A viral propagation model with Holling type-II response function and free boundaries

Lei Li, Siyu Liu, Mingxin Wang

School of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China

Abstract

In this paper we put forward a viral propagation model with Holling type-II response function and free boundaries and investigate the dynamical properties. This model is composed of two ordinary differential equations and one partial differential equation, in which the spatial range of the first equation is the whole space \( \mathbb{R} \), and the last two equations have free boundaries. As a new mathematical model, we prove the existence, uniqueness and uniform estimates of global solution, and provide the criteria for spreading and vanishing, and long time behavior of the solution components \( u, v, w \). Comparing with the corresponding ordinary differential systems, the Basic Reproduction Number \( R_0 \) plays a different role. We find that when \( R_0 \leq 1 \), the virus cannot spread successfully; when \( R_0 > 1 \), the successful spread of virus depends on the initial value and varying parameters.

Keywords: Viral propagation model; Free boundaries; Basic reproduction number; Spreading-vanishing; Long time behavior.

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1 Introduction

Background In order to clarify the pathogenesis of diseases and seek effective treatment measures, viral dynamics have been a hot research topic (cf. [1, 2]), which usually cannot be answered by biological experimental methods alone but require the help of mathematical models. For this reason, a simple model was introduced few decades ago by Nowak and Bangham [3]. See also May [4]. The basic model of viral dynamics is the following set of differential equations

\[
\begin{align*}
    u' &= \theta - au - buw, \\
    v' &= bw - cv, \\
    w' &= kv - qw,
\end{align*}
\]

(1.1)

where \( u, v \) and \( w \) represent the population of uninfected cells, infected cells and viruses, respectively; uninfected cells are produced at a constant rate \( \theta \) and with death rate \( a \); the constant \( c \) is the death rate of infected cells; virus particles \( w \) infect uninfected cells with rate \( b \), and meanwhile virus particles are produced by infected cells with rate \( k \) and have death rate \( q \). It had been shown that if the Basic Reproduction Number \( R_0 = \theta b/(acq) < 1 \), then the system returns to the uninfected state \( (\theta/a, 0, 0) \). If \( R_0 > 1 \), then the system will converge to the unique positive equilibrium state \( \left( \frac{\theta}{ak}, \frac{b}{c}, \frac{\theta c - ab}{ak} \right) \). This indicates that in the initial stage of infection, if each infected cell infects
Mathematical Model  To investigate the impact of spatial dynamics on this model, Stancevic et al. [5] extended this model to include spatially random diffusion and spatially directed chemotaxis. Invoked by their ideas, we give the basic model assumptions as follows:

(i) A nonlinear response in the virus \( w \) could happen due to saturation at high virus concentration, where the infectious fraction is so high that exposure is very likely. Moreover, with the increase of the virus concentration the living environment for cells becomes worse and worse. Thus, it is reasonable for us to assume that the rate of infection for virus and the virion production rate for infected cells are both nonlinear. Here we take the Holling type-II response function and use

\[
    f_1(u, w) = \theta - au - \frac{buw}{1 + w}, \quad f_2(u, v, w) = \frac{buw}{1 + w} - cv, \quad f_3(v, w) = \frac{kv}{1 + w} - qw
\]

instead of the three terms in the right hand side of (1.1).

(ii) We assume that the major spatial dispersal comes from the moving (diffusion) of viruses in vivo, while both the uninfected and infected cells are immobile (do not diffuse). So we add only a diffusion term to the differential equation of viruses;

(iii) Since the infected cells are caused by viruses, their distribution range is the same;

(iv) The distribution of viruses and infected cells is a local range, which is small relative to the distribution of uninfected cells, so we think that uninfected cells are distributed over the whole space. Such kind of assumptions have been used in the species invasion models (cf. [6, 7, 8] for example);

(v) Initially, viruses are distributed over a local range \( \Omega_0 \) (the initial habitat). They will spread from boundary to expand their habitat as a result of the spatial dispersal freely. That is, as time \( t \) increases, \( \Omega_0 \) will evolve into expanding region \( \Omega(t) \) with expanding front \( \partial \Omega(t) \). Initial function \( v_0(x) \), and as a result \( v_0(x) \), will evolve into positive functions \( w(t, x) \) and \( v(t, x) \) which vanish on the moving boundary \( \partial \Omega(t) \);

(vi) For simplicity, we restrict our problem to the one dimensional case. Based on the deduction of free boundary conditions given in [9], we have the following free boundary conditions

\[
    g'(t) = -\mu w_x(t, g(t)), \quad h'(t) = -\beta w_x(t, h(t)).
\]

All of these assumptions (i)-(vi) suggest the following model, which governs the spatial and temporal evolution of viruses and cells, as well as free boundaries:

\[
\begin{align*}
    u_t &= f_1(u, w), & t > 0, \quad -\infty < x < \infty, \\
    v_t &= f_2(u, v, w), & t > 0, \quad g(t) < x < h(t), \\
    w_t - dw_{xx} &= f_3(v, w), & t > 0, \quad g(t) < x < h(t), \\
    v(t, x) &= w(t, x) = 0, & t > 0, \quad x \notin (g(t), h(t)), \\
    g'(t) &= -\mu w_x(t, g(t)), \quad h'(t) = -\beta w_x(t, h(t)), & t \geq 0, \\
    u(0, x) &= u_0(x), & -\infty < x < \infty, \\
    v(0, x) &= v_0(x), \quad w(0, x) = w_0(x), & -h_0 \leq x \leq h_0, \\
    h(0) &= -g(0) = h_0,
\end{align*}
\]
where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, $v(t, x)$ and $w(t, x)$; $d$, $h_0$, $\mu$, $\beta$, $\theta$, $a$, $b$, $c$, $k$, $q$ are positive constants.

Denote $C^1(I)$ the space of Lipschitz continuous functions in $I$. We assume that the initial functions $u_0, v_0, w_0$ satisfy

$$
\begin{cases}
    u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), & v_0 \in C^1(\mathbb{R}^+), & w_0 \in W_p^2([-h_0, h_0]), \\
    v_0(\pm h_0) = w_0(\pm h_0) = 0, & w'(-h_0) > 0, & w'(h_0) < 0, \\
    u_0 > 0 \text{ in } \mathbb{R}, & v_0, w_0 > 0 \text{ in } (-h_0, h_0)
\end{cases}
$$

with $p > 3$. Denote $L_0$ and $L_*$ the Lipschitz constant of $u_0$ and $v_0$ respectively.

Partially degenerate reaction-diffusion systems, which mean that several diffusion coefficients are zeros, have been increasingly applied to epidemiology, population biology etc; see [10, 11], for example. Some researchers have introduced the Stefan type free boundary to the partially degenerate systems, please refer to [12, 13, 14, 15] and the references therein.

**Aims and Main Results** In this paper, we investigate the dynamics of (1.2), and have the conclusion about the global existence, uniqueness, regularity and estimates of solution. Moreover, a spreading-vanishing dichotomy holds for (1.2), i.e., either

(i) **Spreading** (virus persistence): the virus successfully infects the uninfected cells and spreads itself to the uninfected area in the sense that $\lim_{t \to \infty} h(t) = -\lim_{t \to \infty} g(t) = \infty$, and

$$
\limsup_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} > 0, \hspace{0.5cm} \limsup_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} > 0.
$$

Moreover, if we suppose $R_0 + \sqrt{R_0} > b/a$, then

$$
\begin{align*}
    u_\infty &\leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_\infty, \\
v_\infty &\leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_\infty, \\
w_\infty &\leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \bar{w}_\infty
\end{align*}
$$

locally uniformly in $\mathbb{R}$ for some positive constants $u_\infty$, $\bar{u}_\infty$, $v_\infty$, $\bar{v}_\infty$, $w_\infty$ and $\bar{w}_\infty$. Particularly, under a stronger assumption that $b \leq 2a$, we will derive

$$
\begin{align*}
    \lim_{t \to \infty} u(t, x) &= u^*, \hspace{0.5cm} \lim_{t \to \infty} v(t, x) = v^*, \hspace{0.5cm} \lim_{t \to \infty} w(t, x) = w^* \text{ locally uniformly in } \mathbb{R},
\end{align*}
$$

where $(u^*, v^*, w^*)$ is the unique positive root of (4.9), or

(ii) **Vanishing** (virus dies out): the virus $w$ and the infected cells $v$ will vanish in a bounded area, i.e., $-\infty < \lim_{t \to \infty} g(t) < \lim_{t \to \infty} h(t) < \infty$ and

$$
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = 0, \hspace{0.5cm} \lim_{t \to \infty} u = \theta/a \text{ uniformly in } \mathbb{R}.
$$

Moreover, $\lim_{t \to \infty} h(t) - \lim_{t \to \infty} g(t) \leq \pi \sqrt{acd/(\theta kb - acq)}$ if $R_0 > 1$.

As for the Basic Reproduction Number $R_0 = \theta kb/(acq)$, in our results, we shall show that it plays a different role, comparing with the corresponding ordinary differential systems. When $R_0 \leq 1$, vanishing always happens, that is, the virus cannot spread successfully. On the other hand, when
Let $\mathcal{R}_0 > 1$, we have a criterion as follows: if the initial occupying area $[-h_0, h_0]$ is beyond a critical size, namely $2h_0 \geq \pi \sqrt{acd/(\theta kb - acq)}$, then spreading happens regardless of the moving parameter $\mu$, $\beta$ and initial population density $(u_0, v_0, w_0)$. While $2h_0 < \pi \sqrt{acd/(\theta kb - acq)}$, whether spreading or vanishing happens depends on the initial population density $(v_0, w_0)$ and the moving parameter $\mu$ and $\beta$.

The paper is organized as follows. Section 2 concerns with the existence, uniqueness and uniform estimates of solution of (1.2). Before ending this section we mention that in recent years, more and more free boundary problems of reaction diffusion systems have been introduced to describe the dynamics of species after the pioneering work [16]. Interested readers can refer to, except for the above cited papers, [17, 18, 19, 20, 21] for competition models, [22, 23, 24] for prey-predator models.

### 2 Existence, uniqueness and uniform estimates of solution of (1.2)

In this section we prove the global existence and uniqueness of the solution to problem (1.2). For convenience, we first introduce some notations. Denote

\[ A_1 = \max \{\|u_0\|_\infty, \theta/a\}, \quad B_1 = \|v_0\|_\infty + 1, \quad B_2 = \|w_0\|_\infty + 1, \]
\[ A = \{a, b, c, d, k, q, h_0, \mu, \beta, \alpha, A_1, B_1, B_2, \|w_0\|_{W^2_p([-h_0, h_0]), w_0(\pm h_0)}\}, \]
\[ \Pi_T = [0, T] \times \mathbb{R}, \quad \Delta_T = [0, T] \times [-1, 1], \quad D^T_{g,h} = \{0 \leq t \leq T, g(t) < x < h(t)\}. \]

Let $X$ be a Banach space and $\varphi, \psi \in X$. Denote $\|\varphi, \psi\|_X = \max\{\|\varphi\|_X, \|\psi\|_X\}$ for the simplicity.

