ON RATIONAL IN Variant SUMMATION OF P-ADIC POWER SERIES 
WITH BINOMIAL COEFFICIENTS

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Abstract. In the work we have considered p-adic functional series with binomial coefficients 
and discussed its p-adic convergence. Then we have derived a recurrence relation following 
with a summation formula which is invariant for rational argument. More precisely, we have 
investigated a sufficient condition under which the p-adic power series converges to a rational 
invariant sum 0. Finally we have shown application of the invariant summation formula to get 
some interesting relations involving Bernoulli numbers and Bernoulli polynomials.

1. Introduction

During the last three decades, the p-adic number theory played an essential role in number 
theory due to its applications in various physical phenomenon (see [4]) as well as to solve 
many important problems in various fields of mathematics, especially, on number theory (see 
[6],[7]). Again, the investigation of convergence of series plays an important role in the theory 
of mathematical analysis.

The infinite series of rational numbers can be considered both in p-adic and real number field 
as rational numbers are endowed with both real norm and p-adic norm. The most interesting 
fact about the real number series which diverges in real field, may converge in p-adic field and 
hence needs p-adic investigation. It is important to investigate p-adic series having rational 
sum for rational argument because physical measurement are considered as rational values. 
Recently, Dragovich ([1],[2],[3]) introduced p-adic invariant summation of a class of infinite 
p-adic functional series \( \sum n!P_k(n, x)x^n \), \( \sum \epsilon^n(n + v)!P_{k\alpha}(n, x)x^{\alpha n + \beta} \), \( \sum \epsilon^n n!P_{k\epsilon}(n, x)x^n \) each containing factorial coefficients \( n! \) such that for any rational argument the corresponding sum is also a rational number, where \( P_k(n, x) \) are polynomials in \( x \) of degree \( k \), \( \epsilon = \pm 1, k, v, \beta \in \mathbb{N}\cup\{0\}, \alpha \in \mathbb{N}. \) But the case when the above p-adic power series contains binomial coefficients instead 
of factorials coefficients \( n! \), has not been covered so far. In this paper we have considered p-adic 
functional series with binomial coefficients instead of factorial \( n! \) and we have arrived with a 
summation formula having invariant form which gives rational number for rational argument.

Mathematics Subject Classification: 12J25, 32P05, 26E30, 40A05, 40D99, 40A30.
Key words and phrases: p-adic numbers, p-adic series, convergence, binomial coefficient, invariant sum
The outline of the paper is as follows: Section 2 is considered with the main results (see, Theorem (2.2) and Theorem (2.3)). The last section is an application part which deals with a connection of the obtained invariant summation formula (2.26) with Bernoulli numbers and Bernoulli polynomials.

2. Main Results

We consider the following p-adic functional series

\begin{equation}
S_k(x) = \sum_{n=0}^{\infty} \binom{2n}{n} P_k(n, x)x^n = P_k(0, x) + \left(\frac{2}{1}\right)P_k(1, x)x + \left(\frac{4}{2}\right)P_k(2, x)x^2 + \cdots,
\end{equation}

where

\begin{equation}
P_k(n, x) = B_k(n)x^k + B_{k-1}(n)x^{k-1} + \cdots + B_1(n)x + B_0(n),
\end{equation}

\begin{equation}
k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ and } x \in \mathbb{Q}_p.
\end{equation}

Here \(B_k(n)\), \(0 \leq l \leq k\), represents polynomials in \(n\) having integer coefficients. Our goal is to find the polynomials \(P_k(n, x)\) for which the series (2.1) converges p-adically and has invariant sum. In particular, for rational argument \(x \in \mathbb{Q}\), the sum \(S_k(x) \in \mathbb{Q}\).

