Forecasting stock options prices via the solution of an ill-posed problem for the Black–Scholes equation

Michael V Klibanov\textsuperscript{1,*}, Aleksander A Shananin\textsuperscript{2}, Kirill V Golubnichiy\textsuperscript{3} and Sergey M Kravchenko\textsuperscript{2}

\textsuperscript{1} Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, United States of America
\textsuperscript{2} Department of Analysis of Systems and Solutions, Moscow Institute of Physics and Technology, Moscow, 117303, Russia
\textsuperscript{3} Department of Mathematics, University of Washington, Seattle, WA 98195, United States of America

E-mail: mklibanv@uncc.edu, alexshan@yandex.ru, kgolubni@math.washington.edu and kravchukov1998@gmail.ru

Received 27 February 2022, revised 1 September 2022
Accepted for publication 14 September 2022
Published 30 September 2022

Abstract
In the previous paper (2016 Inverse Problems 32 015010), a new heuristic mathematical model was proposed for accurate forecasting of prices of stock options for 1–2 trading days ahead of the present one. This new technique uses the Black–Scholes equation supplied by new intervals for the underlying stock and new initial and boundary conditions for option prices. The Black–Scholes equation was solved in the positive direction of the time variable, this ill-posed initial boundary value problem was solved by the so-called quasi-reversibility method (QRM). This approach with an added trading strategy was tested on the market data for 368 stock options and good forecasting results were demonstrated. In the current paper, we use the geometric Brownian motion to provide an explanation of that effectiveness using computationally simulated data for European call options. We also provide a convergence analysis for QRM. The key tool of that analysis is a Carleman estimate.

Keywords: Black–Scholes equation, European call options, geometric Brownian motion, probability theory, ill-posed problem, quasi-reversibility method, Carleman estimate

*Author to whom any correspondence should be addressed.
1. Introduction

A new heuristic mathematical algorithm designed to forecast prices of stock options was proposed in [9]. This algorithm is based on the so-called quasi-reversibility method (QRM). QRM is a regularization method for an ill-posed problem for the Black–Scholes equation. The goal of this paper is to address both analytically and numerically the following question: Why this algorithm has worked well for real market data in [9, 12]? Our explanations are based on our new analytical results in the probability theory and are supported by our numerical results for the computationally simulated data generated by the geometrical Brownian motion.

A significant advantage of the technique of [9] is that it uses historical data about stock and option prices only over short time intervals. This assumption is a practically valuable one since formations of those prices are random processes. This indicates that the information used in the algorithm possesses stable probabilistic characteristics.

The mathematical model of [9] was supplied by a trading strategy. Results of [9, table 4] for real market data of [3] indicate that a combination of that mathematical model with that trading strategy has resulted in 72.83% profitable options out of 368 options for real market data. More recently, the model of [9] was used in [12] to forecast stock option prices in the case when results of QRM are enhanced by the machine learning approach, which was applied on the second stage of the price forecasting procedure. Market data of [3] for total 169,862 European call stock options were used in [12]. Following the machine learning approach, these data were divided in three sets [12, table 1]: training (132,912 options), validation (13,401 options) and testing (23,549 options). Total 23,549 options were tested by QRM, and good results on predictions of options with profits were obtained in [12, first lines in tables 2 and 3]. Later the authors of [12] have tested the performance of QRM for all those 169,862 options, and results were almost the same as ones of [12, first lines in tables 2 and 3]. However, since the latter results are not yet published, then we do not discuss them here.

Remark 1.1. Without further specifications, we consider in this paper only European call options. The mathematical model of [9] does not use neither the payment function at the expiry time nor the strike price.

We now present in tables 1 and 2 the most recent results of [12], which were obtained using the method of [9] for the data consisting of 23,549 historical trades collected in 2018. The same market data of [3] were used in tables 1 and 2. Option prices for one trading day ahead of the present day were forecasted. Definitions of accuracy, precision and recall are well known, see, e.g. [7].

In table 1, ‘Error’ means the average relative error of predictions of option prices, i.e.

\[
\text{Error} = \frac{1}{N} \sum_{n=1}^{N} \frac{|p_{n,\text{corr}} - p_{n,\text{fc}}|}{p_{n,\text{corr}}} \cdot 100\% ,
\]

where \( N = 23,549 \) is the total number of tested options, \( p_{n,\text{corr}} \) and \( p_{n,\text{fc}} \) are correct and forecasted prices respectively of the option number \( n \).

A perfect financial market does not allow a winning strategy [6]. This means that to address the above question, we need to assume that the market is imperfect. The present article considers a model situation, in which there is a difference between the volatility \( \sigma \) of the underlying stock and traders’ opinion \( \hat{\sigma} \) of the volatility of an European call option generated by this stock.
We prove analytically that, theoretically, this allows one to design a winning strategy. First, we back up this theory numerically for the ideal case when both volatilities are known. In practice, however, only $\hat{\sigma}$ is approximately known from [3], where implied volatility $\sigma_{\text{impl}}$ of option prices is posted. It is reasonable to conjecture that $\hat{\sigma} \approx \sigma_{\text{impl}}$.

Second, to address the question posed in the first paragraph of this section for the nonideal case, we consider a mathematical model, in which the dynamics of the stock prices is generated by the stochastic differential equation of the geometric Brownian motion. This allows us to computationally generate the time series of stock prices. At the same time, we assume that the price of the corresponding stock option is governed by the Black–Scholes equation, in which the volatility coefficient is $\sigma$. Hence, using that time series of stock prices, we apply the Black–Scholes formula to get the time series for prices of the corresponding options. Next, we apply the QRM to predict the prices of these options for one trading day ahead of the current one. Next, we formulate the winning strategy for the nonideal case.

Both the theory and the numerical studies of this paper support our two hypotheses formulated in subsection 6.3. Our first hypothesis that the heuristic algorithm of [9] actually figures out in many cases the sign of the difference $\sigma - \hat{\sigma}$. Our second hypothesis is also based on our results below as well as on the 'Precision' column of Table 1 and the second column of Table 2. More precisely, the second hypothesis is that probably about 56% of tested 23,549 options of [12] with the real market data had $\sigma - \hat{\sigma} < 0$.

This algorithm of [9] is based on the solution of a new initial boundary value problem (IBVP) for the Black–Scholes equation, see, e.g. [2, 19] for this equation. Since the Black–Scholes equation is actually a 1D parabolic partial differential equation (PDE) with the reversed time, then that IBVP is ill-posed, see, e.g. [9] for an example of a high instability of a similar problem. The ill-posedness of that IBVP is the main mathematical obstacle of that algorithm. Therefore, we solve that IBVP both here and in [9] by a specially designed version of QRM. QRM stably solves this problem forwards in time for two consecutive trading days after the current one. QRM is a version of the Tikhonov regularization method [18] being adapted to ill-posed problems for partial differential equations (PDEs). We refer to [15] for the first publication on QRM as well as to [4, 5, 8, 10, 11] for some more recent ones.

We provide a convergence analysis for QRM being applied to the above problem. The main new element of this analysis is that we lift a restrictive assumption of [9] of a sufficiently small time interval. We note that the smallness assumption imposed on the time interval is a traditional one for IBVPs for parabolic PDEs with the reversed time, see [8], [16, theorem 1 of section 2 in chapter 4], where a certain Carleman estimate was used. However, a new Carleman estimate was derived in [10] for a general parabolic operator of the second order with variable
coefficients in the \( n-D \) case. This estimate enables one to lift that smallness assumption. We simplify here the Carleman estimate of [10] as well as some other results of [10] via adapting them to our simpler 1D case, as compared with the \( n-D \) case of [10].

The Black–Scholes equation describes the dependence of the price \( v(s, t) \) of a stock option from the price of the underlying stock \( s \) and time \( t \) [1, 2, 19]. In fact, this is a parabolic PDE with the reversed time. Let \( t = T \) be the maturity time and \( t = 0 \) is the present time [19]. Traditionally, IBVPs for the Black–Scholes equation are solved backwards with respect to time \( t \in (0, T) \) with the initial condition at \( \{ t = T \} \). The latter is a well posed problem, for which the classic theory of parabolic PDEs works, see, e.g. the book [14] for this theory.

However, the maturity time \( T \) is usually a few months away from the present time. It is obviously impossible to accurately predict the future behavior of the volatility coefficient of the Black–Scholes equation on such a large time interval. Since the formations of both stock and option prices are stochastic processes, then it is intuitively clear a good accuracy of forecasting of stock option prices for long time periods is unlikely.

Thus, we focus in this paper on forecasting of option prices for a short time period of just one trading day ahead of the current one. Let the time variable \( t \) counts trading days. Since there are 255 trading days annually, then we introduce the dimensionless time \( t' \) as

\[
t' = \frac{t}{255}. \tag{1.1}
\]

Hence,

\[
\text{one (1) dimensionless trading day} = 1/255 \approx 0.00392 \ll 1. \tag{1.2}
\]

**Remark 1.2.** To simplify notations, we still use everywhere below the notation \( t \) for the dimensionless time \( t' \) of (1.1).

**Remark 1.3.** There are many important questions about the technique of [9], which are not addressed in this paper, such as, e.g. the performance of this technique for some ‘stress’ tests, its performance for significantly larger sets of market data, its performance for the case when the transaction cost is taken into account, and many others. However, addressing any of those questions would require a significant additional effort. Therefore, those questions are outside of the scope of this publication. Still, the question of the transaction cost might probably be addressed using a threshold number \( \eta > 0 \) in our trading strategy for the non-ideal case, see subsection 6.3.

This paper is organized as follows. In section 2 we show that a winning strategy on an infinitesimal time interval might be possible if \( \sigma \neq \hat{\sigma} \). In section 3 we present the heuristic mathematical model of [9]. In section 4 we present a convergence analysis for our version of QRM. In section 5 we use arguments of the probability theory to justify our trading strategy in the ideal case when both volatilities \( \sigma \) and \( \hat{\sigma} \) are known. In section 6 we describe our numerical studies and end up with a trading strategy for the nonideal case when only the volatility \( \hat{\sigma} \) is known. In addition, we formulate in section 6 our two hypotheses mentioned above. Concluding remarks are given in section 7.

**Disclaimer.** This paper is written for academic purposes only. The authors do not provide any assurance that the technique of this paper would result in a successful trading on a real financial market.

## 2. A possible winning strategy

Let \( \sigma \) be the volatility of a certain stock and \( s \) be the price of this stock. Consider an option corresponding to this stock. Let \( \hat{\sigma} \) be an idea of the volatility of that option, which has been
developed among the agents, who trade this option on the market. If \( \sigma \neq \hat{\sigma} \), then the financial market is imperfect, and an opportunity for designing a winning strategy exists.

At a given time \( t \), the time until the maturity will occur is \( \tau \),

\[
\tau = T - t.
\]

(2.1)

Let \( s \) be the stock price and \( f(s) \) be the payoff function of that option at the maturity time \( t = T \). We assume that the risk-free interest rate is zero. Let \( u(s, \tau) \) be the price of that option and the variable \( \tau \) is the one defined in (2.1). We assume that the function \( u(s, \tau) \) satisfies the Black–Scholes equation with the volatility coefficient \( \hat{\sigma} \) [1, chapter 7, theorem 7.7]:

\[
\frac{\partial u(s, \tau)}{\partial \tau} = \frac{\hat{\sigma}^2}{2} \frac{\partial^2 u(s, \tau)}{\partial s^2}, \quad s > 0,
\]

(2.2)

\( u(s, 0) = f(s) \).

