Covariant BRST Quantization of Closed Strings in the PP-Wave Background

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Abstract

We canonically quantize closed string theory in the pp-wave background with a non-zero flux of the three-form field strength by using the covariant BRST operator formalism. In this canonical quantization, we completely construct new covariant free-mode representations, for which it is particularly important to take account of the commutation relations of the zero mode of the light-cone string coordinate $X^-$ with other modes. All covariant string coordinates are composed of free modes. Moreover, employing these covariant string coordinates for the energy-momentum tensor, we calculate the anomaly in the Virasoro algebra and determine the number of dimensions of spacetime and the ordering constant from the nilpotency condition of the BRST charge in the pp-wave background.

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1. Introduction

Understanding the quantization of strings in a variety of backgrounds is of great importance. In particular, this quantization is significant for analyzing the string landscape, the AdS/CFT correspondence, matrix models and string phenomenology. Of course, quantization in a number of interesting backgrounds has already been investigated and has been applied to many models. One of the interesting backgrounds with respect to which quantization has been studied is the \( pp \)-wave background. Bosonic string theory in the \( pp \)-wave background has been quantized through application of the canonical operator formalism in the light-cone gauge,\(^1\)–\(^3\) and it has been investigated by the way of the path integral formalism\(^4\),\(^5\) and from the point of view of the world-sheet conformal field theory.\(^6\) The quantization of superstrings in the \( pp \)-wave background with a non-zero flux of the RR five-form field strength has been carried out in the light-cone gauge,\(^7\),\(^8\) and this quantization has been used in the BMN correspondence.\(^9\) Moreover, in the covariant quantization of superstrings in the NS-NS \( pp \)-wave background, a free field realization of current algebra and the Sugawara construction of the world-sheet conformal field theory have been used.\(^10\),\(^11\)

Although the methods mentioned above are very useful, there is another method important for understanding the structure of spacetime and constructing a covariant string field theory in the \( pp \)-wave background. That is, it is important to consider the covariant BRST quantization in the \( pp \)-wave background from the point of view of the canonical operator formalism. From this covariant formalism, we should be able to understand the covariant BMN correspondence. In addition, it is necessary to elucidate how all approaches are related.

In this paper, we canonically quantize a closed bosonic string in the \( pp \)-wave background with a non-zero flux of the three-form field strength of the antisymmetric two-form field by using the covariant BRST operator formalism. First, we construct new free-mode representations of all the covariant string coordinates. Here, we would like to emphasize that these covariant string coordinates in the free-mode representation must satisfy the condition of the canonical commutation relations and must be general solutions of the Heisenberg equations of motion whose form is that of the Euler-Lagrange equations of motion in the \( pp \)-wave background. Second, by using the free-mode representations of the covariant string coordinates for the energy-momentum tensor, we calculate the anomaly in the Virasoro algebra and determine the number of dimensions of spacetime and the ordering constant from the nilpotency condition of the BRST charge in the \( pp \)-wave background.

This paper is organized as follows. In §2 we briefly review the action and the general solutions of the equations of motion of a closed string in the \( pp \)-wave background and define our notation.
Moreover, we review the quantization of ghosts and antighosts. In §3 we present new free-mode representations of all the covariant string coordinates. The free-mode representation of a light-cone string coordinate $X^-$ is characteristic. In §4 we prove that the free-mode representations satisfy all the equal-time canonical commutation relations among all the covariant string coordinates. In §5 we calculate the anomaly in the Virasoro algebra by using the energy-momentum tensor in the free-mode representation of all the covariant string coordinates. In §6 we determine the number of dimensions of spacetime and the ordering constant from the nilpotency condition of the BRST charge in the $pp$-wave background. Section 7 contains some conclusions. In Appendix A we construct new general classical solutions of a closed string in the $pp$-wave background without the antisymmetric tensor field, while Appendix B contains some details of the special mode expansion.

§2. Notation and review

We begin by defining our notation and reviewing the action, the background, the equations of motion, the general solutions and the quantization of ghosts. In §2.1 we define the total action of the closed bosonic string in the $pp$-wave background with the flux of the antisymmetric tensor field in $D$ spacetime dimensions; here we ignore the dilaton field. In §2.2 we explain the equations of motion of strings and their general solutions in the $pp$-wave background with a flux. In §2.3 we review the equations of motion of ghosts, their general solutions and the quantization of ghosts.

2.1. Action and backgrounds

Our starting point is the total action $S = S_X + S_{GF+gh}$, which is BRST invariant:

$$S_X = \frac{-1}{4\pi\alpha'} \int d\tau d\sigma \left[ \sqrt{-\eta_{ab}} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu} \right] \partial_a X^\mu \partial_b X^\nu,$$

(2.1)

$$S_{GF+gh} = \int d\tau d\sigma \sqrt{-g} \left[ \frac{1}{4\pi} B_{ab} \left( g^{ab} - \eta^{ab} \right) - \frac{i}{2\pi} b_{ab} \nabla^a c^b \right],$$

(2.2)

where $g_{ab}$, $\epsilon^{ab}$, and $\eta^{ab}$ are, respectively, a general world-sheet metric, the totally world-sheet antisymmetric tensor ($\epsilon^{01} = +1$), and the flat world-sheet metric, which is diag($-1, +1$). In the nonlinear sigma model action $S_X$, $G_{\mu\nu}$ and $B_{\mu\nu}$ are, respectively, a general spacetime string metric and an antisymmetric tensor field, and the spacetime indices $\mu$ and $\nu$ run over $+, -, 2, 3, \cdots, D-1$. In the action $S_{GF+gh} = S_{GF} + S_{gh}$, where $S_{GF}$ is the gauge fixing action and $S_{gh}$ is the Faddeev-Popov ghost action, $B_{ab}$, $c^a$, and $b_{ab}$ are, respectively, the auxiliary field to fix the gauge, the ghost field, and the antighost field. Because we construct the covariant BRST quantization for string
theory in this paper, we choose the covariant gauge-fixing condition on the world-sheet, \(g^{ab} = \eta^{ab}\). This covariant gauge-fixing condition is obtained from the equation of motion for the auxiliary field, \(B_{ab}\). After the gauge fixing, we use the world-sheet light-cone coordinates \(\sigma^\pm = \tau \pm \sigma\), so that the components of the world-sheet metric and the world-sheet totally antisymmetric tensor become \(\eta^{+-} = \eta^{-+} = -2\) and \(\epsilon^{+-} = -\epsilon^{-+} = -2\). Also, their partial derivatives are then \(\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)\).

Moreover, we use the spacetime light-cone coordinates \(X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)\).

The condition that string theory be Weyl-invariant in its quantization on the world-sheet requires that the renormalization group \(\beta\)-functions vanish at all loop orders; these necessary conditions correspond to the field equations, which resemble Einstein’s equation, the antisymmetric tensor generalization of Maxwell’s equation, and so on.\(^{12}\) As the background field that which satisfies these field equations, we use the following pp-wave metric and antisymmetric tensor field, whose flux is a constant:

\[
ds^2 = -\mu^2(X^2 + Y^2)dX^+dX^- + 2dX^+dX^- + dXdX + dYdY + dX^kdX^k, \tag{2.3}
\]

\[
B = -\mu YdX^+ \wedge dX + \mu XdX^+ \wedge dY. \tag{2.4}
\]

Here, we define \(X^\mu = X^\mu^2 = X\) and \(X^\mu = Y\), and the index \(k\) runs over \(4, 5, \cdots, D - 1\). Thus, the components of \(G_{\mu\nu}\) and \(B_{\mu\nu}\) are

\[
G_{++} = -\mu^2(X^2 + Y^2), \quad G_{+-} = G_{-+} = -1, \tag{2.5}
\]

\[
G_{ij} = \delta_{ij}, \quad i, j = 2, 3, \cdots, D - 1, \tag{2.6}
\]

\[
B_{+2} = -B_{2+} = -\mu Y, \quad B_{+3} = -B_{3+} = \mu X, \tag{2.7}
\]

with all others vanishing. This is almost identical to the Nappi-Witten background.\(^{13}\)

Finally, we introduce the complex coordinates \(Z = X + iY\) and \(\bar{Z} = X - iY\). Then the action \(S_X\) takes the simple form

\[
S_X = \frac{1}{2\pi \alpha'} \int d\tau d\sigma \left[ -2\partial_+ X^+ \partial_- X^- - 2\partial_- X^+ \partial_+ X^- + \bar{D}_+ \bar{Z} \bar{D}_- Z + \bar{D}_- \bar{Z} \bar{D}_+ Z + 2\partial_+ X^k \partial_- X^k \right], \tag{2.8}
\]

where the world-sheet covariant derivatives are defined as \(\bar{D}_\pm = \partial_\pm \pm i\mu \partial_\pm X^+,\) which have a form similar to the covariant derivatives of quantum electrodynamics. After integrating out the auxiliary field \(B_{ab}\), the gauge fixing action \(S_{GF}\) vanishes, and the Faddeev-Popov ghost action \(S_{gh}\) is reduced to

\[
S_{gh} = \frac{i}{\pi} \int d\tau d\sigma \left[ b_{++} \partial_- c^+ + b_{--} \partial_+ c^- \right]. \tag{2.9}
\]
2.2. The equations of motion of $X^\mu$ and their general solutions

We obtain the equations of motion of $X^\mu$ from the action (2.8). These equations are obviously related to the Heisenberg equations of motion with respect to quantization.

- The equations of motion of $X^+$ and $X^k$ are

$$\partial_+ \partial_- X^+ = 0, \quad \partial_+ \partial_- X^k = 0.$$  \hspace{1cm} (2.10)

- The equations of motion of $Z$ and $\bar{Z}$ are

$$\mathcal{D}_+ \mathcal{D}_- Z = 0, \quad \bar{\mathcal{D}}_+ \bar{\mathcal{D}}_- \bar{Z} = 0.$$  \hspace{1cm} (2.11)

- The equation of motion of $X^-$ is

$$\partial_+ \partial_- X^- + \frac{i\mu}{4} \left[ \partial_+ (\bar{Z} \mathcal{D}_- Z - Z \bar{\mathcal{D}}_- \bar{Z}) - \partial_- (\bar{Z} \mathcal{D}_+ Z - Z \bar{\mathcal{D}}_+ \bar{Z}) \right] = 0.$$  \hspace{1cm} (2.12)

In Eq.(2.10), $X^+$ and $X^k$ satisfy the free field equations. By contrast, $Z$ and $\bar{Z}$ interact with $X^+$ through the covariant derivatives in Eq.(2.11), and $X^-$ interacts with $Z$, $\bar{Z}$ and $X^+$ in Eq.(2.12). Nevertheless, we are able to obtain general solutions for all $X^\mu$ in the following.

