CONVERGENCE OF MILSTEIN BROWNIAN BRIDGE MONTE CARLO METHODS AND STABLE GREEKS CALCULATION

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Abstract. We consider the pricing and the sensitivity calculation of continuously monitored barrier options. Standard Monte Carlo algorithms work well for pricing these options. Therefore they do not behave stable with respect to numerical differentiation. One would generally resort to regularized differentiation schemes or derive an algorithm for precise differentiation. For barrier options the Brownian bridge approach leads to a precise, but non-Lipschitz-continuous, first derivative.

In this work, we will show a weak convergence of almost order one and a variance bound for the Brownian bridge approach. Then, we generalize the idea of one-step survival, first introduced by Glasserman and Staum, to general scalar stochastic differential equations and combine it with the Brownian bridge approach leading to a new one-step survival Brownian bridge approximation. We show that the new technique can be adapted in such a way that its results satisfies stable second order Greeks. Besides studying stability, we will prove unbiasedness, leading to an uniform convergence property and variance reduction.

Furthermore, we derive the partial derivatives which allow to adapt a pathwise sensitivity algorithm. Moreover, we develop an one-step survival Brownian bridge Multilevel Monte Carlo algorithm to greatly reduce the computational cost in practice.

Key words. Monte Carlo, barrier options, pathwise sensitivities, Brownian bridge, one-step survival, second order Greeks

1. Introduction. In computational finance, Monte Carlo methods are used extensively in pricing of financial derivatives and quantitative risk management [16, 3]. We consider Monte Carlo pricing schemes for the prices and sensitivities for different types of exotic options with discontinuous payoffs, especially continuously monitored barrier options. Depending on whether a certain predefined barrier condition is fulfilled or not, a continuously monitored barrier option is knocked in or out the instant the underlying asset crosses this barrier. For an overview over other exotic options we refer to e.g. [22], particularly for options with discontinuous payoff. For an overview about various approaches aiming to price specific types of exotic options through Monte Carlo simulation we refer to the monograph by Glasserman [16].

Being among the most popular exotic derivatives, it is essential to be able to price barrier options in models that are flexible enough to describe the observed market option prices. Even though for some basic models there exist some analytical pricing formulas [19, 5], it is well known that e.g. the classical Black-Scholes model lacks the needed flexibility to fit the observed market data, see e.g. [10]. Therefore, by studying more complex stochastic models and in the case of many uncertain input parameters Monte Carlo simulation remains the preferred approach for pricing. Beside pricing, financial institutions need to evaluate the sensitivities of their portfolios due to regulations. This leads to the main challenge, particularly since some of these portfolios are huge, to compute first and higher order Greeks accurately and effectively, see e.g. [20].

Pricing an option is equal to evaluating the integral of its discounted expected payoff under a risk-neutral probability measure. In this work we are interested in the expected value $\mathbb{E}[P]$ of a quantity that is a functional $P$ of the solution of a stochastic differential equation (SDE) with a general drift and volatility term. It is

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well known, that under certain conditions [18] one obtains accurate discrete solutions for the SDE using certain discretisation schemes convergence for the expectation of a Lipschitz continuous payoff only depending on the time of maturity of this solution. In particular we are interested in the (weak) convergence

\begin{equation}
E[P - \hat{P}] \leq C h^\alpha,
\end{equation}

with a constant $C > 0$ for the approximated expected payoff $\hat{P}$ evaluated on a discretisation of the SDE using step width $h$. For example, one obtains $\alpha = 1$ for European options using the Euler-Maruyama or Milstein scheme, see e.g. [18]. However, for barrier options the payoff is discontinuous over the space of all paths. From Asmussen, Glynn and Pitman [2] we know that for any path dependent payoff using the maximum (minimum) of a discrete approximation as an approximation of the maximum (minimum), as one would do by default for barrier options, the convergence order is bounded by $\alpha = 1/2$. To recover the convergence order for barrier options, the mostly used approach is the Brownian bridge interpolation, see e.g. [16], sampling if the maximum exceeds the barrier between two steps. In a recent work, Giles, Debrabant and Rössler [13] point out that their work on the Multilevel Monte Carlo analysis for the Brownian bridge approach could be modified to show that the Brownian bridge interpolation satisfies $\alpha = 1 - \delta$, for any $\delta > 0$. For sake of completeness the first part of this work will be to prove this property.

Finite difference approximations of the numerically calculated prices to compute option sensitivities is very simple but hard to control since differentiation is generally unstable. Even the smallest numerical errors may have arbitrarily large effects on the finite difference approximation. This property, known as ill-posedness, cf., Engl, Hanke and Neubauer [9], requires further studies. In [1] it was shown, that a Monte Carlo pricing algorithm that uses a Lipschitz continuous payoff allows stable differentiation by simple finite differences. The Brownian bridge estimator leads to stable first order Greeks, since the modified payoff function is Lipschitz continuous. However, differentiating the payoff of the Brownian bridge estimator leads to a non-Lipschitz continuous first derivative, as analytically calculated in [6], not leading to stable second order Greeks.

To overcome this problem, we combine the one-step survival strategy [17] with the Brownian bridge method to obtain a new approach which allows a stable second order Greek computation. Therefore, we extend the technique to the Milstein discretisation scheme and apply this to the Brownian bridge approach. The second main part of this article will be to show that this new one-step survival Brownian bridge estimator has the same expectation and a reduced variance. Not only granting variance reduction, the new estimator allows stable second order Greek calculation, since having Lipschitz continuous first order Greeks. Furthermore, we will extend the result of stable first order differentiation, by finite differences, of [1] to stable second order differentiation, by finite differences, and additionally present the partial derivatives of the new approach. We will compare these derivatives with the Brownian bridge derivatives to study the stability. Since, providing smooth functions but a modified Milstein scheme, the pathwise sensitivity approach of the Brownian bridge estimator, see [6], can not be implemented in a straightforward way. Nevertheless, the approach can be combined with the ideas of [11].

As already mentioned, Monte Carlo methods can be computationally expensive as in the case of stochastic differential equations, particularly since the cost of generating the individual stochastic samples is very high. It is well established that the
computational complexity (cost) is $O(e^{-3})$, see e.g. [8], provided that the stochastic differential equations satisfy certain conditions [18, 4, 21]. Giles [14, 12] shows that multigrid ideas can be used to reduce the computational complexity to $O(e^{-2})$, under certain conditions, using the Multilevel Monte approach by performing most of the simulations with low accuracy at a correspondingly low cost. The Multilevel Monte Carlo method got various generalizations and extensions, see [15] for an overview. Giles, Debrabant and Roessler [13] showed that the Brownian bridge approach for barrier options satisfies the needed convergence properties leading to an efficient Multilevel Monte Carlo method. Even though, the Multilevel Monte Carlo estimator for the Brownian bridge approach is not easily applied to the new one-step survival Brownian bridge estimator, since the coarse path modification would lead to biased one-step survival probabilities. Nevertheless, we will provide a Multilevel Monte Carlo algorithm and show its efficiency in numerical results. For some further Brownian bridge Multilevel Monte Carlo approach results we refer to [7, 6].

The structure of this work is as follows. In section 2 we present the main result on the one-step survival Brownian bridge approximation in the first subsection and the convergence result of the Brownian bridge approximation in the second subsection. Then, in the following two subsections we study first and second order Greeks, including the pathwise sensitivity approximation for the one-step survival Brownian bridge approach and the stability result for second order finite differences. In section 3 we present the Multilevel Monte Carlo algorithm for the new approach. Numerical results for the variance reduction, the stability of Greeks and the efficiency of the Multilevel algorithm are provided in section 4. In section 5 we present the proof to the convergence theorem of the Brownian bridge approach. Section 6 contains some concluding remarks.

2. One-step survival Brownian bridge Monte Carlo estimator for continuously monitored barrier options. We will focus barrier options, which only depend on one underlying asset. We are interested in the expected value of an payoff $P$ that is a functional of the asset price. In specific we suppose to have the following model for the asset price.

**Definition 2.1.** The underlying asset price $S(t)$ is a continuous time stochastic process whose evolution SDE is of the generic form

\[ dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dW(t), \]

on the time interval $t \in [t_0, T]$, with initial data $S_0$, drift $\mu$, volatility $\sigma$ and the Brownian motion $W$.

