Abstract—The article proposes an approach to complete-type and related Lyapunov-Krasovskii functionals that neither requires knowledge of the delay-Lyapunov matrix function nor does it involve linear matrix inequalities. The approach is based on ordinary differential equations (ODEs) that approximate the time-delay system. The ODEs are derived via spectral methods, e.g., the Chebyshev collocation method (also called pseudospectral discretization) or the Legendre tau method. A core insight is that the Lyapunov-Krasovskii theorem resembles a theorem for Lyapunov-Rumyantsev partial stability in ODEs. For the linear approximating ODE, only a Lyapunov equation has to be solved to obtain a partial Lyapunov functional. The latter approximates the Lyapunov-Krasovskii functional. Results are validated by applying Clenshaw-Curtis and Gauss quadrature to a semi-analytical result of the functional, yielding a comparable finite-dimensional approximation. In particular, the article provides a formula for a tight quadratic lower bound, which is important in applications. Examples confirm that this new bound is significantly less conservative than known results.

Index Terms—delay systems, Lyapunov-Krasovskii functional, operator-valued Lyapunov equation, spectral methods, pseudospectral discretization, Gauss quadrature

I. INTRODUCTION

Whenever a control law \( u = \gamma(x) \) is constructed for a system \( \dot{x} = f(x,u) \), the closed loop description \( \dot{x}(t) = f(x(t), \gamma(x(t))) \) hinges on the availability of the instantaneous \( x(t) \). In practice, however, measurements, network communication, computation times, or the actuator response cause a delay. The resulting \( \dot{x}(t) = f(x(t), \gamma(x(t-h))) \), with a time delay \( h > 0 \), is a retarded functional differential equation (RFDE) and can no longer be tackled by the well-known stability theory of finite-dimensional ordinary differential equations (ODEs). What changes?

A. Motivation: Delay-free versus Time-Delay System

For delay-free nonlinear time-invariant ODEs we could consider the linearization about the equilibrium (provided it is hyperbolic and the right-hand side is differentiable) and simply conclude exponential stability from the eigenvalues of \( A \) in the resulting \( \dot{x} = Ax \), \( A \in \mathbb{R}^{n \times n} \). We might be interested in the domain of attraction of the equilibrium. To this end, we could calculate a quadratic Lyapunov function \( V(x) = x^\top P x \) for the linearized system by prescribing a desired Lyapunov function derivative \( \dot{D}^+(x) = -x^\top Q x \). Solving the associated Lyapunov equation \( PA + A^\top P = -Q \) for the matrix \( P \) with a standard algorithm is accomplished in one line of Matlab code. The obtained Lyapunov function also gives a negative Lyapunov function derivative in the nonlinear system – at least in a certain domain around the equilibrium [1]. Let this domain be estimated by a norm ball with radius \( r > 0 \). Then the probably most basic estimation of the domain of attraction, cf. [1, Sec. 8.2], is provided by the set of points \( x \in \mathbb{R}^n \) such that \( V(x) < k_1 r^2 \), where \( k_1 \) is the coefficient of the positive-definiteness bound \( k_1 \|x\|_2^2 \leq V(x) \). Thus, having a non-conservative result for \( k_1 \) is important. It is simply the minimum eigenvalue of \( P \) that provides the largest possible coefficient \( k_1 \).

In time-delay systems, analogous steps become more elaborate. Given a nonlinear system, the principle of linearized stability still holds [2], and we are led to the linear RFDE

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t-h),
\]

with \( A_0, A_1 \in \mathbb{R}^{n \times n} \), a discrete delay \( h > 0 \). The characteristic equation has, generically, an infinite number of roots. Still, to determine stability for a given delay \( h \), we can resort to numerical eigenvalue calculations [3]–[7] or we use other characteristic-equation-based criteria that can prove stability for all delays or all delays smaller than a first critical one [8], [9]. The initial state \( x_0 \) is in fact an initial function, the domain of attraction is a set of initial functions, the state \( x_t \) at time \( t \geq 0 \) represents the solution segment on the past delay interval \( [t-h,t] \), and instead of a Lyapunov function, a Lyapunov-Krasovskii (LK) functional \( V(x_t) \) is required. Analogously to the delay-free case discussed above, we can explicitly prescribe the desired LK functional derivative and determine the corresponding LK functional. The LK functional derivative along trajectories of (1) is commonly [10] set as

\[
D^+_1 V(x_t) = -x^\top(t) Q_0 x(t) - x^\top(t-h) Q_1 x(t-h)
\]

\[
- \int_{-h}^0 x^\top(t+\theta) Q_2 x(t+\theta) d\theta,
\]

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with freely chosen $Q_0, Q_1 \succ 0_{n \times n}, Q_2 \succeq 0_{n \times n}$. This derivative is accomplished by so-called complete-type (if $Q_{0,1,2} \succ 0_{n \times n}$) or related LK functionals [11], [10, Thm. 2.11]. Their determination is far more elaborate than the simple Lyapunov equation for ODEs: the known formula for the solution of (2)

$$V(x_t) =$$

$$x^T(t)\Psi(0; \bar{Q})x(t) + 2\int_{-h}^{0} x^T(\tau)\Psi(-t - \tau; \bar{Q})A_1 x(t - \tau) d\tau + \int_{-h}^{0} \int_{-h}^{0} x^T(\tau + \xi)A_1^T \Psi(-\xi - \tau; \bar{Q})A_1 x(t + \tau) d\tau d\xi + \int_{-h}^{0} x^T(t + \eta) [Q_1 + (h + \eta)Q_2] x(t + \eta) d\eta$$

(3)

requires the so-called delay Lyapunov matrix function

$$\Psi(\cdot; \bar{Q}): [-h, h] \to \mathbb{R}^{n \times n}$$

in association with $\bar{Q} = Q_0 + Q_1 + hQ_2$. This matrix-valued function $\Psi$ is defined via a matrix-valued time-delayed boundary-value problem [10, Def. 2.5] that first has to be solved semi-analytically or numerically. The lower bound of interest on $V(x_t)$, e.g., needed in an estimation of the domain of attraction [12, Thm. 1], is described by

$$k_1 \|x(t)\|^2 \leq V(x_t).$$

(4)

In contrast to the ODE case, where the minimum eigenvalue of $P$ gives the best possible coefficient $k_1$, nothing is reported about the conservativity of known formulae [10], [12] for (4).

**B. Objectives and Related Results**

In light of the previous section, we intend to benefit from the enormous simplification that comes along with the treatment of ODEs in contrast to RFDEs. To this end, we use schemes of ODEs that approximate the RFDE. Based on these, the paper aims to provide a new numerical approach to complete-type or related LK functionals which only requires to solve (a sequence of) Lyapunov equations. Moreover, a main objective is to get an improved coefficient $k_1$ in (4). Additionally, we hope to make the Lyapunov-Krasovskii theory more transparent, by interpreting the results in terms of Lyapunov-Rumyantsev partial stability of the approximating ODE.

Numerical approaches to complete-type and related LK functionals are a recent field of research. However, existing results either rely on the knowledge of the delay Lyapunov matrix function $\Psi$, [13]–[20], or they aim to determine $\Psi$, [21]–[24]. In contrast, the procedure in the present paper directly leads to an approximation of the overall LK functional (3). Our main focus is not to provide a stability criterion, but, as outlined in Section I-A, we are interested in the functional itself and, in particular, in its lower bound (4).

We are going to use so-called discretization of the infinitesimal generator approaches, which are well-established for numerical eigenvalue calculations [3]. These approaches provide an ODE approximation of the RFDE. To analyze that ODE is also the core idea, e.g., in [25]–[27]. The involved ODE can be obtained by various methods. We resort exemplary to the Chebyshev collocation method, also known as pseudospectral discretization [28], and to the Legendre tau method [29].

Even in the context of more general LK functionals, a discretization of the RFDE in whatever form seems to be rarely considered in the literature. An early existence proof for quadratic LK functionals [30], as well as a recent approach to so-called safety functionals [31], also employ discretizations. These, however, do not lead to ODEs, but to difference equations (so-called discretization of the solution operator approaches [32]). Moreover, in [33], a discretization occurs in a proof of a linear-matrix-inequality stability criterion.

The core of the approach in the present paper is a Lyapunov equation from the ODE system matrix. The system matrices from both used discretization schemes are already known to give applicable Lyapunov, or, more generally, Riccati equation solutions. Concerning Chebyshev collocation, the resulting system matrix has successfully been employed for Lyapunov equations in the context of $H_2$-norm computations [22], [34], [35], where the delay Lyapunov matrix $\Psi(0; \bar{Q})$ at $s = 0$ is of interest. Further calculations are mentioned in [22] to obtain, at least under the assumption of an exponentially stable RFDE equilibrium, the matrix-valued function $\Psi$ for the LK functional formula (3). The Lyapunov equation is a common element with the present paper, but only a submatrix of the Lyapunov equation solution is used in [22, Prop. 2.1], the product with a matrix exponential is required for any value of $s$ in $\Psi(s; \bar{Q})$, and the integral expressions in (3) still would have to be evaluated to obtain a LK functional value. Concerning Legendre tau, the system matrix (respectively a similar matrix) has already successfully been used for algebraic Riccati equations in the context of optimal control [36].

**Structure.** The paper is organized as follows. Sec. II describes the numerical approach, and Sec. III gives the formula for the quadratic lower bound, which is applied to an example in Sec. IV. In Sec. V, we interpret the approach in terms of partial stability of the approximating ODE. Finally, Sec. VI addresses convergence, before Sec. VII concludes the paper.

