Queuing with Deterministic Service Times and No Waiting Lines in Machine Type Communications

René Brandborg Sørensen, Student Member, IEEE, Jimmy J. Nielsen, Member, IEEE and Petar Popovski, Fellow, IEEE

Abstract—The growth of Machine-Type Communication (MTC) increases the relevance of queuing scenarios with deterministic service times. In this letter, we present a model for queues without waiting lines and with degenerate service time distributions and show how the framework is extendable to model general service time distributions. Simple bounds and a close approximation of the blocking probability are derived and the results are shown to hold for simulated queues with Markovian and degenerate arrival processes.

Index Terms—Queuing theory, G/D/n/n, G/G/n/n, Finite capacity, Multiple servers, degenerate service time, blocking probability, intermittency, outage

I. INTRODUCTION

A number of emerging use cases in Machine Type Communication (MTC) [1] require queueing models that are significantly different from traditional models used in teletraffic theory. A distinctive feature of many MTC applications is that machines are likely to perform different tasks within an almost-deterministic time. This brings relevance to queues with degenerate service times and non-Markovian, in particular periodical, arrival times. While networks in general have become packet-switched, relying heavily on buffering, there are still switching operations within communications that require immediate service or service within a very short time. Let us, for example, take the case of LoRa, where messages are modulated with one of seven different spreading factors. In LoRaWAN six spreading factors are used to create six quasi-orthogonal sub-channels in each channel within the network. LoRa gateway transceiver chipsets are capable of detecting preambles for every spreading factor simultaneously on multiple channels, but only a finite amount of demodulation paths are available [2], [3]. So messages in excess of the available demodulation paths are lost. Other general examples in telecommunications include: service of critical real-time interrupts [4], scheduling of immediate resources in FDMA networks and packet demodulation in FDMA networks.

We use Kendall’s notation [5], noting that ~/~/h/n refers to queues of finite capacity equivalent to the number of servers. The steady state solution for the M/D/n/n queue is well known as derived by A. K. Erlang. This solution was later shown to comply with simulation for Markovian and Degenerate arrival distributions. In the Markovian case, this entails that it also fits well with well-known exact solutions for M/G/n/n queues.

II. SYSTEM MODEL

We will develop a general analytical model for queues with deterministic service times, but we will treat the concrete problem of reservations of demodulation paths in LoRaWAN gateways as mentioned in the introduction. SX1301 is a chipset meant for usage in LoRaWAN gateways. This chipset is capable of demodulating up to 8 frames in parallel [7].

A. Arrival process

Let the number of messages transmitted within a fixed time $T$ be denoted $k$. The probability of at-least $k_0$ transmissions within a time step of $\tau$ is

$$B_0(\lambda, \tau) = \Pr(k \geq 0 \sum_{x \in \{1\}} t_x \leq \tau, \lambda) := 1,$$  

$$B_k(\lambda, \tau) = \Pr(k \geq k_0 \sum_{x=1}^{k_0} t_x \leq \tau, \lambda) = \iiint_D (f_{1,2,\ldots,p}) \, dD,$$  

$$D \in \{ t_1, \ldots, t_x | x=1 \sum_{x=1}^{k_0} t_x \leq \tau \}.$$  

Then we can find the probability of transmitting exactly $k$ messages in a period $\tau$ by

$$A_k(\lambda, \tau) = \Pr(k = p \sum_{x=1}^{p} t_x \leq \tau, \lambda),$$

$$= \Pr(k \geq p \sum_{x=1}^{p} t_x \leq \tau, \lambda) - \Pr(k \geq p + 1 \sum_{x=1}^{p+1} t_x \leq \tau, \lambda),$$

$$= B_k(\lambda, \tau) - B_{k+1}(\lambda, \tau), \text{ for } 0 \leq k \leq \infty.$$  

The set of received messages is a subset of the set of transmitted messages due to outage caused by for example poor channel conditions, noise or interference. Let $p_o$ be the outage probability for a transmission not to be received. Then the probability for the number of received messages can
A. Bounds on the blocking probability

be found by transforming the probability of the number of transmissions as

\[ A_k^B(\lambda, \tau) = \sum_{x=k}^{\infty} (A_k(\lambda, \tau)(1 - p_0)^{k-x} - k \binom{k}{x}). \]

