The Perturbative Gross Neveu Model Coupled to a Chern-Simons Field: A Renormalization Group Study

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Abstract

In 2+1 dimensions, for low momenta, using dimensional renormalization we study the effect of a Chern-Simons field on the perturbative expansion of fermions self interacting through a Gross Neveu coupling. For the case of just one fermion field, we verify that the dimension of operators of canonical dimension lower than three decreases as a function of the Chern-Simons coupling.

I. INTRODUCTION

Effective field theories is a subject of great interest in theoretical physics not only due to their potential applications but also because they provide new insights into the way we look at field theories [1]. From this perspective nonrenormalizable models have acquired a new status as they may become physically relevant at low energies [2]. The point is that, if the scale of energy one is interested is low enough, the ambiguities due to the virtual states of high energy do not show up or, equivalently, are not meaningful. On the energy interval where this happens the theory proceeds as an usual renormalizable one. Nonetheless, as observed in [3], the use of a mass independent regularization is almost mandatory to guarantee that high order counterterms can be effectively neglected.
The ultraviolet behavior of the Green functions may be changed by a rearrangement of the perturbative series. In fact, the incorporation of vacuum polarization effects in general improves the convergence properties of the resumed series; this mechanism is well known to be operative in the context of the $1/N$ expansion. In particular, Gross Neveu [4] or Thirring [5] like four fermion interactions which in (2+1) dimensions are perturbatively nonrenormalizable become renormalizable within the framework of the $1/N$ expansion [4]. This result has motivated a series of investigations on the properties of these theories [6]. In particular, using renormalization group (RG) methods, it has been proved that the $N$ component Gross Neveu model in 2+1 dimensions is infrared stable at low energies but has also a nontrivial ultraviolet stable fixed point. These facts indicate that the theory could be perturbatively investigated if the momentum is low enough. This actually would be the only remaining possibility for small $N$.

It has recently been conjectured that in 2+1 dimensions, besides the $1/N$ expansion, there is another way to improve the ultraviolet behavior of Feynman amplitudes. By coupling fermion fields to a Chern-Simons field the scale dimension of field operators could be lowered possibly turning non renormalizable interactions into renormalizable or, better, super-renormalizable ones. Using a sharp cutoff to regulate divergences, this idea was tested in [7] where the effect of the CS field over massless self-coupled fermions with a quartic, Gross–Neveu like, interaction was studied.

In this communication we will pursue this study further by considering massive fermions and adopting dimensional renormalization [8] as a tool to render finite the Feynman amplitudes. In this way we evade the ambiguity problem associated with the routing of the momentum flowing through the associated Feynman graphs [4]. Nevertheless, it should be stressed that our calculations are valid insofar, as said above, the effect of the higher order counterterms can be neglected. Otherwise, new couplings should be introduced. Our investigations, restricted to the case of fermions of just one flavor, i. e. $N = 1$, show that, differently to what happens for large $N$, the renormalization group beta function has only a trivial infrared stable fixed point. Moreover, the operator dimensions of the basic field and
composites of canonical dimension lower than three are monotonic decreasing functions of the CS parameter. This indicates that the Feynman amplitudes have a better ultraviolet behavior if the underlying theory is renormalizable. However, no improvement in the ultraviolet behavior seems to occur if the composite operators have canonical dimension bigger than three.

Our work is organized as follows. In section II some basic properties of the model as Feynman rules, ultraviolet behavior of Feynman diagrams and comments on the regularization procedure are presented. The derivation and calculation of the renormalization group parameters are indicated in section III. Section IV contains a discussion of our results as well as our conclusions. Details of the calculations of the pole part of the relevant amplitudes are described in the appendices A and B.

II. A QUARTIC INTERACTION

We consider a self-interacting two–component spinor field minimally coupled to a CS field. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2\pi\alpha} \epsilon^{\mu\nu\alpha} \partial_\mu A_\nu A_\alpha + \bar{\psi}(i\slashed{\partial} - m)\psi + \bar{\psi}\gamma^\mu\psi A_\mu - G(\bar{\psi}\psi)(\bar{\psi}\psi) + \frac{1}{2\lambda}(\partial_\mu A^\mu)^2,$$  \hspace{1cm} (2.1)

The Dirac field $\psi_\alpha$ represents particles and anti-particles of spin up and the same mass $m$ (the parameter $m$ is to be taken positive) \[\text{(1)}\]. The Gross-Neveu term in (2.1) is the most general Lorentz covariant quartic self-interaction, for the Thirring-like vector interaction is not independent but satisfies $: (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) := -3 : (\bar{\psi}\psi)(\bar{\psi}\psi)$. $\lambda$ is a gauge fixing parameter but for simplicity we will always work in the Landau gauge, formally obtained by letting $\lambda \rightarrow 0$. In this gauge, the Green functions may be computed using the Feynman rules depicted in Fig. I. For convenience, we have introduced auxiliary dotted lines, hereafter called auxiliary GN lines, to clarify the structure of the four fermion vertex.

