Saari’s homographic conjecture for general masses in planar three-body problem under Newton potential and a strong force potential

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Abstract
Saari’s homographic conjecture claims that in the $N$-body problem under the homogeneous potential, $U = \alpha^{-1} \sum m_i m_j r_{ij}^\alpha$ for $\alpha \neq 0$, a motion with the constant configurational measure $\mu = I^{\mu/2} U$ is homographic, where $I$ represents the moment of inertia defined by $I = \sum m_i r_{ij}^2 / \sum m_i$, $m_i$ is the mass, and $r_{ij}$ is the distance between particles. We prove this conjecture for general masses, $m_i > 0$, in the planar three-body problem under the Newton potential ($\alpha = 1$) and a strong force potential ($\alpha = 2$).

Keywords: dynamical system, celestial mechanics, few body systems

1. Saari’s homographic conjecture and the construction of this paper

Consider the $N$-body problem described by the Lagrangian, $L = K/2 + U$. The $K$ here represents twice the kinetic energy

$$K = \sum_{k=1,2,\ldots,N} m_k \left| \frac{dq_k}{dt} \right|^2,$$

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and $U$ represents the homogeneous potential function

$$
U = \begin{cases} 
\frac{1}{\alpha} \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|^\alpha} & \text{for } \alpha \neq 0, \\
- \sum_{1 \leq i < j \leq N} m_i m_j \log |q_i - q_j| & \text{for } \alpha = 0.
\end{cases}
$$

(2)

Here, $m_k$ and $q_k \in \mathbb{R}^3$ represent the mass and the position vector, respectively, of the point particle $k = 1, 2, \ldots, N$. The real parameter, $\alpha$, represents the power of the mutual distance of point particles. The values $\alpha = 1, 2, -2$ give the Newton potential, a strong force potential, and a harmonic oscillator potential, respectively. Although the logarithmic potential is not invariant under a scale transformation, we take this potential for $\alpha = 0$. This is because a scale transformation for this potential adds a constant term that has no effect for the equations of motion. Indeed, definition (2) makes the equations of motion

$$
m_k \frac{d^2 q_k}{dt^2} = \sum_{i \neq k} m_i m_k \left( q_i - q_k \right) \left| q_i - q_k \right|^{\alpha-2} + \sum_{i \neq k} m_i m_k \log \left| q_i - q_k \right|
$$

for all $\alpha \in \mathbb{R}$. The moment of inertia, $I$, is defined as follows,

$$
I = \left( \sum_{1 \leq i < j \leq N} m_i m_j |q_i - q_j|^2 \right) \left( \sum_{k=1,2, \ldots, N} m_k \right)^{-1}.
$$

(3)

The configurational measure, $\mu$, is defined as a scale-invariant product of $I$ and $U$, as follows,

$$
\mu = \begin{cases} 
\alpha I^{\alpha/2} U & \text{for } \alpha \neq 0, \\
- \sum_{i \neq j} m_i m_j \log \left| q_i - q_j \right| / \sqrt{I} & \text{for } \alpha = 0.
\end{cases}
$$

(4)

Saari’s homographic conjecture, which is the subject of this paper, may have several expressions. One expression is the following.

**Conjecture 1 (Saari’s homographic conjecture, 2005).** For a homogeneous potential with an arbitrary $\alpha$, where the configurational measure, $\mu$, is not identically constant for all motions, if a motion has a constant value of $\mu$, then the motion is homographic.

This conjecture consists of two parts. The first part states that some exceptional cases should be excluded, and the second part states the body of the conjecture.

The statement for the exceptional cases may need some explanations. If $\mu$ is identically constant—in other words, if $\mu$ is constant for any motion—the constancy of $\mu$ obviously gives no restriction for the motion. This will take place when $U$ is proportional to $I^{-\alpha/2}$. There are two known cases. Case 1: harmonic oscillator ($\alpha = -2$), $U = -I \sum m_k / 2$. Therefore, $\mu = -2I^{-1} U = \sum m_k$ is identically constant. Case 2: three-body equal-mass rectilinear motion in $\alpha = -4$ [1, 21]. In this case, $U = -\sum (x_i - x_j)^4 / 4 = -9I^2 / 8$ is an identity for $x_k \in \mathbb{R}$, $k = 1, 2, 3$. Therefore, $\mu = -4I^{-2} U = 9/2$ is identically constant. We expect that there are no more exceptional cases.

A motion is called homographic if the configuration, $\{q_i(t)\}$, remains similar to the original configuration, $\{q_i(0)\}$. Here, the similarity is defined by scale transformation and rotation. In other words, for a homographic motion in a planar $N$-body problem, there a
complex function, \( z(t) \), exits such that
\[
q_k(t) = z(t)q_k(0) .
\]
(5)

Although the term ‘similarity transformation’ usually contains translations and reflections, we exclude them if we do not explicitly mention otherwise. Translations are excluded because we always consider the center of mass frame in this paper. Reflections are excluded because we are considering a dynamical motion that is always continuous in time.

The converse of conjecture 1 is obviously true. This is because \( \mu \) is invariant under the previously mentioned similarity transformations. Therefore, if the motion is homographic, then \( \mu \) is constant.

The aim of this paper is to prove Saari’s homographic conjecture in a planar three-body problem for general masses with \( \alpha = 1, 2 \). Namely, we will prove the following theorem.

**Theorem 1.** For a planar three-body problem with \( \alpha = 1 \) and 2, if a motion has constant \( \mu \), then the motion is homographic.

The construction of this paper is as follows. In section 2, we give a history of Saari’s conjecture. In section 3, we introduce dynamical variables to describe the motion of size, rotation, and shape. Then, we obtain the Lagrangian and derive the equations of motion for these variables under the potential, \( \alpha \neq 0 \). In section 4, nonhomographic motion with constant \( \mu \) will be assumed to exist. Then, we will obtain a necessary condition for this motion to be compatible with the equations of motion. In section 5, we prove that the necessary condition is not satisfied for \( \alpha = 1, 2 \). This means that there is no nonhomographic motion with constant \( \mu \). This is a proof of Saari’s homographic conjecture for \( \alpha = 1 \) and 2. A summary and discussions are provided in section 6.

