ANALYSIS OF $U_q(sl(m+1))$-SYMMETRIES ON QUANTUM $n$-SPACES

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Abstract. In this paper, the module-algebra structures of $U_q(sl(m+1))$ on the quantum $n$-space $A_q(n)$ are studied. We characterize all module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$ and $A_q(3)$ when $m \geq 2$. The module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ are also considered for any $n \geq 4$.

1. Introduction

The actions of Hopf algebras [10] and their generalizations (see, e.g., [6]) play an important role in the quantum group theory [13, 14] and in its various application in physics [5]. However, it was long time believed that the quantum plane [14] admits only one special symmetry [17] inspired by the action of $U_q(sl(2))$ (in other words the $U_q(sl(2))$-module algebra structure [13]). In [11], the quantum $n$-space is equipped with the special $U_q(sl(m+1))$-module algebra structure via a certain $q$-differential operator realization. Then it was shown [7], that the $U_q(sl(2))$-module algebra structure on the quantum plane is much more diverse and consists of 8 non-isomorphic cases [7, 8]. The full classification was given in terms of the introduced so-called weight. Its introduction follows from the general form of an automorphism of the quantum plane [11]. Some properties of the actions of commutative Hopf algebras on quantum polynomials were studied in [2, 3].

Here we consider the actions of quantum universal enveloping algebra $U_q(sl(m+1))$ on the quantum $n$-space $A_q(n)$. We use the method of weights [7, 8] to classify some actions in terms of the introduced action matrices. Then we present the Dynkin diagrams for the obtained actions and find their classical limit. A special case discussed in this paper was also included in [11].

This work is organized as follows. In Section 2, we give the necessary preliminary information and notation, as well as prove an important lemma about action on generators and any elements of the quantum $n$-space. In Section 3, we present a general idea about how to connect actions of $U_q(sl(2))$ with those of $U_q(sl(m+1))$ and characterize all the module-algebra structures of $U_q(sl(m+1))$ on the quantum plane when $m \geq 2$. Section 4 is devoted to classification of $U_q(sl(m+1))$ actions on the quantum 3-space $A_q(3)$ using the method of weights [7, 8] for $m \geq 2$. Then we present the classical limit of the obtained actions together with the Dynkin diagrams. In Section 5, we study the module-algebra structures of $U_q(sl(m+1))$ on the quantum $n$-space for $n \geq 4$.

2010 Mathematics Subject Classification. 81R50, 16T20, 17B37, 20G42, 16S40.

Key words and phrases. quantum enveloping algebra, quantum space and quantum plane, Hopf action, module algebra, Verma representation, projection, weight.
2. Preliminaries

In this paper, all algebras, modules and linear spaces are over the field \( \mathbb{C} \) of complex numbers.

Let \( H \) be a Hopf algebra whose comultiplication is \( \Delta \), counit is \( \varepsilon \) and antipode is \( S \) and let \( A \) be a unital algebra with unit 1. Using the Sweedler notation, we set \( \Delta(h) = \sum_i h_i^0 \otimes h_i^1 \).

**Definition 2.1.** By a structure of \( H \)-module algebra (or say, \( H \)-symmetry) on \( A \), we mean a homomorphism \( \pi : H \to \text{End}_A \) such that:

1. \( \pi(h)(ab) = \sum_i \pi(h_i^0)(a)\pi(h_i^1)(b) \) for all \( h \in H, a, b \in A \),
2. \( \pi(h)(1) = \varepsilon(h)1 \) for all \( h \in H \).

The structures \( \pi_1, \pi_2 \) are said to be isomorphic, if there exists an automorphism \( \psi \) of \( A \) such that \( \psi\pi_1(h)\psi^{-1} = \pi_2(h) \) for all \( h \in H \).

Throughout this paper we assume that \( q \in \mathbb{C}\setminus\{0\} \) and \( q \) is not a root of unity.

For any integer \( n \), we introduce the \( q \)-integer by \( (n)_q = \frac{q^n - 1}{q - 1} \). They were introduced by Heine \cite{10} and are called the Heine numbers or \( q \)-deformed numbers \cite{12}. If \( n > 0 \),

\[
(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.
\]

First, we introduce the definition of \( U_q(sl(m+1)) \).

**Definition 2.2.** The quantum universal enveloping algebra \( U_q(sl(m+1)) \) \( (m \geq 1) \) as the algebra is generated by \( (e_i, f_i, k_i, k_i^{-1})_{1 \leq i \leq m} \) and the relations

\[
\begin{align*}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad (2.1) \\
k_i e_j k_i^{-1} &= q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad (2.2) \\
[e_i, f_j] &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \quad (2.3) \\
e_i e_j &= e_j e_i \quad \text{and} \quad f_i f_j = f_j f_i, \quad \text{if } a_{ij} = 0, \quad (2.4)
\end{align*}
\]

if \( a_{ij} = -1 \),

\[
\begin{align*}
e_i^2 e_j - (q + q^{-1})e_i e_j e_i + e_j e_i^2 &= 0, \quad (2.5) \\
f_i^2 f_j - (q + q^{-1})f_i f_j f_i + f_j f_i^2 &= 0, \quad (2.6)
\end{align*}
\]

where for any \( i, j \in \{1, 2, \cdots, m\}, a_{ii} = 2 \) and \( a_{ij} = 0 \), if \( |i - j| > 1; a_{ij} = -1 \), if \( |i - j| = 1 \).

The standard Hopf algebra structure on \( U_q(sl(m+1)) \) is determined by

\[
\begin{align*}
\Delta(e_i) &= 1 \otimes e_i + e_i \otimes k_i, \quad (2.7) \\
\Delta(f_i) &= k_i^{-1} \otimes f_i + f_i \otimes 1, \quad (2.8) \\
\Delta(k_i) &= k_i \otimes k_i, \quad \Delta(k_i^{-1}) = k_i^{-1} \otimes k_i^{-1}, \quad (2.9)
\end{align*}
\]

\[
\begin{align*}
\varepsilon(k_i) &= \varepsilon(k_i^{-1}) = 1, \quad (2.10) \\
\varepsilon(e_i) &= \varepsilon(f_i) = 0, \quad (2.11)
\end{align*}
\]
Proof. Let \( A_q(n) \) be a unital algebra generated by \( n \) generators \( x_i \) for \( i \in \{1, \ldots, n\} \) satisfying the relations
\[
x_i x_j = qx_j x_i \quad \text{for any} \ i > j.
\] (2.14)

If \( n = 2 \), \( A_q(2) \) is called a quantum plane (see [13]). In this case, Duplij and Sinel’shchikov studied the classification of \( U_q(sl(2)) \)-module algebra structures on \( A_q(2) \) in [7].

The next proposition is very important for us to study the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(n) \) (see [14]).

Proposition 2.4. Let \( \varphi \) be an automorphism of \( A_q(n) \), then there exist nonzero constants \( \alpha_i \in \mathbb{C} \) for \( i \in \{1, \ldots, n\} \) such that
\[
\varphi : x_i \rightarrow \alpha_i x_i.
\]

Finally, we present a lemma which is useful for us in checking the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(n) \).

Lemma 2.5. Given the module-algebra actions of the generators \( k_i, e_i, f_i \) of \( U_q(sl(m+1)) \) on \( A_q(n) \) for \( i \in \{1, \ldots, m\} \), if an element in the ideal from the relations (2.1)- (2.6) of \( U_q(sl(m+1)) \) acting on the generators \( x_i \) of \( A_q(n) \) produces zero for \( i \in \{1, \ldots, n\} \), then this element acting on any \( v \in A_q(n) \) produces zero.

Proof. Here, we only prove that if \( e_i^2 e_{i+1}(x) - (q + q^{-1}) e_i e_{i+1} e_i(x) + e_{i+1} e_i^2(x) = 0 \) and \( e_i^2 e_{i+1}(y) - (q + q^{-1}) e_i e_{i+1} e_i(y) + e_{i+1} e_i^2(y) = 0 \), then \( e_i^2 e_{i+1}(x y) - (q + q^{-1}) e_i e_{i+1} e_i(x y) + e_{i+1} e_i^2(x y) = 0 \) where \( x, y \) are both generators of \( A_q(n) \). The other relations can be proved similarly.

\[
e_i^2 e_{i+1}(x y) - (q + q^{-1}) e_i e_{i+1} e_i(x y) + e_{i+1} e_i^2(x y) = e_i^2(x e_{i+1}(y) + e_{i+1}(x)k_i(y)) - (q + q^{-1}) e_i e_{i+1}(x e_i(y) + e_i(x)k_i(y)) + e_i e_{i+1}(x e_i(y) + e_i(x)k_i(y)) = e_i(x e_{i+1}(y) + e_{i+1}(x)k_i(y)) + e_i(x e_{i+1}(y) + e_{i+1}(x)k_i(y)) + e_i e_{i+1}(x e_i(y) + e_i(x)k_i(y)) + e_i e_{i+1}(x e_i(y) + e_i(x)k_i(y)) = (e_i^2 e_{i+1}(y) - (q + q^{-1}) e_i e_{i+1} e_i(y) + e_{i+1} e_i^2(y)) + (e_i^2 e_{i+1}(x) - (q + q^{-1}) e_i e_{i+1} e_i(x) + e_{i+1} e_i^2(x)) = -q + q^{-1} e_i(x) e_i e_{i+1}(y) + e_{i+1} e_i^2(x y).
\]
Thus, the lemma holds. □

Therefore, by Lemma 2.3, in checking, whether the relations of \( U_q(sl(m+1)) \), acting on any \( v \in A_q(n) \), produce zero, we only need to check, whether they, acting on the generators \( x_1, \cdots, x_n \), produce zero.

3. Classification of \( U_q(sl(m+1)) \)-Symmetries on \( A_q(2) \)

In this section, we study the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(2) \) where \( m \geq 2 \).

According to the paper [7] by S.Duplij and S.Sinel’shchikov, we have known all module-algebra structures of \( U_q(sl(2)) \) on \( A_q(2) \). For \( U_q(sl(3)) \), there are two sub-Hopf algebras which are isomorphic to \( U_q(sl(2)) \). One is generated by \( k_1, e_1 \) and \( f_1 \). Denote this algebra by \( A \). The other denoted by \( B \) is generated by \( k_2, e_2 \) and \( f_2 \).

By the definition of module algebra of one Hopf algebra, the module-algebra structures on \( A_q(2) \) of these two sub-Hopf algebras are kinds of those in [7]. All kinds of module-algebra structures of \( A \) and \( B \) on \( A_q(2) \) are as follows (here, \( i = 1, 2 \):

\[
(1) \quad k_i(x_1) = \pm x_1, \quad k_i(x_2) = \pm x_2, \\
e_i(x_1) = e_i(x_2) = f_i(x_1) = f_i(x_2) = 0.
\]

\[
(2) \quad k_i(x_1) = qx_1, \quad k_i(x_2) = q^{-1}x_2, \\
e_i(x_1) = 0, \quad e_i(x_2) = \tau_i x_1, \\
f_i(x_1) = \tau_i^{-1} x_2, \quad f_i(x_2) = 0,
\]

where \( \tau_i \in \mathbb{C}\{0\} \).

\[
(3) \quad k_i(x_1) = qx_1, \quad k_i(x_2) = q^{-2}x_2, \\
e_i(x_1) = 0, \quad e_i(x_2) = b_i, \\
f_i(x_1) = b_i^{-1} x_1 x_2, \quad f_i(x_2) = -qb_i^{-1} x_2^2,
\]

where \( b_i \in \mathbb{C}\{0\} \).

\[
(4) \quad k_i(x_1) = q^2 x_1, \quad k_i(x_2) = q^{-1}x_2, \\
e_i(x_1) = -qc_i^{-1} x_2^2, \quad e_i(x_2) = c_i^{-1} x_1 x_2, \\
f_i(x_1) = c_i, \quad f_i(x_2) = 0,
\]

where \( c_i \in \mathbb{C}\{0\} \).

\[
(5) \quad k_i(x_1) = q^{-2} x_1, \quad k_i(x_2) = q^{-1}x_2, \\
e_i(x_1) = a_i, \quad e_i(x_2) = 0, \\
f_i(x_1) = -qa_i^{-1} x_2^2 + t_i x_2^4, \quad f_i(x_1) = -qa_i^{-1} x_1 x_2 + s_i x_2^3,
\]

where \( a_i \in \mathbb{C}\{0\} \), \( t_i, s_i \in \mathbb{C} \).
Theorem 3.1. \( U_q(sl(3)) \)-symmetries on \( A_q(2) \) up to isomorphisms and their classical limits are as follows

| \( U_q(sl(3)) \)-symmetries | Classical limit \( sl_3 \)-actions on \( \mathbb{C}[x_1, x_2] \) |
|---------------------------|-----------------------------|
| \( k_i(x_1) = \pm x_1, \quad k_i(x_2) = \pm x_2, \) | \( h_i(x_1) = 0, \quad h_i(x_2) = 0, \) |
| \( k_j(x_1) = \pm x_1, \quad k_j(x_2) = \pm x_2, \) | \( h_j(x_1) = 0, \quad h_j(x_2) = 0, \) |
| \( e_i(x_1) = e_i(x_2) = f_i(x_1) = f_i(x_2) = 0, \) | \( e_i(x_1) = e_i(x_2) = f_i(x_1) = f_i(x_2) = 0, \) |
| \( f_j(x_1) = x_2, \quad f_j(x_2) = 0 \) | \( f_j(x_1) = x_2, \quad f_j(x_2) = 0 \) |

for any \( i = 1, j = 2 \) or \( i = 2, j = 1 \). Additively, there are no isomorphisms between these four kinds of module-algebra structures.

Proof. Denote the six cases of module-algebra structures of \( A \) (resp. \( B \)) on \( A_q(2) \) by \((A1), \cdots, (A6)\) (resp. by \((B1), \cdots, (B6)\)) respectively. Then, for studying the module-algebra structures of \( U_q(sl(3)) \) on \( A_q(2) \), we only need to check that whether the actions of \( A \) and those of \( B \) are compatible. In other words, we only need to check that

\[
k_1 e_2(u) = q^{-1} e_2 k_1(u), \quad k_1 f_2(u) = q f_2 k_1(u), \quad (3.1)
\]
where $u \in \{x_1, x_2\}$. Here, since the actions of $A$ and $B$ in $U_q(sl(3))$ can be interchanged, we only need to check 21 kinds of cases, i.e., whether $(A_i)$ is compatible with $(B_j)$ for any $1 \leq i \leq j \leq 6$. By some computations, we can find that $(A1)$ is compatible with $(B1)$, $(A2)$ is compatible with $(B5)$ only when $s_2 = 0$, $t_2 = 0$, $(A2)$ is compatible with $(B6)$ only when $u_2 = 0$, $v_2 = 0$, $(A5)$ is compatible with $(B6)$ only when $s_1 = t_1 = u_2 = v_2 = 0$ and any other two are not compatible. Here, for example, we only check whether $(A5)$ and $(B6)$ are compatible. First, we need to check $k_1e_2(x_1) = q^{-1}e_2k_1(x_1)$. In this case, we obtain $k_1e_2(x_1) = -q^{-2}d_1^{-1}x_1x_2 + q^{-6}u_2x_1^3$ and $q^{-1}e_2k_1(x_1) = -q^{-2}d_1^{-1}x_1x_2 + q^{-3}u_2x_1^3$. Therefore, if $k_1e_2(x_1) = q^{-1}e_2k_1(x_1)$, we must have $u_2 = 0$. Similarly, in checking $k_1e_2(x_2) = q^{-1}e_2k_1(x_2)$ and $k_2f_1(u) = qf_1k_2(u)$ for $u \in \{x_1, x_2\}$, we obtain $v_2 = 0$, $s_1 = 0$ and $t_1 = 0$. Then, other relations in (3.1)-(3.7) are easily checked when $u_2 = 0$, $v_2 = 0$, $s_1 = 0$ and $t_1 = 0$. In this case, all actions are isomorphic to that with $a_1=1$, $d_2 = 1$. The desired isomorphism is given by $\varphi : x_1 \rightarrow a_1x_1$, $x_2 \rightarrow d_2x_2$. Other cases can be checked similarly. It should be pointed out that since all the automorphisms of $A_q(2)$ commute with the actions of $k_1$ and $k_2$, the first 16 kinds of actions of $U_q(sl(3))$ in the first case are pairwise nonisomorphic. Thus, the classical limits in the table above are obtained.

Since every automorphism of $A_q(2)$ commutes with the actions of $k_1$ and $k_2$ and the actions of $k_1$ and $k_2$ in the four kinds of actions are different, there are no isomorphisms between these four kinds of module-algebra structures. 

