Convexity of the class of currents with finite relative energy

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Abstract

We prove the convexity of the class of currents with finite relative energy. A key ingredient is an integration by parts formula for relative non-pluripolar products which is of independent interest.

1 Introduction

Let $X$ be a compact Kähler manifold of dimension $n$. Let $T$ be a closed positive current of bi-degree $(p, p)$ on $X$. Let $T_1, \ldots, T_m$ be closed positive $(1, 1)$-currents on $X$. The $T$-relative non-pluripolar product $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$ of $T_1, \ldots, T_m$ was introduced in [14]. The last product is a closed positive current of bi-degree $(p + m, p + m)$. When $T$ is a constant function equal to 1 (i.e., $T$ is the current of integration along $X$), the current $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$ coincides with the usual non-pluripolar product of $T_1, \ldots, T_m$ given in [2, 4, 11].

For every closed positive currents $S$ on $X$, we denote by $\{S\}$ its cohomology class. For two cohomology $(q, q)$-classes $\alpha, \beta$ on $X$, we write $\alpha \leq \beta$ if $\beta - \alpha$ can be represented by a closed positive $(q, q)$-current.
Recall that by \([14, \text{Theorem 1.1}]\) (also \([4, 15, 6]\)), if \(T_j'\) is a closed positive \((1,1)\)-current on \(X\) which is cohomologous to \(T_j\) and less singular than \(T_j\) for \(1 \leq j \leq m\), then we have

\[
\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle \} \leq \{\langle T_1' \wedge \cdots \wedge T_m' \wedge T \rangle \}.
\]

\((1.1)\)

The last inequality allows us to define the notion of \(T\)-relative full mass intersection, see \([14, 4]\). We say that \(T_1, \ldots, T_m\) are of \(T\)-relative full mass intersection if

\[
\{\langle \bigwedge_{j=1}^m T_j \wedge T \rangle \} = \{\langle \bigwedge_{j=1}^m T_{j, \text{min}} \wedge T \rangle \},
\]

where \(T_{j, \text{min}}\) is a current with minimal singularities in the class \(\{T_j\}\) for \(1 \leq j \leq m\).

Let \(\alpha\) be a pseudoeffective \((1,1)\)-class. Denote by \(E_m(\alpha, T)\) the set of currents \(P \in \alpha\) such that \(P, \ldots, P\) \((m\) times \(P\)) are of \(T\)-relative full mass intersection.

Recall that \(W^-\) is the set of convex increasing functions \(\chi\) from \(\mathbb{R}\) to \(\mathbb{R}\) such that \(\chi(-\infty) = -\infty\). Let \(\chi \in W^-\). We can define \(E_{\chi,m}(\alpha, T)\) to be the subclass of \(E_m(\alpha, T)\) consisting of \(P\) such that \(P\) has finite \((T\)-relative\) \(\chi\)-energy. When \(T \equiv 1\) and \(m = n\), the class \(E_{\chi,m}(\alpha, T)\) generalizes the usual class of currents with finite energy in \([4, 11]\), see also \([5]\) for the local setting. We refer to Section \(3\) for details. For the moment, we note here that

\[
E_m(\alpha, T) = \bigcup_{\chi \in W^-} E_{\chi,m}(\alpha, T).
\]

Here is our main result.

**Theorem 1.1.** The sets \(E_{\chi,m}(\alpha, T)\) and \(E_m(\alpha, T)\) are convex.

The last result was proved in \([14, \text{Theorem 1.3}]\) in the case where \(\alpha\) is Kähler. When \(m = n\) and \(T \equiv 1\), the convexity \(E_{\chi,m}(\alpha, T)\) was conjectured in \([4]\). It was later answered affirmatively in \([7, \text{Corollary 2.12}]\) in this setting. The proof in \([7]\) doesn’t extend directly to our setting because it uses, in a crucial way, Monge-Ampère equations in big classes.

We will see that Theorem \(1.1\) is a direct consequence of a more general result (Theorem \(3.4\)) which is in turn deduced from a monotonicity property of joint energy of currents, see Theorem \(3.1\) below. To prove these results, we use ideas from the proof of \([14, \text{Theorem 1.3}]\) and prove an integration by parts formula for relative non-pluripolar products (Theorem \(2.7\)) which is of independent interest. We emphasize that the last formula
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was applied to the study of complex Monge-Ampère equations. It plays a key role in the proof of main results in [10], see Theorem 1.3 there.

Moreover it was also explained in [10] that by using the integration by parts formula obtained in this work and the variational method ([3, 6]), one can solve the Monge-Ampère equation in the prescribed singularity setting without the small unbounded locus assumption. Hence this gives another proof of a main result in [8]. We refer to [10, Theorem 3.8] for details.

In the next section, we will present the above-mentioned integration by parts formula for relative non-pluripolar products. This formula strengthens (and generalizes) recent ones obtained in [12, 16] (see Corollary 2.8 and the paragraph following it). Our main result will be proved in Section 3.

2 Integration by parts

We first recall some basic facts about relative non-pluripolar products. This notion was introduced in [14] as a generalization of the usual non-pluripolar products given in [2, 4, 11].

Let $X$ be a compact Kähler manifold. Let $T_1, \ldots, T_m$ be closed positive $(1,1)$-currents on $X$. Let $T$ be a closed positive current of bi-degree $(p,p)$ on $X$. By [14], the $T$-relative non-pluripolar product $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ is defined in a way similar to that of the usual non-pluripolar product. The product $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ is a well-defined closed positive current of bi-degree $(m+p, m+p)$; and $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ is symmetric with respect to $T_1, \ldots, T_m$ and is homogeneous.

