On face antimagic labeling of double duplication of graphs

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Abstract. A Labeling of a plane graph G is called d-antimagic if every numbers, the set of s-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers $a_s$ and $d$ ($a_s > 0, d \geq 0$), where $f_s$ is the number of $s$-sided faces. We allow different sets $w_s$ of different s. In this paper, we proved the existence of face antimagic labeling of types $(1,0,0), (1,0,1), (1,1,0), (0,1,1)$ and $(1,1,1)$ of double duplication of all vertices by edges of a cycle graph $C_n$: $n \geq 3$ and a tree of order $n$.

1. Introduction
The concept of graph labeling was first introduced by Rosa in 1967. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The notion of face antimagic labeling was investigated by Baca [2]. Let $G=(V,E,F)$ be a finite plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. A bijection $\lambda: V \cup E \cup F \rightarrow \{1,2,3,\ldots, |V| \cup |E| \cup |F|\}$ is called a labeling of type $(1,1,1)$. The weight of a face under a labeling is the sum of the labels called by that face and the edges and vertices surrounding it.

A labeling of a plane graph $G$ is called d-antimagic if every numbers, the set of $s$-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers $a_s$ and $d$ ($a_s > 0, d \geq 0$), where $f_s$ is the number of $s$-sided faces. We allow different sets $w_s$ of different $s$.

The concept of double duplication of a graph is introduced in [6]. The double duplication of a graph is defined as a duplication of a vertex by an edge $e = \{v_k, v_k\}$ in a graph $G$ produces a graph $G'$, in which $N(v'_k) = \{v_k, v_k\}$ and $N(v'') = \{v_k, v_k\}$. Again duplication of a vertex $v'_k, v'^{'}_k$ and $v''_k$ by the edges $e' = \{u'_1u'_2, u'_3u'_4, u'_5u'_6\}$ respectively in $G'$ produces a new graph $G''$ such that $N(u'_1) = \{v'_k, u'_2\}, N(u'_2) = \{v'_k, u'_3\}, N(u'_3) = \{v'_k, u'_4\}, N(u'_4) = \{v'_k, u'_5\}, N(u'_5) = \{v'_k, u'_6\}$ and $N(u'_6) = \{v'_k, u'_5\}$. In this paper, we proved the existence of face antimagic labeling of types $(1,0,0), (1,0,1), (1,1,0), (0,1,1)$ and $(1,1,1)$ of double duplication of vertex by edge of a cycle graph $C_n$: $n \geq 3$ and a tree of order $n$.

2. Main Results
In this section, face antimagic labeling of double duplication of vertex by edge of types $(1,0,0), (1,0,1), (1,1,0), (0,1,1)$ and $(1,1,1)$ of a cycle graph $C_n$: $n \geq 3$ and a tree of order $n$ are discussed.

Theorem 2.1: The graph $G$ obtained by double duplication of all vertices by edges of a tree $T_n$, $n \geq 2$ admits face antimagic labeling of types $(1,0,0), (1,1,0), (1,0,1), (0,1,1)$ and $(1,1,1)$. 
Proof: Let $G(V, E, F)$ be an arbitrary tree $T_n, n \geq 2$ with vertex set $V = \{l_i|1 \leq i \leq n\}$ and the edge set $E = \{l_{i+1}l_i|1 \leq i \leq n - 1\}$. Let $G'(V', E', F')$ be the graph obtained by double duplication of vertex by edge in $G$ with vertex set $V' = \{s_i, t_i, a_i, l_i|1 \leq i \leq 2n\}$ and edge set $E' = \{l_{i+1}l_i|1 \leq i \leq 2n\} \cup V$ and edge set of $G$ and face set $F' = \{f_i = l_i, s_{2i-1}, s_{2i}, \cdots \}$, $f'_i = l_i, t_{2i-1}, t_{2i}, \cdots \}$, $f''_i = s_{2i}, a_{2i}, a_{2i+1}, s_{2i+2}, b_{2i}, b_{2i}, a_{2i+1}, s_{2i+2}, a_{2i}, a_{2i+1}, s_{2i+2}, a_{2i}, a_{2i+1}, l_i \leq i \leq n\}$ and face set $F'' = \{f_i = l_i, s_{2i-1}, s_{2i}, \cdots \}$, $f'_i = l_i, t_{2i-1}, t_{2i}, \cdots \}$, $f''_i = s_{2i}, a_{2i}, a_{2i+1}, s_{2i+2}, b_{2i}, b_{2i}, a_{2i+1}, s_{2i+2}, a_{2i}, a_{2i+1}, l_i \leq i \leq n\}$. The following are the five types of the labeling.