Denote the actions of $U_q(sl(2))$ on $A_q(2)$ in Case (1), Case (2), Case (5) with $t_1 = s_1 = 0$ and Case (6) with $u_1 = v_1 = 0$ by $*1$, $*2$, $*3$ and $*4$ respectively. If $*s$ and $*t$ are compatible, in other words, they determine a $U_q(sl(3))$-module algebra structure on $A_q(2)$, we use an edge connecting $*s$ and $*t$, since $k_1$, $e_1$, $f_1$ and $k_2$, $e_2$, $f_2$ are symmetric in $U_q(sl(3))$. Therefore, we can use the following diagrams to express all the module algebra structures of $U_q(sl(3))$ on $A_q(2)$ in Theorem 3.1

\[ *1 \rightarrow 1 \rightarrow *2 \rightarrow *3. \] 

(3.8)

Here, every two adjacent vertices corresponds to two classes of module-algebra structures of $U_q(sl(3))$ on $A_q(2)$. For example, $*2 \rightarrow *3$ corresponds to the following two kinds of module-algebra structures of $U_q(sl(3))$ on $A_q(2)$: one is that the actions of $k_1$, $e_1$, $f_1$ are of type *2 and the actions of $k_2$, $e_2$, $f_2$ are of type *3; the other is that the actions of $k_1$, $e_1$, $f_1$ are of type *3 and the actions of $k_2$, $e_2$, $f_2$ are of type *2.
Next, we study the module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$ for $m \geq 3$. The corresponding Dynkin diagram of $sl(m+1)$ with $m$ vertices is as follows:

$\circ \quad \circ \quad \cdots \quad \circ \quad \circ \quad \circ$

In $U_q(sl(m+1))$, every vertex corresponds to one Hopf subalgebra isomorphic to $U_q(sl(2))$ and two adjacent vertices correspond to one Hopf subalgebra isomorphic to $U_q(sl(3))$. Therefore, for studying the module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$, we need to endow every vertex one kinds of actions of $U_q(sl(2))$ on $A_q(2)$.

Moreover, there are some important observations:

1. Since two adjacent vertices in the Dynkin diagram correspond to one Hopf subalgebra isomorphic to $U_q(sl(3))$, by Theorem 3.1, the action of $U_q(sl(2))$ on $A_q(2)$ (on every vertex) should be of the following four kinds of possibilities: $*1$, $*2$, $*3$ and $*4$. Moreover, every two adjacent vertices should be of the types in $(3.8)$.

2. Except $*1$, no other type of actions of $U_q(sl(2))$ on $A_q(2)$ can be endowed with two different vertices simultaneously, since the relations (2.2), acting on $x_1$, $x_2$ producing zero, can not be satisfied.

3. If every vertex in the Dynkin diagram of $sl(m+1)$ is endowed an action of $U_q(sl(2))$ on $A_q(2)$ which is not Case $*1$, any two vertices which are not adjacent can not be endowed with the types which are adjacent in $(3.8)$.

By the above discussion, we can obtain the following theorem immediately.

**Theorem 3.2.** For any $m \geq 3$, all module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$ are as follows:

$$k_i(x_1) = \pm x_1, \quad k_i(x_2) = \pm x_2,$$

$$e_i(x_1) = e_i(x_2) = f_i(x_1) = f_i(x_2) = 0,$$

for any $i \in \{1, \cdots, m\}$.

4. **Classification of $U_q(sl(m+1))$-symmetries on $A_q(3)$**

In this section we study the module-algebra structures of $U_q(sl(m+1))$ on $A_q(3)$. As in Section 3, we first study the actions of $U_q(sl(2))$ on $A_q(3)$.

Let us assume that $U_q(sl(2))$ is generated by $k$, $e$, $f$. By the definition of the module algebra, it is easy to see that any action of $U_q(sl(2))$ on $A_q(3)$ is determined by such $3 \times 3$ matrix with entries from $A_q(3)$:

$$M \overset{def}{=} \begin{bmatrix} k(x_1) & k(x_2) & k(x_3) \\ e(x_1) & e(x_2) & e(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{bmatrix},$$  \hspace{1cm} (4.1)

which is called the action matrix (see [7]). Given a $U_q(sl(2))$-module algebra structure on $A_q(3)$. Obviously, the action of $k$ determines an automorphism of $A_q(3)$. Therefore, by Proposition [2.3], we can set

$$M_k \overset{def}{=} \begin{bmatrix} k(x_1) & k(x_2) & k(x_3) \end{bmatrix} = \begin{bmatrix} \alpha x_1 & \beta x_2 & \gamma x_3 \end{bmatrix},$$  \hspace{1cm} (4.2)

where $\alpha, \beta, \gamma$ are non-zero complex numbers. So, every monomial $x_1^m x_2^n x_3^s \in A_q(3)$ is an eigenvector for $k$ and the associated eigenvalue $\alpha^m \beta^n \gamma^s$ is called the weight of this monomial, which will be written as $wt(x_1^m x_2^n x_3^s) = \alpha^m \beta^n \gamma^s$. 


Therefore, we obtain that
\begin{equation}
\mathbf{M}_{ef} \overset{\text{def}}{=} \begin{bmatrix}
e(x_1) & e(x_2) & e(x_3) \\
f(x_1) & f(x_2) & f(x_3)
\end{bmatrix},
\end{equation}
(4.3)
where the relation \(M = (a_{ij}) \times B = (b_{ij})\) means that if for every pair of indices \(i, j\) such that both \(a_{ij}\) and \(b_{ij}\) are nonzero, one has \(a_{ij} = b_{ij}\).

In the following, we denote the \(i\)-th homogeneous component of \(M\), whose elements are just the \(i\)-th homogeneous components of the corresponding entries of \(M\), by \((M)_i\).

Set
\begin{equation}
(M)_0 = \begin{bmatrix}
0 & 0 & 0 \\
a_0 & b_0 & c_0 \\
d_0 & e_0 & f_0
\end{bmatrix}_0,
\end{equation}
where \(a_0, b_0, c_0, d_0, e_0, f_0 \in \mathbb{C}\). Then, we obtain
\begin{equation}
\text{wt}((M)_0) \times \begin{bmatrix}
0 & 0 & 0 \\
q^2\alpha & q^2\beta & q^2\gamma \\
q^{-2}\alpha & q^{-2}\beta & q^{-2}\gamma
\end{bmatrix}_0 \times \begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}_0.
\end{equation}
(4.4)
An application of \(e\) and \(f\) to Equation (2.13) gives the following six equalities:
\begin{align}
x_2e(x_1) - q\beta e(x_1)x_2 &= qx_1e(x_2) - e(x_2)x_1, \\
f(x_1)x_2 - q^{-1}\beta^{-1}x_2f(x_1) &= q^{-1}f(x_2)x_1 - q^{-1}x_1f(x_2), \\
x_3e(x_1) - q\gamma e(x_1)x_3 &= qx_1e(x_3) - e(x_3)x_1, \\
f(x_1)x_3 - q^{-1}\gamma^{-1}x_3f(x_1) &= q^{-1}f(x_3)x_1 - q^{-1}x_1f(x_3), \\
x_3e(x_2) - q\gamma e(x_2)x_3 &= qx_2e(x_3) - \beta e(x_3)x_2, \\
f(x_2)x_3 - q^{-1}\gamma^{-1}x_3f(x_2) &= q^{-1}f(x_3)x_2 - q^{-1}x_2f(x_3).
\end{align}
(4.5-4.10)
After projecting the above six relations to \((A_4(3))_1\), we get
\begin{align}
a_0(1 - q\beta)x_2 &= b_0(q - \alpha)x_1, \\
d_0(q - \beta^{-1})x_2 &= e_0(1 - q\alpha^{-1})x_1, \\
a_0(1 - q\gamma)x_3 &= c_0(q - \alpha)x_1, \\
d_0(q - \gamma^{-1})x_3 &= f_0(1 - q\alpha^{-1})x_1, \\
b_0(1 - q\gamma)x_3 &= c_0(q - \beta)x_2, \\
e_0(q - \gamma^{-1})x_3 &= f_0(1 - q\beta^{-1})x_2.
\end{align}
Therefore, we obtain that
\begin{align}
a_0(1 - q\beta) &= b_0(q - \alpha) = d_0(q - \beta^{-1}) = e_0(1 - q\alpha^{-1}) \\
= a_0(1 - q\gamma) = c_0(q - \alpha) = d_0(q - \gamma^{-1}) = f_0(1 - q\alpha^{-1}) \\
= b_0(1 - q\gamma) = c_0(q - \beta) = e_0(q - \gamma^{-1}) = f_0(1 - q\beta^{-1}) = 0.
\end{align}
Then, we have
\begin{align}
a_0 \neq 0 \Rightarrow \beta = q^{-1}, \gamma = q^{-1}, \\
b_0 \neq 0 \Rightarrow \alpha = q, \gamma = q^{-1}, \\
c_0 \neq 0 \Rightarrow \alpha = q, \beta = q, \\
f_0 \neq 0 \Rightarrow \alpha = q, \beta = q.
\end{align}
(4.11-4.13)
By \[1.4\] and using the above six equalities, we have only seven possibilities as follows:

\[
\begin{bmatrix}
  * & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & *
\end{bmatrix}_0 \Rightarrow \alpha = q^{-2}, \beta = q^{-1}, \gamma = q^{-1}, \quad (4.14)
\]

\[
\begin{bmatrix}
  0 & * & 0 \\
  0 & 0 & 0 \\
  0 & 0 & *
\end{bmatrix}_0 \Rightarrow \alpha = q, \beta = q^{-2}, \gamma = q^{-1}, \quad (4.15)
\]

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  * & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}_0 \Rightarrow \alpha = q, \beta = q, \gamma = q^{-2}, \quad (4.16)
\]

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & * & 0 \\
  0 & 0 & 0
\end{bmatrix}_0 \Rightarrow \alpha = q^2, \beta = q^{-1}, \gamma = q^{-1}, \quad (4.17)
\]

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & *
\end{bmatrix}_0 \Rightarrow \alpha = q, \beta = q, \gamma = q^2, \quad (4.18)
\]

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}_0 \Rightarrow \alpha = q, \beta = q, \gamma = q, \quad (4.19)
\]

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}_0 \Rightarrow \alpha = 0, \beta = q, \gamma = q, \quad (4.20)
\]

Here, in the above matrices, \(\ast\) just means the entry in the corresponding position is nonzero.

For the 1-st homogeneous component, since \(wt(e(x_1)) = q^2 wt(x_1) \neq wt(x_1)\), we have \((e(x_1))_1 = a_1 x_2 + b_1 x_3\), for some \(a_1, b_1 \in \mathbb{C}\). Therefore, we set

\[
(M_{ef})_1 = \begin{bmatrix}
  a_1 x_2 + b_1 x_3 & a_2 x_1 + b_2 x_3 & a_3 x_1 + b_3 x_2 \\
  c_1 x_2 + d_1 x_3 & c_2 x_1 + d_2 x_3 & c_3 x_1 + d_3 x_2
\end{bmatrix},
\]

(4.21)

where \(a_i, b_i, c_i \in \mathbb{C}\) for \(i \in \{1, 2, 3\}\).

After projecting \((A_q(3))_2\) to \((A_q(3))_2\), we can obtain

\[
a_1(1 - q \beta) x_2^2 = b_1(1 - q^2 \beta) x_1 x_2 = a_2(q - \alpha) x_1^2 = b_2(q - qa) x_1 x_3,
\]

\[
c_1(1 - q^{-1} \beta^{-1}) x_2^2 = d_1(q - q^{-1} \beta^{-1}) x_2 x_3 = d_2(1 - \alpha^{-1}) x_1 x_3 = c_2(q^{-1} - \alpha^{-1}) x_1^2,
\]

\[
a_3(q - q \gamma) x_2 x_3 = b_3(1 - q \gamma) x_2^2 = a_3(q - \alpha) x_1^2 = b_3(q - qa) x_1 x_2,
\]

\[
c_3(1 - \gamma^{-1}) x_2 x_3 = d_3(1 - q^{-1} \gamma^{-1}) x_2^2 = d_3(1 - \alpha^{-1}) x_1 x_2 = c_3(q^{-1} - \alpha^{-1}) x_1^2,
\]

\[
a_2(q - q \gamma) x_1 x_3 = b_2(1 - q \gamma) x_3^2 = a_3(q^2 - \beta) x_1 x_2 = b_3(q - \beta) x_2^2,
\]

\[
c_2(1 - \gamma^{-1}) x_1 x_3 = d_2(1 - q^{-1} \gamma^{-1}) x_3^2 = c_3(q^{-1} - q \beta^{-1}) x_1 x_2 = d_3(q^{-1} - \beta^{-1}) x_2^2.
\]

Therefore, we have

\[
a_1 \neq 0 \Rightarrow b_1 = 0, \beta = q^{-1}, \gamma = 1, \quad b_1 \neq 0 \Rightarrow a_1 = 0, \beta = q^{-2}, \gamma = q^{-1},
\]

\[
a_2 \neq 0 \Rightarrow b_2 = 0, \alpha = q, \gamma = 1, \quad b_2 \neq 0 \Rightarrow a_2 = 0, \alpha = 1, \gamma = q^{-1},
\]

\[
a_3 \neq 0 \Rightarrow b_3 = 0, \alpha = q, \beta = q^2, \quad b_3 \neq 0 \Rightarrow a_3 = 0, \alpha = 1, \beta = q,
\]

\[
c_1 \neq 0 \Rightarrow d_1 = 0, \beta = q^{-1}, \gamma = 1, \quad d_1 \neq 0 \Rightarrow c_1 = 0, \beta = q^{-2}, \gamma = q^{-1},
\]

\[
c_2 \neq 0 \Rightarrow d_2 = 0, \alpha = q, \gamma = 1, \quad d_2 \neq 0 \Rightarrow c_2 = 0, \alpha = 1, \gamma = q^{-1},
\]

\[
c_3 \neq 0 \Rightarrow d_3 = 0, \alpha = q, \beta = q^2, \quad d_3 \neq 0 \Rightarrow c_3 = 0, \alpha = 1, \beta = q.
\]
Since \( \text{wt}((M_{ef})_1) = \left[ \begin{array}{ccc} q^2\alpha & q^2\beta & q^2\gamma \\ q^2\alpha & q^2\beta & q^2\gamma \\ q^2\alpha & q^2\beta & q^2\gamma \end{array} \right] \), by the above discussion, for the 1-st homogeneous component, we only have the following possibilities (here, \( \star_i \) in the position of \( e(u) \) means that the \( i \)-th monomial of \( e(u) \) following the \( x_1, x_2, x_3 \) order is nonzero, where \( u \in \{x_1, x_2, x_3\} \)):

\[
\begin{bmatrix}
\star_1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q^{-3}, \beta = q^{-1}, \gamma = 1,
\]

\[
\begin{bmatrix}
\star_2 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q^{-3}, \beta = q^{-2}, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & \star_1 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^{-1}, \gamma = 1,
\]

\[
\begin{bmatrix}
0 & \star_2 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = 1, \beta = q^{-3}, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & \star_1 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^2, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & \star_2 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = 1, \beta = q, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
\star_1 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^{-1}, \gamma = 1,
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
\star_2 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^{-2}, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \star_1 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^2, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \star_2 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = 1, \beta = q, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \star_1
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^{-2}, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \star_2
\end{bmatrix}_{1}
\Rightarrow \alpha = 1, \beta = q, \gamma = q^{-1},
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = q, \beta = q^3, \gamma = q^{-1}.
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{1}
\Rightarrow \alpha = 1, \beta = q, \gamma = q^{-1},
\]

Obviously, if both the 0-th homogeneous component and the 1-th homogeneous component of \( M_{ef} \) are nonzero, there are only two possibilities

\[
\left( \begin{bmatrix}
0 & \star & 0 \\
0 & 0 & 0
\end{bmatrix}_{0}, \begin{bmatrix}
0 & 0 & 0 \\
\star & 0 & 0
\end{bmatrix}_{1} \right) \Rightarrow \alpha = q, \beta = q^{-2}, \gamma = q^{-1}, \quad (4.22)
\]

\[
\left( \begin{bmatrix}
0 & 0 & 0 \\
0 & \star & 0
\end{bmatrix}_{0}, \begin{bmatrix}
0 & 0 & \star \\
0 & 0 & 0
\end{bmatrix}_{1} \right) \Rightarrow \alpha = q, \beta = q^2, \gamma = q^{-1}. \quad (4.23)
\]
Moreover, there are no possibilities when the 0-th homogeneous component of $M_{ef}$ is 0 and the 1-th homogeneous component of $M_{ef}$ has only one nonzero position. The reasons are the same as those in [7].

So, we only need to consider the following 11 possibilities:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & * & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
* & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\]

where $e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0e_0^0$.

For convenience, we denote these 11 kinds of cases in the above order by $(*)_1$, $\cdots$, $(*)_11$ respectively.

**Lemma 4.1.** For Case $(*)_1$, all $U_q(sl(2))$-module algebra structures on $A_q(3)$ are as follows:

\[
k(x_1) = \pm x_1, \ k(x_2) = \pm x_2, \ k(x_3) = \pm x_3,
\]

\[
e(x_1) = e(x_2) = e(x_3) = f(x_1) = f(x_2) = f(x_3) = 0.
\]

**Proof.** The proof is similar to that in Theorem 4.2 in [7].

**Lemma 4.2.** For Case $(*)_2$, all $U_q(sl(2))$-module algebra structures on $A_q(3)$ are

\[
k(x_1) = q^{-2}x_1, \ k(x_2) = q^{-1}x_2, \ k(x_3) = q^{-1}x_3,
\]

\[
e(x_1) = a_0, \ e(x_2) = 0, \ e(x_3) = 0,
\]

\[
f(x_1) = -qa_0^{-1}x_1^2,
\]

\[
f(x_2) = -qa_0^{-1}x_1x_2 + \xi_1x_2x_3^2 + \xi_2x_3^3 + \xi_3x_3^3,
\]

\[
f(x_3) = -qa_0^{-1}x_1x_3 + \xi_4x_2x_3^2 + (1 + q + q^2)\xi_3x_2x_3 - q^{-1}\xi_1x_3^3,
\]

where $a_0 \in \mathbb{C}\backslash\{0\}$, and $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{C}$.

