Bandgaps and directional propagation of elastic waves in 2D square zigzag lattice structures

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Abstract
In this paper we propose various types of two-dimensional (2D) square zigzag lattice structures, and we study their bandgaps and directional propagation of elastic waves. The band structures and the transmission spectra of the systems are calculated by using the finite element method. The effects of the geometry parameters of the 2D-zigzag lattices on the bandgaps are investigated and discussed. The mechanism of the bandgap generation is analyzed by studying the vibration modes at the bandgap edges. Multiple wide complete bandgaps are found in a wide porosity range owing to the separation of the degeneracy by introducing bending arms. The bandgaps are sensitive to the geometry parameters of the systems. The deformed displacement fields of the transient response of finite structures subjected to time-harmonic loads are presented to show the directional wave propagation. The research in this paper is relevant to the practical design of cellular structures with enhanced vibro-acoustics performance.

Keyword: bandgap, directional wave propagation, zigzag lattice

(Some figures may appear in colour only in the online journal)

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a high porosity. In spite of bandgaps, directional propagation of waves in the lattice structures is also a topic which has received considerable attention [13, 14, 20, 21].

In this paper, we study the wave propagation behaviors of a kind of lattice structure with zigzag arms [22, 23]. Such structures can generate complete bandgaps in a wide porosity range even when the porosity is very small. Compared to the system with straight arms, they can generate complete bandgaps at lower frequencies and also present good directional properties. Numerical simulations are conducted by using the finite element method. Both band structures and transmission spectra of the systems are calculated. The effects of the geometry parameters on the bandgaps are investigated and discussed. The vibration modes at the bandgap edges are calculated to analyze the mechanism of the bandgap generation. The phase constant surfaces of some typical bands, as well as the transient propagation of elastic waves in finite structures, are calculated to show the directional properties of the considered systems.

2. Problem statement and computational model

We consider two kinds of 2D square zigzag lattice structures, on which see figure 1(a). They have zigzag arms in one figure 1(c) or two figure 1(d) directions. We name them 1D- and 2D-zigzag lattices, respectively. Suppose that the lattice constant is a, the bending angle of the zigzag arm is θ, and the horizontal (or vertical) distance between two inflection points is \( l \), see figure 1. Then the other geometry parameters shown in the figure can be determined from the above parameters. For instance, the widths of the arms are \( d_0 = a(1 - \sqrt{7}) \), \( d_1 = d_0 \cos \theta \), and \( d_2 = d_0 \cos (\theta - \theta_1) \), and the rotational angle of the zigzag arm is

\[
\theta_1 = \arctan \frac{\sqrt{a^2 + 4l(a-l)\tan^2\theta} - a}{2l\tan\theta}.
\]

Typically, when \( l/a = 0.5 \), we have \( \theta_1 = \theta / 2 \), and thus

\[
d_1 = d_2 = a\left(1 - \sqrt{7}\right) \cos (\theta / 2),
\]

which indicates that the zigzag arms have an identical width. When \( \theta = 0^\circ \), we have the square lattice structure with straight arms.

An alternative profile of the 2D-zigzag lattice may be obtained by perforating a solid with rotational cross holes [7]; see figure 1(b). The cross-section of the unit cell of this alternative profile is shown in figure 1(g), where a is the lattice constant, \( \varphi \) is the rotational angle of the hole, while b and c denote the other geometry sizes of the cross hole, respectively. The rotational angle of the cross hole is limited to the range \( \arcsin(c/a) < \varphi < \arcsin(b/(2a)) \) \( (c < a < b < 2a) \). For simplicity, we name the two parts between the inflection points as the main and the secondary arms; see figure 1. Their arm widths are \( (2ac \cos \varphi - b - c)/2 \) and \( (asin \varphi - c) \), respectively. This structure has the same rotational symmetry as the one in figure 1(d).

