REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY VIA THE NORM
AND REAL WITT VECTORS

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Abstract. We prove that Real topological Hochschild homology can be characterized as the norm from the cyclic group of order 2 to the orthogonal group $O(2)$. From this perspective, we then prove a multiplicative double coset formula for the restriction of this norm to dihedral groups of order $2m$. This informs our new definition of Real Hochschild homology of rings with anti-involution, which we show is the algebraic analogue of Real topological Hochschild homology. Using extra structure on Real Hochschild homology, we define a new theory of $p$-typical Witt vectors of rings with anti-involution. We end with an explicit computation of the degree zero $D_{2m}$-Mackey functor homotopy groups of THR($\mathbb{Z}$) for $m$ odd. This uses a Tambara reciprocity formula for sums for general finite groups, which may be of independent interest.

1. Introduction

In recent years, tremendous advances in the study of algebraic K-theory have been facilitated by the trace method approach, in which algebraic K-theory is approximated by invariants from homological algebra and their topological analogues. For a ring $R$, the Dennis trace map relates its algebraic K-theory to its Hochschild homology,

$$K_q(R) \to \text{HII}_q(R).$$

Classical Hochschild homology of algebras has a topological analogue. This topological Hochschild homology, THH, plays a key role in the trace method approach to algebraic K-theory. Indeed, for a ring (or ring spectrum) $R$ there is a trace map from algebraic K-theory to topological Hochschild homology, lifting the classical Dennis trace,

$$K_q(R) \to \pi_q \text{THH}(R).$$

Topological Hochschild homology is an $S^1$-equivariant spectrum, and by understanding the equivariant structure of THH one can define topological cyclic homology (see, e.g. [5], [36]), which is often a close approximation to algebraic K-theory.

Real algebraic K-theory, KR, defined by Hesselholt and Madsen [21], is an invariant of a ring (or ring spectrum) with anti-involution. It is a generalization of Karoubi’s Hermitian K-theory [29], and an analogue of Atiyah’s topological K-theory with reality [2]. Real topological Hochschild homology, THR($A$), is an $O(2)$-equivariant spectrum that receives a trace map from Real algebraic K-theory [21, 8, 27]. Just as topological Hochschild homology is essential to the trace method approach to algebraic K-theory, THR is essential to computing Real algebraic K-theory.

In Hill, Hopkins, and Ravenel’s solution to the Kervaire invariant one problem [23], they developed the theory of a multiplicative norm functor $N^G_H$ from $H$-spectra to $G$-spectra, for $H \subset G$ finite groups. In the case of non-finite compact Lie groups, however, norms are currently only accessible in a few specific cases. In [1] the authors extend the norm construction to the circle group $T$, defining the equivariant norm $N^T_e(R)$ for an associative ring spectrum.
They further show that this equivariant norm is a model for topological Hochschild homology. In the present paper, we extend the norm construction to an equivariant norm from $D_2$ to $O(2)$, using the dihedral bar construction. We then show that for a ring spectrum with anti-involution $A$, the equivariant norm $N^{O(2)}_{D_2}(A)$ is a model for the Real topological Hochschild homology of $A$.

Before defining the norm, we introduce some notation. We write $B^\text{di}(A)$ for the dihedral bar construction on $A$, as defined in Section 3.2. Let $U$ denote a complete $O(2)$-universe, and let $V$ be the complete $D_2$-universe constructed by restricting $U$ to $D_2$. We write $\tilde{V}$ for the $O(2)$-universe associated to $V$ by inflation along the determinant homomorphism. The input for Real topological Hochschild homology is a ring spectrum with anti-involution, $(A, \omega)$.

Ring spectra with anti-involution can be alternatively described as algebras in $D_2$-spectra over an $E_\sigma$-operad, where $\sigma$ is the sign representation of $D_2$, the cyclic group of order 2. The model structure $\text{Assoc}_\sigma(\text{Sp}_{D_2} V)$ on such $E_\sigma$-algebras is defined in Proposition 3.19.

**Definition 1.1.** We define the functor $N^{O(2)}_{D_2}: \text{Assoc}_\sigma(\text{Sp}_{D_2} V) \to \text{Sp}_{O(2)} U$ to be the composite functor

$$A \mapsto \mathcal{T}_V^U | B^\text{di}_\bullet(A)|.$$

Here $\mathcal{T}_V^U$ denotes the change of universe functor.

We then prove that this functor satisfies one of the fundamental properties of equivariant norms: in the commutative setting it is left adjoint to restriction. Let $\mathcal{R}$ denote the family of subgroups of $O(2)$ which intersect $T$ trivially, and let $\text{Sp}_{O(2), \mathcal{R}}$ denote the $\mathcal{R}$-model structure on genuine $O(2)$-spectra, as defined in 3.20.

**Theorem 1.2.** The restriction

$$N^{O(2)}_{D_2}: \text{Comm}(\text{Sp}_{D_2}) \to \text{Comm}(\text{Sp}_{O(2), \mathcal{R}})$$

of the norm functor $N^{O(2)}_{D_2}$ to genuine commutative $D_2$-ring spectra is left Quillen adjoint to the restriction functor $i^*_D$.

This equivariant norm is a model for Real topological Hochschild homology.

**Proposition 1.3.** Given a flat $E_\sigma$-ring in $\text{Sp}_{D_2}$, there is a natural zig-zag of $\mathcal{R}$-equivalences

$$N^{O(2)}_{D_2}(A) \simeq \text{THR}(A)$$

of $O(2)$-orthogonal spectra. This is also a zig-zag of $\mathcal{F}_{\text{Fin}}^\mathbb{T}$-equivalences, where $\mathcal{F}_{\text{Fin}}^\mathbb{T}$ is the family of finite subgroups of $\mathbb{T} \subset O(2)$.

This result extends a comparison from [13] of the dihedral bar construction and Real topological Hochschild homology as $D_2$-spectra. In [13] the authors also prove that for a flat ring spectrum with anti-involution $(R, \omega)$, there is a stable equivalence of $D_2$-spectra

$$i^*_D \text{THR}(R) \simeq R \wedge_{N^L_{\mathcal{F}_{\text{Fin}}^\mathbb{T}}}^{L \times i^*_D} R.$$

We generalize this result by proving a multiplicative double coset formula for the norm $N^{O(2)}_{D_2}$. 
\textbf{Theorem 1.4} (Multiplicative Double Coset Formula). Let $\zeta$ denote the $2m$-th root of unity $e^{2\pi i/2m}$. When $R$ is a flat $E_\sigma$-ring and $m$ is a positive integer, there is a stable equivalence of $D_{2m}$-spectra

$$t^*_{D_{2m}} N^{O(2)}_{D_2} R \simeq N^{D_{2m}}_{D_2} R \wedge L_{N^{D_{2m}}_{D_2} t^*_{D_{2m}} R} N^{D_{2m}}_{C_{2m}} c_{\zeta R}.$$ 

Ordinary topological Hochschild homology is the topological analogue of the classical algebraic theory of Hochschild homology. Indeed, for a ring spectrum $R$, the two theories are related by a linearization map

$$\pi_k \text{THH}(R) \to \text{HH}_k(\pi_0 R),$$

which is an isomorphism in degree 0. It is natural to ask, then, what is the algebraic analogue of Real topological Hochschild homology? In this paper we define such an analogue: a theory of Real Hochschild homology for rings with anti-involution, or more generally for discrete $E_\sigma$-rings. A discrete $E_\sigma$-ring is a type of $D_{2m}$-Mackey functor that arises as the algebraic analogue of $E_\sigma$-rings in spectra (see Definition 6.8). Indeed, if $R$ is an $E_\sigma$-ring in spectra, $\pi_{D_{2m}}(R)$ is a discrete $E_\sigma$-ring. The definition of Real Hochschild homology follows naturally from the double coset formula above.

\textbf{Definition 1.5.} The Real $D_{2m}$-Hochschild homology of a discrete $E_\sigma$-ring $M$ is the graded $D_{2m}$-Mackey functor

$$HR^{D_{2m}}(M) = H_* \left( HR^{D_{2m}}(M) \right),$$

where

$$HR^{D_{2m}}(M) = B_*(N^{D_{2m}}_{D_2} M, N^{D_{2m}}_{C_{2m}} t^*_{D_{2m}} M, N^{D_{2m}}_{C_{2m}} c_{\zeta M}).$$

This theory of Real Hochschild homology is computable using homological algebra for Mackey functors. We prove that Real topological Hochschild homology and Real Hochschild homology are related by a linearization map.

\textbf{Theorem 1.6.} For any $(-1)$-connected $E_\sigma$-ring $A$, we have a natural homomorphism

$$\pi_k^{D_{2m}} \text{THR}(A) \to HR_k^{D_{2m}}(\pi_0^D A),$$

which is an isomorphism when $k = 0$.

This relationship facilitates the computation of Real topological Hochschild homology. As an example, we compute the degree zero $D_{2m}$-Mackey functor homotopy groups of $\text{THR}(H\mathbb{Z})$, for odd $m$.

\textbf{Theorem 1.7.} Let $m \geq 1$ be an odd integer. There is an isomorphism of $D_{2m}$-Mackey functors

$$\pi_0^{D_{2m}} \text{THR}(H\mathbb{Z}) \cong A_{D_{2m}}^m/(2 - [D_{2m}/\mu_m])$$

where $A_{D_{2m}}^m$ is the Burnside Mackey functor for the dihedral group $D_{2m}$ of order $2m$ and we quotient by the congruence relation generated by $2 - [D_{2m}/\mu_m]$.

When restricted to $\pi_0^{D_{2m}}(\text{THR}(H\mathbb{Z})^D_{2k})$, this computation recovers a computation of [11], proven by different methods.

As part of the above computation, we do some of the first calculations to appear in the literature of dihedral norms for Mackey functors. In doing so, we establish a Tambara reciprocity formula for sums of general finite groups.
Theorem 1.8 (Tambara reciprocity for sums). Let $G$ be a finite group and $H$ a subgroup, and let $R$ be a $G$-Tambara functor. For each $F \in \text{Map}^H(G, \{a, b\})$, let $K_F$ be the stabilizer of $F$. Then for any $a, b \in \mathcal{R}(G/H)$, we have

$$N^G_H(a + b) = \sum_{[F] \in \text{Map}^H(G, \{a, b\})/G} t^G_{K_F} \left( \prod_{[\gamma] \in K_F \cap (G/H)} N^{K_F}_{K_F \cap (G/H)}(\gamma \gamma^{-1} H) \left( \gamma \gamma^{-1} H \right) \left( F(\gamma^{-1}) \right) \right)$$

In the case of dihedral groups, this leads to a very explicit formula for Tambara reciprocity for sums (see Lemma 8.11), facilitating the computation of dihedral norms.

Another important aspect of the theory of Real Hochschild homology is that it leads to a definition of Witt vectors for rings with anti-involution, or more generally for discrete $E_\sigma$-rings. Classically, Witt vectors are closely related to topological Hochschild homology. Indeed, in [20] Hesselholt and Madsen showed that for a commutative ring $R$

$$\pi_0(\text{THH}(R)) \cong \mathcal{W}_{n+1}(R; p)$$

where $\mathcal{W}_{n+1}(R; p)$ denotes the length $n + 1$ $p$-typical Witt vectors of $R$. Further, in [19], Hesselholt generalized the theory of Witt vectors to non-commutative rings and showed that for any associative ring $R$ there is a relationship between Witt vectors and topological cyclic homology,

$$\text{TC}_{-1}(R; p) \cong W(R; p)_F.$$ 

Here $W(R)_F$ denotes the coinvariants of the Frobenius endomorphism on the $p$-typical Witt vectors $W(R; p)$.

In the current work we consider the analogous results for rings with anti-involution. We prove that there is a type of Real cyclotomic structure on Real Hochschild homology, and use this to define Witt vectors of rings with anti-involution $R$, denoted $\mathcal{W}(R; p)$, (Definition 7.10). As a consequence of this work, we show that $\mathcal{W}_{-1} \text{TC}(R)$ can be described purely in terms of equivariant homological algebra.

Theorem 1.9. Let $A$ be an $E_\sigma$-ring and assume $R^1\lim_{k} \pi_0^{C_2} \text{THR}(A)^{p,k} = 0$. There is an isomorphism

$$\mathcal{W}_{-1} \text{TC}(A; p) \cong \mathcal{W}(\mathcal{A}_0^{D_2} A; p)_F.$$ 

where $\mathcal{W}(A; p)_F$ is the coinvariants of an operator

$$F: \mathcal{W}(\mathcal{A}_0^{D_2} A; p) \rightarrow \mathcal{W}(\mathcal{A}_0^{D_2} A; p).$$

1.1. Organization. In Section 2, we recall the theory of dihedral and cyclic objects, and dihedral subdivision. In particular, we offer the perspective that dihedral objects are Real cyclic objects (see Definition 2.13). In Section 3, we recall the definition of $E_\sigma$-rings, the dihedral bar construction, Real topological Hochschild homology, our model categorical framework, and the notion of (genuine) Real $p$ cyclotomic spectra.

The main results begin in Section 4 where we prove the that dihedral bar construction can be regarded as a norm in Theorem 4.12. To do this we first give background on equivariant orthogonal spectra and extend the comparison between the dihedral bar construction and the Bökstedt model of Real topological Hochschild homology from [13] to an identification as $O(2)$-spectra.

In Section 5, we prove a multiplicative double coset formula for Real topological Hochschild homology (see Theorem 5.9). In section 6, we develop the theory Real Hochschild homology and show that it is the algebraic analogue of Real topological Hochschild homology. In Section
we define Witt vectors for rings with anti-involution. We then prove an identification of the lowest nontrivial homotopy group of TCR with the coinvariants of an operator on our theory of Witt vectors for rings with anti-involution in Theorem 7.16.

Finally, in Section 8, we end with a Tambara reciprocity formula for dihedral groups, computations of norms for dihedral groups, and a computation of the degree zero $D_{2m}$-Mackey functor homotopy groups of $THR(\mathbb{Z})$ for an odd integer $m \geq 1$.

1.2. Conventions. Let $\text{Top}$ denote the category of based compactly generated weak Hausdorff spaces. We refer to objects in $\text{Top}$ as spaces and morphisms in $\text{Top}$ as maps of spaces. Let $G$ be a compact Lie group. Then $\text{Top}^G$ denotes the category of based $G$-spaces and based $G$-equivariant maps of spaces. When regarded as a $\text{Top}$-enriched category, we do not distinguish between them notationally. Let $\text{Top}_G$ denote the $\text{Top}^G$-enriched category with the same objects as $\text{Top}^G$ and the mapping spaces given by all maps of $G$-spaces with $G$-action given by conjugation. For a $G$-universe $\mathcal{U}$, let $\text{Sp}^G_\mathcal{U}$ be the category of orthogonal $G$-spectra indexed on $\mathcal{U}$ [34, II. 2.6].

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2. Dihedral Objects

We begin by fixing some conventions. Let $O(2)$ denote the compact Lie group of two-by-two orthogonal matrices. The determinant map determines an extension

$$1 \rightarrow \mathbb{T} \rightarrow O(2) \xrightarrow{\det} \{1,-1\} \rightarrow 1$$

of groups. We choose a splitting by sending $-1$ to the matrix $\tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and write $D_2$ for the subgroup of $O(2)$ generated by $\tau$. We then write $\mu_m \subset \mathbb{T}$ for the subgroup of $m$-th roots of unity generated by $\zeta_m = e^{2\pi i/m}$. Finally, we fix a presentation $D_{2m} = < \tau, \zeta_m | \tau^2 = \zeta_m^m = (\tau \zeta_m)^2 = 1 >$ for the dihedral group of order $D_{2m}$ regarded as a subgroup of $\mathbb{T} \times D_2 = O(2)$.

2.1. Dihedral objects and cyclic objects. Let $\Delta$ denote the category with objects the totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ of cardinality $n+1$ and maps of finite sets preserving the total order. A simplicial object in a category $\mathcal{C}$ is a functor $X_\cdot : \Delta^{\text{op}} \rightarrow \mathcal{C}$ which consists of an object $X_n$ in $\mathcal{C}$ for each $n \geq 0$ and maps

$$d_i : X_n \rightarrow X_{n-1} \text{ and } s_i : X_n \rightarrow X_{n+1}$$

for each $0 \leq i \leq n$ and $n \geq 0$ satisfying the simplicial identities

$$d_id_j = d_{j-1}d_i \text{ if } i < j \quad d_is_j = s_{j-1}d_i \text{ if } i < j \quad d_{j+1}s_j = 1 \quad d_is_j = s_{j+1}s_i \text{ if } i < j \quad d_js_i = 1.$$
Definition 2.1. A cyclic object in a category $C$ is a simplicial object $X_\bullet$ in $C$ together with a $\mu_{n+1}$-action on $X_n$ given by maps

$$t_n: X_n \to X_n$$

satisfying the relations

\begin{align}
(2) & \quad d_it_n = t_{n-1}d_{i-1} \text{ for } 1 \leq i \leq n \\
(3) & \quad s_it_n = t_{n+1}s_{i-1} \text{ for } 1 \leq i \leq n
\end{align}

in addition to the simplicial identities (1).

Definition 2.2. A dihedral object in a category $C$ is a simplicial object in $C$ together with a $D_{2(n+1)}$-action on $X_n$ given on generators $t_n$ and $\omega_n$ of $D_{2(n+1)}$ by structure maps

$$t_n: X_n \to X_n \quad \text{and} \quad \omega_n: X_n \to X_n$$

satisfying the usual dihedral group relations

$$\omega_nt_n = t_n^{-1}\omega_n$$

as well as the relations (2) and (3) and the additional relations

\begin{align}
(4) & \quad d_i\omega_n = \omega_{n-1}d_{n-i} \text{ for } 0 \leq i \leq n \\
(5) & \quad s_i\omega_n = \omega_{n+1}s_{n-i} \text{ for } 0 \leq i \leq n
\end{align}

We will also use the notion of a Real simplicial object.

Definition 2.3. A Real simplicial object $X$ is a simplicial object $X: \Delta^{op} \to C$ together with maps $\omega_n: X_n \to X_n$ for each $n \geq 0$ satisfying the relations (5), (4) and $\omega_n^2 = \text{id}_{X_n}$.

Remark 2.4. The definition of the cyclic category $\Lambda$ [7, §2] and the dihedral category $\Xi$ [31, Definition 3.1] are standard, so we omit them. It will be useful to recall that a cyclic object in a category $C$ is a functor $\Lambda^{op} \to C$ and a dihedral object in a category $C$ is a functor $\Xi^{op} \to C$.

Let $\Delta \Theta \in \{\Lambda, \Xi\}$ and let $X: \Delta \Theta^{op} \to C$ be a functor. We can then produce a simplicial object $t^*X$ in $C$ by precomposition with the functor $\iota: \Delta \to \Delta \Theta$.

Theorem 2.5 (Thm. 5.3 [17]). Let $\Delta \Theta \in \{\Lambda, \Xi\}$. Write $\Theta_n = \text{Aut}_{\Delta \Theta}([n])$. Then $\Theta_\bullet$ is a simplicial set and the geometric realization of $\Theta_\bullet$ has a natural structure of a topological group $\Theta = |\Theta_\bullet|$. Let $X_\bullet: \Delta \Theta^{op} \to \text{Top}$ be a functor. Then $|t^*X_\bullet|$ has a natural continuous left $\Theta$-action.

Corollary 2.6. The realization of a cyclic space has a natural $T$-action and the realization of a dihedral space has a natural $O(2)$-action.

2.2. Dihedral subdivision. We now recall two types of subdivision of a simplicial sets: the Segal-Quillen subdivision [39] denoted $\text{sq}$ and the edgewise subdivision [5] denoted $\text{sd}_r$. Let $\text{sq}: \Delta^{op} \to \Delta^{op}$ be the functor defined on objects by $[n] \mapsto [n] \ast [n] = [2n+1]$ and on morphisms by $\alpha \mapsto \alpha \ast \omega(\alpha)$ where $\alpha: [n] \to [k]$. Here $\omega(\alpha)(i) = k - \alpha(n-i)$, and $\ast$ denotes the join. Given a simplicial set $Y_\bullet$, we define its Segal–Quillen subdivision $\text{sq}Y_\bullet$ by precomposition of the simplicial set with the functor $\text{sq}$. The induced map $|\text{sq}Y_\bullet| \cong |Y_\bullet|$ is a canonical homeomorphism, which is $D_2$-equivariant when $Y_\bullet = t^*X_\bullet$ is the restriction of a dihedral set (or simply a Real simplicial set) to $\Delta^{op}$. Moreover, when $X_\bullet$ is a dihedral set (or Real simplicial set) and $Y_\bullet = t^*X_\bullet$ the simplicial set $\text{sq}(Y_\bullet)$ is equipped with a simplicial $D_2$-action such that there is a canonical homeomorphism $|(\text{sq}Y_\bullet)^{D_2}| \cong |Y_\bullet|^{D_2}$. 
We now recall the $r$th edgewise subdivision. Let $sd_r: \Delta^\op \to \Delta^\op$ be the functor defined by $[n-1] \mapsto [nr-1]$ on objects and by $f \mapsto f^{nr}$ on morphisms. Letting $Y_\bullet$ be a simplicial set, we can form $sd_r Y_\bullet$ by precomposition with the functor $sd_r$. The induced map is a canonical homeomorphism $|sd_r Y_\bullet| \cong |Y_\bullet|$ and it is $\mu_r$-equivariant when $Y_\bullet = \iota^* X_\bullet$ is the restriction of a dihedral set (or simply a cyclic set) to $\Delta^\op$. Moreover, $sd_r(Y_\bullet)$ has a simplicial $\mu_r$-action and there is a canonical homeomorphism $|(sd_r Y_\bullet)^{\mu_r}| \cong |Y_\bullet|^{\mu_r}$ when $X_\bullet$ is a cyclic or dihedral set.

Now, given a simplicial set $Y_\bullet$, we define

$$sd_{D_{2r}} Y_\bullet := sqs sd_r Y_\bullet$$

following [40]. The induced map

$$|sd_{D_{2r}} Y_\bullet| \cong |Y_\bullet|$$

is a homeomorphism and it is $D_{2r}$-equivariant when $Y_\bullet = \iota^* X_\bullet$ is the restriction of a dihedral set $X_\bullet$ to $\Delta^\op$. Moreover, $sd_{D_{2r}} Y_\bullet$ has a simplicial $D_{2r}$-action and there is a canonical homeomorphism

$$|(sd_{D_{2r}} Y_\bullet)^{D_{2r}}| \cong |Y_\bullet|^{D_{2r}}.$$

### 2.3. Dihedral objects as Real cyclic objects.

The goal of this section is to describe dihedral objects and their realizations in terms of $D_2$-diagrams indexed by the cyclic category, which we call Real cyclic objects. This perspective will be used in Section 4 in order to show how work of [13] extends from the Real simplicial setting to the dihedral setting.

Let $BD_2$ be the small category with one object $*$ and morphisms set $BD_2(*,*) = D_2$. Consider a small category $I$ and a functor $b: BD_2 \to \text{Cat}$ such that $b(*) = I$. This data is equivalent to specifying an involution $b(\tau): I \to I$ such that $b(\tau)^2 = b(\tau^2) = b(1) = \text{id}_I$. We will write $\text{Cat}^{BD_2}$ for the category of functors $BD_2 \to \text{Cat}$.

**Example 2.7.** The terminal object in $\text{Cat}^{BD_2}$ is defined by $1(*) = [0]$ where we write $[0]$ for the category with a single object and a single morphism. The action $1(\tau): [0] \to [0]$ is the identity functor.

**Example 2.8.** Let $\delta: BD_2 \to \text{Cat}$ be defined by $\delta(*) = \Delta$ and $\delta(\tau)$ is the functor $\delta(\tau): \Delta \to \Delta$ defined as the identity on objects and by $(\delta(\tau)(f))(i) = k - f(n-i)$ for $f \in \Delta([n],[k])$ (cf. [13, Ex. 1.2]). This induces an involution $\delta(\tau)^\op: \Delta^\op \to \Delta^\op$.

