Occupation Time Fluctuations of Weakly Degenerate Branching Systems

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Abstract
We establish limit theorems for re-scaled occupation time fluctuations of a sequence of branching particle systems in \(\mathbb{R}^d\) with anisotropic space motion and weakly degenerate splitting ability. In the case of large dimensions, our limit processes lead to a new class of operator-scaling Gaussian random fields with non-stationary increments. In the intermediate and critical dimensions, the limit processes have spatial structures analogous to (but more complicated than) those arising from the critical branching particle system without degeneration considered by Bojdecki et al. [5, 6]. Due to the weakly degenerate branching ability, temporal structures of the limit processes in all three cases are different from those obtained by Bojdecki et al. [5, 6].

Keywords: Functional limit theorem; Occupation time fluctuation; Branching particle system; Operator stable Lévy process; Operator-scaling random field

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1. Introduction

Consider a kind of branching particle systems in $\mathbb{R}^d$ as follows. Particles start off at time $t = 0$ from a Poisson random field with Lebesgue intensity measure $\lambda$ and evolve independently. They move in $\mathbb{R}^d$ according to a Lévy process

$$\tilde{\xi} = \{(\xi(t), t \geq 0) = \{((\xi_1(t), \xi_2(t), \cdots, \xi_d(t)), t \geq 0\}$$

with independent stable components as in [22], i.e., for every $1 \leq k \leq d$, $\xi_k = \{\xi_k(t), t \geq 0\}$ is a real-valued symmetric $\alpha_k$-stable Lévy process, and $\xi_1, \cdots, \xi_d$ are independent of each other. In addition, the particles split at a rate $\gamma$ and the branching law at age $t$ has the generating function

$$g(s, t) = \left(1 - e^{-\delta t} \right) + e^{-\delta t} \frac{s^2}{2}, \quad 0 \leq s \leq 1, \ t \geq 0.$$ 

Intuitively, in this model, the particles' motion in $\mathbb{R}^d$ is anisotropic (i.e., the motion in different direction is controlled by different mechanism) and their ability of splitting new particles declines as their ages increase (at time $t$ each particle produces 2 particles with probability $e^{-\delta t} / 2$ and no particles with probability $1 - e^{-\delta t} / 2$). For simplicity of notation, we denote the vector $(\alpha_1, \cdots, \alpha_d)$ by $\vec{\alpha}$, and call this model a $(d, \vec{\alpha}, \delta, \gamma)$-degenerate branching particle system.

Let $N(s)$ denote the empirical measure of the particle system at time $s$, i.e., $N(s)(A)$ is the number of particles in the set $A \subset \mathbb{R}^d$ at time $s$. We call the measure-valued process

$$L(t) = \int_0^t N(s) ds, \quad t \geq 0,$$

the occupation time of the system, and call the process

$$X(t) = \int_0^t (N(s) - \mathbb{E}(N(s))) ds, \quad t \geq 0,$$

the occupation time fluctuation, where $\mathbb{E}(N(s))$ is the expectation functional understood as $\langle \mathbb{E}(N(s)), \phi \rangle = \mathbb{E}(\langle N(s), \phi \rangle)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, the space of smooth rapidly decreasing functions. Here and sometimes in the sequel, we write $\langle \mu, f \rangle = \int f d\mu$, where $\mu$ is a measure and $f$ a measurable function.

The main purpose of this paper is to study the scaling limit of occupation time fluctuations of a sequence of $(d, \vec{\alpha}, \delta, \gamma)$-degenerate branching particle systems (the specific assumptions will be given below). This is motivated by the limit theorems for occupation time fluctuations of branching particle systems established recently by Bojdecki et al. [5, 6, 7, 8] and by the research on anisotropic Gaussian random fields (cf. [21, 22]).

Branching particle systems and their associated superprocess have been widely studied; see for example, [12, 13, 16, 18, 21, 22]. Recently Bojdecki et al. [4, 6, 7, 8, 10] established very interesting limit theorems for occupation time fluctuations of the $(d, \alpha, \beta)$-branching systems, and showed that the limit processes depend on the relations among the parameters $d, \alpha$ and $\beta$ as well as the initial distribution of the particles. Under the assumptions that the initial state $N(0)$ is a Poisson random measure with intensity the Lebesgue measure $\lambda$, particles move independently following isotropic $\alpha$-stable Lévy processes in $\mathbb{R}^d$ and undergo critical binary branching (that is, $\beta = 1$), Bojdecki et al. [5, 6] proved that, if the dimension is intermediate (i.e., $\alpha < d < 2\alpha$), then the limit process has the form $C\lambda \xi$, where $C$ is a constant and $\xi = \{\xi_t, t \geq 0\}$ is a sub-fractional Brownian motion (sub-fBm) defined in [3]; if the dimension is critical (i.e., $d = 2\alpha$), then the limit process has the form $C\lambda \zeta$, where $C$ is a constant and $\zeta = \{\zeta_t, t \geq 0\}$ is a standard Brownian motion; and if the dimension is
large (i.e., \(d > 2\alpha\)), then the limit is a generalized Gaussian process which is essentially a

generalized Wiener process whose spatial structure is determined solely by \(\alpha\).

We observe that the limit process, say \(X\), of occupation time fluctuation is a process valued in \(S'(\mathbb{R}^d)\), the dual space of \(S(\mathbb{R}^d)\). Intuitively, when we consider a multi-parameter process \(Y = \{Y(z), z \in \mathbb{R}^d\}\) defined by \(Y_z = \langle X(1), 1_{D(z)} \rangle\), where \(D(z) = \{(x_1, \ldots, x_d), 0 \leq x_k \leq z_k, 1 \leq k \leq d\}\), we obtain a real-valued random field. For random fields, “anisotropy” is a
distinct property from those of one-parameter processes and is important for many applied
areas such as geophysical, economic and ecological sciences. Several classes of anisotropic
Gaussian random fields have been explicitly constructed by using stochastic integrals and
their properties have been studied. We refer to \([1, 2, 11, 20, 27, 28]\) and the references therein
for further information. Many of the anisotropic random fields \(Y = \{Y(t), t \in \mathbb{R}^d\}\) in the
literature have the following scaling property: There exists a linear operator \(E\) (which may
not be unique) on \(\mathbb{R}^d\) with positive real parts of the eigenvalues such that for all constants
\(c > 0\),

\[
\{Y(c^E t), t \in \mathbb{R}^d\} \overset{f.d.}{=} \{c Y(t), t \in \mathbb{R}^d\}.
\]

Here and in the sequel, \(\overset{f.d.}{=}\) means equality in all finite-dimensional distributions and, for
\(c > 0\), \(c^E\) is the linear operator on \(\mathbb{R}^d\) defined by \(c^E = \sum_{n=0}^{\infty} \frac{\ln(n)^n E^n}{n!}\). If \(Y = \{Y(t), t \in \mathbb{R}^d\}\)
satisfies (1.1), we call \(Y\) an operator-scaling random field with (time-scaling) exponent \(E\).

More general forms of scaling properties are also possible for multivariate random fields (cf.
[20]).

It is known that some anisotropic Gaussian random fields such as the fractional Brownian
sheets can be obtained as scaling limit of partial sums of discrete-time random fields ([17]).
Such results provide physical interpretation for anisotropic random fields. Given the signifi-
cance of branching particle systems, it is interesting to investigate whether operator-scaling
random fields arise naturally in such models. Considering branching particle systems with
anisotropic particle motions is a natural step in this direction.

Now we specify our setting. Because of the sub-critical branching laws at positive ages,
a fixed \((d, \vec{\alpha}, \delta, \gamma)\)-branching particle system with \(\delta > 0\) will go to local extinction as time
elapses. To overcome this difficulty, we borrow the idea of nearly critical branching processes
(see \([15, 19, 20]\)) and consider a sequence of \((d, \vec{\alpha}, \delta_n, \gamma)\)-branching particle systems with
\(\delta_n \to 0\) as \(n \to \infty\). We study the limit process of the re-scaled occupation time fluctuations

\[
X_n(t) = \frac{1}{F_n} \int_0^{nt} (N_n(s) - \mathbb{E}(N_n(s)))ds,
\]

where \(\{F_n, n \geq 1\}\) is a sequence of norming constants to be chosen appropriately. This
setting is different from those in Bojdecki et al. ([5, 6] and other aforementioned references,
where a fixed system is studied. We further assume that \(n\delta_n \to \theta\) for some constant \(\theta \geq 0\)
as \(n \to \infty\), which is referred to as weak degeneration. Our focus is on how the anisotropy of
the space motion \(\vec{\xi}\) and the degeneration of splitting ability affect the limit process of \(X_n\).
Let

\[
\vec{\alpha} = \frac{1}{\sum_{k=1}^{d} \frac{1}{\alpha_k}}
\]

Our results will show that the scaling limit of \(X_n(t)\) depends crucially on \(\vec{\alpha}\). We assume
throughout the paper that \(\vec{\alpha} > 1\). In analogy to the terminology of Bojdecki et al. ([5, 6],
we refer to the three cases \(\vec{\alpha} > 2\), \(\vec{\alpha} = 2\) and \(1 < \vec{\alpha} < 2\) as the large dimension, critical
dimension and intermediate dimension, respectively.

The remainder of this paper is organized as follows. In Section 2, we derive an explicit
expression for the re-scaled occupation time fluctuation \(X_n\) and state our main theorems.
These results show that operator-scaling Gaussian random fields can arise only in the large
dimension case and they generally have non-stationary increments. In the cases \( \alpha \in (1, 2) \)
and \( \alpha = 2 \), the spatial structures of the limit processes are similar to (but more complicated
than) those of the \((d, \alpha, 1)\)-branching systems in \([3, 4]\). However, the temporal structures of
limit processes are always different. See Remark \(2.2\) for more details.

The space-time method employed in this paper for proving our main results are analogous
to those developed by Bojdecki et al. \([5, 6]\), with some nontrivial modifications to handle the
new technical complexities caused by the anisotropy of the space motion and the degenerate
branching ability. Section 3 contains discussions on the Laplace functionals of the occupation
time fluctuations and is devoted to the proofs of several technical lemmas. Finally in Section
4 we provide proofs of the main results stated in Section 2.

Unless stated otherwise, \( K \) denotes an unspecified positive finite constant which may not
necessarily be the same in each occurrence.

2. Main results

Consider a sequence of \((d, \alpha, \delta_n, \gamma)\)-degenerate branching particle systems. For every \( n \geq 1 \),
let \( N_n(t) \) be the corresponding empirical measures and let \( H \) be the \( d \times d \) diagonal matrix
\((1/\alpha_k)_{1 \leq k \leq d} \). The corresponding space motion is denoted by \( \vec{\xi}_n = \{\xi_n(t), t \geq 0\} \). We
assume that \( \{\xi_n, n \geq 1\} \) is a sequence of identically distributed \( \mathbb{R}^d \)-valued Lévy processes
with \( \alpha_k\)-stable components \((1 \leq k \leq d) \). The distribution of \( \vec{\xi}_n \) is completely determined by
its characteristic function

\[
E(e^{i(z, \vec{\xi}_n(t))}) = e^{-t \sum_{k=1}^{d} |z_k|^\alpha_k}, \quad z \in \mathbb{R}^d.
\] (2.1)

It follows that \( \vec{\xi}_n \) has the following operator-self-similarity: For all constants \( c > 0 \),
\[
\{\vec{\xi}_n(ct), t \geq 0\} = \{e^{Ht}\vec{\xi}_n(t), t \geq 0\}.
\]

Hence, \( \vec{\xi}_n \) is an operator-stable Lévy process on \( \mathbb{R}^d \) (cf. \([13, Theorem 7]\) or \([25]\)). We refer to
\([25]\) for a systematic account on Lévy processes. In the following, we denote the semigroup
of \( \vec{\xi}_n \) by \( \{L_t\}_{t \geq 0} \), i.e.,
\[
T_s f(x) := \mathbb{E}(f(\vec{\xi}_n(t + s))) \vec{\xi}_n(t) = x
\]
for all \( s, t \geq 0, x \in \mathbb{R}^d \) and bounded measurable functions \( f \) on \( \mathbb{R}^d \). To avoid misunder-
standing, we sometimes write \( T_s f(x) \) as \( T_s(f(\cdot))(x) \).

