NEW SERIES OF MODULI COMPONENTS OF RANK 2 SEMISTABLE SHEAVES ON $\mathbb{P}^3$ WITH SINGULARITIES OF MIXED DIMENSION

A. IVANOV-VIAZEMSKY

We construct a new infinite series of irreducible components of the Gieseker-Maruyama moduli scheme $M(k)$, $k \geq 3$ of coherent semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = k$, $c_3 = 0$ on $\mathbb{P}^3$ generic points of which correspond to sheaves with singularities of mixed dimension. These sheaves are constructed by elementary transformations of stable and properly $\mu$-semistable reflexive sheaves along disjoint union of collections of points and smooth irreducible curves which are rational or complete intersection curves. As a special member of this series we obtain a new component of $M(3)$.

2010 MSC: 14D20, 14J60

Keywords: Rank 2 stable sheaves, Reflexive sheaves, Moduli space.

1. INTRODUCTION

Let $M(0, k, 2n)$ be the Gieseker-Maruyama moduli scheme of semistable rank-2 sheaves with Chern classes $c_1 = 0$, $c_2 = k$, $c_3 = 2n$ on the projective space $\mathbb{P}^3$. Denote $M(k) = M(0, k, 0)$. By the singular locus of a given $\mathcal{O}_{\mathbb{P}^3}$-sheaf $E$ we understand the set $\text{Sing}(E) = \{x \in \mathbb{P}^3 \mid E \text{ is not locally free at the point } x\}$. $\text{Sing}(E)$ is always a proper closed subset of $\mathbb{P}^3$ and, moreover, if $E$ is a semistable sheaf of nonzero rank, every irreducible component of $\text{Sing}(E)$ has dimension at most 1. Also for simplicity we will not make a distinction between a stable sheaf $E$ and corresponding isomorphism class $[E]$ as a point of moduli scheme.

Any semistable rank-2 sheaf $[E] \in M(k)$ is torsion-free, so it satisfies the exact triple

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q \rightarrow 0,$$

where $E^{\vee\vee}$ is a reflexive sheaf and $\dim \text{Supp}(Q) \leq 1$. Conversely, take a reflexive sheaf $F$, subscheme $X \subset \mathbb{P}^3$, $\mathcal{O}_X$-sheaf $Q$ and surjective map $\phi : F \rightarrow Q$, then it is very likely that the kernel sheaf $E := \ker \phi$ is semistable. We call a sheaf $E$ an elementary transform of $F$ along $X$. In general, an elementary transform of a sheaf $F$ can be defined as follows.

**Definition 1.** An elementary transform of a sheaf $F$ along an element $[F \xrightarrow{\phi} Q] \in \text{Quot}^P(F)$ of corresponding Quot-scheme is a sheaf $E := \ker \phi$.

In fact, all known irreducible components of the moduli schemes $M(k)$ generic points of which correspond to non-locally free sheaves are constructed by using elementary transformations of stable reflexive sheaves.
More precisely, in [7] there were found two infinite series $T(k, n)$ and $C(d_1, d_2, k - d_1 d_2)$ of irreducible components of $\mathcal{M}(k)$ which (generically) parameterize stable sheaves with singularities of dimension 0 and pure dimension 1, respectively. Generic points of components of the first series are elementary transforms of stable reflexive sheaves along unions of $n$ distinct points in $\mathbb{P}^3$, while the ones of the second series are elementary transforms of instanton bundles of charge $k - d_1 d_2$ along smooth complete intersection curves of degree $d_1 d_2$.

Next, in [8] there were constructed three components of $\mathcal{M}(3)$ parameterizing sheaves with singularities of mixed dimension. Generic sheaves of these components are elementary transforms of stable reflexive sheaves with Chern classes $(c_2, c_3) = (2, 2)$, $(2, 4)$ along disjoint union of a projective line and a collection of points in $\mathbb{P}^3$. This approach was generalized in [9] by considering stable reflexive sheaves with other Chern classes to construct infinite series of components of $\mathcal{M}(-1, c_2, c_3)$.

Also it is worth to note that in [6] there were constructed the certain collections of divisors of the boundaries $\partial \mathcal{I}(k) = \overline{\mathcal{I}(k)} \setminus \mathcal{I}(k)$ of instanton components of $\mathcal{M}(k)$ for each $k$. Generic sheaves of these divisors are elementary transforms of instanton bundles along rational curves.

The present paper is devoted to further generalization of these results. Namely, we construct an infinite series of irreducible moduli components which includes the components parameterizing non-locally free sheaves constructed in [6, 7, 8] as special cases. As in loc. cit., the construction is essentially using of elementary transformations of reflexive sheaves. The first new feature of this construction is that we use not only stable sheaves, but also properly $\mu$-semistable reflexive sheaves which are not semistable. The second is that we do elementary transformations along disjoint union of a collection of distinct points and a smooth irreducible curve in $\mathbb{P}^3$, where a curve is rational or complete intersection curve. As a consequence, generic sheaves of these components have singularities of mixed dimension (except those which coincide with components constructed in [6, 7]). Since a complete enumeration of components of $\mathcal{M}(k)$ for small values of $k$ is of particular interest, it is worth to note that this series contains a new component of $\mathcal{M}(3)$. The reason this component was not discovered in [8] is that the properly $\mu$-semistable reflexive sheaves were not considered there.

The paper is organized in the following way. In Section 2 the necessary facts about moduli spaces of stable reflexive sheaves are given. Also in this section the construction of parameter space of properly $\mu$-semistable reflexive sheaves and some formulas for computation of Ext-groups of these
sheaves are presented. Section 3 is devoted to the description of the new series of moduli components. More precisely, there we describe some families of semistable sheaves, then define scheme structure on them and show the existence of natural inclusion of these schemes to $\mathcal{M}(k)$ for some $k$. The proof of the fact that these subschemes are irreducible components is also given in this section. Finally, in Section 4 we enumerate the components of this series and describe particular case of the new component of $\mathcal{M}(3)$.

Acknowledgements. The work was supported in part by Young Russian Mathematics award and by the Russian Academic Excellence Project 5-100. I would like to thank A. S. Tikhomirov and D. Markushevich for usefull discussions.

2. Some properties of reflexive sheaves

Since there are two definitions of stability, the one is due to Mumford and the other is due to Gieseker-Maruyama, and we will use both of them, it would be helpfull to recall the corresponding definitions.

Definition 2. A coherent sheaf $F$ on $\mathbb{P}^3$ is said to be (semi)stable ($\mu$-(semi)stable) if and only if it is torsion free and for all proper coherent subsheaves $G$ the inequality

$$\frac{P(G,m)}{\text{rk } G} (\leq) < \frac{P(F,m)}{\text{rk } F}, \quad \left( \frac{c_1(G)}{\text{rk } G} (\leq) < \frac{c_1(F)}{\text{rk } F} \right)$$

holds for all large integers $m$, where $P$ is Hilbert polynomial of a sheaf.

