A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Complex-valued harmonic functions that are univalent and sense-preserving in the open unit disc $U$ can be written in the form $f = h + \bar{g}$, where $h$ and $g$ are analytic in $U$. In this paper authors introduce the class, $R_{H}(\beta)$, $(1 < \beta \leq 2)$ consisting of harmonic univalent functions $f = h + \bar{g}$, where $h$ and $g$ are of the form $h(z) = z + \sum_{k=2}^{\infty} |a_k|z^k$ and $g(z) = \sum_{k=1}^{\infty} |b_k|z^k$ for which $\Re\{h'(z) + g'(z)\} < \beta$. We obtain distortion bounds extreme points and radii of convexity for functions belonging to this class and discuss a class preserving integral operator. We also show that class studied in this paper is closed under convolution and convex combinations.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [1].

Denote by $S_{H}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{ z : |z| < 1 \}$ for which $f(0) = f'(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_{H}$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \quad (1.1)$$

Note that $S_{H}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its member is zero.

A function $f$ of the form (1.1) is harmonic starlike for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) = \Re \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > 0, \quad |z| = r < 1. \quad \text{See [2].}$$

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Silverman [4] proved that the coefficient conditions \( \sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1 \) and \( \sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1 \) are sufficient conditions for functions \( f = h + \bar{g} \) to be harmonic starlike convex functions, respectively.

Denote by \( V_H \) the subclass of \( S_H \) consisting of functions of the form \( f = h + \bar{g} \), where

\[
h(z) = z + \sum_{k=2}^{\infty} |a_k|z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k|z^k, \quad |b_1| < 1. \tag{1.2}
\]

Recently Yalcin et al. [6] studied the class \( H P(\alpha) \), \( 0 \leq \alpha < 1 \) the subclass of \( S_H \) satisfying the condition

\[
\Re\{h'(z) + g'(z)\} > \alpha. \tag{1.3}
\]

Further let \( V_H P(\alpha) \) be the subclass of \( V_H \) consisting of functions of the form \( \tag{1.2} \) that satisfy condition \( \tag{1.3} \).

Let \( R_H(\beta), 1 < \beta \leq 2 \), denote the subclass of \( V_H \) satisfying the condition

\[
\Re\{h'(z) + g'(z)\} < \beta. \tag{1.4}
\]

We note that the class \( R_H(\beta) \) reduces to class \( R(\beta) \) if co-analytic part of \( f \) is zero i.e. \( g \equiv 0 \) studied by Uralegaddi et al. [5]. Yalcin et al. [6] have studied the functions with negative coefficients that satisfy \( \Re\{h'(z) + g'(z)\} > \alpha, 0 \leq \alpha < 1 \) for \( z \in U \). We need the following Lemma due to Theorem 2.1 of [6].

**Lemma 1.** Let \( f = h + \bar{g} \in V_H \) be given by \( \tag{1.2} \) and \( \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 1 - \alpha, 0 \leq \alpha < 1 \) then \( f \in V_H P(\alpha) \).

2. Main results

**Theorem 2.1.** A function \( f \) of the form \( \tag{1.2} \) is in \( R_H(\beta) \) if and only if

\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1. \tag{2.1}
\]

**Proof.** Let \( \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1 \). It suffices to prove that

\[
\frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)} < 1, \quad z \in U.
\]

We have

\[
\frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)} = \frac{\sum_{k=2}^{\infty} k|a_k|z^{k-1} + \sum_{k=1}^{\infty} k|b_k|z^{k-1}}{\sum_{k=2}^{\infty} k|a_k|z^{k-1} + \sum_{k=1}^{\infty} k|b_k|z^{k-1} - 2(\beta - 1)}
\]
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\[
\left| \sum_{k=2}^{\infty} k|a_k||z|^{k-1} + \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \right| \\
\leq \frac{2(\beta - 1) - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z|^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|}
\]

which is bounded above by 1 by hypothesis and the sufficient part is proved.

Conversely, suppose that
\[ \text{Re}\{h'(z) + g'(z)\} < \beta, \]
i.e.
\[ \text{Re}\left\{ 1 + \sum_{k=2}^{\infty} k|a_k|z^{k-1} + \sum_{k=1}^{\infty} k|b_k|z^{k-1} \right\} < \beta, \quad z \in U. \]

The above condition must hold for all values of \( z, \quad |z| = r < 1 \). Upon choosing the values of \( z \) to be real and let \( z \to 1^- \), we obtain
\[ \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1, \]
which gives the necessary part. The proof of the theorem is complete.

Next we determine bounds for the class \( R_H(\beta) \).

**Theorem 2.2.** If \( f \in R_H(\beta) \), then
\[ |f(z)| \leq (1 + |b_1|)r + \frac{1}{2}(\beta - 1 - |b_1|)r^2, \quad |z| = r < 1 \]
and
\[ |f(z)| \geq (1 - |b_1|)r - \frac{1}{2}(\beta - 1 - |b_1|)r^2, \quad |z| = r < 1. \]
The bounds are sharp for the functions \( f(z) = z + |b_1|z + \frac{1}{2}(\beta - 1 - |b_1|)z^2 \) and \( f(z) = z + |b_1|z + \frac{1}{(\beta - 1 - |b_1|)}z^2 \) for \( |b_1| \leq \beta - 1 \).

