Influence of gluon behavior on heavy-quark pair production

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Abstract – We check the impact of the gluon distribution due to the number of active flavor employed in the calculation of heavy-quark deep-inelastic scattering (DIS) electro-production on the reduced cross section and structure functions determined in the leading order (LO) and next-to-leading order (NLO) quantum chromodynamics (QCD) analysis using the Hadron Electron Ring Accelerator (HERA) combined data. The charm and beauty structure functions are determined and compared with the HERA combined data. The results are in good agreement with respect to the experimental data. We also compare the obtained charm and bottom structure functions with the results from CTEQ6.6 [Phys.Rev.D78, 013004(2008)] and NNPDF3.1 [Eur.Phys.J.C77, 663(2017)] parameterization models. The effect of gluon density on these calculations depends on the number of active flavor at asymptotical values of the momentum transfer $Q^2$. Also, in the HERA kinematic range, the ratio of $F_2^c/F_2$ and $F_2^b/F_2$ are obtained. These results and comparison with the HERA combined data demonstrate that the suggested method can be applied in analysis of new colliders.

I. INTRODUCTION. – Heavy-quark pair production in deeply inelastic scattering (DIS) provides the predictions of quantum chromodynamics (QCD). These predictions can be seen in the combined data collected at HERA [1]. Also, these predictions will also be visible at future accelerators [2]. The new accelerators (LHeC [3] and FCC-eh [4]) will guide predictable data toward ultra-high energies. In DIS processes, the heavy-quark production is dominantly via the boson-gluon fusion (BGF) process, $\gamma^* g \rightarrow Q\bar{Q}$, where $Q$ is heavy quark. The reaction under study is

$$e^- + P \rightarrow e^- + Q\bar{Q} + X,$$

where $P$ is a proton, $Q\bar{Q}$ is a heavy-quark pair and $X$ is any hadronic state allowed. Indeed the heavy-quark production is sensitive to the gluon distribution, as in QCD perturbation theory the cross section for the process $\gamma^* P \rightarrow Q\bar{Q} X$ can be written as a convolution of the gluon distribution function $G(x, \mu^2)$ and the partonic cross section of the photon-gluon fusion process $\delta(\gamma^* g \rightarrow Q\bar{Q})$ [5]. Indeed $G(x, \mu^2) = xg(x, \mu^2)$ which $g(x, Q^2)$ is the gluon density. The gluon momentum in the proton is dependent on the heavy quarks mass. This dependence on the effect of the gluon density can be seen in both Fixed Flavor Number (FFN) [6] and Variable Flavor Number (VFN) [7] schemes. Which DIS nucleon structure functions can be explain using these schemes. In FFNS heavy quarks are not considered as active. In this case, for $Q^2 \sim m^2_Q$ the heavy flavors are generated only by BGF. In the FFNS the heavy-quark structure functions at low values of $x$ are given by

$$F_{2L}(x, Q^2) = C_{2L}^{FF,n_f}(x, \frac{Q^2}{\mu^2}) \otimes G_{n_f}(x, \mu^2),$$

where $n_f = 3$ is the number of light quark flavors when all heavy flavors are considered as massive, and $G_{n_f}$ is the gluon distribution function due to the number of active quark flavors. Here $C_{2L}$ are the Wilson coefficients. $\mu^2$ denotes the factorization scale and the Mellin convolution is give by the integral $[f \otimes g](x) = \int_0^1 (dy/y)f(y)g(y/x)/y$. For $Q^2 > m^2_Q$ VFNs has been introduced, which heavy quarks considered as the light-flavor ones. For realistic kinematics it has to be extended to the case of a General-Mass VFNs (GM-VFNs) which is defined similarly to the Zero-Mass VFNs (ZM-VFNs) in the $Q^2/m^2_Q \rightarrow \infty$ limit [8]. In this scheme, the transition from $n_f$ active flavors to $n_f + 1$ considered in the construction of charm-quark parton distribution function (PDF). Rather at some
large scale the transition with two massive quarks (i.e., $n_f \to n_f + 2$) has been discussed in Ref.[8]. One of the ingredients used in the GM-VFNS in the argument of the respective massless Wilson coefficient functions is rescaling of the parton momentum fractions $z$ by replacing $z \to \chi = z(1 + \frac{4m^2}{Q^2})$. Within these schemes, the heavy quark densities arise via the gluon evolution. Gluon behavior in determining the heavy-quark density depends on the effect of the number of active flavor. In Ref.[9] this behavior is well illustrated. Authors in [9] showed an analytical behavior of the gluon distribution function from the proton structure function due to effects of heavy quarks. Indeed, accurate knowledge of heavy structure function obtained from the fit parameters shown in Ref.[9].

