On the Structure of Conditionally Positive Definite Algebraic Operators

Zenon Jan Jabłoński1 · Il Bong Jung2 · Jan Stochel1

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Abstract
Recently, the authors have introduced and intensively studied a class of bounded Hilbert space operators called conditionally positive definite. Its origins go back to the harmonic analysis on ∗-semigroups, namely to the concept of conditional positive definiteness. Our main aim here is to give a complete description of algebraic conditionally positive definite operators on inner product spaces; we do not assume that the operators under consideration are bounded.

Keywords
Algebraic operator · Conditional positive definiteness · Conditionally positive definite operator · Similarity

Mathematics Subject Classification
Primary 47B20; Secondary 47B90

Dedicated to the memory of our friend Jörg Eschmeier.

Communicated by Mihai Putinar.

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Jan Stochel
Jan.Stochel@im.uj.edu.pl

Zenon Jan Jabłoński
Zenon.Jablonski@im.uj.edu.pl

Il Bong Jung
ibjung@knu.ac.kr

1 Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland
2 Department of Mathematics, Kyungpook National University, Daegu 41566, Korea

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1 Introduction

The concepts of positive and conditional positive definiteness have their origins in stochastic processes [9, 22, 24, 26, 27]. The first concept entered operator theory soon after on the occasion of studying isometries in a group sense and, more generally, subnormal operators in a semigroup sense. One of the main results in this area states that a bounded Hilbert space operator $T$ is subnormal if and only if the sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$ is positive definite on the additive semigroup $\{0, 1, 2, \ldots\}$ for all vectors $h$ (see [1, 16, 23]). The conditional positive definiteness in a semigroup sense appeared in operator theory for the first time in relation to subnormal contractions (see [28]). Later it appeared in the context of complete hyperexpansivity and complete hypercontractivity of finite order [5, 11, 12]. In a recent paper [20] we have provided the fundamentals of the theory of operators called conditionally positive definite (CPD for brevity), that is, bounded Hilbert space operators $T$ for which the sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$ is conditionally positive definite on the additive semigroup $\{0, 1, 2, \ldots\}$ for all vectors $h$. This class of operators contains subnormal operators [14, 17], 2- and 3-isometries [2–4], complete hypercontractions of order 2 [11, 18], certain algebraic operators which are neither subnormal nor $m$-isometric, and much more. Furthermore, the class of CPD weighted shift operators has been characterized in [21]. The aim of the present paper is to provide a complete description of the structure of (not necessarily bounded) CPD algebraic operators on inner product spaces.

Before stating the main result, we recall the necessary concepts.

All vector spaces considered in this paper are complex. Denote by $L(\mathcal{M})$ the algebra of all linear operators on a vector space $\mathcal{M}$ and by $I = I_{\mathcal{M}}$ the identity operator on $\mathcal{M}$. Let $\mathcal{N}(T)$ stand for the kernel of $T \in L(\mathcal{M})$. An operator $T \in L(\mathcal{M})$ is called algebraic if there exists a nonzero polynomial $p$ in one indeterminate over the field of complex numbers such that $p(T) = 0$. An operator $T \in L(\mathcal{M})$ is said to be a nilpotent of index $n$, where $n$ is an integer greater than 1, if $T^{n-1} \neq 0$ and $T^n = 0$.

Let $\mathcal{M}$ be an inner product space. Following [20] we say that an operator $T \in L(\mathcal{M})$ is CPD if

$$\sum_{i,j=0}^{k} \|T^{i+j} h\|^2 \lambda_i \bar{\lambda}_j \geq 0$$

for each $h \in \mathcal{M}$ and all finite sequences $\lambda_0, \ldots, \lambda_k$ of complex numbers such that $\sum_{j=0}^{k} \lambda_j = 0$. Call $T \in L(\mathcal{M})$ an $m$-isometry ($m$ is a positive integer) if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k h\|^2 = 0, \quad h \in \mathcal{M}.$$ 

We write $B(\mathcal{M})$ for the algebra of all bounded linear operators on $\mathcal{M}$.

The following theorem is the main result of this paper. It provides a complete description of the structure of CPD algebraic operators on inner product spaces.
Theorem 1.1 Let $\mathcal{M}$ be an inner product space and $T \in L(\mathcal{M})$. Then the following conditions are equivalent:

(i) $T$ is a CPD algebraic operator,

(ii) there are vector subspaces $\{\mathcal{X}_j,1\}_{j=1}^{k}$, $\{\mathcal{X}_j,2\}_{j=1}^{k}$, $\{\mathcal{Y}_j\}_{j=1}^{l}$, $\{\mathcal{Z}_j\}_{j=1}^{m}$ of $\mathcal{M}$, nilpotent operators $N_j \in L(\mathcal{Z}_j)$ of index 2, $j = 1, \ldots, m$, and distinct complex numbers $x_1,1, \ldots, x_k,1, x_1,2, \ldots, x_k,2, y_1, \ldots, y_l, z_1, \ldots, z_m$ such that

$$
\mathcal{M} = \bigoplus_{j=1}^{k} (\mathcal{X}_j,1 + \mathcal{X}_j,2) \oplus \bigoplus_{j=1}^{l} \mathcal{Y}_j \oplus \bigoplus_{j=1}^{m} \mathcal{Z}_j, \quad (1.1)
$$

$T|_{\mathcal{X}_j,1} = x_j,1 I_{\mathcal{X}_j,1}$ and $T|_{\mathcal{X}_j,2} = x_j,2 I_{\mathcal{X}_j,2}$ for $j = 1, \ldots, k$,

$T|_{\mathcal{Y}_j} = y_j I_{\mathcal{Y}_j}$ for $j = 1, \ldots, l$,

$T|_{\mathcal{Z}_j} = z_j I_{\mathcal{Z}_j} + N_j$ for $j = 1, \ldots, m$,

$x_j,1 x_j,2 = 1$ for $j = 1, \ldots, k$,

$y_i y_j \neq 1$ for $i \neq j$,

$|z_j| = 1$ for $j = 1, \ldots, m$.

Moreover, if (ii) holds, then for all $j$,

$$
\mathcal{X}_j,1 = \mathcal{N}(T - x_j,1 I_{\mathcal{M}}), \quad \mathcal{X}_j,2 = \mathcal{N}(T - x_j,2 I_{\mathcal{M}}),
$$

$$
\mathcal{Y}_j = \mathcal{N}(T - y_j I_{\mathcal{M}}), \quad \mathcal{Z}_j = \mathcal{N}((T - z_j I_{\mathcal{M}})^2).
$$

In particular, if $T \in B(\mathcal{M})$, then each of the spaces $\mathcal{X}_j,1$, $\mathcal{X}_j,2$, $\mathcal{Y}_j$ and $\mathcal{Z}_j$ is closed.

The structure of bounded CPD algebraic operators on a Hilbert space with spectral radius not exceeding 1 is given below. Comparing with Theorem 1.1, the decomposition (1.2) below does not contain the component that corresponds to the space $\bigoplus_{j=1}^{k} (\mathcal{X}_j,1 + \mathcal{X}_j,2)$ appearing in (1.1).

Theorem 1.2 Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Then the following conditions are equivalent:

(i) $T$ is a CPD algebraic operator with spectral radius $r(T) \leq 1$,

(ii) $T$ has an orthogonal decomposition (some of the summands may be absent)

$$
T = (z_1 I_{\mathcal{H}_1} + N_1) \oplus \cdots \oplus (z_n I_{\mathcal{H}_n} + N_n) \quad (1.2)
$$

relative to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, where

$$
\{z_j\}_{j=1}^{n} \text{ are distinct complex numbers,} \quad (1.3)
$$

$$
N_j \in B(\mathcal{H}_j) \text{ for } j = 1, \ldots, n, \quad (1.4)
$$

1 Some components of the form $\bigoplus_{j=1}^{k} (\mathcal{X}_j,1 + \mathcal{X}_j,2)$, $\bigoplus_{j=1}^{l} \mathcal{Y}_j$ and $\bigoplus_{j=1}^{m} \mathcal{Z}_j$ may be absent; if the first component is nonzero, then both spaces $\mathcal{X}_j,1$ and $\mathcal{X}_j,2$ must be nonzero (a similar rule applies to the other two components). The symbols $+$ and $\oplus$ are reserved to denote the algebraic direct sum and the orthogonal direct sum of vector spaces, respectively.
if $N_j \neq 0$, then $N_j^2 = 0$ and $|z_j| = 1$,

\[|z_j| \leq 1 \text{ for } j = 1, \ldots, n,\]

(iii) $T$ has an orthogonal decomposition $T = A \oplus V$, where $A$ is an algebraic normal contraction and $V$ is an algebraic 3-isometry (again some of the summands may be absent).

