Integrated fractional Brownian motion: persistence probabilities and their estimates

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Abstract. The problem is a log-asymptotics of the probability that the Integrated fractional Brownian motion of index 0<H<1 does not exceed a fixed level during long time. For the growing time interval (0,T) the hypothetical log-asymptotics is (H(H-1)+o(1))Log T. In support of the hypothesis, we update our earlier estimates of the probability and give analytical proofs.

1. The problem

Let $x(t), x(0) = 0$ be a real-valued stochastic process with the polynomial asymptotics:

$$P( x(t) < 1, t \in \Delta_T ) = T^{-\theta_x + o(1)}, \ T \to \infty, \ \Delta_T = T \Delta,$$

where $\Delta$ is some bounded interval containing 0. In that case, $\theta_x$ is known as the persistence exponent. The problem of exact values of $\theta_x$ is usually a difficult task. An overview and some new results are presented in (Bray et al., 2013; Aurzada and Simon, 2015; Profeta and Simon, 2015; Molchan, 2017; Aurzada et al., 2018).

Below we consider the process which is obtained by integrating the fractional Brownian motion, i.e.,

$$I_H(t) = \int_0^t w_H(s)ds$$

where $w_H(t), t \in R^1$ is a Gaussian process with the correlation function

$$Ew_H(t)w_H(s) = 1/2(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

As is known (Molchan 1999),

$$\theta_{w_H} = 1-H, \ \Delta_T = (0,T).$$

(1.2)

A similar result holds for the persistence exponent of $I_H(t)$, but in a bilateral growing interval:

$$\theta_{I_H} = 1-H, \ \Delta_T = (-T,T),$$

(1.3)

(Molchan , 2017). In the case $H = 1/2$ the trajectories of $I_H(t)$ are independent in the intervals $t < 0$ and $t > 0$. Therefore, putting $H = 1/2$ in (3), we arrive at the Sinai (1992) result for the integrated Brownian motion in $\Delta_T = (0,T)$, namely,

$$\theta_{I_{1/2}} = 1/4, \ \Delta_T = (0,T).$$

(1.4)
In the general case of $H$, the exact value of the persistence exponent $\theta_{I_H}$ for $\Delta_T = (0,T)$ is unknown, and remains an important unsolved problem. Molchan and Khokhlov (2004) analyzed the exponent $\theta_{I_H}$ theoretically and numerically to put forward the following

**Hypothesis:** $\theta_{I_H} = H(1 - H)$ for $\Delta_T = (0,T)$.

The hypothetical equality $\theta_{I_H} = \theta_{I_{-H}}$ is quite interesting, because the exponents are related to processes with fundamentally different probabilistic properties. This difference is well reflected in the result (1.3) for the bilateral growing interval $\Delta_T = (-T, T)$. In support the hypothesis, below we update our earlier estimates of $\theta_{I_H}$ for $\Delta_T = (0,T)$ (see Molchan, 2012) and give analytical proofs for them. The main result is the following

**Proposition 1.** For the integrated fractional Brownian motion in $\Delta_T = (0,T)$

a) $\theta_{I_H} \geq \theta_{I_{-H}}$, $0 < H \leq 0.5$;

b) $0.5(H \land \overline{H}) \leq \theta_{I_H} \leq H \land \overline{H}$, $\overline{H} = 1 - H$;

c) $\theta_{I_H} \leq 1/4 \sqrt{(1 - H^2)/12}$.

**Remark.** The above estimates are closest to the hypothetical values near the indexes $H = 0, 1/2, 1$ and are well matched with possible symmetry of $\theta_{I_H}$ relative to the point $H = 1/2$.

### 2. Modification of the problem

Any self-similar process $x(t)$ in $\Delta_T = (0,T)$ generates a dual stationary process $\tilde{x}(s) = e^{-ht}x(e^t)$, $s < \ln T = \tilde{T}$, where $h$ is the self-similarity index of $x(t)$. For a large class of Gaussian processes, relation (1.1) induces the dual asymptotics

$$P(\tilde{x}(s) \leq 0.0 < s < \tilde{T}) = \exp(-\tilde{\theta}_x \tilde{T}(1 + o(1))), \quad \tilde{T} \to \infty$$

with the same exponent $\tilde{\theta}_x = \theta_x$. This is particularly true for the processes $I_H(t)$ and $w_H(t)$ [Molchan 1999, 2008]. The equality $\tilde{\theta}_x = \theta_x$ reduces the original problem to the estimation of $\tilde{\theta}_x$. Non-negativity of the correlation function of $\tilde{x}(s)$, $\tilde{B}_x(s)$, guarantees the existence of the exponent $\tilde{\theta}_x$ [Li and Shao, 2004]. This is the case of $I_H(t)$. The inequality of two correlation functions, $\tilde{B}_1(s) \leq \tilde{B}_2(s)$, $\tilde{B}_1(0) = 1$, implies, by Slepian’s lemma, [Lifshits, 1995], the inverted inequality for the relevant exponents: $\tilde{\theta}_1 \geq \tilde{\theta}_2$.

