Black Holes with MDRs and Bekenstein-Hawking and Perelman Entropies for Finsler-Lagrange-Hamilton Spaces

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Abstract

New geometric and analytic methods for generating exact and parametric solutions in generalized Einstein-Finsler like gravity theories and nonholonomic Ricci soliton models are reviewed and developed. We show how generalizations of the Schwarzschild - (anti) de Sitter metric can be constructed for modified gravity theories with arbitrary modified dispersion relations, MDRs, and Lorentz invariance violations, LIVs. Such theories can be geometrized on cotangent Lorentz bundles (phase spaces) as models of relativistic Finsler-Lagrange-Hamilton spaces. There are considered two general classes of solutions for gravitational stationary vacuum phase space configurations and nontrivial (effective) matter sources or cosmological constants. Such solutions describe nonholonomic deformations of conventional higher dimension black hole, BH, solutions with general dependence on effective four dimensional, 4-d, momentum type variables. For the first class, we study physical properties of Tangherlini like BHs in phase spaces with generic dependence on an energy coordinate/ parameter. We investigate also BH configurations on base spacetime and in curved cofiber spaces when the BH mass and the maximal speed of light determine naturally a cofiber horizon. For the second class, the solutions are constructed with Killing symmetry on an energy type coordinate. There are analysed the conditions when generalizations of Beckenstein-Hawking entropy (for solutions with conventional horizons) and/or Grigory Perelman’s W-entropy (for more general generic off-diagonal solutions) can be defined for phase space stationary configurations.

Keywords: higher dimension black holes; modified dispersion relations; modified gravity theories; Finsler-Lagrange-Hamilton geometries; generalized Bekenstein-Hawking entropy; Grigory Perelman entropies.

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6 Summary and Concluding Remarks

1 Introduction and preliminaries

Geometric and phenomenological physical models with modified dispersion relations, MDRs, are studied in quantum gravity, QG, and modified gravity theories, MGTs, encoding global and/or local Lorentz invariance violations, LIVs, noncommutative effects, string and generalized Finsler modifications etc., see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and references therein. Black holes, BHs, consist a most suitable and important example which allows us to study possible physical implications and impose physical constraints on MGTs. One may therefore use a series of previous results on quantum corrections and brane-world black holes with special classes of MDRs, for instance, with even powers of the energy parameter) [13, 14]. Nevertheless, it is not clear if BH objects may exist for theories with general MDRs and LIVs and when nonlinear dispersion relations are derived for QG and MGTs. In general, MDRs result in a generic off-diagonal nonlinear phase space dynamics modelled on (co) tangent Lorentz bundles, when the coefficients of (generalized) metrics and connections depend additionally on velocity/ momentum variables. There are necessary more advanced geometric, analytic and numeric methods for constructing solutions of systems of nonlinear partial differential equations, PDEs, related to modified Einstein equations and generalized metrics and connections. Another important question is that if new classes of stationary and quasi-stationary solutions can be constructed, what type of new physics is described, for instance, by phase space generalized BH configurations? To characterize physical properties of such locally anisotropic BH and cosmological models it would be enough to develop the standard (Bekenstein-Hawking) entropy and BH thermodynamics [15, 16, 17, 18] or we have to elaborate on more general concepts of geometric spacetime and phase space thermodynamics and kinetics (for instance, Perelman type entropies [19, 20, 21] and generalized Einstein-Vlasov systems [22, 23])? We argue that a more general geometric and statistical thermodynamics interpretation of general relativity, GR, and MGTs is possible by developing the concepts of the so-called W- and F-entropies, see reviews of results and motivations in [24, 25, 26, 27].

We develop the anholonomic frame deformation method, AFDM, for constructing quasi-stationary solutions of generalized Einstein equations on nonholonomic cotangent Lorentz bundles and associated models of Finsler-Lagrange-Hamilton gravity and nonholonomic Ricci soliton configurations. For reviews on AFDM and applications, we cite our partner works [28, 29, 30] and references therein. In this paper, we consider that readers are familiar with basic concepts of mathematical relativity and differential geometry of bundle spaces. Such non Riemanninan modified gravity theories can be canonically constructed using lifts and nonholonomic deformations of geometric objects considered in general relativity, GR, and Lorentz spacetime manifolds.

Our purpose is to prove that generalized phase space static and stationary Schwarzschild - de Sitter and/or vacuum like BH like solutions can be constructed for MGTs with general MDRs characterized
by an indicator function \( \varpi(x^i, E, \mathbf{p}, m; \ell_P) \)\(^2\). In any point \( x_0 = \{ x^i_0; i = 1, 2, 3, 4 \} \in V \) with \( x^4 = t \) being a time like coordinate, we can consider
\[
c^2 \mathbf{p}^2 - E^2 + c^4 m^2 = \varpi(x^i_0, E, \mathbf{p}, m; \ell_P),
\]
where an indicator of modifications \( \varpi(\ldots) \) encodes possible contributions of generalized physical theories with LIVs etc. We can always chose a corresponding local system of coordinates both on base and co-fiber coordinates when certain MDRs are parameterized in such a form. For quantum theories, such modifications are proportional to the Planck length \( \ell_P \).

The motivation behind the study of \( \text{BH} \) solutions with MDRs seam from the fact that in such conventional extra dimension gravity theories (in our case, with additional conventional velocity/momentum type coordinates) it is encoded a more rich information on nonlinear classical and quantum interactions. Such models are with noncompact phase space (co) fiber dimensions (for realistic physical models, one should be considered a maximal speed of light with possible warped and trapped configurations) and supposed to explain important scenarios in modern cosmology and particle physics. An indicator \( \varpi \) defines certain fundamental nonholonomic (equivalently, anholonomic, i.e. non-integrable) geometric structures which are typical for generalized relativistic models of Finsler-Lagrange-Hamilton spaces.

In this work, the geometric and physical models are defined by three fundamental and canonical geometric and physical objects on \( T^*V \): 1) the nonlinear connection, N-connection, structure \( \{ \mathbf{N}[\varpi] \) and respective N-adapted dual) frame structure \( \{ \mathbf{e}_\alpha[\varpi] \) and \( \{ \tilde{\mathbf{e}}^\alpha[\varpi] \) as functionals of \( \varpi \); 2) the distinguished metric, d-metric, structure \( \{ \tilde{g}_{\alpha\beta}[\varpi] \) and \( \{ \tilde{g}^{\alpha\beta}[\varpi] \) as functionals of \( \varpi \); 3) the Cartan, \( \mathbf{D}[\varpi] \), or canonical, \( \mathbf{D}(x, p) \), d-connections, both uniquely defined by distortions of the Levi-Civita, LC, which is a metric compatible and torsionless linear connection, \( \nabla \), see details in next section and \([28, 29, 30]\). We emphasize that we have to work with generalized Finsler like geometric and physical objects of type 1)-3) and not only with metric type geometries if MDRs (11) are considered in a MGT. Up to frame/coordinate transforms, we can establish such an equivalence of geometric data
\[
\{ \mathbf{N}; \mathbf{e}_\alpha, \mathbf{e}^\alpha; \mathbf{g}_{\alpha\beta} \} \longleftrightarrow \{ \mathbf{H}; \mathbf{N}; \mathbf{e}_\alpha, \mathbf{e}^\alpha; \tilde{g}^{\alpha\beta}; \tilde{g}_{\alpha\beta}; \mathbf{D} \} \longleftrightarrow \{ \mathbf{N}; \mathbf{e}_\alpha, \mathbf{e}^\alpha; \mathbf{g}_{\alpha\beta} \}_{s=1}^{s=4},
\]
where tilde values are defined by canonical geometric objects and the left low, or up, labels like \( ^\prime \) are used for geometric objects determined by a Hamilton space with nondegenerate Hessian for a conventional Hamiltonian \( \mathbf{H}(x, p) \)\(^3\). The left label \( s = 1, 2, 3, 4 \) is used for a conventional space-time coordinates, for instance, for local constructions in GR. We put "tilde" on a symbol in order to emphasize that such a value is canonically defined by a generating function. Boldface symbols are used for geometric/ physical objects which are constructed/defined in a form adapted to a N-connection structure.

\(^2\)On a cotangent bundle \( T^*V \) on a four dimensional, 4-d, Lorentz manifold \( V \), the local coordinates are labeled \( \{ u = (x, p) \} \) \( \{ u^\alpha = (x^i, p_i) \} \). Such dual phase space coordinates can be related via Legendre transforms to velocity-type coordinates/variables on \( TV \), when \( u = (x, v) = \{ u^\alpha = (x^i, v^i) \} \). Unfortunately, we have to elaborate on very sophisticate systems of labels for frames and coordinates if it is necessary to construct physically important exact solutions of systems of nonlinear partial differential equations with variables in (co) tangent bundles. This imposes us to consider various classes of indices and geometric objects with abstract left/right, tilde and hat labels and boldface symbols. Such systems and conventions on notations were introduced in our previous partner works \([28, 29, 30]\). In this paper, the notations for geometric and physical objects are used in an intuitive geometric form with labels on symbols and indices adapted to conventional shell by shell 2+2 and 3+1 decompositions of spacetime and phase space dimensions.

\(^3\)For a MDR (11), we can introduce an effective Hamiltonian \( H(p) := E = \pm (c^2 \mathbf{p}^2 + c^4 m^2 - \varpi(E, \mathbf{p}, m; \ell_P))^{1/2} \). In general, such values depend also on base curves spacetime coordinates, for instance, for local constructions in GR. We put "tilde" on a symbol in order to emphasize that such a value is canonically defined by a generating function. Boldface symbols are used for geometric/ physical objects which are constructed/defined in a form adapted to a N-connection structure.
nonholonomic diadic formalism with \((2+2)+(2+2)\) splitting of the 8-d total bundle space \(T^*V\) over a nonholonomic Lorentz manifold \(V\). Such a diadic formalism and the canonical shell connection (s-connection, \(\hat{s}D\)) allow a very general decoupling and integration via generating functions, effective generating sources of \(\hat{\gamma}_{\alpha_2\beta_2}\), and integration functions of generalized Einstein equations for MGTs with MDRs,

\[
\hat{R}_{\alpha_2\beta_2}[^sD] = \hat{\gamma}_{\alpha_2\beta_2}.
\] (2)

In this formula, \(\hat{R}_{\alpha_2\beta_2}\) is the Ricci tensor for a shell adapted canonical d-connection \(\hat{s}D\).

Generalize Finsler like BH solutions have been studied in a series of our works \cite{31, 32, 33}, see also and references therein, on commutative and noncommutative, supersymmetric, fractional, string/brane, nonholonomic and/or Finsler like generalizations of the GR theory. For certain classes of nonholonomic deformations, such generic off-diagonal solutions can define black ellipsoid/torus/wormhole configurations self-consistently embedded in higher dimension (super) spaces with noncommutative and/or Finsler like variables \cite{34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46}.

Similar constructions can be performed on tangent bundles \(TV\) endowed with structures of Lagrange spaces (Finsler spaces consisting some particular homogenous examples) and higher order and/or nonholonomic jet and/or algebroid generalizations.

In this paper, there are two main goals: The first one is to prove that stationary and quasi-stationary generalizations of the Schwarzschild metric with extra dimensional momentum like variables can be constructed as exact and parametric solutions of modified Einstein equations on \(T^*V\). We consider that such a study of BHs with MDRs may bring about new information on MGTs, QG and possible generalized uncertainty principle etc. Former our results and the AFDM are reviewed in the directions of research 7, 9-12, 14, 18-20 discussed in Appendix B4 of \cite{29}. The second main goal is to study how a statistical thermodynamic geometric approach can be elaborated for stationary configurations and BH solutions on generalized spacetimes and phase spaces. Here we emphasize that analogous constructions involving modifications and generalizations of the Bekenstein-Hawking definitions for BH entropy and spacetime thermodynamics \cite{15, 16, 17, 18} (see, for instance, further developments and alternative constructions in \cite{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28}) can be elaborated only for very special subclasses to solutions. Such higher symmetry generalized metrics are with conventional horizons when a corresponding hypersurface area can be computed (for instance, black ellipsoid/torus, holographic and/or emergent configurations in phase spaces) for physical objects imbedded into certain phase space backgrounds with high symmetry and flat space asymptotic structure.

In our previous works \cite{53, 54, 55, 24, 56, 57, 58, 59, 60, 61, 62, 63, 64, 27, 65} (see also a review of directions of research 10 and 17 in Appendix B4 of \cite{29}), we proved that for generic off-diagonal exact solutions in GR and MGTs with nonholonomic, noncommutative, supersymmetric, fractional and Finsler like variables, we can elaborate a more general approach to the geometric and statistical thermodynamics of gravitational fields using G. Perelman’s concepts of \(W\)- and \(F\)-entropy \cite{19, 20, 21}. On mathematics of Ricci flows and certain physical applications, we cite \cite{66, 67, 68, 69, 70, 71, 72, 73, 74, 75}.

This work is organized as follow: In section 2 we study generalized vacuum phase space and Schwarzschild - de Sitter metrics with generic dependence of coefficients on an energy type variable. There are constructed exact and parametric solutions for BHs in energy depending phase backgrounds. As typical examples, we construct and analyse some important physical properties of nonholonomic deformations of Tangherlini’s BHs \cite{76} phase analogs which can be considered on (co) tangent Lorentz bundles. There are studied also double BH metrics describing black hole configura-
2 BHs with dependence on an energy type variable

2.1 Geometric preliminaries

We follow the conventions for indices and coordinates of geometric objects on $TV$ and $T^*V$ enabled with nonholonomic $(3+1) + (3+1)$ and/or diadic $(2+2) + (2+2)$ splitting in details in Section 3. When boldface symbols are used for spaces and geometric/physical objects enabled with respective nonlinear connection, N-connection, structures $N$ and $\mathcal{N}$. Here we note that the N-connections are defined by certain Whitney sums $\oplus$ of conventional $h$- and $v$-distributions, or $h$ and $cv$-distributions, (for 4+4 splitting), when

$$N : TTV = hTV \oplus vTV \text{ or } \mathcal{N} : T^*TV = h^*V \oplus v^*V. \quad (3)$$

Additionally, $(2+2) + (2+2)$ splitting can be stated if diadic decompositions are considered both for the base and fiber spaces when the N-connections are defined and labelled by a low left label

For local coordinates on a 8-d tangent bundle $TV$, we consider $u^\alpha = (x^i, v^a)$, (or in brief, $u = (x, v)$), when $i, j, k, ... = 1, 2, 3, 4$ and $a, b, c, ... = 5, 6, 7, 8$; and for cumulative indices $\alpha, \beta, ... = 1, 2, ... 8$. Similarly, on a cotangent bundle $T^*V$, we write $'u'^\alpha = (x^i, p_a)$, (in brief, $'u' = (x, p)$), where $x = \{x^i\}$ are considered as coordinates for a base Lorentz manifold $V$. The coordinate $x^i = t$ is considered as time like one and $p_8 = E$ is an energy type one. If necessary, we shall work with $3+1$ decompositions when, for instance, $x^i$, for $i = 1, 2, 3$, are used for space coordinates; and $p_9$, for $a = 5, 6, 7$, are used for momentum like coordinates.

To construct generic off-diagonal exact solutions is useful to introduce diadic indices with a conventional $(2+2) + (2+2)$ splitting. Such indices are labeled as follow: $\alpha_1 = (i_1), \alpha_2 = (i_1, i_2 = a_2), \beta_2 = (j_1, j_2 = b_2); \alpha_3 = (i_3, a_3), \beta_3 = (j_3, b_3), ... \alpha_4 = (i_4, a_4), \beta_4 = (j_4, b_4)$, for $i_1, j_1 = 1, 2; i_2, j_2 = 1, 2, 3, 4; i_3, j_3 = 1, 2, 3, 4, 5, 6; \text{ and } a_2, b_2 = 3, 4; a = (a_3, a_4), b = (b_3, b_4), a_3, b_3 = 5, 6 \text{ and } a_4, b_4 = 7, 8$. A diadic splitting can be adapted to the splitting of h-space into 2-d horizontal and vertical subspaces, $h$ and $v$; or of (co) vertical spaces $v$ into $v_3$ and $v_4$ into conventional four 2-d shells labeled with left up or low abstract indices like $s^v$, or $\alpha_s = (i_s, a_s)$ for $s = 1, 2, 3, 4$ referring to ordered shells. We shall put shell labels on the left up, or left low, to symbols for certain geometric objects if it will be necessary. Indices can be contracted on corresponding shells using ordered diadic groups $\alpha_2 = (i_1, a_2) = 1, 2, 3, 4; \alpha_3 = (a_2, a_3) = 1, 2, 3, 4, 5, 6; \alpha = \alpha_4 = (a_3, a_4) = 1, 2, ..., 8$. In diadic form, the shell coordinates split in the form $x^i = (x^{i_1}, y^{a_2}), u^a = (u^{a_3}, v^{a_4}); p_a = (p_{a_3}, p_{a_4})$. We shall write also $s^v u = \{ u^{\alpha^s} = (x^{i_1}, v^{a_4}) \}$ and $\alpha_s = (x^i, p_a)$ for cumulative indices on corresponding s-shell. For certain classes of solutions, we shall adapt that system of notations in order to consider higher dimensional spherical variables.
solutions in explicit form as we proved in [30, 28, 29]. Such diadic splitting are important for decoupling (modified) Einstein equations and generating exact solutions in explicit form as we proved in [30, 28, 29].