First, we simply solve the equations of motion for the free fields $X^+$ and $X^k$ under the periodic condition of closed string theory. In this way, we find that the general solutions to these equations are the normal d’Alembert solutions,

$$X^+ = X^+_L(\sigma^+) + X^+_R(\sigma^-), \quad X^k = X^k_L(\sigma^+) + X^k_R(\sigma^-),$$  \hspace{1cm} (2.13)

where L and R indicate the left-moving and right-moving parts, respectively. Second, we solve the equation of motion for $Z$ under the periodic condition of closed string theory. To accomplish this, we define $\tilde{X}^+ \equiv X^+_L - X^+_R$ and multiply the equation of motion for $Z$, Eq.(2.11), by $e^{i\mu \tilde{X}^+}$ from the left. Then, the equation of motion for $Z$ becomes

$$\partial_+ \partial_- \left[ e^{i\mu \tilde{X}^+} Z \right] = 0.$$  \hspace{1cm} (2.14)

This equation shows that $e^{i\mu \tilde{X}^+} Z$ is similar to a free field. Therefore, the quantity $e^{i\mu \tilde{X}^+} Z$ can be expressed as a sum of arbitrary functions $f(\sigma^+)$ and $g(\sigma^-)$ that satisfy the twisted boundary conditions, and the general form of $Z$ is

$$Z = e^{-i\mu \tilde{X}^+} \left[ f(\sigma^+) + g(\sigma^-) \right].$$  \hspace{1cm} (2.15)

Since the general form of $\bar{Z}$ can be obtained from the complex conjugate of $Z$, we have

$$\bar{Z} = e^{i\mu \tilde{X}^+} \left[ \bar{f}(\sigma^+) + \bar{g}(\sigma^-) \right].$$  \hspace{1cm} (2.16)
Finally, in order to solve the equation of motion for $X^-$ under the periodic condition of closed string theory, we substitute the solutions $Z$ given in Eq. (2.15) and $\bar{Z}$ given in Eq. (2.16) into the equation of motion for $X^-$, Eq. (2.12). In this way, the equation of motion for $X^-$ is reduced to a simpler form,

$$\partial_+ \partial_- X^- - \frac{i\mu}{2} \left[ \partial_+ f \partial_- \bar{g} - \partial_+ \bar{f} \partial_- g \right] = 0.$$  \hspace{1cm} (2.17)

The important point here is that Eq. (2.17) does not include $X^+$. Moreover, because $f$ is an arbitrary function of $\sigma^+$ and $g$ is an arbitrary function of $\sigma^-$, we have $\partial_+ f \partial_- \bar{g} - \partial_+ \bar{f} \partial_- g = \partial_+ \partial_-(f \bar{g} - \bar{f} g)$. In terms of this relation, Eq. (2.17) becomes

$$\partial_+ \partial_- \left[ X^- - \frac{i\mu}{2} (f \bar{g} - \bar{f} g) \right] = 0.$$  \hspace{1cm} (2.18)

Because this equation is of a classical free field type, similar to the equation of motion for $Z$ Eq. (2.14), we can easily solve Eq. (2.18). Doing so, we obtain its general solution,

$$X^- = X_L^- (\sigma^+) + X_R^- (\sigma^-) + \frac{i\mu}{2} (f \bar{g} - \bar{f} g),$$  \hspace{1cm} (2.19)

where $X_L^-$ is an arbitrary function of $\sigma^+$, $X_R^-$ is an arbitrary function of $\sigma^-$, and $X_L^- + X_R^-$ is a periodic function of $\sigma$. Note that $X_L^-$ and $X_R^-$ are not free fields in the quantized theory, although $X_L^+, X_R^+, X_L^k$ and $X_R^k$ are completely free fields in the quantized theory. Moreover, it is important that $X_L^-$ and $X_R^-$ can be divided into almost free parts and completely non-free parts. We give detailed discussion of these points in §3.

We explain closed string theory in the $pp$-wave background without a non-zero flux of the antisymmetric tensor field (i.e., $B_{\mu\nu} = 0$) in Appendix A. There, constructing the action and the equations of motion in the covariant gauge, we find the general solutions and the energy momentum tensor. In particular, the general solutions $X^\mu$ given in that appendix are new solutions, and for this reason, the new mode expansion is significant.

2.3. The equations of motion and the quantization of the ghost system

We obtain the equations of motion for ghosts and antighosts from the action (2.9):

$$\partial_- c^+ = 0, \quad \partial_+ c^- = 0, \quad \partial_- b_{++} = 0, \quad \partial_+ b_{--} = 0.$$  \hspace{1cm} (2.20)

The general solutions $c^+$ and $b_{++}$ are purely left-moving, whereas the general solutions $c^-$ and $b_{--}$ are purely right-moving. For closed string theory, the ghosts and antighosts satisfy a periodic condition, which is simply periodicity in $\sigma$ of period $2\pi$. Therefore, $c^+$ and $c^-$ have independent
mode expansions. Similarly, $b_{++}$ and $b_{--}$ also have independent mode expansions. Thus the mode expansions of the ghosts and the antighosts are

\begin{align*}
    c^+ &= \sum_n \tilde{c}_n e^{-i\sigma^+}, \\
    c^- &= \sum_n c_n e^{-i\sigma^-}, \\
    b_{++} &= \sum_n \tilde{b}_n e^{-i\sigma^+}, \\
    b_{--} &= \sum_n b_n e^{-i\sigma^-}.
\end{align*}

The ghost system is quantized according to the following equal-time canonical anticommutation relations:

\begin{align*}
    \{c^+(\tau, \sigma), b_{++}(\tau, \sigma')\} &= 2\pi\delta(\sigma - \sigma'), \\
    \{c^-(\tau, \sigma), b_{--}(\tau, \sigma')\} &= 2\pi\delta(\sigma - \sigma'),
\end{align*}

with all other anticommutators vanishing. In terms of the modes, the anticommutation relations are

\begin{align*}
    \{\tilde{c}_m, \tilde{b}_n\} &= \{c_m, b_n\} = \delta_{m+n}, \\
    \{\tilde{c}_m, c_n\} &= \{c_m, c_n\} = 0, \\
    \{\tilde{b}_m, \tilde{b}_n\} &= \{b_m, b_n\} = 0,
\end{align*}

with the anticommutators of the left modes with the right modes vanishing.\(^{14}\)

\section*{§3. Free-mode representation}

In this section we derive the free-mode representations of all the covariant string coordinates in closed string theory in the $pp$-wave background with a non-zero flux of the three-form field strength of the antisymmetric tensor field. In particular, we place special emphasis on the new free-mode representations of $Z$, $\bar{Z}$ and $X^-$. These new free-mode representations satisfy the quantum condition that all the equal-time canonical commutation relations and the Heisenberg equations of motion must be satisfied. The proof of the free-mode representations in the canonical quantization is given in the next section. When constructing the free-mode representations, it is important to express the general solutions in terms of free-mode expansions and to consider the commutation relations of the zero mode of $X^-$ with other modes. Of course, the commutators between the modes of different fields must completely vanish in the free-mode expansions. The free-mode representations are useful for the calculation of the anomaly of the Virasoro algebra, the construction of the BRST quantization, and so on. Moreover, the free-mode representations may be effective for the purpose of investigating the exact formation of the free-field representation.

We now present the free-mode representations of $X^+$, $X^k$, $Z$, $\bar{Z}$ and $X^-$ in a clear manner.
First, the free-mode representations of $X^+$ and $X^k$ are

$$X^+ = x^+ + \frac{\alpha'}{2} p^+ (\sigma^+ + \sigma^-) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^+ e^{-i n \sigma^+} + \alpha_n^+ e^{-i n \sigma^-} \right], \quad (3.1)$$

$$X^k = x^k + \frac{\alpha'}{2} p^k (\sigma^+ + \sigma^-) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^k e^{-i n \sigma^+} + \alpha_n^k e^{-i n \sigma^-} \right], \quad (3.2)$$

where $\alpha_n^+$ and $\alpha_n^k$ are left-moving free modes, and $\alpha_n^+$ and $\alpha_n^k$ are right-moving free modes. These representations are the same as those of the usual free fields, and $X^+$ and $X^k$ satisfy Eq. (2.10).

Of course, $X^+$ and $X^k$ can also be divided into left-moving fields and right-moving fields, as in Eq. (2.13).

Second, we present the free-mode representations of $Z$ and $\tilde{Z}$, which must satisfy the periodic boundary condition of closed string theory. Because the factor $e^{-i \mu \tilde{X}^+ (\tau, \sigma)}$ in $Z$ appearing in Eq. (2.15) is transformed into $e^{-2 \pi i \mu \alpha p^+} e^{-i \mu \tilde{X}^+ (\tau, \sigma)}$ under the shift $\sigma \rightarrow \sigma + 2 \pi$, $f(\sigma^+)$ and $g(\sigma^-)$ in $Z$ are twisted fields. In other words, these fields satisfy the twisted boundary conditions,

$$f(\sigma^+ + 2 \pi) = e^{2 \pi i \mu \alpha p^+} f(\sigma^+), \quad (3.3)$$

$$g(\sigma^- - 2 \pi) = e^{2 \pi i \mu \alpha p^+} g(\sigma^-). \quad (3.4)$$

Moreover, in order for the construction of the free-mode representation of $Z$ to be possible, it is necessary that $Z$ satisfy the exact quantum condition. In this paper, we use the momentum representation for $p^+$ and $x^-$, which is a zero mode of $X^-$. In this representation, $p^+$ is a real variable and $x^-$ is the differential operator $-i \frac{\partial}{\partial p^+}$, and hence the canonical commutation relation between $p^+$ and $x^-$ is satisfied. We thus obtain the following free-mode representation of $Z$:

$$Z(\sigma^+, \sigma^-) = e^{-i \mu \tilde{X}^+} \left[ f(\sigma^+) + g(\sigma^-) \right], \quad (3.5)$$

$$f(\sigma^+) = \sqrt{\alpha'} \sum_{N = -\infty}^{\infty} \frac{A_N}{\sqrt{|N - \mu \alpha p^+|}} e^{-i(N - \mu \alpha p^+) \sigma^+}, \quad (3.6)$$

$$g(\sigma^-) = \sqrt{\alpha'} \sum_{N = -\infty}^{\infty} \frac{B_N}{\sqrt{|N + \mu \alpha p^+|}} e^{-i(N + \mu \alpha p^+) \sigma^-}. \quad (3.7)$$

Here, we assume that $\mu \alpha p^+$ is not an integer. (We present the free-mode representations in the case that $\mu \alpha p^+$ is an integer in Appendix B. That case includes the case $p^+ = 0$.) The reason that the modes $A_N$ and $B_N$ in the free-mode representations are divided by $\sqrt{|N \pm \mu \alpha p^+|}$ is that the equal-time canonical commutation relations between $X^-$ and other fields, for example $[X^-(\tau, \sigma), Z(\tau, \sigma') = 0$, must be satisfied, and in particular, the commutators between the zero mode of $X^-$ and the modes of $Z$ must vanish. A proof of this is given in §4. Similarly, taking the
Hermitian conjugate of $Z$, we obtain the free-mode representation of $\bar{Z}$:

$$
\bar{Z}(\sigma^+, \sigma^-) = e^{i\mu \hat{X}^+} \left[ \bar{f}(\sigma^+) + \bar{g}(\sigma^-) \right],
$$

(3.8)

$$
\bar{f}(\sigma^+) = \sqrt{\alpha'} \sum_{N=-\infty}^{\infty} \frac{A^+_N}{\sqrt{|N - \mu \alpha' p^+|}} e^{i(N-\mu \alpha' p^+)\sigma^+},
$$

(3.9)

$$
\bar{g}(\sigma^-) = \sqrt{\alpha'} \sum_{N=-\infty}^{\infty} \frac{B^+_N}{\sqrt{|N + \mu \alpha' p^+|}} e^{i(N+\mu \alpha' p^+)\sigma^-}.
$$

(3.10)

Third, we present the free-mode representation of $X^-$. To construct this representation, we divide $X^-_L + X^-_R$ in Eq.(2.19) into an almost free part, $X^-_0$, and a completely non-free part, $X^-_1$, which is constructed from $f$ and $g$. Furthermore, we define $X^-_2$ as $\frac{\mu}{2} (\bar{f} \bar{g} - \bar{f} \bar{g})$ in Eq.(2.19). Here, $X^-_0$ satisfies the canonical commutation relations with $X^+$ and the momentum of $X^-$, and it does not contain the modes $A_N, B_N$ and $p^+$. Moreover, $X^-_1$ commutes with $X^+$ and the momentum of $X^-$, and it contains the modes $A_N, B_N$ and $p^+$. In this setting, $X^-$ must satisfy all the canonical commutation relations with the string coordinates and the string momentum; the important point is the canonical commutation relation $[X^- (\tau, \sigma), Z(\tau, \sigma')] = 0$, from which we can determine the free-mode representation of $X^-_1$. Thus, using $X^-_0, X^-_1, X^-_2$, the free-mode representation of $X^-$ is obtained as

$$
X^- = X^-_0 + X^-_1 + X^-_2,
$$

(3.11)

where

$$
X^-_0 = x^- + \frac{\alpha' p^-}{2} (\sigma^+ + \sigma^-) + i \sqrt{\alpha'} \sum_{N=0}^{\infty} \frac{1}{n} \left[ \hat{a}^-_n e^{-i n \sigma^+} + \alpha_n^- e^{-i n \sigma^-} \right],
$$

