Holonomic Quantum Computation with Electron Spins in Quantum Dots

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With the help of the spin-orbit interaction, we propose a scheme to perform holonomic single qubit gates on the electron spin confined to a quantum dot. The manipulation is done in the absence (or presence) of an applied magnetic field. By adiabatic changing the position of the confinement potential, one can rotate the spin state of the electron around the Bloch sphere in semiconductor heterostructures. The dynamics of the system is equivalent to employing an effective non-Abelian gauge potential whose structure depends on the type of the spin-orbit interaction. As an example, we find an analytic expression for the electron spin dynamics when the dot is moved around a circular path (with radius R) on the two dimensional electron gas (2DEG), and show that all single qubit gates can be realized by tuning the radius and orientation of the circular paths. Moreover, using the Heisenberg exchange interaction, we demonstrate how one can generate two-qubit gates by bringing two quantum dots near each other, yielding a scalable scheme to perform quantum computing on arbitrary N qubits. This proposal shows a way of realizing holonomic quantum computers in solid-state systems.

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I. INTRODUCTION

The emergence of geometrical phases in quantum mechanical systems and their physical and geometrical consequences were first recognized by Berry and Simon in their works on cyclic quantum evolution [1, 2]. Soon after that, Wilczek and Zee discovered the connection between (non-Abelian) gauge fields and the adiabatic dynamics of such systems, where the dimension of non-Abelian geometric phases is given by the n-fold degeneracy of the eigenstates of the Hamiltonian [3]. Moreover, Aharonov and Anandan [4] generalized Berry’s idea to non-adiabatic evolutions, however, the resulting geometric phase is not given anymore by the holonomy in the parameter space of the Hamiltonian but in the projective Hilbert space. Although, at the time, geometrical phases were already known in classical systems [5], their quantum mechanical counterparts are physically richer and more subtle.

Based on the above mentioned ideas, a variety of schemes for holonomic (HQC) and geometric (GQC) quantum computation have been proposed, which recently attained a considerable attention, and believed to be promising candidates to implement quantum computers using topological transformations as qubit gates [6, 7]. In HQC, for example, one can perform quantum computing by encoding quantum information in the degenerate levels of the Hamiltonian and adiabatically traversing closed loops (holonomies) in the parameter space of the Hamiltonian. So far, many theoretical (and experimental) investigations have been made to implement such Hamiltonians in physical systems, for example, confined ions in a linear Pauli trap [8]. The experiment by Jones et al. [9] was the first attempt in this direction where they realized geometric two-qubit gates between a pair of nuclear spins. Note that geometrical phases are generally small compared to dynamical phases. Being a small effect on top of a large effect, makes it challenging for the experimentalists to identify and employ them for quantum computation.

Among several proposals for HQC and GQC, solid state matrix is usually more desirable due to its potential in realizing large scale qubit systems. Specifically, spin of an electron in a quantum dot, as a two level system, has been shown to be a suitable qubit [10], meanwhile, rapid experimental progress in the field of semiconductor spintronics made it possible to access individual electron spin in low dimensional systems [11]. Manipulating the spin of the electrons/holes in semiconductors is, therefore, one of the objectives of spintronics [11, 12]. Among different tools to achieve this goal is to apply an external magnetic field, in combination with the spin-orbit interaction, in a controlled way. Recently, there has been a great progress in developing techniques to manipulate electrically the electron/hole spins in two dimensional electron/hole gases (2DEGs/2DHGs) and quantum dots (QDs) [13–22]. However, most of the previous works on confined electrons are based on the assumption that the quantum dot itself is almost frozen in real space. Moreover, the presence of an applied magnetic field is usually assumed in order to break the time reversal symmetry, which is essential in electron spin resonance (ESR) schemes. The question is: what is the dynamics of the spin sector if we move the quantum dot in the absence (or presence) of the magnetic field? If time reversal symmetry is not broken, a convenient way to study the dynamics of such a system is to employ non-Abelian gauge fields [5]. In spite of the fact that the origin of non-Abelian gauge fields in classical/quantum field theories and the current
problem is somewhat different, as long as the dynamics is concerned, they play the same role.