Therefore, Proposition 1 follows from

**Proposition 2.** Let $\tilde{B}_{I_H}(t)$ and $\tilde{B}_{w_H}(t)$ be the correlation functions of the stationary processes that are dual to $I_H$ and $w_H$, respectively. Then

a) $\tilde{B}_{I_H}(t) \leq \tilde{B}_{I_{-H}}(t)$, $0 < H \leq 0.5$. (2.1)
b) \( \widetilde{B}_{t, \mu} (t) \geq \widetilde{B}_{w_{t, \mu}} (t), \quad 0 < H \leq 0.5, \) \hfill (2.2)

c) \( \widetilde{B}_{t, 1/2} (t) \leq \widetilde{B}_{t, \mu} (t) \leq \widetilde{B}_{t, 1/2} (2(1 - H)t), \quad 0.5 < H < 1, \) \hfill (2.3)

d) \( \widetilde{B}_{t, \mu} (t) \geq \widetilde{B}_{t, 1/2} (2t\sqrt{(1 - H^2)/3}), \quad 0.25 \leq H \leq 0.5. \) \hfill (2.4)

The relationship between Propositions 1 and 2.

Assume that Proposition 2 holds, then

1) (2.1) entails \( \tilde{\theta}_{t, \mu} \geq \tilde{\theta}_{t, w_{t, \mu}} \) for \( 0 < H \leq 0.5 \), which supports Proposition 1(a);  

2) (2.2) and (1.2) entail \( \tilde{\theta}_{t, \mu} \geq \tilde{\theta}_{w_{t, \mu}} = H \) for \( 0 < H \leq 0.5 \), which supports Proposition 1(b,right) for the case \( 0 < H \leq 0.5 \);  

3) (2.3, right) and (1.4) entail \( \tilde{\theta}_{t, \mu} \geq p(H)/4 \) with \( p(H) = 2(1 - H) \) for \( 0.5 < H < 1 \). Here we use the obvious fact that the exponent for \( \tilde{x}(pt) \) is \( p\tilde{\theta}_x \). Using the inequality \( \tilde{\theta}_{t, \mu} \geq \tilde{\theta}_{w_{t, \mu}} \) for \( 0 < H \leq 0.5 \), we shall also have \( \tilde{\theta}_{t, \mu} \geq p(1 - H)/4 = H/2 \) for \( H \leq 0.5 \). Finally, one has \( \tilde{\theta}_{t, \mu} \geq 0.5H \wedge (1 - H) \), which supports Proposition 1(b,left);  

4) (2.3, left) and (1.4) entail \( \tilde{\theta}_{t, \mu} \leq \tilde{\theta}_{t, 1/2} = 1/4 \) for \( 0.5 < H < 1 \), which supports Proposition 1(c) for \( 0.5 < H < 1 \).  

5) (2.4) and (1.4) entail \( \tilde{\theta}_{t, \mu} \leq p(H)/4 \) with \( p(H) = 2\sqrt{(1 - H^2)/3} \), which supports Proposition 1(c) for \( 0.25 \leq H \leq 0.5 \). This estimate is trivial for the interval \( 0 \leq H \leq 0.25 \) because \( \tilde{\theta}_{t, \mu} \leq H \) according to Proposition 1(b).  

6) It remains to show that \( \theta_{t, \mu} \leq 1 - H \) (Proposition 1(b)). This fact obviously follows from (1.3), because \( \theta_{t, \mu} (\Delta_T^x) \leq \theta_{t, \mu} (\Delta_T^x) = 1 - H \) for \( \Delta_T^x = (0, T) \subseteq \Delta_T^x = (T, T) \). In addition, \( \theta_{t, \mu} = \tilde{\theta}_{t, \mu} \).

3. Proof of Proposition 2.

Below we use the dual processes to the integrated fractional Brownian motion \( I_{\mu} (t) \) and to the fractional Brownian motion \( w_{\mu} (t) \). They are stationary and have the following correlation functions:

\[
\widetilde{B}_{t, \mu} (t) = (2 + 4H)^{-1}[(4 + 4H)\cosh(2Ht) - 2\cosh((1 + H)t) + (2\sinh(t/2))^{2H+2}],
\]

\[
\widetilde{B}_{t, 1/2} (t) = 1/2(3\exp(-|t|/2) - \exp(-3|t|/2)),
\]

\[
\widetilde{B}_{w_{\mu}} (t) = \cosh(tH) - (2\sinh(t/2))^{2H}/2.
\]
Proof of the relation (2.1): $\widetilde{B}_{I_{n}}(t) \leq \widetilde{B}_{I_{n}-i}(t), \quad 0 < H \leq 0.5$.

Notation: $x = \exp(-t), \alpha = 1 - 2H; \bar{x} = 1 - x$.

By (3.1),

$$2(4 - \alpha^2)x^{3/2}(\widetilde{B}_{I_{n}}(t) - \widetilde{B}_{I_{n}-i}(t)) = U(x, \alpha) - U(x,-\alpha) := \Delta(x, \alpha) \quad (3.4)$$

where

$$U(x, \alpha) = (2 + \alpha)[\bar{x}^{3-\alpha} - 1 + (3 - \alpha)x + (3 - \alpha)x^{2-\alpha} - x^{3-\alpha}]x^{\alpha/2}$$

We have to show that $\Delta(x, \alpha) \leq 0$ on $S = [0,1]^2$.

Step 1: $0 \leq x \leq 0.5, 0 \leq \alpha \leq 1$.