Divergences show up, the degree of superficial divergence of a generic graph $\gamma$ being

$$d(\gamma) = 3 - N_A - N_F + V,$$  \hspace{1cm} (2.2)
where \( N_A \) and \( N_F \) are the number of external lines associated with the propagators for the Chern–Simons and the fermion fields, respectively; \( V \) denotes the number of quartic vertices in \( \gamma \). The model is of course nonrenormalizable and the number of counterterms necessary to render the amplitudes finite increases with the order of perturbation but, to a given order, the number of counterterms is finite. To do calculations we will employ dimensional renormalization starting at the space-time dimension \( d \). It is therefore convenient to introduce a dimensionless coupling \( g \) and a renormalization parameter \( \mu \) through \( G = (g/\Lambda)\mu^\epsilon \) and \( \alpha \rightarrow \alpha\mu^\epsilon \), where \( \epsilon = 3 - d \) must be set zero at the end. The massive parameter \( \Lambda \) must be considered much bigger than any typical momenta and than the fermion mass \( m \); it sets the scale which limits the region where our results are valid. Divergences will appear as poles in \( \epsilon \) and a renormalized amplitude is given by the \( \epsilon \) independent term in the Laurent expansion of the corresponding regularized integral. However, at one loop level no infinities will remain after the remotion of the regulator. This is so because the poles for a graph \( \gamma \) may occur only at even values of the degree of superficial divergence of \( \gamma \) \[^{11}\]. Moreover, it is easy to check that asymptotically, i. e., for zero external momenta, one loop graphs with even degree of divergence are odd functions of the loop momentum and therefore vanishes, after symmetric integration.

### III. RENORMALIZATION GROUP

As known, Green functions of renormalizable models, which have been made finite by the subtraction of pole terms in the dimensionally regularized amplitudes, satisfy a ’t Hooft-Weinberg type renormalization group equation \[^{12}\]. Nonrenormalizable models require special consideration since the form of the effective Lagrangian changes with the order of perturbation. However, at sufficient small momenta, such that the effect of the new counterterms may be neglected, the Green functions will still approximately satisfy the RG equation. Thus, although being nonrenormalizable, for small enough momenta, the Green function of the theory \(^{2,4}\) satisfy the following renormalization group equation
\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} + \mu \frac{\partial}{\partial \mu} + \delta m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial g} - N\gamma \right] \Gamma^{(N)}(p_1, \ldots, p_N) \approx 0, \tag{3.1}
\]

where \( \Gamma^{(N)}(p_1, \ldots, p_N) \) denotes the vertex function of \( N \) fermion fields (since \( A_\mu \) is not a dynamical field we shall not consider vertex functions having external vector fields). The symbol \( \approx \) means equality in the region where all counterterms different from those terms already present in (2.1) can be neglected. As a consequence of the Coleman–Hill theorem, which states that all radiative corrections to the CS term are finite, \([13]\), the beta function for the CS coupling \( \alpha \) vanishes identically; that explains why the term with a derivative with respect to \( \alpha \) is absent from (3.1).

The coefficients \( \delta, \beta \) and \( \gamma \) in equation (3.1) may be obtained by formally computing the action of the differential operator over the two point and four point Green functions. For the two point function, up to second order in the coupling constants, we have

\[
\begin{align*}
\Gamma^{(2)}(p) = i\hat{p} - m + \frac{g}{\Lambda} & \mu \epsilon^I_1^{(2)} + \alpha \mu \epsilon^I_2^{(2)} + \alpha^2(1 - \mathcal{T}) \mu^{2\epsilon} I_3^{(2)} \\
+ \frac{g\alpha}{\Lambda} & (1 - \mathcal{T}) \mu^{2\epsilon} I_4^{(2)} + \frac{g^2}{\Lambda^2} (1 - \mathcal{T}) \mu^{2\epsilon} I_5^{(2)},
\end{align*}
\tag{3.2}
\]

