REMARK ON THE CHAIN RULE OF FRACTIONAL DERIVATIVE IN THE SOBOLEV FRAMEWORK

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Abstract. A chain rule for power product is studied with fractional differential operators in the framework of Sobolev spaces. The fractional differential operators are defined by the Fourier multipliers. The chain rule is considered newly in the case where the order of differential operators is between one and two. The study is based on the analogy of the classical chain rule or Leibniz rule.

1. Introduction

The chain rule or Leibniz rule is an essential tool to study nonlinear differential equations. In the study of nonlinear partial differential equations (PDEs), fractional differential operators are also known as powerful tools. So, in order to analyze nonlinear PDEs, chain rules for fractional differential operators are naturally required. Even though fractional differential operators may be non-local unlike classical operators, estimates for fractional derivative has been studied on the analogy of classical chain rules. The history of this study can go back at least to the work of Kato and Ponce [10].

In this paper, we consider a chain rule corresponding to the identity:

\[
\frac{d}{dx} F(u) = F'(u) u',
\]

in the framework of Riesz potential space \( \dot{H}^s_p = D^{-s} L^p(\mathbb{R}^d) \), where \( s \in \mathbb{R}, d \geq 1 \), and \( L^p \) is the usual Lebesgue space. \( \dot{H}^s_p \) is also called as homogeneous Sobolev space. The fractional differential operator \( D^s = (-\Delta)^{s/2} \) is recognized as a Fourier multiplier by \( D^s = \mathcal{F}^{-1} |\cdot|^s \mathcal{F}, \) where \( \mathcal{F} \) is the standard Fourier transform. Especially, we study the case where \( F \) behaves like power product, that is, \( F(z) \sim |z|^{p-1} z \).

In [2], Christ and Weinstein showed the following estimate:

**Proposition 1.1** ( [2, Proposition:3.1]). Let \( d \geq 1 \) and \( s \in (0, 1) \). Let \( F \in C(\mathbb{C}) \) and \( G \in C(\mathbb{C} : [0, \infty)) \) satisfy

\[
|F(u) - F(v)| \leq (G(u) + G(v))|u - v|.
\]

Let \( 1 < p < \infty \), \( 1 < r, q < \infty \) satisfy

\[
\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.
\]

The estimate

\[
\|F(u)\|_{\dot{H}^s_p} \leq \|G(u)\|_{L^q} \|u\|_{\dot{H}^s_r}.
\]

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holds for any \( u \in \dot{H}^s_p \) with \( G(u) \in L^q \).

Roughly speaking, Proposition 1.1 asserts that \( D^s F(u) \) behaves like \( F'(u) D^s u \). Since the Riesz operator \( R = D^{-1} \nabla \) is bounded on \( L^p \) when \( p \in (1, \infty) \), \( \| F(u) \|_{\dot{H}^s_p} \) may be estimated even when \( s \geq 1 \) by combining the classical chain rule, Proposition 1.1, and the Hölder inequality.

On the other hand, when \( F(z) = |z|^{\rho - 1} z \), Proposition 1.1 may be insufficient to handle general cases. For example, when \( \rho \in (1, 2) \) and \( s \in (1, \rho) \), one may regard

\[
D^s F(u) \sim D^{s-1} (|u|^{\rho-1} \nabla u).
\]

By using the fractional Leibniz rule (for example, see [8, Theorem 1], [2], [5], and references therein), one can distribute \( \nabla \) to \( \nabla u \) and \( |u|^{\rho-1} \). Since \( f(z) = |z|^{\rho-1} \) is not differentiable, one cannot control \( D^{s-1} |u|^{\rho-1} \) by applying Proposition 1.1 directly. However, on the analogy of the fact that \( f^p \in C^1 \) when \( f \in C[\rho] \), \( D^s (|u|^{\rho-1} u) \) is expected to be controlled similarly when \( s < \rho \). Here \( \lceil \cdot \rceil \) is the ceiling function defined by \( \lceil s \rceil = \min\{ m \in \mathbb{Z} : m \geq s \} \). In order to see this natural expectation, one may need to extend Proposition 1.1 with \( s \in (1, 2) \).

