Quantum causality relations and the emergence of reality from coherent superpositions

Holger F. Hofmann
Graduate School of Advanced Sciences of Matter, Hiroshima University, Kagamiyama 1-3-1, Higashi Hiroshima 739-8530, Japan

The Hilbert space formalism describes causality as a statistical relation between initial experimental conditions and final measurement outcomes, expressed by the inner products of state vectors representing these conditions. This representation of causality is in fundamental conflict with the classical notion that causality should be expressed in terms of the continuity of intermediate realities. Quantum mechanics essentially replaces this continuity of reality with phase sensitive superpositions, all of which need to interfere in order to produce the correct conditional probabilities for the observable input-output relations. In this paper, I investigate the relation between the classical notion of reality and quantum superpositions by identifying the conditions under which the intermediate states can have real external effects, as expressed by measurement operators inserted into the inner product. It is shown that classical reality emerges at the macroscopic level, where the relevant limit of the measurement resolution is given by the variance of the action around the classical solution. It is thus possible to demonstrate that the classical notion of objective reality emerges only at the macroscopic level, where observations are limited to low resolutions by a lack of sufficiently strong intermediate interactions. This result indicates that causality is more fundamental to physics than the notion of an objective reality, which means that the apparent contradictions between quantum physics and classical physics may be resolved by carefully distinguishing between observable causality and unobservable sequences of hypothetical realities “out there”.

I. INTRODUCTION

Since the earliest days of quantum mechanics, the representation of quantum states as a superposition of possible measurement outcomes has caused much confusion and controversy. On the one hand side, it is natural to assume that measurement outcomes should somehow correspond to “elements of reality,” independent of whether they are actually observed or not. On the other hand side, quantum superpositions do not allow any intermediate measurements of the Hilbert space components that make up the quantum state, lest the measurement destroy the phase relations that are needed to fully define the distribution of future measurement outcomes. It may be tempting to treat the quantum state as a description of reality, but it should be recognized that it is merely a representation of the initial conditions that allows us to calculate the probabilities of future events caused by these initial conditions. A scientific discussion of the physical meaning of quantum theory should therefore focus on the causality relations between the initial conditions determined by the external manipulation of the system and the final conditions observed as a result of a measurement performed on the same system. A number of approaches have been proposed within the framework of quantum information theory, mostly based on an operational approach to measurement theory [1–7]. Unfortunately, most of these approaches are based on a somewhat excessive level of abstraction and therefore fail to answer a number of important questions regarding the nature of causality in physical systems. Specifically, we would like to know something about the laws that govern the dynamics of a system, and these laws should be sufficiently objective so that they can be formulated without any reference to the measurement context. This is precisely the purpose of the Hilbert space formalism: it provides an objective description of causality that can be applied to any combinations of state preparation and measurement. The main difference to the corresponding classical description of causality in phase space is the use of quantum superpositions which prevents us from attributing reality to intermediate observations. This seems to be at odds with our macroscopic experience, since we can normally watch objects as they move. In order to understand the relation between classical theories and quantum mechanics properly, it would seem to be important to understand how quantum mechanics reconciles the experience of objective causality as an undisturbed sequence of observations with the impossibility of precise intermediate measurements between an initial condition and a final outcome.

The problem of intermediate measurements and the associated resolution-disturbance trade-off have been widely discussed in the literature, and recent advances in quantum information technologies have led to a number of successful experimental demonstrations of some of the stranger aspects of quantum measurements [8–16]. As a result, we now
have a wide range of theoretical and experimental tools that may allow us to address fundamental questions regarding
the objective physics described by the quantum formalism. Of particular interest is the question of whether we can
observe physical properties of a quantum system without disturbing the time evolution of the system between its
preparation and a final measurement. An important breakthrough in that direction has been achieved using weak
measurements [17], which has resulted in a direct observation of quantum coherences between an initial preparation
and a final measurement [18, 29]. In these weak measurement, quantum coherences are observed as an average shift
of the meter system, lending some credibility to the idea that we might be looking at an intermediate reality that
manifests itself as a well-defined external effect. The problem with the weak measurement limit is the low resolution
of each individual measurement outcome, which makes it highly problematic to associate the average result with a
physical property of the individual system [30–50]. It is therefore of great interest to consider alternative approaches
that can provide meaningful insights into the causal origin of intermediate observations while keeping the disturbance
of the measurement interaction at a negligibly low level.