For every closed positive $(1,1)$-current $P$, we denote by $I_P$ the set of $x \in X$ so that local potentials of $P$ are equal to $-\infty$ at $x$. Note that $I_P$ is a locally complete pluripolar set. The following is deduced from [14, Proposition 3.5].

**Proposition 2.1.** (i) Given a locally complete pluripolar set $A$ such that $T$ has no mass on $A$, then $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ also has no mass on $A$.

(ii) Let $T'_1$ be a closed positive $(1,1)$-current on $X$ and $T_j, T$ as above. Assume that $T$ has no mass on $I_{T_1} \cup I_{T'_1}$. Then we have

\begin{equation}
\langle (T_1 + T'_1) \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle = \langle T_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle + \langle T'_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle.
\end{equation}
Let $1 \leq l \leq m$ be an integer. Then for $R := \langle \bigwedge_{j=l+1}^{m} T_j \wedge T \rangle$, there holds $\langle \bigwedge_{j=1}^{m} T_j \wedge T \rangle = \langle \bigwedge_{j=1}^{l} T_j \wedge R \rangle$.

The equality $\langle \bigwedge_{j=1}^{m} T_j \wedge T \rangle = \langle \bigwedge_{j=1}^{m} T_j \wedge T' \rangle$ holds, where $T' := 1_{X \setminus \bigcup_{j=1}^{m} r_{T_j} T}$. We also need the following result.

**Theorem 2.2.** ([14, Theorem 2.6, Remark 2.7]) Let $u_j$ be a locally bounded plurisubharmonic (psh) function on an open subset $U$ of $\mathbb{C}^n$ for $1 \leq j \leq m$. Let $(u_{jk})_{k \in \mathbb{N}}$ be a sequence of locally bounded psh functions increasing to $u_j$ almost everywhere as $k \to \infty$. Let $T$ be a closed positive current on $U$. Then, the convergence $u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \wedge T \to u_{1} dd^c u_{2} \wedge \cdots \wedge dd^c u_{m} \wedge T$ (2.2) as $k \to \infty$ holds provided that $T$ has no mass on $A_j := \{x \in U : u_j(x) \neq \lim_{k \to \infty} u_{jk}(x)\}$ for every $1 \leq j \leq m$ and the set $A_j$ is locally complete pluripolar for every $j$.

Recall that a dsh function on $X$ is the difference of two quasiplurisubharmonic (quasi-psh for short) functions on $X$ (see [9]). These functions are well-defined outside pluripolar sets. Let $v$ be a dsh function on $X$. Write $v = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2$ are quasi-psh function. Hence $v(x)$ is well-defined for $x \in X \setminus A$, where $A := \{\varphi_1 = -\infty\} \cup \{\varphi_2 = -\infty\}$. The function $v$ is said to be bounded in $X$ if there exists a constant $C$ such that $|v(x)| \leq C$ for every $x \in X \setminus A$.

We say that $v$ is $T$-admissible (or admissible with respect to $T$) if there exist quasi-psh functions $\varphi_1, \varphi_2$ on $X$ such that $v = \varphi_1 - \varphi_2$ and $T$ has no mass on $\{\varphi_j = -\infty\}$ for $j = 1, 2$. In particular, if $T$ has no mass on pluripolar sets, then every dsh function is $T$-admissible. Assume now that $v$ is $T$-admissible. The following is a direct consequence of Proposition 2.1 (i).

**Lemma 2.3.** If $v$ is $T$-admissible, then $v$ is also admissible with respect to $\langle \bigwedge_{j=1}^{m} T_j \wedge T \rangle$.

Recall that if $v = \varphi_1 - \varphi_2$ for some bounded quasi-psh functions $\varphi_1, \varphi_2$ on $X$ (note $X$ is compact), the current $dv \wedge d^c v \wedge T$ is, by definition, equal to $\frac{1}{2} dd^c (\varphi_1 - \varphi_2)^2 \wedge T - (\varphi_1 - \varphi_2) dd^c (\varphi_1 - \varphi_2) \wedge T$. (2.3)
We notice that in the above formula the function \((\varphi_1 - \varphi_2)^2\) is locally the difference of two bounded psh functions. More precisely, we can assume \(\varphi_j\) is \(\omega\)-psh function for \(j = 1, 2\), where \(\omega\) is a Kähler form on \(X\), and consider an open local chart \(U\) such that \(\omega = dd^c\phi\) for some smooth psh function \(\phi\) on \(U\). By adding to \(\phi\) a big constant, we can even assume that \(\varphi_j + \phi \geq 0\) on \(U\) for \(j = 1, 2\). Thus \((\varphi_j + \phi)^2\) and \((\varphi_1 + \varphi_2 + 2\phi)^2\) are psh on \(U\) for \(j = 1, 2\), and

\[(\varphi_1 - \varphi_2)^2 = (2(\varphi_1 + \phi)^2 + 2(\varphi_2 + \phi)^2) - (\varphi_1 + \varphi_2 + 2\phi)^2\]

which is the difference of two bounded psh functions on \(U\).

Consider another dsh function \(w\) which is equal to the difference of two locally bounded psh functions and \(T\) is of bi-degree \((n - 1, n - 1)\), we have

\[2dv \wedge d^cw \wedge T = d(v + w) \wedge d^c(v + w) \wedge T - dv \wedge d^cv \wedge T - dw \wedge d^cw \wedge T.\]

However, in general, even when \(v\) is bounded, \(v\) might not be the difference of two bounded quasi-psh functions. Hence, the current \("dv \wedge d^cv \wedge T"\) is not well-defined in the above sense. We will introduce below the current \(\langle dv \wedge d^cv \wedge T \rangle\) in the spirit of non-pluripolar products. Before going into details, we need the following auxiliary estimate.