We generalize this expectation slightly. With \( \rho > 1 \), we put \( F_\rho \in C^1(\mathbb{C}) \) satisfying \( F(0) = F'(0) = 0 \) and

\[
|F_\rho(u) - F_\rho(v)| \lesssim \max(|u|, |v|)^{\rho-1}|u - v|, \quad |F_\rho'(u) - F_\rho'(v)| \lesssim \begin{cases} 
\max(|u|, |v|)^{\rho-2}|u - v| & \text{if } \rho \geq 2, \\
|u - v|^{\rho-1} & \text{if } \rho < 2,
\end{cases}
\]

where \( a \lesssim b \) stands for \( a \leq Cb \) with some positive constant \( C \). We also denote \( a \sim b \) when \( a \lesssim b \) and \( b \lesssim a \). We use these notation though this paper.

In [8], Ginibre, Ozawa, and Velo showed the expectation above in the framework of homogeneous Besov spaces \( \dot{B}^{s,q}_p \).

**Proposition 1.2** ([8, Lemma 3.4]). Let \( d \geq 1 \), \( \rho \geq 1 \), and \( s \in (0, \min(2, \rho)) \). Let \( 1 \leq p, r \leq \infty \) and \( (\rho - 1)^{-1} \leq q \leq \infty \) satisfy (1.1). If \( u \in \dot{B}^{s,2}_r \cap L^{(\rho-1)q} \), then

\[
\| F_\rho(u) \|_{\dot{B}^{s,2}_r} \lesssim \| u \|_{L^{(\rho-1)q}} \| u \|_{\dot{B}^{s,2}_r}. \quad (1.3)
\]

Since Besov spaces may play a role in useful auxiliary spaces, Proposition 1.2 has been used to study even \( \dot{H}^s_p \)-valued solutions to some nonlinear PDEs. The advantage to consider the fractional chain rule in the framework of Besov space is the following representation of homogeneous Besov norms:

\[
\| u \|_{\dot{B}^{s,q}_r} \sim \left( \int_0^\infty \lambda^{-sq-1} \sup_{|y| < \lambda} \| (\tau_y - 2 + \tau_{-y}) u \|_{L^r}^q d\lambda \right)^{1/q},
\]

where \( \tau_y u = u(\cdot + y) \), \( s \in (0, 2) \), and \( 1 \leq p, q \leq \infty \) (See [1] 6.2.5. Theorem, for example.). With this representation, the term \( (\tau_y - 2 + \tau_{-y}) u \) gives a clear explanation of the connection between the classical and fractional chain rules. We remark that even though the fractional differential operators are defined by Fourier multipliers, it is nontrivial why \( F'(u) \sim |u|^{\rho-1} \) appears as an upper bound of (1.3) from the viewpoint of Fourier multipliers. We further remark that when \( \rho = 2 \), the chain rule may be shown by the argument of Fourier multipliers. See [8].

Proposition 1.2 is an effective estimate but it seems more handy to close the argument only with Sobolev spaces. So one of the main purpose of this paper is to
show a similar estimate to Proposition 1.2 in the framework of $\dot{H}^s_p$. This is one of the main statement of this paper.

**Proposition 1.3.** Let $p > 1$ and $s \in (1, p)$. Let $1 < p, r < \infty$ and $(\rho - 1)^{-1} < q < \infty$ satisfy (1.1). The estimate

$$\|F_p(u)\|_{\dot{H}^s_p} \lesssim \|u\|_{L^q(\rho - 1)} \|u\|_{\dot{H}^s_p}$$

holds for any $u \in \dot{H}^s_p \cap L^q(\rho - 1)$. We remark that the case when $p$ or $r = 1, \infty$ is included in the assumption of Proposition 1.2 but not in that of Proposition 1.3. It is because our proof is based on the boundedness of Hardy-Littlewood maximal operator. See next section.

