Experimental Study on Parametrically Excited Oscillation of a Spar-Buoy Under Mathieu Instability

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Summary

The occurrence of the Mathieu-type instability was investigated theoretically and experimentally. The main aim of this paper is utilization of auto-parametrically excited oscillation to increase efficiency of wave energy converters. In this paper, the subject is the auto-parametrically excited oscillation of a spar-buoy type point absorber with the coupled motions of pitch and heave. A spar-buoy model with innovative ballast control system was made and the model experiments were carried out to realize the large oscillating motion based on the Mathieu-type instability. The ballast control system installed in the buoy model can make the ballast move in the vertical direction to realize a certain change of the pitch natural period. Free-decay experiments were performed to determine the natural frequencies and damping ratios with different ballast positions. During the experiments in regular waves, it was observed that the pitch motion became unstable and the large pitch occurred suddenly. Based on time histories of experimental results, the points (peaks and troughs) are extracted near the large pitch occurring and the periods of transient states were calculated to derive the parametrical values of stability. Considering these values with plotted stability chart of Mathieu equation, the occurrence of the large pitch motion is discussed.

1. Introduction

Wave energy is an intermittent energy source and widely considered as one of the most alternative renewable and sustainable energy. In the ocean engineering, it is one of the most attractive research to convert wave energy to the electricity. Many types of the energy-extracting technologies have been developed and some full-scale devices were built 1-3). According to the working principle, the technologies can be classified into three main types, the oscillating water column, the oscillating bodies and the over-topping. The greatest advantage of the oscillating body type wave energy converters is the huge kinetic energy. To maximize the energy, the oscillation characteristics of the floating device, such as point absorbers, should be optimized to the most frequent waves at the local site. However, the bandwidth of the frequency response is relatively narrow, and the natural frequency is much higher than the typical ocean wave frequencies. Therefore, several phase-control methods have been proposed. 4)

One of the prominent measures to improve the efficiency is phase control by latching 5, 6). The control by latching is applied to the two-body devices and shows the practicality 7).

On the other hand, it is well known that the spar-type platform shows extreme heave motions at resonance. Moreover, parametric resonance of the pitch motion has been confirmed. When the frequency of the heave motion is approximately twice the pitch natural frequency, the very large amplitude oscillatory motion will occur. In this case, it is essential to analyze the motions from the viewpoint of Mathieu-type instability 8) and the preventive measures should be investigated 9).

By changing the viewpoint, a new concept is proposed to utilize the Mathieu-type instability for WEC 10-13). The subject of study was limited to the pure heave motion of a spar-buoy type point absorber. Based on the numerical simulations, the possibility of utilization of the auto-parametrically excited oscillation was shown. Based on the model test, the device of changing the restoring force were made and the utilization of the Mathieu-type instability was shown, but it revealed that the efficiency couldn’t satisfy or was far from practical applications. Therefore, it was concluded that the controlling energy had to be supplied from the oscillating response itself like the parametric rolling in order to induce a real auto-parametrically excited oscillation.

As the next step of the study, the subject was extended to a coupled motion of heave and pitch. A new spar buoy model with innovative ballast control system was made. The system was installed in the buoy model and the vertical movement of the ballast produces a certain change of the pitch natural period. In the previous report 14, 15), the system was not controlled dynamically but used to realize several initial conditions efficiently. The experiments of spar-type model were conducted to realize the large oscillatory motion based on the Mathieu-type instability. In some experiments, it was observed that the large pitch motion occurred suddenly, and the time histories showed different excitation patterns from the theoretical Mathieu-type instability. The energy time histories of the pitch and heave motion were discussed. When the large pitch motion occurred, the kinetic energy was transferred from the heave mode to the pitch mode and shrunken by the pitch damping, the optimum ballast position can be determined from this point of view. The relationship between the pitch angle and the righting lever under the Mathieu-type instability was also theoretically discussed in
In this study, the stability chart of Mathieu equation is plotted for experimental results. During the experiment, it can be observed that the large pitch motion occurred suddenly and decayed at a shot time, after which large pitch motion occurred again. In order to explain this phenomenon, the stability results of the pitch motion at peaks and troughs are investigated with stability chart of Mathieu equation in detail. In this paper, there is no intention of discussing how the wave power is taken off but the occurrence of the large pitch motion is intensively discussed based on the model experiments.

