THE PRINCIPAL INDECOMPOSABLE MODULES
OF THE DILUTE TEMPERLEY-LIEB ALGEBRA

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ABSTRACT. The Temperley-Lieb algebra $\text{TL}_n(\beta)$ can be defined as the set of rectangular diagrams with $n$ points on each of their vertical sides, with all points joined pairwise by non-intersecting strings. The multiplication is then the concatenation of diagrams. The dilute Temperley-Lieb algebra $\text{dTL}_n(\beta)$ has a similar diagrammatic definition where, now, points on the sides may remain free of strings. Like $\text{TL}_n$, the dilute $\text{dTL}_n$ depends on a parameter $\beta \in \mathbb{C}$, often given as $\beta = q + q^{-1}$ for some $q \in \mathbb{C}^\times$. In statistical physics, the algebra plays a central role in the study of dilute loop models. The paper is devoted to the construction of its principal indecomposable modules.

Basic definitions and properties are first given: the dimension of $\text{dTL}_n$ and its break up into two even and odd subalgebras. The standard modules $S_{n,k}$ are then introduced and their behaviour under restriction and induction is described. A bilinear form, the Gram product, is used to identify their (unique) maximal submodule $R_{n,k}$ which is then shown to be irreducible or trivial. The structure of $\text{dTL}_n$ as a left module over itself is given for all values of the parameter $q$, that is, for both $q$ generic and a root of unity. Complete sets of irreducible and principal indecomposable modules are constructed explicitly.

Keywords dilute Temperley-Lieb algebra · Temperley-Lieb algebra · principal indecomposable modules · dilute loop models · Nienhuis weights · O(N) models

1. INTRODUCTION

Since its introduction in the 1970s [1], the Temperley-Lieb algebra has played a central role in several domains of mathematical physics, mainly in the statistical physics description of lattice models and in conformal field theory. But, since its “rediscovery” by mathematicians — Jones’ seminal paper [2] comes here to mind, — algebraists have contributed significantly to its understanding. Its representation theory was first described independently by Goodman and Wenzl [3] and by Martin [4] and is now widely used.

Several generalizations have been introduced, many suggested by physical problems: the periodic (affine) Temperley-Lieb algebra [5, 6, 7, 8, 9], polychromatic algebras [12], the Birman-Wenzl-Murakami algebra [10, 11] and the dilute Temperley-Lieb algebra [13]. Their role in mathematical physics has developed over the years, particularly since their intimate relationship with infinite-dimensional Lie algebras appearing in the description of continuum limits of lattice models have been recognized. The fact that some hamiltonians or transfer matrices could be seen as representatives, within given modules, of an abstract element of the Temperley-Lieb algebra was already in Temperley and Lieb’s work. But the following fact is Pasquier and Saleur’s crucial observation [14]: the representation theory of the Temperley-Lieb algebra can be used to understand the Virasoro representations appearing in the limit, when the mesh goes to zero, of the finite-size lattice models.

The origin of the dilute Temperley-Lieb algebra $\text{dTL}_n$ can be tied to Nienhuis’ work [15]. It was known since early works by Yang and Baxter that some algebraic conditions

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on Boltzmann weights of statistical lattice models assure some form of integrability. Trying to find integrable $O(N)$ models, Nienhuis introduced a family of such weights satisfying these conditions. He noticed soon after that these weights, labeled by two parameters $\lambda$ and $u$, were part of a larger family defined by Izergin and Korepin [16]. With Blöte he explored the large $n$ limit through numerical simulations [17]. (Note that we use small $n \geq 1$ for the size of the lattice. This integer $n$ is independent of the $O(N)$ model.) Under the hypothesis that such lattice models would go to conformal field theories in the limit $n \to \infty$, they found a simple relation between the parameter $\lambda$ and the central charge of these continuum theories. Nienhuis’ weights are attached to the tiles forming the lattice. The various states of the tiles of these models are described by non-intersecting links joining their edges pairwise, exactly as in the Temperley-Lieb description of (fully-packed) loop models. But contrarily to the Temperley-Lieb case, some of the edges of the tiles may be left free of links in dilute models. Generalisations of these dilute models [18, 19, 20] and sets of integrable boundary conditions [21, 22] to match the (bulk) Boltzmann weights were found in the years that followed.

Even though the representation theory of the (original) Temperley-Lieb algebra [3, 4] and that of the periodic version [5, 9] are well-established, that of the dilute Temperley-Lieb lags behind. A few years ago the dichromatic Temperley-Lieb algebra has been studied [23] and one might be able to retrieve, at least partially, some properties of the dilute $dTL_n$ from some quotient of the dichromatic one. But the dilute Temperley-Lieb algebra $dTL_n(\beta), \beta \in \mathbb{C}$, has now become such an important tool in mathematical physics that a direct and systematic description of its properties is necessary. The structure uncovered and tools developed should be powerful enough to study questions like, for example, the computation of the fusion ring of its standard and projective modules, the possible existence of a Schur-Weyl duality with some other (quantum) algebra, or the identification of modules in which transfer matrices have non-trivial Jordan structure. The present paper is a first step toward this goal. It gives an explicit construction of all its principal indecomposable modules, for both cases when the algebra $dTL_n(\beta)$ is semisimple and non-semisimple.

Several approaches surrounding the families of Temperley-Lieb algebras are based on diagrammatic techniques. Several rigorous mathematical works resort to them and they are used to define many lattice models. So it is not surprising that the early construction of the principal indecomposable modules of the (original) Temperley-Lieb algebra $TL_n$ by Martin has been reformulated through methods based on link diagrams [24, 25]. It is this approach that we choose to follow here. Both the elements and the multiplication of the dilute Temperley-Lieb algebra $dTL_n(\beta)$ are defined through diagrams in section 2. (Another parameter, $q \in \mathbb{C}^\times$, is also used. It is related to the first by $\beta = q + q^{-1}$.) These definitions lead to the identification of a natural subalgebra $S_n \subset dTL_n$ and several copies of the usual Temperley-Lieb algebras $TL_{n'}$, $n' \leq n$, the computation of its dimension and its decomposition into even and odd parts. Section 3 is devoted to the construction of standard modules. Their basic characteristics are there established: they are cyclic and indecomposable and their dimensions are expressed in terms of those of the standard modules of $TL_n$. Restriction and induction are used to probe their inner structure. Section 4 introduces another classical tool of representation theory. A bilinear form, called the Gram product, is defined on the standard modules. The radical of this form, that is the subspace of vectors with vanishing Gram coupling with all others, is shown to be the unique maximal submodule of the standard module. The determinant of the Gram matrix representing the bilinear form in some basis is easily computed. Its zeroes occur when the parameter $q$ is a root of unity and, consequently, the algebra $dTL_n$ is semisimple when $q$ is generic, that
is when it is not a root of unity. Finally the radical, when it is non-trivial, is shown to be irreducible and isomorphic to the irreducible quotient of another standard module. The last step, the construction of the principal indecomposable modules when the algebra is non-semisimple, is achieved in section 5. Each of these modules is given a fairly explicit description in terms of some of the standard modules that are recursively induced from one algebra \( dTL_n \) to the next one \( dTL_{n+1} \). The conclusion reviews the main results and discusses possible extensions. Some results of this paper are based on the analogous ones for the algebra \( TL_n \). These are reviewed in appendix A. Appendices B and C contain more technical computations and proofs. Finally appendix D reviews the algebraic tools that are used throughout the paper, but particularly in section 5.

2. BASIC PROPERTIES OF THE DILUTE ALGEBRA \( dTL_n \)

This section introduces the dilute Temperley-Lieb algebra whose elements and product are defined diagrammatically. It is shown to split naturally into a direct sum of two ideals, its even and odd parts. Another subalgebra \( S_n \subset dTL_n \) will play a role in the subsequent section and it is also defined. The section ends with the computation of the dimension of \( dTL_n \). Several techniques used here are borrowed from previous studies of the (original) Temperley-Lieb algebra. Appendix A gathers some basic results for this algebra. Reading this appendix in parallel will ease the understanding of this section and of the next one.

2.1. Definition of \( dTL_n(\beta) \). The basic objects, \( n \)-diagrams, are first introduced. Draw two vertical lines, each with \( n \) points on it, \( n \) being a positive integer. Choose first \( 2m \) points, \( 0 \leq m \leq n \) an integer, and put a \( \circ \) on each of them. A point with a \( \circ \) will be called a vacancy. Now connect the remaining points, pairwise, with non-intersecting strings. The resulting object is called a *dilute n-diagram*.

On the set of formal linear combinations of all dilute \( n \)-diagrams a product is defined by extending linearly the product of two \( n \)-diagrams obtained as follows. The two diagrams are put side by side, the inner borders and the points on them are identified, then removed. A string which no longer ties two points is called a *floating string*. A floating string that closes on itself is called a *closed loop*. If all floating strings are closed loops, the result of the product of the two dilute \( n \)-diagrams is then the diagram obtained by reading the vacancies on the left and right vertical lines and the strings between them multiplied by a factor of \( \beta \) for each closed loop. Otherwise, the product is the zero element of the algebra.

The three following products give examples of these definitions. The second contains two floating strings that are not closed and the product is therefore zero, and the third has one closed floating string leading to the factor \( \beta \):

\[
\begin{align*}
\text{prod} & = \text{prod} \\
\text{prod} & = \text{prod} \\
\text{prod} & = 0
\end{align*}
\]
A dashed string represents the formal sum of two diagrams: one where the points are linked by a regular string, and one where the points are both vacancies. For example,

\[ \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array} \]

Note that the diagram where each point is linked by a dashed line to the corresponding point on the opposite side acts as the identity on all dilute \( n \)-diagrams. It is a sum of \( 2^n \) diagrams. For example, when \( n = 3 \)

\[ \text{id}_3 = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array} = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \]

Note finally that the product is clearly associative: the reading of how the left and right sides are connected in a product of three diagrams is blind to the order of gluing, and so is the number of closed loops. The set of \( n \)-diagrams with the formal sum and product just introduced is the dilute Temperley-Lieb algebra \( \text{dTL}_n = \text{dTL}_n(\beta) \). We also define \( \text{dTL}_0 = \mathbb{C} \). When the parameter \( \beta \) is chosen to be a formal one, then the algebra is over \( \mathbb{C}[\beta] \). We shall be interested mostly in the case \( \beta \in \mathbb{C} \) for which the algebra is over \( \mathbb{C} \).

Several generating sets for \( \text{dTL}_n \) can be found. For instance, the set \{ \( a_i, a'_i, b_i, b'_i, e_j, x_j, i \in [1, n-1], j \in [1, n] \) \} where

\[ e_i = \begin{array}{c}
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\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \quad x_i = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \quad a_i = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \quad d'_i = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \quad b'_i = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \quad b_i = \begin{array}{c}
\vspace{-0.5cm}
\begin{array}{c}
\vspace{-0.5cm}
\end{array}
\end{array}, \]

generates the algebra. However, they do not form a minimal set, as for all \( 1 \leq i \leq n \), \( e_i + x_i = \text{id}_n \). Making the identification \( u_i = b'_ib_i \), the connection with the regular \( n \)-diagram algebra \( \text{TL}_n \) should be clear. A set of relations was proposed by Grimm [13] to define \( \text{dTL}_n \) through generators and relations. The equivalence between the diagrammatic definition, the one used here, and that with relations is stated there without proof.
The numbers of vacancies on either side of a dilute $n$-diagram always have the same parity. If these numbers are even (odd), the diagram will be called an even (odd) diagram. The subset spanned by only even (odd) diagrams is closed under the product and this subalgebra will be called the even (odd) dilute Temperley-Lieb subalgebra, noted by $\text{edTL}_n$ ($\text{odTL}_n$). Clearly any dilute $n$-diagram is either even or odd. Since the product of two diagrams of distinct parities is zero, it is clear that the even and odd subalgebras are two-sided ideals of $\text{dTL}_n$ and

$$\text{dTL}_n = \text{edTL}_n \oplus \text{odTL}_n.$$  

For example

$$\text{dTL}_2 \simeq \text{span}\{ \begin{array}{c}
\begin{array}{c}
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\vline \\
\vline
\end{array} \\
\begin{array}{c}
\vline \\
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\end{array}
\end{array} , \begin{array}{c}
\begin{array}{c}
\vline \\
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\vline
\end{array} \\
\begin{array}{c}
\vline \\
\vline \\
\vline
\end{array} \\
\begin{array}{c}
\vline \\
\vline \\
\vline
\end{array} \\
\begin{array}{c}
\vline \\
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\end{array}
\end{array} \} \oplus \text{span}\{ \begin{array}{c}
\begin{array}{c}
\vline \\
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\vline \\
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\begin{array}{c}
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\begin{array}{c}
\vline \\
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\begin{array}{c}
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\begin{array}{c}
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\vline \\
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\begin{array}{c}
\vline \\
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\vline
\end{array}
\end{array} \} .$$  

(1)

The unit $\text{id} \in \text{dTL}_n$ decomposes into $\text{id} = \text{eid} + \text{oid}$ with $\text{eid} \in \text{edTL}_n$ and $\text{oid} \in \text{odTL}_n$. The odd and even units are orthogonal idempotents: $\text{eid}^2 = \text{eid}$, $\text{oid}^2 = \text{oid}$ and $\text{oid} \cdot \text{eid} = \text{oid} \cdot \text{oid} = 0$. (On the previous example of $\text{id}_3$, the four 3-diagrams of the first line of the rhs form $\text{eid}$ and the last line is $\text{oid}$.) Let $M$ be a $\text{dTL}_n$-module and decompose it, as vector space, into $M = \text{eid} \cdot M \oplus \text{oid} \cdot M$. Clearly $\text{eid} \cdot (\text{oid} \cdot M) = 0$ and $\text{oid} \cdot (\text{eid} \cdot M) = 0$. But $a = a \cdot \text{eid}$ for any $a \in \text{edTL}_n$ and therefore $\text{edTL}_n$ acts trivially on $\text{oid} \cdot M$ and, similarly, so does $\text{odTL}_n$ on $\text{eid} \cdot M$. The decomposition into a direct sum of subspaces is thus a direct sum of modules. The two summands $\text{oid} \cdot M$ and $\text{eid} \cdot M$ will be called the odd and even submodules of $M$. If the odd submodule of $M$ is trivial, $M$ will be said to be even and vice versa. An indecomposable module $M$ is either odd or even.

Consider now $S_n$, the subset of $\text{dTL}_n$ spanned by dilute $n$-diagrams having symmetric vacancies, that is, a position on one of their sides is a vacancy if and only if it is also on their other side. Multiplying two symmetric $n$-diagrams gives either zero if the vacancies do not match perfectly or is a symmetric diagram. The subset $S_n$ is therefore a subalgebra of $\text{dTL}_n$. Now, choose a subset $A \subset \{1,2,\ldots,n\}$ of $t$ integers and define $\pi_A = \prod_{i \in A, j \notin A} x_j e_i$. Note that $\pi_A^2 = \pi_A$ and thus $\pi_A(\text{dTL}_n)\pi_A$ is a subalgebra of $S_n$. It is spanned by all $n$-diagrams with links starting and ending at positions labeled by $A$ and vacancies at all other positions. Therefore $\pi_A(\text{dTL}_n)\pi_A$ is isomorphic to $\text{TL}_t$ and any $n$-diagram in $S_n$ belongs to precisely one of these subalgebras. For a given $t$, there are $\binom{n}{t}$ distinct such subalgebras in $S_n$, all isomorphic to $\text{TL}_t$. Finally, since the product of two diagrams with different vacancies is always zero, it follows that $S_n$ is isomorphic to the direct sum of all subalgebras obtained from subsets of $\{1,2,\ldots,n\}$. We arrive at the following proposition.

**Proposition 2.1.** The subalgebra $S_n \subset \text{dTL}_n$ is isomorphic to

$$S_n \simeq \bigoplus_{0 \leq t \leq n} \left( \bigoplus_{1 \leq p \leq \binom{n}{t}} \text{TL}_t \right)$$  

(2)

where $\text{TL}_0 = \mathbb{C}$.

2.2. **The dimension of $\text{dTL}_n$.** The ressemblance with the Temperley-Lieb algebra $\text{TL}_n$ provides a fairly straightforward method to obtain the dimension of $\text{dTL}_n$. In fact, the same technique of “slicing diagrams” can be used here. The procedure goes as follows: first, take a dilute $n$-diagram and rotate its right side so that it sits below its left side,
stretched the strings so that the points remain connected. Second, connect the two-sides together. For example,

Now, consider the dilute \( n \)-diagrams whose vacancies are all at the same places, apply the procedure, then remove the points where the vacancies are. For \( n = 3 \) and two vacancies located as below, the result looks like this:

One should recognize in the results two elements of the link basis of the standard module \( V_{4,0} \) of \( TL_4 \) or, in general, of the \( TL_{2m} \)-module \( V_{2m,0} \) with no defects. (See section [3] for a formal definition of links and standard modules for \( dTL_n \) and also appendix A for their \( TL_n \) analogues.) By the reverse of the procedure just described, it was shown in [25] that \( \dim TL_n = \dim V_{2n,0} \). This leads to the following expression for the dimension of \( dTL_n \).

Proposition 2.2. The dimension of the associative algebra \( dTL_n \) is

where \( TL_0 = \mathbb{C} \).

Proof. Choose \( 2m \leq 2n \) positions and form the subset of dilute \( n \)-diagrams that have vacancies at (and only at) these fixed positions. The previous procedure applied to this subset will lead to the link basis of \( V_{2(n-m),0} \), irrespective of the chosen positions. Since there are \( \binom{2n}{2m} \) different ways of choosing these positions, it follows that the space of dilute \( n \)-diagrams with \( 2m \) vacancies has dimension \( \binom{2n}{2m} \dim V_{2(n-m),0} \). The proof is completed by recalling that, for all \( n \), \( \dim V_{2n,0} = \dim TL_n \).

Motzkin numbers \( M_n, n \geq 0 \), are defined as the number of ways of drawing any number of nonintersecting chords joining \( n \) (labeled) points on a circle. The first Motzkin numbers are:

Clearly each \( n \)-diagram of \( dTL_n \) with its \( 2n \) points leads to such a drawing of non-intersecting chords on a circle with \( 2n \) points and vice versa. The dimension of \( dTL_n \) is thus the Motzkin number \( M_{2n} \) and, for example, \( \dim dTL_8 = M_{16} = 853467 \).

3. LEFT (AND RIGHT) \( dTL_n \)-MODULES

This section introduces some of the basic modules over the dilute Temperley-Lieb algebra \( dTL_n \): the link modules \( A_n \) and then the standard modules \( S_{n,k} \). The latter will turn out to form a complete set of non-isomorphic irreducible modules when \( q \) is not a root of unity. Some of their properties will be proved here. The modules \( S_{n,k} \) are cyclic and
indecomposable, their dimensions can be computed, and both their restriction to $\text{dTL}_{n-1}$ and induction to $\text{dTL}_{n+1}$ satisfy short exact sequences.

3.1. **The link modules** $A_n$ and $H_{n,k}$. A left (right) $n$-link diagram, with $n \geq 1$, is built in the following way. First, take a dilute $n$-diagram and remove its right (left) side as well as the points that were on it. An object, whether it is a string or a vacancy that no longer touches any point, is simply removed. The other floating strings are straightened out and called *defects*. For example,

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example1.png}}
\end{array} \]

The resulting diagram is called a *left $n$-link (right $n$-link)*. It is seen that a dilute $n$-diagram induces a unique pair of one left and one right $n$-link diagrams and that, given such a pair, there can be at most one $n$-diagram, if any, that could have induced them. It will thus be useful to denote an $n$-diagram by its induced $n$-links, which we will denote by $b = |lr|$, where $l$ ($r$) is the left (right) link diagram induced from $b$. This notation can also be used for linear combinations of $n$-diagrams as in $b = |l + j)r| + |uv|$ where $l, j, u$ are left $n$-links and $r, v$ right ones. If $u$ is a left link, then $\bar{u}$ will denote its (right) mirror image.

A natural action can be defined of $\text{dTL}_n$ on left (and right) $n$-link diagrams. We start with the left action. Draw the $n$-diagram on the left side of the left $n$-link, identify the points on its right side with those on the link and remove them. Each floating string that is not connected to the remaining side is removed and yields a factor $\beta$ if it is closed and zero if it touches a vacancy. If a floating string starting on the remaining side is connected to a defect in the $n$-link diagram, it becomes a defect. Finally, if a floating string contains two distinct defects of the original diagram, it is simply removed, as any remaining vacancies. The remaining drawing is the resulting $n$-link diagram, weighted by factors of $\beta$, one for each closed floating strings. For example

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example2.png}}
\end{array} \]
This action can be extended linearly to any element of dTLₙ. Let Aₙ be the vector space of all formal linear combinations of n-link diagrams. Again the above action can be extended linearly to any element of this space. This action is associative. (The connectivities of each floating string in (ab)v and a(bv), for a, b ∈ dTLₙ and v ∈ Aₙ, are the same.) The vector space Aₙ is therefore a left dTLₙ-module for this action. Right modules can be defined similarly by putting the elements of dTLₙ to the right of right n-links. A general element of Aₙ will be called a n-link state. The modules Aₙ extend the link modules of the Temperley-Lieb algebra. One should note that, unlike for Temperley-Lieb link modules, the number of arcs in dTLₙ-link modules can vary freely. However, as in Temperley-Lieb link modules, the action of an element of dTLₙ on a link diagram cannot increase its number of defects. The submodule of Aₙ spanned by n-link diagrams having at most k defects is called Hₙ,k, 0 ≤ k ≤ n, and these submodules Hₙ,k form a filtration of Aₙ:

\[ H_{n,0} \subset H_{n,1} \subset \cdots \subset H_{n,n} = Aₙ. \] (4)

The submodules Hₙ,k and the module Aₙ will be called link modules.

3.2. The standard modules Sₙ,k. The filtration (4) leads to the definition of another family of left modules, obtained simply by the quotient of two consecutive link modules Hₙ,k and Hₙ,k−1, namely:

\[ S_{n,k} \simeq H_{n,k}/H_{n,k−1}, \text{ for } 1 \leq k \leq n, \quad \text{and} \quad S_{n,0} = H_{n,0}. \]

It will also be useful to set S₀ₙ = {0} for integers k ∈ ℤ not in the set \{0, 1, \ldots, n\}. The left dTLₙ-modules Sₙ,k are called the standard modules and extend those of the Temperley-Lieb algebra. (In [25], the standard modules of TLₙ were denoted by \( V_{n,p} \) where p stands for the number of arcs. The number of defects is then \( n - 2p \), as there are no vacancies in the diagrammatic definition of TLₙ. As noted before, the number of arcs is not constant in Sₙ,k. This explains the discrepancy in labelling between the present text and [25]. From now on, we shall use defects instead of arcs even for objects related to TLₙ and will translate results of [25] accordingly.)

By construction the number of defects is always conserved in Sₙ,k. More precisely, a basis of \( H_{n,k}/H_{n,k−1} \) can be chosen to be the set of equivalence classes of n-links with precisely k defects. If v is such an n-link diagram, then the class \([v]\) ∈ \( H_{n,k}/H_{n,k−1} \) contains a unique n-link with k defects and it is precisely v. For that reason we shall write v for \([v]\).

