A FAMILY OF REDUCTIONS FOR SCHUBERT INTERSECTION PROBLEMS

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Abstract. We produce a family of reductions for Schubert intersection problems whose applicability is checked by calculating a linear combination of the dimensions involved. These reductions do not alter the Littlewood-Richardson coefficient, and they lead to an explicit solution of the intersection problem when this coefficient is 1.

1. Introduction

Given integers \( n > r \geq 1 \), we denote by \( G(r, \mathbb{C}^n) \) the Grassmannian manifold consisting of all \( r \)-dimensional subspaces in \( \mathbb{C}^n \). For every flag \( \mathcal{E} = \{\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n\} \), where \( E_j \) is a subspace of dimension \( j \), \( G(r, \mathbb{C}^n) \) can be written as a union of Schubert varieties described as follows. For each set \( I = \{i_1 < i_2 < \cdots < i_r\} \subset \{1, 2, \ldots, n\} \) one defines the Schubert variety

\[
S(\mathcal{E}, I) = \{M \in G(r, \mathbb{C}^n) : \text{dim}(M \cap E_{i_x}) \geq x, x = 1, 2, \ldots, r\}.
\]

Schubert calculus allows one to find the number of points in the intersection of several Schubert varieties \( S(\mathcal{E}_\ell, I) \), \( \ell = 1, 2, \ldots, p \), when the flags \( (\mathcal{E}_\ell)_{\ell=1}^p \) are in generic position. We will be mostly concerned with the case \( p = 3 \) where the classical Littlewood-Richardson rule applies (cf. [4]). Thus, given sets \( I, J, K \subset \{1, 2, \ldots, n\} \) of cardinality \( r \) such that

\[
\sum_{\ell=1}^r (i_\ell + j_\ell + k_\ell - 3\ell) = 2r(n - r),
\]

the Littlewood-Richardson rule (which will be reviewed below) provides a non-negative integer \( c_{IJK} \) with the property that the set

\[
S = S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)
\]

contains \( c_{IJK} \) elements for generic flags \( \mathcal{E}, \mathcal{F}, \mathcal{G} \). For nongeneric flags, this intersection is still certain to be nonempty if \( c_{IJK} > 0 \).

Thompson and Therianos [6] pointed out that under certain circumstances one can reduce the problem of finding elements in the set \( S \) to a problem where \( n \) is replaced by a smaller number. In order to explain their reductions, it will be convenient to set \( I = \{i_1 < i_2 < \cdots < i_r\} \), define \( i_0 = 0 \), and similarly \( j_0 = k_0 = 0 \). Assume that the indices \( x, y, z \in \{0, 1, 2, \ldots, r\} \) are such that \( x + y + z = r \) and

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The question arises naturally whether
$c$

and this condition
$i
j
k

are generically independent, and for any space
$M \in S$ we have

$$r = \dim(M) \geq \dim(M \cap E_i) + \dim(M \cap F_j) + \dim(M \cap G_k) \geq x + y + z = r.$$  

Therefore
$M$
is contained in
$E_i + F_j + G_k$. Replace now
$C^n
$by the space
$X = E_i + F_j + G_k$
and the spaces
$E_i, F_j, G_k$
by their intersections with
$X$. Observe that generically

$$\dim(E_i \cap X) = \begin{cases} i & \text{if } i \leq i_x \\ i_x & \text{if } i_x < i \leq i_x + p \\ i - p & \text{if } i_x + p < i \leq n, \end{cases}$$

and these spaces will form (after the repeating spaces of dimension
$i
are deleted) a flag
$E
in
$X$. Flags
$F
and
$G$
are defined similarly. Finding the spaces in
$S$
amounts to finding the spaces in

$$S' = \mathcal{G}(E', I') \cap \mathcal{G}(F', J') \cap \mathcal{G}(G', K') \subset G(r, X),$$

where

$$i'_\ell = \begin{cases} i_\ell & \text{if } 1 \leq \ell \leq x \\ i_\ell - p & \text{if } x < \ell \leq r, \end{cases}$$

with similar definitions for
$J
and
$K
. (The sequence
$(i'_\ell)_{\ell=1}^r
$is still strictly increasing because the condition
$i_x + j_y + k_z = n - p$
actually implies that
$i_{x+1} > i_x + p$.) The question arises naturally whether
$c_{I'J'K'} \neq 0$ if
$c_{IJK} \neq 0$
so that the reduced problem is still guaranteed to have a solution. That is indeed the case was shown by Collins and Dykema [3] who proved that in fact
$c_{I'J'K'} = c_{IJK}$.

The purpose of this paper is to identify a much larger family of reductions associated with various inequalities satisfied by
$I, J, K$
. This family is sufficient for the complete solution of the intersection problem when
$c_{IJK} = 1$
. The simplest of these new reductions is as follows. Assume that
$x, y, z \in \{1, 2, \ldots, r\}$
satisfy
$x + y + z = 2r$
and

$$i_x + j_y + k_z = 2n - p < 2n.$$  

In this case the space

$$X = (E_i \cap F_j) + (E_i \cap G_k) + (F_j \cap G_k)$$

has generically codimension
$2p$
and it contains all the spaces in
$S$
. The reduced problem in
$G(r, X)$
corresponds with the sets
$I', J', K'$
defined by

$$i'_\ell = \begin{cases} i_\ell - p & \text{if } 1 \leq \ell \leq i_x \\ i_\ell - 2p & \text{if } i_x < \ell \leq r, \end{cases}$$

with analogous definitions for
$j'_\ell
and
$k'_\ell
. As in the result of [3] just mentioned, we have
$c_{I'J'K'} = c_{IJK}$.

The general reduction we propose can be described as follows. We are given
$r$
tuples
$a = (a_\ell)_{\ell=1}^r
,b = (b_\ell)_{\ell=1}^r
,c = (c_\ell)_{\ell=1}^r
$of nonegative integers such that

$$\sum_{\ell=1}^r (\ell a_\ell + \ell b_\ell + \ell c_\ell) = \omega r$$

for some positive integer
$\omega$
; 
$a, b, c$
are subject to other conditions which will be discussed later. Assume that the sets
$I, J, K \subset \{1, 2, \ldots, n\}$
have cardinality
$r,$
\[ \sum_{\ell=1}^{r} (a\ell i + bj\ell + ck\ell) = \omega n - p, \]

where \( p \) is some integer. The reduction corresponding to \( a, b, c \) can be applied when \( p < 0 \). Namely, if \( p < 0 \), we necessarily have \( \omega p \leq n \). Moreover, there exist

1. a space \( X \subset C^n \) with \( \text{dim} \ X = n - \omega p \),
2. flags \( E, F, G \) in \( X \),
3. sets \( I', J', K' \subset \{1, 2, \ldots, n-\omega p\} \) of cardinality \( r \) such that \( c_{I'J'K'} = c_{IJK} \) and

\[ \mathcal{S}(E', I') \cap \mathcal{S}(F', J') \cap \mathcal{S}(G', K') \subset \mathcal{S}(E, I) \cap \mathcal{S}(F, J) \cap \mathcal{S}(G, K). \]

In addition, the space \( X \) can be constructed (when the flags \( E, F, G \) are in ‘general position’) explicitly from \( E, F, G \) by applying a finite number of sums and intersections. The sequences \( a, b, c \) which appear here are themselves related to the Littlewood-Richardson rule.

Our proofs deepen some of the results in [1]. Even though we review the relevant results of [1], familiarity with that paper would be helpful in reading this one.

The remainder of the paper is organized as follows. In Section 2 we describe the formulation of the Littlewood-Richardson rule in terms of measures. This is essentially the puzzle formulation of [5], and was also used in [1]. We also introduce the linear combinations of dimensions which serve as witnesses for the possibility of reductions. In Section 3 we discuss a special class of measures, the tree measures. It was implicit in the results of [1] that rigid extremal measures have an underlying tree structure, and this is made explicit here. Section 4 reviews the construction of a puzzle from a measure, and uses the results of Section 3 to deduce the identity \( c_{I'J'K'} = c_{IJK} \). In Section 5 we prove the essential technical result needed to show in Section 6 that the analogues of the reductions of [6] can indeed be performed. It seems practically impossible to describe all rigid tree measures in a uniform manner. We provide in Section 7 a description of a fairly large class of such measures.

### 2. The Littlewood-Richardson Rule

We will give the description of the Littlewood-Richardson rule in terms of measures. This is equivalent with the puzzle description of [5]. Choose unit vectors \( u, v, w \) in the plane such that \( u + v + w = 0 \).

\[
\begin{array}{c}
\text{u} \\
\text{v} \\
\text{w}
\end{array}
\]

The points \( iu + jv \) with integer \( i, j \) will be called lattice points, and a segment joining two nearest lattice points will be called a small edge. We consider positive measures \( m \) which are supported by the union of the small edges, whose restriction
to each small edge is a multiple of arclength measure, and which satisfy the balance
condition (called zero tension in [5])

\[ m(AB) - m(AB') = m(AC) - m(AC') = m(AD) - m(AD') \]

whenever \( A \) is a lattice point and the neighboring lattice points \( B, C', D, B', C, D' \)
are in cyclic order around \( A \).

\[ \begin{array}{ccc}
D & B' & C' \\
A & & C \\
& B & D'
\end{array} \]

If \( e \) is a small edge, the value \( m(e) \) is equal to the density of \( m \) relative to
arclength measure on that edge.