Moreover, if (i) holds and $r(T) < 1$, then $T$ is a diagonal operator, that is, the nilpotent summands in the decomposition (1.2) are absent.

Another consequence of Theorem 1.1 is the following result, which is closely related to Theorem 1.2.

Proposition 1.3 Suppose that $K$ is a Hilbert space and $S \in B(K)$ is a CPD algebraic operator. Then $S$ is similar to an operator $T \in B(H)$ of the form (1.2) which satisfies the conditions (1.3)–(1.5); the operator $T$ takes the form $T = A \oplus V$, where $A$ is an algebraic normal operator and $V$ is an algebraic 3-isometry (some of the summands may be absent).

The proofs of Theorems 1.1 and 1.2 and Proposition 1.3 are given in Sect. 4. Sects. 2 and 3 prepare the ground for the proofs of these results.

2 Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. Set $\mathbb{R}_+ = \{t \in \mathbb{R}: t \geq 0\}$ and $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$. Write $\mathbb{Z}_+$ and $\mathbb{N}$ for the sets of nonnegative and positive integers, respectively. Denote by $\mathcal{B}(\Omega)$ the $\sigma$-algebra of Borel subsets of a topological space $\Omega$. For $z \in K$, where $K \in \{\mathbb{R}, \mathbb{C}\}$, the symbol $\delta_z$ stands for the Borel probability measure on $K$ concentrated on $\{z\}$. The closed support of a complex Borel measure $\rho$ on $K$ is denoted by $\text{supp}(\rho)$. As usual, the symbol $\mathbb{C}[X]$ stands for the ring of polynomials in indeterminate $X$ with complex coefficients. The derivative of $p \in \mathbb{C}[X]$ is denoted by $p'$. If $p \in \mathbb{C}[X]$ is a polynomial of degree at least 1, then there is a unique polynomial $q \in \mathbb{C}[X]$ such that

$$(2.1) \quad q(x) = \frac{p(x) - p(0)}{x} \text{ for } x \in \mathbb{R} \setminus \{0\} \text{ and } q(0) = p'(0).$$

In what follows, the rational function $\frac{p(x) - p(0)}{x}$ is identified with the polynomial $q$.

The following result of classical nature will be used in the proof of Lemma 3.8.

Lemma 2.1 Let $b, w \in \mathbb{C}$ be such that $|w| = 1$ and $w \neq \pm 1$. Assume that the sequence $(\text{Re } (w^n b))_{n=0}^{\infty}$ is convergent. Then $b = 0$.

Proof Suppose, to the contrary, that $b \neq 0$.

Consider first the case when $w^m \neq 1$ for all $m \in \mathbb{N}$. Then, by Jacobi’s theorem (see [15, Theorem I.3.13]), the sequence $(w^n)_{n=0}^{\infty}$ is dense in the unit circle $\mathbb{T}$. Hence,
there exist two subsequences \( \{w^{k_n}\}_{n=0}^\infty \) and \( \{w^{l_n}\}_{n=0}^\infty \) of the sequence \( \{w^n\}_{n=0}^\infty \) such that

\[
\lim_{n \to \infty} w^{k_n} = \frac{\bar{b}}{|b|} \quad \text{and} \quad \lim_{n \to \infty} w^{l_n} = -\frac{\bar{b}}{|b|}.
\]

As a consequence, we have

\[|b| = \lim_{n \to \infty} \Re (w^{k_n} b) = \lim_{n \to \infty} \Re (w^{l_n} b) = -|b|,
\]

which contradicts \( b \neq 0 \).

Suppose now that \( w^m = 1 \) for some \( m \in \mathbb{N} \). By our assumption on \( w \), \( m \) must be greater than or equal to 3. It is easily seen that \( \Re (b), \Re (wb), \ldots, \Re (w^{m-1} b) \) are the accumulation points of the sequence \( \{\Re (w^n b)\}_{n=0}^\infty \). Hence, we have

\[\Re (b) = \Re (w^j b), \quad j \in \mathbb{N}.
\]

This implies that \( b(1 - w^j) = \bar{b}(\bar{w}^j - 1) \) for \( j \in \mathbb{N} \). Since \( b \neq 0 \) and \( w \neq \pm 1 \), we get

\[
\frac{1 - w}{1 - \bar{w}} = -\frac{\bar{b}}{b} = \frac{1 - w^2}{1 - \bar{w}^2} = \frac{(1 - w)(1 + w)}{(1 - \bar{w})(1 + \bar{w})},
\]

and so \( w = \bar{\bar{w}} \). As \( |w| = 1 \), this yields \( w = \pm 1 \), which is a contradiction. \( \square \)

Let \( \gamma = \{\gamma_n\}_{n=0}^\infty \) be a sequence of complex numbers. Define the sequence \( \triangle \gamma \) of complex numbers by

\[(\triangle \gamma)_n = \gamma_{n+1} - \gamma_n, \quad n \in \mathbb{Z}_+,
\]

and write \( \triangle^2 \gamma := \triangle (\triangle \gamma) \). In this paper we need the Leibnitz formula for discrete differentiation of products of sequences

\[
(\triangle (\alpha \cdot \beta))_n = (\triangle \alpha)_n \beta_{n+1} + \alpha_n (\triangle \beta)_n, \quad n \in \mathbb{Z}_+,
\]  \hspace{1cm} (2.2)

where \( \alpha = \{\alpha_n\}_{n=0}^\infty \) and \( \beta = \{\beta_n\}_{n=0}^\infty \) are sequences of complex numbers and \( (\alpha \cdot \beta)_n = \alpha_n \beta_n \) for \( n \in \mathbb{Z}_+ \).

Let \( \gamma = \{\gamma_n\}_{n=0}^\infty \) be a sequence of real numbers. We say that \( \gamma \) is positive definite (PD for brevity) if for all finite sequences \( \lambda_0, \ldots, \lambda_k \in \mathbb{C} \),

\[
\sum_{i, j=0}^k \gamma_{i+j} \lambda_i \bar{\lambda}_j \geq 0.
\]  \hspace{1cm} (2.3)

If (2.3) holds for all finite sequences \( \lambda_0, \ldots, \lambda_k \in \mathbb{C} \) such that \( \sum_{j=0}^k \lambda_j = 0 \), then we call \( \gamma \) conditionally positive definite (CPD for brevity). Using this terminology, one
can rephrase the notion of a CPD operator as follows: an operator $T \in L(M)$ on an inner product space $M$ is CPD if and only if the sequence $\{ \| T^n h \|_2 \}_{n=0}^\infty$ is CPD for every $h \in M$. The following fact is a consequence of [20, Proposition 2.2.9].

If $\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty$, then $\gamma$ is CPD if and only if $\Delta^2 \gamma$ is PD. \hfill (2.4)

The CPD sequences of exponential growth have the following integral representation.

Theorem 2.2 (see [20, Theorem 2.2.5]) Let $\gamma = \{ \gamma_n \}_{n=0}^\infty$ be a sequence of real numbers. Then the following conditions are equivalent:

(i) $\gamma$ is CPD and $\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty$,

(ii) there exist $b \in \mathbb{R}$, $c \in \mathbb{R}_+$ and a finite Borel measure $\nu$ on $\mathbb{R}$ with compact support such that $\nu(\{1\}) = 0$ and

$$
\gamma_n = \gamma_0 + bn + cn^2 + \int_{\mathbb{R}} Q_n(x) d\nu(x), \quad n \in \mathbb{Z}_+,
$$

where $Q_n$ is the polynomial given by

$$
Q_n(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{R} \text{ and } n = 0, 1, \\
\sum_{j=0}^{n-2} (n-j-1)x^j & \text{if } x \in \mathbb{R} \text{ and } n \geq 2.
\end{cases}
$$

Moreover, the triplet $(b, c, \nu)$ satisfying (ii) is unique.

If $\gamma = \{ \gamma_n \}_{n=0}^\infty$ is a CPD sequence such that $\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty$ and $(b, c, \nu)$ satisfies Theorem 2.2(ii), then $(b, c, \nu)$ is called the representing triplet of $\gamma$.