For a dual Lorentz bundle enabled with nonholonomic diadic structure, $sT^*V$, we can parameterize a prime metric $\tilde{\mathfrak{g}}$ in the form

$$
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_i = \tilde{g}_{\alpha_i \beta_i}(x^i, p_a) d^{\alpha_i} \otimes d^{\beta_i} = \tilde{g}_{\alpha_i \beta_i}(\tilde{u}) \tilde{\mathfrak{e}}^{\alpha_i} \otimes \tilde{\mathfrak{e}}^{\beta_i}
$$

We consider splitting into conventional 2-dim nonholonomic distributions of $TTV$ and $TT^*V$ if

$$
\dim(1hTV) = \dim(2hTV) = \dim(3vTV) = \dim(4vTV) = 2,
$$

where left up labels like $1h$, $3v$ etc. state that [using nonholonomic (equivalently, anholonomic and/or non-integrable) distributions] the respective 8-d total spaces split into 2-d shells 1, 2, 3, and 4.

We can consider local formulas for [3] when $N = N_a \frac{\partial}{\partial x^a}$ and/or $\tilde{N} = \tilde{N}_{ia} \frac{\partial}{\partial x^a}$, with respective N-connection coefficients $N = \{N^a\}$ or $\tilde{N} = \{\tilde{N}_{ia}\}$; the up label bar "\" will be used in order to emphasize that certain geometric/physical objects are defined on cotangent bundles. For a nonholonomic diadic splitting with N-connections [4] there are used such parameterizations of coefficients:

$$
\begin{align*}
\mathfrak{N} & = \left\{ N^{a_2} (x^1, y^{a_2}), N^{a_3} (x^1, y^{a_2}, y^{b_3}), N^{a_4} (x^1, y^{a_2}, v^{b_3}, v^{b_4}) \right\}, \\
\tilde{\mathfrak{N}} & = \left\{ \tilde{N}^{a_2} (x^1, y^{a_2}), \tilde{N}^{a_3} (x^1, y^{a_2}, p_{b_3}), \tilde{N}^{a_4} (x^1, y^{a_2}, p_{b_3}, p_{b_4}) \right\}
\end{align*}
$$
of $\eta$-polarizations can be parameterized

$$\delta \hat{g} \rightarrow \delta g = g_{is}(x^{k})dx^{i}s \otimes dx^{i}s + g_{as}(x^{i}, p_{b}) \delta e^{a}s \otimes \delta e^{a}s$$

(6)

The $s$-coefficients of metrics and $N$-elongated dual basis $\delta e^{a}s [\eta]$ used in (6) are respectively defined by formulas

$$\delta e^{a}s = dy^{a}s + \eta_{is}(x^{i}, y^{a}, p_{b}, p_{a}) N^{a}s(x^{i}, y^{a}, p_{b}, p_{a}) dx^{i}s.$$  

(7)

Multiples of type $\eta$ $\delta \hat{g}$ in (7) may depend, in principle, on extra shell coordinates. Nevertheless, such products are subject to the condition that the target $s$-metrics (with the coefficients in the left sides) are adapted to the shell coordinates ordered form $s = 1, 2, 3, 4$.

Here we note that for any prescribed prime $s$-metric $\delta \hat{g}$ we can consider as generating functions a subclass of $\eta$-polarizations $\eta_{i}(x^{i}, y^{3}), \eta_{j}(x^{i}, y^{3}, p_{b}, p_{a})$ which should be defined from the condition that the target $s$-metric $\delta g$ is a quasi-stationary solution of (2). Variants of (3+1) decompositions on the base space and in the typical fiber can be additionally prescribed if we want to define conventional space and time and, respectively, momentum and energy, variables in certain forms which also Adapted to corresponding $N$-connection structures. To generate BH solutions, there will be considered also spherical coordinate systems of respective 3-d, 5-d and 6-d space and phase spaces of Euclidean signature and embedding such configurations in 8-d $T^*V$.

2.2 Prime and target metrics for nonholonomic deformations of BHs

We consider on $T^*V$ a prime metric $\hat{g} = \{\delta g_{\alpha\beta}\}$ (5) which in corresponding local coordinates describe an embedding or it is an analogous of the Tangherlini BH solution for $4 + m$ dimensional spacetimes ($m = 1, 2, ...$), see (76) and (52). In further sections a "math ring, or double ring" labels like $\hat{g}$ and/or $\otimes g$ will be used in order to emphasized that certain classes of solutions involve a prime metric with well defined and important physical properties like a BH solution, an imbedding of such a lower dimensional metric, or certain well defined limits of more general classes of solutions.

2.2.1 Cotangent bundle analogs and imbedding of prime 6-d Tangherlini solutions

In this subsection, an extra dimensional index $m' = 5, 6$ is used for coordinates $p_{m}$ in cofiber spaces. We can multiply momenta to an additional constant and use total space coordinates of the
same dimension as the base space coordinates, \([x^i] = [p_a]\). For spherical symmetry coordinates on the third shell \(s = 3\), the radial coordinate

\[ r = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2 + (p_5)^2 + (p_6)^2} \]

is defined for a 5-d phase subspace with signature \((+++++)\) and some Cartesian coordinates \((x^1, x^2, y^3, p_5, p_6)\). The prime quadratic line element is parameterized

\[
ds^2 = \hat{g}_{\alpha \beta}(x^i, p_a)dx^\alpha dx^\beta = \hat{g}_{\alpha \beta}(x)\hat{e}^\alpha \hat{e}^\beta
\]

\[= h^{-1}(r)dr^2 - h(r)dt^2 + (r)^2d\Omega_4^2 + (dp_7)^2 - dE^2\]

(8)

In these formulas,

\[ h(r) = 1 - \frac{\mu}{r^2} - \frac{\kappa_0^2}{10}\Lambda (r)^2, \]

the area of the 5-dimensional unit sphere is given by

\[ d\Omega_4^2 = d\theta_3^2 + \sin^2 \theta_3 (d\theta_2^2 + \sin^2 \theta_2 (d\theta_1^2 + \sin^2 \theta_1 d\varphi^2)) , \]

(9)

when the shell \(s = 3\) coordinates are \((p_5 = \theta_2, p_6 = \theta_3)\) and shell \(s = 4\) coordinates are \((p_7, p_8 = E)\). Identifying \(x^1 = r, x^2 = \theta = \theta_1, y^3 = \varphi, y^4 = t\) on the shell \(s = 2\), we obtain instead of (9) a 2-d unite sphere for a base spacetime \(V\),

\[ d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2 = (dx^2)^2 + \sin^2(x^2)(dy^3)^2 \]

considered for a 4-d Lorentz manifold. For some fixed coordinates \((p_T, E)\), the prime metric \(\hat{g}_{\alpha \beta}\) describes a 6-d Schwarzschild-de Sitter phase space with effective cosmological constant \(\Lambda\) and BH mass \(M\) through the relation

\[ \mu = \frac{\kappa_0^2 M \Gamma[5/2]}{4 \pi^{5/2}}. \]

Depending on the values of the parameters \(M\) and \(\Lambda\), this metric may have two, one, or zero horizons corresponding to the real, positive roots of the equation \(h(r) = 0\). For \(\Lambda\), a metric (8) may define a solution of gravitational field equations for MGTs with MDRs (2).

In a similar form, we can consider prime metrics and embedding in \(T^*V\), i.e. \(\hat{g} = \{\hat{g}_{\alpha \beta}\}\), for other types higher dimension BH solutions studied, for instance, in [77, 51]. In this work the prime and target metrics will be (quasi) stationary ones of signature \((+++;+--+\)), which is different from standard higher dimension BH solutions with signature \((+++--++++-\+)

In principle, such a \(\hat{g}_{\alpha \beta}\) may be or not a solution of certain gravitational field equations but BH target configurations \(g = \{g_{\alpha \beta}\}\) positively can be generated if certain "small" nonholonomic deformations \(\hat{g} \to g\) are considered for a BH \(\hat{g}\) or embedding in \(T^*V\) and respective relativistic Finsler-Lagrange-Hamilton model.

### 2.2.2 Target stationary 8-d phase space s-metrics

We parameterize the quadratic line element for a target stationary s-metric \(g\) written the form

\[
ds^2 = g_\alpha(x^k)dx^\alpha + g_\alpha(x^i, p_a)(e^\alpha)^2
\]

(10)

\[= g_1(\theta, \varphi)(dr)^2 + g_2(r, \theta)(d\theta)^2 + g_3(r, \theta, \varphi)(d\varphi)^2 + g_4(\theta, \varphi, \theta_3)(d\theta_3)^2 + g_5(r, \theta, \varphi, \theta_3, E)(dp_7)^2 + g_6(r, \theta, \varphi, \theta_3, E)(dE)^2, \]
where the coefficients are with respect to N-elongated dual frames $\mathbf{e}^a$,

1. $\mathbf{e}^1 = d'r$, $\mathbf{e}^2 = d\theta$, $\mathbf{e}^3 = \delta \varphi = d\varphi + N^i_1(\,r, \theta, \varphi)d(\,r) + N^i_2(\,r, \theta, \varphi)d\theta$,

2. $\mathbf{e}^4 = \delta t = dt + N^i_1(\,r, \theta, \varphi)d(\,r) + N^i_2(\,r, \theta, \varphi)d\theta$,

3. $\mathbf{e}^5 = \delta \varphi = d\varphi + N^i_1(\,r, \theta, \varphi, \theta_3)d(\,r) + N^i_2(\,r, \theta, \varphi, \theta_3)d\theta + N^i_3(\,r, \theta, \varphi, \theta_3)d\varphi + N^i_4(\,r, \theta, \varphi, \theta_3)d\varphi$,

4. $\mathbf{e}^6 = \delta \theta_3 = d\theta_3 + N^i_1(\,r, \theta, \varphi, \theta_3)d(\,r) + N^i_2(\,r, \theta, \varphi, \theta_3)d\theta + N^i_3(\,r, \theta, \varphi, \theta_3)d\varphi + N^i_4(\,r, \theta, \varphi, \theta_3)d\varphi$,

5. $\mathbf{e}^7 = \delta \varphi_3 = d\varphi_3 + N^i_1(\,r, \theta, \varphi_3, E)d(\,r) + N^i_2(\,r, \theta, \varphi_3, E)d\theta + N^i_3(\,r, \theta, \varphi_3, E)d\varphi + N^i_4(\,r, \theta, \varphi_3, E)d\varphi$,

6. $\mathbf{e}^8 = \delta E = dE + N^i_1(\,r, \theta, \varphi, \theta_3, E)d(\,r) + N^i_2(\,r, \theta, \varphi, \theta_3, E)d\theta + N^i_3(\,r, \theta, \varphi, \theta_3, E)d\varphi + N^i_4(\,r, \theta, \varphi, \theta_3, E)d\varphi$.

Nonholonomic transforms of prime to target s-metrics, $\dot{s}_g (8) \rightarrow s_g (10)$ can be parameterized in terms of gravitational $\eta$-polarization functions following Convention 5.1 with formulas (66) and (67) in [30]. The N-adapted coefficients of s-metrics are related by such formulas:

\begin{equation}
\begin{aligned}
\dot{g}_{i1}(x^{k1}) &= g_{i1}(\,r, \theta) = \eta_{i1}(x^{k1}, y^{a2}, p_{a3}, p_{a4}) \dot{g}_{i1}(x^{k1}, y^{a2}, p_{a3}, p_{a4}) \text{, where } i_1 = 1, 2 \\
\quad \text{and } \dot{g}_{1} &= \dot{1}(\,r) = (\,r)^2 \sin^2 \theta_3 \sin^2 \theta_2 \text{ for } \eta_{1}(\,r, \theta), \eta_{2}(\,r, \theta, \theta_2, \theta_3); \\
\dot{g}_{i2}(x^{i2}, y^{a2}) &= g_{i2}(\,r, \theta, \varphi) = \eta_{i2}(x^{i2}, y^{a2}, p_{a3}, p_{a4}) \dot{g}_{i2}(x^{i2}, y^{a2}, p_{a3}, p_{a4}) \text{, where } i_2 = 3, 4 \\
\quad \text{and } \dot{g}_{3} &= \dot{1}(\,r) = (\,r)^2 \sin^2 \theta_3 \sin^2 \theta_2 \sin^2 \theta, \dot{g}_{4} = -h(\,r) \text{ for } \eta_{3}(\,r, \theta, \varphi, \theta_2, \theta_3), \eta_{4}(\,r, \theta, \varphi); \\
\dot{g}_{i3}(x^{i3}, p_{a3}) &= g_{i3}(\,r, \theta, \varphi, \theta_2) = \eta_{i3}(x^{i3}, y^{b2}, p_{b3}, p_{b4}) \dot{g}_{i3}(x^{i3}, y^{b2}, p_{b3}, p_{b4}) \text{, where } i_3 = 5, 6 \\
\quad \text{and } \dot{g}_{5} &= \dot{1}(\,r) = (\,r)^2 \sin^2 \theta_3, \dot{g}_{6} = -1 \text{ for } \eta_{5}(\,r, \theta, \varphi, \theta_3), \eta_{6}(\,r, \theta, \varphi, \theta_3); \\
\dot{g}_{i4}(x^{i4}, E) &= g_{i4}(\,r, \theta, \varphi, \theta_2, \theta_3, E) = \eta_{i4}(x^{i4}, y^{b2}, p_{b3}, p_{b4}) \dot{g}_{i4}(x^{i4}, y^{b2}, p_{b3}, p_{b4}) \text{, where } i_4 = 7, 8 \\
\quad \text{and } \dot{g}_{7} &= 1, \dot{g}_{8} = -1 \text{ for } \eta_{7}(\,r, \theta, \varphi, \theta_2, \theta_3, E), \eta_{8}(\,r, E); \\
\end{aligned}
\end{equation}

and, for N-connection coefficients,

\begin{equation}
\begin{aligned}
N^a_{i1}(x^{k1}, y^{a2}, E) &= N^a_{i1}(\,r, \theta, \varphi) = \eta_{i1}(x^{k1}, y^{a2}, p_{a3}, p_{a4}) \dot{N}^a_{i1}(x^{k1}, y^{a2}, p_{a3}, p_{a4}) \text{, where } a, i_1 = 1, 2 \\
N^a_{i2}(x^{i2}, y^{b2}, p_{a3}) &= N^a_{i2}(\,r, \theta, \varphi, \theta_3) = \eta_{i2}(x^{i2}, y^{b2}, p_{b3}, p_{b4}) \dot{N}^a_{i2}(x^{i2}, y^{b2}, p_{b3}, p_{b4}), \\
N^a_{i3}(x^{i3}, p_{a3}, E) &= N^a_{i3}(\,r, \theta, \varphi, \theta_2, \theta_3, E) = \eta_{i3}(x^{i3}, y^{b2}, p_{b3}, p_{b4}) \dot{N}^a_{i3}(x^{i3}, y^{b2}, p_{b3}, p_{b4}), \\
\end{aligned}
\end{equation}

where the N-connection coefficients $\dot{N}^a_{i1}, \dot{N}^a_{i2}, \dot{N}^a_{i3}$ are nonzero in arbitrary local coordinates and vanish in spherical coordinates for a prime s-metric (8).