**Proof.** Since $\text{wt}(M_{ef}) \cong \begin{bmatrix} 1 & q & q \qquad q^{-3} \\
q^{-4} & q & q^{-3} \\
q^{-3} & q & q^{-3} \end{bmatrix}$ and $\alpha = q^{-2}$, $\beta = q^{-1}$, $\gamma = q^{-1}$, we must have $e(x_1) = a_0$, $e(x_2) = 0$, $e(x_3) = 0$. With the same reason, $f(x_1)$, $f(x_2)$, $f(x_3)$ must be of the following forms:

\[
f(x_1) = u_1x_1^2 + u_2x_1x_2 + u_3x_1x_3 + u_4x_2x_3 + u_5x_2x_3 + u_6x_2x_3 + u_7x_2x_3 + u_8x_2x_3 + u_9x_3,
\]

\[
f(x_2) = v_1x_1x_2 + v_2x_1x_3 + v_3x_2x_3 + v_4x_2x_3 + v_5x_2x_3 + v_6x_3,
\]

\[
f(x_3) = w_1x_1x_2 + w_2x_1x_3 + w_3x_2x_3 + w_4x_2x_3 + w_5x_3 + w_6x_3,
\]

where these coefficients are in $\mathbb{C}$. Then, we consider (4.5)-(4.10). Taking $e(x_1)$, $e(x_2)$, $e(x_3)$, $f(x_1)$, $f(x_2)$, $f(x_3)$ into the six equalities, by comparing the coefficients, we obtain that

\[
f(x_1) = u_1x_1^2,
\]

\[
f(x_2) = u_1x_1x_2 + v_3x_2x_3 + v_5x_2 + v_6x_3,
\]

\[
f(x_3) = u_1x_1x_3 + v_3x_2x_3 + v_5x_2 + v_6x_3,
\]
where

The proof is similar to that in Lemma 4.2.

For Case \((\ast_3)\), Lemma 4.4.

Using \(ef(u) - fe(u) = \frac{k - k^{-1}}{q - q^{-1}}(u)\), for any \(u \in \{x_1, x_2, x_3\}\), we get \(u_1 = -qa_i^{-1}\). So we proved the lemma. \(\square\)

**Lemma 4.3.** For Case \((\ast_3)\), all \(U_q(\mathfrak{sl}(2))\)-module algebra structures on \(A_q(3)\) are as follows

\[
k(x_1) = qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^2 x_3, \quad (4.31)
\]

\[
e(x_1) = -qf_0^{-1}x_1x_3 + x_1x_2 - q\mu_2 x_1^3 + (1 + q + q^2)\mu_3 x_1 x_2, \quad (4.32)
\]

\[
e(x_2) = -qf_0^{-1}x_2x_3 + \mu_2 x_2^3 + \mu_3 x_2^3 + \mu_4 x_1^3, \quad (4.33)
\]

\[
e(x_3) = -qf_0^{-1}x_3^2, \quad (4.34)
\]

\[
f(x_1) = 0, \quad f(x_2) = 0, \quad f(x_3) = f_0, \quad (4.35)
\]

where \(f_0 \in \mathbb{C}\{0\}\), and \(\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{C}\).

**Proof.** The proof is similar to that inLemma 4.2 \(\square\)

**Lemma 4.4.** For Case \((\ast_4)\), to satisfy \((4.9)-(4.10)\), the actions of \(k, e, f\) must be of the following form

\[
k(x_1) = qx_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = x_3, \quad (4.36)
\]

\[
e(x_1) = \sum_{m \geq 0, p \geq 0, p \neq m+3} a_{m,p} x_1^{m+3} x_2^{m} x_3^{p} + \sum_{m \geq 0} d_{m+1} x_2^{m+3} x_3^{p}, \quad (4.37)
\]

\[
e(x_2) = a_2 x_1 + \sum_{m \geq 0, p \geq 0, p \neq m+3} b_{m,p} x_1^{m+2} x_2^{m+1} x_3^{p}, \quad (4.38)
\]

\[
e(x_3) = \sum_{m \geq 0, p \geq 0, p \neq m+3} c_{m,p} x_1^{m+2} x_2^{m+1} x_3^{p}, \quad (4.39)
\]

\[
f(x_1) = c_1 x_2 + \sum_{m \geq 0, p \geq 0, p \neq m+3} d_{m,p} x_1^{m+2} x_2^{m+1} x_3^{p} + \sum_{m \geq 0} h_{m} x_1^{m+1} x_2^{m} x_3^{m+1}, \quad (4.40)
\]

\[
f(x_2) = \sum_{m \geq 0, p \geq 0, p \neq m+1} e_{m} x_1^{m+3} x_2^{m+1} x_3^{p}, \quad (4.41)
\]

\[
f(x_3) = \sum_{m \geq 0, p \geq 0, p \neq m+1} g_{m} x_1^{m+2} x_2^{m+2} x_3^{p} + \sum_{m \geq 0} g_{m} x_1^{m+1} x_2^{m+2} x_3^{p}, \quad (4.42)
\]

where \(a_2, c_1 \in \mathbb{C}\{0\}\), other coefficients are in \(\mathbb{C}\) and \(\frac{a_{m,p}}{b_{m,p}} = -\frac{(m+p+1)q}{q^{p+1}(m+3-p)q}, \quad \frac{b_{m,p}}{c_{m,p}} = \frac{q^{p+1}(m+3-p)q}{(2m+4)q}, \quad \frac{d_{m,p}}{e_{m}} = \frac{q^{p+1}(m+1-p)q}{(2m+4)q}, \quad \frac{h_{m}}{g_{m}} = \frac{q^{p+1}(m+3-p)q}{(2m+4)q}, \quad \frac{d_{m}}{e_{m}} = \frac{q^{p+1}(m+3-p)q}{(2m+4)q}, \quad \frac{h_{m}}{g_{m}} = \frac{q^{p+1}(m+3-p)q}{(2m+4)q} \) for any \(m, p \geq 0\).

Especially, there are the following \(U_q(\mathfrak{sl}(2))\)-module algebra structures on \(A_q(3)\)

\[
k(x_1) = qx_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = x_3, \quad (4.36)
\]

\[
e(x_1) = 0, \quad e(x_2) = a_2 x_1, \quad e(x_3) = 0, \quad (4.37)
\]

\[
f(x_1) = a_2^{-1} x_2, \quad f(x_2) = 0, \quad f(x_3) = 0, \quad (4.38)
\]

where \(a_2 \in \mathbb{C}\{0\}\).
The proof is similar to that in Lemma 4.4. Using the equalities (4.5), (4.7) and (4.9) and by some computations, we can obtain Lemma 4.5.

Proof. In this case, we get $\alpha = q$, $\beta = q^{-1}$, $\gamma = 1$. Therefore, $wt(M_{ef}) \cong \begin{bmatrix} q^3 & q & q^2 \\ q^{-1} & q^{-3} & q^{-2} \end{bmatrix}$. Since $wt(e(x_1)) = q^3$, $wt(e(x_2)) = q$ and $wt(e(x_3)) = q^2$, using the equalities (4.3), (4.7) and (4.9) and by some computations, we can obtain $e(x_1)$, $e(x_2)$ and $e(x_3)$ as the forms in the lemma. Similarly, we also can determine the forms of $f(x_1)$, $f(x_2)$ and $f(x_3)$ in the lemma.

Moreover, it is easy to check that $\text{(4.39)-(4.38)}$ determine the module-algebra structures of $U_q(sl(2))$ on $A_q(3)$. □

Lemma 4.5. For Case (b), to satisfy (4.39)-(4.40), the actions of $U_q(sl(2))$ on $A_q(3)$ must be of the following form

\[
\begin{align*}
k(x_1) &= x_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= \sum_{m \geq 0, p \geq 0 \atop p \neq m+1} \tilde{d}_{m,p} x_1^{p+1}x_2^{2+m}x_3^m + \sum_{m \geq 0} \tilde{d}_{m} x_1^{m+2}x_2^{m+2}x_3^m, \\
e(x_2) &= \sum_{m \geq 0, p \geq 0 \atop p \neq m+1} \tilde{b}_{m,p} x_1^{p+3+m}x_3^m, \\
e(x_3) &= b_3x_2 + \sum_{m \geq 0, p \geq 0 \atop p \neq m+1} \tilde{c}_{m,p} x_2^{m+2}x_3^{m+1} + \sum_{m \geq 0} \tilde{c}_m x_1^{m+1}x_2^{m+2}x_3^m, \\
f(x_1) &= \sum_{m \geq 0, p \geq 0 \atop p \neq m+3} \tilde{d}_{m,p} x_1^{p+1}x_2x_3^{m+2} + \sum_{m \geq 0} \tilde{d}_m x_1^{m+4}x_2x_3^{m+2}, \\
f(x_2) &= d_2x_3 + \sum_{m \geq 0, p \geq 0 \atop p \neq m+3} \tilde{e}_{m,p} x_1^{p+1}x_2^2x_3^{m+2}, \\
f(x_3) &= \sum_{m \geq 0, p \geq 0 \atop p \neq m+3} \tilde{g}_{m,p} x_1^{p+m+3}x_2x_3^{m+3} + \sum_{m \geq 0} \tilde{g}_m x_1^{m+3}x_2x_3^{m+3},
\end{align*}
\]

where $b_3, d_2 \in \mathbb{C}\setminus\{0\}$, other coefficients are in $\mathbb{C}$ and \[
\begin{align*}
d_{m,p} &= \frac{(2m+2)q}{(m+p+3)q}, \\
\tilde{d}_{m,p} &= -q^2(2m+2)q^{-1}, \\
\tilde{d}_m &= \frac{(2m+2)q}{q^3(q-1)(m+p+3)q}, \\
\tilde{g}_{m,p} &= \frac{(2m+2)q}{(m+p+1)q}, \quad \tilde{g}_m = \frac{(2m+2)q}{(m+p+1)q}, \\
\tilde{c}_{m,p} &= -\frac{q^2(2m+2)q^{-1}}{(2m+4)q}, \\
\tilde{e}_{m,p} &= -\frac{q(2m+2)}{(2m+4)q}. 
\end{align*}
\]

There are the following $U_q(sl(2))$-module algebra structures on $A_q(3)$

\[
\begin{align*}
k(x_1) &= x_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= 0, \quad e(x_2) = 0, \quad e(x_3) = b_3x_2, \\
f(x_1) &= 0, \quad f(x_2) = b_3^{-1}x_3, \quad f(x_3) = 0,
\end{align*}
\]

where $b_3 \in \mathbb{C}\setminus\{0\}$.

Proof. The proof is similar to that in Lemma 4.4. □
Lemma 4.6. For Case \((*)_6\) and Case \((*)_7\), to satisfy \([4.3]-[4.10]\), the actions of \(k, e\) and \(f\) on \(A_q(3)\) are

\[
\begin{align*}
k(x_1) &= qx_1, \quad k(x_2) = q^{-2}x_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= \sigma_1 x_1 + \sum_{n \geq 0} \sigma_n x_1^{2n+5}x_2^{n+1} + \sum_{p \geq 0} \hat{\sigma}_p x_1^{p+4}x_3^{p+1} \\
&\quad + \sum_{n \geq 0, p \geq 0} \sigma_{n,p} x_1^{2n+p+6}x_2^{n+1}x_3^{p+1}, \\
e(x_2) &= b_0 + \rho x_2 x_1 + \sum_{n \geq 0} \rho_n x_1^{2n+4}x_2^{n+2} + \sum_{p \geq 0} \hat{\rho}_p x_1^{p+3}x_2x_3^{p+1} \\
&\quad + \sum_{n \geq 0, p \geq 0} \rho_{n,p} x_1^{2n+p+5}x_2^{n+2}x_3^{p+1}, \\
e(x_3) &= \tau x_1 x_3 + \sum_{n \geq 0} \tau_n x_1^{2n+4}x_2^{n+1}x_3 + \sum_{p \geq 0} \hat{\tau}_p x_1^{p+3}x_3^{p+2} \\
&\quad + \sum_{n \geq 0, p \geq 0} \tau_{n,p} x_1^{2n+p+5}x_2^{n+1}x_3^{p+2}, \\
f(x_1) &= d_1 x_3 + \sum_{p \geq 0} \lambda_p x_1^{p+1}x_3 + \sum_{n \geq 0} \hat{\lambda}_n x_1^{2n+1}x_2^{n+1} + \sum_{n \geq 0} \lambda_n x_1^{2n+2}x_2^{n+1}x_3 \\
&\quad + \sum_{n \geq 0, p \geq 0} \lambda_{n,p} x_1^{2n+p+3}x_2^{n+1}x_3^{p+2}, \\
f(x_2) &= \sum_{n \geq 0} \tilde{\tau}_n x_1^{2n+2}x_2 + \sum_{n \geq 0} \tilde{\tau}_n x_1^{2n+1}x_2^{n+2}x_3 + \sum_{n \geq 0, p \geq 0} \nu_{n,p} x_1^{2n+p+2}x_2^{n+2}x_3^{p+2}, \\
f(x_3) &= \sum_{p \geq 0} \omega_p x_1^{p+3}x_3 + \sum_{n \geq 0} \omega_n x_1^{2n+1}x_2^{n+1}x_3 + \sum_{n \geq 0} \omega_n x_1^{2n+2}x_2^{n+1}x_3^{p+3} \\
&\quad + \sum_{n \geq 0, p \geq 0} \omega_{n,p} x_1^{2n+p+2}x_2^{n+1}x_3^{p+3},
\end{align*}
\]

where \(b_0 \in \mathbb{C} \setminus \{0\}\) and other coefficients are in the \(\mathbb{C}\), and \(\sigma_p = -\frac{q^2}{(3n+6)_q}, \quad \sigma_{n,p} = \frac{-(n+2)_q}{q^{n+1}(3n+6)_q}, \quad \sigma_n = \frac{-(n+3)_q}{q^{n+1}(3n+6)_q}, \quad \sigma_p = \frac{n+3_q}{q^{n+1}(3n+6)_q}, \quad \tau_p = \frac{q(n+2)_q}{q^{n+1}(2n+5)_q}, \quad \tau_{n,p} = \frac{n+3_q}{q^{n+1}(2n+5)_q}, \quad \tau_n = \frac{n+3_q}{q^{n+1}(2n+5)_q}, \quad \tau_p = \frac{n+3_q}{q^{n+1}(2n+5)_q}, \quad \lambda_p = \frac{q(n+2)_q}{q^{n+1}(2n+5)_q}, \quad \lambda_{n,p} = \frac{(n+2)_q}{q^{n+1}(2n+5)_q}, \quad \lambda_n = \frac{(n+2)_q}{q^{n+1}(2n+5)_q}, \quad \lambda_{n,p} = \frac{(n+3)_q}{q^{n+1}(2n+5)_q}, \quad \lambda_n = \frac{(n+3)_q}{q^{n+1}(2n+5)_q}, \quad \omega_p = \frac{q(n+2)_q}{q^{n+1}(2n+2)_q}, \quad \omega_{n,p} = \frac{(n+2)_q}{q^{n+1}(2n+2)_q}, \quad \omega_n = \frac{(n+2)_q}{q^{n+1}(2n+2)_q}, \quad \omega_{n,p} = \frac{(n+2)_q}{q^{n+1}(2n+2)_q}, \quad \omega_n = \frac{(n+3)_q}{q^{n+1}(2n+2)_q}, \quad \omega_{n,p} = \frac{(n+3)_q}{q^{n+1}(2n+2)_q}, \quad \omega_n = \frac{(n+3)_q}{q^{n+1}(2n+2)_q}.

Especially, there are the following \(U_q(sl(2))\)-module algebra structures on \(A_q(3)\):

\[
\begin{align*}
k(x_1) &= qx_1, \quad k(x_2) = q^{-2}x_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= 0, \quad e(x_2) = b_0, \quad e(x_3) = 0, \\
f(x_1) &= d_1x_3 + b_0^{-1}x_1 x_2 + \sum_{p=0}^{n} \hat{d}_p x_1^{p+1}x_3^{p+2}, \quad f(x_2) = -q b_0^{-1}x_2^2, \\
f(x_3) &= -q b_0^{-1} x_2 x_3 - \sum_{p=0}^{n} q(p+1)_q \hat{d}_p x_1^{p+1}x_3^{p+3},
\end{align*}
\]

where \(n \in \mathbb{N}, d_1, \hat{d}_p \in \mathbb{C}\) for all \(p\), \(b_0 \in \mathbb{C} \setminus \{0\}\).
Proof. In these two cases, we have the same values of \( \alpha, \beta \) and \( \gamma \), i.e., \( \alpha = q, \beta = q^{-2}, \gamma = q^{-1} \). Therefore, \( \text{wt}(M_{ef}) \cong \left[ \begin{array}{ccc} q^3 & 1 & q \\ q^{-1} & q^{-4} & q^{-3} \end{array} \right] \). Using the equalities (4.9)-(4.10) and by some computations, we can obtain that \( e(x_1), e(x_2), e(x_3), f(x_1), f(x_2), f(x_3) \) are of the forms in this lemma.

