ON THE PRIMES IN FLOOR FUNCTION SETS

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Abstract

Let \( \lfloor t \rfloor \) be the integral part of the real number \( t \) and let \( \mathbb{1}_p \) be the characteristic function of the primes. Denote by \( \pi_S(x) \) the number of primes in the floor function set \( S(x) := \lfloor x/n \rfloor : 1 \leq n \leq x \) and by \( S_1(x) \) the number of primes in the sequence \( \{ \lfloor x/n \rfloor \}_{n \leq x} \). Improving a result of Heyman [‘Primes in floor function sets’, Integers 22 (2022), Article no. A59], we show

\[
\pi_S(x) = \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x} e^{-c(\log x)^{3/5}(\log \log x)^{-1/5}}) \quad \text{and} \quad S_1(x) = C_1 P x + O(\epsilon x^{9/19 + \epsilon})
\]

for \( x \to \infty \), where \( C_1 := \sum_{p} 1/p(p+1) \), \( c > 0 \) is a positive constant and \( \epsilon \) is an arbitrarily small positive number.

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1. Introduction

The distribution of prime numbers is one of the most important problems in number theory. Denote by \( \pi(x) \) the number of primes \( p \leq x \). The prime number theorem states that

\[
\pi(x) = \frac{x}{\log x} + O\left( \frac{x}{(\log x)^2} \right) \quad (x \to \infty).
\]

A strong form of this theorem is

\[
\pi(x) = \text{Li}(x) + O(x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})) \quad (x \to \infty),
\]

where \( c \) is a positive constant, \( \log_2 \) denotes the iterated logarithm function and

\[
\text{Li}(x) := \int_2^x \frac{dt}{\log t}.
\]
The Riemann hypothesis is equivalent to the asymptotic formula
\[ \pi(x) = \text{Li}(x) + O_e(x^{1/2+\varepsilon}) \quad (x \to \infty), \] (1.2)
where \( \varepsilon \) is an arbitrarily small positive number. More generally, let \( N(x) \) be a set of integers of \([1, x]\) and let \( \mathcal{N}_P(x) \) be the set of prime numbers in \( N(x) \). We expect that
\[ |\mathcal{N}_P(x)| \sim \frac{|N(x)|}{\log |N(x)|} \quad (x \to \infty), \] (1.3)
provided \( N(x) \) is rather regular and is not too sparse. Some well-known examples are
- \( \{qn+1 \leq x\} \), \( \{[n^c] \leq x\} \), \( \{m^2 + n^4 \leq x\} \), \( \{m^3 + 2n^3 \leq x\} \), \( \{x < n \leq x + x^{7/12+\varepsilon}\} \),
the respective densities for which are
\[ \frac{x}{\varphi(q)} \quad (q \leq (\log x)^4, \text{Walfisz–Siegel [3]}), \]
\[ x^{1/c} \quad \left( c \leq \frac{2817}{2426}, \text{Rivat–Sargos [12]} \right), \]
\[ x^{3/4} \quad (\text{Friedlander–Iwaniec [4]}), \]
\[ x^{2/3} \quad (\text{Heath-Brown [5]}), \]
\[ x^{7/12+\varepsilon} \quad (\text{Huxley [8]}), \]
where \([t]\) is the integral part of the real number \( t \), \( \varphi(q) \) is the Euler function, \( A \) is any positive constant and \( \varepsilon > 0 \) is an arbitrarily small positive number.