B. Queue behaviour

We denote the number of transmissions being demodulated at \( t_{i-1} \) by \( K_{i-1} \) and the number of transmissions arriving over the demodulation paths after \( t_{i-1} + \tau \) by \( K_i \). Denote new arrivals in the queue by \( K_i^A \), demodulated transmissions by \( K_i^D \), and the number of blocked transmissions by \( K_i^B \). Transmissions are blocked when they arrive to find all demodulation paths unavailable as depicted in Fig. 1. Then we have

\[ K_i = K_{i-1} + K_i^A - K_i^D - K_i^B, \quad 0 \leq K_i \leq n. \]

where

\[ Z_i = K_i^A - K_i^D, \]

so

\[ f_{Z_i} = f_{K_i^A} * f_{K_i^D}, \]

where

\[ f_{K_i^A}[k] = f_{K_i^A}[-k], \]

and

\[ f_{K_i^D}[k] = A_k^B. \]

All messages that arrived within the prior period and weren’t blocked are served, so

\[ K_i^D = \min(K_{i-1}^A, n) \]

III. ANALYSIS

A. Bounds on the blocking probability

The blocking probability is defined as

\[ P_b = \frac{E[K_i^B]}{E[K_i^A]} + n, \quad (6) \]

where the substitution \( E[K_i^A] = E[K_i^B] + n \) in the denominator of (6) is valid, because the blocking probability is zero until the amount of messages in queue is larger than the total number of demodulation paths \( n \). This also means that

\[ f_{K_i^A}[k] = \begin{cases} 1, & \text{for } k = n \ , \\ 0, & \text{otherwise} \ . \end{cases} \]

Assume that there’s no spill-over between messages in the observed periods \( i-1 \) and \( i \), then \( K_{i-1} = 0 \) and we obtain a lower bound on the blocking probability.

\[ P_{b_{lower}} = \frac{\sum_{k=1}^{\infty} f_{K_i^A}[k] \cdot k}{\sum_{k=1}^{\infty} f_{K_i^A}[k] \cdot k + n}, \]

\[ = \frac{\sum_{k=n+1}^{\infty} (A_k(\lambda, \tau) \cdot (k-n))}{\sum_{k=n+1}^{\infty} (A_k(\lambda, \tau) \cdot (k-n)) + n}. \]

In the same manner, we can assume that there’s complete spill-over between messages in period \( i-1 \) and \( i \), then \( K_{i-1} = n \) and we obtain an upper bound on the blocking probability.

\[ P_{b_{upper}} = \frac{\sum_{k=0}^{\infty} f_{K_i^A}[k] \cdot k}{\sum_{k=0}^{\infty} f_{K_i^A}[k] \cdot k + n}, \]

\[ = \frac{\sum_{k=n}^{\infty} (A_k(\lambda, \tau) \cdot k)}{\sum_{k=n}^{\infty} (A_k(\lambda, \tau) \cdot k) + n}. \]

B. Blocking probability

To describe the exact blocking probability we need to describe the spill-over between observation period \( i-1 \) and \( i \), \( K_{i-1} \). We let \( X_i = K_{i-1} + Z_i \) so that

\[ f_{X_i}[x_i] = \sum_{k_{i-1}=0} f_{K_{i-1}, Z_i}[k_i-1, x_i - k_{i-1}], \]

\[ f_{K_{i-1}, Z_i}[k_{i-1}, z_i] = f_{Z_i}[z_i] f_{K_{i-1}}[k_{i-1}] f_{K_{i-1}}[k_{i-1}]. \]

Then we can describe the probability of blocking \( k \) transmissions as

\[ f_{K_i^A}[k] = \left\{ \begin{array}{ll} \sum_{x=-\infty}^{0} f_{X_i}[x], & \text{for } k = 0 \\ f_{X_i}[k + n], & \text{for } k \geq 1 \\ 0, & \text{otherwise} \end{array} \right. \]

Then we can use a prior for \( f_{K_i^A} \), to approximate \( f_{K_i^A}, f_{Z_i} \), and \( f_{X_i} \approx \sum (f_{X_i} f_{K_i^A}) \) and then approximate the blocking probability, \( P_b \), using (6).

