Nonminimal Maxwell-Chern-Simons-O(3)–σ vortices: asymmetric potential case

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In this work we study a nonlinear gauged O(3)-sigma model with both minimal and nonminimal coupling in the covariant derivative. Using an asymmetric scalar potential, the model is found to exhibit both topological and non-topological soliton solutions in the Bogomol’nyi limit.

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I. THE (2+1) NONMINIMAL MODEL

The $O(N)−\sigma$ model is a generalization of the $O(4)$ model, introduced by Gell-Mann and Levy [1]. Classically, the field configurations are $\varphi$ mappings from a space-time to a target space, which is a multiplet of $N$ scalar real fields $\varphi_i$, $i = 1, \ldots, N$, under the constraint $\varphi^2 = \varphi_i \varphi_i = 1$.

Since solitonic solutions have some importance in Condensed Matter systems [2 3 4], especially in two-dimensional isotropic ferromagnets [5 6], some authors have studied gauge invariance of the $O(3)$-model where the group $U(1)$ is a subgroup of $O(3)$.

However, since these studies, Stern has shown that the inclusion of nonminimal coupling in Maxwell-Chern-Simons electrodynamics tends to mimic anyonic behavior without the pure Chern-Simons limit [7].

In the $O(3)$-σ model, the complex scalar field $\phi$ is a 3-vector constrained to satisfy the relation $\phi \cdot \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$.

We are looking for a Lagrangian invariant under global iso-rotations of the field $\phi$ about a fixed axis $\mathbf{n} \in S^2$. In order to gauge the symmetry, we choose $\mathbf{n} = (0,0,1)$. The 3D-gauge invariant Lagrangian density is then given by

$$L = -\frac{1}{2} F_{\mu \nu}^2 + \frac{\kappa}{4} \epsilon_{\mu \nu \rho} A_\mu F_{\nu \rho} + \frac{1}{2} \nabla \phi \cdot \nabla \phi$$

$$+ \frac{1}{2} \partial_\mu M \partial^\mu M - \frac{1}{2} g \partial_\mu M \partial^\mu (\mathbf{n} \cdot \phi) - U(M, \mathbf{n} \cdot \phi)$$

The first two terms in Eq. (1) are the normal Maxwell and Chern-Simons terms, $M$ is a neutral scalar field and $\phi$ is the multiplet field, as already defined. The introduction of the $M$ field is explained by supersymmetry arguments [8]. The gauge covariant derivative is modified in order to introduce non-minimal coupling, $i.e.$,

$$\nabla \phi = \partial_\mu \phi + \left[ e A_\mu + \frac{\kappa}{2} \epsilon_{\mu \nu \rho} F^{\nu \rho} \right] \mathbf{n} \times \phi$$

The real parameter $g$ introduced in Eq. (1) and in the covariant derivative is a nonminimal coupling constant. The equation of motion for the gauge field $A_\mu$ can be written as

$$\partial_\mu \left[ \epsilon_{\mu \nu \rho} \left( \frac{g}{e} J_\nu + F_{\nu \rho} \right) \right] = J^\alpha - \kappa F^\alpha, \quad (3)$$

where we have used the $F^{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho} F^{\rho \sigma}$ dual field and the matter current $J^\mu = -\epsilon n^\mu \phi \times \nabla \phi$. Assuming a critical coupling $g_c = -\frac{\kappa}{e}$, we obtain a topological first order equation [10 11],

$$F^\alpha = \frac{1}{\kappa} J^\alpha. \quad (4)$$

The above equation was first used by Stern [7] and later by Torres [12 13].

As in [14], we can now construct a general topological gauge invariant current

$$J^\mu_{top} = \frac{1}{8\pi} \epsilon_{\mu \nu \sigma} \phi \left[ D_\nu \phi \times D_\sigma \phi - \frac{e}{2} F_{\nu \sigma} (v - \mathbf{n} \cdot \phi) \right]$$

(5)

with $v$ as a real parameter. This current yields a topologically conserved charge given by

$$Q_{top} = \frac{1}{4\pi} \int d^3 x \left[ \phi \cdot D_\mu \phi \times D_\nu \phi + \frac{eB}{2} (v - \mathbf{n} \cdot \phi) \right]. \quad (6)$$

We can impose different boundary conditions at spatial infinity for finite energy field configurations. Depending on such conditions, the solutions belong to a broken phase or to a symmetric one. In the $v = 1$ symmetric phase we have $\lim_{|x| \to \infty} \phi(t,x) = \pm \mathbf{n}$. On the other hand, for $|v| < 1$, we can impose $\lim_{|x| \to \infty} -\mathbf{n} \cdot \phi(t,x) = v$ which characterizes the asymmetric (broken) phase.