Recently, Bordellès et al. [2] investigated the asymptotic behaviour of the summative function
\[ S_f(x) := \sum_{n \leq x} f\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \]
under some simple hypothesis on the growth of \( f \) and there are a number of further developments on this theme. If we use \( \Lambda(n) \) to denote the von Mangoldt function, then [13, Theorem 1.2(ii)] or [15, Theorem 1] give us immediately
\[ S_\Lambda(x) = C_\Lambda x + O_e(x^{1/2+\varepsilon}), \] (1.4)
for any \( \varepsilon > 0 \) and \( x \to \infty \), where \( C_\Lambda := \sum_{n \geq 1} \Lambda(n)/n(n+1) \). Ma and Wu [11] applied the Vaughan identity and the technique of one-dimensional exponential sums to break the \( \frac{1}{2} \)-barrier by establishing
\[ S_\Lambda(x) = C_\Lambda x + O_e(x^{35/71+\varepsilon}). \] (1.5)
This result seems rather interesting if we compare it with (1.2). The exponent 35/71 has been improved to 97/203 by Bordellès [1] and 9/19 by Liu et al. [10], using more sophisticated techniques of multiple exponential sums. Obviously, (1.5) is the prime number theorem for the floor function set
\[ S(x) := \left\{ \left\lfloor \frac{x}{n} \right\rfloor : 1 \leq n \leq x \right\}. \]
considered as the weighted count of prime powers. Very recently, Heyman [7] examined the number of primes in the floor function set \( S(x) \) without the multiplicity. The principal result of Heyman [7, Theorem 1] is the asymptotic formula

\[
\pi_S(x) := \sum_{p \leq x} 1 = \frac{4\sqrt{x}}{\log x} + O\left(\frac{\sqrt{x}}{\log x^2}\right) \quad (x \to \infty).
\]  

(1.6)

Since Heyman [6, Theorems 1 and 2] proved that

\[
|S(x)| = 2\sqrt{x} + O(1) \quad (x \to \infty).
\]  

(1.7)

it follows from (1.6) that (1.3) holds for this sparse set \( S(x) \). This may be the first example of such a sparse subset of \([1, x] \cap \mathbb{N}\) (of density \( x^{-1/2} \)) for which the prime number theorem holds.

It seems natural and interesting to establish an analogue of the strong form of the prime number theorem in (1.1) for the set \( S(x) \). We prove such a result.

**Theorem 1.1.** (i) For \( x \to \infty \),

\[
\pi_S(x) = \text{Li}_S(x) + O(\sqrt{x} \exp(-c'(\log x)^{3/5} (\log_2 x)^{-1/5})),
\]  

where \( c' > 0 \) is a positive constant and

\[
\text{Li}_S(x) := \int_2^x \frac{dt}{\log t} + \int_2^x \frac{dt}{\log(x/t)}.
\]

(ii) There is a real sequence \( \{a_n\}_{n \geq 1} \) with \( a_1 = 4 \) such that for any positive integer \( N \geq 1 \),

\[
\pi_S(x) = \sqrt{x} \sum_{n=1}^N \frac{a_n}{(\log x)^n} + O_N\left(\frac{\sqrt{x}}{(\log x)^N} \right) \quad (x \to \infty).
\]

Let \( \mathbb{P} \) be the set of all primes and let \( \mathbb{P}_{\text{power}} \) be the set of all prime powers. Denote by \( \mathbb{1}_\mathbb{P} \) and \( \mathbb{1}_{\mathbb{P}_{\text{power}}} \) their characteristic functions. Define

\[
S_{\mathbb{1}_\mathbb{P}}(x) := \sum_{n \leq x} \mathbb{1}_\mathbb{P}\left(\left\lfloor \frac{x}{n} \right\rfloor\right), \quad S_{\mathbb{1}_{\mathbb{P}_{\text{power}}}}(x) := \sum_{n \leq x} \mathbb{1}_{\mathbb{P}_{\text{power}}}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\]

Theorems 5 and 7 of [7] can be stated as follows:

\[
S_{\mathbb{1}_\mathbb{P}}(x) = C_{\mathbb{1}_\mathbb{P}}x + O(x^{1/2}),
\]  

(1.9)

\[
S_{\mathbb{1}_{\mathbb{P}_{\text{power}}}}(x) = C_{\mathbb{1}_{\mathbb{P}_{\text{power}}}}x + O(x^{1/2}),
\]  

