AN EXPLICIT ERROR TERM IN THEOREM A

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1. INTRODUCTION

Recall that Theorem A above ensures the existence of a constant \( \delta > 0 \) such that the number \( N(V) \) of sLag fibrations with volume \( \leq V \) in a generic twistor family of K3 surfaces is

\[
N(V) = C \cdot V^{20} + O(V^{20-\delta})
\]

where \( C > 0 \) is the ratio of volumes of two concrete homogenous spaces.

The goal of this appendix is to prove that \( \delta \) can be taken to be \( \left( \frac{4}{697633} \right) \): 

**Theorem 1.1.** In the same setting as Theorem A above, one actually has

\[
N(V) = C \cdot V^{20} + O_\varepsilon\left(V^{\frac{10352656}{697633}+\varepsilon}\right)
\]

for all \( \varepsilon > 0 \).

2. REDUCTION OF THEOREM 1.1 TO DYNAMICS IN HOMOGENOUS SPACES

Filip derived his counting formula (1.1) from certain equidistribution results. More precisely, let \( \Lambda \subset H^2(S, \mathbb{Z}) \) be a lattice isomorphic to \( H^2(S, \mathbb{Z}) \), where \( S \) is a K3 surface. Fix \( P \subset \Lambda_R \) a positive-definite 3-plane. Denote by \( \Lambda^0 \subset \Lambda \) the set of primitive isotropic integral vectors and fix \( e \in \Lambda^0 \). For each \( v \in \Lambda_R = P \oplus P^\perp \), let \( v = (v)P \oplus (v)P^\perp \) with \( (v)P \in P \) and \( (v)P^\perp \in P^\perp \). Consider the orthogonal group \( G := O(\Lambda_R) \), the lattice \( \Gamma := O(\Lambda) \) and the maximal compact subgroup \( K := O(P) \times O(P^\perp) \) of \( G \), and, for a fixed \( e \in \Lambda^0 \), denote by \( H_e := Stab_G(e) \) and \( \Gamma_e := Stab_\Gamma(e) \).

The volumes of the locally homogenous spaces \( X := \Gamma \setminus G \) and \( Y := \Gamma_e \setminus H_e \) are finite. As it is observed in [3, pp. 4], the constants \( C > 0 \) and \( \delta > 0 \) in (1.1) are the constant described in [3, Theorem 3.1.3]. In particular,

\[
C = \frac{\text{Vol} \ Y}{20 \cdot \text{Vol} \ X}
\]

The constant \( \delta > 0 \) is related to the dynamics of a certain one-parameter subgroup \( a_t \) of \( G \approx SO(3, 19)(\mathbb{R}) \). More concretely, given \( e \) and \( P \) as above, let \( e' \) be the isotropic vector given by

\[
e' := (e)P \oplus -(e)P^\perp \quad \text{where} \quad e := (e)P \oplus (e)P^\perp
\]

In this context, we denote by \( \{a_t\}_{t \in \mathbb{R}} \subset G \) the one-parameter subgroup defined as

\[
a_t \cdot e = \exp(-t) \cdot e, \quad a_t \cdot e' = \exp(t) \cdot e', \quad a_t |_{e \oplus e'} = \text{id}.
\]

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It is explained in [3] Subsection 3.6.9 that the quantity $\delta$ in (1.1) is

$$
\delta = \frac{\delta_0}{d_{l_0} + 1}
$$

where $(d_l)_{l \in \mathbb{N}}$ are the exponents in [3] Proposition 3.5.10 (ii), and $\delta_0 > 0$, $l_0 \in \mathbb{N}$ are the constants in the following equidistribution statement in [3] Theorem 4.3.1:

$$
\int_{Y_{\alpha t}} w \, d\mu_{\gamma_{\alpha t}} = \frac{\text{Vol} Y}{\text{Vol} X} \int_X w \, d\mu_X + O(\|w\|_l \varepsilon^{-\delta_0 t})
$$

for all Sobolev scales $l \geq l_0$ (see [3] §4.2.2 for the definition of the Sobolev norms in this context).