We note that the power series \(\sum a_n x^n\) converges p-adically if and only if \(|a_n x^n|_p \to 0\) as \(n \to \infty\). Since the p-adic functional series (2.1) contains the binomial coefficients \(\binom{2n}{n}\), we must know the p-adic valuation of \(\binom{2n}{n}\) in order to check p-adic convergence of it. In 1852, Kummer ([5]) showed a method to calculate p-adic valuation of binomial coefficients given as follows:

**Theorem 2.1.** (Kummer [5]) Given integers \(n \geq m \geq 0\) and a prime number \(p\), the p-adic valuation \(\binom{n}{m}\) is equal to the number of carries when \(m\) is added with \(n - m\) in base \(p\).

For \(n = n_0 + n_1p + \cdots + n_rp^r\), the sum of digits is denoted as \(\delta_p(n)\). By Legendre’s formula we have

\begin{equation}
v_p(n!) = \sum_{k=1}^{r} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - \delta_p(n)}{p - 1} \text{ and } |n|_p = p^{-\frac{n - \delta_p(n)}{p - 1}}.
\end{equation}

Then we have

\begin{equation}
v_p\left(\binom{n}{m}\right) = \sum_{k=0}^{r} \left( \left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{m}{p^k} \right\rfloor - \left\lfloor \frac{n - m}{p^k} \right\rfloor \right)
= \frac{\delta_p(m) + \delta_p(n - m) - \delta_p(n)}{p - 1}.
\end{equation}
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where each of the summands \( \left\lfloor \frac{n}{p^k} \right\rfloor, \left\lfloor \frac{m}{p^k} \right\rfloor, \left\lfloor \frac{n-m}{p^k} \right\rfloor \) is \( \leq 1 \). Therefore,

\[
(2.5) \quad \left| \binom{n}{m} \right|_p = p^{\frac{\delta_p(m) + \delta_p(n-m) - \delta_p(n)}{p-1}}.
\]

For \( x = 1 \), the \( p \)-adic norm of the general term in (2.1) is given by

\[
\left| \binom{2n}{n} P_k(n, x) x^n \right|_p \leq \left| \binom{2n}{n} x^n \right|_p = p^{\frac{-\delta_p(n) - \delta_p(2n)}{p-1}} |x^n|_p = \left( p^{\frac{-\delta_p(n) - \delta_p(2n)}{n(p-1)}} |x|_p \right)^n.
\]

Thus the power series \( \sum \binom{2n}{n} P_k(n, 1) x^n \) converges for \( |x|_p \leq 1 \). Since \( |P_k(n, x)|_p \leq |P_k(n, 1)|_p \) for \( |x|_p \leq 1 \), the power series (2.1) converges \( p \)-adically on the set of rationals

\[
\{ x \in \mathbb{Q} : x = \frac{u}{v}, u, v \in \mathbb{Z}, (u, v) = 1, |u|_\infty \leq |v|_\infty, p |u, p \nmid v \}.
\]

Now we prove the following theorem:

Theorem 2.2. Let

\[
(2.6) \quad A_k(n, x) = b_k(n)x^k + b_{k-1}(n)x^{k-1} + \cdots + b_1(n)x + b_0(n)
\]

be a polynomial with polynomial coefficients \( b_j(n), 0 \leq j \leq k \) in \( n \) with rational coefficients. Then there exists such a polynomial \( A_{k-1}(N, x), N \in \mathbb{N} \) satisfying the equality

\[
(2.7) \quad \sum_{n=0}^{N-1} \binom{2n}{n} \left[ n^k (4x - 1)^k + U_k(x) \right] x^n = \binom{2N}{N} x^N A_{k-1}(N, x),
\]

where

\[
(2.8) \quad U_k(x) = 2xA_{k-1}(1, x) - A_{k-1}(0, x), k \geq 1, x \in \mathbb{Q} \setminus \left\{ \frac{1}{4} \right\}.
\]

Proof. Let us consider the finite power series with binomial coefficients:

\[
(2.9) \quad S_k(N, x) = \sum_{n=0}^{N-1} \binom{2n}{n} n^k x^n; k, n \in \mathbb{N} \cup \{0\}
\]