Type (i):(1, 0, 0)-face antimagic.

Define a mapping $f: V' \rightarrow \{1, 2, 3, \ldots, 9n\}$ as follows:

$$f(l_i) = i; 1 \leq i \leq n \quad f(t_{2i-1}) = 6n + i; 1 \leq i \leq n \quad f(t_{2i}) = 6n - i + 1; 1 \leq i \leq n$$

$$f(s_{2i-1}) = 4n - i + 1; 1 \leq i \leq n \quad f(s_{2i}) = 8n - i + 1; 1 \leq i \leq n \quad f(b_{2i-1}) = 8n + i; 1 \leq i \leq n$$

$$f(a_{2i-1}) = 2n + i; 1 \leq i \leq n \quad f(b_{2i}) = 4n + i; 1 \leq i \leq n \quad f(a_{2i}) = 2n - i + 1; 1 \leq i \leq n$$

Clearly, the face weights are calculated as follows:

$$\lambda(f_1) = f(l_i) + f(s_{2i-1}) + f(s_{2i}); 1 \leq i \leq n \quad \lambda(f_2) = f(l_i) + f(t_{2i-1}) + f(t_{2i}); 1 \leq i \leq n$$

$$= 12n - i + 2; 1 \leq i \leq n = 12n + i - 1; 1 \leq i \leq n$$

$$\lambda(f_3) = f(s_{2i-1}) + f(b_{2i-1}) + f(a_{2i-1}); 1 \leq i \leq n \quad \lambda(f_4) = f(s_{2i}) + f(b_{2i}) + f(a_{2i}); 1 \leq i \leq n$$

$$= 14n + i + 1; 1 \leq i \leq n = 14n - i + 2; 1 \leq i \leq n$$

The weight of all 3-sided faces form an arithmetic progression

$$\{a + (1 \ast d), a + (2 \ast d), \ldots, a + ((4n - 1) \ast d)\}$$

where $a = l_1 n + 2$ and $d = 1$.

Type (ii):(1, 1, 0)-face antimagic.

Define a mapping $f: V' \cup E' \rightarrow \{1, 2, 3, \ldots, 22n - 1\}$ as follows:

The vertex labeling of the type (1,1,0) is same as the vertex labeling of the type (1,0,0). We define the edge labeling as:

$$f(l_{i+1}l_i) = 12n - 3i + 3; 1 \leq i \leq n \quad f(s_{2i-1}s_{2i}) = 12n - 3i + 2; 1 \leq i \leq n$$

$$f(l_i, s_{2i}) = 12n - 3i + 1; 1 \leq i \leq n \quad f(l_{i+1}, t_{2i}) = 12n + 3i - 2; 1 \leq i \leq n$$

$$f(t_{2i-1}t_{2i}) = 12n + 3i - 1; 1 \leq i \leq n \quad f(l_{i+1}, t_{2i}) = 12n + 3i; 1 \leq i \leq n$$

$$f(s_{2i-1}, b_{2i-1}) = 18n + 3i - 2; 1 \leq i \leq n \quad f(b_{2i-1}, a_{2i-1}) = 18n + 3i - 1; 1 \leq i \leq n$$

$$f(s_{2i-1}, a_{2i-1}) = 18n + 3i; 1 \leq i \leq n \quad f(s_{2i}, b_{2i}) = 18n - 3i + 3; 1 \leq i \leq n$$

