\textbf{\textsc{A¹-invariants in Galois cohomology and a claim of Morel}}

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\textbf{Abstract}

We establish variants of the results in [1] for invariants taking values in a strictly homotopy invariant sheaf. As an application, we prove the folklore result of Morel that $\hat{\text{et}}(B_\text{et}\text{Fin}^{\text{et}})^+ = GW$.

\section{\textbf{A¹-invariants of Étale Algebras}}

This entire section is basically a minor variation of arguments from [1]. Related results were also obtained by Morel (unpublished) and Hirsch [2, Theorem 2.3.12].

Throughout, we fix a field $k$. Let $Sm(k)$ denote the category of smooth $k$-schemes and $\text{Pre}(Sm(k))$ the category of presheaves on $Sm(k)$. As usual, if $F \in \text{Pre}(Sm(k))$ and $X$ is an essentially smooth $k$-scheme, then $F(X)$ makes sense.

\textbf{Definition.} We call $F \in \text{Pre}(Sm(k))$ homotopy invariant if $F(X) = F(X \times \A^1)$ for all $X \in Sm(k)$.

\textbf{Definition.} We call $F \in \text{Pre}(Sm(k))$ unramified if $F(X \coprod Y) = F(X) \times F(Y)$, for all connected $X \in Sm(k)$ the canonical map $F(X) \to F(k(X))$ is injective, and moreover

$$F(X) = \bigcap_{x \in X^{(1)}} F(x)$$

(intersection in $F(k(X))$).

\textbf{Remark 1.} Suppose that $k$ is perfect and $F \in \text{Pre}(Sm(k))$ is a strictly homotopy invariant sheaf of abelian groups. Then $F$ is unramified [3, Lemma 6.4.4].

In this section we are interested in the presheaf $\text{Et}_n \in \text{Pre}(Sm(k))$ which assigns to $X \in Sm(k)$ the set of isomorphism classes of étale $X$-schemes, everywhere of rank $n$. (This is neither homotopy invariant nor unramified, of course.)

\textbf{Definition.} A rank $n$ versal étale scheme is an étale morphism $X \to Y$, everywhere of rank $n$, such that if $T \to \text{Spec}(l)$ is any rank $n$ étale algebra over a finitely generated field extension $l/k$, then $T$ is obtained from $X \to Y$ by pullback along a morphism $\text{Spec}(l) \to Y$.

We will make good use of the following result.

\textbf{Theorem 2} ([1], Proposition 24.6(2)). There exists a smooth $k$-scheme $X$, an irreducible divisor $\Delta \subset \A^n$ and a finite étale morphism $p : X \to \A^n \setminus \Delta$ of rank $n$, such that $p$ is versal.

Moreover, let $\eta$ be the generic point of $\A^n$, $L_\Delta^\eta$ the Henselization of $\A^n$ in the generic point of $\Delta$, and $\eta_\Delta$ the generic point of $L_\Delta^\eta$. The finite étale $\eta_\Delta$-scheme $X_{\eta_\Delta}$ splits as a disjoint union $X_{\eta_\Delta} = X_1 \coprod X_2$ with $X_1$ of rank two (unless $n = 1$).

\textbf{Proof.} Let us review the construction of $p$. Let $p_0 : X_0 = \text{Spec}(k[x_1, \ldots, x_n][t]/(t^n + x_1 t^{n-1} + \cdots + x_n)) \to \A^n$ be the evident map. Let $\Delta \subset \A^n$ be the branch locus of $p_0$; the reference proves that this is an irreducible divisor. Since étale algebras over fields are simple, it is clear that $X := X_0 \setminus p_0^{-1}(\Delta)$ is versal.

Let $L_\Delta^\eta$ be the completion of $L_\Delta^\eta$ and write $\hat{\eta}$ for the generic point of $L_\Delta^\eta$. The reference proves that $X_{\hat{\eta}} = Y_1 \coprod Y_2$, with $Y_1$ of rank two and $Y_2$ unramified. What this means is that there exists a finite étale morphism $Y_2' \to L_\Delta^\hat{\eta}$ with $Y_2 \cong Y_2' \times_{L_\Delta^\hat{\eta}} \hat{\eta}$ [1, beginning of Section 11, Proposition 24.2(3) and Definition 24.3]. Since $L_\Delta^\eta \to L_\Delta^\hat{\eta}$ is a morphism of henselian local rings inducing an isomorphism on residue fields, it follows that there exists a finite étale morphism $Y_2'' \to L_\Delta^\hat{\eta}$ with $Y_2'' \cong Y_2'' \times_{L_\Delta^\hat{\eta}} L_\Delta^\eta$ [5, Tag 04GK]. Lemma 3(2) below then furnishes us with a retraction $(Y_2'')_{\eta_\Delta} \to X_{\eta_\Delta} \to (Y_2'')_{\eta_\Delta}$. Thus $X_{\eta_\Delta}$ splits as $X_1 \coprod (Y_2'')_{\eta_\Delta}$, and $X_1$ must have the same degree as $Y_1$, i.e. 2. This concludes the proof. \qed
We used the following result, which is surely well-known.

