# Supplementary Materials

## Weight Function Method for Stress Intensity Factors of Semi-Elliptical Surface Cracks on Functionally Graded Plates Subjected to Non-Uniform Stresses

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### Nomenclature

| Symbol | Description |
|--------|-------------|
| $a$    | crack depth of a semi-elliptical surface crack |
| $c$    | half crack length of a semi-elliptical surface crack |
| $D_1$, $D_2$, $D_3$ | weight function coefficients |
| $D_{D1}$, $D_{D2}$, $D_{D3}$ | weight function coefficients for deepest point |
| $D_{S1}$, $D_{S2}$, $D_{S3}$ | weight function coefficients for surface point |
| $D_{P1}$, $D_{P2}$, $D_{P3}$ | weight function coefficients for general point |
| $E$    | Young’s modulus |
| $E_0$, $E_1$ | Young’s modulus of starting face constituent, ending face constituent |
| $E_{tip}$ | Young’s modulus at the crack tip |
| $E_{up}$ | modified Young’s modulus at the crack tip |
| $F$    | boundary correction factors |
| $F_0$, $F_1$ | boundary correction factors for reference stress intensity factors $K^s_{r1}$ and $K^s_{r2}$ |
| $h$    | half height of functionally graded plate |
| $K$    | stress intensity factor |
| $K_r(a)$ | reference stress intensity factor related to crack length |
| $K^D_{r1}$, $K^D_{r2}$ | reference stress intensity factors of deepest point |
| $K^S_{r1}$, $K^S_{r2}$ | reference stress intensity factors of surface point |
| $K^p_{r1}$, $K^p_{r2}$ | reference stress intensity factors of general point |
| $Q$    | shape factor for an ellipse |
| $t$    | thickness of functionally graded plate |
| $w$    | half width of functionally graded plate |
| $Y_0$, $Y_1$ | boundary correction factors for reference stress intensity factors $K^D_{r1}$ and $K^D_{r2}$ |
| $Z_0$, $Z_1$ | boundary correction factors for reference stress intensity factors $K^D_{r1}$ and $K^D_{r2}$ |
| $\sigma_0$ | nominal or characteristic stress |
| $\sigma(x)$ | local stress distribution normal to the prospective crack face |
| $\nu$  | Poisson’s ratio |
| $\phi$ | parametric angle of an elliptical surface crack |

### 1. Detailed Derivation Process and Explanation

#### 1.1 Detailed Explanation of Equation (15) in the Manuscript.
The relationship between the crack opening displacement \( u(x,a) \) and the weight function \( m(x,a) \) was derived by Rice [5], and it is expressed as follows:

\[
m(x,a) = \frac{E_{\text{yp}}}{K_r(a)} \frac{\partial u(x,a)}{\partial x} \quad \text{(S1)}
\]

Thus, the first derivative of the weight function with respect to \( x \) can be written in the following form.

\[
\frac{\partial m(x,a)}{\partial x} = \frac{E_{\text{yp}}}{K_r(a)} \frac{\partial}{\partial x} \left[ \frac{\partial u(x,a)}{\partial x} \right] \quad \text{(S2)}
\]

The second derivative of the weight function with respect to \( x \) can be written in the following form.

\[
\frac{\partial^2 m(x,a)}{\partial x^2} = \frac{E_{\text{yp}}}{K_r(a)} \frac{\partial}{\partial x} \left[ \frac{\partial^2 u(x,a)}{\partial x^2} \right] \quad \text{(S3)}
\]

The following derivation of the additional condition for a surface crack with depth \( a \) is from the reference [24]. Let us consider displacements \( u \) and \( v \) in the vicinity of the point \((0,0)\). Because the \( x \)-axis is an axis of symmetry, no shear stresses will be acting there.

\[ \tau(x,0) = 0 \quad \text{(S4)} \]

Along the free surface of the plate it holds:

\[ \tau(0,y) = \sigma_y(0,y) = 0 \quad \text{(S5)} \]

For stresses \( \sigma_x \) and \( \tau \) developable by power series as:

\[
\sigma_x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n \quad \tau = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} x^m y^n \quad \text{(S6)}
\]

The following equation is obtained according to conditions (S4) and (S5):

\[ B_{00} = B_{0y} = A_{0y} = 0 \quad \text{(S7)} \]

The relationship between deflections \( u, v \) and shear distortion \( \gamma \) is:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \gamma \quad \text{(S8)} \]

The following equation is obtained by taking the derivative of (S8) with respect to \( x \) and using the strain component \( \varepsilon_x = \partial u / \partial x \).