Remark 1. There are the following implications

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable.}$$

It is known that there exists the moduli scheme $\mathcal{R}(0, m, 2n)$ parameterizing stable reflexive rank-2 sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0$, $c_2 = m$, $c_3 = 2n$. Since $\mathcal{R}(0, m, 2n)$ is open subset of the moduli scheme $\mathcal{M}(0, m, 2n)$ (see [4]), it is quasi-projective. Moreover, according to [3], for $(m,n) = (2,1), (2,2), (3,4)$ this scheme is smooth, irreducible and rational; for $(m,n) = (3,2)$ it is irreducible and reduced at generic point; for $(m,n) = (3,1), (3,3)$ the corresponding reduced scheme is irreducible.

At the paper [7] the infinite series of irreducible components $\mathcal{S}(a,b,c)$ of the moduli schemes $\mathcal{R}(0, m, 2n)$ is described. Sheaves of these components satisfy the following exact triple

$$0 \to a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (a+b+c+2) \cdot \mathcal{O}_{\mathbb{P}^3} \to F(k) \to 0,$$
where \( a, b, c \) are arbitrary non-negative integers such that \( 3a + 2b + c \) is non-zero and even, \( k := \frac{3a + 2b + c}{2} \). The corresponding Chern classes of these sheaves can be expressed through integers \( a, b, c \) by the following way

\[
(1) \quad m = m(a, b, c) = \frac{1}{4}(3a + 2b + c)^2 + \frac{3}{2}(3a + 2b + c) - (b + c),
\]

\[
(2) \quad 2n = 2n(a, b, c) = 27\left(\frac{a + 2}{3}\right) + 8\left(\frac{b + 2}{3}\right) + \left(\frac{c + 2}{3}\right) + 3(3a + 2b + 5)ab + \frac{3}{2}(2a + c + 4)ac + (2b + 3c + 3)bc + 6abc.
\]

The components \( S(a, b, c) \) are smooth and of expected dimension \( 8m - 3 \). In particular, the last property implies that \( \text{Ext}^2(F, F) = 0 \) for any sheaf \( [F] \in S(a, b, c) \).

Moreover, there exists the scheme \( R_{s,s}(0, m, 2n) \) parameterizing reflexive properly \( \mu \)-semistable sheaves with the corresponding Chern classes. This scheme can be constructed by the following way. Firstly, note that any reflexive properly \( \mu \)-semistable sheaf \( F \) has a global section, so it satisfies the exact triple

\[
(3) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow F \rightarrow I_Y \rightarrow 0,
\]

where \( Y \) is Cohen-Macaulay curve which is locally complete intersection at generic point. Conversely, for any such curve \( Y \) with the property that there exists a global section \( \xi \in H^0(\omega_Y(4)) \) which generates the sheaf \( \omega_Y(4) \) except at finitely many points, we can construct the sheaf \( F \) as an extension of the form \( [3] \). This sheaf will be reflexive and properly \( \mu \)-semistable. So considering the Hilbert scheme \( \text{Hilb}_{d,g}(\mathbb{P}^3) \) of algebraic space curves of degree \( d \) and the arithmetic genus \( g \) in \( \mathbb{P}^3 \) we can construct the scheme \( \mathbb{P}(\text{pr}_*\omega_Z(4)) \), where \( Z \hookrightarrow \text{Hilb}_{d,g}(\mathbb{P}^3) \times \mathbb{P}^3 \) is a universal curve and \( \text{pr} : \text{Hilb}_{d,g}(\mathbb{P}^3) \times \mathbb{P}^3 \rightarrow \mathcal{H} \) is a projection on the first term. The closed points of this scheme will be in one-to-one correspondence with properly \( \mu \)-semistable reflexive sheaves with fixed Chern classes.

Now take the irreducible component of the Hilbert scheme \( \text{Hilb}_{d,g}(\mathbb{P}^3) \) generic point of which is reduced and corresponds to a smooth irreducible curve. Then it gives us the irreducible component of \( R_{s,s}(0, m, 2n) \). Let us show that the dimension of \( \text{Ext}^1(F, F) \) is equal to the dimension of this irreducible component. In order to do this apply the functor \( \text{Hom}(-, F) \) to the triple \( [3] \), then we obtain the exact sequence

\[
(4) \quad 0 \rightarrow \text{Hom}(I_Y, F) \rightarrow \text{Hom}(F, F) \rightarrow H^0(F) \rightarrow
\]
\[ \rightarrow \text{Ext}^1(I_Y, F) \rightarrow \text{Ext}^1(F, F) \rightarrow \text{H}^1(F). \]

From the fact that the triple (3) is not splitting we can deduce that \( \text{Hom}(I_Y, F) \simeq \text{Hom}(I_Y, O_{\mathbb{P}^3}) \simeq k \). Next, it is easy to see that \( \text{H}^0(F) \simeq k \). Moreover, it is known that \( \text{Hom}(F, F) \simeq k^2 \) (see [3]). Since the curve \( Y \) is smooth and irreducible we also have \( \text{H}^1(F) \simeq \text{H}^1(I_Y) \simeq 0 \). Therefore, we obtain the isomorphism \( \text{Ext}^1(F, F) \simeq \text{Ext}^1(I_Y, F) \). Now after applying the functor \( \text{Hom}(I_Y, -) \) to the triple (3) we have the exact sequence

\[ 0 \rightarrow \text{Hom}(I_Y, O_{\mathbb{P}^3}) \rightarrow \text{Hom}(I_Y, F) \rightarrow \text{Hom}(I_Y, I_Y) \rightarrow \]
\[ \rightarrow \text{Ext}^1(I_Y, O_{\mathbb{P}^3}) \rightarrow \text{Ext}^1(I_Y, F) \rightarrow \text{Ext}^1(I_Y, I_Y) \rightarrow \text{Ext}^2(I_Y, O_{\mathbb{P}^3}). \]

Taking into account the fact that the structure sheaf \( O_Y \) has homological dimension 2 we conclude that \( \text{Ext}^2(I_Y, O_{\mathbb{P}^3}) \simeq \text{Ext}^3(O_Y, O_{\mathbb{P}^3}) \simeq 0 \). Next, it is easy to check that \( \text{Hom}(I_Y, I_Y) \simeq k \) and \( \text{Ext}^1(I_Y, O_{\mathbb{P}^3}) \simeq \text{H}^0(\text{Ext}^1(I_Y, O_{\mathbb{P}^3})) \simeq \text{H}^0(\text{Ext}^2(O_Y, O_{\mathbb{P}^3})) \simeq \text{H}^0(\omega_Y(4)) \). Moreover, we have \( \text{Ext}^1(I_Y, I_Y) \simeq \text{Ext}^1(O_Y, I_Y) \simeq T_{[Y]} \text{Hilb}_{d,g} \). Finally, we obtain the following exact sequence

\[ 0 \rightarrow k \rightarrow \text{H}^0(\omega_Y(4)) \rightarrow \text{Ext}^1(F, F) \rightarrow T_{[Y]} \text{Hilb}_{d,g} \rightarrow 0 \]

from which it obviously follows that \( \dim \text{Ext}^1(F, F) \) is equal to the dimension of our irreducible component.