Putting \( k = 0 \) in equation (2.9), we have

\[
S_0(N, x) = \sum_{n=0}^{N-1} \binom{2n}{n} x^n, \quad = 1 + \sum_{n=1}^{N-1} \binom{2n}{n} x^n.
\]
We have used the following combinatorics relations:

\[
\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1},
\]

\[
\binom{n}{r} = \frac{n(n-1)}{r(r-1)},
\]

\[
\binom{n}{r} = \frac{n-r+1}{r} \binom{n}{r-1}.
\]

Let us derive the following recurrence formula:

\[
S_k(N, x) = \sum_{n=0}^{N-1} \binom{2n}{n} n^k x^n, \quad k \geq 0
\]

\[
= 2x + \sum_{n=2}^{N-1} \binom{2n}{n} n^k x^n
\]

\[
= 2x + \sum_{n=2}^{N} \binom{2n}{n} n^k x^n - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=2}^{N} \binom{2n-1}{n-1} n^k x^n - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=1}^{N-1} \binom{2n+1}{n} (n+1)^k x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=1}^{N-1} \left[ \binom{2n}{n} + \binom{2n}{n+1} \right] (n+1)^k x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^k x^{n+1} + 2 \sum_{n=1}^{N-1} \binom{2n}{n+1} (n+1)^k x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^k x^{n+1} + 2 \sum_{n=1}^{N-1} \frac{n}{n+1} \binom{2n}{n} (n+1)^k x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 2 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^k x^{n+1} + 2 \sum_{n=1}^{N-1} \frac{n+1}{n+1} \binom{2n}{n} (n+1)^k x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 4 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^k x^{n+1} - 2 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^{k-1} x^{n+1} - \binom{2N}{N} N^k x^N
\]

\[
= 2x + 4 \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^k x^n - 2x \sum_{n=1}^{N-1} \binom{2n}{n} (n+1)^{k-1} x^n - \binom{2N}{N} N^k x^N
\]
or, \( S_k(N, x) = 2x + 4x \sum_{n=1}^{N-1} \binom{2n}{n} \sum_{u=0}^{k} \binom{k}{u} n^u x^n - 2x \sum_{n=1}^{N-1} \binom{2n}{n} \sum_{u=0}^{k-1} \binom{k-1}{u} n^u x^n - \left( \frac{2N}{N} \right)^k x^N \)

\[
= 2x + 4x \sum_{n=1}^{N-1} \binom{2n}{n} x^n + 4x \sum_{u=1}^{k} \binom{k}{u} S_u(N, x) - 2x \sum_{n=1}^{N-1} \binom{2n}{n} x^n - 2x \sum_{v=1}^{k-1} \binom{k-1}{v} S_v(N, x) - \left( \frac{2N}{N} \right)^k x^N
\]

\[
= 2xS_0(N, x) + 4x \sum_{u=1}^{k} \binom{k}{u} S_u(N, x) - 2x \sum_{v=1}^{k-1} \binom{k-1}{v} S_v(N, x) - \left( \frac{2N}{N} \right)^k x^N,\]

since \( S_0(N, x) = \sum_{n=0}^{N-1} \binom{2n}{n} x^n \).

The recurrence formula is given by

(2.10) \( S_k(N, x) = 2xS_0(N, x) + 4x \sum_{u=1}^{k} \binom{k}{u} S_u(N, x) - 2x \sum_{v=1}^{k-1} \binom{k-1}{v} S_v(N, x) - \left( \frac{2N}{N} \right)^k x^N, k \geq 1 \),

which will help us to deduce a summation formula. For, noting that \( \binom{n}{r} = 0 \) if \( n < r \), we have

(2.11) \( S_1(N, x) = 2xS_0(N, x) + 4xS_1(N, x) - \left( \frac{2N}{N} \right) N^1 x^N \)

(2.12) \( S_2(N, x) = 2xS_0(N, x) + 6xS_1(N, x) + 4xS_2(N, x) - \left( \frac{2N}{N} \right) N^2 x^N \)