The main idea for the proof of Proposition 1.3 is to express $\|F(u)\|_{\dot{H}^s_p}$ with $(\tau_y - 2 + \tau_{\cdot -y})F(u)$ like the proof of Proposition 1.2. Since $\dot{H}^s_p$ seems not to be expressed like (1.2), we rewrite each dyadic components of $F(u)$ by $(\tau_y - 2 + \tau_{\cdot -y})F(u).$ Namely, the identity

$$Q_j F(u)(x) = \frac{1}{2} \int (\tau_y - 2 + \tau_{\cdot -y})F(u)(x)\psi_j(y)dy$$

(1.4)

plays a critical role in this paper, where $Q_j$ is the standard $j$-th Littlewood-Paley dyadic operator and $\psi_j$ is the corresponding kernel. The details of $Q_j$ and $\psi_j$ are stated in next section. We note that the identity

$$Q_j F(u)(x) = \int (\tau_y - 1)F(u)(x)\psi_j(y)dy$$

plays an essential role in [2] as well. In this paper, we deploy a similar but more careful approach with (1.4).

Our second purpose is to extend Corollary 1.4 below following from Proposition 1.1. Here we put $a_+ = \max(a, 0)$ and $a_- = \min(a, 0)$ for $a \in \mathbb{R}$.

**Corollary 1.4.** Let $s \in (0, 1), \rho \geq 1$. Let $1 \leq p < \infty$ and $1 < q, r < \infty$ satisfy (1.1). Then the estimate

$$\|F_p(u) - F_p(v)\|_{\dot{H}^s_p} \lesssim (\|u\|_{L^q(\rho - 1)} + \|v\|_{L^q(\rho - 1)})^{s-1} \|u - v\|_{\dot{H}^s_p} + \|u\|_{\dot{H}^s_p} \|u\|_{L^q(\rho - 1)} + \|u\|_{L^q(\rho - 1)}(\rho - 2)^+ \|u - v\|_{L^q(\rho - 1)}^{\min(\rho - 1, 1)}$$

holds for any $u \in \dot{H}^s_p \cap L^q(\rho - 1)$.

Corollary 1.4 corresponds to the identity

$$\nabla(u^p - v^p) = \rho u^{p-1} \nabla(u - v) + \rho (u^{\rho - 1} - v^{\rho - 1}) \nabla v.$$
Proposition 1.5. Let \( \rho \in (1, 2) \) and \( s \in (1, \rho) \). Let \( 1 \leq p < \infty, 1 < r < \infty, (\rho - 1)^{-1} < q < \infty \) satisfy \((1.1)\). Let \( d_{\rho,s} \in (0, \rho - s) \). Then

\[
\| F_\rho(u) - F_\rho(v) \|_{\dot{H}_r^s} \\
\lesssim (\| u \|_{L^q(\rho - 1)} + \| v \|_{L^q(\rho - 1)})^{\rho - 1} \| u - v \|_{\dot{H}_r^s} \\
+ (\| u \|_{\dot{H}_r^s} + \| v \|_{\dot{H}_r^s})(\| u \|_{L^q(\rho - 1)} + \| v \|_{L^q(\rho - 1)})^{\rho - 1 - d_{\rho,s}} \| u - v \|_{L^p(\rho - 1)}^{d_{\rho,s}}.
\]

We note that an extension of Corollary 1.4 in the framework of Besov spaces was given in [11] Proposition 2.1 and [4] Lemma 6.2. We remark that on the analogy of classical chain rules, it is expected that Proposition 1.5 may hold with \( d_{\rho,s} = \rho - s \) but a similar technical difficulty arises in both Sobolev and Besov frameworks. We also remark that the case where \( \rho = s \) is hopeless to extend Proposition 1.5 because of the failure of classical analogy, say \( |x| \notin C^1 \), for example.

In the next section, we collect some notation and basic estimates. In section 3 we revisit the proof of Proposition 1.1 for the completeness. In section 4 we give the proofs of Propositions 1.2 and 1.3.

