The weak limit of Ising models on locally tree-like graphs

Andrea Montanari · Elchanan Mossel · Allan Sly

Received: 3 December 2009 / Revised: 6 June 2010 / Published online: 11 September 2010
© Springer-Verlag 2010

Abstract We consider the Ising model with inverse temperature $\beta$ and without external field on sequences of graphs $G_n$ which converge locally to the $k$-regular tree. We show that for such graphs the Ising measure locally weakly converges to the symmetric mixture of the Ising model with $+$ boundary conditions and the $-$ boundary conditions on the $k$-regular tree with inverse temperature $\beta$. In the case where the graphs $G_n$ are expanders we derive a more detailed understanding by showing convergence of the Ising measure conditional on positive magnetization (sum of spins) to the $+$ measure on the tree.

Mathematics Subject Classification (2000) 60K35 · 82B20 · 82B26
1 Introduction

An Ising model on the finite graph $G$ (with vertex set $V$, and edge set $E$) is defined by the following distribution over $x = \{ x_i : i \in V \}$, with $x_i \in \{ +1, -1 \}$

$$
\mu(x) = \frac{1}{Z(\beta, B)} \exp \left\{ \beta \sum_{(i,j) \in E} x_i x_j + B \sum_{i \in V} x_i \right\}.
$$

(1.1)

The model is ferromagnetic if $\beta \geq 0$ and, by symmetry, we can always assume $B \geq 0$. Here $Z(\beta, B)$ is a normalizing constant (partition function).

The most important feature of the distribution $\mu(\cdot)$ is the ‘phase transition’ phenomenon. On a variety of large graphs $G$, for large enough $\beta$ and $B = 0$, the measure decomposes into the convex combination of two well separated simpler components. This phenomenon has been studied in detail in the case of grids [1–4], and on the complete graph [5]. In this paper we consider sequences of regular graphs $G_n = (V_n, E_n)$ with increasing vertex sets $V_n = [n] = \{ 1, \ldots, n \}$ that converge locally to trees and prove a local characterization of the corresponding sequence of measures $\mu_n(\cdot)$, which corresponds to the phase transition phenomenon.

More precisely, consider the case in which $G_n$ is a sequence of regular graphs of degree $k \geq 3$ with diverging girth. The neighborhood $B_i$ of a vertex $i$ in $G_n$ converges to an infinite regular tree of degree $k$. It is natural to assume that the marginal distribution $\mu_{B_n}(\cdot)$ converges to the marginal of a neighborhood of the root for an Ising Gibbs measure on the infinite tree. For large $\beta$, however, there are uncountably many Gibbs measures on the tree so it is natural to ask which is the limit.

A special role is played by the plus/minus boundary conditions Gibbs measures on the infinite tree, to be denoted, respectively, by $\nu_+ (\cdot)$ and $\nu_-(\cdot)$. It was proved in [6] that, for any $\beta$, and any $B > 0$, $\mu_n(\cdot)$ converges locally to $\nu_+$ as $n \to \infty$ and by symmetry when $B < 0$, $\mu_n(\cdot)$ converges locally to $\nu_- as n \to \infty$.

In this paper we cover the remaining (and most interesting) case proving that

$$
\mu_n(\cdot) \rightarrow \frac{1}{2} \nu_+ (\cdot) + \frac{1}{2} \nu_- (\cdot) \text{ for } B = 0 \text{ and any } \beta \geq 0.
$$

(1.2)

In fact, we prove a sharper result when the graphs $G_n$ are expanders. If $\mu_{n,+}(\cdot)$ and $\mu_{n,-}(\cdot)$ denote the Ising measure (1.1) conditioned to, respectively, $\sum_{i \in V} x_i > 0$ and $\sum_{i \in V} x_i < 0$, then we have

$$
\mu_{n,\pm}(\cdot) \rightarrow \nu_\pm (\cdot) \text{ for } B = 0 \text{ and any } \beta \geq 0,
$$

(1.3)

and moreover the convergence above holds for almost all vertices of the graph. Since $\mu_n = \frac{1}{2} \mu_{n,+} + \frac{1}{2} \mu_{n,-}$ (exactly for $n$ odd and approximately for even $n$), this result implies (1.2).
2 Definitions and main results

2.1 Locally tree-like graphs

We denote by $G_n = (V_n, E_n)$ a graph with vertex set $V_n \equiv [n] = \{1, \ldots, n\}$. The distance $d(i, j)$ between $i, j \in V_n$ is the length of the shortest path from $i$ to $j$ in $G_n$. Given a vertex $i \in V_n$, we denote by $B_i(t)$ the set of vertices whose distance from $i$ is at most $t$ (and with a slight abuse of notation it will also denote the subgraph induced by those vertices). We will let $I$ denote a vertex chosen uniformly from the vertices $V_n$, let $U_n$ denote the measure induced by $I$ and let $J$ denote a uniformly random neighbor of $I$.

This paper is concerned with sequences of graphs $\{G_n\}_{n \in \mathbb{N}}$ of diverging size, that converge locally to $T_k$, the infinite rooted tree of degree $k$. Let $T_k(t)$ be the subset of vertices of $T_k$ whose distance from the root $\emptyset$ is at most $t$ (and, by an abuse of notation, the induced subgraph). For a rooted tree $T$, we write $T \simeq T_k(t)$ if there is a graph isomorphism between $T$ and $T_k(t)$ which maps the root of $T$ to that of $T_k(t)$. The following definition defines what we mean by convergence in the local weak topology.

**Definition 2.1** Consider a sequence of graphs $\{G_n\}_{n \in \mathbb{N}}$, and let $U_n$ be the law of a uniformly random vertex $I$ in $V_n$. We say that $\{G_n\}$ converges locally to the degree-$k$ regular tree $T_k$ if, for any $t$,

$$\lim_{n \to \infty} U_n\{B_I(t) \simeq T_k(t)\} = 1. \quad (2.1)$$

Part of our results hold for sequences of expanders (more precisely, edge expanders), whose definition we now recall. For a graph $G = (V, E)$ and a subset of vertices $S \subset V$, we will denote by $\partial S$ the subset of edges $(i, j) \in E$ having only one endpoint in $S$.

**Definition 2.2** The $k$-regular graph $G = (V, E)$ is a $(\gamma, \lambda)$ (edge) expander if, for any set of vertices $S \subseteq V$ with $|S| \leq n \gamma$, $|\partial S| \geq \lambda |S|$.

2.2 Local weak convergence

In analogy with the definition of locally tree-like graph sequences, we introduce local weak convergence [7] for Ising measures. This is done in two different ways. First one can look at a random vertex and the random configuration in the neighbourhood of the vertex and examine its limiting measure. Alternatively, we may choose a random vertex and consider the marginal distribution of the variables in a neighborhood under the Ising model. This induces (via the random choice of the vertex) a distribution over probability measures. We can therefore ask whether this measure converges to a probability measure over Gibbs measures.

Recall that an Ising measure $\mu$ on the infinite tree $T_k$ may be either defined as a weak limit of Gibbs measures on $T_k(t)$ or in terms of the DLR conditions, see e.g. [8]. An Ising model is in particular a probability measure over $\{-1, +1\}^{T_k}$ endowed with the
σ-algebra generated by cylindrical sets. We let \( G_k \) denote the space of Ising Gibbs measures on \( T_k \) and let \( \mathcal{H}_k \) denote the space of all probability measures on \([+1, -1]^T_k\). We endow both these spaces with the topology of weak convergence. Since \([+1, -1]^T_k\) is compact, \( G_k \) and \( \mathcal{H}_k \) are also compact in the weak topology by Prohorov’s theorem [9, Theorem 16.3].

We define \( \mathcal{M}_k \) (respectively \( \mathcal{M}_k^G \)) to be the space of probability measures over \((\mathcal{H}_k, \mathcal{B}_{\mathcal{H}_k})\) (resp. \((G_k, \mathcal{B}_{G_k})\)), with \( \mathcal{B}_{\Omega} \) the Borel σ-algebra. Also \( \mathcal{M}_k, \mathcal{M}_k^G \) are compact in the weak topology.

