Abstract. We find sufficient conditions for the construction of vertex algebraic intertwining operators, among generalized Verma modules for an affine Lie algebra $\hat{g}$, from $g$-module homomorphisms. When $g = \mathfrak{sl}_2$, these results extend previous joint work with J. Yang, but the method used here is different. Here, we construct intertwining operators by solving Knizhnik-Zamolodchikov equations for three-point correlation functions associated to $\hat{g}$, and we identify obstructions to the construction arising from the possible non-existence of series solutions having a prescribed form.

1. Introduction

This paper extends the results of [MY] on intertwining operators among generalized Verma modules for $\mathfrak{sl}_2$ to general (untwisted) affine Lie algebras $\hat{g}$, where $g$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$. For any level $\ell \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $g$, the generalized Verma $\hat{g}$-module $V_\ell(\ell,0)$, induced from the one-dimensional $g$-module and on which the canonical central element of $\hat{g}$ acts by $\ell$, is a vertex operator algebra [FZ]. Then any generalized Verma module $V_\ell(\ell,U)$ induced from a finite-dimensional $g$-module $U$ is a $V_\ell(\ell,0)$-module; more generally, $V_\ell(\ell,U)$ is an $N$-gradable weak $V_\ell(\ell,0)$-module if $U$ is infinite dimensional.

Intertwining operators among a triple of modules for a vertex operator algebra $V$ are fundamental in the study of tensor categories of $V$-modules (see the review article [HL2]). Indeed, the tensor product of $V$-modules $W_1$ and $W_2$ (if it exists) is the $V$-module $W_1 \otimes V W_2$ such that $\text{Hom}_V(W_1 \otimes W_2, W_3)$ is naturally isomorphic to the space $\text{Hom}_V(W_1, W_3)$ of intertwining operators of type $(W_1, W_2, W_3)$ for any $V$-module $W_3$. While it can be hard to determine when a category of $V$-modules closes under tensor products, a result of Miyamoto [Mi] shows that two $V$-modules satisfying the $C_1$-cofiniteness condition have a $C_1$-cofinite tensor product. For $V = V_\ell(\ell,0)$, generalized Verma modules induced from finite-dimensional $g$-modules are $C_1$-cofinite, so their tensor products do exist.

A first guess for the tensor product of generalized Verma modules $V_\ell(\ell,U_1)$ and $V_\ell(\ell,U_2)$ might be $V_\ell(\ell,U_1 \otimes U_2)$. This would require any $g$-homomorphism $U_1 \otimes U_2 \to U_3$ for $U_3$ a finite-dimensional $g$-module to naturally induce a unique intertwining operator of type $V(\ell, U_1 \otimes U_2)$. However, this paper shows that the reality is more interesting: we only get intertwining operators from $g$-module homomorphisms under certain conditions which at least sometimes are necessary. For example, here is a version of the main Theorem 3.9:

**Theorem 1.1.** Suppose $W_3 = \bigoplus_{n \in \mathbb{N}} W_3(n)$ is an $N$-gradable weak $V_\ell(\ell,0)$-module and $U_1$, $U_2$, $W_3(0)$ are irreducible weight $g$-modules with finite-dimensional weight spaces. Then there
is a linear isomorphism
\[ V^W_{V_g(\ell, U_1)} V_g(\ell, U_2) \rightarrow \text{Hom}_g(U_1 \otimes U_2, W_3(0)) \]
provided that \((\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}\) is invertible on \(U_1 \otimes U_2\) for all \(N \in \mathbb{Z}_+\). Here \(h_3\) is the conformal weight of \(W_3(0)\) and \(C_{U_1 \otimes U_2}\) is a Casimir operator on \(U_1 \otimes U_2\).

In Section 4, we will consider whether non-invertibility of \((\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}\) truly obstructs the existence of intertwining operators. In fact there is no obstruction if \(W_3\) is the contragredient of a generalized Verma module, but we already showed in [MY] that if \(W_3\) is a generalized Verma module for \(\mathfrak{sl}_2\), there can be obstructions arising from singular vectors in \(W_3\). Here, the proof of Theorem 1.1 yields a construction of candidates for singular vectors in \(W_3\) if \((\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}\) is non-invertible for some \(N \in \mathbb{Z}_+\) (see Theorem 4.3).

The proof of Theorem 1.1 is very different from that of the similar [MY, Theorem 6.1] (where \(\mathfrak{g} = \mathfrak{sl}_2\)). In [MY], we adapted the method of [Li], for constructing intertwining operators of type \((V_g(\ell, U_1))_3\) where the third module is the contragredient of a generalized Verma module, to the case of three generalized Verma modules. But for this to work we had to assume that the third module \(V_g(\ell, U_3)\) was not too different from a contragredient (specifically, we assumed \(V_g(\ell, U_3)\) had a composition series of length 2). The method used here is better for modules that are generated by their lowest conformal weight spaces, such as generalized Verma modules. The key observation is that the \(L(-1)\)-derivative property for intertwining operators implies the restriction of an intertwining operator of type \(V^W_{V_g(\ell, U_1)} V_g(\ell, U_2)\) to \(U_1 \otimes U_2\) satisfies a differential equation, essentially a Knizhnik-Zamolodchikov (KZ) equation for three-point correlation functions. Thus, we first try to solve the KZ equation with a series solution ansatz to obtain a linear map
\[ \mathcal{Y} : U_1 \otimes U_2 \rightarrow W_3\{x\}. \]
Potential obstructions arise when the series coefficients cannot be computed recursively from the initial data of a \(g\)-module homomorphism \(U_1 \otimes U_2 \rightarrow W_3(0)\), but if \(\mathcal{Y}\) can be constructed, it uniquely extends to an intertwining operator exactly as in [MY].

We now summarize the remaining contents of this paper. In Section 2, we recall definitions and notation for affine Lie algebras. In Section 3, we recall the definition of intertwining operator and prove our main construction theorems for intertwining operators among \(V_g(\ell, 0)\)-modules. In Section 4, we treat the question of when the conditions of Theorem 1.1 are necessary. Finally in Section 5, we present new examples of intertwining operators when \(g = \mathfrak{sl}_2\) and compare with previous results from [MY].

2. Affine Lie algebras

Let \(g\) be a finite-dimensional simple Lie algebra over \(\mathbb{C}\) with non-zero invariant bilinear form \(\langle \cdot, \cdot \rangle\) scaled so that long roots have square length 2. Then the affine Lie algebra is
\[ \widehat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \]
with \(k\) central and all other brackets determined by
\[(2.1) [g \otimes t^m, h \otimes t^n] = [g, h] \otimes t^{m+n} + m(g, h)\delta_{m+n,0}k\]
for \(g, h \in \mathfrak{g}\) and \(m, n \in \mathbb{Z}\). The Lie algebra \(\widehat{g}\) has the decomposition
\[ \widehat{g} = \widehat{g}_- \oplus \widehat{g}_0 \oplus \widehat{g}_+ \]
where
\[ \hat{g}_\pm = \bigoplus_{n \in \mathbb{Z}_+} g \otimes t^n, \quad \hat{g}_0 = g \otimes t^0 \oplus \mathbb{C}k. \]

For \( g \in g \) and \( m \in \mathbb{Z} \), we will use \( g(m) \) to denote the action of \( g \otimes t^m \) on a \( \hat{g} \)-module.

