Satisfiability Problems on Sums of Kripke Frames

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We consider the operation of sum on Kripke frames, where a family of frames-summands is indexed by elements of another frame. In many cases, the modal logic of sums inherits the finite model property and decidability from the modal logic of summands [Babenyshev and Rybakov 2010; Shapirovsky 2018]. In this paper we show that, under a general condition, the satisfiability problem on sums is polynomial space Turing reducible to the satisfiability problem on summands. In particular, for many modal logics decidability in PSpace is an immediate corollary from the semantic characterization of the logic.

CCS Concepts: • Theory of computation → Modal and temporal logics; Complexity classes;

Additional Key Words and Phrases: Sum of Kripke frames, finite model property, decidability, Turing reduction, PSpace, Japaridze’s polymodal logic, lexicographic product of modal logics, lexicographic sum of modal logics

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1 INTRODUCTION

In classical model theory, there is a number of results (“composition theorems”) that reduce the theory (first-order, MSO) of a compound structure (e.g., sum or product) to the theories of its components, see, e.g., Feferman and Vaught [1959], Gurevich [1979], Mostowski [1952], and Shelah [1975] or Gurevich [1985]. In this paper we use the composition method in the context of modal logic.

Given a family \( \{ F_i \mid i \in I \} \) of frames (structures with binary relations) indexed by elements of another frame \( I \), the sum of the frames \( F_i \)’s over \( I \) is obtained from their disjoint union by connecting elements of \( i \)-th and \( j \)-th distinct components according to the relations in \( I \). Given a class \( \mathcal{F} \) of frames-summands and a class \( I \) of frames-indices, \( \sum_I \mathcal{F} \) denotes the class of all sums of \( F_i \)’s in \( \mathcal{F} \) over \( i \in I \).

In many cases, this operation preserves the finite model property and decidability of the logic of summands [Babenyshev and Rybakov 2010; Shapirovsky 2018]. In this paper we show that transferring results also hold for the complexity of the modal satisfiability problems on sums.
In particular, it follows that for many logics PSpace-completeness is an immediate corollary of semantic characterization.

It is a classical result by R. Ladner that the decision problem for the logic of (finite) preorders S4 is in PSpace [Ladner 1977]. In Shapirovsky [2008], it was shown that the polymodal provability logic GLP is also decidable in PSpace. In spite of the significant difference between these two logics, there is a uniform proof for both these two particular systems, as well as for many other important modal logics: the general phenomenon is that the modal satisfiability problem on sums over Noetherian (in particular, finite) orders is polynomial space Turing reducible to the modal satisfiability problem on summands. In the case of S4 these summands are frames of the form $(W, W \times W)$ with the satisfiability problem being in NP, so in PSpace. And hence, S4 is in PSpace. In the case of GLP, the class of summands is even simpler: the only summand required is an irreflexive singleton [Beklemishev 2010]. (We will discuss these and other examples in Sections 4.5, 5.3, 5.4.)

The paper has the following structure. Section 2 contains preliminary material. Section 3 is about truth-preserving operations on sums of frames; it contains necessary tools for the complexity results, and also quotes recent results on the finite model property. This section is based on Shapirovsky [2018]. The central Section 4 is about complexity. The reduction between sums and summands is described in Theorem 4.1 and in its more technical (but more tunable) version Theorem 4.7; these are the main results of the paper. This section elaborates earlier works [Shapirovsky 2005, 2008] (in particular, our new results significantly generalize and simplify Theorems 22 and 35 in Shapirovsky [2008]). In Section 5 we discuss modifications of the sum operation: (iterated) lexicographic sums of frames, which are important in the context of provability logics [Beklemishev 2010]; lexicographic products of frames, earlier studied in Balbiani [2009, 2010], Balbiani and Fernández-Duque [2016], and Balbiani and Mikulás [2013]; and the operation of refinement of modal logics introduced in Babenyshev and Rybakov [2010]. Further results and directions are discussed in Section 6.

2 PREPARATORY SYNTACTICAL AND SEMANTICAL DEFINITIONS

Let $A \leq \omega$. The set ML$(A)$ of modal formulas over $A$ (or $A$-formulas, for short) is built from a countable set of variables $PV = \{p_0, p_1, \ldots\}$ using Boolean connectives $\bot, \rightarrow$ and unary connectives $\diamond_a$, $a < A$ (modalities). The connectives $\lor, \land, \neg, \top, \Box_a$ are defined as abbreviations in the standard way, in particular $\Box_a \varphi \equiv \neg \diamond_a \neg \varphi$.

An (A-)frame is a structure $F = (W, (R_a)_{a \in A})$, where $W \neq \emptyset$ and $R_a \subseteq W \times W$ for $a < A$. A (Kripke) model on $F$ is a pair $M = (F, \theta)$, where $\theta : PV \rightarrow 2^W$. We write $\text{dom}(F)$ for $W$, which is called the domain of $F$. We write $u \in F$ for $u \in \text{dom}(F)$. For $u \in W$, $V \subseteq W$, we define $R_a(u) = \{v \mid uR_a v\}$, $R_a[V] = \cup_{v \in V} R_a(v)$.

The truth relation in a model is defined in the usual way, in particular $M, v \models \varphi$ means that $M, v \models \varphi$ for some $v \in R_a(w)$. A formula $\varphi$ is satisfiable in a model $M$ if $M, w \models \varphi$ for some $w \in M$. A formula is satisfiable in a frame $F$ (in a class $\mathcal{F}$ of frames) if it is satisfiable in some model on $F$ (in some model on a frame in $\mathcal{F}$). $\varphi$ is valid in a frame $F$ (in a class $\mathcal{F}$ of frames) if $\neg \varphi$ is not satisfiable in $F$ (in $\mathcal{F}$). Validity of a set of formulas means validity of every formula in this set.

The notions of p-morphism, generated subframe, and submodel are defined in the standard way, see e.g., [Gabbay et al. 2003, Section 1.4]. The notation $F \rightarrow G$ means that $G$ is a p-morphic image of $F$; $F \equiv G$ means that $F$ and $G$ are isomorphic.

A (propositional normal modal) logic is a set $L$ of formulas that contains all classical tautologies, the axioms $\neg \diamond_a \bot$ and $\diamond_a (p_0 \lor p_1) \rightarrow \diamond_a p_0 \lor \diamond_a p_1$ for each $a$ in $A$, and is closed under the rules of modus ponens, substitution, and monotonicity (if $\varphi \rightarrow \psi \in L$, then $\diamond_a \varphi \rightarrow \diamond_a \psi \in L$, for each $a$ in $A$).

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The set $\log F$ of all formulas that are valid in a class $F$ of A-frames is a logic (see, e.g., Chagrov and Zakharyaschev [1997]); it is called the logic of $F$; such logics are called Kripke complete. A logic has the finite model property if it is the logic of a class of finite frames (a frame is finite, if its domain is). $\text{Fr} L$ denotes the class of all frames validating $L$.

Remark that for a Kripke complete logic, its decision problem is the validity problem on the class of all its frames (or on any other class $F$ such that $L = \log (F)$). The dual problem is the satisfiability: $\text{Sat}(F)$ is the set of all formulas satisfiable in $F$ (in the signature of $F$). Remark that for the class PSpace (as well as for any other deterministic complexity class), $\text{Sat} F \in \text{PSpace}$ iff $\log F \in \text{PSpace}$.

Natural numbers are considered as finite ordinals. Given a sequence $\nu = (\nu_0, \nu_1, \ldots)$, we write $\nu(i)$ for $\nu_i$.

### 3 SUMS

We fix $A \leq \omega$ for the alphabet and consider the language $\text{ML}(A)$.

Consider a non-empty family $(F_i)_{i \in I}$ of A-frames $F_i = (W_i, (R^a_i)_{a \in A})$. The disjoint union of these frames is the A-frame $\bigcup_{i \in I} F_i = (\bigcup_{i \in I} W_i, (R^a_i)_{a \in A})$, where $\bigcup_{i \in I} W_i = \bigcup_{i \in I} (\{i\} \times W_i)$ is the disjoint union of sets $W_i$, and

$$(i, w)R^a_{i}(j, v) \iff i = j \& wR^a_{iij}v.$$ 

Suppose that $I$ is the domain of another A-frame $l = (I, (S^a)_{a \in A})$.

**Definition 3.1.** The sum of the family $(F_i)_{i \in I}$ of A-frames over the A-frame $l = (I, (S^a)_{a \in A})$ is the A-frame $\sum_{i \in I} F_i = (\bigcup_{i \in I} W_i, (R^a_{i})_{a \in A})$, where

$$(i, w)R^a_{i}(j, v) \iff (i = j \& wR^a_{iij}v) \text{ or } (i \neq j \& iS^a_{ij}).$$

The sum of models $\sum_{i \in I} (F_i, \theta_i)$ is the model $(\sum_{i \in I} F_i, \theta)$, where $(i, w) \in \theta(p)$ if and only if $w \in \theta_i(p)$.

For classes $I$, $F$ of A-frames, let $\sum_I F$ be the class of all sums $\sum_{i \in I} F_i$ such that $I \in I$ and $F_i \in F$ for every $i \in I$.

**Remark 3.2.** We do not require that $S^a$’s are partial orders or even transitive relations. Also, we note that the relations $R^a_{i}$ are independent of reflexivity of the relations $S^a$: if $l' = (I, (S^a')_{a \in A})$, where $S^a'$ is the reflexive closure of $S^a$ for each $a \in A$, then $\sum_{i \in I} F_i = \sum_{i \in I} F_i$.

**Example 3.3 (Skeleton and Clusters).** A cluster is a frame of the form $(W, W \times W)$. Let $F = (W, R)$ be a preorder. The skeleton of $F$ is the partial order $skF = (\bar{W}, \leq_R)$, where $\bar{W}$ is the quotient set of $W$ by the equivalence $R \cap R^{-1}$, and for $C, D \in \bar{W}$, $C \leq_R D$ iff $\exists w \in C \exists v \in D wRv$. The restriction of $R$ on an element of $\bar{W}$ is called a cluster in $F$.

It is easy to see that every preorder $F$ is isomorphic to the sum $\sum_{C \in skF} (C, C \times C)$ of its clusters over its skeleton.

**Example 3.4.** Suppose that a frame $F = (W, R)$ satisfies the property of weak transitivity $xRzRy \Rightarrow xRy \vee x = y$. Let $I$ be the skeleton of the preorder $(W, R^*)$, where $R^*$ is the transitive reflexive closure of $R$. Then $F$ is isomorphic to a sum $\sum_{i \in I} F_i$ such that every $F_i = (W_i, R_i)$ satisfies the property $x \neq y \Rightarrow xR_i y$.

We shall be mainly interested in the polymodal case where the indexing frame has only one non-empty relation.

**Definition 3.5.** Consider a unimodal frame $l = (I, S)$ and a family $(F_i)_{i \in I}$ of A-frames (or A-models). For $a \in A$, the $a$-sum $\sum^a_{i} F_i$ is the sum $\sum_{i} F_i$, where $l'$ is the A-frame whose domain is $I$, the $a$-th relation is $S$ and other relations are empty. If $F$ is a class of A-frames, $I$ is a class of 1-frames, then $\sum_{I} F$ is the class of all sums $\sum^a_{i} F_i$, where $I \in I$ and all $F_i$ are in $F$. 

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3.1 Basic Truth-preserving Operations

The theorem below is a collection of facts illustrating how sums interact with p-morphisms, generated subframes, and disjoint unions. Their proofs are straightforward from definitions, see [Shapirovsky 2018, Section 3] for the details.

**Theorem 3.6 ([Shapirovsky 2018]).**

1. Let $I$ be an $A$-frame and $(F_i)_{i \in I}$ a family of $A$-frames. If $J$ is a generated subframe of $I$, then $\sum_{i \in I} F_i$ is a generated subframe of $\sum_{i \in I} F_i$.

2. Let $I$, $J$ be $A$-frames, $(F_i)_{i \in I}$, $(G_j)_{j \in J}$ families of $A$-frames. Suppose that all the relations in $J$ are irreflexive.
   a. If $f : 1 \to J$ and $F_i \to G_{f(i)}$ for all $i \in 1$, then $\sum_{i \in I} F_i \to \sum_{j \in J} G_j$.
   b. If $f \equiv 1$ and $F_i \to G_i$ for all $i \in 1$, then $\sum_{i \in I} F_i \to \sum_{i \in I} G_i$.
   c. If $f : 1 \to J$, then $\sum_{i \in I} F_{f(i)} \to \sum_{j \in J} G_j$.

3. Let $I$ be an $A$-frame, $(J_i)_{i \in I}$ a family of $A$-frames, and $(F_{ij})_{i \in I, j \in J_i}$ a family of $A$-frames. Then
   \[
   \sum_{i \in I} \sum_{j \in J_i} F_{ij} \equiv \sum_{(i, j) \in \sum_{k \in K} J_k} F_{ij},
   \]
   where $(\emptyset)_A$ denotes the sequence of length $A$ in which every element is the empty set.

3.2 Sums and Universal Modality

For an $A$-frame $F = (W, (R_0, R_1, \ldots))$, let $F^{(0)}$ be the $(1 + A)$-frame $(W, (W \times W, R_0, R_1, \ldots))$. For a class $\mathcal{F}$ of $A$-frames, $\mathcal{F}^{(0)} = \{F^{(0)} \mid F \in \mathcal{F}\}$.

In [Shapirovsky 2018], it was shown that if the classes $\mathcal{F}^{(0)}$ and $\mathcal{G}^{(0)}$ have the same logic, then for any class $I$ of $A$-frames, the logics of sums $\sum_I \mathcal{F}$ and $\sum_I \mathcal{G}$ are equal; moreover, the logics of the classes $\sum_I \mathcal{F}^{(0)}$ and $\sum_I \mathcal{G}^{(0)}$ are equal, thus we have $\Log \sum_J (\sum_I \mathcal{F}) = \Log \sum_J (\sum_I \mathcal{G})$ for any other class of frames-indices $J$, and so on.