2. Preliminary

2.1. Notation. Here we collect some notation.

Let \( S \) be the Schwartz class. Let \( \psi \in S \) be radial and satisfy

\[
supp \tilde{\psi} \subset \{ \xi \mid 1/2 \leq |\xi| \leq 2 \}
\]

and

\[
\sum_j \tilde{\psi}(2^{-j}\xi) = 1
\]

for \( \xi \neq 0 \). For \( j \in \mathbb{Z} \), let \( \psi_j = 2^n \psi(2^j \cdot) \) so that \( \| \psi_j \|_{L^1} = \| \psi \|_{L^1} \) and let \( Q_j = \psi_j * \).

We also put \( \bar{Q}_j = Q_{j-1} + Q_j + Q_{j+1} \) and \( \tilde{\psi}_j = \psi_{j-1} + \psi_j + \psi_{j+1} \). We remark that \( \bar{Q}_j Q_j = Q_j \) holds for any \( j \). It is known that for \( s \in \mathbb{R} \) and \( 1 < p, q < \infty \), the homogeneous Sobolev and Triebel-Lizorkin norms are equivalent, that is,

\[
\| f \|_{\dot{H}_r^s} \sim \| 2^j Q_j f \|_{L^p(\mathbb{R}^d)}.
\]

For the details of this equivalence, we refer the reader to [7] Theorem 5.1.2, for example. For \( f \in L^1_{loc} \), we define the Hardy-Littlewood maximal operator by

\[
M f(x) = \sup_{r > 0} \frac{1}{|B(r)|} \int_{B(r)} |f(x + y)| \, dy
\]

and

\[
M^{(\eta)} f(x) = M(\| f \|^{\eta})^{1/\eta}
\]

for \( \eta > 0 \), where \( B(r) \subset \mathbb{R}^d \) is the ball with radius \( r \) centered at the origin. It is known that \( M \) is bounded operator on \( L^p \) and \( L^p(\mathbb{R}^d) \) for \( 1 < p, q < \infty \) (See [7] Theorems 2.1.6 and 4.6.6 and reference therein). Moreover, for \( 1 \leq p < \infty \), we define weighted \( L^p \) norm with weight function \( w \) by

\[
\| f \|_{L^p_w} = \left( \int |f|^p w \, dx \right)^{1/p}.
\]
2.2. Basic Estimates. Here we collect some estimates.

**Lemma 2.1** ([7, Theorem 2.1.10]). Let \( w \in L^1 \) be a positive radially decreasing function. Then the estimate
\[
|w * g(x)| \leq \|w\|_{L^1} Mg(x)
\]
holds for any \( x \in \mathbb{R}^n \).

**Lemma 2.2.**
\[
|\tilde{Q}_j g(x + y) - \tilde{Q}_j g(x)| \lesssim Mg(x) + Mg(x + y),
\]
\[
|\tilde{Q}_j g(x + y) - 2\tilde{Q}_j g(x) + \tilde{Q}_j g(x - y)| \lesssim Mg(x + y) + Mg(x) + Mg(x - y).
\]
Moreover, if \( |y| < 2^{-j} \), then we have
\[
|\tilde{Q}_j g(x + y) - \tilde{Q}_j g(x)| \lesssim 2^j |y| Mg(x),
\]
\[
|\tilde{Q}_j g(x + y) - 2\tilde{Q}_j g(x) + \tilde{Q}_j g(x - y)| \lesssim 2^{2j} |y|^2Mg(x).
\]

**Proof.** The estimates for \( |\tilde{Q}_j g(x + y) - \tilde{Q}_j g(x)| \) is given in [2] but for completeness, we give the proof.

The first two estimate follows directly from Lemma 2.1.

By the fundamental theorem calculus implies that the following identities hold:
\[
\psi_j(x + y) - \psi_j(x) = \int_0^1 (\nabla \psi)_j(x + \theta y) d\theta \cdot 2^j y
\]
and
\[
\psi_j(x + y) - 2\psi_j(x) + \psi_j(x - y)
\]
\[
= \int_0^1 ((\nabla \psi)_j(x + \theta y) - (\nabla \psi)_j(x - \theta y)) d\theta \cdot 2^j y
\]
\[
= \sum_{|\alpha| = 2} \int_0^1 \int_0^1 (\partial^\alpha \psi)_j(x + (2\theta' - 1)\theta y)) d\theta' d\theta \cdot 2^{2j} y^2.
\]