A quick inspection of the proof of [3] Proposition 3.5.10 (ii) (related to the thickening of $K$) reveals that the exponents $d_l$ depend linearly on $l$. In fact, the constant $c_1(l)$ in [3] Equation (3.5.15) gives the power of $\varepsilon$ associated to the volumes of $\varepsilon$-balls at the origin of $p_m \times n^+ \times a$, that is, $c_1(l) = \dim(G) - \dim(K)$ (and, hence, $c_1(l)$ indepdends of $l$). Since the $l$-th derivative of $\chi_\varepsilon$ is bounded by a multiple of $\varepsilon^{-c_1(l) - l}$ and it is supported in a $\varepsilon$-neighborhood of $K$, the $l$-Sobolev norm of $\chi_\varepsilon$ is bounded by a multiple of $\varepsilon^{-(l-c_1(l))/2}$. Therefore,

$$
d_l := l + \frac{\dim(G) - \dim(K)}{2}.
$$

3. Equidistribution and rates of mixing

The constants $\delta_0 > 0$ and $l_0 \in \mathbb{N}$ in (2.2) are described in [3] pp. 36 and they are related to the geometry of $Y \subset X$ and the rate of mixing of $\alpha_t$.

3.1. Injectivity radius. We denote by $\text{inj}(x)$ the local injectivity radius at a point $x \in X$ and we let $Y_\varepsilon := \{y \in Y : \text{inj}(y) \geq \varepsilon\}$. By [3] Proposition 4.1.3, we know that the arguments of [1] Lemma 11.2 provide a constant $p > 0$ such that $\mu_Y(Y \setminus Y_\varepsilon) = O(\varepsilon^p)$. Actually, a close inspection of these arguments (of integration over Siegel sets) reveal that $p = 1$ in our specific setting (of $G \simeq SO(3,19)(\mathbb{R})$):

$$
\mu_Y(Y \setminus Y_\varepsilon) = O(\varepsilon)
$$

3.2. Thickening of $Y$. Let us fix some parameter $0 < \varepsilon' < 1$ (very close to one in practice) and consider [3] Proposition 4.1.6] (of thickening of $Y'$) where it is constructed a family of smooth versions $\phi_\varepsilon$ of the characteristic function of $Y$. As it turns out, $\phi_\varepsilon$ is the product of two functions: $\tau_\varepsilon$ is a bump function supported on $Y_{\varepsilon'}$ and $\rho_\varepsilon$ is a bump function supported on the $\varepsilon$-neighborhood of the identity in a certain Lie group $N'$ of dimension $\dim(N') = \dim(X) - \dim(Y)$.

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1Indeed, [3] pp. 29] says that the optimal choice of $\delta$ occurs precisely when the terms $\varepsilon e^{20T} = e^{(20-\delta)T}$ and $e^{-d_{l_0} e^{(20-\delta_0)T}} = e^{(20-\delta_0+4d_{l_0})T}$ have the same order in $T$.

2In fact, Filip sets $\varepsilon' = 1/2$ for his construction of $\tau_\varepsilon$, but any value of $0 < \varepsilon' < 1$ can be taken here: indeed, the construction of $\tau_\varepsilon$ can be made as soon as the local product structure statement [3] Proposition 4.1.5] holds (and this is the case for any choice of $0 < \varepsilon' < 1$ because $e^{\varepsilon'} \gg 2\varepsilon$ for all sufficiently small $\varepsilon > 0$).
The bump function $\rho_\varepsilon$ is obtained by rescaling of a fixed smooth bump function on $N'$, so that its $l$-th Sobolev norm satisfies $\|\rho_\varepsilon\|_l = O(\varepsilon^{-l - \frac{\dim(X) - \dim(Y)}{2}})$.

The function $\tau_\varepsilon$ is

$$\tau_\varepsilon = \frac{\sum_{y_j \in \mathcal{F}} \beta_{y_j, \varepsilon}}{\sum_{y_i \in \mathcal{G}} \beta_{y_i, \varepsilon}},$$

where $\{y_k\} \subset Y_{\varepsilon'}$ is a maximal collection of points such that the balls $B(y_k, \varepsilon^3) \subset Y$ are mutually disjoint, $\mathcal{F} = \{y_k\} \cap Y_{4\varepsilon'}$, $\mathcal{G} = \{y_k\} \cap Y_{2\varepsilon'}$, and the functions $\beta_{y, \varepsilon}$ are translates of a bump function $\beta_\varepsilon$ whose $l$-th Sobolev norm is $\|\beta_\varepsilon\|_l = O(\varepsilon^{-l - \frac{\dim(Y)}{2}})$.

