Extremal results for odd cycles in sparse pseudorandom graphs

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Abstract

We consider extremal problems for subgraphs of pseudorandom graphs. Our results implies that for $(n, d, \lambda)$-graphs $\Gamma$ satisfying

$$\lambda^{2k-1} \ll \frac{d^{2k}}{n} \left( \log n \right)^{-2(k-1)/(2k-1)}$$

any subgraph $G \subseteq \Gamma$ not containing a cycle of length $2k+1$ has relative density at most $\frac{1}{2} + o(1)$. Up to the polylog-factor the condition on $\lambda$ is best possible and was conjectured by Krivelevich, Lee and Sudakov.

Keywords: odd cycles, extremal graph theory, pseudorandom graphs
1 Introduction and main result

For two graphs $G$ and $H$, the generalized Turán number, denoted $	ext{ex}(G,H)$, is defined to be the largest number of edges an $H$-free subgraph of $G$ may have. Here, a graph $G$ is $H$-free if it contains no copy of $H$ as a (not necessarily induced) subgraph. With this notation, the well known Erdős-Stone theorem reads

$$\text{ex}(K_n,H) = \left(1 - \frac{1}{\chi(H)} + o(1)\right) \binom{n}{2}$$

where $\chi(H)$ denotes the chromatic number of $H$.

The systematic study of extensions of the Erdős-Stone theorem arising from replacing $K_n$ in (1) with a sparse random or a pseudorandom graph was initiated by Kohayakawa and collaborators (see, e.g., [9,10,11]). For random graphs such extensions were obtained recently in [7,15] (see also [4,14,6,13] for more recent developments).

Here, we continue the study for pseudorandom graphs. Roughly speaking, a pseudorandom graph is a graph whose edge distribution closely resembles that of a truly random graph of the same edge density. One way to formally capture this notion of pseudorandomness is through eigenvalue separation. A graph $G$ on $n$ vertices may be associated with a Boolean $n \times n$ adjacency matrix $A$. This matrix is symmetric and, hence, all its eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are real. If $G$ is $d$-regular, then $\lambda_1 = d$ and $|\lambda_n| \leq d$ by the Perron-Frobenius theorem. The difference in order of magnitude between $d$ and the second eigenvalue $\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$ of $G$ is often called the spectral gap of $G$. It is well known that the spectral gap provides a measure of control over the edge distribution of $G$. Roughly, the larger is the spectral gap the stronger is the resemblance between the edge distribution of $G$ and that of the random graph $G(n,p)$, where $p = d/n$. This phenomenon led to the notion of $(n,d,\lambda)$-graphs by which we mean $d$-regular $n$-vertex graphs satisfying $\lambda(G) \leq \lambda$.

Turán type problems for sparse pseudorandom graphs were studied, e.g. in [11,16,5]. In this paper, we continue in studying extensions of the Erdős-Stone theorem for sparse host graphs and determine upper bounds for the

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generalized Turán number for odd cycles in sparse pseudorandom host graphs, i.e., \( \text{ex}(G, C_{2k+1}) \) where \( G \) is a pseudorandom graph and \( C_{2k+1} \) is the odd cycle of length \( 2k + 1 \).

Our work is related to work of Sudakov, Szabó, and Vu [16] who determined \( \text{ex}(G, K_t) \) for a pseudorandom graph \( G \) and \( t \geq 3 \). Their result may be viewed as the pseudorandom counterpart of Turán’s theorem [19].

For any graph \( G \), the trivial lower bound \( \text{ex}(G, C_{2k+1}) \geq e(G)/2 \), where \( e(G) = |E(G)| \), follows from the fact that every graph \( G \) contains a bipartite subgraph with at least half the edges of \( G \). For \( G \cong K_n \), this bound is tight and our result asserts that this bound remains essentially tight for sufficiently pseudorandom graphs.

**Theorem 1.1** Let \( k \geq 1 \) be an integer. If \( \Gamma \) is an \((n, d, \lambda)\)-graph satisfying

\[
\lambda^{2k-1} \ll \frac{d^{2k}}{n} \left( \log n \right)^{-2(2k-1)},
\]

then

\[
\text{ex}(\Gamma, C_{2k+1}) = \left( \frac{1}{2} + o(1) \right) \frac{dn}{2}.
\]

For \( k = 1 \), the same problem was studied in [16]. In this case, we obtain the same result which is known to be best possible due to the construction of Alon [1]. For \( k \geq 2 \), Alon’s construction can be extended as to fit for general odd cycles [2]. This implies that for any \( k \geq 2 \), up the polylog-factor, the condition (2) is best possible and confirms a conjecture of Krivelevich, Lee and Sudakov [12]. Theorem 1.1 is a consequence of Theorem 1.2 stated below for the so called *jumbled* graphs. We recall this notion of pseudorandomness which can be traced back to Thomason [18].