**Theorem 2.1.** (Local solution) For any given $\alpha \in (0, 1)$ and $p > 3/(1 - \alpha)$, there exists a $T > 0$ such that the problem (1.2) has a unique local solution $(u, v, w, g, h) \in C^{1,1-}(\Pi_T) \times C^{1,1-}(D^T_{g,h}) \times W^{1,2}_p(D^T_{g,h}) \times [C^{1+\frac{2}{p}}([0,T])]^2$. Moreover,

\[ u > 0 \text{ in } \Pi_T; \quad v, w > 0 \text{ in } D^T_{g,h}; \quad g'(t) < 0, \quad h'(t) > 0 \text{ in } [0, T], \]

where $u \in C^{1,1-}(D^T_{g,h})$ means that $u$ is differentiable continuously in $t \in [0, T]$ and is Lipschitz continuous in $x \in [g(t), h(t)]$ for all $t \in [0, T]$.

**Proof.** Invoked by the proof of [13] Theorem 2.1 and [14] Theorem 1.1, we divide the proof into several steps. Unless otherwise specified in the proof, the positive $C_i$ depend only on $A$.

**Step 1:** Given $T > 0$, we say $u \in C^1_x(\Pi_T)$ if there is a constant $L_u(T)$ such that

\[ |u(t, x_1) - u(t, x_2)| \leq L_u(T)|x_1 - x_2|, \quad \forall \; x_1, x_2 \in \mathbb{R}, \; 0 < t \leq T. \]

For $s > 0$, define

\[ X^*_u = \{\phi \in C(\Pi_s) : \phi(0, x) = u_0(x), \; 0 \leq \phi \leq A_1\}. \]
For any given \( u \in X^1_{u_0} \cap C^1_x(\Pi_1) \) we consider the following problem

\[
\begin{aligned}
&v_t = f_2(\mu(t,x), v, w), \\
&w_t - dw_{xx} = f_3(v, w), \\
&v(t, x) = w(t, x) = 0, \\
&g'(t) = -\mu w_x(t, g(t)), h'(t) = -\beta w_x(t, h(t)) & t \geq 0,
\end{aligned}
\]

(2.1)

By [14] Theorem 1.1, we know that for some \( 0 < T \ll 1 \), (2.1) has a unique solution \( (v, w, g, h) \in C^{1,1}((\Omega^T_{g,h}) \times C^{1,1,1,1,1}((\Omega^T_{g,h}) \times [C^{1+\frac{\alpha}{2}}([0, T]])^2. \) Moreover,

\[
\begin{aligned}

&-w_x(t, h(t)), w_x(t, g(t)) > 0 \text{ in } [0, T]; \ 0 < v \leq B_1, 0 \leq w \leq B_2 \text{ in } D^T_{g,h}, \\
&\|w\|_{C^{1,1}((\Omega^T_{g,h})} + \|w\|_{C^{1,1}((\Omega^T_{g,h})} + \|g, h\|_{C^{1+\frac{\alpha}{2}}([0, T])} \leq M,
\end{aligned}
\]

(2.2)

where \( M \) depends only on \( A \).

Step 2: For the function \( w(t, x) \) obtained in Step 1, we consider the following parameterized ODE problem, for every \( x \in \mathbb{R} \),

\[
\begin{aligned}
&\ddot{u}_t = f_1(\ddot{u}, w(t, x)), \quad (t, x) \in (0, T) \times \mathbb{R}, \\
&\ddot{u}(0; x) = u_0(x) > 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

(2.3)

By the standard ODE theory, (2.3) has a unique solution \( \ddot{u} \in C^{1,1}((\Pi_T) \) and \( 0 < \ddot{u} \leq A_1 \).

Now we estimate the Lipschitz constant of \( \ddot{u} \) in \( x \). Since it can be easily derived from (2.2) that

\[
|w(t, x_1) - w(t, x_2)| \leq M|x_1 - x_2| \text{ for any given } (t, x_1), (t, x_2) \in \Pi_T, \text{ we have}
\]

\[
|\ddot{u}(t, x_1) - \ddot{u}(t, x_2)| = \left| \int_0^t \ddot{u}_t(s, x_1) - \ddot{u}_t(s, x_2)ds + u_0(x_1) - u_0(x_2) \right|
\]

\[
\leq \int_0^t |\ddot{u}_t(s, x_1) - \ddot{u}_t(s, x_2)|ds + L_0|x_1 - x_2|
\]

\[
\leq L_0|x_1 - x_2| + \int_0^t |f_1(\ddot{u}(s, x_1), w(s, x_1)) - f_1(\ddot{u}(s, x_2), w(s, x_2))|ds
\]

\[
\leq \int_0^t (a + bB_2)|\ddot{u}(s, x_1) - \ddot{u}(s, x_2)|ds + (bA_1 TM(B_2 + 1) + L_0)|x_1 - x_2|.
\]

Then noticing \( 0 < T \leq 1 \) and making use of the Gronwall inequality, we obtain

\[
|\ddot{u}(t, x_1) - \ddot{u}(t, x_2)| \leq (bA_1 M(B_2 + 1) + L_0)|x_1 - x_2|e^{(a + bB_2)}.
\]

This shows that \( L_{\ddot{u}} = (bA_1 M(B_2 + 1) + L_0)e^{(a + bB_2)} \) is the Lipschitz constant of \( \ddot{u} \). Define

\[
\mathbb{Y}^T_{u_0} = \{ \phi \in C(\Pi_T) : \phi(0, x) = u_0(x), \ 0 \leq \phi \leq A_1, \ |\phi(t, x) - \phi(t, y)| \leq L_{\ddot{u}}|x - y| \}.
\]

Obviously, \( \mathbb{Y}^T_{u_0} \) is complete with the metric \( d(\phi_1, \phi_2) = \sup_{\Pi_T} |\phi_1 - \phi_2| \). The above analysis allows us to define the map \( F(u) = \ddot{u} \), and \( F \) maps \( \mathbb{Y}^T_{u_0} \) into itself.
Step 3: We are in the position to prove that \( \mathcal{F} \) is a contraction mapping in \( Y^T_{u_0} \) for \( T \) small sufficiently. In fact, for \( i = 1, 2 \), let \( v_i, w_i, g_i, h_i \) be the unique solution of (2.1) with \( u = u_i \). By arguing as in the proof of [14, Theorem 1.1], we can show that there exists a constant \( L_v \), which only depends on \( \mathcal{A} \), such that for any given \( (t, x_1), (t, x_2) \in \overrightarrow{T}_{g_i, h_i} \), there holds:

\[
|v_i(t, x_1) - v_i(t, x_2)| \leq L_v|x_1 - x_2|.
\]  

Denote \( U = u_1 - u_2, \tilde{U} = \tilde{u}_1 - \tilde{u}_2, V = v_1 - v_2 \) and \( W = w_1 - w_2 \). Since \( \tilde{u}_i \) satisfy

\[
\begin{align*}
\tilde{u}_{i,t} &= f_1(\tilde{u}_i, w_i), & (t, x) &\in (0, T] \times \mathbb{R}, \\
\tilde{u}_i(0, x) &= u_0(x) > 0, & x &\in \mathbb{R},
\end{align*}
\]

it follows that, for any \( (t, x) \in \Pi_T \),

\[
|\tilde{U}(t, x)| \leq \int_0^t (a + bB_2)|\tilde{U}(s, x)|ds + bA_1(B_2 + 1)T\|W\|_{L^\infty(\Pi_T)}.
\]

By virtue of the Gronwall inequality again, it yields

\[
|\tilde{U}(t, x)| \leq bA_1(B_2 + 1)e^{a(T + bT + 1)}\|W\|_{L^\infty(\Pi_T)}.
\]  

The followings are devoted to the estimate of \( \|W\|_{L^\infty(\Pi_T)} \). Evidently, \( w_i \) satisfy

\[
\begin{align*}
w_{i,t} - dw_{i,xx} &= f_3(v_i, w_i), & 0 < t \leq T, & g_i(t) < x < h_i(t), \\
w_i(t, x) &= 0, & 0 < t \leq T, & x \notin (g_i(t), h_i(t)), \\
w_i(0, x) &= w_0(x), & |x| &\leq h_0.
\end{align*}
\]

We straighten the boundaries and define

\[
x_i(t, y) = \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2}, \quad z_i(t, y) = w_i(t, x_i(t, y)), \quad r_i(t, y) = v_i(t, x_i(t, y)).
\]

For simplicity, we introduce the following notations \( \xi = \xi_1 - \xi_2, \zeta = \zeta_1 - \zeta_2, z = z_1 - z_2, r = r_1 - r_2, h = h_1 - h_2, g = g_1 - g_2, \) where

\[
\begin{align*}
\xi_i(t) &= 4\left(\frac{h_i(t) - g_i(t)}{h_i(t) - g_i(t)}\right)^2, \\
\zeta_i(t, y) &= \frac{h_i(t) + g_i(t)}{h_i(t) - g_i(t)}, \\
r_i(t, y) &= \frac{h_i(t) - g_i(t)}{h_i(t) - g_i(t)}y.
\end{align*}
\]

Then \( z \) satisfies

\[
\begin{align*}
z_t - d\xi_1z_{yy} - \xi_1z_y &= d\xi z_{2,yy} + \zeta z_{2,y} + \frac{kr}{1 + z_1} - \frac{kr_2}{(1 + z_1)(1 + z_2)} - qz, & 0 < t \leq T, |y| < 1, \\
z(t, \pm 1) &= 0, & 0 \leq t \leq T, \\
z(0, y) &= 0, & |y| \leq 1.
\end{align*}
\]

By the \( L^p \) estimates for parabolic equations, we see

\[
\|z\|_{W^{1,2}_p(\Delta_T)} \leq C_1(\|g\|_{C^1(\overrightarrow{T})} + \|h\|_{C^1(\overrightarrow{T})}).
\]

We now estimate \( \|r\|_{C(\Delta_T)} \). For any given \( (t, y) \in \Delta_T \), it follows that

\[
|r(t, y)| = |v_1(t, x_1(t, y)) - v_2(t, x_2(t, y))| \leq |V(t, x_1(t, y))| + |v_2(t, x_1(t, y)) - v_2(t, x_2(t, y))|.
\]
It follows from the inequality (2.4) that
\[ |v_2(t, x_1(t, y)) - v_2(t, x_2(t, y))| \leq C_2 \|g, h\|_{C^1([0, T])}. \]

Additionally, we can prove the following inequality:
\[ |V(t, x_1(t, y))| \leq C_3 \left( \|g, h\|_{C^1([0, T])} + T \|U, W\|_{L^\infty(\Pi_T)} \right). \quad (2.6) \]

Its proof will be put in the next step on account of the length. Thus we have
\[ \|r\|_{C(\Delta_T)} \leq C_4 \left( \|g, h\|_{C^1([0, T])} + T \|U, W\|_{L^\infty(\Pi_T)} \right). \]

Then it follows that
\[ \|z\|_{W_p^{1,2}(\Delta_T)} \leq C_5 \left( \|g, h\|_{C^1([0, T])} + T \|U, W\|_{L^\infty(\Pi_T)} \right). \]

By utilizing the similar methods in Step 2 of [25, Theorem 2.1] and the embedding theorem:
\[ \|z\|_{C^{0,\alpha}(\Delta_T)} \leq C \|z\|_{W_p^{1,2}(\Delta_T)} \]
for some positive constant $C$ independent of $T^{-1}$ ( [26, Theorem 1.1]),
we can show that
\[ \|W\|_{L^\infty(\Pi_T)} \leq C_6 \left( T \|W\|_{L^\infty(\Pi_T)} + \|U\|_{L^\infty(\Pi_T)} \right). \]

Hence
\[ \|W\|_{L^\infty(\Pi_T)} \leq 2C_6 \|U\|_{L^\infty(\Pi_T)} \text{ if } 0 < T \ll 1. \]

This combined with (2.5) arrives at
\[ \|\bar{U}\|_{L^\infty(\Pi_T)} \leq C_7 T \|U\|_{L^\infty(\Pi_T)} \leq \frac{1}{2} \|U\|_{L^\infty(\Pi_T)} \text{ if } 0 < T \ll 1. \]

As a consequence, $F$ is a contraction mapping and there exists the unique local solution $(u, v, w, g, h)$. Moreover the desired properties of the local solution can be obtained from the above arguments.

