Grand Lebesgue Spaces are really Banach algebras relative to the convolution on unimodular locally compact groups equipped with Haar measure

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Abstract
We prove that the Grand Lebesgue Space built on a unimodular locally compact topological group, equipped with bi-invariant Haar measure, forms a Banach algebra relative to the convolution.

KEYWORDS
beta-function, convolution, Grand Lebesgue Spaces, Haar measure, Lebesgue–Riesz space, locally compact topological groups, modulus of continuity, unimodular group, Young inequality

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1 | INTRODUCTION

Haar measures on locally compact groups are used in many fields as statistics, probability, ergodic theory and harmonic analysis, particularly in the theory of Pontryagin duality.

Let \( G \) be a unimodular locally compact topological group equipped with bi-invariant Borelian Haar measure \( \mu \). Define, as usual, the convolution between two measurable integrable functions \( f, g : G \to \mathbb{R} \) by

\[
(f * g)(x) = \int_G f(y)g(y^{-1}x) \, d\mu(y).
\]

The usual Lebesgue–Riesz spaces \( L^p = L^p(G, \mu) \) are the spaces of all measurable functions \( f : G \to \mathbb{R} \) defined by the norms

\[
\|f\|_p := \left( \int_G |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad 1 \leq p < \infty,
\]

\[
\|f\|_\infty := \text{ess sup}_{x \in G} |f(x)|, \quad p = \infty.
\]

The well-known Young (or Hausdorff–Young) inequality has the form

\[
\|f * g\|_r \leq C \|f\|_p \|g\|_q, \quad 1 \leq p, q, r \leq \infty,
\] (1.1)

where

\[
1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}
\] (1.2)
see e.g. [4, p. 199], [27, 49] and [3, 5, 21] for the computation of the best possible constant $C$ in (1.1).

Generalized convolution inequalities and applications have been obtained in [33] and fractional convolutions in [34].

**Remark 1.1.** Let us prove the necessity of the condition (1.2) at least in the case $G = \mathbb{R}^d$, $d = 1, 2, 3, \ldots$. Suppose that for arbitrary non-zero and nonnegative numerical valued smooth functions $f, g$, defined on $\mathbb{R}^d$ and having compact support, inequality (1.1) holds for $p, q, r \geq 1$.

Actually, one can use the so-called scaling method, see e.g. [48]. Define, for any positive constant $\lambda > 0$, the dilation operator

$$T_\lambda f(x) = f(\lambda x), \quad x \in \mathbb{R}^d.$$ 

We deduce, from (1.1),

$$\|T_\lambda f \ast T_\lambda g\|_r \leq \|T_\lambda f\|_p \|T_\lambda g\|_q, \quad p, q, r \geq 1, \quad (1.3)$$

and, after simple calculations,

$$\lambda^{-d - d/r} \|f \ast g\|_r \leq \lambda^{-d/p - d/q} \|f\|_p \|g\|_q.$$ 

Since $\lambda$ is an arbitrary positive value, we conclude $-d - d/r = -d/p - d/q$, which is quite equally to (1.2).

**Remark 1.2.** In [3, 5, 21] the best possible constant in the inequality (1.1) was obtained. Namely,

$$\|f \ast g\|_r \leq \left( \frac{C_p C_q}{C_r} \right)^d \|f\|_p \|g\|_q, \quad 1 + 1/r = 1/p + 1/q, \quad (1.4)$$

where $p, q, r \geq 1$ and

$$C_m = \left( \frac{m^{1/m}}{(m')^{1/m'}} \right)^{1/2}, \quad m' = m/(m-1).$$

This estimate is essentially non-improbable. Indeed, the equality in (1.4) is attained if and only if both functions $f, g$ are proportional to Gaussian densities, namely there exists positive constants $c_1, c_2, c_3, c_4$ such that

$$f(x) = c_1 \exp\left(-c_2\|x\|^2\right), \quad g(x) = c_3 \exp\left(-c_4\|x\|^2\right), \quad x \in \mathbb{R}^n, \quad (1.5)$$

where $\| \cdot \|$ is the euclidean norm. Then the convolution $f \ast g$ is also Gaussian and

$$\|f \ast g\|_r = \left( \frac{C_p C_q}{C_r} \right)^d \|f\|_p \|g\|_q, \quad 1/p + 1/q = 1 + 1/r, \quad p, q, r > 1. \quad (1.6)$$

**Remark 1.3.** For $p = r$ and $q = 1$, inequality (1.1) is sharp

$$\|f \ast g\|_p \leq \|f\|_p \|g\|_1, \quad (1.7)$$

with constant $C = 1$.