$$f(b_{2i}, a_{2i}) = 18n - 3i + 2; 1 \leq i \leq n \quad f(s_{2i}, a_{2i}) = 18n - 3i + 1; 1 \leq i \leq n$$

$$f(l_{i+1}, l_i) = 21n + i; 1 \leq i \leq n - 1$$

We calculate the face weights as follows:

$$\lambda(f_1) = f(l_i) + f(s_{2i-1}) + f(s_{2i}) + f(l_i, s_{2i-1}) + f(s_{2i-1}, s_{2i}) + f(l_i, s_{2i}); 1 \leq i \leq n$$

$$= 12n - i + 2 + 36n - 9i + 6; 1 \leq i \leq n$$

$$= 48n - 10i + 8; 1 \leq i \leq n$$

$$\lambda(f_2) = f(l_i) + f(t_{2i-1}) + f(t_{2i}) + f(l_{i+1}, t_{2i}) + f(t_{2i-1}, t_{2i}) + f(l_{i+1}, t_{2i}); 1 \leq i \leq n$$
The weight of all 3-sided faces form an arithmetic progression
\{a, a+(1*d), a+(2*d),..., a+((4n-1)*d)\} where a=38n+8 and d=10.

Type (iii):(1, 0, 1)-face antimagic:

Define a mapping \( f : V \cup F' \rightarrow \{1,2,3,.......,3n\} \) as follows:
The vertex labeling of the type (1,0,1) is same as the vertex labeling of the type (1,0,0). We define the face labeling as

\[ f(i_1) = 10n - i + 1; 1 \leq i \leq n \quad f(i_2) = 10n + i; 1 \leq i \leq n \quad f(i_3) = 12n - i + 1; 1 \leq i \leq n \]

Clearly, the face weights are calculated as follows:

\[ \lambda(f_1) = f(s_{2i-1}) + f(b_{2i-1}) + f(a_{2i-1}) + f(s_{2i-1}b_{2i-1}) + f(b_{2i-1}a_{2i-1}) + f(s_{2i-1}a_{2i-1}); 1 \leq i \leq n \]

\[ = 12n + i + 1 + 36n + 9i - 3; 1 \leq i \leq n \]

\[ = 48n + 10i - 2; 1 \leq i \leq n \]

\[ \lambda(f_2) = f(s_{2i}) + f(b_{2i}) + f(a_{2i}) + f(s_{2i}b_{2i}) + f(b_{2i}a_{2i}) + f(s_{2i}a_{2i}); 1 \leq i \leq n \]

\[ = 14n - i + 2 + 54n - 9i + 3; 1 \leq i \leq n \]

\[ = 68n + 10i - 6; 1 \leq i \leq n \]

The weight of all 3-sided faces form an arithmetic progression
\{a, a+(1*d), a+(2*d),..., a+((4n-1)*d)\} where a=38n+8 and d=10.

Type (iv):(0, 1, 1)-face antimagic:

Define a mapping \( f : E \cup F' \rightarrow \{1,2,3,.......,17n-1\} \) as follows:
Clearly, the face weights are calculated as follows:

\[ \lambda(f_1) = f(l_{12},t_{21}) + f(t_{21},t_{22}) + f(l_{12},t_{22}):1 \leq i \leq n \]
\[ = 36i - 30 + 13n + 4i - 4;1 \leq i \leq n \]
\[ = 13n + 40i - 34;1 \leq i \leq n \]
\[ \lambda(f_2) = f(l_{12},s_{21}) + f(s_{21},s_{22}) + f(l_{12},s_{22});1 \leq i \leq n \]
\[ = 36i - 21 + 13n + 4i - 3;1 \leq i \leq n \]
\[ = 13n + 40i - 24;1 \leq i \leq n \]
\[ \lambda(f_3) = f(s_{21},b_{21}) + f(b_{21},a_{21}) + f(s_{21},a_{21}) + f(f'_{1});1 \leq i \leq n \]
\[ = 36i - 12 + 13n + 4i - 2;1 \leq i \leq n \]
\[ = 13n + 40i - 14;1 \leq i \leq n \]
\[ \lambda(f_4) = f(s_{22},b_{22}) + f(b_{22},a_{22}) + f(s_{22},a_{22}) + f(f'_{1});1 \leq i \leq n \]
\[ = 36i - 3 + 13n + 4i - 1;1 \leq i \leq n \]
\[ = 13n + 40i - 4;1 \leq i \leq n \]

The weight of all 3-sided faces form an arithmetic progression
\[ \{a, a+(1*d), a+(2*d),..., a+((4n-1)*d)\} \] where \(a=13n+6\) and \(d=10\).

