Localized tadpoles of anomalous heterotic U(1)’s

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\textbf{Abstract}

We investigate the properties of localized anomalous U(1)’s in heterotic string theory on the orbifold $T^6/\mathbb{Z}_3$. We argue that the local four dimensional and original ten dimensional Green–Schwarz mechanisms can be implemented simultaneously, making the theory manifestly gauge invariant everywhere, in the bulk and at the fixed points. We compute the shape of the Fayet–Iliopoulos tadpoles, and cross check this derivation for the four dimensional auxiliary fields by a direct calculation of the tadpoles of the internal gauge fields. Finally we study some resulting consequences for spontaneous symmetry breaking, and derive the profile of the internal gauge field background over the orbifold.
1 Introduction

The present paper is the follow up investigation of our recent work [1] on localized anomalies in heterotic orbifold models. Let us therefore briefly summarize the general context and the main findings of that article. We considered the effective field theory description of ten dimensional heterotic string theory compactified on the six dimensional orbifold $T^6/\mathbb{Z}_3$. Strings on orbifolds have been discussed first by the authors of refs. [2, 3] and with the inclusion of non–trivial gauge field backgrounds, so–called Wilson lines, in [4, 5, 6]. Recently there has been a strong effort to understand the shape of anomalies on orbifolds. First in ref. [7] the anomalies on $S^1/\mathbb{Z}_2$ were computed and it was found that they localize at the fixed points of this orbifold. Afterwards, various groups computed anomalies on the orbifolds $S^1/\mathbb{Z}_2$, $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ and $T^2/\mathbb{Z}_2$ [8, 9, 10, 11, 12]. These results, and the questions on anomaly cancellation in heterotic orbifold models raised in [13], led us to calculate the gaugino anomaly in ref. [1]. The following two observations form the main conclusions of that investigation:

1. First of all, the untwisted bulk gaugino states lead to localized anomalies at the fixed points of $T^6/\mathbb{Z}_3$. These anomalies are entirely determined by the local spectra of those untwisted states, that survive the orbifold projections at the corresponding fixed points. By taking the twisted states at the fixed points into account, we showed that no non–Abelian anomalies arise at any of the fixed points.

2. However, the structure of the localized anomalous U(1)'s turned out to be more complicated. Using the fact that the spectrum of a model with Wilson lines at each fixed point is equivalent to the spectrum of a model without Wilson lines, it followed, that at most one of essentially two types of anomalous U(1)'s can be present locally at each fixed point. The sum of the local anomalous U(1) generators corresponds to the possible anomalous U(1) generator of the zero mode theory. If this sum vanishes, no anomalous U(1) appears at the zero mode level.

The appearance of global anomalous U(1)'s in heterotic orbifold compactifications has been studied extensively in the past and we would like to remind the reader of the most important results (see ref. [14] for details). In heterotic models at most one anomalous U(1) exists at the zero mode level. Gauge invariance is restored by a four dimensional remnant of the Green–Schwarz mechanism [15], which leads to the coupling of the model independent axion to the anomalous Abelian gauge field [16, 17, 18]. However, as observed in [14], the sum of the charges does not vanish for the anomalous U(1), and therefore a quadratically divergent Fayet–Iliopoulos (FI) tadpole arises at one-loop [19]. By direct calculations [20, 21] of scalar masses it has been confirmed, that this tadpole arises in string theory as well. However, in that case the string scale $M_s$ provides the cut–off for the quadratic divergence. In $N = 1$ supersymmetric field theories in four dimensions, the Fayet–Iliopoulos D–term can either lead to supersymmetry or gauge symmetry breaking [23]. For heterotic orbifold models only the latter possibility seems to be realized: the anomalous U(1) is spontaneously broken; its gauge field acquires a mass of the order of the string scale, which effectively removes it from the low–energy spectrum.

With these introductory remarks in mind, we are now in the position to formulate the central issues we wish to address in this work. Comparing the situation of the zero mode anomalous U(1) in heterotic orbifold models to the structure of localized anomalous U(1)'s at the orbifold fixed points, the following questions naturally arise:

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5 Similar tadpoles in open string theory turn out to vanish [22].
• How is local gauge invariance restored at the fixed points of $T^6/Z_3$?
• What is the profile of the Fayet–Iliopoulos tadpoles over this orbifold?
• What are the consequences of these tadpoles?

As for the first question, we will show, that by a local version of the four dimensional Green–Schwarz mechanism the local Abelian anomalies are canceled at the various fixed points.

The structure of Fayet–Iliopoulos tadpoles on orbifolds has received a lot of attention recently. The existence of quadratically divergent tadpoles on five dimensional supersymmetric orbifolds, like $S^1/Z_2$ and $S^1/Z_2 \times Z_2'$, was realized in [24] and the shapes of these tadpoles over such orbifolds have been computed in refs. [25, 8, 9]. These tadpoles of the even auxiliary field components of the five dimensional gauge super multiplets possess both quadratically and logarithmically divergent parts. The latter are proportional to the double derivative of the fixed point delta functions. As noticed in ref. [26], at these branes the auxiliary field of the four dimensional gauge multiplet is shifted by the derivative of the odd real scalar of the gauge multiplet. Therefore it has the same tadpole structure as the even auxiliary field component. In refs. [27, 11] it was shown, that such localized tadpoles lead to peculiar shapes of the corresponding real scalar background, which, in turn, often gives rise to strong localization of bulk states to one or both fixed points. This effect appears in particular due to the double derivatives of the fixed point delta functions. For gauge theories in six dimensions compactified on two dimensional orbifolds, like $T^2/Z_2$ and $T^2/Z_2 \times Z_2'$, tadpoles were found for the internal part of the gauge field strength $F_{56}$ at the fixed points [28, 29].

With these results in mind, one can speculate on the properties of tadpoles in heterotic models on $T^6/Z_3$. One complication is, that the ten dimensional super Yang–Mills theory is only known on–shell. Therefore, one cannot directly identify the Fayet–Iliopoulos tadpoles. However, with respect to the remaining $N = 1$ supersymmetry in four dimensions, one may introduce the appropriate auxiliary fields by hand. In addition, as mentioned above, for the anomalous U(1)’s one may expect tadpoles of the internal gauge field strengths at the fixed points. In this paper we introduce such a four dimensional off–shell formulation, and explicitly compute the tadpoles of the corresponding auxiliary components and the internal gauge fields. The comparison of the results for these two types of tadpoles provides an important cross check of our computations. Motivated by the results in five dimensions, we also investigate some consequences of these localized tadpoles.

Paper organization

In section 2 we introduce the basic elements of heterotic $E_8 \times E_8'$ theory on the orbifold $T^6/Z_3$ with Wilson lines. We explain how the four dimensional $N = 1$ supersymmetry, which survives the $Z_3$ orbifolding, can be realized off–shell in the full ten dimensional theory. (The necessary spinor algebra is reviewed in appendix A.) This off–shell formulation makes the coupling of the twisted string multiplets at the fixed points straightforward. A review of the possible fixed point equivalent models, that contain (anomalous) U(1)’s, concludes this section.

Section 3 is devoted to the question how the Green–Schwarz mechanism is realized on the orbifold, such that the local Abelian anomalies are canceled at the fixed points. Important factorization properties and trace relations needed to check that our modifications of the Green–Schwarz action cancel these anomalies, are provided in appendices B and C respectively.

Section 4 is devoted to the computation of tadpoles. The Fayet–Iliopoulos tadpoles, corresponding to the auxiliary $D^f$ fields introduced in subsection 2.3, are computed in subsection 4.1. (Properties of
wave functions on the torus $T^6$ can be found in appendix D). To confirm these results, we calculate
the tadpoles for the internal gauge fields in the following subsection.

In section 5, we investigate the consequences of the modifications of the BPS background equations.

The question of spontaneous symmetry breaking is addressed, and we show that the internal Cartan
gauge fields in general have non-trivial profiles over the orbifold $T^6/\mathbb{Z}_3$.

Finally, we conclude with a summary of the main results and give an outlook on possible further
research directions.

2 Heterotic $\mathbb{Z}_3$ models with anomalous U(1)’s

2.1 Heterotic $E_8 \times E_8'$ supergravity on $T^6/\mathbb{Z}_3$

The low energy description of heterotic $E_8 \times E_8'$ string theory consists of ten dimensional $N = 1$
supergravity coupled to the super Yang–Mills gauge theory of this group. (For a textbook introduction
see [30, 31].) The supergravity multiplet contains the vielbein $e_a^M$, the dilaton $\phi$, the anti–symmetric
2–tensor $B_{MN}$, the left–handed gravitino $\psi_M$, and the right–handed dilatino $\lambda$. (Here $M,N$ are ten
dimensional spacetime indices, and $a$ is a corresponding tangent space index.) The super Yang–Mills
theory consists of a ten dimensional gauge field $A_M$ and a left–handed gaugino $\chi$. Their adjoint indices
$\alpha = (I,w)$ correspond to the generators $T_\alpha = (H_I,E_w)$ and are often repressed for notational simplicity.
Here $H_I$ represent the generators of the Cartan subalgebra, and $E_w$ the remaining generators of
$E_8 \times E_8'$ labeled by the root vectors $w$. Their components are given by the structure constants in the
Cartan–Weyl basis: $[H_I,E_w] = w_I E_w$. We introduce the notation $[T_\alpha,T_\beta] = f_{\alpha\beta\gamma} T_\gamma$, $\text{tr} T_\alpha T_\beta = \eta_{\alpha\beta}$,
and $\text{tr}[T_\alpha,T_\beta] T_\gamma = f_{\alpha\beta\gamma}$. Notice that this implies that $f_{Iw'w} = w_I \delta_{w'w'}$. Furthermore we assume that
the algebra is normalized such that $\eta_{ww'} = \text{tr} E_w E_{w'} = \delta_{w'w}$.

This theory can be compactified on an orbifold $T^6/\mathbb{Z}_3$, which is constructed as follows: The torus
$T^6 = \mathbb{C}^3/\Gamma$ is obtained from the complex three plane, parameterized by complex coordinates $z_i$, by
modding out the lattice $\Gamma$, generated by the identifications $z_i \sim z_i + R_i$ and $z_i \sim z_i + \theta_i R_i$. (For the
definition of complex coordinates and their conjugates, $\bar{z}_i$, $i = 1,2,3$, in terms of real coordinates,
see [34] in appendix A). Here $R_i$ denote three real radii of the torus, and $\theta_i = \exp(2\pi i \phi_i)$ are third
roots of unity: $3\phi_i \equiv 0$. (The equivalence relation $a \equiv b$ means that $a = b \mod 1$.) The orbifold twist
$\Theta$ acts component wise on the coordinates of the torus $T^6$ as $\Theta(z_i) = \theta_i z_i$. Modding out this twist
defines the orbifold $T^6/\mathbb{Z}_3$. From now on we make the convenient choice $\phi_i = \frac{1}{3}(1^2,-2)$. Then these
third roots of unity are equal $\theta_i = \theta = \exp(2\pi i/3)$. (Notice that $\theta + \bar{\theta} = -1$, where $\bar{\theta} = \theta^{-1} = \theta^2$ is
the complex conjugate of $\theta$.) This orbifold twist does not act freely, and therefore results in orbifold
fixed points. In each of the three complex tori we have three fixed points: $\zeta_0 = 0, \zeta_1 = \frac{1}{3}(2 + \theta)$ and
$\zeta_2 = \frac{1}{3}(1 + 2\theta)$. They are fixed points using shifts over the lattice of the torus:

$$\theta \zeta_0 = \zeta_0, \quad \theta \zeta_1 = \zeta_1 - 1, \quad \theta \zeta_2 = \zeta_2 - 1 - \theta.$$  \hspace{1cm} (1)

Collectively, the 27 fixed points are denoted by $3_s = 3_{s_1s_2s_3} = (R_1 \zeta_{s_1}, R_2 \zeta_{s_2}, R_3 \zeta_{s_3})$ with the integers
$s_1, s_2, s_3 = 0, 1, 2$.

Since gauge fields are only defined up to group transformations, the 1–form gauge potential $A_1 = A_M dx^M$ is not necessarily invariant under the torus periodicities and the orbifolding twist. This leads
to the introduction of the Wilson lines $a_j$ ($j = 1, 2, 3$) and the gauge shift vector $v$ by

$$A_1(z + j) = A_1(z + \theta j) = T_j A_1(z) T_j^{-1}, \quad T_j = e^{2\pi i a_j^I H_I},$$

$$A_1(\Theta z) = U A_1(z) U^{-1}, \quad U = e^{2\pi i v^I H_I},$$

with $\forall w : 3a_j^I w_I \equiv 0$ and $\forall w : 3v^I w_I \equiv 0$. Here, $j$ and $\theta j$ denote the generators of the torus lattice. The three vectors $j$ have length $R_j$ and are mutually orthogonal. This is the Hosotani mechanism which implements the Scherk–Schwarz boundary conditions for the gauge symmetries. By combining these conditions with the relations, it is not hard to show, that the following four dimensional untwisted states

$$A_i^{\mathbf{R}_\mathbf{s}} : \mathbf{R}_s = \{ w | v_i^I w_I + \frac{1}{3} \equiv 0 \},$$
$$A_i^{\mathbf{R}_\mathbf{a}} : \mathbf{R}_a = \{ w | v_i^I w_I + \frac{2}{3} \equiv 0 \},$$

$$A_i^{\mathbf{Ad}} : \mathbf{Ad} = \begin{cases} \{ I \in \text{Cartan} \}, \\ \{ w | v_s^I w_I \equiv 0 \}. \end{cases}$$

survive the orbifold projection at fixed point $3_s$. Here the local shift vector $v_s^I = v^I + s_j a_j^I$ is introduced. The gauge group corresponding to $\mathbf{Ad}_s$ is denoted as $G_s$. It is important to note, that the local shift vectors $v_s$ of all fixed points together decide whether a consistent string model corresponding to the gauge shift $v$ and the Wilson lines $a_j$ exists: Modular invariance requires that the level matching conditions are satisfied

$$\forall s : \frac{3}{2} v_s^2 \equiv 0.$$  

We close this subsection with a few words concerning the conventions, we employ in the remainder of this work. As $A_i^{\mathbf{R}_\mathbf{s}}$ is conjugated to $A_i^{\mathbf{R}_\mathbf{a}}$, we may take the latter as fundamental. (We will see in the next subsection that the $A_i^{\mathbf{R}_\mathbf{s}}$ become the $N = 1$ supersymmetric partners of left-handed fermions in chiral multiplets.) From the four dimensional point of view at fixed point $3_s$ the states $A_i^{\mathbf{R}_\mathbf{s}}$ can be interpreted as scalar matter fields in the representation $(3_H, \mathbf{R}_s)$. The representation $3_H$ is with respect to the holonomy group SU(3)$_H \subset SO(6) \subset SO(1,9)$. (To be precise, the holonomy group of the blow up, the holonomy of the orbifold is $\mathbb{Z}_3$.) Finally, unless otherwise stated, expressions like $F_{ij}$ implicitly assume, that Einstein’s summation convention is employed.