We assume (2.1) to be scalar and that the drift $\mu \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ and the volatility $\sigma \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ satisfy the following standard conditions:

- **A1** (uniform Lipschitz condition): There exists $K_1 > 0$ such that
  \[ |\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| + |L_1 \sigma(x, t) - L_1 \sigma(y, t)| \leq K_1 |x - y|. \]

- **A2** (linear growth bound): There exists $K_2 > 0$ such that
  \[ |\mu(x, t)| + |L_0 \mu(x, t)| + |L_1 \mu(x, t)| + |\sigma(x, t)| + |L_0 \sigma(x, t)| + |L_1 \sigma(x, t)| \leq K_2 (1 + |x|). \]

- **A3** (additional Lipschitz condition): There exists $K_3 > 0$ such that
  \[ |\sigma(x, t) - \sigma(x, s)| \leq K_3 (1 + |x|) \sqrt{|t - s|}, \]
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for all \(x, y, t, s\) and with \(L_0 = \partial / \partial t + \mu \partial / \partial S\) and \(L_1 = \sigma \partial / \partial S\).

In this section we will introduce continuously monitored barrier options and a new algorithm for pricing these barrier options, i.e. approximating the present value. We address up-and-out barrier call options, however the conversion to put or down-and-out options is straightforward.

**Definition 2.2.** The payoff of a continuously observed knock-up-out barrier call option is given by

\[
P(S) := \begin{cases} 
(S(T) - K)^+ =: q(S(T)) & \max_{t \in [t_0, T]} S(t) \leq B \\
0 & \text{otherwise,}
\end{cases}
\]

with barrier value \(B\), strike price \(K\), time of maturity \(T\) and current time \(t_0\).

As stated above, we are interested in the expected value of such an instrument, which is defined as follows.

**Definition 2.3.** The present value of an option with payoff (2.2) is given by the discounted expected payoff

\[
PV_{t_0} = e^{-r(T-t_0)}E[P],
\]

at the current time \(t_0\) and at the time of maturity \(T\).

In order to keep the equations clearer, we will not include the discount factor \(e^{-r(T-t_0)}\) in the following work and therefore only study the expected value. The same results hold for the present value.

It is well known, provided assumptions A1-A3 are satisfied, see e.g. [18], that the Milstein scheme

\[
\begin{aligned}
\hat{S}_{n+1} &= \hat{S}_n + \mu(\hat{S}_n, t_n)h + \sigma(\hat{S}_n, t_n)\sqrt{h}\Delta Z_n \\
&\quad + \frac{1}{2} \sigma(\hat{S}_n, t_n)\sigma'(\hat{S}_n, t_n)(\sqrt{h}\Delta Z_n)^2 - h,
\end{aligned}
\]

with \(n = 0, \ldots, N\) discretization steps, \(\Delta Z_n \sim N(0, 1)\), \(\hat{S}_0 = S_0\), \(h = T/N\), \(t_n\) the time \(t\) at step \(n\), converge, in the sense of (1.1), with \(\alpha = 1\) for Lipschitz continuous payoff functions only depending on the time of maturity, see e.g. [16]. To recover the convergence order for barrier options the difficulty can be circumvented by sampling if the maximum exceeds the barrier between two discretisation steps, instead of sampling the maximum itself, see e.g. Glasserman [16] for an derivation. Thereby we condition the approximation on the Brownian increments using

\[
\hat{p}_n = \exp\left(\frac{-2(B - \hat{S}_n)^+(B - \hat{S}_{n+1})^+}{\sigma(\hat{S}_n, t_n)^2 h} - h\right),
\]

with \(n = 0, \ldots, N - 1\), which is the probability of crossing the barrier between two steps. This approach leads to the Brownian bridge approximation of the payoff (2.2) defined by

\[
\hat{P} = \prod_{n=0}^{N-1} (1 - \hat{p}_n) \cdot q(\hat{S}_N),
\]

with \(q\) defined in (2.2).
2.1. One-step survival Brownian bridge approximation. We now combine the Brownian bridge approximation with the one-step survival idea of Glasserman and Staum [17] to define the one-step survival Brownian bridge approximation and provide some results in the following theorem.

Theorem 2.4. Provided assumptions A1-A3 are satisfied, the one-step survival Brownian bridge approximation defined by

\[ \tilde{P} := \prod_{n=0}^{N-1} (1 - \tilde{p}_n^n) \cdot \prod_{n=0}^{N-1} (\tilde{p}_n^n) \cdot q(\tilde{S}_N) \]  

satisfies

\[ E[\tilde{P}] = E[\tilde{P}], \]  
\[ \text{Var}[\tilde{P}] \leq \text{Var}[\tilde{P}]. \]

For the path simulation the following modified Milstein scheme is used:

\[ \tilde{S}_{n+1}(u^{(n+1)}) = \tilde{S}_n + \mu(\tilde{S}_n, t_n) h + \sigma(\tilde{S}_n, t_n) \sqrt{h} \Phi^{-1}(\tilde{p}_n^{(2)}) + \tilde{p}_n^{(1)} u^{(n+1)} \]  

\[ + \frac{1}{2} \sigma(\tilde{S}_n, t_n)(\sigma'(\tilde{S}_n, t_n) - (\sqrt{h} \Phi^{-1}(\tilde{p}_n^{(2)}) + \tilde{p}_n^{(1)} u^{(n+1)}))^2 - h, \]

with \( u^{(n+1)} \sim U(0, 1), \tilde{S}_0 = S_0 \), the survival probabilities

\[ \tilde{p}_n^{(1), (2)} = \Phi \left( -1 \pm \sqrt{1 + \frac{4}{\sigma(\tilde{S}_n, t_n)^2} \left( \frac{B - \tilde{S}_n - \mu(\tilde{S}_n, t_n) h + \frac{1}{2} \sigma(\tilde{S}_n, t_n) \sigma'(\tilde{S}_n, t_n) h}{\sigma(\tilde{S}_n, t_n)} \right)} \right) \]  

\[ \tilde{p}_n = \tilde{p}_n^{(1)} - \tilde{p}_n^{(2)} \]

and with the modified, i.e. without the characteristic functions, crossing probabilities

\[ \tilde{p}_n = \exp \left( \frac{-2(B - \tilde{S}_n)(B - \tilde{S}_n+1)}{\sigma(\tilde{S}_n, t_n)^2 h} \right). \]

Proof. For (2.7) we have to verify the equivalence of the expected values of (2.5) and (2.6). We start at the first step of the expectation of (2.5) given by

\[ E[\tilde{P}_{S_0}] = \int_{-\infty}^{\infty} \phi(z) (1 - \tilde{p}_0) E[\tilde{P}_{\tilde{S}_1(z)}] dz, \]

with the crossing probability

\[ \tilde{p}_0 = \exp \left( \frac{-2(S_0 - B)^+(\tilde{S}_1 - B)^+}{\sigma(S_0, t_0)^2 h} \right) \]

and the first step of the Milstein Scheme

\[ \tilde{S}_1(z) = S_0 + \mu(S_0, t_0) h + \sigma(S_0, t_0) \sqrt{h} z + \frac{1}{2} \sigma(S_0, t_0) \sigma'(S_0, t_0) \left( (\sqrt{h} z)^2 - h \right). \]
Since we know, that the payoff will become zero if the asset price \( \tilde{S}_1 \) crosses the barrier \( B \) we split the integral such that

\[
E[\tilde{P}_{S_0}] = \int_{\tilde{S}_1(z) < B} \phi(z) \left( 1 - \tilde{p}_0 \right) E[\tilde{P}_{\tilde{S}_1(z)}] \, dz + 0.
\]

We can formulate analogue formulas for \( E[\tilde{P}_{S_1}], \ldots, E[\tilde{P}_{S_{N-2}}] \) and for the last discretisation step we obtain

\[
E[\tilde{P}_{S_{N-1}}] = \int_{\tilde{S}_N(z) < B} \phi(z) \left( 1 - \tilde{p}_{N-1} \right) q(\tilde{S}_N(z)) \, dz + 0,
\]

with the Lipschitz payoff \( q(S(t)) \) of (2.2).