As an example, the equivalence classes corresponding to the following 4-links form a basis of S₄,2:

\[
\begin{align*}
\vcenter{\hbox{\includegraphics[scale=0.3]{example1.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example2.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example3.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example4.png}}} \\
\vcenter{\hbox{\includegraphics[scale=0.3]{example5.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example6.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example7.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example8.png}}} \\
\vcenter{\hbox{\includegraphics[scale=0.3]{example9.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example10.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example11.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example12.png}}} \\
\vcenter{\hbox{\includegraphics[scale=0.3]{example13.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example14.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example15.png}}} & \quad \vcenter{\hbox{\includegraphics[scale=0.3]{example16.png}}}
\end{align*}
\]

Note that if \( n - k \) is even (odd), then only edTLₙ (odTLₙ) can act non-trivially on it. That is, only an element of edTLₙ (odTLₙ) may lead to a non-zero result. For that reason, the standard module Sₙ,k has a given parity, that of the number \( n - k \). Also, note that the number of vacancies on a link diagram restricts the elements of dTLₙ that can act non-trivially on it. For example, n-diagrams with more than \( n - k \) vacancies on either of their sides act as zero on Sₙ,k.
It is useful to define the subset $X_{n,k}$ of $n$-links having precisely $k$ defects and $n-k$ vacancies. In general the subspace $\text{span} X_{n,k}$ is not a $\text{dTL}_n$-submodule, but it will be important for the analysis now to be carried.

Let $z \in X_{n,k}$, $u$ and $v$ be any left $n$-link diagrams in $S_{n,k}$. For the action in $S_{n,k}$, the element $|u\vec{w}|$ of $\text{dTL}_n$ acts as zero on $z$ unless $v$ and $z$ are equal. Similarly, if $|u\vec{z}|v$ is non-zero in $S_{n,k}$, then again $v$ and $z$ are equal. (Note that this fails to be true if $v$ is a general link state and not a link diagram. We will see how this property generalizes to link states soon.)

Note finally that, for all link states $u \in S_{n,k}$, $|u\vec{z}z = u$. This property leads to the following result.

**Proposition 3.1.** $S_{n,k}$ is cyclic, with any non-zero element of $\text{span} X_{n,k}$ being a generator.

**Proof.** The property just outlined means that any element $z$ in $S_{n,k}$ is a generator: $(\text{dTL}_n)z = S_{n,k}$. Let $v$ be a non-zero element in $\text{span} X_{n,k}$. Since the elements of $X_{n,k}$ are linearly independent, $v \in \text{span} X_{n,k}$ has a non-zero component along some $n$-link $z$ and $|z\vec{z}|v$ is equal to $z$ up to a non-zero constant. Therefore $v$ is also a generator of $S_{n,k}$. \(\square\)

This property is also used in the following propositions.

**Proposition 3.2.** $S_{n,k}$ is indecomposable.

**Proof.** Recall that, for any pair of $n$-link diagrams $u \in S_{n,k}$ and $z \in X_{n,k}$, $|z\vec{z}|u = z$ if $u = z$ and zero otherwise. So, suppose that $S_{n,k} \simeq A \oplus B$ for some submodules $A$ and $B$. Since $z$ generates the whole module, it cannot belong to either $A$ or $B$, unless one of them is trivial. There must be two non-zero link states $a \in A$ and $b \in B$ such that $z = a + b$, with $z = |z\vec{z}|z = |z\vec{z}|(a + b) = a' + b'$ with $a' = |z\vec{z}|a \in A$ and $b' = |z\vec{z}|b \in B$. If $a'$ is zero, then $b' = z \in B$ and $B = S_{n,k}$, and $A = \{0\}$. If $a'$ is not zero, it must have a non-zero component along $z$ in the basis of $n$-links. Therefore $a' = |z\vec{z}|a = \alpha z$ for some $\alpha \in \mathbb{C}^\times$. Again this implies that $A = S_{n,k}$ and $B = \{0\}$. So $S_{n,k}$ is indecomposable. \(\square\)

**Proposition 3.3.** $S_{n,k} \simeq S_{n,j} \Leftrightarrow k = j$.

**Proof.** Only the statement “$\Rightarrow$” is non-trivial. Choose $k \leq j$ and let $\theta : S_{n,k} \to S_{n,j}$ be a $\text{dTL}_n$-isomorphism. Choose $x \in X_{n,k}$ and a $\sigma = |u\vec{x}| \in \text{dTL}_n$, with a non-zero $u \in S_{n,k}$. Then $\sigma x$ is non-zero and, since $\theta$ is an isomorphism, so is $\theta(\sigma x) = \sigma \theta(x)$. This means that $\theta(x)$ is a linear combination of states, one of which must have precisely $n-k$ vacancies, all of them coinciding with those of $x$. Since $j \geq k$, all other positions of this state must be occupied by defects, and $j$ and $k$ must actually be equal. \(\square\)

Proposition 3.1 has shown that any vector in $\text{span} X_{n,n}$ generates the standard module $S_{n,k}$. But, for the special case $k = n - 1$ or $n$, any $n$-link diagrams must have precisely 1 and 0 vacancy respectively and $S_{n,n} = \text{span} X_{n,n}$ and $S_{n,n-1} = \text{span} X_{n,n-1}$. Therefore any non-zero vector in these modules generates them and the following result follows.

**Corollary 3.4.** $S_{n,n}$ and $S_{n,n-1}$ are irreducible.

### 3.3. The dimension of $S_{n,k}$

The next step is the computation of the dimensions of the standard modules $S_{n,k}$. This task is made easy by the following ordering of their basis of $n$-link diagrams. (See below for an example.) First start by ordering the $n$-link basis by their number of vacancies $t$, where $0 \leq t \leq n-k$ and $t \equiv n-k \mod 2$. Second, among those with the same number $t$ of vacancies, gather those whose vacancies are at the same positions. The ordering does not need to be specified any further. Now, for a given number of vacancies and their fixed locations, note that the $(n-t)$-link diagrams obtained by omitting the vacant positions are in one-to-one correspondence with elements...
of the link basis of the Temperley-Lieb standard module $V_{n-\iota,k}$ or, equivalently, $V_{k+2p,k}$ if the number of arcs $p = (n - t - k)/2$ is used. The number of arcs must then be in the range $0 \leq p \leq \lfloor (n - k)/2 \rfloor$. For a fixed $\iota$ or $p$, the number of possible positions of the $\iota$ vacancies among the $n$ positions is $\binom{n-\iota}{p} = \binom{n}{k+2p}$. Also, recalling the structure of the subalgebra $S_n$, the action of this algebra will never change the vacancies of a $n$-link diagrams. We have therefore proved the following proposition and corollary.

**Proposition 3.5.** As vector spaces,

$$S_{n,k} \simeq \bigoplus_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{k+2p} V_{k+2p,k}. \quad (5)$$

Furthermore, if we consider $S_{n,k} \downarrow_{\mathcal{TL}_n}$, the restriction of $S_{n,k}$ to the subalgebra $S_n$, then this isomorphism is also a $S_n$-module isomorphism.

**Corollary 3.6.** The dimension of the standard module $S_{n,k}$ is

$$\dim S_{n,k} = \sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{k+2p} \dim V_{k+2p,k} \quad (6)$$

where $\dim V_{n,k} = \binom{n}{(n-k)/2} - \binom{n}{(n-k)/2-1}$.

Here is an example, for the module $S_{5,1}$, of the ordering used in the proof. The subset of 5-links without any vacancy ($p = 2$) form a basis of the $\mathcal{TL}_5$-module $V_{5,1}$:

$\begin{array}{cccc}
\begin{array}{cccc}
\ldots & | & | & \\
| & | & | & \\
\end{array}
\end{array}$

Now the subset with $t = 2$ vacancies ($p = 1$) contains 20 link diagrams:

$\begin{array}{cccc}
\begin{array}{cccc}
\ldots & | & | & \\
| & | & | & \\
\end{array}
\end{array}$

Even though some have been omitted, it is clear that, for fixed vacancy positions, the occupied positions are 3-link diagrams and these form a basis of $V_{3,1}$. Finally the subset with $t = 4$ vacancies ($p = 0$) is

$\begin{array}{cccc}
\begin{array}{cccc}
\ldots & | & | & \\
| & | & | & \\
\end{array}
\end{array}$

This subset contains $\binom{5}{2} = 5$ copies of the 1-link state with a single defect, that is a basis of $V_{1,0}$. The dimension of $S_{5,1}$ is 30.
The same method of slicing and unfolding \( n \)-diagrams used in section 2.2 to obtain the dimensions of \( \text{dTL}_n \) can be used again while keeping track of the number of defects. This leads to another expression for the dimension of the algebra.

**Proposition 3.7.** The dimension of the dilute Temperley-Lieb algebra \( \text{dTL}_n \) is also given by

\[
\dim \text{dTL}_n = \sum_{k=0}^{n} (\dim S_{n,k})^2.
\]

### 3.4. The restriction of \( S_{n,k} \)

The next two subsections are devoted to the restriction and induction of the standard modules \( S_{n,k} \). The first step, for the study of the restriction, is to decide how the subalgebra \( \text{dTL}_{n-1} \) is embedded into \( \text{dTL}_n \). The embedding that we use is realized by adding a pair of points at the bottom of all \((n-1)\)-diagrams (the \( n \)-th points) and connecting this pair by a dashed line. As the dashed line is seen to act as the identity on the \( n \)-th points, this is a natural embedding, similar to the one used for the Temperley-Lieb algebra [24, 25]. Any \((n-1)\)-diagram of \( \text{dTL}_{n-1} \) is then embedded as the sum of two \( n \)-diagrams of \( \text{dTL}_n \). The module \( S_{n,k} \) seen as a \( \text{dTL}_{n-1} \)-module will be called the restriction of \( S_{n,k} \) and denoted by \( S_{n,k} \downarrow \).

**Proposition 3.8.** With the embedding of \( \text{dTL}_{n-1} \) in \( \text{dTL}_n \) described above, the short sequence

\[
0 \rightarrow S_{n-1,k} \oplus S_{n-1,k-1} \rightarrow S_{n,k} \downarrow \rightarrow S_{n-1,k+1} \rightarrow 0
\]

is exact for all \( n \geq 2 \) and \( k \in \{0, 1, \ldots, n\} \) and therefore

\[
S_{n,k} \downarrow / (S_{n-1,k} \oplus S_{n-1,k-1}) \cong S_{n-1,k+1}.
\]

Again \( S_{m,j} = \{0\} \) if \( j \not\in \{0, 1, \ldots, m\} \).

**Proof.** To show exactness at \( S_{n-1,k} \oplus S_{n-1,k-1} \), an injective map \( \phi : S_{n-1,k} \oplus S_{n-1,k-1} \rightarrow S_{n,k} \downarrow \) needs to be constructed. Consider the operation, defined on \((n-1)\)-links with \( k \) or \( k-1 \) defects, that consists in adding a point at the bottom of the diagram and putting a defect there if the diagram had \( k-1 \) defects, and a vacancy otherwise. The result is an \( n \)-link with precisely \( k \) defects. Let \( \phi \) be the map that extends linearly this operation to \( S_{n-1,k} \oplus S_{n-1,k-1} \). Since the elements of \( \text{dTL}_{n-1} \) do not act on the \( n \)-th point, this is a homomorphism. It should also be clear that it is injective.

To define a homomorphism \( \psi : S_{n,k} \downarrow \rightarrow S_{n-1,k+1} \) such that \( \ker \psi = \text{im} \phi \), we again start by defining a diagrammatic operation on \( n \)-links. If an \( n \)-link diagram has a defect or a vacancy at its \( n \)-th position, it is sent to zero in \( S_{n-1,k+1} \). Otherwise, its \( n \)-th point is simply removed and the arch which ended at this point is replaced by a defect at its entry (top) point. For example,

\[
\begin{array}{c}
\text{0} \\
\text{ψ}
\end{array}
\]

\[
\begin{array}{c}
\text{0} \\
\end{array}
\]

The map \( \psi \) is defined as the linear extension to \( S_{n,k} \downarrow \) of this operation defined on links. To see that this is a homomorphism, suppose that an \( n \)-diagram in \( \text{dTL}_{n-1} \subset \text{dTL}_n \) transforms the bubble ending at position \( n \) of an \( n \)-link into a defect or a vacancy. This can only be achieved if the opening point of the bubble is linked to a defect by a bubble on the right side of the \( n \)-diagram. The same diagram applied to the image of the link would then link two of its defects together and would thus correspond to the zero element in \( S_{n-1,k+1} \). So \( \psi \) is indeed a homomorphism. The map has been constructed so that \( \text{im} \phi \subset \ker \psi \).
To see that $\psi$ is surjective, we construct a pre-image for a general $(n-1)$-link in \( S_{n-1,k+1} \). Any such a link has at least one defect since \( k+1 \) is a positive integer. Then add an \( n \)th point to the diagram and close the lowest defect in the link onto this new position \( n \). This is then an element of \( S_{n,k} \) whose image by \( \psi \) is the original \((n-1)\)-link. This construction also shows that there is a one-to-one correspondence between \( n \)-links in \( S_{n,k} \) that have a bubble ending at \( n \) and \((n-1)\)-links in \( S_{n-1,k+1} \). Therefore \( \text{im} \phi \) and \( \ker \psi \) must coincide.

Note finally that the previous constructions for \( \phi \) and \( \psi \) remain valid when \( k = 0, n-1 \) or \( n \) if the modules \( S_{n-1,-1}, S_{n-1,n} \) and \( S_{n-1,n+1} \) are taken to be the trivial ones.

Note that the exact sequence gives a simple relationship between the dimensions of the \( S_{n,k} \):

\[
\dim S_{n,k} = \dim S_{n-1,k} + \dim S_{n-1,k-1} + \dim S_{n-1,k+1}.
\]

This property could also be proved using the dimension \( 9 \) of \( S_{n,k} \). The module \( S_{n-1,k} \oplus S_{n-1,k-1} \) is a direct sum of two submodules of distinct parities. Since \( S_{n-1,k+1} \) has the parity of \( S_{n-1,k-1} \), the submodule \( S_{n-1,k} \) of \( S_{n,k} \) is the largest of its parity.

**Proposition 3.9.** Let \( \beta = q + q^{-1} \) with \( q \in \mathbb{C}^\times \). If \( q^{2(k+1)} \neq 1 \), the sequence

\[
0 \to S_{n-1,k-1} \to S_{n,k} \to S_{n-1,k} \to 0
\]

splits and therefore \( S_{n,k} \cong S_{n-1,k-1} \oplus S_{n-1,k+1} \).

**Proof.** This proof uses the central element \( F_{n-1} \) defined in appendix \( \mathcal{B} \). Since \( F_{n-1} \) is central, its (generalized) eigenspaces in a given \( dTL_{n-1} \)-module are submodules. The appendix shows that \( F_k \) acts on the standard module \( S_{n,k} \) as \( \delta_k \times \text{id} \) with \( \delta_k = q^{k+1} + q^{-k} \). If \( \delta_{n-1,k-1} \) and \( \delta_{n-1,k+1} \) are different, then \( S_{n,k} \cong S_{n-1,k} \) will contain two eigenspaces of \( F_{n-1} \) of dimensions \( \dim S_{n-1,k-1} \) and \( \dim S_{n-1,k+1} \) respectively. The exercise consists then in deciding when the two eigenvalues \( \delta_{n-1,k-1} \) and \( \delta_{n-1,k+1} \) are distinct. Their difference is:

\[
\delta_{n-1,k+1} - \delta_{n-1,k-1} = q^{k+2} - q^k + q^{-(k+2)} - q^{-k} = (q^2 - 1)\left(q^{2(k+1)} - 1\right)q^{-k-2}
\]

and vanishes if and only if \( q^{2(k+1)} = 1 \). The condition that \( q^{2(k+1)} \neq 1 \) will be fundamental for the rest of the text. An integer \( k \) will be called **critical** if \( q^{2(k+1)} = 1 \), and **generic** otherwise. We also say that \( S_{n,k} \) is critical if \( k \) is.

### 3.5. The induction of \( S_{n,k} \)

After studying the restriction of the \( dTL_n \)-module \( S_{n,k} \), it is natural to ask whether its induction is also part of an exact sequence similar to that satisfied by its restriction. This subsection answers this question.

The induction of \( S_{n,k} \), denoted by \( S_{n,k}^\uparrow \), is defined by the tensor product

\[
S_{n,k}^\uparrow = dTL_{n+1} \otimes_{dTL_n} S_{n,k}
\]

where the subscript on the tensor product symbol means that the elements of \( dTL_n \) (embedded in \( dTL_{n+1} \) as in the previous subsection) may pass freely from one of its sides to the other.

The first task is to find a finite generating set for \( S_{n,k}^\uparrow \) of manageable size. Proposition \( 3.1 \) provides a first simplification. Let \( z \) be an \( n \)-link diagram in \( X_{n,k} \). Since \( S_{n,k} = dTL_nz \), then

\[
S_{n,k}^\uparrow = dTL_{n+1} \otimes_{dTL_n} (dTL_nz) = dTL_{n+1} \otimes dTL_nz.
\]
A further simplification is possible. We introduce for this purpose three “surgeries” \( \theta_i \), \( i \in \{-1, 0, 1\} \), that transforms an \( n \)-link diagram \( u \in \mathbb{S}_{n,k} \) into an \((n+1)\)-link one. The first \( \theta_1 \) adds to the \( n \)-link \( u \) a defect in the bottom, at position \( n+1 \), and the second \( \theta_0 \) adds there a vacancy. The last one, \( \theta_{-1} \), closes the lowest defect of \( u \) into an arc ending at \( n+1 \), if such a defect exists. If there is none, \( \theta_{-1} \) sends the \( n \)-link to zero. The index on the \( \theta_i \) indicates how the number of defects changes. Here are some examples.

\[
\begin{align*}
\theta_1 \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= \begin{array}{c}
\vdots \\
\end{array}, & \theta_1 \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= \begin{array}{c}
\vdots \\
\end{array}, \\
\theta_0 \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= \begin{array}{c}
\vdots \\
\end{array}, & \theta_0 \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= \begin{array}{c}
\vdots \\
\end{array}, \\
\theta_{-1} \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= \begin{array}{c}
\vdots \\
\end{array}, & \theta_{-1} \left( \begin{array}{c}
\vdots \\
\end{array} \right) &= 0 .
\end{align*}
\]

We now argue that any non-zero element of \( \mathbb{S}_{n,k} \) can be written as a sum of terms of the form \( |a\theta_i(z)| \otimes z \) where \( i \in \{-1, 0, 1\} \) and \( u \in \mathbb{S}_{n+1,k+1} \). It is sufficient to study elements of \( \text{dTL}_{n+1} \) of the form \( |u\vec{v}| \) with \( u \) and \( v \) left \((n+1)\)-link diagrams.

The first case to study is when \( v \) is in \( X_{n+1,j} \) for some \( j \). It is then possible to write \( |u\vec{v}| = |u\vec{v}||v\vec{v}| \). Let \( v' \) be the \( n \)-link diagram obtained from \( v \) by deleting its position \( n+1 \) and the vacancy or the defect at this position. Then

\[
|v\vec{v}| = a|v'\vec{v}'|_{n+1}
\]

where \( a \) stands for the generator \( x_{n+1} \) if \( v \) has a vacancy at \( n+1 \) and for the generator \( e_{n+1} \) if instead it has a defect there. (The elements \( x_i \) and \( e_i \) were defined in subsection 2.1.)

Therefore

\[
|u\vec{v}| \otimes z = |u\vec{v}|a \otimes |v'\vec{v}'|z
\]

with the appropriate \( a \). This tensor product is zero unless \( v' \) is equal to \( z \). That is, when \( v \) is an \((n+1)\)-link with only defects and vacancies, the vector \( |u\vec{v}| \otimes z \) is non-zero only when \( v = \theta_0(z) \) if position \( n+1 \) of \( v \) is vacant and when \( v = \theta_1(z) \) if it bears a defect.

The second case to study is when \( v \) contains an arc between two positions above \( n+1 \). It is then always possible to find an arc between \( i \) and \( j \) such that \( 1 \leq i < j \leq n \) and that all positions \( k \) in \( v \) with \( i < k < j \) are vacant. Then

\[
|u\vec{v}| = \begin{bmatrix}
\bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_i & \vdots & \bar{v}_j & \cdots & \bar{v}_n \\
i & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_j & \vdots & \bar{v}_i & \cdots & \bar{v}_n
\end{bmatrix}
\]

where \( \bar{v}_i \) (\( \bar{v}_k \)) contains the pattern of positions above \( i \) in \( v \) (below \( j \)). There might be arcs going from \( \bar{v}_i \) to \( \bar{v}_k \) as well as arcs between \( u \) and the \( \bar{v}_i \) and \( \bar{v}_k \). Consider the rightmost factor of (13). All positions corresponding to those of \( \bar{v}_i \) and \( \bar{v}_k \) are occupied by dashed lines. The \( n \) top positions of this factor is an element of \( \text{dTL}_n \) and annihilates \( z \), because either the arc joins two defects in \( z \) or there is a mismatch between the vacancies and
A factorization similar to that used in the first case leads to

\[ |u\bar{v}| = \begin{bmatrix} u & v' \end{bmatrix} \begin{bmatrix} v'' \end{bmatrix} \]

where \( v' \) is obtained from \( v \) by deleting its position \( n+1 \) and putting a defect at position \( i \).

Then \( |u\bar{v}| \otimes z = |u\bar{v}^1 \otimes |v''^1 z| \) and \( |u\bar{v}| \otimes z \) is non-zero only if \( v'' = z \). Hence, when \( v \) has a single arc ending at \( n+1 \), the element \( |u\bar{v}| \otimes z \) is non-zero only if \( \theta_{-1}(z) = v \). The analysis of the above three cases is summed up by saying that the defects of the above three cases is summed up by saying that

\[ B_{n,k} = \{ |u\theta_{i}(z)| \otimes z \mid i \in \{-1, 0, 1\} \text{ and } u \text{ a link diagram in } S_{n+1,k+1} \} \]

The analysis does not prove that \( B_{n,k} \) is a basis however. It does not even rule out some of the elements in \( B_{n,k} \) being zero. The main remaining result of the present subsection is that \( B_{n,k} \) is indeed a basis.