**Lemma 2.4.** Let \(\omega\) be a Kähler form on \(X\). Let \(\varphi_1, \varphi_2\) be bounded \(\omega\)-psh functions on \(X\) and \(v := \varphi_1 - \varphi_2\). Let \(T\) be a closed positive current of bi-dimension \((1, 1)\) on \(X\). Then, there exists a constant \(C\) independent of \(\varphi_1, \varphi_2\) such that

\[\int_X dv \wedge d^cv \wedge T \leq C\|v\|_{L^\infty}.\]

**Proof.** We have

\[I := \int_X dv \wedge d^cv \wedge T = -\int_X vdd^cv \wedge T\]
\[= -\int_X v(dd^c\varphi_1 - dd^c\varphi_2) \wedge T\]
\[= -\int_X v(dd^c\varphi_1 + \omega) \wedge T + \int_X v(dd^c\varphi_2 + \omega) \wedge T\]
\[\leq \|v\|_{L^\infty} \sum_{j=1}^2 \int_X (dd^c\varphi_j + \omega) \wedge T\]
\[= 2\|v\|_{L^\infty} \int_X T \wedge \omega\]
by Stokes’ theorem. The desired estimate follows. The proof is finished.

Assume now that \( v \) is \( T \)-admissible. Let \( \varphi_1, \varphi_2 \) be quasi-psh functions such that \( v = \varphi_1 - \varphi_2 \) and \( T \) has no mass on \( \{ \varphi_j = -\infty \} \) for \( j = 1, 2 \). Let \( \varphi_{j,k} := \max\{\varphi_j, -k\} \) for every \( j = 1, 2 \) and \( k \in \mathbb{N} \). Put \( v_k := \varphi_{1,k} - \varphi_{2,k} \).

Since \( v_k \) is the difference of two bounded quasi-psh functions, using (2.3), we obtain

\[
Q_k := dv_k \wedge d^c v_k \wedge T = dd^c v_k^2 \wedge T - v_k dd^c v_k \wedge T.
\]

Let \( \omega \) be a Kähler form so that \( \varphi_j \) is \( \omega \)-psh for \( j = 1, 2 \). Let \( U \) be a local chart on \( X \) such that \( \omega = dd^c \phi \) on \( U \) for some psh function \( \phi \) such that \( \varphi_{j,k} + \phi \geq 0 \) on \( U \) for \( j = 1, 2 \) (we fix \( k \)). By (2.4) applied to \( \varphi_{j,k} + \phi \), we have

\[
Q_k = 2dd^c(\varphi_{1,k} + \phi)^2 \wedge T + 2dd^c(\varphi_{2,k} + \phi)^2 \wedge T - dd^c(\varphi_{1,k} + \varphi_{2,k} + 2\phi)^2 \wedge T
- (v_k dd^c \varphi_{1,k} \wedge T - v_k dd^c \varphi_{2,k} \wedge T)
\]
on \( U \). By the plurifine locality with respect to \( T \) ([14, Theorem 2.9]) applied to each term in the right-hand side of the last equality, we have

\[
\langle Q, \Phi \rangle = \lim_{k \to \infty} \langle 1_{\{\varphi_j > -k\}} Q_k, \Phi \rangle.
\]

We say that \( \langle dv \wedge d^c v \wedge T \rangle \) is well-defined if the mass of \( 1_{\{\varphi_j > -k\}} Q_k \) on \( X \) is bounded uniformly in \( k \). In this case, using (2.7) implies that there exists a positive current \( Q \) on \( X \) such that for every bounded Borel form \( \Phi \) on \( X \), we have

\[
\langle Q, \Phi \rangle = \lim_{k \to \infty} \langle 1_{\{\varphi_j > -k\}} Q_k, \Phi \rangle.
\]

We define \( \langle dv \wedge d^c v \wedge T \rangle \) to be the current \( Q \). This agrees with the classical definition if \( v \) is the difference of two bounded quasi-psh functions. This definition is independent of the choice of \( \varphi_1, \varphi_2 \) by Lemma 2.5 below. If \( w \) is another \( T \)-admissible dsh function and \( T \) is of bi-dimension \((1, 1)\) such that the currents \( \langle dv \wedge d^c v \wedge T \rangle \), \( \langle dw \wedge d^c w \wedge T \rangle \), and \( \langle d(v + w) \wedge d^c (v + w) \wedge T \rangle \) are all well-defined, we define \( \langle dv \wedge d^c w \wedge T \rangle \) using (2.5) formally.

**Lemma 2.5.** Let \( \varphi_1, \varphi_2' \) be quasi-psh functions on \( X \) such that \( v = \varphi_1 - \varphi_2' \) and \( T \) has no mass on \( \{ \varphi_j' = -\infty \} \) for \( j = 1, 2 \). Let \( \varphi_{j,k}, Q'_k \) be the function and current associated to \( \varphi_j' \) defined similarly as \( \varphi_{j,k} \) and \( Q_k \), respectively. Then if \( 1_{\{\varphi_j > -k\}} Q_k \) is of mass bounded uniformly on \( k \) then so is

\[
1_{\{\varphi_j' > -k\}} Q'_k.
\]
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\[ 1_{\bigcap_{j=1}^{2} \{ \phi_j' > -k \}} Q' \]

and

\[ Q = \lim_{k \to \infty} \left( 1_{\bigcap_{j=1}^{2} \{ \phi_j' > -k \}} Q'_k \right) \Phi \]

(2.8)

for every bounded Borel form \( \Phi \) on \( X \).