**2. Saari’s conjectures**

By now, three conjectures are named after Saari. In 1969 Donald Saari stated his conjecture on the \( N \)-body problem under the Newton potential, which we will call ‘Saari’s original conjecture’.

### Table 1. Summary of Saari’s conjectures.

| Conjecture | Original | Generalized | Homographic |
|------------|----------|-------------|-------------|
| Range of \( \alpha \) | \( \alpha = 1 \) | \( \alpha \in \mathbb{R} \) | |
| Assumption | \( I = \text{const.} \) | \( \alpha = -2 \) and equal-mass rectilinear three-body in \( \alpha = -4 \), where \( \mu \) is trivially constant. | \( \mu = \text{const.} \) |
| Known exceptions | | \( \alpha = 2 \) | |
| Proof | Three-body in spacial dimension \( \geq 2 \). | For \( \alpha \neq 2 \), ‘generalized’ is contained in ‘homographic.’ | Collinear \( N \)-body for any \( \alpha \) and equal-mass\(^a\) planar three-body for \( \alpha = 1, 2 \). |

\(^a\) Saari’s homographic conjecture is expected to be true for \( \alpha = 2 \). Actually, it is proven for an equal-mass planar three-body problem, and we will prove it for a general-mass case in this paper.

\(^b\) In this paper, we will extend the proof for Saari’s homographic conjecture to a general-mass planar three-body problem for \( \alpha = 1, 2 \).
potential, which we will call the ‘generalized Saari’s conjecture.’ Finally, in 2005, Saari extended his conjecture, which we will call ‘Saari’s homographic conjecture.’

In this section, we will describe each conjecture, its brief history, and the current status of known exceptions and proofs (see table 1).

2.1. Saari’s original conjecture

In 1969, Donald Saari [18] gave a conjecture.

**Conjecture 2** (Saari’s original conjecture, 1969). Under the Newton potential ($\alpha = 1$), if a motion has a constant moment of inertia, then the motion is a relative equilibrium. Namely, the only possible motion with constant moment of inertia is a rotation around the center of mass, as if the $N$-bodies were fixed to a rigid body.

People tried to prove this conjecture for more than 30 years without any positive results. However, Chenciner and Montgomery’s [2] discovery in 2000 of the figure-eight solution in the three-body problem under the Newton potential makes us attend to this conjecture because this solution has an almost constant moment of inertia, but is not a relative equilibrium.

The first successful achievement came from Christopher McCord [13] in 2004. He proved this conjecture for an equal-masses case in a planar three-body problem under the Newton potential ($\alpha = 1$). Finally, at the ‘Saarifest 2005’ conference in Guanajuato, Mexico, Richard Moeckel [14, 15] proved this conjecture for a three-body problem with general masses in any spatial dimension greater than or equal to 2.

2.2. Generalized Saari’s conjecture

Saari’s original conjecture was generalized to homogeneous potentials given by (2).

**Conjecture 3** (Generalized Saari’s conjecture). Saari’s original conjecture can be extended to a homogeneous potential with $\alpha \neq -2$ and 2. The rectilinear equal-mass three-body problem under $\alpha = -4$ is also excluded.

The harmonic oscillator, $\alpha = -2$, is excluded because there are trivial counterexamples for this potential. Actually, we can simply construct motions with constant moments of inertia while each body moves on each ellipse. For example, $q_t = (a_k \cos(\omega t), b_k \sin(\omega t))$, 

$\omega^2 = \sum m_k$ at the center of mass frame is a solution of the equation of motion for $\alpha = -2$. Then, the parameters that satisfy $\sum m_k a_k^2 = \sum m_k b_k^2 = c = \text{constant}$ make $I = c$.

In 2006, Gareth E Roberts [17] found a counterexample of this conjecture in the strong force potential ($\alpha = 2$). The figure-eight solution in the strong force potential ($\alpha = 2$) is also a counterexample. These two motions have constant moments of inertia, but not relative equilibria. This exceptional behaviour of the $N$-body problem in $\alpha = 2$ was already pointed out by Alain Chenciner [1] in 1997. Actually, he noticed that the Lagrange-Jacobi identity for $\alpha \neq 0$ yields

$$\frac{d^2 I}{dt^2} = 4E + 2(2\alpha^{-1} - 1)U.$$  \hspace{1cm} (6)

Therefore, $I = \text{constant}$ makes $U = \text{constant}$ for $\alpha \neq 2$, while $U$ can vary in time for $\alpha = 2$. For $\alpha = 2$, integrating $d^2 I/dt^2 = 4E$, we get $I = 2E\tau^2 + c_1 t + c_2$ with constant parameter, $c_1$.
and $c_2$. So, any motion with initial conditions $E = 0$ and $c_1 = 0$ has a constant moment of inertia.

One more known counterexample for the generalized Saari’s conjecture is rectilinear motion in the equal-mass three-body problem under the potential $\alpha = -4$, which was described in section 1. For this case, $U = -9t^2/8$ is an identity. Therefore, $\partial U/\partial x_i = -(9t/2)x_i$ makes the solution with constant $I$ a harmonic oscillation,

$$x_1 = A \cos(\omega t + 2\pi/3), \quad x_2 = A \cos(\omega t - 2\pi/3), \quad x_3 = A \cos(\omega t),$$

(7)

with $A = \sqrt{2I/3}$ and $\omega = 3\sqrt{I/2}$ [1, 6, 21, 22]. This solution is a rectilinear harmonic oscillation—neither a relative equilibrium rotation nor a homographic motion.

2.3. Saari’s homographic conjecture

At the same conference where Moeckel proved Saari’s original conjecture for a three-body problem, the next day, Saari gave a talk and extended his conjecture in another way [19, 20], which is conjecture 1.