**Notation.** The space of continuous $\mathbb{R}^n$-valued functions on the interval $[a, b]$ is denoted by $C([a, b], \mathbb{R}^n)$, in short $C$, and square integrable functions by $L_2([a, b], \mathbb{R}^n)$ or $L_2$. We write $(w_k)_{k \in \mathbb{N}}$ for a vector with entries $w_k$, e.g., $(w_k)_{k \in \{0, \ldots, N\}} = [w_0, \ldots, w_N]^T$, or $(w_k)_k$ if the index set is clear from the context. Similar holds for matrices. The set of eigenvalues $A \in \mathbb{R}^{n \times n}$ is $\sigma(A)$, and $A$ is said to be Hurwitz if all eigenvalues have negative real parts. Moreover, $Q \succeq 0_{n \times n}$ ($Q \succeq 0_{n \times n}$) denotes positive (semi)definiteness of $Q \in \mathbb{R}^{n \times n}$, implicitly requiring that $Q = Q^T$. The zero vector in $\mathbb{R}^n$ is $0_n$, the vector-valued zero function on $[a, b]$ is $0_{n \times [a,b]}$; the $m \times n$ zero matrix $0_{m \times n}$, and the identity matrix in $\mathbb{R}^{n \times n}$ is $I_n$. Given $x \in \mathbb{R}^n$, we write $\|x\|_2$ for the Euclidean norm, whereas $\|x\|$ can be any arbitrary norm in $\mathbb{R}^n$. The Kronecker product of two matrices $A$ and $B$ is $A \otimes B$, and $A^*$ denotes a generalized inverse. To emphasize the structure of a block matrix, e.g., $A = [A_1 \ A_2]$, with differently sized submatrices, $A_1 \in \mathbb{R}^{n \times N}, A_2 \in \mathbb{R}^{m \times n}$, we write $A = \begin{bmatrix} -A_1 & -A_2 \end{bmatrix}$. We use $\dot{}$ to mark a requirement, and $\overset{\circ}{\sim}$ if the relation is explained by $(\ldots)$. The set of class-K functions is defined by $\mathcal{K} = \{ \kappa \in C([0, \infty), \mathbb{R}_{\geq 0}) : \kappa(0) = 0, \text{ strictly increasing} \}$. For the formal definition of $D^{\frac{1}{2}}_{(\text{eq})} V$, see, e.g., [37, Sec. 5.2].
II. THE NUMERICAL APPROACH
A. ODE-Approximation Schemes of RFDEs

Given a continuous initial function \( x_0 \in C([-h, 0], \mathbb{R}^n) \), the state \( x_t \in C([-h, 0], \mathbb{R}^n) \) of the RFDE at time \( t \geq 0 \) is defined by \( x_t(\theta) = x(t + \theta), \theta \in [-h, 0] \). Thus, it represents the solution segment on \([t-h, t]\), cf. Fig. 1a/1b. An ODE approximation has to address a finite-dimensional state vector instead. In the simplest case, this state vector \( y(t) \) at time \( t \) approximates the values of the segment \( x_t \) in \( N+1 \) ordered points \( \theta_0 = -h, \ldots, \theta_N = 0 \).

\[
\begin{bmatrix}
x(t-h)
\ x(t+\theta_1)
\vdots
\ x(t+\theta_{N-1})
\ x(t)
\end{bmatrix}
= \begin{bmatrix}
x_t(-h)
\ x_t(\theta_1)
\vdots
\ x_t(\theta_{N-1})
\ x_t(0)
\end{bmatrix}
\ R \begin{bmatrix}
y^0(t)
y^1(t)
\vdots
y^{N-1}(t)
y^N(t)
\end{bmatrix}
= \begin{bmatrix}
z^0(t)
z^1(t)
\vdots
z^{N-1}(t)
z(t)
\end{bmatrix}
\tag{5}
\]

Henceforth, upper indices \( k \in \{0, \ldots, N\} \) address vector-valued components \( y^k(t) \in \mathbb{R}^n \). Whenever the special interest in \( y^N \) shall be emphasized, we use the indicated decomposition \( y = [z^1, z^T]^T \). For \( \theta_k \) in (5), a non-equidistant grid

\[
\hat{\theta}_k = \frac{h}{2}(\theta_k - 1), \quad \text{with} \quad \hat{\theta}_k = -\cos\left(\frac{k}{N}\pi\right), \tag{6}
\]

\( k \in \{0, \ldots, N\} \), built from shifting and scaling classical Chebyshev nodes \(^2\) \( \hat{\theta}_k \in [-1,1] \) to \( \theta_k \in [-h, 0] \), has proven to be advantageous [38]. The latter is also at the core of the open-source Matlab toolbox Chebfun by Trefethen and co-workers [39], from which we can benefit in the implementations.

It remains to find the ODE

\[
y(t) = A_y y(t), \tag{7}
\]

\( A_y \in \mathbb{R}^{(N+1)\times n} \), that describes the dynamics of \( y \). To this end, we use exemplarily the Chebyshev collocation method and the Legendre tau method combined with a change of basis. The resulting system matrices \( A_y \) are given by (63) and (68) in the appendix. Fig. 1c shows how a solution of (7) looks like, provided the initial condition \( y(0) \), given by the blue points, is a discretization of the initial function \( x_0 \in C([-h, 0], \mathbb{R}^n) \), cf. (12) with \( \phi = x_0 \).  

B. An Approximation Scheme for the LK Functional

We are going to set up a Lyapunov function \( V_y \colon \mathbb{R}^{n(N+1)} \to \mathbb{R} \) for the approximating ODE (7) (in fact, a partial Lyapunov function, see Sec. V). To this end, we make the quadratic ansatz

\[
V_y(y) = y^T P_y y, \tag{8}
\]

with \( P_y = P_y^T \in \mathbb{R}^{n(N+1)\times n(N+1)} \) to be determined. The derivative of \( V_y \) along solutions shall be \(-y^T Q_y y\) with a prescribed symmetric matrix \( Q_y \)

\[
D^+_{(7)} V_y(y) = y^T (P_y A_y + A^T_y P_y)\frac{1}{2} y = -y^T Q_y y, \tag{9}
\]

\(^2\)also called Gauss–Lobatto Chebyshev nodes (cf. Table II) or Chebyshev points of the second kind (despite of referring to extrema of the ‘Chebyshev polynomials of the first kind’) or endpoints-and-extrema Chebyshev nodes.

See Appendix A.3.a for a description in Legendre coordinates (indicated by a subscript \( \zeta \) at the matrices). We construct the right-hand side of (9) according to a discretization of the right-hand side of (2) with freely chosen matrices \( Q_0, Q_1 \succ 0_{n\times n}, Q_2 \succeq 0_{n\times n} \). Hence, a straightforward choice of \( Q_y \) in (10) becomes visible from

\[
D^+_{(7)} V_y(y) = -\frac{1}{y^N}Q_y y^N - (y^0)^T Q_1 y^0 - \sum_{k=0}^{N} (y^k)^T Q_2 y^k w_k = -y^T \begin{bmatrix}
Q_1 & 0_{n\times n} & \cdots & 0_{n\times n} \\
0_{n\times n} & Q_0 & \cdots & 0_{n\times n} \\
\vdots & \vdots & \ddots & \vdots \\
0_{n\times n} & Q_0 & \cdots & Q_n
\end{bmatrix}
\begin{bmatrix}
w_0 Q_2 \\
w_1 Q_2 \\
\vdots \\
w_N Q_2
\end{bmatrix} y
\tag{11}
\]

where \( w_k \in \mathbb{R} \) are integration weights, see Appendix B. Sec. VI-B will present discretization-scheme-dependent modifications of \( Q_y \) that aim at improved convergence properties. Altogether, we solve a discretization of the original problem (2), and thus \( V_y(y) \) in (8) is intended to be an approximation of the LK functional \( V(\phi) \). Convergence aspects will be addressed in Sec. VI. Hence, given a prescribed argument \( \phi \in C([-h, 0], \mathbb{R}^n) \), which might be \( \phi = x_t \) for some \( t \geq 0 \), or, without loss of generality, \( \phi = x_0 \) at \( t = 0 \), we can obtain a numerical approximation for the evaluation \( V(\phi) \). To this end, the argument \( y \) in \( V_y(y) \) must be chosen correspondingly. Such a discretization \( y \) of \( \phi \) can be obtained by evaluating the
vector-valued function $\phi$ at the gridpoints (6) and stacking these $(N+1)$ vectors in

$$y = \begin{bmatrix} \phi(-h) \\ \phi(\theta_1) \\ \vdots \\ \phi(\theta_{N-1}) \\ \phi(0) \end{bmatrix}.$$  (12)

Strictly speaking, (12) is the interpolatory discretization presupposed in the Chebyshev collocation method. If $\phi$ is a polynomial of order $N$ or less, (12) also agrees with the coordinate transform (67) of the discretization in the Legendre tau method (73), but otherwise the latter might give a slightly deviating vector $y$ (pointwise evaluations of the approximating polynomial).

To sum up, we only have to solve the Lyapunov equation (10), to obtain the approximation $V(\phi) \approx V_y(y)$.

C. Existence, Uniqueness, and Non-Negativity

Note that $Q_y$ in (11) is a positive semidefinite, but not necessarily positive definite matrix. Let us revisit some properties of the Lyapunov equation (10) in this rather uncommon semidefinite case, without further assumptions on the involved matrices. See [40, p. 284], and [41, Thm. 1] for Lemma 2.1c.

**Lemma 2.1:** Consider $PA + A^T P = -Q$, $A, Q \in \mathbb{R}^{n \times n}$.

(a) If $\sigma(A) \cap (-\sigma(A)) = \emptyset$, then a unique solution $P$ exists.

(b) If $Q = Q^T$ and $P$ is a solution, then $P^T$ is also a solution.

(c) If $Q \succeq 0$, then $P = P^T$, and $i_0(A) = 0$, then $i_+(P) \leq i_-(P) \leq i_+(A)$, where $i_-(),i_+()$ are the numbers of eigenvalues with negative, zero, and positive real parts.

**Remark 2.1:** Existence of the Lyapunov functional $V$ in (2) is analogously ensured by the time-delay counterpart of Lemma 2.1a, the so-called Lyapunov condition [10, Def. 2.6].

**Proposition 2.1:** Let $Q_y \succeq 0_{N(N+1)\times N(N+1)}$ be given. If $A_y$ is Hurwitz, then there exists a unique solution $P_y$ in (10). Moreover, $P_y = P_y^T$ is positive semidefinite.

**Proof:** Lemma 2.1a with $\sigma(A) \subset \mathbb{C}^-$, Lemma 2.1b, and Lemma 2.1c with $i_0(A) = i_+(A) = 0$.

Consequently, if the zero equilibrium of the ODE approximation (7) is asymptotically stable, and $D_{(T)}V_y(y)$ is chosen according to (11) and thus nonpositive, then existence, uniqueness, and nonnegativity of $V_y(y)$ in Sec. II-B are ensured.

D. Structure of the Result

To get an impression of how the Lyapunov equation solution $P_y$ looks like, we consider an example with $n = 1$. As will be demonstrated, only little implementation effort is required.

**Example 2.1:** Let $\dot{x}(t) = -0.5x(t) - x(t-2.2)$ and $Q_0 = Q_1 = 1$, $Q_2 = 0$ in (11). We get the solution $P_y$ of (10) via\(^3\)

$$Q = \text{blkdiag}(Q_1, \text{zeros}(n(N-1)), Q_0); P = \text{lyap}(A_y', Q);$$ in Matlab, provided $A_y$ is assigned to $A$ (see Rem. 1.1 or Rem. 1.3 in the appendix). The structure of $P_y$ for $N = 40$ is depicted in Fig. 2. It stems from the Legendre tau method,\(^3\)

\[\begin{bmatrix} P_{y,zz} & P_{y,xz}^T \\ P_{y,zz} & P_{y,xx} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z \end{bmatrix}, \quad \begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix} x^T \\ P_{y,xx} - P_{y,zz} \end{bmatrix} \begin{bmatrix} z^T \\ P_{y,xz} \end{bmatrix} + \sum_{j=0}^{N-1} (z^j)^T P_{y,kj} z^k + \sum_{k=0}^{N-1} P_{y,kj} (z^j)^T z^k.\]  (13)

describes, through the (discrete $\mapsto$ continuous) correspondences indicated by (12) and by $k$ vs. $\theta$ in Fig. 1

$$z^k = \phi(\theta_k), \quad z^j = \phi(\theta_j), \quad k \in \{0, \ldots, N-1\}, \quad j \in \{0, \ldots, N-1\}, \quad \dot{x} = \phi(0) \mapsto \phi(0),$$

the discrete version of some

$$V(\phi) = \phi^T(0) P_{xx} \phi(0) + 2 \int_{-h}^0 \phi^T(\theta) P_{xx} \phi(\theta) d\theta + \int_{-h}^0 \int_{-h}^0 \phi^T(\xi) P_{xx} \phi(\xi) d\theta d\xi + \int_{-h}^0 \phi^T(\theta) P_{xx,\text{diag}} \phi(\theta) d\theta.\]  (15a)

Note that the latter exactly reflects the known structure of complete-type and related LK functionals given in (3).