(2.13) \( S_3(N, x) = 2xS_0(N, x) + 8xS_1(N, x) + 10xS_2(N, x) + 4xS_3(N, x) - \left( \frac{2N}{N} \right) N^3 x^N \)

(2.14) \( S_4(N, x) = 2xS_0(N, x) + 10xS_1(N, x) + 18xS_2(N, x) + 14xS_3(N, x) + 4xS_4(N, x) - \left( \frac{2N}{N} \right) N^4 x^N \)

(2.15) \( S_5(N, x) = 2xS_0(N, x) + 12xS_1(N, x) + 28xS_2(N, x) + 40xS_3(N, x) + 20xS_4(N, x) + 4xS_5(N, x) - \left( \frac{2N}{N} \right) N^5 x^N \)

\[ \cdots \cdots \text{so on} \]
Thus the recurrence formula (2.10) helps us to calculate the sum \( S_k(N, x) \) provided all preceding sums \( S_i(N, x) \), \( i = 1, 2, \cdots, k \) as function of \( S_0(N, x) \). Then in view of the relations (2.11), (2.12), (2.13), (2.14), (2.15) and by straightforward calculation, we obtain the following summation formula

\[
N - \sum_{n=0}^{N-1} \binom{2n}{n} [n^k(4x-1)^k + U_k(x)] x^n = \left( \frac{2N}{N} \right) x^N A_{k-1}(N, x),
\]

which can be written as

\[
S_k(N, x) = -(4x-1)^{-k}U_k(x)S_0(N, x) + (4x-1)^{-k}\left( \frac{2N}{N} \right) x^N A_{k-1}(N, x),
\]

where \( U_k(x) \) is a polynomial in \( x \) of degree \( k \) and \( A_{k-1}(N, x) \) is a polynomial in \( x \) of degree \( k - 1 \) and each coefficients in \( A_{k-1}(N, x) \) are itself polynomials in \( N \). Using equation (2.17) in (2.10), we get recurrence formulas for \( U_k(x) \) and \( A_k(N, x) \):

\[
\sum_{u=1}^{k} 4x \binom{k}{u} (4x-1)^{k-u} U_u(x) - \sum_{v=1}^{k-1} 2x \binom{k-1}{v} (4x-1)^{k-v} U_v(x) - U_k(x) - 2x(4x-1)^k = 0
\]

\[
\sum_{u=1}^{k} 4x \binom{k}{u} (4x-1)^{k-u} A_{u-1}(N, x) - \sum_{v=1}^{k-1} 2x \binom{k-1}{v} (4x-1)^{k-v} A_{v-1}(N, x) - A_k(N, x) = 0.
\]

The polynomial \( A_k(N, x) \) has some assigned relation with \( U_k(x) \). If \( N = 0 \), then equation (2.19) takes the following form:

\[
\sum_{u=1}^{k} 4x \binom{k}{u} (4x-1)^{k-u} A_{u-1}(0, x) - \sum_{v=1}^{k-1} 2x \binom{k-1}{v} (4x-1)^{k-v} A_{v-1}(0, x) - A_k(0, x) = 0.
\]

For \( N = 1 \), the equation (2.19) takes the following form:

\[
\sum_{u=1}^{k} 4x \binom{k}{u} (4x-1)^{k-u} A_{u-1}(1, x) - \sum_{v=1}^{k-1} 2x \binom{k-1}{v} (4x-1)^{k-v} A_{v-1}(1, x) - A_k(1, x) - (4x-1)^k = 0.
\]
Multiplying (2.21) by $2x$ and subtracting (2.20) from the result, we get

\[
\sum_{u=1}^{k} 4x \binom{k}{u} [2x A_{u-1}(1, x) - A_{u-1}(0, x)] - \sum_{v=1}^{k-1} 2x \binom{k-1}{v} [2x A_{v-1}(1, x) - A_{v-1}(0, x)] \\
- [2x A_{k-1}(1, x) - A_{k-1}(0, x)] - 2x(4x - 1)^k = 0.
\]

Comparing the relations (2.18) and (2.22), we get the required relation

\[
U_k(x) = 2x A_{k-1}(1, x) - A_{k-1}(0, x), \quad k \geq 1.
\]

This completes the proof. \qed

We now state and prove the main result of our paper about the invariance of the p-adic sum of the p-adic functional power series (2.1) using the above result.

