The effective QCD phase diagram and the critical end point

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We study the QCD phase diagram on the plane of temperature $T$ and quark chemical potential $\mu$, modelling the strong interactions with the linear sigma model coupled to quarks. The phase transition line is found from the effective potential at finite $T$ and $\mu$ taking into accounts the plasma screening effects. We find the location of the critical end point (CEP) to be $(\mu_{\text{CEP}}/T_c, T_{\text{CEP}}/T_c) \sim (1.2, 0.8)$, where $T_c$ is the (pseudo)critical temperature for the crossover phase transition at vanishing $\mu$. This location lies within the region found by lattice inspired calculations. The results show that in the linear sigma model, the CEP’s location in the phase diagram is expectedly determined solely through chiral symmetry breaking. The same is likely to be true for all other models which do not exhibit confinement, provided the proper treatment of the plasma infrared properties for the description of chiral symmetry restoration is implemented. Similarly, we also expect these corrections to be substantially relevant in the QCD phase diagram.

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The different phases in which matter, made up of quarks and gluons, arranges itself depends, as for any other substance, on the temperature and density, or equivalently, on the temperature and chemical potentials. Under the assumptions of beta decay equilibrium and charge neutrality, the representation of the QCD phase diagram is two dimensional. This is customary plotted as the vertical variable and the temperature $T$ as the vertical one. $\mu$ is related to the baryon chemical potential $\mu_B$ by $\mu_B = 3\mu$.

Most of our knowledge of the phase diagram is restricted to the $\mu = 0$ axis. The phase diagram is, by and large, unknown. For physical quark masses and $\mu = 0$, lattice calculations have shown \cite{1} that the change from the low temperature phase, where the degrees of freedom are hadrons, to the high temperature phase described by quarks and gluons, is an analytic crossover. The phase transition has a dual nature: on the one hand the color-singlet hadrons break up leading to deconfined quarks and gluons; this is dubbed as the deconfinement phase transition. On the other hand, the dynamically generated component of quark masses within hadrons vanishes; this is referred to as chiral symmetry restoration.

Lattice calculations have provided values for the crossover (pseudo)critical temperature $T_c$ for $\mu = 0$ and $2+1$ quark flavors using different types of improved rooted staggered fermions \cite{2}. The MILC collaboration \cite{3} obtained $T_c = 169(12)(4)$ MeV. The RBC-Bielefeld collaboration \cite{4} reported $T_c = 192(7)(4)$ MeV. The Wuppertal-Budapest collaboration \cite{5} has consistently obtained smaller values, the latest being $T_c = 147(2)(3)$ MeV. The HotQCD collaboration \cite{6} has computed $T_c = 154(9)$ MeV. The differences could perhaps be attributed to different lattice spacings.

The picture presented by lattice QCD for $T \geq 0, \mu$ cannot be easily extended to the case $\mu \neq 0$, the reason being that standard Monte Carlo simulations can only be applied to the case where either $\mu = 0$ or is purely imaginary. Simulations with $\mu \neq 0$ are hindered by the sign problem, see, for example, \cite{7}, though some mathematical extensions of lattice techniques \cite{8} can probe this region. Schwinger-Dyson equation studies support these findings and can successfully explore all region of the phase space \cite{9}.

On the other hand a number of different model approaches indicate that the transition along the $\mu$ axis, at $T = 0$, is strongly first order \cite{10}. Since the first order line originating at $T = 0$ cannot end at the $\mu = 0$ axis which corresponds to the starting point of the crossover line, it must terminate somewhere in the middle of the phase diagram. This point is generally referred to as the critical end point (CEP). The location and observation of the CEP continue to be at the center of efforts to understand the properties of strongly interacting matter under extreme conditions. The mathematical extensions of lattice techniques place the CEP in the region $(\mu_{\text{CEP}}/T_c, T_{\text{CEP}}/T_c) \sim (1.0 - 1.4, 0.9)$.

In the first of Refs. \cite{8}, it is argued that the theoretical location of the CEP depends on the size of the confining length scale used to describe strongly interacting matter at finite density/temperature. This argument is supported by the observation that the models which do not account for this scale \cite{11,12} produce either a CEP closer...
to the $\mu$ axis ($\mu_{\text{CEP}} / T_c$ and $T_{\text{CEP}} / T_c$ larger and smaller, respectively) or a lower $T_c$ [15] than the lattice based approaches or the ones which consider a finite confining length scale. Given the dual nature of the QCD phase transition, it is interesting to explore whether there are other features in models which have access only to the chiral symmetry restoration facet of QCD that, when properly accounted for, produce the CEP’s location more in line with lattice inspired results.