Using an effective non-Abelian gauge potential \[ \vec{A} \] [12, 23], we propose a novel technique to manipulate topologically the spin of an electron inside a quantum dot without using any applied magnetic field. Although the rotation of the electron spin in a moving quantum dot has been studied in previous works [12, 23], our goal here is to implement systematically all necessary single qubit gates for quantum information processing. We consider a setup, where the quantum dot can be moved on the substrate at distances comparable to \( \lambda_{SO} \), the spin-orbit length. In addition, we assume that the electron is strongly confined to the quantum dot at a length scale \( \lambda_d \ll \lambda_{SO} \), where \( \lambda_d \) is the dot size. We study the case in which the confining potential is only displaced parallel to itself by a vector \( \vec{r}_0(t) \) without changing its shape (see Fig. 2). Moreover, we assume that the driving electric field is classical and only quantize the electron dynamics (for the discussion of the quantum fluctuations of the electromagnetic fields which lead to the decoherence of the Kramers doublets, see Ref. 22). For a moving quantum dot around a circular path, we derive an exact solution of the Schrödinger equation for the spin sector, in the first order in spin-orbit interaction. In addition, we show how one can generate different single- and two-qubit rotations and perform quantum computing on \( N \) spins.

II. THE MODEL AND BASIC RELATIONS

We consider a lateral quantum dot [24] formed by depleting the 2DEG via a set of metallic gates that allow the quantum dot position \( \vec{r}_0 \) to be changed at will to distances comparable to the spin-orbit length in the 2DEG. Such rolling quantum dots can be defined using, for example, a set of gates shown in Fig. 1. Two layers of finger-like gates (separated by an insulator) form a grid, which construct the dot confining potential at virtually any position under the grid by simultaneously pulsing several gates. A relatively different setup is shown in Fig. 3 which makes use of a quantum ring and allows the quantum dot to be moved along a circular trajectory.

The electron motion in the plane of the 2DEG, and in the presence of a time-dependent dot confining potential, is governed by the Hamiltonian

\[
H(t) = H_d(t) + H_Z + H_{SO},
\]

where \( H_d(t) \) describes the moving dot with one electron,

\[
H_d(t) = \frac{\hbar^2}{2m_e} + U(\vec{r} - \vec{r}_0(t)),
\]

with \( \vec{p} = -i\hbar \partial / \partial \vec{r} + (e/c)\vec{A}(\vec{r}) \) and \( \vec{r} = (x, y) \) being the electron momentum and coordinates, respectively. For the vector potential \( \vec{A}(\vec{r}) \), we choose the cylindric gauge, \( \vec{A}(\vec{r}) = B_z (-y/2, x/2, 0) \), where \( B_z \) is the component of the magnetic field normal to the 2DEG plane. The dot confinement potential \( U(\vec{r}, t) = U(\vec{r} - \vec{r}_0(t)) \) changes adiabatically with respect to the size-quantization energy in the dot and, thus, no transitions between orbital levels occur. For simplicity, we also assume that the shape of the dot confinement does not change in time, while the dot is moved along its trajectory \( \vec{r}_0(t) \). In Eq. (1), the Zeeman interaction reads \( H_Z = \frac{\mu}{2} \vec{E}_Z \cdot \vec{\sigma} \), with \( |\vec{E}_Z| \) being the Zeeman energy and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) the Pauli matrices. We note that the quantization axis is generally not along the magnetic field and one has \( E_{Zi} = \mu_B g_i B_j \), where \( \mu_B \) is the Bohr magneton, \( g_{ij} \) is the g-factor tensor in the 2DEG, and \( B \) is the magnetic field. The last term in Eq. (1), \( H_{SO} \), denotes the spin-orbit interaction which has the following general form \( H_{SO} = \frac{\hbar}{2} \vec{h}(\vec{p}) \cdot \vec{\sigma} \), where \( \vec{h}(\vec{p}) = -\vec{h}(-\vec{p}) \) is an odd-power polynomial in \( \vec{p} \). In GaAs 2DEG with the [001] growth direction, for example, the leading order (lowest power in \( \vec{p} \)) spin-orbit interaction terms read

\[
H_{SO} = \alpha(p_z \sigma_y - p_y \sigma_z) + \beta(-p_z \sigma_x + p_y \sigma_y),
\]

where \( \alpha \) and \( \beta \) are the Rashba and Dresselhaus coupling constants, respectively [23, 24].

Considering first a stationary quantum dot potential \( U(\vec{r}) \), we set \( \vec{r}_0(t) \rightarrow 0 \) in Eq. (2) and denote the time-independent Hamiltonian by \( H \) with the following eigenvalue equation, \( H |\psi_{ns}\rangle = E_{ns} |\psi_{ns}\rangle \), where \( n = 0, 1, 2 \ldots \) and \( s = \pm 1/2 \) are the orbital and spin quantum numbers, respectively.

The eigenstates of \( H \) are related to the eigenstates of
The suitable qubit is then defined as \( n = 0 \) (for the discussion of the spin dynamics in the presence of an applied magnetic field, see Appendix).