where the limit \( \epsilon \to 0 \) must be understood. In the above expression \( I_i, i = 1, \ldots, 5, \) denote the regularized Feynman amplitudes. In particular, the graphs ascribed to \( I_3, I_4 \) and \( I_5 \) have been depicted in the figures \([2, 3]\) and \([4]\) respectively; \( \mathcal{T} \) is an operator to remove the pole term in the amplitudes to which it is applied. As mentioned earlier, the amplitudes \( I_1^{(2)} \) and \( I_2^{(2)} \) which are associated with one loop diagrams are finite. Inserting (3.2) into (3.1) allows us to determine the coefficients \( \delta \) and \( \gamma \) as follows. Initially notice that as \( \Lambda \) enters into the perturbative expansion only in the combination \( g/\Lambda \), fixing \( \beta \) in lowest order as being equal to \( g \) eliminates all contributions of the term with the derivative with respect to \( \Lambda \) in (3.1). After that, up to the order we will study, in (3.1) there will be no mixing of higher order contribution to \( \beta \) with those to \( \gamma \) and \( \delta \).

Using the expansions,

\[
\delta = \sum_{i,j} \delta_{i,j} g^i \alpha^j, \tag{3.3}
\]
\[ \gamma = \sum_{i,j} \gamma_{i,j} g^i \alpha^j, \]  

(3.4)

where the sum is restricted to \( i + j \leq 2 \), we get

\[ \delta_{1,0} = \delta_{0,1} = \gamma_{1,0} = \gamma_{0,1} = 0, \]  

(3.5)

\[ \delta_{0,2} = -2i(A_3 + B_3) \quad \gamma_{0,2} = -iB_3 \]  

(3.6)

\[ \delta_{1,1} = -\frac{2m}{\Lambda} (A_4 + B_4) \quad \gamma_{1,1} = -\frac{m}{\Lambda} B_4 \]  

(3.7)

\[ \delta_{2,0} = \frac{2im^2}{\Lambda^2} (A_5 + B_5) \quad \gamma_{2,0} = \frac{im^2}{\Lambda^2} B_5 \]  

(3.8)

\[ \delta_{2,2} = -\frac{8}{3} \alpha^2 - \frac{11m}{8\pi \Lambda} g\alpha + \frac{7}{12\pi^2 \Lambda^2 g^2} \]  

(3.13)

\[ \gamma = -\frac{1}{12} \alpha^2 - \frac{1}{8\pi \Lambda} g\alpha + \frac{5}{48\pi^2 \Lambda^2 g^2} \]  

(3.14)

\[ \Gamma^{(4)}(p_1, p_2, p_3, p_4) = \mu^\varepsilon (-i \frac{g}{\Lambda} + \frac{g\alpha}{\Lambda} \mu^\varepsilon I_1^{(4)} + \frac{g^2}{\Lambda^2} \mu^\varepsilon I_2^{(4)} + \frac{g}{\Lambda} \alpha^2 (1 - \mathcal{T}) \mu^{2\varepsilon} I_3^{(4)} + \frac{g^2}{\Lambda^2} (1 - \mathcal{T}) \mu^{2\varepsilon} I_4^{(4)} + \frac{g^3}{\Lambda^3} (1 - \mathcal{T}) \mu^{2\varepsilon} I_5^{(4)}). \]  

(3.15)
Here, again, the one loop amplitudes $I_1^{(4)}$ and $I_2^{(4)}$ are finite because of the use of the dimensional regularization. Inserting (3.13), the expansion $\beta = \sum_{i,j} \beta_{i,j} g^i \alpha^j$, and $\delta$ and $\gamma$ given in (3.3) and (3.4) into (3.14), we obtain $\beta_{1,0} = 1$, $\beta_{0,1} = \beta_{1,1} = \beta_{2,0} = \beta_{0,2} = 0$ and

\[
\begin{align*}
\beta_{1,2} &= -4iB_3 - 2C_3, \\
\beta_{2,1} &= \frac{2im}{\Lambda} C_4 - \frac{4m}{\Lambda} B_4, \\
\beta_{3,0} &= \frac{2m^2}{\Lambda^2} C_5 + \frac{4im^2}{\Lambda^2} B_5.
\end{align*}
\]

In the above expressions, $C_i$ for $i = 3, 4, 5$ are related to the pole part of the amplitudes $I_i^{(4)}$ through,

\[
\begin{align*}
\text{Pole part of } I_3^{(4)} &= -iC_3/\epsilon, \\
\text{Pole part of } I_4^{(4)} &= -(mC_4 + O(p))/\epsilon, \\
\text{Pole part of } I_5^{(4)} &= i(m^2C_5 + O(p))/\epsilon.
\end{align*}
\]