For convenience, we use the notation
\[
\mathbb{E}_x(f(N_n)) = \mathbb{E}(f(N_n)|N_n(0) = \epsilon_x)
\]
for any measurable function \( f \) on \( D([0, \infty), \mathcal{S}(\mathbb{R}^d)) \), where \( \epsilon_x \) denotes the unit measure
concentrated at \( x \in \mathbb{R}^d \). For all \( \phi \in \mathcal{S}(\mathbb{R}^d) \), define
\[
F_{n, \phi}(x, s) = \mathbb{E}_x((N_n(s), \phi)).
\] (2.2)

Note that all particles split at the rate \( \gamma \) and evolve independently. By the renewal method,
it is not hard to verify that for all \( \phi \in \mathcal{S}(\mathbb{R}^d) \),
\[
F_{n, \phi}(x, s) = e^{-\gamma s} T_s \phi(x) + \int_0^s e^{-\gamma u} e^{-\delta_n u} T_u F_{n, \phi}(\cdot, s - u)(x) du
\]
\[
= e^{-\gamma s} T_s \phi(x) + \int_0^s e^{-(\gamma + \delta_n)(s - u)} T_{s-u} F_{n, \phi}(\cdot, u)(x) du.
\]

Therefore,
\[
\frac{\partial F_{n, \phi}}{\partial s} = (-\gamma + A) F_{n, \phi} + \gamma F_{n, \phi} - \delta_n (F_{n, \phi} - e^{-\gamma s} T_s \phi(x))
\]
\[
= (A - \delta_n) F_{n, \phi} + \delta_n e^{-\gamma s} T_s \phi(x),
\] (2.3)
where $A$ denotes the infinitesimal generator of $T$. Note that $F_{n,\phi}(x,0) = \phi(x)$. From \(2.4\), it follows that

$$F_{n,\phi}(x,s) = e^{-\delta_n s} T_s \phi(x) + \delta_n \int_0^s e^{-\delta_n (s-u)} T_{s-u} (e^{-\gamma u} T_u \phi)(x) du$$

$$= e^{-\delta_n s} T_s \phi(x) \left( 1 + \frac{\delta_n}{\gamma - \delta_n} (1 - e^{-(\gamma - \delta_n)s}) \right).$$

(2.4)

Let

$$f_n(s) := 1 + \frac{\delta_n}{\gamma - \delta_n} (1 - e^{-(\gamma - \delta_n)s}) \quad \text{and} \quad \bar{f}_n(s) e^{-\delta_n s}.$$

(2.5)

Then, for all $n \geq 1, x \in \mathbb{R}^d$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, \(2.4\) can be written as

$$F_{n,\phi}(x,s) = f_n(s) T_s \phi(x).$$

(2.6)

Because the Lebesgue measure is an invariant measure for symmetric stable Lévy processes and all components of $\xi_n$ are symmetric stable Lévy processes and independent of each other, we use Fubini's theorem to derive that

$$\int_{\mathbb{R}^d} T_s \phi(x) dx = \int_{\mathbb{R}^d} \phi(x) dx$$

for all $s > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore, from the fact that $N_n(0)$ has a Poisson distribution with Lebesgue intensity measure $\lambda$, we find that

$$\mathbb{E}(\langle N_n(s), \phi \rangle) = \int_{\mathbb{R}^d} F_{n,\phi}(s,x) dx = f_n(s) \int_{\mathbb{R}^d} \phi(x) dx = f_n(s) \langle \lambda, \phi \rangle.$$

Now we define the occupation time fluctuation process $X_n = \{X_n(t), t \geq 0\}$ as follows

$$\langle X_n(t), \phi \rangle = \frac{1}{F_n} \int_0^t \langle N_n(s) - f_n(s), \phi \rangle ds$$

(2.8)

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $F_n$ is a suitable scaling constant.

Below, we always assume that $N_n(s)$ is the empirical measure of a $(d, \alpha, \delta_n, \gamma)$-degenerate branching particle system with $\alpha = \sum_{k=1}^d \alpha_k^{-1} > 1$ and there is a constant $\theta \in [0, \infty)$ such that $n \delta_n \to \theta$ as $n \to \infty$. Let $\hat{\phi}(z) (z \in \mathbb{R}^d)$ denote the Fourier transform of function $\phi \in L^1(\mathbb{R}^d)$, i.e.,

$$\hat{\phi}(z) = \int_{\mathbb{R}^d} e^{i z \cdot x} \phi(x) dx.$$

We distinguish three cases: (i) $\alpha > 2$ (the large dimension case); (ii) $\alpha = 2$ (the critical dimension case) and (iii) $1 < \alpha < 2$ (the intermediate dimension case). The main results of this paper are stated as follows.

**Theorem 2.1** When $\alpha > 2$, let $F_n^2 = n$. Then for every $t > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$, $\int_0^t \langle X_n, \psi(\cdot, s) \rangle ds$ converges in distribution to $\int_0^t \langle X, \psi(\cdot, s) \rangle ds$, where $X$ is a centered Gaussian process with covariance function

$$\text{Cov}(\langle X(r), \phi_1 \rangle, \langle X(t), \phi_2 \rangle)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{2}{\sum_{k=1}^d |z_k|^{\alpha_k}} \frac{\gamma}{\left( \sum_{k=1}^d |z_k|^{\alpha_k} \right)^2} \hat{\phi}_1(z) \overline{\hat{\phi}_2(z)} dz \int_0^{r \wedge t} e^{-\theta s} ds.$$
and  
\[
\text{Cov}(\langle X_2(r), \phi_1 \rangle, \langle X_2(t), \phi_2 \rangle) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\gamma}{(\sum_{k=1}^{d} |z_k|^{\alpha_k})^2} \hat{\phi}_1(z) \hat{\phi}_2(z) dz \int_0^{r \wedge t} e^{-\theta u} du.
\]

Due to the nuclear property of $\mathcal{S}(\mathbb{R}^d)$, we can define $Y_1(u) = \langle \bar{X}_1(1), 1_{D(u)} \rangle$ and $Y_2(u) = \langle \bar{X}_2(1), 1_{D(u)} \rangle$ for all $u = (u_k)_{1 \leq k \leq d} \in [0, \infty)^d$, where $D(u) = \{(x_1, \cdots, x_d), 0 \leq x_k \leq u_k, 1 \leq k \leq d\}$. The following proposition characterize the operator-scaling properties of the Gaussian random fields $Y_1$ and $Y_2$.

**Proposition 2.1** Suppose $\bar{\alpha} > 2$. (1) When $d = 1$, denote $\bar{\alpha}$ by $\alpha$. Then $Y_1 = \{Y_1(u), u \geq 0\}$ and $Y_2 = \{Y_2(u), u \geq 0\}$, up to some multiplicative constants, are the fractional Brownian motion with Hurst indices $H_1 = \frac{1+\alpha}{2}$ and $H_2 = \frac{1+2\alpha}{2}$, respectively.

(2) When $d \geq 2$, $Y_1 = \{Y_1(u), u \in [0, \infty)^d\}$ and $Y_2 = \{Y_2(u), u \in [0, \infty)^d\}$ are operator-scaling Gaussian random fields with exponent $H$ (which is the $d \times d$ diagonal matrix $\left(1/\alpha_k\right)_{1 \leq k \leq d}$) and have non-stationary increments.

From the proof of Proposition 2.1 in Section 4, one can see that, if $\alpha_1, \ldots, \alpha_d$ are not the same, then the Gaussian random fields $Y_1$ and $Y_2$ are anisotropic. However, they are different from the fractional Brownian sheets or the operator-scaling Gaussian fields constructed by Bierné et al. [2], and can be further studied by applying the general methods in Xiao [27].

**Theorem 2.2** When $\bar{\alpha} = 2$, let $F_n^2 = n \ln n$. Then for any $t > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$,  
\[
\int_0^t \langle X_n, \psi(\cdot, s) \rangle ds \text{ converges in distribution to } \int_0^t \langle X, \psi(\cdot, s) \rangle ds,
\]

where
\[
X = \sqrt{\frac{\gamma}{(2\pi)^d}} \lambda \xi_\theta.
\]

In the above $\xi_\theta = \{\xi_\theta(t), t \geq 0\}$ is a centered Gaussian process with covariance function
\[
C_\theta(r, t) = \operatorname{E}(\xi_\theta(r) \xi_\theta(t)) = \int_0^{r \wedge t} e^{-\theta u} du \int_{\mathbb{R}^d} \varphi(\theta, r-s, t-s, \sum_{k=1}^d \lambda_k y_k \alpha_k) dy,
\]

where for any $x, u, v \geq 0$ and $z > 0$,
\[
\varphi(x, u, v, z) = \frac{u z e^{-u(x+z)}(1 - e^{-u(x+z)})}{(x+z)^2} + \frac{v z e^{-v(x+z)}(1 - e^{-v(x+z)})}{(x+z)^2}
\]
\[
+ \frac{2 \pi (1 - e^{-u(x+z)}) (1 - e^{-v(x+z)})}{(x+z)^3}.
\]

**Theorem 2.3** When $1 < \bar{\alpha} < 2$, let $F_n = n^{(3-\bar{\alpha})/2}$. Then $X_n \Rightarrow X$ in $C([0, 1], \mathcal{S}(\mathbb{R}^d))$ as $n \to \infty$, where $X = \{X(t), t \in [0, 1]\}$ is a centered Gaussian process with covariance function
\[
\text{Cov}(\langle X(r), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) = \frac{\gamma}{\pi} \prod_{k=1}^d \frac{\Gamma(1/\alpha_k)}{\alpha_k} C(r, t) \langle \lambda, \phi_1 \rangle \langle \lambda, \phi_2 \rangle
\]

with
\[
C(r, t) = \int_0^r e^{-\theta u} du \int_0^t e^{-\theta v} dv \int_0^{u \wedge v} e^{\theta s} (u + v - 2s)^{\alpha} ds.
\]

**Remark 2.1** According to Definition 2.1 in Bojdecki et al. [8], the convergence in Theorem 2.2 and Theorem 2.3 is called the convergence in the integral sense. We point out that it is possible to prove the tightness of the processes $X_n$ in $C([0, 1], \mathcal{S}(\mathbb{R}^d))$ by using the method
By (2.13) and (2.14) one can verify that for all \( r,t,s \) substitutions \( \phi \) namely, for any identifying the limit processes in Theorems 2.1 and 2.2, in \([0,1],S'(\mathbb{R}^d)\). Since this approach is lengthy and tedious, in order to save the space of this paper, we do not prove this stronger sense of convergence and focus on identifying the limit processes in Theorems 2.1 and 2.2.

Comparing our results with those of Bojdecki et al. [5, 6] on the critical binary branching particle systems in \( \mathbb{R}^d \) with symmetric stable Lévy motion, we have the following comments.

**Remark 2.2**  
(1) Let \( \hat{X} \) denote a \( S'(\mathbb{R}^d) \)-valued Gaussian process with covariance function

\[
\text{Cov}(\{\hat{X}(r), \phi_1 \}, \{\hat{X}(t), \phi_2 \}) = (r \wedge t) \int_{\mathbb{R}^d} \left( \frac{2}{\sum_{k=1}^{d} |z_k|^\alpha_k} + \frac{\gamma}{(\sum_{k=1}^{d} |z_k|^\alpha_k)^2} \right) \phi_1(z) \phi_2(z) \, dz.
\]

Then \( \hat{X} \) is an analogue of the limit process in [6, Theorem 2.2 (a)]. The limit process in Theorem 2.1 can be written as

\[
X(t) = \int_0^t e^{-\frac{r}{2}} d\hat{X}(s),
\]

namely, for any \( \phi \in S(\mathbb{R}^d) \), \( \langle X(t), \phi \rangle = \int_0^t e^{-\frac{r}{2}} d\langle \hat{X}(s), \phi \rangle \).

