Degeneration of Groups of Type $E_7$
and Minimal Coupling in Supergravity

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Abstract

We study properties of $D = 4, \mathcal{N} \geq 2$ extended supergravities (and related compactifications of superstring theory) and their consistent truncation to the phenomenologically interesting models of $\mathcal{N} = 1$ supergravity. This involves a detailed classification of the “degenerations” of the duality groups of type $E_7$, when the corresponding quartic invariant polynomial built from the symplectic irreducible representation of $G_4$ “degenerates” into a perfect square. With regard to cosmological applications, minimal coupling of vectors in consistent truncation to $\mathcal{N} = 1$ from higher-dimensional or higher-$\mathcal{N}$ theory is non-generic. On the other hand, non-minimal coupling involving vectors coupled to scalars and axions is generic. These features of supergravity, following from the electric-magnetic duality, may be useful in other applications, like stabilization of moduli, and in studies of non-perturbative black-hole solutions of supergravity/string theory.
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1 Introduction

In the present investigation, we relate a physical property of supergravity couplings to a mathematical property of the underlying electric-magnetic duality symmetric of $\mathcal{N} \geq 2$ extended supergravity in $D = 4$ space-time dimensions.

In the textbook [3], the coupling of $\mathcal{N} = 1$ vector and chiral multiplets to supergravity is presented in its minimal form, i.e. it is assumed that the vector kinetic term

$$-\frac{1}{4} \delta_{\alpha \beta} F_{\mu \nu}^{\alpha} \tilde{F}^{\beta \mu \nu}$$

is scalar independent. However, supersymmetry allows for the replacement of the constant kinetic vector matrix $\delta_{\alpha \beta}$ by an holomorphic function of the scalar fields $z$, $\delta_{\alpha \beta} \rightarrow f_{\alpha \beta}(z)$, such that kinetic vector term reads

$$-\frac{1}{4} \left( \text{Re} \, f_{\alpha \beta}(z) \right) F_{\mu \nu}^{\alpha} \tilde{F}^{\beta \mu \nu} + \frac{i}{4} \left( \text{Im} \, f_{\alpha \beta}(z) \right) F_{\mu \nu}^{\alpha} \tilde{F}^{\beta \mu \nu}.$$  \hspace{1cm} (1.2)

Here the function $f_{\alpha \beta}(z)$ is holomorphic, so that a non-minimal coupling is introduced. For example, for one vector, in the simplest case, $f(z) = \phi + ia$ and we have a vector-vector-scalar, $\phi F^2$, and a vector-vector-axion, $aFF$, couplings.

In theories with global supersymmetry the choice of the minimal coupling is often preferred since only for constant, scalar independent $f_{\alpha \beta}$ the theory is renormalizable. It is the same consideration which suggested that a preferred Kähler potential is canonical. In the context of supergravity, however, the requirement of renormalizability is less relevant, the issue we address here is: what kind of vector coupling is preferred in the models originating from higher supersymmetries/higher dimensions.

Non-minimal vector scalar couplings may play an important rule in inflationary cosmology, because a direct coupling of the inflaton scalar field to matter vector fields (as heavy vector bosons, or photons) may provide the only way to complete the creation of matter in the early Universe. This problem was recently addressed in [4], where it was pointed out that in $\mathcal{N} = 1$ supergravity obtained by reduction from higher-dimensional and/or higher-supersymmetric theories the non-minimal vector scalar couplings [1,2] are generic.

The present paper is intended to generalize the results of [4], because we believe that the issue of minimal coupling in $\mathcal{N} \geq 2$ extended supergravities deserves some attention. Indeed, such theories never exhibit a constant $f_{\alpha \beta}$, and in [4] this fact was pointed out to be a consequence of electric-magnetic-duality, which requires a special coupling of the non-linear sigma model of scalars to the vector sector [3]. The kinetic vector matrix $\mathcal{N}_A \Sigma$ which occurs in $\mathcal{N} \geq 2$, $D = 4$ extended supergravities is not holomorphic,

$$\text{Im} \mathcal{N} \Sigma F_{\mu \nu}^{\beta} \tilde{F}^{\alpha \Sigma} + i \text{Re} \mathcal{N} \Sigma F_{\mu \nu}^{\alpha} \tilde{F}^{\beta \Sigma \mu \nu}.$$  \hspace{1cm} (1.3)

Here the kinetic term for vectors $\mathcal{N}_A \Sigma$ in general depends on scalars. The matrix $\text{Im} \mathcal{N}_A \Sigma$ is a metric in the vector moduli space. Comparing the Maxwell term, $\mathcal{N}_A \Sigma$ should reduce to $-\frac{1}{4} \mathcal{F}_{\alpha \beta}(\Sigma)$ in the $\mathcal{N} = 1$ theory [7]. Consistent truncations of $\mathcal{N} \geq 2$ extended supergravities to $\mathcal{N} = 1$ have been studied in [8,9], where it was shown how the non-holomorphic $\mathcal{N}_A \Sigma$ reduces to an anti-holomorphic $f_{\alpha \beta}$ in the corresponding truncated theories.

Let us remind that in $\mathcal{N} = 2$ special Kähler geometry, in a symplectic frame in which an holomorphic prepotential function $F(X)$ exists (such that $X^A \partial_A F = 2F$), the kinetic vector matrix is given by (see e.g. [6], and Refs. therein):

$$\mathcal{N}_A \Sigma = \mathcal{F}_{A \Sigma} - 2 i \mathcal{T}_A \mathcal{T}_\Sigma (L^\Xi \text{Im} F_{\Xi \Omega} L^\Omega),$$  \hspace{1cm} (1.4)

1Further below, we use the term $U$-duality, meaning the “continuous” symmetries of [1]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries [2].

2With exception of “pure” $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity theories, which have no scalars, with $U(1)$ and $U(3)$ $U$-duality group, respectively, consistent with the analysis of [5].
where \( F_{\Lambda \Sigma} = \partial_{\Lambda} \partial_{\Sigma} F \), \( L^\Lambda = e^{K/2} X^\Lambda \) is the covariantly holomorphic contravariant symplectic section, and
\[
T_\Lambda = 2 i \text{Im} \mathcal{N}_{\Lambda \Sigma} L^\Sigma
\]
is the projector on the graviphoton \( (T^\mu_{\nu} = T_\Lambda F^\Lambda_{\mu \nu} \big|_{\Lambda = 0}) \), whose “flux” define the \( \mathcal{N} = 2 \) central charge \( Z \) (see e.g. [10, 11] and Refs. therein). Note that \( \mathcal{N}_{\Lambda \Sigma} \) is not anti-holomorphic because of the presence of the second term in the r.h.s. of (1.4). In order to have a consistent \( \mathcal{N} = 1 \) reduction, one needs to impose \( T_\Lambda = 0 \), i.e. that the graviphoton projection vanishes (when \( \Lambda \) is restricted to the index running on \( \mathcal{N} = 1 \) vector multiplets). One then obtains that minimal coupling demands \( F(X) \) to be quadratic in the truncated scalars of the corresponding would-be \( \mathcal{N} = 1 \) vector multiplets.

It is here worth observing that, while minimal coupling seems natural in \( \mathcal{N} = 1 \) supergravity [3], its relaxation is actually natural if one considers \( \mathcal{N} = 1 \) theories coming from supergravity theory [12] or from higher dimensions [13]. In the systematic approach of the present paper, we will provide a detailed list of examples in which minimal coupling is impossible in the higher-dimensional or higher-\( \mathcal{N} \) theory, but it can be achieved by a further suitable consistent truncation to \( \mathcal{N} = 1 \).

This is related to the mathematical property of the U-duality group \( G_4 \) of type \( E_7 \). Simple, non-degenerate groups \( G_4 \) are related to Freudenthal triple systems \( \mathfrak{M} \) (\( J_3 \)) on simple rank-3 Jordan algebras \( J_3 \). In general, \( G_4 \equiv \text{Conf}(J_3) = \text{Aut}(\mathfrak{M}(J_3)) \) (see e.g. [31-33] for a recent introduction, and a list of Refs.). When considering a consistent reduction to a subgroup, \( G_4 \) groups of type \( E_7 \) may admit a “degeneration” in which the rank-4 invariant symmetric structure \( q \) is reducible, namely it is the product of two symmetric invariant tensors. As a consequence, the corresponding quartic invariant polynomial built from the symplectic irrep. \( R \) of \( G_4 \) “degenerates” into a perfect square. Here \( R \) denotes the symplectic representation of the U-duality group \( G_4 \) formed by a the chiral (or anti-chiral) vector field strengths \( F^{\Lambda \pm} \) and their duals \( G_{\Lambda}^\pm \equiv \mp \frac{i}{2} \delta \mathcal{L} / \delta F^{\Lambda \pm} \):
\[
R = \left( F^{\Lambda \pm}, G_{\Lambda}^\pm \right),
\]
such that “fluxes” of suitably defined projections defines the central charge (matrix) and matter charges (if any; see (see e.g. [16] and Refs. therein). Sometimes, in order to simplify the analysis, in the treatment below we will switch to the basis of the fluxes of the corresponding field strengths, defining the dyonic vector of magnetic and electric charges ([5]: see e.g. the treatment of [6]):
\[
R = (p^\Lambda, q_\Lambda) \equiv Q,
\]
even if our analysis does not only restrict to charged states, such as black holes. By truncation of the charged fluxes \( Q \) we here mean the reduction of the group \( G_4 \) and its irrep. \( R \) (\( G_4 \)) to some proper subgroup \( G_4' \) and its irrep. \( R' \).

Since \( \mathcal{N} > 2 \) theories are related to scalar manifolds which are symmetric spaces, we will consider \( \mathcal{N} = 2 \) theories with symmetric cosets. Therefore, \( \mathcal{N} = 1 \) truncations are simpler to investigate, because the \( \mathcal{N} = 2 \) theory leading to \( \mathcal{N} = 1 \) minimal coupling are the so-called \( \mathcal{N} = 2 \) minimally coupled Maxwell-Einstein supergravities [19], whose scalar manifold is a (non-compact) \( \mathbb{CP}^n \) space. In a scalar-dressed symplectic frame of \( \mathcal{N} = 2 \) special Kähler geometry, the “degeneration” of the quartic polynomial invariant to a quadratic one corresponds to setting the C-tensor to zero \( \langle C_{ijk} \rangle = 0 \). Also for \( \mathcal{N} > 2 \), we will then consider those cases in which the reduction to \( \mathcal{N} = 2 \) gives rise to a \( \mathbb{CP}^n \) special Kähler geometry \( \langle C_{ijk} \rangle = 0 \), in which the U-duality group \( G_4 = U(1, n) \) is a degenerate group of type \( E_7 \) [17], with the rank-4 completely symmetric invariant \( q \)-structure reducible, as pointed out above.

As recalled in Example 1.2 of [17] and proved in [14, 20], all degenerate Freudenthal triple systems are isomorphic to the degenerate triple system in which the resulting quartic invariant polynomial \( I_4 \)

\footnote{An analysis at the level of quartic invariant polynomial, and dependent on charge configurations, has been considered in [15].}

\footnote{In [4] these groups were called “not of type \( E_7 \).”}

3
is the square of a quadratic invariant polynomial $I_2$ which, as pointed out above, also corresponds to the case relevant for $D = 4$ supergravity with symmetric scalar manifold (see the treatment of Sec. 2 as well). The degeneration of a $U$-duality group $G_4$ of type $E_7$ is also confirmed by the fact that the fundamental identity characterizing simple, non-degenerate groups of type $E_7$ (proved in Sec. 2 of [17] for $E_7$, and generalized in formula (2.19) further below at least for all groups listed in Table 1) does not hold in these cases; see Sec. 2. The cases of $U$-duality groups as semi-simple, non-degenerate groups of type $E_7$ relevant to $D = 4$ supergravity theories with symmetric (vector multiplets') scalar manifolds are also analyzed in Subsec. 2.4.

Simple, degenerate groups of type $E_7$ relevant to $D = 4$ supergravity (namely, $U(1,n)$ or $U(3,n)$) share the property that the dyonic charge vector $\mathcal{Q}$ (1.7) (element of the Freudenthal triple system) fits into the sum of the fundamental and anti-fundamental irreps.

$$\mathcal{Q} \in \mathbf{R} \equiv \mathbf{Fund} + \overline{\mathbf{Fund}}, \quad (1.8)$$

thus naturally admitting a complex representation, endowed with an invariant Hermitian quadratic structure (see e.g. [21, 22]), whose real part gives rise to the aforementioned quadratic invariant polynomial $I_2$; see the discussion in Sec. 2.

It should be stressed that the conditions on truncations of fluxes and embeddings of scalar manifolds, under consideration in the treatment below, are generally only necessary, but not sufficient for minimal coupling. An analysis of the consistency of the truncations at the level of supersymmetry transformations, along the lines exploited in [8] and [9] (this latter on the further truncation $N = 2 \to 1$) is required to determine also a sufficient condition; we give a brief general account of this analysis at the start of Sec. 3 and we provide an explicit example in Subsec. 3.1.

The plan of the paper is as follows.

After axiomatically introducing groups of type $E_7$ in Sec. 2 we analyze various truncations to minimal coupling models in subsequent Sections. It is here worth pointing out that by truncation of a theory we here mean a sub-theory obtained from the original one by reducing the amount of supersymmetry. For “pure” ($N \geq 5$) supergravities, this means to consistently truncate away the extra gravitino multiplet(s); these cases are considered in Secs. 3 and 4. On the other hand, for matter-coupled ($2 \leq N \leq 4$) theories the truncation also requires to consistently truncate the matter multiplets’ sector; such cases are analyzed in Secs. 5, 6 and 7. In presence of matter coupling, there is another way of obtaining sub-theories, namely to consistently reduce the matter sector but not the gravitino multiplet(s); Sec. 8 deals with such cases. The list of examples produced by the systematic approach of the present investigation is much larger than the ones given in [8, 9, 4], and it is of some interest also because some truncations correspond to orbifolds and orientifolds of string theories with larger supersymmetry, as discussed in Sec. 9 in which the further truncation $N = 2 \to N = 1$ is considered. Comments on the “degeneration” of the so-called Freudenthal duality are then given in Sec. 10. Sec. 11 contain some remarks on fermions and minimal coupling. Conclusive remarks and an outlook are given in Sec. 12. Appendix A containing some details on the structure of Pauli terms, concludes the paper.

# 2 On Groups of Type $E_7$

## 2.1 Axiomatic Characterization

The first axiomatic characterization of groups “of type $E_7$” through a module (irreducible representation) was given in 1967 by Brown [14].

A group $G$ of type $E_7$ is a Lie group endowed with a representation $\mathbf{R}$ such that:

1. $\mathbf{R}$ is symplectic, i.e. (the subscripts “s” and “a” stand for symmetric and skew-symmetric
was observed \[26\].

It is intriguing to notice that the first paper on groups of type $E_7$ in extended supergravities in \( D = 3 \) is also for a list of related Refs.\).

Remarkably, groups of type $E_7$, appearing in $D = 4$ supergravity as $U$-duality groups, admit a $D = 5$ uplift to groups of type $E_6$, as well as a $D = 3$ downlift to groups of type $E_8$. It should also be recalled that split form of exceptional $E_7$ - Lie groups appear in the exceptional Cremmer-Julia $[1]$ sequence $E_{11-\cdot-D(11-\cdot-D)}$ of $U$-duality groups of $M$-theory compactified on a $D$-dimensional torus, in $D = 3, 4, 5$. Other sequences, composed by non-split, non-compact real forms of exceptional groups, are also relevant to non-maximal supergravity in various dimensions (see e.g. the treatment in $[23]$, also for a list of related Refs.).

The connection of groups of type $E_7$ to supergravity can be summarized by stating that all $2 \leq N \leq 8$-extended supergravities in $D = 4$ with symmetric scalar manifolds $\frac{G_4}{H_4}$ have $G_4$ of type $E_7$ \[15\] \[63\].

It is intriguing to notice that the first paper on groups of type $E_7$ was written about a decade before the discovery of of extended ($N = 2$) supergravity \[25\], in which electromagnetic duality symmetry was observed \[26\].

An example of Lie group which is not of type $E_7$ is the exceptional Lie group $E_6$ in its fundamental representation $[27]$; this is relevant to both maximal ($N = 8$) and exceptional ($N = 2$) supergravity theories in $D = 5$. The representation $27$ is not symplectic, but rather it is conjugated to its contragradient counterpart ($a = 1, \ldots, 27$):

\[
\exists \delta^a_8 \equiv 1 \in 27 \times \overline{27}. \tag{2.8}
\]

\[5\]Strictly speaking, the pair $(G, \mathbb{R}) = (E_6, 27)$ is the prototype of the so-called groups “of type $E_6$.”
Furthermore, 27 admits a unique rank-3 completely symmetric primitive $E_6$-invariant structure, usually named $d$-tensor

$$\exists d_{abc} \ni 1 \in \mathbb{Z}_{27} \times \mathbb{Z}_{27} \times \mathbb{Z}_{27};$$  

(2.9)

thus, by contracting the $d$-tensor with the same charge vector $Q$ in 27, one can construct a rank-3 homogeneous $E_6$-invariant polynomial (whose $\vartheta$ is the normalization constant):

$$d(Q) \equiv d_{abc}Q^aQ^bQ^c,$$

(2.10)

which corresponds to the evaluation of the rank-3 symmetric invariant $d$-structure induced by the $d$-tensor on four identical modules 27:

$$d(Q) \equiv d(Q_x,Q_y,Q_z)|_{Q_x=Q_y=Q_z=Q} \equiv \varsigma \left[ d_{abc}Q_x^aQ_y^bQ_z^c \right]_{Q_x=Q_y=Q_z=Q}.$$  

(2.11)

Focussing on the relevance to supergravity theories in $D = 4$, in the remaining part of this Section we will characterize various classes of groups of type $E_7$ in terms of (tensor and) scalar identities, along the lines of [17] and exploiting results of previous investigations, such as [23] and [27].