**Proof.** Since \( v = \phi_1' - \phi_2' \), we get

\[ \varphi_1 + \varphi_2 = \varphi_1' + \varphi_2' \]

Put \( \nu_k' = \varphi_1' - \varphi_2' \), \( A'_k := \bigcap_{j=1}^{2} \{ \varphi_j' > -k \} \), and \( A_k := \bigcap_{j=1}^{2} \{ \varphi_j > -k \} \).

We have \( Q'_k = dv\wedge dt\wedge T \), and \( v_k = v'_k \) on \( A_k \cap A'_k \) which is open in the plurifine topology.

We claim that

\[ \left( 1_{A'_k} Q'_k = 1_{A_k \cap A'_k} Q_k \right). \]

(2.9)

This is a sort of plurifine locality statement and can be essentially deduced from the plurifine locality for bounded psh functions (here we have \( v_k = v'_k \) on \( A_k \cap A'_k \) but \( v_k, v'_k \) are only dsh). We give details for readers’ convenience. Before doing so, we will show that the desired assertion is a direct consequence of (2.9). First observe that

\[ 1_{A'_k} Q'_k \]

has no mass on the pluripolar set \( \{ \varphi_j' = -\infty \} \) for \( j = 1, 2 \) by Proposition 2.1 (i) and the fact that \( T \) has no mass on \( \{ \varphi_j = -\infty \} \). It follows that

\[ \left( 1_{A'_k} Q'_k \right) \leq \lim_{s \to \infty} \left( 1_{A'_k} Q'_s \right) \leq \lim_{s \to \infty} 1_{A_k} Q_s = Q, \]

where we used (2.7) applied to \( Q'_k \) in the second equality and (2.9) in the third equality. By exchanging the role of \( Q'_k \) and \( Q_k \), we also obtain that

\[ 1_{A_k} Q_k \leq R, \]

for every limit current \( R \) of \( \left( 1_{A'_k} Q'_k \right) \) as \( k \to \infty \). Hence (2.8) follows.

We go back to the proof of (2.9). Write

\[ dv_k \wedge dt v_k \wedge T = dv_k \wedge d' (v_k - v'_k) \wedge T + d (v_k - v'_k) \wedge dv'_k \wedge T + dv'_k \wedge d' v'_k \wedge T. \]

Denote by \( R_1, R_2 \) the first and second currents in the right-hand side of the last equality. In order to obtain (2.9), it suffices to check that \( R_j = 0 \)
on \(A_k \cap A'_k\) for \(j = 1, 2\). Observe
\[
R_1 = dv_k \wedge d^c(\varphi_{1,k} + \varphi'_{2,k} - \varphi'_{1,k} - \varphi_{2,k}) \wedge T
= \left[d\varphi_{1,k} \wedge d^c(\varphi_{1,k} + \varphi'_{2,k}) \wedge T - d\varphi_{1,k} \wedge d^c(\varphi'_{1,k} + \varphi_{2,k}) \wedge T\right] -
\left[d\varphi_{2,k} \wedge d^c(\varphi_{1,k} + \varphi'_{2,k}) \wedge T - d\varphi_{2,k} \wedge d^c(\varphi'_{1,k} + \varphi_{2,k}) \wedge T\right].
\]
Each term in the right-hand side of the above equality is equal to 0 on \(A_k \cap A'_k\) thanks to the plurifine locality and the fact that
\[
\varphi_{1,k} + \varphi'_{2,k} = \varphi_{1} + \varphi'_{2} = \varphi'_{1,k} + \varphi_{2,k}
\]
on \(A_k \cap A'_k\). Hence \(R_1 = 0\) on \(A_k \cap A'_k\). Similarly we get \(R_2 = 0\) on \(A_k \cap A'_k\).
This finishes the proof.

Lemma 2.6. Assume that \(v\) is bounded. Then, the current \(\langle dv \wedge d^c v \wedge T \rangle\) is well-defined.

Proof. Let the notation be as in the proof of Lemma 2.4. Let \(\omega\) be a Kähler form on \(X\) such that \(\varphi_1, \varphi_2\) are \(\omega\)-psh. Note that \(\varphi_{j,k}\) is also \(\omega\)-psh for every \(j, k\). Observe that since \(v\) is bounded, there exists a constant \(C\) such that
\[
\varphi_2 - C \leq \varphi_1 \leq \varphi_2 + C.
\]
Thus, there exists a constant \(C\) so that
\[
\|v_k\|_{L^\infty} \leq C
\]
for every \(k\). Using this and Lemma 2.4, one gets
\[
\|Q_k\| \lesssim \|v_k\|_{L^\infty} \leq C
\]
for some constant \(C\) independent of \(k\). Hence, the desired assertion follows. This finishes the proof.

Let \(T_1, \ldots, T_m\) be closed positive \((1, 1)\)-currents on \(X\) and \(R := \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\). We define
\[
\langle dv \wedge d^c v \wedge T_1 \wedge \cdots \wedge T_m \wedge T \rangle := \langle dv \wedge d^c v \wedge T \rangle.
\]
When \(T \equiv 1\), we write the left-hand side of the last equality simply as \(\langle dv \wedge d^c v \wedge T_1 \wedge \cdots \wedge T_m \rangle\).