The plane of the present paper is as follows. In section II we review the heavy flavors influence in gluon density determination, and we present the gluon distribution functions used in the present study. In section III we describe the various schemes with respect to the gluon behavior. The section IV is devoted to the results and discussion. Finally, we give our conclusions in Sec.V.

II. Theory. – In this section we briefly present the theoretical analysis of the gluon distribution function due to the effect of heavy quarks. The reader can be referred to the Ref.[9] for more details. The DGLAP evolution equations [10] must be modified to take into account the effect of production thresholds for pairs of charm and beauty quarks. The proton structure function is defined into the quark distribution functions $q_i(x, Q^2)$ as $i \in \{u, \bar{u}, d, \bar{d}, s, \bar{s}, c, \bar{c}, b, \bar{b}\}$. The evolution equation for the proton structure function

$$\frac{1}{x} \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dz}{z} \frac{1}{\ln \frac{z}{x}} F_2(z, Q^2) K_{gg}(\frac{x}{z}) + \sum_i c_i^2 \frac{1}{\eta_i} \int_x^1 \frac{dz}{z} G(H_i, Q^2) K_{gg}(\frac{x}{z})],$$  

where $c_i^2$ is the squares of the quark charges, $K$’s are the splitting functions and $\eta_i(Q^2) = 1 + 4M_i^2/Q^2$. For massless quarks the value of $\eta_i$ defined to be 1. In Eq.(3) the gluon distribution function shift from $z$ to $\eta_i z$ for activation of charm and beauty quarks. The gluon distribution has been defined by

$$G(x, Q^2) = \frac{1}{3} \frac{1}{\eta_c} G(x_c, Q^2) + \frac{1}{6} \frac{1}{\eta_b} G(x_b, Q^2) = \mathcal{H}_3(x, Q^2),$$  

where

$$\mathcal{H}_3(x, Q^2) = \frac{1}{\omega} \int_x^1 \frac{dz}{z} \left(\frac{z}{x}\right)^k \sin(\omega \ln \frac{z}{x}) \mathcal{G}_3(x, Q^2),$$  

and

$$\mathcal{G}_3(x, Q^2) = -\frac{3}{4} \frac{4\pi}{\alpha_s(Q^2)} x^4 \frac{\partial^3 F_2(x, Q^2) / \partial x^3}{\partial x^3}. (6)$$  

Here $x_c = x(1 + \frac{M_c^2}{Q^2})$ and $x_b = x(1 + \frac{M_b^2}{Q^2})$. In Eq.(5), $k$ and $\omega$ are the real and imaginary parts of the roots of the factored form of the differential operator, with $k = -3/2$ and $\omega = \sqrt{7}/2$. By considering the Laplace transform method, $\mathcal{H}_3(x, Q^2)$ has been defined by

$$\mathcal{H}_3(x, Q^2) = \frac{3}{4} \frac{4\pi}{\alpha_s(Q^2)} \left(3 \int_x^1 \frac{dz}{z} \frac{F_2(z, Q^2)}{z} - \frac{\partial}{\partial x} F_2(x, Q^2)

- \int_x^1 \frac{dz}{z} \frac{F_2(z, Q^2)}{z} \left(\frac{z}{x}\right)^{3/2} \frac{3}{\omega} \sin(\omega \ln \frac{z}{x}) + 2 \cos(\omega \ln \frac{z}{x})\right). (7)$$  

where shows that high derivatives of the structure functions do not appear in $\mathcal{H}_3(x, Q^2)$. Here $\mathcal{H}_3(x, Q^2)$ converted to the $\nu$ form $\mathcal{H}_3(\nu, Q^2)$ by the substitution $x = e^{-\nu}$, $\nu = \ln(1/x)$. In the case of four massless quarks, the gluon distribution is given by the following form

$$G_4(x, Q^2) = \frac{3}{5} \mathcal{H}_3(x, Q^2) = \mathcal{H}_4(x, Q^2)$$  

When taking mass effects into account, the exact solution for the gluon distribution at $n_f = 5$ has the following form

$$G(x, Q^2) = \mathcal{H}_3(x, Q^2) + \sum_{n=1}^N (-1)^n \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \beta^k \mathcal{H}_3(\eta_c^{-n-k} \eta_b^k x, Q^2). (9)$$  