In this paper, the bandgap properties and directional wave propagation in the systems shown in figure 1 will be studied with emphasis on the effects of the bending arms. The finite element method based on COMSOL 3.5a [7, 19] is used to calculate the band structures of the considered systems. Due to the periodicity, only one unit cell is used in the calculation. In addition to the antisymmetric zigzag lattices, we also consider the symmetric systems figures 1(e) and (f) because the difference between the symmetric and antisymmetric lattices may lead to different wave propagation properties. The default triangular mesh with Lagrange quadratic elements provided by COMSOL is used with a double refinement. Bloch conditions [7] are applied on the opposite boundaries of the unit cell and traction-free boundary conditions are applied on the other boundaries. All of the dispersion relations can be obtained by sweeping the wave vector \( k \) along the edges of the irreducible Brillouin zone. A detailed description of the procedure can be found in [7]. It is worth noting that the irreducible Brillouin zones are different for different unit cells. Figure 1(h) shows the irreducible Brillouin zone for the antisymmetric 1D-zigzag lattice. For the symmetric 1D- and 2D-zigzag lattices and the antisymmetric 2D-zigzag lattice, the irreducible Brillouin zones are \( \Gamma-X-Y-M \), \( \Gamma-Y(X)-M-M' \), and \( \Gamma-X(Y)-M \), respectively.

3. Mechanism of the bandgap generation

This section will present some detailed numerical results for various zigzag lattices and analyze the mechanism of the bandgap generation. The elastic parameters of the aluminum matrix (shadowed parts in figure 1) are \( \rho = 2700 \text{ kg/m}^3 \), \( E = 20 \text{ GPa} \) and \( \nu = 0.25 \) [7]. For convenience, the reduced frequency \( \Omega = o a / (2\pi c_v) \) is introduced, with the lattice constant \( a = 0.02 \text{ m} \) and the transverse wave velocity of the matrix \( c_v = 1721 \text{ m/s} \). The white parts in figure 1 represent vacuums.

The dispersion curves for the 1D- and 2D-zigzag lattices with \( \theta = 40^\circ \) are shown in figure 2. For comparison, the results for the system with straight arms (\( \theta = 0^\circ \)) are presented in figure 2(a). The transmission spectra for the above systems are also calculated to validate the calculation of the band structures. The detailed calculation models and the corresponding results are shown in figures S1 and S2 (stacks.iop.org/JphysD/47/485102) in the supplementary materials. The bandgaps predicted by the transmission spectra and the band structures are in good agreement.

It is noted that no complete bandgap is observed in figure 2(a) for the system with straight arms, as demonstrated in [7]. In contrast, for the antisymmetric 1D-zigzag lattice in figure 2(b), multiple directional bandgaps rather than complete bandgaps appear dominantly in the bending arm direction, but barely in the straight arm direction. In the symmetric 1D-zigzag lattice, the directional bandgap between the 6th and 7th bands gets wider and one narrow complete bandgap appears in the frequency range of 0.67 < \( \Omega < 0.71 \); see figure 2(c). However, three complete bandgaps are observed in the antisymmetric 2D-zigzag lattice, as shown in figure 2(d). The lowest one exists between the 4th and 5th bands in the frequency range of 0.35 < \( \Omega < 0.45 \). For the symmetric 2D-zigzag lattice in figure 2(e), one complete bandgap between the 6th and 7th
bands is observed in the frequency range of $0.51 < \Omega < 0.78$, which is wider but higher than that of the antisymmetric structure. The bandgap width is about 7 times larger than that of the symmetric 1D-zigzag lattice, as shown in figure 2(c).

The above results imply that a system with bending arms is favorable to open multiple wide directional or complete bandgaps at relatively low frequencies. Bending arms in two directions can open complete bandgaps more easily than those in one direction. For the system with straight arms, no complete bandgap, except for a few directional bandgaps, exists. This is because there are many crossover regions [24] which occur either on the high symmetry points [e.g. B, G and H in figure 2(a)] where folding of bands is expected due to the periodicity, or inside the first Brillouin zone [e.g. D, E, F, I and J in figure 2(a)]. In contrast to the straight arm lattice, the symmetric (or antisymmetric) zigzag lattices possess a reflection (or central) symmetry. Different types of symmetry are the cause of the degeneracy lifting and result in new directional, or even complete, bandgaps. For instance, the degeneracy at the crossover region I is separated for either symmetric or antisymmetric 1D-zigzag lattice in both the bending arm direction and the straight arm direction, which directly creates a new directional bandgap between the 9th and 10th bands. In the antisymmetric 1D-zigzag lattice figure 2(b), the new directional bandgap appearing in the $\mu_3M$ (or $\mu_3M'$) direction between the 5th and 6th bands is a result of the separation of the degeneracy at the crossover regions C and D. Furthermore, the degeneracy at the crossover regions D, E and F is separated in the bending arm direction, but not in the straight arm direction and therefore results in the bandgaps in the $\mu_3X$ direction: see figure 2(b). However, for the symmetric 1D-zigzag lattice in figure 2(c), the degeneracy at the crossover region D (but not at the crossover region E) is separated only in the bending arm direction, leading to the bandgap in the $\mu_3X$ direction.