We are mainly interested here in spin dynamics in the absence of the applied magnetic field, therefore, we set \( A(r) = 0 \), and focus on the lowest-energy subspace \( n = 0 \) (for the discussion of the spin dynamics in the presence of an applied magnetic field, see Appendix). The suitable qubit is then defined as \( \mid \uparrow \rangle = \mid \psi_{0,1/2} \rangle \) and \( \mid \downarrow \rangle = \mid \psi_{0,-1/2} \rangle \). In the absence of magnetic fields, the quantization axis can, therefore, be chosen arbitrarily. However, once it is chosen, all subsequent spin rotations are then with respect to this axis.

Given a linear in momentum spin-orbit interaction in Eq. \( \text{(3)} \), and a symmetric confining potential \( U(r) = U(-r) \), we have explicitly found the generator of the rotation for the Kramers doublet (see Appendix), as one moves the dot along a given path

\[
\Delta = 1 - i \sigma \cdot \lambda_{SO}^{-1} \cdot \delta r_0, \tag{5}
\]

\[
\lambda_{SO}^{-1} \equiv \begin{pmatrix} 0 & 1/\lambda_- & 0 \\ 1/\lambda_+ & 0 & 0 \\ 0 & 0 & \lambda_- \end{pmatrix}, \quad \lambda_{SO} = \frac{\hbar}{m_e(\beta \pm \alpha)}. \tag{6}
\]

where \( \sigma \) are Pauli matrices acting on the Kramers doublet states and \( S = i \sigma \cdot \lambda_{SO}^{-1} \cdot r \) \([16]\). According to Eq. \( \text{(3)} \), the electron state, in the space of a Kramers doublet, is rotated during the displacement by an angle \( \sim \delta r_0/\lambda_{SO} \). This interpretation of the spin-orbit interaction effect is identical to the standard interpretation given to semiclassical electrons. In the latter, the electron with the momentum \( p \) travels a distance \( l = pt/m_e \) during a time \( t \) and changes its spin by an angle proportional to \( l/\lambda_{SO} \). This coincidence is not accidental, because in the semiclassical picture one also assumes that the electron moves in a wave packet of an extension that is much smaller than \( \lambda_{SO} \). The speed at which the electron moves is unimportant at \( \delta r_0 \), because the path \( r_0(t) \) is the only information that determines the spin rotation. In particular, if the electron travels along some path forward and then returns the same way, but not necessarily at the same speed, then the initial and final spin states coincide.

We note that, for the linear in momentum \( H_{SO} \), the tensor \( \lambda_{SO}^{-1} \) in Eq. \( \text{(4)} \) is independent of the electron orbital state \( \mid \psi_n \rangle \), which means Eq. \( \text{(4)} \) is valid for all symmetric wave packets at the zeroth order of \( \lambda_{SO} \). Therefore, we can consider also a point-like electron, for which the orbital wave function reads \( \psi_{r_0}(r) = \delta(r-r_0) \). Integrating Eq. \( \text{(4)} \) over an arbitrary path we obtain an exact expression for this case,

\[
\psi_{r_0}(r, \sigma) = e^{-i \int \sigma \cdot \lambda_{SO}^{-1} \cdot dr_0 \delta(r-r_0)} \chi_s(\sigma), \tag{7}
\]

where the exponent is ordered (to the left) along the path of integration. To simplify notations, we use \( \int dr_0 \) to denote the contour integral \( \int_0^{r_0} dr^0 \) and agree that any exponent of an integral to be ordered along the path of integration. The radius vector \( r_0(t) \) gives us the path where we choose the beginning of the path at \( r_0 = 0 \), and denote the running (present) point of the path by \( r_0 \). Eq. \( \text{(4)} \) determines how the spin of an electron is transformed (at \( B = 0 \)) as the electron is moved along an arbitrary path. The difference between the spin and the Kramers doublet disappears here, since for point-like electrons we can take \( S \to 0 \). However, the transformation rule in Eq. \( \text{(4)} \) arises from the fact that \( \partial S/\partial r \) remains constant while taking \( S \to 0 \). Moreover, it holds exactly for the linear in momentum spin-orbit interaction, because \( \lambda_{SO}^{-1} \) is independent of the orbital state. For the \( p^2 \) terms, \( 1/\lambda_{SO} \) is proportional to the electron energy and Eq. \( \text{(7)} \) can be written only if \( H_{SO} \) is first linearized around a given energy.