We will use generically \( \mu \) for Ising measures on \( G_n \) and \( v \) for Ising measure on \( T_k \). For a finite subset of vertices \( S \subseteq V_n \), we let \( \mu^S \) be the marginal of \( \mu \) on the variables \( x_j, j \in S \). We use the shorthand \( \mu^t \) for when \( S = B_j(t) \) is the ball of radius \( t \) about \( i \) (\( i \) should be clear from the context). For a measure \( v \in G_k \) we let \( v^i \) denote its marginal over the variables \( x_j, j \in T_k(t) \). In other words \( v^i \) is the projection of \( v \) on \([+1, -1]^T_k(t)\). For a measure \( m \in \mathcal{M}_k \) we let \( m^i \) denote the measure on the space of measures on \([+1, -1]^T_k(t)\) induced by such projections.

**Definition 2.3** Consider a sequence of graphs/Ising measures pairs \( \{(G_n, \mu_n)\}_{n \in \mathbb{N}} \) and let \( \mathbb{P}^t_n(i) \) denote the law of the pair \((B_j(t), x_{B_j(t)})\) when \( x \) is drawn with distribution \( \mu \) and \( i \in [n] \) is a vertex in the graph. Let \( U_n \) denote the uniform measure over a random vertex \( I \in [n] \). Let \( \mathbb{P}^t_n = \mathbb{E}_{U_n}(\mathbb{P}^t_n(I)) \) denote the average of \( \mathbb{P}^t_n(I) \).

A. The first mode of convergence concerns picking a random vertex \( I \) and a random local configuration in the neighbourhood of \( I \). Formally, for \( \bar{v} \in G_k \) we say that \( \{\mu_n\}_{n \in \mathbb{N}} \) converges locally on average to \( \bar{v} \) if for any \( t \),

\[
\lim_{n \to \infty} d_{tv}(\mathbb{P}^t_n, \delta_{T_k(t)} \times \bar{v}^i) = 0. \tag{2.2}
\]

B. A stronger form of convergence involves picking a random vertex \( I \) and the associated random local measure \( \mathbb{P}^t_n(I) \) and asking if this distribution of distributions converges. Formally, we say that the local distributions of \( \{\mu_n\}_{n \in \mathbb{N}} \) converge locally to \( m \in \mathcal{M}_k^G \) if it holds that the law of \( \mathbb{P}^t_n(I) \) converges weakly to \( \delta_{T_k(t)} \times m^i \) for all \( t \).

C. If \( m \) is a point mass on \( \bar{v} \in G_k \) and if the local distributions of \( \{\mu_n\}_{n \in \mathbb{N}} \) converge locally to \( m \) then we say that \( \{\mu_n\}_{n \in \mathbb{N}} \) converges in probability locally to \( \bar{v} \). Equivalently convergence in probability locally to \( \bar{v} \) says that for any \( t \) and any \( \epsilon > 0 \) it holds that

\[
\lim_{n \to \infty} U_n \left( d_{tv}(\mathbb{P}^t_n(I), \delta_{T_k(t)} \times v^i) > \epsilon \right) = 0. \tag{2.3}
\]

It is easy to verify that \( C \Rightarrow B \Rightarrow A \).

Similar notions of the convergence has been studied before under the name metastates for Gibbs measures. Aizenman and Wehr [10], while investigating the quenched behaviour of lattice random field models, introduced the notion of a metastate which is a probability measures over Gibbs measures as a function of the disorder (the random field). Here, rather than taking a finite graph and choosing a random vertex they take
a fixed random environment in $\mathbb{Z}^d$, and study the measure over increasing finite volumes. Rather than prove convergence (which depending on the model may not hold) they take subsequential limits and study the properties of these limiting distributions of Gibbs measures (metastates). Another, similar notion of convergence to metastates was developed by Newman and Stein [11] where they took the empirical measure over Gibbs measures at over increasing volumes to study spin-glasses. More references and discussions can be found in [12].

In order to state our main result formally, we recall that an Ising measure on $T_k$ is Gibbs if, for any integer $t \geq 0$

$$\mu_{T_k(t) \setminus T_k(t)}(x_{T_k(t)}) = \frac{1}{Z_{t,\pm}(\beta)} \exp \left\{ \beta \sum_{(i, j) \in E(T_k(t+1))} x_i x_j \right\},$$  

(2.4)

where $Z_{t,\pm}(\beta)$ is a normalization function that depends on the conditioning, namely on $x_{T_k(t) \setminus T_k(t)}$.

It is well known that if $(k-1) \tanh \beta \leq 1$, there exist only one Gibbs measure on a $k$-regular tree while for $(k-1) \tanh \beta > 1$ the Gibbs measures form a non-trivial convex set (see e.g. [8]). Two of its extreme points, $\nu_+$ and $\nu_-$ play a special role in the following. The ‘plus-boundary conditions’ measure $\nu_+$ is defined as the monotone decreasing limit (with respect to the natural partial ordering on the space of configurations $\{+1, -1\}^{T_k}$) of $\nu^t_+$ as $t \to \infty$, where $\nu^t_+$ is the measure on $x_{T_k(t)}$ defined by

$$\nu^t_+(x_{T_k(t)}) = \frac{1}{Z^t_+(\beta)} \exp \left\{ \beta \sum_{(i, j) \in E_n} x_i x_j \right\} \prod_{i \in T_k(t) \setminus T_k(t-1)} \mathbb{I}(x_i = +1).$$

(2.5)

The measure $\nu_-$ is defined analogously, by forcing spins on the boundary to take value $-1$ instead of $+1$. The two measures are obviously related through spin reversal. Further it is well known (and easy to prove) that for any Gibbs measure $\nu$ we have $\nu_- \leq \nu \leq \nu_+$ (with $\leq$ the stochastic ordering induced by the partial ordering on $\{+1, -1\}$ configurations, see e.g. [13]). Our main result may be now stated as follows:

**Theorem 2.4** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of $k$-regular graphs that converge locally to the tree $T_k$. For $(k-1) \tanh \beta > 1$, define the sequence $\{\mu_n\}_{n \in \mathbb{N}}, \{\mu_{n,+}\}_{n \in \mathbb{N}}$ by

$$\mu_{n,+}(x) = \frac{1}{Z_{n,+}(\beta)} \exp \left\{ \beta \sum_{(i, j) \in E_n} x_i x_j \right\} \mathbb{I} \left\{ \sum_{i \in V_n} x_i > 0 \right\},$$

(2.6)

$$\mu_{n}(x) = \frac{1}{Z_{n}(\beta)} \exp \left\{ \beta \sum_{(i, j) \in E_n} x_i x_j \right\}.$$

(2.7)

Then

I. $\mu_n$ converges locally in probability to $\frac{1}{2}(\nu_+ + \nu_-)$
II. If the graphs \( \{G_n\} \) are \((1/2, \lambda)\) edge expanders for some \( \lambda > 0 \), then \( \mu_{n,+} \) converges locally in probability to the plus-boundary Gibbs measure on the infinite tree \( \nu_+ \).

This characterization has a number of useful consequences. In particular, ‘spatial’ averages of local functions are roughly constant under the conditional measure \( \mu_{n,+} \).

To be more precise, for each \( i \in V_n \), let

\[
 f_i, n : \{+1, -1\}^{B_i(\ell)} \to [-1, 1],
\]

be a function of its neighborhood \( B_i(\ell) \).

**Theorem 2.5** Let \( \{G_n\}_{n \in \mathbb{N}} \) be a sequence of \( k \)-regular \((1/2, \lambda)\) edge expanders, for some \( \lambda > 0 \), that converge locally to the tree \( T_k \). For each \( n \), let \( \{f_i, n\}_{i=1}^n \) be a collection of local functions as above. Then, for any \( \varepsilon > 0 \)

\[
 \lim_{n \to \infty} \mu_{n,+} \left\{ \left| \frac{1}{n} \sum_{i \in V_n} f_i, n(x_{B_i(\ell)}) - \mu_{n,+} \left( \frac{1}{n} \sum_{i \in V_n} f_i, n(x_{B_i(\ell)}) \right) \right| \geq \varepsilon \right\} = 0.
\]

(2.8)

The proof can be found in Sect. 5.

2.3 Examples and remarks

Notice that, for \((k - 1) \tanh \beta \leq 1\), the set of Ising Gibbs measures on \( T_k \) contains a unique element, that can be obtained as limit of free boundary measures. Therefore, the local limits of \( \{\mu_n\}_{n \in \mathbb{N}}, \{\mu_{n,+}\}_{n \in \mathbb{N}} \) coincide trivially with this unique Gibbs measure.

