The Yamabe flow on incomplete manifolds

YUANZHEN SHAO

Abstract. This article is concerned with developing an analytic theory for second-order nonlinear parabolic equations on singular manifolds. Existence and uniqueness of solutions in an \(L^p\)-framework are established by maximal regularity tools. These techniques are applied to the Yamabe flow. It is proven that the Yamabe flow admits a unique local solution within a class of incomplete initial metrics.

1. Introduction

Nowadays, there is a rising interest in the study of differential operators on manifolds with singularities, which is motivated by a variety of applications from applied mathematics, geometry and topology. All the work is more or less related to the seminal paper by Kondrat’ev [26]. Among the tremendous amount of the literature on this topic, I would like to mention two lines of research on the study of differential operators of Fuchs type, which have been introduced independently by Melrose [36,37], Nazaikinskii et al. [38], Schulze [45,46] and Schulze and Seiler [47].

One important direction of research on singular analysis is connected with the so-called \(b\)-calculus and its generalizations on manifolds with cylindrical ends. See [36,37]. Many authors have been very active in this direction.

Research along another line, known as cone differential operators, has also been known for a long time. Operators in this line of research are modeled on conical manifolds. During the recent decade, many mathematicians have applied analytic tools like bounded imaginary powers, \(H^\infty\)-calculus and \(\mathcal{R}\)-sectoriality, c.f. Sect. 4.1 for precise definitions, to study the closed extensions of cone differential operators in Mellin–Sobolev spaces and to investigate many interesting nonlinear parabolic problems on conical manifolds. See for instance [13,42–44]. A comparison between the \(b\)-calculus and the cone algebra can be found in [27].

There has been more recent progress in understanding elliptic operators on manifolds with higher-order singularities, e.g., manifolds with edge ends. The reader may refer to [7,30–33,46–48] for more details. The amount of research on differential

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operators of Fuchs type is enormous, and thus it is literally impossible to list all the work.

Geometric evolution equations by now are a well-established tool in the analysis of geometric and topological problems, and thus forms probably the most important class of differential equations on Riemannian manifolds. In this article, our focus will mainly be the Yamabe flow on incomplete manifolds. As an alternative approach to the Yamabe problem, Hamilton introduced the Yamabe flow, which asks whether one can drive a given metric by this flow conformally to a metric with constant scalar curvature. More precisely, on a Riemannian manifold \((M, g_0)\), the Yamabe flow studies the conformal evolution of metrics driven by the following rule.

\[
\frac{\partial g}{\partial t} = -R_g g \quad \text{on } M_T, \quad g(0) = g_0 \quad \text{on } M, \tag{1.1}
\]

where \(R_g\) is the scalar curvature with respect to the metric \(g\), and \(M_T := M \times [0, T]\). Let \(g = u^{\frac{4}{m-2}} g_0\). This flow is equivalent to the following scalar quasilinear parabolic equation.

\[
\frac{\partial u}{\partial t} = u^{-\frac{4}{m-2}} \Delta g_0 u - c(m) u^{\frac{m-6}{2}} R_{g_0} \quad \text{on } M_T, \quad u(0) = 1_M \quad \text{on } M, \tag{1.2}
\]

where \(c(m) := \frac{m-2}{4(m-1)}\). On a compact closed manifold \((M, g_0)\), the short-time existence of the Yamabe flow is just a consequence of the positivity of the conformal factor \(u\) and the compactness of \((M, g_0)\). Nevertheless, the theory for the Yamabe flow on non-compact manifolds is far from being settled. Even local well-posedness is only established for restricted situations. Its difficulty can be observed from the fact that, losing the compactness of \((M, g_0)\), Eq. (1.2) can exhibit degenerate and singular behaviors simultaneously. The investigation of the Yamabe flow on non-compact manifolds was initiated by Ma and An [28]. Later, conditions on extending local in time solutions were explored in [29]. In [28], the authors showed that for a complete closed non-compact Riemannian manifold \((M, g_0)\) with Ricci curvature bounded from below and with a uniform bound on the scalar curvature in the sense that:

\[
\text{Ric}_{g_0} \geq -K g_0, \quad |R_{g_0}| \leq C,
\]

Equation (1.1) has short-time solution on \(M \times [0, T(g_0)]\) for some \(T(g_0) > 0\). If in addition \(R_{g_0} \leq 0\), then this solution is global. Here \(\text{Ric}_{g_0}\) is the Ricci curvature tensor with respect to \(g_0\). The proof is based on the widely used technique consisting of exhausting \(M\) with a sequence of compact manifolds with boundary and studying the solutions to a sequence of initial boundary value problems. Then uniform estimates of these solutions and their gradients are obtained by means of the maximum principle on manifolds with Ricci curvature bounded from below in the sense given above.

The more challenging case is to study the Yamabe flow on incomplete manifolds. Bahuaud and Vertman started the research in this direction in [8] by considering the Yamabe flow on a compact manifold with asymptotically simple edge singularities. Their proof for short-time existence is based on a careful analysis of the mapping