If \( U \) is a \( g \)-module, then \( U \) becomes a \( \hat{g}_0 \oplus \hat{g}_+ \)-module on which \( \hat{g}_+ \) acts trivially and \( k \) acts as some scalar \( \ell \in \mathbb{C} \). The generalized Verma \( \hat{g} \)-module is then the induced module
\[ V_\ell(\ell, U) = U(\hat{g}) \otimes_{U(\hat{g}_0 \oplus \hat{g}_+)} U. \]

We say that \( \ell \) is the level of \( V_\ell(\ell, U) \). Since the Poincaré-Birkhoff-Witt Theorem implies
\[ V_\ell(\ell, U) \cong U(\hat{g}_-) \otimes_{\mathbb{C}} U \]
as vector spaces, \( V_\ell(\ell, U) \) is spanned by vectors of the form
\[ g_1(-n_1) \cdots g_k(-n_k)u \]
for \( g_i \in g, n_i \in \mathbb{Z}_+, \) and \( u \in U \).

**Remark 2.1.** If \( \lambda \) is a weight of \( g \) and \( L_\lambda \) is the associated irreducible highest-weight \( g \)-module, we use \( V_\ell(\ell, \lambda) \) to denote the generalized Verma module induced from \( L_\lambda \). In particular, \( V_\ell(\ell, 0) \) denotes the generalized Verma module induced from the one-dimensional \( g \)-module \( \mathbb{C}1 \).

For any level \( \ell \), \( V_\ell(\ell, 0) \) is a vertex algebra with vacuum \( 1 \) [FZ] (see also [LL, Section 6.2]). The vertex algebra \( V_\ell(\ell, 0) \) is generated by the vectors \( g(-1)1 \) for \( g \in g \), with vertex operators
\[ Y(g(-1)1, x) = g(x) = \sum_{n \in \mathbb{Z}} g(n) x^{-n-1}. \]

Moreover, the same vertex operators acting on any generalized Verma module \( V_\ell(\ell, U) \) give it the structure of an \( \mathbb{N} \)-gradable weak \( V_\ell(\ell, 0) \)-module: \( V_\ell(\ell, U) = \bigoplus_{n \in \mathbb{N}} V_\ell(\ell, U)(n) \) for
\[ V_\ell(\ell, U)(n) = \operatorname{span}\{ g_1(-n_1) \cdots g_k(-n_k)u \mid u \in U, g_i \in g, n_i \in \mathbb{Z}_+, n_1 + \ldots + n_k = n \}. \]

Let \( \{ \gamma_i \}_{i=1}^{\dim g} \) be an orthonormal basis for \( g \) with respect to the nondegenerate form \( \langle \cdot, \cdot \rangle \). The Casimir element \( \sum_{i=1}^{\dim g} \gamma_i^2 \) associated to \( \langle \cdot, \cdot \rangle \) acts in the adjoint representation \( g \) by \( 2h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( g \). Then if \( \ell \neq -h^\vee \), \( V_\ell(\ell, 0) \) is a vertex operator algebra with conformal vector
\[ \omega = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim g} \gamma_i(-1)^2 1. \]

Writing \( Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2} \) as usual, we have (see [LL, Theorem 6.2.16]) that
\[ [L(m), g(n)] = -ng(m + n) \]
for any \( g \in g \) and \( m, n \in \mathbb{Z} \). From the definition of \( \omega \), it also follows that
\[ L(0) = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim g} \gamma_i(0)^2 + \frac{1}{\ell + h^\vee} \sum_{i=1}^{\dim g} \sum_{n > 0} \gamma_i(-n) \gamma_i(n) \]
and
\[ L(-1) = \frac{1}{\ell + h^\vee} \sum_{i=1}^{\dim g} \sum_{n \geq 0} \gamma_i(-n - 1) \gamma_i(n). \]
Remark 2.2. By (2.2) and (2.3), any vector of the form
\[ g_1(-n_1) \cdots g_k(-n_k)1 \in V_\ell(\ell, 0) \]
for \( g_i \in \mathfrak{g} \) and \( n_i \in \mathbb{Z}_+ \) has conformal weight \( n_1 + \cdots + n_k \). More generally, for any weight \( \lambda, (2.3) \) shows that \( L(0) \) acts on \( L_\lambda = V_\ell(\ell, \lambda)(0) \) by the scalar
\[ h_{\lambda, \ell} = \frac{1}{2(\ell + h)}(\lambda, \lambda + 2\rho) \]
where \( \rho \) is the sum of the fundamental weights of \( \mathfrak{g} \) (see [Hu, Section 22]). Then by (2.2), \( V_\ell(\ell, \lambda)(n) \) is the conformal weight space of \( V_\ell(\ell, \lambda) \) with \( L(0) \)-eigenvalue \( h_{\lambda, \ell} + n \).

Remark 2.3. The \( m = n = 0 \) case of (2.2) implies that the \( L(0) \)-generalized eigenspaces of a weak \( V_\ell(\ell, 0) \)-module are \( \mathfrak{g} \)-modules.

3. Construction of Intertwining Operators

For a general vector space \( W \), we use \( W \{x\} \) to denote the vector space of formal series of the form \( \sum_{n \in \mathbb{C}} w_n x^n \), \( w_n \in W \). We recall from [FHL] (see also [HL1, HLZ]) the definition of intertwining operator among a triple of modules for a vertex operator algebra:

Definition 3.1. Let \( W_1, W_2 \) and \( W_3 \) be weak modules for a vertex operator algebra \( V \). An intertwining operator of type \( W_3 \) is a linear map
\[ \mathcal{Y} : W_1 \otimes W_2 \to W_3 \{x\} \]
\[ w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{h \in \mathbb{C}} (w_1)_h w_2 x^{-h-1} \in W_3 \{x\} \]
satisfying the following conditions:

1. **Lower truncation:** for \( w_1 \in W_1, w_2 \in W_2 \), and \( h \in \mathbb{C}, (w_1)_h w_2 = 0 \) for \( n \in \mathbb{N} \) sufficiently large.

2. **The Jacobi identity:** for \( v \in V \) and \( w_1 \in W_1 \),
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{W_3}(v, x_1)\mathcal{Y}(w_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_1, x_2)Y_{W_2}(v, x_1) \]
\[ = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \mathcal{Y}(Y_{W_1}(v, x_0)w_1, x_2). \]
\[ (3.1) \]

3. **The \( L(-1) \)-derivative property:** for \( w_1 \in W_1 \),
\[ \mathcal{Y}(L(-1)w_1, x) = \frac{d}{dx} \mathcal{Y}(w_1, x). \]
\[ (3.2) \]

Remark 3.2. We denote the vector space of intertwining operators of type \( W_3 \) by \( \mathcal{V}_{W_3} \) and the corresponding fusion rule is \( \mathcal{N}_{W_3} = \dim \mathcal{V}_{W_3} \).

