A New Approach to the Constructions of Braided T-Categories

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Abstract. The aim of this paper is to construct a new braided T-category via the generalized Yetter-Drinfel’d modules and Drinfel’d codouble over a Hopf algebra, an approach different from that proposed by Panaite and Staic [1]. Moreover, in the case of finite dimensional, we will show that this category coincides with the corepresentation of a certain coquasitriangular Turaev group algebra which we construct. Finally we apply our theory to the case of group algebra.

1. Introduction

Braided T-categories introduced by Turaev [2] are of interest due to their applications in homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories. Braided T-category gives rise to 3-dimensional homotopy quantum field theory and plays a key role in the construction of Hennings-type invariants of flat group-bundles over complements of link in the 3-sphere, see [3]. As such, they are interesting to different research communities in mathematical physics (see [4, 5]).

The quantum double of Drinfel’d [6] is one of the most celebrated Hopf constructions, which associates to a Hopf algebra $H$ a quasitriangular Hopf algebra $D(H)$. Unlike the Hopf algebra axioms themselves, the axioms of a dual quasitriangular (coquasitriangular) Hopf algebra are not self-dual. Thus the axioms and ways of working with these coquasitriangular Hopf algebras look somewhat different in practice and so it is surely worthwhile to study and write them out explicitly in dual form. Moreover, the corepresentation categories of coquasitriangular Hopf algebras can give rise to a braided monoidal category which is different from one coming from the representation categories of quasitriangular Hopf algebras. It is these ideals which many authors studied these notions (cf.[7–17]).

In [1], the authors found a wise method to construct braided T-category $\mathcal{YD}(H)$ over the group $G = Aut_{\text{top}}(H) \times Aut_{\text{top}}(H)$, where $H$ is a Hopf algebra. This category $\mathcal{YD}(H)$ is the disjoint union of all these categories $\mathcal{YD}^{\mathcal{J}}(\alpha, \beta)$ (the categories of $(\alpha, \beta)$-Yetter-Drinfel’d modules) over $H$ for all $\alpha, \beta \in Aut_{\text{top}}(H)$. The authors also proved that, if $H$ is finite dimensional, then $\mathcal{YD}(H)$ coincides with the representations of a certain quasitriangular $T$-coalgebra $DT(H)$. 

\begin{flushright}
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Our motivation is the following: Can we use \((\alpha, \beta)\)-Yetter-Drinfel’d modules and Drinfel’d codouble to construct a new braided \(T\)-category? And in the case of \(H\) being finite dimensional, can we prove that this new braided \(T\)-category is isomorphic to the corepresentation category of a certain coquasitriangular Turaev group algebra?

In this paper, we give a positive answer to the above question. The paper is organized as follows:

In section 1, we recall the notions of braided \(T\)-category, Turaev group algebra and generalized Yetter-Drinfel’d module. In section 2, we introduce the diagonal crossed coproduct \(H^{op} \cong C\), where \(H\) is a Hopf algebra and \(C\) is an \(H\)-bimodule coalgebra. In section 3, we firstly recall the definition of \((\alpha, \beta)\)-Yetter-Drinfel’d module, then we construct braided \(T\)-category \(\mathcal{YD}(H)\) over \(G\) whose multiplication is \((\alpha, \beta) \ast (\gamma, \delta) = (\delta a^{-1} \gamma, \delta \beta)\) for all \(\alpha, \beta, \gamma, \delta \in Aut_{\text{H-hop}}(H)\). We also prove that category \(\mathcal{YD}(H)\) coincides with the corepresentation category of a certain coquasitriangular crossed Turaev group algebra in the sense of [18].

2. Preliminary

Throughout this paper, let \(k\) be a fixed field, and all vector spaces and tensor product are over \(k\). All vector spaces are assumed to be finite dimensional, although it should be clear when this restriction is not necessary.

In this section we recall some basic definitions and results related to our paper.

2.1. Crossed \(T\)-category

Let \(G\) be a group with the unit 1. Recall from [19–21] that a crossed category \(C\) (over \(G\)) is given by the following data:

- \(C\) is a strict monoidal category.
- A family of subcategory \(\{C_{\alpha}\}_{\alpha \in G}\) such that \(C\) is a disjoin union of this family and that \(U \otimes V \in C_{\alpha, \beta}\) for any \(\alpha, \beta \in G\), \(U \in C_{\alpha}\), and \(V \in C_{\beta}\).
- A group homomorphism \(\varphi : G \rightarrow \text{aut}(C), \beta \mapsto \varphi_{\beta}\), the conjugation, where \(\text{aut}(C)\) is the group of the invertible strict tensor functors from \(C\) to itself, such that \(\varphi_{\beta}(C_{\alpha}) = C_{\alpha \beta}\) for any \(\alpha, \beta \in G\).

