SEMICONDUCTOR BOLTZMANN-DIRAC-BENNEY EQUATION
WITH A BGK-TYPE COLLISION OPERATOR:
EXISTENCE OF SOLUTIONS VS. ILL-POSEDNESS

MARCEL BRAUKHOFF
Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstrasse 8-10, 1040 Wien, Austria

Abstract. A semiconductor Boltzmann equation with a non-linear BGK-type collision operator is analyzed for a cloud of ultracold atoms in an optical lattice:

$$\partial_t f + \nabla_x \epsilon(p) \cdot \nabla_x f - \nabla_x n_f \cdot \nabla_p f = n_f (1 - n_f)(F_f - f), \quad x \in \mathbb{R}^d, p \in \mathbb{T}^d, t > 0.$$ 

This system contains an interaction potential $n_f(x, t) := \int_{\mathbb{T}^d} f(x, p, t) dp$ being significantly more singular than the Coulomb potential, which is used in the Vlasov-Poisson system. This causes major structural difficulties in the analysis. Furthermore, $\epsilon(p) = -\sum_{i=1}^d \cos(2\pi p_i)$ is the dispersion relation and $F_f$ denotes the Fermi-Dirac equilibrium distribution, which depends non-linearly on $f$ in this context.

In a dilute plasma—without collisions (r.h.s. = 0)—this system is closely related to the Vlasov–Dirac–Benney equation. It is shown for analytic initial data that the semiconductor Boltzmann equation possesses a local, analytic solution. Here, we exploit the techniques of Mouhout and Villani by using Gevrey-type norms which vary over time. In addition, it is proved that this equation is locally ill-posed in Sobolev spaces close to some Fermi–Dirac equilibrium distribution functions.

1. Introduction. In the last decades, the theory of charge transport in semiconductors has become a thriving field in applied mathematics. Due to the complexity of semiconductors consisting of some $10^{23}$ atoms, there are several effective equations describing different phenomenological properties of semiconductors. Recently, the description of charge transport in semiconductors was extended by an experimental model [21]: a cloud of ultracold atoms in an optical lattice. In this model, the ultracold atoms stand for the charged electrons and the optical lattice describes the periodic potential of the crystal, formed by the ions of the semiconductor. Using the interference of optical laser beams, the atoms are trapped in an optical standing wave [8]. In contrast to a solid lattice, the geometry of an optical lattice as well as the strength of the potential can easily be changed during the experiment. Moreover, the time scale slows down to milliseconds while working with temperatures of a few nanokelvin. Therefore, this experimental model is particularly suited to...
understand the physical behavior of solid materials and of great interest. In addition, it may have the potential to accomplish quantum information processors [16] as well as very precise atomic clocks [2].

The main difference between a cloud of ultracold atoms and a system of electrons is the interaction potential. Assuming that the atoms are uncharged, the interaction potential is significantly more singular than the Coulomb potential of the electrons causing major structural difficulties in the analysis.

In this paper we investigate the ill-posedness of the following Boltzmann equation for the distribution function
\[
\partial_t f + u \cdot \nabla_x f + \nabla_x V_f \cdot \nabla_p f = Q(f),
\] (1)
where \(x \in \mathbb{R}^d\) is the spatial variable, \(p\) is the crystal momentum, defined on the \(d\)-dimensional torus \(T^d\) with unit measure, and \(t > 0\) is the time. The velocity \(u\) is defined by \(u(p) = \nabla_p \epsilon(p)\) with the energy \(\epsilon(p)\), \(V_f(x,t)\) is the lattice potential, and \(Q(f)\) is the collision operator. Compared to the standard semiconductor Boltzmann equation, there are two major differences.

First, we assume that the dispersion relation, i.e. the band energy, is given by
\[
\epsilon(p) = -2\epsilon_0 \sum_{i=1}^d \cos(2\pi p_i), \quad p \in T^d,
\] (2)
where \(\epsilon_0\) denotes the tunneling rate of a particle from one lattice site to a neighboring one [20]. This dispersion relation is typically used in semiconductor physics as for an approximation of the lowest band [3]. In contrast to this, a parabolic band structure is given by \(\epsilon(p) = \frac{1}{2} |p|^2\) [17], which also occurs in kinetic gas theory as the microscopic kinetic energy of free particles.

Second, the potential \(V_f\) is supposed to be proportional to the particle density \(n_f = \int_{T^d} f dp\) with
\[
V_f(x,t) = U n_f(x,t) = U \int_{T^d} f(x,p,t) dp, \quad x \in \mathbb{R}^d, p \in T^d, t > 0.
\] (3)
Here, \(U \neq 0\) describes the strength of the on-site interaction between spin-up and spin-down components [21]. However, in semiconductor physics, the interaction potential is often given by the Coulomb potential \(\Phi_f\) of the electric field which fulfills \(\Delta \Phi_f = n_f\) [17]. Due to this Poisson equation, the Coulomb potential is more regular than the particle density \(n_f\) in contrast to the potential \(V_f\) defined in (3). Therefore, we expect a more “singular behavior” of (1) compared to the standard semiconductor Boltzmann equation; see the discussion below.

Similar to [21], we use the following relaxation-time approximation
\[
Q(f) = \gamma n_f (1 - \eta n_f)(F_f - f)
\] (4)
for the collision operator, where \(1/\gamma > 0\) denotes the relaxation time and
\[
F_f(x, p, t) = \left( \eta + \exp(-\lambda_0(x,t) - \lambda_1(x,t)\epsilon(p)) \right)^{-1}, \quad x \in \mathbb{R}^d, \quad p \in T^d, \quad t > 0
\]
is the generalized Fermi-Dirac distribution function depending on \(f\) through the Lagrange multipliers \((\lambda_0, \lambda_1)\): We define \(\lambda_0\) and \(\lambda_1\) by the mass and energy constraints
\[
\int_{T^d} (F_f - f) dp = 0, \quad \int_{T^d} (F_f - f) \epsilon(p) dp = 0.
\]
Note that \(\eta = 1\) leads to the original Fermi-Dirac distribution as in [21] and \(\eta = 0\) entails that \(F_f\) equals the Maxwell-Boltzmann distribution.
Physically, $\lambda_1$ can be interpreted as the negative inverse (absolute) temperature, while $\lambda_0$ is related to the so-called chemical potential [17]. Since the dispersion relation is bounded, the equilibrium $\mathcal{F}_f$ is well-defined and integrable for all $\lambda_1 \in \mathbb{R}$, which includes negative absolute temperatures. These negative absolute temperatures can actually be realized in experiments with ultracold atoms [20]. Negative temperatures occur in equilibrated (quantum) systems that are characterized by an inverted population of energy states. The thermodynamical implications of negative temperatures are discussed in [19].

So far, there are some results for this type of equation using $\epsilon(p) = \frac{1}{2} |p|^2$ and that $Q(f)$ either vanishes or is quadratic in $f$.

Combining this with the Vlasov equation yields the Vlasov-Dirac-Benney equation

$$\partial_t f(x,u,t) + u \cdot \nabla_x f(x,u,t) - \nabla \rho_f(x,t) \cdot \nabla_u f(x,u,t) = 0 \quad (5)$$

for $x \in \mathbb{R}^d, u \in \mathbb{R}^d$ and $t > 0$. In spatial dimension one, this equation can be used to describe the density of fusion plasma in a strong magnetic field in direction of the field [7]. It can be derived as a limit of a scaled non-linear Schrödinger equation [6]. Comparing the Vlasov-Poisson equation to the Equation (5), we see that the interaction potential $\Phi$ is long ranged (i.e., the support is the whole space) in contrast to the delta distribution with $\text{supp}(\delta_0) = \{0\}$. Therefore, we can understand (5) as a version of the classical Vlasov-Poisson system with a short-ranged Dirac potential, which motivated the “Dirac” in the name of the Vlasov-Dirac-Benney equation. The name Benney is due to its relation to the Benney equation in dimension one (for details see [4]).

However, the analysis of a Vlasov-Dirac-Benney equation is more delicate as in [15] only local in time solvability was shown for analytic initial data in spatial dimension one. Moreover, it is shown in [4] that this system is not locally weakly ($H^m - H^1$) well-posed in the sense of Hadamard. In [13] it is shown that the Vlasov-Dirac-Benney equation is ill-posed in $d = 3$, requiring that the spatial domain is restricted to the 3-dimensional torus $\mathbb{T}^3$. More precisely, they show that the flow of solutions does not belong to $C^\alpha(H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3), L^2(\mathbb{R}^3 \times \mathbb{T}^3))$ for any $s \geq 0, \alpha \in (0,1)$ and $m \in \mathbb{N}_0$. Here, $H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3)$ denotes the weighted Sobolev space of order $s$ with weight $(x,u) \mapsto (u)^m := (1 + |u|^2)^{m/2}$. Even more precisely, they prove that there exist a stationary solution $\mu = \mu(u)$ of (5) and a family of solutions $(f_{\epsilon})_{\epsilon > 0}$, times $t_\epsilon = O(\epsilon |\log \epsilon|)$ and $(x_0, u_0) \in \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$\lim_{\epsilon \to 0} \frac{\|f_{\epsilon} - \mu\|_{L^2([0,t_{\epsilon}] \times B_\epsilon(x_0) \times B_\epsilon(u_0))}}{\|(u)^m f_{\epsilon}|_{t=0} - \mu\|_{H^{s,m}(\mathbb{T}^3 \times \mathbb{R}^3)}} = \infty,$$

where $B_\epsilon(x_0)$ denotes the ball with radius $\epsilon$ centered at $x_0$. In addition, [13] covers also equation (5) with a non vanishing r.h.s.: The authors consider

$$\partial_t f + u \cdot \nabla_x f - \nabla \rho_f(x,t) \cdot \nabla_u f = Q(f,f)$$

for a bilinear operator $Q$.

Moreover, the Vlasov-Dirac-Benney equation can also be derived by a quasi-neutral limit of the Vlasov-Poisson equation [14]. Han-Kwan and Rousset are also able to provide uniform estimates on the solution of the scaled Vlasov-Poisson equation. By taking the quasi-neutral limit, they prove the existence of a unique local solution $f \in C([0,T], H^{2m - 1,2r}(\mathbb{R}^3 \times \mathbb{T}^3))$ of the Vlasov-Dirac-Benney equation. For this, they require that the initial data $f_0 \in H^{2m,2r}(\mathbb{R}^3 \times \mathbb{T}^3)$ satisfies the Penrose
stability condition
\[ \inf_{x \in \mathbb{T}^d} \inf_{(\gamma, \tau, \eta) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}} \left| 1 - \int_0^\infty e^{-(\gamma + \tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot (F_v \nabla_v f)(x, \eta s) \, ds \right| > 0, \]
where \( F_v \) denotes the Fourier Transform in \( v \).

**Focus of this article.** We introduce a concrete BGK-type collision operator (see Equation (4)) arising from semiconductor physics [21], which depends nonlinearly on \( f \). Since a Vlasov equation with collisions is in general called a semiconductor Boltzmann equation, we may call our system a semiconductor Boltzmann-Dirac-Benney equation with a BGK-type collision operator:

Let \( \gamma > 0, \ U \neq 0 \), we consider
\[ \partial_t f + u(p) \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f (1 - \eta n_f) (F_f - f) \]
with \( f(x, p, 0) = f_0(x, p) \), where \( F_f(x, p, t) = (\eta + \exp(-\lambda_0(x, t) - \lambda_1(x, t)\epsilon(p)))^{-1} \), for \( x \in \mathbb{R}^d, \ p \in \mathbb{T}^d \) and \( t > 0 \). Here, \( \lambda_0, \lambda_1 \) shall be chosen in such a way that
\[ n_f(x, t) = n_{F_f}(x, t) \quad \text{and} \quad E_f(x, t) = E_{F_f}(x, t), \]
where \( n_f(x, t) := \int_{\mathbb{T}^d} f(x, p, t) \, dp \) and \( E_f(x, t) := \int_{\mathbb{T}^d} \epsilon(p) f(x, p, t) \, dp \). Moreover, we have \( u(p) = \nabla_p \epsilon(p) \) with
\[ \epsilon(p) = -2\epsilon_0 \sum_{i=1}^d \cos(2\pi p_i), \quad p \in \mathbb{T}^d, \]
for some \( \epsilon_0 > 0 \).

In the first theorem, we prove the local existence of a solution for analytic initial data. It therefore extends the existence results of [15] and [13] to our setting.

**Theorem 1.1.** Let \( \eta > 0, \ \gamma > 0, \ U \neq 0 \) and \( f_0 : \mathbb{T}^d \times \mathbb{T}^d \to (0, \eta^{-1}) \) be analytic. Then there exists a time \( T > 0 \) such that (6) admits a unique analytic solution \( f : \mathbb{T}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R} \) with \( f(x, p, 0) = f_0(x, p) \).

Physically, the BGK-collision operator shall drive the system into an equilibrium given by the generalized Fermi-Dirac distribution and one would expect some nicer results than in [13]. However, the following theorem tells us that this is not always the case since some Fermi-Dirac equilibria are unstable, leading to an ill-posedness result.

**Theorem 1.2.** Let \( k \in \mathbb{N}, \ \theta > 0 \) and \( \gamma > 0, \ U \neq 0 \). There exist \( \lambda \in \mathbb{R}^2 \) and a time \( \tau > 0 \) such that there exist solutions \( f_\lambda : \mathbb{R}_+^d \times \mathbb{T}^d \times [0, \tau] \to [1, \epsilon^{-1}] \) of (6) such that
\[ \lim_{\delta \to 0} \frac{\| f_\lambda(\cdot, \cdot, t) - F_\lambda \|_{L^1(B_v(x, p))}}{\| f_\lambda(\cdot, \cdot, 0) - F_\lambda \|_{W^{k, \infty}(\mathbb{R}^d \times \mathbb{T}^d)}} = \infty \quad \text{for all } x \in \mathbb{R}^d, p \in \mathbb{T}^d, t \in (0, \tau), \]
where \( F_\lambda(p) := 1/(\eta + e^{\lambda_0 - \lambda_1 \epsilon(p)}) \) is a steady-state solution of (6).

**Remark 1.3.** The theorem can easily be extended to all \( \gamma \in \mathbb{R} \). A sufficient condition for the critical \( \lambda \) is given by
\[ 1 < U \lambda_1 \int_{\mathbb{T}^d} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp. \]

It is still an open problem, whether this condition is necessary. However, a similar condition also appears in a different context of semiconductor physics for ultra cold
atoms: In [10], a formal drift-diffusion limit of (6) was considered. The formal analysis indicates degeneracies of the limiting diffusion equation, whenever

\[ n = 0 \] (and in see \[9\] section 5.5. Thus, we can see that the second derivative has a singularity in

Moreover, we suppose there exist \( \mathcal{C} \) the Fermi-Dirac distributions \( \mathcal{F}_f \) are not analytic in \( f = 0 \) as we can see in the following remark.

**Remark 1.4.** According to the definition of the BGK-collision operator, \( \mathcal{F}_f \) is uniquely determined by the constraints from (7) and can be rewritten as a function

\[ \mathcal{F}^0 : U \subset \mathbb{R}^2 \times \mathbb{T}^d \rightarrow [0, \eta^{-1}] \]

with

\[ \mathcal{F}^0(f(x,t), E_f(x,t); p) = \mathcal{F}_f(x,p,t). \]

For this function, one can compute that

\[ \partial^2_z \mathcal{F}^0(n,0; p) = \frac{1 - 2\eta_n}{8\epsilon_0^2 d^2 n(1 - \eta n)}(\epsilon(p)^2 - 2\epsilon_0^2 d) \]

see \[9\] section 5.5. Thus, we can see that the second derivative has a singularity in \( n = 0 \) (and in \( n = \eta^{-1} \)). In particular, there exist a \( g = g(n,p) \) with \( g(0,\cdot) \neq 0 \) such that

\[ \partial^i_n \partial^j_k \mathcal{F}^0(n,0; p) = \frac{g(n,p)}{n^{i+1}(1 - \eta n)^{j+1}}. \]

Clearly, this implies that \( \mathcal{F}^0 \) is not analytic in \( (n,E) = 0 \). Fortunately, we are only interested in the composition of \( \mathcal{F}^0 \) with \( n_f \) and \( E_f \). The idea is to assume enough regularity on \( f \) such that \( \mathcal{F}_f \) is analytic.

This leads to a first version of the local existence theorem for the whole space:

**Theorem 1.5.** Let \( \eta > 0 \), \( \gamma \geq 0 \), \( U \geq 0 \) and \( \lambda^0 = (\lambda^0_0, \lambda^0_1) : \mathbb{R}^d \rightarrow \mathbb{R}^2 \) be analytic such that \( \lambda^0_1 \in L^\infty(\mathbb{R}^d) \) and let

\[ F_{\lambda^0}(x,p) := \frac{1}{\eta + e^{-\lambda^0_0(x)-\lambda^0_1(x)p}}. \]

Moreover, we suppose there exist \( C_0 > 0 \) and \( \nu \geq 0 \) such that

\[ |\partial^2_z n_{F_{\lambda^0}}(x)| + |\partial^2_\xi E_{F_{\lambda^0}}(x)| \leq C_0 n_{F_{\lambda^0}}(x)(1 - \eta n_{F_{\lambda^0}}(x))a!\nu^{-|a|} \]

(9)

for all \( 0 \neq a \in \mathbb{N}_0 \) and all \( x \in \mathbb{R}^d \). Then there exists \( T > 0 \) such that (6) admits a unique analytic solution \( f : \mathbb{R}^d \times \mathbb{T}^d \times [0,T) \rightarrow \mathbb{R} \) with \( f(x,p,0) = F_{\lambda^0}(x,p) \).

**Example 1.6.** In this version of the local existence result, we allow also initial data which may approach zero as \( |x| \rightarrow \infty \). Let \( \lambda^0_1 = 0 \) and

\[ \lambda^0_0(x) := -\log(1 + x^2). \]

Then

\[ F_{\lambda^0}(x) = \frac{1}{\eta + 1 + x^2} \]
and hence $E_{F_{\lambda^0}}$ vanishes and $n_{F_{\lambda^0}} = F_{\lambda^0}(x)$. We will prove in example B.5 in the appendix that

$$|F_{\lambda^0}^{(a)}(x)| \leq \frac{a^1}{\nu^a} F_{\lambda^0}(x) \quad \text{for} \quad \nu = \frac{1}{2} \min\{\sqrt{\eta}, 1\}$$

Using that $n_{F_{\lambda^0}} = F_{\lambda^0}(x) \leq 1/(1 + \eta)$ yields

$$|n_{F_{\lambda^0}}^{(a)}(x)| \leq \frac{\eta + 1}{\eta} n_{F_{\lambda^0}}(x)(1 - \eta n_{F_{\lambda^0}}(x))a!\nu^{-|a|}.$$ 

Finally, we can conclude that

$$F_{\lambda^0}(x) = \frac{1}{\eta + 1 + x^2}$$

satisfies the hypothesis of the foregoing theorem. Thus, there exists $T > 0$ such that (6) admits a unique analytic solution $f : \mathbb{R}^d \times T^d \times [0, T) \rightarrow \mathbb{R}$ with $f(x, p, 0) = F_{\lambda^0}(x, p)$.