Here, \( \phi(z) \) is no longer a probability density and we normalize the integral with

\[
\tilde{p}_0 := \int_{\tilde{S}_1(z) < B} \phi(z) \, dz
\]

resulting in

\[
E[\tilde{P}_{S_0}] = \tilde{p}_0 \int_{\tilde{S}_1(z) < B} \frac{\phi(z)}{\tilde{p}_0} \left( 1 - \tilde{p}_0 \right) E[\tilde{P}_{\tilde{S}_1(z)}] \, dz.
\]

With (2.3) we know that \( \tilde{S}_{n+1}(z) < B \) equals

\[
0 > -\frac{B - \tilde{S}_n - \mu(\tilde{S}_n, t_n) h + \frac{1}{2} \sigma(\tilde{S}_n, t_n) \sigma'(\tilde{S}_n, t_n) h}{\sigma(\tilde{S}_n, t_n) \sqrt{h}} + z + \frac{1}{2} \sigma'(\tilde{S}_n, t_n) \sqrt{h} z^2.
\]

We have

\[
z < \frac{-1 + \sqrt{1 + 4 \left( \frac{1}{2} \sigma'(\tilde{S}_n, t_n) \sqrt{h} \right) \left( \frac{B - \tilde{S}_n - \mu(\tilde{S}_n, t_n) h + \frac{1}{2} \sigma(\tilde{S}_n, t_n) \sigma'(\tilde{S}_n, t_n) h}{\sigma(\tilde{S}_n, t_n) \sqrt{h}} \right)}}{2 \left( \frac{1}{2} \sigma'(\tilde{S}_n, t_n) \sqrt{h} \right)} := \Phi^{-1}(\tilde{p}_n^{(1)})
\]

and

\[
z > \frac{-1 - \sqrt{1 + 4 \left( \frac{1}{2} \sigma'(\tilde{S}_n, t_n) \sqrt{h} \right) \left( \frac{B - \tilde{S}_n - \mu(\tilde{S}_n, t_n) h + \frac{1}{2} \sigma(\tilde{S}_n, t_n) \sigma'(\tilde{S}_n, t_n) h}{\sigma(\tilde{S}_n, t_n) \sqrt{h}} \right)}}{2 \left( \frac{1}{2} \sigma'(\tilde{S}_n, t_n) \sqrt{h} \right)} := \Phi^{-1}(\tilde{p}_n^{(2)})
\]

with \( \tilde{p}_n := \tilde{p}_n^{(1)} - \tilde{p}_n^{(2)} \). Using \( t = 0 \) leads to \( \Phi^{-1}(\tilde{p}_0^{(1)}) \) and \( \Phi^{-1}(\tilde{p}_0^{(2)}) \) and we have

\[
E[\tilde{P}_{S_0}] = \tilde{p}_0 \int_{\Phi^{-1}(\tilde{p}_0^{(1)})}^{\Phi^{-1}(\tilde{p}_0^{(2)})} \frac{\phi(z)}{\tilde{p}_0} \left( 1 - \tilde{p}_0 \right) E[\tilde{P}_{\tilde{S}_1(z)}] \, dz.
\]
Now, by substituting with \( z = \Phi^{-1}(\tilde{p}_0^{(2)} + (\tilde{p}_0^{(1)} - \tilde{p}_0^{(2)}) \cdot u) \) we can follow

\[
(2.17) \quad \mathbb{E}[\tilde{P}_{S_0}] = \tilde{p}_0 \int_0^1 (1 - \tilde{p}_0^u) \mathbb{E}[	ilde{P}_{S_1(u)}] \, du,
\]

with the modified asset price

\[
\tilde{S}_1(u) = S_0 + \mu(S_0, t_0) h + \sigma(S_0, t_0) \Phi^{-1}(\tilde{p}_0^{(2)} + u \cdot \tilde{p}_0) + \frac{1}{2} \sigma(S_0, t_0) \sigma'(S_0, t_0) \left( (\sqrt{h} \Phi^{-1}(\tilde{p}_0^{(2)} + u \cdot \tilde{p}_0))^2 - h \right)
\]

and with

\[
\tilde{p}_0 := \exp \left( \frac{-2(S_0 - B)(\tilde{S}_1 - B)}{\sigma(S_0, t_0)^2 h} \right),
\]

since now for \( \tilde{p}_0 \) no indicator functions are needed, while not allowing a barrier crossing for \( S_0 \) and \( \tilde{S}_1 \).

Continuing by replacing \( \mathbb{E}[\tilde{P}_{S_1(u)}] \) of (2.17) with a similar formula as in (2.15), the expected value is given by

\[
\mathbb{E}[\tilde{P}_{S_0}] = \tilde{p}_0 \int_0^1 (1 - \tilde{p}_0^u) \left[ \int_{\tilde{S}_2(z) < B} \phi(z) (1 - \tilde{p}_1) \mathbb{E}[\tilde{P}_{\tilde{S}_2(z,u)}] \, dz \right] \, du,
\]

with

\[
\tilde{S}_2(z, u) = \tilde{S}_1(u) + \mu(\tilde{S}_1(u), t_1) h + \sigma(\tilde{S}_1(u), t_1) z + \frac{1}{2} \sigma(\tilde{S}_1(u), t_1) \sigma'(\tilde{S}_1(u), t_1) \left( (\sqrt{z}h)^2 - h \right).
\]

Again by splitting, substituting and with \( \tilde{p}_1 = \tilde{p}_1^{(2)} - \tilde{p}_1^{(1)} \) we obtain

\[
\mathbb{E}[\tilde{P}_{S_0}] = \tilde{p}_0 \int_0^1 (1 - \tilde{p}_0^u) \left[ \int_{\tilde{S}_2(z) < B} \phi(z) (1 - \tilde{p}_1) \mathbb{E}[\tilde{P}_{\tilde{S}_2(z,u)}] \, dz \right] \, du
\]

\[
= \tilde{p}_0 \int_0^1 (1 - \tilde{p}_0^u) \left[ \int_{\Phi^{-1}(\tilde{p}_1^{(1)})} \phi(z) (1 - \tilde{p}_1) \mathbb{E}[\tilde{P}_{\tilde{S}_2(z,u)}] \, dz \right] \, du
\]

\[
= \tilde{p}_0 \int_0^1 (1 - \tilde{p}_0^u) \tilde{p}_1 \int_0^1 (1 - \tilde{p}_1) \mathbb{E}[\tilde{P}_{\tilde{S}_2(u^{(2)}, u^{(1)})}] \, P \, du^{(2)} \, du^{(1)}.
\]

with

\[
\tilde{S}_2(u^{(2)}, u^{(1)}) = \tilde{S}_1(u^{(1)}) + \mu(\tilde{S}_1(u^{(1)}), t_1) h + \sigma(\tilde{S}_1(u^{(1)}), t_1) \Phi^{-1}(\tilde{p}_1^{(2)} + \tilde{p}_1 u^{(2)}) + \frac{1}{2} \sigma(\tilde{S}_1(u^{(1)}), t_1) \sigma'(\tilde{S}_1(u^{(1)}), t_1) \left( (\sqrt{\eta h} \Phi^{-1}(\tilde{p}_1^{(2)} + \tilde{p}_1 u^{(2)}))^2 - h \right)
\]
and
\[ \tilde{p}_t = \exp \left( \frac{-2(\tilde{S}_1 - B)(\tilde{S}_2 - B)}{\sigma(S_1, t_1)^2 h} \right). \]

By iteratively splitting and substituting till the last discretisation step, while using (2.16) in the last step, we obtain
\[
\mathbb{E}[\widetilde{P}] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(z^{(1)}) \cdots \phi(z^{(N)}) \cdot \prod_{n=0}^{N-1} (1 - \tilde{p}_n)
\cdot q(\tilde{S}_N(z^{(N)}, \ldots, z^{(1)})) \, dz^{(N)} \cdots dz^{(1)},
\]
(2.18)
\[
= \int_{0}^{1} \cdots \int_{0}^{1} \prod_{n=0}^{N} (1 - \tilde{p}_n)^2 \prod_{n=0}^{N} (\tilde{p}_n) \cdot q(\tilde{S}_N(u^{(N)}, \ldots, u^{(1)})) \, du^{(N)} \cdots du^{(1)}
\]
\[= \mathbb{E}[\widetilde{P}], \]
with (2.10), (2.11), (2.9) and (2.12).

