Research Article

Global Finite-Time and Fixed-Time Synchronization for Discontinuous Complex Dynamical Networks with Semi-Markovian Switching and Mixed Delays

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This paper is concerned with the global finite-time and fixed-time synchronization for a class of discontinuous complex dynamical networks with semi-Markovian switching and mixed time-varying delays. The novel state-feedback controllers, which include integral terms and discontinuous facts, are designed to realize the global synchronization between the drive system and response system. By applying the Lyapunov functional method and matrix inequality analysis technique, the global finite-time and fixed-time synchronization conditions are addressed in terms of linear matrix inequalities (LMIs). Finally, two numerical examples are provided to illustrate the feasibility of the proposed control scheme and the validity of theoretical results.

1. Introduction

In the past ten years, complex dynamical networks (CDNs) have aroused a great attention owing to their extensive applications in many areas such as the Internet, biological system, the World Wide Web, power grid networks, and so on [1–3]. CDNs consist of many subnetworks, which are called nodes, and each of the nodes has the similar functions. Different connection modes between nodes form different networks. Thus, such a complexity makes that the study of CDNs become a hot topic. Up to now, many significative works with respect to CDNs are reported in the literature, see [4–10] and references therein. It should be mentioned that the analysis of dynamical behaviors of CDNs has become a research hotspot in recent years.

The global synchronization, as a dynamical behavior, is the most important. It means the dynamical behavior of the network systems achieve the identical state in time-spatial. Nowadays, the considerable attention is being devoted to the analysis of the global synchronization of CDNs, and some effective synchronization criteria with respect to CDNs have been established in the existing papers [11–15]. In [11], the synchronization problem of complex dynamical networks was addressed under the way of impulsive control. Reference [12] introduced the synchronization problem of complex dynamical networks with switching topology via adaptive control. Also, global synchronization of complex dynamical networks with nonidentical nodes was illustrated in Ref. [13]. Moreover, synchronization of complex dynamical networks with nonidentical nodes was illustrated in Ref. [14], and the obtained result global synchronization for discrete-time stochastic complex networks in Ref. [15] was complex, which has randomly shown nonlinearities and mixed time delays. To the best of our knowledge, most of the existing works with regard to the synchronization of CDNs is taking the continuous networks system into account. But, in practice, it is not applicable to all the synchronization conditions. Thus, the study of discontinuous CDNs has attracted the interests of many scholars. Reference [16] investigated the finite-time synchronization of complex networks with nonidentical discontinuous nodes. Also, exponential synchronization of a discontinuous complex-
valued complex dynamical network is realized in [17]. Reference [18] illustrated the synchronization of time-delayed complex dynamical networks with discontinuous coupling.

As we all know that, among the different types of synchronization, global finite-time synchronization is one of the most optimal synchronization types, only the maximum synchronization time for global finite-time synchronization can be computed. Moreover, some efforts have been spent on solving the global finite-time synchronization of discontinuous CDNs. The finite-time synchronization of chaotic complex networks with stochastic disturbance was obtained in [19]. In Ref. [20], the finite-time hybrid projective synchronization of the drive-response complex networks with distributed delay was considered by the researchers, based on the adaptive intermittent control. Reference [21] introduces finite-time cluster synchronization of Markovian switching complex networks with stochastic perturbations. In Ref. [22], the finite-time synchronization of Markovian jump complex networks with partially unknown transition rates was introduced by the author. It is also necessary to point out that the settling time of finite-time synchronization heavily depends on the initial conditions which would result in different convergence rates under different initial conditions. But, the initial conditions may be ineffective or invalid in practice application. For overcoming these difficulties, a new concept named fixed-time synchronization is firstly taken into consideration in [23]. Then, the future study on fixed-time synchronization can be found in [24,25]. By means of the sliding mode control technique, the fixed-time synchronization of complex dynamical networks is realized in [24]. In [25], the fixed-time cluster synchronization for complex networks is derived under the pinning control scheme. Furthermore, in [25], accompanying with nonidentical nodes and stochastic noise perturbations, the fixed-time synchronization of complex networks was investigated. Also, the fixed-time synchronization of hybrid coupled networks was addressed in Ref. [26].

By adding the Markovian process into the network systems of CDNs, a new network model is developed. Up to now, the study concerning synchronization of CDNs with Markovian switching, especially the global finite-time synchronization, has received wide attention from the scholars, and many efforts have been made for analyzing the synchronization of CDNs with Markovian switching, see [21, 27–30]. Since the sojourn time of the Markovian process obeys exponential distribution, it results in the transition rate to be a constant. However, the sojourn-time of the semi-Markovian process can obey to some other probability distributions, such as Weibull distribution, Gaussian distribution, and so on. Therefore, the investigation for the global finite-time and fixed-time synchronization of CDNs with semi-Markovian switching is of great theoretical value and application value. In [31], the finite-time H∞ synchronization for complex networks with semi-Markov jump topology was addressed. The finite-time H∞ control for linear systems with semi-Markovian switching was discussed in [32–34]. The finite-time nonfragile synchronization of stochastic complex dynamical networks with semi-Markov switching outer coupling was investigated in Ref. [35]. The global fixed-time synchronization of CDNs with semi-Markovian switching is relatively complex. In addition, it is also difficult to apply the semi-Markovian switching to complex dynamic network (CDNs) and use appropriate control methods to derive the corresponding global synchronization conditions. As a result, the global fixed-time synchronization of CDNs with semi-Markovian switching is seldom studied before.

Motivated by the abovementioned discussions, we intend to consider the global finite-time and fixed-time synchronization for discontinuous CDNs with semi-Markovian switching and mixed time-varying delays. The main contributions of this paper, different from the existing works, can be summarized as follows:

1. The novel discontinuous state-feedback controllers, which include integral terms, are designed
2. The mixed delay term with integral and discontinuous function is handled by the approach of the contraction of inequalities
3. The mixed delay and semi-Markovian switching are introduced in the construction of the discontinuous CDNs model
4. The global finite-time and fixed-time synchronization conditions are addressed in terms of LMIs

The rest of this article is arranged as follows. Some preliminaries and model description are described in Section 2. In Section 3, we introduce the main results: finite time and fixed-time synchronization with different nonlinear controllers. In Section 4, two examples are presented which are in order to show the correctness of our main results. In Section 5, also the last part, the conclusion of this paper is given.

Notation: $I_n$ denotes the identity matrix with proper dimension. $R^n$ represents the sets of real numbers. $R^{n\times n}$ denotes the set of all $n \times n$ matrices, and $R^n$ denotes the n-dimensional Euclidean space. $\|x\|$ stands for the Euclidean norm of the vector $x \in R^n$, and $\|x\| = \sqrt{x^T x}$. $D = (d_{ij})_{n \times n}$ denotes an $n \times n$-dimension matrix $[D] = \sqrt{\lambda_{\max} (D^T D)}$, and the superscript $T$ means transposition. For all the matrices, $\lambda_{\min} (\cdot)$ and $\lambda_{\max} (\cdot)$ stand for the minimum and the maximum eigenvalue of the matrices, respectively. $X < Y (X > Y)$, where $X$ and $Y$ are both symmetric matrices, means that $X - Y$ is negative (positive) definite. $B = (b_{ij})_{n \times n}$ and $|B| = (|b_{ij}|)_{n \times n}$; $E$ stand for mathematical expectation. $L' = (1/2) (L + L^T)$, $\Gamma V (x(t), r(t), t)$ denotes the infinitesimal generator of $V (x(t), r(t), t)$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operation.

2. Preliminaries and Model Description

2.1. Preliminaries. Let $(\Theta, F, [F]_{t \geq 0}, \mathcal{F})$ be the complete probability space with filtration $F_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while
Definition 2. (See [37]). The set-valued map of Definition 3 is said to satisfy the basic conditions in the domain $W$ if it is $C$-regular if it is

\begin{align}
P(r(t + h) = k|r(t) = s) = \begin{cases} 
\pi_{sk} (h) + o(h), & \text{if } s \neq k, \\
1 + \pi_{ss} (h) + o(h), & \text{if } s = k,
\end{cases}
\end{align}

where $h > 0$, $\lim_{h \to 0} \pi_{sk}(h)/h = 0$, $\pi_{sk}(h) \geq 0$ for any state or mode, and it satisfies

\begin{equation}
\pi_{sk} (h) = - \sum_{k=1}^{N} \pi_{sk} (h).
\end{equation}

**Remark 1.** In practice, the transition rate $\pi_{sk}(h)$ is general bounded, i.e., $\sum_{sk} \leq \pi_{sk}(h) \leq \bar{\omega}_{sk}$, where $\sum_{sk}$ and $\bar{\omega}_{sk}$ are real constant scalars. Then, $\pi_{sk}(h) = \bar{\omega}_{sk} \pm \Delta \pi_{sk}$, where $\pi_{sk} = 1/2(\pi_{sk} + \bar{\omega}_{sk})$ and $|\pi_{sk}| \leq \bar{\omega}_{sk}$ with $\bar{\omega}_{sk} = 1/2(\pi_{sk} - \bar{\omega}_{sk})$.