Also it will be useful for us to compute \( \dim \text{Ext}^1(F, F) \). At first, note that for any reflexive sheaf \( F \) we have the following formula which can be derived using the Local-to-Global spectral sequence

\[ \sum_{i=0}^{3} \dim \text{Ext}^i(F, F) = -8c_2(F) + 4. \]

Considering the remaining part of the exact sequence (3) we can deduce that \( \text{Ext}^3(F, F) \simeq 0 \) because of \( \text{H}^3(F) \simeq \text{H}^3(I_Y) \simeq 0 \) and \( \text{Ext}^3(I_Y, F) \simeq \text{Ext}^3(I_Y, I_Y) \simeq 0 \). On the other hand, we know that \( \text{Hom}(F, F) \simeq k^2 \) and \( \dim \text{Ext}^1(F, F) \) is equal to the dimension of irreducible component, i.e. \( h^0(N_{Y/\mathbb{P}^3}) + h^0(\omega_Y(4)) - 1 \). Taking into account easy verifiable equality \( c_2(F) = \deg(Y) \) we obtain the following formula

\[ \dim \text{Ext}^2(F, F) = 1 + h^0(N_{Y/\mathbb{P}^3}) + h^0(\omega_Y(4)) - 8\deg(Y). \]

Further we will be interested at irreducible components of \( \mathcal{R}_{s,s}(0, m, 2n) \) for generic point of which we have \( \text{Ext}^2(F, F) = 0 \). This condition is satisfied by components of \( \mathcal{R}_{s,s}(0, d', 4d' - 2) \) which are constructed by rational
curves of degree $d'$. More precisely, there exists the open smooth irreducible subset of $\text{Hilb}_{d',0}(\mathbb{P}^3)$ which parameterizes rational curves $Y$ of degree $d'$ with the property $N_Y \simeq 2\mathcal{O}_Y((2d'-1)pt)$ (see [2]). After substitution such curve to the formula (6) we easily obtain that $\text{Ext}^2(F,F) = 0$ for corresponding properly $\mu$-semistable reflexive sheaf $F$. We will denote these components by $\mathcal{P}(d') \subset \mathcal{R}_{s,s}(0,d',4d'-2)$.

3. CONSTRUCTION OF COMPONENTS

Fix some component $\mathcal{R}$ from the series $\mathcal{S}(a,b,c)$ or $\mathcal{P}(d')$. Next, denote by $\mathcal{H}_0 = \text{Sym}^s_*(\mathbb{P}^3)$ open smooth subset of the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^3$ parameterizing unions $W = \{x_1, \ldots, x_s \mid x_i \neq x_j\}$ of $s$ distinct points in $\mathbb{P}^3$. Similarly, denote by $\mathcal{H}_1$ open smooth irreducible subset of the Hilbert scheme parameterizing smooth irreducible curves $C$ which are rational or complete intersection curves of degree $d$ and genus $g$ in $\mathbb{P}^3$. On the integers we impose the following restrictions

\begin{equation}
\mathcal{R} = \mathcal{S}(a,b,c) \Rightarrow \begin{cases} 
  s < n(a,b,c) & \text{if } C \text{ is rational}, \\
  s \leq n(a,b,c) & \text{if } C \text{ is complete intersection}, 
\end{cases}
\end{equation}

\begin{equation}
\mathcal{R} = \mathcal{P}(d') \Rightarrow d > d' \text{ and } \begin{cases} 
  s < 2d' - 1 & \text{if } C \text{ is rational}, \\
  s \leq 2d' - 1 & \text{if } C \text{ is complete intersection}.
\end{cases}
\end{equation}

Since for a smooth curve invertible sheaves and rank-1 stable sheaves are the same objects, the relative Jacobian functor $J^P : (\text{Sch}/\mathcal{H}_1) \rightarrow (\text{Sets})$ defined as

$$J^P(T) = \{T\text{-flat invertible sheaves } F \text{ on } \mathcal{Z} \times_{\mathcal{H}_1} T \text{ such that}
$$

all restrictions $F|_{C_t \times t}$ have Hilbert polynomial $P}\}/\text{Pic}(T)$$

is corepresented by the $\mathcal{H}_1$-scheme $\mathbb{J}^P$ because it is equal to the Maruyama moduli functor. Further we will consider the following Hilbert polynomial

$$P(k) = g-1+2d+n-s+dk,$$

and will denote $\mathbb{J}^P$ just by $\mathbb{J}$. From the set-theoretical point of view the scheme $\mathbb{J}$ has the following form

$$\mathbb{J} = \{(C,L) \mid C \in \mathcal{H}_1, \ L \in \text{Pic}^{g-1+2d+n-s}(C)\}.$$

For the case of smooth rational curves we obviously have $\mathbb{J} \simeq \mathcal{H}_1$.

**Theorem 1.** The points of $\mathcal{R} \times \mathbb{J} \times \mathcal{H}_0$ satisfying the following conditions

\begin{equation}
\end{equation}

$$C \cap W = \emptyset, \ Sing(F) \cap (C \cup W) = \emptyset,$$

$$C \cap W = \emptyset, \ Sing(F) \cap (C \cup W) = \emptyset,$$
NEW SERIES OF MODULI COMPONENTS OF RANK 2 SEMISTABLE SHEAVES ON $\mathbb{P}^3$ WITH SINGULARITIES OF MIXED DIMENSION

(9) $h^1(\operatorname{Hom}(F, L)) = 0,$

(10) $\operatorname{Hom}_e(F, L \oplus \mathcal{O}_W) \neq 0,$

(11) $h^0(\omega_C(4) \otimes L^{-2}) = 0$

form open dense subset $\mathcal{B} \subset \mathcal{R} \times \mathcal{J} \times \mathcal{H}_0.$

Proof: Note that all these conditions are open, so we only need to prove that each of them is non-empty because of irreducibility of the scheme $\mathcal{R} \times \mathcal{J} \times \mathcal{H}_0.$ Since singularities of reflexive sheaves have dimension 0, the first condition is obviously non-empty. Now let us prove that there exist a sheaf $[F] \in \mathcal{R}$ and a pair $(C, L) \in \mathcal{J}$ satisfying the equality (9).