### 2.2 Effective four dimensional supersymmetry

The $\mathbb{Z}_3$ orbifold twist is chosen such that only $N = 1$ supersymmetry in four dimensions is preserved at the zero mode level. The twist acts on a six dimensional internal spinor as

$$\Theta : \eta_{\kappa_3 \kappa_2 \kappa_1} \to e^{-\pi i \phi^I \kappa_i} \eta_{\kappa_3 \kappa_2 \kappa_1},$$

where $\kappa_i = \pm$ represent the internal two dimensional chiralities. (Conventions and properties of the six dimensional spinors used in this work have been collected in appendix A.) The components of the original 10–dimensional supersymmetry parameter $\epsilon_{10}$, corresponding to the supersymmetry which remains unbroken by the orbifolding, can be represented as

$$\epsilon_4 = \eta_{+++} \otimes \epsilon_L - \eta_{---} \otimes \epsilon_R, \quad \epsilon^{C-} = \epsilon.$$
condition of the supersymmetry parameter $\epsilon_{10}$ in ten dimensions (see (78) of appendix A). Notice, that this decomposition can be applied, even if $\epsilon = \epsilon(x, z)$ is a function of both the four dimensional Minkowski and orbifold coordinates, $x$ and $z$ respectively.

Following the method of ref. [20] we can decompose the ten dimensional supersymmetry transformation in terms of the unbroken four dimensional supersymmetry. Contrary to the five dimensional situation under investigation in ref. [20], for the ten dimensional theory there is no off–shell formulation available. However, by rewriting the ten dimensional super Yang–Mills such that only the remaining four dimensional supersymmetry is manifest, it becomes rather straightforward to infer the $N = 1$ four dimensional off–shell formulation of the theory. As we will see, this approach is particularly useful to describe the interactions with the twisted states (see section 2.4).

A ten dimensional supersymmetry variation $\epsilon_{10}$ of the gauge field $A_M$ and the gaugino $\chi$ read

$$
\delta A_M = \frac{1}{2} \epsilon_{10} \Gamma_M \chi,
\delta \chi = -\frac{1}{4} F^{MN} \Gamma_{MN} \epsilon_{10} + \ldots,
$$

(7)

where the dots represent terms of higher order in the fields,$^6$ and the field strength is defined as

$$
iF_{MN} = \partial_M iA_N - \partial_N iA_M + [iA_M, iA_N].
$$

(8)

By substituting $\epsilon_{10} = \epsilon_4$ given in (6) and the decomposition of the gaugino (78) of appendix A and using table 3 together with the multiplication rules (73) of appendix A, we find the following four dimensional supersymmetry transformations

$$
\delta A_\mu = \frac{1}{2} \epsilon_L \gamma_\mu \chi^{+++} + \frac{1}{2} \epsilon_R \gamma_\mu \chi^{++},
\delta \chi^{+++} = -\frac{1}{4} F^{\mu \nu} \gamma_\mu \epsilon_L - \frac{1}{2} F_{21} \epsilon_L
$$

(9)

and

$$
\delta A_1 = \frac{1}{2} \sqrt{2} \epsilon_R \chi^{---},
\delta \chi^{---} = \frac{1}{2} \sqrt{2} F^{\mu}_\gamma \epsilon_R + \frac{1}{2} F_{23} \epsilon_L,
\delta A_2 = \frac{1}{2} \sqrt{2} \epsilon_R \chi^{--},
\delta \chi^{--} = \frac{1}{2} \sqrt{2} F^{\mu}_\gamma \epsilon_R + \frac{1}{2} F_{31} \epsilon_L,
\delta A_3 = \frac{1}{2} \sqrt{2} \epsilon_R \chi^{--},
\delta \chi^{--} = \frac{1}{2} \sqrt{2} F^{\mu}_\gamma \epsilon_R + \frac{1}{2} F_{12} \epsilon_L.
$$

(10)

Using the linear part of the supersymmetry variation of the fermion the off–shell multiplet structure can be (re)constructed. For a vector multiplet ($B_\mu, \rho, D$) and a chiral multiplet ($Z, \zeta_L, f$) the supersymmetry transformations read

$$
\delta B_\mu = \frac{1}{2} \epsilon_L \gamma_\mu \rho_L + \frac{1}{2} \epsilon_R \gamma_\mu \rho_R,
\delta \rho_L = -\frac{1}{4} F^{\mu \nu} \gamma_\mu \epsilon_L - \frac{i}{2} D \epsilon_L,
\delta \zeta_L = \frac{i}{2} \sqrt{2} \gamma_\mu D_\mu Z \epsilon_R + \frac{1}{2} \sqrt{2} f \epsilon_L.
$$

(11)

(Taken from [33], with $\zeta \rightarrow \zeta/\sqrt{2}$ and some sign changes.) Comparing this with the result we obtained above, we can read off the multiplet structures and the equations of motion of the auxiliary fields:

$$
\left( A_\mu, \chi^{+++}, D \right), \quad (A_1, \chi^{---}, f_1), \quad (A_2, \chi^{--}, f_2), \quad (A_3, \chi^{--}, f_3),
\quad D = iF_{21}, \quad f_1 = \frac{1}{2} \sqrt{2} F_{23}, \quad f_2 = \frac{1}{2} \sqrt{2} F_{31}, \quad f_3 = \frac{1}{2} \sqrt{2} F_{12}.
$$

(12)

$^6$Notice that we have absorbed a dilaton factor into the definition of the gaugino as compared to [30]. This means that the normalization of the gauge field and the gaugino is the same; the modifications in the supersymmetry transformation are higher order in the fermion fields.
As these are ordinary \( N = 1 \) off–shell multiplets in four dimensions, the standard multiplet calculus, see for example [34, 35, 36], or superspace methods [37], can be applied. This holds true even though all these fields are still functions of the internal dimensions. Alternatively, we could perform Fourier decompositions of the internal dimensions, but then one has to keep track of many Kaluza–Klein towers. Of course, both approaches are equivalent, but in order to avoid writing complicated sums and to be able to trace local effects easily, we choose to work in coordinate space.

### 2.3 Elements of the super Yang–Mills Action

The ten dimensional Yang–Mills action takes the form

\[
L_{YM} = -\frac{1}{4} \text{tr} F_{MN} F^{MN} = -\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \text{tr} F_{ij} F_{ij} - \frac{1}{2} \text{tr} F_{ij} F_{ij},
\]

in the decomposition to four dimensions. (We have made the simplifying assumption that the dilaton is constant.) It is instructive to interpret this action from a four dimensional point of view. The first term in this equation represents the four dimensional gauge field Lagrangian. The second term gives the kinetic action for the four dimensional scalars \( A_i \).

\[
-\text{tr} F_{\mu\nu} F^{\mu\nu} = -\text{tr} D_\mu A_i D^\mu A_i - \text{tr} \partial_\mu A_\mu \partial^\mu A_i + \text{tr} \partial_i A_\mu D^\mu A_i + \text{tr} \partial_\mu A_\mu D^\mu A_i,
\]

with the covariant derivative \( D^\mu A_i = \partial^\mu A_i + i[A^\mu, A_i] \). The second term in this expression corresponds to Kaluza–Klein masses if one would choose to work in momentum space. The last two terms constitute the mixing between the massive Kaluza–Klein excitations of \( A_\mu \) and \( A_i \). The final two terms in (13) can be expressed as

\[
-\frac{1}{2} \text{tr} F_{ij} F_{ij} - \frac{1}{2} \text{tr} F_{ij} F_{ij} = -\frac{1}{2} \text{tr} [A_i, A_j][A_j, A_i] + \text{itr}(\partial_i A_j - \partial_j A_i)[A_i, A_j]
\]

\[
-\text{tr} [A_i, A_j][A_j, A_i] - \text{itr}(\partial_i A_j - \partial_j A_i)[A_i, A_j] - \text{itr}(\partial_i A_j - \partial_j A_i)[A_i, A_j] - \text{itr}(\partial_i A_j - \partial_j A_i)[A_i, A_j)
\]

(15)

Here we have used the Jacobi identity to rewrite \( \text{tr} [A_i, A_j][A_j, A_i] \), and applied partial integrations to the term \( \text{itr}\partial_i A_j[A_i, A_j] \) and its conjugate. (In ref. [35] it was first realized that by using the Jacobi identity, the dimensional reduced heterotic theory could be formulated as \( N = 1 \) supergravity in four dimensions.) Clearly the first line resembles the structure of \( D \)–terms in \( N = 1 \) supersymmetry, while the second line takes the form of an \( F \)–term potential. To make this four dimensional off–shell structure explicit, we use the auxiliary fields \( f_i, f_i \) and \( D \), which were introduced in (12), and rewrite that part of the action as

\[
-\frac{1}{2} \text{tr} F_{ij} F_{ij} - \frac{1}{2} \text{tr} F_{ij} F_{ij} = \text{tr} \left( f_i f_i - \frac{1}{2} \sqrt{2} \epsilon_{ijk} f_i f_{jk} - \frac{1}{2} \sqrt{2} \epsilon_{ijk} f_i f_{jk} \right) + \frac{1}{2} \text{tr} D^2 - i\text{tr} D F_{ij}.
\]

In this work we do not attempt to obtain the full superpotential and Kähler potential of the heterotic theory, which is fully equivalent to the original ten dimensional description.

Finally we give some other parts of the super Yang–Mills action we need in the calculation of tadpoles below. The gaugino Lagrangian is given by

\[
L_{\text{gaugino}} = -\frac{1}{2} \bar{\chi} \Gamma^M D_M \chi. \tag{17}
\]
This fermionic action can also be decomposed into four dimensional fields. However, for the tadpole calculations we present later, it is more convenient to keep the ten dimensional structure manifest. As usual computations of loop corrections involving gauge fields, require a gauge fixing prescription in order to be able to define their propagators. All loop computations in this work are performed using the ten dimensional Feynman gauge:

\[
L_{\text{g.f.}} = -\frac{1}{2} \text{tr}(\partial M A^M)^2 = -\frac{1}{2} \text{tr}(\partial_\mu A_\mu)^2 + \text{tr}(\partial_i A_i)^2
\]

\[
+ 2\text{tr}(\partial_\mu A_\mu)(\partial_\mu A_\mu + \partial_i A_i) + 2\text{tr}(\partial_i A_i)(\partial_j A_j).
\]

(18)

The resulting ghost action is

\[
L_{\text{ghost}} = \text{tr}\partial_\mu \bar{\eta} D^\mu \eta = \text{tr}\partial_\mu \bar{\eta} D^\mu \eta + \text{tr}\partial_i \bar{\eta} D_i \eta + \text{tr}\partial_j \bar{\eta} D_i \eta.
\]

(19)

This completes our description of the ten dimensional gauge theory, decomposed into an \(N = 1\) four dimensional language.

### 2.4 Twisted fixed point states

In addition to the requirement of modular invariance, which resulted in the stringent conditions (4), string theory also gives definite predictions of the states present at the orbifold fixed points. These twisted states can be thought of as originally open strings, which only become closed upon non-trivial orbifold twist identifications. For the \(\mathbb{Z}_3\) orbifold this leads to the following spectrum of chiral multiplets

\[
(1_H, S_s : (w^I + v^I_s)^2 = \frac{4}{3}), \quad (3_H, T_s : (w^I + v^I_s)^2 = \frac{2}{3}),
\]

(20)

at fixed point \(3_s\).

In the previous subsection the ten dimensional super Yang–Mills action has been (partly) decomposed into four dimensional states. Only the four dimensional \(N = 1\) supersymmetry, which is preserved by the orbifolding, was left manifest. This four dimensional \(N = 1\) language was extended to an off–shell formulation, involving the auxiliary fields \(f^I, f_i\) and \(D\) as functions of the ten dimensional coordinates. Therefore, the standard rules of \(N = 1\) multiplet calculus can be used to obtain the action for the twisted chiral multiplets \((c_s, \psi_s L, h_s)\), residing at fixed point \(3_s\) in the representations \(20\), and their interactions with the off–shell untwisted multiplets \(12\). For the purpose of the tadpole calculations later, it is sufficient to give only the scalar part of their action

\[
S_{\text{tw}} = \int \left( -D_\mu \bar{c}_s D^\mu c_s + \bar{h}_s h_s - \bar{c}_s D c_s + \ldots \right) \delta^6(z - 3_s - \Gamma) d^6z d^4x.
\]

(21)

The dots here may represent \(F–\)term contributions linear in the auxiliary fields \(f^I, h_s\) and their conjugates. We have introduced the delta function on a fixed point of the orbifold, which satisfies

\[
\delta(z - \theta^{-k}z - \Gamma) = \frac{1}{27} \sum_s \delta^6(z - 3_s - \Gamma), \quad k = 1, 2.
\]

(22)
2.5 Models with anomalous U(1)’s

In a previous publication [11] we have computed the local anomalies at the fixed points of the orbifold $T^6/\mathbb{Z}_3$. We found that the anomaly at fixed point $3_s$ was fully determined by the local spectrum at this fixed point (given by [13] and (20)), and hence ultimately by its local shift vector $v_s$. This naturally leads to the introduction of the concept of fixed point equivalent models, which allows one to identify the local spectrum of this model at fixed point $3_s$, with the spectrum of a pure orbifold model (i.e. with no Wilson lines) with shift vector $v_s$. The advantage of this is, that only a few inequivalent pure $\mathbb{Z}_3$ orbifold models exist. Therefore, the investigation of local anomalies reduces to the analysis of those pure orbifold models.

