q-Peano Kernel and Its Applications

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Abstract

We introduce a $q$-analogue of the Peano kernel theorem by replacing ordinary derivatives and integrals by quantum derivatives and quantum integrals. In the limit $q \to 1$, the $q$-Peano kernel reduces to the classical Peano kernel. We also give applications to polynomial interpolation and construct examples in which classical remainder theory fails whereas $q$-Peano kernel works. Furthermore we derive a relation between $q$-B-splines and divided differences via the $q$-Peano kernel.

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1 Introduction

The Peano kernel theorem provides a useful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. The errors are represented by a linear functional that operates on functions $f \in C^{n+1}[a, b]$ and annihilates all polynomials of degree at most $n$.

Namely, if $L(f) = 0$ for all $f \in P_n$, the space of polynomials of degree $n$, then

$$L(f) = \int_a^b f^{(n+1)}(t)K(x,t)dt,$$

where $K(x,t) = \frac{1}{n!}L((x-t)^n).$

An important application of this result is the Kowalewski’s interpolating polynomial remainder. Let $t_0, t_1, \ldots, t_n \in [a, b]$ be fixed and distinct, and

$$L(f) = f(x) - \sum_{k=0}^n f(t_k)I_{nk}(x)$$

where $I_{nk}(x) = \prod_{v \neq k} \frac{x-t_v}{t_k-t_v}$. If $f \in C^{m+1}[a, b]$, then

$$L(f) = \frac{1}{m!} \sum_{k=0}^n I_{nk}(x) \int_{t_k}^x (t-t)^m f^{(m+1)}(t)dt,$$

for each $m = 0, 1, \ldots, n$.

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is the error functional, see [5]. Our purpose is to extend the Peano kernel when classical derivatives are replaced by $q$-derivatives. This extension is important because there are functions whose $q$-derivatives exist but whose classical derivatives fail to exist.

Section 2 contains definitions and properties of the quantum calculus which we use in the next sections. In Section 3, we give the $q$-Taylor theorem and develop a $q$-analogue of the Peano kernel ($q$-Peano kernel). Furthermore, we present a simple way to find the kernel under some conditions. Section 4 demonstrates how the $q$-Peano kernel is used to find the error of Lagrange interpolation. A $q$-analogue of the trapezoidal rule is also given. Moreover, we discuss the error bounds of quadrature formula on the remainder. Finally, we establish a relation between the $q$-B-splines and the $q$-Peano kernel in Section 5.

### 2 Preliminaries

We begin by giving basic definitions and theorems of the $q$-calculus that are required in the next section. For a fixed parameter $q \neq 1$, the $q$-derivatives are defined by,

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}$$

$$D^n_q f(t) = D_q(D^n_{q-1} f(t)), \quad n \geq 2.$$  

Note that if $f$ is a differentiable function, then

$$\lim_{q \to 1} D_q f(x) = D f(x).$$

For polynomials the $q$-derivative is easy to compute. Indeed it follows easily from the definition of the $q$-derivative that

$$D_q x^n = [n]_q x^{n-1},$$

where the $q$-integers $[n]_q$ are defined by,

$$[n]_q = \begin{cases} 
(1-q^n)/(1-q), & q \neq 1, \\
 n, & q = 1.
\end{cases}$$

Moreover, the $q$-factorial is defined by

$$[n]_q! = [1]_q \cdots [n]_q.$$  

Quantum integrals are the analogues of classical integrals for the quantum calculus. Quantum integrals satisfy a quantum version of the fundamental theorem of calculus, see [7] for details.

**Definition 2.1.** Let $0 < a < b$. Then the definite $q$-integral of a function $f(x)$ is defined by

$$\int_0^b f(x)d_q x = (1-q)b \sum_{i=0}^\infty q^i f(q^i b)$$

and

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x.$$  

**Theorem 2.2.** [Fundamental Theorem of Calculus]

If $F(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q F(x)d_q x = F(b) - F(a)$$

where $0 \leq a < b \leq \infty$.