Since for \( |\alpha| \leq 2 \), if \( |y| < 2^{-j} \),
\[
(\partial^\alpha \psi)_j(x + y) \lesssim 2^j (2 + 2^j |x + y|)^{-n - 1} \leq 2^j (1 + 2^j |x|)^{-n - 1},
\]
Then the third and fourth estimates are obtained by combining this and Lemma 2.1. \( \square \)

**Lemma 2.3.** For \( 0 < S < S' \), \( P \geq 1 \), and \( \alpha \in \ell^P \), the following estimate holds:
\[
\|2^jS \sum_k 2^{(k-j)-S'} a_k\|_{\ell^P} \leq \left( \frac{1}{2^S - 1} + \frac{1}{2^{S' - 1}} \right) \|2^{ks} a_k\|_{\ell^P}.
\]

**Proof.** By the Minkowski inequality, we have
\[
\|2^jS \sum_{k<j} 2^{(k-j)-S'} a_k\|_{\ell^P} = \|2^jS \sum_{\ell<0} 2^{\ell} \alpha_{\ell+j}\|_{\ell^P} + \|2^jS \sum_{\ell \geq 0} \alpha_{\ell+j}\|_{\ell^P}
\]
\[
\leq \sum_{\ell < 0} 2^{(S' - S)} \|2^{(\ell+j)} \alpha_{\ell+j}\|_{\ell^P} + \sum_{\ell \geq 0} 2^{-\ell} \|2^{S(j)} \alpha_{\ell+j}\|_{\ell^P}
\]
\[
\leq \frac{2}{2^{S' - S} - 1} \|2^{S'k} a_k\|_{\ell^P} + \frac{1}{2^S - 1} \|2^{Sk} a_k\|_{\ell^P}.
\]
\( \square \)
Lemma 2.4. Let $0 < S < 1$ and $1 \leq P, Q < \infty$. The following estimate holds:

$$
\|2^{jS}(v(y + x)(u(y + x) - u(x))\|_{L_{\psi_j,y}^P}^P \lesssim \|M^{(P)}v(x)\|_{L_{\psi_j,y}^P}^P \|kMQ_k u\|_{L_{\psi_j,y}^Q}^Q + \|2^{kS}M^{(P)}(vMQ_ku(x))\|_{L_{\psi_j,y}^Q}^Q
$$

Proof. Let $w \in L^1$ be smooth radially decreasing function satisfying

$$
\|w(x) - \psi(x)\| \lesssim (1 + |x|)^P \|\psi(x)\|
$$

and put $w_j = 2^{jn}w(2^j \cdot)$. Lemmas 2.1 and 2.2 imply that we have

$$
\|v(y + x)(u(y + x) - u(x))\|_{L_{\psi_j,y}^P}^P
\leq \sum_k \|v(y + x)(\overline{Q}_k u(y + x) - \overline{Q}_k u(x))\|_{L_{\psi_j,y}^P}^P
\leq \sum_{k<j} 2^k \|v(y + x)\|_{L_{\psi_j,y}^P}^P \|MQ_k u(x)\|
\leq \sum_{k<j} \left(\|v(y + x)MQ_k u(x + y)\|_{L_{\psi_j,y}^P}^P + \|MQ_k u(x)\|_{L_{\psi_j,y}^P}^P \|v(y + x)\|_{L_{\psi_j,y}^P}^P\right)
\leq \|w\|_{L^1} \sum_{k<j} 2^{k-j} M^{(P)} v(x)MQ_k u(x)
\leq \|w\|_{L^1} \sum_{k<j} \left(M^{(P)}(vMQ_k u(x)) + M^{(P)}(vMQ_k u(x))\right).
$$

Therefore, Lemma 2.4 follows from these estimates above and Lemma 2.3. □

Corollary 2.5. Let $0 < S < 1$, $1 \leq P_0 < 2$. Let $P_0 < P < \infty$ and $1 < Q, R < \infty$ satisfy

$$
\frac{1}{P} = \frac{1}{Q} + \frac{1}{R}.
$$

The estimates

$$
\|2^{jS}\|_{L_{\psi_j,y}^{P_0}}^{P_0} \|u(y + \cdot) - u(\cdot)\|_{L_{\psi_j,y}^P}^P \lesssim \|u\|_{H^S_P}
$$

and

$$
\|2^{jS}\|_{L_{\psi_j,y}^{P_0}}^{P_0} \|u(y + \cdot) - u(\cdot)\|_{L_{\psi_j,y}^P}^P \lesssim \|v\|_{L^Q} \|u\|_{H^S_R}
$$

hold.