(3.12)

$$
X^-_1 = \mu \alpha' \sum_{N=-\infty}^{\infty} \left[ \text{sgn}(N - \mu \alpha' p^+) : A^+_N A_N : -\text{sgn}(N + \mu \alpha' p^+) : B^+_N B_N : \right] \tau
$$

$$
- \frac{i \mu \alpha'}{2} \sum_{M \neq N} \frac{1}{M - N} \left[ M + N + 2 \mu \alpha' p^+ \right. \left. \frac{A^+_M A_N e^{i(M-N)\sigma^+}}{\omega^+_M \omega^+_N} - M + N + 2 \mu \alpha' p^+ \right. \left. \frac{B^+_M B_N e^{i(M-N)\sigma^-}}{\omega^+_M \omega^+_N} \right],
$$

(3.13)

$$
X^-_2 = \frac{i \mu \alpha'}{2} \sum_{M,N=-\infty}^{\infty} \frac{1}{\omega^+_M \omega^+_N} \left[ A_M B^+_N e^{-i(M-N-2 \mu \alpha' p^+ ) \tau - i(M-N)\sigma} \right.
$$

$$
- A^+_M B_N e^{i(M-N-2 \mu \alpha' p^+ ) \tau + i(M-N)\sigma} \right].
$$

(3.14)

Here we have $\omega^+_N = \sqrt{|N \mp \mu \alpha' p^+|}$, and the notation $:\text{ }$ represents the normal ordered product, whose definition is given in the final part of this section. In the summation with $M \neq N$ in Eq.(3.13), $M$ and $N$ run from $-\infty$ to $\infty$, excluding $M = N$. Note that the terms in this summation are not
influenced by the normal ordered product. We can also write \( X^- \) in \( X^- \) in terms of \( f \) and \( g \) as

\[
X^-_1 = \frac{i\mu}{2} \left[ \int d\sigma^+ : (f \partial_+ f - \partial_+ \bar{f} f) : - \int d\sigma^- : (\bar{g} \partial_- g - \partial_- \bar{g} g) : \right] - \mu J \sigma, \tag{3.15}
\]

where

\[
J = \frac{i}{4\pi} \left[ \int_0^{2\pi} d\sigma^+ : (f \partial_+ f - \partial_+ \bar{f} f) : + \int_0^{2\pi} d\sigma^- : (\bar{g} \partial_- g - \partial_- \bar{g} g) : \right]. \tag{3.16}
\]

Here, the integrals in Eq.(3.15) are indefinite integrals, and we choose the constants of integration to be zero. In the free-mode representation, \( J \) in Eq.(3.16) is given by

\[
J = \alpha' \sum_{N=\infty}^{\infty} \left[ \text{sgn}(N - \mu \alpha' p^+) : A^\dagger_N A_N : + \text{sgn}(N + \mu \alpha' p^+) : B^\dagger_N B_N : \right]. \tag{3.17}
\]

We now explicitly present the commutation relations for all the modes, in order to demonstrate that they are perfectly free-modes:

- The nonvanishing commutation relations between the modes of \( X^+ \) and \( X^- \) are

\[
[x^+, p^-] = [x^-, p^+] = -i, \quad [\tilde{\alpha}^+_m, \tilde{\alpha}^-_n] = [\alpha^+_m, \alpha^-_n] = -m \delta_{m+n}. \tag{3.18}
\]

- The nonvanishing commutation relations between the modes of \( Z \) and \( \bar{Z} \) are

\[
[A_M, A^\dagger_N] = \text{sgn}(M - \mu \alpha' p^+) \delta_{MN}, \tag{3.19}
\]

\[
[B_M, B^\dagger_N] = \text{sgn}(M + \mu \alpha' p^+) \delta_{MN}. \tag{3.20}
\]

- The nonvanishing commutation relations between the modes of \( X^k \) are

\[
[x^k, p^l] = i \delta^{kl}, \quad [\tilde{\alpha}^k_m, \tilde{\alpha}^l_n] = [\alpha^k_m, \alpha^l_n] = m \delta^{kl} \delta_{m+n}. \tag{3.21}
\]

All the other commutators between the modes of \( X^+ \), \( X^- \), \( Z \), \( \bar{Z} \) and \( X^k \) vanish. Among the vanishing commutators, those of \( x^- \) given in Eq.(3.12) with \( A_N \) and \( B_N \) have an especially important meaning. We confirm in the next section that these commutation relations of the modes are consistent with the canonical commutation relations of string coordinates.

Let us consider the normal ordering of the modes \( A_M \) and \( B_M \). The definitions of creation and annihilation for these modes are determined by the signs of \( M \pm \mu \alpha' p^+ \) in the commutation relations (3.19) and (3.20). For example, if \( M \) is larger then \( \mu \alpha' p^+ \) \( (M > \mu \alpha' p^+) \), Eq.(3.19) is positive \( ([A_M, A^\dagger_N] > 0) \), and thus in this case we find that \( A_M \) are annihilation operators and \( A^\dagger_M \) are creation operators. Contrastingly, if \( M \) is smaller than \( \mu \alpha' p^+ \) \( (M < \mu \alpha' p^+) \), Eq.(3.19) is negative \( ([A_M, A^\dagger_N] < 0) \), and thus in this case we find that \( A^\dagger_M \) are annihilation operators and \( A_M \)
are creation operators. We can determine the definitions of creation and annihilation for $B_M$ and $B_M^\dagger$ similarly. Therefore, the normal orderings of $A_M^\dagger A_M$ and $B_M^\dagger B_M$ are found to be

$$
: A_M^\dagger A_M : = \begin{cases} 
A_M^\dagger A_M & (M > \mu \alpha p^+) \\
A_M A_M^\dagger & (M < \mu \alpha p^+) 
\end{cases} \tag{3.22}
$$

$$
: B_M^\dagger B_M : = \begin{cases} 
B_M^\dagger B_M & (M > -\mu \alpha p^+) \\
B_M B_M^\dagger & (M < -\mu \alpha p^+) 
\end{cases} \tag{3.23}
$$

Of course, the normal ordering of the modes $\tilde{\alpha}_n^\pm$, $\alpha_n^\pm$, $\tilde{\alpha}_n^k$ and $\alpha_n^k$ is exactly the same as that in the usual case of free fields. The normal ordering plays an important role in the calculation of the anomaly of the Virasoro algebra given in §5.

Finally, we comment on a free-field representation which is not identical to the free-mode representation. Although $x^-$ is a free-mode, $x^-$ does not commute with $f$ and $g$, which contain $p^+$, and therefore the commutators of $X_0^-$ with $f$ and $g$ do not vanish. Therefore $X_0^-$ is not a free field. The derivative of $X_0^-$, however, is a free field, because of the cancellation of $x^-$. A field $X_0^-$ that is not a free field should be important in the vertex operators, the physical states, and so on.

§4. Proof of the free-mode representation

In this section we prove that the free-mode representations appearing in §3 satisfy the canonical commutation relations for all the covariant string coordinates. Because we need the canonical momentum to quantize the string coordinates, we obtain the canonical momentum from the action (2.8) using $P_\mu = \frac{\partial S}{\partial (\partial_\tau X^\mu)}$, $P_Z = \frac{\partial S}{\partial (\partial_\tau \bar{Z})}$ and $P_{\bar{Z}} = \frac{\partial S}{\partial (\partial_\tau Z)}$:

$$
P_+ = -\frac{1}{2\pi \alpha'} \partial_\tau X^- + \frac{i\mu}{2} (Z \partial_\sigma \bar{Z} - \bar{Z} \partial_\sigma Z) + \mu^2 \partial_\tau X^+ \bar{Z} \bar{Z}, \tag{4.1}
$$

$$
P_- = -\frac{1}{2\pi \alpha'} \partial_\tau X^+, \quad P_k = \frac{1}{2\pi \alpha'} \partial_\tau X^k, \tag{4.2}
$$

$$
P_Z = \frac{1}{4\pi \alpha'} (\partial_\tau Z - i\mu \partial_\sigma X^+ \bar{Z}), \quad P_{\bar{Z}} = \frac{1}{4\pi \alpha'} (\partial_\tau Z + i\mu \partial_\sigma X^+ Z). \tag{4.3}
$$

Although the momentum appears complicated, it can be put into a simpler form by using the fields $X^+$, $\bar{X}^+$, $X_0^-$, $f$ and $g$ which appear in the free-mode representation discussed in §3. The field $P_+$ becomes the most simplified:

$$
P_+ = -\frac{1}{2\pi \alpha'} \partial_\tau X_0^-, \tag{4.4}
$$

$$
P_- = -\frac{1}{2\pi \alpha'} \partial_\tau X^+, \quad P_k = \frac{1}{2\pi \alpha'} \partial_\tau X^k, \tag{4.5}
$$

$$
P_Z = \frac{1}{4\pi \alpha'} e^{i\mu \bar{X}^+} (\partial_+ \bar{f} + \partial_- \bar{g}), \quad P_{\bar{Z}} = \frac{1}{4\pi \alpha'} e^{-i\mu X^+} (\partial_+ f + \partial_- g). \tag{4.6}
$$
There are no constraints on the momentum, and thus we can quantize the string coordinates using the ordinary method of canonical quantization. The canonical commutation relations are given by

\[
[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = i\delta^\mu_\nu\delta(\sigma - \sigma'),
\]

and

\[
[X^\mu(\tau, \sigma), X'^\nu(\tau, \sigma')] = 0, \quad [P_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = 0.
\]

First, let us explain almost self-evident parts of the proof.

1. Because \(X^k\) and \(P_k\) are constructed from usual free modes even in our free-mode representation, it is evident that \(X^k\) and \(P_k\) satisfy the canonical commutation relations between all string coordinates.

2. Clearly, the canonical commutators between \(Z(\tau, \sigma)\) and \(Z(\tau, \sigma')\), between \(Z(\tau, \sigma)\) and \(\bar{P}Z(\tau, \sigma')\) and between \(PZ(\tau, \sigma)\) and \(\bar{P}Z(\tau, \sigma')\) vanish, because the modes \(A_N, B_N, p^+, \tilde{\alpha}_n^+\) and \(\alpha_n^+\) commute in the free-mode representation. The same relations are satisfied in the case of the Hermitian conjugate of \(Z\) and \(\bar{P}Z\), namely \(\bar{Z}\) and \(PZ\).

3. Clearly, the canonical commutators between \(X^+\) and \(P_-\), \(Z\), \(\bar{Z}\), \(PZ\) and \(\bar{P}Z\), between \(P_-\) and \(Z\), \(\bar{Z}\), \(PZ\) and \(\bar{P}Z\), between \(X^+(\tau, \sigma)\) and \(X^+(\tau, \sigma')\) and between \(P_- (\tau, \sigma)\) and \(P_- (\tau, \sigma')\) vanish, because the modes \(A_N, B_N, p^+, \tilde{\alpha}_n^+\) and \(\alpha_n^+\) commute in the free-mode representation.

4. \(X^+\) and the fields constructed from only \(f, \bar{f}, g\) and \(\bar{g}\) commute, because the modes \(x^+, p^+, \tilde{\alpha}_n\) and \(\alpha_n\) commute with \(p^+, A_N, B_N, \tilde{A}_N\) and \(\tilde{B}_N\) in the free-mode representation. Furthermore, \(X^+\) and \(X_0^-\) commute, in analogy to usual free fields, and hence \(X^+\) and \(X_-\) commute.

5. Because \(P_+\) contains only \(\partial_\tau X_0^-\) in Eq.(4.14), it can be shown that the canonical commutation relation between \(X^+(\tau, \sigma)\) and \(P_+(\tau, \sigma')\) is satisfied by using Eq.(3.18). That between \(X^-(\tau, \sigma)\) and \(P_- (\tau, \sigma')\) is also satisfied, because only \(X_0^-\) in \(X^+\) is effective in \(\partial_\tau X^+\), which \(P_-\) contains. Moreover, as in the case of usual free fields, the commutator between \(P_+(\tau, \sigma)\) and \(P_- (\tau, \sigma')\) vanishes, and that between \(P_+(\tau, \sigma)\) and \(P_+(\tau, \sigma')\) also vanishes.

6. Because \(P_+\) does not contain \(x^-,\) all the modes in \(P_+\) and all the modes in \(X^-\) commute, and thus the canonical commutator between \(P_+\) and \(X^-\) vanishes.