**Theorem 2.3.** The p-adic functional power series (2.1) has the following p-adic invariant sum

\[
\sum_{n=0}^{\infty} \left(\begin{array}{c} 2n \\ n \end{array}\right) P_k(n, x) x^n = 0
\]

provided

\[
P_k(n, x) = \sum_{j=1}^{k} B_j [n^j(4x - 1)^j + U_j(x)], \quad \forall x \in \mathbb{Q} \setminus \left\{\frac{1}{4}\right\},
\]

where $B_j \in \mathbb{Q}$.

**Proof.** Taking $N \to \infty$ in (2.16), the term $A_{k-1}(N, x)$ vanishes with respect to p-adic absolute value and gives the sum of following p-adic infinite functional series

\[
\sum_{n=0}^{\infty} \left(\begin{array}{c} 2n \\ n \end{array}\right) \left[n^k(4x - 1)^k + U_k(x)\right] x^n = 0.
\]

Now taking $P_k(n, x) = \sum_{j=1}^{k} B_j [n^j(4x - 1)^j + U_j(x)]$, the relation (2.24) is obtained, for all $x \in \mathbb{Q} \setminus \left\{\frac{1}{4}\right\}$ and the constants $B_j \in \mathbb{Q}$. This completes the proof. \qed

The equation (2.26) is valid for any $k \in \mathbb{N}$ and has same form for any $k \in \mathbb{N}$. Indeed, the equality is independent of p-adic properties for $x \in \mathbb{Q} \setminus \left\{\frac{1}{4}\right\}$. In other words, it is a p-adic invariant expression. It gives rational sum 0 for all rational arguments. The polynomials $U_k(x)$ and $A_{k-1}(N, x)$ plays plays important roles throughout the work. We can investigate more deeply about the properties of $U_k(x)$ and $A_{k-1}(N, x)$ in the above results. The recurrence formulas (2.18) and (2.19) are the tools to compute the polynomials $U_k(x)$ and $A_{k-1}(n, x)$.
The first few of $U_k(x)$ and $A_{k-1}(n, x)$ are given below:

For $k = 1$,
\[
\begin{align*}
U_1(x) &= 2x \\
A_0(n, x) &= n.
\end{align*}
\]

For $k = 2$,
\[
\begin{align*}
U_2(x) &= -2x - 4x^2 \\
A_1(n, x) &= (4n^2 - 4n)x - n^2.
\end{align*}
\]

For $k = 3$,
\[
\begin{align*}
U_3(x) &= 8x^3 + 20x^2 + 2x \\
A_2(n, x) &= (16n^3 - 40n^2 + 72n)x^2 - (8n^3 - 10n^2 + 8n)x + n^3.
\end{align*}
\]

For $k = 4$,
\[
\begin{align*}
U_4(x) &= -16x^4 - 144x^3 - 60x^2 - 2x \\
A_3(n, x) &= (64n^4 - 224n^3 + 272n^2 - 880n)x^3 \\
&\quad - (48n^4 - 112n^3 - 4n^2 - 120n)x^2 \\
&\quad + (12n^4 - 14n^3 - 18n^2 - 10n)x - n^4.
\end{align*}
\]

For $k = 5$,
\[
\begin{align*}
U_5(x) &= 800x^5 + 352x^4 + 1176x^3 + 136x^2 + 2x \\
A_4(n, x) &= (256n^5 - 1152n^4 + 1984n^3 - 1568n^2 + 7648n)x^4 \\
&\quad - (256n^5 - 864n^4 + 480n^3 + 1216n^2 - 3008n)x^3 \\
&\quad + (96n^5 - 216n^4 - 132n^3 + 308n^2 - 108n)x^2 \\
&\quad - (16n^5 - 18n^4 - 32n^3 - 28n^2 - 12n)x + n^5.
\end{align*}
\]

