Adaptive Hierarchical Clustering Using Ordinal Queries

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Abstract

In many applications of clustering (for example, ontologies or clusterings of animal or plant species), hierarchical clusterings are more descriptive than a flat clustering. A hierarchical clustering over \( n \) elements is represented by a rooted binary tree with \( n \) leaves, each corresponding to one element. The subtrees rooted at interior nodes capture the clusters. In this paper, we study active learning of a hierarchical clustering using only ordinal queries. An ordinal query consists of a set of three elements, and the response to a query reveals the two elements (among the three elements in the query) which are “closer” to each other than to the third one. We say that elements \( x \) and \( x' \) are closer to each other than \( x'' \) if there exists a cluster containing \( x \) and \( x' \), but not \( x'' \).

Our main result is a deterministic algorithm that learns the underlying hierarchical clustering using at most \( n \log_2(n) \) adaptive ordinal queries. We generalize our algorithm to be robust in a model in which each query response is correct independently with probability \( p > \frac{1}{2} \), and adversarially incorrect with probability \( 1 - p \). We show that in the presence of noise, our algorithm outputs the correct hierarchical clustering with probability at least \( 1 - \delta \), using \( O(n \log n + \log(1/\delta)) \) adaptive ordinal queries. For our results, adaptivity is crucial: we prove that even in the absence of noise, every non-adaptive algorithm requires \( \Omega(n^3) \) ordinal queries in the worst case.

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1 Introduction

One of the most useful and frequent computational tasks in interacting with a set of elements is to cluster them, inferring groups (clusters) of elements that are “similar to each other.” Clustering is often an essential step in “learning” facts about the items, inferring generative models, deriving scientific or other insights, or letting users interact with large data sets meaningfully. The wide range of applications has led to an enormous amount of research spanning many disciplines, driven by different applications, different notions of “similarity,” different constraints on allowable groups of elements, and different models of user interaction, to name just a few.

Two particularly common restrictions on the allowable groups are flat clustering, in which each item belongs to exactly (or at most) one cluster, and hierarchical clustering, in which the groups form a nested structure. Both restrictions have myriad natural applications. Among the prototypical applications of hierarchical clusterings are hierarchies of animal or plant species [18], ontologies of words or other entities, or social networks, viewed at different levels of granularity. Hierarchical clusterings are particularly well suited for human data exploration, in that they allow a person to zoom in/out to understand the elements and their relationships at different levels. In this paper, we are interested in inferring a hierarchical clustering on a set \( \mathcal{X} \) of \( n \) elements.

A second key question is how the similarities between elements are defined and obtained. In much of the work on clustering, the similarities are derived from observable features of the elements (including their position in an observable metric space), and finding a clustering is cast as an optimization problem for a suitably chosen objective function. Such an approach frequently identifies clusters quite similar — but not identical — to the ground truth. An alternative approach is to interact with human users, and obtain input to guide the search for the true clustering. These interactions are typically expensive; thus, minimizing the number of interactions with human users is often a primary goal in designing algorithms. When interacting with humans, it is also important to keep in mind that even when there is an objective meaning to numerical similarity values, humans are notoriously bad at estimating such numerical quantities. This has led to a search for suitable models of interaction, advocated recently in [8, 7, 6, 35], in which human users are only given queries they may reasonably be in a position to answer.

Inspired by the equivalence query model for learning a classifier [3], Balcan and Blum [8] and Awasthi et al. [7, 6] defined a model of interaction (specifically for flat clustering) in which an algorithm repeatedly proposes a clustering, and a user can request corrections, such as splitting or merging clusters. In this model, the user has a lot of leeway in determining what feedback the algorithm receives. The ordinal query model ([32, 22, 35, 27], for example) is more reminiscent of traditional “active learning” models. Here, a user is given queries comprising triplets of elements, and asked to identify the pair from each triplet that is closest. For example, a user may be asked which pair out of \{lion, tiger, shark\} is most similar. Topics such as social computing and a computational lens on political and other decision making processes have led to increasing interest in ordinal query responses. Not only have they been analyzed in the context of clustering or inferring distances [32, 1, 33, 22, 35], but also in social choice scenarios such as voting [5, 4] and political and group decision making [19, 20]. Learning of a hierarchical clustering from ordinal queries is the topic of the present work.

More formally, we assume that there is an unknown ground truth hierarchical clustering to be learned. It is represented by an (unknown) tree \( \mathcal{T}^* \) whose \( n \) leaves are the elements. For each node \( v \) of the tree, the leaves in the subtree rooted at \( v \) form a cluster in the hierarchical clustering. For most of the paper (except Section 6), we assume that \( \mathcal{T}^* \) is binary. This implies that for each triplet \( \{x, x', x''\} \) of leaves/elements, there is a unique pair that shares a closer common ancestor
than any other pair: this pair is the correct response to the query \( \{x, x', x''\} \). The goal is to learn \( T^* \) (or an isomorphic tree) using few ordinal queries.

For some of this paper, we assume that the answers to all ordinal queries are correct. However, in particular in light of the motivation of interaction with humans, it is natural to assume that responses may be noisy. In the independent noise mode (which we use in Section 5), each response is correct independently with probability \( p > \frac{1}{2} \), and adversarially incorrect otherwise.

It is not difficult to show (see Section 2.2) that even without incorrect responses, when the ordinal queries are non-adaptive (i.e., have to be chosen ahead of time), \( \Theta(n^3) \) queries are necessary and sufficient. We therefore focus on adaptive queries. For adaptive queries, when \( T^* \) has height \( O(\log n) \), as pointed out by Tamuz et al. [33], there is a simple top-down algorithm to learn \( T^* \) using \( O(n \log n) \) adaptive ordinal queries (with no incorrect responses). A matching lower bound follows easily from a counting argument like the one for sorting (see Section 3).

**Our Results and Techniques**

The analogy with sorting is also useful in the design of algorithms for inferring a hierarchy. We begin (in Sections 3 and 4) by studying the model without incorrect responses.

Our first algorithm (in Section 3) is modeled after randomized QuickSort. It picks two random pivots elements \( x_A, x_B \), and partitions the elements into those closest to \( x_A \), those closest to \( x_B \), and those which are further from both pivots than the two pivots are from each other. We prove that with constant probability, each of the three sets is only a constant fraction of the original set. Then, we present an algorithm that, once the hierarchies for all three sets have been (recursively) computed, can merge them using a linear number of queries. As a result, the algorithm uses \( O(n \log n) \) ordinal queries.

In order to obtain a deterministic algorithm, and to improve the constant in the number of queries, in Section 4, we turn to a variant of InsertionSort. Our main observation is that, given the true hierarchy over \( i \) elements, even when the tree is arbitrarily unbalanced, a new element can be inserted using at most \( 1 + \log_2 i \) ordinal queries, giving an overall query complexity of \( n \log_2 n \). The algorithm to insert an element is a close variant of the generalization of Binary Search to trees or arbitrary graphs [31, 30, 15]. While the work on Binary Search in trees/graphs assumes a different query model called vertex queries (see Section 2.3), the algorithm is easily adapted.

