Exact Analytical Solutions in Inhomogeneous Magnetic Fields for Linear Asteroseismic Waves

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Abstract

We solve for waves in an isothermal, stratified medium with a magnetic field that points along a direction perpendicular to that of gravity and varies exponentially in the direction of gravity. We find exact analytical solutions for two different cases: (a) waves propagating along the direction of the magnetic field and (b) waves propagating along the direction of the gravity. In each case, we find solutions in terms of either the hypergeometric functions or their confluent cousins. We solve the resultant transcendental dispersion relation numerically. The eigenfrequencies decrease with increasing degree of the spatial inhomogeneity of the magnetic field. Further, the nodes of the eigenfunctions leak toward regions of lower Alfvén wave speed due to softened wave-reaction in such regions. Such changes in the dispersion relation and the mode structures may allow the detection of magnetic fields buried in the stellar interior.

Unified Astronomy Thesaurus concepts: Magnetic fields (994); Helioseismology (709); Solar physics (1476); Magnetohydrodynamics (1964); Asteroseismology (73); Analytical mathematics (38); Stellar pulsations (1625); Solar magnetic fields (1503)

1. Introduction

The subsurface properties of the Sun have been inferred in great detail by analyzing the waves on its surface. This method of helioseismic inversion has revolutionized solar physics by providing otherwise inaccessible details of the solar interior, including its interior rotation profiles (see, e.g., the review by Christensen-Dalsgaard 2002). To create such reconstructions, observational studies commonly use hydrodynamic properties of the waves. Although magnetic fields can have additional imprints on the observed surface oscillations, nonmagnetic analyses have offered a great wealth of knowledge in solar physics (Basu & Antia 2008; Gizon et al. 2010). With advances in more precise instruments and calculations, however, observational analysis with considerations to (evolving) magnetic fields become more important and thus have received more attention (Braun 1995; Hindman et al. 1997; Crouch & Cally 2005; Jain 2007; Hanson & Cally 2014).

Theoretical investigations, on the other hand, in global helioseismology, have been carried out extensively over several decades to predict the effect of magnetic fields on helioseismic waves (see, e.g., Nye & Thomas 1976; Adam 1977; Thomas 1983; Campos 1983; Lee & Roberts 1986; Miles & Roberts 1992; Jain & Roberts 1991; Cally 2007; Foullon & Roberts 2005; Pintér & Erdélyi 2011; Campos & Marta 2015; Pintér 2018). It is well established that magnetic effects upshift the frequencies of acoustic modes of stellar oscillations (Gough & Thompson 1990; Goldreich et al. 1991; Dziembowski & Goode 2004). Although these signatures themselves are clear, other studies suggest that these frequency shifts may be much smaller than the actually observed range of oscillation frequencies (Foullon & Roberts 2005). This leaves uncertainty as to whether and how the magnetic fields buried inside the stellar interior can be reliably estimated.

In local helioseismology, there has been several recent attempts (Schunker et al. 2005; Ilonidis et al. 2011; Singh et al. 2014, 2015, 2016, 2020) to predict the emergence of active regions via the detection of large subsurface magnetic fields through their seismic signatures. Signatures from numerical simulations (Singh et al. 2014, 2015, 2016, 2020) and observational studies such as acoustic travel-time perturbations (Schunker et al. 2005; Ilonidis et al. 2011) have been reported to be insightful and in some cases predictive about the emergence of subsurface magnetic fields on the solar surface and about their subsurface field strengths. In the context of other stars, interior magnetic fields have been assigned bounds, based upon analysis of depressed dipole oscillation modes due to magnetic fields (Fuller et al. 2015). Nevertheless, how to decipher the background profiles of magnetic fields via inversion procedure of astro- and helioseismic data remains an unsolved problem.