For $k = 6$,
\[
\begin{align*}
U_6(x) &= -4032x^6 - 3616x^5 - 32992x^4 - 2244x^3 - 332x^2 - 2x \\
A_5(n, x) &= (1024n^6 - 5632n^5 + 12544n^4 - 384n^3 - 2630n^2 - 14768n)x^5 \\
&\quad - (128n^6 - 5632n^5 + 6208n^4 - 15684n^3 + 8688n^2 + 28304n)x^4 \\
&\quad + (640n^6 - 2112n^5 - 19552n^4 + 6408n^3 - 384n^2 - 2792n)x^3 \\
&\quad - (160n^6 - 352n^5 - 404n^4 + 444n^3 + 2436n^2 - 280n)x^2 \\
&\quad + (20n^6 - 22n^5 - 50n^4 - 60n^3 - 40n^2 - 14n)x - n^6.
\end{align*}
\]
3. Application of the Invariant Summation Formula

In this section we will connect the summation formula (2.26) with Bernoulli numbers and Bernoulli polynomials.

3.1. Connection with Bernoulli Numbers. Let $C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$ be the set of all p-adic continuous differentiable functions i.e,

$$C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p) = \{ f \mid f : \mathbb{Z}_p \rightarrow \mathbb{C}_p, f(x) \text{ is differentiable and } \frac{d}{dx} f(x) \text{ is continuous} \}.$$

Then the Volkenborn integral [8] or p-adic Bosonic integral of the function $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$ is given by

$$\int_{\mathbb{Z}_p} x^n dx = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x).$$

The above integral has the following property

(3.1) $$\int_{\mathbb{Z}_p} x^n dx = B_n, \ n = \mathbb{N} \cup \{0\},$$

where $B_n$ is the $n^{th}$ Bernoulli number. The Bernoulli numbers satisfy the following recurrence relation

$$\sum_{i=0}^{m} \binom{m+1}{i} B_i = 0, \ B_0 = 1, \ m = 1, 2, 3, \cdots$$

Now we show the connection of the invariant summation formula (2.26) with Bernoulli numbers as follows:

For $k = 1$

$$\sum_{n=0}^{\infty} \binom{2n}{n} [n(4x - 1) + 2x] x^n = 0, \ x \in \mathbb{Z}_p,$$

(3.2) $$\sum_{n=0}^{\infty} \binom{2n}{n} [(4n + 2)B_{n+1} - nB_n] = 0.$$

For $k = 2$

$$\sum_{n=0}^{\infty} \binom{2n}{n} [n^2(4x - 1)^2 - (2x + 4x^2)] x^n = 0, \ x \in \mathbb{Z}_p,$$

(3.3) $$\sum_{n=0}^{\infty} \binom{2n}{n} [(16n^2 + 12)B_{n+2} - (8n^2 - 2)B_{n+1} - n^2B_n] = 0.$$
For $k = 3$,
\begin{equation}
\sum_{n=0}^{\infty} \binom{2n}{n} [n^3(4x - 1)^3 + (8x^3 + 20x^2 + 2x)]x^n = 0, \quad x \in \mathbb{Z}_p,
\end{equation}
\begin{equation}
\sum_{n=0}^{\infty} \binom{2n}{n} [(64n^3 + 8)B_{n+3} - (48n^3 - 20)B_{n+2} + (12n^3 + 2)B_{n+1} - n^3B_n] = 0.
\end{equation}

For $k = 4$,
\begin{equation}
\sum \binom{2n}{n} [n^4(4x - 1)^4 + (-16x^4 - 144x^3 - 60x^2 - 2x)] = 0,
\end{equation}
\begin{equation}
\sum \binom{2n}{n} [(256n^4 - 16)B_{n+4} - (256n^4 + 144)B_{n+3} + (96n^4 - 60)B_{n+2} - (16n^4 + 2)B_{n+1} + n^4B_n] = 0.
\end{equation}