**Theorem 3.7 ([Shapirovsky 2018], Theorem 4.11).** Let $I$, $\mathcal{F}$, $\mathcal{G}$ be classes of $A$-frames. If $\Log \mathcal{F}^{(0)} = \Log \mathcal{G}^{(0)}$, then $\Log \sum_I \mathcal{F}^{(0)} = \Log \sum_I \mathcal{G}^{(0)}$, and hence $\Log \sum_I \mathcal{F} = \Log \sum_I \mathcal{G}$.

In particular, it follows that if the logic of the class $\mathcal{F}^{(0)}$ has the finite model property, then the logic of the class of sums $\sum_I \mathcal{F}$ is equal to the logic of the class of sums $\sum_I \mathcal{G}$, where $\mathcal{G}$ is a class of finite frames.

3.3 Decomposition of Sums

To reduce satisfiability in sums to the satisfiability in summands, we will use an auxiliary notion: *satisfiability under conditions*.

**Definition 3.8.** A sequence $\Gamma = (\Gamma_a)_{a \in A}$, where $\Gamma_a$ are sets of $A$-formulas, is called a condition (in the language $ML(A)$).
Consider a model $M = (W, (R_a)_{a \in A}, \theta)$, $w$ in $M$. By induction on the length of an $A$-formula $\phi$, we define the relation $M, w \models_{\Gamma} \phi$ (‘under the condition $\Gamma$, $\phi$ is true at $w$ in $M$’): as usual, $M, w \not\models_{\Gamma} \bot$, $M, w \models_{\Gamma} p$ iff $M, w \models p$ for a variable $p$, $M, w \models_{\Gamma} \phi \rightarrow \psi$ iff $M, w \not\models_{\Gamma} \phi$ or $M, w \models_{\Gamma} \psi$; for $a \in A$,

$$M, w \models_{\Gamma} \circ_a \phi \iff \phi \in \Gamma_a \text{ or } \exists v \in R_a(w) \ M, v \models_{\Gamma} \phi.$$  

In particular, if all $\Gamma_a$ are empty, then we have the standard notion of truth in a Kripke model:

$$M, w \models_{(\circ)_A} \phi \iff M, w \models \phi,$$

where $(\circ)_A$ denotes the condition consisting of empty sets. The truth under conditions is respected by the standard operations on Kripke models:

**Proposition 3.9.** Let $\Gamma$ be a condition and $\phi$ a formula.

1. If $M'$ is a generated submodel of $M$, then $M', w \models_{\Gamma} \phi$ iff $M, w \models_{\Gamma} \phi$ for every $w$ in $M'$.
2. If $M = \bigcup_{i \in I} M_i$, then $M, (i, w) \models_{\Gamma} \phi$ iff $M_i, w \models_{\Gamma} \phi$ for every $i$ in $I$ and every $w$ in $M_i$.
3. If $f : M \rightarrow M'$, then $M, w \models_{\Gamma} \phi$ iff $M', f(w) \models_{\Gamma} \phi$ for every $w$ in $M$.

**Proof.** This proof completely reflects the proof of these facts for the standard truth relation in Kripke models and can be obtained by a straightforward induction on the length of $\phi$. \qed

Let $\text{sub}(\phi)$ be the set of all subformulas of $\phi$. We put

$$\phi[M, \Gamma] = \{ \psi \in \text{sub}(\phi) \mid M, v \models_{\Gamma} \psi \text{ for some } v \}.$$  

In particular, $\phi[M, (\circ)_A] = \{ \psi \mid \text{sub}(\phi) \mid M, v \models_{\Gamma} \psi \text{ for some } v \}$. Hence, we have:

**Proposition 3.10.** A formula $\phi$ is satisfiable in a class $\mathcal{F}$ of frames iff there exists $\Phi \subseteq \text{sub}(\phi)$ such that $\phi \in \Phi$ and the tie $(\phi, \Phi, (\circ)_A)$ is satisfiable in $\mathcal{F}$.

Classes of $A$-frames $\mathcal{F}$ and $\mathcal{G}$ are said to be interchangeable, in symbols $\mathcal{F} \equiv \mathcal{G}$, if the same ties are satisfiable in classes $\mathcal{F}$ and $\mathcal{G}$. From the above proposition, it follows that if $\mathcal{F} \equiv \mathcal{G}$, then the logics of these classes are equal. Moreover, the following holds:

**Proposition 3.11 ([Shapirovsky 2018], Proposition 4.10).** Let $\mathcal{F}$ and $\mathcal{G}$ be classes of $A$-frames. The following are equivalent:

- A tie is satisfiable in $\mathcal{F}$ iff it is satisfiable in $\mathcal{G}$.
- A formula is satisfiable in $\mathcal{F}^{(v)}$ iff it is satisfiable in $\mathcal{G}^{(v)}$.

The latter condition means that the logics of $\mathcal{F}^{(v)}$ and $\mathcal{G}^{(v)}$ are equal. The term ‘interchangeable’ is motivated by Theorem 3.7.

Hence, the language of ties is as expressible as the modal language with the universal modality. In Section 4.2, we will present explicit reductions between these languages.

**Definition 3.12.** Let $V$ be a set of elements of a model $M = (W, (R_a)_{a \in A}, \theta)$. Given a formula $\phi$ and a condition $\Gamma$, let $\Delta$ be the condition defined as follows: for $a \in A$,

$$\Delta(a) = \Gamma(a) \cup \{ \chi \in \text{sub}(\phi) \mid \exists w \in R_a[V], w \in M, w \models_{\Gamma} \chi \}.$$  

$\Delta$ is called the external condition of $V$ in $M$ with respect to $\phi$ and $\Gamma$.

**Remark 3.13.** If $\Gamma \subseteq \text{sub}(\phi)^A$, then $\Delta \subseteq \text{sub}(\phi)^A$ as well (this will be important in the next section, where we consider the conditional satisfiability problem).
Lemma 3.14 ([Shapirovsky 2018], Lemma 4.5). Consider a sum of models $M = \sum_i M_i$, $i$ in $I$, and the set $V = \{i\} \times \text{dom}(M_i)$. If $\Delta$ is the external condition of $V$ in $M$ with respect to some given $\varphi$, $\Gamma$, then for all $v$ in $M_i$, $\chi$ in sub($\varphi$),

$$M_i \models (i, v) \models_{\Gamma} \chi \iff M_i \models (1, v) \models_{\Delta} \chi. \quad (1)$$

Lemma 3.15 below is a particular corollary of Lemma 3.14. It will be important for the proofs of our complexity results.

Consider $a \in A$ and models $M_0, M_1$. The model $M_0 +^a M_1$ is obtained from the disjoint union of $M_0$ and $M_1$ by adding all the pairs of form $((0, w), (1, v))$ to the $a$-th relation; that is, in our general notation, $M_0 +^a M_1 = \sum_i (\Gamma_i \cup M_i)$.

For an $A$-condition $\Gamma = (\Gamma_0, \ldots, \Gamma_{\lambda-1})$ and a set of $A$-formulas $\Psi$, we put $\Gamma \cup^a \Psi = (\Gamma_0', \ldots, \Gamma_{\lambda-1}')$, where $\Gamma_i' = \Gamma_i \cup \Psi$, and $\Gamma_b' = \Gamma_b$ for $b \neq a$.

Lemma 3.15. For $A$-models $M_0, M_1$, an $A$-formula $\varphi$, an $A$-condition $\Gamma$, and $a \in A$, we have

$$\varphi[M_0 +^a M_1, \Gamma] = \varphi[M_0, \Gamma] \cup^a \varphi[M_1, \Gamma] \cup \varphi[M_1, \Gamma]. \quad (2)$$

Proof. Let $V$ be the bottom part of the sum $M_0 +^a M_1$: $V = \{(0, w) \mid w \text{ is in } M_0\}$. Then $\Delta = \Gamma \cup^a \varphi[M_1, \Gamma]$ is the external condition of $V$ w.r.t. $\varphi$ and $\Gamma$. The external condition of the top part $\{(1, v) \mid v \text{ is in } M_1\}$ w.r.t. $\varphi$ and $\Gamma$ is just $\Gamma$. By Lemma 3.14, we have for every $w$ in $M_0$, every $v$ in $M_1$, and every $\chi \in \text{sub}(\varphi)$:

$$M_0 +^a M_1, (0, w) \models_{\Gamma} \chi \iff M_0, w \models_{\Delta} \chi$$

$$M_0 +^a M_1, (1, v) \models_{\Gamma} \chi \iff M_1, v \models_{\Gamma} \chi.$$  

Now (2) follows. \hfill $\Box$

3.4 Sums Over Noetherian Orders

We say that $I$ is a Noetherian (or converse well-founded) order if $I$ is a strict partial order which has no infinite ascending chains. Let NPO be the class of all non-empty Noetherian orders (we say that a partial order is non-empty, if its domain is).

A strict partial order $(I, <)$ is called a (transitive irreflexive) tree if it has a least element (the root) and for all $i \in I$ the set $\{j \mid j < i\}$ is a finite chain. Let $\text{Tr}_f$ be the class of all finite trees.

Consider a finite tree $I = (I, <)$. The branching of $i$ in $I$, denoted by $br(i, I)$, is the number of immediate successors of $i$ ($i$ is an immediate successor of $i$, if $i < j$ and there is no $k$ such that $i < k < j$); the branching of $I$, denoted by $br(I)$, is max $\{br(i, I) \mid i \text{ in } I\}$. The height of $I$, denoted by $ht(I)$, is max $\{|V| \mid V \text{ is a chain in } I\}$. For $h, b \in \omega$, let $\text{Tr}(h, b)$ be the class of all finite trees with height $\leq h$ and branching $\leq b$:

$$\text{Tr}(h, b) = \{l \in \text{Tr}_f \mid ht(l) \leq h \& br(l) \leq b\}.$$

Let $\biguplus_i F_i$ be the class of frames of form $\biguplus_i F_i$, where $I$ is a non-empty set, $F_i \in F$ for all $i \in I$, and let $\biguplus_{<k} F$ be the class of such frames with $0 < |I| \leq k$. Likewise for $\biguplus_{\leq k} F$.

Let $\#\varphi$ denote the number of subformulas of a formula $\varphi$.

Theorem 3.16 ([Shapirovsky 2018]). Let $F$ be a class of $A$-frames, $a \in A$, and $I$ a class of Noetherian orders containing all finite trees.

(1) We have

$$\log \sum_{NPO}^a F = \log \sum_{I}^a F = \log \sum_{\text{Tr}_f}^a F.$$
Moreover, for every $A$-formula $\varphi$ we have:

\[ \varphi \text{ is satisfiable in } \sum_{\text{NPO}}^a \mathcal{F} \text{ iff } \varphi \text{ is satisfiable in } \sum_{\text{Tr}(\#\varphi, \#\lnot\varphi)}^a \mathcal{F}. \]

(2) Assume that $\mathcal{I}$ is closed under finite disjoint unions. Then

\[ \sum_{\text{NPO}}^a \mathcal{F} \equiv \sum_{\mathcal{I}}^a \mathcal{F} \equiv \bigsqcup_{\varphi} \sum_{\text{Tr}(\#\varphi, \#\lnot\varphi)}^a \mathcal{F}. \]

Moreover, for every $A$-tie $\tau = (\varphi, \Phi, \Gamma)$ we have:

\[ \tau \text{ is satisfiable in } \sum_{\text{NPO}}^a \mathcal{F} \text{ iff } \tau \text{ is satisfiable in } \bigsqcup_{\varphi} \sum_{\text{Tr}(\#\varphi, \#\lnot\varphi)}^a \mathcal{F}. \] (3)

In view of Proposition 3.10, the first statement of the theorem is a corollary of the second statement. For the key equivalence (3), see Theorem 5.2(i) in Shapirovsky [2018].

Theorem 3.16 will be the crucial semantic tool for the complexity results.

4 COMPLEXITY

The main goal of this section is to show that the modal satisfiability problem on sums over Noetherian orders is polynomial space Turing reducible to the modal satisfiability problem on summands.

For problems $A$ and $B$, we put $A \leq_{PSpace}^T B$ if there exists a polynomial space bounded oracle deterministic machine $M$ with oracle $B$ that decides $A$ [Simon and Gill 1977] (it is assumed that every tape of $M$, including the oracle tape, is polynomial space bounded).

**Theorem 4.1.** Let $a < A < \omega$, $\mathcal{F}$ a class of $A$-frames, and $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then:

1. \( \text{Sat} \sum_{\mathcal{I}}^a \mathcal{F} \leq_{PSpace}^T \text{Sat} \mathcal{F}(V) \).
2. \( \text{If also } \mathcal{I} \text{ is closed under finite disjoint unions, then } \text{Sat} (\sum_{\mathcal{I}}^a \mathcal{F})(V) \leq_{PSpace}^T \text{Sat} \mathcal{F}(V) \).

This theorem will be proven in Section 4.3. For technical reasons, first we will address complexity of the conditional satisfiability problem. Let $A$ be finite, and let $\mathcal{F}$ be a class of $A$-frames. We shall be interested in whether a given tie $(\varphi, \Phi, \Gamma)$ is satisfiable in $\mathcal{F}$. The following simple observation shows that, w.l.o.g., we may assume that every $\Gamma(a)$, $a < A$, consists of subformulas of $\varphi$, and hence that $\Gamma$ is a finite sequence of finite sets:

**Proposition 4.2.** A tie $(\varphi, \Phi, \Gamma)$ is satisfiable in a class $\mathcal{F}$ iff $(\varphi, \Phi, \text{sub}(\varphi) \cap \Gamma(a))_{a \in A}$ is.