**Definition 1.** The complex dynamical networks model (17) is said to be synchronized with the networks model (18), if for any initial condition, we have

\begin{equation}
\lim_{t \to t_i} E\left(\|x_i(t) - y_i(t)\|\right) = 0,
\end{equation}

for $t > t_i$, $i = 1, 2, \ldots, N$.

**Definition 2 (See [36]).** The Filippov set-valued map of $f_i(x)$ at $x \in R^n$ is defined as follows:

\begin{equation}
F(x) = \cap_{\mathcal{F} > 0} \cap_{\mu(N) = 0} \mathcal{C}o\left(\{ f(B(x, \mathcal{F}) \cap \Pi)\right),
\end{equation}

where $\mathcal{C}o[E]$ is the closure of the convex hull of the set $E$, $B(x, \mathcal{F}) = \{ y : \| y - x \| \leq \mathcal{F} \}$, and $\mu(\Pi)$ is the Lebesgue measure of set $\Pi$.

**Definition 3 (See [37]).** The set-valued map of $F : R^n \to R^n$ is said to satisfy the basic conditions in the domain $W \in R^n$, if for any $x \in W$, $F(x)$ is upper semicontinuous in $W$.

**Definition 4 (See [38]).** A function $V(x) : R^n \to R$ is $C$-regular if it is

(1) regular in $R^n$

(2) positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$

(3) radially unbounded, i.e., $V(x) \to +\infty$, as $\|x\| \to \infty$

$S = \{1, 2, \ldots, N\}$. The evolution of the semi-Markovian process $r(t)$ is governed by the following probability transitions:

\begin{equation}
W + W^T \leq \epsilon W + \epsilon^{-1} U^T W^T.
\end{equation}

**Lemma 1 (See [39]).** Given any scalar $\epsilon$ and matrix $W \in R^{n \times n}$, there exists symmetric positive definite matrices $U \in R^{n \times n}$, such that

\begin{equation}
W + W^T \leq \epsilon W + \epsilon^{-1} U^T W^T.
\end{equation}

**Lemma 2 (See [40]).** Let $\xi_1, \xi_2, \ldots, \xi_{n} \geq 0$, $0 < s < \omega, r > 1$. Then, the following two inequalities hold:

\begin{equation}
\sum_{i=1}^{n} \xi_i^{(1/s)} \leq \sum_{i=1}^{n} \xi_i^{1/s},
\end{equation}

\begin{equation}
\sum_{i=1}^{n} \xi_i^{(1/r)} \leq \left( \sum_{i=1}^{n} \xi_i \right)^{r}. \quad \sum_{i=1}^{n} \xi_i^{(1/r)} \leq \left( \sum_{i=1}^{n} \xi_i \right)^{r}.
\end{equation}

**Lemma 3.** Let $P \in R^{n \times n}$ be a symmetric matrix, and let $x \in R^n$; then, we have the following inequality:

\begin{equation}
\lambda_{min}(P)x^T x \leq x^T Px \leq \lambda_{max}(P)x^T x.
\end{equation}

**Lemma 4 (See [41]).** Given constant matrices $\Phi_1, \Phi_2,$ and $\Phi_3$, where $\Phi_3 = \Phi_2^T$ and $\Phi_2 > 0$, we have

\begin{equation}
\Phi_1 + \Phi_2 \Phi_3 \Phi_2^{-1} \Phi_3 < 0,
\end{equation}

if and only if

\begin{equation}
\begin{bmatrix} \Phi_1 & \Phi_2^T \\ \Phi_3 & -\Phi_2 \end{bmatrix} \leq 0.
\end{equation}

**Lemma 5 (See [42]).** Let $y_1, y_2, \ldots, y_n \in R^n$ be any vectors and $0 < r < 2$ be a real number which satisfies the following condition:

\begin{equation}
\|y_1\|^r + \|y_2\|^r + \cdots + \|y_n\|^r \geq \left( \|y_1\|^2 + \|y_2\|^2 + \cdots + \|y_n\|^2 \right)^{r/2}.
\end{equation}

2.2. Model Description. The model we consider in the present paper is the complex dynamical network model which consists of $N$ identical nodes with semi-Markovian switching in the probability space $(\Omega, F, \{F\}_{h=0}, \mathcal{F})$. The dynamics of $i$th nodes of drive systems is described by the following differential equations:
\[ \dot{x}(t) = D(r(t))x(t) + A(r(t))f_i(x(t)) + B(r(t))u_i(t) \]
\[ + \sum_{j=1}^{N} L_{ij}(r(t))Ax_j(t) + c_i(r(t)) \int_{t-d(t)}^{t} f_i(x(s))ds, \]
\[ \text{with the initial condition } x(t_0) = \psi(t_0, r(t_0)), -\tau \leq t \leq 0. \]

Also, the corresponding response system is
\[ \dot{y}(t) = D(r(t))y(t) + A(r(t))f_i(y(t)) + B(r(t))u_i(t) \]
\[ + \sum_{j=1}^{N} L_{ij}(r(t))Ay_j(t) + c_i(r(t)) \int_{t-d(t)}^{t} f_i(y(s))ds + u_i(t), \]
\[ \text{with the initial condition } y(t_0) = \chi(t_0, r(t_0)), -\tau \leq t \leq 0. \]

Here, \( \{r(t), t \geq 0\} \) is the continuous-time semi-Markovian process, and \( r(t) \) describes the evolution of the mode at time \( t \). \( x(t) \in \mathbb{R}^n, \dot{x}(t) \in \mathbb{R}^n \) denotes the state vector of \( ith \) network at time \( t \); \( D(r(t)) \in \mathbb{R}^{nnx} \) is a positive definite diagonal matrix; \( A(r(t)) \in \mathbb{R}^{nnx} \) and \( B(r(t)) \in \mathbb{R}^{nnx} \) are matrices with real values in mode \( r(t) \); \( N \) is the number of coupled nodes; \( f_i(x) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))) \in \mathbb{R}^n \) is the nonlinear vector-valued function; and \( u_i(t) \in \mathbb{R}^n \) is the control input of the \( ith \) node to be determined. \( c_i(r(t)) \) represents the distributively delayed connection weight; the diagonal matrix \( \Lambda > 0 \) represents the inner-coupling matrix of the complex networks; and \( \{L_{ij}\}_nnx \) stands for the coupling configuration matrix which represents the topological structures of the considered networks. For matrix \( L_{ij}(r(t)) \), if there is a connection from node \( j \) to \( i (j \neq i) \), then \( L_{ij}(r(t)) > 0 \); otherwise, \( L_{ij}(r(t)) = 0 (j \neq i) \).

\[ L_{ij}(r(t)) = -\sum_{j=1,j\neq i}^{N} L_{ij}(r(t)), i = 1, 2, \ldots, N. \]

The function \( d(t) \) and \( d(t) \) denote the discrete delay and distributed delay and satisfy
\[ 0 < \tau(t) \leq \tau, \]
\[ 0 < d(t) \leq d, \]
\[ 0 < t(t) \leq r, \quad 0 < d(t) \leq \omega, \]

where \( \tau > 0, d > 0 \) and \( \omega < 1, r < 1 \) are some known constants.

Throughout this paper, we list the following assumptions:
\[ (H_1): F_i(x_i), i = 1, 2, \ldots, N \text{ satisfy the general conditions defined in Definition 3, where} \]
\[ F_i(x_i) = \cap_{\tau > 0} \{ \| (u_N) \|_\infty \text{ is measurable} \}; \]
\[ (H_2): \text{there exists two positive constants } \alpha_i \text{ and } \beta_i, i = 1, 2, \ldots, N, \text{ and then, the following inequalities hold:} \]
\[ |y(t) - y(t)| \leq \alpha_i |y_i(t) - x_i(t)| + \beta_i, \]
\[ \text{where } y(t) \in \mathcal{O}[f_i(x_i(t))], \quad \text{and} \quad \gamma(t) \in \mathcal{O}[f_i(y_i(t))], \]
\[ i = 1, 2, \ldots, N \]

For notation simplicity, we replace \( D(r(t)), A(r(t)), \) and \( B(r(t)) \) with \( D_s, A_s, \) and \( B_s \) for \( r(t) = s \in S \). Then, the complex dynamical systems can be rewritten as follows:
\[ \dot{x}(t) = D_sx(t) + A_s\mathcal{O}[f_i(x_i(t))] + B_s\mathcal{O}[f_i(x_i(t) - \tau(t))] \]
\[ + \sum_{j=1}^{N} L_{ij}Ax_j(t) + c_i(r(t)) \int_{t-d(t)}^{t} \mathcal{O}[f_i(x(s))]ds, \]
\[ x(t_0) = \overline{\psi}(t_0), -\tau \leq t \leq 0, \]

where \( \overline{\psi}(t) \) is a measurable function and \( \overline{\psi}(t) \in \mathcal{O}[f_i(\chi_i(t))] \).