We developed a code to calculate numerically the spin dynamics for an arbitrary path, however as an example, we consider a point like quantum dot (strong confining potential) moving around a circle with radius \( R \) (see Fig. \( \text{3} \)). Using Eq. \( \text{(4)} \), the dynamical equation for the electron spin is then given by

\[
\frac{d}{d\phi} \chi_s(\phi) = iM_{DR} \chi_s(\phi), \tag{8}
\]

\[
M_{DR} \equiv R(\sin \phi \sigma_y - \cos \phi \sigma_z), \tag{9}
\]

where \( \chi_s(\phi) = (v_1, v_2) \) is the spinor, \( \phi \) is the angle between the starting point vector \( r_0(0) \) and the \( x \) axis, and \( M_{DR} \) is a Hermitian matrix due to both Dressel-
III. SINGLE-QUBIT ROTATIONS: HADAMARD AND PHASE GATES

The transformation (10) is nothing but a rotation along the vector \( \eta \) at point \( P \) which is in the \( x-z \) plane and makes an angle \( \theta = \arctan(\frac{\beta}{\lambda}) \) with the \( z \) axis, see Fig. 2. Therefore, for large values of \( \lambda \) (small circles), \( \theta \approx 0 \) and one would be able to realize arbitrary rotations around \( z \) axis (phase gate) with reasonable precision. Moreover, moving counterclockwise along the red (2) circle in Fig. 2 leads to the same result as in the blue (2) circle, but now the rotation takes place in the \( y-z \) plane. Therefore, depending on the orientation of the circles and their corresponding radii, we can, in principle, achieve all kinds of rotations around the Bloch sphere.

To be more specific, we show how to generate the Hadamard gate by rotations around two non-orthogonal axes (in our case \( z \) and \( \eta \) directions) and, for convenience, we only consider the Dresselhaus term (\( \alpha = 0 \)). As shown in Eq. (10), circles with different radii and/or orientations lead to different rotations. In particular, if we go counterclockwise along a full circle from point \( P \) which makes an angle \( \phi \) with \( x \) axis (see Fig. 2), the electron spin will transform as follows

\[
\begin{align*}
U_{11} &= -\cos \pi \epsilon + \frac{i}{\epsilon} \sin \pi \epsilon, \\
U_{12} &= \frac{2ie^{i\phi}}{\lambda \epsilon} \sin \pi \epsilon, \\
U_{11} &= U_{22}, \quad U_{12} = -U_{21}^*, 
\end{align*}
\]

where \( U_{ij} \) is the unitary transformation which acts on the initial spin state. Geometrically, \( U \) is the matrix corresponding to a rotation around \( \eta \) axis which lies in the same plane as \( z \) axis and \( P \) (see Fig. 2c).

Hadamard gate can be achieved (up to a global phase) by a clockwise \( \frac{\pi}{2} \) rotation around \( y \) axis followed by a counterclockwise \( \pi \) rotation around \( z \) axis,

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = i U_z(\pi) U_y(-\frac{\pi}{2}). \quad (12)
\]

The rotation around \( z \) with an arbitrary angle has already been discussed above, therefore, we show here how one can implement \( U_z(\pi) \). The main problem is that, according to Eqs. (10,11), the magnitude and the direction of the vector \( \eta \) are not independent variables (both are functions of the variable \( \lambda \)). Moreover, the vector \( \eta \) is not in general orthogonal to \( z \) axis. To obtain an arbitrary rotation around, say, \( y \) axis, we need to perform 3 rotations (two around \( \eta \) and one around \( z \) axis). Specifically, we want to know for which values of \( \lambda, \pm \frac{\pi}{2} \), rotations around \( y \) axis can be achieved. One can show that

\[
U_y(\gamma) = U_\eta(\theta) U_z(\phi) U_\eta(\theta), \quad (13)
\]

where for our purpose, \( U_y(\frac{\pi}{2}) \), we need to evaluate Eq. (13) at \( \gamma = \pi/2 \). Obviously, there are an infinite number of solutions corresponding to different values of...
from the position $P$ ($P'$) to $G$ ($G'$), as shown in Fig. 4. The top gates are introduced to control over the wave functions of the confined electrons at the touching points of the circles. By lowering the potential barrier ($V_g$) between two quantum quantum dots, the residing electron spins can couple to each other (due to the overlap of their orbital wave functions) via the Heisenberg exchange interaction $[10, 31]$. It has already been shown that by electrically engineering the gate potentials, one can generate the SWAP gate, and eventually the CNOT gate, between two spins $[10]$. However, this additional step, i.e. moving two quantum dots towards each other to perform two-qubit gates, leads to a spin rotation of each electron, see Eqs. $[8,9]$, and therefore this partial spin dynamics should also be taken into account. For practical purposes, we assume that the radius of the (holonomic) single-qubit gates ($R$) is smaller than the radius of the two-qubit circles ($R'$), to avoid spatial overlap of different local single-qubit operations, see Figs. 4, 5. The readout part of the scheme is accomplished by applying an external magnetic field, using different techniques such as spin-to-charge conversion $[24]$. As an example, we observe that our scheme is able to produce the so called cluster states on $N$ qubits $[32]$. The peculiar properties of cluster states made them a suitable candidate for realizing quantum computers in quantum optical and solid state systems $[32–35]$. One of the main advantages of one-way quantum computation is that these set of entangled states (cluster states) are produced once, and then quantum computation is done by local (adaptive) measurements of the qubits. Therefore, there is no need to perform two-qubit gates during the information processing. In Fig. 4 we showed a one dimensional linear chain of qubits, however, this scheme can be easily generalized to higher dimensional lattices, and in principal be used to generate cluster states and perform holonomic one-way quantum computing in solid state environments.