**Proof.** It is immediate from Definition 3.8 that for any conditions $\Gamma, \Delta$ such that $\Gamma(a) \cap \text{sub}(\varphi) = \Delta(a) \cap \text{sub}(\varphi)$ for all $a \in A$,

we have

\[ M, w \models_{\Gamma} \chi \text{ iff } M, w \models_{\Delta} \chi \]

for every model $M$, every $w$ in $M$, and every $\chi \in \text{sub}(\varphi)$.

The statement of the proposition is a particular case of this observation, where $\Delta(a) = \text{sub}(\varphi) \cap \Gamma(a)$ for all $a \in A$. \qed

The conditional satisfiability problem on $\mathcal{F}$ is to decide whether a given tie $(\varphi, \Phi, \Gamma)$ such that $\Gamma \subseteq \text{sub}(\varphi)^A$ is satisfiable in $\mathcal{F}$. In symbols,

\[ \text{CSat} \mathcal{F} = \{ (\varphi, \Phi, \Gamma) \mid \varphi \text{ is an A-formula } \& \Phi \subseteq \text{sub}(\varphi) \& \Gamma \subseteq \text{sub}(\varphi)^A \& \text{ the tie } (\varphi, \Phi, \Gamma) \text{ is satisfiable in } \mathcal{F} \}. \]
In Section 4.1 we will describe a decision procedure for the conditional satisfiability problem on sums over Noetherian orders $\sum_{n \in \mathbb{N}} F$ with the oracle $\text{CSat } F$. Next, in Section 4.2, we will describe reductions between $\text{CSat } F$ and $\text{Sat } F^{(\forall)}$, which will complete the proof of Theorem 4.1.

### 4.1 Arithmetic of Conditional Satisfiability

It will be convenient to encode subformulas of a given $\varphi$ as Boolean vectors of length $\# \varphi$, considered as characteristic functions on $\text{sub}(\varphi)$. For $A \subseteq \omega$, the set of modal formulas is linearly ordered by a polynomial time computable relation $\not\equiv$ such that if $\psi$ is a subformula of $\varphi$, then $\varphi \equiv \psi$ (e.g., put $\varphi \equiv \psi$ if $\psi$ is shorter than $\varphi$, and assume that formulas of the same length are ordered lexicographically). Let $(\psi_0, \ldots, \psi_{\#\varphi})$ be the $\not\equiv$-chain of all subformulas of $\varphi$ (hence, $\psi_0 = \varphi$); for $v \in 2^{\#\varphi}$, we write $\varphi[v]$ for $\{\psi_i \mid v(i) = 1\}$; similarly, a sequence $U = (u_a)_{a \in A}$ of such vectors represents the condition $G = (\varphi[u_a])_{a \in A}$. Hence, for a finite $A$, every tie $\tau = (\varphi, \Phi, \Gamma)$ with $\Gamma \subseteq \text{sub}(\varphi)^A$ is represented by a triple $\tau' = (\varphi, v, (u_a)_{a \in A})$, where $v, u_0, \ldots, u_{\#\varphi} \in 2^{\#\varphi}$; this triple is also called a tie. In this case, by the satisfiability of $\tau'$ we mean the satisfiability of $\tau$.

Let $0_\varphi$ denote the sequence of length $A$ of zero vectors of length $\#\varphi$ (that is, $0_\varphi$ represents the condition, consisting of empty sets). In view of Proposition 4.2, we have the following reformulation of Proposition 3.10:

**Proposition 4.3.** $\varphi$ is satisfiable in $F$ iff there exists $v \in 2^{\#\varphi}$ such that $v(0) = 1$ and the tie $(\varphi, v, 0_\varphi)$ is satisfiable in $F$.

For Boolean vectors $v = (v_0, \ldots, v_{l-1})$, $u = (u_0, \ldots, u_{l-1})$, let $v + u$ be their element-wise disjunction ($\max\{v_0, u_0\}, \ldots, \max\{v_{l-1}, u_{l-1}\}$).

**Lemma 4.4.** Let $G$ be a class of $A$-frames and $0 < b < \omega$. A tie $(\varphi, v, U)$ is satisfiable in $\bigsqcup_{i \leq b} G$ iff there exist a positive $k \leq b$, $v_0, \ldots, v_{k-1} \in 2^{\#\varphi}$ such that $v = \sum_{0 < k} v_i$ and for every $i < k$ the tie $(\varphi, v_i, U)$ is satisfiable in $G$.

**Proof.** This proposition is a corollary of Proposition 3.9 formulated in our new vector notation for ties. Indeed, by Proposition 3.9, we have

$$\varphi\left[\bigsqcup_{i < k} M_i, \Gamma\right] = \bigcup_{i < k} \varphi[M_i, \Gamma]$$

for every $A$-models $M_0, \ldots, M_{k-1}$ and every $A$-condition $\Gamma$. We are interested in the situation when $0 < k \leq b$ and frames of $M_0, \ldots, M_{k-1}$ are in $G$. Assuming that $\Gamma$ is the condition represented by $U$ and vectors $v, v_1, \ldots, v_{k-1}$ are given by the identities $\varphi[v] = \varphi[\bigsqcup_{i < k} M_i, \Gamma], \varphi[v_0] = \varphi[M_0, \Gamma], \ldots, \varphi[v_{k-1}] = \varphi[M_{k-1}, \Gamma]$, we see that (4) takes the form $v = \sum_{0 < k} v_i$. \hfill \Box

For Boolean vectors $v, u_0, \ldots, u_{A-1}$ of the same length and $a \in A$, we put $(u_0, \ldots, u_{A-1}) +^a v = (u_0', \ldots, u'_{A-1})$, where $u'_0 = u_a + v$, and $u'_b = u_b$ for $b \neq a$.

Similarly to models, for $A$-frames $F_0$ and $F_1$ and $a \in A$ we define $F_0 +^a F_1$ as $\sum_{2, <}^a F_i$; for classes $F$ and $G$ of $A$-frames, let $F +^a G = \{ F +^a G \mid F \in F \land G \in G \}$.

**Lemma 4.5.** Let $F$ and $G$ be classes of $A$-frames and $a \in A$. Then a tie $(\varphi, v, U)$ is satisfiable in $F +^a G$ iff there exist $v_0, v_1 \in 2^{\#\varphi}$ such that

1. $v = v_0 + v_1$, and
2. $(\varphi, v_0, U +^a v_1)$ is satisfiable in $F$, and
3. $(\varphi, v_1, U)$ is satisfiable in $G$. 
Proof. Let $M_0$ be a model on a frame in $\mathcal{F}$, and let $M_1$ be a model on a frame in $\mathcal{G}$. Let $\Gamma$ be the condition represented by $U$, i.e., $\Gamma = (\varphi[U(a)])_{a \in A}$, and let $\Psi = \varphi[M_1, \Gamma]$. By Lemma 3.15,

$$\varphi[M_0 + a M_1, \Gamma] = \varphi[M_0, \Gamma \cup a \Psi] \cup \Psi.$$ (5)

For the “only if” part, assume that $\varphi[v] = \varphi[M_0 + a M_1, \Gamma]$. Consider tuples $v_0, v_1 \in 2^{\Psi}$ such that

$$\varphi[v_0] = \varphi[M_0, \Gamma \cup a \Psi], \text{ and}$$ (6)

$$\varphi[v_1] = \varphi[M_1, \Gamma].$$ (7)

The identity (6) says that $(\varphi, v_0, U + a v_1)$ is satisfiable in $\mathcal{F}$, and the identity (7) says that $(\varphi, v_1, U)$ is satisfiable in $\mathcal{G}$. Since $\Psi = \varphi[M_1, \Gamma]$, by (5) we obtain $\varphi[v] = \varphi[v_0] \cup \varphi[v_1]$, that is $v = v_0 + v_1$.

For the “if” part, assume that for some $v_0, v_1$ with $v = v_0 + v_1$ we have (6) (this can be assumed since $(\varphi, v_0, U + a v_1)$ is satisfiable in $\mathcal{F}$), and (7) (this can be assumed since $(\varphi, v_1, U)$ is satisfiable in $\mathcal{G}$). Since $v = v_0 + v_1$, we have $\varphi[v] = \varphi[v_0] \cup \varphi[v_1]$. Now by (5) we obtain that $\varphi[v] = \varphi[M_0 + M_1, \Gamma]$. The latter proves that $(\varphi, v, U)$ is satisfiable in $\mathcal{F} + a \mathcal{G}$. $\square$

Lemma 4.6. Let $\mathcal{F}$ be a class of $A$-frame, $a \in A$, and $0 < h, b < \omega$. A tie $(\varphi, v, \Psi)$ is satisfiable in $\sum_{\Gamma(h+1, b)}^a \mathcal{F}$ if it is satisfiable in $\mathcal{F}$, or there exist a positive $k \leq b$ and $u, v_0, \ldots, v_{k-1} \in 2^{\Psi}$ such that

1. $v = u + \sum_{i<k} v_i$, and
2. the tie $(\varphi, u, U + a \sum_{i<k} v_i)$ is satisfiable in $\mathcal{F}$, and
3. for all $i < k$, the tie $(\varphi, v_i, U)$ is satisfiable in $\sum_{\Gamma(h, b)}^a \mathcal{F}$.

Proof. By the definition of $\sum_{\Gamma(h+1, b)}^a \mathcal{F}$, the tie $\tau = (\varphi, v, U)$ is satisfiable in $\sum_{\Gamma(h+1, b)}^a \mathcal{F}$ iff $\tau$ is satisfiable in $\mathcal{F}$ or in $\mathcal{F} + a \mathcal{G}$, where $\mathcal{G} = \bigcup_{0 \leq h} \sum_{\Gamma(h, b)}^a \mathcal{F}$.

By Lemma 4.5, $\tau$ is satisfiable in $\mathcal{F} + a \mathcal{G}$ iff there exist $u, u' \in 2^{\Psi}$ such that $v = u + u'$, $(\varphi, u, U + a u')$ is satisfiable in $\mathcal{F}$, and $(\varphi, u', U)$ is satisfiable in $\mathcal{G}$. By Lemma 4.4, $(\varphi, u', U)$ is satisfiable in $\mathcal{G}$ iff there exist $0 < k \leq b$ and $v_0, \ldots, v_{k-1} \in 2^{\Psi}$ such that $u' = \sum_{i<k} v_i$, and for all $i < k$ the tie $(\varphi, v_i, U)$ is satisfiable in $\sum_{\Gamma(h, b)}^a \mathcal{F}$.

This lemma allows to describe the procedure CSatSum$\mathcal{F}$ (Algorithm 1) which using an oracle for CSat $\mathcal{F}$ decides whether a given tie is satisfiable in $\sum_{\Gamma(h, b)}^a \mathcal{F}$. Namely, we have:

Theorem 4.7. Let $a \in A < \omega$, $\mathcal{F}$ a class of $A$-frames, $(\varphi, v, U)$ an $A$-tie, and $0 < h, b < \omega$. Then

$$(\varphi, v, U) \text{ is satisfiable in } \sum_{\Gamma(h, b)}^a \mathcal{F} \text{ iff } \text{CSatSum}_\mathcal{F}(\varphi, v, U, h, b) \text{ returns true.}$$

This theorem will be our main technical tool for complexity results. For the first of its corollaries, we show how to reduce the conditional satisfiability on sums to the conditional satisfiability on summands.

Theorem 4.8. Let $a \in A < \omega$, $\mathcal{F}$ a class of $A$-frames, and $I$ a class of Noetherian orders containing all finite trees. Then:

1. $\text{Sat} \sum_I^a \mathcal{F} \leq_{T}^{\text{PSPACE}} \text{CSat} \mathcal{F}$.
2. If also $I$ is closed under finite disjoint unions, then $\text{CSat} \sum_I^a \mathcal{F} \leq_{T}^{\text{PSPACE}} \text{CSat} \mathcal{F}$.

Proof. By Theorem 3.16(1), a formula $\varphi$ is satisfiable in $\sum_I^a \mathcal{F}$ iff $\varphi$ is satisfiable in $\sum_{\Gamma(\varphi, \varphi)}^a \mathcal{F}$, and by Proposition 4.3, this means that there exists a satisfiable in $\sum_{\Gamma(\varphi, \varphi)}^a \mathcal{F}$ tie $(\varphi, v, 0_\varphi)$ with $v(0) = 1$. From Theorem 4.7 we obtain
such that $\sum$. This fact is closest to a result obtained in [Hemaspaandra and sur 1996] that $\sum = \sum + b$. If $h > 1$ then for $k$ such that $1 \leq k \leq b$ do for $u, v_0, \ldots, v_{k-1} \in 2^{\phi}$ such that $v = u + \sum_{i < k} v_i$ do if $(\phi, u, u + ^{\sum_{i \leq k}} v_i)$ is satisfiable in $F$ then if $\bigwedge_i CSatSum(\phi, v_i, U, h-1, b)$ then return true; return false.

**Lemma 4.9.** A formula $\phi$ is satisfiable in $\sum^a F$ iff there exists $v \in 2^{\phi}$ such that $v(0) = 1$ and $CSatF(\phi, v, 0, \# \phi, \# \phi)$ returns true.

Assume that $I$ is closed under finite disjoint unions. By Theorem 3.16(2), a tie $(\phi, v, U)$ is satisfiable in $\sum I F$ iff it is satisfiable in $\sum_{i \leq k}^a F$. By Lemma 4.4, this means that there exists $k \leq \# \phi$ and tuples $v(0), \ldots, v(k-1) \in 2^{\phi}$ such that $u = \sum_{i < k} v(i)$ and $(\phi, v(0), \# \phi, \# \phi)$ is satisfiable in $\sum_{i \leq k}^a F$ for every $i < k$. Using Theorem 4.7 again, we obtain

**Lemma 4.10.** If $I$ is closed under finite disjoint unions, then a tie $(\phi, v, U)$ is satisfiable in $\sum I F$ iff there exist $k \leq \# \phi$ and tuples $v(0), \ldots, v(k-1) \in 2^{\phi}$ such that $u = \sum_{i < k} v(i)$ and $CSatF(\phi, v(0), \# \phi, \# \phi)$ returns true for every $i < k$.