Choose \( z \in X_{n,k} \) and let \( \phi = \phi_z \) be the linear map \( dTL_{n+1} \otimes_S S_{n,k} \rightarrow S_{n+2,k} \) defined by the following action on elements of the form \( |u\bar{v}^1 \otimes_C y| \) where \( u \) and \( v \) are \((n+1)\)-links with the same numbers of defects and \( y \in S_{n,k} \). (The index on the tensor product sign will be omitted only if it is \( dTL_n \).) To compute \( \phi(|u\bar{v}^1 \otimes_C y|) \), first draw

\[ \begin{bmatrix} u & \bar{v} \end{bmatrix} \begin{bmatrix} y \bar{z} \end{bmatrix} \]

and then detach the dashed line ending at position \((n+1)\) on the far right to attach it at the bottom of \( u \):

\[ \begin{bmatrix} u \bar{v} y \bar{z} \end{bmatrix} \]

The object created has \( n+2 \) positions on its left edge, \( n \) on its right one. There are \( n+1 \) positions on both sides of the central line and one can use the usual rules to multiply diagrams for this new object. If vacancies do not match, then \( \phi(|u\bar{v}^1 \otimes_C y|) \) is set to zero. If they match, then there exists \( w \in S_{n+2,j} \) and \( x \in S_{n,j} \) such that the above diagram is \( \beta y \otimes w \bar{y} \) where \( \# \) is the number of closed loops in \( (\ref{14}) \). The image \( \phi(|u\bar{v}^1 \otimes_C y|) \) is non-zero only if \( j = k \) and it is then \( \beta \otimes w \). Note that, if \( u', v' \) are some \( n \)-links with the same numbers of defects, then

\[ \phi\left( \begin{bmatrix} u \bar{v} u' \bar{v}' \end{bmatrix} \otimes_C y \right) = \phi(|u\bar{v}| \otimes_C |u'\bar{v}'| y) \]

because the computation of the resulting image is based in both cases on the diagram

\[ \begin{bmatrix} u \bar{v} u' \bar{v}' \bar{z} \end{bmatrix} \]

Therefore \( \phi \) maps to zero the subspace spanned by \( \{ ab \otimes_C y - a \otimes_C by \mid a \in dTL_{n+1}, b \in dTL_n, y \in S_{n,k} \} \). The linear map \( \phi \) thus induces a well-defined linear map

\[ \Phi : S_{n,k} \mapsto \frac{dTL_{n+1} \otimes_S S_{n,k}}{\langle ab \otimes_C y - a \otimes_C by \rangle} \rightarrow S_{n+2,k} \]

where \( dTL \) is the number of closed loops in \( (\ref{14}) \). The image \( \phi(|u\bar{v}^1 \otimes_C y|) \) is non-zero only if \( j = k \) and it is then \( \beta \otimes w \). Note that, if \( u', v' \) are some \( n \)-links with the same numbers of defects, then

\[ \phi\left( \begin{bmatrix} u \bar{v} u' \bar{v}' \end{bmatrix} \otimes_C y \right) = \phi(|u\bar{v}| \otimes_C |u'\bar{v}'| y) \]

because the computation of the resulting image is based in both cases on the diagram

\[ \begin{bmatrix} u \bar{v} u' \bar{v}' \bar{z} \end{bmatrix} \]

Therefore \( \phi \) maps to zero the subspace spanned by \( \{ ab \otimes_C y - a \otimes_C by \mid a \in dTL_{n+1}, b \in dTL_n, y \in S_{n,k} \} \). The linear map \( \phi \) thus induces a well-defined linear map

\[ \Phi : S_{n,k} \mapsto \frac{dTL_{n+1} \otimes_S S_{n,k}}{\langle ab \otimes_C y - a \otimes_C by \rangle} \rightarrow S_{n+2,k} \]
Proposition 3.10. Let \( n \geq 1 \) and \( k \in \{0, 1, \ldots, n\} \). Then

(i) the set \( B_{n,k} \) is a basis of \( S_{n,k} \uparrow \) and

(ii) \( S_{n,k} \uparrow \cong S_{n+2,k} \downarrow \) as \( dTL_{n+1} \)-modules.

Proof. The linear map \( \Phi \) defined in (15) is a \( dTL_{n+1} \)-homomorphism. This follows by the observation that, if \( \Phi(\|u\| \otimes y) = \beta^* w \), then \( \Phi(a\|u\| \otimes y) \) and \( a\Phi(\|u\| \otimes y) \) will both give \( \beta^* aw \) for all \( a \in dTL_{n+1} \) as can be verified diagrammatically.

No elements of the spanning set \( B_{n,k} \) is zero. To see this, it is sufficient to note that their images by \( \Phi \) are non-zero. Indeed a direct computation shows that, if \( u \in S_{n+1,k+1} \) with \( i \in \{-1, 0, 1\} \), then \( \phi(\|u\| (\|\| z) \otimes z) = \theta_{-i}(u) \in S_{n+2,k} \) which is non-zero.

To end the proof, it remains to show that the spanning set is linearly independent. Since \( |B_{n,k}| = \dim S_{n+2,k} \), it is sufficient to show that any link diagram in \( S_{n+2,k} \) has a pre-image in \( B_{n,k} \). To find the pre-image of \( u \), a \((n+2)\)-link in \( S_{n+2,k} \), simply construct \( \|u\| \) and detach the bottom position of \( u \) to attach it to \( z \). The result is \( \|u\| (\|\| z) \otimes z \) for some \( i \in \{-1, 0, 1\} \). Then \( \Phi(\|u\| (\|\| z) \otimes z) = u \). The spanning set is therefore linearly independent and \( \Phi \) is a \( dTL_{n+1} \)-isomorphism. \( \square \)

The following corollaries are immediate consequences of proposition 3.10 and the properties of the restriction of \( S_{n,k} \) obtained in the last subsection.

Corollary 3.11. The short sequence

\[ 0 \rightarrow S_{n,k} \oplus S_{n,k-1} \rightarrow S_{n-1,k} \uparrow \rightarrow S_{n,k+1} \rightarrow 0 \quad (16) \]

is exact for all \( n \geq 2 \) and \( k \in \{0, 1, \ldots, n-1\} \).

Corollary 3.12. For all \( n \geq 2 \) and \( k \) generic in \( \{0, 1, \ldots, n-1\} \)

\[ S_{n-1,k} \uparrow \cong S_{n,k-1} \oplus S_{n,k} \oplus S_{n,k+1} \cdot \quad (17) \]

Proposition 3.10 and the analogous result for TL\(_n\) differs on one small point. For the latter the isomorphism \( V_{n,k} \uparrow \cong V_{n+2,k} \downarrow \) fails in one particular case, namely when \( \beta = 0 \), then \( V_{2,0} \uparrow \neq V_{4,0} \downarrow \). Instead \( \dim V_{2,0} \uparrow = 3 > \dim V_{4,0} \downarrow = 2 \). The difficulty can easily be seen to occur only at \( \beta = 0 \) because, if \( \beta \neq 0 \), then \( u_1 u_2 \otimes b = \frac{u_1 u_2 u_1 \otimes b}{b u_1 \otimes b} = \frac{u_1 \otimes b}{b} \) where \( u_i = b^i h_i \). For \( \beta = 0 \) the vectors \( u_1 u_2 \otimes b \) and \( \frac{u_1 \otimes b}{b} \) are linearly independent. The problem does not occur for the dilute modules. For example, the analogous situation is resolved as follows:

\[
\begin{align*}
\vdots \otimes b & = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\otimes b \\
\vdots \otimes b & = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\otimes b \\
\vdots \otimes b & = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\otimes b \\
\vdots \otimes b & = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\otimes b \\
\vdots \otimes b & = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\otimes b
\end{align*}
\]

4. The Gram Product

This section introduces a bilinear form on standard modules that is invariant under (some dual of) the action of the algebra \( dTL_n \) (see lemma 4.1). It is a familiar tool of representation theory since the radical of this bilinear form is a submodule. For \( dTL_n \), the radical will be the (unique) maximal submodule. Such a submodule can be non-trivial only if the Gram matrix, representing the bilinear form into some basis, is singular. The Gram determinant and its zeroes can be easily computed. These zeroes occur only when \( q \) is a root of unity. The structure of \( dTL_n \) is then semisimple when \( q \) is generic (not a root of unity) and a complete set of non-isomorphic irreducible modules can be identified.
The central result of the section concerns non-trivial radicals at \( q \) a root of unity. Proposition 4.16 shows that they are then irreducible and isomorphic to the irreducible quotients of another standard modules. The section ends with the description of what the irreducible modules \( I_{n,k} \) become under restriction and induction.

### 4.1. The bilinear form \( \langle \ast, \ast \rangle_{n,k} \)

The Gram product \( \langle \ast, \ast \rangle_{n,k} : S_{n,k} \times S_{n,k} \rightarrow \mathbb{C} \) is a bilinear form defined on \( n \)-link diagrams and extended linearly. To compute the pairing of two (left) link diagrams, first reflect the first link diagram along its vertical axis and then glue it on the left side of the second one, identifying the corresponding points on both diagrams. If a point containing a string in one of the diagrams is identified with a point containing a vacancy in the other, the result is 0. Otherwise, the result is non-zero if and only if every defect of the first diagram is linked to a defect of the second. In such cases, the result is \( \beta^m \), where \( m \) is the number of closed loops formed by the glueing of the two links. For example:

\[
\langle \begin{array}{c} \ast \end{array} \rangle \rightarrow \beta^1, \quad \langle \begin{array}{c} \ast \end{array} \rangle \rightarrow \beta^1,
\langle \begin{array}{c} \ast \end{array} \rangle \rightarrow 0, \quad \langle \begin{array}{c} \ast \end{array} \rangle \rightarrow 0.
\]

This bilinear form extends that defined on standard modules \( V_{n,k} \) of the Temperley-Lieb algebra \( TL_n \) (see appendix A). One difference between the two bilinear forms for \( TL_n \) and \( dTL_n \) is worth mentioning. It concerns the bilinear form on the standard modules \( S_{n,0} \) and \( V_{n,0} \) when \( \beta = 0 \). For \( V_{n,0} \) with \( n \) even, the bilinear form is strictly zero, as the pairing of link diagrams always closes at least one loop. A special definition has to be introduced to counter this difficulty [25]. The bilinear form on \( S_{n,0} \) as described above is not zero, even when \( \beta = 0 \), as the pairing of the link diagram with \( n \) vacancies with itself gives 1.

The bilinear form is symmetric since exchanging the two arguments amounts to a reflection through a vertical mirror when written in terms of diagrams. We shall say that two elements of \( S_{n,k} \) are orthogonal if their Gram product is zero, even though \( \langle \ast, \ast \rangle_{n,k} \) can be degenerate.

**Lemma 4.1.** If \( x, y \in S_{n,k} \) and \( u \in dTL_n \) then

\[
\langle x, uy \rangle_{n,k} = \langle u'x, y \rangle_{n,k}
\]

where \( u' \) is the diagram obtained by reflecting \( u \) along its vertical axis. If \( u \) is a sum of diagrams, the reflection is done on each diagram of the linear combination separately.

**Proof.** The proof consists in writing the two sides of the equality in terms of diagrams. \( \square \)

**Lemma 4.2.** If \( x, y, z \in S_{n,k} \), then

\[
\langle xy \rangle_{n,k} \rangle z = \langle y, z \rangle_{n,k} x.
\]

**Proof.** It is sufficient to verify the relation for link diagrams \( x, y, z \in S_{n,k} \), by linearity. Equation (19) is then non-trivial only if all defects and vacancies of \( z \) are respectively linked to defects and vacancies of \( y \). In this case, all defects, arcs and vacancies of \( x \) will be preserved and remain at their places, so that \( \langle xy \rangle_{n,k} \rangle z \) is proportional to \( x \). The proportionality constant is the number of closed loops formed which is precisely \( \langle y, z \rangle_{n,k} \).

\( \square \)

The link diagrams in \( X_{n,k} \) enjoy a particular property: the Gram product of any pair is 1 if the two diagrams are the same and 0 otherwise. Proposition 4.1 showed that any
link diagram in $X_{n,k}$ (or even any non-zero element in its span) is a generator of $S_{n,k}$. The
next lemma explains, in terms of the bilinear form $\langle \cdot, \cdot \rangle_{n,k}$, why these link diagrams are
generators and identifies a larger set of generators.

**Lemma 4.3.** An element $x$ is a generator of $S_{n,k}$ if there exist $y \in S_{n,k}$ such that $\langle x, y \rangle_{n,k} \neq 0$.

*Proof.* Let $y \in S_{n,k}$ be such that $\langle y, x \rangle_{n,k} = \alpha \neq 0$. For any $z \in S_{n,k}$, both $z$ and $\bar{y}$ have
the same number of defects and $|z\bar{y}|$ is thus an element of $dTL_n$. Therefore $\frac{1}{\alpha}|z\bar{y}|x = z$ and
$(dTL_n)x = S_{n,k}$.

Hence any link state that is not orthogonal to all others is a generator. Those that are
orthogonal to all others are known to be as important. Their set

$$R_{n,k} = \{ x \in S_{n,k} | \langle y, x \rangle_{n,k} = 0, \text{ for all } y \in S_{n,k} \}$$

is called the (dilute) radical of $S_{n,k}$. It is easy to see that it is a submodule. Lemma 4.3
actually shows that it is its maximal submodule, that is, every proper submodule of $S_{n,k}$ is
a submodule of its radical. Moreover the module $I_{n,k} = S_{n,k}/R_{n,k}$ is irreducible, since any
of its non-zero elements generates it.

The Gram product can also be used to restrict morphisms between quotients of standard
modules.

**Lemma 4.4.** Let $N, N'$ be submodules of $S_{n,k}$ and $S_{n,k'}$, respectively, with $k < k'$. Then the
only homomorphism $S_{n,k}/N \to S_{n,k'}/N'$ is the zero homomorphism.

*Proof.* Let $\gamma$ be the canonical homomorphism from $S_{n,k}$ to $S_{n,k}/N$ and $\theta$ be a homomor-
phism from $S_{n,k}/N$ to $S_{n,k'}/N'$. Choose $y, z \in S_{n,k}$ such that $\langle y, z \rangle_{n,k} = 1$. Then for all
$x \in S_{n,k}$,

$$|y\bar{z}|\theta(\gamma(z)) = \theta(\gamma(|y\bar{z}|z)) = \theta(\gamma(x)).$$

(20)

Since $\theta(\gamma(z)) \in S_{n,k'}/N'$, the usual representative of this conjugacy class has $k'$ defects.
But $|y\bar{z}|\theta(\gamma(z))$ can have at most $k < k'$ defects and the left side of (20) must be zero.
Therefore $\theta(\gamma(x))$ is zero for all $x$ and, since $\gamma$ is surjective, $\theta$ is zero. □

4.2. **The structure of the radical.** Let $dG_{n,k}$ be the matrix representing the bilinear form
$\langle \cdot, \cdot \rangle_{n,k}$ in the basis of link diagrams. Similarly denote by $G_{n,k}$ the matrix for the bilinear
form for the corresponding standard TL$_n$-module, also in its link basis. These matrices
will be called Gram matrices and, if need be, the adjective dilute will be added to the
first one. The Gram product of two link diagrams in $S_{n,k}$ may be non-zero only if their
vacancies coincide. In that case, the product does not depend on their positions and it
is equal to the Gram product defined for standard modules of TL$_{n'}$ applied to the two
link diagrams obtained from the original ones by deleting their vacancies. (Then $n'$ is
$n - \#(\text{vacancies}).$) It is then clear that the matrix $dG_{n,k}$ is block-diagonal if the link basis
is ordered, first, by gathering links with the same number of vacancies and, second, those
with the same positions for these vacancies. The shape of the Gram matrix $dG_{n,k}$ then
appears as a consequence of the decomposition of the dilute standard modules into a direct
sum of $S_{n}$-modules (see proposition 3.3). The next result then follows immediately. (The
direct sum symbol is used to indicate the block diagonal decomposition of $dG_{n,k}$ and the
binomial factors give the multiplicity of each block or vector space.)

**Proposition 4.5.** The dilute Gram matrix for the $dTL_n$-modules $S_{n,k}$ is

$$dG_{n,k} = \bigoplus_{p=0}^{\frac{n}{k}} \binom{n}{k+2p} G_{k+2p,k}$$

(21)
Corollary 4.6. The determinant of the Gram matrix is

$$\det dG_{n,k} = \prod_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} (\det G_{k+2p,k})^{(n-k)/2+2p}$$

(22)

Corollary 4.7. The dilute radical \( R_{n,k} \) decomposes as

$$R_{n,k} \cong \bigoplus_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \left( \begin{array}{c} n \\ k + 2p \end{array} \right) R_{k+2p,k}$$

as vector spaces,

(23)

where \( R_{n,k} \) is the radical of the Gram bilinear form on \( V_{n,k} \) and the “\( \oplus' \)” indicates that the trivial radicals (= \{0\}) are omitted of the direct sum. Furthermore this decomposition for \( R_{n,k} \subset S_{n,k} \downarrow d_{TL_n} \) holds as \( S_n \)-modules.

Corollary 4.8.

$$\dim R_{n,k} = \sum_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \left( \begin{array}{c} n \\ k + 2p \end{array} \right) \dim R_{k+2p,k}.$$  

(24)

The last corollary leads to various recurrence relations for the dimensions of the dilute radicals and the irreducible modules. They are simple, though neither compact nor particularly enlightening. They will be presented along with their proofs in appendix C.

A distinction between the two algebras \( TL_n \) and \( dTL_n \) at \( \beta = 0 \) follows from the above proposition and corollaries. When \( \beta = 0 \) (and therefore \( q = \pm i \)), the determinant \( G_{n,k} \) vanishes for all even \( ks \) and is otherwise non-zero. It follows that \( TL_n(\beta = 0) \) is semisimple if \( n \) is odd, because then all its standard modules \( V_{n,k} \) have odd \( ks \), and \( TL_n(\beta = 0) \) is non-semisimple if \( n \) is even. It will be shown that the dilute \( dTL_n(\beta = 0) \) is non-semisimple for all \( n > 1 \).

The previous results show that the dilute radical \( R_{n,k} \) is trivial if the radicals \( R_{k+2p,k}, 0 \leq p \leq \lfloor (n-k)/2 \rfloor \), are all trivial. Since the determinant of \( G_{n,k} \) can vanish only at a root of unity distinct from \( \pm 1 \) (see (23)), then the following corollaries are straightforward.

Corollary 4.9. The dilute standard module \( S_{n,k} \) is irreducible if \( q \) is not a root of unity.

Corollary 4.10. The dilute standard module \( S_{n,k} \) is irreducible if \( k \) is critical.

Proof. We recall that the radical \( R_{n,k} \) of the standard \( TL_n \)-module \( V_{n,k} \) is trivial whenever \( k \) is critical, that is when \( q^{2(k+1)} = 1 \). (See proposition A.3.) This result is independent of \( n \) and all vector spaces \( R_{k+2p,k} \) appearing in (23) are trivial. □

Theorem 4.11 (Structure of \( dTL_n \) for \( q \) generic). If \( q \) is not a root of unity, then \( dTL_n \) is semisimple, the set \( \{S_{n,k}, 0 \leq k \leq n\} \) forms a complete set of non-isomorphic irreducible modules and, as a left module, the algebra \( dTL_n \) decomposes as

$$dTL_n = \bigoplus_{0 \leq k \leq n} (\dim S_{n,k}) S_{n,k}.$$  

Proof. Corollary 4.9 states that the \( S_{n,k} \) are irreducible when \( q \) is not a root of unity and proposition A.3 that they are non-isomorphic. Weddeburn’s theorem D.14 and its generalization D.14 show that, given a subset \( \{I_k, k \in K\} \) of its non-isomorphic irreducible modules, the dimension of an algebra is bounded from below by \( \sum_{k \in K} (\dim I_k)^2 \). In the
Hence \( \text{gram} \) and \( \pi \) if it is an isomorphism, then similar decompositions hold for all submodules of their sides, located where the vacancies of the present case \( \sum_{0 \leq k \leq n} (\dim S_{n,k})^2 = \dim \text{dTL}_n \) by proposition \([5.7]\). The three statements then appear as a consequence of Wedderburn’s theorem.

Our next step is to show that, like the radicals of the standard modules of the Temperley-Lieb algebra, the dilute radicals \( R_{n,k} \) are either irreducible or trivial. With this goal in mind, we introduce a few useful tools. Let \( z \in X_{n,k} \) be an \( n \)-link diagram and set \( \pi_z = [z] \in \text{dTL}_n \).

(Nota that \( \pi_z \) coincides with the projectors \( \pi_{\pi} \) introduced in subsection \([2.1]\).) If \( A \) is taken to be the set of positions of the defects of \( z \).) Here are some simple observations about \( \pi_z \). The set \( T_z = \pi_z \cdot \text{dTL}_n \cdot \pi_z \) is spanned by \( n \)-diagrams that have precisely \( (n-k) \) vacancies on their sides, located where the vacancies of \( z \) are. The vector space \( T_z \) is a submodule of \( \text{dTL}_n \) isomorphic to \( \text{TL}_k \). This leads to a reformulation of proposition \([2.1]\), namely:

\[
S_n \simeq \bigoplus_{0 \leq k \leq n} \bigoplus_{\pi \in \pi_{\pi}} \pi_z \cdot \text{dTL}_n \cdot \pi_z.
\]

Similarly the identity \( \text{id} \in \text{dTL}_n \) can be written as \( \text{id} = \sum_{0 \leq k \leq n} \sum_{\pi \in \pi_{\pi}} \pi_z \cdot \pi_z \).

Note that \( \pi_{\pi} \cdot \pi_{\pi} = \pi_{\pi} \) so that \( \pi_{\pi} \cdot z \in X_{n,k+2p} \). acts as a projector on \( S_{n,k} \). Moreover, for two distinct link diagrams \( z \in X_{n,k+2p} \) and \( z' \in X_{n,k+2p} \), \( 0 \leq p, p' \leq \lfloor \frac{n-k}{2} \rfloor \), \( \pi_z \cdot \pi_{\pi} = \pi_{\pi} \cdot \pi_z = 0 \) and \( \pi_z \cdot S_{n,k} \cap \pi_{\pi} \cdot S_{n,k} = \{0\} \). Therefore \( S_{n,k} \) decomposes as

\[
S_{n,k} = \bigoplus_{0 \leq p \leq \lfloor \frac{(n-k)}{2} \rfloor} \bigoplus_{z \in X_{n,k+2p}} \pi_z \cdot S_{n,k}, \quad \text{as vector spaces. (25)}
\]

Similar decompositions hold for all submodules of \( S_{n,k} \).

For a standard module \( S_{n,k} \) and an integer \( k' \) of the parity of \( k \), choose a link diagram \( z \in X_{n,k'} \). Define a linear map \( \phi_z : \pi_z \cdot S_{n,k} \rightarrow V_{k',k} \) by its action on link diagrams \( u \) in \( \pi_z \cdot S_{n,k} \). The image \( \phi_z(u) \) is obtained by deleting from \( u = [z]u \) its vacancies. Since \( z \) has \( (n-k') \) vacancies, the result has \( k' \) positions and, since \( u \) has \( k \) defects, so does \( \phi_z(u) \). Hence \( \phi_z(u) \in V_{k',k} \). Again here are some simple facts about \( \phi_z \). The map \( \phi_z \) is a \( T_z \)-homomorphism (recall that \( T_z \simeq \text{TL}_{k'} \).) It is zero if \( k' < k \) and an isomorphism otherwise. If it is an isomorphism, then \( \phi_z \) preserves the Gram product, that is,

\[
(u,v)_{n,k} = (\phi_z(u), \phi_z(v))_{k',k}, \quad \text{for all } u, v \in \pi_z \cdot S_{n,k},
\]

where, on the left, the bilinear form on \( S_{n,k} \) is used and, on the right, that on \( V_{k',k} \).

Let \( B \) be a submodule of the radical \( R_{n,k} \) different than \( \{0\} \). Showing that \( R_{n,k} \) is irreducible amounts to proving that \( B = R_{n,k} \) for any such \( B \). The next two lemmas use the map \( \phi_z \) to construct several subspaces that \( B \) automatically contains. The irreducibility of the radical will follow from the observation that the sum of these disjoint subspaces is precisely \( R_{n,k} \).

**Lemma 4.12.** Let \( B \subset R_{n,k} \subset S_{n,k} \) be a \( \text{dTL}_n \)-submodule. If there exists an \( n \)-link diagram \( z \in X_{n,k+2p} \) such that \( \pi_z \cdot B \neq \{0\} \), then \( B \) contains the \( \text{TL}_{k+2p} \)-submodule

\[
G = \bigoplus_{z' \in X_{n,k+2p}} \phi_{z'}^{-1}(R_{k+2p,k}).
\]

If \( u \in R_{n,k} \) is such that \( \pi_z \cdot u = u \), then \( u \in B \).