Fix now an integer \( r \geq 1 \), and denote by \( \triangle_r \) the (closed) triangle with vertices
0, \( ru \), and \( ru + rv = -rw \). We will use the notation \( A_j = ju, B_j = ru + jv, \) and
\( C_j = (r - j)w \) for the lattice points on the boundary of \( \triangle_r \). We also set

\[ X_j = A_j + w, Y_j = B_j + u, Z_j = C_j + v \]

for \( j = 0, 1, 2, \ldots, r + 1 \). The following picture represents \( \triangle_5 \) and the points just
defined; the labels are placed on the left.

\[ \begin{array}{ccc}
X_0 & & \\
X_1 & A_0 & \\
X_2 & A_1 & \\
& Z_2 & \\
& C_1 & Z_1 & \\
B_0 & B_1 & \cdots & C_0 & Z_0 & \\
Y_0 & Y_1 & Y_2 & & & \\
\end{array} \]

to at least three edges in the support of \( m \). We will only consider measures with at
least one branch point. This excludes measures whose support consists of one or
more parallel lines. We denote by \( \mathcal{M}_r \) the collection of all measures \( m \) satisfying
the balance condition above, whose branch points are contained in \( \triangle_r \), and such that

\[ m(A_jX_{j+1}) = m(B_jY_{j+1}) = m(C_jZ_{j+1}) = 0, \quad j = 0, 1, \ldots, r. \]

The numbers \( \alpha_j = m(A_jX_j), \beta_j = m(B_jY_j) \) and \( \gamma_j = m(C_jZ_j) \) will be called the
exit densities of \( m \). The weight \( \omega(m) \) of a measure \( m \in \mathcal{M}_r \) is defined as

\[ \omega(m) = \sum_{j=0}^{r} \alpha_j = \sum_{j=0}^{r} \beta_j = \sum_{j=0}^{r} \gamma_j; \]

the equality of the three sums follows from the balance condition.
Assume that \( m \in \mathcal{M}_r \) assigns integer densities to all small edges. We can then define an integer
\[
n = r + \omega(m),
\]
and sets \( I, J, K \subset \{1, 2, \ldots, n\} \) of cardinality \( r \) by setting \( I = \{i_1, i_2, \ldots, i_r\} \), where
\[
(2.2) \quad i_\ell = \ell + \sum_{j=0}^{\ell-1} \alpha_j, \quad \ell = 1, 2, \ldots, r,
\]
with similar formulas for \( J \) and \( K \). These are precisely the triples of sets \( (I, J, K) \) which satisfy the Littlewood-Richardson rule. The Littlewood-Richardson coefficient \( c_{IJK} \) equals the number of measures \( m \in \mathcal{M}_r \) with integer densities which satisfy (2.2). (See [5], or [2, Appendix] for a direct proof of this fact.) We will also write \( c_m = c_{IJK} \) when \( I, J, K \) are obtained from \( m \). When \( c_m = 1 \), we will say that \( m \) is rigid. In other words, \( m \) is rigid if there is no other measure with the same exit densities. Note that knowledge of \( n \) and of the sets \( I, J, K \) determines entirely the numbers \( \alpha_j, \beta_j, \gamma_j \). The Littlewood-Richardson rule requires these numbers to be the actual exit densities of some measure.

One of the advantages of this formulation of the Littlewood-Richardson rule is that it displays an underlying convexity structure. Thus, the set \( \mathcal{M}_r \) is a convex polyhedral cone, and therefore each measure \( 0 \neq m \in \mathcal{M}_r \) can be written as a sum of extremal measures. Recall that \( m \neq 0 \) is extremal if every measure \( m' \leq m \) is a multiple of \( m \). This decomposition into extremal summands is unique (except for the order of the terms) if \( m \) is a rigid measure (see [11] Corollary 3.6)). In the proof of Theorem 3.1 we will describe briefly the result of [11] showing how the extremal summands of a rigid measure are obtained.

The results of [5] imply that, given a measure \( m \in \mathcal{M}_r \) with exit densities \( \alpha_j, \beta_j, \gamma_j \), there exist Hermitian \( r \times r \) matrices \( X, Y, Z \) such that \( X + Y + Z = 2\omega(m)1_r \), and the eigenvalues of \( X, Y, Z \) are, respectively, the numbers
\[
\sum_{j=0}^{\ell-1} \alpha_j, \quad \sum_{j=0}^{\ell-1} \beta_j, \quad \sum_{j=0}^{\ell-1} \gamma_j, \quad \ell = 1, 2, \ldots, r;
\]
here \( 1_r \) denotes the \( r \times r \) identity matrix. The sum of the traces of \( X, Y, Z \) must then be \( 2r\omega(m) \), and this can be written in the equivalent form
\[
\sum_{\ell=0}^{r} \ell(\alpha_\ell + \beta_\ell + \gamma_\ell) = r\omega(m).
\]

As seen in the introduction, the possibility of reductions for the Schubert intersection problem defined by the sets \( I, J, K \subset \{1, 2, \ldots, n\} \) is tested by calculating an appropriate sum of the indices in these sets. We are now ready to discuss these sums in full generality. Assume therefore that \( r \) is fixed, \( I, J, K \subset \{1, 2, \ldots, n\} \) and \( I', J', K' \subset \{1, 2, \ldots, n'\} \) are sets of cardinality \( r \) such that \( c_{IJK} > 0 \) and \( c_{I'J'K'} > 0 \). Let us set \( \omega = n - r \) and \( \omega' = n' - r \). Choose measures \( m, m' \in \mathcal{M}_r \) such that \( \omega(m) = \omega, \omega(m') = \omega' \), and \( I, J, K \) (resp. \( I', J', K' \)) are derived from \( m \) (resp. \( m' \)) via (2.2). We denote by \( \alpha_\ell, \beta_\ell, \gamma_\ell \) (resp. \( \alpha'_\ell, \beta'_\ell, \gamma'_\ell \)) the exit densities of \( m \) (resp. \( m' \)). The sum we are interested in is
\[
\Sigma_m(m) = \sum_{\ell=1}^{r} (\alpha'_\ell i_\ell + \beta'_\ell j_\ell + \gamma'_\ell k_\ell) - \omega' n.
\]
Observe that $\Sigma_{m'}(m)$ depends only on the exit densities of $m$ and $m'$, and therefore it can be calculated directly from the sets $I, J, K$ and $I', J', K'$.

The general reduction will proceed as follows. Assume that we want to solve the Schubert problem associated to a measure $m \in M_r$. We calculate the sum $\Sigma_{m'}(m)$ for a certain kind of measure $m'$ (a rigid tree measure in the terminology introduced below). If this sum is equal to $-p < 0$, then one can effectively reduce the intersection problem to solving first an intersection problem for a stretched version of $m'$, followed by the intersection problem for $m - pm'$, for which we have $c_{m - pm'} = c_m$; see Theorem 6.2. The problem corresponding to the stretched version of $m'$ can be solved algorithmically, as seen in [1].

Since $n = \omega(m) + r$, we can rewrite

$$\Sigma_{m'}(m) = \sum_{\ell < \ell'} (\alpha_\ell \alpha'_{\ell'} + \beta_\ell \beta'_{\ell'} + \gamma_\ell \gamma'_{\ell'}) - \omega(m) \omega(m') + \left[ \sum_{\ell=0}^r \ell (\alpha'_{\ell} + \beta'_{\ell} + \gamma'_{\ell}) - r \omega(m') \right].$$

We have seen earlier that the sum inside the brackets is equal to zero, and thus

$$\Sigma_{m'}(m) = \sum_{\ell < \ell'} (\alpha_\ell \alpha'_{\ell'} + \beta_\ell \beta'_{\ell'} + \gamma_\ell \gamma'_{\ell'}) - \omega(m) \omega(m').$$

This formula has several advantages: it does not depend explicitly on $r$, and by including the branch points of $m$ and $m'$ in a triangle of a different size we do not alter the sum. More precisely, if we enlarge the triangle containing the branch points of the measures, the value of $r$ changes, but the nonzero values $\alpha_\ell, \alpha'_{\ell'}$ remain the same, and they appear in the same order, leaving the sum $\Sigma_{m'}(m)$ unchanged. The arguments in the remainder of the paper are easier to visualize when all the branch points are contained in the interior of $\Delta_r$, and the reader is free to make this additional assumption at any point. Another change which does not affect the value of $\Sigma_{m'}(m)$ is homothety. Denote by $S$ and $S'$ the supports of $m$ and $m'$, and let $q$ be a positive integer. It is then possible to define measures $\mu$ and $\mu'$ supported by $qS$ and $qS'$, respectively, and such that the density of each segment of the form $qe$ is the original density of $e$. It is obvious that $\Sigma_{\mu'}(\mu) = \Sigma_{m'}(m)$. Taking, for instance, $q = 2$, each small edge in the support of $m$ turns into two collinear small edges in the support of $\mu$. It is thus possible to assume that for every small edge $e$ in the support of $m$ there is a second, collinear, edge $e'$ which meets $e$ in a vertex $V$ which is not a branch point. This is a formal way to perform an operation which is referred to as ‘breaking an edge in half’ later on.

The fact that $\omega(m) \omega(m') = \sum_{\ell, \ell'=1}^r \alpha_\ell \alpha'_{\ell'}$ implies easily that

$$\Sigma_{m'}(m) + \Sigma_m(m') = \omega(m) \omega(m') - \sum_{\ell=1}^r (\alpha_\ell \alpha'_{\ell} + \beta_\ell \beta'_{\ell} + \gamma_\ell \gamma'_{\ell}).$$

In particular, when $m = m'$ we have

$$\Sigma_m(m) = \frac{1}{2} \left[ \omega(m)^2 - \sum_{\ell=1}^r (\alpha_\ell^2 + \beta_\ell^2 + \gamma_\ell^2) \right],$$

a formula requiring fewer multiplications.
3. Trees and Measures

Some measures \( m \in \mathcal{M}_r \) have an underlying tree structure which we describe next. We start with a special class of planar trees. We consider trees embedded in the usual Euclidean plane such that

1. each edge of the tree is a straight line segment of unit length,
2. each vertex has order 2 or 3, and
3. there are only finitely many vertices of order 3.

These conditions imply that the tree is infinite, but it has a finite number of ends. These are sequences of vertices of the form \( V_0V_1\cdots \) such that \( V_0 \) has order 3, \( V_j \) has order 2 for \( j \geq 1 \), and \( V_jV_{j+1} \) is an edge for each \( j \geq 0 \). We will require one more condition on our trees.

4. The shortest path joining two different ends contains an odd number of vertices of order 3.

All the trees we use will satisfy these four properties, and therefore we will not introduce a special name for this particular species. An immersion of a tree \( T \subset \mathbb{R}^2 \) is simply a continuous map \( \varphi : T \to \mathbb{R}^2 \) which

- is isometric on each edge,
- if \( VA \) and \( VB \) are the two edges meeting at a vertex of order 2, then \( 2\varphi(V) = \varphi(A) + \varphi(B) \), and
- if \( VA, VB, VC \) are the three edges meeting at a vertex of order 3, then \( 3\varphi(V) = \varphi(A) + \varphi(B) + \varphi(C) \), and the restriction of \( \varphi \) to \( VA \cup VB \cup VC \) preserves the orientation.