E. Validation via Numerical Integration

To be more precise, the structure of complete-type and related LK functionals is the one in (15a), and the kernel functions can be identified in (3) as

$$P_{xx}(\xi, \theta) = A_y^T \Psi(\xi - \theta; \tilde{Q}) A_1, \quad P_{xx}(\theta) = \Psi(-h - \theta; \tilde{Q}) A_1, \quad P_{xx,\text{diag}}(\theta) = Q_1 + (h + \theta)Q_2, \quad P_{xx} = \Psi(0; \tilde{Q}).\]  (15b)
For the sake of validation, we also go the other way around and discretize the known formula of $V(\phi)$ by interpolatory quadrature rules (cf. Table II). That is, replacing the integrals in (15a) by weighted sums from evaluations at the grid points. In Appendix B, we write the result as a quadratic form

$$V(\phi) \approx y^T P_y^{\text{quad}} y$$

(16)

like (13). Taking for $\Psi$ in (15b) the semi-analytical solution approach from [23], the picture of the resulting $P_y^{\text{quad}}$ for Example 2.1 is indeed hardly distinguishable from Fig. 2. See Sec. IV for further numerical comparisons.

Remark 2.2: Both the ODE-based approach from Sec. II-B and the numerical-integration-based approach from Sec. II-E provide an approximation $V_y(y) = y^T P_y y$. The former seeks for an approximative solution of the defining equation (2). In contrast, the latter already starts with the exact knowledge of the LK functional (3), presupposing knowledge of $\Psi$. In contrast, the latter already starts with the exact knowledge of the LK functional (3), presupposing knowledge of $\Psi$. As a result of the preceding section, we have an approximation of the LK functional (3), presupposing knowledge of $\Psi$ like (13). Taking for

$$\min_{z \in \mathbb{R}^n \setminus \{0_n\}} \frac{1}{\|x\|_2^2} z^T [Z \quad B^T \quad X] z = \lambda_{\min}(P/Z),$$

(18)

where $P/Z = X - B^T Z^{-1} B$. (19)

The minimum is attained by $[z] = [-Z^{-1} B^T]$, with $v$ being an eigenvector in $(P/Z) v = v \lambda_{\min}(P/Z)$.

Proof: Let us replace $z$ by $w := z - Z^{-1} B x$, which amounts to the coordinate transformation

$$[z \quad x] \left[ \begin{array}{cc} I_p & -Z^{-1} B^T \\ 0_{n \times p} & I_n \end{array} \right] [w \quad x] =: T_{yz} [w \quad x].$$

(20)

We arrive at the so-called generalized Aitken block-diagonalization of $P$ in

$$[z \quad x]^T \left[ \begin{array}{ccc} Z & B \\ B^T & X \end{array} \right] [z \quad x] = [w \quad x]^T T_{yz} \left[ \begin{array}{ccc} Z & B \\ B^T & X \end{array} \right] T_{yz} [w \quad x]$$

$$= [w \quad x]^T [Z \quad Z^T Z + B^T \\ -B^T Z^{-1} B + B \\ X - B^T Z^{-1} B] [w \quad x]$$

$$= w^T Z w + x^T (P/Z) x \geq x^T (P/Z) x \geq \lambda_{\min}(P/Z) \|x\|_2^2.$$

The bound is attained for $w = 0_n$ and $x = v$.

The following theorem is not only useful for the ODE-based approach from Sec. II-B. It is as well applicable to the numerical-integration-based results from Sec. II-E.

Theorem 3.1: If $P_y y = y^T P_y y$ is positive semidefinite, then the largest possible coefficient in (17) is

$$k_1 = \lambda_{\min}(P_y/P_{y,zz}),$$

(22)

where $P_{y,zz}$ denotes the left upper $nN \times nN$ submatrix of $P_y$ and $(-/-)$ is the generalized Schur complement (19).

Proof: Lemma 3.1 applied to $P = P_y$ with $Z = P_{y,zz}$ as in (13).

Testing whether $P_y$ is positive semidefinite is not even required if $V_y$ originates from the ODE-based approach in Sec. II-B. If $A_y$ is Hurwitz, the only thing to do is to evaluate (22).

Corollary 3.1: Let $V_y(y) = y^T P_y y$, where $P_y$ is a solution of (10) for a given positive semidefinite matrix $Q_y$. If $A_y$ is Hurwitz, then (17) holds with $k_1$ from (22).

Proof: By Prop. 2.1, $P_y$ is positive semidefinite. Consequently, Thm. 3.1 applies. See Appendix A.3.d for an evaluation in other coordinates.

IV. EXAMPLE AND COMPARISON

We compare the thus obtained bound with known quadratic lower bounds (4) on the LK functional (15). These known formulae for the coefficient in $k_1 \|x(t)\|^2 \leq V(x(t))$ are

$$k_1 = \max \alpha$$

(10, Lem. 2.10)

s.t. \begin{align*}
\frac{Q_0 \cdot 0_{n \times n}}{A_0} + \alpha A_1 \cdot A_0 \cdot A_1 \cdot 0_{n \times n} & \geq 0_{2n \times 2n}, \\
\lambda_{\min}(Q_0) & \geq 2 \|A_0\|_2^2 + \|A_1\|^2_2, \\
\lambda_{\min}(Q_1) & \geq \|A_0\|_2^2.
\end{align*}

(12, Prop. 1)
provided the equilibrium is exponentially\(^5\) stable and \(Q_0, Q_1 > 0_{n \times n}, Q_2 \preceq 0_{n \times n}\). Two issues should be noted.

Firstly, since the LK functional satisfies by construction the monotonicity condition of the common LK theorem, cf. (29), the existence of a quadratic lower bound with \(k_1 > 0\) (or actually even \(k_1 \geq 0\), cf. Thm. 5.4) is also the crucial missing step that proves asymptotic stability via the LK functional. However, the above formulae are only valid if exponential (equivalently, asymptotic) stability has been proven beforehand. Hence, the stability analysis must already be done by other means in a separate step. For instance, this can be achieved via frequency-domain based methods, e.g., via the eigenvalues of \(A_y\). Having thus \(A_y\) already at hand, the approach in the present paper becomes even more convenient.

**Remark 4.1:** As a consequence of the above issue, how at all to conclude stability from the LK functional (15) or the involved delay Lyapunov matrix function \(\Psi\) has long been an open question. It has only recently been resolved by Egorov et al. [14] and Gomez et al. [13]. The criterion is equivalent to requiring that, for some \(Q > 0_{n \times n}\),

\[
P_{xx}(\xi, \eta) := \Psi(\xi - \eta; Q)
\]

(23)
is a positive definite kernel, in the sense that the block matrix \((P_{xx}(\theta_j, \theta_k))_{jk}\) must be positive semidefinite, with an a priori bound on the discretization resolution of the grid \((\xi, \eta) \in \{\theta_j, \theta_k\}_{jk} \subset [-h, 0] \times [-h, 0]\). Despite of a completely different framework, the result can be brought in relation to Sec. II-E by rewriting the matrix in (16) as

\[
P_y^{quad} = S^T (P_{xx}(\theta_j, \theta_k))_{jk} S + D,
\]

(24)
with \(S = \text{diag}((w_k)_{k}) \otimes A_1 + \left[\begin{array}{cc} 0_{n \times n} & I_n \\ 0_{n \times n} & 0_{n \times n} \end{array}\right]\) and \(D = \text{blkdiag}((w_k(Q_1 + (h + \theta_k)(Q_2)))_{k})\), cf. (79) with (15b). The first term in (24) clearly preserves the positive semidefiniteness of \((P_{xx}(\theta_j, \theta_k))_{jk}\), and \(D\) is only an added block diagonal matrix that inherits positive semidefiniteness from \(Q_1, Q_2\).

Secondly, of course the LK functional changes as the delay changes. Note that, however, the above stated formulae for \(k_1\) do not depend on the value of the delay.

**Example 4.1:** For all delay values \(h\) that are smaller than \(h_c := \arccos(-0.9) / \sqrt{1 - 0.9^2} \approx 6.17\), the equilibrium of

\[
\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h)
\]

(25)
is asymptotically stable [9, Example 3.2]. Let \(Q_0 = Q_1 = I_2, Q_2 = 0_{2 \times 2}\). For any given \(h > 0\) (affecting \(A_y\)), the Lyapunov equation solution \(P_0\) can be computed as in Example 2.1. We get \(k_1 = 22\) in (22) via the additional lines

\[
p = \text{mat2cell}(P, n \times [N, 1], n \times [N, 1]);
k1 = \text{min}((eig(p(2, 2) - p(2, 1) + (p(1, 1) - p(1, 2)))
\]

in Matlab (as \(P_{yxz}\) is nonsingular). We also consider the numerical-integration-based \(P_y^{quad}\) from (79) and (80). Fig. 3a shows the convergence of \(k_1\) for all approaches.

---

\(^5\) equivalently, asymptotically since (1) is a linear autonomous RFDE. In linear RFDEs with bounded delays, uniform asymptotic stability and uniform exponential stability are equivalent [37, Thm. 5.3 in Ch. 6]. Moreover, in autonomous or periodic RFDEs (in contrast to neutral FDEs), asymptotic stability is always uniform [37, Lemma 1.1 in Ch. 5].
Remark 4.2: If \( Q_1 = Q_2 = 0_{n \times n} \), only a local cubic lower bound on \( V \) is known to exist, and non-existence\(^4\) of a positive quadratic one is proven for [10, Example 2.1]. Indeed, for this example, \( k_1 \) from (22) converges to zero as \( N \) increases.

Finally, the reduced conservativity of \( k_1 \), already indicated by Fig. 3e, is confirmed by other examples in Table I.

V. INTERPRETATION IN TERMS OF PARTIAL STABILITY

Note that \( V_y \) obtained in Sec. II-B does not necessarily qualify as a Lyapunov function for the ODE (7) since, if \( Q_2 = 0_{n \times n} \), the matrix \( Q_y \) in the Lyapunov function derivative (11) is not positive definite. Even if \( Q_2 > 0_{n \times n} \), the involved \( Q_y \) is theoretically positive definite for any finite \( N \), but the smallest eigenvalue of \( Q_y \) converges to zero as \( N \) increases (the denser the grid, the smaller the integration weights \( w_k \)). Moreover, the lower bound (17) on \( V_y \) does not fit with the classical Lyapunov theory. The present section explains why \( V_y \) is still meaningful for a stability analysis of the approximating ODE. Within the presented approach, the lower bound (17) is exactly what is required. First, we clarify what we are actually looking for when we target stability in a RFDE.