By the Taylor expansions,

$$\bar{x}^{3-\alpha} = 1 - (3 - \alpha)x + (3 - \alpha)(2 - \alpha)x^2 / 2 - (3 - \alpha)(2 - \alpha)(1 - \alpha)x^3 / 6 - r(x|\alpha)x^{\alpha} / 6, \quad (3.5)$$

where

$$r(x|\alpha) = (3 - \alpha)(2 - \alpha)(1 - \alpha)x^2 \int_0^x(1 - u)^3(1 - ux)^{1-\alpha} du .$$

Since $(1 - ux)^q \geq 1$ for $q < 0$,

$$r(x|\alpha) \geq (3 - \alpha)(2 - \alpha)(1 - \alpha)x^2 / 4, \quad |\alpha| < 1 \quad (3.6)$$

and

$$\Delta(x, \alpha) = -\alpha \bar{x}^3 [f(x|\alpha) + R(x|\alpha)] / 6, \quad (3.7)$$

where

$$f(x|\alpha) = 3(2 - \alpha)(1 + \alpha)x^{\alpha} - (2 - \alpha)(5 + 2\alpha - \alpha^2)x^{1+\alpha} + 3(2 + \alpha)(1 - \alpha) - (2 + \alpha)(5 - 2\alpha - \alpha^2)x \quad (3.8)$$

and

$$R(x|\alpha) = (2 + \alpha)r(x|\alpha) + (2 - \alpha)r(x|\alpha). \quad (3.9)$$

Due to (3.6), we have

$$R(x|\alpha) \geq (4 - \alpha^2)(3 + \alpha^2)x^2 / 2 \quad (3.10)$$

Therefore
Now we are going to show that $\tilde{f}(x|\alpha) \geq 0$.

- The function $x \to f(x|\alpha)$ is concave, and $f(0|\alpha) > 0$. In addition,

$$2f(0.5|\alpha) = (2 - \alpha)(1 + 4\alpha + \alpha^2)2^{-\alpha} + (2 + \alpha)(1 - 4\alpha + \alpha^2)$$

is decreasing because

$$2f''(0.5|\alpha) = -(7 - 4\alpha - 3\alpha^2)(1 - 2^{-\alpha}) - (2 - \alpha)(1 + 4\alpha + \alpha^2)2^{-\alpha} \ln 2 - 8\alpha \leq 0$$

Hence,

$$f(0.5|\alpha) \geq f(0.5|0.65) = 0.34 \geq 0 \text{ for } \alpha \in [0,0.65].$$

As a result,

$$\tilde{f} \geq f \geq 0 \text{ for } (x, \alpha) \in [0,0.5] \times [0,0.65].$$

- Assume that $\alpha > 0.65$. In this case the function $x \to \tilde{f}(x|\alpha)$ is concave. Indeed, the expression

$$\varphi(x, \alpha) := -x^{-2\alpha}(2 - \alpha)^{-1} \tilde{f}_x^\alpha = \alpha(1 + \alpha)[3(1 - \alpha) + (5 + 2\alpha - \alpha^2)x] - (2 + \alpha)(3 + \alpha^2)x^{2-\alpha}$$

is nonnegative because $\varphi(0, \alpha) \geq 0$, $x \to \varphi(x, \alpha)$ is concave, and $\varphi(0.5, \alpha) \geq 0$. Only the last property is not obvious and needs to be verified.

Using here and below the symbol $\beta = 1 - \alpha$, one has

$$2\beta^{-1}\varphi(0.5, \alpha) = 4 - 19\beta + 10\beta^2 - \beta^3 + (3 - \beta)(4 - 2\beta + \beta^2)u(\beta), \quad (3.11)$$

where

$$u(\beta) = (1 - 2^{-\beta})/\beta \geq u(0.35) \geq 0.61 \quad 0 \leq \beta \leq 0.35.$$ 

Hence we can continue (3.11) as follows

$$\geq 11.32 - 25.1\beta + 13.05\beta^2 - 1.61\beta^3 \geq 11.32 - 25.1\beta \geq 0 \quad 0 \leq \beta \leq 0.35.$$ 

Consequently, the function $\tilde{f}(x|\alpha)$ is concave.

- Since $\tilde{f}(0|\alpha) > 0$, we will have $\tilde{f}(x|\alpha) \geq 0$ for $0 \leq x \leq 0.5$ if $\tilde{f}(0.5|\alpha) > 0$. One has

$$8\tilde{f}(0.5|\alpha) = 2(2 - \alpha)(1 + 4\alpha + \alpha^2)2^\beta + 4(2 + \alpha)(1 - 4\alpha - \alpha^2) + (4 - \alpha^2)(3 - \alpha^2).$$

Due to $2^\beta \geq 1$, one has

$$[f(x|\alpha) + R(x|\alpha)] \geq \tilde{f}(x|\alpha) = f(x|\alpha) + (4 - \alpha^2)(3 + \alpha^2)x^2 / 2$$
The proof of the case \(0 \leq x \leq 0.5\) is complete.