Figure 1. (a) Sketch of 2D-zigzag lattice and (b) an alternative profile of 2D-zigzag lattice. Panels (c)–(f) show the unit cells of the antisymmetric and symmetric 1D- and 2D-zigzag lattices, respectively. Panel (g) shows the unit cell of panel (b). Panel (h) shows the irreducible Brillouin zones for panel (c).
Actually, the crossover regions D, E and I in figure 2(a) are real cross points, i.e. the dispersion curves are the true intersection [25]. As defined in [26], the amount of the polarization along the x-axis can be represented by one positive number given by

$$\frac{\int_S |\psi|^2 \, dS}{\int_S (|\psi|^2 + |\phi|^2) \, dS},$$

with the integral taken over the whole unit cell S. Here, \(\psi\) and \(\phi\) are the co-existing horizontal and vertical displacements for the mixed plane wave modes. The amount of the polarization along the x-axis is 0.97 at point B\(_1\) and 0.06 at point B\(_2\). This means that these two modes are dominantly \(x\)- and \(y\)-polarized, respectively. A similar behavior is observed at the crossover regions C and D, while the amount of the polarization along the x-axis is 0.5 at point E\(_1\) and 0.99 at point E\(_2\). So the coupling effect between the two modes at the crossover region E is stronger than that at the crossover regions B, C and D. For both symmetric and antisymmetric 1D-zigzag lattices, the weak degeneracy at the crossover regions B, C and D is separated in the bending direction. However, only the
antisymmetric 1D-zigzag lattice separates the relatively strong degeneracy at the crossover region E. So the weak degeneracy is simple to separate when the arms are bent; and the decrease of the symmetry of the system can result in the separation of a relatively strong degeneracy. The amount of the polarization along the x-axis is 0.97 at point I1 and 0.46 at point I2. The coupling effect between the modes at the crossover region I is nearly the same as that at the crossover region E, whereas the degeneracy at the crossover region I is easier to separate because it exists at a higher frequency compared to that at the crossover region E.

In the 2D-zigzag lattices, the bending effects are extended into two directions and the effect of the degeneracy separation generally gets stronger. The directional bandgaps (e.g. the ones generated because of the separation of the degeneracy at the crossover region I) get wider compared to those for the 1D-zigzag lattices. In particular, multiple wide complete bandgaps appear in the band structures. For instance, in the antisymmetric 2D-zigzag lattice, the lowest bandgap occurs owing to the separation of the degeneracy at the crossover region E: see figure 2(d). It should be pointed out that, by introducing the antisymmetric 1D-zigzag lattice, the natural mode changes from point X\_2 to point X\_1, the eigenfrequency of which is smaller than that at point X\_1, and results in the decrease of the 4th band in the bending arm direction. Thus, the degeneracy at the crossover region E is separated, even if no new directional bandgap appears. For the antisymmetric 2D-zigzag lattice, the corresponding mode changes to be point X\_2, the eigenfrequency of which becomes even smaller. The 4th band decreases in all the bending arm directions and gives rise to the lowest complete bandgap. In the symmetric 2D-zigzag lattice figure 2(e), the separation of the degeneracy at the crossover region B becomes more distinct compared with the symmetric 1D-zigzag lattice in figure 2(c). This pronounced effect is due to the larger decrease of the eigenfrequencies from point M\_0 to point M\_3 in comparison with that from point M\_0 to point M\_6. Then the corresponding band decreases and the associated complete bandgap becomes wider. These results also imply that different symmetry can result in different vibration modes and consequently different bandgaps. However, in some cases, the effect of the degeneracy separation becomes weaker. For example, a directional bandgap is generated owing to the degeneracy separation at the crossover region E. So the weak degeneracy is separated, even if no new directional bandgap appears. For the antisymmetric 2D-zigzag lattice figure 2(e), but not for the antisymmetric one figure 2(d).

To more clearly understand the mechanism of the bandgap generation, we illustrate the band structures of the antisymmetric 2D-zigzag lattices with different bending angles in figure 3. It is shown that no complete bandgap appears in the case of straight arms, i.e. when $\theta = 0^\circ$. When the arms are bended slightly ($\theta = 10^\circ$), the degeneracy at the crossover region C is separated and one complete bandgap appears in a relatively high frequency range. The degeneracy at the crossover regions A and B is also separated. When $\theta = 30^\circ$, the lowest complete bandgap appears due to the increase of the 5th band. The upper edge mode changes from point D to point E when $\theta = 40^\circ$. These two modes have a rotational vibration. However, there are four complete standing points at the ends of the arms at point D, but four incomplete standing points in the middle parts of the arms at point E. Thus, these two modes will play different roles. The former one will result in the generation of the bandgap, while the latter one can narrow the bandgap. When continuing to increase the bending angle, some bandgaps in a high frequency appear once again.