The full four dimensional anomaly $I^1_{6|s}$ at fixed point $3_s$, is given by the solution [11] (c.f. appendix C.1.) of the descent equation from the anomaly polynomial

$$I_{6|s} = \frac{3}{27} \text{ch}_{R_s}[iF_2] + 3 \text{ch}_{T_s}[iF_2] + \text{ch}_{S_s}[iF_2] \bigg| \hat{A}[R_2] \bigg|_6|s|,$$  

where wedge products are implicitly understood. The subscript 6$|s$ indicates that this formal expression is restricted to the 6–form part, and refers in particular to the anomaly at fixed point $3_s$. This requires that both the field strength $F_2$ of $E_8 \times E_8'$ and the curvature 2–form $R_2$ are restricted to this fixed point. Here the Chern character $\text{ch}_r[iF_2] = \text{tr}_r \exp(iF_2/2\pi)$ is computed in representation $r$, and $\hat{A}(R_2)$ denotes the roof (Dirac) genus. (For an exposition of some useful properties of (Chern) characters, see appendix C.1.) We have used that pure gravitational anomalies can never arise in four dimensions.

The factor of $1/27$ appears because the bulk fields constitute at a given fixed point of $T^6/\mathbb{Z}_3$ only 1/27 part of the usual anomaly.

As was shown in [11] using fixed point equivalent models, the non–Abelian gauge anomalies always cancel. Therefore we only need to consider the possible Abelian anomalies (both pure and mixed) more carefully. At a given fixed point $3_s$ we may have at most one anomalous U(1). Its gauge field is denoted by $A'_1|s$, while the other gauge fields that exist at this fixed point are collectively referred to as $\tilde{A}_1|s$. Employing similar notation for the corresponding field strengths, the anomaly polynomial becomes

$$I_{6|s} = \frac{i}{48} \text{tr}_{L_s} \left( \frac{F'_1}{2\pi} \right) \text{tr} \left( \frac{R'_2}{2\pi} \right)^2 |_{s} - i \text{tr}_{L_s} \left[ \frac{1}{6} \left( \frac{F'_1}{2\pi} \right)^3 + \frac{1}{2\pi} \left( \frac{F'_2}{2\pi} \right)^2 \right] |_{s}$$  

Here we have utilized the symbolic short–hand notation $L_s = \frac{3}{27} R_s + S_s + 3T_s$ to denote the local matter representations with their relevant normalizations at fixed point $3_s$. In addition, from the fixed point model analysis it followed, that there are only two types of anomalous U(1)’s at a given fixed point $11$. The relevant $E_8$ branching rules [39] are given by

$$\begin{align*}
E_8 &\rightarrow E_7 \times SU(2) & \rightarrow E_7 \times U(1), \\
248 &\rightarrow \{133, 1\} + (1, 3) + (56, 2) & \rightarrow 133_0 + 10_1 + 12 + 1_{-2} + 56_1 + 56_{-1}; \\
E_8 &\rightarrow SO(16) & \rightarrow SO(14) \times U(1)', \\
248 &\rightarrow 120 + 128 & \rightarrow 91_0 + 10 + 14_2 + 14_{-2} + 64_1 + 64_{-1}; \\
E_8 &\rightarrow SU(9), \\
248 &\rightarrow 80 + 84 + \bar{84}.
\end{align*}$$

The spectra corresponding to the pure orbifold models have been summarized in table 1. The last column of this table gives the traces over $L_s$ of the possible anomalous charges $(q_s, q'_s)$. Since $q_s$ in the
SU(9) model is part of the generators of SU(9) it is of course tracesless over each representation. The one but last column gives the traces over the untwisted, bulk, representations $R_s$. As can be deduced by comparing these two final columns, in the $E_7$ model the generator $q'_s$ of the $U(1)'$ is traceless, and therefore anomaly free, only if both untwisted and twisted states are taken into account. The anomalous $U(1)$’s of these models are “universal”, in the sense that the following relation holds \[^{[40]}\]

\[
\frac{1}{6} k_{q_s} \text{tr}_{L_s}(q_s^3) = \frac{1}{4} \sum_a q_s(L_s^{(a)}) I_2(L_s^{(a)}) = \frac{1}{48} \text{tr}_{L_s}(q_s).
\]

(26)

The sum is over the irreducible representations $L_s^{(a)}$ of the gauge group factors $G_s$ in $G_s$. The quadratic indices $I_2(L_s^{(a)})$ are normalized w.r.t. to these factor groups, and $q_s(L_s^{(a)})$ is the $U(1)$ charge of $L_s^{(a)}$. (For a more detailed discussion of the indices and their normalization, see appendix \[^{[3]}\] and refs. \[^{[40, 41, 42]}\].) Because of the inclusion of the level $k_{q_s} = 2q_s^2$ of $q_s$ this formula is valid for any normalization of this local $U(1)$ generator.

### 3 Green–Schwarz mechanism on the orbifold $T^6/\mathbb{Z}_3$

In \[^{[2]}\] we have derived the full structure of the gauge anomaly on $T^6/\mathbb{Z}_3$. In a similar fashion, also pure gravitational and mixed gauge–gravitational anomalies can be obtained. The full ten dimensional anomaly of this orbifolded theory is given by

\[
\int A_{10}\! = \! \int \frac{1}{3} I_{10} + \sum_s I_{4|s} \delta^6(z - 3s - \Gamma) d^6z.
\]

(27)

The factor $1/3$ results from the orbifold projection; only $1/3$ of the ten dimensional states on the torus $T^6$ survive the orbifold twist. The anomaly $I_{10}$ is determined by the descent equations \[^{[30]}\] of appendix \[^{[3]}\] from the anomaly polynomial

\[
I_{12} = \hat{A}_{3/2}[R_2] - \hat{A}[R_2] + \text{ch}_{E_8 \otimes E_8}[F_2] \hat{A}[R_2]_{12}.
\]

(28)

The first term results from the left–handed (spin 3/2) gravitino $\psi_M$, the second term is due to the right–handed dilatino $\lambda$, and the final term is the consequence of the gaugino $\chi$ of the $E_8 \times E_8'$ super
Yang–Mills gauge theory. The anomaly polynomial $I_{6|s}$ of the four dimensional anomaly $I_{4|s}$ at fixed point $s$ has already been discussed in section 2.5. As the non–Abelian anomalies cancel, $I_{6|s}$ reduces to $I_{14|s}$ and is non–vanishing, only if the spectrum at this fixed point is equivalent to the $E_7$ or SU(9) spectra (given in table 1). The aim of this section is to show, that the ten dimensional anomaly and the four dimensional Abelian anomalies at the fixed points can be canceled simultaneously by an anomalous variation of the anti–symmetric tensor.

The theory of $N = 1$ supergravity in ten dimensions has two equivalent formulations, using either the anti–symmetric tensors residing in the 2–form $B_2$, or the 6–form $C_6$ potential [43, 44, 45]. Their 1–form and 5–form gauge transformations $\delta \Lambda B_2 = d\Lambda$ and $\delta \Lambda C_6 = d\Lambda$ leave their actions

$$S_2 = \int \left( -\frac{1}{2} d^2 B_2 + (X_3 + X_7) dX_3 - \frac{1}{2} \ast X_3 X_3 \right),$$

$$S_6 = \int \left( -\frac{1}{2} \ast (dC_6 + \ast X_3 + X_7)(dC_6 + \ast X_3 + X_7) - \frac{1}{2} \ast X_3 X_3 \right),$$

(29)

invariant. Here the 3– and 7–forms $X_3, X_7$ are derived from arbitrary closed 4– and 8–forms, $X_4, X_8$, by Poincaré’s lemma (i.e. we have locally $dX_3 = X_4$ and $dX_7 = X_8$). To show that these two actions are equivalent, start with $S_2$ for example: Introduce the canonical field strength $H_3 = dB_2 - X_3 - \ast X_7$ and a 6–form Lagrange multiplier $C_6$ to enforce the Bianchi identity $dH_3 + X_4 + d\ast X_7 = 0$. Eliminate field strength $H_3$, using its algebraic equation of motion, to obtain action $S_6$.

We now determine for which $X_3$ and $X_7$ the variations of these actions under gauge and local Lorentz transformations, with infinitesimal parameters $\Lambda$ and $L$, respectively, cancel all anomalies of (27). The non–Abelian variations of the gauge connection $A_1$ and the spin–connection $\omega_1$

$$\delta \Lambda A_1 = d\Lambda + [\Lambda, A_1], \quad \delta L \omega_1 = dL + [L, \omega_1],$$

(30)

lead to the transformations

$$\delta \Lambda B_2 = X_2^1, \quad \delta \Lambda C_6 = -X_6^1,$$

(31)

and similarly for $L$. Here we have assumed that $X_4$ and $X_8$ are gauge invariant; hence $\delta X_3 = dX_2^1$ and $\delta X_7 = dX_6^1$ locally. These variations follow because the anomaly (27), and hence the variations of the actions (29), do not contain any dependence on either of these higher tensor fields. Since the anomaly (27) does not contain any Hodge dualization, the final term $-\frac{1}{2} \ast X_3 X_3$ is needed in (29). Therefore, the variations of both actions $S_2$ and $S_6$ are equal to

$$\delta \Lambda S_2 = \delta \Lambda S_6 = \int X_7 \delta \Lambda X_3.$$  

(32)

Since the ten dimensional part of the anomaly on the orbifold (27) is one third of the anomaly in ten dimensional Minkowski space, we expect that the original Green–Schwarz mechanism will be relevant here as well. For that reason we briefly review it here.

The crucial observation by Green and Schwarz [15] for anomaly cancellation in ten uncompact dimensions is, that the anomaly can be cancelled if the corresponding anomaly polynomial (28) factorizes:

$$I_{12} = X_4GSX_8GS,$$

(33)

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The normalization of $X_{4GS}$ is fixed by supersymmetry \(^7\) since the gauge Chern–Simons term $\omega_{3Y}$ appears in the supergravity Lagrangian when it is coupled to super Yang–Mills theory, hence

$$X_{3GS} = \omega_{3L} - \frac{1}{30} \omega_{3Y}, \quad X_{4GS} = d X_{3GS} = \text{tr} R_2^2 - \frac{1}{30} \text{Tr} F_2^2,$$

(34)

with the standard notation $\text{Tr} = \text{tr}_{E_6 \times E_6'}$ for the trace in the adjoint of $E_6 \times E_6'$. (The factor of $1/30$ can be thought of as the normalization of the $\text{tr}_{E_6 \times E_6'}$ trace in $\omega_{3Y}$ w.r.t. the vector representation of $SO(1,9)$.)

For the group $E_8 \times E_8'$ the factorization equation \([33]\) can be satisfied, with

$$X_{8GS} = \frac{1}{(2\pi)^6} \left[ \frac{1}{24} \text{Tr} F_2^2 - \frac{1}{7200} (\text{Tr} F_2^2)^2 - \frac{1}{240} \text{Tr} F_2^2 \text{tr} R^2 + \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2 \right].$$

(35)

With these ingredients we return to the local anomaly cancellation on the orbifold.

As the $T^6/\mathbb{Z}_3$ anomaly is $\frac{1}{3}$ of the original ten dimensional anomaly, and the Chern–Simons term $\omega_{3Y}$ in the field strength $H_3$ is required by the ten dimensional supersymmetry in the bulk, we infer that $X_7 = \frac{1}{3} X_{7GS} + \ldots$ and $X_3 = X_{3GS} + \ldots$; the dots refer to additional terms which are relevant for the cancellation of the four dimensional fixed point anomalies. Since the fixed points of the orbifold $T^6/\mathbb{Z}_3$ are isolated and have codimension six, the corresponding orbifold delta function cannot be factorized, and hence should reside completely within $X_7$. (Otherwise, we would have lower dimensional hyper planes.) Hence, we conclude that

$$X_3 = X_{3GS}, \quad X_7 = \alpha X_{7GS} + \sum_s \alpha_s A'_1|_s \delta^6(z - 3_s - \Gamma) d^6 z,$$

(36)

with $\alpha = \frac{1}{7}$ and $\alpha_s$ some constants. As the fixed point anomalies only involve mixed and pure $U(1)$ anomalies of the anomalous $U(1)$’s, the anomaly polynomials $I_{6|s}$ have to factorize like

$$I_{6|s} = \alpha_s X_{4GS}|_s F'_2|_s, \quad X_{4GS}|_s = \text{tr} R_2^2|_s - 2 \sum_a \text{tr} F'_2|_s.$$  

(37)

Here $X_{4GS}|_s$ denote the restrictions of $X_{4GS}$ to the groups $G_s$ present at the fixed point $3_s$, the sum is over the gauge group factors in $G_s$, and traces $\text{tr} F'_2(a)$ are normalized with respect to the quadratic indices of the respective gauge group factors. For details and a proof of the second equation in \([37]\), we refer the reader to appendix \([C]\), where the relevant calculations are performed. Now, precisely because of the “universality” relation \([26]\) for the anomalous $U(1)$’s of the two anomalous pure orbifold models, the above expression for $I_{6|s}$ is proportional to that given by eq. \([24]\). The normalization factor is easily determined to be $\alpha_s = -\frac{1}{48 (2\pi)^3}$ and turns out to be fixed point independent. The full Green–Schwarz action reads

$$S_{GS} = S_+ + \left( \beta X_{7GS} + \sum_s \beta_s A'_1|_s \delta^6(z - 3_s) d^6 z \right) X_{3GS},$$

(38)

where $S_+$ may either be $S_2$ or $S_6$, depending whether one uses the 2– or 6–form formulation of supergravity. The coefficients $\beta$ and $\beta_s$ are determined below. The gauge variation of this action is given by

$$\delta A S_{GS} = \int \beta d X^1_{6GS} X_{3GS} + (\alpha + \beta) X_{7GS} d X^1_{2GS}$$

$$+ \left( \beta_s A^s X_{3GS} + (\alpha_s + \beta_s) A'_1|_s d X^1_{2GS} \right) \delta^6(z - 3_s) d^6 z.$$  

(39)

\(^7\)This has only been explicitly checked for the Yang–Mills part because the Lorentz part is of higher order in derivatives.
Here we have used equation (87) of appendix B to determine the actual form of the anomaly given by the factorization of the anomaly polynomials [46]. This, in fact, fixes the coefficients: $\beta = -2/3$ and $\beta_s = \alpha_s$.