where $\eta_c = 1 + \frac{M_c^2}{Q^2}$, $\eta_b = 1 + \frac{M_b^2}{Q^2}$, $\alpha = (2/3\eta_c)$, $\beta = (1/6\eta_b)$ and $\binom{n}{k}$ is a binomial coefficient. The summations in Eq.(9) are finite as the sum on $n$ terminates at $N$ such that $(N+1) \ln \eta_c \geq \ln(1/x)$. Indeed $G_5$ was found in the form

$$G_5(x, Q^2) = \frac{6}{11} \mathcal{H}_3(x, Q^2) = \mathcal{H}_5(x, Q^2)$$  

The fractional uncertainty in $\mathcal{H}_3(x, Q^2)$ was determined by the fractional statistical error in $F_2(x, Q^2)$, where $F_2(x, Q^2)$ determined [11] from the ZEUS structure function data [12]. This fractional statistical error is defined as

$$\Delta \mathcal{H}_3(x, Q^2) = \mathcal{H}_3(x, Q^2) \frac{\Delta F_2(x, Q^2)}{F_2(x, Q^2)}$$  

where the fractional statistical error of the proton structure function obtained from the fit parameters shown in Ref.[9].
III. Method. – The deeply inelastic heavy-quark structure functions \( F_k(x, Q^2, m_Q^2) \) for \( k = 2, L \) are given by [13] the following forms as

\[
F_k^{Q \bar{Q}}(x, Q^2, m_Q^2) = \frac{Q^2 \alpha_s(\mu^2)}{4m_Q^2\pi^2} \int_x^{\infty} \frac{dz}{z} \left[ \frac{Q^2}{2} \frac{g(z)}{\mu^2} c_k^{(0)} \right] + \frac{Q^2 \alpha_s^2(\mu^2)}{m_Q^2\pi} \int_x^{\infty} \frac{dz}{z} \left[ \frac{Q^2}{2} \frac{g(z)}{\mu^2} c_k^{(1)} + \frac{\tau_{k,g}}{\tau_{k,\bar{g}}} \ln \frac{\mu^2}{m_Q^2} \right] + \sum_{i=g,q} \left[ \frac{2}{3} \frac{Q^2}{z} q_i(z, \mu^2) c_i^{(1)} + \frac{1}{3} \frac{\tau_{i,q}}{\tau_{i,\bar{q}}} \ln \frac{\mu^2}{m_Q^2} \right] + \frac{c_{\text{Light},i} q_i(z, \mu^2) c_i^{(1)}}{2} \right] \right). \]

The functions \( q_i \) for quark and anti-quark denote the parton densities in the proton. The coefficient functions, defined by \( c_k^{(i)}(z) = c_{k,g}^{(i)}(z) = (i = g, q, \overline{q}, l = 0, 1) \) and by \( d_k^{(i)}(z) = d_{k,\bar{g}}^{(i)}(z) = (i = q, \overline{q}, l = 0, 1) \) are represented in the \( \overline{\text{MS}} \)-scheme and calculated in [13]. The gluon density (i.e., \( g(x, \mu^2) \)) is dominant at low values of \( x \), therefore the simplest form after some manipulation with the expression for \( F_k^{Q \bar{Q}}(x, Q^2, m_Q^2) \) can be written as

\[
F_k^{Q \bar{Q}}(x, Q^2, m_Q^2) = 2x e^2 \frac{\alpha_s(\mu^2)}{2\pi} \int_x^{1} \frac{dz}{z^2} C_{g,k}(\frac{x}{z}) \frac{Q^2}{\mu^2} G(z, \mu^2), \tag{12}
\]

where \( \chi = \eta \), and \( Q(\mu^2) = 1 + 4 \frac{m_s^2}{Q^2} \). The renormalization scale is assumed to be \( \mu^2 = 4m_Q^2 + Q^2 \). In Eq.(12), \( C_{g,k}(\frac{Q^2}{\mu^2}) \) are the scale dependent Wilson coefficients in the \( \overline{\text{MS}} \) scheme with \( C_{g,k}(\frac{Q^2}{\mu^2}) = c_{g,k} \) for \( Q^2 = \mu^2 \). Here \( C_{Q,g,k}^{Q \bar{Q}}(\frac{Q^2}{\mu^2}) \) are heavy-flavor coefficient functions to next-to-leading order (NLO) approximation [6,8,13,14] and are represented in the \( \overline{\text{MS}} \) scheme and presented in the following form

\[
C_{g,k} = C_{Q,g,k}^{Q \bar{Q}} + \frac{\alpha_s(\mu^2)}{4\pi} C_{g,k}^{Q \bar{Q}} \left[ \frac{m_Q^2}{2} \ln \frac{\mu^2}{m_Q^2} \right]. \tag{13}
\]