The above results indicate that complete bandgaps can be induced due to the separation of the degeneracy when the arms are bent. Different types of the symmetry can give rise to different directional or complete bandgaps. The generation of the lowest bandgap is due to the increase of the frequency of the rotational mode of the unit cell. The appearance of the modes with incomplete standing still points or big vibration-less parts narrows or eventually closes, the bandgaps.

Figure 4 presents the band structures for different rotational angles of the perforated 2D-zigzag lattice with cross holes. The eigenfrequency of the mode at point X\_1 is smaller than
that at points $X_0^2 - X_3^2$. Therefore, this system can generate a complete bandgap at a lower frequency. It is also shown that no complete bandgap appears in the first ten bands when $\varphi = 23.68^\circ$. When $\varphi = 23.7^\circ$, one small complete bandgap appears between the 8th and 9th bands due to the increase of the $l_1$- and $l_2$-bands. The vibration mode at point A is also presented, where the shape of the cross holes may not be easily recognized because the secondary arm is very thin. Both the main and secondary arms vibrate and the whole system has a rotational vibration, similar to that shown in figure 3. The natural frequency of the system is dominated by the vibration of the secondary arm. With the increase of the rotational angle, the width of the secondary arm increases and thus the natural frequency increases. When $\varphi = 23.8^\circ$, one complete bandgap between the 6th and 7th bands arises due to the increase of the degenerate $l_{4,5}$-bands. A complete bandgap between the 4th and 5th bands appears when $\varphi = 24.13^\circ$ as a consequence of the increase of the degenerate $l_{6,7}$-bands. When $\varphi = 25^\circ$, a complete bandgap appears between the 7th and 8th bands due to the increase of the $l_8$-band. The vibration mode of the $l_1$-band changes from point A to point B when $\varphi = 28^\circ$. With the increase of the rotational angle, the natural frequency of this mode decreases because the width of the main arm gets smaller. When $\varphi = 32^\circ$, the vibration mode of the $l_8$-band changes and this band decreases with the increase of the rotational angle.

Generally speaking, the generation of the complete bandgaps is attributed to the rotational vibration of the whole unit cell, as with the antisymmetric 2D-zigzag lattice discussed in the last section. When the vibration of the main arm plays a dominant role, the natural frequency of the mode decreases with the increase of the rotational angle due to the decrease of the main arm width and thus results in the descent of the corresponding bandgap edges.

It is also noted that the mechanism of the bandgap generation for the proposed system is different from that for a system with circular holes, where the locally translational resonance of the large lumps dominates the appearance of the complete bandgaps [7]. For the present system, the bandgaps are generally induced by the rotational vibration which is usually more complicated than the translational resonance. Because there is more freedom in selecting geometrical parameters, the bandgaps of the present system can be easily engineered. Furthermore, it will be proven in what follows that the proposed system can exhibit complete bandgaps in a wider porosity range at lower frequencies than the system with circular holes.

4. Effects of the geometry parameters on bandgaps

In this part, we will mainly discuss the effects of the geometry parameters on complete bandgaps for the antisymmetric 2D-zigzag lattice figure 1(d) and the perforated 2D-zigzag lattice with cross holes figure 1(g).

4.1. The antisymmetric 2D-zigzag lattice

To further demonstrate the dependence of the geometry parameters on the bandgaps of the antisymmetric 2D-zigzag lattice, the variations of the bandgap edges with the geometry parameters are demonstrated in figure 5. It is shown that multiple complete bandgaps appear in the systems. The first (lowest) one appears between the 4th and 5th bands, the second one between 6th and 7th bands, the third one between 8th and 9th bands and the fourth one between the 9th and 10th.

