Calculation of Transport Coefficients of Ultra Cold Normal Dipolar Bose Gas

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Abstract. We derive the Boltzmann equation for relaxation rate, viscosity and thermal conductivity in ultra cold normal dipolar bose gas. We use anisotropic dipole-dipole interaction. We obtain the temperature dependence of the relaxation rate and give formal expressions for the shear viscosity and thermal conductivity coefficients. These coefficients are proportional to $T^2$. The different form of potentials in calculations show themselves in the coefficients of the viscosity and thermal conductivity. The ratio of the viscosity coefficient for dipole-dipole interaction to contact potential is three order, and this ratio for thermal conductivity is two order.

1. Introduction
The interactions between two particles in polarized gases include two parts, a short-range van der waals [1] and a long-rang dipole-dipole interaction [2]. Recent experimental development in trapping and cooling of polar molecules [3,4] show that the dominant interactions are dipole-dipole. Bose gases follow Bose Einestein distribution function. In this paper we explicitly calculate the transport coefficients of a trapped normal dipolar Bose gas. This work appears to be the first time and these transport coefficients have been explicitly obtained.

2. Transition Probability
The transport properties are described by the semi-classical kinetic equation for the distribution function $f(\vec{r}, \vec{p}, t)$ of a Bose gas [5]

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_{\vec{r}} - \vec{\nabla} U(\vec{r}, t) \cdot \nabla_{\vec{p}}\right) f(\vec{r}, \vec{p}, t) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}},$$

(1)

to lowest order we can determine $f$ by demanding that $\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = 0$.

The solution to $\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = 0$ must be of the form

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\[
f(\tilde{r}, \tilde{p}, t) = \langle \exp \beta (\tilde{p} \cdot m^* \tilde{v}(\tilde{r}, t)) \rangle - \mu(\tilde{r}, t) + U(\tilde{r}, t) \rangle - 1 \rangle^{-1}.
\]

Here \( U = U_0 + \overline{U} \), \( U_0 \) is the external potential and \( \overline{U} \) is the average potential on point \( \tilde{r} \) which is induced by the magnetic dipolar in neighbor of it. The value of \( \overline{U} \) is nearly zero. \( m^* \) is effective mass and \( \beta = 1/k_B T \).

The collision integral is

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = \int \frac{d\tilde{p}_1}{(2\pi)^3} \int \frac{d\tilde{p}_2}{(2\pi)^3} \int \frac{d\tilde{p}_3}{(2\pi)^3} W(\Theta, \Phi) \times (2\pi)^3 \delta(\tilde{p}_1 + \tilde{p}_2 - \tilde{p}_3 - \tilde{p}_4) \\
\times (2\pi) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) [(1 + f_i)(1 + f_2)f_3f_4 - f_1f_2f_3f_4],
\]

where \( f_i = f(\tilde{r}_i, \tilde{p}_i, t) \).

\( W(\Theta, \Phi) \) is binary transition probability that defined as follows [5]

\[
W(\Theta, \Phi) = \frac{1}{2} [V(\tilde{p}_1 - \tilde{p}_2) + V(\tilde{p}_1 - \tilde{p}_3)]^2.
\]

It should be noted that above equation are not appropriate for long ranged potential like coulomb potential, \( V(r) = e^2/r \), whereas is suitable with weak, short-ranged potentials like contact and probably dipole-dipole potentials [5]. The range of polarized dipolar potential is between the above mentioned potentials, and approximation (4) can be used for calculation of the transition probability.

If the uniform magnetic field is in the \( z \) direction and all atoms are polarized along this axis, Figure 1, the dipole-dipole interparticle interaction reads

\[
V_\perp(\tilde{r}) = \frac{m^*}{r^3} (1 - 3\cos^2(\Theta)),
\]

Figure 1. View of the dipolar interaction.