A. Stability in RFDEs

Having in mind the classical Lyapunov theorem for ODEs, one might wonder why the lower bound in (4) relies on \( \| x(t) \| \) and not on the norm of the RFDE state \( x_t \). The latter addresses the norm in \( C([-h,0],\mathbb{R}^n) \) defined by

\[
\| x(t) \|_C = \max_{\theta \in [-h,0]} \| x_t(\theta) \|. \quad (26)
\]

For Lyapunov functions in ODEs, both the positivity-definiteness bound (\( \kappa_1(\| x \|) \leq V(x), \kappa_1 \in \mathcal{K} \)) and the monotonicity requirement (\( D^+ V(x) \leq -\kappa_3(\| x \|), \kappa_3 \in \mathcal{K} \)) refer to the norm of the ODE state. Thus, one would expect (26) at these places when transferring Lyapunov’s results from \( x(t) \in \mathbb{R}^n \) to \( x_t = \phi \in C([-h,0],\mathbb{R}^n) \). However, this is not the case in the following common LK theorem—neither in the left inequality of (28) nor in (29). Instead of \( \| \phi \|_C = \| x_t \|_C \), only \( \| \phi(0) \| = \| x_t(0) \| = \| x(t) \| \) occurs. As usual, the theorem refers to general autonomous RFDEs

\[
\dot{x}(t) = f(x_t), \quad (27)
\]

with \( f(0_{n \times n,0}) = 0_n \) and \( f \) locally Lipschitz.

Theorem 5.1 (LK Theorem [37, Thm. 5.2.1]): If there is a continuous \( V : C([-h,0],\mathbb{R}^n) \rightarrow \mathbb{R} \geq 0 \) such that, for all \( \phi \) in a domain \( G \subseteq C([-h,0],\mathbb{R}^n) \), \( 0_{n \times n,0} \in G \), it holds

\[
\kappa_1(\| \phi(0) \|) \leq V(\phi) \leq \kappa_2(\| \phi \|_C) \quad (28)
\]

\[
D^+ V(\phi) \leq -\kappa_3(\| \phi(0) \|), \quad (29)
\]

with some class-K functions \( \kappa_{1,2,3} \in \mathcal{K} \), then the zero equilibrium of (27) is asymptotically stable.

The key to the above question is that there are two, obviously equivalent, definitions of asymptotic stability in the RFDE. Starting from the same norm ball for the initial function \( x_0 \), they differ in the condition on the implication side: Either the state \( x_t \) with the norm (26) is taken into account (Def. 5.1a), or the pointwise solution \( x(t) \in \mathbb{R}^n \) is considered (Def. 5.1b).

Definition 5.1 (Lyapunov stability in RFDEs): The zero equilibrium of (27) is asymptotically stable if

a) \( \forall \varepsilon > 0, \exists \delta > 0 : \| x_0 \|_C < \delta \Rightarrow \forall t \geq 0 : \| x_t \|_C < \varepsilon \)

and \( \exists r > 0 : \| x_0 \|_C < r \Rightarrow \| x_t \|_C \rightarrow 0 \) as \( t \rightarrow \infty \), or, equivalently,

b) \( \forall \varepsilon > 0, \exists \delta > 0 : \| x_0 \|_C < \delta \Rightarrow \forall t \geq 0 : \| x(t) \| < \varepsilon \)

and \( \exists r > 0 : \| x_0 \|_C < r \Rightarrow \| x(t) \| \rightarrow 0 \) as \( t \rightarrow \infty \).

In terms of the whole state \( x_t \), the pointwise consideration in Def. 5.1b refers only to the boundary value \( x(t) = x_t(0) \) in Fig. 1b. The classical LK theorem, Thm. 5.1, addresses Def. 5.1b since, \( \forall t \geq 0 \),

\[
\kappa_1(\| x(t) \|) \leq V(x_t) \leq V(x_0) \leq \kappa_2(\| x_0 \|_C) \quad (30)
\]

gives a pointwise estimation \( \| x(t) \| \leq \kappa_1^{-1}(\kappa_2(\| x_0 \|_C)) \) to indicate stability. A theorem that addresses Def. 5.1a would instead rely on \( \kappa_1(\| \phi \|_C) \) in (28) and \( \kappa_3(\| \phi \|_C) \) in (29), as has been expected above. Such a theorem is also valid [45, Thm. 30.1], but these bounds are quite restrictive and not satisfied by common LK functionals.

B. Partial Stability in ODEs

In the approximating ODE, cf. Fig. 1c, the state \( y(t) \in \mathbb{R}^{n(N+1)} \) represents the RFDE state \( x_t \in C([-h,0],\mathbb{R}^n) \), and its last vector-valued component \( y^n(t) = \hat{x}(t) \in \mathbb{R}^n \) represents the pointwise solution value \( x(t) \in \mathbb{R}^n \). While Def. 5.1a translates to the usual\(^6\) definition of asymptotic stability in the ODE, Def. 5.1b amounts to the concept of partial asymptotic stability with respect to (w.r.t.) \( \hat{x} \). Again, we give the definition for a general class of systems. These are ODEs where \( y(t) \) is partitioned into two parts, \( z(t) \in \mathbb{R}^p \) and \( \hat{x}(t) \in \mathbb{R}^n \), with \( \dim(y(t)) = p+n \), and the latter part \( \hat{x}(t) \) is of special interest. That is, we consider autonomous ODEs

\[
\frac{dz(t)}{dt} = f^z(z(t),\hat{x}(t)) \quad \frac{d\hat{x}(t)}{dt} = f^{\hat{x}}(z(t),\hat{x}(t)) \quad (31)
\]

with \( f^{z,0}(0_{p,n}) = 0_{p+n} \) and \( f^{\hat{x}:x} \) locally Lipschitz.

Definition 5.2 (Lyapunov-Rumyantsev partial stability): The zero equilibrium of (31) is partially stable w.r.t. \( \hat{x} \) if

\[
\forall \varepsilon > 0, \exists \delta > 0 : \| z(0) \| < \delta \Rightarrow \forall t \geq 0 : \| \hat{x}(t) \| < \varepsilon \}
\]

It is partially asymptotically stable w.r.t. \( \hat{x} \) if, additionally,

\[
\exists r > 0 : \| z(0) \| < r \Rightarrow \| \hat{x}(t) \| \rightarrow 0 \text{ as } t \rightarrow \infty \}
\]

\(^6\)The choice of the norm \( \| y \|_\infty = \max_{k \in \{0,\ldots,N\}} \| y^k \| \) is irrelevant due to the equivalence of norms in finite-dimensional spaces.
For an in-depth discussion of this stability concept, see [46]. As in Def. 5.1b for stability in RFDEs, the initial value deviations consider the whole state, but the implications address only the part \( \hat{x} \) that is of special interest.

The following partial stability theorem fits well with Thm. 5.1 (note that an upper bound \( V_\gamma(\|\hat{x}\|) \leq \kappa_2(\|\hat{x}\|) \) always exists).

**Theorem 5.2 (Peiffer and Rouche 1969 [47, Thm. II]):** If there is a continuous \( V_\gamma: \mathbb{R}^{p+n} \to \mathbb{R}_0^+, V_\gamma(0_{p+n}) = 0 \), such that, for all \( \|\hat{z}\| \) in a domain \( G \subseteq \mathbb{R}^{p+n}, \hat{v}_{p+n} \in G \), it holds

\[
\kappa_1(\|\hat{x}\|) \leq V_\gamma(\|\hat{z}\|),
\]

with \( \kappa_1 \in \mathbb{K} \), and \( D_+(31) V_\gamma(\|\hat{z}\|) \leq 0 \), then the zero equilibrium of (31) is partially stable w.r.t. \( \hat{x} \). If, additionally, \( \forall \|\hat{z}\| \in G: D_+(31) V_\gamma(\|\hat{z}\|) \leq -\kappa_3(\|\hat{x}\|) \)

(33)

with \( \kappa_3 \in \mathbb{K} \), and there exists \( r > 0 \) such that \( \|f^T(z(t), \hat{x}(t))\| < r \) implies that \( \|f^T(z(t), \hat{x}(t))\| \) is bounded for all \( t \geq 0 \), then it is partially asymptotically stable w.r.t. \( \hat{x} \).

As in the classical LK theorem for RFDEs (Thm. 5.1), both the (partial) positive-definiteness condition (32) and the monotonicity requirement \( \tau \) (33) consider only the part of special interest \( \hat{x}(t) = y^N(t) \approx x(t) = x_1(0) \). We call \( V \) in Thm. 5.2 a partial Lyapunov function.

To sum up, the discretization of Def. 5.1b for RFDE stability is exactly the definition of Lyapunov-Rumyantsev partial stability w.r.t. \( \hat{x} \) (Def. 5.2). Moreover, the Lyapunov-Krasovskii theorem for stability in the RFDE (Thm. 5.1) becomes Peiffer and Rouche’s theorem for partial stability (Thm. 5.2).

**C. Equivalence of Stability and Partial Stability in the Approximating ODE**

In general ODEs, the concept of partial stability is a weaker concept than stability. We can still focus without doubt on partial stability if the equivalence between Def. 5.1a and 5.1b is reflected by the ODE approximation, so that proving partial stability w.r.t. \( \hat{x} \) is already sufficient for proving stability.

**Condition 5.1:** The zero equilibrium of the ODE approximation (7) is (asymptotically) stable if and only if it is partially (asymptotically) stable w.r.t. \( \hat{x} \).

To verify this condition for the discretization schemes at hand, we consider a result from the realm of total stability.

**Lemma 5.1:** [49, Thm. 3.11.3]. If the zero equilibrium of the auxiliary system

\[
\dot{z} = f^T(z, 0_n)
\]

(34)

is asymptotically stable, then, in (31), partial (asymptotic) stability w.r.t. \( \hat{x} \) of the zero equilibrium implies (asymptotic) stability of the zero equilibrium.

Loosely speaking, for reasonable approximations the latter seems to be a matter of course since, if \( x(t) \) for \( t \geq 0 \) could be forced to remain zero, then, for \( t \geq h \), the solution segment \( x_i \) is zero, which should at least asymptotically be reflected by \( z(t) \to 0_p \), as \( t \to \infty \). In terms of the linear ODE (7), Lemma 5.1 only refers to the submatrix \( \hat{A}_{y,zz} := (A_y^{jk})_{j,k \in \{0,\ldots,N-1\}} \). For collocation schemes like \( A_y = A_y^c \) in Appendix A.1, stability of this submatrix is clearly neither affected by the RFDE coefficient matrices \( A_0, A_1 \) (occurrence only in the last block-row), nor the delay \( h \) (scalar factor), nor the dimension \( n \) (Kronecker product with \( F_0 \)). For tau methods, an analogous independence can be achieved by first applying a change of basis w.r.t. the \( z \)-coordinates. In appropriate coordinates, setting, e.g., \( A_0 = A_1 = 0_{n \times n} \) does not alter the submatrix eigenvalues. The next lemma formulates the thus motivated coordinate invariant form of Lemma 5.1 for the linear ODE. Whether it applies is, consequently, no question of \( A_0, A_1, h, \) but it is rather a question of the discretization scheme.

