-1, \omega_l \) | \( GL_l/\mathbb{Z}_2 \) | \( \pi'_2, \pi'_4, \ldots \) | Yes | \( n = 6 \) |
| \( G_2 \) | \( \omega_1 \) | \( GL_2 \) | \( \pi'_1, \pi'_2, \pi'_1 + \pi'_2, 2\pi'_2 \) | Yes | Yes |
| \( \omega_2 \) | \( GL_2 \) | \( 2\pi'_1, 3\pi'_1, 2\pi'_2 \) | No | No |
| \( F_4 \) | \( \omega_1 \) | \( \mathbb{C} \times \text{Spin}_7 \) | \( \epsilon'_1, 2\epsilon'_1, 4\epsilon'_1 \) | \( 2\epsilon'_1 + \omega''_2, 3\epsilon'_1 + \omega''_3 \) | Yes | No |
| \( \omega_2 \) | \( \text{GL}_2 \times \text{SL}_3 \) | \( \pi'_1 + \pi'_1 \) | Yes | No |
| \( \omega_3 \) | \( \text{SL}_3 \times GL_2 \) | \( 2\pi'_1 + \pi'_1, \ldots \) | No | \( \neq \pi'_2 + \pi''_1 + 2\pi''_1 \) |
| \( \omega_4 \) | \( \mathbb{C} \times Sp_6 \) | \( \epsilon'_1 + \omega'_5, 2\epsilon'_1, 4\epsilon'_1 \) | \( 2\epsilon'_1 + 2\omega''_2, 3\epsilon'_1 + \omega''_3 \) | No | \( \neq \epsilon'_1 + \omega''_1, \exists 2\epsilon'_1 + 2\omega''_1 \) |
| \( E_6 \) | \( \omega_1, \omega_5 \) | \( \mathbb{C} \times \text{Spin}_{10} \) | \( \epsilon'_1 + \omega'_5, 2\epsilon'_1 + \omega''_1 \) | Yes | No |
| \( \omega_2, \omega_4 \) | \( \text{GL}_2 \times \text{SL}_5 \) | \( \pi'_1 + \pi'_1 \) | Yes | No |
| \( \omega_3 \) | \( \text{GL}_2 \times \text{SL}_3 \times \text{SL}_3 \) | \( \pi'_1 + \pi'_1 + \pi'_1 \) | Yes | No |
Table 2: (continued)

| Group | Module | $L$ | $L$-weights | Normality | Smoothness |
|-------|--------|-----|-------------|-----------|------------|
| $\omega_6$ | GL$_6$/Z$_3$ | $\pi_1, \pi_2$ | $\pi_3, \pi_4, 2\pi_5, \pi_6$ | Yes | No |
| & | | & | $\pi_3 + \pi_4, \pi_1 + \pi_2$ | Yes | No |
| E$_7$ | $\omega_1$ | $\mathbb{C}^\times \times E_6$ | $\epsilon + \omega_1', \omega_2'$ | Yes | No |
| & | | & | $\pi_1 + \omega_2'$ | Yes | No |
| & | $\omega_3$ | GL$_2 \times Spin_{10}$ | $\pi_1 + \omega_2''$ | Yes | No |
| & | | & | $\pi_1 + \pi_2''$ | Yes | No |
| & | $\omega_4$ | GL$_3 \times SL_3$ | $\pi_1 + \pi_2''$ | Yes | No |
| & | | & | $\pi_1 + \pi_2''$ | Yes | No |
| & | $\omega_5$ | GL$_2 \times SL_3 \times SL_4$ | $\pi_1 + \pi_2''$ | Yes | No |
| & | | & | $\pi_1 + \pi_2''$ | Yes | No |
| & | $\omega_6$ | $\mathbb{C}^\times \times Spin_{12}/Z_2$ | $\epsilon + \omega_1', 2\epsilon', \omega_2'$ | Yes | No |
| & | | & | $2\epsilon' + \omega_2', 3\epsilon' + \omega_2''$ | Yes | No |
| E$_8$ | $\omega_1$ | $\mathbb{C}^\times \times E_7$ | $\epsilon + \omega_1', 2\epsilon', 3\epsilon', 4\epsilon'$ | Yes | No |
| & | | & | $2\epsilon' + \omega_2', 3\epsilon' + \omega_2''$ | Yes | No |
| & | $\omega_2$ | GL$_2 \times E_6$ | $\pi_1 + \omega_2'$ | Yes | No |
| & | | & | $\pi_1 + \omega_2'$ | Yes | No |
| & | $\omega_3$ | GL$_3 \times Spin_{10}$ | $\pi_1 + \omega_2'$ | Yes | No |
| & | | & | $\pi_1 + \omega_2'$ | Yes | No |
| & | $\omega_4$ | GL$_4 \times SL_5$ | $\pi_1 + \omega_2'$ | Yes | No |
| & | | & | $\pi_1 + \omega_2'$ | Yes | No |
| & | $\omega_5$ | GL$_2 \times SL_3 \times SL_5$ | $\pi_1 + \omega_2'$ | Yes | No |
| & | | & | $\pi_1 + \omega_2'$ | Yes | No |
| & | $\omega_6$ | GL$_2 \times SL_7$ | $\pi_1 + \omega_2$ | Yes | No |
| & | | & | $\pi_1 + \omega_2$ | Yes | No |
| & | $\omega_7$ | $\mathbb{C}^\times \times Spin_{14}$ | $\epsilon + \omega_2$ | Yes | No |
| & | | | $\pi_3, \pi_4$ | Yes | No |

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