It is clear that each tree has a unique immersion up to rigid motions. Immersions are generally not one-to-one. A tree \( T \) is endowed with arclength measure. Given an immersion \( \varphi \) of \( T \), we consider the push-forward \( m_{\varphi} \) of this measure. Thus, if we arrange our immersion such that \( \varphi(T) \) is contained in the small edges of the triangular lattice determined by the vectors \( u, v, w \), then \( m \) assigns to each edge a density equal to the number of its preimages in \( T \). The resulting measure clearly satisfies the balance condition (2.1) at all vertices. Condition (4) implies that we can arrange \( \varphi \) so that \( m_{\varphi} \in \mathcal{M}_r \), provided that \( r \) is sufficiently large (so that \( \triangle_r \) contains \( \varphi(V) \) whenever \( V \) is a vertex of order 3 of \( T \)). A measure \( m \in \mathcal{M}_r \) will be called a tree measure if \( m = m_{\varphi} \) for some immersion \( \varphi \) of a tree. The following illustration shows a tree, and the range of one of its immersions. The arrows indicate ends of the tree, and the asterisk indicates where one of these ends is mapped by the immersion.

In the second illustration, some edges of the immersion have multiplicity two (i.e., they have two preimages under the corresponding immersion). They are represented by thicker lines.
Here is one more figure illustrating the fact that a tree measure need not be extremal.

In this case, the measure $m_{\omega}$ has two summands with unit densities; the support of one of them is pictured below.

If $m \in M_r$ is a tree measure, it is fairly easy to see that the number of ends of the corresponding tree $T$ is $3\omega(m)$. We will write $\omega(T) = \omega(m)$. For the trees above, the value of $\omega(T)$ is 3 or 4.

**Theorem 3.1.** Assume that $m \in M_r$ is a rigid extremal measure. Then there exists a tree measure $m' \in M_r$ such that $m = cm'$ for some constant $c > 0$.

**Proof.** Assume, more generally, that $m \in M_r$ is a rigid measure. Given two adjacent small edges $AB, BC$ in the support of $m$, we write $AB \rightarrow m BC$ if either

(a) $A, B, C$ are collinear and one of the edges $BX$ such that $\angle XBC = 60^\circ$ satisfies $m(BX) = 0$, or
(b) $\angle ABC = 120^\circ$, and the edge $BX$ opposite $AB$ satisfies $m(BX) = 0$.

Given an edge $e = AB$, there exist at most two edges $f$ adjacent to $B$ such that $e \rightarrow_m f$. More generally, if $e, f$ are two small edges, we write $e \Rightarrow_m f$ if either $e = f$, or

$$e = e_1 \rightarrow_m e_2 \rightarrow_m \cdots \rightarrow_m e_k = f$$

for some chain $\gamma = \{e_1, e_2, \ldots, e_k\}$, $e_j = X_{j-1}X_j$, of small edges. This relation is called *descendance*, and it was proved in [1] that each edge in the support of $m$ is the descendant of a minimal (or root) edge contained in $\Delta_r$. Moreover, the descendants of a root edge form the support of an extremal measure. Here minimality is defined up to the equivalence relation $e \Leftrightarrow_m f$ if $e \Rightarrow_m f$ and $f \Rightarrow_m e$. A chain $\gamma$ as above is called a *descendance path* from $e$ to $f$. 
Assume now that $m$ is extremal and $e$ is a root edge for $m$ contained in $\triangle r$. Dividing $m$ by $c = m(e)$, we may assume that $m(e) = 1$. If $f$ is any edge in the support of $m$, $m(f)$ equals the number of descendence paths from $e$ to $f$ (cf. [1]). Note that $m$ may have several (often, infinitely many) root edges $f$; they are characterized by the equality $m(f) = 1$.

The construction of the required tree $T$ is somewhat analogous to the construction of a universal covering space. Abstractly, the vertices of $T$ are sequences $X_0X_1 \cdots X_n$ such that either $n = 0$ and $X_0$ is an endpoint of $e$, or $n \geq 1$ and $\gamma = \{X_0X_1, X_1X_2, \ldots, X_{n-1}X_n\}$ is a descendence path from $e$. The vertices $X_0, X_1$ are identified with $X_1X_0, X_0X_1$, respectively, if $X_0$ and $X_1$ are the endpoints of $e$. Two vertices of the form $X_0X_1 \cdots X_n, X_0X_1 \cdots X_nX_{n+1}$ are joined by an edge. Assigning unit length to the edges of $T$, there is a map $\varphi : T \to \mathbb{R}^2$ which sends a vertex $X_0X_1 \cdots X_n$ to $X_n$. We embed the tree $T$ into the plane in such a way that this map $\varphi$ preserves orientation at each triple vertex of $T$. It should be clear now that $m = m_\varphi$.  

Let $\varphi$ be the immersion of $T$ described in the preceding proof, and let $e$ be an edge of $T$ such that $\varphi(e)$ is a root edge for the measure $m$. We can orient all other edges of $T$ away from $e$. It was shown in [1] that the map $\varphi$ has the following additional property: if $g$ and $h$ are two edges such that $\varphi(g) = \varphi(h)$, then $\varphi$ induces the same orientation on this common image. In other words, the edges in the support of $m$, other than $\varphi(e)$, can be consistently oriented in the direction of a descendence path from $\varphi(e)$. The following lemma is also proved in [1] (see the discussion following Theorem 3.5 in [1]).

**Lemma 3.2.** Let $m$ be a rigid extremal measure, and orient the edges in its support away from a fixed root edge. Each lattice point meets at most four edges in the support of $m$, and the possible positions of these edges, including their orientations, are as follows

![Diagram](image)

up to rotations.

In order to study the sums $\Sigma_{m'}(m)$, we will also need some maps which are closely related to immersions, but are discontinuous. Assume that $T$ is a tree, and $\varphi$ is an immersion of $T$ such that the induced measure is in $\mathcal{M}_r$ for some $r$. Denote by $T_0$ the set of points in $T$ which are not vertices. A function $\psi : T_0 \to \mathbb{R}^2$ will be called a fractured immersion if

1. the range of $\psi$ is contained in the small edges of the triangular lattice determined by $u, v, w$,
2. there is an immersion $\varphi$ of $T$ such that $\psi(t) - \varphi(t)$ is constant on the interior of every edge, and
3. $\psi$ extends continuously to all except finitely many vertices of $T$.

Let $\psi$ be a fractured immersion of a tree $T$. We will associate to each vertex $V$ of $T$ an integer $\delta_\psi(V)$ which measures how badly fractured $\psi$ is at $V$. If $\psi$ extends continuously to the point $V$ we set $\delta_\psi(V) = 0$. Assume next that the order of $V$ is 2 and the two edges $AV, VB$ are mapped to $AV', VB'$, respectively, with $V' \neq V''$. We will set $\delta_\psi(V) = q$ if the point $V''$ lies $q$ lattice units to the left of the line.
joining \( A' \) and \( V' \), where this line is oriented so that \( A'V' \) points toward \( V' \). Note that \( V'' \) could be to the right of this line, in which case \( q < 0 \), and \( V'' \) (as well as \( B' \)) could be on this line, in which case \( q = 0 \). Finally, let \( V \) be a vertex of order 3, assume that the three edges \( AV, BV, CV \) are mapped to \( A'V', B'V'', C'V''' \), and note that these three segments still form 120\(^\circ\) angles. If the lines containing these three segments are concurrent, we set \( \delta_\psi(V) = 0 \). Otherwise, these three lines form an equilateral triangle \( \Delta \) with sidelength \( q \). Orient the sides of this triangle so that the segments \( A'V', B'V'', C'V''' \) point toward \( V', V'', V''' \), respectively. If the boundary of \( \Delta \) is oriented clockwise, set \( \delta_\psi(V) = -q \), and in the contrary case set \( \delta_\psi(V) = q \). The following figures shows three cases in which the values of \( \delta_\psi(V) \) are 0, \(-2\) and 1. The dotted lines represent small edges.

The orientations indicated above are used exclusively for the calculation of the numbers \( \delta_\psi(V) \). In the proofs below we will need to orient all the edges of a tree \( T \) (not just the ones adjacent to \( V \)), and this will generally be the orientation away from a fixed vertex or edge.

In the following statement, the segment \( A_0X_0 \) is deemed to exit \( \Delta_r \) at the point \( A_0 \), rather than \( C_r \), while \( C_rZ_r \) is deemed to exit at \( C_r \). Of course, this issue does not arise when the corners of \( \Delta_r \) are not exit points, and this can be achieved by enlarging the triangle.

**Theorem 3.3.** Let \( \psi \) be a fractured immersion of a tree \( T \) such that all the limits of \( \psi \) at discontinuity points are contained in \( \Delta_r \). For each end \( E \) of \( T \), denote by \( \ell(E) \) the rank of the exit point of \( \psi(E) \) from \( \Delta_r \). In other words, \( \ell(E) = \ell \) if the closure of \( \psi(E) \) intersects \( \partial\Delta_r \) in \( A_\ell, B_\ell, \) or \( C_\ell \). Then we have

\[
\sum_{\text{all ends } E \text{ of } T} \ell(E) = r\omega(T) + \sum_{\text{all vertices } V \text{ of } T} \delta_\psi(V).
\]

**Proof.** We proceed by induction on the number of vertices where \( \psi \) does not extend continuously. When this number is equal to zero, \( g \) is an immersion, and the sum in the left hand side is nothing but

\[
\sum_{\ell} \ell(\alpha_\ell + \beta_\ell + \gamma_\ell) = r\omega(m),
\]

where \( \alpha_\ell, \beta_\ell, \gamma_\ell \) are the exit densities of the corresponding measure \( m \). This is precisely the desired identity because \( \omega(m) = \omega(T) \). Assume then that the theorem has been proved for all fractured immersions with fewer discontinuity points than \( \psi \), and there exists at least one vertex \( V \) where \( \psi \) does not extend continuously. Consider first the case when \( V \) is of order 2, and the vertices \( AV, VB \) are mapped by \( \psi \) to \( A'V', B'V'' \), which we will assume to be horizontal for definiteness. By transposing the points \( A, B \), we can also assume that \( A' \) is to the left of \( V'' \) and \( B' \) is to the right of \( V'' \). There is then a point \( A_\ell \) such that the segment \( A_\ell V'' \) is horizontal; denote by \( a \) its length. Similarly, there is a point \( C_k \) such that the segment \( V'C_k \) is horizontal. The definition of \( \delta \) implies that

\[
(3.1) \quad \ell + k + \delta_\psi(V) = r.
\]
We now form two trees in the following way. Cut the tree $T$ at the point $V$, and add to the part containing $AV$ an end $VV_1V_2\cdots$, thus forming a tree $T'$. Analogously, add to the part containing $BV$ a path $VW_1W_2\cdots W_a$, where $W_1, W_2, \ldots, W_{a-1}$ have order 2, and two ends meeting at $W_a$, thus forming a tree $T''$. The map $\psi$ gives rise to two fractured immersions $\psi'$ and $\psi''$ of $T'$ and $T''$ as follows: $\psi'(VV_1V_2\cdots)$ is the half line starting with $A'V'$, $\psi''(VW_1W_2\cdots W_a) = V''A_\ell$, and the two ends meeting at $W_a$ are mapped onto the two half lines starting at $A_\ell$ and pointing left. It is clear that $\psi'$ and $\psi''$ have fewer vertices of discontinuity than $\psi$, and therefore the desired formula is true for $\psi'$ and $\psi''$. It is clear that

$$\sum_{\text{vertices } W \text{ of } T} \delta_\psi(W) = \sum_{\text{vertices } V' \text{ of } T'} \delta_{\psi'}(V') + \sum_{\text{vertices } V'' \text{ of } T''} \delta_{\psi''}(V'') + \delta_\psi(V),$$

while

$$\sum_{\text{all ends } E' \text{ of } T'} \ell(E') + \sum_{\text{all ends } E'' \text{ of } T''} \ell(E'') = \sum_{\text{all ends } E \text{ of } T} \ell(E) + \ell + k.$$ 

The desired equality follows then from (3.1) because $\omega(T') + \omega(T'') = \omega(T) + 1$.