**Corollary 5.1:** Consider the linear ODE (7). If there exists a change of coordinates w.r.t. \( z \), where \( [z^T, \hat{x}^T]^T = T[v^T, \hat{x}^T]^T \), such that the upper \( nN \times nN \) submatrix of \( T^{-1}A_yT \) is Hurwitz, then Condition 5.1 holds.

**Proof:** Lemma 5.1 with (31) given by \( \frac{d}{dt} \|\hat{y}\| = T^{-1}A_yT[\|\hat{y}\|] \).

For \( A_y = A_y^c \) from the Chebyshev collocation method (A.1), the submatrix \( \hat{A}_{y,zz} \) can indeed proven to be Hurwitz for any discretization resolution \( N \) [26, Prop. 2], [50]. Thus, by Corollary 5.1 (with \( v = z \)), Condition 5.1 holds. For other discretization schemes we refer to [51, Sec. 4.3.2]. For the Legendre tau method, we consider the coordinates described in (76), where \( v = [\zeta^0, \ldots, \zeta^{N-1}]^T \) consists of the first \( N \) of the \( N + 1 \) Legendre coordinates. Thus, Corollary 5.1 (with \( T^{-1}A_yT = T_{\zeta^N}A\zeta^{N-1}\zeta^N \) and \( T_{\zeta^N} \) from (76)) can numerically be shown to be true for relevant values of \( N \).

Consequently, Condition 5.1 is not only a reasonable assumption for an ODE that approximates an RFDE, but, regarding the discretization of a RFDE, it can even be confirmed as a property of the underlying discretization schemes.

**D. Proving Stability in the ODE via \( V_\gamma(y) \)**

The main result of this section, Thm. 5.3, shows that \( V_y \) from Sec. II-B indeed always qualifies as a partial Lyapunov function for (7) if the equilibrium is asymptotically stable. As a side effect, Thm. 5.4 gives a necessary and sufficient stability criterion in terms of \( P_y \). We introduce the following wording.

**Definition 5.3:** Let \( \hat{x} \)-pd be a necessary abbreviation for ‘(partially) positive definite w.r.t. the components \( \hat{x} \).’ We call

- (a) a function \( U : \mathbb{R}^{p+n} \to \mathbb{R}; \ y = [\hat{z}] \to U(y) \)-pd on \( \Omega \subseteq \mathbb{R}^{p+n} \) if it is positive semidefinite (\( \forall y \in \Omega : U(y) \geq 0 \), \( U(0_{p+n}) = 0 \)) and \( \forall \hat{y} = [\hat{z}] \in \Omega \) with \( \|\hat{y}\| \neq 0 \) (\( U(y) > 0 \)).

- (b) a symmetric matrix \( \hat{M} = M^T \in \mathbb{R}^{(p+n) \times (p+n)} \)-pd if \( U(y) = y^T \hat{M}y \) is \( \hat{x} \)-pd on \( \mathbb{R}^{p+n} \).

Analogously to local, or in terms of \( U(y) = y^T \hat{M}y \) even global, positive definiteness, cf. [1, Lemma 4.3], partial positive definiteness can be expressed via a class-K function.

**Lemma 5.2:** \( \hat{M} = M^T \in \mathbb{R}^{(p+n) \times (p+n)} \)-pd if and only if \( \exists \kappa \in \mathbb{K} \) such that \( \forall \|\hat{z}\| \in \mathbb{R}^{p+n} : \kappa(\|\hat{x}\|) \leq \|\hat{z}\|^2 \|\hat{M}\|_2 \).

Regarding \( y^T Q_\gamma y = -D_+(31) V_\gamma(y) \). Lemma 5.2 refers to the class-K function in (33). For \( Q_\gamma \) in (11) or (37), we can choose

\[
\kappa_3(\|\hat{x}\|) := \lambda_{\min}(Q_0) \|\hat{x}\|^2 \leq y^T Q_\gamma y.
\]
Rather decisive is whether the Lyapunov equation solution $P_y$ is also $\dot{x}$-pd, as it is required in (32) with $V_y(y) = y^TP_yy$.

**Lemma 5.3:** Let $P_y = P_y^T$ be a solution of (10) for a $\dot{x}$-pd $Q_y$. If $P_y$ is positive semidefinite, then it is even $\dot{x}$-pd.

**Proof:** The result is shown by contradiction. Assume there exists a $y = [\hat{x}^T; \xi]$ with $\|\xi\| \neq 0$ such that $y^TP_y\xi = 0$. Then $P_y\xi = 0_n(n+1)$ (cf. a decomposition $P_y = C^TC$ in $y^TP_y\xi = \|C\xi\|_2^2 = 0$, $C^TC = 0_{n(N+1)}$), which leads by (10) to $y^T\dot{Q}_y\xi = 0$, contradicting that $Q_y$ is $\dot{x}$-pd.

**Lemma 5.4:** Let $P_y = P_y^T$ be a solution of (10) for a $\dot{x}$-pd $Q_y$. Consider Thm. 5.2 in terms of partial asymptotic stability w.r.t. $y^N = \dot{x}$ for the zero equilibrium in (7). If $P_y$ is positive semidefinite, then, under Cond. 5.1, $V_y(y) = y^TP_yy$ satisfies the conditions on a partial Lyapunov function in Thm. 5.2.

**Proof:** In Thm. 5.2, (32) and (33) hold by Lemma 5.3 and 5.2. The boundedness condition on $\|f^x(z(t), \dot{x}(t))\|$ in Thm. 5.2 is also ensured: due to Cond. 5.1, the already provable partial stability implies stability, which is accompanied by compactness of the trajectories, and the image under the continuous mapping $f^x$ remains compact.

We are led to the desired interpretation of the function $V_y$ whenever the ODE equilibrium is asymptotically stable.

**Theorem 5.3:** If $A_y$ is Hurwitz and Cond. 5.1 applies, then $V_y$ from Sec. II-B is a partial Lyapunov function for (7).

**Proof:** If $A_y$ is Hurwitz, $P_y$ is positive semidefinite by Prop. 2.1. As $Q_y$ in Sec. II-B is $\dot{x}$-pd, Lemma 5.4 applies.

Our focus is not preliminary on a stability criterion in terms of $P_y$ because we can simply compute the eigenvalues of $A_y$ to conclude stability. Nevertheless, the following result might still be of interest since it shows that $V_y$ must only be tested for positive semidefiniteness. Proving existence of $\kappa_1$ in (32) is not required due to Lemma 5.3.

**Theorem 5.4:** Assume Cond. 5.1 holds. Let $P_y = P_y^T$ be a solution of (10) for a $\dot{x}$-pd matrix $Q_y$ e.g., (11) or (37)). The zero equilibrium of the approximating ODE (7) is asymptotically stable if and only if $P_y$ is positive semidefinite.

**Proof:** If $P_y \succeq 0_{n(N+1)\times n(N+1)}$, then Lemma 5.4 applies. Thus, partial asymptotic stability w.r.t. $\dot{x}$ can be proven by Thm. 5.2. The latter implies asymptotic stability by Cond. 5.1. Conversely, if $A_y$ is Hurwitz, then $P_y \succeq 0_{n(N+1)\times n(N+1)}$ because of Prop. 2.1.

We conclude Sec. V as follows. The function $V_y$ obtained in Sec. II-B does not necessarily qualify as a classical Lyapunov function. Instead, it is a partial Lyapunov function for a system in which proving partial stability is already sufficient for proving stability.

**VI. CONVERGENCE**

A sequence of refined results with an enlarged $N$ should always be considered. It remains to discuss convergence aspects.

**A. Stability Properties of the Approximating ODEs**

The discretization scheme used in the proposed ODE-based approach should be stability preserving in the following sense.

**Condition 6.1:** Provided the discretization resolution $N$ is chosen sufficiently large, the zero equilibrium of the approximating ODE is exponentially stable if and only if the zero equilibrium of the RFDE is exponentially stable.

**B. Scheme-Dependent Improvements**

1) **Chebyshev collocation:** Consider the ODE-based approach with the Chebyshev collocation method. To improve the convergence properties (indicated in Fig. 3), we transform the problem of approximating $V(\phi)$ to a problem of approximating a modified $V_0(\phi)$ with $Q_1$ and $Q_2$ being zero. To this end, we choose the shift matrices $Q_1 = Q_1$ and $Q_2 = Q_2$ in the following splitting lemma. The idea is closely related to the derivation of complete-type functionals in [10, Thm. 2.11].

![Fig. 4: Characteristic roots (A₀, A₁, h from Example 2.1)](image)
Lemma 6.1 (Splitting): For \( Q_0, Q_1, Q_2 \in \mathbb{R}^{n \times n} \), let \( V(\phi) = V(\phi; Q_0, Q_1, Q_2) \) denote a solution of (2). Then

\[
V(\phi; Q_0, Q_1, Q_2) = V(\phi; Q_0 + Q_1 + hQ_2),
\]

\( (Q_1 - Q_2), \quad (Q_2 - \tilde{Q}_2) \)

for arbitrarily chosen shifts \( \tilde{Q}_1, \tilde{Q}_2 \in \mathbb{R}^{n \times n} \). and

Motivated by the numerical results in Sec. IV, we focus in this section on the Legendre tau method. Moreover, for this discretization scheme, we benefit from existing convergence proofs for the approximation of algebraic Riccati equations from the context of optimal control [7], [36], [53].

1) Operator-based description: Henceforth, we use any argument \( \phi \in C \) for \( V(\phi) \) gives rise to an element \( [\phi_{(0)}] \in C \times \mathbb{R}^n \subset L_2 \times \mathbb{R}^n = M_2 \)

2) Legendre tau: A separate numerical treatment of \( V_1 \) and \( V_2 \) in (36) is not required if the Legendre-tau-based approach is used. However, if \( Q_2 \) is nonzero, the following modification of \( Q_2 \) in (11) should be used in (71) or (10)

\[
Q_y = \text{blkdiag}(Q_1, 0_{n(N-1) \times n(N-1)}), Q_0) + T^T \rho \epsilon_{Q_2, \cdot} T^{\epsilon_{Q_2, \cdot}}
\]

with \( Q_1 \), (11) and (37)

(37)

the right lower component \( hQ_2 \) in \( Q_2 \) is motivated by Lemma 1.2 in the appendix). Despite of not being treated separately in the numerical approach, the arising contributions for \( V_1(\phi) \) and \( V_2(\phi) \) within the approximation of \( V(\phi) \) are still of interest for the proofs in the next sections. They can be obtained by solving Lyapunov equations with \( Q_{0,1,2} \) being replaced by the matrices behind the semicolon in \( V_1(\phi) = V(\phi; \cdot) \) and \( V_2(\phi) = V(\phi; \cdot) \) from Lemma 6.1. Appendix A.3.c shows that the resulting Legendre-tau-based approximations of \( V_1(\phi) \) and \( V_2(\phi) \) give even the exact value for any \( \phi \) that is a polynomial of order \( N - 1 \) or less.