Note that (9) is a local conditions for the particle and energy densities. This is a consequence of the fact that the BGK-collision operator is local in space.

**Theorem 1.7.** Let $\eta > 0$, $\gamma > 0$, $U > 0$ and let $\lambda^0 = (\lambda_{\gamma}^{(0)}, \lambda_{\eta}^{(0)}) : \mathbb{R}^d \rightarrow \mathbb{R}^2$, $C_0$, $\nu$ be as in Theorem 1.5. Then there exist $\delta > 0$ and $T > 0$ such that (6) admits a unique analytic solution $f : \mathbb{R}^d \times T^d \times [0, T) \rightarrow \mathbb{R}$ with

$$f(x, p, 0) = f_0(x, p) := \frac{1}{\eta + e^{-\lambda_{\gamma}^{(0)}(x) - \lambda_{\eta}^{(0)}(x)\epsilon(p)}} + g_0(x, p),$$

if $g_0 : \mathbb{R}^d \times T^d \rightarrow \mathbb{R}$ is analytic with $0 \leq f_0(x, p) \leq \eta^{-1}$ and satisfies

$$\int_{T^d} |\partial_x^a \partial_p^b g_0(x, p)| dp \leq C_0 n_{f_0}(x)(1 - \eta n_{f_0}(x))a!b!\nu^{-|a + b|}$$

as well as

$$|\partial_x^a n_{f_0}(x)| + |\partial_x^a E_{f_0}(x)| \leq \delta n_{f_0}(x)(1 - \eta n_{f_0}(x))a!b!\nu^{-|a|}$$

for all $x \in \mathbb{R}^d$ and $a, b \in \mathbb{N}_0^d$ with $a + b \neq 0$.

Moreover, there exist $\tilde{C}, \tilde{\nu} > 0$ and $T \in (0, T)$ such that

$$\int_{T^d} |\partial_x^a \partial_p^b f(x, p, t)| dp \leq \tilde{C}a!b!\tilde{\nu}^{-|a + b|}$$

for all $x \in \mathbb{R}^d$, $t \in [0, T]$.

**Remark 1.8.** The solution is well-posed in the following sense: There exist $\tilde{\nu} > 0$ and $\tilde{C} > 0, \tilde{T} \in (0, T)$ such that two solutions $f^1, f^2$ of (6) fulfill

$$\int_{T^d} |\partial_x^a \partial_p^b (f^2(x, p, t) - f^1(x, p, t))| dp$$

$$\leq \tilde{C}a!b!\tilde{\nu}^{-|a + b|} \sum_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\nu^{(|a + \beta|)}}{a!b!} \sum_{\alpha + \beta > 0} \int_{T^d} |\partial_x^a \partial_p^b (g_0^2(x, p) - g_0^1(x, p))| dp$$

for all $x \in \mathbb{R}^d$, $t \in [0, \tilde{T}]$, where

$$f_0^1(x, p) := f^1(x, p, 0) \quad \text{and} \quad g_0^1(x, p) := f_0^1(x, p) - \frac{1}{\eta + e^{-\lambda_{\gamma}^{(0)}(x) - \lambda_{\eta}^{(0)}(x)\epsilon(p)}}$$

satisfy the same conditions as $f_0$ and $g_0$ from Theorem 1.7 for $i = 1, 2$. 
2. Analytic norms. Our strategy to solve (6) will be applying a fixed-point argument. Therefore, we require suitable functions spaces: we use the following analytic norms, which are similar to those from [18].

**Definition 2.1.** Let $\nu > 0$, $d \in \mathbb{N}$. We define

$$
\| f \|_{C^\nu} := \sum_{a,b \in \mathbb{N}_0^d, |a|+|b| \leq 1} \frac{\nu |a|+|b|!}{a!b!} \| \partial_x^a \partial_p^b f \|_{W^{1,1}_1} 
$$

for $f : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}^k$ being analytic, where we use the notation

$$
\| f \|_{W^{1,1}_1} := \sum_{a,b \in \mathbb{N}_0^d, |a|+|b| \leq 1} \| \partial_x^a \partial_p^b f \|_{L^\infty_1 L^1_1} \quad \text{and} \quad \| f \|_{L^\nu_1 L^1_1} := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{T}^d} |f(x,p)| \, dp.
$$

Moreover, we define the semi-norm

$$
\| Df \|_{C^\nu} := \sum_{a,b \in \mathbb{N}_0^d, |a|+|b| = 1} \| \partial_x^a \partial_p^b f \|_{C^\nu}
$$

and we set

$$
\| u \|_{C^\nu,\infty} := \max_{i=1,\ldots,d} \sum_{b \in \mathbb{N}_0^d} \frac{\nu |b|!}{b!} \| \partial_x^b u_i \|_{W^{1,1}(\mathbb{T}^d)}.
$$

Comparing these norms to the analytic norms

$$
| f |_{C^\nu} := \sum_{a,b \in \mathbb{N}_0^d} \frac{\nu |a|+|b|!}{a!b!} \| \partial_x^a \partial_p^b f \|_{L^\infty_1 L^1_1} \quad \text{and} \quad | u |_{C^\nu,\infty} := \sum_{b \in \mathbb{N}_0^d} \frac{\nu |b|!}{b!} \| \partial_x^b u \|_{L^\infty(\mathbb{T}^d)}
$$

from [18], we have the trivial estimate $| \cdot |_{C^\nu} \leq \| \cdot \|_{C^\nu}$. For the inverse estimate, we can only compare $| \cdot |_{C^\nu}$ with $\| \cdot \|_{C^\nu}$ if $\mu > \nu$ as the following lemma suggests. As we will see later on, the norm $\| \cdot \|_{C^\nu}$ is suited better for treating semiconductor Boltzmann–Dirac–Bennett type equations. The idea is to do the analysis with our tailor-made norms $\| \cdot \|_{C^\nu}$. We only use the more “standard” analytic norms $| \cdot |_{C^\nu}$ afterward for the statements by using the following comparison estimate.

**Lemma 2.2.** Let $\mu > \nu > 0$ and $d \in \mathbb{N}$. Then there exists $C_{\mu,\nu} > 0$ such that

$$
\| f \|_{C^\nu} \leq C_{\mu,\nu} | f |_{C^\nu}
$$

for all analytic $f : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$.

**Proof.** It suffices to show that we have $\| \partial f \|_{C^\nu} \leq C | f |_{C^\nu}$ for $\partial \in \{ \partial_x, \partial_p \}$ for some $C > 0$. Let $\partial = \partial_x$ and compute

$$
\| \partial_x f \|_{C^\nu} = \sum_{i,j \in \mathbb{N}_0} \frac{\nu^{i+j}}{i!j!} \| \partial_x^i \partial_p^j f \|_{L^\infty_1 L^1_1} \leq \frac{1}{\nu} \sum_{i,j \in \mathbb{N}_0} \frac{\nu^{i+j}}{i!j!} \int_{\mathbb{T}^d} \| \partial_x^i \partial_p^j f \|_{L^\infty_1 L^1_1} 
$$

for $C = \sup_{a \in \mathbb{N}} \nu^{a-1} \mu^a < \infty$. The estimate for $\partial = \partial_p$ can be proved similarly. □

The equation (10) consists of terms which involve product. Therefore, the following algebraic properties are particularly useful for treating equation (6).
Lemma 2.3. Let $f : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$, $n : \mathbb{R}^d \to \mathbb{R}$ and $u : \mathbb{T}^d \to \mathbb{R}^d$ be analytic. Let $\nu \geq 0$. Then it holds
\[ \|fn\|_{C^\nu} \leq \|f\|_{C^\nu} \|n\|_{C^\nu} \quad \text{and} \quad \|u \cdot \nabla_x f\|_{C^\nu} \leq \|u\|_{C^{\nu,\infty}} \|Df\|_{C^\nu} \]
as well as
\[ \| \nabla_x n \cdot \nabla_p f \|_{C^\nu} \leq \|n\|_{C^\nu} \|Df\|_{C^\nu} + \|Dn\|_{C^\nu} \|f\|_{C^\nu}. \]

Proof. First, we try to rewrite the norm $\|\cdot\|_{C^\nu}$ in such a way that we can use the results of [18, section 4]. Then we can easily show using the Leibniz rule that $|fn|_{C^\nu} \leq |f|_{C^\nu} |n|_{C^\nu}$ and $|fu|_{C^\nu} \leq |f|_{C^\nu} |u|_{C^{\nu,\infty}}$ (see [18, section 4]). Using this and the chain rule, we have
\[ \|fn\|_{C^\nu} = \sum_{|a+b| \leq 1} |\partial_x^a \partial_p^b (fn)|_{C^\nu} \leq \sum_{|a+b| \leq 1} |\partial_x^a \partial_p^b f|_{C^\nu} |\partial_x^a \partial_p^b n|_{C^\nu} = \|f\|_{C^\nu} \|n\|_{C^\nu}. \]
Likewise,
\[ \|u \cdot \nabla f\|_{C^\nu} \leq \sum_{i=1}^d \sum_{a,b \in \mathbb{N}_0^d:|a+b| \leq 1} \left( |u_i|_{C^{\nu,\infty}} |\partial_x^a \partial_p^b \partial_x f|_{C^\nu} + |\partial_p^b u_i|_{C^{\nu,\infty}} |\partial_x^a \partial_x f|_{C^\nu} \right) \]
\[ \leq \|u\|_{C^{\nu,\infty}} \sum_{i=1}^d \|\partial_x f\|_{C^\nu} \]
and
\[ \| \nabla_x n \cdot \nabla_p f \|_{C^\nu} \leq \sum_{i=1}^d \sum_{a,b \in \mathbb{N}_0^d:|a+b| \leq 1} \left( |\partial_x n|_{C^\nu} |\partial_x^a \partial_p^b \partial_p f|_{C^\nu} \right. \]
\[ + |\partial_x^a \partial_x n|_{C^\nu} |\partial_p^b \partial_p f|_{C^\nu} \right) \]
\[ \leq \sum_{i=1}^d \|n\|_{C^\nu} \|\partial_p f\|_{C^\nu} + \sum_{i=1}^d \|\partial_x n\|_{C^\nu} \|f\|_{C^\nu}. \]

\[ \square \]

In [18], Mouhot and Villani unleashed the full potential of these analytic norms by varying the index $\nu$ over time. Motivated by their results, we define the following norm and derive the proceeding lemma.

Definition 2.4. For $\nu, T > 0$, $\mu \in (0, \nu/T)$, we define
\[ \|f\|_{\nu,\mu} := \sup_{0 \leq t < T} \left( \|f(t)\|_{C^{\nu-\mu}} + \mu \int_0^t \|Df(s)\|_{C^{\nu-\mu}} \, ds \right) \]
for $f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R}$ being analytic in $(x, p)$ and continuous in $t$ writing $f(t) = f(\cdot, \cdot, t)$.

Lemma 2.5. For $\nu, T > 0$, $\mu \in (0, \nu/T)$ and $f : \mathbb{R}^d \times \mathbb{T}^d \times (0, T) \to \mathbb{R}$ be analytic in $x, p$ and continuously differentiable in $t$. Then
\[ \|f\|_{\nu,\mu} \leq \|f(0)\|_{C^\nu} + \int_0^T \|\partial_t f(\cdot, \cdot, t)\|_{C^{\nu-\mu}} \, dt. \]
Proof. Let $0 < t \leq T$. Throughout this proof, we write $f(t) := f(\cdot, \cdot, t)$. Without loss of generality, we assume that

$$\tau \mapsto \| \partial_t f(\tau) \|_{C^{\nu-\nu^*}} \in L^1(0, t),$$

because otherwise, the assertion is trivial. Setting

$$P_{f,N}(\lambda, t) := \sum_{|a|, |b| \leq N} \frac{\lambda^{a+b}}{a!b!} \left\| \partial_x^{i+a} \partial_p^{j+b} f(t) \right\|_{W^{1,\infty}_p},$$

and

$$Q_N(\lambda, t) := \sum_{|i+j| = 1} \sum_{|a|, |b| \leq N} \frac{\lambda^{a+b}}{a!b!} \left\| \partial_x^{i+a} \partial_p^{j+b} f(t) \right\|_{W^{1,\infty}_p},$$

we have $P_{f,N}(\lambda, t) \to \| f(t) \|_{C^{\lambda}}$ and $Q_N(\lambda, t) \to \| Df(t) \|_{C^{\lambda}}$ as $N \to \infty$. Let $i, j, a, b \in \mathbb{N}_0^d$ and $0 < s < t$. Then

$$\left\| \partial_x^{i+a} \partial_p^{j+b} f(t) \right\|_{W^{1,\infty}_p} \leq \sup_{s \leq \tau \leq t} \left\| \partial_x^{i+a} \partial_p^{j+b} \partial_t f(x, \cdot, \tau) \right\|_{W^{1,\infty}_p}(t - s).$$

implies

$$|P_{f,N}(\lambda, t) - P_{f,N}(\lambda, s)| \leq \sup_{s \leq \tau \leq t} P_{\partial_t f,N}(\lambda, \tau)(t - s).$$

Next, let $\lambda = \lambda(t) = \nu - \mu t$. Using the estimate

$$\left( \frac{\lambda(t)^n - \lambda(s)^n}{a!} = \left( \frac{\lambda(s) + \mu(s - t))}{a!} \right)^n - \lambda(s)^n \right)$$

$$= \mu(s - t) \sum_{j=0}^{a-1} \frac{\lambda(s)^j \lambda^{a-1-j}}{j!(a-1-j)!(a-j)} \left\{ \begin{array}{ll} \leq \mu(s - t) \frac{\lambda(s)^{n-1}}{(a-1)!} & \text{if } a \geq \frac{\lambda(s)^{n-1}}{(a-1)!}, \\ \geq \mu(s - t) \frac{\lambda(s)^{n-1}}{(a-1)!} & \end{array} \right.$$ we can derive that

$$|P_{f,N}(\nu - \mu t, t) - P_{f,N}(\nu - \mu s, s)| \leq \sup_{s \leq \tau \leq t} P_{\partial_t f,N}(\nu - \mu t, \tau)(t - s) + \mu \sup_{s \leq \tau \leq t} Q_N(\nu - \mu \tau, s)(t - s).$$

Thus, $P_{f,N}(\nu - \mu t, t)$ is Lipschitz continuous w.r.t. $t$ and belongs to $W^{1,\infty}((0, T))$ with

$$\frac{d}{dt} P_{f,N}(\nu - \mu t, t) \leq P_{\partial_t f,N}(\nu - \mu t, t) - \mu Q_N(\nu - \mu t, t),$$

since $P_{f,N}, P_{\partial_t f,N}$ and $Q_N$ are continuous.

Since $P_{\partial_t f,N}(\nu - \mu \tau, \tau) \leq \| \partial_t f(\tau) \|_{C^{\nu-\nu^*}} \in L^1(0, T)$, the dominated convergence theorem implies that

$$\int_0^T P_{\partial_t f,N}(\nu - \mu \tau, \tau) d\tau \to \int_0^T \| \partial_t f(\tau) \|_{C^{\nu-\nu^*}} d\tau \quad \text{as } N \to \infty.$$
Moreover, we can utilize the monotone convergence theorem in order to obtain that \( \int_0^T Q_N(\nu - \mu \tau, \tau) d\tau \to \int_0^T \|DF(\tau)\|_{C^{\nu-\mu}} d\tau \). Thus, we summarize
\[
\|f(t)\|_{C^{\nu-\mu}} + \int_0^t (\mu \|DF(\tau)\|_{C^{\nu-\mu}} - \|\partial_t f(\tau)\|_{C^{\nu-\mu}}) d\tau
\]
\[
\leq \sum_{n \leq \nu} P_{f, N}(t, t) + \int_0^t (\mu Q_N(\nu - \mu \tau, \tau) - P_{f, N}(\nu - \mu \tau, \tau)) d\tau
\]
finishing the proof.

Remark 2.6. Let \( T, \nu > 0 \) and let \( f_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R} \) be analytic. Then
\[
\|f_0\|_{\nu, \mu} = \|f_0\|_{C^\nu} \quad \text{for every } \mu \in (0, \nu/T).
\]

3. Local well-posedness in analytic norms. In this section, we analyze the semiconductor Boltzmann equation (1) for ultracold atoms (setting \( V_f := -Un_f \) for \( U \in \mathbb{R} \)) in combination with a relaxation time approximation with fixed equilibrium. We consider
\[
\partial_t f + u(\rho) \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f (1 - \eta n_f) (F - f),
\]
with \( f(x, p, 0) = f_0(x, p) \) for some given \( F = F(x, p, t) \) and \( \gamma \geq 0 \).

Theorem 3.1. Let \( C, R, \nu > 0 \) and \( f_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R} \) and \( F : \mathbb{R}^d \times \mathbb{T}^d \times [0, T'] \to \mathbb{R} \) be analytic such that
\[
\|f_0\|_{C^\nu} < R.
\]
Then if \( \mu > 0 \) is sufficiently large, \( T \in (0, \nu/\mu) \) and \( F : \mathbb{R}^d \times \mathbb{T}^d \times [0, T'] \to \mathbb{R} \) is analytic such that
\[
\|F(t)\|_{C^{\nu-\mu}} \leq C
\]
for all \( 0 \leq t \leq T \), then equation (10) admits a unique analytic solution \( f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R} \) with \( \|f\|_{\nu, \mu} \leq R \) and \( f(x, p, 0) = f_0(x, p) \) for all \( x \in \mathbb{R}^d \) and \( p \in \mathbb{T}^d \).

Moreover, let \( \Psi : (f_0, F) \to f \) be defined by the unique solution of
\[
\partial_t f + u \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f (1 - \eta n_f) (F - f)
\]
with \( f(x, p, 0) = f_0(x, p) \). If \( \mu > 0 \) is sufficiently large, the mapping \( \Psi \) is Lipschitz continuous, i.e.,
\[
\|\Psi(f_0, F) - \Psi(g_0, G)\|_{\nu, \mu} \leq 2 \|(f_0, F) - (g_0, G)\|,
\]
where
\[
\|(f_0, F)\| := \|f_0\|_{C^\nu} + \mu \frac{1}{2} \sup_{0 \leq t < T} \|F\|_{C^{\nu-\mu}}.
\]
for \( f_0, g_0 \) and \( F, G \) satisfying (25) and (26), respectively.

Remark 3.2. A sufficient condition for \( \mu \) is given by
\[
\mu \geq \frac{C}{R - \|f_0\|_{C^\nu}} + C \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}}
\]
for some \( C > 0 \) independent from \( f_0 \) and \( F \).
The key idea for the proof relies on the contraction mapping principle/Banach’s fixed-point theorem. We define the mapping $\Phi$

$$
\Phi(f) := f_0 - \int_0^1 (u \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f - \gamma n_f (1 - \eta n_f)(F - f)) dt .
$$

(13)

for $f$ being analytic in $(x, p)$ and continuous in time. In order to prove that $\Phi$ admits a (unique) fixed-point, we require the next lemmas.