Saari’s homographic conjecture is actually an extension of the original and generalized conjectures. Indeed, for $\alpha \neq 2$, $I = \text{constant}$ makes $U = \text{constant}$, and thus makes $\mu = aI^{n-2}U, \ = \text{constant}$. So, Saari’s homographic conjecture contains the original conjecture and generalized conjecture for $\alpha \neq 2$. We expect that this conjecture is true for all $\alpha \neq -2, -4$.

Both Roberts’s counterexample and the figure-eight solution in $\alpha = 2$ have constant $I$ and nonconstant $U$, and therefore nonconstant $\mu$. So, these two examples do not satisfy the assumption of the homographic conjecture. Therefore, they are not the counterexamples, for this conjecture. We expect that Saari’s homographic conjecture is true for $\alpha = 2$. Actually, in this paper, we will prove the conjecture for $\alpha = 2$ in the planar three-body problem with general masses.

On the other hand, the potentials for $\alpha = -2, -4$ are really exceptions to Saari’s homographic conjecture, because $\mu$ is identically equal to a constant value in these potentials.

In 2005, Florin Diacu, Ernesto Pérez-Chavela, and Manuele Santoprete [4] proved this conjecture for a collinear $N$-body problem for any $\alpha$. In 2008, Florin Diacu, Toshiaki Fujiwara, Ernesto Pérez-Chavela, and Manuele Santoprete [5] showed that the conjecture is true for many sets of initial conditions for a planar three-body problem. In this paper [5], the authors call this conjecture ‘Saari’s homographic conjecture’ to distinguish it from two other similar ‘Saari’s conjectures’. In 2012, the present authors proved the conjecture for a planar equal-mass three-body problem under both the strong force [7] and Newton potentials [8]. In this paper, we extend our proof to the general masses case.

3. Dynamical variables and equations of motion

3.1. Notations

In this paper, we consider the planar three-body problem. We identify a two-dimensional vector, $a = (a_x, a_y) \in \mathbb{R}^2$, and a complex number, $a = a_x + ia_y \in \mathbb{C}$. Inner and outer products are defined by $a \cdot b = a_xb_x + a_yb_y$ and $a \wedge b = a_xb_y - a_yb_x$, respectively. The partial differentiation by $a$ is defined by
\[ \frac{\partial}{\partial a} = \frac{\partial}{\partial a_x} + i \frac{\partial}{\partial a_y}. \]  

For example, for \(|a|^2 = a_x^2 + a_y^2\), \(\partial|a|^2/\partial a = 2(a_x + ia_y) = 2a\).

### 3.2. Dynamical variables

We take the center of mass frame. So the position vectors, \(q_k\), always satisfy

\[ m_1 q_1 + m_2 q_2 + m_3 q_3 = 0, \]  

and the moment of inertia is expressed by \(I = \sum m_k |q_k|^2\).

According to Richard Moeckel and Richard Montgomery [16], we define the ‘shape variable’ \(\zeta \in \mathbb{C}\), by the ratio of two Jacobi vectors, \(J_1\) and \(J_2\),

\[ J_1 = q_2 - q_1, \quad J_2 = q_3 - \left( \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \right) q_3, \]  

\[ \zeta = \frac{J_2}{J_1} = \left( \frac{m_1 + m_2 + m_3}{m_1 + m_2} \right) \left( \frac{q_3}{q_2 - q_1} \right). \]  

The variable \(\zeta\) has a simple geometric interpretation. Consider a similarity transformation that involves a parallel transformation

\[ z \mapsto \frac{z - q_1}{q_2 - q_1} = \frac{m_2}{m_1 + m_2}. \]  

The points \(q_1, q_2\) are mapped to fixed points

\[ q_1 \mapsto -\frac{m_2}{m_1 + m_2}, \quad q_2 \mapsto \frac{m_1}{m_1 + m_2}, \]  

so the image of \(q_3 \mapsto \zeta\) represents the shape of the triangle (see figure 1). It is convenient to use the following rescaled ‘shape variable’, \(\eta\), instead of \(\zeta\),
Let us define \( \xi_k = q_k/(q_2 - q_1) \), which satisfies \( \xi_2 - \xi_1 = 1 \) and \( m_1 \xi_1 + m_2 \xi_2 + m_3 \xi_3 = 0 \). The explicit expression for \( \xi_k \) by \( \eta \) is

\[
\begin{align*}
\xi_1 &= -\frac{m_2}{m_1 + m_2} - \frac{\eta}{m_1 + m_2} \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}}, \\
\xi_2 &= \frac{m_1}{m_1 + m_2} - \frac{\eta}{m_1 + m_2} \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}}, \\
\xi_3 &= \eta \sqrt{\frac{m_3 m_2}{m_1 + m_2 + m_3}}.
\end{align*}
\]

(16)

Obviously, the triangles made \( \{q_k\} \) and \( \{\xi_k\} \) are similar, with a common center of mass. Therefore, there are \( r \geq 0 \) and \( \phi \in \mathbb{R} \), such that

\[
q_k = re^{i\phi} \frac{\xi_k}{\sqrt{\sum m_\ell |\xi_\ell|^2}}.
\]

(17)

We take the variables \( r, \phi, \) and \( \eta \) as the dynamical variables.