Therefore, the claim I is proved under the weakest possible, hypothesis, namely local convergence of the graphs to \( T_k \). An important class of graphs for which Theorem 2.4 is applicable are random \( k \)-regular graphs. These are known to converge locally to \( T_k \) [14,15].

The expansion condition (or an analogous ‘connectedness’ condition) is needed to obtain the convergence of the conditional measures \( \mu_{n,+} \). For example, consider \( r \) identical but disjoint graphs on \( n/r \) vertices. Then conditioning on the sum of the spins being positive the probability that the sum of spins in a specific component is positive is of order \( 1 + O(1/r) \). Therefore in this case we have:

\[
 \mu_{n,+} \to (1 - q) \nu_+ + q \nu_-,
\]

with \( q = 1/2 - O(1/r) \). A similar construction may be repeated with a small number of edges connecting different components, e.g., when the components are connected in a cyclic fashion.

In order to identify the limit for \( \mu_n \) and obtain our results, there are a number of challenges that need to be overcome. First, while soft compactness arguments imply that subsequential limits exist, such arguments do not imply the existence of a proper limit. Second, recalling that there are uncountably many extremal Gibbs measures for \( T_k \), it is remarkable we are able to identify precisely those that appear in the limit. Finally,
for conditional measures such as $\mu_{n,+}$ it is not even a priori clear that (subsequential) limits are in fact Gibbs measures.

2.4 Proof strategy

The basic idea of the proof is the following. Look at a ball of radius $t$ around a vertex $i$ in $G_n$. Since $G_n$ is tree like, the ball is with high probability a tree. The measure $\mu_n$ restricted to the ball is clearly a Gibbs measure on a tree of radius $t$. The same is true (although less obvious) for $\mu_{n,+}$.

In order to characterize the limit of this measure as $n \to \infty$, we proceed as follows:

1. The probability of agreement between neighboring spins in the ball is asymptotically the same as in the measure $\nu_+$ on the infinite tree.
2. We further show that $\nu_+$ maximizes the probability of agreement between neighboring spins among all Gibbs measures on the tree. These two facts together imply that any local limit must converge to a convex combination of $\nu_+$ and $\nu_-$. 
3. By symmetry this already implies converges of $\mu_n$ to $\frac{1}{2}(\nu_+ + \nu_-)$. Note that this step does not require expansion, just the local weak convergence to the tree.
4. In order to deal with the conditional measure, we use expansion to show that it is unlikely that simultaneously a positive fraction of the vertices have their neighborhood “in the $+$ state” and another positive fraction “in the $-$ state”.

3 Proof of the main theorem

We now proceed with the proof. For each of claims I and II we break the proof into three steps:

(i) We consider a subsequence of sizes $\{n(m)\}_{m \in \mathbb{N}}$ along which $\mu_{n(m)}$ or $\mu_{n(m),+}$ converge locally in average to a limit $\bar{\nu}$ or $\bar{\nu}_+$ (respectively).
(ii) We prove that any such limit is in fact always the same and is $\bar{\nu} = (1/2)(\nu_+ + \nu_-)$ for $\mu_{n(m)}$ and (using expansion) $\bar{\nu}_+ = \nu_+$ for $\mu_{n(m),+}$. As a consequence the sequences themselves converge.
(iii) Finally we show how it is possible to deduce local convergence from convergence in average.

3.1 Subsequential limits

The construction of subsequential weak limits is based on a standard diagonal argument, for similar results see [16]. For the sake of simplicity we refer to the measures $\mu_{n,+}$, and construct the subsequential limit $\bar{\nu}_+$, but the same procedure works for $\mu_n$ with limit $\bar{\nu}$. Let $B_I(t)$ be the ball of radius $t$ centered at a uniformly random vertex $I$ in $V_n$, and $x$ be an Ising configuration with distribution $\mu_{n,+}$. If $P_n$ denotes the joint distribution of $(B_I(t), x_{B_I(t)})$, we define

$$
\mu_{n,+}^t (x_{T_k(t)}^+) := P_n \left\{ (B_I(t), x_{B_I(t)}) \simeq (T_k(t), x_{T_k(t)}^+) \right\}
$$

(3.1)
where the isomorphism indicates first that $B_I(t)$ is isomorphic to $T_k(t)$ and if that holds that there is equality between $\mathcal{L}_{B_I(t)}$ and $\mathcal{L}_{T_k(t)}$ (determined according the fixed isomorphism between $B_I(t)$ and $T_k(t)$). Since this is a sequence of measures over a finite state space, it converges over some subsequence $\{n_t(m)\}_{m \geq 0}$. Further, since by hypothesis $P_n(B_I(t) \simeq T_k(t)) \to 1$, the limits of $\mu^t_{n_t(m),+}$ and $\mu^t_{n_t(m)}$ are in fact probability measures. We call the limit $\bar{\nu}^t_+$.  

Fix one of these subsequences $\{n_{t_0}(m)\}_{m \geq 0}$ for $t = t_0$, leading to the limit $\bar{\nu}^0_+$, and recursively refine it to $\{n_{t_0}(m)\}_{m \geq 0} \supseteq \{n_{t_0+1}(m)\}_{m \geq 0} \supseteq \{n_{t_0+2}(m)\}_{m \geq 0} \supseteq \cdots$ leading to limits $\bar{\nu}^t_+$ for all $t \geq t_0$. Notice that, for any graph $G_n$, any vertex $i$ and any $t$ we have

$$\mu^t_{n_t,+}(\mathcal{L}_{B_i(t)}) = \sum_{\mathcal{L}_{B_i(t) \setminus B_i(t)}} \mu^{t+1}_{n_t,+}(\mathcal{L}_{B_i(t+1)}).$$

(3.2)

As a consequence, for any $t$, the measures limit $\bar{\nu}^t_+$ measure satisfies

$$\bar{\nu}^t_+(\mathcal{L}_{T_k(t)}) = \sum_{\mathcal{L}_{T_k(t+1) \setminus T_k(t)}} \bar{\nu}^{t+1}_+(\mathcal{L}_{T_k(t+1)}).$$

(3.3)

By Kolmogorov extension theorem, there exist measures $\bar{\nu}_+$ over $\{+1, -1\}^{T_k}$ such that $\bar{\nu}_+$ are the marginals of $\bar{\nu}^t_+$ over the variables in the subtree $T_k(t)$. By taking the diagonal subsequence $n(m) = n_m(m)$ we obtain the desired subsequence $\{n(m)\}_{m \in \mathbb{N}}$ such that $\mu_{n(m),+}$ converges locally on average to $\bar{\nu}_+$.  

$3.2 \; \bar{\nu} = \frac{1}{2}(\nu_+ + \nu_-)$  

In this section, we carry out our program in the case of the unconditional measures $\mu_n$. It is immediate that, since each of the measures $\mu^t_{n_0} \cdot P_n(B_i(t) \simeq T_k(t))^{-1}$ is a Gibbs measure on $T_k$ (although with a complicated boundary condition) and that $P_n(B_i(t) \simeq T_k(t)) \to 1$, the limit measure $\bar{\nu}$ is also a Gibbs measure on $T_k$ (i.e. $\bar{\nu} \in \mathcal{G}_k$).  

For proving convergence of the unconditional measure we need two lemmas. The first one establishes that the $+$ (equivalently $-$) Gibbs measure $\nu_+$ has the correct expected number of edge disagreements (in physics terms, the correct internal energy density).

**Lemma 3.1** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of $k$-regular graphs converging locally to $T_k$, let $I$ be a uniformly random vertex in $G_n$, and $J$ be chosen uniformly among its $k$ neighbors. Then

$$\lim_{n \to \infty} E_{U_n} \mu_{n,+}(x_I \cdot x_J) = \lim_{n \to \infty} E_{U_n} \mu_n(x_I \cdot x_J) = \nu_+(x_0 \cdot x_1) = \nu_-(x_0 \cdot x_1),$$

(3.4)

where 1 is one of the neighbors of the root in $T_k$, and $E_{U_n}$ denotes the expectation over the random edge $(I, J)$ in $G_n$.  

† Springer
For the proof of this lemma we refer to Sect. 4.2. Notice that $\nu_+$ and $\nu_-$ have the same expectation of the product $x_\varnothing x_1$ by symmetry under inversion $\{x_i\} \to \{-x_i\}$. The probability that the spins at $\varnothing$ and 1 agree is simply $(1 + \nu(x_\varnothing \cdot x_1))/2$. The second Lemma shows that $\nu_+, \nu_-$ are uniquely characterized by this agreement probability among all Ising Gibbs measures on $T_k$.