Set $n = \# \phi$. Let us estimate the amount of space used by $CSatSumF$ for the case $n = h = b$. At each call $CSatSumF$ needs $O(n^2)$ space to store new variables. The depth of recursion is bounded by $n$. Thus, we can reduce $Sat \sum I F$ to the conditional satisfiability problem on $F$ in $O(n^3)$ space by Lemma 4.9. This proves the first statement of the theorem. In the case when $I$ is closed under finite disjoint unions, $CSat \sum I F$ is also reducible to the conditional satisfiability problem on $F$ in $O(n^3)$ space by Lemma 4.10. This proves the second statement of the theorem. □

### 4.2 Reductions between $Sat F$, $CSat F$, and $Sat F^{(I)}$

By Proposition 4.3, the satisfiability problem on a class $F$ is polynomial space Turing reducible to the conditional satisfiability problem on $F$. Let $\leq_{PTime}$ denote the polynomial-time many-to-one reduction. Below we show that in many cases $CSatF$ is polynomial time reducible to $SatF$, that is $CSatF \leq_{PTime}^m SatF$; hence, in these cases, the above two problems are equivalent with respect to $\leq_{PTime}$. This fact is close to a result obtained in [Hemaspaandra 1996], where it was shown that in many situations there exists a (stronger than $\leq_{PTime}^m$, but weaker$^1$ than $\leq_{PTime}^m$) reduction of $Sat F^{(I)}$ to $Sat F$. Let us discuss reductions between the above two problems in more details.

For a binary relation $R$ on a set $W$, put $R^{\leq m} = \bigcup_{n \leq m} R^n$, where $R^0$ is the identity relation on $W$, $R^{m+1} = R \cap R^n$ (operation denotes the composition of relations). Recall that $R^*$ denotes the transitive reflexive closure of $R$: $R^* = \bigcup_{n \geq 0} R^n$. $R$ is said to be $m$-transitive if $R^{\leq m}$ includes $R^{m+1}$, or equivalently, $R^{\leq m} = R^*$ (e.g., every transitive relation is 1-transitive). For a frame $F = (W, \{R_a\}_{a \in A})$, put $R_F = \bigcup_{a \in A} R_a$. We say that $F$ is $m$-transitive if the relation $R_F$ is. In particular, if one of the relations of $F$ is universal (i.e., is equal to $W \times W$), then $F$ is 1-transitive.

For $w$ in $F$, let $F[w]$ denote the subframe of $F$ generated by the singleton $\{w\}$ (that is, $F[w]$ is the restriction of $F$ on the set $\{v \mid w R_a v\}$); such frames are called cones. A class $F$ of frames is closed under taking cones if for every $F$ in $F$ and for every $w$ in $F$, the cone $F[w]$ is in $F$.

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$^1$I am using ”stronger” and ”weaker” in a non-strict sense.

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A class \( \mathcal{F} \) of A-frames is said to be preconical, if

- \( \mathcal{F} \) is closed under taking cones, and
- there exists \( m < \omega \) such that every frame in \( \mathcal{F} \) is \( m \)-transitive, and
- for every \( F \) in \( \mathcal{F} \), the relation \( R^*_F \) is downward directed (i.e., for every \( w, v \) in \( F \) there exists \( u \) such that \( uR^*_Fw \) and \( uR^*_Fv \)).

Proposition 4.11. Let \( A \) be finite and \( \mathcal{F} \) a class of A-frames. If \( \mathcal{F} \) is preconical, then we have \( \text{CSat} \mathcal{F} \leq_{\text{mPTime}} \text{Sat} \mathcal{F} \).

Proof. Given a condition \( \Gamma \) and a formula \( \phi \), we define the formula \([\phi]_{\Gamma}\) as follows: \([\bot]_{\Gamma} = \bot\), \([p]_{\Gamma} = p\) for variables, \([\phi_1 \to \phi_2]_{\Gamma} = [\phi_1]_{\Gamma} \to [\phi_2]_{\Gamma}\), and

\[
[\square_a \phi]_{\Gamma} = \begin{cases} \top, & \text{if } \phi \in \Gamma(a), \\ [\square_a [\phi]]_{\Gamma}, & \text{otherwise.} \end{cases}
\]

For every A-model \( M \), we have:

\[
M, w \models_\Gamma \phi \iff M, w \models [\phi]_{\Gamma} \tag{8}
\]

The proof is straightforward, see [Shapiroovsky 2018, Lemma 4.7] for the details.

Let \( \lozenge \phi \) abbreviate the A-formula \( \bigvee_{a \in A} \lozenge_a \phi \), and let \( \lozenge^0 \phi = \phi, \lozenge^{m+1} \phi = \lozenge \lozenge^m \phi, \lozenge^{\omega} \phi = \bigvee_{n \leq m} \lozenge^n \phi \).

For a tie \((\phi, v, U)\) with \( U = (u_a)_{a \in A} \), we put

\[
\delta_m(\phi, v, U) = \bigwedge_{\psi \in \phi[v]} \lozenge^{\leq m}[\psi]_{\Gamma} \land \bigwedge_{\psi \in \text{sub}(\phi) \setminus \phi[v]} \neg \lozenge^{\leq m}[\psi]_{\Gamma},
\]

where \( \Gamma \) is the condition represented by \( U \), i.e., \( \Gamma = (\phi[u_a])_{a \in A} \).

Since \( \mathcal{F} \) is preconical, there exists a finite \( m \) such that every frame in \( \mathcal{F} \) is \( m \)-transitive. We claim that

\[
(\phi, v, U) \in \text{CSat} \mathcal{F} \iff \delta_m(\phi, v, U) \in \text{Sat} \mathcal{F}. \tag{10}
\]

First, assume that \((\phi, v, U)\) is satisfiable in a frame \( F \in \mathcal{F} \). This means that for a model \( M \) based on \( F \) we have \( \phi[v] = \phi[M, \Gamma] \), where \( \Gamma \) is the condition represented by \( U \). For every \( \psi \in \phi[v] \) we choose a point \( w_\psi \) such that \( M, w_\psi \models_\Gamma \phi \), and then put \( V = \{ w_\psi \mid \psi \in \phi[v] \} \). The relation \( R^*_F \) is downward directed, hence there exists a point \( w \) in \( M \) such that \( wR^*_F v \) for all \( v \) in \( V \); by \( m \)-transitivity, \( wR^*_{\leq m} V \). It follows that if \( \psi \in \phi[v] \), then \( M, w \models \lozenge^{\leq m}[\psi]_{\Gamma} \); indeed, we have \( M, w_\psi \models [\psi]_{\Gamma} \) by (8) and \( wR^*_{\leq m} w_\psi \). On the other hand, if a subformula \( \psi \) of \( \phi \) is not in \( \phi[v] \), then \([\psi]_{\Gamma}\) is false at every point in \( M \) by (8), and so \( M, w \models \neg \lozenge^{\leq m}[\psi]_{\Gamma} \). It follows that the formula \( \delta_m(\phi, v, U) \) is satisfiable in \( \mathcal{F} \).

Now assume that \( \delta_m(\phi, v, U) \) is true at a point \( w \) in a model \( M \) over a frame \( F \in \mathcal{F} \). Since \( \mathcal{F} \) is closed under taking cones, we may assume that \( F = F[w] \) (recall that if a formula is true at a point in a model, then it is true at this point in the model generated by this point). Let \( \psi \) be a subformula of \( \phi \). Suppose that \( \psi \in \phi[v] \). Then \( M, w \models \lozenge^{\leq m}[\psi]_{\Gamma} \). Hence, the formula \([\psi]_{\Gamma}\) is true at a point \( u \) of \( M \), which means that \( M, u \models_\Gamma \psi \) by (8). Thus, \( \psi \in \phi[M, \Gamma] \). On the other hand, if \( \psi \notin \phi[v] \), then \( M, w \models \neg \lozenge^{\leq m}[\psi]_{\Gamma} \); by \( m \)-transitivity, we obtain that \([\psi]_{\Gamma}\) is false at every point of \( M \); using (8) again, we obtain \( \psi \notin \phi[M, \Gamma] \). Thus, \( \phi[v] = \phi[M, \Gamma] \). Since \( \Gamma = (\phi[u_a])_{a \in A} \), the tie \((\phi, v, U)\) is satisfiable in \( \mathcal{F} \).

Proposition 4.12. If \( A \) is finite and \( \mathcal{F} \) is a class of A-frames, then \( \text{CSat} \mathcal{F} \leq_{\text{mPTime}} \text{Sat} \mathcal{F} \).

Proof. It is trivial that \( \text{CSat} \mathcal{F} \leq_{\text{mPTime}} \text{CSat} \mathcal{F} \) (the reduction increases by one the indexes of modalities occurring in formulas of a given tie).
The class $F^{(v)}$ is preconical: if $F \in F$, then $F$ is 1-transitive, $F[w] = F$ for every $w \in F$, and $R^F$ is downward-directed, since $R_F$ is the universal relation on $F$. Hence, CSat $F^{(v)} \leq_{\text{m}} \text{PTime}$ Sat $F^{(v)}$ by Proposition 4.11. 

Now let us describe a reduction of Sat $F^{(v)}$ to CSat $F$. This reduction is based on the following construction proposed in [Hemaspaandra 1996]. Let $\varphi$ be a formula in the language ML(1+A). For subformulas $\varphi_0 \psi$ of $\varphi$ starting with $\varphi_0$, we choose distinct variables $p_\psi$ not occurring in $\varphi$, and put for subformulas of $\varphi$: $\perp' = \perp$; $p' = p$ for variables; $(\varphi_1 \rightarrow \varphi_2)' = \varphi_1' \rightarrow \varphi_2'$; $(\varphi_a)' = \varphi_a \psi'$ for $a > 0$; and $(\varphi_0 \psi)' = p_\psi$. We have for a class $F$ of A-frames:

$$\varphi \in \text{Sat } F^{(v)} \iff \varphi' \land \bigwedge_{\varphi_0 \psi \in \text{sub}(\varphi)}\left((\varphi_0 \psi' \leftrightarrow \Box_0 p_\psi) \land (\Box_0 p_\psi \lor \Box_0 \neg p_\psi)\right) \in \text{Sat } F^{(v)},$$

(11)

see [Hemaspaandra 1996, Lemma 4.5] for details.²

Let $\eta$ denote the formula in the right-hand side of the above equivalence. Let us show how the satisfiability of $\eta$ in $F^{(v)}$ can be expressed as the satisfiability of a tie in $F$. Consider the formula

$$\xi_\varphi = \varphi' \land \bigwedge_{\varphi_0 \psi \in \text{sub}(\varphi)}(\psi' \land \neg p_\psi)$$

(the only role of the second conjunct is to have $\psi'$, $\neg p_\psi$, $p_\psi$ that occur in $\eta$ as subformulas of $\xi_\varphi$).

Let $M$ be a model on a frame in $F^{(v)}$, and let $\Phi$ be the set of subformulas of $\xi_\varphi$ that are satisfiable in $M$, that is, $\Phi = \xi_\varphi[M, (\varnothing)_{1+A}]$. Then $\eta$ is true at a point in $M$ iff

$$\varphi' \in \Phi \text{ and for every } \psi \text{ with } \varphi_0 \psi \in \text{sub}(\varphi), \psi' \in \Phi \text{ if } p_\psi \in \Phi \text{ iff } \neg p_\psi \notin \Phi.$$ (12)

It follows that $\eta$ is satisfiable in $F^{(v)}$ iff there exists $\Phi \subseteq \text{sub}(\xi_\varphi)$ satisfying (12) and the tie $(\xi_\varphi, \Phi, (\varnothing)_{1+A})$ is satisfiable in $F^{(v)}$. The formula $\xi_\varphi$ and so, the formulas in $\Phi$, do not contain $\varphi_0$. Hence, the satisfiability of $(\xi_\varphi, \Phi, (\varnothing)_{1+A})$ in $F^{(v)}$ is equivalent to the satisfiability of $(\xi_\varphi, \Phi, (\varnothing)_{\lambda})$ in $F$, where $\xi_\varphi$ and $\Phi$ are obtained by decreasing indexes of modalities by 1. Putting everything together, we have shown that

$$\varphi \in \text{Sat } F^{(v)} \iff \text{there exists } \Phi \subseteq \text{sub}(\xi_\varphi) \text{ satisfying (12) such that } (\xi_\varphi, \Phi, (\varnothing)_{\lambda}) \text{ is satisfiable in } F.$$ This proves

**Proposition 4.13.** If $A$ is finite and $F$ is a class of A-frames, then Sat $F^{(v)} \leq_{\text{TSpace}}$ CSat $F$.

**Remark 4.14.** In fact, Proposition 4.13 provides a stronger reduction than $\leq_{\text{TSpace}}$.

### 4.3 Proof of Theorem 4.1

In view of the above propositions, Theorem 4.1 is an easy corollary of Theorem 4.8.

**Proof of Theorem 4.1.** We obtain Sat $\sum^a_I F \leq_{\text{TSpace}}$ CSat $F$ by Theorem 4.8 (1), and then CSat $F \leq_{\text{m}} \text{PTime}$ Sat $F^{(v)}$ by Proposition 4.12. Since $\leq_{\text{m}} \text{PTime}$ is stronger than $\leq_{\text{TSpace}}$, we obtain CSat $F \leq_{\text{TSpace}}$ Sat $F^{(v)}$. Hence, Sat $\sum^a_I F \leq_{\text{TSpace}}$ Sat $F^{(v)}$, which proves the first statement of the theorem.