**Step 4:** In this step, we are going to tackle the estimate (2.6), which will be divided into several cases. By the definition of $x_1(t, y)$, it is easy to see that $g_1(t) \leq x_1(t, y) \leq h_1(t)$. We denote $x_1 = x_1(t, y)$ for the simplicity.

- **Case 1:** $x_1 \notin (g_2(t), h_2(t))$. In this case $v_2(t, x_1) = 0$, and either $g_1(t) \leq x_1 \leq g_2(t)$ or $h_2(t) \leq x_1 \leq h_1(t)$. We only deal with the former case. Hence

\[
V(t, x_1) = |v_1(t, x_1) - v_1(t, g_1(t))| \leq L_v |x_1 - g_1(t)| \\
\leq L_v |g_2(t) - g_1(t)| \leq L_v \|g\|_{C^1([0, T])}.
\]

- **Case 2:** $x_1 \in (g_2(t), h_2(t))$ and either $x_1 > h_0$ or $x_1 < -h_0$. We deal with only the case $x_1 > h_0$. Then we can uniquely find $0 < t_{x_1}, t'_{x_1} \leq t$ such that $h_1(t_{x_1}) = x_1$ and $h_2(t'_{x_1}) = x_1$. Then $v_1(t_{x_1}, x_1) = v_2(t'_{x_1}, x_1) = 0$. Without loss of generality, we assume $t'_{x_1} > t_{x_1}$. Then $h_1(t'_{x_1}) > h_1(t_{x_1}) = x_1 = h_2(t'_{x_1})$, $x_1 \in (g_1(s), h_1(s)) \cap (g_2(s), h_2(s))$ for all $t'_{x_1} < s \leq t$ and $x_1 \in (g_1(t_{x_1}), h_1(t_{x_1})) \setminus (g_2(t'_{x_1}), h_2(t'_{x_1}))$. Hence,

\[
V(t_{x_1}, x_1) = v_1(t'_{x_1}, x_1) \leq L_v \|g, h\|_{C^1([0, T])}
\]
by the conclusion of Case 1. Integrating the differential equation of $v_1$ from $t_{x_1}'$ to $s$ $(t'_{x_1} < s \leq t)$ we obtain
\[
v_1(s, x_1) = v_1(t_{x_1}', x_1) + \int_{t_{x_1}'}^{s} f_2(u_1, v_1, w_1) \big|_{x=x_1} d\tau,
\]
\[ v_2(s, x_1) = \int_{t_1}^{s} f_2(u_2, v_2, w_2) \big|_{x=x_1} \, dt. \]

It then follows that
\[
|V(s, x_1)| \leq v_1(t_1, x_1) + \int_{t_1}^{s} \left| \frac{b_1 u_1}{1 + w_1} - \frac{b_2 u_2}{1 + w_2} + c(v_2 - v_1) \right| \bigg|_{x=x_1} \, dt \\
\leq L_v \| g, h \|_{C([0,T])} + TC_8 \left( \| V(\cdot, x_1) \|_{C([t_1, t])} + \| U, W \|_{L^\infty(\Pi_T)} \right),
\]
where \( C_8 = \max \{ c, bA_1(B_2 + 1), bB_2 \} \). It follows from that
\[
|V(t, x_1)| \leq C_{10} \left( \| g, h \|_{C^1([0,T])} + T \| U, W \|_{L^\infty(\Pi_T)} \right)
\]
if \( T > 0 \) is sufficiently small.

- Case 3: \( x_1 \in (g_2(t), h_2(t)) \) and \( x_1 \in [-h_0, h_0] \). In this case we can derive
\[
|V(t, x_1)| \leq C_{11} T \| U, W \|_{L^\infty(\Pi_T)}
\]
by using similar methods. Since it is actually much simpler, we omit the details. In conclusion, we have proved the estimate \( \text{(2.6)} \). \( \blacksquare \)

**Theorem 2.2.** (Global solution) The problem \( \text{(1.2)} \) has a unique global solution \((u, v, w, g, h)\), and there exist four positive constants \( A_i, \ i = 1, 2, 3, 4, \) such that
\[
(u, v, w, g, h) \in C^{1, 1} (\Pi_\infty) \times C^{1, 1} (\Pi_\infty) \times W^{1, 2}_p (D^\infty_{g, h}) \times [C^{1 + \frac{d}{2}}([0, \infty)])^2; \\
0 < u \leq A_1 \text{ in } \Pi_\infty; \ 0 < v \leq A_2, \ 0 < w \leq A_3 \text{ in } D^\infty_{g, h}; \ 0 < -g'(t), h'(t) \leq A_4 \text{ in } [0, \infty),
\]
where \( A_1 = \max \{ \| u_0 \|_{\infty}, \theta/a \} \).

**Proof.** It follows from Theorem 2.1 that the problem \( \text{(1.2)} \) has the unique local solution \((u, v, w, g, h)\) for some \( 0 < T \ll 1 \) and \( g'(t) < 0, h'(t) > 0 \) for \( 0 < t \leq T \).

It is easy to show that \( 0 < u \leq A_1 \) in \( \Pi_T \). Recalling the equations of \((v, w)\) we can readily conclude that there exists \( A_2, A_3 > 0 \) such that \( 0 < v \leq A_2, 0 < w \leq A_3 \text{ in } D^\infty_{g, h} \). Making use of the similar arguments in the proof of 2.7, Lemma 2.1, we can show that there exists constant \( A_4 > 0 \), which only depends on the initial data, such that \( 0 < -g'(t), h'(t) \leq A_4 \text{ in } [0, T] \).

With these above estimates, we can extend the unique local solution uniquely to the global solution, and
\[
(u, v, w, g, h) \in C^{1, 1} (\Pi_\infty) \times C^{1, 1} (\Pi_\infty) \times W^{1, 2}_p (D^\infty_{g, h}) \times [C^{1 + \frac{d}{2}}([0, \infty)])^2;
\]
see 2.6, Corollary 1.1) for the details. It follows from the standard parabolic regularity theory that \((u, v, w, g, h)\) is the unique classical solution of \( \text{(1.2)} \). Combining \( v(t, x) = 0 \) for \( x \notin (g(t), h(t)) \) and the equation satisfied by \( v \), we easily derive that \( v \in C^{1, 1} (\Pi_\infty) \). The proof is ended. \( \blacksquare \)

Since \( g'(t) < 0, h'(t) > 0 \), there exist \( g_\infty \in [-\infty, 0) \) and \( h_\infty \in (0, \infty] \) such that
\[
\lim_{t \to \infty} g(t) = g_\infty, \quad \lim_{t \to \infty} h(t) = h_\infty.
\]
The case \( h_\infty = -g_\infty = \infty \) is called *Spreading*, and the case \( h_\infty - g_\infty < \infty \) is called *Vanishing*. 8
Theorem 2.3. (Uniform estimates) Let \((u, v, w, g, h)\) be the unique global solution of (1.2). Then there exists a constant \(C > 0\) such that

\[
\|w(t, \cdot)\|_{C^1([g(t), h(t)])} \leq C, \quad \|g', h'\|_{C^{\alpha/2(1, \infty)}} \leq C, \quad \forall \ t \geq 1.
\] (2.7)

Proof. Remember \(0 \leq v \leq A_2^2, 0 \leq w \leq A_3^3\). The estimates (2.7) can be proved by using analogous methods in [28, Theorem 2.1] for the case \(h_\infty - g_\infty < \infty\) and [24, Theorem 2.2] for the case \(h_\infty - g_\infty = \infty\). We omit the details here.

3 Preliminaries

In this section, we will show some preliminaries which are crucial in the later parts. First we will investigate an eigenvalue problem and analyze the properties of its principal eigenvalue which will pave the ground for later discussion. It is well known that the eigenvalue problem

\[
\begin{cases}
d\hat{\phi}_{xx} + a_{11}\hat{\phi} = \rho\hat{\phi}, & l_1 < x < l_2, \\
\hat{\phi}(l_i) = 0, & i = 1, 2
\end{cases}
\]

has a principle eigenpair \((\rho_1, \hat{\phi}_1)\), where

\[
\rho_1 = a_{11} - \frac{d\pi^2}{(l_2 - l_1)^2}, \quad \hat{\phi}_1(x) = \cos \frac{\pi(2x - l_2 - l_1)}{2(l_2 - l_1)}.
\]

Now we consider the following eigenvalue problem

\[
\begin{cases}
d\phi_{xx} + a_{11}\phi + a_{12}\psi = \lambda\phi, & l_1 < x < l_2, \\
a_{21}\phi + a_{22}\psi = \lambda\psi, & l_1 < x < l_2, \\
\phi(l_i) = \psi(l_i) = 0, & i = 1, 2
\end{cases}
\]

(3.1)

with \(a_{12}, a_{21} > 0\) and \(a_{11}, a_{22} < 0\). Define

\[
\mathcal{L} = \begin{pmatrix} d\partial_{xx} + a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\]

and choose the domain of \(\mathcal{L}\):

\[
\mathcal{D}(\mathcal{L}) = \{(\phi, \psi) \in H^2((l_1, l_2)) \times L^2((l_1, l_2)) : \phi(l_i) = \psi(l_i) = 0, i = 1, 2\}.
\]

Similar to the proof of [15, Theorem 3.1], we can prove the following results by means of [20, Theorem 2.3, Remark 2.2]. The details are omitted here.

Theorem 3.1. Let \(\sigma(\mathcal{L})\) be the spectral set of \(\mathcal{L}\) and \(s(\mathcal{L}) := \sup\{\text{Re}\lambda : \lambda \in \sigma(\mathcal{L})\}\). Then the followings hold true:

(i) \(s(\mathcal{L})\) is the principal eigenvalue of (3.1) with positive eigenvectors \((\phi_1, \psi_1)\);

(ii) \(s(\mathcal{L}) = \frac{1}{2}[\rho_1 + a_{22} + \sqrt{(\rho_1 - a_{22})^2 + 4a_{12}a_{21}}]\) and has the same sign with \(\rho_1 - a_{12}a_{21}/a_{22}\);

(iii) \(s(\mathcal{L})\) is strictly monotone increasing in the length of the interval \((l_1, l_2)\) and strictly monotone decreasing in \(d\).
By Theorem 3.1 we can easily deduce the following results.

**Corollary 3.2.** Define $\Gamma = a_{11} - \frac{a_{12}a_{21}}{a_{22}}$. Let $\lambda_1$ be the principle eigenvalue of the problem [3.1]. Then the followings are valid:

(i) If $\Gamma \leq 0$, then $\lambda_1 < 0$ for any $d > 0$ and ($l_1, l_2$);

(ii) If $\Gamma > 0$, we fix the domain ($l_1, l_2$) and let $d^*(l_1, l_2) = \Gamma(l_2 - l_1)^2/2\pi^2$. Then $\lambda_1 > 0$ when $0 < d < d^*(l_1, l_2)$, $\lambda_1 = 0$ when $d = d^*(l_1, l_2)$, and $\lambda_1 < 0$ when $d > d^*(l_1, l_2)$;

(iii) If $\Gamma > 0$, we fix $d > 0$ and set $L^*(d) = \pi\sqrt{d/\Gamma}$. Then $\lambda_1 > 0$ when $l_2 - l_1 > L^*(d)$, $\lambda_1 = 0$ when $l_2 - l_1 = L^*(d)$ and $\lambda_1 < 0$ when $l_2 - l_1 < L^*(d)$.