Taking \( W_1, W_2 \), and \( W_3 \) to be \( N \)-gradable weak \( V_\ell(\ell, 0) \)-modules for some level \( \ell \), suppose \( \mathcal{Y} \) is an intertwining operator of type \( W_3 \). We will need some consequences of the Jacobi identity and \( L(-1) \)-derivative property in this setting. First, the coefficient of \( x_0^{-1} x_1^{-n-1} \) for \( n \in \mathbb{Z} \) in (3.1), when \( v = g(-1)1, g \in \mathfrak{g} \), is the commutator formula
\[ [g(n), \mathcal{Y}(w_1, x_2)] = \text{Res}_{x_0} (x_2 + x_0)^n \mathcal{Y}(g(x_0)w_1, x_2) = \sum_{i \geq 0} \binom{n}{i} x_2^{n-i} \mathcal{Y}(g(i)w_1, x_2). \]
If \( w_1 \in W_1 \) satisfies \( g(i)w_1 = 0 \) for all \( i > 0 \), we get

\[
(3.3) \quad [g(n), \mathcal{Y}(w_1, x)] = x^n \mathcal{Y}(g(0)w_1, x).
\]

Next, the iterate formula is the coefficient of \( x_0^{n-1}x_1^{i-1} \) in (3.1):

\[
\mathcal{Y}(g(n)w_1, x_2) = \text{Res}_{x_1} \left( (x_1 - x_2)^n g(x_1) \mathcal{Y}(w_1, x_2) - (-x_2 + x_1)^n \mathcal{Y}(w_1, x_2)g(x_1) \right).
\]

for \( v = g(-1)1 \) and \( n \in \mathbb{Z} \). The case \( n = -1 \) yields

\[
\mathcal{Y}(g(-1)w, x_2) = \sum_{i \geq 0} g(-i - 1)x_1^i \mathcal{Y}(w, x_2) + \sum_{i \geq 0} x_2^{-i-1} \mathcal{Y}(w, x_2)g(i)
\]

(3.4)

\[
= g(x_2)^{+} \mathcal{Y}(w, x_2) + \mathcal{Y}(w, x_2)g(x_2)^{-},
\]

where \( g(x)^{\pm} \) denote the non-singular and singular parts of \( g(x) \), respectively.

Now for the \( L(-1) \)-derivative property: when \( w_1 \in W_1 \) satisfies \( g(i)w_1 = 0 \) for \( g \in \mathfrak{g} \) and \( i > 0 \), (2.4), (3.2), and (3.4) imply

\[
(\ell + h^\gamma) \frac{d}{dx} \mathcal{Y}(w_1, x)w_2 = \sum_{i=1}^{\dim \mathfrak{g}} \mathcal{Y}(\gamma_i(-1)\gamma_i(0)w_1, x)w_2
\]

\[
= \sum_{i=1}^{\dim \mathfrak{g}} (\gamma_i(x)^+ \mathcal{Y}(\gamma_i(0)w_1, x)w_2 + \mathcal{Y}(\gamma_i(0)w_1, x)\gamma_i(x)^- w_2)
\]

for any \( w_2 \in W_2 \). If also \( g(i)w_2 = 0 \) for \( g \in \mathfrak{g} \) and \( i > 0 \), we then have:

**Proposition 3.3.** If \( w_1 \in W_1, w_2 \in W_2 \) satisfy \( g(i)w_1, g(i)w_2 = 0 \) for \( g \in \mathfrak{g} \) and \( i > 0 \), then

\[
(\ell + h^\gamma) \frac{d}{dx} \mathcal{Y}(w_1, x)w_2 = \sum_{i=1}^{\dim \mathfrak{g}} (x^{-1} \mathcal{Y}(\gamma_i(0)w_1, x)\gamma_i(0)w_2 + \gamma_i(x)^+ \mathcal{Y}(\gamma_i(0)w_1, x)w_2).
\]

**Remark 3.4.** We shall construct intertwining operators of type \( \left( \frac{W_3}{W_1, W_2} \right) \) from solutions to the differential equation of Proposition 3.3, which is basically a Knizhnik-Zamolodchikov equation [KZ] for three-point correlation functions in conformal field theory based on \( \hat{\mathfrak{g}} \).

For \( i = 1, 2, 3 \), let \( U_i \) denote the degree-zero subspace of the \( \mathbb{N} \)-gradable weak \( V \)-module \( W_i \); each \( U_i \) is a \( \mathfrak{g} \)-module. Then if \( \mathcal{Y} \) is an intertwining operator of type \( \left( \frac{W_3}{W_1, W_2} \right) \), Proposition 3.3 shows that

\[
\mathcal{Y}|_{U_1 \otimes U_2} = \mathcal{Y}(\cdot \otimes \cdot, x) : U_1 \otimes U_2 \rightarrow W_3\{x\}
\]

satisfies

\[
(\ell + h^\gamma) \frac{d}{dx} \mathcal{Y}(u_1 \otimes u_2, x) = x^{-1} \mathcal{Y}(C_{U_1 \otimes U_2}(u_1 \otimes u_2), x) + \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i(x)^+ \mathcal{Y}(\gamma_i \cdot u_1 \otimes u_2, x),
\]

where for \( u_1 \in U_1, u_2 \in U_2 \),

\[
(3.5) \quad C_{U_1 \otimes U_2}(u_1 \otimes u_2) = \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i \cdot u_1 \otimes \gamma_i \cdot u_2
\]

Using \( C_U \) to denote the action of the Casimir element of \( (\mathfrak{g}, \langle \cdot, \cdot \rangle) \) on a \( \mathfrak{g} \)-module \( U \), we have

\[
(3.6) \quad C_{U_1 \otimes U_2} = \frac{1}{2} (C_{U_1 \otimes U_2} - C_{U_1} \otimes 1_{U_2} - 1_{U_1} \otimes C_{U_2}),
\]

so \( C_{U_1 \otimes U_2} \) is a \( \mathfrak{g} \)-endomorphism of \( U_1 \otimes U_2 \).
We now present our construction theorems for intertwining operators among $V_{\ell}(\ell, 0)$-modules. The first is from [MY]; it does not require the $\mathfrak{g}$-modules $U_i$ to be finite dimensional:

**Theorem 3.5** ([MY], Theorem 6.2). Suppose $\mathcal{Y}(\cdot, x) : U_1 \otimes U_2 \to W_3\{x\}$ is a lower-truncated linear map satisfying

\begin{equation}
[g(n), \mathcal{Y}(u, x)] = x^n \mathcal{Y}(g(0)u, x)
\end{equation}

for $g \in \mathfrak{g}$, $n \geq 0$, and

\begin{equation}
[L(0), \mathcal{Y}(u, x)] = x \frac{d}{dx} \mathcal{Y}(u, x) + \mathcal{Y}(L(0)u, x).
\end{equation}

Then $\mathcal{Y}$ has a unique extension to an intertwining operator of type $(V_{\ell}(\ell, U_1), V_{\ell}(\ell, U_2))$.

Conversely, by the commutator and $L(-1)$-derivative formulas, any intertwining operator of type $(V_{\ell}(\ell, U_1), V_{\ell}(\ell, U_2))$ restricted to $U_1 \otimes U_2$ satisfies (3.7) and (3.8). To construct lower-truncated linear maps as in Theorem 3.5, an ansatz for the shape of the formal series $\mathcal{Y}$ will help. Thus, we now assume that $L(0)$ acts on each $\mathfrak{g}$-module $U_i$ as a scalar $h_i \in \mathbb{C}$; equivalently by (2.3), $C_{U_i}$ acts as $2(\ell + h^\vee)h_i$. For example, $U_i$ could be a not-necessarily-finite-dimensional irreducible weight $\mathfrak{g}$-module with finite-dimensional weight spaces. Set $h = h_3 - h_1 - h_2$; under our assumption, (3.8) is equivalent to

\[ \mathcal{Y}(u_1, x)u_2 = \sum_{m \in \mathbb{Z}} \mathcal{Y}_m(u_1 \otimes u_2)x^{h+m} \]

for $u_1 \in U_1$, $u_2 \in U_2$, with $\mathcal{Y}_m : U_1 \otimes U_2 \to W_3\{m\}$ for $m \in \mathbb{Z}$. Then (3.7) is equivalent to

\begin{equation}
\label{eq:3.9}
g(n)\mathcal{Y}_m(u_1 \otimes u_2) = \mathcal{Y}_{m-n}(g(0)u_1 \otimes u_2) + \mathcal{Y}_m(u_1 \otimes g(n)u_2)
\end{equation}

for $g \in \mathfrak{g}$, $u_1 \in U_1$, $u_2 \in U_2$, $m \in \mathbb{Z}$, and $n \geq 0$.

