NON-LOCAL SUBLINEAR PROBLEMS: EXISTENCE, COMPARISON, AND RADIAL SYMMETRY

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Abstract. We establish a symmetry result for a non-autonomous overdetermined problem associated to a sublinear fractional equation. To this purpose we prove, in particular, that the solution of the corresponding Dirichlet problem is monotonically increasing with respect to the domain. We also obtain a strong minimum principle and a boundary-point lemma for linear fractional equations that may have an independent interest.

1. Introduction. We consider the boundary-value problem

\[
\begin{cases}
    (-\Delta)^s u = f(x,u), & u > 0 \text{ in } \Omega, \\
    u = 0 & \text{ in } \mathbb{R}^N \setminus \Omega, 
\end{cases}
\]

where \( \Omega \) is a sufficiently smooth, bounded open set in \( \mathbb{R}^N \), \( N \geq 1 \), and the operator \((-\Delta)^s, s \in (0, 1)\), is the fractional Laplacian (see (2)). Due to the variational structure of the equation in (1), the existence of a bounded weak solution follows from the direct method of the calculus of variations under the assumptions mentioned in Proposition 1. The solution is sought in the function space \( X^s_0(\Omega) \), which is the set of all functions \( u \in L^2(\mathbb{R}^N) \) vanishing identically in all of \( \mathbb{R}^N \setminus \Omega \) and such that

\[
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dx \, dy < +\infty.
\]

The definition of such a function space (denoted there by \( X_0 \)) is found, for instance, in [22, p. 70], and some of its properties are outlined in [22, Sections 2.2 and 2.3]. In the definition of \( X^s_0(\Omega) \), the set \( \Omega \) is allowed to be any open set (and may coincide with the whole space \( \mathbb{R}^N \)). Furthermore, it is apparent that if \( \Omega_1 \subseteq \Omega_2 \) then \( X^s_0(\Omega_1) \subset X^s_0(\Omega_2) \). The space \( X^s_0(\Omega) \) is a Hilbert space whose scalar product is given by

\[
(u, v) = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

Note that the set \( \Omega \) does not enter explicitly in the integral above, hence we write simply \((u, v)\) instead of \((u, v)_{X^s_0(\Omega)}\). However, the norm of \( u \) in \( X^s_0(\Omega) \) is denoted

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by \( \|u\|_{X_0^s(\Omega)} \) to keep it distinct from the norm in \( L^2(\Omega) \), which is also used in the sequel. We note in passing the useful equality

\[
\|u\|_{X_0^s(\Omega)}^2 = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \int_{\Omega^2} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy + 2\int_{\Omega \times (\mathbb{R}^N \setminus \Omega)} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy.
\]

A weak solution of problem (1) is a function \( u \in X_0^s(\Omega) \) such that (the Lebesgue integral below is well defined and)

\[
\frac{C_{N,s}}{2} (u, v) = \int_{\Omega} f(x, u(x)) v(x) \, dx
\]

for every \( v \in X_0^s(\Omega) \). The constant \( C_{N,s} \) is given by

\[
C_{N,s} = \frac{2^{2s} \Gamma(\frac{N}{2} + 1)}{\pi^{\frac{N}{2}} \Gamma(1 - s)} = \left( \int_{\mathbb{R}^N} \frac{1 - \cos x_1}{|x|^{N+2s}} \, dx \right)^{-1}, \tag{3}
\]

where \( \Gamma \) denotes Euler’s gamma function (see the Appendix). The value of \( C_{N,s} \) is chosen to make the fractional Laplacian the pseudodifferential operator whose symbol is \( |\xi|^{2s} \); see [2, Section 3.1] and [4, Proposition 3.3] for details.

**Remark 1.** A well-known result by Stein [23, Theorem 2] (see also [24]) implies, in particular, that for every \( u \in X_0^s(\mathbb{R}^N) \) the function of \( x \in \mathbb{R} \) given by

\[
C_{N,s} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+s}} \, dy
\]

converges in \( L^2(\mathbb{R}^N) \) when \( \varepsilon \to 0^+ \). The limiting function is denoted by \((-\Delta)^{s/2} u\), and its \( L^2 \)-norm is related to the norm \( ||u||_{X_0^s(\mathbb{R}^N)} \) by the identity

\[
||(-\Delta)^{s/2} u||_{L^2(\mathbb{R}^N)}^2 = \frac{C_{N,s}}{2} ||u||_{X_0^s(\mathbb{R}^N)}^2,
\]

which is found in [4, Proposition 3.6]. Replacing \( u \) with \( u \pm v \) in the identity above, for \( u, v \in X_0^s(\Omega) \), we obtain two equalities whose difference yields

\[
\int_{\mathbb{R}^N} ((-\Delta)^{s/2} u(x)) \, ((-\Delta)^{s/2} v(x)) \, dx = \frac{C_{N,s}}{2} (u, v),
\]

therefore equation (2) can be equivalently rewritten as

\[
\int_{\mathbb{R}^N} ((-\Delta)^{s/2} u(x)) \, ((-\Delta)^{s/2} v(x)) \, dx = \int_{\Omega} f(x, u(x)) \, v(x) \, dx.
\]

A similar setting is adopted, for instance, in [21, Definition 2.1].

At the beginning of the present paper we establish some useful tools that will be used later and may have an independent interest. The first two tools are a strong minimum principle and a boundary-point lemma of Hopf’s type for weak solutions \( w \) of the linear inequality

\[
(-\Delta)^s w(x) \geq b(x) \, w(x) \tag{4}
\]

in an open set \( \Omega \) (possibly non-smooth, unbounded, and disconnected), where the coefficient \( b: \Omega \to \mathbb{R} \) is a measurable function, bounded from below. The results extend [16, Theorem 2.5 and Lemma 2.7], respectively, where the case \( b \equiv 0 \) is considered (see also [17]). Corresponding results for functions \( w \) satisfying (4) pointwise
are found in [15, Theorem 2.1 and Lemma 1.2]. A function \( w \in X^s_0(\mathbb{R}^N) \) satisfies (4) in the weak sense in \( \Omega \) if
\[
\frac{C_{N,s}}{2} (w, \eta) \geq \int_\Omega b(x) w(x) \eta(x) \, dx
\]
for every non-negative \( \eta \in X^s_0(\Omega) \).

**Theorem 1.1** (Strong minimum principle). Let \( \Omega \) be an open set in \( \mathbb{R}^N \), \( N \geq 1 \), and let \( w \in X^s_0(\mathbb{R}^N) \) be a weak solution of (4) in \( \Omega \) satisfying \( w \geq 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \). Suppose that the coefficient \( b \) is bounded from below. If either \( w \geq 0 \) in \( \Omega \) or \( b \leq 0 \) in \( \Omega \), then either \( w \) vanishes a.e. in \( \mathbb{R}^N \) or for every compact subset \( K \subset \Omega \) there exists a positive constant \( \varepsilon_K > 0 \) such that \( w \geq \varepsilon_K \) a.e. in \( K \).

**Lemma 1.2** (Fractional Hopf’s boundary-point lemma). Let \( \Omega \) be an open set in \( \mathbb{R}^N \), \( N \geq 1 \), and let \( z \in \partial \Omega \) be a boundary point such that \( z \in \partial B_R \) for some ball \( B_R = B_R(x_0) \subset \Omega \) centered at \( x_0 \in \Omega \). Let \( w \in X^s_0(\mathbb{R}^N) \) be a weak solution of (4) in \( \Omega \) satisfying \( w \geq 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \). Suppose that the coefficient \( b \) is bounded from below. If either \( w \geq 0 \) in \( \Omega \) or \( b \leq 0 \) in \( \Omega \), then either \( w \) vanishes a.e. in \( \mathbb{R}^N \) or \( w \) can be modified on a negligible set to achieve
\[
\liminf_{B_R \ni z \rightarrow x} \frac{w(x)}{(\text{dist}(x, \partial B_R))^s} > 0.
\]  

(5)

Theorem 1.1 and Lemma 1.2 are proved in Section 2. The last tool is the following comparison principle, ensuring that the weak solution of the boundary-value problem (1) is monotone with respect to set inclusion.