The specific formula for the payoff function is \( f(s) = \max(s - K, 0) \), where \( K \) is the strike price [1]. Then the price function \( u(s, \tau) \) of the option is given by the Black–Scholes formula [1]:

\[
u(s, \tau) = s \Phi(\Theta_+(s, \tau)) - e^{-\tau r} K \Phi(\Theta_-(s, \tau)),
\]

(2.3)

where \( r = 0 \) and

\[
\Theta_{\pm}(s, \tau) = \frac{1}{\hat{\sigma} \sqrt{\tau}} \left[ \ln \left( \frac{s}{K} \right) \pm \frac{\hat{\sigma}^2 \tau}{2} \right],
\]

(2.4)

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2 / 2} \, dt, \quad z \in \mathbb{R}.
\]

Let \( u(s, t) = u(s, T - t) \). The stochastic equation of the geometric Brownian motion for the stock price \( s \) with the volatility \( \sigma \) has the form \( ds = \sigma s \, dW \), where \( W \) is the Wiener process.

The Itô formula implies

\[
\frac{dv}{dt} = \left( -\frac{\partial u(s, T - t)}{\partial \tau} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 u(s, T - t)}{\partial s^2} \right) \, dt + \sigma s \frac{\partial u(s, T - t)}{\partial s} \, dW,
\]

(2.5)

where \( dv \) is the option price change on an infinitesimal time interval and \( dW \) is the Wiener process.

Replacing in (2.5) \( \partial u(s, T - t) \) with the right-hand side of (2.2), we obtain

\[
\frac{dv}{dt} = \frac{\left( \sigma^2 - \hat{\sigma}^2 \right)}{2} s^2 \frac{\partial^2 u(s, T - t)}{\partial s^2} \, dt + \sigma s \frac{\partial u(s, T - t)}{\partial s} \, dW.
\]

(2.6)

The mathematical expectation of \( dW \) is zero [1, chapter 4]. Therefore, we find that the expected value of the increment of the option price on an infinitesimal time interval is

\[
\frac{\left( \sigma^2 - \hat{\sigma}^2 \right)}{2} s^2 \frac{\partial^2 u(s, T - t)}{\partial s^2} \, dt.
\]

(2.7)

In the mathematical finance, the second derivative

\[
\frac{\partial^2 u(s, \tau)}{\partial s^2}
\]

(2.8)
is called Greek $\Gamma(s, \tau)$. For an European call option [1, chapter 9]

$$
\Gamma(s, \tau) = \frac{1}{\sigma s \sqrt{2\pi \tau}} \exp\left[-\frac{(\Theta(s, \tau))^2}{2}\right] > 0.
$$

Therefore, it follows from (2.7)–(2.9) that the sign of the mathematical expectation of the increment of the option price on an infinitesimal time interval is determined by the sign of the difference $\sigma^2 - \hat{\sigma}^2$. Thus, if $\sigma^2 > \hat{\sigma}^2$, then a possible winning strategy involves buying an option at the present time and selling it in the next trading period. If $\sigma^2 < \hat{\sigma}^2$, then the winning strategy is to take the short position at the present time and to close the short position in the next trading period.

3. The mathematical model

3.1. The model

We now describe the mathematical model of [9]. We use this model here for computationally simulated data. Also, it was used in [12] for real market data to obtain the above tables 1 and 2. We do not differentiate in this model between volatilities $\sigma$ and $\hat{\sigma}$ and just use the time dependent volatility $\sigma(t)$.

Everywhere below, as the dimensionless time, we still use the notation $t$ for $t'$ in (1.1) for brevity. Let $\sigma(t)$ be the volatility of the option at the moment of time $t$. When working with the market data in [9,12], we have used the historical implied volatility listed on the market data of [3]. Let $v_b(t)$ and $v_a(t)$ be respectively the bid and ask prices of the option and $s_b(t)$ and $s_a(t)$ be the bid and ask prices of the stock. It is known that $v_b(t) < v_a(t)$ and $s_b(t) < s_a(t)$.

For brevity, we simplify notations as $s_b = s_b(0)$, $s_a = s_a(0)$. We impose a natural assumption that $0 < s_b < s_a$.

It was observed on the market data in [9, formulas (2.3)–(2.6)] that the relative differences are usually small,

$$
\left|\frac{s_b(t)}{s_a(t)} - 1\right| \leq 0.03, \quad \left|\frac{v_b(t)}{v_a(t)} - 1\right| \leq 0.27.
$$

Hence, we define the initial condition $q(s)$ at $t = 0$ of the function $v(s, t)$ as the linear interpolation on the interval $s \in (s_b, s_a)$ between $v_b(0)$ and $v_a(0)$,

$$
v(s, 0) = q(s) = -\frac{s - s_a}{s_a - s_b} v_b(0) + \frac{s - s_b}{s_a - s_b} v_a(0), \quad s \in (s_b, s_a).
$$

Define the domain $Q_T = \{(s, t) \in (s_b, s_a) \times (0, T)\}$. We assume that the volatility of the option depends only on $t$, i.e. $\sigma = \sigma(t) \equiv \sigma_0 = \text{const.} > 0$. Let $L$ be the partial differential operator of the Black–Scholes equation,

$$
L v = \frac{\partial v}{\partial t} + \frac{\sigma^2(t)}{2} \frac{\partial^2 v}{\partial s^2} = 0 \quad \text{in} \ Q_T.
$$

We impose the following initial and boundary conditions on the function $v(s, t)$:

$$
v(s, 0) = q(s), \quad s \in (s_b, s_a),
$$
Conditions (3.3)–(3.5) represent the heuristic mathematical model of [9, formulas (2.3)–(2.6)]. Also, (3.3)–(3.5) is our IBVP for the Black–Scholes equation. We now formulate this as problem 1:

**Problem 1.** Find the function \( v \in H^2(Q_T) \) satisfying conditions (3.3)–(3.5).

Problem 1 is ill-posed since we need to solve equation (3.3) forwards in time.

**Remarks 3.1.**

(a) The conventional model for the Black–Scholes equation stresses on the maturity time \( T \) via considering the function \( u(s, t) = v(s, T - t) \) instead of the function \( v(s, t) \). Unlike this, we are not doing so in (3.3)–(3.5) since we do not need the maturity time, also, see remark 1.1.

(b) As it is a conventional way in the theory of ill-posed problems, we increase here the required smoothness of the solution from \( H^{2,1}(Q_T) \) to \( H^2(Q_T) \).

3.2. Three steps

In order to solve problem 1, we need first to define the time dependent option’s volatility \( \sigma(t) \), and boundary conditions \( v_b(t), v_a(t) \). Then the initial condition \( q(s) \) in (3.4) would be found via (3.4). We explain these in steps 1 and 2 of this subsection 3.2.

In our computations of [9, 12] we have used the implied volatility of the options in the last trade price of the day for \( \sigma(t) \) [3]. As to \( s_b \) and \( s_a \), we have used the end of the day underlying price ask and the end of day underlying price bid of [3]. Similarly for \( v_b(t) \) and \( v_a(t) \), in which case the end of the day option price ask and the end of day option price bid of [3] were used. The moment of time \( t = 0 \) is the end of the present day time, and similarly for the following two trading days of \( t = y, 2y \) and for the preceding two trading days \( t = -y, -2y \). Naturally, the question can be raised here on how did we find future values of boundary conditions \( v_b(t) \) and \( v_a(t) \) for \( t \in (0, 2y) \) in (3.5), and the same for \( \sigma(t) \). This question is addressed in step 2 below. Our method for the solution of problem 1 consists of three steps:

**Step 1 (introducing dimensionless variables).** First, we make equation (3.3) dimensionless. Recall that \( s_b < s_a \). Introduce the dimensionless variable \( x \) for \( s \) as:

\[
s \leftrightarrow x = \frac{s - s_b}{s_a - s_b}.
\]

Let \( y \) denotes one dimensionless trading day. By (1.2)

\[
y = \frac{1}{255} \approx 0.00392. \tag{3.6}
\]

By (3.2) the function \( q(s) \) is transformed in the function \( g(x) \),

\[
g(x) = (1 - x)v_b(0) + xv_a(0). \tag{3.7}
\]

And the operator \( L \) in (3.3) is transformed in the operator \( M \),

\[
Mv = v_t + \sigma^2(t)A(x)v_{xx}, \tag{3.8}
\]
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\[
A(x) = \frac{255}{2} \frac{[x(s_a - s_b) + s_b]^2}{(s_a - s_b)^2},
\]

(3.9)

\[
G_{2y} = \{(x, t) \in (0, 1) \times (0, 2y)\}.
\]

(3.10)

Problem is transformed in problem 2:

Problem 2. Assume that functions

\[
v_b(t), v_a(t) \in H^2[0, 2y], \sigma(t) \in C^1[0, 2y].
\]

(3.11)

Find the solution \(u \in H^2(G_{2y})\) of the following IBVP:

\[
Mv = 0 \quad \text{in } G_{2y},
\]

(3.12)

\[
v(0, t) = v_b(t), v(1, t) = v_a(t), \quad t \in (0, 2y),
\]

(3.13)

\[
v(x, 0) = g(x), \quad x \in (0, 1),
\]

(3.14)

where the partial differential operator \(M\) is defined in (3.8), the function \(A(x)\) is defined in (3.9), the initial condition \(g(x)\) is defined in (3.7), and the domain \(G_{2y}\) is defined in (3.10).

**Step 2 (interpolation and extrapolation).** Having the historical market data for an option up to ‘today’, we forecast the option price for ‘tomorrow’ and ‘the day after tomorrow’, with 255 trading days annually. ‘One day’ corresponds to \(y = 1/255\). ‘Today’ means \(t = 0\). ‘Tomorrow’ means \(t = y\). ‘The day after tomorrow’ means \(t = 2y\). We forecast these prices for the \(s\)-interval as \(s \in [s_b(0), s_a(0)]\) via the solution of problem (3.12) and (3.13). To do this, however, we need to know functions \(v_b(t), v_a(t)\) and \(\sigma(t)\) in the ‘future’, i.e. for \(t \in (0, 2y)\). We obtain approximate values of these functions via interpolation and extrapolation procedures described in the next paragraph.

Let \(t = -2y\) be ‘the day before yesterday’, \(t = -y\) be ‘yesterday’ and \(t = 0\) be ‘today’. Let \(d(t)\) be any of three functions \(v_b(t), v_a(t), \sigma(t)\). First, we interpolate the function \(d(t)\) by the quadratic polynomial for \(t \in [-2y, 0]\) using the values \(d(-2y), d(-y), d(0)\). We obtain

\[
d(t) = at^2 + bt + c \quad \text{for } t \in [-2y, 0].
\]

(3.15)

Next, we extrapolate (3.15) on the interval \(t \in [0, 2y]\) via setting

\[
d(t) = at^2 + bt + c \quad \text{for } t \in [0, 2y].
\]

The so defined functions \(v_b(t), v_a(t), \sigma(t)\) were used to numerically solve problem (3.12) and (3.13) for both the computationally simulated data below and for real market data of tables 1 and 2 above as well as in [9].