Each $\mathcal{Y}_m$ is a $\mathfrak{g}$-module homomorphism by the $n = 0$ case of (3.9), so $\mathcal{Y} \mapsto \mathcal{Y}_0$ defines a linear map $\mathcal{Y}_{W_3}^{V_{\ell}(\ell, U_1), V_{\ell}(\ell, U_2)} \to \text{Hom}_{\mathfrak{g}}(U_1 \otimes U_2, U_3)$. Conversely, we will construct intertwining operators starting from such $\mathfrak{g}$-module homomorphisms using the following theorem:

**Theorem 3.6.** Suppose that for all $N \in \mathbb{Z}_+$, the $\mathfrak{g}$-module endomorphism

\[ (\ell + h^\vee)(h + N) - C_{U_1, U_2} \]

of $U_1 \otimes U_2$ is invertible. Then for any $f \in \text{Hom}_{\mathfrak{g}}(U_1 \otimes U_2, U_3)$, there are unique linear maps

\[ \mathcal{Y}_m : U_1 \otimes U_2 \to W_3\{m\} \]

for $m \in \mathbb{Z}$ such that $\mathcal{Y}_0 = f$ and (3.9) holds.

**Proof.** Since $W_3$ is $\mathbb{N}$-graded, we must have $\mathcal{Y}_m = 0$ for $m < 0$. Now, if the desired linear maps exist for $m \geq 0$, then Theorem 3.5 implies that

\[ \mathcal{Y}(\cdot, x) = \sum_{m \geq 0} \mathcal{Y}_m x^{h+m} : U_1 \otimes U_2 \to W_3\{x\} \]

extends to an intertwining operator of type $(V_{\ell}(\ell, U_1), V_{\ell}(\ell, U_2))$. So by Proposition 3.3,

\[ (\ell + h^\vee)\frac{d}{dx} \mathcal{Y}(u_1, x)u_2 = \sum_{i=1}^{\dim g} (x^{-1} \mathcal{Y}(\gamma_i \cdot u_1, x)(\gamma_i \cdot u_2) + \gamma_i(x)^+ \mathcal{Y}(\gamma_i \cdot u_1, x)u_2) \]
for $u_1 \in U_1$ and $u_2 \in U_2$. In component form, this is

$$
(3.10) \quad \mathcal{Y}_m\left(\left[(\ell + h^\vee)(h + m) - C_{U_1,U_2}\right](u_1 \otimes u_2)\right) = \sum_{i=1}^{\dim g} \sum_{k=1}^{m} \gamma_i(-k)\mathcal{Y}_{m-k}(\gamma_i \cdot u_1 \otimes u_2)
$$

for $m \geq 0$. To prove the theorem, it is enough to show that (3.10) has a unique solution for $m > 0$ given $\mathcal{Y}_0 = f$, and that this solution satisfies (3.9).

We first show that $\mathcal{Y}_0 = f$ satisfies the $m = 0$ case of (3.10). Since $C_{U_i} = 2(\ell + h^\vee)h_i$ for $i = 1, 2, 3$ (recall (2.3)), (3.6) implies

$$
(3.11) \quad (\ell + h^\vee)h - C_{U_1,U_2} = (\ell + h^\vee)(h_3 - h_1 - h_2) - \frac{1}{2}(C_{U_1 \otimes U_2} - C_{U_1} \otimes 1_{U_2} - 1_{U_1} \otimes C_{U_2})
$$

Then because Casimir operators commutes with $\mathfrak{g}$-homomorphisms, we have

$$
(3.12) \quad f\left(\left[(\ell + h^\vee)h - C_{U_1,U_2}\right](u_1 \otimes u_2)\right) = (\ell + h^\vee)h_3f(u_1 \otimes u_2) - \frac{1}{2}C_{U_3}f(u_1 \otimes u_2) = 0
$$

for $u_1 \in U_1$ and $u_2 \in U_2$, as required. Now we can use (3.10) to construct $\mathcal{Y}_m$ recursively, since by assumption $(\ell + h^\vee)(h + m) - C_{U_1,U_2}$ is invertible for all $m > 0$:

$$
(3.13) \quad \mathcal{Y}_m(u_1 \otimes u_2) = \sum_{i=1}^{\dim g} \sum_{k=1}^{m} \gamma_i(-k)\mathcal{Y}_{m-k}\left((\gamma_i \otimes 1)\left[(\ell + h^\vee)(h + m) - C_{U_1,U_2}\right]^{-1}(u_1 \otimes u_2)\right)
$$

for $u_1 \in U_1$, $u_2 \in U_2$. This shows (3.10) has a unique solution for each $m > 0$ given $\mathcal{Y}_0 = f$.

We need to show that $\mathcal{Y}_m$ as given by (3.13) satisfies (3.9) for $m \geq 0$. As both sides of (3.9) are zero for $n > m$, we may assume $0 \leq n \leq m$ and prove (3.9) by induction on $m$. The base case $m = 0$ is clear because $\mathcal{Y}_0 = f$ is a $\mathfrak{g}$-module homomorphism, so we assume (3.9) holds for all $m$ less than some fixed $M > 0$ and prove (3.9) for $M$. Since the $\mathfrak{g}$-homomorphism $C_{U_1,U_2}^{(M)} = (\ell + h^\vee)(h + M) - C_{U_1,U_2}$ is invertible, it is enough to prove that

$$
(3.14) \quad g(0)\mathcal{Y}_M(C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)) = \mathcal{Y}_M(C_{U_1,U_2}^{(M)}(g \otimes 1 + 1 \otimes g)(u_1 \otimes u_2)),
$$

$$
(3.15) \quad g(n)\mathcal{Y}_M(C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)) = \mathcal{Y}_{M-n}\left((g \otimes 1)C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)\right)
$$

for $g \in \mathfrak{g}$ and $1 \leq n \leq M$. For (3.14), we use (3.10), the induction hypothesis, the commutation relations (2.1), and Lemma 3.7 below to obtain

$$
g(0)\mathcal{Y}_M(C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)) = \sum_{i=1}^{\dim g} \sum_{k=1}^{M} g(0)\gamma_i(-k)\mathcal{Y}_{M-k}(\gamma_i \cdot u_1 \otimes u_2)
$$

$$
= \sum_{i=1}^{\dim g} \sum_{k=1}^{M} \left(\gamma_i(-k)\mathcal{Y}_{M-k}\left((g \otimes 1 + 1 \otimes g)(\gamma_i \otimes 1)(u_1 \otimes u_2)\right) + [g, \gamma_i](-k)\mathcal{Y}_{M-k}(\gamma_i \cdot u_1 \otimes u_2)\right)
$$

$$
= \sum_{i=1}^{\dim g} \sum_{k=1}^{M} \gamma_i(-k)\mathcal{Y}_{M-k}\left((\gamma_i \otimes 1)(g \otimes 1 + 1 \otimes g)(u_1 \otimes u_2)\right)
$$

$$
+ \sum_{i=1}^{\dim g} \sum_{k=1}^{M} ([g, \gamma_i](-k)\mathcal{Y}_{M-k}(\gamma_i \otimes 1)(u_1 \otimes u_2) + \gamma_i(-k)\mathcal{Y}_{M-k}(([g, \gamma_i] \otimes 1)(u_1 \otimes u_2)))
$$
\[
\mathcal{Y}_M(C^{(M)}_{U_1,U_2}(g \otimes 1 + 1 \otimes g)(u_1 \otimes u_2))
\]
for any \(u_1 \in U_1\) and \(u_2 \in U_2\).

