Study of the pressure distribution along a concentric annular channel

O V Germider 1,* and V N Popov 2

1,2 Northern (Arctic) Federal University named after M.V. Lomonosov, 17, Severnaya Dvina Emb., Arkhangelsk, 163002, Russia
E-mail: *o.germider@narfu.ru

Abstract. A study of an isothermal rarefied gas flow through a long concentric annular channel is carried out. The solution is based on the linearized BGK model with diffuse boundary conditions and it is valid in the whole range of the rarefaction parameter. The pressure distribution along the channel is obtained and investigated depending on the values of the pressure maintained at the channel ends.

1. Introduction
During the last years it is possible by development of the kinetic approach made to solve in a computationally multidimensional problems in complex geometries [1-3]. The main advantage of the kinetic approach is that the solution is valid in the whole range of the rarefaction parameter from the free molecular to the hydrodynamic regimes. Using the solution is critical when carefully and systematically studying of the pressure distribution. However, finding solution of the model kinetic equation and analysis are challenging computational tasks in a rarefied gas dynamics [1]. The present work is devoted to investigating the gas pressure along a concentric annular channel applying kinetic approach. Here, we consider the Maxwell diffuse reflection model with complete accommodation at the bounding walls of the channel and assume, that the rarefied gas flow is due to a pressure gradient imposed in the longitudinal direction. The methodology of obtaining the pressure distribution along the channel is proposed and results are provided due to a ratio of the pressure maintained at the channel ends in the wide range of gas rarefaction. The proposed approach is accomplished through Chebyshev approximation [4] for the solution of the linearized BGK (Bhatnagar, Gross and Krook [5]) model kinetic equation and applying the discrete orthogonality relations and Kronecker and Hadamard products [6] to minimize computational error. The realization of the polynomial approach during the study can be applied to any cross section.

2. Materials and methods
Consider a long concentric annular channel of length \(L'\) and radius \(R'_1\) and \(R'_2\) (\(R'_2 > R'_1 > 0\)), connecting two reservoirs maintained at pressures \(p'_1\) and \(p'_2\) (\(p'_2 > p'_1 > 0\)). Under the influence of the pressure gradient there is an isothermal (Poiseuille) flow from the high towards the low pressure reservoir. Assume that the radius \(R'_2 \ll L'\) and the non-dimensionless pressure gradient \(|G_p| = R'_2|dp'/p'_0dz'| \ll 1\), where \(p'_0\) the gas pressure at a certain point taken as the origin. The pressure \(p'\) vary only along the flow direction \(z'\). Since a kinetic approach is followed, the
main unknown is the distribution function \( f'(r', v) \), where \( r' \) is the radius-vector and \( v \) is the molecular velocity. In the configuration and velocity spaces are use the cylindrical coordinates \( r' = (\rho', r''_x, r''_z) \) and \( v = (v_\perp, v_\psi, v_z) \). Here \( \psi \) and \( \psi \) are polar angles.

The dimensionless radius-vector \( r' \), radii \( R'_1 \) and \( R'_2 \), length \( L' \), distribution function \( f' \), molecular velocity \( v \), mass velocity \( u' \), gas number density \( n' \), temperature \( T' \) and mass flux \( J'_M \) are written as

\[
\begin{align*}
    r &= r'/R'_2, \quad R_1 = R'_1/R'_2, \quad R_2 = 1, \quad L = L'/R'_2, \quad f = f'/n'_0^{3\beta/2}, \\
    C &= \beta^{1/2}v, \quad u = \beta^{1/2}u', \quad n = n'/n'_0, \quad T = T'/T'_0, \quad J_M = J'_M/\pi(R'_2 - R'_1)^{\beta/2}.
\end{align*}
\]

Here \( n'_0 \) and \( T'_0 \) are the gas number density and temperature at a certain point taken as the origin, \( \beta = m'/(2k_BT'_0) \), \( k_B \) is the Boltzmann constant, \( m' \) is mass of a molecule, \( T = 1 \). Below, the non-dimensional variables are denoted by the same letters as the dimensional ones.