On one hand, since a ball $B$ of radius $\varepsilon$ at a point of $Y_{\varepsilon'}$ has volume $O(\varepsilon^{\dim(Y)})$, the cardinality of $\mathcal{G} \cap B$ is $O(\varepsilon^{-2 \dim(Y)})$, the arguments in [1] pp. 1928 imply that the $L^\infty$-norm of the first $l$ derivatives of $1/\beta_{\mathcal{G}, \varepsilon}$ is $O(\varepsilon^{-l - 2 \dim(Y)})$. On the other hand, the cardinality of $\mathcal{F}$ is $O(\varepsilon^{-3 \dim(Y)})$ and $\|\beta_{y_j, \varepsilon}\|_l = \|\beta_\varepsilon\|_l$. It follows that

$$\|\tau_\varepsilon\|_l = O(\varepsilon^{-l - \frac{\dim(Y)}{2}}).$$

By inserting these facts into the definition of $\phi_\varepsilon$ in [3] Equation (4.1.7), we deduce from Sobolev’s lemma that

$$\|\phi_\varepsilon\|_l = O(\varepsilon^{-2l - \frac{\dim(Y)}{2}}),$$

for all $l > \dim(X)/2$, that is, the constant $C_l$ in [3] Proposition 4.1.6 (iii)] is

$$C_l := 2l + 4 \dim(Y) + \frac{\dim(X)}{2}.$$

For later use, notice that $\phi_\varepsilon$ verifies $\int_X \phi_\varepsilon \, d\mu_X = \text{Vol } Y + O(\text{Vol}(Y \setminus Y_{\varepsilon'})).$

By combining this estimate with (3.1), we get

$$\int_X \phi_\varepsilon \, d\mu_X = \text{Vol } Y + O(\varepsilon^{3'}).$$

3.3. Wavefront lemma. The proof of Lemma 4.1.10 in [3] says that

$$\int_X w \cdot (\phi_\varepsilon \cdot a_t) \, d\mu_X = \int_Y w(ya_t \varepsilon) \, d\mu_Y + O(\varepsilon \text{Lip}(w)) + O(\varepsilon^{3p'} \|w\|_{L^\infty})$$

where $p > 0$ is the parameter such that $\mu_Y(Y \setminus Y_{\varepsilon'}) = O(\varepsilon^{3p'})$. Therefore, we deduce from (3.1) and Sobolev’s lemma that

$$\int_X w \cdot (\phi_\varepsilon \cdot a_t) \, d\mu_X = \int_Y w(ya_t \varepsilon) \, d\mu_Y + O(\varepsilon^{3p'} \|w\|_l)$$

for all $l > 1 + \dim(X)/2$. 

3.4. Reduction of equidistribution to rate of mixing. By following [3, pp. 36], let us compute the constants $\delta_0 > 0$ and $l_0 \in \mathbb{N}$ in (2.2) in terms of the following quantitative mixing statement: there exists $\delta'_0 > 0$ such that

\[
\left| \int_X \alpha \cdot (\beta \cdot g) d\mu - \left( \int_X \alpha d\mu \right) \left( \int_X \beta d\mu \right) \right| = O\left( \|\alpha\|_l \|\beta\|_l \|g\|^{-\delta'_0} \right)
\]

for all $l \geq l'_0$. (Here, $\mu = \mu_X / \text{Vol} X$ is the normalized Haar measure.)

By (3.2) and (3.3), the previous estimate implies

\[
\int_X w \cdot (\phi \cdot a_t) d\mu_X = \frac{\text{Vol} Y}{\text{Vol} X} \left( \int_X wd\mu_X \right) + O(\|w\|_l) + O(\|w\|_l e^{-C_1 e^{-t\delta'_0}})
\]

for all $l \geq \max\{l'_0, \lfloor \text{dim}(X)/2 \rfloor + 1\}$.

By plugging (3.4) into the estimate above, we conclude that

\[
\int_{Y a_t} wd\mu_{Y a_t} = \frac{\text{Vol} Y}{\text{Vol} X} \left( \int_X wd\mu_X \right) + O(\|w\|_l) + O(\|w\|_l e^{-C_1 e^{-t\delta'_0}})
\]

for all $l \geq l_0 := \max\{l'_0, \lfloor \text{dim}(X)/2 \rfloor + 2\}$.

By taking $\varepsilon := e^{-\delta'_0}$ and by optimizing\(^\text{3}\) the value of $\delta'_0$, we obtain that

\[
\int_{Y a_t} wd\mu_{Y a_t} = \frac{\text{Vol} Y}{\text{Vol} X} \left( \int_X wd\mu_X \right) + O(\|w\|_l e^{-t\delta_0})
\]

for $l_0 := \max\{l'_0, \lfloor \text{dim}(X)/2 \rfloor + 2\}$ and $\delta_0 = \frac{\delta'_0}{p + 2l_0}$.

Since $0 < p' < 1$ is an arbitrary parameter, we deduce that (2.2) holds for $l_0 := \max\{l'_0, \lfloor \text{dim}(X)/2 \rfloor + 2\}$ and any choice of $\delta_0$.