The polynomials $Q_n$ defined in (2.6) have the following property:

$$
(\Delta^2 Q_n(x))_n = x^n, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R},
$$

where $\Delta^2 Q_n(x)$ denotes the action of the transformation $\Delta^2$ on the sequence $\{Q_n(x)\}_{n=0}^\infty$ (see [20, Lemma 2.2.1]).

Let $\gamma = \{ \gamma_n \}_{n=0}^\infty$ be a sequence of real numbers. We say that $\gamma_n$ is a polynomial in $n$ of degree $k$ if there exists a polynomial $p \in \mathbb{C}[X]$ of degree $k$ such that $\gamma_n = p(n)$ for all $n \in \mathbb{Z}_+$. By the Fundamental Theorem of Algebra such $p$ is unique and its coefficients are real. We say that $\gamma$ is a Stieltjes moment sequence if there exists a positive Borel measure $\mu$ on $\mathbb{R}_+$, called a representing measure of $\gamma$, such that

$$
\gamma_n = \int_{\mathbb{R}_+} t^n d\mu(t), \quad n \in \mathbb{Z}_+.
$$

It is clear that any Stieltjes moment sequence is PD. The celebrated Stieltjes moment theorem states that $\gamma$ is a Stieltjes moment sequence if and only if the sequences $\gamma$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are PD (see [7, Theorem 6.2.5]). The following result is a counterpart of the Stieltjes moment theorem for CPD sequences.
Theorem 2.3  Suppose that $\gamma = \{\gamma_n\}_{n=0}^\infty$ is a CPD sequence such that

$$\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty.$$ 

Let $(b, c, \nu)$ be the representing triplet of $\gamma$. Then the following conditions are equivalent:

(i) $\text{supp}(\nu) \subseteq \mathbb{R}_+$,

(ii) $\{\gamma_{n+1}\}_{n=0}^\infty$ is CPD.

Proof  Note that $\limsup_{n \to \infty} |\gamma_{n+1}|^{1/n} < \infty$ and

$$(\Delta^2 \gamma^{(1)})_n = (\Delta^2 \gamma)_{n+1}, \quad n \in \mathbb{Z}_+,$$ 

where $\gamma^{(1)} := \{\gamma_{n+1}\}_{n=0}^\infty$. This combined with [20, Proposition 2.2.9] completes the proof.

The following fact is a basic characterization of algebraic operators that will be needed in this paper (see e.g., [13, Section 6]).

Lemma 2.4  Let $\mathcal{M}$ be a vector space and $T \in L(\mathcal{M})$. Then the following conditions are equivalent:

(i) $T$ is algebraic,

(ii) there exist positive integers $\iota_1, \ldots, \iota_n$, distinct complex numbers $w_1, \ldots, w_n$ and nonzero vector subspaces $\mathcal{M}_1, \ldots, \mathcal{M}_n$ of $\mathcal{M}$ such that

(ii-a) $\mathcal{M} = \mathcal{M}_1 + \cdots + \mathcal{M}_n$,

(ii-b) $T|_{\mathcal{M}_j} = w_j I_{\mathcal{M}_j} + N_j$ with $N_j \in L(\mathcal{M}_j)$ such that $N_j^{\iota_j} = 0$ for all $j = 1, \ldots, n$.

Moreover, if (ii) holds, then $\mathcal{M}_j = \mathcal{N}((T - w_j I_{\mathcal{M}})^{\iota_j})$ for $j = 1, \ldots, n$ and

$$\mathcal{N}((T - w_{j_1} I_{\mathcal{M}})^{\iota_{j_1}} \cdots (T - w_{j_s} I_{\mathcal{M}})^{\iota_{j_s}}) = \mathcal{M}_{j_1} + \cdots + \mathcal{M}_{j_s}$$ (2.8)

for all finite sequences of integers $1 \leq j_1 < \cdots < j_s \leq n$. In particular, if $\mathcal{M}$ is an inner product space and $T \in B(\mathcal{M})$, then each space $\mathcal{M}_j$ is closed.

3 Preparatory Lemmas

The proof of the main result of this paper will be preceded by a series of lemmas.

Lemma 3.1  Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of complex numbers such that

$$\gamma_n = \int_C z^n d\rho(z), \quad n \in \mathbb{Z}_+,$$ (3.1)

where $\rho$ is a complex Borel measure on $\mathbb{C}$ with finite $\text{supp}(\rho)$. Then the following two statements are equivalent:
(i) $\gamma$ is PD, 
(ii) $\text{supp}(\rho) \subseteq \mathbb{R}$ and $\rho$ is a positive measure.

**Proof** (i)$\Rightarrow$(ii) By the Hamburger theorem (see [7, Theorem 6.2.2]), there exists a finite positive Borel measure $\mu$ on $\mathbb{R}$ such that 
\[
\gamma_n = \int_{\mathbb{R}} t^n d\mu(t), \quad n \in \mathbb{Z}_+.
\]
Then, by (3.1), we have 
\[
\int_{\mathbb{R}} p(t) d\mu(t) = \int_{\mathbb{C}} p(z) d\rho(z), \quad p \in \mathbb{C}[X]. \tag{3.2}
\]
Consider the polynomial $p \in \mathbb{C}[X]$ given by 
\[
p(z) = \prod_{w \in \text{supp}(\rho)} (z - w)(z - \overline{w}), \quad z \in \mathbb{C}.
\]
Setting $p^{-1}([0]) = \{\xi \in \mathbb{C}: p(\xi) = 0\}$, we see that 
\[
 p^{-1}([0]) = \text{supp}(\rho) \cup \{\overline{w}: w \in \text{supp}(\rho)\}, \tag{3.3}
\]
\[
p(t) = \prod_{w \in \text{supp}(\rho)} |t - w|^2, \quad t \in \mathbb{R}.
\]
Substituting this polynomial into (3.2), we deduce from (3.3) that 
\[
\int_{\mathbb{R}} \prod_{w \in \text{supp}(\rho)} |t - w|^2 d\mu(t) = \int_{\text{supp}(\rho)} p(z) d\rho(z) = 0.
\]
This implies that 
\[
\text{supp}(\mu) \subseteq \left\{ t \in \mathbb{R}: \prod_{w \in \text{supp}(\rho)} |t - w|^2 = 0 \right\} \subseteq \mathbb{R} \cap \text{supp}(\rho). \tag{3.4}
\]
We claim that $\text{supp}(\rho) \subseteq \mathbb{R}$. Suppose, to the contrary, that there exists $z_0 \in \text{supp}(\rho) \setminus \mathbb{R}$. Let $p \in \mathbb{C}[X]$ be the Lagrange interpolating polynomial such that 
$p^{-1}([0]) = \text{supp}(\rho) \setminus \{z_0\}$ and $p(z_0) = 1$. Substituting this polynomial into (3.2), we obtain 
\[
0 = \int_{\mathbb{R}} p(t) d\mu(t) = \int_{\text{supp}(\rho)} p(z) d\rho(z) = \rho([z_0]).
\]
Since $\text{supp}(\rho)$ is finite, this contradicts $z_0 \in \text{supp}(\rho)$.

Summarizing, we have proved that $\text{supp}(\rho) \subseteq \mathbb{R}$. Combined with (3.4) and [10, Lemma 4.1], this implies that $\rho(\Delta) = \mu(\Delta \cap \mathbb{R})$ for all $\Delta \in \mathcal{B}(\mathbb{C})$, which yields (ii).
(ii)⇒(i) This implication is easily seen to be true. □

It turns out that the implication (i)⇒(ii) of Lemma 3.1 is no longer true if supp(ρ) is infinite, even if it is a compact subset of \( \mathbb{C} \). Note that the converse implication is always true.