In order to do this we consider a flat family of smooth curves $C_t \in \mathcal{H}_1,$ $t \neq 0$ degenerating to a configuration of $d$ lines $C_0 = \bigcup_{i=1}^{d} l_i.$ For the cases of rational curves and complete intersection curves such families do exist. In other words, it means that there exists a curve $Y \hookrightarrow \operatorname{Hilb}_{d,g}(\mathbb{P}^3)$ with marked point $0 \in Y,$ such that $Y \setminus \{0\} \subset \mathcal{H}_1$ and the point 0 corresponds to a configuration of lines $C_0.$ In [7] it was shown using Esteves' compactification of a relative Jacobian that over the universal curve $Z|_Y$ there exists the line bundle $\tilde{L}$ satisfying the following properties:

- for $t \neq 0$: $\tilde{L}_t \in \operatorname{Pic}^{g-1}(C_t)$ and
  
  $$ h^0(\tilde{L}|_{C_t}) = h^1(\tilde{L}|_{C_t}) = 0, \quad \tilde{L}|_{C_t} \otimes 2 \neq \omega_C, $$

- $\tilde{L}_0$ is a line bundle over $C_0$ which can be defined by the following sequence of extensions

$$ 0 \longrightarrow (\tilde{L}_0)_{i-1} \longrightarrow (\tilde{L}_0)_i \longrightarrow \mathcal{O}_{l_i}(-1) \longrightarrow 0, $$

where $(\tilde{L}_0)_1 := \mathcal{O}_{l_i}(-1), (\tilde{L}_0)_d = \tilde{L}_0.$

According to the Grauert-Mullich theorem (see [5]), for any stable reflexive sheaf $[F] \in \mathcal{R}$ there exists an open dense subset of Grassmanian of lines in $\mathbb{P}^3$ such that all restrictions of the sheaf $F$ on its lines are trivial. So it is obvious that there is the sheaf $[F] \in \mathcal{R}$ which is trivial on every line $l_i$ of configuration $C_0,$ i.e.

$$ F|_{l_i} \simeq 2\mathcal{O}_{l_i}, \quad i = 1, \ldots, d.$$
From this it is easy to see that
\[ h^1(\text{Hom}(F, \mathcal{O}_{H_i}(-1))) = 0, \quad i = 1, \ldots, d. \]

Using the sequence of extensions (12) it immediately implies the equality
\[ h^1(\text{Hom}(F, \tilde{\mathcal{L}}_0)) = 0. \]

Taking into account the upper-semicontinuity of the rank of cohomologies we can deduce that this equality holds also for sheaves \( \tilde{\mathcal{L}}_t \), where \( t \) belongs to some open subset \( U \subset Y, \quad 0 \in U \). Now fix a plane \( H \subset \mathbb{P}^3 \) which intersects \( C_0 \) at \( d \) points, then \( H \) transversally intersects the curve \( C_t \) for any \( t \) from some open subset of \( U \). After corresponding etale base change, we can take a cross-section \( \{x_t\} \) and define a line bundle \( L \) such that \( L_t \cong \tilde{\mathcal{L}}_t((H \cap C_t) + (d + n - s) x_t) \). Obviously, the line bundle \( L_0 \) over \( C_0 \) satisfies the following exact triple
\[ 0 \to \tilde{\mathcal{L}}_0 \to L_0 \to \mathcal{O}_{H \cap C_0} \oplus (d + n - s)\mathcal{O}_{x_0} \to 0, \]
from which we immediately obtain the equality
\[ h^1(\text{Hom}(F, L_0)) = 0. \]

Again using the upper-semicontinuity we can conclude that this equality holds for \( L_t \), where \( t \) belongs to some open subset. Since \( L_t \in \mathcal{J} \) for \( t \neq 0 \) it proves that the property (9) is non-empty.

Now proceed to the condition (10). It is easy to see that the sheaf \( L_0 \) constructed above also satisfies the triple
\[ 0 \to \tilde{\mathcal{L}}_0(1) \to L_0 \to (d + n - s)\mathcal{O}_{x_0} \to 0. \]

Using the sequence of extensions (12) we obtain that \( \text{Hom}_e(F, \tilde{\mathcal{L}}_0(1)) \neq 0 \). Since \( L_0 \) is a locally-free \( \mathcal{O}_{C_0} \)-sheaf it implies that \( \text{Hom}_e(F, L_0) \neq 0 \) as well. So it is also true for some open subset of \( Y \). Obviously, for any \( W \in \mathcal{H}_0 \) not intersecting \( C_t \) we also have \( \text{Hom}_e(F, \mathcal{O}_W \oplus \mathcal{L}_t) \neq 0 \).

Finally, let us prove that the condition (11) is non-empty. Note that for any pair \((C, L) \in \mathcal{J}\) we have the following equality
\[ \deg(\omega_C(4) \otimes L^{-2}) = 2g - 2 + 4d - 2(g - 1 + 2d + n - s) = 2(s - n). \]

If \( C \) is a rational curve than the condition (7) implies that the line bundle \( \omega_C(4) \otimes L^{-2} \) has negative degree, so it has no global sections. On the other hand, if \( C \) is a complete intersection curve and \( s = n \), then the line bundle \( L \) can be chosen in such way that \( L \cong \tilde{L}(2) \), where \( \tilde{L} \) is not a
theta-characteristic, i.e. $\widetilde{L}^\otimes 2 \neq \omega_C$. Then the line bundle $\omega_C(4) \otimes L^{-2}$ has degree 0, but it is not trivial, so it has no global sections also.

Therefore, the statement of the theorem is proven. □

Now for any triple $(F, W \sqcup C, L) \in \mathcal{B}$ and surjective morphism $\phi \in \text{Hom}_e(F, \mathcal{O}_W \oplus L)$ we can consider the sheaf $E := \ker \phi$. Let us show that this sheaf is stable.

For this purpose it is enough to consider the subsheaves $G \subset E$ without torsion, such that the quotient sheaf $E/F$ also has no torsion. Moreover, since the sheaf $F$ is $\mu$-semistable, we can consider only the case $c_1(G) = 0$. Hence, the sheaf $G$ is a sheaf of ideals (because $G \hookrightarrow G^{\vee \vee} \simeq \mathcal{O}_{\mathbb{P}^3}$) of some subscheme $\Delta \subset \mathbb{P}^3$ of the dimension no more than 1. Now the condition $\text{Sing}(F) \cap (W \sqcup C) = \emptyset$ implies that $E^{\vee \vee} \simeq F$. On the other hand, taking of double dual sheaf is a natural transformation, so we have the following diagram

$$
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
0 & \to I_\Delta & \to \mathcal{O}_{\mathbb{P}^3} \\
\downarrow & \downarrow \\
0 & \to E & \to F
\end{array}
$$

(14)

From this diagram the strict inequality $h^0(F) > 0$ follows, which is impossible for stable reflexive sheaves. On the other hand, the properly $\mu$-semistable sheaves have the global sections, but from the triple (3) it follows that $h^0(F) = 1$. So the diagram above can be written in the following form

$$
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to I_\Delta & \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_\Delta \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to E & \to F & \to \mathcal{O}_W \oplus L & \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to T & \to I_Y \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