We will consider gas flow in the middle part of channel \( (z'_0 = 0, \ z' = z'/L') \). In the case, when the pressure and temperature gradients are small then, we may assume linear distribution along the channel and write

\[
    p'_0 = p_{av}', \quad p'(z) = p_{av}'(1 + G_p z), \quad G_p = \frac{p'_2 - p'_1}{L' p_{av}'}.
\]

where \( p_{av}' = (p'_2 + p'_1)/2 \) and \( G_p \) is the dimensionless pressure gradient.

However, when the pressure ratio \( p'_2 = p'_2/p'_1 \) is large, the pressure distribution along the channel is not linear. Also, the rarefaction parameter \( \delta = \text{Kn}^{-1} \) varies significantly along the channel. Here \( \text{Kn} = v'_g/R'_2 \) is the Knudsen number, \( v'_g \) is the mean free path of gas molecules. For the hard-sphere model it depends on the local pressure and temperature according to [7]

\[
    \delta = \frac{R'_2 p'}{n'_g(T')^{3/2}} \sqrt{\frac{m'}{2k_BT'}},
\]

Here \( n'_g \) is the dynamic gas viscosity. Using the equation of continuity (the total mass flux is a constant) we introduce the new reduced mass flux whose value does not vary along the channel as [7]

\[
    J'_M = \frac{L}{\pi(R'_2^2 - R'_1^2)^{1/2}} \sqrt{2k_BT'} m'/m'.
\]

Linearize the distribution function as [8]

\[
    f(r, C) = f_0(C)(1 + G_p(z + h(r, C))), \quad f_0(C) = \pi^{-3/2}\exp\left(-\frac{C^2}{2}\right),
\]

where \( \zeta = \cos \psi \). The component \( u_z \) can be expressed in terms of the function \( f(r, C) \) as follows:

\[
    u_z = G_p U_z, \quad U_z(\rho) = \frac{2}{\pi} \int_{-1}^{+\infty} \frac{1}{\sqrt{1 - \zeta^2}} Z(\rho, C_\perp, \zeta) d\zeta dC_\perp,
\]

\[
    Z(\rho, C_\perp, \zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-C_z^2) h(r, C) dC_z.
\]
Then the dimensionless mass flux may be computed according to

\[ J_M = G_p J_{M,p}, \quad J_{M,p} = \frac{4}{(1 - R_1^2)} \int_{R_1}^1 U_z(\rho)d\rho . \]  

(6)

Substituting (1) and (6) in (4) and taking into account that \( G_p = p_1 dp^*/(Ldz^*) \), we have

\[ J_M^* dz^* = J_{M,p} dp^* , \]  

(7)

where \( J_M^* \) is the parameter of the differential equation; \( p^* = p'/p'_1 \). At the inlet \( z^* = -1/2 \) and outlet \( z^* = 1/2 \) boundaries the pressure is equal to the inlet \( p'_1 = 1 \) and to the outlet \( p'_2 \) reservoirs pressure accordingly.

Expressing \( p^* = p'/p'_1 \) from (3) and substituting in (7), we will integrate the equation (7). As a result, we get

\[ J^*_M = \int_1^{p'_2} J_{M,p} dp^* = \frac{1}{\delta_1} \int J_{M,p} d\delta . \]  

(8)

Using the expression of the local rarefaction parameter (3), the values \( J_{M,p}(\delta) \) and the value of the parameter \( J^*_M \) from (8) we can numerically obtain a solution off the differential equation (7) with the boundary condition \( p^*(-1/2) = 1 \).