To prove the second statement, we start with Proposition 4.13 and obtain Sat$(\sum^a_I F)^{(v)} \leq_{\text{TSpace}}$ CSat$(\sum^a_I F)$. Now by Theorem 4.8 (2) and Proposition 4.12, we obtain CSat $\sum^a_I F \leq_{\text{TSpace}}$ Sat $F^{(v)}$. So Sat$(\sum^a_I F)^{(v)} \leq_{\text{TSpace}}$ Sat $F^{(v)}$, which completes the proof. 

² The formula in (11) is an equivalent form of the formula $\varphi_{flar}$ used in [Hemaspaandra 1996, Lemma 4.5].
Recall that in the preconical case, the conditional satisfiability problem is reducible to the standard modal satisfiability problem (Proposition 4.11), and so CSat $\mathcal{F}$, Sat $\mathcal{F}$, and Sat $\mathcal{F}(\forall)$ are $\leq_{\text{PSpace}}$-equivalent by Propositions 4.12, 4.13 (in general, the decidability of Sat $\mathcal{F}$ does not imply the decidability of Sat $\mathcal{F}(\forall)$, see [Spaan 1993, Theorem 4.2.1]). In this case, Theorem 4.1 can be reformulated in the following way:

**Corollary 4.15.** Let $a < A < \omega$, $\mathcal{F}$ a class of A-frames, and $I$ a class of Noetherian orders containing all finite trees. If $\mathcal{F}$ is preconical, then:

1. Sat $\sum_{\mathcal{F}}^a \leq_{\text{PSpace}}$ Sat $\mathcal{F}$.
2. If also $I$ is closed under finite disjoint unions, then Sat $(\sum_{\mathcal{F}}^a)^{(\forall)} \leq_{\text{PSpace}}$ Sat $\mathcal{F}$.

### 4.4 PSPACE-hardness: Some Corollaries of Ladner’s Construction

According to Ladner’s theorem, every unimodal logic contained in S4 is PSpace-hard [Ladner 1977]. In fact, Ladner’s proof yields PSpace-hardness for a wider class of modal logics (e.g., for logics contained in the Gödel-Löb logic GL or in the Grzegorczyk logic Grz), see [Spaan 1993]. With minor modifications, Ladner’s construction also works for logics contained in S4.1, Grz2 etc., for the polymodal case, and in particular – for sums.

To illustrate this, let us briefly discuss the proof. Consider a quantified Boolean formula $\eta = Q_1 p_1 \ldots Q_m p_m \theta$, where $Q_1, \ldots, Q_m \in \{\exists, \forall\}$, and $\theta$ is a propositional Boolean formula in variables $p_1, \ldots, p_m$. Choose fresh variables $q_0, \ldots, q_m$. Let $[\eta]_L$ be the following unimodal formula$^3$:

$$q_0 \land \bigwedge_{i < m} \Box^{\leq m}(q_i \rightarrow \Diamond q_{i+1}) \land \bigwedge_{i \neq j, i, j \leq m} \Box^{\leq m}(q_i \rightarrow \neg q_j) \land \Box^m(q_m \rightarrow \theta) \land$$

$$\bigwedge_{\{i < m | Q_i = \forall\}} \Diamond^i(q_i \rightarrow \Diamond(q_{i+1} \land p_{i+1}) \land \Diamond(q_{i+1} \land \neg p_{i+1})) \land$$

$$\bigwedge_{1 \leq i \leq m-1} \Diamond^i(q_i \rightarrow \Box^{\leq m} p_i \lor \Box^{\leq m} \neg p_i)$$

where $\Box^m = \neg \Diamond^m \neg$; likewise for $\Box^{\leq m}$.

Let $T_\eta = (W_\eta, R_\eta)$ be the quantifier tree of $\eta$: $W_\eta = \bigcup_{k \leq m} \{ \sigma \in 2^k \mid \sigma(i) = 0 \text{ if } Q_{i+1} = \exists\}; \sigma_1 R_\eta \sigma_2$ iff $\sigma_1 \subset \sigma_2$ and $\text{dom}(\sigma_2) = \text{dom}(\sigma_1) + 1$. We remark that $T(\eta)$ is antitransitive (that is, $xRyRz \Rightarrow \neg(xRz)$ holds in $T_\eta$), and so is irreflexive. We have:

$$\eta \text{ is valid } \Rightarrow [\eta]_L \text{ is satisfiable in } (W_\eta, R_\eta^\ast), \quad (13)$$

$$[\eta]_L \text{ is satisfiable in a Kripke frame } \Rightarrow \eta \text{ is valid}; \quad (14)$$

see [Blackburn et al. 2002, Section 6.7] for the details.

Consider a unimodal logic $L \subseteq S4$. Then $(W_\eta, R_\eta^\ast)$ is an $L$-frame, hence every satisfiable in this frame formula is $L$-consistent. So we have:

$$\eta \text{ is valid } \Rightarrow [\eta]_L \text{ is satisfiable in } (W_\eta, R_\eta^\ast) \Rightarrow [\eta]_L \text{ is } L\text{-consistent } \Rightarrow \eta \text{ is valid},$$

that is, $\eta$ is valid iff $[\eta]_L$ is $L$-consistent. Thus, the $L$-consistency problem is PSpace-hard; synonymously, the (provability problem for the) logic $L$ is PSpace-hard.

This proves Ladner’s theorem, in its classical formulation. And, in fact, it proves more. Let Grz.Bin be the logic of the class of all finite transitive reflexive trees with branching $\leq 2$. We

---

$^3$This variant of reduction is a slight modification of the one used in [Blackburn et al. 2002].
have the following proper inclusions

\[ S_4 \subseteq S_4.1 \subseteq \text{GRZ} \subseteq \text{GRZ.BIN}, \]

see, e.g., [Chagrov and Zakharyaschev 1997], and [Gabbay and De Jongh 1974] for the latter inclusion. Observe that \((W_\eta, R_\eta^-)\) is a finite transitive reflexive tree. It immediately follows from the above reasonings that every logic contained in \text{GRZ} is PSpace-hard. Moreover, the branching of \((W_\eta, R_\eta^-)\) is \(\leq 2\); thus, the result holds for logics contained in \text{GRZ.BIN}:

\[ L \subseteq \text{GRZ.BIN} \implies \text{L is PSpace-hard}. \]

This formulation does not include an important logic \(\text{GL} \): its frames are irreflexive. However, we can reformulate (13) in the following way: if \(R_\eta \subseteq R \subseteq R_\eta^\ast\), then

\[ \eta \text{ is valid } \implies [\eta]_L \text{ is satisfiable in } (W_\eta, R); \quad (15) \]

the proof is straightforward and is an immediate analog of the proof of (13) given in [Blackburn et al. 2002].

The condition \(R_\eta \subseteq R \subseteq R_\eta^\ast\) can be reformulated as \(R^\ast = R_\eta^\ast\), because \(T_\eta\) is a tree.

**Definition 4.16.** A class \(\mathcal{F}\) of unimodal frames is **thick** if for every finite transitive reflexive tree \(T = (T, \leq)\) whose branching is bounded by 2 there exist a relation \(R\) on \(T\) and a frame \(F \in \mathcal{F}\) such that \(R^\ast = \leq\) and \(F\) is isomorphic to \((T, R)\).

From (15) we obtain that if \(\mathcal{F}\) is thick then

\[ \eta \text{ is valid } \implies [\eta]_L \text{ is satisfiable in } \mathcal{F}. \quad (16) \]

Thus, for a class \(\mathcal{F}\) of unimodal frames and a unimodal logic \(L\), from (16) and (14) we have:

\[ \mathcal{F} \text{ is thick and } L \subseteq \log \mathcal{F} \implies \text{L is PSpace-hard}. \quad (17) \]

In particular, this formulation allows to apply Ladner’s construction for logics below \(\text{GL}\), since \(\text{GL}\) is the logic of a thick class (consisting of finite irreflexive transitive trees).

The logic \(S_4.2\) is not the logic of a thick class: it does not have trees of height \(> 1\) within its frames. However, trees are contained as subframes in \(S_4.2\)-frames. And this is also enough for PSpace-hardness due to the following relativization argument proposed by E. Spaan [Spaan 1993].

Recall that a **subframe** of a frame \(F = (W, (R_a)_{a \in A})\) is the restriction \(F \upharpoonright V = (V, (R_a \cap (V \times V))_{a \in A})\), where \(V \neq \emptyset\). For a class \(\mathcal{F}\) of frames, let \(\text{Sub} \mathcal{F}\) be its closure under the subframe operation: \(\text{Sub} \mathcal{F} = \{F \upharpoonright V \mid F \in \mathcal{F} \text{ and } \emptyset \neq V \subseteq \text{dom}(F)\}\). The satisfiability in \(\text{Sub} \mathcal{F}\) can be reduced to the satisfiability in \(\mathcal{F}\) (see the proof of Theorem 2.2.1 in [Spaan 1993]). Namely, for an \(A\)-formula \(\varphi\) and a variable \(q\), let \([\varphi]_q\) be the \(A\)-formula inductively defined as follows: \([\bot]_q = \bot\), \([p]_q = p\) for variables, \([\psi_1 \rightarrow \psi_2]_q = [\psi_1]_q \rightarrow [\psi_2]_q\), and \([\varphi_a \psi]_q = \varphi_a([\psi]_q \land q)\) for \(a \in A\). We put \([\varphi]_{\rel} = q \land [\varphi]_q\), where \(q\) is the first variable not occurring in \(\varphi\). If \(M = (F, \theta)\) is a model and \(V = \theta(q)\), then for every \(v \in V\) we have: \(M \upharpoonright V, v \models \varphi\) iff \(M, v \models [\varphi]_{\rel}\); the proof is by induction on \(\varphi\). This immediately yields

**Proposition 4.17 ([Spaan 1993]).** \(\varphi\) is satisfiable in \(\text{Sub} \mathcal{F}\) iff \([\varphi]_{\rel}\) is satisfiable in \(\mathcal{F}\).

Thus, \(\text{SatSub} \mathcal{F}\) is polynomial time reducible to \(\text{Sat} \mathcal{F}\).

It immediately follows from (17) and Proposition 4.17 that if \(\text{Sub} \mathcal{F}\) is thick, then the logic of \(\mathcal{F}\) is PSpace-hard. Moreover, this holds for every \(L \subseteq \log \mathcal{F}\): a quantified Boolean formula \(\eta\) is valid iff \([\eta]_L\) is \(L\)-consistent. Indeed, if \(\eta\) is valid, then \([\eta]_L\) is satisfiable in \(\text{Sub} \mathcal{F}\) by (16), so \([\eta]_L\) is satisfiable in \(\mathcal{F}\) by Proposition 4.17, hence \([\eta]_L\) is \(L\)-consistent. The converse implication follows from (14). Thus, we have
Theorem 4.18 (A corollary of [Ladner 1977] and [Spaan 1993]). Let \( F \) be a class of unimodal frames. If Sub \( F \) is thick, then every unimodal \( L \subseteq \text{Log} F \) is \( \text{PSpace}\)-hard.

For an A-frame \( F = (W, (R_a)_{a \in A}) \) and \( a \in A \), let \( F^{[a]} \) be its reduct \( (W, R_a) \). A class \( F \) of A-frames is said to be thick if for some \( a \in A \) the class \( \{F^{[a]} \mid F \in F\} \) is thick.

Proposition 4.19. Let \( F \) and \( I \) be classes of A-frames. If \( F \) is non-empty and Sub \( I \) is thick, then \( \text{Sub} \sum I F \) is thick.

Proof. Follows from Definitions 3.1 and 4.16. □

Corollary 4.20. Let \( F \) and \( I \) be classes of A-frames. If \( F \) is non-empty and Sub \( I \) is thick, then \( \text{Sat} \sum I F \) is \( \text{PSpace}\)-hard.

4.5 Examples

For many logics, Theorem 4.1 gives a uniform proof of decidability in \( \text{PSpace} \) (\( \text{PSpace}\)-completeness in view of Corollary 4.20). Let us illustrate it with certain examples.

Example 4.21. It is well-known that the logic S4 as well as its expansion with the universal modality, are \( \text{PSpace}\)-complete [Ladner 1977], [Hemaspaandra 1996].

Let us prove it via sums. Recall that clusters are frames of the form \((C, C \times C)\). Every preorder \( F \) is isomorphic to the sum \( \sum_{C \in \text{sk} F} (C, C \times C) \) of its clusters over its skeleton \( \text{sk} F \). The logic S4 has the finite model property, so it is enough to consider only finite indices, and hence S4 is the logic of the class

\[
\sum_{\text{finite partial orders}} \text{clusters}
\]  

Thus, we have:

\[
\text{Sat( preorders) } \leq_{\text{T}} \text{PSpace} \quad \text{Sat( clusters)}
\]

or dually

\[
\text{S4 } \leq_{\text{T}} \text{PSpace} \quad \text{S5}.
\]

The satisfiability problem in clusters is in \( \text{NP} \) (one can easily check that a formula \( \varphi \) is satisfiable in a cluster iff it is satisfiable in a cluster of size \( \#\varphi \); see, e.g., [Chagrov and Zakharyaschev 1997, Section 18.3]). Now \( \text{PSpace}\)-completeness of S4 follows from (18): we have \( \text{PSpace}\)-upper bound by Theorem 4.1(1); \( \text{PSpace}\)-hardness is given by Corollary 4.20. Moreover, Theorem 4.1(2) gives \( \text{PSpace}\)-upper bound of the logic \( \text{S4}^{[V]} = \text{Log}(\{((W, W \times W, R) \mid (W, R) \text{ is a preorder})} \).