**Lemma 3.2** Let $\nu$ be a Gibbs measure for the Ising model on $T_k$. Then

$$\nu(x_\varnothing \cdot x_1) \leq \nu_+(x_\varnothing \cdot x_1) = \nu_-(x_\varnothing \cdot x_1), \quad (3.5)$$

and the inequality is strict unless $\nu$ is a convex combination of $\nu_+$ and $\nu_-$. The proof of this lemma can be found in Sect. 4.3. We can now prove the following:

**Proposition 3.3** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of $k$-regular graphs that converge locally to the tree $T_k$. Then for $(k-1) \tanh \beta > 1$, it holds that $\mu_n$ converges locally in average to $(1/2)(\nu_+ + \nu_-)$.

**Proof** By Lemma 3.1 and weak convergence, we have $\bar{\nu}(x_\varnothing \cdot x_1) = \nu_+(x_\varnothing \cdot x_1)$. By Lemma 3.2, $\nu = (1-q)\nu_+ + q\nu_-$ for some $q \in [0, 1]$. On the other hand, $\mu_{n,t}$ is symmetric under spin inversion for each $n$, and therefore $\bar{\nu}$ must be symmetric as well, whence $q = 1/2$. \qed

We can now prove the first part of our main result.

**Proof** (Theorem 2.4, part I) By a similar construction to the one recalled in Sect. 3.1, and compactness of $M_k$, we can construct a subsequence $\{n(m)\}_{m \in \mathbb{N}}$ such that $\mu_{n(m)}$ converges locally (not only in average) to a distribution $\mu$ over $H_k$. By the arguments above, $\mu$ is in fact a measure over the space of Ising Gibbs measures $G_k$.

We claim that any such subsequential weak limit $\mu$ is in fact a point mass at $(1/2)(\nu_+ + \nu_-)$. Since $\nu \mapsto \nu(x_\varnothing \cdot x_1)$ is continuous in the weak topology it follows that

$$\lim_{m \to \infty} E_{U_n} \mu_{n(m)}(x_I \cdot x_J) = \int \nu(x_\varnothing \cdot x_1) \mu(d\nu). \quad (3.6)$$

By Lemma 3.1, this implies

$$\int \nu(x_\varnothing \cdot x_1) \mu(d\nu) = \nu_+(x_\varnothing \cdot x_1), \quad (3.7)$$

and therefore, by Lemma 3.2, $\mu$ is supported on Ising Gibbs measures $\nu$ that are convex combinations of $\nu_+$ and $\nu_-$. Finally, $\mu_n$ is almost surely symmetric for any $n$. Here “symmetric” means that, for any configuration $x_{B_1(t)}$, $\mu_n'(x_{B_1(t)}) = \mu_n'(-x_{B_1(t)})$. Therefore $\mu$ is supported on Ising Gibbs measures that are symmetric.

There is only one Ising Gibbs measure that is a convex combination of $\nu_+$ and $\nu_-$ and is symmetric, namely $\nu = (1/2)(\nu_+ + \nu_-)$. Hence $\mu$ is a point mass on this distribution. \qed
We now turn to the subsequence of conditional measures $\{\mu_{n(m),+}\}_{m\in\mathbb{N}}$ converging locally in average to $\bar{\nu}_+$. The goal of this section is to show that $\bar{\nu}_+$ is equal to $\nu_+$. For this we repeat the previous proof with two additional ingredients. First, we need to show that $\bar{\nu}_+$ is a Gibbs measure on the tree $T_k$. This requires proof since the conditioning on $\{\sum_{i\in V_n} x_i > 0\}$ implies that the measures $\mu_{n,+}^t$ are not Gibbs measures. The Gibbs property is only recovered in the limit.

Second even after we have established that $\bar{\nu}_+$ is a Gibbs measure, this measure is not symmetric with respect to spin flip. Therefore the argument above only implies that $\bar{\nu}_+= (1-q)\nu_+ + q \nu_-$. It remains to show that $q = 0$. This is where the expansion assumption is used. The first lemma we prove is the following:

**Lemma 3.4** Any subsequential limit $\bar{\nu}_+$ constructed as above is an Ising–Gibbs measure on $T_k$.

We defer the proof to Sect. 4.1. Given Lemma 3.4 the following lemma follows immediately from Lemmas 3.1 and 3.2.

**Lemma 3.5** For any subsequential limit $\bar{\nu}_+$ there exists a $q \in [0, 1]$ such that

$$\bar{\nu}_+ = (1-q)\nu_+ + q \nu_-. \quad (3.8)$$

**Proof** By Lemma 3.4 the measure $\bar{\nu}_+$ is an Ising–Gibbs measure on $T_k$. If it was not a convex combination of $\nu_+$ and $\nu_-$ a contradiction to Lemma 3.2 would be derived.

The last step consists of arguing that $q = 0$. Given a vertex $i$ (either in a graph $G_n$ of the sequence or of $T_k$), an integer $\ell \geq 1$ and a random Ising configuration $x$, let

$$F_i(\ell, \delta, x) \equiv \mathbb{I} \left\{ \sum_{j \in B_i(\ell)} x_j \leq -\delta |B_i(\ell)| \right\}, \quad (3.9)$$

where $\delta \in (0, 1)$ will be chosen below. Roughly speaking $F_i$ indicates which vertices are in the “$-$ state”. We will drop reference to $\delta$ and to the configuration $x$ when clear from the context. The following lemmas will be proven in Sect. 4.4.

**Lemma 3.6** Let $\{G_n\}$ be a sequence of graphs converging locally to $T_k$, and, for each $n$, $x = x(n)$ be a configuration in the support of $\mu_{n, +}$. Then there exists $n_0$, depending on $\delta, \ell$ and the graph sequence, but not on $x$, such that, for all $n \geq n_0$,

$$\mathbb{E}_{U_n}(F_I(\ell, \delta, x)) \leq \frac{1}{1 + \delta/2}, \quad (3.10)$$

where $\mathbb{E}_{U_n}$ denotes expectation with respect to the uniformly random vertex $I$ in $V_n$.
The following lemma is an immediate consequence of the definition of local weak convergence.

**Lemma 3.7** Consider a uniformly random vertex $I$ in $G_n$, let $J$ be one of its neighbors (again uniformly random), and let $(n(m))_{m \in \mathbb{N}}$ a subsequence of graph sizes along which $\mu_{n(m),+}$ converges locally on average to $\bar{v}_+$. Then we have

$$
\lim_{m \to \infty} \mathbb{E}_{U_n(m)} \mu_{n(m),+}(F_{I}(\ell)) = \bar{v}_+(F_{\phi}(\ell)),
$$

(3.11)

$$
\lim_{m \to \infty} \mathbb{E}_{U_n(m)} \mu_{n(m),+}(F_{I}(\ell) \neq F_{J}(\ell)) = \bar{v}_+(F_{\phi}(\ell) \neq F_{1}(\ell)),
$$

(3.12)

with $\mathbb{E}$ denoting expectation with respect to the law $U_{n(m)}$ of vertices $I$ and $J$, and 1 one of the neighbors of $\phi$.

Now the limit quantities can be estimated as follows.

**Lemma 3.8** Assume $(k - 1) \tanh \beta > 1$ and let $\nu = (1 - q) \nu_+ + q \nu_-$ be a mixture of the plus and minus measures for the Ising model on $T_k$. Then there exist $\delta = \delta(\beta) > 0$ such that, letting $F_i(\ell) = F_{i}(\ell, \delta; \chi)$,

$$
\lim_{\ell \to \infty} \nu(F_{\phi}(\ell) = 1) = q,
$$

(3.13)

$$
\lim_{\ell \to \infty} \nu(F_{\phi}(\ell) \neq F_{1}(\ell)) = 0.
$$

(3.14)

We can now prove the following:

**Proposition 3.9** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of $k$-regular graphs that are $(1/2, \lambda)$ expanders for some $\lambda > 0$ and converge locally to the tree $T_k$. Then for $(k - 1) \tanh \beta > 1$, it holds that $\mu_{n,+}$ converges locally on average to $\nu_+$.