**Example 2.9.** Let $\lambda: BD_2 \to \text{Cat}$ be defined by $\lambda(*) = \Lambda^\op$ and $\lambda(\tau): \Lambda^\op \to \Lambda^\op$ is defined as the identity on objects and by $\lambda(\tau)(f) = \delta(\tau)^\op(f)$ for $f \in \Delta^\op([n],[k])$ and by $\lambda(\tau)(\iota^k_{n+1}) = \iota^k_{n+1}$ for $\iota^k_{n+1} \in \mu_{n+1} = \text{Aut}_{\Lambda^\op}(\{n\})$ the canonical generator.

**Lemma 2.10** (cf. [15, Prop. I.19, I.35]). There is an isomorphism of categories

$$\Xi^\op \cong BD_2 \int \lambda$$

where $BD_2 \int \lambda$ is the Grothendieck construction associated to the functor $\lambda$ defined in Example 2.9.

**Proof.** This is straightforward as the objects and morphism are clearly in one-to-one correspondence and there is an evident functor inducing this correspondence. \(\square\)

**Definition 2.11** ([28, 44, 10]). For a category $C$, a $G$-diagram in $C$ consists of an object $b$ in $\text{Cat}^{BG}$ with $b(*) = I$, a functor $X: I \to C$ and natural transformations $g_X: X \Rightarrow X \circ b(g)$ for each $g \in G$, compatible with the group structure in the sense that the composite natural transformation

$$h_{X\circ b(h)} \circ g_X: X \Rightarrow (X \Rightarrow X \circ b(h)) \circ b(g)$$
is \((hg)_X\) for any \(h, g \in G\).

A morphism of \(G\)-diagrams is a natural transformation of functors from \(I\) to \(\mathcal{C}\), \(f : X \to Y\), compatible with all the structure. Following \[10\], we write \(\mathcal{C}_G^I\) for the category of \(G\)-diagrams in \(\mathcal{C}\) with respect to the functor \(b : BG \to \text{Cat}\) with \(b(*) = I\) and with morphisms the maps of \(G\)-diagrams.

We record the following notation for later use.

**Notation 2.12.** Given a \(G\)-diagram in \(\text{Sp}\), \(X : I \to \text{Sp}\) we write \(\text{hocolim}_I X\) and \(\text{holim}_I X\) for the homotopy colimit and the homotopy limit in \(\text{Sp}^G\) as defined in \[10\], Definition 1.16.

**Definition 2.13.** A **Real cyclic** object in a category \(\mathcal{C}\) is a functor \(BD_2/\lambda \to \mathcal{C}\) where \(\lambda : BD_2 \to \text{Cat}\) is the functor in Example 2.9. We write \(\mathcal{C}^\lambda_{\Lambda^{\text{op}}}\) for the category of Real cyclic objects in \(\mathcal{C}\).

We now identify the category of dihedral objects in \(\mathcal{C}\) and the category of Real cyclic objects in \(\mathcal{C}\).

**Corollary 2.14.** There is an isomorphism of categories
\[\mathcal{C}^{\Xi^{\text{op}}} \cong \mathcal{C}^\lambda_{\Lambda^{\text{op}}}\]
where \(\lambda : BD_2 \to \text{Cat}\) is defined as in Example 2.9.

**Proof.** There is an isomorphism of categories
\[\mathcal{C}^{BD_2/\lambda} \cong \mathcal{C}^{\Xi^{\text{op}}}\]
induced by the isomorphism of categories of Lemma 2.10. Then by \[10\], Lemma 1.9, there is an isomorphism of categories
\[\Phi : \mathcal{C}^{BD_2/\lambda} \cong \mathcal{C}^{\Lambda^{\text{op}}}\]
Recall that this isomorphism sends an object \(X\) in \(\mathcal{C}^{BD_2/\lambda}\) to the the object
\[\Phi(X) := X|_{\Lambda^{\text{op}}}\]
Here \(X|_{\Lambda^{\text{op}}}\) denotes the restriction of \(X\) along the natural inclusion into the Grothendieck construction, \(i : \Lambda^{\text{op}} \to BD_2/\lambda\). For an element \(g \in D_2\), the natural transformation
\[g\Phi(X) : X|_{\Lambda^{\text{op}}} \to X|_{\Lambda^{\text{op}}} \circ \lambda(g)\]
is defined at an object \([n]\) in \(\Lambda^{\text{op}}\) by
\[(g\Phi(X))[n] := X(g, \text{id} : g[n] \to g[n])\]
and all remaining conditions follow by functoriality. \(\square\)

**Remark 2.15.** There is also an equivalence of categories between the category of Real simplicial objects in \(\mathcal{C}\) and \(\mathcal{C}^\delta_{\Lambda^{\text{op}}}\) where \(\delta : BD_2 \to \text{Cat}\) is defined in Example 2.8. This motivates the naming convention in Definition 2.13.

Recall from \[10, 13\] that we can form the geometric realization of a \(D_2\)-diagram \(\Delta^{\text{op}} \to \text{Sp}\) and this geometric realization has a genuine \(D_2\)-action. By abuse of notation, write \(|X_\bullet| \in \text{Sp}_{D_2}\) for the geometric realization of a \(D_2\)-diagram \(X_\bullet : \Delta^{\text{op}} \to \text{Sp}\). Let
\[F : \mathcal{C}_\delta^{\Delta^{\text{op}}} \to \mathcal{C}^\lambda_{\Lambda^{\text{op}}}\]
denote the left adjoint to the forgetful functor \(i^* : \mathcal{C}^\lambda_{\Lambda^{\text{op}}} \to \mathcal{C}^{\Delta^{\text{op}}}\).
Lemma 2.16. Let \( Y_\bullet \in \text{Set}^{A_\omega}_\delta \). Then

\[
|t^* F(Y)| \cong \mathbb{T} \times |Y_\bullet|
\]

with diagonal \( D_2 \)-action given by the action of \( D_2 \) on \( \mathbb{T} \) by complex conjugation and the action of \( D_2 \) on \( |Y_\bullet| \).

Proof. The proof is a straightforward generalization of the proof of the classical result (cf. [17, Theorem 5.3 (iii)]) so we omit it. \( \square \)

Remark 2.17. Note that specifying an \( O(2) \)-action \( O(2) \times X \to X \) on a topological space \( X \) is equivalent to specifying a \( D_2 \)-action on \( X \) as well as a \( \mathbb{T} \)-action \( \mathbb{T} \times X \to X \) which is \( D_2 \)-equivariant with respect to the diagonal action on the source, where \( D_2 \) acts on \( \mathbb{T} \) by complex conjugation.

Corollary 2.18. The geometric realization \(|t^* X_\bullet|\) of a Real cyclic set \( X_\bullet \) has an \( O(2) \)-action.

Note that we also know that the realization of an object \( X_\bullet \) in \( \text{Set}^{A_\omega}_\lambda \) has an \( O(2) \)-action by Corollary 2.14. From this perspective, however, it is more clear that the \( O(2) \)-action restricts to the \( D_2 \)-action on \(|t^* X_\bullet|\) from [13]. We use this perspective in Section 4.

3. Real topological Hochschild homology

In this section, we recall the definition of Real topological Hochschild homology. To begin we review the theory of \( E_\sigma \)-rings, which are the input for Real topological Hochschild homology.

3.1. Algebras over equivariant little disks. Equivariant norms will play an important role here and in subsequent sections, so we begin by recalling their definition.

Definition 3.1. Let \( G \) be a finite group and \( H \) a subgroup of \( G \) of index \( n \). An ordered set of coset representatives of the \( G \)-set \( G/H \) defines a group homomorphism \( \alpha: G \to \Sigma_n \wr H \). Given a choice of ordering of the coset representatives of the \( G \)-set \( G/H \), we define the indexed smash \( \wedge^G_H \) as the composite

\[
\wedge^G_H: \text{Sp}^{BH} \xrightarrow{\wedge^n} \text{Sp}^{B\Sigma_n \wr H} \xrightarrow{\alpha^*} \text{Sp}^{BG}
\]

and we define the norm \( N^G_H \) as the composite

\[
N^G_H = T_{E_\infty} \circ \wedge^G_H \circ T_{W_\infty} \circ \text{Sp}^{H} \xrightarrow{\iota_H^* W} \text{Sp}^{G}
\]

where \( W \) is a complete \( G \)-universe and \( \iota_H^* W \) is the restriction of \( W \) to \( H \).

Remark 3.2. One may also define norms more generally for finite \( G \)-sets as in [4]. For example, when \( X \) is a \( D_2 \)-spectrum and \( S \) is a finite \( D_2 \)-set with orbit decomposition

\[
S = (\bigsqcup_{i \in I} D_2/e) \cup (\bigsqcup_{j \in J} D_2/D_2),
\]

then there is an isomorphism

\[
N^S X \cong (\bigwedge_{i \in I} N^{D_2/e} \iota_e^* X) \wedge (\bigwedge_{j \in J} X)
\]

by [4, Proposition 6.2]. When there is no subscript in the notation of the norm, we will always mean the construction above.
Definition 3.3. Let \( n \) be the \( D_2 \)-set \( \{1, 2, \ldots, n\} \) with the generator \( \tau \) of \( D_2 \) acting by \( \tau(i) = n - i + 1 \). This produces a group homomorphism \( D_2 \to \Sigma_n \) and consequently a graph subgroup \( \Gamma_n \) of \( \Sigma_n \times D_2 \), where a graph subgroup is a subgroup of \( \Sigma_n \times D_2 \) that intersects \( \Sigma_n \) trivially. The \( \text{Assoc}_{\sigma} \)-operad is an operad in \( D_2 \)-spaces with \( n \)-th space

\[
\text{Assoc}_{\sigma,n} = (\Sigma_n \times D_2)/\Gamma_n.
\]

See [24] for the definition of the structure maps in this operad.

Definition 3.4. By an \( E_\sigma \)-operad, we mean any operad \( O_\sigma \) in \( \text{Top}^{D_2} \) such that there is a map of operads in \( D_2 \)-spaces

\[
O_\sigma \to \text{Assoc}_{\sigma,n}
\]

inducing a weak equivalence \( O_{\sigma,n} \simeq (\Sigma_n \times D_2/\Gamma_n) \) in \( \text{Top}^{D_2} \).

Definition 3.5. Let \( O_\sigma \) be an \( E_\sigma \)-operad. Define a monad \( P_\sigma \) on \( \text{Sp}^{D_2} \) by the formula

\[
P_\sigma(-) = \bigoplus_{n \geq 1} O_{\sigma,n} \otimes_{\Sigma_n} (-)^{\otimes n}.
\]

By an \( E_\sigma \)-ring, we mean an algebra in \( \text{Sp}^{D_2} \) over the monad \( P_\sigma \) associated to the operad \( O_{\sigma,n} = \text{Assoc}_{\sigma,n} \).

Proposition 3.6. Let \( X \in \text{Sp}^{D_2} \). Then

\[
P_\sigma(X) = T(N^{D_2}X) \land (S \lor X).
\]

Proof. This follows from the equivariant twisted James splitting (cf. [24, Thm. 4.3]). Write \( n = \{1, \ldots, n\} \) for the \( D_2 \)-set where the generator \( \tau \) acts by \( i \mapsto n+1-i \). There is an isomorphism

\[
n \cong \begin{cases} D_2^{n/2} & \text{if } n \text{ is even} \\ D_2/D_2 \cup D_2^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}
\]

of \( D_2 \)-sets. The action of \( D_2 \) on \( n \) is a homomorphism \( D_2 \to \Sigma_n \) and we define the associated graph subgroup to be \( \Gamma_n \). Thus, when \( n \) is even

\[
I_{\ell}^{V} \left( \left( \text{Assoc}_{\sigma,n} \right)_{*} \land \Sigma_{n} I_{\ell}^{V} X \right) = \left( N^{D_2}(X) \right)^{\land n/2}
\]

and when \( n \) is odd

\[
I_{\ell}^{V} \left( \left( \text{Assoc}_{\sigma,n} \right)_{*} \land \Sigma_{n} I_{\ell}^{V} X \right) = X \land \left( N^{D_2}(X) \right)^{\land n/2}.
\]

The result then follows by regrouping smash factors according to the number of smash factors involving the norm.

Definition 3.7. By an \( E_1 \)-ring \( A \), we mean an algebra over the associative operad \( \text{Assoc} \) in the category of orthogonal spectra \( \text{Sp} \). By an \( E_0 \)-\( A \)-algebra we mean an \( A \)-bimodule \( M \) equipped with a map \( A \to M \) of \( A \)-bimodules (cf. [33, Rem. 2.1.3.10]).

Example 3.8. An \( E_1 \)-ring \( A \) is an \( E_0 \)-ring in \( A^\land A^{\text{op}} \)-bimodules with unit map \( \mu : A^\land A^{\text{op}} \to A \) given by multiplication and the bimodule structure given by

\[
\varphi_L : A^\land A^{\text{op}} \land A \xrightarrow{1 \land B_{A^\land A^{\text{op}}}} A \land A \land A^{\text{op}} \xrightarrow{\mu \land 1 \land \mu} A
\]

\[
a_1 \land a_2 \land a \mapsto a_1 \cdot a \cdot a_2
\]

\[
\varphi_R : A^\land A \land A^{\text{op}} \xrightarrow{B_{A^\land A^{\text{op}}}} A \land A \land A^{\text{op}} \xrightarrow{1 \land \mu \land 1} A
\]

\[
a \land a_1 \land a_2 \mapsto a_1 \cdot a \cdot a_2
\]
which we call the *standard structure* of an \( E_0 \)-\( A \wedge A^{\text{op}} \)-algebra on \( A \). Here \( B_{X,Y} : X \wedge Y \to Y \wedge X \) denotes the natural braiding in the symmetric monoidal structure.

**Remark 3.9.** Given an \( E_2 \)-ring \( R \), then \( \iota_e^* R \) is an \( E_1 \)-ring with anti-involution given by the action of the generator of the Weyl group \( D_2 \). Throughout, we will let \( \tau : \iota_e^* R^{\text{op}} \to \iota_e^* R \) denote this anti-involution.

**Corollary 3.10.** An \( E_\sigma \)-ring \( R \) is exactly a \( D_2 \)-spectrum \( R \) such that

1. the spectrum \( \iota_e^* R \) is an \( E_1 \)-ring with anti-involution, always denoted \( \tau : \iota_e^* R^{\text{op}} \to \iota_e^* R \), given by the action of the generator of \( D_2 \) group,
2. the spectrum \( R \) is an \( E_0 \)-\( N^{D_2} \)-algebra and applying \( \iota_e^* \) to the \( E_0 \)-\( N^{D_2} \)-algebra structure map gives \( \iota_e^* R \) the standard \( E_0 \)-\( \iota_e^* R \wedge \iota_e^* R^{\text{op}} \)-algebra structure.

**Example 3.11.** Given an \( E_1 \)-ring with anti-involution, regarding \( R \) as an object in \( \text{Sp}^{D_2}_Y \) produces an \( E_\sigma \)-ring structure on \( R \).

**Example 3.12.** If \( R \) is in \( \text{Comm}(\text{Sp}^{D_2}_Y) \), then \( R \) is an \( E_\sigma \)-ring.

### 3.2. The dihedral bar construction.

We first define the relevant input for Real topological Hochschild homology with coefficients.

**Definition 3.13.** Suppose \( A \) is an \( E_\sigma \)-ring. As observed in Remark 3.9, the underlying spectrum \( \iota_e^* A \) is an \( E_1 \)-ring with anti-involution \( \tau : \iota_e^* A^{\text{op}} \to \iota_e^* A \). An \( A \)-bimodule \( M \) with *involution* \( j \) is an \( \iota_e^* A \)-bimodule \( M \) along with a map of \( \iota_e^* A \)-bimodules \( j : M^{\text{op}} \to M \), such that \( j^{\text{op}} \circ j = \text{id} \). Here \( M^{\text{op}} \) is the \( \iota_e^* A^{\text{op}} \)-bimodule with module structures given by:

\[
\iota_e^* A^{\text{op}} \otimes M \xrightarrow{B_{\iota_e^* A^{\text{op}},M}} M \otimes \iota_e^* A^{\text{op}} \xrightarrow{1 \otimes \tau} M \otimes \iota_e^* A \xrightarrow{\psi_R} M
\]

and

\[
M \otimes \iota_e^* A^{\text{op}} \xrightarrow{B_{M,\iota_e^* A^{\text{op}}}} \iota_e^* A^{\text{op}} \otimes M \xrightarrow{\tau \otimes 1} \iota_e^* A \otimes M \xrightarrow{\psi_L} M
\]

where \( \psi_L \) and \( \psi_R \) are the bimodule structure maps of \( M \) and \( B_{-,-} \) is braiding in the symmetric monoidal structure on \( \text{Sp}^{D_2}_Y \).

**Remark 3.14.** In the definition below, we will drop the notation \((-)^{\text{op}}\) in the maps \( \tau \) and \( j \) when we just consider the maps as maps in \( \text{Sp} \).

**Definition 3.15 (Dihedral bar construction).** The dihedral bar construction \( B_{\bullet}^\text{dl}(A;M) \) has \( k \)-simplices

\[
B_{\bullet}^\text{dl}(A;M) = M \otimes A^{\otimes k}.
\]

It has the same structure maps as the cyclic bar construction with coefficients, \( B_{\bullet}^\text{cy}(A;M) \), with the addition of a levelwise involution \( \omega_k \) acting on the \( k \)-simplices.

To specify the levelwise involution, let \( k \) be the \( D_2 \)-set \( \{1,2,\ldots,k\} \) with the generator \( \alpha \) of \( D_2 \) acting by \( \alpha(i) = k-i+1 \) for \( 1 \leq i \leq k \). Then we define the action of \( \omega_k \) by the composite

\[
\omega_k : M \otimes A^k \xrightarrow{M \otimes \alpha} M \otimes A^k \xrightarrow{j \otimes \tau \otimes \ldots \otimes \tau} M \otimes A^k
\]

where \( A^\alpha \) is the automorphism of \( A^k \) induced by \( \alpha : k \to k \). It is straightforward to check that the structure maps satisfy the usual simplicial identities (and cyclic identities when \( M = A \)), together with the additional relations

\[
d_i \omega_k = \omega_{k-1} d_{k-i} \text{ for } 0 \leq i \leq k, \\
s_i \omega_k = \omega_{k+1} s_{k-i} \text{ for } 0 \leq i \leq k, \\
\omega_k t_k = t_{k-1} \omega_k \text{ when } M = A.
\]
Therefore, the dihedral bar construction is a Real simplicial object and, when \( M = A \), it is a dihedral object. We define the dihedral bar construction of the pair \((A; M)\) by the formula
\[
B^\text{di}_\bullet(A; M) = |B^\text{di}_\bullet(A; M)|.
\]

3.3. **Real topological Hochschild homology.** We now recall the Bökstedt model for \( \text{THR}(A) \), following [21, 13]. Let \( \mathcal{I} \) be the category with objects the finite sets \( k = \{1, 2, \ldots, k\} \) and injective morphisms, where 0 is the empty set. Equip this category with a \( D_2 \)-action \( \tau: \mathcal{I} \to \mathcal{I} \) where \( \tau \) acts trivially on objects and for an injection \( \alpha: i \to j \), we let
\[
\tau(\alpha)(s) = j - \alpha(i - s + 1) + 1,
\]
following [21]. Observe that \( \mathcal{I} \) is a monoid in \( \text{Cat} \) with operation
\[
+: \mathcal{I} \times \mathcal{I} \to \mathcal{I}
\]
defined by sending \((i, j)\) to \( i + j \) on objects and by sending a pair of morphisms \( \alpha: i \to j \), \( \beta: i' \to j' \) to
\[
(\alpha + \beta)(s) = \begin{cases} 
\alpha(s) & \text{if } 1 \leq s \leq i \\
\beta(s - i) + j & \text{if } i < s \leq i + i'
\end{cases}
\]
given by disjoint union and neutral element 0. The unit of \( \mathcal{I} \) regarded as a monoid is therefore the functor \( \eta: [0] \to \mathcal{I} \) that sends the unique object in the terminal category \([0]\) to 0 and the unique morphism in the terminal category to the identity map \( \text{id}_0: 0 \to 0 \). The functor \( \tau \) is strong monoidal with respect to this monoidal structure on the target and the opposite monoidal structure on the source. In other words, there is a strong monoidal functor
\[
\tau: \mathcal{I}^{-\text{rev}} \to \mathcal{I}
\]
where \( \mathcal{I}^{-\text{rev}} \) has the same underlying category \( \mathcal{I} \), but the monoidal structure is defined by
\[
\mathcal{I} \times \mathcal{I} \xrightarrow{B^\text{di}_\bullet} \mathcal{I} \times \mathcal{I} \xrightarrow{\tau} \mathcal{I}
\]
where \( B^\text{di}_\bullet: \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C} \) is the natural braiding in the symmetric monoidal category \( \text{Cat} \).

We can therefore view \( \mathcal{I} \) as a monoid with anti-involution in \( \text{Cat} \) and let \( B^\text{di}_\bullet(\mathcal{I}) \) be its associated dihedral Bar construction. We write \( \mathcal{I}^{1+k} \) for the \( k \)-simplices of \( B^\text{di}_\bullet(\mathcal{I}) \) regarded as an object in \( \text{Cat}^{BD_2(k+1)} \).

By Corollary 2.14, we may regard \( B^\text{di}_\bullet \mathcal{I} \) as a \( D_2 \)-diagram
\[
B^\text{di}_\bullet \mathcal{I}: \Lambda^{\text{op}} \to \text{Cat}
\]
and take the Grothendieck construction [43, Def. 1.1] of this functor, denoted \( \Lambda^{\text{op}} \int B^\text{di}_\bullet \mathcal{I} \).

Let \( A \) be an \( E_\sigma \)-ring and let \( S \) be the \( D_2 \)-equivariant sphere spectrum with \( S_k = S^k \). We write \([k]\) for an object in \( \Lambda^{\text{op}} \) and \( j = (j_0, \ldots, j_k) \) for an object in \( \mathcal{I}^{1+k} \). We define \( D_2 \)-diagrams in spectra
\[
\Omega^\bullet_\mathcal{I}(A; S): \Lambda^{\text{op}} \int B^\text{di}_\bullet \mathcal{I} \to \text{Sp}
\]
on objects, by \(([k]; j) \mapsto \Omega^{j_0+\ldots+j_k}(S \wedge A_{j_0} \wedge \ldots \wedge A_{j_k}) \) as in [37]. Given a symmetric monoidal category \((\mathcal{C}, \otimes)\), let \( \gamma_k \) denote the natural transformation between the functor \( - \otimes \ldots \otimes -: \mathcal{C}^k \to \mathcal{C} \) and itself that reverses the order of the entries. We abuse notation and write \( \gamma_k := (\gamma_k)(A_{j_1}, \ldots, A_{j_k}) \) and \( \gamma_{ji} = (\gamma_{ji})(s^1, \ldots, s^1) \) in \( \text{Top} \). We also write \( \gamma_{ji} = A((\gamma_{ji})(\mathbb{R}, \ldots, \mathbb{R})) \), where
\( \gamma_{j_i} \) is defined on the symmetric monoidal category of orthogonal representations with respect to direct sum, for \( 1 \leq i \leq k \). Let \( W_k \) be the composite map

\[
S \wedge A_{j_0} \wedge A_{j_1} \wedge \cdots \wedge A_{j_k} \xrightarrow{S \wedge \tau^{\wedge k+1}} S \wedge A_{j_0} \wedge A_{j_1} \wedge \cdots \wedge A_{j_k} \\
\xrightarrow{S \wedge A_{j_0} \wedge \gamma_k} S \wedge A_{j_0} \wedge A_{j_k} \wedge \cdots \wedge A_{j_1} \\
\xrightarrow{W_k} S \wedge A_{j_0} \wedge A_{j_k} \wedge \cdots \wedge A_{j_1}
\]

of orthogonal spectra and let \( G_k \) be the composite map

\[
S^{j_0} \wedge S^{j_1} \wedge \cdots S^{j_k} \xrightarrow{S^{j_0} \wedge \gamma_k} S^{j_0} \wedge S^{j_k} \wedge \cdots S^{j_1} \\
\xrightarrow{G_k} S^{j_0} \wedge S^{j_1} \wedge \cdots S^{j_k}
\]

of topological spaces. The diagram \( \Omega_+^\bullet(A;S)([k];j) \) has \( D_2 \)-action

\[
\Omega_+^\bullet(A;S) \xrightarrow{\tau_j} \Omega_+^\bullet(A;S)
\]

where \( \tau_j = (j_0, j_k, \ldots, j_1) \), defined by sending a map

\[
f: S^{j_0+\cdots+j_k} \to S \wedge A_{j_0} \wedge \cdots A_{j_k}
\]

to \( W_k \circ f \circ G_k \).