For (2.8) we first see that with analogue techniques as used in the first part of the proof we obtain
(2.19)
\[
\mathbb{E}[\tilde{P}^2] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(z^{(1)}) \cdots \phi(z^{(N)}) \cdot \prod_{n=0}^{N-1} (1 - \tilde{p}_n)^2
\cdot q(\tilde{S}_N(z^{(N)}, \ldots, z^{(1)}))^2 \, dz^{(N)} \cdots dz^{(1)},
\]
\[
= \int_{0}^{1} \cdots \int_{0}^{1} \prod_{n=0}^{N} (1 - \tilde{p}_n)^2 \prod_{n=0}^{N} (\tilde{p}_n) \cdot q(\tilde{S}_N(u^{(N)}, \ldots, u^{(1)}))^2 \, du^{(N)} \cdots du^{(1)}
\]
\[\geq \int_{0}^{1} \cdots \int_{0}^{1} \prod_{n=0}^{N} (1 - \tilde{p}_n)^2 \prod_{n=0}^{N} (\tilde{p}_n)^2 \cdot q(\tilde{S}_N(u^{(N)}, \ldots, u^{(1)}))^2 \, du^{(N)} \cdots du^{(1)}
\]
\[= \mathbb{E}[\tilde{P}^2], \]
which holds since we have \(\tilde{p}_n \in [0, 1]\) for every \(t = 0, \ldots, N - 1\) implying
(2.20)
\[\prod_{n=0}^{N} (\tilde{p}_n)^2 \leq \prod_{n=0}^{N} (\tilde{p}_n).\]

All in all we obtain (2.8)
\[\text{Var}[\tilde{P}] = \mathbb{E}[\tilde{P}^2] - \mathbb{E}[\tilde{P}]^2 = \mathbb{E}[\tilde{P}^2] - \mathbb{E}[\tilde{P}]^2 \leq \mathbb{E}[\tilde{P}^2] - \mathbb{E}[\tilde{P}]^2 = \text{Var}[\tilde{P}], \]
since we have \(\mathbb{E}[\tilde{P}^2] = \mathbb{E}[\tilde{P}]^2\) as a consequence of (2.7).

We want to remark that the variance reduction is a consequence of (2.20), which is, as it was to be expected, most significant near the barrier.

By sampling a sequence of possible realizations \((\tilde{s}_{1,m}, \ldots, \tilde{s}_{N,m})\), \(m = 1, \ldots, M\), of the random variables \((\tilde{S}_1, \ldots, \tilde{S}_N)\), we obtain the unbiased one-step survival Brownian bridge Monte Carlo estimator for \(PV_{0}\), see e.g. [16].
Corollary 2.5. The new one-step survival Brownian bridge Monte Carlo estimator

\[ P_M := \frac{1}{M} \sum_{m=1}^{M} \prod_{n=0}^{N-1} (1 - \hat{p}_{n,m}^*) \prod_{n=0}^{N-1} (\hat{p}_{n,m}) q(s_{N,m}), \]  

is unbiased.

The extension to knock-down out barrier options is done by splitting and with a modified substitution, see [11] for further information. The extension to knock-in options is not straightforward, but with the in-out parity the pathwise sensitivities can be calculated through knock-out and plain vanilla options.

2.2. Convergence and variance bound for the Brownian bridge approximation. In this section we prove a convergence property of the Brownian bridge approximation and we will show a variance bound.

Theorem 2.6. Provided assumptions A1-A3 are satisfied and using the Milstein Scheme (2.3), the Brownian bridge approximation (2.5) satisfies

\[ \mathbb{E}[P - \hat{P}] \leq C_1 h^{1-\delta}, \]  
\[ \text{Var}[\hat{P}] \leq C_2 \]  

for a constant \( C_1 \), any \( \delta > 0 \) and a constant \( C_2 \) which does not depend on \( h \).

To prove the theorem we will use the following known results, see e.g. [18, 13].

Theorem 2.7. Provided assumptions A1-A3 are satisfied, then for all positive integers \( m \) there exists a constant \( C_m \) such that

\[ \mathbb{E}\left[ \sup_{0 \leq t \leq T} |S(t)|^m \right] < C_m. \]  

Definition 2.8. We define the Kloeden & Platen continuous time interpolant of the Milstein scheme (2.3) for \( t_n \leq t \leq t_{n+1} \) by

\[ \hat{S}_{KP}(t) = \hat{S}_n + \mu(\hat{S}_n, t_n)(t - t_n) + \sigma(\hat{S}_n, t_n)(W(t) - W_{t_n}) + \frac{1}{2} \sigma(\hat{S}_n, t_n) \sigma'(\hat{S}_n, t_n) \left( (W(t) - W_{t_n})^2 - (t - t_n) \right). \]  

Theorem 2.9. Provided assumptions A1-A3 are satisfied, then for all positive integers \( m \) there exists a constant \( C_m \) such that

\[ \mathbb{E}\left[ \sup_{0 \leq t \leq T} |S(t) - \hat{S}_{KP}(t)|^m \right] < C_m h^m, \]  
\[ \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\hat{S}_{KP}(t)|^m \right] < C_m, \]

whereas \( C_m \) does not depend on the step size \( h \).

Theorem 2.10. If \( Y \) is a scalar random variable, \( \mathbb{E}[Y^2] \) is uniformly bounded, and for each \( p > 0 \), the indicator function \( 1_E \) (which takes value 1 or 0 depending whether or not a path lies within some set \( E \)) satisfies

\[ \mathbb{E}[1_E] = o(h^p), \]  

then for each \( p > 0 \),

\[ \mathbb{E}[|Y| 1_E] = o(h^p). \]
Definition 2.11. We define the Brownian bridge interpolation for \( t_n \leq t \leq t_{n+1} \)

\[
\hat{S}(t) = \hat{S}_n + \frac{(t-t_n)}{h} (\hat{S}_{t_{n+1}} - \hat{S}_{t_n}) \\
+ \sigma(\hat{S}_{t_n}, t_n) (W(t) - \hat{W}(t))
\]

with the piecewise linear interpolant \( \hat{W}(t) = W_{t_n} - \frac{(t-t_n)}{h} (W_{t_{n+1}} - W_{t_n}) \) of the discrete values \( W_{t_n} \).

Theorem 2.12. Provided assumptions A1-A3 are satisfied, then for any \( \gamma > 0 \), the probability that a Brownian path \( W(t) \), its increments \( \Delta W_n \equiv W((n+1)h) - W(nh) \), and the corresponding SDE solution \( S(t) \) and its path approximations \( \hat{S}_n \) satisfy any of the following extreme conditions

\[
\max_n \left( \max(|S(nh)|, |\hat{S}_n|) \right) > h^{-\gamma} \\
\max_n \left( \max(|S(nh) - \hat{S}_n|) \right) > h^{1-\gamma} \\
\max_n |\Delta W_n| > h^{1/2-\gamma} \\
\sup_{[0,T]} |\hat{S}(t) - S(t)| > h^{1-\gamma} \\
\sup_{[0,T]} |W(t) - \hat{W}(t)| > h^{1/2-\gamma}
\]

is \( o(h^p) \) for all \( p > 0 \). If none of these extreme conditions is satisfied, and \( \gamma < 1/2 \) then

\[
\max_n |\hat{S}_n - \hat{S}_{n-1}| \prec h^{1/2-2\gamma} \\
\max_n |\sigma_n - \sigma_{n-1}| \prec h^{1/2-2\gamma} \\
\max_n (\max(|\sigma_n|)) \prec h^{-\gamma},
\]

with \( u :< h^\alpha \) when \( u > 0 \) and there exists a constant \( c > 0 \) such that \( u < ch^\alpha \), for sufficiently small \( h \).

Furthermore, we will make use of the following lemmata.

Lemma 2.13. Provided assumptions A1-A3 are satisfied, then for the approximation

\[
\tilde{P} \text{euro}p. := q(\hat{S}_N)
\]

of a European plain vanilla option with the Milstein scheme, we have that for all integers \( m \) there exists a constant \( C_m \) such that

\[
\limsup_{h \downarrow 0} \mathbb{E} \left[ |\tilde{P} \text{euro}p.|^m \right] < C_m
\]

whereas \( C_m \) does not depend on the step size \( h \).
Proof. Since $\hat{P}_{\text{europe}}$ is Lipschitz and with Theorem 2.9 we have
\[
\limsup_{h \downarrow 0} E \left[ |\hat{P}_{\text{europe}}|^m \right] \leq \limsup_{h \downarrow 0} L^m \cdot E \left[ |\hat{S}_N|^m \right] 
\leq L^m \cdot E \left[ \limsup_{h \downarrow 0} |\hat{S}_N|^m \right] 
\leq L^m \cdot E \left[ \lim_{h \downarrow 0} \left( \sup_{0 \leq t \leq T} |\hat{S}_{KP}(T)|^m \right) \right] 
= L^m \cdot E \left[ \sup_{0 \leq t \leq T} |\hat{S}_{KP}(T)|^m \right] < C_m
\]
since on the fixed discretisation steps the Milstein scheme is equal to the Kloeden & Platen interpolant for which the estimate is independent of $h$. 