**Step 2:** \(0.5 \leq x \leq 1, 0 \leq \alpha \leq 0.6\).

**Notation:** \(\beta = 1 - \alpha, \ y = 1 - x\).

By (3.4),

\[
\hat{f}(x|\alpha) = -(1 - x)^{-1}x^{\alpha/2}\Delta(x, \alpha)
\]

\[
= [\alpha^2 + \alpha(2 - \alpha)y + (2 - \alpha)y^2 - (2 + \alpha)y^{2-\alpha}](1 - y)^\alpha
\]

\[
= -\alpha^2 + \alpha(2 + \alpha)y - (2 + \alpha)y^2 + (2 - \alpha)y^{2+\alpha}.
\] (3.12)

Define two functions:

\[
\hat{R}(y|\alpha) = [1 - (1 - y)^\alpha]/y = \alpha \sum_{k \geq 0} \frac{(\beta)_k}{(2)_k} y^k \geq 0,
\] (3.13)

where \((\beta)_k = \beta(\beta + 1)\cdots(\beta + k - 1)\), and

\[
A(y|\alpha) = (2 + \alpha)y^{2-\alpha} - (2 - \alpha)y^2.
\] (3.14)

We may rewrite (3.12) as follows

\[
y^{-1} \hat{f}(x|\alpha) = 4\alpha - [(2 + \alpha)y^{1-\alpha} - (2 - \alpha)y^{1+\alpha}]_{(1)} - 2\alpha y
\]

\[
[ -\alpha^2 - \alpha(2 - \alpha)y + A(y|\alpha)]_{(2)} \hat{R}(y|\alpha)
\] (3.15)

where the subscripts at the square brackets are used to number the grouped summands.

- Consider the term \(\ldots_{(1)} = \psi_1(y|\alpha)\). It is concave, \(\psi_1(0|\alpha) = 0\), and \(\psi_1(1|\alpha) = 2\alpha\). Hence \(y \to \psi_1(y|\alpha)\) increases up to the critical point (the extreme point of the function):

\[
y^* = \left[\frac{\beta(2 + \alpha)(1 - \alpha)}{(2 - \alpha)(1 + \alpha)}\right]^{1/(2\alpha)}.
\]

If \(u(\alpha) := (2 - \alpha)(1 - \alpha)(1 - 4^{-\alpha}) - 2\alpha > 0\), then \(y^* \geq 0.5\) and

\[
\ldots_{(1)} = \psi_1(y|\alpha) \leq \psi(0.5|\alpha) \quad \text{for} \quad 0 \leq y \leq 0.5.
\] (3.16)
The function \( u(\alpha) \) is concave, \( u(0) = 0 \), and \( u(0.6) = 0.065 > 0 \).

Hence (3.16) holds for \( 0 \leq \alpha \leq 0.6 \).

- Consider the second term \( [\ldots]_{2} = \psi_{2}(y|\alpha) \) in (3.15):

By (3.14),
\[
A(y|\alpha) = (2 - \alpha)(y^{2-\alpha} - y^{2}) + 2\alpha y^{2-\alpha} \geq 2\alpha y^{2-\alpha}.
\]
Therefore,
\[
\psi_{2}(y|\alpha) \geq -\alpha[\alpha + (2 - \alpha)y - 2y^{2-\alpha}] = -\alpha \varphi(y|\alpha).
\]
The function \( \varphi(y|\alpha) \) is concave on \( 0 \leq y \leq 1 \), and \( \varphi(0|\alpha) = \varphi(1|\alpha) = 0 \). Therefore \( \varphi(y|\alpha) \geq 0 \).

The critical point of \( y \to \varphi(y|\alpha) \) is \( y^{*} = 2^{-1/(1-\alpha)} \leq 1/2 \). Hence,
\[
[\ldots]_{2} = \psi_{2}(y|\alpha) \geq -\alpha \varphi(y^{*}|\alpha) = -\alpha[\alpha + (1 - \alpha)2^{-1/(1-\alpha)}].
\]

(3.17)

- We now return to (3.15). Using (3.16), (3.17), and the simple relations
\[
2\alpha y \leq \alpha \quad \text{for} \quad y \leq 0.5 \quad \text{and} \quad \hat{R}(y|\alpha) \leq \hat{R}(0.5|\alpha),
\]
we obtain \((\alpha y)^{-1} \hat{f}(x|\alpha) \geq 3 - w(\alpha)\), where
\[
w(\alpha) = [(2 + \alpha)2^{\alpha} - (2 - \alpha)2^{-\alpha}](2\alpha)^{-1} + 2(\alpha + (1 - \alpha)2^{-1/(1-\alpha)})(1 - 2^{-\alpha})
\]
\[
= 2\alpha^{-1}\sinh(\alpha \ln 2) + \cosh(\alpha \ln 2) + 2(1 - 2^{-\alpha})(1 - e^{-\lambda})/\lambda, \quad \lambda = (1 - \alpha)^{-1}.
\]
We can see that \( w(\alpha) \) is an increasing function. Therefore,
\[
(\alpha(1 - x))^{-1} \hat{f}(x|\alpha) \geq 3 - w(\alpha) \geq 3 - w(0.6) = 0.03 > 0, \quad (x, \alpha) \in [0.5,1] \times [0,0.6].
\]
In other words, see (3.12), \( \Delta(x, \alpha) \leq 0 \) for \( (x, \alpha) \in [0.5,1] \times [0,0.6] \).

