ONLY FINITELY MANY ALTERNATING KNOTS CAN YIELD A GIVEN MANIFOLD BY SURGERY

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Abstract. We show that given a 3-manifold \( Y \) there is only a finite number of alternating knots \( K \subset S^3 \) such that \( Y \) can be obtained by surgery on \( K \). A very similar but somewhat not complete statement has been recently obtained in [1, Theorem 1.3].

1. Introduction

Dehn surgery is a fundamental technique in 3-manifold topology. Indeed, we can construct any 3-manifold beginning with any other 3-manifold and performing Dehn surgery enough times. However, it is a highly non-trivial and widely open problem to understand what manifolds can be obtained by doing Dehn surgery once, even starting from the ‘simplest’ 3-manifold, namely \( S^3 \). It is also a very difficult problem to determine which knots can produce a given manifold by surgery.

It turns out that there are manifolds which are Dehn surgeries on infinitely many distinct knots (see [2] or [3]). There is still hope, however, that perhaps this does not happen for some nice classes of knots.

One interesting and well-studied class of knots is that of alternating knots. At first sight their diagrammatic definition seems to have little to do with the geometric-topological properties of these knots. However, this is not so - see [1] and references therein. In particular, the following is [1, Theorem 1.3]

**Theorem 1** (Lackenby-Purcell). For any closed 3-manifold \( M \) with sufficiently large Gromov norm, there are at most finitely many prime alternating knots \( K \) and fractions \( p/q \) such that \( M \) is obtained by \( p/q \) surgery along \( K \).

In fact, the statement about fractions \( p/q \) can be deduced, for example, from [4]. The author has also demonstrated in [5] that given any manifold \( Y \) there is a universal bound on \( q \) for such fractions, which also implies that they are finite in number. Using techniques that are very different from those used in [1] we are able to establish the following improvement of this theorem.

**Theorem 2.** Let \( Y \) be a 3-manifold. There are at most finitely many alternating knots \( K \subset S^3 \) such that \( Y = S^3_{p/q}(K) \).

Our proof uses Heegaard Floer homology and the mapping cone formula for the Dehn surgeries - see [6] (or [4] for a very accessible exposition).

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1In this paper, whenever we say ‘3-manifold’ we mean ‘closed connected orientable 3-manifold’.
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2. Background

In this section, we set up notation and list certain properties coming from Heegaard Floer homology. For references and a more detailed discussion, please refer to [5].

Heegaard Floer homology is a powerful 3-manifold invariant, first defined in [7]. There are different ‘flavours’ of Heegaard Floer homology — we will be mainly using the + version. It assigns to a 3-manifold $Y$ equipped with a Spin$^c$-structure $s$ an absolutely $\mathbb{Z}/2\mathbb{Z}$-graded Abelian group $HF^+(Y,s)$. In this paper we use coefficients in the field with two elements $\mathbb{F}_2$, thus all homology groups are in fact $\mathbb{F}_2$-modules.

Connection with surgeries is obtained via knot Floer homology defined in [8] and independently in [9]. While there are actual homology theories associated to a knot, even more important is the doubly filtered complex $C = CFK\infty(K)$, the doubly filtered chain homotopy type of which is an invariant of $K$. Indeed, all knot Floer homologies are obtained from taking homologies of quotients and sub complexes of $C$.

We can form the quotient complex

$$A^+_k(K) = C \{ i \geq 0 \text{ or } j \geq k \} = C/Q,$$

where $Q = C \{ i < 0 \text{ and } j < k \}$, which means the subcomplex of $C$ generated by the elements filtration level $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ of which satisfies the condition in curly brackets. We denote by $A^+_k(K)$ the homology of $A^+_k(K)$.

Let $T^+ = \mathbb{F}_2[U,U^{-1}]/U \cdot \mathbb{F}_2[U]$. Elements of $T^+$ are simply polynomials in the variable $U^{-1}$, $\mathbb{F}_2[U]$ acts on $T^+$ by multiplication and positive powers of $U$ are set equal to zero.

It turns out that the groups $A^+_k(K)$ can be decomposed as

$$A^+_k(K) = A^+_k(K) \oplus A^{red}_k(K),$$

where $A^+_k(K) \cong T^+$ and $A^{red}_k(K)$ is a finite dimensional vector space in the kernel of some power of $U$.

Since $C$ is relatively $\mathbb{Z}/2\mathbb{Z}$-graded (in fact it is $\mathbb{Z}$-graded and the $\mathbb{Z}/2\mathbb{Z}$ grading we are referring to is the reduction modulo 2 of this $\mathbb{Z}$-grading), we can endow $A^+_k(K)$ with relative $\mathbb{Z}/2\mathbb{Z}$-grading. Furthermore $A^+_k(K)$ is all concentrated in the same grading (in fact, the map $U$ is homogeneous and does not change this grading). We can now fix an absolute grading on $A^+_k(K)$ (and thus on $A^{red}_k(K)$) by requiring that $A^+_k(K)$ is in grading 0.

