On the distance eigenvalues of design graphs

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Abstract

A design graph is a regular bipartite graph in which any two distinct vertices of the same part have the same number of common neighbors. This class of graphs have a close relationship to strongly regular graphs. In this paper, we study the distance eigenvalues of the design graphs. Also, we will explicitly determine the distance eigenvalues of a class of design graphs, and determine the values for which the class is distance integral, that is, its distance eigenvalues are integers.

1 Introduction and Preliminaries

In this paper, a graph $G = (V, E)$ is considered as an undirected simple graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For all the terminology and notation not defined here, we follow [4, 6, 7, 8, 20].

Let $G = (V, E)$ be a graph and $A = A(G)$ be an adjacency matrix of $G$. The characteristic polynomial of $G$ is defined as $P(G; x) = P(x) = \det(xI - A)$. A zero of $P(x)$ is called an eigenvalue of the graph $G$. A graph is called integral if all its eigenvalues are integers. The study of integral graphs was initiated by Harary and Schwenk in 1974 (see [9]). A survey of papers up to 2002 has been appeared in [3].

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Let $n$ be the number of vertices of the graph $G$. The distance matrix $D = D(G)$ is an $n \times n$ matrix indexed by $V$ such that $D_{uv} = d_G(u, v) = d(u, v)$ where $d_G(u, v)$ is the distance between the vertices $u$ and $v$ in the graph $G$. The characteristic polynomial $P(D; x) = \det(xI - D) = D_G(x)$ is the distance characteristic polynomial of $G$. Since $D$ is a real symmetric matrix, the distance characteristic polynomial $D_G(x)$ has real zeros. Every zero of the polynomial $D_G(x)$ is called a distance eigenvalue of the graph $G$. Two surveys on the distance spectra of graphs are [1,10]. An interested reader may see some of the recent works on distance spectra of graphs in [2,5,11,18,20].

A graph $G$ is called distance integral (briefly, $D$-integral) if all the distance eigenvalues of $G$ are integers. The $D$-integral graphs are studied only in a few number of papers (see [17,18,20]).

Let $G = (V, E)$ be a graph with the vertex-set $V = \{v_1, \ldots, v_n\}$ and distance matrix $D(G) = D = (d_{ij})_{n \times n}$, where $d_{ij} = d(v_i, v_j)$. Let $H \leq \text{Aut}(G)$ and $\pi = \{w_1^H = C_1, \ldots, w_m^H = C_m\}$ be the orbit partition of $H$, where $\{w_1, \ldots, w_m\} \subset V$. Let $Q = Q_\pi = (q_{ij})_{m \times m}$ be the matrix which its rows and columns are indexed by $\pi$ such that,

$$q_{ij} = \sum_{w \in C_j} d(v, w), \quad (*)$$

where $v$ is a fixed element in the cell $C_i$. It is easy to check that this sum is independent of $v$, that is, if $u \in C_i$, then $q_{ij} = \sum_{w \in C_j} d(v, w) = \sum_{w \in C_j} d(u, w)$. Hence, the matrix $Q$ is well defined. We call the matrix $Q$ the quotient matrix of $D$ over $\pi$. We claim that every eigenvalue of $Q$ is an eigenvalue of the distance matrix $D$. In fact we have the following facts.

**Theorem 1.1.** [17] Let $G = (V, E)$ be a graph with the distance matrix $D$. Let $\pi$ be an orbit partition of $V$ and $Q$ be a quotient matrix of $D$ over $\pi$. Then every eigenvalue of $Q$ is an eigenvalue of the distance matrix $D$.

**Proposition 1.2.** [17] Let $G = (V, E)$ be a graph with the distance matrix $D$. Let $\pi$ be an orbit partition of $V$ and $Q$ be a quotient matrix of $D$ over $\pi$. Let $\theta$ be an eigenvalue of the distance matrix $D$ with the non zero eigenvector $f$ such that $f$ is constant on every cell of $\pi$. Then $\theta$ is an eigenvalue of the matrix $Q$.