An important clue is provided by the behavior of the critical temperature as a function of an applied magnetic field. Lattice calculations have found that this temperature decreases when the field strength increases [10–18]. It has been recently shown that this phenomenon, dubbed \textit{inverse magnetic catalysis}, is not due exclusively to confinement but instead that chiral symmetry restoration plays an important role. This result is born out of the decrease of the coupling constant with increasing field strength and is obtained within effective models that do not have confinement such as the Abelian Higgs model or the linear sigma model with quarks. The novel feature implemented in these calculations is the handling of the screening properties of the plasma, which effectively make the treatment go beyond the mean field approximation [15, 20].

The importance of accounting for screening in plasmas where massless bosons appear has been pointed out since the pioneering work in Ref. [21] and implemented in the context of the Standard Model to study the electroweak phase transition [22]. Screening is also important to obtain a decrease of the coupling constant with the magnetic field strength in QCD in the Hard Thermal Loop approximation [23]. In this work we explore the consequences of the proper handling of the plasma screening properties in the description of the effective QCD phase diagram within the linear sigma model with quarks. We find that for certain values of the model parameters, obtained from physical constraints, the CEP’s location agrees with lattice inspired calculations. Since the linear sigma model does not have confinement, we argue that it is the adequate description of the plasma screening properties for the chiral symmetry breaking within the model which determines the CEP’s location.

We start from the linear sigma model coupled to quarks. It is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sigma \partial_{\mu} \partial^{\mu} \sigma + \frac{1}{2} \bar{\pi} \partial_{\mu} \pi^{\mu} + \frac{\lambda}{4} \sigma^2 + \bar{\pi} \pi - \frac{\lambda}{4} (\sigma^2 + \bar{\pi} \pi)^2$$

$$+ i \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - g \bar{\psi} \psi,$$

where $\psi$ is an SU(2) isospin doublet, $\bar{\pi} = (\pi_1, \pi_2, \pi_3)$ is an isospin triplet and $\sigma$ is an isospin singlet. The neutral pion is taken as the third component of the pion isovector, $\bar{\pi}^0 = \pi_3$ and the charged pions as $\pi_{\pm} = (\pi_1 \mp i \pi_2) / 2$. The squared mass parameter $a^2$ and the self-coupling $\lambda$ and $g$ are taken to be positive.

To allow for the spontaneous breaking of symmetry, we let the $\sigma$ field develop a vacuum expectation value $v$ with $\sigma \to \sigma + v$, (2) which can later be taken as the order parameter of the theory. After this shift, the Lagrangian density can be rewritten as

$$\mathcal{L} = -\frac{1}{2} \sigma \partial_{\mu} \partial^{\mu} \sigma - \frac{1}{2} \bar{\pi} \partial_{\mu} \pi^{\mu} - \frac{\lambda}{2} (\sigma^2 + a^2)^2$$

$$+ \frac{\lambda}{4} v^4 + i \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - g v \bar{\psi} \psi + \mathcal{L}^0 + \mathcal{L}^I,$$ (3)

where $\mathcal{L}^0$ and $\mathcal{L}^I$ are given by

$$\mathcal{L}^0 = -\frac{\lambda}{4} \left[ (\sigma^2 + (\pi^0)^2)^2 + 4 \pi^+ \pi^- (\sigma^2 + (\pi^0)^2 + \pi^+ \pi^-) \right],$$

and describe the interactions among the fields $\sigma$, $\bar{\pi}$ and $\psi$, after symmetry breaking. From Eq. (3) we see that the $\sigma$, the three pions and the quarks have masses

$$m_{\sigma}^2 = 3 \lambda v^2 - a^2,$$

$$m_{\pi}^2 = \lambda v^2 - a^2,$$

$$m_f = g v,$$ (5)

respectively.

Including the $v$-independent terms, choosing the renormalization scale as $\mu = e^{-1/2} a$ and after mass renormalization, it is straightforward to show that the effective potential up to the ring diagrams contribution for a finite chemical potential and in the limit where the masses are small compared to $T$, can be written as

$$V^{(\text{eff})} = -\frac{a^2}{2} v^2 + \frac{\lambda}{4} v^4$$

$$+ \sum_{i=\sigma, \bar{\pi}} \left\{ m_i^4 \ln \left( \frac{(4 \pi T)^2}{2 a^2} \right) - 2 \gamma_E + 1 \right\}$$

$$- \frac{\pi^2 T^4}{90} + \frac{m_{\pi}^2 T^2}{24} - \frac{T}{12 \pi} (m_{\pi}^2 + \Pi)^{3/2}$$

$$- \frac{N_c}{16 \pi^2} \sum_{f=u, d} \left\{ m_f^4 \ln \left( \frac{(4 \pi T)^2}{2 a^2} \right) + \psi^0 \left( \frac{1}{2} + \frac{i \mu}{2 \pi T} \right) + \psi^0 \left( \frac{1}{2} - \frac{i \mu}{2 \pi T} \right) \right\}$$

$$+ 8 m_{\pi}^2 T^2 \left[ L_{i_2}(e^{\mu/T}) + L_{i_2}(e^{-\mu/T}) \right]$$

$$- 32 T^4 \left[ L_{i_4}(e^{\mu/T}) + L_{i_4}(e^{-\mu/T}) \right],$$ (6)