**Proof** Let $n(m)$ be a subsequence along which $\mu_{n,+}$ converges locally on average to some $\bar{v}_+$. By Lemma 3.5 we can write this in the form $\bar{v}_+ = (1 - q) \nu_+ + q \nu_-$. Then by Eqs. (3.11), (3.12), for any $\varepsilon > 0$, there exists $\ell$, such that for large enough $n(m)$,

$$
\mathbb{E}_{\mu_{n(m),+}}(F_{I}(\ell)) \geq \nu - \varepsilon,
$$

(3.15)

$$
\mathbb{E}_{\mu_{n(m),+}}(1[F_{I}(\ell) \neq F_{J}(\ell)]) \leq \varepsilon.
$$

(3.16)

On the other hand, since $G_n$ is a $(1/2, \lambda)$ expander, and using Eq. (3.10), we have

$$
\sum_{(i,j) \in E_n} 1[F_{I}(\ell) \neq F_{J}(\ell)] \geq \lambda \min \left( \sum_{i \in V_n} F_{i}(\ell), \sum_{i \in V_n} (1 - F_{i}(\ell)) \right)
$$

(3.17)

$$
\geq \lambda \min \left( \sum_{i \in V_n} F_{i}(\ell), n\delta/(2 + \delta) \right)
$$

(3.18)

$$
\geq \frac{\lambda \delta}{2 + \delta} \sum_{i \in V_n} F_{i}(\ell).
$$

(3.19)
Recalling (3.15), (3.16), taking expectation of both sides with respect to $\mu_n, +$ and representing the sums over $E_n, V_n$ as expectations, we get

$$ k^2 \varepsilon \geq k \mathbb{E}_{\mu_n, +} \left( \mathbb{I}[\mathcal{F}_I(\ell) \neq \mathcal{F}_J(\ell)] \right) \geq \frac{\lambda \delta}{2 + \delta} \mathbb{E}_{\mu_n, +} (\mathcal{F}_I(\ell)) \geq \frac{\lambda \delta}{2 + \delta} (q - \varepsilon). \quad (3.20) $$

Since $\varepsilon > 0$ is arbitrary, we derive a contradiction unless $q = 0$. The proof follows.

We can now complete the proof of Theorem 2.4.

**Proof** (Theorem 2.4, part II) Let $n(m)$ be a subsequence along which the local distributions of $\mu_n, +$ converge locally to some $m$ (by the same compactness arguments used in the previous section, one always exists). Now by Proposition 3.9, it follows that $\nu_+ = \int_{G_n} \nu m(d\nu)$ which implies that $m$ is a point measure on $\nu_+$ since it is extremal. This implies local convergence in probability to $\nu_+$, which completes the proof. $\square$

**4 Proofs of Lemmas**

**4.1 Proof of Lemma 3.4**

We start from a very general remark, which is implicit in [17] holding for a general Markov random field on a graph $G = (V, E)$

$$ \mu(x) = \frac{1}{Z} \prod_{(i, j) \in E} \psi_{i, j}(x_i, x_j) \quad (4.1) $$

where $x = \{x_i\}_{i \in V} \in \mathcal{X}^V$ for a finite spin alphabet $\mathcal{X}$, and $\psi_{ij} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a collection of potentials. Recall that a subset $S$ of the vertices of $G$ is an independent set if, for any $i, j \in S$, $(i, j) \notin E$.

**Lemma 4.1** Assume $0 < \psi_{\min} \leq \psi_{ij}(x_i, x_j) \leq \psi_{\max}$, let $k$ be the maximum degree of $G$, and $I(G)$ the maximum size of an independent set of $G$. Then there exists a constant $C = C(k, \psi_{\max}/\psi_{\min}) > 0$ such that, for any $x \in \mathcal{X}$ and any $\ell \in \mathbb{N}$,

$$ \mu \left( \sum_{i \in V} I_{x_i = x} = \ell \right) \leq \frac{C}{\sqrt{I(G)}}. \quad (4.2) $$

**Proof** Let $S$ be a maximum size independent set and $S^c = V \setminus S$ its complement. Further, let $Y_U = \sum_{i \in U} I_{x_i = x}$ for $U \subseteq V$. Conditioning on $x_{S^c} = \{x_i : i \in S^c\}$

$$ \mu \left( \sum_{i \in V} I_{x_i = x} = \ell \right) = \mathbb{E}_\mu \left\{ \mu \left( Y_S = \ell - Y_{S^c} \mid x_{S^c} \right) \right\}. \quad (4.3) $$
Conditional on $x_{S'}$, the variables $\{x_i\}_{i \in S}$ are independent with $\delta \leq \mu(x_i = x| x_{S'}) \leq 1 - \delta$ for some $\delta > 0$ depending on $k$ and $\psi_{\text{max}} / \psi_{\text{min}}$. As a consequence $Y_S$ is the sum of $|S| = I(G)$ independent Bernoulli random variables with expectation bounded away from 0 and 1. By the Berry-Esseen Theorem

$$\mu \left( Y_S = \ell - Y_{S'} | x_{S'} \right) \leq \frac{C}{\sqrt{T(G)}}, \quad (4.4)$$

which implies the thesis.

**Proof** (Lemma 3.4) Recall that for $T_k$, the infinite rooted $k$-regular tree, we denote by $T_k(t)$ the subtree induced by nodes with distance at most $t$ from the root $\varnothing$. Also, denote $T_k(t, t+) = T_k(t_+) \setminus T_k(t)$, the subgraph induced by nodes $i$ with distance $t + 1 \leq d(i, \varnothing) \leq t_+$. Let $\tilde{\nu}_+$ denote a subsequential limit of the measures $\mu_{n,+}$ constructed as in Sect. 2.4. For any $t \geq 1$ and $t_+ > t$ we will prove that the conditional distribution of $x_{T_k(t)}$ given $x_{T_k(t, t_+)}$ is given by (here and below we adopt the convention of writing $p(x|y) \equiv f(x, y)$ for a conditional distribution $p$, whenever $p(x|y) = f(x, y) / \sum_{x'} f(x', y)$):

$$\tilde{\nu}_+^{T_k(t)|T_k(t, t_+)}(x_{T_k(t)}|x_{T_k(t, t_+)}) \equiv \exp \left\{ \beta \sum_{(i,j) \in E(T_k(t+1))} x_i x_j \right\}. \quad (4.5)$$

Since the sigma-algebra generated by $\bigcup_{t_+ \geq t} \sigma(\{x_{T_k(t, t_+)}\})$ coincides with the one $\sigma(\{x_{T_k(t, \infty)}\})$, this establishes the DLR conditions and implies that $\tilde{\nu}_+$ is a Gibbs measure as required.

In analogy with the notation introduced above (and recalling that $B_i(t)$ is the ball of radius $t$ around vertex $i$ in $G_n$), we let $B_i(t, t_+) = B_i(t_+) \setminus B_i(t)$ be the subgraph induced by vertices $j$ such that $t + 1 \leq d(i, j) \leq t_+$. Also $E_i(t) = E_i(t) = E_i \setminus E_i(t)$, the marginal distribution of $x_{B_i(t_+)}$ under $\mu_{n,+}$ is given by

$$\mu_{n,+}^{T_k+(x_{B_i(t_+)})} \equiv F_{B_i(t_+)}(x_{B_i(t_+)}) Z_{B_i(t_+)}(x_{B_i(t_+)}) \quad (4.6)$$

$$F_{B_i(t_+)}(x_{B_i(t_+)}) \equiv \exp \left\{ \beta \sum_{(l,j) \in E_i(t_+)} x_l x_j \right\}, \quad (4.7)$$

$$Z_{B_i(t_+)}(x_{B_i(t_+)}) \equiv \sum_{x_{V \setminus B_i(t_+)}} \exp \left\{ \beta \sum_{(l,j) \in E_i(t_+)} x_l x_j \right\} \mathbb{1} \left( \sum_{j \in B_i(t_+)} x_j > - \sum_{j \in B_i(t_+)} x_j \right). \quad (4.8)$$

We, therefore, have the following expression for the conditional distribution of $x_{B_i(t_+)}$, given $x_{B_i(t, t_+)}$:
\[
\mu_{B_i(t_+)|B_i(t_+)}(x_{B_i(t_+)}) = \frac{F_{B_i(t_+)}(x_{B_i(t_+)})Z_{B_i(t_+)}(x_{B_i(t_+)})}{\sum_{x_{B_i(t_+)}} F_{B_i(t_+)}(x_{B_i(t_+)})Z_{B_i(t_+)}(x_{B_i(t_+)})}.
\] (4.9)