The moment of inertia (3) is given by

\[
I = \sum m_k |q_k|^2 = r^2.
\]

(18)

The kinetic energy is expressed by the variables \( r, \phi, \) and \( \eta \),

\[
\frac{K}{2} = \frac{r^2}{2} + \frac{r^2}{2} \left( \phi + \frac{\eta \wedge \dot{\eta}}{1 + |\eta|^2} \right)^2 + \frac{r^2}{2} \frac{|\eta|^2}{(1 + |\eta|^2)^2},
\]

(19)

where the dots placed over the variables represent the derivative with respect to time. The terms on the right-hand side of (19) represent the kinetic energy for the size motion, rotation, and the motion in shape, respectively. The potential function (2) for \( a \neq 0 \) is expressed as

\[
U = \frac{\mu(\eta)}{ar^a},
\]

\[
\mu(\eta) = \left( \frac{m_1 m_2}{m_1 + m_2} \left( 1 + |\eta|^2 \right) \right)^{\alpha/2} \times \left( \frac{m_2 m_3}{m_1/(m_1 + m_2) - \eta/\sqrt{\eta}} + \frac{m_3 m_1}{m_2/(m_1 + m_2) + \eta/\sqrt{\eta}} \right).
\]

(20)

Thus, we obtained expressions for the kinetic energy (19); the potential function (20); thus the Lagrangian, \( L \) and the total energy, \( E \), are represented by the variables \( r, \phi, \) and \( \eta \).
3.3. The equations of motion

Since the variable $\phi$ is cyclic, the angular momentum $C$ is constant of motion.

\[
C = \frac{\partial L}{\partial \dot{\phi}} = r^2 \left( \dot{\phi} + \frac{(\eta \wedge \dot{\eta})}{1 + |\eta|^2} \right) = \text{constant.} \tag{21}
\]

The equation of motion for $r$ is

\[
\dot{r} = \frac{C^2}{r^3} + \frac{r |\eta|^2}{(1 + |\eta|^2)^2} - \frac{\mu(\eta)}{r^{\alpha+1}}. \tag{22}
\]

Multiplying both sides of (22) by $\dot{r}$, we obtain

\[
\frac{d}{dr} \left( \frac{\dot{r}^2}{2} \right) = -\frac{d}{dr} \left( \frac{C^2}{2r^2} \right) + \frac{|\eta|^2}{(1 + |\eta|^2)^2} \frac{d}{dr} \left( \frac{r^2}{2} \right) + \frac{\mu}{\alpha} \frac{d}{dr} \left( \frac{1}{r^\alpha} \right). \tag{23}
\]

Then, using this equation and the energy conservation $dE/dr = 0$, we obtain the following relation, which was first derived by Saari [20],

\[
\frac{d\mu}{dr} = \frac{ar^{\alpha-2}}{2} \frac{d}{dr} \left( r^4 \frac{|\eta|^2}{(1 + |\eta|^2)^2} \right). \tag{24}
\]

This equation shows that the variation in $\mu$ is proportional to the variation in the kinetic energy of the shape motion multiplied by $r^2$. Let us define $v^2$ as

\[
v^2 = r^4 \frac{|\eta|^2}{(1 + |\eta|^2)^2}. \tag{25}
\]

Then the total energy is given by

\[
E = \frac{\dot{r}^2}{2} + \frac{C^2}{2r^2} + v^2 - \frac{\mu}{ar^\alpha} \tag{26}
\]

and $v = \text{constant}$ if and only if $\mu = \text{constant}$. Inspired by Saari’s relation, let us introduce a new ‘time’ variable, $\tau$, by

\[
\left( \frac{\dot{r}^2}{1 + |\eta|^2} \right) \frac{d}{dr} = \frac{d}{d\tau}. \tag{27}
\]

The equation of motion for $\eta$ in the time variable $\tau$ is

\[
\frac{d^2\eta}{d\tau^2} = \frac{2i}{1 + |\eta|^2} \frac{d\eta}{d\tau} + \frac{r^{2-\alpha}}{\alpha} \frac{d\mu}{d\eta}. \tag{28}
\]

Now, consider a motion that has a constant value of $\mu$. Then, by Saari’s relation, the motion must have a constant value of $v^2$. We have two cases.

\[
v^2 = \left| \frac{d\eta}{d\tau} \right|^2 = \begin{cases} 0 & \text{(homographic motion),} \\ > 0 & \text{(nonhomographic motion).} \end{cases} \tag{29}
\]
For homographic motion, the equation of motion (28) demands that the shape variable must satisfy \( \frac{\partial \mu}{\partial \eta} = 0 \). We know five solutions: two Lagrange configurations and three Euler configurations.

4. Necessary condition for nonhomographic motion

Saari’s homographic conjecture claims that a nonhomographic motion with constant \( \mu \) is not realized. In this section, we assume the existence of a nonhomographic motion with constant \( \mu \), and we will derive a necessary condition for the motion to satisfy the equation of motion.

4.1. Necessary condition in the Cartesian coordinates

Since such motions satisfy

\[
\frac{d\mu}{d\tau} = \frac{d\nu}{d\tau} \cdot \frac{\partial \mu}{\partial \eta} = 0 \quad \text{and} \quad \left| \frac{d\eta}{d\tau} \right|^2 = \nu^2 > 0.
\]  

(30)

The ‘velocity’ in the shape variable \( d\eta/d\tau \) must be orthogonal to the gradient vector \( \partial \mu/\partial \eta \), and must have the magnitude \( \nu \). Therefore, the ‘velocity’ is uniquely determined by the gradient vector and \( \nu \),

\[
\frac{d\eta}{d\tau} = -i\nu \frac{\partial \mu}{|\partial \mu/\partial \eta|}.
\]  

(31)

Here, \( \nu \in \mathbb{R} \) and \( \nu \neq 0 \). In the \( \eta \) plane, the motion may pass through a critical point, \( \partial \mu/\partial \eta = 0 \). However, we assume a motion with nonzero \( \nu \); the set of critical points is discrete. Therefore, we can find a part of the motion with finite length where \( \partial \mu/\partial \eta \neq 0 \). In the following arguments, we assume \( \partial \mu/\partial \eta \neq 0 \) without loss of generality.