**Lemma 3.3.** Let $\|f_0\|_{C^0} \leq R$. For every sufficient large $\mu$, there exists a $T \in (0, \nu / \mu)$ such that $\|f\|_{\nu, \mu} \leq R$ implies $\|\Phi(f)\|_{\nu, \mu} \leq R$. Here, a sufficient condition for $\mu$ is given by $\mu \geq C/(R - \|f_0\|_{C^0})$ for some $C > 0$ independent from $f_0$.

**Proof.** First we fix $\mu > 0$ and $T \in (0, \nu / \mu)$. According to Lemma 2.5, we have

$$
\|\Phi(f)\|_{\nu, \mu} - \|f_0\|_{C^0} \leq \int_0^T \|\partial_t \Phi(f)\|_{C^{\nu - \mu}} dt
$$

$$
\leq \int_0^T \left( \|u \cdot \nabla_x f\|_{C^\lambda} + \|U\|_{C^\lambda} \|\nabla_x n_f \cdot \nabla_p f\|_{C^\lambda} + \gamma \|n_f (1 - \eta n_f)(f - F)\|_{C^\lambda} \right) dt .
$$

Using the submultiplicativity obtained by Lemma 2.3 and $\|n_f\|_{C^\lambda} \leq \|f\|_{C^\lambda}$ as well as $\|Dn_f\|_{C^\lambda} \leq \|Df\|_{C^\lambda}$, we obtain

$$
\int_0^T \|\partial_t \Phi(f)\|_{C^{\nu - \mu}} dt \leq \frac{1}{\mu} \left( \|u\|_{C^{\nu, \infty}} + \|U\|_{\nu, \mu} \right) \|f\|_{\nu, \mu}
$$

$$
+ \gamma T \|f\|_{\nu, \mu} \left( 1 + \eta \|f\|_{\nu, \mu} \right) \left( \|f\|_{\nu, \mu} + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu - \mu}} \right) .
$$

(14)

Thus, assuming $\|f\|_{\nu, \mu} \leq R$ entails that

$$
\|\Phi(f)\|_{\nu, \mu} \leq \|f_0\|_{C^0} + \frac{1}{\mu} \left( \|u\|_{C^{\nu, \infty}} + \|U\|_{\nu, \mu} \right) R
$$

$$
+ T\nu R \left( 1 + \eta R \right) \left( \frac{\|f\|_{\nu, \mu}}{R} + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu - \mu}} \right) .
$$

Let $\mu_R := 2 \left( \frac{\|u\|_{C^{\nu, \infty}} + \|U\|_{\nu, \mu}}{R - \|f_0\|_{C^0}} \right) > 0$. Then for all $\mu \geq \mu_R$, we have

$$
\|\Phi(f)\|_{\nu, \mu} \leq \|f_0\|_{C^0} + \frac{1}{2} \left( \frac{R - \|f_0\|_{C^0}}{R} \right)
$$

$$
+ T\nu R \left( 1 + \eta R \right) \left( \frac{\|f\|_{\nu, \mu}}{R} + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu - \mu}} \right) \leq R
$$

if $T \in (0, \nu / \mu)$ satisfies

$$
T\nu R \left( 1 + \eta R \right) \left( \frac{\|f\|_{\nu, \mu}}{R} + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu - \mu}} \right) \leq \frac{1}{2} \left( \|f_0\|_{C^0} \right) .
$$

(15)

Therefore, for every sufficient large $\mu$, i.e. $\mu \geq \mu_R$, every $T < \nu / \mu$ satisfies condition (15). Thus, $\|\Phi(f)\|_{\nu, \mu} \leq R$.

**Lemma 3.4.** Let $R > \|f_0\|_{C^0}$. For every sufficient large $\mu$, there exists a $T \in (0, \nu / \mu)$ such that $\|f_1\|_{\nu, \mu}, \|f_2\|_{\nu, \mu} \leq R$ imply

$$
\|\Phi(f_1) - \Phi(f_2)\|_{\nu, \mu} \leq \frac{1}{2} \|f_1 - f_2\|_{\nu, \mu} .
$$
Here, a sufficient condition for $\mu$ is given by
\[ \mu \geq \frac{C}{R - \|f_0\|_{C^\nu}} + C \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}} \]
for some $C > 0$ independent from $f_0$ and $F$.

Proof. The difference $g := \Phi(f_2) - \Phi(f_1)$ is given by
\[ g(t) = \int_0^t \left( -u \cdot \nabla_x g + U \nabla_x n_{f_1} \cdot \nabla_p g + U \nabla_x n_{f_2} \cdot \nabla_p f_2 + Q(f_2) - Q(f_1) \right) ds \]
where $n_{f_j} = \int_{\mathbb{R}^d} f_j dp$, $n_g = \int_{\mathbb{R}^d} g dp$ and
\[ Q(f_j) := \gamma_{n_j}(1 - \eta_{n_j})(F - f_j) \quad \text{for } j = 1, 2. \]
Since $Q(f_j)$ is affine in $f_j$ and quadratic in $n_{f_j}$, we use the submultiplicativity properties of the norm $\|\cdot\|_{C^{\nu-\mu}}$ from Lemma 2.3 to ensure that
\[ \|Q(f_2) - Q(f_1)\|_{C^{\nu-\mu}} \leq \gamma(2 + 3\eta_R)(\|F\|_{C^{\nu-\mu}} + R)\|g\|_{C^{\nu-\mu}} \]
for $\|f_1\|_{C^{\nu-\mu}}, \|f_2\|_{C^{\nu-\mu}} \leq R$. We derive similarly to the proof of Lemma 3.3 that
\[ \|\partial_t(\Phi(f_2) - \Phi(f_1))\|_{C^{\nu-\mu}} \leq \left( \|u\|_{C^{\nu,\infty}} + |U| \right) \left( \|f_1\|_{C^{\nu-\mu}} + \|f_2\|_{C^{\nu-\mu}} \right)\|Dg\|_{C^{\nu-\mu}} \]
\[ + \left( |U|\|Df_2\|_{C^{\nu-\mu}} + \|U\|\|Df_1\|_{C^{\nu-\mu}} \right) \]
\[ + \gamma(2 + 3\eta_R)(R + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}})\|g\|_{C^{\nu-\mu}}. \]
By Lemma 2.5, we obtain for all $\|f_j\|_{\nu,\mu} \leq R$
\[ \|\Phi(f_2) - \Phi(f_1)\|_{\nu,\mu} \leq \int_0^T \|\partial_t(\Phi(f_2) - \Phi(f_1))\|_{C^{\nu-\mu}} dt \]
\[ \leq \frac{1}{\mu} \left( \|u\|_{C^{\nu,\infty}} + 4|U|R \right)\|g\|_{\nu,\mu} \]
\[ + T\gamma(2 + 3\eta_R) \left( R + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}} \right)\|g\|_{\nu,\mu} \]
\[ \leq \frac{C_{\nu,R}}{\mu} \|g\|_{\nu,\mu} = \frac{C_{\nu,R}}{\mu} \|f_2 - f_1\|_{\nu,\mu}, \]
where $C_{\nu,R} = \|u\|_{C^{\nu,\infty}} + 4|U|R + \gamma(2 + 3\eta_R)(R + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}})$ using $T < \nu/\mu$. Finally, we obtain the assertion by assuming that $\mu \geq 2C_{\nu,R}$. \hfill $\square$

Proof of Theorem 3.1. Let $X$ consist of all functions $f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R}$ being analytic in $x, p$ and continuous in $t$ such that $\|f\|_{\nu,\mu} \leq R$. Combining the previous two lemmata, we directly obtain that $\Phi : X \to X$ defined by (13) is a contraction requiring that $\mu$ is sufficiently large and $T \in (0, \nu/\mu)$. Thus, Banach’s fixed-point theorem implies that equation (10) admits a unique mild solution in the space $X$. Using a bootstrap argument yields that $f$ is also analytic in $t$ and satisfies equation (10) classically.

For the second part of the assertion, let $f = \Psi(f_0, F)$, $g = \Psi(g_0, G)$. There exists a $\tilde{\mu} > 0$ such that for $T \in [0, \nu/\tilde{\mu})$ the functions $f, g$ are both defined on $[0, T)$ and satisfy $\|f\|_{\tilde{\mu},\nu}, \|g\|_{\tilde{\mu},\nu} \leq R$. Defining $h = f - g$, $h_0 = f_0 - g_0$ as well as $H = F - G$, we have
\[ \partial_t h + u \cdot \nabla_x h - U \nabla_x n_h \cdot \nabla_p f - U \nabla_x n_g \cdot \nabla_p h = Q(f, F) - Q(g, G) \]
with \( h(x,p,0) = h_0(x,p) \), where \( Q(f,F) = \gamma n_f(1 - \eta n_f)(F - f) \). Similar to the proof of Lemma 3.4, we estimate
\[
\|h\|_{\nu,\mu} \leq \|h_0\|_{C^\nu} + \int_0^T \|\partial_t h\|_{C^{\nu-\mu}} \, dt \\
\leq \|h_0\|_{C^\nu} + \frac{1}{\mu} (\|u\|_{C^{\nu,\infty}} + 4|U|\, R) \|h\|_{\nu,\mu} \\
+ \frac{\nu}{\mu} \gamma(2 + 3\eta) \left( R + \sup_{0 \leq t \leq T} \|F\|_{C^{\nu-\mu}} \right) \|h\|_{\nu,\mu} \\
+ \frac{\nu}{\mu} \sup_{0 \leq t \leq T} \|n_f(1 - \eta n_f)\|_{C^{\nu-\mu}} \|H\|_{C^{\nu-\mu}}
\]
for \( \mu \geq \tilde{\mu} \). Using \( \|F(t)\|_{C^{\nu-\mu}} \leq C \) and choosing \( \mu > 0 \) sufficiently large, we obtain
\[
\|h\|_{\nu,\mu} \leq \frac{1}{2} \|h\|_{\nu,\mu} + \|h_0\|_{C^\nu} + \frac{\nu}{\mu} \sup_{0 \leq t \leq T} \|n_f(1 - \eta n_f)\|_{C^{\nu-\mu}} \|H\|_{C^{\nu-\mu}}
\]
(16)
implying
\[
\|f - g\|_{\nu,\mu} \leq 2\|f_0 - g_0\|_{C^\nu} + \frac{\nu}{\mu} \sup_{0 \leq t \leq T} \|n_f(1 - \eta n_f)\|_{C^{\nu-\mu}} \|F - G\|_{C^{\nu-\mu}}
\]
(17)
for \( \mu > \tilde{\mu} \) being sufficiently large. Moreover, we can again use the submultiplicative property of the norm \( \|\cdot\|_{C^{\nu-\mu}} \) and the fact that
\[
\|n_f\|_{C^{\nu-\mu}} \leq \|f\|_{C^{\nu-\mu}} \leq \|f\|_{\nu,\mu} \leq R
\]
to obtain that
\[
\frac{\nu}{\mu} \|n_f(1 - \eta n_f)\|_{C^{\nu-\mu}} \leq \frac{\nu}{\mu} R (1 + \eta R) \leq \frac{1}{\sqrt{\mu}}
\]
if \( \mu > \tilde{\mu} \) is sufficiently large. This finishes the proof. \( \square \)

4. BGK-type collision operator. In this section, we focus on the semiconductor Boltzmann–Dirac–Benney equation
\[
\partial_t f + u(p) \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f(1 - \eta n_f)(F_f - f)
\]
(6)
with \( f(x,p,0) = f_0(x,p) \) for given \( U \neq 0 \) and \( \gamma \geq 0 \).

It can also be understood as a version of Eq. (10) with a self-consistent equilibrium distribution function \( F = F_f(x,p,t) = \left( \eta + \exp(-\lambda_0(x,t) - \lambda_1(x,t)\epsilon(p)) \right)^{-1} \), for \( x \in \mathbb{R}^d \), \( p \in \mathbb{T}^d \) and \( t > 0 \). Here, \( \lambda_0, \lambda_1 \) shall be chosen in such a way that
\[
n_f(x,t) := n_{F_f}(x,t) \quad \text{and} \quad E_f(x,t) = E_{F_f}(x,t),
\]
(18)
where \( n_f(x,t) := \int_{\mathbb{T}^d} f(x,p,t) \, dp \) and \( E_f(x,t) = \int_{\mathbb{T}^d} \epsilon(p)f(x,p,t) \, dp \). This is well-defined according to [9] section 5.1.

**Theorem 4.1.** Let \( \eta, \nu, R > 0 \) and \( \alpha > 0 \). There exist \( \delta > 0 \) and \( \mu_0 > 0 \) such that the following is true:

Let \( \bar{n}, \bar{E} \in \mathbb{R} \) and \( f_0 : \mathbb{R}^d \times \mathbb{T}^d \to [2\alpha, \eta^{-1} - 2\alpha] \) be analytic such that
\[
\|f_0\|_{C^\nu} \leq \frac{R}{2} \quad \text{and} \quad \|n_{f_0} - \bar{n}\|_{C^\nu} + \|E_{f_0} - \bar{E}\|_{C^\nu} \leq \frac{1}{2} \delta.
\]
(19)
Lemma 4.4. Let $\mu \geq \mu_0$, $T \in (0, \nu/\mu)$, equation (10) admits a unique analytic solution $f : \mathbb{R}^d \times T^d \times [0, T) \rightarrow [\alpha, \eta^{-1} - \alpha]$ satisfying $f|_{t=0} = f_0$ and
\[\|f(t)\|_{C^{\nu-\mu}} \leq R \text{ and } \|n_f(t) - \bar{n}\|_{C^{\nu-\mu}} + \|E_f(t) - \bar{E}\|_{C^{\nu-\mu}} \leq \delta\]
for all $0 \leq t < T$. Moreover, let $f, g$ be the unique solution of (6) for with $f(x, p, 0) = f_0(x, p)$ and $g(x, p, 0) = g_0(x, p)$, where $f_0$ and $g_0$ satisfy both the hypothesis of this theorem. Then there exists a $C > 0$ such that
\[\|f(t) - g(t)\|_{C^{\nu-\mu}} \leq C\|f_0 - g_0\|_{C^\nu}\]
for all for all $0 \leq t < T$.

In order to prove that Eq. (6) admits a local, analytic solution, we basically require Theorem 3.1 and the following Lipschitz estimate from Proposition 7.3.

Let $\Psi : (f_0, F) \rightarrow f$ be the mapping as in Theorem 3.1 defined by the solution of
\[\partial_t f + u \cdot \nabla_x f - U \nabla_x n_f : \nabla_p f = \gamma n_f(1 - \eta n_f)(F - f)\]
with $f(x, p, 0) = f_0(x, p)$. With this, we define the mapping
\[\Theta(g) := \Psi(f_0, F_g)\]
Therefore, every fixed-point of $\Theta$ is a classical solution of (6). At first, we need to show that $\Theta$ is well-defined.

**Proposition 4.2.** Let $\eta, \nu, R > 0$ and $\alpha > 0$. There exists an $C, \delta > 0$ such that the following is true.

Let $f, g : \mathbb{R}^d \times T^d \rightarrow [\alpha, \eta^{-1} - \alpha]$ be analytic satisfying $\|f\|_{C^\nu}, \|g\|_{C^\nu} \leq R$ and
\[\|n_h - \bar{n}\|_{C^\nu} + \|E_h - \bar{E}\|_{C^\nu} \leq \delta \text{ for } h \in \{f, g\}\]
and some $\bar{n}, \bar{E} \in \mathbb{R}$, it holds
\[\|F_f\|_{C^\nu}, \|F_g\|_{C^\nu} \leq C\]
and
\[\|F_f - F_g\|_{C^\nu} \leq C\|f - g\|_{C^\nu}\]

**Proof.** See appendix. \qed

Using this proposition, we can define the metric space $Y$ on which $\Theta$ is a contraction.

**Definition 4.3.** For $R, \nu, \eta, \alpha > 0$, let $\delta > 0$ be as in Proposition 4.2. Moreover, let $\bar{n}, \bar{E} \in \mathbb{R}$, $\mu > 0$ and $T \in (0, \nu/\mu)$.

We define $Y$ as the space of all analytic functions $f : \mathbb{R}^d \times T^d \times [0, T) \rightarrow [\alpha, \eta^{-1} - \alpha]$ satisfying
1. $\|f\|_{\nu, \mu} \leq R$,
2. $\|n_f(\cdot, t) - \bar{n}\|_{C^{\nu-\mu}} + \|E_f(\cdot, t) - \bar{E}\|_{C^{\nu-\mu}} \leq \delta$
for all $t \in [0, T)$. Thus, $Y$ is a complete if the metric is induced by the norm $\|\cdot\|_{\nu, \mu}$.

As we plan to apply the Banach fixed-point theorem, we need to show that $\Theta$ is a contraction, i.e., the image of $\Theta$ is included in $Y$ and $\Theta$ is Lipschitz continuous with Lipschitz constant $L < 1$.

**Lemma 4.4.** Let $\bar{n}, \bar{E} \in \mathbb{R}$, $f_0 : \mathbb{R}^d \times T^d \rightarrow [2\alpha, \eta^{-1} - 2\alpha]$ be analytic such that
\[\|f_0\|_{C^\nu} \leq \frac{R}{2}\text{ and } \|n_{f_0} - \bar{n}\|_{C^\nu} \text{, } \|E_{f_0} - \bar{E}\|_{C^\nu} \leq \frac{1}{2}\delta\]
If $\mu > 0$ is sufficiently large and $g \in Z$, then $\Theta(g) \in Z$. 

Proof. By definition, we have
\[ \Theta(g) = \Psi(f_0, \mathcal{F} g). \]
For \( g \in Z \), we know from Proposition 4.2 that \( \| \mathcal{F} g \|_{C^{0, \mu}} \leq C \) for some \( C > 0 \) and all \( t \in [0, T) \). Hence, \( f := \Theta(g) \) is well-defined for sufficiently large \( \mu > 0 \) and \( \| f \|_{\nu, \mu} \leq R \). Clearly, by continuity, if \( \mu \) sufficiently large and thus \( T > 0 \) sufficiently small, then the image of \( f \) belongs to \( [0, \eta^{-1} - \alpha] \).

Therefore, it remains to show that \( \| n_f(\cdot, t) \|_{C^{0, \mu}} + \| E_f(\cdot, t) \|_{C^{0, \mu}} \leq \delta \) for all \( t \in [0, T) \). Using Lemma 2.5 entails that
\[ \| n_f(\cdot, t) - \bar{n} \|_{C^{0, \mu}} \leq \int_0^T \| \partial_t n_f \|_{C^{0, \mu}} dt \]
and likewise
\[ \| E_f(\cdot, t) - \bar{E} \|_{C^{0, \mu}} \leq \int_0^T \| \partial_t E_f \|_{C^{0, \mu}} dt. \]
As in the proof of Lemma 3.3 (see inequality 14), we can show that
\[ \int_0^T \| \partial_t f \|_{C^{0, \mu}} dt \leq \frac{1}{\mu} \left( \| u \|_{C^{\infty}} + |U| \right) \| f \|_{\nu, \mu} + \gamma TR(1 + \eta R)(R + C) \leq \frac{\bar{C}}{\mu} \]
for some \( \bar{C} > 0 \). Thus,
\[ \| n_f(\cdot, t) - \bar{n} \|_{C^{0, \mu}} + \| E_f(\cdot, t) - \bar{E} \|_{C^{0, \mu}} \leq \| n_f(\cdot, t) \|_{C^{0, \mu}} + \| E_f(\cdot, t) \|_{C^{0, \mu}} \leq \frac{\bar{C}}{\mu} \leq \delta \]
for all \( t \in [0, T) \) and some \( \bar{C} > 0 \) if \( \mu \geq \frac{\bar{C}}{\delta} \), which proves the assertion.