Waves in magnetized gases, with particular emphasis on solar physics, have been widely studied; see, e.g., Campos (1987) for a review. A major bottleneck in progress is the fact that there are very few exact analytical results for these waves in the presence of gravity and an inhomogeneous magnetic field. The first study of MHD waves in an isothermal atmosphere (Yu 1965) with an inhomogeneous magnetic field, assumed a space dependence such that the Alfvén speed remained uniform in space. This led to sinusoidal solutions for the linear waves. Later work by Nye & Thomas (1976) obtained exact solutions for waves in an isothermal atmosphere with a spatially varying Alfvén wave speed, arising from an exponential profile of the mass density and a uniform horizontal magnetic field. Even this solution is not completely general—it is valid only for waves that propagate along the direction of the magnetic field. Note that the inhomogeneity in the magnetic field is undeniably present in real stars. Roberts and the collaborators (Lee & Roberts 1986; Jain & Roberts 1991; Miles & Roberts 1992; Foullon & Roberts 2005).
have considered cases where the magnetic field is uniform in one layer—in which they use the solution obtained by Nye & Thomas (1976)—and zero in the other layer(s) and matched the solutions across the boundary of the layers. Gough & Thompson (1990) have used perturbation theory to obtain leading-order changes to the dispersion relations for the pressure-dominated ($p$) modes due to nonuniform magnetic fields. Singh et al. (2015) have approached the problem numerically by considering a two-layer model—each layer assumed to be isothermal, with different scale heights and with a uniform horizontal magnetic field. Singh et al. (2014) considered the same two-layer isothermal model, but with an inhomogeneous magnetic field and found that the inhomogeneity significantly changes the amplitude of the fundamental ($f$) mode of oscillations. A recent observational study (Korpi-Lagg et al. 2022) suggests that these $f$-mode oscillations may have imprints of the poloidal magnetic field as the study exposes anticorrelation of mode-integrated energy with the solar cycle, and imprints of the poloidal magnetic field as the study exposes anticorrelation of mode-integrated energy with the solar cycle. The frequencies decrease with increasing heights and with a uniform horizontal magnetic field. Singh et al. (2015) have used perturbation theory to obtain leading-order changes to inhomogeneous magnetic fields. This causes the eigenfunctions to become broader. This property may be useful in detection of magnetic fields buried deep inside a star.

Our principal result is that magnetic imprints on the waves, due to inhomogeneous fields, are quite appreciable, both on the frequencies and the eigenfunctions of the pressure-dominated modes of oscillations. The frequencies decrease with increasing degree of the inhomogeneity. The effects on the eigenfunctions are also identified and node leakages or node shifts are found with spatially varying magnetic fields. This causes the eigenfunctions to become broader. This property may be useful in detection of magnetic fields buried deep inside a star.

This article is organized in the following manner. Section 2 entails the MHD model of the wave. A general wave equation is presented in Section 3. Section 3.1 contains the details of the background profiles. Exact solutions for the waves are obtained in Section 4. In Section 5, effects of an inhomogeneous magnetic fields on the eigenfrequencies and the eigenfunctions are detailed. Concluding remarks are made in Section 6.

2. Model

Consider the equations of motion for magnetized plasmas within the framework of MHD (see, e.g., Choudhuri 1998), with a uniform gravity $g$ acting vertically downward:

$$\partial_t \rho + \nabla \cdot (\rho \, U) = 0,$$

$$\rho \left[ \partial_t U + (U \cdot \nabla) U \right] = -\nabla P + \rho g + J \times B,$$

$$\partial_t B = \nabla \times (U \times B),$$

where $\rho$, $U$, $P$, and $B$ are the density, the velocity, the pressure, and the magnetic field, respectively. The current density is $J = \nabla \times B/\mu_0$ with $\mu_0$ denoting the permittivity of the vacuum. Equations (1a) and (1c) are respectively the continuity equation and the induction equation. These equations are supplemented with

$$\nabla \cdot B = 0,$$

Dissipative effects such as plasma viscosity and resistivity are ignored for asteroseismic waves. These equations must be augmented by an equation of state, which we consider to be an isothermal condition. Consequently, the ideal gas law suggests that the sound speed $c_s$, given by

$$c_s^2 = \frac{\gamma_{\text{ad}} P}{\rho},$$

is a constant. The factor $\gamma_{\text{ad}}$ is the adiabatic index of the gas.

2.1. Static Stationary State

The plasma with no background flow $U = 0$ is permeated by a magnetic field $B = B_0$ that varies in space. Here, a horizontal magnetic field that decreases vertically with height is considered (see Figure 1). A similar scenario is relevant in the Sun and Sun-like stars where the magnetic fields—toroidal in direction (horizontal in local analysis)—decreases with height. This vertically varying magnetic field imposes a density profile $\rho_0(z)$ where $z$ is the vertical coordinate. Note a uniform magnetic field or the absence thereof plays no role in the stationary state. In such cases, the background density $\rho_0(z)$ is a pure exponential function of $z$, varying with a scale height of $\ell_0 = c_s^2/\left(g \gamma_{\text{ad}}\right)$.