For restoration of \( Z(\rho, C_{\perp}, \zeta) \) the equation takes the form [8]:

\[ \left( \frac{\partial Z}{\partial \rho} + \frac{\partial Z(1-\zeta^2)}{\rho} \right) C_{\perp} + \delta Z(\rho, C_{\perp}, \zeta) + \frac{1}{2} = \delta U_z(\rho) . \]  

(9)

As the boundary conditions on the channel walls, we will use the diffuse reflection model. Then, for the function \( Z(\rho, C_{\perp}, \zeta) \) boundary conditions can be written as

\[ Z(1, C_{\perp}, \zeta) = 0, \quad \zeta < 0, \quad Z(R_1, C_{\perp}, \zeta) = 0, \quad \zeta > 0 . \]  

(10)

We will obtain the solution to the equation (9) with the boundary conditions (10). We expand the unknown function \( Z(\rho, C_{\perp}, \zeta) \) in the truncated series in the Chebyshev polynomials \( T_{k_i} \) \((k_i \leq n_i; i = 1, 3)\) of the first kind defined on \( \rho \in [0, 1], C_{\perp} \in [0, +\infty) \) and \( \zeta \in [-1, 1] \) as

\[ Z(\rho, C_{\perp}, \zeta) = T_1(x_1) \otimes T_2(x_2) \otimes T_3(x_3) \mathbf{A} , \]  

(11)

where \( x_1 = (2\rho - 1 - R_1)/(1 - R_1), \ x_2 = (C_{\perp} - 1)/(C_{\perp} + 1), \ x_3 = \zeta, \ T_1 \) is the \( 1 \times n'_i \) matrix \((n'_i = n_i + 1, i = 1, 3)\) as

\[ T_i(x_i) = (T_0(x_i) T_1(x_i) \ldots T_{n_i-1}(x_i) T_{n_i}(x_i)) , \]  

(12)

and \( \otimes \) denotes the Kronecker product defined for two matrices \( \mathbf{T}_1(x_1) \) and \( \mathbf{T}_2(x_2) \) as [6]

\[ \mathbf{T}_1(x_1) \otimes \mathbf{T}_2(x_2) = (T_0(x_1)T_0(x_2)T_0(x_1)T_1(x_2) \ldots T_{n_1}(x_1)T_{n_2-1}(x_2)T_{n_1}(x_1)T_{n_2}(x_2)) , \]  

(13)

and \( \mathbf{A} \) is the unknown \( n'_1n'_2n'_3 \times 1 \) matrix as

\[ \mathbf{A} = (a_{000} a_{001} \ldots a_{n_1n_2n_3} a_{n_1n_2n_3})^T . \]  

(14)

Substituting (5) and (11) in (9), we have

\[ \mathbf{B}(x_1, x_2, x_3) \mathbf{A} = -\frac{1}{2} , \]  

(15)
\begin{align}
T_1(-1) \otimes T_2(x_2) \otimes T_3(x_3) A &= 0, \quad x_2 > 0, \\
T_1(1) \otimes T_2(x_2) \otimes T_3(x_3) A &= 0, \quad x_2 < 0, \\
\end{align}

where

\begin{align}
B(x_1, x_2, x_3) &= \frac{2}{1-R_1} \left( \frac{dT_1(x_1)}{dx_1} \otimes T_2(x_2)x_2 + \\
&+ T_1(x_1) \otimes \frac{dT_2(x_2)}{dx_2} \left( \frac{1-x_2^2}{x_1+(1+R_1)/(1-R_1)} \right) \right) \otimes \\
&\otimes T_3(x_3) \frac{1+x_3}{1-x_3} + \delta T_1(x_1) \otimes (T_2(x_2) \otimes T_3(x_3) - P_2 \otimes P_3), \\
\end{align}

\begin{align}
P_2 &= \frac{2}{\pi} \int_{-1}^{1} \frac{T_2(x_2)}{\sqrt{1-x_2^2}} dx_2 = (20 \ldots 00), \\
P_3 &= 2 \int_{-1}^{1} \frac{1+x_3}{(1-x_3)^3} T_3(x_3) \exp \left( -\frac{(1+x_3)^2}{(1-x_3)^2} \right) dx_3. \\
\end{align}