C. Convergence Towards the Functional

We are interested in the following convergence statement.

Condition 6.2: For any given \( \phi \in C([-h, 0], \mathbb{R}^n) \), the scalar value \( V_y(y) \) converges to \( V(\phi) \) as \( N \) increases.

More formally, we use the notation \( y = \pi_y(\phi) \) to emphasize that the discretization \( y \in \mathbb{R}^{n(N+1)} \) is uniquely determined from \( \phi \in C \) (depending on the discretization scheme). Additionally, to keep track of the discretization resolution \( N \), a superscript \( [N] \) is added, e.g., in \( V_y^{[N]}(\cdot) = V_y(\cdot) \) and \( \pi_y^{[N]}(\cdot) = \pi_y(\cdot) \).

Thus, Cond. 6.2 can be rewritten as

\[
\forall \phi \in C : \quad V_y^{[N]}(\pi_y^{[N]}(\phi)) \rightarrow V(\phi), \quad (N \rightarrow \infty).
\]

(38)

8The term \( \phi^{(-1)}(\cdot)Q_1(\phi) \) would require an unbounded operator \( \mathcal{B} \) in the Lyapunov equation (51). Moreover, \( \mathcal{B}_{12} \) is not compact.
2) Convergence towards $V_0$: The operator $\mathcal{P}_0$ in (42) satisfies a no-operator-valued Lyapunov equation, c.f. [54], [55]. Its right-hand side is based on the right-hand side of (2). Because of the splitting approach, the latter is $D_{t}(1) V_0(x_0) = x^T(t) \dot{Q} x(t)$ with $\dot{Q} = Q_0 + Q_1 + h Q_2$, or, for $x_1 = \phi$, 
\[
D_{t}(1) V_0(\phi) = -\phi^T(0) \ddot{Q} \phi(0) = - \left[ \frac{\phi(0)}{[\phi(0)]_M} \right] , \quad (50)
\]
$\mathcal{D}\left[ \frac{\phi(0)}{[\phi(0)]_M} \right] = \left[ a_{0,-h,n}^0 \right] \ddot{Q}(0,0)$. Therefore, the operator-valued Lyapunov equation for the self-adjoint operator $\mathcal{P}_0 = \mathcal{P}_0^*$ reads 
\[
\langle \phi, \mathcal{P}_0 \phi, \psi, \mathcal{P}_0 \psi \rangle_M = \langle \phi, \mathcal{D} \phi,M \rangle , \quad (51)
\]
$\forall \psi \in D(\mathcal{A}) \subset M_2$, c.f. [54], [55], where $\mathcal{A}$ is the infinitesimal generator of the $2_0$-semigroup of solution operators on $M_2$ (which for linear RFDEs is as well an appropriate state space), and $D(\mathcal{A})$ is its domain. See, e.g., [54] for background on $\mathcal{A}$.

From the ODE-based approach in Sec. II-B, we obtain an approximation $V_0(\phi) \approx V_{0,y}(y) = \tilde{P} y_{0,y}$, or, in the notation of (38), $V_{0,y}^N(\pi_y^N(\phi)) = \tilde{P} y_{0,y}$, which, for the exact operator $V_0(\phi)$ in (42), this approximation can be described via 
\[
V_{0,y}^N(\pi_y^N(\phi)) = \left\{ \phi, \phi(x_0) \right\}_M \]
with an approximated operator $\mathcal{P}_0^N$. Moreover, similarly to the exact operator $\mathcal{P}_0$ from (51), this approximated operator $\mathcal{P}_0^N$ also satisfies an operator-valued Lyapunov equation, 
\[
2 \langle \phi, \mathcal{P}_0^N \phi, \psi, \mathcal{P}_0^N \psi \rangle_M = - \langle \phi, \mathcal{D} \phi, \psi, \mathcal{P}_0^N \phi \rangle_M , \quad (53)
\]
which, however, only relies on an approximation $\mathcal{A}_N$ instead of $\mathcal{A}$. See [36] for details. The matrices $A_2$ or, equivalently, $A_y$ in Sec. II are coordinate representations of that $\mathcal{A}^N$.

It has to be shown that, $\forall \phi \in C$, the scalar value $V_0(\phi)$ in (42) is indeed the limit of its approximations in (52) as $N \to \infty$. In terms of the operators, weak* operator convergence $\mathcal{P}_0^N \rightharpoonup \mathcal{P}_0$ suffices for that objective.

Lemma 6.2: Let (52) describe a Legendre-tau-based result for $V_0(\phi)$. Assume $\left\{ \mathcal{P}_0^N \right\}_N$ is bounded and, the existence and uniqueness conditions from Lemma 2.1 and Rem. 2.1 hold. Then $\mathcal{P}_0^N$ converges weakly to $\mathcal{P}_0$ as $N \to \infty$.

Proof: See [36, Thm. 5.1 (i)] with zero input operator and the uniqueness conditions from Sec. II-C.

In fact, this result is not at all special to the Legendre tau method. An alternative proof from [53, Thm. 6.7] applies to any discretization scheme that satisfies standard conditions proving convergence of numerical solutions for (41) in $M_2$. Lemma 6.2 relies on uniform boundedness and existence of operator $\mathcal{P}_0$.

Theorem 6.1: If the RFDE equilibrium is exponentially stable, then $\mathcal{P}_0^N$ converges in operator norm to $\mathcal{P}_0$, i.e., it holds $\left\| \mathcal{P}_0^N - \mathcal{P}_0 \right\| \to 0$ as $N \to \infty$.

Proof: See [53, Thm. 6.9], where even convergence in the trace norm [53, p. 111] is proven. The result requires that not only the approximations of the solution operator $\mathcal{A}(t)$ converge strongly on $\mathcal{P}_0$ with the assumptions of Lemma 6.2 hold, or, more generally, if the assumptions of Lemma 6.2 hold, then $\mathcal{P}_0^N$ converges weakly to $\mathcal{P}_0$ as $N \to \infty$.

D. Quadratic Lower Bound on the Functional

We are going to prove that, for $N \to \infty$, the quadratic lower bound on the approximation gives also a valid quadratic lower bound on the functional. This holds for any discretization scheme satisfying Cond. 6.2. Moreover, for the Legendre approximations. In the following we show that these can be ignored in the case of an exponentially stable RFDE equilibrium. Nevertheless, while simplifying the considerations, stability of the equilibrium is no necessary condition in the derivations.

Lemma 6.3: If the RFDE equilibrium is exponentially stable, then the assumptions in Lemma 6.2 hold.

Proof: Let $\mathcal{F}(t) : M_2 \to M_2$, $\left[ x_0 \right] \mapsto \left[ x \right]$ be the solution operator, and $\mathcal{F}(t)$ its approximation (represented by $e^{t K}$). Due to the stability preservation property from [7, Thm. 5.3], $\exists M \geq 1, \beta > 0, N \in \mathbb{N}$, such that $\forall N \geq N : \left\| \mathcal{F}(t) \right\| \leq M e^{-\beta t}$. Therefore, the improper integral formula $\mathcal{F}(t) = \int_0^{\infty} (\mathcal{F}(t))^\star(s) \mathcal{D}(\mathcal{F}(t))^\star(s) ds$ is applicable, see, e.g., [53]. Thus, with $\left\| \mathcal{F} \right\| = \left\| Q_2 \right\|$, the operators $\mathcal{F}(t)$ are uniformly bounded by $\left\| \mathcal{F}(t) \right\| \leq \int_0^{\infty} \left\| Q_2 \right\| \mathcal{D}(\mathcal{F}(t))^\star(s) ds \leq \left\| Q_2 \right\|^2 M_2$. Moreover, the existence and uniqueness assumptions hold by Prop. 2.1.

The convergence towards $V_0(\phi)$ does not require more than the thus established weak convergence $\mathcal{P}_0^N \rightharpoonup \mathcal{P}_0$. However, the following stronger result will become helpful in Sec. VI-D.

Lemma 6.4: Let (52) describe a Legendre-tau-based result for $V_0(\phi)$. If the RFDE equilibrium is exponentially stable, then $\mathcal{P}_0^N$ converges in operator norm to $\mathcal{P}_0$, i.e., it holds $\left\| \mathcal{P}_0^N - \mathcal{P}_0 \right\| \to 0$ as $N \to \infty$.

Proof: See [53, Thm. 6.9], where even convergence in the trace norm [53, p. 111] is proven. The result requires that not only the approximations of the solution operator $\mathcal{F}(t)$ converge strongly, but also those of its adjoint $\mathcal{F}^*(t)$, which for the Legendre tau method is proven in [7, Thm. 2.2].

3) Convergence towards $V$: To prove Cond. 6.2 on convergence towards $V = V_0 + V_1$, it only remains to include $V_1$.

We are going to prove that, for $N \to \infty$, the quadratic lower bound on the approximation gives also a valid quadratic lower bound on the functional. This holds for any discretization scheme satisfying Cond. 6.2. Moreover, for the Legendre approximations. In the following we show that these can be ignored in the case of an exponentially stable RFDE equilibrium. Nevertheless, while simplifying the considerations, stability of the equilibrium is no necessary condition in the derivations.
tau method, the thus obtained bound will be shown to be tight, meaning that the largest possible coefficient $k_1$ in (4) is obtained.

For any discretization resolution $N$, the largest possible coefficient $k_1^{(N)}$ for the bound (17) on the approximation $V_y^{(N)}$ is given by (22). Note that $k_1^{(N)}$ and, similarly, the largest possible coefficient $k_1 = k_1^{\text{opt}}$ for the bound (4) on the functional $V$ are defined by

$$k_1^{(N)} = \min_{\phi \in \mathcal{C}} \frac{1}{\lVert \phi \rVert_2} V_y^{(N)}(\{z/\varepsilon\}), \quad k_1^{\text{opt}} = \inf_{\phi \in \mathcal{C}} \frac{1}{\lVert \phi \rVert_2} V(\phi).$$

(54)

However, since both the functional and its approximation are quadratic, with $V(c\phi) = c^2V(\phi)$ for any $c \in \mathbb{R}$ in (15a) and $V_y^{(N)}(c\phi) = c^2V_y^{(N)}(\chi)$ in (8), definition (54) simplifies to

$$k_1^{(N)} = \min_{\phi \in \mathcal{C}} V_y^{(N)}(\{z/\varepsilon\}), \quad k_1^{\text{opt}} = \inf_{\phi \in \mathcal{C}} V(\phi).$$

(55)

Theorem 6.2: If Cond. 6.2 holds, then $k_1 = \limsup_{N \to \infty} k_1^{(N)}$ is a valid quadratic lower bound coefficient in (4).