With a nonuniform magnetic field, the stationary state is given by

$$\left(\frac{c_s^2}{\gamma_{\text{ad}}}\right) \nabla \rho_0 = \rho_0 g + J_0 \times B_0,$$

$$\nabla \cdot B_0 = 0,$$

$$J_0 = (\nabla \times B_0)/\mu_0.$$

2.2. Linearized MHD Equations

Linear waves are studied by perturbing the governing equations of motion about a background state as

$$(\rho, \, U, \, B) = (\rho_0, \, 0, \, B_0) + \varepsilon (\rho, \, u, \, b) + \mathcal{O} (\varepsilon^2),$$

where $\varepsilon$ is a small parameter.
We substitute Equation (5) in Equation (1), expand it, and keep terms up to first order in $\varepsilon$. Algebraic simplifications then reduce the problem to a wave equation for the perturbed velocity:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho_0} \nabla(\rho_0 c_s^2 \nabla \cdot u) + \nabla(u \cdot g) - g(\nabla \cdot u)
$$

$$\quad - \frac{1}{\mu_0 \rho_0} \nabla \left[ u \cdot \left( \frac{B_0^2}{2} - B_0 \cdot \nabla B_0 \right) \right]
$$

$$\quad + \frac{1}{\mu_0 \rho_0} \left[ \nabla \times \left( \nabla \times (u \times B_0) \right) \right] \times B_0
$$

$$\quad + \frac{1}{\mu_0 \rho_0} \left[ \left( \nabla \times B_0 \right) \times \left( \nabla \times (u \times B_0) \right) \right].$$

(6)

This is the standard method of analysis (see, e.g., Chandrasekhar 1961).

2.3. Nondimensionalization

To simplify the calculations, all the variables henceforth are nondimensionalized. Consider $l'_\rho$ to be the characteristic length scale, the speed of sound $c_s$ as the characteristic velocity scale, and the density $\rho_0$ and the magnetic field $B_0$ at the bottom of the domain as their respective characteristic scales. An important dimensionless number $M_\Lambda$, the Alfvénic Mach number, emerges (see Table 1). Other nondimensional variables are denoted therein by symbols with asterisks ($^*$). In the rest of this article, as we shall use nondimensional units only, the asterisks are omitted.

3. General Treatment

As the background profiles are homogeneous along the horizontal directions ($x,y$), Fourier modes are used along those directions to expand the velocity perturbations,

$$u(x, y, z, t) \sim \hat{u}(k_x, k_y, z, \omega) e^{i(k_x x + k_y y - \omega t)},$$

(7)

where $\hat{u}(k_x, k_y, z, \omega)$ is the amplitude of the perturbation at the wavenumbers $(k_x, k_y)$ with frequency $\omega$. This expansion when substituted in Equation (6) yields a set of three coupled ordinary differential equations for the three components of the velocities $\hat{u}_x, \hat{u}_y,$ and $\hat{u}_z$. Using the Einstein convention (summing repeated indices), we find

$$M_{ij} \hat{u}_j + N_{ij} D \hat{u}_j + \delta_{i,3} \left( 1 + \frac{B_0^2}{\rho_0 M_\Lambda^2} \right) D^2 \hat{u}_3 = 0,$$

(8)

where

$$M = \begin{bmatrix}
\omega^2 - k_x^2 & 0 & -k_x k_y / \gamma \rho_0 \\
0 & \omega^2 - k_y^2 & -k_y k_x / \gamma \rho_0 \\
-i k_x / \gamma \rho_0 & i k_y / \gamma \rho_0 & \omega^2 - \frac{B_0^2}{2 \rho_0 M_\Lambda^2} k_x^2 + \frac{D(B_0)^2}{\rho_0}
\end{bmatrix},$$

(9)

with $D \equiv d/dz$ and

$$N = \begin{bmatrix}
0 & 0 & i k_x \\
0 & 0 & -k_x / \gamma \rho_0 \\
-i k_y / \gamma \rho_0 & i k_x / \gamma \rho_0 & \omega^2 - \frac{B_0^2}{2 \rho_0 M_\Lambda^2} - 1
\end{bmatrix}.\tag{10}$$

Equation (8) is general for the case where the background magnetic field points along the $x$-axis and varies arbitrarily along the $z$-axis. If it is restricted to be uniform in space, the equations simplify further. The exact solution of waves for such a profile has already been found by Nye & Thomas (1976), Adam (1977), Thomas (1983), and Campos (1983). Here we consider a general case where the background magnetic field $B_0$ is an exponential function of the $z$-coordinate.

In particular, we list the following cases:

1. Waves propagating along the $x$-axis (along the background magnetic field);
2. Waves propagating along the $z$-axis (along the direction of the gravity); and
3. Waves propagating along the $y$-axis (orthogonal to the direction of both the gravity and the background magnetic field).

For waves traveling along the gravity (i.e., case 2 above), Campos & Marta (2015) presented an exact analytical solution for the special case of an inhomogeneous magnetic field—an exponential function whose scale height is twice the density scale height, $B = \exp(-z/q) \hat{x}$ with $q = 2$. In this article, we present exact analytical solutions for both the first and the second case for magnetic fields that vary exponentially in space with arbitrary scale height, i.e., any value of $q$. For waves propagating along any arbitrary direction, the problem remains unsolved, although writing the linearized equations themselves is straightforward.