**Proof.** Suppose \( x \in B \) is such that \( \pi_z \cdot x \neq 0 \). Then \( \pi_z \cdot x \in B \) since \( B \) is a \( \text{dTL}_n \)-module. Moreover, since \( \phi_z \) is an \( T_z \)-isomorphism and preserves the Gram product, \( \phi_z(\pi_z \cdot x) \) is non-zero and belongs to the radical \( R_{k+2p,k} \). Therefore \( \phi_z(T_z \cdot x) \) is a \( T_z \)-submodule of \( R_{k+2p,k} \). Since the (non-trivial) radicals of the standard modules of \( \text{TL}_n \) are irreducible, \( \phi_z(T_z \cdot x) \) (and \( T_z \cdot x \)) must be isomorphic to \( R_{k+2p,k} \). Moreover, if \( z' \) is a different link diagram in
$X_{n,k+2p}$, then $|\mathcal{E}| : \pi_n S_{n,k} \to \pi_n S_{n,k}$ is an isomorphism between two subspaces of $S_{n,k}$ whose intersection is $\{0\}$. The first statement follows.

For the second statement, note that $u$ must be in $\pi_n(R_{n,k})$. By the above argument, $R_{k+2p,k} \simeq \pi_n B \subset \pi_n(R_{n,k}) \simeq R_{k+2p,k}$. Therefore $u$ must be also in $B$. \hfill \square

**Lemma 4.13.** Let $B \subset R_{n,k} \subset S_{n,k}$ be a $dTL_n$-submodule. If there exists an $n$-link diagram $z \in X_{n,k+2p}$ such that $\pi_n B \neq \{0\}$, then, for all $0 < p' \leq \lfloor n/2 \rfloor$ such that $R_{k+2p',k}$ is non-trivial, there exists $w \in X_{n,k+2p'}$ such that $\pi_n B \neq \{0\}$.

**Proof.** Choose $p'$ such that $R_{k+2p',k}$ is non-trivial and suppose first that $0 < p' < p$. From the link diagram $z \in X_{n,k+2p}$ of the statement, construct three link diagrams: $w \in X_{n,k+2p'}$, and $y, y' \in S_{n,k+2p'}$. The link $w$ is obtained by replacing the top $2(p-p')$ defects of $z$ by vacancies. For $y$, these top defects are replaced by $(p-p')$ unested bubbles. Finally, for $y'$, the first defect of $w$ is left untouched and the following $2(p-p')$ ones are replaced by $(p-p')$ unested bubbles. Here are examples with $p = 2, p' = 1$ and $k = 0$:

\[
\begin{align*}
z &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} 
\in X_{7,4}, \quad w = \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} 
\in X_{7,2}, \quad y = \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array}, \quad y' = \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} \in S_{7,2}.
\end{align*}
\]

Choose a non-zero $a \in R_{k+2p',k}$. Since $\phi_w$ preserves the Gram product, $\phi_w^{-1}(a) \in \pi_n S_{n,k}$ is in $R_{n,k}$. Moreover $|\mathcal{E}|\phi_w^{-1}(a)$ is non-zero since $2(p-p')$ vacancies in (each link of) $\phi_w^{-1}(a)$ were simply replaced by arcs. Note that $\pi_n|\mathcal{E}|\phi_w^{-1}(a) = |\mathcal{E}|\phi_w^{-1}(a)$. For the previous example, this equality is simply

\[
\begin{align*}
\pi_n|\mathcal{E}|\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} \\
\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} \\
\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array}.
\end{align*}
\]

By the previous lemma $|\mathcal{E}|\phi_w^{-1}(a)$ is therefore in $B$. So is the element $|\mathcal{E}|\phi_w^{-1}(a)$, since $B$ is a submodule. The expression for this element can be simplified:

\[
|\mathcal{E}|\phi_w^{-1}(a) = (y,y)_{n,k+2p'}\pi_n\phi_w^{-1}(a) = |\mathcal{E}|\phi_w^{-1}(a) = \phi_w^{-1}(a).
\]

The first equality follows from lemma 4.13, the second by the choice of $y$ and $y'$ and the third because $\pi_n = |\mathcal{E}|$ is the identity on $\text{im} \phi_w^{-1}$. Again, for the above example,

\[
\begin{align*}
|\mathcal{E}|\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} \\
\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array} \\
\phi_w^{-1}(a) &= \begin{array}{|c|c|c|} \hline \\
& & \\
& & \\
& & \\
\end{array}.
\end{align*}
\]

Hence $\phi_w^{-1}(a)$ is a non-zero vector in $B \subset R_{n,k}$ and $\pi_n B \neq \{0\}$ with $w \in X_{n,k+2p'}$ as desired.

Suppose now $p' > p$, and let $z' \in S_{n,k+2p'}$ be the $n$-link diagram having $p' - p$ bubbles in the first positions followed by $k+2p$ defects, and then by $(n-k-2p')$ vacancies for the remaining positions. If $x \in B$ and $\pi_n x \neq 0$, the $|\mathcal{E}|x$ is in $B$, because $B$ is a module, and non-zero (again some vacancies were replaced by bubbles). Then the link diagram
Proposition 4.14. The radical $R_{n,k}$ is either irreducible or $\{0\}$.

\begin{proof} Suppose that the radical is nontrivial and let $B \neq \{0\}$ be a submodule of $R_{n,k}$. There must be an integer $\ell$ and a link diagram $z \in X_{n,k+2\ell}$ such that $\pi_{\ell}B \neq \{0\}$. By Lemma 4.13, there are then $w \in X_{n,k+2\ell}$ with the property $\pi_{\ell}B \neq \{0\}$ for all integers $p$ in the range $0 < p \leq \left\lfloor \frac{(n-k)}{2} \right\rfloor$ and such that the radical $R_{k+2p,k}$ is non-trivial. By Lemma 4.12, this implies that the submodule $B$ must be at least of dimension

$$\sum_{0 \leq p \leq \left\lfloor \frac{(n-k)}{2} \right\rfloor} \binom{n}{k+2p} \dim R_{k+2p,k}$$

because there are $\binom{n}{k+2p}$ distinct link diagrams in $X_{n,k+2p}$. Note that the $p$s such that $R_{k+2p,k}$ are trivial were included in the sum, but then $\dim R_{k+2p,k} = 0$. This is the case for $p = 0$. By Corollary 4.8, this is the dimension of the radical $R_{n,k}$ which must then be irreducible. \end{proof}

4.3. Symmetric pairs of standard modules. Let $q$ be a root of unity other than $\pm 1$ and let $l$ be the smallest integer such that $q^{2l} = 1$. Then $l \geq 2$. Two non-negative integers $k$ and $k'$ form a symmetric pair if they satisfy

$$(k + k')/2 + 1 \equiv 0 \mod l \quad \text{and} \quad 0 < |k - k'|/2 < l. \quad (27)$$

The Bratteli diagram in Figure 4.3 explains the meaning of these two conditions. The first equations implies that the average of $k$ and $k'$ falls on a critical line, that is, $k_c = (k + k')/2$ satisfies $q^{2(k_c+1)} = 1$. (On the Bratteli diagram with $l = 4$, the critical lines are through $k = 3, 7, \ldots$) Consider now the closest critical lines to that going through $k_c$. (If the latter one is the leftmost, the critical line to its left would be one passing through $k = -1$.) The second condition above means that the integers $k$ and $k'$ are strictly between these two closest critical lines. Hence $k$ and $k'$ fall symmetrically on each side of the line $k_c = (k + k')/2$. A pair of standard modules $S_{n,k}$ and $S_{n,k'}$ is also said symmetric if $k$ and $k'$ form a symmetric pair. Note, finally, that there are always a pair of positive integers $a, b$ with $0 < b < l$ such that $k$ and $k'$ are $k_{\pm} = al - 1 \pm b$. When the pair $S_{n,k_{+}}$ and $S_{n,k_{-}}$ is symmetric, then the eigenvalues of $F_n$ on these modules coincide (see Proposition 3.3).

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
(1,1) & (2,0) & (2,2) & | & \quad (3,3) \\
(4,0) & (4,2) & (4,4) & | & \quad (5,5) \\
(6,0) & (6,2) & (6,4) & (6,6) & | & \quad (7,7) \\
(8,0) & (8,2) & (8,4) & (8,6) & (8,8) & \\
\hline
\end{tabular}
\caption{The indices $(n,k)$ of even standard modules are presented on a Bratteli diagram. Each line corresponds to a given $n$ and therefore a given $dTL_n$. The vertical lines are the critical lines when $l = 4$. Symmetric pairs for $n = 8$ are joined by dashed lines.}
\end{figure}
The main result of this subsection relates the radical and the irreducible quotient of a pair of symmetric standard modules.

**Lemma 4.15.** Let $q$ be a root of unity other than $\pm 1$ and $k$ be critical for this $q$. Let $\psi$ be the endomorphism of $S_{n-1,k}/S_{n,k}$ defined by left multiplication by the central element $F_n - \delta_{k-1} \cdot \text{id}$. Then $\psi$ is non-zero.

**Proof.** The proof builds on that for the Temperley-Lieb algebra. To make contact with this previous result, we need to choose a link diagram $z \in X_{n-1,k}$. The actual one is irrelevant, but the explanations are simpler when $z$ has all its vacancies at the top and its defects at the bottom positions. The vector $v = \pi_{\psi_1(z)} \otimes dTL_{n-1} \cdot z$ is then an element of the basis constructed in subsection 3.3 for the induced module $S_{n-1,k}^\uparrow$. We claim that $(F_n - \delta_{k-1} \cdot \text{id})v$ is non-zero. Note first that

$$(F_n - \delta_{k-1} \cdot \text{id})v = (F_n - \delta_{k-1} \cdot \text{id})\pi_{\psi_1(z)}v = (\pi_{\psi_1(z)}(F_n - \delta_{k-1} \cdot \text{id})\pi_{\psi_1(z)})v$$

since $\pi_{\psi_1(z)}\pi_{\psi_1(z)} = \pi_{\psi_1(z)}$. Due to proposition 3.2. $\pi_{\psi_1(z)}F_n\pi_{\psi_1(z)}$ corresponds to the action of $F_{k+1}$ on the bottom $k+1$ positions, the top ones being forced to be vacancies. The fact that these vacancies do not play any role is useful. Recall that $T_{\pi_{\psi_1(z)}} = \pi_{\psi_1(z)}dTL_n \pi_{\psi_1(z)}$ is a subalgebra isomorphic to $TL_{k+1}$. Similarly $T_z = \pi_{\psi_1(z)}dTL_{n-1} \pi_{\psi_1(z)} \simeq TL_k$ and $\pi_{\psi_1(z)}S_{n-1,k}$ is a $TL_k$-module (with the restricted action) isomorphic to $V_{k,k}$. With these isomorphisms, the computation of $(F_n - \delta_{k-1} \cdot \text{id})v$ amounts to computing the action of $(F_k - \delta_{k-1} \cdot \text{id})_{\psi_1(z)}$ on $idTL_{k+1} \otimes TL_{n-1} z_k \in V_{k,k}^\uparrow$ where $z_k$ is the $k$-link state with $k$ defects. Note that the criterion for criticality does not depend on $n$ and the $TL_k$-module $V_{k,k}$ also sits on the critical line. Proposition 3.2 then states readily that $(F_k - \delta_{k-1} \cdot \text{id})_{\psi_1(z)}$ is non-zero. One can then conclude that $(F_n - \delta_{k-1} \cdot \text{id})\pi_{\psi_1(z)} \otimes dTL_{n-1} \cdot z$ is non-zero since $T_{\pi_{\psi_1(z)}}v$ and $V_{k,k}^\uparrow$ are isomorphic as modules over the subalgebra $T_{\pi_{\psi_1(z)}} \subset dTL_n$. Clearly the vector $v \in S_{n,k}^\uparrow$ lies in the submodule of $S_{n-1,k}^\uparrow$ that has the parity of $S_{n,k+1}$ and thus projects onto a non-zero vector in $S_{n-1,k}^\uparrow/S_{n,k}$. \hfill \Box

**Proposition 4.16.** Let $q$ be a root of unity other than $\pm 1$ and let $S_{n,k_-}$ and $S_{n,k_+}$ be two standard $dTL_n$-modules where $k_-$ and $k_+$ form a symmetric pair ($k_- < k_+$). Then

$$R_{n,k_-} \simeq I_{n,k_+}.$$  

**Proof.** Let $k = (k_- + k_+)/2$ be the critical $k$ between $k_-$ and $k_+$ and let $b$ such that $k_{\pm} = k \pm b$. If $b = 1$, the short sequence

$$0 \to S_{n,k-1} \xrightarrow{\alpha} S_{n-1,k}^\uparrow/S_{n,k} \xrightarrow{\beta} S_{n,k+1} \to 0$$

is exact by corollary 3.11. Let $\psi$ be the endomorphisms obtained by left multiplying a $dTL_n$-module by $(F_n - \delta_{k-1} \cdot \text{id})$. By the previous lemma, this is a non-zero endomorphism on $S_{n-1,k}^\uparrow/S_{n,k}$. But it does act as zero on $S_{n,k-1}$ and therefore $\ker \alpha \subset \ker \psi$. It also acts as zero on $S_{n,k+1}$ and $\ker \psi \subset \ker \gamma = \ker \alpha$. Since $\gamma$ is surjective, for any $w \in S_{n,k+1}$, there is a $v \in S_{n-1,k}^\uparrow/S_{n,k}$ such that $\gamma(v) = w$. If $v' \in S_{n-1,k}^\uparrow/S_{n,k}$ is another vector satisfying $\gamma(v') = w$, then $v - v' \in \ker \gamma \subset \ker \psi$. It thus follows that the map $w \mapsto \psi(v)$ is well-defined. It can be seen to be a module homomorphism $\Psi : S_{n,k+1} \to \ker \alpha \subset S_{n-1,k}^\uparrow/S_{n,k}$. Since $\alpha$ is injective, it has an inverse on $\ker \psi \subset \ker \alpha$. Therefore $\alpha^{-1} \circ \Psi : S_{n,k+1} \to S_{n,k-1}$ is a non-zero homomorphism and $\text{Hom}_{dTL_n}(S_{n,k+1}, S_{n,k-1}) \neq 0$.\hfill \Box
Let $b$ be an integer such that $1 < b < l$ where $l$ is the smallest integer such that $q^{2l} = 1$. Then

$$\text{Hom}_{\text{TL}_{n+b}}(S_{n+b,k+b}, S_{n+b,k-b}) = \text{Hom}_{\text{TL}_{n+b}}(S_{n+b,k+b} \oplus S_{n+b,k+b-1} \oplus S_{n+b,k+b-2}, S_{n+b,k-b})$$

$$= \text{Hom}_{\text{TL}_{n+b}}(S_{n+b-1,k+b-1} \oplus S_{n+b,k-b})$$

$$= \text{Hom}_{\text{TL}_{n+b}}(S_{n+b-1,k+b-1} \oplus S_{n+b-1,k+b-2} \oplus S_{n+b-1,k+b-1} \oplus S_{n+b-1,k+b-1})$$

$$= \text{Hom}_{\text{TL}_{n+b}}(S_{n+b-1,k+b-1} \oplus S_{n+b-1,k-b}) .$$

The third equality is due to Frobenius reciprocity theorem and the second and the fourth follow from corollary 3.12 and the fact that neither $(k + b - 1)$ nor $(k - b)$ are critical. The first equality rests upon two slightly different observations. Lemma B.4 shows that act upon the two modules $S_{n+b,k+b-2}$ and $S_{n+b,k-b}$ with distinct eigenvalues and therefore any homomorphism between them is zero. Similarly, there cannot be a homomorphism between two standard modules of distinct parities and $\text{Hom}_{\text{TL}_{n+b}}(S_{n+b,k+b-1}, S_{n+b,k-b}) = 0$. The last equality follows from the same two observations. Therefore

$$\text{Hom}_{\text{TL}_{n+b}}(S_{n+b,k+b}, S_{n+b,k-b}) = \text{Hom}_{\text{TL}_{n}}(S_{n,k+1}, S_{n,k-1}) \neq \{0\} . \quad (30)$$

Let $k_-$ and $k_+$ be a symmetric pair and $f : S_{n,k_-} \to S_{n,k_+}$ a non-zero homomorphism. Its kernel is a proper submodule of $S_{n,k_+}$, and, since the radical of a standard module is a maximal and irreducible submodule, $\ker f$ is either $R_{n,k_+}$ or $\{0\}$. Suppose that it is $\{0\}$. Because $S_{n,k_+}$ and $S_{n,k_-}$ are non-isomorphic (proposition 3.3), then $S_{n,k_+}$ must be isomorphic to $R_{n,k_+}$ and therefore irreducible, that is $R_{n,k_+} = \{0\}$. Hence $\ker f = R_{n,k_-}$, even when $\ker f = \{0\}$. Similarly $\text{im} f$ is not zero and is therefore either $S_{n,k_-}$ or $R_{n,k_-}$, again by the maximality and irreducibility of the radical. However it cannot be $S_{n,k_-}$, as then $S_{n,k_-} \simeq S_{n,k_+} / \ker f$ which would contradict lemma 4.4. So $\text{im} f = R_{n,k_-}$ and thus $I_{n,k_-} = S_{n,k_+} / R_{n,k_-} \simeq R_{n,k_-}$. 

Suppose that $(k_-, k_+)$ is a symmetric pair with $k_- \leq n < k_+$. Then the radical $R_{n,k_-}$ is trivial and $S_{n,k_-}$ irreducible. This can be proved either by extending the previous proof (allowing $S_{n,j} = \{0\}$ whenever $j > n$), or by a careful analysis of the zeroes of $\det dG_{n,k}$ (corollary 4.8), or by checking with (75) which radicals of $\text{TL}_{n}$ occurring in corollary 4.8 are non-trivial.

**Corollary 4.17.** If $f \in \text{Hom}(S_{n,k}, S_{n,k})$, then $f$ is an isomorphism or zero.

**Proof.** If $S_{n,k}$ is irreducible, the result is trivial. If $S_{n,k}$ is reducible, it is then part of a symmetric pair with $k_- = k$. Choose a non-zero element $f \in \text{Hom}(S_{n,k}, S_{n,k})$. If $\ker f$ is non-zero, then $\ker f = \text{im} f = R_{n,k}$, since $R_{n,k}$ is the only non-trivial proper submodule. Then the first isomorphism theorem says $S_{n,k} / R_{n,k_-} \simeq S_{n,k_-} \simeq I_{n,k_-} = S_{n,k_+} / R_{n,k_-}$, contradicting lemma 4.4. So $f$ must be an isomorphism. 

A similar argument gives the following corollary.

**Corollary 4.18.** If $S_{n,k}$ is reducible, then

$$\text{Hom}(I_{n,k}, S_{n,k}) \simeq \text{Hom}(S_{n,k}, R_{n,k}) \simeq 0 . \quad (31)$$
4.4. Restriction and induction of irreducible modules. We complete the analysis of the restriction and induction of the fundamental modules by giving those of the radicals and the irreducible quotients. The results are simple and elegant and will be needed in the next section. Their proofs are straightforward but somewhat long and repetitive.

Proposition 4.19. If \( R_{n+1,k} \neq 0 \), then

\[
R_{n+1,k} \downarrow \simeq R_{n,k-1} \oplus R_{n,k} \oplus \begin{cases} S_{n,k+1} & \text{if } k+1 \text{ is critical} \\ R_{n,k+1} & \text{otherwise} \end{cases}
\]  

(32)

Some of the direct summands may be trivial.

Proof. If \( R_{n+1,k} \neq 0 \), proposition 4.16 gives the exactness of the following short sequence of \( dTL_{n+1} \)-modules:

\[
0 \rightarrow R_{n+1,k} \rightarrow S_{n+1,k} \rightarrow l_{n+1,k} \rightarrow 0
\]  

(33)

and therefore of its restriction to \( dTL_n \):

\[
0 \rightarrow R_{n+1,k} \downarrow \rightarrow S_{n+1,k} \downarrow \rightarrow l_{n+1,k} \downarrow \rightarrow 0.
\]  

(34)

It follows that \( R_{n+1,k} \downarrow \) is isomorphic to a submodule of \( S_{n+1,k} \downarrow \) which splits in a direct sum of three modules which are distinct eigenspaces of \( F_n \) of different parity: \( R_{n+1,k} \downarrow \simeq R_0 \oplus R_- \oplus R_+ \) where \( R_0 \) and the \( R_\pm \) are submodules of \( S_{n,k} \) and \( S_{n,k;1} \) respectively. One or more of the \( R_\pm \) may vanish. (See propositions 3.9 and B.3 and the beginning of subsection 3.1 where a similar argument is detailed.)

We first study \( R_0 \). Consider the (restriction of) the injective homomorphism \( \phi : S_{n,k} \rightarrow S_{n+1,k} \downarrow \) introduced in the proof of proposition 3.8 that simply adds a vacancy at the bottom of every link diagram. Let \( u \in S_{n+1,k} \downarrow \) and write it as \( u' + v' \) where all terms in \( u' \) have a vacancy at the position \( n+1 \) while those in \( v' \) do not. Then, if \( r \) is in the radical \( R_{n,k} \subset S_{n,k} \)

\[
\langle \phi(r), u \rangle_{n+1,k} = \langle \phi(r), u' \rangle_{n+1,k} = (r, \phi^{-1}(u'))_{n,k} = 0.
\]  

(35)

The image \( \phi(R_{n,k}) \) is thus in \( R_{n+1,k} \downarrow \). Since \( R_0 \) is the only summand of \( R_{n+1,k} \downarrow \) having the parity of \( R_{n,k} \), it must contain a submodule isomorphic to \( R_{n,k} \).

We turn to the other two submodules \( R_- \) and \( R_+ \). Corollary 3.9 has established the exactness of the short sequence

\[
0 \rightarrow S_{n+1,k-1} \oplus S_{n+1,k} \rightarrow S_{n,k} \uparrow \rightarrow S_{n+1,k+1} \rightarrow 0
\]  

(36)

which implies the exactness of (see proposition D.4)

\[
0 \rightarrow \text{Hom}(S_{n+1,k+1}, R_{n+1,k}) \rightarrow \text{Hom}(S_{n,k} \uparrow, R_{n+1,k}) \rightarrow \text{Hom}(S_{n+1,k-1} \oplus S_{n+1,k}, R_{n+1,k}).
\]  

(37)

Corollary 4.18 the linearity of Hom and the parity of the modules involved lead to

\[
\text{Hom}(S_{n+1,k-1} \oplus S_{n+1,k}, R_{n+1,k}) = 0 \quad \text{and} \quad \text{Hom}(S_{n+1,k+1}, R_{n+1,k}) = 0.
\]  

(38)

Frobenius theorem then gives

\[
\text{Hom}(S_{n,k} \uparrow, R_{n+1,k}) \simeq \text{Hom}(S_{n,k}, R_{n+1,k} \downarrow) \simeq 0.
\]  

(39)

Therefore \( R_{n+1,k} \downarrow \) has no (non-trivial) submodule isomorphic to a quotient of \( S_{n,k} \). This proves that \( R_0 \) is isomorphic to \( R_{n,k} \).