Lemma 4.5. Let \( \bar{n}, \bar{E} \in \mathbb{R} \), \( f_0 : \mathbb{R}^d \times T^d \rightarrow [2\alpha, \eta^{-1} - 2\alpha] \) be analytic such that
\[ \| f_0 \|_{C^\nu} \leq \frac{R}{2} \quad \text{and} \quad \| n_{f_0} - \bar{n} \|_{C^\nu}, \| E_{f_0} - \bar{E} \|_{C^\nu} \leq \frac{1}{2} \delta. \]
If \( \mu > 0 \) is sufficiently large, then for \( f, g \in Y \) it holds
\[ \| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq \frac{1}{2} \| f - g \|_{\nu, \mu}. \]
Proof. According to the previous Lemma, we can apply Theorem 3.1 entailing for sufficiently large \( \mu > 0 \)
\[ \| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq 2\mu^{-\frac{1}{2}} \sup_{t} \| \mathcal{F} f - \mathcal{F} g \|_{C^{0, \mu}}. \]
Then the second statement of Proposition 4.2 yields that
\[ \| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq C\mu^{-\frac{3}{2}} \sup_{t} \| f - g \|_{C^{0, \mu}} \leq C\mu^{-\frac{3}{2}} \| f - g \|_{\nu, \mu} \]
for some \( C > 0 \). This implies the assertion for sufficiently large \( \mu \) satisfying \( \mu \geq 4C^2 \).

Proof of Theorem 4.4. The contraction mapping theorem ensures that \( \Psi \) has a unique fixed-point implying that equation (6) admits a unique solution. Finally, the Lipschitz estimate is a direct consequence of Theorem 3.1.

With Theorem 4.1 we can now easily prove the following weaker version of Theorem 1.1.
Theorem 4.6. Let $\eta > 0$, $\gamma \geq 0$, $U \neq 0$ and $f_0 : \mathbb{T}^d \times \mathbb{T}^d \to (0, \eta^{-1})$ be analytic such that

$$n_{f_0}(x) = \int_{\mathbb{T}^d} f_0(x, p) dp = \text{const.} \quad \text{and} \quad E_{f_0} = \int_{\mathbb{T}^d} \epsilon(p) f_0(x, p) dp = \text{const.}$$

w.r.t. $x \in \mathbb{T}^d$. Then there exists a time $T > 0$ such that (6) admits a unique analytic solution $f : \mathbb{T}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R}$ with $f(x, p, 0) = f_0(x, p)$.

Proof. Since $f_0$ is analytic and hence continuous, there exists a $\alpha > 0$ such that $2\alpha < f_0 < \eta^{-1} - 2\alpha$. The key difference to Theorem 4.1 is that now the spacial domain is essentially restricted to a compact set $\mathbb{T}^d$, which can be extended periodically to $\mathbb{R}^d$. Any analytic function $f_0$ on a compact domain has a minimal radius $r$ of convergence, i.e. a number $r > 0$ such that for all $(x, p)$ the series

$$\sum_{i,j \in \mathbb{N}_0^d} \frac{\partial_i^j \partial_p^j f_0(x, p)}{i!j!} x^i p^j$$

converges absolutely for $|x| + |p| \leq r$. This implies that

$$M(x, p) := \sum_{i,j \in \mathbb{N}_0} \frac{(r/2)^{|i+j|}}{i!j!} |\partial_x^i \partial_p^j f_0(x, p)| < \infty$$

for all $(x, p) \in (\mathbb{T}^d)^2$. Now, choose $(x_1, p_1), \ldots, (x_N, p_N) \in (\mathbb{T}^d)^2$ such that

$$\bigcup_{i=1}^N B_{r/4}(x_i, p_i) \supseteq (\mathbb{T}^d)^2$$

and define

$$K := \max_{i=1,\ldots,N} M(x_i, p_i).$$

Let $\nu := r/8$. Then for every $(x, p) \in (\mathbb{T}^d)^2$ there exists an $i \in \{1, \ldots, N\}$ such that $|(x - x_i, p - p_i)| \leq \nu$ and

$$\sum_{k,j \in \mathbb{N}_0^d} \frac{(2\nu)^{|k+j|}}{k!j!} |\partial_x^k \partial_p^j f_0(x, p)| \leq \sum_{k,j \in \mathbb{N}_0^d} \frac{(4\nu)^{|k+j|}}{k!j!} |\partial_x^k \partial_p^j f_0(x_i, p_i)| \leq K.$$

This directly implies that $|f_0|_{C^{2\nu}} < \infty$. Moreover, by Lemma 2.2, we can see that also $R := \|f_0\|_{C^r} + 1$ is finite as $\nu < 2\nu$. As $n := n_{f_0}$ and $E := E_{f_0}$ are constant, we have shown all the hypothesis of Theorem 4.1 and finally obtain a analytic solution on a small time interval.

For a full proof of Theorem 1.1, we refer to section 6 and section 7, in which we refine the presented technique using that the collision operator is local in space. The next section is devoted to an application of Theorem 4.1 showing the ill-posedness of equation (6).

5. On the ill-posedness of the semiconductor Boltzmann-Dirac-Benney equation. This section is motivated by the ill-posedness result of [13] and [7] for the Vlasov-Dirac-Benney equation. Similar to [7], we linearize the equation around an equilibrium. Let $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2$. Then

$$F_{\lambda} : \mathbb{T}^d \to \mathbb{R}, \quad p \mapsto \frac{1}{\eta} e^{-\lambda_0 - \lambda_1 \epsilon(p)}$$

is a stationary analytic solution of (6), which is constant in $x$. 


5.1. **Linearized equation.** Now let us formally linearize the left-hand side of (6) around $F_\lambda$ and consider
\[ \partial_t g + u(p) \cdot \nabla_x g - U \nabla_x n_g \cdot \nabla F_\lambda(p) = \gamma n_{F_\lambda}(1 - \eta n_{F_\lambda}) \left( G(\lambda; p) \cdot \left( \frac{n_g}{E_g} \right) - g \right) \tag{20} \]
with $g(x,p,0) = g_0(x,p)$ and $G(\lambda; p) := \partial_{(n_f,E_f)} F_f(p)|_{f=F_\lambda}$. Recall that $u(p) = \nabla_p \epsilon(p)$ with
\[ \epsilon(p) = -2\epsilon_0 \sum_{i=1}^d \cos(2\pi p_i), \quad p \in \mathbb{T}^d, \]
for some $\epsilon_0 > 0$.

**Remark 5.1.** The definition of $G(\lambda; p)$ has to be understood according to Definition A.1: $F_f$ can be written by $F_f = F_0(n_f,E_f;p)$ for some analytic $F_0 : V \subset \mathbb{R}^2 \times \mathbb{T}^d \to [0, \eta^{-1}]$. By Lemma A.2 from the appendix, it holds
\[ G(\lambda; p) = \frac{F_\lambda(p)(1 - \eta F_\lambda(p))}{\int_{\mathbb{T}^d} e^2 \mu_1 \int_{\mathbb{T}^d} 1 \mu - \left( \int_{\mathbb{T}^d} e \mu \right)^2} \int_{\mathbb{T}^d} \left( \frac{-\epsilon(p')}{1} \right) (\epsilon(p) - \epsilon(p')) d\mu', \]
where $d\mu := F_\lambda(p)(1 - \eta F_\lambda(p)) dp$.

In the following, we will denote the components of $G$ as $G_1, G_2$ and write $p = (p_1, \ldots, p_d), x = (x_1, \ldots, x_d)$ and $u(p) = (u_1(p), \ldots, u_d(p))$.

**Lemma 5.2.** For $\lambda \in \mathbb{R}^2$ we abbreviate $\gamma_\lambda := \gamma n_{F_\lambda}(1 - \eta n_{F_\lambda})$. Assume that there exists a bounded set $K \subset (\mathbb{R} \setminus \{0\})^2$ with $K \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}$ such that the eigenvalues of
\[ B = B(\alpha, \beta) := \int_{\mathbb{T}^d} \frac{(U u_1(p) \partial_{p_1} F_\lambda(p) + \beta G_1(\lambda; p), \beta G_2(\lambda; p))}{u_1(p)^2 + \alpha^2} \left( \frac{1}{\epsilon(p)} \right) dp \]
are 0 and 1 for $(\alpha, \beta) \in K$. Let $(\hat{n}_{\alpha,\beta}, \hat{E}_{\alpha,\beta})$ denote the eigenvector to the eigenvalue 1 and define
\[ A_{\alpha,\beta}(p) := \frac{1}{u_1(p)} \left( U \partial_{p_1} F_\lambda(p) \hat{n}_{\alpha,\beta} - \hat{E}_{\alpha,\beta} \left( \frac{\partial n_{F_\lambda}}{\partial E_{G_{\alpha,\beta}}}, \frac{1}{G_{\alpha,\beta}} \right) \right). \]
Then
\[ g_{\alpha,\beta}(x,p) := A_{\alpha,\beta}(p) e^{i \gamma n_{F_\lambda}(1 - \eta n_{F_\lambda}) \frac{x_1}{2}} \]
is a solution of
\[ u(p) \cdot \nabla_x g_{\alpha,\beta} - U \nabla_x n_{\alpha,\beta} \cdot \nabla F_\lambda(p) = \gamma n_{F_\lambda}(1 - \eta n_{F_\lambda}) \left( G(\lambda; p) \cdot \left( \frac{n_{\alpha,\beta}}{E_{\alpha,\beta}} \right) - \frac{\alpha^2}{\beta} g_{\alpha,\beta} \right). \]
Moreover, let $N \in \mathbb{N}$. There exists $C_N > 0$ such that
\[ \sup_{(\alpha,\beta) \in K} \| g_{\alpha,\beta} \|_{W^{N,\infty}(\mathbb{R}^d \times \mathbb{T}^d)} \leq C_N(1 + |\beta|^{-N}) \]
for all $(x,p) \in \mathbb{R}^d \times \mathbb{T}^d$. In addition, there exists a $\nu_0 > 0$ and such that
\[ \| g_{\alpha,\beta} \|_{C^\nu} \leq C_{\nu_0}(1 + |\beta|^{-1}) e^{\frac{\nu_0}{C_{\nu_0}}} \]
and for all $\nu \leq \nu_0$ and some $c, C_{\nu_0} > 0$ being independent from $\alpha, \beta$. 
Proof. Note that $G$ is symmetric and $\partial_p$ is anti-symmetric, i.e., $G(\bar{\lambda}; -p) = G(\bar{\lambda}; p)$ and $\partial_p F_{\bar{\lambda}}(-p) = -\partial_p F(p)$, which is a consequence of $u(-p) = -u(p)$ as well as $F_{\bar{\lambda}}(-p) = F_{\bar{\lambda}}(p)$ for $p \in \mathbb{T}^d$. Therefore, since the denominator is even, we may add an odd function to the denominator without changing the integral. Thus, we can divide the integrand by $u_1(p) + i\alpha$ and obtain

$$B(\alpha, \beta) = \int_{\mathbb{T}^d} \frac{(U \partial_p F_{\bar{\lambda}}(p) - i\beta G_1(\bar{\lambda}; p), -i\alpha G_2(\bar{\lambda}; p))}{u_1(p) - i\alpha} \left( \frac{1}{\epsilon(p)} \right) dp.$$ 

Since $(\hat{n}_A, \bar{E}_A)$ is the eigenvector to the eigenvalue 1 of $B$, we infer $\int_{\mathbb{T}^d} A_{\alpha, \beta}(p) dp = \hat{n}_{\alpha, \beta}$ and $\int_{\mathbb{T}^d} \epsilon(p) A_{\alpha, \beta}(p) dp = \bar{E}_{\alpha, \beta}$. Finally, we directly compute

$$\gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d^2 \bar{g}_{\alpha, \beta} + u(p) \cdot \nabla g_{\alpha, \beta}}{\beta}$$

$$= i\gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d}{\beta} \left( -i\alpha + u_1(p) \right) A_{\alpha, \beta}(p) e^{i\gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d}{\beta} x_1}$$

$$= i\gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d}{\beta} \left( U \partial_p F_{\bar{\lambda}}(p) \hat{n}_{\alpha, \beta} - i\frac{\beta}{\alpha} G(\bar{\lambda}; p) \cdot \left( \hat{n}_{\alpha, \beta} \right) e^{i\gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d}{\beta} x_1} \right)$$

$$= U \nabla_x g_{\alpha, \beta} \cdot \nabla F_{\bar{\lambda}}(p) + \gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) G(\bar{\lambda}; p) \cdot \left( \hat{n}_{\alpha, \beta} \right).$$

Since $p \mapsto g_{\alpha, \beta}(0, p)$ is analytic on $\mathbb{T}^d$ and $\mathbb{T}^d$ is compact and $\bar{K} \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}$ is compact, there exists a $\nu > 0$ such that

$$C_\nu := \sup_{(\alpha, \beta) \in K} \sum_{j \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \nu^{1 + j} \left\| \partial^j_p g_{\alpha, \beta}(0, p) \right\|_{W^{1,1}_p(\mathbb{T}^d)} < \infty.$$ 

Thus, \n
$$\|g_{\alpha, \beta}\|_{C_\nu} = \sum_{j \in \mathbb{N}_0} \left( \sum_{j \in \mathbb{N}_0} \nu^{1 + j} \left\| \partial^j_p g_{\alpha, \beta}(x, p) \right\|_{W^{1,\infty}_p(\mathbb{T}^d)} \right)$$

$$= \sum_{j \in \mathbb{N}_0} \left( \sum_{j \in \mathbb{N}_0} \nu^{1 + j} \left\| \partial^j_p g_{\alpha, \beta}(0, p) \right\|_{W^{1,1}_p(\mathbb{T}^d)} \right)$$

$$= C_\nu \left( 1 + |\varphi| e^{\nu |\varphi|} \right)$$

setting $\varphi := \gamma_n F_{\bar{\lambda}}(1 - \eta \lambda) \frac{d}{\beta}$ for all $(\alpha, \beta) \in K$. If we want to estimate only a finite number of derivatives, we see that for all $N > 0$ there exists a $C_N > 0$ such that $\sup_{(\alpha, \beta) \in K} \sum_{|j| \leq N} \left\| \partial^j_p g_{\alpha, \beta}(0, p) \right\|_{L^\infty(\mathbb{T}^d)} \leq C_N$ since $g_{\alpha, \beta}$ is smooth. This yields

$$\sum_{|j| \leq N} \left\| \partial^j_p g_{\alpha, \beta} \right\|_{L^\infty(\mathbb{R}^d \times \mathbb{T}^d)}$$

$$\leq C_N \sum_{|\varphi| \leq N} |\varphi| \leq NC_N \left( 1 + |\varphi| N \right). \square$$
In order to prove that the hypothesis of the previous lemma can be fulfilled, we start with an easier case, where \( \beta = 0 \). Then the condition simplifies to
\[
1 = U\bar{\lambda}_1 \int_{\mathbb{T}^d} \frac{u_1(p)^2}{u_1(p)^2 + \alpha_0^2} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp
\] (21)
for some \( \alpha_0 \neq 0 \).

**Lemma 5.3.** Let \( U \neq 0 \). Then there exist \( \bar{\lambda} \in \mathbb{R}^2 \) and an \( \alpha_0 > 0 \) such that (21) is satisfied. In addition, the solution \( \alpha_0 \) of (21) is unique (up to its sign) for fixed \( \bar{\lambda} \).

**Proof.** At first, we define
\[
\kappa(\lambda) := \lambda_1 \int_{\mathbb{T}^d} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp.
\]
According to [9] section 5.3, it holds that \( \sup_{\lambda \in \mathbb{R}^2} \kappa(\lambda) = \infty \) and by symmetry \( \inf_{\lambda \in \mathbb{R}^2} \kappa(\lambda) = -\infty \). Thus, there exists \( \bar{\lambda} \in \mathbb{R}^2 \) such that
\[
U\bar{\lambda}_1 \int_{\mathbb{T}^d} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp = U\kappa(\bar{\lambda}) > 1.
\]
Finally, by
\[
1 < U\kappa(\bar{\lambda}) \xrightarrow{c \to 0} U\bar{\lambda}_1 \int_{\mathbb{T}^d} \frac{u_1(p)^2}{u_1(p)^2 + c^2} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp \xrightarrow{c \to \infty} 0,
\]
the intermediate value theorem yields the first assertion. The uniqueness is a consequence of the monotonicity of \( U\bar{\lambda}_1 \int_{\mathbb{T}^d} \frac{u_1(p)^2}{u_1(p)^2 + \lambda^2} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp \) w.r.t. \( c \). \( \square \)

**Remark 5.4.** We used in the proof that Equation (21) admits a solution if
\[
1 < U\bar{\lambda}_1 \int_{\mathbb{T}^d} F_\lambda(p)(1 - \eta F_\lambda(p)) \, dp
\]
is satisfied.

Now, we go back to the general case, where \( \beta \neq 0 \).

**Lemma 5.5.** Let \( \bar{\lambda} \) and \( \alpha_0 \) be as in Lemma 5.3. There exist an open Interval \( 0 \in I \subset \mathbb{R} \) and a function \( \alpha : I \to \mathbb{R} \) with \( \alpha(0) = \alpha_0 \) such that \( B(\alpha(\beta), \beta) \) possesses the eigenvalues 0 and 1 for all \( \beta \in I \).