As the collocation points in the equation (15) we will use the extreme points of the polynomial \( T_{n_1}(x_1) \) on the segment \([-1; 1]\)

\begin{align}
x_{1,k_1} &= \cos \left( \frac{\pi(n_1-k_1)}{n_1} \right), \quad k_1 = 0, n_1, \\
\end{align}

and the roots of the polynomials \( T_{n_2'}(x_2) \) and \( T_{n_3'}(x_3) \) on the segment \([-1; 1]\)

\begin{align}
x_{i,k_i} &= \cos \left( \frac{\pi(2n_i-2k_i+1)}{2(n_i+1)} \right), \quad k_i = 0, n_i, i = 2, 3. \\
\end{align}

Substituting (21) and (22) in (15), we arrive at the system of the \( n_1' n_2' n_3' \) linear equations, in which we replace the equations with \( x_{1,0} \) and \( x_{2,k_2} \) \( (k_2 = n_2'/2, n_2) \) by the equations following from the boundary condition (10) as

\begin{align}
T_1(-1) \otimes T_2(x_{2,k_2}) \otimes T_3(x_{3,k_3}) A &= 0, \quad k_3 = 0, n_3, \\
\end{align}

with \( x_{1,n_1} \) and \( x_{2,k_2} \) \( (k_2 = 0, n_2'/2-1) \) by the equations following from the boundary condition (10) as

\begin{align}
T_1(1) \otimes T_2(x_{2,k_2}) \otimes T_3(x_{3,k_3}) A &= 0, \quad k_3 = 0, n_3. \\
\end{align}

Here and below, we assume that, \( n_2 \) is an odd number. To find the values of the Chebyshev polynomials at points (23) and (24), we use the geometric form: \( T_j(x_i) = \cos(j_i \arccos x_i), \) where \( x_i \in [-1, 1] \) [5]. Then

\begin{align}
T_{j_1}(x_{1,k_1}) &= \cos \left( \frac{\pi j_1(n_1-k_1)}{n_1} \right), \quad j_1, k_1 = 0, n_1, \\
T_{j_i}(x_{i,k_i}) &= \cos \left( \frac{\pi j_i(2n_i-2k_i+1)}{2(n_i+1)} \right), \quad j_i, k_i = 0, n_i, i = 2, 3. \\
\end{align}

Now, if we consider the extreme points (23) of the polynomial \( T_{n_1}(x_1) \), then we have the discrete orthogonality relation [4]

\begin{align}
\frac{2}{n_1} \sum_{k_1=0}^{n_1} T_{j_1}(x_{1,k_1}) T_{l_1}(x_{1,k_1}) = \gamma_{j_1} \delta_{j_1,l_1}, \\
\end{align}
where \( \sum_{k=1}^{n_i} \) is meant the sum, in which the first and last terms are multiplied by 1/2, the points \( x_{1,k_i} \) \((k_i = 0, n_i)\) are given by (23),

\[
\gamma_{j_i} = \begin{cases} 2, & j_i = 0 \lor (i = 1 \land j_i = n_i) \\ 1, & j_i > 0 \land (i = 1 \land j_i < n_i) \end{cases}, \quad \delta_{j_i,i_1} = \begin{cases} 1, & j_i = i_1, \\ 0, & j_i \neq i_1, \quad i = 1, 3. \end{cases}
\] (26)

If we now consider the roots (24) of the polynomials \( T_{n_i}(x_i) \) \((i = 2, 3)\), then we obtain

\[
\frac{2}{n_i + 1} \sum_{k_i=0}^{n_i} T_{j_i}(x_{i,k_i})T_{i_1}(x_{i,k_i}) = \gamma_{j_i} \delta_{j_i,i_1}, \quad i = 2, 3,
\] (27)

\[
\frac{2}{n_i + 1} \sum_{k_i=0}^{n_i} T_{k_i}(x_{i,j_i})T_{k_i}(x_{i,i_1}) = \delta_{j_i,i_1}, \quad i = 2, 3,
\] (28)

where \( \sum_{k=0}^{n_i} \) is meant the sum in which the first term is multiplied by 1/2.