Type (v)\((1, 1, 1)\)-face antimagic:

Define a mapping \(f: V \setminus \{E' \cup F'\} \rightarrow \{1,2,3,\ldots,26n-1\}\) as follows:

The vertex labeling and edge labeling of the type \((1,1,1)\) is same as the vertex labeling and edge labeling of the type \((1,1,0)\). We define the face labeling as

\[ f(f_1') = 23n - i;1 \leq i \leq n \]
\[ f(f_1) = 23n + i - 1;1 \leq i \leq n \]
\[ f(f_2') = 25n - i;1 \leq i \leq n \]
\[ f(f_2) = 25n + i - 1;1 \leq i \leq n \]

Clearly, the face weights are calculated as follows:

\[ \lambda(f_1) = f(l_{12}) + f(s_{21}) + f(s_{21},s_{22}) + f(l_{12},s_{22}) + f(l_{12},t_{22}) + f(f_1);1 \leq i \leq n \]
\[ = 12n + i + 2 + 36n - 9i + 6 + 23n - 9i;1 \leq i \leq n \]
\[ = 71n - 11i + 8;1 \leq i \leq n \]
\[ \lambda(f_2) = f(l_{12}) + f(t_{21}) + f(t_{21}) + f(l_{12},t_{21}) + f(l_{12},t_{22}) + f(f_2);1 \leq i \leq n \]
The weight of all 3-sided faces form an arithmetic progression
\{a, a+(1*d), a+(2*d),..., a+((4n-1)*d)\} where a=60n+8 and d=11.

Figure 1: The double duplication of all vertices by edges of tree \( T_n \)

**Theorem 2.2:** The graph \( G \) obtained by double duplication of all vertices by edges of a cycle graph \( C_n, n \geq 3 \) admits face antimagic labeling of types \( (1,0,0), (1,1,0), (1,0,1), (0,1,1) \) and \( (1,1,1) \).

**Proof:** Let \( G(V, E, F) \) be a cycle graph \( C_n, n \geq 3 \) with vertex set \( V = \{l_i | 1 \leq i \leq n\} \) and the edge set \( E = \{l|_i l_{i+1} | 1 \leq i \leq n-1\} \cup \{l_n\} \). Let \( G'(V', E', F') \) be a graph obtained by double duplication of all vertices by edges in \( G \) with vertex set \( V' = \{s, b, r, k, l | 1 \leq i \leq 2n\} \cup V \), and edge set
\[ E' = \{l_{b_{2i-1}}, b_{2i-1}, b_{2i}, l_{s_{2i-1}}, s_{2i-1}, s_{2i}, l_{r_{2i-1}}, r_{2i-1}, k_{2i-1}, s_{2i-1}, k_{2i-1}, s_{2i}, r_{2i}, k_{2i}, s_{2i}, 1 \leq i \leq n\} \cup E \]
and face set $F' = \{ f_i : l_i s_{2i-1} s_{2i}, f'_{i} : l_i b_{2i-1} b_{2i}, f''_{i} : s_{2i} r_{2i} k_{2i}, f'''_{i} : s_{2i-1} r_{2i-1} k_{2i-1}, 1 \leq i \leq n \} \cup \{ f''_{i} : l_i l_{m} \cup l_i l_{n}, 1 \leq i \leq n - 1 \}$. The following are the five types of the labeling.