(2) For \( x, u, v \geq 0 \) and \( z > 0 \), let

\[
\varphi_1(x, u, z) = \frac{uze^{-u(x+z)}}{(x+z)^2} \quad \text{and} \quad \varphi_2(x, u, v, z) = \frac{2x(1-e^{-u(x+z)})(1-e^{-v(x+z)})}{(x+z)^3}.
\]

Then

\[
\varphi(x, u, v, z) = \varphi_1(x, u, z) + \varphi_1(x, v, z) - \varphi_1(x, u + v, z) + \varphi_2(x, u, v, z). \tag{2.13}
\]

For \( x \geq 0 \), define

\[
\Phi(x) := \int_{\mathbb{R}^d} \varphi_1(x, 1, \sum_{k=1}^{d} |y_k|^\alpha_k) \, dy.
\]

Then

\[
\Phi(x) \leq \int_{\mathbb{R}^d} e^{-\sum_{k=1}^{d} |y_k|^\alpha_k} \sum_{k=1}^{d} |y_k|^\alpha_k \, dy < \infty,
\]

where the finiteness follows from Remark 2.3 below. Then for \( x \geq 0 \) and \( s < r \), using the substitutions \( y' = (r-s)^H y \) and the condition \( \alpha = 2 \), we obtain that

\[
\int_{\mathbb{R}^d} \varphi_1(x, r-s, \sum_{k=1}^{d} |y_k|^\alpha_k) \, dy = \int_{\mathbb{R}^d} \varphi_1((r-s)x, 1, \sum_{k=1}^{d} |y_k'|^\alpha_k) \, dy' = \Phi((r-s)x), \tag{2.14}
\]

Let

\[
\Phi_1(\theta, r, s, t) = \Phi((t-s)\theta) + \Phi((r-s)\theta) - \Phi((t+r-2s)\theta).
\]

By (2.13) and (2.14) one can verify that for all \( r, t, s \in [0,1] \) with \( s < r \wedge t \),

\[
\int_{\mathbb{R}^d} \varphi(\theta, r-s, t-s, \sum_{k=1}^{d} |y_k|^\alpha_k) \, dy = \Phi_1(\theta, r, s, t) + \Phi_2(\theta, r, s, t),
\]

where

\[
\Phi_2(\theta, r, s, t) = \int_{\mathbb{R}^d} \varphi_2(\theta, r-s, t-s, \sum_{k=1}^{d} |y_k|^\alpha_k) \, dy < \infty.
\]
So the Gaussian process $\xi_\theta$ in Theorem 2.2 has the covariance function
\[ C_\theta(r, t) = \int_0^{r+t} e^{-\theta s} \Phi_1(\theta, r, s, t) ds + \int_0^{r+t} e^{-\theta s} \Phi_2(\theta, r, s, t) ds. \]

When $\theta = 0$, $C_\theta(r, t) = \Phi(0)(r \wedge t)$. In this case, up to a multiplicative constant, the limit process $X$ is Brownian motion, which is the same as that in [5, Theorem 2.2 (b)] However, when $\theta \neq 0$, the limit process has complicated covariance function and can be expressed as the sum of two independent $S'(\mathbb{R}^d)$-valued Gaussian processes—the first one is an extension of that in [6, Theorem 2.2 (b)] and the second is a new process.

(3) In Theorem 2.3, if $\theta = 0$, then, up to a multiplicative constant, the limit process is the same as that in Theorem 2.2 of [5], i.e. the limit process can be written as $K\lambda\xi^h$, where $\xi^h$ is a sub-fractional Brownian motion with covariance function
\[ s^{3-\alpha} + t^{3-\alpha} - \frac{1}{2}(s + t)^{3-\alpha} + |t-s|^{3-\alpha}. \]

This sub-fBm is self-similar with index $(3-\alpha)/2$. When $\theta \neq 0$, the Gaussian process with covariance (2.12) in the limit process in Theorem 2.3 is not self-similar and is a new process.

We end this section with some preliminary facts which will be useful for our proofs. The proof of Lemma 2.1 is elementary and thus is omitted.

**Lemma 2.1** (1) If $\bar{\alpha} > 2$, then
\[ \int_{[0,1]^d} \frac{1}{\sum_{k=1}^d z_k^{\bar{\alpha}}} dz < \infty. \]
(2) If $\bar{\alpha} < 2$, then
\[ \int_{\mathbb{R}^d \setminus [0,1]^d} \frac{1}{\sum_{k=1}^d z_k^{\bar{\alpha}}} dz < \infty. \]

**Remark 2.3** Lemma 2.1 implies that for $r \in (0, \bar{\alpha})$, $\int_{[-1,1]^d} \frac{1}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz < \infty$, and for $r > \bar{\alpha}$, $\int_{\mathbb{R}^d \setminus [-1,1]^d} \frac{1}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz < \infty$. Therefore, if $\tau(z)$ is bounded and $\int_{\mathbb{R}^d} \tau(z) dz < \infty$, then $\int_{\mathbb{R}^d} \frac{\tau(z)}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz < \infty$ for all $r \in (0, \bar{\alpha})$.

We will repeatedly use the following formulas involving Fourier transforms. Let $\phi_1$, $\phi_2$ and $\phi_3$ be functions from $\mathbb{R}^d$ to $\mathbb{R}$, bounded and integrable. Then
\[ \int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}_1(z) \hat{\phi}_2(z) dz, \tag{2.15} \]
(the Plancherel formula). If $\hat{\phi}_1$ and $\hat{\phi}_2$ are integrable, then
\[ \int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) \phi_3(x) dx = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \hat{\phi}_1(z) \hat{\phi}_2(z_1) \hat{\phi}_3(z + z_1) dz dz_1, \tag{2.16} \]
(the inverse Fourier transform) and moreover, by the Riemann-Lebesgue Lemma, $\hat{\phi}_1(z)$ is bounded and goes to 0 as $|z| \to \infty$. It follows from (2.1) and Fubini’s theorem that for any $t > 0$,
\[ \overline{\mathcal{T}_t \phi_1}(z) = \hat{\phi}_1(z) e^{-t \sum_{k=1}^d |z_k|^{\alpha_k}}. \tag{2.17} \]

From now on, we define a sequence of random variables $X_n$ in $S'((\mathbb{R}^d)^{d+1})$ as follows: For any $n \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$,
\[ \langle X_n, \psi \rangle = \int_0^1 \langle X_n(t), \psi(\cdot, t) \rangle dt. \tag{2.18} \]
3. Laplace functionals of the occupation time fluctuations

As in Bojdecki et al. [3, 6], the method for proving Theorems 2.1, 2.3 and 2.3 rely on the Laplace functionals of the occupation time fluctuations. In this section, we establish the main technical lemmas which will be needed for the proofs in Section 4. For simplicity, we always assume \( \psi(x, t) = \phi(x)h(t) \), where \( \phi \in \mathcal{S}(\mathbb{R}^d) \) and \( h \in \mathcal{S}(\mathbb{R}) \) are nonnegative functions. Hence from (3.4) and (3.3), we have that

\[
\langle \hat{X}_n, \psi \rangle = \frac{1}{F_n} \int_0^1 \left( \int_0^{nt} \langle N_n(s), \phi \rangle ds - \int_0^{nt} f_n(s) \langle \lambda, \phi \rangle ds \right) h(t) dt.
\]

Let

\[
\tilde{h}(s) = \int_s^1 h(t) dt \quad \text{and} \quad \psi_n(x, s) = \frac{1}{F_n} \phi(x) \tilde{h}(\frac{s}{n}).
\]

Then (3.1) can be rewritten as

\[
\langle \hat{X}_n, \psi \rangle = \int_0^n \langle N_n(s), \psi_n(\cdot, s) \rangle ds - \int_0^n f_n(s) \langle \lambda, \psi_n(\cdot, s) \rangle ds.
\]

Define

\[
H_{n, \psi_n}(x, t, r) = \mathbb{E}_x \left\{ \exp \left\{ - \int_0^t \langle N_n(s), \psi_n(\cdot, r + s) \rangle ds \right\} \right\}.
\]

Since \( N_n(0) \) is a Poisson random measure with Lebesgue intensity measure, it follows from (3.3) that

\[
\mathbb{E}(e^{-\langle \hat{X}_n, \psi \rangle}) = \exp \left\{ \int_{\mathbb{R}^d} [H_{n, \psi_n}(x, n, 0) - 1] dx + \int_{\mathbb{R}^d} dx \int_0^n f_n(s) \psi_n(x, s) ds \right\}.
\]

Note that for any \( n \geq 1 \), \( N_n \) is a Markov process. By using the renewal argument, we can rewrite (3.4) as

\[
H_{n, \psi_n}(x, t, r) = e^{-\gamma t} \mathbb{E}_x \left\{ \exp \left( - \int_0^t \psi_n(\tilde{\xi}_n(s), r + s) ds \right) \right\}
\]

\[+ \int_0^t \gamma e^{-\gamma s} \mathbb{E}_x \left\{ \exp \left( - \int_0^s \psi_n(\tilde{\xi}_n(u), r + u) du \right) \right\}
\]

\[\times \left[ \mathbb{E}_{\tilde{\xi}_n(s)} \exp \left( - \int_0^{t-s} \langle N_n(u), \psi_n(\cdot, r + s + u) \rangle du \right) \right] \lambda_n(s) \right\} ds,
\]

where \( \lambda_n(s) \) denotes the number of particles generated at the first splitting time. Since the process \( \lambda_n \) is independent of the process \( \xi_n \) and by the assumptions, for any \( 0 \leq z \leq 1 \)

\[
\mathbb{E}(e^{\lambda_n(s)}) = g_n(z, s) = \left( 1 + \frac{z^2}{2} \right) e^{-\delta_n s} + (1 - e^{-\delta_n s}),
\]

(3.6) yields that

\[
H_{n, \psi_n}(x, t, r) = e^{-\gamma t} I_{n, \psi_n}(x, t, r) + \int_0^t \gamma e^{-\gamma(t-s)} K_{n, \psi_n}(x, t-s, r, s) ds,
\]

where for any \( x \in \mathbb{R}^d, t, r \geq 0, \)

\[
I_{n, \psi_n}(x, t, r) = \mathbb{E}_x \left\{ \exp \left( - \int_0^t \psi_n(\tilde{\xi}_n(u), r + u) du \right) \right\},
\]

\[
K_{n, \psi_n}(x, t, r, s) = \mathbb{E}_x \left\{ \exp \left( - \int_0^t \psi_n(\tilde{\xi}_n(u), r + u) du \right) g_n \left( H_{n, \psi_n}(\tilde{\xi}_n(t), s, r + t), t \right) \right\}.
\]
Define \( g_{n,t}(z,t) = \frac{\partial g_n(z,t)}{\partial t} = -\delta_n g_n(z,t) + \delta_n \) and \( I_{n,\psi_n}(x,dt,r) = \frac{\partial I_{n,\psi_n}(x,t,r)}{\partial t} dt \). By the Feynman-Kac formula,

\[
\frac{\partial I_{n,\psi_n}}{\partial t} = (A + \frac{\partial}{\partial r} - \psi_n(x,r) I_{n,\psi_n},
\frac{\partial K_{n,\psi_n}}{\partial t} = (A + \frac{\partial}{\partial r} - \psi_n(x,r) K_{n,\psi_n},
+ \mathbb{E}_x \left( e^{-\int_0^t \psi_n(\vec{\xi}_n(u),r+u) du} g_{n,t}'(H_{n,\psi_n}(\vec{\xi}_n(t),s,r,t),t) \right)
= \left( A + \frac{\partial}{\partial r} - \psi_n(x,r) \right) K_{n,\psi_n} - \delta_n K_{n,\psi_n} + \delta_n I_{n,\psi_n}.
\]

Therefore, (3.7) indicates that

\[
\frac{\partial H_{n,\psi_n}(x,t,r)}{\partial t} = \int_0^t \left( A + \frac{\partial}{\partial r} - \psi_n(x,r) - \gamma \right) e^{-\gamma(t-s)} K_{n,\psi_n}(x,t-s,r,s) ds
+ \left( A + \frac{\partial}{\partial r} - \psi_n(x,r) - \gamma \right) e^{-\gamma t} I_{n,\psi_n}(x,t,r) + \delta_n \int_0^t \gamma e^{-\gamma s} I_{n,\psi_n}(x,s,r) ds
- \delta_n \int_0^t \gamma e^{-\gamma(t-s)} K_{n,\psi_n}(x,t-s,r,s) ds + \gamma \mathbb{E}_x \left( g_n(H_{n,\psi_n}(\vec{\xi}_n(0),t,r),0) \right).
\]

which, combined with the fact that \( g_n(z,0) = \frac{1+z^2}{2} =: g(z) \), yields that

\[
\frac{\partial H_{n,\psi_n}(x,t,r)}{\partial t} = \left( A + \frac{\partial}{\partial r} - \psi_n(x,r) - \gamma - \delta_n \right) H_{n,\psi_n}(x,t,r) + \gamma g(H_{n,\psi_n}(x,t,r))
+ \delta_n \int_0^t e^{-\gamma s} I_{n,\psi_n}(x,ds,r) + \delta_n.
\]