**Step 3:** \( 0.6 \leq \alpha \leq x \leq 1 \).

Regroup the elements in (3.15) as follows
\[
y^{-1} \hat{f}(x|\alpha) = \alpha y(1 - y^{\beta}) - (\hat{R} - \alpha) - [(2 + \alpha)\beta^{2} + (3 - \beta^{2})\alpha y - \alpha(2 + \alpha)y^{1+\beta}]_{(1)}
\]
\[
\quad + [(\beta - y + \beta(1 - \beta) + \beta^{2}y + (2 + \alpha)y^{1+\beta})(\hat{R}(y|\alpha) - \alpha)]_{(2)}.
\]

(3.18)

- By (3.13),
\[
\hat{R} - \alpha \geq \alpha \beta y / 2 \quad , \quad \alpha + \beta = 1 = x + y
\]  
\hspace{1cm} (3.19)

Consider
\[
\Phi(y|\beta) = \frac{\hat{R} - \alpha}{\alpha \beta y} = \sum_{k \geq 0} \frac{(\beta + 1)_k}{(2)_{k+1}} y^k \geq 0.
\]

This function is increasing in both arguments. Hence
\[
\frac{\hat{R} - \alpha}{\alpha \beta y} \leq \Phi(y_0|\beta_0) = \frac{1 - x_0^\alpha - \alpha_0 y_0}{\alpha_0 \beta_0 y_0}, \quad 0 \leq y \leq y_0, \quad 0 \leq \beta \leq \beta_0 .
\]  
\hspace{1cm} (3.20)

In particular
\[
\hat{R} - \alpha \leq \tilde{k} \beta y, \quad \tilde{k} = 0.625, 0 \leq y, \beta \leq 0.4.
\]  
\hspace{1cm} (3.21)

\bullet Consider (3.18). We have \([\ldots]_{21} \geq 0\) as a consequence of \(\beta \geq y\) and (3.19). Using (3.21), one has
\[
(2 + \alpha)^{-1} r^{-1} f(x|\alpha) \geq 1 - [(1 - \alpha y)^{\beta} + \beta^2 + (2 + \alpha)^{-1}(\tilde{k} \beta + 3 - \beta^2) \alpha y] := \varphi(y|\beta).
\]

It is easy to show that \(\varphi(y|\beta)\) is decreasing on \(0 \leq y \leq 1\). Hence, \(\varphi(y|\beta) \geq \varphi(\beta|\beta)\) for \(0 \leq y \leq \beta\).

The non-negativity of \(\varphi(\beta|\beta)\) is equivalent to the relation
\[
u(\beta) := [(3 - 4 \beta + 3 \beta^2 + (1 - \beta) \beta^2] \nu(\beta) - (\tilde{k} + 4) + (\tilde{k} + 3) \beta - \beta^2 \geq 0 .
\]  
\hspace{1cm} (3.22)

Here \(\nu(\beta) = (1 - \beta^2)/(\beta^2)\) is convex on \(0 \leq \beta \leq 1\) because
\[
\beta^4 \nu^*(\beta) = 6 - \beta^6 [(\beta \ln \beta + \beta - 2)^2 - \beta - 2] \geq 8 + \beta - (\beta \ln \beta + \beta - 2)^2 \geq 8 + \beta - (2 + e^{-3}) \geq 0 .
\]

Therefore,
\[
\nu(\beta) \geq \nu(0.4) + \nu^*(0.4)(\beta - 0.4) = 5.898 - 9.952 \beta .
\]

Using this inequality, we can continue (3.22) as follows
\[
u(\beta) \geq 13.069 - 49.823 \beta + 56.502 \beta^2 - 29.856 \beta^3
\]
\[
\geq 13 - 50 \beta + 26 \beta^2 \geq 0 .
\]

Here we use the \(\tilde{k}\) -value from (3.21). The final inequality is obvious for \(0 \leq \beta \leq 0.4\).

Hence \(\hat{f}(x|\alpha) \geq 0\). The proof of the case \(0.6 \leq \alpha \leq x \leq 1\) is complete.

**Step 4:** \(0.5 \leq x \leq \alpha, \alpha \geq 0.6\) or \(0.5 \leq y \geq \beta, \beta \leq 0.4\).

Regroup the elements in (3.18) again
\[ y^{-1} \hat{f}(x|\alpha) = (2 + \alpha)(1 - y^\beta) x^\alpha + \beta (1 + \alpha \hat{R} - \alpha) + (1 + \alpha) \beta^2 y - (1 - 2y)(\hat{R} - \alpha) - \beta[(2 + \alpha) \beta + y] . \]  

(3.23)

- Due to (3.19), (3.20), we have

\[ \hat{R} - \alpha = K\alpha \beta y, \ K \in [k, \overline{k}], \]

where

\[ \overline{k} = \Phi(0.5 | 0.4) = 0.681 \text{ for } 0 \leq \beta \leq 0.4, 0 \leq y \leq 0.5, \]

\[ k = \Phi(y_0 | \beta_0) \quad \text{for} \quad \beta \geq \beta_0, y \geq y_0 . \]  

(3.24)

Therefore

\[ y^{-1} \hat{f}(x|\alpha) \geq [(2 + \alpha)(1 - y^\beta) x^\alpha]_{(1)} - [((1 - 2y)k + 1) y\beta + (2 + \alpha) \beta^2]_{(2)}, \]

\[ m \beta^2 y \quad \text{where} \]

\[ m = (1 + \alpha \alpha k + \alpha + 1 . \]

Note that

\[ m \geq m = 5k / 8 + 1.5 \quad \text{for} \quad \alpha \geq 0.5, x \geq 0.5 . \]

The term \([...]_{(1)} \) in (3.25) is a decreasing function of \( y \). The same is true for \((-1)[...]_{(2)} \) in the \( y \)-interval \( I = [0, (k + 1) / 4k] \). We have \( k \leq 1 \) in the case under consideration. Therefore, \( I \supset [0,0.5] \).