If, in addition, the group $G$ is also compact, one can suppose $\mu(G) = 1$ so that $\|g\|_1 \leq \|g\|_p$. Therefore

$$\|f \ast g\|_p \leq \|f\|_p \|g\|_p, \quad p \in [1, \infty]. \quad (1.8)$$

The last inequality shows that the space $L^p(G, \mu)$, relative to the convolution bilinear operator, forms in this case a Banach algebra, see [49].
In a recent paper ([26, Theorem 1]) T. Gürkanlı extended inequality (1.8) to the Grand Lebesgue Spaces $L^{p,\lambda}(G)$, $\lambda \geq 0$ (see Section 2 for the definition). Namely, he proved that the inequality
\[
\|f \ast g\|_{L^{p,\lambda}} \leq \|f\|_{L^{p,\lambda}} \|g\|_{L^{p,\lambda}}, \quad p > 1,
\]
holds if and only if $G$ is compact (see also [24, 25]).

In [11] P. Fernandez-Martinez and E. Brandani da Silva proved new Young inequalities in the context of grand and small Lebesgue spaces, where the underlying measure space is the one-dimensional torus, with the Lebesgue measure. Some applications of these results concern the study of Fourier multipliers.

In this paper we improve and extend further inequality (1.9) to the more general Grand Lebesgue Spaces $G\psi$ (see Section 2 for the definition). In particular, we point out here that the following inequality, appearing in the proof of Theorem 1 in the preprint [26],
\[
\sup_{0 < \epsilon \leq p - 1} \epsilon^{\frac{\lambda}{p - \epsilon}} \|f \ast g\|_{p - \epsilon} \leq \sup_{0 < \epsilon \leq p - 1} \epsilon^{\frac{\lambda}{p - \epsilon}} \|f\|_{p - \epsilon} \cdot \sup_{0 < \epsilon \leq p - 1} \epsilon^{\frac{\lambda}{p - \epsilon}} \|g\|_{p - \epsilon}
\]
is, in general, false (see Remark 3.4).

In more details, in Section 3, we will find a sufficient condition on the generating function $\psi$ of $G\psi$ spaces, under which they form also a Banach algebra relative to the convolution. Our assumption on the group $G$ is weaker than that in [26], since we do not require that $G$ is compact.

In Section 4 we exhibit an application of the theory of the Banach algebra $G\psi$ to a linear convolution integral equation of renewal type.

2 GRAND LEBESGUE SPACES

We recall here some known definitions and facts about the theory of the Grand Lebesgue Spaces which we will use. We consider only the case of spaces on the set $G$ equipped with the Haar measure $\mu$.

Definition 2.1. Let $\psi(p) = \psi[b](p)$, $p \in [1, b)$, $b \in (1, \infty]$, be a numerical valued nonnegative function, not necessarily finite in every point, such that
\[
\inf_{p \in [1, b)} \psi(p) > 0
\]
and
\[
\psi(1) \in (0, \infty).
\]
We denote by $\Psi = \Psi[b] = \{\psi(\cdot)\}$ the set of all such functions. If $b < \infty$, one can assume $p \in [1, b]$.

The (Banach) Grand Lebesgue Space $G\psi = G\psi[b]$ consists of all the real (or complex) numerical valued measurable functions $f : G \to \mathbb{R}$ having finite norm, defined by
\[
\|f\|_{G\psi} = \|f\|_{G\psi[b]} := \sup_{p \in [1, b]} \left[ \frac{\|f\|_p}{\psi(p)} \right].
\]

The function $\psi$ is named as the generating function for this space.

We will assume, without loss of generality, $\psi(1) = 1$ for all such functions $\psi(\cdot)$. For instance, the following functions
\[
\psi(p) := p^{1/m}, \quad p \in [1, \infty), \quad m > 0,
\]

are members of $G\psi[b]$ for all $b > 1$. In particular, this includes the case $b = \infty$. The functions $\psi(p)$ for $p = 1/m$, $m = 1, 2, 3, \ldots$, are the so-called power functions.

For the case $b = \infty$, we obtain the Grand Lebesgue Spaces $L^{p,\lambda}(G)$, $\lambda \geq 0$.
or
\[
\psi(p) := (b - 1)^{\beta} (b - p)^{-\beta}, \quad p \in [1, b), \quad b \in (1, \infty), \quad \beta \geq 0,
\] (2.5)
satisfy (2.1) and (2.2) and \( \psi(1) = 1 \).

If
\[
\psi(p) = \begin{cases} 
  1, & p = r, \\
  +\infty, & p \neq r,
\end{cases} \quad r \in [1, \infty),
\]
where \( C/\infty := 0, C \in \mathbb{R} \) (extremal case), then the corresponding \( G\psi \) space coincides with the classical Lebesgue–Riesz space \( L^r = L^r(G) \).