**Lemma 2.14.** Provided assumptions A1-A3 are satisfied, then the Brownian bridge approximation satisfies that for all integers $m$ there exists a constant $C_m$ such that
\[
(2.24) \quad \limsup_{h \downarrow 0} E \left[ |\hat{P}|^m \right] < C_m,
\]
whereas $C_m$ does not depend on the step size $h$.

Proof. With (2.4) we see that $\prod_{n=0}^{N-1} (1 - \hat{p}_n) \leq 1$, since $\hat{p}_n \in [0,1]$, which implies that $\limsup_{h \downarrow 0} \prod_{n=0}^{N-1} (1 - \hat{p}_n) \leq 1$. With the monotonicity of the expected value and Lemma 2.13 we obtain
\[
\limsup_{h \downarrow 0} E \left[ |\hat{P}|^m \right] = \limsup_{h \downarrow 0} E \left[ \left( \prod_{n=0}^{N-1} (1 - \hat{p}_n) \right) \hat{P}_{\text{europe}} \right]^m 
\leq \limsup_{h \downarrow 0} E \left[ |\hat{P}_{\text{europe}}|^m \right] < C_m.
\]

**Lemma 2.15.** Provided the assumptions A1-A3 the expected value of (2.2) satisfies
\[
E[|P|^m] < C_m,
\]
for all positive integers $m$.

Proof. With Theorem 2.7 and the Lipschitz property we follow:
\[
E[|P|^m] \leq E \left[ |(S(T) - K)^+|^m \right] \leq L^m E[|S(T)|^m] < C_m.
\]

**Proof of Theorem 2.6.** First we will show
\[
(2.25) \quad E[|P - \hat{P}|] \leq Ch^{1/2 - \delta}.
\]
Therefore we divide the paths into the following three subsets:
(i) extreme paths
(ii) paths which are not extreme and for which $|S_{\text{max}} - B| \geq h^{1/2 - \gamma}$ for $0 < \gamma < \frac{1}{8}$
(iii) the rest
and see
\[ E[P - \hat{P}] = E[(P - \hat{P})1_{(i)}] + E[(P - \hat{P})1_{(ii)}] + E[(P - \hat{P})1_{(iii)}], \]
with the indicator functions to be unit value for paths in the respected subset. Each of these is considered and their contributions to \( E[P - \hat{P}] \) are bounded.

(i) Paths satisfying any of the conditions of Theorem 2.12 are defined to be extreme for \( 0 < \gamma < \frac{1}{8} \). Lemma 2.14 and Lemma 2.15 deliver a uniform bound for \( E[|\hat{P}|^2] \) and \( E[|P|^2] \) and therefore also for \( E[(P - \hat{P})^2] \). By Theorem 2.10 we see \( E[(P - \hat{P})1_{(i)}] = o(h^p) \) for all \( p > 0 \), since \( E[1_{(i)}] = o(h^p) \).

(ii) We suppose \( S(t) \) attains its maximum at \( \tau \in [t_n, t_{n+1}] \). First we consider the case \( S_{\text{max}} > B + h^{1/2-4\gamma} \), where we have to study the interpolant, since \( S_{\text{max}} \) could be between two discretisation steps. The first summand of the right hand side of
\[ |\hat{S}_n - S_{\text{max}}| \leq |\hat{S}_n - \hat{S}(\tau)| + |\hat{S}(\tau) - S(\tau)| \]
can be written as
\[ \hat{S}(\tau) - \hat{S}_n = \frac{\tau - t_n}{h} \left( \hat{S}_{n+1} - \hat{S}_n \right) + \sigma(\hat{S}_n, t_n) \left( W(t) - \bar{W}(t) \right) \]
and together with Theorem 2.12 we can conclude that \( |\hat{S}_n - S_{\text{max}}| \prec h^{1/2-2\gamma} \).

Hence, for sufficiently small \( h \) we have \( |\hat{S}_n - S_{\text{max}}| \prec h^{1/2-4\gamma} \) and therefore \( \hat{S} \) is guaranteed to be greater than \( B \) and hence \( \hat{P} - P = 0 \).

Considering \( S_{\text{max}} < B - h^{1/2-4\gamma} \) we don’t have to study the interpolant but the conditioning probabilities. We have
\[ \max_n \max |\hat{S}_n| < B - h^{1/2-4\gamma} + h^{1-\gamma}, \]
since it is not extreme. Since \( h^{1-\gamma} \prec h^{1/2-4\gamma} \) it follows that \( \prod_t (1 - \rho_t) \) is equal to \( 1 - o(h^p) \) for all \( p > 0 \). Hence, with the Lipschitz condition and the bound on \( S_N - S(T) \) for non extreme paths from Theorem 2.12 we obtain
\[ E[(P - \hat{P})1_{(ii)}] \] is at most \( O(h^{1-\gamma}) \), since \( E[1_{(ii)}] = 1 \).

(iii) Since \( E[1_{(iii)}] = o(h^{1-3\gamma}) \) and \( E[|\hat{P}|^2] \) and \( E[|P|^2] \) are bounded, it follows
\[ E[(P - \hat{P})1_{(iii)}] \] is at most \( O(h^{1-4\gamma}) \). Finally we obtain (2.25) by choosing \( \gamma < \min(\frac{1}{8}, \delta/4) \).

In the proof of Theorem 3.16 in [13] we see that for set (iii) we have \( \hat{P}_t - \hat{P}_{t-1} \prec h_t^{1/2-6\gamma} \), with \( h_t = 2^{-t} \) implying that \( E[(\hat{P}_t - \hat{P}_{t-1})1_{(iii)}] \) is at most \( O(h_t^{1-10\gamma}) \). Adding the results of the cases (i) and (ii), which are \( E[(\hat{P}_t - \hat{P}_{t-1})1_{(ii)}] \) is at most \( O(h_t^{1-1\gamma}) \) and \( E[(\hat{P}_t - \hat{P}_{t-1})1_{(i)}] \) is \( O(h^p) \) and by choosing \( \gamma < \min(\frac{1}{8}, \delta/10) \) we obtain
\[ E[\hat{P}_t - \hat{P}_{t-1}] \leq C h_t^{1-\delta}, \]
for \( C \) and any \( \delta > 0 \). Hence, \( E[\hat{P}_t] \) forms a Cauchy series with \( E[\hat{P}_{t+k} - \hat{P}_{t+k-1}] \leq C_1 h_t^{1-\delta} \) which is converging to \( E[P] \), due to (2.25). It has an error of \( O(h^{1-\delta}) \) leading to \( E[P - \hat{P}] \leq C_1 h^{1-\delta} \).

To see (2.23) we use Lemma 2.14 which delivers
\[ \lim_{h \downarrow 0} \sup E \left| \hat{P}_t^2 \right| < C_2 \]
and hence
\[ \limsup_{h \downarrow 0} \text{Var}[\hat{P}] = \limsup_{h \downarrow 0} (E[\hat{P}^2] - E[\hat{P}]^2) \leq \limsup_{h \downarrow 0} E[\hat{P}^2] < C_2, \]
which completes the proof.

We know, see e.g. [16], that if a Monte Carlo estimator is unbiased it satisfies
\[ \text{Var} [\hat{P}_N] = \frac{1}{N} \text{Var} [\hat{P}]. \]

**Corollary 2.16.** The Brownian bridge Monte Carlo estimator for the present value of a knock-up-out barrier option given by the average
\[ \hat{P}_M := \frac{1}{N} \sum_{m=1}^{m} \prod_{n=1}^{N-1} (1 - \hat{p}_{n,m}) \cdot q(\hat{s}_{N,M}), \]
is unbiased and satisfies (2.22) and \( \text{Var}[\hat{P}_M] \leq C(h)/M, \) with \( C(h) \) bounded for \( h \downarrow 0. \)

Combining this result with the one-step survival Brownian bridge approximation we obtain the following corollary.