(15)
From this we immediately conclude that $\Delta \subset W \sqcup C$. Since $C$ is irreducible curve, then only following cases are possible: $\Delta \cap C = \emptyset$ or $C$. The first case leads to contradiction because there is no surjection $I_Y \twoheadrightarrow L$. The second case is not destabilizing due to the computation

$$\frac{1}{2}P(E) - P(I_{C \sqcup W'}) = \frac{d - d'}{2}k + \text{const} > 0.$$ 

Therefore, the sheaf $E$ is stable and defines the point $[E] \in \mathcal{M}(m + d)$ of the moduli scheme. Moreover, two different triples $(F, W \sqcup C, L)$, $(F', W' \sqcup C', L') \in \mathcal{B}$ give the same isomorphism class $[E \cong E'] \in \mathcal{M}(m + d)$ if and only if $W = W'$, $C = C'$, and there exist automorphisms $\phi \in \text{Aut}(F)$, $\psi \in \text{Aut}(\mathcal{O}_W \oplus L)$ which complete the commutative diagram

$$
\begin{array}{c}
0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_W \oplus L \longrightarrow 0 \\
\downarrow \cong \downarrow \phi \downarrow \psi \\
0 \longrightarrow E' \longrightarrow F' \longrightarrow \mathcal{O}_{W'} \oplus L' \longrightarrow 0
\end{array}
$$

It means that the elements of the following set

$$Q := \left\{ ([F], C \sqcup W, L, \phi) \mid ([F], C \sqcup W, L) \in \mathcal{B}, \phi \in \text{Hom}_e(F, \mathcal{O}_W \oplus L)/\text{Aut}(F) \times \text{Aut}(\mathcal{O}_W \oplus L) \right\}$$

are in one-to-one correspondence with the points of some subset of closed points of the moduli scheme $\mathcal{M}(m + d)$.

**Theorem 2.** The set $Q$ can be endowed with structure of algebraic scheme. Moreover, there exists the open inclusion of schemes $Q \hookrightarrow \mathcal{M}(k)$.

**Proof:** In short, we can define a scheme structure on $Q$ considering it as a quotient bundle of morphisms between some universal sheaves. The problem is that a Gieseker-Maruyama moduli scheme $\mathcal{M}$ is only coarse moduli scheme, so it has no universal sheaf in general. However, by construction $\mathcal{M}$ is a GIT-quotient of certain Quot-scheme which is a fine moduli space. Moreover, it is known that over subset of stable sheaves of $\mathcal{M}$ the corresponding restriction of Quot-scheme is a principal bundle. In particular, it means that for the case $R \in \{S(a, b, c)\}$ there exists etale base change $\widetilde{R} \twoheadrightarrow R$ such that there is a universal sheaf $\mathbf{F}$ over $\widetilde{R} \times \mathbb{P}^3$. The same can be said about the moduli scheme $\mathbb{J}$, i.e. there is a universal sheaf $\mathbf{L}$ over $\widetilde{\mathbb{J}} \times \mathcal{H}_1 \mathcal{Z}_1$, where $\widetilde{\mathbb{J}}$ is some etale covering of $\mathbb{J}$. On the other hand, for the case $R \in \{P(d')\}$ we always have a universal sheaf $\mathbf{F}$ which comes from the canonical section of the sheaf $\mathcal{E}xt^1(\pi^*I_Y, \text{pr}^*\mathcal{O}_{P(d')}(1))$, where
\( \pi : \mathcal{P}(d') \times \mathbb{P}^3 \to \mathcal{H} \times \mathbb{P}^3 \) and \( \text{pr} : \mathcal{P}(d') \times \mathbb{P}^3 \to \mathcal{P}(d') \) are corresponding projections.

Denote the fiber product \( \mathcal{B} \times_{\mathbb{P}^3} \mathcal{R} \times_{\mathbb{P}^3} \mathcal{J} \) by \( \tilde{\mathcal{B}} \) and consider the lifts \( \mathbf{F}_{\tilde{\mathcal{B}}}, \mathbf{L}_{\tilde{\mathcal{B}}} \) of the sheaves \( \mathbf{F} \) and \( \mathbf{L} \) to \( \tilde{\mathcal{B}} \times \mathbb{P}^3 \) and \( \tilde{\mathcal{B}} \times_{\mathbb{P}^3} \mathcal{H}_1 \subset \tilde{\mathcal{B}} \times \mathbb{P}^3 \), respectively. Then we can define the vector bundle \( \tau = \text{pr}_* \mathcal{H}om(\mathbf{F}_{\tilde{\mathcal{B}}}, \text{p}^* \mathcal{O}_{\mathcal{Z}_0} \oplus \mathbf{L}_{\tilde{\mathcal{B}}}) \) on \( \tilde{\mathcal{B}} \), where \( \text{pr} : \tilde{\mathcal{B}} \times \mathbb{P}^3 \to \tilde{\mathcal{B}} \) and \( \text{p} : \tilde{\mathcal{B}} \times_{\mathbb{P}^3} \mathcal{H}_0 \to \mathcal{H}_0 \) are corresponding projections. If the point \( u \in \tilde{\mathcal{B}} \) lies over the triple \( (F, W \sqcup C, L) \in \mathcal{B} \), then the fiber of the vector bundle \( \tau \) over that point is isomorphic to the vector space \( \mathcal{H}om(F, L \oplus \mathcal{O}_W) \).

The obstruction for the vector bundle \( \tau \) to be descended on \( \mathcal{B} \) is the scalar multiplier in the cocycle condition for transition functions. So after taking quotient by group generated by automorphisms of sheaves \( \mathcal{O}_{\mathcal{Z}_0} \) and \( \mathbf{L}_{\tilde{\mathcal{B}}} \), this scalar multiplier does not matter. Therefore, the quotient bundle \( \tilde{\mathcal{Q}} := \tau/ (k^*)^1 \times (k^*)^{s+1} \times (k^*)^1 \to \tilde{\mathcal{B}} \) descends to the bundle over \( \mathcal{B} \).