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**Appendix: The Non-Abelian Spin Rotation Generator**

The transformation matrix $S = -S^\dagger$ in Eq. (4) can be evaluated by perturbation theory in $H_{SO}$ or by diagonalizing the Hamiltonian $H$ for a specific potential $U(r)$. At the leading order in $H_{SO}$, $S$ satisfies the operator equation

$$[H_d + H_Z, S] = H_{SO}, \quad (A.1)$$
whereas the energy levels of $H$ coincide, at this order, with the energy levels of $H_x + H_Z$. A detailed analysis of the transformation in Eq. (A.3) is given in Refs. 16, 36.

Next we consider the Schrödinger equation,

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = H(t)\Psi(r, t),$$  \hspace{1cm} (A.2)

in the presence of a time-dependent displacement-vector $r_0(t)$. At each moment in time, the Hamiltonian $H(t)$ has an instantaneous basis of states, which we denote by $|\Phi_{n,s,r_0}\rangle$, where the index $r_0$ indicates that the dot is centered at $r_0$. Obviously, the states $|\Phi_{n,s,r_0}\rangle$ can be obtained from Eq. (4) by means of a displacement by the vector $-r_0$. In the presence of a magnetic field, the instantaneous eigenstates read

$$\Phi_{n,s,r_0}(r) = e^{i(e/\hbar)A(r-r_0)t}T_{-r_0}\psi_{ns}(r),$$ \hspace{1cm} (A.3)

where $f(r, r_0)$ is a gauge function satisfying the equation

$$-\frac{\partial f(r, r_0)}{\partial r} = A(r - r_0) - A(r).$$ \hspace{1cm} (A.4)

In Eq. (A.3), $T_\alpha = \exp(a \cdot \partial / \partial r)$ denotes the translation operator by a vector $a$ and for the cylindric gauge $A(r) = B_z (-y/2, x/2, 0)$, we can choose $f(r, r_0) = r_0 \cdot A(r)$.

The solutions of the Schrödinger equation (A.2) can be looked for in terms of the instantaneous basis in Eq. (A.3),

$$\Psi(r, t) = \sum_{ns} a_{ns}(t) \Phi_{n,s,r_0}(t) \langle r, t),$$ \hspace{1cm} (A.5)

where the coefficients $a_{ns}(t)$ satisfy the normalization condition $\sum_{ns} |a_{ns}(t)|^2 = 1$ and we have used the notation $\Phi_{n,s,r_0}(t) \langle r, t) = \exp(-iE_{ns}t/\hbar)\Phi_{n,s,r_0}(t) \langle r$ for the Schrödinger picture of Eq. (A.3). By substituting Eq. (A.5) into Eq. (A.2) we obtain a set of equations for $a_{ns}(t)$

$$\frac{da_{ns}}{dt} = \frac{i}{\hbar} \sum_{n',s'} \langle \psi_{ns} | \hat{\mathbf{p}}(t) | \psi_{n's'} \rangle e^{i\omega_{nsn's'}t}a_{n's'},$$ \hspace{1cm} (A.6)

where $\psi_{n,s}(t) = dr_0(t)/dt$ is the velocity of the slipping dot and $\omega_{nsn's'} = (E_{ns} - E_{n's'})/\hbar$. The quantity $\hat{\mathbf{p}}(t)$ depends on $t$ only through $r_0(t)$ and is defined as follows

$$\hat{\mathbf{p}} = -i\hbar \frac{\partial}{\partial r} - \frac{e}{c} T_{-r_0} \frac{\partial f(r, r_0)}{\partial r} T_{-r_0}.\hspace{1cm} (A.7)$$

For our choice of gauge, i.e. cylindrical gauge, we obtain $\hat{\mathbf{p}} = -i\hbar \partial / \partial r - (e/c)A(r + r_0)$. Note that the choice of $f(r, r_0)$ is not unique; In general, one can also include terms of the form $g_0(r_0) + sg_2(r_0)|\psi_{ns}\rangle |\psi_{n,s}\rangle$ and, at $B = 0$, additionally terms of the form $g_1(r_0) |\psi_{ns}\rangle |\psi_{n,-s}\rangle + isg_2(r_0) |\psi_{ns}\rangle |\psi_{n,-s}\rangle$, where $g_f(r_0)$ are arbitrary real functions of $r_0$.