**Corollary 2.17.** The one-step survival Brownian bridge approximation (Theorem 2.4) satisfies the (weak) convergence \( E[\hat{P} - \hat{\tilde{P}}] \leq Ch^{1-\delta} \) as a consequence of (2.7) and (2.22). Furthermore, the unbiased one-step survival Brownian bridge Monte Carlo estimator (Corollary 2.5) satisfies \( \text{Var}[\hat{P}_M] \leq C(h)/M, \) with \( C(h) \) bounded for \( h \downarrow 0. \)

**2.3. Partial derivatives, pathwise sensitivities and finite difference: first order Greeks.** In this section we will study different ways of calculating first order Greeks for barrier options with payoff (2.2). From Burgos [6] we know that the Brownian bridge partial derivatives represented by the Brownian bridge approximation (2.5) of a knock-up-out barrier option with payoff (2.2) with respect to \( \Theta, \) with \( \hat{P}(\Theta) \) sufficiently regular in \( \Theta, \) are given by
\[ \frac{\partial \hat{P}}{\partial \Theta_i} = \left( \begin{array}{c}
1_{\hat{S}_n > K} \frac{\partial \hat{S}_n}{\partial \Theta_i} \prod_{n=1}^{N-1} (1 - \hat{p}_n) \\
\left( \hat{S}_N - K \right)^+ \sum_{n=0}^{N-1} \left[ \prod_{k=0, k \neq t}^{N-1} \left( 1 - \hat{p}_k(\Theta, u) \right) \frac{\partial \hat{p}_n}{\partial \Theta_i} \right] \right). \]

(2.26)

See Burgos [6] for explicit formulas of \( \frac{\partial \hat{S}_n}{\partial \Theta_i} \) and \( \frac{\partial \hat{p}_n}{\partial \Theta_i} \). Here, we just want to motivate for the next theorem, by mentioning that the derivative of \( \frac{\partial \hat{P}}{\partial \Theta_i} \) in (2.26) is not a Lipschitz continuous function, i.e. it is a characteristic function depending on whether or not \( \hat{S}_n \) or \( \hat{S}_{n+1} \) did cross the barrier.

For simplicity, we want to introduce a more general notation for (2.10), (2.9), (2.11) and (2.12), see [11] for similar ideas. Let \( \Theta \) be a vector of inputs than (2.10), (2.9), (2.11) and (2.12) can be written as
\[ \hat{p}_n(\Theta, u) = h(\ast_1)_{\ast_1}, \quad \hat{p}_n^{(2)}(\Theta, u) = f_2(\ast_5)_{\ast_2}, \quad \hat{p}_n(\Theta, u) = f(\ast_5)_{\ast_2}, \quad \hat{S}_{n+1}(\Theta, u) = g(\ast_6)_{\ast_3}. \]
with

\[ *_1 := (\varsigma = \sigma(\tilde{S}_n(\Theta, u), t_n), s_1 = \tilde{S}_n(\Theta, u), s_2 = \tilde{S}_{n+1}(\Theta, u), \vartheta = \Theta) \]
\[ *_2 := (\nu = \mu(\tilde{S}_n(\Theta, u), t_n), \varsigma = \sigma(\tilde{S}_n(\Theta, u), t_n), \varsigma' = \sigma'(\tilde{S}_n(\Theta, u), t_n), \]
\[ s = \tilde{S}_n(\Theta, u), \vartheta = \Theta) \]
\[ *_3 := (\nu = \mu(\tilde{S}_n(\Theta, u), t_n), \varsigma = \sigma(\tilde{S}_n(\Theta, u), t_n), \varsigma' = \sigma'(\tilde{S}_n(\Theta, u), t_n), \pi = \tilde{p}_n(\Theta, u), \]
\[ \pi_2 = \tilde{p}_n^{(2)}(\Theta, u), s = \tilde{S}_n(\Theta, u), \vartheta = \Theta, \omega = u^{(n)} \]
\[ *_4 := (\varsigma, s_1, s_2, \vartheta), *_5 := (\nu, \varsigma', \varsigma, s, \vartheta), *_6 := (\nu, \varsigma, \varsigma', \pi, \pi_2, s, \vartheta, \omega) \]

and with \( u = (u^{(T)}, \ldots, u^{(1)}) \).

We want to remark, that the indicator function at the final step can be smoothed out by forcing the path to stay between \( B \) and \( K \), see [11] for further information, which can be done by similar calculations as in Theorem 2.4.

**Theorem 2.18.** The one-step survival Brownian bridge partial derivatives of a knock-up-out barrier option with payoff (2.2) represented by the one-step survival Brownian bridge approximation (Theorem 2.4) with respect to a vector of inputs \( \Theta \), with \( \tilde{S}(\Theta_i) \) sufficiently regular in \( \Theta_i \) for all \( i \) and with a vector \( u = (u_1, \ldots, u_N) \) are given by

\[
\frac{\partial \tilde{P}}{\partial \Theta_i} = \left( 1_{\tilde{S}_N > K} \frac{\partial \tilde{S}_N}{\partial \Theta_i} \prod_{j=0}^{N-1} \tilde{p}_j \prod_{n=0}^{N-1} (1 - \tilde{p}_n^*) \right)
\]
\[
+ q(\tilde{S}_N) \sum_{j=0}^{N-1} \left[ \frac{\partial \tilde{p}_j}{\partial \Theta_i} \prod_{k \neq j} \tilde{p}_k \right] \prod_{n=0}^{N-1} (1 - \tilde{p}_n^*)
\]
\[
- q(\tilde{S}_N) \sum_{n=0}^{N-1} \left[ \prod_{k=0, k \neq t}^{N-1} (1 - \tilde{p}_k^*) \frac{\partial \tilde{p}_n}{\partial \Theta_i} \prod_{j=0}^{N-1} \tilde{p}_j \right].
\]

whereas all of \( \tilde{P}, \tilde{p}_n, \tilde{p}_n^* \) and \( \tilde{S} \) depend on \( (\Theta, u) \). The derivatives of \( \tilde{p}_n^{(2)}(\Theta, u) \) and \( \tilde{p}_n(\Theta, u) \) are recursively given by

\[
\frac{\partial \tilde{p}_n^{(2)}}{\partial \Theta_i}(\Theta, u) = \frac{\partial f_2}{\partial s} (s_{(5)})_{(5)} \frac{\partial \tilde{S}_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f_2}{\partial m} (s_{(5)})_{(5)} \frac{\partial m_n}{\partial \Theta_i}(\Theta, u)
\]
\[
+ \frac{\partial f_2}{\partial \varsigma} (s_{(5)})_{(5)} \frac{\partial \sigma_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f_2}{\partial \varsigma'} (s_{(5)})_{(5)} \frac{\partial \sigma'_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f_2}{\partial \Theta_i} (s_{(5)})_{(5)}
\]
\[
\frac{\partial \tilde{p}_n}{\partial \Theta_i}(\Theta, u) = \frac{\partial f}{\partial s} (s_{(5)})_{(5)} \frac{\partial \tilde{S}_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f}{\partial m} (s_{(5)})_{(5)} \frac{\partial m_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f}{\partial \varsigma} (s_{(5)})_{(5)} \frac{\partial \sigma_n}{\partial \Theta_i}(\Theta, u)
\]
\[
+ \frac{\partial f}{\partial \varsigma'} (s_{(5)})_{(5)} \frac{\partial \sigma'_n}{\partial \Theta_i}(\Theta, u) + \frac{\partial f}{\partial \Theta_i} (s_{(5)})_{(5)}
\]
The derivatives of \( \tilde{S}_n(\Theta, u) \) and \( \tilde{p}_n(\Theta, u) \) are recursively given by

\[
\frac{\partial \tilde{p}_n}{\partial \Theta_i}(\Theta, u) = \frac{\partial h}{\partial \Theta_i}(\Theta, u) + \frac{\partial h}{\partial s_2}(\Theta, u) + \frac{\partial \tilde{S}_{n+1}}{\partial \Theta_i}(\Theta, u)
\]

\[
+ \frac{\partial h}{\partial \Theta_i}(\Theta, u) + \frac{\partial h}{\partial \Theta_i}(\Theta, u)
\]

\[
\frac{\partial \tilde{S}_{n+1}}{\partial \Theta_i}(\Theta, u) = \frac{\partial g}{\partial \Theta_i}(\Theta, u) + \frac{\partial g}{\partial \Theta_i}(\Theta, u) + \frac{\partial g}{\partial \Theta_i}(\Theta, u) + \frac{\partial g}{\partial \Theta_i}(\Theta, u)
\]

The derivatives of the local drift and volatility are given by

\[
\frac{\partial \mu_n}{\partial \Theta_i}(\Theta, u) = \frac{\partial k}{\partial s}(s, \vartheta)_{(s, \vartheta)} + \frac{\partial k}{\partial \vartheta}(s, \vartheta)_{(s, \vartheta)}
\]

\[
\frac{\partial \sigma_n}{\partial \Theta_i}(\Theta, u) = \frac{\partial l}{\partial s}(s, \vartheta)_{(s, \vartheta)} + \frac{\partial l}{\partial \vartheta}(s, \vartheta)_{(s, \vartheta)}
\]

\[
\frac{\partial \sigma_n'}{\partial \Theta_i}(\Theta, u) = \frac{\partial m}{\partial s}(s, \vartheta)_{(s, \vartheta)} + \frac{\partial m}{\partial \vartheta}(s, \vartheta)_{(s, \vartheta)}
\]

with \( \vartheta = \tilde{S}_n(\Theta, u), \vartheta = \Theta \).