For the first two cases outlined above ($k_x, k_y = (k, 0)$, which simplifies Equation (8) to

$$\left( \omega^2 - k^2 \right) \hat{u}_x - i k \left( \frac{1}{\gamma \rho_0} - D \right) \hat{u}_z = 0,$$

(11a)

$$\left( \omega^2 - k^2 \frac{B_0^2}{\rho_0 M_\Lambda^2} \right) \hat{u}_z = 0,$$

(11b)
Table 1
Summary of the Nondimensional Parameters

| Nondimensional Parameter | Nondimensional Parameter |
|--------------------------|--------------------------|
| $\xi^* = \frac{\xi}{\ell}$ | $\rho_0^* = \frac{\rho_0}{\rho_0^*}$ |
| $D = \frac{\ell_f}{\ell}$ | $B_0^* = \frac{B_0}{B_0^*}$ |
| $k_x^* = k_x \ell$ | $M_\lambda = \frac{\ell_\lambda}{\ell}$ |
| $k_y^* = k_y \ell$ | $\beta = \frac{\beta_0}{\beta_0^*}$ |
| $w^* = \frac{w}{\ell}$ | $q = \frac{q_0}{q_0^*}$ |

Note. Here, $\ell_f$ is the density scale height, $\ell$ is the scale height of magnetic field variation, $\beta$ is the ratio of gas pressure to magnetic pressure, and $M_\lambda$ is the Alfvénic Mach number.

\[

i k \left[ D + \frac{1}{\gamma_{ad}} - 1 - \frac{\gamma_{ad}}{2M_\lambda^2} \frac{D(B_0^2)}{\rho_0} \right] \ddot{u}_x \\
+ \left[ \frac{1}{\rho_0 M_\lambda^2} \frac{B_0^2}{D^2} - \left( 1 + \frac{\gamma_{ad}}{2M_\lambda^2} \frac{D(B_0^2)}{\rho_0} \right) \right] \ddot{u}_z \\
+ \left( \omega^2 - k^2 \frac{B_0^2}{\rho_0 M_\lambda^2} \frac{\omega}{D^2} \right) \ddot{u}_\ell = 0. 
\]

(11c)

Note that $\ddot{u}_x$ in Equation (11b) is decoupled from $\ddot{u}_z$ and $\ddot{u}_\ell$. The same equation informs that there exists a mode of oscillation with frequency $\omega^2 = k^2 q_\lambda^2$ where $q_\lambda = B_0 / (M_\lambda \sqrt{\rho_0})$. These are, of course, the well-known Alfvén waves. Now, eliminating $\ddot{u}_\ell$ in favor of $\ddot{u}_x$ in Equations (11a) and (11c),

\[

\left( \omega^2 - k^2 \frac{B_0^2}{\rho_0 M_\lambda^2} \right)^2 \ddot{u}_x - \omega^2 D^2 \ddot{u}_z \\
+ \left( \omega^2 - k^2 \frac{B_0^2}{\rho_0 M_\lambda^2} \right) \omega \frac{\gamma_{ad}}{2M_\lambda^2} \ddot{u}_z \\
+ \frac{D(B_0^2)}{\rho_0 M_\lambda^2} \left( \omega^2 - k^2 - \frac{\omega^2 \gamma_{ad}}{2} \right) \ddot{u}_x + \frac{k^2}{2} \ddot{u}_z = 0. 
\]

(12)

In Equation (12), the expressions for $B_0$ and $\rho_0$ are not yet prescribed, which we do next. If the background magnetic field is uniform, Equation (12) reduces to the one considered by Nye & Thomas (1976).

3.1. Background Profiles

Rewriting the force-balance equation for the background state in Equation (4a) in nondimensional form, we obtain

\[

D \rho_0 + \rho_0 + \frac{1}{2M_\lambda^2} \gamma_{ad} D(B_0^2) = 0. 
\]

(13)

Integrating Equation (13), we find the background density depends on the background magnetic field via

\[

\rho_0(z) = -\frac{\gamma_{ad}}{2M_\lambda^2} e^{-z} \int_0^z dz' e^{z'} \frac{d(B_0^2)}{dz'}. 
\]

(14)

Equation (14) is valid for an arbitrary background magnetic field that depends on the vertical coordinate $z$. Most previous studies considered a uniform magnetic field, except, for instance, Campos & Marta (2015) who considered a nonuniform magnetic field. They obtained the magnetohydrostatic equilibrium solution for a horizontal magnetic field that varies with height. For the solution of waves, they chose a specific profile that decreases exponentially at twice the density scale height, $q = 2$ in our notation. Such a solution is a special case of this study. In what follows now, the background magnetic field is a generic exponential function of the height $z$ such that

\[

B_0 = \dot{x} \exp(-z/q), 
\]

where $q = \ell_b / \ell_p$ is the ratio of the characteristic length scale of magnetic field variation to the characteristic length scale of density variation. When $q \to \infty$, the case of the uniform magnetic field is recovered (Nye & Thomas 1976).