Next, we turn our attention to the noisy model. It is easy to see that any algorithm that learns the true hierarchy in the absence of noise can be made robust to noise by repeating each ordinal query \( \Theta(\log n) \) times and using a majority output. Then, Hoeffding’s inequality and union bounds guarantee that the algorithm succeeds with high probability. However, this approach increases the number of queries by a factor of \( \Theta(\log n) \). In Section 5, we show how to avoid the worse dependence on \( n \) for the InsertionSort-based algorithm, and only increase the number of queries by a constant factor depending solely on the correctness probability \( p \).

The robustness is achieved by replacing the Binary Search subroutine with a robust version. While robust Binary Search algorithms in a vertex query model in the presence of noise had been given in work of Feige et al. [17] and Emamjomeh-Zadeh et al. [15], neither algorithm itself has strong enough guarantees for our purposes: the algorithm of [17] requires a number of queries linear in the diameter of the tree, while the algorithm of [15] has a dependency on the desired error guarantee that would lead to a query complexity of \( \Theta(\log^2 n) \) in our context. Our main contribution in Section 5 is a hybrid algorithm combining the first stage of the algorithm from [15] with the algorithm of [17] to obtain a number of queries that is logarithmic in \( n \). Specifically, we

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1 This does not hold in trees of higher degree.
show how, given the tree over a subset of elements, to correctly insert a new element into the tree with probability $1 - \delta$, using $O(\log n + \log(1/\delta))$ queries. This leads to an algorithm that learns the correct hierarchy $T^*$ with probability at least $1 - \delta$ using $O(n(\log n + \log(1/\delta)))$ ordinal queries.

**Related Work**

In order to infer phylogenies, similarities and dissimilarities of different species can be interpreted as ordinal constraints. The algorithmic question of discovering the “best” hierarchical clustering over species (where “best” is with respect to a fixed objective function) has been surveyed in [18]. In particular, Aho et al. [2] presented an algorithm that, given a set of ordinal constraints ahead of time, finds a tree consistent with the constraints (or determines that none exists).

Vikram and Dasgupta [35] addressed the problem of incorporating hard ordinal constraints into algorithms for hierarchical clusterings. However, contrary to our model, they assume that the algorithm is provided information regarding the geometry of the data, and the hard constraints given by the user are additional information to improve the accuracy of the outcome.

A hierarchical clustering over $n$ elements can be seen as a metric (or in fact, ultra-metric) over the elements. In this sense, learning a hierarchical clustering is related to a recent line of work (mostly in the ML community) [1, 24, 23, 29, 33, 34, 28, 22] on constructing a geometric interpretation of data based on ordinal information. In particular, the “non-metric multidimensional scaling” problem [1, 24, 28, 22] is formulated as finding an embedding of the elements into a (low-dimensional) Euclidean space, such that the distances between the points satisfy a given set of ordinal constraints. Many of the algorithms on the non-metric multidimensional scaling problem are in the non-adaptive model (e.g., [1, 22]). Jamieson and Nowak [24] point out that an adaptive algorithm may be able to find an embedding of the elements into the low-dimensional space using only a few ordinal queries, in a way that the embedding satisfies all the $\Theta(n^3)$ possible queries. They proved a lower bound of $\Omega(dn \log n)$ where $d$ is the dimension, and they conjectured that this lower bound is tight.

Feige et al. [17] studied several problems under the independent noise model. In particular, they showed that for the classical Binary Search problem, if comparisons are noisy (i.e., each comparison is correct independently with probability $p > \frac{1}{2}$ and flipped otherwise), there exists an algorithm that uses $O(\log n)$ queries and succeeds with high probability. Ben-Or and Hassidim [11] improved this result and obtained the information-theoretically optimal query complexity. Braverman and Mossel [12] studied the sorting problem under an independent noise model in a non-adaptive setting. They proposed an algorithm that, given the noisy comparisons for every pair of elements, finds the maximum-likelihood permutation. They also showed that the maximum-likelihood permutation is “close” to the ground truth.

## 2 Definitions and Preliminaries

### 2.1 Hierarchical Clusterings

Let $\mathcal{X}$ be a set of $n \geq 2$ elements. A hierarchical clustering of $\mathcal{X}$ can be equivalently characterized in two ways:

1. A laminar family $\mathcal{H}$ of subsets (called clusters) of $\mathcal{X}$, with the properties that $\emptyset \notin \mathcal{H}$, $\mathcal{X} \in \mathcal{H}$ and $\{x\} \in \mathcal{H}$ for all $x \in \mathcal{X}$.

2. For instance, let the distance between every pair of elements be the size of the smallest cluster in the hierarchy containing both of them.

3. Recall that a family of sets is laminar if $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies that $A \subseteq B$ or $B \subseteq A$. 

2. A rooted tree $\mathcal{T}$ whose leaves are the elements $x \in \mathcal{X}$.

The equivalence is obvious: the clusters in $\mathcal{H}$ exactly correspond to vertices $v$ in $\mathcal{T}$, or more specifically, to the set of leaves in the subtree rooted at $v$. We call the set of all clusters corresponding to rooted subtrees of $\mathcal{T}$ the hierarchical clustering corresponding to $\mathcal{T}$, and denote it by $\mathcal{H}[\mathcal{T}]$. When there is no risk of confusion, we also refer to $\mathcal{T}$ itself as a hierarchical clustering.

We primarily use the representation in terms of the tree $\mathcal{T}$. In keeping with much of the prior literature, ([14] [13] [18], for example) we focus (except in Section 6) on the case when $\mathcal{T}$ is binary. In terms of the representation $\mathcal{H}$, this means that for each $X \in \mathcal{H}$ with $|X| \geq 2$, there exists a set $A \subseteq X$ with $A, X \setminus A \in \mathcal{H}$. We assume without loss of generality that $\mathcal{T}$ has no internal node with one child, since such a node could be removed without changing the corresponding family of sets.

There is an unknown ground truth hierarchical clustering $\mathcal{T}^*$, which an algorithm is to recover using ordinal queries. The algorithm succeeds if it finds a rooted binary tree $\mathcal{T}$ with $\mathcal{H}[\mathcal{T}] = \mathcal{H}[\mathcal{T}^*]$; this corresponds to the rooted binary trees $\mathcal{T}$ and $\mathcal{T}^*$ being isomorphic (when the ordering of left vs. right subtrees is immaterial). In this case, we also write $\mathcal{T} \equiv \mathcal{T}^*$.

For a set $S \subseteq \mathcal{X}$ with $|S| \geq 2$, the induced hierarchy $\mathcal{H}[S]$ is defined in the obvious way: $\mathcal{H}[S] = \{X \cap S \mid X \in \mathcal{H}\}$. We define the induced tree $\mathcal{T}[S]$ as follows: remove from $\mathcal{T}$ all leaves not in $S$, then repeatedly remove all internal nodes without children, and contract all nodes with a single child.\footnote{If the root has one child, it is removed, and its child becomes the root of the tree.}

**Proposition 2.1** $\mathcal{H}[S]$ is the hierarchy corresponding to the tree induced by $S$, i.e., $\mathcal{H}[S] = \mathcal{H}[\mathcal{T}[S]]$.

In reasoning about trees, we use the following notation. For an internal vertex $v$, let $\mathcal{T}^v_L, \mathcal{T}^v_R$ be the subtrees rooted at $v$’s left and right children, and let $\mathcal{T}^v = \mathcal{T} \setminus (\mathcal{T}^v_L \cup \mathcal{T}^v_R)$ be the set of all other vertices (so $v \in \mathcal{T}^v$). As is standard for rooted trees, we consider the degree of a node to be its number of children, i.e., we do not count its parent.