2. Spar-Buoy Model

The new spar-buoy model was made to investigate a coupled motion of heave and pitch. The main body of cylindrical is made of transparent acrylic acid resin and a hemispherical aluminium ballast is attached to the bottom. The appearance of the buoy model is shown in Figure 1 and the principal particulars are listed in Table 1. In order to investigate the coupled motion around the Mathieu-type instability condition, a ballast control mechanism was installed in the buoy model. The mechanism consists of weights, a step motor and a ball screw. The ball screw can shift the weights up and down so that the centre of gravity of the buoy can be changed. Using this mechanism, the pitch natural frequency can be adjusted around half the heave natural frequency. The ballast position can be adjusted by the ball screw from 0 mm to 20 mm. The position is defined by the distance from the upper surface of the hemispherical ballast to the bottom surface of movable ballast. The specifications of buoy under different ballast positions are shown in Table 2.

The ball screw mechanism is controlled by an Arduino Nano Mega328 and the control signals are received through the wireless transmission modules from the control PC. Nine axes accelerometer (Arduino 9 Axes Sensor Shield) is also installed in the buoy model to measure the buoy motions. The measured signals are also sent back through the wireless transmission modules to the control PC.

| Table 1 Principle particulars of the buoy model. |
|-----------------------------------------------|
| Depth                                         | 745 mm |
| Outer diameter of the buoy                     | 160 mm |
| Weight                                        | 11.15 kg |
| Draught                                       | 581.2 mm |
| Centre of buoyancy (from the bottom)          | 303.6 mm |
| Natural heave period (excluding added mass)    | 1.49 s |
| Ballast weight                                | 2.74 kg |
| Ballast movable range                          | 0 ~ 20 mm |

| Table 2 The specifications in the different ballast positions. |
|---------------------------------------------------------------|
| Ballast position (mm)                                        |
| Centre of gravity OG0 (mm)                                   | 316.1 | 314.9 | 313.7 | 312.4 | 311.2 |
| Inertia moment of pitch Iyy (kgm²)                           | 0.74  | 0.73  | 0.73  | 0.72  | 0.72  |
| Metacentric height GMb (mm)                                 | 41.4  | 40.2  | 39.0  | 37.7  | 36.5  |
| Natural pitch period (s) (excluding added inertia moment)    | 2.53  | 2.56  | 2.60  | 2.63  | 2.67  |

Fig. 1 Configuration and co-ordinate systems of the spar-buoy model. (*B*: centre of buoyancy, *G*: centre of gravity when the ballast position is at 0 mm.)

In the experiments, the buoy model is likely to start swaying and yawing by the effect of the vortex induced vibration. Therefore, a flat metal bar was fixed on the top of the buoy model and a restraining frame was also introduced. The actual condition of the experiments is shown in Figure 2.

3. Experimental Results

In order to investigate the coupled motion of heave and pitch of a spar-buoy model, the experiments were conducted in the ship manoeuvring research basin of TUMSAT. The principal dimensions and the photo are shown in Table 3 and Figure 3.
The free-decay experiments were performed to determine the natural periods and damping coefficients of the heave and pitch motions at different ballast position.  

Wireless transmission modules and Arduino 9 Axes Sensor Shield was installed in the buoy to record the data of the buoy motions. The pure pitch and heave motion were performed respectively with beginning of certain heave displacement and pitch angle. It was found that the 5mm ballast position is closest to the Mathieu-type instability condition (the heave natural period is almost half of the pitch’s). The estimated natural period of heave motion was 1.55s. The analysed pitch natural periods at the different ballast positions are listed in Table 5. The estimated damping ratios of heave and pitch were 0.013 and 0.038, respectively.

| Ballast position (mm) | 0   | 5   | 10  | 15  | 20  |
|-----------------------|-----|-----|-----|-----|-----|
| Natural period of pitch (s) | 2.93 | 3.24 | 3.26 | 3.30 | 3.36 |

Figure 4a to 4e show the measured time histories of heave accelerations and pitch angles with respect to the ballast positions 0, 5, 10, 15 and 20mm. The period of the incident waves was set to the heave natural period (1.55s) and the height was 42mm. Looking at the graphs, there are common observed characteristics as follows:

*The heave amplitude becomes larger alone in the first 10 seconds.

**Large pitch motions occur suddenly at around 15 second.**

*The heave amplitude shrinks just after the occurrence of larger pitch motion.