Type (i): (1, 0, 0)-face antimagic:

Define a mapping \( f : V' \to \{1,2,3,...,9n\} \) as follows:

\[
\begin{align*}
    f(l_i) &= i; 1 \leq i \leq n \\
    f(b_{2i-1}) &= 6n + i; 1 \leq i \leq n \\
    f(b_{2i}) &= 6n - i + 1; 1 \leq i \leq n \\
    f(s_{2i-1}) &= 4n - i + 1; 1 \leq i \leq n \\
    f(s_{2i}) &= 8n - i + 1; 1 \leq i \leq n \\
    f(r_{2i-1}) &= 2n + i; 1 \leq i \leq n \\
    f(r_{2i}) &= 4n + i; 1 \leq i \leq n \\
    f(k_{2i-1}) &= 2n - i + 1; 1 \leq i \leq n
\end{align*}
\]

Clearly, the face weights are calculated as follows:

\[
\begin{align*}
    \lambda(f_i) &= f(l_i) + f(s_{2i-1}) + f(s_{2i}); 1 \leq i \leq n \\
    \lambda(f_2) &= f(l_i) + f(r_{2i-1}) + f(r_{2i}); 1 \leq i \leq n \\
    \lambda(f_3) &= f(s_{2i-1}) + f(r_{2i-1}) + f(k_{2i-1}); 1 \leq i \leq n \\
    \lambda(f_4) &= f(s_{2i}) + f(r_{2i}) + f(k_{2i}); 1 \leq i \leq n
\end{align*}
\]

The weight of an \( n \)-sided face of \( C_n \) is given by

\[
\sum_{i=1}^{n} f(l_i) = \frac{n(n+1)}{2}
\]

and the weight of all 3-sided faces form an arithmetic progression \( \{a, a + (1 \cdot d), a + (2 \cdot d), ..., a + ((4n - 1) \cdot d)\} \) where \( a = 11n + 2 \) and \( d = 1 \).

Type (ii): (1, 1, 0)-face antimagic:

Define a mapping \( f : V' \cup E' \to \{1,2,3,...,22n\} \) as follows:

The vertex labeling of the type (1,1,0) is same as the vertex labeling of the type (1,0,0). We define the edge labeling as

\[
\begin{align*}
    f(l_is_{2i-1}) &= 12n - 3i + 3; 1 \leq i \leq n \\
    f(s_{2i-1}s_{2i}) &= 12n - 3i + 2; 1 \leq i \leq n \\
    f(l_is_{2i}) &= 12n - 3i + 1; 1 \leq i \leq n \\
    f(l_is_{2i}) &= 12n + 3i - 2; 1 \leq i \leq n \\
    f(b_{2i-1}b_{2i}) &= 12n + 3i - 1; 1 \leq i \leq n \\
    f(l_is_{2i}) &= 12n + 3i; 1 \leq i \leq n \\
    f(s_{2i-1}r_{2i-1}) &= 18n + 3i - 2; 1 \leq i \leq n \\
    f(r_{2i-1}k_{2i-1}) &= 18n + 3i - 1; 1 \leq i \leq n \\
    f(s_{2i-1}k_{2i-1}) &= 18n + 3i; 1 \leq i \leq n \\
    f(r_{2i}k_{2i}) &= 18n - 3i + 2; 1 \leq i \leq n \\
    f(s_{2i}r_{2i}) &= 18n - 3i + 3; 1 \leq i \leq n \\
    f(r_{2i}k_{2i}) &= 18n - 3i + 1; 1 \leq i \leq n \\
    f(l_is_{2i}) &= 21n + i; 1 \leq i \leq n - 1 \\
    f(l_is_{2i}) &= 22n
\end{align*}
\]