The bandgap edges varying with different bending angle $\theta$ are plotted in figure 5(a) with the fixed porosity $f = 0.8$ and distance $l = 0.5a$. It is shown that no complete bandgap appears when $\theta = 0^\circ$. The first (lowest) bandgap appears between the 4th and 5th bands when $\theta = 26^\circ$. With the increase of the bending angle, the upper edge first increases due to the increase of the natural frequency of the mode at point D. When $\theta > 40^\circ$, the upper edge mode changes to be similar to that at

Figure 4. Band structures for cross holes with different rotation angles and vibration modes at the marked points. For each angle, the horizontal axis represents the wave vector via $\Gamma’-X-M-\Gamma’$. 
point E, the natural frequency of which decreases with the increase of the bending angle, while the lower edge decreases monotonously. The bandgap width first becomes large and then small with the decrease of the bending angle. The second complete bandgap first exists when \( \theta < 24^\circ \) with a very small width. When \( \theta > 50^\circ \), it comes up again. The variations of its bandgap edges and the bandgap width are similar to those of the first complete bandgap. Similar to the case of the second complete bandgap, the third complete bandgap first appears when \( 4^\circ < \theta < 34^\circ \) with a small bandgap width. When \( \theta > 80^\circ \), it comes up again and its bandgap edges decrease with the bending angle increasing. The fourth complete bandgap appears when \( \theta = 40^\circ \). Its bandgap edges vary in a similar trend as the first bandgap with the increase of the bending angles.

Figure 5(b) illustrates the bandgap edges varying with the porosity \( f \) for the fixed arm width \( d_1 = d_2 = d = 0.09a \) and distance \( l = 0.5a \). Four complete bandgaps appear when the system has a relatively high porosity (or similarly when the bending angle determined by equation (1) is relatively big for the fixed arm width). With the porosity decreasing, the widths of all the bandgaps first become larger and then smaller. The lower edges of all these bandgaps decrease monotonously, while the upper edges first increase and then decrease except for the third bandgap. Indeed, the systems with circular holes can also generate complete bandgaps, but only within a small porosity range \( 0.5 < f < 0.79 \) [27]. Also, only one geometry parameter, namely the diameter of the holes, can be used to tune the bandgaps. Moreover, its lowest bandgap edge, which can be obtained when its porosity reaches the limiting value \( f = 0.78 \), is about \( \Omega = 0.3 \). However, the lower edge of the first bandgap is about \( \Omega = 0.2 \) for the proposed system when \( f = 0.78 \) and it can be easily tuned to a smaller value. So the proposed system shows some merits in practical design.

The variations of the bandgap edges with different arm width \( d_1(=d_2) \) for the fixed bending angle \( \theta = 60^\circ \) and distance \( l = 0.5a \) are presented in figure 5(c). The four complete bandgaps exist in a wide range of the arm width (or when the porosity range determined by equation (1) varies from a small to a big value approaching to 1 for the fixed bending angle). The width of all the bandgaps first becomes large and then small. The second and third bandgaps disappear when the

![Figure 5](image-url)
arm width has an intermediate value. With the increase of the arm width, all of these bandgap edges increase monotonously, except the upper edge of the third one.

In figure 5(d), the bending angle $\theta$ and the porosity $f$ are fixed; if the horizontal (vertical) distance $l$ is no longer a constant, the arm width will thus vary. No complete bandgap appears when the distance $l$ is small. The first three complete bandgaps appear when $l/\alpha = 0.3$, 0.27 and 0.22, respectively. With the increase of the bending distance, their lower edges first decrease and then increase. Conversely, their upper edges first increase and then decrease, except for the upper edge of the second one which increases monotonously. All of these bandgaps reach their maximum width when $l/\alpha$ takes certain optimal values.

4.2. The perforated 2D-zigzag lattice with cross holes

The variations of the bandgaps with the rotational angle or the size of the cross hole are shown in figure 6. The first (lowest) complete bandgap locates between the 4th and 5th bands, the second one between the 5th and 6th bands, the third one between the 6th and 7th bands and the fourth one between the 7th and 8th bands.