In this paper, I consider the possibility of sidestepping the very tight relation between resolution and disturbance
for the initial quantum state by considering only the causality relation between an input \( |a\rangle \) and a final result \( |b\rangle \).
It is then possible to identify a quantitative condition for disturbance free measurements that depends on the specific
relation between the initial state and the measurement result described by the inner product of the two. For sufficiently
large quantum systems, this relation can be characterized by the quantum phases of intermediate states which can be
expressed as an action \( S(a, m, b) \). It can then be shown that the classical notion of an intermediate reality emerges
from the contribution \( m \) that minimizes this action. The analysis below thus demonstrates that our classical notion
of reality is an emergent feature of quantum measurements performed with a sufficiently low resolution. Importantly,
the action \( S(a, m, b) \) defines the intermediate measurement resolution with respect to the fundamental constant \( \hbar \),
explaining the classical limit as a natural approximation of the underlying quantum formalism. The results presented
in the following thus demonstrates that the classical notion of reality is based on an emergent phenomenon which is
not fundamental to physics. This means that the notion of objective reality can be discarded as a redundant feature,
even where the classical explanation of physical phenomena is concerned. Instead, the notion of observable reality can
be rooted in fundamental causality relations that are sufficiently accurate to give an objective meaning to subjective
experience.

II. CAUSALITY OF INPUT-OUTPUT RELATIONS

In most cases, the Hilbert space formalism makes it all too easy to treat a quantum system as a black box that
magically produces the measurement outcomes \( b \) from initial conditions \( a \). The reason for this black box thinking
is that both \( a \) and \( b \) can be represented by Hilbert space vectors, \( |a\rangle \) and \( |b\rangle \), which leaves the relation with other
physical properties somewhat ambiguous. The conditional probabilities are given by the absolute value square of the
inner product,

\[
P(b|a) = \langle b | a \rangle^2, \tag{1}
\]

where the inner product \( \langle b | a \rangle \) is an abstract mathematical rule that is only justified in terms of its successful prediction
of the observable statistics \( P(b|a) \). Interestingly, this abstract mathematical rule demands a rather specific logical
relation with any other measurement represented by an orthogonal basis set \{\( |m\rangle \)\} due to the fact that every basis
set is a valid representation of all Hilbert space vectors and their inner products. This is the origin of the widespread
notion that quantum states represent superpositions of “realities,” which seems to be an over interpretation of the
basis vectors \( |m\rangle \). Proceeding more carefully, we can merely say that the causality relation between \( a \) and \( b \) in Eq.(1)
can be related to elements of the causality relations between \( a \) and \( m \) and between \( m \) and \( b \) by means of the inner
products,

\[
\langle b | a \rangle = \sum_m \langle b | m \rangle \langle m | a \rangle. \tag{2}
\]

To fully understand the problematic nature of the quantum formalism, it is important to consider the relation between
Eq.(2) and the traditional attempts to visualize quantum physics in terms of interferences between different “realities.”
Clearly, the inner product is highly sensitive to the complex phases in the sum over \( m \), making it impossible to select
an intermediate value of \( m \) associated with the “path” between \( a \) and \( b \). However, the classical limit of causality
suggests that the combination of initial condition \( a \) and final condition \( b \) should select a specific value of \( m \) as a
function of \( (a, b) \).

In the context of Hilbert space inner products, the relation between classical causality and quantum phases can be
identified using the action of unitary transformations [24, 37–39]. As pointed out in [24], the action \( S(a, m, b) \) can be

\[
\sum_m S(a, m, b) \propto \sum_m \langle b | m \rangle \langle m | a \rangle. \tag{3}
\]
defined as the action of the unitary transformation with eigenstates $|m\rangle$ that maximizes the inner product given by

$$\langle b | \hat{U}_M | a \rangle = \sum \langle b | m \rangle \langle m | a \rangle \exp\left(-\frac{i}{\hbar} S(a, m, b)\right).$$

(3)