Proof. For $P_1 > P_0$, the boundedness of $M$ implies that we have

$$
\|M^{(P_0)}f\|_{L^{P_1}} = \|M[f]^{P_0}\|_{L^{P_1}}^{1/P_0} \lesssim \|f\|_{P_1}^{P_0} \|f\|_{L^{P_1}}^{1/P_0} = \|f\|_{L^{P_1}}.
$$

Since $P_0 < P < Q$, Corollary 2.5 follows from Lemma 2.4 and the Hölder inequality. □

3. Revisit of the Chain rules when $s \in (0, 1)$

In this section, we revisit the proof of Proposition 1.1 and Corollary 1.4. The proofs are essentially given in [2] but for completeness, we give the proofs.
Proof of Proposition 1.1 Since $\int \psi(y) dy = 0$, the identity

$$Q_j F(u)(x) = \int F(u(x + y) - F(u)(y)) \psi_j(y) dy$$

holds for any $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Therefore, Proposition 1.1 follows from Corollary 2.5 with $(v, P_0, P, Q, R, S)$ replaced by $(G(u), 1, p, q, r, s)$.

Proof of Corollary 1.4 The identity

$$Q_j F^j(\rho)(u)(x) = \int (F^j(\rho)(u)(x + y) - F^j(\rho)(u)(y) + F^j(\rho)(v)(x + y) - F^j(\rho)(v)(x)) \psi_j(y) dy$$

holds. The fundamental theorem of calculus implies that the identity

$$F^j(\rho)(u)(x + y) - F^j(\rho)(u)(y) + F^j(\rho)(v)(x + y) - F^j(\rho)(v)(x)$$

$$= \{(u - v)(x + y) - (u - v)(x)\} \int_0^1 F^j(\theta u(x + y) + (1 - \theta)u(x)) d\theta$$

$$+ \{v(x + y) - v(x)\} \cdot \int_0^1 \left\{ F^j(\theta u(x + y) + (1 - \theta)u(x)) - F^j(\theta v(x + y) + (1 - \theta)v(x)) \right\} d\theta.$$}

holds. The identity above and (4.2) imply that the estimate

$$|F^j(\rho)(u)(x + y) - F^j(\rho)(u)(y) + F^j(\rho)(v)(x + y) - F^j(\rho)(v)(x)|$$

$$\lesssim (|u(x + y)|^{\rho - 1} + |u(x)|^{\rho - 1})(|u - v)(x + y) - (u - v)(x)|$$

$$+ (|u(x + y)| + |u(x)| + |v(x + y)| + |v(x)|)^{(\rho - 2)}$$

$$\cdot (|(u - v)(x + y)| + |(u - v)(x)|^{\rho - 1}(|u - v)(x + y) - (u - v)(x)|$$

holds. Therefore, Corollary 1.4 follows from Corollary 2.5 and the estimates above.

4. Proofs of the Chain rules when $s \in (1, 2)$

Proof of Proposition 1.3 We note that it is shown in [6] (3.26) that the estimate

$$|(\tau_y - \tau_{-y}) F^j(\rho)(u)(x)| \lesssim |u(x)|^{\rho - 1}|u(x + y) - 2u(x) + u(x - y)|$$

$$+ |u(x + y) - u(x)|^\rho + |u(x - y) - u(x)|^\rho$$

(4.1)