The work [11] gives the mean value theorem in the $q$-calculus which will be needed in one of our results.
Theorem 2.3. If $F$ is continuous and $G$ is $1/q$-integrable and is nonnegative (or nonpositive) on $[a, b]$, then there exists $\tilde{q} \in (1, \infty)$ such that for all $q > \tilde{q}$ there exists a $\tilde{\xi} \in (a, b)$ for which

$$\int_a^b F(x)G(x)d_{1/q}x = F(\tilde{\xi}) \int_a^b G(x)d_{1/\tilde{q}}x.$$  

We also require a $q$-Hölder inequality and appropriate notions of distance in $q$-integrals, see [2], [4] and [13].

Definition 2.4. We will denote by $L_{p,q}([0,b])$ with $1 \leq p < \infty$ the set of all functions $f$ on $[0,b]$ such that

$$||f||_{p,q} := \left( \int_0^b |f|^q d_{1/q}t \right)^{1/q} < \infty.$$  

Furthermore let $L_{\infty,q}([0,b])$ denote the set of all functions $f$ on $[0,b]$ such that

$$||f||_{\infty,q} := \sup_{x \in [0,b]} |f(x)| < \infty.$$  

Theorem 2.5. Let $x \in [0,b]$, $q \in [1, \infty)$ and $p_1, p_2 > 1$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then

$$\int_0^x |f(x)||g(x)|d_{1/q}t \leq \left( \int_0^x |f(x)|^{p_1}d_{1/q}t \right)^{1/p_1} \left( \int_0^x |g(x)|^{p_2}d_{1/q}t \right)^{1/p_2}.$$  

3 $q$-Peano Kernel Theorem

In this section we derive a generalization of the Peano kernel theorem. This generalization is based on the $q$-Taylor expansion analogous to the proof of the classical Peano kernel Theorem. So we start by giving the $q$-Taylor expansion with integral representation. A detailed treatment of the classical Peano Kernel theorem can be found in [5], [9] and [10].

We use the notation $q$-$C^k[a,b]$ to denote the space of functions whose $q$-derivatives of order up to $k$ are continuous on $[a,b]$.

Theorem 3.1. ($q$-Taylor Theorem) Let $f$ be $n+1$ times $1/q$-differentiable in the closed interval $[a,b]$. Then

$$f(x) = \sum_{k=0}^n q^k \frac{k^{k-1/2}(D_{1/q}^k)(q^k a)}{[k]_q!} (x-a)^k + R_a(f),$$  

where

$$(x-t)^{n,q} = (x-q^{n-1}t) \cdots (x-qt)(x-t)$$  

and

$$R_a(f) = q^n \frac{n^{n+1/2}}{[n]_q!} \int_a^x (D_{1/q}^n f)(q^n t)(x-t)^{n,q}d_{1/q}t.$$  

Another way to express the remainder $R_a f$ is to employ the truncated power function. That is

$$R_a(f) = q^n \frac{n^{n+1/2}}{[n]_q!} \int_a^b (D_{1/q}^n f)(q^n t)(x-t)^{n,q}_+d_{1/q}t,$$  

where

$$(x-t)^{n,q}_+ = (x-q^{n-1}t) \cdots (x-qt)(x-t)_+.$$  

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Here \((x - t)_+\) is the truncated power function
\[
(x - t)_+ = \begin{cases} 
  x - t, & \text{if } x > t \\
  0, & \text{otherwise}.
\end{cases}
\]

There are other forms of \(q\)-Taylor Theorem, see for example [11, 8, 6].

**Theorem 3.2.** Let \(g_t(x) = (x - t)^{n,q}\) and let \(L\) be a linear functional that commutes with the operation of \(q\)-integration and also satisfies the conditions: \(L(g_t)\) exists and \(L(f) = 0\) for all \(f \in \mathcal{P}_n\). Then for all \(f \in 1/q - C^{n+1}[a,b]\)
\[
L(f) = \int_a^b (D^q_{1/q} f)(q^n t) K(x,t) d_{1/q}^t,
\]
where
\[
K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L(g_t).
\]

**Proof.** Recall that here the function \((x - t)^{n,q}\) is a function of \(t\) and \(x\) behaves as a parameter. When we say \(L(g_t)\) we mean that \(L\) is applied to the truncated power function, regarded as a function of \(x\) with \(t\) as a parameter. Hence we find real number that depends on \(t\). We apply \(L\) to the equation \(1\). Since \(L\) is linear and annihilates polynomials, we have
\[
L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D^q_{1/q} f)(q^n t) (x - t)^{n,q} d_{1/q}^t.
\]
Since \(L\) commutes with the operation of \(q\)-integration,
\[
L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D^q_{1/q} f)(q^n t) L((x - t)^{n,q}) d_{1/q}^t.
\]