We now can easily verify the following statement by an easy discussion [17].

**Theorem 1.3.** Let $G = (V, E)$ be a graph and $D$ be a distance matrix for $G$. Let $f \neq 0$ be an eigenvector with the eigenvalue $\lambda$ for $D$. Let $H$ be a subgroup of $\text{Aut}(G)$ and $\pi$ be its orbit partition on $V$ and $Q$ is a quotient matrix of $D$ over $\pi$. If $\lambda$ is not an eigenvalue of $Q$, then the sum of the values of $f$ on each cell of $\pi$ is zero.

The following fact is an important tool for finding some interesting results in the present work.
Theorem 1.4. Let $G = (V, E)$ be a vertex-transitive graph with the distance matrix $D$. Let $H$ be a subgroup of $\text{Aut}(G)$ with the orbit partition $\pi$ on $V$ such that $\pi$ has a singleton cell $\{x\}$. Let $Q = Q_\pi$ be a quotient matrix of $D$ over $\pi$. Then the set of distinct eigenvalues of $D$ is equal to the set of distinct eigenvalues of $Q$.

A partition $V = V_1 \cup V_2 \cup \cdots \cup V_m$ is called a distance equitable partition for the vertex-set of the graph $G = (V, E)$ if for every $v \in V_i$ the sum $\sum_{w \in V_j} d(v, w)$ is constant, that is, $\sum_{w \in V_j} d(v, w)$ is independent of the choice of the vertex $v$. Hence, as we saw, every orbit partition for the vertex-set $V$ is a distance equitable partition. Now if we define by the partition $\Pi = \{V_1, \ldots, V_m\}$ the matrix $P = P_\Pi = (p_{ij})_{m \times m}$ by the rule,

$$p_{ij} = \sum_{w \in V_j} d(v, w),$$

where $v$ is a fixed element in the cell $C_i$, then the matrix $P$ is well defined. We call $P$ a quotient distance matrix over the partition $\Pi$, or a quotient matrix for the distance matrix $D$ over $\Pi$. By a similar argument which have been appeared in the proof of Theorem 2.1, in [17], we can check that the following fact holds.

Theorem 1.5. Let $G = (V, E)$ be a graph with the distance matrix $D$. Let $\Pi$ be a distance equitable partition of $V$ and $P$ be a quotient matrix of $D$ over $\Pi$. Then every eigenvalue of $P$ is an eigenvalue of the distance matrix $D$.

We have a result similar to what is appeared in Theorem 1.1, for the case distance equitable partitions.

Proposition 1.6. Let $G = (V, E)$ be a graph with the distance matrix $D$. Let $\Pi$ be a distance equitable partition of $V$ and $P$ be a quotient matrix of $D$ over $\Pi$. Let $\theta$ be an eigenvalue of the distance matrix $D$ with the non zero eigenvector $f$ such that $f$ is constant on every cell of $\Pi$. Then $\theta$ is an eigenvalue of the matrix $P$.

Proof. The proof is similar to the discussion which has been appeared in [17, Prop 2.2].

We now introduce a class of graphs, one which some classes of graphs with interesting algebraic properties are subclasses of it.

Definition 1.7. [19] A design graph with parameters $(m, d, c)$, $c \neq 0$, is a $d$-regular bipartite graph of order $2m$ in which any two distinct vertices of the same part have $c$ common neighbor(s). The complete bipartite graph $K_{n,m}$ fits the definition, but we exclude it by convention.
It is easy to check (by double counting method) that if \( G = (V, E) \) is a
design graph with parameters \((m, d, c)\), then the following equality holds,

\[
c(m - 1) = d(d - 1).
\]

It follows from Definition 1.7, that every design graph is a connected graph with
diameter 3 and girth 4 or 6 according to whether \( c > 1 \) or \( c = 1 \). Conversely,
a regular bipartite graph of diameter 3 and girth 6 is a design graph with
parameter \( c = 1 \) [19].