holds. Moreover, the symmetry of $\psi$ and $\int \psi = 0$ imply that we have

$$Q_j F^j(\rho)(u)(x) = \int F^j(\rho)(u)(x + y) \psi_j(y) dy$$

$$= \int F^j(\rho)(u)(x - y) \psi_j(y) dy$$

$$= \frac{1}{2} \int (F^j(\rho)(u)(x + y) + F^j(\rho)(u)(x - y)) \psi_j(y) dy$$

$$= \frac{1}{2} \int (\tau_y - \tau_{-y}) F^j(\rho)(u)(x) \psi_j(y) dy$$

(4.2)
From (4.1) and (4.2), the estimate
\begin{align}
|Q_j F(u)(x)| \lesssim |u(x)|^{p-1} \int |u(x+y) - 2u(x) + u(x-y)| |\psi_j(y)| \, dy \\
+ \int |u(x+y) - u(x)|^p |\psi_j(y)| \, dy
\end{align}
(4.3)
follows. Put \( w(x) \) be radially decreasing function with \( w(x) \geq (1+|x|^2) |\psi(x)| \). By Lemma 2.2 when \( k < j \), the first term on the RHS of (4.3) is estimated by
\begin{align*}
&\int_{|y| < C2^{-k}} |\tilde{Q}_k Q_k u(x+y) - 2\tilde{Q}_k Q_k u(x) + \tilde{Q}_k Q_k u(x-y)| |\psi_j(y)| \, dy \\
&\leq MQ_k u(x) \int_{|y| < C2^{-k}} 2^{2(k-j)} \omega_j(y) \, dy \\
&\lesssim 2^{2(k-j)} MQ_k u(x).
\end{align*}
Similarly, when \( k \geq j \), it is estimated by
\begin{align*}
&\int_{|y| > C2^{-k}} |\tilde{Q}_k Q_k u(x+y) - 2\tilde{Q}_k Q_k u(x) + \tilde{Q}_k Q_k u(x-y)| |\psi_j(y)| \, dy \\
&\lesssim 2^{2(k-j)} \int_{|y| > C2^{-k}} (2MQ_k u(x+y) + 2MQ_k u(x) |\omega_j(y)| \, dy \\
&\lesssim 2^{2(k-j)} (MQ_k u(x) + M^2 Q_k u(x)).
\end{align*}
Therefore, the estimates above and Lemma 2.2 imply that we have
\begin{align}
&\|2^{js} |u|^{p-1} \int |u(\cdot + y) - 2u(\cdot) + u(\cdot - y)| |\psi_j(y)| \, dy \|_{L^p(\mathbb{R}^2)} \\
&\lesssim \|2^{ks} |u|^{p-1} (MQ_k u + M^2 Q_k u)\|_{L^p(\mathbb{R}^2)} \\
&\lesssim \|u\|_{L^{q(p-1)}} \|u\|_{H^s_{\mathbb{R}^2}}. \quad (4.4)
\end{align}
Moreover, by the Lemma 2.4, the second term of the RHS of (4.3) satisfy the estimate
\begin{align}
\|2^{js} |u(\cdot + y) - u(\cdot)| |\nu_{\psi_j}^p\|_{L^p(\mathbb{R}^2)} &= \|2^{js/p} |u(\cdot + y) - u(\cdot)| |\nu_{\psi_j}^p\|_{L^p(\mathbb{R}^2)} \\
&\lesssim \|2^{ks/p} M^{(p)} Q_k u \|_{L^p(\mathbb{R}^2)} \\
&\lesssim \|u\|_{H^{s/p}_{\mathbb{R}^2}} \\
&\lesssim \|u\|_{L^{q(p-1)}} \|u\|_{H^s_{\mathbb{R}^2}}. \quad (4.5)
\end{align}
where we have used the Gagliardo-Nirenberg inequality to obtain the last estimate. For the details of the Gagliardo-Nirenberg inequality, we refer the reader to [3] Corollary 2.4 and reference therein. Combining (4.4) and (4.5), we obtain Proposition 1.3 \( \square \)
Proof of Proposition 1.5. For simplicity, we denote $d_{p,s}$ by $d$. We note that the estimate