In the appendix B we have collected the results of the calculations of the pole part of the relevant graphs. Using B6 we obtain, finally,

\[
\beta = g + \frac{20}{3} g^2 \alpha^2 - \frac{21m}{2\pi \Lambda} g^2 \alpha + \frac{161m^2}{12\pi^2 \Lambda^2} g^3.
\]

\[
\text{IV. DISCUSSION AND CONCLUSIONS}
\]

An inspection of Eq. (3.22) shows that the renormalization group beta function has $g = 0$ as a fixed point. As, for $m \ll \Lambda$, $\beta \approx g(1 + 20\alpha^2/3)$ the origin is an infrared stable fixed point. Actually, this is the only existing fixed point. Here we are in disagreement with Ref. [7] where a line of fixed points was found. The diverse conclusions are perhaps due to the use of different regularizations but a more direct comparison of the methods seems unfeasible as the calculations in [7] were not spelled out.

We will examine now the dimensions of some operators. As seen before the basic field $\psi$ has operator dimension $d_\psi = 1 - \alpha^2/12$, and so at $g = 0$ the Green functions of the fermion
field have an improved ultraviolet behavior as $\alpha$ increases. Similar results are obtained if one considers composite operators of canonical dimension less than three. The simplest of them, the mass operator $\bar{\psi}\psi$ has an anomalous dimension given by

$$\gamma_{\bar{\psi}\psi} = 2\gamma - 2\text{Res}, \quad (4.1)$$

where $\text{Res}$ is the residue coming from graphs contributing to the vertex function with the insertion of the mass operator, $\bar{\psi}\psi$ and having two external fermionic lines. For practical purpose this residue may be computed by taking the mass derivative of the contributions calculated in the item 1 of appendix A. The result is $iA_3 = 5/4$. Thus the dimension of $\bar{\psi}\psi$ turns out to be equal to

$$d_{\bar{\psi}\psi} = 2 - \frac{8}{3}\alpha^2. \quad (4.2)$$

From the computation of the anomalous dimension of the operator $\bar{\psi}\psi$ one could easily obtain the dimension of $\bar{\psi} \cdot \partial \psi$. Indeed, the dimension of $\bar{\psi} \cdot \partial \psi$ is given by (4.1) but with the replacement of $\text{Res}$ by $3iA_3 = 15/4$. Thus

$$d_{\bar{\psi} \partial \psi} = 3 - \frac{23}{3}\alpha^2. \quad (4.3)$$

The determination of the anomalous dimension of the operator $(\bar{\psi}\psi)^2$ at $g = 0$ is more complicated due to the fact that renormalization in general produces a mixing with other operators of dimension lower or equal to four. However if we restrict the calculation to the $m = 0$ case, as we will do, only operators of dimension four need to be considered. A further simplification is obtained by considering only (formally) integrated operators. We have,

$$\int d^3x N[(\bar{\psi}\psi)^2] = a_1 \int d^3x (\bar{\psi}\psi)^2 + a_2 \int d^3x \bar{\psi}\psi \partial^2 \psi \quad (4.4)$$

$$\int d^3x N[\bar{\psi}\partial^2 \psi] = b_1 \int d^3x (\bar{\psi}\psi)^2 + b_2 \int d^3x \bar{\psi}\psi \partial^2 \psi \quad (4.5)$$

where the symbol $N$ indicates a normal product prescription corresponding to the subtraction of the pole terms. A direct calculation gives that $a_2 = b_1 = 0$, $a_1 = 1 + \frac{C_2\alpha^2}{\epsilon}$ and $b_2 = 1 - \frac{\alpha^2}{3\epsilon}$. A straightforward analysis shows now that the dimensions of $\int d^3x N[(\bar{\psi}\psi)^2]$ and $\int d^3x N[\bar{\psi}\partial^2 \psi]$ are given by
respectively. One sees that, at least for the operators that we explicitly considered, the operator dimension decreases with $\alpha$ accordingly the canonical dimension is lower or equal to three, in accord with [7]. However, if the canonical dimension is bigger than three, the operator dimension increases with $\alpha$ so that no improvement for nonrenormalizable interactions results.

Our results are valid if the basic fermion field is flavorless. The $N$-flavor case is presently under investigation.