Similarly the short exact sequences

\[
0 \rightarrow S_{n+1,k;1} \rightarrow S_{n,k;1} \rightarrow S_{n+1,k;1+1} \rightarrow 0
\]  

(40)
Note that 

\[ \text{proves that} \]

\[ \text{thus} \]

where the second equality follows from either proposition 4.4 or corollary 4.18. The argument now splits according to whether \( k + 1 \) is critical or not.

If \( k + 1 \) is not critical, the central element \( F_{n+1} \) takes distinct eigenvalues on \( S_{n+1,k+2} \) and \( S_{n+1,k} \), which forces \( \text{Hom}(S_{n+1,k+2}, R_{n+1,k}) = 0 \). Corollary 4.18 also gives \( \text{Hom}(S_{n+1,k}, R_{n+1,k}) = 0 \), so Frobenius theorem leads to

\[ \text{Hom}(S_{n,k+1,1} \uparrow, R_{n+1,k}) \cong \text{Hom}(S_{n,k+1}, R_{n+1,k}) \cong 0. \] (43)

Therefore, the \( S_{n,k+1} \) are not isomorphic to submodules of \( R_{n+1,k} \), and in particular \( R_{\pm} \neq S_{n,k+1} \).

If \( k + 1 \) is critical, proposition 4.16 gives \( R_{n,k} \cong I_{n,k+2} \) so that

\[ \text{Hom}(S_{n+1,k+2}, R_{n+1,k}) \cong \text{Hom}(S_{n+1,k+1} \uparrow, R_{n+1,k}) \cong \text{Hom}(S_{n+1,k}, R_{n+1,k}) \neq 0 \] (44)

by the exactness of (41). Since \( S_{n,k+1} \) is irreducible when \( k + 1 \) is critical, the restriction \( R_{n+1,k} \) has a submodule isomorphic to \( S_{n,k+1} \). But since the parity of \( S_{n,k} \) and \( S_{n,k+1} \) are different, this submodule cannot be in \( R_{0} \). Again \( F_{n} \) takes distinct eigenvalues on \( S_{n,k+1} \) and \( S_{n,k-1} \) so that \( S_{n,k+1} \) cannot be a submodule of \( R_{-} \). (This statement remains true in the special case when \( k - 1 \) is also critical. Then \( l = 2 \), \( q = \pm i \) and \( \delta_{k+1} = -\delta_{k-1} \).) This proves that \( S_{n,k+1} \) must be a submodule of \( R_{+} \), which is itself a submodule of \( S_{n,k+1} \) and thus \( S_{n,k+1} \cong R_{+} \).

So far, we have narrowed down the possible submodules of \( R_{n+1,k} \) to

\[ R_{n+1,k} \cong R_{n,k} \oplus \{ 0 \} \] or \( R_{n,k-1} \oplus \left\{ \begin{array}{ll} S_{n,k+1} & \text{if } k + 1 \text{ is critical} \\ 0 & \text{otherwise} \end{array} \right. \} \] (45)

Equation (79) and proposition C.1 give a formula for the dimension of \( R_{n+1,k} \). The proof ends with a comparison of this dimension with the above possibilities. \( \square \)

Note that equation (34) gives \( I_{n+1,k} \cong S_{n+1,k} / R_{n+1,k} \). Combining this observation with the preceding proposition then gives the following corollary.

**Corollary 4.20.** If \( R_{n+1,k} \neq 0 \) then

\[ I_{n+1,k} \cong I_{n,k-1} \oplus I_{n,k+1} \oplus \left\{ \begin{array}{ll} 0 & \text{if } k + 1 \text{ is critical} \\ 0 & \text{otherwise} \end{array} \right. \] (46)

Now that we have formulas for the restriction of the irreducible modules, we can use them to prove formulas for their induction.

**Proposition 4.21.** If \( R_{n-1,k} \neq 0 \) then

\[ I_{n-1,k} \cong I_{n-1,k} \oplus I_{n,k+1} \oplus \left\{ \begin{array}{ll} 0 & \text{if } k + 1 \text{ is critical} \\ 0 & \text{otherwise} \end{array} \right. \] (47)

**Proof.** The argument is similar to that of proposition 4.19 and uses systematically Frobenius theorem, the parity of the modules and the eigenspaces of the central element \( F_n \) (or \( F_{n-1} \)). If \( R_{n-1,k} \neq 0 \), the exactness of

\[ 0 \to R_{n-1,k} \to S_{n-1,k} \to I_{n-1,k} \to 0. \] (48)
implies the exactness of the sequence of $dT_{n+1}$-modules:
\[ R_{n-1,k} \rightarrow S_{n-1,k} \rightarrow l_{n-1,k} \rightarrow 0. \] 
(49)

Since $S_{n-1,k}$ splits in a direct sum of three modules of distinct parities or on which $F_n$ has different eigenvalues, the module $l_{n-1,k}$ splits accordingly into $L_- \oplus L_0 \oplus L_+$ where $L_0$ and the $L_{\pm}$ are quotients of $S_{n,k}$ and $S_{n,k \pm 1}$ respectively.

We first study $L_0$. Corollary 4.18 gives
\[ \text{Hom} (l_{n-1,k}, l_{n,k}) \simeq \text{Hom} (l_{n-1,k}, l_{n-1,k}) \neq 0. \] 
(50)

Therefore $L_0$, the only submodule of $l_{n-1,k}$ of the parity of $l_{n,k}$, is non-trivial. Moreover proposition 4.18 gives
\[ \text{Hom} (l_{n-1,k}, S_{n,k}) \simeq \text{Hom} (l_{n-1,k}, S_{n-1,k} \oplus S_{n-1,k} \oplus S_{n-1,k+1}) \simeq 0. \] 
(51)

Hence $L_0$ is non-trivial, distinct from $S_{n,k}$ and must be isomorphic to $l_{n,k}$.

We now turn to $L_-$. If $k-1$ is not critical, corollary 4.20 shows again that $\text{Hom} (l_{n-1,k}, l_{n,k-1})$ is non-trivial and $l_{n,k-1}$ must be isomorphic to a quotient of $L_-$. The short exact sequence
\[ 0 \rightarrow S_{n-1,k-2} \oplus S_{n-1,k-1} \rightarrow S_{n,k-1} \rightarrow S_{n-1,k} \rightarrow 0 \] 
(52)
gives rise to the exactness of
\[ 0 \rightarrow \text{Hom} (l_{n-1,k}, S_{n-1,k-2} \oplus S_{n-1,k-1}) \rightarrow \text{Hom} (l_{n-1,k}, S_{n,k-1}) \rightarrow \text{Hom} (l_{n-1,k}, S_{n-1,k}). \] 
(53)

Corollary 4.18 gives $\text{Hom} (l_{n-1,k}, S_{n,k}) \simeq 0$ and therefore
\[ \text{Hom} (l_{n-1,k}, S_{n,k-1}) \simeq 0 \] 
(54)

since the three eigenvalues $\delta_{k-2}, \delta_{k-1}$, and $\delta_k$ of $F_{n-1}$ are distinct if both $k-1$ and $k$ are non-critical. The module $S_{n,k-1}$ is not a quotient of $l_{n-1,k}$ and $L_-$ must therefore be distinct of $S_{n-1,k}$. Hence $L_-$ must be isomorphic to $l_{n,k-1}$. Finally, if $k-1$ is critical, then $l_{n-1,k} \simeq R_{n-1,k-2}$ by proposition 4.18 and $\text{Hom} (l_{n-1,k}, S_{n,k-1}) \neq 0$. Since $S_{n,k-1}$ is then irreducible, $L_- \simeq l_{n,k-1}$.

It remains to study $L_+$. The exact sequence
\[ 0 \rightarrow \text{Hom} (l_{n-1,k}, S_{n,k} \oplus S_{n,k+1}) \rightarrow \text{Hom} (l_{n-1,k}, S_{n,k+1}) \rightarrow \text{Hom} (l_{n-1,k}, S_{n-1,k+2}) \] 
(55)

follows from the exact sequence for $S_{n,k+1}$. The two outer Hom spaces are trivial because of corollary 4.18 and lemma 4.4. This proves that $\text{Hom} (l_{n-1,k}, S_{n,k+1}) \simeq 0$ and that no submodules of $S_{n,k+1}$ are isomorphic to a quotient of $l_{n-1,k}$ and in particular that $L_+ \neq S_{n,k+1}$. Now, if $k+1$ is not critical, then $\text{Hom} (l_{n-1,k}, S_{n,k+1})$ is non-trivial by corollary 4.20 and $L_+$ must therefore be $l_{n,k+1}$. If $k+1$ is critical, then $S_{n,k+1} \simeq l_{n,k+1}$ is irreducible and, since $L_+ \neq S_{n,k+1}$, the submodule $L_+$ must be trivial.

The use of symmetric pairs and proposition 4.16 give the last result of this section.

**Corollary 4.22.** If $k_c$ is critical, $0 < i < l$ and $R_{n-1,k_c+i} \neq 0$, then
\[ R_{n-1,k_c-i} \simeq \begin{cases} 0 & \text{if } i = l - 1 \\ R_{n,k_c-i-1} & \text{if } i = 1 \\ \text{otherwise} & \text{otherwise} \end{cases}. \] 
(56)

Some of these radicals may vanish.

Note that if $R_{n-1,k_c+i} = 0$ we have $R_{n-1,k_c-i} \simeq l_{n-1,k_c+i} \simeq S_{n-1,k_c+i}$. We therefore find simply $R_{n-1,k_c-i} \simeq S_{n-1,k_c+i}$. 

5. THE STRUCTURE OF \textit{dTL}_n AT A ROOT OF UNITY

In this section, \( q \) is a root of unity and \( l \) is the smallest positive integer such that \( q^{2l} = 1 \).

When \( q \) is not a root of unity, every standard module of the dilute Temperley-Lieb algebra is irreducible. The algebra is then semisimple and the standard modules form a complete set of irreducible modules (theorem 4.11). However, when \( q \) is a root of unity, some of them will be reducible, yet indecomposable. That is, if \( q \) is a root of unity, the algebra \( \text{dTL}_n \) is not always semisimple. We describe here the structure of the algebra when \( q \) is a root of unity and construct a complete set of projective indecomposable modules.

5.1. The modules \( P^i_{n,k_c} \). In this subsection we introduce new modules, denoted by \( P^i_{n,k_c} \). The principal indecomposable modules of \( \text{dTL}_n \), when \( q \) is a root of unity, will turn out to be either standard modules \( S_{n,k_c} \) or some of these new ones. The modules \( P^i_{n,k_c} \) are defined recursively by (58) and their crucial properties are described in propositions and corollaries 5.4, 5.8 and 5.9. These will show that, if \( k_c \) is critical and \( 1 \leq i \leq l \), then the modules \( P^i_{n,k_c} \) are projective and belong to the following short exact sequence:

\[
0 \rightarrow S_{n,k_c-i} \rightarrow P^i_{n,k_c} \rightarrow S_{n,k_c+i} \rightarrow 0
\]

which splits only when \( i = l \). Moreover they are indecomposable and tied among themselves by induction:

\[
P^i_{n,k_c} \uparrow \simeq P^{i-1}_{n+1,k_c} \oplus P^i_{n+1,k_c} \oplus P^{i+1}_{n+1,k_c}.
\]

(The concepts of projective and flat modules are equivalent for finite-dimensional algebras like \( \text{dTL}_n \). They are reviewed in appendix D and used throughout.)

In the present subsection, \( k_c \) and \( k'_c \) will denote critical integers \( (q^{2(k_c+1)} = q^{2(k'_c+1)} = 1) \) and the integer \( i \) \((i')\) is chosen such that \( 0 \leq k_c - i < k_c + i \leq n \) \((0 \leq k'_c - i' < k'_c + i' \leq n, \text{resp.})\).

Let \( M \) be a \( \text{dTL}_n \)-module and \( \Lambda = \{ \lambda_i \}_{i \in I} \) be the set of eigenvalues of the central element \( F_n \) acting on \( M \). The generalized eigenspaces \( M_{\lambda} = \{ m \in M \mid (F_n - \lambda \cdot \text{id}_M) m = 0 \} \subseteq M \) for \( \lambda \in \Lambda \) are submodules and \( M = \bigoplus_{\lambda \in \Lambda} M_{\lambda} \). Let \( \hat{\rho}_\lambda \in \text{End}(M) \) be the projector onto the subspace \( M_{\lambda} \) : \( \hat{\rho}_\lambda \hat{\rho}_\mu = \delta_{\lambda,\mu} \hat{\rho}_\lambda \) and \( \sum_{\lambda \in \Lambda} \hat{\rho}_\lambda = \text{id}_M \). Here \( \delta_{\lambda,\mu} \) is the Kronecker delta and is to be distinguished from the eigenvalues \( \delta_i \) of \( F_n \) that bear a single latin index. If \( \mu \) is not an eigenvalue of \( F_n \) on \( M \), we define \( \hat{\rho}_\mu = 0 \). If \( N \) is another module, \( \text{Hom}(\hat{\rho}_\lambda M, \hat{\rho}_\lambda N) \) is zero unless \( \lambda_i = \lambda_j \). (We use somewhat abusively the same letter “\( \hat{\rho} \)” for the projectors on \( M \) and those on \( N \).) In particular \( \text{Hom}(\hat{\rho}_\delta M, S_{n,i}) = 0 \) if \( \delta_i \neq \delta_j \) where \( \delta_j \) is the eigenvalue of \( F_n \) on \( S_{n,j} \) (see lemma B.3).

Parity offers a further refinement of the decomposition into submodules afforded by the \( \hat{\rho}_s \). For some roots of unity, two distinct integers \( i, j \) with \( k_c < i < j < k_c + l \) for some critical \( k_c \) may label standard modules with the same eigenvalues of \( F_n \) : \( \delta_i = \delta_j \). When this happens, \( i \) and \( j \) are of distinct parity (see lemma B.4) and one may use \( \text{eid} \cdot \hat{\rho}_\delta \) and \( \text{oid} \cdot \hat{\rho}_\delta \) instead of simply \( \hat{\rho}_\delta \). We define the projectors

\[
p^j_i = \begin{cases} 
\text{eid} \cdot \hat{\rho}_\delta, & \text{if } j \text{ is even}, \\
\text{oid} \cdot \hat{\rho}_\delta, & \text{if } j \text{ is odd}.
\end{cases}
\]

(57)
Let $k_c$ be critical for $q$, that is $q^{2(k_c+1)} = 1$. Let $P_{n,k_c}^i$ be the $dTL_n$-modules defined through the following recurrence relation:

\[ \begin{align*}
P_{n,k_c}^0 &= S_{n,k_c} \oplus S_{n,k_c} \\
P_{n,k_c}^1 &= S_{n-1,k_c} / S_{n,k_c} = p_{k+1}^{n-k-1}(S_{n-1,k_c}) \\
P_{n,k_c}^i &= p_{k+1}^{n-k-i}(S_{n-1,k_c}), \quad 2 \leq i \leq l.
\end{align*} \]

(58)

**Lemma 5.1.** For $0 \leq i \leq l$, the sequence

\[ S_{n,k_c-i} \rightarrow P_{n,k_c}^i \rightarrow S_{n,k_c+i} \rightarrow 0 \]

(59)

is exact.

**Proof.** Since $k_c$ is critical, then $\delta_{k_c-i} = \delta_{k_c+i}$ for all $i$, $1 \leq i \leq l$, by lemma B.4. We proceed by induction on $i$. The case $i = 0$ is trivial ($S_{n,k_c} \rightarrow S_{n,k_c} \oplus S_{n,k_c} \rightarrow S_{n,k_c} \rightarrow 0$) and, for the case $i = 1$, the exactness of

\[ S_{n,k_c-1} \rightarrow P_{n,k_c}^1 \simeq S_{n-1,k_c} / S_{n,k_c} \rightarrow S_{n,k_c+1} \rightarrow 0 \]

follows from corollary 3.11. Suppose now that it is true for $1 \leq i - 1 \leq l - 1$. Since induction is right-exact, induction from $dTL_{n-1}$ to $dTL_n$ gives the exactness of

\[ S_{n-1,k_c-i+1} \rightarrow P_{n-1,k_c}^{i-1} \rightarrow S_{n-1,k_c+i-1} \rightarrow 0. \]

The exactness must hold for each generalized eigenspace of $F_n$ and for each submodule of a given parity separately. The projection by $p_{k_c+i}^{n-k-i}$, together with corollary 3.11 and lemma B.4, gives the exactness of

\[ S_{n,k_c-i} \rightarrow P_{n,k_c}^i \rightarrow S_{n,k_c+i} \rightarrow 0. \]

\[ \square \]

**Lemma 5.2.** If $1 \leq i < l - 1$, $n \geq k_c + i + 1$ and the short sequences

\[ 0 \rightarrow S_{n+1,k_c-i} \overset{\gamma}{\rightarrow} Q \overset{\alpha}{\rightarrow} S_{n+1,k_c+i} \rightarrow 0 \]

(60)

and

\[ 0 \rightarrow l_{n,k_c-(i+1)} \rightarrow P \rightarrow S_{n+1,k_c+i+1} \rightarrow 0 \]

(61)

are exact for $Q$ and $P$ two projective modules, then $S_{n+1,k_c+i}$ is projective.

**Proof.** Since induction is right-exact, the sequence

\[ l_{n,k_c-(i+1)} \rightarrow P \rightarrow S_{n+1,k_c+i+1} \rightarrow 0 \]

(62)

is exact. Since $l_{n,k_c-(i+1)}$ and $S_{n,k_c-(i+1)}$ decomposes into direct sums of three modules distinguished by their parity and the eigenvalues taken by $F_{n+1}$ upon them, the module $P$ must split similarly. By proposition 4.21 the projection through $p_{k_c+i}^{n+1-k-i}$ gives the exact sequence

\[ l_{n+1,k_c-i} \overset{\alpha}{\rightarrow} P' \overset{\beta}{\rightarrow} S_{n+1,k_c+i} \rightarrow 0 \]

(63)

where $P' = p_{k_c+i}^{n+1-k-i} P$. (Note that $P'$ is projective since it is a direct summand of a projective module. See the end of appendix D) Since $l_{n+1,k_c-i}$ is irreducible, $\alpha$ is either zero or injective. If it is zero, then $P' \simeq S_{n+1,k_c+i}$ and the conclusion follows.
Suppose then that $\alpha \neq 0$. The two horizontal rows of the following diagram are now exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S_{n+1,k_i-i} & \gamma & Q & \sigma & S_{n+1,k_i+i} & \longrightarrow & 0 \\
0 & \longrightarrow & l_{n+1,k_i-i} & \alpha & P' & \beta & S_{n+1,k_i+i} & \longrightarrow & 0 \\
\end{array}
$$

Homomorphisms $f$ and $\tilde{f}$, that make the right square commutes, can be constructed because $Q$ and $P'$ are projective and $\sigma$ and $\beta$ surjective. Homomorphisms $g$ and $\bar{g}$, that make the left square commutes, exist since $\sigma f \alpha = 0$ and $\beta \tilde{f} \gamma = 0$. (See lemma 4.4.) By corollary 4.17, the map $g$ must vanish and therefore $f \alpha = 0$. It follows that $\sigma f \tilde{f} = \text{id} \tilde{f} = \sigma$ and $f \tilde{f} \gamma = f \alpha \bar{g} = 0$. The map $f \tilde{f}$ is thus a non-trivial endomorphism of $Q$ that is neither nilpotent nor an isomorphism. There must then exist $N \in \mathbb{N}$ such that

$$Q \simeq \ker (f \tilde{f})^N \oplus \text{im}(f \tilde{f})^N.$$ 

Since $f \tilde{f} \gamma = 0$, the image $\gamma \subset \ker (f \tilde{f})^N$ and, because $S_{n+1,k_i+i}$ is indecomposable, it must coincide with $\sigma \text{(im}(f \tilde{f})^N)$. We therefore conclude that $\tilde{f} S_{n+1,k_i+i}$ is isomorphic to the direct summand $\text{im}(f \tilde{f})^N$ of the projective $Q$ and is therefore itself projective.

To prove flatness of standard modules, right dTL$\alpha_i$-modules will be needed. We shall use the right modules obtained by considering spans of right $n$-link diagrams, introduced in section 3 together with the action defined by gluing $n$-diagrams to their right. To distinguish them from the left modules, we shall denote them by a bar. Note that a bilinear form on the right standard module $\bar{S}_{n,k}$ can be defined identically to that on its left counterpart and, consequently, its maximal submodule is the mirror reflexion of the (left) radical $R_{n,k}$ and therefore irreducible.

**Lemma 5.3.** if $1 \leq i < l$, then $S_{n,k_i+i}$ is not flat.

**Proof.** The module $S_{n,k_i+i}$ will be flat if, for any right dTL$\alpha$-modules $\tilde{A}$ and $\tilde{B}$ and any injective homomorphism $\phi \otimes \text{id} : \tilde{A} \otimes S_{n,k_i+i} \rightarrow \tilde{B} \otimes S_{n,k_i+i}$ is injective (see section D.2 of appendix D). Let $\tilde{A} = l_{n,k_i+i} \otimes \bar{S}_{n,k_i-i} \otimes \tilde{B} = \bar{S}_{n,k_i-i}$. Recall that $l_{n,k_i+i} \simeq R_{n,k_i-i}$ and there is then an injective homomorphism $\phi : A \rightarrow B$. Note that $A \otimes S_{n,k_i+i}$ is non-zero. Indeed, for $\tilde{a} \in \tilde{A}$, there is always an element $u \in S_{n,k_i+i}$ such that $\langle a, u \rangle_{n,k_i+i} = 1$ since the bilinear form is non-degenerate on $l_{n,k_i+i}$. Then $\tilde{a} \otimes u = \tilde{a} [u] \otimes z = \langle a, u \rangle z \otimes z$ for any link diagram $z \in X_{n,k_i+i}$. The element $\tilde{z} \otimes z$ will be zero if and only if there exists $b \in \text{dTL} \alpha$ such that $z = bz$ and $\tilde{z}b = 0$ (see proposition D.10). But if $z = bz$, then $1 = \langle z, z \rangle = \langle z, bz \rangle = \langle \tilde{z}b, z \rangle = (\tilde{z}b, z)$ and $\tilde{z}b$ cannot be zero. Therefore $\tilde{a} \otimes u$ is non-zero. The action of $\phi \otimes \text{id}$ on this non-zero element is

$$\phi \otimes \text{id}(\tilde{a} \otimes u) = \phi(\tilde{a}) \otimes u = \bar{y}[\phi(\tilde{a})] \otimes u = \bar{y} \otimes |\phi(\tilde{a})| u = 0$$

for any link diagram $y$ in $X_{n,k_i-i}$. The second equality follows from lemma D.2 and the last one from the fact that $|\phi(\tilde{a})|u$ is now a multiple of a state with $k_i - i$ defects and is therefore zero in $S_{n,k_i+i}$. The homomorphism $\phi \otimes \text{id}$ is therefore not injective (it is in fact zero) and the module $S_{n,k_i+i}$ is not flat.

**Proposition 5.4.** If $S_{k_i,k_i}$ is projective and $1 \leq i \leq l$, then $P_{n,k_i}^i$ is projective for all $n \geq k_i + i$. Furthermore, the sequence

$$0 \longrightarrow S_{n,k_i-i} \longrightarrow P_{n,k_i}^i \longrightarrow S_{n,k_i+i} \longrightarrow 0$$

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is exact for all \( n \geq k_c + i \) and splits if and only if \( i = 1 \).