**Example 1.8.** Let \( n \geq 3 \) be an integer. Let \( V \) be the set of all \( 1 \)-subsets and
\((n - 1)\)-subsets of \([n] = \{1, \ldots, n\}\). The bipartite Kneser graph \( H(n, 1) \) has
\( V \) as its vertex-set, and two vertices \( v, w \) are adjacent if and only if \( v \subset w \)
or \( w \subset v \). It is easy to see that \( H(n, 1) \) is a bipartite graph of diameter 3.
Some of properties of the graph \( H(n, 1) \) and a generalization of it have been
appeared in [13,14,15,16]. We can easily show that \( H(n, 1) \) is a design graph
with parameters \((n, n - 1, n - 2)\).

A strongly regular graph with parameters \((n, k, a, c)\) is a \( k \)-regular graph of
order \( n \) in which every pair of adjacent vertices has exactly \( a \) common neighbors
and every pair of nonadjacent vertices has exactly \( c \) common neighbors. For
example, the cycle \( C_5 \) is a strongly regular graph with parameters \((5, 2, 0, 1)\)
and the Petersen graph is a strongly regular graph with parameters \((10, 3, 0, 1)\).

Let \( G = (V, E) \) be a graph. The bipartite double cover of \( G \) which we denote
it by \( B(G) \) is a graph with the vertex-set \( V \times \{0, 1\} \), in which vertices \((v, a)\) and
\((w, b)\) are adjacent if and only if \( a \neq b \) and \( \{v, w\} \in E \). For example, if \( n > 2 \) is
an odd integer, then the bipartite double cover of the cycle \( C_n \) is (isomorphic
with) the cycle \( C_{2n} \). Also, the bipartite double cover of the complete graph \( K_n \)
is (isomorphic with) the bipartite Kneser graph \( H(n, 1) \).

**Example 1.9.** Let \( G = (V, E) \) be a strongly regular graph with parameters
\((n, d, a, c)\) with \( a \neq 0 \) and \( a = c \). It is not hard to check that the bipartite
double cover of \( G \), that is, the graph \( B(G) \) is a design graph with parameters
\((n, d, c)\).

In this paper, we will study the distance eigenvalues of design graphs. Also
we will explicitly determine the distance eigenvalues of an important class of
design graphs and determine the values for which the class is integral. The
main tool which we use in our work is the distance equitable partition and
the orbit partition method in algebraic graph theory, which we have already
employed it in determining the adjacency eigenvalues of a particular family of
graphs [12] and later in determining the sets of distance eigenvalues of some
other families of graphs [17,18]. We will show how we can find, by using this
method, the set of all distinct distance eigenvalues of some classes of design
graphs.
2 Main Results

Let \( G = (V = V_1 \cup V_2, E) \), \( V_1 \cap V_2 = \emptyset \) be a design graph with parameters \((m, d, c)\), the adjacency matrix \( A = (a_{ij}) \) and distance matrix \( D = (d_{ij}) \). Consider the partition \( \pi = \{V_1, V_2\} \). If \( v \in V_1 \), then the number of its neighbors in \( V_2 \) is \( d \). If \( w \in V_2 \) is not adjacent to \( v \) then it is at distance 3 from \( v \). Thus we have,

\[
\sum_{w \in V_2} d(v, w) = d + 3(m - d) = 3m - 2d.
\]

On the other hand, if \( w \in V_1 \), since \( v \) and \( w \) have at least a common neighbor, then \( d(v, w) = 2 \). Hence we have,

\[
\sum_{w \in V_1} d(v, w) = 2(m - 1).
\]

We now deduce that \( \pi = \{V_1, V_2\} \) is a distance equitable partition for the vertex-set \( V \). Hence, the following matrix \( Q \) is a quotient matrix of the distance matrix \( D \) over the partition \( \pi \),

\[
Q = \begin{pmatrix}
2(m - 1) & 3m - 2d \\
3m - 2d & 2(m - 1)
\end{pmatrix}.
\]