Changing the class of summands, we obtain \( \text{PSpace}\)-completeness for other logics. Let \( S_0 \) and \( S_1 \) be an irreflexive and a reflexive singleton, respectively.

If we add \( S_0 \) to the class of summands-clusters, then we obtain \( \text{PSpace}\)-completeness for the logics K4 and \( K4^{[V]} \).

Letting the class of summands be \( \{S_0\} \), we obtain \( \text{PSpace}\)-completeness for the logics GL. If the class of summands consists of a reflexive singleton \( S_1 \), the above reasonings give \( \text{PSpace}\)-completeness of the Grzegorczyk logic, and if the class of summands is \( \{S_0, S_1\} \) — for the weak Grzegorczyk logic (recall that the latter logic is characterized by frames whose reflexive closures are non-strict Noetherian orders [Litak 2007]; this class can be represented as \( \sum_{\text{NPO}} \{S_0, S_1\}\)).

The results discussed in Example 4.21 are well-known [Ladner 1977], [Spaan 1993], [Hemaspaandra 1996]. To the best of our knowledge, the following result has never been published before.

Example 4.22. The weak transitivity logic \( \text{wK4} \) is the logic of frames satisfying the condition \( xRzRy \Rightarrow xRy \lor x = y \). The difference logic \( DL \) is the logic of the frames such that \( xRy \) whenever
$x \neq y$; let $\mathcal{F}_x$ be the class of such frames. The logic $\text{wK4}$ has the finite model property [Esakia 2001], hence it is the logic of the class

$$\sum_{\text{finite partial orders}} \mathcal{F}_x,$$

see Example 3.4. It is not difficult to check that $\text{Sat} \mathcal{F}_x$ is in NP (like in the case of clusters, the size of a countermodel is linear in the length of a formula), and by Theorem 4.1 we obtain that

$$\text{wK4} \leq_{\text{PSpace}} \text{DL} \in \text{PSpace},$$

so wK4 is in PSpace (and PSpace-complete: PSpace-hardness follows from [Ladner 1977], since $\text{wK4} \subseteq S_4$, or from the representation (19) and Corollary 4.20). Moreover, since the class $\mathcal{F}_x$ is preconical, $\text{wK4}$ is in PSpace.

Example 4.23. The logic $\text{wK4}_2$ is the logic of weakly transitive frames (considered in the above example) satisfying the Church-Rosser property $xRy_1 \& xRy_2 \Rightarrow \exists z (y_1Rz \& y_2Rz)$. Observe that every frame validating $\text{wK4}_2$ is either in $\mathcal{F}_x$, or is isomorphic to the sum of a frame validating $\text{wK4}$ and a frame in $\mathcal{F}_x$.

$$\text{Fr}(\text{wK4}_2) = \mathcal{F}_x \cup (\text{Fr}(\text{wK4}) +^0 \mathcal{F}_x).$$

Now that $\text{wK4}_2$ is decidable in PSpace follows from the previous example and the following theorem.

**Theorem 4.24.** Let $\mathcal{F}$ and $\mathcal{G}$ be classes of $A$-frames, and $a \in A$. If $\text{Sat} \mathcal{G}^{(v)} \leq_{\text{T}} \text{Sat} \mathcal{F}^{(v)}$ (in particular, if $\text{Sat} \mathcal{G}^{(v)}$ is in PSpace), then $\text{Sat}(\mathcal{F} +^a \mathcal{G})^{(v)} \leq_{\text{T}} \text{Sat} \mathcal{F}^{(v)}$, and consequently $\text{Sat}(\mathcal{F} +^a \mathcal{G}) \leq_{\text{PSpace}} \text{Sat} \mathcal{F}^{(v)}$.

**Proof.** By Propositions 4.12 and 4.13, from $\text{Sat} \mathcal{G}^{(v)} \leq_{\text{T}} \text{Sat} \mathcal{F}^{(v)}$ we obtain $\text{CSat} \mathcal{G} \leq_{\text{T}} \text{CSat} \mathcal{F}$. Using Lemma 4.5, from the latter reduction we obtain $\text{CSat}(\mathcal{F} +^a \mathcal{G}) \leq_{\text{T}} \text{CSat} \mathcal{F}$. Using Propositions 4.12 and 4.13 again, we obtain that $\text{Sat}(\mathcal{F} +^a \mathcal{G})^{(v)} \leq_{\text{T}} \text{PSpace} \text{Sat} \mathcal{F}^{(v)}$. □

**Remark 4.25.** This theorem can be strengthened in two aspects. First, it can be formulated for reductions stronger than $\leq_{\text{T}}$. Another observation is that instead of the class $\mathcal{F} +^a \mathcal{G}$, i.e., a class of sums over $I = (2, <)$, one can consider sums over an arbitrary finite indexing frame $I$; in this case, a reduction (with several oracles) from sums to summands would follow from Lemma 3.14.

We conclude this section with the following example. In topological semantics ($\Diamond$ is for the derivation), wK4 is the logic of all topological spaces [Esakia 2001]. In [Bezhanishvili et al. 2011] it was shown that the logic $\text{wK4T}_0$ of all $T_0$-spaces is equal to the logic of all finite weakly transitive frames where clusters contain at most one irreflexive point. The satisfiability for such clusters is in NP, and we obtain

**Corollary 4.26.** $\text{wK4T}_0$ is PSpace-complete.

## 5 Variations

In this section we are interested in polymodal logics which can be characterized by models obtained via multiple applications of the sum operation. An important example of such logic is Japaridze’s polymodal logic [Beklemishev 2010; Japaridze 1986]; we also consider lexicographic products of modal logics introduced in Balbiani [2009] and refinements of logics introduced in Babenyshev and Rybakov [2010]; see Sections 5.3 and 5.4 below. We anticipate them with some general observations.
5.1 Iterated Sums Over Unimodal Indexes

Let \( \mathcal{F} \) be a class of \( A \)-frames and \( \mathcal{I} \) a class of 1-frames. Let \( a = (a_0, \ldots, a_{s-1}) \) be a finite sequence of elements of \( A \). If \( a \) is the empty sequence, let \( a \)-iterated sums of \( \mathcal{F} \) be the elements of \( \mathcal{F} \). For \( 0 < s < \omega \), let \( a \)-iterated sums of \( \mathcal{F} \) be \( a_0 \)-sums of \( (a_1, \ldots, a_{s-1}) \)-iterated sums: that is, an \( a \)-iterated sum of frames in \( \mathcal{F} \) over frames in \( \mathcal{I} \) is a frame of form \( \sum_i a_i H_i \), where \( i \in \mathcal{I} \) and every \( H_i \) is an \( (a_1, \ldots, a_{s-1}) \)-iterated sum of frames in \( \mathcal{F} \) over frames in \( \mathcal{I} \). The class of all such sums is denoted by \( \sum_a^\mathcal{F} \).

Since \( \sum_a^\mathcal{F} \) is \( \sum_0 a_i \sum_0^{(a_1, \ldots, a_{s-1})} \mathcal{F} \), by Theorem 4.1 we obtain

**Corollary 5.1.** Let \( A < \omega \), \( \mathcal{F} \) a class of \( A \)-frames, and \( a \) a finite sequence of elements of \( A \). If \( \mathcal{I} \) is a class of Noetherian orders that contains all finite trees and is closed under finite disjoint unions, then \( \text{Sat}(\sum_a^\mathcal{F})(\varphi) \leq_{\text{PSpace}} \text{Sat}(\mathcal{F})(\varphi) \).

Theorem 3.16 reduces satisfiability on sums over Noetherian orders to sums over finite trees. This theorem can also be extended for the case of iterated sums [Shapirovsky 2018]. Namely, let \( a = (a_0, \ldots, a_{s-1}) \in A^s \), \( 0 < s < \omega \). Then for every \( A \)-tie \( \tau = (\varphi, \Phi, \Gamma) \) we have:

\[
\tau \text{ is satisfiable in } \sum_a^\mathcal{F} \iff \tau \text{ is satisfiable in } \bigcup_{\leq \varphi} \sum_a^\mathcal{F}.
\]

(20)

This can be proven by induction of the length of \( a \) with the help of the following lemma (see the proof of [Shapirovsky 2018, Theorem 5.2] for the details):

**Lemma 5.2.** Let \( \mathcal{F} \) be a class of \( A \)-frames and \( a \in A \). Then every frame in \( \sum_a^\mathcal{F} \) is isomorphic to a frame in \( \sum_a^{\text{NPO}} \mathcal{F} \).

**Proof.** By Theorem 3.6 (3c), a sum of form \( \sum_{i \in I} \bigcup_{j \in J_i} F_{ij} \) is isomorphic to the sum \( \sum_{(i, j) \in \sum_{k \in \mathcal{K}} (J_k, \varnothing)} F_{ij} \). It remains to observe that if \( l = (I, <) \in \text{NPO} \) and \( (J_i)_{i \in I} \) is a family of non-empty sets, then \( \sum_l (J_i, \varnothing) \in \text{NPO} \).

In view of Proposition 3.10, from (20) we obtain

**Corollary 5.3.** Let \( \mathcal{F} \) be a class of \( A \)-frames, \( s < \omega \), and \( a = (a_0, \ldots, a_{s-1}) \in A^s \). Then for every \( A \)-formula \( \varphi \) we have:

\[
\varphi \text{ is satisfiable in } \sum_a^\mathcal{F} \iff \varphi \text{ is satisfiable in } \sum_a^\mathcal{F}.
\]

**Remark 5.4.** The operation of iterated sum can result in very tangled structures. In this remark, we expand the formal definition of this operation.

Let \( 0 \leq s < \omega \). Consider a tree \( (T, <) \) such that every maximal chain in \( T \) has \( s + 1 \) elements. Let \( I \) be the set of maximal elements of \( (T, <) \), \( J = T \setminus I \), and for \( i \in J \), let \( \text{sc}(i) \) be the set of immediate successors of \( i \). A structure \( T = (T, <, (S_i)_{i \in I}) \) such that \( S_i \) is a binary relation on \( \text{sc}(i) \) is called an indexing tree. If also \( \text{sc}(i), S_i \in I \) for every \( i \in J \), then \( T \) is an indexing tree for \( I \). For a sequence \( a = (a_0, \ldots, a_{s-1}) \in A^s \), and a family \( (F_i)_{i \in I} \) of \( A \)-frames, we define the \( a \)-sum of \( (F_i)_{i \in I} \) over the indexing tree \( T \), in symbols \( \sum_a^T F_i \), as the following \( A \)-frame \( (W, (R_a)_{a \in A}) \). The domain of this structure is the disjoint union of the domains of \( F_i \):

\[
W = \{(i, w) \mid i \text{ is maximal in } T \text{ and } w \text{ is in } F_i\}
\]

To define the relations \( R_a \), consider \( (i, w), (j, v) \in W \). If \( i = j \), then, as usual, we put \((i, w)R_a(j, v)\) iff \( w R_{i,a} v \), where \( R_{i,a} \) denotes the \( a \)-th relation in \( F_i \). Assume that \( i \neq j \). Let \( k \) be the infimum \( \inf \{i, j\} \), and let \( h \) be the height of \( k \) in \( T \), that is the number of elements in \( \{l \in T \mid l < k\} \). There
are unique immediate successors $i'$ and $j'$ of $k$ such that $i' \leq i$ and $j' \leq j$. Hence $0 \leq h < s, i' \neq j'$, and $i', j' \in \text{sc}(k)$. We put $(i, w)R_a(j, u)$ iff $iS_kj'$ and $a$ is the $h$-th element of $a$.

The class $\sum^+_I F$ of \textit{iterated sums of frames in $F$ over frames in $I$} consists of such structures where $T$ is an indexing tree for $I$ and all $F_i$ are in $F$. One can see that for a fixed $a$, the elements of $\sum^+_I F$ are (up to isomorphisms) $a$-iterated sums $\sum^a_I F$.

5.2 Lexicographic Sums

The sum operation does not change the signature. In many cases (see below) it is convenient to characterize a polymodal logic via the following modification of $a$-sums.

\textbf{Definition 5.5.} Let $l = (I, S)$ be a unimodal frame, $(F_i)_{i \in I}$ a family of A-frames, and $F_i = (W_i, (R_i,a)_{a \in A})$. The \textit{lexicographic sum} $\sum^\text{lex} I F_i$ is the $(1 + A)$-frame $\left(\bigsqcup_{i \in I} W_i, S^\text{lex}_i, (R_i)_{a < N}\right)$, where

\[
(i, w)S^\text{lex}_i(j, u) \iff iS_j,
\]

\[
(i, w)R_i(a, j, u) \iff i = j & wR_i(a, u).
\]

For a class $F$ of A-frames and a class $I$ of 1-frames, we define $\sum^\text{lex} I F$ as the class of all sums $\sum^\text{lex} I F_i$, where $l \in I$ and all $F_i$ are in $F$.

Notice that lexicographic sums depend on reflexivity of the indexing frame (unlike sums considered in previous sections, see Remark 3.2.) This difference is explained in Propositions 5.6 and 5.7 below.

Recall that for an A-frame $F = (W, (R_i)_{i \in A})$, $F^{(V)}$ is the $(1 + A)$-frame $(W, W \times W, (R_i)_{i \in A})$. Let $F^{(\emptyset)}$ be the $(1 + A)$-frame $(W, \emptyset, (R_i)_{a \in A})$; for a class $F$ of frames, let $F^{(\emptyset)} = \{F^{(\emptyset)} \mid F \in F\}$.

The following is immediate from the definitions:

\textbf{Proposition 5.6.} \textit{For a unimodal frame} $l = (I, S)$ and a family $(F_i)_{i \in I}$ of A-frames,

\[
\sum^\text{lex} I F_i = \sum^\text{lex} I F'_i = \sum^0 I F_i',
\]

where $V'$ is the $(1 + A)$-frame $(I, S, (\emptyset)_{A})$, and for $i \in I$, $F'_i = F^{(V)}_i$ whenever $i$ is reflexive in $l$, and $F'_i = F^{(\emptyset)}_i$ otherwise.