We close this section with some comments to link these results to the well–known situation of the zero mode theory of heterotic models on $T^6/\mathbb{Z}_3$ with an anomalous U(1). As discussed in [40, 41] for all orbifold models with Wilson lines the “universality” relation [26] holds if the model contains an anomalous U(1). With the local anomaly cancellation presented here, this result can be understood easily: For the two pure orbifold models with an anomalous U(1) (the E7 and SU(9) models) this relation holds; hence it holds for all localized anomalous U(1)’s in all $\mathbb{Z}_3$ models with Wilson lines as well, since the model at an ‘anomalous’ fixed point is equivalent to one of the two pure orbifold models with an anomalous U(1). Moreover, we know that the zero mode anomalous U(1) is a linear combination of the local anomalous U(1)’s, see [1]. Therefore, also the anomaly of the zero mode U(1) is canceled, and zero mode factorization is implied.

Finally, we turn to the issue of the model independent axion(s). Notice that the second term of the second equation in (36) leads to the interaction (in the 2–form formulation)

$$
\int \alpha_s A_1'\delta^6(z - 3s - \Gamma) d^6 z dB = \sum_s \int \alpha_s A'_\mu|s \partial_\mu b_s \delta^6(z - 3s - \Gamma) d^6 z d^4 x,
$$

of the local anomalous U(1) gauge field $A'_{\mu}|s$ and the anti–symmetric tensor. (This coupling is precisely the local version of the zero mode interaction $A_\mu \partial^\mu b_s$, discussed in [14].) Because of the delta function $\delta^6(z-3s)d^6z$ the exterior derivative on $B$ acts only in the four non–compact dimensions: $d = d_{(4)}$. (The subscript (4) emphasizes, that manipulations like Hodge dualization and exterior differentiation are performed in four dimensions.) By performing local dualization in four dimensions we have introduced the fixed point axions $b_s$ by

$$
d_{(4)}B_2(x, z)|s = d_{(4)}B_2(x, 3s) = \ast_{(4)}d_{(4)}b_s(x).
$$

Notice, that these fixed point axions $b_s$ are only defined on the fixed points. The model independent axion $b(x)$ is the dual of the zero mode of the four dimensional anti–symmetric tensor $B_{\mu
u}(x)$. After substituting this in the above equation and considering the zero mode anomalous U(1) gauge field, it follows that the model independent axion is identified as the sum of all local axions: $b = \sum_s b_s$.

## 4 Tadpoles

In four dimensional $N = 1$ supersymmetric U(1) gauge theories coupled to chiral multiplets, one can show that the auxiliary field of the gauge multiplet acquires a quadratically divergent tadpole at one loop, which is proportional to the sum of charges [19]. In section 2.2 we showed that also the full ten dimensional super Yang–Mills theory can be cast in the form of an $N = 1$ off–shell theory in four dimensions. Therefore, it is natural to consider the possibility of tadpoles for the auxiliary fields $D$ introduced in section 2.2. These tadpoles are computed in the next subsection. Because of the supersymmetry structure of the $D$–term scalar potential in [16], one would expect that a tadpole for $D$ arises if and only if there is also a tadpole for $\partial_i A_i$. Therefore (as a cross check) we compute all tadpoles for $A_i$ in section 1.2.

Before, we turn to the details of these tadpole calculations, we first describe the basic technique to perform the loop calculation for the bulk states on the orbifold $T^6/\mathbb{Z}_3$. Obviously, it is much more
Table 2: This table gives the symmetrization factors $\sigma(\ldots)$ needed for the computation of the tadpole diagrams in dependence of the fields and their components.

| State      | $\sigma(\ldots)$ | Value                                      |
|------------|-------------------|--------------------------------------------|
| Gauge field | $\sigma(M, w)$    | $\begin{cases} 3v^I w_I & M = \mu \\ 2 + 3v^I w_I & M = i \\ 1 + 3v^I w_I & M = \bar{i} \end{cases}$ |
| Gaugino    | $\sigma(\kappa, w)$ | $3(\frac{1}{2}\phi^i \kappa_i + v^I w_I)$ |
| Ghost      | $\sigma$          | $3v^I w_I$                                 |

convenient to perform the whole computation on the torus $T^6$. To take the orbifolding into account we insert an explicit orbifold projection operator that projects onto orbifold covariant states. (This method has been applied for the related anomaly calculation in ref. [1], see also ref. [22] for a string computation of Fayet–Iliopoulos tadpoles in type I models.) To explain the method, consider an operator $\mathcal{O}(z)$ that acts on a Hilbert space associated to a scalar field $S$ on the torus $T^6$. Let $\{\phi_q(z)\}$ be an orthonormal basis for this Hilbert space. For example, the basis (107) defined in appendix D, can be used. However, it should be stressed that our results are independent of the basis chosen for this Hilbert space. The expectation value of $\mathcal{O}(z)$ on the torus reads

$$\langle \mathcal{O}(z) \rangle_{T^6} = \sum_q \phi_q^\dagger(z) \mathcal{O}(z) \phi_q(z). \quad (42)$$

For the computation on the orbifold, it is essential that this scalar $S$ transforms covariantly under the orbifold twist: $S(\Theta z) = \theta^\sigma S(z)$, where the eigenvalue $\sigma = 0, 1, 2$ is defined modulo 3. The expectation value of the same operator on the orbifold is defined as

$$\langle \mathcal{O}(z) \rangle_{T^6/\mathbb{Z}_3} = \frac{1}{3} \sum_k \theta^{-\sigma k} \phi_q^\dagger(\theta^{-k} z) \mathcal{O}(z) \phi_q(z). \quad (43)$$

The part of this expression with $k = 0$ gives the same contribution as the torus expectation value, up to the normalization factor $\frac{1}{3}$. It is straightforward to extend this procedure to other fields on the orbifold in more complicated representations. In the tadpole calculations below we apply this technique to the homogeneous twist components of the gauge fields, gauginos and ghosts. Their symmetrization factors have been collected in table 2.

Before we turn to the explicit tadpole calculation, we make one technical comment: All our integrals are taken over Euclidean momentum space; i.e. the Wick rotation from Minkowskian momentum space has been performed implicitly.

4.1 Fayet–Iliopoulos tadpole for $\mathcal{D}$

In figure 1 we have given the possible tadpole diagrams for $\mathcal{D}$. We have employed the following notation for the first diagram of figure 1 which has internal gauge fields $A^a_j$ in the loop (corresponding to the second term in the second line of equation (16)). The dotted lines refer to gauge index contractions using the inverse Killing metric $\eta_{\alpha\beta}$ and a vertex of dotted lines refers to the structure coefficient $f_{\alpha\beta\gamma}$.  

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Figure 1: The following diagrams give rise to FI–tadpoles of $\mathcal{D}$. In the loop we have internal gauge fields $A_j$, and fixed point states $c_s$.

This means that loops of dotted lines imply that we have to sum over all generators of $E_8 \times E_8'$. The solid lines refer to contractions of spacetime indices. Since $T^6$ is described as a complex manifold, these solid lines carry a complex orientation, which we indicate using open arrows. In the second diagram of figure 1 the fixed point twisted scalars $c_s$ run around in the loop, see the interaction term in (21).

Only auxiliary $\mathcal{D}$ with a Cartan gauge index ($I$) can develop a tadpole: The propagators are diagonal in the gauge indices, therefore, it is not possible to form a closed tadpole diagram with a root index ($w$) on the external line. Since both the Wilson lines and the orbifold twist are generated by the Cartan subalgebra, such tadpoles are allowed by the boundary conditions of the heterotic orbifold theory.

The diagram with the internal gauge fields (untwisted states) in the loop gives rise to

$$\xi_{I\,un} = \frac{1}{3} \int \frac{d^4 p}{(2\pi)^4} \sum_{k, w, q, j} \theta^{-\sigma(j, w)k} \phi_{qw}^\dagger(\theta^{-k}z) \frac{1}{p^2 + \Delta} \phi_{qw}(z) f_{Iw}^w,$$

where we have introduced the internal Laplacian $\Delta = -2 \sum \partial_i \partial_i$. The case $k = 0$ does not contribute, since it is proportional to the trace of the Cartan generator $H_I$ over the full adjoint of $E_8 \times E_8'$. For $k \neq 0$ we would like to rewrite the sum over mode functions as fixed point delta functions, using identity (112) of appendix D. Clearly, we are only able to do so, if the Laplacian acts on the product of the mode functions $\phi_{qw}$. This can be achieved with the help of (111) of the same appendix, and we obtain

$$\xi_{I\,un} = \frac{1}{3} \sum_{k, s, w, j} \theta^{-\sigma(j, w, s)k} f_{Iw}^w \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + \frac{\Delta}{4}} \frac{1}{2\pi} \delta(z - Z_s - \Gamma).$$

Here we have introduced the fixed point dependent symmetrization factor $\sigma(j, w, s) = 1 + 3 v^I_s w_I$ corresponding to the local shift vector $v_s$ at the fixed point $Z_s$. This can be rewritten further, as a sum over the local representations $r = R_s, \overline{R}_s$ and $Ad_s$ defined in (3). Of course, the trace of $H_I$ in the local adjoint $Ad_s$ vanishes, and hence will be dropped. Furthermore, we have that $\text{tr}_{R_s}(H_I) = -\text{tr}_{\overline{R}_s}(H_I) = \sum_{w \in R_s} f_{Iw}^w$. We use the notation: $(-)^r = +, -$ for $r = R_s$ and $r = \overline{R}_s$, respectively. With these definitions the expression above becomes

$$\xi_{I\,un} = \frac{1}{3} \sum_{k, s, j} \text{tr}_{R_s}(H_I) \sum_{r=R_s, \overline{R}_s} \theta^{-\sigma(j, r)k} (-)^r \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + \frac{\Delta}{4}} \frac{1}{2\pi} \delta(z - Z_s - \Gamma).$$
To evaluate this we need to compute the sum:

\[
\sum_{k=1,2} \sum_{r=R_s,\bar{R}_s} \theta^{-\sigma(x)k}(-)^r = 3(2 - \theta^2 - \theta) = 9.
\]  

(47)

By Taylor expanding to first order in \(\frac{1}{3}\Delta\), and performing the resulting divergent integrals using the cut–off scheme, we find that the FI–parameter takes the form

\[
\xi_{I\text{un}} = \sum_s 3\text{tr}_{R_s} (H_I) \left( \frac{\Lambda^2}{16\pi^2} + \frac{\ln \Lambda^2}{16\pi^2} \frac{1}{3\Lambda} \right) \frac{1}{27} \delta(z - 3_s - \Gamma),
\]

(48)

where \(\Lambda\) denotes the cut–off scale.

The brane contributions are easier to obtain, as they are already confined to the four dimensional orbifold planes. Their effect on the FI–parameter can be read off straightforwardly from (21). As the complex scalars \(c_s\) of fixed point \(3_s\) reside in the representations (20), their tadpole contribution reads

\[
\xi_{I\text{tw}} = \sum_s (\text{tr}_{S+s} + 3\text{tr}_{T_s}) (H_I) \frac{\Lambda^2}{16\pi^2} \delta(z - 3_s - \Gamma).
\]

(49)

Combining these results, we find the expression for the full FI-term

\[
L_{FI} = -\xi_{I\text{D}I}, \quad \xi_{I} = \sum_s \left( \frac{\Lambda^2}{16\pi^2} \text{tr}_{L_s} (H_I) + \frac{1}{27} \frac{\ln \Lambda^2}{16\pi^2} \text{tr}_{R_s} (H_I) \Delta \right) \delta(z - 3_s - \Gamma).
\]

(50)

The sign in this Fayet–Iliopoulos action is dictated by the Wick rotation. Here we have again used the notation \(\text{tr}_{L_s}\) which has been introduced in eq. (21). The quadratically divergent part of the FI–parameter \(\xi_I\) is proportional to precisely the same trace which determines the localized anomalous U(1)’s, see (21). The logarithmically divergent part of this expression is proportional to the trace over the bulk states only. As can be seen from the one but last column of table (11) for all local U(1) factors, not just the anomalous ones, this logarithmically divergent part is present.

It is not difficult to see that this expression reduces to the well–known result at the zero mode level, by integrating out the internal dimensions of the orbifold. In particular the second term, with the Laplacian \(\Delta\), then drops out. In fact, since we are considering the low energy regime of a heterotic string model, the cut–off \(\Lambda\) should be related to the string scale. The calculation of the zero mode Fayet–Iliopoulos terms has been performed in full heterotic string theory, see refs. [20, 21]. From these calculations we infer that \(\Lambda = 1/\sqrt{3\alpha'}\) with \(\alpha'\) the string tension.

4.2 Tadpoles of the internal gauge fields

This section is devoted to the computation of tadpoles of internal gauge fields. As for the auxiliary fields, it is not possible to have tadpole diagrams of internal gauge fields with non–Cartan indices. The computation of tadpoles for \(A^I_j\) and \(A^I_\alpha\) are completely analogous and hence we focus on the tadpoles of \(A^I_j\) only. As the contributions of the internal gauge fields \(A_j\) to the tadpoles of \(A^I_j\) are rather subtle, we discuss them first and in more detail.