Proof: Let $\phi_\delta$ give a $V(\phi_\delta)$ that is arbitrarily close to the infimum in (55) according to

$$\forall \delta > 0, \exists \phi_\delta \in \mathcal{C}, \lVert \phi_\delta \rVert_2 = \delta : V(\phi_\delta) < k_1^{\text{opt}} + \delta.$$ 

(56)

The assumed convergence (38), i.e., $\forall \phi \in \mathcal{C}, \forall \epsilon > 0, \exists N(\epsilon, \phi) \in \mathbb{N}, \forall N \geq N(\epsilon, \phi) : \lVert V_y^{(N)}(\pi_y(\phi)) \rVert - V(\phi) < \epsilon,$

shows that

$$\forall N \geq \tilde{N}(\varepsilon, \phi) : \lVert V_y^{(N)}(\pi_y(\phi)) \rVert - V(\phi) < \varepsilon,$$

(57)

and thus,

$$\forall N \geq \tilde{N}(\varepsilon, \phi) : \begin{cases} k_1^{(N)} \leq \min_{\phi \in \mathcal{C}} V_y^{(N)}(\{z/\varepsilon\}) \quad (55) \\ k_1^{\text{opt}} \leq \inf_{\phi \in \mathcal{C}} V(\phi) \end{cases}$$

(58)

Choosing $\delta = \varepsilon/2$, (58) becomes $k_1^{(N)} < k_1^{\text{opt}} + \varepsilon$. Hence, $\limsup_{N \to \infty} k_1^{(N)} \leq k_1^{\text{opt}}$. Any $k_1 \leq k_1^{\text{opt}}$ is admissible in (4).

For the Legendre tau method, we are going to prove that $k_1^{(N)}$ converges to the largest admissible coefficient $k_1^{\text{opt}}$. The proof involves the following assumption on the arguments of the minimum in (55): For any $N$, we can consider a vector $\{z/\varepsilon\}$, with $\lVert z/\varepsilon \rVert_2 = 1$, such that $V_y^{(N)}(\{z/\varepsilon\}) = k_1^{(N)}$. By (77), any $\{z/\varepsilon\}$ represents a function $\phi(\xi)$ (we use (77) since the minimizing argument is not expected to be continuous at $\theta = 0$). The assumption below is that $\phi(\xi)$ remains uniformly bounded in $L_2$, which, however, could numerically12 be confirmed for all tested examples that give a nonzero $k_1$.

Theorem 6.3: Consider the Legendre tau method with (37). As described above, for $\phi(\xi)$ being related to $k_1^{(N)}$, assume that $\exists \beta > 0, \forall N : \lVert \phi(\xi) \rVert_{L_2} < \beta$. Then the quadratic lower bound coefficient $k_1^{(N)}$ from Cor. 3.1 converges to the largest possible quadratic lower bound coefficient on the functional in (4).

Proof: We denote by $C_d$ the set of functions $\phi : [-h,0) \to \mathbb{R}$ that are continuous on $[-h,0)$ and possibly have a jump discontinuity at the end point $\phi(0^-) \neq \phi(0)$. Note that $\phi(\xi) \in C_d$. The functional $V : C \to \mathbb{R}$ can straightforwardly be extended to arguments in $C_d$ since $V(\phi) = V_{M_2}((\phi, \phi(0)))$ holds by (41), which, in fact, is defined for all $(\phi, \phi(0)) \in L_2 \times \mathbb{R}$. Also on this extended set of arguments, the value of interest from (55) is still the infimum $k_1^{\text{opt}} = \inf_{\phi \in C_d} V(\phi)$ (even on $L_2 \times \mathbb{R}$ it would be since $V_{M_2}$ is continuous11 in $M_2 = L_2 \times \mathbb{R}$ and $C$ is dense in $L_2$). With a slight abuse of notation we do not alter the name $V$ for the extension on $C_d$. By construction, the discretization $\pi_y(\phi(\xi)) = \{z/\varepsilon\}$ yields an argument of the minimum in (55). First, we have to show that $\forall \varepsilon > 0, \exists N_1(\varepsilon) \in \mathbb{N}$, such that

$$\forall N \geq N_1(\varepsilon) : \lVert V_y^{(N)}(\pi_y(\phi(\xi))) \rVert - V(\phi) < \varepsilon.$$ 

(59)

According to the splitting approach (Lemma 6.1 with $\hat{Q}_1 = Q_1, \hat{Q}_2 = Q_2$), we decompose $V$ into three parts $V(\phi(\xi)) = V_0(\phi(\xi)) + V_1(\phi(\xi)) + V_2(\phi(\xi))$ and its approximation correspondingly. The second and third term, $V_1(\phi(\xi))$ and $V_2(\phi(\xi))$, do not contribute to the error in (59) since $\phi(\xi)$ is an $(N - 1)$-th order polynomial on $\theta \in [-h,0)$ for which the approximation is exact, according to the lemmata of Appendix A.3.c. Therefore, it suffices to show uniform convergence of $\pi_y(\phi(\xi))$ for the approximations of $V_0$. Let $\psi(\xi) = (\psi_y(\xi), \phi(0)) \in M_2$. By assumption, $\lVert \psi(\xi) \rVert_{M_2} = \lVert \phi(\xi) \rVert_2^2 + \lVert \phi(0) \rVert_2^2 \leq \beta^2 + 1$. Thus, using (42) and (52), the error in (59) becomes $\lVert \psi(\xi) - \phi(\xi) \rVert_{M_2} \leq \lVert \phi(\xi) \rVert_{M_2} \leq \lVert \phi(\xi) \rVert_{M_2} \leq \lVert \phi(\xi) \rVert_{M_2} \leq \lVert \phi(\xi) \rVert_{M_2}$.

By Lemma 6.4, the latter converges to zero, and thus (59) holds. Consequently, $\forall N \geq N_1(\varepsilon)$,

$$k_1^{(N)} - V(\phi(\xi)) - \varepsilon \geq \inf_{\phi \in C_d} V(\phi) - \varepsilon \geq k_1^{\text{opt}}.$$ 

(60)

Choosing $\delta = \varepsilon/2$ from Thm. 6.2, we obtain

$$\forall N \geq \max\{N_0(\varepsilon), N_1(\varepsilon)\} : k_1^{\text{opt}} - \varepsilon < k_1^{(N)} < k_1^{\text{opt}} + \varepsilon,$$

completing the proof of $|k_1^{(N)} - k_1^{\text{opt}}| \to 0$ $(N \to \infty)$.

VII. CONCLUSION

The present paper shows that the counterpart of LK functionals for RFDEs are not classical Lyapunov functions for ODEs, but rather they correspond to partial Lyapunov functions, i.e., Lyapunov functions that prove partial stability. The latter are still simply obtained by solving a Lyapunov equation. Using the system matrix of an approximating ODE, the result gives an approximation of the LK functional $V(\phi)$. Note that Fig. 2 yields the structure of complete-type LK functionals without any prior knowledge. For an appropriate ODE approximation with a sufficiently large discretization resolution $N$, the involved matrix $P_y$ is positive semidefinite if and only if the RFDE equilibrium is asymptotically stable. A formula for a partial positive-definiteness bound on the functional approximation is derived. When it is applied to the Legendre-tau ODE-based result, a rapid convergence of the
resulting lower bound coefficient is observed as $N$ increases. Its limit is shown to be the best possible quadratic lower bound coefficient $k_1$ on the LK functional. Examples demonstrate that the latter significantly improves known results. In particular, the obtained $k_1$ depends on the delay, which is not the case in existing formulae. For the sake of validation, the present paper also proposes a numerical integration of the LK functional formula by Clenshaw-Curtis and Gauss quadrature rules. For these, the lower bound formula is purposeful as well. However, the ODE-based approach is expected to provide approximations of LK functionals even in more general cases where the LK functional is not known analytically.

APPENDIX

Table II classifies the employed polynomial methods. The following appendix also includes some implementation hints.

A. ODEs that Approximate RFDEs

We consider ODE approximations for (1) from two spectral methods: Chebyshev collocation and Legendre tau.

1) Chebyshev collocation method: By interpolation, the vector $y(t)$ at time $t$ in (5), cf. Fig. 1, determines an $N$-th order approximating polynomial for $x_t$. More specifically,

$$x_t(\theta) \approx \sum_{k=0}^{N} y^k(\theta) \ell_k(\theta),$$  

(61)

where $\ell_k: [-1, 1] \to \mathbb{R}$ are interpolating Lagrange basis polynomials w.r.t. the (Gauss-Lobatto) Chebyshev nodes $\{\theta_k\}_{k \in \{0, \ldots, N\}}$ on $[-1, 1]$, and where $\theta: [-h, 0] \to [-1, 1]; \ \theta \to \theta(\theta) := \frac{2}{h} \theta + 1$ maps the argument $\theta \in [-h, 0]$ to this interval.

The exact evolution of $x_t$ in Fig. 1b can be described by an abstract ODE in $C([-h,0], \mathbb{R}^n)$, see [3] for details. This abstract ODE can be discretized via the collocation method. The result describes the dynamics of the unknown coefficients $y^k(t)$ in (61). It is the ODE (7) with $A_y = A_y^C$,

$$A_y^C := \begin{bmatrix} \frac{2}{h} \ell'_0(\theta_0)I_n & \cdots & \cdots & \frac{2}{h} \ell'_N(\theta_0)I_n \\ \vdots & \ddots & \ddots & \vdots \\ \frac{2}{h} \ell'_0(\theta_{N-1})I_n & \cdots & \cdots & \frac{2}{h} \ell'_N(\theta_{N-1})I_n \\ I_n & \cdots & \cdots & I_n \end{bmatrix},$$  

(63)

cf. [25]. The upper part of $A_y^C$ that is given by $\frac{2}{h} \ell'_k(\theta_0) \in \mathbb{R}$ requires the first $N$ rows of the $(N+1) \times (N+1)$ differentiation matrix $(\ell'_k(\theta))_{k \in \{0, \ldots, N\}} \otimes I_n$ requires the first $N$ rows of the $(N+1) \times (N+1)$ differentiation matrix $(\ell'_k(\theta))_{k \in \{0, \ldots, N\}} \otimes I_n$. See [38, p. 54] (with $x_k = -\theta_0$).