Consider next the case that $V$ is of order three, and the three edges $AV, BV, CV$ of $T$ are mapped to $A'V', B'V'', C'V'''$. Assume that $A, B, C$ are arranged clockwise around $V$. A cyclic permutation allows us to assume that $A'V'$ is horizontal, and we must consider the two cases where $A'$ is to the left or to the right of $V'$. These two situations are illustrated below.
Assume first that $A'$ is on the left. The half lines $A'V'$, $B'V''$, $C'V'''$ exit $\triangle r$ at points $C_k, B_\ell, A_p$, respectively. As in the preceding proof, we cut $T$ at the point $V$, and form three trees $T', T'', T'''$ by attaching to the part of $T$ which contains $A, B, C$, respectively, an end attached at $V$. The map $\psi$ gives rise to three fractured immersions $\psi', \psi'', \psi'''$ of these trees. For instance, $\psi'$ maps the additional end at $V$ to the half line starting with $V'C_k$. Moreover, the new fractured immersions have fewer discontinuity points than $\psi$, and therefore the inductive hypothesis applies to them. As in the preceding case, we have
\[ \omega(T') + \omega(T'') + \omega(T''') = \omega(T) + 1, \]
and
\[ k + \ell + p + \delta_\psi(V) = r. \]
The desired formula follows now easily. Finally, consider the case in which $A'$ is to the right of $V'$. In this case, the half lines $A'V', B'V'', C'V'''$ exit $\triangle r$ at points $A_\ell, B_k, C_p$, respectively, and the trees $T', T'', T'''$ must be constructed by attaching at $V$ a few edges followed by two ends. In this case we have $\omega(T') + \omega(T'') + \omega(T''') = \omega(T) + 2$ and the reader can verify easily that $k + \ell + p + \delta_\psi(V) = 2r$. The conclusion follows as before.

\[ \square \]

4. Inflations and Fractured Immersions

We recall from [5] (see also [1]) that every measure $\nu \in \mathcal{M}_r$ has an associated puzzle obtained by inflating $\nu$. The inflation of $\nu$ is defined as follows. Cut the plane along the edges in the support of $\nu$ to obtain a collection of puzzle pieces, and translate these pieces away from each other in the following way: the parallelogram formed by the two translates of a side $AB$ of a white puzzle piece has two sides of length equal to the density of $\nu$ on $AB$ and $60^\circ$ clockwise from $AB$. The balance condition (2.1) implies that the original puzzle pieces and these parallelograms fit together, and leave a space corresponding to each branch point in the support of $\nu$. Here is an illustration of the process with $r = 3$; the thinner lines in the support of the measure have density one, and the thicker ones density 2. The original pieces of the triangle $\triangle r$ are white, the added parallelogram pieces are dark gray, and the branch points become light gray pieces. Each light gray piece has as many sides as there are branches at the original branch point.
The dotted lines indicating the boundary of $\triangle_r$ have been translated so that they now outline a triangle with sides $r + \omega(\nu)$, which we may assume is precisely $\triangle_{r+\omega(\nu)}$. The decomposition of this triangle into white, dark gray, and light gray pieces is the puzzle associated to $\nu$. The white regions in the puzzle are called ‘zero regions’, and the light gray ones ‘one regions’, and the dark gray parallelograms ‘0-1 regions’ in [5].

The main use of inflations will be to produce fractured immersions from a given immersion of a tree $T$. Assume indeed that $T$ is a tree, $\varphi$ is an immersion of $T$ such that the corresponding measure $m' = m_{\varphi}$ is in $M_r$, and let $\nu \in M_r$ be another measure. Assume that each edge of $T$ has been given an orientation, and that all the edges belonging to an end of $T$ have been oriented outward (i.e., towards the infinite part of that end). For each edge $e$ in $T$ such that $\varphi(e)$ is in the support of $\nu$, we attach $\varphi(e)$ to the white puzzle piece on the right of $\varphi(e)$ when $\varphi(e)$ is given the orientation induced by the orientation of $e$. For edges $e$ with $\nu(e) = 0$, $\varphi(e)$ is contained in a white puzzle piece, and it moves along with that piece. If we denote now by $\psi(e)$ the translate of $\varphi(e)$ in the puzzle construction, we obviously obtain a fractured immersion. The following figure illustrates the process as applied to a measure $m' = m_{\varphi}$ whose support is pictured below, and $\nu$ is the measure whose inflation was depicted in the preceding figure. We have oriented all the edges away from the branch point inside $\triangle_3$, and completed the outline of $\triangle_6$.

Note that all the fractures of $\psi$ are contained in $\triangle_{r+\omega(\nu)}$, and therefore the formula in Theorem 3.3 applies. Let $\alpha_j, \beta_j, \gamma_j$ be the exit densities of $\nu$, and let
\( \alpha_j', \beta_j', \gamma_j' \) be the exit densities of \( m' \). Then it is easy to see that

\[
\sum_{\text{all ends } E \text{ of } T} \ell(E) = \sum_{\ell=0}^r \left[ \alpha'_\ell \left( \ell + \sum_{k < \ell} \alpha_k \right) + \beta'_\ell \left( \ell + \sum_{k < \ell} \beta_k \right) + \gamma'_\ell \left( \ell + \sum_{k < \ell} \gamma_k \right) \right] = r \omega(m') + \sum_{k < \ell} \left( \alpha'_\ell \alpha_k + \beta'_\ell \beta_k + \gamma'_\ell \gamma_k \right).
\]

Indeed, this follows from the fact that an end \( E \) such that \( \varphi(E) \) exits at \( A_\ell \) is translated to \( \psi(E) \) which exits at \( A_{\ell+\alpha_0+\ldots+\alpha_{\ell-1}} \).

**Lemma 4.1.** With the notation above, we have

\[
\Sigma_{m'}(\nu) = \sum_{\text{all vertices } V \text{ of } T} \delta_\psi(V).
\]

**Proof.** Theorem 3.3 yields

\[
\sum_{\text{all ends } E \text{ of } T} \ell(E) = (r + \omega(\nu)) \omega(m') + \sum_{\text{all vertices } V \text{ of } T} \delta_\psi(V).
\]

Combining this with the identity preceding the statement, we obtain

\[
\sum_{\text{all vertices } V \text{ of } T} \delta_\psi(V) = \sum_{k < \ell} \left( \alpha'_\ell \alpha_k + \beta'_\ell \beta_k + \gamma'_\ell \gamma_k \right) - \omega(m') \omega(\nu),
\]

and this is precisely the formula (2.3) for \( \Sigma_{m'}(\nu) \). \( \square \)

**Theorem 4.2.** Assume that \( \nu, m' \in \mathcal{M}_r \), and \( m' \) is a tree measure.

1. If the support of \( m' \) is not contained in the support of \( \nu \), then \( \Sigma_{m'}(\nu) \geq 0 \).
2. If \( m' \) is not rigid, we also have \( \Sigma_{m'}(m') \geq 0 \).
3. If \( m' \) is an extremal rigid measure assigning unit density to its root edges, we have \( \Sigma_{m'}(m') = -1 \).

**Proof.** Let \( \varphi \) be an immersion of a tree \( T \) such that \( m' = m'_{\varphi} \). To prove (1), fix an edge \( e_0 \) such that \( \varphi(e_0) \) is not contained in the support of \( \nu \), and orient all the other edges of \( T \) away from \( e_0 \). Construct a fractured immersion \( \psi \) using the above construction associated with the inflation of \( \nu \). It is easy to verify that in this case we have \( \delta_\psi(V) \geq 0 \) for every vertex \( V \) of \( T \). Indeed, \( \delta_\psi(V) \) can be calculated explicitly in terms of the values of \( \nu \) on one of the edges adjacent to \( \varphi(V) \). To see this, assume first that \( V \) is of order two, \( AV \) and \( VB \) are the two adjacent edges, and they are mapped by \( \varphi \) to \( A'V' \) and \( V'B' \). These two edges are shown below, with the arrows indicating their orientation, and the dotted extensions are drawn to indicate the value of \( \delta_\psi(V) \).
Clearly, we have $\delta_\varphi(V) = \nu(V'X)$, with $X$ as in the figure, i.e. on the right side of $AV'$, and $\angle XV'B' = 60^\circ$. If $V$ has order three, let $AV, BV, CV$ be the three adjacent edges, with $AV$ oriented toward $V$. Assume that $\varphi(AV) = A'V'$, and $X$ is symmetric to $A'$ relative to $V'$. We have again $\delta_\varphi(V) = \nu(V'X)$.

Assertion (1) follows now from Lemma 4.1. (In both illustrations we assumed that $\nu$ assigns nonzero densities to all six edges adjacent to $V'$. More precisely, these densities were taken to be 1, 7, 4, 3, 5 and 6 in clockwise order.)