Does this motion satisfy the equation of motion? To give an answer, we calculate the component of the acceleration, \( d^2\eta/d\tau^2 \), in the orthogonal component to the velocity \( d\eta/d\tau \), because the parallel component to the velocity is always zero, both in the equation of motion (28) and in the motion (31). From the velocity (31) and its derivative by \( \tau \), using \( d/d\tau = d\eta/d\tau \cdot d/d\eta \), we obtain

\[
\left( \frac{d\eta}{d\tau} \right)^2 \frac{d^2\eta}{d\tau^2} = \frac{\nu^3}{(\mu_x^2 + \mu_y^2)^{3/2}} \left( \mu_x^2 \mu_y^2 - 2\mu_x \mu_y \mu_{xy} + \mu_y^2 \mu_{xx} \right).
\]  

(32)

where \( x, y \in \mathbb{R} \) are defined by \( \eta = x + iy \) and \( \mu_x = \partial \mu/\partial x, \mu_y = \partial \mu/\partial y \), etc.... On the other hand, the equation of motion and the velocity (31) yield

\[
\left( \frac{d\eta}{d\tau} \right)^2 \frac{d^2\eta}{d\tau^2} = \left( \frac{2\nu^2}{1 + (x^2 + y^2)^2} \right) \left( -C + \frac{\nu}{\sqrt{\mu_x^2 + \mu_y^2}} \left( x\mu_x + y\mu_y \right) - \frac{r^2 - u^2}{\alpha} \right) \frac{1}{\sqrt{\mu_x^2 + \mu_y^2}}.
\]  

(33)

Two expressions in (32) and (33) must be the same. Thus, we get a necessary condition that must be satisfied by a nonhomographic motion with constant \( \mu \),
\[ r^{2-a} \alpha = \frac{-2Cv}{(1 + x^2 + y^2)(\mu_x^2 + \mu_y^2)} + \frac{2v^2}{(1 + x^2 + y^2)(\mu_x^2 + \mu_y^2)}(x\mu_x + y\mu_y) \]

\[ -\frac{v^2}{(\mu_x^2 + \mu_y^2)}(\mu_x^2\mu_y - 2\mu_x\mu_y\mu_x + \mu_x^2\mu_x). \quad (34) \]

The right-hand side of the necessary condition is written in the Cartesian coordinate \((x, y)\). It is convenient to write the right-hand side in a coordinate-free form. The kinetic energy for the shape motion in equation (19) naturally defines the distance squared, \(ds^2\), and metric tensor, \(g_{ij}\), as follows,

\[ ds^2 = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} = g_{ij}dx^idx^j, \quad g_{ij} = \frac{\delta_{ij}}{(1 + x^2 + y^2)^2}. \quad (35) \]

Here the repeated indices are understood to be summed. The vector \((dx^1, dx^2)\) is identified to be \((dx, dy)\) and \(\delta_{ij}\) represents the Kronecker symbol,

\[ \delta_{ij} = \delta^{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \quad (36) \]

This metric space is called the ‘shape sphere’. This sphere is exactly the Riemann sphere of the complex plane, \(x + iy\). This fact was first noticed by George Lemaître [12] and used by Hsiang and Straume [9, 10], Chenciner and Montgomery [2], Montgomery and Moeckel [16], and Kuwabara and Tanikawa [11].

The inverse and the determinant of the metric are

\[ g^{ij} = \left(1 + x^2 + y^2\right)^2\delta^{ij}, \quad |g| = \text{det}(g_{ij}) = \frac{1}{(1 + x^2 + y^2)^4}. \quad (37) \]

Let us define the following three scalars,

\[ |V\mu|^2 = g^{ij}(\partial_i\mu)(\partial_j\mu) = \left(1 + x^2 + y^2\right)^2\left(\mu_x^2 + \mu_y^2\right), \quad (38) \]

\[ \Delta\mu = \frac{1}{|V\mu|^2}\partial_i\left(\frac{g^{ij}}{|V\mu|^2}(\partial_j\mu)\right) = \left(1 + x^2 + y^2\right)^{\frac{1}{2}}(\mu_x\mu_x + \mu_y\mu_y), \quad (39) \]

\[ \lambda = g^{ij}(\partial_i\mu)(\partial_j|V\mu|^2) \]

\[ = 4\left(1 + x^2 + y^2\right)^2\left(x\mu_x + y\mu_y\right)\left(\mu_x^2 + \mu_y^2\right) \]

\[ + 2\left(1 + x^2 + y^2\right)^{\frac{3}{2}}\left(\mu_x^2\mu_x + 2\mu_x\mu_y\mu_y + \mu_y^2\mu_y\right). \quad (40) \]

In each equality, the first step is the definition of each scalar, and the last step is a representation in \((x, y)\) coordinates. The derivative with respect to time, \(t\), is given by

\[ \frac{d}{dt} = \frac{d\eta}{dt} \cdot \frac{d}{d\eta} = \frac{(1 + x^2 + y^2)}{r^2|V\mu|^2}\left((\partial_i\mu)\partial_j - (\partial_j\mu)\partial_i\right) \]

\[ = \frac{v}{r^2|V\mu|}D_i. \quad (41) \]
where $D$ is a differential operator defined by
\[
D = \frac{1}{\sqrt{|\mathbf{s}|}} e^{ij}(\partial_{i}\mu)\partial_{j} = \left(1 + x^2 + y^2\right)^2 \left( (\partial_{i}\mu)\partial_{j} - (\partial_{j}\mu)\partial_{i} \right). \tag{42}
\]
and $e^{ij}$ is the Lévi-Cività antisymmetric symbol
\[
e^{ij} = \begin{cases} 
1 & \text{for } i = 1, j = 2, \\
-1 & \text{for } i = 2, j = 1, \\
0 & \text{for } i = j.
\end{cases} \tag{43}
\]
Using these scalars, the necessary condition (34) is written in the coordinate-free expression,
\[
\frac{r^2-a}{\alpha} = \frac{-2C\nu}{|V\mu|} + \frac{v^2\Delta}{2|V\mu|} - \frac{v^2\Delta}{|V\mu|^2}. \tag{44}
\]

4.2. Necessary condition in two-center bipolar coordinates

In this section, we will show a method for rewriting the necessary condition (44) in the two-center bipolar coordinates defined by
\[
\eta = \zeta - \frac{m_1}{m_1 + m_2} = \frac{q_1 - q_3}{q_1 - q_2}, \quad r_2 = \zeta + \frac{m_2}{m_1 + m_2} = \frac{q_3 - q_4}{q_1 - q_2}. \tag{45}
\]
(See figure 1.) Although the coordinates $\eta = x + iy = \sqrt{n} \zeta$ are useful for describing the Lagrangian and to get the equations of motion, they are not convenient for expressing the necessary condition. The expression of the condition in $(x, y)$ coordinates is lengthy and complex, while in $r_1$ and $r_2$ it is relatively short and simple.