Remark 1.1 (Implementation of $A_y^C$): A Matlab implementation of the skew-centrosymmetric differentiation matrix is available from diffmat in the Chebfun toolbox [59]. Based on the latter, $A_y^C = \mathbb{A}$ is obtained from

$$D = \text{diffmat}(N+1, [-\text{delay}, 0]); \ A = \text{kron}(D, \text{eye}(n)); \ A(\text{end}+n+1:end, :) = [A_0, \text{zeros}(n, n \times (N-1))], A_0$$

(if $A_0, A_1, h, n, N$ are assigned to $A_0, A_1, \text{delay}, n, N$). \footnote{Orthogonal w.r.t. the (weighted) inner product in which the chosen basis polynomials are orthogonal. In (73), the modified $\tilde{c}^N$ makes the projection non-orthogonal, unless the discretization is interpreted in terms of (77).}

2) Legendre tau method: Let $p_k: [-1, 1] \to \mathbb{R}$ denote the $k$-th Legendre polynomial. See, e.g., [57] for formulae and plots. Using $p_k(\theta(\cdot))_{k=0}^N$ as basis, an $N$-th order approximating polynomial for $x_t$ becomes

$$x_t(\theta) \approx \sum_{k=0}^{N} \zeta^k(t) p_k(\theta(\theta)).$$  

(64)

The evolution of the coefficients $\zeta^k(t) \in \mathbb{R}^n$, stacked as $\zeta := [(\zeta^0)^T, \ldots, (\zeta^N)^T]^T$, shall again be described by

$$\dot{\zeta}(t) = A_{\zeta} \zeta(t).$$  

(65)

In [29], this is achieved via Lanczos’ tau method (considering a Hilbert space setting, cf. Sec. VI-C). A general introduction to the tau method is given in [57]. We only state the result, which is (65) with $A_{\zeta} = A_{\zeta}^L$ having the block entries

$$A_{\zeta}^L_{jk} = \begin{cases} \frac{2}{h} (2j + 1) I_n, & \text{if } j \in \{0, \ldots, N-1\}, \\
A_0 + (-1)^k A_1 - \frac{2}{h} k(k+1) \frac{I_n}{2}, & \text{if } j = N \\
0_{n \times n}, & \text{else.} \end{cases}$$  

(66)

Thus, $A_{\zeta}^L$ exhibits the structure (exemplarily for $N$ even)

$$A_{\zeta}^L = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} \\
A_0 + A_1, A_0 - A_1, A_0 + A_1 & \cdots & A_0 - A_1 \end{bmatrix} + \frac{2}{h} I_n,$$

Remark 1.2 (Implementation of $A_{\zeta}^L$): Written in standard Matlab code, we obtain $A_{\zeta}^L = \mathbb{A}$ from

$$D = \text{zeros}(N+1, N+1); \ D(\text{end}, :) = (0:N) \times (1:N+1)/2; \ \text{for } j=0:N; \ D(j+1, j+1:end) = 2j+1; \end{D}$$

$$A = 2/\text{delay} \ast \text{kron}(D, \text{eye}(n)); \ \text{if } j \in \{0, \ldots, N-1\}, \text{else } A_j = \text{kron}(\text{ones}(1,N+1), A_0) + \text{kron}((-1)^j, A_1)$$

(cf. $A_0, A_1, \text{delay}, n, N$ as above).

Hence, we have the dynamics (65) of the Legendre coordinates $\zeta(t) \in \mathbb{R}^n(N+1)$ that, in (64), describe the approximating polynomial for $x_t$. However, we can equivalently express the polynomial (64) in interpolation coordinates $y(t) \in \mathbb{R}^n(N+1)$, referring to the interpolation basis $\{\ell_k(\theta(\cdot))\}_{k=0}^N$ of (61). Let $T_{y\zeta}$ denote the transformation matrix of this change of basis

$$y(t) = T_{y\zeta} \zeta(t).$$  

(67)

The thus computed $y(t)$ obeys the ODE (7) with $A_y = A_y^L$,

$$A_y^L = T_{y\zeta} A_{\zeta}^L T_{y\zeta}^{-1}.$$  

(68)

Note that the first and last block row of $T_{y\zeta}$ in (67) are simply

$$\begin{bmatrix} y^0(t) \\
y^N(t) \end{bmatrix} = \begin{bmatrix} I_n & -I_n & \cdots & (-1)^{N-1} I_n \\
I_n & I_n & \cdots & I_n \end{bmatrix} \zeta(t)$$  

(69)

since $p_k(-1) = (-1)^k$, $p_1(1) = 1$ (and $T_{y\zeta}^{jk} = p_k(\theta(\theta)) I_n$).

Remark 1.3 (Implementation of $A_{\zeta}^L$): Efficient conversion algorithms [59] are found in the Chebfun toolbox. Applying
these to the identity matrix yields \( T_{y\zeta} \) and \( T_{y\zeta} := T_{y\zeta}^{-1} \). Thus, 
\[ A_y = A_L \] can be derived by adding the lines 
\[ \text{Ty} = \text{ Kron(legcoefs2chebvals(eye(N+1)), eye(n))}; \]
\[ \text{Tcy} = \text{ Kron(chebvals2legcoefs(eye(N+1)), eye(n))}; \]
\[ A = \text{Ty} \times \text{Tcy} \]
to the code given in Remark 1.2.

3) Further notes on the Legendre tau method:

a) Lyapunov equation in Legendre coordinates: To obtain an
approximation of \( V(\zeta) \) via the Legendre tau approach, we
can resort to (10) with \( A_y = A_L \) from (68). However, from a
numerical point of view, it might be preferable to remain in
Legendre coordinates \( \zeta \) and use \( A_L = A_L^\zeta \), (66), in

\[ V(\zeta) := \zeta^T P \zeta, \quad P = P^\zeta \in \mathbb{R}^{(N+1)\times(N+1)} \]  

(70)

\[ D^T \zeta V(\zeta) = \zeta^T (P \zeta A + A^T P \zeta) \zeta = -\zeta^T T_{y\zeta}^{-1} Q_{y\zeta} T_{y\zeta} \zeta, \quad \forall \zeta \in \mathbb{R}^{(N+1)} \text{ with } T_{y\zeta} \text{ from (67). That is, we solve} \]

\[ P \zeta A + A^T P \zeta = -T_{y\zeta}^{-1} Q_{y\zeta} T_{y\zeta} \zeta \]  

(71)

for \( P \zeta \) and, if desired, express the result in \( y \) coordinates

\[ V(y) = V(\zeta) = V(\zeta(T_{y\zeta}^{-1} y)) = y^T (T_{y\zeta}^{-1})^T P \zeta (T_{y\zeta}^{-1} y) = p \]  

(72)

With \( Q_y \) from (37), only the first and last block rows (69) of
\( T_{y\zeta} \) are required in (71).

b) Discretization: The discretization \( \zeta = (\zeta^k)_{k \in \{0, \ldots, N\}} \)
of a numerical function \( \phi \), e.g., an initial condition \( x_0 = \phi \) or an
argument of the functional \( V(\zeta) \), is chosen as

\[ \zeta^k = \tilde{\zeta}^k, \quad \text{if } k < N, \quad \text{and } \zeta^N = \phi(0) - \sum_{k=0}^{N-1} \tilde{\zeta}^k, \]  

(73)

where \( \{\tilde{\zeta}^0, \ldots, \tilde{\zeta}^{N-1}\} \) stem from a truncation of the Legendre
series representation \( \phi(\theta) = \sum_{k=0}^\infty \tilde{\zeta}^k p_k(\theta(\theta)) \), [29]. The last
component \( \zeta^N \) in (73) is such that the \( N \)-th order approximating
polynomial \( \sum_{k=0}^N \tilde{\zeta}^k p_k(\theta(\theta)) \approx \phi(\theta) \), at \( \theta = 0 \), exactly
matches \( \phi(0) \) (note that \( \theta(1) = 1 \) and \( p_k(1) = 1, \forall k \)).

c) \( V_1 \) and \( V_2 \): For two important cases of the right-hand
side \( -Q_1 = -T_{y\zeta}^{-1} Q_{y\zeta} T_{y\zeta} \) in (71), we can give the solution \( P \zeta \),
respectively the resulting \( V_1(y) = V_1(\zeta) \approx V(\phi) \), analytically.

\[ \text{Lemma 1.1: The Legendre-based approximation of} \]

\( V_1(\phi) \) in Lemma 6.1 becomes \( V_1(\tilde{\phi}^{(N-1)}) \), where
\( \tilde{\phi}^{(N-1)}(\theta) = \sum_{k=0}^{N-1} \tilde{\zeta}^k p_k(\theta(\theta)) \) is the \( (N-1) \)-th order
Legendre series truncation of \( \phi(\theta) = \sum_{k=0}^\infty \tilde{\zeta}^k p_k(\theta(\theta)) \). ▶
at \( \theta = 0 \) is of interest, it is convenient to consider as approximating function instead the piecewise defined \((N - 1)\)-th order polynomial with a discontinuous end point

\[
\phi(\theta) \approx \left\{ \begin{array}{ll}
\sum_{k=0}^{N-1} \varepsilon_k p_k(\phi(\theta)), & \text{if } \theta < 0 \\
\sum_{k=0}^{N} \varepsilon_k p_k(\phi(\theta)), & \text{if } \theta = 0
\end{array} \right.
\] (77)

(which, in (73), has the same discretization).

**B. Numerical Integration of LK Functionals**

Sec. II-E proposes to apply interpolatory quadrature rules to the LK functional. We consider Clenshaw-Curtis and Gauss quadrature. See, e.g., [58] for convergence statements.

1) **Clenshaw-Curtis quadrature**: A numerical integration of (15a) by an interpolatory quadrature rule replaces integrals by weighted sums from values at certain grid points. If these grid points are the (Gauss-Lobatto) Chebyshev nodes \( \{\theta_k\}_{k=0}^{N} \) introduced in (6), this amounts to a Clenshaw-Curtis quadrature, cf. [56, Sec. 3.7]. The weights \( w_k \) are, e.g., available \(^{16}\) in the Chebfun toolbox [39]. For (15a), we obtain

\[
V(\phi) \approx \sum_{k=0}^{N} w_k \phi^T(\theta_k) \sum_{j=0}^{N} w_j \phi(\theta_j) + \sum_{k=0}^{N} w_k \phi^T(\theta_k) P_{xk}(\theta_j) \phi(\theta_k)
\]

(78)

Let \( y^k = \phi(\theta_k), k \in \{0, \ldots, N\} \), where \( y^N = \phi(\theta_N) = \phi(0) \). As in (13), the result (78) can be written as a quadratic form (with \( p = \text{dim}(z) : = \text{dim}(\begin{bmatrix} y^0 & \ldots & y^{N-1} \end{bmatrix}^T) = nN \))

\[
V(\phi) \approx y^T P_{quad} y = y^T \begin{bmatrix}
0_{p \times p} & 0_{p \times n} \\
0_{n \times p} & P_{xx}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0_{p \times (p+n)} \\
(P_{xx}(\theta_k) w_k)_k
\end{bmatrix} + \begin{bmatrix}
0_{(p+n) \times p} \\
(P_{xz}(\theta_j) w_j)_j
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
(\phi^T(\theta_k) P_{xk}(\theta_j) \phi(\theta_k))_k \\
(\phi^T(\theta_k) P_{xk}(\theta_j) \phi(\theta_k))_j
\end{bmatrix} + \text{blkdiag} \left( \begin{bmatrix}
(\phi^T(\theta_k) P_{xz}(\theta_j) \phi(\theta_k))_k \\
(\phi^T(\theta_k) P_{xz}(\theta_j) \phi(\theta_k))_j
\end{bmatrix} \right)
\]

(79)

See (24) for a factorization taking (15b) into account. Note that the right lower component of \( P_{quad} \) approximately becomes \( P_{xx} \) since the other contributions are weighted by \( w_N \), which is quite small in the non-equidistant grid.

2) **Gauss quadrature**: As an alternative, we apply (Legendre) Gauss quadrature. Thus, the integral of a function is approximated by weighted sums from the function values at (Gauss) Legendre nodes. Being Gauss nodes, cf. Table II, they do not contain the boundary points of the domain \([-h, 0]\). That is why we take \( N \) (Gauss) Legendre nodes \( \theta_k \), and add the zero end point with zero weight to get the

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