We present all the important parts of the proof in next subsections. In §4.1, we prove that our free-mode representation satisfies all the important canonical commutation relations between \(Z, \bar{Z}, PZ\) and \(\bar{P}Z\). In §4.2 we prove that our free-mode representation satisfies all the canonical commutation relations between \(X^-\) and \(Z, \bar{Z}, PZ, P\bar{Z}\) and \(X^-\). In addition, we prove that all the
canonical commutation relations between \( P_+ \) and \( Z, \bar{Z}, P_Z \) and \( P_Z \) are satisfied in our free-mode representation.

4.1. Canonical commutation relations between \( Z, \bar{Z}, P_Z \) and \( P_Z \)

In this subsection, we prove the equal-time canonical commutation relations involving \( Z \). In particular, we prove the non-trivial commutation relations \([Z(\tau, \sigma), P_Z(\tau, \sigma')] = i\delta(\sigma - \sigma')\), \([Z(\tau, \sigma), \bar{Z}(\tau, \sigma')] = 0\) and \([P_Z(\tau, \sigma), P_Z(\tau, \sigma')] = 0\). The commutation relation \([\bar{Z}(\tau, \sigma), P_Z(\tau, \sigma')] = i\delta(\sigma - \sigma')\) is obtained by taking the Hermitian conjugate of \([Z(\tau, \sigma), P_Z(\tau, \sigma')] = i\delta(\sigma - \sigma')\), and therefore we need not calculate it directly. Further, we prove the equal-time commutation relations in arbitrary world-sheet time \( \tau \).

4.1.1. Proof of \([Z(\tau, \sigma), P_Z(\tau, \sigma')] = i\delta(\sigma - \sigma')\)

Let us calculate the commutator \([Z(\tau, \sigma), P_Z(\tau, \sigma')]\). We have

\[
[Z(\tau, \sigma), P_Z(\tau, \sigma')] = \frac{1}{4\pi\alpha'} e^{-i\mu [\bar{X}^{+}(\sigma) - \bar{X}^{+}(\sigma')] - i\mu} \left\{ [f(\sigma^+), \partial_+\bar{f}(\sigma'^+) + [g(\sigma^-), \partial_-\bar{g}(\sigma'^-)] \right\},
\]

where \( \sigma'^\pm = \sigma \pm \sigma' \) and \( \partial_\pm = \partial_\tau \pm \partial_{\sigma'} \). In §4.1, we use \( \bar{X}^{+}(\tau, \sigma) \) to represent \( \bar{X}^{+}(\tau, \sigma) \). The commutation relations \([f(\sigma^+), \partial_+\bar{f}(\sigma'^+)\) and \([g(\sigma^-), \partial_-\bar{g}(\sigma'^-)\]

\[
[f(\sigma^+), \partial_+\bar{f}(\sigma'^+)] = i\alpha' \sum_{M,N=-\infty}^{\infty} \frac{N - \mu \alpha'^+}{\sqrt{|M - \mu \alpha'^+||N - \mu \alpha'^+|}} [A_M, A_N^\dagger] e^{-i(M - \mu \alpha'^+)\sigma^+ + i(N - \mu \alpha'^+)\sigma'^+},
\]

\[
[g(\sigma^-), \partial_-\bar{g}(\sigma'^-) = i\alpha' \sum_{M,N=-\infty}^{\infty} \frac{N + \mu \alpha'^+}{\sqrt{|M + \mu \alpha'^+||N + \mu \alpha'^+|}} [B_M, B_N^\dagger] e^{-i(M + \mu \alpha'^+)\sigma^- + i(N + \mu \alpha'^+)\sigma'^-}.
\]

Using the commutation relation \([3.19]\), the commutation relation \([4.10]\) is reduced to

\[
[f(\sigma^+), \partial_+\bar{f}(\sigma'^+)] = i\alpha' \sum_{M=-\infty}^{\infty} e^{-i(M - \mu \alpha'^+)\sigma^- - i(M + \mu \alpha'^+)\sigma'^+}.
\]

Similarly, using the commutation relation \([3.20]\), the other commutation relation \([4.11]\) is reduced to

\[
[g(\sigma^-), \partial_-\bar{g}(\sigma'^-)] = i\alpha' \sum_{M=-\infty}^{\infty} e^{i(M + \mu \alpha'^+)\sigma^- - i(M - \mu \alpha'^+)\sigma'^+}.
\]

\[
= i\alpha' \sum_{M=-\infty}^{\infty} e^{-i(M - \mu \alpha'^+)\sigma^- - i(M + \mu \alpha'^+)\sigma'^+}.
\]

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In the last line of this calculation, we have changed the notation $M$ to $-M$. Here, we note that we cannot use the normal formulas for the delta function given in Eqs. (4.12) and (4.13) because of the twisted factor $e^{i\mu\alpha'p^+(\sigma-\sigma')}$, which is not periodic, although the delta function is obtained through the summations in Eqs. (4.12) and (4.13), respectively. However, multiplying $e^{i\mu\alpha'p^+(\sigma-\sigma')}$ by $e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')]2}$ in the commutation relation (4.19), we can use the delta function, because the factor $e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')-\alpha'p^+(\sigma-\sigma')]}$ satisfies the periodic condition:

$$[Z(\tau, \sigma), P_Z(\tau, \sigma')] = \frac{i}{2\pi} \sum_{M=-\infty}^{\infty} e^{-iM(\sigma-\sigma')} e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')-\alpha'p^+(\sigma-\sigma')]} = i\delta(\sigma - \sigma') e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')-\alpha'p^+(\sigma-\sigma')].}$$ (4.14)

Using the property of the delta function, the factor $e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')-\alpha'p^+(\sigma-\sigma')]}$ becomes 1. Therefore, the commutation relation becomes $[Z(\tau, \sigma), P_Z(\tau, \sigma')] = i\delta(\sigma - \sigma')$.

### 4.1.2. Proof of $[Z(\tau, \sigma), \tilde{Z}(\tau, \sigma')] = 0$

Let us calculate the commutator $[Z(\tau, \sigma), \tilde{Z}(\tau, \sigma')]$. It is given by

$$[Z(\tau, \sigma), \tilde{Z}(\tau, \sigma')] = e^{-i\mu[\hat{X}^+(\sigma)-\hat{X}^+(\sigma')]} \left\{ [f(\sigma^+), \tilde{f}(\sigma'^+)] + [g(\sigma^-), \tilde{g}(\sigma'^-)] \right\}. \quad (4.15)$$

Using the commutation relation (3.19), we can calculate $[f(\sigma^+), \tilde{f}(\sigma'^+)]$:

$$[f(\sigma^+), \tilde{f}(\sigma'^+)] = \alpha' \sum_{M,N=-\infty}^{\infty} \frac{[A_M, A^+_N]}{|M - \mu \alpha' p^+| |N - \mu \alpha' p^+|} e^{-i(M-\mu \alpha' p^+)\sigma^+ + i(N-\mu \alpha' p^+)\sigma'^+}$$

$$= \alpha' \sum_{M=-\infty}^{\infty} \frac{1}{M - \mu \alpha' p^+} e^{-i(M-\mu \alpha' p^+)\sigma^+} \left\{ [f(\sigma^+), \tilde{f}(\sigma'^+)] \right\}. \quad (4.16)$$

Similarly, using the commutation relation (3.20), we can calculate $[g(\sigma^-), \tilde{g}(\sigma'^-)]$:

$$[g(\sigma^-), \tilde{g}(\sigma'^-)] = \alpha' \sum_{M=-\infty}^{\infty} \frac{1}{M + \mu \alpha' p^+} e^{i(M+\mu \alpha' p^+)\sigma^-} \left\{ [g(\sigma^-), \tilde{g}(\sigma'^-)] \right\}$$

$$= -\alpha' \sum_{M=-\infty}^{\infty} \frac{1}{M - \mu \alpha' p^+} e^{-i(M-\mu \alpha' p^+)\sigma^-} \left\{ [f(\sigma^+), \tilde{f}(\sigma'^+)] \right\}. \quad (4.17)$$

In the second line of this calculation, we have changed the notation $M$ to $-M$. Thus, because of the relation $[f(\sigma^+), \tilde{f}(\sigma'^+)] + [g(\sigma^-), \tilde{g}(\sigma'^-)] = 0$, the commutation relation $[Z(\tau, \sigma), \tilde{Z}(\tau, \sigma')]$ becomes zero.
4.1.3. Proof of $[P_Z(\tau, \sigma), P_{\bar{Z}}(\tau, \sigma')] = 0$

Let us calculate the commutator $[P_Z(\tau, \sigma), P_{\bar{Z}}(\tau, \sigma')]$. It takes the form

$$[P_Z(\tau, \sigma), P_{\bar{Z}}(\tau, \sigma')] = \frac{1}{(4\pi \alpha')^2} e^{-i\mu[\bar{X}^+(\sigma) - \bar{X}^+(\sigma')]} \times \left\{ [\partial_+ \tilde{f}(\sigma^+), \partial_+ f(\sigma'^+)] + [\partial_- g(\sigma^-), \partial_- g(\sigma'^-)] \right\}. \quad (4.18)$$

By a calculation similar to that in Eq. (4.17), we obtain the following relation:

$$[\partial_- g(\sigma^-), \partial_- g(\sigma'^-)] = -[\partial_+ \tilde{f}(\sigma^+), \partial_+ f(\sigma'^+)]. \quad (4.19)$$

Thus, because of the relation (4.19), the commutation relation $[P_Z(\tau, \sigma), P_{\bar{Z}}(\tau, \sigma')]$ becomes zero.

Finally, from the results of §§4.1.1, 4.1.2 and 4.1.3, we find that the canonical quantization of $Z$ is consistent.

4.2. Canonical commutation relations of $X^-$ and $P_+$ with $Z$, $\bar{Z}$, $P_Z$, $P_{\bar{Z}}$ and $X^-$

In this subsection, we prove the equal-time canonical commutation relations of $X^-$ and $P_+$ with $Z$, $\bar{Z}$, $P_Z$, $P_{\bar{Z}}$, $X^-$ and $P_+$. This proof is the most important one in this paper, and it demonstrates the consistency of the free-mode representations. The commutation relations $[X^-(\tau, \sigma), Z(\tau, \sigma')] = 0$, $[X^-(\tau, \sigma), P_Z(\tau, \sigma')] = 0$ and $[X^-(\tau, \sigma), X^-(\tau, \sigma')] = 0$, and their Hermitian conjugates, are non-trivial because $X^-$ manifestly contains $f$ and $g$. Therefore, we must prove these relations. Here, it is useful to prove the commutation relations in the case $\tau = 0$ by applying the Heisenberg formalism, because there is reason to believe that $X^-$ takes an elegant form in this case.

At the end of this subsection, although they are trivial, we prove the relations $[P_+(\tau, \sigma), Z(\tau, \sigma')] = 0$ and $[P_+(\tau, \sigma), P_Z(\tau, \sigma')] = 0$ and their Hermitian conjugates.

4.2.1. Preparation for the proof

Before beginning the proof of the commutation relations involving $X^-$ in the free-mode representations, we define new notation for the modes in order to avoid complication in the coefficients, such as the sign function. We also present some useful formulas.