Note that there is a \( \mu_{k+1} \)-action on

\[
\Omega_{j_0+\cdots+j_k}^\bullet(S \wedge A_{j_0} \wedge \cdots A_{j_k})
\]

by conjugation where we act on source and target of a loop by cyclic permutation of the indices \((j_0, \ldots, j_k)\). The composite

\[
I^{1+k} \xrightarrow{t_k} \Lambda^\text{op} \int B^\text{di}_{\bullet} I \xrightarrow{\Omega_{i}^\bullet(A;S)} \mathbf{Sp}
\]

sending \( j \) to \( \Omega_+^\bullet(A;S)([k],j) \) is in fact a \( D_{2(k+1)} \)-diagram in spectra.

**Lemma 3.16.** Given a functor \( X: \Lambda^\text{op} \int B^\text{di}_{\bullet} I \to \mathbf{Sp} \) and the canonical \( D_{2(k+1)} \)-equivariant functor

\[
t_k: I^{1+k} \longrightarrow \Lambda^\text{op} \int B^\text{di}_{\bullet} I,
\]

the functor \( \Lambda^\text{op} \to \mathbf{Sp}^{C_2} \) defined on \( k \)-simplices by \( D_{2(k+1)} \)-spectra

\[
\text{THR}(X)_k := \text{hocolim}_{I^{1+k}} t^*_k X,
\]

using Notation 2.12 defines a Real cyclic object in \( \mathbf{Sp} \) (see Definition 2.13).

**Proof.** By [13, Prop. 2.15], we know that this construction defines a natural \( D_2 \)-diagram \( \Delta^\text{op} \to \mathbf{Sp} \), so in order to elevate this to a natural \( D_2 \)-diagram \( \Lambda^\text{op} \to \mathbf{Sp} \), it will suffice to describe the additional cyclic structure maps and their compatibility with the \( D_2 \)-diagram structure \( \Delta^\text{op} \to \mathbf{Sp} \). The cyclic structure comes from cyclically permuting the coordinates in \( I^{1+k} \) and acting via the \( \mu_{k+1} \)-action on \( t_k X \). Then taking the homotopy colimit

\[
\text{hocolim}_{I^{1+k}} t^*_k X
\]
of the $D_{2(k+1)}$-diagram $\iota^*_n X$ in the sense of [10, Definition 1.16], we have a $D_{2(k+1)}$-spectrum, which restricts to the $D_2$-spectrum defined in [13, Prop. 2.15]. The restriction to the $\mu_{k+1}$-action produces the standard $\mu_{k+1}$-action on the Bökstedt construction so this is compatible with the face and degeneracy maps. Since it is in fact the restriction of a $D_{2(k+1)}$-action, the compatibility of the cyclic structure maps and the additional dihedral structure maps is clear except for the compatibility of the additional dihedral structure maps with the face and degeneracy maps and this compatibility is exactly what is proven in [13, Prop. 2.15].

Definition 3.17 (Real topological Hochschild homology: Bökstedt model). When $X = \Omega^*_{\mathcal{D}}(A; \mathbb{S})$ then the Bökstedt model for Real topological Hochschild homology is defined as

$$\text{THR}(A) := |\text{THR}(\Omega^*_{\mathcal{D}}(A; \mathbb{S}))_s|$$

and it is an $O(2)$-spectrum by Lemma 2.16. It also restricts to a genuine $H$-spectrum for each cyclic group $H$ of order 2 such that $H \cap \mathbb{T} = e$.

3.4. Model structures. We now set up the basic model categorical constructions we need. We refer the reader to [1], [34], and [13] for a more thorough survey of the model categorical considerations used in this paper.

First, we fix some notation from [18]. Given a normal subgroup $N$ of a compact Lie group $G$, we write $\mathcal{F}(N, G) = \{H < G : H \cap N = e\}$ for the family of subgroups of $G$ that intersect $N$ trivially and write $\mathcal{A}_{\mathcal{F}}(e, G)$ for the family of all subgroups of $G$. Following Hogenhaven [27], we will write $\mathcal{R}$ for the family $\mathcal{F}(\mathbb{T}, O(2))$. We also write $\mathcal{F}[N]$ for the family of subgroups of $G$ that don’t contain $N$. For any family $\mathcal{F}$ of subgroups of $O(2)$, we say an $O(2)$-equivariant map is an $\mathcal{F}$-equivalence if it induces isomorphisms on homotopy groups $\pi^H_*(\mathcal{F})$ for all $H \in \mathcal{F}$.

We write $\mathbb{R}^\infty$ for the trivial $G$-universe for any compact Lie group $G$. We fix a complete $O(2)$-universe $\mathcal{U}$ throughout this section and let $\mathcal{V}_n := \iota^*_{D_{2n}}(\mathcal{U})$ be a complete $D_{2n}$-universe constructed as the restriction of $\mathcal{U}$ to $D_{2n}$. When $n = 1$, we simply write $\mathcal{V} = \mathcal{V}_1$. We note that the determinant homomorphism $O(2) \to \{-1, 1\}$ has a (non-canonical) splitting producing $D_2$ as a subgroup of $O(2)$ (see Section 2 for details on our choice of splitting). We write $\overline{\mathcal{V}}$ for the $O(2)$-universe associated to $\mathcal{V}$ by inflation along the determinant homomorphism.

The following statement combines results from [23, Appendix B] (cf. [1, Theorem 2.26,2.29]).

Proposition 3.18. Let $G$ be a compact Lie group. There is a positive complete stable compactly generated model structure on orthogonal $G$-spectra indexed on a complete universe $\mathcal{U}$, denoted $\text{Sp}^{G, \mathcal{F}}_\mathcal{U}$, where the weak equivalences are $\mathcal{F}$-equivalences, the cofibrations the positive complete stable $\mathcal{F}$-cofibrations, and the fibrations are then determined by the right lifting property. There is also a positive complete stable compactly generated $\mathcal{F}$-model structure on the category of commutative monoids in orthogonal $G$-spectra indexed on a complete universe $\mathcal{U}$, denoted $\text{Comm}(\text{Sp}^{G, \mathcal{F}}_\mathcal{U})$. The weak equivalences and fibrations are exactly those maps which are weak equivalences and fibrations after applying the forgetful functor

$$\text{Comm}(\text{Sp}^{G, \mathcal{F}}_\mathcal{U}) \to \text{Sp}^{G, \mathcal{F}}_\mathcal{U}$$

and the cofibrations are determined by the left-lifting property. When $\mathcal{F}$ is the family of all subgroups, we drop $\mathcal{F}$ — from the notation and refer to the equivalences as the stable equivalences.

Proposition 3.19 (Proposition A.2 [13]). When $G = D_2$, there is also a model structure on the category of $\text{Assoc}_\sigma$-algebras in $\text{Sp}^{D_2}_\mathcal{U}$, which we denote $\text{Assoc}_\sigma(\text{Sp}^{D_2})$ where the weak
equivalences are the stable equivalences and the cofibrations are the positive complete stable cofibrations.

**Definition 3.20.** We will refer to each of the model structures on $\text{Sp}_U^{G, \mathcal{F}}$ and $\text{Comm}(\text{Sp}_U^{G, \mathcal{F}})$ defined in Proposition 3.18 and the model structure on $\text{Assoc}_\sigma(\text{Sp}_V^{D_2})$ of Proposition 3.19 as the $\mathcal{F}$-model structure for brevity.

**Definition 3.21.** Let $G$ be a compact Lie group. A map of $G$-spaces is a $G$-cofibration if it has the left lifting property with respect to all maps of $G$-spaces $f: X \to Y$ such that $X^H \to Y^H$ is a weak equivalence and a Serre fibration for every closed subgroup $H$ of $G$.

We use of another collection of cofibrations in orthogonal $D_2$-spectra indexed on a complete universe $V$, which we call flat cofibrations. These are also the cofibrations of a model structure on orthogonal $D_2$-spectra [6].

**Definition 3.22.** We say a map $X \to Y$ of orthogonal $D_2$-spectra is flat if the latching map

$$L_n Y \coprod_{L_n X} X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$$

is a $(D_2 \times O(n))$-cofibration for all $n \geq 0$. In particular, $Y$ is a flat orthogonal $D_2$-spectrum if the latching map

$$L_n(Y) \to Y(\mathbb{R}^n)$$

is a $(D_2 \times O(n))$-cofibration for all $n \geq 0$.

**Remark 3.23.** If $Y$ is cofibrant in $\text{Assoc}_\sigma(\text{Sp}_V^{D_2})$ with the positive complete stable model structure, then it is flat in the sense of Definition 3.22 (cf. Remark A.3 [13]). Therefore, we can replace any spectrum $Y$ in $\text{Assoc}_\sigma(\text{Sp}_V^{D_2})$ by a flat spectrum $\tilde{Y}$ in $\text{Assoc}_\sigma(\text{Sp}_V^{D_2})$ by cofibrantly replacing in the positive complete stable model structure of Proposition 3.18.

Finally, in order to make sure that certain simplicial spectra are Reedy cofibrant, we need the following definition.

**Definition 3.24.** We say an orthogonal $G$-spectrum $X$ indexed on a complete universe $U$ is well-pointed if $X(V)$ is well-pointed in $\text{Top}^G$ for all finite dimensional orthogonal $G$-representations $V$. We say an $E_\sigma$-ring in $\text{Sp}_V^{D_2}$ is very well-pointed if it is well-pointed and the unit map $S^0 \to X(\mathbb{R}^0)$ is a Hurewicz cofibration in $\text{Top}^{D_2}$.

### 3.5. Real $p$-cyclotomic spectra.

Here we briefly recall the theory of Real cyclotomic spectra following [27, 38]. We include this section to motivate our choice of the family of subgroups $\mathcal{R}$ in Section 4. In Section 7, we will prove that our construction of Real Hochschild homology has a notion of Real cyclotomic structure that allows one to define Witt vectors for rings with anti-involution, so this section is also relevant for that setup.

First, recall that given a family of closed subgroups $\mathcal{F}$ of a topological group $G$, there is a $G$-CW complex $E\mathcal{F}$ with the property that $E\mathcal{F}^H \simeq \ast$ if $H \in \mathcal{F}$ and $E\mathcal{F} = \varnothing$ otherwise. We define $E\mathcal{F}$ as the homotopy cofiber of the map $E\mathcal{F}^+ \to S^0$ induced by the collapse map.

Recall that there is a (derived) geometric fixed point functor

$$\Phi^\mu: \text{Sp}_U^{O(2)} \to \text{Sp}_U^{O(2)}$$

for each odd prime $p$ defined by first cofibrantly replacing and then applying the functor $(E\mathcal{R} \wedge \cdot)^{\mu_p}$. Here we identify $O(2)/\mu_p$ with $O(2)$ via the root homeomorphism $O(2) \to O(2)/\mu_p$ and we post-compose with a change of universe functor so that we do not need to distinguish between $U^{\mu_p}$ and $U$. We write $D_{2p^\infty}$ for the semi-direct product $\mu^\infty_p \rtimes D_2 \subset O(2)$ where $\mu^\infty_p$ is the Prüfer $p$-group. For the rest of this section fix an odd prime $p$. 


**Definition 3.25** (cf. [27, Definition 2.6]). We say $X$ in $\text{Sp}_{\text{Id}}^{O(2)}$ is genuine Real $p$-cycloplatomic if there is a $\mathcal{F}_{\text{Fin}}^{O(2)}$-equivalence $\Phi^\mu p X \simeq X$, where $\mathcal{F}_{\text{Fin}}^{O(2)}$ is the family of finite subgroups of $O(2)$.

**Example 3.26.** By [27, Theorem 2.9], the spectrum $\text{THR}(A)$ is genuine Real $p$-cycloplatomic for any $E_\sigma$-ring $A$.

As a consequence, there are $D_2$-equivariant structure maps

$$R_k: \text{THR}(A)^{\mu_p k} \to (\tilde{E} \wedge \text{THR}(A))^{\mu_p k} \to \text{THR}(A)^{\mu_p k-1}$$

called restriction maps for all primes $p$.

**Definition 3.27** (cf. [27, Definition 3.6.]). Real topological restriction homology is defined as

$$\text{TRR}(A;p) = \text{holim}_{k,R} \text{THR}(A)^{\mu_p k}$$

in the category $\text{Sp}_{D_2}^{D_2}$. There are also $D_2$-equivariant Frobenius maps

$$F_k: \text{THR}(A)^{\mu_p k} \to \text{THR}(A)^{\mu_p k-1},$$

induced by inclusion of fixed points, which in turn induce a map

$$F: \text{TRR}(A;p) \to \text{TRR}(A;p)$$

on Real topological restriction homology. We define $\text{TCR}(A;p)$ as the homotopy equalizer of the diagram

$$\begin{array}{ccc}
\text{TRR}(A;p) & \xrightarrow{F} & \text{TRR}(A;p) \\
\downarrow{id} & & \downarrow{id} \\
\text{TRR}(A;p) & & \text{TRR}(A;p)
\end{array}$$

in the category $\text{Sp}_{D_2}^{D_2}$.

Let $X$ be an object in $\text{Sp}_{\text{Id}}^{O(2),\mathcal{R}}$ and write $E = E \mathcal{R}$ and $\tilde{E} = \tilde{E} \mathcal{R}$. We consider the isotropy separation sequence associated to $\mathcal{R}$, as in [18] (cf. [38, Remark 5.32])

$$E_+ \wedge F(E_+,X)^{\mu_p k} \to F(E_+,X)^{\mu_p k} \to (\tilde{E} \wedge F(E_+,X))^{\mu_p k}$$

where $(-)^{\mu_p k}$ denotes the categorical fixed points functor (denoted $\Psi^{\mu_p k}$ in [38, Remark 3.2]). We write

$$X^{tD_2 \mu_p k} = (\tilde{E} \wedge F(E_+,X))^{\mu_p k} \quad \text{and} \quad X^{hD_2 \mu_p k} = F(E_+,X)^{\mu_p k}.$$ 

We now recall the definition of Real $p$-cycloplatomic spectrum. Our definition will differ from [38, Definition 3.15] in that we use $O(2)$ instead of $D_2 \circ \infty$.

**Definition 3.28** (cf. [38, Definition 6.5]). A object $X$ in $\text{Sp}_{\mathcal{V}}^{O(2),\mathcal{R}}$ is Real $p$-cycloplatomic if there is a Tate-valued Frobenius map

$$\varphi_p: X \to X^{tD_2 \mu_p}$$

in $\text{Sp}_{\mathcal{V}}^{O(2),\mathcal{R}}$. 


Identifying \( N^O(D_2) \) and \( \text{THR}(A) \) as Real \( p \)-cyclotomic spectra in the sense above therefore only requires showing that there is a zig-zag of \( \mathcal{R} \)-equivalences between \( N^O(D_2) \) and \( \text{THR}(A) \) and compatible Tate valued Frobenius maps \( \varphi_p \) and \( \varphi'_p \)

\[
N^O(D_2) \simeq \text{THR}(A)
\]

\[
N^O(D_2)(A)^{t_{D_2} \mu_p} \simeq \text{THR}(A)^{t_{D_2} \mu_p}
\]

in \( \text{Sp}^{O(2), \mathcal{R}} \). Finally, we note that it suffices that \( \text{THR}(A) \) is Real \( p \)-cyclotomic in the sense of Definition 3.28 in order to define \( p \)-typical Real topological cyclic homology. For this discussion, we assume \( t^*_e \text{THR}(A) \) is bounded below. Let

\[
\text{can}_p : \text{THR}(A)^{h_{D_2} \mu_p} \rightarrow \text{THR}(A)^{t_{D_2} \mu_p}
\]
denote the right map in the diagram (7) for \( X = \text{THR}(A) \). Using the identification

\[
(\text{THR}(A)^{h_{D_2} \mu_p})^{h_{D_2} \mu_p} \simeq (\text{THR}(A)^{h_{D_2} \mu_p})^{h_{D_2} \mu_p},
\]

and the identification

\[
((\text{THR}(A))^{t_{D_2} \mu_p})^{h_{D_2} \mu_p} \simeq (\text{THR}(A))^{t_{D_2} \mu_p},
\]

which uses the hypothesis that \( t^*_e \text{THR}(A) \) is bounded below and the Real Tate orbit lemma [38, Lemma 7.19], we have maps

\[
\text{can} = (\text{can}_p)^{h_{D_2} \mu_p} : TCR^- (A; p) \rightarrow TPR(A; p)
\]

and

\[
\varphi = (\varphi_p)^{h_{D_2} \mu_p} : TCR^- (A; p) \rightarrow TPR(A; p).
\]

Finally, let \( TCR(A; p) \) denote the homotopy equalizer of the diagram

\[
\xymatrix{T CR^- (A; p) \ar[r]^-{\text{can}} & TPR(A; p)}
\]

Then [38, Corollary 7.3] implies that the definition of \( TCR(A; p) \) as the homotopy equalizer of the diagram (9) and the definition of Real topological cyclic homology \( TCR(A; p) \) of Definition 3.27 agree after \( p \)-completion.

4. **Real topological Hochschild homology via the norm**

Recall that the third author with Hopkins and Ravenel [23] define multiplicative norm functors from \( H \)-spectra to \( G \)-spectra

\[
N^T_H : \text{Sp}^H \rightarrow \text{Sp}^G
\]

for a finite group \( G \) and subgroup \( H \) (see Definition 3.1). It is shown in [1] and [6] that one can extend this construction to a norm \( N^T_e \) on associative ring orthogonal spectra. The functor

\[
N^T_e : \text{Assoc}(\text{Sp}) \rightarrow \text{Sp}^T_U
\]

is defined via the cyclic bar construction, \( N^T_e(R) = T^U_{R^\infty} B^e_{\Lambda'} R \). In [1] the authors then show that this functor satisfies the adjointness properties that one would expect from a norm. In particular they show that in the commutative setting the norm functor is left adjoint to the forgetful functor from commutative ring orthogonal \( T \)-spectra to commutative ring orthogonal
spectra. It follows from this construction of $N^\Sigma_T$ that topological Hochschild homology can be viewed as a norm from the trivial group to $\mathbb{T}$.

In this section, we consider the analogous story for Real topological Hochschild homology. In particular, we define a norm functor $N^{O(2)}_{D_2}$ using the dihedral bar construction, and prove that it satisfies the adjointness properties that one would expect from an equivariant norm. We then characterize Real topological Hochschild homology as the norm $N^{O(2)}_{D_2}$.

4.1. The norm and the relative tensor. We begin by observing that the dihedral bar construction takes values in $O(2)$-spectra.

**Lemma 4.1.** There is a functor

$$|B^\text{di}_*(-)|: \text{Assoc}_{\sigma}(\text{Sp}^{D_2}) \to \text{Sp}^{O(2)}.$$ 

**Proof.** By Theorem 2.5 the realization of a dihedral space has an $O(2)$-action. Since geometric realization in orthogonal spectra is computed level-wise, we know $|B^\text{di}_*(A)|$ has an $O(2)$-action and this orthogonal $O(2)$-spectrum is indexed on $\mathcal{V}$. This construction is also clearly functorial.

We then define the norm as the dihedral bar construction.

**Definition 4.2.** Let $A$ be an $E_\sigma$-ring in $\text{Sp}^{D_2}$. We define

$$N^{O(2)}_{D_2} A = T^d_{\mathcal{V}}|B^\text{di}_*(A)|$$

to be the the norm from $D_2$ to $O(2)$. This defines a functor

$$N^{O(2)}_{D_2}: \text{Assoc}_{\sigma}(\text{Sp}^{D_2}) \to \text{Sp}^{O(2)}.$$ 

**Remark 4.3.** In particular, we produce a functor

$$N^{O(2)}_{D_2}: \text{Comm}(\text{Sp}^{D_2}) \to \text{Comm}(\text{Sp}^{O(2)})$$

by restriction to the subcategory $\text{Comm}(\text{Sp}^{D_2}) \subset \text{Assoc}_{\sigma}(\text{Sp}^{D_2})$.

Note that the category of commutative monoids in $\text{Sp}^{D_2}$ is tensored over the category of $D_2$-sets. This follows because $\text{Sp}^{D_2}$ is tensored over $D_2$-sets and the forgetful functor $\text{Comm}(\text{Sp}^{D_2}) \to \text{Sp}^{D_2}$ creates all indexed limits and the category $\text{Comm}(\text{Sp}^{D_2})$ contains all $\text{Top}^{D_2}$-enriched equalizers, by a generalization of [35, Lem. 2.8].

We simply write $\otimes$ for this tensoring, viewed as a functor

$$- \otimes -: \text{Comm}(\text{Sp}^{D_2}) \times D_2\text{-Set} \to \text{Comm}(\text{Sp}^{D_2}).$$

Note that this naturally extends to a functor

$$- \otimes -: \text{Comm}(\text{Sp}^{D_2}) \times (D_2\text{-Set})^{\Delta^{op}} \to \text{Comm}(\text{Sp}^{D_2}).$$

Consider the minimal simplicial model

$$D_2 \rightarrow D_4 \rightarrow D_6 \rightarrow \ldots$$

for $O(2)$, where $D_{2(n+1)} = \text{Aut}_{\Sigma}(\{n\})$. Note that it is standard that $D_{2(\bullet+1)}$ is in fact a functor $\Sigma^{op} \to \text{Set}$ [32, Lemma 6.3.1]. From the discussion in Section 2.2, we know that $\text{sq}(D_{2(\bullet+1)})$ is equipped with the structure of a simplicial $D_2$-set.

**Definition 4.4.** We define $O(2)_\bullet = \text{sq}(D_{2(\bullet+1)})$ regarded as a simplicial $D_2$-set.
Definition 4.5. Let $A$ be an object in $\text{Comm}(\text{Sp}_V^{D_2})$. We may form the coequalizer of the diagram

$$A \otimes D_2 \otimes O(2) \xrightarrow{\text{id}_A \otimes \psi} \frac{N \otimes \text{id}_{O(2)} \otimes 1}{N \otimes \text{id}_{O(2)}},$$

where

$$N: A \otimes D_2 = N^{D_2} A \to A$$

is the $E_0N^{D_2}A$-algebra structure map and $\psi: D_2 \times O(2) \rightarrow O(2)$ is the left $D_2$-action on $O(2)$. We define the coequalizer in the category of simplicial objects in $\text{Comm}(\text{Sp}_V^{D_2})$ to be $A \otimes_{D_2} O(2)$.

and we consider the geometric realization $|A \otimes_{D_2} O(2)|$. This construction has an $O(2)$-action on the right so we may regard $|A \otimes_{D_2} O(2)|$ as an object in $\text{Sp}_V^{O(2)}$. We therefore make the following definition

$$A \otimes_{D_2} O(2) = T^d_{\mu} |A \otimes_{D_2} O(2)|.$$

4.2. Comparison of the norm and the Bökstedt model. We will now show that for an $E_\sigma$-ring $A$ the construction $N_{D_2}^{O(2)} A$ recovers the Bökstedt model for Real topological Hochschild homology. In [13, Thm. 2.23], the authors prove that for a flat $E_\sigma$-ring $A$ there is a zig-zag of stable equivalences of $D_2$-spectra

$$|B_{\text{sp}}(A)| \simeq \text{THR}(A),$$

where the righthand side is the Bökstedt model for THR. We extend this to a zig-zag of $R$-equivalences of $O(2)$-spectra

$$N_{D_2}^{O(2)}(A) \simeq \text{THR}(A).$$

Proposition 4.6 (Equivalence of norm model and Bökstedt model). Given a flat $E_\sigma$-ring in $\text{Sp}_V^{D_2}$, there is a natural zig-zag of $R$-equivalences

$$N_{D_2}^{O(2)} A \simeq \text{THR}(A)$$

of $O(2)$-orthogonal spectra. This is also a zig-zag of $F_{\text{Fin}}^{T}$-equivalences, where $F_{\text{Fin}}^{T}$ is the family of finite subgroups of $T \subset O(2)$.