Figure 6(a) plots the variation of the bandgaps with the rotational angles $\varphi$ for the fixed size of the cross hole (bla = 1.2, clal = 0.4). No complete bandgap exists when the rotational angle is of a small or big value. Complete bandgaps appear when the rotational angle is of some intermediate value. The third bandgap exists in a small range of $\varphi$ ($23.79 < \varphi < 24.9^\circ$), and is very narrow. The first and fourth bandgap first enlarges, then narrows and finally disappears. With the increase of the rotational angle, the upper edge of the first bandgap increases first due to the increase of the $l_0$-bands. Then it begins to decrease due to the change of the edge mode from point A to point B. The lower edge hardly changes when $\varphi < 30^\circ$, and then decreases with the increase of the rotational angle due to the decrease of the $l_0$-band. The upper edge of the fourth bandgap first increases and then decreases. The lower edge first increases with the rotational angle increasing till $\varphi = 26^\circ$, then it begins to decrease due to the change of the edge modes. When $\varphi = 29^\circ$, the lower edge mode changes once more and the lower edge tends to increase with the rotational angle again. When $\varphi = 24.9^\circ$, the second bandgap appears and its upper and lower edges first increase and then decrease with the increase of the rotational angle, owing to the increase/decrease of the $l_0$, $\gamma$- and $l_3$-bands. The width of this bandgap first becomes big, later becomes small and then big again. The existence of this wide bandgap when $\varphi$ is near $36^\circ$ is due to the resonance of the main arms which are quite thin in this case.

Figure 6(b) and (c) plots the variation of the bandgaps with the geometrical size $b/\alpha$ for the fixed $clal = 0.4$ and rotational angle $\varphi = 30^\circ$ is presented in figure 6(b). All of the bandgap edges decrease monotonously with the increase of $b/\alpha$. The first, second and forth complete bandgaps vary in a similar trend with $b/\alpha$. They first enlarge, then narrow and finally disappear. Similar to the second complete bandgap in figure 6(a), when $\varphi$ is big, the third complete bandgap exists when $b/\alpha$ is small due to the resonance of the thin main arms. With the increase of $b/\alpha$, this bandgap becomes narrow and disappears finally.

The bandgap edges varying with the geometry size $clal$ for the fixed $b/\alpha = 1.2$ and rotational angle $\varphi = 30^\circ$ are illustrated.
in figure 6(c). The bandgap edges decrease monotonously with the increase of $c/a$, except for the upper edge of the first complete bandgap (which first increases slightly and then decreases) and the lower edge of the third bandgap (which increases slightly and monotonously). The variation of the bandgap width with $c/a$ is similar to that with $b/a$ in figure 6(b). The existence of the complete bandgap when $c/a$ is of a small or big value is also due to the local resonance of the system with large lumps connected with narrow connectors [7].

In general, the bandgaps are sensitive to the geometry parameters of the zigzag lattice, which is quite useful in the practical design of the lattice structures.

5. Directional wave propagation

Here we will demonstrate that the proposed structures can exhibit inspiring directional wave propagation.

5.1. The 1D- and 2D-zigzag lattices

We first consider the symmetric 1D-zigzag lattice which has different configurations along the various directions. The 6th phase constant surface is chosen to estimate the direction of the elastic wave propagation. It is well known that the direction of the elastic wave propagation is identical to that of the group velocity. In a 2D system, the group velocity is given by [28]

$$c_{gx} = \frac{\partial \omega}{\partial k_x}, \quad c_{gy} = \frac{\partial \omega}{\partial k_y},$$

(3)

where $c_{gx}$ and $c_{gy}$ are the group velocity components along the $x$- and $y$-axis, respectively. Equation (3) implies that the direction of the group velocity can be evaluated by calculating the normal to the isofrequency curves in the phase constant surfaces. In our numerical computation, 120 rays from the contour center are supposed to be evenly circularly distributed. The cross points between the contour and the 120 rays are chosen. Then, the direction of the elastic wave propagation can be established by evaluating the angle of the corresponding normal.

Isofrequency curves are shown in figure 7(a). For a wide frequency range, the curves are noncircular, which generally implies that the direction of the elastic wave propagation does not coincide with the wave vector. Similar results can
be found in the grid-like structures [13, 14, 20]. When the vibration is excited in a given region of a grid structure, the displacement amplitude or the energy flux will become high in some directions and low in the others with respect to the average. It is also noted here that the isofrequency curve of a particular mode in a particular direction in figure 7(a) is open, which means that this mode cannot propagate in this direction. This phenomenon is in agreement with the directional bandgap along the /FX direction in figure 2(c). The polar plot of the directional propagation of the elastic wave [20, 21], or the angle of the normal to the isofrequency curve, at /Omega; = 0.6 is also shown in figure 7(a). The main directions of the elastic wave propagation are along the ±y-directions. This directional wave propagation can also exist in a frequency bandwidth near Ω = 0.6.