**Corollary 3.3.** If the conditions in Theorem 3.2 are satisfied and also the kernel \(K(x,t)\) does not change sign on \([a,b]\), then
\[
L(f) = \left( D^q_{1/q} f \right) \left( \frac{\xi}{[n+1]_q} \right) q^{n(n+1)/2} L(x^{n+1})
\]

**Proof.** Since \(D^q_{1/q} f\) is continuous and \(K(x,t)\) does not change sign on \([a,b]\), we can apply the Mean Value Theorem 2.3. Thus we have
\[
L(f) = \left( D^q_{1/q} f \right) \left( \frac{\xi}{[n+1]_q} \right) \int_a^b K(x,t) d_{1/q}^t, \quad a < \xi < b.
\]
Replacing \(f(x)\) by \(x^{n+1}\) gives
\[
L(x^{n+1}) = \frac{[n+1]_q!}{q^{n(n+1)/2}} \int_a^b K(x,t) d_{1/q}^t,
\]
so
\[
\int_a^b K(x,t) d_{1/q}^t = \frac{q^{n(n+1)/2}}{[n+1]_q!} L(x^{n+1}),
\]
and this completes the proof.
4 Application to polynomial interpolation

The main idea in this section is to apply the $q$-Peano kernel Theorem on the remainder of polynomial interpolation. Findings demonstrate the advantage of using the $q$-Peano kernel Theorem where the classical theorem does not work.

**Proposition 4.1.** Suppose $t_0, t_1, \ldots, t_n \in [a, b]$ are distinct points. For a fixed $x \in [a, b]$, define the corresponding error functional by

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x).$$

Then

$$L(f) = \frac{q^{m(m+1)/2}}{|m|_q^1} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} \quad \text{for each } m = 0, 1, \ldots, n.$$

**Proof.** Since $\sum_{k=0}^{n} l_{nk}(x) = 1$, by the $q$-Peano kernel Theorem 3.2 we get,

$$\frac{|m|_q^1}{q^{m(m+1)/2}} K(x, t) = L((x - t)^{m,q} +) = (x - t)^{m,q} - \sum_{k=0}^{n} (t_k - t)^{m,q} l_{nk}(x) = \sum_{k=0}^{n} [(x - t)^{m,q} - (t_k - t)^{m,q} -] l_{nk}(x).$$

From the fact that

$$\int_{a}^{b} [(x - t)^{m,q} - (t_k - t)^{m,q} -] \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} = \int_{a}^{x} [(x - t)^{m,q} - (t_k - t)^{m,q} -] \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} + \int_{t_k}^{x} (t_k - t)^{m,q} \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q}$$

we have

$$\frac{|m|_q^1}{q^{m(m+1)/2}} \int_{a}^{b} K(x, t) \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} = \int_{a}^{x} \left( D_{1/q}^{m+1} f \right) (q^m t) \sum_{k=0}^{n} [(x - t)^{m,q} - (t_k - t)^{m,q} -] l_{nk}(x) dt_{1/q} + \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q}.$$

For each $m \leq n$, since the interpolation operator is a projection, it reproduces polynomials and the term in the square brackets vanishes in the last equation for $f(x) = (x - t)^{m,q}$. Accordingly,

$$L(f) = \int_{a}^{b} K(x, t) \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} = \frac{q^{m(m+1)/2}}{|m|_q^1} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left( D_{1/q}^{m+1} f \right) (q^m t) dt_{1/q} \quad \text{for each } m = 0, 1, \ldots, n.$$