We will use the left index notation in Turaev: Given \(\beta \in G\) and an object \(V \in C_{\alpha}\), the functor \(\varphi_{\beta}\) will be denoted by \(\beta(\cdot)\) or \(V(\cdot)\) and \(\beta^{-1}(\cdot)\) will be denoted by \(\overline{V}(\cdot)\). Since \(V(\cdot)\) is a functor, for any object \(U \in C\) and any composition of morphism \(g \circ f\) in \(C\), we obtain \(V(id_U) = id_{VU}\) and \(V(g \circ f) = Vg \circ Vf\). Since the conjugation \(\varphi : \pi \rightarrow \text{aut}(C)\) is a group homomorphism, for any \(V, W \in C\), we have \(V(id_W) = \overline{V}(id_V)\) and \(V(f) = \overline{V}(V) = \overline{V}(id_V) = id_C\). Since for any \(V \in C\), the functor \(V(\cdot)\) is strict, we have \(V(f \otimes g) = Vf \otimes Vg\) for any morphism \(f \in \text{mor} C\) and \(g \in C\), and \(V(1) = 1\).

A Turaev braided \(G\)-category is a crossed \(T\)-category \(C\) endowed with a braiding, i.e., a family of isomorphisms

\[
c = \{c_{UV} : U \otimes V \rightarrow \overline{V}U \otimes \overline{V}V\}_{UV \in C}\n\]

obeying the following conditions:

- For any morphism \(f \in \text{Hom}_{C_{\alpha}}(U, U')\) and \(g \in \text{Hom}_{C_{\beta}}(V, V')\), we have
  \[
  (\overline{V}g \otimes f) \circ c_{UV} = c_{U'V'} \circ (f \otimes g),
  \]

- For all \(U, V, W \in C\), we have
  \[
  c_{UWCV} = (c_{VW} \otimes id_U)(id_U \otimes c_{UV}),
  \]
  \[
  c_{ULWV} = (id_U \otimes c_{LV})(c_{UV} \otimes id_W).
  \]

- For any \(U, V \in C\) and \(\alpha \in G\), \(\varphi_{\alpha}(c_{UV}) = c_{\alpha U \alpha V}^{-1}\).
2.2. Turaev Group Algebras

Let $G$ be a group with unit 1. Recall from [18, 22] that a $G$-algebra is a family $A = \{A_a\}_{a \in G}$ of $k$-spaces together with a family of $k$-linear maps $m = \{m_{a,b} : A_a \otimes A_b \to A_{ab}\}_{a,b \in G}$ (called multiplication) and a $k$-linear map $\eta : k \to A_1$ (called unit) such that $m$ is associative in the sense that, for all $a, b, c \in G$

$$m_{a,b}(m_{a,b} \otimes id) = m_{a,b}((id \otimes m_{b,c})), \quad m_{a,1}(id \otimes \eta) = id = m_{1,a}(\eta \otimes id).$$

A Turaev $G$-algebra is a $G$-algebra $H = \{H_a\}_{a \in G}$ such that each $H_a$ is a coalgebra with comultiplication $\Delta_a$ and counit $\varepsilon_a$, the map $\eta : k \to H_1$ and the maps $m_{a,b} : H_a \otimes H_b \to H_{ab}$ are coalgebra maps, with a family of $k$-linear maps $S = \{S_a : H_a \to H_{a^{-1}}\}_{a \in G}$ (called the antipode) such that for all $a \in G$

$$m_{a,a^{-1}}(id \otimes S_a)\Delta_a = \varepsilon_a 1 = m_{a^{-1},a}((S_a \otimes id)\Delta_a).$$

Furthermore, a crossed Turaev $G$-algebra is a Turaev $G$-algebra with a family of coalgebra isomorphisms $\psi = \{\psi_{ab} : H_a \otimes H_{b^{-1}} \to H_{ab}\}_{a,b \in G}$ (called coassociativity), satisfying the following conditions: for all $a, b, c \in G$

(i) $\psi$ is multiplicative, i.e., $\psi_a \psi_b = \psi_{ab}$,

(ii) $\psi$ is compatible with $m$, i.e., $m_{a,1} = m_{a,1} \psi_{ab} = m_{a,1} \psi_{bc}$,

(iii) $\psi$ is compatible with $\eta$, i.e., $\eta = \eta \psi_{ab}$,

(iv) $\psi$ preserves the antipode, i.e., $\psi_{abc} = S_{b^{-1}ab} \psi_{ab}$.