In the FFNS at low \( Q^2 \), the heavy-flavor structure functions are given by

\[
F_k^{Q \bar{Q}}(x, Q^2 \leq m_Q^2) = 2x e^2 \frac{\alpha_s(\mu^2)}{2\pi} \int_x^{1} \frac{dz}{z^2} C_{g,k}(\frac{x}{z}) \frac{Q^2}{\mu^2} \left[ \frac{m_Q^2}{H_3(z, \mu^2)} \right] \tag{14}
\]

In the GM-VFNS at high \( Q^2 \), the heavy-flavor structure functions are dependent on the active flavor number as

\[
F_k^{Q \bar{Q}}(x, Q^2 \gg m_Q^2) = 2x e^2 \frac{\alpha_s(\mu^2)}{2\pi} \int_x^{1} \frac{dz}{z^2} C_{g,k}(\frac{x}{z}) \frac{Q^2}{\mu^2} \left[ \frac{m_Q^2}{H_3(z, \mu^2)} \right] \tag{15}
\]

where we take \( n_f = 4 \) for \( m_c^2 < \mu^2 < m_b^2 \) and \( n_f = 5 \) for \( m_b^2 < \mu^2 < m_t^2 \).

In these approaches the observation of heavy-quark pair production represents a sensitive probe of the gluon density in the proton. The calculation of \( F_k^{Q \bar{Q}}(x, Q^2) \) is used to test different gluon distributions. The charm and beauty structure functions (i.e., \( F_2^{Q \bar{Q}} \) and \( F_2^q \)) are also compared to the prediction from the ZEUS \( F_2(x, Q^2) \) experimental data [12]. The global parameterization of the ZEUS data for the proton structure function made by authors in [12] and updated by using the data as combined by the ZEUS and H1 [15] groups in [16]. The explicit expression for the \( F_2 \) parameterization in a wide range of the kinematical variables \( x \) and \( Q^2 \) obtained by the following general form

\[
F_2(x, Q^2) = D(Q^2)(1-x)^n \sum_{m=0}^{2} A_m(Q^2)L^m, \tag{15}
\]

where the parameters with their statistical errors are given in Refs.[11,16]. We can investigate the ratio \( F_2^{Q \bar{Q}}(x, Q^2)/F_2(x, Q^2) \) which this behavior is very interesting in some evidence from the EMC experiments [17]. It will be very interesting to study this effect in the range of available HERA energy and its extension to future energies in LHeC and FCC-eh.

The direct effect of the gluon density with respect to the active flavor number on the heavy-flavor structure functions can indirectly express the reduced cross section behavior. The deep inelastic heavy-quarks structure functions related to the reduced cross section are given by

\[
\sigma_{Q \bar{Q}}^r(x, Q^2) = F_2^{Q \bar{Q}}(x, Q^2) - \frac{y^2}{Y} F_L^{Q \bar{Q}}(x, Q^2), \tag{16}
\]

where \( y^2 = Q^2/sx \) is the inelasticity with \( s \) the ep center of mass energy squared and \( Y = 1 + (1-y)^2 \). With respect to the mass-threshold, the reduced cross section for heavy quarks at low \( Q^2 \) is given by

\[
\sigma_{Q \bar{Q}}^r(x, Q^2) = 2x e^2 \frac{\alpha_s(\mu^2)}{2\pi} \int_x^{1} \frac{dz}{z^2} \left[ \frac{C_{g,2}(x, z)}{z^2} \frac{m_Q^2}{Q^2} \right] \tag{17}
\]

and at high \( Q^2 \) values we have

\[
\sigma_{Q \bar{Q}}^r(x, Q^2) = 2x e^2 \frac{\alpha_s(\mu^2)}{2\pi} \int_x^{1} \frac{dz}{z^2} \left[ C_{g,2}(x, z) \frac{m_Q^2}{Q^2} \right] \tag{18}
\]