The network model (17) is a differential equation with a discontinuous right-hand side, and the traditional definition of the solution is not applicable here. Hence, we introduce the Filippov solution for system (17).

\[ \text{Definition 5 (See [43]). A function } x: [-\tau, T] \rightarrow \mathbb{R}^n, \]
\[ T \in (0, +\infty) \text{ is a solution (in the sense of Filippov) of the discontinuous system (17) on } [-\tau, T] \text{ if} \]
\[ (i) x_i(t) \text{ is absolutely on } [-\tau, T] \]
\[ (ii) \text{There exists a measurable function } y(t) = (y_1(t), y_2(t), \ldots, y_N(t)) \in \mathbb{R}^n, \text{ such that } y(t) \in \mathcal{O}[f_i(x_i(t))], \text{ for a.a.t } t \in (0, T) \]

(19)
In this paper, the error dynamics system between drive-response system (17) and (18) attracts us. Hence, let \( e_i(t) = y_i(t) - x_i(t) \) be the synchronization error system, and it can be expressed as

\[
\begin{align*}
\dot{e}_i(t) &= D_i e_i(t) + A_i \bar{f}_i(e_i(t)) + B_i \bar{f}_i(e_i(t - \tau(t))) \\
&+ \sum_{j=1}^{n} L_{ij} \beta_i e_j(t) + c_i(r(t)) \int_{t-d(t)}^{t} \bar{f}_i(e_i(s)) \, ds + u_i(t), \\
\end{align*}
\]

where \( \bar{f}_i(e_i(t)) = y_i(t) - \bar{y}_i(t) \in \text{co}[f_i(y_i(t))] - \text{co}[f_i(x_i(t))], \)

\( \bar{f}_i(e_i(t - \tau(t))) = y_i(t - \tau(t)) - y_i(t - \tau(t)) \in \text{co} \left[ f_i(y_i(t - \tau(t))) \right] - \text{co} \left[ f_i(x_i(t - \tau(t))) \right], \)

and \( e_i(t) = \bar{y}_i(t) - \bar{y}_i(t), \) \( -\tau \leq t \leq 0, \)

\( (20) \)

Then the origin is global fixed-time stable for (20), and the estimated value of the settling time function satisfies

\[
T(e_0) \leq T_{max} = \frac{\pi \rho}{\sqrt{pq}}, \quad \forall e_0 \in \mathbb{R}^n.
\]

3. Main Results

3.1. Global Finite-Time Synchronization. Based on the linear matrix inequalities technique [45] and the Lyapunov functional method [38, 46], this section aims to develop some new conditions which can ensure the synchronization between complex dynamical network models (17) and (18) over a finite-time interval.

For the purpose of global finite-time synchronization, a state-feedback controller is designed as follows:

\[
u_i(t) = \left( \xi_i(r(t)) + \kappa_i(r(t)) \right) \text{sign}(e_i(t)) - 2\eta_i(r(t)) e_i(t),
\]

\[
-\kappa_i(r(t)) e_i(t) - \left( \int_{t-d(t)}^{t} \hat{c}_i(s) G e_i(s) \, ds \right)^{(1/2)}
\]

\[
- c_i(r(t)) \left( \int_{t-d(t)}^{t} \hat{c}_i(s) \, ds \right),
\]

in which \( \text{sign}(e_i(t)) = (\text{sign}(e_{i1}(t)), \ldots, \text{sign}(e_{in}(t)))^T, \)

\( e_i(t) = (e_{i1}(t), e_{i2}(t), \ldots, e_{in}(t))^T, \) \( \kappa_i(r(t)) > 0 \) and is a tunable parameter, and \( \eta_i(r(t)) \) is a parameter to be determined.

Theorem 1. Let the assumptions \((H_1)\) and \((H_2)\) hold. Then, the drive system (17) is synchronized with the response system (18) in finite time, if there exist symmetric positive definite matrices \( P_i, U, G, \) such that

\[
\begin{align*}
\bar{D} + \bar{B} \bar{\Omega} - \eta_i(r(t)) I_{n_i} < 0, \\
\beta_i \left( |P_i A_i| + |P_i B_i| \right) + \Delta \omega P_i - P_i \xi_i(r(t)) < 0, \\
\tilde{\alpha} < 1, \\
\Omega \left( \begin{array}{c} P_i B_i \end{array} \right) \leq 0,
\end{align*}
\]

where \( \bar{D} = \text{diag}(|D_1|, |D_2|, \ldots, |D_n|), \)

\( \bar{\Omega} = \left( \tilde{\eta}_i \right)_{n_i \times n_i}, \)

\( \tilde{\eta}_j = \xi_{ij}, j \neq i, \)

\( \tilde{\eta}_i = \left( \lambda_{min} / \theta \right), \)

\( \theta = \lambda_{max}, \) and \( \lambda_{max} \) is the eigenvalue of \( \bar{A}^T \bar{A}. \)

\( \Delta = \max (c_i(r(t)) \beta_i), \)

\( \tilde{\alpha} = \max (\alpha_i). \)

\( \Omega = \sum_{i=1}^{n} \pi_i \pi_k^T P_i + \sum_{j=1}^{n} \xi^2 \lambda_j^2 / 4 U_{\lambda j} (P_k - P_i) U_{\lambda j}^T (P_k - P_i) + \alpha_i |P_i A_i| + \alpha_i |P_i B_i| + G - 2 \rho \eta_i(r(t)). \)

Proof. To develop the finite-time synchronization criterion, we consider a stochastic Lyapunov function as follows:
where $\Delta t$ is a small positive number. Hence, for every $r(t) = i \in S$, it can be deduced that

$$\mathcal{L}V(t) = \lim_{\Delta t \to 0} \frac{\mathcal{E}[V(e_1(t + \Delta t), r(t + \Delta t), t + \Delta t)] - V(e_1(t), i, t)}{\Delta t}$$

Calculating $\mathcal{L}V(e_1(t), t, r)$ along the trajectory of the error system (20) based on the condition $H_0$ formula, we have

$$2e_i^T(t)P_sB_x\tilde{f}_i(e_1(t)) \leq 2\sum_{i=1}^{N} \sum_{j=1}^{N} \|e_i(t)\| p_{ui}b_{ui}\|\tilde{f}_i(e_1(t))\|$$

$$\leq 2\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} |e_i(t)| |p_{ui}b_{ui}| + \beta_i$$

$$\leq 2\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} |e_i(t)| |p_{ui}a_{ui}|$$

and similarly, we have

$$2e_i^T(t) P_s A_x \tilde{f}_i(e_1(t)) \leq 2\sum_{i=1}^{N} \sum_{j=1}^{N} |e_i(t)| |p_{ui}b_{ui}| |\tilde{f}_i(e_1(t))|$$

$$\leq 2\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} |e_i(t)| |p_{ui}a_{ui}|$$

Then, based on (37), we have the following inequality:
Noting the condition (29) and Lemma 3, we have

\[ \mathcal{L}V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_i D_i e_i(t) + 2 \beta \sum_{i=1}^{N} |e_i^T(t)\left(\left|P_i A_i\right| + |P_i B_i| \right)\]

\[ + 2 \alpha \sum_{i=1}^{N} e_i^T(t) P_i A_i e_i(t) + 2 \sum_{i=1}^{N} e_i^T(t) P_i B_i e_i(t - \tau(t))\]

\[ + 2 \sum_{i=1}^{N} e_i^T(t) P_i \eta_i e_i(t) + 2 \Delta \omega \sum_{i=1}^{N} |e_i^T(t) P_i|\]

\[ + 2 c_i (r(t)) (\bar{a} - 1) \sum_{i=1}^{N} |e_i^T(t)| P_i \int_{t-d(t)}^{t} e_i(s) ds\]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) Ge_i(t) + (r(t) - 1) e_i^T(t - \tau(t)) Ge_i(t - \tau(t)) \right]\]

\[ - 2 \sum_{i=1}^{N} e_i^T(t) P_i \eta_i (r(t)) + \kappa_i(r(t))) - 2 e_i^T(t) P_i \eta_i (r(t)) e_i(t)\]

\[ - 2 \sum_{i=1}^{N} e_i^T(t) P_i \frac{e_i(t)}{|e_i(t)|} \int_{t-d(t)}^{t} e_i^T(s) Ge_i(s) ds \]

\[ + \sum_{i=1}^{N} e_i^T(t) \left[ \sum_{k=1}^{N} \pi_{ik}(h) P_k \right] e_i(t)\]