The current \(\langle dv \wedge d^c w \wedge T_1 \wedge \cdots \wedge T_m \wedge T \rangle\) is defined similarly if \(p + m = n - 1\), where \(T\) is of bi-degree \((p, p)\). We put
\[
\langle dd^c v \wedge T \rangle := \langle dd^c \varphi_1 \wedge T \rangle - \langle dd^c \varphi_2 \wedge T \rangle.
\]
Define
\[
\langle dd^c v \wedge T_1 \wedge \cdots \wedge T_m \wedge T \rangle := \langle dd^c v \wedge T \rangle.
\]
By Proposition \[2.1\](iii), this definition agrees with the \(T\)-relative non-pluripolar product of \(dd^c v, T_1, \ldots, T_m\) if \(v\) is quasi-psh. When \(T \equiv 1\), we write \(\langle dd^c v \wedge T_1 \wedge \cdots \wedge T_m \rangle\) for \(\langle dd^c v \wedge T_1 \wedge T_2 \wedge \cdots \wedge T_m \wedge \hat{T} \rangle\). In this case the product \(\langle dd^c v \wedge T_1 \wedge \cdots \wedge T_m \rangle\) is the one defined in the paragraph right after Theorem 1.2 in [12].

By admissibility and Proposition \[2.1\](ii), we can check that if \(v, w\) are dsh functions which are admissible with respect to \(T\), then

\[
\langle dd^c (v + w) \wedge \hat{T} \rangle = \langle dd^c v \wedge \hat{T} \rangle + \langle dd^c w \wedge \hat{T} \rangle.
\]

Here is an integration by parts formula for relative non-pluripolar products.

**Theorem 2.7.** Let \(T\) be a closed positive current of bi-degree \((n - 1, n - 1)\) on \(X\). Let \(v, w\) be bounded \(T\)-admissible dsh functions on \(X\). Then, we have

\[
\int_X w \langle dd^c v \wedge \hat{T} \rangle = \int_X v \langle dd^c w \wedge \hat{T} \rangle = -\int_X \langle dw \wedge d^c v \wedge \hat{T} \rangle.
\]

(2.10)

The last result was proved in [4, Theorem 1.14] if \(v, w\) can be written as the differences of psh functions which are locally bounded outside a closed locally complete pluripolar set; see also [1, 13].

**Proof.** We use ideas from the proof of [14, Proposition 4.2]. Let \(\varphi_1, \varphi_2, \varphi_3, \varphi_4\) be negative quasi-psh functions on \(X\) such that \(v = \varphi_1 - \varphi_2\) and \(w = \varphi_3 - \varphi_4\) and \(T\) has no mass on \(\bigcup_{j=1}^{4} \{\varphi_j = -\infty\}\). Let \(\omega\) be a Kähler form on \(X\) such that \(\varphi_j\) is \(\omega\)-psh for every \(1 \leq j \leq 4\). Put

\[
\psi := \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4, \quad \psi_k := k^{-1} \max \{\psi, -k\} + 1.
\]

and \(\varphi_{jk} := \max \{\varphi_j, -k\}\) for \(1 \leq j \leq 4\). Observe that \(0 \leq \psi_k \leq 1\). Let \(x \in X\) such that \(\psi_k(x) > 0\). We have

\[
\varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x) = \psi(x) > -k.
\]

This combined with the property that \(\varphi_j \leq 0\) for every \(1 \leq j \leq 4\) yields that \(\varphi_j(x) > -k\) for every \(1 \leq j \leq 4\). We infer that

\[
\{\psi_k \neq 0\} \subset \bigcap_{j=1}^{4} \{\varphi_j > -k\}.
\]

(2.11)

Put \(v_k := \varphi_{1k} - \varphi_{2k}\) and \(w_k := \varphi_{3k} - \varphi_{4k}\). Since \(v\) and \(w\) are bounded, the functions \(v_k, w_k\) are bounded uniformly in \(k\).
Let \( A := \bigcup_{j=1}^{4} \{ \varphi_j = -\infty \} \). By admissibility and Proposition 2.1 (i), we see that

\[
(2.12) \quad 1_A ((dd^c \varphi_j + \omega) \wedge T) = 0.
\]

Using (2.12), we can consider \( w \) as a bounded function with respect to the trace measure of \( ((dd^c \varphi_j + \omega) \wedge T) \). Using (2.11), we have

\[
w \psi_k dd^c \varphi_{jk} \wedge T = w \{ \varphi_j > -k \} \psi_k (dd^c \varphi_j \wedge T) + w (1_{\{ \varphi_j > -k \} \psi_k - 1}) (dd^c \varphi_j \wedge T).
\]

The second term in the right-hand side of the last equality converges weakly to 0 as \( k \to \infty \) by the fact that \( \psi_k \to 1 \) pointwise outside \( A \) as \( k \to \infty \) and Lebesgue’s dominated convergence theorem. Hence

\[w (dd^c \varphi_j \wedge T) = \lim_{k \to \infty} w \psi_k dd^c \varphi_{jk} \wedge T.\]

Applying the last equality to \( j = 1, 2 \), and using \( v = \varphi_1 - \varphi_2 \), we obtain

\[w (dd^c v \wedge T) = \lim_{k \to \infty} w \psi_k dd^c v_k \wedge T = \lim_{k \to \infty} w_k \psi_k dd^c v_k \wedge T.\]

Here in the second equality we used the fact that \( w = w_k \) on \( \{ \varphi_3 > -k \} \cap \{ \varphi_4 > -k \} \) which contains \( \{ \psi_k \neq 0 \} \). We also have an analogous formula by exchanging the roles of \( v, w \). Thus,

\[w (dd^c v \wedge T) - v (dd^c w \wedge T) = \lim_{k \to \infty} \psi_k (w_k dd^c v_k - v_k dd^c w_k) \wedge T.\]