Given \( p = p(n) \) and \( \gamma = \gamma(n) \), we say that an \( n \)-vertex graph \( \Gamma \) is \((p, \gamma)\)-*jumbled* if for all disjoint \( X, Y \subset V(\Gamma) \) we have

\[
|e(X, Y) - p|X||Y|\| \leq \gamma \sqrt{|X||Y|}.
\]

The following is our main result.

**Theorem 1.2** For every integer \( k \geq 1 \) and every \( \delta > 0 \) there exists a \( \gamma > 0 \) such that for every sequence of densities \( p = p(n) \) there exists an \( n_0 \) such that for any \( n \geq n_0 \) the following holds.

If \( \Gamma \) is an \( n \)-vertex \((p, \beta)\)-*jumbled* graph satisfying

\[
\beta \leq \gamma p^{\frac{1}{1+\frac{1}{2k+1}}} n \log^{-2(k-1)} n,
\]

Then...
then
\[ \text{ex}(\Gamma, C_{2k+1}) < \left( \frac{1}{2} + \delta \right) p \left( \frac{n}{2} \right). \]

By the so called \textit{expander mixing lemma} [3,17] an \((n, d, \lambda)-graph\) is \((p, \beta)-jumbled\) with \(p = d/n\) and \(\beta = \lambda\). Hence, Theorem 1.2 indeed implies Theorem 1.1.

2 Sketch of the proof of Theorem 1.2

Theorem 1.2 easily follows from Lemmas 2.1 and 2.2 stated below. To state Lemma 2.1, we employ the following notation.

For a graph \(G\) and disjoint vertex sets \(X, Y \subseteq V(G)\), we write \(G[X, Y]\) to denote the bipartite subgraph of \(G\) induced by the bipartition \(X \cup Y\). For a graph \(R\) and a positive integer \(m\), we write \(R(m)\) to denote the graph obtained by replacing every vertex \(i \in V(R)\) with a set of vertices \(V_i\) of size \(m\) and adding the complete bipartite graph between \(V_i\) and \(V_j\) whenever \(ij \in E(R)\). A spanning subgraph of \(R(m)\) is called an \(R(m)\)-graph. In addition, such a graph, say \(G \subseteq R(m)\), is called \((\alpha, p, \varepsilon)\)-degree-regular if \(\deg_{G[V_i, V_j]}(v) = (\alpha \pm \varepsilon)pm\) holds whenever \(ij \in E(R)\) and \(v \in V_i \cup V_j\).

The following lemma essentially asserts that under a certain assumption of jumbledness, a relatively dense subgraph of a sufficiently large \((p, \beta)\)-jumbled graph contains a degree-regular \(C_\ell(m)\)-graph with large \(m\).

\textbf{Lemma 2.1} For any integer \(\ell \geq 3\), all \(q > 0, \alpha_0 > 0\) and \(0 < \varepsilon < \alpha_0\) there exist a \(\nu > 0\) and a \(\gamma > 0\) such that for every sequence of densities \(p = p(n) \gg \log n/n\) there exists an \(n_0\) such that for every \(n \geq n_0\) the following holds.

Let \(\Gamma\) be an \(n\)-vertex \((p, \beta)\)-jumbled graph with \(\beta = \beta(n) \leq \gamma p^{1+\varepsilon}n\) and let \(G \subseteq \Gamma\) be a subgraph of \(\Gamma\) satisfying \(e(G) \geq \alpha_0 p(n_0)\). Then, there exists an \(\alpha \geq \alpha_0\) such that \(G\) contains an \((\alpha, p, \varepsilon)\)-degree-regular \(C_\ell(\nu n)\)-graph as a subgraph.  \(\square\)

Equipped with Lemma 2.1, we focus on large degree-regular \(C_\ell(m)\)-graphs hosted in a sufficiently jumbled graph \(\Gamma\). In this setting, we shall concentrate on odd cycles in \(\Gamma\) that have all but one of their edges in the hosted \(C_\ell(m)\)-graph. The remaining edge belongs to \(\Gamma\). The first part of Lemma 2.2 stated below provides a lower bound for the number of such configurations (see (5)). We now make this precise.