**APPENDIX A:**

In this appendix we shall present a detailed analysis of the contributions to the pole part of the two point vertex function. Due to the use of the dimensional regularization the poles only appear at the two loop level, beginning at second order in the coupling constants. We will examine separately each order of perturbation, i. e., the orders $\alpha^2$, $g\alpha$ and $g^2$. We have,

1. Order $\alpha^2$. In this order there are three diagrams which are shown in Fig. 2. These diagrams give the contributions,

$$I_3(a) = 4\pi^2 i \epsilon_{\mu}\lambda \epsilon_{\nu} \theta \epsilon \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} k_1^\lambda k_2^\nu \Tr [\gamma^\mu (k_2 + m) \gamma^\nu (k_2 - k_1 + m)]$$

$$\times \frac{\epsilon_0 \gamma^0 (\not p - \not k_1 + m) \epsilon_0}{(k_2 - m^2) [(k_2 - k_1)^2 - m^2] [(p - k_1)^2 - m^2] (k_1^2)^2}$$

$$= (m A_3(a) + \not p B_3(a)) \frac{1}{\epsilon}.$$  \hspace{1cm} (A1)

where, on the right hand side of the first equality, we have introduced the operator $\mathcal{T}$ to extract the pole part of the expression to which it is applied. From (A1) we obtain

$$\frac{A_3(a)}{\epsilon} = \frac{1}{2m} \Tr I_3(a)|_{p=0} = \frac{2i\pi^2}{m} \mathcal{T} \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d}$$

$$\times [\gamma^\mu (k_2 + m) \gamma^\nu (k_2 - k_1 + m)]$$

$$\times \frac{\epsilon_0 \gamma^0 (\not p - \not k_1 + m) \epsilon_0}{(k_2 - m^2) [(k_2 - k_1)^2 - m^2] [(p - k_1)^2 - m^2] (k_1^2)^2}$$

$$= (m A_3(a) + \not p B_3(a)) \frac{1}{\epsilon}.$$  \hspace{1cm} (A1)
\[
\times \frac{-8m^3k_1^2 - 8m(k_1^2)^2 - 8mk_2^2(k_1,k_2) + 8m(k_1,k_2)^2}{(k_2^2 - m^2)[(k_2 - k_1)^2 - m^2][k_1^2 - m^2](k_1^2)^2}.
\]

(A2)

For simplicity, the trace was taken in the integrand of (A1) and calculated directly at \(d = 3\). This, of course, does not affect the result for the pole part of the integrals. Analogously,

\[
\frac{B_3(a)}{\epsilon} = \frac{1}{2p^2} \operatorname{Tr}(I_3(a) \not{p}) = \frac{2i\pi^2}{p^2} \mathcal{T} \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{\text{Numerator}}{(k_2^2 - m^2)[(k_2 - k_1)^2 - m^2](k_1^2)^2},
\]

where

\[
\text{Numerator} = -16\epsilon_{\mu\nu\lambda} \epsilon_{\alpha\beta\sigma} k_1^\mu k_2^\nu k_1^\lambda k_2^\sigma p^\alpha - 16m^2k_1^2(k_1 \cdot p) - 8k_1^2(k_1 \cdot k_2)(k_1 \cdot p) + 8m^2(k_1 \cdot p)^2 + 8(k_1 \cdot k_2)(k_1 \cdot p)^2 - 8(k_1 \cdot k_2)^2k_2^2 - 8(k_1 \cdot k_2)^2p^2 + 8k_1^2k_2^2p^2.
\]

(A4)

Here and in what follows we shall adopt the following procedure for performing the integrals.

We first consider the \(k_2\) integral and use Feynman’s trick,

\[
\frac{1}{a_1^{\alpha_1}a_2^{\alpha_2}} = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \frac{x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1}}{[xa_1 + (1 - x)a_2]^{\alpha_1 + \alpha_2}}
\]

(A5)

to reduce the denominators containing the variable of integration to only one denominator. After integrating in \(k_2\) we use again Feynman’s formula (A5) to combine the denominators that depend on \(k_1\). We then integrate over \(k_1\) and, finally, perform the parametric integrations. In the present case, after integrating on \(k_2\), we get

\[
\frac{A_3(a)}{\epsilon} = \mathcal{T} \frac{\pi^{(2 - \frac{d}{2})}}{2^{d-3}} \int_0^1 dx \int \frac{d^dk_1}{(2\pi)^d} \frac{1}{k_2^2 - m^2 k_1^2}
\]

\[
\times \left[ \Gamma(1 - \frac{d}{2}) \Delta^{\frac{d}{2} - 1} + \Gamma(2 - \frac{d}{2}) k_2^2 \Delta^{\frac{d}{2} - 2} (1 + x - x^2) \right]
\]

