Type II Fluid Solutions to Einstein’s Field Equations in N-Dimensional Spherical Spacetimes

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Abstract

A large class of Type II fluid solutions to Einstein field equations in N-dimensional spherical spacetimes is found, which includes most of the known solutions. A family of the generalized collapsing Vaidya solutions with homothetic self-similarity, parametrized by a constant $\lambda$, is studied, and found that when $\lambda > \lambda_c(N)$, the collapse always forms black holes, and when $\lambda < \lambda_c(N)$, it always forms naked singularities, where $\lambda_c(N)$ is function of the spacetime dimension $N$ only.

PACS numbers: 04.20Jb, 04.40.+c, 97.60.Lf.

I. INTRODUCTION

The Vaidya solutions found in 1951 represent an imploding (exploding) null dust fluid with spherical symmetry and has been intensively studied in gravitational collapse in General Relativity. Papapetrou first showed that these solutions can give rise to the formation of naked singularities, and thus provide one of the earlier counterexamples to the cosmic censorship conjecture. Other generalized Vaidya solutions with remarkable physically properties are, for example, the charged ones and the Husain solutions of a null fluid with a particular equation of state, which have been lately used as the formation of
black holes with short hair [7]. Recently, Wang and Wu [8] generalized the Vaidya solutions to a more general case, which include most of the known solutions to the Einstein field equations, such as, the de Sitter-charged Vaidya solutions, and the Husain solutions. More recently, a class of the Wang-Wu solutions was used to study the collapse of strange quark matter in a Vaidya background [9].

On the other hand, Waugh and Lake [10] studied the Vaidya solutions in double null coordinates and found the explicit close form of the metric coefficients for the linear and exponential mass functions. As a result, the global structure of the spacetime was showed clearly. In particular, they showed that for the homothetic case, where the mass function takes the form \( m(v) = \lambda v \), when \( \lambda > 1/16 \) the collapse forms black holes and when \( \lambda < 1/16 \) the collapse always forms naked singularities. This is quite similar to the critical phenomena in gravitational collapse found recently by Choptuick [11], but now the “critical” solution \( \lambda = 1/16 \) separates black holes from naked singularities. To have a definite answer to this problem, one needs to study the perturbations of the “critical” solutions and shows that it indeed has only one unstable mode. Otherwise, this “critical” solutions is not critical by definition [12].

In this paper, we first generalize the results obtained in [8] to N dimensional spherically symmetric spacetimes in Eddington-Finkelstein (radiation) coordinates, and then study the physical properties of generalized Vaidya ingoing solutions with homothetic self-similarity. In particular, we shall show that, similar the four dimensional case [10], there always exists a “critical” solution \( \lambda = \lambda_c(N) \), which separates the formation of black holes from the formation of naked singularities. It is remarkable that \( \lambda_c(N) \) is function of the spacetime dimension \( N \), only.

II. TYPE II FLUID SOLUTIONS IN N-DIMENSIONAL SPHERICAL SPACETIMES

Let us begin with the general spherically symmetric line element [13]
\[ ds^2 = e^{\psi(v,r)} dv \left[ f(v,r)e^{\psi(v,r)} dv + 2\epsilon dr \right] - C^2(v,r)d\Omega_{N-2}^2, \tag{1} \]

where \( d\Omega_{N-2}^2 \) is the line element on the unit \((N-2)\)-sphere, given by

\[
d\Omega_{N-2}^2 = \left( d\theta^2 \right) + \sin^2(\theta^2) \left( d\theta^3 \right)^2 + \sin^2(\theta^2) \sin^2(\theta^3) \left( d\theta^4 \right)^2
+ \ldots + \sin^2(\theta^2) \sin^2(\theta^3) \ldots \sin^2(\theta^{N-2}) \left( d\theta^{N-1} \right)^2
= \sum_{i=2}^{N-1} \left[ \prod_{j=2}^{i-1} \sin^2(\theta^j) \right] \left( d\theta^i \right)^2 \tag{2} \]

and \( \epsilon = \pm 1 \). When \( \epsilon = +1 \), the radial coordinate \( r \) increases toward the future along a ray \( v = \text{Const.} \), i.e., the light cone \( v = \text{Const.} \) is expanding. When \( \epsilon = -1 \), the radial coordinate \( r \) decreases toward the future along a ray \( v = \text{Const.} \), and the light cone \( v = \text{Const.} \) is contracting.