**Proof.** According to Lemma 5.3, we know that 1 is an eigenvalue of \( B(\alpha(0), 0) \) which is equivalent to \( \det(B(\alpha(0), 0) - \text{Id}) = 0 \). Since
\[
\phi : \mathbb{R} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}, \quad (a, b) \mapsto \det(B(a, b) - \text{Id})
\]
is smooth, there exists an \( \alpha : I \ni 0 \to \mathbb{R} \) with \( \alpha(0) = \alpha_0 \) if the derivative of \( \phi \) has full rank at \( (a, b) = (\alpha_0, 0) \). In order to show this, we only need to look at the derivative w.r.t. \( a \):
\[
\partial_a \phi = \partial_a ((B_{11} - 1)(B_{22} - 1) - B_{12}B_{21}) = \partial_a B_{11}(B_{22} - 1) + (B_{11} - 1)\partial_a B_{22} - \partial_a B_{12}B_{21} - B_{12}\partial_a B_{21}
\]
\[
\partial_a \phi(\alpha_0, 0) = -\partial_a B_{11}(\alpha_0, 0) = 2\alpha_0 U \int_{\mathbb{T}^d} \frac{u_1(p)^2 F_\lambda(p)(1 - \eta F_\lambda(p))}{(u_1(p)^2 + \alpha_0^2)^2} \, dp \neq 0.
\]
Thus, the derivative of \( \phi \) has at \( (\alpha_0, 0) \) full rank and therefore the zero-set of \( \phi \) is locally a one-dimensional manifold at \( (\alpha_0, 0) \). According to Lemma 5.3, \( \phi(a, 0) = 0 \).
has only one positive solution at \( a = \alpha_0 \). Finally, the fact that \( B \) has rank 1 implies directly the trivial eigenvalue and finishes the proof. \( \square \)

**Proposition 5.6.** Let \( \alpha_0 > 0, \lambda \in \mathbb{R}^2 \) be a solution of

\[
1 = U \hat{\lambda}_1 \int_{\mathbb{T}^d} \frac{u_1(p)^2}{u_1(p)^2 + \alpha_0^2} F_{\hat{\lambda}}(p)(1 - \eta F_{\hat{\lambda}}(p)) dp
\]

(see Lemma 5.3). Then there exist an open interval \( I \ni 0 \) and function \( \omega : I \setminus \{0\} \to \mathbb{R} \) and analytic \( g_\beta : \mathbb{T}^d \times \mathbb{T}^d \times [0, \infty) \to \mathbb{R} \) solutions of (20) for \( \beta \in I \setminus \{0\} \) such that the following holds:

- \( \beta \mapsto \beta \omega(\beta) \) can be extended on \( I \) to a positive continuous function.
- \( g_\beta(x, p, t) = g_\beta(x, p, 0) e^{\omega(\beta)t} \).
- There exists a \( \nu_0 > 0 \) and all \( \nu > \nu_0 \) and some \( c, C_{\nu_0} > 0 \) being independent from \( x \) and \( \beta \).
- There exists \( C_\nu > 0 \) such that
  \[
  \|g_\beta(\cdot, \cdot, t)\|_{W_\nu(\mathbb{T}^d \times \mathbb{T}^d)} \leq C_\nu (1 + |\beta|^{-N}) e^{\omega(\beta)t} \quad \text{for all } \beta \in I \setminus \{0\}. \tag{23}
  \]

**Proof.** Let \( \alpha : I \to \mathbb{R} \) with \( \alpha(0) = \alpha_0 \) be given by Lemma 5.5. For \( \beta \in I \setminus \{0\} \), we define

\[
\beta(x, p, t) := \Re(g_{\alpha(\beta, \beta}(x, p)) e^{\omega(\beta)t} \quad \text{and} \quad \omega(\beta) := \gamma n F_{\hat{\lambda}} \left( 1 + \eta F_{\hat{\lambda}} \right) \frac{\alpha(\beta)^2 - \beta}{\beta}
\]

and where \( g_{\alpha, \beta} \) is given by Lemma 5.2. Then \( g_\beta \) is a solution of (20) fulfilling \( g_\beta(x, p, 0) = \Re(g_{\alpha(\beta, \beta}(x, p)) \) for all \( \beta \in I \setminus \{0\} \). The remaining parts are a direct consequence of Lemma 5.2. \( \square \)

### 5.2. Nonlinear equation

Fix \( \bar{\lambda} \) and \( \alpha_0 \) such that (21) is fulfilled (see Lemma 5.3). We now choose \( \nu > 0 \) such that \( \|F_{\bar{\lambda}}\|_{C_\nu} < \infty \). Let \( g_\beta \) be as in Proposition 5.6 and let \( c > 0 \) be given such that (22) is fulfilled. We set

\[
f_{\bar{\lambda}}(x, p, t) = F_{\bar{\lambda}}(p) + \beta e^{-\frac{\eta}{\beta}} g_\beta(x, p, 0).
\]

Then \( \|f_{\bar{\lambda}}\|_{C_\nu} \) is uniformly bounded w.r.t. \( \beta > 0 \). Since \( F_{\bar{\lambda}}(p) \in [b, \eta^{-1} - b] \) for some \( b > 0 \) and all \( p \in \mathbb{T}^d \) and \( g_\beta(x, p, 0) \) is uniformly bounded w.r.t. \( x, p \) and \( \beta \), we can apply Theorem 3.1: there exists a \( \beta_0 > 0 \) and a \( T > 0 \) such that

\[
\partial_t f_{\beta} + u(p) \cdot \nabla_x f_{\beta} - U \nabla_x n_{f_{\beta}} \cdot \nabla_p f_{\beta} - \gamma n_{f_{\beta}}(1 - \eta n_{f_{\beta}})(f_{f_{\beta}} - f_{\beta})
\]

has a unique analytic solution \( f_{\beta} : \mathbb{T}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R} \) for each \( \beta \in (-\beta_0, \beta_0) \setminus \{0\} \) with

\[
f_{\beta}(x, p, 0) = f_{\bar{\lambda}}(x, p, 0)
\]

By shrinking \( T > 0 \), the theorem moreover implies that there exist \( \bar{\nu} \in (0, \nu) \) and \( \bar{C} > 0 \) such that \( \|f_{\beta}(t)\|_{C_{\bar{\nu}}} \leq \bar{C} \) for all \( \beta \in (-\beta_0, \beta_0) \setminus \{0\} \) and \( t \in [0, T) \). Define \( h_{\beta} \) by the equation

\[
f_{\beta}(x, p, t) = F_{\bar{\lambda}}(p) + \beta e^{-\frac{\eta}{\beta}} (g_{\beta}(x, p, t) + h_{\beta}(x, p, t)).
\]

Then \( h_{\beta} \) solves

\[
\partial_t h_{\beta} + u(p) \cdot \nabla_x h_{\beta} - U \nabla_p f_{\beta} \cdot \nabla_x n_{h_{\beta}} - U \beta e^{-\frac{\eta}{\beta}} \nabla_x n_{g_{\beta}} \cdot \nabla_p h_{\beta} = Q_{\beta} + U \beta e^{-\frac{\eta}{\beta}} \nabla_x n_{g_{\beta, \varphi}} \cdot \nabla_p g_{\beta, \varphi},
\]
Thus, for \( t \) and let \( \mu \) with
\[
\left| \nabla_c \mu \right| \leq \left( G(\lambda) \cdot \left( \frac{n_{g_3}}{E_{g_3}} \right) - g_3 \right)
\]
and \( h(x, p, 0) = 0 \). Note that \( c \) is the constant provided by Proposition 5.6.

**Lemma 5.7.** There exist \( C, \tau > 0 \) such that
\[
\left\| h^\beta(\cdot, \cdot, t) \right\|_{L^\infty(\mathbb{R}^d \times T^d)} \leq Ct \quad \text{for } 0 \leq t \leq \tau \text{ and all } |\beta| < \beta_0.
\]

**Proof.** Recall the norms
\[
|f|_{C^\nu} := \sum_{a,b \in \mathbb{N}_0^d} \frac{\nu(\nu + h)}{a!b!} \left\| \partial_x^a \partial_p^b f \right\|_{L^\infty L^p}, \quad \text{and} \quad |u|_{C^\nu, \infty} := \sum_{b \in \mathbb{N}_0^d} \frac{\nu(\nu + h)}{b!} \left\| \partial^b u \right\|_{L^\infty(T^d)}
\]
and let \( \mu \in (0, \nu/2) \) and \( M > 0 \). Similar to the proof of Lemma 2.5, we see that
\[
\partial_t |h^\beta|_{C^{\nu-Mt}} \leq \left| \partial_t h^\beta \right|_{C^{\nu-Mt}} - M \left| \partial_p h^\beta \right|_{C^{\nu-Mt}}
\]
\[
\leq |Q^\beta|_{C^{\nu-Mt}} + |U \nabla_p f^\beta|_{C^n} \left( |\nabla x|_{C^n} - M \right) \left| \partial_p h \right|_{C^{\nu-Mt}}
+ e^{-\frac{T}{\infty}} \left| U \nabla x n_{g_3} \right|_{C^n} \left| \nabla p g_3 \right|_{C^{\nu-Mt}}.
\]
Using Proposition 5.6, we note that there exists a constant \( C_0 > 0 \) independent from \( \beta \) such that
\[
|\beta U \nabla x n_{g_3}|_{C^n} \leq C_0 \exp \left( \frac{\beta \omega(\beta) t + c \mu}{|\beta|} \right).
\]
Thus, for \( t \leq \tau := \min \{ c(\nu/2 - \mu)/\max_{|\beta| \leq \beta_0} \beta \omega(\beta), T \} \) we have that
\[
|U \nabla x n_{g_3}|_{C^n} \leq C_0 \exp \left( \frac{c \nu}{2 |\beta|} \right).
\]
Similarly to the proof of Lemma 5.2 Proposition 5.6, we can show that \( |\nabla p g_3|_{C^n} \leq C_1 e^{\frac{\nu}{\infty}} \) for some \( C_1 > 0 \) which does not depend on \( \beta \). Choosing now
\[
M := C_0 + \sup_{|\beta| \leq \beta_0} \sup_{t \in [0, \tau]} \{ |U \nabla_p f^\beta|_{C^n} + |u|_{C^n, \infty} \} < \infty.
\]
We note that \( M \) is finite due to the choice of \( \bar{\tau} \) and Lemma 6.2, because \( \| f^\beta \|_{C^\nu} \) is uniformly bounded and \( \mu < \nu \). This choice of \( M \) implies that
\[
\partial_t |h^\beta|_{C^{\nu-Mt}} \leq |Q^\beta|_{C^{\nu-Mt}} + C_0 C_1
\]
for \( 0 \leq t \leq \min \{ \tau, \mu/2M \} \). In order to show that the first term on the r.h.s. is also bounded for small \( t \), we define \( H_s := F_\lambda(p) + s \left( g_3(x, p, t) + h^\beta(x, p, t) \right) \) and
\[
\phi : s \mapsto \gamma n_{H_s}(1 - \eta m_{H_s})(F_{H_s} - H_s)/s.
\]
Then, we have
\[
Q^\beta = \phi(\beta^{-1}e^{-\frac{\nu}{\infty}}) - \lim_{s \to 0} \phi(s^{-1}e^{-\frac{\nu}{s}}) = \int_0^{\beta} \phi'(s^{-1}e^{-\frac{\nu}{s}}) e^{-\frac{\nu}{s}} (cv - s) \frac{ds}{s^3}.
\]
with
\[ \phi'(s) = \gamma(1 - 2\eta n_{H_s}) \partial_s n_{H_s} \frac{F_{H_s} - H_s}{s} 
- \gamma n_{H_s} (1 - \eta n_{H_s}) \frac{F_{H_s} - s\partial_s H_s, \partial_s H_s} {s^2} 
+ \gamma n_{H_s} (1 - \eta n_{H_s}) \frac{F_{H_s} - s\partial_s H_s} {s^2} \]

for all \( s \in (0, 1) \) and \( s \), \( \mu, \nu > 0 \). Thus, \( f \) is twice differentiable (see appendix) and \( H_s \) is linear in \( s \), one can prove that \( |\phi'(s)|_{C^{1,1}_s} \) is uniformly bounded for small \( s > 0 \). Thus,
\[ |Q^\beta|_{C^{1,1}_s} \leq \sup_{0 \leq s \leq \beta} |\phi'(s)|_{C^{1,1}_s} |\beta|^{-1} e^{-\frac{\beta}{\mu}} \leq C_2 \]
for \( 0 < \beta \leq \beta_0 \) some \( C_2 > 0 \) depending only \( \beta_0 \). Therefore,
\[ |h^2|_{C^{1,1}_s} \leq (C_2 + C_0 C_1) t \]
for \( t \leq \min\{\tau, \mu/2M\} \). Finally, we can use \( ||\cdot||_{L^\infty(R^d \times T^d)} \) for some \( C > 0 \) and all \( 0 \leq t \leq \mu/2M \) in order to finish the proof.

**Remark 5.8.** For every \( \delta > 0 \) there exists a constant \( C_\delta > 0 \) such that
\[ ||q_\delta(\cdot, t)||_{L^1(B_\delta(x,p))} \geq 2C_\delta e^{\omega(\beta)t} \]
for all \( \beta \leq \beta_0 \) and \( (x, p) \in R^d \times T^d \) and small \( t \). Thus, by Lemma 5.7, there exists a \( \tau_\delta > 0 \) such that
\[ ||f^\beta(\cdot, t) - F_\lambda||_{L^1(B_\delta(x,p))} \geq C_\delta e^{\omega(\beta)t - \frac{\beta}{\mu}} \]
for all \( t < \tau_\delta \), \( x \in R^d \), \( p \in T^d \) and \( \beta \leq \beta_0 \), where \( \omega \) is given by Proposition 5.6 satisfying
\[ \beta \omega(\beta) \geq \hat{c} \text{ for some } \hat{c} > 0 \text{ and all } \beta \leq \beta_0. \]

**Proof.** The first part is clear due to the definition of \( q_\delta \). The second assertion is then a consequence of Lemma 5.7, which guarantees for sufficiently small \( t \) that
\[ ||h^2(\cdot, 0)||_{L^1(B_\delta(x,p))} \leq C t \].

**Proof of Theorem 1.2.** Let \( \theta > 0 \), \( \delta > 0 \) and \( \nu < 0 \). If we combine (23) with (24), we see that there exists a constant \( C_{\delta, k, \nu} > 0 \) such that
\[ ||f^\beta(\cdot, t) - F_\lambda||_{L^1(B_\delta(x,p))} \geq ||f^\beta(\cdot, 0) - F_\lambda||_{L^1(B_\delta(x,p))} e^{\omega(\beta)t - \frac{\beta}{\mu}} \]
for all \( t < \tau_\delta, x \in R^d \), \( p \in T^d \) and \( \beta \leq \beta_0 \). We recall \( \omega \) from Proposition 5.6 and see that
\[ \inf_{|\beta| \leq \beta_0} \frac{e^{\omega(\beta)t - \frac{\beta}{\mu}}}{\beta (1 + |\beta|^{-k})^\theta > 0 \text{ if } t > \tau_{\min}(\nu) := \frac{c \nu}{\inf_{|\beta| < \beta_0} \beta \omega(\beta)} \]
assuming that \( \beta_0 \) is sufficiently small such that \( \beta \omega(\beta) \) is positive for all \( \beta \leq \beta_0 \). Since the parameter \( \nu > 0 \) was arbitrary, we may choose \( \tau_{\min}(\nu) < \delta/2 \). Therefore,
Lemma 6.2. Let The proof is similar to that of Lemma 2.2 and will be omitted.

and for all $x \in \mathbb{R}^d$, $p \in \mathbb{T}^d$ and $t \in (\delta, \tau)$. This implies the assertion of the theorem as $\beta \to 0$.

6. Space local method. In order to improve the existence results we have obtained so far, we need to make use of the fact that the collision operator of the semiconductor-Boltzmann-Dirac-Benny equation is local in space. Therefore, we are now focusing on a space local version of the method presented in sections 2 and 3. For this we replace the analytic norms $|\cdot|_{C^r}$ to space-local semi-norms, i.e. we define for every point $x$ in the physical space a semi-norm $\|f\|_{C^r_x}$ that only consists of all the derivatives of $f$ evaluated at the point $x$.

Definition 6.1. Let $\nu > 0$, $d \in \mathbb{N}$ and fix $x \in \mathbb{R}^d$. We consider the space-local semi-norms

$$\|f\|_{C^r_x} := \sum_{i,j \in \mathbb{N}_0^d} \sum_{a,b \in \mathbb{N}_0^d} \frac{\nu^{a+b}}{a!b!} \int_{\mathbb{R}^d} |\partial_x^a \partial_p \partial_t^j f(x,p)| \, dp$$

and

$$\|Df\|_{C^r_x} := \sum_{a,b \in \mathbb{N}_0^d} \|\partial_x^a \partial_p \partial_t^j f\|_{C^r_x}$$

as well as

$$\|f\|_{C^r_x} := \|f\|_{C^r_x} - \|f\|_{C^r_{-\infty}}$$

for $f : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ being analytic.

Let $\nu, T > 0$, $\mu \in [0, \nu/T)$. Using the semi-norms from above, we define

$$\|f\|_{\nu, \mu} := \sup_{x \in \mathbb{R}^d} \|f\|_{\nu, \mu, x}$$

and

$$\|f\|_{\nu, \mu} := \sup_{0 \leq t < T} \left( \|f(t)\|_{C^r_x} + \mu \int_0^T \|Df(s)\|_{C^r_x} \, ds \right)$$

for $f : \mathbb{R}^d \times \mathbb{T}^d \times [0,T) \to \mathbb{R}$ being analytic in $(x,p)$ and continuous in $t$ writing $f(t) = f(\cdot, t)$.

Note that we can prove the following version of Lemma 2.2 for these semi-norms. The proof is similar to that of Lemma 2.2 and will be omitted.

Lemma 6.2. Let $\mu_2 > \mu_1 > 0$ and $d \in \mathbb{N}$. Then there exists a constant $C = C_{\mu_1, \mu_2} > 0$ such that for all analytic $f : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ and all $x \in \mathbb{R}^d$, it holds

$$\|f\|_{C^r_x} \leq C_{\mu_1, \mu_2} |f|_{C^r_x}$$

for all $\nu \in [0, \mu_1]$, where

$$|f|_{C^r_x} := \sum_{0 \neq (a,b) \in \mathbb{N}_0^d} \frac{\nu^{a+b}}{a!b!} \int_{\mathbb{R}^d} |\partial_x^a \partial_p^b f(x,p)| \, dp.$$

With the same arguments as in the previous section, one can prove the following counterpart to Theorem 3.1.
Theorem 6.3. Let $C, R, \nu > 0$ and $f_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ be analytic such that
\[
\sup_{x \in \mathbb{R}^d} \|f_0\|_{C^\nu_x} < R.
\] (25)
Then if $\mu > 0$ is sufficiently large, $T \in (0, \nu/\mu)$ and $F : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R}$ is analytic such that
\[
\|F(t)\|_{C^\nu-x} \leq C
\] (26)
for all $0 \leq t \leq T$ and $x \in \mathbb{R}^d$, then the equation
\[
\partial_t f + u \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f (1 - \eta n_f)(F - f)
\] (27)
admits a unique analytic solution $f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R}$ with $\|f\|_{\nu, \mu, x} \leq R$ and $f(x, p, 0) = f_0(x, p)$ for all $x \in \mathbb{R}^d$ and $p \in \mathbb{T}^d$.
Moreover, let $\Psi : (f_0, F) \mapsto f$ be defined by the unique solution of (27) with $f(x, p, 0) = f_0(x, p)$. If $\mu > 0$ is sufficiently large, the mapping $\Psi$ is Lipschitz continuous, i.e., for all $x \in \mathbb{R}^d$
\[
\|\Psi(f_0, F) - \Psi(g_0, G)\|_{\nu, \mu, x} \leq 2 \|f_0, F) - (g_0, G)\|_x,
\]
where
\[
\|(f_0, F)\|_x := \|f_0\|_{C^\nu_x} + \mu^{-\frac{1}{2}} \sup_{0 \leq t < T} \|F\|_{C^{\nu-x}t}.
\]
for $f_0, g_0$ and $F, G$ satisfying (25) and (26), respectively.

Similarly as in estimate (17) in the proof of Theorem 3.1, we can improve the Lipschitz estimate.

Lemma 6.4. Let $f := \Psi(f_0, F)$ and $g := \Psi(g_0, G)$. We have
\[
\|f - g\|_{\nu, \mu, x} \leq 2 \|f_0 - g_0\|_{C^\nu_x} + \frac{4\gamma\nu\mu}{\mu} \sup_{0 \leq t \leq T} \|n_f(1 - \eta n_f)\|_{C^\nu_x} \|F - G\|_{C^{\nu-x}t}.
\] (28)
if $\mu > 0$ is sufficiently large.