II. BOGOMOLO'NYI EQUATIONS

The derivative of the Lagrangian (1) with respect to the space-time metric tensor $g_{\mu \nu}$ produces a naturally symmetric tensor in the $\mu$ and $\nu$ indices, namely,

$$T_{\mu \nu} = G(g, \phi) F^\alpha_{\mu \nu} F_{\alpha \nu} - \frac{1}{2} \left[ \partial_\mu M \partial_\nu (\mathbf{n} \cdot \phi) + \partial_\nu M \partial_\mu (\mathbf{n} \cdot \phi) \right] + D_\mu \phi \cdot D_\nu \phi + \partial_\mu M \partial_\nu M - \eta_{\mu \nu} L_{ntop}, \quad (7)$$

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where \( G(g, \phi) = 1 - g^2 \langle n \times \phi \rangle^2 \) and \( \eta_{\mu\nu} \) stands for flat metric \((1, -1, -1, -1)\). \( \mathcal{L}_{\text{top}} \) differs from \( \mathcal{L} \) only in the topological terms. The energy functional is given by

\[
T_{00} = \frac{1}{2}G \left( B^2 + E^2 \right) + \frac{1}{2}g_{\mu0}M\partial_\mu M + \frac{1}{2}g_{\nu0}M\partial_\nu M - g_{\mu0}M\partial_\mu (n \cdot \phi) - g_{\nu0}M\partial_\nu (n \cdot \phi) + \frac{1}{2}D_\nu \phi \cdot D_\nu \phi + \frac{1}{2}D_\mu \phi \cdot D_\mu \phi + U(M, n \cdot \phi). \tag{8}
\]

We can establish the following expression

\[
\frac{1}{2}D_\mu \phi \cdot D_\nu \phi = -\frac{1}{2}g^2 \langle n \times \phi \rangle^2 E^2 + \frac{1}{2}g^2 F_i k_i + gE_i \partial_i (n \cdot \phi) + \frac{1}{2}(\nabla_i \phi \pm \nabla_2 \phi)^2 \pm \phi \cdot D_1 \phi \cdot D_2 \phi. \tag{9}
\]

where we have used \( k^\mu \equiv -cn \cdot \phi \times D^\mu \phi \), which contains no explicit contribution of the nonminimal coupling. Using Eq. 1 in Eq. 8 and integrating it over the whole space, we obtain the total energy \( \mathcal{E} \),

\[
\mathcal{E} = \int d^2 x \left\{ \frac{1}{2}G \left[ B \pm \frac{e}{G} \left( v - n \cdot \phi + \frac{\kappa}{e} M \right) \right]^2 \right\} \pm \epsilon B \left[ v - n \cdot \phi + \frac{\kappa}{e} M \right] + g \langle n \times \phi \rangle^2 M^2 \pm \frac{1}{2} (E_i \pm \partial_\mu M - 1) g_{\mu\nu} \partial_\nu (n \cdot \phi) - g_{\mu0} M \partial_\mu (n \cdot \phi) + \frac{1}{2}D_\mu \phi \pm \epsilon M \left( J_0 - \epsilon g B (n \times \phi) \right) - \frac{1}{2}g^2 \langle n \times \phi \rangle^2 E^2 \pm \frac{1}{2}g^2 F_i k_i + gE_i \partial_i (n \cdot \phi) + \frac{1}{2}(\nabla_i \phi \pm \nabla_2 \phi) \pm \phi \cdot D_1 \phi \cdot D_2 \phi + U(M, n \cdot \phi). \tag{10}
\]

In order to achieve the Bogomol’nyi limit, we must choose an adequate potential given by

\[
U = \frac{e^2}{2G} \left[ v - n \cdot \phi + \frac{\kappa}{e} M + g \langle n \times \phi \rangle^2 M \right]^2 + \frac{1}{2}e^2 M^2 \langle n \times \phi \rangle^2. \tag{11}
\]