Proof. We write $\Omega^2 := \Omega^{0_0+\ldots+i_0}$ when $\bar{i} \in T^{k+1}$. Let $\text{sh}^i X$ denote the shifted spectrum where $(\text{sh}^i X)_n = X_{i+n}$. Recall that there is a stable equivalence

$$t: A \to \mathop{\text{hocolim}}_{i \in T} \Omega^i \text{sh}^i A$$

and a canonical isomorphism

$$\text{can}: \bigwedge_{j \in k+1} \left( \mathop{\text{hocolim}}_{i \in T} \Omega^{i_j} \text{sh}^{i_j} A \right) \to \mathop{\text{hocolim}}_{i \in T^{1+k}} \Omega^i (\text{sh}^{i_0} A \wedge \ldots \wedge \text{sh}^{i_k} A)$$

produced by commuting smash products with loops [13, Thm. 2.23]. We write

$$\overline{id}: \Sigma^{\infty} (\prod A_i \wedge \ldots \wedge A_i) \to \text{sh}^{i_0} A \wedge \ldots \wedge \text{sh}^{i_k} A$$

for the map adjoint to the identity map

$$\text{id}: A_{i_0} \wedge \ldots \wedge A_{i_k} = (\text{sh}^{i_0} A)_0 \wedge \ldots \wedge (\text{sh}^{i_k} A)_0 = (\text{sh}^{i_0} A \wedge \ldots \wedge \text{sh}^{i_k} A)_0$$

and we consider the geometric realization $|A \otimes_{D_2} O(2)|$. This construction has an $O(2)$-action on the right so we may regard $|A \otimes_{D_2} O(2)|$ as an object in $\text{Sp}_V^{O(2)}$. We therefore make the following definition

$$A \otimes_{D_2} O(2) = T^d_{\mu} |A \otimes_{D_2} O(2)|.$$
and we write $\overline{\text{id}}_x := \text{hocolim}_{\mathcal{I}^{1+k}} \Omega^x(\text{id})$ for the induced map.

We consider the zigzag

$$A^{1+k} \rightarrow \text{hocolim}_{\mathcal{I}^{1+k}} \Omega^x(A \wedge \ldots \wedge A_{i_{k}}) \rightarrow \text{hocolim}_{\mathcal{I}^{1+k}} \Omega^x\Sigma^\infty(A_{i_{0}} \wedge \ldots \wedge A_{i_{k}}).$$

Note that the spectrum on the right is the $k$-simplicies of the B"okstedt model for THR($A$). To prove the theorem, it will suffice to show that this zig-zag is $\mu_{k+1}$-equivariant. The right-hand side has a $\mu_{k+1}$-action by the cyclic permutation action on $\mathcal{I}^{1+k}$ and the conjugation action on the loop space where $\mu_{k+1}$ acts on $\mathcal{I}$ and $A_{i_{0}} \wedge \ldots \wedge A_{i_{k}}$ by cyclic permutation. The left-hand-side has $\mu_{k+1}$-action induced by the cyclic permutation action on the set $1+k$.

The maps $t^{1+k}$ and can are both clearly $\mu_{k+1}$-equivariant. These $\mu_{k+1}$-actions are also compatible with the $D_2$-action so the composite map can $\sigma t^{1+k}$ is a $D_{2(k+1)}$-equivariant map. This composite is a stable equivalence on $D_2$-fixed points by [13, Thm. 2.23]. The map $\overline{\text{id}}_x$ is also clearly $\mu_{k+1}$-equivariant because the identity map (10) is $\mu_{k+1}$-equivariant and consequently its adjoint is as well. This action is also clearly compatible with the $D_2$-action, so it produces a $D_{2(k+1)}$-equivariant map and it is an equivalence on $D_2$-fixed points by [13, Thm. 2.23]. These $\mu_{k+1}$-equivariant maps are also compatible with the face and degeneracy maps and therefore this is a zig-zag of maps of $D_2$-diagrams $\Delta^{\text{op}} \rightarrow \text{Sp}$, which restricts to the zig-zag of maps of $D_2$-diagrams $\Delta^{\text{op}} \rightarrow \text{Sp}$ of [13, Thm. 2.23]. The maps in the zig-zag of $D_2$-diagrams are also $D_2$-equivalences on $k$-simplicies for all $k$ by [13, Thm. 2.23]. By Corollary 2.14, these maps are also maps of dihedral objects. By the flatness hypotheses, each of the objects in the zig-zag are Real cyclic spectra, which restrict to good Real simplicial spectra in the sense of [13, Definition 1.5] so we produce a zig-zag of $O(2)$-equivariant maps which are equivalences on $D_2$-fixed points by [13, Lemma 1.6]. Consequently, the zig-zag of $O(2)$-equivariant maps

$$N_{D_2}^{O(2)} A \simeq \text{THR}(A)$$

is a zig-zag of $\mathcal{R}$-equivalences as desired because all the groups in $\mathcal{R}$ are conjugate to $D_2$ in $O(2)$. Since this entire argument is natural, this is in fact a natural $\mathcal{R}$-equivalence.

Since these maps are exactly the ones used to prove an equivalence of genuine $p$-cyclotomic structures between $B^{\text{cyc}} A$ and THR($A$) on underlying $T$-spectra in [9], this is also an $\mathcal{F}_{\text{Fin}}^\mathcal{R}$-equivalence.

**Remark 4.7.** It is claimed in [11, Remark 3.6] and [12, p.8] that the identification of the dihedral bar construction and the B"okstedt model is an identification of Real $p$-cyclotomic spectra. Proposition 4.6 is the first step towards proving this claim using the theory of Real cyclic objects. We will not carry out a complete proof of this claim from our perspective since it is not the main thrust of this work.

**Corollary 4.8.** Given a stable equivalence of very well-pointed $E_\sigma$-rings $A \rightarrow A'$ in $D_2$-orthogonal spectra indexed on $\mathcal{V}$, there is an $\mathcal{R}$-equivalence

$$N_{D_2}^{O(2)} A \rightarrow N_{D_2}^{O(2)} A'$$

of $O(2)$-spectra.

**Proof.** The induced map $N_{D_2}^{O(2)} A \rightarrow N_{D_2}^{O(2)} A'$ is $O(2)$-equivariant by construction, so it suffices to check that after restricting to $D_2$-spectra it is a stable equivalence of $D_2$-spectra. Note that this is independent of the choice of subgroup in $\mathcal{R}$ of order 2 because all of the subgroups...
of order two in \( \mathcal{R} \) are conjugate. After restricting to \( D_2 \)-spectra, there is a zigzag of stable equivalences of \( D_2 \)-spectra
\[
{\tau}^*_{D_2} N_{D_2}^{O(2)} A \simeq {\tau}^*_{D_2} \THR(A) \xrightarrow{\simeq} {\tau}^*_{D_2} \THR(A') \simeq {\tau}^*_{D_2} N_{D_2}^{O(2)} A',
\]
by [13, Thm. 2.20] and this agrees with the restriction of the map induced by \( A \to A' \) by naturality of Proposition 4.6.

4.3. Comparison of the norm and the tensor. We now identify the norm of Definition 4.2 with the relative tensor of Definition 4.5 in the commutative case.

**Proposition 4.9.** Let \( A \) be an object in \( \Comm(\Sp^D_2) \) and assume that \( A \) is very well pointed. There is a natural map
\[
N_{D_2}^{O(2)} A \to A \otimes_{D_2} O(2),
\]
in \( \Comm(\Sp^O_{D_2}) \), which is a weak equivalence after forgetting to \( \Sp^D_2 \).

**Proof.** We first prove that there is an equivalence of Real cyclic objects in \( \Comm \Sp^D_2 \)
\[
\text{sq}(B^0(A)) \simeq A \otimes_{D_2} O(2).$
\]
We consider the \( k \) simplices on each side. On the left, the \( k \)-simplices, are \( A \wedge A^{2k+1} \) with \( D_4^{(k+1)} = (\omega, t)^{2(k+1)} = \omega^2 = t \omega t \omega = 1 \) action given by letting \( t \) cyclically permute the \( 2k + 2 \) copies of \( A \), and letting \( \tau \) act on \( k = \{1, \ldots, 2k + 1\} \) by \( \tau(\{i\}) = 2k + 1 - i + 1 \). The \( k \)-simplices on the right hand side are given by the coequalizer of the diagram
\[
A \otimes D_2 \otimes D_4^{(k+1)} \xrightarrow{id_A \otimes \psi} A \otimes D_4^{(k+1)}
\]
\[
\xrightarrow{N \otimes id_{D_4^{(k+1)}}} A \otimes D_4^{(k+1)}
\]
in the category of \( D_4^{(k+1)} \)-spectra. This is \( A \otimes_{D_2} D_4^{(k+1)} \) and therefore the result on \( k \)-simplices follows from the \( D_4^{(k+1)} \)-equivariant map
\[
A \otimes_{D_2} D_4^{(k+1)} \simeq A \wedge A^{2k+1},$
which is clearly an equivalence on underlying commutative \( D_2 \)-spectra. To see that this map is \( D_4^{(k+1)} \)-equivariant, note that \( D_4^{(k+1)}/D_2 \simeq \mu_{2k+2} \) as \( D_4^{(k+1)} \)-sets and the right-hand-side can also be considered as a tensoring with the \( D_4^{(k+1)} \)-set \( \mu_{2k+2} \). Since this map is \( D_4^{(k+1)} \)-equivariant it is compatible with the automorphisms in the dihedral category. It is also easy to check that this is compatible with the face and degeneracy maps. Since \( A \) is very well-pointed, both sides are good in the sense of [13, Definition 1.5], this level equivalence induces an equivalence on geometric realizations in the category of \( D_2 \)-spectra by [13, Lemma 1.6].

**Remark 4.10.** Note that in the \( \mathcal{F} \)-model structure on \( \Sp^G_W \) where \( G \) is a compact Lie group, \( \mathcal{F} \) is a family of subgroups, and \( W \) is a universe, the \( \mathcal{F} \)-equivalences can either be taken to be the maps \( X \to Y \) that induce isomorphism on homotopy groups \( \pi_\ast(X^H) \to \pi_\ast(Y^H) \) for all \( H \in \mathcal{F} \) or the maps that induce isomorphisms \( \pi_\ast(\Phi^H X) \to \pi_\ast(\Phi^H Y) \), for all \( H \in \mathcal{F} \), where \( \Phi^H \) denotes the \( H \)-geometric fixed points.

In order to show that the identification from Proposition 4.9 is in fact an \( \mathcal{R} \)-equivalence it suffices to check that the map is an equivalence on geometric fixed points. Since all nontrivial groups in \( \mathcal{R} \) are conjugate to \( D_2 \), to check that the map is an \( \mathcal{R} \)-equivalence it suffices to check the condition \( D_2 \)-geometric fixed points.
Corollary 4.11. Let $A$ be an object in $\text{Comm}(\mathcal{S}_\mathcal{D}_2)$ and assume that $A$ is very well pointed and flat. Then there is an $\mathcal{R}$-equivalence

$$N_{D_2}^{O(2)} A \simeq A \otimes_{D_2} O(2).$$

Proof. Since there is a zig-zag of stable equivalences of orthogonal spectra

$$\Phi_{D_2}(N_{D_2}^{O(2)} A) \simeq \Phi_{D_2}(\text{THR}(A))$$

by Proposition 4.6, we know by [13] that there is a zig-zag of stable equivalences of orthogonal spectra

$$\Phi_{D_2}(N_{D_2}^{O(2)} A) \simeq \Phi_{D_2}(A) \wedge_A^{L} \Phi_{D_2}(A)$$

where the right-hand-side is the derived smash product. Since $\Phi_{D_2}(-)$ sends homotopy colimits of $O(2)$-orthogonal spectra indexed on $\mathcal{U}$ to homotopy colimits of orthogonal spectra, $\Phi_{D_2}(A \otimes_{D_2} O(2)_*)$ is the homotopy coequalizer of

$$\Phi_{D_2}(A \otimes D_2 \otimes O(2)_*) \rightarrowtail \Phi_{D_2}(A \otimes O(2)_*)$$

which is level-wise equivalent to

$$\Phi_{D_2} N_{D_2}^{D_{2(n+1)}}(A) \simeq \Phi_{D_2} \left( \bigwedge_\gamma N_{D_2}^{D_{2(\gamma \gamma')}}(t^{*}_{D_2} \gamma \gamma' \gamma^{-1} c \gamma A) \right)$$

where $\gamma$ ranges over the representatives of double cosets $D_2 \backslash \gamma / D_2$ in the set of double cosets $D_2 \backslash D_{2(n+1)}/D_2$, which in turn is equivalent to

$$B_n(\Phi_{D_2}(A), t^*_{e}(A), \Phi_{D_2}(A)).$$

It is tedious, but routine, to check that this equivalence is compatible with the face and degeneracy maps. Since both source and target are Reedy cofibrant by our assumptions (cf. [13, Definition 1.5, Lemma 1.6]), this produces a stable equivalence

$$\Phi_{D_2}(A \otimes_{D_2} O(2)) \simeq \Phi_{D_2}(A) \wedge^{L}_{t^*_{e}(A)} \Phi_{D_2}(A)$$

of orthogonal spectra on geometric realizations. Since all the groups of order 2 in $\mathcal{R}$ are conjugate in $O(2)$, we have proven the claim.

4.4. The dihedral bar construction as a norm. We now show that our definition of the norm from $D_2$ to $O(2)$ satisfies one of the fundamental properties that one would expect of a norm: in the commutative case it is left adjoint to restriction.

Theorem 4.12 (The dihedral bar construction as a norm). The restriction

$$N_{D_2}^{O(2)} \colon \text{Comm}(\mathcal{S}_{\mathcal{D}_2}) \rightarrow \text{Comm}(\mathcal{S}_{\mathcal{U}^{(2), \mathcal{R}}})$$

of the norm functor $N_{D_2}^{O(2)}$ to genuine commutative $D_2$-ring spectra is left Quillen adjoint to the restriction functor $t^*_{D_2}$, where

$$\text{Comm}(\mathcal{S}_{\mathcal{D}_2}) \text{ and } \text{Comm}(\mathcal{S}_{\mathcal{U}^{(2), \mathcal{R}}})$$

are equipped with the All-model structure and the $\mathcal{R}$-model structure of Definition 3.20 respectively.

Using the Bousfield–Kan formula for homotopy colimits, we can write any homotopy colimit as the geometric realization of a simplicial spectrum and then use the fact that genuine geometric fixed points commute with sifted colimits.
Proof. By Corollary 4.11, there is a natural $\mathcal{R}$-equivalence
$$N_{D_2}^{O(2)}(A) \simeq A \otimes_{D_2} O(2),$$
of $O(2)$-orthogonal spectra. If $A \to A'$ is an $\mathcal{R}$-equivalence of $D_2$-spectra where both source and target are very well-pointed then there is an $\mathcal{R}$-equivalence
$$N_{D_2}^{O(2)}(A) \simeq N_{D_2}^{O(2)}(A')$$
of $O(2)$-spectra by Corollary 4.8. In particular, if $A$ and $A'$ are cofibrant in the positive complete stable model structure on $\text{Assoc}_{\mathcal{R}}(\text{Sp}_{\mathcal{R}}^{D_2})$, then they are in particular very well-pointed. This shows that both the functors $N_{D_2}^{O(2)}(-)$ and $(-) \otimes_{D_2} O(2)$ induce well-defined functors between the homotopy categories
$$N_{D_2}^{O(2)} : \text{Ho}(\text{Comm}(\text{Sp}_{\mathcal{R}}^{D_2})) \to \text{Ho}\left(\text{Comm}(\text{Sp}^{O(2),\mathcal{R}}_{\mathcal{R}})\right)$$
and
$$- \otimes_{D_2} O(2) : \text{Ho}(\text{Comm}(\text{Sp}_{\mathcal{R}}^{D_2})) \to \text{Ho}\left(\text{Comm}(\text{Sp}^{O(2),\mathcal{R}}_{\mathcal{R}})\right)$$
and they are naturally isomorphic on the homotopy categories. It is clear that $- \otimes_{D_2} O(2)$ is left adjoint to the restriction functor
$$t^*_{D_2} : \text{Ho}\left(\text{Comm}(\text{Sp}^{O(2),\mathcal{R}}_{\mathcal{R}})\right) \to \text{Ho}\left(\text{Comm}(\text{Sp}_{\mathcal{R}}^{D_2})\right).$$
Moreover, the restriction functor sends cofibrations and weak equivalences to cofibrations and weak equivalences by definition of the stable $\mathcal{R}$-equivalences and the positive stable $\mathcal{R}$-cofibrations. Consequently it also preserves all fibrations and acyclic fibrations. \(\square\)

5. A MULTIPlicative DOUBLE COSET FORMULA

Classically, the multiplicative double coset formula for finite groups gives an explicit formula for the restriction to $K$ of the norm from $H$ to $G$ where $H$ and $K$ are subgroups of $G$. For compact Lie groups, no such multiplicative double coset formula is known in general. In this section, we present a multiplicative double coset formula for the restriction to $D_{2m}$ of the norm from $D_2$ to $O(2)$.

Conventions 5.1. When the integer $m$ is understood from context, let
$$\zeta = \zeta_{2m} = e^{2i\pi/2m} \in \mathbb{T} \subset O(2).$$
We consider the element $\zeta$ as a lift of the element $-1$ along a chosen homeomorphism
$$D_{2m} \setminus O(2)/D_2 \cong \mu_m \setminus \mathbb{T} \cong \mathbb{T}.$$
We make this choice of homeomorphism simply so that the formula for $\zeta$ can be chosen consistently for all $m$ independent of whether $m$ is odd or even. We observe that $\zeta D_2 \zeta^{-1} = \zeta_m \tau$.

We fix total orders on the $D_{2m}$-sets $D_{2m}/e$, $D_{2m}/D_2$, and $D_{2m}/\zeta D_2 \zeta^{-1}$. Let
$$D_{2m}/e = \{1 \leq \zeta_m \tau \leq \zeta_m \leq \zeta_m^2 \tau \leq \zeta_m^2 \leq \cdots \leq \zeta_m^{m-1} \leq \tau\},$$
$$D_{2m}/D_2 = \{D_2 \leq \zeta_m D_2 \leq \cdots \leq \zeta_m^{m-1} D_2\},$$
$$D_{2m}/\zeta D_2 \zeta^{-1} = \{\zeta D_2 \zeta^{-1} \leq \zeta_m \cdot \zeta D_2 \zeta^{-1} \leq \cdots \leq \zeta_m^{m-1} \cdot \zeta D_2 \zeta^{-1}\}.$$These choices of total orderings on the $D_{2m}$-sets $D_{2m}/e$, $D_{2m}/D_2$, and $D_{2m}/\zeta D_2 \zeta^{-1}$ also fix group homomorphisms
$$\lambda_\zeta : D_{2m} \to \Sigma_{2m},$$
\[ \lambda_{D_2} : D_{2m} \rightarrow \Sigma_m \circlearrowleft D_2 \]
\[ \lambda_{\zeta D_2 \zeta^{-1}} : D_{2m} \rightarrow \Sigma_m \circlearrowleft \zeta D_2 \zeta^{-1}. \]

We denote the associated norms by \( N_{D_2}^{D_{2m}}, N_{D_2}^{D_{2m}} \) and \( N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \), respectively. The choice of ordering does not matter for our norm functors up to canonical natural isomorphism, but remembering the choice of ordering clarifies our constructions later.

**Remark 5.2.** Any finite subset \( F \) of \( \mathbb{T} \subset \mathbb{C} \) can be equipped with a total order by considering \( 1 \leq e^{2i\pi \theta} \leq e^{2i\pi \theta'} \) for all \( 0 \leq \theta \leq \theta' < 1 \). In particular, the subset of \( m \)-th roots of unity \( \mu_m \) in \( \mathbb{T} \) can be equipped with a total order in this way.

**Lemma 5.3.** There is an isomorphism of totally ordered \( D_{2m} \)-sets
\[ f_{m,k} : \mu_{2m(k+1)} \cong D_{2m}/D_2 \cup D_{2m}^{2k} \cup D_{2m}/\zeta D_2 \zeta^{-1} \]
where \( \zeta = \zeta_{2m} \), the \( D_{2m} \)-sets on the right have the total orders from Convention 5.1, and \( \mu_{2m(k+1)} \) is equipped with a total order by Remark 5.2.

**Proof.** Without the total ordering this isomorphism is clear. We chose the total ordering on the right hand side in Convention 5.1 so that this lemma would be true. \( \square \)

**Remark 5.4.** We will also write \( f_{m,k} \) for the underlying map of totally ordered sets from Lemma 5.3 after forgetting the \( D_{2m} \)-set structure.

Given an \( E_\sigma \)-ring \( R \), then \( \iota_e R \) is an \( E_1 \)-ring and \( R \) is a \( N_{D_2}^{D_{2m}} \iota_e^* R \)-bimodule with right action
\[ \overline{\psi}_R : R \wedge N_{D_2}^{D_{2m}} \iota_e^* R \rightarrow R \]
and left action
\[ \overline{\psi}_L : \iota_e^* R \wedge R \rightarrow R. \]
We also note that there is an equivalence of categories
\[ c_\zeta : \mathbf{Sp}^{D_2} \rightarrow \mathbf{Sp}^{\zeta D_2 \zeta^{-1}} \]
which is symmetric monoidal and therefore sends \( E_\sigma \)-rings in \( \mathbf{Sp}^{D_2} \) to \( E_\sigma \)-rings in \( \mathbf{Sp}^{\zeta D_2 \zeta^{-1}} \).

In particular, \( c_\zeta R \) is a left \( N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \iota_e^* R \)-module with action map
\[ c_\zeta(\overline{\psi}_L) : N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \iota_e^* R \wedge c_\zeta R \rightarrow c_\zeta R. \]

**Definition 5.5.** Let \( R \) be an \( E_\sigma \)-ring. We define a right \( N_{D_2}^{D_{2m}} \iota_e^* R \)-module structure on \( N_{D_2}^{D_{2m}} R \) as the composite
\[ \psi_R : N_{D_2}^{D_{2m}} R \wedge N_{D_2}^{D_{2m}} \iota_e^* R \rightarrow N_{D_2}^{D_{2m}} (R \wedge N_{D_2}^{D_{2m}} \iota_e^* R) \rightarrow N_{D_2}^{D_{2m}} R. \]

We define a left \( N_{D_2}^{D_{2m}} \iota_e^* R \)-module structure on \( N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \)
\[ \psi_L : N_{D_2}^{D_{2m}} \iota_e^* R \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \rightarrow N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \]
as the composite of the isomorphism
\[ N_{D_2}^{D_{2m}} \iota_e^* R \wedge N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R \rightarrow N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \left( N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \iota_e^* R \wedge c_\zeta R \right) \]
with the map
\[ N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} \iota_e^* R \wedge c_\zeta R \rightarrow N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta R. \]
Example 5.6. When \( m = 2 \), we note that \( \zeta_4 D_2 \zeta_4^{-1} \) in \( D_8 \) can be identified with the diagonal subgroup \( \triangle \) of \( D_4 \). In this case, \( \triangle \) and \( D_2 \) are conjugate in \( D_8 \) even though they are not conjugate in \( D_4 \). We still define a left \( N^D_{\triangle} t^*_e R \)-module structure on \( N^D_{\triangle} c_\zeta R \) by composing the map
\[
N_e^D t^*_e R \wedge N^D_{\triangle} c_\zeta R \xrightarrow{\Delta} N^D_{\triangle} \left( N_e^D t^*_e R \wedge c_\zeta R \right)
\]
with the map
\[
N^D_{\triangle} \left( N_e^D t^*_e R \wedge c_\zeta R \right) \xrightarrow{N^D_{\triangle} (c_\zeta (\tilde{\psi}_L))} N^D_{\triangle} c_\zeta R.
\]

Remark 5.7. Note that there is an isomorphism of simplicial \( D_{2m} \)-sets
\[
\mu_{2m}(\bullet) \to D_{2m} / D_2 \sqcup D_{2m}^{\op} \sqcup D_{2m} / \zeta D_2 \zeta^{-1}
\]
which is given by the isomorphism \( f_{m,k} \) of totally ordered \( D_{2m} \)-sets of Lemma 5.3 on \( k \)-simplices. The simplicial maps on the left are given by the simplicial maps in the simplicial set \( \text{sd}_{D_{2m}} S^1_{\bullet} \) where \( S^1_{\bullet} \) is the minimal model of \( S^1 \) as a simplicial set. On the right, the face maps
\[
D_{2m} / D_2 \sqcup D_{2m}^{\op} \sqcup D_{2m} / \zeta D_2 \zeta^{-1} \to D_{2m} / D_2 \sqcup D_{2m}^{\op} \sqcup D_{2m} / \zeta D_2 \zeta^{-1}
\]
are given by the canonical quotient composed with the fold map
\[
D_{2m} / D_2 \sqcup D_{2m} \xrightarrow{D_{2m} / D_2^{\upq}} D_{2m} / D_2 \sqcup D_{2m} / D_2 \xrightarrow{\updownarrow} D_{2m} / D_2
\]
for the first face map, the fold map
\[
D_{2m} \sqcup D_{2m} \xrightarrow{\updownarrow} D_{2m}
\]
for the middle maps, and for the last face map it is given by the composite
\[
D_{2m} \sqcup D_{2m} / \zeta D_2 \zeta^{-1} \xrightarrow{\updownarrow} D_{2m} / \zeta D_2 \zeta^{-1} \sqcup D_{2m} / \zeta D_2 \zeta^{-1} \xrightarrow{\updownarrow} D_{2m} / \zeta D_2 \zeta^{-1}.
\]
The degeneracy maps are given by the canonical inclusions.