In the following, we shall consider the particular case where \( \psi(v,r) = 0 \) and \( \epsilon = -1 \) then the non-vanishing components of the Einstein tensor are given by

\[
G_0^0 = G_1^1 = -\frac{(N-2)}{2r^2} \left\{ (N-3)[1 - f(v,r)] - rf'(v,r) \right\}, \tag{3} \\
G_0^1 = -\frac{(N-2)}{2r} \hat{f}(v,r), \tag{4} \\
G_i^i = \frac{1}{2r^2} \left\{ r^2 f''(v,r) + (N-3)[2rf'(v,r) + (N-4)(1-f(v,r))] \right\}, \tag{5} 
\]

where \( \{x^\mu\} = \{v, r, \theta^2, ..., \theta^{N-1}\}, \ (\mu = 0, 1, 2, ...N-1) \), and

\[
\hat{f}(v,r) \equiv \frac{\partial f(v,r)}{\partial v}, \quad f'(v,r) \equiv \frac{\partial f(v,r)}{\partial r}. \tag{6} 
\]

Then, from the Einstein field equations \( G_{\mu\nu} = \kappa T_{\mu\nu} \), we find that the corresponding EMT can be written in the form

\[
T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)}, \tag{7} 
\]

where

\[
T_{\mu\nu}^{(n)} = \mu l_\mu l_\nu, \\
T_{\mu\nu}^{(m)} = (\rho + P) (l_\mu n_\nu + l_\nu n_\mu) + Pg_{\mu\nu}. \tag{8} 
\]

3
\[ \mu = - \frac{(N - 2)}{2\kappa r} f(v, r), \]
\[ \rho = \frac{(N - 2)}{2\kappa r^2} \left\{ (N - 3)[1 - f(v, r)] - rf'(v, r) \right\}, \]
\[ P = \frac{1}{2\kappa r^2} \left\{ r^2 f''(v, r) + (N - 3)[2rf'(v, r) - (N - 4)(1 - f(v, r))] \right\}, \]

with \( l_\mu \) and \( n_\mu \) being two null vectors,

\[ l_\mu = \delta_\mu^0, \quad n_\mu = \frac{1}{2} f(v, r) \delta_\mu^0 + \delta_\mu^1, \]
\[ l_\lambda n_\lambda = 0, \quad l_\lambda n_\lambda = -1. \] (10)

The part \( T^{(n)}_{\mu\nu} \) of the EMT, can be considered as the component of the matter field that moves along the null hypersurfaces \( v = \text{Const.} \). In particular, when \( \rho = P = 0 \), the solutions reduce to the the N-dimensional Vaidya solutions with \( m = m(v) \) \cite{13,14}. Therefore, for the general case we consider the EMT of Eq.(7) as a generalization of the Vaidya solutions in N-dimensional spacetimes.

Projecting the EMT of Eq.(7) to the orthonormal basis, defined by the unit vectors,

\[ E^\mu_{(0)} = \frac{l_\mu + n_\mu}{\sqrt{2}}, \]
\[ E^\mu_{(1)} = \frac{l_\mu - n_\mu}{\sqrt{2}}, \]
\[ E^\mu_{(2)} = \frac{1}{r} \delta^\mu_2, \]
\[ E^\mu_{(i)} = \frac{1}{r \left[ \prod_{l=2}^{i-1} \sin^2 (\theta^l) \right]} \delta^\mu_i, \quad (i = 3, 4, ..N - 1), \] (11)

we find that \( T_{(a)(b)} \equiv e^\mu_{(a)} e^\nu_{(b)} T_{\mu\nu} \) takes the form
\[
\begin{bmatrix}
\frac{\mu}{2} + \rho & \frac{\mu}{2} & 0 & 0 & \ldots & 0 \\
\frac{\mu}{2} & \frac{\mu}{2} - \rho & 0 & 0 & \ldots & 0 \\
0 & 0 & P & 0 & \ldots & 0 \\
0 & 0 & 0 & P & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & P
\end{bmatrix},
\]

which belongs to the Type II fluids defined in [15]. The null vector \( l^\mu \) is the double null eigenvector of the EMT. Physically acceptable solutions have to satisfy several energy conditions [15].