Since the maximal value is obtained when all of the phases are equal, the action $S(a, m, b)$ is determined by the quantum phases of $\langle b | m \rangle \langle m | a \rangle$. To obtain a geometric phase, it is convenient to define $S(a, m, b)$ as

$$S(a, m, b) = \hbar (\text{Arg}(\langle b | m \rangle \langle m | a \rangle) - \text{Arg}(\langle b | a \rangle))$$

(4)

In a sufficiently large Hilbert space, $S(a, m, b)$ will be a slowly varying function of the eigenvalues $x_a$, $x_b$ and $x_m$ associated with the eigenstates $|a\rangle$, $|b\rangle$ and $|m\rangle$. An approximate solution of the inner product in Eq. (2) can then be obtained by omitting all contributions with phases that oscillate rapidly in $x_m$, leaving only a small region around the least action value of $x_m$, given by

$$\frac{\partial}{\partial x_m} S(x_a, x_m, x_b) = 0.$$ 

(5)

It is then possible to recover the classical form of causality defined by the principle of least action. Specifically, Eq. (5) defines a deterministic relation between $x_m$ and $(x_a, x_b)$, so that the intermediate property can be expressed as a function $x_m(x_a, x_b)$ of the initial and final conditions.

$S(x_a, x_m, x_b)$ can also be derived from the dynamics of states with finite uncertainties. As shown in \[39\], the application of a unitary transformation that modifies the quantum phases by a factor of $\exp(-ix_m t/\hbar)$ moves a wave packet of energy $x_m$ from $x_a$ to $x_b$ within a propagation time of

$$t(x_a, x_m, x_b) = \frac{\partial}{\partial x_m} S(x_a, x_m, x_b).$$

(6)

Derivatives of the action $S(x_a, x_m, x_b)$ thus describe propagation times between initial and final conditions. Importantly, these propagation times constitute a macroscopic effect of the precise coherences in Hilbert space defined by Eq. (4). Likewise, the principle of least action in Eq. (5) is merely the macroscopic limit of the actual quantum interference effects in Eq. (2). Specifically, the principle of least action states that the value of $x_m$ for $(x_a, x_b)$ is identified with the value of $x_m$, for which the transformation distance between $x_a$ and $x_b$ is zero. However, the vicinity of this value of $x_m$ must be included in the inner product given by Eq. (2). Classical causality can only be recovered if $x_m$ is defined with a sufficiently low precision, so that the necessary quantum coherence in $x_m$ can be maintained. To understand this constraint, it is necessary to take a closer look at the possibility of verifying the causality relation implied by the principle of least action using an intermediate measurement.

III. INTERMEDIATE MEASUREMENTS WITH NEGLIGIBLE DISTURBANCE

The main reason why the notion of a microscopic reality is so problematic in quantum mechanics is the observation that there is no non-invasive measurement by which intermediate realities could be looked up without any changes to the state of the system. In terms of the causality relation between $|a\rangle$ and $|b\rangle$, this means that any intermediate measurement relating to $|m\rangle$ will change the probabilities $P(b|a)$ that define the causality relation. It is therefore important to consider the conditions under which we can approximately neglect the changes to $P(b|a)$ while still obtaining meaningful information about $x_m$.

According to the rules of quantum mechanics, an intermediate measurement of $x_m$ will change the probability amplitudes of the eigenstates $|m\rangle$ in accordance with the Bayesian probability update associated with the measurement outcome $r$. As discussed in \[40\], this probability update is directly responsible for the decoherence in the system caused by the measurement interaction. At the most fundamental level, a measurement is therefore represented by the conditional probabilities $P(r|x_m)$, and the minimal decoherence is represented by an operator $\hat{M}(r)$ with

$$\langle b | \hat{M}(r) | a \rangle = \sum_m \langle b | m \rangle \langle m | a \rangle \sqrt{P(r|x_m)}.$$ 

(7)