We use the isomorphism of simplicial \( D_{2m} \)-sets above to keep track of the smash factors in the proof of the following result.

Proposition 5.8. Suppose \( R \) is an \( E_\sigma \)-ring in \( D_2 \)-spectra indexed on the complete universe \( \mathcal{V} = \iota^*_D \mathcal{U} \) where \( \mathcal{U} \) is a fixed complete \( O(2) \)-universe. More generally, let \( \mathcal{V}_n = \iota^*_D \mathcal{U} \) for \( n \geq 1 \). There is an isomorphism of simplicial \( D_{2m} \)-spectra
\[
\mathcal{V}^m \left( \text{sd}_{D_{2m}} B^D_{\mathcal{V}}(R) \right) \cong B_{\bullet} \left( N^D_{D_2} R, N^D_{D_{2m}} t^*_e R, N^D_{D_{2m}} c_\zeta c_\zeta R \right)
\]
where we write \( \widetilde{\mathcal{V}} \) for the \( D_2 \)-universe \( \mathcal{V} \) regarded as a \( D_{2m} \)-universe via inflation along the canonical quotient \( D_{2m} \to D_2 \).

Proof. When \( R \) is an \( E_\sigma \)-ring, we explicitly define the simplicial map
\[
\mathcal{V}^m \left( \text{sd}_{D_{2m}} B^D_{\mathcal{V}}(R) \right) \to B_{\bullet} \left( N^D_{D_2} R, N^D_{D_{2m}} t^*_e R, N^D_{D_{2m}} c_\zeta c_\zeta R \right)
\]
on \( k \)-simplices. There is an isomorphism given by composition of two maps. The first map
\[
f_{m,k} : R^{\mu_{2m(k+1)}} \to R^{\wedge m} \wedge \left( R^{\wedge m} \wedge (R^{\op})^{\wedge m} \right)^{\wedge k} \wedge (R^{\op})^{\wedge m}
\]
is the isomorphism induced by $f_{m,k}$ of Remark 5.4 regarded simply as a map of totally ordered sets. The second map is

$$R^\wedge m \wedge ( (R \wedge R^{op})^\wedge m )^\wedge k \wedge (R^{op})^\wedge m$$

$$\downarrow 1^\wedge m \wedge (1^{\wedge r})^\wedge mk \wedge \tau^\wedge m$$

$$N_{D_2}^{D_{2m}} R \wedge (N_e^{D_{2m}} R)^\wedge k \wedge N_{\zeta D_2\zeta^{-1}} c \zeta R,$$

where we use the indexing conventions of Conventions 5.1.

This is a $D_{2m}$-equivariant isomorphism on $k$-simplices essentially because it comes from the isomorphism of ordered $D_{2m}$-sets of Lemma 5.3. It follows that the map is compatible with the simplicial structure maps by comparing the structure maps in the isomorphism of simplicial $D_{2m}$-sets in Remark 5.7 to the structure maps on either side.

Consequently, for a flat $E_\sigma$-ring, we have the following multiplicative double coset formula.

**Theorem 5.9** (Multiplicative Double Coset Formula). When $R$ is a flat $E_\sigma$-ring and $m$ is a positive integer, there is a stable equivalence of $D_{2m}$-spectra

$$i_{{D_2}^{D_{2m}}}^{*} N_{D_2}^{O(2)} R \simeq N_{D_2}^{D_{2m}} R \wedge_{N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R}^{L} N_{\zeta D_2\zeta^{-1}} c \zeta R.$$

**Proof.** By Proposition 5.8, we know there is an equivalence

$$\mathcal{D}_v^v (sd_{D_2m} B_{\Sigma}^\wedge (R)) \simeq B_{\bullet} (N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R, N_{\zeta D_2\zeta^{-1}} c \zeta R).$$

When $R$ is a flat $E_\sigma$-algebra in $D_2$-spectra, then $N_{D_2}^{D_{2m}} R$ is a flat $N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R$-module by [41, Theorem 3.4.22-23] and [6] and therefore we may identify

$$N_{D_2}^{D_{2m}} R \wedge_{N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R}^{L} N_{\zeta D_2\zeta^{-1}} c \zeta R$$

with the realization of the bar resolution of $N_{D_2}^{D_{2m}} R$ by free $N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R$-modules then smashed with the right $N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R$-module $N_{\zeta D_2\zeta^{-1}} c \zeta R$. This is exactly the realization of the simplicial spectrum

$$B_{\bullet} (N_{D_2}^{D_{2m}} R, N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R, N_{\zeta D_2\zeta^{-1}} c \zeta R).$$

This generalizes a result of Dotto, Moi, Patchkoria, and Reeh, in [13], where they prove that for $R$ a flat $E_\sigma$-ring there is a stable equivalence of $D_2$-spectra

$$i_{{D_2}^{D_{2m}}}^{*} \text{THR}(R) \simeq R \wedge_{N_e^{D_{2m}} i_{{D_2}^{D_{2m}}}^{*} R}^{L} R.$$

6. **Real Hochschild homology**

The remaining goal of this work is to develop an algebraic analogue of Real topological Hochschild homology and use it to define a notion of Witt vectors for rings with antiautinvolution. Ordinary topological Hochschild homology (THH) is a topological analogue of the classical algebraic theory of Hochschild homology. For a ring $R$, there is a linearization map relating the topological and algebraic theories:

$$\pi_k (\text{THH}(HR)) \rightarrow \text{HH}_k (R).$$

Here $HR$ denotes the Eilenberg–MacLane spectrum of $R$. Further, this linearization map is an isomorphism in degree 0.
For generalizations of topological Hochschild homology, it is natural to ask, then, for their algebraic analogues. In [1], for \( H \subset T \) the authors define the \( H \)-twisted topological Hochschild homology of an \( H \)-ring spectrum \( R \):

\[
\text{THH}_H(R) = N^T_H R.
\]

In [3] an algebraic analogue of this topological theory is constructed. In particular, the authors define a theory of Hochschild homology for Green functors, \( \text{HH}^G_H(R) \), for \( H \subset G \subset T \), and \( R \) an \( H \)-Green functor. They prove that for an \( H \)-ring spectrum \( R \) there is a linearization map:

\[
\pi^G_k \text{THH}_H(R) \to \text{HH}^G_H(\pi^H_0 R)_k,
\]

which is an isomorphism in degree 0.

In this section we address the question: What is the algebraic analogue of \( \text{THR} \)? We do this by defining a theory of Real Hochschild homology for discrete \( E_\sigma \)-rings. We then show how this leads to a theory of Witt vectors for rings with anti-involution.

To begin, we recall some basic terminology in the theory of Mackey functors, we define norms in the category of Mackey functors, and \( E_\sigma \)-algebras in \( D_2 \)-Mackey functors, which we call discrete \( E_\sigma \)-rings.

### 6.1. Representable Mackey functors

For a more thorough review of the theory of Mackey functors, we refer the reader to [3, §2]. Here we simply recall the constructions and notation we use in the present paper. Let \( G \) be a finite group. Write \( A \) for the Burnside category of \( G \). For a finite \( G \)-set \( X \),

\[
\underline{A}^G_X := \mathcal{A}(X, -)
\]
denotes the representable \( G \)-Mackey functor represented by \( X \). This construction forms a co-Mackey functor object in Mackey functors, by viewing it also as a functor in the variable \( X \), so in particular

\[
\underline{A}^G_{X \sqcup Y} = \mathcal{A}(X \sqcup Y, -) = \mathcal{A}(X, -) \oplus \mathcal{A}(Y, -) = \underline{A}^G_X \oplus \underline{A}^G_Y.
\]

We write \( \underline{A}^G \) for the Burnside Mackey functor associated to \( G \), which can be identified with \( \underline{A}^G \) where \( * = G/G \), so there is no clash in notation. This Mackey functor has the property that \( \underline{A}^G(G/H) = A(H) \) where \( A(H) \) denotes the Burnside ring for a finite group \( H \). Recall that as an abelian group \( A(H) \) is free with basis \( \{ [H/K] \} \) where \( K \) ranges over all conjugacy classes of subgroups \( K < H \). When \( K = H \) we simply write \( 1 = [H/H] \) and when \( K \) is the trivial group we simply write \( [H] = [H/\{ e \}] \). The transfer and restriction maps in \( \underline{A}^G \) are given by induction and restriction maps on finite sets.

**Example 6.1.** The Burnside Mackey functor \( \underline{A}^{D_2} \) can be described by the following diagram.

\[
\begin{array}{cccc}
1 & [D_2] & M^{D_2}(D_2/D_2) = \mathbb{Z}\langle [D_2] \rangle & [D_2] \\
\downarrow & & \downarrow & \\
1 & 2 & M^{D_2}(D_2/*) = \mathbb{Z}\langle 1 \rangle & 1
\end{array}
\]

Given a finite group \( G \), a subgroup \( H \subset G \), and a \( H \)-set \( X \) we write \( \text{Map}^H(G, X) \) for the \( G \)-set of \( H \)-equivariant maps from \( G \) to \( X \), which is a functor in the variable \( X \) known as coinduction.
6.2. Norms for Mackey functors. In this section, we recall briefly the definition and properties of the norm in \( G \)-Mackey functors for a finite group \( G \). Let \( \mathcal{SP}^G_U \) denote the category of \( G \)-spectra indexed on a complete universe \( U \) and let \( \text{Mack}_G \) denote the category of \( G \)-Mackey functors. Recall that these categories are both symmetric monoidal and the symmetric monoidal structures are compatible in the following sense.

**Proposition 6.2.** [30] For \( X \) and \( Y \) cofibrant, (-1)-connected orthogonal \( G \)-spectra, there is a natural isomorphism
\[
\pi_0(X \wedge Y) \cong \pi_0 X \square \pi_0 Y.
\]

A \( G \)-Mackey functor \( M \) has an associated Eilenberg Mac Lane \( G \)-spectrum, \( HM \). The defining property of this spectrum is that
\[
\pi^G_k(HM) = \begin{cases} M & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}
\]

It then follows from Proposition 6.2 above that the box product of Mackey functors has a homotopical description:
\[
M \square N \cong \pi_0(HM \wedge HN).
\]

The category \( \mathcal{SP}^G_U \) has an equivariant enrichment of the symmetric monoidal product, a \( G \)-symmetric monoidal category structure [23, 22]. Such a \( G \)-symmetric monoidal structure requires multiplicative norms for all subgroups \( H \subset G \). In \( \mathcal{SP}^G_U \) these are given by the Hill–Hopkins–Ravenel norm. The \( G \)-symmetric monoidal structure on \( \mathcal{SP}^G_U \) induces such a structure on \( \text{Mack}_G \) as well. In particular, one can define norms for \( G \)-Mackey functors.

**Definition 6.3** (cf. [22]). Given a finite group \( G \) with subgroup \( H \) and an \( H \)-Mackey functor \( M \), the norm in Mackey functors is defined by
\[
N^G_H M = \pi^G_0(HM \wedge HN).
\]

From this definition, the following lemma is immediate.

**Lemma 6.4.** The norm in Mackey functors commutes with sifted colimits.

6.3. Discrete \( E_\sigma \)-rings. In Section 3.1, we discussed \( E_\sigma \)-rings in \( D_2 \)-spectra, which serve as the input for Real topological Hochschild homology. We now define their algebraic analogues, discrete \( E_\sigma \)-rings. These discrete \( E_\sigma \)-rings will be the input for our construction of Real Hochschild homology.

**Definition 6.5.** Let \( V \) be a finite dimensional representation of a finite group \( G \). We define an \( E_V \)-algebra in \( G \)-Mackey functors to be a \( P_V \)-algebra in \( G \)-Mackey functors, where \( P_V \) is the monad
\[
P_V(-) = \bigoplus_{n \geq 0} E^G_n \left( ((E_{V,n})^+ \wedge_{\Sigma_n} H(-)^{\wedge n}) \right)
\]
and \( HM \) is the Eilenberg–MacLane \( G \)-spectrum associated to the Mackey functor \( M \).

When \( V = \sigma \), this monad is particularly simple, since the spaces in the \( E_\sigma \)-operad are homotopy discrete.

**Proposition 6.6.** For \( D_2 \)-Mackey functors, the monad \( P_\sigma \) is given by
\[
P_\sigma(M) = T \left( N^{D_2}_{i^*_e M} \wedge (A \oplus M) \right),
\]
where \( T(-) \) is the free associative algebra functor.
Proof. Recall that we have an equivariant equivalence
\[ E_{\sigma,n} \cong (D_2 \times \Sigma_n)/\Gamma_n. \]
If \( n \) is even, then we have natural isomorphisms
\[ \pi_0\left( ((E_{\sigma,n})_+ \wedge \Sigma_n H \underline{M}^{\wedge n}) \right) \cong \pi_0\left( (N^{D_2} H \underline{M})^{\wedge n/2} \right) \cong (N^{D_2} \underline{M})^{\wedge n/2}. \]
If \( n \) is odd, then since fixed points contributes a box-factor of \( M \) itself:
\[ \pi_0\left( ((E_{\sigma,n})_+ \wedge \Sigma_n H \underline{M}^{\wedge n}) \right) \cong (N^{D_2} \underline{M})^{\wedge \lfloor n/2 \rfloor} \square M. \]
The result follows from grouping the terms according to the number of box-factors involving the norm. \( \square \)

Remark 6.7. If we compose with the forgetful functor, then we actually recover the tensor algebra, but in a slightly curious presentation. This is the tensor algebra as
\[ T(V \otimes V) \otimes (\mathbb{Z} \oplus V). \]
We view the two tensor factors as left and right multiplication, and use this to “unfold” the various tensor powers. Now the multiplication is the usual concatenation product, but we see that it fails to be equivariant (which reflects the anti-automorphism property).

Definition 6.8. By a discrete \( E_{\sigma}-\text{ring} \), we mean an algebra over the monad \( P_\sigma \) in the category of \( D_2\)-Mackey functors.

We can further unpack this structure to describe the monoids.

Lemma 6.9. A discrete \( E_{\sigma}\)-ring is the following data:

1. A \( D_2\)-Mackey functor \( M \), together with an associative product on \( M(D_2/e) \) for which the Weyl action is an anti-homomorphism,
2. a \( N_{\sigma}^{D_2}M \)-bimodule structure on \( M \) that restricts to the standard action of \( M(D_2/e) \otimes M(D_2/e)^{\text{op}} \) on \( M(D_2/e)^{\text{op}} \).
3. an element \( 1 \in M(D_2/D_2) \) that restricts to the element \( 1 \in M(D_2/e) \).

Remark 6.10. These conditions are almost identical to those of a Hermitian Mackey functor in the sense of [14]: the only difference is that a discrete \( E_{\sigma}\)-ring includes the additional assumption that there is a fixed unit element \( 1 \in M(D_2/D_2) \), or in other words, an \( E_0\)-\( N_{\sigma}^{D_2}M(D_2/e)\)-ring structure on \( M \).

Example 6.11. Given an \( E_{\sigma}\)-ring \( R \) it is clear that \( \pi_0^{D_2}(R) \) is a discrete \( E_{\sigma}\)-ring. In fact, this does not depend on our choice of \( E_{\sigma}\)-operad. By Example 3.12, we also conclude that if \( R \) is a commutative monoid in \( S_2^{D_2} \), then \( \pi_0^{D_2}R \) is a discrete \( E_{\sigma}\)-ring.

Example 6.12 (Rings with anti-involution). Let \( R \) be a discrete ring with anti-involution \( \tau: R^{\text{op}} \to R \), regarded as the action of the generator of \( D_2 \). Then there is an associated Mackey functor \( M \) with \( M(D_2/e) = R \) and \( M(D_2/D_2) = R^{C_2} \). The restriction map \( \text{res}_{e}^{D_2} \) is the inclusion of fixed points, the transfer \( \text{tr}_{e}^{D_2} \) is the map \( 1 + \tau \), and the Weyl group action of \( D_2 \) on \( M(D_2/e) = R \) is defined on the generator of \( D_2 \) by the anti-involution \( \tau: R^{\text{op}} \to R \). Since \( R^{\text{op}} \to R \) is a ring map, there is an element \( 1 \in R^{C_2} \) that restricts to the multiplicative unit in \( 1 \in R \). This specifies a discrete \( E_{\sigma}\)-ring structure on the Mackey functor \( M \).
6.4. Real Hochschild homology of discrete $E_\sigma$-rings. In this section, we define the Real Hochschild homology $\text{HR}^{D_{2m}}(M)$ of a discrete $E_\sigma$-ring $M$, which takes values in graded $D_{2m}$-Mackey functors. To give our construction, we first need to specify a right $N^D_{2m} \iota_e^* M$-action on $N^D_{2m} M$, and a left $N^D_{2m} \iota_e^* M$-action on $N^D_{2m} \iota_c^* M$. Here $c_\iota$ is the symmetric monoidal equivalence of categories

$$c_\iota: \text{Mack}D_2 \to \text{Mack}\iota\text{D}_{2,\iota}^{-1}.$$ For $M$ a discrete $E_\sigma$-ring, $\iota_e^* M$ is an (associative unital) ring so $N^D_{2m} \iota_e^* M$ is an associative Green functor. Recall from Lemma 6.9 that there is a left action

$$\overline{\psi}_L: N^D_{2m} \iota_e^* M \Box M \to M$$

and a right action

$$\overline{\psi}_R: M \Box N^D_{2m} \iota_e^* M \to M.$$ **Definition 6.13.** We define a right $N^D_{2m} \iota_e^* M$-module structure on $N^D_{2m} M$ as the composite

$$\psi_R: N^D_{2m} M \Box N^D_{2m} \iota_e^* M \xrightarrow{\iota_e} N^D_{2m} \iota_e^* M \Box N^D_{2m} M \xrightarrow{\overline{\psi}_R} N^D_{2m} M.$$ We define a left $N^D_{2m} \iota_e^* M$-module structure $\psi_L$ on $N^D_{2m} \iota_c^* M$ as the composite of the map

$$N^D_{2m} \iota_e^* M \Box N^D_{2m} \iota_c^* M \xrightarrow{\iota_c} N^D_{2m} \iota_c^* M \Box N^D_{2m} \iota_e^* M \xrightarrow{\overline{\psi}_L} N^D_{2m} \iota_c^* M$$

with the map

$$N^D_{2m} \iota_e^* M \Box N^D_{2m} \iota_c^* M \xrightarrow{\iota_e \Box c_\iota} N^D_{2m} \iota_c^* M$$

where $c_\iota(\overline{\psi}_L)$ is the left action of $N^D_{2m} \iota_c^* M$ on $c_\iota M$ coming from the fact that $c_\iota$ is symmetric monoidal and therefore sends $E_\sigma$-rings in Mack$_D$ to $E_\sigma$-rings in Mack$_{\iota\text{D}_{2,\iota}}$.

**Definition 6.14.** Given Mackey functors $R$, $M$, and $N$ where $R$ is an associative Green functor, $N$ is a right $R$-module and $M$ is a left $R$-module, we define the two-sided bar construction

$$B_*(M, R, N)$$

with $k$-simplices

$$B_k(M, R, N) = M \Box R^k \Box M$$

and the usual face and degeneracy maps.

**Definition 6.15.** The Real $D_{2m}$-Hochschild homology of a discrete $E_\sigma$-ring $M$ is defined to be the graded $D_{2m}$-Mackey functor

$$\text{HR}_{D_{2m}}(M) = H_*(\text{HR}_{D_{2m}}(M)).$$

where

$$\text{HR}_{D_{2m}}(M) = B_*(N^D_{2m} M, N^D_{2m} \iota_e^* M, N^D_{2m} \iota_c^* M).$$

Recall that the homology of a simplicial Mackey functor is defined to be the homology of the associated normalized dg Mackey functor, as in [3].

**Lemma 6.16.** If $M$ is a $D_2$-Tambara functor, then $\text{HR}_{0}^{D_{2m}}(M)$ is a $D_{2m}$-Tambara functor.

**Proof.** This follows since reflexive coequalizers in the category of Tambara functors are computed as the reflexive coequalizer of the underlying Mackey functors (cf. [42]).
**Proposition 6.17.** There is an isomorphism of $D_{2m}$-Mackey functors

$$ \text{HHR}_0^{D_{2m}}(M) \cong N_{D_{2m}, M}^{D_{2m}} \square N_{\zeta D_{2m} - 1}^{D_{2m} c M} $$

*Proof.* Both sides are given by the coequalizer

$$ N_{D_{2m}, M}^{D_{2m}} \square N_{\zeta D_{2m} - 1}^{D_{2m} c M} \xrightarrow{\text{id} \otimes \psi} N_{D_{2m}, M}^{D_{2m}} \square N_{\zeta D_{2m} - 1}^{D_{2m} c M} $$

and are hence isomorphic. $\square$

**Definition 6.18.** We say that a Mackey functor $M$ is **flat** if the derived functors of the functor $M \square -$ vanish.

**Proposition 6.19.** For any discrete $E_\sigma$-ring $M$ that can be written as a filtered colimit of representable $D_2$-Mackey functors, there is an isomorphism of $D_{2m}$-Mackey functors

$$ \text{HHR}_0^{D_{2m}}(M) = \text{Tor}^M(N_{D_{2m}, M}^{D_{2m}}, N_{\zeta D_{2m} - 1}^{D_{2m} c M}) $$

*Proof.* Norms send representable $D_2$-Mackey functors to representable $D_{2m}$-Mackey functors by [3, Proposition 3.7]. Norms also commute with sifted colimits by Lemma 6.4. The result then follows because filtered colimits of of representable $D_{2m}$-Mackey functors are flat. $\square$

### 6.5. Comparison between Real Hochschild homology and THR

We will now show our definition of Real Hochschild homology serves as the algebraic analogue of Real topological Hochschild homology. In particular, the two theories are related by a linearization map which is an isomorphism in degree 0.

**Theorem 6.20.** For any $(-1)$-connected $E_\sigma$-ring $A$, we have a natural homomorphism

$$ \pi_{D_{2m}}^\ast \text{THR}(A) \to \text{HHR}_k^{D_{2m}}(\pi_{D_{2m}}^\ast A), $$

which is an isomorphism when $k = 0$.

*Proof.* From [16, X.2.9], recall that for a simplicial spectrum $X_\bullet$, there is a spectral sequence

$$ E^2_{p, q} = H_p(\pi_q(X_\bullet)) \Longrightarrow \pi_{p+q}(X_\bullet), $$

from filtering by the skeleton. The same proof yields an equivariant version of this spectral sequence.