The Fourier transform of (5) is

\[
V_\perp(\tilde{q}) = \frac{2m^*}{3\sqrt{\pi}} (3\cos^2(\Theta) - 1),
\]

with \( \Theta \) being the angle between the momentum \( \tilde{q} \) and the \( z \)-axis. Choosing \( \tilde{p}_i \) along the \( z \)-axis, \( \Theta_i \) which makes the angle between \( \tilde{p}_i \) and \( \tilde{p}_1 \) (i = 2, 3, 4), \( \Theta_2 = \Theta \), and \( \Phi \) is the angle between the plane of \( \tilde{p}_1 \) and \( \tilde{p}_2 \), and the plane of \( \tilde{p}_3 \) and \( \tilde{p}_4 \). The relations \( \cos \Theta_3, \cos \Theta_4 \) according to the momentum conservation are [6],

\[
\cos \Theta_3 = \cos^2(\frac{\Theta}{2}) + \sin^2(\frac{\Theta}{2}) \cos \Phi,
\]

(7)

\[
\cos \Theta_4 = \cos^2(\frac{\Theta}{2}) - \sin^2(\frac{\Theta}{2}) \cos \Phi,
\]

(8)

with replacement (6), (7) and (8) in (4), transition probability is obtained in the form below:

\[
W(\Theta, \Phi) = \frac{4m^*}{9\pi} [3(\cos^2(\Theta/2) + \sin^2(\Theta/2) \cos \Phi) - 1].
\]

3. Relaxation Rate

Relaxation rate of Bose gases is [7]
\[ \tau_p = \sum_{\ell=3}^{4} W(\bar{p}_\ell) \delta(\bar{p}_1 + \bar{p}_2 - \bar{p}_3 - \bar{p}_4) \delta(e'_\ell + e'_\ell - e'_\ell - e'_\ell) f_{\ell} \delta(1 + f_{\ell}) (1 + f_{\ell}), \] (10)

the \( \delta \) functions of the above equation have implications on momentum and energy conservation.

With transforming summation to integral relaxation rate is

\[ \tau_p = \frac{m^3}{2(2\pi)^6} \int \frac{W(\Theta, \phi) \sin \theta \frac{d\phi d\phi_d \delta(e'_\ell + e'_\ell - e'_\ell - e'_\ell) f_{\ell} \delta(1 + f_{\ell}) (1 + f_{\ell})}{1 - \exp[-\beta(e_p - \mu + U)]}. \] (11)

With replacement the \( f = [\exp[\beta e - 1]]^{-1} \) where \( e' = e + (\gamma/4) \alpha \), \( \exp(\alpha) = z^{-1}, z = \exp[\beta(\mu - U)] \) and using the Fourier transform of \( \delta \) function \( \delta(\xi) = \int_{-\infty}^{\infty} e^{-i\xi \phi} \frac{d\phi}{2\pi} \) and calculation of the integral over energy, the relation to the relaxation rate for Bose gases is as follows

\[ \tau_p = \frac{m^3}{32\pi^5 h^6} \int \frac{W(\Theta, \phi) \sin \theta \frac{d\phi d\phi_d \delta(e'_\ell + e'_\ell - e'_\ell - e'_\ell) f_{\ell} \delta(1 + f_{\ell}) (1 + f_{\ell})}{1 - \exp[-\beta(e_p - \mu + U)]}}. \] (12)

Relaxation rate for special case in ultra cold normal dipolar Bose gas may be obtained by substituting (9) in above equation:

\[ \tau_p = \frac{m^3}{32\pi^5 h^6} \left[ \frac{(e_p - \mu + U)^2 + 10\pi^2 k^2 T^2}{1 - \exp[-\beta(e_p - \mu + U)]} \right]. \] (13)

4. The transport coefficients for Bose gases

For calculation of the transport coefficients, we can use (1)-(3). The conservation laws are obtained by multiplying (1) by \( 1, \bar{p} \) and \( p' \) and integrating over \( \bar{p} \). In all three cases, the integrals of the collision term in (3) are vanishing. The first approximation for the distribution function is the local equilibrium form

\[ f^{(0)}(\bar{p}, \bar{p}, \tau) = [\exp[\beta(\bar{p} - m^2 \bar{v})^2 - \mu + U - 1^{-1}]. \] (14)

a solution of the Boltzmann equation (1) of the form [8]

\[ f(\bar{p}, \bar{p}, \tau) = f^{(0)}(\bar{p}, \bar{p}, \tau) + f^{(0)}(\bar{p}, \bar{p}, \tau)(1 + f^{(0)}(\bar{p}, \bar{p}, \tau) \psi(\bar{p}, \bar{p}, \tau), \] (15)

shows the deviation of the system from local equilibrium and \( \psi \) expresses a small deviation. The collision integral (3) can reduce the following equation to first order in \( \psi \)

