An experimental study of algorithms for obtaining a singly connected subgraph

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Abstract

A directed graph $G = (V, E)$ is singly connected if for any two vertices $v, u \in V$, the directed graph $G$ contains at most one simple path from $v$ to $u$. In this paper, we study different algorithms to find a feasible but necessarily optimal solution to the following problem. Given a directed acyclic graph $G = (V, E)$, find a subset $H \subseteq E$ of minimum size such that the subgraph $(V, E \setminus H)$ is singly connected. Moreover, we prove that this problem can be solved in polynomial time for a special kind of directed graphs.

keywords

Depth First Search, Singly Connected Graph, cross edges, forward edges, backward edges, Sources, Sinks

1. Introduction

A directed graph $G = (V, E)$ is singly connected if for any two vertices $v, u \in V$, the directed graph $G$ contains at most one simple path from $v$ to $u$ [2, 3]. Buchsbaum and Carlisle [1], Khuller [5, 6], and Karlin [7] provided algorithms to check whether an input directed graph is singly connected or not in $O(n^2)$ time. In 2015, Dietzfelbinger and Jaberi [4] presented a better algorithm for testing whether a directed graph is singly connected or not. Moreover, they proved that the following problem is NP-hard. Given a directed graph $G = (V, E)$, find a subset $H \subseteq E$ of minimum size such that the subgraph $(V, E \setminus H)$ is singly connected. In this paper, we prove that this problem can be solved in polynomial time for a special kind of graphs. Furthermore, we study different algorithms to find a feasible but necessarily optimal solution to this problem when the input graph has no cycles.
2. An optimal solution for special kind of graphs

In this section we study a special case when the input graph \( G = (V, E) \) has the following properties:

- \( G \) has no cycles
- the number of sources in \( G \) is one
- the number of vertices is equal to the number of edges in \( G \)

The following Theorem shows that removing only one edge leaves a singly connected subgraph. This means we can find an optimal solution which contains only one edge.

**Theorem 1:** Let \( G = (V, E) \) be a directed acyclic graph such that \( |E| = |V| \) and the number of sources in \( G \) is one. Then an optimal solution can be found in polynomial time.

**proof.** Let \( h \) be the source in \( G \). Then all the vertices in \( V \) are reachable from the vertex \( h \). Therefore, performing depth first search algorithm starting from vertex \( h \) produces exactly \( |V| - 1 \) tree edges and one non-tree edge which is a cross or forward edge. Moreover, these tree edges form a tree called dfs tree. This tree is a subgraph of \( G \) and for any two vertices \( v, u \in V \), there is at most one simple path from \( v \) to \( u \) in this tree. Notice that the non-tree edge produced by depth first search is an optimal solution.

The following pseudocode (DFS_once) shows an algorithm that is able to compute an optimal solution when the input graph is a directed acyclic graph with one source and the number of edges is equal to the number of vertices.

**DFS_once(DirectedGraph):**
1. Let \( h \) be the source source in \( G \)
2. tree ← DFS(DirectedGraph)
3. for each edge in tree:
   3.1 if(edge.class.is_equal_to("cross") or edge.class.is_equal_to("forward"))
      
   3.1.1 cross_edge ← edge
   
4. DirectedGraph.remove_edges_from(cross_edge)
5. return DirectedGraph
The correctness of this algorithm follows from Theorem 1 and the following theorem shows that DFS\_once algorithm runs in linear time.

**Theorem 2.** DFS\_once algorithm runs in $O(n)$ time.

**proof.** DFS runs in linear time and the number of edges is $n$.

### 3. Algorithms for directed graphs that has no cycles

In this section we describe different algorithms for obtaining a singly connected subgraph when the input graph contains no cycles.