Let $\lambda_1$ be the principle eigenvalue of (3.1), that is, two components of the corresponding eigenfunction are both positive or negative. Then we have

$$\lambda_1 > a_{22}, \rho_1 = \lambda_1 - \frac{a_{12}a_{21}}{a_{11} - a_{22}}.$$ 

Thus by the uniqueness of $\rho_1$ we easily derive the uniqueness of the principle eigenvalue of (3.1).

Let $(\mu_1, u_1)$ be the first eigenpair of $-\Delta$ with homogeneous Dirichlet boundary on $(l_1, l_2)$ and $(\lambda_1, \phi_1, \psi_1)$ be the principal eigenpair of the problem (3.1). The direct calculation yields

$$\begin{pmatrix} -d\mu_1 + a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \phi_1, u_1 \rangle \\ \langle \psi_1, u_1 \rangle \end{pmatrix} = \lambda_1 \begin{pmatrix} \langle \phi_1, u_1 \rangle \\ \langle \psi_1, u_1 \rangle \end{pmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2((l_1, l_2))$.

The following lemma will play an important role in the study of long time behaviors of $(u, v, w)$ when $h_\infty - g_\infty < \infty$.

**Lemma 3.3.** Let $m(t, x)$ be a bounded function, $d, C, \mu$ and $\eta_0$ be positive constants, and constant $x_0 < \eta_0$. Let $\eta \in C^1([0, \infty))$, $w \in W_p^1,2((0, T) \times (x_0, \eta(t)))$ and $w_0 \in W_p^2((x_0, \eta_0))$ for some $p > 1$ and any $T > 0$, and $w_x \in C([0, \infty) \times (x_0, \eta(t)))$. If $(w, \eta)$ satisfies

$$\begin{cases} w_t - dw_{xx} + m(t, x)w_x \geq -Cw, & t > 0, \ x_0 < x < \eta(t), \\
w \geq 0, & t > 0, \ x = x_0, \\
w = 0, & t > 0, \ x = \eta(t), \\
w(0, x) = w_0(x) \geq 0, & x \in (x_0, \eta_0), \\
\eta(0) = \eta_0, \
\end{cases}$$

and $\lim_{t \to \infty} \eta(t) = \eta_\infty < \infty$, $\lim_{t \to \infty} \eta(t) = 0$,

$$\|w(t, \cdot)\|_{C^1(x_0, \eta(t))} \leq M, \ \forall t \geq 1$$

for some constant $M > 0$. Then $\lim_{t \to \infty} \max_{x_0 \leq x \leq \eta(t)} w(t, x) = 0$.

**Proof.** When $x_0 = 0$ and $m(t, x) = 0$, this lemma is exactly [30 Proposition 2]; when $x_0 = 0$ and $m(t, x) = \gamma$ is a constant, this lemma is exactly [31 Lemma 3.3]. For our present case, by the maximum principle we have $w(t, x) > 0$ for $t > 0$ and $x_0 < x < \eta(t)$. Follow the proof of [20 Theorem 2.2] word by word we can prove this lemma. We will leave out the details because the advection term and boundary condition at $x = x_0$ do not influence the availability of the argument in [20 Theorem 2.2]. \[\square\]
Lemma 3.4. (Comparison principle) Let $T > 0$, $\bar{g}, \bar{h} \in C^1([0,T])$ and $\bar{g} < \bar{h}$ in $[0,T]$. Let $\tilde{u} \in C^{1,0}([0,T] \times \mathbb{R})$, $\bar{v} \in C^{1,0}(\mathcal{O})$, $\bar{w} \in C(\mathcal{O}) \cap C^{1,2}(O)$ with $O = \{0 < t \leq T, \bar{g}(t) < x < \bar{h}(t)\}$. Assume that $(\tilde{u}, \bar{v}, \bar{w}, \bar{g}, \bar{h})$ satisfies

$$
\begin{align*}
\tilde{u}_t &\geq \theta - a\tilde{u}, \\
\bar{v}_t &\geq f_2(\bar{u}, \bar{v}, \bar{w}), \\
\bar{w}_t - d\bar{w}_{xx} &\geq f_3(\bar{v}, \bar{w}), \\
\bar{v}(t,x) &= \bar{w}(t,x) = 0, \\
g'(t) &\leq -\mu \bar{w}_x(t, \bar{g}(t)),
\end{align*}
$$

If $\bar{g}(0) \leq -\bar{h}, \bar{h}(0) \geq \bar{h}, u_0(x) \leq \tilde{u}(0,x)$ in $\mathbb{R}$, and $v_0(x) \leq \bar{v}(0,x), w_0(x) \leq \bar{w}(0,x)$ on $[-\bar{h}, \bar{h}]$. Then the solution $(u,v,w,g,h)$ of (1.2) satisfies

$$g \geq \bar{g}, \ h \leq \bar{h} \ \text{on} \ [0,T]; \ u \leq \tilde{u} \ \text{on} \ [0,T] \times \mathbb{R}; \ v \leq \bar{v}, w \leq \bar{w} \ \text{on} \ \Omega_{g,h},$$

where $\Omega_{g,h}$ is defined as in the beginning of Section 2.

Proof. Take $0 < \rho < 1$ and let $(u_\rho, v_\rho, w_\rho, g_\rho, h_\rho)$ be the corresponding unique solution of (1.2) with $(h_0, v_0, w_0)$ replaced by $(\rho h_0, v_0, w_0, \rho h_0)$, where $v_0, w_\rho(x)$ satisfy (1.3) in there $h_0$ replaced by $\rho h_0$, and satisfy

$$0 < v_{0,\rho}(x) \leq v_0(x), \ 0 < w_{0,\rho}(x) \leq w_0(x) \ \text{on} \ (-\rho h_0, \rho h_0),$$

as well as

$$\lim_{\rho \to 1} v_{0,\rho}(\rho x) = v_0(x) \ \text{in} \ W^1_\infty((-\rho h_0, \rho h_0)), \ \lim_{\rho \to 1} w_{0,\rho}(\rho x) = w_0(x) \ \text{in} \ W^2_p((-\rho h_0, \rho h_0)).$$

By a simple comparison consideration, we have $u_\rho \leq \tilde{u}$ on $[0, T] \times \mathbb{R}$. Thus $(v_\rho, w_\rho)$ satisfies

$$
\begin{align*}
v_{\rho,t} &\leq f_2(\bar{u}, v_\rho, w_\rho), \\
w_{\rho,t} - d w_{\rho,xx} &\leq f_3(v_\rho, w_\rho), \\
v_\rho(t,x) &= w_\rho(t,x) = 0, \\
g_\rho(t) &\leq -\mu w_\rho(t,g_\rho(t)), \ h_\rho(t) = -\beta w_\rho(t,h_\rho(t)),
\end{align*}
$$

Similar to (10) Lemma 3.5], by use of the indirect arguments and strong maximum principle we can show that $g_\rho(t) > \bar{g}(t), h_\rho(t) < \bar{h}(t)$ for $0 \leq t \leq T$. Thus $v_\rho(t,x) < \bar{v}(t,x), w_\rho(t,x) < \bar{w}(t,x)$ for $0 < t \leq T$ and $g_\rho(t) \leq x \leq h_\rho(t)$ by the standard comparison principle. Letting $\rho \to 1$ and using the continuous dependence of solution on parameters we have $(u_\rho, v_\rho, w_\rho, g_\rho, h_\rho) \to (u, v, w, g, h)$.

The details are omitted.
4 Long time behavior of \((u, v, w)\)

This section concerns with the long time behavior of \((u, v, w)\). We first study the vanishing case \((h_\infty - g_\infty < \infty)\).

**Theorem 4.1.** Let \((u, v, w, g, h)\) be the unique global solution of (1.2). If \(h_\infty - g_\infty < \infty\), then

\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0,
\]

\[
\lim_{t \to \infty} u(t, x) = \theta/a \quad \text{uniformly in } \mathbb{R}.
\]

**Proof.** Recall the second estimate in (2.7). It is easy to deduce that \(\lim g'(t) = \lim h'(t) = 0\). Then, using the first estimate of (2.7), and Lemma 3.3 in \([0, h(t)]\) and a similar version of Lemma 3.3 in \((g(t), 0]\), one can arrive at \(\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = 0\). For any \(\varepsilon > 0\), there exists \(T > 0\) such that \(\frac{b\varepsilon w}{1+w} \leq \varepsilon\) for \(t \geq T\) and \(x \in \mathbb{R}\). Thus \(v\) satisfies

\[
\begin{cases}
vt \leq \varepsilon - cv, & t \geq T, g(t) < x < h(t), \\
v(t, g(t)) = v(t, h(t)) = 0, & t \geq T, \\
v(T, x) \geq 0.
\end{cases}
\]

By the comparison principle, we have \(\limsup_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} \leq \varepsilon/c\). The arbitrariness of \(\varepsilon\) implies \(\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0\). Similarly, we can easily deduce that

\[
\limsup_{t \to \infty} u(t, x) \leq \theta/a \quad \text{uniformly in } \mathbb{R}. \tag{4.1}
\]

On the other hand, for any \(\varepsilon_1 > 0\), there exists \(T_1 > 0\) such that \(w/(1+w) \leq \varepsilon_1\) for \(t \geq T_1\) and \(x \in \mathbb{R}\). So \(u\) satisfies

\[
\begin{cases}
u_t \geq \theta - (a + b\varepsilon_1)u, & t \geq T_1, x \in \mathbb{R}, \\
u(T_1, x) > 0, & x \in \mathbb{R}.
\end{cases}
\]

Let \(\underline{u}\) be the unique solution of the problem

\[
\begin{cases}
\underline{u}_t = \theta - (a + b\varepsilon_1)\underline{u}, & t \geq T_1, \\
\underline{u}(T_1) = 0.
\end{cases}
\]

By using the comparison principle and the fact that \(\lim_{t \to \infty} u(t) = \theta/(a + b\varepsilon_1)\), we have that \(\liminf_{t \to \infty} u \geq \theta/(a + b\varepsilon_1)\) uniformly in \(\mathbb{R}\). Thanks to the arbitrariness of \(\varepsilon_1\) and the inequality (4.1), we derive the desired result. \(\square\)

In the following we study the spreading case \((h_\infty - g_\infty = \infty)\). To get the accurate limits of the solution components \((u, v, w)\) of (1.2), we first give a proposition which concerns the existence, uniqueness and asymptotic behavior of positive solution of a boundary value problem.

**Proposition 4.2.** Let \(m, l\) be positive constants and consider the following problem

\[
\begin{cases}
f_2(m, v, w) = 0, & -l < x < l, \\
-dw_{xx} = f_3(v, w), & -l < x < l, \\
v(x) = w(x) = 0, & x = \pm l.
\end{cases} \tag{4.2}
\]
Let $\lambda_1$ be the principle eigenvalue of

\[
\begin{cases}
  d\phi_{xx} - q\phi + k\psi = \lambda \phi, & -l < x < l, \\
  bm\phi - c\psi = \lambda \psi, & -l < x < l, \\
  \phi(\pm l) = \psi(\pm l) = 0.
\end{cases}
\]  

(4.3)

Then (4.2) has a positive solution if and only if $\lambda_1 > 0$. Moreover, the positive solution of (4.2) is unique when it exists.