The relations (25)-(28) allow for the matrix \( A \), depend on the unknown matrix \( Z \) as

\[
Z = (Z_{000}, Z_{001}, \ldots, Z_{n_in_n-1})^T, \quad Z_{k_1,k_2,k_3} = Z(p_{k_1}, C_{k_2,k_3}, \zeta_{k_3}),
\] (29)

In this case the matrix \( A \) depend on the unknown matrix \( Z \) as

\[
A = \varphi J_1 \otimes H_1 \otimes G_1 Z, \quad \varphi = \frac{8}{n_1n_2n_3}.
\] (30)

where \( J_1 \) is the \( n_1' \times n_1' \) matrix, in which

\[
J_{1,i_1,i_2} = T_{i_1}(x_{1,i_2})/4, \quad (i_1, i_2 = 0, n_1),
\]

\[
J_{1,i_1,i_2} = T_{i_1}(x_{1,i_2})/2, \quad (i_1 = 0, n_1 \land i_2 \neq 0, n_1) \lor (i_2 = 0, n_1 \land i_1 \neq 0, n_1),
\]

and \( J_{1,i_1,i_2} = T_{i_2}(x_{1,i_2}) \) for other values of the indices \( i_1 \) and \( i_2 \); \( H_1 \) and \( G_1 \) are the \( n_k' \times n_k' \) matrices, respectively \((k = 2, 3)\), in which

\[
H_{i_1,i_2,i_3} = T_{i_1,i_2}(x_{2,i_3})/2, \quad G_{i_1,i_2,i_3} = T_{i_1,i_2}(x_{3,i_3})/2,
\]

else \( H_{i_1,i_2,i_3} = T_{i_1,i_2}(x_{2,i_3}), G_{i_1,i_2,i_3} = T_{i_1,i_2}(x_{3,i_3}). \)

Taking into consideration (23)-(30), the system of the \( n_1'n_2'n_3 \) linear equations can be written in the form:

\[
LZ = F,
\] (31)

\[
L = \frac{2}{1-R_1}B_1 + \delta I - \delta B_2, \quad F = -\frac{1}{2}E_{JH_1} \otimes G_2,
\]

\[
B_1 = E_{JH} \circ \left( \frac{2}{n_1} J_2 J_1 \otimes I_{d_2} + \frac{2}{n_2^2} I_{d_1} \otimes (H_2 H_1) \right) \otimes I_{d_3},
\]

\[
B_2 = \frac{4}{n_2^3} E_{JH} \circ (I_1 \otimes E_H) \otimes E_G.
\]

Here, \( \circ \) denotes the Hadamard product of two matrices [6], \( I_1 \) and \( I \) are \( n_1' \times n_1' \) and \( n_1'n_2'n_3 \times n_1'n_2'n_3 \) identity matrices, respectively; \( I_{d_1}, I_{d_2} \) and \( I_{d_3} \) are \( n_k \times n_k \) diagonal matrices \((k = 1, 3)\) as

\[
I_{d_1,i_1,i_1} = \frac{1}{x_{1,i_1} + (1 + R_1)/(1 - R_1)}, \quad I_{d_2,i_2,i_2} = x_{2,i_2}, \quad I_{d_3,i_3,i_3} = \frac{1 + x_{3,i_3}}{1 - x_{3,i_3}},
\]
The differences between the results for \( R \) are curves 2. It is seen that in figure 2 the agreement between the results for \( J_0 \leq n_0 \) is almost identical (figures 5 and 6).

Numerical results for the dimensionless mass flux \( J_1 = J/\rho \).