\[ \int \frac{4(m^*)^3 m^2}{45\pi^4} \delta(\bar{p}, \bar{p}, \tau) = \frac{5g_{\gamma}(z)}{2g_{\gamma}(z)} \left[ \frac{m^*}{k_b T} D_{\mu \nu} (\mu \nu \bar{v} - \frac{1}{3} \delta_{\mu \nu} \bar{v}^2) \right] f^{(0)}(1 + f^{(0)}) \psi. \] (16)

by using the lowest-order hydrodynamic equations and (14) the linearized equation for \( \psi \) is [8]

\[ \left[ \frac{\tilde{u} \cdot \hat{\nabla} T}{T} - \frac{5g_{\gamma}(z)}{2g_{\gamma}(z)} \left[ \frac{m^*}{k_b T} D_{\mu \nu} (\mu \nu \bar{v} - \frac{1}{3} \delta_{\mu \nu} \bar{v}^2) \right] f^{(0)}(1 + f^{(0)}) \psi. \] (17)

where the thermal velocity \( \tilde{u} \) is defined by \( m^* \tilde{u} = \bar{p} - m^2 \bar{v} \), \( D_{\mu \nu} \) is the strain tensor,

\[ z(\bar{p}, \tau) = \exp(\beta(\bar{p}, \tau)(\mu(\bar{p}, \tau) - U(\bar{p}, \tau))) \] is the local fugacity and \( g_{\gamma}(z) = \sum_{z=1}^{\infty} \frac{z^\gamma}{\Gamma(z)} \).

We impose the following constraints and have a unique solution for \( \psi \)

\[ \int d\bar{p} f^{(0)}(1 + f^{(0)}) \psi = \int d\bar{p} f^{(0)}(1 + f^{(0)}) \psi = \int d\bar{p} f^{(0)}(1 + f^{(0)}) \psi = 0, \] (18)

the dimensionless velocity variables is introducing by \( \bar{m} / (2k_b T)^{1/2} \tilde{u} = \bar{z} \) and (17) becomes

\[ \int \frac{45\pi^4}{4(m^*)^3 m^2 k_b T^4} \left[ \tilde{u} \cdot \hat{\nabla} T \right] - \frac{5g_{\gamma}(z)}{2g_{\gamma}(z)} \left[ \frac{m^*}{k_b T} D_{\mu \nu} (\mu \nu \bar{v} - \frac{1}{3} \delta_{\mu \nu} \bar{v}^2) \right] f^{(0)}(\bar{z})(1 + f^{(0)}(\bar{z})) = L[\tilde{u}], \] (19)

where \( f^{(0)}(\bar{z}) = (\bar{z}^{-1} e^{\bar{z} - 1} - 1)^{-1} \) and
By using the Uehling method [10] our final expression for the thermal conductivity \( k \) and the shear viscosity \( \eta \) are

\[
k = -\frac{45\pi}{8} \frac{1}{m^3} \int d\xi d\eta d\zeta \frac{2kT}{m^3} \frac{1}{m^3} A(\xi) f^{\text{ro}}(1 + f^{\text{ro}}),
\]

\[
\eta = -\frac{45\pi}{8} \frac{1}{m^3} \int d\xi d\eta d\zeta \frac{2kT}{m^3} \frac{1}{m^3} B(\xi) f^{\text{ro}}(1 + f^{\text{ro}}).
\]

Eqs. (24) and (25) are general expressions for the shear viscosity and heat conductivity. The temperature dependence of fugacity functions \( g_\sigma(z) \) and \( (f_\sigma(z) - 1) \) at low temperatures are proportional to \( e^{-\beta |\phi|^2} \) and \( (e^{-2\beta |\phi|^2}) \) respectively. With the insertion of these functions in (24) and (25) \( k \) and \( \eta \) are proportional to \( T^{\frac{1}{2}} \) at low temperatures. With considering the contact potential the temperature dependency of transition are proportional to \( T^{\frac{1}{2}} \) [8]. The ratio of shear viscosity coefficient for dipole-dipole interaction to contact potential is three order, and this ratio for thermal conductivity is two order. The entropy of excitation particles is usually proportional to \( T \), hence, the extrapolation of experimental data [11] may give \( \eta \propto T \), which is fairly in agreement with our result.

Dobado et al. [12] obtained the viscosity in the hard sphere approximation proportional to \( T^{\frac{1}{2}} \).

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