Moreover, using (4.12)-(4.15), it is easy to check that \( ef(u) - fe(u) = \frac{k-k^{-1}}{q-q^{-1}}(u) \), where \( u \in \{x_1, x_2, x_3 \} \). Therefore, they determine the module-algebra structures of \( U_q(\mathfrak{sl}(2)) \) on \( A_q(3) \). \( \square \)

**Lemma 4.7.** For Case \( (*_8) \) and Case \( (*_9) \), to satisfy (4.5)-(4.10), the actions of \( k, e, f \) are of the form

\[
\begin{align*}
k(x_1) &= qx_1, \quad k(x_2) = q^2x_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= \sum_{p \geq 0} \alpha_{p}x_1^{p+3}x_3^{p+3} + \sum_{m \geq 0} \alpha_{m}x_1x_2x_1^{m+1}x_3^{2m} + \sum_{m \geq 0} \alpha_{m}x_1x_2x_1^{m+1}x_3^{2m+1} \\
&\quad + \sum_{p \geq 0, m \geq 0} \alpha_{m,p}x_1^{p+3}x_2x_1x_3^{m+2}x_3^{2m+2}, \\
e(x_2) &= \sum_{m \geq 0} \beta_{m}x_1x_2x_1^{m+2}x_3^{2m} + \sum_{m \geq 0} \beta_{m}x_1x_2x_1^{m+2}x_3^{2m+1} \\
&\quad + \sum_{p \geq 0, m \geq 0} \beta_{m,p}x_1^{p+2}x_2x_1x_3^{m+2}x_3^{2m+2}, \\
e(x_3) &= \sum_{p \geq 0} \gamma_{p}x_1x_3^{p+1}x_3^{p+3} + \sum_{m \geq 0} \gamma_{m}x_2x_1x_3^{m+1}x_3^{2m+1} + \sum_{m \geq 0} \gamma_{m}x_2x_1x_3^{m+1}x_3^{2m+2} \\
&\quad + \sum_{p \geq 0, m \geq 0} \gamma_{m,p}x_2x_1x_3^{m+2}x_3^{2m+3}, \\
f(x_1) &= c_{x_1}x_3^{2} + \sum_{p \geq 0} \gamma_{p}x_1^{p+2}x_3^{p+3} + \sum_{m \geq 0} \gamma_{m}x_2x_1x_3^{m+1}x_3^{2m+4} \\
&\quad + \sum_{m \geq 0, p \geq 0} \gamma_{m,p}x_1x_3^{m+2}x_3^{2m+5}, \\
f(x_2) &= \sum_{p \geq 0} \theta_{p}x_2x_1x_3^{p+1}x_3^{p+3} + \sum_{m \geq 0} \theta_{m}x_2x_1x_3^{m+1}x_3^{2m+4} \\
&\quad + \sum_{m \geq 0, p \geq 0} \theta_{m,p}x_2x_1x_3^{m+2}x_3^{2m+5}, \\
f(x_3) &= \sum_{p \geq 0} \eta_{p}x_3^{p+1}x_3^{p+4} + \sum_{m \geq 0} \eta_{m}x_2x_1x_3^{m+1}x_3^{2m+5} \\
&\quad + \sum_{m \geq 0, p \geq 0} \eta_{m,p}x_2x_1x_3^{m+1}x_3^{2m+6},
\end{align*}
\]

where \( e_0 \in \mathbb{C} \setminus \{0\} \) and other coefficients are in the \( \mathbb{C} \), and \( \frac{c_{x_1}}{\gamma_p} = \frac{q(p+1)a}{\gamma(m+2)}, \frac{\alpha_{m}}{\gamma_m} = \frac{q(3m+2)a}{(p+2)m}, \frac{\alpha_{m,p}}{\gamma_m} = \frac{q(3m+2)p}{(2m+3)}, \frac{\alpha_{m}}{\gamma_m} = \frac{q(3m+2)a}{(p+2)m}, \frac{\alpha_{m}}{\gamma_m} = \frac{q(3m+2)a}{(p+2)m} \), \( \frac{\gamma_{m,p}}{\gamma_m} = \frac{q(3m+2)p}{(2m+3)}, \frac{\gamma_{m,p}}{\gamma_m} = \frac{q(3m+2)p}{(2m+3)}, \frac{\gamma_{m,p}}{\gamma_m} = \frac{q(3m+2)p}{(2m+3)}, \frac{\gamma_{m,p}}{\gamma_m} = \frac{q(3m+2)p}{(2m+3)} \), \( \frac{\eta_{m,p}}{\eta_m} = \frac{q(3m+2)p}{(2m+3)}, \frac{\eta_{m,p}}{\eta_m} = \frac{q(3m+2)p}{(2m+3)} \).
There are the following $U_q(sl(2))$-module algebra structures on $A_q(3)$

\[ k(x_1) = qx_1, \quad k(x_2) = q^2x_2, \quad k(x_3) = q^{-1}x_3, \]

\[ e(x_1) = -q e_0^{-1}x_1x_2 - \sum_{p=0}^{n} \frac{q(p+1)}{(p+3)q} \alpha_p x_1^{p+3} x_3^p, \quad e(x_2) = -q e_0^{-1}x_2^2, \]

\[ e(x_3) = a_3 x_1 + e_0^{-1}x_2x_3 + \sum_{p=0}^{n} \alpha_p x_1^{p+2} x_3^{p+1}, \]

\[ f(x_1) = 0, \quad f(x_2) = e_0, \quad f(x_3) = 0, \]

where $n \in \mathbb{N}$, $a_3, \alpha_p \in \mathbb{C}$ for all $p$, $e_0 \in \mathbb{C}\setminus\{0\}$.

**Proof.** The proof is similar to that in Lemma 4.6.

**Lemma 4.8.** For Case ($*_{10}$), to satisfy (4.3)-(4.10), the actions of $k, e, f$ on $A_q(3)$ are

\[ k(x_1) = qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-2}x_3, \]

\[ e(x_1) = \sum_{n \geq 0, p \geq 0 \atop 2+2p-n \geq 0 \atop n \neq p+1} r_{n,p} x_1^{3+2p-n} x_2^n x_3^p + \sum_{p=0}^{\infty} r_{p} x_1^{2+p-n} x_2 x_3^{p+1}, \]

\[ e(x_2) = \sum_{n \geq 0, p \geq 0 \atop 2+2p-n \geq 0 \atop n \neq p+1} s_{n,p} x_1^{2+2p-n} x_2^n x_3^p, \]

\[ e(x_3) = c_0 + \sum_{n \geq 0, p \geq 0 \atop 2+2p-n \geq 0 \atop n \neq p+1} t_{n,p} x_1^{2+2p-n} x_2^n x_3^p + \sum_{p=0}^{\infty} t_{p} x_1^{p+1} x_2 x_3^{p+1}, \]

\[ f(x_1) = \sum_{n \geq 0, p \geq 0 \atop 2p-n \geq 0 \atop p \neq n+2} u_{n,p} x_1^{2p-n} x_2^n x_3^{p+1} + \sum_{n \geq 0} u_n x_1^{n+5} x_2^n x_3^{n+3}, \]

\[ f(x_2) = \sum_{n \geq 0, p \geq 0 \atop 2p-n \geq 0 \atop p \neq n+2} v_{n,p} x_1^{2p-n} x_2^n x_3^{p+1}, \]

\[ f(x_3) = \sum_{n \geq 0, p \geq 0 \atop 2p-n \geq 0 \atop p \neq n+2} w_{n,p} x_1^{2p-n} x_2^n x_3^{p+2} + \sum_{n \geq 0} w_n x_1^{n+4} x_2^n x_3^{n+4}, \]

where $c_0 \in \mathbb{C}\setminus\{0\}$ and other coefficients are in the $\mathbb{C}$, and $r_{n,p} = \frac{(n+p+1)q}{q^{2p+1}(p+1-n)q}$, $r_{p} = \frac{q^2(2p+1)q}{(2p+4)q}$, $u_{n,p} = \frac{(n+p+2)q}{q^{2p+3}(p-2-n)q}$, $w_{n,p} = \frac{(n+p+2)q}{q^{2p+3}(p-2-n)q}$, $w_n = \frac{(n+p+2)q}{q(2n+4)+q}$. Specifically, there are the following $U_q(sl(2))$-module algebra structures on $A_q(3)$

\[ k(x_1) = qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-2}x_3, \]

\[ e(x_1) = 0, \quad e(x_2) = 0, \quad e(x_3) = c_0, \]

\[ f(x_1) = c_0^{-1}x_1x_3, \quad f(x_2) = c_0^{-1}x_2x_3, \quad f(x_3) = -q c_0^{-1}x_3^2, \]
where $c_0 \in \mathbb{C}\{0\}$.

**Proof.** In this case, we have $\alpha = q, \beta = q, \gamma = q^{-2}$. Therefore, $wt(M_{cf}) \propto \begin{bmatrix} q^3 & q^3 & 1 \\ q^{-1} & q^{-1} & q^{-4} \end{bmatrix}$. Then, the proof is similar to those in the above lemmas. \hfill \Box

**Lemma 4.9.** For Case $(+1)$, to satisfy (4.5)-(4.10), the actions of $k, e$ and $f$ on $A_q(3)$ are

\[
\begin{align*}
  k(x_1) &= q^2 x_1, \\
  k(x_2) &= q^{-1} x_2, \\
  k(x_3) &= q^{-1} x_3, \\
  e(x_1) &= \sum_{m \geq 0, p \geq 0} r_{m,p} x_1^{m+2} x_2^p x_3^{2m-p} + \sum_{p \geq 0} t_p x_1^{p+4} x_2^p x_3^{p+4}, \\
  e(x_2) &= \sum_{m \geq 0, p \geq 0} s_{m,p} x_1^{m+1} x_2^{p+1} x_3^{2m-p}, \\
  e(x_3) &= \sum_{m \geq 0, p \geq 0} \tilde{t}_{n,p} x_1^{m+1} x_2^p x_3^{2m-p} + \sum_{p \geq 0} \tilde{t}_p x_1^{p+3} x_2^p x_3^{p+5}, \\
  f(x_1) &= d_0 + \sum_{m \geq 0, p \geq 0} \bar{u}_{m,p} x_1^{m+1} x_2^p x_3^{2m+2-p} + \sum_{m \geq 0} \bar{u}_m x_1^{m+1} x_2^{m+1} x_3^{m+1}, \\
  f(x_2) &= \sum_{m \geq 0, p \geq 0} \bar{u}_{m,p} x_1^{m+1} x_2^p x_3^{2m-p+1}, \\
  f(x_3) &= \sum_{m \geq 0, p \geq 0} \bar{u}_{m,p} x_1^{m+1} x_2^p x_3^{2m-p+3} + \sum_{m \geq 0} \bar{u}_m x_1^{m+1} x_2^{m+1} x_3^{m+2},
\end{align*}
\]

where $d_0 \in \mathbb{C}\{0\}$ and other coefficients are in $\mathbb{C}$, and \( r_{m,p} = \frac{(2m+2)x_q^{m+1}}{(m+1)^3}, \) \( r_p = \frac{(2p+6)x_q^{p+1}}{(2p+4)^3}, \) \( u_{m,p} = \frac{(2m+4)x_q^{m+1}}{(m+1)^3}, \) \( u_m = \frac{(2m+4)x_q^{m+1}}{(m+1)^3} \), \( \bar{u}_{m,p} = \frac{(2m+2)x_q^{m+1}}{(m+1)^3}, \) \( \bar{u}_m = \frac{(2m+2)x_q^{m+1}}{(m+1)^3} \).

There are the following $U_q(\mathfrak{sl}(2))$-module algebra structures on $A_q(3)$

\[
\begin{align*}
  k(x_1) &= q^2 x_1, \\
  k(x_2) &= q^{-1} x_2, \\
  k(x_3) &= q^{-1} x_3, \\
  e(x_1) &= -q d_0^{-1} x_1^2, \\
  e(x_2) &= d_0^{-1} x_1 x_2, \\
  e(x_3) &= d_0^{-1} x_1 x_3, \\
  f(x_1) &= d_0, \\
  f(x_2) &= 0, \\
  f(x_3) &= 0,
\end{align*}
\]

where $d_0 \in \mathbb{C}\{0\}$.

**Proof.** The proof is similar to that in Lemma 4.8. \hfill \Box

Now, we begin to classify all module-algebra structures of $U_q(\mathfrak{sl}(3))$ on $A_q(3)$.

Denote the nine cases of the actions of $k_1, e_1, f_1$ (resp. $k_2, e_2, f_2$) in Lemma 4.4 and Lemma 4.9 by $(A_1), \ldots, (A_9)$ (resp. $(B_1), \ldots, (B_9)$). To determine all module-algebra structures of $U_q(\mathfrak{sl}(3))$ on $A_q(3)$, we have to find all the actions of $k_1, e_1,$
all the automorphisms of Lemma 4.11.

module-algebra structures are pairwise non-isomorphic.

Proof. which are pairwise non-isomorphic. Therefore, they are module-algebra structures of $A$

For Case $(2)$ we have to make $(3.1)$-$(3.7)$ hold for any $i \neq j$.

We use $(Ai)|(Bj)$ to denote that the actions of $k_1, e_1, f_1$ are those in $(Ai)$ and the actions of $k_2, e_2, f_2$ are those in $(Bj)$. Moreover, in Case $(A_j)|(B_j)$ ($j \geq 2$), since the actions of $e_i, f_i$ are not zero simultaneously for $i \in \{1, 2\}$, $(3.1)$ and $(3.2)$ can not be satisfied simultaneously. Therefore, Cases $(A_j)|(B_j)$ ($j \geq 2$) are excluded.

First, let us consider Case $(A_2)|(B_5)$. Since $k_2e_1(x_1) = k_2(a_0) = a_0$, $q^{-1}e_1k_2(x_1)$ does not hold. Therefore, $(A_2)|(B_5)$ should be excluded. For the same reason, we exclude $(A_2)|(B_6), (A_2)|(B_8), (A_2)|(B_9), (A_3)|(B_4), (A_3)|(B_7), (A_3)|(B_8), (A_3)|(B_9), (A_4)|(B_6), (A_4)|(B_7), (A_4)|(B_8), (A_4)|(B_9), (A_5)|(B_7), (A_5)|(B_8), (A_5)|(B_9), (A_6)|(B_7), (A_6)|(B_8), (A_6)|(B_9), (A_7)|(B_8), (A_7)|(B_9), (A_8)|(B_9).

Second, we consider $(A_1)|(B_2)$. Since $k_1f_2(x_1) = -q^{-1}a_0^{-1}x_1^2$ and $qf_2k_1(x_1) = \mp q^{-1}a_0^{-1}x_1^2$, we have $k_1f_2(x_1) \neq qf_2k_1(x_1)$. Thus, $(A_1)|(B_2)$ should be excluded. Similarly, $(A_1)|(B_1)$ should be excluded for $i \geq 3$.

Therefore, we only need to consider the following cases: $(A_1)|(B_1), (A_2)|(B_3), (A_2)|(B_4), (A_2)|(B_7), (A_3)|(B_5), (A_3)|(B_6), (A_4)|(B_5), (A_4)|(B_7), (A_5)|(B_6).

Lemma 4.10. For Case $(A_1)|(B_1)$, all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows

$k_1(x_1) = \pm x_1, k_1(x_2) = \pm x_2, k_1(x_3) = \pm x_3, k_2(x_1) = \pm x_1, k_2(x_2) = \pm x_2, k_2(x_3) = \pm x_3, e_1(x_1) = e_1(x_2) = e_1(x_3) = f_1(x_1) = f_1(x_2) = f_1(x_3) = 0, e_2(x_1) = e_2(x_2) = e_2(x_3) = f_2(x_1) = f_2(x_2) = f_2(x_3) = 0,$

which are pairwise non-isomorphic.

Proof. It can be seen that $(3.1)$-$(3.7)$ are satisfied for any $u \in \{x_1, x_2, x_3\}$ in this case. Therefore, they are module-algebra structures of $U_q(sl(3))$ on $A_q(3)$. Since all the automorphisms of $A_q(3)$ commute with the actions of $k_1$ and $k_2$, all these module-algebra structures are pairwise non-isomorphic.

Lemma 4.11. For Case $(A_2)|(B_3)$, all $U_q(sl(3))$-module algebra structures on $A_q(3)$ are as follows:

$k_1(x_1) = q^{-2}x_1, k_1(x_2) = q^{-1}x_2, k_1(x_3) = q^{-1}x_3, k_2(x_1) = qx_1, k_2(x_2) = qx_2, k_2(x_3) = q^2x_3, e_1(x_1) = a_0, e_1(x_2) = 0, e_1(x_3) = 0, e_2(x_1) = -qf_0^{-1}x_1, e_2(x_2) = -qf_0^{-1}x_2, e_2(x_3) = -qf_0^{-1}x_3^2, f_1(x_1) = -qa_0^{-1}x_1^2, f_1(x_2) = -qa_0^{-1}x_1x_2, f_1(x_3) = -qa_0^{-1}x_1x_3, f_2(x_1) = 0, f_2(x_2) = 0, f_2(x_3) = f_0,$

where $a_0, f_0 \in \mathbb{C}\setminus \{0\}$.

All these structures are isomorphic to that with $a_0 = f_0 = 1$.

Proof. By Lemma 4.2 and Lemma 4.3 to determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$, we have to make $(3.1)$-$(3.7)$ hold for any $u \in \{x_1, x_2, x_3\}$.
using the actions of \( k_1, e_1, f_1 \) in Lemma 4.2 and the actions of \( k_2, e_2, f_2 \) in Lemma 4.3.

Since \( k_1 e_2(x_1) = q^{-1} e_2 k_1(x_1) = q^{-3} e_2(x_1) \), we have \( e_2(x_1) = -q f_0^{-1} x_1 x_3 \), i.e., \( \mu_1 = \mu_2 = \mu_3 = 0 \). Using \( k_1 e_2(x_2) = q^{-1} e_2 k_1(x_2) = q^{-2} e_2(x_2) \), we obtain \( e_2(x_2) = -q f_0^{-1} x_2 x_3 \). Similarly, by \( k_2 f_1(x_2) = q f_1 k_2(x_2) = q^2 f_1(x_2) \) and \( k_2 f_1(x_3) = q f_1 k_2(x_3) = q^3 f_1(x_3) \), we get \( f_1(x_2) = -q a_0^{-1} x_1 x_2 \) and \( f_1(x_3) = -q a_0^{-1} x_1 x_3 \). Then, it is easy to check that (3.1)-3.2 hold for any \( u \in \{x_1, x_2, x_3\} \).