### 2.2 Simple, Non-Degenerate

In simple, non-degenerate groups $G_4$ of type $E_7$ [14] relevant to $D = 4$ (super)gravity with symmetric scalar manifolds (listed in Table 16), the following identity holds (cfr. (5.18) of [23]):

$$K_{MNPQ}K_{RSTU}C^{PT}C^{QU} = \xi \left[ (2\tau - 1) K_{MNR} + \xi \tau (\tau - 1) C_{M(R}C_{SN)I} \right].$$  

(2.12)

$C_{MN}$ is the symplectic metric, and $K_{MNPQ}$ denotes the completely symmetric, rank-4 invariant “$K$-tensor” in the relevant symplectic irrep. $R(G_4)$ ($M$ is an index in $R$):

$$C \equiv \exists \mathbb{1} \in [R \times R]_a;$$

(2.13)

$$K \equiv \exists \mathbb{1} \in [R \times R \times R \times R]_s,$$

(2.14)

where the subscript “$s$” (“$a$”) denotes the (anti)symmetric part of the tensor product. Moreover, the $G_4$-dependent parameters are defined as [23, 30]

$$\tau \equiv \frac{2d}{f(f+1)};$$

(2.15)

$$\xi \equiv -\frac{1}{3\tau},$$

(2.16)

where

$$f \equiv \dim_R (R(G_4));$$

(2.17)

$$d \equiv \dim_R (\text{Adj}(G_4)).$$

(2.18)

By using (2.12), one can show that the following identity holds:

$$\text{tr} \left( p(x \otimes x) p(y \otimes y) \right) = \beta \left[ q(x,x,y,y) - 2b(y,x)^2 \right],$$

(2.19)

where (recall definition (2.20))

$$b(x,y) \equiv -C_{MN}Q^M_xQ^N_y = -\langle Q_x,Q_y \rangle,$$

(2.20)

$$q(x,y,z,w) \equiv -6K_{MNPQ}Q^M_xQ^N_yQ^P_zQ^Q_w;$$

(2.21)

$$\beta \equiv \frac{2}{\tau},$$

(2.22)

We only consider rank-3 Jordan algebras related to locally supersymmetric theories of gravity.
and $p$ denotes the following vector space map (cfr. Sec. 2 of [17] for further detail)
\begin{equation}
p(x \otimes y) z \equiv t(x, y, z) - b(z, x) y - b(z, y) x,
\end{equation}
where $t(x, y, z)$ is the trilinear product related to $q(x, y, z, w)$ as
\begin{equation}
q(x, y, z, w) \equiv b(x, t(y, z, w)) .
\end{equation}
The scalar identity (2.19) holds at least for all simple, non-degenerate groups $G_4$ of type $E_7$ listed in Table 1 (and for all their other non-compact forms, as well as for the corresponding compact Lie group $G_{4,c}$), and it is a consequence of the tensor identity (2.12), which in turn follows from the identity for the $K$-tensor given by (5.17) of [23]. In the particular case of $E_7$ (see Tables 1 and 2), it holds $\tau = 1/12 \Rightarrow \beta = 24$, and the identity proved in Theorem 2.3 of [17] is retrieved.

It is worth remarking that, by defining the parameter $q$ as specified in Table 2, the values of $f$, $d$, $\tau$, $\xi$ and $\beta$ can be easily parametrized as follows (2.26) was noticed in [23]:
\begin{align}
f &= 2 (3q + 4) ; \\
d &= \frac{3 (3q + 4) (2q + 3)}{q + 4} ; \\
\tau &= \frac{1}{q + 4} ; \\
\xi &= -\frac{(q + 4)}{3} ; \\
\beta &= 2 \left( q + 4 \right).
\end{align}
The specific values for the groups listed in Table 1 are reported in Table 2. Note that, speaking in terms of compact form $G_{4,c}$ of $G_4$, for $G_{4,c} = E_7$, $SO(12)$, $SU(6)$ and $USp(6)$, $q$ can be defined as
\begin{equation}
q \equiv \dim_{\mathbb{R}} A,
\end{equation}
where $A$ denotes the division algebra on which the corresponding rank-3 simple Jordan algebra $J_3^A$ is constructed ($q = 8, 4, 2, 1$ for $A = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively). Note that the triality symmetric so-called $\mathcal{N} = 2$ STU model [34], based on $J_3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, can be obtained by setting $q = 0$; however, since the corresponding $G_4$ is semi-simple, it will be considered further below.

Also, note that the dimensions $f$ and $d$ of $G_4$’s listed in Table 1 satisfy the relation [23]
\begin{equation}
d = \frac{3f(f + 1)}{f + 16} .
\end{equation}

### 2.3 Simple, Degenerate

As pointed out in Sec. 2 of [17], the story changes for degenerate groups of type $E_7$.

Confining ourselves to the ones relevant in $D = 4$ supergravity with symmetric scalar manifold, they are nothing but $G_4 = U(r, s)$ with $r = 1$ ($\mathcal{N} = 2$ minimally coupled to $s$ vector multiplets [19]) or $r = 3$ ($\mathcal{N} = 3$ coupled to $s$ vector multiplets [39]), and the relevant (complex) symplectic representation is $\mathbb{R}(G_4) = r + s$. In these cases, it can be computed that
\begin{equation}
\mathbb{E}_{MNPQ} = \frac{\zeta^2}{3} S_{MN} S_{PQ} ,
\end{equation}
where $\zeta$ is a real constant, and the rank-2 symmetric invariant symplectic tensor $S$ ($S^T = S$, $SCS = C$) is defined by the following formula:
\begin{equation}
Q_x^\dagger Q_y^\dagger \eta_{ij} = S_{MN} Q_x^M Q_y^N + i C_{MN} Q_x^M Q_y^N ,
\end{equation}
Table 1: Simple, non-degenerate groups $G_4$ related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras $J_3$. The relevant symplectic irrep. $\mathbf{R}$ of $G_4$ is also reported. $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ are the corresponding split forms. Note that the $G_4$ related to split forms $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ is the maximally non-compact (split) real form of the corresponding compact Lie group. The corresponding scalar manifolds are the symmetric cosets $\frac{G_4}{H_4}$, where $H_4$ is the maximal compact subgroup (with symmetric embedding) of $G_4$. The number of supercharges of the resulting supergravity theory in $D = 4$ is also listed. $M_{1,2}(\mathbb{O})$ is the Jordan triple system generated by $2 \times 1$ vectors over $\mathbb{O}$ [42]. The $D = 5$ uplift of the $T^3$ model based on $J_3 = \mathbb{R}$ is the pure $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^\mathbb{H}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are “twin”, namely they share the very same bosonic sector [42, 35, 37, 38].

| $J_3$          | $G_4$    | $\mathbf{R}$ | $\mathcal{N}$ |
|---------------|----------|--------------|---------------|
| $J_3^\mathbb{O}$ | $E_7(-25)$ | 56           | 2             |
| $J_3^{\mathbb{O}_s}$ | $E_7(7)$   | 56           | 8             |
| $J_3^\mathbb{H}$   | $SO^*(12)$ | 32           | 2, 6          |
| $J_3^\mathbb{C}$   | $SU(3,3)$  | 20           | 2             |
| $M_{1,2}(\mathbb{O})$ | $SU(1,5)$  | 20           | 5             |
| $J_3^\mathbb{R}$   | $Sp(6,\mathbb{R})$ | 14'     | 2             |
| $\mathbb{R}$ (T$^3$ model) | $SL(2,\mathbb{R})$ | 4        | 2             |

Table 2: The parameter $q$ and the related $q$-parametrized quantities $f$ (2.25), $d$ (2.26), $\tau$ (2.27), $\xi$ (2.28) and $\beta$ (2.29). The corresponding compact form $G_{4,c}$ of $G_4$ is listed.
where $\eta_{ij}$ is the invariant metric of the fundamental irrep. $r+s$ of $U(r,s)$, and $Q^i_x$ and $Q^i_y$ are the charge vectors in the complex (manifestly $U(r,s)$-covariant) symplectic frame. By introducing

$$I_2(x,y) \equiv \zeta^{S_{MN}}Q^M_x Q^N_y,$$

it is immediate to check the degenerate nature of the quartic invariant $q$-structure (2.21):

$$q(x,y,z,w) = -6K_{MNPQ}Q^M_x Q^N_y Q^P_z Q^Q_w$$

$$= -2[I_2(x,y)I_2(z,w) + I_2(x,z)I_2(y,w) + I_2(x,w)I_2(y,z)];$$

$$\downarrow$$

$$q(x,y,z,w) = -2\left[2I_2(x,y)^2 + I_2(x,x)I_2(y,y)\right];$$

$$\downarrow$$

$$\frac{-1}{6}q(x,x,x,x) = I_2(x,x)^2. \quad (2.37)$$

The analogue of identity (2.19) for such degenerate groups of type $E_7$ enjoys a very simple form ($\mathbb{C}_{MN}\mathbb{C}^{MN} = 2(r+s)$):

$$K_{QPQR}K_{SMTU}C^{RS} = \zeta^4S_{(QP)C_N(MS_{TU})};$$

$$\downarrow$$

$$K_{QPQR}K_{SMTU}C^{NM}C^{RS} = \frac{\zeta^4}{9}[(2(r+s) + 4)S_{PQ}S_{TU} + 2C_{PT}C_{QU} + 2C_{PU}C_{QT}]. \quad (2.39)$$

By exploiting (2.39), one can thus compute:

$$tr(p(x \otimes x)p(y \otimes y)) = 4\left[q(x,x,y,y) - (4\zeta^4 + 1)b(y,x)^2\right] - 4\zeta^2[2(r+s)+4]I_2(x,x)I_2(y,y), \quad (2.40)$$

which can be considered the analogue of (2.19) for the degenerate groups of type $E_7$ under consideration. The validity of the postulate (2.7) implies $\zeta^2 = 1/2$.

It should be remarked that, according to the discussion in Example 1.2 of [17] (and to the whole treatment therein), the invariant $q$-structure of any degenerate Freundental triple system enjoys the form (2.37), up to isomorphisms. Therefore, the simple, degenerate groups of type $E_7$ mentioned above (relevant to $\mathcal{N} = 2$ minimally coupled and $\mathcal{N} = 3$ supergravity in $D = 4$; see also the treatment below) can be regarded as "prototypes" (up to isomorphisms) of (simple) degenerate groups of type $E_7$.

### 2.4 Semi-Simple, Non-Degenerate

Let us now consider semi-simple, non-degenerate groups of type $E_7$.

Confining ourselves to the ones relevant in $D = 4$ supergravity with symmetric scalar manifold, they are nothing but $G_4 = SL(2,\mathbb{R}) \times SO(m,n)$ with $m = 2$ ($\mathcal{N} = 2$ coupled to $n + 1$ vector multiplets) or $m = 6$ ($\mathcal{N} = 4$ coupled to $n$ vector multiplets), and the relevant symplectic representation is the bi-fundamental $R(G_4) = (2,m + n)$. They are respectively related to semi-simple rank-3 Jordan algebras $\mathbb{R} \oplus \Gamma_{m-1,n-1}$, where $\Gamma_{m-1,n-1}$ is a Jordan algebra with a quadratic form of pseudo-Euclidean $(m-1,n-1)$ signature, i.e. the Clifford algebra of $O(m-1,n-1)$ [51]. The aforementioned $\mathcal{N} = 2$ $STU$ model [34], based on $J_3 = \mathbb{R} \oplus \Gamma_{1,1} \sim \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, is recovered by setting $m = n = 2$.

In these cases, electro-magnetic splitting of the symplectic representation $R$ can be implemented in a manifestly $G_4$-covariant fashion. Namely, $Q$ is an electro-magnetic doublet of the $SL(2,\mathbb{R})$ factor of $G_4$ itself. The symplectic index $M$ thus splits as follows (cfr. Eq. (3.7) of [28])

$$M = \alpha \Lambda,$$

$$\alpha = 1, 2, \quad \Lambda = 1, ..., m + n - 2. \quad \Rightarrow \quad Q^M = Q^\Lambda_{\alpha}.$$  \quad (2.41)
and it should be pointed out that in the $N = 2$ case usually $\Lambda = 0, 1, \ldots, n - 1$, with "0" pertaining to the $D = 4$ graviphoton vector. The manifestly $G_4$-covariant symplectic frame (2.41) is usually dubbed Calabi-Vesentini frame [29], and it was firstly introduced in supergravity in [10].

The symplectic metric $C_{MN} = C_{\Lambda\Sigma}^{\alpha\beta}$ and rank-4 completely symmetric $\mathbb{K}$-tensor $K_{MNPQ} = K_{\Lambda\Sigma\Xi\Omega}^{\alpha\beta\gamma\delta}$ enjoy the following expression in term of the invariant structures $\epsilon^{\alpha\beta}$ and $\eta_{\Lambda\Sigma}$ of $SL_\nu(2, \mathbb{R})$ and of $SO(m, n - 2)$, respectively:

$$C_{\Lambda\Sigma}^{\alpha\beta} = \eta_{\Lambda\Sigma} \epsilon^{\alpha\beta};$$
$$K_{\Lambda\Sigma\Xi\Omega}^{\alpha\beta\gamma\delta} = \frac{1}{12} \left[ (\epsilon^{\alpha\beta} \epsilon^{\gamma\delta} + \epsilon^{\alpha\delta} \epsilon^{\beta\gamma}) \eta_{\Lambda\Xi} \eta_{\Sigma\Omega} + (\epsilon^{\alpha\beta} \epsilon^{\delta\gamma} + \epsilon^{\alpha\gamma} \epsilon^{\beta\delta}) \eta_{\Lambda\Omega} \eta_{\Sigma\Xi} + (\epsilon^{\alpha\gamma} \epsilon^{\beta\delta} + \epsilon^{\alpha\delta} \epsilon^{\beta\gamma}) \eta_{\Lambda\Sigma} \eta_{\Xi\Omega} \right].$$

(2.42)

(2.43)

From this, one can compute the analogue of identities (2.12) and (2.39) for the semi-simple, non-degenerate groups of type $E_7$, under consideration (since $\epsilon_{\alpha\beta} \epsilon^{\alpha\beta} = 2$, $\eta_{\Lambda\Sigma} \eta^{\Lambda\Sigma} = m + n$):

$$K_{MNPQ} K_{RSTU} C^{PT} C^{QU} = \sum_{\alpha} K_{\Lambda\Sigma\Xi\Omega}^{\alpha\beta\gamma\delta} \sum_{\alpha} K_{\Pi\Omega\Xi\Psi}^{\mu\nu\rho\sigma} C^\Lambda C^\Sigma C^\Xi C^\Omega C^\Psi C^\Omega C^\Xi C^\Lambda$$

$$= \frac{1}{6} C^{\Lambda\pi\Sigma\Phi} C^{\mu\nu\rho\sigma} \left[ \frac{\eta_{\Lambda\Xi} \eta_{\Sigma\Omega} \eta_{\Pi\Omega} \eta_{\Xi\Psi} \eta_{\Pi\Omega} \eta_{\Xi\Psi}}{2} \left( \epsilon^{\alpha\beta} \epsilon^{\rho\sigma} + \epsilon^{\alpha\rho} \epsilon^{\beta\sigma} \right) + \frac{1}{72} \left( \eta_{\Lambda\Xi} \eta_{\Sigma\Omega} \eta_{\Pi\Omega} \eta_{\Xi\Psi} \eta_{\Pi\Omega} \eta_{\Xi\Psi} \right) \right].$$

(2.44)

By exploiting (2.41), one can thus compute:

$$tr (p (x \otimes x) p (y \otimes y)) = 5 q (x, x, y, y) - 4 b (x, y)^2$$

$$- \left[ \epsilon^{\alpha\beta} \epsilon^{\rho\delta} \eta_{\Lambda\Phi} \eta_{\Xi\Psi} + (m + n - 2) \epsilon^{\alpha\rho} \epsilon^{\beta\delta} \eta_{\Lambda\Sigma} \eta_{\Phi} \right] Q^\Lambda_{\alpha|x} Q^\Sigma_{\beta|x} Q^\Phi_{\alpha|y} Q^\Psi_{\beta|y},$$

(2.45)

where we recall that the quartic invariant form $q$ is defined by (2.21). The identity (2.45) can be considered the analogue of (2.19) and (2.40) for the semi-simple, non-degenerate groups of type $E_7$, under consideration, and it is different from them both.