We use the Sweedlers notation for a comultiplication $\Delta_a$ on $H_a$: for all $h \in H_a$

$$\Delta_a(h) = h_1 \otimes h_2.$$ 

Recall from [18], a Turaev $G$-algebra $H$ is called coquasitriangular if there exists a family of $k$-linear maps $\sigma = \{\sigma_{ab} : H_a \otimes H_b \to k\}$ such that $\sigma_{ab}$ is convolution invertible for all $a, b \in G$ and the following conditions are satisfied:

(TCT1) $\sigma_{ab,c}(xy,z) = \sigma_{a,c}(x,z\sigma_{b,c}(y,z))$

(TCT2) $\sigma_{a,b,c}(x,yz) = \sigma_{a,b}(x,y)\sigma_{b^{-1},a,c}^{-1}(\psi_{b^{-1}}(x),z)$

(TCT3) $\sigma_{a,b}(x_1,y_1)\psi_{b^{-1}}(x_2) = x_1y_1\sigma_{a,b}(x_2,y_2)$

(TCT4) $\sigma_{a,b}(x,y) = \sigma_{b^{-1},a,c}^{-1}(\psi_{b^{-1}}(x),\psi_{b^{-1}}(y))$,

for all $x \in H_a, y \in H_b, z \in H_c$.

Note that if Turaev $G$-algebra $H$ is coquasitriangular, then $(H_1, \sigma_{1,1})$ is a coquasitriangular Hopf algebra.

2.3. Yetter-Drinfeld’s module

Let $H$ be a Hopf algebra and $C$ an $H$-bimodule coalgebra, with module structures $H \otimes C \to C$, $h \otimes c \mapsto h \cdot c$ and $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$. Recall from [23, 24], we can consider the Yetter-Drinfeld’s datum $(H,C,H)$ and the Yetter-Drinfeld’s category $\mathcal{M}DF^H$, whose object $M$ is a left $H$-module (with the action $h \otimes m \mapsto h \cdot m$) and right $C$-comodule (with the coaction $m \mapsto m(0) \otimes m(1)$) such that for all $h \in H, m \in M$,

$$h_1 \cdot m(0) \otimes h_2 \cdot m(1) = (h_2 \cdot m)(0) \otimes (h_2 \cdot m)(1) \cdot h_1,$$

or equivalently

$$(h \cdot m)(0) \otimes (h \cdot m)(1) = h_2 \cdot m(0) \otimes h_3 \cdot m(1) \cdot S^{-1}(h_1).$$
3. Diagonal Crossed Coproduct

As the dual of diagonal crossed product (for details, see [1]), we have the following result.

**Proposition 3.1.** Let $H$ be a Hopf algebra with a bijective antipode $S$, and $C$ a bimodule coalgebra with the actions $H \otimes C \rightarrow C$, $h \otimes c \mapsto h \cdot c$ and $C \otimes H \rightarrow C$, $c \otimes h \mapsto c \cdot h$. Then we have a coalgebra $H^{op} \otimes C$ (denoted by $H^{op} \triangleright C$) with the comultiplication and counit

$$
\Delta(p \triangleright c) = \sum_{i,j} p_1 \triangleright h_i \cdot h_j \cdot S^{-1}(h_i) \otimes h_j^\prime p_2 h_i^\prime \triangleright c_2,
$$

(3.1)

$$
\varepsilon(p \triangleright c) = p(1) \varepsilon(c),
$$

(3.2)

for all $p \in H^{op}$, $c \in C$, where $\{h_i\}$ and $\{h_i^\prime\}$ are basis and dual basis of $H$. $H^{op} \triangleright C$ is called diagonal crossed coproduct.

**Proof.** For all $p \in H^{op}$, $c \in C$, on one hand

$$(\Delta \otimes id)\Delta(p \triangleright c) = \sum_{i,j} \Delta(p_1 \triangleright h_i \cdot c_1 \cdot S^{-1}(h_i)) \otimes h_j^\prime p_2 h_i^\prime \triangleright c_2$$

$$= \sum_{i,j,k} p_1 \triangleright h_i \cdot (h_j \cdot c_1 \cdot S^{-1}(h_i))_1 S^{-1}(h_i) \otimes h_j^\prime p_3 h_i^\prime \triangleright c_2$$

$$= \sum_{i,j,k} p_1 \triangleright h_i h_j h_k \cdot c_1 \cdot S^{-1}(h_i h_j) \otimes h_j^\prime p_3 h_i^\prime \triangleright c_3.$$