Since \( \pi_*(\mathcal{O}_{\tilde{\mathcal{Q}}}(1)) \simeq \tau^\vee \), there exists the canonical section \( \sigma \in \Gamma(\tilde{\mathcal{Q}}, \pi^* \tau \otimes \mathcal{O}_{\tilde{\mathcal{Q}}}(1)) \) which defines the surjective morphism

\[
\phi : \pi^* \mathbf{F}_{\tilde{\mathcal{B}}} \twoheadrightarrow \pi^*(\text{p}^* \mathcal{O}_{\mathcal{Z}_0} \oplus \mathbf{L}_{\tilde{\mathcal{B}}}) \otimes \mathcal{O}_{\tilde{\mathcal{Q}}}(1).
\]

It is easy to see that all fibers of the kernel sheaf \( \mathbf{E} := \ker \phi \) have the same Hilbert polynomial, so the sheaf \( \mathbf{E} \) is flat over \( \mathcal{Q} \). Moreover, every fiber of \( \mathbf{E} \) is stable. Therefore, the existence of the Gieseker-Maruyama moduli scheme \( \mathcal{M}(k) \) provides us the classifying map \( \tilde{\mathcal{Q}} \to \mathcal{M}(k) \) which factors through the map \( \mathcal{Q} \to \mathcal{M}(k) \). The last map is an open inclusion of schemes because the fibers of the sheaf \( \mathbf{E} \) over different points of \( \mathcal{Q} \) are not isomorphic as it was shown above. \( \square \)

Consider now the scheme-theoretical image \( \overline{\mathcal{C}} = \text{im}(\ker) \subset \mathcal{M}(m + d) \) as a closed subscheme. By construction the subscheme \( \overline{\mathcal{C}} \) is irreducible, and its dimension can be computed by the following formula

\[
\dim \overline{\mathcal{C}} = \dim \mathcal{R} + \dim \mathcal{H}_0 \times \mathcal{H}_1 + \dim \text{Jac}(C) + \dim \text{Hom}(F, \mathcal{O}_W \oplus L)/\text{Aut}(F) \times \text{Aut}(\mathcal{O}_W \oplus L),
\]

where \( (F, W \sqcup C, L) \) is a generic point of \( \mathcal{B} \).

**Theorem 3.** For generic sheaf \( E \) of the subscheme \( \overline{\mathcal{C}} \subset \mathcal{M}(m + d) \) we have the equality

\[
\dim T_E \mathcal{M}(m + d) = \dim \mathcal{C}.
\]

Therefore, the subscheme \( \overline{\mathcal{C}} \) is irreducible component of the moduli scheme \( \mathcal{M}(m + d) \).
Proof: For computation of the dimension of the tangent space of the moduli scheme $\mathcal{M}(m + d)$ at the point $[E]$ defined above, we use standard fact of the deformation theory, $T_{[E]}\mathcal{M}(m + d) \cong \text{Ext}^1(E, E)$, and also the Local-to-Global spectral sequence $H^p(\text{Ext}^q(E, E)) \Rightarrow \text{Ext}^{p+q}(E, E)$, from which the following exact sequence follows

\begin{equation}
0 \rightarrow H^1(\text{Hom}(E, E)) \rightarrow \text{Ext}^1(E, E) \rightarrow H^0(\text{Ext}^1(E, E)) \overset{\phi}{\rightarrow} H^2(\text{Hom}(E, E)) \rightarrow \text{Ext}^2(E, E).
\end{equation}

All components of the moduli schemes of reflexive sheaves which we consider have expected dimension, so for generic sheaf $F$ we have $\text{Ext}^2(F, F) \cong 0$. Hence, from to the following triple

\begin{equation}
0 \rightarrow \text{Ext}^1(F, Q) \rightarrow \text{Ext}^2(F, E) \rightarrow \text{Ext}^2(F, F) \rightarrow 0,
\end{equation}

we obtain that $\text{Ext}^2(F, E) \cong 0$. It implies the commutative diagram

\[
\begin{array}{c}
\text{s} \, \text{H}^0(\text{Ext}^1(F, E)) \rightarrow \text{H}^2(\text{Hom}(F, E)) \rightarrow 0 \\
\downarrow \quad \downarrow \\
\text{H}^0(\text{Ext}^1(E, E)) \rightarrow \text{H}^2(\text{Hom}(E, E)) \rightarrow \text{Ext}^2(E, E)
\end{array}
\]

from which it follows that the morphism $\phi$ is surjective. Therefore, we have the following formula

\begin{equation}
\dim \text{Ext}^1(E, E) = h^0(\text{Ext}^1(E, E)) + h^1(\text{Hom}(E, E)) - h^2(\text{Hom}(E, E)),
\end{equation}

and analogous formula for the sheaf $F$.

According to construction, the generic sheaf $[E] \in \mathcal{C}$ satisfies the exact triple of the following form

\begin{equation}
0 \rightarrow E \rightarrow F \rightarrow Q \rightarrow 0.
\end{equation}

Apply the functor $\text{Hom}(-, E)$ to this triple and consider the following part of resulting long exact sequence

\begin{equation}
\text{Ext}^1(Q, E) \overset{0}{\rightarrow} \text{Ext}^1(F, E) \rightarrow \text{Ext}^1(E, E) \rightarrow \\
\rightarrow \text{Ext}^2(Q, E) \overset{0}{\rightarrow} \text{Ext}^2(F, E).
\end{equation}

Note at once that the leftmost and rightmost morphisms are identically equal to zero because by condition $\text{Supp}(Q) \cap \text{Sing}(F) = \emptyset$, but $\text{Supp}\left(\text{Ext}^1(F, E)\right)$,
Supp\( (\mathcal{E}xt^2(F, E)) \subset \text{Sing}(F) \). By the same reasons, we have the isomorphism

\begin{equation}
\mathcal{E}xt^1(E, E) \simeq \mathcal{E}xt^1(F, E) \oplus \mathcal{E}xt^2(Q, E).
\end{equation}

Applying the functor \( \mathcal{H}om(F, -) \) to the triple \( (22) \) we obtain the exact sequence

\begin{equation}
\mathcal{H}om(F, Q) \xrightarrow{0} \mathcal{E}xt^1(F, E) \xrightarrow{} \mathcal{E}xt^1(F, F) \xrightarrow{} \mathcal{E}xt^1(F, Q).
\end{equation}

The first morphism of this sequence is equal to zero and \( \mathcal{E}xt^1(F, Q) \simeq 0 \) due to the condition on the supports and singularities of the sheaves \( Q \) and \( F \). From this we immediately obtain that \( \mathcal{E}xt^1(F, E) \simeq \mathcal{E}xt^1(F, F) \). Therefore, we have

\begin{equation}
\mathcal{E}xt^1(E, E) \simeq \mathcal{E}xt^1(F, F) \oplus \mathcal{E}xt^2(Q, E).
\end{equation}

Further we apply the functor \( \mathcal{H}om(Q, -) \) to the triple \( (22) \) which gives us the exact sequence

\begin{equation}
\mathcal{E}xt^1(Q, F) \longrightarrow \mathcal{E}xt^1(Q, Q) \longrightarrow \mathcal{E}xt^2(Q, E) \longrightarrow \rightarrow \mathcal{E}xt^2(Q, F) \xrightarrow{\phi} \mathcal{E}xt^2(Q, Q) \longrightarrow \mathcal{E}xt^3(Q, E).
\end{equation}

Since smooth curve \( C \) and 0-dimensional subscheme \( W \) are locally complete intersections, the following isomorphisms are true

\begin{equation}
\text{Ext}^1_{\mathcal{O}_{\mathbb{P}^3, x}}(\mathcal{O}_{C, x}, \mathcal{O}_{\mathbb{P}^3, x}) \simeq 0, \quad \text{Ext}^2_{\mathcal{O}_{\mathbb{P}^3, x}}(\mathcal{O}_{W, x}, \mathcal{O}_{\mathbb{P}^3, x}) \simeq 0.
\end{equation}