Then, we check that (3.3) holds. Obviously, \( e_1 f_2(u) = f_2 e_1(u) \) for any \( u \in \{x_1, x_2, x_3\} \). Now, we check \( e_2 f_1(x_1) - f_1 e_2(x_1) = 0 \). In fact,

\[
e_2 f_1(x_1) - f_1 e_2(x_1) = e_2(-q a_0^{-1} x_1^2) - f_1(-q f_0^{-1} x_1 x_3)
= -q a_0^{-1}(x_1 e_2(x_1) + e_2(x_1) k_2(x_1)) + q f_0^{-1} k_1^{-1}(x_1) f_1(x_3) + f_1(x_1) x_3
= (q^2 + q^4) a_0^{-1} f_0^{-1} x_1^2 x_3 - (q^4 + q^2) a_0^{-1} f_0^{-1} x_1^2 x_3
= 0.
\]

Similarly, other equalities in (3.3) can be checked.

Next, we check that (3.3)-3.7 hold for any \( u \in \{x_1, x_2, x_3\} \). Here, we only check \( e_2^2 e_1(x_1) - (q + q^{-1}) e_2 e_1 e_2(x_1) + e_1 e_2^2(x_1) = 0 \). The other equalities can be checked similarly. In fact,

\[
e_2^2 e_1(x_1) - (q + q^{-1}) e_2 e_1 e_2(x_1) + e_1 e_2^2(x_1)
= 0 - (q + q^{-1}) e_2 e_1 e_2(x_1) + e_1 e_2 e_1 e_2(x_1)
= (q + q^{-1}) f_0^{-1} e_2(a_0 x_3) - q f_0^{-1} e_1(-q f_0^{-1} x_1 x_2 - q^3 f_0^{-1} x_1 x_3)
= -q(q + q^{-1}) a_0 f_0^{-2} x_3^2 + a_0 f_0^{-2}(1 + q^2) x_3^2
= 0.
\]

Finally, we claim that all the actions with nonzero \( a_0 \) and \( f_0 \) are isomorphic to the specific action with \( a_0 = 1, f_0 = 1 \). The desired isomorphism is given by \( \psi_{a_0, f_0} : x_1 \mapsto a_0 x_1, x_2 \mapsto x_2, x_3 \mapsto f_0 x_3 \).

**Lemma 4.12.** For Case (A2)/(B4), all \( U_q(sl(3)) \)-module algebra structures on \( A_q(3) \) are as follows

\[
k_1(x_1) = q^{-1} x_1, k_1(x_2) = q^{-1} x_2, k_1(x_3) = q^{-1} x_3,
k_2(x_1) = q x_1, k_2(x_2) = q^{-1} x_2, k_2(x_3) = x_3,
e_1(x_1) = a_0, e_1(x_2) = 0, e_1(x_3) = 0,
e_2(x_1) = 0, e_2(x_2) = a_2 x_1, e_2(x_3) = 0,
f_1(x_1) = -q a_0^{-1} x_1^2, f_1(x_2) = -q a_0^{-1} x_1 x_2, f_1(x_3) = -q a_0^{-1} x_1 x_3,
f_2(x_1) = a_2^{-1} x_2, f_2(x_2) = 0, f_2(x_3) = 0,
\]

where \( a_0, a_2 \in \mathbb{C} \setminus \{0\} \).

All these module-algebra structures are isomorphic to that with \( a_0 = a_2 = 1 \).

**Proof.** By the above actions of \( k_1, e_1, f_1 \) and \( k_2, e_2, f_2 \), it is easy to check that (3.1)-3.7 hold for any \( u \in \{x_1, x_2, x_3\} \). Therefore, by Lemma 4.2 and Lemma 4.3 they determine the module-algebra structures of \( U_q(sl(3)) \) on \( A_q(3) \).

Next, we prove that there are no other actions except these in this lemma.

Using \( k_1 e_2(x_1) = q^{-1} e_2 k_1(x_1) = q^{-3} e_2(x_1) \), we can obtain \( e_2(x_1) = 0 \). By Lemma 4.3 we also have \( e_2(x_2) = a_2 x_1 \) and \( e_2(x_3) = 0 \). Similarly, by \( k_1 f_2(x_1) = \)
The action $q f_2 k_1(x_1) = q^{-1} f_2(x_1)$, we get $f_2(x_1) = c_1 x_2$. Therefore, $f_2(x_2) = f_2(x_3) = 0$.

Then, using $e_2 f_2(x_i) - f_2 e_2(x_i) = k_3^{-1} - q^{-1} k_1(x_i)$ for any $i \in \{1, 2, 3\}$, we obtain $c_1 = a_2^{-1}$. Since $k_2 f_1(x_2) = q f_1 k_2(x_2) = f_1(x_2)$ and $k_2 f_1(x_3) = q f_1 k_2(x_3) = f_1(x_3)$, by Lemma 4.2, we have $f_1(x_2) = -qa_0^{-1} x_1 x_2 + \xi x_3^2$, $f_1(x_3) = -qa_0^{-1} x_1 x_3$.

Due to the condition of module algebra, it is easy to see that we have to let $f_2^2 f_2(x_3) - (q + q^{-1}) f_1 f_2 f_1(x_3) + f_2 f_2^2(x_3) = 0$ hold. On the other hand, we have

$$f_2^2 f_2(x_3) - (q + q^{-1}) f_1 f_2 f_1(x_3) + f_2 f_2^2(x_3) = -(q + q^{-1}) f_1 f_2(-qa_0^{-1} x_1 x_3) + f_2 f_1(-qa_0^{-1} x_1 x_3)$$

$$= q(q + q^{-1}) a_0^{-1} a_2^{-1} f_1(x_2 x_3) - qa_0^{-1} f_2 f_1(x_1 x_3 + q^2 x_1 f(x_3))$$

$$= q(q + q^{-1}) a_0^{-1} a_2^{-1} (f_1(x_2 x_3) + q x_2 f_1(x_3)) + qa_0^{-2} (q + q^3) f_2(x_1^2 x_3)$$

$$= (q^2 + 1) a_0^{-1} a_2^{-1} (-(q + q^3) a_0^{-1} x_1 x_2 x_3 + \xi x_3^2))$$

$$+ (q^2 + 1)(q + q^3) a_0^{-1} a_0^{-2} x_1 x_2 x_3$$

$$= (q^2 + 1) a_0^{-1} a_2^{-1} \xi x_3^2.$$
Next, let us consider the condition $e_2f_1(x_3) - f_1e_2(x_3) = 0$. Since $e_2f_1(x_3) - f_1e_2(x_3)$

\[\begin{align*}
e_2(-qa_0^{-1}x_1x_3 + \xi_4x_2x_3^2) - f_1(a_3x_1 + e_0^{-1}x_2x_3) \\
= -qa_0^{-1}(x_1e_2(x_3) + e_2(x_1)k_2(x_3)) + \xi_4(x_2x_3e_2(x_3) + x_2e_2(x_3)k_2(x_3)
+ e_2(x_2)k_2(x_3)k_2(x_3)) + qa_0^{-1}x_1^2 - e_0^{-1}(k_1^{-1}(x_2)f_1(x_3) + f_1(x_2)x_3)
= \xi_4a_3(q^2 + 1)x_1x_2x_3 + a_0^{-1}e_0^{-1}(q^3 + q)x_1x_2x_3
= 0,
\end{align*}\]

we obtain $\xi_4a_3 = -qa_0^{-1}e_0^{-1}$.

Therefore, there are no other actions except these in this lemma.

Finally, we show that all the actions with nonzero $a_0$, $a_3$, $e_0$ and $\xi_4$ are isomorphic to the specific action with $a_0 = e_0 = a_3 = 1$ and $\xi_4 = -q$. The desired isomorphism is given by $\psi_{a_0,a_3,e_0} : x_1 \mapsto a_0x_1, x_2 \mapsto e_0x_2, x_3 \mapsto a_0a_3x_3$.

\[\square\]

**Lemma 4.14.** For Case (A3)(B5), all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows:

\begin{align*}
k_1(x_1) &= qx_1, \quad k_1(x_2) = qx_2, \quad k_1(x_3) = q^2x_3, \\
k_2(x_1) &= x_1, \quad k_2(x_2) = qx_2, \quad k_2(x_3) = q^{-1}x_3, \\
e_1(x_1) &= -q f_0^{-1}x_1x_3, \quad e_1(x_2) = -q f_0^{-1}x_2x_3, \quad e_1(x_3) = -q f_0^{-1}x_3^2, \\
e_2(x_1) &= 0, \quad e_2(x_2) = e_2(x_3) = 0, \\
f_1(x_1) &= 0, \quad f_1(x_2) = 0, \quad f_1(x_3) = f_0, \\
f_2(x_1) &= 0, \quad f_2(x_2) = b_3^{-1}x_3, \quad f_2(x_3) = 0,
\end{align*}

where $b_3, f_0 \in \mathbb{C}\setminus\{0\}$.

All module-algebra structures above are isomorphic to that with $b_3 = f_0 = 1$.

\[\square\]

**Lemma 4.15.** For Case (A3)(B6), all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows:

\begin{align*}
k_1(x_1) &= qx_1, \quad k_1(x_2) = qx_2, \quad k_1(x_3) = q^2x_3, \\
k_2(x_1) &= qx_1, \quad k_2(x_2) = q^{-2}x_2, \quad k_2(x_3) = q^{-1}x_3, \\
e_1(x_1) &= -q f_0^{-1}x_1x_3 + \mu_1x_1x_2, \quad e_1(x_2) = -q f_0^{-1}x_2x_3, \quad e_1(x_3) = -q f_0^{-1}x_3^2, \\
e_2(x_1) &= 0, \quad e_2(x_2) = b_0, \quad e_2(x_3) = 0, \\
f_1(x_1) &= 0, \quad f_1(x_2) = 0, \quad f_1(x_3) = f_0, \\
f_2(x_1) &= d_1x_3 + b_0^{-1}x_1x_2, \quad f_2(x_2) = -q b_0^{-1}x_2^2, \quad f_2(x_3) = -q b_0^{-1}x_2x_3,
\end{align*}

where $d_1, b_0, \mu_1, f_0 \in \mathbb{C}\setminus\{0\}$ and $\mu_1 d_1 = -q b_0^{-1} f_0^{-1}$.

All module-algebra structures above are isomorphic to that with $d_1 = b_0 = f_0 = 1$ and $\mu_1 = -q$.

\[\square\]
Lemma 4.16. For Case (A4)|(B5), all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows:

\[
\begin{align*}
 k_1(x_1) &= qx_1, \quad k_1(x_2) = q^{-1}x_2, \quad k_1(x_3) = x_3, \\
k_2(x_1) &= x_1, \quad k_2(x_2) = qx_2, \quad k_2(x_3) = q^{-1}x_3, \\
e_1(x_1) &= 0, \quad e_1(x_2) = a_2x_1, \quad e_1(x_3) = 0, \\
e_2(x_1) &= 0, \quad e_2(x_2) = 0, \quad e_2(x_3) = b_3x_2, \\
f_1(x_1) &= a_2^{-1}x_2, \quad f_1(x_2) = 0, \quad f_1(x_3) = 0, \\
f_2(x_1) &= 0, \quad f_2(x_2) = b_3^{-1}x_3, \quad f_2(x_3) = 0,
\end{align*}
\]

where $a_2, b_3 \in \mathbb{C}\setminus\{0\}$.

All the above module-algebra structures are isomorphic to that with $a_2 = b_3 = 1$.

Proof. By the actions of $k_1, e_1, f_1$ and $k_2, e_2, f_2$, it is easy to check that (3.7) hold for any $u \in \{x_1, x_2, x_3\}$. Therefore, by Lemma 4.4 and Lemma 4.5, they determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Next, we prove that there are no other actions except these in this lemma.

By Lemma 4.5 and using (3.7) holding for any $u \in \{x_1, x_2, x_3\}$, we can obtain that $e_2(1) = \sum_{n \geq 0} a_n x_1^{n+1} x_2^n x_3, \quad e_2(x_2) = 0, \quad e_2(x_3) = b_3 x_2 + \sum_{n \geq 0} c_n x_1^{n+1} x_2^n x_3$.

Next, we consider $f_2(1) = \sum_{m \geq 0} d_{m,m+1} x_1^{m+1} x_2^m x_3, \quad f_2(x_2) = d_2 x_2 + \sum_{m \geq 0} e_{m,m+1} x_1^m x_2^{m+1} x_3, \quad f_2(x_3) = c_{m,m+1} x_1^m x_2^{m+1} x_3$. By Lemma 4.5 we know that

\[
\begin{align*}
\text{by Lemma } 4.5 \text{ we have that} & \quad a_n = \frac{q^{2n+2} - 1}{(2n+3)q - q^2}, \\
\text{by Lemma } 4.5 \text{ we have that} & \quad \kappa_m = \frac{d_{m,m+1}}{e_{m,m+1}} = -1.
\end{align*}
\]

Next, we consider $e_2 f_2(x_2) - f_2 e_2(x_2) = \frac{k_2 - k_2^{-1}}{q - q^{-1}}(x_2) = x_2$. By some computations, we obtain

\[
e_2 f_2(x_2) - f_2 e_2(x_2) = b_3 d_2 x_2 + \sum_{n \geq 0} (q^{2n+4} - 1) a_n x_1^{n+1} x_2^n x_3^{n+1} - \sum_{m \geq 0} (q^{-1} - q^{2m+3}) e_{m,m+1} x_1^m x_2^{m+1} x_3 + \sum_{m,n \geq 0} q^{3m+3n+2} x_1^{m+1} x_2^m x_3^{n+1} + \sum_{m,n \geq 0} q^{-2m+2n+6} e_{m,m+1} x_1^{m+1} x_2^m x_3^{n+1}.
\]

If there exist $v_n$ and $\kappa_m$ not equal to zero, we can choose the terms with coefficients $v_n$ and $\kappa_m$ in $(e_2 f_2)(x_2)$, $(e_2)(x_2)$, $(e_2)(x_3)$, $(f_2)(x_1)$, $(f_2)(x_2)$, $(f_2)(x_3)$ such that the degrees of them are the highest. Then, the unique monomial of the highest degree in $(e_2 f_2 - f_2 e_2)(x_2)$ is

\[
q^{3m_f + 3n_f + 2} (1 - q^{-2}) (1 - q^{2m_f + 2n_f + 6}) v_n \kappa_m x_1^{m_f + n_f + 2} x_2^{m_f + n_f + 3} x_3^{m_f + n_f + 2}.
\]

Since the degree of this term is larger than 1, this case is impossible. Similarly, all cases except that all $v_n, \kappa_m$ are equal to zero should be excluded. Therefore, we obtain that $e_2(x_1) = 0, e_2(x_2) = 0, e_2(x_3) = b_3 x_2, f_2(x_1) = 0, f_2(x_2) = b_3^{-1} x_3$ and $f_2(x_3) = 0$.
Similarly, using (3.2), Lemma 4.4 and $e_1 f_1 (u) - f_1 e_1 (u) = \frac{k_1 - k^{-1}}{q-q^{-1}} (u)$ for any $u \in \{x_1, x_2, x_3\}$, we can get $e_1 (x_1) = 0$, $e_1 (x_2) = a_2 x_1$, $e_1 (x_3) = 0$, $f_1 (x_1) = a_2^{-1} x_2$, $f_1 (x_2) = 0$, $f_1 (x_3) = 0$.

Therefore, there are no other actions except ones in this lemma.

Finally, we claim that all the actions with nonzero $a_2$, $b_3$ are isomorphic to the specific action with $a_2 = b_3 = 1$. The desired isomorphism is given by $\psi_{a_2, b_3} : x_1 \mapsto x_1, x_2 \mapsto a_2 x_2, x_3 \mapsto a_2 b_3 x_3$.

\( \square \)

Lemma 4.17. For Case (A5)(B6), all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are

\[
\begin{align*}
k_1 (x_1) &= x_1, \quad k_1 (x_2) = q x_2, \quad k_1 (x_3) = q^{-1} x_3, \\
k_2 (x_1) &= qx_1, \quad k_2 (x_2) = q^{-2} x_2, \quad k_2 (x_3) = q^{-1} x_3, \\
e_1 (x_1) &= 0, \quad e_1 (x_2) = 0, \quad e_1 (x_3) = b_3 x_2, \\
e_2 (x_1) &= 0, \quad e_2 (x_2) = b_0, \quad e_2 (x_3) = 0, \\
f_1 (x_1) &= 0, \quad f_1 (x_2) = b_3^{-1} x_3, \quad f_1 (x_3) = 0, \\
f_2 (x_1) &= b_0^{-1} x_1 x_2, \quad f_2 (x_2) = -q b_0^{-1} x_2^2, \quad f_2 (x_3) = -q b_0^{-1} x_2 x_3,
\end{align*}
\]

where $b_0, b_3 \in \mathbb{C} \setminus \{0\}$.

All module-algebra structures are isomorphic to that with $b_0 = b_3 = 1$.

Proof. It is easy to check that the above actions of $k_1, e_1, f_1$ and $k_2, e_2, f_2$ determine module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Then, we prove that there are no other actions except these in this lemma.