For $k = 5$,
\begin{equation}
\sum \binom{2n}{n} [n^5(4x - 1)^5 + (800x^5 + 352x^4 + 1176x^3 + 136x^2 + 2x)] = 0,
\end{equation}
\begin{equation}
\sum \binom{2n}{n} [(1024n^5 + 800)B_{n+5} - (1280n^5 - 352)B_{n+4} + (640n^5 - 1176)B_{n+3} - (160n^5 + 136)B_{n+2} + (20n^5 + 2)B_{n+1} - n^5B_n] = 0.
\end{equation}

For $k = 6$,
\begin{equation}
\sum \binom{2n}{n} [n^6(4x - 1)^6 + (-4032x^6 - 3616x^5 - 32992x^4 - 2244x^3 - 332x^2 - 2x)] = 0,
\end{equation}
\begin{equation}
\sum \binom{2n}{n} [(4096n^6 - 4032)B_{n+6} - (6144n^6 + 3616)B_{n+5} + (3840n^6 - 32992)B_{n+4} - (1280n^6 + 2244)B_{n+3} + (240n^6 - 332)B_{n+2} - (24n^6 + 2)B_{n+1} + n^6B_n] = 0.
\end{equation}

The important fact to notice is that all the above series (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) of the Bernoulli numbers are p-adic convergent because $|B_n|_p \leq p$, [8].

3.2. Connection with Bernoulli Polynomials: Bernoulli polynomials satisfy the following relation
\begin{equation}
B_n(x+1) - B_n(x) = nx^{n-1}.
\end{equation}

Then the summation formula (2.26) produces the following relations with Bernoulli polynomials:
For $k = 1$,
\begin{equation}
\sum \binom{2n}{n} \left[\frac{4n + 2}{n + 2} \left\{ B_{n+2}(x+1) - B_{n+2}(x) \right\} - \frac{n}{n + 1} \left\{ B_{n+1}(x+1) - B_{n+1}(x) \right\} \right] = 0,
\end{equation}
For $k = 2$,

$$\sum \binom{2n}{n} \left[ \frac{16n^2 - 4}{n+3} \left\{ B_{n+3}(x+1) - B_{n+3}(x) \right\} - \frac{8n^2 + 2}{n+2} \left\{ B_{n+2}(x+1) - B_{n+2}(x) \right\} \right] + \frac{n^2}{n+1} \left\{ B_{n+1}(x+1) - B_{n+1}(x) \right\} = 0,$$

(3.10)

For $k = 3$,

$$\sum \binom{2n}{n} \left[ \frac{64n^3 + 8}{n+4} \left\{ B_{n+4}(x+1) - B_{n+4}(x) \right\} - \frac{48n^3 - 20}{n+3} \left\{ B_{n+3}(x+1) - B_{n+3}(x) \right\} \right] + \frac{12n^3 + 2}{n+2} \left\{ B_{n+2}(x+1) - B_{n+2}(x) \right\} - \frac{n^3}{n+1} \left\{ B_{n+1}(x+1) - B_{n+1}(x) \right\} = 0,$$

(3.11)

For $k > 3$, there are similar relations involving Bernoulli polynomials. Putting $x = 0$ in the relations (3.9), (3.10) and (3.11), we will come back to the relations involving Bernoulli numbers. In other words, putting $x = 0$ in the relations (3.9), (3.10) and (3.11), we will get some p-adic convergent series involving Bernoulli numbers.

4. Conclusion

The paper deals with the p-adic convergence of the constructed p-adic functional series with binomial coefficient. We have then deduced a summation formula of p-adic power series and shown that the summation formula is an invariant, i.e., for rational argument, the sum is rational and in fact the invariant sum is 0. Then we make a direct application of the invariant summation formula with Bernoulli polynomials and Bernoulli numbers producing some relations involving Bernoulli numbers and Bernoulli polynomials.

Acknowledgement: The second author is grateful to The Council Of Scientific and Industrial Research (CSIR), Government of India, for the award of JRF (Junior Research Fellowship).

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