For this class of target stationary s-metrics with Killing symmetry on $\partial/\partial p_\tau$, it is important that the N-adapted coefficients are parameterized on coordinates $(\,r, \theta, \varphi, \theta_2, \theta_3, E)$ as it is respectively stated in above formulas. In principle, the coefficients of $\dot{s}_g$ and $\eta$-polarizations may depend on all phase space coordinates (including dependencies on time like variables etc. which can be always introduced by arbitrary frame and coordinate transforms). Nevertheless, respective products of such coefficients have to satisfy the conditions (11) and (12) with respect to certain N-adapted frames in order to generate target s-metrics which will possess a decoupling property of modified Einstein equations.
2.3 Off-diagonal ansatz for radial and energy dependence

The $\eta$-polarization functions are subjected to the condition that nonholonomic deformations

$\dot{s}_g \rightarrow \dot{\eta}_s g = [\dot{g}_{\alpha s} = \dot{\eta}_s g_{\alpha s}, \dot{N}^s_{i s-1} = \dot{\eta}_s N^s_{i s-1}]$

generate target metrics $\dot{s}_g$ as solutions of gravitational field equations with MDRs. In this section, we construct in explicit form nonholonomic deformations of 8-d imbedding of prime 6-d Tangherlini like solutions to stationary phase BH configurations with explicit dependence on energy type coordinate $E$.

2.3.1 Configurations with nontrivial shell matter sources

Such exact generic off-diagonal solutions can be generated following the conditions of Corollary 5.2 with formulas (66) and (67) in [30]. In spherical variables with additional $p_7$ and $E$ coordinates, such stationary configurations with Killing symmetry on $\partial/\partial p_7$ and explicit dependence on $E$ are defined by such quadratic line elements:

The quasi-stationary phase configurations on cotangent Lorentz bundles in terms of $\eta$-polarization functions are described by an asatz of type (10),

$$ds^2 = g_{\alpha s, B} d\nu^a ds du^\alpha = e^\psi(r, \theta) [(d'r)^2 + (\cdot r)^2 (d\theta)^2]$$

$$- \int \int d\varphi \left[ \frac{\partial}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \partial \varphi (\dot{\eta}^5 \dot{g}^5) \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5)$$

$$\left\{ d\varphi + \frac{\partial^1}{\partial^1} \left[ \int \int d\varphi \left( \frac{\dot{\eta}^5 \dot{g}^5}{\dot{\eta}^5 \dot{g}^5} \right) \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right\}$$

$$\{ dt + \int_{n_k_1}^{2 m_k_1} \int_{n_k_2}^{2 m_k_2} \int d\varphi \left[ \frac{\partial}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \}$$

$$\left\{ d\varphi + \frac{\partial^1}{\partial^1} \left[ \int \int d\varphi \left( \frac{\dot{\eta}^5 \dot{g}^5}{\dot{\eta}^5 \dot{g}^5} \right) \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right\}$$

$$\{ dp_7 + \int_{n_k_1}^{2 m_k_1} \int_{n_k_2}^{2 m_k_2} \int dE \left[ \frac{\partial}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \}$$

$$\left\{ d\varphi + \frac{\partial^1}{\partial^1} \left[ \int \int d\varphi \left( \dot{\eta}^5 \dot{g}^5 \right) \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right\}$$

$$\{ dt + \int_{n_k_1}^{2 m_k_1} \int_{n_k_2}^{2 m_k_2} \int dE \left[ \frac{\partial}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \right] \frac{\partial \varphi}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5) \}$$

where $n_{k_1}(r, \theta), n_{k_2}(r, \theta, \varphi)$, and $n_{k_3}(r, \theta, \varphi, \theta_2, \theta_3)$, are integration functions on respective shells. The coefficients of such s-metrics are written in terms of $\eta$-polarization functions and determined by generating functions $\dot{\eta}_1 (r, \theta), \dot{\eta}_2 (r, \varphi, \theta_2, \theta_3), \dot{\eta}_3 (r, \varphi, \theta_2, \theta_3)$, and prime s-metric coefficients $\dot{\dot{g}}_{\alpha, B}$ following such formulas

$$\dot{\eta}_1 \dot{g}_1 = \eta_2 \dot{g}_2 = e^{\psi(x^{k_1})}, \eta_3 \dot{g}_3 = -\frac{\partial}{\partial \varphi} (\dot{\eta}_1 \dot{g}_1),$$

$$\dot{\eta}_6 \dot{g}_6 = -\frac{\partial}{\partial \varphi} (\dot{\eta}^5 \dot{g}^5), \dot{\eta}^8 \dot{g}^8 = -\frac{\partial}{\partial \varphi} (\dot{\eta}^7 \dot{g}^7);$$

11
\[ \iota_{i_1}^{N_{i_1}} = \frac{\partial_i \int d\varphi (\hat{2}_i \hat{V}) \partial_\varphi (\eta_4 \hat{g}_4)}{(\hat{2}_i \hat{V}) \partial_\varphi (\eta_4 \hat{g}_4)}, \tag{14} \]
\[ \iota_{k_1}^{N_{k_1}} = 1^{n_{k_1} + 2n_{k_1}} \int d\varphi \frac{[\partial_\varphi (\eta_4 \hat{g}_4)]^2}{|\int d\varphi (\hat{2}_i \hat{V}) \partial_\varphi (\eta_4 \hat{g}_4) (\eta_4 \hat{g}_4)^{5/2}}. \]
\[ \iota_{k_25}^{N_{k_25}} = 1^{n_{k_2} + 2n_{k_2}} \int dp_6 \frac{[\partial^6 (\eta_5 \hat{g}_5)]^2}{|\int dp_6 (\beta_5 \eta_5 \hat{g}_5) (\eta_5 \hat{g}_5)^{5/2}}, \]
\[ \iota_{k_26}^{N_{k_26}} = \frac{\partial_2 \int dp_6 (\hat{2}_i \hat{V}) \partial_\varphi (\eta_5 \hat{g}_5)}{\hat{2}_i \hat{V} \partial_\varphi (\eta_5 \hat{g}_5)}, \]
\[ \iota_{k_37}^{N_{k_37}} = 1^{n_{k_3} + 2n_{k_3}} \int dE \frac{[\partial_E (\eta_7 \hat{g}_7)]^2}{|\int dE (\hat{2}_i \hat{V}) \partial_E (\eta_7 \hat{g}_7) (\eta_7 \hat{g}_7)^{5/2}}, \]
\[ \iota_{k_38}^{N_{k_38}} = \frac{\partial_3 \int dE (\hat{2}_i \hat{V}) \partial_\varphi (\eta_7 \hat{g}_7)}{\hat{2}_i \hat{V} \partial_\varphi (\eta_7 \hat{g}_7)}. \]

A solution of type (13) defines nonholonomic deformations of the 6-d Tangherlini BH into a 8-d phase space. It contains a nontrivial nonholonomic induced torsion which can be constrained to Levi Civita, LC, configurations by imposing additional constraints on integration and generating functions, see details in [30] and subsection [2.3]. For nonsingular small deformations on a parameter \( \varepsilon \), such a s-metric describes a BH embedded self-consistently in a locally anisotropic polarized phase space media. We note that the N-connection coefficients (14) result in generic off-diagonal phase space metrics written in higher dimension spherical coordinates. Such solutions can not be diagonalized for nontrivial anholonomy coefficients. Nevertheless, we can choose respective subcases of generating and integration functions when nonholonomic deformations may result in certain diagonal configurations. Such generalized Tangherlini type solutions describe certain BH configurations with phase space degrees of freedom and contributions from effective and matter field sources on shall. All such sources result in MDRs of type (1).

### 2.3.2 Stationary vacuum off-diagonal configurations

Let us consider an example of target vacuum solution of the generalized Einstein equations (2) with zero source describing a stationary phase space as in Definition 5.4 and Consequence 5.4 of [30] (other types solutions can be generated similarly as in section 5.5 of that work).

Type 1 vacuum off-diagonal quasi-stationary phase space configurations are defined by the conditions \( \partial_\varphi g_4 = 0 \) but \( g_4 \neq 0 \), \( \partial_\varphi g_3 \neq 0 \) and \( g_3 \neq 0 \); \( \frac{\partial}{\partial \varphi} (\eta^5) = 0 \) but \( \eta^5 \neq 0 \), \( \frac{\partial}{\partial \varphi} (\eta^6) \neq 0 \) and \( \eta^6 \neq 0 \); and \( \partial_E (\eta^7) = 0 \) but \( \eta^7 \neq 0 \), \( \partial_E (\eta^8) \neq 0 \) and \( \eta^8 \neq 0 \). The target quadratic line element can be parameterized in a shell adapted form

\[ d s_{v1,8}^2 = d s_{v1,s1}^2 + d s_{v1,s2}^2 + d s_{v1,s3}^2 + d s_{v1,s4}^2 \] where

\[ d s_{v1,s1}^2 = e^{\psi(r,\theta)}[(d'r)^2 + (r)^2(d\theta)^2], \]
\[ d s_{v1,s2}^2 = g_3(r,\varphi)[d\varphi + w_{k1}(r,\theta,\varphi)dx^{k1}]^2 \]
\[ + g_4(r,\theta)[dt + (n_{k1}(r,\theta) + 2n_{k1}(r,\theta)\int d\varphi/g_3)dx^{k1}]^2, \]

12
\[ ds_{v,3}^2 = g^5(r, \theta, \varphi) [d\theta^2 + \left( 1 \eta_k(r, \theta, \varphi) + 2 \mu_k(r, \theta, \varphi) \int d\theta / g^6 \right) dx^2] \]
\[ + \left( \gamma^k(r, \theta, \varphi, \theta_3) \right)_k \left( \eta_{\gamma} g_\gamma(r, \theta, \varphi, \theta_3) \right)_k \right) \] \[ + \left( \gamma^7(r, \theta, \varphi, \theta_2, \theta_3) \right)_k \left( \eta_{\gamma} g_\gamma(r, \theta, \varphi, \theta_2, \theta_3) \right)_k \right) \right) \]
\[ + \left( \gamma^8(r, \theta, \varphi, \theta_2, \theta_3, E) \right)_k \left( \eta_{\gamma} g_\gamma(r, \theta, \varphi, \theta_2, \theta_3, E) \right)_k \right) \] \[ + \left( \gamma^9(r, \theta, \varphi, \theta_2, \theta_3, E) \right)_k \left( \eta_{\gamma} g_\gamma(r, \theta, \varphi, \theta_2, \theta_3, E) \right)_k \right) \]

This vacuum phase solution possesses Killing symmetries on \( \partial_t \) and \( \partial^\theta \) being determined by arbitrary generating functions

\[ g_3(r, \theta, \varphi) = \eta_3 \gamma_3, g_4(r, \theta) = \eta_4 \gamma_4 \text{ and } w_k(1, \gamma_1, \gamma_2, \gamma_3) \]
\[ g^5(r, \theta, \varphi) = \eta^5 \gamma^5, g^6(r, \theta, \varphi, \theta_3) = \eta^6 \gamma^6 \text{ and } w_k(r, \theta, \varphi, \theta_3) \]
\[ g^7(r, \theta, \varphi, \theta_2, \theta_3) = \eta^7 \gamma^7, g^8(r, \theta, \varphi, \theta_2, \theta_3, E) = \eta^8 \gamma^8 \]
\[ \text{and } w_k(r, \theta, \varphi, \theta_2, \theta_3, E) = \eta_{k8} f_{k8} \]

and when the integration functions can be parameterized

\[ \begin{align*}
1 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^4 \gamma^4 k_4; \\
2 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^5 \gamma^5 k_5; \\
1 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^6 \gamma^6 k_6; \\
2 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^7 \gamma^7 k_7; \\
1 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^8 \gamma^8 k_8; \\
2 \eta_k(1, \gamma_1, \gamma_2, \gamma_3) & \sim \eta^9 \gamma^9 k_9; \\
\end{align*} \]

where \( \psi(1, \gamma_r) \) is a solution of 2-d Laplace equation. Such generating and integration functions can be prescribed to be singular or of necessary smooth class. If the \( \eta \)-polarizations are smooth, a corresponding solution (11) describes a 6-d Schwarzschild-de Sitter phase space BH with effective cosmological constant \( \Lambda \) and BH mass \( M \) defined by data \( \eta \) nonholonomically deformed and embedded self-consistently in a vacuum phase space aether. Re-defining the coordinates and for small \( \varepsilon \)-parametric deformations (see details in Theorem 5.1 in [30]), we can always prove that this class of solutions are similar to extra dimensional BHs but with certain polarizations of constants and horizons to dependencies on momentum type coordinates. Here we note the value of \( \Lambda \) should be chosen respectively for different types for Finsler gravity and nonholonomic gravity models, for instance, as in the early works [34, 35] (for off-diagonal corrections to BH solutions) or in more recent approaches with induced cosmological constants in a class of Finsler-Randers and DGP gravity theories [85, 86].

### 2.3.3 Diagonal s-metrics with energy depending polarization functions

We can generate exact solutions with diagonal phase spaces following the Definition 5.3 and Corollary 5.3 in [30] if the N-connection coefficients are zero. A s-metric is diagonal on a shell \( s \) if there are satisfied the conditions \( N_{i-1} = \eta_{i-1} N_{i-1} \). Choosing a prime diagonal metric, for instance \( \eta \) in diagonal coordinates and prescribing a corresponding subclass of data \( (\eta, \eta^5, \eta^7) \), we generate diagonal configurations. Such exact solutions depend on the type of generating and polarization functions (singular, or smooth ones) used for explicit constructions.

By straightforward computations using spherical phase space coordinates, we can prove that there are generated diagonal quasi-stationary phase space configurations if the \( \eta \)-polarization functions for
a s-metric $\eta_{13}$ are chosen in the form

$$
\eta_4 = (\hat{g}_4)^{-1} \int d\phi \eta_4^{[1]}(\phi)/(\hat{\eta}_4^{[2]}(\phi,\hat{\eta}^{[1]}(\phi)),
\eta_5 = (\hat{g}_5)^{-1} \int d\theta_3 \eta_5^{[1]}(\theta_3)/(\hat{\eta}_5^{[2]}(\phi,\hat{\eta}^{[1]}(\phi)),
\eta_7 = (\hat{g}_7)^{-1} \int dE \eta_7^{[1]}(E)/(\hat{\eta}_7^{[2]}(\phi,\hat{\eta}^{[1]}(\phi)),
$$

for integration functions $\eta_4^{[1]}(\phi)$ and $\eta_5^{[2]}(\phi,\theta)$, where $n_{k_1} = 2n_{k_1} = 0$;

$$
\eta_5^{[2]} = \eta_5^{[2]}(\phi,\theta,\varphi)
\eta_7^{[2]} = \eta_7^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3),
\eta_8^{[2]} = \eta_8^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3),
$$

for integration functions $\eta_5^{[2]}(\phi,\theta,\varphi)$, where $n_{k_2} = 2n_{k_2} = 0$;

$$
\eta_7^{[2]} = \eta_7^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3),
\eta_8^{[2]} = \eta_8^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3),
$$

for integration functions $\eta_7^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3)$, where $n_{k_3} = 2n_{k_3} = 0$.

Diagonal stationary configurations consist a special class of phase space nonlinear and nonholonomically systems when the generalized gravitational dynamics is defined by diagonal s-metrics self-consistently embedded into diagonal phase spaces backgrounds. The dependence on the energy type coordinate $E$ is determined by a generating function $\eta^{[1]}(\phi,\theta,\varphi,\theta_2,\theta_3,\theta)$. In particular, we can generate exact solutions for a $\eta^{[1]}(E)$ or certain re-defined systems of coordinates resulting in dependencies on an energy type parameter. Such "rainbow" type models are known due to [12] where we can consider corresponding systems of coordinates resulting in de-formations of horizons and nonlinear polarizations of physical constants. In a more general context, we can consider that certain $\eta$-polarization functions $\eta^{[1]}$ with decompositions on small parameter $\varepsilon$ following the conditions of Consequence 5.3 and formulas (77) in [30]. Formulas with decompositions on small parameters allow an obvious physical interpretation of such generalized BH solutions using analogies from GR and extra dimensions with small deformations of horizons and nonlinear polarizations of physical constants.

In a more general context, we can consider that certain $\eta$-polarization functions define a topologically nontrivial phase space structure with possible filaments, anisotropies, fractional configurations etc. which may model BHs, for instance, when the sources $\eta^{[1]}$ are approximated to cosmological constants.

### 2.3.4 Levi-Civita off-diagonal energy phase space solutions

The stationary solutions constructed above are with nonholonomically induced torsions. We can impose additional constraints on the generating and integration functions and extract zero torsion configurations following the conditions of Consequence 4.2 and formulas (55) in [30].