**Step 3 (Numerical solution of problem 2. Regularization).** Since problem (3.3)–(3.5) is ill-posed, then we apply a regularization method to obtain an approximate solution of this problem. More precisely, we solve the following problem:

**Minimization problem 1.** Let \(J_{\alpha} : H^2(G_{2y}) \to \mathbb{R}\) be the regularization Tikhonov functional defined as:
\[
J_\alpha(v) = \int_{G_2} (Mv)^2 ds \, dt + \alpha \|v\|_{H^2(G_2)}^2, \tag{3.16}
\]

where \( \alpha \in (0,1) \) is the regularization parameter. Minimize functional (3.16) on the set \( S \), where

\[
S = \{ v \in H^2(G_2) : v(0,t) = v_b(t), v(1,t) = v_a(t), v(x,0) = g(x) \}. \tag{3.17}
\]

Minimization problem 1 is a version of QRM for problem 2. This version is an adaptation of the QRM for problem (3.12) and (3.13). In section 4 we present the theory of this specific version of the QRM. In particular, theorem 4.2 of section 4 implies uniqueness of the solution \( u \in H^2(G_2) \) of problem 2 and provides an estimate of the stability of this solution with respect to the noise in the data. Theorem 4.3 of section 4 implies existence and uniqueness of the minimizer \( v_\alpha \in H^{2,1}(G_2) \) of the functional \( J_\alpha(v) \) on the set \( S \) defined in (3.17). Following the theory of ill-posed problems, we call such a minimizer ‘regularized solution’ [18]. Theorem 4.4 estimates convergence rate of regularized solutions to the exact solution of problem 2 with the noiseless data. These estimates depend on the noise level in the data.

4. Convergence analysis

In this section, we provide convergence analysis for problem 2 of subsection 3.2. This problem is the IBVP for parabolic equation (3.12) with the reversed time, see (3.8). The QRM for this problem for a more general parabolic operator in \( \mathbb{R}^n \) with arbitrary variable coefficients was proposed in [8] and convergence analysis was also carried out there. Then corresponding theorems were reformulated in [9]. Although a stability estimate was not a part of [9], such an estimate was proven in [8]. It was pointed out in introduction, however, that traditional stability estimates for this problem were proven, using a certain Carleman estimate, only under the assumption that the time interval is sufficiently small. The same is true for the convergence theorems of QRM in [8, 9]. Unlike this, the smallness assumption was lifted in [10] via a new Carleman estimate. In this section, we significantly modify results of [10] for a simpler 1D case. Recall (see introduction) that this modification allows us to obtain more accurate estimates in the 1D case, as compared with the \( n-D \) case of [10]. We note that even though we work in our computations below on a small time interval \((0, 2y) = (0, 0.00784)\) (see (3.6) and (3.10)), the smallness assumption of [8, 9], [16, theorem 1 of section 2 in chapter 4] might result in the requirement of even a smaller length of that interval.

4.1. Problem statement

Consider a number \( T_1 > 0 \) and denote

\[
Q_{T_1} = \{(x, t) \in (0, 1) \times (0, T_1)\}.
\]

Let two numbers \( a_0, a_1 > 0 \) and \( a_0 < a_1 \). Let the function \( a(x, t) \in C^1(\overline{Q_{T_1}}) \) satisfies:

\[
\|a\|_{C^1(\overline{Q_{T_1}})} \leq a_1, \quad a(x, t) \geq a_0 \quad \text{in} \quad Q_{T_1}. \tag{4.1}
\]

Let functions \( \varphi_0(t), \varphi_1(t) \in H^2(0, T_1) \). In the above case of subsection 3.2,

\[
T_1 = 2y, a(x, t) = \sigma^2(t)A(x), \varphi_0(t) = v_b(t), \varphi_1(t) = v_a(t).
\]

We now formulate problem 3, which is a slight generalization of problem 2.
**Problem 3.** Find a solution \( w \in H^2(Q_{T_1}) \) of the following IBVP:

\[
P w = w_t + a(x,t)w_{xx} = 0 \quad \text{in } Q_{T_1},
\]

\[
w(0, t) = \varphi_0(t), \ w(1, t) = \varphi_1(t), \quad t \in (0, T_1),
\]

\[
w(x, 0) = q(x) = \varphi_0(0)(1 - x) + \varphi_1(0)x, \quad x \in (0, 1).
\]

**Remark 4.1.** Since problem 3 is a more general one than problem 2, then our convergence analysis for problem 3, which we provide below, is also valid for problem 2.

The reason why we use the linear function for \( w(x, 0) \) in (4.4) is our desire to simplify the presentation by using the fact that, in the case of problem 2, the initial condition in (3.14) is the linear function defined in (3.7). Problem 3 is an IBVP for the parabolic equation (4.2) with the reversed time. Therefore, this problem is ill-posed. Just as it is always the case in the theory of ill-posed problems [18], we assume that the boundary in (4.3) are given with a noise of the level \( \delta > 0 \), where \( \delta \) is a sufficiently small number, i.e.

\[
\| \varphi_0 - \varphi_0^\ast \|_{H^1(0,T_1)} < \delta, \quad \| \varphi_1 - \varphi_1^\ast \|_{H^1(0,T_1)} < \delta,
\]

where functions \( \varphi_0^\ast, \varphi_1^\ast \in H^2(0,T_1) \) are ‘ideal’ noiseless data. Following to one of postulates of the theory of ill-posed problems, we assume that there exists an exact solution \( w^* \in H^2(Q_{T_1}) \) of problem (4.2)–(4.4) with these noiseless data. We will estimate below how this noise affects the accuracy of the solution of problem 3 (if this solution exists) and also will establish the convergence rate of numerical solutions obtained by QRM to the exact one as \( \delta \to 0 \).

Consider the following analog of functional (3.16):

\[
I_\alpha(w) = \int_{Q_{T_1}} (Pw)^2 \, dx \, dt + \alpha \| w \|_{H^2(Q_{T_1})}^2.
\]

Introduce the set \( Y \subset H^2(Q_{T_1}) \),

\[
Y = \{ w \in H^2(Q_{T_1}) : w(0, t) = \varphi_0(t), \ w(1, t) = \varphi_1(t), \ w(x, 0) = q(x) \}.
\]

We construct an approximate solution of problem 3 via solving the following problem:

**Minimization problem 2.** Minimize the functional \( I_\alpha(w) \) on the set \( Y \) given in (4.7).

Similarly with the minimization problem 1, minimization problem 2 means QRM for problem 3.

**4.2. Theorems**

In this subsection, we formulate four theorems for problem 3. Let \( \lambda > 2 \) be a parameter. Introduce the Carleman weight function \( \psi_\lambda(t) \) for the operator \( \partial_t + a(x,t)\partial_x^2 \) as:

\[
\psi_\lambda(t) = e^{(T_1+1-t)^\lambda}, \quad t \in (0, T_1).
\]

Hence, the function \( \psi_\lambda(t) \) is decreasing on \([0, T_1]\), \( \psi_\lambda'(t) < 0 \),

\[
\max_{[0, T_1]} \psi_\lambda(t) = \psi_\lambda(0) = e^{(T_1+1)^\lambda}, \quad \min_{[0, T_1]} \psi_\lambda(t) = \psi_\lambda(T_1) = e.
\]
Denote
\[ H^2_0(\Omega_{T_1}) = \{ u \in H^2(\Omega_{T_1}) : u(0, t) = u(1, t) = 0 \}. \]  
(4.10)

\[ H^2_{0,0}(\Omega_{T_1}) = \{ u \in H^2_0(\Omega_{T_1}) : u(x, 0) = 0 \}. \]  
(4.11)

**Theorem 4.1 (Carleman estimate).** Let the coefficient \( a(x, t) \) of the operator \( P \) satisfies conditions (4.1). Then there exist a sufficiently large number \( \lambda_0 = \lambda_0(T_1, a_0, a_1) > 2 \) and a constant \( C = C(T_1, a_0, a_1) > 0 \), both depending only on listed parameters, such that the following Carleman estimate holds for the operator \( P \):

\[
\int_{\Omega_{T_1}} (P u)^2 \psi^2 \, dx \, dt \geq C \sqrt{\lambda} \int_{\Omega_{T_1}} u^2 \psi^2 \, dx \, dt + C \lambda^2 \int_{\Omega_{T_1}} u^2 \psi^2 \, dx \, dt - C \sqrt{\lambda} \|u\|_{H^2(\Omega_{T_1})}^2 \\
- C (\lambda(T_1 + 1))^{3/2} \exp[\lambda] \|w(x, 0)\|_{L_2(0,1)}^2 \\
\forall \lambda \geq \lambda_0, \forall u \in H^2_0(\Omega_{T_1}).
\]  
(4.12)

Carleman estimate (4.12) is the key to proofs of theorems 4.2 and 4.4.

**Theorem 4.2 (Hölder stability estimate for problem 3 and uniqueness).** Let the coefficient \( a(x, t) \) of the operator \( P \) satisfies conditions (4.1). Assume that the functions \( w \in H^2(\Omega_{T_1}) \) and \( u^* \in H^2(\Omega_{T_1}) \) are solutions of problem 3 with the vectors of data \( (\varphi_0(t), \varphi_1(t)) \) and \( (\varphi_0^*(t), \varphi_1^*(t)) \) respectively, where \( \varphi_0, \varphi_1, \varphi_0^*, \varphi_1^* \in H^2(0, T_1) \). Also, assume that error estimates (4.5) of the boundary data hold. Choose an arbitrary number \( \rho \in (0, T_1) \). Denote

\[ \mu = \mu(T_1, \rho) = \frac{\ln(T_1 + 1 - \rho)}{\ln(T_1 + 1)} \in (0, 1). \]  
(4.13)

Then there exists a sufficiently small number \( \delta_0 = \delta_0(T_1, a_0, a_1) \in (0, 1) \) and a constant \( C_1 = C_1(T_1, a_0, a_1, \rho) > 0 \), both depending only on listed parameters, such that the following stability estimate holds for all \( \delta \in (0, \delta_0) \):

\[
\|w_t - u^*_t\|_{L_2(\Omega_{T_1-\rho})} + \|w - u^*\|_{L_2(\Omega_{T_1-\rho})} \\
\leq C_1 \left( 1 + \|w - u^*\|_{H^2(\Omega_{T_1})} \right) \exp\left[ - \left( \ln \delta^{-1/2} \right)^{\mu} \right].
\]  
(4.14)

Below \( C = C(T_1, a_0, a_1) > 0 \) and \( C_1 = C_1(T_1, a_0, a_1) > 0 \) denote different constants depending only on listed parameters.

**Corollary 4.1 (Uniqueness).** Let the coefficient \( a(x, t) \) of the operator \( P \) satisfies conditions (4.1). Then problem 3 has at most one solution (uniqueness).