**Example 3.2** Let \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) be the sequence given by

\[
\gamma_n = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{if } n \geq 1.
\end{cases}
\]

It is clear that \( \gamma \) is a Stieltjes moment sequence with a representing measure \( \delta_0 \) and consequently it is PD. Define the Borel probability measure \( \rho \) on \( \mathbb{C} \) by

\[
\rho(\Delta) = \frac{1}{2\pi} \int_0^{2\pi} \chi_\Delta(e^{it}) dt, \quad \Delta \in \mathcal{B}(\mathbb{C}),
\]

where \( \chi_\Delta \) stands for the characteristic function of \( \Delta \). Then a standard measure-theoretic argument gives

\[
\int_{\mathbb{C}} f d\rho = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt, \quad f \in L^1(\rho).
\] (3.5)

It is a matter of routine to verify that

\[
\text{supp}(\rho) = \mathbb{T}.
\] (3.6)

It follows from (3.5) that

\[
\int_{\mathbb{C}} z^n d\rho(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{int} dt = \gamma_n, \quad n \in \mathbb{Z}_+,
\]

so the measure \( \rho \) satisfies (3.1). However, by (3.6), \( \rho \) is not supported in \( \mathbb{R} \). This means that the implication (i)⇒(ii) of Lemma 3.1 does not hold in this case. ◊

**Lemma 3.3** Suppose that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are vector subspaces of an inner product space \( \mathcal{M} \) and \( z_1, z_2 \) are distinct complex numbers. Then the following conditions are equivalent:

(i) the sequence \( \{\|z_1^n h_1 + z_2^n h_2\|^2\}_{n=0}^{\infty} \) is CPD for all \( (h_1, h_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \),

(ii) either \( z_1 \bar{z}_2 = 1 \), or \( z_1 \bar{z}_2 \neq 1 \) and \( \mathcal{M}_1 \perp \mathcal{M}_2 \).

**Proof** Fix temporarily \( (h_1, h_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \) and define \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) by

\[
\gamma_n = \|z_1^n h_1 + z_2^n h_2\|^2, \quad n \in \mathbb{Z}_+.
\]

Clearly \( \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \). Since

\[
\gamma_n = |z_1|^{2n} \|h_1\|^2 + 2\text{Re}(\langle z_1 \bar{z}_2 \rangle^n \langle h_1, h_2 \rangle) + |z_2|^{2n} \|h_2\|^2, \quad n \in \mathbb{Z}_+,
\]
we deduce that
\[
(\Delta^2 \gamma)_n = (|z_1|^2 - 1)^2 \|h_1\|^2 |z_1|^{2n} + (|z_2|^2 - 1)^2 \|h_2\|^2 |z_2|^{2n} \\
+ (z_1 \bar{z}_2 - 1)^2 \langle h_1, h_2 \rangle (z_1 \bar{z}_2)^n \\
+ (z_2 \bar{z}_1 - 1)^2 \langle h_2, h_1 \rangle (z_2 \bar{z}_1)^n, \quad n \in \mathbb{Z}_+.
\]  
(3.7)

It follows from (3.7) that
\[
(\Delta^2 \gamma)_n = \int_{\mathbb{C}} z^n d\rho(z), \quad n \in \mathbb{Z}_+,
\]  
(3.8)

where \( \rho \) is the complex Borel measure on \( \mathbb{C} \) defined by
\[
\rho = (|z_1|^2 - 1)^2 \|h_1\|^2 \delta_{|z_1|^2} + (|z_2|^2 - 1)^2 \|h_2\|^2 \delta_{|z_2|^2} \\
+ (z_1 \bar{z}_2 - 1)^2 \langle h_1, h_2 \rangle \delta_{z_1 \bar{z}_2} \\
+ (z_2 \bar{z}_1 - 1)^2 \langle h_2, h_1 \rangle \delta_{z_2 \bar{z}_1}.
\]  
(3.9)

Combining (3.8) and (3.9) with (2.4) and Lemma 3.1, we get the following.

\( \gamma \) is CPD if and only if supp(\( \rho \)) \( \subseteq \mathbb{R} \) and \( \rho \) is a positive measure.  
(3.10)

If \( z_1 \bar{z}_2 = 1 \), then, in view of (3.9), we have
\[
\rho = (|z_1|^2 - 1)^2 \|h_1\|^2 \delta_{|z_1|^2} + (|z_2|^2 - 1)^2 \|h_2\|^2 \delta_{|z_2|^2},
\]
which together with (3.10) implies that the sequence \( \gamma \) is CPD. Therefore, we can assume that \( z_1 \bar{z}_2 \neq 1 \). We will consider three cases.

**Case 1.** \( z_1 \bar{z}_2 \notin \mathbb{R} \).

Observe that \( z_2 \bar{z}_1 \notin \mathbb{R} \) and \( z_1 \bar{z}_2 \neq z_2 \bar{z}_1 \). Combined with (3.9) and (3.10), this implies that \( \gamma \) is CPD if and only if \( h_1 \perp h_2 \).

**Case 2.** \( z_1 \bar{z}_2 \in \mathbb{R} \setminus \{0\} \).

Note that \( z_2 \bar{z}_1 \neq 1, z_1 \bar{z}_2 = z_2 \bar{z}_1, z_1 \bar{z}_2 \neq |z_1|^2 \) and \( z_1 \bar{z}_2 \neq |z_2|^2 \). Thus, by (3.9) and (3.10), \( \gamma \) is CPD if and only if \( \text{Re} \langle h_1, h_2 \rangle \geq 0 \). Hence, replacing \( h_1 \) by \( -h_1 \) and then by \( ih_1 \), we conclude that (i) holds if and only if \( \mathcal{M}_1 \perp \mathcal{M}_2 \).

**Case 3.** \( z_1 \bar{z}_2 = 0 \).

Since \( z_1 \neq z_2 \), by symmetry, it suffices to consider the situation where \( z_1 = 0 \) and \( z_2 \neq 0 \). Then by (3.9), we have
\[
\rho = (\|h_1\|^2 + 2\text{Re} \langle h_1, h_2 \rangle) \delta_0 + (|z_2|^2 - 1)^2 \|h_2\|^2 \delta_{|z_2|^2},
\]
which together with (3.10) shows that \( \gamma \) is CPD if and only if
\[
\|h_1\|^2 + 2\text{Re} \langle h_1, h_2 \rangle \geq 0.
\]
Hence, applying the above to $t h_2$ in place of $h_2$ with $t \in \mathbb{R}$ implies that (i) holds if and only if $\text{Re}(h_1, h_2) = 0$ for all $(h_1, h_2) \in \mathcal{M}_1 \times \mathcal{M}_2$. Finally, arguing as in Case 2, we conclude that (i) holds if and only if $\mathcal{M}_1 \perp \mathcal{M}_2$. This completes the proof. \hfill \Box

The multi-vector version of Lemma 3.3 takes the following form.

**Lemma 3.4** Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be vector subspaces of an inner product space $\mathcal{M}$, and $z_1, \ldots, z_k$ be distinct complex numbers. Then the following conditions are equivalent:

(i) the sequence $\{\| \sum_{j=1}^{k} z_j^n h_j \|^2\}_{n=0}^\infty$ is CPD for all $h_1 \in \mathcal{M}_1, \ldots, h_k \in \mathcal{M}_k$.

(ii) $\mathcal{M}_i \perp \mathcal{M}_j$ for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$ and $z_i \bar{z}_j \neq 1$.

**Proof** (i)$\Rightarrow$(ii) Fix $i, j \in \{1, \ldots, k\}$ such that $i \neq j$ and $z_i \bar{z}_j \neq 1$. It follows from (i) that the sequence $\{\| z_i^n h_i + z_j^n h_j \|^2\}_{n=0}^\infty$ is CPD for all $h_i \in \mathcal{M}_i$ and $h_j \in \mathcal{M}_j$. Hence by Lemma 3.3, $\mathcal{M}_i \perp \mathcal{M}_j$. Thus (ii) holds.

(ii)$\Rightarrow$(i) Take $h_j \in \mathcal{M}_j$ for $j = 1, \ldots, k$. Define the sequence $\gamma = \{\gamma_n\}_{n=0}^\infty$ by

$$\gamma_n = \left\| \sum_{j=1}^{k} z_j^n h_j \right\|^2 = \sum_{i, j=1}^{k} (z_i \bar{z}_j)^n \langle h_i, h_j \rangle, \quad n \in \mathbb{Z}_+.$$ 

Then $\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty$ and

$$\Delta^2 \gamma_n = \sum_{i, j=1}^{k} (z_i \bar{z}_j - 1)^2 (z_i \bar{z}_j)^n \langle h_i, h_j \rangle$$

$$= \sum_{i=1}^{k} (|z_i|^2 - 1)^2 |z_i|^{2n} \| h_i \|^2 + \sum_{i \neq j, z_i \bar{z}_j \neq 1} (z_i \bar{z}_j - 1)^2 (z_i \bar{z}_j)^n \langle h_i, h_j \rangle$$

$$= \sum_{i=1}^{k} (|z_i|^2 - 1)^2 |z_i|^{2n} \| h_i \|^2, \quad n \in \mathbb{Z}_+.$$ 

This implies that $\Delta^2 \gamma$ is PD. Applying (2.4) completes the proof. \hfill \Box

We also need an extension of Lemma 3.3.