The quadratic line nonlinear elements for stationary LC-configurations with spherical phase space symmetry and generic dependence on $E$ can be parameterized in such a form:

$$
d^2 s_{\text{Lcst}}^2 = \hat{g}_{\alpha\beta}(x^k, E) du^\alpha du^\beta = e^\psi((\phi,\theta)[(\phi,\theta)^2 + \eta_7^{[2]}(\phi,\theta,\varphi,\theta_2,\theta_3)] +
\left[\partial_\phi [\hat{\psi}^2(\hat{\eta}_4)^{[2]}]\right]^2\left[\partial_\phi [\hat{\psi}^2(\hat{\eta}_4)^{[2]}]\right]^{-2}\left\{d\phi + [\partial_\phi [\hat{\psi}^2(\hat{\eta}_4)^{[2]}]]dx^{i_1}\right\} +
\left\{g_4^{[0]} - \int d\phi \partial_\phi \left[\hat{\psi}^2(\hat{\eta}_4)^{[2]}\right]dx^{i_1}\right\}\left\{dt + \partial_\phi [2n(x^k)]dx^{i_1}\right\} +
$$
functions with respective functional dependence

\[ \{g_5^{[0]} \} - \int d\theta_3 \frac{\partial}{\partial \theta_3} \left[ \frac{(\frac{3}{3}\Psi)^2}{4(\frac{3}{3}\Psi)} \right] \{d\theta_2 + \partial_{i_2} \left[ \frac{3n(x^{k_2})}{d} \right] \} + \frac{[\frac{\partial}{\partial \theta_3} (\frac{3}{3}\Psi)]^2}{4(\frac{3}{3}\Psi)(\frac{3}{3}\Psi)} \{d\theta_3 + \partial_{i_2} (\frac{3}{3}\Delta) \} dx^{i_2} \} + \frac{[\frac{\partial}{\partial \theta_3} (\frac{3}{3}\Psi)]^2}{4(\frac{3}{3}\Psi)(\frac{3}{3}\Psi)} \} \{d\theta_3 + \partial_{i_2} (\frac{3}{3}\Delta) \} dx^{i_2} \} + \frac{[\frac{\partial}{\partial \theta_3} (\frac{3}{3}\Psi)]^2}{4(\frac{3}{3}\Psi)(\frac{3}{3}\Psi)} \} \{d\theta_3 + \partial_{i_2} (\frac{3}{3}\Delta) \} dx^{i_2} \} + \frac{[\frac{\partial}{\partial \theta_3} (\frac{3}{3}\Psi)]^2}{4(\frac{3}{3}\Psi)(\frac{3}{3}\Psi)} \} \{d\theta_3 + \partial_{i_2} (\frac{3}{3}\Delta) \} dx^{i_2} \}.

In above formulas, there are considered generating functions, generating sources and integration functions with respective functional dependence [...] :

\[ s = 2 : \quad \frac{3}{2}\Psi = \frac{3}{2}\Psi(\frac{r}{r}, \theta, \varphi), \partial_{\varphi}(\varphi_{i_2} \frac{3}{2}\Psi) = \varphi_{i_2} \frac{3}{2}\Psi, \]
\[ \varphi_{i_2} = \varphi_{i_2}(\frac{3}{2}\Psi)|_{\frac{3}{2}\Psi}; \quad n_{i_2} = \partial_{i_2} [\frac{3}{2}n(\frac{r}{r}, \theta)]; \]
\[ \frac{3}{2}\Psi = \frac{3}{2}\Psi(\frac{r}{r}, \theta, \varphi), \frac{\partial}{\partial \theta_3} |_{\frac{3}{2}\Psi} \] = \frac{3}{2}\Psi, \text{ or } \frac{3}{2}\Psi = \text{const} ;
\[ s = 3 : \quad \frac{3}{3}\Psi = \frac{3}{3}\Psi(\frac{r}{r}, \theta, \varphi, \theta_3), \frac{\partial}{\partial \theta_3} |_{\frac{3}{3}\Psi} \] = \frac{3}{3}\Psi, \text{ or } \frac{3}{3}\Psi = \text{const} ;
\[ s = 4 : \quad \frac{4}{4}\Psi = \frac{4}{4}\Psi(\frac{r}{r}, \theta, \varphi, \theta_3, E), [\partial_{i_3} \partial_{E}\frac{4}{4}\Psi] = \partial_{i_3} \partial_{E}(\frac{4}{4}\Psi); \]
\[ \varphi_{i_3} = \varphi_{i_3}(\frac{4}{4}\Psi)|_{\frac{4}{4}\Psi}; \quad n_{i_3} = \partial_{i_3} [\frac{4}{4}n(\frac{r}{r}, \theta, \varphi, \theta_3)]; \]
\[ \frac{4}{4}\Psi = \frac{4}{4}\Psi(\frac{r}{r}, \theta, \varphi, \theta_3, E), \text{ or } \frac{4}{4}\Psi = \text{const} .

We note that quasi-stationary s-metrics of type (17) are generic off-diagonal if there are nontrivial anholonomy relations for respective N-adapted bases on a shell. Such exact solutions possess a Killing symmetry on \( \mathcal{O}^7 \) for respective canonical N-adapted frames and coordinate systems, i.e. when the s-metrics do not depend on coordinate \( p_7 \). In principle, we can always extract LC-configurations by considering respective subclasses of generating functions and (effective) sources and integration functions. Other type conditions (additional or special ones) allows us to construct diagonal LC-configurations. In general, such stationary solutions may be not stable. But we can chose special classes of nonholonomic constraints which may stabilize BH nonholonomic LC-configurations.

2.4 Spacetime and cofiber phase space double BH configurations

On \( T^*TV \), we can construct BHs which are different from the Tangherlini type one [5] and nonholonomic deformations studied in previous subsections.

2.4.1 Prime phase metrics for double Schwarzschild - de Sitter BHs

Let us consider two 4-d spherical systems of local coordinates on a base manifold \( V \) and a typical fiber of \( T^*V \), parameterized respectively

\[ \begin{align*}
  x^1 &= r, x^2 = \theta, x^3 = \varphi, x^4 = t; p_5 &= p_r, p_6 = \rho_\theta, p_7 = \rho_\varphi, p_8 = E,
\end{align*} \]
where \( r = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2} \), \( p_r = \sqrt{(p_{\gamma})^2 + (p_{\theta})^2 + (p_\varphi)^2} \).

For such coordinates, prime indices are used for some Cartesian coordinates and \( \theta, \varphi \) and \( p_\theta, p_\varphi \) are respective angular ones (we use a left label 'p' in order to distinguish such spherical type coordinated from the previous type ones used in previous subsections).

A prime quadratic line element

\[
\begin{align*}
\text{ds}^2 &= g_{\alpha\beta}du^\alpha du^\beta = \\
&= f^{-1}(r)dr^2 + r^2d\Omega^2 - f(r)dt^2 + \frac{p_f^{-1}(p_r)d(p_r)^2 + (p_r)^2d\Omega^2 - p_f(p_r)dE^2}{1 - \frac{p_\mu}{p_r} - \frac{(p_\kappa)^2(p_\Lambda)(p_r)^2}{3}}
\end{align*}
\]

defines an exact solution of modified Einstein equations (2) with sources \( \hat{\Gamma}_{\alpha\beta} = \Lambda \), \( \hat{\Gamma}_{ab} = p_\Lambda \). For a \( g_{\alpha\beta} \) (we use the left label "\( \circ \circ \)" in order to emphasize that this is a phase metric with two BH configurations), the areas of respective 2-dimensional unite spheres are given by

\[
\begin{align*}
\text{d}^2p_\Omega^2 &= \text{d}^2\theta^2 + \sin^2\theta\text{d}\varphi^2 = (dx^2)^2 + \sin^2(x^2)(dy^3)^2, \\
\text{d}p_\Omega^2 &= \text{d}(p_\theta)^2 + \sin^2(p_\theta)\text{d}(p_\varphi)^2 = (dp_\theta)^2 + \sin^2(p_\theta)(dp_\varphi)^2.
\end{align*}
\]

Such an exact solution defines on the base Lorentz manifold \( V \) a Schwarzschild - de Sitter spacetime determined by \( f(r) \) with standard BH mass \( M \) via formula \( \mu = \frac{\kappa^2M^2/(3/2)}{2\pi^3/2} \), cosmological constant \( \Lambda \) and \( \kappa^2 \) determined by the Newton constant in GR, see details in [52]. There is also an analogous Schwarzschild - de Sitter configuration on a typical cofiber space determined by \( p_\mu \), where the constants \( p_\mu, p_\Lambda, p_\kappa \) are integration parameters which should be determined experimentally or computed for certain generalized MGTs. For instance, we can take \( p_\Lambda = 0 \) and consider that \( p_\mu \) determines the horizon, i.e. the maximal momentum determined by the maximal speed of light \( c \) for a BH with mass \( M \). Such a double BH configuration describe, for instance, a generalized Schwarzschild phase space with additional cofiber singularity and horizon configurations. Here we note that there are two types of singularities in [18], on the base manifold and in the cofiber space with horizons similar to those of the Schwarzschild - de Sitter BHs. There are not such solutions, for instance, for the Finsler-Randers models [85] even a nontrivial \( p_\Lambda \) can be considered as induced by certain Finsler background fluctuations. To generate BH configurations in a (co) fiber space we need a more rich Finsler structure with canonical d-connections than in the case of minimal extensions of GR by Randers nonlinear quadratic elements.

### 2.4.2 Nonholonomic deformations of phase space double BH s-metrics

We can use \( g_{\alpha\beta} \) (18) as a prime metric instead of 6-d Tangherlini prime metric \( \hat{g} \) (8). A new class of generic off-diagonal solutions can be constructed. Such phase space metrics are very different from those defined by (13) and correlate BH configurations on cofibers with BH solutions on the base spacetime manifold.

The quasi-stationary phase configurations on cotangent Lorentz bundles written in terms of \( \eta-\)
polarization functions are described by an ansatz of type \([10]\),

\[
ds^2 = g_{\alpha \beta} du^\alpha du^\beta = e^{b(r, \theta)} [(dr)^2 + r^2 (d\theta)^2] - \frac{[\partial_\varphi (\eta^4_{\circ \circ} g_4)]^2}{\int d\varphi (\lambda^5_{\circ \circ} g_5) } \{d\varphi + \partial_\varphi [\int d\varphi (\lambda^5_{\circ \circ} g_5) \} dx^{i_1}\}^2 + \frac{[\partial_\varphi (\eta^4_{\circ \circ} g_4)]^2}{\int d\varphi (\lambda^5_{\circ \circ} g_5) } \{d\varphi + \partial_\varphi [\int d\varphi (\lambda^5_{\circ \circ} g_5) \} dx^{i_1}\} + \nonumber 
\]

\[
(\eta^4_{\circ \circ} g_4) \{dt + [1 n_{k_1} + 2 n_{k_1}] \int d\varphi \frac{[\partial_\varphi (\eta^4_{\circ \circ} g_4)]^2}{\int d\varphi (\lambda^5_{\circ \circ} g_5) } \{d\varphi + \partial_\varphi [\int d\varphi (\lambda^5_{\circ \circ} g_5) \} dx^{i_2}\}^2 
\]

\[
(\eta^5_{\circ \circ} g^5) \{d (p r) + \int d (p \theta) \frac{[\partial_\varphi (\eta^5_{\circ \circ} g^5)]^2}{\int d\varphi (\lambda^5_{\circ \circ} g^5) } \{d\varphi + \partial_\varphi [\int d\varphi (\lambda^5_{\circ \circ} g^5) \} dx^{i_2}\}^2 
\]

\[
(\eta^7_{\circ \circ} g^7) \{d p \varphi + \int d (p \theta) \frac{[\partial_\varphi (\eta^7_{\circ \circ} g^7)]^2}{\int d\varphi (\lambda^7_{\circ \circ} g^7) } \{d\varphi + \partial_\varphi [\int d\varphi (\lambda^7_{\circ \circ} g^7) \} dx^{i_3}\}^2 
\]

where \(1 n_{k_1}(r, \theta), \ 2 n_{k_1}(r, \theta), \ 1 n_{k_2}(r, \theta, \varphi), \ 2 n_{k_2}(r, \theta, \varphi), \ 1 n_{k_3}(r, \theta, \varphi, \ p_r, \ p_\theta), \ 2 n_{k_3}(r, \theta, \varphi, \ p_r, \ p_\theta), \) are integration functions on respective shells. The coefficients of such s-metrics for nonholonomic deformations of double BH pase space solutions are written in terms of \(\eta\)-polarization functions and determined by generating functions \([\psi (r, \theta), \eta_4 (r, \theta, \varphi), \ \eta^5 (r, \theta, \varphi, \theta), \ \eta^7 (r, \theta, \varphi, \ p_r, \ p_\theta, \ E)\); generating sources \([\lambda^5_{\circ \circ} g^5] (r, \theta, \varphi, \ p_r, \ p_\theta, \ E)\].

We note that formulas for \(\eta\)-polarization functions in \([19]\) are similar to \([14]\) but with respective re-definition of coordinates and coefficients of prime metrics. General frame and coordinate transforms on \(T^*V\) mix the conventional spacetime-momentum like variables and it is not clear what physical meaning may have such solutions for general \(\eta\)-polarization. Nevertheless, we can construct nonholonomic deformations of the 6-d Tangherlini metric, or of double Schwarzschild - de Sitter configurations, for small parametric \(\epsilon\)-deformations. In principle, we can consider two independent such parameters: one for spacetime deformations and another one for typical co-fiber configurations. For small parameters, the BH solutions with nonholonomic deformations possess the same singular properties as in GR but with certain small deformations of horizons and polarizations of constants.

2.5 Stationary configurations with spherical symmetries and BHs in energy depending phase spaces

There are more general classes of exact stationary solutions with 6-d spherical symmetry nonholonomically deformed on a 8-d \(T^*V\) then those constructed in previous sections. Such generic off-diagonal metrics consist explicit examples of nonholonomic stationary phase spaces in MGT with MDRs defined by Consequence 5.1 and Remark 5.1 in \([30]\). For simplicity, we construct and analyse solutions of type \([10]\) when the conventional radial coordinate depend both on space and fiber.
2.5.1 Solutions with energy depending generating functions & (effective) sources

Considering certain generated data \([g_1(\dot{r}, \theta, \varphi), g^5(\dot{r}, \theta, \varphi, \theta_3), g^7(\dot{r}, \theta, \varphi, \theta_2, \theta_3, E)]\) and nonlinear symmetries involving generating sources (see next subsection) \(_s\tilde{\Psi} = [\dot{\tilde{\Psi}}(\dot{r}, \theta), \dot{\tilde{\Psi}}(\dot{r}, \theta, \varphi), \dot{\tilde{\Psi}}(\dot{r}, \theta, \varphi, \theta_3), \dot{\tilde{\Psi}}(\dot{r}, \theta, \varphi, \theta_2, \theta_3, E)]\), we construct such spherical symmetric nonlinear quadratic elements,

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = e^{\psi(\dot{r}, \theta)}[(\dot{r})^2 + (\dot{r})^2(\dot{\theta})^2] - \frac{(\partial_\varphi g_4)^2}{\int d\varphi \partial_\varphi (\dot{\tilde{\Psi}} g_4) g_4 (\dot{\tilde{\Psi}} g_4)} \{d\varphi + \partial_{i_1} \left[ \int d\varphi (\dot{\tilde{\Psi}} g_4) \partial_\varphi d\varphi \right] dx^{i_1} \}^2 + \]

\[
g_4 \{dt + [1 n_{k_1} + 2 n_{k_1}] \int d\varphi \frac{(\partial_\varphi g_4)^2}{\int d\varphi \partial_\varphi (\dot{\tilde{\Psi}} g_4) [g_4]^2} dx^{i_1} \} + \]

\[
g^5 \{d\theta_2 + [1 n_{k_2} + 2 n_{k_2}] \int d\theta_3 \frac{[\partial_\varphi (g^5)]^2}{\int d\varphi \partial_\varphi (\dot{\tilde{\Psi}} g^5) [g^5]^2} dx^{i_2} \} - \]

\[
\frac{[\partial_\varphi (g^5)]^2}{\int d\varphi \partial_\varphi (\dot{\tilde{\Psi}} g^5) [g^5]^2} \{d\theta_3 + \partial_{i_2} \left[ \int d\varphi (\dot{\tilde{\Psi}} g^5) \partial_\varphi d\varphi \right] dx^{i_2} \}^2 + \]

\[
g^7 \{dp_7 + [1 n_{k_3} + 2 n_{k_3}] \int dE \frac{[\partial_\varphi (g^7)]^2}{\int dE \partial_\varphi (\dot{\tilde{\Psi}} g^7) [g^7]^2} dx^{i_3} \} - \]

\[
\frac{[\partial_\varphi (g^7)]^2}{\int dE \partial_\varphi (\dot{\tilde{\Psi}} g^7) [g^7]^2} \{dE + \partial_{i_3} \left[ \int dE (\dot{\tilde{\Psi}} g^7) \partial_\varphi dE \right] dx^{i_3} \}^2.\]

Such solutions are also determined by generating functions

\(1 n_{k_1} (\dot{r}, \theta), 2 n_{k_1} (\dot{r}, \theta), 1 n_{k_2} (\dot{r}, \theta, \varphi), 2 n_{k_2} (\dot{r}, \theta, \varphi), 1 n_{k_3} (\dot{r}, \theta, \varphi, \theta_3), 2 n_{k_3} (\dot{r}, \theta, \varphi, \theta_3).\)

The class of solutions (20) is nonsingular if the values \(_s\tilde{\Psi}\) are not zero and the generating and integration functions are not singular. If there are encoded nonholonomic deformations of certain prime singular metrics, such target solutions (in general) are singular and describe BHs with phase space off-diagonal stationary configurations. With singular generating and integration functions, we can transform BH solutions into nonsingular ones and inversely. We can apply the AFDM for vacuum phase space or vacuum Lorentz manifold configurations as in subsection 2.3.2.