By (3.1) for any $u \in \{x_1, x_2, x_3\}$ and Lemma 4.6, we have

\[
\begin{align*}
e_2 (x_1) &= (q - q^3) u x_1^4 x_3 + \sum_{n \geq 0} (q - q^{2n+5}) u a^3_n x_1^{3n+7} x_2^{n+1} x_3^{n+2}, \\
e_2 (x_2) &= b_0 + (q^4 - 1) u x_1^3 x_2 x_3 + \sum_{n \geq 0} (q^{3n+7} - q^{n+1}) u a^3_n x_1^{3n+6} x_2^{n+2} x_3^{n+2}, \\
e_2 (x_3) &= (q^4 - 1) u x_1^3 x_2^2 + \sum_{n \geq 0} (q^{4n+8} - 1) u a^3_n x_1^{3n+6} x_2^{n+1} x_3^{n+3}, \\
f_2 (x_1) &= g x_1 x_2 + (q^3 - q^{-1}) \varepsilon x_1^4 x_2 x_3 + \sum_{p \geq 0} (q^{2p+5} - q^{-1}) \mu_p x_1^{3p+7} x_2^{p+3} x_3^{p+2}, \\
f_2 (x_2) &= -q g x_2^3 + (q - q^5) \varepsilon x_1^3 x_2^2 x_3 + \sum_{p \geq 0} (q^{p+2} - q^{3p+8}) \mu_p x_1^{3p+6} x_2^{p+4} x_3^{p+2}, \\
f_2 (x_3) &= -q g x_2 x_3 + (1 - q^6) \varepsilon x_1^3 x_2^2 x_3 + \sum_{p \geq 0} (1 - q^{4p+10}) \mu_p x_1^{3p+6} x_2^{p+3} x_3^{p+3}.
\end{align*}
\]

Then, we consider the condition $e_2 f_2 (u) - f_2 e_2 (u) = \frac{k_2 - k^{-1}}{q-q^{-1}} (u)$ for any $u \in \{x_1, x_2, x_3\}$.

Let us assume that there exist some $u$ or $v_n$ which are not equal to zero. Then, we can choose the monomials in $e_2 (x_1)$, $e_2 (x_2)$, $e_2 (x_3)$ with the highest degree. Obviously, these monomials are unique. It is also easy to see that $f (x_1)$, $f (x_2)$, $f (x_3)$ can not be equal to zero simultaneously. Therefore, there are some nonzero $g$, $\varepsilon$ or $\mu_p$. Similarly, those monomials in $f_2 (x_1)$, $f_2 (x_2)$, $f_2 (x_3)$ with the highest degree are chosen. Then, by some computations, we can obtain a monomial with
the highest degree, whose degree is larger than 1. Then, we get a contradiction with \( e_2 f_2(x_1) - f_2 e_2(x_1) = x_1 \). For example, if the coefficient of the monomials in \( e_2(x_1), e_2(x_2) \) and \( e_2(x_3) \) is \( v_n \), and the coefficient of the monomials in \( f_2(x_1), f_2(x_2), f_2(x_3) \) with the highest degree is \( \mu_p \), then the monomial with the highest degree in \( e_2 f_2(x_1) - f_2 e_2(x_1) \) is

\[
q^{7n_e p_e + 15n_e + 11p_e + 22(1 - q^{2n_e + 2p_e + 10})^2 v_n \mu_p x_1^{3n_e + 3p_e + 13} x_2^{n_e + p_e + 4} x_3^{n_e + p_e + 4}.
\]

Therefore, all \( u, v_n \) are equal to zero. Then, \( e_2(x_1) = 0, e_2(x_2) = b_0 \) and \( e_2(x_3) = 0 \). Thus, we can obtain

\[
e_2 f_2(x_1) - f_2 e_2(x_1) = gb_0 x_1 + (1 - q^{-1})(1 + q^2)z b_0 x_1^2 x_2 x_3
\]

\[
+ \sum_{p \geq 0} \frac{1 - q^{2p+6}}{1 - q^2} (q^{-1-p} - q^{-3p-7}) b_0 \mu_p x_1^{3p+7} x_2^{p+2} x_3^{p+2}.
\]

Thus, we obtain \( f_2(x_1) = b_0^{-1} x_1 x_2, f_2(x_2) = -qb_0^{-1} x_2^2 \) and \( f_2(x_3) = -qb_0^{-1} x_2 x_3 \).

On the other hand, with a similar discussion, by [3,2], Lemma 1.5 and \( e_1 f_1(u) - f_1 e_1(u) = \frac{k_1 - k_2^{-1}}{q - q^{-1}}(u) \) for any \( u \in \{x_1, x_2, x_3\} \), we can obtain \( e_1(x_1) = 0, e_1(x_2) = 0, e_1(x_3) = b_3 x_2, f_1(x_1) = 0, f_1(x_2) = b_3^{-1} x_3, f_1(x_3) = 0 \).

Therefore, there are no other actions except these in this lemma.

Finally, we claim that all module algebra structures with nonzero \( b_0, b_3 \) are isomorphic to that with \( b_0 = b_3 = 1 \). The desired isomorphism is given by

\[
\psi_{b_0, b_3} : x_1 \mapsto x_1, x_2 \mapsto b_0 x_2, x_3 \mapsto b_0 b_3 x_3.
\]

\[\square\]

**Lemma 4.18.** For Case (A4)|(B7), all module-algebra structures of \( U_q(sl(3)) \) on \( A_q(3) \) are as follows

\[
\begin{align*}
k_1(x_1) &= q x_1, \quad k_1(x_2) = q^{-1} x_2, \quad k_1(x_3) = x_3, \\
k_2(x_1) &= q x_1, \quad k_2(x_2) = q^2 x_2, \quad k_2(x_3) = q^{-1} x_3, \\
e_1(x_1) &= 0, \quad e_1(x_2) = a_2 x_2, \quad e_1(x_3) = 0, \\
e_2(x_1) &= -q e_0^{-1} x_1 x_2, \quad e_2(x_2) = -q e_0^{-1} x_2^2, \quad e_2(x_3) = e_0^{-1} x_2 x_3, \\
f_1(x_1) &= a_2^{-1} x_2, \quad f_1(x_2) = 0, \quad f_1(x_3) = 0, \\
f_2(x_1) &= 0, \quad f_2(x_2) = e_0, \quad f_2(x_3) = 0,
\end{align*}
\]

where \( a_2, e_0 \in \mathbb{C} \setminus \{0\} \).

All module-algebra structures are isomorphic to that with \( a_2 = e_0 = 1 \).

**Proof.** The proof is similar to that in Lemma 4.17. \[\square\]

Since the actions of \( k_1, e_1, f_1 \) and \( k_2, e_2, f_2 \) in \( U_q(sl(3)) \) are symmetric, by Lemma 4.18, Lemma 4.19, and the discussion above, we obtain the following theorem.

**Theorem 4.19.** \( U_q(sl(3)) \)-symmetries up to isomorphisms on \( A_q(3) \) and their classical limits, i.e., Lie algebra \( sl_3 \)-actions by differentiations on \( \mathbb{C}[x_1, x_2, x_3] \) are as follows:
### $U_q(sl(3))$-symmetries

| $k_i(x_1) = \pm x_1, k_i(x_2) = \pm x_2,$ | $h_i(x_1) = 0, h_i(x_2) = 0,$ |
| $k_i(x_1) = \pm x_3, k_j(x_1) = \pm x_1,$ | $h_i(x_3) = 0, h_j(x_1) = 0,$ |
| $k_j(x_2) = \pm x_2, k_j(x_3) = \pm x_3,$ | $h_j(x_2) = 0, h_j(x_3) = 0,$ |
| $e_i(x_1) = e_i(x_2) = e_i(x_3) = 0,$ | $e_i(x_1) = e_i(x_2) = e_i(x_3) = 0,$ |
| $f_i(x_1) = f_i(x_2) = f_i(x_3) = 0,$ | $f_j(x_1) = f_j(x_2) = f_j(x_3) = 0,$ |
| $e_j(x_1) = e_j(x_2) = e_j(x_3) = 0,$ | $e_j(x_1) = e_j(x_2) = e_j(x_3) = 0,$ |
| $f_j(x_1) = f_j(x_2) = f_j(x_3) = 0$ | $f_j(x_1) = f_j(x_2) = f_j(x_3) = 0$ |

### Classical limit

| $k_i(x_1) = q^{-2}x_1, k_i(x_2) = q^{-2}x_2,$ | $h_i(x_1) = -2x_1, h_i(x_2) = -x_2,$ |
| $k_i(x_3) = q^{-1}x_3, k_j(x_1) = qx_1,$ | $h_i(x_3) = -x_3, h_j(x_1) = x_1,$ |
| $k_j(x_2) = qx_2, k_j(x_3) = q^2x_3,$ | $h_j(x_2) = 2x_2, h_j(x_3) = 2x_3,$ |
| $e_i(x_1) = 1, e_i(x_2) = 0, e_i(x_3) = 0,$ | $e_i(x_1) = 1, e_i(x_2) = 0, e_i(x_3) = 0,$ |
| $e_j(x_1) = -qx_1x_3, e_j(x_2) = -qx_2x_3,$ | $e_j(x_1) = -x_1x_3, e_j(x_2) = -x_2x_3,$ |
| $e_j(x_3) = -q^{-2}x_3, f_i(x_1) = -q^{-2}x_3,$ | $e_j(x_3) = -x_3^2, f_i(x_1) = -x_3^2,$ |
| $f_i(x_2) = -qx_1x_2, f_i(x_3) = -qx_1x_2,$ | $f_i(x_2) = -x_1x_2, f_i(x_3) = -x_1x_2,$ |
| $f_j(x_1) = 0, f_j(x_2) = 0, f_j(x_3) = 1$ | $f_j(x_1) = 0, f_j(x_2) = 0, f_j(x_3) = 1$ |

### Additional symmetries

| $k_i(x_1) = qx_1, k_i(x_2) = qx_2,$ | $h_i(x_1) = x_1, h_i(x_2) = x_2,$ |
| $k_i(x_3) = q^{-3}x_3, k_j(x_1) = x_1,$ | $h_i(x_3) = 2x_3, h_j(x_1) = 0,$ |
| $k_j(x_2) = qx_2, k_j(x_3) = q^{-1}x_3,$ | $h_j(x_2) = x_2, h_j(x_3) = -x_3,$ |
| $e_i(x_1) = q^{-2}x_1x_3, e_i(x_2) = -qx_1x_2,$ | $e_i(x_1) = -x_1x_3, e_i(x_2) = -x_2x_3,$ |
| $e_i(x_3) = -q^{-2}x_3, e_j(x_1) = 0,$ | $e_i(x_3) = -x_3^2, e_j(x_1) = 0,$ |
| $e_j(x_2) = 0, e_j(x_3) = x_2,$ | $e_j(x_2) = 0, e_j(x_3) = x_2,$ |
| $f_i(x_1) = 0, f_i(x_2) = 0, f_i(x_3) = 1,$ | $f_i(x_1) = 0, f_i(x_2) = 0, f_i(x_3) = 1,$ |
| $f_j(x_1) = 0, f_j(x_2) = x_3, f_j(x_3) = 0$ | $f_j(x_1) = 0, f_j(x_2) = x_3, f_j(x_3) = 0$ |

### Additional symmetries

| $k_i(x_1) = qx_1, k_i(x_2) = q^{-1}x_2,$ | $h_i(x_1) = x_1, h_i(x_2) = -x_2,$ |
| $k_i(x_3) = x_3, k_j(x_1) = x_1,$ | $h_i(x_3) = 0, h_j(x_1) = 0,$ |
| $k_j(x_2) = qx_2, k_j(x_3) = q^{-1}x_3,$ | $h_j(x_2) = x_2, h_j(x_3) = -x_3,$ |
| $e_i(x_1) = 0, e_i(x_2) = x_1, e_i(x_3) = 0,$ | $e_i(x_1) = 0, e_i(x_2) = x_1, e_i(x_3) = 0,$ |
| $e_j(x_1) = 0, e_j(x_2) = 0, e_j(x_3) = x_2,$ | $e_j(x_1) = 0, e_j(x_2) = 0, e_j(x_3) = x_2,$ |
| $f_i(x_1) = x_2, f_i(x_2) = 0, f_i(x_3) = 0,$ | $f_i(x_1) = x_2, f_i(x_2) = 0, f_i(x_3) = 0,$ |
| $f_j(x_1) = 0, f_j(x_2) = x_3, f_j(x_3) = 0$ | $f_j(x_1) = 0, f_j(x_2) = x_3, f_j(x_3) = 0$ |

### Additional symmetries

| $k_i(x_1) = qx_1, k_i(x_2) = q^{-1}x_2,$ | $h_i(x_1) = x_1, h_i(x_2) = -x_2,$ |
| $k_i(x_3) = x_3, k_j(x_1) = x_1,$ | $h_i(x_3) = 0, h_j(x_1) = 0,$ |
| $k_j(x_2) = qx_2, k_j(x_3) = q^{-1}x_3,$ | $h_j(x_2) = x_2, h_j(x_3) = -x_3,$ |
| $e_i(x_1) = 0, e_i(x_2) = x_1, e_i(x_3) = 0,$ | $e_i(x_1) = 0, e_i(x_2) = x_1, e_i(x_3) = 0,$ |
| $e_j(x_1) = -qx_1x_2, e_j(x_2) = -qx_2x_3,$ | $e_j(x_1) = -x_1x_2, e_j(x_2) = -x_2x_3,$ |
| $e_j(x_3) = -q^{-2}x_3, f_i(x_1) = -q^{-2}x_3,$ | $e_j(x_3) = -x_3^2, f_i(x_1) = -x_3^2,$ |
| $f_i(x_2) = -qx_1x_2, f_i(x_3) = -qx_1x_2,$ | $f_i(x_2) = -x_1x_2, f_i(x_3) = -x_1x_2,$ |
| $f_j(x_1) = 0, f_j(x_2) = 1, f_j(x_3) = 0$ | $f_j(x_1) = 0, f_j(x_2) = 1, f_j(x_3) = 0$ |
\[
\begin{array}{c|c}
\begin{array}{l}
k_1(x_1) = x_1, \quad k_1(x_2) = qx_2, \\
k_1(x_3) = q^{-1}x_3, \quad k_1(x_1) = qx_1, \\
k_1(x_2) = q^{-2}x_2, \quad k_1(x_3) = q^{-1}x_3, \\
e_i(x_1) = 0, \quad e_i(x_2) = 0, \\
e_i(x_3) = x_2, \quad e_j(x_1) = 0, \\
e_j(x_2) = 1, \quad e_j(x_3) = 0, \\
f_i(x_1) = 0, \quad f_j(x_2) = x_3, \quad f_i(x_3) = 0, \\
f_j(x_1) = x_1x_2, \quad f_j(x_2) = -qx_2^2, \\
f_j(x_3) = -qx_2x_3 \\
\end{array} & \begin{array}{l}
h_1(x_1) = 0, \quad h_1(x_2) = x_2, \\
h_1(x_3) = x_3, \quad h_1(x_1) = x_1, \\
h_1(x_2) = -2x_2, \quad h_1(x_3) = -x_3, \\
e_i(x_1) = 0, \quad e_i(x_2) = 0, \\
e_i(x_3) = x_2, \quad e_j(x_1) = 0, \\
e_j(x_2) = 1, \quad e_j(x_3) = 0, \\
f_i(x_1) = 0, \quad f_i(x_2) = x_3, \quad f_i(x_3) = 0, \\
f_j(x_1) = x_1x_2, \quad f_j(x_2) = -x_2^2, \\
f_j(x_3) = -x_2x_3 \\
\end{array} \\
\end{array}
\]

for any \( i = 1, j = 2 \) or \( i = 2, j = 1 \). Moreover, there are no isomorphisms between these nine kinds of module-algebra structures.

**Remark 4.20.** Case (5) when \( i = 1, j = 2 \) in Theorem 4.19 is the case discussed in (11) when \( n = 3 \).

Let us denote the actions of \( U_q(sl(2)) \) on \( A_q(3) \) in (A1), those in (A2) and (B3) in Lemma 4.11, those in (B4) in Lemma 4.12, those in (B5) in Lemma 4.14, those in (B6) in Lemma 4.17 and those in (B7) in Lemma 4.18 by *1, *2, *3, *4, *5, *6, *7 respectively. In addition, denote the actions of \( U_q(sl(2)) \) on \( A_q(3) \) in (A2) and (B7) in Lemma 4.13, those in (A3) and (B6) in Lemma 4.14 by *2’, *7’, *3’, *6’ respectively. Then, as in Section 3, we can use the following diagrams to denote all actions of \( U_q(sl(3)) \) on \( A_q(3) \):

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
*1 \quad *1', \quad *2' \quad *3' \quad *6', \\
\end{array} & \begin{array}{c}
\begin{array}{c}
*2 \quad *3, \\
\end{array} \\
\end{array} \\
\end{array}
\end{array}
\]

Just like that in Section 3, every two adjacent vertices corresponds to two classes of the module-algebra structures of \( U_q(sl(3)) \) on \( A_q(3) \).

Then, we begin to study the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(3) \) for \( m \geq 3 \). For studying the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(3) \),
we have to endow every vertex in the Dynkin diagram of \( sl(m + 1) \) an action of \( U_q(sl(2)) \) on \( A_q(3) \). As in Section 3, there are some rules which we should obey:

1. Since every two adjacent vertices in the Dynkin diagram corresponds to one Hopf subalgebra isomorphic to \( U_q(sl(3)) \), by Theorem 4.19 the action of \( U_q(sl(2)) \) on \( A_q(3) \) on every vertex should be of the following 11 kinds of possibilities: \(*1, *2, *3, *4, *5, *6, *7, *2', *3', *6' \). Moreover, every two adjacent vertices should be of the types in \((4.56)\) and \((4.57)\).