In the previous paper,[14][15] the curves of the transient periods of pitch and heave motions at timing were plotted with different ballast positions. It can be observed that the period of pitch motion is twice of the transient period of the heave motion when the occurrence of the large pitch motion. Therefore, it may be able to find some conditions in which the large pitch motion occurs suddenly by considering the theoretical condition of the Mathieu-type instability.
4. Theoretical Consideration

4.1 Mathieu Equation

In this section, the Mathieu equation is derived from the equation of pitch motion in order to discuss the auto-parametrically excited oscillation. For simplicity, the pitch motion is assumed to be a single harmonic oscillation without wave exciting moment. Expressing the pitch angle by $\phi$, the equation of pitch motion can be expressed as follows:

$$ (I_{yy} + j_{yy})\ddot{\phi} + N\phi + \Delta G\phi = 0 $$ \hspace{1cm} (1)

Where,

- $I_{yy}$: virtual inertia moment of the buoy
- $N$: equivalent linear pitch damping coefficient
- $G$: metacentric height
- $\Delta$: displacement of the buoy

In order to take the effect of heaving motion into account, $\Delta G$ is assumed to be a function of time. Assuming a resonant condition of the heaving motion, the incident wave and the heave motion can be defined by $h(t) = h_0 \sin(\omega_d t)$ and $z(t) = z_0 \sin(\omega_d t - \pi/2)$, respectively. The relative water level $\zeta(t)$ is also defined as follows:

$$ \zeta(t) = z(t) - h(t) = -\zeta_0 \sin(\omega_d t + \theta_\zeta) $$

$$ \zeta_0 = \sqrt{z_0^2 + h_0^2} \quad \theta_\zeta = \tan^{-1}(z_0/h_0) $$ \hspace{1cm} (2)

Therefore, the concrete form of $\Delta G(t)$ can be written as follows:

$$ \Delta G(t) \equiv \Delta_0 G(t) [1 - S_{\phi} \sin(\omega_d t + \theta_\zeta)] $$ \hspace{1cm} (3)

where the numerical subscript 0 denotes the value in still water and $S_{\phi}$ is expressed as follows:

$$ S_{\phi} = \frac{\rho g A_W \beta G}{\Delta_0 m^2 c_{\phi}^2} $$ \hspace{1cm} (4)

where $\rho$ denotes the density of water, $g$ the acceleration of gravity, $A_W$ the water plane area of the buoy.

Substituting Equation (3) into Equation (1) and dividing both sides by $I_{yy} + j_{yy}$, the equation of motion can be rewritten as follows:

$$ \phi + 2\nu_{\phi} \omega_{\phi n} \phi + \omega_{\phi n}^2 (1 - S_{\phi} \sin(\omega_d t + \theta_\zeta))\phi = 0 $$ \hspace{1cm} (5)

where the pitch natural frequency $\omega_{\phi n}$ and the damping ratio $\nu_{\phi}$ are expressed as follows:

$$ \omega_{\phi n} = \frac{\Delta G(t)}{\sqrt{I_{yy} + j_{yy}}} \quad \nu_{\phi} = \frac{N}{2\omega_{\phi n} (I_{yy} + j_{yy})} $$ \hspace{1cm} (6)

In order to remove the damping component, the following variable is introduced:

$$ \phi = q e^{-\nu_{\phi} \omega_{\phi n} t} $$ \hspace{1cm} (7)

Substituting this expression for $\phi$ in Equation (5), the equation can be expressed as follows:

$$ \dot{\phi} + [p - q \sin(\omega_d t + \theta_\zeta)]\phi = 0 $$ \hspace{1cm} (8)

where

$$ p = \omega_{\phi n}^2 = \omega_{\phi n}^2 (1 - \nu_{\phi}^2) \quad q = \omega_{\phi n}^2 S_{\phi} $$

In order to understand the Mathieu-type instability in the time domain, the solution of Equation (5) is examined next. As is well known, the solution becomes unstable when $\omega_d = 2\omega_{\phi n}$ and can be approximated by the following form:

$$ \phi = Ce^{\sqrt{1-4\nu_{\phi}^2} \omega_{\phi n} t} \sin\{\omega_{\phi n} t + \theta_{\phi} \} $$ \hspace{1cm} (9)

where $C$ is a constant to be determined by the initial conditions.

Therefore, looking at the exponential part of Equation (9), if the condition expressed by

$$ S_{\phi} > 4 \Delta G(t) m^2 \nu_{\phi} \sqrt{1 - \nu_{\phi}^2} $$ \hspace{1cm} (10)

is satisfied, the pitch amplitude becomes larger and larger as time proceeds.