On the other hand, we have
\[
Z_{B_i(t_+)}^{-}(x_{B_i(t_+)}) \leq Z_{B_i(t_+)}(x_{B_i(t_+)}) \leq Z_{B_i(t_+)}^{+}(x_{B_i(t_+)})
\]
where we define
\[
Z_{B_i(t_+)}^{\pm}(x_{B_i(t_+)}) = \sum_{x_{B_i(t_+)}} \exp \left\{ \beta \sum_{(l,j) \in E_i(t_+)} x_i x_j \right\} \mathbb{I}_n \left( \sum_{j \in B_i(t_+)} x_j > -|B_i(t_+)| \right).
\] (4.10)

Notice that \(Z_{B_i(t_+)}^{\pm}(x_{B_i(t_+)})\) depend on \(x_{B_i(t_+)}\) only through \(x_{B_i(t_+)}\). Using the expression (4.9) for the conditional probability (and dropping subscripts on \(\mu\) to lighten the notation), we have

\[
\mu_{n,+}(x_{B_i(t_+)}) \leq \mu^*(x_{B_i(t_+)}) \leq \mu_{n,+}(x_{B_i(t_+)})
\]
\[
\mu_{n,+}(x_{B_i(t_+)}) \geq \mu^*(x_{B_i(t_+)}) \leq \mu_{n,+}(x_{B_i(t_+)})
\]
\[
\mu^*(x_{B_i(t_+)}) \equiv \exp \left\{ \beta \sum_{(l,j) \in E_i(t+1)} x_i x_j \right\}.
\] (4.11)

The claim (4.5) thus follows from the fact that \(B_i(t_+) \sim \mathbb{T}_k(t_+)\) with probability going to 1 as \(n \to \infty\), if we can show that
\[
\frac{Z_{B_i(t_+)}^{-}(x)}{Z_{B_i(t_+)}^{+}(x)} \to 1
\] (4.14)
for all \(x \in \{+1, -1\}^{T_k(t_+)}\) as \(n \to \infty\).

Let \(\hat{\mu}\) denote the Ising measure on \(x_{B_i(t_+)}\) with boundary conditions \(x_{B_i(t_+)}\)
\[
\hat{\mu}(x_{B_i(t_+)}) = \frac{1}{Z(x_{B_i(t_+)})} \exp \left\{ \beta \sum_{(l,j) \in E_i(t_+)} x_i x_j \right\}.
\] (4.15)
Now

\[
1 - \frac{Z_{B_i}(t_+)}{Z_{B_i}^+(t_+)(x)} = \frac{\hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > -|B_i(t_+)|\right) - \hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > |B_i(t_+)|\right)}{\hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > -|B_i(t_+)|\right)}.
\]

Observe that by the Gibbs construction of $\mu$ for any $x_{B_i^c(t_+)}$, we have that

\[
\hat{\mu}(x_{B_i^c(t_+)}) \geq \exp(-2\beta k |B_i(t_+)|)\mu_n(x_{B_i^c(t_+)}),
\]

as this is the maximum effect that conditioning on a set of size $|B_i(t_+)|$ can have on the measure $\mu$. By symmetry of the measure $\mu_n$ with respect to the sign of $x$,

\[
\hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > -|B_i(t_+)|\right) \geq \exp(-2\beta k |B_i(t_+)|)\mu_n\left(\sum_{j \in B_i^c(t_+)} x_j > 0\right)
\]

\[
\geq \frac{1}{2} \exp(-2\beta k |B_i(t_+)|).
\]

Now applying Lemma 4.1 to the measure $\hat{\mu}$ we have that

\[
\hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > -|B_i(t_+)|\right) - \hat{\mu}\left(\sum_{j \in B_i^c(t_+)} x_j > |B_i(t_+)|\right)
\]

\[
= \hat{\mu}\left(\left|\sum_{j \in B_i^c(t_+)} x_j\right| \leq |B_i(t_+)|\right)
\]

\[
\leq \frac{2C|B_i(t_+)|}{\sqrt{n - |B_i(t_+)|}} \to 0 \quad (4.17)
\]

for some $C = C(k, \beta)$ as $n \to \infty$. Combining Eqs. (4.17) and (4.16) we establish Eq. (4.14) which completes the proof.

4.2 Proof of Lemma 3.1

For the convenience of the reader, we restate the main result of [6] in the case of $k$-regular graphs, with no magnetic field $B$. This provides an asymptotic estimate of
the partition function (we restrict ourselves to the case of vanishing external field)

\[ Z_n(\beta) = \sum_{x} \exp \left\{ \beta \sum_{(i,j) \in E} x_i x_j \right\}. \]  

(4.18)

**Theorem 4.2** Let \( \{G_n\}_{n \in \mathbb{N}} \) be a sequence of graphs that converges locally to the \( k \)-regular tree \( T_k \). For \( \beta > 0 \), let \( h \) be the largest solution of

\[ h = (k-1) \text{atanh}[\tanh(\beta) \tanh(h)]. \]  

(4.19)

Then

\[ \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta) = \phi(\beta), \]  

where

\[ \phi(\beta) \equiv k \log \cosh(\beta) - \frac{k}{2} \log \left( 1 + \tanh(\beta) \tanh(h)^2 \right) + \log \left\{ [1 + \tanh(\beta) \tanh(h)]^k + [1 - \tanh(\beta) \tanh(h)]^k \right\}, \]  

(4.20)

For the proof of Lemma 3.1 we start by noticing that, by symmetry under change of sign of the \( x_i \)'s, we have \( \mu_{n,+}(x_i \cdot x_j) = \mu_n(x_i \cdot x_j) \). Simple calculus yields

\[ \frac{1}{n} \frac{\partial}{\partial \beta} \log Z_n(\beta) = \frac{1}{n} \sum_{(i,j) \in E_n} \mu_n(x_i \cdot x_j) = \frac{k}{2} \mathbb{E} \mu_n(x_I \cdot x_J), \]  

(4.21)

where the expectation \( \mathbb{E} \) is taken with respect to \( I \) uniformly random vertex, and \( J \) one of its neighbors taken uniformly at random.

On the other hand, differentiating Eq. (4.20) with respect to \( \beta \), and using the fixed point condition (4.19), we get after some algebraic manipulations

\[ \frac{\partial}{\partial \beta} \phi(\beta) = \frac{k}{2} \frac{\tan \beta + (\tanh h)^2}{1 + \tan \beta (\tanh h)^2} = \frac{k}{2} v_+(x_\emptyset \cdot x_1). \]  

(4.22)

The last identification comes from the fact that the joint distribution of \( x_\emptyset \) and \( x_1 \) on a \( k \)-regular tree under the plus-boundary Gibbs measure is \( v_+(x_\emptyset, x_1) \propto \exp\{\beta x_\emptyset x_1 + h x_\emptyset + h x_1\} \) (see [6]).

Further \( \beta \mapsto \frac{1}{n} \log Z_n(\beta) \) is convex because its second derivative is proportional to the variance of \( \sum_{(i,j)} x_i x_j \) with respect to the measure \( \mu_n \). Therefore, its derivative \( (k/2) \mathbb{E} \mu_n(x_I \cdot x_J) \) converges to \( (k/2) v_+(x_\emptyset \cdot x_1) \) for a dense subset of values of \( \beta \). Since the limit \( \beta \mapsto v_+(x_\emptyset \cdot x_1) \) is continuous, convergence takes place for every \( \beta \).

### 4.3 Proof of Lemma 3.2

Recalling that \( T_k \) denotes the infinite \( k \)-regular tree rooted at \( \emptyset \) let \( T^\emptyset \) and \( T^1 \) be the subtrees obtained by removing the edge (\( \emptyset, 1 \)) where 1 is a neighbor of \( \emptyset \). It is sufficient to prove the claim when \( v \) is an extremal Gibbs measure on \( T_k \) since of course we may
decompose any Gibbs measure into a mixture of extremal measures. For \( i \in \{\emptyset, 1\} \) define

\[
m_i^\nu = \lim_{\ell \to \infty} \mathbb{E}_{T_i}(x_i \mid x_{\mathcal{B}_i(\ell) \cap T_i})
\]

where \( \mathbb{E}_{T_i} \) denotes expectation with respect to the Ising model on the tree \( T_i \) and the boundary condition \( x_{\mathcal{B}_i(\ell) \cap T_i} \) is chosen according to \( \nu \). The limit exists by the Backward Martingale Convergence Theorem. Further it is a constant almost surely, because it is measurable with respect to the tail \( \sigma \)-field, and \( \nu \) is extremal.