### 2.2 Ordinal Queries

An ordinal query is a set of three distinct elements $\{x, x', x''\} \subseteq \mathcal{X}$. The response to the query reveals which two elements are “closest” to each other. We say that $x$ and $x'$ are closer to each other than to $x''$ (with respect to $\mathcal{T}$) if there exists a cluster (subtree rooted at an internal node) in $\mathcal{T}$ which contains $x$ and $x'$, but not $x''$. In other words, for every set of three elements, the two elements which have the lowest common ancestor are closest. Because $\mathcal{T}$ is a binary tree, such a cluster — and hence a valid response — always exists and is unique.

**Proposition 2.2** Two hierarchical clusterings $\mathcal{T}, \mathcal{T}'$ of the same elements $\mathcal{X}$ satisfy $\mathcal{T} \equiv \mathcal{T}'$ if and only if for every triplet $\{x, x', x''\}$, the query responses with respect to $\mathcal{T}$ and $\mathcal{T}'$ are the same.

Thus, using at most $\binom{n}{3} = \Theta(n^3)$ ordinal queries, an algorithm can uniquely recover the underlying hierarchy $\mathcal{T}^*$. (The actual algorithm is a simple bottom-up algorithm.) For non-adaptive algorithms, the following easy theorem (proved in Appendix A) gives a matching lower bound.

**Theorem 2.3** Any non-adaptive algorithm requires $\Omega(n^3)$ ordinal queries to learn the hierarchical clustering over $n$ elements (even in the absence of noise).
Our goal is to reduce the number of queries to $O(n \log n)$ using adaptive ordinal queries. The algorithm has access to an oracle (for example, a human user) that will answer ordinal queries based on the (otherwise unknown) ground truth hierarchy $T^*$. For the next two sections, we assume that these answers are correct; in Section 5 we turn our attention to the independent noisy model. In this model, there is a known constant $p > \frac{1}{2}$ such that each query response is correct independently with probability $p$, and adversarially incorrect with probability $1 - p$. Because the query responses are independent, if an algorithm queries the same triplet multiple times, it may get different responses.

In several of our algorithms, we will try to locate where within an existing (partial) hierarchy $T$ another element $x$ should be inserted. In doing so, it is particularly useful to “query” an internal vertex $v$, and learn whether $x$ should be in the left subtree of $v$, the right subtree of $v$, or neither. This can be accomplished as follows. Let $x_L \in T^v_L, x_R \in T^v_R$ be arbitrary elements in the left and right subtrees of $v$. ($x_L, x_R$ exist because $v$ is an internal vertex, and thus has exactly two children.) We call the query $\{x_L, x_R, x\}$ an ordinal query for $x$ with pivot $v$. Its possible outcomes can be interpreted as follows.

**Proposition 2.4** If the response to $\{x_L, x_R, x\}$ is that $x_L$ and $x$ are closest, then $x$’s correct location is in $T^v_L$ (the left subtree of $v$). If the response to $\{x_L, x_R, x\}$ is that $x_R$ and $x$ are closest, then $x$’s correct location is in $T^v_R$. If $x_R$ and $x_L$ are closest, then $x$’s correct location is not in the subtree rooted at $v$, i.e., $x$ belongs in $\overline{T}$.

Ordinal queries with pivot $v$ are only defined when $v$ is an internal node, and we will only use them in that case. When notationally convenient, we will consider the left/right child of $v$ itself as the response to an ordinal query with pivot $v$.

### 2.3 Vertex Queries

Ordinal queries with pivot $v$ provide “directional” information regarding the location of $x$ with respect to $v$. As such, they are similar to (though subtly different from) vertex queries defined in past work [31, 15]. In the vertex query model, the goal is to identify an a priori unknown target node $t$ in a known (undirected, not necessarily rooted) tree $T$, by repeatedly querying nodes $v$ of $T$. In response to a vertex query, the algorithm is told that $t = v$, or otherwise is given the unique neighbor that is closer to the target.

Vertex queries are subtly more powerful than ordinal queries with a pivot, because they can identify $v$ as the target. We will show later how to simulate (for our purposes) vertex queries with a constant number of ordinal queries with pivots. In fact, our analysis of the algorithm inspired by INSERTIONSORT is based on ideas from prior work in the vertex query model, and the analysis for the independent noisy model explicitly uses a reduction from the ordinal query model to the vertex query model.

### 3 A QuickSort-like Randomized Algorithm

In this section, we propose a simple randomized algorithm (Algorithm 1), reminiscent of QUICKSORT. The algorithm uses $O(n \log n)$ ordinal queries in expectation to learn the hierarchy over $n$ elements. Algorithm 1 randomly partitions the elements into three groups $A$, $B$, and $C$ (lines 4–10), until all of the partitions are small enough. Once they are, the algorithm recursively determines the hierarchy for each partition, and finally merges them.

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5Of course, the adversary could give a correct answer if an algorithm were to rely on it always answering incorrectly.
The partitioning is akin to the partitioning stage in QuickSort: it draws a pair of distinct pivot elements \( x_A, x_B \in X \), uniformly at random. It then partitions the elements of \( X \) into three sets. These sets can be characterized as follows: Let \( \overline{v} \) be the lowest common ancestor of \( x_A, x_B \) in the ground truth hierarchy \( T^* \), with children \( v_A, v_B \). Then, \( A (B) \) consists of all elements in the subtree of \( T^* \) rooted at \( v_A (v_B) \), while \( C \) consists of all remaining elements: exactly the elements outside of the subtree of \( T^* \) rooted at \( \overline{v} \). Therefore, after the recursive calls, \( T_A \) and \( T_B \) are the subtrees of \( T^* \) rooted at \( v_A \) and \( v_B \), while \( T_C \) is the result of removing \( \overline{v} \) and its subtree from \( T^* \) (and contracting its parent).

The role of the \texttt{Merge} function (Algorithm 2) is then to insert \( \overline{v} \) and its subtree in the correct place in \( T_C \); the primary task here is to correctly identify the sibling \( v \) of \( \overline{v} \). The algorithm does this by starting at the root and repeatedly using ordinal queries for a representative element \( x \) of \( T_A \cup T_B \) with \( v \) as a pivot, and following the direction returned by the ordinal query. That inserting \( \overline{v} \) as a sibling of \( v \) (with a new parent \( v' \)) is correct follows from Proposition 2.1.