To estimate the third term in (3.25), recall that \( \beta \leq y \) and \( k \geq \Phi(0|0) = 0.5 \). Hence

\[ m \beta^2 y \geq m \beta^4 , \quad m = 5 / 16 + 1.5 = 1.8125 . \]

As a result,

\[ y^{-1} \hat{f}(x|\alpha) \geq \varphi(y_0|\beta) , \quad \beta \leq 0.4, \beta \leq y \leq y_0 \leq 0.5 , \]  

(3.26)

where

\[ \varphi(y|\beta) = [((2 + \alpha)(1 - y^\beta) x^\alpha]_{(1)} - [((1 - 2y)k + 1) y\beta + (2 + \alpha) \beta^2]_{(2)} + m \beta^2 y \]  

(3.27)

- For \( y_0 = 0.5 \), we have \( \varphi(y_0|\beta) \geq 0 \) if

\[ \psi(\beta) := (3 - \beta) \left( (2\beta - 1) / \beta - 2\beta \right) - 1 + 2m \beta^2 \geq 0 . \]  

(3.28)

But


\[
(2^\beta - 1) / \beta \geq \ln 2 + \beta (\ln 2)^2 / 2.
\]

Therefore,

\[
\psi(\beta) \geq 1.079 - 5.97 \beta + 5.38 \beta^2 \geq 0, \quad 0 \leq \beta \leq 0.2.
\]

Hence, \( \hat{f} \geq 0 \) for \( \beta \leq 0.2, \beta \leq y \leq 0.5 \).

- For \( y_0 = 0.4 \), \( \varphi(y_0 | \beta) \geq 0 \) if

\[
\beta^{-1}(1 - y_0^\beta) x_0^\alpha \geq \beta + (3 - \beta)^{-1}(a - m\beta^2), \quad x_0 = 1 - y_0,
\]

where \( a = (1 - 2y_0 \bar{k} + 1)y_0 = 0.4545, \quad \bar{k} = \Phi(0.5 | 0.4) = 0.681, \quad m = 1.8125 \).

To prove (3.29), note that the left part of (3.29) can be represented as

\[
\beta^{-1}(1 - y_0^\beta) x_0^\alpha = x_0 \int_{\min(1/y_0)}^{\min(1/\bar{k})} e^{x^\alpha} dt := u(\beta).
\]

Since \( u(\beta) \) is convex, \( u(\beta) \geq u(0) + u'(0)\beta \) and (3.29) will be true if

\[
u(0) + u'(0)\beta \geq \beta + (3 - \beta)^{-1}(a - m\beta^2),
\]

or

\[
1.649 - 1.463\beta + 1.783\beta^2 \geq 0.
\]

The last relation is obviously true for \( \beta \leq 0.4 \).

Consequently, we have proved the case: \( \beta \leq y \leq 0.4, \quad \beta \leq 0.4 \).

- Consider the last case: \( 0.4 \leq y \leq 0.5, 0.2 \leq \beta \leq 0.4 \). Then \( \bar{k} = \Phi(0.4 | 0.2) = 0.6039 \) and

\[
m = (1 + \alpha\alpha)\alpha\bar{k} + \alpha + 1 \geq 1.9925 =: \underline{m}. \quad \text{We need to verify (3.28) for } 0.2 \leq \beta \leq 0.4 \text{ using the new parameter } \underline{m}. \text{ This can be done as above.}
\]

**Proof of Relation (2.2):**

\[
\widetilde{B}_{t_0}(H) \geq \widetilde{B}_{x_0}^{-\alpha}(H), \quad 0 < H \leq 0.5.
\]

To prove this relation, consider

\[
\Delta(t, H) = (2 + 2H) \exp(-(1 + H)(\widetilde{B}_{t_0}(H) - \widetilde{B}_{x_0}^{-\alpha}(H))).
\]

Using (3.1, 3.3) and the notation: \( x = \exp(-t), \alpha = 2H; \quad x = 1 - x \), we can rewrite our problem as follows:

\[
\widetilde{\Delta}(x, \alpha) = x^{2+\alpha} - 1 + (2 + \alpha)x + (2 + \alpha)x^{1+\alpha} - x^{2+\alpha}
\]

\[
+ (1 + \alpha)x^{2-\alpha} x^\alpha - (1 + \alpha)x^\alpha - (1 + \alpha)x^{2} \geq 0, \quad 0 \leq \alpha, x \leq 1.
\]
One has
\[ \tilde{\Delta}(x, \alpha) / \bar{x} = x^{1-\alpha} - 1 + (1 + \alpha)x - x^\alpha(\bar{x}^\alpha + \alpha) + (1 + \alpha)x^{1-\alpha}x^\alpha \]
\[ = (1 - \bar{x}^\alpha) x^{1-\alpha} + \alpha x^\alpha(\bar{x}^{1-\alpha} - \bar{x}) + (1 - x^\alpha)(1 - (1 - \alpha)x - \bar{x}^{1-\alpha}). \]

Obviously
\[ (1 - (1 - \alpha)x - \bar{x}^{1-\alpha}) = x^2 \int_0^1 (1 - v)(1 - vx)^{-\alpha-1} dv > 0. \]

Therefore \( \Delta(t, H) \) is nonnegative.