By the monotonicity of the Ising model if \( \nu \leq \nu' \), then \( m_i^\nu \leq m_i^{\nu'} \). The following calculation gives the effect of \( m_\emptyset^\nu \), \( m_1^\nu \) and the edge \( (\emptyset,1) \) on the expectation of \( x_\emptyset \) which is given by

\[
v(x_\emptyset) = \frac{\sum_{(z_0,z_1) \in [-1,1]^2} x_\emptyset \mathbb{P}_{T_\emptyset}(x_\emptyset) \mathbb{P}_{T_1}(x_1) e^{\beta x_\emptyset z_1}}{\sum_{(z_0,z_1) \in [-1,1]^2} \mathbb{P}_{T_\emptyset}(x_\emptyset) \mathbb{P}_{T_1}(x_1) e^{\beta x_\emptyset z_1}}
\]

\[
= \frac{\sum_{(z_0,z_1) \in [-1,1]^2} x_\emptyset (1 + x_\emptyset m_\emptyset^\nu)(1 + x_1 m_1^\nu) e^{\beta x_\emptyset z_1}}{\sum_{(z_0,z_1) \in [-1,1]^2} (1 + x_\emptyset m_\emptyset^\nu)(1 + x_1 m_1^\nu) e^{\beta x_\emptyset z_1}}
\]

\[
= \frac{2(e^\beta + e^{-\beta}) m_\emptyset^\nu + 2(e^\beta - e^{-\beta}) m_1^\nu}{2(e^\beta + e^{-\beta}) + 2(e^\beta - e^{-\beta}) m_\emptyset^\nu m_1^\nu}
\]

\[
= \frac{m_\emptyset^\nu + \tanh(\beta) m_1^\nu}{1 + \tanh(\beta) m_\emptyset^\nu m_1^\nu}.
\]

Now if \( \nu \neq \nu_+ \) then \( v(x_\emptyset = 1) < v_+ (x_\emptyset = 1) \). Under the plus measure \( m_\emptyset^{\nu_+} \), \( m_1^{\nu_+} \) which by the monotonicity of the system is the maximal such value. Since the right hand side of Eq. (4.23) is increasing in \( m_\emptyset, m_1 \) it follows that \( m_\emptyset^\nu = m_1^\nu = m^+ \) if and only if \( \nu = \nu_+ \).

An easy tree calculation shows that the expectation of \( x_\emptyset \cdot x_1 \) is

\[
v(x_\emptyset \cdot x_1) = \frac{\tanh(\beta) + m_\emptyset^\nu m_1^\nu}{1 + \tanh(\beta) m_\emptyset^\nu m_1^\nu}.
\]

which is strictly increasing in \( m_\emptyset^\nu \) when \( m_1^\nu > 0 \). By symmetry it is also strictly increasing in \( m_1^\nu \) when \( m_\emptyset^\nu > 0 \). Hence amongst measures \( \nu \) with \( m_\emptyset^\nu \geq 0 \), the expectation \( v(x_\emptyset \cdot x_1) \) is uniquely maximized when \( m_\emptyset^\nu = m_1^\nu = m^+ \), that is when \( \nu = \nu_+ \). Similarly amongst measures \( \nu \) with \( m_\emptyset^\nu \leq 0 \) the agreement probability is uniquely maximized by \( \nu_- \), which completes the proof.

4.4 Proof of Lemma 3.6

Observe first by the local weak convergence of the graphs \( \{G_n\} \) that all but \( o(n) \) vertices appear in \( |\mathcal{B}_i(\ell)| \) balls \( \mathcal{B}_i(\ell) \). Hence given a configuration \( \chi \) with \( \sum_i x_i \geq 0 \), we have
\[
\sum_{i \in V_n} \left( \frac{1}{|B_i(\ell)|} \sum_{j \in B_i(\ell)} x_j \right) \geq -o(n). \tag{4.24}
\]

By Markov’s inequality (applied to the uniform choice of \(i \in V_n\)) we have
\[
\frac{1}{n} \sum_{i \in V_n} F_i(\ell) \leq \frac{1}{1 + \delta} + o_n(1) \leq \frac{1}{1 + \delta/2}, \tag{4.25}
\]
where the second inequality holds for all \(n\) large enough.

### 4.5 Proof of Lemma 3.8

Setting \(\rho = \nu_+(x_0)\) note that by invariance of \(\nu_+\) under graph homomorphisms of \(T_k\), we have
\[
\nu_+ \left( \sum_{j \in B_i(\ell)} x_j \right) = \rho |B_i(\ell)|.
\]
Moreover, under \(\nu_+\), along any path of vertices in \(T_k\) the states are distributed as a 2-state homogenous Markov chain and hence
\[
\nu_+(x_j \cdot x_{j'}) - \nu_+(x_j) \nu_+(x_{j'}) = A b^{d(j,j')}
\]
where \(d(j, j')\) is the graph distance between vertices \(i\) and \(j\), and \(b \in (0, 1)\) is a constant depending on \(\beta\).

This in particular implies that
\[
\Var_{\nu_+} \left( \sum_{j \in B_i(\ell)} x_j \right) = o \left( |B_i(\ell)|^2 \right),
\]
and therefore, using Chebychev inequality,
\[
\frac{1}{|B_i(\ell)|} \sum_{j \in B_i(\ell)} x_j \text{ converges in probability to } \rho \text{ as } \ell \to \infty.
\]
Similarly under the measure \(\nu_-\) we have that
\[
\frac{1}{|B_i(\ell)|} \sum_{j \in B_i(\ell)} x_j \text{ converges in probability to } -\rho.
\]
Now taking \(0 < \delta < \rho\) we have that
\[
\lim_{\ell \to \infty} \nu_+(F_0(\ell) = 1) = 0,
\]
\[
\lim_{\ell \to \infty} \nu_-(F_0(\ell) = 1) = 1.
\]
Therefore, for \(\nu = (1 - q)\nu_+ + q \nu_-\), we have \(\nu_+(F_\emptyset(\ell) = 1) \to q\).

Moreover, by translation invariance
\[
\nu_+(F_\emptyset(\ell) \neq F_1(\ell)) = 2\nu_+(F_\emptyset(\ell) = 1, \ F_1(\ell) = 0) \leq 2\nu_+(F_\emptyset(\ell) = 1) \to 0.
\]
By applying the same argument to $\nu_-$, we deduce that the probability that $F_\theta(\ell)$ and $F_1(\ell)$ differ goes to 0 under any mixture of $\nu_+$ and $\nu_-$. Since $\nu$ is a mixture of $\nu_+$ and $\nu_-$ this completes the lemma.

5 Proof of Theorem 2.5

To simplify notation we will write $f_i$ or $f_i(x)$ for $f_i,n(x_{B_i(\ell)})$. We will prove that, denoting by $\text{Var}_n,\nu_+$ and $\text{Cov}_n,\nu_+$ variance and covariance under $\mu_n,\nu_+$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i \in V_n} f_i(x_{B_i(\ell)}) = \lim_{n \to \infty} \mathbb{E}_{U_n} \text{Cov}_n,\nu_+(f_i(x_{B_i(\ell)}), f_L(x_{B_L(\ell)})) = 0.$$ 

Here $\mathbb{E}_{U_n}$ denotes expectation with respect to two independent and uniformly random vertices $I, L$ in $V_n$. The thesis then follows by Chebyshev inequality.