**Lemma 3.7.** In \(\mathfrak{g} \otimes \mathfrak{g}\), \(\sum_{i=1}^{\dim \mathfrak{g}} [g, \gamma_i] \otimes \gamma_i = -\sum_{i=1}^{\dim \mathfrak{g}} \gamma_i \otimes [g, \gamma_i]\) for any \(g \in \mathfrak{g}\).

**Proof.** For \(g \in \mathfrak{g}\), we have \([g, \gamma_i] = \sum_{j=1}^{\dim \mathfrak{g}} c_i^j \gamma_j\) for each \(1 \leq i \leq \dim \mathfrak{g}\), where \(c_i^j \in \mathbb{C}\). Then
\[
\sum_{i=1}^{\dim \mathfrak{g}} [g, \gamma_i] \otimes \gamma_i = \sum_{i,j=1}^{\dim \mathfrak{g}} c_i^j (\gamma_j \otimes \gamma_i),
\]
while
\[
\sum_{i=1}^{\dim \mathfrak{g}} \gamma_i \otimes [g, \gamma_i] = \sum_{i,j=1}^{\dim \mathfrak{g}} c_i^j (\gamma_i \otimes \gamma_j) = \sum_{i,j=1}^{\dim \mathfrak{g}} c_i^j (\gamma_j \otimes \gamma_i).
\]
But by the invariance of the form \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{g}\), we have
\[
c_i^j = \langle \gamma_j, [g, \gamma_i] \rangle = \langle [\gamma_j, g], \gamma_i \rangle = -c_j^i,
\]
proving the lemma. \(\Box\)

For (3.15), we use (3.10), the induction hypothesis, (2.1), and Lemma 3.7:
\[
g(n)\mathcal{Y}_M(C^{(M)}_{U_1,U_2}(u_1 \otimes u_2)) = \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M} g(n)\gamma_i(-k)\mathcal{Y}_{M-k}(\gamma_i \cdot u_1 \otimes u_2)
\]
\[
= \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} \gamma_i(-k)\mathcal{Y}_{M-n-k}(g \cdot (\gamma_i \cdot u_1) \otimes u_2)
\]
\[
+ \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} ([g, \gamma_i](n - k) + n\langle g, \gamma_i \rangle \delta_{n,k} \ell)\mathcal{Y}_{M-k}(\gamma_i \cdot u_1 \otimes u_2)
\]
\[
= \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} \gamma_i(-k)\mathcal{Y}_{M-n-k}(\gamma_i \cdot (g \cdot u_1) \otimes u_2)
\]
\[
+ \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} (\gamma_i(-k)\mathcal{Y}_{M-n-k}([g, \gamma_i] \cdot u_1 \otimes u_2) + [g, \gamma_i](-k)\mathcal{Y}_{M-n-k}(\gamma_i \cdot u_1 \otimes u_2))
\]
\[
+ \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} [g, \gamma_i](n - k)\mathcal{Y}_{M-k}(\gamma_i \cdot u_1 \otimes u_2) + n\ell\mathcal{Y}_{M-n}(g \cdot u \otimes v)
\]
\[
(3.16) = \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k=1}^{M-n} \gamma_i(-k)\mathcal{Y}_{M-n-k}(\gamma_i \cdot (g \cdot u_1) \otimes u_2) + n \sum_{i=1}^{\dim \mathfrak{g}} \mathcal{Y}_{M-n}([g, \gamma_i] \cdot (\gamma_i \cdot u_1) \otimes u_2)
\]
\[
+ \sum_{i=1}^{\dim \mathfrak{g}} \mathcal{Y}_{M-n}(\gamma_i \cdot u_1 \otimes [g, \gamma_i] \cdot u_2) + n\ell\mathcal{Y}_{M-n}(g \cdot u_1 \otimes u_2).
\]
First consider the case \(1 \leq n \leq M - 1\). Using (3.10) and (3.5), the first term on the right of (3.16) becomes
\[
\mathcal{Y}_{M-n}(((\ell + h^v)(h + M - n) - C_{U_1,U_2})(g \cdot u_1 \otimes u_2)) = \mathcal{Y}_{M-n}((g \otimes 1)C^{(M)}_{U_1,U_2}(u_1 \otimes u_2))
\]
(3.17) $$+ \sum_{i=1}^{\dim g} \mathcal{Y}_{M-n}([g, \gamma_i] \cdot u_1 \otimes \gamma_i \cdot u_2) - n(\ell + h^\vee)\mathcal{Y}_{M-n}(g \cdot u_1 \otimes u_2).$$

To analyze the second term on the right of (3.16), we use another lemma:

Lemma 3.8. In $U(g)$, $\sum_{i=1}^{\dim g} [g, \gamma_i] \gamma_i = h^\vee g$ for any $g \in g$.

Proof. We know from Lemma 3.7 that $\sum_{i=1}^{\dim g} [g, \gamma_i] \gamma_i = - \sum_{i=1}^{\dim g} \gamma_i [g, \gamma_i]$. Therefore

$$\sum_{i=1}^{\dim g} [g, \gamma_i] \gamma_i = \frac{1}{2} \sum_{i=1}^{\dim g} ([g, \gamma_i] \gamma_i - \gamma_i [g, \gamma_i]) = \frac{1}{2} \sum_{i=1}^{\dim g} [[g, \gamma_i], \gamma_i] = h^\vee g,$$

recalling that the Casimir operator on the adjoint representation $g$ is the scalar $2h^\vee$. \qed

Now we insert (3.17) back into (3.16) and cancel terms using Lemmas 3.7 and 3.8. Only the first term to the right of the equality in (3.17) survives, completing the proof of the case $1 \leq n \leq M - 1$.

Finally for the case $n = M$, the first term on the right of (3.16) vanishes. We calculate the remaining terms using Lemmas 3.7 and 3.8, and then (3.5) and (3.12):

$$M(\ell + h^\vee)\mathcal{Y}_0(g \cdot u_1 \otimes u_2) - \sum_{i=1}^{\dim g} \mathcal{Y}_0([g, \gamma_i] \cdot u_1 \otimes \gamma_i \cdot u_2)$$

$$= \mathcal{Y}_0((g \otimes 1)(\ell + h^\vee)M - C_{U_1,U_2}(u_1 \otimes u_2)) + \mathcal{Y}_0(C_{U_1,U_2}(g \cdot u_1 \otimes u_2))$$

$$= \mathcal{Y}_0((g \otimes 1)C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)) + \mathcal{Y}_0([C_{U_1,U_2} - (\ell + h^\vee)h](g \cdot u_1 \otimes u_2))$$

$$= \mathcal{Y}_0((g \otimes 1)C_{U_1,U_2}^{(M)}(u_1 \otimes u_2)).$$

This completes the proof of the theorem. \qed

Now our main theorem combines Theorems 3.5 and 3.6:

Theorem 3.9. Suppose $U_1$, $U_2$ are $g$-modules and $W_3$ is an $\mathbb{N}$-gradable weak $V_\theta(\ell, 0)$-module such that $L(0)$ acts on $U_1$, $U_2$, and $W_3(0)$ by scalars $h_1$, $h_2$, and $h_3$, respectively. If moreover $(\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}$ is invertible on $U_1 \otimes U_2$ for all $N \in \mathbb{Z}_+$, then the linear map

$$\mathcal{Y}_{V_\theta(\ell,U_1)}^{W_3} \otimes \mathcal{Y}_{V_\theta(\ell,U_2)} \to \mathcal{Y}_{V_\theta(U_1 \otimes U_2)}^{W_3} \otimes \mathcal{Y}_{V_\theta(U_1 \otimes U_2)}^{W_3}$$

is an isomorphism.