Now we give examples that show how we can find the $q$-Peano kernel.
Example 4.2. Suppose that we interpolate a function $f \in 1/q - C^3[-1,1]$ by a polynomial $p \in \mathcal{P}_2$. Here $n = 2$ and $m = 2$. Let $t_0 = -1$, $t_1 = 0$, $t_2 = 1$. Then the error function becomes

$$L(f) = \frac{q^3}{2^q} \sum_{k=0}^{2} I_{2k}(x) \int_{\frac{x}{t_k}}^{1} (l_k - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t$$

with $l_{20}(x) = \frac{1}{2} x (x - 1)$, $l_{21}(x) = (1 - x^2)$, $l_{22}(x) = \frac{1}{2} x (x + 1)$. Then,

$$\frac{[2]^1}{q^2} L(f) = \int_{-1}^{x} (-1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t - l_{21}(x) \int_{x}^{1} (-1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t + l_{22}(x) \int_{0}^{1} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t.$$

Now if $x \leq 0$, then

$$\frac{[2]^1}{q^2} L(f) = \int_{-1}^{x} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t - l_{22}(x) \int_{0}^{1} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t.$$

Hence,

$$L(f) = \frac{q^3}{2^q} \int_{-1}^{1} K(x,t) \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t$$

where

$$K(x,t) = \begin{cases} l_{20}(x)(-1 - t)^{2q}, & -1 \leq t \leq x \\ -l_{21}(x)(1 - t)^{2q} - l_{22}(x)(1 - t)^{2q}, & x \leq t \leq 0 \\ -l_{22}(x)(1 - t)^{2q}, & 0 \leq t \leq 1. \end{cases}$$

Similarly for $x \geq 0$,

$$\frac{[2]^1}{q^2} L(f) = \int_{-1}^{x} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t + l_{20}(x) \int_{0}^{1} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t + l_{21}(x) \int_{0}^{1} (1 - t)^{2q} \left(D_{1/4}^q f\right) (q^2 t) d_{1/q} t$$

and the Peano kernel becomes

$$K(x,t) = \begin{cases} l_{20}(x)(-1 - t)^{2q}, & -1 \leq t \leq 0 \\ l_{20}(x)(-1 - t)^{2q} + l_{21}(x)(-t)^{2q} - l_{22}(x)(1 - t)^{2q}, & 0 \leq t \leq x \\ -l_{22}(x)(1 - t)^{2q}, & x \leq t \leq 1. \end{cases}$$
Example 4.3. Let

\[
f(x) = \begin{cases} 
\frac{q^3 x^3}{6}, & 0 \leq x < 1 \\
\frac{1}{6} (4 - 4[3]_q x + 4q[3]_q x^2 - 3q^3 x^3), & 1 \leq x < 2 \\
\frac{1}{6} (-44 + 20[3]_q x - 8q[3]_q x^2 + 3q^3 x^3), & 2 \leq x < 3 \\
-\frac{1}{6} (-4 + x)(-4 + qx)(-4 + q^2 x), & 3 \leq x < 4 \\
0, & \text{otherwise.}
\end{cases}
\]

It is obvious that for \( q \neq 1 \), \( f \in C[0,4] \) but \( f \notin C^1[0,4] \). However, one may check that \( f \in 1/q - C^2[0,4] \). Classical error functionals cannot work but we may find the error via the \( q \)-Peano kernel theorem. Let \( t_0 = 0, t_1 = 1 \) and \( t_2 = 4 \). Then the error functional

\[
L(f) = q \sum_{k=0}^{2} l_{2k}(x) \int_{a}^{x} (t_k - t) \left( D^{2}_{1/q} f \right) (qt) dt/q
\]

where \( l_{20}(x) = \frac{1}{8} (x - 2)(x - 4) \), \( l_{21}(x) = -\frac{1}{4} x(x - 4) \) and \( l_{22}(x) = \frac{1}{8} x(x - 2) \). Then,

\[
\frac{1}{q} L(f) = l_{20}(x) \int_{0}^{x} (-t) \left( D^{2}_{1/q} f \right) (qt) dt/q + l_{21}(x) \int_{2}^{x} (2 - t) \left( D^{2}_{1/q} f \right) (qt) dt/q + l_{22}(x) \int_{4}^{x} (4 - t) \left( D^{2}_{1/q} f \right) (qt) dt/q.
\]

Now we will find the kernel. If \( 0 \leq x < 2 \), then

\[
K(x,t) = \begin{cases} 
-l_{20}(x)t, & 0 \leq t < x \\
l_{21}(x)(2 - t) - l_{22}(x)(4 - t), & x \leq t < 2 \\
l_{22}(x)(4 - t), & 2 \leq t < 4.
\end{cases}
\]