Clearly, the face weights are calculated as follows:

\[
\begin{align*}
    \lambda(f_i) &= f(l_i) + f(s_{2i-1}) + f(s_{2i}) + f(l_is_{2i-1}s_{2i}); 1 \leq i \leq n \\
    \lambda(f_2) &= f(l_i) + f(b_{2i-1}) + f(b_{2i}) + f(l_is_{2i-1}b_{2i}); 1 \leq i \leq n
\end{align*}
\]
\[= 12n + i + 1 + 36n + 9i - 3; 1 \leq i \leq n\]
\[= 48n + 10i - 2; 1 \leq i \leq n\]
\[\lambda(f_1) = f(s_{2i-1}) + f(r_{2i-1}) + f(k_{2i-1}) + f(s_{2i-1}r_{2i-1}k_{2i-1}); 1 \leq i \leq n\]
\[= 14n + i + 1 + 54n + 9i - 3; 1 \leq i \leq n\]
\[= 68n + 10i - 2; 1 \leq i \leq n\]
\[\lambda(f_1) = f(s_{2i}) + f(r_{2i}) + f(k_{2i}) + f(s_{2i}r_{2i}k_{2i}); 1 \leq i \leq n\]
\[= 14n - i + 2 + 54n - 9i + 6; 1 \leq i \leq n\]
\[= 68n - 10i + 8; 1 \leq i \leq n\]

The weight of an \(n\)-sided face of a cycle graph is given by
\[\lambda(f_1^{nv}) = \sum_{i=1}^{n} f(l_i) + \sum_{i=1}^{n} f(l_{i+1}) + f(l_{i_l}) = \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} 2i + \sum_{i=1}^{n-1} i + 22n = 22n^2 + n\]

where \(f_1^{nv}\) is the \(n\) sided face of \(C_n\).

and the weight of all 3-sided faces form an arithmetic progression
\{\(a, a + (1*d), a + (2*d), ..., a + ((4n-1)*d)\}\) where \(a=38n+8\) and \(d=10\).

Type (iii): (1, 0, 1)-face antimagic:

Define a mapping \(f: V \cup F' \rightarrow \{1, 2, 3, ..., 13n + 1\}\) as follows:
The vertex labeling of the type (1,0,1) is same as the vertex labeling of the type (1,0,0). We define the face labeling as
\[f(f_1) = 10n - i + 1; 1 \leq i \leq n\]
\[f(f_1') = 10n + i; 1 \leq i \leq n\]
\[f(f_1^{++}) = 12n - i + 1; 1 \leq i \leq n\]
\[f(f_1^{--}) = 12n + i; 1 \leq i \leq n\]
\[f(f_1^{++}) = 13n + 1\]

Clearly, the face weights are calculated as follows:
\[\lambda(f_1) = f(l_i) + f(s_{2i-1}) + f(s_{2i}) + f(f_1); 1 \leq i \leq n\]
\[= 12n - i + 2 + 10n - i + 1; 1 \leq i \leq n\]
\[= 22n - 2i + 3; 1 \leq i \leq n\]
\[\lambda(f_2) = f(l_i) + f(b_{2i-1}) + f(b_{2i}) + f(f'_1); 1 \leq i \leq n\]
\[= 12n + i + 1 + 10n + i; 1 \leq i \leq n\]
\[= 22n + 2i + 1; 1 \leq i \leq n\]
\[\lambda(f_3) = f(s_{2i-1}) + f(r_{2i-1}) + f(k_{2i-1}) + f(f_1^{--}); 1 \leq i \leq n\]
\[= 14n + i + 1 + 12n + i; 1 \leq i \leq n\]
\[= 26n + 2i + 1; 1 \leq i \leq n\]
\[\lambda(f_4) = f(s_{2i}) + f(r_{2i}) + f(k_{2i}) + f(f_1^{--}); 1 \leq i \leq n\]
\[= 14n - i + 2 + 12n - i + 1; 1 \leq i \leq n\]
\[= 26n - 2i + 3; 1 \leq i \leq n\]
The weight of an n-sided face of a cycle graph is given by

\[ \lambda(f_i^n) = \sum_{i=1}^{n} f(l_i) + f(f_i^{iv}) \]

\[ = (n^2 + 27n + 2)/2 \]

where \( f_i^{iv} \) is the n sided face of \( C_n \).

Clearly, the weight of all 3-sided faces form an arithmetic progression \( \{a, a+(1*d), ..., a+((4n-1)*d)\} \) where \( a=20n+3 \) and \( d=2 \).