**Proof.** The proof proceeds recursively on \( i \), establishing for each \( i \) the result for all \( n \). For \( i = 1 \), the induced module \( S_{n,k_c,1} \) splits into \( S_{n+1,k_c} \oplus P^1_{n+1,k_c} \). Since \( S_{n,k_c} \) is projective, so are the direct summands (see appendix D). This argument can then be repeated for all \( n \geq k_c \): the two summands in \( S_{n,k_c} \) are projective. Therefore \( P^1_{n,k_c} \) is projective for all \( n \geq k_c + 1 \) and satisfies the required exact sequence by corollary 3.11. If the sequence splits for a certain \( n \), then \( S_{n,k_c+1} \) is projective and flat, contradicting lemma 5.3.

Now assume the result for \( 1 \leq i < l - 1 \) and \( n \geq k_c + i \). Since \( P^n_{n,k_c} \) is projective by hypothesis, then so is \( P^{i+1}_{n+1,k_c} \) by definition and \( P^{i+1}_{n,k_c} \) is projective for all \( n \geq k_c + i + 1 \). Proposition 5.3 established that the sequence

\[
S_{n,k_c-i-1} \xrightarrow{f} P^{i+1}_{n,k_c} \rightarrow S_{n,k_c+i+1} \rightarrow 0
\]

is exact. To establish the result, \( \text{ker } f \) must be shown to be \( \{0\} \) or, equivalently, the other possibilities for \( \text{ker } f \) must be ruled out.

If \( i < l - 1 \), then the submodules of \( S_{n,k_c-i-1} \) are \( 0, S_{n,k_c-i-1} \) and \( S_{n,k_c-i-1} \), since the radical of a standard module is the unique maximal proper submodule. If \( \text{ker } f = S_{n,k_c-i-1} \), then \( P^{i+1}_{n,k_c} \simeq S_{n,k_c+i+1} \) and \( S_{n,k_c+i+1} \) would be flat, contradicting lemma 5.3. If \( \text{ker } f = S_{n,k_c-i-1} \) then the two sequences

\[
0 \rightarrow S_{n+1,k_c-i} \rightarrow P^{i+1}_{n,k_c} \rightarrow S_{n+1,k_c+i} \rightarrow 0
\]

are exact, the first because of the induction hypothesis. Lemma 5.2 then gives the flatness of \( S_{n+1,k_c+i} \) which contradicts lemma 5.3. We thus conclude that \( \text{ker } f = 0 \) and that \( P^{i+1}_{n,k_c} \) satisfies the required short exact sequence for all \( n \geq k_c + i + 1 \). Finally the sequence cannot split, as otherwise \( S_{n,k_c+i+1} \) would be flat, contradicting lemma 5.3.

If \( i = l \), then \( S_{n,k_c-l} \) is irreducible and \( \text{ker } f \) is either \( \{0\} \) or \( S_{n,k_c-l} \). Assume the latter. Then \( P^n_{n,k_c} \simeq S_{n,k_c-l} \). If \( k_c = l - 1 \), then \( S_{n,k_c-l} = \{0\} \) by definition and the sequence (66) splits (trivially) with \( P^n_{n,k_c} \simeq S_{n,2l-1} \). If \( k_c > l \), then since \( k_c \geq i > 0 \) and \( k_c + i \) are critical, the two modules \( S_{n,k_c} \) are irreducible and non-isomorphic (proposition 5.3). Therefore \( \text{Hom}(S_{n,k_c+l}, S_{n,k_c+l}) = 0 \). Because \( P^n_{n,k_c} \simeq S_{n,k_c+l} \), this observation leads to

\[
0 = \text{Hom}(S_{n,k_c+l}, S_{n,k_c+l}) = \text{Hom}(P^n_{n,k_c}, S_{n,k_c+l}) = \text{Hom}(P^n_{n,k_c}, \text{Hom}(P^n_{n-1,k_c}, S_{n,k_c+l}))
\]

The definition of \( P^n_{n,k_c} \) gives the first equality. The projector \( P^n_{n,k_c} \) can then be removed because \( \text{Hom}(M, S_{n,k_c+l}) \) can only be non-zero if \( M \) contains a non-trivial submodule in the eigenspace of \( F_n \) with eigenvalue \( \delta_{k_c-l} \). Frobenius reciprocity theorem then gives the second equality. The third follows from proposition 3.10 and the fourth from the definition of \( P^n_{n-1,k_c-l} \).

By lemma 5.1 the sequence \( S_{n-1,k_c-l-1} \rightarrow P^n_{n-1,k_c-l} \rightarrow S_{n-1,k_c-l+1} \rightarrow 0 \) is exact and proposition D.3 gives the exactness of

\[
\text{Hom}(P^{n-1}_{n-1,k_c}, S_{n-1,k_c-l-1}) \rightarrow \text{Hom}(P^{n-1}_{n-1,k_c}, P^n_{n-1,k_c-l-1}) \rightarrow \text{Hom}(P^{n-1}_{n-1,k_c}, S_{n-1,k_c-l+1}) \rightarrow 0
\]
Since $P_{-1, k_i}^{l-1}$ is projective. Since $\text{Hom}(P_{-1, k_i}^{l-1}, P_{-1, k_i}^1)$ is zero by the above equalities, so is $\text{Hom}(P_{-1, k_i}^{l-1}, S_{n-1, k_i-l+1})$. Therefore

$$0 = \text{Hom}(P_{-1, k_i}^{l-1}, S_{n-1, k_i-l+1}) = \text{Hom}(P_{-1, k_i}^{l-2}(P_{n-2, k_i}^{l-1}), S_{n-1, k_i-l+1})$$

$$5 = \text{Hom}(P_{-2, k_i}^{l-2}, S_{n-1, k_i-l+1}) = \text{Hom}(P_{n-2, k_i}^{l-2}, S_{n-2, k_i-l+2}).$$

The fifth equality uses again the definition of the $P_{-1, k_i}^{l-1}$. Frobenius theorem gives the sixth and, since $k_i - l + 1$ is not critical, the last follows from proposition 3.9. Repeating the steps 5, 6 and 7 leads to

$$0 = \text{Hom}(P_{-l-1, k_i}^{l}, S_{n-1, k_i-l+1}) = \cdots = \text{Hom}(P_{n-(l-1), k_i}, S_{n-(l-1), k_i-l+1}) = \cdots$$

$$= \text{Hom}(S_{n-1, k_i-l}, S_{n-1, k_i}).$$

Since $\text{Hom}(S_{n-1, k_i-l}, S_{n-1, k_i})$ contains the identity map, this is a contradiction and the case ker $f = S_{n, k_i-l}$ must be rejected.

It remains to prove that the exact sequence splits when $i = l$. Note now that the above argument established that $\text{Hom}(P_{-1, k_i}^{l-1}, P_{-1, k_i-l}) \neq 0$. Therefore

$$0 \neq \text{Hom}(P_{-1, k_i}^{l-1}, P_{-1, k_i-l}) = \text{Hom}(P_{-1, k_i}^{l-1}, P_{-1, k_i-l} S_{n, k_i-l+1})$$

$$= \text{Hom}(P_{-1, k_i}^{l-1}, P_{n-1, k_i}^{l+1} S_{n, k_i-l}) = \text{Hom}(P_{n, k_i}^{l+1}, S_{n, k_i-l}).$$

by the same arguments as above. Since $S_{n, k_i-l}$ is irreducible, there must be a surjective homomorphism from $P_{n, k_i-l}$ to $S_{n, k_i-l}$. Since $S_{n, k_i-l}$ and $S_{n, k_i-l+1}$ are not isomorphic, the module $S_{n, k_i-l}$ cannot be inside the kernel of this map. Corollary 4.17 and proposition D.2 completes the proof. □

**Corollary 5.5.** If $S_{l-1, l-1}$ is projective, then $P_{n, k_i}^{l}$ is projective for all $n \geq k_c + i$ and all $k_c$.

**Proof.** Simply use the fact that $P_{n, k_i}^{l} \simeq S_{n, k_i-l} \oplus S_{n, k_i}$ with the preceding proposition. □

**Corollary 5.6.** If $1 \leq i, i' < l$ and $S_{k_i, k_i'}, S_{k_i', k_i}$ are both projective, then

$$P_{n, k_i}^{l} \simeq P_{n, k_i'}^{l} \iff i = i' \text{ and } k_c = k_c'.$$

(67)

**Proof.** The projectivity of $P_{n, k_i}^{l}$ and $P_{n, k_i'}^{l}$ follows from proposition 5.4 and the hypothesis. Now choose $k_c - i \leq k_i' - i'$ and assume that $P_{n, k_i}^{l} \xrightarrow{f} P_{n, k_i'}^{l}$ is an isomorphism. The following diagram, without the dashed arrows, describes the situation

$$\begin{array}{c}
0 & \longrightarrow & S_{n, k_i-l} & \longrightarrow & P_{n, k_i}^{l} & \longrightarrow & S_{n, k_i+l} & \longrightarrow & 0 \\
\downarrow g_i & & \downarrow f & & \downarrow h_i & & \downarrow & & \downarrow & \\
0 & \longrightarrow & S_{n, k_i'-l'} & \longrightarrow & P_{n, k_i'}^{l} & \longrightarrow & S_{n, k_i'+l'} & \longrightarrow & 0
\end{array}$$

The two rows are exact by proposition 5.4. Since $k_c - i \leq k_i' - i' < k_i' + i'$ and $S_{n, k_i-l} \simeq f \circ \alpha(S_{n, k_i-l})$, lemma 4.4 shows that $\beta \circ f \circ \alpha = 0$. Proposition D.1 then insures the existence of two homomorphisms $g$ and $h$ and the commutativity of the resulting diagram. Since the left part of the diagram commutes and because $f$ is an isomorphism, $g$ is injective. Moreover, $S_{n, k_i-l}$ being reducible, $\text{im } g$ is also. The only non-trivial submodule of a standard module is its irreducible radical. The image $\text{im } g$ must therefore be $S_{n, k_i-l}$ and $g$ an isomorphism. The short five lemma D.3 then implies that $h$ is an isomorphism.
Proposition 3.3 gives \( k_c - i = k'_c - i' \) and \( k_c + i = k'_c + i' \) which are simply \( i = i' \) and \( k_c = k'_c \).

Lemma 5.7. Let \( 0 < i < l \) and \( P \) and \( P' \) be two projective \( d \text{TL}_n \)-modules. If the sequence

\[
0 \longrightarrow S_{n,k_c-i} \xrightarrow{f} P \oplus P' \xrightarrow{g} S_{n,k_c+i} \longrightarrow 0
\]

is exact, then it splits.

Proof. Consider the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & S_{n,k_c-i} & \xrightarrow{f} & P \oplus P' & \xrightarrow{g} & S_{n,k_c+i} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{id} & & \downarrow g & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & P \oplus P' & \xrightarrow{\beta} & B & \longrightarrow & 0
\end{array}
\]

whose first row is exact by hypothesis. Let \( q : P \oplus P' \to P \) and \( q' : P \oplus P' \to P' \) denote the projections on each summand and set \( A = \text{im} qf \oplus \text{im} q'f \) and \( B = \text{coker} qf \oplus \text{coker} q'f \). Now define \( \alpha \) as the inclusion of the two summands \( \text{im} qf \) and \( \text{im} q'f \) of \( A \) into \( P \) and \( P' \) of the direct sum \( P \oplus P' \) and \( \beta \) to be the surjective map on each cokernel. The bottom row is then exact. We now construct two maps \( \tilde{f} \) and \( \tilde{g} \) so that the two boxes of the diagram commute. Since \( \text{im} f \subset A \), then \( \text{im} f \subset \text{im} \alpha \) and therefore \( \beta f = 0 \). For all \( y \in S_{n,k_c-i} \), there exists a unique \( x \in A \) such that \( \alpha(x) = f(y) \). Define \( \tilde{f}(y) = x \). The map \( \tilde{f} \) makes the left box commutes and is injective since both \( f \) and \( \alpha \) are. To construct \( \tilde{g} \), let \( z \in S_{n,k_c+i} \) and choose \( u \in P \oplus P' \) such that \( g(u) = z \). Define \( \tilde{g}(z) = \beta(u) \). Another choice \( u' \in P \oplus P' \) with \( g(u') = z \) would be such that \( u - u' \in \text{im} f \subset \text{im} \alpha = \ker \beta \) and \( \tilde{g} \) is well-defined. This map makes the right box commute. It can be seen to be surjective as follows. Let \( v \in B \) be non-zero. There is a \( w \in P \oplus P' \) such that \( \beta(w) = v \) since \( \beta \) is surjective. This \( w \) cannot be in \( \ker \beta = \ker \alpha = 0 \) and therefore is not in \( \text{im} f \). Therefore \( g(w) \neq 0 \). Because the right box commute \( \tilde{g}(g(w)) = \beta(w) = v \).

Since \( k_c + i \) is non-critical, \( \ker \tilde{g} \) can only be \( 0, R_{n,k_c+i} \) or \( S_{n,k_c+i} \). Suppose first that \( \ker \tilde{g} = 0 \). Then \( \tilde{g} \) is an isomorphism and, by the short five lemma, so is \( \tilde{f} \). Therefore \( \text{im} \tilde{f} \) and \( \text{im} \tilde{g} \) are isomorphic to \( S_{n,k_c-i} \) and \( S_{n,k_c+i} \) respectively. Since the standard modules are indecomposable, the only possibilities are either

\[
S_{n,k_c-i} \simeq \text{im} qf \simeq P \quad \text{and} \quad S_{n,k_c+i} \simeq \text{coker} q'f \simeq P'
\]

or \( S_{n,k_c-i} \simeq \text{im} q'f \simeq P' \) and \( S_{n,k_c+i} \simeq \text{coker} qf \simeq P \)

and the sequence splits.

Suppose, for the second case, that \( \ker \tilde{g} = R_{n,k_c+i} \). Then \( \text{im} \tilde{g} = I_{n,k_c+i} \) is irreducible and one of \( \text{coker} qf \) and \( \text{coker} q'f \) must be trivial. Suppose, for example, that it is \( \text{coker} qf \). This implies that \( \text{im} qf \simeq P \) and therefore \( qf \) maps \( S_{n,k_c-i} \) onto \( P \). Is this map injective on \( P \)? (Note that \( S_{n,k_c-i} \xrightarrow{f} P \oplus P' \) is injective, but this does not imply in itself that \( qf \) is.) If it is not, then \( 0 \to \ker \alpha \to S_{n,k_c-i} \xrightarrow{\alpha} P \to 0 \) is exact and a nontrivial extension of \( P \) as \( S_{n,k_c-i} \) is indecomposable. But this contradicts the fact that \( P \) is projective. Therefore \( S_{n,k_c-i} \simeq P \).
Let \( \phi \) be such an isomorphism. Again the two rows of the following diagram are exact

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & S_{n,k_{c-i}} & \xrightarrow{\phi} & P \oplus P' & \xrightarrow{\phi'} & P' & \longrightarrow & 0 \\
\downarrow \phi & & \downarrow \phi' & & \downarrow \text{id} & & \downarrow \phi' & & \\
0 & \longrightarrow & S_{n,k_{c-i}} & \xrightarrow{f} & P \oplus P' & \xrightarrow{g} & S_{n,k_{c+i}} & \longrightarrow & 0 \\
\end{array}
\]

with \( g \phi = 0 \) because \( \text{Hom}(S_{n,k_{c-i}}, S_{n,k_{c+i}}) = 0 \). (The map \( \phi' \) is simply \( q' \).) Like at the beginning of the proof, \( \bar{\phi} \) and \( \phi' \) can be constructed to make the boxes commute and such that \( \bar{\phi} \) is injective and \( \phi' \) is surjective. Since \( \phi \) is injective, it is an isomorphism and so must be \( \phi' \). So again, the (bottom) sequence splits.

Suppose, finally, that \( \ker \bar{g} = S_{n,k_{c+i}} \). Then \( \bar{g} = 0 \) and \( \text{coker} \ q f \simeq \text{coker} \ q f' \simeq 0 \). The second row gives \( \text{im} q f \simeq P \) and \( \text{im} q f' \simeq P' \) and, as for the previous case, \( P \simeq P' \simeq S_{n,k_{c-i}} \). But \( \text{Hom}(S_{n,k_{c-i}}, S_{n,k_{c+i}}) = 0 \) and \( g = 0 \) and the first row cannot be exact. The case \( \ker \bar{g} = S_{n,k_{c+i}} \) must thus be ruled out. \( \square \)

**Corollary 5.8.** If \( S_{k_{c},k_{c}} \) is projective and \( 1 \leq i < l \), then \( P_{n,k_{c}} \) is indecomposable.

**Proof.** By proposition 5.4, \( P_{n,k_{c}} \) is then projective. If it were not indecomposable, the preceding lemma would give the flatness of \( S_{n,k_{c+i}} \), contradicting lemma 5.3. \( \square \)

**Proposition 5.9.** If \( 1 \leq i < l \) and \( S_{k_{c},k_{c}} \) is projective, then

\[
P_{n,k_{c}} \uparrow \simeq P_{n+1,k_{c}} \uplus P_{n+1,k_{c}} \uplus P_{n+1,k_{c}} \uplus P_{n+1,k_{c}} \uplus \cdots
\]

for all \( n \geq k_{c} + i \).

**Proof.** The induction of the exact sequence (65) breaks down into three exact ones by the use of corollary 3.11 and the usual parity argument:

\[
\begin{align*}
S_{n+1,k_{c}-i} & \longrightarrow P_{n,k_{c}}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i-1} \longrightarrow S_{n+1,k_{c}+i-1} \longrightarrow 0 \\
S_{n+1,k_{c}} & \longrightarrow P_{n,k_{c}}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \longrightarrow S_{n+1,k_{c}+i} \longrightarrow 0 \\
S_{n+1,k_{c}+i} & \longrightarrow P_{n,k_{c}}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \longrightarrow S_{n+1,k_{c}+i+1} \longrightarrow 0.
\end{align*}
\]

(For the special case \( i = 2 \) and \( i = 1 \), the three eigenvalues of \( F_{n} \) are still distinct even though \( k_{c} + i \pm 1 \) are then both critical. This is because \( q \) is \( \pm i \) for this \( l \) and the eigenvalues of \( F_{n} \) on critical lines alternate between \( +1 \) and \( -1 \).) The injectivity of the homomorphism in each sequence can be deduced by the same argument used in the proof of the proposition 5.4. By definition the module \( P_{n+1,k_{c}}^{i} \) is precisely \( P_{n,k_{c}}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \uparrow \) appearing in the last sequence and is thus one of the direct summands of \( P_{n,k_{c}}^{i} \uparrow \). This settles the analysis of the third sequence above and the last summand in (69).

By the last comment of appendix D since \( P_{n,k_{c}}^{i} \) is projective for all \( n \geq k_{c} + k_{c} \), so will be the direct summands of \( P_{n,k_{c}}^{i} \uparrow \). Therefore both modules \( P_{n,k_{c}+i}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \uparrow \) and \( P_{n,k_{c}+i}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \uparrow \), appearing in the first two sequences above, are projective. These two exact sequences (with the initial “\( 0 \rightarrow \)” added) are precisely those for \( P_{n+1,k_{c}}^{i} \) and \( P_{n+1,k_{c}}^{i} \). Since \( n + 1 \geq k_{c} + i + 1 \geq k_{c} + i \), corollary 5.5 shows that \( P_{n+1,k_{c}}^{i} \) and \( P_{n+1,k_{c}}^{i} \) are both projective and proposition D.7 shows that they are precisely \( P_{n,k_{c}+i}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \uparrow \) and \( P_{n,k_{c}+i}^{n-k_{c}-i} \uparrow P_{n+1,k_{c}}^{i} \uparrow \). (Note that, to use proposition D.7 in the case \( i = 1 \), \( P_{n+1,k_{c}}^{0} \) must satisfy the exact sequence \( 0 \rightarrow S_{n+1,k_{c}} \rightarrow P_{n+1,k_{c}}^{0} \rightarrow S_{n+1,k_{c}} \rightarrow 0 \). This justifies a posteriori the definition of \( P_{n,k_{c}}^{0} \).) \( \square \)
5.2. The structure of $dTL_n$ at root of unity. The regular module of $dTL_n$ is the algebra itself seen as module for the action given by left multiplication. Wedderburn’s theorem states that this module is reducible as a direct sum of irreducible modules, if the algebra is semisimple, and principal indecomposable modules if it is not. (See theorems D.14 and D.15 in appendix D) Theorem 4.11 gave a complete set of non-isomorphic irreducible modules for $q$ generic. This subsection completes the analysis for $q$ a root of unity.

We start by giving examples, for $l = 3$, of the procedure using the results of the previous subsections, mainly propositions and corollaries 5.4 to 5.9. It copies the technique used in subsections, mainly propositions and corollaries 5.4 to 5.9. It copies the technique used in [25] based on the simple observation that $dTL_n \simeq dTL_n \otimes_{dTL_{n-1}} dTL_{n-1} \uparrow$. Since $dTL_n$ seen as a module over itself is a free module, its direct summands will all be projective modules. (See the end of appendix D).

So let $l = 3$. The parameter $\beta$ is then $\pm 1$. The algebra $dTL_4$ is spanned by the two elements

$$\begin{align*}
\uparrow \\
\uparrow
\end{align*}$$

and is isomorphic to the direct sum

$$dTL_4 \simeq S_{1,1} \oplus S_{1,0}.$$

The modules $S_{1,1}$ and $S_{1,0}$ are therefore the principal indecomposable ones for $dTL_4$ when $l = 3$. The next case $dTL_2$ is straightforward. The computation relies solely on corollaries 3.11 and 3.12 and on the indecomposability of the standard modules (proposition 3.2):

$$dTL_2 \simeq dTL_1 \uparrow \simeq S_{1,1} \uparrow \oplus S_{1,0} \uparrow \simeq 2S_{2,0} \oplus 2S_{2,1} \oplus S_{2,2}.$$

Note that $dTL_1$ and $dTL_2$ are semisimple at $\beta = \pm 1$. Moreover $S_{2,2}$ is then projective and the hypothesis of corollary 5.5 is then verified.