Proof. By (3.11), $(\ell + h^\vee)(h + N) - C_{U_1,U_2}$ is invertible if $(\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}$ is invertible. Then Theorems 3.5 and 3.6 imply that given $f \in \mathcal{Y}_{V_\theta(U_1 \otimes U_2,W_3(0))}$, there is a unique intertwining operator $\mathcal{Y}$ of type $(V_\theta(\ell,U_1) \to V_\theta(\ell,U_2))$ such that $\mathcal{Y}_0|_{U_1 \otimes U_2} = f$. That is, $\mathcal{Y} \mapsto \mathcal{Y}_0|_{U_1 \otimes U_2}$ is both injective and surjective. \qed

4. Analysis of the obstructions

In this section, we discuss whether non-invertibility of $(\ell + h^\vee)(h_3 + N) - \frac{1}{2}C_{U_1 \otimes U_2}$ for some $N \in \mathbb{Z}_+$, as in Theorem 3.9, is truly an obstruction to constructing intertwining operators. For simplicity, we will assume that $U_1$, $U_2$, and $U_3 = W_3(0)$ are finite-dimensional irreducible $g$-modules corresponding to dominant integral weights $\lambda_1$, $\lambda_2$, and $\lambda_3$, respectively; this will
guarantee that $C_{U_1 \otimes U_2}$ is diagonalizable. We will focus on the cases that $W_3$ is a generalized Verma module or its contragredient dual (see [FHL, Section 5.2]). In this setting, Theorem 3.9 reads:

**Theorem 4.1.** Suppose $\lambda_1$, $\lambda_2$, and $\lambda_3$ are dominant integral weights of $\mathfrak{g}$ and $W_3$ is an $\mathbb{N}$-gradable weak $V_\mathfrak{g}(\ell, 0)$-module such that $W_3(0) = L_{\lambda_3}$. If moreover $2(\ell + h^\vee)(h_{\lambda_3, \ell} + N)$ is not an eigenvalue of $C_{L_{\lambda_1} \otimes L_{\lambda_2}}$ for any $N \in \mathbb{Z}_+$, then there is a linear isomorphism

$$\mathcal{Y}_{V_\mathfrak{g}(\ell, \lambda_1) V_\mathfrak{g}(\ell, \lambda_2)} \rightarrow \text{Hom}_\mathfrak{g}(L_{\lambda_1} \otimes L_{\lambda_2}, L_{\lambda_3})$$

$$\mathcal{Y} \mapsto \mathcal{Y}_0|_{L_{\lambda_1} \otimes L_{\lambda_2}}$$

The easiest case to analyze is that $W_3$ is the contragredient $V_\mathfrak{g}(\ell, \lambda_3^*)'$ of a generalized Verma module, where $\lambda_3^*$ is the dominant integral weight of $\mathfrak{g}$ such that $L_{\lambda_3^*} \cong L_{\lambda_3}$. Then there are no obstructions: $V_\mathfrak{g}(\ell, \lambda_3^*)' \cong \text{Hom}_\mathfrak{g}(L_{\lambda_1} \otimes L_{\lambda_2}, L_{\lambda_3})$ unconditionally. This follows from [Li, Theorem 2.11], or can be proved using Theorem 3.5. To produce the maps $\mathcal{Y}_m : L_{\lambda_1} \otimes L_{\lambda_2} \rightarrow V_\mathfrak{g}(\ell, \lambda_3^*)(m)$ satisfying (3.9) required by Theorem 3.5, one starts with $\mathcal{Y}_0 = f$ for any $f \in \text{Hom}_\mathfrak{g}(L_{\lambda_1} \otimes L_{\lambda_2}, L_{\lambda_3})$ and then recursively defines $\mathcal{Y}_m$ by

$$\left(\mathcal{Y}_m(u_1 \otimes u_2), g(-n)w_3\right) = -\left(\mathcal{Y}_{m-n}(g \cdot u_1 \otimes u_2), w_3\right)$$

for $u_1 \in L_{\lambda_1}$, $u_2 \in L_{\lambda_2}$, $g \in \mathfrak{g}$, $1 \leq n \leq m$, and $w_3 \in V_\mathfrak{g}(\ell, \lambda_3^*)(m-n)$.

Although we do not need Theorems 3.5 and 3.6 to determine $V_\mathfrak{g}(\ell, \lambda_3^*)'$, they still provide information. If the conditions of Theorem 4.1 hold, the construction given by Theorems 3.5 and 3.6 shows that the image of every intertwining operator of type $(V_\mathfrak{g}(\ell, \lambda_3^*)')$ is contained in the $V_\mathfrak{g}(\ell, 0)$-submodule of $V_\mathfrak{g}(\ell, \lambda_3^*)'$ generated by the lowest weight space $V_\mathfrak{g}(\ell, \lambda_3^*)(0) = L_{\lambda_3^*} \cong L_{\lambda_3}$. This submodule is the radical of the unique maximal proper submodule $J_\mathfrak{g}(\ell, \lambda_3^*)$ of $V_\mathfrak{g}(\ell, \lambda_3^*)$ and thus is isomorphic to the contragredient $L_\mathfrak{g}(\ell, \lambda_3)$ of the irreducible quotient $L_\mathfrak{g}(\ell, \lambda_3^*) = V_\mathfrak{g}(\ell, \lambda_3^*)/J_\mathfrak{g}(\ell, \lambda_3^*)$. So we have:

**Theorem 4.2.** Suppose $\lambda_1$, $\lambda_2$, and $\lambda_3$ are dominant integral weights of $\mathfrak{g}$. If $2(\ell + h^\vee)(h_{\lambda_3, \ell} + N)$ is not an eigenvalue of $C_{L_{\lambda_1} \otimes L_{\lambda_2}}$ for any $N \in \mathbb{Z}_+$, then every intertwining operator of type $(V_\mathfrak{g}(\ell, \lambda_3^*)')$ factors through the inclusion $L_\mathfrak{g}(\ell, \lambda_3) \hookrightarrow V_\mathfrak{g}(\ell, \lambda_3^*)'$.