**Lemma 3.5** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be vector subspaces of an inner product space $\mathcal{M}$, $N_j \in \mathcal{L}(\mathcal{M}_j)$ for $j = 1, 2$, and $z_1, z_2$ be distinct complex numbers. Suppose that the sequence

$$\{\| (z_1 I_{\mathcal{M}_1} + N_1)^n h_1 + (z_2 I_{\mathcal{M}_2} + N_2)^n h_2 \|^2\}_{n=0}^\infty$$

is CPD for all $h_1 \in \mathcal{N}(N_1)$ and $h_2 \in \mathcal{N}(N_2)$. Then the following assertions hold:

(i) if $z_1 \bar{z}_2 \neq 1$, then $\mathcal{N}(N_1) \perp \mathcal{N}(N_2)$,

(ii) if $|z_1|, |z_2| \leq 1$, then $\mathcal{N}(N_1) \perp \mathcal{N}(N_2)$. 

\textbf{Proof} (i) Fix $h_1 \in \mathcal{N}(N_1)$ and $h_2 \in \mathcal{N}(N_2)$. It follows from Newton’s binomial formula that

$$(z_j I + N_j)^n h_j = \sum_{k=0}^{n} \binom{n}{k} z_j^{n-k} N_j^k h_j = z_j^n h_j, \quad n \in \mathbb{Z}_+, \quad j = 1, 2.$$ 

This implies that for all $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\| (z_1 I + N_1)^n \lambda_1 h_1 + (z_2 I + N_2)^n \lambda_2 h_2 \|^2 = \| z_1^n \lambda_1 h_1 + z_2^n \lambda_2 h_2 \|^2, \quad n \in \mathbb{Z}_+.$$ 

Applying Lemma 3.3 to the vector spaces spanned by $h_1$ and $h_2$, we conclude that $h_1 \perp h_2$.

(ii) Note that $|z_1|, |z_2| \leq 1$ implies that $z_1 \bar{z}_2 \neq 1$. Indeed, otherwise $|z_1||z_2| = 1$, so $|z_1| = 1$, and thus $z_1 \bar{z}_1 = 1 = z_1 \bar{z}_2$ and finally $z_1 = z_2$, which is a contradiction. Therefore, (ii) follows from (i).

\hfill \Box

\textbf{Lemma 3.6} Suppose that $\mathcal{M}$ is a nonzero inner product space, $z$ is a complex number and $N \in L(\mathcal{M})$ is a nilpotent of index $k \geq 2$. Then the following conditions are equivalent:

(i) $zI + N$ is CPD,
(ii) $|z| = 1$ and $k = 2$.

\textbf{Proof} (i)$\Rightarrow$(ii) Fix a vector $h \in \mathcal{M}$ and define the sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ by

$$\gamma_n = \| (zI + N)^n h \|^2, \quad n \in \mathbb{Z}_+. \quad (3.11)$$

If $z = 0$, then

$$\gamma_n = 0 \text{ for all integers } n \geq k, \quad (3.12)$$

and consequently $\limsup_{n \to \infty} \gamma_n^{1/n} = 0$. Applying [20, Theorem 2.2.13(iv)] to $\theta = 1$, we see that $\gamma$ is PD. Hence, the following Cauchy-Schwarz inequality holds (cf. [25, p. 19, inequality (6)])

$$|\gamma_n|^2 \leq \gamma_0 \gamma_{2n}, \quad n \in \mathbb{Z}_+. \quad (3.13)$$

Combined with (3.12), this implies that $\gamma_1 = 0$, so $\|Nh\| = 0$. Since $h$ is arbitrarily fixed, $N = 0$, which contradicts $N^{k-1} \neq 0$.

Suppose now that $z \neq 0$. Since $N^{k-1} \neq 0$, we can choose the above $h$ so that $N^{k-1}h \neq 0$. Using Newton’s binomial formula, we obtain (cf. [19, Eq. (3.2)])

$$\gamma_n = \left\| \sum_{j=0}^{k-1} \frac{(n)_j}{j!} z^{n-j} N^j h \right\|^2 = |z|^{2n} \left\| \sum_{j=0}^{k-1} \frac{(n)_j}{j!} z^j N^j h \right\|^2, \quad n \in \mathbb{Z}_+. \quad (3.13)$$
where \((n)_j\) is the polynomial in \(n\) of degree \(j\) given by

\[
(n)_j = \begin{cases} 
1 & \text{if } j = 0 \text{ and } n \in \mathbb{Z}_+, \\
\prod_{i=0}^{j-1} (n - i) & \text{if } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}_+.
\end{cases}
\]

Note that \((n)_j = 0\) for all integers \(j > n \geq 0\). It follows from (3.13) that there are real numbers \(\{a_j\}_{j=0}^{2(k-1)}\) such that

\[
\gamma_n = |z|^{2n} \sum_{j=0}^{2(k-1)} a_j n^j, \quad n \in \mathbb{Z}_+,
\]

(3.14)

where

\[
a_{2(k-1)} = \frac{\|N^{k-1}h\|^2}{((k-1)!|z|^{k-1})^2}.
\]

(3.15)

Clearly, by (3.14), \(\limsup_{n \to \infty} \gamma_n^{1/n} < \infty\).

We now show that \(|z| = 1\). Suppose, to the contrary, that \(|z| \neq 1\). It follows from (3.14) that \(\frac{(\Delta^2 \gamma_n)}{|z|^{2n}}\) is a polynomial in \(n\) with real coefficients of degree \(2(k-1)\) with the leading coefficient given by

\[
(\frac{|z|^2 - 1}{|z|^2})^2 a_{2(k-1)} \overset{(3.15)}{=} \frac{(\frac{|z|^2 - 1}{|z|^2})^2 \|N^{k-1}h\|^2}{((k-1)!|z|^{k-1})^2}.
\]

(3.16)

Since \(\gamma\) is CPD, it follows from (2.4) that the sequence \(\Delta^2 \gamma\) is PD. This in turn implies that the sequence \(\{\frac{(\Delta^2 \gamma_n)}{|z|^{2n}}\}_{n=0}^{\infty}\) is PD. By [10, Lemma 4.7] and the Hamburger theorem (see [7, Theorem 6.2.2]), \(\frac{(\Delta^2 \gamma_n)}{|z|^{2n}}\) is a constant polynomial in \(n\), so by (3.16) we must have \(k = 1\), which contradicts the assumption that \(k \geq 2\). This means that \(|z| = 1\). Therefore, by (3.14) and (3.15), \(\gamma_n\) is a polynomial in \(n\) with real coefficients of degree \(2(k-1)\). This, combined with [20, Proposition 2.2.11] and the assumption that \(k \geq 2\), implies that \(k = 2\).

(ii)\(\Rightarrow\)(i) Suppose that \(|z| = 1\) and \(k = 2\). Let \(h \in \mathcal{M}\) and \(\gamma\) be as in (3.11). It is easy to see that \(\gamma_n\) is a polynomial in \(n\) with real coefficients, which is constant if \(Nh = 0\), and of degree \(2\) with a positive leading coefficient if \(Nh \neq 0\). A routine computation now shows that \(\gamma\) is CPD in both cases. This completes the proof. \(\square\)

Lemma 3.7 Let \(\mathcal{M}_1\) and \(\mathcal{M}_2\) be vector subspaces of an inner product space \(\mathcal{M}\), \(N_2 \in L(\mathcal{M}_2)\) be a nilpotent of index 2, and \(z_1, z_2\) be distinct complex numbers such that \(|z_2| = 1\). Suppose that the sequence

\[
\{\|z_1^n h_1 + (z_2 I_{\mathcal{M}_2} + N_2)^n h_2\|^2\}_{n=0}^{\infty}
\]

(3.17)

is CPD for all \(h_1 \in \mathcal{M}_1\) and \(h_2 \in \mathcal{M}_2\). Then \(\mathcal{M}_1 \perp \mathcal{M}_2\).
Proof Noting that $\alpha := z_1\bar{z}_2 \neq 1$, we deduce from Lemma 3.5 that

$$\mathcal{M}_1 \perp \mathcal{N}(N_2).$$

(3.18)