Only three cubic terms in (15) are relevant for scalar contributions to the tadpole of \(A^I_j\). (The reason is, that for cubic terms with two \(A_j\) one cannot close the loop if \(A_j\) represents an external leg.) In figure 2 we have collected these three terms and drawn the corresponding vertices. In addition to
\[ i(\partial_i A_\gamma^i) A_\beta^j A_\alpha^\beta f_{\alpha\beta\gamma} \quad -i(\partial_i A_\gamma^j - \partial_j A_\gamma^i) A_\alpha^\alpha A_\beta^\beta f_{\alpha\beta\gamma} \]

\[ j \quad \alpha \quad \beta \quad \gamma \]

\[ i \quad \alpha \quad \gamma \quad j \]

\[ i \quad \gamma \quad j \]

\[ if_{\alpha\beta\gamma} \quad -if_{\alpha\beta\gamma} \quad if_{\alpha\beta\gamma} \]

Figure 2: The vertices relevant for the tadpoles of \( A_I^j \) involving internal gauge fields \( A_I^w \) and \( A_j^w \). The adjoint indices \( \alpha, \beta \) and \( \gamma \) can refer to both Cartan indices \( I \) as well as root \( w \).

the Feynman rules introduced in the previous section, a solid arrow at the end of a solid line indicates differentiation w.r.t. a holomorphic coordinate \( z_i \). Using these vertices one can draw four different diagrams which give rise to tadpole contributions of \( A_I^j \). They are depicted in figure 3. The first and the second tadpole diagrams result from the first vertex given in figure 2. The last two diagrams, both, have a multiplicity of two, since they can be obtained from the middle as well as the last vertex of figure 2. The expressions for these tadpole diagrams are given by

\[ A = \partial_i A_I^j(z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,j,w,q} \phi_{qw}^\dagger(\theta^{i-k}z) \frac{1}{p^2 + \Delta} \phi qw(z) (if_{j w}^w) \theta^{-\sigma(j,w)k}, \]

\[ B = A_I^j(z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,j,w,q} \phi_{qw}^\dagger(\theta^{i-k}z) \frac{\partial_i}{p^2 + \Delta} \phi qw(z) (if_{j w}^w) \theta^{-\sigma(j,w)k}(-\bar{\theta}^k), \]

\[ C = A_I^j(z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,j,w,q} \phi_{qw}^\dagger(\theta^{i-k}z) \frac{\partial_i}{p^2 + \Delta} \phi qw(z) (-if_{j w}^w) \theta^{-\sigma(j,w)k}(2), \]

\[ D = A_I^j(z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,j,w,q} \phi_{qw}^\dagger(\theta^{i-k}z) \frac{\partial_i}{p^2 + \Delta} \phi qw(z) (-if_{j w}^w) \theta^{-\sigma(j,w)k}(2), \]

(51)

For diagrams \( A \) and \( D \) of fig. 3 we have to sum over \( j \), since all of the complex components of the internal gauge field contribute. The last three diagrams of figure 3 all have \( z_i \) derivative inside the loop. It is important to realize, that only for diagram \( B \) the orientation of the differentiation arrow and the complex index arrow are opposite. This signifies, that the derivative is not hitting \( \phi qw(z) \) but rather \( \phi_{qw}^\dagger(\theta^{i-k}z) \). As can be inferred from the first formula in (111) of appendix D, this implies that this diagram picks up an additional factor \(-\bar{\theta}^k\).

In addition to these four dimensional scalar loop diagrams, we have contributions from the four dimensional vector \( A_\mu \), the ten dimensional gaugino \( \chi \) and the ghost \( \eta \). As the bulk theory is non–Abelian, the resulting ghost sector does not decouple from the calculation. In ten dimensional Feynman gauge the ghost has a ten dimensional propagator. In figure 4 these other tadpole diagrams are
collected; their contributions read

\[
E = \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,w,q,\mu} \phi_{qw}(\theta - k z) \frac{\partial_i}{p^2 + \Delta} \phi_{qw}(z) (-i f_{1w}^w) \theta - \sigma(\mu,w) k,
\]

\[
F = -\frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,w,q,\alpha} \phi_{qw}(\theta - k z) \frac{\partial_i}{p^2 + \Delta} \phi_{qw}(z) (-i f_{1w}^w) \theta - \sigma(\alpha,w) k(2),
\]

\[
G = -\frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \sum_{k,w,q} \phi_{qw}(\theta - k z) \frac{\partial_i}{p^2 + \Delta} \phi_{qw}(z) (-i f_{1w}^w) \theta - \sigma(w) k(2).
\]

Before we discuss the details of the evaluation of these tadpoles, we first turn to the following cancellations, which are ultimately due to supersymmetry. Symbolically they may be represented as

\[
E + F_{++} + G \propto 4 - 2 - 2 = 0;
\]

\[
D + F_{(-,-)} \propto 3 * 2 - 3 * 2 = 0.
\]

Here \(F_{++}\) denotes the expression of \(F\) in equation (52) (or diagram \(F\) of fig. 4) with the internal gaugino chirality +++. Similarly, \(F_{(-,-)}\) refers to the sum over the three cyclic permutations of the chirality indices +−− of expression \(F\). We have used table 2 to conclude that the corresponding expressions are equal up to the given multiplicities. This shows that we are left with the three diagrams \(A, B\) and \(C\) (see figure 3). From this point onwards it is important to distinguish between the cases \(k = 0\) and \(k \neq 0\); we will denote the expressions for the corresponding diagrams with subscripts. Let us first consider the remaining diagrams for \(k = 0\).

\(k = 0\): bulk tadpoles

As can be inferred from the discussion below definition (43), the case \(k = 0\) corresponds to the calculation on the torus. It is not difficult to see that \(A_{k=0}\) vanishes: The sum is over all positive and negative \(q\) and \(w\), by taking \(q \rightarrow -q\) and \(w \rightarrow -w\) the resulting expression remains the same, except that the structure coefficients change sign: \(f_{1w}^w = -f_{1w}^w\). But since the summation indices \(q\) and \(w\) are dummy indices, this implies that \(A_{k=0} = -A_{k=0} = 0\). For \(B_{k=0}\) and \(C_{k=0}\) a similar argument does not hold: Because of the extra derivative \(\partial_i\) sandwiched between \(\phi_{qw}^\dagger\) and \(\phi_{qw}\), those expressions
do not vanish. However, the expression for the tadpole can be represented as a derivative w.r.t. $a^I_i$

$$B_{k=0} + C_{k=0} = -iA^I_i \frac{R_i}{16\pi} \frac{\bar{\theta} - \theta}{\theta - \frac{i}{\partial a^I_i}} \frac{1}{V_6} \sum_{q,w} \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + (2\pi)^2 |q_i + b_{i,w}|^2 / R^2_i \right] f_{Iw}^w, \quad (54)$$

where $V_6$ is the volume of the torus and $b_{i,w}$ is defined in (110) of appendix D. This completes the computation of the case $k = 0$. The interpretation of this tadpole is the following: Because we have allowed for Wilson lines in the model, it is not surprising, that these constant gauge backgrounds will receive quantum corrections. To see that this interpretation makes sense, we observe that if there are no Wilson lines: $b_{i,w} = 0$. Hence, the whole expression vanishes due to the derivatives $\partial / \partial a^I_i$. Since our main interest in this paper is to investigate the new counter terms at one loop, we will ignore this contribution from now on.

$k \neq 0$: localized tadpoles

For the case $k \neq 0$ similar methods can be employed as in the previous section for the derivation of (45). We need a subsequent partial integration to put the single derivative $\partial_i$ on the external line of $A^I_{i\perp}$, which gives

$$A_{k\neq 0} = \partial_i A^I_{i\perp} (z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + \frac{3}{4}\Delta} \sum_{k,j,w,s} \delta(z - 3s - \Gamma) \frac{2}{27} (i f_{Iw}^w \theta - \sigma(j,w,s)^k);$$

$$B_{k\neq 0} = \partial_i A^I_{i\perp} (z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + \frac{3}{4}\Delta} \sum_{k,w,s} \delta(z - 3s - \Gamma) \frac{2}{27} (i f_{Iw}^w \theta - \sigma(j,w,s)^k) \left( \frac{\bar{\theta}}{1 - \theta^k} \right);$$

$$C_{k\neq 0} = \partial_i A^I_{i\perp} (z) \frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + \frac{3}{4}\Delta} \sum_{k,w,s} \delta(z - 3s - \Gamma) \frac{2}{27} (i f_{Iw}^w \theta - \sigma(j,w,s)^k) \left( \frac{2}{1 - \theta^k} \right).$$

(55)

Here we encounter another, more subtle, cancellation: The sum of the contributions $B$ and $C$ is proportional to

$$\sum_{k=1,2} \sum_{r=R_s,R_s} \frac{2 + \bar{\theta}}{1 - \theta^k} \theta^{-\sigma(j,r)^k(-)^r} = - \sum_{k=1,2} \sum_{r=R_s,R_s} \theta^{-\sigma(j,r)^{-1}k(-)^r} = 0. \quad (56)$$
In the last step we have used that \( \sigma(\bar{i}_s R_s) = 0 \mod 3 \) and \( \sigma(\bar{i}_s \bar{R}_s) = 2 \mod 3 \). Therefore, the only non–vanishing contribution for \( k \neq 0 \) comes from diagram A of figure 3. As can be seen from the expression of \( A_{k \neq 0} \), the sum over the twist factors is the same as the one already computed in (17). Hence we obtain

\[
A_{k \neq 0} = \partial_\bar{i} A^I_{\bar{i}} \sum_s 3i \text{tr}_{R_s}(H_I) \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + \frac{1}{3} \Delta} \frac{1}{27} \delta(z - 3s - \Gamma) \tag{57}
\]

Again, we use the cut–off scheme to compute the divergent integrals. Clearly, the calculation of the tadpole of \( A^I_{\bar{i}} \) is completely analogous to the one just presented, except that in the whole calculation \( i \leftrightarrow \bar{i} \). Therefore the two expressions are related by Hermitian conjugation, and we obtain a relative minus sign. Hence the full expression for the gauge field tadpoles on the orbifold takes the form:

\[
L_{\text{tadp un}} = -\sum_s 3i \text{tr}_{R_s}(\partial_\bar{i} A^I_{\bar{i}} - \partial_i A^I_i) \left( \frac{\Lambda^2}{16\pi^2} + \frac{\ln \Lambda^2}{16\pi^2} \right) \frac{1}{27} \delta(z - 3s - \Gamma). \tag{58}
\]

In addition to the contributions of the gauge multiplet in the bulk, the twisted states at the fixed points supply us with an additional source of a tadpole for \( A^I_{\bar{i}} \), see (21). As we discussed there, the inclusion of the twisted states is performed most conveniently using a four dimensional off–shell formulation. This results in the last tadpole diagram of figure 4 in which the auxiliary field \( D \) is exchanged; the tadpole for \( A^I_i \) due to twisted states, therefore, becomes

\[
L_{\text{tadp tw}} = -\sum_s i \text{tr}_{S_s + 3T_s}(\partial_\bar{i} A^I_{\bar{i}} - \partial_i A^I_i) \frac{\Lambda^2}{16\pi^2} \delta(z - 3s - \Gamma). \tag{59}
\]

Hence the total expression for the tadpoles of \( A^I_{\bar{i}} \) reads

\[
L_{\text{tadp}} = -i (\partial_\bar{i} A^I_{\bar{i}} - \partial_i A^I_i) \sum_s \left[ \text{tr}_{L_s}(H_I) \frac{\Lambda^2}{16\pi^2} + \frac{1}{27} \text{tr}_{R_s}(H_I) \frac{\ln \Lambda^2}{16\pi^2} \Delta \right] \delta(z - 3s - \Gamma). \tag{60}
\]

This is precisely the expression for the localized tadpoles one would expect on supersymmetry grounds from the tadpole for the auxiliary field \( D \), as computed in (50). An additional cross check of the off–shell \( D^I \)–tadpoles calculated in section 4.1, can be provided by a direct computation of the mass terms of \( A^I_{\bar{i}} \). We will not present this in this paper.

### 5 Consequences of vanishing \( D^I \)–terms

We investigate some consequences of localized FI–tadpoles in heterotic models. Similar methods will be pursued in this analysis as those that were used in the five dimensional case of U(1) gauge fields on the orbifold \( S^1/\mathbb{Z}_2 \) [27, 11]. However, there are various reasons why the analysis in the present case is in principle more involved: there are more fields in the game, in particular the gravitational interactions, as well as the anti–symmetric tensor may be relevant. Additionally, the (local) Green–Schwarz mechanism has introduced various interaction terms involving (non–Abelian) gauge fields.
5.1 Cartan symmetry breaking

We turn our attention to one phenomenologically important issue: spontaneous breaking of gauge symmetries due to one–loop induced FI–terms. Since the FI–tadpoles only arise for the gauge symmetries of the Cartan subalgebra, we investigate when spontaneous breaking of the Cartan gauge symmetries is inevitable. Cartan symmetry breaking occurs, if a field that is charged under the Cartan subalgebra of $E_8 \times E_8'$ acquires a non–vanishing VEV. This can be either an internal gauge field component $\langle A^w_{\mu} \rangle$, a twisted state $\langle c_s \rangle$ at a fixed point, or some combination. As usual we are looking for a supersymmetric minimum of the theory from the global four dimensional point of view. This means that we can exploit various BPS–like equations, which simplify the analysis of the equations of motion considerably. Here we do not perform a full analysis, but rather we are only concerned with the BPS–equations that result from the supersymmetry transformations of the gauginos in the Cartan subalgebra (see [11] and [12])

\[
\delta \chi^{+++I} = -\frac{1}{4} F^{\mu\nu I}_{\mu\nu} \gamma^I \epsilon - \frac{i}{2} \bar{D}^I \bar{\gamma} \epsilon,
\]

where we have used the auxiliary field $D^I$ to encode the modifications to the supersymmetry transformation rule due to the twisted states and the FI–tadpoles. As we are looking for vacuum configurations that preserve $N = 1$ supersymmetry at the global four dimensional level, i.e. $\epsilon(x, z) = \epsilon(x)$ is constant over the internal dimensions, we find that

\[
\langle D^I \rangle = i \langle F^I_{\mu\nu} \rangle + \sum_s \{ (\bar{c}_s) H_I \langle c_s \rangle + \frac{\Lambda^2}{16 \pi^2} \text{tr}_L (H_I) + \frac{1}{27} \frac{\ln \Lambda^2}{16 \pi^2} \text{tr}_R (H_I) \Delta \} \delta(z - 3s - \Gamma) = 0.
\]

Here we have assumed that the four dimensional vacuum does not break Lorentz invariance and hence the VEV of $F^{\mu\nu}$ vanishes. In general this equation may be viewed as a BPS equation of motion for $\langle A^I_w \rangle$ and its conjugate, which reside in the field strength $\langle F^I_{\mu\nu} \rangle$.