In the variables $x$ and $y$,
\[
r_1^2 = \left(\frac{x}{\sqrt{n}} - \frac{m_1}{m_1 + m_2}\right)^2 + \frac{y^2}{n}, \quad r_2^2 = \left(\frac{x}{\sqrt{n}} + \frac{m_2}{m_1 + m_2}\right)^2 + \frac{y^2}{n}. \tag{46}
\]
Inversely,
\[
\begin{align*}
x &= \frac{m_1 - m_2}{2(m_1 + m_2)} \sqrt{n} + \frac{\sqrt{n}}{2} (r_2^2 - r_1^2), \\
y &= \pm \frac{\sqrt{n}}{2} \sqrt{\left(1 - (r_1^2 - r_2^2)\right) \left((r_1^2 - r_2^2) - 1\right)}. \tag{47}
\end{align*}
\]
Then, the distance squared, $ds^2 = (dx^2 + dy^2)/(1 + x^2 + y^2)^2$, is given by
\[
\begin{align*}
\frac{4m_1 m_2 m_3 (m_1 + m_2 + m_3) \eta r_2}{(1 - (r_1 - r_2)^2)(r_1 + r_2)^2 - 1}\left(m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2\right)^2.
\end{align*} \tag{48}
\]
Then the metric tensor for these coordinates is defined by
\[
g_{ij} = \frac{4m_1 m_2 m_3 (m_1 + m_2 + m_3) \eta r_2}{(1 - (r_1 - r_2)^2)(r_1 + r_2)^2 - 1}\left(m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2\right)^2 \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{49}
\]
with \( a = r_1 r_2 \) and \( b = -(r_1^2 + r_2^2 - 1)/2 \). The inverse metrics, \( g^{ij} \) and \( |g|^{1/2} \), are

\[
g^{ij} = \left( \frac{m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2}{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \right)^2 \left( \begin{array}{cc} 1 & c \\ c & 1 \end{array} \right)
\]

with \( c = (r_1^2 + r_2^2 - 1)/(2r_2) \). (50)

\[
|g|^{1/2} = \left( \frac{2m_1 m_2 m_3 (m_1 + m_2 + m_3) r_2}{(m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2)^2 (1 - (r_1 - r_2)(r_1 + r_2)^2 - 1)} \right)^{1/2}.
\]

(51)

Using \( \mu \) for \( \alpha \neq 0 \) expressed as functions of \( r_\ell \), \( \ell = 1, 2 \),

\[
\mu = \left( \frac{m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2}{m_1 + m_2 + m_3} \right)^{1/2} \left( \frac{m_1 m_2 + m_2 m_3}{r_\ell^a} + \frac{m_3 m_1}{r_\ell^a} \right),
\]

(52)

the three scalars (38)–(40), and thus the necessary condition (44), are expressed as a function of \( r_\ell \).

5. Proof of the conjecture

5.1. Proof for the strong force potential

For the strong force potential \( \alpha = 2 \), the left-hand side of the necessary condition (44) is constant and the right-hand side is a function of \( r_\ell \), \( \ell = 1, 2 \),

\[
\frac{1}{2} = \frac{-2C\nu}{|\nabla\nu|} + \frac{\nu^2 A\mu}{2 |\nabla\mu|^2} - \frac{\nu^2 A\mu}{|\nabla\mu|^2}.
\]

(53)

Two variables, \( r_1^2 \) and \( r_2^2 \), are not independent, because we are considering a motion that has \( \mu(r_1^2, r_2^2) = \text{constant} \). We have only one independent variable (see figure 2). A possible choice of one independent variable is, say, \( r_1^2 \). Solving \( \mu(r_1^2, r_2^2) = \mu \) for \( r_2^2 \), we obtain \( r_2^2 = r_2^2(m_1, \mu, r_1^2) \). Then, the necessary condition (53) will be in the form of \( 1/2 = F(m_1, C, \nu, \mu, r_1^2) \). This is a condition for the independent variable, \( r_1^2 \), with constants \( m_1, C, \nu, \mu \). However, this choice breaks the invariance of the condition (53) under the simultaneous exchange of \( m_1 \leftrightarrow m_2 \) and \( r_1^2 \leftrightarrow r_2^2 \). Breaking this symmetry will make our analysis complex. Let us write desirable variables \( \{\nu, \rho\} \) that will keep this symmetry, an easy-to-solve variable change \( \{r_1^2, r_2^2\} \leftrightarrow \{\nu, \rho\} \), and simple to eliminate one variable using \( \mu = \text{constant} \).

Our choice for \( \{\nu, \rho\} \) is

\[
\left\{ \begin{array}{l}
\nu = m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2; \\
\rho = m_1 m_2 + m_2 m_3 \frac{m_3 m_1}{r_1^2} \frac{m_3 m_1}{r_2^2}.
\end{array} \right.
\]

(54)

Obviously, these variables keep the symmetry. Equation (54) is easy to solve for \( \{r_1^2, r_2^2\} \) because this equation is quadratic. Moreover, we simply eliminate \( \nu \) by

\[
\nu = \frac{\mu}{\rho}, \quad \text{where} \quad \tilde{\mu} = (m_1 + m_2 + m_3)\mu.
\]

(55)

We have two solutions of \( r_1^2 = r_1^2(\mu, \rho) \) for equation (54). Substituting a solution into the necessary condition (53), we obtain a necessary condition for \( \rho \) as follows,
If there is a nonhomographic motion with constant \( \mu = \frac{\hat{\mu}}{m_1 + m_2 + m_3} \), there is some finite physical interval of \( \rho \) where the condition (56) is satisfied (see figure 2). Since the right-hand side of the condition (56) is an analytic function of \( \rho \), this condition must be satisfied for the whole complex plane of \( \rho \in \mathbb{C} \). Therefore, the condition (56) must be satisfied near the origin of \( \rho \), although this region is unphysical.