The case that $W_3 = V_\mathfrak{g}(\ell, \lambda_3)$ is more interesting. Example 5.3 in the next section will show that the conditions of Theorem 4.1 are not always necessary. However, [MY, Section 8] showed some examples of genuine obstructions to the existence intertwining operators in the case $\mathfrak{g} = \mathfrak{sl}_2$, arising from singular vectors in $V_\mathfrak{g}(\ell, \lambda_3)$. We now show why singular vectors can be a problem when $C_{L_{\lambda_1} \otimes L_{\lambda_2}}$ has eigenvalue(s) $2(\ell + h^\vee)(h_{\lambda_3, \ell} + N)$:

**Theorem 4.3.** Suppose $\lambda_1$, $\lambda_2$, and $\lambda_3$ are dominant integral weights of $\mathfrak{g}$ and $N$ is the smallest positive integer such that $2(\ell + h^\vee)(h_{\lambda_3, \ell} + N)$ is an eigenvalue of $C_{L_{\lambda_1} \otimes L_{\lambda_2}}$. Given $f \in \text{Hom}_\mathfrak{g}(L_{\lambda_1} \otimes L_{\lambda_2}, L_{\lambda_3})$, let

$$\mathcal{Y}_m : L_{\lambda_1} \otimes L_{\lambda_2} \rightarrow V_\mathfrak{g}(\ell, \lambda_3)(m)$$

for $1 \leq m \leq N - 1$ denote the maps defined by the recursive formula (3.13), starting from $\mathcal{Y}_0 = f$. Then if $\sum_j u_j^{(1)} \otimes u_j^{(2)}$ is an eigenvector of $C_{L_{\lambda_1} \otimes L_{\lambda_2}}$ with eigenvalue $2(\ell + h^\vee)(h_{\lambda_3, \ell} + N)$,

$$\sum_{i=1}^{\dim \mathfrak{g}} \sum_j \sum_{k=1}^{N} \gamma_i(-k) \mathcal{Y}_{N-k}(\gamma_i \cdot u_j^{(1)} \otimes u_j^{(2)})$$

(4.1)
lies in the maximal proper submodule $J_g(\ell, \lambda_3)$. Moreover, if (4.1) is non-zero for some eigenvector, then there is no intertwining operator of type $\left(V_0(\ell, \lambda_3) V_0(\ell, \lambda_2)\right)$ such that $Y_0|_{L_{\lambda_1} \otimes L_{\lambda_2}} = f$.

Proof. Because $N$ is the smallest positive integer such that $(\ell + h^\vee)(h + N) - C_{L\lambda_1, L\lambda_2}$ is not invertible, (3.13) is well defined for $1 \leq m \leq N - 1$. So if $W_3$ is any $N$-gradable weak $V_0(\ell, 0)$-module with $W_3(0) = L_{\lambda_3}$ and $Y$ is an intertwining operator of type $\left(V_0(\ell, \lambda_1) V_0(\ell, \lambda_2)\right)$ such that $Y_0|_{L_{\lambda_1} \otimes L_{\lambda_2}} = f$, then (3.10) implies that $Y_m|_{L_{\lambda_1} \otimes L_{\lambda_2}}$ for $1 \leq m \leq N - 1$ must be given by (3.13) and that (4.1) must vanish. The second assertion of the theorem then follows from taking $W_3 = V_0(\ell, \lambda_3)$.

For the first assertion, take $W_3 = V_0(\ell, \lambda_3)'$. Then (4.1) is an expression in $U(\hat{\mathfrak{g}}_-) \otimes L_{\lambda_3}$ that vanishes in $V_0(\ell, \lambda_3)'$ since in this case the intertwining operator $Y$ exists. Since we have observed that the $V_0(\ell, 0)$-submodule of $V_0(\ell, \lambda_3)'$ generated by $L_{\lambda_3}$ is isomorphic to $L(\ell, \lambda_3) = (U(\hat{\mathfrak{g}}_-) \otimes L_{\lambda_3})/J_g(\ell, \lambda_3)$, it follows that (4.1) viewed as a vector in $V_0(\ell, \lambda_3)$ lies in $J_g(\ell, \lambda_3)$. 

Remark 4.4. Theorem 4.3 provides a recipe for producing candidates for singular vectors in $V_0(\ell, \lambda_3)$. However, Example 5.3 will show that (4.1) can vanish even if $f \neq 0$.

Remark 4.5. Since the eigenvalues of $C_{L\lambda_1 \otimes L\lambda_2}$ and $C_{L\lambda_3}$ are rational when the $\lambda_i$ are dominant integral weights of $\mathfrak{g}$, obstructions to intertwining operators of type $\left(V_0(\ell, \lambda_3) V_0(\ell, \lambda_2)\right)$ never arise if $\ell \notin \mathbb{Q}$. This is no surprise because in this case generalized Verma modules induced from finite-dimensional irreducible $\mathfrak{g}$-modules are themselves irreducible and thus isomorphic to contragredients of generalized Verma modules. However, obstructions might occur for $\ell \notin \mathbb{Q}$ if the $\lambda_i$ are not dominant integral.

5. The case $\mathfrak{g} = \mathfrak{sl}_2$ revisited

We conclude by comparing Theorem 4.1 in the case $\mathfrak{g} = \mathfrak{sl}_2$ with the results of [MY], and by demonstrating some new examples of intertwining operators among generalized Verma modules for $\mathfrak{sl}_2$. For $p \in \mathbb{N}$, we let $L_p$ denote the $(p + 1)$-dimensional irreducible $\mathfrak{sl}_2$-module of highest weight $p^2 \frac{\mathbb{Z}}{2}$ and let $V_{\mathfrak{sl}_2}(\ell, p)$ denote the corresponding generalized Verma module for $\mathfrak{sl}_2$. In this setting, we express Theorem 4.1 as follows:

Theorem 5.1. For $p, q, r \in \mathbb{N}$, the fusion rule $N^{V_{\mathfrak{sl}_2}(\ell, r)}_{V_{\mathfrak{sl}_2}(\ell, p)V_{\mathfrak{sl}_2}(\ell, q)} = 1$ under the following conditions:

1. $r = p + q - 2n$ with $0 \leq n \leq \min(p, q)$, and
2. $m(m + r + 1) \notin (\ell + 2)\mathbb{Z}_+$ for $1 \leq m \leq n$.

Proof. The first condition guarantees that $\dim \text{Hom}_{\mathfrak{sl}_2}(L_p \otimes L_q, L_r) = 1$, so we just need to check that the second condition guarantees that $2(\ell + h^\vee)(h_r, \ell + N)$ is not an eigenvalue of $C_{L_p \otimes L_q}$ for any $N \in \mathbb{Z}_+$. Recalling (2.5) and using $h^\vee = 2$ for $\mathfrak{sl}_2$ and

$$L_p \otimes L_q \cong \bigoplus_{k=0}^{\min(p, q)} L_{p+q-2k},$$

the conditions of Theorem 4.1 amount to

$$\frac{(p + q - 2k)(p + q - 2k + 2)}{2} \neq 2(\ell + 2) \left(\frac{(p + q - 2n)(p + q - 2n + 2)}{4(\ell + 2)} + N\right)$$
for $0 \leq k \leq \min(p, q)$ and any $N \in \mathbb{Z}_+$. This simplifies to

$$(n - k)(p + q - n - k + 1) = (n - k)(r + n - k + 1) \notin (\ell + 2)\mathbb{Z}_+,$$

where we may now assume $0 \leq k \leq n - 1$ since otherwise the left side is non-positive. Setting $m = n - k$, we get

$$m(m + r + 1) \notin (\ell + 2)\mathbb{Z}_+$$

for $1 \leq m \leq n$, as desired. \hfill \qed

Example 5.2. We determine when obstructions to intertwining operators can possibly occur in the cases $n = 0, 1, 2$ of Theorem 5.1:

- For any $\ell \in \mathbb{C} \setminus \{-2\}$ and $p, q \in \mathbb{N}$, $N_{V_{\mathfrak{sl}_2}(\ell, p+q)} = V_{\mathfrak{sl}_2}(\ell, p+q-2)$ = 1 unconditionally.