However, there may not always be a solution of this BPS equation of motion. If there is no solution, this implies that supersymmetry is spontaneously broken. Therefore, it is important to investigate under which condition the BPS equation can be satisfied, and what the consequences of this condition are. To investigate these questions we integrate over the extra internal dimensions

\[
\int_{T^6/Z_3} d^6 z \left\{ \sum_s \{ (\bar{c}_s) H_I \langle c_s \rangle + \frac{\Lambda^2}{16 \pi^2} \text{tr}_L (H_I) + \frac{1}{27} \frac{\ln \Lambda^2}{16 \pi^2} \text{tr}_R (H_I) \Delta \} \delta(z - 3s - \Gamma) + i \langle F^I_{\mu\nu} \rangle \right\} = 0.
\]

Since the orbifold singularities have codimension six, we can simply use Stoke’s theorem to remove the $\partial_i A^I_w - \partial_i A^I_{\bar{w}}$ part in the field strength $F^I_{\mu\nu}$ and the term proportional to $\Delta$. Hence we are left with the constraint

\[
\sum_s \langle c_s \rangle H_I \langle c_s \rangle + w_I \int_{T^6/Z_3} d^6 z \langle A^{-w}_{\bar{I}} \rangle \langle A^w_I \rangle = -\frac{\Lambda^2}{16 \pi^2} \sum_s \text{tr}_L (H_I).
\]

Now, since $\sum_s \text{tr}_L (H_I) H_I$ precisely identifies the global anomalous U(1) generator, it follows that if $\sum_s \text{tr}_L (H_I) \neq 0$, at least either an internal gauge field $A^w_i$ with $\sum_s \text{tr}_L (H_I) w_I \neq 0$ or a twisted scalar $c_s$ with $\sum_s \text{tr}_L (H_I) q_I \neq 0$ has to get a non–vanishing VEV to cancel the FI–tadpole. Observe that the sign of the charge of that field must be opposite to that of $\sum_s \text{tr}_L (H_I)$. This in turn implies
Anomalous pure orbifold models

It is instructive to see how Cartan symmetry breaking works for the pure orbifold models: both in its own right, and because they exemplify some typical symmetry breaking patterns also to be expected to appear in models with Wilson lines. In table 1 we have given various characteristics of the two pure orbifold models, \(E_7\) and \(SU(9)\): the trace over \(\mathbf{L}_s\), which appears as a factor in the global (zero mode) FI–term, and the untwisted and twisted matter representations are given. Hence we can read of which field(s) may develop a VEV to cancel the quadratically divergent FI–tadpole.

In the \(E_7\) model, there are two types of states which have negative anomalous charge and therefore would break the anomalous \(U(1)\): the untwisted state \((3_H, (1)\cdot 2(1)')_0\) and the twisted states \((1_H, (1)\cdot 4(1)')_2\). Both types are non–Abelian gauge singlets and will therefore not induce further spontaneous non–Abelian symmetry breaking. However, in addition to the charges under the anomalous \(U(1)\), a twisted singlet is charged under the non–anomalous \(U(1)'\) as well; therefore its VEV leads to spontaneous breaking of both \(U(1)'\)'s, the anomalous and the non–anomalous one. Let us assume that only one type of states has a non–vanishing VEV and in such a way that the zero mode FI–term is cancelled. In the \(E_7\) model the twisted spectra at all fixed points are the same, therefore it is possible that all quadratically divergent tadpoles can be canceled locally. However, since the only condition is the cancellation of the zero mode (quadratically divergent) FI–tadpole, it might just be one twisted state at one of the fixed points that cancels the zero mode tadpole. Likewise the untwisted state \((3_H, (1)\cdot 2(1)')_0\) may cancel the quadratically (and even the logarithimically) divergent tadpoles locally, but again this is not necessary. Observe that contrary to the untwisted states, the twisted states can never cancel any of the logarithmic divergences. As these tadpoles are proportional to the trace \(\text{tr}_{\mathbf{R}_s}(H_I)\) over the untwisted matter representations \(\mathbf{R}_s\) only, it follows that both \(U(1)'\)'s (not just the anomalous one) have non–vanishing logarithimically divergent tadpoles, as can be easily confirmed by consulting the one but last column of table 1.

The \(SU(9)\) model has only one representation with a charge opposite to the quadratically divergent FI–tadpole: the untwisted state \((3_H, (1)\cdot 14(1)')_1\). Therefore, like in the case of the untwisted states in the \(E_7\) model cancelling the global Fayet–Iliopoulos tadpole, the shape of the untwisted states of the \(SU(9)\) model may be such that all localized tadpoles are cancelled. However, since this state is charged under the \(SO(14)\), we infer that the model exhibits spontaneous symmetry breaking: \(SO(14) \rightarrow SO(13)\). Like for the \(E_7\) model the \(U(1)'\) has a non–vanishing logarithimically divergent tadpole (c.f. table 1).

5.2 Background profile of \(A^I_\perp\)

From now on we assume that either \(\sum_s \text{tr}_{\mathbf{L}_s}(H_I) = 0\) or that some untwisted states \(\langle \mathbf{A}^w \rangle\) and/or twisted states \(\langle \mathbf{c} \rangle\) have acquired a VEV such that (64) is satisfied. Then we know that there exists a solution for \(\langle A^I_\perp \rangle\) to (62). In this subsection we wish to construct it explicitly. Locally we can introduce a potential \(\langle P^I \rangle\) for \(\langle A^I_\perp \rangle\) defined by the following equations

\[
\langle A^I_\perp \rangle = i \partial_\perp \langle P^I \rangle, \quad \langle A^I_\perp \rangle = -i \partial_i \langle P^I \rangle.
\]

(65)
Substituting this in (62) leads to the equation

\[
\Delta \langle P^I \rangle = \sum_s \left( \langle c_s \rangle H_I \langle c_s \rangle + \frac{\Lambda^2}{16\pi^2} \text{tr}_{L_s}(H_I) \right) \delta(z - 3_s - \Gamma) + w_I \langle A^{-w}_I \rangle \langle A^w_1 \rangle.
\]

(66)

To solve this equation, consider first the Green’s function \( G(z - y) \) on \( \mathbb{C}^3 \) defined by \( \Delta G(z - y) = \delta^6(y) \). Since the delta function on the torus is \( \delta^6(y - \Gamma) \), it follows that the Green’s function of the torus \( T^6 \) reads \( G(z - y - \Gamma) \). Using this Green’s function, it is straightforward to obtain the solution for \( \langle P^I \rangle \), it reads

\[
\langle P^I \rangle = \sum_s \left( \langle c_s \rangle H_I \langle c_s \rangle + \frac{\Lambda^2}{16\pi^2} \text{tr}_{L_s}(H_I) \right) G(z - 3_s - \Gamma) + \frac{1}{27} \ln \frac{\Lambda}{16\pi^2} \sum_s \text{tr}_{R_s}(H_I) \delta(z - 3_s - \Gamma) + w_I \int_{T^6/Z_3} d^6y \langle A^{-w}_I(y) \rangle \langle A^w_1(y) \rangle G(z - y - \Gamma).
\]

(67)

Let us make a couple of comments: It might seem that it always provides us with a solution. But that is only a local statement, which ignores the crucial compactness that resulted in the constraint (64). If a bulk state \( A^{-w}_I \) is required to get a VEV to satisfy the global BPS condition (64), its shape over the extra dimensions might be quite complicated. (It will not be determined in this paper.) However, whatever precisely its profile is, this formula gives the resulting shape of \( \langle A^{-w}_I \rangle \).

Suppose another situation, where all anomalous fixed points are equivalent to an \( E_7 \) model, then the discussion of the previous subsection tells us that it is possible to cancel all quadratically divergent tadpoles locally by giving VEVs to the twisted singlets \( (1_H, (1), (1)) \). This means that the first and the final line are zero, but the middle line will still be present. But still, as observed in the previous section, at the anomalous fixed points, the logarithmically divergent tadpoles are present. From the analysis here we infer that they lead to the profile

\[
\langle A^I \rangle = \frac{i}{27} \ln \frac{\Lambda^2}{16\pi^2} \sum_s \text{tr}_{R_s}(H_I) \partial_2 \delta(z - 3_s - \Gamma),
\]

(68)

with a derivative of the fixed point delta function.

6 Conclusions

In this paper we investigated the role of localized anomalous U(1)’s, which appear at the fixed points of heterotic orbifold compactifications with Wilson lines, for the case \( T^6/Z_3 \). The main results of this work are summarized as follows:

The first question we addressed was how gauge invariance at the fixed points (with anomalous U(1)’s) is restored. We showed that by using a local version of the Green–Schwarz mechanism at the fixed points, the localized pure and mixed U(1) anomalies, that arise due to the ten dimensional gauginos, are canceled. Also 1/3 of the original ten dimensional anomaly is present on the orbifold. We checked explicitly that the ten dimensional and the four dimensional local fixed point Green–Schwarz mechanisms are compatible with each other.
Next we investigated whether these localized anomalous U(1)'s are associated with Fayet–Iliopoulos tadpoles, as is the case for the well-know situation at the zero mode level. To this end it proved useful to construct an off–shell formulation of the full ten dimensional super Yang–Mills theory with respect to the four dimensional $N = 1$ supersymmetry, which is left unbroken by the orbifolding. The Fayet–Iliopoulos tadpole diagrams with gauge fields in the loop were computed with the help of the orbifold projector method. Using cut–off regularization, we found that the quadratically divergent part of these tadpoles are proportional to the same traces as the anomalous U(1)'s (twisted states taken into account). However, we found also logarithmically divergent terms, which scale with the traces over the untwisted states only, and appear together with the double derivative of the orbifold delta functions. (These results are very similar to the ones obtained previously for five dimensional orbifold models [9, 27, 11, 8].) Because of supersymmetry, one would expect similar tadpoles to arise for the internal part of the Cartan gauge field strengths. We confirm this by a direct (on–shell) calculation of the tadpoles of the internal gauge fields.

In the final part of this article we investigated some consequences of such tadpoles. First we studied the BPS equations $D^I = 0$, which are required for unbroken $N = 1$ supersymmetry in four dimensions. We found that they can only be solved, if the global Fayet–Iliopoulos tadpole is canceled by a VEV of at least one charged bulk or twisted field. In this way we rederived the standard four dimensional zero mode conditions for unbroken $N = 1$ supersymmetry, and spontaneous breaking of the zero mode anomalous U(1). However, it turned out that these global BPS conditions can be solved in numerous ways, corresponding to different profiles of the charged untwisted states over the orbifold and different VEVs of the charged twisted states. (We have discussed some of these possibilities for the pure orbifold models $E_7$ and SU(9), that are defined in table 1.) The background profiles of the internal Cartan gauge fields $A^I_5$ have been determined in general, using the only constraint, that the global BPS condition was satisfied.

**Outlook**

We would like to make a couple of remarks on further developments along the lines of the present work.

First of all we should stress that all the results presented in this paper were obtained using pure field theory arguments. But since we are really describing the low energy limit of heterotic string theory, it would be interesting to see if our results can be confirmed by full string computations as well. In particular the localized version of the Green–Schwarz mechanism, and the structure of the tadpoles of the internal field strength may also be calculated using string theory techniques.

As compared to the five dimensional situation, one may wonder whether also (strong) localization effects of untwisted states can arise. (For the localization effects in five dimensional orbifold models see [27, 11, 37, 15].) In this work we have obtained the profile of the background of the bulk gauge field. Therefore, in principle localization effects can be investigated. However, there is one technical hurdle to overcome here: not only the gauge connection, but also the spin connection appears in the equation of motion of the gaugino. Because of the curvature singularities, this spin connection is also strongly peaked at the orbifold fixed points. It is therefore questionable, whether it suffices to take only the gauge connection background into account to investigate the possibility of localization effects.

Finally, the methods and the results obtained in this paper for the $\mathbb{Z}_N$ orbifold with $N = 3$, can be generalized for higher $N$. Especially for non–prime $N$ it may be interesting to see what kind of tadpoles can arise and to compare the results with localized anomalies on such orbifolds.
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A Decomposition of ten dimensional spinors

This appendix provides some useful properties of ten dimensional spinors and their decomposition to a four dimensional Minkowski space times an three dimensional complex internal manifold. (Details can be found in [49, 31].) To change from the six dimensional real coordinates, \( \{x^4, \ldots, x^9\} \), to the three dimensional complex coordinates, \( \{z_1, z_2, z_3\} \), we use the following redefinitions

\[
x^{2i+2} = x_{2i+2} = \frac{1}{\sqrt{2}} (z_i + \bar{z}_i), \quad x^{2i+3} = x_{2i+3} = \frac{-i}{\sqrt{2}} (z_i - \bar{z}_i).
\]

and the induced transformations on covariant vectors. Here, we have used that we work with a metric with the signature: \( \text{diag}(-1, 1^9) \). Hence for the six dimensional part of the ten dimensional Clifford algebra we get

\[
\Gamma^{2i+2} = \Gamma_{2i+2} = \frac{1}{\sqrt{2}} (\Gamma_i + \Gamma_{\bar{i}}), \quad \Gamma^{2i+3} = \Gamma_{2i+3} = \frac{-i}{\sqrt{2}} (\Gamma_i - \Gamma_{\bar{i}}), \quad \{\Gamma_i, \Gamma_{\bar{j}}\} = 2\delta_{ij}.
\]

For the decomposition of the ten dimensional supersymmetry transformations in section 2.2 it will be convenient to rewrite this six dimensional internal Clifford algebra in terms of two dimensional Clifford algebras.