Figure 5 shows the relationship between the pitch angle and restoring moment. Here, the pitch angle is expressed by Equation (9) under Mathieu instability condition expressed by Equation (10), and the restoring moment is expressed by Equation (3). For simplicity, the parameter $C$ in Equation 9 is set to constant and the ballast position was set to 20mm and the amplitude of heave are indicated by 15mm in the calculation. Looking at the phase relationship between the pitch motion (dashed line) and the restoring moment (full line), it can be observed that the restoring moment takes the value less than 4 (N·m) during the pitch motion is returning to the equilibrium position. On the contrary, the restoring moment takes more than 4 (N·m) when the pitch motion is leaving to the equilibrium position. It can be recognized as the physical explanation of the Mathieu-type instability.
4.2 Stability Chart of Mathieu Equation

The stability of pitch motion is analyzed based on Equation (8). Firstly, Equation (8) can be transformed to the standard form of Mathieu equation as follows:

\[ \ddot{\varphi} + (\delta + \varepsilon \cos(t))\varphi = 0 \]  

(11)

\[ \delta = \frac{\omega_m^2}{\omega_0^2} \left(1 - \nu^2\right), \varepsilon = \frac{\omega_m^2 \nu^2}{\omega_0^2}, \tau = \omega_c t + \theta_c + \frac{\pi}{2} \]  

(12)

where

\[ \varphi = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{n\pi}{2}\right) + b_n \sin\left(\frac{n\pi}{2}\right)\right) \]  

(13)

Substituting Equation (13) into Mathieu Equation (11) and collecting terms using a method called harmonic balance, four sets of algebraic equations on the coefficient \( \delta \) and \( \varepsilon \) can be obtained. Here, \( a_{even} \) and \( b_{even} \) are defined as the even coefficients of \( a_n \) and \( b_n \), respectively. \( a_{odd} \) and \( b_{odd} \) are defined as the odd coefficients of \( a_n \) and \( b_n \), respectively. For a non-zero solution, the infinite determinants are formed as follows:

\[
\begin{bmatrix}
\delta & \varepsilon/2 & 0 & 0 \\
\varepsilon & \delta - 1 & \varepsilon/2 & 0 \\
0 & \delta - 4 & \varepsilon/2 & \cdots
\end{bmatrix} = 0
\]

(14)

\[
\begin{bmatrix}
\delta - 4 & 0 & 0 & 0 \\
0 & \varepsilon/2 & \delta - 9/4 & \varepsilon/2 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} = 0
\]

(15)

\[
\begin{bmatrix}
\delta - 1/4 + \varepsilon/2 & \varepsilon/2 & 0 & 0 \\
\varepsilon/2 & \delta - 9/4 & \varepsilon/2 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} = 0
\]

(16)

\[
\begin{bmatrix}
\delta - 1/4 + \varepsilon/2 & \varepsilon/2 & 0 & 0 \\
\varepsilon/2 & \delta - 9/4 & \varepsilon/2 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} = 0
\]

(17)

Finally, the stable and unstable regions of the Mathieu equation (11) are determined. The above equations are tridiagonal matrix determinants. In order to increase the accuracy of calculation, the order of the determinants is valued by 30. For a given value \( \varepsilon \), the corresponding value \( \delta \) is calculated by solving the eigenvalues of the determinants, computing of which was programmed by MATLAB.

Figure 6 shows the stability chart of the Mathieu equation (11). ‘U’ and ‘S’ in the figure mean the unstable regions and the stable regions, respectively. According to the region where the result of \((\delta, \varepsilon)\) locates, the solution of Equation (11) can be determined to be stable or unstable. Especially when \( \delta \) equals 0.25 (dashed line in Figure 6), that is the natural frequency of pitch is half the natural frequency of the heave, it is shown the result of \((\delta, \varepsilon)\) locates are all in the unstable region. Hence, all the solutions of Equation (11) become unstable and the large pitch motion will occur. That’s the reason why to consider the initial condition when the natural frequency of pitch is half the natural frequency of the heave in the experiment. Under this condition in theoretical calculation (assume that the natural frequency of heave is twice of the pitch motion’s), the circle marks in Figure 6 are plotted to indicate the results of \((\delta, \varepsilon)\) with the ballast positions at 0mm, 5mm, 10mm, 15mm and 20mm. It is shown that all the calculated results of \((\delta, \varepsilon)\) by Equation (12) in the unstable region, which can result in the instability of pitch motion occurring in theory.