Proof. (2.27) is the derivative the integrand of (2.6). For (2.10), (2.9), (2.11) and (2.12) we have

\[
f_2(5) = \Phi\left(-1 - \sqrt{1 + 4 \left(\frac{2 - s - \nu \vartheta_1 + \frac{1}{2} \vartheta \vartheta_1}{\sqrt{\vartheta_1}}\right)}\right) \frac{\phi_2(s - \nu \vartheta_1 + \frac{1}{2} \vartheta \vartheta_1)}{2 \left(\frac{1}{2} \vartheta \sqrt{\vartheta_1}\right)}
\]

\[
f(5) = \Phi\left(\frac{-1 + \sqrt{1 + 4 \left(\frac{2 - s - \nu \vartheta_1 + \frac{1}{2} \vartheta \vartheta_1}{\sqrt{\vartheta_1}}\right)}}{2 \left(\frac{1}{2} \vartheta \sqrt{\vartheta_1}\right)}\right) - \Phi\left(\frac{-1 - \sqrt{1 + 4 \left(\frac{2 - s - \nu \vartheta_1 + \frac{1}{2} \vartheta \vartheta_1}{\sqrt{\vartheta_1}}\right)}}{2 \left(\frac{1}{2} \vartheta \sqrt{\vartheta_1}\right)}\right)
\]

\[
g(5) = s + \nu \vartheta_1 + \varsigma \sqrt{\vartheta_1} \Phi^{-1}(\pi_2 + \pi \omega) + \frac{1}{2} \vartheta \sqrt{\vartheta_1} \Phi^{-1}(\pi_2 + \pi \omega)^2 - \vartheta_1
\]

\[
h(5) = \exp\left(-2(\vartheta_2 - s_1)(\vartheta_2 - s_2)\right)
\]

The recursive formulas follow through differentiation with product rule.
For both (2.26) and (2.27) one could formulate a corollary with an unbiased pathwise sensitivity Monte Carlo estimator. However, the new one-step survival Brownian bridge pathwise sensitivity estimator is Lipschitz-continuous and therefore we expect it to have reduced variance, which we will study later in the numerical section. As an alternative the first order Greeks can be calculated with finite differences under a certain stability condition introduced by the following definition.

**Definition 2.19.** We say that a Monte Carlo estimator allows for stable differentiation by finite differences if there exists $C > 0$ such that

$$\text{Var} \left( D_h P_M \right) \leq \frac{1}{M} C(h)$$

and $C$ is bounded independent of $h$.

In [1] it is shown, that, if both $\hat{P}$ and the Monte Carlo payoff $\hat{Q}$ depend Lipschitz continuously on $\Theta$, the estimator allows for stable differentiation with respect to $\Theta$. This is the case for both the Monte Carlo estimators of $P_M$ of the Brownian bridge approximation $\hat{P}$ and the one-step survival Brownian bridge approximation $\tilde{P}$.

### 2.4. Partial derivatives, pathwise sensitivities and finite difference: second order Greeks

In this section we study three different ways to obtain second order Greeks for barrier options with payoff (2.2). As we can easily see (2.26) is not a Lipschitz-continuous function and hence it is first, not differentiable and second, doesn’t apply for Theorem 2.2 of [1], i.e. for calculating second order Greeks stable through finite differences of first order Greeks. However, for the one-step survival Brownian bridge approximation we have an at least two times continuously differentiable payoff function and one could calculate the Greeks through pathwise sensitivities, which can be done by a straightforward extension of Theorem 2.18. Furthermore, an alternative one could use finite differences of the first order Greeks, gained through a pathwise approach or finite differences, since we know that (2.27) is Lipschitz. With the following result we present a third alternative using second order finite differences.

**Theorem 2.20.** If $\tilde{P}$, $\tilde{P}'$ and the Monte Carlo payoffs $\tilde{Q}$ and $\tilde{Q}'$, depend Lipschitz continuously on $\Theta$, then the estimator $P_M$, with $m = 1, \ldots, M$ simulations $\tilde{Q}_m$ of $\tilde{Q}$, allows for stable second order differentiation with respect to $\Theta$ by second order finite differences.

**Proof.** We have that

$$\text{Var} \left( D_h^{(2)} P_M \right) = \text{Var} \left( D_h^{(2)} \frac{1}{N} \sum \tilde{Q}_i \right) = \frac{1}{N} \text{Var} \left( D_h^{(2)} \tilde{Q} \right)$$

$$\leq \frac{1}{N} \int \left( D_h^{(2)} \tilde{Q} - D_h^{(2)} \tilde{P} \right)^2 du \leq \frac{1}{N} \int \left( \left| D_h^{(2)} \tilde{Q} \right| + \left| D_h^{(2)} \tilde{P} \right| \right)^2 du.$$
For (1) we have
\[ D_h^{(2)} \tilde{Q} = \frac{\tilde{Q}(\Theta + h) - 2\tilde{Q}(\Theta) + \tilde{Q}(\Theta - h)}{h^2} = \frac{1}{h} \int_0^1 \tilde{Q}'(\Theta + ht)h \, dt - \frac{1}{h} \int_0^1 \tilde{Q}'(\Theta - ht)h \, dt \]
\[ \leq \frac{1}{h} \int_0^1 \left| \tilde{Q}'(\Theta + ht) - \tilde{Q}'(\Theta - ht) \right| \, dt \leq \frac{L |2h|}{h} \leq C. \]
Analog relations hold for (2). All in all we obtain
\[ \frac{1}{N} \int \left( |D_h^{(2)} \tilde{Q}| + |D_h^{(2)} \tilde{P}| \right)^2 \, du \leq \frac{1}{N} C. \]

3. Multilevel one-step survival Monte Carlo method. In this section we will present a Multilevel approach for the one-step survival Brownian bridge Monte Carlo estimator. For the complexity theorem of Giles [12] one must of all wishes to have \( \beta > 1 \) in \( V[Y_1] \leq cM^{-1}h^\beta \), for a positive constant \( c \) and \( M \) simulations on level \( l \). From Giles [12] we know that for the Brownian bridge approximation using the Milstein scheme, leading to \( \beta \approx 0.5 \) without modification and that there exists a coarse path modification leading to \( \beta \approx 1.5 \). For the one-step survival Brownian bridge approximation, without coarse modification, we determined \( \beta \approx 0.5 \). The issue we here have to overcome is, that we can’t use the same coarse path modification technique used for the Brownian bridge estimator, since a midterm interpolation would lead to a biased one-step survival probability. Nevertheless, we found a way to modify the one-step survival Brownian bridge approximation yielding to \( \beta \approx 1.5 \) in numerical experiments. In the following we present the used scheme, whereas the approximated \( \beta \)’s can be viewed in the numerical section. For the first coarse step we use
\[ p_n = \Phi \left( \frac{\frac{1}{2} \sigma \sigma' h \sqrt{\frac{\sigma^2 h - 2 \sigma \sigma' h (S_n + \mu h - \frac{1}{2} \sigma \sigma' h - B)}{(2 \sigma \sigma')^2} - \sigma \sqrt{B}}}{\sigma \sigma' h} \right) \]
\[ \tilde{S}_{n+\frac{1}{2}} = S_n + \mu h + \sigma \sqrt{B} \Phi^{-1}(p_n U_n) + \sigma \sigma' \left( (\sqrt{\sigma} \Phi^{-1}(p_n U_n))^2 - h \right), \]
with \( \sigma := \sigma(S_n, t_n), \mu := \mu(S_n, t_n) \) and \( h \) the step-width of the fine level. For the second step of the coarse path we use the following scheme:
\[ p_{n+\frac{1}{2}} = \]
\[ \Phi \left( \frac{- \sigma \sqrt{h} - \sigma \sigma' h \Phi^{-1}(U_n) + \sqrt{\left( \sigma \sqrt{h} + \sigma \sigma' h \right)^2 - \sigma \sigma' h \left( S_{n+\frac{1}{2}} + \mu h - B + \frac{1}{2} \sigma \sigma' h \right)}}{\sigma \sigma' h} \right) \]
\[ S_{n+1} = S_{n+\frac{1}{2}} + \mu h + \sigma S(t) \sqrt{h} \Phi^{-1}(U_{n+\frac{1}{2}}) \]
\[ + \frac{1}{2} \sigma \sigma' \left( (2 \sqrt{h} \Phi^{-1}(U_n) \sqrt{h} \Phi^{-1}(p_{n+\frac{1}{2}} U_{n+\frac{1}{2}}))^2 - h \right). \]
Table 4.1