**Theorem 1.3** (Comparison principle for sublinear equations). Let \( \Omega \) be an open set in \( \mathbb{R}^N \), \( N \geq 1 \), and let \( f: \Omega \times (0, +\infty) \rightarrow (0, +\infty) \) be a positive Carathéodory function (i.e., \( f(x, t) \) is measurable as a function of \( x \) for each given \( t \), and continuous in \( t \) for almost all \( x \)) such that
\[
\text{the ratio } \frac{f(x, t)}{t} \text{ is strictly decreasing in } t \in (0, +\infty) \text{ for almost every } x.
\]  

(6)

Let \( u_i \in X^s_0(\mathbb{R}^N) \), \( i = 1, 2 \), be such that the integral below is finite and
\[
(-1)^i \left( \frac{C_{N,s}}{2} (u_i, \eta) - \int_\Omega f(x, u_i(x)) \eta(x) \, dx \right) \geq 0, \quad i = 1, 2
\]  

(7)

for all non-negative functions \( \eta \in X^s_0(\Omega) \). Suppose that
1. both \( u_1 \) and \( u_2 \) are positive in \( \Omega \);
2. \( u_1 \in L^\infty(\mathbb{R}^N) \), and \( u_2 \geq 0 \) almost everywhere in \( \mathbb{R}^N \).
If \( u_1 \leq u_2 \) a.e. in \( \mathbb{R}^N \setminus \Omega \), then \( u_1 \leq u_2 \) a.e. in \( \mathbb{R}^N \).

The proof is found in Section 3. The result extends Theorem 20 of [18], where both \( u_1 \) and \( u_2 \) are supposed to be bounded and to vanish outside \( \Omega \). In Section 4 we then give a detailed existence and uniqueness proof of a weak solution of problem (1) assuming that the set \( \Omega \) is bounded and sufficiently smooth, and the function \( f: \mathbb{R}^N \times (0, +\infty) \rightarrow (0, +\infty) \) is a positive Carathéodory function satisfying (6) and having the properties listed below. Now the variable \( x \) is let vary in the whole space \( \mathbb{R}^N \) in view of the subsequent application, where the domain \( \Omega \) is an unknown of the problem. For every compact \( K \subset \mathbb{R}^N \) and every \( \varepsilon > 0 \) we require that there exists a constant \( A_{K,\varepsilon} > 0 \) such that
\[
0 < f(x, t) \leq A_{K,\varepsilon} + \varepsilon t \text{ for every } (x, t) \in K \times (0, +\infty).
\]  

(8)
We also require that
\[
\lim_{t \to 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{locally uniformly with respect to } x.
\]  
(9)
As a consequence of (8) we have
\[
\lim_{t \to +\infty} \frac{f(x,t)}{t} = 0 \quad \text{locally uniformly with respect to } x.
\]
Model cases are \( f(x,t) = t^m, \ m \in (0,1), \) as well as \( f(x,t) \equiv 1. \) Note that (8) prevents \( f(x,t) \) from becoming infinite as \( t \to 0. \) When the function \( f \) is subject to different structural conditions, multiplicity results have been established: see, for instance, [20] and the references therein.

In Section 5 we turn to consider an overdetermined problem. Recall that if \( \Omega \) is a sufficiently smooth, bounded domain, then the weak solution \( u \) of (1) is Hölder continuous in \( \mathbb{R}^N \) and satisfies
\[
\lim_{x \to z} \frac{u(x)}{(\text{dist}(x, \partial \Omega))^s} = \varphi(z) \quad \text{for each } z \in \partial \Omega,
\]  
(10)
where \( \varphi \in C^0(\overline{\Omega}) \) is a convenient Hölder continuous function (see Proposition 1). Denoting by \( \nu \) the inner normal to \( \partial \Omega \) at \( z, \) (10) implies
\[
(\partial_\nu)^s u(z) = \varphi(z) \quad \text{for each } z \in \partial \Omega,
\]  
(11)
where the fractional inner derivative \( (\partial_\nu)^s \) is defined as
\[
(\partial_\nu)^s u(z) = \lim_{\varepsilon \to 0^+} \frac{u(z + \varepsilon \nu)}{\varepsilon^s}, \quad z \in \partial \Omega.
\]
If we do not know the shape of \( \Omega, \) but we do know that the weak solution of (1) satisfies (11) for some radial function \( \varphi(z) = q(|z|), \) can we infer that \( \Omega \) is a ball centered at the origin? Counterexamples show that the answer is negative, in general. Clearly, if \( \varphi \equiv \text{constant} \) and \( f \) does not depend on \( x, \) then the overdetermined problem (1)-(11) is autonomous. In such a case, which has been considered in [7], any solution \( u \) in a given domain \( \Omega \) is readily transformed in the solution \( v(x) = u(x - x_0) \) of the corresponding problem in the domain \( \Omega + x_0, \) for every fixed \( x_0 \in \mathbb{R}^N. \) Hence we cannot say that the problem is solvable only if \( \Omega \) is a ball centered at a prescribed point. Furthermore, a non-spherical domain \( \Omega \) such that the solution of (1) satisfies (11) although \( f(\cdot, t) \) and \( \varphi(\cdot) \) are radial functions can be constructed as follows.

**Example 1.4.** Let \( \Omega \) be the ellipse \( \Omega = \{ (x_1, x_2) \mid x_1^2/a^2 + x_2^2/b^2 < 1 \} \subset \mathbb{R}^2, \) with \( 0 < a < b, \) and let \( f(x,t) = f(|x|, t) \) be any function satisfying (6), (8) and (9). For instance, we may simply take \( f \equiv 1. \) Then by Proposition 1 we have that problem (1) has a unique weak solution \( u. \) We now construct a convenient function \( \varphi(x) = q(|x|) \) such that the same solution \( u \) also satisfies (11) although \( u \) is clearly non-radial. Since \( f(x,t) \) is a radial function of the variable \( x = (x_1, x_2), \) it is also symmetric with respect to each component \( x_i, \ i = 1, 2. \) Hence the four functions obtained by arbitrarily selecting the two signs in \( u(\pm x_1, \pm x_2) \) are solutions of problem (1). By uniqueness, it follows that \( u \) is symmetric with respect to both \( x_1 \) and \( x_2. \) But then it is legitimate to define
\[
q(r) = (\partial_\nu)^s u(z) \quad \text{for any } z \in \partial \Omega \text{ satisfying } |z| = r
\]
because the derivative does not depend on the particular choice of \( z, \) by symmetry reasons. Thus, letting \( \varphi(z) = q(|z|), \) we may assert that \( u \) solves the overdetermined
problem (1)-(11) with radial functions \( f(\cdot, t) \) and \( \varphi(\cdot) \) although the domain of the problem is not a disc.

Similar examples for local problems are found in [13, p. 242] and [14, Section 5]. Here we establish a sufficient condition to exclude counterexamples as above. Concerning regularity, we require that \( f(x, t) \) satisfies locally a Lipschitz condition from below in \( t \) in the sense that for every \( R_0, t_0 > 0 \) there exists \( C_0 \in \mathbb{R} \) such that
\[
f(x, t_2) - f(x, t_1) \geq C_0 (t_2 - t_1)
\]
for almost every \( x \in B_{R_0}(0) \) and for all \( t_1, t_2 \in (0, t_0) \) satisfying \( t_1 < t_2 \). We prove:

**Theorem 1.5** (Radial symmetry). Let \( \Omega \) be a bounded open set of class \( C^{1,1} \) in \( \mathbb{R}^N, N \geq 2 \), containing the origin, and let \( f: \mathbb{R}^N \times (0, +\infty) \to (0, +\infty) \) be a positive Carathéodory function having the form \( f(x, t) = \tilde{f}(|x|, t) \) and satisfying (6), (8), (9) and (12). Moreover, let \( q(r) \) be a positive function of the variable \( r > 0 \) such that for every \( \rho, \lambda > 0 \) the compound function
\[
\tilde{f}\left(\frac{\rho}{r}, \frac{\lambda}{r^{s} q(r)}\right)
\]
is monotone non-decreasing in \( r > 0 \).