It is easy to see that each of the functions (vectors) \( f_1 = (1, 1)^t \) and \( f_2 = (1, -1)^t \) are the eigenvectors for the matrix \( Q \) with the eigenvalues,

\[
\gamma_1 = 2(m - 1) + 3m - 2d = 5m - 2d - 2 \quad \text{and} \quad \gamma_2 = 2(m - 1) - (3m - 2d) = -m + 2d - 2,
\]

respectively. Hence, from Theorem 1.5, we have the following result.

**Proposition 2.1.** Let \( G = (V_1 \cup V_2, E) \), \( V_1 \cap V_2 = \emptyset \) be a design graph with parameters \((m, d, c)\). Then, the integers \( \gamma_1 = 5m - 2d - 2 \) and \( \gamma_2 = -m + 2d - 2 \) are distance eigenvalues of \( G \).

**Remark 2.2.** Concerning Proposition 2.1, we can easily check that the eigenvector corresponding to the eigenvalue \( \gamma_1 \) for the distance matrix \( D \) of the graph \( G \) is the function \( e_1 \), defined by the rule \( e_1(v) = 1 \), for every \( v \in V_1 \cup V_2 = V(G) \). Also, the eigenvector corresponding to the eigenvalue \( \gamma_2 \) for the distance matrix \( D \) of the graph \( G \) is the function \( e_2 \), defined by the rule \( e_2(v) = 1 \), for every \( v \in V_1 \), and \( e_2(v) = -1 \), for every \( v \in V_2 \).

We now proceed to construct another distance equitable partition for the vertex-set of the design graph \( G = (V, E) \), \( V = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \), with parameters \((m, d, c)\). Let \( v_1 \) be an arbitrary vertex in the part \( V_1 \). We let,

\[
O_1 = \{v_1\}.
\]
\[ O_2 = V_1 - \{ v_1 \}, \]
\[ O_3 = N(v_1), \text{ the set of neighbors of } v \text{ in } V_2, \text{ and} \]
\[ O_4 = V_2 - N(v_1) = V_2 - O_3. \]

We claim that \[ \pi_2 = \{ O_1, O_2, O_3, O_4 \}, \] is a distance equitable partition for the vertex-set \( V \) of \( G \). Let \( v_i \in O_i, \ i \in \{ 2, 3, 4 \} \) be arbitrary vertices. Then we have the following equalities.

\[ p_{11} = \sum_{w \in O_1} d(v_1, w) = 0. \]
\[ p_{12} = \sum_{w \in O_2} d(v_1, w) = 2(m - 1). \]
\[ p_{13} = \sum_{w \in O_3} d(v_1, w) = d. \]
\[ p_{14} = \sum_{w \in O_4} d(v_1, w) = 3(m - d). \]
\[ p_{21} = \sum_{w \in O_1} d(v_2, w) = 2. \]
\[ p_{22} = \sum_{w \in O_2} d(v_2, w) = 2(m - 2). \]
\[ p_{23} = \sum_{w \in O_3} d(v_2, w) = c + 3(d - c) = 3d - 2c. \]

Because vertices \( v_1 \) and \( v_2 \) have \( c \) common neighbors which are in \( N(v_1) = O_3 \). Moreover, if the vertex \( w \) in \( O_3 \) is not adjacent to \( v_2 \), then \( d(v_2, w) = 3 \).

\[ p_{24} = \sum_{w \in O_4} d(v_2, w) = (d - c) + 3((m - d) - (d - c)) = 3(m - d) - 2d + 2c = 3m - 5d + 2c. \]

Because \( c \) neighbors of \( v_2 \) are in \( O_3 \), and hence \( d - c \) neighbors of it are in \( O_4 \).