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We can also associate a sequence \( \{V_k\}_{k \geq 0} \) of non-negative numbers to a knot \( K \subset S^3 \). This sequence is defined as follows: let \( B^+ = C\{i \geq 0\} \). Then the homology of \( B^+ \) is \( B^+ \cong T^+ \). Define the chain map \( v_k : A^+_k(K) \to B^+ \) by projection, and denote the map induced by \( v_k \) in homology by \( v_k \). The restriction of \( v_k \) to \( A^+_k(K) \cong T^+ \) turns out to be multiplication by a power of \( U \), and we denote this power by \( V_k \).

The numbers \( V_k \) have the following properties:

- \( V_k \geq V_{k+1} \);
- \( V_k = 0 \) for \( k \geq g(K) \), where \( g(K) \) is the genus of \( K \).

If \( Y \) is a rational homology sphere and \( s \) is a Spin\(^c\)-structure on \( Y \), we can associate a rational number \( d(Y, s) \) to it, termed its \( d \)-invariant (in Spin\(^c\)-structure \( s \)). Let \( L(p, q) \) denote the lens space obtained by \( p/q \)-surgery on the unknot.\(^3\)

If \( Y = S^3_{p/q}(K) \) for \( p \neq 0 \), then it has \( p \) Spin\(^c\)-structures which can be numbered simply by \( \{0, 1, \ldots, p-1\} \), as in [6]. It is not important for us here how this numbering is constructed. From now on we will denote the Spin\(^c\)-structures simply by numbers in this range.

Let \( p > 0, q > 0 \). In [4] Ni and Wu prove the following formula:

\[
d(S^3_{p/q}(K), i) = d(L(p, q), i) - 2 \max \{V_{\lfloor \frac{i}{p} \rfloor}, V_{-\lfloor \frac{i-1}{q} \rfloor} \}.
\]  

For a knot \( K \subset S^3 \), let \( \Delta_K \) denote its symmetrised Alexander polynomial. That is, it is required to satisfy \( \Delta_K(1) = 1 \) and \( \Delta_K(T) = \Delta_K(T^{-1}) \). Suppose

\[
\Delta_K(T) = a_0 + \sum_{i \geq 0} a_i(T^i + T^{-i}).
\]

For \( i \geq 0 \) we define the torsion coefficients of \( K \) by

\[
t_i(K) = \sum_{j \geq 0} ja_{i+j};
\]

it is straightforward to verify that

\[
a_i = t_{i-1}(K) - 2t_i(K) + t_{i+1}(K).
\]  

In [5 Lemma 21] the following equation was shown:

\[
t_k(K) = V_k + \chi(A^+_k(K)),
\]

where for taking the Euler characteristic we use the absolute \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( A^+_k(K) \) discussed above.

### 3. Proof of Theorem 2

The strategy of our proof is as follows. We first want to restrict the possible Alexander polynomials of knots that yield a given 3-manifold \( Y \) by surgery. We then want to show, that out of this restricted set, only finitely many can be Alexander polynomials of alternating knots. This will finish the proof, due to the next Proposition, which can be found in [10 Proposition 14]. We provide the proof for the reader’s convenience (and since it is nice and short).

**Proposition 3** (Moore-Starkston). There is only a finite number of alternating knots with a given Alexander polynomial.
Proof. By the Bankwitz Theorem [11, Theorem 5.5] the determinant \( \det(K) \) of an alternating knot \( K \) is greater than or equal to the minimal crossing number of \( K \). Thus there are only finitely many alternating knots with a given determinant. The classical result [12, page 213] (or definition) \( \det(K) = |\Delta_K(-1)| \) finishes the proof. \qed

For a knot \( K \subset S^3 \), let \( m(K) \) be its mirror image. Clearly, \( K \) is alternating if and only if \( m(K) \) is. Since \( S^3_{p/q}(K) = -S^3_{-p/q}(m(K)) \) we can assume that the surgery slope is positive (if non-zero).

For \( Y \) a rational homology sphere and \( q > 0 \) a natural number define

\[
M(Y, q) = \frac{1}{2} \left( \sum_{0 \leq i \leq p-1} d(L(p, q), i) - \sum_{s \in Spin^c(Y)} d(Y, s) \right),
\]

where \( p = |H_1(Y)| \).

In [5, Theorem 5], it is shown that for any rational homology sphere \( Y \) there is some number \( n(Y) \) such that \( Y \neq S^3_{p/q}(K) \) for any \( K \) and \( |q| > n(Y) \).

If \( Y \) is obtained by \( p/q > 0 \) surgery on \( K \), then by (1) the numbers \( V_k \) for \( K \) satisfy

\[
M(Y, q) = \sum_{i=0}^{p-1} \max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{i-p}{q} \rfloor}\}.
\]

To a rational homology sphere \( Y \) and a Spin\(^c\)-structure \( s \) on it we can associate a homology group \( HF_{red}(Y, s) \), called the reduced Floer homology of \( Y \) in Spin\(^c\)-structure \( s \). For each Spin\(^c\)-structure, the reduced Floer homology is a finite dimensional vector space. The sum over all Spin\(^c\)-structures is called the reduced Floer homology of \( Y \) and is denoted

\[
HF_{red}(Y) = \bigoplus_{s \in Spin^c(Y)} HF_{red}(Y, s).
\]