Results and discussion

and \( \mathbf{E}_{\mathbf{H}} \) is \( n_1' \times n_1 \) matrix such that the null rows have numbers \( l \) \((= \lceil n_2/2, n_2 \rceil)\) and all other rows consist of ones, \( \mathbf{E}_{\mathbf{JH},1} \) is the first column of the matrix \( \mathbf{E}_{\mathbf{JH}} \), \( J_2 \) and \( H_2 \) are \( n_k' \times n_k \) matrices \((k = 1, 2)\) respectively as

\[
J_{2,i_1,j_1} = \frac{dT_{1,j_1}(x_{1,i_1})}{dx_1}, \quad (i_1, j_1 = 0, n_1),
\]

\[
H_{2,i_2,j_2} = \frac{dT_{2,j_2}(x_{2,i_2})}{dx_2}(1 - x_{2,i_2}^2) \quad (i_2, j_2 = 0, n_2),
\]

and \( \mathbf{E}_{\mathbf{H}} \) and \( \mathbf{G}_2 \) are \( n_2' \times n_2 \) and \( n_3' \times 1 \) matrices with ones; \( \mathbf{E}_{\mathbf{G}} \) is \( n_3' \times n_3' \) matrix as

\[
\mathbf{E}_{\mathbf{G}} = \mathbf{G}_2 \otimes (\mathbf{P}_2 \mathbf{G}_1).
\]

The system of the \( n_1' \times n_1' \) linear equations (31) with unknown matrix \( \mathbf{Z} \) has been solved by the \( LU \)-method. Based on the obtained elements of the matrix, we restore the dimensionless component of the mass velocity \( U_z(\rho) \) and mass flux \( J_{M,p} \):

\[
U_z(\rho) = \frac{\rho}{l + R_1} \mathbf{T}_1 \left( \frac{2\rho - 1 - R_1}{1 - R_1} \right) \mathbf{J}_1 \otimes \mathbf{E}_{\mathbf{H},r} \otimes \mathbf{E}_{\mathbf{G},r} \mathbf{Z}, \quad (32)
\]

\[
J_{M,p} = \frac{\rho}{l + R_1} \mathbf{P}_1 \mathbf{J}_1 \otimes \mathbf{E}_{\mathbf{H},1} \otimes \mathbf{E}_{\mathbf{G},1} \mathbf{Z}, \quad (33)
\]

\[
\mathbf{P}_1 = \int_{-1}^{1} \mathbf{T}_1(x_1) \left( x_1(1 - R_1) + 1 + R_1 \right) dx_1.
\]

Here, \( \mathbf{E}_{\mathbf{H},r} \) and \( \mathbf{E}_{\mathbf{G},r} \) are first rows of the matrices \( \mathbf{E}_{\mathbf{H}} \) and \( \mathbf{E}_{\mathbf{G}} \) respectively.

3. Results and discussion

Numerical results for the dimensionless mass flux \( J_{M,p} \) obtained by the \( LU \)-method using the formula (33) are presented figures 1 and 2 for \( R_1 = 0.1 \) and \( R_1 = 0.5 \) consideration, with \( 0 \leq \delta \leq 10 \). For the sake of comparison, in the figures 1 and 2 we have given the values of the \( J_{M,p} \) borrowed from [9] by the method of discrete velocities with the use of cubic splines. These are curves 2. It is seen that in figure 2 the agreement between the results for \( R_1 = 0.5 \) is good. The differences between the results for \( R_1 = 0.5 \) are due to the hydraulic diameter \( 2(R_2 - R_1) \) of the concentric annular channel has been taken as the dimensional length scale in [9]. In figures 3 and 4 we have given the distributions \( p^*(z^*) \) obtained by the Runge-Kutta method for \( \delta_1 = 0.1 \) and \( \delta_1 = 1 \) with \( R_1 = 0.1 \) and \( p_{2}^* = 10 \). These are curves 1. From the figure 3 it can be seen that for \( \delta_1 = 0.1 \) the distribution \( p^*(z^*) \) approaches the linear distribution. The results for \( R_1 = 0.5 \) are almost identical (figures 5 and 6).