\begin{algorithm}
\caption{QuickClustering (\( X \))}
\begin{algorithmic}[1]
\State \textbf{if} \( |X| \leq 2 \) \textbf{then}
\State \textbf{return} the obvious hierarchy.
\EndIf
\State \textbf{repeat}
\State \textbf{for} each \( x \in X \setminus \{x_A, x_B\} \) \textbf{do}
\State \textbf{Make an ordinal query of} \( \{x_A, x_B, x\} \).
\State \textbf{Let} \( A \leftarrow \{x_A\} \cup \{x \in X \mid x \text{ and } x_A \text{ are closer to each other than to } x_B\} \).
\State \textbf{Let} \( B \leftarrow \{x_B\} \cup \{x \in X \mid x \text{ and } x_B \text{ are closer to each other than to } x_A\} \).
\State \textbf{Let} \( C \leftarrow \{x \in X \mid x_A \text{ and } x_B \text{ are closer to each other than to } x\} \).
\State \textbf{until} \( \max(|A|, |B|, |C|) \leq \frac{15}{16} |X| \).
\State \textbf{Return} \( T_A \leftarrow \text{QuickClustering}(A) \).
\State \textbf{Return} \( T_B \leftarrow \text{QuickClustering}(B) \).
\State \textbf{Return} \( T_C \leftarrow \text{QuickClustering}(C) \).
\State \textbf{return} \( \text{Merge} (T_A, T_B, T_C) \).
\end{algorithmic}
\end{algorithm}

\begin{algorithm}
\caption{Merge (\( T_A, T_B, T_C \))}
\begin{algorithmic}[1]
\State Add a vertex \( \overline{v} \) whose children are the roots of \( T_A \) and \( T_B \).
\State Let \( v \) be an arbitrary element (i.e., leaf) in \( T_A \) or \( T_B \).
\State Let \( \overline{v} \) be the root of \( T_C \).
\State \textbf{while} \( v \) is not a leaf, and the ordinal query for \( x \) with pivot \( v \) does not return \( T^* \) \textbf{do}
\State \textbf{Update} \( v \) to the child of \( v \) returned by that ordinal query for \( x \)
\State \textbf{Insert} a new vertex \( v' \) as parent of \( v \), and make \( \overline{v} \) its other child.
\end{algorithmic}
\end{algorithm}

We next analyze the expected number of ordinal queries required by the QuickClustering algorithm. The number of queries required for \texttt{Merge} is at most \(|X| \), and the number of queries for each iteration of the partitioning loop (lines 4–10 in Algorithm 1) is \(|X| - 2\). The key observation is that in expectation, the loop is executed only a constant number of times before succeeding,

\[ \text{In fact, using the techniques from Section 4, the bound could be improved to } 1 + \log_2 |X|; \text{ but this improvement is drowned out by the number of queries for the partitioning.} \]
and therefore \( n \) elements in an arbitrary order \( X \) of \( \mathcal{H}[T] \) such that \( \frac{n}{15} < |C'| \leq \frac{n}{15} \) and \( \frac{n}{15} < |C| \leq \frac{n}{15} \). This follows because \( T \) is binary: starting from \( C = X \), we can repeatedly consider a bipartition of \( C \) into two disjoint subclusters. While \( C \) has more than \( \frac{n}{15} \) elements, we can replace it by the larger of the two subclusters, which must have at least half the size of \( C \). We end up with a set \( C \) of the required size. Continuing from \( C' = C \) in the same way until \( C' \) has size at most \( \frac{n}{15} \) gives us the claimed sets.

The probability that one of \( x_A, x_B \) is in \( C' \) and the other in \( C \setminus C' \) is at least

\[ \frac{|C'| \cdot |C \setminus C'|}{\binom{n}{2}} > 2 \cdot \frac{1}{16} \cdot \frac{3}{16} = \frac{3}{128}. \]

When this event happens, we assume w.l.o.g. that \( x_A \in C' \) and \( x_B \in C \setminus C' \). Then, \( C' \subseteq A \subseteq C \), and therefore \( \frac{n}{15} \leq |A| \leq \frac{n}{2} \leq \frac{15n}{16} \). Because \( |A| \geq \frac{n}{15} \), we also get that \( |B|, |C| \leq \frac{15n}{16} \), completing the proof.

By Lemma 3.1 and the preceding discussion, the total number of ordinal queries for both partitioning and merging is \( O(|X|) \). Letting \( Q(n) \) denote the expected number of comparisons for \( \text{QuickClustering} \) on \( n \) elements, we obtain the recurrence \( Q(n) \leq Q(|A|) + Q(|B|) + Q(|C|) + O(n) \). Because \( |A| + |B| + |C| = n \) and \( \max(|A|, |B|, |C|) \leq \frac{15n}{16} \), the solution of this recurrence is \( Q(n) = O(n \log n) \). Hence, we have proved the following theorem.

Theorem 3.2 Algorithm 2 learns the hierarchy using \( O(n \log n) \) ordinal queries in expectation.

The query complexity of Algorithm 2 is asymptotically optimal. Because the hierarchical clusterings include all permutations of \( \mathcal{X} \), there are at least \( 2^{\Omega(n \log n)} \) different hierarchical clusterings of \( n \) elements (see also [35]). More precisely, as pointed out in [18], there are \( (2n - 3) \cdot (2n - 5) \cdot (2n - 7)! \cdot (n - 2)! \cdot 2^{n-2} \) hierarchical clusterings. By Stirling’s approximation, at least \( n \log_2 n - O(n) \) bits of information are necessary to uniquely identify \( T^* \). Since each query response reveals at most \( \log_2 (3) \) bits of information, the number of queries required is at least \( n \log_2 n - O(n) \).

4 An InsertionSort-like Algorithm without Noise

In this section, we analyze an algorithm for learning a hierarchy that is reminiscent of INSERTIONSORT. While the analysis is somewhat longer than for the QUICKSORT-like algorithm in Section 3, the algorithm has three advantages: (1) it is deterministic, (2) its number of queries can be bounded more tightly, and (3) it can be generalized naturally to work in the noisy model. To keep the presentation clean, we defer the analysis for the case of noisy responses to Section 5.

The algorithm (Algorithm 3) is based on the classic INSERTIONSORT algorithm: it considers the elements in an arbitrary order \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \). In each iteration \( i \), element \( x_i \) is inserted into the hierarchy \( T_{i-1} \) for the preceding \( i - 1 \) elements. Proposition 4.1 (whose proof follows directly from Proposition 2.1) shows that \( x_i \) will always be inserted as a leaf sibling of an existing (leaf or internal) node \( v_i \), by inserting a new common parent of \( x_i \) and \( v_i \).

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Proposition 4.1  Let $C \in \mathcal{H}[x_1, \ldots, x_{i-1}]$ be minimal with the property that $C \cup \{x_i\} \in \mathcal{H}[x_1, \ldots, x_i]$. (Informally speaking, $C \subseteq \{x_1, \ldots, x_{i-1}\}$ is the cluster corresponding to the sibling of $x_i$ in $T^*[x_1, \ldots, x_i]$.) In $T^*[x_1, \ldots, x_{i-1}]$, let $v_i$ be the root of the subtree corresponding to the cluster $C$. Then, $T^*[x_1, \ldots, x_i]$ is obtained from $T^*[x_1, \ldots, x_{i-1}]$ by inserting a new internal node $v'_i$ as the parent of $v_i$, and making the leaf $x_i$ its other child.

Algorithm 3 INSERTIONCLUSTERING ($x_1, \ldots, x_n$)

1: Let $\mathcal{T}_2$ be the unique (trivial) hierarchical clustering of elements $x_1, x_2$. 
2: for $i = 3, \ldots, n$ do 
3:   Let $v_i = \text{FINDSIBLING}(\mathcal{T}_{i-1}, x_i)$. 
4:   Let $v'_i$ be the parent of $v_i$ (or $\emptyset$ if $v_i$ is the root) 
5:   Let $\mathcal{T}_i$ be the tree obtained from $\mathcal{T}_{i-1}$ by adding a new vertex $v'$ with children $v_i$ and $x_i$ and parent $v'_i$. 
6: return $\mathcal{T}_n$

Algorithm 3 is essentially the same as one proposed by Tamuz et al. [33]. However, Tamuz et al. assumed that $T^*$ is balanced, so that its height is $O(\log n)$. In that case, FINDSIBLING can be simply implemented as a top-down search similar to the MERGE algorithm in Section 3. This approach can take linear time if the tree is not balanced, and our main focus in this section is on improving the time to $\log n$ for arbitrary trees.