**Proof.** If \( \delta = 0 \), then (4.14) implies that \( w(x, t) = u^*(x, t) \) in \( \Omega_{T_1-\rho} \). Since \( \rho \in (0, T_1) \) is an arbitrary number, then \( w(x, t) \equiv u^*(x, t) \) in \( \Omega_{T_1}. \)

**Theorem 4.3 (Existence and uniqueness of the minimizer).** Let functions \( \varphi_0(t), \varphi_1(t) \in H^2(0, T_1) \). Let \( Y \) be the set defined in (4.7). Then there exists unique minimizer \( w_{\min} \in Y \) of functional (4.6) and

\[
\|w_{\min}\|_{H^2(\Omega_{T_1})} \leq \frac{C}{\sqrt{\alpha}} \left( \|\varphi_0\|_{H^0(0, T_1)} + \|\varphi_1\|_{H^0(0, T_1)} \right). \]  
(4.15)
In the theory of ill-posed problems, this minimizer \( w_{\min} \) is called ‘regularized solution’ of problem 3 [18]. According to the theory of ill-posed problems, it is important to establish convergence rate of regularized solutions to the exact one \( w^* \). In doing so, one should always choose a dependence of the regularization parameter \( \alpha \) on the noise level \( \delta \), i.e. \( \alpha = \alpha(\delta) \in (0, 1) \) [18].

**Theorem 4.4 (Convergence rate of regularized solutions).** Let \( w^* \in H^2(Q_{T_1}) \) be the solution of problem 3 with the noiseless data \((\phi_0^*, \phi_1^*, \phi_0^*, \phi_1^*) \). Let functions \( \phi_0, \phi_1, \phi_0^*, \phi_1^* \in H^2(0, T_1) \). Let \( w_{\min} \in Y \) be the unique minimizer of functional (4.6) on the set \( Y \). Assume that error estimates (4.5) hold. Choose an arbitrary number \( \rho \in (0, T_1) \). Let \( \mu = \mu(T_1, \rho) \in (0, 1) \) be the number defined in (4.15) and let

\[
\alpha = \alpha(\delta) = \delta^2. \tag{4.16}
\]

Then there exists a sufficiently small number \( \delta_0 = \delta_0(T_1, a_0, a_1) \in (0, 1) \) depending only on listed parameters such that the following convergence rate of regularized solutions \( w_{\min} \) holds for all \( \delta \in (0, \delta_0) \):

\[
\|\partial_x w_{\min} - \partial_x w^*\|_{L^2(Q_{T_1-\rho})} + \|w_{\min} - w^*\|_{L^2(Q_{T_1-\rho})} \leq C_1 \left( 1 + \|w^*\|_{L^2(Q_{T_1})} + \|\phi_0^*\|_{H^2(0,T_1)} + \|\phi_1^*\|_{H^2(0,T_1)} \right) \exp \left[ - \left( \ln \delta^{-1/2} \right)^{\mu} \right]. \tag{4.17}
\]

### 4.3. Proof of theorem 4.1

We assume in this proof that \( u \in C^1(\overline{Q_{T_1}}) \). The case \( u \in H^2(Q_{T_1}) \) can be obtained via density arguments. It is assumed in this proof that \( \lambda \geq \lambda_0 = \lambda_0(T_1, a_0, a_1) > 2 \) and \( \lambda_0 \) is sufficiently large. We remind that \( C = C(T_1, a_0, a_1) > 0 \) denotes different constants depending only on listed parameters. Change variables as

\[
v(x, t) = u(x, t)\psi_3(t) = u(x, t)e^{(T_1 + 1 - t)^{\lambda_0}}. \tag{4.18}
\]

Hence,

\[
u(x, t) = v(x, t)e^{-(T_1 + 1 - t)^{\lambda_0}},
\]

\[
u_t = (v_t + \lambda(T_1 + 1 - t)^{\lambda_0 - 1} v)e^{-(T_1 + 1 - t)^{\lambda_0}},
\]

\[
u_x = v_x e^{-(T_1 + 1 - t)^{\lambda_0}}, \quad u_{xx} = v_{xx} e^{-(T_1 + 1 - t)^{\lambda_0}}.
\]

Hence,

\[
(Pu)_x^2 = v_t^2 + 2v_t (a(x, t)v_{xx} + \lambda(T_1 + 1 - t)^{\lambda_0 - 1} v) \]

\[
\geq v_t^2 + 2v_t (a(x, t)v_{xx} + \lambda(T_1 + 1 - t)^{\lambda_0 - 1} v). \tag{4.19}
\]

We have used here \( (a + b)^2 \geq a^2 + 2ab, \forall a, b \in \mathbb{R} \). We now estimate from the below terms in the second line of (4.19).

**Step 1.** Estimate from the below \( 2a(x, t)v_{xx}v_t \). We have:

\[
2a(x, t)v_{xx}v_t = (2a(x, t)v_{xx}v_t)_x - 2a(x, t)v_{xx}v_t \]

\[
= (2a(x, t)v_{xx}v_t)_x + (a(x, t)v_t^2) - a_x(x, t)v_t^2 - 2a(x, t)v_{xx}v_t.
\]
Thus,

\[ 2a(x, t)v_{xx}v_t \geq (2a(x, t)v_xv_t)_x + \left( -a(x, t)v_t^2 \right)_x - C\psi t^2 - C|v_x|v_t. \]  

(4.20)

**Step 2.** Estimate from the below \( 2\lambda(T_1 + 1 - t)^{\lambda - 1}v_t \). We have:

\[
2\lambda(T_1 + 1 - t)^{\lambda - 1}v_t = \left( \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t + \lambda(\lambda - 1)(T_1 + 1 - t)^{\lambda - 2}v^2
\[
\geq \left( \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t + \frac{\lambda^2}{2}(T_1 + 1 - t)^{\lambda - 2}v^2.
\]

(4.21)

**Step 3.** Estimate from the below the entire second line of (4.19). Using (4.20) and (4.21) and Cauchy–Schwarz inequality 'with \( \epsilon \)',

\[
2ab \geq -\epsilon a^2 - \frac{1}{\epsilon}b^2, \quad \forall a, b \in \mathbb{R}, \quad \forall \epsilon > 0,
\]

we obtain

\[
v_t^2 + 2v_t\left( a(x, t)v_{xx} + \lambda(T_1 + 1 - t)^{\lambda - 1}v \right)
\]

\[
\geq v_t^2 - C\psi t^2 - C|v_x|v_t + \frac{\lambda^2}{2}(T_1 + 1 - t)^{\lambda - 2}v^2
\]

\[
+ (2a(x, t)v_xv_t)_x + \left( -a(x, t)v_t^2 + \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t,
\]

\[
\geq \frac{1}{2}v_t^2 - C\psi t^2 + \frac{\lambda^2}{2}(T_1 + 1 - t)^{\lambda - 2}v^2
\]

\[
+ (2a(x, t)v_xv_t)_x + \left( -a(x, t)v_t^2 + \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t.
\]

Thus, we have obtained that

\[
v_t^2 + 2v_t\left( a(x, t)v_{xx} + \lambda(T_1 + 1 - t)^{\lambda - 1}v \right)
\]

\[
\geq \frac{1}{2}v_t^2 - C\psi t^2 + \frac{\lambda^2}{2}(T_1 + 1 - t)^{\lambda - 2}v^2
\]

\[
+ (2a(x, t)v_xv_t)_x + \left( -a(x, t)v_t^2 + \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t.
\]

(4.23)

Using (4.19) and (4.23) as well as dropping the non-negative term \( v_t^2/2 \) in the right-hand side of (4.23), we obtain

\[
(Pu)^2\psi^2 \geq -C\psi t^2 + \frac{\lambda^2}{2}(T_1 + 1 - t)^{\lambda - 2}v^2
\]

\[
+ (2a(x, t)v_xv_t)_x + \left( -a(x, t)v_t^2 + \lambda(T_1 + 1 - t)^{\lambda - 1}v^2 \right)_t.
\]

(4.24)

**Step 4.** Using (4.18), change variables in the right-hand side of (4.24). We have \( v^2 = u^2\psi^2, v_t^2 = u_t^2\psi^2 \). Thus,
\[(Pu)^2 \psi_\lambda^2 \geq - Cu^2 \psi_\lambda^2 + \frac{\lambda^2}{2} (T_1 + 1 - t)^{\lambda - 2} u^2 \psi_\lambda^2 \tag{4.25} \]
\[+ \left( 2a(x,t) u_x (u_t - \lambda(T_1 + 1 - t)^{\lambda - 2} u) \psi_\lambda^2 \right)_x \]
\[+ \left( (-a(x,t) u_x^2 + \lambda(T_1 + 1 - t)^{\lambda - 1} u^2) \psi_\lambda^2 \right)_x. \]

**Step 5.** Estimate from the below \[- Pu \cdot u \psi_\lambda^2.\] We have
\[- Pu \cdot u \psi_\lambda^2 = (-u_t - a(x,t) u_{xx}) u e^{2(T_1 + 1 - t^\lambda)} \]
\[= \left( -\frac{1}{2} u^2 e^{2(T_1 + 1 - t^\lambda)} \right)_t - \lambda(T_1 + 1 - t)^{\lambda - 1} u^2 e^{2(T_1 + 1 - t^\lambda)} \tag{4.26} \]
\[+ \left( -a(x,t) u_x u e^{2(T_1 + 1 - t^\lambda)} \right)_x + a(x,t) u_x^2 e^{2(T_1 + 1 - t^\lambda)} + a_x(x,t) u_x u e^{2(T_1 + 1 - t^\lambda)}. \]

Using (4.1) and (4.22), we obtain
\[a(x,t) u_x^2 + a_x(x,t) u_x u \geq a_0 a_x^2 - a_1 |u_x| |u| \geq \frac{a_0}{2} a_x^2 - Cu^2 \]
\[\geq \frac{a_0}{2} a_x^2 - \lambda(T_1 + 1 - t)^{\lambda - 2} u^2.\]

Hence, multiplying (4.26) by \(\sqrt{\lambda}\), we obtain
\[- \sqrt{\lambda} Pu \cdot u \psi_\lambda^2 \geq \frac{a_0}{2} \sqrt{\lambda} a_x^2 e^{2(T_1 + 1 - t^\lambda)} - 2 \lambda^{\lambda/2} (T_1 + 1 - t)^{\lambda - 2} u^2 e^{2(T_1 + 1 - t^\lambda)} \tag{4.27} \]
\[+ \left( -\frac{\sqrt{\lambda}}{2} u^2 e^{2(T_1 + 1 - t^\lambda)} \right)_t + \left( -\sqrt{\lambda} a(x,t) u_x u e^{2(T_1 + 1 - t^\lambda)} \right)_x. \]

**Step 6.** Estimate from the below \((Pu)^2 \psi_\lambda^2 - \sqrt{\lambda} Pu \cdot u \psi_\lambda^2).\] Using (4.25) and (4.27), we obtain
\[(Pu)^2 \psi_\lambda^2 \geq \frac{a_0}{2} \sqrt{\lambda} \left( 1 - \frac{2C}{\sqrt{\lambda}} \right) a_x^2 \psi_\lambda^2 + \frac{\lambda^2}{2} (T_1 + 1 - t)^{\lambda - 2} \left( 1 - \frac{4}{\sqrt{\lambda}} \right) u^2 \psi_\lambda^2 \]
\[+ \frac{\partial}{\partial t} \left[ \left( -a(x,t) u_x^2 + \lambda(T_1 + 1 - t)^{\lambda - 1} u^2 - \frac{\sqrt{\lambda}}{2} u^2 \right) \psi_\lambda^2 \right] \tag{4.28} \]
\[+ \frac{\partial}{\partial x} \left[ \left( 2a(x,t) u_x (u_t - \lambda(T_1 + 1 - t)^{\lambda - 2} u) - \sqrt{\lambda} a(x,t) u_x u \right) \psi_\lambda^2 \right]. \]