2. Except \(*1\), any other type of actions of \( U_q(sl(2)) \) on \( A_q(3) \) cannot be endowed with two different vertices simultaneously, since the relations \((2.2)\) acting on \( x_1, x_2, x_3 \) producing zero cannot be satisfied.

3. If every vertex in the Dynkin diagram of \( sl(m + 1) \) is endowed an action of \( U_q(sl(2)) \) on \( A_q(3) \) which is not Case \(*1\), any two vertices which are not adjacent cannot be endowed with the types which are adjacent \((4.56)\) and \((4.57)\).

**Theorem 4.21.** If \( m \geq 4 \), all module-algebra structures of \( U_q(sl(m + 1)) \) on \( A_q(3) \) are as follows

\[
\begin{align*}
k_i(x_1) &= \pm x_1, \quad k_i(x_2) = \pm x_2, \quad k_i(x_3) = \pm x_3, \\
e_i(x_1) &= e_i(x_2) = e_i(x_3) = f_i(x_1) = f_i(x_2) = f_i(x_3) = 0,
\end{align*}
\]

for any \( i \in \{1, 2, \ldots, m\} \).

For \( m = 3 \), all module-algebra structures of \( U_q(sl(4)) \) on \( A_q(3) \) are given by

\[
\begin{align*}
k_i(x_1) &= \pm x_1, \quad k_i(x_2) = \pm x_2, \quad k_i(x_3) = \pm x_3, \\
e_i(x_1) &= e_i(x_2) = e_i(x_3) = f_i(x_1) = f_i(x_2) = f_i(x_3) = 0,
\end{align*}
\]

for any \( i \in \{1, 2, 3\} \). All these module-algebra structures are not pairwise non-isomorphic.

\[
\begin{align*}
k_i(x_1) &= q x_1, \quad k_i(x_2) = q^{-1} x_2, \quad k_i(x_3) = x_3, \\
e_i(x_1) &= 0, \quad e_i(x_2) = a_2 x_1, \quad e_i(x_3) = 0, \\
f_i(x_1) &= a_2^{-1} x_2, \quad f_i(x_2) = 0, \quad f_i(x_3) = 0, \\
k_j(x_1) &= q^{-2} x_1, \quad k_j(x_2) = q^{-1} x_2, \quad k_j(x_3) = q^{-1} x_3, \\
e_j(x_1) &= a_0, \quad e_j(x_2) = 0, \quad e_j(x_3) = 0, \\
f_j(x_1) &= -q a_0^{-1} x_2^2, \quad f_j(x_2) = -q a_0^{-1} x_1 x_2, \quad f_j(x_3) = -q a_0^{-1} x_1 x_3, \\
k_s(x_1) &= q x_1, \quad k_s(x_2) = q x_2, \quad k_s(x_3) = q^2 x_3, \\
e_s(x_1) &= -q f_0^{-1} x_1 x_3, \quad e_s(x_2) = -q f_0^{-1} x_2 x_3, \quad e_s(x_3) = -q f_0^{-1} x_3^2, \\
f_s(x_1) &= 0, \quad f_s(x_2) = 0, \quad f_s(x_3) = f_0,
\end{align*}
\]

where \( a_0, a_2, f_0 \in \mathbb{C} \setminus \{0\} \) and \( i = 1, j = 2, k = 3 \) or \( i = 3, j = 2, k = 1 \). All these module-algebra structures are isomorphic to that with \( a_0 = a_2 = f_0 = 1 \).
where $a$, $b$, $s$ are not adjacent in this path have no edge connecting them in (4.56) and
(3)
\[
\begin{align*}
  k_i(x_1) &= q x_1, \ k_i(x_2) = q^{-1} x_2, \ k_i(x_3) = x_3, \\
  e_i(x_1) &= 0, \ e_i(x_2) = a_2 x_1, \ e_i(x_3) = 0, \\
  f_i(x_1) &= a_2^{-1} x_2, \ f_i(x_2) = 0, \ f_i(x_3) = 0, \\
  k_j(x_1) &= x_1, \ k_j(x_2) = q x_2, \ k_j(x_3) = q^{-1} x_3, \\
  e_j(x_1) &= 0, \ e_j(x_2) = 0, \ e_j(x_3) = b_3 x_2, \\
  f_j(x_1) &= 0, \ f_j(x_2) = b_3^{-1} x_3, \ f_j(x_3) = 0, \\
  k_s(x_1) &= q x_1, \ k_s(x_2) = q x_2, \ k_s(x_3) = q^2 x_3, \\
  e_s(x_1) &= -q f_0^{-1} x_1 x_3, \ e_s(x_2) &= -q f_0^{-1} x_2 x_3, \ e_s(x_3) &= -q f_0^{-1} x_3, \\
  f_s(x_1) &= 0, \ f_s(x_2) = 0, \ f_s(x_3) = f_0,
\end{align*}
\]
where $a_2$, $b_3$, $f_0 \in \mathbb{C}\setminus\{0\}$ and $i = 1$, $j = 2$, $s = 3$ or $i = 3$, $j = 2$, $s = 1$. All these module-algebra structures are isomorphic to that with $a_2 = b_3 = f_0 = 1$.

(4)
\[
\begin{align*}
  k_i(x_1) &= q^{-2} x_1, \ k_i(x_2) = q^{-1} x_2, \ k_i(x_3) = q^{-1} x_3, \\
  e_i(x_1) &= a_0, \ e_i(x_2) = 0, \ e_i(x_3) = 0, \\
  f_i(x_1) &= -qa_0^{-1} x_1^2, \ f_i(x_2) = -qa_0^{-1} x_1 x_2, \ f_i(x_3) = -q a_0^{-1} x_1 x_3, \\
  k_j(x_1) &= q x_1, \ k_j(x_2) = q x_2, \ k_j(x_3) = q^2 x_3, \\
  e_j(x_1) &= -q f_0^{-1} x_1 x_3, \ e_j(x_2) &= -q f_0^{-1} x_2 x_3, \ e_j(x_3) &= -q f_0^{-1} x_3, \\
  f_j(x_1) &= 0, \ f_j(x_2) = 0, \ f_j(x_3) = f_0, \\
  k_s(x_1) &= x_1, \ k_s(x_2) = q x_2, \ k_s(x_3) = q^{-1} x_3, \\
  e_s(x_1) &= 0, \ e_s(x_2) = 0, \ e_s(x_3) = b_3 x_2, \\
  f_s(x_1) &= 0, \ f_s(x_2) = b_3^{-1} x_3, \ f_s(x_3) = 0,
\end{align*}
\]
where $b_3$, $a_0$, $f_0 \in \mathbb{C}\setminus\{0\}$ and $i = 1$, $j = 2$, $k = 3$ or $i = 3$, $j = 2$, $k = 1$. All these module-algebra structures are isomorphic to that with $a_0 = b_3 = f_0 = 1$.

(5)
\[
\begin{align*}
  k_i(x_1) &= q^{-2} x_1, \ k_i(x_2) = q^{-1} x_2, \ k_i(x_3) = q^{-1} x_3, \\
  e_i(x_1) &= a_0, \ e_i(x_2) = 0, \ e_i(x_3) = 0, \\
  f_i(x_1) &= -qa_0^{-1} x_1^2, \ f_i(x_2) = -qa_0^{-1} x_1 x_2, \ f_i(x_3) = -q a_0^{-1} x_1 x_3, \\
  k_j(x_1) &= q x_1, \ k_j(x_2) = q^{-1} x_2, \ k_j(x_3) = x_3, \\
  e_j(x_1) &= 0, \ e_j(x_2) = a_2 x_1, \ e_j(x_3) = 0, \\
  f_j(x_1) &= a_2^{-1} x_2, \ f_j(x_2) = 0, \ f_j(x_3) = 0, \\
  k_s(x_1) &= x_1, \ k_s(x_2) = q x_2, \ k_s(x_3) = q^{-1} x_3, \\
  e_s(x_1) &= 0, \ e_s(x_2) = 0, \ e_s(x_3) = b_3 x_2, \\
  f_s(x_1) &= 0, \ f_s(x_2) = b_3^{-1} x_3, \ f_s(x_3) = 0,
\end{align*}
\]
where $b_3$, $a_0$, $a_2 \in \mathbb{C}\setminus\{0\}$ and $i = 1$, $j = 2$, $k = 3$ or $i = 3$, $j = 2$, $k = 1$. All these module-algebra structures are isomorphic to that with $a_0 = a_2 = b_3 = 1$.

Proof. First, we consider the case when $m \geq 5$. By the above discussion, since there are no paths in (4.56) whose length is larger than 4 and any two vertices which are not adjacent in this path have no edge connecting them in (4.56) and
\[ (4.57) \text{the unique possibility of putting the actions of } U_q(sl(2)) \text{ on the } m \text{ vertices in the Dynkin diagram is as follows:} \]

\[
\begin{array}{ccccccc}
\ast 1 & \ast 1 & \cdots & \ast 1 & \ast 1.
\end{array}
\]

Obviously, the above case determines the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(3) \).

Second, let us study the case when \( m = 3 \). By the above rules, and because the Dynkin diagram of \( sl(4) \) is symmetric, we only need to check the following cases:

\[
\begin{array}{ccccccc}
\ast 7 & \ast 4 & \ast 5, & \ast 7 & \ast 4 & \ast 2, & \ast 4 & \ast 2 & \ast 3, \\
\ast 4 & \ast 5 & \ast 3, & \ast 4 & \ast 5 & \ast 6, & \ast 2 & \ast 3 & \ast 5, \\
\ast 2 & \ast 4 & \ast 5, & \ast 3 & \ast 5 & \ast 6, & \ast 1 & \ast 1 & \ast 1.
\end{array}
\]

To determine the module-algebra structures of \( U_q(sl(4)) \) on \( A_q(3) \), we still have to check the following equalities:

\[
\begin{align*}
k_1e_3(u) &= e_3k_1(u), k_1f_3(u) = f_3k_1(u), \\
k_3e_1(u) &= e_1k_3(u), k_3f_1(u) = f_1k_3(u), \\
e_1f_3(u) &= f_3e_1(u), e_3f_1(u) = f_1e_3(u), \\
e_1c_3(u) &= e_3c_1(u), f_1f_3(u) = f_3f_1(u),
\end{align*}
\]

for any \( u \in \{x_1, x_2, x_3\} \). For \( \ast 7 \longrightarrow \ast 4 \longrightarrow \ast 5 \), since \( k_1e_3(z) = k_1(b_3y) = q^2b_3y \) and \( e_3k_1(z) = q^{-1}b_3y, k_3e_3(z) \neq e_3k_1(z) \). Therefore, \( \ast 7 \longrightarrow \ast 4 \longrightarrow \ast 5 \) is excluded. Similarly, we exclude \( \ast 7 \longrightarrow \ast 4 \longrightarrow \ast 2, \ast 4 \longrightarrow \ast 5 \longrightarrow \ast 6, \) and \( \ast 3 \longrightarrow \ast 5 \longrightarrow \ast 6 \). Moreover, it is easy to check the five remaining cases determine the module-algebra structures of \( U_q(sl(4)) \) on \( A_q(3) \).

Thirdly, we consider the case when \( m = 4 \). By the discussion above, we only need to check the cases:

\[
\begin{array}{ccccccc}
\ast 7 & \ast 4 & \ast 2 & \ast 3, & \ast 7 & \ast 4 & \ast 5 & \ast 3, \\
\ast 7 & \ast 4 & \ast 5 & \ast 6, & \ast 2 & \ast 3 & \ast 5 & \ast 6, \\
\ast 1 & \ast 1 & \ast 1 & \ast 1.
\end{array}
\]

Since the three adjacent vertices in the Dynkin diagram of \( sl(5) \) corresponds to one Hopf algebra isomorphic to \( U_q(sl(4)) \), by the results of the module-algebra structures of \( U_q(sl(4)) \) on \( A_q(3) \), there is only one possibility:

\[
\begin{array}{ccccccc}
\ast 1 & \ast 1 & \ast 1 & \ast 1.
\end{array}
\]

Finally, we consider the isomorphism classes. Here, we only show that all module-algebra structures of \( U_q(sl(4)) \) on \( A_q(3) \) in Case (2) are isomorphic to that with \( a_0 = a_2 = f_0 = 1 \). The desired isomorphism is given by \( \psi_{a_0, a_2, f_0} : x_1 \to a_0x_1, x_2 \to a_0a_2x_2, x_3 \to f_0x_3 \). Other cases can be considered similarly. \( \Box \)

**Remark 4.22.** By Theorem [4.21] the classical limits of the above actions, i.e., the Lie algebra \( sl_{m+1} \)-actions by differentiations on \( \mathbb{C}[x_1, x_2, x_3] \) can also be obtained as before.
5. Structures of $U_q(sl(m+1))$-symmetries on $A_q(n)$

In this section, we will study the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$.

We also consider the module-algebra structures of $U_q(sl(2))$ on $A_q(n)$ first. Let

$$M \overset{\text{def}}{=} \begin{bmatrix} k(x_1) & k(x_2) & \cdots & k(x_n) \\ e(x_1) & e(x_2) & \cdots & e(x_n) \\ f(x_1) & f(x_2) & \cdots & f(x_n) \end{bmatrix}. \quad (5.1)$$

As usual, we can set

$$M_k \overset{\text{def}}{=} \begin{bmatrix} k(x_1) & k(x_2) & \cdots & k(x_n) \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 & \alpha_2 x_2 & \cdots & \alpha_n x_n \end{bmatrix},$$

where $\alpha_i$ for $i \in \{1, \cdots, n\}$ are non-zero complex numbers. So, every monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} \in A_q(n)$ is an eigenvector for $k$ and the associated eigenvalue $\alpha_1^{m_1}\alpha_2^{m_2}\cdots \alpha_n^{m_n}$ is called the weight of this monomial, which will be written as $wt(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}) = \alpha_1^{m_1}\alpha_2^{m_2}\cdots \alpha_n^{m_n}$.

Let

$$M_{ef} \overset{\text{def}}{=} \begin{bmatrix} e(x_1) & e(x_2) & \cdots & e(x_n) \\ f(x_1) & f(x_2) & \cdots & f(x_n) \end{bmatrix}. \quad (5.2)$$

Then, we have

$$wt(M_{ef}) \overset{\text{def}}{=} \begin{bmatrix} wt(e(x_1)) & wt(e(x_2)) & \cdots & wt(e(x_n)) \\ wt(f(x_1)) & wt(f(x_2)) & \cdots & wt(f(x_n)) \end{bmatrix} \overset{\propto}{=} \begin{bmatrix} q^2\alpha_1 & q^2\alpha_2 & \cdots & q^2\alpha_n \\ q^{-2}\alpha_1 & q^{-2}\alpha_2 & \cdots & q^{-2}\alpha_n \end{bmatrix}.$$

Set

$$(M)_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}_0.$$

Then, we obtain

$$wt((M)_0) \overset{\propto}{=} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ q^2\alpha_1 & q^2\alpha_2 & \cdots & q^2\alpha_n \\ q^{-2}\alpha_1 & q^{-2}\alpha_2 & \cdots & q^{-2}\alpha_n \end{bmatrix}_0 \overset{\propto}{=} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}_0.$$ 

An application of $e$ and $f$ to $[2.14]$ gives the following equalities

$$x_i e(x_j) - q\alpha_i e(x_j)x_i = qx_j e(x_i) - \alpha_j e(x_i)x_j \text{ for } i > j, \quad (5.4)$$

$$f(x_i)x_j - q\alpha_i^{-1}x_j f(x_i) = qf(x_j)x_i - \alpha_i^{-1}x_i f(x_j) \text{ for } i > j. \quad (5.5)$$

After projecting the equalities above to $(A_q(n))_1$, we obtain

$$a_j(1 - q\alpha_i)x_i = a_i(q - \alpha_j)x_j \text{ for } i > j,$$

$$b_i(1 - q\alpha_i^{-1})x_j = b_j(q - \alpha_i^{-1})x_i \text{ for } i > j.$$
Therefore, for \( i > j \), we obtain
\[
a_j \neq 0 \Rightarrow \alpha_i = q^{-1}, \quad a_i \neq 0 \Rightarrow \alpha_j = q, \quad (5.6)
b_i \neq 0 \Rightarrow \alpha_i = q, \quad b_j \neq 0 \Rightarrow \alpha_i = q^{-1}, \quad (5.7)
\]
Then, we have that for any \( j \in \{1, \ldots, n\} \),
\[
a_j \neq 0 \Rightarrow \alpha_i = q^{-1} \text{ for } \forall \ i > j, \quad \alpha_i = q \text{ for } \forall \ i < j, \quad (5.8)
b_j \neq 0 \Rightarrow \alpha_i = q^{-1} \text{ for } \forall \ i > j, \quad \alpha_i = q \text{ for } \forall \ i < j. \quad (5.9)
\]
By \cite{5.3} and using the above equalities, we get
\[
a_j \neq 0 \Rightarrow \alpha_1 = q, \ldots, \alpha_{j-1} = q, \alpha_j = q^{-2}, \alpha_{j+1} = q^{-1}, \ldots, \alpha_n = q^{-1},
b_j \neq 0 \Rightarrow \alpha_1 = q, \ldots, \alpha_{j-1} = q, \alpha_j = q^2, \alpha_{j+1} = q^{-1}, \ldots, \alpha_n = q^{-1}.
\]
So, there are \( 2n + 1 \) cases as follows: \( a_j \neq 0, \alpha_i = 0 \) for \( i \neq j \) and all \( b_i = 0 \) for any \( j \in \{1, \ldots, n\} \); \( b_j \neq 0, b_i = 0 \) for \( i \neq j \) and all \( a_i = 0 \) for any \( j \in \{1, \ldots, n\} \); \( a_j = 0 \) and \( b_j = 0 \) for any \( j \in \{1, \ldots, n\} \).