Assuming that the function FINDSIBLING (to be specified below) correctly identifies $v_i$, the algorithm maintains the following invariants:

1. Each internal node of $\mathcal{T}_i$ has two children; this holds because every time an internal node is added, it is explicitly given two children.
2. $\mathcal{T}_i$ is always the correct hierarchy for $\{x_1, x_2, \ldots, x_i\}$, in the sense that $\mathcal{T}_i \equiv T^*[x_1, \ldots, x_i]$; this follows inductively from Proposition 4.1.

Thus, it remains to specify an efficient implementation of the function FINDSIBLING and analyze the overall number of ordinal queries.

FINDSIBLING is very similar to “Binary Search in Trees” [30, 31, 15] in the vertex query model. Recall that in the vertex query model, an unknown node $t$ is to be located in a given and known tree $\mathcal{T}$. When an algorithm queries node $v$ which is not the target, it learns the first edge of the unique $v$-$t$ path (equivalently, the connected component of $\mathcal{T} \setminus \{v\}$ in which the target lies). The “Binary Search” algorithm is based on the following observation of Jordan [25]: each tree $\mathcal{T}$ has a separator node $v$ with the property that each connected component of $\mathcal{T} \setminus \{v\}$ contains at most half the nodes of $\mathcal{T}$. Repeatedly querying such a separator node of the remaining subtree is easily shown to find the target $t$ in a tree of $n$ nodes using at most $\log_2 n$ vertex queries [31].

FINDSIBLING treats the (unknown) sibling $v_i$ as the target to discover, and uses ordinal queries with internal vertices as pivots (see Section 2.2) to essentially simulate vertex queries. It maintains a set $S$ of candidate vertices $v$ that have not been ruled out from being $v_i$.

Lemma 4.2 When all answers to ordinal queries are correct, FINDSIBLING($\mathcal{T}, x$) finds $v_i$ in $\mathcal{T}$ using at most $\log |\mathcal{T}|$ ordinal queries.

Proof. Inductively, the induced subgraph $\mathcal{T}[S]$ always forms a binary tree in which each internal node has two children. In verifying the correctness of the algorithm, notice first that the pivot
vertex $v$ is indeed never a leaf. That is because for any leaf $v$, we would have $T' \cap S = S$, whereas for $v$’s parent $v'$, we would have that $\max(|T'_L \cap S|, |T'_R \cap S|) < |S| = |T' \cap S|$. The fact that the update of $S$ is correct follows from Proposition \ref{prop4.1}.

Next, we analyze the number of ordinal queries made. By the result of Jordan \cite{Jordan}, $T[S]$ has a separator node $v$ with the property that each of $T'_L \cap S$, $T'_R \cap S$, $(T' \cap S) \setminus \{v\}$ contains at most $|S|/2$ nodes. Because in the third case ($x_L$ and $x_R$ are closest), the node $v$ remains in $S$, the size of $S$ might change to $1 + |S|/2$ instead of $|S|/2$. However, notice that this can only happen when $|S|$ is odd. For when $|S|$ is even, if $T'$ contained $1 + |S|/2$ nodes from $S$, then choosing the parent $v'$ of $v$ instead would strictly decrease the size of $T'$, while the size of each of the subtrees of $v'$ is bounded by $|S|/2$. Thus, we obtain that the new size of $S$ is at most $\lceil |S|/2 \rceil$.

When $|S| = 3$, a single query suffices to identify $v_i$; the case $|S| = 4$ cannot arise, as there is no binary tree with 4 nodes in which all internal nodes have degree 2. Thus, the number $Q(k)$ of required ordinal queries satisfies the recurrence $Q(k) \leq 1$ for $k \leq 4$ and $Q(k) \leq 1 + Q(\lceil k/2 \rceil)$ for $k \geq 5$. An easy induction proof shows that $Q(k) \leq \log k$, so the number of ordinal queries required to find $v_i$ in $T$ is at most $\log |T|$.

**Theorem 4.3** When the answers to all ordinal queries are correct, Algorithm 4 learns the correct hierarchical clustering of $n$ elements using at most $n \log_2 n$ ordinal queries.

**Proof.** The correctness of the algorithm follows by repeatedly applying Proposition \ref{prop4.1} and using that by Lemma \ref{lemma4.2}, each iteration uses the correct $v_i$.

By Lemma \ref{lemma4.2}, because $T_i$ has $2i - 1$ nodes ($i$ element leaves and $i - 1$ internal nodes), inserting element $x_i$ into $T_{i-1}$ for $i = 3, \ldots, n$ requires at most $\log_2(2i - 3)$ ordinal queries. Thus, the total number of ordinal queries is at most

$$\sum_{i=3}^{n} \log_2(2i - 3) < \sum_{i=2}^{n-1} (1 + \log_2 i) < n + \log_2((n - 1)!) \leq n + (n \log_2 n - n \log_2 e + O(\log n)) \leq n \log_2 n.$$

The step labeled (*) used Stirling’s inequality.
5 Dealing with Noisy Feedback

In this section, we turn our attention to the noisy model: the response to any ordinal query is correct independently with probability \( p > \frac{1}{2} \), and adversarially incorrect otherwise. Our main result is the following theorem:

**Theorem 5.1** When each query response is correct with constant probability \( p > \frac{1}{2} \), and adversarially incorrect with probability \( 1 - p \), there is an algorithm that uses \( O(n(\log n + \log(1/\delta))) \) ordinal queries, and learns the correct hierarchical clustering with probability at least \( 1 - \delta \).

Since ordinal queries are only used in the call to FindSibling (in Line 3) inside the algorithm InsertionClustering, it suffices to show how to find \( v_i \) in each iteration with high probability using \( O(\log n) \) ordinal queries. This is guaranteed by the following lemma, the focus of this section.

**Lemma 5.2** Assume that each response to an ordinal query is correct independently with probability \( p > \frac{1}{2} \), and adversarially incorrect with probability \( 1 - p \). There exists an adaptive algorithm with the following property: Given a hierarchical clustering \( T \) with \( i \) leaves, an element \( x \) to insert, and an error guarantee \( \delta > 0 \), the algorithm finds the sibling \( \overline{v} \) of \( x \) using at most \( O(\log i + \log(1/\delta)) \) ordinal queries.

**Proof of Theorem 5.1.** Using Lemma 5.2, the proof of Theorem 5.1 is straightforward: set \( \delta' = \delta/n \) and apply Lemma 5.2 to each invocation of FindSibling. Then, by the union bound, all iterations will succeed with probability at least \( 1 - \delta \). The total number of ordinal queries will be at most

\[
\sum_{i=3}^{n} O(\log(2i - 3) + \log(1/\delta')) \leq n \cdot O(\log n + \log(n/\delta)) = O(n(\log n + \log(1/\delta))).
\]

5.1 Simulating Vertex Queries with Ordinal Queries

To avoid essentially repeating a large amount of prior analysis, we begin by making the connection between ordinal queries and vertex queries more explicit: we show how to simulate one vertex query using a constant number of ordinal queries.