2.5 The Unified Limit

The different structure exhibited by the scalar identities (2.19) (holding for simple, non-degenerate groups of type $E_7$), (2.40) (holding for simple, degenerate groups of type $E_7$) and (2.43) (holding for semi-simple, non-degenerate groups of type $E_7$) is manifest: the structure of (2.19) is the same as the structure of the first line of (2.40) and of (2.43), but the second line of (2.40) and of (2.45) is not compatible with such a structure.

Therefore, along the lines of [17], the scalar identities (2.19), (2.40) and (2.45) (or the corresponding tensor identities) can be considered as defining identities for simple non-degenerate, simple degenerate, and semi-simple non-degenerate groups of type $E_7$, respectively.

However, it should be also noted that (2.19), (2.40) and (2.45) share the very same $s \equiv y$ limit:

$$tr (p (x \otimes x) p (x \otimes x)) = \beta q (x, x, x, x),$$

(2.46)

modulo the renamings

$$\beta = 4 \left[ 1 + \frac{c^2}{3} (r + s + 2) \right] \equiv \left[ 5 + \frac{1}{3} (m + n) \right].$$

(2.47)
Before proceeding to the analysis of various truncation patterns to minimal coupling models, it is worth stressing a peculiar feature of the $\mathcal{N} = 2$ theory among $D = 4$ extended supergravity theories. $\mathcal{N} = 2$ supergravity is the unique extended supergravity which admits two different types of matter multiplets, namely vector and hyper multiplets. Thus, out of the three classes (simple non-degenerate, simple degenerate and semi-simple non-degenerate, respectively treated in Subsecs. 2.2, 2.3 and 2.4) of groups $G_4$ of type $E_7$ treated above, one can always construct a semi-simple group of type $E_7$ with the following structure:

$$G_4 \times G_4; \qquad (\mathcal{R}(G_4), \mathcal{R}(G_4) = 1). \tag{2.48}$$

As pointed out above, $G_4$ is the $U$-duality group of the $\mathcal{N} = 2$ theory (which is also the global isometry group of the special Kähler vector multiplets’ scalar manifold), whereas $G_4$ is the global isometry group of the quaternionic Kähler hypermultiplets’ scalar manifold. The various truncations analyzed in subsequent Sections provide a number of examples of truncations of simple non-degenerate groups of the quaternonic Kähler hypermultiplets’ scalar manifold. The various truncations analyzed in subsequent Sections provide a number of examples of truncations of simple non-degenerate groups of the quaternonic Kähler hypermultiplets’ scalar manifold, whereas $G_4$ is the global isometry group of the quaternionic Kähler hypermultiplets’ scalar manifold. The various truncations analyzed in subsequent Sections provide a number of examples of truncations of simple non-degenerate groups of the quaternonic Kähler hypermultiplets’ scalar manifold.

### 3 Maximal Truncations from $\mathcal{N} = 8$ ($J_3^{(1)}$)

One can perform the kinematical reduction of $\mathcal{N}$-extended supergravity multiplets down to $\mathcal{N}' < \mathcal{N}$ multiplets (massless multiplets in $\mathcal{N}$-extended $D = 4$ supergravity are reported in Tables 3 and 4). The reduction is subjected to the following dynamical conditions: the inclusion of $U$-duality groups: $G_4 \supset G_4'$, as well as of the stabilizers of the scalar manifold: $H_4 \supset H_4'$, such that the scalar manifold of the truncated theory is a proper sub-manifold of the scalar manifold of the starting theory: $G_4/H_4 \supset G_4'/H_4'$. At the level of electric and magnetic fluxes, the branching $\mathcal{R}(G_4) = \mathcal{R}(G_4') + \ldots$ has to hold, where $\mathcal{R}(G_4)$ is the relevant symplectic representation of $G_4$ itself.

If $\mathcal{N}, \mathcal{N}' \geq 4$, the kinematical multiplet truncation actually coincides with the dynamical truncation, because there is a unique choice of matter multiplets in these cases. On the other hand, already for $\mathcal{N}' = 3$ this is no longer true for $\mathcal{N} \geq 6$, and for $\mathcal{N} = 8 \to \mathcal{N}' = 2$ many possibilities exist; the maximal truncations (in the sense of $G_4 \supset G_4'$ specified above) are listed in Table 5. Kinematical truncations $\mathcal{N} = 6 \to 5, \mathcal{N} = 6 \to 4$ and $\mathcal{N} = 5 \to 4$ actually coincide with the corresponding dynamical reduction. The two latter cases yield 2 and no matter multiplets, respectively. Further truncation of these theories down to $\mathcal{N} = 1$ reduces to some of the general examples we consider further below.

Before proceeding with the analysis of the various truncations $\mathcal{N} = 8 \to \mathcal{N}' < 8$, we would like here to add a brief discussion of the general consistency conditions yielded by the supersymmetry transformations of the $\mathcal{N} = 8$ fermionic fields, along the lines of the treatment in Sec. 5 of [8] (a similar discussion related to the Attractor Mechanism has been given also in Sec. 5 of [18]). Neglecting three fermion terms, the transformations of the gravitinos $\psi_A$ and of spin $\frac{1}{2}$ fermions $\chi_{ABC}$ read as follows ($A = 1, \ldots, 8$; cfr. e.g. (5.2)-(5.3) of [8]):

$$\delta \psi_{A\mu} = \nabla_{\mu} \epsilon_A + T_{AB}^{-} \gamma_{\mu} \epsilon^B; \tag{3.1}$$
$$\delta \chi_{ABC} = P_{ABCD,\alpha} \partial_\mu \phi^\alpha \gamma^\mu \epsilon^D + T_{AB}^{-} \gamma_{\mu\nu} \epsilon^B; \tag{3.2}$$

where $\nabla_{\mu} \epsilon_A = \partial_\mu \epsilon_A + \omega_A^\alpha \gamma^\nu \epsilon_B$, $T_{AB}^{-}$ is the (dressed) graviphotonic field strengths’ 2-form, $P_{ABCD}$ is the Vielbein 1-form, and $\phi^\alpha$ are the 70 real scalars of the rank-7 symmetric $\mathcal{N} = 8$ scalar manifold $E_7(7)/SU(8)/\mathbb{Z}_2$.

When considering a truncation $\mathcal{N} = 8 \to \mathcal{N}' < 8$, it holds that

$$SU(8) \supset SU(\mathcal{N}') \times SU(8 - \mathcal{N}') \times U(1); \tag{3.3}$$
$$8 = (\mathcal{N}', 1)_{\mathcal{N}'-8} + (1, 8-\mathcal{N}')_{\mathcal{N}'}. \tag{3.4}$$
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$N$ & massless $\lambda_{\text{MAX}} = 2$ multiplet & massless $\lambda_{\text{MAX}} = 3/2$ multiplet \\
\hline
8 & $[(2), 8(\frac{3}{2}), 28(1), 56(\frac{1}{2}), 70(0)]$ & none \\
6 & $[(2), 6(\frac{3}{2}), 16(1), 26(\frac{1}{2}), 30(0)]$ & $[(\frac{3}{2}), 6(1), 15(\frac{1}{2}), 20(0)]$ \\
5 & $[(2), 5(\frac{3}{2}), 10(1), 11(\frac{1}{2}), 10(0)]$ & $[(\frac{3}{2}), 6(1), 15(\frac{1}{2}), 20(0)]$ \\
4 & $[(2), 4(\frac{3}{2}), 6(1), 4(\frac{1}{2}), 2(0)]$ & $[(\frac{3}{2}), 4(1), 7(\frac{1}{2}), 8(0)]$ \\
3 & $[(2), 3(\frac{3}{2}), 3(1), (\frac{1}{2})]$ & $[(\frac{3}{2}), 3(1), 3(\frac{1}{2}), 2(0)]$ \\
2 & $[(2), 2(\frac{3}{2}), (1)]$ & $[(\frac{3}{2}), 2(1), (\frac{1}{2})]$ \\
1 & $[(2), (\frac{3}{2})]$ & $[(\frac{3}{2}), (1)]$ \\
\hline
\end{tabular}
\caption{Massless multiplets with maximal helicity $\lambda_{\text{MAX}} = 2, 3/2$ \cite{36}.}
\end{table}

Correspondingly, the supersymmetry parameters, the gravitinos and the spin $1/2$ fermions branch as $(a = 1, ..., N', i = 1, ..., 8 - N')$:

\begin{align}
\epsilon_A &= \epsilon_a, \quad \epsilon_i; \\
\psi_A &= \psi_a, \quad \psi_i; \\
\chi_{ABC} &= \chi_{abc}, \quad \chi_{abi}, \quad \chi_{aij}, \quad \chi_{ijk}.
\end{align}

The conditions of consistent truncation read

\begin{equation}
\begin{cases}
\epsilon_i = 0; \\
\psi_i = 0; \\
\chi_{abi} = \chi_{aij} = \chi_{ijk} = 0,
\end{cases}
\quad \text{such that} \quad
\begin{cases}
\delta\psi_i = 0; \\
\delta\chi_{abi} = \delta\chi_{aij} = \delta\chi_{ijk} = 0,
\end{cases}
\end{equation}

with the exception of the case $N' = 6$ (discussed in Subsec. \cite{3.1}), for which $\chi_{aij}$, as well as its corresponding supersymmetry variation, does not vanish.
By exploiting (3.8) and (3.1)-(3.2), one obtains the following general consistency conditions:

\[
\begin{align*}
&i) \, : \omega_i^a = 0; \\
&ii) \, : T_{ai} = 0; \\
&iii) \, : P_{abc} = P_{aijk} = 0; \\
&iv) \, : P_{abij} = 0; \\
v) \, : T_{ij} = 0,
\end{align*}
\]

(3.9)

where conditions iv) and v) do not hold for \( \mathcal{N} = 6 \). Condition i) (on the spin connection \( \omega \)) confirms the consistency condition on the reduction of the holonomy (\( R \)-symmetry) group, as discussed in Secs. 3 and 4 of [8]. Conditions iii) and iv) (on the Vielbein \( P \)) confirm the consistency conditions from the embedding of the scalar manifold of the \( \mathcal{N} \)-extended supergravity sub-theory into the scalar manifold of \( \mathcal{N} = 8 \) theory, as discussed in Sec. 4 of [8]. Furthermore, conditions ii) and v) (on the graviphotonic field strengths \( T \)), which are the (necessary but not necessary) conditions which we discuss in the present investigation, are needed in order for \( T_{ab} \) to consistently parametrize the (dressed) graviphotons which survive the truncation under consideration (in the case \( \mathcal{N} = 6 \), one should consider also \( T_{ij} \) non-vanishing).

3.1 \( \rightarrow \mathcal{N} = 6 \)

In this Subsection, we discuss, at the level of the consistency conditions yielded by supersymmetry, the case \( \mathcal{N} = 8 \rightarrow 6 \) (for the “twin” case \( \mathcal{N} = 2 \) \( (n_V, n_H) = (15, 0) \), and further decomposition, see point 1 of Subsubsec. 3.4.1):

\[
J^{O_3} = J^3: \mathcal{N} = 8 \rightarrow J^3 =: \mathcal{N} = 6
\]

(3.10)

\[
E_{7(7)} \supset SO^*(12) \times SU(2) \supset SU(6) \times SU(2) \times U(1):
\]

(3.11)

\[
SU(8) \supset SU(6) \times SU(2) \times U(1):
\]

(3.12)

\[
\frac{E_{7(7)}}{SU(8)} \supset \frac{SO^*(12)}{SU(6) \times U(1)} \supset \frac{SL(2, \mathbb{R}) \times SO(6, 2)}{SO(6) \times SO(2)}.
\]

(3.13)

In particular, the decomposition of the 28 of \( SU(8) \) yields the following branching of the \( \mathcal{N} = 8 \) dressed graviphotonic field strengths \((a = 1, \ldots, 6, \, i = 1, 2)\):

\[
T_{AB}^{-} = T_{ab}^{-} \cdot T_{ai}^{-} \cdot T_{ij}^{-}.
\]

(3.14)

The \( SU(2) \) commuting factor in (3.11) and (3.12) is the \( R \)-symmetry truncated away in the supersymmetry reduction \( \mathcal{N} = 8 \rightarrow 6 \) (a further truncation \( \mathcal{N} = 6 \rightarrow \mathcal{N} = 3 \) is considered in Subsec. 4.1). The truncation condition on the two-form Abelian field strengths’ fluxes reads

\[
(12, 2) = (6, 2) + (\bar{6}, 2) = 0 \iff T_{ab}^{-} = 0.
\]

(3.15)
We anticipate that this truncation condition on fluxes is complementary to the condition (3.44) considered in Subsec. 3.3; indeed, the embedding (3.42)-(3.44) is a different non-compact, real form of the embedding (3.12).

As a consequence of the decomposition (3.12) of the 8 of SU(8), the supersymmetry parameters, the gravitinos and the spin 1/2 fermions respectively branch as:

\[ \epsilon_A^8 = \epsilon_a^{(6,1)}, \epsilon_i^{(1,2)}; \tag{3.16} \]
\[ \psi_A^8 = \psi_a^{(6,1)}, \psi_i^{(1,2)}; \tag{3.17} \]
\[ \chi_{ABC}^{56} = \chi_{abc}^{(20,15,2)}, \chi_{aij}^{(6,1)}; \tag{3.18} \]

Correspondingly, the conditions of consistent truncation read

\[ \begin{cases} 
\epsilon_i = 0; \\
\psi_i = 0; \\
\chi_{abi} = 0,
\end{cases} \quad \delta \psi_i = 0; \quad \delta \chi_{abi} = 0. \tag{3.19} \]

By evaluating the supersymmetry transformations (3.1)-(3.2) on the truncation conditions (3.19), one obtains

\[ \delta \psi_{a\mu} = \nabla_\mu \epsilon_a + T_{ab\nu\rho}^i \gamma^\nu \gamma^\rho \epsilon^b; \tag{3.20} \]
\[ \delta \chi_{abc} = P_{abcd,\alpha} \partial_\mu \phi^{\alpha} \gamma^\mu \epsilon^d + \frac{1}{3} \left( T_{ab\nu\rho}^{\mu} \gamma^{\mu} \epsilon_c + T_{ca\mu\nu}^{\rho} \gamma^{\rho} \epsilon^b + T_{bc\lambda\mu}^{\nu} \gamma^{\nu} \epsilon^a \right); \tag{3.21} \]
\[ \delta \chi_{aij} = P_{abij,\alpha} \partial_\mu \phi^{\alpha} \gamma^\mu \epsilon^b + \frac{1}{3} T_{ij\mu\nu}^{\gamma} \gamma^{\mu} \epsilon^a \tag{3.22} \]

in the untruncated sector, and

\[ \begin{cases} 
\omega_i^a = \omega_i^{\alpha} \epsilon_a + T_{ai\nu\rho}^\alpha \gamma^\nu \gamma^\rho \epsilon^a; \\
\delta \chi_{abi} = -P_{abc,\alpha} \partial_\mu \phi^{\alpha} \gamma^\mu \epsilon^c + \frac{2}{3} T_{abc}^{\mu\nu} \gamma^{\mu} \epsilon^d \end{cases} \tag{3.23} \]

in the truncated sector.

By then imposing (3.19) on (3.23) and (3.24), one obtains

\[ \omega_i^a = 0 = T_{ia}^-= P_{abc,\alpha} \partial_\mu \phi^{\alpha} \gamma^\mu \epsilon^c. \tag{3.25} \]

In particular, \( T_{ia}^- = 0 \) is nothing but the condition (3.15).

Thus, one can conclude that the truncation (3.25) is fully consistent.

A similar analysis at the level of supersymmetry can be performed in all cases. We observe that whenever the truncation \( \mathcal{N} = 8 \rightarrow \mathcal{N}' < 8 \) is consistent with supersymmetry, and thus it actually exists, there occurs an \( SU(8-\mathcal{N}') \) factor commuting with the the \( \mathcal{R} \)-symmetry \( U(\mathcal{N}) \) of the truncated sub-theory inside the \( \mathcal{N} = 8 \) \( \mathcal{R} \)-symmetry \( SU(8) \).