(1.10)

where \( C_{\mathbb{1}_\mathbb{P}} := \sum_p 1/p(p+1) \) and \( C_{\mathbb{1}_{\mathbb{P}_{\text{power}}}} := \sum_{p, \nu \geq 1} 1/p\nu(p^\nu + 1) \). Similar to (1.4), these are immediate consequences of [13, Theorem 1.2(i)] or [15, Theorem 1]. Heyman [7, Theorem 6] also proved that there is a positive constant \( B > 0 \) such that the inequality

\[
S_{\mathbb{1}_\mathbb{P}}(x) \geq C_{\mathbb{1}_\mathbb{P}}x - \frac{Bx^{1/2}}{\log x}
\]  

(1.11)

for \( x \geq 2 \). We improve these results by breaking the \( 1/2 \)-barrier in the error terms of (1.9), (1.10) and (1.11).
THEOREM 1.2. For any $\varepsilon > 0$,
\begin{align*}
S_{\lfloor x \rfloor} &= C_1 x + O((x^{9/19+\varepsilon}), \quad (1.12) \\
S_{\lfloor x \rfloor} \text{Power} &= C_1 \text{Power} x + O((x^{9/19+\varepsilon}), \quad (1.13)
\end{align*}
as $x \to \infty$, where the implied constants depend on $\varepsilon$.

REMARK 1.3. It is possible to improve the error terms in (1.12) and (1.13). It seems interesting to prove $\Omega$-results for the error terms in (1.8), (1.12) and (1.13). We shall return to this problem in forthcoming work.

Very recently, Yu and Wu [14] generalised Heyman’s (1.7) by showing
\begin{equation}
S(x; q, a) := \sum_{m \in S(x)} 1 = \frac{2\sqrt{x}}{q} + O((x/q)^{1/3} \log x)
\end{equation}
uniformly for $x \geq 3$, $1 \leq q \leq x^{1/4}/(\log x)^{3/2}$ and $1 \leq a \leq q$, where the implied constant is absolute. This confirms a numerical test of Heyman.

2. Proof of Theorem 1.1

We begin by following the argument of [7]. First, we note that
\begin{equation}
S(x) = \left\{ p \in \mathbb{P} : \exists n \in [1, x] \text{ such that } \left\lfloor \frac{x}{n} \right\rfloor = p \right\}.
\end{equation}
Further, if $[x/n] = p \in \mathbb{P}$, then $x/(p + 1) < n \leq x/p$. Thus, we can write
\begin{equation}
\pi_S(x) = \sum_{p \leq x} \mathbb{1}(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p + 1} \right\rfloor > 0) = G_1(x) + G_2(x), \quad (2.1)
\end{equation}
where $\mathbb{1}(Q) = 1$ if the statement $Q$ is true and 0 otherwise, and
\begin{align*}
G_1(x) := \sum_{p \leq \sqrt{x}} \mathbb{1}(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p + 1} \right\rfloor > 0), \\
G_2(x) := \sum_{\sqrt{x} < p \leq x} \mathbb{1}(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p + 1} \right\rfloor > 0).
\end{align*}
For $p \leq \sqrt{x} - 1$,
\begin{equation}
\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p + 1} \right\rfloor > \frac{x}{p(p + 1)} - 1 > 0.
\end{equation}
Thus, the prime number theorem (1.1) gives us
\begin{equation}
G_1(x) = \pi(\sqrt{x}) + O(1) = \text{Li}(\sqrt{x}) + O(\sqrt{x} \exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5}) (2.2)
\end{equation}
for $x \geq 3$, where $c' > 0$ is a positive constant.