First, let us define new notation for the modes:

$$\hat{A}_N \equiv \frac{1}{\sqrt{|N - \mu \alpha' p^+|}} A_N, \quad (4.20)$$

$$\hat{B}_N \equiv \frac{1}{\sqrt{|N + \mu \alpha' p^+|}} B_N. \quad (4.21)$$
The new modes $\hat{A}_N$ and $\hat{B}_N$ interact with $x^-$, because they contain $p^+$. The commutation relations between these modes and $x^-$ are
\begin{align}
[x^-, \hat{A}_N] &= -\frac{i\mu\alpha'}{2} N - \mu\alpha' p^+ \hat{A}_N, \tag{4.22}
[x^-, \hat{B}_N] &= \frac{i\mu\alpha'}{2} N + \mu\alpha' p^+ \hat{B}_N. \tag{4.23}
\end{align}

Next, let us construct the string coordinates in the case $\tau = 0$ using the Heisenberg formalism. We can describe the time evolution using the Hamiltonian of the system, which is defined by
\begin{align}
H = &\frac{\alpha'}{2} (-2p^+ p^- + p^k p^k) + \frac{1}{2} \sum_{n \neq 0} \left[ -2(\hat{\alpha}_n^+ \hat{\alpha}_n^-) + : \alpha_n^+ \alpha_n^- : + : \alpha_n^k \alpha_n^k : + : \alpha_n \alpha_n : \right] \\
&+ \sum_{N = -\infty}^{\infty} \left[ (N - \mu\alpha' p^+)^2 : \hat{A}_N^\dagger \hat{A}_N : + (N + \mu\alpha' p^+)^2 : \hat{B}_N^\dagger \hat{B}_N : \right]. \tag{4.24}
\end{align}

Thus, we obtain the relations
\begin{align}
Z(\tau, \sigma) &= e^{iH\tau} Z(\sigma) e^{-iH\tau}, \quad P_Z(\tau, \sigma) = e^{iH\tau} P_Z(\sigma) e^{-iH\tau}, \\
X^-(\tau, \sigma) &= e^{iH\tau} X^-(\sigma) e^{-iH\tau}, \tag{4.25}
\end{align}
where we define $Z(\sigma) \equiv Z(\tau = 0, \sigma)$, $P_Z(\sigma) \equiv P_Z(\tau = 0, \sigma)$ and $X^-(\sigma) \equiv X^-(\tau = 0, \sigma)$. In addition, we have the following example of the commutation relations:
\begin{align}
[X^-(\tau, \sigma), Z(\tau, \sigma')] = e^{iH\tau} [X^-(\sigma), Z(\sigma')] e^{-iH\tau}. \tag{4.26}
\end{align}
Therefore we have only to prove the relations $[X^-(\sigma), Z(\sigma')] = 0$, $[X^-(\sigma), P_Z(\sigma')] = 0$ and $[X^-(\sigma), X^-(\sigma')] = 0$. In the case $\tau = 0$, $Z(\sigma)$ and $P_Z(\sigma)$ are given by
\begin{align}
Z(\sigma) &= e^{-i\mu \hat{X}^+(\sigma)} [f(\sigma) + g(-\sigma)], \tag{4.27}
P_Z(\sigma) &= e^{-i\mu \hat{X}^+(\sigma)} [\partial_\sigma f(\sigma) - \partial_\sigma g(-\sigma)], \tag{4.28}
\end{align}
where
\begin{align}
f(\sigma) &= \sqrt{\alpha'} \sum_{N = -\infty}^{\infty} \hat{A}_N e^{-i(N - \mu\alpha' p^+)^\sigma}, \tag{4.29}
g(-\sigma) &= \sqrt{\alpha'} \sum_{N = -\infty}^{\infty} \hat{B}_N e^{i(N + \mu\alpha' p^+)^\sigma}. \tag{4.30}
\end{align}
Here, the minus sign on $g(-\sigma)$ is due to the origin of $\sigma^-$. In the case $\tau = 0$, $X^-(\sigma)$ takes the elegant form
\begin{align}
X^-(\sigma) &= X_0^-(\sigma) - \frac{i\mu\alpha'}{2} [U + V(\sigma)], \tag{4.31}
\end{align}
Moreover, it is useful to define the operator
\[ \hat{A}_N \] by
\[ \hat{B}_N \] and their Hermitian conjugates:
\[ P_{MN} = \left( \hat{A}_M^\dagger - \hat{\beta}_M \right) \left( \hat{A}_N + \hat{\beta}_N \right), \quad Q_{MN} = \left( \hat{A}_M + \hat{\beta}_M \right) \left( \hat{A}_N - \hat{\beta}_N \right). \] (4.34)

Moreover, it is useful to define the operator
\[ W = x^- - \frac{i\mu\alpha'}{2} U. \] (4.35)

The operator \( W \) plays an important role in Eq. (4.44). In the calculation of the commutation relations between \( X_0^- \) and \( f \) and between \( X_0^- \) and \( g \), only \( x^- \) in \( X_0^- \) survives. Adding \( x^- \) to \( U \), we can divide \( X^- (\sigma) \) into a part with \( \sigma \) dependence and the other part, and thus it is useful to define \( W \).

We now present several useful commutation relations that we use in the following subsection. First, the commutation relations between \( W \) and the others are
\[ [W, p^+] = -i, \] (4.36)
\[ [W, \hat{A}_N] = -\frac{i\mu\alpha'}{2} \frac{1}{N - \mu\alpha' p^+} \left( \hat{A}_N - \hat{\beta}_N \right), \quad [W, \hat{B}_N] = -\frac{i\mu\alpha'}{2} \frac{1}{N + \mu\alpha' p^+} \left( \hat{A}_N - \hat{\beta}_N \right), \] (4.37)
\[ [W, P_{MN}] = -\frac{i\mu\alpha'}{M - \mu\alpha' p^+} P_{MN}, \quad [W, Q_{MN}] = -\frac{i\mu\alpha'}{N - \mu\alpha' p^+} Q_{MN}. \] (4.38)

Second, the commutation relations between \( P_{MN} \) and \( \hat{A}_K, P_{MN} \) and \( \hat{B}_K, Q_{MN} \) and \( \hat{A}_K \) and \( Q_{MN} \) and \( \hat{B}_K \) are
\[ [P_{MN}, \hat{A}_K] = -\frac{\delta_{MK}}{M - \mu\alpha' p^+} \left( \hat{A}_N + \hat{\beta}_N \right), \quad [Q_{MN}, \hat{A}_K] = -\frac{\delta_{MK}}{M - \mu\alpha' p^+} \left( \hat{A}_N - \hat{\beta}_N \right), \] (4.39)
\[ [P_{MN}, \hat{B}_K] = \frac{\delta_{K,-M}}{K + \mu\alpha' p^+} \left( \hat{A}_N + \hat{\beta}_N \right), \quad [Q_{MN}, \hat{B}_K] = -\frac{\delta_{K,-M}}{K + \mu\alpha' p^+} \left( \hat{A}_N - \hat{\beta}_N \right). \] (4.40)

Finally, the commutation relations between \( P_{MN} \) and \( Q_{MN} \) are
\[ [P_{MN}, Q_{KL}] = 0, \] (4.41)
\[ [P_{MN}, P_{KL}] = 2 \left( \frac{\delta_{NK}}{N - \mu\alpha' p^+} P_{ML} - \frac{\delta_{ML}}{M - \mu\alpha' p^+} P_{KN} \right), \] (4.42)
\[ [Q_{MN}, Q_{KL}] = 2 \left( \frac{\delta_{NK}}{N - \mu\alpha' p^+} Q_{ML} - \frac{\delta_{ML}}{M - \mu\alpha' p^+} Q_{KN} \right). \] (4.43)

Below we prove the canonical commutation relations using these useful relations.
4.2.2. Proof of $[X^-(\sigma), Z(\sigma')] = 0$

Let us calculate the commutation relation $[X^-(\sigma), Z(\sigma')]$. We first note that $X_0^-(\sigma)$ and $\tilde{X}^+(\sigma')$ do not commute. Therefore, $X_0^-(\sigma)$ and $\exp(-i\mu\tilde{X}^+(\sigma'))$ do not commute. Calculating this commutator, the following term survives:

$$[X^-(\sigma), Z(\sigma')] = \exp(-i\mu\tilde{X}^+(\sigma')) \left\{ -i\mu[X_0^-(\sigma), \tilde{X}^+(\sigma')] (f(\sigma') + g(-\sigma')) + [W, f(\sigma') + g(-\sigma')] - \frac{i\mu\alpha'}{2} [V(\sigma), f(\sigma') + g(-\sigma')] \right\}, \quad (4.44)$$

where we use $W$ defined above. Using the fact that the modes $\tilde{\alpha}_n^-$ and $\alpha_n^-$ of $X^-$ commute with $f$ and $g$, we can construct the operator $W$. Here, the commutator between $X_0^-(\sigma)$ and $\tilde{X}^+(\sigma')$ is

$$[X_0^-(\sigma), \tilde{X}^+(\sigma')] = -i\alpha'\sigma' + \alpha' \sum_{n \neq 0} \frac{1}{n} e^{in(\sigma-\sigma')} \quad (4.45)$$

First, using the relation (4.37), we can calculate the commutation relation $[W, f(\sigma') + g(-\sigma')]$. Although this commutator is simply given by $f + g$ multiplied by the factor $\mu\alpha'\sigma'$, $x^-$ is very important in $W$:

$$[W, f(\sigma') + g(-\sigma')] = \mu\alpha'\sigma' \left[ f(\sigma') + g(-\sigma') \right]. \quad (4.46)$$

Second, using the commutation relations given in Eqs. (4.39) and (4.40), we can calculate the commutators $[V(\sigma), f(\sigma')]$ and $[V(\sigma), g(-\sigma')]$, and we obtain

$$[V(\sigma), f(\sigma')] = -\sqrt{\alpha'} \sum_{M \neq N} \frac{1}{M - N} \left\{ (\hat{A}_N + \hat{B}_{-N}) + \frac{N - \mu\alpha'p^+}{M - \mu\alpha'p^+} (\hat{A}_N - \hat{B}_{-N}) \right\} \times e^{i(M-N)(\sigma-\sigma')-i(N-\mu\alpha'p^+)(\sigma')}, \quad (4.47)$$

and

$$[V(\sigma), g(-\sigma')] = -\sqrt{\alpha'} \sum_{M \neq N} \frac{1}{M - N} \left\{ (\hat{A}_N + \hat{B}_{-N}) - \frac{N - \mu\alpha'p^+}{M - \mu\alpha'p^+} (\hat{A}_N - \hat{B}_{-N}) \right\} \times e^{i(M-N)(\sigma-\sigma')-i(N-\mu\alpha'p^+)(\sigma')}. \quad (4.48)$$

Adding Eq.(4.47) to Eq.(4.48), the factor $\frac{N - \mu\alpha'p^+}{M - \mu\alpha'p^+} (\hat{A}_N - \hat{B}_{-N})$ vanishes. Therefore $[V(\sigma), f(\sigma') + g(-\sigma')]$ becomes

$$[V(\sigma), f(\sigma') + g(-\sigma')] = -2\sqrt{\alpha'} \sum_{M \neq N} \frac{1}{M - N} (\hat{A}_N + \hat{B}_{-N}) \times e^{i(M-N)(\sigma-\sigma')-i(N-\mu\alpha'p^+)(\sigma')} \quad (4.49)$$

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In addition, setting $M - N = n$ ($n \neq 0$), we obtain

$$[V(\sigma), f(\sigma') + g(-\sigma')] = -2 \sum_{n \neq 0} \frac{1}{n} e^{in(\sigma - \sigma')} \cdot \sqrt{\alpha'} \sum_{N=-\infty}^{\infty} \left( \hat{A}_N + \hat{B}_N \right) e^{-i(N-\mu\sigma'\pi)\sigma'}$$

$$= -2 \sum_{n \neq 0} \frac{1}{n} e^{in(\sigma - \sigma') \left[ f(\sigma') + g(-\sigma') \right]}. \quad (4.50)$$

Finally, substituting Eqs. (4.45), (4.46) and (4.50) into Eq. (4.44), we find that the commutator $[X^{-}(\sigma), Z(\sigma')]$ is zero.

4.2.3. Proof of $[X^{-}(\sigma), P_{\tilde{Z}}(\sigma')] = 0$

Let us calculate the commutator $[X^{-}(\sigma), P_{\tilde{Z}}(\sigma')]$. We have

$$[X^{-}(\sigma), P_{\tilde{Z}}(\sigma')] = e^{-i\mu \tilde{X}^{+(\sigma')}} \left\{ -i\mu [X^{-0}(\sigma), \tilde{X}^{+(\sigma')}] \left( \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(\sigma') \right) + [W, \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma')] - \frac{i\mu\alpha'}{2} [V(\sigma), \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma')] \right\}. \quad (4.51)$$

This can be calculated in a manner similar to that used in the proof of $[X^{-}(\sigma), Z(\sigma')] = 0$. First, the commutation relation of $W$ is similar to (4.46):

$$[W, \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma')] = \mu \alpha' [\partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma')] \quad (4.52)$$

The commutators $[V(\sigma), \partial_{\sigma'} f(\sigma')]$ and $[V(\sigma), \partial_{\sigma'} g(-\sigma')]$ are obtained by taking partial derivatives of Eqs. (4.47) and (4.48):

$$[V(\sigma), \partial_{\sigma'} f(\sigma')] = \partial_{\sigma'} [V(\sigma), f(\sigma')], \quad (4.53)$$

$$[V(\sigma), \partial_{\sigma'} g(-\sigma')] = \partial_{\sigma'} [V(\sigma), g(-\sigma')]. \quad (4.54)$$

Thus, subtracting Eq. (4.51) from Eq. (4.53) and replacing $M - N$ by $n$, we obtain the following relation:

$$[V(\sigma), \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma')] = -2 \sum_{n \neq 0} \frac{1}{n} e^{in(\sigma - \sigma') \left[ \partial_{\sigma'} f(\sigma') - \partial_{\sigma'} g(-\sigma') \right]} \quad (4.55)$$

Thus, similarly to the case of $[X^{-}(\sigma), Z(\sigma')]$, the commutation relation $[X^{-}(\sigma), P_{\tilde{Z}}(\sigma')]$ is found to be zero.