By integration by parts for bounded psh functions, we have

\[\int_X \psi_k (w_k dd^c v_k - v_k dd^c w_k) \wedge T = - \int_X w_k d\psi_k \wedge d^c v_k \wedge T + \int_X v_k d\psi_k \wedge d^c w_k \wedge T.\]

Denote by \( I_1, I_2 \) the first and second term in the right-hand side of the last equality. We will check that \( I_j \to 0 \) as \( k \to \infty \) for \( j = 1, 2 \). Using the Cauchy-Schwarz inequality, the boundedness of \( v_k, w_k \) and Lemma 2.4, we infer

\[
|I_1| \leq \left( \int_X d\psi_k \wedge d^c \psi_k \wedge T \right)^{\frac{1}{2}} \times \left( \int_X |w_k|^2 dv_k \wedge d^c v_k \wedge T \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \int_X d\psi_k \wedge d^c \psi_k \wedge T \right)^{\frac{1}{2}}.
\]
Recall that \( \{ \lim_{k \to \infty} \psi_k < 1 \} \) is equal to the complete pluripolar set \( \{ \psi = -\infty \} \). Using this, Theorem 2.2 and the fact that \( T \) has no mass on \( \{ \psi = -\infty \} \), we get
\[
\lim_{k \to \infty} d\psi_k \wedge d^c \psi_k \wedge T = \lim_{k \to \infty} (dd^c \psi_k^2 - \psi_k dd^c \psi_k) \wedge T = 0
\]
Thus we obtain
\[
(2.16) \quad \lim_{k \to \infty} I_1 = 0.
\]
By similarity, we also get \( I_2 \to 0 \) as \( k \to \infty \). Combining this with (2.15) and (2.13) gives the first desired equality of (2.10). We prove the second one similarly as follows. Put \( u := v + w \), and
\[
u_k := \max\{\varphi_1 + \varphi_3, -k\} - \max\{\varphi_2 + \varphi_4, -k\}.
\]
By (2.11) observe that
\[
1_{\{\psi_k > 0\}} \max\{\varphi_1 + \varphi_3, -2k\} = 1_{\{\psi_k > 0\}} (\varphi_{1k} + \varphi_{3k})
\]
and a similar equality for \( \varphi_2, \varphi_4 \) also holds. Thus, by plurifine locality, we get
\[
2\langle dv \wedge d^c w \wedge T \rangle = \langle du \wedge d^c u \wedge T \rangle - \langle dv \wedge d^c v \wedge T \rangle - \langle dw \wedge d^c w \wedge T \rangle
\]
\[
= \lim_{k \to \infty} \psi_k \left( \langle dw_{2k} \wedge d^c w_{2k} \wedge T \rangle - \langle dv_k \wedge d^c v_k \wedge T \rangle - \langle dw_k \wedge d^c w_k \wedge T \rangle \right)
\]
\[
= \lim_{k \to \infty} \psi_k \left( \langle d(v_k + w_k) \wedge d^c (v_k + w_k) \wedge T \rangle - \langle dv_k \wedge d^c v_k \wedge T \rangle - \langle dw_k \wedge d^c w_k \wedge T \rangle \right).
\]
Consequently
\[
\langle dv \wedge d^c w \wedge T \rangle = \lim_{k \to \infty} \psi_k \langle dv_k \wedge d^c w_k \wedge T \rangle.
\]
It follows that
\[
\int_X v \langle dd^c w \wedge T \rangle + \langle dv \wedge d^c w \wedge T \rangle = \lim_{k \to \infty} \int_X \psi_k (v_k dd^c w_k + dv_k \wedge d^c w_k) \wedge T
\]
\[
= - \lim_{k \to \infty} \int_X v_k d\psi_k \wedge d^c w_k \wedge T
\]
which is equal to 0 by analogous arguments as in the proof of (2.16). This finishes the proof.  

\[\square\]
**Corollary 2.8.** Let $v, w$ be bounded dsh functions on $X$. Then, for every closed smooth form $\Phi$ of right bi-degree, we have

$$\int_X w \langle dd^c v \wedge \bigwedge_{j=1}^m T_j \rangle \wedge \Phi = \int_X v \langle dd^c w \wedge \bigwedge_{j=1}^m T_j \rangle \wedge \Phi = -\int_X \langle dv \wedge d^c w \wedge \bigwedge_{j=1}^m T_j \rangle \wedge \Phi.$$  

**Proof.** By writing $\Phi$ as the difference of two closed positive forms, we can assume that $\Phi$ is positive. The desired formula is a direct consequence of Theorem 2.7 applied to $T := \langle T_1 \wedge \cdots \wedge T_m \rangle \wedge \Phi$.

We recall that the first inequality of (2.17) was proved in [12, Theorem 1.2] and [16] when $m = n$ and the cohomology classes of $T_j$’s are big. One should notice a crucial point that the integration by parts formulae obtained in [12, 16] contain no term involving $dv \wedge d^c w$. Such a term is essential in applications, especially, in the pluricomplex energy theory. The following result is more general than Theorem 2.7. We will need it later.

**Theorem 2.9.** Let $T$ a closed positive current of bi-degree $(n-1, n-1)$ on $X$. Let $v, w$ be bounded $T$-admissible dsh functions on $X$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^3$ function. Then we have

$$\int_X \chi(w) \langle dd^c v \wedge T \rangle = \int_X v \chi''(w) \langle dw \wedge d^c w \wedge T \rangle + \int_X v \chi'(w) \langle dd^c w \wedge T \rangle. \tag{2.18}$$

A quick heuristic reason explaining why (2.18) should hold is because $dd^c \chi(w) = \chi''(w) dw \wedge d^c w + \chi'(w) dd^c w$ if $w$ is a bounded quasi-psh function.