Let

\[
V_{n,\psi_n}(x,t,r) = 1 - H_{n,\psi_n}(x,t,r).
\]

Then \( V_{n,\psi_n}(x,0,r) = 0 \) and from (3.8), we have that

\[
\frac{\partial V_{n,\psi_n}(x,t,r)}{\partial t} = \left( A + \frac{\partial}{\partial r} - \delta_n \right) V_{n,\psi_n}(x,t,r) - \delta_n \int_0^t e^{-\gamma s} I_{n,\psi_n}(x,ds,r)
- \frac{\gamma}{2} V_{n,\psi_n}^2(x,t,r) + \psi_n(x,r)\left( 1 - V_{n,\psi_n}(x,t,r) \right),
\]

where we have used the elementary fact that \( g(1 - z) - (1 - z) = \frac{z^2}{2} \). Therefore,

\[
V_{n,\psi_n}(x,t,r) = \int_0^t e^{-\delta_n s} T_s \left( \psi_n(\cdot, r + s) \left( 1 - V_{n,\psi_n}(\cdot, t-s, r + s) \right) \right) ds
+ \delta_n \int_0^t e^{-\gamma u} \mathbb{E}_x \left( e^{-\int_0^u \psi_n(\vec{\xi}_n(v),r+s+v) du} \psi_n(\vec{\xi}_n(u), r + s + u) \right) du \bigg) (x) ds. \tag{3.10}
\]

Some simple calculation shows

\[
\delta_n \int_0^t e^{-\delta_n s} T_s \left( \int_0^t e^{-\gamma u} T_u \psi_n(\cdot, r + s + u) du \right) (x) ds
= \int_0^t \left( f_n(s) - e^{-\delta_n s} \right) T_s \psi_n(x, r + s) ds. \tag{3.11}
\]

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In addition, by using (2.7), (2.6), (3.4), (3.9), and the fact $1 - e^{-x} \leq x$ for all $x \in \mathbb{R}$, we derive that

$$V_{n,\psi_n}(x, t, r) \leq \int_0^t f_n(s) T_s \psi_n(\cdot, r + s)(x)ds =: J_{n,\psi_n}(x, t, r). \quad (3.12)$$

It follows from (3.10), (3.11) and (3.12) that

$$J_{n,\psi_n}(x, t, r) - V_{n,\psi_n}(x, t, r) = \delta_n \int_0^t e^{-\delta_n s} T_s \left( \int_0^{t-s} e^{-\gamma u} \chi_{n,\psi_n}(\cdot, u, r + s)du \right)(x)ds$$

$$+ \int_0^t e^{-\delta_n s} T_s \left( \psi_n(\cdot, r + s)V_{n,\psi_n}(\cdot, t - s, r + s) \right)(x)ds$$

$$+ \int_0^t e^{-\delta_n s} T_s \left( \frac{\gamma^2}{2} n,\psi_n(\cdot, t - s, r + s) \right)(x)ds, \quad (3.13)$$

where for any $x \in \mathbb{R}^d, u \geq 0$ and $s \geq 0$

$$\chi_{n,\psi_n}(x, u, s) = \mathbb{E}_x \left[ \left( 1 - e^{-\int_0^1 \tilde{\psi}_n(\tilde{\xi}(v), s + v)dv} \right) \tilde{\psi}_n(\tilde{\xi}(u), s + u) \right]. \quad (3.14)$$

By (2.7), (3.6) and (3.13), we obtain that

$$\mathbb{E}(e^{-\langle X_n, \psi \rangle}) = \exp \left( \int_{\mathbb{R}^d} dx \int_0^n f_n(s) T_s \psi_n(x, s)ds - \int_{\mathbb{R}^d} [1 - H_{n,\psi_n}(x, n, 0)] dx \right)$$

$$= \exp \left( \int_{\mathbb{R}^d} [J_{n,\psi_n}(x, n, 0) - V_{n,\psi_n}(x, n, 0)] dx \right)$$

$$= \exp \left( I_1(n, \psi_n) + I_2(n, \psi_n) + I_3(n, \psi_n) \right). \quad (3.15)$$

where

$$I_1(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} V_{n,\psi_n}^2(x, n - s, s)ds; \quad (3.16)$$

$$I_2(n, \psi_n) = \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s)V_{n,\psi_n}(x, n - s, s)ds; \quad (3.17)$$

$$I_3(n, \psi_n) = \delta_n \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} \chi_{n,\psi_n}(x, u, s)du. \quad (3.18)$$

Below, we evaluate the limits of $I_1(n, \psi_n), I_2(n, \psi_n)$ and $I_3(n, \psi_n)$, respectively. Recall that $H$ is the $d \times d$ diagonal matrix $(1/\alpha_k)_{1 \leq k \leq d}$. For all $t > 0$ and $y = t^H z$, $dy = t^5 dz$.

We first study the limit of $I_1(n, \psi_n)$. By using (3.16), we can write

$$I_1(n, \psi_n) = I_{11}(n, \psi_n) + I_{12}(n, \psi_n), \quad (3.19)$$

where

$$I_{11}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} f_{n,\psi_n}^2(x, n - s, s)ds, \quad (3.20)$$

$$I_{12}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} (V_{n,\psi_n}^2(x, n - s, s) - J_{n,\psi_n}^2(x, n - s, s))ds. \quad (3.21)$$

The following lemma determines the limit of $I_{11}(n, \psi_n)$. 

$$I_2(n, \psi_n) = I_{21}(n, \psi_n), \quad (3.22)$$

where

$$I_{21}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} J_{n,\psi_n}^2(x, n - s, s)ds, \quad (3.23)$$

$$I_{22}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} (V_{n,\psi_n}^2(x, n - s, s) - J_{n,\psi_n}^2(x, n - s, s))ds. \quad (3.24)$$
Lemma 3.1 (1) If $\bar{\alpha} > 2$ and $F^2_n = n$, then as $n \to \infty$,
\[
I_{11}(n, \psi_n) \to \frac{\gamma/2}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(z)|^2}{\sum_{k=1}^d |z_k|^\alpha_k} \, dz \int_0^1 h(u)du \int_0^1 h(v)dv \int_0^{u^\wedge v} e^{-\theta_s} \, ds. \tag{3.22}
\]

(2) If $\bar{\alpha} = 2$ and $F^2_n = n \ln n$, then
\[
I_1(n, \psi_n) \to C \int_0^1 h(r)dr \int_0^1 h(t)dt \int_0^{t^\wedge r} e^{-\theta_s} \, ds \int_{\mathbb{R}^d} \varphi(\theta, r - s, t - s, \sum_{k=1}^d |y_k|^\alpha_k) \, dy, \tag{3.23}
\]
where $C = \frac{\gamma/2}{(2\pi)^d} (\int_{\mathbb{R}^d} \phi(x)dx)^2$ and $\varphi(x, u, v, z)$ is defined by (2.11).

(3) If $\bar{\alpha} \in (1, 2)$ and $F_n = n^{(3-\bar{\alpha})/2}$, then
\[
I_1(n, \psi_n) \to \frac{\gamma}{2\pi^d} \prod_{k=1}^d \left( \frac{\Gamma(1/\alpha_k)}{\alpha_k} \right) \left( \int_{\mathbb{R}^d} \phi(x)dx \right)^2 \int_0^1 \int_0^1 h(s)h(t)C(s, t)dsdt, \tag{3.24}
\]
where $C(s, t)$ is as (2.11).

**Proof.** Since all components of $\xi$ are symmetric stable Lévy processes and independent of each other, we have
\[
\int_{\mathbb{R}^d} T_s f(x)g(x)dx = \int_{\mathbb{R}^d} f(x)T_s g(x)dx \tag{3.25}
\]
for any $s \geq 0$ and bounded measurable functions $f$ and $g$. By using (2.5), (2.15), (2.17), (3.2), (3.12) and (3.20), we derive that
\[
I_{11}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n \left( e^{-\delta_n s} \int_0^{n-s} e^{-\delta_n u} f_n(u)T_u \psi_n(x, s + u)du \right. \\
\left. \times \int_0^{n-s} e^{-\delta_n v} f_n(v)T_v \psi_n(x, s + v)dv \right) \, ds \\
= \frac{\gamma/2}{F^2_n(2\pi)^d} \int_0^n \left( \int_0^{n-s} f_n(u) \tilde{h} \left( \frac{s + u}{n} \right) du \right) \int_0^{n-s} e^{-\delta_n(u+v)} f_n(v) \tilde{h} \left( \frac{s + v}{n} \right) dv \\
\times \int_{\mathbb{R}^d} |\hat{\varphi}(z)|^2 e^{-n(u+v)\sum_{k=1}^d |z_k|^\alpha_k} \, dz \, e^{-\delta_n s} \, ds. \tag{3.26}
\]
Equation (3.26) and the fact that $\tilde{f}_n$ converges uniformly to 1 imply that
\[
\lim_{n \to \infty} I_{11}(n, \psi_n) = \lim_{n \to \infty} \left\{ \frac{n^3 \gamma/2}{(2\pi)^d F^2_n} \int_0^1 \left[ \int_0^{1-s} \tilde{h}(s + u)du \right] \int_0^{1-s} e^{-n\delta_n(u+v)} \tilde{h}(s + v)dv \right. \\
\left. \times \int_{\mathbb{R}^d} |\hat{\varphi}(z)|^2 e^{-n(u+v)\sum_{k=1}^d |z_k|^\alpha_k} \, dz \] e^{-n\delta_n s} \, ds \right\}. \tag{3.27}
\]

(1) Suppose $\bar{\alpha} > 2$ and $F^2_n = n$. It follows from (3.27) that
\[
\lim_{n \to \infty} I_{11}(n, \psi_n) = \lim_{n \to \infty} \left\{ \frac{\gamma/2}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\varphi}(z)|^2 \, dz \int_0^1 \left[ \int_0^{1-s} ne^{-n(u-s)\sum_{k=1}^d |z_k|^\alpha_k} \tilde{h}(u)du \right. \\
\left. \times \int_0^{1-s} ne^{-n(u-v)\sum_{k=1}^d |z_k|^\alpha_k} \tilde{h}(v)dv \right] e^{-n\delta_n s} \, ds \right\}. \tag{3.28}
\]
Since $\bar{h}(t)$ is a continuous and bounded function, it is easy to check that for every $z \in \mathbb{R}^d \setminus \{0\}$, as $n \to \infty$,
\[
\frac{\tilde{h}(s)}{\sum_{k=1}^{d} |z_k|^{\alpha_k}} \geq \int_{s}^{1} \tilde{h}(u)e^{-n(u-s)(\delta_n + \sum_{k=1}^{d} |z_k|^{\alpha_k})} ndu \to \frac{\tilde{h}(s)}{\sum_{k=1}^{d} |z_k|^{\alpha_k}}.
\]
By using \[3.23\] and the dominated convergence theorem, we deduce that as $n \to \infty$,
\[
I_{11}(n, \psi_n) \to \frac{\gamma/2}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \begin{array}{c} \phi(z) \\ \sum_{k=1}^{d} |z_k|^{\alpha_k} \end{array} \right)^2 dz \int_{0}^{1} e^{-\theta s} \tilde{h}(s)^2 ds. \quad (3.29)
\]
Notice that, by Remark 2.3, the right hand side of (3.29) is finite. Substituting $\tilde{h}(s) = \int_{s}^{1} h(t)dt$ into (3.29) yields (3.22).