4.2.4. Proof of $[X^{-}(\sigma), X^{-}(\sigma')] = 0$

Let us calculate the commutation relation $[X^{-}(\tau, \sigma), X^{-}(\tau, \sigma')]$. Here, it is useful to use the method of the Fourier series; because $X^{-}(\sigma)$ is a periodic function under the shift $\sigma \to \sigma + 2\pi$, we
can expand $X^-(\sigma)$ into a Fourier series, $X^-(\sigma) = \sum_n \hat{X}_n e^{i n \sigma}$. The Fourier coefficients are

$$\hat{X}_n = \int_0^{2\pi} \frac{d\sigma}{2\pi} e^{-i n \sigma} X^-(\sigma).$$  \hspace{1cm} (4.56)

Therefore, the Fourier coefficients of the commutator are the following:

$$[\hat{X}_n, \hat{X}_m] = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_0^{2\pi} \frac{d\sigma'}{2\pi} e^{-i m \sigma} e^{-i n \sigma'} [X^-(\sigma), X^-(\sigma')].$$  \hspace{1cm} (4.57)

We have only to prove $[\hat{X}_m, \hat{X}_n] = 0$ below. Calculating the commutator $[X^-(\sigma), X^-(\sigma')]$, the following term survives:

$$[X^-(\sigma), X^-(\sigma')] = [W - \frac{i \mu \alpha'}{2} V(\sigma), W - \frac{i \mu \alpha'}{2} V(\sigma')]$$

$$= \frac{i \mu \alpha'}{2} [W, V(\sigma) - V(\sigma')] - \frac{\mu^2 \alpha'^2}{4} [V(\sigma), V(\sigma')].$$  \hspace{1cm} (4.58)

However, using the relations (4.36), (4.37) and (4.38), the first term of Eq. (4.58) is found to be zero:

$$[W, V(\sigma)] = \sum_{M \neq N} \frac{1}{M - N} \left[ i \mu \alpha' (P_{MN} + Q_{MN}) \right.$$  

$$\left. + (M - \mu \alpha' p^+)[W, P_{MN}] - (N - \mu \alpha' p^+)[W, Q_{MN}] \right] e^{i (M - N) \sigma}$$

$$= 0. \hspace{1cm} (4.59)$$

Next, using the relations (4.41), (4.42) and (4.43), we can calculate the second $[V(\sigma), V(\sigma')]$ term in Eq. (4.58):

$$[V(\sigma), V(\sigma')] = \sum_{M \neq N} \sum_{K \neq L} \frac{1}{M - N} \frac{1}{K - L} e^{i (M - N) \sigma + i (K - L) \sigma'}$$

$$\times \left[ (M - \mu \alpha' p^+) \delta_{NK} P_{ML} - (K - \mu \alpha' p^+) \delta_{ML} P_{KN} \right.$$  

$$\left. + (M - \mu \alpha' p^+) \delta_{NK} Q_{ML} - (K - \mu \alpha' p^+) \delta_{ML} Q_{KN} \right].$$  \hspace{1cm} (4.60)

It seems that the commutator $[X^-(\sigma), X^-(\sigma')]$ is not zero. However, we can prove that this Fourier coefficient is zero. First, we have

$$[\hat{X}_m, \hat{X}_n] = -\frac{\mu^2 \alpha'^2}{4} \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_0^{2\pi} \frac{d\sigma'}{2\pi} e^{-i m \sigma} e^{-i n \sigma'} [V(\sigma), V(\sigma')]$$

$$= -\frac{\mu^2 \alpha'^2}{2} \sum_{M \neq N} \sum_{K \neq L} \frac{1}{M - N} \frac{1}{K - L} \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_0^{2\pi} \frac{d\sigma'}{2\pi} e^{i (M - N - m) \sigma + i (K - L - n) \sigma'}$$

$$\times \left[ (M - \mu \alpha' p^+) \delta_{NK} P_{ML} - (K - \mu \alpha' p^+) \delta_{ML} P_{KN} \right.$$  

$$\left. + (M - \mu \alpha' p^+) \delta_{NK} Q_{ML} - (K - \mu \alpha' p^+) \delta_{ML} Q_{KN} \right].$$  \hspace{1cm} (4.61)
Calculating this integral, the Kronecker delta appears in Eq. (4.61). Furthermore, carrying out the summation over $M$ and $K$, Eq. (4.61) can be put into a simpler form, from which it can be seen to vanish:

$$\left[ \hat{X}^{-n}, \hat{X}^{-m} \right] = -\frac{\mu^2 \alpha'^2}{2 mn} \left[ \sum_{L=-\infty}^{\infty} \left( L + m + n - \mu \alpha' p^+ \right) \left( P_{L+m+n,L} + Q_{L+m+n,L} \right) \right. \\
\left. - \sum_{N=-\infty}^{\infty} \left( N + m + n - \mu \alpha' p^+ \right) \left( P_{N+m+n,N} + Q_{N+m+n,N} \right) \right] = 0.$$  

(4.62)

Therefore the Fourier coefficient of $[X^-(\sigma), X^-(\sigma')]$ is zero. This is consistent with the canonical quantization.

4.2.5. Proof of $[P_+ (\tau, \sigma), Z(\tau, \sigma')] = 0$ and $[P_+ (\tau, \sigma), P_Z (\tau, \sigma')] = 0$

It is necessary to note that the momentum $P_+$ contains $\partial_\tau X_0^-$, and that $Z$ and $P_Z$ contain $\tilde{X}^+$, in order to prove the commutation relations $[P_+ (\tau, \sigma), Z(\tau, \sigma')] = 0$ and $[P_+ (\tau, \sigma), P_Z (\tau, \sigma')] = 0$. Although the modes of $\partial_\tau X_0^-$ commute with the modes of $f$ and $g$, the modes of $\partial_\tau X_0^-$ do not commute with the modes of $\tilde{X}^+$, and thus the commutation relation becomes

$$[P_+ (\tau, \sigma), Z(\tau, \sigma')] = -\frac{1}{2\pi \alpha'} [\partial_\tau X_0^- (\tau, \sigma), \tilde{X}^+ (\tau, \sigma')] Z(\tau, \sigma').$$

(4.63)

Therefore we have only to prove $[\partial_\tau X_0^- (\tau, \sigma), \tilde{X}^+ (\tau, \sigma')] = 0$. The commutation relation becomes

$$[\partial_\tau X_0^- (\tau, \sigma), \tilde{X}^+ (\tau, \sigma')] = i\frac{\alpha'}{2} \sum_{n \neq 0} \left[ e^{in(\sigma-\sigma')} - e^{-in(\sigma-\sigma')} \right].$$

(4.64)

Replacing $n$ with $-n$ in the first term on the right-hand side of this equation, the commutator is found to be zero, and hence the commutator $[P_+ (\tau, \sigma), Z(\tau, \sigma')]$ is zero. Using the same type of calculation, we can also prove the relation $[P_+ (\tau, \sigma), P_Z (\tau, \sigma')] = 0$. In addition, taking the Hermitian conjugates of these commutation relations, we can also prove $[P_+ (\tau, \sigma), \bar{Z}(\tau, \sigma')] = 0$ and $[P_+ (\tau, \sigma), \bar{P}_Z (\tau, \sigma')] = 0$.

We have thus completed the proofs of all the equal-time canonical commutation relations in the free-mode representations.
§5. The energy-momentum tensor, Virasoro algebra and Virasoro anomaly

In this section we define the energy-momentum tensors and the Virasoro operators using the normal procedure. Moreover, we exactly calculate the commutators between the Virasoro operators and obtain the Virasoro anomaly.

First, the variation of the action with respect to the world-sheet metric $g^{ab}$ defines the energy-momentum tensor. Because the term containing $B_{\mu \nu}$ in the action (2.1) does not have the world-sheet metric, this term does not influence the energy-momentum tensor. After the variation, we fix the covariant gauge as $g^{ab} = \eta^{ab}$. The energy-momentum tensor of the string coordinates, $T^{X}_{ab}$, and that of the ghosts, $T^{gh}_{ab}$, generally take the following forms:

\begin{align}
T^{X}_{ab} &= \frac{1}{\alpha'} G_{\mu \nu} \left[ \partial_{a} X^{\mu} \partial_{b} X^{\nu} - \frac{1}{2} \eta_{ab} \partial_{\rho} X^{\mu} \partial_{\rho} X^{\nu} \right], \\
T^{gh}_{ab} &= 2i \left[ b_{ac} \partial_{b} c^{c} + b_{ca} \partial_{b} c^{c} + \frac{1}{2} \eta_{ab} b_{cd} \partial_{c} c^{d} - \frac{1}{2} \eta_{ab} b_{cd} \partial_{c} c^{d} \right].
\end{align}

Using the world-sheet light-cone coordinates, the energy-momentum tensors become simpler. Moreover, substituting the solutions for the string coordinates, (3.1), (3.2), (3.5), (3.8) and (3.11), and those for the ghosts, (2.21) and (2.22), into these energy-momentum tensors, they become

\begin{align}
T^{X}_{++} &= \frac{1}{\alpha'} \left[ -2 : \partial_{+} X^{+} \partial_{+} X_{0}^{-} + : \partial_{+} f \partial_{+} f : + \mu \partial_{+} X^{+} J^{+} + \partial_{+} X^{k} \partial_{+} X^{k} : \right], \\
T^{X}_{--} &= \frac{1}{\alpha'} \left[ -2 : \partial_{-} X^{+} \partial_{-} X_{0}^{-} + : \partial_{-} g \partial_{-} g : - \mu \partial_{+} X^{+} J^{+} + \partial_{-} X^{k} \partial_{-} X^{k} : \right], \\
T^{gh}_{++} &= 2i \left[ : b_{+} \partial_{+} c^{c} + : \partial_{+} b_{+} c^{c} : \right].
\end{align}

We define the total energy-momentum tensor as $T_{\pm \pm} \equiv T^{X}_{\pm \pm} + T^{gh}_{\pm \pm}$. Although the interaction appears in the term $\partial_{+} X^{+} J$, we can calculate the anomaly in almost the same manner as the free fields. When we calculate the anomaly, we divide the fields into those which consist of creation operators and those which consist of annihilation operators. Here we assume $0 < \mu \alpha' p^{+} < 1$.