For the 1-st homogeneous component, since \( wt(e(x_i)) = q^2 wt(x_i) \neq wt(x_i) \), we have \( (e(x_i))_1 = \sum_{s \neq j} c_{is} x_s \) for some \( c_{is} \in \mathbb{C} \). Similarly, we set \( (f(x_i))_1 = \sum_{s \neq i} d_{is} x_s \) for some \( d_{is} \in \mathbb{C} \).

After projecting Equations \( 5.4-5.5 \) to \((A_q(n))_2\), we can obtain that for any \( i > j \),
\[
\sum_{s \neq j, s < i} (q - q \alpha_i) c_{js} x_s x_i + (1 - q \alpha_i) c_{ji} x_i^2 + \sum_{s > i} (1 - q \alpha_i) c_{js} x_i x_s =
\sum_{s < j} (q^2 - \alpha_j) c_{is} x_s x_j + (q - \alpha_j) c_{ij} x_j^2 + \sum_{s \neq j, s > j} (q - q \alpha_j) c_{is} x_j x_s,
\sum_{s < j} (1 - q \alpha_j^{-1}) d_{is} x_s x_j + (1 - q \alpha_j^{-1}) d_{ij} x_j^2 + \sum_{s \neq j, s > j} (q - q \alpha_j^{-1}) d_{is} x_j x_s =
\sum_{s < i, s \neq j} (q - q \alpha_i^{-1}) d_{js} x_s x_i + (q - \alpha_i^{-1}) d_{ji} x_i^2 + \sum_{s > i} (q^2 - \alpha_i^{-1}) d_{js} x_i x_s.
\]
Therefore, we have
\[
c_{js} \neq 0 \ (s < i, s \neq j) \Rightarrow \alpha_i = 1, \ c_{js} = 0 \text{ for all } s \geq i,
c_{ji} \neq 0 \Rightarrow \alpha_i = q^{-1}, \ c_{js} = 0 \text{ for any } s \neq i,
c_{js} \neq 0 \ (s > i) \Rightarrow \alpha_i = q^{-2}, \ c_{js} = 0 \text{ for all } s \leq i,
c_{is} \neq 0 \ (s < i) \Rightarrow \alpha_j = q^2, \ c_{is} = 0 \text{ for all } s \geq j,
c_{ij} \neq 0 \Rightarrow \alpha_j = q, \ c_{is} = 0 \text{ for all } s \neq j,
c_{is} \neq 0 \ (s > j, s \neq i) \Rightarrow \alpha_j = 1, \ c_{is} = 0 \text{ for all } s \leq j,
d_{is} \neq 0 \ (s < j) \Rightarrow \alpha_j = q^2, \ d_{is} = 0 \text{ for all } s > j,
d_{ij} \neq 0 \Rightarrow \alpha_j = q, \ d_{is} = 0 \text{ for all } s \neq j,
d_{is} \neq 0 \ (s > j, s \neq i) \Rightarrow \alpha_j = 1, \ d_{is} = 0 \text{ for all } s \leq j,
\]
\[d_j \neq 0 \ (s < i, \ s \neq j) \Rightarrow \alpha_s = 1, \ d_j = 0 \ \text{for all} \ s \geq i,\]
\[d_j \neq 0 \Rightarrow \alpha_s = q^{-1}, \ d_j = 0 \ \text{for all} \ s \neq i,\]
\[d_j \neq 0 \ (s > i) \Rightarrow \alpha_s = q^{-2}, \ d_j = 0 \ \text{for all} \ s \leq i.\]

Therefore, we have that for any \(j \in \{1, \cdots, n\},\)
\[c_j \neq 0 \ (s > j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_{j+1} = q^{-2}, \cdots,\]
\[\alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\]
\[c_j \neq 0 \ (s < j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\]
\[\alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1,\]
\[d_j \neq 0 \ (s > j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_{j+1} = q^{-2}, \cdots,\]
\[\alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\]
\[d_j \neq 0 \ (s < j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\]
\[\alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1.\]

Since \(\text{wt}(\{M_{ij}\}) = \begin{bmatrix} q^2a_1 & q^2a_2 & \cdots & q^2a_n \\ q^{-2}a_1 & q^{-2}a_2 & \cdots & q^{-2}a_n \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix},\)
we obtain that for any \(j \in \{1, \cdots, n\},\)
\[c_j \neq 0 \ (s > j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_{s-1} = q^{-3}, \alpha_{j+1} = q^{-2}, \cdots,\]
\[\alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\]
\[c_j \neq 0 \ (s < j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\]
\[\alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1,\]
\[d_j \neq 0 \ (s > j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_s = q, \alpha_{s+1} = q^{-2}, \cdots,\]
\[\alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\]
\[d_j \neq 0 \ (s < j) \Rightarrow \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\]
\[\alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1.\]

By the above discussion, we have only the following possibilities for the 1-st homogeneous component: \(c_j \neq 0\) for some \(i \neq j,\) other \(c_{st}\) equal to zero and all \(d_{st} = 0;\) \(d_{ij} \neq 0\) for some \(i \neq j,\) other \(d_{st}\) equal to zero and all \(c_{st} = 0;\) \(c_{j+1} \neq 0,\) \(d_{j,j+1} \neq 0\) for some \(j \in \{1, \cdots, n\}.\)

Obviously, if both the 0-th homogeneous component and the 1-th homogeneous component of \(M_{ef}\) are nonzero, there are no possibilities except when \(n = 3.\) But, the case when \(n = 3\) has been considered in Section 4. Therefore, we assume that \(n \geq 4\) in the following paper.

Moreover, there are no possibilities when the 0-th homogeneous component of \(M_{ef}\) is 0 and the 1-th homogeneous component of \(M_{ef}\) has only one nonzero position. The reasons are the same as those in [7].

Therefore, we have to consider the following cases
\[
\left[\begin{array}{cccccc}
0 & 0 & \cdots & a_i & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right], \left[\begin{array}{cccccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{array}\right]_1
\] for any \(i \in \{1, \cdots, n\},\)
structures on $b$ isomorphic to that with $a$ for any $j$, if $j$ is fixed, then all these structures are isomorphic to that with $a_j = 1$.

(4) $k(x_i) = x_i$ for all $i < j$, $k(x_j) = qx_j$, $k(x_{j+1}) = q^{-1}x_{j+1}$, $k(x_i) = x_i$ for all $i > j+1$, $e(x_i) = 0$ for all $i \neq j + 1$, $e(x_{j+1}) = c_{j+1,j}x_{j+1}$, $f(x_i) = 0$ for all $i \neq j$, $f(x_j) = c_{j+1,j}^{-1}x_{j+1}$, for any $j \in \{1, \cdots, n-1\}$ and $c_{j+1,j} \in \mathbb{C} \setminus \{0\}$. If $j$ is fixed, then all these structures are isomorphic to that with $c_{j+1,j} = 1$.

Remark 5.2. In Proposition 5.1, we present only the simplest module-algebra structures. It is also complicated to give the solutions of (5.4) and (5.5) for all cases. For example, by a very complex computation, we can obtain that in Case $\left[ \begin{array}{lll} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \end{array} \right]$, all $U_q(sl(2))$-module algebra structures on $A_q(n)$ are given by $k(x_1) = q^{-2}x_1$, $k(x_i) = q^{-1}x_i$ for all $i > 1$, $e(x_1) = a_1$, $e(x_i) = 0$ for all $i > 1$, $f(x_1) = -qa_1^{-1}x_2^2$, 

By the above cases and the discussions in Section 4, we can obtain the following proposition.

**Proposition 5.1.** For $n \geq 4$, there are the module-algebra structures of $U_q(sl(2))$ on $A_q(n)$ as follows:

1. $k(x_i) = \pm x_i$, $e(x_i) = f(x_i) = 0$, for any $i \in \{1, \cdots, n\}$. All these structures are pairwise nonisomorphic.

2. $k(x_i) = qx_i$ for all $i < j$, $k(x_j) = q^{-2}x_j$, $k(x_i) = q^{-1}x_i$ for all $i > j$, $e(x_i) = 0$ for all $i \neq j$, $e(x_j) = a_j$, $f(x_i) = a_j^{-1}x_i x_j$ for all $i < j$, $f(x_j) = -qa_j^{-1}x_j^2$, $f(x_i) = -qa_j^{-1}x_j x_i$ for all $i > j$, for any $j \in \{1, \cdots, n\}$ and $a_j \in \mathbb{C} \setminus \{0\}$. If $j$ is fixed, then all these structures are isomorphic to that with $a_j = 1$.
\[ f(x_2) = -qa_1^{-1}x_1x_2 + \sum_{2<s<n} \hat{\nu}_{2s}x_2x_s^2 + \sum_{2<s<k<l<n} \alpha_{22kl}x_2x_kx_l + \beta_{22}x_2^3, \]

\begin{align*}
 f(x_i) &= -qa_1^{-1}x_1x_i + (3)_q\beta_{22}x_2^2x_i - \sum_{2<s<i<n} q^{-1}(3)_q \hat{\nu}_{2s}x_s^2x_i \\
 &+ \frac{(2)_q}{(3)_q} \alpha_{2n2n}x_2x_i^2 + \sum_{2<s<i\leq n} \hat{\nu}_{2s}x_s^2x_i^2 - \sum_{2<s<i<n} q^{-1}(2)_q \alpha_{22si} \\
 &\cdot x_sx_i^2 + \frac{q}{(2)_q} \tilde{\nu}_{nn2}x_2x_i x_n + \sum_{2<k<i\leq n} \alpha_{2n2k}x_2x_kx_i \\
 &+ \sum_{2<s<k<i\leq n} \frac{q}{(3)_q} \alpha_{2n2k}x_2x_sx_k - \sum_{2<s<i<k\leq n} \alpha_{22sk}x_sx_i x_k \\
 &+ \sum_{2<s<k<i\leq n} \alpha_{22kl}x_sx_kx_l - \sum_{2<s<k<i<n} q^{-1}(3)_q \alpha_{22sk}x_sx_kx_i \\
 &-q^{-1}\hat{\nu}_{2i}x_i^3,
\end{align*}

where \(2<i<n\),

\[ f(x_n) = -qa_1^{-1}x_1x_n + (3)_q\beta_{22}x_2^2x_n - \sum_{2<s<n} q^{-1}(3)_q \hat{\nu}_{2s}x_s^2x_n \\
+ \hat{\nu}_{nn2}x_2x_n^2 - \sum_{2<k<n} q^{-1}(2)_q \alpha_{22kn}x_kx_n^2 + \sum_{2<k<n} \alpha_{2n2k}x_2x_kx_n \\
- \sum_{2<s<k<n} q^{-1}(3)_q \alpha_{22sk}x_sx_kx_n - q^{-1}\hat{\nu}_{2n2n}x_n^3, \]

where \(a_1 \in \mathbb{C} \setminus \{0\}\) and \(\hat{\nu}_{22}, \alpha_{22kl}, \beta_{22}, \tilde{\nu}_{nn2}, \alpha_{2n2k} \in \mathbb{C}\).

Let us denote the module-algebra structures of Case (1), those in Case (2), Case (3) and Case (4) in Proposition 5.1 by \(D\), \(A_j\), \(B_j\) and \(C_j\) respectively. For determining the module-algebra structures of \(U_q(\mathfrak{sl}(3))\) on \(A_q(n)\), we only need to check whether (5.11) hold for any \(u \in \{x_1, \ldots, x_n\}\). For convenience, we introduce a notation: if the actions of \(k_s, e_s, f_s\) are of the type \(A_i\) and the actions of \(k_t, e_t, f_t\) are of the type \(B_j\), they determine a module-algebra structure of \(U_q(\mathfrak{sl}(3))\) on \(A_q(n)\) for \(s = 1, t = 2\) or \(s = 2, t = 1\), then we say \(A_i\) and \(B_j\) are compatible. By some computations, we can obtain that \(D\) and \(D\) are compatible, \(A_i\) and \(B_j\) are compatible if and only if \(i = 1\) and \(j = n\), \(A_i\) and \(C_j\) are compatible if and only if \(i = j\), \(B_i\) and \(C_j\) are compatible if and only if \(j = i + 1\), \(C_i\) and \(C_j\) are compatible if and only if \(i = j + 1\) or \(i = j - 1\), and any two other cases are not compatible.

Therefore, by the above discussion, similar to that in Section 4, we can obtain the following proposition.

**Proposition 5.3.** For \(n \geq 4\), there are the module-algebra structures of \(U_q(\mathfrak{sl}(3))\) on \(A_q(n)\) as follows

\[ D \rightleftharpoons D, \quad (5.10) \]
Here, every two adjacent vertices determines two classes of module-algebra structures of $U_q(sl(3))$ on $A_q(n)$. For example, $A_1 \longrightarrow C_1$ corresponds to the following two kinds of module-algebra structures of $U_q(sl(3))$ on $A_q(n)$: one is that the actions of $k_1, e_1, f_1$ are of type $A_1$ and the actions of $k_2, e_2, f_2$ are of type $C_1$; the other is that the actions of $k_1, e_1, f_1$ are of type $C_1$ and the actions of $k_2, e_2, f_2$ are of type $A_1$.

Then, for determining the module-algebra structures of $U_q(sl(m + 1))$ on $A_q(n)$, we have to find the pairs of vertices which are not adjacent in $[5.11]$ and satisfy the following relation: $k_i e_j(x_s) = e_j k_i(x_s)$, $k_j e_i(x_s) = e_i k_j(x_s)$, $k_i f_j(x_s) = f_j k_i(x_s)$, $k_j f_i(x_s) = f_i k_j(x_s)$ where one vertex corresponds to the actions of $k_i, e_i$ and $f_i$ and the other vertex corresponds to the actions of $k_j, e_j$ and $f_j$, $s \in \{1, \ldots, n\}$. It is easy to check that $A_i$ and $C_j$ satisfy the above relations if and only if $i < j$ or $i > j + 1$, $B_i$ and $C_j$ satisfy the above relations if and only if $i < j$ or $i > j + 1$, $C_i$ and $C_j$ satisfy the above relations if and only if $i \neq j + 1$ or $j \neq i + 1$, and any other two vertices do not satisfy the above relations.

Therefore, we obtain the following theorem.

**Theorem 5.4.** For $m \geq 3$, $n \geq 4$, there are the module-algebra structures of $U_q(sl(m + 1))$ on $A_q(n)$ as follows:

\[ D \longrightarrow D \longrightarrow \cdots \longrightarrow D \text{,} \quad (5.12) \]

\[ A_1 \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_{i+m-2} \text{,} \quad (5.13) \]

\[ C_i \longrightarrow C_{i+1} \longrightarrow \cdots \longrightarrow C_{i+m-2} \longrightarrow B_{i+m-1} \text{,} \quad (5.14) \]

\[ C_i \longrightarrow C_{i+1} \longrightarrow \cdots \longrightarrow C_{i+m-1} \text{,} \quad (5.15) \]

\[ A_1 \longrightarrow B_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_{n+2-m} \text{,} \quad (5.16) \]

where $n + 2 - m > 1$,

\[ B_n \longrightarrow A_1 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_{m-2} \text{,} \quad (5.17) \]

where $m - 2 < n - 1$.

Here, every such diagram corresponds to two classes of the module-algebra structures of $U_q(sl(m + 1))$ on $A_q(n)$. For instance, there are the two module-algebra structures of $U_q(sl(m + 1))$ on $A_q(n)$ corresponding to $[5.17]$: the one is that the actions of $k_1, e_1, f_1$ are of the type $B_n$, those of $k_2, e_2, f_2$ are of the type $A_1$ and those of $k_i, e_i, f_i$ are of the type $C_{i-2}$ for any $3 \leq i \leq m$. The other is that the actions of $k_1, e_1, f_1$ are of the type $C_{m-1-i}$ for any $1 \leq i \leq m - 2$, those of $k_{m-1}, e_{m-1}, f_{m-1}$ are of the type $A_1$ and those of $k_m, e_m, f_m$ are of the type $B_n$. 

\[ \frac{A_1}{A_2} \cdots \frac{A_{n-2}}{A_{n-1}} \]
Remark 5.5. When \( m = n - 1 \) and the indexes of the vertices of the Dynkin diagram are given \( 1, \cdots, n - 1 \) from the left to the right, the actions corresponds to (5.15), i.e.,

\[
C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_{n-1},
\]

is the case discussed in [11]. In addition, we are sure that when \( m \geq n + 1 \), all the module-algebra structures of \( U_q(sl(m+1)) \) on \( A_q(n) \) are of the type in (5.12), since there are no paths whose length is larger than \( n + 1 \) and any two vertices which are not adjacent in this path have no edge connecting them in (5.11). The detailed proof may be similar to that in Section 4. Moreover, the module-algebra structures of the quantum enveloping algebras corresponding to the other semisimple Lie algebras on \( A_q(n) \) can be considered in the same way.

Acknowledgments. The first author (S.D.) is grateful to L. Carbone and E. Karolinsky for fruitful discussions. The second and third authors express their gratitude to the support from the projects of the National Natural Science Foundation of China (No.11271318, No.11171296 and No. J1210038) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20110101110010).

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ANALYSIS OF $U_q(sl(m+1))$-SYMMETRIES ON QUANTUM $n$-SPACES

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