Evaluating the first, the third and the fifth factors at $h, h', h'' \in H$ respectively, we have

$$
\sum_{i,j,k} p_1(h) h_i h_j h_k \cdot c_1 \cdot S^{-1}(h_i h_j) \otimes h_j^\prime p_3 h_i^\prime \triangleright c_2
$$

$$= p_1(h) h_i^\prime h_j^\prime \cdot c_1 \cdot S^{-1}(h_i^\prime h_j^\prime) \otimes p_2(h_i^\prime) h_j^\prime \cdot c_2 \cdot S^{-1}(h_j^\prime) \otimes p_3(h_j^\prime) c_3$$

$$= p(h h_i^\prime h_j^\prime) h_j^\prime h_k^\prime \cdot c_1 \cdot S^{-1}(h_i^\prime h_j^\prime) \otimes h_j^\prime h_k^\prime \cdot c_2 \cdot S^{-1}(h_j^\prime) \otimes c_3.$$

On the other hand

$$(id \otimes \Delta)(p \triangleright c) = \sum_{i,j,k} p_1 \triangleright h_j \cdot c_1 \cdot S^{-1}(h_j) \otimes \Delta(h_j p_2 h_i^\prime \triangleright c_2)$$

$$= \sum_{i,j,k} p_1 \triangleright h_j \cdot c_1 \cdot S^{-1}(h_j) \otimes h_j^\prime p_2 h_i^\prime \otimes h_i \cdot c_2 \cdot S^{-1}(h_i) \otimes h_j^\prime h_k^\prime p_3 h_i^\prime \triangleright c_3.$$

Evaluating the first, the third and the fifth factors at $h, h', h'' \in H$ respectively, we have

$$
\sum_{i,j,k} p_1(h) h_j \cdot c_1 \cdot S^{-1}(h_j) \otimes h_j^\prime p_2 h_i^\prime \otimes h_i \cdot c_2 \cdot S^{-1}(h_i) \otimes h_j^\prime h_k^\prime p_3 h_i^\prime \triangleright c_3
$$

$$= \sum_{i,j,k} p_1(h) h_j \cdot c_1 \cdot S^{-1}(h_j) \otimes h_j^\prime p_2 h_i^\prime h_j^\prime h_k^\prime \cdot c_2 \cdot S^{-1}(h_i) \otimes h_j^\prime(h_k^\prime) p_3(h_k^\prime) h_j^\prime h_k^\prime(h_j^\prime) c_3$$

$$= \sum_{i,j,k} p_1(h) h_j h_j^\prime h_k^\prime \cdot c_1 \cdot S^{-1}(h_i) \otimes h_j^\prime h_k^\prime \cdot c_2 \cdot S^{-1}(h_i) \otimes c_3.$$ 

Thus $\hat{\Delta}$ is coassociative. Easy to check that $\hat{\varepsilon}$ is counit. The proof is completed. $\square$

**Remark 3.2.** In particular when $C = H$ and the module action is multiplication, we can recover the Drinfel’d codouble $\hat{D}(H)$ introduced in [12, Proposition 10.3.14].
Proposition 3.3. Diagonal crossed coproduct $H^{op} \triangleright C$ is a $\hat{D}(H)$-bimodule coalgebra with structures

\[
\begin{align*}
\hat{D}(H) \otimes H^{op} \triangleright C & \rightarrow H^{op} \triangleright C, \ (p \otimes h) \triangleright (q \triangleright c) = qp \triangleright h \cdot c, \\
H^{op} \triangleright C \otimes \hat{D}(H) & \rightarrow H^{op} \triangleright C, \ (q \triangleright c) \triangleleft (p \otimes h) = pq \triangleright c \cdot h,
\end{align*}
\]
for all $p, q \in H^{op}, h \in H, c \in C$.

Proof. Obviously $H^{op} \triangleright C$ is a left $\hat{D}(H)$-module. And for all $p, q \in H^{op}, h \in H, c \in C,$

\[
\check{\lambda}((p \otimes h) \triangleright (q \triangleright c)) = \check{\lambda}(qp \triangleright h \cdot c)
\]

\[
= \sum_{i,j} q_ip_1 \triangleright h_j \cdot (h \cdot c)_1 \cdot S^{-1}(h_i) \otimes h'^2q_2h^i \triangleright (h \cdot c)_2
\]

\[
= \sum_{i,j} q_ip_1 \triangleright h_jh_1 \cdot c_1 \cdot S^{-1}(h_i) \otimes h'^2q_2h^i \triangleright h_2 \cdot c_2
\]

\[
= \sum_{i,j} q_ip_1 \triangleright h_jh_1S^{-1}(h_i)h_4 \cdot c_1 \cdot S^{-1}(h_i) \otimes h'^2q_2h^i \triangleright h_2 \cdot c_2
\]

\[
= (p \otimes h)_1 \triangleright (q \triangleright c)_1 \otimes (p \otimes h)_2 \triangleright (q \triangleright c)_2.
\]

Thus $H^{op} \triangleright C$ is a left $\hat{D}(H)$-module coalgebra. Similarly one can check that $H^{op} \triangleright C$ is also a right $\hat{D}(H)$-module coalgebra. The proof is completed. \(\square\)