#### 3.1. Algorithm 1

The first algorithm is about doing DFS (Depth First Search) on each node with indegree zero (called a source node) and classify each edge during the search then we combine the edges classified as cross or forward edges and delete them from graph, here is the pseudo code:

```
DFS\_From\_Sources(DirectedGraph):
{
1. non_tree_edges[] ← [ ]
2. sources[] ← [ ]
3. for each node in Directedgraph do
   {
      3.1 deg ← node.caldegree(node)
      3.2 if(deg.is\_equal\_to(0)
         {
            3.2.1 sources.add(node)
         }
   }
4. for each source in sources do
   {
      4.1 tree← DFS(Source) /*this classifies edges of tree to cross,forward and tree edges*/
      4.2 for each edge in tree do 
         {
            4.2.1 if(edge.class.is\_equal\_to("cross") or edge.class.is\_equal\_to("forward"))
         }
   }
```
4.2.1.1 non_tree_edges.add(edge)

}

5. DirectedGraph.remove_edges_from(non_tree_edges[])
6. return DirectedGraph

**Theorem 3.** The running time of DFS\_From\_Sources algorithm is $O(n(n + m))$

**Proof.** Calculating the indegree for all vertices requires $O(n + m)$ time. Furthermore, DFS runs in linear time and edges can be classified using dfs in linear time.

**Theorem 4:** The first algorithm (DFS\_from\_sources) returns a singly connected subgraph.
**Proof.** Let $H$ be the set of non-tree edges computed in DFS\_from\_sources algorithm. Suppose that the subgraph $(v, E \setminus H)$ is not singly connected. Then there are two vertices $u, v \in V$ such that the subgraph $(V, E \setminus H)$ contains two edge-disjoint paths $P, \hat{P}$ from $u$ to $v$ in $G$. Consequently, since the subgraph $(V, E \setminus H)$ is a directed acyclic graph there is a source $s \in V$ such that all the vertices and edges of the paths $P, \hat{P}$ are reachable from the source $s$. Notice that the DFS tree rooted at $s$ can’t have two tree edges entering $u$.

### 3.2. Algorithm 2

The second algorithm is similar to the first one but the difference is that the execution of DFS will depend on the number of sources and sinks so if there are more sinks than sources about reversing the direction of all the edges and then executing DFS (Depth First Search) on each node with outdegree of the node equal to zero (a sink node) and classify each edge during the search then we combine the edges classified as cross or forward edges and delete corresponding original edges from the graph, here is the pseudo code:

```python
DFS\_From\_Sources\_Sinks(DirectedGraph):
{
    1. non\_tree\_edges[] ← [ ]
    2. sinks[] ← [ ]
    3. sources[] ← [ ]
    4. for each node in Directedgraph do
    {
        4.1 indeg ← node.caldegree()
        4.2 outdeg ← node.cal_outdegree()
        4.3 if(indeg == 0)
        {
```
4.3.1 sources.add(node)

} 4.4 if(outdeg.is_equal_to(0)
{

4.4.1 sinks.add(node)

}

5. max ← max(sources, sinks)
6. for each node in max do
{

6.1 tree←− DFS(node) /*this classifies edges of tree to cross,forward and tree edges*/
6.2 for each edge in tree do 
{

6.2.1 if(edge.class.is_equal_to("cross") or edge.class.is_equal_to("forward"))
{

6.2.1.1 non_tree_edges.add(edge)

}

}

7. DirectedGraph.remove_edges_from(non_tree_edges[])
8. DirectedGraph.reverse_edges
9. return DirectedGraph

}  

Theorem 5. Algorithm DFS_From_Sources|Sinks runs in O(max {i1,i2}.(n+m)) time wherei1 is the number of sources and i2 is the number of sinks, the second algorithm pretty similar to the first one but the only difference is that edges may be reversed based on the biggest number between sources and sinks(if the sinks are more than the sources)

The correctness of the second algorithm (DFS from max(sinks,sources))follows from theorem 4 and the following observation, G is singly connected if and only if $G^R$ is singly connected.