Next, we treat $G_2(x)$. Noticing that
\begin{equation}
0 < \frac{x}{p} - \frac{x}{p + 1} = \frac{x}{p(p + 1)} < 1
\end{equation}
for \( p > \sqrt{x} \), the quantity \( \lfloor x/p \rfloor - \lfloor x/p + 1 \rfloor \) can only equal 0 or 1. However, for \( p > x^{10/19} \), we have \( p = \lfloor x/n \rfloor \) for some \( n \leq x^{9/19} \). Thus, we can write

\[
G_2(x) = \sum_{x^{1/2} < p \leq x^{10/19}} \left( \left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p + 1} \right\rfloor \right) + O(x^{9/19})
\]

\[
= \sum_{x^{1/2} < p \leq x^{10/19}} \left( \frac{x}{p} - \frac{x}{p + 1} - \psi\left( \frac{x}{p} \right) + \psi\left( \frac{x}{p + 1} \right) \right) + O(x^{9/19})
\]

\[
= G_{2,1}(x) - G_{2,2}^{(0)}(x) + G_{2,2}^{(1)}(x) + O(x^{9/19}),
\]

where \( \psi(t) := t - \lfloor t \rfloor - \frac{1}{2} \) and

\[
G_{2,1}(x) := \sum_{x^{1/2} < p \leq x^{10/19}} \left( \frac{x}{p} - \frac{x}{p + 1} \right),
\]

\[
G_{2,2}^{(\delta)}(x) := \sum_{x^{1/2} < p \leq x^{10/19}} \psi\left( \frac{x}{p + \delta} \right) \quad (\delta = 0, 1).
\]

With the help of the prime number theorem (1.1), a simple partial integration gives

\[
G_{2,1}(x) = \sum_{x^{1/2} < p \leq x^{1/2}} \frac{x}{p^2} + O(x^{9/19}) = x \int_{\sqrt{x}}^{x/2} \frac{d\pi(t)}{t^2} + O(x^{9/19})
\]

\[
= x \int_{\sqrt{x}}^{x/2} \frac{dt}{t^2 \log t} + O(\sqrt{x} \exp(-c'(\log x)^3/5 (\log_2 x)^{-1/5}),
\]

where \( c' > 0 \) is a positive constant. Making the change of variables \( t \to x/t \) in the last integral, it follows that

\[
G_{2,1}(x) = \int_{2}^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x} \exp(-c'(\log x)^3/5 (\log_2 x)^{-1/5})
\]

(2.4)

for \( x \to \infty \).

It remains to bound \( G_{2,2}^{(\delta)}(x) \). Similar to [10], define

\[
\Xi_\delta(x; D, D') := \sum_{D < d < D'} \Lambda(d) \psi\left( \frac{x}{d + \delta} \right).
\]

According to [10, (4.3)], for any \( \varepsilon > 0 \),

\[
\Xi_\delta(x; D, 2D) \ll \varepsilon (x^2 D^7)^{1/12} x^\varepsilon
\]

uniformly for \( x \geq 3 \) and \( x^{6/13} \leq D \leq x^{2/3} \). The same proof shows that for any \( \varepsilon > 0 \),

\[
\Xi_\delta(x; D, D') \ll \varepsilon (x^2 D^7)^{1/12} x^\varepsilon
\]

(2.5)
uniformly for $x \geq 3$, $x^{6/13} \leq D \leq x^{2/3}$ and $D < D' \leq 2D$. Since we have trivially
\[
\sum_{D < p \leq D', \nu \geq 2} \Lambda(p^\nu) \psi\left(\frac{x}{p^\nu + \delta}\right) \ll \sum_{p \leq (2D)^{1/2}} \sum_{\nu < (\log 2D)/\log p} \log p \ll D^{1/2},
\]
the inequality (2.5) implies that the bound
\[
\sum_{D < p \leq D'} (\log p) \psi\left(\frac{x}{p + \delta}\right) \ll \varepsilon (x^2 D^7)^{1/12} x^\varepsilon \tag{2.6}
\]
holds uniformly for $x \geq 3$, $x^{6/13} \leq D \leq x^{2/3}$ and $D < D' \leq 2D$. Using (2.6),
\[
G_{2,2}^{(\delta)}(x) \ll \max_{x^{1/2} < D \leq x^{10/19}} \sum_{D < p \leq 2D} \psi\left(\frac{x}{p + \delta}\right)
\ll \max_{x^{1/2} < D \leq x^{10/19}} \int_D^{2D} \frac{1}{\log t} d\left(\sum_{D < p \leq t} (\log p) \psi\left(\frac{x}{p + \delta}\right)\right) \tag{2.7}
\ll \varepsilon \max_{x^{1/2} < D \leq x^{10/19}} (x^2 D^7)^{1/12} x^\varepsilon
\ll \varepsilon x^{9/19 + \varepsilon}.
\]
Inserting (2.4) and (2.7) into (2.3), we find that
\[
G_2(x) = \int_2^\sqrt{x} \frac{dt}{\log(x/t)} + O(\sqrt{x} \exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5}). \tag{2.8}
\]
Now the required result (1.8) follows from (2.1), (2.2) and (2.8).