\textbf{Proposition 5.7.} \textit{Let $F$ be a class of A-frames.}

(1) If $I$ is a class of irreflexive 1-frames, then $\sum^\text{lex} I F = \sum^0 I F^{(\emptyset)}$.

(2) If $I$ is a class of reflexive 1-frames, then $\sum^\text{lex} I F = \sum^0 I F^{(V)}$.

\textbf{Proof.} Immediate from Proposition 5.6. \hfill $\square$

Given a class $F$ of frames in an alphabet $A$ and a class of unimodal frames $I$, let $0$-\textit{iterated lexicographic sums} be elements of $F$, and for $n < \omega$, let $(n + 1)$-\textit{iterated lexicographic sums} be lexicographic sums of $n$-iterated sums. The class of $n$-iterated lexicographic sums of frames in $F$ over frames in $I$ is denoted by $\sum^{\text{lex}_n} I F$.

The next fact is the iterated version of the above proposition. For a class $F$ of A-frames and $n < \omega$, let $F^{(\emptyset)_n}$ be the class of $(n + A)$-frames $(W, (\emptyset)_{n}, (R_a)_{a \in A})$ such that $(W, (R_a)_{a \in A}) \in F$. Likewise, let $F^{(V)_n}$ be the class of frames $(W, (S_a)_{a \in n+1})$ such that $S_a = W \times W$ for $a < n$, and $(W, (S_{n+a})_{a \in A}) \in F$.

\textbf{Proposition 5.8.} \textit{Let $A \leq \omega$, $0 < n < \omega$, and let $F$ be a class of A-frames.}

(1) If $I$ is a class of irreflexive 1-frames, then $\sum^{\text{lex}_n} I F = \sum I F^{(\emptyset)_n}$.

(2) If $I$ is a class of reflexive 1-frames, then $\sum^{\text{lex}_n} I F = \sum I F^{(V)_n}$.
Proof. Follows from Proposition 5.6 by a straightforward induction on \( n \).

Propositions 5.7 and 5.8 allow to expand our results on the finite model property and complexity to the logics of (iterated) lexicographic sums.

5.3 Japaridze’s Polymodal Logic

In this section we show that Japaridze’s polymodal provability logic GLP is decidable in PSpace.

First, this theorem was proven in [Shapirovsky 2008]. Here we provide a version of the proof based on Theorem 4.7.

GLP is a normal modal logic in the language \( \text{ML}(\omega) \). This system was introduced in [Japaridze 1986] and plays an important role in proof theory (see, e.g., [Beklemishev 2004]). GLP is known to be Kripke incomplete, so we cannot directly apply our tools to analyze it. However, in [Beklemishev 2010], L. Beklemishev introduced a modal logic \( J \), a Kripke complete approximation of GLP. Semantically, \( J \) is characterised as the logic of frames called stratified, or hereditary partial orderings ([Beklemishev 2010, Section 3]). They are defined as follows.

Definition 5.9. For \( A \leq \omega \), let \( S_A \) be \((\{0\}, (\emptyset)_A)\), a singleton \( A \)-frame with empty relations. For \( A \leq \omega \), let \( J(A) \) be the class of \( A \)-iterated lexicographic sums

\[
\sum_{\text{NPO}}^\text{lex} \{S_\omega\} = \sum_{\text{NPO}}^\text{lex} \cdots \sum_{\text{NPO}}^\text{lex} \{S_\omega\}.
\]

A times

The class \( J \) of hereditary partial orderings is the class \( \bigcup_{A < \omega} J(A) \). The logic \( J \) is defined as the logic of the class \( J \).

In [Beklemishev 2010], it was shown that GLP is polynomial time many-to-one reducible to \( J \): there exists a polynomial time computable \( f : \text{ML}(\omega) \rightarrow \text{ML}(\omega) \) such that \( \varphi \in \text{GLP} \) iff \( f(\varphi) \in J \) (for an explicit description of \( f \), see [Beklemishev et al. 2014, Lemma 3.4]).

Our aim is to show that \( J \) is in PSpace. For our purposes, it is more convenient to work with \( a \)-sums. Since \( S_\omega^{(\emptyset)_A} = S_\omega \), from Proposition 5.8(1) we have:

Proposition 5.10. For every \( A < \omega \), \( J(A) = \sum_{\text{NPO}}^0 \cdots \sum_{\text{NPO}}^{A-1} \{S_\omega\} \).

Formally, for every \( A < \omega \), each frame in \( J(A) \) has infinitely many relations. Next facts allow us to consider frames of finite signatures. Let

\[
\hat{J}(A) = \sum_{\text{NPO}}^0 \cdots \sum_{\text{NPO}}^{A-1} \{S_A\}.
\]

For an alphabet \( C \), a \( C \)-frame \( F = (W, (R_a)_{a \in C}) \), and \( B \leq C \), let \( F^{\restriction B} \) be the reduct \((W, (R_a)_{a \in B})\); for a class \( \mathcal{F} \) of \( C \)-frames, \( \mathcal{F}^{\restriction B} = \{F^{\restriction B} \mid F \in \mathcal{F}\} \).

Proposition 5.11. For every \( A < \omega \), we have:

1. \( J(A)^{\restriction A} = \hat{J}(A) \);
2. If \( A > 0 \), then every frame in \( J^{\restriction A} \) is isomorphic to a frame in \( \hat{J}(A) \).

Proof. From Definition 3.1 it is immediate that for alphabets \( B \leq C \) and classes of \( C \)-frames \( \mathcal{F}, \mathcal{I} \),

\[
\left( \sum_{\mathcal{I}} \cdots \sum_{\mathcal{I}}^{\mathcal{F}} \right)^{\restriction B} = \sum_{\mathcal{I}^{\restriction B}} \cdots \sum_{\mathcal{I}^{\restriction B}} \left( \mathcal{F}^{\restriction B} \right).
\]
Clearly, $S_\omega \upharpoonright A$ is $S_A$. Now we obtain (1) from Proposition 5.10 and Definition 3.5.

To prove (2), let us fix $B < \omega$ and show that every frame in $\mathcal{J}(B)^\upharpoonright A$ is isomorphic to a frame in $\hat{\mathcal{J}}(A)$.

First, observe that $\mathcal{J}(B + 1)$ contains copies of all frames in $\mathcal{J}(B)$. Indeed, every frame in the class $\sum_{NPO}^0 \ldots \sum_{NPO}^{B-1} \{S_\omega\}$ is isomorphic to a frame in $\sum_{NPO}^0 \ldots \sum_{NPO}^{B-1} \sum_{NPO}^B \{S_\omega\}$, since $\sum_{NPO}^B \{S_\omega\}$ contains a copy of $S_\omega$. Hence, if $B < A < \omega$, $\mathcal{J}(A)$ contains copies of all frames in $\mathcal{J}(B)$.

The case $A < B$ is more interesting. Consider the class $\mathcal{G} = \sum_{NPO}^{A-1} \ldots \sum_{NPO}^{B-1} \{S_\omega\}$. By (21) and Proposition 5.10, we have:

$$\mathcal{J}(B)^\upharpoonright A = \left(\sum_{NPO}^0 \ldots \sum_{NPO}^{B-1} \{S_\omega\}\right)^\upharpoonright A = \sum_{NPO}^{A-2} \sum_{NPO}^B \mathcal{G}^\upharpoonright A.$$ 

By (21) again, $\mathcal{G}^\upharpoonright A$ can be represented as sums of disjoin unions of singletons:

$$\left(\sum_{NPO}^0 \ldots \sum_{NPO}^{B-1} \{S_\omega\}\right)^\upharpoonright A = \sum_{NPO}^{A-1} \bigcup_{B-A \text{ times}} \{S_A\},$$

since a sum over the structure with all relations empty is just a disjoint sum. By Lemma 5.2, every frame in $\mathcal{G}^\upharpoonright A$ is isomorphic to a frame in the class $\sum_{NPO}^{A-1} \{S_A\}$. It follows that $\mathcal{J}(A)$ contains copies of all frames in $\mathcal{J}(B)^\upharpoonright A$. $\square$

**Proposition 5.12.** For every $A < \omega$, $\mathcal{J} \cap \text{ML}(A) = \text{Log}(\hat{\mathcal{J}}(A))$.

**Proof.** By Proposition 5.11(1), $\mathcal{J}(A) \subseteq \mathcal{J}^\upharpoonright A$, so if $\varphi$ is satisfiable in $\hat{\mathcal{J}}(A)$, then it is satisfiable in $\mathcal{J}$. Conversely, if $\varphi \in \text{ML}(A)$ is satisfiable in $\mathcal{J}$, then it is satisfiable in $\mathcal{J}^\upharpoonright A$, and hence — in $\hat{\mathcal{J}}(A)$ by Proposition 5.11(2). $\square$

It is trivial that the satisfiability problem on the singleton $S_A^{(\varphi)}$ is in PSpace (in fact, it is in NP). From Corollary 5.1 and Proposition 5.11(1), we obtain

**Corollary 5.13.** For every $A < \omega$, $\text{Sat}(\mathcal{J}(A)^{\varphi}) \in \text{PSpace}$.

Hence, every fragment of $\mathcal{J}$ with finitely many modalities is in PSpace. The above corollary does not directly imply that $\text{Sat} \mathcal{J}$ is in PSpace. But the latter fact is a corollary of Theorem 4.7 and the observations below.

**Lemma 5.14.** Let $A < \omega$. An $A$-formula $\varphi$ is satisfiable in $\mathcal{J}$ iff $\varphi$ is satisfiable in $\sum_{\text{Tr}(\# \varphi, \# \varphi)}^{(0, \ldots, A-1)} \{S_A\}$.

**Proof.** By Proposition 5.12, $\varphi$ is satisfiable in $\mathcal{J}$ iff it is satisfiable in $\hat{\mathcal{J}}(A)$. By Corollary 5.3, the latter is equivalent to the satisfiability of $\varphi$ in $\sum_{\text{Tr}(\# \varphi, \# \varphi)}^{(0, \ldots, A-1)} \{S_A\}$. $\square$

Consider a formula $\varphi$ in the language ML$_{\omega, \omega}$. Let $a_0 < \cdots < a_{N-1}$ be the increasing sequence of indices of all occurring in $\varphi$ modalities. Put $N(\varphi) = N$. Let $\hat{\varphi}$ be the result of replacing $a_b$-modalities by $b$-modalities in $\varphi$. Note that

$$\hat{\varphi} \text{ is an } N(\varphi) \text{-formula, and } N(\varphi) < \# \varphi = \# \hat{\varphi}. \quad (22)$$

**Lemma 5.15** ([Beklemishev et al. 2014, Lemma 3.5]). $\varphi \in \mathcal{J}$ iff $\hat{\varphi} \in \mathcal{J}$.

**Theorem 5.16.** $\mathcal{J}$ is in PSpace.
Algorithm 2: Decision procedure for CSat $S(h, b, a, A)$

**Input:** An A-tie $(\varphi, v, U)$; positive integers $h, b; a \leq A < \omega$.

1. If $a = A$ then return $(\varphi, v, U)$ is satisfiable in $S_A$;
2. If CSat$(\varphi, v, U, A, a + 1, A)$ then return true;
3. If $h > 1$ then
   - For $k$ such that $1 \leq k \leq b$ do for $u, v_0, \ldots, v_{k-1} \in 2^{\#\varphi}$ such that $v = u + \sum_{i<k} v_i$
     - If CSat$(\varphi, u, U + a \sum_{i<k} v_i, A, a + 1, A)$ then return true;
   - If $\lor_{i<k}$ CSat$(\varphi, v_i, U, h - 1, b, a, A)$ then return true;
4. Return false.

**Proof.** We describe a decision procedure for the conditional satisfiability on iterated sums of trees. For positive $h, b$ and $a \leq A < \omega$, we define sums $S(h, b, a, A)$ as follows: if $a = A$, let $S(h, b, a, A)$ denote $\{S_A\}$ for all $h, b$; if $a < A$, let $S(h, b, a, A)$ denote $\sum_{\text{Tr}(h, b)} (a+1, \ldots, A-1) |S_A|$. In particular, $S(h, b, a - 1, A)$ is $\sum_{\text{Tr}(h, b)} (a+1, \ldots, A-1) |S_A|$, and $S(A, a, 0, A) = \sum_{\text{Tr}(A, A)} (0, \ldots, A-1) |S_A|$. Using Theorem 4.7, we describe the procedure CSatJ (Algorithm 2) that decides whether a given A-tie $(\varphi, v, U)$ is satisfiable $S(h, b, a, A)$.

**Lemma 5.17.** Let $a \leq A < \omega$, $(\varphi, v, U)$ an A-tie, and $0 < h, b < \omega$. Then $(\varphi, v, U)$ is satisfiable in $S(h, b, a, A)$ iff CSatJ$(\varphi, v, U, h, b, a, A)$ returns true.

**Proof.** By induction on $A - a$. The case $A = a$ is trivial: $S(h, b, a, A)$ consists of a single singleton $S_A$. If $a < A$, then $S(h, b, a, A)$ is $\sum_{\text{Tr}(h, b)} a |S_A|$. By induction hypothesis, CSatJ$(\tau, A, a, a + 1, A)$ decides whether a tie $\tau$ is satisfiable in $S(A, a, a + 1, A)$. Observe that for $a < A$, the algorithm CSatJ is an instance of the algorithm CSatSum (Algorithm 1 in Section 4), where CSatJ$(\tau, A, a, a + 1, A)$ is the oracle for the summands $\sum_{\text{Tr}(A, A)} (a+1, \ldots, A-1) |S_A|$. Now the induction step follows from Theorem 4.7.