4. The Construction of Braided $T$-Category $\mathcal{YD}(H)$

Definition 4.1. [1, Definition 2.1] Let $H$ be a Hopf algebra and $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$. An $(\alpha, \beta)$-Yetter-Drinfel’d module over $H$ is a vector space $M$ such that $M$ is a left $H$-module and right $H$-comodule with the following compatible condition

\[ h_1 \cdot m_0 \otimes \beta(h_2)m_1 = (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1\alpha(h_1), \]

for all $h \in H, m \in M$. We denote by $H\mathcal{YD}^H(\alpha, \beta)$ the category of $(\alpha, \beta)$-Yetter-Drinfel’d modules, morphisms being the $H$-linear and $H$-co-linear.

Example 4.2. For any Hopf algebra $H$ and $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$, define $H_{\alpha, \beta}$ as follows: $H_{\alpha, \beta} = H$ with regular left $H$-module structure and right $H$-comodule structure given by

\[ \rho(h) = h_2 \otimes \beta(h_3)S^{-1}\alpha(h_1), \]

for all $h \in H$. Then $H_{\alpha, \beta} \in H\mathcal{YD}^H(\alpha, \beta)$.

Let $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$. We define an $H$-bimodule coalgebra $H(\alpha, \beta)$ as follows: $H(\alpha, \beta) = H$ as coalgebra with module structures

\[ H \otimes H(\alpha, \beta) \rightarrow H(\alpha, \beta), \quad h \otimes h' \mapsto \beta(h)h', \]

\[ H(\alpha, \beta) \otimes H \rightarrow H(\alpha, \beta), \quad h' \otimes h \mapsto h'\alpha(h), \]

for all $h, h' \in H$.

Now consider the Yetter-Drinfel’d datum $(H, H(\alpha, \beta), H)$ and its Yetter-Drinfel’d category $H\mathcal{YD}^{H(\alpha, \beta)}$.

Proposition 4.3. With the above notations, we have the relation:

\[ H\mathcal{YD}^{H(\alpha, \beta)} = H\mathcal{YD}^H(\alpha, \beta). \]
Consider now the diagonal crossed coproduct $C(\alpha, \beta) = H^{\otimes 2} \otimes H(\alpha, \beta)$ with the comultiplication

$$∆(p \otimes h) = \sum_{i,j} p_i \otimes β(h_i)h_1S^{-1}a(h_i) \otimes h'p_2h' \otimes h_2,$$

for all $p \in H^{\otimes 2}, h \in H$. Moreover $C(\alpha, \beta)$ is a $D(H)$-bimodule coalgebra with module structures

$$\mathcal{D}(H) \otimes H^{\otimes 2} \otimes H(h, \alpha) \to H^{\otimes 2} \otimes H(h, \alpha), \quad \rho \otimes h \otimes q \mapsto h' \mapsto \rho \otimes β(h)h',$$

$$H^{\otimes 2} \otimes H(h, \alpha) \otimes \mathcal{D}(H) \to H^{\otimes 2} \otimes H(h, \alpha), \quad q \mapsto h' \otimes p \otimes h \mapsto pq \otimes h'α(h).$$

Since $H$ is finite dimensional, we have a category isomorphism $\mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta) \cong \mathcal{M}^{H^{\otimes 2} \otimes H(h, \alpha)}$, hence $\mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta) \cong \mathcal{M}^{H^{\otimes 2} \otimes H(h, \alpha)}$. The correspondence is given as follows. If $M \in \mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta)$, then $M \in \mathcal{M}^{H^{\otimes 2} \otimes H(h, \alpha)}$ with structure

$$m_{[0]} \otimes m_{[1]} = \sum_i h_i \cdot m_{(0)} \otimes h'_i \otimes m_{(1)}.$$

Conversely if $M \in \mathcal{M}^{H^{\otimes 2} \otimes H(h, \alpha)}$, then $M \in \mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta)$ with structures

$$h \cdot m = m_{[0]}(h \otimes ε)m_{[1]},$$

$$m_{[0]} \otimes m_{[1]} = m_{[0]} \otimes (ε \otimes id)m_{[1]}.$$

**Proposition 4.4.** Let $H$ be a Hopf algebra and $\alpha, \beta, γ, δ$ be $Aut_{\text{Hopf}}(H)$. If $M \in \mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta)$, $N \in \mathcal{M}^{\mathcal{D}(H)}(γ, δ)$, then $M \otimes N \in \mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta, γ, δ)$ with the following structures:

$$h \cdot (m \otimes n) = h_2 \cdot m \otimes h_1 \cdot n,$$

$$(m \otimes n)_{[0]} \otimes (m \otimes n)_{[1]} = m_{[0]} \otimes n_{[0]} \otimes δ(m_{[1]}), δαδ^{-1}(n_{[1]}),$$

for all $h \in H, m \in M, n \in N$.