(A6)

and

\[
\frac{B_3(a)}{\epsilon} = -\mathcal{T} \frac{1}{p^2} \frac{\pi^{(2 - \frac{d}{2})}}{2^{d-1}} \int_0^1 dx \int \frac{d^dk_1}{(2\pi)^d} \frac{k_1 \cdot p}{[(p - k_1)^2 - m^2](k_1^2)^2}
\]

\[
\times \left[ 4\Gamma(1 - \frac{d}{2}) \Gamma(d - 2)(k_1 \cdot p - k_1^2) \Delta^{\frac{d}{2} - 1} + 8x(1 - x) \right.
\]

\[
\times \Gamma(2 - \frac{d}{2}) \left( (k_1 \cdot p)k_1^2 - (k_1^2)^2 \right) \Delta^{\frac{d}{2} - 2} \right] ,
\]

(A7)
where \( \Delta = m^2 - x(1-x)k_1^2 \). Continuing our calculation, we would introduce two new parametric integrations as there are now three different denominators (we take \( 1/\Delta \) as a new denominator) depending on \( k_1 \) in each of the terms of the above expressions. However, as the result does not depend on \( m \) and we are looking only for the pole part of the amplitudes, we can speed up the calculation by modifying the dependence on \( m \) of some denominators.

For example, without changing the final result, we can replace the first term on the right hand side of (A6) by

\[
T \frac{\pi^{(2-d)/2}}{2^{d-3}} \int_0^1 dx \int \frac{d^d k_1}{(2\pi)^d} \Gamma(1-d/2) \Delta^{d-1} \frac{1}{(k_1^2 - m^2)^2}
\]

Similarly, in the computation of \( B_3(a) \) one can set \( m = 0 \) in the expression for \( \Delta \) so that one has to use only one parametric integral. Following this recipe, after integrating in \( k_1 \) we obtain \( (a = 1 - y - yx(1-x)) \),

\[
\frac{A_3(a)}{\epsilon} = T \frac{\pi^{2-d}}{2^{2d-3}} \Gamma[3-d] \int_0^1 dx \int_0^1 dy (1-y) \left[ \frac{1 - 2y}{\alpha} \right]^{d-3} \left[ \frac{y^{-d/2}}{\alpha^{3-d/2}} + \frac{d}{2} (1 - x^2) \frac{y^{1-d/2}}{\alpha^{4-d/2}} \right] = -\frac{5i}{8\epsilon}
\]

and

\[
\frac{B_3(a)}{\epsilon} = -T \frac{\pi^{2-d}}{2^{2d-3}} \Gamma[3-d] \int_0^1 dx \int_0^1 dy \frac{(1-y)^2-d/2}{x(x-1)^{1-d/2}} \left[ \frac{(1-5y)\Gamma[1-d/2]}{\Gamma[3-d/2]} - \frac{2\Gamma[2-d/2] \Gamma[5-d/2]}{\Gamma[4-d/2]} (1-y)(1-7y) \right] = -\frac{i}{24\epsilon}.
\]

Now, for the remaining graphs of order \( \alpha^2 \), we have,

Figure 2(b) :

\[
I_3(b) = -4i\pi^2 \epsilon_{\mu\nu\lambda} \epsilon_{\alpha\beta\rho} T \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} k_1^\lambda k_2^\rho \left[ \gamma^\mu (\not{p} - \not{k}_1 + m) \gamma^\alpha (\not{p} - \not{k}_1 - \not{k}_2 + m) \gamma^\beta (\not{p} - \not{k}_1 + m) \gamma^\nu \right] \left[ (p-k_1)^2 - m^2 \right] \left[ (p-k_2)^2 - m^2 \right] k_1^2 k_2^2 = (m A_3(b) + \not{p} B_3(b)) \frac{1}{\epsilon}
\]
Following the same steps as in previous case, we obtain

\[
\frac{A_3(b)}{\epsilon} = \frac{1}{2m} \text{Tr} I_3(b)\big|_{p=0} = -\frac{i}{4\epsilon} \\
\frac{B_3(b)}{\epsilon} = \frac{1}{2p^2} \text{Tr}(I_3(b) \not{p}) = -\frac{i}{12\epsilon}
\]

(A12) (A13)

Figure 2(c):

\[
I_3(c) = -4i\pi^2 \epsilon_{\mu\beta\lambda} \epsilon_{\alpha\nu\rho} \mathcal{T} \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} k_1^\lambda k_2^\rho \\
\times \left[ \gamma^\mu (p-\not{k}_1+m) \gamma^\alpha (p-\not{k}_1+\not{k}_2+m) \gamma^\nu (p-\not{k}_2+m) \right] \\
\times \frac{1}{[(p-k_1)^2 - m^2] [(p-k_1-k_2)^2 - m^2] [(p-k_2)^2 - m^2]} k_1^\lambda k_2^\rho
\]

\[
= (m A_3(c) + \not{p} B_3(c)) \frac{1}{\epsilon}.
\]