Substituting controller (28) into (39),

\[ \mathcal{L}V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_i D_i e_i(t) \]

\[ + 2 \sum_{i=1}^{N} e_i^T(t) P_i A_i e_i(t) + 2 \sum_{i=1}^{N} e_i^T(t) P_i B_i e_i(t - \tau(t))\]

\[ + 2 \sum_{i=1}^{N} e_i^T(t) P_i \eta_i e_i(t) + 2 \Delta \omega \sum_{i=1}^{N} |e_i^T(t) P_i|\]

\[ + 2 c_i (r(t)) (\bar{a} - 1) \sum_{i=1}^{N} |e_i^T(t)| P_i \int_{t-d(t)}^{t} e_i(s) ds\]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) Ge_i(t) + (r(t) - 1) e_i^T(t - \tau(t)) Ge_i(t - \tau(t)) \right]\]

\[ - 2 \sum_{i=1}^{N} e_i^T(t) P_i \eta_i (r(t)) + \kappa_i(r(t))) - 2 e_i^T(t) P_i \eta_i (r(t)) e_i(t)\]

\[ - 2 \sum_{i=1}^{N} e_i^T(t) P_i \frac{e_i(t)}{|e_i(t)|} \int_{t-d(t)}^{t} e_i^T(s) Ge_i(s) ds \]

\[ + \sum_{i=1}^{N} e_i^T(t) \left[ \sum_{k=1}^{N} \pi_{ik}(h) P_k \right] e_i(t)\]

Noting the condition (29) and Lemma 3, we have
\[ \mathcal{L} V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_i D_i e_i(t) + 2 \beta \sum_{i=1}^{N} e_i^T(t) \left( |P_i A_i| + |P_i B_i| \right) + 2 \alpha \sum_{i=1}^{N} e_i^T(t) \left( P_i e_i(t) - e_i(t) \right) \]

Then, combining (41) and (42), we get

\[ \mathcal{L} V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_i D_i e_i(t) + 2 \alpha \sum_{i=1}^{N} e_i^T(t) P_i e_i(t) \]

In view of \( \pi_{sk}(h) = \pi_{sk} + \Delta \pi_{sk} \) and \( \Delta \pi_{sk} = - \sum_{k=1, k \neq k}^{N} \Delta \pi_{sk}^k \) according to Lemma 1, we can obtain that

\[ \sum_{k=1}^{N} \pi_{sk} P_k = \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq k}^{N} \Delta \pi_{sk} P_k + \Delta \pi_{sk} P_k, \]

\[ = \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq k}^{N} \Delta \pi_{sk} (P_k - P_k), \]

\[ = \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq k}^{N} \Delta \pi_{sk} \left( P_k - P_k \right), \]

\[ \cdot \left[ \frac{1}{2} \Delta \pi_{sk} (P_k - P_k) + \frac{1}{2} \Delta \pi_{sk} (P_k - P_k) \right] \]

\[ \leq \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq k}^{N} \left[ \frac{1}{4} \pi_{sk}^2 \right] (P_k - P_k) \]

By virtue of the condition of the theorem, we get from inequality (43) that
\[ \mathcal{L}V(t) \leq 2 \sum_{i=1}^{N} \left| e_i(t) \right| \left| P_s \right| \left| D \right| \left| e_i(t) \right| \\
\quad + 2 \theta \sum_{i,j=1, j \neq i}^{N} l_{ij} \left| e_i(t) \right| \left| P_s \right| \left| e_j(t) \right| \\
\quad + 2 \sum_{i=1}^{N} \lambda_{\min} l_i \left| e_i(t) \right| \left| P_s \right| \left| e_i(t) \right| \\
\quad - 2 \sum_{i=1}^{N} \eta_i (r(t)) \left| e_i(t) \right| \left| P_s \right| \left| e_i(t) \right| \\
\quad + \sum_{i=1}^{N} \left| e_i(t) \right| \left| \alpha_i P_s A_i + \alpha_i A_i^T P_s \right| \left| e_i(t) \right| \\
\quad + \sum_{i=1}^{N} \left| e_i(t) \right| \left[ \alpha_i P_s B_i + \alpha_i B_i^T P_s \right] \left| e_i(t - \tau(t)) \right| \\
\quad + \sum_{i=1}^{N} \left| e_i(t) \right| \left[ \beta_i \left( \left| P_s A_i \right| + \left| P_s B_i \right| \right) + \Delta \omega I_n - \xi_i (r(t)) I_n \right] \\
\quad + \sum_{i=1}^{N} \left| e_i(t) \right| \left( t - 1 \right) \left| e_i(t - \tau(t)) \right| \left| G e_i(t - \tau(t)) \right| \\
\quad + \sum_{i=1}^{N} \left| e_i(t) \right| \left| P_s \right| \left| \kappa_i (r(t)) \right| - 4 \sum_{i=1}^{N} \left| e_i(t) \right| \left| P_s \right| \left| \eta_i (r(t)) \right| \left| e_i(t) \right| \\
\quad - 2 \kappa_i (r(t)) \lambda_{\min} \left( P_s \right) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \left| e_i(s) \right| G e_i(s) ds \right]^{1/2} \] (44)

According to Lemma 3, (44) can be rewritten as

\[ \mathcal{L}V(t) \leq 2 \left\| P_s \right\| \left( \left| D \right| + t \theta n \bar{L} q - h \eta_i (r(t)) \lambda_{\min} (P_s) \right) \bar{e}_i(t) \\
\quad + 2 \sum_{i=1}^{N} \left| e_i(t) \right| \left[ \beta_i \left( \left| P_s A_i \right| + \left| P_s B_i \right| \right) + \Delta \omega P_s - \lambda_{\min} \left( P_s \right) \xi_i (r(t)) \right] \\
\quad + \left[ \left| e_i(t) \right| \left| e_i(t - \tau(t)) \right| \left( \left( r(t) \right) \bar{G} \right) \left[ \left| e_i(t - \tau(t)) \right| \right] \right] \\
\quad - 2 \lambda_{\min} \left( P_s \right) \kappa_i (r(t)) \sum_{i=1}^{N} \left| e_i(t) \right| \\
\quad - 2 \kappa_i (r(t)) \lambda_{\min} \left( P_s \right) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \left| e_i(s) \right| G e_i(s) ds \right]^{1/2} \] (45)

where \( \bar{e}_i(t) = (\left\| e_1(t) \right\|, \left\| e_2(t) \right\|, \ldots, \left\| e_n(t) \right\|)^T \).

Exploiting condition (29) and (30), we have the following inequality:

\[ \sum_{i=1}^{N} \left| e_i(t) \right| \geq \left( \sum_{i=1}^{N} \left| e_i(t) \right|^2 \right)^{1/2} \geq \left( \sum_{i=1}^{N} \left| e_i(t) \right|^2 \right)^{1/2} \] (48)

Combining (46), (48), and Lemma 3, we get
\[ L V(t) \leq -2 \frac{\lambda_{\min}(P_s)}{\lambda_{\max}(P_s)} \kappa_i(r(t)) \left( \sum_{i=1}^{N} e_i^T(t)P_se_i(t) \right)^{1/2} \]

\[-2\kappa_i(r(t))\lambda_{\min}(P_s) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s)G e_i(s) ds \right]^{1/2} . \]

By Lemma 5, (49) is equal to

\[ L V(t) \leq -2\varrho \left[ V(t) \right]^{1/2}, \]

that is,

\[ L V(t) \leq -2\varrho \left[ V(t) \right]^{1/2}, \]

where

\[ \varrho = \min \left\{ \frac{\lambda_{\min}(P_s)}{\lambda_{\max}(P_s)}, \kappa_i, \lambda_{\min}(P_s) \kappa_i \right\}, \]

\[ \kappa_i(r(t)) = \text{diag}(\kappa_i). \]

As we all know, \( \mathcal{L}[V(t)]^{1/2} = (\mathcal{L}[V(t)])^{1/2} \), for any \( t > 0 \). Thus, we have the following inequality:

\[ \mathcal{L}[L V(t)] \leq -2\varrho \left[ V(t) \right]^{1/2}. \]

Then, based on Lemma 6, the finite-time synchronization of the complex network is finally achieved. At the same time, the settling time is estimated as

\[ t_1 \leq \frac{1}{\varrho} V^{1/2}(0). \]

Thus, we can conclude that the drive system (17) synchronizes with the response system (18) in finite time by adopting the controller (28). Also, the settling time \( t_1 \) is given in (54). The proof of the global finite-time synchronization is finished. \( \Box \)

Remark 2. The global finite-time synchronization of discontinuous CDNs with semi-Markovian switching and mixed time-varying delays is considered for the first time in Theorem 1. The approach we get the global finite-time synchronization condition is the matrix inequalities but not the linear matrix inequalities because \( \Omega \) has a nonlinear term \( \sum_{k=1,k \neq s}^{N} (P_k - P_s)U_{sk}^{-1} (P_k - P_s). \)

3.2. Global Fixed-Time Synchronization. In this part, the global fixed-time synchronization criteria are developed under the following state-feedback controller:

\[ u_i(t) = -\xi_i(r(t)) \text{sign}(e_i(t)) - 2\eta_i(r(t))e_i(t) \]

\[ -\kappa_i(r(t))e_i^T(\tilde{\mathcal{C}}(\tilde{m})) - \delta_i(r(t))e_i^T(\tilde{\mathcal{D}}+\tilde{P}) - c_i(r(t)) \]

\[ \left[ \int_{t-d(t)}^{t} e_i(s) ds \right]^T (\int_{t-d(t)}^{t} e_i(s)G e_i(s) ds)^{1/2}, \]

where \( \xi_i(r(t)) > 0 \) and \( \eta_i(r(t)) > 0 \) are the parameters to be determined. \( \kappa_i(r(t)) > 0 \) and \( \delta_i(r(t)) > 0 \) are tunable parameters. 0 < \( \tilde{m} \), 0 < \( p < q \).