(ii) To stress the dependence on $l$, we denote the unique positive solution of (4.2) by $(v_l, w_l)$. Then $(v_l, w_l)$ is nondecreasing in $l$, and converges to $(\tilde{v}, \tilde{w})$ locally uniformly in $\mathbb{R}$ as $l \to \infty$, where $(\tilde{v}, \tilde{w})$ is the unique positive root of

\[
f_2(m, v, w) = 0, \quad f_3(v, w) = 0.
\]  

(4.4)

Proof. (i) Clearly, the problem (4.2) is equivalent to

\[
\begin{cases}
  -dw_{xx} + qw = \frac{kbmw}{c(1 + w)^2}, \quad v = \frac{bmw}{c(1 + w)}, & -l < x < l, \\
  w(x) = 0, & x = \pm l.
\end{cases}
\]  

For clarity of exposition, we will always use the problem (4.2) in later discussion.

If (4.2) has a positive solution $(v, w)$, it is easy to show that $q < \lambda_1(q) < \frac{kbm}{c}$, where $\lambda_1(q)$ is the principle eigenvalue of

\[
\begin{cases}
  -d\phi_{xx} + q\phi = \lambda \phi, & -l < x < l, \\
  \phi(x) = 0, & x = \pm l.
\end{cases}
\]  

(4.5)

Moreover, since $\lambda_1(q) < \frac{kbm}{c}$, and the function $bm/(c + x) - x$ is decreasing in $x > -c$, we can show that there exists the unique $\lambda^* > 0$ such that $bm/(c + \lambda^*) - \lambda^* = \lambda_1(q)$. Substituting this into (4.5), one can easily see that $\lambda^*$ is the principle eigenvalue of (4.3), that is, $\lambda^* = \lambda_1$.

If $\lambda_1 > 0$, by the standard upper and lower solution methods we can show that (4.2) has at least one positive solution. Thanks to the structure of nonlinear terms of (4.2), the uniqueness is easily derived.

(ii) It follows from the above analysis and Corollary 3.2 that for the large $l$, (4.2) has the unique positive solution $(v_l, w_l)$ provided that $kbm > qc$. A comparison argument shows that $(v_l, w_l)$ is nondecreasing in $l$, and there exists $C > 0$ such that $v_l, w_l < C$ for all large $l$. Making use of the standard elliptic regularity theory, we have that $(v_l, w_l) \to (\tilde{v}, \tilde{w})$ in $C^2_{loc}(\mathbb{R})$, where $(\tilde{v}, \tilde{w})$ is the positive solution of

\[
\begin{cases}
  f_2(m, v, w) = 0, & -\infty < x < \infty, \\
  -dw_{xx} = f_3(v, w), & -\infty < x < \infty.
\end{cases}
\]  

(4.6)

Obviously, $\tilde{w}$ satisfies

\[-d\tilde{w}_{xx} = \frac{kbm}{c(1 + \tilde{w})^2}\tilde{w} - q\tilde{w}, \quad -\infty < x < \infty.
\]  

Since $\frac{kbm}{c(1 + w)^2} - q$ is decreasing in $w > 0$, the possible positive solution of (4.6) is the unique positive root of $kbm = qc(1 + w)^2$. Thus $\tilde{w} = \hat{w}$, and consequently $\tilde{v} = \hat{v}$. The proof is finished. \qed
Theorem 4.3. Suppose that \( h_\infty = -g_\infty = \infty \). If \( R_0 + \sqrt{R_0} > b/a \), then there are six positive constants \( \underline{u}_\infty, \bar{u}_\infty, \underline{v}_\infty, \bar{v}_\infty, \underline{w}_\infty \) and \( \bar{w}_\infty \) such that
\[
\begin{align*}
\underline{u}_\infty \leq \liminf_{t \to \infty} u(t, x) & \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_\infty, \\
\underline{v}_\infty \leq \liminf_{t \to \infty} v(t, x) & \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_\infty, \\
\underline{w}_\infty \leq \liminf_{t \to \infty} w(t, x) & \leq \limsup_{t \to \infty} w(t, x) \leq \bar{w}_\infty
\end{align*}
\]
locally uniformly in \( \mathbb{R} \). Particularly, if we assume \( b \leq 2a \), then
\[
\lim_{t \to \infty} u(t, x) = u^*, \quad \lim_{t \to \infty} v(t, x) = v^*, \quad \lim_{t \to \infty} w(t, x) = w^* \quad \text{locally uniformly in } \mathbb{R},
\]
where \((u^*, v^*, w^*)\) is the unique positive root of
\[
f_1(u, w) = 0, \quad f_2(u, v, w) = 0, \quad f_3(v, w) = 0.
\]

Proof. The condition \( h_\infty - g_\infty = \infty \) implies \( R_0 > 1 \) (cf. Theorem 5.1). One can easily see that (4.9) has a unique positive root \((u^*, v^*, w^*)\). The following proof is actually an iterative process, the idea of which comes from [27, 32].

Step 1: Clearly,
\[
\limsup_{t \to \infty} u(t, x) \leq \theta/a =: \bar{u}_1 \quad \text{uniformly in } \mathbb{R}.
\]
Then for any \( \varepsilon > 0 \), there exists \( T > 0 \) such that \( u \leq \theta/a + \varepsilon \) with \( t \geq T \) and \( x \in \mathbb{R} \). Thus \((v, w)\) satisfies
\[
\begin{align*}
v_t & \leq f_2(\theta/a + \varepsilon, v, w), \quad t > T, \quad g(t) < x < h(t), \\
w_t - dw_{xx} & = f_3(v, w), \quad t > T, \quad g(t) < x < h(t), \\
v(t, x) & = 0, \quad w(t, x) = 0, \quad t > T, \quad x = g(t) \text{ or } h(t), \\
v(T, x) & \geq 0, \quad w(T, x) \geq 0, \quad g(T) \leq x \leq h(T).
\end{align*}
\]
Consider the ODEs problem
\[
\begin{align*}
\bar{v}_t & = f_2(\bar{u}_1 + \varepsilon, \bar{v}, \bar{w}), \quad \bar{w}_t = f_3(\bar{v}, \bar{w}), \quad t > T, \\
\bar{v}(T) & = A_2, \quad \bar{w}(T) = A_3.
\end{align*}
\]
Since \( R_0 > 1 \), the problem (4.10) has a unique positive equilibrium \((\bar{v}_1^\varepsilon, \bar{w}_1^\varepsilon)\) which is globally asymptotically stable. By a simple comparison consideration, we have \( v \leq \bar{v} \) and \( w \leq \bar{w} \) for \( t \geq T \) and \( x \in \mathbb{R} \). And so, \( \limsup_{t \to \infty} v(t, x) \leq \bar{v}_1^\varepsilon \) and \( \limsup_{t \to \infty} w(t, x) \leq \bar{w}_1^\varepsilon \) uniformly in \( \mathbb{R} \). By the arbitrariness of \( \varepsilon \), we have
\[
\limsup_{t \to \infty} v(t, x) \leq \bar{v}_1, \quad \limsup_{t \to \infty} w(t, x) \leq \bar{w}_1 \quad \text{uniformly in } \mathbb{R},
\]
where \((\bar{v}_1, \bar{w}_1)\) is the unique positive root of the algebraic system (4.4) with \( m \) replaced by \( \bar{u}_1 \).

Step 2: For small \( \varepsilon > 0 \), there exists \( T > 0 \) such that \( w \leq \bar{w}_1 + \varepsilon \) for \( t \geq T \) and \( x \in \mathbb{R} \). Hence \( u \) satisfies
\[
\begin{align*}
u_t & \geq f_1(u, \bar{w}_1 + \varepsilon), \quad t > T, \quad x \in \mathbb{R}, \\
u(T, x) & > 0, \quad x \in \mathbb{R}.
\end{align*}
\]
Using the comparison argument with the solution having initial value 0 we can deduce that
\[
\liminf_{t \to \infty} u(t, x) \geq \frac{\theta(1 + \bar{w}_1)}{a + aw_1 + bw_1} =: u_1 \text{ uniformly in } \mathbb{R}.
\]

Direct calculation shows that \( \bar{w}_1 = \sqrt{R_0} - 1 \) and
\[
kbu_1 > qc \text{ if and only if } R_0 + \sqrt{R_0} > b/a.
\]

By our assumptions, we have \( kb \bar{u}_1 > qc \), and then \( kb(u_1 - \varepsilon) > qc \) for the small \( \varepsilon > 0 \).

Recall Proposition 4.2. For any large \( l \), let \((v_l, w_l)\) and \((\bar{v}^l, \bar{w}^l)\) be the unique positive solution of (4.2) and (4.3) with \( m \) replaced by \( u_1 - \varepsilon \) respectively, then \((v_l, w_l) \to (\bar{v}^l, \bar{w}^l)\) locally uniformly in \( \mathbb{R} \) as \( l \to \infty \). For the given \( N \gg 1 \) and \( 0 < \sigma < 1 \), there exists a large \( l > N \) such that \( v_l > \bar{v}^l - \sigma/2 \) and \( w_l > \bar{w}^l - \sigma/2 \) for \( x \in [-N, N] \).

For such a fixed \( l > N \), let \((\lambda_1, \phi, \psi)\) be the principle eigenpair of (4.3) with \( m \) replaced by \( u_1 - \varepsilon \). We can verify that for small \( \delta > 0 \), \((\delta \psi, \delta \phi)\) is a lower solution of (4.2) with \( m \) replaced by \( u_1 - \varepsilon \) (see the proof of Theorem 5.2 for details). Moreover, we may choose \( T \gg 1 \), \( 0 < \delta \ll 1 \) such that \(-l, l \subseteq (g(t), h(t))\) for \( t \geq T \), \( u \geq u_1 - \varepsilon \) on \([T, \infty) \times [-l, l]\), and \( \delta \psi(\cdot) \leq v(T, \cdot), \delta \phi(\cdot) \leq w(T, \cdot) \) on \([-l, l]\). Hence \((v, w)\) satisfies
\[
\begin{align*}
v_t &\geq f_2(u_1 - \varepsilon, v, w), \quad t > T, \quad -l < x < l, \\
w_t - dw_{xx} &\leq f_3(v, w), \quad t > T, \quad -l < x < l, \\
v(t, x) &> 0, \quad w(t, x) > 0, \quad t > T, \quad x = \pm l, \\
v(T, x) &\geq \delta \psi(x), \quad w(T, x) \geq \delta \phi(x), \quad -l \leq x \leq l.
\end{align*}
\]

Let \((\tilde{v}, \tilde{w})\) be the unique positive solution of the following problem
\[
\begin{align*}
\tilde{v}_t &\geq f_2(u_1 - \varepsilon, \tilde{v}, \tilde{w}), \quad t > T, \quad -l < x < l, \\
\tilde{w}_t - dw_{xx} &\geq f_3(\tilde{v}, \tilde{w}), \quad t > T, \quad -l < x < l, \\
\tilde{v}(t, x) &\geq 0, \quad \tilde{w}(t, x) = 0, \quad t > T, \quad x = \pm l, \\
\tilde{v}(T, x) &\geq \delta \psi(x), \quad \tilde{w}(T, x) = \delta \phi(x), \quad -l \leq x \leq l.
\end{align*}
\]

Then \( \tilde{v} \) and \( \tilde{w} \) are nondecreasing in \( t \). By the standard parabolic regularity we can show that
\[
\lim_{t \to \infty} (\tilde{v}, \tilde{w}) = (v_1(x), w_1(x)) \text{ uniformly in } [-l, l].
\]