**Proof.** We first note that [14, Lemma 5.7] still holds for dsh functions which are the differences of two bounded quasi-psh functions. Now, to obtain the desired equality, we just follow the proof of Theorem 2.7 verbatim with $\chi(w)$ in place of $w$. The only thing we need to clarify is the computation concerning $dd^c \chi(w_k) \wedge T$. To this end, it suffices to use [14, Lemma 5.7] because $w_k$ is the difference of two bounded quasi-psh functions. This finishes the proof.

### 3 Currents with finite relative energy

Let $X$ be a compact Kähler manifold. Let $\alpha_1, \ldots, \alpha_m$ be pseudoeffective $(1, 1)$-classes of $X$ and $T$ a closed positive current on $X$. Let $P_j$ be a closed
positive \((1, 1)\)-current in the class \(\alpha_j\) for \(1 \leq j \leq m\). Put \(P := (P_1, \ldots, P_m)\). We define \(E(P)(T)\) to be the set of \(m\)-tuple \((T_1, \ldots, T_m)\) of closed positive \((1, 1)\)-currents such that \(T_j \in \alpha_j\) and \(T_j\) is more singular than \(P_j\) and

\[
\{(\bigwedge_{j=1}^m T_j \wedge T)\} = \{(\bigwedge_{j=1}^m P_j \wedge T)\}.
\]

Notice that for every current \(P'_j\) in \(\alpha_j\) such that \(P'_j\) has the same singularities as \(P_j\) for \(1 \leq j \leq m\), by the monotonicity of relative non-pluripolar products (see (1.1)), we have

\[
E(P)(T) = E(P')(T).
\]

Hence, when \(P_j\) has minimal singularities in \(\alpha_j\) for \(1 \leq j \leq m\), we recover the class \(E(\alpha_1, \ldots, \alpha_m, T)\) of currents of full mass intersection introduced in [14, 4] because we have

\[
E(P)(T) = E(\alpha_1, \ldots, \alpha_m, T)
\]

in this case.

Let \(\chi \in W^-\). Write \(P_j = dd^c\varphi_j + \theta_j\), where \(\theta_j\) is a smooth form and \(\varphi_j\) is a negative \(\theta_j\)-psh function. Let \((T_1, \ldots, T_m) \in E(P)(T)\). Let \(u_j\) be a negative \(\theta_j\)-psh function so that \(T_j = dd^c u_j + \theta_j\) and \(u_j \leq \varphi_j\) for \(1 \leq j \leq m\). For a negative Borel function \(\xi\), we put

\[
E_{\xi, P}(T_1, \ldots, T_m; T) := \sum_j \int_X -\xi \langle \bigwedge_{j \in J} T_j \wedge \bigwedge_{j \not\in J} P_j \wedge T \rangle,
\]

where the sum is taken over every subset \(J\) of \(\{1, \ldots, m\}\). The \((T, P)\)-relative joint \(\chi\)-energy of \(T_1, \ldots, T_m\) is, by definition, \(E_{\xi, P}(T_1, \ldots, T_m; T)\), where

\[
\xi := \chi((u_1 - \varphi_1) + \cdots + (u_m - \varphi_m)).
\]

The last energy depends on the choice of \(u_j, \varphi_j\) but its finiteness does not. That notion generalizes those in [4, 11, 14], see also [5] for the local setting.

For every closed positive \((1, 1)\)-current \(P\), let \(I_P\) be the set of \(x \in X\) so that the potentials of \(P\) are equal to \(-\infty\) at \(x\). Note that \(I_P\) is a complete pluripolar set. By Proposition 2.1 (iv), the right-hand side of (3.1) remains unchanged if we replace \(T\) by \(1_{X \setminus \bigcup_{j=1}^m I_{P_j}} T\). Hence, in practice, we can assume \(T\) has no mass on \(\bigcup_{j=1}^m I_{P_j}\). We denote by \(E_{\chi, P}(T)\) the subset of \(E_P(T)\) containing every \((T_1, \ldots, T_m)\) such that their \((T, P)\)-relative joint \(\chi\)-energy is finite.
Here is a monotonicity for the class $\mathcal{E}_{\chi, P}(T)$ when $P_j = P$ for every $1 \leq j \leq m$. This generalizes [14, Theorem 5.8].

**Theorem 3.1.** Let $P = d\bar{d}\varphi + \theta$ be a closed positive $(1,1)$-current and $P := (P, \ldots, P)$ ($m$ times $P$). Let $\chi \in \mathcal{W}^-$ with $|\chi(0)| \leq 1$. Let $(T_1, \ldots, T_m) \in \mathcal{E}_{\chi, P}(T)$ and $(T_1', \ldots, T_m') \in \mathcal{E}_{P}(T)$ such that $T_j = d\bar{d}u_j + \theta$, $T_j' = d\bar{d}u_j' + \theta$ such that $u_j, u_j'$ are $\theta$-psh and $u_j \leq u_j' \leq \varphi$. Put 

$$\xi := \chi \left( (u_1 - \varphi) + \cdots + (u_m - \varphi) \right).$$

Then we have 

$$E_{\xi}(T_1', \ldots, T_m'; T) \leq c_1 E_{\xi}(T_1, \ldots, T_m; T) + c_2,$$

for some constants $c_1, c_2 > 0$ independent of $\chi$. In particular, $(T_1', \ldots, T_m') \in \mathcal{E}_{\chi, P}(T)$.