(2) Next we consider the case $\bar{\alpha} = 2$ and $F_n^2 = n \ln n$. Define
\[
\bar{I}_{11}(n, \psi_n) := \frac{n^2 \gamma/2}{(2\pi)^d} F_n \int_{0}^{1} \left[ \int_{0}^{1-s} \tilde{h}(s + u)du \int_{0}^{1-s} e^{-\theta(u+v)} \tilde{h}(s + v)dv \right. \times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-n(u+v)\sum_{k=1}^{d} |z_k|^{\alpha_k}} dz \left. \right] e^{-\theta s} ds. \quad (3.30)
\]
Since $n\delta_n \to \theta$, for sufficiently large $n$ and all $t \in [0, 1]$,\[
e^{-\theta t}(1 - |\theta - n\delta_n|) \leq e^{-\theta t} e^{-|\theta - n\delta_0|t} \leq e^{-\theta t} e^{\theta - n\delta_0|t} \leq e^{-\theta t}(1 + 3|\theta - n\delta_n|),\]
we see from \[3.27\] that, in order to prove \[3.23\], it suffices to prove that $\bar{I}_{11}(n, \psi_n)$ converges to
\[
\mathcal{C} \int_{0}^{1} h(r)dr \int_{0}^{1} h(t)dt \int_{0}^{t+s} e^{-\theta \gamma} \int_{\mathbb{R}^d} \varphi(\theta, r - s, t - s, \sum_{k=1}^{d} |y_k|^{\alpha_k}) dy, \quad (3.31)
\]
where $\mathcal{C} = \frac{\gamma/2}{(2\pi)^d} (\int_{\mathbb{R}^d} \phi(x) dx)^2$ and $\varphi$ is defined by (2.11). Substituting $\tilde{h}(u) = \int_{u}^{1} h(t)dt$ and $F_n^2 = n \ln n$ into (3.30) and changing the order of integration, we obtain that
\[
\lim_{n \to \infty} \bar{I}_{11}(n, \psi_n) = \lim_{n \to \infty} \left\{ \frac{\gamma/2}{(2\pi)^d} \int_{0}^{1} e^{-\theta s} \int_{0}^{1} h(t)dt \int_{0}^{1} h(r)dr \right. \times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-(u+v-2s)(\theta + n \sum_{k=1}^{d} |z_k|^{\alpha_k})} dz \left. \right\}
= \lim_{n \to \infty} \left\{ \frac{\gamma/2}{(2\pi)^d} \int_{0}^{1} e^{-\theta s} \int_{0}^{1} h(t)dt \int_{0}^{1} h(r)dr \right. \times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 dz \left. \right\}
= \lim_{n \to \infty} \left\{ \frac{\gamma/2}{(2\pi)^d} \int_{0}^{1} h(r)dr \int_{0}^{1} h(t)dt \int_{0}^{t+s} e^{-\theta \gamma} \right. \times \left. \int_{\mathbb{R}^d} e^{-(u+v-2s)(\theta + n \sum_{k=1}^{d} |z_k|^{\alpha_k})} du dv \right\}
= \lim_{n \to \infty} \frac{\gamma/2}{(2\pi)^d} \int_{0}^{1} h(r)dr \int_{0}^{1} h(t)dt \int_{0}^{t+s} e^{-\theta \gamma} W_{n,r,t,s}(\theta) ds, \quad (3.32)
\]
where for any $x \geq 0$ and $0 \leq s \leq r \land t$,
\[
W_{n,r,t,s}(x) = \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 \frac{\varpi(x, r - s, t - s, \sum_{k=1}^{d} |z_k|^{\alpha_k}, n)}{\ln n} dz, \quad (3.33)
\]
and for $x, u, v \geq 0$ and $y, w > 0$,
\[
\varpi(x, u, v, w, y) = y^2 \frac{(1 - e^{-u(x+yw)})(1 - e^{-v(x+yw)})}{(x+yw)^2}.
\]

\[13\]
Since the function $r \mapsto (1 - e^{-r})/r$ is decreasing in $r \in (0, +\infty)$, we have that
\[
\varpi(x, u, v, w, y) \leq \varpi(0, 1, 1, w, y)
\]  
(3.34) for all $u, v \in [0, 1]$. By applying L'Hôpital's rule and the dominated convergence theorem, we derive from (3.33) that
\[
\lim_{n \to \infty} W_{n, r, t, s}(\theta) = \lim_{n \to \infty} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \hat{\varphi}_n(\theta, r, t, s, z) dz,
\]
where
\[
\hat{\varphi}_n(\theta, r, t, s, z) = n \varpi' \left( \frac{\varpi}{\varpi'} \right) \left( \frac{r-s}{r-t-s} \right) \left( \frac{t-s}{t-r-s} \right) \left( \frac{n}{n} \right) \left( \frac{|z_k|^{\alpha_k}}{n} \right),
\]
and $\varphi$ is defined by (2.11). By using the substitution $y = n^H z$, we get that
\[
\int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \hat{\varphi}_n(\theta, r, t, s, z) dz = \int_{\mathbb{R}^d} |\hat{\phi}(n^{-H} y)|^2 \varphi \left( \theta, r-s, t-s, \sum_{k=1}^d |y_k|^{\alpha_k} \right) dy.
\]
Therefore,
\[
\lim_{n \to \infty} W_{n, r, t, s}(\theta) = |\hat{\phi}(0)|^2 \int_{\mathbb{R}^d} \varphi \left( \theta, r-s, t-s, \sum_{k=1}^d |y_k|^{\alpha_k} \right) dy. 
\]  
(3.35)

On the other hand, let
\[
\hat{W}(n) = W_{n, 0, 0, 0}(0) = \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \varpi(0, 1, 1, \sum_{k=1}^d |z_k|^{\alpha_k}, n) dz.
\]

Then (3.33) yields that
\[
\lim_{n \to \infty} \hat{W}(n) = |\hat{\phi}(0)|^2 \int_{\mathbb{R}^d} \varphi \left( 0, 1, 1, \sum_{k=1}^d |y_k|^{\alpha_k} \right) dy
\]
\[= 2|\hat{\phi}(0)|^2 \int_{\mathbb{R}^d} \frac{e^{-\sum_{k=1}^d |z_k|^{\alpha_k}}(1 - e^{-\sum_{k=1}^d |z_k|^{\alpha_k}})}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz < \infty. 
\]  
(3.36)

Hence $\{\hat{W}(n)\}_{n \geq 1}$ is bounded. Combining (3.33), (3.34) and (3.36) gives
\[
|W_{n, r, t, s}(\theta)| \leq |\hat{W}(n)| \leq K,
\]
for some constant $K < \infty$. By the dominated convergence theorem, we derive (3.31) from (3.32) and (3.36).

(3) To consider the case $\alpha \in (1, 2)$, we substitute $y = (nu + nv)^H z$ into (3.27) to obtain that
\[
\lim_{n \to \infty} I(11, \psi_n) = \lim_{n \to \infty} \left\{ \frac{n^{3-\alpha}/2}{(2\pi)^d F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}((nu + nv)^H y)|^2 e^{-\sum_{k=1}^d |y_k|^{\alpha_k}} dy \int_0^{1-s} du \times \int_0^{1-s} e^{-n\delta_n(u+v)} \hat{h}(s+v) \left( \frac{dv}{(u+v)^3} \right) e^{-n\delta_n s} ds \right\}. 
\]  
(3.37)
Since \((nu + nv)^{-H} y \to 0\) for every \(y\), substituting \(F_n = n^{(3-\alpha)/2}\) into (3.37), we see that \(I_{11}(n, \psi_n)\) converges to
\[
\frac{\gamma/2}{(2\pi)^d} \int_{\mathbb{R}^d} |\phi(0)|^2 e^{-\sum_{k=1}^d |y_k|^\alpha} dy \int_0^1 e^{\theta s} \left[ \int_s^1 e^{-\theta u} \tilde{h}(u) du \right] \int_0^1 e^{-\theta v} \tilde{h}(v) dv \frac{e^{\theta s} ds}{(u + v - 2s)^\alpha} ds
\]
\[
= \frac{\gamma/2}{\pi^d} |\phi(0)|^2 \prod_{k=1}^d \Gamma(1/\alpha_k) \int_0^1 e^{-\theta u} \tilde{h}(u) du \int_0^1 e^{-\theta v} \tilde{h}(v) dv \int_0^{u+v} \frac{e^{\theta s} ds}{(u + v - 2s)^\alpha}. \quad (3.38)
\]
Substituting \(\tilde{h}(t) = \int_1^t h(s) ds\) into (3.38) yields (3.24).

For \(I_{12}(n, \psi_n)\) in (3.21) we have

**Lemma 3.2** If \(\alpha > 1\) and \(F_n\) takes values according to Theorem 2.1, 2.2 and 2.3 respectively, then
\[
\lim_{n \to \infty} I_{12}(n, \psi_n) = 0. \quad (3.39)
\]

**Proof.** By (3.12) and (3.13), we have that
\[
J_{\alpha}^2(\psi_n) = 2 \int_{\mathbb{R}^d} e^{-\delta_n u T_n \psi_n(x, n-s, s)} ds
\]
\[
\leq 2 \int_{\mathbb{R}^d} e^{-\delta_n u T_n \psi_n(x, n-s, s)} ds \int_0^u e^{-(\delta_n - \gamma) v} dv
\]
\[
+ \int_0^u e^{-\delta_n u T_n \psi_n(x, n-s, s, u)} du
\]
\[
+ \frac{\gamma}{2} \int_0^u e^{-\delta_n u T_n J_{\alpha}^2(\psi_n)} ds
\]
This and (3.21) imply that
\[
|I_{12}(n, \psi_n)| \leq \frac{\gamma}{2} I_{121}(n, \psi_n) + \frac{\gamma}{2} I_{122}(n, \psi_n) + \frac{\gamma^2}{2} I_{123}(n, \psi_n), \quad (3.40)
\]
where
\[
I_{121}(n, \psi_n) = \int_{\mathbb{R}^d} J_{\alpha}^2(\psi_n) ds
\]
\[
= \int_{\mathbb{R}^d} \int_0^u e^{-\delta_n u T_n \psi_n(x, n-s, s)} ds
\]
\[
\times \int_0^u e^{-\delta_n u \left(\int_0^v T_n \psi_n(x, s) ds\right)} dv, \quad (3.41)
\]
\[
I_{122}(n, \psi_n) = \int_{\mathbb{R}^d} J_{\alpha}^2(\psi_n) ds
\]
\[
= \int_{\mathbb{R}^d} \int_0^u e^{-\delta_n u T_n \psi_n(x, n-s, s)} ds
\]
\[
\times \int_0^u e^{-\delta_n u \left(\int_0^v T_n \psi_n(x, s) ds\right)} dv, \quad (3.42)
\]
and
\[
I_{123}(n, \psi_n) = \int_{\mathbb{R}^d} J_{\alpha}^2(\psi_n) ds
\]
\[
= \int_{\mathbb{R}^d} \int_0^u e^{-\delta_n u T_n \psi_n(x, n-s, s)} ds
\]
\[
\times \int_0^u e^{-\delta_n u T_n J_{\alpha}^2(\psi_n)} ds, \quad (3.43)
\]
Substituting (3.12) and (3.22) into (3.41) gives that
\[
I_{121}(n, \psi_n) = \frac{1}{F_n^2} \int_0^n e^{-\delta_n s} ds \int_0^n (\int_0^s T_n \psi_n) ds
\]
\[
\times \int_0^n e^{-\delta_n \left(\frac{s+u}{n}\right)} dv, \quad (3.44)
\]
Using (2.16), (2.17), (3.2) and (3.25), we derive from (3.46) that

\[
\int_{\mathbb{R}^d} T_u \phi(x) T_v \phi(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-(u+v) \sum_{k=1}^{d} |z_k|^2 \alpha_k} dz.
\]

Comparing (3.44) with (3.26), we find that as

\[
\text{Note that from (2.15), (2.17) and (3.25) we have that}
\]

\[
\text{where we have used the fact 0} < \frac{\tilde{f}_n(u) - 1}{\tilde{f}_n(u)} \leq \frac{d_n}{\gamma} \frac{(n+1)}{\delta_n} \text{ which follows from (2.5).}
\]

In addition, substituting (2.16) and (3.12) into (3.12) we obtain that

\[
I_{121}(n, \psi_n) \leq \frac{2\delta_n}{\gamma} f_{11}(n, \psi_n),
\]

where we have used the fact that as 

\[
\text{In (3.12), we derive from (3.46) that}
\]

\[
I_{122}(n, \psi_n) = \int_{\mathbb{R}^d} e^{-\delta_n \alpha_k} ds \int_{0}^{\infty} e^{-\delta_n \alpha_k} du \int_{0}^{\infty} \tilde{f}_n(u) e^{-\delta_n \alpha_k} du \int_{0}^{\infty} \tilde{f}_n(w) e^{-\delta_n \alpha_k} dw 
\]

\[
\times \int_{\mathbb{R}^d} \psi_n(x, s + u) T_u \psi_n(x, s + u + w) T_v \psi_n(x, s + v) dx
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\phi(z)\phi(z')| e^{-\delta_n \alpha_k} dxdz' \int_{0}^{\infty} ds \int_{0}^{\infty} \tilde{f}_n(u) e^{-\delta_n \alpha_k} du \int_{0}^{\infty} \tilde{f}_n(w) e^{-\delta_n \alpha_k} dw 
\]

\[
\times e^{-(u+v) \sum_{k=1}^{d} |z_k|^2 \alpha_k} e^{-\delta_n \alpha_k} \sum_{k=1}^{d} |z_k|^2 \alpha_k dw.
\]