There exists \( T_1 > T \) such that \( \tilde{v}(t, x) \geq v_1(x) - \sigma/2 \), \( \tilde{w}(t, x) \geq w_1(x) - \sigma/2 \) for \( t > T_1 \) and \( x \in [-l, l] \). Furthermore, by the comparison principle, \( v \geq \tilde{v}, w \geq \tilde{w} \) for \( t > T \) and \( x \in [-l, l] \). So we have
\[
v(t, x) \geq \bar{v}^l - \sigma, \quad w(t, x) \geq \bar{w}^l - \sigma \quad \text{for } t > T_1, \quad |x| \leq N.
\]

This combined with the arbitrariness of \( \varepsilon, \sigma \) and \( N \) arrives at
\[
\liminf_{t \to \infty} v(t, x) \geq \underline{u}_1, \quad \liminf_{t \to \infty} w(t, x) \geq \underline{w}_1 \text{ locally uniformly in } \mathbb{R},
\]
where \((\underline{u}_1, \underline{w}_1)\) is the unique positive root of (4.4) with \( m \) replaced by \( u_1 \).
Step 3: For any given $N > 0$ and $0 < \varepsilon \ll 1$, there exists $T > 0$ such that $w \geq w_1 - \varepsilon$ for $t > T$ and $-N \leq x \leq N$. So we have

$$
\begin{align*}
    u_t &\leq f_1(u, w_1 - \varepsilon), \quad t > T, \quad -N \leq x \leq N, \\
    u(T, x) &> 0, \quad -N \leq x \leq N.
\end{align*}
$$

Comparing with the following ODE problem

$$
\bar{u}_t = f_1(\bar{u}, w_1 - \varepsilon), \quad t > T; \quad \bar{u}(T) = A_1,
$$

we can show that $u(t, x) \leq \bar{u}(t)$ for $t \geq T$ and $-N \leq x \leq N$. Similar to the above, we have

$$
\limsup_{t \to \infty} u(t, x) \leq \frac{\theta(1 + w_1)}{a + aw_1 + bw_1} =: \bar{u}_2 \quad \text{locally uniformly in } \mathbb{R},
$$

and $\bar{u}_2 > w_1$. Moreover, the direct calculation yields $kb\bar{u}_2 > qc$.

For the fixed $0 < \varepsilon \ll 1$, take $K > \max \{ A_3, \frac{kb(\bar{u}_2 + \varepsilon)}{qc} \}$ and consider the problem

$$
\begin{align*}
    -dw_{xx} &= \frac{kb(\bar{u}_2 + \varepsilon)w}{c(1 + w)^2} - qw, \quad -l < x < l, \\
    w(\pm l) &= K.
\end{align*}
$$

Clearly, $kb(\bar{u}_2 + \varepsilon) > qc$. By the standard method we can show that (4.11) has a unique positive solution $w^I$ for the large $l$. Moreover, $0 < w^I \leq K$. The comparison principle gives that $w^I$ is nonincreasing in $l$ and $w^I \geq w_1$. Same as the proof of Proposition 4.2 (ii) we can derive $\lim_{l \to \infty} w^I(x) = \bar{w}^2$ locally uniformly in $\mathbb{R}$, where $\bar{w}^2$ is the unique positive root of $kb(\bar{u}_2 + \varepsilon) = qc(1 + w)^2$. Take

$$
v^I(x) = \frac{b(\bar{u}_2 + \varepsilon)w^I(x)}{c(1 + w^I(x))}, \quad \bar{v}^2 = \frac{b(\bar{u}_2 + \varepsilon)\bar{w}^2}{c(1 + \bar{w}^2)}.
$$

Then $\lim_{l \to \infty} v^I(x) = \bar{v}^2$ locally uniformly in $\mathbb{R}$, and $((\bar{v}^2, \bar{w}^2))$ is the unique positive root of (4.12) in there $m$ is replaced by $\bar{u}_2 + \varepsilon$.

For the given $N \gg 1$ and $0 < \sigma \ll 1$, there exists a large $l > N$ such that $v^I(x) \leq \bar{v}^2 + \sigma$ and $w^I(x) \leq \bar{w}^2 + \sigma$ for $-N \leq x \leq N$. Moreover, there exists $T > 0$ such that $u \leq \bar{u}_2 + \varepsilon$ for $(t, x) \in [T, \infty) \times [-l, l]$, and $h(T) > 1$, $g(T) < -l$. Thanks to the equation of $v$ and $K > A_3$, we can find $T_1 > T$ such that $v \leq \frac{b(\bar{u}_2 + \varepsilon)K}{c(1 + K)} := A^*_2$ on $[T_1, \infty) \times [-l, l]$. Therefore, $(v, w)$ satisfies

$$
\begin{align*}
    v_t &\leq f_2(\bar{u}_2 + \varepsilon, v, w), \quad t > T_1, \quad -l < x < l, \\
    w_t - dw_{xx} &= f_3(v, w), \quad t > T_1, \quad -l < x < l, \\
    v(t, x) &\leq A^*_2, \quad w(t, x) \leq K, \quad t > T_1, \quad x = \pm l, \\
    v(T_1, x) &\leq A^*_2, \quad w(T_1, x) \leq K, \quad -l \leq x \leq l.
\end{align*}
$$

Let $(\bar{v}, \bar{w})$ be the unique positive solution of the problem

$$
\begin{align*}
    \bar{v}_t &= f_2(\bar{u}_2 + \varepsilon, \bar{v}, \bar{w}), \quad t > T_1, \quad -l < x < l, \\
    \bar{w}_t - d\bar{w}_{xx} &= f_3(\bar{v}, \bar{w}), \quad t > T_1, \quad -l < x < l, \\
    \bar{v}(t, x) &= A^*_2, \quad \bar{w}(t, x) = K, \quad t > T_1, \quad x = \pm l, \\
    \bar{v}(T_1, x) &= A^*_2, \quad \bar{w}(T_1, x) = K, \quad -l \leq x \leq l.
\end{align*}
$$
Then we can deduce that \((\tilde{v}(t,x), \tilde{w}(t,x)) \to (v^I(x), w^I(x))\) uniformly in \([-l,l]\) as \(t \to \infty\). Thus there exists \(T_2 > T_1\) such that \(\tilde{v}(t,x) \leq v^I(x) + \sigma, \tilde{w}(t,x) \leq w^I(x) + \sigma\) for \(t > T_2\) and \(x \in [-l,l]\). A comparison consideration yields that \(v(t,x) \leq \tilde{v}(t,x)\) and \(w(t,x) \leq \tilde{w}(t,x)\) on \([T_1, \infty) \times [-l,l]\).

Recalling our early conclusion we immediately derive that

\[
v(t,x) \leq \tilde{v}_2^2 + 2\sigma, \quad w(t,x) \leq \tilde{w}_2^2 + 2\sigma, \quad t > T_2, -N \leq x \leq N.
\]

The arbitrariness of \(\varepsilon, \sigma\) and \(N\) implies

\[
l\limsup_{t \to \infty} v(t,x) \leq \tilde{v}_2, \quad \limsup_{t \to \infty} w(t,x) \leq \tilde{w}_2 \quad \text{locally uniformly in } \mathbb{R},
\]

where \((\tilde{v}_2, \tilde{w}_2)\) is the unique positive root of the equations (4.4) with \(m\) replaced by \(\bar{u}_2\).

We may argue as in Step 2 to conclude that

\[
l\liminf_{t \to \infty} u(t,x) \geq \bar{u}_2, \quad \liminf_{t \to \infty} v(t,x) \geq \bar{v}_2, \quad \liminf_{t \to \infty} w(t,x) \geq \bar{w}_2 \quad \text{locally uniformly in } \mathbb{R},
\]

where \(\bar{u}_2 = \frac{\theta(1+\bar{w}_2)}{a+\bar{w}_2+b\bar{w}_2}\), and \((\bar{v}_2, \bar{w}_2)\) is the unique positive root of (4.4) with \(m\) replaced by \(\bar{u}_2\).

**Step 4:** According to the above arguments we have

\[
\bar{u}_1 < \bar{u}_2 < \bar{u}_1, \quad \bar{v}_1 < \bar{v}_2 < \bar{v}_1, \quad \bar{w}_1 < \bar{w}_2 < \bar{w}_1.
\]

Repeating the above procedures we can find six sequences \(\{\bar{u}_n\},\{\bar{v}_n\},\{\bar{v}_n\},\{\bar{w}_n\}\) and \(\{\bar{w}_n\}\) satisfying

\[
\bar{u}_1 < \bar{u}_2 < \cdots < \bar{u}_n < \cdots < \bar{u}_2 < \bar{u}_1, \\
\bar{v}_1 < \bar{v}_2 < \cdots < \bar{v}_n < \cdots < \bar{v}_2 < \bar{v}_1, \\
\bar{w}_1 < \bar{w}_2 < \cdots < \bar{w}_n < \cdots < \bar{w}_2 < \bar{w}_1,
\]

so that

\[
\bar{u}_n \leq \liminf_{t \to \infty} u(t,x) \leq \limsup_{t \to \infty} u(t,x) \leq \bar{u}_n, \\
\bar{v}_n \leq \liminf_{t \to \infty} v(t,x) \leq \limsup_{t \to \infty} v(t,x) \leq \bar{v}_n, \\
\bar{w}_n \leq \liminf_{t \to \infty} w(t,x) \leq \limsup_{t \to \infty} w(t,x) \leq \bar{w}_n
\]

locally uniformly in \(\mathbb{R}\). The limits of the above six sequences are well defined, and denoted by \(\bar{u}_\infty, \bar{v}_\infty, \bar{w}_\infty, \bar{w}_\infty\) and \(\bar{w}_\infty\) respectively. It is clear that (4.7) holds.

Now we assume \(b \leq 2a\) and prove (4.8). By the careful calculations one can obtain

\[
\bar{u}_1 = \frac{\theta}{a}, \quad \frac{b\bar{u}_n \bar{w}_n}{1+\bar{w}_n} = c\bar{v}_n, \quad \frac{k\bar{v}_n}{1+\bar{w}_n} = q\bar{w}_n, \quad \frac{\bar{w}_n}{1+\bar{w}_n} = \frac{\theta(1+\bar{w}_n)}{a+b\bar{w}_n+2\bar{w}_n}.
\]

Consequently, \(\bar{u}_\infty, \bar{v}_\infty, \bar{w}_\infty, \bar{w}_\infty\) and \(\bar{w}_\infty\) satisfy

\[
\frac{b\bar{u}_\infty \bar{w}_\infty}{1+\bar{w}_\infty} = c\bar{v}_\infty, \quad \frac{k\bar{v}_\infty}{1+\bar{w}_\infty} = q\bar{w}_\infty, \quad \frac{\bar{w}_\infty}{1+\bar{w}_\infty} = \frac{\theta(1+\bar{w}_\infty)}{a+2\bar{w}_\infty+2\bar{w}_\infty}.
\]

Using our assumptions \(R_0 > 1\) and \(b/a \leq 2\), by a series of careful calculations we can derive

\[
\bar{u}_\infty = \bar{u}_\infty = u^*, \quad \bar{v}_\infty = \bar{v}_\infty = v^*, \quad \bar{w}_\infty = \bar{w}_\infty = w^*.
\]

Thus (4.8) holds and the proof is ended.
5 Criteria for spreading and vanishing

In this section we study the criteria governing spreading \((h_\infty - g_\infty = \infty)\) and vanishing \((h_\infty - g_\infty < \infty)\). In the following, we divide our discussion into two cases based on the Basic Reproduction Number \(R_0 = \theta kb/(acq)\). For convenience, we denote \(\gamma = \max \{\mu, \beta\}\).