**Remark 2.2.** We observe that \( G\psi \subset L^1 \) since, for any nonnegative function \( \psi(p), p \in [1, b), b \in (1, \infty) \), satisfying (2.1) and (2.2), we have
\[
\|f\|_1 = \psi(1) \cdot \frac{\|f\|_1}{\psi(1)} \leq \psi(1) \cdot \sup_{p \in [1, b]} \left( \frac{\|f\|_p}{\psi(p)} \right) = \psi(1) \cdot \|f\|_{G\psi}.
\]

**Remark 2.3.** Note that the space \( L^{b,\theta} = L^{b,\theta}(G), b > 1, \theta \geq 0, \) with Lebesgue measure \( \mu(G) < \infty \), introduced in [22] (see [30] for \( \theta = 1 \)), through the norm
\[
\|f\|_{L^{b,\theta}} = \|f\|_{L^{b,\theta}(G)} := \sup_{0 < \varepsilon \leq b-1} \left[ \varepsilon^{\theta/(b-\varepsilon)} \|f\|_{b-\varepsilon} \right], \quad \theta \geq 0,
\] (2.6)
quite coincides, up to equivalence of the norms, with the appropriate one \( G\psi \), where \( \psi \) is defined in (2.5) with \( \beta = \theta/b \). Namely, there exists two positive constants \( C_1 = C_1(b, \theta, G), C_2 = C_2(b, \theta, G) \) such that
\[
C_1 \|f\|_{G\psi} \leq \|f\|_{L^{b,\theta}} \leq C_2 \|f\|_{G\psi},
\]
where
\[
\psi(p) = (b - 1)^{\theta/b} (b - p)^{-\theta/b}, \quad p \in [1, b), \quad b \in (1, \infty).
\] (2.7)
In fact, it suffices to choose \( \varepsilon \in (0, b - 1) \), replace \( p \) with \( b - \varepsilon \) in (2.7) and take into account that \( \varepsilon^{\theta/b} \sim e^{\theta/(b-\varepsilon)} \).

For \( \theta = 0 \) the space \( L^{b,0}(G) \) reduces to the Lebesgue–Riesz space \( L^b(G) \). The inclusion \( L^b(G) \subset L^{b,\theta}(G), b > 1, \theta \geq 0, \) holds (see e.g. [22]).

**Definition 2.4.** Let \( f : G \to \mathbb{R} \) be a measurable function and \( b \in (1, \infty] \) such that
\[
\|f\|_p < \infty, \quad \text{for all } p \in [1, b);
\]
the so-called natural function \( \psi_f(p) \) for \( f \) is defined by
\[
\psi_f(p) := \|f\|_p.
\]
Obviously,
\[
\|f\|_{G\psi_f} = 1.
\]

**Proposition 2.5.** Let \( f : G \to \mathbb{R} \) be a measurable non-zero function such that its natural function
\[
\psi_f(p) = \|f\|_p, \quad p \in [1, b), \quad b \in (1, \infty],
\]
is finite for a non-trivial domain, i.e.

$$\forall p \in [1, b) \Rightarrow \psi_f(p) < \infty.$$ 

Then

$$\inf_{p \in (1, b)} \psi_f(p) > 0.$$ 

The case $p \in (a, b), 1 < a < b \leq \infty$ may be considered quite analogously.

**Proof.** Suppose on the contrary that $\inf_{p \in [1, b)} \psi_f(p) = 0$. Then there exists a sequence $p(k), p(k) \in [1, b)$, for which $\|f\|_{p(k)} \to 0$ as $k \to \infty$, and such that there exists the limit (finite or not) $q := \lim_{k \to \infty} p(k)$.

Let us consider the two possibilities:

I. $p(k) \to q \in [1, b), \quad q < \infty,$

II. $p(k) \to \infty, \quad q = \infty.$

In the first case I, we deduce

$$\|f\|_q = \lim_{k \to \infty} \|f\|_{p(k)} = 0$$

and, consequently, $f = 0$ a.e., in contradiction with the assumption $f$ non-zero.

In the second case II, we apply the well-known Tchebychev–Markov inequality,

$$\mu\{x : |f(x)| > z\} \leq \frac{\|f\|_{p(k)}}{z^{p(k)}}, \quad z > 0.$$ 

Let us choose $z = 1$, then

$$\mu\{x : |f(x)| > 1\} \leq \|f\|_{p(k)}^{p(k)}.$$ 

As long as $p(k) \geq 1$, passing to the limit as $k \to \infty$, we conclude

$$\mu\{x : |f(x)| > 1\} = \lim_{k \to \infty} \|f\|_{p(k)}^{p(k)} = 0. \quad (2.8)$$