**Lemma 5.18.** $\varphi$ is satisfiable in $J$ iff $\hat{\varphi}$ is satisfiable in $S(\#\varphi, 0, 0, \#\varphi)$.

**Proof.** We have from Lemmas 5.15 and 5.14: $\varphi$ is satisfiable in $J$ iff $\hat{\varphi}$ is satisfiable in $S(0, \ldots, A-1) |S_A|$, where $A$ can be assumed equal to $\#\varphi$ by (22). The latter class is $S(A, 0, A, 0)$. Hence, by Proposition 4.3, we obtain

$\varphi$ is satisfiable in $J$ iff $\exists v \in 2^{\#\varphi} \left( v(0) = 1 \& (\hat{\varphi}, v, 0) \in \text{CSat}S(\#\varphi, 0, 0, \#\varphi) \right)$,

where 0 represents the empty condition.

Let us estimate the amount of space used by CSatJ on the input $(\hat{\varphi}, v, 0, 0, 0, 0, 0)$. On each recursive call, either the argument $a$ increases by 1, or $a$ does not change and the parameter $b$ decreases by 1. Since $1 \leq h \leq \#\varphi$ and $0 \leq a \leq \#\varphi$, we obtain that the depth of recursion is bounded by $\#\varphi \cdot (\#\varphi + 1)$. For $n = \#\varphi$, at each call CSatJ needs $O(n^2)$ space to store new variables. Hence, to check the satisfiability of $\varphi$ in $J$ we need $O(n^3)$ space. This completes the proof of the theorem.

**Corollary 5.19.** Japaridze’s polymodal provability logic GLP is decidable in PSpace.

**Remark 5.20.** It is immediate that GLP and J are PSpace-hard (e.g., it follows from the fact that the 1-modal fragments of these logics are the logic GL). From [Chagrov and Rybakov 2003], it
follows that one-variable fragment of GLP is PSpace-hard. In [Pakhomov 2014], it was shown that even the constant (closed) fragment of GLP is PSpace-hard.

5.4 Refinements and Lexicographic Products of Modal Logics

In this paragraph we discuss algorithmic properties of logics obtained via the refinement and the lexicographic product operations.

The operation of refinement of modal logics was introduced in [Babenyshev and Rybakov 2010].

Definition 5.21. Let $F = (W, R)$ be a preorder and $\text{sk}F = (\overline{W}, \subseteq)$ its skeleton. Consider a family $(F_C)_{C \in \overline{W}}$ of A-frames such that $\text{dom}(F_C) = C$ for all $C \in \overline{W}$. The refinement of $F$ by $(F_C)_{C \in \overline{W}}$ is the $(1 + A)$-frame $(W, R, (R^+_a)_{a \in A})$, where

$$R^+_a \subseteq \bigcup_{C \in \overline{W}} C \times C \quad \text{for all } a \in A,$$

$$(W, (R^+_a)_{a \in A}) \mid C = F_C \quad \text{for all } C \in \overline{W}.$$  \hfill (23)

For a class $I$ of preorders and a class $G$ of A-frames let $\text{Ref}(I, F)$ be the class of all refinements of frames from $I$ by frames in $F$. For logics $L_1 \supseteq S4, L_2$, we put $\text{Ref}(L_1, L_2) = \log \text{Ref}(\text{Fr}L_1, \text{Fr}L_2)$.

In [Babenyshev and Rybakov 2010] it was shown that in many cases the refinement operation preserves the finite model property and decidability.

Refinements can be considered as sums according to the following fact:

Proposition 5.22. The refinement of $F$ by frames $(F_C)_{C \in \text{sk}F}$ is isomorphic to the sum $\sum_{C \in \text{sk}F} F_C^{(V)}$.

Proof. The isomorphism is given by $w \mapsto (C, w)$, where $w \in C$. \hfill $\square$

In view of Theorems 3.7 and 3.16, this observation provides another way to prove the finite model property of refinements (see [Shapirovsky 2018, Section 5.2] for more details). Moreover, representation of refinements as sums allows to obtain complexity results according to our theorems in Section 4.

Let us illustrate this with the logic Ref($S4, S4$). In [Babenyshev and Rybakov 2010], it was proven that Ref($S4, S4$) is decidable and that $\text{Ref}(S4, S4) = \log \text{Ref}(\text{QO}_f, \text{QO}_f)$, where $\text{QO}_f$ is the class of all finite non-empty preorders. The latter fact in combination with above proposition yields

$$\text{Ref}(S4, S4) = \log \sum_{\text{Fr}_f} \text{QO}_f^{(V)}.$$  \hfill (24)

Since the satisfiability problem for the class $\text{QO}_f^{(V)}$ is in PSpace ([Hemaspaanda 1996]; see also Example 4.21), from Theorem 4.1 (and Corollary 4.20) we obtain

Corollary 5.23. $\text{Ref}(S4, S4)$ is PSpace-complete.

A related operation is the lexicographic product of modal logics introduced in [Balbiani 2009] by Ph. Balbiani.

Definition 5.24. Consider frames $I = (I, S)$ and $F = (W, (R_a)_{a \in A})$. Their lexicographic product $I \times F$ is the $(1 + A)$-frame $(I \times W, S^\wedge, (R^\wedge_a)_{a \in A})$, where

$$(i, w)S^\wedge (j, u) \iff \text{iSj},$$

$$(i, w)R^\wedge_a (j, u) \iff \text{i = j \& wR_a u}.$$  \hfill (25)

In other words, $I \times F$ is the lexicographic sum $\sum_{i \in I} F_i$, where $F_i = F$ for all $i \in I$.

For a class $I$ of 1-frames and a class $G$ of A-frames, the class $I \times G$ is the class of all products $I \times F$ such that $I \in I$ and $F \in G$. For logics $L_1, L_2$, we put $L_1 \times L_2 = \log (\text{Fr}L_1 \times \text{Fr}L_2)$. 

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As examples, consider the logics $S4 \land S4$ and $GL \land S4$ and show that they are in PSpace (PSpace-complete). In [Shapirovsky 2018, Theorem 5.13], it was shown that $S4 \land S4$ is equal to $\text{Ref}(S4, S4)$. Also, it was shown that $GL \land S4 = \text{Log}(\text{Tr}_f \land \text{QO}_f) = \text{Log} \sum_{\text{Tr}_f}^0 \sum_{\text{Tr}_f}^1 C$ where $C$ is the class of finite frames of form $(C, \emptyset, C \times C)$ (in terms of lexicographic sums, $\sum_{\text{Tr}_f}^0 \sum_{\text{Tr}_f}^1 C$ is the class $\sum_{\text{Tr}_f}^\text{lex} \text{QO}_f$); see [Shapirovsky 2018, Theorem 5.14] for the proof. Hence, this logic is in PSpace by Corollary 5.1.

**Corollary 5.25.** The lexicographic products $S4 \land S4, GL \land S4$ are PSpace-complete.

This result contrasts with the undecidability results for modal products of transitive logics [Gabelaia et al. 2005].

### 6 CONCLUSION

In many cases, sum-like operations preserve the finite model property and decidability [Babenyshev and Rybakov 2010; Shapirovsky 2018]. In this paper we showed that transferring results can be obtained for the complexity of the modal satisfiability problems on sums. In particular, it follows that for many logics PSpace-completeness is immediate from their semantic characterizations.

Let us indicate some further results and directions.

- **Sums and products with linear indices:** In the linear case, modal products are typically (highly) undecidable [Reynolds and Zakharyaschev 2001]. However, the modal satisfiability problem on the lexicographic squares of dense unbounded linear orders is in NP [Balbiani and Mikulás 2013]. This positive result seems to be scalable according to the following observation: in many cases, $\varphi$ is satisfiable in sums over linear (pre)orders iff $\varphi$ is satisfiable in such sums that the length of indices is bounded by $\#\varphi$. In this situation there is a stronger than $\leq_{\text{PSpace}}$ reduction between the sums and the summands.

- **Further results on the finite model property and decidability of sums:** In many cases, the finite model property of a modal logic $L$ can be obtained by a filtration method; in this case we say that $L$ admits filtration. It follows from Babenyshev and Rybakov [2010] that filtrations of refinements can be reconstructed from filtrations of components. This result can be extended for a more general setting of lexicographic sums.

Also, the results obtained in Babenyshev and Rybakov [2010] in a combination with Theorem 3.7 suggest the following conjecture: in the case of finitely many modalities, if $\text{Log} F^{(v)}$ has the finite model property and $\text{Log} I$ admits filtration, then the logic of $\sum I F$ has the finite model property.

Another fact that we announce relates to the property of local finiteness of logics: if both the logics of indices and summands are locally finite, then the logic of lexicographic sums (a fortiori, of lexicographic products) is locally finite.

- **Axiomatization of sums:** For unimodal logics $L_1, L_2$, let $\sum_{L_1}^\text{lex} L_2$ be the logic of the class $\sum_{\text{Fr}_L}^\text{lex} L_2$. Consider the following 2-modal formulas:

  $\alpha = \diamond_1 \diamond_0 p \rightarrow \diamond_0 p, \quad \beta = \diamond_0 \diamond_1 p \rightarrow \diamond_0 p, \quad \gamma = \diamond_0 p \rightarrow \square_1 \diamond_0 p$

One can see that these formulas are valid in every lexicographic sum $\sum_{L_1}^\text{lex} F_i$ (and hence, in every product $1 \land F$) of 1-frames. In many cases, these axioms provide a complete axiomatization of $\sum_{L_1}^\text{lex} L_2$. In particular, the logic $\sum_{L_1}^\text{lex} GL = \text{Log}(\sum_{\text{NPO}}^\text{lex} \text{NPO})$, the bimodal fragment of the logic $J$ considered in Section 5.3, is the logic $GL \ast GL + \{\alpha, \beta, \gamma\}$ [Beklemishev 2010] ($L_1 \ast L_2$...
denotes the fusion of $L_1$ and $L_2$, $L + \Psi$ is for the least logic containing $L \cup \Psi$). Analogous results hold for various lexicographic products [Balbiani 2009]; e.g., $S_4 \triangleright S_4 = S_4 \ast S_4 + \{\alpha, \beta, \gamma\}$. We announce the following results:

(1) If $L_1 \ast L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, and the class $\text{Fr} L_1$ (considered as a class of models in the classical model-theoretic sense) is first-order definable without equality, then

$$\sum_{L_1}^{\text{lex}} L_2 = L_1 \ast L_2 + \{\alpha, \beta, \gamma\}.$$ 

(2) Assume that $\text{Fr} L_2$ is closed under direct products and validates the property $\forall x \exists y xRy$, and that the class $\text{Fr} L_1$ is first-order definable without equality. If the logic $L_1 \ast L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, then

$$L_1 \triangleright L_2 = L_1 \ast L_2 + \{\alpha, \beta, \gamma\}.$$ 

At the same time, no general axiomatization results are known for the logics of sums in the sense of Definition 3.1.

- **Sum-based operations in the Kripke-incomplete case**: The operations we considered so far lead to Kripke complete logics. What could be definition of sums of modal algebras (models, general Kripke frames)? E.g., can we give a semantic characterization of an important Kripke-incomplete logic GLP by sums of form $\sum_{\text{NPO}} \cdots \sum_{\text{NPO}} C$?

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REFERENCES

Sergey Babenyshev and Vladimir Rybakov. 2010. Logics of Kripke meta-models. *Logic Journal of the IGPL* 18, 6 (2010), 823–836.

Philippe Balbiani. 2009. Axiomatization and completeness of lexicographic products of modal logics. In *Frontiers of Combining Systems*, Silvio Ghilardi and Roberto Sebastiani (Eds.). Lecture Notes in Computer Science, Vol. 5749. Springer, 165–180.

P. Balbiani, 2010. Axiomatizing the temporal logic defined over the class of all lexicographic products of dense linear orders without endpoints. In 2010 17th International Symposium on Temporal Representation and Reasoning, 19–26.

Philippe Balbiani and David Fernández-Duque. 2016. Axiomatizing the lexicographic products of modal logics with linear temporal logic. In *Advances in Modal Logic, Vol. 11*, Lev Beklemishev, Stéphane Demri, and András Máté (Eds.). College Publications, 78–96.

Philippe Balbiani and Szabolcs Mikulás. 2013. Decidability and complexity via mosaics of the temporal logic of the lexicographic products of unbounded dense linear orders. In *Frontiers of Combining Systems*, Pascal Fontaine, Christophe Ringeissen, and Renate A. Schmidt (Eds.). Springer Berlin, Berlin, 151–164.

Lev D. Beklemishev. 2004. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic* 128, 1 (2004), 103–123. https://doi.org/10.1016/j.apal.2003.11.030

Lev D. Beklemishev. 2010. Kripke semantics for provability logic GLP. *Annals of Pure and Applied Logic* 161, 6 (2010), 756–774. The Proceedings of the IPM 2007 Logic Conference.

Lev D. Beklemishev, David Fernández-Duque, and Joost J. Joosten. 2014. On provability logics with linearly ordered modalities. *Studia Logica: An International Journal for Symbolic Logic* 102, 3 (2014), 541–566.

Guram Bezhanishvili, Leo Esakia, and David Gabelaia. 2011. Spectral and $T_0$-spaces in $d$-semantics. In *Logic, Language, and Computation*, Nick Bezhanishvili, Sebastian Löbner, Kerstin Schwabe, and Luca Spada (Eds.). Springer Berlin, Berlin, 16–29.

Patrick Blackburn, Maarten de Rijke, and Yde Venema. 2002. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science, Vol. 53. Cambridge University Press.

Alexander Chagrov and Michael Zakharyaschev. 1997. *Modal Logic*. Oxford Logic Guides, Vol. 35. Oxford University Press.

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