By Proposition 5.8, there is a weak equivalence

$$ \iota_{D_{2m}}^\ast \text{THR}(A) \cong [B_\bullet(N_{D_{2m}}^A, N_{\zeta D_{2m} - 1}^{D_{2m} c M} A)], $$

so the spectral sequence in this case will be of the form:

$$ E^2_{p, q} = H_p(\pi_q^{D_{2m}}(N_{D_{2m}}^A \wedge N_{\zeta D_{2m} - 1}^{D_{2m} c M} A)) \to \pi_{p+q}(\text{THR}(A)), $$

The edge homomorphism of this spectral sequence is a map

$$ \pi_{D_{2m}}^\ast (\text{THR}(A)) \to H_p(\pi_0^{D_{2m}}(N_{D_{2m}}^A \wedge N_{\zeta D_{2m} - 1}^{D_{2m} c M} A)). $$

We can identify the right hand side as

$$ H_p(\pi_0^{D_{2m}} N_{D_{2m}}^A \square (\pi_0^{D_{2m}} N_{\zeta D_{2m} - 1}^{D_{2m} c M} A)^\square \square (\pi_0^{D_{2m}} N_{\zeta D_{2m} - 1}^{D_{2m} c M} A)), $$

(11)
using the collapse of the Künneth spectral sequence [30] in degree 0. By Definition 6.3, there are isomorphisms of $D_{2m}$-Mackey functors

$$\varpi_0^{D_{2m}}(N_e^{D_{2m}} i_e^* A) \cong N_e^{D_{2m}} i_e^* \varpi_0^D(A),$$

$$\varpi_0^{D_{2m}}(N_e^{D_{2m}} \zeta \zeta_1 A) \cong N_e^{D_{2m}} \zeta \zeta_1 \varpi_0^D(A),$$

$$\varpi_0^{D_{2m}}(N_e^{D_{2m}} A) \cong N_e^{D_{2m}} \varpi_0^D(A).$$

We can therefore identify (11) as

$$H_p\left(N_e^{D_{2m}} D_2 \varpi_0^D(A) \boxtimes (N_e^{D_{2m}} i_e^* \varpi_0^D(A)) \boxtimes N_e^{D_{2m}} \zeta \zeta_1 \varpi_0^D(A) \right),$$

and hence the edge homomorphism gives a linearization map

$$\pi_0^{D_{2m}}(\text{THR}(A)) \to \text{HR}_p^{D_{2m}}(\varpi_0^D(A)).$$

To prove the claim that this map is an isomorphism in degree zero, we note that only contribution to $t + s = 0$ is

$$E_0^{2,0} \cong N_e^{D_{2m}} D_2 \varpi_0^D(A) \boxtimes N_e^{D_{2m}} \zeta \zeta_1 \varpi_0^D(A) \cong N_e^{D_{2m}} \zeta \zeta_1 \varpi_0^D(A)$$

concentrated in degree $s = t = 0$. By Proposition 6.17 this is $\text{HR}_p^{D_{2m}}(\varpi_0^D(A))$. Since this is a first quadrant spectral sequence, we observe that

$$E_0^{2,0} \cong E_0^{\infty} \cong \pi_0^{D_{2m}} \text{THR}(A).$$

\[\blacksquare\]

**Remark 6.21.** Forthcoming work of Chloe Lewis constructs a Bökstedt spectral sequence for Real topological Hochschild homology, which computes the equivariant homology of THR($A$). The $E_2$-term of this spectral sequence is described by Real Hochschild homology, further justifying that $\text{HR}$ is the algebraic analogue of THR.

### 7. Witt vectors of rings with anti-involution

Hesselholt–Madsen [20] proved that for a commutative ring $A$, there is an isomorphism

$$\pi_0(\text{THH}(A))^{\mu_p^n} \cong W_{n+1}(A;p)$$

where $W_{n+1}(A;p)$ denotes the length $n+1$ $p$-typical Witt vectors of $A$. This can be reframed as an isomorphism

$$\varpi_0^{\mu_p^n}(\text{THH}(A))^{\mu_p^n/\mu_p^{n-1}} \cong W_{n+1}(A;p).$$

This was extended to associative rings in [19]. Recall that topological Hochschild homology is a cyclotomic spectrum, which yields restriction maps

$$R_n: \text{THH}(A)^{\mu_p^n} \rightarrow \text{THH}(A)^{\mu_p^{n-1}}.$$

One can then define

$$\text{TR}(A;p) := \lim_{n,R_n} \text{THH}(A)^{\mu_p^n},$$

and it follows from the Hesselholt-Madsen result above that

$$\pi_0 \text{TR}(A;p) \cong W(A;p),$$

where $W(A;p)$ denotes the $p$-typical Witt vectors of $A$. This was then extended by Hesselholt to non-commutative Witt vectors. From [19, Thm. A], for an associative ring $A$ there is an isomorphism

$$\text{TC}_{-1}(A;p) \cong W(A;p)_F.$$
Here $W(A; p)$ denotes the non-commutative $p$-typical Witt vectors of $A$, and

$$W(A; p)_F = \text{coker} \left( 1 - F: W(A; p) \rightarrow W(A; p) \right),$$

where $F$ is the Frobenius map.

Analogously, one would like to have a notion of (non-commutative) Witt vectors for discrete $E$-rings, such that for an $E$-ring $A$, $\widetilde{D}_{m}^{2m} \text{THR}(A)$ is closely related to the Witt vectors of $\widetilde{D}_{m}^{2m} A$. In this section, we define such a notion of Witt vectors.

Real topological Hochschild homology also has restriction maps

$$R_n: \text{THR}(A)^{\mu_{p^n}} \rightarrow \text{THR}(A)^{\mu_{p^{n-1}}},$$

defined in Section 3.5. To begin, we want to understand the algebraic analogue of these restriction maps, which in particular, requires defining a Real cyclotomic structure on Real Hochschild homology. To do this, we first recall from [3] the definition of geometric fixed points for Mackey functors. Let $G$ be a finite group and let $A$ be the Burnside Mackey functor for the group $G$. For $N$ a normal subgroup of $G$, let $F[N]$ denote the family of subgroups of $G$ such that $N \notin H$.

**Definition 7.1.** Fix a finite group $G$ and let $N < G$ be a normal subgroup. Let $E F[N](A)$ be the subMackey functor of the Burnside Mackey functor $A$ for $G$ generated by $A(G/H)$ for all subgroups $H$ such that $H$ does not contain $N$. Then define

$$\bar{E} F[N](A) = A/(E F[N](A)).$$

If $M$ is a $G$-Mackey functor and $N$ is a normal subgroup of $G$, then one can define

$$E F[N](M) := M \square E F[N](A),$$

and

$$\bar{E} F[N](M) := \bar{E} F[N](A).$$

More generally, if $M_n$ is a dg-$G$-Mackey functor we define

$$(E F[N](M_n))_n := E F[N](M_n).$$

Note that the $G$-Mackey functor $\bar{E} F[N](A)$ has the property that

$$\bar{E} F[N](A)(G/H) = \begin{cases} 0 & N \notin H \\ \bar{A}/((G/N)/(N/H)) & N < H \end{cases}$$

which is the desired property for isotropy separation. There is a fundamental exact sequence

$$E F[N](A) \rightarrow A \rightarrow \bar{E} F[N](A)$$

which models the isotropy separation sequence.

We now recall the definition of geometric fixed points for Mackey functors, as in [3]. Let $M$ be a $D_{2m}$-Mackey functor. By [3, Prop. 5.8], we know $\bar{E} F[\mu_d](M)$ is in the image of $\pi_d^*$, the pullback functor from $D_{2m}/\mu_d$-Mackey functors to $D_{2m}$-Mackey functors. Consequently, we may produce a $D_{2m}/\mu_d \cong D_{2m}/\mu_d$-Mackey functor $\pi_d^{-1}(\bar{E} F[\mu_d](M))$.

**Definition 7.2** (Definition 5.10 [3]). Let $M$ be a $D_{2m}$-Mackey functor, and $\mu_d$ a normal subgroup of $D_{2m}$. We define the $D_{2m/d}$-Mackey functor of $\Phi^{\mu_d}$-geometric fixed points to be

$$\Phi^{\mu_d}(M) := (\pi_d^*)^{-1}(\bar{E} F[\mu_d](M))$$

We now show that Real Hochschild homology has a type of genuine Real cyclotomic structure.
Proposition 7.3. Given $D_2 \subset D_{2m} \subset O(2)$, $\mu_d$ a normal subgroup of $D_{2m}$ where $d | m$ and $M$ a discrete $E_\sigma$-ring, there is a natural isomorphism

$$\Phi^{\mu_d}(HR^{D_{2m}}(M)) \cong HR^{D_{2m}/d}(M)$$

of simplicial $D_{2m/d}$-Mackey functors and consequently an isomorphism

$$\Phi^{\mu_d}(HR^{D_{2m}}(M)) \cong HR^{D_{2m/d}}(M)$$

of $D_{2m/d}$-Mackey functors.

Proof. We apply $\Phi^{\mu_d}$ level-wise to the bar construction

$$(HR^{D_{2m}}(M))_* = B_*(N_{D_2}^{D_{2m}} M, N_e^{D_{2m}} t_e^* M, N_{\zeta_{2m}}^{D_{2m}} D_{2m} \zeta_{2m}^{-1} \zeta_{2m} M).$$

By [3, Proposition 5.13], the functor $\Phi^{\mu_d}$ is strong symmetric monoidal so on $k$-simplices there is an isomorphism

$$\Phi^{\mu_d}(N_{D_2}^{D_{2m}} M) \square (\Phi^{\mu_d}(N_e^{D_{2m}} t_e^* M)) \square \Phi^{\mu_d}(N_{\zeta_{2m}}^{D_{2m}} D_{2m} \zeta_{2m}^{-1} \zeta_{2m} M)$$

of $D_{2m/d}$-Mackey functors. The interaction of the geometric fixed points and the norm is described in [3, Theorem 5.15]. It follows that

$$\Phi^{\mu_d}(HR^{D_{2m}}(M)_k) \cong N_{D_2}^{D_{2m/d}} M \square (N_e^{D_{2m/d}} t_e^* M) \square N_{\zeta_{2m/d}}^{D_{2m/d}} \zeta_{2m} M$$

where we use the fact that $D_{2m/d} \cong D_{2m/d}$ by an isomorphism sending $\zeta_m$ to $\zeta_{m/d}$ and $\tau$ to $\tau$. Similarly,

$$(\zeta_{2m} D_{2m} \cdot \mu_d)/\mu_d \cong \zeta_{2m/d} D_{2m} \zeta_{2m/d},$$

since $\zeta_{2m} D_{2m} \zeta_{2m}^{-1} = < \zeta_m \tau >$ and the isomorphism sends $\zeta_m \tau$ to $\zeta_{m/d} \tau$. It therefore suffices to check that the simplicial structure maps commute with the isomorphisms (12) and (13). To see this we consider the isomorphism of totally ordered $D_{2m}$-sets

$$\mu_{2m(n+1)} : D_{2m}/D_2 \sqcup D_{2m}^{\mu_d} \sqcup D_{2m}/\zeta_{2m} D_{2m} \zeta_{2m}^{-1}$$

from Lemma 5.3. We observe that in fact these maps form an isomorphism of simplicial $D_{2m}$-sets, as shown in Remark 5.7. Applying $\mu_d$-orbits to each side we get an isomorphism

$$\mu_{2m(n+1)/d} : D_{2m/d}/D_2 \sqcup D_{2m/d}^{\mu_d} \sqcup D_{2m/d}/\zeta_{2m/d} D_{2m/d} \zeta_{2m/d}^{-1}$$

of simplicial $D_{2m/d}$-sets. If our discrete $E_\sigma$-ring $M$ is actually the restriction of a $D_{2m}$-Mackey functor then the isomorphism is simply induced by tensoring with the isomorphism (14) of simplicial $D_{2m/d}$-sets. To see this, note that given a finite group $G$ with normal subgroup $N \subset G$, the geometric fixed points $\Phi^N$ of the norm $N_T$ of a $G$-set $T$ is given by taking the norm $N_T/N$ of the orbits $T/N$. The more general statement also holds since we described the isomorphism level-wise and the compatibility with the face and degeneracy maps can be described on each box product factor indexing by the isomorphism of simplicial $D_{2m/d}$-sets (14) and using the compatibility of that isomorphism with the face and degeneracy maps. \; \; \; \Box
Definition 7.4. Given a normal subgroup $N$ in $G$ we define a functor

$$(-)^N: \text{Mack}_G \to \text{Mack}_{G/N}$$

on objects by

$$M \mapsto \pi_{G/N}^0 \left( (HM)_N^1 \right)$$

and on morphisms in the evident way.

Given a $D_2m$-Mackey functor $M$, the $D_2m$-Mackey functor $M^\mu_m$ is the data

$$M(D_2m/D_2m) \to M(D_2m/\mu_m)$$

regarded as a $D_2$-Mackey functor with the action of the Weyl group $W_{D_2}$ given by the action of the Weyl group $W_{D_2m}(\mu_m) = D_{2m}/\mu_m \cong D_2$.

Remark 7.5. This Mackey “fixed points” functor is the functor denoted $q^*$ in [26], where it was shown to preserve Green and Tambara functors.

These fixed points in Mackey functors perfectly connect to the categorical fixed points in $G$-spectra.

Proposition 7.6. For a $G$-spectrum $E$ and for all integers $k$, we have

$$\pi_{G/N}^0 (E^N) \cong (\pi_0^G (E))^k.$$  

Remark 7.7. This gives an alternate characterization of the geometric fixed points $\Phi^\mu_d M$ of a $D_2p_k$-Mackey functor $M$ for $d|p$ as

$$\Phi^\mu_d M \cong (\tilde{E} \mathcal{F}[\mu_d] M)^\mu_d$$

since it is clear in this case that there is a natural isomorphism

$$(\tilde{E} \mathcal{F}[\mu_d] (M))^\mu_d \cong (\pi^*_d)^{-1} (\tilde{E} \mathcal{F}[\mu_d] (M)).$$

Construction 7.8. Given a simplicial $D_2p_k$-Mackey functor $M$, there is a natural map

$$M \mapsto \tilde{E} \mathcal{F}[\mu_p](M)$$

and then an induced natural map

$$((M)^{\mu_p})^{\mu_{kp}} \to ((\tilde{E} \mathcal{F}[\mu_p](M))^{\mu_p})^{\mu_{kp}}$$

Note that we can identify

$$((M)^{\mu_p})^{\mu_{kp}} = (M)^{\mu_{kp}}$$

by unraveling the definition. In other words, there is a natural transformation

$$R_k: (-)^{\mu_{kp}} \to (\Phi^{\mu_p} (-))^{\mu_{kp}}$$

of functors $s \text{Mack}_{D_2p_k} \to s \text{Mack}_{D_2}$ for each $k \geq 1$. On corresponding Lewis diagrams, these produce maps

$$\xymatrix{ M_\bullet (D_2p_k/D_2p_k) \ar[r]^{R_k} & \Phi^{\mu_p} M_\bullet (D_2p_{kp}/D_2p_{kp}) \ar@{-->}[d] \ar[d] \ar@{-->}[r] \ar@{-->}[dr] & \Phi^{\mu_p} M_\bullet (D_2p_{kp}/\mu_{kp}) \ar[r]^{R_k} & \Phi^{\mu_p} M_\bullet (D_2p_{kp}/\mu_{kp}) }$$

of simplicial $D_2$-Mackey functors for each $k$ when evaluated at a simplicial $D_2p_k$-Mackey functor $M_\bullet$.  


We give this map the name $R_k$ because in our case of interest, where $\underline{M} = \underline{HR}^{D_{2p^k}}(\underline{A})$ for an $E_\sigma$-ring $A$, it is an algebraic analogue of the restriction map $R_k$ on THR$(A)$.

**Construction 7.9.** If $\underline{M}$ is a discrete $E_\sigma$-ring, it follows from Proposition 7.3 that there is an isomorphism of $D_{2p^k-1}$-Mackey functors

$$\phi^{\mu, p} \underline{HR}^{D_{2p^k}}(\underline{M}) \cong \underline{HR}^{D_{2p^{k-1}}} (\underline{M}).$$

The above construction therefore produces restriction maps

$$R_k: \left(\underline{HR}^{D_{2p^k}}(\underline{M})\right)^{\mu, p} \to \left(\underline{HR}^{D_{2p^{k-1}}}(\underline{M})\right)^{\mu, p-1},$$

which are maps of $D_2$-Mackey functors. The maps $R_k$ can be described explicitly on Lewis diagrams by

$$\begin{align*}
\underline{HR}^{D_{2p^k}}(\underline{M})/(D_{2p^k}/D_{2p^{k-1}}) &\xrightarrow{R_k} \underline{HR}^{D_{2p^{k-1}}}(\underline{M})/(D_{2p^{k-1}}/D_{2p^{k-2}}) \\
\underline{HR}^{D_{2p^k}}(\underline{M})/(D_{2p^k}/\mu, p^k) &\xrightarrow{R_k} \underline{HR}^{D_{2p^{k-1}}}(\underline{M})/(D_{2p^{k-1}}/\mu, p^{k-1}).
\end{align*}$$

**Definition 7.10.** Given a discrete $E_\sigma$-ring $\underline{M}$, we define the truncated $p$-typical Real Witt vectors of $\underline{M}$ by the formula

$$\underline{W}_{k+1}(\underline{M}; p) = \underline{HR}^{D_{2p^k}}_0(\underline{M})^{\mu, p^k}$$

and the $p$-typical Real Witt vectors of $\underline{M}$ as

$$\underline{W}(\underline{M}; p) := \varprojlim_{k, R_k} \underline{HR}^{D_{2p^k}}_0(\underline{M})^{\mu, p^k}$$

where the limit is computed in the category of $D_2$-Mackey functors. This construction is entirely functorial in $\underline{M}$ by naturality of the restriction maps $R$ so we produce a functor

$$\underline{W}(\underline{M}; p): \text{Alg}_\sigma(\text{Mack}_{D_2}) \to \text{Mack}_{D_2}.$$  

**Remark 7.11.** If $\underline{M}$ is a $D_2$-Tambara functor, then by Lemma 6.16 and [26, Prop 5.16], we know $\underline{HR}^{D_{2p^n}}_0(\underline{M})^{\mu, p^n}$ is also a $D_2$-Tambara functor and applying the limit in the category of $D_2$-Tambara functors, we produce a functor

$$\underline{W}(\underline{M}; p): \text{Tamb}_{D_2} \to \text{Tamb}_{D_2}.$$  

We now consider how the $p$-typical Real Witt vectors are related to Real topological Hochschild homology. Let $A$ be an $E_\sigma$-ring. Recall that there is a restriction map (6). Since the family $\mathcal{R}$ is defined in Section 3.4 does not contain $\mu, p^k$ and the family $\mathcal{F}[\mu, p]$ does not contain $\mu, p^k$ for any $k \geq 1$, on $\mu, p^k$-fixed points, we do not need to distinguish between these two families.

**Theorem 7.12.** Let $A$ be an $E_\sigma$-ring and $\underline{M} = \underline{HR}^{D_2}A$. There is an isomorphism of $D_2$-Mackey functors

$$\underline{HR}^{D_2}_0 \text{TRR}(A; p) \cong \underline{W}(\underline{M}; p)$$

whenever $R^1 \varprojlim_k \underline{HR}^{D_2}(\text{THR}(A))^{\mu, p^k} = 0.$
Proof. We note that there is an isomorphism
\[ \pi_{0}^{D_{2}p^{k}} \text{THR}(A) \cong \text{HR}_{0}^{D_{2}p^{k}}(M) \]
of \( D_{2}p^{k} \)-Mackey functors by Theorem 6.20 and consequently a natural isomorphism of \( D_{2} \)-Mackey functors
\[ \pi_{0}^{D_{2}}(\text{THR}(A)^{\mu_{p^{k}}}) \cong (\text{HR}_{0}^{D_{2}p^{k}}(M))^{\mu_{p^{k}}}. \]

By construction, the natural isomorphism (15) is compatible with the natural transformation
\[ R_{k} : \pi_{0}^{D_{2}}(\text{THR}(-)^{\mu_{p^{k}}}) \to \pi_{0}^{D_{2}}(\text{THR}(-)^{\mu_{p^{k-1}}}) \]
of functors \( \text{Alg}_{\sigma}(\text{Sp}_{D_{2}V}) \to \text{Mack}_{D_{2}} \) in the sense that the diagram
\[ \begin{array}{ccc}
\pi_{0}^{D_{2}}(\text{THR}(-)^{\mu_{p^{k}}}) & \xrightarrow{\cong} & \pi_{0}^{D_{2}}(\text{THR}(-)^{\mu_{p^{k-1}}}) \\
\text{HR}_{0}^{D_{2}p^{k}}(\pi_{0}^{D_{2}}(-))^{\mu_{p^{k}}} & \xrightarrow{\cong} & \text{HR}_{0}^{D_{2}p^{k-1}}(\pi_{0}^{D_{2}}(-))^{\mu_{p^{k-1}}} \\
\end{array} \]
commutes. To see this, we note that the edge homomorphism in the spectral sequence associated to the skeletal filtration of a simplicial spectrum (see Proof of 6.20) is compatible with the restriction maps \( R_{k} \). Consequently, there are isomorphisms
\[ R_{1} \lim_{k} \pi_{0}^{D_{2}} \text{THR}(A;p) \cong \text{lim}_{R} \pi_{0}^{D_{2}} \text{THR}(A)^{\mu_{p^{k}}} \]
\[ \cong \text{lim}_{R} \left( \text{HR}_{0}^{D_{2}p^{k}}(\pi_{0}^{D_{2}}(A))^{\mu_{p^{k}}} \right) \]
\[ = \mathbb{W}(\pi_{0}^{D_{2}}A;p). \]
where the first isomorphism holds by our assumption that
\[ R_{1} \lim_{k} \pi_{0}^{D_{2}} \text{THR}(A)^{\mu_{p^{k}}} = 0. \]

\[ \square \]

Construction 7.13. We now define a Frobenius map
\[ F : \mathbb{W}(M;p) \to \mathbb{W}(M;p). \]
We note that the restriction maps \( \text{res}_{D_{2}p^{k-1}}^{\mu_{p^{k}}} \) and \( \text{res}_{D_{2}p^{k-1}}^{D_{2}p^{k}} \) on the Mackey functor \( \text{HR}_{0}^{D_{2}p^{k}}(M) \) induce a map of \( D_{2} \)-Mackey functors
\[ \begin{array}{ccc}
\left( \text{HR}_{0}^{D_{2}p^{k}}(M) \right)^{\mu_{p^{k}}} (D_{2}/D_{2}) & \xrightarrow{\cong} & \left( \text{HR}_{0}^{D_{2}p^{k}}(M) \right)^{\mu_{p^{k-1}}} (D_{2p}/D_{2}) \\
\left( \text{HR}_{0}^{D_{2}p^{k}}(M) \right)^{\mu_{p^{k}}} (D_{2}/e) & \xrightarrow{\cong} & \left( \text{HR}_{0}^{D_{2}p^{k}}(M) \right)^{\mu_{p^{k-1}}} (D_{2p}/e) \\
\end{array} \]
that we call $F$. We observe that the target of this map can be identified with the $D_2$-Mackey functor \( \left( \text{HR}_0^{D_2^{p^k-1}}(M) \right)^{\mu_{p^{k-1}}} \), so we abbreviate and simply write
\[
F_k: \left( \text{HR}_0^{D_2^{p^k}}(M) \right)^{\mu_{p^k}} \to \left( \text{HR}_0^{D_2^{p^k-1}}(M) \right)^{\mu_{p^{k-1}}}
\]
for this map. Note that this construction is natural so we can apply it to the map
\[
\text{HR}_0^{D_2^{p^k}}(M) \to \mathcal{E}[\mu_p] \left( \text{HR}_0^{D_2^{p^k}}(M) \right).
\]
Therefore, by construction and Proposition 7.3 this map is compatible with the restriction maps in the sense that there are commutative diagrams
\[
\begin{array}{ccc}
\left( \text{HR}_0^{D_2^{p^k}}(M) \right)^{\mu_{p^k}} & \xrightarrow{F_k} & \left( \text{HR}_0^{D_2^{p^k-1}}(M) \right)^{\mu_{p^{k-1}}} \\
\downarrow R_k & & \downarrow R_k \\
\left( \text{HR}_0^{D_2^{p^k-1}}(M) \right)^{\mu_{p^{k-1}}} & \xrightarrow{F_k} & \left( \text{HR}_0^{D_2^{p^k-2}}(M) \right)^{\mu_{p^{k-2}}} 
\end{array}
\]
Therefore, we have an induced map
\[
F_k: \mathbb{W}(M; p) \to \mathbb{W}(M; p).
\]

Remark 7.14. Note that there is a slight clash of notation here. The Mackey functor restriction maps \( \text{res}^{D_2^{p^k}}_{\mu_{p^{k-1}}} \) and \( \text{res}^{D_2^{p^k}}_{D_2^{p^k-1}} \) on the Mackey functor \( \text{HR}_0^{D_2^{p^k}}(M) \) induce the Frobenius maps \( F_k \), not the maps \( R_k \), which are also called restriction maps. Indeed the maps \( R_k \) are not induced by any of the structure maps in the Mackey functor. While the use of these terms in this paper is consistent with the literature on topological Hochschild homology, we point out the notation clash to avoid confusion.