\[ p_{31} = \sum_{w \in O_1} d(v_3, w) = 1. \]
\[ p_{32} = \sum_{w \in O_2} d(v_3, w) = (d - 1) + 3((m - 1) - (d - 1)) = 3m - 2d - 1. \]

Because, \( (d - 1) \) neighbors of \( v_3 \) are in \( O_2 \).

\[ p_{33} = \sum_{w \in O_3} d(v_3, w) = 2(d - 1). \]
\[ p_{34} = \sum_{w \in O_4} d(v_3, w) = 2(m - d). \]
\[ p_{41} = \sum_{w \in O_1} d(v_4, w) = 3. \]
\[ p_{42} = \sum_{w \in O_2} d(v_4, w) = d + 3(m - 1 - d) = 3m - 2d - 3. \]
\[ p_{43} = \sum_{w \in O_3} d(v_4, w) = 2d. \]
\[ p_{44} = \sum_{w \in O_4} d(v_4, w) = 2(m - d - 1). \]

Now from our argument it follows that \( \pi_2 \) is a distance equitable partition for the vertex-set \( V \). Hence, we have the following result.

**Theorem 2.3.** Let \( G = (V, E) \), \( V = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \), be a design graph with parameters \((m, d, c)\). Let \( v \) be a vertex in \( V_1 \). Then the partition

\[ \pi_2 = \{O_1 = \{v\}, O_2 = V_1 - O_1, O_3 = N(v), O_4 = V_2 - O_3\}, \quad (*) \]

is a distance equitable partition for the vertex-set \( V \) and the following matrix \( P = (p_{ij})_{4 \times 4} \) is a distance quotient matrix over \( \pi_2 \),

\[
P = P(m, d, c) = \begin{pmatrix} 0 & 2m - 2 & d & 3m - 3d \\
2 & 2m - 4 & 3d - 2c & 3m - 5d + 2c \\
1 & 3m - 2d - 1 & 2d - 2 & 2m - 2d \\
3 & 3m - 2d - 3 & 2d & 2m - 2d - 2 \end{pmatrix}. \quad (**)\]

**Remark 2.4.** From Proposition 2.1, remark 2.2, and proposition 1.6, we can deduce that \( \gamma_1 = 5m - 2d - 2 \) and \( \gamma_2 = -m + 2d - 2 \) are distance eigenvalues of the matrix \( P \) which is defined in Theorem 2.3. Now one can find the other eigenvalues of \( P \) by some handy calculations.

We now conclude from Theorem 1.4, Theorem 1.5, and Theorem 2.3, the following important result.

**Theorem 2.5.** Let \( G = (V, E) \), \( V = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \), be a design graph with parameters \((m, d, c)\). Then, every eigenvalues of the matrix \( P \) defined in Theorem 2.3, is a distance eigenvalue for the graph \( G \). Moreover, if the design graph \( G \) is vertex-transitive and the partition \( \pi_2 \) defined in Theorem 2.3, is an orbit partition, then the set of eigenvalues of \( P \) is equal to the set of distance eigenvalues of the graph \( G \).

In the next section, we will see how Theorem 2.5, can help us to determine the distance eigenvalues of a design graph.

### 3 An Application

Let \( q \) be a power of a prime \( p \) and \( \mathbb{F}_q \) be a finite field of order \( q \). Let \( V(q, n) \) be a vector space of dimension \( n \) over \( \mathbb{F}_q \). We define the graph \( S(q, n, 1) \) as a graph with the vertex-set \( V = V_1 \cup V_{n-1} \), where \( V_1 \) and \( V_{n-1} \) are the family of subspaces in \( V(q, n) \) of dimension 1 and \( n - 1 \), respectively, in which two vertices \( v \) and \( w \) are adjacent whenever \( v \) is a subspace of \( w \) or \( w \) is a subspace of \( v \). It is clear that \( S(q, n, 1) \) is a bipartite graph with partition \( V = V_1 \cup V_{n-1} \).
We know that the number of $k$-subspaces of a vector space $U$ of dimension $n$ over the field $\mathbb{F}_q$ is the Gaussian number

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\ldots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\ldots(q^k-q^{k-1})} = \frac{(q^n-1)\ldots(q^{n-k+1}-1)}{(q^k-1)\ldots(q-1)}.$$ 

Thus, $|V_1| = \binom{n}{1}$ and $|V_{n-1}| = \binom{n}{n-1} = \binom{n}{1}$.