In case a prior is not evident for an arrival process \( G \), then we may use

\[ f_{K_{i-1}}[k] = \left\{ \begin{array}{ll} \sum_{x=-\infty}^{0} f_{K_{i-1}}[X_i = x], & \text{for } k = 0 \\ f_{X_i}[k^A = k], & \text{for } 0 < k < n \end{array} \right. \]

\[ = \sum_{x=-\infty}^{0} f_{K_{i-1}}[X_i = x], \quad \text{for } k = n, \]

\[ = \sum_{x=-\infty}^{0} f_{K_{i-1}}[X_i = x], \quad \text{for } k = n, \]

\[ = 0, \quad \text{otherwise} \ . \]

C. Server state probability and server utilization

The server utilization can be found by considering the timing within the queue. Consider the case of \( n = 1 \), then the time spent without messages in queue can be described
as the time between completion of service of one message till the arrival of the next. Hence:
\[ T_0 = \sum_{k=0}^{\infty} C_k \quad \text{and} \quad T_1 = \tau , \]  
(14)
where \( C_k \) is the mean time spent without a message in queue if the \( k \)'th message is the first one received after the demodulation of another message finishes,
\[ C_k = \int_{t=\tau}^{\infty} \text{Pr}\left( \sum_{x=1}^{k} t_x = t \right) (t - \tau) \, dt , \]
for \( k \geq 0 \).
The complexity of describing the timing in this way increases greatly as \( n \) increases. The state ratio for state \( y \) is given by
\[ q_y = \frac{T_y}{\sum_{y'} T_{y'} } . \]  
(16)
Based on the state ratios we can compute the average number of messages being served as (17) and the server utilization as (18).
\[ L = \sum_{y=0}^{n} q_y \cdot y . \]  
(17)
\[ \eta = \frac{L}{n} . \]  
(18)
\[ \zeta = \eta (1 - P_k) , \]  
(19)
where \( \zeta \) is the non-blocking server utilization, which is the probability that the demodulation path is being used and message demodulation is not being blocked.

D. Jobs with non-homogeneous service times
We divide messages into classes \( C_1 \) through \( C_m \) corresponding to the different spreading factors in LoRaWAN. The service time of class \( x \) is given as
\[ \tau_x \in \tau = \{ \tau_1, \tau_2, ..., \tau_m \} \]  
for corresponding mean arrival rates of \( \lambda_x \in \lambda = \{ \lambda_1, \lambda_2, ..., \lambda_m \} \).
Denoting the number of messages from class \( y \) by \( k_y \), the probability of \( k_y \) messages arriving within the service period, \( \tau_y \), is \( A_k(\lambda_y, \tau_y) \) and the arrival distribution \( f_{K_x}(\lambda, \tau) \), we have
\[ f_{K_x}(\lambda, \tau) = \sum_{y=1}^{m} f_{K_y}(\lambda_y, \tau_y) \frac{\lambda_y}{\sum_{x=1}^{m} \lambda_x} . \]  
(20)
Any service time distribution can be represented by binning with infinitesimally small bins, so that we obtain two infinite sets for \( \lambda \) and \( \tau \), where \( \lambda_x = p_x \cdot \lambda \) and \( p_x = f_{\text{ServiceDist}}(\tau_x) \). Practically, the service time distribution can be accurately approximated by binning with an appropriately small finite bin size. In this way we may be able to represent any G/G/n/n queue in the described framework by binning the service time distribution.

IV. DEGENERATE AND MARKOVIAN ARRIVAL PROCESSES
In this section the transmission count probability, \( A_k \), is derived for degenerate and Markovian arrival processes by solving (2). The mean arrival-rate, \( \lambda' \), and priors, \( f_{K_i-1}[k] \), are also discussed for each arrival process.