Assume next that $m'$ is not rigid, and choose a different measure $\nu$ with the same exit densities. Then $\nu$ can be written as a sum of distinct extreme measures, say $\nu = \sum_j m_j$. If the support of $m_j$ contains the support of $m'$, then $m_j$ is a positive multiple of $m'$ by extremality. Thus there is at most one $j$ such that the support of $m_j$ contains the support of $m'$. Assume for definiteness that $m_1 = \kappa m'$, where $0 \leq \kappa < 1$. Then part (1) of the theorem implies that $\Sigma m_1(m_1) \geq 0$ for $j \neq 1$, hence

$$\Sigma m_1(m') = \Sigma m_1(\nu) = \sum_j \Sigma m_1(m_j) \geq \Sigma m_1(m_1) = \kappa \Sigma m_1(m'),$$

and therefore $\Sigma m_1(m') \geq 0$, as claimed.

Finally, assume that $m'$ is rigid, and choose an edge $e_0$ such that $\varphi(e_0)$ is a root edge for $\nu = m'$ contained in $\triangle_r$. Orient the other edges $T$ away from $e_0$, and
also give \(c_0\) some orientation, say it is oriented away from one of its endpoints \(V_0\). In this case we have \(\delta_\psi(V_0) = -1\) and \(\delta_\psi(V) = 0\) for all other vertices. To verify this fact one must observe that in the pictures above we must have \(m'(V'X) = 0\) because of the rigidity of \(m'\). This follows from Lemma 4.2. The only exception is the orientation at the point \(V_0\) which produces a nonzero \(\delta_\psi(V_0)\). To calculate the value of \(\delta_\psi(V_0)\), we will further assume that \(V_0\) is a vertex of order 2 and both edges \(A_0V_0B_0\) adjacent to \(V_0\) are mapped by \(\varphi\) to root edges of \(m'\). This can be achieved by applying a homothety, as seen in the introduction. Assuming, for instance, that \(\varphi(A_0) = A, \varphi(B_0) = B\) and \(\varphi(V_0) = V\), we have \(m'(AV) = m'(VB) = 1\). If we orient \(A_0V_0\) and \(B_0V_0\) away from \(V_0\), the inflation process looks as follows:

![Diagram of orientations](image)

The width of the dark gray parallelogram is \(m'(AV) = 1\), hence \(\delta_\psi(V_0) = -1\). The theorem follows.

**Corollary 4.3.** Assume that \(m, m' \in \mathcal{M}_r\) and \(m'\) is a rigid extreme tree measure. If \(\Sigma_{m'}(m) = -p < 0\) then \(pm' \leq m\) and \(c_{m-pm'} = c_m\).

**Proof.** Let \(s\) be the largest number such that \(sm' \leq m\). Then the support of \(m-sm'\) does not contain the support of \(m'\), and therefore \(\Sigma_{m'}(m-sm') \geq 0\) by Theorem 4.2 (1). Thus

\[-p = \Sigma_{m'}(m) = \Sigma_{m'}(m-sm') + s\Sigma_{m'}(m') \geq -s,
\]

so that \(s \geq p\). If \(m''\) is any other measure with the same exit densities as \(m\), it follows that \(pm' \leq m''\) as well, and the exit densities for \(m-sm'\) the same. This yields a bijection \(m'' \leftrightarrow m'' - pm'\) between measures with the exit densities of \(m\) and measures with the exit densities of \(m - pm'\).

Corollary 4.3 extends [3] Proposition 3.10] which, in our terminology, states that \(c_{m-m''} = c_m\) if \(m'\) is a tree measure with \(\omega(m') = 1\), and \(\Sigma_{m'}(m) < 0\). That result was stated in terms of the sets \(I, J, K\), and the proof proceeds through a very explicit construction of Littlewood-Richardson tableaux.

We can now give a general method for the construction of rigid measures, thus completing [1] Theorem 3.8. First, we need to review that result. Let \(m \in \mathcal{M}_r\) be a rigid measure, and let \(m_1, m_2 \in \mathcal{M}_r\) be two tree measures with support contained in the support of \(m\). The relation \(m_1 \prec_0 m_2\) was defined in [1] as follows: there exist four small edges \(AX, XB, CX\) and \(XD\) such that

1. \(AX\) and \(XB\) are collinear edges in the support of \(m_1\),
2. \(CX\) and \(XD\) are collinear edges in the support of \(m_2\), and
3. \(XB\) is 60° clockwise from \(XD\).

It was shown in [1] that \(\prec_0\) can be extended to an order relation on the set of extremal rigid measures with support contained in the support of \(m\). As noted earlier, each extremal rigid measure is a positive multiple of a tree measure. The following result allows us to extend \(\prec_0\) to the collection of all rigid tree measures; this extension is no longer contained in an order relation.

**Lemma 4.4.** Let \(m\) be a rigid measure, and let \(m_1, m_2\) be extremal measures with support contained in the support of \(m\). We have \(m_1 \prec_0 m_2\) if and only if \(\Sigma_{m_2}(m_1) > 0\).
Proof. Observe that $m_1$ and $m_2$ are also rigid. Let $\varphi$ be an immersion of some tree $T$ such that $m_2 = m_\varphi$; such an immersion exists by Theorem 5.1. Orient all the edges of $T$ away from some edge $e_0$ such that $\varphi(e_0)$ is a root edge for $m_2$ not contained in the support of $m_1$. Assume first that $m_1 \prec_0 m_2$, and the small edges $AX, XB, CX, XD$ satisfy conditions (1-3) above. We may assume that $CX = \varphi(e_1), XD = \varphi(e_2)$, where $e_1$ and $e_2$ are adjacent edges, and $e_1$ is oriented toward $e_2$. The proof of Theorem 4.2 implies that $\Sigma_{m_2}(m_1) \geq m_1(XB) > 0$.

Conversely, assume that $\Sigma_{m_2}(m_1) > 0$. Let $e_1, e_2, e_3$ be three edges of $T$ adjacent to a vertex $V$, and assume that $e_1$ is oriented toward $V$. These edges are mapped by $\varphi$ to $A_jX, j = 1, 2, 3$, and we must have $A_1X \rightarrow m XA_2$ and $A_1X \rightarrow m XA_3$. It follows that the edge $XB$ opposite $A_1X$ satisfies $m(XB) = 0$, and therefore $m_1(XB) = 0$, so that this vertex $V$ contributes nothing to $\Sigma_{m_2}(m_1)$. We conclude that there must exist some vertex $V$ of order 2 which contributes to $\Sigma_{m_2}(m_1)$. Let $e_1, e_2$ be the two edges adjacent to $V$, and assume that $e_1$ is oriented toward $V$. Then $\varphi$ maps these two edges to collinear edges $CX, XD$ so that $CX \rightarrow m XD$. The fact that $V$ contributes to $\Sigma_{m_2}(m_1)$ means simply that the edge $XB$ which is 60° clockwise from $XD$ is in the support of $m_1$. We claim that the edge $AX$ opposite $XB$ is also in the support of $m_1$. Indeed, the fact that $CX \rightarrow m XD$ implies that the edge $XB'$ which is 60° counterclockwise from $XD$ is not in the support of $m_1$, hence not in the support of $m_1$. The balance condition for $m_1$ implies that $m_1(AX) > 0$.

Thus the vertices $AX, XB, CX, XD$ witness the fact that $m_1 \prec_0 m_2$. □

Corollary 3.6 of [1] allows us to write any rigid measure $m \in \mathcal{M}_r$ under the form

$$m = \sum_{j=1}^{n} p_j m_j,$$

where $p_j > 0$, and the $m_j$ are distinct extremal tree measures. Moreover, Theorem 3.8 of that paper allows us to arrange the terms of this sum in such a way that $m_i \prec_0 m_j$ implies that $i \leq j$. According to Lemma 4.4, $m_i \prec_0 m_j$ is equivalent to $\Sigma_{m_i}(m_j) > 0$ for these measures, Thus, the following result can be viewed as a converse of [1, Corollary 3.6].

**Corollary 4.5.** Let $m_1, m_2, \ldots, m_n$ be extremal, rigid tree measures such that $\Sigma_{m_i}(m_j) = 0$ for $i < j$. For every $p_1, p_2, \ldots, p_n > 0$, the measure $m = \sum_{j=1}^{n} p_j m_j$ is rigid.

Proof. We proceed by induction, observing that the result is trivial for $n = 1$. For the inductive step, the hypothesis implies $\Sigma_{m_1}(m) = -p_1$, and therefore $c_m = c_{m-p_1}m_1$ by Corollary 4.3. □

5. Mending Fractured Immersions

We will analyze in more detail the main result of the preceding section. This analysis is a necessary preliminary for the results in Section 6. Let us fix a tree $T$ and an immersion $\varphi$ of $T$ which maps all the triple vertices of $T$ to $\Delta_r$. Let $m' = m_\varphi$ be the corresponding measure in $\mathcal{M}_r$, and let $\nu \in \mathcal{M}_r$ be another measure. Fix for the moment an edge $e_0$ in $T$, and orient all the other edges of $T$ away from $e_0$. We define for every vertex $V$ of $T$ a number $\delta_{e_0}(V, \nu)$. Assume first that $V$ has order 2 and the corresponding edges are $AV, VB$, oriented toward $B$. Setting
$A' = \varphi(A), V' = \varphi(V), B' = \varphi(B)$, we set
$$
\delta_{e_0}(V, \nu) = \nu(V'X),
$$
where $X$ is on the right side of $A'V'$, and $\angle XV'B' = 60^\circ$. On the other hand, if $V$ has order 3 and the corresponding edges are $AV, BV, CV$, with $AV$ oriented toward $V$, then
$$
\delta_{e_0}(V, \nu) = \nu(V'X),
$$
where $A' = \varphi(A), V' = \varphi(V)$, and $X$ are collinear. When $V$ is one of the endpoints of $e_0$, we orient $e_0$ toward that endpoint in this definition. Theorem 4.2 can now be given a more precise form.

**Theorem 5.1.** With $\nu$ and $m'$ as above, we have
$$
\Sigma_{m'}(\nu) + \nu(\varphi(e_0)) = \sum_V \delta_{e_0}(V, \nu).
$$

**Proof.** The easiest way to see this is to cut $e_0$ in half, and orient the two halves away from its midpoint $Y$. Construct a fractured immersion $\psi$ of $T$ as in the proof of Theorem 4.2 For this immersion we have $\delta_{\psi}(V) = \delta_{e_0}(V, m)$ for each $V$, and $\delta_{\psi}(Y) = -\nu(\varphi(e_0))$.