5. Analysis of the Experimental Results

In this section, the measured experimental results in Figure 4a–4e are investigated in the time history with the stability chart of the Mathieu equation. In order to analyze the phenomenon why the large pitch motions occurred suddenly at around 15 second, the corresponding peaks and troughs of pitch and heave motions are extracted out. The time histories of transient periods in the vicinity intervals where the large pitch motion occurring are calculated by the time difference of the adjacent peaks or troughs.

Figure 7 shows the measured partial time histories of pitch angle and heave acceleration with the ballast position at 0mm. The data of peaks and troughs in the vicinity of the large motions occurring are extracted to calculate the transient periods, and extracted times are defined as ‘Time1’ to ‘Time10’ (the circle marks in Figure 7). The same work is also applied to the ballast positions at 5mm, 10mm, 15mm and 20mm. The extracted times at different ballast positions keep the same characteristics as follows: When the times locate at ‘Time4’ and ‘Time5’, the large pitch motions became unstable and the large pitch motion occurred suddenly. After ‘Time5’, the pitch motions decayed, and the large pitch motions disappeared. After ‘Time10’, the large pitch motions occurred again. In order to investigate these common characteristics of time histories of transient periods under experimental results, the transient frequencies of pitch and heave are assumed to be corresponding \( \omega_{m1} \) and \( \omega_{m2} \) to calculate the results of \((\delta, \varepsilon)\) defined by Equation (12), and the results of \((\delta, \varepsilon)\) are marked into the stability charts of Mathieu equation to perform the analysis. Meanwhile, the amplitude of heave motion was assumed to be 0.1m in calculation for simplicity (the maximum amplitude of heave motions is about 0.1m in experiment results). Hence, \( S_0 \) in Equation (4) is constant, and \( \varepsilon \) is the linear function relative to \( \delta \).

Figure 8a to 8e show the time histories (‘Time1’ to ‘Time10’) of the experimental results of \((\delta, \varepsilon)\) with different ballast positions in the stability charts of Mathieu equation. The arrows are added to the graph in order to indicate the movement of stability results with the time progress. The square marks of ‘Time1’ to ‘Time4’ are always in the unstable region, under the condition of which the pitch motion becomes unstable. It is also indicated that the large pitch motions occurred suddenly when that the times at ‘Time4’ and ‘Time5’ (large pitch motions occur suddenly at around 15 second). It can be concluded that the large pitch motions can be induced when the condition of Mathieu-type instability in satisfied (the stability results of \((\delta, \varepsilon)\) is in unstable...
The first large pitch motion occurred until ‘Time5’, when at the time of which, it can be observed that all the results of stability are in the stable region with different ballast positions. The stability result of ‘Time5’ is marked by triangle marks. After ‘Time5’, it was shown in Figures 8 that all the large pitch motions disappeared (decayed) at a short time. And all the stability marks of ‘Time5’ are in the stable region (it is considered as that the condition of the Mathieu-type instability is lost at timing), which is the reason why pitch motion disappeared shortly just after the large pitch motion occurred. After large pitch motions disappeared (from ‘Time5’), all the stability marks of ‘Time6’ to ‘Time10’ return to the unstable region again, it is indicated that the large pitch motion occurred again, which is also indicated to be induced by the Mathieu-type instability.

The large pitch motions disappear (decayed) at the timing when the condition of the Mathieu-type instability is lost. Based on it, considering protecting the buoy device from damaging in the extreme weather, it is necessary to keep results of $(\delta, \varepsilon)$ always in stable region, which can reduce the overlarge pitch motion occurring. For example, the circle marks in Figures 8 is in the unstable region, if there is enough inner space for ballast to move up and down and change the value of $(\delta, \varepsilon)$ to the stable region, for example, change value of $(\delta, \varepsilon)$ to the value of triangle marks, which can reduce the occurring of the large pitch motion to some extent.

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**Fig. 7** The extracted times in the time history of heave and pitch motions. (ballast position = 0mm)

**Fig. 8a** Stability analysis of buoy motions with experiment in time history (ballast position = 0mm).

**Fig. 8b** Stability analysis of buoy motions with experiment in time history (ballast position = 5mm).

**Fig. 8c** Stability analysis of buoy motions with experiment in time history (ballast position = 10mm).

**Fig. 8d** Stability analysis of buoy motions with experiment in time history (ballast position = 15mm).

**Fig. 8e** Stability analysis of buoy motions with experiment in time history (ballast position = 20mm).
6. Conclusions

In this work, the auto-parametrically excited oscillation of a spar-buoy type point absorber with two degrees of freedom was investigated mathematically and experimentally. In order to realize the large oscillating motion based on the Mathieu-type instability, a spar buoy model with a ballast controlling system was made and the model experiments were carried out. The pitch motion in regular waves under the heave resonant period was measured to determine the natural frequencies and damping ratios with different ballast positions. In some experiments, it was observed that the large pitch motions became unstable and occurred suddenly. Based on the plotted stability charts of Mathieu equation, the occurrence of the large pitch motion was discussed. The results are summarized below:

(1) The stability chart of Mathieu equation was plotted for the experiment with the method of the harmonic balance.