If the overdetermined problem
\[
\begin{align*}
(-\Delta)^s u &= \tilde{f}(|x|, u), \quad u > 0 \quad \text{in } \Omega; \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega; \\
(\partial_{\nu})^s u(z) &= q(|z|) \quad \text{on } \partial \Omega
\end{align*}
\]
has a weak solution, then \( \Omega \) is a ball centered at the origin.

The proof is given in Section 5, and it is based on the comparison of \( u \) with suitable radial functions. Here we point out some cases in which assumption (13) is satisfied. The simplest case occurs when \( f(x, t) \equiv 1 \) and the ratio \( q(r)/r^s \) is monotone non-decreasing in \( r > 0 \). This case is considered in [15], which deals with classical solutions. Theorem 1.5, instead, applies to weak solutions and does not require \( \Omega \) to be connected, which is an assumption of [15, Theorem 1.3]. To give another example, assumption (13) is satisfied when \( f(x, t) = t^m, m \in (0, 1) \), and
\[
\frac{(q(r)/r^s(1+m))^{1-m}}{r^{s(1+m)}}
\]
is monotone non-decreasing in \( r > 0 \). This case is considered in [19] (where strict monotonicity is required). As a consequence of Theorem 1.5, we thus have:

**Corollary 1.** Let \( \Omega \) be a bounded open set of class \( C^{1,1} \) in \( \mathbb{R}^N, N \geq 2 \), containing the origin, and let \( q(r) \) be a positive function of the variable \( r > 0 \) satisfying (15) for some \( m \in (0, 1) \). If the overdetermined problem
\[
\begin{align*}
(-\Delta)^s u &= a^m, \quad u > 0 \quad \text{in } \Omega; \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega; \\
(\partial_{\nu})^s u(z) &= q(|z|) \quad \text{on } \partial \Omega
\end{align*}
\]
has a weak solution, then \( \Omega \) is a ball centered at the origin.

As a final example, observe that if
1. \( \tilde{f}(r, t) \) is monotone non-increasing in \( r > 0 \);
2. the ratio \( \tilde{f}(r, t)/t^{1-\varepsilon_0} \) is monotone non-increasing in \( t > 0 \) for almost every \( r > 0 \) and some \( \varepsilon_0 \in (0, 1) \);
3. the ratio \( q(r)/r^{s\left(\frac{2}{n}-1\right)} \) is monotone non-decreasing in \( r > 0 \),
then assumption (13) is satisfied. This is easily checked, taking into account that
the product \( r^s q(r) \) is strictly increasing as a consequence of the third condition
listed above. A similar setting is found in the paper [12], which deals with the
subdiffusive \( p \)-Laplacian (a local, degenerate/singular operator). In particular,
the third condition above may be considered as a fractional counterpart of condition (5)
in [12].

2. Strong minimum principle and Hopf’s lemma. The proofs of Theorem 1.1
and Lemma 1.2 rely on the following technical lemma. Recall that the weak solution
\( v_R \) of

\[
\begin{align*}
(-\Delta)^s v_R &= 1 \quad \text{in} \ B_R(0), \\
v_R &= 0 \quad \text{in} \ \mathbb{R}^N \setminus B_R(0)
\end{align*}
\]

is known explicitly (see [9, 21]) and it is given by

\[ v_R(x) = \gamma_{N,s} ((R^2 - |x|^2)^+)^s, \quad x \in \mathbb{R}^N, \]

where the exponent \( + \) denotes the positive part, and \( \gamma_{N,s} \) is the following constant:

\[ \gamma_{N,s} = \frac{\Gamma\left(\frac{N}{2}\right)}{4^s \Gamma(1 + s) \Gamma\left(\frac{N}{2} + s\right)}. \]

**Lemma 2.1.** Let \( w \in X_0^s(\mathbb{R}^N), \ N \geq 1, \) be a weak solution of (4) in the ball
\( B_R(x_0), \) and let the coefficient \( b : B_R(x_0) \to \mathbb{R} \) be a measurable function, bounded
from below. Suppose that \( w \geq 0 \ a.e. \ in \ \mathbb{R}^N \setminus B_R(x_0), \) and the set \( B_R^+(x_0) = \{ x \in B_R(x_0) \mid w(x) > 0 \} \) has a positive measure. If at least one of the following
conditions holds true:

(i) \( w \geq 0 \ a.e. \ in \ B_R(x_0); \)

(ii) \( b \leq 0 \ a.e. \ in \ B_R(x_0), \)

then there exists a positive integer \( n \) such that \( w(x) \geq \frac{1}{n} v_R(x - x_0) \ a.e. \ in \ B_R(x_0). \)

**Proof.** Without loss of generality we may assume \( R = 1 \) and \( x_0 = 0. \) Let us write
\( B_1 \) in place of \( B_1(0) \) and let \( v_n = \frac{1}{n} v_1, \) for shortness. Define \( \eta_n = (v_n - w)^+ \) and
suppose, by contradiction, that for all \( n > 0 \) the set \( \Omega_n = \{ x \in \mathbb{R}^N \mid \eta_n(x) > 0 \} \subset \)
\( B_1 \) has a positive measure. Since \( v_1 \) is the solution of (16) we have

\[ \frac{C_{N,s}}{2} (v_n, \eta_n) = \frac{1}{n} \int_{\Omega_n} \eta_n(x) \, dx. \]

Furthermore, since \( w \) satisfies (4) we also have

\[ \frac{C_{N,s}}{2} (w, \eta_n) \geq \int_{\Omega_n} b(x) w(x) \eta_n(x) \, dx. \]

Since the function \( u_n = v_n - w \) coincides with \( \eta_n \) in the set \( \Omega_n, \) by subtracting the
last inequality from the preceding equality we obtain

\[ \frac{C_{N,s}}{2} (u_n, \eta_n) \leq \int_{\Omega_n} \left( \frac{1}{n} - b(x) w(x) \right) u_n(x) \, dx. \]

Since \( b \) is bounded from below we have \( -b(x) \leq M \) in \( \Omega_n \) for a convenient constant
\( M > 0. \) Furthermore, since \( w(x) < v_n(x) \leq \frac{1}{n} \gamma_{N,s} \) in \( \Omega_n, \) using assumption (i)
or (ii) we deduce \( -b(x) w(x) \leq \frac{M}{n} \gamma_{N,s} \) in \( \Omega_n \) and therefore

\[ \frac{C_{N,s}}{2} (u_n, \eta_n) \leq \frac{1 + M \gamma_{N,s}}{n} \int_{\Omega_n} u_n(x) \, dx. \]
To reach a contradiction with (17), we will estimate from below the scalar product

\[(u_n, \eta_n) = \int_{\Omega_n^+} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N+2s}} \, dx \, dy + 2 \int_{\Omega_n \times (\mathbb{R}^N \setminus \Omega_n)} \frac{(u_n(x) - u_n(y)) u_n(x)}{|x - y|^{N+2s}} \, dx \, dy.\]

Now the assumption that \(B_1^+ = \{ x \in B_1 \mid w(x) > 0 \} \) has a positive measure comes into play. Indeed, the assumption implies the existence of a set \(S\) of positive measure such that \(w \geq \varepsilon\) in \(S\) and \(S \subset B_1 \setminus \{0\}\) for some \(\varepsilon \in (0, 1)\). In particular, \(S \subset \mathbb{R}^N \setminus \Omega_n\) for \(n\) large because \(v_n\) tends uniformly to 0 in \(B_1 \setminus \{0\}\). Since \(u_n(y) \leq 0 < u_n(x)\) for \((x, y) \in \Omega_n \times (\mathbb{R}^N \setminus \Omega_n)\) (see also (19)) we may estimate \((u_n, \eta_n)\) as follows:

\[(u_n, \eta_n) \geq \frac{1}{2N+2s-1} \int_{\Omega_n \times S} (-u_n(y) u_n(x)) \, dx \, dy,\]

where we have taken into account that \(|x - y| < 2\) for \((x, y) \in \Omega_n \times S\). Finally, since \(-u_n(y) \geq \varepsilon - v_n(y) \geq \varepsilon/2\) in \(S\) for \(n\) large, we arrive at

\[(u_n, \eta_n) = \int_{\Omega_n^+} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N+2s}} \, dx \, dy \geq \varepsilon |S| \frac{\varepsilon}{2N+2s},\]

By comparing the last inequality with (17) we obtain

\[C_{N,s} \varepsilon |S| \frac{\varepsilon}{2N+2s+1} < \frac{1 + M \gamma_{N,s}}{n},\]

which cannot hold as \(n \to +\infty\). Hence the set \(\Omega_n\) must be negligible for \(n\) large, and the claim follows.