**Proof of Relation (2.3, left):**
\[ \tilde{B}_{_{2/1}} \leq \tilde{B}_{_{1/2}} (t), \quad 0.5 < H < 1, \]

**Notation:** \( x = \exp(-t); \alpha = 2H - 1, 0 \leq \alpha \leq 1. \)

Consider
\[ \Delta(t, H) = (2 + 4H)[\tilde{B}_{_{1/2}} (t) - \tilde{B}_{_{2/1}} (t)] \exp(-(1 + H)t). \]

Due to (3.1, 3.2), the problem looks in terms of the new variables \((x, \alpha) \in S = (0,1) \times (0,1)\) as follows:
\[ \tilde{\Delta}(x, \alpha) = (1 - x)^{\alpha+3} - 1 + (3 + \alpha)x + (3 + \alpha)x^{\alpha+2} - x^{\alpha+3} - (\alpha + 2)(3 - x)x^{2+\alpha/2} \geq 0. \]

Since \( \tilde{\Delta}(x,0) = 0 \), we shall have
\[ \tilde{\Delta}(x, \alpha) = \tilde{\Delta}(x, \alpha) - \tilde{\Delta}(x,0) \]
\[ = (1 - x)^2[(1 - x)^{\alpha+1} - (1 - x) + \alpha x^{\alpha+1/2} + \alpha(1 - x^{\alpha+2})(1 - x^{1+\alpha/2}) + (3 - x)x^2(1 - x^{\alpha/2})^2]. \]

Using the inequality
\[ (1 - x)^{\alpha+1} - (1 - x) \geq -\alpha x + (1 + \alpha)x\alpha^2 / 2, \]
we can continue
\[ \tilde{\Delta}(x, \alpha) \geq (1 - x)^2\alpha x(-1 + (1 + \alpha)x / 2 + x^{\alpha/2}) + \alpha(1 - x^{\alpha+2})(1 - x^{1+\alpha/2}) \]
\[ = \alpha x^2[(1 + \alpha)(1 - x)^2 / 2 + (1 - x^{\alpha+2})(2 - x - x^{\alpha+2})] \geq 0. \]

**Proof of Relation (2.3, right):**
\[ \tilde{B}_{_{1/2}} (t) \leq \tilde{B}_{_{1/2}} (pt), \quad H \geq 1/2, \quad p = 2(1 - H). \]

**Notation:** \( x = \exp(-t); \alpha = 2H - 1, 0 \leq \alpha \leq 1; \beta = 1 - \alpha, y = 1 - x. \)
Consider
\[ \Delta(t, H) = (2 + 4H) \left[ \widetilde{B}_{t_n} (t) - \widetilde{B}_{t_{1/2}} (2(1 - H)t) \right] \exp(- (1 + H)t). \]
Due to (3.1, 3.2), the problem looks in terms of the new variables \((x, \alpha) \in S = (0,1) \times (0,1)\) as follows:
\[ \widetilde{\Delta}(x, \alpha) = (3 + \alpha)(x + x^{\alpha+2}) - 1 - x^{\alpha+3} + (1 - x)^{\alpha+3} - 3(\alpha + 2)x^2 + (\alpha + 2)x^{3-\alpha} \geq 0. \]
By simple algebra one gets
\[ -(xy)^{-2} \widetilde{\Delta}(x, \alpha) = [(2 + \alpha) \hat{R}(y | \beta) + 1] \hat{R}(y | \alpha) - I(x | \alpha), \]
where
\[ \hat{R}(y | \alpha) = [1 - (1 - y)^{\alpha}] / y = \alpha \sum_{k=0} \frac{(\beta)_k}{(2)_k} y^k \geq 0, \]
\[ I(y | \alpha) = [y^{1+\alpha} - 1 + (1 + \alpha)x]x^{-2} = \alpha(1 + \alpha) \int_0^y (1 - v)(1 - xv)^{\alpha-1} dv \] (3.30)
Since \(\hat{R}(y | \alpha) \geq \alpha\) (see 3.13)), we get
\[ -\alpha^{-1} (xy)^{-2} \widetilde{\Delta}(x, \alpha) \geq [(2 + \alpha) \beta + 1] - I(y | \alpha) / \alpha. \] (3.31)
The integral representation of \(I(y | \alpha)\) leads to the following relation
\[ I(y | \alpha) \leq 0.5\alpha(1 + \alpha)(1 - x_0)^{\alpha-1}, 0 \leq x \leq x_0 \]
and, as a result, to the conclusion:
the right part of (3.31) is non-negative for \(0 \leq x \leq x_0 = 5/6\), if
\[ 6^{-\alpha} \leq (1 + \alpha)^{-1} - \alpha / 3. \] (3.32)
Using the piecewise linear approximation of the convex function \(6^{-\alpha}\) with interpolation between the points 0, 0.2, 0.5, and 1, we can easily verify (3.32). This proves the case \(0 \leq x \leq 5 / 6, 0 \leq \alpha \leq 1\).
Consider the case \(5 / 6 \leq x \leq 1\). By (3.30),
\[ I(y | \alpha) = [y^{1+\alpha} - (1 + \alpha)y + \alpha]x^{-2} \leq \alpha(6 / 5)^2. \]
Hence
\[ -\alpha^{-1} (xy)^{-2} \widetilde{\Delta}(x, \alpha) \geq [(2 + \alpha) \beta + 1] - 36 / 25 \geq 0, 0 \leq \alpha \leq 0.845. \]
Suppose now that \(\alpha \geq 0.84\). Combining (3.31) with (3.30), one has
\[ -\beta^{-1} y^{-2} \widetilde{\Delta}(x, \alpha) \geq x^2(\alpha(2 + \alpha) - 1) + 1 - [y^2(y^{-\beta} - 1)\beta^{-1} + y],_{(i)}. \] (3.33)
The function \( u(y, \beta) := \left[ \ldots \right]_{(i)} \) is increasing as a function of two variables. This property for the variable \( x \) follows from the inequality:

\[
u'_y(y, \beta) = 2(y^{1-\beta} - y)\beta^{-1} + (1 - y^{1-\beta}) \geq 0.
\]

Hence

\[ u(y, \beta) \leq u(1/6, 0.16) = 0.90975 < 1, \]

i.e., (3.33) is positive for \( \alpha \geq 0.84, x \geq 5/6 \). The proof is complete.