Next, a comparison between the kinetic and free molecular solutions is performed. In the free molecular regime the equation (9) can be solved analytically. In this case we obtain

\[
Z(\rho, \zeta, \zeta) = -\frac{\rho \zeta \pm \sqrt{1 - \rho^2(1 - \zeta^2)}}{2C_\perp}, \quad -1 \leq \zeta \leq \frac{\sqrt{R_2^2 - \rho^2}}{\rho}, \quad (34)
\]

\[
Z(\rho, \zeta, \zeta) = -\frac{\rho \zeta - \sqrt{R_1^2 - \rho^2(1 - \zeta^2)}}{2C_\perp}, \quad \frac{\sqrt{R_1^2 - \rho^2}}{\rho} < \zeta \leq 1. \quad (35)
\]
Substituting (34) and (35) in (5), we have

$$U_z(\rho) = \frac{1}{2\sqrt{\pi}} \left( \int_{\rho_0}^{1} \frac{\sqrt{R_1^2 - \rho^2(1 - \zeta^2)}}{\sqrt{1 - \zeta^2}} d\zeta - \int_{-1}^{\rho} \frac{\sqrt{R_1^2 - \rho^2}}{\sqrt{1 - \zeta^2}} d\zeta \right).$$

(36)

We will integrate (8). In the figures 1 and 2, for $R_1 = 0.1$ and $R_1 = 0.5$, values of the free molecular mass flux $J_{M,p}$ shown by points M. Taking (3) and (36) into account, we obtain $J^*_M = (p^*_2 - 1)J_{M,p}$ and the solution of the differential equation (7) with the boundary conditions $p^*(-1/2) = 1$ and $p^*(1/2) = p^*_2$ takes the form

$$p^*(z) = \frac{1 + p^*_2}{2} + (p^*_2 - 1)z^*,$$

(37)

which corresponds to the expression (2). For comparison, let us consider the values of $J^*_M$ found by means of the equation (31) using the formulas (8) and (33) at $\delta_1 = 0.1$ and the linearized Boltzmann equation for collisionless gas in the channels with $R_1 = 0.1$ and $R_1 = 0.5$ at $p^*_2 = 10$, taking $J_M = (p^*_2 - 1)J_{M,p}$ into account in this case. For $R_1 = 0.1$, these values are $-11.3970$ and $-12.5010$, while the corresponding values for $R_1 = 0.5$ are $-6.7940$ and $-7.7922$, respectively.

It can be seen that for $\delta_1 = 0.1$ the obtained values is close to corresponding free molecular limits. From figures 3 and 5 it follows that for $\delta_1 = 0.1$ and $p^*_2 = 10$ the kinetic solutions (curves 1) almost coincide free molecular ones (lines 2).

In the hydrodynamic flow regime ($\delta^{-1} \ll 1$) the gas flow is described by the Stokes equation. The analytical solution of this equation for $U_z$ with the nonslip boundary conditions may be found in [10]

$$U_z(\rho) = \frac{\delta (\rho^2 - 1)}{4} \left( 1 - \frac{\ln \rho}{\ln R_1} \right),$$

(38)
Therefore the function $Z(\rho, C_\perp, \zeta)$ take the form

$$Z(\rho, C_\perp, \zeta) = \frac{\delta (\rho^2 - 1)}{4} \left( 1 - \frac{\ln \rho}{\ln R_1} \right). \quad (39)$$

Substituting (38) in (6), we obtain

$$J_{M,p} = \frac{\delta (R_1^2 (1 - \ln R_1) - 1 - \ln R_1)}{4 \ln R_1}. \quad (40)$$