Next, we define the Virasoro operators, which are the Fourier coefficients of the energy-momentum tensors:

\begin{align}
\hat{L}^{X}_{n} &= \int_{0}^{2\pi} \frac{d\sigma}{2\pi} e^{i n \sigma} T^{X}_{++}, \\
\bar{L}^{X}_{n} &= \int_{0}^{2\pi} \frac{d\sigma}{2\pi} e^{-i n \sigma} T^{X}_{--}, \\
\hat{L}^{gh}_{n} &= \int_{0}^{2\pi} \frac{d\sigma}{2\pi} e^{i n \sigma} T^{gh}_{++}, \\
\bar{L}^{gh}_{n} &= \int_{0}^{2\pi} \frac{d\sigma}{2\pi} e^{-i n \sigma} T^{gh}_{--}.
\end{align}

The Virasoro operator $\hat{L}^{X}_{n}$ ($L^{X}_{n}$) can be divided into $\hat{L}^{(+k)}_{n}$ ($L^{(+k)}_{n}$), which is a part of $X^{+}$, $X_{0}^{-}$ and $X^{k}$, and $\bar{L}^{f}_{n}$ ($\bar{L}^{g}_{n}$), which is a part of $f$ ($g$), as $\hat{L}^{X}_{n} = \hat{L}^{(+k)}_{n} + \hat{L}^{f}_{n}$, $\bar{L}^{X}_{n} = L^{(+k)}_{n} + L^{g}_{n}$. In the
We must pay attention to the convergence of the infinite sum; the anomalies are

\[ \tilde{\mathcal{L}}_{n}^{(+ - k)} = \frac{1}{2} \sum_{m, n = -\infty}^{\infty} \left[ -2 : \tilde{\alpha}_{n-m}^{+} \tilde{\alpha}_{m}^{-} : + : \tilde{\alpha}_{n-m}^{k} \tilde{\alpha}_{m}^{k} : \right], \quad (5.8) \]

\[ \tilde{L}_{n}^{(+ - k)} = \frac{1}{2} \sum_{m, n = -\infty}^{\infty} \left[ -2 : \alpha_{n-m}^{+} \alpha_{m}^{-} : + : \alpha_{n-m}^{k} \alpha_{m}^{k} : \right], \quad (5.9) \]

\[ \tilde{\mathcal{L}}_{n}^{f} = \frac{\mu}{\sqrt{2} \alpha'} \tilde{\alpha}_{n}^{+} J + \sum_{N = -\infty}^{\infty} (N - n - \mu \alpha' p^+) (N - \mu \alpha' p^+) : \tilde{A}_{N-n}^{\dagger} \tilde{A}_{N} : , \quad (5.10) \]

\[ \tilde{L}_{n}^{g} = -\frac{\mu}{\sqrt{2} \alpha'} \alpha_{n}^{+} J + \sum_{N = -\infty}^{\infty} (N - n + \mu \alpha' p^+) (N + \mu \alpha' p^+) : \tilde{B}_{N-n}^{\dagger} \tilde{B}_{N} : , \quad (5.11) \]

where we define \( \tilde{\alpha}_{0}^{\pm} = \alpha_{0}^{\pm} = \sqrt{\frac{\alpha'}{2}} p^{\pm} \) and \( \tilde{\alpha}_{0}^{k} = \alpha_{0}^{k} = \sqrt{\frac{\alpha'}{2}} p^{k} \). Furthermore, the Virasoro operators of ghosts are given by

\[ \tilde{\mathcal{L}}_{n}^{gh} = \sum_{m = -\infty}^{\infty} (n - m) : \tilde{b}_{m+n} \tilde{c}_{m} : , \quad \tilde{L}_{n}^{gh} = \sum_{m = -\infty}^{\infty} (n - m) : b_{m+n} c_{m} : . \quad (5.12) \]

Calculating the commutators between the Virasoro operators \( \tilde{\mathcal{L}}_{n}^{f} \) and \( \tilde{L}_{n}^{g} \), we obtain

\[ [\tilde{\mathcal{L}}_{m}^{f}, \tilde{\mathcal{L}}_{n}^{f}] = (m - n) \tilde{\mathcal{L}}_{m+n}^{f} + \tilde{\mathcal{A}}^{f}(m) \delta_{m+n, 0}, \quad (5.13) \]

\[ [\tilde{L}_{m}^{g}, \tilde{L}_{n}^{g}] = (m - n) \tilde{L}_{m+n}^{g} + \tilde{A}^{g}(m) \delta_{m+n, 0}, \quad (5.14) \]

where \( \tilde{\mathcal{A}}^{f}(m) \) and \( \tilde{A}^{g}(m) \) represent the anomalies of \( \tilde{\mathcal{L}}_{n}^{f} \) and \( \tilde{L}_{n}^{g} \). When we calculate these anomalies, we must pay attention to the convergence of the infinite sum; the anomalies are

\[ \tilde{\mathcal{A}}^{f}(m) = \frac{1}{6} (m^3 - m) - \mu \alpha' p^{+} (\mu \alpha' p^{+} - 1) m, \quad (5.15) \]

\[ \tilde{A}^{g}(m) = \frac{1}{6} (m^3 - m) - \mu \alpha' p^{+} (\mu \alpha' p^{+} - 1) m. \quad (5.16) \]

Because the twisted fields \( f \) and \( g \) are complex fields, and each of them has two degrees of freedom, the coefficient \( \frac{1}{6} \) appears in Eqs. \( (5.15) \) and \( (5.16) \). In spite of the fact that the sign of the coefficient of the term \( \mu \alpha' p^{+} \) in \( \tilde{\mathcal{L}}_{n}^{f} \) is different from that in \( \tilde{L}_{n}^{g} \), the anomaly of \( \tilde{\mathcal{L}}_{n}^{f} \) corresponds to the anomaly of \( \tilde{L}_{n}^{g} \).

As a known case, the anomalies of \( \tilde{\mathcal{L}}_{n}^{(+ - k)} \) and \( \tilde{L}_{n}^{(+ - k)} \), which contain \( D - 2 \) string coordinates \( X^{+}, X_{0}^{-} \) and \( X^{k} \), are \( \tilde{\mathcal{A}}^{(+ - k)}(m) = \mathcal{A}^{(+ - k)}(m) = \frac{D-2}{12} (m^3 - m) \), and the anomalies of \( \tilde{\mathcal{L}}_{n}^{gh} \) and \( \tilde{L}_{n}^{gh} \) are \( \tilde{\mathcal{A}}^{gh}(m) = \mathcal{A}^{gh}(m) = \frac{1}{6} (m - 13m^3) \). Let us define the total Virasoro operator in terms of an ordering constant \( a \):

\[ \tilde{\mathcal{L}}_{m} \equiv \tilde{\mathcal{L}}_{m}^{X} + \tilde{\mathcal{L}}_{m}^{gh} - a \delta_{m, 0}, \quad (5.17) \]

\[ L_{m} \equiv L_{m}^{X} + L_{m}^{gh} - a \delta_{m, 0} . \quad (5.18) \]
Then, the Virasoro commutation relations become

\[
[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \tilde{A}(m)\delta_{m+n,0}, \quad (5.19)
\]
\[
[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}. \quad (5.20)
\]

From this calculation, we find the total anomalies to be

\[
\tilde{A}(m) = A(m) = \frac{D-26}{12}(m^3 - m) + 2\left[a - 1 - \frac{1}{2}\mu\alpha'p^+(\mu\alpha'p^+ - 1)\right]. \quad (5.21)
\]

If \( D = 26 \) and \( a = 1 + \frac{1}{2}\mu\alpha'p^+(\mu\alpha'p^+ - 1) \), these vanish, and the theory becomes conformally invariant. We consider the details of this conclusion from the standpoint of BRST quantization in the following section.

§6. The nilpotency of the BRST charge

It is very important to understand the BRST quantization of string theory in a flat spacetime. In this case, the nilpotency condition of the BRST charge gives the number of spacetime dimensions \( D = 26 \) and the ordering constant \( a = 1 \).\(^{15,16} \) Moreover, we have obtained a better understanding of BRST quantization. For example, we now understand BRST cohomology, the no-ghost theorem, and the equivalence between BRST quantization, the old covariant quantization and the light-cone gauge quantization. On the basis of these studies, it is important to understand the BRST quantization of string theory in a non-flat background. Here we treat the pp-wave background. In this section, first, we define the BRST charge in terms of the Virasoro operators defined in the previous section. Second, we calculate the square of the BRST charge and impose the nilpotency condition on it. Then we determine the number of dimensions of the spacetime and the ordering constant.

In closed string theory, the left modes and the right modes are independent. Therefore, the BRST charge can be decomposed into left modes and right modes as

\[
Q_B = Q_B^L + Q_B^R, \quad (6.1)
\]

where

\[
Q_B^L = \sum_{m=-\infty}^{\infty} :L^X_m + \frac{1}{2}L^\text{gh}_m - a\delta_{m,0}: \tilde{c}_{-m} :, \quad (6.2)
\]
\[
Q_B^R = \sum_{m=-\infty}^{\infty} :L^X_m + \frac{1}{2}L^\text{gh}_m - a\delta_{m,0}: c_{-m} :, \quad (6.3)
\]
Concentrating our attention on the normal ordering of the mode operators, especially with regard to the ghosts, we obtain the square of $Q_B$:

\[
Q_B^2 = \frac{1}{2} \{ Q_B^L, Q_B^R \} + \{ Q_B^R, Q_B^R \} \\
= \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \left[ \left( [\tilde{L}_m, L_n] - (m-n)\tilde{L}_{n+m} \right) \tilde{c}_{-m} \tilde{c}_{-n} \\
+ \left( [L_m, \tilde{L}_n] - (m-n)L_{n+m} \right) c_{-m} c_{-n} \right] \\
= \frac{1}{2} \sum_{m=-\infty}^{\infty} A(m) \left( \tilde{c}_{-m} \tilde{c}_m + c_{-m} c_m \right),
\]

(6.4)

where $A(m)$ is given in Eq. (5.21). We note that the ghosts and the anomaly survive. If the anomaly is zero, the square of the BRST charge vanishes. Because the BRST charge must have the property of nilpotency, the anomaly must be zero. Thus, according to the results of the previous section, we can determine the number of spacetime dimensions and the ordering constant:

\[
D = 26, \quad a = 1 + \frac{1}{2} \mu \alpha' p^+ \left( \mu \alpha' p^+ - 1 \right).
\]

(6.5)

This ordering constant corresponds to a constant that has been determined using the method of the $\zeta$-function in the light-cone gauge quantization.\(^3\) However, the number of spacetime dimensions cannot be determined in the case of the operator formalism of the light-cone gauge quantization.\(^{17}\)

Considering the spectrum of a closed string in the $pp$-wave background, the physical state must satisfy $Q_B |\text{phys} \rangle = 0$. From this condition, we can obtain the structure of the physical state. For example, the mass of the lightest string state, which is tachyonic, is

\[
m_0^2 = -\frac{4}{\alpha'} \left[ 1 + \frac{1}{2} \mu \alpha' p^+ \left( \mu \alpha' p^+ - 1 \right) \right],
\]

(6.6)

and, moreover, the mass of the first excited state, which contains a massive graviton, etc., is

\[
m_1^2 = -\frac{2}{\alpha'} \mu \alpha' p^+ \left( \mu \alpha' p^+ - 1 \right).
\]

(6.7)

In the case $0 < \mu \alpha' p^+ < 1$, the first excited state is stable, because $m_1^2 > 0$; the maximum mass of this state is $m_1^2 = \frac{\mu \alpha'}{2 \alpha'}$ at $\mu \alpha' p^+ = \frac{1}{2}$. Here, defining the mass, we use the mass-shell condition of the particle which exhibits behavior similar to harmonic oscillation in non-flat directions of the $pp$-wave background.

§7. Conclusion

In this paper we have canonically quantized a closed bosonic string in the $pp$-wave background with a non-zero $B_{\mu \nu}$ field using the covariant BRST operator formalism. In this $pp$-wave back-
ground, we have constructed the free-mode representations of all the covariant string coordinates. Moreover, we proved that the free-mode representations satisfy both the equal-time canonical commutation relations between all the covariant string coordinates and the Heisenberg equations of motion, whose form is the same as that of the Euler-Lagrange equations of motion in this pp-wave background. It is worth noting that the zero mode $x^{-}$ of $X_{0}^{-}$ has played important rules in this study. In particular, the coefficients of the expansion modes in $Z$ and $\bar{Z}$ are determined by the condition that $x^{-}$ must be a free mode. It is also interesting that $X_{0}^{-}$ is not a free field and the derivative of $X_{0}^{-}$ is a free field. Moreover, $X_{0}^{-}$ should play an important role in the vertex operators, the physical states, and so on.

Since the energy-momentum tensor takes a very simple form in the free-mode representations of the covariant string coordinates, we have been able to calculate the anomaly in the Virasoro algebra. Using this anomaly, we have determined the number of dimensions of spacetime and the ordering constant from the nilpotency condition of the BRST charge in the pp-wave background.

From a new point of view regarding the free-mode representation in the pp-wave background, we should be able to obtain important information concerning the no-ghost theorem, all the physical states, and the exact quantization of the covariant superstring. Moreover, it would be interesting to construct free-mode representations in other backgrounds, for example the shock wave. On the basis of these and other new concepts, it may be possible to elucidate the background.