**Step 7.** Estimate from the below
\[\int_{\Omega_1} (Pu)^2 \psi_\lambda^2 \, dx \, dt.\]
We have
\[(Pu)^2 \psi_\lambda^2 - \sqrt{\lambda} Pu \cdot u \psi_\lambda^2 \leq \frac{3}{2} (Pu)^2 \psi_\lambda^2 + \frac{1}{2} \sqrt{\lambda} u^2 \psi_\lambda^2.\]
Combining this with (4.28), we obtain
\[(Pu)^2 \psi^2 \geq C \sqrt{\lambda} u^2 \psi^2 + C \lambda u^2 \psi^2 \]
\[+ \frac{\partial}{\partial t} \left[ -a(x,t)u^2 + \lambda (T_1 + 1 - t)^{1/2} u^2 - \frac{\sqrt{\lambda}}{2} u^2 \right] \psi^2 \] (4.29)
\[+ \frac{\partial}{\partial x} \left[ 2a(x,t)u_x \left( u_t + \lambda (T_1 + 1 - t)^{1/2} u \right) \right] \psi^2.\]

Integrate (4.29) using \(u \in H^2_0(Q_{T_1})\) and also using (4.9). We obtain
\[
\int_{Q_{T_1}} (Pu)^2 \psi^2 \, dx \, dt \geq C \sqrt{\lambda} \int_{Q_{T_1}} u^2 \psi^2 \, dx \, dt + C \lambda \int_{Q_{T_1}} u^2 \psi^2 \, dx \, dt
- C \sqrt{\lambda} \|u(x,T_1)\|_{H^1(0,1)}^2
- C \lambda (T_1 + 1)^{1/2} \|u(x,0)\|_{L^2(0,1)}^2.
\] (4.30)

Finally, applying the trace theorem to the second line of (4.30), we obtain desired estimate (4.12) of this theorem. \(\square\)

4.4. Proof of theorem 4.2

Introduce the following functions:
\[\tilde{\varphi}_0(t) = \varphi_0(t) - \varphi^*_0(t), \tilde{\varphi}_1(t) = \varphi_1(t) - \varphi^*_1(t),\] (4.31)
\[F(x,t) = \varphi_0(t)(1 - x) + \varphi_1(t)x, F^*(x,t) = \varphi^*_0(t)(1 - x) + \varphi^*_1(t)x,\] (4.32)
\[\tilde{F}(x,t) = F(x,t) - F^*(x,t) = \tilde{\varphi}_0(t)(1 - x) + \tilde{\varphi}_1(t),\] (4.33)
\[\tilde{w}(x,t) = w(x,t) - F(x,t), \tilde{w}^*(x,t) = w^*(x,t) - F^*(x,t),\] (4.34)
\[\tilde{\psi}(x,t) = \tilde{w}(x,t) - \tilde{w}^*(x,t).\] (4.35)

It follows from (4.4), (4.5) and (4.31)–(4.35) that:
\[\tilde{w}(x,0) = \tilde{w}^*(x,0) = \tilde{\psi}(x,0) = 0,\] (4.36)
\[F_{xx}(x,t) = F^*_{xx}(x,t) = \tilde{F}_{xx}(x,t) = 0,\] (4.37)
\[\|\tilde{F}\|_{L^2(Q_{T_1})} \leq C \delta.\] (4.38)

By (4.2)–(4.4) and (4.31)–(4.36)
\[\tilde{\psi} + a(x,t)\tilde{\psi}_{xx} = -\tilde{F}_t \text{ in } Q_{T_1}.\] (4.39)
\( w(0,t) = w(1,t) = 0, \quad t \in (0, T_1), \)  
(4.40)

\( w(x,0) = 0, \quad x \in (0,1). \)  
(4.41)

Also, by (4.10), (4.11) and (4.36)
\[ \hat{w}, \hat{w}^*, \hat{w} \in H^2_{0,0}(Q_{T_1}). \]  
(4.42)

Square both sides of equation (4.39), multiply by the function \( \psi_2^2(t) \) and integrate over the domain \( Q_{T_1} \). Using (4.9) and (4.38), we obtain
\[ \int_{Q_{T_1}} (\hat{w}_t + a(x,t)\hat{w}_{xx})^2 \psi_2^2(t)dx \leq C \delta^2 e^{2(T_1 + 1)\lambda}. \]  
(4.43)

Hence, applying Carleman estimate (4.12) to the left-hand side of (4.43) and taking into account (4.9)–(4.11), we obtain
\[ \int_{Q_{T_1}} \hat{w}_x^2 \psi_2^2 dx + \lambda^{3/2} \int_{Q_{T_1}} \hat{w}_x^3 \psi_2^2 dx \leq C \delta^2 e^{2(T_1 + 1)\lambda} + C \| \hat{w} \|^2_{H^2} \]  
(4.44)

Since \( Q_{T_1-\rho} \subset Q_{T_1} \) and also since by (4.8) \( \psi_2^2(t) \geq e^{2(T_1+1-\rho)\lambda} \) in \( Q_{T_1-\rho} \), then (4.44) implies
\[ \| \hat{w}_x \|^2_{L^2} + \| \hat{w}_x \|^2_{L^2} \leq C \delta e^{(T_1+1)\lambda} + C e^{-(T_1+1-\rho)\lambda} \| \hat{w} \|^2_{H^2} \]  
(4.45)

Choose \( \delta_0 = \delta_0(T_1, a_0, a_1) \in (0,1) \) so small that
\[ \ln \left( \ln \left( \frac{1}{\ln(T_1 + 1)} \right)^{1/\ln(T_1 + 1)} \right) > \lambda_0. \]  
(4.46)

Let \( \delta \in (0, \delta_0) \). We now choose \( \lambda = \lambda(\delta) \) so large that
\[ e^{(T_1+1)\lambda} = \frac{1}{\sqrt{\delta}} \]  
(4.47)

Hence,
\[ \lambda = \lambda(\delta) = \ln \left( \ln \left( \frac{1}{\ln(T_1 + 1)} \right)^{1/\ln(T_1 + 1)} \right) > \lambda_0, \quad \forall \delta \in (0, \delta_0). \]  
(4.48)

Then
\[ e^{-(T_1+1-\rho)\lambda} = \exp \left[ - \left( \ln \left( \frac{1}{\ln(T_1 + 1)} \right)^{1/\ln(T_1 + 1)} \right) \right], \]  
(4.49)

where the number \( \mu \in (0,1) \) is defined in (4.13). We have
\[ \frac{e^{-\left( \ln \left( \frac{1}{\ln(T_1 + 1)} \right)^{1/\ln(T_1 + 1)} \right) \mu}}{\sqrt{\delta}} = \exp \left[ \frac{1}{2} \ln \delta \left( 1 + \frac{2 \ln \left( \frac{1}{\ln(T_1 + 1)} \right)^{1/\ln(T_1 + 1)} \mu}{\ln \delta} \right) \right]. \]  
(4.50)

Since \( \mu \in (0,1) \), then the Hospital’s rule implies
\[ \lim_{\delta \to 0} \frac{2 \ln \delta^{-1/2}}{\ln \delta} = \lim_{\delta \to 0} \left( -\mu \left( \ln \delta^{-1/2} \right)^{\mu-1} \right) = 0. \]

Hence,
\[ \lim_{\delta \to 0} \left[ -\frac{1}{2} \ln \delta \left( 1 + \frac{2 \ln \delta^{-1/2}}{\ln \delta} \right) \right] = \lim_{\delta \to 0} \left( \ln \delta^{-1/2} \right) = \infty. \tag{4.51} \]

It follows from (4.50) and (4.51) that
\[ \lim_{\delta \to 0} \frac{e^{-\left( \ln \delta^{-1/2} \right)^{\mu}}}{\sqrt{\delta}} = \infty. \]

Hence,
\[ \sqrt{\delta} \leq C_1 e^{-\left( \ln \delta^{-1/2} \right)^{\mu}}, \quad \forall \delta \in (0, 1). \tag{4.52} \]

Using (4.45)–(4.49) and (4.52), we obtain
\[ \|\bar{\mu} - \mu\|_{L^2(Q_{1-\rho})} + \|\bar{\mu}\|_{L^2(Q_{1-\rho})} \leq C_1 \left( 1 + \|\bar{\mu}\|_{H^2(Q_{1-\rho})} \right) \exp\left( -\left( \ln \delta^{-1/2} \right)^{\mu} \right), \quad \forall \delta \in (0, \delta_0). \tag{4.53} \]

By (4.31)–(4.35) \( \bar{\mu} = (w - w^*) - \bar{F} \). Hence, the triangle inequality, (4.5), (4.31)–(4.33), (4.38) and (4.53) imply (4.14), which is the target estimate of this theorem. \( \square \)

4.5. Proof of theorem 4.3

Denote \([,]\) the scalar product in the space \( H^2(Q_{T_1}) \). Let \( \hat{w} \in H^2_{0,0}(Q_{T_1}) \) be the function defined in (4.34). Then, using (4.6) and (4.37), consider the functional
\[ I_a(\hat{w} + F) = \int_{Q_{T_1}} (P\hat{w} + F)^2 \, dx \, dt + \alpha \|\hat{w} + F\|_{H^2(Q_{T_1})}^2. \tag{4.54} \]

Suppose that the function \( \hat{w}_{\min} \in H^2_{0,0}(Q_{T_1}) \) is a minimizer of the functional \( I_a(\hat{w} + F) \) on the space \( H^2_{0,0}(Q_{T_1}) \), i.e.
\[ I_a(\hat{w}_{\min} + F) \leq I_a(\hat{w} + F), \quad \forall \hat{w} \in H^2_{0,0}(Q_{T_1}). \tag{4.55} \]

Consider the function \( w_{\min} = \hat{w}_{\min} + F \). Then it follows from (4.11) and (4.32) that \( w_{\min} \in Y \), where the set \( Y \) is defined in (4.7). Also, \( w = \hat{w} + F \in Y, \forall \hat{w} \in H^2_{0,0}(Q_{T_1}) \). Hence, (4.55) implies that the function \( w_{\min} \) is a minimizer of the functional \( I_a(w) \) on the set \( Y \).

We now prove the reverse. Suppose that a function \( w_{\min} \in Y \) is a minimizer of the functional \( I_a(w) \) on the set \( Y \), i.e.
\[ I_a(w_{\min}) \leq I_a(w), \quad \forall w \in Y. \tag{4.56} \]

Consider the function \( \hat{w}_{\min} = w_{\min} - F \). And for every function \( \hat{w} \in H^2_{0,0}(Q_{T_1}) \) consider the function \( \tilde{w} = \hat{w} + F \in Y \). Then by (4.56)
\[ I_a(\hat{w}_{\min} + F) = I_a(\hat{w}_{\min}) \leq I_a(\hat{w} + F), \quad \forall \hat{w} \in H^2_{0,0}(Q_{T_1}). \]
Hence, $\hat{w}_{\text{min}}$ is a minimizer of functional (4.54) on the space $H^2_{0,0}(Q_T)$. Therefore, it is sufficient to find a minimizer of the functional $I_w(\hat{w} + F)$ on the space $H^2_{0,0}(Q_T)$.