Next we consider a specific situation for which we can further simplify Eq. (A.6).

We can further define a resting qubit at a position $r_0$ using the transformation in Eq. (A.3). Let the quantum dot be driven along a trajectory $r_0(t)$ between two points $r_A = 0$ and $r_B$ during a time interval $T$, such that

$$r_0(0) = r_A, \hspace{1cm} r_0(T) = r_B, \hspace{1cm} \text{for } t \in [0, T]\hspace{1cm} (A.8)$$

$$v_0(0) = 0, \hspace{1cm} v_0(T) = 0. \hspace{1cm} (A.9)$$

The probability for the qubit to leak out of its subspace by the end of the pulse is given by

$$P_{\text{leak}} = \sum_{n,s \neq 0} |a_{ns}(T)|^2. \hspace{1cm} (A.10)$$

The coefficients $a_{ns}(T)$ can, therefore, be found by solving Eq. (A.6) with the initial condition $\sum_s |a_{ns}(0)|^2 = 1$. At the leading order in the driving, we have

$$a_{ns}(T) \approx \frac{i}{\hbar} \int_0^T dt \psi_{n,s}(t) \langle \psi_{ns} | \hat{\mathbf{p}}(t) | \sigma \rangle e^{i\omega_{nsn's'}t}. \hspace{1cm} (A.11)$$

where $|\sigma\rangle = |\psi_{ns}\rangle$ denotes the qubit state at $t = 0$. Since the matrix elements $\langle \psi_{ns} | \hat{\mathbf{p}}(t) | \sigma \rangle$ do not depend on time for $n \neq 0$, the coefficients $a_{ns}(T)$ in Eq. (A.11) are, thus, proportional to the Fourier transform of the quantum dot velocity $v_0(t)$ evaluated at the orbital transition frequency $\omega_{nsn's'} \approx \omega_0$. It is, therefore, sufficient to devise pulses of $v_0(t)$ with the spectral weight below the orbital frequency $\omega_0$ in order to avoid leakage from the qubit subspace.

It is convenient to have an adiabaticity criterion based on the differential properties of $v_0(t)$. We note that Eq. (A.6) can be rewritten in terms of the new unknowns $\tilde{a}_{ns} = a_{ns} \exp(-itE_{ns}/\hbar)$ as follows

$$\frac{d\tilde{a}_{ns}}{dt} = \frac{i}{\hbar} \sum_{n',s'} H_{nns's'}(t) \tilde{a}_{n's'},$$ \hspace{1cm} (A.12)

$$H_{nns's'}(t) = E_{ns} \delta_{n,n's'} - v_0(t) \cdot \langle \psi_{ns} | \hat{\mathbf{p}}(t) | \psi_{n's'} \rangle. \hspace{1cm} (A.13)$$

One can identify Eq. (A.13) with $H(t) = H(t) - i\hbar \partial / \partial t$, expressed in the time-dependent basis of Eq. (A.3). For the case of $B = 0$, Eq. (A.13) has been previously obtained in Ref. 23. The virtue of Eq. (A.13) is that $H_{nns's'}(t)$ depend on time only through a perturbation $\propto v_0(t)$, which vanishes at $t = 0, T$. Applying the adiabaticity criterion to the Hamiltonian in Eq. (A.13) for the orbital transitions out of the qubit subspace, we obtain that the condition

$$\left| \frac{dv_0}{dt} \cdot \langle \psi_{ns} | \hat{\mathbf{p}} | \sigma \rangle \right| \ll \hbar \omega_0^2 \hspace{1cm} (A.14)$$

must be satisfied at any moment in time in order for the pulse to be adiabatic.