Similarly, for \( 2 \leq x < 4 \),

\[
K(x,t) = \begin{cases} 
-l_{20}(x)t, & 0 \leq t < 2 \\
-l_{20}(x)t + l_{21}(x)(2 - t), & 2 \leq t < x \\
l_{21}(x)(2 - t) - l_{22}(x)(4 - t), & x \leq t < 4.
\end{cases}
\]

The function \( f(x) \) given above is indeed a cubic \( q \)-B-spline. \( q \)-B-splines form a basis for quantum splines which are piecewise polynomials whose quantum derivatives agree up to some order at the joins, see [12].

### 4.1 Trapezoidal rule in \( q \)-integration

Consider the \( 1/q \)-integral of a function \( f \) on the interval \([a,b]\). We want to evaluate the \( q \)-integral approximately using linear interpolant formula. That is,
Let us define the operator $L$ as
\[ L(f) = \int_a^b f(x) \frac{d_{1/q}x}{x} - b - aq f(a) - b - aq f(b). \]

Since $L(f) = 0$ for all functions $f \in \mathcal{P}_1$, for all $f \in 1/q - C^2[a,b]$ we have
\[ L(f) = \int_a^b \left(D_{1/q}^2 f\right)(qt)K(x,t)\frac{d_{1/q}t}{t} \]

and
\[ K(x,t) = qL((x-t)^+). \]

What follows we find the kernel $K(x,t)$. First,
\[ K(x,t) = q \left\{ \int_t^b (x-t)_{+}d_{1/q}x - b - aq (a-t)_{+} - b - aq (b-t)_{+} \right\}. \]

Then for $t \in [a,b],$
\[ \int_a^b (x-t)_{+}d_{1/q}x = \int_t^b (x-t)d_{1/q}x, \quad (a-t)_{+} = 0 \quad \text{and} \quad (b-t)_{+} = (b-t). \]

Thus,
\[ K(x,t) = q \left\{ \int_t^b (x-t)d_{1/q}x - b - aq (b-t) \right\} = q \left\{ \int_t^b (b-t)(b - \xi)_{+} - b - aq (b-t) \right\} = q \frac{[2]_q}{[2]_q}(b-t)(a-t) \]

for $a \leq t \leq b$.

Notice that $K(x,t) < 0$ on $[a,b]$. Then we can apply Mean Value Theorem. So, we have
\[ L(f) = D_{1/q}^2 f(\xi), \]

where
\[ L(x^2) = \int_a^b x^2d_{1/q}x - b - aq a^2 - b - aq b^2 \]
\[ = \frac{b^3 - a^3}{[3]_q!} - b - aq a^2 - b - aq b^2 \]
\[ = \frac{-(b-a)(bq-a)(b-aq)}{[3]_q!}. \]
Finally, we derive

\[
L(f) = \int_a^b f(x) \frac{d_{1/q}x}{2q} - \frac{b-aq}{2q} f(a) - \frac{bq-a}{2q} f(b)
\]

\[
= -q(b-a)(bq-a)(b-aq) \frac{1}{[3]q!} [2]q! D_{1/q}^2 f(\xi)
\]

where \(a < \xi < b\).

When \(q = 1\), the above equation reduces to the well-known trapezoidal rule, see [9].

### 4.2 Remainder on quadrature

We now discuss error bounds of quadrature formulas on remainders given by

\[
R_n(f; q) = \int_0^b f(x) d_{1/q}x - \sum_{k=0}^n \gamma_k f(t_k)
\]

which appear in numerical integration. Assuming \(f \in 1/q - C^{m+1}[0,b]\) and \(R_n(f; q) = 0\) for all \(f \in \mathcal{P}_m\), we can apply the \(q\)-Peano kernel theorem. Hence

\[
R_n(f; q) = \int_0^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^{m} t) d_{1/q} t.
\]

By applying the \(q\)-Hölder inequality, we have

\[
|R_n(f; q)| \leq \left[ \int_0^b \left|D_{1/q}^{m+1} f\right|^p (q^{m} t) d_{1/q} t \right]^{1/p} \left[ \int_0^b |K(x,t)|^q d_{1/q} t \right]^{1/q}
\]

for all \(1 \leq p, q \leq \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Since the second integral in the above equation is independent of \(f\), by choosing coefficients and nodes appropriately we can minimize the remainder.