Substituting Equation (15) in Equation (14) and simplifying, we obtain

\[

\rho_0(z) = \left[ 1 - \frac{\gamma_{ad}}{2M_\lambda^2} \right] e^{-z} + \frac{\gamma_{ad}}{2M_\lambda^2} e^{-2z/q} 
\]

for $q \neq 2$, (16a)

\[

\rho_0(z) = \left[ 1 + \frac{\gamma_{ad}}{2M_\lambda^2} \right] e^{-z} 
\]

for $q = 2$. (16b)

This means that the density drops exponentially with height for $q \neq 2$. However, when $q = 2$, it increases linearly with height for $z \ll 1$ and then drops exponentially for $z \gg 1$.

4. Exact Solutions

Clearly, the case of $q = 2$ is special and turns out to be somewhat simpler. First, we deal with the case of $q \neq 2$.

4.1. Alfvén Speed for $q \neq 2$

For this case, the density profile is given by Equation (16a).

So, the Alfvén speed, $v_A(z)/\rho_0(z)$, can be written as

\[

v_A^2 = \left[ \frac{\gamma_{ad}}{2M_\lambda^2} \right] \left[ 1 - \frac{\gamma_{ad}}{2M_\lambda^2} \right] e^{-2z/q}. 
\]

(17)

Straightforward algebra shows

\[

\frac{\gamma_{ad}}{M_\lambda^2} = \frac{2}{\beta}, 
\]

where $\beta$ is the plasma beta (the ratio of the gas pressure to the magnetic pressure) at the bottom of the considered domain.

4.2. Alfvén Speed for $q = 2$

The Alfvén speed for this case is

\[

v_A^2(z) = \frac{B_0^2}{\rho_0} = \left[ 1 + \frac{\gamma_{ad}}{2M_\lambda^2} \right] e^{-2z/(\gamma_{ad} M_\lambda^2)}. 
\]

(19)

4.3. Waves along the Magnetic Field: $q \neq 2$

Here, we obtain exact solutions for waves that propagate along the magnetic field for the case of $q \neq 2$. 

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Substituting Equation (17) in Equation (12), simplifying it, and performing the following change of variable
\[ s = \left(1 - \frac{2}{q}\right) \xi, \]  
(20)

Equation (12) assumes the form of
\[ [A_2 e^{s} + B_2] \frac{d^2 \hat{u}_c}{ds^2} + [A_1 e^{s} + B_1] \frac{d \hat{u}_c}{ds} + [A_0 e^{s} + B_0] \hat{u}_c = 0, \]  
(21)

where the expressions for \(A_0, A_1, A_2\) and \(B_0, B_1, B_2\) are given in Appendix A.

Next, following Campos & Marta (2015), we perform two more changes of variables,
\[ \xi = -\frac{B_2}{A_2} e^{-s}, \]  
(22a)
\[ W = \hat{u}_c e^{-\theta s}, \]  
(22b)

where \(\theta\) is a yet undetermined constant. Upon further simplifications, we obtain
\[ \xi (1 - \xi) \frac{d^2 W}{d\xi^2} + \left[ \left(2\theta + 1 - \frac{A_1}{A_2}\right) - \left(2\theta + 1 - \frac{B_1}{B_2}\right) \xi \right] \frac{dW}{d\xi} - (\theta^2 - \frac{A_1}{A_2} \theta + \frac{A_0}{A_2} W) \xi = 0. \]  
(23)

The crucial step now is to choose \(\theta\) such that the last term in Equation (23) vanishes, i.e.,
\[ \theta^2 - \frac{A_1}{A_2} \theta + \frac{A_0}{A_2} = 0. \]  
(24)

Consequently, Equation (23) turns into the standard hypergeometric differential equation
\[ \xi (1 - \xi) \frac{d^2 W}{d\xi^2} + \left[ C - (A + B + 1) \xi \right] \frac{dW}{d\xi} - ABW = 0, \]  
(25)

where the parameters \(A, B,\) and \(C\) are given as
\[ C = 2\theta + 1 - \frac{A_1}{A_2}, \]  
(26a)
\[ A + B + 1 = 2\theta + 1 - \frac{B_1}{B_2}, \]  
(26b)
\[ AB = \theta^2 - \frac{B_1}{B_2} \theta + \frac{B_0}{B_2}, \]  
(26c)

and \(\theta\) is given by Equation (24).