A. Degenerate inter-arrival distribution
Let transmissions occur with inter-arrival times \( t_x \) that are distributed according to a degenerate distribution; \( f_{t}(t) = p_x \) for \( t = t_x \) where \( t_x \in \{ t_1, t_2, ..., t_y \} \) for a corresponding set \( p_x \in \{ p_1, p_2, ..., p_y \} \) where \( \sum_{y=1}^{y} p_x = 1 \). Solving (1) and (2) for the D/D/n/n queue we obtain
\[ B_0(\lambda, \tau) = \sum_{D} p_x , \quad \text{for} \quad D = \{ x | t_x \geq \tau \} , \]
\[ B_1(\lambda, \tau) = \sum_{D} p_x , \quad \text{for} \quad D = \{ x | t_x \leq \tau \} , \]
\[ B_2(\lambda, \tau) = \sum_{D} p_x , \quad \text{for} \quad D = \{ x_1, x_2 \} \]
\[ \sum_{y=1}^{2} \tau_{xy} \leq \tau , \]
\[ B_k(\lambda, \tau) = \sum_{D} p_x , \quad \text{for} \quad D = \{ x_1, x_2, ..., x_k | \sum_{y=1}^{k} \tau_{xy} \leq \tau \} , \]
\[ A_k(\lambda, \tau) = B_k(\lambda, \tau) - B_{k+1}(\lambda, \tau) . \]  
(21)
The prior of (13) gives good approximations for \( n > 1 \). When \( n=1 \), it is clear that \( K_{i-1} = 0 \) holds, so our prior should be \( f_{K_{i-1}}[k] = 1 \) for \( k = 0 \) where \( f_{K_{i-1}}[k] = 0 \). The mean arrival-rate is \( \lambda' = \frac{1}{\sum_{x=1}^{y} \frac{p_x}{T_x}} \). \( C_k \) can be found to be
\[ C_k(\lambda, \tau) = \sum_{D} p_x \cdot \left( \sum_{D} \tau_{xy} - \tau_y \right) , \]  
(22)
for \( D = \{ x_1, x_2, ..., x_k+1 | \sum_{y=1}^{k} \tau_{xy} \leq \tau, \sum_{y=1}^{k+1} \tau_{xy} > \tau \} , \]

B. Exponential inter-arrival distribution
Let transmissions occur with inter-arrival times \( t_x \) that are distributed according to an exponential distribution; \( f_{t}(t) = \lambda \exp(-\lambda t) \) for \( t \geq 0 \). Solving (2) for this arrival process we obtain
\[ A_0(\lambda, \tau) = \exp(-\lambda \tau) , \]
\[ A_1(\lambda, \tau) = \lambda \tau \cdot \exp(-\lambda \tau) , \]
\[ A_2(\lambda, \tau) = \frac{(\lambda \tau)^2}{2} \cdot \exp(-\lambda \tau) , \]
\[ A_3(\lambda, \tau) = \frac{(\lambda \tau)^3}{6} \cdot \exp(-\lambda \tau) , \]
\[ A_k(\lambda, \tau) = \frac{(\lambda \tau)^k}{k!} \cdot \exp(-\lambda \tau) . \]  
(23)
The mean arrival-rate is \( \lambda \). We shall use the prior in (13) when approximating \( P_k \). \( C_k \) can be found to be
\[ C_k(\lambda, \tau) = \frac{\exp(-\lambda \tau) \lambda^{k-1} \tau^k}{k!} . \]  
(24)

V. RESULTS
In this section we present results for the accuracy of this framework for modelling M/D/n/n and D/D/n/n queues for a fixed service time and a set of heterogeneous service times. Then we discuss the impact of the results on the example case of blockage in the demodulation paths of a LoRaWAN gateway.
The approximated blocking probability and bounds for the M/D/n/n queue can be found in Fig. 2. The approximation is close to the exact value, i.e. Erlang-B result. The server efficiency and non-blocking server efficiency are plotted in
The arrival count as a function of the service time in a D/D/n/n queue changes as a nontrivial step-wise function of the service time \( \tau \) and inter-arrival rate as depicted in Fig. 4. We observe that the index \( x \) of the smallest negligible \( A_x \) grows with the offered traffic load in Erlang. This also applies to the M/D/n/n queue, but in that case the count probability is a smooth function. The blocking probability exhibits the same behaviour as depicted in Fig. 5. The approximated blocking probability, here is also very close to simulated values. The Server efficiency and non-blocking server efficiency, \( \eta \), for the M/D/1/1 queue is fixed at \( \frac{1}{3} \)

The results of our investigation are straightforward to interpret as trade-offs in our motivating example of the blocking probability of demodulation paths in a LoRa receiver; Clearly a larger number of demodulation paths \( n \) yields a lower blocking probability, however the cost of increasing \( n \) should yield an equivalent gain in performance. In general the distribution of the arrival-process is of course of dire importance. We observe a higher blocking probability \( P_b = 0.25 \) in the M/D/2/2 queue than \( P_b = 0.068 \) in the D/D/2/2 queue (configured as ID 2 in 1) for \( \tau = 1 \) and \( \lambda' = \frac{1}{3} \). It is also evident that given Degenerate arrival times tuning the service time relatively little can yield large gains.