A convenient complex basis for the two dimensional Clifford algebra is defined by

\[
\sigma_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

where \( \sigma_\pm = (\sigma_1 \pm i\sigma_2)/\sqrt{2} \). It has the properties

\[
\sigma_3 \sigma_\pm = -\sigma_\pm \sigma_3 = \pm \sigma_\pm, \quad \sigma_\pm^2 = 0, \quad \sigma_+ \sigma_- = 2\pi_+ = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \sigma_+ = 2\pi_- = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \( \eta_\kappa \) with \( \kappa = \pm \) form the basis of two dimensional spinors, with the properties

\[
\sigma_0 \eta^\kappa = \eta^\kappa, \quad \sigma_3 \eta^\kappa = \kappa \eta^\kappa, \quad \sigma_\pm \eta^\mp = \sqrt{2} \eta^\pm, \quad \sigma_\pm \eta^\pm = 0, \quad \eta^\kappa \eta^{\kappa'} = \delta^{\kappa \kappa'}.
\]

By introducing the notation \( \sigma_{\alpha_3 \alpha_2 \alpha_1} = \sigma_{\alpha_3} \otimes \sigma_{\alpha_2} \otimes \sigma_{\alpha_1} \), with \( \alpha_i = 0, \pm, 3 \), we can represent the six dimensional Clifford algebra in complex coordinates (here and below the tensor products are understood).
The embedding of this six dimensional Clifford algebra in the ten dimensional one can then be represented as

\[ \Gamma_\mu = I_6^\gamma_\mu, \quad \Gamma_1 = \sigma_{00+} \tilde{\gamma}, \quad \Gamma_2 = \sigma_{0+3} \tilde{\gamma}, \quad \Gamma_3 = \sigma_{+33} \tilde{\gamma}, \]

\[ \Gamma_1 = \sigma_{00-} \tilde{\gamma}, \quad \Gamma_2 = \sigma_{0-3} \tilde{\gamma}, \quad \Gamma_3 = \sigma_{-33} \tilde{\gamma}, \]

(74)

where \( \tilde{\gamma} \) is the four dimensional chirality operator. The (anti–symmetric) products of the six dimensional Clifford algebra generators have been collected in table 3. The main advantage of this basis is that the action of the Clifford algebra elements on the six dimensional spinors \( \eta_{\kappa_1 \kappa_2 \kappa_3} \) can be worked out straightforwardly.

\[ \Gamma_{ij}, \Gamma_{i} \quad \begin{array}{ccc}
\Gamma_1 &=& \sigma_{00+} \tilde{\gamma} \\
\Gamma_2 &=& \sigma_{0+3} \tilde{\gamma} \\
\Gamma_3 &=& \sigma_{+33} \tilde{\gamma} \\
\end{array} \]

\[ \Gamma_{ij} \quad \begin{array}{ccc}
\Gamma_{12} &=& -\sigma_{0++} \\
\Gamma_{23} &=& -\sigma_{+++} \\
\Gamma_{13} &=& -\sigma_{++3} \\
\end{array} \]

\[ \Gamma_{ij} \quad \begin{array}{ccc}
\Gamma_{11} &=& \sigma_{003} \\
\Gamma_{12} &=& -\sigma_{0--+} \\
\Gamma_{13} &=& -\sigma_{--3} \\
\Gamma_{21} &=& -\sigma_{0+-} \\
\Gamma_{22} &=& \sigma_{030} \\
\Gamma_{23} &=& -\sigma_{-0+} \\
\Gamma_{31} &=& -\sigma_{+3-} \\
\Gamma_{32} &=& -\sigma_{+-0} \\
\Gamma_{33} &=& \sigma_{300} \\
\end{array} \]

\[ \Gamma_{ijk} \quad \begin{array}{ccc}
\Gamma_{123} &=& -\sigma_{+++} \tilde{\gamma} \\
\Gamma_{121} &=& -\sigma_{00+} \tilde{\gamma} \\
\Gamma_{123} &=& -\sigma_{0+3} \tilde{\gamma} \\
\Gamma_{131} &=& -\sigma_{+30} \tilde{\gamma} \\
\Gamma_{132} &=& -\sigma_{+03} \tilde{\gamma} \\
\Gamma_{133} &=& -\sigma_{+33} \tilde{\gamma} \\
\Gamma_{231} &=& -\sigma_{-3+} \tilde{\gamma} \\
\Gamma_{232} &=& -\sigma_{-0+} \tilde{\gamma} \\
\Gamma_{233} &=& -\sigma_{-33} \tilde{\gamma} \\
\Gamma_{213} &=& -\sigma_{-03} \tilde{\gamma} \\
\Gamma_{231} &=& -\sigma_{-30} \tilde{\gamma} \\
\Gamma_{232} &=& -\sigma_{-33} \tilde{\gamma} \\
\end{array} \]

Table 3: The complete basis for the six dimensional internal Clifford algebra within the 10 dimensional Clifford algebra is given, up to Hermitian conjugation.

Using the six dimensional \( \eta_{\kappa_1 \kappa_2 \kappa_3} \), a ten dimensional Majorana–Weyl spinor \( \chi \) can be decomposed in terms of four dimensional spinors \( \tilde{\chi}_{\kappa_1 \kappa_2 \kappa_3} \) as

\[ \chi = \sum_\kappa \eta_{\kappa_1 \kappa_2 \kappa_3} \tilde{\chi}_{\kappa_1 \kappa_2 \kappa_3}. \]

(75)
The Majorana–Weyl conditions then lead to the following relations on the four dimensional spinors:

$$\kappa_1\kappa_2\kappa_3 \tilde{\gamma} \chi^{\kappa_3\kappa_2\kappa_1} = (-\kappa_1\kappa_3)(\tilde{\chi}^{\kappa_3\kappa_2\kappa_1})C_- = \tilde{\chi}^{\kappa_3\kappa_2\kappa_1}. \tag{76}$$

We can define a basis of four Majorana fermions in $D = (1,3)$ dimensions: $\chi^{\kappa_1\kappa_2\kappa_3}$ with $\kappa_3\kappa_2\kappa_1 = 1$, such that

$$\chi^{\kappa_3\kappa_2\kappa_1}_L = (\kappa_1\kappa_3) \tilde{\chi}^{\kappa_3\kappa_2\kappa_1}_L \quad \chi^{\kappa_3\kappa_2\kappa_1}_R = (\kappa_1\kappa_3)(\tilde{\chi}^{\kappa_3\kappa_2\kappa_1})C_. \tag{77}$$

The expansion of the ten dimensional spinor then takes the form

$$\chi = \sum_{\kappa_1\kappa_2\kappa_3 = +} (\kappa_1\kappa_3) \left( \eta_{\kappa_3\kappa_2\kappa_1} \chi^{\kappa_3\kappa_2\kappa_1}_L - \kappa_2 \eta_{\kappa_3\kappa_2\kappa_1} \chi^{\kappa_3\kappa_2\kappa_1}_R \right). \tag{78}$$

The factor $(\kappa_1\kappa_3)$ has been included for notational convenience: it ensures that the signs appearing in (10) are all the same.

\section*{B Anomaly polynomials and factorization}

It is well known, that the anomaly is determined by the Wess–Zumino consistency condition \cite{50} up to an overall normalization factor. The solution to this consistency condition can be obtained from the characteristic class whose integral gives the index of the Dirac operator in two dimensions higher \cite{51}. Let $F_2$ be the curvature 2–form of a connection $A_1$ (for example a Yang–Mills gauge connection and/or a spin connection), and $\tilde{I}$ any analytic function. By taking the trace and restricting to a $2n + 2$–form, we obtain a closed and invariant form $I_{2n+2}$ defined as

$$\tilde{I}_{2n+2}(F_2) = \tilde{I}(F_2) \bigg|_{2n+2}, \quad I_{2n+2}(F_2) = \text{tr} \tilde{I}_{2n+2}(F_2) \quad dI_{2n+2}(F_2) = 0, \quad \delta \Lambda I_{2n+2}(F_2) = 0, \tag{79}$$

which exists locally because of Poincaré’s lemma. Here $\Lambda$ represents a gauge or local Lorentz transformation. Using the descent equations

$$I_{2n+2}(F_2) = dI_{2n+1}(A_1), \quad \delta \Lambda I_{2n+1}(A_1) = dI_{2n}^1(\Lambda, A_1), \tag{80}$$

the $2n + 1$ and $2n$ forms, $I_{2n+1}$ and $I_{2n}^1$, can be determined explicitly. For example $I_{2n}^1$ is given by the integral expression

$$I_{2n}^1(\Lambda, A) = \left( \frac{i}{2\pi} \right)^2 \int_0^1 dt (1 - t) \text{str} \left[ \Lambda d \left\{ A \tilde{I}_{2n-2}(t \text{d}A + t^2 A^2) \right\} \right]. \tag{81}$$

Of course, all these anomaly polynomials are still dependent on the chiral matter content under consideration. This is encoded in the trace $\text{tr}$ (str denotes the fully symmetrized trace.)

The Green–Schwarz mechanism relies on the factorization of anomaly polynomials. This special property can be stated as

$$I_{2p+2q+4}(F_2) = b_{p,q} I_{2p+2}(F_2) I_{2q+2}(F_2), \tag{82}$$

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for some integers \( p, q \geq 0 \), and a representation dependent proportionality factor \( b_{p,q} \). When applying the descent relations on the product of two anomaly polynomials, we find an additional free parameter \( \alpha_{p,q} \) because

\[
d\left( \alpha_{p,q} I_{2p+1}(A_1) I_{2q+2}(F_2) + (1 - \alpha_{p,q}) I_{2p+2}(F_2) I_{2q+1}(A_1) \right) = I_{2p+2}(F_2) I_{2q+2}(F_2). \tag{83}
\]

This seems to lead to an ambiguity in the definition of \( I_{2n} \). However, as has been discussed in [46], the constant \( \alpha_{p,q} \) is in fact fixed by the following observations.

The form of the anomaly, given in [S1], applies for any (gauge) connection \( A_1 \). The factorization relies on the underlying property of the trace of the anomaly polynomials:

\[
\text{tr}[T_1 \ldots T_{p+1} T_{p+2} T_{p+q+2}] = c_{p,q} \text{tr}[T_1 \ldots T_{p+1}] \text{tr}[T_{p+2} \ldots T_{p+q+2}] + \text{cyclic perm.} \tag{84}
\]

Since the trace is cyclic, we have to account for all cyclic permutations on the right hand side. As we only want to determine the constant \( \alpha_{p,q} \), we can safely restrict ourselves to the Abelian situation and apply this relation directly to the anomaly (81). In this way, we obtain the expression

\[
I_{2p+2q+2}^1(\Lambda, A_1) = c_{p,q} \frac{(p+1)!(q+1)!}{(p+q+2)!} \left( (p+1) I_{2q}^2(\Lambda, A_1) I_{2q+2}(F_2) + (q+1) I_{2p+2}(F_2) I_{2q}^1(\Lambda, A_1) \right), \tag{85}
\]

by performing the integral over the variable \( t \). Comparing this with the expressions [S2], [S1] and [S1] given above, we conclude that that

\[
b_{p,q} = c_{p,q} \frac{(p+1)!(q+1)!}{(p+q+2)!}, \quad \text{and} \quad \alpha_{p,q} = \frac{p+1}{p+q+2}. \tag{86}
\]

Since there was just one parameter \( \alpha_{p,q} \) to be fixed in the Abelian as well as the non–Abelian case, it follows that these results hold in general. Therefore, we obtain that if an anomaly polynomial factorizes like [S2], then the anomaly takes the form

\[
I_{2p+2q+2}^1(\Lambda, A_1) = b_{p,q} \left( \frac{p+1}{p+q+2} I_{2q}^2(\Lambda, A_1) I_{2q+2}(F_2) + \frac{q+1}{p+q+2} I_{2p+2}(F_2) I_{2q}^1(\Lambda, A_1) \right). \tag{87}
\]

## C trace decompositions of \( E_8 \)

In this appendix we verify, that for both equivalent models with an anomalous \( U(1) \) (the \( E_7 \) and \( SU(9) \) models discussed section 2.5), the following relation is valid

\[
\frac{1}{60} \text{tr}_{E_8 \times E_8'} F^2_\epsilon \big|_s = \sum_a \text{tr} F^2_\epsilon(a) \tag{88}
\]

when we restrict the quadratic \( E_8 \times E_8' \) trace to gauge group \( G_s \) of one of those anomalous models. (That is \( G_s = E_7 \times U(1) \times SO(14)' \times U(1)' \) or \( G_s = SU(9) \times SO(14)' \times U(1)' \).) The sum \( a \) is over the gauge group factors in \( G_s \). Here we have defined

\[
\text{tr} F^2_\epsilon(a) = \frac{1}{I_2^a} \text{tr}_{\text{fund}} F^2_\epsilon(a), \quad \text{tr} F^2_{U(1)} = 2 F^2_{U(1)}, \quad \text{tr} F^2_{U(1)'} = 4 F^2_{U(1)'} . \tag{89}
\]

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for the non–Abelian $G_{(a)}$ and Abelian group factors, respectively. $I_q^a$ denotes the quadratic indices for the non–Abelian group factors, given in table [4]. The normalizations for the U(1)s stem from the levels $k_q = 2$ and $k_q' = 4$, which we get in our conventions.