**Proof of Theorem 1.1.** In the case when \(b \leq 0\) in \(\Omega\), let us check that \(w \geq 0\) a.e. in \(\Omega\). To this aim we suppose, by contradiction, that the negative part \(\eta(x) = w^{-}(x)\) is positive in a set \(\Omega_- \subset \Omega\) of positive measure. Then we have

\[0 \leq \int_{\Omega_-} b(x) w(x) \eta(x) \, dx \leq \frac{C_{N,s}}{2} (w, \eta).\]

However, we also have

\[(w, \eta) = - \int_{\Omega_+^2} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} \, dx \, dy - 2 \int_{\Omega_- \times (\mathbb{R}^N \setminus \Omega_-)} \frac{(w(x) - w(y)) w(x)}{|x - y|^{N+2s}} \, dx \, dy.\]

Since the last integrand is positive in the domain of integration, and since \(\Omega_-\) has a positive measure by assumption, the inequality above implies that \(|\mathbb{R}^N \setminus \Omega_-| = 0\) and \(w(x) = w(y)\) a.e. in \(\Omega_+^2\). But then \(\Omega_-\) would have an infinite measure, and \(w\) would be a negative constant there, which is in contrast with \(w \in X_0^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)\). Hence we may assume \(w \geq 0\) a.e. in \(\Omega\) in the sequel. To proceed in the proof of the theorem, we use the fact that \(\Omega\) is open and write

\[\Omega = \bigcup_{k=0}^{+\infty} B_R(x_k)\]
for some open balls $B_{R_k}(x_k) \subset \Omega$. By Lemma 2.1, for each $k$ the function $w$ must either vanish a.e. or be positive a.e. in $B_{R_k}(x_k)$. Hence we may write $\Omega = \Omega_0 \cup \Omega_+$, where $\Omega_0$ is the union of those $B_{R_k}(x_k)$ where $w$ vanishes almost identically, and $\Omega_+$ the union of the remaining balls. Clearly, $w = 0$ a.e. in $\Omega_0$ and $w > 0$ a.e. in $\Omega_+$, and therefore $\Omega_0 \cap \Omega_+ = \emptyset$. Let us check that if $\Omega_0 \neq \emptyset$ then $\Omega_+ = \emptyset$. Assume that $\Omega_0 \neq \emptyset$ and choose a non-negative $\eta \in X^s_0(\Omega_0)$ with a nonempty support. By (4) we have
\[ \frac{C_{N,s}}{2} \langle w, \eta \rangle \geq \int_{\Omega_0} b(x) w(x) \eta(x) \, dx = 0. \]
Furthermore, the scalar product $\langle w, \eta \rangle$ reduces to
\[ \langle w, \eta \rangle = 2 \int_{\Omega_0 \times (\mathbb{R}^N \setminus \Omega_0)} -w(y) \eta(x) \frac{1}{|x-y|^{N+2s}} \, dx \, dy. \]
Since $w(y) \geq 0$ in $\mathbb{R}^N \setminus \Omega_0$, we must have either $\mathbb{R}^N \setminus \Omega_0 = \emptyset$ or $w = 0$ a.e. in $\mathbb{R}^N \setminus \Omega_0$. In both cases we have $w = 0$ a.e. in $\mathbb{R}^N$ and consequently $\Omega_+ = \emptyset$, as claimed. Hence, either $w = 0$ a.e. in $\mathbb{R}^N$ or $w > 0$ a.e. in $\Omega$. In the last case, every compact subset $K \subset \Omega$ admits a finite covering made up of a finite number $n_K$ of balls, still denoted by $B_{R_k}(x_k)$, satisfying $B_{2R_k}(x_k) \subset K$ for $k = 1, \ldots, n_K$. By applying Lemma 2.1 to each ball $B_{R_k}(x_k)$ we get a positive integer $n_k$ such that $w(x) \geq \varepsilon_k = \varepsilon_{N,s}(4R_k^2 - R_k^2)^s = \varepsilon_{N,s}(3R_k^2)^s$. Letting $\varepsilon_K = \min_{k=1,\ldots,n_K} \varepsilon_k$ the conclusion follows.

**Proof of Lemma 1.2.** By Theorem 1.1, either $w = 0$ a.e. in $\mathbb{R}^N$ or $w > 0$ a.e. in $\Omega$. In the last case, by Lemma 2.1 there exists a positive integer $n$ such that $w(x) \geq \frac{1}{n} v_R(x-x_0)$ a.e. in $B_R(x_0)$ and (5) follows.

### 3. Comparison principle

In order to establish Theorem 1.5 we need to compare a weak subsolution $u_1$ of problem (1) in an open set $\Omega_1$ with a weak supersolution of the same problem in a possibly larger set $\Omega_2$. To this purpose we use Theorem 1.3, whose proof is given here. The argument is based on the following inequality.

**Lemma 3.1.** For every $a_1, b_1 \in \mathbb{R}$ and every $a_2, b_2 > 0$ we have
\[ (a_2 - b_2) \left( \frac{a_1}{a_2} (a_1 - a_2)^+ - \frac{b_1}{b_2} (b_1 - b_2)^+ \right) \leq (a_1 - b_1) ((a_1 - a_2)^+ - (b_1 - b_2)^+). \] (18)

**Proof.** Since the mapping $t \mapsto t^+$ is monotone non-decreasing, we have
\[ (q^+ - p^+) \, (q - p) \geq 0 \text{ for all } p, q \in \mathbb{R}. \] (19)
Letting $p = (a_1 - a_2)/a_2$ and $q = (b_1 - b_2)/b_2$ and multiplying the inequality by $a_2 b_2 > 0$ we obtain
\[ (a_2 b_1 - a_1 b_2) \, \frac{(a_1 - a_2)^+}{a_2} \leq (a_2 b_1 - a_1 b_2) \, \frac{(b_1 - b_2)^+}{b_2} \]
which is equivalent to (18).
Proof of Theorem 1.3. For $\varepsilon \geq 0$ define $G_{\varepsilon} = \{ x \in \mathbb{R}^N \mid u_1(x) > u_2(x) + \varepsilon \} \subset \Omega$ and assume, contrary to the claim, that $G_0$ has a positive measure. In order to reach a contradiction we use the test functions $v_{\varepsilon}(x) = (u_1 - u_2 - \varepsilon)^+(x)$ for $\varepsilon \geq 0$, and

$$w_{\varepsilon}(x) = \frac{u_1(x)}{u_2(x) + \varepsilon} v_{\varepsilon}(x) \quad \text{for } \varepsilon > 0.$$ 

Both $v_{\varepsilon}$ and $w_{\varepsilon}$ are essentially bounded because $u_1 \in L^\infty(\mathbb{R}^N)$ by assumption, and $u_2 \geq 0$ a.e. in $\mathbb{R}^N$. Let us check that

$$v_{\varepsilon} \in X_0^s(G_{\varepsilon}) \quad \text{for all } \varepsilon \geq 0. \quad (20)$$