Type(iv): (0, 1, 1)-face antimagic:

Define a mapping \( f : E \cup F' \rightarrow \{1, 2, 3, ..., 17n + 1\} \) as follows

\[ f(l_{2i-1}) = 12i - 1; \quad 1 \leq i \leq n \]

\[ f(l_{2i}) = 12i - 10; \quad 1 \leq i \leq n \]

\[ f(l_{s_{2i-1}}) = 12i - 8; \quad 1 \leq i \leq n \]

\[ f(l_{s_{2i}}) = 12i - 7; \quad 1 \leq i \leq n \]

\[ f(s_{2i-1}r_{2i-1}) = 12i - 5; \quad 1 \leq i \leq n \]

\[ f(r_{2i-1}k_{2i-1}) = 12i - 4; \quad 1 \leq i \leq n \]

\[ f(s_{2i-1}k_{2i-1}) = 12i - 3; \quad 1 \leq i \leq n \]

\[ f(l_{l_{i+1}}) = 12n + i; \quad 1 \leq i \leq n - 1 \]

\[ f(l_{l_{n}}) = 13n \]

\[ f(f_i) = 13n + 4i - 2; \quad 1 \leq i \leq n \]

\[ f(f'_i) = 13n + 4i - 3; \quad 1 \leq i \leq n \]

\[ f(f''_i) = 13n + 4i - 1; \quad 1 \leq i \leq n \]

\[ f(f'''_i) = 17n + 1 \]

Clearly, the face weights are calculated as follows:

\[ \lambda(f_i) = f(l_{b_{2i}}) + f(b_{2i-1}b_2) + f(l_{b_2}) + f(f'_i); 1 \leq i \leq n \]

\[ = 36i - 30 + 13n + 4i - 3; \quad 1 \leq i \leq n \]

\[ = 13n + 40i - 33; \quad 1 \leq i \leq n \]

\[ \lambda(f_2) = f(l_{s_{2i}}) + f(s_{2i-1}s_2) + f(l_{s_2}) + f(f'_i); 1 \leq i \leq n \]

\[ = 36i - 21 + 13n + 4i - 2; \quad 1 \leq i \leq n \]

\[ = 13n + 40i - 23; \quad 1 \leq i \leq n \]

\[ \lambda(f_3) = f(s_{2i-1}r_{2i-1}) + f(r_{2i-1}k_{2i-1}) + f(s_{2i-1}k_{2i-1}) + f(f''_i); 1 \leq i \leq n \]

\[ = 36i - 12 + 13n + 4i - 1; \quad 1 \leq i \leq n \]

\[ = 13n + 40i - 13; \quad 1 \leq i \leq n \]

\[ \lambda(f_4) = f(s_2r_2) + f(r_2k_2) + f(s_2k_2) + f(f'''_i); 1 \leq i \leq n \]

\[ = 36i - 3 + 13n + 4i; \quad 1 \leq i \leq n \]

\[ = 13n + 40i - 3; \quad 1 \leq i \leq n \]

The weight of an n-sided face of a cycle graph is given by
\[ \lambda(f_i^{iv}) = \sum_{i=1}^{n} f(l_i^{i_{1+1}}) + f(l_i^{i_{1}}) + f(f_i^{iv}) \]
\[ = (25n^2 + 35n + 2)/2 \]

where \( f_i^{iv} \) is the \( n \) sided face of \( C_n \).

Clearly, the weight of all 3-sided faces form an arithmetic progression \( \{a, a + (1*d), \ldots, a + ((4n-1)*d)\} \) where \( a=13n+7 \) and \( d=10 \).