2.5.2 Nonlinear spherical symmetries of generating functions and sources

A large class of exact solutions in MGTs with MDRs which are constructed following the AFDM possess a new type of nonlinear symmetries stated by Theorem 4.3 with formulas (60) in [30]. For nonholonomic deformations of the 6-d Tangherlini metric \(_s\hat{g}\) [8], such symmetries are stated for respective shells when the generated functions, \(_s\hat{\Psi}\), and sources, \(_s\hat{\Psi}\), are transformed into equivalent
data for other types generating functions, \( \frac{1}{3} \Phi \), with effective cosmological constants, \( \frac{1}{4} \Lambda \):

\[
\begin{align*}
\left[ \frac{1}{2} \Psi'(r, \theta, \varphi), \frac{1}{3} \Psi'(r, \theta, \varphi, \theta_3), \frac{1}{4} \Psi'(r, \theta, \varphi, \theta_2, \theta_3, E), \frac{1}{4} \tilde{T}(r, \theta, \varphi, \theta_3) \right] \\
\rightarrow \left[ \frac{1}{2} \Phi'(r, \theta, \varphi), \frac{1}{3} \Phi'(r, \theta, \varphi, \theta_3), \frac{1}{4} \Phi'(r, \theta, \varphi, \theta_2, \theta_3, E), \frac{1}{4} \Lambda \right].
\end{align*}
\]

Such transforms are determined by formulas

\[
\begin{align*}
s = 2 : & \quad \frac{\partial \varphi[\frac{1}{2} \Psi^2]}{\frac{1}{2} \tilde{T}} = \frac{\partial \varphi[\frac{1}{2} \Phi^2]}{\frac{1}{2} \Lambda}, \\
\text{i.e.} \quad (\frac{1}{2} \Phi)^2 = \frac{1}{2} \Lambda \int d\varphi(\frac{1}{2} \tilde{T})^{-1} \partial \varphi[\frac{1}{2} \Psi^2] \quad \text{and/or} \quad (\frac{1}{2} \Psi)^2 = (\frac{1}{2} \Lambda)^{-1} \int d\varphi(\frac{1}{2} \tilde{T}) \partial \varphi[\frac{1}{2} \Phi^2]. \\
s = 3 : & \quad \frac{\partial \varphi[\frac{1}{3} \Psi^2]}{\frac{1}{3} \tilde{T}} = \frac{\partial \varphi[\frac{1}{3} \Phi^2]}{\frac{1}{3} \Lambda}, \\
\text{i.e.} \quad (\frac{1}{3} \Phi)^2 = \frac{1}{3} \Lambda \int d\theta_3(\frac{1}{3} \tilde{T})^{-1} \partial \theta_3[\frac{1}{3} \Psi^2] \quad \text{and/or} \quad (\frac{1}{3} \Psi)^2 = (\frac{1}{3} \Lambda)^{-1} \int d\theta_3(\frac{1}{3} \tilde{T}) \partial \theta_3[\frac{1}{3} \Phi^2]. \\
s = 4 : & \quad \frac{\partial E[\frac{1}{4} \Psi^2]}{\frac{1}{4} \tilde{T}} = \frac{\partial E[\frac{1}{4} \Phi^2]}{\frac{1}{4} \Lambda}, \\
\text{i.e.} \quad (\frac{1}{4} \Phi)^2 = \frac{1}{4} \Lambda \int dE(\frac{1}{4} \tilde{T})^{-1} \partial E[\frac{1}{4} \Psi^2] \quad \text{and/or} \quad (\frac{1}{4} \Psi)^2 = (\frac{1}{4} \Lambda)^{-1} \int dE(\frac{1}{4} \tilde{T}) \partial E[\frac{1}{4} \Phi^2].
\end{align*}
\]

There are nonlinear symmetries which are similar to (21) nonholonomic deformations of prime metrics of type \( g_{\alpha \beta} \) (18) for double 4-d BH configurations on spacetime and typical fiber space.

### 2.5.3 Off-diagonal solutions with effective constants and zero torsion

To extract LC-configurations as in (17) the nonlinear symmetries (21) have to constrained to certain subclasses of generating functions defined by data
g\( \tilde{g}_4(\ r, \theta, \varphi), \tilde{g}_5(\ r, \theta, \varphi, \theta_3), \tilde{g}_7(\ r, \theta, \varphi, \theta_2, \theta_3, E); \)
on general results, see Remark 5.1 in [30]. The generating functions (we use inverse hat labels emphasizing that certain additional integrability conditions are imposed on generating functions) are redefined following formulas:

\[
\begin{align*}
\text{shell } s = 2 : & \quad \partial \varphi[\frac{1}{2} \Psi^2] = \int d\varphi(\frac{1}{2} \tilde{T}) \partial \varphi \tilde{g}_4, \quad (\frac{1}{2} \Phi)^2 = -4 \frac{1}{2} \Lambda \tilde{g}_4; \\
\text{shell } s = 3 : & \quad \partial \varphi[\frac{1}{3} \Psi^2] = \int d\theta_3(\frac{1}{3} \tilde{T}) \partial \theta_3 \tilde{g}_5, \quad (\frac{1}{3} \Phi)^2 = -4 \frac{1}{3} \Lambda \tilde{g}_6; \\
\text{shell } s = 4 : & \quad \partial E[\frac{1}{4} \Psi^2] = \int dE(\frac{1}{4} \tilde{T}) \partial E \tilde{g}_7, \quad (\frac{1}{4} \Phi)^2 = -4 \frac{1}{4} \Lambda \tilde{g}_8.
\end{align*}
\]
Corresponding quadratic line nonlinear elements for exact solutions are parameterized in the form

\[ ds^2_{LC} = \hat{g}_{\alpha\beta} du^\alpha du^\beta = e^{\psi (\frac{\partial}{\partial t})} [(d\ 'r)^2 + (\ 'r)^2 (d\theta)^2] \]

\[ - \frac{\partial \hat{g}_4 (\partial \hat{g}_4)}{\int d\phi (\hat{g}_5)(\partial \hat{g}_4) g_4} \{ d\phi + \partial_n [\hat{\nabla} \phi(\hat{r}, \theta, \varphi)] dx^{i_1} \} + \hat{g}_4 \{ dt + \partial_n [2n (\ 'r, \theta)] dx^{i_2} \} + \hat{g}_4 \{ dp_7 + \partial_n [3n (x^{k_3})] dx^{i_3} \} - \frac{[\partial_{E}(\hat{g}^5)]^2}{\int dE (\hat{g}^5)} \{ d\theta_3 + \partial_n [\hat{\nabla} \phi(\hat{r}, \theta, \varphi)] dx^{i_2} \} + \hat{g}_7 \{ dp_7 + \partial_n [3n (x^{k_3})] dx^{i_3} \} - \frac{[\partial_{E}(\hat{g}^7)]^2}{\int dE (\hat{g}^7)} \{ dE + \partial_n [\hat{\nabla} \phi(\hat{r}, \theta, \varphi, \theta_2, \theta_3, E)] dx^{i_3} \}. \] (22)

Here we note that the effective shell sources \( \hat{\nabla} \phi \) can be absorbed into certain classes of integration functions if such values do not depend on vertical shell coordinates on respective shells. The sources \( \hat{\nabla} \phi \) are encoded correspondingly in \( \hat{\nabla} \phi \) for (22). For nonsingular generating functions and sources, such solutions define stationary phase space configurations with 6-d spherical symmetry. If such phase space solutions encode prime BH configurations, we can consider that this quadratic line nonlinear elements define new classes of BH solutions with additional cofiber symmetries determined by MDRs.

3 Phase space stationary BHs with energy type Killing symmetry

Applying the AFDM, we can construct various classes of stationary and BH solutions of modified Einstein equations (22). For example, we can generate stationary configurations with a fixed energy phase coordinate, \( E = E_0 \) (for instance, we can take \( E_0 = 0 \)) and possessing a Killing symmetry on \( \partial / \partial E \). In this section, we study nonholonomic deformations of prime BH metrics to target stationary phase configurations when the s-metric and N-connection coefficients considered for N-adapted frames depend on space coordinates \( (x^1, x^2, y^3) \) and momentum variables \( (p_5, p_6, p_7) \) but not on \( y^4 = t \) and \( p_8 = E \).

3.1 7-d Tangherlini like BHs embedded into 8-d phase spaces

As a prime configuration, we can consider a 7-d Tangherlini BH solution (instead of a 6-d one (8)). For a 6-d phase space with signature \((+++++)\), we consider a new radial coordinate

\[ \tau = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2 + (p_5)^2 + (p_6)^2 + (p_7)^2}. \]

The prime quadratic line element is parameterized

\[ ds^2 = \hat{g}_{\alpha\beta} (x^{i_3}, p_\alpha) \frac{d' u^\alpha d' u^\beta}{\hat{g}_{\alpha\beta}} = \hat{g}_{\alpha\beta} (y^u e^\alpha e^\beta) \hat{g}_{\alpha\beta} \]

\[ = \mathcal{h}^{-1}(\tau) d(\tau)^2 - \mathcal{h}(\tau) dt^2 + (\tau)^2 d\Omega_5^2 - dE^2, \] (23)

where certain symbols are "overlined" in order to emphasize that respective values are defined by different formulas from those considered in previous section. In \( \hat{g}_{\alpha\beta} \) (23), the 7-d Schwarzschild-de
Sitter phase space BH solution is determined by

\[ \mathbf{\Pi}(r) = 1 - \frac{\mu}{(r)^4} - \frac{\gamma_7^2}{15} \left( \frac{\Lambda(r)}{r} \right)^2, \]

where the effective gravitational constant \( \gamma_7 \), cosmological constant \( \Lambda \) and BH mass \( \mu \) are related through a relation with Gamma function \( \Gamma[3] \), \( \frac{\mu}{r^4} = \frac{\gamma_7^2}{5} \frac{\Lambda(r)}{\mu^2} \). For such solutions, the area of the 5-dimensional unit sphere with four angular coordinates \((\theta_1 = \theta, \varphi, \theta_2, \theta_3, \theta_4)\) is given by

\[ d\Omega^2 = d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 (d\theta_3^2 + \sin^2 \theta_3 (d\theta_4^2 + \sin^2 \theta_4 d\varphi^2))), \]

when the respective shell \( s = 3 \) coordinates are \((p_5 = \theta_2, p_6 = \theta_3)\) and shell \( s = 4 \) coordinates are \((p_7 = \theta_4, p_8 = \varphi)\).

We emphasize that the 7-d BH metric \( \hat{g}_{\alpha\beta} \) does not contain as a particular case certain double BH configurations of type \( \gamma_\infty g_{\alpha\beta} \). Such exact solutions are characterized by different topological, local and asymptotic properties. This type of phase space BH metrics can not be constructed in the Finsler-Randers models but exist in other types of generalized Finsler-Lagrange-Hamilton gravity theories, see examples in [28, 31, 32, 33, 41, 44].

### 3.2 Nonholonomic deformations of BHs with fixed energy parameter

Using \( \eta \)-polarization functions and a prime BH s-metric \( \hat{g}_{\alpha\beta} \), we can generate stationary phase configurations on cotangent Lorentz bundles described by an ansatz with Killing symmetries on \( \partial_4 = \partial_7 \) and \( \partial_8 = \partial_5 \) which is different from \( \lambda_{10} \) being with generic Killing symmetry on \( \partial_4 \) and \( \partial_7 \). The AFDM can be applied similarly to solutions \( \lambda_{13} \) and \( \lambda_{14} \) when the local coordinates, prime metric data and Killing symmetries are redefined correspondingly. We construct nonholonomic deformations \( \hat{g} \rightarrow g \) defined by such target nonlinear quadratic elements:

\[
\begin{align*}
\hat{g}_{\alpha\beta} &= \hat{g}_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} \partial_\gamma, \\
\hat{s}^2 &= \epsilon_{\alpha\beta\gamma} \partial_\gamma d\hat{s}_{\alpha\beta} = e^{\psi(\tau, \theta)} [(d\mathbf{r})^2 + (\gamma_4)^2 (d\theta)^2] \\
&- \frac{\partial_\gamma (\gamma_4 \hat{g}_{\gamma 4})}{\partial_\gamma} \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\gamma_4 \hat{g}_{\gamma 4}) \right] \{d\phi + \int d\gamma \hat{g}_{\gamma 4} \partial_\gamma (\gamma_4 \hat{g}_{\gamma 4}) \} \\
&+ \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\gamma_4 \hat{g}_{\gamma 4}) \right] \{dt + \int d\gamma \hat{g}_{\gamma 4} \partial_\gamma (\gamma_4 \hat{g}_{\gamma 4}) \} \\
&+ \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\gamma_5 \hat{g}_{\gamma 5}) \right] \{d\gamma_2 + \int d\gamma_3 \hat{g}_{\gamma 5} \partial_\gamma (\gamma_5 \hat{g}_{\gamma 5}) \} \\
&- \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \right] \{d\gamma_4 + \int d\gamma_5 \hat{g}^{\gamma 8} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \} \\
&- \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \right] \{d\gamma_5 + \int d\gamma_6 \hat{g}^{\gamma 8} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \} \\
&+ \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \right] \{d\gamma_6 + \int d\gamma_7 \hat{g}^{\gamma 8} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \} \\
&+ \left[ \epsilon_{\alpha\beta\gamma} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \right] \{d\gamma_7 + \int d\gamma_8 \hat{g}^{\gamma 8} \partial_\gamma (\eta^8 \hat{g}^{\gamma 8}) \}.
\end{align*}
\]

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where \(1 n_k(\tau, \theta)\), \(2 n_k(\tau, \theta)\), \(1 n_k(\tau, \varphi)\), \(2 n_k(\tau, \varphi)\), \(1 n_k(\tau, \varphi, \theta_2, \theta_3)\), \(2 n_k(\tau, \varphi, \theta_2, \theta_3)\), are integration functions on respective shells with new types of radial and angular coordinates. The coefficients of (24) are written in terms of integration functions on respective shells with new types of radial and angular coordinates. The conditions:

\[
\eta^c \text{ponents of (24) and determined by generating functions } [\psi(\tau, \theta), \eta(\tau, \theta, \varphi), \eta^5(\tau, \theta, \varphi, \theta_3), \eta^8(\tau, \theta, \varphi, \theta_2, \theta_3, \theta_4)]; \text{ generating sources } \tilde{\nabla}(\tau, \theta, \varphi), \tilde{\nabla}(\tau, \theta, \varphi, \theta_3), \tilde{\nabla}(\tau, \theta, \varphi, \theta_2, \theta_3), \tilde{\nabla}(\tau, \theta, \varphi, \theta_2, \theta_3, \theta_4)]; \text{ and prime s-metric coefficients } \tilde{g}_{\alpha \beta} \text{ subjected to such conditions:}
\]

\[
\eta_1 \tilde{g}_{\alpha 1} = \eta_2 \tilde{g}_{\alpha 2} = e^{\psi(x_k)}, \eta_3 \tilde{g}_{\alpha 3} = -\frac{[\partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)]^2}{|\int d\varphi \partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)|} \eta_4 \tilde{g}_{\alpha 4},
\]

\[
\eta^6 \tilde{g}^6 = -\frac{[\partial_\varphi \left( \eta^5 \tilde{g}^5 \right)]^2}{|\int d\varphi \partial_\varphi \left( \eta^5 \tilde{g}^5 \right)|} \eta^5 \tilde{g}^5,
\]

\[
\eta^7 \tilde{g}^7 = -\frac{[\partial_\varphi \left( \eta^8 \tilde{g}^8 \right)]^2}{|\int d\varphi \partial_\varphi \left( \eta^8 \tilde{g}^8 \right)|} \eta^8 \tilde{g}^8,
\]

\[
\eta_1 \tilde{\nabla}_{\alpha 1} = \eta_2 \tilde{\nabla}_{\alpha 2} = \frac{\partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)}{(\eta_4 \tilde{g}_{\alpha 4})},
\]

\[
\eta_4 \tilde{\nabla}_{\alpha 4} = \frac{1 n_k + 2 n_k \int d\varphi \partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)}{|\int d\varphi \partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)|} \eta_4 \tilde{g}_{\alpha 4},
\]

\[
\eta_5 \tilde{\nabla}_{\alpha 5} = 1 n_k + 2 n_k \int d\varphi \partial_\varphi \left( \eta_5 \tilde{g}_{\alpha 5} \right) \frac{[\partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)]^2}{|\int d\varphi \partial_\varphi \left( \eta_4 \tilde{g}_{\alpha 4} \right)|} \eta_4 \tilde{g}_{\alpha 4},
\]

\[
\eta_6 \tilde{\nabla}_{\alpha 6} = \frac{\partial_\varphi \left( \eta_6 \tilde{g}_{\alpha 6} \right)}{(\eta_6 \tilde{g}_{\alpha 6})},
\]

\[
\eta_7 \tilde{\nabla}_{\alpha 7} = \frac{\partial_\varphi \left( \eta_7 \tilde{g}_{\alpha 7} \right)}{(\eta_7 \tilde{g}_{\alpha 7})},
\]

\[
\eta_8 \tilde{\nabla}_{\alpha 8} = \frac{\partial_\varphi \left( \eta_8 \tilde{g}_{\alpha 8} \right)}{(\eta_8 \tilde{g}_{\alpha 8})}.
\]

A solution of type (24) with off-diagonal conditions (25) defines nonholonomic deformations of the 7-d Tangherlini BH (23) into a 8-d phase space on \(T^*V\). It should be noted that such stationary solutions are different from all classes of solutions constructed and considered in [30] and above sections because of fixed energy conditions \(p_8 = E = \text{const}\) and Killing symmetry on \(\partial^8 = \partial_E\).