Following [16], let us define the mass function \( m(v, r) \) as
\[
m(v, r) \equiv \frac{B_N}{2} r^{N-3} (1 + r_{,\alpha} r_{,\beta} g^{\alpha\beta})
\]

where \( ()_\alpha = \partial() / \partial x^\alpha \) and \( B_N \) is a constant defined by
\[
B_N = \frac{\kappa \Gamma \left( \frac{N-1}{2} \right)}{2(N - 2) \pi^{(N-1)/2}}.
\]

In the present case, it can be shown that \( m(v, r) \) takes the form
\[
m(v, r) = \frac{B_N}{2} r^{N-3} [1 - f(r, v)],
\]

or inversely,
\[
f(v, r) = 1 - 2 \frac{m(v, r)}{B_N r^{N-3}}.
\]

Clearly, from the above definition, we find that the apparent horizon is located on the hypersurface defined by
\[
r_{,\alpha} r_{,\beta} g^{\alpha\beta} = 0.
\]

In terms of \( m(r, v) \), the Kretschmann Scalar reads
\[ R \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \]
\[ = \frac{6}{B_N^2 r^{2(N-1)}} \left\{ \left[ (N - 3)^2 (N - 2)^2 + 4 \sum_{k=1}^{N-3} k \right] m^2 + 4r^4 m''^2 \right. \]
\[ + 4(N - 3)^2 \left[ (N - 2) rm'' + r^2 m'^2 \right] \]
\[ - 2(N - 3) \left[ (N - 2)m + 2rm' \right] r^2 m'' \right\}, \tag{18} \]

while Eqs. (9) takes the form
\[
\mu = \frac{N - 2}{\kappa B_N} \frac{\dot{m}(v, r)}{r^{N-2}} \]
\[
\rho = \frac{N - 2}{\kappa B_N} \frac{m'}{r^{N-2}} \]
\[
P = -\frac{1}{\kappa B_N} \frac{m''(v, r)}{r^{N-3}}. \tag{19} \]

Without loss of generality, we expand \( m(v, r) \) in the powers of \( r \),
\[
m(v, r) = \sum_{k=-\infty}^{+\infty} a_k(v) r^k, \tag{20} \]
where \( a_k(v) \) are arbitrary functions of \( v \) only. Note that the sum of the above expression should be understood as an integral, when the “spectrum” index \( k \) is continuous. With this definition, from Eq. (19) we have
\[
\mu = \frac{(N - 2)}{\kappa B_N} \sum_{k=-\infty}^{+\infty} \dot{a}_k(v) r^{k-N+2}, \quad \rho = \frac{(N - 2)}{\kappa B_N} \sum_{k=-\infty}^{+\infty} k a_k(v) r^{k-N+1}, \]
\[
P = -\frac{1}{\kappa B_N} \sum_{k=-\infty}^{+\infty} k(k - 1) a_k(v) r^{k-N+1}. \tag{21} \]

Once we have the general solutions, let us consider some particular cases.

i) The Null Fluid Solution: Let us consider first the null dust case, defined by the conditions
\[
P = 0 = \rho \tag{22} \]
which in terms of (21) can be written as
\[
a_k(v) = \begin{cases} m(v), & k = 0, \\ 0, & k \neq 0. \end{cases} \tag{23} \]
These are the N-Dimensional Vaidya solutions \([13,14]\), in which the three energy conditions, weak, strong and dominant, all reduce to \(\mu \geq 0\). For these solutions, the Kretschmann Scalar \([15]\) reads

\[
\mathcal{R} = \frac{6}{B^2} \left\{ (N - 3)^2 (N - 2)^2 + 4 \sum_{k=1}^{N-3} k \right\} \frac{h^2(v)}{r^{2(N-1)}},
\]

which is singular when \(r = 0\).

**ii) The de Sitter and the Anti-de Sitter solutions:** Another simple case is the consideration of the vacuum equations in the presence of the cosmological constant \(\Lambda\). The solutions of the Einstein equations for this case are given by

\[
a_l(v) = \begin{cases} 
\frac{B \Lambda}{(N-2)(N-1)}, & k = N - 1, \\
0, & k \neq N - 1.
\end{cases}
\]

**iii) The Charged Vaidya Solutions:** The consideration of the electromagnetic field \(F_{\mu\nu}\), given by

\[
F_{\mu\nu} = \frac{q(v)}{r^{N-2}} (\delta^0_\mu \delta^1_\nu - \delta^1_\mu \delta^0_\nu),
\]

yields

\[
a_k(v) = \begin{cases} 
f(v), & k = 0, \\
-\frac{B N q^2(v)}{(N-2)(N-3)}, & k = -(N - 3), \\
0, & k \neq 0, -(N - 3).
\end{cases}
\]