To illustrate the reduction most easily, consider the case \( p = 1 \) in which all queries are answered correctly. To simulate a vertex query to the root, simply make an ordinal query for \( x \) with the root as pivot. This one query suffices, since the response reveals the subtree that \( x \) belongs to, or otherwise that the root is the sibling of \( x \) (if \( x \) does not belong to either of the subtrees rooted at the root’s children). To simulate a vertex query to a leaf node \( v \), instead make an ordinal query with the parent \( v' \) of \( v \) as the pivot. If the response points to \( v \), then return \( v \) as the sibling; otherwise, \( x \) is not the sibling of \( v \).

Finally, to simulate a vertex query to an internal vertex \( v \) that is not the root, let \( v' \) be the parent of \( v \) in \( T \), and make two ordinal queries for \( x \), one with pivot \( v \) and the other with pivot \( v' \). If in response to the ordinal query with pivot \( v \), a child of \( v \) is identified as containing \( x \) in its subtree, then this case directly corresponds to the corresponding vertex query response at \( v \). If \( T' \) is known to contain \( x \), then the response to the query with pivot \( v' \) clarifies whether the target is in fact \( v \) itself, or contained in \( T' \setminus \{v\} \). Thus, the responses to the two queries together contain (at

\footnote{The big-O notation hides constants depending on \( p \), but not on any other parameter.}
least) all the information obtained from the vertex query. (Notice that applying this reduction, we could have obtained the result of Lemma 4.2 with a loss of a factor of 2 in the number of queries required.)

We now show how to generalize this idea to \( p \in (\frac{1}{2}, 1) \). The cases of the root vertex and a leaf node are handled exactly as in the case \( p = 1 \); in particular, using only a single ordinal query. To simulate a vertex query to an internal vertex \( v \), we choose a suitably large constant \( k_p \) (determined below) and make \( k_p \) ordinal queries for \( x \) with pivot \( v \). If a strict majority of these queries places \( x \) in one of the subtrees below \( v \), then that subtree is returned. If a strict majority of the queries with pivot \( v \) places \( x \) places \( x \) in \( T_v \), then \( v \)'s parent \( v' \) is queried \( k_p \) times to determine if the desired target is equal to \( v \) or contained in \( T_v \setminus \{v\} \). The former is returned as the answer if a strict majority of the responses place \( x \) in the subtree of \( v' \) also containing \( v \). The latter is returned when a strict majority places \( x \) in the other subtree below \( v' \), or a strict majority place it in \( T_{v'} \). Finally, in all other cases, the responses are inconclusive, and the simulation algorithm returns an arbitrary response. Formally, the algorithm (which has a number of tedious case distinctions) is given as Algorithm 5.

**Algorithm 5** SimulateVertexQuery \((T, v, x)\)

1: if \( v \) is the root of \( T \) then
2: Make one ordinal query for \( x \) with pivot \( v \), and return its result.
   (If \( T_v = \{v\} \), the response reveals the root as the proposed answer.)
3: else if \( v \) is a leaf then
4: Let \( v' \) be the parent of \( v \) in \( T \).
5: Make one ordinal query for \( x \) with pivot \( v' \).
6: if the response places \( x \) at \( v \) then
7: return the vertex \( v \) as a proposed answer.
8: else
9: return the direction towards \( v' \).
10: else
11: Let \( v' \) be the parent of \( v \) and \( k_p \) be an odd constant such that \( 1 - e^{-k_p(2p-1)^2/2} \geq \sqrt{p} \).
12: Make \( k_p \) ordinal queries for \( x \) with pivot \( v \).
13: if a strict majority of responses place \( x \) in \( T_{v'}^L \) then
14: return the direction towards the left child of \( v \).
15: else if a strict majority of responses place \( x \) in \( T_{v'}^R \) then
16: return the direction towards the right child of \( v \).
17: else if a strict majority of responses place \( x \) in \( T_v \) then
18: Make \( k_p \) ordinal queries for \( x \) with pivot \( v' \).
19: if a strict majority of responses place \( x \) in the subtree of \( v' \) containing \( v \) then
20: return the vertex \( v \) as a proposed answer.
21: else if a strict majority of responses place \( x \) in the other subtree of \( v' \) or \( T_{v'} \) then
22: return the direction towards \( v' \).
23: else
24: return an arbitrary response.
25: else
26: return an arbitrary response.
Lemma 5.3 If each ordinal query is answered correctly independently with probability $p$, then (for a suitable choice of $k_p$, specified below), \textsc{SimulateVertexQuery}$(T, v, x)$ returns the correct response (either $v$ or the direction in which to search) to the vertex query for $v$, with probability at least $p$.

Proof. By the additive Hoeffding Bound [21], the probability that a majority out of $k$ query responses is correct for every $p > \frac{1}{2}$ is at least $q(k) := 1 - e^{-k(2p-1)^2/2}$. For odd $k$, any majority is strict. For any $p > \frac{1}{2}$, we have that $e^{-k(2p-1)^2/2} < 1$, so $q(k) \to 1$ as $k \to \infty$. In particular, $q(k) \geq \sqrt{p}$ for $k$ sufficiently large. Fixing such a $k_p$, the probability that a strict majority of the queries both with pivots $v, v'$ are correct is at least $(\sqrt{p})^2 = p$. Whenever strict majorities of both sets of ordinal queries are correct, the output of \textsc{SimulateVertexQuery} is correct, completing the proof.

5.2 Reanalyzing Binary Search with Noisy Vertex Queries

In light of Lemma 5.3 for the purpose of analysis in this section, we can ignore ordinal queries, and instead focus on finding $x$’s sibling $v$ using vertex queries: in response to querying $v$, the algorithm is told either that $v = \overline{v}$ is the correct sibling of $x$, or is given a subtree (rooted at one of the children of $v$, or obtained by removing $v$ and its subtrees) that must contain $x$. This information is correct with probability (at least) $p$.

[15] gave an algorithm for Binary Search on undirected graphs with noisy responses that solves this problem with probability at least $1 - \delta$, using at most $O(\log n + \log^2(1/\delta))$ vertex queries [15, Theorem 8]. Since we need $\delta = O(1/n)$ to be able to take a Union Bound, applying this result directly would only give us an overall guarantee of $O(n \log^2 n)$. Hence, in this section, we show how to combine algorithms of [15, Algorithm 2] and of Feige et al. [17] (see also related algorithms for noisy Binary Search on the line [11, 26]) to obtain an algorithm finding $\overline{v}$ in $O(\log n + \log(1/\delta))$ queries. In contrast to [15], the dependence of this new algorithm’s number of queries on $p$ is not information-theoretically optimal, but it obtains an improved dependence on $\delta$ in return.