Theorem 2. Consider network models (17) and (18) with the hypotheses (H1) and (H2). For any given constants scalars 0 < \( \tilde{m} \), 0 < \( p < q \), the drive system (17) is said to be synchronized with the response system (18) in fixed-time, if there exist symmetric positive definite matrix \( P_s \) and \( U_m \) and matrices G > 0 such that

\[ D + \theta L - \eta_i(r(t))I_n < 0, \]

\[ \beta_i([P_sA_i] + [P_sB_i]) + \Delta \omega P_s - P_s \tilde{\kappa}_i(r(t)) < 0, \]

\[ \tilde{\alpha} < 1, \]

where \( D = \text{diag}(\|D_1\|, \|D_2\|, \ldots, \|D_n\|), \quad \tilde{\alpha} (\tilde{P}_{ij}^{D}) \), \( \tilde{L} = (\tilde{L}_{ij})_{n \times n} \), \( \tilde{L}_{ij} = 1_{ij} i \neq j, \) \( \tilde{\kappa}_i = \lambda_{\min}(\tilde{\alpha}) \), \( \theta = \|A\| \), and \( \lambda_{\min} \) is the eigenvalue of \( \Lambda' \). \( \Delta = max(c_i(r(t)))B_i, \tilde{\alpha} = max(\alpha_i). \)

\[ \begin{array}{c}
\tilde{\Omega} \\
\alpha_i [P_sB_i] \\
* (r - 1) G \\
* \\
* -U_s
\end{array} \]

\[ (57) \]

where

\[ \Omega = \sum_{k=1}^{N} \sigma_{sk} P_k + \sum_{k=1,k \neq s}^{N} \frac{\lambda_{sk}^2}{4} U_{sk} + (P_k - P_s)U_{sk}^{-1} (P_k - P_s) \]

\[ + \alpha_i [P_sA_i] + \alpha_i [A_i^T P_s] + G - 2P_s \eta_i (r(t)) \]

\[ \tilde{\Omega} = \sum_{k=1}^{N} \sigma_{sk} P_k + \sum_{k=1,k \neq s}^{N} \frac{\lambda_{sk}^2}{4} U_{sk} + \alpha_i [P_sA_i] + \alpha_i [A_i^T P_s] \]

\[ + G - 2P_s \eta_i (r(t)), \]

\[ \Xi = [P_s - P_1, \ldots, P_s - P_{s-1}, P_s - P_{s+1}, \ldots, P_s - P_N], \]

\[ U_s = \text{diag} \{ U_{s1}, U_{s2}, \ldots, U_{s(s-1)}, U_{s(s+1)}, \ldots, U_{sN} \}. \]

(58)
Taking the same stochastic Lyapunov functional with Theorem 1,

\[ V(t) = \sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}e_{i}(t) + \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \varepsilon_{i}^{T}(s)Ge_{i}(s)ds. \tag{59} \]

Then, based on (20), (33), and (42), we can obtain the derivative of \( V(e_{i}(t), t, r(t)) \) as follows:

\[
\mathcal{L}V(t) \leq 2\sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}[D_{e_{i}}(t) + A_{i}f_{i}(e_{i}(t)) + B_{i}\tilde{f}_{i}(e_{i}(t - \tau(t))]
\]
\[
+ \sum_{i=1}^{N} c_{i}^{T}(t)\int_{t-d(t)}^{t} \bar{f}_{i}(e_{i}(s))ds
\]
\[
+ \sum_{i=1}^{N} \varepsilon_{i}^{T}(t)Ge_{i}^{T}(t) + \sum_{i=1}^{N} \varepsilon_{i}^{T}(t) \left[ \sum_{k=1}^{N} \pi_{ik}(h)P_{k} \right] e_{i}(t)
\]
\[
- (r - 1)\sum_{i=1}^{N} \varepsilon_{i}^{T}(t - \tau(t))Ge_{i}(t - \tau(t))
\]
\[
- 2c_{i}(r(t))\sum_{i=1}^{N} \varepsilon_{i}^{T}(t)[P_{i}]^{\frac{1}{2}}[P_{i}]^{\frac{1}{2}} e_{i}(s)ds
\]
\[
- \frac{2}{N} \sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}e_{i}(r(t)) - 4\sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}\eta_{i}(r(t))e_{i}(t)
\]
\[
- 2\sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}\kappa_{i}(r(t))e_{i}(t) + 2\sum_{i=1}^{N} \varepsilon_{i}^{T}(t)P_{i}\delta_{i}(r(t))e_{i}(t)
\]
\[
- 2\lambda_{\text{min}}(P_{i})\kappa_{i}(r(t))[\sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \varepsilon_{i}^{T}(s)Ge_{i}(s)ds]^{\frac{1}{2}}\]
\[
- 2\lambda_{\text{min}}(P_{i})\delta_{i}(r(t))[\sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \varepsilon_{i}^{T}(s)Ge_{i}(s)ds]^{\frac{1}{2}}.
\]

Substituting (55) into (60) and adopting Lemma 3, (60) changes into
From (35)–(37), (42), and (61), we have

\[ \mathcal{L} V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_s D_s e_i(t) + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^T(t) P_s L_j^T \Lambda e_j(t) \]

\[ + 2 \alpha \sum_{i=1}^{N} \left[ e_i^T(t) P_s B_i e_i(t - \tau(t)) \right] + e_i^T(t) P_s A_i e_i(t) \]

\[ + 2 \sum_{i=1}^{N} \left[ e_i^T(t) \left[ \beta_i \left( |P_s A_i| + |P_s B_i| \right) \right] + \Delta \omega P_s - P_s \xi_i(\tau(t)) \right] \]

\[ + 2 c_1 (r(t)) (\bar{a} - 1) \sum_{i=1}^{N} e_i^T(t) \left| P_s \left[ \int_{t-d(t)}^{t} e_i(s) ds \right] \right| \]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) G e_i^T(t) + (r - 1) e_i^T(t - \tau(t)) G e_i(t - \tau(t)) \right] \]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) \left[ \sum_{k=1}^{N} \pi_{s,k} P_k \right] + \sum_{k=1}^{N} \lambda_k^2 U_{s,k} \right] \]

\[ + (P_k - P_s) U_{s,k}^{-1} \left( P_k - P_s \right) e_i(t) - 2 \sum_{i=1}^{N} \left[ 2 e_i^T(t) P_s \xi_i(\tau(t)) e_i(t) \right] \]

\[ + e_i^T(t) P_s \xi_i(r(t)) e_i^{(\text{imm})}(t) + e_i^T(t) P_s \delta_i(r(t)) e_i^{(\text{imp})}(t) \]

\[ - 2 \lambda_{\min} (P_s) \kappa_i(r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) G e_i(s) ds \right]^{(\text{imm})/2m} \]

\[ - 2 \lambda_{\min} (P_s) \delta_i(r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) G e_i(s) ds \right]^{(\text{imp})/2p} . \]

Based on the (56), it yields that

\[ \mathcal{L} V(t) \leq 2 \sum_{i=1}^{N} e_i^T(t) P_s D_s e_i(t) + 2 \alpha \sum_{i=1}^{N} \left[ e_i^T(t) P_s B_i e_i(t - \tau(t)) \right] \]

\[ + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^T(t) P_s L_j^T \Lambda e_j(t) + 2 \alpha \sum_{i=1}^{N} \left[ e_i^T(t) P_s A_i e_i(t) \right] \]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) G e_i^T(t) + (r - 1) e_i^T(t - \tau(t)) G e_i(t - \tau(t)) \right] \]

\[ + \sum_{i=1}^{N} \left[ e_i^T(t) \left[ \sum_{k=1}^{N} \pi_{s,k} P_k \right] + \sum_{k=1}^{N} \lambda_k^2 U_{s,k} \right] \]

\[ + (P_k - P_s) U_{s,k}^{-1} \left( P_k - P_s \right) e_i(t) - 2 \sum_{i=1}^{N} \left[ 2 e_i^T(t) P_s \xi_i(r(t)) e_i(t) \right] \]

\[ + e_i^T(t) P_s \xi_i(r(t)) e_i^{(\text{imm})}(t) + e_i^T(t) P_s \delta_i(r(t)) e_i^{(\text{imp})}(t) \]

\[ - 2 \lambda_{\min} (P_s) \kappa_i(r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) G e_i(s) ds \right]^{(\text{imm})/2m} \]