**Proof.** Clearly $M \otimes N$ is a left $H$-module and right $H$-comodule. We need only to verify the compatible condition.

$$h_1 \cdot (m \otimes n)_{[0]} \otimes δβ(h_2)(m \otimes n)_{[1]}$$

$$= h_2 \cdot m_{[0]} \otimes h_1 \cdot n_{[0]} \otimes δ(h_2m_{[1]}) \deltaαδ^{-1}(n_{[1]})$$

$$= (h_3 \cdot m_{[0]} \otimes h_1 \cdot n_{[0]} \otimes δ(h_3 \cdot m_{[1]}) \deltaαδ^{-1}(h_2m_{[1]}))$$

$$= (h_3 \cdot m_{[0]} \otimes (h_2 \cdot n_{[0]} \otimes δ((h_3 \cdot m_{[1]}) \deltaαδ^{-1}(h_2 \cdot n_{[1]})))$$

$$= (h_2 \cdot (m \otimes n)_{[0]} \otimes (h_2 \cdot (m \otimes n)_{[1]}) \deltaαδ^{-1}(h_1)).$$

The proof is completed. □

Note that if $M \in \mathcal{M}^{\mathcal{D}(H)}(\alpha, \beta)$, $N \in \mathcal{M}^{\mathcal{D}(H)}(γ, δ)$ and $P \in \mathcal{M}^{\mathcal{D}(H)}(\mu, ν)$, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$ as an object in $\mathcal{M}^{\mathcal{D}(H)}(δαδ^{-1}γ, δβ, ν)$.

Denote $G = Aut_{\text{Hopf}}(H) \times Aut_{\text{Hopf}}(H)$, a group with multiplication

$$(\alpha, \beta) \cdot (γ, δ) = (δαδ^{-1}γ, δβ).$$

The unit is $(id, id)$ and $(α, β)^{-1} = (β^{-1}α^{-1}, β^{-1})$.

**Proposition 4.5.** Let $N \in \mathcal{M}^{\mathcal{D}(H)}(γ, δ)$ and $(α, β) \in G$. Define $(α, β)N = N$ as vector space with structures

$$h \rightarrow n = α^{-1}β(h) \cdot n,$$

$$n_{<0>} \otimes n_{<1>} = n_{[0]} \otimes β^{-1}δαδ^{-1}(n_{[1]}).$$

Then $(α, β)N \in \mathcal{M}^{\mathcal{D}(H)}(β^{-1}δαδ^{-1}γα^{-1}β, β^{-1}δβ) = \mathcal{M}^{\mathcal{D}(H)}((α, β) \cdot (γ, δ) \cdot (α, β)^{-1})$. 
Proof. Easy to see that \((a, b)N\) is a left \(H\)-module and right \(H\)-comodule. We check the compatible condition.

\[
h_1 \to n_{<0>} \otimes \beta^{-1} \delta \beta(h_2) n_{<1>}
\]

\[
= a^{-1} \beta(h_1) \cdot n_{<0>} \otimes \beta^{-1} \delta \beta(h_2) \beta^{-1} \delta a^{-1} \Delta(h_1)
\]

\[
= (a^{-1} \beta(h_2) \cdot n_{<0>}) \otimes \beta^{-1} \delta a^{-1} \Delta \beta(h_1)
\]

\[
= (a^{-1} \beta(h_2) \cdot n_{<0>}) \otimes \beta^{-1} \delta a^{-1} \Delta \beta(h_1)
\]

\[
= (a^{-1} \beta(h_2) \cdot n_{<0>}) \otimes (a^{-1} \beta(h_2) \cdot n_{<1>}) \beta^{-1} \delta a^{-1} \gamma \alpha^{-1} \Delta \beta(h_1)
\]

\[
= (a^{-1} \beta(h_2) \cdot n_{<0>}) \otimes (a^{-1} \beta(h_2) \cdot n_{<1>}) \beta^{-1} \delta a^{-1} \gamma \alpha^{-1} \Delta \beta(h_1)
\]

\[
= (h_2 \to n)_{<0>} \otimes (h_2 \to n)_{<1>} \beta^{-1} \delta a^{-1} \gamma \alpha^{-1} \Delta \beta(h_1)
\]

The proof is completed. □

Remark 4.6. Let \(M \in H YD^H(a, b, N) \in H YD^H((\gamma, \delta) \cdot (\mu, \nu) \in G). \)

We have

\[
(a, b)N = (a, b)N
\]

as an object in \(H YD^H((\gamma, \delta) \cdot (\mu, \nu) \in G). \)