5.1 The case \(R_0 \leq 1\)

**Theorem 5.1.** Let \((u, v, w, g, h)\) be the unique solution of \((1.2)\). If \(R_0 \leq 1\), then \(h_\infty - g_\infty < \infty\).

**Proof.** By a simple comparison argument, we have

\[
u(t, x) \leq \theta/a + \|u_0\|_{\infty} e^{-at} =: \hat{u}(t) \quad \text{for } t \geq 0, \ x \in \mathbb{R}.
\]

Hence \(v\) satisfies

\[
\begin{cases}
v_t \leq f_2(\hat{u}(t), v, w), & t > 0, \ g(t) < x < h(t), \\
v(t, g(t)) = v(t, h(t)) = 0, & t > 0, \\
v(0, x) = v_0(x), & |x| \leq h_0.
\end{cases}
\]

Notice that \(R_0 = \theta kb/acq \leq 1\). It follows that by simple calculations

\[
\frac{d}{dt} \int_{g(t)}^{h(t)} (cw + kv) dx = \int_{g(t)}^{h(t)} (cw_t + kv_t) dx \\
\leq \int_{g(t)}^{h(t)} \left[ cw_{xx} + kb\|u_0\|_{\infty} e^{-at} w + (k\theta b/a - qc)w \right] dx \\
\leq \int_{g(t)}^{h(t)} \left( cw_{xx} + kb\|u_0\|_{\infty} e^{-at} w \right) dx \\
\leq -cd\gamma^{-1}(h'(t) - g'(t)) + kbA_3\|u_0\|_{\infty} e^{-at}(h(t) - g(t)).
\]

Set

\[
f(t) = \int_{g(t)}^{h(t)} (cw + kv) dx, \quad \ell(t) = h(t) - g(t), \quad \varphi(t) = kbA_3\|u_0\|_{\infty} e^{-at}.
\]

Then we have

\[
cd\ell'(t) \leq -\gamma f'(t) + \gamma \varphi(t) \ell(t).
\]

Integrating the above differential inequality from 0 to \(t\) yields

\[
\ell(t) \leq \ell(0) + \gamma(cdf)^{-1}f(0) + \gamma(cdf)^{-1} \int_0^t \varphi(s) \ell(s) ds.
\]

By virtue of the Gronwall inequality,

\[
\ell(t) \leq [\ell(0) + \gamma(cdf)^{-1}f(0)] \exp \left\{ \gamma(cdf)^{-1} \int_0^t \varphi(s) ds \right\} < \infty.
\]

Thus, \(h_\infty - g_\infty < \infty\). \(\square\)
5.2 The case $\mathcal{R}_0 > 1$

In this subsection, we always assume that $\mathcal{R}_0 > 1$, and consider $d$, $h_0$, $\mu$ and $\beta$ as varying parameters to depict the criteria for spreading and vanishing.

**Theorem 5.2.** Let $(u,v,w,g,h)$ be the solution of (1.2). If $h_\infty - g_\infty < \infty$, then we have

$$h_\infty - g_\infty \leq \pi \sqrt{acd/(kb\theta - acq)} =: \Lambda.$$

This implies that if $h_0 \geq \Lambda/2$, then $h_\infty - g_\infty = \infty$. Moreover, if $h_\infty - g_\infty = \infty$, then

$$\limsup_{t \to \infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} > 0, \quad \limsup_{t \to \infty} \|w(t,\cdot)\|_{C([g(t),h(t)])} > 0. \quad (5.1)$$

**Proof.** Due to Theorem 4.4 and $h_\infty - g_\infty < \infty$, we have that $\lim_{t \to \infty} u = \theta/a$ uniformly in $\mathbb{R}$, and

$$\lim_{t \to \infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0, \quad \lim_{t \to \infty} \|w(t,\cdot)\|_{C([g(t),h(t)])} = 0. \quad (5.2)$$

Arguing indirectly, if $h_\infty - g_\infty > \Lambda$, then there exists $T > 0$ such that for any small $\varepsilon > 0$ satisfying $\frac{kb(\theta/a - \varepsilon)}{\varepsilon} - q > 0$, we have

$$u > \theta/a - \varepsilon, \quad \forall \, t \geq T, \, x \in \mathbb{R};$$

$$h(t) - g(t) > \pi \sqrt{d} [kb(\theta/a - \varepsilon)/c - q]^{-1/2} =: \Lambda_\varepsilon, \quad \forall \, t \geq T.$$

Then for any $[l_1, l_2] \subseteq (g(T), h(T))$ and $l_2 - l_1 > \Lambda_\varepsilon$, we have

$$\begin{align*}
  v_t &\geq f_2(\theta/a - \varepsilon, v,w), \quad t > T, \ l_1 < x < l_2, \\
  w_t - dw_{xx} &\geq f_3(v,w), \quad t > T, \ l_1 < x < l_2, \\
  v(t,x) &> 0, \ w(t,x) > 0, \quad t > T, \ x = l_i, \ i = 1,2, \\
  v(T,x) &> 0, \ w(T,x) > 0, \quad l_1 \leq x \leq l_2.
\end{align*} \quad (5.3)$$

Consider the following eigenvalue problem

$$\begin{align*}
  d\phi_{xx} - q\phi + k\psi &= \lambda \phi, \quad l_1 < x < l_2, \\
  b(\theta/a - \varepsilon)\phi - cv\psi &= \lambda \psi, \quad l_1 < x < l_2, \\
  \phi(l_i) &= \psi(l_i) = 0, \quad i = 1,2.
\end{align*} \quad (5.4)$$

Denote the principal eigenpair of (5.4) by $(\lambda_1, \phi, \psi)$. It follows from Corollary 3.2 that $\lambda_1 > 0$ due to $l_2 - l_1 > \Lambda_\varepsilon$. Let

$$\overline{v}(x) = \delta \psi(x), \quad \overline{w}(x) = \delta \phi(x)$$

with $\delta > 0$ to be determined later.

We claim that there exists $\delta_0 > 0$ sufficiently small such that for any $0 < \delta < \delta_0$, we have

$$\begin{align*}
  0 &\leq f_2(\theta/a - \varepsilon, \overline{v}, \overline{w}), \quad l_1 < x < l_2, \\
  -dw_{xx} &\geq f_3(\overline{v}, \overline{w}), \quad l_1 < x < l_2.
\end{align*} \quad (5.5)$$

In fact, by the Hopf lemma we have

$$\phi_x(l_1) > 0, \quad \phi_x(l_2) < 0; \quad \psi_x(l_1) > 0, \quad \psi_x(l_2) < 0.$$
Moreover,

\[-f_2(\theta/a - \varepsilon, v, w)\delta^{-1} = b(\theta/a - \varepsilon)\phi - c\psi - \lambda_1\psi - b(\theta/a - \varepsilon)\delta\phi(1 + \delta\phi)^{-1} + c\psi\]

\[= b(\theta/a - \varepsilon)\phi [1 - (1 + \delta\phi)^{-1}] - \lambda_1\psi\]

\[:= H(x; \delta).\]

Direct computations show that \(H(l_i; \delta) = 0\) and \(H_x(l_i; \delta) = -\lambda_1\psi_x(l_i)\). Additionally, for any \(x \in (l_1, l_2)\), \(H(x; \delta)\) is increasing in \(\delta > 0\). Hence there exist \(0 < \sigma, \delta_1 < 1\) such that \(H(x; \delta) < 0\) for \(x \in (l_1, l_1 + \sigma) \cup [l_2 - \sigma, l_2)\) and \(\delta \in (0, \delta_1)\). On the other hand, since \(\lambda_1\psi\) has a positive lower bound on \([l_1 + \sigma, l_2 - \sigma]\) and

\[b(\theta/a - \varepsilon)\phi [1 - (1 + \delta\phi)^{-1}] \leq b(\theta/a - \varepsilon)||\phi||_{\infty} [1 - (1 + \delta||\phi||_{\infty})^{-1}],\]

we can find \(0 < \delta_0 < \delta_1\) \(H(x; \delta) < 0\) for \(x \in [l_1 + \sigma, l_2 - \sigma]\) and \(\delta \in (0, \delta_0)\). Thus the first inequality of (5.5) holds.

We may argue as above to conclude that the second inequality of (5.5) is valid if \(0 < \delta_0 < 1\). The claim is proved.

Furthermore, one may choose \(\delta > 0\) small such that \(v(T, x) \geq \delta\psi(x)\) and \(w(T, x) \geq \delta\phi(x)\) for \(x \in [l_1, l_2]\). Then \((v, w)\) satisfies

\[
\begin{align*}
v_t &\leq f_2(\theta/a - \varepsilon, v, w), & t > T, & l_1 < x < l_2, \\
w_t - dw_{xx} &\leq f_3(v, w), & t > T, & l_1 < x < l_2, \\
v(t, x) &\equiv 0, w(t, x) = 0, & t > T, & x = l_i, i = 1, 2, \\
v(T, x) &\leq v(T, x), w(T, x) \leq w(T, x), & l_1 < x \leq l_2.
\end{align*}
\]

By virtue of the comparison principle,

\[v(t, x) \geq v(x), w(t, x) \geq w(x), t \geq T, l_1 \leq x \leq l_2. \tag{5.6}\]

This is a contradiction with (5.2).

We now assume \(h_\infty - g_\infty = \infty\) and prove (5.1). By the comparison principle, it is easy to see that \(\lim_{t \to \infty} ||v(t, \cdot)||_{\infty} = 0\) if and only if \(\lim_{t \to \infty} ||w(t, \cdot)||_{\infty} = 0\). Hence if we assume that one of the two limits in (5.1) does not hold, we can similarly obtain \(\lim w = \theta/a\) uniformly in \(\mathbb{R}\). By \(h_\infty - g_\infty = \infty\), we can derive the analogous contradictions as above. The proof is ended. \(\square\)

Obviously, \(h_0 \geq \Lambda/2\) is equivalent to \(d \leq 4h_0^2q(R_0 - 1)^{\pi^{-2}} =: D\). So the above result suggests that when \(R_0 > 1\), the larger initial habitat \([-h_0, h_0]\) or the lower dispersal rate \(d\) of the virus is, the more possible it will spread successfully.