**Proof.** Let \( f \in R_H(\beta) \). Taking the absolute value of \( f \), we have
\[ |f(z)| \leq (1 - |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k \]
\[
\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2
\]
\[
\leq (1 + |b_1|)r + \frac{1}{2} \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^2
\]
\[
\leq (1 + |b_1|)r + \frac{1}{2} (\beta - 1 - |b_1|)r^2
\]

and

\[
|f(z)| \geq (1 + |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k
\]
\[
\geq (1 + |b_1|)r - \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^2
\]
\[
\geq (1 + |b_1|)r - \frac{1}{2} \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^2
\]
\[
\geq (1 + |b_1|)r + \frac{1}{2} (\beta - 1 - |b_1|)r^2.
\]

The functions \(z + |b_1| \bar{z} + \frac{1}{2} (\beta - 1 - |b_1|) z^2\) and \(z + |b_1| \bar{z} + \frac{1}{2} (\beta - 1 - |b_1|) z^2\) for \(|b_1| \leq \beta - 1\) show that the bounds given in Theorem 2.2 are sharp.

The following result follows from the left hand inequality in Theorem 2.2.

**Corollary 2.1.** If \(f \in R_H(\beta)\), then

\[
\{ \omega : |\omega| < \frac{1}{2} (3 - \beta - |b_1|) \} \subset f(U).
\] (2.2)

Next we determine the extreme points of the closed convex hulls of \(R_H(\beta)\), denoted by \(\text{clo} R_H(\beta)\).

**Theorem 2.3.** \(f \in \text{clo} R_H(\beta)\), if and only if

\[
f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k + \gamma_k g_k)
\] (2.3)

where \(h_1(z) = z\), \(h_k(z) = z + \frac{\beta - 1}{k} z^k (k = 2, 3, 4, \ldots)\), \(g_k(z) = z + \frac{\beta - 1}{k} \bar{z}^k (k = 1, 2, 3, \ldots)\) and \(\sum_{k=1}^{\infty} (\lambda_k + \gamma_k) = 1\), \(\lambda_k \geq 0\) and \(\gamma_k \geq 0\). In particular the extreme points of \(R_H(\beta)\) are \(\{h_k\}\) and \(\{g_k\}\).

**Proof.** For functions \(f\) of the form (2.3) write

\[
f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k + \gamma_k g_k)
\]
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\[ = z + \sum_{k=2}^{\infty} \left( \frac{\beta - 1}{k} \right) \lambda_k z^k + \sum_{k=1}^{\infty} \left( \frac{\beta - 1}{k} \right) \gamma_k \bar{z}^k. \]

Then
\[ \sum_{k=2}^{\infty} \frac{k}{\beta - 1} \left( \frac{\beta - 1}{k} \lambda_k \right) + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} \left( \frac{\beta - 1}{k} \gamma_k \right) \]
\[ = \sum_{k=2}^{\infty} \lambda_k + \sum_{k=1}^{\infty} \gamma_k \]
\[ = 1 - \lambda_1 \leq 1, \]
and so \( f \in \text{clco}R_H(\beta). \)

Conversely, suppose that \( f \in \text{clco}R_H(\beta). \) Set \( \lambda_k = \frac{k}{\beta - 1} |a_k|, (k = 2, 3, 4, \ldots) \) and \( \gamma_k = \frac{k}{\beta - 1} |b_k|, (k = 1, 2, 3, \ldots). \) Then note that by Theorem 2.1, \( 0 \leq \lambda_k \leq 1, (k = 2, 3, 4, \ldots) \) and \( 0 \leq \gamma_k \leq 1, (k = 1, 2, 3, \ldots). \) We define \( \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k \) and note that by Theorem 2.1, \( \lambda_1 \geq 0. \) Consequently, we obtain \( f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k + \gamma_k g_k) \) as required.

**Theorem 2.4.** If \( f \in R_H(\beta) \) then \( f \in V_H P(2 - \beta). \)

**Proof.** The inclusion relation is a direct consequence of Lemma 1 and Theorem 2.1.

Next we give the interrelation between the class \( R_H(\beta) \) and \( S_H^*, \) where \( S_H^* \) is the class of harmonic starlike function in \( U. \)

**Theorem 2.5.** \( R_H(\beta) \subseteq S_H^*, \) where \( 1 < \beta \leq 2. \)

**Proof.** Let \( f \in R_H(\beta). \) Then by Theorem 2.1
\[ \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \leq 1. \] (2.4)

Now
\[ \sum_{k=2}^{\infty} k |a_k| + \sum_{k=1}^{\infty} k |b_k| \]
\[ \leq \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \]
\[ \leq 1. \quad \text{[using (2.4)]} \]

Thus \( f \in S_H^*. \)

This completes the proof of Theorem 2.5.
Theorem 2.6. Each function in the class $R_H(\beta)$ maps a disks $U_r$ where $r < \inf_k \left\{ \frac{1}{k(\beta - 1 - |b_1|)} \right\}^{\frac{1}{\beta-1}}$ onto convex domains for $\beta > 1 + |b_1|$. 