Proposition 4.7. Let \(M \in H YD^H(a, b, N) \in H YD^H((\gamma, \delta) \cdot (\mu, \nu) \in G). \)

Denote \(M N = (a, b)N\) as an object in \(H YD^H((\gamma, \delta) \cdot (\mu, \nu) \in G). \)

Define the map

\[
c_{MN} : M \otimes N \to MN \otimes M, \quad m \otimes n \mapsto \alpha^{-1}(m_{<1>}) \cdot n \otimes m_{<0>},
\]

for all \(m \in M, n \in N. \)

Then \(c_{MN}\) is \(H\)-linear \(H\)-colinear and satisfies the relations (1.1) and (1.2). And \(c_{MN} = c_{MN}.

Moreover \(c_{MN}\) is bijective with inverse \(c_{MN} = c_{MN}. \)

Proof. We prove that \(c_{MN}\) is \(H\)-linear \(H\)-colinear. Indeed

\[
c_{MN}(h \cdot (m \otimes n)) = c_{MN}(h_2 \cdot m \otimes h_1 \cdot n)
\]

\[
= a^{-1}((h_2 \cdot m_{<1>}) \alpha(h_1)) \cdot n \otimes (h_2 \cdot m_{<0>})
\]

\[
= a^{-1}(\beta(h_2) m_{<1>}) \cdot n \otimes h_1 \cdot m_{<0>}
\]

\[
= h \cdot c_{MN}(m \otimes n).
\]

And

\[
c_{MN}(m \otimes n)_{<0>}) \otimes c_{MN}(m \otimes n)_{<1>}
\]

\[
= (a^{-1}(m_{<1>}) \cdot n)_{<0>}) \otimes m_{<0>}) \otimes \beta((a^{-1}(m_{<1>}) \cdot n)_{<1>}) \delta a^{-1} \gamma a^{-1}(m_{<0>})
\]

\[
= (a^{-1}(m_{<1>}) \cdot n)_{<0>}) \otimes m_{<0>}) \otimes \delta a^{-1}((a^{-1}(m_{<1>}) \cdot n)_{<1>}) \gamma a^{-1}(m_{<0>})
\]

\[
= a^{-1}(m_{<1>}) \cdot m_{<0>}) \otimes m_{<0>}) \otimes \delta a^{-1}((m_{<1>}) \cdot m_{<1>}) \gamma a^{-1}(m_{<0>})
\]

\[
= c_{MN}((m \otimes n)_{<0>}) \otimes (m \otimes n)_{<1>}).
\]

Furthermore

\[
(c_{MN} \otimes \text{id})((m \otimes n) \otimes p)
\]

\[
= (c_{MN} \otimes \text{id})(m \otimes y(a^{-1}(n_{<1>})) \cdot p \otimes n_{<0>})
\]

\[
= a^{-1}(m_{<1>}) \otimes y(a^{-1}(n_{<1>})) \cdot p \otimes n_{<0>}
\]

\[
= y^{-1} \delta a^{-1}(m_{<1>}) \gamma(a^{-1}(n_{<1>})) \cdot p \otimes n_{<0>}
\]

\[
= y^{-1} \delta a^{-1}((m \otimes n)_{<1>}) \cdot p \otimes (m \otimes n)_{<0>}
\]

\[
= c_{MN}(m \otimes n \otimes p).
\]

Similarly we can prove (1.2). The proof is completed. □
Define $\mathcal{M}(H)$ as the disjoint union of all $H\mathcal{YD}^H(\alpha, \beta)$ with $(\alpha, \beta) \in G$. If we endow $\mathcal{M}(H)$ with monoidal structure given in Proposition 4.4, then it becomes a strict monoidal category with the unit $k$ as an object in $H\mathcal{YD}^H$ (with trivial structure).

The group homomorphism $\psi : G \to Aut(\mathcal{M}(H))$, $(\alpha, \beta) \mapsto \psi_{(\alpha,\beta)}$ is defined on components as

$$\psi_{(\alpha,\beta)}(\beta)^*YD(\gamma, \delta) \to H\mathcal{YD}^H((\alpha, \beta) \ast (\gamma, \delta) \ast (\alpha, \beta)^{-1}),$$

where $\psi_{(\alpha,\beta)}(N) = (\alpha,\beta)^*N$.

and the functor acts on morphisms as identity. The braiding in $\mathcal{M}(H)$ is given by the family $c = \{c_{MN}\}$. Hence we have

**Proposition 4.8.** $\mathcal{M}(H)$ is a braided $T$-category over $G$.

It is well known that for a Hopf algebra with a bijective antipode, the subcategory $H\mathcal{YD}^H_{fd}$ of all finite dimensional objects in $H\mathcal{YD}^H$ is rigid, i.e., every object has left and right dualities. For the category $\mathcal{M}(H)$, we have the following result.