(i) For \(p_1 = \infty\) and \(p_2 = 1\),

\[
|R_n(f; q)| \leq \left| D_{1/q}^{m+1} f \right|_\infty \int_0^b |K(x,t)| d_{1/q} t
\]

(ii) For \(p_1 = p_2 = 2\),

\[
|R_n(f; q)| \leq \left| D_{1/q}^{m+1} f \right|_2 \left[ \int_0^b |K(x,t)|^2 d_{1/q} t \right]^{1/2}
\]

The Peano kernel \(K(x,t)\) can be written as

\[
K(x,t) = q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1} q}{[m+1]q^q} - s(t; q),
\]

where \(s(t; q) = \frac{q^{m(m+1)/2}}{[m]q^q} \sum_{k=0}^n \gamma_k (t_k - t)^m q\) is a quantum spline with the knot sequence \(\{t_k\}_{k=0,...,n}\).

Eventually, the problem of minimizing the \(q\)-integral

\[
\left[ \int_0^b |K(x,t)|^p d_{1/q} t \right]^{1/p}
\]

is equivalent to finding the best approximation of the polynomial \(q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1} q}{[m+1]q^q} \) in \(t\) by a quantum spline with respect to the norm \(||||_{p_1}\).
5 Application to divided differences

For about a half century, B-splines have played a central role in approximation theory, geometric modeling and wavelets. Recently their $q$-analogues or quantum B-splines has been introduced and studied in [2], [3].

In this section we establish certain relations between $q$-B-splines and $q$-Peano kernels. When $q = 1$, Theorem 5.1 reduces to its classical counterpart which can be found in [10].

The work [3] finds that $q$-B-splines of degree $n$ are essentially divided differences of $q$-truncated power functions. That is, the $q$-B-splines are given by

$$N_{k,n}(t;q) = (t_{k+n} - t_k)[t_k, \ldots, t_{k+n}](x-t)_+^{n,q}. $$

Now recall the fact that a divided difference $f[t_0, t_1, \ldots, t_{n+1}]$ can be represented as symmetric sum of $f(t_j)$, see [10],

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} f(t_i) \prod_{j \neq i} (t_i - t_j). \tag{3} $$

Hence we can readily derive

$$N_{k,n}(t;q) = (t_{k+n} - t_k) \sum_{i=k}^{k+n} (t_i - t_k)_+^{n,q} \prod_{j \neq i} \frac{1}{(t_i - t_j)}. $$

The following theorem is also derived in [3] by a different method.

**Theorem 5.1.**

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{|n|q!} \int_a^b N_{0,n}(t;q) \left( \frac{D_{1/q}^{n+1} f}{t_{n+1} - t_0} \right) (q^2 t) d_{1/q} t. $$

**Proof.** We first set $L$ as

$$f[t_0, t_1, \ldots, t_{n+1}] = \sum_{i=0}^{n+1} f(t_i) \prod_{j \neq i} (t_i - t_j) = L(f). $$

We see that, for any fixed and distinct points $\{t_i : i = 0, 1, \ldots, n+1\}$, $L$ is a bounded linear operator. From the $q$-Peano Kernel Theorem [5,2] we have

$$L(f) = \int_a^b K(x,t) (D_{1/q}^{n+1} f)(q^2 t) d_{1/q} t, $$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{|n|q!} L \left( (x-t)_+^{n,q} \right) = \frac{q^{n(n+1)/2}}{|n|q!} \sum_{i=0}^{n+1} (t_i - t)_+^{n,q} \prod_{j \neq i} (t_i - t_j). $$

Thus

$$K(x,t) = \frac{q^{n(n+1)/2} N_{0,n}(t;q)}{|n|q!} \frac{1}{t_{n+1} - t_0} $$

Combining the last equation with (3) we derive

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{|n|q!} \int_a^b N_{0,n}(t;q) \left( \frac{D_{1/q}^{n+1} f}{t_{n+1} - t_0} \right) (q^2 t) d_{1/q} t. $$

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