But it is well known that, by virtue of (2.8),

$$\lim_{p \to \infty} \|f\|_p = \text{ess sup}_{x \in G} |f(x)| = \|f\|_{\infty},$$

therefore

$$\|f\|_{\infty} = \lim_{k \to \infty} \|f\|_{p(k)} = 0 \quad \Leftrightarrow \quad f = 0 \text{ a.e.}$$

again in contradiction with the assumption. \qed

The Grand Lebesgue Spaces have been widely investigated, see, e.g., [6, 7, 13, 19, 31, 32, 37, 38, 42] and references therein. They play an important role in the theory of partial differential equations (PDEs) (see, e.g., [1, 15, 17, 18, 22]), in interpolation theory (see, e.g., [2, 12, 14, 16]), in the theory of probability ([10, 20, 36, 43–45]), in statistics and in the theory of random fields (see, e.g., [35], [41, Chapter 5]), in functional analysis and so on.
These spaces are rearrangement invariant (r.i.) Banach function space; their fundamental function has been studied in [44]. They do not coincide, in the general case, with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz, etc., see [37, 42].

3 | MAIN RESULT

**Theorem 3.1.** Let $G$ be a unimodular locally compact topological group equipped with bi-invariant Haar measure $\mu$ and let $G\psi$ be the (Banach) Grand Lebesgue space built on $G$. If $\psi(1) = 1$, then $G\psi$ is also a Banach algebra under the convolution as a multiplicative operation, that is

$$
\|f * g\|_{G\psi} \leq \|f\|_{G\psi} \cdot \|g\|_{G\psi}
$$

for all $f, g : G \to \mathbb{R}$ in $G\psi$.

*Proof.* Let $f, g$ be two non-zero functions from the space $G\psi$; we assume, without loss of generality, that

$$
\|f\|_{G\psi} = \|g\|_{G\psi} = 1.
$$

From the definition of the norm in the Grand Lebesgue Spaces it follows immediately that

$$
\|f\|_p \leq \psi(p), \quad \|g\|_p \leq \psi(p), \quad p \in [1, b).
$$

We apply once again the Young inequality (1.7)

$$
\|f * g\|_p \leq \|f\|_p \|g\|_1 \leq \psi(p) \|g\|_1,
$$

thus

$$
\frac{\|f * g\|_p}{\psi(p)} \leq \|g\|_1 \leq \psi(1) = 1
$$

and, taking the supremum over $p \in [1, b)$, we have

$$
\sup_{p \in [1, b)} \frac{\|f * g\|_p}{\psi(p)} \leq \|g\|_1 \leq \psi(1) = 1.
$$

Therefore

$$
\|f * g\|_{G\psi} \leq 1 = \|f\|_{G\psi} \cdot \|g\|_{G\psi}.
$$

□

In the next example we will see that equality in (3.1) can be achieved.

**Example 3.2.** Let $G = \mathbb{R}$ with the classical Lebesgue measure $d\mu = dx$ and the ordinary convolution

$$
(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x - y) \, dy,
$$

for all measurable integrable functions $f, g$ defined on $\mathbb{R}$. 


Define the Gaussian density

\[ z_\sigma(x) := \left( \sigma \sqrt{2\pi} \right)^{-1} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\}, \quad \sigma = \text{const} > 0; \]  

(3.3)

then its natural function \( \psi_{z_\sigma}(p) \) has the form

\[ \psi_{z_\sigma}(p) = \| z_\sigma(\cdot) \|_p = (2\pi)^{1/(2p)-1/2} p^{-1/(2p)} \sigma^{1/p-1}, \quad p \in [1, \infty). \]  

One can define formally the value \( \psi_{z_\sigma}(p) \), for \( p = \infty \), as

\[ \psi_{z_\sigma}(\infty) := \lim_{p \to \infty} \psi_{z_\sigma}(p) = \left( \sigma \sqrt{2\pi} \right)^{-1} = \max_{x \in \mathbb{R}} z_\sigma(x). \]

Notice that \( \psi_{z_\sigma}(1) = 1 \). By virtue of Theorem 3.1 we conclude that the Grand Lebesgue Space \( G_{\psi_{z_\sigma}} \), built over the ordinary real line \( \mathbb{R} \), relative to the convolution operation, forms also a Banach algebra, despite the (commutative) group \((\mathbb{R}, +)\) is not compact.