Remark 7.15. We can also define a Verschiebung operator
\[
V: \mathbb{W}(M; p) \to \mathbb{W}(M; p)
\]
in exactly the same way as in Construction 7.13 by replacing the restriction maps in the Mackey functor with the transfer maps in the Mackey functor.

There are also topological analogues of the maps $F$, $V$ and $R$ on THR(A)\(^{\mu_{p^k}}\) when $A$ is an $E_\sigma$-ring, which satisfy certain relations (cf. [27, §3]). In particular, $R_k$ and $F_k$ are compatible in the sense that $R_{k-1} \circ F_k = F_{k-1} \circ R_k$. The cokernel of the map id\(\mathbb{W}(A;p)\)\(^{-F}\) is defined to be the coinvariants \(\mathbb{W}(A;p)_F\). As a consequence of our setup, we have the following refinement of [19, Theorem A].

**Theorem 7.16.** Let $A$ be an $E_\sigma$-ring, and suppose that
\[
R^1 \lim_k \mathbb{W}^{D_2}(A) = 0.
\]
Then there is an isomorphism
\[
\mathbb{W}(A;p)_F \cong \mathbb{W}(\mathbb{W}^{D_2}(A); p)_F.
\]
Proof. We compute the homotopy fiber of the topological map \( \text{id} - F \) by the long exact sequence in homotopy groups. By Theorem 7.12, we can identify \( \pi_0 \text{TRR}(A) \). By inspection, the topological map \( F \): \( \text{TRR}(A) \to \text{TRR}(A) \) induces the algebraic map \( F: \mathbb{W}(\pi_0 A; p) \to \mathbb{W}(\pi_0 A; p) \) and therefore \( \pi_1 \text{TCR}(A; p) \) is the cokernel of the algebraic map \( \text{id} - F \).

7.1. Relation to existing work. In [11], Dotto, Moi, and Patchkoria give a definition of \( p \)-typical Witt vectors for a \( D_2 \)-Tambara functor at odd primes \( p \). In particular they show that for an odd prime \( p \) and a \( D_2 \)-Tambara functor \( T \), the classical \( p \)-typical Witt vectors of the commutative rings \( T(D_2/D_2) \) and \( T(D_2/e) \) can be assembled into a \( D_2 \)-Tambara functor which they denote \( W(T; p) \). The involution, restriction, and norm maps in \( W(T; p) \) are induced from the analogous maps in \( T \). The transfer is determined by the Tambara reciprocity relation. By [11, Theorem 3.7], this notion of Witt vectors recovers \( \overline{\pi}_0^{D_2} \text{THR}(E)^p_{p^n} \) when \( p \) is odd, \( E \) is a connective flat commutative orthogonal \( D_2 \)-ring spectrum, and \( M := \overline{\pi}_0^{D_2} E \) is a \( \mathbb{Z} \)-module (equivalently, when \( M \) is a cohomological Mackey functor). Our work extends this description to the non-commutative setting, i.e. \( E_\sigma \)-rings. We also give a description of the full dihedral Mackey functor which is accessible using tools from homological algebra for Mackey functors.

**Proposition 7.17.** Fix an odd prime \( p \). When \( E \) is a connective flat commutative orthogonal \( D_2 \)-ring spectrum such that \( M := \pi_0^{D_2} E \) is a \( \mathbb{Z} \)-module and \( R^1 \lim_k \pi_0^{D_2} \text{THR}(A)^{p^k} = 0 \), then there is an isomorphism
\[
\mathbb{W}(M; p) \cong W(HR_0^{D_2}(M); p)
\]
of Tambara functors.

**Proof.** This is a direct corollary of Theorem 7.12 and [11, Corollary 3.14].

More generally, we have the following corollary of our work and [11].

**Corollary 7.18.** Fix an odd prime \( p \). When \( E \) is a connective flat commutative orthogonal \( D_2 \)-ring spectrum, \( M = \pi_0^{D_2} E \), and
\[
R^1 \lim_k \pi_0^{D_2} \text{THR}(A)^{p^k} = 0,
\]
there are isomorphisms
\[
\mathbb{W}(M; p)(D_2/D_2) \cong \mathbb{W}(HR_0^{D_2}(M)(D_2/D_2); p)
\]
and
\[
\mathbb{W}(M; p)(D_2/e) \cong W(HR_0^{D_2}(M)(D_2/e); p),
\]
where \( W(-; p) \) denotes the classical \( p \)-typical Witt vectors and \( \mathbb{W}(-; p) \) is defined as in [11].

**Proof.** We first note that as a direct consequence of Theorem 7.12 and [11, Thm. 3.15] there is an isomorphism
\[
\mathbb{W}(M; p)(D_2/D_2) \cong \mathbb{W}(B \otimes_{\phi} B; p),
\]
where \( B = M(D_2/D_2) \) and \( B \otimes_{\phi} B \) from [11] is exactly \( HR_0^{D_2}(M)(D_2/D_2) \) in our notation.

For the second isomorphism, recall from Theorem 7.12 that
\[
\mathbb{W}(M; p)(D_2/e) \cong \pi_0 \text{TRR}(E; p).
\]
By [20], this is isomorphic to \( W(\pi_0 E; p) = W(HR_0^{D_2}(M)(D_2/e); p) \).

\[\square\]
8. Computations

In this section, we use the new algebraic framework from Section 6.4 to do some concrete calculations. In particular, we present a computation of $\mathbb{Z}_n^{D_{2m}} \text{THR}(H\mathbb{Z})$ where $m \geq 1$ is an odd integer and $\mathbb{Z}_n$ is the constant Mackey functor. The first step in this computation is a Tambara reciprocity formula for sums, which we present for a general finite group and may be of independent interest.

8.1. The Tambara reciprocity formulae. The most difficult relations in Tambara functors tend to be the interchange describing how to write the norm of a transfer as a transfer of norms of restrictions. These are described by the condition that if

$$U \xleftarrow{h} T \xleftarrow{f'} U \times_S \prod_g(T) \xrightarrow{g} S \xrightarrow{k'} \prod_g(T)$$

is an exponential diagram, then we have

$$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}. $$

The formulae called “Tambara reciprocity” unpack this in two basic cases:

(1) $h: G/H \cup G/H \to G/H$ is the fold map or

(2) $h: G/J \to G/H$ is a map of orbits.

These respectively describe universal formulae for $N^K_H(a + b)$ and $N^K_{H \triangleright H}(a)$.

In general, these can be tricky to specify, since we have to understand the general form of the dependent product (or equivalently here, coinduction).

**Lemma 8.1** ([25, Proposition 2.3]). If $h: T \to G/H$ is a morphism of finite $G$-sets, with $T_0 = h^{-1}(eH)$ the corresponding finite $H$-set, and if $g: G/H \to G/K$ is the quotient map corresponding to an inclusion $H \subset K$, then we have an isomorphism of $G$-sets

$$\prod_g(T) \cong G \times_K \text{Map}^H(K, T_0).$$

This entire argument is induced up from $K$ to $G$, so it suffices to study the case $K = G$. In this formulation, the map $f'$ along which we restrict is the map

$$G/H \times \text{Map}^H(G, T_0) \to G \times_H T_0$$

given by

$$(gH, F) \mapsto [g, F(g)].$$

Since the transfer along the fold map is the sum, we can understand the exponential diagram by further pulling back along the inclusions of orbits in $\text{Map}^H(G, T)$. Let $F \in \text{Map}^H(G, T_0)$ be an element, let $G \cdot F$ be the orbit, and let $K = \text{Stab}(F)$ be the stabilizer. We can now unpack the orbit decomposition of $G/H \times G/K$ and the maps to $T$ and to $G/K \cong G \cdot F$. We depict the exponential diagram, together with the pullback along the inclusion of the orbit $G \cdot F$ and orbit decompositions of the relevant pieces in Figure 1.

The map labeled $\nabla_{K,H}$ is the coproduct of the canonical projection maps

$$G/(K \cap \gamma^{-1} H \gamma) \to G/K,$$
and the norm along this is, by definition
\[
N_{\gamma,K,H} = \prod_{[\gamma] \in K \setminus G/H} N^K_{K \cap \gamma^{-1} H \gamma}.
\]
Also by definition, the restriction along \( \prod c_\gamma \) is
\[
\prod_{[\gamma] \in H \setminus G/K} M(G/(H \cap \gamma K \gamma^{-1})) \prod_{[\gamma] \in K \setminus G/H} M(G/(K \cap \gamma^{-1} H \gamma))
\]
for a Mackey or Tambara functor \( M \), where \( \gamma \) here is the Weyl action. These give all the tools needed to understand the Tambara reciprocity formulae. We spell out the formula for a norm of a sum in general; we will not need the formula for the norm of a transfer here.

**Theorem 8.2.** Let \( G \) be a finite group and \( H \) a subgroup, and let \( R \) be a \( G \)-Tambara functor. For each \( F \in \text{Map}^H(G,\{a,b\}) \), let \( K_F \) be the stabilizer of \( F \). Then for any \( a,b \in R(G/H) \), we have
\[
N^G_H(a+b) = \sum_{[F] \in \text{Map}^H(G,\{a,b\})/G} t_{tr}_{K_F} \left( \prod_{[\gamma] \in K_F \cap \gamma^{-1} H \gamma} N^K_{K_F \cap (\gamma^{-1} H \gamma)}(\gamma \in H \cap (\gamma^{-1} H \gamma)) \right)
\]

*Proof.* This follows immediately from the proceeding discussion. The only step to check is the identification of the restriction. This follows from the identification of the map \( f' \) with the evaluation map. In the case \( T_0 = \{a,b\} \), where here we blur the distinction between \( a \) and \( b \) as elements of \( R(G/H) \) and as dummy variables, the map
\[
G/(H \cap (\gamma K \gamma^{-1})) \to G/H \times \{a,b\}
\]
coincides with the canonical quotient onto the summand specified by evaluating \( F \) at \( \gamma^{-1} \). \( \square \)

Here we only need the Tambara reciprocity formulae for dihedral groups, so we now restrict attention to these cases.

### 8.2. Formulae for dihedral groups

We use the following two lemmas to describe the coinductions needed for the Tambara reciprocity formulae for
\[
N^D_{2m}(a+b) \quad \text{and} \quad N^D_{2m} tr_{D_{2m}}(a),
\]
where \( n/m \) is an odd prime.

**Lemma 8.3.** Let \( H, K \) be subgroups of a finite group \( G \) and \( T \) be a \( G \)-set. Then there is a natural bijection
\[
\text{Map}^H(G,T)^K \cong \prod_{K \gamma H \in K \setminus G/H} \text{Map}^H(K \gamma H,T)^K \cong \prod_{K \gamma H \in K \setminus G/H} T^{(\gamma^{-1}K \gamma^{-1}H)}.
\]
When $T$ has trivial action this simplifies to

$$\text{Map}^H(G,T)^K \cong \prod_{K\gamma H \in K\backslash G/H} T.$$  

**Proof.** Regarding $G$ as a $K \times H^{\text{op}}$-space, there is an isomorphism

$$t_{K \times H^{\text{op}}}^* G \cong \bigsqcup_{\gamma} K\gamma H$$

where $\gamma$ ranges over representatives for double cosets $K\backslash G/H$ in $K\backslash G/H$. The $K$ fixed points of coinduction up from $H$ to $G$ are the same as the $K \times H^{\text{op}}$-fixed points of just the set of maps out of $G$. This gives a natural (in $T$) bijection

$$\text{Map}^H(G,T)^K \cong \text{Map}^H\left(\bigsqcup_{K\backslash G/H} K\gamma H, T\right)^K \cong \prod_{K\gamma H \in K\backslash G/H} \text{Map}^H(K\gamma H, T)^K$$

as desired.

To understand each individual factor, we use the quotient map $\pi_K : K \times H \to H$ to rewrite the fixed points:

$$\left(\text{Map}^H(K\gamma H, T)^K\right)^K \cong \text{Map}^{K \times H}(K\gamma H, \pi_K^* T).$$

Since by definition of the pullback, $K$ acts trivially on $T$, the map factors through the orbits $K\backslash K\gamma H$, which is an $H$-orbit. The classical double coset formula identifies this with $H/(H \cap \gamma^{-1} K\gamma)$, and the result follows.

The simplification follows from the action being trivial, and hence all points being fixed. \qed

**Lemma 8.4.** Let $p$ be an odd prime. There are isomorphisms of $D_{2p}$-sets

$$\text{Map}^{D_2}(D_{2p}, \{a, b\}) \cong \ast \ast \amalg \prod_{i=1}^{\lfloor (p+1)/2 \rfloor} D_{2p}/D_2 \amalg \prod_{i=1}^{\lfloor (2p-1)/2 \rfloor} D_{2p}.$$  

**Proof.** We observe that the only fixed points with respect to $\mu_p$ and any subgroup of $D_{2p}$ containing $\mu_p$ are the constant maps $f_a$ and $f_b$ sending $D_{2p}$ to $a$ or $D_{2p}$ to $b$, respectively. This gives the first two summands. The $D_2$-fixed points are given by a product of $(p+1)/2 = |D_2\backslash D_{2p}/D_2|$ copies of $\{a, b\}$ with an additional copy corresponding to the constant maps. Combining this information, we have $2^{\lfloor (p+1)/2 \rfloor} - 2$ copies of $D_{2p}/D_2$ each contributing one $D_2$-fixed point. The remaining summands must be given by copies of the $D_{2p}$-set $D_{2p}$ and examining the cardinality the number of copies of $D_{2p}$ must be $\lfloor (2p-1)/2 \rfloor$. \qed

**Remark 8.5.** There is a geometric interpretation of this. The $D_{2p}$-set $\text{Map}^{D_2}(D_{2p}, \{a, b\})$ can be thought of as the ways to label the vertices of the regular $p$-gon with labels $a$ or $b$. The stabilizer of a function is the collection of those rigid motions which preserve the labeling. Those with stabilizer $D_2$ are the ones that are symmetric with respect to the reflection through some fixed vertex and passing through the center. These then depend only on the label at the chosen vertex, and then $\lfloor \frac{p-1}{2} \rfloor$ labels for the the next vertices, moving either clockwise or counterclockwise from that vertex. Of these, there are two that are special: the two where everything has a fixed label.

**Notation 8.6.** For an odd prime $p$, let

$$c_p = 2^{\frac{p+1}{2}} - 1 \text{ and } d_p = \frac{2^{p-1} - 1}{p} - c_p.$$
Lemma 8.7. Let \( p \) be an odd prime. We have an isomorphism of \( D_{2p} \)-sets

\[
\text{Map}^{D_2}(D_{2p}, D_2) \cong D_{2p}/\mu_p \sqcup D_{2p}.
\]

Proof. Since \( D_2 \) is a free \( D_2 \)-set, there are no \( D_{2p} \)-fixed points, by the universal property of coinduction. On the other hand, \( D_2 \) as a \( D_2 \)-set is actually in the image of the restriction from \( D_{2p} \)-sets via the quotient map \( D_{2p} \to D_2 \), and this is compatible with the inclusion of \( D_2 \) into \( D_{2p} \). This allows us to rewrite our \( D_{2p} \)-set as

\[
\text{Map}^{D_2}(D_{2p}, D_2) \cong \text{Map}(D_{2p}/D_2, D_{2p}/\mu_p).
\]

Since \( \mu_p \) acts trivially in the target, the \( \mu_p \)-fixed points are

\[
\text{Map}(D_{2p}/D_2, D_{2p}/\mu_p)^{\mu_p} = \text{Map}(D_{2p}/\mu_p D_2, D_{2p}/\mu_p) \cong D_{2p}/\mu_p.
\]

Finally, since \( D_2 \)-acts freely in the target, there are no \( D_2 \)-fixed points (and hence for any of the conjugates). Counting gives the desired answer. \( \square \)

We need two much more general versions of these identifications, both of which follow from the preceding lemmas.

Lemma 8.8. If \( N \subset H \subset G \) with \( N \) a normal subgroup of \( G \), then for any \( H \)-set \( T \) with \( T = T^N \), we have a natural bijection of \( G \)-sets

\[
\text{Map}^H(G, T) \cong \text{Map}^{H/N}(G/N, T).
\]

Proof. Since \( N \) is a normal subgroup of \( G \), for any \( g \in G \), \( n \in N \), and \( f \in \text{Map}^H(G, T) \), we have

\[
f(gn) = f(c_g(n)g) = c_g(n)f(g) = f(g),
\]

where the first equality is by normality of \( N \), the second is by \( H \)-equivariance of \( f \), and the third is by the condition that \( T^N = T \). \( \square \)

Corollary 8.9. Let \( p \) be an odd prime. For any \( m \geq 1 \), there are isomorphisms of \( D_{2pm} \)-sets

\[
\text{Map}^{D_{2m}}(D_{2pm}, \{a, b\}) \cong \{f_a, f_b\} \sqcup \bigcup_{i=1}^{2p} D_{2pm}/D_{2m} \sqcup \bigcup_{i=1}^{d_p} D_{2pm}/\mu_m.
\]

and

\[
\text{Map}^{D_{2m}}(D_{2pm}, D_{2m}/\mu_m) \cong D_{2pm}/\mu_m \sqcup D_{2pm}/\mu_m.
\]

Proof. This follows from Lemma 8.8. The subgroup \( N = \mu_m \). The quotient \( D_{2m}/\mu_m \) is \( D_2 \); the quotient \( D_{2pm}/\mu_m \) is \( D_{2p} \), and the result follows from the previous lemmas. \( \square \)

We will now produce a Tambara reciprocity formula for sums for the group \( D_{2p} \) when \( p \) is an odd prime.

Notation 8.10. Let

\[
X = \left( \text{Map}^{D_2}(D_{2p}, \{a, b\}) \right)^{D_2} - \{f_a, f_b\}
\]

be a set of representatives for the \( D_2 \)-fixed points, and for a point \( x \in X \), let \( x_i = x(\zeta_p^i) \).

Let

\[
Y = \left( \text{Map}^{D_2}(D_{2p}, \{a, b\}) - D_{2p} \cdot X \right)/D_{2p}
\]

be the set of free orbits in \( \text{Map}^{D_2}(D_{2p}, \{a, b\}) \), and for an equivalence class \( [y] \in Y \), let \( y_i = y(\zeta_p^i) \).
Lemma 8.11 (Tambara Reciprocity for Sums for Dihedral groups). Let $D_{2p}$ be the dihedral group where $p$ is an odd prime, with a generator $\tau$ of order 2 and $\zeta_p$ of order $p$, and let $D_2$ be the cyclic subgroup generated by $\tau$. Let $\mathbb{S}$ be a $D_{2p}$-Tambara functor. Then for all $a$ and $b$ in $\mathbb{S}(D_{2p}/D_2)$

$$N_{D_2}^{D_{2p}}(a_{D_2} + b_{D_2}) = N_{D_2}^{D_{2p}}(a_{D_2}) + N_{D_2}^{D_{2p}}(b_{D_2}) + \sum_{x \in X} \text{tr}_{D_2}^{D_{2p}}(x_0 \prod_{i=1}^{(p-1)/2} N_{e}^{D_2}(\zeta_p \text{res}_e^{D_2}(x_i))) + \sum_{[y] \in Y} \text{tr}_{e}^{D_{2p}}( \prod_{i=1}^{p} \zeta_p^i \text{res}_e^{D_2}(y_i))$$

where $X$ and $Y$ are as in Notation 8.10.

Proof. This follows from Theorem 8.2, using the identification of coinduction given by Lemma 8.4. 

Example 8.12. Explicitly, in the case of $p = 3$, we have the formula

$$N_{D_2}^{D_6}(a + b) = N_{D_2}^{D_6}(a_{D_2}) + N_{D_2}^{D_6}(b_{D_2}) + \text{tr}_{D_2}^{D_6}(b_{D_2} \cdot N_{e}^{D_2}(\zeta_5 \cdot \text{res}_e^{D_2}(a_{D_2}))) + \text{tr}_{D_2}^{D_6}(a_{D_2} \cdot N_{e}^{D_2}(\zeta_5 \cdot \text{res}_e^{D_2}(b_{D_2})))$$

because in this case the set $Y$ is empty. When $p = 7$, abbreviating $a_e = \text{res}_e^{D_2} a$ and $b_e = \text{res}_e^{D_2} b$ there is a summand

$$\text{tr}_{e}^{D_{14}}(\xi_7 b_e \cdot \xi_7^2 b_e \cdot \xi_7^3 b_e \cdot \xi_7^4 a_e \cdot \xi_7^5 a_e \cdot \xi_7^6 a_e \cdot \xi_7^7 a_e)$$

8.3. Truncated $p$-typical Real Witt vectors of $\mathbb{Z}$. For $p$ an odd prime, we compute the $D_{2p^k}$-Tambara functor $\pi_0^{D_{2p^k}} \text{THR}(H\mathbb{Z})$, using the formula

$$\pi_0^{D_{2p^k}} \text{THR}(H\mathbb{Z}) \simeq N_{D_2}^{D_{2p^k}} \mathbb{Z} \square_{N_{e}^{D_{2p^k} \zeta^e \mathbb{Z}}} N_{\zeta_2 \mathbb{Z}}^{D_{2p^k-1}} \mathbb{Z}$$

from Theorem 6.20 and Proposition 6.17. We therefore begin by computing the Mackey functor norm, $N_{D_{2p^k}}^{D_{2p^k}} \mathbb{Z}$.

Since we will be working both with dihedral groups as groups and with them as representatives of isomorphism classes of $D_{2m}$-sets in the corresponding Burnside ring, we will use distinct notation to keep track.

Notation 8.13. If $T$ is a finite $G$-set, then let $[T]$ denote the isomorphism class of $T$ as an element of the Burnside ring. When $T = G/G$, we will also simply write this as 1.

We also need some notation for generation of a Mackey functor, especially representable ones.

Notation 8.14. If $T$ is a finite $G$-set, let $A_T: f$ be $A_T$, with the canonical element $T \xleftarrow{\epsilon} T \xrightarrow{\epsilon} T$ named $f$.