**Theorem 5.3.** If \(h_\infty - g_\infty = \infty\), then \(h_\infty = \infty\) and \(g_\infty = -\infty\).

**Proof.** By way of contradiction, we assume that \(h_\infty < \infty\) and \(g_\infty = -\infty\). Take \(L > \Lambda + 2\), where \(\Lambda\) is defined in Theorem 5.2. there exists \(T_0 > 0\) such that \(g(T_0) < -L\). Then \(w\) satisfies

\[
\begin{align*}
w_t - dw_{xx} &= f_3(v, w), & t > T_0, & -L < x < h(t), \\
w(-L) &> 0, w(t, h(t)) = 0, & t > T_0, \\
h' &= -\beta w_x(t, h(t)), & t > T_0, \\
w(T_0, x) &\geq 0, & -L \leq x \leq h(T_0).
\end{align*}
\]
As $h_\infty < \infty$, using the second estimate in (2.7) we have $\lim_{t \to \infty} h'(t) = 0$. Then, using the first estimate in (2.7) and Lemma 3.3, one can arrive at
\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([-L, h(t)])} = 0. \tag{5.7}
\]

Then we may argue as in the proof of Theorem 4.1 with minor modifications to derive that
\[
\lim_{t \to \infty} \max_{[1-L, h(T_0)]} v(t, \cdot) = 0, \quad \lim_{t \to \infty} \max_{[1-L, h(T_0)]} u(t, \cdot) = \theta/a.
\]

There exists $T > T_0$ such that $u \geq \theta/a - \varepsilon$ for $(t, x) \in [T, \infty) \times [1 - L, h(T_0)]$. Let $\varepsilon > 0$ be small enough satisfying $L - 1 > \Lambda_\varepsilon$, where $\Lambda_\varepsilon$ is defined as in Theorem 5.2. Choose an interval $[l_1, l_2] \subset (1 - L, h(T_0))$ with $l_2 - l_1 > \Lambda_\varepsilon$. Then $(v, w)$ satisfies (5.3). As in the proof of Theorem 5.2, we can conclude that (5.6) holds. This is a contradiction with (5.7).

Analogously, we can prove that the case with $h_\infty = \infty$ and $g_\infty > -\infty$ also does not hold. Therefore, we must have $h_\infty = \infty$ and $g_\infty = -\infty$. \hfill \Box

The following implies that although the initial habitat is small or the dispersal rate is fast, the spreading also can occur if the expanding rate $\mu$ or $\beta$ is appropriately large. By using similar method in the proof of Lemma 3.2 with some modifications, we can prove the following lemma.

Lemma 5.4. If $h_0 < \Lambda/2$ (or $d > D$), then there exists $\mu^0 > 0$ ($\beta^0 > 0$) such that if $\mu \geq \mu^0$ ($\beta \geq \beta^0$), then $h_\infty - g_\infty = \infty$.

The above lemma indicates that if $\gamma = \max \{\mu, \beta\} \geq \max \{\mu^0, \beta^0\}$, then $h_\infty - g_\infty = \infty$. Instinctively, we deem that if $R_0 > 1$, $h_0 < \Lambda/2$ (or $d > D$), $\mu$ and $\beta$ both are small, then the vanishing will happen. The lemma listed below supports our belief.

Lemma 5.5. Assume $h_0 < \Lambda/2$ (or $d > D$). Then there exists $\mu_0 > 0$ such that when $\gamma = \max \{\mu, \beta\} \leq \mu_0$, we must have $h_\infty - g_\infty < \infty$.

Proof. Let $\tilde{u}$ be the unique solution of the problem
\[
\tilde{u}_t = \theta - a\tilde{u}, \quad t > 0; \quad \tilde{u}(0) = \max \{\|u_0\|_\infty, \theta/a\}.
\]
Then $\tilde{u}(t) \geq \theta/a$ and $\lim_{t \to \infty} \tilde{u}(t) = \theta/a$. By the comparison principle, $u(t, x) \leq \tilde{u}(t)$ in $[0, \infty) \times [0, \infty)$. For the fixed $h_0 < l < \Lambda/2$, we consider the following eigenvalue problem
\[
\begin{cases}
-\frac{d^2\phi}{dx^2} - q\phi + k\psi = \lambda\phi, & -l < x < l, \\
(b\theta/a)\phi - c\psi = \lambda\psi, & -l < x < l, \\
\phi(\pm l) = \psi(\pm l) = 0.
\end{cases}
\]
In view of Corollary 3.2 the principal eigenvalue $\lambda_1 < 0$ since $2l < \Lambda$. Moreover, by (3.2), there exists a positive constant $\tilde{\phi}$ such that
\[
\tilde{\phi}_{xx} - q\tilde{\phi} + k = \lambda_1\tilde{\phi}, \quad \frac{b\theta}{a} \tilde{\phi} - c = \lambda_1.
\]
Define
\[
f(t) = M \exp \left\{ \int_0^t \left[ \tilde{\phi}b(\tilde{u}(s) - \theta/a) + \lambda_1 \right] ds \right\},
\]
21
\[
\begin{align*}
    r(t) &= \left( h_0^2 + \gamma \pi \phi \int_0^t f(s) \, ds \right)^{1/2}, \\
    \dot{v}(t, x) &= f(t) \cos \frac{\pi x}{2r(t)}, \quad \dot{w}(t, x) = \tilde{\phi} f(t) \cos \frac{\pi x}{2r(t)}, \quad -r(t) \leq x \leq r(t),
\end{align*}
\]

where \( \gamma = \max \{ \mu, \beta \}, M > 0 \) is taken large so that

\[
v_0(x) \leq M \cos \frac{\pi x}{2h_0} = \dot{v}(t, 0), \quad w_0(x) \leq \tilde{\phi} M \cos \frac{\pi x}{2h_0} = \dot{w}(t, 0) \text{ in } [-h_0, h_0].
\]

As \( \lim_{t \to \infty} \dot{u}(t) = \theta/a \) and \( \lambda_1 < 0 \), it follows that \( \int_0^\infty f(s) \, ds < \infty \). Clearly, \( r'(t) > 0 \) for \( t \geq 0 \). Set

\[
    \mu_0 = \frac{l^2 - h_0^2}{\pi \phi \int_0^\infty f(s) \, ds}.
\]

Then \( r(t) < l \) for \( t \geq 0 \) provided \( 0 < \gamma \leq \mu_0 \).

Using \( \ref{5.3} \) and \( \dot{u}(t) \geq \theta/a, r(t) < l \) for all \( t \geq 0 \), by a series of calculations we have

\[
    \begin{align*}
    &\dot{v}_t - f_2(\dot{u}, \dot{v}, \dot{w}) \geq f(t) \cos \frac{\pi x}{2r(t)} \left( \lambda_1 + c - \frac{\theta}{a} \beta \right) = 0, \\
    &\dot{w}_t - d\dot{w}_{xx} - f_3(\dot{v}, \dot{w}) \geq f(t) \cos \frac{\pi x}{2r(t)} \left[ \phi^2 b \left( \dot{u} - \frac{\theta}{a} \right) + d\tilde{\phi} \left( \frac{\pi}{2r(t)} \right)^2 + \lambda_1 \tilde{\phi} + q\tilde{\phi} - k \right] \\
    &= \tilde{\phi} f(t) \cos \frac{\pi x}{2r(t)} \left[ \phi^2 b \left( \dot{u} - \frac{\theta}{a} \right) + d \left( \frac{\pi}{2r(t)} \right)^2 - d \left( \frac{\pi}{2l} \right)^2 \right] \geq 0
    \end{align*}
\]

for \( t > 0 \) and \( -r(t) < x < r(t) \). And we easily see

\[
r'(t) = -\gamma \dot{w}_x(t, -r(t)), \quad r'(t) = -\gamma \dot{w}_x(t, r(t)).
\]

Thus for any \( 0 < \gamma \leq \mu_0 \), \((\dot{u}, \dot{v}, \dot{w}, -r, r)\) satisfies \( r(0) = h_0 \) and

\[
\begin{align*}
    \dot{u}_t &= \theta - a\dot{u}, \quad t > 0, \\
    \dot{v}_t \geq f_2(\dot{u}, \dot{v}, \dot{w}), \quad t > 0, \quad -r(t) < x < r(t), \\
    \dot{w}_t - d\dot{w}_{xx} \geq f_3(\dot{v}, \dot{w}), \quad t > 0, \quad -r(t) < x < r(t), \\
    \dot{v}(t, \pm r(t)) = \dot{w}(t, \pm r(t)) = 0, \quad t > 0, \\
    -r'(t) \leq -\mu \dot{w}_x(t, -r(t)), \quad r'(t) \geq -\beta \dot{w}_x(t, r(t)), \quad t \geq 0, \\
    \dot{u}(0) \geq u_0(x), \quad -\infty < x < \infty, \\
    \dot{v}(0, x) \geq v_0(x), \dot{w}(0, x) \geq w_0(x), \quad |x| \leq h_0.
\end{align*}
\]

By the comparison principle (Lemma \( \ref{5.3} \)), \(-r(t) \leq g(t), h(t) \leq r(t)\) for \( t \geq 0 \). As a result, we have

\[
g_\infty \geq -\lim_{t \to \infty} r(t) \geq -l, \quad h_\infty \leq \lim_{t \to \infty} r(t) \leq l,
\]

which implies \( h_\infty - g_\infty < \infty \). This completes the proof. \( \square \)

According to the above proof, we can see that \( \mu_0 \) is independent of \( v_0 \) and \( w_0 \) and strictly decreasing in \( M \). Thus for any given \( \mu \) and \( \beta \), there exists \( M > 0 \) small sufficiently such that \( \gamma \leq \mu_0 \). Meanwhile, for this \( M \) if \( v_0 \) and \( w_0 \) are both small enough such that

\[
v_0(x) \leq M \cos \frac{\pi x}{2h_0}, \quad w_0(x) \leq \tilde{\phi} M \cos \frac{\pi x}{2h_0}, \quad \forall x \in [-h_0, h_0],
\]

we still can derive \( h_\infty - g_\infty < \infty \) by the above arguments. Hence we have the following conclusion.
Remark 5.6. If $h_0 < \Lambda/2 (d > D)$, then for any $(v_0, w_0)$ satisfying (1.3), vanishing happens if $v_0$ and $w_0$ are both small enough.

Combining the above two lemmas, by the similar arguments in [27, Theorem 5.2] we can show the following conclusion.

**Theorem 5.7.** If $h_0 < \Lambda/2 (d > D)$. There exists $0 < \mu_* \leq \mu^*$ such that $h_\infty - g_\infty < \infty$ if $\gamma \leq \mu_*$ or $\gamma = \mu^*$, and $h_\infty - g_\infty = \infty$ if $\gamma > \mu^*$.

### 6 Discussion

In this paper we proposed a viral propagation model with Holling type-II response function and free boundaries and investigated the dynamical properties. This model is composed of two ordinary differential equations and one partial differential equation, in which the spatial range of the first equation is the whole space $\mathbb{R}$, and the last two equations have free boundaries. As a new mathematical model, we proved the existence, uniqueness and uniform estimates of global solution, and provide the criteria for spreading and vanishing, and long time behavior of the solution components $u, v, w$.

Comparing with the corresponding ordinary differential systems, the *Basic Reproduction Number* $\mathcal{R}_0 = \theta kb/(acq)$ plays a different role:

(i) For the corresponding ordinary differential systems, by the Lyapunov function method we can prove that if $\mathcal{R}_0 < 1$ then the infection can not spread successfully, while if $\mathcal{R}_0 > 1$ then the infection will spread successfully. When $\mathcal{R}_0 = 1$ the dynamical property is not clear;

(ii) For our present model, the results indicate that when $\mathcal{R}_0 \leq 1$, the virus cannot spread successfully; when $\mathcal{R}_0 > 1$, the successful spread of virus depends on the initial value and varying parameters. If the initial occupying area $[-h_0, h_0]$ is beyond a critical size, namely $2h_0 \geq \pi \sqrt{acd/(bk\theta - acq)}$, then spreading happens regardless of the moving parameter $\mu, \beta$ and initial population density $(u_0, v_0, w_0)$. While $2h_0 < \pi \sqrt{acd/(bk\theta - acq)}$, whether spreading or vanishing happens depends on the initial population density $(v_0, w_0)$ and the moving parameter $\mu$ and $\beta$.

From a biological point of view, our model and results seem closer to the reality. On the other hand, our model shows more complex and precise dynamical properties from a mathematical point of view.

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