### 3.2 \( \rightarrow \mathcal{N} = 5 \rightarrow \mathcal{N} = 3, 2 \)

Next, we consider the maximal non-symmetric embedding:

\[ J_3^{D_8} : \quad \mathcal{N} = 8 \rightarrow M_{1,2} (\mathbb{O}) : \mathcal{N} = 5; \tag{3.26} \]
\[ E_7(7) \supset SU (1,5) \times SU (3); \tag{3.27} \]
\[ 56 = (6,3) + (6,\bar{3}) + (20,1); \tag{3.28} \]
\[ E_7(7) \quad SU (8) \supset SU (1,5) \tag{3.29} \]

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$M_{1,2}(\mathbb{O})$ is the Jordan triple system (not upliftable to $D = 5$) generated by $2 \times 1$ matrices over $\mathbb{O}$ \cite{12}. The $20$ is the rank-3 antisymmetric self-real irrep. of $SU(1,5)$. The commuting $SU(3)$ factor can be interpreted as the part of the $R$-symmetry truncated away in the supersymmetry reduction $N = 8 \rightarrow N = 5$. On the two-form Abelian field strengths’ fluxes, the truncation condition reads

$$\langle 6,3 \rangle = 0.$$  \hspace{1cm} (3.30)

As discussed in Sec. 8 of \cite{37}, the quartic invariant of the $R = 20$ of $SU(1,5)$, after skew-diagonalization in the scalar-dressed $R$-symmetry $U(5)$-basis and use of the Hua-Bloch-Messiah-Zumino Theorem \cite{43}, is a perfect square. On this respect the couples $(SU(1,5), R = 20)$ and $(SL(2,\mathbb{R}) \times SO(6), R = (2,6))$ (this latter pertaining to $N = 4$ “pure” supergravity) stand on a particular footing among simple and respectively semisimple groups “of type $E_7$” \cite{14}. Thus, this embedding does not concern a proper “degeneration” of a group of type $E_7$, but it is however noteworthy.

In turn, the “pure” $N = 5$ theory admits two maximal “degenerative” truncations, which precisely match the kinematical decomposition of the $N = 5$ gravity multiplet into matter $N = 2$ multiplets.

1. The first reads:

$$N = 5 \rightarrow N = 3, \; n_V = 1 \text{ “twin” } N = 2 \mathbb{C}P^3;$$ \hspace{1cm} (3.31)

$$SU(1,5) \supset SU(1,3) \times SU(2) \times U(1);$$ \hspace{1cm} (3.32)

$$20 = (4,1)_+ + (\overline{7},1)_{-3} + (6,2)_0;$$ \hspace{1cm} (3.33)

$$SU(1,5) \over U(5) \supset SU(1,3) \over U(3).$$ \hspace{1cm} (3.34)

and it admits two possible interpretations, due to the fact that $N = 3$ supergravity coupled to 1 vector multiplet and $N = 2$ supergravity minimally coupled to 3 vector multiplets share the very same bosonic sector (namely, they are “twin” theories; see the discussion in Sec. 9 of \cite{37}). In the $N = 3$ interpretation, one gets a theory with 1 vector multiplets, and the $SU(2)$ commuting factor can be interpreted as the part of the $R$-symmetry truncated away in the supersymmetry reduction $N = 5 \rightarrow N = 3$. On the other hand, in the $N = 2$ interpretation, one gets a theory with 3 minimally coupled vector multiplets without hypermultiplets, and the $SU(2)$ commuting factor is the global $N = 2$ hyper $R$-symmetry. In both cases, on the two-form Abelian field strengths’ fluxes the truncation condition reads

$$\langle 6,2 \rangle_0 = 0.$$ \hspace{1cm} (3.35)

One can also prove that the quartic invariant of the $R = 20$ of $SU(1,5)$, under the truncation \cite{33} becomes the square of the quadratic invariant of the $R = 4$ of $SU(1,3)$.

2. The second maximal “degenerative” truncation of the “pure” $N = 5$ theory reads

$$N = 5 \rightarrow N = 2, \; n_V = 0, \; n_H = 1;$$ \hspace{1cm} (3.36)

$$SU(1,5) \supset SU(1,2) \times SU(3) \times U(1);$$ \hspace{1cm} (3.37)

$$20 = (1,1)_+ + (1,1)_{-3} + (3,3)_+ + (\overline{3},3)_+;$$ \hspace{1cm} (3.38)

$$SU(1,5) \over U(5) \supset SU(1,2) \over U(2).$$ \hspace{1cm} (3.39)

The $N = 2$ theory is coupled to the universal hypermultiplet, in absence of vector multiplets. The $SU(3)$ commuting factor can be interpreted as the part of the $R$-symmetry truncated away in the supersymmetry reduction $N = 5 \rightarrow N = 2$, whereas the commuting $U(1)$ factor is the global $N = 2$ vector $R$-symmetry. On the two-form Abelian field strengths’ fluxes, the truncation condition reads

$$\langle 3,\overline{3} \rangle_{-1} = 0.$$  \hspace{1cm} (3.40)
such that only the graviphoton charges \((\mathbf{1}, \mathbf{1}),_{+3} + (\mathbf{1}, \mathbf{1}),_{-3}\) survive the truncation. One can also prove that the \textit{quartic} invariant of the \(R = 20\) of \(SU(1, 5)\), under the truncation \((3.40)\) becomes nothing but the \textit{square} of the Reissner-Nördstrom entropy

\[
\frac{S_{RN}}{\pi} = \frac{1}{2} \left[(p^0)^2 + q_0^2\right].
\]  

(3.41)

It is here worth pointing out that a consistent truncation to an hypermultiplet(s)-coupled \(\mathcal{N} = 2\) theory with no vector multiplets should necessarily contain two real singlets (namely the electric and magnetic charge of the graviphoton) in the branching of the original flux representation, as it holds \textit{e.g.} for \((3.38)\) and \((3.70)\) respectively pertaining to truncations \((3.36)\) and \((3.68)\). However, such truncations are not interesting for our investigation, because they yield no vectors when further reduced down to \(\mathcal{N} = 1\) models (the \(\mathcal{N} = 2\) graviphoton is contained in the \(\mathcal{N} = 1\) gravitino multiplet, which is truncated away).

### 3.3 \(\mathcal{N} = 4\) \(\mathbb{R} \oplus \Gamma_{5, 5}\)

Let’s consider now the embedding:

\[
\begin{align*}
J_3^{\mathcal{O}_3} : & \quad \mathcal{N} = 8 \longrightarrow \mathbb{R} \oplus \Gamma_{5, 5} : \mathcal{N} = 4, \ n_V = 6; \\
E_7(7) & \supset SL(2, \mathbb{R}) \times SO(6, 6); \\
56 & = (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}); \\
E_7(7) & \supset SL(2, \mathbb{R}) \times U(1) \times SO(6, 6) \times SO(6).
\end{align*}
\]  

(3.42)

(3.43)

The \(\mathcal{N} = 4\) theory is coupled to 6 vector multiplets and, on the two-form Abelian field strengths’ fluxes, the truncation condition reads

\[
(\mathbf{1}, \mathbf{32}) = 0. 
\]  

(3.44)

It still exhibits a \textit{quartic} \(U\)-invariant \(I_4\), but it can be further truncated to a theory with \(U\)-duality group \(U(3, 3)\) with \textit{quadratic} invariant.

### 3.4 \(\mathcal{N} = 2\)

We now consider the reduction of \(\mathcal{N} = 8\) supergravity to an \(\mathcal{N} = 2\) theory with \(n_V\) vector and \(n_H\) hypermultiplets:

\[
(n_V, n_H) \equiv \left( \dim \left( \frac{G_V}{H_V} \right), \ dim \left( \frac{G_H}{H_H} \right) \right), \ n_V \leq 15, \ n_H \leq 20,
\]  

(3.45)

where \(\frac{G_V}{H_V}\) and \(\frac{G_H}{H_H}\) respectively stand for the special Kähler and quaternionic Kähler scalar manifolds, where \(H_V = \text{mcs}(G_V)\) and \(H_H = \text{mcs}(G_H)\). \(H_V\) always contains a factorized commuting \(U(1)\) subgroup, which is promoted to global symmetry when \(n_V = 0\); on the other hand, \(H_H\) always contains a factorized commuting \(SU(2)\) subgroup, which is promoted to global symmetry when \(n_H = 0\) \cite{14}.

We consider only \(\mathcal{N} = 2\) \textit{maximal} supergravities, \textit{i.e.} \(\mathcal{N} = 2\) theories (obtained by consistent truncations of \(\mathcal{N} = 8\) supergravity) which cannot be obtained by a further reduction from some other \(\mathcal{N} = 2\) theory, which are also \textit{magic}. They are called \textit{magic}, since their symmetry groups are the groups of the famous \textit{Magic Square} of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries \cite{15} \cite{16}. From the analysis performed in \cite{8}, only six \(\mathcal{N} = 2, \ d = 4\) \textit{maximal magic} supergravities\footnote{By \(E_7(p)\) we denote a non-compact form of \(E_7\), where \(p \equiv (\# \text{ non-compact} - \# \text{ compact})\) generators of the group \cite{22} \cite{21}. In such a notation, the compact form of \(E_7\) is \(E_7(-133)\) \((\dim \mathbb{R}E_7 = 133)\).} exist which can be obtained by consistently truncating \(\mathcal{N} = 8, \ d = 4\) supergravity;
Table 5: $\mathcal{N}=2$ supergravities obtained as consistent maximal truncation of $\mathcal{N}=8$ supergravity

| $J^H_3$ | $SO^*(12)$ | $SU(2)$ | $SU(6) \times U(1)$ | $-$ | $\frac{SO^*(12)}{SU(6) \otimes SU(1)}$ | $(15, 0)$ |
| $J^C_3$ | $SU(3, 3)$ | $SU(2, 1)$ | $SU(3) \times SU(3) \times U(1)$ | $SU(2) \times U(1)$ | $\frac{SU(3, 3)}{SU(3) \times U(1)} \times \frac{SU(2, 1)}{SU(2) \times U(1)}$ | $(9, 1)$ |
| $J^R_3$ | $Sp(6, \mathbb{R})$ | $G_2(2)$ | $SU(3) \times U(1)$ | $SU(2) \times SU(2)$ | $\frac{Sp(6, \mathbb{R})}{SU(3) \times U(1)} \times \frac{G_2(2)}{SO(4)}$ | $(6, 2)$ |
| $STU$ | $SU(1, 1) \times SO(4, 4)$ | $U(1) \times SO(2) \times SO(2)$ | $SO(4) \times SO(4)$ | $\frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SO(4, 4)}{SO(4) \otimes SO(4)}$ | $(3, 4)$ |
| $J^R_{3,M}$ | $SU(1, 1)$ | $SU(6) \times SU(2)$ | $USp(6) \times SU(2)$ | $\frac{SU(1, 1)}{U(1)} \times \frac{USp(6)}{USp(6)} \times \frac{SU(6)}{SU(6) \otimes SU(2)}$ | $(1, 7)$ |
| $J^E_{3,M}$ | $U(1)$ | $E_6(2)$ | $-$ | $SU(6) \times SU(2)$ | $\frac{E_6(2)}{SU(6) \otimes SU(2)}$ | $(0, 10)$ |

they are given by Table 5. After [47], we also include the case of $STU$ model [48, 49, 50] with $n_H = 4$ hypermultiplets; see below.

The models have been denoted by referring to their special geometry. $J^H_3$, $J^C_3$ and $J^R_3$ stand for three of the four $\mathcal{N}=2$, $d=4$ magic supergravities which, as their 5-dim. versions, are respectively defined by the three simple Jordan algebras $J^H_3$, $J^C_3$ and $J^R_3$ of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian $3 \times 3$ matrices over the division algebras of quaternions $\mathbb{H}$, complex numbers $\mathbb{C}$ and real numbers $\mathbb{R}$ [42, 51, 52, 53, 54].

In Table 5, the subscript “$M$” denotes the model obtained by performing a $D=4$ mirror map (i.e. the composition of two $c$-maps [55] in $D=4$) from the original manifold; such an operation maps a model with content $(n_V, n_H)$ to a model with content $(n_H - 1, n_V + 1)$, and thus the mirror $J^R_{3,M}$ of $J^R_3$, with $(n_V, n_H) = (-1, 16)$ and quaternionic manifold $\mathbb{E}_7(-5)$, does not exist, at least in $D=4$. The $STU$ model is self-mirror: $STU = STU_M$. 

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3.4.1 Further Truncation to Minimal Coupling

Then, we consider further truncations to \( \mathcal{N} = 2 \) theories exhibiting scalar-vector minimal coupling; since hyperscalars are always minimally coupled, we study only truncations of the vector multiplets’ scalar sector.

Out of the cases reported in Table 5, some deserve immediate comments:

- The case pertaining to the self-mirror \( STU_M \) model is included in the treatment of Sec. 6 starting from \( \mathcal{N} = 4 \) theory coupled to \( n = 6 \) vector multiplets (which in turn is maximally embedded into \( \mathcal{N} = 8 \) theory), and considering the splitting \((n_1, n - n_1) = (2, 4)\).

- The case pertaining to the mirror model \( J_{3,M}^R \) is not interesting in our investigation: indeed, in the vector multiplets sector, \( J_{3,M}^R \) is nothing but the so-called \( \mathcal{N} = 2 \) \( T^3 \) model, in which the complex scalar field \( T \) is not minimally coupled to vectors, and no further truncation to minimally coupled \( \mathcal{N} = 2 \) or \( \mathcal{N} = 1 \) models is possible.

Let’s now list the various relevant possibilities from the models reported in Table 5:

1. Also by recalling the treatment of Subsec. 3.1 (and in particular Eqs. (3.10) and (3.12)), one can consider:

\[
J_{3}^{\Omega}: \quad \mathcal{N} = 8 \rightarrow J_{3}^{\mathcal{H}}: \left\{ \begin{array}{l}
\mathcal{N} = 2 \quad (n_V, n_H) = (15, 0) \\
\mathcal{N} = 6 \\
\mathcal{N} = 4
\end{array} \right.
\]

\[
\begin{array}{l}
\rightarrow \mathbb{R} \oplus \Gamma_{1,5}
\end{array}
\]

\[
N = 2 \quad (n_V, n_H) = (7, 0)
\]

\[
\overset{\text{“twin”}}{\mathcal{N} = 4}
\]

\[
E_{7(7)} \supset SO^* (12) \times SU (2)
\]

\[
\supset SO^* (8) \times SO^* (4) \times SU (2) \sim SO (6, 2) \times SL (2, \mathbb{R}) \times SU (2) \times SU (2)
\]

\[
56 = (32, 1) + (12, 2)
\]

\[
= (8_s, 2, 1, 1) + (8_c, 1, 2, 1) + (1, 2, 2, 2) + (8_v, 1, 1, 2)
\]

\[
E_{7(7)} \left/ SU (8) \right. \supset \frac{SO^* (12)}{SU (6) \times U (1)} \supset \frac{SL (2, \mathbb{R})}{U (1)} \times \frac{SO (6, 2)}{SO (6) \times SO (2)}
\]

The \( J_{3}^{\mathcal{H}} \)-based theory can either be interpreted as \( \mathcal{N} = 2 \) or as its “twin” \( \mathcal{N} = 6 \); in the former case, the \( SU (2) \) commuting factor is the global hyper \( R \)-symmetry, whereas in the latter case we already mentioned that the \( R \)-symmetry truncated away in the supersymmetry reduction \( \mathcal{N} = 8 \rightarrow \mathcal{N} = 6 \) (a further truncation \( \mathcal{N} = 6 \rightarrow \mathcal{N} = 3 \) is considered in Subsec. 4.1). In both cases, the truncation condition on the two-form Abelian field strengths’ fluxes is given by (3.15). Thence, one can proceed by truncating to the \( \left( \mathbb{R} \oplus \Gamma_{1,5} \right) \)-based theory still enjoys a “twin”\( \mathcal{N} = 2 \) or \( \mathcal{N} = 4 \) supergravity; in the former case, the second \( SU (2) \) commuting factor also be interpreted as the global hyper \( R \)-symmetry, whereas in the latter case it is the \( R \)-symmetry truncated away in the supersymmetry reduction \( \mathcal{N} = 6 \rightarrow \mathcal{N} = 4 \). In both cases, the truncation condition is

\[
(8_c, 1, 2, 1) = 0 \quad \text{or} \quad (8_s, 2, 1, 1) = 0.
\]

The resulting theory still exhibits a quartic \( U \)-invariant \( \mathcal{L}_4 \), but it can be further truncated to a theory with \( U \)-duality group \( U (1, 3) \) with quadratic invariant. It is here worth remarking that
such a theory still admits a “twin” interpretation\(^\text{[37]}\), namely either as \(\mathcal{N} = 3\) with \(n_V = 1\) vector multiplet or as \(\mathcal{N} = 2\) minimally coupled to \(n_V = 3\) vector multiplets (and no hypermultiplets).

2. \(J_3^{O_{}\alpha} : \mathcal{N} = 8 \rightarrow J_3^{C} : \mathcal{N} = 2\) \((n_V, n_H) = (9, 1) \rightarrow \mathcal{N} = 2\) \(\mathbb{CP}^3\) \((n_V, n_H) = (4, 1)\); (3.51)

\[
E_{7(7)} \supset SU(3, 3) \times SU(2, 1) \supset SU(1, 3) \times SU(2) \times SU(2, 1) \times U(1) ;
\]

\[
56 = (6, 3) + (\overline{6}, \overline{3}) + (20, 1)
\]

\[
= (1, 2, 3)_2 + (4, 1, 3)_{-1} + (1, 2, \overline{3})_{-2} + (\overline{4}, 1, \overline{3})_1 + (4, 1, 0)_{+3} + (\overline{4}, 1, 0)_{-3} + (6, 2)_0 ;
\]

\[
\frac{E_{7(7)}}{SU(8)} \supset \frac{SU(3, 3)}{SU(3) \times U(3)} \times \frac{SU(2, 1)}{U(2)} \supset \frac{SU(1, 3)}{U(3)} \times \frac{SU(2, 1)}{U(2)}. \quad (3.54)
\]

The \(J_3^{C}\)-based theory is magic \(\mathcal{N} = 2\) with 9 vector multiplets and 1 universal hypermultiplet. The truncation condition reads

\[
(6, 3) = 0. \quad (3.55)
\]

A different realization of this truncation has been studied in Subsec. 3.2. Thence, one can proceed by truncating to \(\mathcal{N} = 2\) minimally coupled to 3 vector multiplets (hyper sector untouched); the further truncation condition is

\[
(6, 2)_0 = 0. \quad (3.56)
\]

Through this chain of truncation, one can also prove that the quartic invariant of the \(R = 20\) of \(SU(3, 3)\) becomes the square of the quadratic invariant of the \(R = 4\) of \(SU(1, 3)\).