To this aim it suffices to observe that

$$\left| (u_1 - u_2 - \varepsilon)^+(x) - (u_1 - u_2 - \varepsilon)^+(y) \right| \leq \left| (u_1 - u_2 - \varepsilon)(x) - (u_1 - u_2 - \varepsilon)(y) \right| \leq |u_1(x) - u_1(y)| + |u_2(x) - u_2(y)|$$

for all $x, y \in \mathbb{R}^N$, and therefore

$$\frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{(v_{\varepsilon}(x) - v_{\varepsilon}(y))^2}{|x - y|^{N+2}} \, dx \, dy \leq \int_{\mathbb{R}^{2N}} \frac{((u_1(x) - u_1(y))^2}{|x - y|^{N+2}} \, dx \, dy + \int_{\mathbb{R}^{2N}} \frac{((u_2(x) - u_2(y))^2}{|x - y|^{N+2}} \, dx \, dy.$$ 

Since the right-hand side is finite, and since $v_{\varepsilon} = 0$ in $\mathbb{R}^N \setminus G_{\varepsilon}$, (20) follows. We also claim that

$$w_{\varepsilon} \in X_0^s(G_{\varepsilon}) \quad \text{for every } \varepsilon > 0. \quad (21)$$

To check this, let us consider the function $g(z_1, z_2) = z_1 z_2^{-1} (z_1 - z_2)^+$, which is Lipschitz continuous in the strip $S = [0, M] \times [\varepsilon, +\infty)$ for $M, \varepsilon > 0$. Since $|\partial g/\partial z_1| \leq 2M/\varepsilon$ and $|\partial g/\partial z_2| \leq M^2/\varepsilon^2$ a.e. in $S$, by the mean value theorem we have

$$|g(a_1, a_2) - g(b_1, b_2)| \leq \frac{2M}{\varepsilon} |a_1 - b_1| + \frac{M^2}{\varepsilon^2} |a_2 - b_2|$$

for all couples $(a_1, a_2)$ and $(b_1, b_2)$ in $S$. Letting $a_1 = u_1(x)$, $b_1 = u_1(y)$, $a_2 = u_2(x) + \varepsilon$, and $b_2 = u_2(y) + \varepsilon$ for $x, y \in \mathbb{R}^N$, as well as $M = \|u_1\|_{L^\infty(\Omega)}$, by a similar argument as before we obtain $\|w_{\varepsilon}\|_{X_0^s(\Omega)} < +\infty$, hence (21) holds true. Since $X_0^s(G_{\varepsilon}) \subset X_0^s(G_0) \subset X_0^s(\Omega)$, we may choose $i = 1$ and $\eta = v_{\varepsilon}$ in (7), thus getting

$$\frac{C_{N,s}}{2} (u_1, v_{\varepsilon}) \leq \int_{G_0} f(x, u_1(x)) v_{\varepsilon}(x) \, dx \quad \text{for all } \varepsilon \geq 0. \quad (22)$$

Furthermore, choosing $a_1, b_1, a_2, b_2$ as before and using (18) we obtain

$$(u_2(x) - u_2(y)) (w_{\varepsilon}(x) - w_{\varepsilon}(y)) \leq (u_1(x) - u_1(y)) (v_{\varepsilon}(x) - v_{\varepsilon}(y)) \quad (23)$$

for every $x, y \in \mathbb{R}^N$ and every $\varepsilon > 0$. By definition of the scalar product $(u_2, w_{\varepsilon})$, and using inequalities (23) and (22) we may write

$$(u_2, w_{\varepsilon}) \leq (u_1, v_{\varepsilon}) \leq \frac{2}{C_{N,s}} \int_{G_0} f(x, u_1(x)) v_{\varepsilon}(x) \, dx \quad \text{for all } \varepsilon > 0. \quad (24)$$

On the other side, since $w_{\varepsilon}$ is an admissible test function, by (7) we have

$$(u_2, w_{\varepsilon}) \geq \frac{2}{C_{N,s}} \int_{G_0} f(x, u_2(x)) \frac{u_1(x)}{u_2(x) + \varepsilon} v_{\varepsilon}(x) \, dx \quad \text{for all } \varepsilon > 0.$$
By comparison with (24) we obtain
\[ \int_{G_0} f(x, u_2(x)) \frac{u_1(x)}{u_2(x)} v_\varepsilon(x) \, dx \leq \int_{G_0} f(x, u_1(x)) v_\varepsilon(x) \, dx \quad \text{for all } \varepsilon > 0. \]
Hence, by the monotone convergence theorem we get
\[ \int_{G_0} f(x, u_2(x)) \frac{u_1(x)}{u_2(x)} v_0(x) \, dx \leq \int_{G_0} f(x, u_1(x)) v_0(x) \, dx, \tag{25} \]
where the right-hand side is finite by assumption. However, by (6) we may write
\[ \int_{G_0} f(x, u_1(x)) v_0(x) \, dx = \int_{G_0} f(x, u_1(x)) u_1(x) v_0(x) \, dx < \int_{G_0} f(x, u_2(x)) \frac{u_1(x)}{u_2(x)} v_0(x) \, dx, \]
which contradicts (25). Hence the set \( G_0 \) must have measure zero, and the proof is complete. \( \square \)

4. Existence for the Dirichlet problem. The existence proof of a weak solution of problem (1) is based on the fact that equation (2) is the Euler equation in integral form of the functional
\[ J[u] = \frac{C_{N,s}}{4} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy - \int_\Omega F(x, u(x)) \, dx, \tag{26} \]
where \( F(x, u) \) is given by
\[ F(x, u) = \int_0^u f(x, t) \, dt, \]
which is well defined for a.e. \( x \in \Omega \) and for every \( u \in [0, +\infty) \) by virtue of (8). In the negligible set of all \( x \in \Omega \) such that \( f(x, \cdot) \) is not a continuous function, we let \( F(x, u) = 0 \) for all \( u \in [0, +\infty) \). The domain of the functional \( J \) is the set
\[ (X_0^s(\Omega))^+ = \{ u \in X_0^s(\Omega) \mid u \geq 0 \text{ a.e. in } \Omega \}. \]

**Proposition 1.** Let \( \Omega \) be a bounded open set of class \( C^{1,1} \) in \( \mathbb{R}^N \), \( N \geq 2 \), and let \( f: \mathbb{R}^N \times (0, +\infty) \to (0, +\infty) \) be a Carathéodory function satisfying (6), (8) and (9). Then

(i) Problem (1) has a unique weak solution \( u \in X_0^s(\Omega) \cap L^\infty(\Omega) \).

(ii) The solution \( u \) belongs to the Hölder class \( C^s(\mathbb{R}^N) \).

(iii) The ratio \( u(x)/(\text{dist}(x, \partial\Omega))^s \) is in the Hölder class \( C^{\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \).

**Proof.** **Step 1.** Two useful estimates. Observe that for every function \( u \in (X_0^s(\Omega))^+ \subset L^2(\Omega) \) the compound function \( F(x, u(x)) \) is measurable: see [3, Section 8.3 C, pp. 284-285]. As a consequence of (8) and the boundedness of \( \Omega \), for every \( \varepsilon > 0 \) there exists a constant \( A_\varepsilon \) such that
\[ 0 < f(x, t) \leq A_\varepsilon + \varepsilon t \quad \text{for every } (x, t) \in \Omega \times (0, +\infty), \tag{27} \]
and therefore
\[ F(x, t) \leq A_\varepsilon t + \frac{\varepsilon}{2} t^2 \quad \text{for every } (x, t) \in \Omega \times [0, +\infty). \]
Letting \( t = u(x) \geq 0 \) we get the pointwise estimate
\[ F(x, u(x)) \leq A_\varepsilon u(x) + \frac{\varepsilon}{2} u^2(x) \text{ in } \Omega. \tag{28} \]
By integration over $\Omega$, and since $\|u\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|u\|_{L^2(\Omega)}$ we obtain

$$
\int_{\Omega} F(x, u(x)) \, dx \leq A_\varepsilon \|u\|_{L^1(\Omega)} + \frac{\varepsilon}{2} \|u\|_{L^2(\Omega)}^2
$$

$$
\leq A_\varepsilon |\Omega|^{1/2} \|u\|_{L^2(\Omega)} + \frac{\varepsilon}{2} \|u\|_{L^2(\Omega)}^2.
$$

The estimate above implies, in particular, that the functional $J[u]$ in (26) has a finite value for each $u \in (X^s_0(\Omega))^\ast$.