Hence, the first sheaf of the exact sequence \( (27) \) is identically equal to zero and support of the sheaf \( \mathcal{E}xt^2(Q, F) \) is the curve \( C \), i.e. \( \mathcal{E}xt^2(Q, F) \simeq \mathcal{E}xt^2(L, F) \). On the other hand, the homological dimension of the structure sheaf \( \mathcal{O}_C \) is equal to 2, so the support of the last sheaf of the sequence \( (27) \) lies on the 0-dimensional subscheme \( W \). From this it immediately follows that the morphism \( \phi|_C : \mathcal{E}xt^2(Q, F) \longrightarrow \mathcal{E}xt^2(Q, Q)|_C \) is a surjection because otherwise we should have \( \text{Supp}(\mathcal{E}xt^3(Q, E)) \cap C \neq \emptyset \) which is contradiction. Taking into account the isomorphisms

\[
\mathcal{E}xt^2(Q, Q)|_C \simeq \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_C) \simeq \omega_C(4),
\]

\[
\mathcal{E}xt^1(Q, Q) \simeq N_{C/\mathbb{P}^3} \oplus N_{W/\mathbb{P}^3},
\]

we obtain the following exact sequence

\begin{equation}
0 \longrightarrow N_{C/\mathbb{P}^3} \oplus N_{W/\mathbb{P}^3} \longrightarrow \mathcal{E}xt^2(Q, E) \longrightarrow \mathcal{E}xt^2(L, F) \xrightarrow{\phi|_C} \omega_C(4) \longrightarrow 0.
\end{equation}
Moreover, since $\det F \otimes \mathcal{O}_C \simeq \mathcal{O}_C$ the following exact triple holds

$$
0 \rightarrow L^{-1} \rightarrow F \otimes \mathcal{O}_C \rightarrow L \rightarrow 0.
$$

After tensoring this triple on the sheaf $\mathcal{E}xt^2(L, \mathcal{O}_C)$ we obtain the following exact triple

$$
0 \rightarrow \mathcal{E}xt^2(L, L^{-1}) \rightarrow \mathcal{E}xt^2(L, F) \rightarrow \mathcal{E}xt^2(L, L) \rightarrow 0.
$$

Taking into account the fact that this triple can be naturally included to the exact sequence (29), we obtain the exact triple

$$
0 \rightarrow N_{C/P^3} \oplus N_{W/P^3} \rightarrow \mathcal{E}xt^2(Q, E) \rightarrow \omega_C(4) \otimes L^{-2} \rightarrow 0.
$$

According to the construction of the subscheme $C$, the equality $h^0(\omega_C(4) \otimes L^{-2}) = 0$ holds, so we have the following formula

$$
h^0(\mathcal{E}xt^2(Q, E)) = h^0(N_{C/P^3}) + h^0(N_{W/P^3}) = \dim \mathcal{H}_0 \times \mathcal{H}_1.
$$

Hence, due to the isomorphism (26) we have the equality

$$
h^0(\mathcal{E}xt^1(E, E)) = \dim \mathcal{H}_0 \times \mathcal{H}_1 + h^0(\mathcal{E}xt^1(F, F)).
$$

Now let us compute the dimensions of cohomologies of the sheaf $\mathcal{H}om(E, E)$. In order to do this we again apply the functor $\mathcal{H}om(\cdot, E)$ to the exact triple (22) and as a result we obtain the following exact sequence

$$
0 \rightarrow \mathcal{H}om(Q, E) \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{H}om(E, E) \rightarrow
\rightarrow \mathcal{E}xt^1(Q, E) \rightarrow \mathcal{E}xt^1(F, E)
$$

As it was already mentioned the morphism $\mathcal{E}xt^1(Q, E) \rightarrow \mathcal{E}xt^1(F, E)$ in this exact sequence is identically equal to zero. Moreover, the sheaf $E$ has no torsion, so $\mathcal{H}om(Q, E) \simeq 0$. Therefore, we have the following exact triple

$$
0 \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{H}om(E, E) \rightarrow \mathcal{E}xt^1(Q, E) \rightarrow 0.
$$

After applying the functor $\mathcal{H}om(\cdot, F)$ to the exact triple (22) we obtain another long exact sequence of sheaves

$$
0 \rightarrow \mathcal{H}om(Q, F) \rightarrow \mathcal{H}om(F, F) \rightarrow \mathcal{H}om(E, F) \rightarrow \mathcal{E}xt^1(Q, F).
$$

Due to torsion-freeness of the sheaf $F$ and local isomorphisms (28) we obtain the isomorphism $\mathcal{H}om(E, F) \simeq \mathcal{H}om(F, F)$. 

Finally, after applying the functor $\mathcal{H}om(F, -)$ to the triple (22) we obtain the following exact sequence

$$(37) \ 0 \to \mathcal{H}om(F, E) \to \mathcal{H}om(F, F) \to \mathcal{H}om(F, Q) \to \mathcal{E}xt^1(F, E).$$

As it was already mentioned, due to the conditions on the singularities of the sheaf $F$ and the support of the sheaf $Q$, the morphism $\mathcal{H}om(F, Q) \to \mathcal{E}xt^1(F, E)$ in this exact sequence is identically equal to zero. Hence, we have the following exact triple

$$(38) \ 0 \to \mathcal{H}om(F, E) \to \mathcal{H}om(F, F) \to \mathcal{H}om(F, Q) \to 0.$$ 

Note that the exact sequences (34), (36) and (37) are included to the ambient commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}om(Q, E) & \mathcal{H}om(Q, F) \\
\downarrow & \downarrow \\
0 \to \mathcal{H}om(F, E) \to \mathcal{H}om(F, F) \to \mathcal{H}om(F, Q) \to \mathcal{E}xt^1(F, E) \\
\downarrow & \downarrow \\
0 \to \mathcal{H}om(E, E) \to \mathcal{H}om(E, F) \\
\downarrow & \downarrow \\
\mathcal{E}xt^1(Q, E) \to \mathcal{E}xt^1(Q, F) \\
\downarrow & \downarrow \\
\mathcal{E}xt^1(F, E)
\end{array}
$$

Due to the above arguments this diagram can be simplified to diagram of the following form

$$
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 \to \mathcal{H}om(F, E) \to \mathcal{H}om(F, F) \to \mathcal{H}om(F, Q) \to 0 \\
\downarrow & \downarrow \\
0 \to \mathcal{H}om(E, E) \xrightarrow{\tau} \mathcal{H}om(E, F) \\
\downarrow & \downarrow \\
\mathcal{E}xt^1(Q, E) \to 0 \\
\downarrow & \\
0
\end{array}
$$
Applying the snake lemma to this diagram we obtain the following exact triple

\[(39) \quad 0 \longrightarrow \mathcal{E}xt^1(Q, E) \longrightarrow \mathcal{H}om(F, Q) \longrightarrow \text{coker } \tau \longrightarrow 0.\]