Moreover, let us choose \( f(x) = g(x) = z_1(x) (\sigma = 1) \), then

\[ \| f \|_{G_{\psi_{z_1}}} = \| g \|_{G_{\psi_{z_1}}} = 1. \]

Furthermore, it is easily seen that

\[ h(x) := (f * g)(x) = z_\sqrt{2}(x), \quad (\sigma = \sqrt{2}); \]

\[ \| h \|_p = (2\pi)^{1/(2p)-1/2} p^{-1/(2p)} \left[ \sqrt{2} \right]^{1/p-1}, \quad p \in [1, \infty), \]

so that

\[ \frac{\| h \|_p}{\psi_{z_1}(p)} = \left[ \sqrt{2} \right]^{1/p-1}, \]

and

\[ \| h \|_{G_{\psi_{z_1}}} = \sup_{p \geq 1} \left[ \sqrt{2} \right]^{1/p-1} = 1. \]

Thus, in this case, we have

\[ \| h \|_{G_{\psi_{z_1}}} = \| f * g \|_{G_{\psi_{z_1}}} = \| f \|_{G_{\psi_{z_1}}} \cdot \| g \|_{G_{\psi_{z_1}}}. \]

Remark 3.3. Note that the case \( G = \mathbb{R}^d \) or \( G = [0, 2\pi)^d \) or \( G = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \) may be investigated analogously.

Remark 3.4. As announced in Section 1 we observe that the inequality,

\[ \sup_{0 < \varepsilon \leq \rho \leq 1} \varepsilon^{p-\varepsilon} \| f * g \|_{p-\varepsilon} \leq \sup_{0 < \varepsilon \leq \rho \leq 1} \varepsilon^{p-\varepsilon} \| f \|_{p-\varepsilon} \cdot \sup_{0 < \varepsilon \leq \rho \leq 1} \varepsilon^{p-\varepsilon} \| g \|_{p-\varepsilon}, \]

appearing in the proof of Theorem 1 in the preprint [26] and equivalent to

\[ \| f * g \|_{L_p(G)} \leq \| f \|_{L_p(G)} \cdot \| g \|_{L_p(G)}, \]

(3.5)
is, in general, false. Indeed

\[ \sup_{\varepsilon \in (0,1)} (\varepsilon A B) \leq \sup_{\varepsilon \in (0,1)} (\varepsilon A) \cdot \sup_{\varepsilon \in (0,1)} (\varepsilon B), \]

is false, for instance, if \( A = B = e^{-3/4} \).

The inequality (3.5) is, in fact, true up to a multiplicative constant. This follows from our Theorem 3.1 and the equivalence of the norm \( \| \cdot \|_{G^\psi} \) and \( \| \cdot \|_{L_0^\beta} \). So, also \( \| \cdot \|_{L_0^\beta} \) forms a Banach algebra, at least up to renorming the generating function.

\section*{4 | AN APPLICATION OF THE BANACH ALGEBRA \( G^\psi \)}

Let us consider an application of the theory of the Banach algebra \( G^\psi \), described in the previous section.

Introduce the following linear convolution integral equation of renewal type relative to the function \( P \) defined through

\[ P = f + \lambda g * P, \quad \lambda \in \mathbb{R}, \quad \text{(4.1)} \]

which appeared in particular in the probability theory, theory of reliability etc. (see, e.g., [9], [40]), where both the (known) functions \( f, g \) belong to the Banach algebra \( G^\psi \).

Define the affine, i.e. non-homogeneous, linear operator

\[ U(P) \overset{\text{def}}{=} f + \lambda g * P. \quad \text{(4.2)} \]

Assume that

\[ \alpha := |\lambda| \| g \|_{G^\psi} < 1, \quad \text{(4.3)} \]

then, from the algebraic property of the space \( G^\psi \), it follows

\[ \| U(Q_1) - U(Q_2) \|_{G^\psi} \leq |\lambda| \cdot \| g * (Q_1 - Q_2) \|_{G^\psi} \leq |\lambda| \cdot \| g \|_{G^\psi} \cdot \| Q_1 - Q_2 \|_{G^\psi} \]

\[ = \alpha \| Q_1 - Q_2 \|_{G^\psi}, \quad \text{for all } Q_1, Q_2 \in G^\psi. \]

Thus, the operator \( U \) satisfies the contraction property and consequently, by fixed point theory, the solution of Equation (4.1) there exists, is unique and may be computed by means of the classical iterations:

\[ P_0 := f, \ldots, P_n := U(P_{n-1}), \quad n = 1, 2, 3, \ldots, \quad \text{(4.4)} \]

and

\[ \| P \|_{G^\psi} \leq \| f \|_{G^\psi} \frac{1}{1 - \alpha}. \]

An error estimate is given by

\[ \| P - P_n \|_{G^\psi} \leq \frac{\alpha^n \| P_0 - P_1 \|_{G^\psi}}{1 - \alpha}, \]

(see [46]).

A concrete embodiment of this method in a particular case is described, e.g., in [23], but the application of the above technique (4.4), i.e. the theory of \( G^\psi \) algebras, allows to obtain a more exact confidence region for the solution, as long as the \( G^\psi \) estimations give the exponential decreasing tails of distributions.
5 | CONCLUDING REMARKS

A. Let $E$ be an arbitrary translation invariant Banach function space with norm $\| \cdot \|_E$ built over a unimodular locally compact topological group $G$ and such that

$$\forall g \in L^1(G) \Rightarrow \| g \|_1 \leq \| g \|_E. \quad (5.1)$$

Then $E$ is a Banach algebra.