**Step 2.** Let us check that $J$ is coercive. Since $\Omega$ is bounded and has a Lipschitz boundary, the immersion $X^s_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see [22, Lemma 9]), and in particular the norm $\|u\|_{X^s_0(\Omega)}$ dominates the $L^2$-norm $\|u\|_{L^2(\Omega)}$ in the sense that $\|u\|_{L^2(\Omega)} \leq C_\Omega \|u\|_{X^s_0(\Omega)}$ for all $u \in X^s_0(\Omega)$ and some $C_\Omega > 0$. But then

$$
\int_{\Omega} F(x, u(x)) \, dx \leq A_\varepsilon |\Omega|^{1/2} C_\Omega \|u\|_{X^s_0(\Omega)} + \frac{\varepsilon}{2} C^2_\Omega \|u\|_{X^s_0(\Omega)}^2
$$

which implies

$$
J[u] \geq \frac{C_{N,s}}{4} \|u\|_{X^s_0(\Omega)}^2 - A_\varepsilon |\Omega|^{1/2} C_\Omega \|u\|_{X^s_0(\Omega)} - \frac{\varepsilon}{2} C^2_\Omega \|u\|_{X^s_0(\Omega)}^2.
$$

Fixing $\varepsilon < C_{N,s}/(2C^2_\Omega)$ we see that $J[u] \to +\infty$ when $\|u\|_{X^s_0(\Omega)}^2 \to +\infty$, hence $J$ is coercive, as claimed.

**Step 3.** Existence of a minimizer in $(X^s_0(\Omega))^\ast$. Since $J$ is coercive, any minimizing sequence (made up of non-negative functions) for the functional $J$ must be bounded in $X^s_0(\Omega)$ and therefore it must also have a converging subsequence in $L^2(\Omega)$. By the weak compactness theorem in the Hilbert space $X^s_0(\Omega)$ we may also extract a sub-subsequence $(u_n)$ converging to some $u_0 \in X^s_0(\Omega)$ in the weak topology of $X^s_0(\Omega)$. Since the dual space $(L^2(\Omega))^\ast$ is included in $(X^s_0(\Omega))^\ast$, we also have $u_n \rightharpoonup u_0$ in $L^2(\Omega)$, hence $u_0$ must coincide with the strong limit of $u_n$ in $L^2(\Omega)$ whose existence was asserted above. Passing to a further subsequence $(u_{nk})_{k \geq 1}$ we may assume that $u_{nk}(x) \to u_0(x)$ a.e. in $\Omega$ and there exists a function $h \in L^2(\Omega)$ such that $u_{nk}(x) \leq h(x)$ a.e. in $\Omega$ for all $k \geq 1$, and consequently there exists a summable function $\psi(x)$ such that

$$
A_\varepsilon u_{nk}(x) + \frac{\varepsilon}{2} u^2_{nk}(x) \leq \psi(x) \text{ a.e. in } \Omega \text{ for all } k \geq 1
$$

(cf. (28)). Hence, Lebesgue’s dominated convergence theorem applies and we have

$$
\lim_{k \to +\infty} \int_{\Omega} F(x, u_{nk}(x)) \, dx = \int_{\Omega} F(x, u_0(x)) \, dx.
$$

Recalling that the norm in a Hilbert space is weakly lower semicontinuous, we also have $\|u_0\|_{X^s_0(\Omega)} \leq \liminf_{k \to +\infty} \|u_{nk}\|_{X^s_0(\Omega)}$, and consequently

$$
J[u_0] \leq \liminf_{k \to +\infty} J[u_{nk}] = \lim_{k \to +\infty} J[u_{nk}] = \inf_{X^s_0(\Omega)} J.
$$

Thus, $u_0$ is a minimizer of $J$. 


Step 4. The minimum of $J$ over $(X^+_0(\Omega))^+$ is not a minimizer. To see this, choose a bounded function $u_1$, not vanishing identically, in the space $(X^+_0(\Omega))^+$. Let $u(x) = tu_1(x)$ for $t > 0$. We aim to verify that $J[tu_1] < 0$ when $t$ is small. By (9), for every $M > 0$ there exists $t_M > 0$ such that $f(x,t) > Mt$ for almost all $x \in \Omega$ and for all $t \in (0,t_M)$. But then $F(x, tu_1(x)) \geq Mt^2 u_1(x)/2$ whenever $tu_1(x) \in [0,t_M)$ (apart from a negligible subset of $\Omega$) and therefore

$$J[u_t] \leq t^2 \left( \| u_1 \|^2_{X^+_0(\Omega)} - \frac{M}{2} \| u_1 \|_{L^1(\Omega)} \right) \text{ for } t < \frac{t_M}{\sup_{x \in \Omega} u_1(x)}.$$ 

We may choose $M$ so large that the difference in parentheses is negative, thus getting $J[u_t] < 0$ for $t$ in a small interval $(0,t_0)$, hence

$$\min_{u \in (X^+_0(\Omega))^+} J[u] < 0.$$

Step 5. Every minimizer $u \in (X^+_0(\Omega))^+$ satisfies the variational inequality $(-\Delta)^s u \geq 0$ in the weak sense in $\Omega$. To check this, take an arbitrary function $\eta \in (X^+_0(\Omega))^+$ and a real number $t > 0$. We have

$$0 \leq J[u + t\eta] - J[u] = \frac{C_{N,s}}{4} \left( 2t(u,\eta) + t^2 \| \eta \|^2_{X^+_0(\Omega)} \right) - \int_{\Omega} \left( F(x,u(x) + t\eta(x)) - F(x,u(x)) \right) dx.$$

Since the function $F(x, u)$ is strictly increasing in the variable $u$ for almost every $x$, the integral above is non-negative. Dividing the inequality by $t$ we arrive at

$$0 \leq \frac{C_{N,s}}{4} \left( 2(u,\eta) + t \| \eta \|^2_{X^+_0(\Omega)} \right).$$

Letting $t \to 0^+$ we obtain $(u,\eta) \geq 0$, which is what we had to prove.

Step 6. Every minimizer $u \in (X^+_0(\Omega))^+$ keeps far from zero on each compact subset $K \subset \Omega$. This follows by letting $w = u$ and $b = 0$ in the strong minimum principle (Theorem 1.1), taking into account that $u$ does not vanish a.e. in $\Omega$.

Step 7. Every minimizer $u$ of the functional $J$ in $X^+_0(\Omega)$ satisfies (2). To prove this, we first take $\eta$ in the space $C^\infty_c(\Omega)$ of all functions $\eta \in C^\infty(\mathbb{R}^N)$ whose support is a compact subset of $\Omega$. Since the minimizer $u$ keeps far from zero in the support of $\eta$, the function $u_t = u + t\eta$ keeps positive as long as $|t|$ is sufficiently small, hence the function $\varphi(t) = J[u_t]$ has a minimum at $t = 0$. We have

$$\varphi(t) = \frac{C_{N,s}}{4} \left( \| u \|^2_{X^+_0(\Omega)} + 2t(u,\eta) + t^2 \| \eta \|^2_{X^+_0(\Omega)} \right) - \int_{\Omega} F(x,u(x) + t\eta(x)) dx.$$

By (27), and since $u,\eta \in L^2(\Omega)$, we may compute $\varphi'(0)$ by differentiating under the sign of integral, which yields

$$\varphi'(0) = \frac{C_{N,s}}{2} (u,\eta) - \int_{\Omega} f(x,u(x))\eta(x) dx.$$

But since $\varphi'(0) = 0$, we obtain (2) restricted to the case when $v = \eta \in C^\infty_c(\Omega)$. Finally, since the space $C^\infty_c(\Omega)$ is a dense subset of $X^+_0(\Omega)$ (see [8, Theorem 6]), equality (2) follows in its full generality.
Step 8. Conclusion. The preceding steps show the existence of a weak solution $u$ of problem (1) in the space $X^s_0(\Omega)$. Moreover, by (8) the assumption in [16, Lemma 3.2] is satisfied with $q = 2$, and therefore we have $u \in L^\infty(\Omega)$. Uniqueness follows now from the comparison principle (Theorem 1.3), and the proof of Claim (i) is complete. Finally, since the compound function $f(x,u(x))$ is also essentially bounded in $\Omega$, the results in [21, Proposition 1.1 and Theorem 1.2] apply, whence Claim (ii) and Claim (iii) follow. To this purpose recall that by Remark 1 a solution of (2) is also a weak solution of $(-\Delta)^s u = f(x,u(x))$ in the sense of [21, Definition 2.1].