The case $dTL_3$ proceeds as before except for the induction of $S_{2,2}$. Since $k_e = 2$ is critical, corollary 3.12 cannot be used for $S_{n,k_e} = S_{2,2}$. Instead, by definition of the $P_{n,k_e}$ and the usual parity argument, $S_{2,2} \uparrow \simeq S_{1,2} \oplus P_{1,2}^1$ and corollaries 5.8 and 3.12 show that the two summands are indecomposable. Therefore, for $l = 3$,

$$dTL_3 \simeq dTL_2 \uparrow \simeq 2S_{2,0} \uparrow \oplus 2S_{2,1} \uparrow \oplus S_{2,2} \uparrow \simeq 4S_{3,0} \oplus 4S_{3,1} \oplus 3S_{3,2} \oplus P_{3,2}^1.$$

Again the summands are the principal indecomposables of $dTL_3$. Now $dTL_3$ is not semisimple, since its decomposition into indecomposables includes reducible modules. As will be shown in this section, all $dTL_n(\pm 1), n \geq 3$, are non-semisimple. The next step $dTL_4$ shows the use of yet another result of the previous section. The induced module $P_{3,2}^1 \uparrow$ needs to be written in terms of indecomposable summands. Proposition 5.9 gives $P_{3,2}^1 \uparrow \simeq P_{4,2}^0 \oplus P_{4,2}^1 \oplus P_{4,2}^2$. Recalling the definition of $P_{n,k_e}^i$ (equation (58)), we obtain the principal indecomposables for $dTL_4$:

$$dTL_4 \simeq dTL_3 \uparrow \simeq 4S_{3,0} \uparrow \oplus 4S_{3,1} \uparrow \oplus 3S_{3,2} \uparrow \oplus P_{3,2}^1 \uparrow$$

$$= 8S_{4,0} \oplus 8S_{4,1} \oplus 9S_{4,2} \oplus 4P_{4,2}^1 \oplus P_{4,2}^2.$$

The example ends with the computation of $dTL_5$. Again a new tool is used. Proposition 5.9 gives $P_{4,2}^2 \uparrow \simeq P_{5,2}^1 \oplus P_{5,2}^2 \oplus P_{5,2}^3$. Now the module $P_{5,2}^3$ escapes corollary 5.8 as its index $i$ has reached the value $l = 3$. Instead proposition 5.4 says that the exact sequence (65) for $P_{5,2}^3$ splits but, because $S_{5,1}$ is the trivial module, $P_{5,2}^3 \simeq S_{5,5}$. The principal indecomposables of $dTL_5$ are readable from

$$dTL_5 \simeq dTL_4 \uparrow \simeq 8S_{4,0} \uparrow \oplus 8S_{4,1} \uparrow \oplus 9S_{4,2} \uparrow \oplus 4P_{4,2}^1 \uparrow \oplus P_{4,2}^2 \uparrow$$

$$\simeq 16S_{5,0} \oplus 16S_{5,1} \oplus 25S_{5,2} \oplus 14P_{5,2}^1 \oplus 5P_{5,2}^2 \oplus S_{5,5}.$$
No new tools are required to proceed to \( \text{dTL}_n, n \geq 6 \), but patience. A check with table 2 of appendix C shows that the multiplicities in the above direct sums coincide with the dimensions of the irreducible \( l_{n,k} \) when \( l = 3 \), as it should be, by Wedderburn’s theorem D.15 for non-semisimple algebras. The general case can now be tackled.

**Theorem 5.10** (Structure of \( \text{dTL}_n \) for a root of unity). Let \( q \) be a root of unity other than \( \pm 1 \) and \( l \) the smallest positive integer such that \( q^{2l} = 1 \). Let \( K \) be the set of critical integers smaller or equal to \( n \). Then

\[
\text{dTL}_n \cong \left( \bigoplus_{0 \leq k < l - 1} i_{n,k} S_{n,k} \right) \bigoplus \left( \bigoplus_{k \in K} \left( i_{n,k} S_{n,k} \bigoplus \bigoplus_{1 \leq i < l} \left( \bigoplus_{k \in K} P_{n,k}^i \right) \right) \right) \tag{71}
\]

where \( i_{n,k} = \dim l_{n,k} \) if \( k \in \{0, 1, \ldots, n\} \) and 0 otherwise.

**Proof.** For \( n = 1 \) the set \( K = \{1\} \) if \( l = 2 \) and \( \emptyset \) otherwise. In both cases the above formula give \( \text{dTL}_1 = S_{1,0} \oplus S_{1,1} \) which is the right answer.

As long as \( n \) is smaller than \( l - 1 \), only the first sum of (71) for \( \text{dTL}_n \) contributes and the decomposition of \( \text{dTL}_{n+1} \) from \( \text{dTL}_n \) relies only on corollary 5.12 and proposition 5.2. The multiplicities can easily be checked to be the right ones using (81). This argument allows one to proceed inductively until \( \text{dTL}_{n+l-1} \). The multiplicity of the last summand is \( S_{n+k} \), and all \( P_{n,k}^i \) by corollary 5.5 are projective.

To pursue the induction beyond \( n = l - 1 \), all terms of (71) need to be considered.

Assuming the result for \( \text{dTL}_n \), the decomposition of \( \text{dTL}_{n+1} \) is obtained by inducing each summand of (71). Again corollary 5.12 can be used for the first sum, and the same corollary together with the definition of \( P_{n,k}^i \) takes care of the terms appearing in the second.

The hypothesis of corollary 5.5 has now been checked and, thus, so are those of propositions 5.3 and 5.9. These results assure the decomposition of the \( P_{n,k}^i \)'s of the last sum. All the terms in the decomposition are now indecomposable by proposition 5.2 and corollary 5.8 except the \( P_{n,k}^i \)'s that may have occurred. These can be further decomposed using proposition 5.4. With this further step, all direct summands are indecomposable, and all terms thus obtained are, by definition, principal indecomposable modules. It is straightforward to check that all these summands appear in the decomposition (71), now written for \( n + 1 \). The last step is to check their multiplicities. It can be done using the recurrence relations (80) and appendix C gives some details of the exercise. \( \square \)

**Corollary 5.11.** Under the hypothesis of the previous theorem and with its notation, the set \( \{l_{n,k}, 0 \leq k \leq n\} \) is a complete set of non-isomorphic irreducible modules, and the set

\[
\{S_{n,k}, 0 \leq k < l - 1\} \cup \{S_{n,k}, k_\in K \} \cup \{P_{n,k}, k_\in K, l \leq i < l, i + k_\in K \}
\]

is a complete set of non-isomorphic principal indecomposable modules.

6. **Conclusion**

The main results of this paper are now overviewed. The dilute Temperley-Lieb algebras \( \text{dTL}_n(\beta), n \geq 0 \), form a family of algebras parameterized by a complex (or formal) parameter \( \beta \), often written as \( \beta = q + q^{-1} \) with \( q \in \mathbb{C}^\times \). The dimension of \( \text{dTL}_n(\beta) \) is the Motzkin number \( M_{2n} \). These algebras decompose into a sum of even and odd parts, and so do their modules.
Their representation theory is largely based on the study of the standard modules $S_{n,k}$, $0 \leq k \leq n$. These are shown to be indecomposable (proposition 3.2) and cyclic (proposition 3.1). Their diagrammatic definition is a technical advantage: it allows for quick computations and observing several of their properties. For example, in the link basis, the matrices representing the generators have at most one non-zero element per column and this element is then a power of $\beta$. It is also easy to observe that any link state with only defects and vacancies is actually a generator. Finally the standard modules are all distinct ($S_{n,k} \cong S_{n,j} \Leftrightarrow k = j$) and, for neighbouring $n$s, they are related by restriction and induction, namely $S_{n,k} \uparrow \cong S_{n+2,k} \downarrow$ for all $n$ and $0 \leq k \leq n$ (proposition 3.10).

It is the latter property, together with the natural bilinear form $\langle *, * \rangle_{n,k}$ and a particular central element $F_{n,k}$, that is used to unravel the structure of the algebra $dTL_n$. If the complex number $q$ is generic, that is not a root of unity, then $dTL_n(\beta = q + q^{-1})$ is semisimple and the standard modules form a complete set of non-isomorphic irreducible modules (theorem 4.11).

If however $q$ is a root of unity, distinct from $\pm 1$, a finer analysis is required. Let $l$ be the smallest positive integer such that $q^{2l} = 1$. An integer $k_c$ is said critical if $k_c + 1 \equiv 0 \mod l$ and a pair $(k_-, k_+)$ of distinct integers form a symmetric pair if their average is critical and $0 < (k_+ - k_-)/2 < l$. With this notation the standard module $S_{n,k}$ is reducible, but indecomposable, if $k$ is the smallest element $k_-$ of a symmetric pair with $0 \leq k_- < k_+ \leq n$. In that case, its maximal proper submodule $R_{n,k} \subset S_{n,k}$ is the radical of the Gram pairing $\langle *, * \rangle_{n,k}$ and is irreducible. In fact, if $k$ is the $k_-$ of a symmetric pair $(k_-, k_+)$, then $R_{n,k} = l_{n,k}$, where $l_{n,k}$ is the irreducible quotient $S_{n,k} / R_{n,k}$.

The indecomposable projective modules, that is the principal indecomposable ones, can be constructed by induction starting from a standard $S_{n,k}$ with critical $k$. More precisely, they are obtained by inducing repetitively a critical $S_{n',k_{n'}}$, $n' < n$. The principal indecomposable modules of $dTL_n$ are characterized as a submodule of $(S_{n',k_{n'}}) \uparrow \ldots \uparrow$ (with $n - n' < l$) completely determined by its $F_{n'}$-eigenvalue and parity. It is then possible to decompose the non-semisimple algebra $dTL_n$, seen as a left module over itself as a direct sum of some standard modules and the principal ones obtained by the induction process (theorem 5.10).

![Figure 2. The Loewy diagrams of the principal indecomposable modules](image-url)
It is useful to draw the Loewy diagrams of the principal indecomposable modules. (The construction of the Loewy diagrams for dTL\(_n\) is identical to that for TL\(_n\) which is described in [25].) If \(k\) is critical, the projective is simply the (irreducible) standard module \(S_{n,k}\) and its Loewy diagram contains a single node (figure 6 (a)). For \(k\) non-critical, let \(k_−, k_+\) and \(k_++\) be such that \(k_− < k_+ = k < k_++\) and both \((k_−, k_+)\) and \((k_+, k_++\) are symmetric pairs. Then, if \(k_−, k_+, k_++ \in \{0, 1, \ldots, n\}\), the Loewy diagrams of the principal modules with irreducible quotient \(I_{n,k}\) has the form (b) in the figure. If \(k_++ > n\), then the right node is deleted and the resulting Loewy diagram is of type (c) on the figure. Finally, if \(k\) is at the left of the first critical line, then its Loewy diagram is that of the standard \(S_{n,k}\) and appears as (d) on the figure.

What can one learn from these results about limiting structures appearing in physical models like conformal field theories (CFT)? The original Temperley-Lieb algebras, whose representation theory the dilute ones mimic so closely, has been used to understand the representation theory of the Virasoro algebra appearing in the continuum limit of lattice models whose transfer matrix is an element of TL\(_n\). The fusion ring, defined formally in [26], is a natural outcome of the representation theory of these finite-dimensional associative algebras. The result announced there for the TL\(_n\) fusion ring is paralleled to the CFT fusion for Virasoro modules, with staggered ones sharing the Loewy structure of the principal indecomposable modules of type (b) in figure 6. We hope that the results reported here may help reveal the fusion ring of the dilute Temperley-Lieb algebras.

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APPENDIX A. THE TEMPERLEY-LIEB ALGEBRA

The original Temperley-Lieb algebras [1] were introduced much before the dilute ones. Since the present text studies the latter, the former will be presented starting from the definitions for the dilute objects. Only the results needed here are recalled. They are taken from an article by Ridout and one of the authors [25]. It must be underlined that their paper uses the number \(p\) of arcs instead of the number \(k\) of defects to characterize link states and modules. The results stated below have been adapted to the labelling in terms of defects.

The one-parameter family of Temperley-Lieb algebras TL\(_n(\beta)\) is spanned by all \(n\)-diagrams, defined in subsection 2.1 that contain no vacancies. The product is given by the same rules as for dTL\(_n\), the factor \(\beta\) also weighting each closed loop generated through concatenation of diagrams. The algebra TL\(_n\) has a compact definition in terms of generators \(u_i, 1 \leq i \leq n - 1\), and the unit \(\text{id}\). These correspond to the following diagrams:

\[
\begin{align*}
\text{id} &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
n
\end{array}, \\
\begin{array}{c}
i-1 \\
i \\
i+1 \\
n
\end{array}, \\
\begin{array}{c}
i-1 \\
i \\
i+1 \\
n
\end{array}
\end{align*}
\]

(73)
and satisfy the relations:
\[
\begin{align*}
u_i^2 &= \beta u_i, \\
u_i &= u_i u_{i \pm 1}, & \text{if } 1 \leq i, i \pm 1 \leq n - 1, \\
u_i u_j &= u_j u_i, & |j - i| > 1.
\end{align*}
\]

The dimension of TLₙ is the Catalan number \(C_{n+1} = \frac{1}{n+1} \binom{2n}{n}\).

The standard module \(V_{n,k}\) is spanned by the basis of \(S_{n,k}\) from which all \(n\)-link states that bear vacancies are discarded. They are defined only when \(n \) and \(k \) have the same parity.

The action of TLₙ on \(V_{n,k}\) is defined as that of dTLₙ on \(S_{n,k}\). Their dimension is given by \(\dim V_{n,k} = \binom{n}{(n-k)/2} - \binom{n}{(n-k)/2-1}\).

The Gram bilinear form \(\langle \ast, \ast \rangle_{n,k} : V_{n,k} \times V_{n,k} \to \mathbb{C}\) is introduced exactly as the Gram product on \(S_{n,k}\). Now the rule stating that \(\langle u, v \rangle_{n,k}\) is zero whenever unmatched vacancies arise upon gluing of \(\bar{u}\) and \(v\) can be ignored safely as no link states with vacancies occur in \(V_{n,k}\). (The same observation holds for the multiplication in TLₙ and the action of TLₙ on the standard modules \(V_{n,k}\), discussed above.) It is possible to compute the determinant of the matrix \(G_{n,k}\) representing the bilinear form \(\langle \ast, \ast \rangle_{n,k}\) in the basis of \(n\)-link states with \(k\) defects. (See for example [24][25].)

**Proposition A.1.** The Gram determinant for the bilinear form on \(V_{n,k}\) when \(\beta = q + q^{-1}\) is given, up to a sign, by
\[
\det G_{n,k} = \prod_{j=1}^{(n-k)/2} \left[ \frac{\mid k+j+1\mid q}{\mid j\mid q}\right]^\dim V_{n,(n-k)/2-j}
\]
(74)
where \(q\)-numbers are used: \(\mid m\rangle_q = (q^m - q^{-m})/(q - q^{-1})\).

Note that \(\det G_{n,k}\), \(n \geq 1, 0 \leq k \leq n\), does not vanish at \(\beta = \pm 2\), that is, at \(q = \pm 1\).

The radical \(R_{n,k} = \{ v \in V_{n,k} \mid \langle v, w \rangle_{n,k} = 0 \text{ for all } w \in V_{n,k} \}\) is a submodule of the standard module \(V_{n,k}\). It has the following properties.

**Proposition A.2.** The radical \(R_{n,k}\) is the maximal proper submodule of \(V_{n,k}\). It is either trivial (\(\simeq \{0\}\)) or irreducible.

Like for the dilute ones, the radicals of the Temperley-Lieb standard modules are nontrivial only when \(q\) is a root of unity distinct than \(\pm 1\). Let \(l\) be the smallest positive integer such that \(q^{2l} = 1\). An integer \(k\) is called critical if \(k + 1 \equiv 0 \mod l\) and non-critical otherwise. Let \(I_{n,k}\) stand for the irreducible quotient \(V_{n,k}/R_{n,k}\) of the standard module \(V_{n,k}\).

**Proposition A.3.** With the notation just introduced, the dimensions of the irreducible quotients can be obtained from the following recurrence equations:
\[
\dim I_{n,k} = \begin{cases} 
\dim V_{n,k}, & \text{if } k \text{ is critical}, \\
\dim I_{n-1,k-1}, & \text{if } k + 1 \text{ is critical}, \\
\dim I_{n-1,k-1} + \dim I_{n-1,k+1}, & \text{otherwise},
\end{cases}
\]
(75)

with initial conditions \(\dim I_{n,n} = 1\) for all \(n\) and \(\dim I_{n,0} = 0\) when \(n\) is odd.

The algebra TLₙ(\(\beta\)) has a central element \(F_{n}\) whose eigenvalues can distinguish any pair of standard modules whose labels \(k\) and \(k'\) fall between two consecutive critical lines. It is defined diagrammatically as the analogous element in dTLₙ (see appendix [B]) by equation
Here, however, the building tiles are defined by

\[
\begin{align*}
\text{tile 1} & : \sqrt{q} \quad - \frac{1}{\sqrt{q}} \\
\text{tile 2} & : \sqrt{q} \quad - \frac{1}{\sqrt{q}}
\end{align*}
\]

Here are the basic properties of \( F_n \).

**Proposition A.4.** (i) The element \( F_n \in \mathbb{T}L_n \) is central, satisfies \( F^i = F \) and acts on \( V_{n,k} \) as the identity times \( \delta_k = q^{k+1} + q^{-(k+1)} \).

(ii) Let \( q \) be a root of unity distinct from \( \pm 1 \) and \( k \in \{0, 1, \ldots, n\} \) be critical for this \( q \). Let \( z_k \in V_{n,k} \) be the link state with \( k \) defects at the lowest positions and arcs between positions \( 2i - 1 \) and \( 2i \) for \( 1 \leq i \leq (n-k)/2 \). Then the action of \( F_{n+1} \) on \( \text{id} \otimes z_k \in V_{n,k} \uparrow \) has a non-zero component along the vector \( y_k = u_1u_2 \ldots u_n \otimes z_k \) in a basis containing both linearly independent elements \( y_k \) and \( z_k \in V_{n,k} \uparrow \).

A basis \( S_{n,k} \) of the induced module \( V_{n,k} \uparrow \) is constructed explicitly in [25]. It is with this basis that the above result (ii) is stated in that paper. The simpler statement above establishes that the action of \( F_{n+1} \) is not a multiple of the identity on \( V_{n,k} \uparrow \). This is what will be used in the proof of lemma 4.15.

**Appendix B. The Central Element \( F_n \)**

One central element of \( d\mathbb{T}L_n \) plays an important role in the text, starting with the proof of proposition 3.9. If \( q \) is generic, it has distinct eigenvalues on non-isomorphic standard modules. If \( q \) is a root of unity, for certain indecomposable modules, it is not a multiple of the identity, a property that allows one to probe their structure.

The central element \( F_n \) is defined graphically through the following tiles

\[
\begin{align*}
\text{tile 1} & : \sqrt{q} \quad - \frac{1}{\sqrt{q}} \\
\text{tile 2} & : \sqrt{q} \quad - \frac{1}{\sqrt{q}}
\end{align*}
\]

which are multiplied according to the rules used for diagrams. Then, with these definitions, \( F_n \) is written as

\[
F_n = \begin{pmatrix}
\ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots \\
\ldots & \ldots & \ldots
\end{pmatrix}
\]
The expansion of the $2n$ tiles leads to $3^{2n}$ different diagrams, most of them being zero. The two first $F_n$ are

$$F_1 = (q^2 + q^{-2}) \begin{array}{c} \hline \end{array} + \beta \begin{array}{c} \hline \end{array}$$

$$F_2 = (q^3 + q^{-3}) \begin{array}{c} \hline \end{array} - (q - q^{-1})^2 \begin{array}{c} \hline \end{array} \begin{array}{c} \hline \end{array} + (q^2 + q^{-2}) \begin{array}{c} \hline \end{array} + \beta \begin{array}{c} \hline \end{array} \begin{array}{c} \hline \end{array}.$$ 

To verify that it is indeed a central element, we compute its products with the generators of $dTL_n$. We start by expanding the tiles of the left column:

Note that a sign appears during the commutation of the two last generators. The computation for the right column is obtained from that for the left by the exchange $\sqrt{q} \leftrightarrow -1/\sqrt{q}$.

The following result is thus proved.

**Proposition B.1.** $F_n$ is a central element of $dTL_n$.

The eigenvalues of $F_n$ on standard modules $S_{n,k}$ are easily computed. Before doing so, it is useful to note that rows of the defining tiles, acting on vacancies or arcs of a link state,
have the following properties

\[ \begin{array}{c|c}
1 & 1 \\
-1 & -1 \\
\end{array} = \begin{array}{c|c}
1 & 1 \\
-1 & -1 \\
\end{array} \quad \text{and} \quad \begin{array}{c|c}
0 & 0 \\
0 & 0 \\
\end{array} = \begin{array}{c|c}
0 & 0 \\
0 & 0 \\
\end{array} \quad (77)

as a direct expansion of the tiles shows. The first property above indicates that \( F_n \) is actually an element of the subalgebra \( S_n \). To see this, let \( u \in dTL_n \) be an \( n \)-diagram. If \( I = \{i_1,i_2,\ldots,i_{n-k}\} \) is the set of positions of its vacancies on its left side, then \( u = \pi_i u \) where \( \pi \in X_{n,k} \) is the link diagram with vacancies at these same positions. (The element \( \pi_i = [\pi] \in dTL_n \) is introduced and discussed in Section [3].) The product \( F_n u = F_n \pi_i u \) is simplified by the observation that all vacancies of \( \pi_i \) go through \( F_n \) due to (77) and \( F_n u = (\pi_i F_n \pi_i) u \). The sums in the remaining tiles may omit the tile \( \mathbb{I} \), as a link in \( \pi_i \) is connected on either side of each tile to be summed. These sums are then precisely those intervening in the definition of \( F_k \) of \( TL_k \).

**Proposition B.2.** The central element \( F_n \in S_n \subset dTL_n \) can be written as

\[ F_n = \sum_{0 \leq k \leq n} \sum_{\pi \in X_{n,k}} \pi F_n \pi \]  

where each summand \( \pi F_n \pi \) is constructed by insertion in \( F_k \) of \( (n-k) \) lines of vacancies to match those in \( z \).

**Proposition B.3.** On \( S_{n,k} \), the element \( F_n \) acts as \( \delta_k \cdot \text{id} \) where \( \delta_k = q^{k+1} + q^{-(k+1)} \).

**Proof.** Since \( F_n \) is central and the modules \( S_{n,k} \) are indecomposable (proposition [3.2]), the endomorphism defined by left multiplication by \( F_n \) can only have one eigenvalue. By the previous proposition and the properties (77), the tiles on lines of \( F_n \) acting on vacancies or on arcs of a link state are therefore completely determined, they contribute an overall factor of 1 and can be left out of the computation. For \( z \) a link diagram in \( S_{n,k} \), the computation of \( F_n z \) thus reduces to that of \( F_k z_0 \) where \( z_0 \) is the unique \( k \)-link diagram with \( k \) defects. Moreover the computation of \( F_k z_0 \) does not involve anymore the tile \( \mathbb{I} \) and it becomes identical to that for the action of the central element \( F_k \) on \( V_{k,k} \). This computation was done in [25]: \( F_k z_0 = F_k z_0 = (q^{k+1} + q^{-(k+1)}) z_0 \). (See proposition [A.4]) Since the link diagrams form a basis of \( S_{n,k} \), the element \( F_n \) acts as a multiple of the identity and the result follows.

These eigenvalues of the central element \( F_n \) provide a good way to distinguish between standard modules. More precisely:

**Lemma B.4.** Let \( n \) be a positive integer.

(i) If \( q \) is not a root of unity, then \( \delta_j \neq \delta_k \) if \( j \neq k \).

Let \( q \) be a root of unity other than \( \pm 1 \) and \( l \) be the smallest positive integer such that \( q^{2l} = 1 \).

Let \( k_c \) be critical or \( -1 \) and \( K_c \) (resp. \( K_{-c} \)) denote the set of \( k \)'s such that \( k_c < k \leq k_c + l \) and \( k \) has the parity of \( n - k_c \) (resp. of \( n - k_c - 1 \)).

(ii) If \( j \) and \( k \) are distinct and both in \( K_c \) (or both in \( K_{-c} \)), then \( \delta_j \neq \delta_k \).

(iii) The intersection \( K_c \cap K_{-c} \) is non-empty if and only if \( q \) is of the form \( e^{2i\pi m/l} \) with \( \gcd(m,l) = 1 \) and \( l \) odd.