Indeed, we use the following inequality (see, e.g., [47, p. 309, Lemma 27.10])

$$\left\| \int_G h(y) d\nu(y) \right\|_E \leq \int_G \| h(y) \|_E d\nu(y), \quad (5.2)$$

where $\nu(\cdot)$ is a Borel measure on $G$ and $h \in E$ is a strongly measurable function on $G$.

Inequality (5.2) may be proved first for the simple functions, i.e. functions of the form

$$h(y) = \sum_{j=1}^m c_j I(A_j),$$

where $A_j$ are measurable subsets of the whole set $G$, $I(A_j)$ are the indicator functions. For such functions (5.2) follows immediately from the classical (repeated) triangle inequality.

Using (5.2), (5.1) and taking into account the translation invariance, we obtain

$$\| f \ast g \|_E = \left\| \int_G f(y^{-1}x) g(y) d\mu(y) \right\|_E$$

$$\leq \int_G \| f \|_E |g(y)| d\mu(y)$$

$$= \| f \|_E \| g \|_1 \leq \| f \|_E \cdot \| g \|_E,$$

which proves the statement.

**B. A note about ideals in the Banach algebra $G\psi_{z_1}$**

Let $\psi_{z_1}(p)$, $p \in [1, \infty)$, be the function defined by (3.4), with $\sigma = 1$, and let $G = \mathbb{R}^d$. As long as the Grand Lebesgue Space $G\psi_{z_1}$, built on $\mathbb{R}^d$, forms a Banach algebra, it is interesting to consider its ideals, maximal or not. We show here two examples. Some well-known classical results from [8, 28, 29, 39] are used.

**Example 5.1.** For every function $f$ belonging to the space $G\psi_{z_1}$ define, as usual, the Fourier transform

$$\hat{f}(\xi) = F[f](\xi) := \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} f(x) dx, \quad (5.3)$$

where $(\xi, x)$ denotes the inner (scalar) product and $|x| := \sqrt{\langle x, x \rangle}$.

Since $f \in L^1(\mathbb{R}^d)$, the integral in (5.3) converges uniformly and it represents a uniformly continuous bounded function on the whole space $\mathbb{R}^d$

$$\sup_{\xi \in \mathbb{R}^d} |F[f](\xi)| \leq \| f \|_1 \leq \| f \|_{G\psi_{z_1}} < \infty. \quad (5.4)$$

We observe that one can replace $\psi_{z_1}(\cdot)$ with many other such functions, for instance, with the more general $\psi_{z_\sigma}(\cdot)$, $\sigma \in (0, \infty)$, defined in (3.4).
Recall that for \( f, g \in \mathcal{G}\psi_{z_1} \), if \( h(x) = (f \ast g)(x) \), then
\[
F[h](\xi) = F[f](\xi) \cdot F[g](\xi).
\]
(5.5)

Let \( \eta \) be an arbitrary fixed point of the space \( \mathbb{R}^d \); define the set
\[
J(\eta) = \{ f : f \in \mathcal{G}\psi_{z_1}, \quad F[f](\eta) = 0 \}.
\]
(5.6)

It follows immediately, from the relation (5.5), that the set \( J(\eta) \) forms a well-known classical (maximal) ideal in the Banach algebra \( \mathcal{G}\psi_{z_1} \).

In fact, for every \( \xi \), the mapping \( f \to F[f](\xi) \) is a multiplicative linear functional, since convolution goes into pointwise product under the Fourier transform; therefore it is a maximal ideal in the Banach algebra \( \mathcal{G}\psi_{z_1} \) (see, for example, [50, p. 36]).

**Example 5.2.** Denote by \( UC = U(\mathbb{C}^{\mathbb{R}^d}) \) the set of all uniformly continuous bounded functions \( f : \mathbb{R}^d \to \mathbb{R} \) or \( f : \mathbb{R}^d \to \mathbb{C} \). This set forms also an ideal in the Banach algebra \( \mathcal{G}\psi_{z_1} \). Namely, define the usual modulus of continuity for a function \( f \in UC(\mathbb{R}^d) \) by
\[
\omega[f](\delta) := \sup_{|x-y| \leq \delta} |f(x) - f(y)|, \quad \delta \in [0, \infty),
\]
so that
\[
f \in UC(\mathbb{R}^d) \Rightarrow \lim_{\delta \to 0^+} \omega[f](\delta) = 0.
\]