Remark 2. Problem (1) may be seen as a non-local counterpart of the problems investigated in the classical papers [1, 5]. However, assumptions (5) and (6) of [1] as well as (1) and (2) in [5] involve the first eigenvalue $\lambda_1$ of the domain $\Omega$ and therefore they are domain-dependent. Proposition 1, instead, asserts the existence of a unique solution in every bounded open set $\Omega$. This is particularly useful in the proof of Theorem 1.5, which makes use of radial solutions in balls whose radii are arbitrary.

5. Constrained symmetry. The proof of Theorem 1.5 is based on the comparison with suitable radial functions. The argument and some of its variants have been used in a series of papers on local problems (see, for instance, [11, 12, 13, 14] and the references listed there) and in [15] for a non-local problem. Here we present a further improvement, which is effective under more general conditions. See [10] for a characterization of the ellipsoid through a (local) overdetermined problem.

Proof of Theorem 1.5. Assume that the overdetermined problem (14) is solvable in some bounded open set $\Omega \ni 0$, and define

$$R_1 = \min_{x \in \partial B_1} |x|, \quad R_2 = \max_{x \in \partial B_1} |x|.$$  

Suppose, contrary to the claim, that $R_1 < R_2$. In order to reach a contradiction, denote by $u_1 \in X^s_0(B_1) \cap L^\infty(\Omega) \subset X^s_0(\Omega)$ the weak solution of

$$\begin{cases}
(-\Delta)^s u_1 = f(x,u_1), & u_1 > 0 \text{ in } B_1, \\
u_1 = 0 & \text{ in } \mathbb{R}^N \setminus B_1,
\end{cases}$$

where $B_1$ is the ball of radius $R_1$ centered at 0. Existence and uniqueness of such a solution follow from Proposition 1. Since $f(x,t)$ is a radial function of $t$ by assumption, problem (29) is invariant under rotations: this and uniqueness imply that $u_1$ is a radial function. Using the comparison principle (Theorem 1.3) in $\Omega = B_1$ we get $u_1 \leq u$ in $\mathbb{R}^N$. Now let us consider a point $z_1 \in \partial \Omega \cap \partial B_1$. The last condition in (14) implies that the solution $u$ satisfies $(\partial_\nu)^s u(z_1) = q(R_1)$. Note that the inner normal $\nu_1 = -|z_1|^{-1} z_1$ to $\partial B_1$ at $z_1$ coincides with the inner normal $\nu$ to $\partial \Omega$. Since the fractional derivative $(\partial_\nu)^s u_1$ exists by Proposition 1, and both $u$ and $u_1$ are continuous at $z_1$, we arrive at

$$(\partial_\nu)^s u_1(z_1) \leq q(R_1).$$

To proceed further, let $B_2$ be the ball of radius $R_2$ centered at 0. Instead of considering the radial solution $u_2 \in X^s_0(B_2)$ which solves (29) where the subscript 1 is replaced with 2, define the rescaled radial function $v(x) = u_1(R_1 x/R_2)$ in $\mathbb{R}^N$, where the coefficient $a$ is given by

$$a = \frac{R_2^2 q(R_2)}{R_1^2 q(R_1)}.$$
The use of $v$ in place of $u_2$ is a refinement of the argument in the literature (cf. [12, 13, 14, 15]) and allows for less demanding assumptions. Since the inner fractional derivative of $v$ on $\partial B_2$ is given by

\[(\partial_{v_2})^s v(z) = a \left( \frac{R_1}{R_2} \right)^s ((\partial_{v_1})^s u_1)(R_1 z / R_2) \]

\[= \frac{q(R_2)}{q(R_1)} ((\partial_{v_1})^s u_1)(R_1 z / R_2), \]

where $\nu_2 = -|z|^{-1} z$ denotes the inner normal to $\partial B_2$ at $z$, from (30) we get

\[(\partial_{v_2})^s v(z) \leq q(R_2) \text{ for } z \in \partial B_2. \tag{31} \]

Furthermore, since $u_1$ is the weak solution of (29), by computation we find that $v$ is the weak solution of the problem

\[\begin{cases} (-\Delta)^s v = \frac{R_1^s q(R_2)}{R_2^s q(R_1)} f \left( \frac{R_1}{R_2} x, \frac{R_1^s q(R_1)}{R_2^s q(R_2)} v \right), & v > 0 \text{ in } B_2, \\ v = 0 & \text{in } \mathbb{R}^N \setminus B_2. \end{cases} \tag{32} \]

To complete the proof, we have to compare $v$ and $u$. Choose $x \in B_2$ and define $\rho = R_1 |x|$ and $\lambda = R_1^s q(R_1) v(x)$. By assumption (13) we get

\[\frac{q(R_1)}{R_1^s} f(x, v(x)) \leq \frac{q(R_2)}{R_2^s} f \left( \frac{R_1}{R_2} x, \frac{R_1^s q(R_1)}{R_2^s q(R_2)} v(x) \right).\]

This and (32) show that $v$ satisfies $(-\Delta)^s v \geq f(x, v)$ in the weak sense in $B_2$. But then by Theorem 1.3 we must have $u \leq v$ in $\mathbb{R}^N$, and therefore

\[(\partial_v)^s u(z_2) \leq (\partial_v)^s v(z_2) \]

at every point $z_2 \in \partial \Omega \cap \partial B_2$. Of course, the inner normal $\nu$ to $\partial \Omega$ coincides with the inner normal $\nu_2$ to $\partial B_2$ at $z_2$. Now the third condition in (14), together with (31), shows that $(\partial_v)^s u(z_2) = (\partial_v)^s v(z_2)$: hence we are in a position to apply Hopf’s lemma (Lemma 1.2). The function $w = v - u \geq 0$ in $\mathbb{R}^N$ satisfies $(\partial_v)^s w(z_2) = 0$ as well as $(-\Delta)^s w \geq b(x) w(x)$ in the weak sense in $\Omega$, where $b: \Omega \to \mathbb{R}$ is given by

\[b(x) = \begin{cases} \frac{f(x, v(x)) - f(x, u(x))}{v(x) - u(x)} & \text{if } v(x) > u(x); \\ 0 & \text{if } v(x) = u(x). \end{cases} \]

Since $f$ satisfies (12), and $v$ is bounded, the function $b(x)$ is bounded from below, and by Lemma 1.2 we deduce that $u = v$ in $\mathbb{R}^N$. But since the solution $u$ vanishes in $\mathbb{R}^N \setminus \Omega$, and $v$ is positive in $B_2 \supset \Omega$, we deduce $\Omega = B_2$ and the proof is complete.

**Appendix. Two equivalent expressions of $C_{N,s}$.** The constant $C_{N,s}$ is written in different forms by different authors: compare, for instance, [2, Remark 3.11] and [4, (3.2)]. Here we prove that the two expressions in (3) coincide. The proof is based on some identities involving the Bessel functions $J_\nu$ and the gamma function, which are found, for instance, in the classical reference [6].