7. BGK-type collision operator - space local method. In this section, we consider again equation
\[
\partial_t f + u(p) \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f (1 - \eta n_f)(F_f - f)
\] (6)
with $f(x, p, 0) = f_0(x, p)$ for given $U \neq 0$ and $\gamma \geq 0$. As before, we use the self-consistent equilibrium distribution function
\[
F_f(x, p, t) = \left(\eta + \exp(-\lambda_0(x, t) - \lambda_1(x, t)e(p))\right)^{-1}
\]
for $x \in \mathbb{R}^d, p \in \mathbb{T}^d$ and $t > 0$, where $\lambda_0, \lambda_1$ satisfy
\[
n_f(x, t) := n_{F_f}(x, t) \quad \text{and} \quad E_f(x, t) = E_{F_f}(x, t),
\] (29)
for $n_f(x, t) := \int_{\mathbb{T}^d} f(x, p, t)dp$ and $E_f(x, t) = \int_{\mathbb{T}^d} e(p)f(x, p, t)dp$.

The main goal is to improve the existence result from Theorem 4.1 using the space local semi-norms. Similar as before, the key ingredient will Theorem 6.3 and the Lipschitz estimate (28).

Definition 7.1. Let $a \geq 1$ and $\delta > 0$. We define
\[
M_a := \left\{ \int_{\mathbb{T}^d} \frac{(1, e(p))dp}{\eta + e^{-\lambda_0 - \lambda_1 e(p)}} : \lambda_0, \lambda_1 \in \mathbb{R} \right\} \subset \mathbb{R}^2
\]
Let \( \mathcal{U}_{a,\delta} := \bigcup_{(m_0,m_1) \in M_a} B_{5m_0(1-\eta m_0)}(m_0, m_1) \supset M_a \), where \( B_y(r) \) denotes the ball in \( \mathbb{R}^2 \) centered at \( y \) with radius \( \theta \).

**Proposition 7.2.** Let \( \eta, \nu_0, R > 0, \gamma \geq 0, a \geq 1 \). Then there exist \( \alpha, \beta, \mu > 0 \) such that the following holds:

Let \( f_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R} \) is analytic such that \( \| f_0 \|_{C^r_\gamma} \leq R/2 \) for some \( \nu \in (0, \nu_0) \) and

\[
(n_{f_0}(x)) := \int_{\mathbb{T}^d} \left( \frac{1}{\epsilon(p)} \right) f_0(x, p) dp \in \mathcal{U}_{a,\alpha/2}.
\]

is well-defined for all \( x \in \mathbb{R}^d \). Moreover, suppose that

\[
\| f_0 \|_{C^r_\gamma} \leq \beta n_{f_0}(x)(1 - \eta n_{f_0}(x)) \quad \text{for} \quad x \in \mathbb{R}^d.
\]

Then equation (6) with \( f \big|_{t=0} = f_0 \) admits an analytic solution \( f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to \mathbb{R} \) with \( \| f \|_{\nu, \mu} \leq R \) for \( T < \nu/\mu \).

The theorem will also be proved using the Banach fixed-point theorem. In order to define the right metric space, we require some properties of the equilibrium distribution.

**Proposition 7.3.** Let \( \eta > 0, a \geq 1 \) and \( R, \nu > 0 \). Then there exist \( \alpha > 0 \) such that for all \( f, g : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R} \) being analytic with \( \| f \|_{C^r_\gamma}, \| g \|_{C^r_\gamma} \leq R \) and

\[
\| n_h \|_{C^r_\gamma} + \| E_h \|_{C^r_\gamma} \leq \alpha n_h(x)(1 - \eta n_h(x)) \quad \text{for} \quad h \in \{ f, g \}
\]

and \( (n_h(x), E_h(x)) \in \mathcal{U}_{a,\alpha} \) for \( h \in \{ f, g \} \), it holds

\[
\| \mathcal{F}_f \|_{C^r_\gamma}, \| \mathcal{F}_g \|_{C^r_\gamma} \leq C
\]

and

\[
\| \mathcal{F}_f - \mathcal{F}_g \|_{C^r_\gamma} \leq C \| f - g \|_{C^r_\gamma},
\]

for some \( C > 0 \) and all \( x \in \mathbb{R}^d \).

**Proof.** See appendix. \( \Box \)

**Remark 7.4.** According to the proof in the appendix, the parameter \( \alpha \) only depends on \( a \). More precisely, it can be written as \( \alpha = 1/(2B_a) \) for \( B_a \) from Lemma B.3.

**Definition 7.5.** For \( R, \nu, \eta > 0, a \geq 1 \) let \( \alpha > 0 \) be as in Proposition 7.3. Moreover, let \( \mu > 0 \) and \( T \in (0, \nu/\mu) \). We assume that

\[
\| f_0 \|_{C^r_\gamma} \leq \beta n_{f_0}(x)(1 - \eta n_{f_0}(x))
\]

Let \( Z \) space of all analytic functions \( f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T) \to [0, \eta^{-1}] \) satisfying

1. \( \| f \|_{\nu, \mu} \leq R \),
2. \( \| n_f(\cdot, t) \|_{C^{r-\mu} \gamma} + \| E_f(\cdot, t) \|_{C^{r-\mu} \gamma} \leq \alpha n_f(x, t)(1 - \eta n_f(x, s)) \)
3. \( (n_f(x, t), E_f(x, t)) \in \mathcal{U}_{a,\alpha} \)

for all \( x \in \mathbb{R}^d \) and \( t \in [0, T) \). Thus, \( Z \) is a complete if the metric is induced by the norm \( \| \cdot \|_{\nu, \mu} \).
Let \( f_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R} \) be analytic such that \( \| f_0 \|_{C^\infty} \leq R/2 \) and
\[
(n_{f_0}(x), E_{f_0}(x)) := \int_{\mathbb{T}^d} (1, (\epsilon(p))) f_0(x, p) dp \in U_{a, \alpha/2}
\]
is well-defined for all \( x \in \mathbb{R}^d \). Moreover, suppose that
\[
\| f_0 \|_{C^\infty} \leq \beta n_{f_0}(x)(1 - \eta n_{f_0}(x)) \quad \text{for } x \in \mathbb{R}^d
\]
for some \( \beta > 0 \). For sufficiently large \( \mu > 0 \), we define the mapping
\[
\Theta : Z \ni g \mapsto f,
\]
where \( f \) is the solution of
\[
\partial_t f + u(p) \cdot \nabla_x f - U \nabla_x n_f \cdot \nabla_p f = \gamma n_f(1 - \eta n_f)(F_g - f)
\]
with \( f|_{t=0} = f_0 \). This is well-defined for large \( \mu > 0 \) according to Theorem 6.3 and Proposition 7.3. As we plan to apply the Banach fixed-point theorem, we need to show that \( \Theta \) is a contraction, i.e., the image of \( \Theta \) is included in \( Z \) and \( \Theta \) is Lipschitz continuous with Lipschitz constant \( L < 1 \). We start with the Lipschitz estimate, which is in this case the easier assertion.

**Lemma 7.6.** Let \( \mu > 0 \) be sufficiently large. Then for \( f, g \in Z \) it holds
\[
\| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq \frac{1}{2} \| f - g \|_{\nu, \mu}.
\]

**Proof.** Using \( \Psi \) from Theorem 6.3, we can rewrite \( \Theta \) as
\[
\Theta(f) = \Psi(f_0, F_f).
\]
For \( f \in Z \), we know from Proposition 7.3 that \( \| F_f \|_{C^{\nu, \mu}} \leq C \) for some \( C > 0 \) and all \( x \in \mathbb{R}^d \) and \( t \in [0, T) \). Thus, Theorem 6.3 entails that for sufficiently large \( \mu > 0 \),
\[
\| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq 2\mu^{-\frac{1}{2}} \sup_{t, x} \| F_f - F_g \|_{C^{\nu, \mu}}.
\]
Then the second statement of Proposition 7.3 yields that
\[
\| \Theta(f) - \Theta(g) \|_{\nu, \mu} \leq C \mu^{-\frac{1}{2}} \| f - g \|_{C^{\nu, \mu}} \leq C \mu^{-\frac{1}{2}} \| f - g \|_{\nu, \mu}
\]
for some \( C > 0 \). This implies the assertion for sufficiently large \( \mu \) satisfying \( \mu \geq 4C^2 \).

**Lemma 7.7.** Let \( \mu > 0 \) be sufficiently large, \( (1 + \nu^2)\beta > 0 \) sufficiently small and \( g \in Z \). Then \( \Theta(g) \in Z \).

**Proof.** Let \( g \in Z \) and define \( f := \Theta(g) \).

**Claim 1:** \( \| f \|_{\nu, \mu} \leq R \) if \( \mu \) is sufficiently large.

This is a direct consequence of Theorem 6.3 combined with Proposition 7.3.

**Claim 2:** We have
\[
\| n_f(\cdot, t) \|_{C^{\nu, \mu}} + \| E_f(\cdot, t) \|_{C^{\nu, \mu}} \leq \alpha n_f(x, t)(1 - \eta n_f(x, t)) \quad \text{for } x \in \mathbb{R}^d, t \in [0, T).
\]
Fix \( x \in \mathbb{R}^d \) and define
\[
h_0 : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}, \ (y, p) \mapsto f_0(x, p) + \partial_x f_0(x, p)y
\]
as well as
\[
h(y, p, t) = h_1(p, t) + h_2(p, t)y.
\]
where \( h_1, h_2 \) solve
\[
\partial_t h_1 + h_2 u(p) = U \nabla_x n_{f_0}(x) \cdot \nabla_p h_1 \quad \text{and} \quad \partial_t h_2 = U \nabla_x n_{f_0}(x) \cdot \nabla_p h_2.
\]
with \( h_1(p, 0) = f_0(x, p) \) and \( h_2(p, 0) = \partial_x f(x, p) \). Then it holds
\[
\partial_t h + u \cdot \nabla_y h = \partial_t h_1 + \partial_t h_2 y + h_2 u = U \nabla_x n_{f_0}(x) \cdot \nabla_p (h_1 + h_2 y) = U \nabla_x n_{h} \cdot \nabla_p h.
\]

Note that the equations for \( h_1 \) and \( h_2 \) are linear transport equation. We thus can solve them explicitly, e.g.
\[
h_2(p, t) = \partial_x f_0(x, p - t U \nabla_x n_{f_0}(x)).
\]
With this, we can easily compute the density \( n_{h_1} = \int_{\mathbb{R}_d} h_1(p, \cdot) dp \) by
\[
n_{h_1}(t) = n_{f_0}(x) - \int_0^t \int_{\mathbb{T}^d} \partial_x f_0(x, p - s U \nabla_x n_{f_0}(x)) u(p) dp ds
\]
and estimate
\[
|n_{h_1}(t) - n_{f_0}(x)| \leq t \|u\|_{L^\infty(\mathbb{T}^d)} \|\partial_x f^0(x, p)\|_{L^1(\mathbb{T}^d)}.
\]

Next, we infer from the Lipschitz estimate (28) that
\[
\|f - h\|_{\nu, \mu, x} \leq 2 \|f_0 - h_0\|_{C^\nu_x} + \frac{4\gamma \nu}{\mu} \sup_{0 \leq t \leq T} \|n_h(1 - \eta n_h)\|_{C^\nu_x} \|F_\mu\|_{C^{\nu - \mu}_x}.
\]
for sufficiently large \( \mu > 0 \). At first, we note that \( \|F_\mu\|_{C^{\nu - \mu}_x} \) and \( \|h\|_{C^{\nu - \mu}_x} \) are uniformly bounded. Then, we see by the definition of \( h \) that we can estimate the r.h.s. using that \( \|f_0 - h_0\|_{C^\nu_x} \leq \|f_0\|_{C^\nu_x} \) and obtain
\[
\|f - h\|_{\nu, \mu, x} \leq 2 \|f_0\|_{C^\nu_x} + \frac{C \nu}{\mu} \sup_{0 \leq t \leq T} |n_{h_1}(1 - \eta n_{h_1}(t))|.
\]
for some \( C > 0 \) independent from \( \nu \). Moreover, it holds
\[
\sup_{0 \leq t \leq T} |n_{h_1}(1 - \eta n_{h_1})| \leq |n_{f_0}(x)(1 - \eta n_{f_0}(x))| + C T \|\partial_x f^0(x, p)\|_{L^1(\mathbb{T}^d)}
\]
\[
\leq |n_{f_0}(x)(1 - \eta n_{f_0}(x))| + \frac{C \nu}{\mu} \|f^0\|_{C^\nu_x}
\]
because \( T < \nu/\mu \). Thus, there exists a constant \( C > 0 \) independent from \( \nu \) such that for all \( t \leq \tau_0 \), we have
\[
\|f - h\|_{\nu, \mu, x} \leq \left(2 + \frac{C \nu^2}{\mu^2}\right) \|f_0\|_{C^\nu_x} + \frac{C \nu}{\mu} |n_{f_0}(x)(1 - \eta n_{f_0}(x))|.
\]

Note that \( h \) is affine in \( y \), hence \( \partial_y^a h = 0 \) for \( |i| \geq 2 \) and
\[
\sum_{|i|, |j| = 0, a, b \in \mathbb{N}_0^d \atop |i + a| \geq 2} \frac{(\nu - \mu s)^a + b}{a! b!} \|\partial_x^{i+a} \partial_p^j f(x, p, t)\|_{L^1(\mathbb{T}^d)} \leq \|f - h\|_{\nu, \mu, x}
\]
for \( 0 \leq t < T \). In particular,
\[
\|n_f\|_{C^{\nu - \mu}_x} + \|E_f\|_{C^{\nu - \mu}_x} \leq (1 + \|\epsilon\|_{L^\infty(\mathbb{T}^d)}) \|f - h\|_{\nu, \mu, x} + \nu (|\partial_x n_f(x, t)| + |\partial_x E_f(x, t)|).
\]
Moreover, we can estimate the latter two terms by
\[ |\partial_x n_f(x,s)| \leq \|\partial_y (n_f(y,s) - n_h(y,s)) + \partial_y n_h(y,s)\|_{y=x} \]
\[ \leq \|f - h\|_{t,x} + |n_h(s)| = \|f - h\|_{t,x} + |\partial_x f_0(x,p)|_{L^1_p(T^d)} \]
and likewise,
\[ |\partial_x E_f(x,s)| \leq \|\partial_x f_0(x,p)|_{L^1_p(T^d)} \]
Since \(\nu|\partial_x f_0(x,p)|_{L^1_p(T^d)} \leq \|f_0\|_{C^2_x}\), there exist a constant \(C > 0\) such that for sufficiently large \(\mu > 0\) it holds
\[ \|n_f\|_{C^{\alpha}} + \|E_f\|_{C^{\alpha}} \leq C(1 + \nu^2)\|f_0\|_{C^2_x} + \frac{C}{\mu} n_f_0(x)(1 - \eta f_0(x)) \tag{32} \]
for all \(0 \leq t < T\). By the hypothesis, we have
\[ |\partial_t (n_f(1 - \eta m_f))| \leq \|v\|_{L^\infty(T^d)} \int_{T^d} |\nabla f| dp \]
\[ \leq \|u\|_{L^\infty} \left( \|f - h\|_{t,x} + |\partial_x f_0(x,p)|_{L^1_p(T^d)} \right) \]
which entails
\[ n_f(x,t)(1 - \eta m_f(x,t)) \geq (1 - Ct)n_f_0(x)(1 - \eta m_f(x)) \tag{33} \]
for some \(C > 0\) and all \(t \leq T < \nu/\mu\) if \(\mu\) is sufficiently large. Thus, we even have
\[ n_f(x,t)(1 - \eta m_f(x,t)) \geq \frac{1}{2} n_f_0(x)(1 - \eta m_f(x)) \]
if \(\mu > 0\) is sufficiently large implying
\[ \|n_f\|_{C^{\alpha}} + \|E_f\|_{C^{\alpha}} \leq \delta n_f(x,s)(1 - \eta m_f(x,s)). \]
This proves the claim. Finally, there is only one assertion left:

**Claim 3:** \((n_f(x,t), E_f(x,t)) \in \overline{U_{a,\alpha}}\) for all \(x \in \mathbb{R}^d\) and \(0 \leq t < T\) if \(\mu > 0\) is sufficiently large.

Recall that
\[ U_{a,\alpha} = \bigcup_{(m_0,m_1) \in M_a} B_{\delta m_0(1 - \eta m_0)}(m_0,m_1). \]
Similar to (33), we obtain that
\[ |n_f(x,t) - n_0(x)| + |E_f(x,t) - E_0(x)| \leq C t n_f_0(x)(1 - \eta m_f(x)) \]
for some \(C > 0\) independent from \(x\) and small \(t > 0\). Since by assumption \((n_f_0(x), E_f_0(x)) \in U_{a,\alpha/2}\), there exits \((m_0,m_1) \in M_a\) such that
\[ |(n_f_0(x), E_f_0(x)) - (m_0,m_1)| < \frac{\alpha}{2} m_0(1 - \eta m_0). \]
Thus, we compute that
\[ |(n_f(x, t), E_f(x, t)) - (m_0, m_1)| \leq |(n_{f_0}(x), E_{f_0}(x)) - y| + C t n_{f_0}(x)(1 - \eta m_{f_0}(x)) \]
\[ \leq \frac{a}{2} m_0(1 - \eta m_0) + C t n_{f_0}(x)(1 - \eta m_{f_0}(x)). \]
Hence, for sufficiently large \( \mu > 0 \), it holds
\[ |(n_f(x, t), E_f(x, t)) - (m_0, m_1)| < \delta m_0(1 - \eta m_0). \]
for all \( 0 \leq t < T < \nu / \mu \).

**Proof of Theorem 1.7.** The idea is to adjust the parameter such that the hypothesis \( \theta > 0 \) such that \( \theta < f_0 < \eta^{-1}(1 - \theta) \). Likewise to the proof of Theorem 4.6, we can use the analyticity of \( f_0 \) to show that \( R := 2 \sup_{x \in \mathbb{T}^d} \| f_0 \|_{C^{\nu_0}} + 1 < \infty \) for sufficiently small \( \nu_0 > 0 \). Moreover, it is easy to check that
\[ \| f_0 \|_{C^{\nu}} \leq \frac{R}{2} \quad \text{and} \quad \| f_0 \|_{C_2^\nu} \leq \frac{R \nu}{2 \nu_0} \]
holds for all \( 0 < \nu \leq \nu_0 \) and \( x \in \mathbb{T}^d \). Using the bounds for \( f_0 \), we see that
\[ n_{f_0}(1 - \eta m_{f_0}) \geq \theta^2 \]
and thus given \( \beta \), we have
\[ \| f_0 \|_{C^{\nu}} \leq \frac{R}{2} \quad \text{and} \quad \| f_0 \|_{C_2^{\nu}} \leq \beta n_{f_0}(x)(1 - \eta m_{f_0}(x)) \]
for all \( x \in \mathbb{T}^d \) if \( \nu \leq \min\{\nu_0, \beta \nu_0/(K \theta^2)\} \).