(2) It could be found that all the experimental results of ($\delta$, $\epsilon$) were in the unstable region before the large pitch motion disappeared (until ‘Time5’), which inducing the Mathieu-type instability of pitch motion.

(3) Before the large pitch motion decayed, that is at ‘Time5’, all the stability results were in the stable region of the stability chart of Mathieu equation.

(4) It has been indicated that the large pitch motions disappeared (decayed) at the timing when the condition of the Mathieu-type instability is lost.

(5) It has been shown stability results returned to the unstable region, and large pitch motion occurred again after large pitch motion disappeared, which is induced by the condition of Mathieu instability.

Much more detailed experiments and the numerical simulations must be conducted as the next work in order to investigate the actual energy transfer phenomena. And the effective mechanism for the power take off also must be discussed.

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References

1) Falnes, J., 2002, “Ocean waves and oscillating systems”, Cambridge University Press.
2) Falnes, J., 2007, “A review of wave-energy extraction”, Marine Structures 20, 185-201.
3) Falcão, A. F. de O., 2010, ‘Wave energy utilization: A review of the technologies’, Renewable and Sustainable Energy Reviews, 14, pp899-918.
4) Falnes, J., 2002, “Optimum control of oscillation of wave energy converters”, International Journal of Offshore and Polar Engineering 12 (2), 147-155.
5) Falnes, J. and Budal, K., 1978, “Wave-power conversion by power absorbers, Norwegian Maritime Research 6 (4), 2-11.
6) Budal, K. and Falnes, J., 1980, “Interacting point absorbers with controlled motion”, in: B. Count (Ed.), Power form the Waves, Academic Press, London, 381-399.
7) Henriques, J. C. C., Falcão, A. F. O., Gomes, R. P. F. and Gato, L. M. C., 2012, “Latching Control of an OWC Spar-buoy Wave Energy Converter in Regular Waves”, Proc. OMAE2012-83631.
8) Rho, J.B., Choi, H.S., Shin, H. S. and Park I. K., 2005, “A Study on Mathieu-type Instability of Conventional Spar Platform in Regular Waves”, Int. J. Offshore Polar Eng., 15(2), pp. 104-108.
9) Koo, B., Kim, M. and Randall, R., 2006, “Mathieu instability of a spar platform with mooring and risers,” Ocean Engineering, 31(2), 249–256.
10) Iseki, T. 2014, “Optimization Method for Oscillation Characteristics of a Spar-buoy”, Proc. OMAE2014-23223.
11) Iseki, T. 2015, “Dynamic Control of Oscillation Characteristics of a Spar-buoy”, Maritime Technology and Engineering – Guedes Soares & Santos (Eds), 2, CRC Press, Taylor & Francis Group (London), 1243-1250.
12) Iseki, T., 2017, “Experimental Study on Dynamic Control of Oscillation Characteristics of a Spar-buoy”, Proc.OMAE2017-61612.
13) Iseki, T., 2018 “Experimental Study on Auto-parametrically Excited Heaving Motion of a Spar-buoy”, Progress in Maritime Technology and Engineering-Guedes Soares & Santos (Eds), CRC Press, Taylor &Francis Group (London), 677-684.
14) Xu, P. and Iseki, T., 2018, “Experimental Study on Oscillation Characteristics of a Spar-buoy under Mathieu Instability”, Proceedings of 3rd International Conference on Offshore Renewable Energy (CORE 2018), 2, 170-178.
15) Iseki, T. and XU P., 2019, “Experimental Study on coupled motions of a spar-buoy under Mathieu instability”, Proc. OMAE2019-95937.
16) Van der Pol, B. and Strutt, M. J. O., 1928, “On the stability of the solutions of Mathieu’s equation”, Phil. Mag. J. Sci. 5, 18-38.
17) Kovacic I, Rand R, Sah S M. Mathieu's Equation and Its Generalizations: Overview of Stability Charts and Their Features[J]. Applied Mechanics Reviews, 2018, 70(2): 02080.