The above potential is a generalization of potentials found in the literature, involving Maxwell and Chern-Simons terms \[14, 15, 16, 17\]. In the present case, the following equations are necessary to achieve the energy bound limit,

\[
B \pm \frac{e}{G} \left[ v - n \cdot \phi + \frac{\kappa}{e} M + g \langle n \times \phi \rangle^2 M \right] = 0 \tag{12a}
\]

\[
E_i \pm \partial_\mu M = 0 \tag{12b}
\]

\[
D_\nu \phi \pm \epsilon M \langle n \times \phi \rangle = 0 \tag{12c}
\]

\[
\nabla_1 \phi \pm \nabla_2 \phi = 0. \tag{12d}
\]

These are the Bogomol’nyi (or selfdual) equations \[18\] for the MCS-O(3)-\( \sigma \) model with nonminimal coupling in the asymmetric potential case. As usual, if we consider static solutions, the total energy is simplified, namely,

\[
\mathcal{E} = 4\pi |Q_{\text{top}}|. \tag{13}
\]

III. CRITICAL CASE \( g = g_c \)

The selfdual Eqs. \[12a, 12d\] are strongly coupled and not easily solvable. However, they are simplified by considering the critical limit \( g = g_c \). In this case, \( M \) and \( B \) fields decouple and can be written as

\[
M = g_c (v - n \cdot \phi) \tag{14a}
\]

\[
B = \pm \epsilon g_c^2 \left( 1 - \frac{1}{g_c^2 (n \times \phi)^2} \right) (v - n \cdot \phi) \tag{14b}
\]

As a consequence, the potential \[11\] can be written only in terms of \( \phi \) as

\[
U = \frac{e^2 g_c^2}{2} \frac{(v - n \cdot \phi)^2}{1 - \frac{1}{g_c^2 (n \times \phi)^2}} (n \times \phi)^2, \tag{15}
\]

as illustrated in Fig. 1 for \( g = g_c = -0.5 \). For arbitrary \( v \), when we substitute \( g_c = -\epsilon/\kappa \) and then take \( \kappa > 1 \), the above expression reduces to the case studied in Ref. \[14\],

\[
U \approx U_{CS} = \frac{e^4}{2\kappa^2} (v - n \cdot \phi)^2 (n \times \phi)^2. \tag{16}
\]

IV. STATIC VORTICES IN THE \( g = g_c \) LIMIT

In order to solve the remaining Eqs. \[12d\] and \[14b\], we use a well-known parameterization for \( \phi \), which is valid for invariant solutions under both rotations and reflections in space-time and target space manifold \[19\]. Explicitly,

\[
\phi(r, \theta) = (\sin f \cos N\theta, \sin f \sin N\theta, \cos f) \tag{17}
\]

where \( f = f(r) \) and \( N \) is an integer which defines the topological grade (vorticity) of the solution. For the gauge field we also use a parameterization which takes into account the symmetries of the model, i.e.,

\[
A = \frac{a(r) - N}{er} \hat{\theta}. \tag{18}
\]
With this choice, we obtain two coupled nonlinear first order differential equations,

\[ f'(r) = \pm \frac{1}{r} \frac{\sin f(r)}{1 + g_c^2 \sin^2 f(r)} a(r) \]

\[ a'(r) = \pm r \left( \frac{g_c^2 \sin^2 f(r)}{1 - g_c^2 \sin^2 f(r)} \right) \left[ v - \cos f(r) \right], \]

where we have used \( A_0 = \mp M \) and \( r \to \frac{1}{e^r} \).

V. ANALYSIS AND NUMERICAL RESULTS

Equations (19a) and (19b) have no analytical solution. However, we can analyze their asymptotic behavior. Considering Eq. (18), non singular solutions at the origin always require that, for small values of \( r, a(r) \) approaches \( N \). Assuming \( f(r \to 0) \ll 1 \) then, from Eqs. (19), \( f(r) \approx C_N r^N \) and \( a(r) \approx N + (v - 1)C_N^2 g_c^2 r^{2N+2}/(2N+2) \), \( N \in \mathbb{Z}^* \).

When \( r \to \infty, a'(r) \) cannot diverge, and thus we have three situations satisfying Eqs. (19), namely \( f(\infty) = (0, \pi, f_v) \), where we follow Ref. 14 in labelling \( f(\infty) = f_v \) such that \( \cos f_v = v \). These values correspond to the minima of the potential \( g \) in the interval \([-1, 1]\).

The \( C_N \in \mathbb{R} \) constants are determined by the behavior of \( f(r) \) at spatial infinity. Our numerical results indicate that there exists a critical value for \( C_N \) such that \( f(\infty) \to 0 \left( C_N < C_{\text{crit}} \right); f(\infty) \to \pi \left( C_N > C_{\text{crit}} \right) \); and \( f(\infty) \to f_v \left( C_N = C_{\text{crit}} \right) \).