Set $M_2 = \bar{z}_2N_2$. Fix $h_1 \in \mathcal{M}_1$ and $h_2 \in \mathcal{M}_2$. Denote the sequence (3.17) by $\gamma$. Since $M_2h_2 \in \mathcal{N}(N_2)$, we infer from (3.18) that $\langle h_1, M_2h_2 \rangle = 0$. Combined with Newton’s binomial formula, this yields

$$\gamma_n = \|z_1^n h_1 + z_2^n (I_2 + M_2)^n h_2\|^2$$

$$= \|z_1^n h_1 + z_2^n (I_2 + nM_2)h_2\|^2$$

$$= \|h_1\|^2|\alpha|^{2n} + \|h_2\|^2 + 2nRe\langle h_2, M_2h_2 \rangle$$

$$+ n^2\|M_2h_2\|^2 + \alpha^n \langle h_1, h_2 \rangle + \bar{\alpha}^n \langle h_2, h_1 \rangle, \quad n \in \mathbb{Z}_+.$$  

(3.19)

This implies that $\limsup_{n \to \infty} \gamma_n^{1/n} < \infty$. Let $(b, c, \nu)$ be the representing triplet of $\gamma$. Applying $\Delta^2$ to (2.5) and (3.19) and using (2.7), we get

$$\int_{\mathbb{R}} x^n d\rho(x) = (\Delta^2 \gamma)_n = \|h_1\|^2|\alpha|^{2n} - 1^2 |\alpha|^{2n} + 2\|M_2h_2\|^2$$

$$+ (\alpha - 1)^2 \alpha^n \langle h_1, h_2 \rangle + (\bar{\alpha} - 1)^2 \bar{\alpha}^n \langle h_2, h_1 \rangle, \quad n \in \mathbb{Z}_+,$$

where $\rho$ is the finite positive Borel measure on $\mathbb{R}$ given by $\rho = \nu + 2c\delta_1$. This yields

$$\int_{\mathbb{R}} \rho d\rho = \|h_1\|^2|\alpha|^{2n} - 1^2 p(|\alpha|^2) + 2\|M_2h_2\|^2 p(1)$$

$$+ (\alpha - 1)^2 \langle h_1, h_2 \rangle p(\alpha) + (\bar{\alpha} - 1)^2 \langle h_2, h_1 \rangle p(\bar{\alpha}), \quad p \in \mathbb{C}[X].$$  

(3.20)

Substituting the polynomial $p_0(x) := (x - |\alpha|^2)^2(x-1)^2(x-\alpha)(x-\bar{\alpha})$ into (3.20) and using the fact that $p_0(x) \geq 0$ for all $x \in \mathbb{R}$, we obtain

$$\rho(\{x \in \mathbb{R}: (x - |\alpha|^2)^2(x-1)^2|x-\alpha|^2 > 0\}) = 0.$$  

(3.21)

We now consider two cases.

CASE 1. $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

Since $\alpha \not\in \mathbb{R}$, $|x-\alpha|^2 > 0$ for all $x \in \mathbb{R}$. This together with (3.21) implies that $\text{supp}(\rho) \subseteq \{|z|^2\} \cup \{1\}$, so $\rho = u\delta_{|z|^2} + v\delta_1$ for some $u, v \in \mathbb{R}_+$. Hence,

$$up(|z|^2) + vp(1) \overset{(3.20)}{=} \|h_1\|^2|\alpha|^{2n} - 1^2 p(|\alpha|^2) + 2\|M_2h_2\|^2 p(1)$$

$$+ (\alpha - 1)^2 \langle h_1, h_2 \rangle p(\alpha) + (\bar{\alpha} - 1)^2 \langle h_2, h_1 \rangle p(\bar{\alpha}), \quad p \in \mathbb{C}[X].$$  

(3.22)

Substituting the polynomial $p_1(x) := (x - |z|^2)(x-1)(x-\bar{\alpha})$ into (3.22), we see that $0 = (\alpha - 1)^2 \langle h_1, h_2 \rangle p_1(\alpha)$. Since $p_1(\alpha) \neq 0$, we conclude that $h_1 \perp h_2$.

CASE 2. $\alpha \in \mathbb{R}$.
It follows from (3.21) that supp(\(\rho\)) \(\subseteq \{|z_1|^2\} \cup \{1\} \cup \{\alpha\}\). Thus, there exist \(u, v, w \in \mathbb{R}_+\) such that \(\rho = u\delta_{|z_1|^2} + v\delta_1 + w\delta_{\alpha}\), so

\[
up(|z_1|^2) + vp(1) + wp(\alpha) \overset{(3.20)}{=} \|h_1\|^2(|z_1|^2 - 1)^2 p(|z_1|^2) + 2\|M_2h_2\|^2p(1) + 2(\alpha - 1)^2p(\alpha)\text{Re}\langle h_1, h_2 \rangle, \quad p \in \mathbb{C}[X].
\]

(3.23)

Consider first the case when \(\alpha = 0\). Substituting the polynomial \(p_2(x) = 1 - x^2\) into (3.23) and observing that \(z_1 = 0\), we get

\[
\|h_1\|^2 + 2\text{Re}\langle h_1, h_2 \rangle = u + w \geq 0.
\]

Arguing as in the proof of Lemma 3.3, Cases 2 and 3, we see that \(h_1 \perp h_2\). Suppose now that \(\alpha \in \mathbb{R} \setminus \{0\}\). Then it is easy to see that \(|z_1|^2 \neq \alpha\). Substituting the polynomial \(p_3(x) = (x - |z_1|^2)(x - 1)\) into (3.23) and noting that \(p_3(\alpha) \neq 0\) (recall that \(\alpha \neq 1\)), we deduce that

\[
2(\alpha - 1)^2\text{Re}\langle h_1, h_2 \rangle = w \geq 0.
\]

The familiar argument yields \(h_1 \perp h_2\). This completes the proof. \(\square\)

**Lemma 3.8** Let \(\mathcal{M}_1\) and \(\mathcal{M}_2\) be vector subspaces of an inner product space \(\mathcal{M}\), \(N_j \in \mathcal{L}(\mathcal{M})\) be a nilpotent of index 2 for \(j = 1, 2\), and \(z_1, z_2\) be distinct complex numbers such that \(|z_j| = 1\) for \(j = 1, 2\). Suppose that the sequence

\[
\{(z_1I_{\mathcal{M}_1} + N_1)^n h_1 + (z_2I_{\mathcal{M}_2} + N_2)^n h_2\}^\infty_{n=0}
\]

(3.24)
is CPD for all \(h_1 \in \mathcal{M}_1\) and \(h_2 \in \mathcal{M}_2\). Then \(\mathcal{M}_1 \perp \mathcal{M}_2\).

**Proof** First observe that \(\alpha := z_1\bar{z}_2 \in \mathbb{C} \setminus \{0, 1\}\) and \(|\alpha| = 1\). Set \(M_j = \bar{z}_jN_j\) for \(j = 1, 2\). Fix \(h_1 \in \mathcal{M}_1\) and \(h_2 \in \mathcal{M}_2\). Let us denote by \(\gamma\) the sequence (3.24). Since \(M_j h_j \in \mathcal{N}(N_j)\) for \(j = 1, 2\), we deduce from Lemma 3.5 that \(\langle M_1 h_1, M_2 h_2 \rangle = 0\). Hence, using Newton’s binomial formula, we get

\[
\gamma_n = \|z_1^n(I_{\mathcal{M}_1} + M_1)^n h_1 + z_2^n(I_{\mathcal{M}_2} + M_2)^n h_2\|^2 \\
= \|h_1\|^2 + 2n\text{Re}\langle h_1, M_1 h_1 \rangle + n^2\|M_1 h_1\|^2 \\
+ \|h_2\|^2 + 2n\text{Re}\langle h_2, M_2 h_2 \rangle + n^2\|M_2 h_2\|^2 \\
+ \alpha^n\langle h_1, h_2 \rangle + \bar{\alpha}^n\langle h_2, h_1 \rangle \\
+ n\alpha^n\langle M_1 h_1, h_2 \rangle + n\bar{\alpha}^n\langle M_2 h_2, h_1 \rangle \\
+ n\alpha^n\langle M_1 h_1, h_2 \rangle + n\bar{\alpha}^n\langle M_2 h_2, M_1 h_1 \rangle, \quad n \in \mathbb{Z}_+.
\]

(3.25)