(3.6) 

\[
0 < \delta_0 < \frac{\delta'_0}{1 + 2l_0 + 4 \text{ dim}(Y) + \frac{\text{dim}(X)}{2}}
\]

4. Rates of mixing and representation theory

Definition 4.1. 1. A unitary representation $\pi$ of $G$ in a (separable) Hilbert space $\mathcal{H}_\pi$ is a morphism $G \to U(\mathcal{H}_\pi)$ such that for any $v \in \mathcal{H}_\pi$ the map $G \to \mathcal{H}_\pi; g \mapsto \pi(g)v$ is continuous. If this map is smooth one says that $v$ is a $C^\infty$-vector of $\pi$. We denote by $\mathcal{H}_\pi^c$ the set of $C^\infty$-vectors of $\pi$.

2. Given two vectors $v, w \in \mathcal{H}_\pi$, we define the matrix coefficient $c_{v,w} : G \to \mathbb{C}$ of $\pi$ as the continuous map $g \mapsto \langle \pi(g)v, w \rangle$. The coefficient $c_{v,w}$ is said to be $K$-finite if both the vector spaces generated by $\pi(K) \cdot v$ and $\pi(K) \cdot w$ are finite dimensional.

3. Let $p(\pi)$ be the infimum of the set of real numbers $p \geq 2$ such that all $K$-finite matrix coefficients of $\pi$ are in $L^p(G)$.

\(^3\text{i.e., we choose } \delta'_0 > 0 \text{ so that } e^{p'} = e^{-C_1 e^{-t\delta'_0}}.\)
4. Say that a unitary representation $\sigma$ of $G$ is \textit{weakly contained} in $\pi$ if any matrix coefficient of $\sigma$ can be obtained as the limit, with respect to the topology of uniform convergence on compact subsets, of a sequence of matrix coefficients of $\pi$.

Given an element $g = nak \in G$, we write $a = e^{H(g)}$. The \textit{Harish-Chandra} function is $\Xi = \Xi_G : G \to \mathbb{R}$ defined by

$$\Xi(g) = \int_K e^{-\rho(H(kg^{-1}))} dk$$

where $\rho$ is half the sum of the positive restricted roots counting multiplicities. The Harish-Chandra function decreases exponentially fast along $A^+$; modulo a polynomial factor of a logarithmic argument, it decreases like $e^{-\rho(H)}$.

Let $d = \dim(K)$ be the dimension of $K$ and fix a basis $B$ of the Lie algebra $\mathfrak{k}$ of $K$. Given a smooth vector $v \in \mathcal{H}_\pi$ we set

$$S(v) = \sum_{\text{ord}(D) \leq [d/2]+1} ||\pi(D)v||,$$

where $D$ varies among all monomials in elements of $B$ of degree $\leq [d/2] + 1$ and, if $X_1, \ldots, X_r$ are elements of $B$, we have $\pi(X_1 \cdots X_r) = \pi(X_1) \cdots \pi(X_r)$ and each $\pi(X_i)$ acts by derivation.

\textbf{Proposition 4.2.} For all positive $\varepsilon$ and $k \in \mathbb{N}^*$, there exists a constant $C = C(\varepsilon, k)$ such that if $\pi$ is a unitary representation of $G$ with $p(\pi) \leq 2k$, then for all $v, w \in \mathcal{H}_\pi$ and for all positive $t$ we have:

$$|\langle \pi(a_t)v, w \rangle| \leq CS(v)S(w)e^{-p/k-\varepsilon}t,$$

where $p = \rho(H)$ and $H$ is the infinitesimal generator of the one-parameter subgroup $(a_t)$.

\textit{Proof.} Up to replacing $\pi$ by the tensor product $\pi^\otimes k$ we may suppose that $k = 1$; see [2, p. 108]. It then follows from [2, Theorem 1] that $\pi$ is weakly contained in the (right) regular representation $L^2(G)$. We are then reduced to prove the proposition in the case where $\pi$ is the regular representation of $G$ (and $k = 1$); see the proof of [2, Theorem 2] for more details on this last reduction.