Moreover, after applying the functor \( \mathcal{H}om(Q, -) \) to the triple (22) we obtain the isomorphism

\[(40) \quad \mathcal{E}xt^1(Q, E) \simeq \mathcal{H}om(Q, Q).\]

Therefore, we have the following exact triple

\[(41) \quad 0 \longrightarrow \mathcal{H}om(Q, Q) \longrightarrow \mathcal{H}om(F, Q) \longrightarrow \text{coker } \tau \longrightarrow 0.\]

On the other hand, the sheaf \( \text{coker } \tau \) satisfies the exact triple

\[(42) \quad 0 \longrightarrow \mathcal{H}om(E, E) \longrightarrow \mathcal{H}om(F, F) \longrightarrow \text{coker } \tau \longrightarrow 0.\]

Consequently, taking into account the fact that the sheaf \( E \) is simple due to stability, we obtain the following equality

\[(43) \quad h^1(\mathcal{H}om(E, E)) = 1 - h^0(\mathcal{H}om(F, F)) - h^0(\mathcal{H}om(Q, Q)) +
\quad + h^0(\mathcal{H}om(Q, Q)) + h^1(\mathcal{H}om(Q, Q)) + h^1(\mathcal{H}om(F, F)) =
\quad = h^1(\mathcal{H}om(F, F)) + \dim \text{Jac}(C) + \dim \mathcal{H}om(F, Q)/\text{Aut}(F) \times \text{Aut}(Q)
\]

Therefore, from this and the equality (33) we have the following

\[(44) \quad \dim \mathcal{E}xt^1(E, E) = h^0(\mathcal{E}xt^1(E, E)) + h^1(\mathcal{H}om(E, E)) =
\quad = \dim \mathcal{R}(0, n, 2m) + \dim \mathcal{H}_0 \times \mathcal{H}_1 + \dim \text{Jac}(C) +
\quad + \dim \mathcal{H}om(F, Q)/\text{Aut}(F) \times \text{Aut}(Q) = \dim \mathcal{C}.\]

\[\square\]

4. Conclusion

In order to enumerate the components of constructed series we introduce the following notation for them (all parameters are non-negative integers)

- \( \mathcal{C}_{d,s}^{(a,b,c)} \) is a component of \( \mathcal{M}(m(a, b, c) + d) \) constructed by \( \mathcal{R} = \mathcal{S}(a, b, c) \), rational curves of degree \( d \) and \( s \) distinct points, such that
  \[\frac{3a + 2b + c}{2} \in \mathbb{Z}_{>0}, \quad 0 \leq s < n(a, b, c);\]
NEW SERIES OF MODULI COMPONENTS OF RANK 2 SEMISTABLE SHEAVES ON $\mathbb{P}^3$ WITH SINGULARITIES OF MIXED DIMENSION 17

- $C^{(a,b,c)}_{(d_1,d_2),s}$ is a component of $\mathcal{M}(m(a, b, c) + d_1d_2)$ constructed by $\mathcal{R} = \mathcal{S}(a, b, c)$, complete intersection curves of the form $S_{d_1} \cap S_{d_2}$, where $S_{d_i}$ is a surface of degree $d_i$, and $s$ distinct points, such that
  $$\frac{3a + 2b + c}{2} \in \mathbb{Z}_{>0}, \quad 0 \leq s \leq n(a, b, c);$$

- $C^d_{d,s}$ is a component of $\mathcal{M}(d + d')$ constructed by $\mathcal{R} = \mathcal{P}(d')$, rational curves of degree $d$ and $s$ distinct points, such that
  $$d' < d, \quad 0 \leq s < 2d' - 1;$$

- $C^{d'}_{(d_1,d_2),s}$ is a component of $\mathcal{M}(d_1d_2 + d')$ constructed by $\mathcal{R} = \mathcal{P}(d')$, complete intersection curves of the form $S_{d_1} \cap S_{d_2}$, where $S_{d_i}$ is a surface of degree $d_i$, and $s$ distinct points, such that
  $$d' < d_1d_2, \quad 0 \leq s \leq 2d' - 1.$$

Since a complete enumeration of components of $\mathcal{M}(k)$ for small values of $k$ is of particular interest, we point out that the series described above contains the new component from $\mathcal{M}(3)$, namely, $C^1_{2,0}$. By construction the generic sheaf $[E]$ of the component $C^1_{2,0} \subset \mathcal{M}(3)$ satisfies the exact sequence
  $$0 \rightarrow E \rightarrow F \rightarrow \mathcal{O}_C(2) \rightarrow 0,$$
where $C$ is a smooth conic and $[F] \in \mathcal{P}(1) = \mathcal{R}_{s,s}(0, 1, 2)$. More precisely, the reflexive sheaf $F$ can be defined as the following extension
  $$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow F \rightarrow I_l \rightarrow 0,$$
where $l$ is a projective line in $\mathbb{P}^3$. Therefore, the number of components of $\mathcal{M}(3)$ is at least 11.

References

[1] E. Esteves, Compactifying the relative Jacobian over families of reduced curves, Trans. Amer. Math. Soc., 353, 2001, 3045 – 3095.

[2] D. Eisenbud, A. Van de Ven, On the normal bundles of smooth rational space curves, Math. Ann., 256 (1981), 453 – 463.

[3] M.-C. Chang, Stable rank 2 reflexive sheaves on $\mathbb{P}^3$ with small $c_2$ and applications, Trans. Amer. Math. Soc., 284 (1984), 57–89.

[4] R. Hartshorne, Stable Reflexive Sheaves, Math. Ann. 254 (1980), 121–176.

[5] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, 2nd ed., Cambridge Math. Lib., Cambridge University Press, Cambridge, 2010.
[6] M. Jardim, D. Markushevich, A. S. Tikhomirov, New divisors in the boundary of the instanton moduli space, Moscow Mathematical Journal, 2018, Vol. 18, No. 1, P. 117-148.

[7] M. Jardim, D. Markushevich, A. S. Tikhomirov, Two infinite series of moduli spaces of rank 2 sheaves on $\mathbb{P}^3$, Annali di Matematica Pura ed Applicata (4), 196(4):15731608, 2017.

[8] A. N. Ivanov, A. S. Tikhomirov, Semistable rank 2 sheaves with singularities of mixed dimension on $\mathbb{P}^3$, Journal of Geometry and Physics, 2018, Vol. 129, p. 90–98.

[9] C. Almeida, M. Jardim, A. S. Tikhomirov, Irreducible components of the moduli space of rank 2 sheaves of odd determinant on $\mathbb{P}^3$, 2019. In preparation.

Department of Mathematics, National Research University Higher School of Economics, 6 Usacheva Street, 119048 Moscow, Russia

E-mail address: anivanov_1@edu.hse.ru