| Parameter | Value |
|-----------|-------|
| $t_0$     | 0     |
| $T$       | 1     |
| $S_0$     | 1     |
| $B$       | $1.1$ |
| $r$       | 5 %   |
| $b$       | 0 %   |
| $\sigma$ | 20 %  |
| $K$       | 1     |

Parameters of the up-and-out barrier option.

with $\sigma' := \sigma'(\tilde{S}_n, t_n)$. Furthermore the Brownian bridge probability is applied on both $S_{n+\frac{1}{2}}$ and $S_{n+1}$. For a better understanding we will give a short derivation of these formulas. Starting with the unmodified Milstein scheme on the coarse path with step width $2h$ we have:

$$\hat{S}_{n+1} = \hat{S}_n + \mu 2h + \sigma \sqrt{2h} Z_{n+1} + \frac{1}{2} \sigma \sigma' \left( \left( \sqrt{2h} Z_{n+1} \right)^2 - 2h \right)$$

Using the same Brownian path we have the relation $Z_{n+1} = Z_n + \frac{1}{2} \sqrt{2h}$ and obtain:

$$\hat{S}_{n+1} = \hat{S}_n + \mu 2h + \sigma \sqrt{2h} Z_n + \frac{1}{2} \sigma \sigma' \left( \left( \sqrt{2h} Z_n \right)^2 - 2h \right)$$

$$\quad = \hat{S}_n + \mu h + \sigma \sqrt{h} Z_n + \frac{1}{2} \sigma \sigma' \left( \left( \sqrt{h} Z_n \right)^2 - h \right)$$

$$\quad \quad + \mu h + \sigma \sqrt{h} Z_{n+\frac{1}{2}} + \frac{1}{2} \sigma \sigma' \left( h \left( 2Z_n Z_{n+\frac{1}{2}} + Z_{n+\frac{1}{2}}^2 \right) - h \right).$$

Now by summarizing the first line to $S_{n+\frac{1}{2}}$ we obtain:

$$\hat{S}_{n+1} = S_{n+\frac{1}{2}} + \mu h + \sigma \sqrt{h} Z_{n+\frac{1}{2}} + \frac{1}{2} \sigma \sigma' \left( h \left( 2Z_n Z_{n+\frac{1}{2}} + Z_{n+\frac{1}{2}}^2 \right) - h \right)$$

$$\quad = S_{n+\frac{1}{2}} + \mu h + \sigma \sqrt{h} Z_{n+\frac{1}{2}} + \frac{1}{2} \sigma \sigma' \left( 2\sqrt{h} Z_n \sqrt{h} Z_{n+\frac{1}{2}} + (\sqrt{h} Z_{n+\frac{1}{2}})^2 - h \right).$$

Applying the one-step survival probability to $S_{n+\frac{1}{2}}$ with $Z_n$ and to $S_{n+1}$ with $Z_{n+\frac{1}{2}}$ while assuming that $Z_n$ is a constant, delivers the final scheme from above.

4. Numerical Results. Within this section, we will provide some numerical results for the new Brownian bridge one-step survival estimator and its derivatives. Therefore, we consider a simple continuously observed up-and-out barrier option. We will use the presented parameters of Table 4.1, whereas the example is fictitious.

In the first column of Figure 4.1 we see the estimated mean squared error of the options present value in dependence of the Monte Carlo samples. In the second column we see the estimated absolute error in dependence of the calculation time. The results of the Brownian bridge estimator and the one-step survival Brownian bridge estimator are plotted in a blue line, and in a red line, respectively. We observe the proven variance reduction, which depends on (2.20) as mentioned above.
Next, we want to take a deeper look at the comparison of the two estimators for sensitivity calculation. Therefore, we analogously compare the mean squared error and the calculation time in Figure 4.2, but on this occasion for the options Delta, calculated through the pathwise sensitivity approach, plotted again in a blue line for the Brownian bridge estimator, and in a red line for the one-step survival Brownian bridge estimator, respectively.

Here, we clearly see the first strength and advantage of the Lipschitz continuous first derivative of the one-step survival Brownian bridge estimator, leading to a significant variance reduction and time savings.

Now we study the stability for the second order Greeks. Figure 4.3 shows the second derivative of the barrier options present value with respect to the underlying price (the Gamma) calculated by applying second order finite differences as in Theorem 2.20, to both the Brownian bridge estimator and the one-step survival Brownian bridge estimator plotted in a blue line, and in a red line, respectively. The plot clearly demonstrates the instability of the Brownian bridge estimator with respect to second order numerical differentiation and the stability of the Brownian bridge one-step survival estimator.

Lastly, we want to observe the variance of the Multilevel Monte Carlo estimator.
Figure 4.3. This Figure shows the second order Greek (Gamma) of the one-step survival Brownian bridge estimator (red line) and the Brownian bridge estimator (blue line) depending on initial values of $S_0$. Gamma was calculated through second order finite differences with step width $h = 10^{-3}$ of $S_0$ and $M = 10^5$ Monte Carlo simulations.

Figure 4.4 shows the behavior of both $\tilde{P}_l$ and $\tilde{P}_l - \tilde{P}_{l-1}$, with the logarithmic base 2 as quantity versus the grid level. The slope of the line for $\tilde{P}_l - \tilde{P}_{l-1}$ is approximately 1.5, indicating the wished $V_l = \text{Var}[\tilde{P}_l - \tilde{P}_{l-1}] = O(h^{1.5-\delta})$.

5. Conclusion. We showed weak convergence of almost one and a variance bound for the Brownian bridge approach. The proven (weak) convergence holds under certain assumptions to the stochastic differential equations by using the Milstein scheme.

Next, since we were interested in second order Greeks, we faced the problem that the Brownian bridge approximation leads to a non Lipschitz continuous first derivative. We clearly demonstrated the instability of the Brownian bridge estimator for second order Greeks with numerical results. To overcome this issue, we adapted the idea of the one-step survival Monte Carlo method to the Milstein scheme and the Brownian bridge approach. This resulted in a new one-step survival Brownian bridge approximation with a modified Milstein scheme and slightly modified/smoothed crossing probabilities of the Brownian bridge interpolation. We showed that this new one-step survival Brownian bridge approach is unbiased and leads to a variance reduction, depending on the survival probabilities. Furthermore, we presented the one-step survival Brownian bridge partial derivatives and saw a huge variance reduction compared to the Brownian bridge estimator. In addition the new approach satisfies the wished
smoothness and can be stably differentiated by a simple second order finite difference scheme, which we demonstrated in a theoretical result and a numerical example.

To simplify the presentation we only presented first order pathwise sensitivity results but one could straightforward extend these results to second order pathwise sensitivities of the new approximation, see [11] for some thoughts on complexity of the derivatives.

To improve the computational complexity, we also provided a Multilevel Monte Carlo algorithm, even though we didn’t study the theoretical background. In a numerical test we demonstrated that it achieves similar variance behavior as the coarse modified Brownian bridge approach.

Even if restricted to up-and-out barrier call options the conversion to put or down-and-out options is straightforward. Furthermore, we expect that the algorithms can be expended to the multivariate case similar to the ideas of [13] and [1].

At last, it should be mentioned that the new approach can be combined with other variance reduction methods as well, such as antithetic sampling or control variates [17].

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