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial \gamma}{\partial x} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial \gamma}{\partial x} \frac{\partial \varepsilon_x}{\partial y} \quad \text{(S9)}
\]

By using Hooke’s law, we obtain:

\[
\varepsilon_x = \sigma_{\varepsilon_x} + n \sigma_y; \quad \gamma = \tau y / G \quad \text{(S10)}
\]

\[
m = \begin{cases} 1/E & \text{for plane stress} \\ (1-v^2)/E & \text{for plane strain} \end{cases} \quad \text{and} \quad n = \begin{cases} -v/E & \text{for plane stress} \\ -(1+v)/E & \text{for plane strain} \end{cases}
\]
The equilibrium condition gives:

\[ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0 \]  

(S11)

The following expression is obtained from Equations (S9), (S10) and (S11).

\[ \frac{\partial^2 u}{\partial x^2} = \left( \frac{1}{G} + n \right) \frac{\partial \tau}{\partial x} - m \frac{\partial \sigma_y}{\partial y} \]  

(S12)

The second derivative of \( u \) at point \((0, 0)\) is as follows:

\[ \frac{\partial^2 u}{\partial x^2} \bigg|_{x=0} = \left( \frac{1}{G} + n \right) B_{10} - m A_{10} = 0 \]  

(S13)

That is, directly at the surface of the plate, the curvature of the crack contour disappears [24].

Based on Equations (S5), (S6), (S7) and (S12), it can be proved that the curvature of the crack contour at the surface \((x = 0)\) vanishes, as follows.

\[ \frac{\partial^2 u(x,a)}{\partial x^2} \bigg|_{x=0} = 0 \]  

(S14)

Consequently, the second derivative of the weight function at \( x = 0 \) must also be zero. Thus, in the case of an edge or surface crack, Equation (S3) can be written as follows [6,24]:

\[ \frac{\partial^2 m_u(x,a)}{\partial x^2} \bigg|_{x=0} = 0 \]  

(S15)

1.2. Detailed Explanation of Equation (17) in the Manuscript.

The explanation of the sentence “Due to the weight function for the surface point of a semi-elliptical surface crack is derived from the weight function for the embedded penny-shape crack, therefore, the weight function in Equation (11) must vanish at \( x=a \) [25]” in the manuscript is as follows.

The closed form weight function for an embedded circular crack (embedded penny-shape crack) is given [25]:

\[ m_x(a,x,\theta) = \frac{1}{\pi \sqrt{\pi a}} \sqrt{\frac{(a^2 - x^2)}{a^2 + x^2 - 2ax \cos \theta}} \]  

(S16)

Shen et al. [25] derived the weight function for the surface point B of a semi-elliptical surface crack from Equation (S16); it is expressed as follows.

\[ m_u(x,a) = \frac{2}{\sqrt{\pi x}} \left\{ 1 - \sqrt{\frac{x}{a}} \right\} = \frac{2}{\sqrt{\pi x}} \left[ 1 - \left( \frac{x}{a} \right)^{1/2} \right] \]  

(S17)

The weight function for the surface point in the manuscript is given by analogy with the equation (21) in reference [25].

\[ m_x(x,a) = \frac{2}{\sqrt{\pi x}} \left( \frac{x}{a} \right)^{1/2} \left\{ 1 + D_{51} \left( \frac{x}{a} \right) + D_{52} \left( \frac{x}{a} \right)^2 + D_{53} \left( \frac{x}{a} \right)^3 \right\} \]  

(S18)

Since the equation (21) in the reference must satisfy the condition that the weight function is zero at the crack tip \((x = a)\) [25]; therefore, the weight functions in Equations (S17) and (S18) are equal to zero at \( x=a \), leading to:

\[ m_x(x,a) \bigg|_{x=a} = 0 \]  

(S19)
1.3. Detailed Explanation of Equations (19) and (20) in the Manuscript.