By the variational principle the function $\hat{w}_{\text{min}} \in H^2_{0,0}(Q_T)$ is a minimizer of the functional $I_w(\hat{w} + F)$ if and only if the following integral identity is satisfied:

$$
\int_{Q_T} (P\hat{w}_{\text{min}} \cdot Ph) \, dx \, dt + \alpha [\hat{w}_{\text{min}}, h] = - \int_{Q_T} F_t \cdot Ph \, dx \, dt - \alpha [F, h],
$$

(4.57)

$\forall h \in H^2_{0,0}(Q_T)$.

Consider a new scalar product in $H^2_{0,0}(Q_T)$ defined as

$$\{ u, v \} = \int_{Q_T} (Pu \cdot Pv) \, dx \, dt + \alpha [u, v], \quad \forall u, v \in H^2_{0,0}(Q_T).$$

Recall that $\alpha \in (0, 1)$. Obviously,

$$\alpha \|u\|_{H^2(Q_T)}^2 \leq \{u, u\} \leq C \|u\|_{H^2(Q_T)}^2, \quad \forall u \in H^2_{0,0}(Q_T).$$

Hence, norms $\sqrt{\{u, u\}}$ and $\|u\|_{H^2(Q_T)}$ are equivalent. Hence, one can consider the scalar product $\{u, v\}$ as the scalar product in $H^2_{0,0}(Q_T)$.

Hence, we can rewrite (4.57) as

$$\{\hat{w}_{\text{min}}, h\} = - \int_{Q_T} F_t \cdot Ph \, dx \, dt - \alpha [F, h], \quad \forall h \in H^2_{0,0}(Q_T).$$

(4.58)

The right-hand side of (4.58) can be estimated as

$$\left| - \int_{Q_T} F_t Ph \, dx \, dt - \alpha [F, h] \right| \leq C \|F\|_{H^2(Q_T)} \{h\}, \quad \forall h \in H^2_{0,0}(Q_T).$$

Hence, the right-hand side of (4.58) can be considered as a bounded linear functional of $h \in H^2_{0,0}(Q_T)$. Hence, by Riesz theorem there exists unique function $W \in H^2_{0,0}(Q_T)$ such that

$$- \int_{Q_T} F_t Ph \, dx \, dt - \alpha [F, h] = \{W, h\}, \quad \forall h \in H^2_{0,0}(Q_T).$$

Comparing this with (4.58), we obtain

$$\{\hat{w}_{\text{min}}, h\} = \{W, h\}, \quad \forall h \in H^2_{0,0}(Q_T).$$

Therefore, $\hat{w}_{\text{min}} = W$. Thus, we have proven existence and uniqueness of the minimizer $\hat{w}_{\text{min}}$ of the functional $I_w(\hat{w} + F)$ on the space $H^2_{0,0}(Q_T)$. Therefore, it follows from the discussion in the beginning of this proof that there exists unique minimizer of the functional $I_w(w)$ on the set $Y$ and this minimizer is $w_{\text{min}} = \hat{w}_{\text{min}} + F$.

We now estimate the norm $\|w_{\text{min}}\|_{H^2(Q_T)}$. Setting in (4.57) $h = \hat{w}_{\text{min}}$ and using Cauchy–Schwarz inequality, we obtain

$$\|w_{\text{min}}\|_{H^2(Q_T)} \leq C \|F\|_{H^2(Q_T)}.$$
\[
\int_{Q_{T_1}} (P\tilde{\omega}_{\text{min}})^2 \, dx \, dt + \alpha \|\tilde{\omega}_{\text{min}}\|_{H^2(Q_{T_1})}^2 \\
\leq \frac{1}{2} \|F^*_t\|_{L^2(Q_{T_1})}^2 + \frac{1}{2} \|P\tilde{\omega}_{\text{min}}\|_{H^2(Q_{T_1})}^2 + \frac{\alpha}{2} \|F\|_{H^2(Q_{T_1})}^2 + \frac{\alpha}{2} \|\tilde{\omega}_{\text{min}}\|_{H^2(Q_{T_1})}^2.
\]

Hence,
\[
\|\tilde{\omega}_{\text{min}}\|_{H^2(Q_{T_1})} \leq \frac{C}{\sqrt{\alpha}} \|F\|_{H^2(Q_{T_1})}.
\]

This estimate, triangle inequality and (4.32) imply the target estimate (4.15) of theorem 4.3. □

### 4.6. Proof of theorem 4.4

We still use notations (4.31)–(4.35). By corollary 4.1 problem 3 has at most one solution. Hence, there exists unique exact solution \(w^* \in H^2(Q_{T_1})\) of problem 3 with the data \(\varphi^*_0, \varphi^*_1 \in H^2(0,T_1)\) in (4.3) and (4.4). Hence, we have the following analog of integral identity (4.57)
\[
\int_{Q_{T_1}} (P\tilde{\omega}^* \cdot Ph) \, dx \, dt + \alpha [\tilde{\omega}^*, h] = -\int_{Q_{T_1}} F^*_t \cdot Ph \, dx \, dt + \alpha [\tilde{\omega}^*, h], \quad \forall h \in H^2_{0,0}(Q_{T_1}).
\]

(4.59)

Subtract (4.59) from (4.57). Then, using (4.33) and (4.35), we obtain
\[
\int_{Q_{T_1}} (P\overline{\omega} \cdot Ph) \, dx \, dt + \alpha [\overline{\omega}, h] = -\int_{Q_{T_1}} F^*_t \cdot Ph \, dx \, dt + \alpha [\tilde{\omega}, h], \quad \forall h \in H^2_{0,0}(Q_{T_1}).
\]

(4.60)

Set in (4.60) \(h = \overline{\omega}.\) Then, using (4.38) and Cauchy–Schwarz inequality, we obtain
\[
\int_{Q_{T_1}} (P\overline{\omega})^2 \, dx \, dt \leq C \left( \delta^2 + \alpha \|\tilde{\omega}^*\|_{H^2(Q_{T_1})}^2 \right).
\]

(4.61)

\[
\|\overline{\omega}\|_{H^2(Q_{T_1})} \leq C \left( \delta \sqrt{\frac{\delta}{\alpha}} + \|\tilde{\omega}^*\|_{H^2(Q_{T_1})} \right).
\]

(4.62)

Inequality (4.61) is equivalent with
\[
\int_{Q_{T_1}} (P\overline{\omega})^2 \psi^2 \psi^{-2} \, dx \, dt \leq C \left( \delta^2 + \alpha \|\tilde{\omega}^*\|_{H^2(Q_{T_1})}^2 \right).
\]

(4.63)

Since by (4.9) \(\psi^{-2}(t) \geq e^{-2(T_1+1)^4} \) in \(Q_{T_1}\), then (4.61) implies
\[
\int_{Q_{T_1}} (P\overline{\omega})^2 \, dx \, dt \leq C \left( \delta^2 + \alpha \|\tilde{\omega}^*\|_{H^2(Q_{T_1})}^2 \right) e^{2(T_1+1)^4}.
\]
Hence, applying Carleman estimate (4.12) to the left-hand side of (4.63) and recalling again that \( \alpha \in (0, 1) \), we obtain
\[
\int_{Q_{T_1}} \overline{w}_t^2 \psi_\delta^2 \, dx \, dt + \lambda^{3/2} \int_{Q_{T_1}} \overline{w}_t^2 \psi_\lambda^2 \, dx \, dt \\
\leq C \left( \delta^2 + \alpha \| \hat{\omega} \|^2_{H^2(\Omega_r)} \right) e^{2(T_1 + 1)\lambda} + C \| \bar{w} \|^2_{H^2(\Omega_r)}, \quad \forall \lambda \geq \lambda_0.
\]
Hence, we obtain similarly with (4.45)
\[
\| \bar{w} \|^2_{L^2(Q_{T_1}, \omega)} + \| \bar{w} \|^2_{L^2(Q_{T_1}, \sigma)} \leq C \left( \delta^2 + \alpha \| \hat{\omega} \|^2_{H^2(\Omega_r)} \right) e^{2(T_1 + 1)\lambda} \\
+ C e^{-2(T_1 + 1 - \rho)\lambda} \| \bar{w} \|^2_{H^2(\Omega_r)}, \quad \forall \lambda \geq \lambda_0.
\]
Combining this with (4.62), we obtain
\[
\| \bar{w} \|^2_{L^2(Q_{T_1}, \omega)} + \| \bar{w} \|^2_{L^2(Q_{T_1}, \sigma)} \\
\leq C \left( \delta + \sqrt{\alpha} \| \hat{\omega} \|^2_{H^2(\Omega_r)} \right) e^{(T_1 + 1)\lambda} \\
+ C e^{-\rho \lambda} e^{-(T_1 + 1 - \rho)\lambda} \| \bar{w} \|^2_{H^2(\Omega_r)} e^{-(T_1 + 1 - \rho)\lambda}, \quad \forall \lambda \geq \lambda_0.
\]
Suppose now that \( \alpha = \alpha(\delta) = \delta^2 \), as stated in (4.16). Choose \( \delta_0 = \delta_0(T_1, \alpha_0, a_1) \in (0, 1) \) as in (4.46) and \( \lambda = \lambda(\delta) \) as in (4.48). Then (4.47), (4.49), (4.52) and (4.64) imply
\[
\| \bar{w} \|^2_{L^2(Q_{T_1}, \omega)} + \| \bar{w} \|^2_{L^2(Q_{T_1}, \sigma)} \\
\leq C_1 \left( 1 + \| \hat{\omega} \|^2_{H^2(\Omega_r)} \right) \exp \left[ - \left( \frac{\ln \delta^{-1/2}}{\mu} \right) \right], \quad \forall \delta \in (0, \delta_0).
\]
The target estimate (4.17) of this theorem follows immediately from the triangle inequality, (4.5), (4.31)–(4.35) and (4.65).

5. Probabilistic arguments for a trading strategy

A heuristic algorithm of section 3 can be used as the basis for a trading strategy of options. The algorithm predicts the option price change relatively to the current price. The fact that this algorithm uses the information about stock and option prices only over a small time period makes realistic the assumptions of the model of section 2 about the volatilities being independent on time. Formulas (2.6) and (2.9) indicate that the sign of the mathematical expectation of the option price increment should likely define the trading strategy. In addition to the mathematical expectation of the option price increment, it is necessary to take into account indicators that reflect the risk of using that trading strategy. This is because the option price dynamics is described by a random process. Based on the model of section 2, we construct in this section such indicators for a ‘perfect’ trading strategy, which always correctly predicts the sign of the mathematical expectation of the option price increment.

We assume in this section that both the volatility \( \sigma \) of the stock and the idea \( \hat{\sigma} \) of the volatility of the call option, which has been developed among the participants involved in trading of
this option, are known. Recall that we have assumed in section 2 that the dynamics of the stock price is described by a stochastic differential equation of the geometric Brownian motion $d\hat{s}(t) = \sigma \hat{s}(t) \, dW_t$ with the initial condition $\hat{s}(t_0) = s_0$, and also that the corresponding option price is $v(s(t), t) = u(s(t), T - \tau)$, where $\tau = T - t \in (0, T)$ and the function $u(s, \tau)$ can be found by the Black–Scholes formula (2.5). The option price expected by option market participants is described by a stochastic process $v(\hat{s}(t), t) = u(\hat{s}(t), T - t)$, where the expected stock price satisfies the stochastic differential equation of the geometric Brownian motion $d\hat{s} = \hat{s} \, dW_1$ with the initial condition

$$\hat{s}(t_0) = s_0 = s(t_0).$$

(5.1)

Here $W_1$ is a Wiener process, and the processes $W$ and $W_1$ are independent.