The remainder of the appendix is composed as follows: in appendix [C], a number of useful features of characters are reviewed. These properties are then used in the subsequent subappendices to compute the quadratic traces for the gauge groups that appear in the anomalous U(1) models.

### C.1 General properties of characters

To relate traces of the field strength $iF$ in various representations of different groups to each other, a convenient tool is the Chern character

$$\text{ch}_r[iF]_{\text{Ad}} = \text{tr}_r \exp \left[ \frac{iF}{2\pi} \right]_{\text{Ad}}.$$  \hspace{1cm} (90)

Here $\text{Ad}$ denotes the algebra in which the field strength $iF$ lives, and $r$ denotes the representation of this algebra over which the trace is taken. From the definition of the character it follows, that the dimension of a representation is given by $|r| = \dim r = \text{ch}_r[0]_{\text{Ad}} = \text{tr}_r[1]_{\text{Ad}}$. (Many useful properties of characters and indices are collected in [12].) The following properties of the Chern character are very useful

$$\text{ch}_{r_1 \times r_2}[iF]_{\text{Ad}} = \text{ch}_{r_1}[iF]_{\text{Ad}} \cdot \text{ch}_{r_2}[iF]_{\text{Ad}}, \quad \text{ch}_{r_1 + r_2}[iF]_{\text{Ad}} = \text{ch}_{r_1}[iF]_{\text{Ad}} + \text{ch}_{r_2}[iF]_{\text{Ad}}.$$  \hspace{1cm} (91)

For example, the trace over the adjoint $\text{Ad}_N = N^2 - 1$ of SU($N$) over the field strength squared, can be expressed as:

$$\text{Ad}_N + 1 = N \times N \quad \Rightarrow \quad \text{tr}_{\text{Ad}_N}[(iF)^2]_{\text{Ad}} = 2N \text{ tr}_{N}[(iF)^2]_{\text{Ad}}.$$  \hspace{1cm} (92)

Next we obtain the characters for anti–symmetric representations obtained from a representation $r$. We denote the $k$th totally anti–symmetric product of $r$ by $[r]_k$. (Of course, we set $[r]_0 = 1$ and $[r]_1 = r$.) Because the determinant, in the representation $r$, is fully anti–symmetric, we can define the generating function of the characters of the anti–symmetric products $[r]_k$ as

$$\sum_{k=1}^{|r|} x^k \text{ch}_{[r]_k}[iF] = \det_r \left( 1 + xe^iF \right) = \exp G_r(x, iF).$$  \hspace{1cm} (93)

The function $G_r(x, iF)$ has the properties:

$$G_r(x, iF) = \sum_{n \geq 1} \frac{(-i)^{n-1}}{n} x^n \text{ch}_r[n iF], \quad G_r^p = \left. \left( \frac{\partial}{\partial x} \right)^p G_r(x, iF) \right|_{x=0} = (-1)^{p-1}(p-1)!\text{ch}_r[p iF].$$  \hspace{1cm} (94)

In the following we only need the first (non–trivial) characters of fully anti–symmetric representations $[r]_k$ explicitly:

$$\text{ch}_{[r]_2}[iF] = \frac{1}{2} \left[ (\text{ch}_r[iF])^2 - \text{ch}_r[2iF] \right],$$

$$\text{ch}_{[r]_3}[iF] = \frac{1}{3!} \left[ (\text{ch}_r[iF])^3 - 3\text{ch}_r[iF]\text{ch}_r[2iF] + 2\text{ch}_r[3iF] \right],$$

$$\text{ch}_{[r]_4}[iF] = \frac{1}{4!} \left[ (\text{ch}_r[iF])^4 - 6(\text{ch}_r[iF])^2\text{ch}_r[2iF] + 8\text{ch}_r[iF]\text{ch}_r[3iF] + 3(\text{ch}_r[2iF])^2 - 3!\text{ch}_r[4iF] \right].$$  \hspace{1cm} (95)
| group      | fund. repr. | quadr. index |
|------------|-------------|--------------|
| $E_8$      | 248         | 60           |
| $E_7$      | 56          | 12           |
| SO(14)     | 14          | 2            |
| SU(9)      | 9           | 1            |

Table 4: The relevant quadratic indices of the fundamental representations of the (simple) gauge groups that arise in the models with an anomalous $U(1)$.

By substituting $iF = 0$ these characters give the dimensions of the representations:

\[ \dim[r]_2 = \frac{|r| - 1}{2}|r|, \quad \dim[r]_3 = \frac{|r|^2 - 3|r| + 2}{3!}|r|, \quad \dim[r]_4 = \frac{|r|^3 - 6|r|^2 + 11|r| - 6}{4!}|r|. \quad (96) \]

Furthermore, for a simple Lie group the traces of $(iF)^2$ over these anti-symmetric representations read

\[ \begin{align*}
\text{tr}_{[r]_2}(iF)^2 &= (|r| - 2)\text{tr}_r(iF)^2, \\
\text{tr}_{[r]_3}(iF)^2 &= \frac{|r|^2 - 5|r| + 6}{2}\text{tr}_r(iF)^2, \\
\text{tr}_{[r]_4}(iF)^2 &= \frac{|r|^3 - 9|r|^2 + 26|r| - 24}{6}\text{tr}_r(iF)^2.
\end{align*} \quad (97) \]

It is important to note that these formulae can be applied for any representation $r$ not necessarily the fundamental one.

### C.2 Quadratic traces of the anomalous fixed point models

The quadratic indices and reference representations of the relevant groups are given in table 4. It is conventional to normalize the indices w.r.t. the fundamental representation of $SU(n)$. We now compute $(1/60)\text{Tr}(iF)^2|_{G_s}$ where $G_s$ is the local gauge group of one of the two local anomalous equivalent models: $G_s = E_7 \times U(1) \times SO(14)' \times U(1)'$ and $G_s = SU(9) \times SO(14)' \times U(1)'$.

#### $E_7$ quadratic trace

From the branching rules (25) we see that we obtain two times the reference representation 56 and once the adjoint representation 133 of $E_7$. To relate the traces of these two representations, we use their decompositions under the branching

\[ E_7 \rightarrow SU(8): \quad 56 \rightarrow 28 + 28, \quad 133 \rightarrow 70 + 63. \quad (98) \]

To be able to use the general formulae derived above, we identify these representations as follows: 56 = $Ad_8$, 28 = 82 and 70 = 84. (Using the dimension formulae 48 for the anti-symmetrized representations these identifications can be confirmed easily.) Moreover, for the quadratic traces we find

\[ \begin{align*}
\text{tr}_{56}(iF)^2 &= 12 \text{tr}_8(iF)^2, \\
\text{tr}_{133}(iF)^2 &= 36 \text{tr}_8(iF)^2.
\end{align*} \quad (99) \]

(This confirms that the quadratic index of $E_7$ equals 12.) We conclude that

\[ \frac{1}{60}\text{tr}_{248}[(iF)^2]_{E_7} = \frac{1}{12}\text{tr}_{56}[(iF)^2]_{E_7}. \quad (100) \]
U(1) quadratic trace

For the U(1) factor in the first E_8 the computation is more straightforward, since we only have to take the dimensions of the E_7 representations and their charges into account. This gives

$$\frac{1}{60}\text{tr}_{248}[(iF)^2]_{U(1)} = \frac{1}{60}(2 \cdot 56 \cdot (\pm 1)^2 + 2 \cdot (\pm 2)^2)((iF)^2)]_{U(1)} = 2[(iF)^2]_{U(1)}.$$  \hfill (101)

SU(9) quadratic trace

For the quadratic trace in the adjoint of E_8 the relevant branching is given in [25]. We identify 80 = Ad_9 and 84 = [9]_4, hence we find their traces can be expressed as

$$\text{tr}_{80}[(iF)^2]_{SU(9)} = 18(18 + 2 \cdot 21)\text{tr}_{9}[(iF)^2]_{SU(9)}, \quad \text{tr}_{84}[(iF)^2]_{SU(9)} = 21\text{tr}_{9}[(iF)^2]_{SU(9)},$$  \hfill (102)

in terms of the reference representation 9 of SU(9). This confirms that the index of E_8 is 60:

$$\frac{1}{60}\text{tr}_{248}[(iF)^2]_{SU(9)} = \frac{1}{60}(18 + 2 \cdot 21)\text{tr}_{9}[(iF)^2]_{SU(9)} = \text{tr}_{9}[(iF)^2]_{SU(9)}.$$  \hfill (103)

SO(14)' quadratic trace

Using the expression for the character of the spinor representation SO(2n), given in [12], we find for the quadratic trace of the spinorial representation 64 of SO(14)

$$\text{tr}_{64}[(iF)^2]_{SO(14)} = 27^2 \frac{2^2 - 1}{12}B_2 \text{tr}_{14}[(iF)^2]_{SO(14)} = 8 \text{tr}_{14}[(iF)^2]_{SO(14)},$$

$$\text{tr}_{91}[(iF)^2]_{SO(14)} = 12 \text{tr}_{14}[(iF)^2]_{SO(14)},$$

with the Bernoulli number: $B_2 = 1/6$. For the second relation, we used that the adjoint of SO(14) is obtained as the anti–symmetric representation 91 = [14]_2. Following the branching [25] of E_8 to SO(14) representations gives

$$\frac{1}{60}\text{tr}_{248}[(iF)^2]_{SO(14)} = \frac{1}{60}(12 + 2 \cdot 8)\text{tr}_{14}[(iF)^2]_{SO(14)} = \frac{1}{2}\text{tr}_{14}[(iF)^2]_{SO(14)}.$$  \hfill (105)

U(1)' quadratic trace

Finally, we compute the quadratic traces of the U(1)' factor in the second E_8' gauge group. As for the previous U(1) factor, we can use the charges given in the branching rules [25]

$$\frac{1}{60}\text{tr}_{248}[(iF)^2]_{U(1)'} = \frac{1}{60}(2 \cdot 14 \cdot (\pm 2)^2 + 2 \cdot 64 \cdot (\pm 1)^2)((iF)^2)]_{U(1)'} = 4[(iF)^2]_{U(1)}. \hfill (106)$$

D Torus wavefunctions with Wilson lines

Here we collect some of the properties of bosonic torus wave functions that we use in the main text to compute the tadpoles of the gauge fields. The mode functions $\phi_q(z)$ of the torus are the periodic scalar functions on $\mathbb{C}^3$

$$\phi_q(z + i) = \phi_q(z) \quad \phi_q(z + \theta i) = \phi_q(z) \quad \Rightarrow \quad \phi_q(z) = N_q e^{2\pi i(q_i z_i + q_{i-1} z_{i-1})/R_i}, \quad \left(\begin{array}{c} q_i \\ q_{i-1} \end{array}\right) = \frac{1}{\theta - \theta}(\begin{array}{c} \phi n_i - m_i \\ -\phi n_i + m_i \end{array}),$$  \hfill (107)
with \( n_i, m_i \in \mathbb{Z} \). The normalization \( N_q^{-2} = \prod_i R_i^2 \frac{\theta - \theta_i}{q_i^2} \) is chosen such that these wave functions are orthonormal and form a complete set on the torus \( T^6 \):

\[
\int_{T^6} dz \phi_i^*(z) \phi_j(z) = \delta_{ij}, \quad \sum_q \phi_i(z) \phi_i^*(z') = \delta(z - z' - \Gamma). \tag{108}
\]

From these mode functions it is not difficult to obtain algebra valued mode functions that are periodic up to global gauge transformations.

\[
\phi_{qa}(z) = \phi_q(z) T(z) T^{-1}(z) = \phi_q(z) e^{2\pi i a_\alpha(z)} T^\alpha
\]

\[
\phi_{qa}^\dagger(z) = \phi_q^\dagger(z) T(z) T^\dagger(z) T^{-1}(z) = \phi_q^\dagger(z) e^{-2\pi i a_\alpha(z)} T^\dagger T^\alpha, \tag{109}
\]

using the notation

\[
a_\alpha(z) = \sum_i \frac{1}{R_i} (b_{\alpha i} z^i + b_{\alpha i} \bar{z}^i), \quad \text{with} \quad b_{\alpha i} = \frac{1 - \bar{\theta}}{\theta - \bar{\theta}} a^i, \quad w_I(T_\alpha), \quad b_{\alpha i} = (b_{\alpha i})^* . \tag{110}
\]

These mode functions satisfy the following properties

\[
\partial_i \left( \phi_{qa}^\dagger(\theta^{-k}z) \right) \phi_{qa}(z) = -\bar{\theta}^k \phi_{qa}^\dagger(\theta^{-k}z) \partial_i \phi_{qa}(z),
\]

\[
\partial_i \left( \phi_{qa}(\theta^{-k}z) \phi_{qa}(z) \right) = (1 - \bar{\theta}^k) \phi_{qa}^\dagger(\theta^{-k}z) \partial_i \phi_{qa}(z),
\]

\[
\partial_\mathbf{\mathbf{z}} \left( \phi_{qa}(\theta^{-k}z) \phi_{qa}(z) \right) = (1 - \theta^k) \phi_{qa}^\dagger(\theta^{-k}z) \partial_\mathbf{\mathbf{z}} \phi_{qa}(z),
\]

\[
\Delta \left( \phi_{qa}(\theta^{-k}z) \phi_{qa}(z) \right) = 3 \phi_{qa}^\dagger(\theta^{-k}z) \Delta \phi_{qa}(z),
\]

where the Laplacian \( \Delta = -2 \sum_i \partial_i \partial_i \). By combining (108) and (109), we obtain the following identity

\[
\sum_q \phi_{qw}(\theta^{-k}z) \phi_{qw}(z) = \sum_s e^{-2\pi i ks} a^i_{s} w_I(T_\alpha) \frac{1}{27} \delta(z - 3s - \Gamma), \tag{112}
\]

where the orbifold delta function is given by (22).

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