**Lemma 3.** Let $R$ be a henselian DVR with completion $\hat{R}$, fraction field $K$ and completed fraction field $\hat{K}$.

1. Let $A/K$ be an étale algebra with completion $\hat{A}/\hat{K}$. If $x \in \hat{A}$ satisfies a separable polynomial with coefficients in $A$, then $x \in A$.

2. Let $A, B$ be étale $K$-algebras. Then $\text{Hom}_K(A, B) = \text{Hom}_{\hat{K}}(\hat{A}, \hat{B})$.

**Proof.** (1) We may assume that $A = L$ is a field. The normalization $R'$ of $R$ in $L$ is finite [5, Tag 032L], and hence $R'$, being a domain, is local henselian [5, Tag 04GH(1)]. We may thus replace $R$ by $R'$ and assume that $A = K$. Let $\pi$ be a uniformizer of $R$; then $\hat{K} = R[1/\pi]$. It follows that $\pi^n x \in R$ for $n$ sufficiently large and still satisfies a separable polynomial; hence we may assume that $x \in R$. This reduced statement is a well-known characterisation of henselian DVRs.¹

(2) We may assume that $A = K[T]/P$, where $P$ is a separable polynomial. Then $\text{Hom}_{\hat{K}}(\hat{A}, \hat{B})$ is the set of elements $t \in \hat{B}$ with $P(t) = 0$. By (1), such $t$ lie in $B$. It follows that $\text{Hom}_K(A, B) \to \text{Hom}_{\hat{K}}(\hat{A}, \hat{B})$ is surjective. Injectivity is clear since $A \to A$ etc. are all injective.

**Lemma 4.** Let $X$ be the localisation of a smooth scheme in a point of codimension one. Write $X^h$ for the Henselization, $\eta \in X$ for the generic point and $\eta^h$ for the generic point of $X^h$. If $F \in \text{Pre}(\text{Sm}(k))$ is a Nisnevich sheaf, then the following diagram is cartesian:

$$
\begin{array}{ccc}
F(X) & \longrightarrow & F(\eta) \\
\downarrow & & \downarrow \\
F(X^h) & \longrightarrow & F(\eta^h).
\end{array}
$$

**Proof.** Let $X' \to X$ be an étale neighbourhood of the closed point, and $\eta'$ the generic point of $X'$. Then

$$
\eta' \longrightarrow X' \\
\downarrow & & \downarrow \\
\eta \longrightarrow X
$$

is a distinguished Nisnevich square; hence applying $F$ yields a cartesian square. Since $X^h$ is obtained as the filtered inverse limit of the $X'$ and filtered colimits (of sets) commute with finite limits, the result follows.

Recall that an étale algebra $A/k$ is called *multiquadratic* if it is a (finite) product of copies of $k$ and quadratic separable extensions of $k$.

**Corollary 5.** Let $F \in \text{Pre}(\text{Sm}(k))$ be a homotopy invariant, unramified Nisnevich sheaf of sets and $a : \text{Et}_n \to F$ any morphism (of presheaves of sets). Assume there exists $* \in F(*)$ such that for any field $l/k$ and any multiquadratic étale algebra $A/l$ we have $a(A) = *|_l$. Then for any $Y \in \text{Sm}(k)$ and any $A \in \text{Et}_n(Y)$ we have $a(A) = *|_l$.

**Proof.** This is essentially the same as the proof of [1, Theorem 24.4].

Since $F$ is unramified, it suffices to prove the claim when $Y$ is the spectrum of a field. We proceed by induction on $n$. If $n \in \{1, 2\}$ there is nothing to do.

Suppose now that $l/k$ is a field, $A/l \in \text{Et}_n(l)$ and $A \approx A_1 \times A_2$ with $A_2$ multiquadratic (but non-zero). Define an invariant $a'$ of $\text{Et}_{n-2}$ over $l$ via $a'(B) = a(B \times A_2)$. By assumption $a'(B) = *$ if $B$ is multiquadratic, hence $a' = *$ by induction. We conclude that $a(A) = *$.

Now let $X \to \mathbb{A}^n \setminus \Delta$ be the versal morphism from Theorem 2. We consider $a(X|_\eta) \in F(K)$, where $K = k(\mathbb{A}^n)$. I claim that if $x \in \mathbb{A}^n$ is a point of codimension one, then $a(X|_\eta) = a(x)$. By definition of the versal morphism, $x$ is a point of codimension one, hence $a(x)$ is in the image of $F(\mathbb{A}^n) \to F(K)$. Thus $a(x)\mid_{\eta} = a(x)(\eta) = *$, by the previous step. The claim now follows from Lemma 4.

Since $F$ is unramified, it follows that $a(X|_\eta) \in F(K)$ lies in the image of $F(\mathbb{A}^n) \to F(K)$. But $F(\mathbb{A}^n) = F(*)$, and so there exists $a_0 \in F(*)$ such that $F(X|_\eta) = a_0|_K$. Since $X\eta$ is versal this implies that $a(A/l) = a_0|_l$ for any $l$ and any $A$. But $a(k/k) = *$, so $a_0 = *$. This concludes the induction step.