The quantities \([19]\) are now

\[
\rho = P = \frac{q^2(v)}{\kappa r^{2(N-2)}},
\]

\[
\mu = \frac{1}{\kappa B N (N - 3) r^{2N-5}} \left[ (N - 2)(N - 3) r^{N-3} \dot{f}(v) - 2B_N q(v) \dot{q}(v) \right].
\]

The condition \(\mu \geq 0\) gives the main restriction on the choice of the functions \(f(v)\) and \(q(v)\). In particular, if \(df/dq > 0\), we can see that there always exists a critical radius \(r_c\),

\[
r_c = \left[ \left( \frac{2B_N}{(N-2)(N-3)} \right) \frac{q}{f} \right]^{\frac{1}{N-3}}.
\]

When \(r < r_c\), we have \(\mu < 0\), and the energy conditions are always violated.
In this case, the apparent horizon is given by

\[ r_{ah} = \left\{ B_N^{-1} \left[ f \pm \sqrt{\frac{(N - 2)(N - 3)f^2(v) - 2q(v)^2}{(N - 2)(N - 3)}} \right] \right\}^{\frac{1}{N - 3}}. \quad (30) \]

It can be show that the corresponding Kretschmann Scalar is proportiol to \( r^{-4(N-2)} \).

iv) The Solutions with Linear Equation of State: We will consider now the case for the equation of state

\[ P = \alpha \rho, \quad (31) \]

where \( \alpha \) is a constant. Solving the equations (19) and (31) we found two irreductible cases: \( \alpha \neq 1/(N - 2) \) and \( \alpha = 1/(N - 2) \).

When \( \alpha \neq 1/(N - 2) \), we have

\[ a_k(v) = \begin{cases} h_1(v), & k = 0, \\
-\frac{h_0(v)}{[\alpha(N - 2) - 1]}, & k = 1 - \alpha(N - 2), \\
0, & k \neq 0, 1 - \alpha(N - 2). \end{cases} \quad (32) \]

Then the apparent horizon is given by

\[ r^{\alpha(N-2)+(N-4)} \frac{2h_1(v)}{B_N} r^{\alpha(N-2)-1} + \frac{4h_0}{B_N[\alpha(N - 2) - 1]} = 0. \quad (33) \]

The corresponding Kretschmann scalar is proportional to \( r^{-2[(N-2)(\alpha+1)]} \), and singularities at \( r = 0 \) will occur for \( \alpha \geq -1 \). From Eqs. (39) we find

\[ P = \alpha \rho = \frac{(N - 2) h_0(v)}{\kappa B_N^2 r^{(N-2)(\alpha+1)}}, \]

\[ \mu = \frac{(N - 2)}{[\alpha(N - 2) - 1]B_N r^{(N-2)(\alpha+1)-1}} \times \left\{ \alpha[(N - 2) - 1]B_N r^{(N-2)(\alpha+1)-1} h_1(v) - \dot{h}_0(v) \right\}. \quad (34) \]

When \( \alpha = 1/(N - 2) \) the solution for Eqs. (39) and (31) is given by

\[ a_k(v) = \begin{cases} h_1(v) + h_2(v) \ln[r], & k = 0, \\
0, & k \neq 0, \end{cases} \quad (35) \]

while Eqs. (39) then yield
\[ P = \frac{1}{N-2} \rho = \frac{h_2(v)}{\kappa B_N r^{N-1}} \quad (36) \]

\[ \mu = \frac{(N-2)}{\kappa B_N r^{(N-2)}} \left\{ \dot{h}_1(v) - \dot{h}_2(v) \ln[r] \right\} \]

and the corresponding apparent horizon is given by

\[ \frac{B_N}{2} r^{N-3} - h_1(v) - h_2(v) \ln[r] = 0. \quad (37) \]

v) **The Solutions with the Polytropic Equation of State:** A polytropic equation of state is defined as

\[ P = \alpha \rho^\beta \quad (38) \]

where \( \alpha \) and \( \beta \) are constants.