In the preceding proof, when $\varphi(e_0)$ is not contained in the support of $\nu$, the edge $\varphi(e_0)$ is simply translated along with the white puzzle piece which contains it. For our next result, it will be important that $m'$ be a rigid measure and $\varphi(e_0)$ be a root edge for the measure $m'$ with $m'(\varphi(e_0)) = 1$. With this choice, Lemma 3.2 implies the equality
$$
\delta_{e_0}(V, m') = 0
$$
for every vertex $V$.

Let $T$ be a tree, and let $AV, VB$ be two edges meeting at a vertex $V$ of order 2. One can stretch the tree to a tree $T'$ replacing $V$ by a path $V_1V_2\cdots V_k$ of consecutive vertices of order 2 and the edges $AV$ and $BV$ are replaced by $AV_1$ and $BV_k$. Analogously, if $AV, BV, CV$ are three edges meeting at $V$, we can stretch $T$ by replacing $V$ with a ‘tripod’ formed by edges $V_1V_2\cdots V_k, W_1W_2\cdots W_k, U_1U_2\cdots U_kX$, where all new vertices except $X$ have order 2, and $AV, BV, CV$ are replaced by $AV_1, BW_1, CU_1$. If $T'$ is obtained from $T$ by a finite number of such stretch operations, we will say that $T'$ is a stretch of $T$. If $\varphi$ is an immersion of a stretch $T'$ of $T$, the restriction of $f$ to the original edges of $T$ determines a fractured immersion $\psi$ of $T$ with the property that $\delta_{\psi}(V) = 0$ for every vertex $V$ of $T$. Such a fractured immersion of $T$ is said to be stretchable. If $\psi$ is a stretchable fractured immersion and it is obtained as the restriction of an immersion $\varphi$, we will also write $m_{\psi}$ for the measure $m_{\varphi}$. The condition $\delta_{\psi}(V) = 0$ for all $V$ is not sufficient for stretchability. For instance, assume that $V$ has degree 2, $AV, VB$ are the two adjacent edges mapped by $\psi$ to $A'V'$ and $V''B'$. The condition $\delta_{\psi}(V) = 0$ implies that the points $A', B', V', V''$ are collinear, but stretchability requires that $V''$ should be between $A'$ and $B'$; the distance from $V'$ to $V''$ is precisely the number of additional edges one must add at the point $V$. Similarly, if $AV, BV, CV$ are mapped to $A'V', B'V'', C'V'''$, the condition $\delta_{\psi}(V) = 0$ implies that these three lines intersect in a point $Z$, and stretchability requires that $V'$ (resp. $V''$, $V'''$) be between $A'$ (resp. $B'$, $C'$) and $Z$.

Part of the following argument (namely, the case $q = 0$) amounts to a simplified proof of [1, Theorem 4.3].
Theorem 5.2. Let \( \mu, \mu' \in \mathcal{M}_r \), where \( \mu' \) is an extremal rigid measure assigning unit density to its root edges; in particular \( \mu' = m_\varphi \) for some immersion \( \varphi \) of a tree \( T \). Assume further that \( \Sigma_{m_\varphi}(\mu) = 0 \). Denote by \( \alpha', \beta', \gamma' \) and \( \tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}' \) the exit densities of \( \mu \) and \( \mu' \), respectively. There exists a stretchable fractured immersion \( \psi \) of \( T \) such that

1. all the limits of \( \psi \) at discontinuity points are contained in \( \Delta_{r+\omega(\mu)} \),
2. the exit densities \( \tilde{\alpha}'_i \) of the corresponding measure \( \tilde{m}_\psi = m_\psi \in \mathcal{M}_{r+\omega(\mu)} \) are only different from zero for \( i = \ell + \sum_{s=1}^{r-1} \alpha_s, \ell = 1, 2, \ldots, r \), in which case

\[
\tilde{\alpha}'_i = \alpha'_i, \text{ with similar formulas for } \tilde{\beta}'_i \text{ and } \tilde{\gamma}'_i.
\]

Proof. Denote by \( q \) the largest integer with the property that \( qm' \leq \mu \), and set \( \nu = \mu - qm' \). It is clear that the support of \( \mu' \) is not contained in the support of \( \nu \).

Assume first that \( q = 0 \), and choose an edge \( e_0 \) such that \( \varphi(e_0) \) is contained in \( \Delta_r \) and \( \nu(\varphi(e_0)) = 0 \). Theorem 5.1 implies that \( \delta_{e_0}(V, \nu) = 0 \) for every vertex \( V \) of \( T \). Orient all the edges of \( T \) away from \( e_0 \), and construct a fractured immersion \( \psi \) of \( T \) by attaching each \( \varphi(e) \) to the white puzzle piece of \( \nu \) on its right. The condition \( \delta_{e_0}(V) = 0 \) insures that \( \psi \) is stretchable at \( V \), so that (1) holds. Since all the ends of \( T \) are oriented outward, condition (2) is satisfied as well.

Consider now the case \( q > 0 \), fix an edge \( e_0 \) such that \( \varphi(e_0) \) is a root edge of \( m' \) contained in \( \Delta_r \), and orient all the edges away from \( e_0 \). Give \( e_0 \) either orientation, and construct a fractured immersion \( \psi_0 \) of \( T \) by attaching each \( \varphi(e) \) to the white puzzle piece of \( \mu = \nu + qm' \) on its right. To conclude the proof, it will suffice to construct a stretchable fractured immersion \( \psi \) which coincides with \( \psi_0 \) on the ends of \( T \). Note that \( \psi_0(e) \) is now an edge of a dark gray parallelogram whose other side has length \( \mu(e) \). We construct \( \psi(e) \) by moving \( \psi_0(e) \) inside this parallelogram a number of units equal to

\[
\sum_{V \geq e} \delta_{e_0}(V, \nu) = \sum_{V \geq e} \delta_{e_0}(V, \mu),
\]

away from the white piece to which \( \psi_0(e) \) was attached, where the sum is extended over the vertices \( V \) which are descendants of \( e \) in the chosen orientation. In other words, the sum is extended over those vertices \( V \) for which the shortest path from \( e_0 \) to \( V \) passes through \( e \). It is important to note that \( \psi(e) \) really is contained in this (closed) gray parallelogram, and for this purpose it suffices to show that

\[
\sum_{V \geq e} \delta_{e_0}(V, \mu) \leq \mu(e).
\]

This follows from the fact that \( \delta_{e_0}(V, \mu) = \delta_e(V, \mu) \) if \( V \geq e \), and therefore

\[
\sum_{V \geq e} \delta_{e_0}(V, \mu) = \sum_{V \geq e} \delta_e(V, \mu) \leq \sum_{V} \delta_e(V, \mu) = \mu(e)
\]

by Theorem 5.1 since \( \Sigma_{m'}(\mu) = 0 \). Also observe that the position of \( \psi(e_0) \) does not depend on the orientation chosen for \( e_0 \) because (with either orientation)

\[
\sum_{V \geq e_0} \delta_{e_0}(V, \mu) + \sum_{V \leq e_0} \delta_{e_0}(V, \mu) = \sum_{V} \delta_{e_0}(V, \mu) = \mu(e_0),
\]

and this is precisely the width of the dark gray parallelogram of which \( \psi_0(e_0) \) is a side. It remains now to verify that \( \psi \) is stretchable. Consider first two edges \( e_1 = AB, e_2 = BC \) adjacent to a vertex \( B \) of order 2, oriented toward \( B \) and \( C \).
respectively. Assume that $\varphi(e_1) = A'B'$, $\varphi(e_2) = B'C'$, and the small edge $B'X$ is on the right of $A'B'$ such that $\angle XBC' = 60^\circ$. We have then
\[
\sum_{V \geq e} \delta_{e_0}(V, \mu) = \mu(B'X) + \sum_{V \geq f} \delta_{e_0}(V, \mu),
\]
so that $\psi_0(e_1)$ must be moved left $\mu(B'X)$ more units than $\psi_0(e_1)$. This is precisely what is needed to align $\psi(e)$ and $\psi(f)$, as illustrated in the figure below, where the solid lines represent $\psi_0(e_1)$ and $\psi_0(e_2)$, the dashed lines represent $\psi(e)$ and $\psi(f)$, and the dotted line represents the range of the stretch of $\psi$.

Assume now that $e_1 = AV$, $e_2 = BV$, $e_3 = CV$ are three edges adjacent to $V$, such that $e_1$ is oriented toward $V$. These edges are mapped by $\varphi$ to $A'V'$, $B'V'$, $C'V'$. Let $V'X$ be the small edge opposite $A'V'$. We have
\[
\sum_{V \geq e_1} \delta_{e_0}(V, \mu) = \mu(V'X) + \sum_{V \geq e_2} \delta_{e_0}(V, \mu) + \sum_{V \geq e_3} \delta_{e_0}(V, \mu).
\]
This relation is precisely what is needed to insure that the break of $\psi$ at $V$ is stretchable, as in the illustration.

The preceding theorem produces a measure $\tilde{m}'$ which is again a rigid tree measure. Indeed, $\tilde{m}'$ has the same nonzero densities as $m'$, and therefore $\Sigma_{\tilde{m}'}(\tilde{m}') =$
Assume further that we are given flags defined by (2.2). The index attachment index for \( I_m \), \( \Sigma_m(m') = -1 \). In fact, it is easy to see that \( \hat{m}' \) is homologous to \( m' \) in the sense defined in [1] and discussed in the following section. Indeed, using the notation in the proof above, this follows because two edges \( e, e' \) of \( T \) such that \( \varphi(e) = \varphi(e') \) will satisfy

\[
\sum_{V \geq e} \delta_{m_0}(V, m) = \sum_{V \geq e'} \delta_{m_0}(V, m),
\]

and therefore their translates \( \psi(e) \) and \( \psi(e') \) will coincide as well.

6. REDUCTION OF THE INTERSECTION PROBLEM

We are now ready to discuss the reduction procedures mentioned in the introduction. We recall first some facts from [1]. Fix a measure \( m \in \mathcal{M}_r \) with integer densities. A point \( A_\ell \) (resp. \( B_\ell, C_\ell \)) is called an attachment point of \( m \) if \( \ell \geq 1 \) and \( m(A_\ell X_\ell) > 0 \) (resp. \( m(B_\ell Y_\ell) > 0, m(C_\ell Z_\ell) > 0 \)). We denote by \( \text{att}_I(m) \) (resp. \( \text{att}_J(m), \text{att}_K(m) \)) the collection of indices \( \ell \in \{1, 2, \ldots, r\} \) such that \( A_\ell \) (resp. \( B_\ell, C_\ell \)) is an attachment point for \( m \).