For \( f \in UC(\mathbb{R}^d) \) and \( g \in \mathcal{G}\psi_{z_1} \), the function \( x \to h(x) = (g \ast f)(x) \) is bounded and uniformly continuous, in fact if \( x_1, x_2 \in \mathbb{R}^d : |x_1 - x_2| \leq \delta \)
we have
\[
|h(x_1) - h(x_2)| \leq \int_{\mathbb{R}^d} |g(y)| \cdot |f(x_1 - y) - f(x_2 - y)| \, dy
\]
\[
\leq \|g\|_1 \cdot \omega[f](\delta) \leq \|g\|_{\mathcal{G}\psi_{z_1}} \cdot \omega[f](\delta) \to 0, \quad \delta \to 0^+.
\]

Thus, \( h(\cdot) \in UC(\mathbb{R}^d) \).

**C. The role of the condition (2.2).**

Let us show now that the condition (2.2) is, in the general case, essential. Namely, suppose temporarily that \( \psi(1) = 0 \), then
\[
\forall f \in \mathcal{G}\psi \Rightarrow \|f\|_1 = 0
\]
and, consequently, \( f = 0 \), i.e. \( \mathcal{G}\psi = \{0\} \) (trivial case).

Furthermore, let us consider the following counterexample. Choose \( G = \mathbb{R} \) with the usual convolution. Define the function
\[
f(x) = f_{3/2}(x) := x^{-2/3} I(x \geq 1),
\]
where \( I(A) \) denotes the indicator function of the predicate \( A \).
We deduce, after simple calculations,

$$\hat{\psi}(p) := \| f \|_p = \begin{cases} \left( \frac{3}{2p - 3} \right)^{1/p}, & p > 3/2, \\ +\infty, & p \in [1, 3/2]. \end{cases}$$

Here $\hat{\psi}(1) = +\infty$. Evidently,

$$\| f \|_{G\hat{\psi}} = 1,$$

as long as the function $\hat{\psi}(p)$ is the natural function for $f$.

Let us now evaluate the convolution

$$h(x) = (f \ast f)(x)$$

as $x \to \infty$; of course, $h(x) = 0, x < 1$. For $x \geq 3$ we have

$$h(x) = \int_1^{x^{-1}} y^{-2/3} (x - y)^{-2/3} \, dy = x^{-1/3} \int_{1/x}^{1} z^{-2/3} (1 - z)^{-2/3} \, dz,$$

so as $x \to \infty$,

$$h(x) \sim B(1/3, 1/3)x^{-1/3} = C_1 x^{-1/3},$$

where $B(\cdot, \cdot)$ denotes the ordinary beta function and $C_1 = B(1/3, 1/3) \in (0, \infty)$.

More precisely note that, as $x \geq 3$,

$$h(x) \geq x^{-1/3} \int_{1/3}^{2/3} z^{-2/3} (1 - z)^{-2/3} \, dz = C_2 x^{-1/3}, \quad C_2 = \text{const} \in (0, \infty);$$

$$|h|_p^p \geq C_2^p \int_{1/3}^{2/3} x^{-p/3} \, dx = C_2^p \frac{3}{p - 3} \cdot \frac{1 - 2^{1-p/3}}{3^{1-p/3}} =: K^p(p), \quad K(p) \in (0, +\infty);$$

$$|h|_p \geq K(p), \quad p > 3,$$

$$|h|_p = \infty, \quad 1 \leq p \leq 3.$$

It remains to note that the function $h(\cdot)$ does not belong to the Grand Lebesgue Space $G\hat{\psi}$. In fact,

$$\| h \|_{G\hat{\psi}} = \sup_{p > 3/2} \left\{ \frac{|h|_p}{\hat{\psi}(p)} \right\} \geq \frac{|h|_2}{\hat{\psi}(2)} = \infty,$$

or equivalently

$$\infty = \| f \ast f \|_{G\hat{\psi}} > \| f \|_{G\hat{\psi}} \cdot \| f \|_{G\hat{\psi}} = 1.$$

Thus, the (Banach) Grand Lebesgue Space $G\hat{\psi}$ is not a Banach algebra.

The case when

$$f(x) = f_\alpha(x) = x^{-1/\alpha} I (x \geq 1), \quad \alpha = \text{const} \in (1, 2),$$

may be investigated quite analogously.
By the way note that, despite $f_\alpha(\cdot) \not\in L_1(\mathbb{R})$, its Fourier transform $F[f_\alpha](\xi)$ exists as an element of the space $L_2(\mathbb{R})$ because $f_\alpha(\cdot) \in L_2(\mathbb{R})$, by Plancherel–Parseval theorem. Furthermore, $F[f_\alpha](\xi)$ exists for all non-zero values of the variable $\xi$. More exactly, as $\xi \to 0+$

$$F[f_\alpha](\xi) \sim \xi^{1/\alpha-1} \cdot \int_0^\infty e^{2\pi i y^{-1/\alpha}} dy$$

and

$$\xi \to \infty \Rightarrow F[f_\alpha](\xi) \sim \frac{e^{2\pi i \xi}}{2\pi i \xi}.$$

D. An open and interesting problem, in our opinion, is to find the necessary conditions imposed on the generating function $\psi(\cdot)$ under which $G\psi$ forms a Banach algebra and to find, in this case, all its maximal ideals.