Finally we note that new classes of generic off-diagonal solutions in MGT with MDRs constructed in this section contain (in general) a nontrivial nonholonomical induced torsion. As in previous section, we can state additional nonholonomic constraints to LC-configurations or to extract diagonal metrics. The BH properties can not be preserved for general nonholonomic deformations on the total phase space. It is important to analyse additionally if certain topological and singularity conditions allow projections into a base spacetime black hole configuration. Nevertheless, we argue that at least for nonsingular small deformations on a parameter \(\varepsilon\), such a s-metric describes a BH embedded self-consistently in a locally anisotropic polarized phase space media with fixed energy conditions.
4 Finsler-Lagrange-Hamilton symmetries of phase space stationary and BH configurations

Any BH solution in GR and extra dimension gravity \(^6\) can be nonholonomically extended as exact and/or parametric solutions in MGTs with MDRs. Such geometric and physical models are elaborated on (co) tangent Lorentz bundles. Corresponding gravity theories and their evolution and field equations, and their solutions, can be described equivalently in Finsler-Lagrange-Hamilton variables, see an axiomatic approach formulated in section 2 of [29]. For quasi-stationary solutions, such constructions are outlined in section 5.6 and Appendix B of [30].

The goal of this section is to study quasi-stationary Finsler like symmetries of nonholonomic deformations of BH solutions. As a typical example of prime metric, we shall consider \( \tilde{g}_8 \) and generalizations with an energy like variable \( E \) using local coordinates \(( \bar{r}, \theta, \phi, \ell )\) on a 6-d spherical symmetric phase space \( T^*V \).

In Finsler like variables on total (co) bundles, such quasi-stationary solutions may depend on a time like variable \( t \) and other types of space and (co) fiber coordinates. Geometric and analytic constructions can be performed in a more simplified form if the generating functions are taken with a Killing symmetry on \( \partial_7 = \partial/\partial p_7 \) and (for projections on base spaces) on \( \partial_t \).

4.1 Lagrange-Hamilton variables and distortion of connections on cotangent Lorentz bundles

Fixing a point \( x_0^i = \{ \bar{r}_0, \theta_0, \varphi_0 \} \in V \) (for \( i' = 1, 2, 3 \)) on a BH configuration and MDR \((1)\), we can construct an effective Hamiltonian

\[
H_0(p) := E = \pm(c^2 \bar{p}^2 + c^4 m^2 - \varpi(x_0^i, \bar{p}, E, m; \ell_P))^{1/2} \tag{26}
\]

describing the motion of a relativistic point particle propagating in a typical co-fiber of a \( T^*_x V \) and \( \bar{p} = (p_{a'}) = (\theta_2, \theta_3, p_7) \), for \( a' = 5, 6, 7 \). Globalizing the constructions for a BH with an effective phase space endowed with local coordinates \(( r, \theta, \varphi, t, \theta_2, \theta_3, p_7, E \)\), we obtain indicators \( \varpi \) depending both on spherical spacetime and phase space coordinates. In result, the motion of probing particles and linearized interactions of scalar fields in \( T^*V \), can be conventionally described by a Hamiltonian \((26)\) generalized as \( H(x, p) = H( r, \theta, \varphi, \theta_2, \theta_3, p_7, E ) \) for a stationary relativistic mechanical model.

4.1.1 BH solutions with associated Lagrange and Hamilton geometries

The Legendre transforms and the concept of \( L \)-duality are introduced in standard form \( L \rightarrow H \) following definition

\[
H( r, \theta, \varphi, p_{a'}) := p_{a'} v^{a'} - L( r, \theta, \varphi, v^{a'}), \tag{27}
\]

where the 'spherical' velocities \( v^{a'} \) are defined as solutions of the equations \( p_{a'} = \partial L/\partial v^{a'} \). The inverse Legendre transforms, \( H \rightarrow L \), are constructed

\[
L := p_{a'} v^{a'} - H, \tag{28}
\]

\(^6\)for instance, the double phase space Schwarzschild solution \( \tilde{g}_{\alpha \beta} \) \((18)\) and the Tangherlini 6-d and/or 7-d s-metrics, see \( \tilde{g}_8 \) \((8)\) and/or \( \tilde{g}_{s, \beta} \) \((23)\).
where \( p_{a'} \) are solutions of the equations \( y'' = \partial H / \partial p_{a'} \).

Nonholonomic deformations of BH solutions determined by MDRs \([1]\) are characterized by non-Riemannian total phase space geometries with nonlinear quadratic line elements

\[
d s^2_{L} = L( 'r, \theta, \varphi, v^{a'}), \text{ for models on } TV; \tag{29}
\]
\[
d s^2_{H} = H( 'r, \theta, \varphi, \theta_2, \theta_3, p_7, E), \text{ for models on } T^*V. \tag{30}
\]

Correspondingly, the values \( L \) \((29)\) and \( H \) \((30)\) are called the Lagrange and Hamilton fundamental (equivalently, generating) functions for associated mechanical systems. For stationary configurations, such a relativistic 4-d model is geometrized as a Lagrange space \( L^{3,1} = (TV, L(x,y)) \) with vertical phase space metric structure when \( L \) can be also considered as a generating Lagrange function, \( TV \ni ( 'r, \theta, \varphi, v^{a'}) \rightarrow L( 'r, \theta, \varphi, v^{a'}) \in \mathbb{R} \).

We model self-consistent mechanical models for a conventional \( L \) considered as a real valued and differentiable function on \( TV := TV/\{0\} \), for \( \{0\} \) being the null section of \( TV \), and continuous on the null section of \( \pi : TV \rightarrow V \). Such a model is regular for stationary configurations if the vertical metric \((v\text{-metric, defined as a Hessian function})\)

\[
\tilde{g}_{a'b'}( 'r, \theta, \varphi, y^{a'}) := \frac{1}{2} \frac{\partial^2 L}{\partial y^{a'} \partial y^{b'}} \tag{31}
\]

is non-degenerate, i.e. \( \det |\tilde{g}_{a'b'}| \neq 0 \), and of constant local Euclidean signature.

A BH configuration with MDRs can be described alternatively by a 4-d relativistic model of Hamilton space \( H^{3,1} = (T^*V, H( 'r, \theta, \varphi, \theta_2, \theta_3, p_7, E)) \) which is determined by a fundamental Hamilton function on a respective base spacetime Lorentz manifold \( V \). Such a real valued generating function is defined as a map \( T^*V \ni ( 'r, \theta, \varphi, \theta_2, \theta_3, p_7, E) \rightarrow H( 'r, \theta, \varphi, \theta_2, \theta_3, p_7, E) \in \mathbb{R} \) for which there are satisfied the conditions that it is differentiable on \( T^*V := T^*V/\{0\} \), for \( \{0\} \) being the null section of \( T^*V \), and continuous on the null section of \( \pi^* : T^*V \rightarrow V \). We define an analogous regular mechanical model if the co-vertical metric \((cv\text{-metric, defined as a Hessian})\)

\[
\tilde{g}^{a'b'}( 'r, \theta, \varphi, p_{c'}) := \frac{1}{2} \frac{\partial^2 H}{\partial p_{a'} \partial p_{b'}} \tag{32}
\]

is non-degenerate, i.e. \( \det |\tilde{g}^{a'b'}| \neq 0 \), and of constant signature.

Here we note that a model of Finsler phase space is an example of relativistic Lagrange mechanics when a regular Lagrangian \( L = F^2 \) is defined by a fundamental \((\text{generating})\) Finsler function subjected to additional conditions: 1) a fundamental Finsler function \( F \) is a real positive valued one which is differential on \( TV \) and continuous on the null section of the projection \( \pi : TV \rightarrow V \), where \( V \) is a Lorentz manifolds; 2) a generating function \( F \) satisfies the homogeneity condition \( F(x, \lambda v) = |\lambda| F(x, v) \), for a nonzero real value \( \lambda \); and a Hessian \((31)\) is defined by \( F^2 \) in such a form that in any point \((x_{(0)}, v_{(0)})\) the v-metric is of signature \((+++\ldots)\). In a similar form, we can define relativistic 4-d Cartan spaces \( C^{3,1} = (V, C(x, p)) \), when \( H = C^2(x, p) \) is 1-homogeneous on co-fiber coordinates \( p_{a} \);

\[\text{\textsuperscript{7}}\text{In our works, we use tilde } "^\sim" \text{ in order to emphasize that certain geometric objects are defined canonically by respective Lagrange and/or Hamilton generating functions. For instance, we write } \tilde{g}_{ab} \text{ and } \tilde{g}^{ab}. \text{ Such (co) vertical metrics may encode various types of MDRs and LIVs terms etc. Considering general frame/coordinate transforms on } TV \text{ and/or } T^*V, \text{ we can express any } "\text{tilde}" \text{ value in a } "\text{non-tilde}" \text{ form. In such cases, we shall write } g_{ab}, \text{ for a v-metric, and } g^{ab}, \text{ for a cv-metric.}\]
such Cartan spaces are Finsler ones but on cotangent bundles when additions conditions for Legendre transforms and possible almost symplectic structures can be considered, see details and references in [29]. In this work, for simplicity, we shall analyse properties of BHs with associated Hamilton geometries determined by MDRs.

4.1.2 N-connections and stationary frames in Lagrange-Hamilton phase spaces

Using the formula for MDR [11], we can define $L$–dual (related via Legendre transforms) canonical stationary configurations of Hamilton space $\widetilde{H}^{3,1} = (T^*V, \widetilde{H}(x, p))$ and Lagrange space $\widetilde{L}^{3,1} = (TV, \widetilde{L}(x, y)).$ We can consider general Lagrange or Hamilton variables which may depend explicitly on time and/or energy like coordinates. In such phase space BH and nonholonomically deformed configurations, the dynamics of a probing point particle is described by fundamental generating functions $\widetilde{H}$ and $\widetilde{L}$ subjected to respective via Hamilton-Jacobi and Lagrange equations

$$\frac{dx^i}{d\tau} = \frac{\partial\widetilde{H}}{\partial p_i} \text{ and } \frac{dp_i}{d\tau} = -\frac{\partial\widetilde{H}}{\partial x^i},$$

$$\frac{d}{d\tau} \frac{\partial\widetilde{L}}{\partial y^i} - \frac{\partial\widetilde{L}}{\partial x^i} = 0.$$  

These equations are equivalent to some nonlinear geodesic (semi-spray) equations

$$\frac{d^2 x^i}{d\tau^2} + 2\widetilde{G}^i(x, y) = 0, \text{ for } \widetilde{G}^i = \frac{1}{2}\widetilde{g}^{ij}(\frac{\partial^2 \widetilde{L}}{\partial y^j y^k} - \frac{\partial \widetilde{L}}{\partial x^j})^k,$$  

where $\widetilde{g}^{ij}$ is inverse to $\widetilde{g}_{ij}$ [31]. The value (33) can be used for defining canonical N–connection structures determined by MDRs in $L$–dual form when

$$\widetilde{N} = \left\{ \widetilde{N}_i^a := \frac{\partial\widetilde{G}}{\partial y^i} \right\} \text{ and } \widetilde{\tilde{N}} = \left\{ \widetilde{\tilde{N}}_{ij} := \frac{1}{2} \left\{ \widetilde{g}_{ij}, \widetilde{H} \right\} - \frac{\partial^2 \widetilde{H}}{\partial p_k \partial x^i} \widetilde{g}_{jk} - \frac{\partial^2 \widetilde{H}}{\partial p_k \partial x^j} \widetilde{g}_{ik} \right\}$$  

and $\widetilde{\tilde{g}}_{ij}$ is inverse to $\widetilde{\tilde{g}}^{ab}$ [32].

The canonical N–connections $\widetilde{N}$ and $\widetilde{\tilde{N}}$ from [31] define respective systems of N–adapted (co) frames

$$\widetilde{e}_a = (\widetilde{e}_i = \frac{\partial}{\partial x^i} - \widetilde{N}_i^a(x, y) \frac{\partial}{\partial y^a} e_b = \frac{\partial}{\partial y^b}), \text{ on } TV;$$

$$\widetilde{e}^a = (\widetilde{e}^i = dx^i, \widetilde{e}^a = dy^a + \widetilde{N}_a^a(x, y) dx^i), \text{ on } (TV)^*;$$

$$\widetilde{\tilde{e}}_a = \left( \widetilde{\tilde{e}}_i = \frac{\partial}{\partial x^i} - \widetilde{\tilde{N}}_{ia}(x, p) \frac{\partial}{\partial p_a}, \widetilde{\tilde{e}}^b = \frac{\partial}{\partial p_b} \right), \text{ on } T^*V;$$

$$\widetilde{\tilde{e}}^a = \left( \widetilde{\tilde{e}}^i = dx^i, \widetilde{\tilde{e}}_a = dp_a + \widetilde{\tilde{N}}_{ia}(x, p) dx^i \right) \text{ on } (T^*V)^*.$$  

The canonical frames and spherical coordinates can be prescribed in such forms that the coefficients of Finsler like stationary d-metrics and N-connections do not depend on a respective time like coordinate.

We conclude that BH configurations in GR and extra dimension gravity are nonholonomically deformed by MDR [11] and can be described in equivalent forms using canonical data ($L$, $\widetilde{N}; \widetilde{e}_a, \widetilde{e}^a$) and/or ($\widetilde{H}$, $\widetilde{N}; \widetilde{e}_a, \widetilde{e}^a$) for Lagrange and/or Hamilton spaces. Considering general frame and coordinate transforms, we omit tilde and work with a general N-splitting and geometric data ($N; e_a, e^a$) and/or ($N; e_a, e^a$).
4.1.3 Canonical d-metrics and d-connections for Lagrange-Hamilton spaces and BHs

There are canonical d-metric structures \( \tilde{g} \) and \( \tilde{\tilde{g}} \) completely determined by a MDR \(^{(1)}\) and respective "tilde" data,

\[
\tilde{g} = \tilde{g}_{\alpha \beta}(x,y)\tilde{e}^\alpha \otimes \tilde{e}^\beta = \tilde{g}_{ij}(x,y)e^i \otimes e^j + \tilde{g}_{ab}(x,y)\tilde{e}^a \otimes \tilde{e}^b \quad \text{and/or}
\]

\[
\tilde{\tilde{g}} = \tilde{\tilde{g}}_{\alpha \beta}(x,p)\tilde{e}^\alpha \otimes \tilde{e}^\beta = \tilde{g}_{ij}(x,p)e^i \otimes e^j + \tilde{g}^{ab}(x,p)\tilde{e}_a \otimes \tilde{e}_b,
\]

where \( \tilde{e}_a = (\tilde{e}_i, e_b) \) and \( \tilde{e}_a = (\tilde{\tilde{e}}_i, \tilde{\tilde{e}}^b) \). Such nonholonomic frame bases are characterized by corresponding anholonomy relations, for instance, of type

\[
[ e_\alpha, e_\beta ] = e_\alpha e_\beta - e_\beta e_\alpha = \tilde{W}_{\alpha \beta}^\gamma e_\gamma,
\]

with anholonomy coefficients \( \tilde{W}_{ab}^i = \partial \tilde{N}_a / \partial p_b \) and \( \tilde{W}_{ija} = \tilde{\Omega}_{ija} \), see explicit definitions and formulas in \([28]\).