(iv) The function \( \delta_k \) is even with respect to a mirror reflection through a critical line.

**Proof.** If \( \theta \in \mathbb{C} \) is chosen such that \( q = e^{i\theta} \), then \( \delta_j = \delta_k \) is equivalent to \( \cos((j + 1)\theta) = \cos((k + 1)\theta) \) which in turn amounts to either (a) \( (k+1)\theta = (j+1)\theta + 2\pi p \) or (b) \( (k+1)\theta = (j+1)\theta - 2\pi p \) where \( p \) is a positive integer.
\[ \theta = -(j+1)\theta + 2\pi p \] for some integer \( p \). If \( j \neq k_c \), then \( \theta \) must be a (real) rational multiple of \( \pi \) and (i) follows. If \( q^{2l} = 1 \) with \( l \) the smallest possible, then either (c) \( q = e^{2\pi m/l} \) with \( \gcd(m,l) = 1 \) and \( l \) odd or (d) \( q = e^{\pi(2m+1)/l} \) with \( \gcd(2m+1,l) = 1 \). The equation (a) requires that \( (k-j)\theta \) be an integer multiple of \( 2\pi \). But \( (k-j)\theta \) is either \( 2\pi(k-j)m/l \) or \( \pi(k-j)(2m+1)/l \). Both forms require that the difference \( k-j \) be a multiple of \( l \) which is impossible since \( k < j, j \leq k_c + l \).

To study the case (b), write \( k = k_c + \bar{k} \) and \( j = j_c + \bar{j} \) with \( 0 < \bar{k}, \bar{j} \leq l \). Since \( k_c + 1 \equiv 0 \mod l \), the equation (b) forces \((k+j+2)\theta \) (or equivalently \((\bar{k} + \bar{j})\theta \)) to be an integer multiple of \( 2\pi \). For the case (d), this is impossible since \((k+j)(2m+1)/l = 2p \) would mean that \( (\bar{k} + \bar{j}) \) is an even multiple of \( l \). However, in the case (c), \( l \) is always odd (and \( \geq 3 \)) and the equation \((\bar{k} + \bar{j})m/l = p \) has always the solution \( k = 1 \) and \( \bar{j} = l - 1 (\neq \bar{k}) \). Note that this solution and all others \((k = i \text{ and } j = l - i) \) have \( k \) and \( l \) of distinct parity. This proves both (ii) and (iii).

If \( k_c = k_c \pm m \), then \( q^{k+c+1} = q^{k+c+m+1} = q^{-k+c+m-1} = q^{-(k+c+1)} \) where the criticality of \( k_c \) was used. The last statement follows. \( \square \)

### Appendix C. The Dimensions of the Irreducible Modules \( l_{n,k} \)

The dimensions of the irreducible quotients \( l_{n,k} \) satisfy recurrence relations used in the proof of theorem (5.10). We gather here these relations and their proofs. Tables containing the dimensions of standard modules and of irreducible quotients for \( l = 3 \) and \( l = 4 \) are also given.

As usual \( q \) is a root of unity other than \( \pm 1 \) and \( l \) is the smallest positive integer such that \( q^{2l} = 1 \) (and \( l \geq 2 \)). The notation

\[ i_{n,k} = \dim l_{n,k}, \quad r_{n,k} = \dim R_{n,k}, \quad s_{n,k} = \dim S_{n,k} \quad \text{and} \quad \tilde{i}_{n,k} = \dim I_{n,k}, \]

is used throughout.

The module \( l_{n,k} \) is defined to be the irreducible quotient \( S_{n,k}/R_{n,k} \). Corollaries 3.6 and 4.8 then give a simple formula for its dimension in terms of those for the irreducibles \( I_{n,k} \) of the (original) Temperley-Lieb algebra:

\[ \tilde{i}_{n,k} = \dim S_{n,k} - \dim R_{n,k} = \sum_{p=0}^{\lfloor (n-k)/2 \rfloor} \binom{n}{k+2p} \tilde{s}_{n,k}. \quad (79) \]

**Proposition C.1.** Let \( n \geq 1 \) and \( k_c \) be an integer critical for \( q \). Then the three following recurrence relations hold:

\[ i_{n+1,k_c+i} = i_{n,k_c+i+1} + i_{n,k_c+i+1}, \quad 1 \leq i \leq l - 2, \quad (81) \]

where any \( i_{n,j} \) with a \( j \) outside the set \( \{0,1,\ldots,n\} \) is zero. With the latter convention, the second equation also holds for \( k_c = -1 \).

**Proof.** For the first of these recurrences, use (79) to write \( i_{n+1,k_c+i} \) in terms of the \( \tilde{i}s \) and split the sum into two using the binomial identity \( \binom{n+i}{j} = \binom{n}{j} + \binom{n}{j-1} \). The summation index of the second sum, that containing the binomial \( \binom{n}{j-1} \), is then shifted using (75) of proposition A.2. Terms that are missing at either end of the sums can be added as they are weighted by a binomial that vanishes. The proof of the second recurrence follows the same lines.
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The last recurrence is proved as follows:

\[ i_{n+1,k} = s_{n+1,k} \]
\[ = s_{n,k-1} + i_{n,k} + s_{n,k+1} \]
\[ = i_{n,k-1} + r_{n,k-1} + i_{n,k} + i_{n,k+1} + r_{n,k+1} \]
\[ = i_{n,k-1} + i_{n,k+1} + i_{n,k} + i_{n,k+1} + i_{n,k+2l-1} \]

The first equation is simply the irreducibility of \( S_{n+1,k} \), the second line follows from the restriction of \( S_{n+1,k} \) (see (10)), the last line is a consequence of proposition (4.16). □

The dimensions of \( S_{n,k} \) are showed in table 1 for \( n \leq 10 \), and the dimension of \( I_{n,k} \) are showed in table 2 and 3 for \( l = 3 \) and \( l = 4 \), respectively.

We end this appendix by showing how to check the multiplicities of (71). This last step was left out of the proof of theorem 5.10. As described in the proof, each summand of (71) can be written after induction as a sum of indecomposable modules. Here are the decomposition of the three sums:

\[
\bigoplus_{0 \leq k \leq l-2} i_{n,k} s_{n,k} \uparrow \simeq i_{n,0} (S_{n+1,0} \oplus S_{n+1,1}) \oplus \bigoplus_{1 \leq k \leq l-3} i_{n,k} (S_{n+1,k-1} \oplus S_{n+1,k} \oplus S_{n+1,k+1})
\]
\[ \oplus i_{n,l-2} (S_{n+1,l-3} \oplus S_{n+1,l-2} \oplus S_{n+1,l-1}) \]
where the last two standard modules are trivial if $k_i \pm l$ is out of range. For each of the distinct modules, the contributing multiplicities are to be summed. Several cases have to be considered. The exercise is simple and repetitive and relies on the recurrences (80–82).

Here are a few examples.

D.1. Short exact sequences. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be two module homomorphisms. The sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is said to be exact (or exact at $M$) if the kernel of $g$ is equal to the image of $f$. A short exact sequence is a sequence of homomorphisms

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$

Table 3. Dimensions $i_{n,k} = \dim i_{n,k}$ for $l = 4$.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 |   |   |   |   |   |   |   |   |   |
| 2 | 2 | 2 | 1 |   |   |   |   |   |   |   |   |
| 3 | 4 | 5 | 3 | 1 |   |   |   |   |   |   |   |
| 4 | 9 | 12 | 8 | 4 | 1 |   |   |   |   |   |   |
| 5 | 21 | 29 | 20 | 14 | 5 | 1 |   |   |   |   |   |
| 6 | 50 | 70 | 49 | 44 | 20 | 6 | 1 |   |   |   |   |
| 7 | 120 | 169 | 119 | 133 | 70 | 27 | 7 | 1 |   |   |   |
| 8 | 289 | 408 | 288 | 392 | 230 | 104 | 34 | 8 | 1 |   |   |
| 9 | 697 | 985 | 696 | 1140 | 726 | 368 | 138 | 44 | 9 | 1 |   |
| 10 | 1682 | 2378 | 1681 | 3288 | 2234 | 1232 | 506 | 200 | 54 | 10 | 1 |

where the last two standard modules are trivial if $k_i \pm l$ is out of range. For each of the distinct modules, the contributing multiplicities are to be summed. Several cases have to be considered. The exercise is simple and repetitive and relies on the recurrences (80–82).

Here are a few examples.

For the term $S_{n+1,l-2}$ on the last line of the first sum, there are only two contributions: $i_{n,l-2} + i_{n,l-3}$ which is precisely $i_{n+1,l-2}$. The multiplicity of the critical $S_{n,k_c}$ is $i_{n,k_c} + 2i_{n,k_c+1} + i_{n,k_c+2} + i_{n,k_c-1}$ which is $i_{n+1,k_c}$ and the intermediate terms $P_{n+1,k_c}^l$ (for $1 \leq i \leq l-2$) have multiplicities $i_{n,k_c} + i_{n,k_c+1} + i_{n,k_c+2}$ which gives $i_{n+1,k_c+1}$. The other cases are similar.

APPENDIX D. Tools from algebra

We review here concepts and results in algebra that are used in the article and might not be familiar to some readers. We start by presenting short exact sequences and proceed to projective and flat modules. The interplay between induction and the tensor product is then recalled. We finally recall Wedderburn’s theorem, and its generalization, and Frobenius reciprocity theorem.

Throughout the appendix, $A$ is a unital associative algebra over $C$, $B$ a subalgebra of $A$. Unless otherwise stated, $L, M, N$ and $P$ are $A$-modules.

D.1. Short exact sequences. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be two module homomorphisms. The sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is said to be exact (or exact at $M$) if the kernel of $g$ is equal to the image of $f$. A short exact sequence is a sequence of homomorphisms

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$
that is exact at \( L, M \) and \( N \). This is equivalent to saying that the sequence is exact at \( M \) with \( f \) and \( g \) being injective and surjective, respectively.

**Proposition D.1.** A sequence

\[
0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0
\]

is exact if and only if it verifies the three following conditions:

(i) \( gf = 0 \)

(ii) If there is a module \( U \) and a homomorphism \( u : U \to M \) such that \( gu = 0 \), then there is a unique homomorphism \( \bar{u} : U \to L \) such that \( f \bar{u} = u \);

(iii) If there is a module \( V \) and a homomorphism \( v : M \to V \) such that \( vf = 0 \), then there is a unique homomorphism \( \bar{v} : N \to V \) such that \( \bar{v}g = v \).

The short exact sequence of proposition D.1 is called **split** if \( M \cong L \oplus N \).

**Proposition D.2.** If the short sequence

\[
0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0
\]

is exact, the three following statements are equivalent:

(i) the sequence splits;

(ii) there is a homomorphism \( \bar{f} : M \to L \) such that \( \bar{f}f = \text{id}_L \);

(iii) there is a homomorphism \( \bar{g} : N \to M \) such that \( \bar{g}g = \text{id}_N \).

We end this section with two useful results.

**Lemma D.3** (The short five lemma). Let

\[
\begin{array}{ccc}
0 & \longrightarrow & L \rightarrow M \rightarrow N \rightarrow 0 \\
& \downarrow{g} & \downarrow{f} & \downarrow{h} \\
0 & \longrightarrow & L' \rightarrow M' \rightarrow N' \rightarrow 0
\end{array}
\]  
(83)

be a commuting diagram with exact rows. If any two of \( f, g, \) and \( h \) are isomorphisms, then the third one is an isomorphism too.

Write \( \text{Hom}(M,N) \) for the vector space of \( A \)-homomorphisms of \( M \) into \( N \).

**Proposition D.4.** If \( 0 \to M \to N \to P \to 0 \) is exact, then, for any other module \( L \), so are

\[
0 \longrightarrow \text{Hom}(L,M) \rightarrow \text{Hom}(L,N) \rightarrow \text{Hom}(L,P),
\]

and

\[
0 \longrightarrow \text{Hom}(P,L) \rightarrow \text{Hom}(N,L) \rightarrow \text{Hom}(M,L).
\]

Moreover, if \( L \) is projective, then the last homomorphism in the first sequence above is surjective.

**D.2. Projective and flat modules.** A module \( P \) is said to be **projective** if for all modules \( M \) and \( N \) and all homomorphisms \( f : M \to N \) and \( g : P \to N \) with \( f \) surjective, there is a homomorphism \( h : P \to M \) such that \( f \circ h = g \). In other words, given homomorphisms \( f \) and \( g \) as in the diagram below with an exact horizontal row, then there exist \( h \) that makes the diagram commute.
Because of proposition (D.2), the above definition (with \( P = N \) and \( g = \text{id}_P \)) gives:

**Proposition D.5.** A module \( P \) is projective if and only if all short exact sequences
\[
0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0
\]
split.

Direct sums and direct summands of projective modules are also projective. Note also that an algebra seen as a module over itself is always projective.

Recall that a module \( L \) is indecomposable if it cannot be written as the direct sum of two modules. Schur’s lemma of group theory can be extended to them as follows.

**Lemma D.6** (Schur’s lemma). If \( L \) is a finite-dimensional indecomposable \( A \)-module, than every endomorphism of \( L \) is an isomorphism or is nilpotent.

**Proposition D.7.** If there are two short exact sequences
\[
0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow L \longrightarrow \bar{M} \longrightarrow N \longrightarrow 0
\]
with \( L \) a finite-dimensional indecomposable module and \( M \) and \( \bar{M} \) two projective modules, then \( M \simeq \bar{M} \).

**Proof.** Because \( M \) (resp. \( \bar{M} \)) is projective and \( \beta_2 \) (\( \alpha_2 \)) is surjective, there exists a map \( f \) (\( \bar{f} \)) that makes the right square of the following diagram commutes:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
& & g & \swarrow \bar{g} & f & \swarrow \bar{f} & \downarrow \text{id}_N & & \\
0 & \longrightarrow & L & \longrightarrow & \bar{M} & \longrightarrow & N & \longrightarrow & 0
\end{array}
\]

Proposition [D.3] then gives the existence of \( g \) (\( \bar{g} \)) and the commutativity of the diagram, because
\[
\beta_2 \circ f \circ \alpha_1 = \text{id}_N \circ \alpha_2 \circ \alpha_1 = 0 \quad \text{and} \quad \alpha_2 \circ \bar{f} \circ \beta_1 = \text{id}_N \circ \beta_2 \circ \beta_1 = 0
\]
and the top (bottom) row is exact. Since \( L \) is indecomposable, it follows by Schur’s lemma that \( gg \) is an isomorphism or is nilpotent. Suppose that it is nilpotent, that is, there exists an integer \( n \) such that \((gg)^n = 0\). We then have,
\[
\alpha_1 (gg)^n = \bar{f} \beta_1 g (gg)^{n-1} = \bar{f} f \alpha_1 (gg)^{n-1} = (\bar{f} f)^n \alpha_1 = 0. \tag{84}
\]

Proposition [D.2] then gives the existence of a unique homomorphism \( f_* : N \rightarrow M \) such that \( f_*, \alpha_2 = (\bar{f} f)^n \). Left multiplying both sides by \( \alpha_2 \) gives
\[
\alpha_2 f_* \alpha_2 = \alpha_2 (\bar{f} f)^n = \beta_2 f (\bar{f} f)^{n-1} = \alpha_2 (\bar{f} f)^{n-1} = \alpha_2 \tag{85}
\]
and, since \( \alpha_2 \) is surjective, we conclude that \( \alpha_2 f_* = \text{id}_N \), that is, the sequence at the top splits (see statement (iii) of proposition [D.2]). The two rows can be exchanged in the argument and the bottom must split as well. Thus \( M \simeq L \oplus N \simeq \bar{M} \).

If \( gg \) is an isomorphism then both \( g \) and \( \bar{g} \) are too. The short five lemma leads to the conclusion. \( \square \)
Proposition D.8. Let $V$ be a left module. The sequence
\[ L \otimes V \longrightarrow M \otimes V \longrightarrow P \otimes V \longrightarrow 0 \]
of abelian groups is exact if the sequence
\[ L \longrightarrow M \longrightarrow P \longrightarrow 0 \]
of right modules is.

A left module $F$ is called flat if for every exact sequence
\[ 0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0 \]
of right modules, the induced sequence of abelian groups
\[ 0 \longrightarrow L \otimes F \longrightarrow M \otimes F \]
is exact. This means that the tensor product with $F$ preserves the injectivity of homomorphisms. Direct sums and direct summands of flat modules are also flat.

Proposition D.9. For finite-dimensional modules over Noetherian rings, flatness and projectivity are equivalent.

The right-exactness of the tensor product (proposition D.8) is used in the proof of the next result. Let $N$ be a left $A$-module. The annihilator $\text{Ann } n \subset A$ for $n \in N$ is $\{a \in A | an = 0\}$. A similar definition holds for right modules.

Lemma D.10. Let $U$ and $V$ be cyclic left- and right- $A$-modules with generators $u_0$ and $v_0$ respectively. For $v \in V$, the element $u_0 \otimes_A v$ is zero in $U \otimes_A V$ if and only if there exists $a \in \text{Ann } u$ such that $v = av_0$.

Proof. The statement \(\leftarrow\) follows directly from the definition of the tensor product $U \otimes_A V$.

For \(\Rightarrow\), note that the sequence of right modules $0 \longrightarrow \text{Ann } u_0 \longrightarrow A \longrightarrow U \longrightarrow 0$ with $f : a \mapsto u_0 a$ is exact since $u_0$ is a generator of $U$. (The map $i : \text{Ann } u_0 \rightarrow A$ is the inclusion.) Therefore the top row of the following diagram of abelian groups
\[
\begin{array}{ccccccccc}
\text{Ann } u_0 \otimes_A V & \overset{i \otimes \text{id}}{\longrightarrow} & A \otimes_A V & \overset{f \otimes \text{id}}{\longrightarrow} & U \otimes_A V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker f^* & \longrightarrow & V & \overset{f^*}{\longrightarrow} & U \otimes_A V & \longrightarrow & 0 \\
\end{array}
\]
is exact. Similarly the bottom row where $f^* : v \mapsto u_0 \otimes v$ is exact. With this diagram, the statement amounts to prove that all elements $v$ in $\ker f^*$ can be written as $av_0$ for some $a \in \text{Ann } v_0$. Define $g$ by $a \otimes v \mapsto av$. It is clearly surjective and makes the right box commute: $f \otimes \text{id} = f^* g$. This allows for the construction of the unique (surjective) map $\tilde{g}$ that makes the left box commute. Hence, for any $v \in \ker f^*$, there exists $x = \sum a_i \otimes v_i \in \text{Ann } u_0 \otimes_A V$ such that $\tilde{g}(x) = v$. Then $\tilde{g}(x) = g(i \otimes \text{id}(\sum a_i \otimes v_i)) = \sum a_i v_i$. Since $v_0$ is a generator of $V$, there exists a $b_1 \in A$ for each $v_i$ such that $v_i = b_1 v_0$ and thus $\tilde{g}(x) = (\sum a_i b_i)v_0$ and $\sum a_i b_i \in \text{Ann } u_0$ since $\text{Ann } u_0$ is a right ideal. Therefore, if $u_0 \otimes_A v$ is zero, the vector $v$ can be written as $av_0$ for some $a \in \text{Ann } u_0$. \(\square\)
D.3. **Restriction and induction.** If an algebra $A$ has a subalgebra $B$, it is natural to ask how a given $A$-module $M$ would behave as a $B$-module. Since $B$ is a subalgebra of $A$, the space $M$ can be seen as a $B$-module for the same action and the $B$-module thus obtained is called the **restriction** of $M$ to $B$, and is noted $M \downarrow_B$ (or $M \downarrow_B^A$). It can be shown that restriction preserves short exact sequences, that is:

**Proposition D.11.** Let $A$ be an associative algebra and $B$ a subalgebra of $A$, the sequence

$$0 \to L \to M \to P \to 0$$

of $A$-modules is exact if and only if the sequence

$$0 \to L \downarrow_B^A \to M \downarrow_B^A \to P \downarrow_B^A \to 0$$

of $B$-modules is exact.

It is also natural to do the “reverse process”, that is, to transform a $B$-module into an $A$-module. This process is slightly more complex, and the resulting module is called the **induction** of $M$ to $A$, noted $M \uparrow_B^A$ (or $M \uparrow_B^A$). It is defined as the tensor product $A \otimes_B M$. The regular module structure then carries over to the tensor product: $a(a \otimes m) = (a')a \otimes m$ for all $a,a' \in A$ and $m \in M$. It can be shown that the induction preserves parts of exact short sequences. More precisely, one has:

**Proposition D.12.** Let $A$ be an associative algebra and $B$ a subalgebra of $A$, the sequence

$$A \otimes L \to A \otimes M \to A \otimes P \to 0$$

of $A$-modules is exact if the sequence

$$L \to M \to P \to 0$$

of $B$-modules is exact.

There are cases when the homomorphism $A \otimes L \to A \otimes M$ fails to be injective even when the $B$-module homomorphism $L \to M$ is.

D.4. **Frobenius reciprocity theorem and Wedderburn’s theorem.** The operations of restriction and induction were presented as “reverse processes”. This is particularly meaningful in view of the next result.

**Proposition D.13** (Frobenius reciprocity theorem). Let $A$ be a finite-dimensional associative algebra over $\mathbb{C}$ and $B$ a subalgebra of $A$. Let $M$ be an $A$-module and $N$ be a $B$-module. Then, as vector spaces

$$\text{Hom}_B(N, M \downarrow) \cong \text{Hom}_A(N \uparrow, M).$$

(86)

The algebra $A$ can be seen as a left $A$-module where the action is simply left multiplication. This module is called the **regular** module and one may write $A^A$ to emphasize the left module structure. The algebra is called **semisimple** if its regular module is completely reducible, that is, it is isomorphic to a direct sum of irreducible modules. A key property of semisimple algebras is the following.

**Theorem D.14** (Wedderburn’s theorem). Let $A$ be a finite-dimensional associative algebra over $\mathbb{C}$. $A$ is semisimple if and only if the regular module decomposes as

$$A^A \cong \bigoplus_i (\dim L_i) L_i$$

where the set $\{L_i\}$ forms a complete set of non-isomorphic irreducible $A$-modules.
It can also be shown that $A$ is semisimple if and only if every $A$-module is projective. If $A$ is not semisimple, Wedderburn’s theorem no longer holds, and it is replaced by the following generalisation.

**Theorem D.15.** Let $A$ be a finite-dimensional associative algebra over $\mathbb{C}$. The regular module decomposes as

$$AA \cong \bigoplus (\dim L_i) P_i$$

where the set $\{P_i\}$ forms a complete set of non-isomorphic projective indecomposable $A$-modules, and $L_i$ is the unique irreducible quotient of $P_i$.

The projective indecomposables in this last proposition are called *principal indecomposable modules*. It can be shown that any projective module is a direct sum of principal indecomposable ones.

A last comment will be useful. Note that induction of the regular module $B \uparrow^A_B$ is simply

$$B \uparrow^A_B = AA \otimes_B BB \cong AA.$$ If $B = \bigoplus B_i$ is the decomposition of $B$ into its principal indecomposable modules, then $AA \cong \bigoplus B_i \uparrow$ and, since they appear as direct summands of the free module $AA$, the $A$-modules $B_i \uparrow$ are projective. (They might not be indecomposable.) Therefore the induction of a projective module is projective.

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