The second assertion is an immediate consequence of the first one thanks to a simple partial integration.

3. Proof of Theorem 1.2

We begin by following the argument of [9]. Let $f = 1_p$ or $1_{p_{\text{even}}}$ and let $N \in [x^{1/3}, x^{1/2})$ be a parameter which can be chosen later. First, we write
\[
S_f(x) = \sum_{n \leq x} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right) = S_f^f(x) + S_f^g(x) \tag{3.1}
\]
with
\[
S_f^f(x) = \sum_{n \leq N} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right), \quad S_f^g(x) = \sum_{N < n \leq x} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right).
\]
We have trivially
\[
S_f^f(x) \ll N. \tag{3.2}
\]
To bound $S_f^g(x)$, we put $d = [x/n]$. Noticing that
\[
x/n - 1 < d \leq x/n \Leftrightarrow x/(d + 1) < n \leq x/d,
\]
we see that
\[
S_f^\delta(x) = \sum_{d \leq x/N} f(d) \sum_{\nu \leq \nu(x) \leq \delta x/N} 1
= \sum_{d \leq x/N} f(d) \left( \frac{x}{d} - \frac{x}{d+1} + \psi\left( \frac{x}{d+1} \right) \right)
= x \sum_{d \geq 1} \frac{f(d)}{d(d+1)} + R_1^f(x, N) - R_0^f(x, N) + O(N),
\]  
(3.3)
where we have used the bounds
\[
x \sum_{d > x/N} \frac{f(d)}{d(d+1)} \ll N,
\]
\[
\sum_{d \leq N} f(d) \left( \psi\left( \frac{x}{d+1} \right) - \psi\left( \frac{x}{d} \right) \right) \ll N
\]
and
\[
R_0^f(x, N) = \sum_{N < d \leq x/N} f(d) \psi\left( \frac{x}{d+\delta} \right).
\]

Combining (3.1), (3.2) and (3.3), it follows that
\[
S_f(x) = x \sum_{d > 1} \frac{f(d)}{d(d+1)} + O_\varepsilon(|R_1^f(x, N)| + |R_0^f(x, N)| + N).
\]

However,
\[
R_\varepsilon^{\text{power}}(x, N) = \sum_{N < d \leq \nu(N)} \psi\left( \frac{x}{p^\delta} \right) = R_\delta^{\text{power}}(x, N) + O((x/N)^{1/2}).
\]

Thus, to prove Theorem 1.2, it suffices to show that
\[
R_\delta^{\text{power}}(x, N) \ll_\varepsilon N x^\varepsilon \quad (x \geq 1)
\]
for $N = x^{9/19}$. This can be done exactly as for (2.7) by using (2.6):
\[
R_\delta^{\text{power}}(x, N) \ll_\varepsilon x^\varepsilon \max_{x^{9/19} < D \leq x^{10/19}} \sum_{D < p \leq 2D} \psi\left( \frac{x}{p+\delta} \right)
\ll_\varepsilon x^\varepsilon \max_{x^{9/19} < D \leq x^{10/19}} \int_0^{2D} \frac{1}{\log t} \left( \sum_{D < p \leq t} (\log t) \psi\left( \frac{x}{p+\delta} \right) \right)
\ll_\varepsilon x^{9/19+\varepsilon}.
\]

This completes the proof.

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