Algorithm 2 of [15] is a variant of a multiplicative weights algorithm which starts with a candidate set $S$ of vertices, and within essentially $O(\log |S|)$ rounds reduces it to only $O(\log |S|)$ “likely” remaining candidates. The overall algorithm in [15] works by first reducing the candidate set of all nodes to just $O(\log n)$ nodes, then reducing that further to $O(\log \log n)$ nodes with a second invocation of the multiplicative weights algorithm, and finally querying each remaining candidate frequently enough to identify the correct node with sufficiently high probability. (Dependencies on $\delta$ are omitted in our high-level overview.) As part of our algorithm here, we just use a single invocation of the Multiplicative Weights algorithm (Algorithm 2 of [15]), with parameters modified to obtain a better dependence on $\delta$ at the cost of a worse dependence on $p$. The performance of this algorithm (used as a Black Box here) is summarized by Lemma 5.4, which is obtained from Lemma 5 of [15] by setting $\lambda = \frac{1 + p \log p + (1 - p) \log (1 - p)}{2 \log (p/(1 - p))}$.

Lemma 5.4 There exists an algorithm which uses at most $O(\log n + \log(1/\delta))$ vertex queries and returns a set $S$ of $O(\log n + \log(1/\delta))$ nodes such that the unknown target is contained in $S$ with probability at least $1 - \delta$.

Once the number of candidate targets has been reduced to $O(\log n)$, due to the $\log(1/\delta)$ dependence, a further reduction using the techniques of [15] seems difficult. However, at this point, we can apply a modification of a beautiful algorithm of Feige et al. [17]. Translated into our nomenclature, Lemma 3.1 of [17] about noisy Binary Search can be phrased as follows:
Lemma 5.5 (Lemma 3.1 of [17], restated) Let $\mathcal{T}$ be a binary tree, and assume that one of the leaves of $\mathcal{T}$ is the unknown target. In response to a vertex query, assume that the algorithm is given the correct answer with probability $p$, and an adversarially incorrect one otherwise. Then, there is an algorithm which finds the correct target with probability at least $1 - \delta$, using $O(D + \log(1/\delta))$ queries where $D$ is the diameter of $\mathcal{T}$.

Notice two minor inconveniences in applying Lemma 5.5 for our purposes:

1. In the specific application of [17], the binary tree is complete and thus, $D$ is logarithmic in size of the tree. In our application, on the other hand, after invocation of the Multiplicative Weights algorithm, the number of candidates (and hence diameter of the tree) is bounded by $(\log n)$.

2. [17] assume that the target is always a leaf node, whereas in our application, the unknown sibling $v$ may be an internal node. It is very straightforward to modify the algorithm of Feige et al. to handle non-leaf targets.

For completeness, we present the modified algorithm and corresponding analysis in Appendix B. Its guarantee is summarized by the following lemma:

Lemma 5.6 Assume that each query response is correct independently with probability $p > \frac{1}{2}$, and adversarially incorrect with probability $1 - p$. There exists an adaptive algorithm with the following property: Given a tree $\mathcal{T}$ of diameter $D$, and error parameter $\delta > 0$, the algorithm finds the target with probability at least $1 - \delta$, using at most $O(D + \log(1/\delta))$ vertex queries.

Using Lemmas 5.4 and 5.6 it is easy to see how to find a target in $O(\log n + \log(1/\delta))$ queries. Set $\delta' = \delta/2$ and apply Lemma 5.4. Within $O(\log n + \log(1/\delta')) = O(\log n + \log(1/\delta))$ iterations, we obtain a set $S$ of at most $O(\log n + \log(1/\delta))$ nodes that contains the target with probability at least $1 - \delta'$. Then run the algorithm of Lemma 5.6 with parameter $\delta'$ on the tree $\mathcal{T}[S]$. If the target was in $S$, the algorithm will find it with probability at least $1 - \delta'$. Since the height of $\mathcal{T}[S]$ is at most $|S| = O(\log n + \log(1/\delta))$, the number of queries required in this phase is at most $O(\log n + \log(1/\delta))$. Thus, the total number of queries is $O(\log n + \log(1/\delta))$, and the failure probability is at most $2\delta' = \delta$.

6 Conclusion and Open Problems

We presented adaptive algorithms for learning a hierarchical clustering from $O(n \log n)$ ordinal queries. When answers to queries are always correct, the number of queries is actually upper-bounded by $n \log_2 n$; when answers are correct only with probability $p$, the constant in $O(n \log n)$ depends on $p$. An immediate question is whether the constant in $n \log_2 n$ is correct in the noiseless case, or could be improved. We only gave an $n \log_3 n - O(n)$ lower bound, leaving a multiplicative gap of $\log_2(3) \approx 1.58$.

For all our results, we assumed that the tree $\mathcal{T}$ capturing the hierarchical clustering is binary. When nodes can have larger degrees, our analysis does not continue to hold. One problem is that for an ordinal query, another response of $\perp$ may be required: when $x, x', x''$ are leaves in three distinct subtrees of the same internal node $v$, no pair of them is closer to each other than the third. This issue also manifests itself in a higher query complexity: when $\mathcal{T}$ is a star in which one leaf has been replaced with an internal node with two new leaves, and the assignment of elements to leaves is uniformly random, it is not difficult to show that any non-adaptive or adaptive algorithm (even randomized) requires $\Omega(n^2)$ ordinal queries to learn $\mathcal{T}$.
On the other hand, when the maximum degree $d$ of $\mathcal{T}$ is a constant, then positive results can still be obtained. In the absence of noise, an algorithm of [15] for the edge query model can be adapted (using techniques similar to the ones in Section 4) to insert a new element $x$ into $\mathcal{T}$ using $O\left(\frac{d}{\log d} \cdot \log n\right)$ ordinal queries. Hence, a hierarchical clustering of maximum degree $d$ over $n$ elements can be learned using $O\left(\frac{d}{\log d} \cdot n \log n\right)$ ordinal queries. An interesting open question is whether there is a matching lower bound for every $d$. For $d = 2$ (and more generally $d = O(1)$), our counting argument from Section 3 gives a matching lower bound up to a constant factor, and for $d = n - 1$ (and more generally $d = \Omega(n)$), the example mentioned in the preceding paragraph does. We do not have a matching lower bound for intermediate $d$. Another natural question is whether the upper bound of $O\left(\frac{d}{\log d} \cdot \log n\right)$ ordinal queries can be achieved in the noisy model.

A different interpretation of our result is that we show how to learn a (very restricted) ultrametric on the elements in $\mathcal{X}$ using ordinal queries. This raises the natural question of how well more general metrics can be learned using ordinal queries. Indeed, as mentioned earlier, some past work has considered this question, with some assumption on the geometry of the metric (e.g., [22]). Naturally, even for very simple metrics (such as points on a line), we cannot hope to learn all distances accurately: for example, if one element is further from all other elements than any other pair is from each other, no algorithm can learn its distance to other elements to within any approximation guarantee. Is there a natural ordinal approximation to a (not necessarily Euclidean) metric that can be obtained efficiently? Could techniques such as [9, 10, 16] of probabilistic tree embeddings be useful in this context?

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Figure 1: A full binary tree and one of its clusters of size 4.

A Proof of Theorem 2.3

Here, we prove Theorem 2.3 restated for convenience.

Theorem 2.3 Any non-adaptive algorithm requires $\Omega(n^3)$ ordinal queries to learn the hierarchical clustering over $n$ elements (even in the absence of noise).