Substituting Eqs. (38) into Eqs. (19) we find that for \( \beta \neq 1 \) the function \( m(v, r) \) is given by

\[ m(v, r) = \int \left\{ [1 - \beta] h_0(v) - \alpha \left[ \frac{B_N}{N-2} r^{N-2} \right]^{1-\beta} \right\} \frac{r^\beta}{r^{1-\beta}} dr + h_1(v), \quad (39) \]

while Eqs. (19) now read

\[ P = \alpha \rho^\beta = \frac{\alpha}{\kappa^{\beta-1}} \left[ \frac{(N-2)}{\kappa B_N r^{N-2}} \right]^\beta \left\{ [1 - \beta] h_0(v) - \alpha \left[ \frac{B_N}{N-2} r^{N-2} \right]^{1-\beta} \right\} \frac{r^\beta}{r^{1-\beta}}, \]

\[ \mu = \frac{\beta (N-2) \dot{h}(v)}{\kappa B_N r^{N-2}} \int \left\{ [1 - \beta] h_0(v) - \alpha \left[ \frac{B_N}{N-2} r^{N-2} \right]^{1-\beta} \right\} \frac{r^\beta}{r^{1-\beta}} dr + \dot{h}_1(v). \quad (40) \]

The apparent horizon is given by the equation

\[ r_{ah}^{N-3} - \frac{2}{B_N} \int \left\{ [1 - \beta] h_0(v) - \alpha \left[ \frac{B_N}{N-2} r_{AH}^{N-2} \right]^{1-\beta} \right\} \frac{r^\beta}{r^{1-\beta}} dr + \]

\[ - \frac{2}{B_N} h_1(v) = 0. \quad (41) \]
III. N-DIMENSIONAL VAIDYA SOLUTIONS IN DOUBLE-NULL COORDINATES

In double null coordinates, we can write the spherically symmetric N-dimensional space-time as \[ \text{(42)} \]

\[ ds^2 = 2f(u,v)dudv - r^2(u,v)d\Omega_{N-2}^2, \]

where \( u \) and \( v \) are the double null coordinates and \( d\Omega_{N-2}^2 \), is the line element on the unit (N-2)-sphere, defined by \((2)\).

The EMT for a null dust fluid is given now by \[ \text{(43)} \]

\[ T_{\mu\nu} = \mu(u,v)\delta_{\mu}^v\delta_{\nu}^v. \]

Then the Einstein’s field equations take the form \[ \text{(44)} \]

\[ \frac{(N-2)}{rf} (fr_{uu} - f_u r_u) = 0 \]

\[ \frac{2}{f} [rr_{uv} + (N-3)r_{u} r_{v}] + (N-3) = 0 \]

\[ - \frac{(N-2)}{rf} (fr_{vv} - f_v r_v) = \mu(u,v) \]

\[ r(fff_{uv} - f_u f_v) + (N-2)f^2 r_{uv} = 0. \]

Integrating Eq. \((44)\) with respect to \( u \), we obtain \[ \text{(48)} \]

\[ f = u_0 h_1(v)r_u \]

where \( u_0 \) is an arbitrary constant and \( h_1 \) an arbitrary function.

Differentiating Eq. \((44)\) with respect to \( u \), considering Eq.(44), then integrating the result with respect to \( u \), we find \[ \text{(49)} \]

\[ r^{N-2}r_{uv} f + h_0(v) = 0, \]

where \( h_0(v) \) is another arbitrary function. Setting now Eqs. \((48)\) and \((49)\) in \((47)\) we have \[ \text{(50)} \]

\[ r_{,v} = -h_1 \left[ 1 - \frac{2}{N-3} \frac{h_0}{r^{N-3}} \right], \]
where in writing the above expression, we set \( u_0 = 2 \). Using this definition, Differentiating Eq. (15) with relation to \( v \), and considering Eqs. (46), (47), (49), and (50) we have

\[
\mu = 2 \frac{N - 2}{N - 3} \frac{h_{0,v}}{r^{N-2}}.
\]  

(51)

Equations (48), (49), (50) and (51) are, respectively, the N-dimensional generalization of equations (11), (9), (12) and (13) in [10].

\[\text{From Eq.(13), we find}
\]

\[
m(u,v) = \frac{B_N}{2} r^{N-3} \left[ 1 + 2 \frac{r_{u,v}}{f} \right] = \frac{B_N}{N - 3} h_0(u,v)
\]

or inversely,

\[
h_0(u,v) = \frac{N - 3}{B_N} m(u,v).
\]