\[ - 2 \lambda_{\min} (P_s) \delta_i(r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) G e_i(s) ds \right]^{(\text{imp})/2p} . \]
Similar to (44), we have the following inequality:

\[
\mathcal{L}V(t) \leq 2 \sum_{i=1}^{N} \|e_i^T(t)\|P_i\|D_i\|\|e_i(t)\|
\]

\[
+ 2 \left( \sum_{i,j=1, i \neq j}^{N} \theta_{ij} + \sum_{i=1}^{N} \lambda_{\min} \eta_i - \eta_i (r(t))I_n \right) \|e_i^T(t)\|P_i\|e_i(t)\|
\]

\[
+ \sum_{i=1}^{N} e_i^T(t) \left[ \alpha_i |P_i A_i| + \alpha_i |A_i^T P_i| \right] e_i(t)
\]

\[
+ \sum_{i=1}^{N} e_i^T(t) \left[ \alpha_i |P_i B_i| + \alpha_i |B_i^T P_i| \right] e_i(t - \tau(t))
\]

\[
+ \sum_{i=1}^{N} e_i^T(t) \left[ \sum_{k=1}^{N} \pi_{ik} P_k \right] + \sum_{k=1, k \neq s}^{N} \left[ \lambda_{\min} \eta_k + (P_k - P_s)U_{ik}^{-1} (P_k - P_s) \right] e_i(t)
\]

\[
+ (r - 1) \sum_{i=1}^{N} e_i^T(t - \tau(t))Ge_i(t - \tau(t))
\]

\[
+ \sum_{i=1}^{N} e_i^T(t)Ge_i^T(t) - 2 \sum_{i=1}^{N} e_i^T(t)P_i \eta_i (r(t))e_i(t)
\]

\[
- 2 \sum_{i=1}^{N} \left[ e_i^T(t)P_j \kappa_i (r(t)) e_i^{(\bar{h}m)} (t) + e_i^T(t)P_j \delta_i (r(t)) e_i^{(q+q)} (t) \right]
\]

\[
- 2\lambda_{\min} (P_j) \kappa_i (r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T (s) Ge_i (s) ds \right] \frac{(\bar{h}m/2m)}{(p+q/2p)}
\]

\[
- 2\lambda_{\min} (P_j) \delta_i (r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T (s) Ge_i (s) ds \right] \frac{(\bar{h}m/2m)}{(p+q/2p)}
\]

Then, (64) can be rewritten as follows:

\[
\mathcal{L}V(t) \leq 2P_i\|e_i^T(t)\|e_i^T(t - \tau(t))) + \left[ e_i^T(t - \tau(t)) \right] \left[ \begin{array}{c} \Omega \alpha_i |P_i B_i| \\ (r - 1)G \end{array} \right] \left[ \begin{array}{c} e_i(t) \\ e_i(t - \tau(t)) \end{array} \right]
\]

\[
- 2 \sum_{i=1}^{N} \left[ e_i^T(t)P_j \kappa_i (r(t)) e_i^{(\bar{h}m)} (t) - 2 \sum_{i=1}^{N} e_i^T(t)P_j \delta_i (r(t)) e_i^{(q+q)} (t) \right]
\]

\[
- 2\lambda_{\min} (P_j) \kappa_i (r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T (s) Ge_i (s) ds \right] \frac{(\bar{h}m/2m)}{(p+q/2p)}
\]

\[
- 2\lambda_{\min} (P_j) \delta_i (r(t)) \left[ \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T (s) Ge_i (s) ds \right] \frac{(\bar{h}m/2m)}{(p+q/2p)}
\]
where \( \bar{e}_i(t) = (\|e_{1i}(t)\|, \|e_{2i}(t)\|, \ldots, \|e_{ni}(t)\|)^T \).

Based on the condition (56), we obtain that

\[
\mathcal{L}V(t) \leq -2 \sum_{i=1}^{N} \bar{e}_i^T(t) P_i k_i(r(t)) \bar{e}_i(t) + 2 \lambda_{\min}(P_i) \sum_{i=1}^{N} \int_{t}^{t^-} e_i^T(s) G e_i(s) ds + 2 \lambda_{\min}(P_i) \delta_i(r(t)) \sum_{i=1}^{N} \int_{t}^{t^-} e_i^T(s) G e_i(s) ds
\]

where

\[
\Omega = \sum_{k=1}^{N} \pi_{ik} P_k + \sum_{k=1, k \neq s}^{N} \frac{\lambda_{\max}^2}{4} U_{sk} + (P_k - P_s) U^{-1}_{sk} (P_k - P_s)
\]

\[
+ \alpha_i [P_i A_i] + \alpha_i [A_i^T P_i] + G - 2 \eta_i P_s (r(t)).
\]

Because of the nonlinear terms, \((P_k - P_s) U^{-1}_{sk} (P_k - P_s)\) in \(\Omega\) is difficult to be handled, and we need to turn it into a linear matrix inequality.

Hence, we define

\[
\tilde{\Omega} = \sum_{k=1}^{N} \pi_{ik} P_k + \sum_{k=1, k \neq s}^{N} \frac{\lambda_{\max}^2}{4} U_{sk} + \alpha_i [P_i A_i] + \alpha_i [A_i^T P_i]
\]

\[
+ G - 2 \eta_i P_s (r(t)).
\]

Then, according to the (56), (30) can be rewritten as follows:

\[
\begin{bmatrix}
\Omega & \alpha_i [P_i B_i] \\
* & (r - 1)G
\end{bmatrix} + \text{diag}
\begin{bmatrix}
\tilde{\Omega} & \alpha_i [P_s B_i] \\
* & (r - 1)G
\end{bmatrix} \leq 0.
\]

We note that Lemma 4 implies that

\[
\begin{bmatrix}
\tilde{\Omega} & \alpha_i [P_s B_i] \\
* & (r - 1)G
\end{bmatrix} \leq 0.
\]

Hence, from (70), (66) turns into

\[
\mathcal{L}V(t) \leq -2 \sum_{i=1}^{N} \bar{e}_i^T(t) P_i k_i(r(t)) \bar{e}_i(t) + 2 \lambda_{\min}(P_i) \sum_{i=1}^{N} \int_{t}^{t^-} e_i^T(s) G e_i(s) ds + 2 \lambda_{\min}(P_i) \delta_i(r(t)) \sum_{i=1}^{N} \int_{t}^{t^-} e_i^T(s) G e_i(s) ds
\]

By using Lemmas 2 and 3, we can obtain that

\[
\mathcal{L}V(t) \leq -2 \lambda_{\min}(P_i) \lambda_{\max}^{-((\bar{\lambda}_{\max} + 2)/2)} (P_i) k_i(r(t)) \sum_{i=1}^{N} \left( \bar{e}_i^T(t) P_i \bar{e}_i(t) \right)^{-((\bar{\lambda}_{\max} + 2)/2)}
\]

\[
- 2 \lambda_{\min}(P_i) \lambda_{\max}^{-((p+q)/2)} (P_i) \delta_i(r(t)) \sum_{i=1}^{N} \left( \bar{e}_i^T(t) P_i \bar{e}_i(t) \right)^{-((p+q)/2)}
\]

\[
- 2 \lambda_{\min}(P_i) k_i(r(t)) \sum_{i=1}^{N} \int_{t}^{t^-} \bar{e}_i^T(s) G e_i(s) ds
\]

\[
- 2 \lambda_{\min}(P_i) \delta_i(r(t)) \sum_{i=1}^{N} \int_{t}^{t^-} \bar{e}_i^T(s) G e_i(s) ds.
\]
\[ \mathcal{L} V(t) \leq -2\bar{k} \left[ \sum_{i=1}^{N} e_i^T(t) P_i e_i(t) + \sum_{i=1}^{N} \left( t - (r_i) \right) e_i^T(s) G_{r_i} e_i(s) ds \right] \left( (\bar{\tau} + m)/2m \right) \]
\[ - 2\tilde{\delta} \left[ \sum_{i=1}^{N} e_i^T(t) P_i e_i(t) + \sum_{i=1}^{N} \left( t - (r_i) \right) e_i^T(s) G_{r_i} e_i(s) ds \right] \left( (p + q)/2p \right). \]  

(73)

Taking the expectation on both sides of (74), we can get the following inequality:
\[ \mathbb{E} [\mathcal{L} V(t)] \leq -2\bar{k} \mathbb{E} \left[ V(t) \right] \left( (\bar{\tau} + m)/2m \right) - 2\tilde{\delta} \mathbb{E} \left[ V(t) \right] \left( (p + q)/2p \right). \]  

(76)

Similar with Theorem 1, the differential coefficients of \( V(t) \) are given as follows:
\[ \mathbb{E} [\mathcal{L} V(t)] \leq -2\bar{k} \mathbb{E} \left[ V(t) \right] \left( (\bar{\tau} + m)/2m \right) - 2\tilde{\delta} \mathbb{E} \left[ V(t) \right] \left( (p + q)/2p \right). \]  

(77)

Because \( 0 < \bar{\tau} < m \), \( 0 < p < q < (\bar{\tau} + m)/2m \), and \( (p + q)/2p \) > 1, (77) satisfies Lemma 7, and we are in a position to complete the proof of Theorem 2. In other words, the drive system (17) synchronizes with the response system (18) in fixed-time. Also, the settling time is estimated as
\[ T(e_0) \leq T_{\text{max}} \leq \frac{m}{\bar{k} (m - \bar{\tau})} + \frac{\bar{p}}{\delta (q - p)}. \]  

(78)

Thus, the global fixed-time synchronization problem is finally addressed under the controller (55). The proof of this theorem is completed.