3.3. Algorithm 3

The third algorithm calculates dfs tree rooted at each source. The edges of these dfs trees form a subgraph which is not necessarily singly connected. Then the algorithm performs the first algorithm on the obtained subgraph, here is the pseudo code:

DFS_on_tree_edges(DirectedGraph):
{  
1. non_tree_edges[ ] ← [ ]
2. sinks[ ] ← [ ]
3. sources[ ] ← [ ]
4. for each node in Directedgraph do
   {
       4.1 indeg ← node.caldegree()
       4.2 outdeg ← node.cal_outdegree()
       4.3 if(indeg.is_equal_to(0)
       {
           4.3.1 sources.add(node)
       }
   }
5. tree_edges_set← {} 
6. for each source in sources do 
   {
       6.1 tree← DFS(Source)
       6.2 for each edge in tree do:
       {
           6.2.1 tree_edges_set ← tree_edges_set + edge
       }
   }
7. DirectedGraph.remove_edges_from(tree_edges_set)
8. for each source in sources do 
   {
       8.1 tree ← DFS(Source)
       8.2 for each edge in tree do:
       {
           8.2.1 if(edge.class.is_equal_to("cross") or edge.class.is_equal_to("forward"))
           {
               8.2.1.1 non_tree_edges.add(edge)
           }
       }
   }
}
9. DirectedGraph.remove_edges_from(non_tree_edges)
10. return DirectedGraph

**Theorem 6** The running time of the algorithm (DFS_on_tree_edges) is \(O(i(n+m))\).

Notice that the tree edges calculated by performing dfs on each source form a subgraph but necessarily singly connected. But when we perform the first algorithm on this subgraph we can obtain a singly connected subgraph by removing cross and forward edges. Therefore, the correctness of algorithm DFS_on_tree_edges follows from Theorem 4.

### 3.4. Algorithm 4:

Algorithm 4 executes DFS search on every medial node (medial means that it’s not a source or a sink), after that we perform algorithm1 on the resulting subgraph, the pseudo code:

```python
def DFS_from_medials(DirectedGraph):
    non_tree_edges = []
    medials = []
    for each node in DirectedGraph do
        indeg ← node.cal_degree()
        outdeg ← node.cal_outdegree()
        if (indeg.is_not_equal_to(0) & outdeg.is_not_equal_to(0))
        {
            medials.add(node)
        }
    
    for each medial in medials do
    {
        tree ← DFS(medial)
        for each edge in tree do
        {
            if (edge.class.is_equal_to("cross") or edge.class.is_equal_to("forward"))
            {
                non_tree_edges.add(edge)
            }
        }
    }
```

4. for each medial in medials do

```python
    tree ← DFS(medial)
```
5. TempGraph ← Graph(DirectedGraph.nodes(), non_tree_edges)
6. execute Algorithm 1 on TempGraph
7. return TempGraph

Theorem 7. algorithm DFS_from_medials runs in $O(n(n + m))$ time.

4. Study Results:

the following table contains the results for the execution of the algorithms on three graphs
two of them were taken from SNAP, the first one is p2p-Gnutella04 [8, 9, 10] and the second
one is soc-Epinions1 [8, 11] (we removed all cycles from p2p-Gnutella04 and soc-Epinions1),
the third one it’s a simple random acyclic graph that we built to try the algorithms on
small inputs, the cpu used to execute code on the data is Intel core i7 11370h and 16GBs of
memory, here are the results:

| algo                  | nodes | edges  | final edges | deleted edges | time(ms) |
|-----------------------|-------|--------|-------------|---------------|----------|
| dfs from sources      | 8     | 10     | 6           | 4             | 16       |
| dfs from sources or sinks | 8   | 10     | 6           | 4             | 5        |
| dfs on tree edges     | 8     | 10     | 6           | 4             | 0        |
| dfs from medials      | 8     | 10     | 3           | 7             | 5        |
| dfs from sources      | 10876 | 31478  | 10853       | 20625         | 278      |
| dfs from sources or sinks | 10876 | 31478  | 4784       | 26694         | 340      |
| dfs on tree edges     | 10876 | 31478  | 10853       | 20625         | 226      |
| dfs from medials      | 10876 | 31478  | 10828       | 20650         | 219      |
| dfs from sources      | 75888 | 258570 | 50715       | 207855        | 21304    |
| dfs from sources or sinks | 75888 | 258570 | 46653       | 211917        | 18880    |
| dfs on tree edges     | 75888 | 258570 | 50715       | 207855        | 22988    |
| dfs from medials      | 75888 | 258570 | 48303       | 210267        | 14753    |

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