To verify the above prediction, we also calculate the transient response of the structure to a time-harmonic (sinusoidal) wave. The finite structure is composed of 21 × 21 unit cells. A point source is located at the center of the central unit cell of the structure. In other words, a time-harmonic displacement field with unit amplitude is imposed on the central node of the central unit cell. It operates at a reduced frequency of Ω = 0.6. Here, both horizontal (1, 0) figure 7(b) and vertical (0, 1) figure 7(c) point sources are considered [29]. Their deformed displacement fields at typical time instants are shown in figure 7 please refer to Supplementary materials for the recordings. For elastic waves with either horizontal or vertical polarization, the wave propagation is confirmed mainly within two branches along the ±y-direction, which validates the prediction of the polar plot in figure 7(a). It is also noted that the amplitude of the displacement field of the x-polarized wave is smaller than that of the y-polarized wave. To explain this, we calculate the amount of the polarization equation (2) of the cross point A in figure 2(c). The value along the x-direction is 0.03, which is smaller than that along the y-direction. So this is a dominantly y-polarized wave and the radiation of the y-polarized source is preferred. The transmission spectra in figure S2(c) (stacks.iop.org/JphysD/47/485102) also illustrate this phenomenon. The transmission coefficient of the x-polarized wave (the solid line) is smaller than that of the y-polarized wave (the dashed line) at Ω = 0.6. The time-harmonic response of the finite symmetric 1D-zigzag lattice is also calculated and presented in figure S3 (stacks.iop.org/JphysD/47/485102) in the supplementary materials. A good agreement between the transient response and the time-harmonic response is observed. The slight difference of the wave propagation in +y- and −y-directions is due to the symmetry of the system.

Next, we examine the directional wave propagation of the antisymmetric 2D-zigzag lattice. Isofrequency curves of the 7th and 8th phase constant surfaces, associated with the band structures for θ = 60° in figure 3, are illustrated in figure 8(a). The curves are nearly parallel to the axes. To clearly show the directional propagation of the elastic wave, we present the direction of the group velocity at Ω = 0.68 in the polar plot in figure 8(a), which shows that the main directions of the elastic wave propagation deviate slightly from the x- and y-directions by a small angle of about 3°. The corresponding results for the 9th phase constant surface are presented in figure 8(b). The slopes of the curves are higher than those of the 7th and 8th phase constant surfaces. Thus, the main directions of the wave propagation at Ω = 0.84 are near ±45° and far from the bending arm directions.

We further calculate the transient response of a finite 2D-zigzag lattice with 21 × 21 unit cells subjected to a time-harmonic load. The horizontal source (1, 0) is located at the center of the structure. The turgidly deformed displacement fields at typical time instants are shown in figures 8(a) (Ω = 0.68) and 8(b) (Ω = 0.84) to highlight the directionality. The results show that the directions of the elastic wave propagation are in general agreement with the evaluation predicted by the polar plots. However, one may also note that the deformation is mainly localized along the bending arm direction in figure 8(a), while in figure 8(b) this effect is less pronounced. So the concentricity of the wave propagation along the bending arm direction figure 8(a) is better than that along the diagonal figure 8(b). Moreover, it is noted that the directional wave propagation in figure 8(b) is not precisely along the diagonal, but with a small deviation about 3°. So the wave propagates along the +48° direction other than the −42° direction for the horizontally exciting source (1, 0). If a vertically exciting source (0, 1) is applied, the wave will mainly propagate along the −42° direction.

It is also noted that the x-direction is preferred to some extent in figure 8(a) because the source is x-polarized. The coexistence of the longitudinal and transverse wave modes results in this different phenomenon compared to the flexural wave [20, 21] or the acoustic wave [30, 31]. Actually, the x- and y-directions are identical if the sources have the same symmetry as the system, on which see figure S4 (stacks.iop.org/JphysD/47/485102) in the supplementary materials.

5.2. The perforated 2D-zigzag lattice with cross holes

Next we consider the directional wave propagation in the perforated 2D-zigzag lattice with cross holes. The isofrequency curves of the 6th, 7th and 5th phase constant surfaces, associated with the band structures for θ = 30° in figure 4, are shown in figures 9(a) and 9(b), respectively. The directional wave propagations at Ω = 0.44 and Ω = 0.3 in the polar plot are also presented in figure 9. It is shown in the polar plot that the main directions of the elastic wave propagation at Ω = 0.44 are near ±x and ±y-directions. In contrast, the directions at Ω = 0.3 are close to ±45°.