The solutions to Equation (25) are the hypergeometric functions (for \(|\xi| < 1\))
\[ W(A, B; C; \xi) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)} \sum_{n=0}^{\infty} \frac{\Gamma(A + n)\Gamma(B + n)}{\Gamma(C + n)} \frac{\xi^n}{n!}. \]  
(27)

Transforming back to the original variables, the final solution is
\[ \hat{u}_c(z) = D_1 e^{-\theta(1-2/q)} {}_2F_1 \left( A, B; C; -\frac{B_2}{A_2} e^{-z(1-2/q)} \right) + D_2 e^{\theta(1-2/q)(1-\theta)B_0/A_2} \left( -\frac{B_2}{A_2} \right)^{1-\theta} \times {}_2F_1 \left( A - C + 1, B - C + 1; 2 - C; -\frac{B_2}{A_2} e^{-z(1-2/q)} \right). \]  
(28)

where \(D_1\) and \(D_2\) are constants. They are constrained by the boundary conditions. The function \( {}_2F_1 \) is the Gauss hypergeometric function. All constants \(A, B, C, A_0, A_1, A_2, B_0, B_1, B_2,\) and \(\theta\) are functions of \(\omega, k, \gamma_{ad}, M_A,\) and \(q\) (which are given in Appendix A).

4.4. Waves along the Magnetic Field: \(q = 2\)

Following a similar procedure as in Section 4.3, we obtain
\[ \hat{u}_c(k, z, \omega) = A_+ {}_1F_1(m_+; 1; -\eta B_0(k_1 + z)) \left[ \frac{e^{\eta z}}{\eta^{m_+}} \right] \left( \eta_3 - \eta_4 \right), \]  
(29)

where the function \( {}_1F_1 \) is the confluent hypergeometric function of the first kind. The relation between the constants \(A_+\) and \(A_-\) are determined by the choice of the boundary conditions. The constants \(m_+\) and \(m_-\) are
\[ m_{\pm} = \frac{1}{2} \left[ 1 \pm \frac{1}{\eta B_0} \right] \pm \frac{\eta_4}{\eta_4} \left( \eta_3 - \eta_4 \right). \]  
(30)

The functions represented by \(\eta_4\) with \(j = 1, 2, 3,\) and \(4\) are functions of \(\omega, k, \gamma_{ad},\) and \(\beta\) as given in Appendix B.

4.5. Waves along the Magnetic Field: \(q \to \infty\) (Uniform \(B_0\))

If \(B_0\) is uniform in space, we recover the solution of Nye & Thomas (1976) from our solution by taking the limit \(q \to \infty\) in Equation (28). In this limit, the parameters simplify to
\[ A_2 = a_2 = (\omega^2 - k^2)/M_A^2, \]  
\[ A_1 = a_1 = 0, \]  
\[ A_0 = a_0 = -k^2(\omega^2 - k^2)/M_A^2. \]  
(31)

Equation (24) also reduces to
\[ 0 = k. \]  
(32)

This set of parameters in Equation (31), when substituted in Equation (28), reproduces the solution obtained by Nye & Thomas (1976).

4.6. Waves Propagating along the Direction of the Gravity

Next we consider the problem of waves propagating perpendicular to the magnetic field and along the direction of the gravity. We separate the two cases: \(q = 2\) and \(q = 2.\) The latter has already been solved exactly by Campos & Marta (2015). Below we present the solution for the former.
As the waves propagate along the z-axis, \( k_x = k_y = 0 \). This, when substituted in Equation (8), yields
\[
\begin{align*}
\omega^2 \ddot{u}_x &= 0, \\
\omega^2 \ddot{u}_y &= 0,
\end{align*}
\]

\[
\left[ \omega^2 + D^2 \left( 1 + \frac{B_0^2}{\rho_0 M_A^2} \right) - D + \frac{2 - \gamma_{ad}}{2M_A^2} \frac{D(B_0)^2}{\rho_0} \right] \ddot{u}_z = 0.
\]

The solution to Equation (33c) can be found as a special case by substituting \( k_x = k = 0 \) in the solution presented in Section 4.3. The solution is
\[
\ddot{u}_z(z) = D_1 e^{-\theta(1-2/q)} _2F_1 \left( A, B; C; \frac{-B_2}{A_2} \right) + D_2 e^{\theta(1-2/q)(1-\delta_t/A_2)} \left[ \frac{-B_2}{A_2} \right]^{1-C} \times _2F_1 \left( A - C + 1, B - C + 1; 2 - C; \frac{-B_2}{A_2} e^{-\theta(1-2/q)} \right),
\]

where all the arguments of the hypergeometric functions are redefined in Appendix C.