Two solutions of (54) are

\[
\begin{align*}
\frac{1}{2} &= F(C, v^2, m_k, \tilde{\mu}, \rho).
\end{align*}
\]  

(56)

Figure 2. Contours of \( \mu = \text{constant} \) for \( a = 2 \). If a nonhomographic motion having constant \( \mu \) exists, the shape variable \( \zeta \) moves on one of the contours.

If there is a nonhomographic motion with constant \( \mu = \frac{\mu}{m_1 + m_2 + m_3} \), there is some finite physical interval of \( \rho \) where the condition (56) is satisfied (see figure 2). Since the right-hand side of the condition (56) is an analytic function of \( \rho \), this condition must be satisfied for the whole complex plane of \( \rho \in \mathbb{C} \). Therefore, the condition (56) must be satisfied near the origin of \( \rho \), although this region is unphysical.

Two solutions of (54) are

\[
\begin{align*}
\rho &= -\frac{m_3}{m_1} + \frac{m_3 (m_1^2 m_2^2 - \hat{\mu})}{m_1 m_2} \left( \frac{\rho}{\hat{\mu}} \right) + O(\rho^2), \quad \rho^2 = \frac{\hat{\mu}}{m_1 m_3 \rho} + \frac{m_2 (m_1^2 - m_2^2)}{m_1 m_2^2} + O(\rho),
\end{align*}
\]  

(57)

and

\[
\begin{align*}
\rho^2 &= \frac{\rho}{m_2 m_3 \rho} + \frac{m_1 (m_2^2 - m_3^2)}{m_2 m_3} + O(\rho), \quad \rho^2 = -\frac{m_3}{m_2} + \frac{m_3 (m_2^2 m_3^2 - \hat{\mu})}{m_1 m_2^2} \left( \frac{\rho}{\hat{\mu}} \right) + O(\rho^2).
\end{align*}
\]  

(58)

The latter is given by the simultaneous exchange of \( \eta \leftrightarrow r_2 \), and \( m_1 \leftrightarrow m_2 \) in the former. Substituting the solution (57) into the condition (53), we obtain

\[
\begin{align*}
\frac{1}{2} &= \frac{(m_1 + m_2 + m_3) v}{2m_1^2 m_2^2} \left\{ -v \left( \hat{\mu} + m_1^2 \left( m_2^2 + m_2 m_3 - m_3^2 \right) + m_1 m_2 m_3^2 + m_2^2 m_3^2 \right) + 2i C m_1 m_3 \sqrt{m_2^3 (m_1 + m_2 + m_3)} \left( \frac{\rho}{\hat{\mu}} \right)^2 + O(\rho^3) \right\}.
\end{align*}
\]  

(59)

Each of the three terms on the right-hand side of (53) contributes to \( O(\rho^2) \). Note that there is no term of order \( \rho^0 \), on the right-hand side of (59), while on the left-hand side, is \( 1/2 \). Therefore, this condition cannot be satisfied by the solution (57). For the solution (58), we have a similar result. The only difference from equation (59) is the exchange of \( m_1 \) and \( m_2 \). Thus, the condition (53) cannot be satisfied. Namely, there is no nonhomographic motion
with constant $\mu$. This is a proof of Saari’s homographic conjecture for the strong force potential $a = 2$.

5.2. Proof for the Newton potential

For the Newton potential $a = 1$, the necessary condition (44)

$$r = -\frac{2Cv}{|V\mu|} + \frac{\nu^2\lambda}{2|V\mu|^2} - \frac{\nu^2\Delta\mu}{|V\mu|^2}$$  \hspace{1cm} (60)

determines the size variable in the form $r = r(C, v, m_k, r_l)$. Then, by equation (41), $\dot{r}$ is also given in the form $\dot{r} = \dot{r}(C, v, m_k, r_l)$. Thus the total energy (26) is written in the form $E = E(C, v, m_k, r_l)$.

For the Newton potential, let us take new variables

$$\nu = m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2,$$

$$\rho = m_1m_2 + \frac{m_2m_3}{r_1} + \frac{m_3m_1}{r_2}.$$ \hspace{1cm} (61)

Then, we eliminate $\nu$ by

$$\nu = \left(\frac{\bar{\rho}}{\rho}\right)^2, \text{ where } \bar{\rho} = \sqrt{m_1 + m_2 + m_3}.$$ \hspace{1cm} (62)

Equation (61) is a quartic equation for $r_l$. Let one of the solutions be $r_l = r_l(m_k, \bar{\mu}, \rho)$. Substituting this solution into the expression of $E$, we will obtain the total energy in the form

$$E = E(C, v, m_k, \bar{\mu}, \rho).$$ \hspace{1cm} (63)

Let us assume that there is a physical value of $C, v, m_k, \bar{\mu}$ and a finite physical interval of $\rho$ where the right-hand side of equation (63) is constant. For the physical region, $\bar{\mu}$ is always greater than $(m_1m_3)^{3/2}$ and $(m_2m_3)^{3/2}$. This is because

$$\bar{\mu} > \sqrt{m_3m_1} \left(\frac{m_3m_1}{r_2}\right) = (m_1m_3)^{3/2},$$ \hspace{1cm} (64)

and a similar inequality $\bar{\mu} > (m_2m_3)^{3/2}$. Since the right-hand side of equation (63) is an analytic function of $\rho$, the right-hand side must be constant for the whole region of the complex plane, $\rho \in \mathbb{C}$. Therefore, expression (63) must be constant near the origin of $\rho$ for some physical value of $C, v, m_k > 0$, and $\bar{\mu} > (m_1m_3)^{3/2}, (m_2m_3)^{3/2}$.