3. From \(\mathcal{N} = 2\) \(J_3^{C}\) theory another truncation is possible, namely:

\[
J_3^{O_{}\alpha} : \mathcal{N} = 8 \rightarrow J_3^{C} : \mathcal{N} = 2\) \((n_V, n_H) = (9, 1) \rightarrow \mathcal{N} = 2\) \(\mathbb{R} \oplus \Gamma_{1, 3}\) \((n_V, n_H) = (5, 1)\); (3.57)

\[
E_{7(7)} \supset SU(3, 3) \times SU(2, 1)
\]

\[
\supset SU(1, 1) \times SU(2, 2) \times SU(2, 1) \times U(1) \sim SL(2, \mathbb{R}) \times SO(2, 4) \times SU(2, 1) \times U(1) ;
\]

\[
56 = (6, 3) + (\overline{6}, \overline{3}) + (20, 1)
\]

\[
= (2, 1, 3)_2 + (1, 4, 3)_{-1} + (1, 2, \overline{3})_{-2} + (\overline{4}, 1, \overline{3})_1 + (1, 4, 1)_{+3} + (1, \overline{4}, 1)_{-3} + (2, 6)_0 ;
\]

\[
\frac{E_{7(7)}}{SU(8)} \supset \frac{SU(3, 3)}{SU(3) \times U(3)} \times \frac{SU(2, 1)}{U(2)} \supset \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(2, 4)}{SO(2) \times SO(4)} \times \frac{SU(2, 1)}{U(2)}. \quad (3.60)
\]

As for the point 2 above, the first truncation condition is given by \(3.55\), but the second one is the very opposite of \(3.56\): only \((2, 6)_0\) does not vanish, or equivalently:

\[
(1, 4, 1)_3 = 0. \quad (3.61)
\]

The resulting theory still exhibits a quartic \(U\)-invariant \(\mathcal{I}_4\), but it can be non-maximally further truncated to an \(\mathcal{N} = 2\) \(\mathbb{CP}^2\) model with quadratic invariant through the procedure considered in Sec. 8.3 to which we address the reader for further elucidation.
4.

\[ J_3^{\mathbb{O}^*} : \mathcal{N} = 8 \rightarrow J_3^{\mathbb{R}} : \mathcal{N} = 2 \quad (n_V, n_H) = (6, 2) \rightarrow \mathcal{N} = 2 \ \mathbb{R} \oplus \Gamma_{1,2} \quad (n_V, n_H) = (4, 2) ; \]

\[ E_7(7) \supset S p (6, \mathbb{R}) \times G_{2(2)} \supset S p (2, \mathbb{R}) \times S p (4, \mathbb{R}) \times G_{2(2)} \sim S L (2, \mathbb{R}) \times S O (2, 3) \times G_{2(2)} ; \]

\[ 56 = (14', 1) + (6, 7) = (1, 4, 1) + (2, 5, 1) + (2, 1, 7) + (1, 4, 7) ; \]

\[ E_7(7) \supset E_6(2) \times U (1) \supset U (1) ; \]

\[ SU (8) \quad 56 = 27 + 1 + 27' - 1 + 1_{+3} + 1'_{-3} ; \]

\[ E_7(7) \supset SU (8) \supset SU (6) \times SU (2) . \]

The resulting theory still exhibits a quartic \( U \)-invariant \( I_4 \), but it can be non-maximally further truncated to an \( \mathcal{N} = 2 \ \mathbb{CP}^1 \) model with quadratic invariant through the procedure considered in Sec. 8.3 (see also comment in Subsec. 8.3.1), to which we address the reader for further elucidation.

5.

\[ J_3^{\mathbb{O}^*} : \mathcal{N} = 8 \rightarrow J_3^{\mathbb{C}^*} : \mathcal{N} = 2 \quad (n_V, n_H) = (0, 10) ; \]

\[ E_7(7) \supset E_6(2) \times U (1) \supset U (1) ; \]

\[ 56 = 27 + 1 + 27' - 1 + 1_{+3} + 1'_{-3} ; \]

\[ SU (8) \quad 27 + 1 = 0, \]

such that only the graviphoton charges \( 1_{+3} + 1'_{-3} \) survive the truncation. One can also prove that the quartic invariant of the \( R = 56 \) of \( E_7(7) \), under the truncation (3.72) becomes nothing but the square of the Reissner-Nordstrom entropy (3.41).
4 Maximal Truncations from $\mathcal{N} = 6$ ($J_3^H$)

4.1 $\rightarrow \mathcal{N} = 3$

From $\mathcal{N} = 6$ “pure” theory, one can consider the following maximal “degenerate” truncation:

$$J_3^H : \mathcal{N} = 6 \rightarrow \mathcal{N} = 3, \ n_V = 3; \quad (4.1)$$

$$SO^* (12) \supset SU (3, 3) \times U (1); \quad (4.2)$$

$$32 = 6_{-2} + \overline{6}_{+2} + 20_0; \quad (4.3)$$

$$\frac{SO^* (12)}{SU (6) \times U (1)} \supset \frac{SU (3, 3)}{SU (3) \times SU (3) \times U (1)}. \quad (4.4)$$

The $\mathcal{N} = 3$ theory is coupled to 3 vector multiplets and, on the two-form Abelian field strengths’ fluxes, the truncation condition reads

$$20_0 = 0. \quad (4.5)$$

One can prove that the quartic invariant of the $\mathbf{R} = 32$ of $SO^* (12)$, under the truncation (4.5) becomes the square of the quadratic invariant of the $\mathbf{R} = 6$ of $SU (3, 3)$. 

4.2 $\mathcal{N} = 5$

Note that, one might consider another truncation by setting

$$6_{-2} = 0 \quad (4.6)$$

in (4.3); this corresponds to a truncation $\mathcal{N} = 6 \rightarrow \mathcal{N} = 2$ based on $J_3^H$ or, equivalently (due to the fact that $\mathcal{N} = 6$ and $\mathcal{N} = 2$ based on $J_3^H$ are “twin”, i.e. they share the very same bosonic sector $[35, 59, 37, 60]$) to $\mathcal{N} = 2 \ J_3^H \rightarrow \mathcal{N} = 2 \ J_3^C$. However, the resulting $\mathcal{N} = 2$ “magic” complex theory exhibits a generally “non-degenerate” quartic $U$-invariant $I_4$.

On the other hand, if in (4.2) $SU (3, 3)$ is changed into $SU (1, 5)$, another, complementary, realization of the above truncation reads

$$\mathcal{N}' = 6 \rightarrow \mathcal{N}' = 5; \quad (4.7)$$

$$SO^* (12) \supset SU (1, 5) \times U (1); \quad (4.8)$$

$$32 = 6_{-2} + \overline{6}_{+2} + 20_0; \quad (4.9)$$

$$\frac{SO^* (12)}{SU (6) \times U (1)} \supset \frac{SU (1, 5)}{U (5)}. \quad (4.10)$$

The $\mathcal{N}' = 5$ theory is “pure” and the commuting $U (1)$ factor corresponds to the part of the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N} = 6 \rightarrow \mathcal{N} = 5$. On the two-form Abelian field strengths’ fluxes, the truncation condition is

$$6_{-2} = 0. \quad (4.11)$$

In turn, the “pure” $\mathcal{N} = 5$ theory admits two maximal “degenerate” truncations, treated in Sec. 3.2 which precisely match the kinematical decomposition of the $\mathcal{N} = 5$ gravity multiplet into matter $\mathcal{N} = 2$ multiplets.

5 $\mathcal{N}' = 4 \ \mathbf{R} \oplus \Gamma_{5,2n-1} \rightarrow \mathcal{N} = 3$

We start with $\mathcal{N}' = 4$ supergravity coupled to $n_V = 2n$ matter (vector) multiplets, which is based on the rank-3 Jordan algebra $\mathbf{R} \oplus \Gamma_{1,2n-1}$, with data

$$\frac{G_4}{H_4} = \frac{SL_v (2, \mathbb{R}) \times SO (6, 2n)}{U (1) \times SO (6) \times SO (n)}, \quad (5.1)$$

$$\mathbf{R} = (2, 6 + n). \quad (5.2)$$
The relevant products of electric and magnetic charges read

\[ p^2 \equiv p^\Lambda p^\Sigma \eta_{\Lambda\Sigma} = \sum_{a=1}^{6} (p^a)^2 - \sum_{I=1}^{2n} (p^I)^2; \]
\[ q^2 \equiv q_{\Lambda\Sigma} q^{\Lambda\Sigma} = \sum_{a=1}^{6} q_a^2 - \sum_{I=1}^{2n} q_I^2; \quad (5.3) \]
\[ p \cdot q \equiv p^\Lambda q_{\Lambda}, \]

where \( \eta \) is the symmetric invariant structure of the vector (Fund) irrep. \( 6 + 2n \) of \( SO(6, 2n) \), with \( \Lambda = 1, \ldots, 2n + 6 \), where the indices \( 1, \ldots, 6 \) pertain to the 6 graviphotons.

We consider a complexification of the electric and magnetic charge vectors \( p^\Lambda \) and \( q_{\Lambda} \) as follows:

\[
\begin{cases}
  P^1 = p^1 + ip^2; \\
  P^2 = p^3 + ip^4; \\
  P^3 = p^5 + ip^6; \\
  P^4 = p^7 + ip^8; \\
  \ldots \\
  P^{n+3} = p^{2n+5} + ip^{2n+6},
\end{cases}
\]

and analogously for the electric charges. Thus (5.3) can be rewritten as

\[ p^2 = \sum_{A=1}^{3} |P_A|^2 - \sum_{A=4}^{n+3} |P_A|^2 = P^T \overline{P} \eta_{\overline{7}}; \quad (5.5) \]
\[ q^2 = \sum_{A=1}^{3} |Q_A|^2 - \sum_{A=4}^{n+3} |Q_A|^2 = \eta^7 Q_i \overline{Q}_j; \quad (5.6) \]
\[ p \cdot q = \sum_{i=1}^{n+3} \text{Re} \left( P^i \overline{Q}_i \right); \quad (5.7) \]

with \( \eta \) here denoting the invariant rank-2 structure in the product \( (3 + n) \times (3 + n) \) of \( U(3, n) \), with \( i = 1, \ldots, n + 3 \) (in Sec. 2 the complex charge vector \( (P^n, Q_i) \) has been indicated by \( Q \)). Therefore:

\[ \frac{1}{4} T_{4, \mathbb{R} \in \Gamma_{5, 2n-1}} = p^2 q^2 - (p \cdot q)^2 = \eta_{\overline{7}} \eta_7 P^T \overline{P} Q_k \overline{Q}_k - \left( \sum_{i=1}^{n+3} \text{Re} \left( P^i \overline{Q}_i \right) \right)^2 \]
\[ = \frac{1}{4} \left( S_1^2 - |S_2|^2 \right), \quad (5.9) \]

where the following quantities have been introduced [35 [56]:

\[ S_1 \equiv p^2 + q^2 = (P^T \overline{P} + Q^T \overline{Q}) \eta_{\overline{7}}; \quad (5.11) \]
\[ S_2 \equiv (p^2 - q^2) + 2ip \cdot q = (P^T \overline{P} - Q^T \overline{Q}) \eta_{\overline{7}} + 2i \sum_{i=1}^{n+3} \text{Re} \left( P^i \overline{Q}_i \right). \quad (5.12) \]

The “degeneration” condition we exploit reads as follows:

\[ S_2 = 0 \iff \begin{cases} 
  \text{Re} S_2 = 0 \iff \left( P^T \overline{P} - Q^T \overline{Q} \right) \eta_{\overline{7}} = 0; \\
  \text{Im} S_2 = 0 \iff \sum_{i=1}^{n+3} \text{Re} \left( P^i \overline{Q}_i \right) = 0, \end{cases} \]
whose a solution is
\[ Q_j = \pm iP_j \quad \forall j, \]  
with \( j \)-dependent “±” branches. One thus obtains:
\[ I_{4,R\oplus \Gamma_{5,2n-1}}|_{S_2=0} = (S^1)^2 = 4\left(P^\alpha \bar{T}^j \eta_{ij}\right)^2 = (I_{2,N=3})^2. \]  

Namely, the quartic invariant \( I_{4,R\oplus \Gamma_{5,2n-1}} \) of the real irrep. \( R = (2, 6 + 2n) \) of the semisimple group of type \( E_7 \) \( G_4 = SL_v(2, \mathbb{R}) \times SO(6, 2n) = \text{Conf}(\mathbb{R} \oplus \Gamma_{5,2n-1}) \) “degenerates” into the square of the quadratic invariant \( I_{2,N=3} \) of the complex irrep. \( R' = 3 + n \) of the “degenerate” group of type \( E_7 \) \( G'_4 = U(3, n) \). This latter is the U-duality group of \( \mathcal{N} = 3 \) supergravity coupled to \( n \) vector multiplets.

In a manifestly \( U(3, n) \)-covariant symplectic basis, \( I_{2,N=3} \) reads:
\[ I_{2,N=3} = 3 \sum_{\alpha=1}^{\alpha} \left[ (p^\alpha)^2 + q_{\alpha}^2 \right] - \sum_{\alpha=1}^{\alpha} \left( (p^\alpha)^2 + q_{\alpha}^2 \right). \]  

In order to make (5.10) consistent with (5.15), the following dyonic identification of charges can be performed:
\[ P^A = \frac{1}{\sqrt{2}} (p^\alpha + i q_{\alpha}) ; \]
\[ P^A = \frac{1}{\sqrt{2}} (p^\alpha - i q_{\alpha}). \]  

In group-theoretical terms, the “degeneration” procedure under consideration goes as follows:
\[ SL_v(2, \mathbb{R}) \times SO(6, 2n) \supset SL_v(2, \mathbb{R}) \times U(3, n) \supset U(3, n) ; \]
\[ (2, 6 + 2n) = (2, (3 + n)_{+1}) + (2, (3 + n)_{-1}) \]
\[ = 2: \left[ (3 + n)_{+1} + (3 + n)_{-1} \right]. \]  

with the double-counting eventually removed by the “degeneration” truncating condition (5.13)–(5.14), which in this case sets to zero \( n + 3 \) complex, i.e., \( 2n + 6 \) real, charge combinations. Notice that, also in this case, (5.14) breaks \( SL_v(2, \mathbb{R}) \), and its various branches, generated by the various possibilities in the choice of “±” for each index \( i \), are all inter-related by suitable \( U(3, n) \)-transformations. At the level of the vector multiplets’ scalar manifolds, it holds
\[ \frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(6, 2n)}{SO(2) \times SO(2n)} \supset \frac{SU(3, n)}{U(3) \times SU(n)} \]  

6 \( \mathcal{N} = 4 \mathbb{R} \oplus \Gamma_{5,n-1} \longrightarrow \mathcal{N} = 2 \mathbb{R} \oplus \Gamma_{1,n-1} + \text{Hypermultiplets} \)

\( \mathcal{N} = 2 \) hypermultiplets can be added to the “degenerative” truncation procedures (starting from the \( \mathcal{N} = 2 \) factorized sequence) treated above, by considering the following truncation:
\[ \mathcal{N} = 4 \mathbb{R} \oplus \Gamma_{5,n-1} \longrightarrow \mathcal{N} = 2 \mathbb{R} \oplus \Gamma_{1,n-1} + (n - n_1) \text{ hypermults.} \]  
\[ SL_v(2, \mathbb{R}) \times SO(6, n) \supset SL_v(2, \mathbb{R}) \times SO(2, n_1) \times SO(4, n - n_1); \]
\[ (2, 6 + n) = (2, 2 + n_1, 1) + (2, 1, 4 + n - n_1), \]  
where the hyperscalars fit into the quaternionic Kähler symmetric space
\[ \frac{SO(n-n_1)}{SO(4) \times SO(n-n_1)}. \]  

23
Thus, the \( \mathcal{N} = 2 \) theory is obtained by setting

\[
(2, 1, 4 + n - n_1) = 0.
\]  

(6.5)

At the level of the scalar manifolds, the truncation (6.11)-(6.3) corresponds to
\[
\begin{align*}
\frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(6, n)}{SO(6) \times SO(n)} & \supset \frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n_1)}{SO(2) \times SO(n_1)} \times \frac{SO(4, n-n_1)}{SO(4) \times SO(n-n_1)}.
\end{align*}
\]  

(6.6)

It is worth recalling that the case \( n = 0 \) of the truncation (6.6) has been considered in Sec. 5 of [58] (see also the considerations in Subsec. 8.3).