Proof. Let $f \in R_H(\beta)$ and let $r$, be fixed is that $0 < r < 1$, then $r^{-1}f(rz) \in R_H(\beta)$ and we have

$$\sum_{k=2}^{\infty} k^2 |a_k| + |b_k| r^{k-1} = \sum_{k=2}^{\infty} |a_k| + |b_k| (kr^{k-1})$$

$$\leq \sum_{k=2}^{\infty} k |a_k| + |b_k|$$

$$\leq \beta - 1 - |b_1| \leq 1,$$

provided

$$kr^{k-1} \leq \frac{1}{\beta - 1 - |b_1|}$$

or, $r < \inf_k \left\{ \frac{1}{k(\beta - 1 - |b_1|)} \right\}^{\frac{1}{\beta-1}}$.

The proof of Theorem 2.6 is complete.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

and

$$F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k$$

we define their convolution

$$(f \ast F)(z) = f(z) \ast F(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k. \quad (2.5)$$

Using this definition, we show that the class $R_H(\beta)$ is closed under convolution.

Theorem 2.7. For $1 < \beta \leq \alpha \leq 2$ let $f \in R_H(\alpha)$ and $F \in R_H(\beta)$. Then $f \ast F \in R_H(\beta) \subseteq R_H(\alpha)$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ be in $R_H(\beta)$ and $F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k$ be in $R_H(\alpha)$. Then the convolution $f \ast F$ is given by (2.5). We wish to show that the coefficients of $f \ast F$ satisfy the required condition given in
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Theorem 2.1. For $F(z) \in R_H(\alpha)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f \ast F$, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k B_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k|$$

$$\leq 1. \quad \text{(Science } f \in R_H(\beta)).$$

Therefore $f \ast F \in R_H(\beta) \subseteq R_H(\alpha)$.

Next, we show that $R_H(\beta)$ is closed under convex combinations of its members.

**Theorem 2.8.** The class $R_H(\beta)$ is closed under convex combination.

**Proof.** For $i = 1, 2, 3, \ldots$ let $f_i(z) \in R_H(\beta)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_k i z^k + \sum_{k=1}^{\infty} b_k i \bar{z}^k.$$

Then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) \bar{z}^k.$$

Then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |b_k| \right)$$

$$= \sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \right)$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

Therefore

$$\sum_{i=1}^{\infty} t_i f_i(z) \in R_H(\beta).$$
The δ-neighborhood of \( f \) is the set
\[
N_\delta(f) = \left\{ F : F(z) = z + \sum_{k=2}^{\infty} |A_k|z^k + \sum_{k=1}^{\infty} |B_k|z^k \\
\text{and } \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \leq \delta \right\}.
\]
See [3].

**Theorem 2.9.** Let \( f \in R_H(\beta) \) and \( \delta \leq 2 - \beta \). If \( F \in N_\delta(f) \), then \( F \) is harmonic starlike function.

**Proof.** Let \( F(z) = z + \sum_{k=2}^{\infty} |A_k|z^k + \sum_{k=1}^{\infty} |B_k|z^k \) belong to \( N_\delta(f) \). We have
\[
\sum_{k=2}^{\infty} k|A_k| + \sum_{k=1}^{\infty} k|B_k| \leq \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |b_1 - B_1| + |b_1| \\
\leq \delta + \beta - 1 \leq 1.
\]
Hence, \( F(z) \) is harmonic starlike function.

### 3. A family of class preserving integral operator

Let \( f(z) = h(z) + g(z) \in S_H \) be given by (1.1) then \( F(z) \) defined by relation
\[
F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1}h(t)dt + \frac{c+1}{z^c} \int_0^z t^{c-1}g(t)dt, \quad (c > -1).
\]

**Theorem 3.1.** Let \( f(z) = h(z) + g(z) \in S_H \) be given by (1.2) and \( f(z) \in R_H(\beta) \) then \( F(z) \) be defined by (3.1) also belong to \( R_H(\beta) \).

**Proof.** Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k|z^k + \sum_{k=1}^{\infty} |b_k|z^k \) be in \( R_H(\beta) \) then by Theorem 2.1, we have
\[
\sum_{k=2}^{\infty} \frac{k}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1} |b_k| \leq 1. \quad (3.2)
\]
By definition of \( F(z) \), we have
\[
F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k|z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k|z^k.
\]
Now
\[
\sum_{k=2}^{\infty} \frac{k}{\beta-1} \left( \frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{k}{\beta-1} \left( \frac{c+1}{c+k} |b_k| \right)
\]
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\[ \leq \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \leq 1. \]

Thus \( F(z) \in R_H(\beta) \).

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