Note that \(\{\tilde{f}_n\}\) and \(|\tilde{\phi}|\) are bounded and, for \(h \in \mathcal{S}(\mathbb{R})\), \(\tilde{h}\) is bounded as well. There exists a constant \(K > 0\) such that

\[
I_{122}(n, \psi_n) \leq \frac{K n^4}{F_{\alpha}(\gamma)} \int_{\mathbb{R}^d} |\phi(z)\phi(z')| dxdz' \int_{0}^{\infty} e^{-\delta_n \alpha_k} ds \int_{0}^{\infty} du \int_{0}^{1-s} dw 
\]

\[
\times \int_{0}^{1-s-u} e^{-\delta_n \alpha_k} \sum_{k=1}^{d} |z_k|^2 \alpha_k e^{-\delta_n \alpha_k} \sum_{k=1}^{d} |z_k|^2 \alpha_k dw.
\]

Furthermore, substituting (2.25) and (3.12) into (3.12), we obtain that

\[
I_{123}(n, \psi_n) = \int_{\mathbb{R}^d} e^{-\delta_n \alpha_k} ds \int_{0}^{\infty} e^{-\delta_n \alpha_k} du \int_{0}^{\infty} e^{-\delta_n \alpha_k} \tilde{f}_n(t) dt 
\]

\[
\times \int_{0}^{\infty} e^{-\delta_n \alpha_k} \tilde{f}_n(t') dt' \int_{0}^{\infty} e^{-\delta_n \alpha_k} \tilde{f}_n(v) dv 
\]

\[
\times \int_{\mathbb{R}^d} T_u \psi_n(x, s + u + t) T_v \psi_n(x, s + u + t') T_u \psi_n(x, s + u) dx.
\]
By the same argument used to get (3.47), we can find $K > 0$ such that
\[
I_{123}(n, \psi_n) \leq \frac{K n^5}{\sqrt{n}} \int_{\mathbb{R}^{2d}} |\hat{\phi}(z + z')| |\hat{\phi}(z')| \, dz \, dz' \int_0^1 e^{-n\delta_n s} \, ds \int_0^{1-s} e^{-n u \sum_{k=1}^d |z_k|^{\alpha_k}} \, du \\
\times \int_0^{1-s} e^{-n v \sum_{k=1}^d |z_k|^{\alpha_k}} \, dv \int_0^{1-s-u} e^{-nt' \sum_{k=1}^d |z_k'|^{\alpha_k}} \, dt' \\
\times \int_0^{1-s-u} e^{-nt \sum_{k=1}^d |z_k + z_k'|^{\alpha_k}} \, dt.
\] (3.48)

Note that Lemma 3.1 and (3.45) imply
\[
\lim_{n \to \infty} I_{121}(n, \psi_n) = 0.
\]
Thus we see from (3.40) that, in order to prove (3.39), it suffices to show that $I_{122}(n, \psi_n)$ and $I_{123}(n, \psi_n)$ all converge to 0 as $n \to \infty$. Below we divide the proof of these facts into three cases.

**Case (1) $\bar{\alpha} \in (2, \infty)$ and $F_n^2 = n$.** From (3.47), we have that for some $K > 0$
\[
I_{122}(n, \psi_n) \leq \frac{K}{\sqrt{n}} \int_{\mathbb{R}^{2d}} \frac{|\hat{\phi}(z)|}{\sum_{k=1}^d |z_k|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k}} \, dz \, dz'.
\]
By Remark 2.3 the last integral is finite. Hence we have
\[
\lim_{n \to \infty} I_{122}(n, \psi_n) = 0, \quad (3.49)
\]
From (3.48), it follows that
\[
I_{123}(n, \psi_n) \leq \frac{K}{\sqrt{n}} \int_{\mathbb{R}^{2d}} \frac{|\hat{\phi}(z + z')|}{\sum_{k=1}^d |z_k + z_k'|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k} (\sum_{k=1}^d |z_k|^{\alpha_k})^2} \, dz \, dz'. \quad (3.50)
\]
By the same argument as those used in [6] p.27, we can verify that
\[
\int_{\mathbb{R}^{2d}} \frac{|\hat{\phi}(z + z')|}{\sum_{k=1}^d |z_k + z_k'|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k} (\sum_{k=1}^d |z_k|^{\alpha_k})^2} \, dz \, dz' < \infty.
\]
Hence (3.50) implies that
\[
\lim_{n \to \infty} I_{123}(n, \psi_n) = 0. \quad (3.51)
\]

**Cases (2) $\bar{\alpha} = 2$ and $F_n^2 = n \ln n$.** (3.47) implies that for some $K > 0$,
\[
I_{122}(n, \psi_n) \leq \frac{K}{(n \ln n)^{1/2}} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2 \ln n} \, dz \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z')|}{\sum_{k=1}^d |z_k'|^{\alpha_k}} \, dz'. \quad (3.52)
\]
By applying L'Hôpital's rule and substituting $y = n^{1/2}z$, we derive that as $n \to \infty$,
\[
\int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|(1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |y_k|^{\alpha_k})^2 \ln n} \, dz \to 2 |\hat{\phi}(0)| \int_{\mathbb{R}^d} \frac{e^{-\sum_{k=1}^d |y_k|^{\alpha_k}}}{(\sum_{k=1}^d |y_k|^{\alpha_k})^2} \, dy.
\]
By Remark 2.3 the last integral and $\int_{\mathbb{R}^d} \frac{|\hat{\phi}(z')|}{\sum_{k=1}^d |z_k'|^{\alpha_k}} \, dz'$ are finite. Therefore (3.52) implies (3.49).
Substituting $F_n^2 = n \ln n$ into (3.48), we get that for some constant $K > 0$,

$$I_{123}(n, \psi_n) \leq \frac{Kn}{n^{1/2} \ln n^{3/2}} \int_{\mathbb{R}^d} \left[ (1 - e^{-n \sum_{k=1}^d |z_k + z_k'|^{\alpha_k}})(1 - e^{-n \sum_{k=1}^d |z_k'|^{\alpha_k}}) \right] \frac{\sum_{k=1}^d |z_k + z_k'|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k}}{\sum_{k=1}^d |z_k|^{\alpha_k}} \times \frac{1}{\sum_{k=1}^d |z_k|^{\alpha_k}} |\hat{\phi}(z)| |\hat{\phi}(z')||\hat{\phi}(z + z')| \, dz \, dz'. $$

Furthermore, by using the inequality $1 - e^{-x} \leq x^{1/8}$ for $x \geq 0$ and the similar argument to that in [3, p.29 lines 8-15], we arrive at (3.51). The details are omitted.

**Case (3)** $\bar{\alpha} \in (1, 2)$ and $F_n^2 = n^{3-\bar{\alpha}}$. It follows from the boundedness of $|\hat{\phi}(z)|$ and (3.47) that

$$I_{122}(n, \psi_n) \leq \frac{K}{F_n} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z')| dz'}{\sum_{k=1}^d |z_k'|^{\alpha_k}} \left[ 1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}} \right]^2 \, dy. \quad (3.53)$$

Substituting $y = n^H z$ in (3.53) yields

$$I_{122}(n, \psi_n) \leq \frac{K}{F_n} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z')| dz'}{\sum_{k=1}^d |z_k'|^{\alpha_k}} \left[ 1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}} \right]^2 \, dy. \quad (3.54)$$

Note that, for $\bar{\alpha} \in (1, 2)$, Lemma 2.1 and Remark 2.3 imply

$$\int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|}{\sum_{k=1}^d |z_k|^{\alpha_k}} \, dz \int_{\mathbb{R}^d} \left[ 1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}} \right]^2 \, dy < \infty. $$

Thus, (3.54) implies (3.49).

On the other hand, (3.48) implies that

$$I_{123}(n, \psi_n) \leq \frac{Kn}{F_n^3} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z) + z'| |\hat{\phi}(z')|}{\sum_{k=1}^d |z_k + z_k'|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k}} \hat{\phi}(z) |\hat{\phi}(z')| |\hat{\phi}(z + z')| \, dz \, dz' \times \int_0^1 \left[ 1 - e^{-n \sum_{k=1}^d |z_k'|^{\alpha_k}} \right]^2 \, ds$$

$$\leq \frac{Kn}{F_n^3} \int_{\mathbb{R}^d} \frac{1}{\sum_{k=1}^d |z_k + z_k'|^{\alpha_k} \sum_{k=1}^d |z_k'|^{\alpha_k}} \left[ 1 - e^{-n \sum_{k=1}^d |z_k'|^{\alpha_k}} \right]^2 \, dz \, dz'. \quad (3.55)$$

Letting $y = n^H z$ and $y' = n^H z'$ and substituting $F_n^2 = n^{3-\bar{\alpha}}$ into (3.54) lead to

$$I_{123}(n, \psi_n) \leq \frac{Kn^{1/2}}{n^{\bar{\alpha}/2}} \int_{\mathbb{R}^d} \frac{1}{\sum_{k=1}^d |y_k + y_k'|^{\alpha_k} \sum_{k=1}^d |y_k'|^{\alpha_k}} \left[ 1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}} \right]^2 \, dy \, dy'. \quad (3.56)$$

Since $\bar{\alpha} \in (1, 2)$, we can use the same argument as in [3, p.17] to verify that

$$\int_{\mathbb{R}^d} \frac{1}{\sum_{k=1}^d |y_k + y_k'|^{\alpha_k} \sum_{k=1}^d |y_k'|^{\alpha_k}} \left[ 1 - e^{-n \sum_{k=1}^d |y_k|^{\alpha_k}} \right]^2 \, dy < \infty. $$

Therefore, (3.50) follows from (3.56).

For $I_2(n, \psi_n)$ in (3.17), we have the following lemma.
Lemma 3.3 (1) If $\alpha > 2$ and $F_n^2 = n$, then

$$
\lim_{n \to \infty} I_2(n, \psi_n) = \frac{1}{(2\pi)^d} \int_0^1 \int_0^1 h(t)dt \int_0^1 \int_0^1 \sum_{k=1}^d |z_k|^\alpha_k dz. 
$$

(2) If $\alpha \in (1, 2]$ and $F_n$ takes values according to Theorems 2.2 and 2.3 respectively, then

$$
\lim_{n \to \infty} I_2(n, \psi_n) = 0.
$$

Proof. To prove (3.57), we write

$$I_2(n, \psi_n) = I_{21}(n, \psi_n) - I_{22}(n, \psi_n),$$

where

$$I_{21}(n, \psi_n) = \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s) J_n, \psi_n(x, n - s, s) ds,$$

$$I_{22}(n, \psi_n) = \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s) (J_n, \psi_n(x, n - s, s) - V_n, \psi_n(x, n - s, s)) ds.$$

By (2.54), (2.57), (3.2) and (3.12), $I_{21}(n, \psi_n)$ equals

$$\frac{n^2}{F_n^2(2\pi)^d} \int_0^1 e^{-\delta_n s} ds \int_0^{1-s} f_n(nu) \hat{h}(s) \hat{h}(s + u) du \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-n - \sum_{k=1}^d |z_k|^\alpha_k} dz.$$

Substituting $\hat{h}(t) = \int_t^1 h(s) ds$ and (2.55) into the above formula gives that

$$I_{21}(n, \psi_n) = \frac{n^2}{F_n^2(2\pi)^d} \int_0^1 e^{-\delta_n s} ds \int_0^{1-s} e^{-\delta_n u} f_n(nu) du \int_0^1 h(t) dt$$

$$\times \int_{s+u}^1 h(t) dt' \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-n - \sum_{k=1}^d |z_k|^\alpha_k} dz$$

$$= \frac{n^2}{F_n^2(2\pi)^d} \int_0^1 e^{-\delta_n s} ds \int_0^1 h(t) dt \int_0^1 h(t') dt'$$

$$\times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 dz \int_0^{1-s} f_n(nu) e^{-n - \sum_{k=1}^d |z_k|^\alpha_k} du. \quad \quad \quad \text{(3.59)}$$

Since $f_n(u) \to 1$ uniformly as $n \to \infty$, (3.59) implies that

$$\lim_{n \to \infty} I_{21}(n, \psi_n) = \lim_{n \to \infty} \tilde{I}_{21}(n, \psi_n),$$

where

$$\tilde{I}_{21}(n, \psi_n) = \frac{n^2}{F_n^2(2\pi)^d} \int_0^1 e^{-\delta_n s} ds \int_0^1 h(t) dt \int_0^1 h(t') dt'$$

$$\times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 dz \int_0^{1-s} e^{-n \left(t - \delta_n \sum_{k=1}^d |z_k|^\alpha_k\right)} du$$

$$= \frac{n}{F_n^2(2\pi)^d} \int_0^1 e^{-\delta_n s} ds \int_0^1 h(t) dt \int_0^1 h(t') dt'$$

$$\times \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 \frac{1 - e^{-n \left(t - \delta_n \sum_{k=1}^d |z_k|^\alpha_k\right)}}{\delta_n + \sum_{k=1}^d |z_k|^\alpha_k} dz.$$
Since $n\delta_n \to \theta \in [0, \infty)$ and $F_n^2 = n$, from the above formula, it follows that

$$\lim_{n \to \infty} \tilde{I}_{21}(n, \psi_n) = \frac{1}{(2\pi)^d} \int_0^1 h(t) dt \int_0^1 h(t') dt' \int_0^{t \land t'} e^{-\theta s} ds \int_{\mathbb{R}^d} \frac{1}{1 + |z|^2} dz. \quad (3.61)$$

To determine the limit of $I_{22}(n, \psi_n)$ in (3.58), we note the similarity between $I_{22}(n, \psi_n)$ and $I_{12}(n, \psi_n)$. By using similar (but much simpler) arguments, we can show

$$\lim_{n \to \infty} I_{22}(n, \psi_n) = 0; \quad (3.62)$$

the details are omitted here. Combining (3.58) with (3.60)–(3.62) yields (3.61).