The disturbance of the causality relation between $a$ and $b$ is negligible if the probability $P(b|a)$ is not changed by the measurement and the joint probability $P(r, b|a)$ can be written as

$$P(r, b|a) \approx P(r|a, b)P(b|a).$$ 

(8)
This relation between probabilities corresponds to an analogous relation between probability amplitudes in the Hilbert space inner product of \( \langle \psi | \psi \rangle \). Specifically, the expectation is that the application of the operator \( \hat{M}(r) \) will not change the inner product of \( |a\rangle \) and \( |b\rangle \), and return only one value of \( \sqrt{P(r|x_m)} \) for each result \( r \),

\[
\langle b | \hat{M}(r) | a \rangle \approx \sqrt{P(r|x_m)} \langle b | a \rangle.
\] (9)

The most important aspect of this approximation is the selection of the value of \( x_m \), which corresponds to the classical notion of an intermediate reality of \( x_m \) determined by the causality relation between \( a \) and \( b \). According to the discussion in the previous section, most of the contributions the sum over \( m \) in Eq.(7) cancel out because of the rapidly oscillating phases associated with the action gradient \( \partial S/\partial x_m \). The multiplication with \( \sqrt{P(r|x_m)} \) has no effect on this cancellation as long as \( P(r|x_m) \) changes only little over one period of the phase oscillation. Likewise, the separation in Eq.(9) is possible if \( P(r|x_m) \) changes only slowly in the relevant region of nearly stationary action, where the summation over neighboring states \( |m\rangle \) does not vanish. Eq.(7) then effectively selects the least action value \( x_m(x_a, x_b) \), seemingly confirming the classical notion of causality according to which the combination of physical properties \( x_a \) and \( x_b \) determines the precise value of \( x_m \). However, the precision of \( P(r|x_m) \) is now constrained by the need to maintain quantum coherence in a sufficiently wide range of eigenstates \( |m\rangle \). The identification of this range of eigenstates is the main purpose of the present paper.

In order to show that the approximation in Eq.(9) is indeed justified, we have to make use of the slow variation of \( S(x_m) \), which makes it possible to approximate the sum in Eq.(2) by an integral,

\[
\langle b | a \rangle \approx \int \sqrt{\rho(x_m|b)\rho(x_m|a)} \exp\left(\frac{i}{\hbar} S(x_a, x_m, x_b)\right) dx_m,
\] (10)

where the conditional probability densities are obtained by multiplying the conditional probabilities \( P(m|a) \) and \( P(m|b) \) with the density of states in \( \hat{M} \) given by the inverse of the eigenvalue difference \( \Delta x_m \) between subsequent states \( |m\rangle \),

\[
\rho(x_m|\psi) = \frac{\langle m | \psi \rangle^2}{\Delta x_m},
\] (11)

where

\[
\Delta x_m = x_{m+1} - x_m.
\] (12)

The solution of the integral in Eq.(10) can now be performed in the immediate vicinity of the least action solution \( x_m(x_a, x_b) \). The quantum interference effects in Eq.(2) can then be represented by

\[
\langle b | a \rangle \approx \frac{\langle b | m \rangle \langle m | a \rangle}{\Delta x_m} \int \exp\left(\frac{i}{\hbar} \left( S(x_a, x_m, x_b) + \frac{1}{2} \frac{\partial^2}{\partial x_m^2} S(x_a, x_m, x_b)(x' - x_m)^2 \right)\right) dx',
\] (13)

where \( m \) and \( x_m \) are the values at which the action is stationary \( (\partial S/\partial x_m = 0) \) and the variable \( x' \) is used to express small variations of \( x_m \) around the value at which the action is minimal. Due to the slow variation of the absolute values of \( \langle b | m \rangle \) and of \( \langle m | a \rangle \), the second derivative of the action \( S(x_m) \) is fully determined by the inner products of the state vectors,

\[
\frac{\partial^2}{\partial x_m^2} S(x_a, x_m, x_b) = \frac{2\pi \hbar}{\Delta x_m^2} \left( \frac{\langle b | m \rangle \langle m | a \rangle}{\langle b | a \rangle} \right)^2.
\] (14)

It should be noted that the Hilbert space inner products enter this relation in the form of a weak value for the projector \( |m\rangle \langle m| \), highlighting the fundamental role of such weak values in defining the relations between the physical properties \( x_a \), \( x_b \) and \( x_m \) associated with the eigenstates \( |a\rangle \), \( |b\rangle \) and \( |m\rangle \) \cite{21, 37, 38}.