The deepest point \((\phi = \pi/2)\) and surface point \((\phi = 0)\) are special cases of general points [26]. Fett et al. [24] pointed out that for a surface crack with depth \(a\), the curvature of the crack contour at the surface \((x = 0)\) vanishes. That is, the condition that the curvature of the crack contour at \(x = 0\) is zero should be satisfied if the general point infinitely approaches the deepest point \((\phi \rightarrow \pi/2)\).

\[
\frac{\partial^2 u(x, a, \phi)}{\partial x^2} \bigg|_{x=0, \phi \rightarrow \pi/2} = 0
\]  

(S20)

Consequently, the second derivative of the weight function of the general point at \(x = 0\) should also be zero.

\[
\frac{\partial^2 m_{p1}(x, a, \phi)}{\partial x^2} \bigg|_{x=0, \phi \rightarrow \pi/2} = 0
\]  

(S21)

Finally, we obtain:

\[
\frac{\partial^2 m_{p1}(x, a)}{\partial x^2} \bigg|_{x=0} = 0
\]  

(S22)

As explained in the previous derivation, the weight function for the surface point must be zero at the crack tip \((x = a)\) [25]. The condition that the weight function of the surface point is zero at \(x = a\) should be satisfied if the general point is infinitely approaching the surface point \((\phi \rightarrow 0)\).

\[
m_{p2}(x, a, \phi) \bigg|_{x=a, \phi \rightarrow 0} = 0
\]  

(S23)

Consequently, the Equation (19) in the manuscript is obtained:

\[
m_{p2}(x, a) \bigg|_{x=a} = 0
\]  

(S24)

Notice: For references cited in supplementary materials, please refer to the corresponding references in the manuscript.

2. Detailed Derivation Process of Equations (41), (42), (43) and (44)

\[
\frac{\partial}{\partial x^2} \left[ \frac{2}{\pi a \sin \phi} \left[ \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{2}} + D_{p1} \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{2}} + D_{p2} \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{2}} \right] \right] \bigg|_{x=0} = 0
\]

\[
= \left[ \frac{2}{\pi a \sin \phi} \left( \frac{3}{4} \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{5}{2}} - \frac{1}{4} D_{p1} \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{5}{2}} - \frac{3}{4} D_{p2} \left(1 - \frac{x}{a \sin \phi} \right)^{\frac{5}{2}} \right) \right] \bigg|_{x=0} = 0
\]

\[
\Rightarrow 3 - D_{p1} + 3D_{p2} = 0
\]  

(41)
\[
\begin{align*}
&\left\{ \frac{2}{\sqrt{\pi a \sin \phi}} \left[ \left( \frac{x}{a \sin \phi} \right)^{\frac{1}{3}} + D_{p3} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{3}} + D_{p4} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{2}{3}} \right] \right\} \\
&= \frac{2}{\sqrt{\pi a \sin \phi}} \left[ \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{1}{3}} + D_{p3} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{1}{3}} + D_{p4} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{2}{3}} \right] = 0
\end{align*}
\]

\[
\Rightarrow 1 + D_{p3} \left( \frac{1}{\sin \phi} - 1 \right) + D_{p4} \left( \frac{1}{\sin \phi} - 1 \right)^2 = 0 \tag{42}
\]

\[
K_v^p = \sigma_0 \int_0^{\sin \phi} \left[ \frac{2}{\sqrt{\pi a \sin \phi}} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{3}} + D_{p3} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{3}} + D_{p4} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{2}{3}} \right] dx
\]

\[
+ \sigma_0 \int_{\sin \phi}^{\pi} \left[ \frac{2}{\sqrt{\pi a \sin \phi}} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{3}} + D_{p3} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{3}} + D_{p4} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{2}{3}} \right] dx
\]

\[
= \sigma_0 \sqrt{\frac{2}{\pi a \sin \phi}} \left[ -2a \sin \phi \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{2}{3}} + \frac{2}{3} a \sin \phi D_{p3} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{3}} \right]_{\sin \phi}^{\pi}
\]

\[
+ \frac{2}{5} a \sin \phi D_{p4} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{2}{3}} \bigg|_{\sin \phi}
\]

\[
= \sigma_0 \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2a \sin \phi + \frac{2}{3} a \sin \phi D_{p3} + \frac{2}{5} a \sin \phi D_{p4} \right]
\]

\[
+ \sigma_0 \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2a \sin \phi \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{1}{3}} + \frac{2}{3} a \sin \phi D_{p3} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{2}{3}} \right]
\]

\[
+ \frac{2}{5} a \sin \phi D_{p4} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{2}{3}} = \sigma_0 \sqrt{\frac{\pi a}{Q}} Z_0
\]

\[
\Rightarrow \left[ 1 + \frac{1}{3} D_{p3} + \frac{1}{5} D_{p4} \right] + \left[ \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{1}{3}} + \frac{1}{3} D_{p3} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{2}{3}} + \frac{1}{5} D_{p4} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{2}{3}} \right] = \pi \sqrt{\frac{1}{8Q \sin \phi} Z_0 = \frac{1}{Q \sin \phi} V_0} \tag{43}
\]
\[ K_{2r}^{a} = \int_{0}^{\sin \phi} \sigma_{i} \left( 1 - \frac{x}{a} \right) \left\{ \frac{2}{\pi a \sin \phi} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} + D_{p_{1}} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{1}{2}} + D_{p_{2}} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} \right\} \, dx \]