To prove Part (2), let

$$\tilde{I}_2(n, \psi_n) := \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s) J_n,\psi_n(x, n - s, s) ds. \quad (3.63)$$

Then (3.12) and (3.17) imply that

$$I_2(n, \psi_n) \leq \tilde{I}_2(n, \psi_n) = \int_0^n e^{-\delta_n s} ds \int_0^{n-s} f_n(v) dv \int_{\mathbb{R}^d} \psi_n(x, s) T_v \psi_n(x, s + v) dx. \quad (3.64)$$

By (2.5), (3.2), (2.15) and (2.17), there exists a constant $K > 0$ such that

$$I_2(n, \psi_n) \leq K \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\delta_n v} dv \int_{\mathbb{R}^d} |\hat{\psi}(z)|^2 e^{-v \sum_{k=1}^d |z_k|^\alpha_k} dz
\leq \frac{nK}{F_n^2} \int_{\mathbb{R}^d} \sum_{k=1}^d |z_k|^\alpha_k \int_0^1 e^{-\delta_n s} ds \int_0^n e^{-\delta_n v} dv \int_{\mathbb{R}^d} |\hat{\psi}(z)|^2 e^{-v \sum_{k=1}^d |z_k|^\alpha_k} dz. \quad (3.65)$$

By Remark 2.3, $\int_{\mathbb{R}^d} \sum_{k=1}^d |z_k|^\alpha_k < \infty$ for $\alpha > 1$. Substituting $F_n^2 = n^{3-\alpha}$ as $\alpha \in (1, 2)$, or $F_n^2 = n \ln n$ as $\alpha = 2$ into (3.64), we can readily see that $I_2(n, \psi_n) \to 0$ as $n \to \infty$. This finishes the proof.

The last lemma is concerned with $I_3(n, \psi_n)$ in (3.18).

**Lemma 3.4** If $F_n \to \infty$ as $n \to \infty$, then

$$\lim_{n \to \infty} I_3(n, \psi_n) = 0. \quad (3.66)$$

**Proof.** From (3.18) and (3.14), we can obtain that

$$I_3(n, \psi_n) \leq \tilde{I}_3(n, \psi_n), \quad (3.67)$$

where $\tilde{I}_3(n, \psi_n)$ is defined as

$$\delta_n \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} \psi_n(\xi_n(u), s + v) dv \psi_n(\xi_n(u), s + u) du
\leq \delta_n \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \int_{\mathbb{R}^d} \psi_n(x, s + v) T_{u-v} \psi_n(x, s + u) dx. \quad (3.68)$$

By using (2.15), (2.17) and (3.2), we get from (3.64) that

$$\tilde{I}_3(n, \psi_n) \leq \frac{\delta_n}{F_n^2 (2\pi)^d} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \int_0^u h(s + u) \hat{h}(\frac{u}{n}) dv \int_{\mathbb{R}^d} |\hat{\psi}(z)|^2 e^{-(u-v) \sum_{k=1}^d |z_k|^\alpha_k} dz. \quad (3.69)$$
The boundedness of $|\hat{\psi}(z)|$ and $\hat{h}$ implies that

$$I_3(n, \psi_n) \leq K \frac{\delta_n}{s_n} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \leq \frac{K}{\gamma s_n}. \quad (3.69)$$

Therefore, (4.1) follows from (3.65) and (3.69).

4. Proofs of the main results

In this section, we give the proofs of the main results stated in Section 2.

Proof of Theorem 2.1. Without loss of generality, we prove the conclusion for $t = 1$, namely, we prove that $\langle X_n, \psi \rangle$ converges in distribution to $\langle X, \psi \rangle$ for all $\psi \in S(\mathbb{R}^d)$, where $X_n$ and $X$ are defined as in (4.18). As explained in Bojdecki et al. [5, p.9], it suffices to show that

$$\lim_{n \to \infty} \mathbb{E}(e^{-\langle X_n, \psi \rangle}) = \exp \left\{ \frac{1}{2} \int_0^1 \int_0^1 \text{Cov}\left( \langle X(s), \psi(\cdot, s) \rangle, \langle X(t), \psi(\cdot, t) \rangle \right) ds dt \right\} \quad (4.1)$$

for every non-negative $\psi \in S(\mathbb{R}^d)$. We only consider the case of $\psi(x, t) = \phi(x)h(t)$.

It follows from (3.15), (3.19)–(3.21) and Lemma 3.1–3.4 that

$$\lim_{n \to \infty} \mathbb{E}(e^{-\langle X, \psi \rangle}) = \exp \left\{ \frac{1}{2(2\pi)^d} \int_0^1 h(t) dt \int_0^1 h(t) dt' \int_0^{t+t'} e^{-\theta s} ds \right. \times \left. \int_{\mathbb{R}^d} \frac{1}{\sum_{k=1}^d |z_k|^\alpha_k + \gamma/2 (\sum_{k=1}^d |z_k|^\alpha_k)^2} |\tilde{\phi}(z)|^2 d\tilde{z} \right\}. \quad (3.69)$$

Note that for the $S'(\mathbb{R}^d)$-valued process $X$ with covariance (2.9), we have

$$\text{Cov}\left( \langle X(s), \psi(\cdot, s) \rangle, \langle X(t), \psi(\cdot, t) \rangle \right) = \frac{1}{(2\pi)^d} \left[ \int_{\mathbb{R}^d} \left( \frac{2}{\sum_{k=1}^d |z_k|^\alpha_k + \gamma/2 (\sum_{k=1}^d |z_k|^\alpha_k)^2} \right) \tilde{\phi}_1(z) \tilde{\phi}_2(z) d\tilde{z} \right] \int_0^{t+t'} e^{-\theta s} ds \right] h(t) h(t').$$

Therefore, (4.1) holds.

For general $\psi \in S(\mathbb{R}^d)$, the proof is the same with slightly more complicated notation. The details are omitted and hence the proof of Theorem 2.1 is complete.

□

Proof of Theorem 2.2. The idea is same as that of Theorem 2.1. The details are omitted. □

Below we prove Theorem 2.3.

Proof of Theorem 2.3. We employ the space-time method formulated in Bojdecki et al. [1]. Following Bojdecki et al. [1], it suffices to show the following two claims.

(i) $\langle \tilde{X}_n, \psi \rangle$ converges in distribution to $\langle \tilde{X}, \psi \rangle$ for all $\psi \in S(\mathbb{R}^d)$ as $n \to \infty$.

(ii) $\{\langle X_n, \phi \rangle; n \geq 1\}$ is tight in $C([0, 1], \mathbb{R})$ for all $\phi \in S(\mathbb{R}^d)$, where the theorem of Mitoma [23] is used.

The proof of (i) is similar to that of Theorem 2.1. We sketch it briefly as follows. By (3.15), (3.19)–(3.21) and Lemma 3.1–3.4 we can readily get

$$\lim_{n \to \infty} \mathbb{E}(e^{-\langle \tilde{X}_n, \psi \rangle}) = \exp \left\{ \frac{\gamma}{2(2\pi)^d} \prod_{k=1}^d \Gamma(1/\alpha_k) \left( \int_{\mathbb{R}^d} \phi(x) dx \right)^2 \times \int_0^1 e^{-\theta u \tilde{h}(u)} du \int_0^1 e^{-\theta v \tilde{h}(v)} dv \int_0^{u+v} \frac{e^{\theta s} ds}{(u + v - 2s)^\alpha} \right\} \quad (3.69)$$

$$= \exp \left\{ \frac{\gamma}{2(2\pi)^d} \prod_{k=1}^d \Gamma(1/\alpha_k) \left( \int_{\mathbb{R}^d} \phi(x) dx \right)^2 \times \int_0^1 \theta(s) h(t) C(s, t) ds dt \right\}.$$
The last term is the right hand side of (4.1) for the process $X$ in Theorem 2.3 and $\psi(x, t) = \phi(x)h(t)$. This verifies (i).

Next we prove (ii). By Theorem 3.1 of Mimato [27] and the same argument as that used in the proof of [7, Proposition 3.3], it suffices to prove that for all $\phi \in \mathcal{S}(\mathbb{R}^d), 0 \leq t_1 < t_2 \leq 1$ and $\eta > 0$, there exist constants $a \geq 1, b > 0$ and $K > 0$, which is independent of $t_1, t_2$, such that for all $n \geq 1$.

$$
\int_{0}^{1/\eta} \left( 1 - \text{Re} \left( \mathbb{E} \left[ \exp \left( -i\omega \langle \tilde{X}_n, \phi \rangle \right) \right] \right) \right) d\omega \leq \frac{K}{\eta} (t_2 - t_1)^{1+b},
$$

where $h \in \mathcal{S}(\mathbb{R})$ is an approximation of $1_{\{t_2\}}(t) - 1_{\{t_1\}}(t)$ supported on $[t_1, t_2]$ such that $\tilde{h}(t)$ satisfies

$$
\tilde{h} \in \mathcal{S}(\mathbb{R}), \quad 0 \leq \tilde{h} \leq 1_{[t_1, t_2]}.
$$

We now repeat the argument in Section 3 with $\phi$ replaced by $i\omega \phi, \omega > 0$ and $\tilde{h}$ satisfying (4.3) and derive

$$
\mathbb{E} \left[ \exp \left\{ -i\omega \langle \tilde{X}_n, \phi \rangle \right\} \right] = \exp \left\{ I_1(n, i\omega \psi_n) + I_2(n, i\omega \psi_n) + I_3(n, i\omega \psi_n) \right\}
$$

and

$$
|V_{n, i\omega \psi_n}| \leq J_{n, i\omega \psi_n}.
$$

Consequently, from the expressions of $I_1, I_2, I_3$ and $I_{11}, (3.63)$ and (3.66), we can verify that

$$
|I_1(n, i\omega \psi_n)| \leq I_{11}(n, \omega \psi_n); \quad |I_2(n, i\omega \psi_n)| \leq \tilde{I}_2(n, \omega \psi_n);
$$

$$
|I_3(n, i\omega \psi_n)| \leq \tilde{I}_3(n, \omega \psi_n).
$$

In the following we estimate $I_{11}(n, \omega \psi_n), \tilde{I}_2(n, \omega \psi_n)$ and $\tilde{I}_3(n, \omega \psi_n)$ separately.