### VI. Conclusion

In this paper, a framework for modelling G/D/n/n queues was presented and we looked at the case of blocking demodulation paths in LoRaWAN receivers. The proposed framework assesses the arrival count in service periods to model blocking probabilities in G/D/n/n quite accurately. Bounds valid for the...
blocking probability in G/D/n/n queues were also presented. In essence, we showed how blocking probability depends on the arrival process, which is therefore essential for describing the exact blocking probability for e.g. demodulation in LoRaWAN receivers. In general, increasing \( n \) decreases the blocking probability as one would expect intuitively. Exponential inter-arrival times are often assumed for the arrival process of communication networks, but for example in the case of the LoRaWAN receiver, this transmission process will be filtered by capture effect, yielding a non-Markovian distribution at the demodulation paths.

The framework depends on finding counting functions, which may be relatively easy to find by induction given the tools available today for integrating symbolic expressions. Moreover, the framework is directly applicable when the distribution of the inter-arrival times is given numerically as ‘binned’ or Degenerate approximation of the arrival process, but it is not available on an explicit analytical form. The

**Fig. 5:** The blocking probability in D/D/n/n queues as a stepwise function of the service time \( \tau \) for the four configurations of the degenerate arrival process outlined in Tab. I.

**Fig. 6:** State probabilities, \( q_0 \) and \( q_1 \), server efficiency, \( \eta = q_1 \), and non-blocking server efficiency, \( \zeta \), for the M/D/1/1 queue for ID 1 in Tab. I.

**Fig. 7:** \( p_b \) for Markovian (above) and degenerate (below) arrival processes and a service process that is defined by \( p_x = \{\frac{1}{2}, \frac{1}{3}\} \) and \( \tau_x = \{\frac{2}{3}, \frac{4}{3}\} \). The degenerate arrival processes in this example is ID 1 & ID 2 where \( \tau \) is scaled to achieve the mean service rate.

methodology was shown to yield close approximates of the well known results for the M/G/n/n queue along with close approximates for the D/D/n/n queue. The framework was extended to G/G/n/n-queues, but verification was limited to a small set of service times due to simulation complexity.

**REFERENCES**

[1] T. Hofeld, F. Metzger, and P. E. Heegaard, “Traffic modeling for aggregated periodic iot data,” in 2018 21st Conference on Innovation in Clouds, Internet and Networks and Workshops (ICIN), 2018, pp. 1–8.

[2] A. Rahmadhani and F. Kuipers, “When lorawan frames collide,” Proc. of the 12th International Workshop on Wireless Network Testbeds, Experimental Evaluation & Characterization (ACM WINE) 2018, 2018.

[3] R. B. Sørensen, N. Razmi, J. J. Nielsen, and P. Popovski, “Analysis of lorawan uplink with multiple demodulating paths and capture effect,” in ICC 2019 - 2019 IEEE International Conference on Communications (ICC), May 2019, pp. 1–6.

[4] J. Kreuzinger, A. Schulz, M. Pfeffer, T. Ungerer, U. Brinkschulte, and C. Krakowski, “Real-time scheduling on multithreaded processors,” in Proceedings Seventh International Conference on Real-Time Computing Systems and Applications, 2000, pp. 155–159.

[5] D. G. Kendall, “Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded markov chain,” Annals of Mathematical Statistics, vol. 24, no. 3, pp. 338–354, 09 1953. [Online]. Available: https://doi.org/10.1214/aoms/117772975

[6] L. Takacs, “On erlang’s formula,” The Annals of Mathematical Statistics, vol. 40, no. 1, pp. 71–78, 1969. [Online]. Available: http://www.jstor.org/stable/2239199

[7] Semtech, SX1301 Datasheet, v2.4 ed., June 2017.