Lemma 8.15. Let $p$ be an odd prime. There is an isomorphism of $D_{2p}$-Tambara functors

$$N_{D_2}^{D_{2p}}(\mathbb{Z}) \simeq A_{D_{2p}}/(2 - [D_{2p}/\mu_p]).$$
Proof. The constant Mackey functor $\mathbb{Z}$ for $D_2$ is the quotient of $A^{D_2}$ by the element $2 - [D_2] \in A^{D_2}(D_2/D_2)$ using the conventions of Section 6.1. Equivalently, we can rewrite this as a coequalizer of maps, both of which are represented by multiplication by a fixed $D_2$-set:

$$A^{D_2} \xrightarrow{D_2} A^{D_2} \xrightarrow{2} A^{D_2}.$$  

We can extend this to a reflexive coequalizer by formally putting in the zeroth degeneracy, and this represents $\mathbb{Z}$ as a sifted colimit of free Mackey functors:

$$A^{D_2} \cdot a \oplus A^{D_2} \cdot b \xrightarrow{d_1} A^{D_2} \xrightarrow{d_0} A^{D_2} \xrightarrow{s_0} A^{D_2} \xrightarrow{s_1} \mathbb{Z},$$

where $s_0(1) = b$, and where

$$d_0(b) = d_1(b) = 1 \text{ and } d_i(a) = \begin{cases} [D_2] & i = 0 \\ 2 & i = 1. \end{cases}$$

The norm commutes with sifted colimits, so we deduce that we have a reflexive coequalizer diagram

$$N_{D_2}^{D_2p} (A^{D_2} \cdot a \oplus A^{D_2} \cdot b) \xrightarrow{N(d_0)} N_{D_2}^{D_2p} (A^{D_2}) \xrightarrow{N(d_1)} N_{D_2}^{D_2p} (\mathbb{Z}).$$

The norm is defined by the left Kan extension of coinduction, so we have a canonical isomorphism for representable functors:

$$N_{D_2}^{D_2p} (A^{D_2} \oplus A^{D_2}) \cong N_{D_2}^{D_2p} (A^{D_2}_{(a,b)}) \cong A_{\text{Map}^{D_2}(D_2p, \{a,b\})},$$

and the norm of the Burnside Mackey functor for $D_2$ is the Burnside Mackey functor for $D_2p$. Here and from now on we simply write $A_T$ for $A_{T}^{D_2p}$.

Lemma 8.4 determines the $D_2p$-set we see here:

$$\text{Map}^{D_2} (D_2p, \{a,b\}) = \{f_a\} \cup \{f_b\} \cup \bigcup_{x \in X} D_2p/D_2 \cdot x \cup \bigcup_{[y] \in Y} D_2p \cdot [y].$$

This decomposition gives a decomposition of the representable:

$$A_{\text{Map}^{D_2}(D_2p, \{a,b\})} \cong A \cdot f_a \oplus A \cdot f_b \oplus \bigoplus_{x \in X} A_{D_2p/D_2} \cdot x \oplus \bigoplus_{[y] \in Y} A_{D_2p} \cdot [y].$$

Since the direct sum is the coproduct in Mackey functors, we can view each summand in the coequalizer as independently introducing a relation on $A$. We can therefore work one summand at a time, keeping track of the added relations.

By the Yoneda Lemma, maps from a representable Mackey functor $A_T \cdot f$ to $A$ are in bijective correspondence with elements of $A(T)$, and the bijection is given by evaluating a map of Mackey functors on the canonical element $f$. To determine these, we work directly, using the definition of the representables.

For a general summand parameterized by the orbit of a function $D_2p/D_2 \rightarrow \{a,b\}$, the value of the corresponding face map is built out of the functions values at the points of $D_2p/D_2$. The slogan here is that this is simply a “decategorification” of the Tambara reciprocity formula we already described.
The first case is the constant functions. Here, we have
\[ d_i(f_*) = N_{D_2}^{D_{2p}}(d_i(*)), \]
for \( * = a, b \). Both \( d_0 \) and \( d_1 \) agree on \( b \) with value 1, so the summand \( A \cdot f_b \) contributes no relation. For the summand \( A \cdot f_a \), we use that the norms in the Burnside Tambara functor are given by coinduction:
\[ d_0(f_a) = N_{D_2}^{D_{2p}}([D_2]) = [D_{2p}/\mu_p] + (d_p + c_p)[D_{2p}] \]
and
\[ d_1(f_a) = N_{D_2}^{D_{2p}}(2) = 2 + 2c_p[D_{2p}/D_2] + d_p[D_{2p}], \]
where
\[ c_p = 2^{\frac{p-1}{2}} - 1 \quad \text{and} \quad d_p = \frac{2^{p-1} - 1}{p} - c_p \]
are as defined in Notation 8.6. Coequalizing these two maps introduces a relation
\[ 2 + 2c_p[D_{2p}/D_2] + d_p[D_{2p}] - ([D_{2p}/\mu_p] + (d_p + c_p)[D_{2p}]), \]
which simplifies to
\[ (2 - [D_{2p}/\mu_p]) + c_p(2[D_{2p}/D_2] - [D_{2p}]) \]
in \( A(D_{2p}/D_2) \).

The second case we consider is the easiest one: the summands parameterized by \( Y \). Maps from \( A_{D_{2p}} \cdot y \) to \( A \) are in bijection with elements of \( A(D_{2p}/e) = \mathbb{Z} \). The explicit value is the corresponding summand from the Tambara reciprocity formula:
\[ d_i(y) = \prod_{j=0}^{p} \text{res}_{e} D_2 (d_i(y_j)), \]
since the Weyl action on the underlying abelian group in the Burnside Mackey functor is trivial. Since \( d_0(b) = d_1(b) = 1 \) and since
\[ \text{res}_{e} D_2 (d_0(a)) = \text{res}_{e} D_2 (d_1(a)) = 2, \]
we find that both face maps always agree on these summands, with value given by
\[ d_i(y) = 2^{(p-1)/2}(a). \]

Finally, the trickiest summands are the ones parameterized by \( X \). Since we are mapping out of
\[ \underline{A}_{D_{2p}/D_2} \cong \text{Ind}_{D_2}^{D_{2p}} \underline{A}_{D_2}, \]
by the induction-restriction adjunction, it suffices to understand instead the restriction to \( D_2 \) of the target. Here we use the multiplicative double coset formula: for a general \( D_2 \)-Mackey functor \( M \), we have
\[ i_{D_{2}}^{D_{2p}} N_{D_{2}}^{D_{2p}} M \cong \underline{M} \square \square_{D_2 \backslash D_{2p}/D_{2} - D_2 \backslash D_{2}} N_{e}^{D_2} i_{e}^{*} M. \]

For the Burnside Mackey functor, there is a confusing collision: every Mackey functor in this expression is the Burnside Mackey functor, so we cannot distinguish between \( \underline{A}_{D_2} \) as itself or as \( N_{e}^{D_2} \mathbb{Z} \). Writing things in terms of the actual norms of a generic Mackey functor, helps disambiguate. To a function \( x \) with stabilizer \( D_2 \), we again have the corresponding summand from the Tambara reciprocity formula:
\[ d_i(x) = d_i(x_0) \cdot \prod_{j=1}^{(p-1)/2} N_{e}^{D_2} (\text{res}_{e}^{*} d_i(x_j)), \]
where again, the triviality of the Weyl group allows us to ignore it. Note also that with the exception of $x_0$, we actually only see the restriction of $d_i(x_j)$. As we saw in the second case, the two face maps here always agree, with value 1 if $x_j = b$ and with value 2 if $x_j = a$.

If $x_0 = b$, then $d_0(x) = d_1(x)$, since the product factors always agreed.

If $x_0 = a$, then we have

$$d_i(f) = d_i(a) \cdot \prod_{j=1}^{(p-1)/2} N^{D_2}_{e} \res_{e} d_i(x_j) = d_j(a)(2 + [D_2])^k,$$

where $k$ is the number of $j$ between 1 and $(p-1)/2$ such that $x_j = 1$. The coequalizer therefore induces the relation

$$\left(2 - [D_2]\right) \cdot \left(2 + [D_2]\right)^k.$$

Since these are multiples of the case $i = 0$, so we deduce that all of these summands contribute exactly one relation:

$$2 - [D_2] \in A(D_{2p}/D_2).$$

Summarizing, we have that the norm $N^{D_{2p}}_{D_2} A$ is

$$A/\left(\left(2[D_{2p}/D_2] - [D_{2p}/\mu_p]\right) + c_p(2[D_{2p}/D_2] - [D_{2p}])(2[D_2/D_2] - [D_2])\right),$$

where to help the reader keep track of where in the Mackey functor the relations are born, we replace $1 \in A(G/H)$ with $H/H$.

This simplifies in several ways, however. Since transfers in the Burnside Mackey functor are given by induction, we have

$$c_p(2[D_{2p}/D_2] - [D_{2p}]) = c_p\tr_{D_2}^{D_{2p}}(2[D_2/D_2] - [D_2]),$$

so we can remove this from the first relations with impunity, giving

$$A/\left(2[D_{2p}/D_2] - [D_{2p}/\mu_p], 2[D_2/D_2] - [D_2]\right).$$

We also have

$$\res_{D_2}^{D_{2p}}(2[D_{2p}/D_2] - [D_{2p}/\mu_p]) = 2[D_2/D_2] - [D_2],$$

so we can now drop the second relation. This yields

$$N^{D_{2p}}_{D_2} A \cong A/\left(2 - [D_{2p}/\mu_p]\right).$$

Since $A$ is a Tambara functor, $N^{D_{2p}}_{D_2} A$ is, and as a quotient of $A$, it has a unique Tambara functor structure.

This has a somewhat surprising consequence: the form of the norm is the same as what we started with, in that we are coequalizing two maps represented by $G$-sets of cardinality 2. Induction gives the following generalization.

**Theorem 8.16.** For any odd integer $m \geq 1$, we have an isomorphism of $D_{2m}$-Tambara functors

$$N^{D_{2m}}_{D_2} A \cong A^{D_{2m}}/(2 - [D_{2m}/\mu_m]).$$

We pause here to unpack this definition some, since the quotient of Mackey functors by a congruence relation might be less familiar than the abelian group case.

**Definition 8.17.** For any odd natural number $m$, let

$$R_m = N^{D_{2m}}_{D_2} A.$$
Lemma 8.18. For any $k$ dividing $m$, we have
\[
\iota_{D_{2k}^*}^* R_m \cong R_k,
\]
and
\[
\iota_{\mu_m}^* R_m \cong A^{\mu_k}.
\]

Proof. Theorem 8.16 writes the norm $R_m$ as the coequalizer of
\[
\begin{array}{ccc}
\mathbb{A}^{D_{2m}} & \xrightarrow{2} & \mathbb{A}^{D_{2m}} \\
\quad & \downarrow & \quad \\
\left[D_{2m}/\mu_m\right] & & \left[D_{2m}/\mu_m\right]
\end{array}
\]
Since the restriction functor on Mackey functors is exact, for any subgroup $H$, the restriction of the norm is the coequalizer of 2 and
\[
\text{res}^{D_{2m}}_{H} [D_{2m}/\mu_m] = [i^*_H D_{2m}/\mu_m],
\]
When $H = D_{2k}$, we have
\[
iota_{D_{2k}}^* D_{2m}/\mu_m = D_{2k}/\mu_k,
\]
since there is a single double coset and $D_{2k} \cap \mu_m = \mu_k$. This gives the first part.
When $H = \mu_m$, we have
\[
iota_{\mu_m}^* D_{2m}/\mu_m = 2\mu_m/\mu_m,
\]
since $\mu_m$ is normal. This implies that the restriction to $\mu_m$ is $A^{\mu_m}$, and hence the restriction to $\mu_k$ is $A^{\mu_k}$.

Since we are coequalizing two maps from the Burnside Mackey functor to itself, the value at $D_{2m}/D_{2m}$ can also be readily computed. We need a small lemma about the products of certain $D_{2m}$-orbits.

Proposition 8.19. Let $m$ be an odd natural number. Let $H \subset D_{2m}$ be a subgroup, and let $\ell = \gcd(m, |H|)$. Then we have
\[
D_{2m}/H \times D_{2m}/\mu_m \cong \begin{cases} 
D_{2m}/\mu_{\ell} & |H| \text{ even}, \\
D_{2m}/H \cup D_{2m}/H & |H| \text{ odd}.
\end{cases}
\]

Proof. Since $m$ is odd, any subgroup $H$ is conjugate to either $D_{2k}$ or $\mu_k$, for $k$ dividing $m$, and the two cases are distinguished by the parity of the cardinality. Hence $D_{2m}/H$ is isomorphic to either $D_{2m}/D_{2k}$ or to $D_{2m}/\mu_k$ in exactly the two cases in the statement. The result follows from the isomorphism
\[
D_{2m}/H \times D_{2m}/\mu_m \cong D_{2m} \times \iota_{H}^* D_{2m}/\mu_m,
\]
and our earlier analysis of the restrictions.

Corollary 8.20. For any odd $m$, $R_m(\ast)$ is a free abelian group:
\[
R_m(\ast) \cong \mathbb{Z}\{[D_{2m}/D_{2k}] \mid k|m\}.
\]
The image of $[D_{2m}/D_{2k}] \in A^{D_{2m}}(\ast)$ is $[D_{2m}/D_{2k}]$, while the image of $[D_{2m}/\mu_k] \in A^{D_{2m}}(\ast)$$ is $2[D_{2m}/D_{2k}]$.

Proof. Proposition 8.19 describes the effect of the two maps on the standard basis for the Burnside ring. We see that
\[
(2 - [D_{2m}/\mu_m]) \cdot [D_{2m}/\mu_k] = 0,
\]
while
\[(2 - [D_{2m}/\mu_m]) \cdot [D_{2m}/D_{2k}] = 2[D_{2m}/D_{2k}] - [D_{2m}/\mu_k].\]
This gives both the additive result and the images. \qed

Lemma 8.18 shows then that the same statement is essentially true for the values at dihedral subgroups.

**Corollary 8.21.** For any odd \(m\) and any \(k\) dividing \(m\), we have an isomorphism
\[R_m(D_{2m}/D_{2k}) \cong \mathbb{Z}\{[D_{2k}/D_{2j}] \mid j|k\}.\]
We can also spell out the restriction and transfer maps here. The restriction and transfer to the odd order cyclic subgroups is easier, since there is a unique maximal one.

**Proposition 8.22.** The restriction map
\[R_m(*) \to R_m(D_{2m}/\mu_m) \cong \mathbb{A}^{\mu_m}(\mu_m/\mu_m)\]
is given by
\[[D_{2m}/D_{2k}] \mapsto [\mu_m/\mu_k].\]
The transfer map is given by
\[[\mu_m/\mu_k] \mapsto 2[D_{2m}/D_{2k}].\]

**Proof.** These follow from the restriction and induction in \(D_{2m}\)-sets, together with the relation
\[[D_{2m}/\mu_k] = 2[D_{2m}/D_{2k}]\]
in \(R_m\). \qed

For the restrictions and transfers to dihedral subgroups, we consider a maximal proper divisor. Let \(p\) be a prime dividing \(m\), and let \(k = m/p\).

**Proposition 8.23.** The restriction map
\[R_m(*) \to R_k(*)\]
is given by
\[[D_{2m}/D_{2j}] \mapsto \frac{p\ell}{j}[D_{2k}/D_{2\ell}],\]
where \(\ell = \gcd(k, j)\). The transfer maps are given by
\[[D_{2k}/D_{2j}] \mapsto [D_{2m}/D_{2j}].\]

**Proof.** The transfer maps are immediate. For the restriction, since \(m\) is odd, the normalizer of any dihedral subgroup is itself. The intersection of \(D_{2k}\) with \(D_{2j}\) is the dihedral group \(D_{2\ell}\), while the intersection of \(D_{2k}\) with any conjugate of \(D_{2j}\) is just the intersection \(\mu_j \cap \mu_k = \mu_\ell\). This means that it suffices to count cardinalities. This gives
\[i_{D_{2k}D_{2m}/D_{2j}}^* = D_{2k}/D_{2\ell} \cap \bigsqcup_a D_{2k}/\mu_\ell,\]
where \(a = \frac{p\ell - j}{2j}\). Since \([D_{2k}/\mu_\ell] = 2[D_{2k}/D_{2\ell}]\), the result follows. \qed

**Theorem 8.24.** Let \(m \geq 1\) be an odd integer. There is an isomorphism of \(D_{2m}\)-Mackey functors
\[\mathbb{2}_{m}^{D_{2m}}\text{THR}(H\mathbb{Z}) \cong \mathbb{A}^{D_{2m}/(2 - [D_{2m}/\mu_m])},\]
where \((2 - [D_{2m}/\mu_m])\) is the ideal generated by \(2 - [D_{2m}/\mu_m]\) in the Tambara functor \(\mathbb{A}^{D_{2m}}\).
Proof. By Proposition 6.17 and Theorem 6.20, it suffices to compute the coequalizer
\[ N_{D_2}^{2m} \otimes N_e^{D_2m} \otimes N_{\zeta D_2^{2m-1}} \mathbb{Z} \rightarrow N_{D_2}^{2m} \otimes N_{\zeta D_2^{2m-1}} \mathbb{Z}. \]

For any \( G \), \( N_e^G \mathbb{Z} \) is the Burnside Mackey functor, the symmetric monoidal unit. The \( E_0 \)-structure map here is just the unit \( A^{D_2m} \rightarrow N_{D_2}^{2m} \mathbb{Z} \).

By Theorem 8.16, this is surjective, so \( N_{D_2}^{2m} \mathbb{Z} \) is a “solid” Green functor in the sense that the multiplication map is an isomorphism. Finally, note that the argument we gave to identify \( N_{D_2}^{2m} \mathbb{Z} \) did not depend on the choice of \( D_2 \) inside \( D_2m \), so we have an isomorphism
\[ N_{D_2}^{2m} \mathbb{Z} \cong N_{\zeta D_2^{2m-1}} \mathbb{Z} \]
of Tambara functors. We deduce that all pieces in the coequalizer diagram are just \( N_{D_2}^{2m} \mathbb{Z} \).

When restricted to \( _D^{D_2m}(THR(H\mathbb{Z})^{D_p^k}) \), our computation recovers the computation in [11].

**Corollary 8.25.** Let \( p \) be an odd prime. Then there are isomorphisms of abelian groups
\[
\_D^{D_2m}(THR(H\mathbb{Z})(D_{2p^k}/D_{2p^k})) \cong _D^{D_2m}(THR(H\mathbb{Z})(D_{2p^k}/\mu_{p^k})) \cong W_{k+1}(\mathbb{Z};p).
\]

In fact, since we computed the restriction and transfer maps, we have the following computation.

**Corollary 8.26.** There is an isomorphism of Mackey functors
\[
\mathbb{W}_k(\mathbb{Z};p) = W_k(\mathbb{Z};p)
\]
for odd primes \( p \).

**References**

[1] Vigleik Angeltveit, Andrew J. Blumberg, Teena Gerhardt, Michael A. Hill, Tyler Lawson, and Michael A. Mandell. Topological cyclic homology via the norm. Doc. Math., 23:2101–2163, 2018.
[2] M. F. Atiyah. K-theory and reality. Quart. J. Math. Oxford Ser. (2), 17:367–386, 1966.
[3] Andrew J. Blumberg, Teena Gerhardt, Michael A. Hill, and Tyler Lawson. The Witt vectors for Green functors. J. Algebra, 537:197–244, 2019.
[4] Andrew J. Blumberg and Michael A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. Adv. Math., 285:658–708, 2015.
[5] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. Invent. Math., 111(3):465–539, 1993.
[6] Morten Brun, Bjørn Dundas, and Martin Stolz. Equivariant Structure on Smash Powers. arXiv e-prints, 2018. arxiv:1604.059939.
[7] Alain Connes. Cohomologie cyclique et foncteurs Ext^n. C. R. Acad. Sci. Paris Sér. I Math., 296(23):953–958, 1983.
[8] E. Dotto. Stable real K-theory and real topological Hochschild homology. Ph.D. thesis, University of Copenhagen, 2012.
[9] Emanuele Dotto, Cary Malkiewich, Irakli Patchkoria, Steffen Sagave, and Calvin Woo. Comparing cyclotomic structures on different models for topological Hochschild homology. J. Topol., 12(4):1146–1173, 2019.
[10] Emanuele Dotto and Kristian Moi. Homotopy theory of G-diagrams and equivariant excision. Algebr. Geom. Topol., 16(1):325–395, 2016.
[11] Emanuele Dotto, Kristian Moi, and Irakli Patchkoria. Witt Vectors, Polynomial Maps, and Real Topological Hochschild Homology. arXiv e-prints, page arXiv:1901.02195, January 2019.
[12] Emanuele Dotto, Kristian Moi, and Irakli Patchkoria. On the geometric fixed-points of real topological cyclic homology. arXiv e-prints, page arXiv:2106.04891, June 2021.

[13] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. Real topological Hochschild homology. J. Eur. Math. Soc. (JEMS), 23(1):63–152, 2021.

[14] Emanuele Dotto and Crichton Ogle. K-theory of Hermitian Mackey functors, real traces, and assembly. Ann. K-Theory, 4(2):243–316, 2019.

[15] T. Dyckerhoff and M. Kapranov. Crossed simplicial groups and structured surfaces. In Stacks and categories in geometry, topology, and algebra, volume 643 of Contemp. Math., pages 37–110. Amer. Math. Soc., Providence, RI, 2015.

[16] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory, volume 47 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[17] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. Trans. Amer. Math. Soc., 326(1):57–87, 1991.

[18] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Mem. Amer. Math. Soc., 113(543):viii+178, 1995.

[19] Lars Hesselholt. Witt vectors of non-commutative rings and topological cyclic homology. Acta Math., 178(1):109–141, 1997.

[20] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. Topology, 36(1):29–101, 1997.

[21] Lars Hesselholt and Ib Madsen. Real algebraic K-theory. to appear, 2015.

[22] M. A. Hill and M. J. Hopkins. Equivariant symmetric monoidal structures. arxiv.org: 1610.03114, 2016.

[23] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. Ann. of Math. (2), 184(1):1–262, 2016.

[24] Michael A. Hill. On the algebras over equivariant little disks. arXiv e-prints, page arXiv:1709.02005, September 2017.

[25] Michael A. Hill and Kristen Mazur. An equivariant tensor product on Mackey functors. J. Pure Appl. Algebra, 223(12):5310–5345, 2019.

[26] Michael A Hill, David Mehrle, and J.D. Quigley. Free incomplete Tambara functors are almost never flat. arxiv.org: 2105.11513, 2021.

[27] Amalie Høgenhaven. Real topological cyclic homology of spherical group rings. ArXiv e-prints:1611.01204, 2016.

[28] Stefan Jackowski and Jolanta S/ suppressed lominska. G-functors, G-posets and homotopy decompositions of G-spaces. Fund. Math., 169(3):249–287, 2001.

[29] Max Karoubi. Périodicité de la K-théorie hermitienne. In Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pages 301–411. Lecture Notes in Math., Vol. 343, 1973.

[30] L. Gaunce Lewis, Jr. and Michael A. Mandell. Equivariant universal coefficient and Künneth spectral sequences. Proc. London Math. Soc. (3), 92(2):505–544, 2006.

[31] Jean-Louis Loday. Homologies diédrale et quaternionique. Adv. in Math., 66(2):119–148, 1987.

[32] Jean-Louis Loday. Cyclic homology, volume 301 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[33] Jacob Lurie. Higher algebra. Available at http://people.math.harvard.edu/~lurie/papers/HA.pdf, September 18, 2017.

[34] J. McClure, R. Schwänzl, and R. Vogt. $\text{TTH}(R) \simeq R \otimes S^1$ for $E_{\infty}$ ring spectra. J. Pure Appl. Algebra, 121(2):137–159, 1997.

[35] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. Acta Math., 221(2):203–409, 2018.

[36] Irakli Patchkoria and Steffen Sagave. Topological Hochschild homology and the cyclic bar construction in symmetric spectra. Proc. Amer. Math. Soc., 144(9):4099–4106, 2016.

[37] J. D. Quigley and Jay Shah. On the parametrized Tate construction and two theories of real p-cyclotomic spectra. arXiv e-prints, page arXiv:1909.03920, Sep 2019.

[38] Graeme Segal. Configuration-spaces and iterated loop-spaces. Invent. Math., 21:213–221, 1973.

[39] Jan Spaliński. Homotopy theory of dihedral and quaternionic sets. Topology, 39(3):557–572, 2000.
[41] Martin Stolz. Equivariant structure on smash powers of commutative ring spectra. 2011.
[42] Neil Strickland. Tambara functors. arXiv e-prints, page arXiv:1205.2516, May 2012.
[43] R. W. Thomason. Homotopy colimits in the category of small categories. Math. Proc. Cambridge Philos. Soc., 85(1):91–109, 1979.
[44] Rafael Villarroel-Flores. The Action by Natural Transformations of a Group on a Diagram of Spaces. arXiv:0411502, 2004.

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