The special solution for \( q = 2 \), presented by Campos & Marta (2015), can be obtained by substituting \( k = 0 \) in Equation (29).

5. Effect of an Inhomogeneous Magnetic Fields on Waves

Here, we obtain the dispersion relations of waves in nonuniformly magnetized plasmas and compare them with well-known waves in uniformly magnetized plasmas. By imposing the boundary conditions on the analytical solution found in Equation (28), the dispersion relations are derived. Following Nye & Thomas (1976), we also demand that the energy per unit volume of the perturbed fields remains bounded. This implies \( D_2 = 0 \) in Equation (28). Note that the upper boundary at \( z \rightarrow \infty \) maps to \( \xi = 0 \) in the new coordinate \( \xi \), defined in Equation (22a). Imposing a rigid lower boundary at \( z = 0 \) (\( \xi = -B_2/A_2 \)), where the vertical velocity \( \ddot{u}_z \) vanishes, the dispersion relation is given by
\[
_2F_1 \left( A(\omega, k, q), B(\omega, k, q); C(\omega, k, q); \frac{-B_2(\omega, k, q)}{A_2(\omega, k, q)} \right) = 0,
\]

where all the parameters \( A, B, C, A_2, \) and \( B_2 \) are functions of \( \omega, k, \) and \( q \). Their expressions are given in Equation (26a) and in Appendix A.

In Figure 2, the dispersion relations for the waves in uniformly and nonuniformly magnetized plasmas are compared. Decreasing magnetic field with the increasing height has the effect of lowering the oscillation frequencies of the waves—which is particularly pronounced for the higher oscillation modes.

Now, we show the effect of spatially varying magnetic fields on the eigenfunctions of the oscillations. As seen in Figure 3, the eigenfunctions leak toward (penetrate into) the upper regions, i.e., higher \( z \)-values, where the magnetic fields are weaker. For example, the second and the third modes of oscillations show that their nodes are shifted due to weakening of the magnetic fields. This may alternatively be interpreted as lowering of the Alfvén speed in these regions when the magnetic field is nonuniform \( (q = 10) \), compared to when it is uniform \( (q \rightarrow \infty) \). This lowered Alfvén speed leads to a softened (lowered) wave reflection, thus enhancing oscillations in these regions (Campos & Marta 2015).

In this article, we have considered the background inhomogeneous magnetic field to be constant in time. This is physically justifiable as the timescale of emergence of magnetic flux and the formation of the active regions on the solar surface is very slow compared to the characteristic timescales of the helioseismic waves. Hence it is a good enough approximation to consider the background magnetic field to be static. Typical emergence phenomenon is the formation of a bipolar region, which is expected to form from the underlying magnetic fields that are horizontal. Thus starting with a horizontal magnetic field (to represent the fields lying deep inside) can also be taken as a reasonable approximation. But the assumption of isothermal atmosphere does not allow us to compare our results directly with solar or stellar observations (Ulrich 1970).
6. Conclusions

We have presented hitherto unknown exact analytical solutions for the linear astroseismic waves, propagating through a medium with a gravity and an exponentially varying horizontal magnetic field, whose scale height is arbitrary. These solutions and the wave dispersion relations are found to be sensitive to the degree of the inhomogeneity. To wit, the eigenfrequencies of the waves rapidly decrease when the magnetic field strength decreases faster with height. The eigenfunctions become broader, with their nodes leaking toward the regions of weaker magnetic fields. This is understood as softened wave reflection from a region where the Alfvén wave speed gets lowered.

It may be noted that mathematically similar problem as considered here appears while analyzing the generation of ocean waves by a turbulent wind of exponential profile (Miles 1960). Recently, perturbative solutions for the Miles problem and also for the case of a wind with other profiles (in particular, logarithmic) were obtained using the method of matched asymptotics (Bonfils et al. 2022). A variant of such a technique may allow to obtain perturbative solutions for more complicated profile of inhomogeneous background magnetic field. But, the insight about the interplay of waves, their nodes, and the inhomogeneous Alfvén speed may carry over from the problem discussed in this article.

Apart from informing the observational studies, the solution found here for the linear eigenmodes can be employed to compute nonlinear mode coupling coefficients that can test and verify numerical codes of nonlinear asteroseismology. In other words, having an exact analytical solution with nontrivial magnetic field geometry helps to set stringent constraint on benchmarking numerical codes of helio- or asteroseismology. Further, asymptotics of these exact solutions can offer insights into the imprints of nonlinearity on the observed stellar oscillations. Such an investigation will be the focus of a future study.