IV. Results and Discussion. – We studied the sensitivity of the heavy-quark structure functions and reduced cross sections to the values of the number of active flavor for a wide range of \( Q^2 \) values. We quantified the impact of varied \( n_f \) on the reduced cross section for \( c \bar{c} \) and \( b \bar{b} \) production at the BGF processes at low values of \( x \). The effect of the gluon influence on production of heavy quarks in ep
collisions at LHeC and FCC-eh can be investigated according to the available energy ranges in the future colliders. Based on the available HERA data, we investigate this effect on charm and beauty behaviors. In the hard scale of the scattering processes, the treatment of heavy quarks requires adapting the number of light flavors in QCD to the kinematics under consideration. Indeed the heavy quark production at low values of $x$ via the BGF process is sensitive to the gluon density in the proton. Dependence on the number of flavors $n_f$ in the gluon density behavior, causes different schemes to be used. At low scales (i.e., $Q^2 < m_c^2$), $n_f = 3$ and at high scales (i.e., $Q^2 \gg m_c^2, m_b^2$), $n_f = 4$ and 5 are used respectively. The HERA combined data [1] are obtained with respect to the massive FFNS and different implementations of the VFNS. The running charm and beauty-quark masses are obtained as $m_c = 1.29_{-0.053}^{+0.077}$ GeV and $m_b = 4.049_{-0.118}^{+0.139}$ GeV, where the uncertainties are obtained through adding the experimental fit, model and parameterization uncertainties in quadrature. The strong coupling constant value is chosen to be $\alpha_s(M_Z^2) = 0.118$ [9]. This is matched to the expression for $n_f = 5$ giving $\alpha_s^{LO} = 0.175$ and $\alpha_s^{NLO} = 0.150$ and also it is matched in turn to the expression for $n_f = 4$ giving $\alpha_s^{LO} = 0.182$ and $\alpha_s^{NLO} = 0.160$.

The behavior of the gluon distribution functions with respect to the number of active flavor $n_f$ are shown in Fig.1 at $Q^2$ values 10 and 100 GeV$^2$. In this figure the gluon distribution for $n_f = 3$ massless quarks compared with the massless distributions for $n_f = 4$ in the region $m_c^2 < Q^2 < m_b^2$ and $n_f = 5$ for $Q^2 > m_b^2$. Dash lines are the fractional statistical errors in $G_n(x,Q^2)$ due to the Eq.(11), where we used the fractional statistical errors of proton structure function from the fit parameters shown in Ref.[11]. The results for $F_2^c(x,Q^2)$ and $F_2^b(x,Q^2)$ with LO and NLO coefficient functions are depicted for $Q^2$ values 6.5, 12, 25, 30, 80 and 160 GeV$^2$ in Figs.2 and 3 respectively. In Fig.2, the charm structure functions due to the number of active quark flavors at LO and NLO approximations are shown. In Fig.3, the bottom structure functions at LO and NLO have a similar behavior. These structure functions are shown as a function of $x$ for different values of $Q^2$ and compared with HERA data [18] as accompanied with total errors. The calculations with respect to the gluon influence are consistent with the measurements. These results are also comparable to the phenomenological results of recent years in Refs.[19,20]. These results for $F_2^c(x,Q^2)$ and $F_2^b(x,Q^2)$ are compared with the next-to-leading order analysis of the CTEQ6.6 [21] and NNPDF3.1[22] parameterization models. For $Q^2 < m_b^2$ which it is below the $b\bar{b}$-production threshold, the parameterization models have no data as depicted in Fig.3. These figures indicate that the obtained results from the present analysis are in good agreement with those obtained by global QCD analysis of CTEQ and NNPDF for the heavy quarks structure functions.

In Fig.4 we shown the ratio $F_2^c/F_2$ and $F_2^b/F_2$ where the $F_2(x,Q^2)$ parameterization is taken from Refs.[9,11]. Within the accuracy of the charm data this ratio seems to be comparable to the experimental data. The results presented in the measured range for the three $Q^2$ are comparable to the H1 data [23]. We observe that these ratios are constant at low values of $x$ and increase with $Q^2$ for fixed $x$. We see that this value for the ratio $F_2^c/F_2$ is approximately between 0.2 and 0.5 in a region of $Q^2$ and this prediction is close to the average result