Proof. A (possibly randomized) non-adaptive algorithm poses $k$ queries and receives their responses all at once. Let $n$ be a power of 2 and generate a ground truth hierarchical clustering $T^*$ as a full binary tree in which the leaves form a uniformly random permutation of the elements. For every constant $\delta > 0$, we prove that if the algorithm uses fewer than $n(n-1)(n-2)(1-\delta)/24 = \Omega(n^3)$ ordinal queries, then it fails to uniquely identify the clustering with probability at least $1-\delta$.

Because $T^*$ is a full binary tree, there are exactly $n/4$ disjoint clusters of size 4 each (Figure 1). Let $C$ be one of them. The 4 elements in $C$ are partitioned into two clusters of size 2 each. The algorithm can learn this internal structure of $C$ only if one of its queries consists of 3 (out of the 4) elements in $C$. Because the elements are uniformly randomly shuffled, the probability that the 3 elements of any one query all lie in $C$ is $\frac{24}{n(n-1)(n-2)}$.

Because there are $n/4$ such clusters $C$, if the algorithm queries $k$ triplets, in expectation, it learns the internal structure of at most $k \cdot \frac{n}{4} \cdot \frac{24}{n(n-1)(n-2)} = \frac{6k}{(n-1)(n-2)}$ clusters of size 4. If the algorithm succeeds with probability at least $1-\delta$, then it learns the internal structure of at least $(1-\delta)\frac{n}{4}$ size-4 clusters in expectation. Thus, $\frac{6k}{(n-1)(n-2)} \geq (1-\delta)\frac{n}{4}$ which implies $k \geq (1-\delta)n(n-1)(n-2)/24$.

B Proof of Lemma 5.6

In this section, we provide the proof of Lemma 5.6. We restate the lemma here for convenience.
Lemma 5.6 Assume that each query response is correct independently with probability \( p > \frac{1}{2} \), and adversarially incorrect with probability \( 1 - p \). There exists an adaptive algorithm with the following property: Given a tree \( T \) of diameter \( D \), and error parameter \( \delta > 0 \), the algorithm finds the target with probability at least \( 1 - \delta \), using at most \( O(D + \log(1/\delta)) \) vertex queries.

Our algorithm is a very minor modification of an algorithm of Feige et al. [17]. The algorithm performs a walk on the tree \( T \): in each iteration, a vertex \( v \) is queried, and if the response points to a neighbor \( v' \), the algorithm next “moves to” \( v' \). However, with every response confirming \( v \) as the target, the algorithm moves “deeper” into \( v \) (by incrementing a counter): subsequently, responses pointing to neighbors of \( v \) will only move the walk one step “shallower” in \( v \), and the walk will only leave \( v \) once the counter has been reduced down to 0.

More formally, each vertex \( v \) in \( T \) has a counter \( c(v) \). All the counters are initially 0, and at any point of the algorithm, at most one counter will have a non-zero (positive) value.

In the \( i^{th} \) iteration, the algorithm queries a vertex \( q_i \). The response to the query is either the node \( q_i \) itself or one of its neighbors in \( T \). If the query response is \( q_i \), i.e., the noisy response proposes \( q_i \) as the target, then the algorithm increments \( c(q_i) \) and will next query the same node \( q_{i+1} = q_i \). If the query response is a node other than \( q_i \), the algorithm’s action depends on \( c(q_i) \). If \( c(q_i) > 0 \), the algorithm decrements \( c(q_i) \) and will next query the same node \( q_{i+1} = q_i \). Finally, if \( c(q_i) = 0 \), then \( q_{i+1} \) is simply the vertex given in response to the query. The algorithm is given formally as Algorithm 6.

Algorithm 6 TreeWalk \((T, \delta)\)

1: \( c(v) \leftarrow 0 \) for every vertex \( v \) in \( T \).
2: Let \( q \) be an arbitrary vertex in \( T \).
3: for \( i = 1, \ldots, \max(\frac{2(D+1)}{2p-1}, \frac{8\log(1/\delta)}{(2p-1)^2}) \) do
4: Query the node \( q \), and let \( r \) be the noisy response.
5: if \( r = q \) then
6: \( c(q) = c(q) + 1 \).
7: else if \( c(q) > 0 \) then
8: \( c(q) = c(q) - 1 \).
9: else
10: \( q \leftarrow r \).
11: return \( q \)

We use \( c_i(v) \) to denote the value of the counter \( c(v) \) after the \( i^{th} \) iteration. Let \( d(v, v') \) denote the distance between \( v \) and \( v' \) in \( T \), and define the potential value of \( v \) in iteration \( i \) as

\[
\Phi_i(v) = d(v, q_i) - c_i(v) + \sum_{v' \neq v} c_i(v').
\]

The following lemma states that the potential of the (unknown) target node must decrease with each correct response to a query (and may increase with incorrect responses).

Lemma B.1 Let \( t \) be the (unknown) target. For every \( i \), if the response to the \( i^{th} \) query is correct, then \( \Phi_{i+1}(t) = \Phi_i(t) - 1 \); if the response is incorrect, then \( \Phi_{i+1}(t) \leq \Phi_i(t) + 1 \).

Proof. We distinguish two cases, based on the node queried in the \( i^{th} \) iteration.
• If \( q_i = t \), then a correct response increments \( t \)'s counter, and thus decreases \( t \)'s potential by 1. An incorrect response either decrements \( t \)'s counter or — if the counter was 0 — moves the query node to one of \( t \)'s neighbors. In either case, \( t \)'s potential is only increased by 1.

• If \( q_i \neq t \), then a correct response points towards \( t \). If \( c_i(q_i) > 0 \), then \( c(q_i) \) is decremented; if \( c_i(q_i) = 0 \), then the new query \( q_{i+1} = r_i \) is made to a node one step closer to \( t \). Therefore, \( t \)'s potential is only increased by 1.

An incorrect response could increment the counter of \( q_i \) (if it points to \( q_i \)), decrement the counter of \( q_i \) (if it points to an incorrect neighbor of \( q_i \) and \( c_i(q_i) > 0 \)), or move the next query \( q_{i+1} = r_i \) to an incorrect neighbor of \( q_i \) that is one step further away from \( t \). In all three cases, the potential of \( t \) increases or decreases by 1; in particular, it at most increases by 1.

Proof of Lemma 5.6 Algorithm 6 runs for \( k \) iterations. In expectation, a \( p \) fraction of the query responses are correct. Let \( \epsilon = \frac{2p-1}{4} < p \). Using Hoeffding’s inequality [21], with probability at least \( 1 - e^{-2k\epsilon^2} \), at least \((p - \epsilon)k \) query responses are correct. In that case, by Lemma B.1

\[
\Phi_k(t) \leq \Phi_0(t) - (p - \epsilon)k + (1 - p + \epsilon)k = d(t, q_0) - (2p - 1 - 2\epsilon)k \leq D - \frac{2p - 1}{2}k. \tag{1}
\]

Because \( k \geq \frac{\ln(1/\delta)}{2\epsilon^2} \), the success probability is at least \( 1 - e^{-2\epsilon^2 k} \geq 1 - \delta \). And because \( k \geq \frac{2(D+1)}{2p-1} \), Inequality (1) implies that \( \Phi_k(t) \leq D - (D + 1) < 0 \). Therefore, by definition of the potential value, \( c_k(t) > 0 \). The only node that could have a positive counter in any iteration \( i \) is \( q_i \); therefore, the returned node must be \( t \) in the high-probability case.