¹It is proved for example here: https://mathoverflow.net/q/105891.
2 Application: computing $\mathbb{P}^1_0(B_{\text{et}} \text{Fin}^{bij})^+$

Now let $k$ be a perfect field of characteristic not two.

We write $\text{Shv}_{Nis, A^1}(k)$ for the category of strictly homotopy invariant Nisnevich sheaves (of abelian groups) on $Sm(k)$, $\text{Pre}^{mon}(k)$ for the category of presheaves of monoids, with morphisms the morphisms of monoids. We have obvious forgetful functors

$$\text{Shv}_{Nis, A^1}(k) \rightarrow \text{Pre}^{mon}(k) \rightarrow \text{Pre}(Sm(k)).$$

We write $U : \text{Shv}_{Nis, A^1}(k) \rightarrow \text{Pre}(k)$, and $U_{mon} : \text{Shv}_{Nis, A^1}(k) \rightarrow \text{Pre}^{mon}(k)$. Then the functors $U_{mon}$ and $U$ have (potentially partially defined) left adjoints denoted $F_{mon}$ and $F$. We make use of the following result of Morel.

**Theorem 6** ([4], Theorem 3.46). The morphism of presheaves of sets $\mathbb{G}_m/2 \rightarrow GW$, $[a] \mapsto (a)$ exhibits $GW$ as $F(\mathbb{G}_m/2)$.

Let $Et_* \in \text{Pre}^{mon}(k)$ denote the presheaf of monoids $X \mapsto \prod_{n \geq 0} Et_n(X)$; the monoidal operation is given by disjoint union of étale schemes. For an étale algebra $A/l$, denote by $tr(A) \in GW(l)$ the class of its trace form. We shall now prove the result advertised in the heading, in the following guise.

**Proposition 7.** Let $k$ be a perfect field of characteristic not two.

The morphism of presheaves of monoids $tr : Et_* \rightarrow GW$, $A \mapsto tr(A)$ exhibits $GW$ as $F_{mon}(Et_*)$.

**Proof.** Let $\phi : Et_* \rightarrow U_{mon}F$ be any morphism, where $F \in \text{Shv}_{Nis, A^1}(k)$ is arbitrary. For $a \in \mathcal{O}(X)^\times$ let $X_a := X[t]/(t^2 - a)$. Since we are in characteristic not two, $X_a \rightarrow X$ is étale. Define $t : \mathbb{G}_m/2 \rightarrow UF$ by mapping $a \in \mathcal{O}(X)^\times$ to

$t([a]) := \phi([X_a/2]) - \phi([X_2]) + \phi([X]) \in F(X).$

(The reason for this formula is that $tr([X_a]) = (2) + (2a)$, and hence $tr([X_a/2]) + tr([X_2]) + tr([X]) = (a)$.) By Theorem 6 this induces $t' : GW \rightarrow F$. I claim that the following diagram commutes:

$$\begin{array}{ccc}
Et_* & \xrightarrow{\phi} & F \\
tr \downarrow & & \downarrow \\
GW & \xrightarrow{t'} & F
\end{array}$$

By Corollary 5 it suffices to show this for multi-quadratic algebras over fields $l$. But we are dealing with morphisms of monoids into unramified sheaves, so it suffices to show this for $l/l \in Et_1(l)$ and $l_a/l \in Et_2(l)$, where $l/k$ is a finitely generated field extension. Now

$$t'(tr([l])) = t'([1]) = t([1]) = \phi([l_1/2]) - \phi([l_2]) + \phi([l]) = \phi([l])$$

since $l_2$ and $l_1/2$ are isomorphic. Finally

$$t'(tr([l_a])) = t'((2) + (2a)) = \phi([l_1]) - \phi([l_2]) + \phi([l]) + \phi([l_a]) - \phi([l_2]) + \phi([l]) = \phi([l_a]) + 2(\phi([l_1]) - \phi([l_2]))$$

(note that $[l_1] = 2[l]$). Hence to prove the claim we need to show that $2(\phi([l_1]) - \phi([l_2])) = 0$. Consider $u : \mathbb{G}_m/2 \rightarrow Et_*, a \mapsto [X_a]$. Then, applying Theorem 6 again, we get a commutative diagram

$$\begin{array}{ccc}
\mathbb{G}_m/2 & \xrightarrow{u} & Et_* \\
\downarrow & & \downarrow \\
GW & \xrightarrow{u'} & F
\end{array}$$

Since $2(2) = 2 \in GW(k)$ (indeed $2x^2 + 2y^2 = (x + y)^2 + (x - y)^2$) and $\phi([l_a]) = \phi(u(a)) = u'(a)$, this proves the claim.

Consequently we have proved that any morphism $\phi$ factors through $tr : Et_* \rightarrow GW$. Since the image of $tr$ generates $GW$ (as an unramified sheaf of abelian groups), this factorization is unique. This concludes the proof. $\blacksquare$
References

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