If $r_0(t)$ changes adiabatically with respect also to the Zeeman energy $E_Z$, then $a_{ns}(t)$ is independent of time, i.e. the qubit follows adiabatically the change of its basis states. In the opposite case, when $B = 0$, the states $|\psi_{ns}\rangle$ in Eq. (4) are degenerate with respect to the spin index.
s to all orders of $H_{SO}$, due to the Kramers theorem. In this case, the change of the instantaneous basis can be interpreted as a unitary operation on the qubit. In order to tell what is the qubit instantaneous basis at $B = 0$, one has, in principle, to consider a finite $B$ and follow the energy levels of the quantum dot in the limit of $B \to 0$. Here, it is important to note that the spin-orbit interaction gives rise to an anisotropic Zeeman interaction at the second order of $H_{SO}$. As a result, the spin quantization axis and the magnetic field are not necessarily aligned with each other. To avoid the need of state finding, we denote $\langle ns|e^{-i\hat{H}t}|n,-s\rangle$ by $\alpha_{ns}$ and remark that $\alpha_{ns} = O(H^2_{SO})$.

Returning now to Eq. (A.5), we consider an infinitesimal displacement of the quantum dot in the $(x,y)$-plane by $\delta r_0$ and derive the corresponding generators of the qubit transformation under translations. We encode the qubit into the instantaneous states of the $n$-th orbital level of the Hamiltonian (1).

Let $r_0(t) = r_0$ be the position of the quantum dot center at time $t$ and $r_0(t + \delta t) = r_0 + \delta r_0$ be the new position at time $t + \delta t$. The infinitesimal transformation that takes the state $\Psi(t)$ to a new state $\Psi(t + \delta t)$ is given in Eq. (A.25). Starting from a basis state $\Phi_{ns}(t)$ at time $t$, we obtain the following state at time $t + \delta t$,

$$\Psi_{ns}(t + \delta t) = \Phi_{ns}(t) - \frac{i}{\hbar} H(t) \Phi_{ns}(t) \delta t.$$  

(A.15)

The overlap of this state with the basis state $\Phi_{ns'}(t + \delta t)$, generates an infinitesimal transformation of the Kramers doublet, where from Eq. (A.29), we find the basis state at time $t + \delta t$,

$$\Phi_{ns'}(t + \delta t) = \Phi_{ns'}(t) - \frac{i}{\hbar} \left[ \frac{dr_0}{dt} \cdot \hat{p} + H(t) \right] \Phi_{ns'}(t) \delta t.$$  

(A.16)

Thus, the desired infinitesimal transformation reads

$$\langle \Phi_{ns'}(t + \delta t)|\Psi_{ns}(t + \delta t)\rangle = \delta_{ss'} + \frac{i}{\hbar} \delta r_0 \langle \Phi_{ns'}(t)|\hat{p}|\Phi_{ns}(t)\rangle.$$  

(A.17)

For a qubit that is encoded into the instantaneous states of the $n$-th orbital level of the Hamiltonian (1), the infinitesimal transformation (A.17) can be rewritten as

$$|s(t)\rangle \rightarrow \exp(\mathcal{G} \cdot \delta r_0) |s(t)\rangle.$$  

(A.18)

Here, $|s(t)\rangle$ denotes the qubit state at time $t$ and the $2 \times 2$ matrices

$$\mathcal{G}_{ss'} = \frac{i}{\hbar} \langle \psi_{ns}\rangle \hat{p} |\psi_{ns'}\rangle$$  

(A.19)

are the corresponding generators of the transformations that take place on the qubit under parallel translations of the quantum dot on the substrate. In deriving Eq. (A.19) we made use of our choice of gauge, see the text below Eq. (A.3).

It is important to note that, along with spin-orbit interaction-induced SU(2) transformations on the qubit, Eqs. (A.18) and (A.19) account also for the Aharonov-Bohm phase due to the orbital magnetic field. It is, therefore, convenient to subdivide $\mathcal{G}$ into Abelian and non-Abelian parts,

$$\mathcal{G}_{ss'} = \mathcal{G}^A_{ss'} + \mathcal{G}^B_{ss'}.$$  

(A.20)

$$\mathcal{G}^A_{ss'} = \frac{1}{2} \sum_p \mathcal{G}_{pp}.$$  

(A.21)

For a point-like quantum dot, we obtain the Abelian generators $\mathcal{G}^A_{ss'} = (-i e / \hbar c) \delta_{ss'} A(r_0)$, recovering, thus, the usual expression for the Aharonov-Bohm phase $(e^{i\varphi_{AB}})$

$$\varphi_{AB} = - \frac{e}{\hbar c} \int_C A(r_0) \cdot dr_0$$  

(A.22)

in going around a closed path $C$. Note that it is always possible to sum up independently the phase due to the Abelian generators, because $[\mathcal{G}^A, \mathcal{G}^B] = 0$. In what follows, we focus on the non-Abelian generators $\mathcal{G}^B_{ss'}$ since they give rise to useful unitary operations on the qubit.