If the approximation in Eq.(13) is sufficiently accurate, it can also be applied to any intermediate measurement of \( x_m \), as represented by the measurement operators \( \hat{M}(r) \). The approximation given in Eq.(9) is therefore valid whenever the conditional probabilities \( P(r|x_a) \) that characterize the measurement operators \( \hat{M}(r) \) vary more slowly than the phases in the integration over \( x' \) in Eq.(13). This condition can be expressed in a particularly symmetric form, since both phases and probabilities are dimensionless. The separation of intermediate measurement and propagation causality expressed by Eq.(9) is valid for

\[
\frac{\partial^2}{\partial x_m^2} P(r|x_m) \ll \frac{1}{2\pi \hbar} \frac{\partial^2}{\partial x_m^2} S(x_a, x_m, x_b).
\] (15)
We can use this condition to identify the maximal disturbance-free resolution of \( x_m \),

\[
\frac{1}{\delta x_m} = \sqrt{\frac{1}{2\pi \hbar} \frac{\partial^2}{\partial x_m^2} S(x_a, x_m, x_b)},
\]

where \( \delta x_m \) is the interval around the least action value \( x_m \) that contributes significantly to the inner product \( \langle b | a \rangle \) and hence to the causality relation between \( x_a \) and \( x_b \). At resolutions lower than \( 1/\delta x_m \), the measurement results reveal the least action value \( x_m \) without changing the outcome \( b \) of the experiment. Within this limit, we can therefore think of \( x_m \) as an intermediate reality associated with the propagation of the system from \( x_a \) to \( x_b \).

Clearly, it is a necessary condition for a disturbance-free observation of the property \( x_m \) that the interval \( \delta x_m \) includes a large number of quantum states \( | m \rangle \). Using Eq. (14), we can identify the number of states in an interval of \( \delta x_m \) and determine the limit of quantum state resolution,

\[
\frac{1}{\delta n} = \frac{\Delta x_m}{\delta x_m} = \frac{\langle b | m \rangle \langle m | a \rangle}{\langle b | a \rangle}.
\]

Classical intermediate realities therefore emerge only if the inner products between the different eigenstates are sufficiently low. This observation highlights a fundamental problem of quantum information theory: the focus on individual states and low dimensional Hilbert spaces makes it impossible to relate the results obtained in this extreme limit of quantum mechanics to the more familiar physics of cause and effect that governs the technology used to control the quantum system. It thus remains a challenging task to properly explain the fundamental nature of causality in terms of quantum interference effects, without any redundant references to intermediate realities. The analogy between eigenstates and classical information that is widely used present quantum information technologies as the next generation of conventional computers seems to be rather misguided in that respect.

\[
\frac{\Delta x_m}{\Delta x_m} = \frac{\langle b | m \rangle \langle m | a \rangle}{\langle b | a \rangle}.
\]
written as
\[
\langle b | \hat{M}(r) | a \rangle \approx \frac{\langle b | m \rangle \langle m | a \rangle}{\Delta x_m} \int \sqrt{P(r|x_m)} \exp \left( \frac{i}{\hbar} \frac{\partial}{\partial x_m} S(x_a, x_m, x_b)(x' - x_r) \right) dx'.
\] (18)

Essentially, the integral corresponds to a Fourier transform of the resolution function \( \sqrt{P(r|x_m)} \), where the Fourier component is determined by the gradient of the action at \( x_m = x_r \). What is being resolved in the measurement is not the value of \( x_m \), but the action gradient associated with the external effect \( x_r \). In the case of intermediate measurements with Gaussian resolution, the precise result for the measurement sequence is
\[
\langle b | \hat{M}(r) | a \rangle \approx \langle b | m \rangle \langle m | a \rangle \left( 8\pi \frac{\delta x^2}{\Delta x_m^2} \right)^{1/4} \exp \left( -\frac{(\delta x_r)}{\hbar} \frac{\partial}{\partial x_m} S(x_a, x_m, x_b) \right)^2.
\] (19)