The geometry of associated phase geometries is also characterized by respective Cartan-Lagrange and Cartan-Hamilton d-connections induced directly by an indicator of MDR and determined by corresponding coefficients of Lagrange and Hamilton d-metrics \((37)\) and \((38)\). The coefficients of such values are generated by "tilde" objects with identifications of d-metric coefficients with corresponding base and (co) fiber indices:

- **on** \( TTV \), \( \tilde{\mathbf{D}} = \{ \tilde{\Gamma}_{\alpha \beta}^\gamma = (\tilde{L}_{jk}^i, \tilde{L}_{bk}^a, \tilde{C}_{jc}^a, \tilde{\tilde{C}}_{bc}^a) \}, \) \( \) for \( \tilde{g}_{\alpha \beta} = (\tilde{g}_{jr}, \tilde{g}_{ab}), \) \( \tilde{N}_i^a = \tilde{\tilde{N}}_i^a \),

\[
\tilde{L}_{jk}^i = \frac{1}{2} \tilde{g}^{ir}(\tilde{e}_k \tilde{g}_{jr} + \tilde{e}_j \tilde{g}_{kr} - \tilde{e}_r \tilde{g}_{jk}), \quad \tilde{L}_{bk}^a \text{ as } \tilde{L}_{jk}^i,
\]

\[
\tilde{C}_{bc}^a = \frac{1}{2} \tilde{g}^{ad}(\epsilon_c \tilde{g}_{bd} + \epsilon_b \tilde{g}_{cd} - \epsilon_d \tilde{g}_{bc}), \text{ being similar to } \tilde{\tilde{C}}_{jc}^a.
\]

- **on** \( TTV^* \), \( \tilde{\mathbf{D}} = \{ \tilde{\Gamma}_{\alpha \beta}^\gamma = (\tilde{\tilde{L}}_{jk}^i, \tilde{\tilde{L}}_{bk}^a, \tilde{\tilde{C}}_{jc}^a, \tilde{\tilde{\tilde{C}}}_{bc}^a) \}, \) \( \) for \( \tilde{\tilde{g}}_{\alpha \beta} = (\tilde{g}^{jr}, \tilde{g}^{ab}), \) \( \tilde{\tilde{N}}_a^i = \tilde{N}_a^i \),

\[
\tilde{\tilde{L}}_{jk}^i = \frac{1}{2} \tilde{g}^{ir}(\tilde{e}_k \tilde{g}_{jr} + \tilde{e}_j \tilde{g}_{kr} - \tilde{e}_r \tilde{g}_{jk}), \text{ with similar } \tilde{\tilde{L}}_{bk}^a,
\]

\[
\tilde{\tilde{C}}_{bc}^a = \frac{1}{2} \tilde{g}^{ad}(\epsilon_c \tilde{g}^{bd} + \epsilon_b \tilde{g}^{cd} - \epsilon_d \tilde{g}^{bc}), \text{ being similar to } \tilde{\tilde{\tilde{C}}}_{jc}^a.
\]

In Lagrange-Hamilton geometry, we can introduce different types of metric compatible or incompatible Finsler like d-connections which characterize phase symmetries of BHs deformed by MDRs, see details in \([28]\) and references therein.

Finally, it should be noted that all above formulas considered for Finsler–Lagrange–Hamilton structures (metrics and nonlinear/linear connections and respective torsions and curvatures) can be written in arbitrary frames of references because all geometric constructions are performed on (co) tangent Lorentz bundles. Such formulas can be written both in abstract or coefficient forms which are very similar to respective vielbein, tetraddic and diadic ones in GR but for some generalize metric-affine and N-connection structures. If we wont to adapt the constructions to a N-connection splitting \((3)\) and/or \((4)\), we should consider a subclass of frame transforms preserving the corresponding h- and v-decompositions, see details in Refs. \([29, 30, 31]\).
4.2 Phase space BHs in Einstein-Hamilton gravity & associated Finsler-Lagrange geometry

Any BH and s-metric structure can be re-written equivalently in terms of Finsler like d-metric and d-connection structures using corresponding N-adapted splitting and frame transforms. We can elaborate on physical properties of effective Lagrange-Hamilton spaces which are characterized by different N- and d-connection structures. This results in different models of associated BH phase space relativistic mechanics. Using nonlinear symmetries relating generating functions, generating sources and cosmological constants, we can model various classes of off-diagonal solutions in a MGT with MDRs as certain classes of Einstein-Hamilton spaces.

Let us consider a nonholonomic diadic structure \( \mathbf{e}_{\alpha_s} \) on an open region \( U \subset T^*V \). We can introduce respective vielbein structures, \( \mathbf{e}^\alpha_{\alpha_s} \) and \( \mathbf{e}^\alpha_{\alpha_s} \), determined by values of type \( H(x,p) \) \( \{30\} \) and \( \tilde{e}_\alpha \( \{30\} \) and stated by frame transforms

\[
\mathbf{e}_{\alpha_s} = e^\alpha_{\alpha_s} \, \bar{e}_\alpha, \quad \text{for Lagrange (Finsler) variables, and } \mathbf{e}^\alpha_{\alpha_s} = \mathbf{e}^\alpha_{\alpha_s} \, \bar{e}_\alpha, \quad \text{for Hamilton variables.}
\]

Using such values and dual bases and respective inverse matrices, \( e^\alpha_{\alpha_s} \) and \( e^\alpha_{\alpha_s} \), we can always redefine the geometric data in the form

\[
\left( ^{s \mathbf{N}}; \mathbf{e}_{\alpha_s}, e^\alpha_{\alpha_s} \right) \longleftrightarrow \left( \bar{\mathbf{L}}, \tilde{\mathbf{N}}; \bar{e}_\alpha, \tilde{e}^\alpha \right) \quad \text{and/or } \left( ^{s \tilde{\mathbf{N}}}; \bar{e}_{\alpha_s}, \bar{e}^\alpha_{\alpha_s} \right) \longleftrightarrow \left( \bar{\tilde{\mathbf{H}}}, \tilde{\mathbf{N}}; \tilde{e}_\alpha, \tilde{e}^\alpha \right).
\]

Any BH s-metric and respective off-diagonal metric structure can be described equivalently as a Hamilton canonical d-metric \( \{38\} \) and inversely. This follows from the possibility to consider necessary type frame transforms \( g_{\alpha_s \beta_s} = e^\alpha_{\alpha_s} \, e^\beta_{\beta_s} \, \tilde{g}_{\alpha \beta} \) and work equivalently with data \( g_{\alpha_s \beta_s} \) or \( \tilde{g}_{\alpha \beta} \). Considering distortions \( \hat{s} \, \hat{D} = \nabla + \hat{s} \, \hat{Z} = \bar{s} \, \bar{D} + \hat{s} \, \hat{Z} \), we compute \( \hat{s} \, \hat{D} = \bar{s} \, \bar{D} + \hat{s} \, \hat{Z} \) for any \( \mathbf{g} = \mathbf{g} = \tilde{\mathbf{g}} \). This allows us to express the distortions of the Ricci d-tensor of type \( \hat{s} \, \hat{R}_{\alpha_s \beta_s} \) as distortions

\[
\hat{s} \, \hat{R}_{\alpha_s \beta_s} = \tilde{R}_{\alpha_s \beta_s} + \bar{Z}_{\alpha_s \beta_s} \left( \hat{s} \, \hat{g} , \, \bar{\hat{D}} \right), \quad \text{where } \hat{s} \, \tilde{R}_{\alpha_s \beta_s} = - \bar{Z}_{\alpha_s \beta_s}.
\]

We conclude that the canonical distortion relations \( \hat{s} \, \hat{D} = \bar{s} \, \bar{D} + \hat{s} \, \hat{Z} \) result in effective sources of type

\[
\hat{s} \, \tilde{\Gamma}_{\alpha_s \beta_s} := \varpi \left( \hat{s} \, \tilde{T}_{\alpha_s \beta_s} - \frac{1}{2} \, g_{\alpha_s \beta_s} \, \hat{s} \, \hat{T} \right), \quad \text{where } \varpi \, \hat{s} \, \tilde{T}_{\alpha_s \beta_s} = - \bar{Z}_{\alpha_s \beta_s} \left[ \hat{s} \, g \left( \tilde{g}_{\alpha \beta} \right), \, \bar{\hat{D}} \right].
\]

For effective sources, the modified Einstein equations \( \{2\} \) can be modelled equivalently as solutions of generalized Einstein-Hamilton equations,

\[
\hat{s} \, \tilde{R}_{\alpha \beta} \left[ \, \hat{s} \, \bar{\hat{D}} \right] = \hat{s} \, \tilde{\Gamma}_{\alpha \beta}
\]

where the diadic indices can be omitted. For such an equivalent representation, the sources are correspondingly redefined by formulas

\[
\hat{s} \, \tilde{\Gamma}_{\alpha_s \beta_s} = \hat{s} \, \tilde{\Gamma}_{\alpha_s \beta_s} + \hat{s} \, \tilde{\Lambda}_{\alpha_s \beta_s} \quad \text{and/or} \quad \hat{s} \, \tilde{\Lambda}_{\alpha_s \beta_s} = \varpi \hat{s} \, \tilde{\Lambda}_{\alpha_s \beta_s} + \hat{s} \, \tilde{\Lambda}_{\alpha_s \beta_s},
\]

where \( \varpi \hat{s} \, \tilde{\Lambda} \) is the cosmological constant associated to matter fields and vacuum fluctuations and the term with tilde \( \hat{s} \, \tilde{\Lambda} \) points to possible contributions to the cosmological constant resulting from Hamilton like degrees of freedom.
5 Entropies of BHs with MDRs & Stationary Ricci Solitons

The stationary solutions in MGTs with MDRs constructed in this work can be generated for phase space metrics of signature \((++-);++-\) on \(T^*V\) and/or \(TV\). For certain classes of small parametric deformations (for instance, with spheroid topologies and horizons for black ellipsoids) of prime metrics describing 6-d, (4+4)-d and 7-d BH configurations embedded into 8-d phase spaces, we can elaborate on generalizations of the Bekenstein-Hawking BH entropy and spacetime thermodynamics [15, 16, 17, 18]. We also generated nonholonomic deformations to stationary generic off-diagonal metrics with generalized connections and constraints to LC-configurations. Here it should be noted that applying the anholonomic frame deformation method, AFDM, there were constructed and studied various classes of more general (non) stationary and locally anisotropic cosmological solutions in GR and MGTs. Such (non) commutative, or supersymmetric, fractional and other types Finsler like BH solutions were studied in a series of works [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46], see also recent works on nonholonomic superstrings and branes [28, 31, 32, 33]. Our conclusion was that the concept of Bekenstein-Hawking entropy and related thermodynamic characteristics are not applicable for generalized spacetimes and phase spaces. The motivation is that general exact and parametric solutions with MDRs are generic off-diagonal and determined by generating functions and sources and integration functions with generalized symmetries and depending on various types of space and phase space variables. Standard and generalized Bekenstein-Hawking definitions are possible only for subclasses of solutions with conventional horizons, higher symmetries for duality of gravity and conformal field theories and/or additional assumptions on related holographic principle etc. In general form, MDRs and LIVs result in substantial deformations of BH and cosmological solutions to configurations with nontrivial locally anisotropic vacuum and non-vacuum configurations with rich geometric structure and/or locally anisotropic polarizations of physical constants.

In a series of works [53, 54, 55, 24, 56, 57, 58, 59, 25, 22, 61, 62, 23], we developed a new statistical and geometric thermodynamics approach which allows us to characterize physical properties of generic off-diagonal configurations in GR and MGTs, see recent results in [26, 63, 64, 27, 65]. Such gravitational and matter field geometric flow theories and generalized Ricci solitons can be elaborated following G. Perelman’s definitions of W- and F-entropies [19]. The goal of this section is to elaborate on generalized Bekenstein-Hawking and Perelman thermodynamic models for stationary and BH configurations with MDRs on phase spaces.

5.1 Generalized phase space Bekenstein-Hawking entropy

The formula of the Bekenstein-Hawking entropy \(S_{BH}\) for a 4-d BH can be written in the form

\[
S_{BH} = \frac{c^3 k_B A}{4G\hbar},
\]

where \(A\) is the BH surface area (the event horizon), \(\hbar\) is the reduced Planck constant, \(c\) is the speed of light, \(k_B\) is the Boltzmann constant and \(G\) is the gravitational constant. Via such fundamental physical constants, this formula counts of the BH effective degrees of freedom and ties together notions from gravitation, thermodynamics and quantum theory. Various generalizations the Bekenstein-Hawking approach were considered in MGTs of various dimension and solutions with conventional area of horizons.
5.1.1 The entropy and temperature of higher dimension Schwarzschild BHs

For higher dimensions, the thermodynamics of multidimensional BHs was studied [78, 79, 80, 81, 82, 83] and references therein. The Hawking temperature of a Schwarzschild BH in a spacetime of dimension \( s' \geq 4 \) is given by formulas (see, for instance, [82])

\[
T = \frac{s' - 3}{4\pi} \left( \frac{\Omega_{s'-2}}{4} \right)^{1/(s'-2)} S_0^{-1/(s'-2)},
\]

where \( S_0 \) is the Bekenstein-Hawking entropy is

\[
S_0 = \frac{\Omega_{s'-2}(\hat{r}_0)^{s'-2}}{4},
\]

is determined by the volume of unit -sphere, \( \Omega_{s'-2} = \frac{s'/2}{(s'/2)!} \) and horizon radius \( \hat{r}_0 = (\frac{8\pi \hat{M}}{(s'-2)\Omega_{s'-2}})^{1/(s'-3)} \).

In these formulas, the radial coordinate

\[
\hat{r} = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2 + (y^5)^2 + \ldots + (y^{s'})^2}
\]

is defined for a \( s' - 2 \) dimensional space like hypersurface endowed with Cartesian coordinates \( (x^1, x^2, y^3, y^5, \ldots y^{s'}) \), when \( y^4 = t \). The quadratic line element is parameterized

\[
ds^2 = \hat{h}^{-1}(\hat{r}) \hat{r}^2 d\hat{r}^2 + \hat{h}(\hat{r}) dt^2 + \hat{r}^2 d\Omega_{s'-2}^2,
\]

where

\[
\hat{h}(\hat{r}) = 1 - \frac{16\pi \hat{M}}{(s'-2)\Omega_{s'-2}(\hat{r})^{s'-3}}
\]

is determined by the BH mass \( \hat{M} \).

Above formulas can be modified and applied for computing the entropy and temperature of phase space BHs defined by respective prime metrics \( \hat{g}_{\alpha \beta} \), \( \hat{g}_{\alpha \beta} \), \( \hat{g}_{\alpha \beta} \), and/or \( \hat{g}_{\alpha \beta} \).

5.1.2 The Bekenstein-Hawking thermodynamics for 6-d phase space Schwarzschild BHs

Considering a prime metric \( \hat{g}_{\alpha \beta} \) with \( \hat{\Lambda} = 0 \), for \( s' = 6 \), and with trivial extensions on coordinates \( (p_7, E) \) on \( T^*V \), the analogs of BH temperature and entropy are computed

\[
\hat{T} = \frac{3}{4\pi} \left( \frac{\Omega_4}{4} \right)^{1/4} \hat{S}_0^{-1/4} \text{ for } \hat{S}_0 = \frac{\Omega_4(\hat{r}_0)^4}{4}.
\]

In these formulas, we put a left label \( \hat{\cdot} \) in order to emphasize that all values are defined for a 6-d phase space BH with the volume of unit -sphere \( \Omega_4 = \frac{\pi^4}{3!} \) and horizon radius \( \hat{r}_0 = \left( \frac{2\pi \hat{M}}{\Omega_4} \right)^{1/3} \). In principle, the values \( \hat{T} \) and \( \hat{S}_0 \) are given by respective formulas (11) and (12) when the coordinates and physical constants are redefined for 6-d phase spaces.