Ni and Zhang calculate the total dimension of \( HF_{red}(S^3_{p/q}(K)) \) for \( p/q > 0 \) in [13, Corollary 3.6]. The same calculation is performed in [5, Proposition 14] with the same notation as used in this paper (in the formula of [5, Proposition 14] constants \( H_k \) with \( k \leq 0 \), not defined in this paper, are used — for purposes of this paper replace \( H_k \) with \( V_{-k} \)). Combining this formula with our definitions here, we obtain

\[
\dim(HF_{red}(S^3_{p/q}(K))) + M(S^3_{p/q}(K), q) = q(\delta(K) + V_0 + 2 \sum_{i \geq 1} V_i),
\]

where \( \delta(K) = \dim(A^+_0(K)) + 2 \sum_{k \geq 1} \dim(A^+_k(K)) \). This formula implies the following inequality:

\[
\frac{\dim(HF_{red}(S^3_{p/q}(K))) + M(S^3_{p/q}(K), q)}{q} \geq \sum_{k \geq 0} (V_k + \dim(A^+_k(K))).
\]

Now let

\[
e(Y) = \max_{1 \leq q \leq n(Y)} \left\{ \frac{\dim(HF_{red}(Y)) + M(Y, q)}{q} \right\}.
\]
The inequality above implies, that if a rational homology sphere $Y$ is obtained by surgery on a knot $K$ with associated sequence $\{V_k\}_{k \geq 0}$, then
\[
c(Y) \geq \sum_{k \geq 0} (V_k + \dim(A_k^+(K))). \tag{4}
\]

**Lemma 4.** Suppose $Y$ is a rational homology sphere obtained by a $p/q > 0$ surgery on a knot $K \subset S^3$. Then
\[
\sum_{i \geq 0} |t_i(K)| \leq c(Y). \tag{5}
\]

**Proof.** It follows from (3) that for each $k \geq 0$
\[
|t_k(K)| = |V_k + \chi(A_k^+(K))| \leq V_k + |\chi(A_k^+(K))| \leq V_k + \dim(A_k^+(K)).
\]
Combining with equation (4) yields the result. \qed

Let $S_Y$ be some set of knots in $S^3$ that give a rational homology sphere $Y$ by surgery (not necessarily all such knots and not necessarily alternating). Denote by $g(S_Y)$ ($\Delta(S_Y)$) respectively the set of genera (Alexander polynomials) of knots in $S_Y$.

**Lemma 5.** If $g(S_Y)$ is finite, then so is $\Delta(S_Y)$.

**Proof.** We clearly have $t_i(K) = 0$ for all $K \in S_Y$ and all $i \geq \max(g(S_Y))$. By Lemma (3) $\sum_{i \geq 0} |t_i(K)|$ is bounded above, so we clearly have finitely many sequences $\{t_i(K)\}$ for $K \in S_Y$. Now observe that by equation (2) (and the fact that $\Delta_K(1) = 1$), the torsion coefficients determine the Alexander polynomial, so this results in at most finitely many possible Alexander polynomials. \qed

A theorem of Murasugi [14, Theorem 1.1] is crucial for our proof:

**Theorem 6** (Murasugi). Let $K \subset S^3$ be an alternating knot and
\[
\Delta_K(T) = a_0 + \sum_{i=1}^{g(K)} a_i(T^i + T^{-i})
\]
be its Alexander polynomial. Then $a_i \neq 0$ for $0 \leq i \leq g(K)$.

The next Lemma is the last step before we can prove Theorem 2.

**Lemma 7.** Let $K \subset S^3$ be an alternating knot that gives a rational homology sphere $Y$ by surgery. Then
\[
g(K) \leq 3c(Y).
\]

**Proof.** Suppose $g(K) \geq 3c(Y) + 1$. Note that $a_g = t_{g-1}(K) \neq 0$. We claim that there are three consecutive indices $i, i+1$ and $i+2 \leq g$ with $t_i(K) = t_{i+1}(K) = t_{i+2}(K) = 0$. It then follows by (2) that $a_{i+1} = 0$, which is a contradiction to Theorem 6.

To prove the claim suppose there is no such consecutive triple of zero torsion coefficients. Then
\[
\sum_{i \geq 0} |t_i(K)| = \sum_{k \geq 0} (|t_{3k}(K)| + |t_{3k+1}(K)| + |t_{3k+2}(K)|) \geq \left\lfloor \frac{g-1}{3} \right\rfloor + 1 \geq c(Y) + 1,
\]
which contradicts Lemma (4).

We have thus established that $g \leq 3c(Y)$. \qed
Proof of Theorem 2. Suppose $Y$ is a rational homology sphere. Then by Lemma 7 there is a genus bound for alternating knots that give $Y$ by surgery, so by Lemma 5 the set of Alexander polynomials of such alternating knots is finite.

If $Y$ is obtained by 0-surgery on $K$, then Propositions 10.14 and 10.17 of [15] show that the Alexander polynomial of $K$ can be deduced directly from the Heegaard Floer homology of $Y$.

Proposition 3 now finishes the proof. □

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