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Appendix A

Classical General Solution for a Closed String

in the pp-Wave Background without an Antisymmetric Tensor Field

In this appendix, we consider closed string theory in the pp-wave background without an antisymmetric tensor field, i.e., with $B_{\mu\nu} = 0$. In pp-wave backgrounds, the spacetime metric is generally as follows:

$$ds^2 = FdX^+dX^- - 2dX^+dX^- + dXdX + dYdY + dX^k dX^k .$$  (A·1)
The Einstein equation, from which the spacetime metric is obtained, is \( R_{+ +} = -\frac{1}{2} (\partial_x^2 + \partial_y^2) F = 0 \). Here, \( F \) must satisfy this Einstein equation, and therefore we choose \( F = -\mu^2 (X^2 - Y^2) \) in the case \( B_{\mu \nu} = 0 \). The action is

\[
S_X = -\frac{1}{4\pi\alpha} \int d\tau d\sigma \sqrt{-g} g^{\mu \nu} \partial_a X^\mu \partial_b X^\nu .
\] (A.2)

We can choose the covariant gauge \( g^{ab} = \eta^{ab} \), and then this action becomes

\[
S_X = -\frac{1}{4\pi\alpha} \int d\tau d\sigma \left[ -2\partial_a X^+ \partial^a X^- - \mu^2 (X^2 - Y^2) \partial_a X^+ \partial^a X^- \\
+ \partial_a X^{-} \partial^a X^{-} + \partial_a Y \partial^a Y + \partial_a X^k \partial^a X^k \right].
\] (A.3)

Therefore, the equations of motion obtained with this action are

\[
\partial_a \partial^a X^+ = 0 , \quad \partial_+ \partial_- X^k = 0 ,
\] (A.4)

\[
\partial_a \partial^a X = -\mu^2 \partial_a X^+ \partial^a X^+ ,
\] (A.5)

\[
\partial_a \partial^a Y = +\mu^2 \partial_a X^+ \partial^a X^+ ,
\] (A.6)

\[
\partial_a \partial^a X^- = -\mu^2 \partial_a [(X^2 - Y^2) \partial^a X^+] .
\] (A.7)

In the particle approximation, \( X \) behaves similarly to a harmonic oscillator and \( Y \) behaves similarly to an unstable operator, like a tachyon. We can obtain classical general solutions from these equations of motion under the periodic boundary condition. First, the general solutions of \( X^+ \) and \( X^k \) are well known:

\[
X^+ = x^+ + \frac{\alpha'}{2} p^+ (\sigma^+ + \sigma^-) + i \sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^+ e^{-i n \sigma^+} + \alpha_n^+ e^{-i n \sigma^-} \right],
\] (A.8)

\[
X^k = x^k + \frac{\alpha'}{2} p^k (\sigma^+ + \sigma^-) + i \sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^k e^{-i n \sigma^+} + \alpha_n^k e^{-i n \sigma^-} \right].
\] (A.9)

Here, \( X^+ (X^k) \) can be divided into the left-moving field, \( X^+_L (X^k_L) \), and the right-moving field, \( X^+_R (X^k_R) \). Second, using \( X^+_L \) and \( X^+_R \), we can solve the equations of motion for \( X \) and \( Y \), and the general solutions under the periodic boundary condition of closed string theory are

\[
X = \sqrt{\frac{\alpha'}{2}} \sum_{N = -\infty}^{\infty} \left[ A_N e^{-i (\lambda_N X^+_L + \tilde{\lambda}_N X^+_R)} + A_N^+ e^{i (\lambda_N X^+_L + \tilde{\lambda}_N X^+_R)} \right],
\] (A.10)

\[
Y = \sqrt{\frac{\alpha'}{2}} \sum_{N = -\infty}^{\infty} \left[ B_N e^{-i (\lambda_N X^+_L + \tilde{\lambda}_N X^+_R)} + C_N e^{i (\lambda_N X^+_L + \tilde{\lambda}_N X^+_R)} \right].
\] (A.11)

Here, \( \lambda_N \) and \( \tilde{\lambda}_N \) are defined as

\[
\lambda_N = \frac{1}{\alpha' p^+} \left[ N + \sqrt{N^2 + (\mu \alpha' p^+)^2} \right] , \quad \tilde{\lambda}_N = \frac{1}{\alpha' p^+} \left[ N + \sqrt{N^2 - (\mu \alpha' p^+)^2} \right].
\] (A.12)
We must note that the modes $A_N$ and $B_N$ here are not the same as the modes used in the main text. Moreover, from the relation $Y = Y^\dagger$, $B_N$ and $C_N$ must obey the following conditions:

- When $|N| > |\mu \alpha p^+|$, $C_N = B_N^\dagger$.
- When $|N| < |\mu \alpha p^+|$, $B_N^\dagger = B_{-N}$ and $C_N = C_{-N}$.
- When $|N| = |\mu \alpha p^+|$, $C_N = B_N^\dagger$; or $B_N^\dagger = B_{-N}$ and $C_N = C_{-N}$.

Next, we define

$$u_N(\tau, \sigma) = e^{-i(\lambda_N X_L^+ + \lambda_{-N} X_R^+)}; \quad v_N(\tau, \sigma) = e^{-i(\lambda_{-N} X_L^+ + \lambda_{N} X_R^+)}.$$  \hspace{1cm} (A-13)

Third, using world-sheet light-cone coordinates, the equation of motion for $X^-$ becomes

$$\partial_+ \partial_- X^- + \frac{\mu^2}{2} \{ \partial_+ \left[ (X^2 - Y^2) \partial_- X^+ \right] + \partial_- \left[ (X^2 - Y^2) \partial_+ X^+ \right] \} = 0.$$  \hspace{1cm} (A-14)

Applying the inverse of the operator $\partial_+ \partial_-$ to this equation from the left, we obtain the general solution of $X^-$,

$$X^- = X_{L}^- (\sigma^+) + X_{R}^+ (\sigma^-) - \frac{\mu^2}{2} (K_X - K_Y),$$  \hspace{1cm} (A-15)

where $X_{L}^-$ is a function of $\sigma^+$ and $X_{R}^+$ is a function of $\sigma^-$. Moreover, $X_{L}^-(\sigma^+) + X_{R}^+(\sigma^-)$ and $K_X - K_Y$ are, respectively, periodic functions of $\sigma$, and $K_X$ and $K_Y$ are as follows:

$$K_X = \frac{\alpha'}{2} \left[ \sum_{N,M=-\infty}^{\infty} \frac{i}{A_{NM}^+} \left( A_N A_M^* u_N u_M - A_N^* A_M u_N^* u_M^+ \right) \right.$$  
$$+ \sum_{N \neq M} \frac{i}{A_{NM}^+} \left( A_N A_M^* u_N^* u_M - A_N^* A_M u_N u_M^+ \right) + 2 \sum_{N=-\infty}^{\infty} \left( A_N A_N^* + A_N^* A_N \right) X^+ \right],$$  

$$K_Y = \frac{\alpha'}{2} \left[ \sum_{N,M=-\infty}^{\infty} \frac{i}{A_{NM}^+} \left( B_N B_M v_N v_M - C_N C_M v_N^* v_M^+ \right) \right.$$  
$$+ \sum_{N \neq M} \frac{i}{A_{NM}^+} \left( B_N C_M v_N^* v_M - C_N B_M v_N v_M^+ \right) + 2 \sum_{N=-\infty}^{\infty} \left( B_N C_N + C_N B_N \right) X^+ \right] \hspace{1cm} (A-16)$$

The quantities $A_{NM}^\pm$ and $\tilde{A}_{NM}^\pm$ here are defined as

$$\frac{1}{A_{NM}^\pm} = \frac{1}{\lambda_N \pm \lambda_M} + \frac{1}{\lambda_{-N} \pm \lambda_{-M}},$$  
$$\frac{1}{\tilde{A}_{NM}^\pm} = \frac{1}{\lambda_N \pm \lambda_M} + \frac{1}{\lambda_{-N} \pm \lambda_{-M}}.$$  \hspace{1cm} (A-18)

Finally, we consider the energy-momentum tensor in this background. The energy-momentum tensor is defined as $T_{ab}^X = \frac{1}{\alpha} G_{\mu \nu}(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \eta_{ab} \partial_c X^\mu \partial^c X^\nu)$. Substituting the general solutions
$X^+, X^-, X, Y$ and $X^k$ for the classical energy momentum tensor $T_{ab}^X$, we obtain the following:

$$T_{++}^X = \frac{1}{\alpha'} \left\{ -2\partial_+ X^+ \partial_+ X_- + \partial_+ X^k \partial_+ X^k + \sum_{N=-\infty}^{\infty} \left[ \left( \lambda_N^2 + \mu^2 \right) \left( A_N A_N^\dagger + 2A_N^\dagger A_N \right) + \left( \tilde{\lambda}_N^2 - \mu^2 \right) \left( B_N C_N + C_N B_N \right) \right] \left( \partial_+ X^+ \right)^2 \right\},$$

(A-19)

$$T_{--}^X = \frac{1}{\alpha'} \left\{ -2\partial_- X^+ \partial_- X_- + \partial_- X^k \partial_- X^k + \sum_{N=-\infty}^{\infty} \left[ \left( \lambda_N^2 + \mu^2 \right) \left( A_N A_N^\dagger + 2A_N^\dagger A_N \right) + \left( \tilde{\lambda}_N^2 - \mu^2 \right) \left( B_N C_N + C_N B_N \right) \right] \left( \partial_- X^+ \right)^2 \right\}.$$

(A-20)

Here $T_{++}$ is a function of $\sigma^+$ and $T_{--}$ is a function of $\sigma^-$.  

**Appendix B**

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**The Mode Expansion of Z in the Case of $\mu \alpha' p^+ = n$**

When $\mu \alpha' p^+$ is an integer ($\mu \alpha' p^+ = n$), one coefficient in the mode expansion of $f$ appearing in Eq.(3.6) and one coefficient in that of $g$ appearing in Eq.(3.7) diverge. For this reason, we must consider another mode expansion. In this case, the free-mode expansion of $Z$ becomes the following:

$$Z(\sigma^+, \sigma^-) = e^{-i \mu \tilde{X}^+} \left[ f(\sigma^+) + g(\sigma^-) \right],$$

(B-1)

$$f(\sigma^+) = \frac{\phi_n}{2} + 2\alpha' p_n \sigma^+ + \sqrt{\alpha'} \sum_{N \neq n} \frac{A_N}{\sqrt{|N-n|}} e^{-i(N-n)\sigma^+},$$

(B-2)

$$g(\sigma^-) = \frac{\phi_n}{2} + 2\alpha' p_n \sigma^- + \sqrt{\alpha'} \sum_{N \neq -n} \frac{B_N}{\sqrt{|N+n|}} e^{-i(N+n)\sigma^-}.$$

(B-3)

Of course, we can obtain the free-mode expansion of $\bar{Z}$ by taking the Hermitian conjugate of $Z$. Although the above expansion slightly resembles the ordinary mode expansion of free fields, the above modes are complex operators, and the coefficients of the modes are different from those of free fields. Moreover, $\phi_n$ is not the center-of-mass coordinate operator of $Z$, and $p_n$ is not the total momentum operator of $Z$. This is the case even for $p^+ = 0$.

We can canonically quantize the string coordinate $Z$, and when this is done all the commutation relations between all the modes of $Z$ take the following forms. The commutation relations between $\phi_n, p_n, \phi_n^\dagger$ and $p_n^\dagger$ are

$$[\phi_n, p_n^\dagger] = i, \quad [\phi_n^\dagger, p_n] = i,$$

(B-4)
with all other commutators between them vanishing. The commutation relations between the
modes of $Z$ and $\bar{Z}$, except in the case $M = \pm n$ or $N = \pm n$, correspond to Eqs. (3.19) and (3.20):

$$[A_M, A_N^\dagger] = \text{sgn}(M - n)\delta_{MN}, \quad (M, N \neq n) \quad (B.5)$$

$$[B_M, B_N^\dagger] = \text{sgn}(M + n)\delta_{MN}, \quad (M, N \neq -n) \quad (B.6)$$

with the commutators between the other modes vanishing. Therefore, the definitions of the normal
orderings of $A_N$ and $B_N$ are obviously the same as those given in Eqs. (3.22) and (3.23).

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