Let $t_0 \in (0, T)$ be a certain moment of time and $\varepsilon > 0$ be a sufficiently small number. The true option price at the moment of time $t_0 + \varepsilon$ is $v(s(t_0 + \varepsilon), t_0 + \varepsilon)$. On the other hand, at the same moment of time $t_0 + \varepsilon$ the price of this option expected by the participants of the market is $v(\hat{s}(t_0 + \varepsilon), t_0 + \varepsilon)$. It follows from (2.6) that, on the small time interval $(t_0, t_0 + \varepsilon)$, a winning trading strategy of the options trading should be based on an estimate of the probability that $v(s(t_0 + \varepsilon), t_0 + \varepsilon) > v(\hat{s}(t_0), t_0)$. This probability is given in theorem 5.1.

**Theorem 5.1.** Let $\varepsilon > 0$ be a sufficiently small number and $\Phi(z), z \in \mathbb{R}$ be the function defined in (2.4). The probability that at the time $t_0 + \varepsilon$ the true option price $v(s(t_0 + \varepsilon), t_0 + \varepsilon)$ is greater than the price expected by the participants of the options market $v(\hat{s}(t_0 + \varepsilon), t_0 + \varepsilon)$ is

$$p = \Phi\left(\frac{(\hat{s}^2 - \sigma^2)^{1/2}}{2(\hat{s}^2 + \sigma^2)}\right).$$

(5.2)

**Proof.** The derivative

$$\frac{\partial u(s, \tau)}{\partial s}$$

is called the Greek delta. This parameter for a call option is

$$\Delta = \frac{\partial u(s, \tau)}{\partial s} = \Phi(u_+(s, \tau)) > 0.$$  

(5.3)

Since by (5.3) $\partial u(s, \tau) > 0$, then the inequality $v(s(t_0 + \varepsilon), t_0 + \varepsilon) > v(\hat{s}(t_0 + \varepsilon), t_0 + \varepsilon)$ is equivalent to the inequality $s(t_0 + \varepsilon) > \hat{s}(t_0 + \varepsilon)$. It follows from (5.1) that the latter inequality is equivalent with

$$\ln\left(\frac{s(t_0 + \varepsilon)}{s(t_0)}\right) > \ln\left(\frac{\hat{s}(t_0 + \varepsilon)}{\hat{s}(t_0)}\right).$$

It follows from the properties of the geometric Brownian motion, see, e.g. [17, chapter 5, section 5.1] that the random variables

$$\ln\left(\frac{s(t_0 + \varepsilon)}{s(t_0)}\right) \in N\left(-\frac{\sigma^2}{2}, \sigma^2 \varepsilon\right),$$

$$\ln\left(\frac{\hat{s}(t_0 + \varepsilon)}{\hat{s}(t_0)}\right) \in N\left(-\frac{\hat{s}^2}{2}, \hat{s}^2 \varepsilon\right)$$
are normally distributed. Hence, the difference of these two random variables is also a normally
distributed random variable, see, e.g. [13, chapter 9, section 9.3], i.e.
\[
\ln \left( \frac{s(t_0 + \varepsilon)}{s(t_0)} \right) - \ln \left( \frac{s(t_0 + \varepsilon)}{\hat{s}(t_0)} \right) \in N \left( \frac{\sigma^2 - \hat{\sigma}^2}{2}, (\hat{\sigma}^2 + \sigma^2)\varepsilon \right).
\]

Therefore, the value given by formula (5.2) is indeed the probability that the true option
price \( v(s(t_0 + \varepsilon), t_0 + \varepsilon) \) is greater than the expected option price \( v(\hat{s}(t_0 + \varepsilon), t_0 + \varepsilon) \).

Theorem 5.1 implies that the operation of buying an option at the time moment \( t_0 \) and selling
it at the time moment \( t_0 + \varepsilon \) will be profitable with the probability \( p \) given in (5.2), and this
operation will be non-profitable with the probability \( 1 - p \).

By (2.4) and (5.2)
\[
p = \begin{cases} 
  > 1/2 & \text{if } \sigma > \hat{\sigma}, \\
  \leq 1/2 & \text{if } \sigma \leq \hat{\sigma}.
\end{cases}
\]

Hence, if \( \sigma > \hat{\sigma} \), then it is reasonable to buy an option at the moment of time \( t_0 \) and sell it
at the moment of time \( t_0 + \varepsilon \). Otherwise, it is reasonable to go in the short position on the
option at \( t = t_0 \). Suppose that \( p > 1/2 \). A winning strategy, which takes into account risks,
should involve the repetition operation multiple times with the same independent probabilities
of outcomes. Consider \( n \) non-overlapping small time intervals \([t_j, t_j + \varepsilon]\), \( j = 1, \ldots, n \), of the
same duration \( \varepsilon > 0 \), on which the option purchase operations are carried out at the moment
of time \( t_j \) with the subsequent sale at the moment of time \( t_j + \varepsilon \).

Consider random variables \( \{\xi_j\}_{j=1}^{n} \).
\[
\xi_j = \begin{cases} 
  1, & \text{if } v(s(t_j + \varepsilon), t_j + \varepsilon) \geq v(\hat{s}(t_j), t_j), \\
  0, & \text{if } v(s(t_j + \varepsilon), t_j + \varepsilon) < v(\hat{s}(t_j), t_j),
\end{cases} 
\quad j = 1, \ldots, n. 
\tag{5.4}
\]

If \( \xi_j = 1 \), then the operation of buying that option at the moment of time \( t_j \) and selling it at
the moment of time \( t_j + \varepsilon \) was profitable. If \( \xi_j = 0 \), then that operation was non profitable.
The random variables \( \xi_1, \ldots, \xi_n \) are independent identically distributed random variables
[13, chapter 18, section 18.1]. The frequency of profitable trading operations is characterized
by the random variable \( \zeta \),
\[
\zeta = \frac{1}{n} \sum_{j=1}^{n} \xi_j. 
\tag{5.5}
\]

It follows from [13, chapter 9, section 9.3] that the variable \( \zeta \) has a binomial distribution with
the mathematical expectation \( p \) given in (5.2) and with the dispersion \( D \), where
\[
D = \frac{p(1 - p)}{n}. 
\tag{5.6}
\]

By the central limit theorem of de Moivre–Laplace, the probability that more than half of trades
are profitable is estimated as [13, chapter 2, section 2.2]:
\[
\sum_{k=\left[ \frac{n}{2} \right]}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = \Phi \left( \frac{(1-2p)\varepsilon}{2\sqrt{p(1-p)}} \right) + \delta(n), 
\tag{5.7}
\]
\[
\lim_{n \to \infty} \delta(n) = 0. 
\tag{5.8}
\]
In our trading strategy, we decide to make transactions if and only if the probability of the profitable trading is not less than a given value $\alpha > \frac{1}{2}$. Hence, by (5.7) and (5.8)

$$\frac{(1 - 2p)\sqrt{n}}{2\sqrt{p(1 - p)}} > \Phi^{-1}(\alpha - \delta(n)),$$

where $\Phi^{-1}$ is the inverse function of the function $\Phi$ of (2.4). By (5.9)

$$p^2 - p + \frac{n}{4\{n + [\Phi^{-1}(\alpha - \delta(n))^2]\}^2} > 0.$$

Thus, we should have

$$p \geq \frac{1}{2} \left(1 + \sqrt{\frac{\Phi^{-1}(\alpha - \delta(n))^2}{n + [\Phi^{-1}(\alpha - \delta(n))^2]}\right).$$

(5.10)

To fulfill inequality (5.10), the imperfection of the stock market must be significant. More precisely, it follows from (5.2) and (5.10) that the difference between the volatilities $\sigma$ and $\hat{\sigma}$ must satisfy the following inequality:

$$\sigma - \hat{\sigma} \geq 2\sqrt{\sigma^2 + \hat{\sigma}^2} \left[\frac{1}{2} \left(1 + \sqrt{\frac{\Phi^{-1}(\alpha - \delta(n))^2}{n + [\Phi^{-1}(\alpha - \delta(n))^2]}\right)\right].$$

(5.11)

Based on this estimate of the difference $\sigma - \hat{\sigma}$, we design a trading strategy in the ideal case. ‘Ideal’ means that we know both volatilities $\sigma$ and $\hat{\sigma}$.

Trading strategy for the ideal case:

Let $\beta_1 > 0$ and $\beta_2 < 0$ be two threshold numbers. Our trading strategy considers three possible scenarios:

(a) If $\sigma - \hat{\sigma} \geq \beta_1 > 0$, then it is recommended to buy a call option at the current moment of time $t_j$ with the subsequent sale at the next moment of time $t_j + \varepsilon$.

(b) If $\sigma - \hat{\sigma} \leq \beta_2$, then it is recommended to go short at the current moment of time $t_j$, followed by closing the short position at time $t_j + \varepsilon$.

(c) If $\beta_2 < \sigma - \hat{\sigma} < \beta_1$, then it is recommended to refrain from trading.

The threshold values $\beta_1$ and $\beta_2$ might probably estimated via numerical simulations using the method of section 3, combined with formula (5.11).

6. Numerical studies

6.1. Some numerical details for the algorithm of section 3

We have computationally simulated the market data as described in subsection 6.2. These data gave us initial and boundary conditions (3.2), (3.4) and (3.5), which, in turn, led us to (3.14) and (3.13), see steps 1 and 2 of subsection 3.2. Next, we have solved minimization problem (3.16) and (3.17). To minimize functional (3.16), we wrote $Mv$ and $\|v\|^2_{H^1(G_1)}$ in finite differences and, using the conjugate gradient method, have minimized the resulting discrete functional with respect to the values of the function $v$ at grid points. The starting point of the minimization procedure was $v = 0$. The regularization parameter $\gamma = 0.01$ was the same as in [9], and
it was chosen by trial and error. The step sizes $h_x$ and $h_t$ of the finite difference scheme with respect to $x$ and $t$ were $h_x = 0.01$ and $h_t = 0.0000784$ respectively. Since by (3.6) and (3.10) $G_{2y} = \{(x,t) \in (0,1) \times (0,0.00784)\}$, then we had 100 grid points with respect to each variable $x$ and $t$.

6.2. The data

We construct the stock price trajectory $s(t)$ as a solution to the stochastic differential equation $ds = \sigma s \, dW$ with the initial condition $s(0) = 100$, where $\sigma = 0.2$. We model the stock prices and then the prices of 90 days European call options of this stock during the life of the stock, assuming that the options are reissued many times with the same maturity date of 90 days. Thus, we obtain a time series $\{s(t_k)\}_{k=1}^N$, $N \geq n = 2000$. We set the payoff function for each option $f(s) = (s - 100)_+$. The generated stock price trajectory is shown on figure 1.

The probabilistic analysis of section 5 of the random variable $\zeta$ characterizes the effectiveness of an ‘ideal’ trading strategy. Thus, we consider the ideal case first. Recall that ‘ideal’ means here that this strategy is based on the knowledge of the information about the imperfection of the financial market, i.e. on the knowledge of both volatilities $\sigma$ and $\hat{\sigma}$. However, in the real market data only approximate values of $\hat{\sigma}$ are available [3].