Clearly, the above identity implies that \(\limsup_{n \to \infty} \gamma_n^{\frac{1}{n}} < \infty\). Let \((b, c, \nu)\) be the representing triplet of \(\gamma\). Substituting \((z_jI_{\mathcal{M}_j} + N_j)h_j\) in place of \(h_j\) in (3.24), we see that the sequence \(\{\gamma_{n+1}\}^\infty_{n=0}\) is CPD. Hence, by Theorem 2.3, we have

\[
\text{supp}(\nu) \subseteq \mathbb{R}_+.
\]

(3.26)
It follows from (2.5), (3.25) and (3.26) that
\[
2\text{Re}\langle h_1, h_2 \rangle + \tilde{b} n + \tilde{c} n^2 + \int_{\mathbb{R}^+} Q_n d\nu = \langle h_1, h_2 \rangle \alpha^n + \langle h_2, h_1 \rangle \bar{\alpha}^n \\
+ \mathcal{E}_1 n \alpha^n + \mathcal{E}_1 n \bar{\alpha}^n, \quad n \in \mathbb{Z}_+, \quad (3.27)
\]
where
\[
\tilde{b} := b - 2\text{Re}\langle h_1, M_1 h_1 \rangle - 2\text{Re}\langle h_2, M_2 h_2 \rangle, \\
\tilde{c} := c - \|M_1 h_1\|^2 - \|M_2 h_2\|^2, \\
\mathcal{E}_1 := \langle h_1, M_2 h_2 \rangle + \langle M_1 h_1, h_2 \rangle.
\]
Applying \(\triangle^2\) to both sides of (3.27) and using (2.7) and the Leibnitz formula (2.2), we obtain
\[
\int_{\mathbb{R}^+} x^n d\rho(x) = \mathcal{E}_1 (\alpha - 1)^2 (n + 1) \alpha^n + \mathcal{E}_1 (\bar{\alpha} - 1)^2 (n + 1) \bar{\alpha}^n \\
+ \mathcal{E}_2 \alpha^n + \mathcal{E}_2 \bar{\alpha}^n, \quad n \in \mathbb{Z}_+, \quad (3.28)
\]
where
\[
\rho := \nu + 2\tilde{c} \delta_1, \\
\mathcal{E}_2 := \langle h_1, h_2 \rangle (\alpha - 1)^2 + 2\mathcal{E}_1 (\alpha - 1) + \mathcal{E}_1 (\bar{\alpha} - 1)^2.
\]
Note that \(\rho\) is a signed Borel measure on \(\mathbb{R}_+\) with compact support.

We show now that the following formula is valid (cf. (2.1)):
\[
\int_{\mathbb{R}^+} \frac{p(x) - p(0)}{x} d\rho(x) = \mathcal{E}_1 (\alpha - 1)^2 p'(\alpha) + \mathcal{E}_1 (\bar{\alpha} - 1)^2 p'(\bar{\alpha}) \\
+ \mathcal{E}_2 \frac{p(\alpha) - p(0)}{\alpha} + \mathcal{E}_2 \frac{p(\bar{\alpha}) - p(0)}{\bar{\alpha}}, \quad p \in \mathbb{C}[X], \quad \text{deg } p \geq 1. \quad (3.30)
\]
Indeed, fix \(k \in \mathbb{N}\) and take a polynomial \(p \in \mathbb{C}[X]\) of the form \(p(x) = \sum_{n=0}^{k} \alpha_n x^n\) for \(x \in \mathbb{R}\), where \(\{\alpha_n\}_{n=0}^{k} \subseteq \mathbb{C}\). Observe that
\[
p'(x) = \sum_{n=0}^{k-1} \alpha_{n+1} (n + 1) x^n, \quad x \in \mathbb{R},
\]
and
\[
\frac{p(x) - p(0)}{x} = \sum_{n=0}^{k-1} \alpha_{n+1} x^n, \quad x \in \mathbb{R}.
\]
By making the appropriate summation in (3.28) and using the above two identities, we obtain (3.30).

Substituting the polynomial \( p(x) = (x - \alpha)^2(x - \bar{\alpha})^2xq(x) \) into (3.30), where \( q \in \mathbb{C}[X] \), we deduce that

\[
\int_{\mathbb{R}^+} q(x)|x - \alpha|^4d\rho(x) = 0, \quad q \in \mathbb{C}[X].
\] (3.31)

Since \( \rho \) is a signed Borel measure on \( \mathbb{R}^+ \) with compact support, we deduce from (3.31) and [10, Lemma 4.1] that

\[
\int_{\Delta} |x - \alpha|^4d\rho(x) = 0, \quad \Delta \in \mathfrak{B}(\mathbb{R}^+).
\] (3.32)

As \( \alpha \neq 1 \) and \( |\alpha| = 1 \), we see that \( |x - \alpha| \neq 0 \) for all \( x \in \mathbb{R}^+ \). Combined with (3.32), this implies that \( |\rho| = 0 \), where \( |\rho| \) denotes the total variation measure of \( \rho \), and consequently \( \rho = 0 \). Since \( \nu((1)) = 0 \), we deduce from (3.29) that \( \tilde{c} = 0 \) and consequently that \( \nu = 0 \). It follows now from (3.27) that

\[
\frac{2}{n} \text{Re}\langle h_1, h_2 \rangle + \tilde{b} = \frac{\langle h_1, h_2 \rangle \alpha^n}{n} + \frac{\langle h_2, h_1 \rangle \bar{\alpha}^n}{n} + \Xi_1 \alpha^n + \bar{\Xi}_1 \bar{\alpha}^n, \quad n \in \mathbb{N}.
\] (3.33)

As \( |\alpha| = 1 \), we conclude that the sequence \( \{\text{Re}(\Xi_1 \alpha^n)\}_{n=0}^\infty \) is convergent. Let us consider two cases.

**Case 1.** \( \alpha \neq -1 \).

Since also \( \alpha \neq 1 \), we infer from Lemma 2.1 that \( \Xi_1 = 0 \). Hence, by (3.33), \( \tilde{b} = 0 \), and thus

\[
\text{Re}\langle h_1, h_2 \rangle = \text{Re}(\langle h_1, h_2 \rangle \alpha^n), \quad n \in \mathbb{Z}_+.
\]

Therefore, the sequence \( \{\text{Re}(\langle h_1, h_2 \rangle \alpha^n)\}_{n=0}^\infty \) is convergent. Applying Lemma 2.1 again, we conclude that \( h_1 \) and \( h_2 \) are orthogonal.

**Case 2.** \( \alpha = -1 \).

By passing to the limits in (3.33) when \( n \) is even, we get \( \tilde{b} = 2\text{Re}\Xi_1 \). The same procedure applied to odd \( n \)'s leads to \( \tilde{b} = -2\text{Re}\Xi_1 \). As a consequence, \( \tilde{b} = \text{Re}\Xi_1 = 0 \). Combined with (3.33), this shows that \( \text{Re}\langle h_1, h_2 \rangle = (-1)^n\text{Re}\langle h_1, h_2 \rangle \) for all \( n \in \mathbb{Z}_+ \). Hence, \( \text{Re}\langle h_1, h_2 \rangle = 0 \), which implies that \( h_1 \) and \( h_2 \) are orthogonal. This completes the proof. \( \square \)

**Remark 3.9** It is worth pointing out that one of the direct consequences of Lemma 3.8 is that, under its assumptions, there always exists an operator \( T \in L(\mathcal{M}_1 + \mathcal{M}_2) \) for which the spaces \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are invariant and \( T|_{\mathcal{M}_j} = z_j I_{\mathcal{M}_j} + N_j \) for \( j = 1, 2 \). In fact, the following identity holds:

\[
T = (z_1 I_{\mathcal{M}_1} + N_1) \oplus (z_2 I_{\mathcal{M}_2} + N_2).
\]

A similar observation applies to Lemma 3.7. \( \diamond \)
4 Proofs of the Main Result and its Consequences

Proof of Theorem 1.1 (i)⇒(ii) It follows from Lemma 2.4 that the operator $T$ takes the form described in part (ii) of this lemma. In particular, the complex numbers $w_1, \ldots, w_n$ are distinct and the vector spaces $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are nonzero. Set

$$J = \{1, \ldots, n\}, \quad J^{c} = \{j \in J : N_j \neq 0\} \text{ and } J^{c} = J \setminus J^{c}. $$