Hence, the graph $S(q, n, 1)$ is of order $2^{\binom{n}{1}}$. It can be checked that $S(q, n, 1)$ is a vertex-transitive graph [16,19]. Also, it can be check that the girth of the graph $S(q, n, 1)$ is 6 when $n = 3$ and is 4 when $n \geq 4$. Some properties of $S(q, 3, 1)$ have been investigated in [8, Chapter 5]. It is easy to show that the graph $S(q, n, 1)$ is a design graph with parameters $(m, d, c)$ [19], where

$$m = \frac{q^n-1}{q-1}, \quad d = \frac{q^{n-1}-1}{q-1}, \quad c = \frac{q^{n-2}-1}{q-1}. \quad (1)$$

Let $K$ be a field, $V(K, n)$ a vector space of dimension $n$ over and $K$ and $GL(n, K)$ be the group of non-singular linear mappings on the space $V(K, n)$. This group contains a normal subgroup isomorphic to $K^*$, namely, the group $Z = \{kI_{V(K, n)} | k \in K\}$, where $I_{V(K, n)}$ is the identity mapping on $V(K, n)$. We denote the quotient group $\frac{GL(n, K)}{Z}$ by $PGL(n, K)$.

Note that if $(a + Z) \in PGL(n, K)$ and $x$ is an $m$-subspace of $V(K, n)$, then $(a + Z)(x) = \{a(u) | u \in x\}$ is an $m$-subspace of $V(K, n)$. In the sequel, we also denote $(a + Z) \in PGL(n, K)$ by $a$. Let $V_m$ denote the set of $m$-subspaces of the vector spaces $V(n, k)$. Now, if $a \in PGL(n, K)$, it is easy to see that the mapping $f_a : V_m \rightarrow V_m$, defined by the rule $f_a(v) = a(v)$, is a well defined function. Therefore if we let

$$A(n) = \{f_a | a \in PGL(n, K)\}, \quad (2)$$

then $A(n)$ is a group isomorphic to the group $PGL(n, K)$ (as abstract groups), which acts transitively on the set $V_m$.

Now, it is easy to check that $A(n)$ is a subgroup of automorphism group of the graph $S(q, n, 1)$. The graph $S(q, n, 1)$ has some other automorphisms. In fact, the mapping $t$ defined on its vertex-set by the rule $t(v) = v^+$, where $v^+$ is the orthogonal complement of $v$, is an automorphism of the graph $S(q, n, 1)$. Hence $M = \langle A(n), t \rangle \leq Aut(G)$. Note that the order of $t$ is 2 and hence $\langle t \rangle \cong \mathbb{Z}_2$. Let $w \in V_1$ be a vertex in the graph $S(q, n, 1)$. Let $H$ be a subgroup of $A(n)$ which fixes $w$. Now, it is not hard to check that the following partition $\pi$ is an orbit partition generated by $H$ on the vertex-set of $S(q, n, 1)$,

$$\pi = \{O_1 = \{w\}, O_2 = V_1 - O_1, O_3 = N(w), O_4 = V_2 - O_3\}.$$
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Since the graph $S(q, n, 1)$ is a vertex-transitive graph, it follows from Theorem 2.4, that the matrix $P = P(m, d, c)$ defined in the theorem contains all the distance eigenvalues of the graph $S(q, n, 1)$, where $m, d, c$ are parameters which are defined in (1). We can see from Remark 2.4, (or Theorem 1.3, and Remark 2.2, because $\pi$ is an orbit partition for the vertex-set of the graph $(S(n, q, 1))$ that,