Type (v):(1,1,1)-face antimagic:

Define a mapping \( f : V \cup E \cup F^' \rightarrow \{1,2,3,\ldots,26n+1\} \) as follows:

The vertex labeling of the type (1,1,1) is same as the vertex labeling of the type (1,0,0). We define the edge-face labeling as

\[ f(l_{i}(s_{2i-1})) = 13n - 3i + 3; 1 \leq i \leq n \]
\[ f(s_{2i-1}s_{2i}) = 13n - 3i + 2; 1 \leq i \leq n \]
\[ f(l_{i}b_{2i-1}) = 13n + 3i - 1; 1 \leq i \leq n \]
\[ f(b_{2i-1}b_{2i}) = 13n + 3i+1; 1 \leq i \leq n \]
\[ f(s_{2i-1}r_{2i-1}) = 19n + 3i - 2; 1 \leq i \leq n \]
\[ f(r_{2i-1}k_{2i-1}) = 19n + 3i - 1; 1 \leq i \leq n \]
\[ f(s_{2i-1}k_{2i-1}) = 19n - 3i + 3; 1 \leq i \leq n \]
\[ f(r_{2i}k_{2i}) = 19n - 3i + 2; 1 \leq i \leq n \]
\[ f(l_{i+1}l_{i}) = 9n + i; 1 \leq i \leq n - 1 \]
\[ f(l_{i}l_{i}) = 10n \]

\[ f(f_i^{iv}) = 23n - i + 2; 1 \leq i \leq n \]
\[ f(f_i^{iv}) = 23n + i + 1; 1 \leq i \leq n \]
\[ f(f_i^{iv}) = 25n - i + 2; 1 \leq i \leq n \]
\[ f(f_i^{iv}) = 25n + i + 1; 1 \leq i \leq n \]
\[ f(f_i^{iv}) = 22n + 1 \]

Clearly, the face weights are calculated as follows:

\[ \lambda(f_i^{iv}) = f(l_i) + f(s_{2i-1}) + f(s_{2i}) + f(l_i^{s_{2i-1}s_{2i}}) + f(l_i^{s_{2i}}) + f(f_i^{iv}); 1 \leq i \leq n \]
\[ = 12n - i + 2 + 39n - 9i + 6 + 23n - i + 2; 1 \leq i \leq n \]
\[ = 74n - 11i + 10; 1 \leq i \leq n \]

\[ \lambda(f_i^{v}) = f(s_{2i-1}) + f(r_{2i-1}) + f(k_{2i-1}) + f(s_{2i-1}r_{2i-1}) + f(r_{2i-1}k_{2i-1}) + f(s_{2i-1}k_{2i-1}) + f(f_i^{iv}); 1 \leq i \leq n \]
\[ = 14n + i + 1 + 57n + 9i - 3 + 25n + i + 1; 1 \leq i \leq n \]
\[ = 96n + 11i - i; 1 \leq i \leq n \]

\[ \lambda(f_i^{v}) = f(s_{2i}) + f(r_{2i}) + f(k_{2i}) + f(s_{2i}r_{2i}) + f(r_{2i}k_{2i}) + f(s_{2i}k_{2i}) + f(f_i^{iv}); 1 \leq i \leq n \]
\[ = 14n - i + 2 + 57n - 9i + 6 + 25n - i + 2; 1 \leq i \leq n \]
\[ = 96n - 11i + 10; 1 \leq i \leq n \]
The weight of n-sided face of a cycle graph is given by

$$
\lambda(f_{i}^{iv}) = \sum_{i=1}^{n} f(l_{i}) + \sum_{i=1}^{n-1} f(l_{i}l_{i+1}) + f(l_{i}l_{i}) + f(f_{i}^{iv})
$$

Where $f_{i}^{iv}$ is the n sided face of $C_n$.

$$
= \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} (9n+i) + 10n + (22n+1) = 10n^2 + 23n + 1
$$

Clearly, the weight of all 3-sided faces form an arithmetic progression

$$
\{a, a+(1*d),..., a+((4n-1)*d)\}
$$

where $a=63n+10$ and $d=11$.

![Figure 2: The double duplication of all vertices by edges of the cycle graph $C_n$](image)

3. Conclusions
In this paper, we proved the existence of face antimagic labeling for double duplication of all vertices by edges of a cycle graph and tree of order $n$. Studying the properties of double duplication of graph is our future work.

4. References
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