For $I_{11}(n, \omega \psi_n), (3.2), (3.12)$ and the boundeness of $\{f_k\}_{n \geq 1}$ imply that for some constant $K > 0$,

$$
I_{11}(n, \omega \psi_n) \leq \frac{K \omega^2}{F_n^2} \int_0^n \int_0^{n-s} \tilde{h}(s + u) du \int_0^{n-s} \tilde{h}(v + u) dv \int_{\mathbb{R}^d} T_u \phi(x) T_v \phi(x) dx
$$

$$
= \frac{2K \omega^2 n^3}{F_n^2} \int_0^1 d s \int_s^1 \tilde{h}(u) du \int_0^1 \tilde{h}(v) dv \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 e^{-n(u+v-2s)} \sum_{k=1}^d |z_k|^{\alpha_k} dz
$$

$$
= \frac{2K \omega^2 n^3}{F_n^2} \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 dz \int_0^1 \tilde{h}(u) du \int_0^1 \tilde{h}(v) dv \int_{\mathbb{R}^d} e^{-n(u+v-2s)} \sum_{k=1}^d |z_k|^{\alpha_k} ds,
$$

which, combined with (4.3), implies that $I_{11}(n, \psi_n)$ is bounded from above by

$$
\frac{2K \omega^2 n^3}{F_n^2} \int_{\mathbb{R}^d} |\tilde{\phi}(z)|^2 dz \int_{t_1}^{t_2} du \int_{t_1}^u e^{-n(u+v)} \sum_{k=1}^d |z_k|^{\alpha_k} dv \int_0^u e^{2n u} \sum_{k=1}^d |z_k|^{\alpha_k} ds
$$

$$
\leq \frac{K \omega^2 n^2}{F_n^2} \int_{\mathbb{R}^d} \sum_{k=1}^d |z_k|^{\alpha_k} dz \int_{t_1}^{t_2} du \int_{t_1}^u e^{-n(u-v)} \sum_{k=1}^d |z_k|^{\alpha_k} dv
$$

$$
= \frac{K \omega^2 n}{F_n^2} \int_{\mathbb{R}^d} \sum_{k=1}^d |z_k|^{\alpha_k} dz \int_{t_1}^{t_2} \frac{1 - e^{-n(u-v)}}{\sum_{k=1}^d |z_k|^{\alpha_k}} dv.
$$

Substituting $F_n^2 = n^{3-\alpha}$ and $y = n H z$ into (4.3), we get

$$
I_{11}(n, \omega \psi_n) \leq K \omega^2 \int_{\mathbb{R}^d} \frac{|\tilde{\phi}(n^{-H} y)|^2}{\sum_{k=1}^d |y_k|^{\alpha_k}} dy \int_{t_1}^{t_2} \frac{1 - e^{-n(u-v)}}{\sum_{k=1}^d |y_k|^{\alpha_k}} dv.
$$
By using the inequality $1 - e^{-x} \leq x^r$ for all $x \geq 0$ and $r \in (0, 1]$ and the boundedness of $|\widehat{\phi}|$, we obtain that,

$$I_{11}(n, \omega \psi_n) \leq K \omega^2 \|\widehat{\phi}\|^2 \left[ \int_{[0,1]^d} \frac{1}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r_1}} dz \int_{t_1}^{t_2} (u-t_1)^{r_1} du + K \int_{R^d \setminus [0,1]^d} \frac{1}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r}} dz \int_{t_1}^{t_2} (u-t_1)^{r} du \right],$$

where $\|\widehat{\phi}\| := \sup_{z \in R^d} |\widehat{\phi}(z)|$. Since $\bar{\alpha} \in (1, 2)$, Lemma 2.1 and Remark 2.3 imply that for any $r_1 \in (2 - \bar{\alpha}, 1)$ and $r \in (0, 2 - \bar{\alpha})$,

$$\int_{[0,1]^d} \frac{1}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r_1}} dz + \int_{R^d \setminus [0,1]^d} \frac{1}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r}} dz < \infty.$$

Therefore, there exist a constant $K > 0$ such that

$$I_{11}(n, \omega \psi_n) \leq K \omega^2 \|\widehat{\phi}\|^2 |t_2 - t_1|^{1+r}.$$  \hspace{1cm} (4.6)

Next we estimate $\tilde{I}_2(n, \omega \psi_n)$. Due to the boundedness of $\tilde{f}_n$, (3.64) implies that for some constant $K > 0$

$$\tilde{I}_2(n, \omega \psi_n) \leq \frac{K \omega^2 \nu}{F_{n}^2} \int_0^n \hat{h}(s) ds \int_0^{n-s} \hat{h}(s+u) du \int_{R^d} \phi(x)t_u \phi(x) dx.$$

Then by condition (4.3), we have

$$\tilde{I}_2(n, \omega \psi_n) \leq \frac{K \omega^2 \nu}{F_{n}^2} \int_{t_1}^{t_2} ds \int_{s}^{t_2} du \int_{R^d} |\widehat{\phi}(z)|^2 e^{-n(u-s)} \sum_{k=1} \delta \gamma_k e^{-\nu \delta \gamma_k \gamma_k} \int_{t_1}^{t_2} (u-t_1)^{r} du.$$

Using the inequality $1 - e^{-x} \leq x^r$ for all $x \geq 0$ and $r \in (0, 1]$ again, and substituting $F_{n}^2 = n^{3-\bar{\alpha}}$ into (4.7), we get that,

$$\tilde{I}_2(n, \omega \psi_n) \leq \frac{K \omega^2 n^{3+r}}{n^{3-\bar{\alpha}}} \int_{R^d} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{1-r}} dz \int_{t_1}^{t_2} (u-t_1)^{r} du.$$

Let $r = 2 - \bar{\alpha}$. Then $1 - r = \bar{\alpha} - 1 \in (0, 1)$ and $\int_{R^d} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{1-r}} dz < \infty$, thanks to Remark 2.3. Therefore there exists a constant $K > 0$ such that

$$\tilde{I}_2(n, \omega \psi_n) \leq K \omega^2 \int_{R^d} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{1-r}} dz |t_2 - t_1|^{1+r}.$$  \hspace{1cm} (4.8)

In order to estimate $\tilde{I}_3(n, \omega \psi_n)$, we combine (3.68) and (4.3) to see that for all $r \in (0, 1)$,

$$\tilde{I}_3(n, \omega \psi_n) \leq \frac{\omega^2 \nu}{F_{n}^2 (2\pi)^d} \int_0^n ds \int_0^{n-s} \int_0^u \int_u^{n-s} d\nu \int_0^{n-u} \int_0^{n-u} \hat{h}(s+v) \hat{h}(v) dv du dv.$$
From \( n\delta_n \to \theta \in [0, \infty) \), \( F_n^2 = n^{3-\alpha} \) and \( 1 < \bar{\alpha} < 2 \) we derive that
\[
0 < \frac{\gamma n^{1+r}\delta_n}{\gamma^2 F_n^2} \leq \frac{\gamma n^{3-r-\alpha}}{\gamma^2 n^{3-\alpha}} \to 0.
\]
Therefore, for any \( r \in (0, 1) \) there exists a constant \( K \) such that
\[
\bar{I}_3(n, \omega\psi_n) \leq K\omega^2|t_2 - t_1|^{1+r}.
\] (4.9)

Combining (4.8) with (4.10) and (4.9), we have that for some \( \bar{r} \in (0, 2 - \bar{\alpha}) \), there is a constant \( K \) independent of \( t_1, t_2 \) and \( r > 0 \) such that
\[
|\bar{I}_3(n, \omega\psi_n)| + |\bar{I}_2(n, \omega\psi_n)| + |I_{11}(n, \omega\psi_n)| \leq K(\phi, r)\omega^2|t_2 - t_1|^{1+r}.
\] (4.10)

Note that
\[
|1 - \text{Re}(\mathbb{E}[\exp \{-i\omega(\bar{X}_n, \phi h)\}])| \leq |I_1(n, i\omega\psi_n)| + |I_2(n, i\omega\psi_n)| + |I_3(n, i\omega\psi_n)|,
\]
we derive from (1.4) and (4.10) that
\[
\int_0^{1/\eta} \left(1 - \text{Re}(\mathbb{E}[\exp \{-i\omega(\bar{X}_n, \phi h)\}])\right) d\omega \leq \frac{K(\phi, r)}{3\eta^2}|t_2 - t_1|^{1+r}.
\]
This completes the proof of (1.2) and hence the proof of Theorem 2.1. \( \square \)

At last, we prove Proposition 2.1.

**Proof of Proposition 2.1.** We only prove the statements on \( Y_1 \). The remainder is similar and omitted.

By the definition of \( Y_1 = \{Y_1(u), u \in [0, \infty)^d\} \), it is a centered Gaussian random field with covariance function given by
\[
\mathbb{E}[Y_1(u)Y_1(v)] = \frac{2}{(2\pi)^d} \int_0^1 e^{-\theta s} ds \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{dz}{|z_k|^{\alpha_k}} \int_{D(u)} \int_{D(v)} e^{-\sum_{k=1}^d (x_k - y_k) z_k} dxdy
\]
\[
= K \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{(1 - e^{iu_k z_k})(1 - e^{-iv_k z_k})}{z_k^2} \frac{dz}{\sum_{k=1}^d |z_k|^{\alpha_k}}.
\] (4.11)

Here and below, \( K = 2(1-e^{-\theta})/(2\pi)^d \theta \). Note that the last integral is finite because \( \bar{\alpha} > 2 \).

(1) If \( d = 1 \), then \( \bar{\alpha} = (\alpha_1) =: \alpha < 1/2 \). It follows from (1.11) that for any \( u, v \in \mathbb{R}_+ \),
\[
\mathbb{E}[Y_1(u)Y_1(v)] = K \int_{\mathbb{R}} \frac{(1 - e^{iu z})(1 - e^{-iv z})}{|z|^{2+\alpha}} dz
\]
\[
= K \int_{\mathbb{R}} \frac{1 - \cos(u z) - \cos(v z) + \cos((u - v) z)}{|z|^{2+\alpha}} dz
\]
\[
= K \left[ u^{1+\alpha} + v^{1+\alpha} - |u - v|^{1+\alpha} \right] \int_{\mathbb{R}} \frac{1 - \cos x}{|x|^{2+\alpha}} dx.
\]

It is well-known that the fractional Brownian motion \( B_h \) with Hurst exponent \( h \in (0, 1) \) is a one-parameter centered Gaussian process with covariance
\[
\mathbb{E}(B_h(u)B_h(v)) = \frac{1}{2} \left( u^{2h} + v^{2h} - |u - v|^{2h} \right).
\]

Therefore, up to a multiplicative constant, \( Y_1 \) is the fractional Brownian motion with Hurst exponent \( (1 + \alpha)/2 \in (1/2, 3/4) \).
Therefore, for any $r > 0$, it follows from (4.11) and the substitution $x = r^{\frac{1}{\alpha}}z$ that

$$
E[Y_1(r^H u)Y_1(r^H v)] = K \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{(1 - e^{ir^{1/\alpha_k}u_k z_k})(1 - e^{-ir^{1/\alpha_k}v_k z_k})}{z_k^2} \frac{dz}{\sum_{k=1}^d |z_k|^{\alpha_k}}
$$

$$
= r^{1+\bar{\alpha}} K \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{(1 - e^{iu_k x_k})(1 - e^{-iv_k x_k})}{x_k^2} \frac{dx}{\sum_{k=1}^d |x_k|^{\alpha_k}}
$$

$$
= r^{1+\bar{\alpha}} E[Y_1(u)Y_1(v)].
$$

Therefore, for any $r > 0$,

$$
\{Y_1(r^H u), u \in [0, \infty)^d\} \overset{f.d}{=} \{r^{(\bar{\alpha}+1)/2}Y_1(u), u \in [0, \infty)^d\},
$$

That is, $Y_1$ is an operator scaling random field with exponent $H$. Let $v_1 = (0, 0, \ldots, 0)$, $v_2 = (1, 0, \ldots, 0)$ and $u_1 = (0, 1, \ldots, 1)$, $u_2 = (1, 1, \ldots, 1)$. Then $u_1 - v_1 = u_2 - v_2$ and (4.11) gives

$$
E[(Y_1(v_1))^2] = E[(Y_1(u_1))^2] = E[(Y_1(v_2))^2] = E[Y_1(u_2)Y_1(v_2)] = 0,
$$

Hence

$$
E[(Y_1(u_2) - Y_1(v_2))^2] = E[(Y_1(u_2))^2]
$$

$$
= K \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{(1 - e^{ix_k})(1 - e^{-ix_k})}{x_k^2} \frac{dx}{\sum_{k=1}^d |x_k|^{\alpha_k}}
$$

$$
\neq 0 = E[(Y_1(u_1))^2] = E[(Y_1(u_1) - Y_1(v_1))^2],
$$

which implies that $Y_1$ does not have stationary increments. \hfill \square

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