Monopole moments and nuclear compressibility

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The nuclear compressibility has a role in nuclear physics in several ways. Its relationship to the giant monopole is well known and has been subject of much theoretical work. Less well known is its affect on in nuclear structure, namely monopole transitions between low-lying states in the spectrum. Here I revisit the topic focusing on the low-frequency β transitions in the structure of deformed nuclei. These transitions are prominent in the search for monopole transitions between low-lying states in the structure of deformed nuclei. It has often been assumed without any microscopic justification that the nucleus can be treated as an incompressible fluid. An analytic formula is derived here for the resulting relationship between monopole and quadrupole moments. The formula is shown to be well satisfied within self-consistent mean-field theory calculated with several energy functionals. The formula can be tested when both monopole and quadrupole matrix elements of band-to-band transitions are known. Data is available for the decay of the first excited K"{a} = 0⁺ band to the ground-state band in ¹⁵⁶Gd, and the extracted matrix elements are found to be consistent with the incompressible fluid picture.

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I. INTRODUCTION

P.F. Bortignon had a great interest in the interplay between collective and single-particle aspects of nuclear structure and their relation to the nuclear Hamiltonian. Among the many topics was the giant monopole vibration and its dependence on nuclear compressibility [1][2]. In this note, I want to revisit the topic focusing on the low-lying monopole transitions in the structure of deformed nuclei. These transitions are prominent in the search for low-frequency β-vibrations, reviewed two decades ago by Garrett [3].

The existence of deformed nuclei has been known almost since the beginnings of nuclear physics [5], but the theoretical understanding of nuclear shapes and sizes has greatly evolved over time. It has been commonly assumed that the nucleus is incompressible when undergoing shape changes [6][7]. However, rather simplified assumptions were made about the nucleon density distribution in those early works. A more general way to implement the incompressibility assumption is to treat the quadrupole shape changes as taking place by an irrotational and divergence-free flow. For axially symmetric deformations, a density distribution ρ that starts out as spherically symmetric ρ(r) = ρ₀(|r|) is transformed to a deformed shape ρ(⃗r') by the scaling transformation

\[ \vec{r}' = (x', y', z') = (x e^{\varepsilon}, y e^{\varepsilon}, z e^{\varepsilon}). \] (1)

Here ε is a deformation parameter. The mean square radius

\[ \langle r'^2 \rangle_\varepsilon = \frac{1}{A} \int d^3r' r'^2 \rho_0(\vec{r}') \] (2)

of a deformed nucleus of mass number A is easily expressed in terms of the mean square radius R₀² for the spherical shape. Namely,

\[ \langle r'^2 \rangle_\varepsilon = \frac{1}{3} (e^{2\varepsilon} + 2 e^{-\varepsilon}) R_0^2. \] (3)

The corresponding moment of the mass quadrupole operator

\[ \hat{Q}^2 = z^2 - (x^2 + y^2)/2 \] (4)

can be written similarly as

\[ Q_2 \equiv \langle \hat{Q}_2 \rangle_\varepsilon = \frac{1}{3} (e^{2\varepsilon} - e^{-\varepsilon}) A R_0^2. \] (5)

Combining Eqs. (3) and (5), there is a parameter-free analytic relationship F between \[ q = Q_2/AR_0^2 \] and \[ \langle r'^2 \rangle_\varepsilon/R_0^2 \] which can be expressed

\[ \langle r'^2 \rangle_\varepsilon = R_0^2 F(q). \] (6)

The formula for F is derived in the Appendix as Eq. (15).

II. \( \langle r'^2 \rangle_\varepsilon \) FROM SELF-CONSISTENT MEAN FIELD THEORY

The next question is how well Eq. (6) is satisfied in microscopic nuclear structure theory. In particular, self-consistent mean-field (SCMF) theory is the method of choice for calculating properties of heavy nuclei including giant monopole vibrations. For this study SCMF is employed using three different energy functionals: the Gogny D1S [8], the BCPM [9], and the Skyrme SLy4 [10]. The functionals are similar in that they all require a density-dependent contact interaction to achieve nuclear saturation. However, the details of the interaction are rather different. In the Gogny D1S and the Skyrme SLy4 the form of the density dependence is a power law \[ \rho^\alpha \] with \[ \alpha = 1/3 \] for D1S and \[ 1/6 \] for SLy4. In the BCPM the density dependence is fitted to the equation of state of nuclear matter. Fluctuations in shape are accessed by the Generator Coordinate Method (GCM), which involves minimizing the energy of the system while constraining some shape variable(s).
The calculations were carried out for the nucleus $^{156}$Gd, the subject of a recent experiment \[11\]. The energy minimizations were performed in the Hartree-Fock-Bogoliubov approximation constraining the mass quadrupole moment. The code HFBaxial \[12\] was used for the D1S and the BCPM functionals, while the SLy4 functional was treated by the code ev8 \[13\].