(A14)

The computation of this expression is a bit more complicated because one has to introduce three Feynman parametric integrals. The final result is, nevertheless, simple,

\[
A_3(c) = -\frac{3i}{8} \\
B_3(c) = \frac{i}{24}
\]

(A15) (A16)

Collecting these results, we obtain

\[
A_3 = A_3(a) + A_3(b) + A_3(c) = -\frac{5i}{4} \\
B_3 = B_3(a) + B_3(b) + B_3(c) = -\frac{i}{12}
\]

(A17)

2. Order $g\alpha$ graphs. There are six diagrams which have been drawn in Fig. 3. They give

Figures 3(a) and 3(b). Both diagrams have the structure of a product of two one loop graphs. The corresponding dimensionally regularized amplitudes does not have a pole at $d = 3$.

Figure 3(c). Actually, this diagram does not contribute because the corresponding analytic expression is finite. Indeed, we have

\[
I_4(c) = \mathcal{T} \frac{\partial}{\partial m} \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{\text{Tr}[\gamma^\mu (k_2+m) \gamma^\nu (k_2 + k_1 + m) \epsilon_{\mu\nu\rho\sigma} k_1^\rho]}{(k_2^2 - m^2)((k_1 + k_2)^2 - m^2)} \\
= -4i\mathcal{T} \frac{\partial}{\partial m} \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{m}{(k_2^2 - m^2)((k_1 + k_2)^2 - m^2)} = 0
\]

(A18)
since the integral in the second equality has the structure of a product of two one loop integrals.

Figure 3(d). The same reasoning can be applied to this situation since no external momentum flows through the diagram. We conclude that there is not a pole term.

Figure 3(e). By Furry’s theorem this diagram cancels with its charge conjugated partner.

Figure 3(f). We have the following contribution

\[ I_4(f) = 4\pi \epsilon_{\mu\nu\lambda} \mathcal{T} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \gamma^\mu (p - k_1 + m) k_1^\lambda \]
\[ \times \frac{(k_2 - k_1 + m) \gamma^\nu (k_2 + m)}{[(p - k_1)^2 - m^2] (k_2^2 - m^2) ((k_2 - k_1)^2 - m^2) k_1^2} \]
\[ = -\frac{i}{\epsilon} \left( m^2 A_4(f) + m \not p B_4(f) \right) \]

from which one obtains

\[ A_4 = A_4(f) = \frac{9}{16\pi} \]
\[ B_4 = B_4(f) = \frac{1}{8\pi} \]

3. Order \( g^2 \) graphs. There are only the two diagrams shown in Fig 5. We get,

\[ I_5 = -4i \mathcal{T} \int \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_1}{(2\pi)^d} \frac{\not p + k_1 + m}{(p + k_1)^2 - m^2} \frac{(k_2 - k_1 + m)}{[(k_2 - k_1)^2 - m^2] \text{Tr}[(k_2 + m)(k_2 - k_1 + m)]} \]
\[ \times \left\{ (k_2 + m)(k_2 - k_1 + m) - \text{Tr}[(k_2 + m)(k_2 - k_1 + m)] \right\} \]
\[ = -\frac{(m^2 A_5 + m^2 \not p B_5)}{\epsilon}, \]

where the two terms on the second equality refers to the graphs 5(a) and 5(b), respectively.

After a lengthy calculation one determines,

\[ A_5 = -\frac{3i}{16\pi^2} \]
\[ B_5 = -\frac{5i}{48\pi^2} \]

**APPENDIX B:**

In this appendix we will discuss the calculation of the pole part of the four point vertex function which is needed for fixing the renormalization group beta function. Actually, since
all we need is the constant part of the residue, the calculation of the relevant graphs will be done at zero external momenta.

The first observation is that, up to the order we are interested, i.e., third order, there are too many graphs. To be systematic, we will separate them accordingly they have or have not closed fermionic loops. Moreover, if they do not posses fermionic loops we group them accordingly the number of CS or auxiliary GN lines linking the two fermion lines crossing the diagram. Many diagrams cancel because of Furry’s theorem; this is the case if there is a fermionic loop with an odd number of attached CS lines. Other diagrams have the structure of a product of two one loop graphs and therefore are finite. We shall not consider these two types of graphs any longer. The anti-symmetrized amplitude for a graph $\gamma$ has the generic structure of a product, $(A \otimes B) C$, where A and B refers to the propagators and vertices associated to the two fermion lines and C to the others factors ($\otimes$ indicates the anti-symmetrized direct product). Using this notation, one can verify that

$$\text{Pole Part of } \int d^d k_1 d^d k_2 (A \otimes B) C = \frac{T}{2} \int d^d k_1 d^d k_2 (\text{Tr}[A] \text{Tr}[B] - \text{Tr}[AB]) C,$$  