If additional sources of decoherence are avoided, the measurement outcomes \( x_r \) can provide rather detailed information on the action gradients that govern the causality relation between \( x_a \) and \( x_b \) in Hilbert space. The failure to observe the intermediate realities \( x_r = x_m \) of the undisturbed propagation from \( x_a \) to \( x_b \) therefore originates from the role that small action gradients play at the microscopic level. As shown by Eq. (19), the relevant condition for the failure of the least action approximation is
\[
\frac{1}{\delta x_r} > \frac{1}{\hbar} \frac{\partial}{\partial x_m} S(x_a, x_m, x_b).
\] (20)

Quantum theory thus applies in the limit of resolutions larger than the action gradient evaluated in units of \( \hbar \). This statement is as fundamental to quantum theory as the statement that the theory of relativity applies at velocities approaching the speed of light is fundamental to the theory of relativity. It clearly identifies the magnitude of the effects described by the theory and therefore explains why they can be neglected in the classical limit. I would therefore conclude that quantum mechanics describes the details of causality relations in the limit of high resolution, where the action provides a universal measure of resolution for causality relations in all fields of physics. The main problem that has prevented us from understanding quantum physics as the natural foundation of classical physics is that we are not used to a quantification of causality in terms of the action. It is therefore necessary to carefully consider the role of the action in unitary dynamics and its relation with the concept of quantum coherence as shown in Eqs. (32) and (33). Ultimately, the role of the action as a universal expression of causality is the cornerstone of a proper understanding of the physical world at the microscopic limit.

V. CONCLUSIONS

As I have explained above, it is possible to understand the physics described by Hilbert space inner products as a universal description of causality at the ultimate limit of quantitative precision. The action emerges naturally from the complex phases that appear in Hilbert space products when they are expressed in terms of components that seem to represent possible intermediate realities. However, these intermediate contributions cannot be converted into observable realities without changing the original causality relation between initial conditions and final measurement. Instead, the necessary modification of causality relations caused by any intermediate observation is fully determined by the action \( S(x_a, x_m, x_b) \), which provides a complete description of deterministic causality relations between the different physical properties of a system [26]. By focusing the discussion on causality relations between physical properties that cannot be measured jointly, it is possible to make statements about causality without any speculation about the nature of quantum states. The result is a theory that manages to smoothly connect the approximate description of phenomena by classical physics to the more precise description provided by quantum mechanics without changing the conceptual framework. I would argue that this result can reconcile our classical intuition with the weirder aspects of quantum physics in a constructive manner and provide a consistent description of both quantum mechanics and its classical limit [43]. In particular, it should not come as such a great shock that the naive assumption of a microscopic reality breaks down as a consequence of the fundamental action scale given by \( \hbar \). Even in the classical limit, we merely reconstruct the reality “out there” from observations that are never very precise. The possibility to do so depends entirely on the reliability of causality relations such as the one represented by the Hilbert space inner product in Eqs. (2) and (7). What I have shown here is that the classical versions of causality are robust up to a resolution of \( \delta x_m \) given by the curvature of the action at its minimum. It may be worth noting that even a measurement with a resolution much lower than \( 1/\delta x_m \) modifies the quantum state \( | a \rangle \) significantly by removing all amplitudes \( \langle m | a \rangle \) outside of the interval \( \delta x_m \). The criterion for classical causality is therefore less restrictive than the criterion for quantum state disturbance, and this fact explains why many forms of quantum coherence have no
observable effects whatsoever. For instance, the quantum coherence of a superposition of dead cats and living cats famously suggested by Schrödinger has no observable consequences in the future, and this will even be true for the experience of the cat itself. Quantum corrections of classical causality relations can only be observed if both state preparation and measurement are sufficiently precise, since neither one has any physical meaning of its own. The quantitative nature of quantum corrections of causality can then be observed and quantified in terms of the statistical relations between different measurement outcomes, an example of which has been given in [44–46] for the failure of Newton’s first law in the case of particle propagation in free space. A closer inspection of the relation between causality and quantum coherence can thus result in the systematic development of new means of control beyond the least action approximation.