Now consider $v$ and $w$ in $L^2(G) \cap C^\infty(G)$. The functions

$$\varphi : x \mapsto \sup_{k \in K} |v(xk)| \quad \text{and} \quad \psi : x \mapsto \sup_{k \in K} |w(xk)|$$

are both positive and $K$-invariant, and we have:

$$|\langle \pi(a_t)v, w \rangle_{L^2(G)}| \leq \int_G \varphi(xa_t)\psi(x)dx = |\langle \pi(a_t)\varphi, \psi \rangle_{L^2(G)}|.$$

Now the Sobolev lemma (see [5, Proposition 2.6]) implies that the $L^\infty$ norms of $\varphi$ and $\psi$ can be estimated in terms of their Sobolev norms along $K$. More precisely: there exists a constant $c$ such that the for all $x \in G$,

$$\varphi(x)^2 = \sup_{k \in K} |v(xk)|^2 \leq c \sum_{\text{ord}(D) \leq [d/2]+1} ||(\rho(D)v)(x)\rangle||_{L^2(K)}.$$
Integrating over \( G \) (here we assume for simplicity that the measure of \( K \) is 1) one concludes that \( ||\varphi||_{L^{2}(G)} \leq \sqrt{ES(v)} \) and similarly for \( \psi \). It remains to prove that there exists a constant \( d_e \) such that if \( \varphi, \psi \in L^{2}(G) \) are two \( K \)-invariant, positive functions of norm 1, then

\[
|\langle \pi(a_t)\varphi, \psi \rangle| \leq d_e e^{-p/(k+\varepsilon)t}.
\]

First it follows from the computations of [2, pp. 106-107] that

\[
|\langle \pi(g)\varphi, \psi \rangle| = \int_{K} \left( \int_{NA} \varphi(na)\psi(nakg^{-1})e^{2\rho(\log a)}dn da \right) dk
\]

\[
\leq ||\varphi||_{L^{2}(G)} \int_{K} \left( \int_{NA} \psi(naH(kg^{-1})^{2}e^{2\rho(\log a)}dn da \right)^{1/2} dk
\]

\[
= ||\varphi||_{L^{2}(G)} \cdot ||\psi||_{L^{2}(G)} \int_{K} e^{-\rho(H(kg^{-1}))}dk = ||\varphi||_{L^{2}(G)} \cdot ||\psi||_{L^{2}(G)} \Xi(g).
\]

Now recall that, up to “polynomial factors of logarithmic arguments”, the function \( \Xi(a_t) \) decreases like \( e^{-\rho(H)} = e^{-pt} \). The proposition follows. \( \square \)

We shall apply this proposition to the (quasi-)regular representation \( \pi \) of \( G \) in the subspace \( L^{2}_{0}(\Gamma \backslash G) \) of \( L^{2}(\Gamma \backslash G) \) that is orthogonal to the space of constant functions. It follows from [4] that \( p(\pi) = 20 \). Proposition 4.2 therefore applies with \( k = 10 \). Note that in our case \( p = 10 \).

Now let \( \alpha \) and \( \beta \) be two smooth functions in \( L^{2}(X) \) then

\[
\alpha_{0} := \alpha - \int_{X} \alpha d\mu \quad \text{and} \quad \beta_{0} := \beta - \int_{X} \beta d\mu \in L^{2}_{0}(X)
\]

and we have:

\[
\langle \pi(g)\alpha_{0}, \beta_{0} \rangle_{L^{2}_{0}(X)} = \int_{X} \alpha \cdot (\beta \cdot g)d\mu - \left( \int_{X} \alpha d\mu \right) \left( \int_{X} \beta d\mu \right).
\]

From Proposition 4.2 and the fact that \( S(\alpha) \leq ||\alpha||_{|d/2|+1} \) we conclude that

\[
\left| \int_{X} \alpha \cdot (\beta \cdot a_{t})d\mu - \left( \int_{X} \alpha d\mu \right) \left( \int_{X} \beta d\mu \right) \right| = O(||\alpha||_{l}||\beta||_{l}e^{-\delta_{0}l})
\]

for any \( l \geq l_{0} := \lfloor \dim(K)/2 \rfloor + 1 \) and any \( \delta_{0} < 1 \).

5. END OF PROOF OF THEOREM 1.1

The explicit value of \( \delta \) announced in Theorem 1.1 can be easily derived from the discussion above. Indeed, we just saw in Section 4 that \( \delta_{0} = 1 \) and \( l_{0} = \lfloor \dim(K)/2 \rfloor + 1 \). Because \( 174 = \dim(K) < \dim(X) = 231 \) and \( \dim(Y) = 210 \), we deduce from (3.6) that \( l_{0} = \lfloor \dim(X)/2 \rfloor + 2 = 117 \) and

\[
\delta_{0} = \left( \frac{1}{1 + 2 \times 117 + 4 \times 210 + \frac{231}{2}} \right)^{-} = \left( \frac{2}{2381} \right)^{-}
\]
Finally, by inserting these informations into (2.3) and (2.1), we conclude that

$$\delta = \frac{\delta_0}{l_0 + \frac{37}{2} + 1} = \left(\frac{4}{697633}\right)^{-1} \approx (5.7336737224 \cdots \times 10^{-6})^{-1}.$$