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Appendix A
Expressions for $A_j$ and $B_j$

The expressions for the parameters introduced in Equation (21) of Section 4.3 are presented below:

\[
A_0 = \frac{1}{M_A^2} \left[ (\omega^2 - k^2) \frac{\omega^2 \gamma_{\text{ad}}}{(q - 2)} - k^2 \frac{\gamma_{\text{ad}} - 1}{\gamma_{\text{ad}}(q - 2)} - k^2 \right],
\]

\[
A_1 = \frac{2}{q} \left[ (\omega^4 - k^2 + \frac{\omega^2 \gamma_{\text{ad}}}{2}) + \omega^2 \frac{\gamma_{\text{ad}}}{(q - 2)} \right] \left( 1 - \frac{2}{q} \right),
\]

\[
A_2 = \frac{1}{M_A^2} \left[ \omega^2 - k^2 + \frac{\omega^2 \gamma_{\text{ad}}}{(q - 2)} \right] \left( 1 - \frac{2}{q} \right),
\]

\[
B_0 = \omega^2 \left[ 1 - \frac{\gamma_{\text{ad}}}{(q - 2)M_A^2} \right] \left[ \omega^2 - k^2 + \frac{k^2 (\gamma_{\text{ad}} - 1)}{\omega^2 \gamma_{\text{ad}}^2} \right],
\]

\[
B_1 = -\omega^2 \left[ 1 - \frac{\gamma_{\text{ad}}}{(q - 2)M_A^2} \right] \left( 1 - \frac{2}{q} \right),
\]

\[
B_2 = \omega^2 \left[ 1 - \frac{\gamma_{\text{ad}}}{(q - 2)M_A^2} \right] \left( 1 - \frac{2}{q} \right).
\]
Appendix B
Expressions for $\eta_j$

The expressions for $\eta_j$ used in Equation (30) of Section 4.4 are listed here:

$$
\eta_1 = \frac{2}{\gamma_{ad}} \left(1 - \frac{k^2}{\omega^2} \right) + \beta,
$$

$$
\eta_2 = \frac{k^2 (\gamma_{ad} - 1)}{\omega^2} + (\omega^2 - k^2),
$$

$$
\eta_3 = \frac{\left(1 - \frac{k^2}{\omega^2}\right) \left(\omega^2 \beta - 2k^2\right) + \frac{k^2 (\gamma_{ad} - 1) \beta - k^2}{\omega^2 \gamma_{ad}}}{\eta_2},
$$

$$
\eta_d = (1 - 4\eta_2)^{1/2}.
$$

Appendix C
Expressions for $A_j$, $B_j$, and $C_j$

The parameters that appear in Equation (34) of Section 4.6 are given next:

$$
A_0 = \frac{1}{M_A^2} \frac{\omega^4 \gamma_{ad}}{q - 2},
$$

$$
A_1 = \frac{1}{M_A^2} \left[ \frac{\gamma_{ad} \beta}{q} - \frac{\gamma_{ad}}{q} \left(1 - \frac{2}{q}\right) \right],
$$

$$
A_2 = \frac{1}{M_A^2} \left[ \frac{1}{q} + \frac{\gamma_{ad}}{q} \left(1 - \frac{2}{q}\right)^2 \right],
$$

$$
B_0 = \omega^4 \left[ 1 - \frac{\gamma_{ad}}{(q - 2) M_A^2} \right],
$$

$$
B_1 = -\omega^4 \left[ 1 - \frac{\gamma_{ad}}{(q - 2) M_A^2} \left(1 - \frac{2}{q}\right) \right],
$$

$$
B_2 = \omega^4 \left[ 1 - \frac{\gamma_{ad}}{(q - 2) M_A^2} \left(1 - \frac{2}{q}\right)^2 \right].
$$

Another parameter $\theta$ is evaluated using the Equation (24), which yields

$$
\theta = \frac{1}{2} \left(\frac{A_1}{A_2} \pm \sqrt{\left(\frac{A_1}{A_2}\right)^2 - \frac{4 A_0}{A_2}}\right).
$$

Lastly, the parameters $A$, $B$, and $C$ in Equation (34) are found by solving the following set of equations:

$$
C = 2\theta + 1 - \frac{A_1}{A_2},
$$

$$
A + B + 1 = 2\theta + 1 - \frac{B_1}{B_2},
$$

$$
AB = \theta^2 - \frac{B_1}{B_2} \theta + \frac{B_0}{B_2}.
$$

This completes the specification for obtaining the Gauss hypergeometric function as the solution for the magnetoacoustic gravity waves in a medium permeated by spatially varying magnetic fields.

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