(53)

Now let us consider the case where

\[m = \lambda v^{N-3},\]

(54)

with \( \lambda > 0 \), so the solutions have homothetic self-similarity [17]. Then, from Eq. (54) we find

\[
\int \frac{x^{N-3} dx}{x^{N-2} - \frac{x^{N-3}}{2} + \lambda} + \ln[v] = h_2(u)
\]

(55)

where

\[x = \frac{r}{v},\]

(56)

is the self-similar variable and \( h_2 \) an arbitrary function. From (48) and (53), and setting \( h_2(u) = -u \), we find

\[f = \frac{r}{x^{N-2}} \left[ x^{N-2} - \frac{1}{2} x^{N-3} + \lambda \right] \equiv \frac{r}{x^{N-2}} F(x).
\]

(57)

Depending on the specific value of \( \lambda \), it can be show that there are three distinguishible cases: i) \( \lambda < \lambda_c(N) \); ii) \( \lambda = \lambda_c(N) \); and iii) \( \lambda > \lambda_c(N) \) [cf. Fig.1]. When \( \lambda < \lambda_c \), \( F(x) = 0 \) in
general has two real roots, says $x_1$ and $x_2$. Without loss of generality, we assume $x_2 > x_1$. Similar to the case $N = 4$, the one with $x = x_2$ corresponds to the Cauchy horizon and the corresponding Penrose diagram is given by Fig 2(a). Thus, in this case, the collapse forms naked singularities. When $\lambda = \lambda_c$, the two real roots, $x_1$ and $x_2$ degenerates to one, $x_c = x_1 = x_2$, and we have $F(x_c) = 0$, i.e,

$$x_c - \frac{N - 3}{2(N - 4)} = 0.$$  \hspace{1cm} (58)

Substituting Eq.(58) into the equation $F(x_c) = 0$, we find that

$$\lambda_c = \frac{(N - 3)^{N-3}}{[2(N - 2)]^{N-2}}.$$  \hspace{1cm} (59)

The corresponding Penrose diagram is given by Fig. 2(b). It is remarkable to note that $\lambda_c$ depends only on the spacetime dimension $N$.

When $\lambda > \lambda_c$, we can see that the equation $F(x) = 0$ has no real roots and the function $f(u, v)$ is strictly positive, $f(u, v) > 0$. Similar to the four dimensional case, it can be show that now the collapse always forms black holes and the corresponding Penrose diagram is given by Fig. 2(c).

### IV. CONCLUSIONS

In this paper, we have first generalized the solutions found by Wang and Wu [8] in four-dimensional spherically symmetric spacetimes to N-dimensional spacetimes and found that the most known solutions belong to the large class of solutions presented there. Then, we have restricted our attention to the ingoing Vaidya solutions, which can be interpreted as representing gravitational collapse of null dust fluid in N-Dimensional spacetime. To study the global structure of the corresponding spacetimes, following Waugh and Lake [10], we found the explicit close form of the metric in double null coordinates for the solutions with homothetic self-similarity [17] and shown explicitly that when $\lambda > \lambda_c$, the collapse always forms black holes, and when $\lambda < \lambda_c$, it always forms naked singularities. The critical value
\(\lambda_c\) has been also found and it is remarkable that it depends only on the space dimension \(N\),

\[
\lambda_c = \frac{(N - 3)^{N-3}}{[2(N - 2)]^{N-2}}.
\]

When \(N=4\) we have \(\lambda_c = 1/16\), which is exactly the value found first by Papaetrou \cite{papaetrou} and later confirmed by several authors.

*Note added:* After this paper has been submitted for publication, a preprint appeared in xxx.lanl.gov \cite{vaidya}, in which the self-similar Vaidya solutions were also studied, but by a different method. Moreover, we have been lately also informed that similar considerations of null dust solutions in higher dimensional spacetimes were given in \cite{null_dust}.

**ACKNOWLEDGMENTS**

The author would like very much to thank A. Z. Wang for valuable suggestions and discussions, and CNPq for financial assistance.
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FIG. 1. The qualitative behavior of the function $F(x)$ defined by Eq. (57) in the text.

FIG. 2. Penrose’s diagrams: (a) the $\lambda < \lambda_c$ case; (b) the case $\lambda = \lambda_c$; and (c) the $\lambda > \lambda_c$ case.