\[ + \int_{0}^{\sin \phi} \sigma_{i} \left( 1 - \frac{x}{a} \right) \left\{ \frac{2}{\pi a \sin \phi} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + D_{p_{3}} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + D_{p_{4}} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{3}{2}} \right\} \, dx \]

\[ = \sigma_{i} \left( 1 - \frac{x}{a} \right) \left\{ \frac{2}{\pi a \sin \phi} \left[ 2 \sin \phi \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} - \frac{2}{3} a \sin \phi D_{p_{1}} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} \right] \right\} \left[ \sin \phi \right]_{0}^{\sin \phi} \]

\[ + \frac{\sigma_{i}}{a} \int_{0}^{\sin \phi} \left\{ \frac{2}{\pi a \sin \phi} \left[ -2 \sin \phi \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} - \frac{2}{3} a \sin \phi D_{p_{1}} \left( 1 - \frac{x}{a \sin \phi} \right)^{\frac{3}{2}} \right] \right\} \, dx \]

\[ = \sigma_{i} \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2 \sin \phi + \frac{2}{3} a \sin \phi D_{p_{1}} + \frac{2}{5} a \sin \phi D_{p_{2}} \right] \]

\[ - \frac{\sigma_{i}}{a} \sqrt{\frac{2}{\pi a \sin \phi}} \left[ \frac{4}{3} a^{2} \sin^{2} \phi + \frac{4}{15} a^{2} \sin^{2} \phi D_{p_{1}} + \frac{4}{35} a^{2} \sin^{2} \phi D_{p_{2}} \right] \]

\[ = a \sigma_{i} \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2 \sin \phi + \frac{2}{3} \sin \phi D_{p_{1}} + \frac{2}{5} \sin \phi D_{p_{2}} \right] \]

\[ - a \sigma_{i} \sqrt{\frac{2}{\pi a \sin \phi}} \left[ \frac{4}{3} \sin^{2} \phi + \frac{4}{15} \sin^{2} \phi D_{p_{1}} + \frac{4}{35} \sin^{2} \phi D_{p_{2}} \right] \]

\[ \theta \phi \rho \]
\[ \int_{a \sin \phi}^{b} \sigma_0 \left( 1 - \frac{x}{a} \right) \sqrt{\frac{2}{\pi a \sin \phi}} \left[ \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + D_{p_1} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + D_{p_2} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} \right] \, dx \]

\[ = \sigma_0 \left( 1 - \frac{x}{a} \right) \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2a \sin \phi \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + \frac{2}{3} a \sin \phi D_{p_1} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} \right] \bigg|_{a \sin \phi}^{b} \]

\[ + \sigma_0 \left( 1 - \frac{x}{a} \right) \sqrt{\frac{2}{\pi a \sin \phi}} \left[ 2a \sin \phi \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} + \frac{2}{3} a \sin \phi D_{p_2} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{1}{2}} \right] \bigg|_{a \sin \phi}^{b} \]

\[ = a \sigma_0 \sqrt{\frac{2}{\pi a \sin \phi}} \left[ \frac{4}{3} \sin^2 \phi \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{3}{2}} + \frac{4}{15} \sin^2 \phi D_{p_1} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{3}{2}} \right] \bigg|_{a \sin \phi}^{b} \]

\[ + \frac{4}{35} \sin^2 \phi D_{p_2} \left( \frac{x}{a \sin \phi} - 1 \right)^{\frac{7}{2}} \bigg|_{a \sin \phi}^{b} \]

\[ \Rightarrow \left[ 1 + \frac{1}{3} D_{p_1} + \frac{1}{5} D_{p_2} \right] - \left[ \frac{2}{3} \sin \phi + \frac{2}{15} \sin \phi D_{p_1} + \frac{2}{35} \sin \phi D_{p_2} \right] \]

\[ + \left[ \frac{2}{3} \sin \phi \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{3}{2}} + \frac{2}{15} \sin \phi D_{p_1} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{3}{2}} + \frac{2}{35} \sin \phi D_{p_2} \left( \frac{1}{\sin \phi} - 1 \right)^{\frac{7}{2}} \right] \]

\[ = \pi \sqrt{\frac{1}{8Q \sin \phi}} Z = \sqrt{\frac{1}{Q \sin \phi}} V' \quad (44) \]

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