The four solutions of (61) are

$$r_l = \frac{-m_3}{m_1} \left(1 + \frac{\bar{\mu} \pm (m_1m_3)^{3/2}}{m_1m_2\bar{\mu}} + O(\rho^2)\right),$$

$$r_2 = \frac{1}{m_1m_3} \left(\frac{\bar{\rho}}{\rho} + O(\rho)\right).$$ \hspace{1cm} (65)

and the simultaneous exchange of $r_l \leftrightarrow r_2$ and $m_1 \leftrightarrow m_2$. Then, three quantities in the necessary condition for the solutions in (65) are
Therefore, the dominant term in the necessary condition (60) near the origin of $\rho$ is $v^2\lambda/(2|V\mu|^4)$. Thus, the condition (60) yields

$$r = \frac{3v^2((m_1m_3)^{3/2} \pm \bar{\mu})}{4m_1^2m_2^2}\left(\frac{\rho}{\bar{\mu}}\right)^2 + O\left(\rho^3\right).$$

(69)

Then

$$\frac{1}{r^2} = \frac{16m_1^4m_2^4}{9(m_1 + m_2 + m_3)v^4((m_1m_3)^{3/2} \pm \bar{\mu})^2}\left(\frac{\bar{\mu}}{\rho}\right)^4 + O\left(\rho^{-3}\right).$$

(70)

Using equation (41), we obtain

$$\left(\frac{\rho}{\bar{\mu}}^2 \frac{dp}{d\tau}\right)^2 = -\frac{64m_1^9m_2^6}{81m_1^3(m_1 + m_2 + m_3)^{3/2}((m_1m_3)^{3/2} \pm \bar{\mu})^2}\left(\frac{\bar{\mu}}{\rho}\right)^8 + O\left(\rho^{-7}\right).$$

(71)

Therefore, near the origin of $\rho$, the dominant term in the total energy (26) is the kinetic term for size motion, $v^2/2$. We obtain

$$E = -\frac{16m_1^5m_2^3}{9m_1^3(m_1 + m_2 + m_3)^{3/2}((m_1m_3)^{3/2} \pm \bar{\mu})^2}\left(\frac{\bar{\mu}}{\rho}\right)^8 + O\left(\rho^{-7}\right).$$

(72)

The other two solutions of $\eta$ give the total energy in the exchange of $m_1 \leftrightarrow m_2$. Note that the coefficient of the term $(\bar{\mu}/\rho)^8$ is not zero.

Thus the total energy, $E$, cannot be constant near the origin of $\rho$. This means that there is no nonhomographiic motion with constant $\mu$. This is a proof of Saari’s homographic conjecture.

6. Summary and discussions

We proved Saari’s homographic conjecture for a planar three-body problem under the Newton potential ($\alpha = 1$) and the strong force potential ($\alpha = 2$) for general masses.

To describe the motion in shape, we used the shape variable $\zeta \in \mathbb{C}$ in equation (11), or $\eta \in \mathbb{C}$ in equation (15), introduced by Moeckel and Montgomery. We wrote the Lagrangian in the size variable $r$, the rotation variable $\phi$, and the shape variable $\eta$. The equations of motion for these variables were given.

Then, we assumed the existence of a nonhomographic motion that has a constant configurational measure, $\mu$. This motion must satisfy the necessary condition (44). Finally, we showed that any nonhomographic motion with constant $\mu$ is not able to satisfy the necessary condition. This is our proof.
In the final stage of our proof, we changed the variables, \( \eta \in \mathbb{C} \), to two-center bipolar coordinates \((r_1, r_2)\) defined in equation (45), and then to \((\bar{\mu}, \rho)\) in equations (54) or (61). The variables \((\bar{\mu}, \rho)\) are useful for proving Saari’s homographic conjecture. Because we assume \( \mu = \bar{\mu}/(m_1 + m_2 + m_3)^{\alpha/2} = \text{constant} \), the only free variable is \( \rho \). This choice of the variables makes our proof simple.

We have two comments on the variable \((\bar{\mu}, \rho)\). One is an alternative method to calculate \( |\nabla \mu|, \Delta \mu, \text{and } \lambda \). In this paper, we expressed these quantities in the variables \((r_1, r_2)\). Then, put \( \mu = \rho \bar{\mu}(\tilde{\rho}) \) to get \( |\nabla \mu| \) etc... in a series of \( \rho \). An alternative method is their direct calculation using the metric in \((\bar{\mu}, \rho)\) space, \( G_{ij} = s G_{xx} d^i d^j \) and \( G_{\bar{\mu} \rho} = \mu \rho \). Then, we will get the metric, \( G_{\bar{\mu} \rho}(\bar{\mu}, \rho) \), in a series of \( \rho \). Using this metric, we directly calculated \( |\nabla \mu|^2 = G^{ij}(\partial \mu)(\partial \mu) \) etc... and got the same results in equations (59) and (69).

Another comment is the difficulty of extending our method to general \( \alpha \)—for example, to \( \alpha = \sqrt{2} \). According to this paper, a naive choice of \((\nu, \rho)\) will be

\[
\nu = m_1 m_2 + m_2 m_3 r_1^2 + m_3 m_1 r_2^2,
\]

\[
\rho = m_1 m_2 + m_2 m_3 r_1^\alpha + m_3 m_1 r_2^\alpha,
\]

and

\[
\bar{\mu} = (m_1 + m_2 + m_3)^{\alpha/2} \mu = \nu^{\alpha/2} \mu.
\]

However, it will be difficult to solve these equations to get \( r_1 \) and \( r_2 \) in a power series of \( \rho \). It would be better to find other variables.

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