Comparisons of calculated mean square charge radii with the incompressibility formula are shown in Fig. 1. For all three functionals the mean square radii are close to that predicted by Eq. \[8\]. This provides a theoretical justification of the incompressibility assumption as applied to fluctuations in the quadrupole deformation.

III. TRANSITIONS BETWEEN BANDS

Here I derive a formula for the ratio of monopole to quadrupole matrix elements connecting two bands in a deformed nucleus. It is conventional to rescale the quadrupole field operator to a dimensionless operator $\hat{\beta}$ as $\hat{\beta} = Q_2/kAR_0^2$, where the proportionality constant is $k = 3/(20\pi)^{1/2}$ and the spherical radius parameter is taken is $R_0 = 1.241^{1/3}$ fm. However, the scaling does not affect the physical relationships and so I will use the $Q_2$ directly in the present derivation. The matrix elements to be calculated are between the intrinsic wave functions of two bands $a$ and $b$. The wave functions of both bands are expanded in a basis of deformed Hartree-Fock (HF) configurations, and it is assumed that the $Q_2$ operator is diagonal in that basis. Then the matrix element of $Q_2$ can be expressed

$$\langle a | Q_2 | b \rangle = \sum_k c_{ak}^*c_{bk} \langle k | Q_2 | k \rangle \quad (7)$$

where $|k\rangle$ are the HF configurations and $c_{ak}$, $c_{bk}$ are their amplitudes in the intrinsic states. Under the incompressibility assumption, the matrix elements of the monopole operator are given by the formula

$$\langle a | r^2 | b \rangle = AR_0^2 \sum_k c_{ak}^*c_{bk} F \left( \frac{\langle k | Q_2 | k \rangle}{AR_0^2} \right). \quad (8)$$

If the quadrupole moments of the HF components are all equal, the matrix elements vanishes due to the orthogonality of the two bands. Thus it is only fluctuations in $Q_2$ that permit a nonzero monopole matrix element. Recognizing that the fluctuation is small compared to the average matrix element, the quadrupole operator is written as the sum of the average and the fluctuation,

$$Q_2 = \langle Q_2 \rangle + \delta Q_2. \quad (9)$$

Inserting this into Eq. \[8\], a first-order Taylor expansion of $F$ yields

$$\langle a | r^2 | b \rangle = AF'(\langle Q_2 \rangle/AR_0^2)\langle a | \hat{Q}_2 | b \rangle \quad (10)$$

where $F' = dF/dq$.

IV. EXPERIMENT

To compare Eq. \[10\] with experiment, one has to extract the band matrix elements from the measured spectroscopic transitions between levels. For well-deformed systems, the relationship is given in Eq. (4-68a) in Ref. \[7\]. If there is only one measured transition it is far from
clear how well the band picture is satisfied. However, recently a number of transitions were measured in the nucleus $^{156}$Gd. There the band analysis was carried out for seven quadrupole transitions between the first excited band $|0^+⟩$ and the ground state band $|g⟩$. The entire set of transitions was consistent with a single band-to-band matrix element of the charge quadrupole operator $Q^c_2$, reported as

$$\langle g|Q^c_2|0^+_g⟩ = 30 \text{ e-fm}^2.$$  \hspace{1cm} (11)

We may assume that the charge moments obey the same relation found for the mass moments, replacing $A$ by $Z$ in Eq. (10). We still need an estimate of intrinsic quadrupole moment of $|g⟩$ to apply Eq. (10). In SCMF the moment is about 800 fm$^2$, giving $q ≈ 0.13$ and $F'(q) ≈ 0.22$. Inserting this into Eq. (10), one obtains

$$\langle g|Q^c_2|0^+_g⟩ = 6.6 \text{ e-fm}^2.$$  \hspace{1cm} (12)

For the experimental value, the tabulation gives a range $\langle g|Q^c_2|0^+_g⟩ = 6 - 10 \text{ fm}^2$. Thus experiment is consistent with the incompressibility assumption, but the uncertainties are too large to measure deviations from it. It would require much more accuracy in the monopole measurement to make a really strong test. But it is certainly worth the effort to learn more about nuclear compressibility.

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**VI. APPENDIX**

The incompressibility formula can be easily derived from Eq. (3) and (5). First, change the deformation variable from $ε$ to $w = e^ε$. Then Eq. (5) is a cubic equation for $w$ in terms of $q = Q_2/AR_0^2$. The physical solution is

$$w = \frac{q}{D} + D$$  \hspace{1cm} (13)

where

$$D = \left(1 + \frac{\sqrt{1 - 4q^3}}{2}\right)^{1/3}.$$  \hspace{1cm} (14)

Substituting Eq. (14) in Eq. (3) yields

$$F(q) = \frac{1}{3} \left(\frac{q}{D} + D\right)^2 + \frac{2}{3} \left(\frac{q}{D} + D\right)^{-1}.$$  \hspace{1cm} (15)

The function $F(q)$ is shown in Fig. 2.
FIG. 2: The incompressibility function $F(q)$ in the range $-1/2 < q < 1/2$. 