Soliton solutions are similar to those of Ref. 14. When \( C_N = C_{\text{crit}} \), the \( (N \neq 0) \) vortices are in the asymmetric phase of the potential and the magnetic flux given by \( \Phi = \frac{2\pi}{\varepsilon} N \), becomes quantized.

When \( C_N \neq C_{\text{crit}} \), topological (and nontopological) vortices are in the symmetric phase of the potential and the magnetic flux, now given by \( \Phi = \frac{2\pi}{\varepsilon} (N - \alpha) \), varies continuously, where \( \alpha \) is the asymptotic value of \( a(r) \) at spatial infinity. Numerical analysis shows that for \( C_N < C_{\text{crit}} \), one has \( \alpha < 0 \); for \( C_N > C_{\text{crit}} \), one has \( \alpha > 0 \).

Fig. 2 shows the magnetic field for \( C_N > C_{\text{crit}} \) case, for a few values of \( g_c \) with \( N = 1 \). Other topological solutions \((|N| > 1)\) exhibit similar behavior. Note that the magnetic field changes sign.

![Figure 2](image-url)

Figure 2: The magnetic field \( B(r) \) for a few values of \( g_c \) \((v = 0.5)\) when \( f(\infty) = \pi \) (symmetric phase of the potential), for \( N = 1 \) vorticity. The solid line represents \( B(r) \) for \( g_c = -0.7 \), dashed line \( g_c = -0.5 \), and dotted line \( g_c = -0.3 \).

![Figure 3](image-url)

Figure 3: The magnetic field \( B(r) \) for null vorticity for different values of \( f(0) \) at the origin \((v=0.5)\): \( f(0) = \pi/4 \) (dashed line); \( f(0) = \pi/3 \) (dashed line); \( f(0) = \pi/2 \) (solid line).

![Figure 4](image-url)

Figure 4: Maxima values of the magnetic field \( B(r) \) for different values of \( f(0)/\pi \) when \( g_c = -0.7 \) \((v=0.5)\) and null vorticity \((N=0)\). Note an abrupt change in \( B(r) \) at \( f(0) = \pi/3 \).

Solutions with null vorticity \((N=0)\) have neither quantized topological charge nor quantized flux but they are of some interest. The magnetic field shows some interesting characteristics for the null vorticity. As seen in Fig. 3 \(|B(r)| \) has a maximum at the origin for \( f(0) < \pi/3 \). On the other hand, at \( f(0) = \pi/3 \) the maximum appears away from the origin, returning as \( f(0) \) increases. In general, the values of the maxima depend on \( g_c \) and the conditions on \( f(r) \) at the origin [Fig. 4].
VI. CONCLUSIONS

To summarize, in this work we construct a gauged nonlinear-O(3)-σ model. The gauging of the U(1) subgroup includes nonminimal coupling in the covariant derivative. The gauge-field dynamics are governed by both Maxwell and Chern-Simons terms.

Through the Bogomol’nyi method, we find an asymmetric potential that contains the contribution of the nonminimal term, and the self-dual equations necessary to achieve the energy bound limit.

It is worthwhile to note that when one chooses a specific value for the nonminimal coupling constant, namely, \( g = -\frac{e}{\kappa} \), some fields decouple and the Higgs potential changes, leading the system to behave like a pure Chern-Simons system. However the model proposed here is different since it possesses both Maxwell and Chern-Simons terms.

The solitons found have topological and non-topological characteristics. When we compare our results with those of models without nonminimal coupling, we can see notable differences in the magnetic flux intensity. This implies that all the fields are influenced by the value of the nonminimal coupling constant.

Indeed, the magnetic field shows interesting behavior in our model. The nontopological solutions in the symmetric case show a magnetic field maximum at the origin and moved away from it when \( f(0) \geq \pi/3 \). Topological solutions in symmetric phase show that magnetic flux is not quantized and magnetic field changes sign. This behavior is related to topological lumps [14].

Topological vortices were found in the asymmetric phase where \( \mathbf{n} \cdot \mathbf{\phi}(\infty) \rightarrow v \). In this case, the magnetic field is purely attractive or repulsive and the flux is quantized.

A further investigation could also be performed for two dimensional models obtained from a dimensional reduction. These kinds of reduced models can exhibit interesting domain wall structures.

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