Let now \( I_m, J_m, K_m \subset \{1, 2, \ldots, n = r + \omega(m)\} \) be the sets of cardinality \( r \) defined by (2.2). The index \( \nu \in I_m \) (resp. \( \nu \in J_m, k \in K_m \)) is called an attachment index for \( m \) if \( A_\ell \) (resp. \( B_\ell, C_\ell \)) is an attachment point. We denote by \( \text{att}_m, \text{Jatt}_m, \text{Katt}_m \) the collections of attachment indices; thus \( I^\text{att}_m = \{ \nu \in \text{att}_I(m) \} \).

Assume further that we are given flags \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) in \( \mathbb{C}^n \). The spaces \( \{\mathcal{E}_{\nu} : \ell \in I^\text{att}_m\} \), \( \{\mathcal{F}_{\nu} : \ell \in J^\text{att}_m\} \), \( \{\mathcal{G}_{\nu} : \ell \in K^\text{att}_m\} \) are called the attachment spaces of \( m \).

Let now \( \tilde{m} \in \mathcal{M}_r \) be a second measure with integer densities. The measures \( m \) and \( \tilde{m} \) are said to be homologous if there is a bijection between the white piece edges determined by the support of \( m \) and those determined by the support of \( \tilde{m} \) such that corresponding edges are parallel, and incident edges correspond to incident edges (the intersection point being the one dictated by the correspondence of the edges). If \( m \) and \( \tilde{m} \) are homologous, there clearly exist order preserving bijections \( \varphi_I : I^\text{att}_m \to I^\text{att}_{\tilde{m}}, \varphi_J : J^\text{att}_m \to J^\text{att}_{\tilde{m}}, \varphi_K : K^\text{att}_m \to K^\text{att}_{\tilde{m}} \).

Also recall that a lattice polynomial of a collection \( \mathcal{X} = (\mathcal{X}_\nu)_{\nu \in N} \) of spaces is defined inductively by the requirements that

1. for each \( \nu \), the expression \( P_m(\mathcal{X}) = \mathcal{X}_\nu \) is a lattice polynomial, and
2. if \( P(\mathcal{X}) \) and \( Q(\mathcal{X}) \) are lattice polynomials, then \( (P(\mathcal{X})) + (Q(\mathcal{X})) \) and \( (P(\mathcal{X})) \cap (Q(\mathcal{X})) \) are also lattice polynomials.

More formally, lattice polynomials should be defined as elements of an abstract lattice generated by a set of variables indexed by \( N \). One can then substitute subspaces for the variables to obtain a new subspace. This gives the proper meaning to the last statement in the next theorem.

The following result is a reformulation of results in [1]. The fact that the lattice polynomial is essentially the same for all homologous measures is not explicitly stated there, but it is easily verified using the argument of [1] Proposition 5.1.

**Theorem 6.1.** Assume that \( m \in \mathcal{M}_r \) is a rigid measure with integer densities, and \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) are flags in \( \mathbb{C}^n \), \( n = r + \omega(m) \). There exists a lattice polynomial \( P_m \) of the attachment spaces of \( m \) such that generically

\[
P_m(\mathcal{E}_i, \mathcal{F}_j, \mathcal{G}_k : i \in I^\text{att}_m, j \in J^\text{att}_m, k \in K^\text{att}_m) \in \mathcal{G}(\mathcal{E}, I_m) \cap \mathcal{G}(\mathcal{F}, J_m) \cap \mathcal{G}(\mathcal{G}, K_m).
\]

Moreover, if \( \tilde{m} \in \mathcal{M}_r \) is homologous to \( m \) and \( \tilde{E}, \tilde{F}, \tilde{G} \) are flags in \( \mathbb{C}^n \),

\[
P_{\tilde{m}}(\mathcal{E}_{\varphi_I(i)}, \mathcal{F}_{\varphi_J(j)}, \mathcal{G}_{\varphi_K(k)} : i \in I^\text{att}_m, j \in J^\text{att}_m, k \in K^\text{att}_m)
\]
equals

\[ P_m(\overline{E}_{\varphi(i)}, \overline{E}_{\varphi(j)}, \overline{G}_{\varphi(k)} : i \in I_m^{\text{att}}, j \in J_m^{\text{att}}, k \in K_m^{\text{att}}). \]

Given a measure (rigid or not) \( m \in \mathcal{M}_r \) and an extremal, rigid tree measure \( m' \in \mathcal{M}_r \), we will be able to apply a reduction of the Schubert problem associated to \( m \) provided that \( \Sigma_{m'}(m) = -p < 0 \). More precisely, the Schubert problem will be reduced to the corresponding problem for the measure \( m - pm' \) (which satisfies \( c_{m - pm'} = c_m \)) by Corollary [4.3] in a space \( \mathcal{X} \) of dimension \( n - p\omega(m') \). The space \( \mathcal{X} \) is obtained by applying the lattice polynomial \( P_m \) to the attachment spaces of \( m \) corresponding to the attachment points of \( m' \). The following result describes the procedure in detail. The argument is essentially contained in [1 Proposition 5.1], but we include it here for completeness, and as a practical recipe. Observe that \( \Sigma_{m'}(m) + \omega'n \) is precisely the sum (1.1) mentioned in our initial discussion of reductions.

**Theorem 6.2.** Let \( m, m' \in \mathcal{M}_r \) be two measures with integer densities such that \( m' \) is a rigid tree measure, and \( \Sigma_{m'}(m) = -p < 0 \). Denote by \( i_\ell, j_\ell, k_\ell, \ell = 1, 2, \ldots, r \), the elements of \( I = I_m, J = J_m, K = K_m \), respectively. Given generic flags \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) in \( \mathbb{C}^n \), \( n = r + \omega(m) \), the space

\[ \mathcal{X} = P_m(E_{i_\ell}, F_{j_\ell}, G_{k_\ell} : x \in \text{att}_I(m'), y \in \text{att}_J(m'), z \in \text{att}_K(m')) \]

has dimension \( n - p\omega(m') \). Moreover, denote by \( \mathcal{E}' \) the flag in \( \mathcal{X} \) obtained by intersecting the spaces in \( \mathcal{E} \) with \( \mathcal{X} \) and discarding repeating spaces, with similar definitions for \( \mathcal{F}' \) and \( \mathcal{G}' \). Then we have

\[ \mathcal{G}(\mathcal{E}', I_{m - pm'}) \cap \mathcal{G}(\mathcal{F}', J_{m - pm'}) \cap \mathcal{G}(\mathcal{G}', K_{m - pm'}) \subseteq \mathcal{G}(\mathcal{E}, I_m) \cap \mathcal{G}(\mathcal{F}, J_m) \cap \mathcal{G}(\mathcal{G}, K_m). \]

**Proof.** Denote the exit densities of \( m \) by \( a_\ell, b_\ell, c_\ell, \ell = 0, 1, \ldots, r \). Thus the elements of \( I_m, J_m, K_m \) are given by

\[ i_\ell = \ell + \sum_{\ell' < \ell} a_{\ell'}, \quad j_\ell = \ell + \sum_{\ell' < \ell} b_{\ell'}, \quad k_\ell = \ell + \sum_{\ell' < \ell} c_{\ell'} \]

for \( \ell = 1, 2, \ldots, r \). By Corollary [4.3] we can write \( m = pm' + \mu \), where \( \mu \in \mathcal{M}_r \), and \( \Sigma_{m'}(\mu) = 0 \). Denote the exit densities of \( \mu \) and \( m' \) by \( \alpha_\ell, \beta_\ell, \gamma_\ell \) and \( \alpha'_\ell, \beta'_\ell, \gamma'_\ell \), respectively. We have

\[ a_\ell = \alpha_\ell + pa'_\ell, \quad b_\ell = \beta_\ell + p\beta'_\ell, \quad c_\ell = \gamma_\ell + p\gamma'_\ell \]

for \( \ell = 0, 1, \ldots, r \). Theorem [5.2] yields a rigid tree measure \( \tilde{m}' \in \mathcal{M}_{r + \omega(\mu)} \), homologous to \( m' \), whose only possible nonzero exit densities are \( \tilde{\alpha}'_\ell = \alpha'_\ell \) for

\[ i = \ell + \sum_{k < \ell} \alpha_k = \ell + \sum_{k < \ell} (a_k - pa'_k), \quad \ell = 0, 1, 2, \ldots, r, \]

with analogous formulas for \( \tilde{\beta}'_\ell \) and \( \tilde{\gamma}'_\ell \). The set \( I_{pm'} \) of cardinality \( r + \omega(\mu) \) has elements

\[ \tilde{i}_x = x + \sum_{y < x} p\tilde{\alpha}'_y, \quad x = 1, 2, \ldots, r + \omega(\mu). \]

In particular

\[ (6.1) \quad \tilde{i}_x = i_\ell \text{ when } x = \ell + \sum_{k < \ell} \alpha_k. \]
with similar formulas for \( j_x \) and \( k_x \). We deduce that the attachment spaces of \( \rho \tilde{n}' \) are precisely
\[
\{ E_{i_x}, F_{j_y}, G_{k_z} : x \in \text{att}_x(m'), y \in \text{att}_y(m')z \in \text{att}_z(m') \}.
\]
Now, the measure \( \rho \tilde{n}' \) is homologous to \( m' \), and therefore Theorem 6.1 implies that the space \( X \) in our statement belongs generically to the intersection
\[
\mathcal{S}(E', I_{\rho \tilde{n}'}) \cap \mathcal{S}(F', J_{\rho \tilde{n}'}) \cap \mathcal{S}(G, K_{\rho \tilde{n}'})
\]
Relation (6.1) implies that
\[
\dim(X \cap E_{i_1}) \geq \ell + \sum_{k<\ell} \alpha_k, \quad \ell = 1, 2, \ldots, r,
\]
with similar estimates for \( \dim(X \cap F_{j}) \) and \( \dim(X \cap G_{k}) \). Thus, by intersecting the spaces in the flags \( E, F, G \) with \( X \) we obtain (after eliminating repeating spaces) flags \( E', F', G' \) in \( X \) with the property that
(6.2)
\[
E'_{i_x} \subset X \cap E_{i_x} \, \text{for} \, x = \ell + \sum_{k<\ell} \alpha_k,
\]
and similarly for \( F' \) and \( G' \).