**Proof.** As mentioned above, we can assume that $T$ has no mass on $I_P = \{ \varphi = -\infty \}$. Note here that $\{ \varphi = -\infty \} \subset \{ u_j = -\infty \}$. Put 

$$u_{jk} := \max\{ u_j, \varphi - k \} - \varphi$$

which is a bounded dsh function and 

$$T_{jk} := d\bar{d}u_{jk} + P.$$ 

Observe that $u_{jk}$’s are admissible with respect to $T$. Define $u_{jk}', T_{jk}'$ similarly. Put 

$$v := \sum_{j=1}^{m} (u_j - \varphi_j), \quad v_k := \max\{ v, -k \}, \quad \xi_k = \chi(v_k).$$

Note that $\xi = \chi(v)$. With these notations and a suitable integration by parts ready in our hands (Theorem 2.9 replacing [14, Lemma 5.7]), the proof goes exactly as in the proof of [14, Theorem 5.8]. The only minor modifications are: the Kähler form $\omega$ is substituted by $P$ and the wedge products appearing in the proof of [14, Theorem 5.8] need to be replaced by $T$-relative non-pluripolar products. This finishes the proof.

The following is a direct consequence of Theorem 3.1.

**Corollary 3.2.** Let $P, P'$ be as in Theorem 3.1. Let $P'$ be a current in $\{ P \}$ which is of the same singularity type as $P$. Then, for $P' := (P', \ldots, P')$ ($m$ times $P'$), we have 

$$\mathcal{E}_{\chi, P'}(T) = \mathcal{E}_{\chi, P}(T).$$
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For every closed positive \((1,1)\)-current \(P\), we define the class \(\mathcal{E}_{m,P}(T)\) (resp. \(\mathcal{E}_{X,m,P}(T)\)) to be the set of currents \(T_1 \in \{P\}\) such that \((T_1, \ldots, T_1)\) belongs to \(\mathcal{E}_P(T)\) (resp. \(\mathcal{E}_{X,P}(T)\)), where \(P = (P, \ldots, P)\) \((m\text{ times})\). The last space was introduced in \([6]\) when \(T\) is the constant function equal to 1. As in the case of the usual class of currents of full mass intersection (\([11+, \text{Proposition } 2.2]\)), notice that

\[
\mathcal{E}_{m,P}(T) = \bigcup_{\chi \in \mathcal{W}^-} \mathcal{E}_{\chi,m,P}(T).
\]

Let \(\alpha\) be a pseudoeffective \((1,1)\)-class. By Corollary 3.2, we see that the notion of the weighted class \(\mathcal{E}_{\chi,m,P}(T)\) makes sense if we replace \(P\) by its equivalent class (in terms of singularity type) of \((1,1)\)-currents. Hence, we can define \(\mathcal{E}_{m}(\alpha, T)\) (resp. \(\mathcal{E}_{\chi,m}(\alpha, T)\)) to be the set \(\mathcal{E}_{m,P}(T)\) (resp. \(\mathcal{E}_{X,m,P}(T)\)), where \(P\) is a current with minimal singularities in \(\alpha\).

**Theorem 3.3.** Let \(U\) be an open subset in \(\mathbb{C}^n\). Let \(T\) be a closed positive current on \(U\) and \(u_j, u_j'\) bounded psh functions on \(U\) for \(1 \leq j \leq m\), where \(m \in \mathbb{N}\). Let \(v_j, v_j'\) be psh functions on \(U\) for \(1 \leq j \leq q\). Assume that \(u_j = u_j'\) on \(W := \bigcap_{j=1}^{q} (v_j > v_j')\) for \(1 \leq j \leq m\). Then we have

\[
1_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = 1_W dd^c u_1' \wedge \cdots \wedge dd^c u_m' \wedge T.
\]

**Proof.** If \(v_j, v_j'\) are all bounded, then the desired assertion is Theorem 2.9 in \([14]\). In general, observe that

\[
\{v_j > v_j'\} = \bigcup_{k=1}^{\infty} \{v_{jk} > v_{jk}'\},
\]

where \(v_{jk} := \max\{v_j, -k\}\) and similarly for \(v_{jk}'\). Let \(W_k := \bigcap_{j=1}^{q} \{v_{jk} > v_{jk}'\}\). We have \(W = \bigcup_{k=1}^{\infty} W_k\) and \(u_j = u_j'\) on \(W_k\). Applying \([14, \text{Theorem } 2.9]\) to \(u_j, u_j', W_k\) gives

\[
1_{W_k} dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = 1_{W_k} dd^c u_1' \wedge \cdots \wedge dd^c u_m' \wedge T
\]

for every \(k\). Hence, the desired assertion follows. This finishes the proof. \(\Box\)

Now, using Theorems 3.1 and 3.3 instead of \([14, \text{Theorem } 5.8]\) and \([14, \text{Theorem } 2.9]\) respectively, and following arguments in the proof of \([14, \text{Theorems } 5.9 \text{ and } 5.1]\), we immediately obtain the following result.

**Theorem 3.4.** For \(\chi \in \mathcal{W}^-\), the sets \(\mathcal{E}_{X,m,P}(T)\) and \(\mathcal{E}_{m,P}(T)\) are convex.
Finally, we would like to make the following comment.

**Remark 3.5.** Let $\mathcal{W}_M^+$ be the class of weights introduced in [11, Page 462]. Using arguments from the proof of [11, Lemma 3.5] and that of Theorem [3.4] we can prove the convexity of $E_{\chi,m,P}(T)$ for $\chi \in \mathcal{W}_M^+$.

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