**Proof of relation (2.4):**

\[
\tilde{B}_{1\nu}(t) \geq \tilde{B}_{1\nu}^\nu ptBtB_{1\nu}^\nu, \quad 1/4 \leq H \leq 1/2, \quad p = 2\sqrt{(1 - H^2)/3}.
\]

Let

\[
\Delta(t, H) = (2 + 4H)(\tilde{B}_{1\nu}(t) - \tilde{B}_{1\nu}^\nu ptBtB_{1\nu}^\nu)e^{-(t+H)^\nu}.
\]

As a function of \( x = \exp(-t) \) and \( \alpha = 2H \), it is

\[
\tilde{\Delta}(x, \alpha) = (2 + \alpha)(x + x^{\alpha+1}) - 1 - x^{\alpha+2} + (1 - x)^{\alpha+2}
- 3(\alpha + 1)x^{1+(\alpha+p)/2} + (\alpha + 1)x^{1+(\alpha+3p)/2}.
\] (3.34)

We need to verify that \( \tilde{\Delta}(x, \alpha) \geq 0, (x, \alpha) \in S = (0,1) \times (1/2,1) \).

It is easy to check that

\[
\tilde{\Delta}(1, \alpha) = \partial/\partial x \tilde{\Delta}(1, \alpha) = 0, \quad (\partial/\partial x)^2 \tilde{\Delta}(1, \alpha) = (1 + \alpha)(\alpha^2 + 3p^2 - 4).
\]

Hence, given \( \alpha^2 + 3p^2 = 4 \),

\[
\tilde{\Delta}(x, \alpha) = \tilde{\Delta}(x, \alpha) - \partial/\partial x \tilde{\Delta}(1, \alpha)(x - 1) - (\partial/\partial x)^2 \tilde{\Delta}(1, \alpha)(x - 1)^2 / 2.
\] (3.35)

Using notation \( y = 1 - x \) and the relation

\[
(1 - y)^\gamma = 1 - \gamma y + \gamma(\gamma - 1)y^2/2 + \gamma(\gamma - 1)(\gamma - 2)y^3 r(x, \alpha),
\] (3.36)

where

\[
r(y, \alpha) = 1/2 \cdot \int_0^1 (1 - u)^\gamma (1 - uy)^{\gamma-3} du
\]

and

\[
1/6 \leq r(y, \alpha) \leq 1/(2\gamma), \quad 0 \leq \gamma \leq 3
\] (3.37)

By (3.34) and (3.35),
\[ \tilde{\Delta}(x, \alpha) = (2 + \alpha)(1 + \alpha)\alpha(1 - \alpha)y^3 r(y, 1 + \alpha) + (2 + \alpha)(1 + \alpha)\alpha y^3 r(y, 2 + \alpha) \\
- 3(1 + \alpha)(1 + (\alpha + p)/2)[(\alpha + p)/2](1 - (\alpha + p)/2)y^3 r(y, 1 + (\alpha + p)/2) \\
+ (1 - y)(1 + \alpha)((\alpha + 3p)/2)((\alpha + 3p)/2 - 1)(2 - (\alpha + 3p)/2)y^3 r(y, (\alpha + 3p)/2) \\
+ y^{2+\alpha} \]

The curve \( \alpha \rightarrow p: \alpha^2 + 3p^2 = 4 \) contains the point \((\alpha, p) = (1,1)\) and lies below the tangent \( \alpha + 3p = 4 \) at that point. Moreover,

\[ 1 < 3p(0)/2 \leq (\alpha + 3p(\alpha))/2 \leq 2, \quad 0 \leq \alpha \leq 1. \]

Hence, the last two summands in (3.38) are non-negative. Neglecting these terms, we will have

\[ (1 - x)^{-3}(1 + \alpha)^{-1}\tilde{\Delta}(x, \alpha) \geq (2 + \alpha)\alpha (1 - \alpha)/6 + (2 + \alpha)\alpha/6 - 3(\alpha + p)(2 - \alpha - p)/8 := V(\alpha). \]

Since

\[ 3(\alpha + p)^2 = 2\alpha^2 + 6\alpha p + 4, \quad \text{and} \quad 3p \leq 4 - \alpha, \]

one has

\[ 6V(\alpha) = (4 - \alpha^2)\alpha - (9/2)(\alpha + p) + (3/2)(\alpha^2 + 3\alpha p + 2) \]

\[ \geq (4 - \alpha^2)\alpha - (3/2)(1 - \alpha)(4 - \alpha) - (9/2)\alpha + (3/2)(\alpha^2 + 2) = (1 - \alpha^2)\alpha + 3(2\alpha - 1) \]

Hence \( \tilde{\Delta}(x, \alpha) \geq 0 \) for \((x, \alpha) \in (0,1) \times (1/2,1)\).
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