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REFERENCES
[1] I. Ahmed, A. Fiorenza, M. R. Formica, A. Gogatishvili, and J. M. Rakotoson, Some new results related to Lorentz $G\Gamma$-spaces and interpolation, J. Math. Anal. Appl. 483 (2020), no. 2. Available at https://doi.org/10.1016/j.jmaa.2019.123623.
[2] I. Ahmed, A. Fiorenza, and A. Hafeez, Some interpolation formulae for grand and small Lorentz spaces, Mediterr. J. Math. 17 (2020), Article number 57.
[3] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. (2) 102 (1975), no. 1, 159–182.
[4] C. Bennett and R. Sharpley, Interpolation of operators, Academic Press, New York, 1988.
[5] H. J. Brascamp and E. H. Lieb, Best constants in Young’s inequality, its converse, and its generalization to more than three functions, Adv. Math. 20 (1976), no. 2, 151–173.
[6] C. Capone, M. R. Formica, and G. Giova, Grand Lebesgue Spaces with respect to measurable functions, Nonlinear Anal. 85 (2013), 125–131.
[7] C. Capone and A. Fiorenza, On small Lebesgue Spaces, J. Funct. Spaces Appl. 3 (2005), no. 1, 73–89.
[8] J. Cigler, Normed ideals in $L^1(G)$, Indag. Math. 31 (1969), 273–282.
[9] D. R. Cox, Renewal theory, Methuen & Co. Ltd., London; John Wiley & Sons, Inc., New York, 1962.
[10] V. Ermaakov and E. I. Ostrovsky, Continuity conditions, exponential estimates, and the central limit theorem for Random fields, VINITI, Moscow, 1986. (Russian).
[11] P. Fernández-Martínez and E. Brandani da Silva, New Young inequalities and applications, Z. Anal. Anwend. 38 (2019), no. 4, 419–437.
[12] P. Fernández-Martínez and T. Signes, Limit cases of reiteration theorems, Math. Nachr. 288 (2015), no. 1, 25–47.
[13] A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces, Collect. Math. 51 (2000), no. 2, 131–148.
[14] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani, and J. M. Rakotoson, Characterization of interpolation between grand, small or classical Lebesgue spaces, Nonlinear Anal. 177 (2018), part B, 422–453.
[15] A. Fiorenza, M. R. Formica, and J. M. Rakotoson, Pointwise estimates for $G\Gamma$-functions and applications, Differential Integral Equations 30 (2017), no. 11–12, 809–824.
[16] A. Fiorenza and G. E. Karadzhov, Grand and small Lebesgue spaces and their analogs, Z. Anal. Anwend. 23 (2004), no. 4, 657–681.
[17] A. Fiorenza, A. Mercaldo, and J. M. Rakotoson, Regularity and comparison results in grand Sobolev spaces for parabolic equations with measure data, Appl. Math. Lett. 14 (2001), no. 8, 979–981.
[18] A. Fiorenza, A. Mercaldo, and J. M. Rakotoson, Regularity and uniqueness results in grand Sobolev spaces for parabolic equations with measure data, Discrete Contin. Dyn. Syst. 8 (2002), no. 4, 893–906.
[19] M. R. Formica and R. Giova, Boyd indices in generalized grand Lebesgue spaces and applications, Mediterr. J. Math. 12 (2015), no. 3, 987–995.
[20] M. R. Formica, Y. V. Kozachenko, E. Ostrovsky, and L. Sirota, Exponential tail estimates in the law of ordinary logarithm (LOL) for triangular arrays of random variables, Lith. Math. J. 60 (2020), 330–358.
[21] J. F. Fournier, Sharpness in Young’s inequality for convolution, Pacific J. Math. 72 (1977), no. 2, 383–397.
[22] L. Greco, T. Iwaniec, and C. Sbordone, Inverting the $p$-harmonic operator, Manuscripta Math. 92 (1997), no. 2, 249–258.
[23] M. L. Grigorieva and E. I. Ostrovsky, Calculation of integrals on discontinuous functions by means of dependent experiments, Zh. Vychisl. Mat. i Mat. Fiz. 36 (1996), no. 12, 28–38. (Russian). English translation in: Comput. Math. Math. Phys. 36 (1997), no. 12, 1661–1670.