$$\lambda_1 = \frac{5q^n - 1}{q - 1} - 2 \frac{q^{n-1} - 1}{(q - 1)} - 2 = \frac{5q^n - 2q^{n-1} - 2q - 1}{q - 1},$$

and

$$\lambda_2 = \frac{-q^n - 1}{q - 1} + 2 \frac{q^{n-1} - 1}{q - 1} - 2 = \frac{-q^n + 2q^{n-1} - 2q + 1}{q - 1},$$

are distance eigenvalues of the graph $S(q, n, 1)$. Having these eigenvalues in the hand, we can find two other eigenvalues of the matrix $P = \tilde{P}(m, d, c)$ by some handy calculations. We can also find the eigenvalues of the matrix $P$ by a suitable software program. Using Wolfram Mathematica [21] we have,

$$P := \{\{0, 2m - 2, d, 3m - 3d\},$$

$$\{2, 2m - 4, 3d - 2c, 3m - 5d + 2c\},$$

$$\{1, 3m - 2d - 1, 2d - 2, 2m - 2d\},$$

$$\{3, 3m - 2d - 3, 2d, 2m - 2d - 2\}\}$$

$$m := (q^n - 1)/(q - 1)$$

$$d := (q^{n-1} - 1)/(q - 1)$$

$$c := (q^{n-2} - 1)/(q - 1)$$

$$\text{Eigenvalues}[P] = \{ r_1 = \frac{-q^2 + 2q^2 + 2q^{n-2} - 5q^{1+n}}{(-1+q)q}, r_2 = \frac{-2q^2 - 2q^n + q^{1+n}}{(-1+q)q},$$

$$r_3 = \frac{2q^2 - q^3 - \sqrt{q^{2+n}-2q^{3+n}+q^{4+n}}}{(-1+q)q^2}, r_4 = \frac{2q^2 - q^3 + \sqrt{q^{2+n}-2q^{3+n}+q^{4+n}}}{(-1+q)q^2} \}.$$ 

Note that $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Hence, the set $R = \{r_1, r_2, r_3, r_4\}$ is the set of distance eigenvalues of the graph $S(n, q, 1)$. Note that $r_1$ and $r_2$ are integers, since they are roots of a monic polynomial with integer coefficients, namely, they are algebraic integers, and every rational algebraic integer is an integer. Also, we have $q^{2+n} - 2q^{3+n} + q^{4+n} = q^{n+2}(1 - 2q + q^2) = q^{n+2}(q - 1)^2$.

Hence,

$$r_3 = \frac{2q^2(1-q) - (q-1)\sqrt{q^{n+2}}}{q^2(-1+q)} = \frac{-2q^2 - \sqrt{q^{n+2}}}{q^2},$$

and

$$r_4 = \frac{-2q^2 + \sqrt{q^{n+2}}}{q^2}.$$
Now it follows that if $n$ is an even integer, then the graph $S(n, q, 1)$ is a distance integral graph. We now have the following result.

**Theorem 3.1.** Let $n \geq 3$ be an integer. Then the set of distance eigenvalues of the graph $S(n, q, 1)$ is the set,

$$\left\{ \frac{5q^n - 2q^{n-1} - 2q - 1}{q - 1}, \frac{-q^n + 2q^{n-1} - 2q + 1}{q - 1}, \frac{-2q^2 - \sqrt{q^{n+2}}}{q^2}, \frac{-2q^2 + \sqrt{q^{n+2}}}{q^2} \right\}.$$  

Hence, if $n$ is an even integer, then the graph $S(n, q, 1)$ is a distance integral graph.

4 Conclusion

In this paper, we studied the distance eigenvalues of design graphs. In particular, by finding the set of distinct distance eigenvalues of a class of design graphs, we showed that the class is a distance integral graph (Theorem 3.1). The main tools which we employed were the distance equitable and orbit partition methods in algebraic graph theory.

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