In the light of the results presented above, it seems that the idea that quantum states and their eigenvalues can describe the physical reality of an object is based on the misconception that the reality of an object can be separated from the causality relations by which we know anything about that reality. The answer to all interpretational problems of quantum mechanics should therefore lie in an improved understanding of the causality relation between objects and their observable effects, where the action can take its rightful place as a fundamental scale in all physical theories. The analysis given above evaluates the precise quantitative limits for the emergence of a classical reality in quantum causality relations. I hope that the discussion presented in this paper will thus prove to be the first step towards a deeper understanding of quantum theory as the most fundamental explanation of all observable phenomena.

ACKNOWLEDGMENT

This work has been supported by JST-CREST (JPMJCR1674), Japan Science and Technology Agency.

[1] A. Zeilinger, “A Foundational Principle for Quantum Mechanics,” Found. of Phys. 29, 631 (1999).
[2] C. Brukner and A. Zeilinger, “Operationally Invariant Information in Quantum Measurements,” Phys. Rev. Lett. 83, 3354 (1999).
[3] C. Fuchs, “Quantum mechanics as quantum information, mostly,” J. Mod. Opt. 50, 987 (2003).
[4] C. M. Caves, C. A. Fuchs, and R. Schack, “Subjective probability and quantum certainty,” Studies in History Philos. Sci. Part B: Studies in History Philos. Mod. Phys. 38, 255 (2007).
[5] P. Goyal, “Information-geometric reconstruction of quantum theory,” Phys. Rev. A 78, 052120 (2008).
[6] J.-W. Lee, “Quantum Mechanics Emerges from Information Theory Applied to Causal Horizons,” Found. of Phys. 41, 744 (2011).
[7] M. S. Leifer and R. W. Spekkens, “Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference,” Phys. Rev. A 88, 052130 (2013).
[8] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, “Experimental realization of the quantum box problem,” Phys. Lett. A 324, 125 (2004).
[9] A. N. Jordan, A. N. Korotkov, and M. Büttiker, “Leggett-Garg inequality with a kicked quantum pump,” Phys. Rev. Lett. 97, 026805 (2006).
[10] J. S. Lundeen and A. M. Steinberg, “Experimental joint weak measurement on a photon pair as a probe of Hardy’s paradox,” Phys. Rev. Lett. 102, 020404 (2009).
[11] K. Yokota, T. Yamamoto, M. Koashi, and N. Imoto, “Direct observation of Hardy’s paradox by joint weak measurement with an entangled photon pair,” New J. Phys. 11, 033011 (2009).
[12] M. E. Goggin, M. P. Almeida, M. Barbieri, B. P. Lanyon, J. L. O’Brien, A. G. White, and G. J. Pryde, “Violation of the Leggett-Garg inequality with weak measurements of photons,” Proc. Natl. Acad. Sci. U. S. A. 108, 1256 (2011).
[13] Y. Suzuki, M. Iinuma, and H. F. Hofmann, “Violation of Leggett-Garg inequalities in quantum measurements with variable resolution and back-action,” New J. Phys. 14, 103022 (2012).
[14] T. Denkmayr, H. Geppert, S. Sponar, H. Lemmel, A. Matzkin, J. Tollaksen, and Y. Hasegawa, “Observation of a quantum Cheshire Cat in a matter-wave interferometer experiment,” Nat. Commun. 5, 4492 (2014).
[15] R. Okamoto and S. Takeuchi, “Experimental demonstration of a quantum shutter closing two slits simultaneously,” Sci. Rep. 6, 35161 (2016).
[16] Z. Minev, S. Mundhada, S. Shankar, P. Reinhold, R. Gutierrez-Jauregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael, and M. H. Devoret, “To catch and reverse a quantum jump mid-flight,” Nature 570, 200 (2019).
[17] Y. Aharonov, D. Z. Albert, and L. Vaidman, “How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100,” Phys. Rev. Lett. 60, 1351 (1988).
[18] H. M. Wiseman, “Weak values, quantum trajectories, and the cavity-QED experiment on wave-particle correlation,” Phys. Rev. A 65, 032111 (2002).