Note now that \( \omega(m - pm') = n - p\omega(m') = \dim(X) \), and therefore it makes sense to solve the Schubert problem associated with this measure and the flags \( E', F', G' \). To conclude the proof, let \( M \) be a space in the intersection
\[
\mathcal{S}(E', I_{m - pm'}) \cap \mathcal{S}(F', J_{m - pm'}) \cap \mathcal{S}(G', K_{m - pm'})
\]
To see that \( M \) belongs to
\[
\mathcal{S}(E, I_{m}) \cap \mathcal{S}(F, J_{m}) \cap \mathcal{S}(G, K_{m})
\]
observe that the \( \ell \)th element of \( I_{m - pm'} \) is equal to \( i = \ell + \sum_{k<\ell} \alpha_k \), so that
\[
\dim(M \cap E_{i_1}) = \dim(M \cap (X \cap E_{i_1})) \geq \dim(M \cap E'_i) \geq \ell,
\]
where we used (6.2) in the first inequality. \( \square \)

When the measure \( m \) is itself rigid, it was shown in [1] that it is possible to choose \( m' \) so that \( \Sigma_{m'}(m) = -p < 0 \), and \( m - pm' \) has strictly smaller support than \( m \). Therefore repeated applications of these reduction procedures eventually yield an explicit solution of the intersection problem.

As an illustration, we will see how to deduce the two kinds of reductions mentioned in the introduction. First, consider a measure \( m' \) with \( \omega(m') = 1 \). There are only three nonzero exit densities \( \alpha'_x = \beta'_y = \gamma'_z = 1 \), and we must have \( x + y + z = r \); see the first triangle in the figure below. The sum
\[
\Sigma_{m'}(\mu) = i_x + j_y + k_z - n
\]
corresponds to the original reductions in [6], and a reduction can be applied when this sum is negative. The relevant lattice polynomial is
\[
P_{m'}(E, F, G) = E + F + G,
\]
thus yielding the formula mentioned in the introduction. The second reduction outlined in the introduction corresponds with a measure \( m' \) satisfying \( \omega(m') = 2 \) whose support is shown in the second triangle below.
There are now six exit densities equal to 1, but three of them are $\alpha_0', \beta_0', \gamma_0'$, which do not correspond to attachment points. The others are $\alpha_x', \beta_y', \gamma_z'$, where the numbers $x, y, z$ are the lengths of the dotted segments in the boundary of $\triangle_\tau$. Clearly $x + y + z = 2r$, and

$$\Sigma_m' (\mu) = i_x + j_y + k_z - 2n.$$ 

This time the lattice polynomial is

$$P_m' (E, F, G) = (E \cap F) + (F \cap G) + (G \cap E).$$

We conclude this section with an analysis of the measures in $\mathcal{M}_3$ which do not allow any of the reductions outlined above. There are 11 extremal measures in $\mathcal{M}_3$, and all of them are rigid tree measures. Their supports are depicted below.

Let us call these tree measures $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3, \rho_1, \rho_2, \rho_3, \tau_1$, and $\tau_2$. In addition to the equalities $\Sigma_\mu (\mu) = -1$, the only other nonzero values for $\Sigma_\nu (\nu)$ with $\mu, \nu$ among these measures are equal to one. These are: $\Sigma_\nu_j (\nu_j)$, $\Sigma_\rho_j (\rho_j)$, $\Sigma_\tau_1 (\tau_j)$ for $j = 1, 2, 3$, and the three cycles $\Sigma_\nu (\nu_{j+1})$, $\Sigma_\rho (\rho_{j-1})$, and $\Sigma_\tau (\tau_{j+1})$. An arbitrary measure $m \in \mathcal{M}_3$ with integer densities can be written as

$$m = \sum_{j=1}^3 (a_j \mu_j + b_j \nu_j + c_j \rho_j) + d_1 \tau_1 + d_2 \tau_2,$$

where the coefficients $a_j, b_j, c_j, d_j$ are nonnegative integers. Note that $\Sigma_\mu_j (m) = -a_j$, $\Sigma_\nu_j (m) = a_j + b_{j+1} - b_j$, $\Sigma_\rho_j (m) = a_j + c_{j-1} - c_j$, $\Sigma_\tau_2 (m) = d_1 - d_2$, and $\Sigma_\tau_1 (m) = a_1 + a_2 + a_3 + d_2 - d_1$. A reduction is possible unless $a_1 = a_2 = a_3 = 0$, $b_1 = b_2 = b_3$, $c_1 = c_2 = c_3$, and $d_1 = d_2$. Observe also that $\tau_1 + \tau_2 = \nu_1 + \nu_2 + \nu_3$, and this measure has the same exit densities as $\rho_1 + \rho_2 + \rho_3$. Thus the only intersection problems which cannot be reduced with our methods arise from measures of the form $d(\tau_1 + \tau_2)$ for some integer $d > 0$. The measure $m$ is rigid if and only if $b_1 b_2 b_3 = c_1 c_2 c_3 = d_1 d_2 = 0$. The first ten of the rigid tree measures above correspond with the reductions considered in [6] and [3]. It should be noted that this analysis can be applied, via the duality described in [1], to the analysis of measures $m$ with $\omega (m) = 3$. The intersection problems for such measures can be reduced to duals of measures of the form $d(\tau_1 + \tau_2)$. 
A similar analysis can be carried out for $r = 4$ and $r = 5$, but with many more tree measures. Indeed, for $r \leq 5$ all extremal measures in $M_r$ are rigid. For $r = 6$ there are already some extremal measures which are not tree measures, though their exit densities coincide with those of a sum of extremal rigid measures. An example is provided below, where all solid edges have unit density. The two resulting measures have the same exit densities, but only the first one is extremal; the second one is the sum of three extremal measures.

For larger values of $r$, there exist tree extremal measures which are not rigid, and do not have the same exit densities as any sum of extremal rigid measures. The support of such a tree measure $m$ is pictured below.

Here $r = 13$, and the exit points are $A_0, A_4, A_{10}, B_0, B_4, B_7, C_0, C_4,$ and $C_{10}$. It is easy to verify that one cannot find among these points $A_x, B_y, C_z$ such that $x + y + z = 13$, and therefore the exit densities do not majorize the exit densities of any measure $\mu$ with $\omega(\mu) = 1$. Since $\omega(m) = 3$, it follows that the exit densities of $m$ do not majorize those of any rigid tree measure.

7. An Arboretum of Rigid Tree Measures

The reduction procedure described in the preceding section requires knowledge of the rigid tree measures in $M_r$, and one might hope that a complete description of these is available. We are not aware of the existence of such a description, but we will use Theorem 4.2 to study those rigid measures which have three nonzero exit densities on each side of $\Delta_r$. Assume thus that the rigid tree measure $m \in M_r$ has weight $\omega$, and nonzero densities $\alpha, \alpha', \alpha''$ in the NW direction, $\beta, \beta', \beta''$ in the SW direction, and $\gamma, \gamma', \gamma''$ in the E direction. These integers must satisfy

\begin{equation}
\alpha + \alpha' + \alpha'' = \beta + \beta' + \beta'' = \gamma + \gamma' + \gamma'' = \omega, \tag{7.1}
\end{equation}

and

\begin{equation}
\alpha^2 + \alpha'^2 + \alpha''^2 + \beta^2 + \beta'^2 + \beta''^2 + \gamma^2 + \gamma'^2 + \gamma''^2 = \omega^2 + 2 \tag{7.2}
\end{equation}
by Theorem 4.2(3). Assume first that $\omega = 3k + 1$ for some integer $k \geq 1$. The smallest value allowed by (7.1) for the sum

$$\alpha^2 + \alpha'^2 + \alpha''^2 + \beta^2 + \beta'^2 + \beta''^2 + \gamma^2 + \gamma'^2 + \gamma''^2$$

is achieved when the weights on each side are $k, k$ and $k + 1$, and that value is precisely $\omega^2 + 2$. Thus (7.2) implies that the weights on each side have precisely these values (in some order). Similarly, when $\omega = 3k + 2$, the densities on each side must be $k, k + 1$ and $k + 1$. When $\omega = 3(k + 1)$, relation (7.2) implies

$$\alpha^2 + \alpha'^2 + \alpha''^2 + \beta^2 + \beta'^2 + \beta''^2 + \gamma^2 + \gamma'^2 + \gamma''^2 \geq 9(k + 1)^2 = \omega^2,$$

with equality achieved only when all the exit densities are equal to $k + 1$. It follows easily from (7.2) that on two sides the exit densities will all be equal to $k + 1$, while on the remaining side they must be $k, k + 1, k + 2$. We will now produce actual examples of rigid tree measures with three nonzero exit densities in each direction, and with all possible values of $\omega$. A first series of examples is described in the following figure.

The thinner edges have density one, and the thicker ones have density two, except for one exit density which is equal to three, as labeled. Other such measures can be obtained by applying $120^\circ$ rotations to these measures, or symmetries about a horizontal line. Another way to obtain new measures is to change the lengths of the edges indicated by a dot. These lengths can be chosen arbitrarily; here is an example of this procedure applied to the second measure above.

The three measures above provide examples with $\omega = 3k + 1, 3k + 2$ and $3(k + 1)$ when $k = 1$. For larger values of $k$ one must continue the spiral pattern. A second series of examples is illustrated below.
As in the first series of examples, these measures can be rotated by multiples of $120^\circ$, and reflected in a horizontal line. Their shapes can also be changed by modifying arbitrarily the lengths of six of the edges. Again, the spiral can be continued to yield examples with weights $3k + 1, 3k + 2$ and $3(k + 1)$ for all integers $k \geq 1$.

A third series of examples is illustrated next.

Note that this series has two spiral arms. To obtain measures with higher weight one proceeds by alternately increasing each spiral by $1/3$ of a complete turn.

When $\omega(m) = 3k + 1$ there is one more series of measures which have greater symmetry. The first two in the series are pictured below.

These measures are invariant under $120^\circ$ rotations, but not under reflection relative to a horizontal line.

A similar series is available for $\omega = 3k + 2$. 
Some of these examples have versions for \( k = 0 \), though in that case there will be fewer than three nonzero exit densities in some direction. Using duality of measures, it can be shown that the measures described above (along with their rotations, reflections and stretched versions) are the only measures with exactly three nonzero exit densities in each direction. Thus, for instance, there are no rigid tree measures whose exit densities are (in counterclockwise order, starting with \( \alpha \)) \((k+1,k+1,k),(k+1,k,k+1)\) and \((k,k+1,k+1)\) or \((k+1,k,k),(k,k,k+1)\) and \((k,k+1,k)\).

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