The Stokes equation for $U_z$ with the slip boundary conditions [12]

$$U_z(R_i) = \frac{\sigma_p}{\delta} \frac{\partial U_z}{\partial n_{R_i}}(R_i), \quad i = 1, 2,$$

can also be solved analytically

$$U_z(\rho) = U_{H,z}(\rho) + \sigma_p U_{s,z}(\rho), \quad (42)$$

$$U_{s,z}(\rho) = \frac{(1 + 2R_1^2 \ln R_1 - R_1^2 + 2R_1 \ln R_1 - R_1^2 + R_1 \ln \rho) + 1}{4R_1 \ln^2 R_1} \frac{R_1^2 - 2 \ln R_1 - 1}{4 \ln R_1}. \quad (43)$$

Here, $n_{R_i}$ is the unit vector normal to cylinder with the radius $R_i$ directed towards the gas $(i = 1, 2)$, $U_{H,z}(\rho)$ is the hydrodynamic solution (38) and $U_{s,z}(\rho)$ is the perturbation of the mass velocity and $\sigma_p$ is the viscous slip coefficient. For the diffuse interaction of gas molecules the coefficient $\sigma_p$ is given by 1.016 [12]. Substituting (42) in (6) and using the notations (40) and (43), we have

$$J_M = \frac{\delta (R_1^2 (1 - \ln R_1) - 1 - \ln R_1)}{4 \ln R_1} + \sigma_p \frac{4R_1^3 \ln^2 R_1 - 4R_1^2 \ln^2 R_1 - 4R_1^2 \ln R_1 + R_1^3 + 4R_1 \ln^2 R_1 + 4R_1 \ln R_1 - 2R_1^2 + 1}{4R_1 (R_1 - 1) \ln^2 R_1} \quad (44)$$

In the figures 1 and 2, for $R_1 = 0.1$ and $R_1 = 0.5$, values of the slip mass flux $J_{M,p}$ shown by broken curves 3. By comparing the kinetic and the slip results for $R_1 = 0.5$, it is found that the relative error in the slip solution at $\delta = 5$ and $10$ is 12.5% and 7.2% respectively. Also, as $R_1$ is decreased the discrepancy of the slip solution is decreased. Substituting (44) in the differential equation (7) for $p^*$ with the boundary conditions $p^*(-1/2) = 1$ and $p^*(1/2) = p_2^*$, we can be solved analytically the boundary value problem. From figure 4 it can be seen that even for $\delta_1 = 1$ at $p_2^* = 10$ the agreement between the kinetic and slip solutions is good. This is due to the fact that for $R_1 = 0.1$ the Knudsen minimum of $J_{M,p}$ appears at about $\delta = 0.5$ (figure 1), while for $R_1 = 0.5$ the Knudsen minimum of $J_{M,p}$ appears at about $\delta = 1$ (figure 2). The value of $J_M^*$ calculated from the formula (8) for $R_1 = 0.1$ at $\delta_1 = 1$ and $p_2^* = 10$ is $-17.4534$ and corresponds value $-16.7507$ obtained in the slip regime, while the values for $R_1 = 0.5$ are $-7.6692$ and $-6.7123$, respectively.

The pressure distributions agree qualitatively with the corresponding ones obtained in [12] for microchannels of triangular and trapezoidal cross sections and in [2] for a long cylindrical channel. They approach linear distributions for small values of $\delta_1$ and then, as $\delta_1$ is increased, they become nonlinear.
4. Conclusion

The flow of rarefied gas in a long concentric annular channel due to an imposed pressure gradient has been investigated implementing a kinetic approach. The linearized BGK model kinetic equation with the diffuse boundary conditions has been solved by the method Chebyshev approximation using the discrete orthogonality relations and Kronecker and Hadamard products. Results are provided for the in the pressure distributions through a long concentric annular
channel. It has been found that in the transition regime ($\delta \geq 1$) the pressure distributions obtained on the basis of the BGK model of the kinetic equation at the large pressure drops at the channel ends approach the corresponding distributions in the slip regime. The validity and the accuracy of the kinetic results have been verified in several ways including the recovery of the well known solutions at the hydrodynamic and free molecular limits. The proposed methodology may be applied to other concentric channels, which are of some interest in several technological fields including nano- and micro-fluidics and vacuum technology.

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