Starting from the \( \mathcal{N} = 2 \) theory with \( n_1 \) vector multiplets and \( n - n_1 \) hypermultiplets, with scalar manifolds given by the direct product on the righthand side of (6.6), \( n_1 \) is even (i.e. \( n_1 = 2m \)) one can then consider the further "degenerative" truncation down to \( \mathcal{N} = 2 \) minimally coupled supergravity with \( m \) vector multiplets and \( n - n_1 = n - 2m \) hypermultiplets: in practice, the procedure outlined in Subsec. 8.3 with \( n \to m \), and the hypermultiplets which are insensitive of the truncation:
\[
\begin{align*}
\frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n_1)}{SO(2) \times SO(n_1)} \times \frac{SO(4, n-n_1)}{SO(4) \times SO(n-n_1)} & \supset \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)} \times \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)} \times \frac{SU(2, k)}{SU(2) \times SU(k) \times U(1)}.
\end{align*}
\]  

(6.7)

Let’s finally mention that the quaternionic manifolds (6.3) are maximal in the framework under consideration, but, \( n - n_1 \) is even (i.e. \( n - n_1 = 2k \)) the further following truncation in the hyper sector can be considered:
\[
\begin{align*}
\frac{SO(4, n-n_1)}{SO(4) \times SO(n-n_1)} & \supset \frac{SU(2, k)}{SU(2) \times SU(k) \times U(1)} \times \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)} \times \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)}.
\end{align*}
\]  

(6.8)

Thus, by combining the two above observations, \( n_1 = 2m; 
\]

\[
\begin{align*}
n - n_1 = 2k;
\end{align*}
\]  

(6.9)

one can consider, along the very same lines of Subsec. 8.3 the following further non-maximal "degenerative" truncation down to \( \mathcal{N} = 2 \) minimally coupled supergravity with \( m \) vector multiplets and \( k \) hypermultiplets:
\[
\begin{align*}
\frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n_1)}{SO(2) \times SO(n_1)} \times \frac{SO(4, n-n_1)}{SO(4) \times SO(n-n_1)} & \supset \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)} \times \frac{SU(1, m)}{SU(2) \times SU(k) \times U(1)} .
\end{align*}
\]  

(6.10)

### 7 \( \mathcal{N} = 3 \rightarrow \mathcal{N} = 2 \) \( \mathbb{CP}^n \) + Hypermultiplets

Finally, let us consider the following truncation:
\[
\begin{align*}
\mathcal{N} & = 3 \text{ p vector mults.} \rightarrow \mathcal{N} = 2 \mathbb{CP}^{s_1} + (p - s_1) \text{ hypermults.}
\end{align*}
\]  

(7.1)

\[
\begin{align*}
U(3, p) & \supset U(1, s_1) \times SU(2, p - s_1) \times U(1); \\
(3 + n) & = (1 + s_1) + (2 + p - s_1).
\end{align*}
\]  

(7.2)

(7.3)

which, at the level of scalar manifolds corresponds to the following maximal embedding:
\[
\begin{align*}
\frac{SU(3, p)}{SU(3) \times SU(p) \times U(1)} & \supset \frac{SU(1, s_1)}{U(s_1)} \times \frac{SU(2, p - s_1)}{SU(2) \times SU(p - s_1) \times U(1)}.
\end{align*}
\]  

(7.4)
Thus, the $\mathcal{N} = 2$ minimally coupled theory is obtained by setting

$$\mathbf{(2 + p - s_1)} = 0. \quad (7.5)$$

Notice that the starting $\mathcal{N} = 3$ theory can be seen to be obtained from $\mathcal{N} = 4$ theory coupled to $2p$ matter (vector) multiplets through the “degenerative” truncation procedure outlined in Subsec. 5, with $n \rightarrow p$.

8 Maximal Truncations within $\mathcal{N} = 2$

8.1 $J_3^\mathcal{O} \rightarrow \mathbb{R} \oplus \Gamma_{1,9}$ (FHSV)

$$J_3^\mathcal{O} : \quad \mathcal{N} = 2, \quad n_V = 27 \longrightarrow \mathcal{N} = 2 \mathbb{R} \oplus \Gamma_{1,9} \quad n_V = 11; \quad (8.1)$$

$$E_{7(-25)} \supset SL(2, \mathbb{R}) \times SO(2, 10); \quad (8.2)$$

$$56 = (2, 12) + (1, 32); \quad (8.3)$$

$$\frac{E_{7(-25)}}{E_6 \times U(1)} \supset \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(2, 10)}{SO(2) \times SO(10)}. \quad (8.4)$$

The truncation condition reads

$$(1, 32) = 0. \quad (8.5)$$

The resulting theory, the so-called $\mathcal{N} = 2$ FHSV model [61], still exhibits a quartic $U$-invariant $I_4$, but it can be non-maximally further truncated to an $\mathcal{N} = 2$ CP$^5$ model with quadratic invariant through the procedure considered in Sec. 8.3 to which we address the reader for further elucidation. Note that this case, as well as the cases treated at points 1, 3 and 4 of Sec. 3.4, is based on the maximal (symmetric) Jordan algebraic embedding (see e.g. [62]):

$$J_3^A \supset J_3^A \oplus \mathbb{R}, \quad A = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}; \quad (8.6)$$

$$J_2^A \sim \Gamma_{1,q+1}, \quad q \equiv \dim \mathbb{R}A = 8, 4, 2, 1. \quad (8.7)$$

8.2 $J_3^\mathcal{O} \longrightarrow \mathbb{C}P^6$

Interestingly, the exceptional magic theory admits another relevant truncation:

$$J_3^\mathcal{O} : \quad \mathcal{N} = 2, \quad n_V = 27 \longrightarrow \mathcal{N} = 2 \mathbb{C}P^6; \quad (8.8)$$

$$E_{7(-25)} \supset SU(6, 2) \supset SU(6, 1) \times U(1); \quad (8.9)$$

$$56 = 56 = 28 + \overline{28} = 21_{+1} + 7_{-3} + \overline{21}_{-1} + \overline{7}_{+3}; \quad (8.10)$$

$$\frac{E_{7(-25)}}{E_6 \times U(1)} \supset \frac{SU(1, 6)}{U(6)}. \quad (8.11)$$

The $\mathcal{N} = 2$ theory is minimally coupled to 6 vector multiplets and, on the two-form Abelian field strengths’ fluxes, the truncation condition reads

$$21_{+1} = 0. \quad (8.12)$$

It can also be proved that the quartic invariant $I_4$ of the $\mathbf{R} = 56$ of $E_{7(-25)}$, under the truncation (8.12), becomes the square of the quadratic invariant of the $\mathbf{R} = 7$ of $SU(1, 6)$.
where $\infty$ is the symmetric invariant structure of the vector (Fund) irrep. $2n + 1$ of $SO (2, 2n)$, with $\Lambda = 0, 1, ..., 2n + 1$, where the indices “0” and “1” respectively pertain to the graviphoton and to the axio-dilatonic Maxwell field.

We consider a complexification of the electric and magnetic charge vectors $p^\lambda$ and $q_\Lambda$ as follows:

\[
\begin{align*}
P^1 &= p^1 + ip^1; \\
P^2 &= p^2 + ip^3; \\
... \\
P^{n+1} &= p^{2n} + ip^{2n+1},
\end{align*}
\]

and analogously for the electric charges. Thus (8.15) can be rewritten as

\[
\begin{align*}
p^2 &= |P^1|^2 - \sum_{A=2}^{n+1} |P^A|^2 = P^\lambda \bar{P}^\lambda \eta_{i\bar{j}}; \\
q^2 &= |Q_1|^2 - \sum_{A=2}^{n+1} |Q_A|^2 = \eta_{i\bar{j}} Q_i \bar{Q}_j; \\
p \cdot q &= \sum_{i=1}^{n+1} \text{Re} (P^i \bar{Q}_i),
\end{align*}
\]

with $\eta$ here denoting the invariant rank-2 structure in the product $(1 + n) \times (1 + n)$ of $U (1, n)$, with $i = 1, ..., n + 1$. Therefore:

\[
\begin{align*}
\frac{1}{4} \mathcal{I}_{4, \mathbb{R} \oplus \Gamma_{1,2n-1}} &= p^2 q^2 - (p \cdot q)^2 \\
&= \eta_{i\bar{j}} \eta_{k\bar{l}} P^i \bar{P}^\lambda Q_k \bar{Q}_\lambda - \left( \sum_{i=1}^{n+1} \text{Re} (P^i \bar{Q}_i) \right)^2 \\
&= \frac{1}{4} \left( S_1^2 - |S_2|^2 \right),
\end{align*}
\]

where, mutatis mutandis, $S_1^2$ and $S_2$ are given in (8.11)-(8.12) [35, 56].

By imposing the very same “degeneration” truncating condition (5.13)-(5.14), and evaluating (8.20)-(8.22) on (5.10)-(5.11), one obtains (in Sec. 2) the complex charge vector $(P^i, Q_i)$ has been indicated by $Q$:

\[
\mathcal{I}_{4, \mathbb{R} \oplus \Gamma_{1,2n-1}} |_{S_2=0} = (S^3)^2 = 4 \left( P^\lambda \bar{P}^\lambda \eta_{i\bar{j}} \right)^2 = (I_{2,\mathbb{CP}^n})^2.
\]
Namely, the quartic invariant $I_{4,R\otimes\Gamma_1,2n-1}$ of the real irrep. $R = (2, 2 + 2n)$ of the semisimple group of type $E_7$ $G_4 = SL_v(2, \mathbb{R}) \times SO(2, 2n) = Conf(\mathbb{R} \oplus \Gamma_1,2n-1)$ “degenerates” into the square of the quadratic invariant $I_{2,\mathbb{C}P^n}$ of the complex irrep. $R' = 1 + n$ of the “degenerate” group of type $E_7$ $G'_4 = U(1, n)$. This latter is the $U$-duality group of $\mathcal{N} = 2$ supergravity minimally coupled to $n$ vector multiplets [19].

In a manifestly $U(1, n)$-covariant symplectic basis, $I_{2,\mathbb{C}P^n}$ reads:

$$I_{2,\mathbb{C}P^n} = (p^0)^2 + q_0^2 - \sum_{a=1}^{n} (p^a)^2 + q_a^2.$$  

(8.24)

In order to make (8.24) consistent with (8.23), the following dyonic identification of charges can be performed:

$$P^I = \frac{1}{\sqrt{2}} (p^0 + iq_0);$$  

$$P^A = \frac{1}{\sqrt{2}} (p^a + iq_a).$$  

(8.25)

Note that in this case (5.13) manifestly breaks $SL_v(2, \mathbb{R})$, whereas its solution (5.14) further breaks $SO(2, 2n)$ down to $U(1, n)$.

The “degeneration” of $I_{4,R\otimes\Gamma_1,2n-1}$ can also be considered in the scalar-dressed formalism, in which [35, 56, 57]

$$S_1 = |Z|^2 + |Z_s|^2 - Z_I \bar{Z}_I;$$  

$$S_2 = 2iZ\bar{Z}_s - Z_I \bar{Z}_I,$$

(8.26)

(8.27)

where $Z$, $Z_s$ and $Z_I$ respectively are the central charge, axio-dilatonic matter charge and non-axio-dilatonic matter charges ($I = 1, ..., 2n$ denotes “flatted” local indices, also the index $s$ does). Recall that $Z_s \equiv D_s Z$, $Z_I \equiv D_I Z$, $\bar{Z}_I = \bar{Z}_I$, $Z_I = Z_I$, where $D$ is the Kähler-covariant differential operator in “flatted” local indices. By splitting the index $I$ as $I = \{\tilde{I}, \bar{I}\}$ with $\tilde{I} = 1, ..., n$ and $\bar{I} = 1, ..., n$, the “degeneration” condition (5.13)

$$S_2 = 0 \Leftrightarrow 2iZ\bar{Z}_s = Z_I \bar{Z}_I$$

(8.28)

can be solved by setting

$$Z_s = 0, \quad Z_{\bar{I}} = iZ_{\bar{I}},$$

(8.29)

thus implying (recall (8.22))

$$I_{4,R\otimes\Gamma_1,2n-1} = S_1^2 = \left(|Z|^2 - |Z_I|^2 - |Z_{\bar{I}}|^2\right)^2 = \left(|Z|^2 - 2|Z_I|^2\right)^2 = (I_{2,\mathbb{C}P^n})^2,$$

(8.30)

where the re-writing of the invariant $I_{2,\mathbb{C}P^n}$ in the scalar-dressed formalism reads (see e.g. [35, 56, 57])

$$I_{2,\mathbb{C}P^n} = |Z|^2 - |Z_0|^2,$$

(8.31)

thus yielding the following identification of scalar-dressed charges with $\alpha$-dependent “±” branches:

$$Z_{\tilde{I}} \equiv \pm \frac{i}{\sqrt{2}} Z_\alpha.$$

(8.32)

It should be stressed that (5.14) and (8.29) are different solutions, in two different (respectively “bare” and “scalar-dressed”) formalisms, to the “degeneration” condition (5.13) (or, equivalently, (8.28)). Note that the solution (8.29) to the manifestly $SL_v(2, \mathbb{R})$-breaking “degeneration” condition (5.13) (or, equivalently, (8.28)) consistently breaks $SO(2, 2n)$ down to $U(1, n)$.

Mutatis mutandis, the “degeneration” in the scalar-dressed formalism considered above can also be performed for of $I_{4,R\otimes\Gamma_5,2n-1}$ of Sec. 5 essentially, one has to identify

$$Z \equiv Z_1, \quad i\bar{Z}_s \equiv Z_2,$$

(8.33)

27
where $Z_1$ and $Z_2$ are the skew-eigenvalues of the $\mathcal{N} = 4$ central charge matrix $Z_{AB}$ ($A, B = 1, \ldots, 4$) (see e.g. [35 56 58 57]).

In group-theoretical terms, the “degeneration” truncating procedure under consideration goes as follows:

$$SL_v(2, \mathbb{R}) \times SO(2, 2n) \supset U(1, n);$$

$$\begin{align*}
(2, 2 + 2n) \rightarrow 2 \cdot \left[ (1 + n)_{+1} + (1 + n)_{-1} \right],
\end{align*}$$

with the double-counting eventually removed by the “degeneration” truncating condition [5.13]-[5.14], which sets to zero $n+1$ complex, i.e. $2n+2$ real, charge combinations. As mentioned, [5.13] manifestly breaks $SL_v(2, \mathbb{R})$-invariance, and its various branches, generated by the various possibilities in the choice of “±” for each index $i$, are all inter-related by suitable $U(1, n)$-transformations. At the level of the vector multiplets’ scalar manifolds, it holds

$$\frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2, 2n)}{SO(2) \times SO(2n)} \supset SU(1, n).$$

\textbf{8.3.1 A Remark On $\mathbb{C}P^1$}

It is worth pointing out that the $n = 1$ case of the “degeneration” procedure [8.34]-[8.35] is different from the “usual” truncation of the $\mathbb{R} \oplus \Gamma_1, n-1$ sequence down to the \textit{axio-dilatonic minimally coupled} 1-modulus $\mathbb{C}P^1$ model, achieved by setting $n = 0$:

$$SL_v(2, \mathbb{R}) \times SO(2) \xrightarrow{n=0} SL_v(2, \mathbb{R}) \times SO(2) \sim U(1, 1);$$

$$\begin{align*}
\frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2n)}{SO(2) \times SO(n)} \xrightarrow{n=0} \frac{SL_v(2, \mathbb{R})}{U(1)} \times \frac{SO(2)}{U(1)} \sim \frac{SU(1, 1)}{U(1)} \times \frac{U(1)}{U(1)}.
\end{align*}$$

Thus, the $U$-duality group of the 1-modulus minimally coupled $\mathcal{N} = 2$ theory is the unbroken \textit{axio-dilatonic} $SL_v(2, \mathbb{R})$ group times the factor $SO(2, n = 0) = SO(2)$. On the other hand, the $n = 1$ case of the “degeneration” procedure described in Sec. [8.3] manifestly breaks $SL_v(2, \mathbb{R})$, and it determines the $U$-duality group of the 1-modulus minimally coupled $\mathcal{N} = 2$ theory as the $n = 1$ case of the breaking $SO(2, 2n) \rightarrow U(1, n)$ of the symmetry pertaining to the non-axio-dilatonic matter sector.