$$< F_2^c/F_2 > = 0.237 \pm 0.021^{+0.043}_{-0.039}$$

in ref.[23]. The ratio $F_2^b/F_2$ is also shown in this figure (i.e., Fig.4) as it is approximately between 0.009 and 0.03 in a region of $Q^2$ values. These ratios, especially the ratio $F_2^b/F_2$, will be able to be considered in the LHeC and FCC-eh collisions. In the following, the error bands illustrated in Figs. 4-7 are due to the heavy quark mass uncertainties (as it is connected with the choice of factorization and renormalization scales) and the statistical errors in the parameterization of $F_2(x,Q^2)$. In Figs.5 and 6, phenomenological predictions of the charm and beauty reduced cross sections are compared to the combined HERA data [1] in a wide range of $Q^2$ values. The center-of-mass energy ($\sqrt{s} = 318$ GeV) for charm and beauty-quark production used in the combined HERA data. In these figures the reduced cross sections are determined with respect to the gluon parameterization due to the number of active flavor. Our results compared to the data of the HERA combined [1] at $Q^2 = 2.5, 12, 120$ and 650 GeV$^2$. Because the beauty reduced cross section experimental error in $Q^2 = 2.5$ GeV$^2$ is very large, we did not examine this value in Fig.6. The uncertainty of the reduced cross sections are due to the $F_2$ parameterization as defined in Eq.(11) according to Refs.[11,16]. Consistency between the determined results with respect to the experimental data are excellent.

The HERA data [1] on heavy-flavor DIS production used of the ABMP16 set [24]. The ABMP16 PDF fit is based on the FFN and VFN schemes with respect to the number of active flavor. Here we use the Bjorken scaling variable $x$ by replacing $\chi \rightarrow x$ in $Q^2 = sy\chi$. In Fig.7, we compared the results of charm and beauty reduced cross sections in the 4 and 5-flavor gluon distributions at LO and NLO approximations with HERA combined data [1]. Combined charm and reduced cross sections (i.e., Figs.6 and 7 in Ref.[1]) compared to the NLO QCD FFNS predictions based on the HERAPDF2.0 FF3A [25] and ABMP16 [24] PDF sets. As shown in Fig.7, the charm and beauty reduced cross sections of the current analysis at $Q^2$ values 32, 60 and 200 GeV$^2$ are in good agreement with the HERA combined data. We conclude that influence of the gluon distribution due to the number of active flavor is effective in determining heavy quark structure functions and it will give us hope for the impact of this effect on future available energies in LHeC and FCC-eh colliders.
V. Summary and Conclusion. In this paper, we present an application of the method of extraction of the gluon distribution function due to the number of active flavor suggested in Ref.[9]. We explored the effect of the number of active flavor on the reduced cross section of the heavy-quark production at the BGF processes. The heavy quark structure functions have been extracted with respect to the gluon distribution functions within the leading order and next-to-leading order approximation on the HERA kinematical region. The obtained explicit expressions for the heavy quark reduced cross sections are fairly well the experimental data. These results show that the heavy-quark production at the BGF processes. The heavy quark structure functions have been extracted with

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Fig. 1: A plot of the gluon distribution function vs. \( x \), for two virtualities due to the number of active flavor. The dash lines are accompanied with statistical errors.

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Fig. 2: The structure function \( F_2^\ell \) at LO and NLO approximations as a function of \( x \) for \( Q^2 \) values 6.5, 12, 25, 30, 80 and 160 GeV\(^2\). These results are compared with ZEUS data [18] according to \( n_f = 3 \) and \( n_f = 4 \). The experimental data accompanied with total errors. The NLO results of CTEQ [21] and NNPDF[22] models are also presented (solid and dashed lines).

Fig. 3: The structure function \( F_2^\ell \) as a function of \( x \) for \( Q^2 \) values 6.5, 12, 25, 30, 80 and 160 GeV\(^2\). These results are compared with ZEUS data [18] according to \( n_f = 5 \). The experimental data accompanied with total errors. The NLO results of CTEQ [21] and NNPDF[22] models are also presented (solid and dashed lines).
Fig. 4: The value of the ratio $F_2^\pi/F_2$ and $F_2^\pi/F_2$. Experimental data are taken from the H1 Collaboration Ref.[23] as accompanied with total error in quadrature only for the available value of the ratio $F_2^\pi/F_2$.

Fig. 5: Theoretical predictions for $\sigma^c_{\perp}$ as a function of $x$ at $Q^2 = 2.5, 12, 120$ and 650 GeV$^2$ using the gluon parameterization with respect to the number of active flavor. These curves are associated with statistical errors due to the $F_2$ parameterization. HERA combined data [1] accompanied with total errors.

Fig. 6: The same as Fig.5 for the reduced cross section $\sigma^c_{\parallel}$.

Fig. 7: The charm and beauty reduced cross sections at LO and NLO approximations as a function of Bjorken scaling $x$ for $Q^2 = 32, 60$ and 200 GeV$^2$, compared to the HERA combined data [1].