**Proposition 2.** For all $N \geq 1$ and all $s \in (0, 1)$ the following equality holds:

\[\int_{\mathbb{R}^N} \frac{1 - \cos x_1}{|x|^{N+2s}} dx = \pi^{\frac{N}{2}} \frac{\Gamma(1-s)}{2^{2s} s \Gamma(\frac{N}{2}+s)}. \]  

(33)
Proof. Let us start with special case \( N = 1 \). In this case we have \( x_1 = x \) and, after an integration by parts, the integral in (33) reduces to
\[
\int_{-\infty}^{+\infty} \frac{1 - \cos x}{|x|^{1+2s}} \, dx = \frac{1}{s} \int_{0}^{+\infty} \frac{\sin x}{x^{2s}} \, dx. \tag{34}
\]
The last integral can be expressed in terms of the gamma function. To this aim, recall that the Bessel function \( J_{\frac{1}{2}}(x) \) is related to \( \sin x \) as follows:
\[
J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}}
\]
(see formula (14) in [6, Section 7.11, p. 79]). Hence we can apply the identity
\[
\int_{0}^{+\infty} x^{\alpha-1} J_{\nu}(x) \, dx = \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha+\nu}{2}\right)}
\]
which is valid when \(-\nu < \alpha < \frac{3}{2} \) and \( \alpha + \nu > 0 \) (see formula (19) in [6, Section 7.7.3, p. 49]). Letting \( \alpha = \frac{3}{2} - 2s \) and \( \nu = \frac{1}{2} \) we obtain
\[
\int_{0}^{+\infty} \frac{\sin x}{x^{2s}} \, dx = \sqrt{\frac{\pi}{2s}} \frac{\Gamma(1-s)}{\Gamma\left(\frac{1}{2} + s\right)}.
\tag{35}
\]
Plugging this into (34) we obtain (33) in dimension \( N = 1 \). To manage the case \( N \geq 2 \) we need to integrate over spherical surfaces of different dimensions. To this purpose, let us denote by \( B_N^R \) the \( N \)-dimensional ball in \( \mathbb{R}^N \) centered at 0 and of radius \( R \). By the change of variables \( \rho = |x| \) and \( u = |x|^{-1} x, x \neq 0 \), we get
\[
\int_{\mathbb{R}^N} \frac{1 - \cos x}{|x|^{N+2s}} \, dx = \int_{\partial B_N^1} \left( \int_{0}^{+\infty} \frac{1 - \cos(\rho u_1)}{\rho^{1+2s}} \, d\rho \right) \, d\omega
\]
\[
= 2 \int_{\partial B_N^1 \cap \{ u_1 > 0 \}} \left( \int_{0}^{+\infty} \frac{1 - \cos(\rho u_1)}{\rho^{1+2s}} \, d\rho \right) \, d\omega, \tag{36}
\]
where \( u_1 \) denotes the first component of the unit vector \( u \in \partial B_N^1 \), and \( d\omega \) is the surface element. For fixed \( u_1 > 0 \), integration by parts followed by the change of variable \( t = \rho u_1 \) yields
\[
\int_{0}^{+\infty} \frac{1 - \cos(\rho u_1)}{\rho^{1+2s}} \, d\rho = \int_{0}^{+\infty} \frac{1 - \cos(\rho u_1)}{\rho^{1+2s}} \, d\rho + \frac{u_1}{2s} \int_{0}^{+\infty} \frac{\sin(\rho u_1)}{\rho^{2s}} \, d\rho
\]
\[
= \frac{u_1^2}{2s} \int_{0}^{+\infty} \frac{\sin t}{t^{2s}} \, dt.
\]
Using this expression in (36), and recalling (35) we get
\[
\int_{\mathbb{R}^N} \frac{1 - \cos x_1}{|x|^{N+2s}} \, dx = \frac{\sqrt{\pi}}{2s} \frac{\Gamma(1-s)}{\Gamma\left(\frac{1}{2} + s\right)} \int_{\partial B_N^1 \cap \{ u_1 > 0 \}} u_1^2 \, d\omega. \tag{37}
\]
To manage the last integral, observe that for every \( \phi \in (0, \frac{\pi}{2}) \), the set of all \( u \in \partial B_N^1 \) such that \( u_1 = \sin \phi \) is the Cartesian product \( \{ \sin \phi \} \times \partial B_{\cos \phi}^{N-1} \). Since the area of
the spherical surface is given by
\[ |\partial B_1^{N-1}| = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \]

we may write
\[
\int_{\partial B_1^{N}\cap\{u_1>0\}} u_1^{2s} \, d\omega = \int_0^{\frac{\pi}{2}} (\sin \phi)^{2s} |\partial B^{N-1}_{\cos \phi}| \, d\phi \\
= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2s} (\cos \phi)^{N-2} \, d\phi.
\]

By applying formula (19) in [6, Vol. 1, Section 1.5.1, p. 10] to evaluate the last integral, we obtain
\[
\int_0^{\frac{\pi}{2}} (\sin \phi)^{2s} (\cos \phi)^{N-2} \, d\phi = \frac{\Gamma\left(\frac{1}{2} + s\right) \Gamma\left(\frac{N-1}{2}\right)}{2 \Gamma\left(\frac{N}{2} + s\right)}
\]

and therefore
\[
\int_{\partial B_1^{N}\cap\{u_1>0\}} u_1^{2s} \, d\omega = \frac{\pi^{\frac{N-1}{2}} \Gamma\left(\frac{1}{2} + s\right)}{\Gamma\left(\frac{N}{2} + s\right)}.
\]

Inserting this expression into (37), the claim follows. \(\square\)

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REFERENCES

[1] H. Brézis and L. Oswald, Remarks on sublinear elliptic equations, Nonlinear Anal., 10 (1986), 55–64.
[2] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 23–53.
[3] L. Cesari, Optimization - Theory and Applications, Springer-Verlag, 1983.
[4] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), 521–573.
[5] J. I. Díaz and J. E. Saa, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, C. R. Acad. Sci. Paris Sér. I Math., 305 (1987), 521–524.
[6] A. Erdélyi (Editor), Higher Transcendental Functions, McGraw-Hill, 1953.
[7] M. M. Fall and S. Jarche, Overdetermined problems with fractional Laplacian, ESAIM Control Optim. Calc. Var., 21 (2015), 924–938.
[8] A. Fiscella, R. Servadei and E. Valdinoci, Density properties for fractional Sobolev spaces, Annales Academiae Scientiarum Fennicae Mathematica, 40 (2015), 235–253.
[9] R. K. Getoor, First passage times for symmetric stable processes in space, Trans. Amer. Math. Soc., 101 (1961), 75–90.
[10] A. Greco, A characterization of the ellipsoid through the torsion problem, J. Appl. Math. Phys. (ZAMP), 59 (2008), 753–765.
[11] A. Greco, Boundary point lemmas and overdetermined problems, J. Math. Anal. Appl., 278 (2003), 214–224.
[12] A. Greco, Comparison principle and constrained radial symmetry for the subdiffusive p-Laplacian, Publ. Mat., 58 (2014), 485–498.
A. Greco, Constrained radial symmetry for the infinity-Laplacian, *Nonlinear Analysis: Real World Applications*, **37** (2017), 239–248.

A. Greco, Symmetry around the origin for some overdetermined problems, *Adv. Math. Sci. Appl.*, **13** (2003), 387–399.

A. Greco and R. Servadei, Hopf’s lemma and constrained radial symmetry for the fractional Laplacian, *Math. Res. Lett.*, **23** (2016), 863–885.

A. Iannizzotto, S. Mosconi and M. Squassina, *H* versus *C*0-weighted minimizers, *Nonlinear Differ. Equ. Appl.*, **22** (2015), 477–497.

S. Jarohs and T. Weth, On the strong maximum principle for nonlocal operators, preprint, arXiv:1702.08767.

T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, *Discrete Contin. Dynam. Syst.*, **35** (2015), 6031–6068.

V. Mascia, *Un Problema Sublineare Non Locale* (Italian), Thesis. University of Cagliari, 2017.

G. Molica Bisci and V. D. Rădulescu, Multiplicity results for elliptic fractional equations with subcritical term, *Nonlinear Differ. Equ. Appl.*, **22** (2015), 721–739.

X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *J. Math. Pures Appl.*, **101** (2014), 275–302.

R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, **367** (2015), 67–102.

E. M. Stein, The characterization of functions arising as potentials, *Bull. Amer. Math. Soc.*, **67** (1961), 102–104.

R. L. Wheeden, On hypersingular integrals and Lebesgue spaces of differentiable functions, *Trans. Amer. Math. Soc.*, **134** (1968), 421–435.

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