where $\psi^0(x)$ is the digamma function and $L_{i_n}(x)$ is the polylogarithm function of order $n$. It can also be shown that the boson self-energy $\Pi$ in Eq. (6), computed for a finite chemical potential and also in the limit where the masses are small compared to $T$, is given by

$$\Pi = \frac{\lambda T^2}{2} - \frac{N_f N_c g^2 T^2}{\pi^2} \left[ L_{i_2}(e^{\mu/T}) + L_{i_2}(e^{-\mu/T}) \right],$$ (7)
Furthermore, we can fix the value of $a$ by noting from Eqs. (5) that the vacuum boson masses satisfy

$$a = \sqrt{\frac{m_\sigma^2 - 3m_\pi^2}{2}}. \quad (10)$$

Since in our scheme we consider two-flavor massless quarks in the chiral limit, we take $T_c \simeq 170$ MeV \cite{24} which is slightly larger than $T_c$ obtained in $N_f = 2 + 1$ lattice simulations. Also, in order to allow for a crossover phase transition for $\mu = 0$ (which in our description corresponds to a second order transition) with $g, \lambda < 1$ we need that $g^2 > \lambda$ and that $a$ is not very large. This last condition is obtained for a small $m_\sigma$. The Particle Data Group quotes $400$ MeV $\leq m_\sigma \leq 550$ MeV \cite{23}. There are also analyses that place $m_\sigma$ close to the two-pion threshold \cite{21}. Since $\sigma$ is anyhow a broad resonance, in order to satisfy the above requirements we take for definitiveness $m_\sigma \simeq 300$ MeV, namely, right at the beginning of the two-pion threshold. Therefore, the allowed values for the couplings $\lambda$ and $g$ are restricted by

$$\sqrt{\frac{\lambda}{2} + \frac{N_f N_c g^2}{6}} \simeq 0.8. \quad (11)$$

Equation (11) provides a restriction stemming exclusively from the boson sector. We can attempt to supplement this restriction with information from the fermion sector. The physical quark mass is given by

$$m_f = g v_0 + m_0, \quad (12)$$

where $v_0$ represents the value of $v$ where the effective potential has a minimum and $m_0$ is the current mass. When chiral symmetry is restored (referring to its dynamical component), $v_0 = 0$ and the quark mass is only given by $m_0 \simeq 5$ MeV. To restrict the allowed values of $g$, we need information on $v_0$. A finite value of $v_0$ is obtained at the beginning of the phase transition when this is first order, for otherwise if the phase transition is second order, $v_0$ vanishes. At the beginning of the first order phase transitions the value of $v_0$ must be a small fraction of the energy scale $a$. For definitiveness, let’s take this scale to be one order of magnitude smaller than $a$, namely,

$$v_0 = 0.1a. \quad (13)$$

Let us consider that at the phase transition, the light-quark masses acquire a chiral symmetry broken contribution to their masses of the same order as the current mass, namely $g v_0 \simeq 5$ MeV. Using $a \simeq 130$ MeV, we get

$$g \simeq 0.7. \quad (14)$$

Here it is pertinent to mention that though the linear sigma model is usually regarded only as a rough model to provide qualitative information at finite temperature, more quantitative statements can be made in this realm by estimating the couplings from arguments valid at the phase transition instead of only from vacuum \cite{27}.
Figure 1 shows the phase diagram obtained for a set of allowed values $\lambda$ and $g$. Note that for small $\mu$ the phase transition is second order. In this case the (pseudo)critical temperature is determined from setting the second derivative of the effective potential in Eq. (6) to zero at $v = 0$. When $\mu$ increases, the phase transition becomes first order. The critical temperature is now computed by looking for the temperature where a secondary minimum for $v \neq 0$ is degenerate with a minimum at $v = 0$. From the detailed analysis, we locate the position of the CEP as $(\mu_{\text{CEP}}/T_c, T_{\text{CEP}}/T_c) \sim (1.2, 0.8)$, which is in the same range as the CEP found from lattice inspired analyses [8].

In conclusion, we have shown that the location of the CEP we find is more in line with the location found by mathematical extensions of lattice analyses. Since the linear sigma model does not have confinement we attribute this location to the adequate description of the plasma screening properties for the chiral symmetry breaking at finite temperature and density. These properties are included into the calculation of the effective potential through the boson’s self-energy and in the determination of the allowed range for the coupling constants through the observation that the thermal boson masses vanish at the phase transition and that the chiral symmetry restored fermion mass is proportional to $v_0$ at the onset of the first order phase transitions. These observations determine a relation between the model parameters which is put in quantitative terms by taking physical values for $T_c$ from lattice calculations, for $\alpha$ from the vacuum boson masses and for $g$ from the light-quark mass. We believe this description will play an important role in determining the location of the CEP in QCD.

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