\( \square \)

**Corollary 1.** Suppose that the hypotheses (H1) and (H2) hold. For given scalars \( 0 < \bar{\tau} < m \), \( 0 < p < q \), the drive system (17) synchronizes with the response system (18) in fixed time based on the controller (55), if there exist symmetric positive definite matrices \( P_i, U_{sk} \), and positive definite matrix \( Q \) such that
\[
\begin{align*}
    d_i + \theta_i - \eta_i &< 0, \\
    \beta_i \left( \left| P_i A_i \right| + \left| P_i B_i \right| \right) + \Delta \omega P_i - P_i \tilde{\xi}_i (r(t)) &< 0, \\
    \bar{\alpha} &< 1,
\end{align*}
\]  

(79)

where \( d_i = \lambda_{\max} (D), \theta_i = \lambda_{\max} (\tilde{L}^T), \eta_i = \lambda_{\max} (\eta_i (r(t))) I_n \), \( \bar{\xi}_i = (\bar{I}_{ij})_{n \times n}, \bar{I}_{ij} = I_{ij}, \# \neq j, \bar{I}_{ii} = (\lambda_{\min} / \theta), \theta = \| A \|, \) and \( \lambda_{\min} \) is the eigenvalue of \( A \). \( \Delta \) is max((c_i (r(t)) (\beta_i)), \( \bar{\alpha} = \max, (\bar{\alpha}). \)

Also, (73) is equal to
\[
\mathcal{L} V(t) \leq -2\bar{k} [V(t)] \left( (\bar{\tau} + m)/2m \right) - 2\tilde{\delta} [V(t)] \left( (p + q)/2p \right),
\]  

(74)

where
\[
\begin{pmatrix}
    \bar{\Omega} & \alpha_i P_i B_i \varepsilon \\
    * & (r - 1) G - U_i
\end{pmatrix} \leq 0.
\]  

(80)

where
\[
\begin{align*}
    \bar{\Omega} &= \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq s}^{N} \frac{\lambda_{sk}^2}{4} U_{sk} - (P_k - P_s) U_{sk}^{-1} (P_k - P_s) + \alpha_i [P_i A_i] + \alpha_i [A_i^T P_i] + G - 2P_i \eta_i (r(t)), \\
    &+ \sum_{k=1}^{N} \pi_{sk} P_k + \sum_{k=1, k \neq s}^{N} \frac{\lambda_{sk}^2}{4} U_{sk} - (P_k - P_s) U_{sk}^{-1} (P_k - P_s) + \alpha_i [P_i A_i] + \alpha_i [A_i^T P_i] + G - 2P_i \eta_i (r(t)), \\
    \varepsilon &= [P_s - P_1, \ldots, P_s - P_{s-1}, P_s - P_{s+1}, \ldots, P_s - P_N], \\
    U_i &= \text{diag} \left\{ U_{i1}, U_{i2}, \ldots, U_{i(s-1)}, U_{i(s+1)}, \ldots, U_{iN} \right\}.
\end{align*}
\]  

(81)

Meanwhile, on the condition of Lemma 8, the settling time of global fixed-time synchronization is estimated as
\[
T(e_0) \leq T_{\text{max}} \leq \frac{np}{2 \sqrt{\bar{k} \delta}}.
\]  

(82)

where \( \bar{k} \) and \( \delta \) are the same with the same parameters in Theorem 2. Also, \( (\bar{\tau}/m) = 1 - (1/2p) \), \( (q/p) = 1 - (1/2p), p > 1. \)

\( \text{Remark 3.} \) There exist many kinds of chaotic system with discontinuous functions, such as the discontinuous Chua circuit, discontinuous Chen system, and neural network with discontinuous activation functions. In this paper, we choose the discontinuous condition that it is applicable to most of the chaotic systems.

\( \text{Remark 4.} \) Global fixed-time synchronization is more complex than global finite-time synchronization in the synchronization conditions. For global finite-time
synchronization, a term such as \( -V^a(t), 0 < a < 1 \) is only needed, whereas for fixed-time synchronization, we need \( -V^{(\text{fin})}(t), 0 < \beta < m; -V^{(\text{fix})}(t), q > p > 0 \) to realize it. Comparing with the settling time \( t_1 \) of the global finite-time synchronization in Theorem 1, the settling time \( T(e_0) \) of the global fixed-time synchronization in Theorem 2 is independent of the initial condition \( e_0 \). When the \( e_0 \) is so large, the \( t_1 \) is not reasonable in practice application.

4. Numerical Examples

In this section, we perform two examples to demonstrate the effectiveness of the results obtained in this paper.

Example 1. consider the complex dynamical networks models in the form of (17) and (18) with two modes and two nodes, and each node is a 2-dimensional dynamical system. The system parameters are described as follows: \( x_i(t) = (x_{i1}(t), x_{i2}(t))^T, y_i(t) = (y_{i1}(t), y_{i2}(t))^T \).

\[
\begin{align*}
D_1 &= \begin{pmatrix} 1.4 & 0 \\ 0 & 1.1 \end{pmatrix}, \\
D_2 &= \begin{pmatrix} 1.5 & 0 \\ 0 & 1.0 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} -1.6 & 0.5 \\ -0.8 & -1.0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} -2.0 & 0.6 \\ 0.8 & -1.9 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} -1.2 & -0.4 \\ -0.8 & -0.6 \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 0.7 & -0.5 \\ 0.3 & -0.6 \end{pmatrix}.
\end{align*}
\] (83)

The coupling matrix \( T \) is given as follows:

\[
\begin{align*}
L^1 &= \begin{pmatrix} -1.0 & 1.0 \\ 1.5 & -1.5 \end{pmatrix}, \\
L^2 &= \begin{pmatrix} -0.9 & 0.9 \\ 1.2 & -1.2 \end{pmatrix}.
\end{align*}
\] (84)

The inner-coupling matrix is

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (85)

The scalars we select in this paper are as follows: \( 0 \leq \alpha_i \leq 0.5, 0 \leq \beta_i \leq 0.5, c^{(1)}_1(r(t)) = 0.3 \) and \( c^{(2)}_1(r(t)) = 0.5 \). The mixed time-varying delay is assumed to be \( \tau(t) = 0.8 + 0.2 \cos t, d(t) = 0.75 + 0.25 \sin t \), and the initial value is selected as \( x_i(t) = (\sin t, \tanh t)^T \) and \( y_i(t) = (\cos t, e^t)^T \). We can see that the upper bound of the delay is \( \tau = 1, \omega = 1, \tau(t) \leq r = 0.5 \). Also, the activation function is taken as

\[
\begin{align*}
f^{1}_{11}(x_i(t)) &= f^{2}_{11}(x_i(t)) = 0.4t + \text{sign}(t), \\
f^{1}_{22}(x_i(t)) &= f^{2}_{22}(x_i(t)) = 0.1t + 0.7\text{sign}(t).
\end{align*}
\] (86)

The transition rates of the semi-Markovian jumping system for each mode are given as follows:

For mode 1,

\[
\begin{align*}
\pi_{11}(h) &\in (-0.72, -0.58), \\
\pi_{13}(h) &\in (0.42, 0.62).
\end{align*}
\] (87)

For mode 2,

\[
\begin{align*}
\pi_{21}(h) &\in (0.84, 0.96), \\
\pi_{23}(h) &\in (-0.92, -0.76).
\end{align*}
\] (88)

Then, according to the transition rates, we can get the parameters \( \pi_{sk}, \lambda_{sk} \), where \( s, k \in \mathcal{S} = \{1, 2\} \).

\[\begin{align*}
\pi_{11} &= -0.60, \\
\pi_{12} &= 0.60, \\
\pi_{21} &= 0.90, \\
\pi_{22} &= -0.90, \\
\lambda_{11} &= 0.07, \\
\lambda_{12} &= 0.10, \\
\lambda_{21} &= 0.06, \\
\lambda_{22} &= 0.08.
\end{align*}\] (89)

Through simple computations, we have

\[P_1 = \begin{pmatrix} 4.4850 & 0.3156 \\ 0.3156 & 3.6320 \end{pmatrix},\]

\[P_2 = \begin{pmatrix} 5.0268 & 0.3462 \\ 0.3462 & 4.6459 \end{pmatrix},\] (90)

\[Q = \begin{pmatrix} 0.5646 & 0 \\ 0 & 0.6024 \end{pmatrix}\]

For mode 1,

\[\xi_1(r(t)) = \begin{pmatrix} 1.1228 & 0 \\ 0 & 1.0235 \end{pmatrix},\]

\[\kappa_1(r(t)) = \begin{pmatrix} 0.5500 & 0 \\ 0 & 0.5500 \end{pmatrix},\] (91)

\[\eta_1(r(t)) = \begin{pmatrix} 1.3142 & 0 \\ 0 & 1.3316 \end{pmatrix}.\]

For mode 2,
\[ \xi_2 (r(t)) = \begin{pmatrix} 0.9350 & 0 \\ 0 & 1.1526 \end{pmatrix}, \]
\[ \kappa_2 (r(t)) = \begin{pmatrix} 0.5010 & 0 \\ 0 & 0.5010 \end{pmatrix}, \]
\[ \eta_2 (r(t)) = \begin{pmatrix} 1.3350 & 0 \\ 0 & 1.4226 \end{pmatrix}. \]

and the parameter \( V(0) = 8.6528 \) and the settling time is \( t^* = 3.2475 \).