(B1)

For example, from the analytic expression for the graph shown in Fig. 5,

$$\lambda \int d^d k_1 d^d k_2 [\gamma^\mu S_F(k_2)] \otimes [\gamma^\nu S_F(-k_2) \text{Tr}[S_F(k_1)S_F(k_1-k_2)] \epsilon_{\mu \nu \lambda} \frac{k_2^\lambda}{k_2^2},$$  

(B2)

where $\lambda$ is a combinatorial factor, we determine $A$, $B$ and $C$ as

$$A = [\gamma^\mu S_F(k_2)]$$  

(B3)

$$B = [\gamma^\nu S_F(-k_2)]$$  

(B4)

$$C = \lambda \epsilon_{\mu \nu \lambda} \frac{k_2^\lambda}{k_2^2}$$  

(B5)

The results for the pole parts are summarized in the Tables A and B, which corresponds to the two cases mentioned above. In the table A are listed the results from graphs with one closed fermionic loop; these have been arranged accordingly the number of CS vertices in the loop. Table B exhibits the pole part of graphs without fermionic loop. They have been collected into types $(i,j)$, where $i$ and $j$ are the number auxiliary GN and CS lines, respectively,
linking the two fermion lines crossing the diagram. Notice that there are not contribution from graphs of type (0,1) since they are not proper. Figure 3 furnishes examples of each one of these sets of diagrams. The final result for each order is obtained by summing the corresponding entries in each table. Thus we have,

\[
C_3 = -\frac{7}{2} \quad C_4 = \frac{11i}{2\pi} \quad C_5 = \frac{13}{2\pi^2}
\]  

(B6)

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[10] We use natural units (c = h = 1) and our metric is \( g_{00} = -g_{11} = -g_{22} = 1 \). The fully
antisymmetric tensor $\epsilon^{\mu\nu\lambda}$ is normalized such that $\epsilon^{012} = 1$ and we define $\epsilon^{ij} \equiv \epsilon^{0ij}$.

Repeated greek indices sum from 0 to 2, while repeated Latin indices from the middle of the alphabet sum from 1 to 2. For the $\gamma$-matrices we adopt the representation $\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2$, where $\sigma^i, i = 1, 2, 3$, are the Pauli spin matrices.

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FIGURES

FIG. 1. Feynman rules for the interaction vertices. Continuous and wavy lines represent the fermion and vector propagators, respectively.

FIG. 2. Order $\alpha^2$ graphs contributing to the two point function.

FIG. 3. Fermionic self-energy graphs of order $g\alpha$.

FIG. 4. Order $g^2$ fermionic self-energy graphs.

FIG. 5. Example of a two-loop diagram.

FIG. 6. Diagrams illustrating the various classes of graphs in the four point vertex function.


TABLES

TABLE I. Pole part for four legs graphs with a closed fermionic loop.

| Order of perturbation | Number of diagrams | Pole part          |
|-----------------------|--------------------|--------------------|
| $g\alpha^2$          | 12                 | $-i/2\epsilon$     |
| $g^2\alpha$          | 7                  | $-4i/\pi\epsilon$  |
| $g^3$                 | 8                  | $4i/\pi^2\epsilon$ |

TABLE II. Pole part for four legs graphs without closed fermionic loops. The first column lists different types of diagrams (i,j), where i and j are the number of GN and CS lines joining the two fermion lines crossing the graph; the digit in parenthesis after the pole parts is the number of contributing graphs.

| Diagram type | Order $g\alpha^2$ | Order $g^2\alpha$ | Order $g^3$ |
|--------------|--------------------|--------------------|--------------|
| (0,2)        | $i/2\epsilon$ (8) | —                  | —            |
| (1,0)        | $-5i/2\epsilon$ (12) | $3i/\pi\epsilon$ (18) | $-3i/4\pi^2\epsilon$ (6) |
| (1,1)        | $8i/\epsilon$ (24) | $4i/\pi\epsilon$ (8) | —            |
| (1,2)        | $-2i/\epsilon$ (12) | —                  | —            |
| (2,0)        | —                  | $-i/\pi\epsilon$ (14) | $i/\pi^2\epsilon$ (12) |
| (2,1)        | —                  | $-15i/2\pi\epsilon$ (9) | —            |
| (3,0)        | —                  | —                  | $5i/4\pi^2\epsilon$ (4) |