Suppose that $J^{c} \neq \emptyset$ (the case when $J^{c} = \emptyset$ can be treated in the same way). We can assume that for every $j \in J^{c}$, the exponent $\iota_j$ is the least positive integer for which $N_j^{\iota_j} = 0$. Since the restriction of a CPD operator to its invariant vector subspace is CPD, we deduce from Lemma 3.6 that $|w_j| = 1$ and $\iota_j = 2$ for all $j \in J^{c}$. Combined with Lemma 3.8, this implies that $\mathcal{M}_i \perp \mathcal{M}_j$ for all distinct $i, j \in J^{c}$. Hence, if $J^{c} = \emptyset$, then the decomposition (1.1) holds with $\mathcal{B}_{j, 1} = \{0\}$, $\mathcal{B}_{j, 2} = \{0\}$ and $\mathcal{B}_j = \emptyset$. Assume that $J^{c} \neq \emptyset$. It follows from Lemma 3.4 that $\mathcal{M}_i \perp \mathcal{M}_j$ for all distinct $i, j \in J^{c}$ such that $w_i w_j \neq 1$. According to Lemma 3.7, $\mathcal{M}_i \perp \mathcal{M}_j$ for all $i \in J^{c}$ and $j \in J^{c}$. Therefore, if $J^{c}$ is a singleton set or $w_i w_j \neq 1$ for all distinct $i, j \in J^{c}$, then the decomposition (1.1) holds with $\mathcal{B}_{j, 1} = \{0\}$ and $\mathcal{B}_{j, 2} = \{0\}$.

Assume now that there are indices $i_1, j_1 \in J^{c}$ such that $i_1 \neq j_1$ and $w_{i_1} w_{j_1} = 1$. Using induction one can construct finite sequences $\{i_s\}_{s=1}^{\eta} \subseteq J^{c}$ and $\{j_s\}_{s=1}^{\eta} \subseteq J^{c}$ such that

- $i_s \neq j_s$ for all $s \in K$,
- $w_{i_s} w_{j_s} = 1$ for all $s \in K$,
- $\{i_s, j_s\} \cap \{i_t, j_t\} = \emptyset$ for all distinct $s, t \in K$,
- $\{i_s, j_s\} \cap \{i_t, j_t\} = \emptyset$ for all distinct $s \in K$, \hspace{1cm} (4.1)

where $K = \{1, \ldots, \eta\}$. It is easy to see that $w_{i_s} w_{j_s} \neq 1$ for all $i \in \{i_s, j_s\}$ and $j \in \{i_s, j_s\}$ whenever $s, t \in K$ are distinct. Applying Lemma 3.3, we deduce that

$$\mathcal{M}_i + \mathcal{M}_j \perp \mathcal{M}_i \perp \mathcal{M}_j \quad \text{for all distinct } s, t \in K.$$ \hspace{1cm} (4.1)

Set $\Omega_\eta := \{i_1, \ldots, i_\eta\} \cup \{j_1, \ldots, j_\eta\}$ and $\Omega_\eta^{c} := J^{c} \setminus \Omega_\eta$. Consider two cases. If $\Omega_\eta^{c} = \emptyset$, then the decomposition (1.1) holds with $\mathcal{B}_{j} = \emptyset$. Suppose that $\Omega_\eta^{c} \neq \emptyset$. Note that $w_{i} w_{j} \neq 1$ for all $i \in \Omega_\eta^{c}$ and $j \in \Omega_\eta$ and, by (4.1), $w_{i} w_{j} \neq 1$ for all distinct $i, j \in \Omega_\eta^{c}$. It follows from Lemma 3.3 that $\mathcal{M}_i \perp \mathcal{M}_j$ for all $i \in \Omega_\eta^{c}$ and $j \in \Omega_\eta$ and for all distinct $i, j \in \Omega_\eta$. Therefore, the decomposition (1.1) is valid. This shows that (ii) holds.

(ii)⇒(i) Since the orthogonal direct sum of CPD operators is CPD, we see that Lemmas 3.3 and 3.6 imply (i).

The “moreover” part follows from Lemma 2.4. This completes the proof. \hspace{1cm} $\Box$

Proof of Theorem 1.2 (i)⇒(ii) It follows from Theorem 1.1 that the operator $T$ takes the form described in part (ii) of this theorem with $\mathcal{M} = \mathcal{H}$, where each of the spaces
\( X_{j,1}, \mathcal{X}_{j,2}, \mathcal{Y}_j \) and \( \mathcal{Z}_j \) is closed. We can assume that all these spaces are non-zero (similar arguments can be applied to other cases). Define the polynomial \( p \in \mathbb{C}[X] \) by

\[
p(x) = \prod_{j=1}^k (x - x_{j,1}) \prod_{j=1}^k (x - x_{j,2}) \prod_{j=1}^l (x - y_j) \prod_{j=1}^m (x - z_j)^2, \quad x \in \mathbb{R}.
\]

It follows from Theorem 1.1(ii) that \( p(T) = 0 \). Applying the spectral mapping theorem (see e.g., [6, Lemma 53.3]), we deduce that

\[
\{x_{j,1}\}_{j=1}^k \cup \{x_{j,2}\}_{j=1}^k \cup \{y_j\}_{j=1}^l \cup \{z_j\}_{j=1}^m = \sigma(T), \tag{4.2}
\]

where \( \sigma(T) \) stands for the spectrum of \( T \). Since \( r(T) \leq 1 \), we deduce that \( |x_{j,1}| \leq 1 \), \( |x_{j,2}| \leq 1 \), \( |y_j| \leq 1 \) for all \( j \) (recall that the modulus of each \( z_j \) is equal to 1). However \( x_{j,1}x_{j,2} = 1 \), so \( x_{j,1} = x_{j,2} \), which is a contradiction. Therefore, the decomposition (1.1) reduces to

\[
\mathcal{H} = \bigoplus_{j=1}^l \mathcal{Y}_j \oplus \bigoplus_{j=1}^m \mathcal{Z}_j.
\]

This shows that (ii) holds. The “moreover” part is a direct consequence of (4.2).

(ii) \( \Rightarrow \) (iii) Set

\[
A = \bigoplus_{j: N_j = 0} z_j I_{\mathcal{H}_j} \quad \text{and} \quad V = \bigoplus_{j: N_j \neq 0} (z_j I_{\mathcal{H}_j} + N_j).
\]

Clearly, \( A \) is an algebraic normal contraction. It follows from [8, Theorem 2.2] that \( z_j I_{\mathcal{H}_j} + N_j \) is a 3-isometry for each \( j \) such that \( N_j \neq 0 \). This implies that \( V \) is a 3-isometry. Using (1.2), we see that (iii) is valid.

(iii) \( \Rightarrow \) (i) It follows from [11, Proposition 2.7] (see also [20, Proposition 4.3.1]) that \( V \) is CPD. As a consequence, \( T \) is a CPD algebraic operator. Clearly \( r(A) \leq 1 \). It follows from [2, Lemma 1.21] that \( r(V) \leq 1 \). As a consequence, \( r(T) \leq 1 \). This completes the proof.

\[ \Box \]

**Proof of Proposition 1.3** According to Theorem 1.1, the operator \( S \) takes the form described in part (ii) of this theorem with \( T = S \) and \( \mathcal{M} = \mathcal{K} \), where each of the spaces \( \mathcal{X}_{j,1}, \mathcal{X}_{j,2}, \mathcal{Y}_j, \mathcal{Z}_j \) and \( \mathcal{X}_{j,1} + \mathcal{X}_{j,2} \) is closed (cf. (2.8)). As a consequence, each map

\[
\mathcal{X}_{j,1} + \mathcal{X}_{j,2} \ni x_1 + x_2 \mapsto (x_1, x_2) \in \mathcal{X}_{j,1} \oplus \mathcal{X}_{j,2}
\]

is a topological and linear isomorphism. Now, it is matter of routine to verify that the operator \( S \) is similar to the operator \( T \) defined by

\[
T = \bigoplus_{j=1}^k (x_{j,1} I_{\mathcal{X}_{j,1}} \oplus x_{j,2} I_{\mathcal{X}_{j,2}}) \oplus \bigoplus_{j=1}^l y_j I_{\mathcal{Y}_j} \oplus \bigoplus_{j=1}^m (z_j I_{\mathcal{Z}_j} + N_j).
\]
That the operator $T$ takes the form $T = A \oplus V$, where $A$ is an algebraic normal operator and $V$ is an algebraic 3-isometry, can be justified as in the proof of the implication (ii)$\Rightarrow$(iii) of Theorem 1.2.

\section*{Data Availability}
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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