To calculate the matrix elements $\langle \Phi_{ns'}(t)|\hat{p}|\Phi_{ns}(t)\rangle$, we make use of Eq. (A.23) and the following property $e^{-ie\hat{p}/\hbar c} |\psi\rangle = |p\rangle$, and obtain that

$$\langle \Phi_{ns'}(t)|\hat{p}|\Phi_{ns}(t)\rangle = e^{\pm (E_{ns'} - E_{ns})t} \langle \psi_{ns'}|T_{r_0}pT_{-r_0}|\psi_{ns}\rangle.$$  

(A.23)

where $|\psi_{ns}\rangle$ are the states in Eq. 3 and $E_{ns}$ are the energies corresponding to these states. Obviously, if the Zeeman energy is large, the exponential factor in Eq. (A.23) oscillates rapidly as a function of time and the transformation in Eq. (A.17) averages out to unity. For the latter to take place, it is sufficient that

$$\left| \frac{dr_0}{dt} \left( \langle \psi_{ns}|T_{r_0}pT_{-r_0}|\psi_{ns}\rangle \right) \right| \ll 1.$$  

(A.24)

Estimating further $|\langle \psi_{ns}|T_{r_0}pT_{-r_0}|\psi_{ns}\rangle| \sim h/\lambda_{SO}$ and $E_{ns} - E_{ns-s} \approx E_Z$, we obtain that the spin rotator is inefficient at small speeds of the dot, $\delta r_0 \ll E_Z \lambda_{SO}$.

In the absence of magnetic fields, the transformation in Eq. (A.17) acquires the form

$$\Delta = 1 + \delta r_0 \cdot \langle \psi_{ns'}|\frac{\partial}{\partial r}|\psi_{ns}\rangle.$$  

(A.25)

Note that Eq. (A.25) can as well be derived from the infinitesimal version of the identity $|\Phi_{ns}(t)\rangle = T_{\delta r_0} |\Phi_{ns}(t + \delta t)\rangle = T_{\delta r_0} |\Phi_{ns}(t + \delta t)\rangle$. Thus, the spin rotation takes place (at least at $B = 0$), because the confinement defines local Kramers states, which differ from each other along the dot trajectory. An illustration of the dot trajectory is given in Fig. 2. The radius-vector $r_0(t)$ describes a curve as a function of time and, as the dot is moved along that curve, the local Kramers state changes. The infinitesimal transformation in Eq. (A.25) [or more generally in Eq. (A.17)] has to be ordered along the path of integration when integrated over $\delta r_0$. This ordering occurs because the spin matrices in Eq. (A.25) do not always commute with each other at different points of the
path due to generally different directions of $\delta \mathbf{r}_0$ at these points.

Further, it is convenient to refer to the Kramers doublets $|\Phi_{ns}(t)\rangle$ as to spin states that are locally defined at each point of the dot trajectory. Mathematically, we perform a mapping given by the following canonical transformation,

$$|\Phi_{ns,\mathbf{r}_0}\rangle = e^{i\mathbf{r}_0/\hbar c \mathbf{T}_c} e^{-S} |\psi_n\rangle |\chi_s\rangle,$$  \hspace{1cm} (A.26)

which is obtained by substituting Eq. (4) into Eq. (A.3) and omitting the free evolution factor $e^{-i/t_h}E_{a,s,t}$ from $\psi_{ns}(\mathbf{r},t)$. For a given quantum number $n$, this transformation, obviously, maps the Kramers doublet at position $\mathbf{r}_0$ onto a spin-1/2 space: $|\chi_s\rangle$, $(s = \pm 1/2)$. Equation (A.25) can then be rewritten in an operator form

$$\Delta = 1 - \delta \mathbf{r}_0 \cdot \langle \psi_n | S \frac{\partial}{\partial \mathbf{r}} |\psi_n\rangle,$$ \hspace{1cm} (A.27)

where we used $\langle \psi_n | \partial/\partial \mathbf{r} | \psi_n \rangle = 0$.

For a small quantum dot the transformation matrix $S$ is small, because $\lambda_d \ll \lambda_{SO}$. In this case, one can expand the transformation to the first order, $e^{\pm S} \approx 1 \pm S$. Then, Eq. (A.27) acquires the form ($B = 0$)

$$\Delta = 1 - \delta \mathbf{r}_0 \cdot \langle \psi_n | S \frac{\partial}{\partial \mathbf{r}} |\psi_n\rangle,$$  \hspace{1cm} (A.28)

where $S$ contains Pauli matrices, which should now be regarded as effective operators in the Hilbert space of a local Kramers doublet $(n, \mathbf{r}_0)$.

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