The transient responses of a finite structure with 21 × 21 unit cells to time-harmonic loads are also calculated and the deformed displacement fields at typical times are presented in figure 9. Different patterns of directional wave propagation are observed for a point source. The polar plot in figure 9(b) shows that the main direction of the wave propagation has a small deviation (−0.23°) from ±45°, so the wave propagation along the two diagonals is not identical. The concentricity of the directional wave propagation along the bending
arm direction figure 9(a) is much more pronounced than that along the diagonals figure 9(b). When the symmetry of the point sources is the same as that of the structure, the four branches of the directional wave propagation are identical: see figure S5 (stacks.iop.org/JphysD/47/485102) in the supplementary materials.

In fact, a system with straight arms can also exhibit directional wave propagation. To compare the directional properties
of the proposed systems with the one that has straight arms, we consider the directional wave propagation along the x-axis. A concentration degree of the directional wave is defined as follows. We consider the displacement distribution along the y-axis for a fixed x, which is normalized by its maximal value and then calculate the inertia moment of the normalized...
displacement distribution and average it over the length of the structure in the $x$-direction, which is given as

$$I_x = \frac{1}{l_x} \int_{l_y} \int \left( \frac{\sqrt{l_x^2 + y^2}}{\sqrt{l_x^2 + l_y^2}} |y - y_0|^2 \right) dy dx,$$

where $l_x$ and $l_y$ are the lengths of the finite structure in $x$- and $y$-directions and $y_0$ is the location of the excitation. The value obtained from equation (4) is averaged once more over a number of time period, i.e.

$$\bar{I}_x = \frac{1}{NT} \int_{0}^{NT} (I_x)^2 \, dt,$$

where $N$ is the number of the period $T$. Finally, the concentration degree is defined as

$$C_x = 1 / \bar{I}_x = 1 / \sqrt{\frac{1}{NT} \int_{0}^{NT} (c_j)^2 \, dt}.$$

The bigger the concentration degree, the better the concentricity of the directional wave propagation. Here, $N = 2$ is chosen to avoid the wave reflections at the fictitious boundaries. The concentration degree is obtained as $C_x = 5.6 \times 10^2 m^{-2}$ for the 2D-zigzag lattice in figure 8(b) and $C_y = 1.4 \times 10^2 m^{-2}$ for the system with rotated cross holes in figure 9(b). For the system with straight arms shown in figure 2(a), the average concentration degree is $C_x = 2.5 \times 10^2 m^{-2}$ at $\Omega = 0.66$, which is somewhat smaller than that of the zigzag lattices. Thus, the directional wave propagation in the zigzag lattices is more concentrated.

6. Conclusions

In this paper, the bandgap and directional behaviors of elastic wave propagation in zigzag lattice structures were analyzed by using the finite element method. Two types of structures with the bending arms in one or two directions (termed as 1D- and 2D-zigzag lattices, respectively) were considered. The transmission spectra of the systems were also calculated to validate the computed band structures. The effects of the geometry parameters of the 2D-zigzag lattices on the bandgaps were investigated and discussed. The mechanism of the bandgap generation was analyzed by studying the vibration modes at the bandgap edges. Directional propagation of elastic waves in the considered systems was also investigated. The deformed displacement fields of the dynamic response of different finite systems to different time-harmonic loads (according to the symmetry of the system) were presented. From the obtained results and discussions we can draw the following conclusions:

1) No complete bandgap appears in a 2D square lattice structure with only straight arms. By introducing zigzag lattices, the degeneracy of some bands is separated due to the complex vibration modes (especially the rotational modes) of the bending arms. Different types of the symmetry can lead to different vibration modes and give rise to different directional or complete bandgaps. Multiple wide complete bandgaps can be found in a wide porosity range. Distinguished features can be obtained in various kinds of zigzag lattices. Generally, multiple directional bandgaps are generated in 1D-zigzag lattices and multiple wide complete bandgaps are obtained in 2D-zigzag lattice.

2) Detailed study of the geometry parameters for the 2D-zigzag lattice shows that an intermediate value of the bending angle of the arms, or the rotational angle of the cross holes, is favorable for the generation of a wide bandgap. Therefore, wider and lower bandgaps may be obtained by optimizing the geometry of the zigzag lattice structures.

3) Directional propagation of elastic waves in different directions and at different frequencies can be realized by a careful design of zigzag lattice structures. The directional wave propagation in the proposed structures is more concentrated than that in the system with straight arms. Generally, the concentricity of the directional wave propagation is perfect when the energy propagates along the straight arm direction in 1D-zigzag lattice, or along the bending arm direction in the 2D-zigzag lattice. The proposed structures are potential candidates of acoustic/elastic metamaterials.

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