We test total thirty three (33) values of $\hat{\sigma}$. More precisely, in our computational simulations, we took the discrete values of $\hat{\sigma}$, where

$$\hat{\sigma} \in [0.05, 0.38] \text{ with the step size } h_{\hat{\sigma}} = 0.01. \quad (6.1)$$

We now generate the function, which describes the dependence of the mathematical expectation of the random variable $\zeta$ in (5.5) on $\hat{\sigma}$. Keeping in mind that $\sigma = 0.2$ in all cases, we compute for each of the discrete values of $\hat{\sigma}$ in (6.1) the mathematical expectation $p(\hat{\sigma})$ of the random variable $\zeta$. We use formula (5.2) for $p(\hat{\sigma})$, also, see (2.4). This way we obtain the function $p(\hat{\sigma})$, which is the above dependence for the ideal case.

Second, we consider a non-ideal case. More precisely, we test how our heuristic algorithm works for the computationally simulated data described in this subsection. We choose $n = 2000$
Figure 2. The trajectory of the option price with \( \hat{\sigma} = 0.1 \) generated by the Black–Scholes formula (2.3) for the stock price trajectory of figure 1. In this case, \( \sigma \) should be replaced with \( \hat{\sigma} \) in (2.3).

non-overlapping time intervals \([t_j, t_j + y]\), \(j = 1, \ldots, n\), where \( y = 1/255 \) means one dimensionless trading day, see (1.2) and (3.6). We still use the dimensionless time as in (1.1), while keeping the same notation \( t \) for brevity. For every fixed value of \( \hat{\sigma} \) indicated in (6.1), we calculate the option price \( v(s(t_j), t_j) = u(s(T - t_j), T - t_j) \), where the function \( u(s, \tau) = u(s, T - t) \) is given by Black–Scholes formula (2.3). Thus, numbers \( v(s(t_j), t_j) \) form the option price trajectory. Figure 2 displays a sample of the trajectory of the option price for \( \hat{\sigma} = 0.1 \). Based on (3.1), we set bid and ask stock prices as well as corresponding bid and ask option prices as:

\[
\begin{align*}
  s_b(t_j) &= 0.99 \cdot s(t_j) & s_a(t_j) &= 1.01 \cdot s(t_j), \\
  v_b(s(t_j), t_j) &= 0.99 \cdot v(s(t_j), t_j) & v_a(s(t_j), t_j) &= 1.01 \cdot v(s(t_j), t_j).
\end{align*}
\]

Next, we solve problem 2 of section 3 for each \( j \) on the time interval \([t_j, t_j + 2y] \) by the algorithm of that section. When doing so, we take in (3.8) \( \sigma^2(t) = \hat{\sigma}^2 \) for \( t \in [t_j, t_j + 2y] \) for all \( j = 1, \ldots, 2000 \). In particular, this solution via QRM gives us the function \( v_{\text{comp}}(s, t_j + y) \), \( s \in (s_b(t_j), s_a(t_j)) \). We set the predicted price of the option at the moment of time \( t_j + y \) as:

\[
v_{\text{pred}}(t_j + y) = v_{\text{comp}} \left( \frac{s_b(t_j) + s_a(t_j)}{2}, t_j + y \right).
\]

6.3. Results

For every discrete value of \( \hat{\sigma} \) in (6.1), we introduce the sequence \( \{\xi_j(\hat{\sigma})\}_{j=1}^n \). This sequence is similar with the sequence \( \{\xi_j\}_{j=1}^n \) in (5.4). Recall that \( v(s(t_j), t_j) \) and \( v(s(t_j + y), t_j + y) \) are true prices of the option at the moments of time \( t_j \) and \( t_j + y \) respectively. We set
Figure 3. Results of our computations. The vertical line indicates the value \( \sigma = 0.2 \) of the volatility of the stock in our numerical studies. The middle curve corresponds to the ideal case when both volatilities \( \sigma \) and \( \hat{\sigma} \) are known. This curve depicts the dependence of the mathematical expectation \( p(\hat{\sigma}) \) of the random variable \( \zeta \) on the market’s opinion about the volatility \( \hat{\sigma} \) of the option, see (5.2). Two curves, which are parallel to the middle one, are shifts of the latter by \( \sqrt{D} \), where \( D \) is the dispersion of \( \zeta \), see (5.6). The bold faced curve represents the frequency \( \overline{\zeta}(\hat{\sigma}) \) of correctly predicted profitable cases for trading of this option with the market’s opinion \( \hat{\sigma} \) of the volatility of the option. Our algorithm of section 3 was used to compute \( \zeta(\hat{\sigma}) \). The meaning of \( \overline{\zeta}(\hat{\sigma}) \) is similar with the meaning of the ideal case of the random variable \( \zeta \). Thus, it is natural that the bold faced curve lies in the trust corridor of \( \zeta \).

\[
\overline{\zeta}(\hat{\sigma}) = \begin{cases} 
1 & \text{if } v_{\text{pred}}(t_j + y) \geq v(s(t_j), t_j) \text{ and } v(s(t_j + y), t_j + y) \geq v(s(t_j), t_j), \\
0 & \text{otherwise.} 
\end{cases}
\]

(6.3)

Next, we introduce the function \( \overline{\zeta}(\hat{\sigma}) \) of the discrete variable \( \hat{\sigma} \) as:

\[
\overline{\zeta}(\hat{\sigma}) = \frac{1}{n} \sum_{j=1}^{n} \zeta(\hat{\sigma}), \quad n = 2000.
\]

(6.4)

It follows from (6.3) and (6.4) that \( \overline{\zeta}(\hat{\sigma}) \) is the frequency of correctly predicted profitable cases for trading of this option with the market’s opinion \( \hat{\sigma} \) of the volatility of the option. Predictions are performed by our algorithm of section 3. Comparison of (5.4) and (5.5) with (6.3) and (6.4) shows that \( \overline{\zeta}(\hat{\sigma}) \) is similar with the ideal case of the random variable \( \zeta \). The bold faced curve on figure 3 depicts the graph of the function \( \zeta(\hat{\sigma}) \). The middle non-horizontal curve on figure 3 depicts the graph of the function \( p(\hat{\sigma}) \), which was constructed in subsection 6.2 for the ideal case. The upper and the lowest curves on figure 3 display the shifts of the ideal curve up and down by \( \sqrt{D} \), where \( D \) is the dispersion of \( \zeta \) and \( D \) is given in (5.6). In other words, there is a high probability chance that the values of \( \zeta \) are contained in the trust corridor between these two curves. The vertical line indicates the ‘critical’ value \( \hat{\sigma} = \sigma = 0.2 \), where \( \sigma = 0.2 \) is the volatility of the stock.

One can see from the bold faced curve of figure 3 that as long as \( \hat{\sigma} \) is either rather close to \( \sigma \) or \( \hat{\sigma} < \sigma \), the short position of this option represents a significant risk. However, when
\( \hat{\sigma} \) becomes less than \( \approx 0.7 \sigma \), the probability of the profit in the short position increases. This coincides with the prediction of our theory.

The bold faced curve is an analog of the middle curve of the mathematical expectation of the random variable \( \zeta \) in the ideal case. Since the bold faced curve on figure 3 lies within the trust corridor of the ideal algorithm, then we conclude that our prediction accuracy of profitable cases is comparable with the ideal one.

Unlike the ideal case, in a realistic scenario of the financial market data of, e.g. [3] only the information about the approximate values of \( \hat{\sigma} \) is available. It is this information, which was used in [9, 12] and, in particular, in tables 1 and 2.

Thus, our results support the following trading strategy in the non-ideal case:

**Trading strategy for the non-ideal case:**

Let \( \eta > 0 \) be a threshold number, which should be determined numerically by trial and error. For example, \( \eta \) might probably be linked with the transaction cost. Let \( v_{\text{pred}}(t_j + y) \) be the number defined in (6.2).

(a) If \( v_{\text{pred}}(t_j + y) \geq v(s(t_j), t_j) + \eta \), then it is recommended to buy the option at the current trading day \( t_j \) and sell it on the next trading \( t_j + y \).

(b) If \( v_{\text{pred}}(t_j + y) < v(s(t_j), t_j) - \eta \), then it is recommended to go short at the current trading day \( t_j \), and to follow by closing the short position at the trading day \( t_j + y \).

(c) If \( v(s(t_j), t_j) - \eta \leq v_{\text{pred}}(t_j + y) < v(s(t_j), t_j) + \eta \), then it is recommended to refrain from trading.

We believe that our results support the following two hypotheses:

**Hypothesis 1:** the reason why the heuristic algorithm of [9] and section 3 performs well is that it likely forecasts in many cases the signs of the differences \( \sigma - \hat{\sigma} \) for the next trading day ahead of the current one.

**Hypothesis 2:** since the maximal value of \( \zeta(\hat{\sigma}) \) in the bold faced curve of figure 3 is 0.515, which is rather close to the value of 0.5577 in the 'Precision' column of table 1 and in the second column of table 2, then we probably had in those tested real market data about 56% of options, in which \( \sigma - \hat{\sigma} < 0 \).

7. Concluding remarks

We have considered a mathematical model, in which two markets are in place: the stock market and the options market. We have assumed that the market is imperfect. More precisely, we have assumed that agents of the option market have their own idea about the volatility \( \hat{\sigma} \) of the option, and this idea might be different from the volatility \( \sigma \) of the stock. We have proven that if \( \sigma \neq \hat{\sigma} \), then there is an opportunity for a winning strategy. A rigorous probabilistic analysis was carried out. This analysis has shown that the mathematical expectation of the correctly guessed option price movements can be obtained, and it depends on the difference between \( \sigma \) and \( \hat{\sigma} \).

We have considered both ideal and non-ideal cases. In the ideal case, both volatilities \( \sigma \) and \( \hat{\sigma} \) are known. In the more realistic non-ideal case, however, only the volatility \( \hat{\sigma} \approx \sigma_{\text{impl}} \) of the option is known from the market data, see, e.g. [3]. We have demonstrated in our numerical simulations that the accuracy of our prediction of profitable cases by the algorithm of [9] for the non-ideal case is comparable with that accuracy for the ideal case.

These results led us to two hypotheses. The first hypothesis is that our algorithm of [9] actually forecasts in many cases the signs of the differences \( \sigma - \hat{\sigma} \) for the next trading day.
ahead of the current one. Our second hypothesis is based on our above results as well as on the ‘Precision’ column in table 1 and the second column in table 2. This second hypothesis tells one that probably about 56% out of tested 23 549 options of [12] with the real market data had $\sigma - \hat{\sigma} < 0$.

A new convergence analysis of our algorithm was carried out. To do this, the technique of [10] was modified and simplified for our specific case of the 1D parabolic equation with the reversed time. We have lifted here the assumption of [9] that the time interval $(0, 2\gamma)$ is sufficiently small. Indeed, even though we actually work with a small number $2\gamma$ in our computations, that assumption might require even smaller values. In addition, we have derived a stability estimate for problem 3 of subsection 4.1, which was not done in [9].

Acknowledgments

The work of A A Shaninan was supported by the Russian Foundation for Basic Research, Grant Number 20-07-00285.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Michael V Klibanov https://orcid.org/0000-0001-5179-6879

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