Since the $f_i$’s are bounded, we have for $r > \ell$,

$$\mathbb{E}_{U_n} \text{Cov}_n,\nu_+(f_I, f_L) \leq \mathbb{P}_{U_n}(d(I, L) \leq 2r) + \mathbb{E}_{U_n} \left\{ \text{Cov}_n,\nu_+(f_I, f_L); d(I, L) > 2r \right\}.$$ 

Since $\{G_n\}_{n \in \mathbb{N}}$ are $k$-regular, the probability $d(I, L) \leq 2r$ vanishes as $n \to \infty$. It therefore suffices to show that

$$\lim_{r \to \infty} \lim_{n \to \infty} \mathbb{E}_{U_n} \left\{ \text{Cov}_n,\nu_+(f_U, f_V); d(U, V) > 2r \right\} = 0.$$

Define

$$\hat{f}_i^+(r)(x) = \mathbb{E}_{n,\nu_+}\{ f(x_{B_i(\ell)}) | x_{V_n \setminus B_i(r)} \},$$

the conditional expectation being taken with respect to $\mu_{n,\nu_+}$. Then we have for all $i, j$ that

$$\mathbb{I}(d(i, j) > 2r) \text{Cov}_n,\nu_+(f_i, f_j) = \mathbb{I}(d(i, j) > 2r) \text{Cov}_n,\nu_+(\hat{f}_i^+(r), f_j) \leq \sqrt{\text{Var}_n,\nu_+(\hat{f}_i^+(r))}$$

and therefore

$$\lim_{r \to \infty} \lim_{n \to \infty} \mathbb{E}_{U_n} \left\{ \text{Cov}_n,\nu_+(f_I, f_L); d(I, L) > 2r \right\} \leq \lim_{r \to \infty} \lim_{n \to \infty} \mathbb{E}_{U_n} \sqrt{\text{Var}_n,\nu_+(\hat{f}_I^+(r))} \leq \lim_{r \to \infty} \lim_{n \to \infty} \sqrt{\mathbb{E}_{U_n} \text{Var}_n,\nu_+(\hat{f}_I^+(r))}.$$

(5.1)
Define the modified function

$$\hat{f}_I(r)(x) = \mathbb{E}_n \{ f(\mathbf{x}_{B_I(r)}) | x_{Vn} | B_I(r) \}, \quad (5.3)$$

where the expectation is taken with respect to the measure \( \mu_n \). Since the latter is a Gibbs measure \( \hat{f}_I(r) \) depends on \( x \) only through the variables \( x_j, j \in B_I(r) \setminus B_I(r - 1) \).

Further \( \hat{f}_I^+(r) \) and \( \hat{f}_I(r) \) differ only if \( | \sum_{j \in Vn \setminus B_I(r)} x_j | \leq | B_I(r) | \). Therefore

$$\text{Var}_{n,+}(\hat{f}_I(r)) \leq 2 \text{Var}_{n,+}(\hat{f}_I^+(r)) + 2 \text{Var}_{n,+}(\hat{f}_I^+(r) - \hat{f}_I(r))$$

$$\leq 2 \text{Var}_{n,+}(\hat{f}_I(r)) + 8 \mu_{n,+} \left( \sum_{j \in Vn \setminus B_I(r)} x_j \right) \leq | B_I(r) |.$$

The last term vanishes as \( n \to \infty \) by Lemma 4.1.

We are therefore left with the task of showing that \( \lim_{r \to \infty} \lim_{n \to \infty} \mathbb{E}_{U_n} \text{Var}_{n,+}(\hat{f}_I(r)) = 0 \). For a function \( f : [-1, 1]^{T_k(\ell)} \to [-1, 1] \), let

$$\tilde{f}(r)(x) = \mathbb{E}_{v^+} \{ f(\mathbf{x}_{T_k(\ell)}) | x_{T_k(\ell)} \}. $$

For all functions whose domain is not \( [-1, 1]^{T_k(\ell)} \) we let \( \tilde{f}(r) = 0 \) by convention. Also, with an abuse of notation, we define \( \hat{f}_I(r) = \tilde{g}(r) \) for \( g = \hat{f}_I \). Since \( \hat{f}_I(r) \) depends on \( x \) only through \( \mathbf{x}_{B_I(r)} \), we obtain by Theorem 2.4 for every \( \varepsilon > 0 \) that

$$\lim_{n \to \infty} \mathbb{E}_{U_n} | \text{Var}_{n,+}(\hat{f}_I(r)) - \text{Var}_{v^+}(\hat{f}_I(r)) | \leq 2 \varepsilon$$

$$+ \lim_{n \to \infty} U_n \left( d_{TV} \left( \mathbb{P}_n^I(I), \delta_{T_k(\ell)} \times v^+_I \right) > \varepsilon \right) = 2 \varepsilon,$$

and therefore

$$\lim_{r \to \infty} \lim_{n \to \infty} \mathbb{E}_{U_n} \text{Var}_{n,+}(\hat{f}_I(r)) \leq \lim \sup_{r \to \infty} \left\{ \text{Var}_{v^+}(\tilde{f}(r)) | f : [-1, 1]^{T_k(\ell)} \to [-1, 1] \right\}.$$

By extremality of \( v^+ \), for each \( f : [-1, 1]^{T_k(\ell)} \to [-1, 1] \), \( \tilde{f}(r) \) converges to an almost sure constant as \( r \to \infty \) and since \( f \) is bounded, \( \lim_{r \to \infty} \text{Var}_{v^+}(\tilde{f}(r)) = 0 \). For each \( r \), the map \( f \to \tilde{f}(r) \) is a contraction in \( L^2 \) and therefore the map \( f \to \sqrt{\text{Var}_{v^+}(\tilde{f}(r))} \) is a Lipchitz map with constant 1. Since the set of functions \( f : [-1, 1]^{T_k(\ell)} \to [-1, 1] \) is compact in \( L^2 \) and for each \( f \) we have \( \lim_{r \to \infty} \text{Var}_{v^+}(\tilde{f}(r)) = 0 \) we conclude that

$$\lim_{r \to \infty} \sup \left\{ \text{Var}_{v^+}(\tilde{f}(r)) | f : [-1, 1]^{T_k(\ell)} \to [-1, 1] \right\} = 0,$$

as needed.
Acknowledgments  A.M. was partially supported by a Terman fellowship, the NSF CAREER award CCF-0743978 and the NSF Grant DMS-0806211. E.M. was partially supported by the NSF CAREER award Grant DMS-0548249, by DOD ONR Grant (N0014-07-1-05-06), by ISF Grant 1300/08 and by EU grant PIRG04-GA-2008-239317. Part of this work was carried out while two of the authors (A.M. and E.M.) were visiting Microsoft Research.

References

1. Aizenman, M.: Translation invariance and instability of phase coexistence in the two-dimensional Ising system. Commun. Math. Phys. 73, 83–94 (1980)
2. Dobrushin, R., Shlosman, S.: The problem of translation invariance of Gibbs states at low temperatures. Math. Phys. Rev. 5, 53–195 (1985)
3. Georgii, H.O., Higuchi, Y.: Percolation and number of phases in the two-dimensional Ising model. J. Math. Phys. 41, 1153–1169 (2000)
4. Bodineau, T.: Translation invariant Gibbs states for the Ising model. Probab. Theory Relat. Fields 135, 153–168 (2006)
5. Ellis, R.S., Newman, C.M.: The statistics of Curie–Weiss models. J. Stat. Phys. 19, 149–161 (1978)
6. Dembo, A., Montanari, A.: Ising models on locally tree-like graphs. Ann. Appl. Probab. (2009, in press)
7. Aldous, D., Steele, J.M.: The objective method: probabilistic combinatorial optimization and local weak convergence. In: Probability on Discrete Structures. Springer, Berlin (2003)
8. Georgii, H.O.: Gibbs Measures and Phase Transitions. Walter de Gruyter, Berlin (1988)
9. Kallenberg, O.: Foundations of Modern Probability. Springer, Berlin (2002)
10. Aizenman, M., Wehr, J.: Rounding of first-order phase transitions in systems with quenched disorder. Commun. Math. Phys. 130, 489–530 (1990)
11. Newman, C.M., Stein, D.L.: Spatial inhomogeneity and thermodynamic chaos. Phys. Rev. Lett. 76, 4821–4824 (1996)
12. Külske, C.: Metastates in disordered mean-field models: random field and Hopfield models. J. Stat. Phys. 88, 1257–1293 (1996)
13. Liggett, T.M.: Interacting Particle Systems. Springer, New York (1985)
14. Wormald, N.C.: Models of random regular graphs. In: Surveys in Combinatorics, 1999 (Canterbury). Lecture Note Series. London Mathematical Society, London (1999)
15. Janson, S., Luczak, T., Ruciński, A.: Random Graphs. Wiley, New York (2000)
16. Aldous, D., Lyons, R.: Processes on unimodular random networks. Electron. J. Probab. 12, 1454–1508 (2007)
17. Dobrushin, R.L., Tirozzi, B.: The central limit theorem and the problem of equivalence of ensembles. Commun. Math. Phys. 54, 173–192
18. Lyons, R.: Phase transitions on nonamenable graphs. J. Math. Phys. 41, 1099 (2000)