At the level of invariant polynomials of the symplectic irrep. of the $U$-duality group, the truncation [8.36] works as (recall [8.10] and [8.25]):

$$\mathcal{I}_{4, \mathbb{R} \oplus \Gamma_1, n-1} \bigg|_{n=0} = 4 \left\{ \left[ (p^1)^2 + (p^3)^2 \right] \left[ (q_1^2 + q_2^2) - (p^1 q_1 + p^2 q_2)^2 \right] \right\} = 4 \left( p^1 q_2 - p^2 q_1 \right)^2 = \left( \mathcal{I}_{2, \mathbb{C}P^1} \right)^2.$$  

(8.37)

The $\mathcal{N} = 2$ symplectic basis obtained in this truncation is the one in which the holomorphic prepotential reads $F = -iX^1 X^2$, and it thus differs from the one pertaining to (8.21) with $n = 1$, in which $F = -i \left[ (X^0)^2 - (X^1)^2 \right]$. Indeed, while (8.37) does not vanish \textit{iff} both the graviphoton (index 1) and the matter Maxwell field (index 2) have at least one non-vanishing field strength’s flux (namely, \textit{iff at least} $p^1, q_2 \neq 0$ or $p^2, q_1 \neq 0$), (8.21) can be non-vanishing also when the graviphoton (index 0) or the matter Maxwell field (index 1) has both electric and magnetic zero charges. The $Sp(4, \mathbb{R}) / U(1, 1)$
finite transformation $S$ relating the two symplectic bases under consideration reads

$$
\begin{pmatrix}
X^1 \\
X^2 \\
F_1 \\
F_2 \\
\end{pmatrix}_{F=-iX^1X^2} = S
\begin{pmatrix}
X^0 \\
X^1 \\
F_0 \\
F_1 \\
\end{pmatrix}_{F=-i(X^0)^2-(X^1)^2};
$$

(8.38)

$$
S \equiv \frac{1}{2}
\begin{pmatrix}
2 & 2 & 0 & 0 \\
2 & -2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}
\in Sp(4, \mathbb{R})/U(1,1).
$$

(8.39)

### 8.4 “Generalized” Groups of Type $E_7$ and Special Geometry

As introduced in Sec. 4.3 of [27], special Kähler geometry can be reformulated in order to capture both non-degenerate and degenerate groups of type $E_7$ in a coordinate-independent (i.e. diffeomorphism-invariant) way. This is achieved by introducing “generalized” groups of type $E_7$, based on a quartic “entropy functional”, expressed in terms of the scalar-dressed basis of $\mathcal{N} = 2$ central charge $Z$ (graviphoton) and $\mathcal{N} = 2$ matter charges $Z_i \equiv D_i Z$ (vector multiplets) as follows:

$$
\mathcal{I}_4 = (i_1 - i_2)^2 + 4i_4 - i_5;
$$

(8.40)

$$
i_1 = |Z|^2;
$$

(8.41)

$$
i_2 = Z_i \overline{Z}^i;
$$

(8.42)

$$
i_3 = \frac{i}{6} \left[ ZC_{ijk} \overline{Z}^j \overline{Z}^k + \overline{Z}C_{ijk} Z^j Z^k \right];
$$

(8.43)

$$
i_4 = \frac{i}{6} \left[ ZC_{ijk} \overline{Z}^j \overline{Z}^k Z^k - \overline{Z}C_{ijk} Z^j Z^k \right];
$$

(8.44)

$$
i_5 = g^{ij} C_{ijk} \overline{C}_{ilm} Z^j \overline{Z}^l Z^m.
$$

(8.45)

Note that $\mathcal{I}_4 = (i_1 - i_2)^2$ if $C_{ijk} = 0$; this corresponds to symmetric $\mathcal{N} = 2$ $\mathbb{C}P^n$ models, which upon reduction to $\mathcal{N} = 1$ yield minimal coupling. Another way to obtain $\mathcal{N} = 2$ $\mathbb{C}P^n$ models by truncating an $\mathcal{N} = 2$ theory with $C_{ijk} \neq 0$ is discussed in Subsec. 8.3.

One can make a model-independent analysis holding for any special Kähler geometry, by relating the invariants $i_1, i_2, i_3, i_4$ and $i_5$ defined in (8.41)-(8.45) to the three roots $\lambda_1, \lambda_2, \lambda_3$ of the universal cubic equation (cfr. Eqs. (5.11)-(5.18) of [74])

$$
\lambda^3 - i_2 \lambda^2 + \frac{i_5}{4} \lambda - \frac{(i_3^2 + i_4^2)}{4i_1} = 0.
$$

(8.46)

Within this formalism, the “degeneration” corresponds to truncating the $\mathcal{N} = 2$ vector multiplets such that

$$
i_3 = i_4 = i_5 = 0
$$

(8.47)

$$
\mathcal{I}_4 = (i_1 - i_2)^2.
$$

(8.48)

The condition (8.47) implies that a unique non-vanishing independent root of (8.46) exists, namely $\lambda = i_2$.

All reductions treated in Sec. 8 satisfy the condition (8.47), which can be regarded as a necessary, but not necessarily sufficient, condition for truncating any $\mathcal{N} = 2$ model down to an $\mathcal{N} = 2$ $\mathbb{C}P^n$ model, and thus to $\mathcal{N} = 1$ supergravity models with minimal coupling.
9 $\mathcal{N} = 2 \to \mathcal{N} = 1$ Truncation and Minimal Coupling

Truncation of $\mathcal{N} = 2$ theories to $\mathcal{N} = 1$ theories was studied in [8, 9]. From Eq. (1.4) it is clear that, after projecting out the graviphoton, the anti-holomorphic vector kinetic matrix becomes

$$N_{\alpha\beta} = \bar{F}_{\alpha\beta} = \partial_{\alpha} \bar{\partial}_{\beta} F(\bar{X}), \quad (9.1)$$

where the projective symplectic sections $t^a \equiv X^a/X^0$ have been split as

$$t^a \equiv (t^\alpha, t^i), \quad (9.2)$$

with index $\alpha$ referring to the scalar directions of the would-be $\mathcal{N} = 1$ vector multiplets, whereas index $i$ refers to the would-be $\mathcal{N} = 1$ chiral multiplets. As pointed out above, minimal coupling of vectors requires $F(X)$ to be quadratic in the $\mathcal{N} = 2$ symplectic sections corresponding to $\mathcal{N} = 1$ vector multiplets, such that when truncating down to $\mathcal{N} = 1$, the kinetic vector matrix $N_{\alpha\beta}$ is a scalar-independent symmetric rank-2 tensor. Note that we here use a symplectic frame of special Kähler geometry in which an holomorphic prepotential exists

$$F(X) = (X^0)^2 F \left( \frac{X}{X^0} \right) \equiv (X^0)^2 f(t), \quad (9.3)$$

so that the $C$-tensor of special geometry reads

$$C_{abc} = e^K \partial_a \partial_b \partial_c f \left( t \right). \quad (9.4)$$

In particular, in this basis, $d$-geometries (which include all symmetric special geometries but the $\mathbb{CP}^n$ models) correspond to

$$\partial_a \partial_b \partial_c f = d_{abc} \text{ constant}, \quad (9.5)$$

whereas $\mathbb{CP}^n$ models correspond to

$$f \left( t \right) = -\frac{i}{2} \left[ 1 - \sum_a (t^a)^2 \right]. \quad (9.6)$$

It is worth remarking that minimal coupling requires, in addition to

$$C_{\alpha\beta\gamma} = 0 = C_{\alpha ij}, \quad (9.7)$$

also [8, 9]

$$C_{\alpha\beta i} = 0, \quad (9.8)$$

and thus the only non-vanishing components of the $C$-tensor can lie along the directions $C_{ijk}$ corresponding to the would-be $\mathcal{N} = 1$ chiral multiplets.

For symmetric cosets, this is only possible for $\mathbb{CP}^n$ scalar manifolds, with $n = n_c + n_V$ (with $n_c$ and $n_V$ here denoting the number of $\mathcal{N} = 1$ chiral and vector multiplets, respectively). The only other possibility would consist in taking the models based on the semi-simple $U$-duality group $SL(2, \mathbb{R}) \times SO(2, n)$, and considering only one vector multiplet, but this is nothing but the $\mathbb{CP}^1$ model itself (see the comment in Subsubsec. 8.3.1).

For non-symmetric special geometry, other solutions exist. In Calabi-Yau compactifications, the effective $\mathcal{N} = 2$ prepotential for particular orbifold realizations can have a cubic dependence on the untwisted moduli $X_U$ and a quadratic dependence on the twisted moduli $X_T$ (see e.g. [71], and Refs. therein):

$$F(X_U, X_T) = C_{ijk} X_U^i X_U^j X_U^k + C_{\alpha\beta} X_T^\alpha X_T^\beta. \quad (9.9)$$
If one performs a truncation in which the \( \mathcal{N} = 1 \) chiral multiplets correspond to *untwisted* moduli and \( \mathcal{N} = 1 \) vector multiplets correspond to *twisted* ones (as suggested by the index splitting in \((9.9)\)), one obtains a scalar-independent kinetic vector matrix: \( f_{\alpha \beta} = C_{\alpha \beta} \) (minimal \( \mathcal{N} = 1 \) vector coupling).

Theories which exhibit *minimal coupling* under truncation can for instance be given by suitable projections of an original \( \mathcal{N} = 3 \) theory down to \( \mathcal{N} = 1 \). Indeed, if some vector multiplets survive the truncation down to \( \mathcal{N} = 1 \), they necessarily exhibit a minimal coupling, because the matrix \( f_{\alpha \beta} \) is independent of the remaining \( \mathcal{N} = 1 \) chiral multiplets’ complex scalar fields. This can be understood by considering the intermediate truncation \( \mathcal{N} = 3 \to \mathcal{N} = 2 \), corresponding to the following branching of the \( U \)-duality group (see Sec. 7):

\[
U (3, n) \supset U (1, n_V) \times SU (2, n_H) \times U (1), \quad n = n_V + n_H.
\]

The kinetic matrix of the \( \mathcal{N} = 2 \) \( n_V \) vector multiplets is independent of the \( n_H \) \( \mathcal{N} = 2 \) hyperscalars, and after projecting out the \( \mathcal{N} = 2 \) graviphoton and thus reducing to \( \mathcal{N} = 1 \), it also becomes independent of the scalars corresponding to the \( \mathcal{N} = 2 \) vector multiplets, thus becoming constant and giving rise to an \( \mathcal{N} = 1 \) *minimal* vector coupling.

Other non-symmetric special geometries are obtained in \( \mathcal{N} = 1 \) Calabi-Yau orientifold compactifications \[72, 4\]. The kinetic vector matrix generally depends on the moduli, and in the simplest case reads as

\[
\mathcal{N}_{\alpha \beta} = d_{\alpha \beta i} z^i,
\]

where as above \( \alpha, \beta \) run over \( \mathcal{N} = 1 \) vector multiplets, and \( i \) runs over \( \mathcal{N} = 1 \) chiral multiplets. \((9.11)\) corresponds to orientifold projections of \( \mathcal{N} = 2 \) special \( d \)-geometries \[73\], as they naturally occur in Calabi-Yau compactifications (where the \( d \)-tensor is related to the triple intersection numbers).

## 10 On Freudenthal Duality and its “Degeneration”

All the cases in which \( \mathcal{I}_4 \) degenerates to \((\mathcal{I}_2)^2\) provide instances of the so-called *Freudenthal duality* \[15, 63\], whose manifest invariance (by construction, and apart from possible “hidden” symmetries) is given by the \( U \)-duality group of the theory obtained after truncation.

In the “degenerative” truncations under consideration, the corresponding “degeneration” of the \( (\text{on-shell}, \text{non-polynomial}) \) Freudenthal duality is given by the \( (\text{on-shell}, \text{linear}) \) formula:

\[
\tilde{\mathcal{Q}}^M = \mathcal{C}^{MN} \frac{\partial \mathcal{I}_2}{\partial \mathcal{Q}^N},
\]

where \( \mathcal{Q} \) is the dyonic charge vector, and

\[
\mathcal{C}^{MN} \equiv \begin{pmatrix} 0^\Lambda \Sigma & -\delta^\Lambda_\Sigma \\ \delta^\Sigma_\Lambda & 0_{\Lambda \Sigma} \end{pmatrix}
\]

is the symplectic metric. Due to the very structure of \( \mathcal{I}_2 \), it holds that

\[
\tilde{\mathcal{I}}_2 (\mathcal{Q}) \equiv \mathcal{I}_2 \left( \tilde{\mathcal{Q}} \right) = \mathcal{I}_2 (\mathcal{Q}).
\]

In the manifestly \( U (1, n) \)-covariant \( \mathcal{N} = 2 \) symplectic basis specified by \((8.24)\), the “degenerate” Freudenthal duality \( (10.1) \) can be made explicit as follows:

\[
\tilde{\mathcal{Q}}^M = \mathcal{C}^{MN} \mathcal{A}_{NP} \mathcal{Q}^P; \quad (10.4)
\]

\[
\mathcal{A}_{MN} \equiv \begin{pmatrix} \eta_{\Sigma \Lambda} & 0^\Lambda_\Sigma \\ 0^\Lambda_\Sigma & -\eta_{\Lambda \Sigma} \end{pmatrix},
\]

\[
\begin{array}{ll}
\mathcal{Q} & \equiv (Q^a, \varphi_i) \\
\mathcal{I}_2 & \equiv (I^a_0, \varphi_i) \\
\mathcal{A}_{MN} & \equiv (A_{MN}, \varphi_i)
\end{array}
\]

\[31\]
namely, in components \((Q = (p^\Lambda, q_\Lambda)^T)\), consistent with \((8,24)\):

\[
\begin{pmatrix}
\tilde{p}^\Lambda \\
\tilde{q}_\Lambda
\end{pmatrix}
= \begin{pmatrix}
-\eta^{\Lambda\Sigma} q_\Sigma \\
\eta_{\Lambda\Sigma} p_\Sigma
\end{pmatrix},
\]

where \(\eta\) is the metric of (the fundamental irrep. of) \(SO(1,n)\). Note that this explicit treatment can be generalized to \(N = 3\) supergravity in the manifestly \(U(3,n)\)-covariant symplectic basis specified by \((5,16)\) by simply considering \(\eta\) as the metric of (the fundamental irrep. of) \(SO(3,n)\).

It can be easily checked that the “degenerate” Freudenthal duality transformation \(\mathcal{CA}\) \((10.4)-(10.6)\) is nothing but a particular anti-involutive symplectic transformation of the relevant \(U\)-duality group \(G_4\). Thus, the invariance \((10.3)\) is trivial, and in the simple, degenerate groups of type \(E_7\) relevant to \(D = 4\) supergravity (namely, \(U(1,n)\) or \(U(3,n)\)) the corresponding Freudenthal duality is an anti-involutive \(U\)-duality transformation.

11 Non-Minimal Coupling and Fermions

Certain aspects of non-minimal vector coupling reflect on fermions and their interactions. In particular, one finds that in case that the holomorphic function \(f_{\alpha\beta}(z)\) depends on \(z\) the mass of gaugino’s may have a non-vanishing tree level contribution of the form (in the notation of \([76]\))

\[
\frac{1}{4} f_{\alpha\beta i} g^{-1} j e^{K/2} D\bar{W} \bar{\lambda}_R^\alpha \lambda_R^\beta + (R \leftrightarrow L).
\]

(11.1)

Such a mass term for \(D\bar{W} \neq 0\) may play an important role in particle physics. In the minimal coupling case, \(f_{\alpha\beta i} \equiv \frac{\partial f_{\alpha\beta}}{\partial z_i} = 0\), and the mass of gaugino’s may only come from soft breaking terms and from quantum effects.

Another case of non-minimal coupling in the fermion sector involves a Pauli coupling of a vector to a fermion of the chiral multiplet and a gaugino (see also App. A further below)

\[
\frac{1}{4} f_{\alpha\beta i} \bar{\chi}^\gamma \gamma^{\mu\nu} F_{\mu\nu}^{-\alpha} \lambda_R^\beta + h.c.
\]

(11.2)

This process is interesting in the context of creation of matter in the Universe, after inflation. The bosonic cubic vertices \(\phi F^2\) or \(aF\bar{F}\) provide a possibility of creation of vectors fields from the inflaton (scalar \(\phi\), or the axion \(a\)). A Pauli coupling above will allow the fermionic partner of the inflaton, \(\chi\) to decay and create a vector and a gaugino, standard model particles. Thus the dependence of the vector coupling on scalars due to supersymmetry is present also in the fermionic sector of the theory and may also be useful. Clearly, both terms in (11.1) and in (11.2) are absent in models of \(\mathcal{N} = 1\) supergravity with minimal coupling, but necessarily present in models originating from higher supersymmetries.

12 Conclusion

The minimal vector coupling in \(\mathcal{N} = 1\) supergravity corresponds to the choice of the constant vector kinetic term as shown in eq. \((1.1)\), when instead of a holomorphic function of scalars, \(f_{\alpha\beta}(z)\), as in eq. \((1.2)\), one has \(f_{\alpha\beta} = \delta_{\alpha\beta}\). Meanwhile, there is an interesting possibility to use the couplings like \(\phi F^2\), and \(aF\bar{F}\) and the ones with fermions, for cosmological applications, see for example \([70]\).

It is therefore interesting to study the origin of such couplings, attractive for cosmology and for particle physics, from well motivated superstring theory and their compactification, and related to these four-dimensional supergravities with higher supersymmetries.