Fix a vertex labeling of \(C_{2k+1}\), say, \((u_k, \ldots, u_1, w, v_1, \ldots, v_k)\). For a jumbled
graph \( \Gamma \) (as in Lemma 2.2), let \( H \subseteq \Gamma \) be a \( C_{2k+1}(m) \)-graph with the corresponding vertex partition \((U_k, \ldots, U_1, W, V_1, \ldots, V_k)\). By \( \mathcal{C}(H, \Gamma) \) we denote the set of all cycles of length \((2k+1)\) of the form \((u'_k, \ldots, u'_1, w', v'_1, \ldots, v'_k)\) such that \( w' \in W, v'_i \in V_i, u'_i \in U_i, v'_k u'_k \in E(\Gamma) \), and all edges other than \( v'_k u'_k \in E(H) \). In other words, a member of \( \mathcal{C}(H, \Gamma) \) is a cycle of \( \Gamma \) of length \( 2k+1 \) with the additional requirement that the labeled edge \( v'_k u'_k \) connects the ends of the path of length \( 2k \) in \( H \). If \( v'_k u'_k \) is contained in \( H \), then clearly, \( H \) contains a \( C_{2k+1} \).

For a real number \( \mu > 0 \), an edge of \( \Gamma[V_k, U_k] \) is called \( \mu \)-saturated if such is contained in at least \( p(\mu pm)^{2k-1} \) members of \( \mathcal{C}(H, \Gamma) \). A cycle in \( \mathcal{C}(H, \Gamma) \) containing a \( \mu \)-saturated edge is called a \( \mu \)-saturated cycle. We write \( \mathcal{S}(\mu, H, \Gamma) \) to denote the set of \( \mu \)-saturated cycles in \( \mathcal{C}(H, \Gamma) \). To motivate the definition of \( \mu \)-saturated edges, note that we expect that an edge of \( \Gamma[U_k, V_k] \) extends to \((\alpha p)^{2k}m^{2k-1} \) members of \( \mathcal{C}(H, \Gamma) \). For \( \mu \approx \alpha \), a \( \mu \)-saturated edge overshoots this expectation by a factor of \( 1/\alpha \).

**Lemma 2.2** For any integer \( k \geq 1 \) and all reals \( 0 < \nu, \alpha_0 \leq 1, \) and \( 0 < \varepsilon \leq \alpha_0/3 \) there exists a \( \gamma > 0 \) such that for every sequence of densities \( p = p(n) \) there exists an \( n_0 \) such that for any \( n \geq n_0 \) the following holds.

If \( \Gamma \) is \((p, \beta)\)-jumbled \( n \)-vertex graphs with

\[
\beta = \beta(n) \leq \gamma p^{1+\frac{1}{k+1}} n \log^{-2(k-1)} n,
\]

then for any \( m \geq \nu n \) and any \( \alpha \geq \alpha_0 \) an \((\alpha, p, \varepsilon)\)-degree-regular \( C_{2k+1}(m) \)-graph \( H \subseteq \Gamma \) satisfies

\[
|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\varepsilon)^{2k} (pm)^{2k+1}
\]

and

\[
|\mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)| \leq (3\varepsilon)^{2k} (pm)^{2k+1}.
\]

\( \square \)

With Lemma 2.1 and Lemma 2.2 at hand Theorem 1.2 easily follows. Let \( G \) and \( \Gamma \) be as in Theorem 1.2. Using Lemma 2.1 we find an \((\alpha, p, \varepsilon)\)-degree-regular \( C_\ell(n) \)-graph with vertex partition \((U_k, \ldots, U_1, W, V_1, \ldots, V_k)\) as a subgraph of \( G \) where \( \alpha \geq 1/2 \). By (5) we find at least \((\alpha - 2\varepsilon)^{2k} (pm)^{2k+1} \) cycles of the form \((u'_k, \ldots, u'_1, w', v'_1, \ldots, v'_k)\) such that \( w' \in W, v'_i \in V_i, u'_i \in U_i, v'_k u'_k \in E(\Gamma) \) where all but the edge \( v'_k u'_k \) of the cycle is in \( H \). Call such an edge a forbidden edge and we wish to show that the set \( F \subset \Gamma[V_k, U_k] \) of forbidden edges intersects with \( E(H) \) which would prove the existence of a cycle of length \( 2k + 1 \) in \( H \subset G \). Choosing \( \varepsilon \) sufficiently small depending on \( \delta \) we obtain

\[
|F| \geq \frac{|\mathcal{C}(H, \Gamma) \setminus \mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)|}{p(\alpha + 2\varepsilon)^{2k-1} (pm)^{2k-1}} \geq \frac{(\alpha - 5\varepsilon)^{2k}}{(\alpha + 2\varepsilon)^{2k-1} pm^2} > \left( \alpha - \frac{\delta}{2} \right) pm.
\]
Hence, with \( \alpha \geq 1/2 \), we derive

\[
|F| + e(H[V_k, U_k]) \geq (2\alpha + \delta/2)pm^2 \geq (1 + \delta/2)pm^2 > e(\Gamma[V_k, U_k])
\]

and \( F \) must intersect \( E(H) \).

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