Then, Figure 1 displays the first state trajectories of the drive-response system with controller (28). The second state trajectories of the drive-response system with controller (28) presented in Figures 2 and 3 depicts the state trajectories of the synchronization error system (20) with the controller (28).

To this, the effectiveness of Theorem 1 is proved. The finite-time synchronization problem of discontinuous semi-Markovian switching complex dynamical networks with mixed time-varying delays is realized.

**Example 2.** For Theorem 2, we take network models in the form of (17) and (18) with two nodes, and each node is a 2-dimensional dynamical system. The system parameters are described as follows: \( x_i(t) = (x_{i1}(t), x_{i2}(t))^T \), \( y_i(t) = (y_{i1}(t), y_{i2}(t))^T \).

\[ D_1 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.0 \end{pmatrix}, \]
\[ D_2 = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.2 \end{pmatrix}, \]
\[ A_1 = \begin{pmatrix} -2.8 & 0.5 \\ -0.8 & -2.3 \end{pmatrix}, \]
\[ A_2 = \begin{pmatrix} -2.6 & -0.3 \\ 0.5 & -2.0 \end{pmatrix}, \]
\[ B_1 = \begin{pmatrix} -1.0 & -0.5 \\ -0.6 & 0.7 \end{pmatrix}, \]
\[ B_2 = \begin{pmatrix} 0.8 & -0.5 \\ -0.7 & -0.6 \end{pmatrix}. \]

The coupling matrix \( L \) is given as follows:

\[ L_1 = \begin{pmatrix} -1.2 & 1.2 \\ 2.0 & -2.0 \end{pmatrix}, \]
\[ L_2 = \begin{pmatrix} -1.3 & 1.3 \\ 2.2 & -2.2 \end{pmatrix}. \]

The inner-coupling matrix we choose is

\[ \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

The transition rates of the semi-Markovian switching system for each mode are given as follows.

\[ \pi_{11} (h) \in (-1.02, -0.82), \]
\[ \pi_{12} (h) \in (0.78, 1.10). \]

For mode 2,

\[ \pi_{21} (h) \in (1.23, 1.05), \]
\[ \pi_{22} (h) \in (-1.32, -1.16). \]

Then, according to the transition rates, we can get the parameters \( \pi_{sk}, \lambda_{sk} \), where \( s, k \in S = \{1, 2\} \).
\[ \pi_{11} = -0.90, \quad \pi_{12} = 0.90, \]
\[ \pi_{21} = 1.20, \quad \pi_{22} = -1.20, \]
\[ \lambda_{11} = 0.10, \quad \lambda_{12} = 0.11, \]
\[ \lambda_{21} = 0.09, \quad \lambda_{22} = 0.08. \]  

For Theorem 2, the corresponding scalars we choose are
\[ m = 6, n = 4, p = 2, q = 3, \]
\[ 0 \leq \beta_i \leq 0.5 c_i^{(1)}(r(t)) = 0.3, \quad \text{and} \quad c_i^{(2)}(r(t)) = 0.6. \]

The mixed time-varying delays are assumed to be
\[ \tau(t) = 0.75 + 0.25 \cos t \quad \text{and} \quad d(t) = 0.5 + 0.5 \sin t, \]
and the initial value is selected as \( x_i(t) = (\sin t, \cos t)^T \) and \( y_i(t) = (\cos t, \tanh t)^T. \)

We can see that the upper bound of the delay is \( \tau = 1, \omega = 1, r(t) \leq r = 0.5. \)

Also, the activation function we select is
\[
\begin{align*}
f_1^1(x_{i1}(t)) &= f_1^2(x_{i1}(t)) = 0.5t + 0.8\text{sign}(t), \\
f_2^1(x_{i2}(t)) &= f_2^2(x_{i2}(t)) = t + 0.5\text{sign}(t). 
\end{align*}
\]

Through simple computations, we have
\[ P_1 = \begin{pmatrix} 30.0357 & 0.2356 \\ 0.2356 & 30.8698 \end{pmatrix}, \]
\[ P_2 = \begin{pmatrix} 11.3242 & 0.0262 \\ 0.0262 & 11.0633 \end{pmatrix}, \]
\[ Q = \begin{pmatrix} 1.0645 & 0 \\ 0 & 1.2315 \end{pmatrix}. \]

\[ \xi_1(r(t)) = \begin{pmatrix} 1.3045 & 0 \\ 0 & 1.5425 \end{pmatrix}, \]
\[ \eta_1(r(t)) = \begin{pmatrix} 2.5235 & 0 \\ 0 & 2.5249 \end{pmatrix}, \]
\[ \kappa_1(r(t)) = \begin{pmatrix} 1.4000 & 0 \\ 0 & 1.4000 \end{pmatrix}, \]
\[ \delta_1(r(t)) = \begin{pmatrix} 1.5125 & 0 \\ 0 & 1.5684 \end{pmatrix}. \]

For mode 2,
\[ \xi_2(r(t)) = \begin{pmatrix} 1.2531 & 0 \\ 0 & 1.1626 \end{pmatrix}, \]
\[ \eta_2(r(t)) = \begin{pmatrix} 1.5738 & 0 \\ 0 & 1.6968 \end{pmatrix}, \]
\[ \kappa_2(r(t)) = \begin{pmatrix} 2.2500 & 0 \\ 0 & 2.2500 \end{pmatrix}, \]
\[ \delta_2(r(t)) = \begin{pmatrix} 1.6145 & 0 \\ 0 & 1.6686 \end{pmatrix}. \]

Meanwhile, the parameters of the controller we choose are as follows.

For mode 1,
\[ \xi_1(r(t)) = \begin{pmatrix} 1.3045 & 0 \\ 0 & 1.5425 \end{pmatrix}, \]
\[ \eta_1(r(t)) = \begin{pmatrix} 2.5235 & 0 \\ 0 & 2.5249 \end{pmatrix}, \]
\[ \kappa_1(r(t)) = \begin{pmatrix} 1.4000 & 0 \\ 0 & 1.4000 \end{pmatrix}, \]
\[ \delta_1(r(t)) = \begin{pmatrix} 1.5125 & 0 \\ 0 & 1.5684 \end{pmatrix}. \]

For mode 2,
\[ \xi_2(r(t)) = \begin{pmatrix} 1.2531 & 0 \\ 0 & 1.1626 \end{pmatrix}, \]
\[ \eta_2(r(t)) = \begin{pmatrix} 1.5738 & 0 \\ 0 & 1.6968 \end{pmatrix}, \]
\[ \kappa_2(r(t)) = \begin{pmatrix} 2.2500 & 0 \\ 0 & 2.2500 \end{pmatrix}, \]
\[ \delta_2(r(t)) = \begin{pmatrix} 1.6145 & 0 \\ 0 & 1.6686 \end{pmatrix}. \]

and the settling time is \( T \leq 2.8. \)

Then, Figure 4 introduces the first state trajectories of the drive-response system with controller (55). The second state trajectories of drive-response system with controller (55) shown in Figures 5 and 6 display the state trajectories of the synchronization error system (20) with the controller (55).
Hence, the effectiveness of Theorem 2 is verified. In other words, the global fixed-time synchronization of discontinuous complex dynamical networks with semi-Markovian switching and mixed delays can be addressed.

5. Conclusions

In this paper, the global finite-time and fixed-time synchronization of discontinuous complex dynamical networks with semi-Markovian switching and mixed time-varying delays are investigated. Two novel state-feedback controllers are designed which include the integral term and mixed delay-term. Based on the linear matrix inequality technique (LMIs), Lyapunov functional method, and the proposed control schemes, some sufficient conditions are established to guarantee the global finite-time and fixed-time synchronization of complex dynamical networks. Then, two numerical examples are provided to confirm the effectiveness of the main results. More complex conditions, such as stochastic disturbance, mixed coupling, and the approach of impulsive control, will be taken into consideration in the future study.

Data Availability

The underlying data supporting the results of this study can be found in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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