The next step is to show the hypothesis on the macroscopic densities of \( f_0 \). We claim that
\[ (n_{f_0}(x), E_{f_0}(x)) := \int_{\mathbb{T}^d} (1, (\epsilon(p))) f_0(x, p) dp \in \mathcal{U}_{a, \alpha/2} \]
for some \( a \geq 1 \) and given \( \alpha > 0 \). According to [9] section 5.1 and \( \theta < f_0 < \eta^{-1}(1 - \theta) \), there exists \( \lambda^0 = (\lambda_0^0, \lambda_1^0) : \mathbb{T}^d \rightarrow \mathbb{R}^2 \) analytic and bounded such that
\[ n(\lambda^0) = n_{f_0} \quad \text{and} \quad E(\lambda^0) = E_{f_0}. \]
Thus,
\[ (n_{f_0}(x), E_{f_0}(x)) \in \left\{ \int_{\mathbb{T}^d} \frac{(1, \epsilon(p)) dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} : \lambda_0, \lambda_1 \in \mathbb{R} \text{ with } |\lambda_1| \leq \log a \right\} \subset \mathcal{U}_{a, \alpha} \]
for \( a := \exp(\| \lambda^0 \|_{L^\infty}) \) and all \( \alpha > 0 \). Finally, we can apply Proposition 7.2 and obtain the assertion.

**Proof of Theorem 1.7.** The idea is to adjust the parameter such that the hypothesis of Proposition 7.2 are fulfilled. At first, we see by Lemma B.7 and Proposition 6.2 that there exists an \( R > 0 \) and a \( \nu_0 > 0 \) such that \( \sup_{x \in \mathbb{R}^d} \| f_0 \|_{C^{\nu_0}} \leq R / 2 \) and
\[ \| f_0 \|_{C^{\nu_0}} \leq C \nu_1 n_{f_0}(x)(1 - \eta m_{f_0}(x)) \]
holds for all \( x \in \mathbb{R}^d \) and all \( \nu_1 < \nu_0 \). Now, we set \( \alpha_0 = \alpha / 2 \) and \( \nu_1 := \beta / C \), where \( \alpha \) and \( \beta \) are given by Proposition 7.2. Then Proposition 7.2 guarantees a
unique analytic solution \( f \) on a short time interval. The well-posedness is then a direct consequence of Theorem 6.3. Finally, using Lemma 6.2, we obtain the well-posedness also in the desired norm with a larger constant.

\[ \square \]

Finally, we note that Theorem 1.5 is actually a corollary of Theorem 1.7.

Appendix A. Proof of Proposition 4.2.

Definition A.1. Let \( \lambda_0 = \lambda_0(n, E) \) and \( \lambda_1 = \lambda_1(n, E) \) be functions of the densities \( n, E \) given by

\[
\left( \begin{array}{c} n \\ E \end{array} \right) = \int_{\mathbb{T}^d} \left( \begin{array}{c} 1 \\ \eta + e^{-\lambda_0(n, E) - \lambda_1(n, E)\epsilon(p)} \end{array} \right) dp.
\]

We define

\[
\mathcal{F}^0(n, E; p) := \frac{1}{\eta + e^{-\lambda_0(n, E) - \lambda_1(n, E)\epsilon(p)}}
\]

for \( (n, E) \in \{ \int_{\mathbb{T}^d}(1, \epsilon(p))g(p)dp : g \in L^1(\mathbb{T}^d; (0, \eta^{-1})) \} \) and \( p \in \mathbb{T}^d \).

Our goal is to estimate the norm of \( \mathcal{F}_f \) be means of \( f \). Due to the preceding Definition, we can rewrite \( \mathcal{F}_f \) as a composition by

\[
\mathcal{F}_f(x, p) = \mathcal{F}^0(n_f(x), E_f(x); p),
\]

where \( n_f(x) := \int_{\mathbb{T}^d} f(x, p)dp \) and \( E_f(x) = \int_{\mathbb{T}^d} \epsilon(p)f(x, p)dp \).

Thus, we can easily compute the first derivative of \( \mathcal{F}_f \) w.r.t. \( x \) as \( \partial_x \mathcal{F}_f = \partial_x \mathcal{F}^0(n_f, E_f) \partial_x n_f + \partial_E \mathcal{F}^0(n_f, E_f) \partial_x E_f \) by using the chain rule and the following Lemma.

Lemma A.2.

\[
\frac{\partial \mathcal{F}^0}{\partial(n, E)}(n, E; p) = \frac{\mathcal{F}^0(n, E; p)(1 - \eta \mathcal{F}^0(n, E; p))}{\int_{\mathbb{T}^d} \epsilon^2d\mu \int_{\mathbb{T}^d} 1d\mu - \left( \int_{\mathbb{T}^d} \epsilon d\mu \right)^2} \int_{\mathbb{T}^d} \left( \begin{array}{c} -\epsilon(p') \\ 1 \end{array} \right)(\epsilon(p) - \epsilon(p'))d\mu_p,
\]

where \( d\mu_p := \mathcal{F}^0(n, E; p)(1 - \eta \mathcal{F}^0(n, E; p))dp \).

Proof. For \( \lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2 \), let us denote \( \mathcal{F}(\lambda; p) := 1/(\eta + e^{-\lambda_0 - \lambda_1\epsilon(p)}) \) and

\[
n(\lambda) := \int_{\mathbb{T}^d} \frac{dp}{\eta + e^{-\lambda_0 - \lambda_1\epsilon(p)}}, \quad E(\lambda) := \int_{\mathbb{T}^d} \frac{\epsilon(p)dp}{\eta + e^{-\lambda_0 - \lambda_1\epsilon(p)}}.
\]

We have

\[
\frac{\partial (n, E)}{\partial \lambda}(\lambda) = \int_{\mathbb{T}^d} \left( \begin{array}{cc} 1 & \epsilon(p) \\ \epsilon(p) & \epsilon(p)^2 \end{array} \right) d\mu_p,
\]

where \( d\mu_p := \mathcal{F}(\lambda; p)(1 - \eta \mathcal{F}(\lambda; p))dp \). Since \( d\mu_p \) is a positive measure, we can use the Cauchy-Schwarz inequality to see that \( \frac{\partial (n, E)}{\partial \lambda}(\lambda) \) is invertible. Finally, we easily compute

\[
\frac{\partial \lambda}{\partial (n, E)} = \frac{1}{\int_{\mathbb{T}^d} \epsilon^2d\mu \int_{\mathbb{T}^d} 1d\mu - \left( \int_{\mathbb{T}^d} \epsilon d\mu \right)^2} \int_{\mathbb{T}^d} \left( \begin{array}{c} \epsilon^2 \\ -\epsilon \end{array} \right) d\mu
\]

by the inverse function theorem and the chain rule ensures the assertion.

\[ \square \]

Note that our main techniques are based on the analytic norms

\[
\|f\|_{C^\nu} := \sum_{a,b \in \mathbb{N}_0} \frac{\nu^{a+b}}{a!b!} \|\partial_x^a \partial_{\epsilon}^b f\|_{W^{1,\infty}_x W^{1,1}_\epsilon}.
\]
for \( f : \mathbb{R}^d \times T^d \to \mathbb{R}^k \) being analytic, where we use the notation
\[
\|f\|_{W^{a,b}_p} := \sum_{a,b \in \mathbb{N}_0^d \mid a + b \leq 1} \|\partial^a_p \partial^b f\|_{L^1_p}, \quad \text{and} \quad \|f\|_{L^1_p} := \sup_{x \in \mathbb{R}^d} \int_{T^d} |f(x, p)| dp.
\]

This motivates Proposition 4.2, which we restate for the reader’s convenience.

**Proposition A.3.** Let \( \eta, \nu, R > 0 \) and \( \alpha > 0 \). There exists an \( C, \delta > 0 \) such that the following is true.

Let \( f, g : \mathbb{R}^d \times T^d \to [\alpha, \eta^{-1} - \alpha] \) be analytic satisfying \( \|f\|_{C^\nu}, \|g\|_{C^\nu} \leq R \) and
\[
\|n_h - \bar{n}\|_{C^\nu} + \|E_h - \bar{E}\|_{C^\nu} \leq \delta \quad \text{for } h \in \{f, g\}
\]
and some \( \bar{n}, \bar{E} \in \mathbb{R} \), it holds
\[
\|F_f\|_{C^\nu}, \|F_g\|_{C^\nu} \leq C
\]
and
\[
\|F_f - F_g\|_{C^\nu} \leq C \|f - g\|_{C^\nu}.
\]

The main steps to prove this proposition is again to consider \( F_f \) as the composition
\[
F^0(n, E; p) := \frac{1}{\eta + e^{-\lambda_0(n, E) - \lambda_1(n, E)p}}
\]
for \( (n, E) \in \{\int_{T^d}(1, \epsilon(p)) g(p) dp : g \in L^1(T^d; (0, \eta^{-1}))\} \) and \( p \in T^d \).

In the analytic norm \( \|\|_{C^\nu} \) involves all derivatives. As a first step we consider the derivatives of \( F^0 \). Using the inverse mapping theorem, we can easily see that \( \lambda_0, \lambda_1 \) are analytic in their domain \( \{\int_{T^d}(1, \epsilon(p)) g(p) dp : g \in L^1(T^d; (0, \eta^{-1}))\} \). This proves the following.

**Lemma A.4.** \( F^0 \) is analytic on \( \{\int_{T^d}(1, \epsilon(p)) g(p) dp : g \in L^1(T^d; (0, \eta^{-1}))\} \times T^d \). In particular, for all
\[
(\bar{n}, \bar{E}) \in \left\{\int_{T^d}(1, \epsilon(p)) g(p) dp : g \in L^1(T^d; (0, \eta^{-1}))\right\}
\]
there exist constants \( A \geq 0 \) such that
\[
\left| \partial^i_{(n, E)} \partial^j_p F^0(\bar{n}, \bar{E}; p) \right| \leq i! j! A^{|i|+|j|} \quad (34)
\]
for all \( p \in T^d \) for \( i \in \mathbb{N}_0^d \) and \( j \in \mathbb{N}_0^d \).

**Corollary A.5.** Let \( (\bar{n}, \bar{E}) \in M := \{\int_{T^d}(1, \epsilon(p)) g(p) dp : g \in L^1(T^d; (0, \eta^{-1}))\} \).
Then there exist a \( \delta > 0 \) and an open neighborhood \( \mathcal{U} \subset M \) of \( (\bar{n}, \bar{E}) \) such that
\[
\|F^0\|_{C^\nu(\mathcal{U})} := \sum_{|i| + |j| \leq 1} \sum_{a \in \mathbb{N}_0^d, b \in \mathbb{N}_0^d} \frac{\delta^{a|i|+|j|}}{a!b!} \sup_{(n, E) \in \mathcal{U}} \int_{T^d} |\partial^{a+i}_{(n, E)} \partial^b_p F^0(n, E; p)| dp
\]
is finite.

**Proof.** According to the previous lemma, there exists an \( A > 0 \) such that the estimate (34) is satisfied. Using the Taylor formula for \( F^0 \) w.r.t. \( (n, E) \) makes sure that
\[
\left| \partial^i_{(n, E)} \partial^j_p F^0(n, E; p) \right| \leq i! j! (2A)^{|i|+|j|} \quad (35)
\]
holds true in a neighborhood \( \mathcal{U} \subset M \) of \( (\bar{n}, \bar{E}) \). Thus, summing up all derivatives with the right weight, we can show that \( \|F^0\|_{C^\nu(\mathcal{U})} \) for \( \delta < 1/(2A) \). \( \square \)
The last ingredient for the proof of Proposition A.3 is a formula for the analytic norms of composition of functions which is in fact a corollary of the Faà di Bruno formula. It was firstly derived by [18]. Note that Mouhot and Villani [18] also state a version for $d > 1$. However, in their proof, they use only the one dimensional Faà di Bruno formula such that they leave the multidimensional case to the reader. For $d \geq 1$, we also refer to [9] Lemma 4.2.5, where the definition of the norm $| \cdot |_{C^\nu}$ slightly differs from our case and involves full derivatives. The same techniques can still be used for this case.

**Lemma A.6.** Let $x \in V \subset \mathbb{R}^k$ open and let $g : \mathbb{R}^d \times T^d \to V$, $\phi : V \to \mathbb{R}$ be analytic. Then

$$| \phi \circ g |_{C^\nu} \leq | \phi |_{C^\nu}$$ for $\mu = |g - v|_{C^\nu}$

for $\nu > 0$ and $v \in V$, where

$$|g|_{C^\nu} := \sum_{a,b \in \mathbb{N}_0^d} \frac{\mu^{a+b}}{a!b!} \| \partial_x^a \partial_p^b g \|_{L^\infty L^p}$$

for $f : \mathbb{R}^d \times T^d \to \mathbb{R}^k$ being analytic.

**Corollary A.7.** Given $\bar{n}, \bar{E} \in \mathbb{R}$ and $\nu > 0$. Let $\delta > 0$ and $\mathcal{U}$ be as in Corollary A.5. Then there exists a $C > 0$ such that for all $(n, E) : \mathbb{R}^d \to \mathbb{R}^2$ being analytic such that

$$\|n - \bar{n}\|_{C^\nu} + \|E - \bar{E}\|_{C^\nu} \leq \delta,$$

it holds

$$\| \mathcal{F}^0(n, E) \|_{C^\nu} \leq C.$$

**Proof.** Using the analytic norms from Lemma A.6, we can write

$$\| \mathcal{F}^0(n, E) \|_{C^\nu} = \| \mathcal{F}^0(n, E) \|_{C^\nu} + \sum_{i=1}^d | \partial_x \mathcal{F}^0(n, E) |_{C^\nu} + \sum_{i=1}^d | \partial_p \mathcal{F}^0(n, E) |_{C^\nu}.$$

By Lemma A.6, we obtain

$$| \mathcal{F}^0(n, E) |_{C^\nu} \leq | \mathcal{F}^0(n, E) |_{C_{\delta}(\mathcal{U})},$$

because $|n - \bar{n}|_{C^\nu} + |E - \bar{E}|_{C^\nu} \leq \delta$.

By assumption $| \mathcal{F}^0(n, E) |_{C_{\delta}(\mathcal{U})} < \infty$ and thus, $| \mathcal{F}^0(n, E) |_{C^\nu}$ is bounded. We can do the same trick for the other terms. Her we only need to use the chain rule and the submultiplicativity of $| \cdot |_{C^\nu}$ to split the terms into

$$| \partial_x \mathcal{F}^0(n, E) |_{C^\nu} = | \partial_n(n, E) \mathcal{F}^0(n, E) \partial_x(n, E) |_{C^\nu} \leq | \partial_n(n, E) \mathcal{F}^0(n, E) |_{C^\nu} | \partial_x(n, E) |_{C^\nu} \leq | \partial_n(n, E) \mathcal{F}^0(n, E) |_{C^\nu} \| (n, E) \|_{C^\nu}$$

with slightly abuse of notation. Note that a version of Corollary A.5 for $\partial(n, E) \mathcal{F}^0(n, E)$ holds true. This can be shown in the same manner as for Corollary A.5. \qed

Without loss of generality, we can assume that $\mathcal{U}$ is convex. We can apply the same arguments for $\partial(n, E) \mathcal{F}^0(n, E)$ and obtain by

$$\mathcal{F}^0(n_1, E_1) - \mathcal{F}^0(n_0, E_0) = \left( \frac{n_1 - n_0}{E_1 - E_0} \right) \int_0^1 \partial(n, E) \mathcal{F}^0(n_1 t + (1-t)n_0, E_1 t + (1-t)E_0) dt.$$

This leads to the following statement.
Proposition 7.3, which we restate for the sake of convenience. Let $B$. 

Given Corollary A.8. Let $\mathcal{U} \subset U$ be convex. Then there exists a $C > 0$ such that for all $(n_i, E_i) : \mathbb{R}^d \to \mathcal{U}$, $i = 0, 1$, being analytic such that

$$\|n_i - \bar{n}\|_{C^0} + \|E_i - \bar{E}\|_{C^0} \leq \delta, \quad i = 0, 1,$$

it holds

$$\|\mathcal{F}^0(n_1, E_1) - \mathcal{F}^0(n_0, E_0)\|_{C^0} \leq C (\|n_1 - n_0\|_{C^0} + \|E_1 - E_0\|_{C^0}).$$



Proof of Proposition A.3. The assertion is basically a direct consequence of the foregoing corollaries. The only difference is that we do not want to assume explicitly that $(n, E)(\mathbb{R}^d) \subset \mathcal{U}$. We can neglect this hypothesis by choosing $\delta$ sufficiently small such that there exist a ball $B_\delta(\bar{n}, \bar{E}) \subset \mathcal{U}$ with radius $\delta$. Then

$$\|n - \bar{n}\|_{L^\infty} + \|E - \bar{E}\|_{L^\infty} \leq \|n - \bar{n}\|_{C^0} + \|E - \bar{E}\|_{C^0} \leq \delta.$$

implies that $(n(x), E(x)) \in B_\delta(\bar{n}, \bar{E}) \subset \mathcal{U}$ for all $x \in \mathbb{R}^d$. \qed

Appendix B. Proof of Proposition 7.3. In this section we are going to prove Proposition 7.3, which we restate for the sake of convenience. Let

$$\|f\|_{C^0} := \sum_{i,j \in \mathbb{N}_0^d, |i+j| \leq 1} \|\partial^i_p \partial^j_x f\|_{C^0}, \quad \text{where} \quad |f|_{C^0} := \sum_{a,b \in \mathbb{N}_0^d} \frac{|a+b|}{a!b!} \int_{\mathbb{R}^d} |\partial^a_p \partial^b_x f(x, p)| \, dp$$

and

$$|f|_{C^0} := |f|_{C^0} - |f|_{C^0} \quad \text{and} \quad \|f\|_{C^0} := \|f\|_{C^0} - \|f\|_{C^0}.$$

Proposition B.1. Let $\eta > 0$, $\alpha \geq 1$ and $R, \nu > 0$. Then there exist $\alpha, \delta > 0$ such that for all $f, g : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ being analytic with $\|f\|_{C^0}, \|g\|_{C^0} \leq R$ and

$$\|n_h\|_{C^0} + \|E_h\|_{C^0} \leq \delta n_h(x)(1 - \eta n_h(x)) \quad \text{for} \ h \in \{f, g\}$$

and $(n_h(x), E_h(x)) \in \mathcal{U}_{\alpha, \nu}, \nu$ for $h \in \{f, g\}$, it holds

$$\|\mathcal{F}_f\|_{C^0}, \|\mathcal{F}_g\|_{C^0} \leq C$$

and

$$\|\mathcal{F}_f - \mathcal{F}_g\|_{C^0} \leq C \|f - g\|_{C^0},$$

for some $C > 0$ and all $x \in \mathbb{R}^d$.

Note that this proposition is stronger than its counter part in Proposition A.3. Therefore, we require a more sophisticated analysis of $\mathcal{F}^0$.