As resulting from the present paper, generalizing and refining the investigation carried out in \([3]\), the answer to this question follows from duality symmetry and has a group theoretical origin. The
question is why the vector kinetic matrix $N_{\Lambda\Sigma}(\varphi)$ in Im$N_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\mu\nu\Sigma} + i\text{Re}N_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu}$ (1.3) in $N \geq 2$ depends, generically, or does not depend, in degenerate cases, on scalars, when the theory is reduced to $\mathcal{N} = 1$ case. In $\mathcal{N} \geq 2$ there is a duality symmetry group $G$, embedded into an $Sp(2n_v, \mathbb{R})$, such that the $n_v$ vector 2-form field strengths and their duals fit into a symplectic representation

$$R' = S R, \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad S^T \Omega S = \Omega, \quad \Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$ (12.1)

The gauge kinetic term $\mathcal{N}$ generically depends on scalars since it transforms via fractional transformations

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}.$$ (12.2)

The symplectic symmetric tensor (see e.g. [11], and Refs. therein)

$$\mathcal{M}_{MN}(\mathcal{N}) \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$ (12.3)

$$A \equiv \text{Im}\mathcal{N} + \text{Re}\mathcal{N} (\text{Im}\mathcal{N})^{-1} \text{Re}\mathcal{N}; \quad B \equiv -\text{Re}\mathcal{N} (\text{Im}\mathcal{N})^{-1};$$

$$C \equiv -(\text{Im}\mathcal{N})^{-1} \text{Re}\mathcal{N}; \quad D \equiv (\text{Im}\mathcal{N})^{-1}$$

is never constant (i.e. scalar-independent) in $\mathcal{N} \geq 2$ supergravity, because, as shown in [1], this would imply the existence of an invariant quadratic form with Euclidean signature (due to the negative definiteness of $\mathcal{M}$ itself). However, in the present investigation we exploited a systematic investigation of the cases in which degenerate groups of type $E_7$, when reduced to $\mathcal{N} = 1$, may provide a scalar-independent kinetic vector matrix $\mathcal{N}$, and thus a scalar-independent $\mathcal{M}$. For $\mathcal{N} = 2$ theories, this can only occur when the matrix $\mathcal{F}_{\Lambda\Sigma} \equiv \partial_{\Lambda} \partial_{\Sigma} F$ projected onto the directions pertaining to the would-be $\mathcal{N} = 1$ vector multiplets, is constant, namely when the holomorphic prepotential $F$ is quadratic in the scalar degrees of freedom corresponding to the would-be $\mathcal{N} = 1$ vector multiplets. In symmetric special Kähler geometry, this implies that $\mathcal{M}(F)$ (defined as (12.3)-(12.4) with $N_{\Lambda\Sigma} \rightarrow \mathcal{F}_{\Lambda\Sigma}$) is a scalar-independent matrix with Lorentzian signature, and the corresponding quadratic form $Q\mathcal{M}(F)Q^T$ defines the quadratic symmetric invariant structure of degenerate groups of type $E_7$ (recall (8.31) and Eqs. (34) and (35) of [75])

$$I_{2,\mathbb{CP}^n} = i_1 - i_2 = -\frac{1}{2} Q\mathcal{M}(F) Q^T.$$ (12.4)

For non-degenerate groups of type $E_7$, $\mathcal{M}(F)$ is never scalar-independent, and thus minimal coupling is not allowed.

In the present paper, we carried out a detailed classification and analysis of all cases of degeneration of groups of type $E_7$ responsible for the duality symmetry of extended supergravity: in this way, our investigation provides an explanation for the fact that the minimal coupling case is non-generic in $\mathcal{N} = 1$ supergravity originating from higher supersymmetries, thus supporting the proposal to use a non-minimal vector coupling for applications in particle physics and cosmology.

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A Pauli Terms

A.1 General Structure

In a $D = 4$ $\mathcal{N}$-extended supergravity theory, the general structure of Pauli terms read (we use the notation and conventions of [64], to which the reader is addressed for further elucidation):

$$\left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} = \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} \left( L_{A B}^\Sigma \bar{\psi}^\mu A \psi^B + L_{I A}^\Sigma \bar{\psi}^\mu A \gamma^\nu \lambda^{I} + L_{I J}^\Sigma \bar{\lambda}^{I \mu \nu} \lambda^{J} \right) + \text{h.c.}, \quad (A.1)$$

where $\lambda_I$ and $\psi_{A \mu}$ respectively denote the spin-$\frac{1}{2}$ fermions and the gravitino fields, and $\mathcal{F}_{\mu \nu}^{(\pm)\Lambda}$ are the self-dual/anti-self-dual combinations of the vector field strengths:

$$\mathcal{F}_{\mu \nu}^{(\pm)\Lambda} \equiv \frac{1}{2} \left( \mathcal{F}_{\mu \nu}^\Lambda \mp i \star \mathcal{F}_{\mu \nu}^\Lambda \right);$$

$$\star \mathcal{F}_{\mu \nu}^\Lambda \equiv \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \mathcal{F}^{\rho \sigma \Lambda};$$

$$\star \mathcal{F}_{\mu \nu}^{(\pm)\Lambda} \equiv \mp i \mathcal{F}_{\mu \nu}^{(\pm)\Lambda}. \quad (A.2)$$

$A, B, \ldots$ indices range in the fundamental representation of the $R$-symmetry $SU(\mathcal{N}) \times U(1)$ (the $U(1)$ term is missing in the maximal case $N = 8$), their lower (upper) position denoting left (right) chirality. Besides enumerating the fields, the indices $I$ actually are a short-hand notation, which encompasses various possibilities: if the fermions belong to vector multiplets $I \rightarrow IA$, since they also transform under $R$-symmetry; if they refer to fermions of the gravitational multiplet they are a set of three $SU(\mathcal{N})$ antisymmetric indices: $I \rightarrow [ABC]$. (In the particular case of $\mathcal{N} = 2$ $n_H$ hypermultiplets: $I \rightarrow \alpha$, where $\alpha$ is in the fundamental of $USp(2n_H)$.)

The matrices entering the Lagrangian are in general all dependent on the scalar fields $q^I$. $\mathcal{N}_{\Lambda \Sigma}$ is the kinetic vector matrix, generally depending on (a subset $q^I$ of) the scalar fields $q^u$. According to [5], the indices $\Lambda, \Sigma$ sit in the relevant symplectic representation of the $U$-duality group $G$. The structures $L_{AB}^\Sigma, L_{IA}^\Sigma, L_{IJ}^\Sigma$ are coset representatives of the $\sigma$-model $G/H$ for $N > 2$, while they are objects of special Kähler geometry for $\mathcal{N} = 2$. For $\mathcal{N} = 1$, they are related to the kinetic matrix of the vectors (with $L_{AB}^\Sigma = 0$, because there are no vectors in the $\mathcal{N} = 1$ gravity multiplet).

In the following, we will specify $[A.1]$ to $\mathcal{N} = 8$, to $\mathcal{N} = 2$ (in particular, when $G$ is a “degenerate” group “of type $E_7$”) and to $\mathcal{N} = 1$ theories (also in presence of minimal coupling).

A.2 $\mathcal{N} = 8$

In this case, $A = 1, \ldots, 8$ range in the 8 of the $R$-symmetry $SU(8)$. Only gravitational multiplet is present; the gauginos $\Lambda_{[ABC]}$ are in the rank-3 antisymmetric irrep. 56 of $SU(8)$, whereas the scalars $q^{[ABCD]}$ sit into the rank-4 antisymmetric self-real irrep. 70 of $SU(8)$. [A.1] thus specifies to:

$$\mathcal{N} = 8 : \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} = \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} L_{AB}^\Sigma \bar{\psi}^\mu A \psi^B + \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} L_{AB}^\Sigma \bar{\psi}^\mu A \gamma^\nu \lambda^{ABC} + \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} \epsilon_{ABCDEF} G H L \bar{\psi}^\mu A \gamma^{ABC} \lambda^{\mu \nu} \chi^{DEF} \Sigma^{GH} + \text{h.c.}. \quad (A.3)$$

Thus, by introducing

$$T_{\mu \nu, AB}^\Sigma \equiv \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} L_{AB}^\Sigma; \quad (A.4)$$

$$T_{\mu \nu}^{[AB]} \equiv \mathcal{F}_{\mu \nu}^A \text{Im} \mathcal{N}_{\Lambda \Sigma} L_{[AB]}^\Sigma, \quad (A.5)$$
(A.3) can be rewritten as
\[ \mathcal{N} = 2 : \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} \]
\[ = \mathcal{F}_{\mu}^{\Lambda} \text{Im} \mathcal{N}_{\Lambda \Sigma} \left[ 4 L^\Sigma \psi^A \psi^B \epsilon_{AB} - 4 i \nabla_\mu \nabla_\nu \lambda^\mu \gamma^\nu \psi_\beta \epsilon^{AB} + \epsilon_{ABC} \bar{\epsilon}_{\mu \nu \rho} \lambda^{\beta} \epsilon^{AB} \right] + \text{h.c.}, \] (A.6)

A.3 \quad \mathcal{N} = 2

\[ \mathcal{N} = 2 \] supergravity the scalar manifold is a product manifold \([65, 66, 55]\),
\[ \mathcal{M}_{\text{scalar}} = \mathcal{M}_{\text{vec}} \times \mathcal{M}_{\text{hyper}} \] (A.7)

since there are two kinds of matter multiplets, the vector multiplets and the hypermultiplets. The geometry of \( \mathcal{M}_{\text{vec}} \) is described by the special Kähler geometry \([65, 67]\), while the geometry of \( \mathcal{M}_{\text{hyper}} \) is described by quaternionic geometry \([65, 66, 68]\); for a thorough geometric treatment, see e.g. \([6]\).

With respect to the general case \([A.1]\)

\[ \Lambda = 0, 1, \ldots, n_V; \quad A, B = 1, 2; \quad i = 1, \ldots, 4n_H + 2n_V; \quad I = 1, \ldots, n_H + n_V, \] (A.8)

where the index 0 pertains to the graviphoton.

As it will be the case in \( \mathcal{N} = 1 \) supergravity, we denote the complex scalars parameterizing \((\text{vec})\) by \( z^i, \bar{z}^\alpha \), while the scalars parameterizing \( \mathcal{M}_{\text{hyper}} \) will be denoted by \( q^u \). When the index \( I \) runs over the vector multiplets it must be substituted by \( I_A \) in all the formulae relevant to the vector multiplet, since the fermions \( \lambda^{IA} \) are in the fundamental representation of the \( R \)-symmetry group \( U(2) \).

\[ L^\Lambda (z, \bar{z}) \] and its "magnetic" counterpart \( M_\Lambda (z, \bar{z}) = N_{\Lambda \Sigma} L^\Sigma \) actually form a \( 2n_V \) dimensional covariantly holomorphic section \( V = (L^\Lambda, M_\Lambda) \) of a flat symplectic bundle.

When the index \( I \) runs over the hypermultiplets, we rename them as follows: \( (I, J) \rightarrow (\alpha, \beta) \) and since there are no vectors in the hypermultiplets we have \( \epsilon^{IA} = 0 \).

The \textit{Vielbein} of the quaternionic manifold \( \mathcal{M}_{\text{hyper}} \) are usually denoted by \( U^A_B \equiv U^u_A dq^u \), where \( \alpha = 1, \ldots, 2n_H \) is an index labelling the fundamental representation of \( USp(2n_H) \). The inverse matrix \textit{Vielbein} is \( u^{-1}_A^\alpha \). We raise and lower the indices \( \alpha, \beta, \ldots \) and \( A, B, \ldots \) with the symplectic matrices \( \epsilon^{\alpha \beta} \) and \( \epsilon_{AB} \).

Thus, \([A.1]\) specifies to:
\[ \mathcal{N} = 2 : \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} \]
\[ = \mathcal{F}_{\mu}^{\Lambda} \text{Im} \mathcal{N}_{\Lambda \Sigma} \left[ 4 L^\Sigma \psi^A \psi^B \epsilon_{AB} - 4 i \nabla_\mu \nabla_\rho \lambda^\mu \gamma^\rho \psi_\beta \epsilon^{AB} + \epsilon_{ABC} \bar{\epsilon}_{\mu \nu \rho} \lambda^{\beta} \epsilon^{AB} \right] + \text{h.c.}, \] (A.9)

where \( \zeta_\alpha, \bar{\zeta}_\alpha \) denote the spin-\( \frac{1}{2} \) fermions of the hypermultiplets \((\text{hyperinos})\). The kinetic vector matrix \( \mathcal{N}_{\Lambda \Sigma} \) can be constructed in terms of \( L^\Lambda \) through the procedure \emph{e.g.} given in \([6]\).

By introducing the gravity- and matter- vector projectors
\[ T^-\mu \equiv 2 \text{Im} \mathcal{N}_{\Lambda \Sigma} L^\Sigma \mathcal{F}^-_{\mu \nu}; \]
\[ T^-_{\mu \nu} \equiv -\text{Im} \mathcal{N}_{\Lambda \Sigma} \mathcal{F}^-_{\mu \nu} g^{\nu} \nabla_\mu \nabla_\Sigma; \] (A.10)

\( \mathcal{A.9} \) can be rewritten as
\[ \mathcal{N} = 2 : \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} \]
\[ = -\frac{i}{2} T^-\mu \left[ 4 L^\Sigma \psi^A \psi^B \epsilon_{AB} - \bar{\zeta}_\alpha \gamma^{\mu \nu} \zeta_\beta \epsilon^{AB} \right] \] (A.12)
\[ + \frac{i}{2} T^-_{\mu \nu} \left[ 8 g_\chi \lambda^\mu \gamma^\nu \psi^B \epsilon^{AB} - C_{ijk} \lambda^\mu \gamma^{\nu \rho} \psi_\beta \epsilon^{AB} \right] + \text{h.c.}. \] (A.13)
Note that for $\mathcal{N} = 2$ minimally coupled theories, whose $U$-duality group is a degenerate group of type $E_7 : G_4 = U(1, n_V)$, it holds that $C_{ijk} = 0$, and thus the second Pauli term in the “matter sector” (A.13) is absent.

A.4 $\mathcal{N} = 1$

In order to specify the general formula (A.1) to $\mathcal{N} = 1$, we recall that the scalar manifold is in this case a Kähler-Hodge manifold and that the $R$-symmetry reduces simply to $U(1)$; for a general treatment, see e.g. [69] [7]. It is convenient in this case to use as “Viebeins” the differentials of the complex coordinates $dz^i, d\bar{z}^\bar{i}$, where $z^i(x)$ are the complex scalar fields parameterizing the Kähler-Hodge manifold of (complex) dimension $n_C$; thus, in this case we set $q^\mu \to (z^i, \bar{z}^{\bar{i}})$. The spin $\frac{1}{2}$ fermions are either in chiral or in vector multiplets; so, the index $I$ runs over the number $n_V + n_C$ of vector and chiral multiplets: $I = 1, \ldots, n_V + n_C$. Furthermore, it is convenient to assign the index $\Lambda$, the same as for the vectors, to the fermions of the vector multiplets : we will denote them as $\lambda^\Lambda, \lambda_i^\Lambda \bar{\gamma}^\Lambda$. Thus, (A.1) specifies to:

\[
\mathcal{N} = 1 : \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} = \text{Im} N_{\Lambda \Sigma} F_{\mu \nu}^\Lambda \chi_i^{\Sigma \gamma \mu \nu} - \frac{i}{8} \partial_\gamma N_{\Lambda \Sigma} F_{\mu \nu}^\Lambda \bar{\chi}_i^{\gamma \mu \nu} \lambda^\Lambda_i + \text{h.c.,} \quad (A.14)
\]

where $F_{\mu \nu}^{(\pm)\Lambda}$ are defined in (A.2). Within the adopted conventions, $N_{\Lambda \Sigma}$ is anti-holomorphic in the chiral multiplets’ complex scalars:

\[
\partial_\Lambda N_{\Sigma} = 0. \quad (A.15)
\]

It is instructive to compare (A.14) with its $\mathcal{N} = 2$ counterpart (A.12)–(A.13). When performing the supersymmetry reduction $\mathcal{N} = 2 \to \mathcal{N} = 1$, the “gravity sector” (A.12) of the $\mathcal{N} = 2$ Pauli terms is projected out because, as mentioned, the $\mathcal{N} = 1$ gravity multiplet des not contain any graviphoton. On the other hand, the “matter sector” (A.13) of the $\mathcal{N} = 2$ Pauli terms (simpler in the $\mathcal{N} = 2$ minimally coupled theory due to $C_{ijk} = 0$) becomes (A.1) itself.

Furthermore, it should be noted that when the $\mathcal{N} = 1$ scalars are minimally coupled to the vectors ($\partial_\Lambda N_{\Sigma} = 0$; thus, from (A.15)), the second term in (A.14) vanishes, and the Pauli term (A.14) acquires its minimally coupled form

\[
\mathcal{N} = 1 \text{ minimal coupling : } \left[ (\sqrt{-g})^{-1} \mathcal{L} \right]_{\text{Pauli}} = \text{Im} N_{\Lambda \Sigma} F_{\mu \nu}^\Lambda \chi_i^{\Sigma \gamma \mu \nu} + \text{h.c.} \quad (A.16)
\]

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