The formulas for \( \hat{T} \) and \( \hat{S}_0 \) can be extended for ellipsoidal and/or toroidal configurations (black ellipsoids and black torus with a conventional hypersurface area) on \( E \) as we considered for various
classes of (commutative and noncommutative, or superstring) nonholonomic and Finsler like generalizations \[37, 38, 39, 41, 43, 44, 31, 32, 33\]. Nevertheless, we can not develop the Bekenstein-Hawking thermodynamics for more general classes of stationary phase space solutions constructed in section 2. There are attempts to study properties of rainbow BHs \[14, 12, 84\] with some phenomenological dependencies of physical constants and horizons on \(E\) and on a cutting parameter but those works are not based on finding exact solutions of certain modified Einstein equation. In our approach, we are able to construct exact solutions for generalized gravitational field equations in MTS with MDRs and to elaborate on generalized thermodynamic models of gravitational interactions.

5.1.3 Bekenstein-Hawking thermodynamics for 7-d phase Schwarzschild BHs with energy Killing symmetry

We can consider a prime metric \(\tilde{\mathbf{g}}_{\alpha\beta}\) \[23\] with \(\Lambda = 0\) for a Schwarzschild BH in conventional phase space of dimension \(s' = 7\) with \(E = E_0 = \text{conts.}\). The formulas \[11\] and \[12\] for such phase space configurations with Killing symmetry on \(\partial/\partial E\) allow us to compute the values

\[
S_0 = \frac{\Omega_5 (r_0)^5}{4} \quad \text{and} \quad T = \frac{1}{4\pi} \left( \frac{\Omega_5}{4} \right)^{1/5} S_0^{-1/5},
\]

where \(\Omega_5 = \frac{\pi^{7/2}}{(7/2)!}\) and \(r_0 = \left( \frac{8\pi M}{\Omega_5} \right)^{1/4}\).

For relativistic stationary configurations on \(\text{TV}\), the horizon determining such values has to be restricted, warped or trapped by a maximal speed of light constant. Unfortunately, we can not elaborate on generalizations of formulas for \(S_0\) and \(T\) for new classes of target metrics constructed as nonholonomic deformations of \(\tilde{\mathbf{g}}_{\alpha\beta}\), for instance, to stationary phase configurations of type \[24\].

5.1.4 Bekenstein-Hawking phase space thermodynamics for double 4-d Schwarzschild BHs

Let us consider a double BH \(\tilde{\mathbf{g}}_{\alpha\beta}\) \[18\] defined as an exact solution of modified Einstein equations \[2\] with zero sources when \(\Lambda = 0\) and \(p\Lambda = 0\). For such static configurations, we can define two independent thermodynamical models (one for a base spacetime manifold and another for typical fiber). The formulas \[11\] and \[12\] for \(s' = 4\) result in respective

base spacetime temperature \(T = \frac{1}{4\pi} \left( \frac{\Omega_2}{4} \right)^{1/2} S_0^{-1/2}\),

phase fiber temperature \(pT = \frac{1}{4\pi} \left( \frac{p\Omega_2}{4} \right)^{1/2} pS_0^{-1/2}\),

where respective base and fiber analogs of the Bekenstein-Hawking entropy are computed

\[
S_0 = \frac{\Omega_2 (r_0)^2}{4} \quad \text{and} \quad pS_0 = \frac{p\Omega_2 (pr_0)^2}{4}.
\]

The volumes of unit -spheres are \(\Omega_2 = \frac{\pi^2}{27}\) and \(p\Omega_2 = \frac{\pi^2}{27}\) and the respective horizon radiiuses are found \(r_0 = \frac{8\pi M}{2\Omega_2}\) and \(pr_0 = \frac{8\pi pM}{2p\Omega_2}\).

Such formulas can be generalized for black ellipsoid/torus solutions \[37, 38, 39, 41, 43, 44, 31, 32, 33, 26\], but not for general off-diagonal nonholonomic deformations to phase space stationary configurations of type \[19\].
5.2 Geometric flows and Perelman’s thermodynamics for phase spaces

We can not apply the concepts of Bekenstein-Hawking entropy and BH temperature for generic off-diagonal solution in GR and MGTs, for instance, for stationary and/or cosmological configurations without explicit horizons and/or assumptions on duality quantum fields - gravity models, holographic principles etc. In this subsection, we provide an introduction to the theory of nonholonomic relativistic flows on stationary phase spaces modelled as cotangent Lorentz bundles $T^*V$ of total dimension 8. Such geometric evolution models are certain dual analogs and Hamilton-Finsler-Ricci modifications of theories elaborated in [23, 26, 63, 27, 65], see also our previous works [53, 54, 55, 24, 56, 57, 58, 59, 25, 22, 61, 62]. We generalized for phase spaces the G. Perelman’s definitions of W- and F-entropies [19, 20, 21] (rigorous mathematic results on of Ricci flows of Riemannian and Kähler metrics can be found in [69, 70, 71, 72, 73, 74]).

Let us consider a family of nonholonomic 8-d cotangent Lorentz cotangent bundles $T^*V(\tau)$ enabled with corresponding sets of metrics (and equivalent d-metrics or s-metrics for respective nonholonomic shell distributions) $\mathbf{g}(\tau) = \mathbf{g}(\tau, u)$, or $\mathbf{g}(\tau, u)$, of signature $(+ + + + ; + + + +)$; and N–connection $\mathbf{N}(\tau) = \mathbf{N}(\tau, u)$, or $\mathbf{N}(\tau, u)$, parameterized by a positive parameter $\tau, 0 \leq \tau \leq \tau_0$. Any relativistic nonholonomic phase space $T^*V \subset T^*V(\tau)$ can be enabled with necessary types double nonholonomic $(2+2)+(2+2)$ and $(3+1)+(3+1)$ splitting (see [23, 26, 30] for the geometry and evolution of spacetimes enabled with such double distributions). Additionally to coordinate and index conventions from footnote 4, we label the local $(3+1)+(3+1)$ coordinates in the form

$$u = \{ u^a = u^{a_1} = (x^i, y^a_2; p_{a_3}, p_{a_4}) = (x^i, y^4 = t; p_8 = E) \}$$

for $i_1, j_1, k_1, ... = 1, 2; a_1, b_1, c_1, ... = 3, 4; a_2, b_2, c_2, ... = 5, 6; a_3, b_3, c_3, ... = 7, 8$; and $i, j, k, ... = 1, 2, 3$, respectively, $\hat{a}, \hat{b}, \hat{c}, ... = 5, 6, 7$ can be used for corresponding spacelike hyper surfaces on a base manifold and typical cofiber.

A nonholonomic $(3+1)+(3+1)$ splitting can be chosen in such a form that any open region on a base Lorentz manifold, $U \subset V$, is covered by a family of 3-d spacelike hypersurfaces $\hat{\Xi}_t$ parameterized by a time like parameter $t$. On a typical cofiber of $T^*V$, we can consider similar 3-d hypersurfaces $\hat{\Xi}_E$ of cofiber signature $(+++)$ and parameterized by an energy type parameter $E$. We shall write $\hat{\Xi} = (\hat{\Xi}_t, \hat{\Xi}_E)$ for such nonholonomic distributions of hypersurfaces with conventional splitting 3+3 of signature $(+++; +++)$ on total phase space. For additional shell decompositions, we can use also a s-label, $\hat{s}\hat{\Xi} = (\hat{s}\hat{\Xi}_t, \hat{s}\hat{\Xi}_E) \subset sT^*V$. In general, there are two generic different types of geometric phase flow theories when 1) $\tau(\chi)$ is a re-parametrization of the temperature like parameter used for labeling 4-d Lorentz spacetimes and their phase space configurations and 2) $\tau(t)$ is a time like parameter when $(3+3)$-d spacelike phase configurations evolve relativistically on a "redefined" time like coordinate. We shall elaborate in this section only on theories of type 1.

We consider generalizations of Perelman’s functionals [19] using canonical data ($\mathbf{g}(\tau), \mathbf{D}(\tau)$) on cotangent Lorentz bundles (for Lagrange-Finsler geometric flow evolution, see [58, 59, 23, 22, 61, 62, 55]):

$$\mathbf{\hat{F}} = \int_{t_1}^{t_2} \int_{\hat{\Xi}_t} \int_{E_t} \int_{\hat{\Xi}_E} e^{-\hat{\mathbf{f}}} \sqrt{|\mathbf{g}_{\alpha\beta}|} d^8 u (\hat{\mathbf{R}} + |\mathbf{D} \hat{\mathbf{f}}|^2),$$

and

$$\mathbf{\hat{W}} = \int_{t_1}^{t_2} \int_{\hat{\Xi}_t} \int_{E_t} \int_{\hat{\Xi}_E} \mu \sqrt{|\mathbf{g}_{\alpha\beta}|} d^8 u [\tau (\hat{\mathbf{R}} + \hat{\mathbf{n}} D \hat{\mathbf{f}} + |\hat{\mathbf{f}}|^2 + |\hat{\mathbf{f}} - 16)],$$

for
where the normalizing function \( f(\tau, u) \) satisfies \( \int_{t_1}^{t_2} \int_{E_i}^{E_f} \int_{\Xi_i}^{\Xi_f} \sqrt{g_{\alpha\beta}} d^8 u = 1 \) for \( \hat{\mu} = (4\pi\tau)^{-8} e^{-\hat{f}} \), when the coefficients \( 16 = 2 \times 8 \) is taken for 8-d spaces. Similar functionals can be postulated for nonholonomic geometric flows on TV using data \( (g(\tau), \hat{D}(\tau)) \) and redefined integration measures and normalized functions (conventionally without "\( \)"") do not have a character of entropy for pseudo–Riemannian \( \hat{W} \)-entropy. It should be noted that \( \hat{W} \) do not have a character of entropy for pseudo–Riemannian metrics but can be treated as a value characterizing relativistic geometric hydrodynamic phase space flows, see similar details in [23].

Using \( N \)-adapted diastic shell and/or double 3+1 frame and coordinate transforms of metrics and sources with additional dependence on a flow parameter, we introduce certain canonical parameterizations which will allow us to decouple and solve systems of nonlinear PDEs and/or to compute entropy like values. To define thermodynamic like variables for geometric flow evolution of stationary configurations, we take

\[
\hat{g} = \hat{g}_{\alpha',\beta'}(\tau, u) = \hat{g}_i(\tau, x^k)dx^i \otimes dx^i + \hat{q}_3(\tau, x^k, y^3)\hat{e}^3 \otimes \hat{e}^3 - \hat{N}(\tau, x^k, y^3)\hat{e}^4 \otimes \hat{e}^4 + \hat{q}_{a2}(\tau, x^k, y^3, p_{b2}) \hat{e}_{a2} \otimes \hat{e}_{a2} + \hat{q}(\tau, x^k, y^3, p_{b3}) \hat{e}_7 \otimes \hat{e}_7 - \hat{N}(\tau, x^k, y^3, p_{b3}) \hat{e}_8 \otimes \hat{e}_8,
\]

where \( \hat{e}^{\alpha_2} \) are \( N \)-adapted bases total space of respective cotangent Lorentz bundles. This ansatz is a general one for a 8-d phase space metric which can be written as an extension of a couple of 3-d metrics, \( q_{ij} = diag(q_i) = (q_i, q_3) \) on a hypersurface \( \hat{\Xi}_t \), and \( \hat{q}^{ab} = diag(\hat{q}^a) = (\hat{q}^{a_2}, \hat{q}^7) \) on a hypersurface \( \hat{\Xi}_E \), if

\[
q_3 = g_3, \hat{N} = -g_4 \text{ and } \hat{q}^7 = \hat{q}^7, \quad \hat{N} = -\hat{g}^8,
\]

where \( \hat{N} \) is the lapse function on the base and \( \hat{N}^2 \) is the lapse function in the fiber.

To provide a statistical analogy for thermodynamical models, we can consider a partition function \( Z = \int \exp(-E) d\omega(E) \) for the canonical ensemble at temperature \( \beta^{-1} \) being defined by the measure taken to be the density of states \( \omega(E) \). The thermodynamical values are computed in standard form for the average energy, \( \langle E \rangle := -\partial \log Z/\partial \beta \), the entropy \( S := \beta \langle E \rangle + \log Z \) and the fluctuation \( \sigma := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z/\partial \beta^2 \). Using \( \hat{\cal F} \) and

\[
\hat{Z} = \exp \left\{ \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|} dx^3 [-\hat{f} + 8] \right\},
\]

we can define such thermodynamic values (see [59, 22, 62, 23, 26] proofs of similar theorems for the dimension 8):

\[
\begin{align*}
\hat{E} &= -\tau^2 \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|} dx^3 \left( |\hat{R}| + |\hat{\bar{D}}\hat{f}|^2 - \frac{3}{\tau} \right), \\
\hat{S} &= -\int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|} dx^3 \left[ \tau \left( |\hat{R}| + |\hat{\bar{D}}\hat{f}|^2 \right) + \hat{f} - 6 \right], \\
\hat{\sigma} &= 2 \tau^4 \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|} dx^3 \left[ |\hat{R}_{ij}| + |\hat{\bar{D}}_i \hat{\bar{D}}_j \hat{f} - \frac{1}{2\tau} q_{ij}^2 \right].
\end{align*}
\]

These formulas can be considered for 4–d configurations taking the lapse function \( \hat{N} = 1 \) for \( N \)-adapted Gaussian coordinates.
Finally we note that the formulas for generalized Perelman’s functionals and derived thermodynamic values are written in terms of geometric values like $\hat{D}$ and $\hat{R}$ all defined by a total phase space, or projections on the base spacetime, metric structure. Such values can be defined and computed in pure geometric forms in various types of commutative and noncommutative/ supersymmetric Finsler geometric flow evolution models and MGTs. There is also an indirect dependence on underlying theories of gravity because the metric and (non)linear connection structures are defined differently in MGTs. The values and depend also on the type of solutions (general stationary ones, BH types, locally cosmological ones) are used in an explicit example of MGT with MDRs. In certain sense, such formulas generalize similar ones for the Bekenstein-Hawking thermodynamics which can be defined always for BH solutions in various MGTs and GR. The main difference is that the Perelman functionals can be used for characterizing thermodynamically geometric flow evolution theories, MGTs and nonholonomic Ricci soliton configurations, and for various classes of stationary and nonstationary, and locally anisotropic cosmological solutions, even such geometric configurations are not characterized by horizons or holographic and/or duality properties.

6 Summary and Concluding Remarks

Study of the black hole, BH, solutions and their thermodynamics is one of the important topics of theoretical physics with applications in high energy physics, astrophysics and cosmology, and quantum information theories. BH thermodynamics provides a relation between quantum physics and gravitation.

In this work (a partner in a series of papers), we addressed both fundamental and phenomenological questions on BH physics in Modified Gravity Theories, MGTs, with general modified dispersion relations, MDRs, encoding Lorentz invariance violations, LIVs. Such theories can be geometrized on (co) tangent Lorentz bundles with nonlinear quadratic elements and generalized Finsler like (non)linear connections. In [29], we elaborated an axiomatic approach to such MGTs which can be modelled effectively as a generalized nonholonomic Lagrange-Hamilton dynamics. The general decoupling and integrability of systems of nonlinear partial differential equations, PDEs, describing such stationary configurations is proven in [30]. In this connection, we constructed in explicit form and studied possible physical implications two classes of BH solutions with MDRs.

We provided details on generating solutions and studied most important physical properties of BHs with dependence on an energy type variable (for the 1-st class of solutions). We considered generalizations of six dimensional, 6-d, Tangherlini type solutions (analogs of Schwarzschild – de Sitter BH solutions) and metrics with double BH configurations on relativistic 8-d phase spaces. For the 2-d class of solutions, we have demonstrated that 7-d Tangherlini like BHs can be embedded and nonholonomically generalized to stationary configurations in 8-d phase spaces. There were defined new classes of nonlinear symmetries for BHs with MDRs. Furthermore, we proved that such solutions are characterized by respective Finsler-Lagrange-Hamilton symmetries depending on the type of generating functions and (effective) sources we use.

Another general goal of this article was to study the thermodynamics of BHs with MDRs and LIVs. We proved that in particular we can elaborate analogs of Bekenstein-Hawking thermodynamics on phase spaces modelled as (co) tangent Lorentz bundles with conventional horizons (for instance, of rotoid or toroid symmetry) in the total space. Nevertheless, such an approach can
not applied for more general classes of exact BH and other type solutions with generic off-diagonal metrics and dependence on some (or all) spacetime and total phase space coordinates. We conjectured that we can characterize in a more general statistical thermodynamics form various classes of exact solutions in general relativity and MGTs if we develop for relativistic phase spaces [24, 25, 26, 27] the concepts of W- and F-entropy introduced by Grigory Perelman in his research on geometric flow theory [19, 20, 21].

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