$$|\tau_y - 2 + \tau_{-y}||F_p(u) - F_p(v)|(x)|$$

$$\lesssim \max_{z \in \{x, x+y, x-y\}} |u(z)|^{p-1}|\tau_y - 2 + \tau_{-y}|(z-w)(x)|$$

$$+ \max_{z \in \{x, x+y, x-y\}} |u(z)|^{p-1}|\tau_y - 2 + \tau_{-y}|w(x)|$$

$$+ (|\tau_y - 1|u(x)| + |\tau_{-y} - 1|u(x)|)^{p-1}|\tau_y - 1|(u-v)(x)|$$

$$+ |\tau_{-y} - 1|u(x)| \min(\sigma, \max_{z \in \{x, x+y, x-y\}}|(u-v)(z)|)^{p-1}$$

holds, where

$$\sigma = |(\tau_y - 1)u(x)| + |(\tau_{-y} - 1)v(x)| + |(\tau_{-y} - 1)v(x)|.$$  

For the details of the estimate above, see [4, Lemma A.1]. Therefore, the proof of proposition 1.5 it is sufficient to show

$$\|2^{j_s}B(u, v)(\cdot, y)\|_{L^{1}_{\psi_j,y}} \|L^p(\mathbb{E})}$$

$$\lesssim (\|u\|_{L^q(p-1)} + \|v\|_{L^q(p-1)})^{p-1-d_{p,s}}\|u - v\|_{L^{q_{0}d_{p,s}}}^{d_{p,s}}.$$  

(4.7)

where

$$B(u, v)(x, y) = |u(x+y) - u(x)||v(x+y) - v(x)|^{p-1-d}|u(x+y) - v(x)|^d.$$  

The other terms on the RHS of (4.6) may be treated like the argument above so we omit the detail. Put

$$q_0 = q \frac{\rho - 1}{d} \quad \text{and} \quad r_0 = \frac{pq_0}{q_0 - p}.$$  

Then the Hölder inequality and Lemma 2.1 imply that we have

$$\|B(u, v)(x, y)\|_{L^{1}_{\psi_j,y}}$$

$$\leq \|(u - v)(x+y)\|^{d}_{L^{q_{0}/d_{p,s}}}\|u(x+y) - u(x)||v(x+y) - v(x)|^{p-1-d}\|L^{r_0/p}_{\psi_j,y}$$

$$\lesssim (M(q{-1}/p)|u - v|(x))^{d}$$

$$\cdot \|u(x+y) - u(x)\|^{(p{-1})}_{L^{d_{p,s}}(r_0/p)}\|v(x+y) - v(x)\|^{(p-1)-d}_{L^{d_{p,s}}(r_0/p)}.$$  

Then Lemma 2.1 and Corollary 2.6 imply that the estimates

$$\|2^{j_s}B(u, v)(\cdot, y)\|_{L^{1}_{\psi_j,y}} \|L^p(\mathbb{E})}$$

$$\lesssim \|M(q{-1}/p)|u - v|\|^d_{L^{d_{q_{0}}}}$$

$$\cdot \|2^{j_s/(p-d)}(u + y) - u(\cdot)\|_{L^{q_{0}/d_{p,s}}(r_0/p)}\|L^{r_0/(p-d)}(\mathbb{E})}$$

$$\cdot \|2^{j_s/(p-d)}(v + y) - v(\cdot)\|_{L^{q_{0}/d_{p,s}}(r_0/p)}\|L^{r_0/(p-d)}(\mathbb{E})}$$

$$\lesssim \|u - v\|^{d}_{L^{q_{0}/d_{p,s}}}\|u\|_{L^{q_{0}/d_{p,s}}(r_0/p)}\|v\|^{(p-1)-d}_{L^{d_{p,s}}(r_0/p)}.$$  

hold. Then (4.7) follows from the estimates above and the Gagliardo-Nirenberg inequality.
Remark 4.1. In the proof above, one cannot take \( d = \rho - s \) because
\[
\|2^j (u(\cdot + y) - u(\cdot))\|_{L^{(\rho-d)\gamma_0/p}} L^{r_0(\rho-d)}(\ell^2)
\]
is not controlled by Lemma 2.4.

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