[19] H. F. Hofmann, “Complete characterization of post-selected quantum statistics using weak measurement tomography,” Phys. Rev. A 81, 012103 (2010).
[20] H. F. Hofmann, “Quasi-determinism of weak measurement statistics: Laplace’s demon’s quantum cousin,” e-print arXiv:1005.0654 (2010).
[21] A. Hosoya and Y. Shikano: Strange weak values. J. Phys. A: Math. Theor. 43, 385307 (2010).
[22] A. Bednorz and W. Belzig, “Quasiprobabilistic Interpretation of Weak Measurements in Mesoscopic Junctions,” Phys. Rev. Lett. 105 106803 (2010).
[23] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, “Direct measurement of the quantum wavefunction,” Nature 474, 188 (2011).
[24] H. F. Hofmann, “On the role of complex phases in the quantum statistics of weak measurements,” New J. Phys. 13, 103009 (2011).
[25] J.S. Lundeen and C. Bamber, “Procedure for direct measurement of general quantum states using weak measurement,” Phys. Rev. Lett. 108, 070402 (2012).
[26] H. F. Hofmann, “Complex joint probabilities as expressions of reversible transformations in quantum mechanics,” New J. Phys. 14, 043031 (2012).
[27] T. Morita, T. Sasaki, and I. Tsutsui, “Complex probability measure and Aharonov’s weak value,” Progress of Theoretical and Experimental Physics 2013 (2013).
[28] D. Das and Arvind, “Estimation of quantum states by weak and projective measurements,” Phys. Rev. A 89, 062121 (2014).
[29] J. Dressel, “Weak values as interference phenomena,” Phys. Rev. A 91, 032116 (2014).
[30] H. F. Hofmann, “How Weak Values Emerge in Joint Measurements on Cloned Quantum Systems,” Phys. Rev. Lett. 109, 020408 (2012).
[31] A. Bednorz, K. Franke, and W. Belzig, “Noninvasiveness and time symmetry of weak measurements,” New J. Phys. 15, 023043 (2013).
[32] L. Maccone and C.C. Rusconi, “State estimation: A comparison between direct state measurement and tomography,” Phys. Rev. A 89, 022122 (2014).
[33] R. Mochizuki, “Weak value as an indicator of back-action,” Progress of Theoretical and Experimental Physics 2014, (2014).
[34] A. C. Ipsen, “Disturbance in weak measurements and the difference between quantum and classical weak values,” Phys. Rev. A 91, 062120 (2014).
[35] E. Cohen and E. Pollak, “Determination of weak values of quantum operators using only strong measurements,” Phys. Rev. A 98, 042112 (2018).
[36] A. Matzkin, “Weak Values and Quantum Properties.” Found. Phys. 49, 298 (2019).
[37] H. F. Hofmann, “Derivation of quantum mechanics from a single fundamental modification of the relations between physical properties,” Phys. Rev. A 89, 042115 (2014).
[38] H. F. Hofmann, “On the fundamental role of dynamics in quantum physics,” Eur. Phys. J. D 70,118 (2016).
[39] K. Hibino, K. Fujiwara, J.-Y. Wu, M. Iinuma, and H. F. Hofmann, “Derivation of quantum statistics from the action of unitary dynamics,” Eur. Phys. J. Plus 133, 118 (2018).
[40] K. Patekar and H. F. Hofmann, “The role of system-meter entanglement in controlling the resolution and decoherence of quantum measurements,” New J. Phys. 21, 103006 (2019).
[41] J. B. Hartle, “Quantum mechanics with extended probabilities,” Phys. Rev. A 78, 012108 (2008).
[42] J. Dressel, K. Y. Bliokh, and F. Nori, “Classical Field Approach to Quantum Weak Measurements,” Phys. Rev. Lett. 112, 110407 (2014).
[43] H. F. Hofmann, “Quantum paradoxes originating from the nonclassical statistics of physical properties related to each other by half-periodic transformations,” Phys. Rev. A 91, 062123 (2015).
[44] H. F. Hofmann, “Quantum interference of position and momentum: A particle propagation paradox,” Phys. Rev. A 96, 020101(R) (2017).
[45] H. F. Hofmann, “Control of particle propagation beyond the uncertainty limit by interference between position and momentum,” Phys. Rev. A 98, 052104 (2018).
[46] H. F. Hofmann, “A quantum magic bullet: hitting two targets without a clear line-of-sight,” e-print arXiv:1909.09250 (2019).