Definition B.2. Let $\alpha \geq 1$ and $\delta > 0$. We define

$$M_\alpha := \left\{ \int_{\mathbb{T}^d} \frac{(1, \epsilon(p))\, dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} : \lambda_0, \lambda_1 \in \mathbb{R} \text{ with } |\lambda_1| \leq \log \alpha \right\} \subset \mathbb{R}_2$$

and

$$\mathcal{U}_{\alpha, \delta} := \bigcup_{(m_0, m_1) \in M_\alpha} B_{5\lambda_0(1 - \eta \lambda_0)}(m_0, m_1) \cap M_\alpha,$$

where $B_\theta(y)$ denotes the ball in $\mathbb{R}^2$ centered at $y$ with radius $\theta$. 

Corollary A.8. Given $\bar{n}, \bar{E} \in \mathbb{R}$ and $\nu > 0$. Let $\delta > 0$ and $\mathcal{U}$ be as in Corollary A.5. Let $\mathcal{U} \subset U$ be convex. Then there exists a $C > 0$ such that for all $(n_i, E_i) : \mathbb{R}^d \to \mathcal{U}$, $i = 0, 1$, being analytic such that

$$\|n_i - \bar{n}\|_{C^0} + \|E_i - \bar{E}\|_{C^0} \leq \delta, \quad i = 0, 1,$$
Lemma B.3. Let \( a \geq 1 \). There exist constants \( A_a, B_a > 0 \) such that

\[
\left| D^i_{(n,E)} D^j_p F^0(n,E;p) \right| \leq i! j! A_a^i \left( \frac{B_a}{n(1 - \eta n)} \right)^j F^0(n,E;p) (1 - \eta F^0(n,E;p))
\]

for all \((n,E) \in M_a, p \in T^d \) and \( i + j \geq 1 \). Moreover, if \( \eta = 0 \) these constants may be chosen independently from \( a \), i.e., there exist \( A, B > 0 \) such that

\[
\left| \partial^i_{(n,E)} \partial^j_p F^0(n,E;p) \right| \leq i! j! A^i B^j n^j F^0(n,E;p)
\]

for any \( i + j \geq 1 \) and all \((n,E) \in [0, \infty) \times \mathbb{R} \).

Proof. For a detailed proof see [9] section 5.4.

In the next step, we state the space local version of Lemma A.6, which can be proved exactly like Lemma A.6.

Lemma B.4. Let \( x \in V \subset \mathbb{R}^k \) open and let \( g : \mathbb{R}^d \times T^d \to V, \phi : V \to \mathbb{R} \) be analytic. Then

\[
|\phi \circ g|_{C^\nu_x} \leq |\phi|_{C^\mu_{x^2}} \quad \text{with } \mu = |g|_{C^\nu_x} \text{ and } y = g(x)
\]

for \( \nu > 0 \).

Using this lemma, we can easily find estimates for the derivatives of some functions.

Example B.5. Let

\[
F_{\lambda_0}(x) = \frac{1}{\eta + 1 + x^2}.
\]

We have

\[
|F_{\lambda_0}|_{C^\nu_x} = |\phi \circ (\cdot)^2|_{C^\nu_x} \leq |\phi|_{C^\mu_{x^2}} = 2\nu|x| + \nu^2
\]

for \( \phi(s) = (\eta + 1 + s)^{-1} \) according to Lemma B.4. We have \( \phi^{(i)}(s) = (-1)^i i! (\eta + 1 + s)^{-(i+1)} \) which implies that

\[
|F_{\lambda_0}|_{C^\nu_x} = \sum_{i=1}^N \frac{(\nu + x)^{2i}}{i!} |\phi^{(i)}(x^2)|
\]

\[
= \frac{1}{(\eta + 1 + x^2)} \sum_{i=1}^N \left( \frac{2\nu|x| + \nu^2}{\eta + 1 + x^2} \right)^i.
\]

Let \( \nu := \frac{1}{2} \min \{ \sqrt{\eta}, 1 \} \). Thus,

\[
2\nu|x| - \nu^2 \leq |x| - \eta \leq \frac{1 + \eta + x^2}{2} - \left( \frac{1 - |x|}{2} \right)^2 \leq \frac{1 + \eta + x^2}{2}.
\]

This implies that

\[
|F_{\lambda_0}^{(a)}(x)| \leq \frac{a!}{\nu^a} |F_{\lambda_0}|_{C^\nu_x} \leq \frac{a!}{\nu^a} \frac{1}{\eta + 1 + x^2} \leq \frac{a! 2^{a+1}}{\min \{ \sqrt{\eta}, 1 \}^a} \frac{1}{\eta + 1 + x^2}
\]

for \( \nu = \frac{1}{2} \min \{ \sqrt{\eta}, 1 \} \).
Corollary B.6. Let \( x \in V \subset \mathbb{R}^k \) open and let \( m : \mathbb{R}^d \to V \), \( \phi : V \times T^d \to \mathbb{R} \) be analytic. We have
\[
\| \phi \circ m \|_{C^\nu_y} \leq \| \phi \|_{C^\nu_y} (1 + \| m \|_{C^\nu_y}) + \mu \| \phi \|_{C^\nu_y}
\]
with \( \mu = \| m \|_{C^\nu_y} \) and \( y = m(x) \) for \( \nu > 0 \). Moreover, assume that \( |\phi|_{C^\nu_y} < \infty \) for some \( \mu_0 > 0 \). Let \( M > 0 \) and \( \bar{\mu} \in (0, \mu_0) \). Then there exists a constant \( C > 0 \) such that
\[
\| \phi \circ m \|_{C^\nu_y} \leq C \| m \|_{C^\nu_y}.
\]
for all \( \nu > 0 \) and all \( m : \mathbb{R}^d \to V \) being analytic such that \( \| m \|_{C^\nu_y} \leq M \) and \( \| m \|_{C^\nu_y} \leq \bar{\mu} \).

Proof. Using the chain rule we first compute
\[
|\partial_x \phi(m, \cdot)|_{C^\nu_y} = |\partial_1 \phi(m, \cdot)|_{C^\nu_y} \leq |\partial_1 \phi(m, \cdot)|_{C^\nu_y} |\partial_x m|_{C^\nu_y}.
\]
Since \( |f|_{C^\nu_y} = \| f \|_{L^1(T^d)} \) for \( f : \mathbb{R}^d \times T^d \to \mathbb{R} \) analytic, we have
\[
\| \partial_x \phi(m, \cdot) \|_{C^\nu_y} \leq \| \partial_1 \phi(m, \cdot) \|_{C^\nu_y} |\partial_x m|_{C^\nu_y} - \| \partial_1 \phi(m, p) \|_{L^1(T^d)} |\partial_x m(x)|
\]
\[
= |\partial_1 \phi(m, \cdot)|_{C^\nu_y} |\partial_x m|_{C^\nu_y} + |\partial_1 \phi(m, p) \|_{L^1(T^d)} |\partial_x m(x)|_{C^\nu_y}
\]
Now we can conclude the first part of the assertion by Lemma B.4. With this, the remaining part is a direct consequence of Lemma 6.2.

Lemma B.7. Let \( \eta > 0 \), \( a \geq 1 \), \( C \notin \mathbb{R}^d \) and \( M \) analytic. We have
\[
\| n \|_{C^\nu_y} + \| E \|_{C^\nu_y} \leq \delta n(x)(1 - \eta n(x)),
\]
we have \( \| F^0(n, E) \|_{C^\nu_y} \leq C_0 \) and
\[
\| F^0(n, E) \|_{C^\nu_y} \leq C_0 n(x)(1 - \eta n(x)).
\]

Proof. Let \( \alpha := \frac{1}{2Ba} > 0 \), where \( B_a \) is given by Lemma B.3. For \( y \in U_{a,\alpha} \), we choose \( m = (m_0, m_1) \in M_a \) such that \( |(m_0, m_1) - y| < \frac{m_0(1 - \eta m_0)}{2Ba} \). Note that \( 0 < m_0 < \eta^{-1} \). Writing \( \beta = \frac{m_0(1 - \eta m_0)}{2Ba} \), we use Taylor's formula and see that
\[
|F^0|_{C^\nu_y} \leq \sum_{i+j \geq 1} \frac{\nu^{i+j}}{i!j!} \| \partial_y^i \partial_p^j F^0(y, p) \|_{L^1(T^d)}
\]
\[
\leq \sum_{i+j \geq 1} \frac{(\nu + \beta)^i}{i!j!} \| \partial_y^i \partial_p^j F^0(m, p) \|_{L^1(T^d)}
\]
\[
\leq \sum_{i+j \geq 1} \frac{(\nu A_\alpha)^i}{i!j!} \left( \frac{(\nu + \beta)Ba}{m_0(1 - \eta m_0)} \right)^i \| F^0(m, p)(1 - \eta F^0(m, p)) \|_{L^1(T^d)}
\]
Now by Jensen’s inequality and the Neumann series, we obtain
\[
|F^0|_{C^\nu_m} \leq \left( \frac{1}{1 - \nu A_\alpha} - \frac{m_0(1 - \eta m_0)}{m_0(1 - \eta m_0) - (\nu + \beta)Ba} - 1 \right) m_0(1 - \eta m_0)
\]
if \( \nu A_\alpha < 1 \) and \( (\nu + \beta)Ba < m_0(1 - \eta m_0) \). Thus, defining
\[
\mu_0 = \min \{ \frac{1}{(3A_\alpha)}, \frac{m_0(1 - \eta m_0)}{(3Ba)} \},
\]
we have $|F^0|_{C^{\alpha}_x} \leq 8m_0(1-\eta m_0)$. Therefore, by Lemma 6.2, there exists a constant $C_1 > 0$ such that

$$
\|F^0\|_{C^{\alpha}_x} \leq C_1 \|(n, E)\|_{C^{\alpha}_x} \tag{36}
$$

if $\|(n, E)\|_{C^{\alpha}_x} \leq \min\{1/(4A_\alpha), m_0(1-\eta m_0)/(4B_\alpha)\}$.

For the next step, we suppose that $(n, E)$ from above fulfills the hypothesis. Then $\|(n, E)\|_{C^{\alpha}_x} \leq 1/(\eta A_\alpha n(x)(1-\eta m(x)) \leq 1/(4A_\alpha)$ since $n(x)(1-\eta m(x)) \leq \eta/4$. Moreover, it holds $\|(n, E)\|_{C^{\alpha}_x} \leq 1/(4B_\alpha n(x)(1-\eta m(x))$. Thus, it holds (36) and in particular

$$
\|F^0\|_{C^{\alpha}_x} \leq C_1 \delta n(x)(1-\eta m(x)).
$$

Finally, we can easily show the remaining estimate by using the inequality

$$
\|F^0(n, E)\|_{C^{\alpha}_x} \leq \|F^0(n, E)\|_{C^{\alpha}_x} + \|F^0(n, E); p\|_{L^1(U)} + \|\partial (n, E) F^0(n, E); p\|_{L^1(U)} \tag{37}
$$

and the fact that $F^0$ and $\partial (n, E, p) F^0(n, E)$ are bounded (see Lemma B.3).

**Lemma B.8.** Let $\nu > 0$ and $a \geq 1$. For $C, \nu > 0$ let $\alpha, \delta > 0$ be as in Lemma B.7. Then there exists a constant $C_0 > 0$ such that for all $x \in \mathbb{R}^d$, $(n_1, E_1) : \mathbb{R}^d \to U_{a, \alpha}$, $i = 0, 1$, being analytic in $x$ with $\|(n_1, E_1)\|_{C^{\alpha}_x} \leq C$ and

$$
n_i \|_{C^{\alpha}_x} + \|E_i\|_{C^{\alpha}_x} \leq \frac{\delta}{2} n_i(x)(1-\eta n_i(x)) \quad \text{for} \ i = 0, 1,
$$

we have

$$
\|F^0(n_1, E_1) - F^0(n_0, E_0)\|_{C^{\alpha}_x} \leq C_1 \left( \|n_1 - n_0\|_{C^{\alpha}_x} + \|E_1 - E_0\|_{C^{\alpha}_x} \right).
$$

**Proof.** Throughout this proof, we fix $x \in \mathbb{R}^d$ and make sure that the constants do not depend explicitly on $x$. To start with, we assume w.l.o.g. that

$$
n_0(x)(1-\eta n_0(x)) \geq n_1(x)(1-\eta n_1(x)).
$$

We define

$$
\langle n_\theta, E_\theta \rangle := \begin{cases} (n_0, (1-2\theta)E_0 + 2\theta E_1), & \text{for } \theta \in \left[0, \frac{1}{2}\right], \\ (2-2\theta)n_0 + (2\theta - 1)n_1, E_1), & \text{for } \theta \in \left[\frac{1}{2}, 1\right]. 
\end{cases}
$$

Hence, for $\theta \in \left[0, \frac{1}{2}\right]$

$$
|n_\theta|_{C^{\alpha}_x} + |E_\theta|_{C^{\alpha}_x} \leq |n_0|_{C^{\alpha}_x} + |E_0|_{C^{\alpha}_x} + |E_1|_{C^{\alpha}_x} \leq \delta n_0(x)(1-\eta n_0(x)) = \delta n_\theta(x)(1-\eta n_\theta(x)).
$$

For $\theta \in \left[\frac{1}{2}, 1\right]$ it holds

$$
|n_\theta|_{C^{\alpha}_x} \leq (2-2\theta) |n_0|_{C^{\alpha}_x} + (2\theta - 1) |n_1|_{C^{\alpha}_x} \leq (2-2\theta) C n_0(x)(1-\eta n_0(x)) + (2\theta - 1) C n_1(x)(1-\eta n_1(x)).
$$

Since the mapping $t \mapsto t(1-\eta t)$ is concave, we have

$$
|n_\theta|_{C^{\alpha}_x} \leq \frac{\delta}{2} n_\theta(x)(1-\eta n_\theta(x)).$$
Moreover, we have $n_1(x)(1 - \eta_1(x)) \leq n_\theta(x)(1 - \eta_\theta(x))$ by construction, which implies

$$|E_\theta|_{C_\theta^r} = |E_1|_{C_\theta^r} \leq \frac{\delta}{2} n_1(x)(1 - \eta_1(x)) \leq \frac{\delta}{2} n_\theta(x)(1 - \eta_\theta(x)).$$

We summarize that

$$|n_\theta|_{C^r} + |E_\theta|_{C_\theta^r} \leq \delta n_\theta(x)(1 - \eta_\theta(x))$$

for all $\theta \in [0, 1]$. The formula

$$\mathcal{F}^0(n_1, E_1) - \mathcal{F}^0(n_0, E_0)$$

$$= (E_1 - E_0) \int_0^1 \partial_E \mathcal{F}^0(n_\theta, E_\theta) d\theta + (n_1 - n_0) \int_0^1 \partial_n \mathcal{F}^0(n_\theta, E_\theta) d\theta$$

and the properties of $(n_\theta, E_\theta)$ we can use the same techniques as in the proof of Lemma B.7 in order to finish the proof.

Proof of Proposition B.1. Finally, the proposition is a direct consequence of Lemmas B.7 and B.8, because we have $\|n_f\|_{C^r} + \|E_f\|_{C_\theta^r} \leq (1 + \|\| \|_{L^1(T^d)}) \|f\|_{C_\theta^r}$ as well as $\|n_f\|_{C^r} + \|E_f\|_{C_\theta^r} \leq (1 + \|\| \|_{L^1(T^d)}) \|f\|_{C_\theta^r}$.

REFERENCES

[1] N. B. Abdallah and P. Degond, On a hierarchy of macroscopic models for semiconductors, J. Math. Phys., 37 (1996), 3308–3333.

[2] A. Al-Masoudi, S. Dörscher, S. Häfner, U. Sterr and C. Lisdat, Noise and instability of an optical lattice clock, Phys. Rev. A, 92 (2015), 063814, 7 pages.

[3] N. W. Ashcroft and N. D. Mermin, Solid state physics, Physics Today, 30 (1977), 61.

[4] C. Bardos and N. Besse, The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits, Kinet. Relat. Models, 6 (2013), 899–917.

[5] C. Bardos and N. Besse, Hamiltonian structure, fluid representation and stability for the Vlasov-Dirac-benney equation, In Hamiltonian Partial Differential Equations and Applications, Selected Papers Based on the Presentations at the Conference on Hamiltonian PDEs: Analysis, Computations and applications, Toronto, Canada, January 10–12, 2014, pages 1–30, Toronto: The Fields Institute for Research in the Mathematical Sciences; New York, NY: Springer, 2015.

[6] C. Bardos and N. Besse, Semi-classical limit of an infinite dimensional system of nonlinear Schrödinger equations, Bull. Inst. Math., Acad. Sin. (N.S.), 11 (2016), 43–61.

[7] C. Bardos and A. Nouri, A Vlasov equation with Dirac potential used in fusion plasmas, J. Math. Phys., 53 (2012), 115621, 16pp.

[8] E. Bloch, Ultracold quantum gases in optical lattices, Nature Physics, 1 (2005), 23–30.

[9] M. Braukhoff, Effective Equations for a Cloud of Ultracold Atoms in an Optical Lattice, Ph.D thesis, University of Cologne, Germany, 2017.

[10] M. Braukhoff and A. Jüngel, Energy-transport systems for optical lattices: Derivation, analysis, simulation, Mathematical Models and Methods in Applied Sciences, 28 (2018), 579–614.

[11] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Lühmann, B. Malomed, T. Sowinski and J. Zakrzewski, Non-standard Hubbard models in optical lattices: A review, Rep. Prog. Phys., 78 (2015), 066001, 47 pages.

[12] A. Griffin, T. Nikuni and E. Zaremba, Bose-Condensed Gases at Finite Temperatures, Cambridge University Press, Cambridge, 2009.

[13] D. Han-Kwan and T. T. Nguyen, Ill-posedness of the hydrostatic Euler and singular Vlasov equations, Arch. Rational Mech. Anal., 221 (2016), 1317–1344.

[14] D. Han-Kwan and F. Rousset, Quasineutral limit for Vlasov-Poisson with Penrose stable data, Ann. Sci. cole Norm. Sup., 49 (2016), 1445–1495.

[15] P.-E. Jabin and A. Nouri, Analytic solutions to a strongly nonlinear Vlasov equation, C. R., Math., Acad. Sci. Paris, 349 (2011), 541–546.
[16] A. Jaksch, Optical lattices, ultracold atoms and quantum information processing, Contemp. Phys., 45 (2004), 367–381.
[17] A. Jüngel, Transport Equations for Semiconductors, Lect. Notes Phys., 773. Springer, Berlin, 2009.
[18] C. Mouhot and C. Villani, On Landau damping, Acta Math., 207 (2011), 29–201.
[19] N. Ramsey, Thermodynamics and statistical mechanics at negative absolute temperature, Phys. Rev., 103 (1956), 20–28.
[20] A. Rapp, S. Mandt and A. Rosch, Equilibration rates and negative absolute temperatures for ultracold atoms in optical lattices, Phys. Rev. Lett., 105 (2010), 220405, 4 pages.
[21] U. Schneider, L. Hackermüller, J. Ph. Ronzheimer, S. Will, S. Braun, T. Best, I. Bloch, E. Demler, S. Mandt, D. Rasch and A. Rosch, Fermionic transport and out-of-equilibrium dynamics in a homogeneous Hubbard model with ultracold atoms, Nature Physics, 8 (2012), 213–218.

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E-mail address: marcel.braukhoff@asc.tuwien.ac.at