- For $p, q \in \mathbb{N}$ such that $\min(p, q) \geq 1$, $N_{V_{\mathfrak{sl}_2}(\ell, p+q-2)} = \mathbb{N}_{V_{\mathfrak{sl}_2}(\ell, p+q)} = 1$ except possibly when $p + q \in (\ell + 2)\mathbb{Z}_+$.\n
- For $p, q \in \mathbb{N}$ such that $\min(p, q) \geq 2$, $N_{V_{\mathfrak{sl}_2}(\ell, p+q-4)} = \mathbb{N}_{V_{\mathfrak{sl}_2}(\ell, p+q)} = 1$ except possibly when $2(p + q - 1) \in (\ell + 2)\mathbb{Z}_+$ or $p + q - 2 \in (\ell + 2)\mathbb{Z}_+$.

Theorem 5.1 is similar to but somewhat different from the intertwining operator theorems of [MY]. In [MY, Theorem 6.1], we assumed that the maximal proper submodule of $V_{\mathfrak{sl}_2}(\ell, r)$ was irreducible and isomorphic to some $L_{\mathfrak{sl}_2}(\ell, r')$, and that the maximal proper submodule of $V_{\mathfrak{sl}_2}(\ell, r'')$ was irreducible and isomorphic to some $L_{\mathfrak{sl}_2}(\ell, r''')$. Under these assumptions, we proved that

$$V_{\mathfrak{sl}_2}(\ell, r') \otimes V_{\mathfrak{sl}_2}(\ell, r'') \cong \text{Hom}_{\mathfrak{sl}_2}(L_\ell \otimes L_{r'}, L_{r''}) = 0,$$

that is, provided $2(\ell + 2)h_{r', \ell} + 2(\ell + 2)h_{r'', \ell}$ are not eigenvalues of $C_{L_\ell \otimes L_{r''}}$. Now, since $h_{r', \ell}$ is the lowest conformal weight of a proper non-zero submodule of $V_{\mathfrak{sl}_2}(\ell, r)$, we must have $h_{r', \ell} = h_{r, \ell} + N'$ for some $N' \in \mathbb{Z}_+$, and then similarly $h_{r'', \ell} = h_{r, \ell} + N''$ for some $N''$. Thus [MY, Theorem 6.1] implies Theorem 5.1, but only for $V_{\mathfrak{sl}_2}(\ell, r)$ that satisfy the irreducibility assumptions on maximal proper submodules.

In practice, we know from [MY, Theorem 3.8] that $J_{\mathfrak{sl}_2}(\ell, r) \cong L_{\mathfrak{sl}_2}(\ell, r')$ and $J_{\mathfrak{sl}_2}(\ell, r'') \cong L_{\mathfrak{sl}_2}(\ell, r''')$ when $\ell \in \mathbb{N}$ and $V_{\mathfrak{sl}_2}(\ell, r)$ appears in the Garland-Lepowsky resolutions $[GL]$ of integrable highest-weight $\mathfrak{sl}_2$-modules. Specifically, this means

$$r = m(j, n) = (\ell + 2)j + \frac{\ell}{2}(1 - (-1)^j) + (-1)^j n$$

with $j \geq 0$ and $0 \leq n \leq \ell$ (see [MY, Proposition 8.2]), and then $r' = m(j + 1, n)$, $r'' = m(j + 2, n)$. For such $r$, the next example shows that [MY, Theorem 6.1] can be stronger than Theorem 5.1, that is, the eigenvalue conditions of Theorem 4.1 are not always necessary for generalized Verma modules.

Example 5.3. For $\ell \in 2\mathbb{Z}_+$, take the $\mathfrak{sl}_2$-homomorphism $L_{\ell/2} \otimes L_{\ell/2} \rightarrow L_0 \cong \mathbb{C}$ given by a non-zero invariant bilinear form $\langle \cdot, \cdot \rangle_{\ell/2}$. Then $r = 0$, $r' = 2\ell + 2$, and $r'' = 2\ell + 4$, so

$$\text{Hom}_{\mathfrak{sl}_2}(L_{\ell/2} \otimes L_{\ell/2}, L_{r'}) = \text{Hom}_{\mathfrak{sl}_2}(L_{\ell/2} \otimes L_{\ell/2}, L_{r''}) = 0$$

and by [MY, Theorem 6.1] there is a unique intertwining operator $\mathcal{Y}$ of type $V_{\mathfrak{sl}_2}(\ell, 0)$ such that for $u_1, u_2 \in L_{\ell/2}$,

$$\mathcal{Y}_0(u_1 \otimes u_2) = \langle u_1, u_2 \rangle_{\ell/2}.$$
But condition (2) of Theorem 5.1 fails when \( \ell \equiv 0 \mod 4 \): for \( m = n = \ell/2 \), we have

\[
m(m + r + 1) = \frac{\ell}{2} \left( \frac{\ell}{2} + 0 + 1 \right) = (\ell + 2) \frac{\ell}{4} \in (\ell + 2)\mathbb{Z}_+.
\]

This means that \( 2(\ell + 2) \left( h_{0,\ell} + \frac{q}{4} \right) \) is an eigenvalue of \( C_{L_{\ell/2} \otimes L_{\ell/2}} \) (note that \( h_{0,\ell} = 0 \)).

When \( \ell = 4 \), it is easy to check that the candidate \((4.1)\) for a singular vector in \( V_{s_{2\ell}}(\ell, 0) \) vanishes. Here, \( L_{\ell/2} \cong s_{2\ell} \) with standard basis \( \{ e, h, f \} \); we scale \( \langle \cdot, \cdot \rangle_{\ell/2} = \langle \cdot, \cdot \rangle \) so that \( \langle h, h \rangle = 2 \) and \( \langle e, f \rangle = \langle f, e \rangle = 1 \), and the appropriately-scaled Casimir operator is \( e f + \frac{1}{2} h^2 + f e \). The most interesting eigenvector of \( C_{s_{2\ell} \otimes s_{2\ell}} \) with eigenvalue \( 2(\ell + 2) \left( 0 + \frac{q}{4} \right) = 12 \) to check is \( e \otimes h + h \otimes e \): then \((4.1)\) becomes

\[
e(-1) \left( \langle [f, h], e \rangle + \langle h, [f, e] \rangle \right) + \frac{1}{2} h(-1) \left( \langle [h, h], e \rangle + \langle h, e, h \rangle \right) + f(-1) \left( \langle [e, h], e \rangle + \langle e, e, h \rangle \right) + h(-1) \langle e, h \rangle - 2f(-1) \langle e, e \rangle = 0.
\]

Although this example shows that [MY] can give better results than Theorem 5.1, Theorem 5.1 is usually more versatile. Especially, Theorem 5.1 applies to any level \( \ell \in \mathbb{C} \setminus \{-2\} \) and any weight \( r \in \mathbb{N} \). For example, we can take \( r \) for which \( V_{s_{2\ell}}(\ell, r) \) does not appear in the Garland-Lepowsky resolutions: from \( (5.1) \), these are the positive integers \( (\ell + 2)j - 1 \) with \( j \geq 1 \). Then Example 5.2 provides for instance many new examples of non-zero intertwining operators of type \( \{ V_{s_{2\ell}}(\ell, (\ell/2 + 2)p - 1) \}_{p + q = (\ell + 2)j - 1} \} \) where \( p + q = (\ell + 2)j - 1 \). Moreover, when the conditions of Theorem 5.1 fail, we can use Theorem 4.3 to compute candidates for singular vectors in \( V_{s_{2\ell}}(\ell, (\ell + 2)j - 1) \). For example \( \ell = 0, p = 2, q = 3, r = 1 \) yields a singular vector candidate of conformal weight \( \frac{25}{8} \) in \( V_{s_{2\ell}}(0, 1) \). It would be interesting to check if \((4.1)\) is non-zero in this case, although the calculations would be involved.

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