**Proposition 4.9.** Let $M \in H\mathcal{YD}^H(\alpha, \beta)$ and suppose that $M$ is finite dimensional. Then $M' = Hom(M, k)$ belongs to $H\mathcal{YD}^H(\beta^{-1}\alpha^{-1}, \beta^{-1})$ with

$$(h \cdot f)(m) = f(S^{-1}(h) \cdot m),$$

$$f_{(0)}(m)f_{(1)} = f(m_{(0)})\beta^{-1}\alpha^{-1}S(m_{(1)}),$$

for all $h \in H$, $m \in M$ and $f \in M'$. Then $M'$ is a left dual of $M$. Similarly we can define the right dual $^*M = Hom(M, k)$ of $M$ with

$$(h \cdot f)(m) = f(S(h) \cdot m),$$

$$f_{(0)}(m)f_{(1)} = f(m_{(0)})\beta^{-1}\alpha^{-1}S^{-1}(m_{(1)}).$$

Therefore the category $\mathcal{M}(H)_{fd}$, the subcategory of $\mathcal{M}(H)$ consisting of finite dimensional objects, is rigid.

**Proof.** First of all, $M'$ is an object in $H\mathcal{YD}^H(\beta^{-1}\alpha^{-1}, \beta^{-1})$. Indeed, obviously $M'$ is a left $H$-module and right $H$-comodule. And

$$(h_2 \cdot f)_{(0)}(m)(h_2 \cdot f)_{(1)}\beta^{-1}\alpha^{-1}\beta(h_1) = (h_2 \cdot f)(m_{(0)})S(m_{(1)})\beta^{-1}\alpha^{-1}\beta(h_1) = f(S^{-1}(h_2) \cdot m_{(0)})\beta^{-1}\alpha^{-1}S(m_{(1)})\beta^{-1}\alpha^{-1}\beta(h_1) = f(S^{-1}(h_2) \cdot m_{(0)})\beta^{-1}\alpha^{-1}(\beta S^{-1}(h_1)m_{(1)}) = f((S^{-1}(h_1) \cdot m)_{(0)})S(\beta^{-1}\alpha^{-1}((S^{-1}(h_1) \cdot m)_{(1)})\beta^{-1}S^{-1}(h_2)) = f((S^{-1}(h_1) \cdot m)_{(0)})\beta^{-1}(h_2)S(\beta^{-1}\alpha^{-1}((S^{-1}(h_1) \cdot m)_{(1)})) = f_{(0)}(S^{-1}(h_1) \cdot m)\beta^{-1}(h_2)f_{(1)} = (h_1 \cdot f)_{(0)}(m)\beta^{-1}(h_2)f_{(1)},$$

as required. Define maps

$$b_M : k \to M \otimes M', \quad 1 \mapsto \sum_i m_i \otimes m',$$

$$d_M : M' \otimes M \to k, \quad f \otimes m \mapsto f(m),$$

as required.
where \( \{m_i\} \) and \( \{m'_i\} \) are basis and dual basis of \( M \). We need to prove that \( b_M \) and \( d_M \) are \( H \)-linear. We compute

\[
(h \cdot b_M(1))(m) = (h \cdot \sum_i m_i \otimes m'_i)(m)
\]

\[
= (\sum_i h_2 \cdot m_i \otimes h_1 \cdot m'_i)(m)
\]

\[
= \sum_i h_2 \cdot m_i m'_i(S^{-1}(h_1) \cdot m)
\]

\[
= h_2 S^{-1}(h_1) \cdot m
\]

\[
= \varepsilon(h) b_M(1)(m),
\]

and

\[
d_M(h \cdot (f \otimes m)) = d_M(h_2 \cdot f \otimes h_1 \cdot m)
\]

\[
= (h_2 \cdot f)(h_1 \cdot m)
\]

\[
= f(S^{-1}(h_2)h_1 \cdot m)
\]

\[
= \varepsilon(h) f(m)
\]

\[
= h \cdot d_M(f \otimes m).
\]

They are also \( H \)-colinear. Indeed,

\[
b_M(1)_{(0)}(m) \otimes b_M(1)_{(1)} = \sum_i m_{i(0)} m'_{i(1)} (m_0) \otimes \beta^{-1}(m_{i(1)}) \beta^{-1} \alpha \beta^{-1}(m'_{i(1)})
\]

\[
= \sum_i m_{(0)} m'_{i(1)} \otimes \beta^{-1}(m_{i(1)}) \beta^{-1}(S(m_{(1)}))
\]

\[
= m_{(0)} \otimes \beta^{-1}(m_{(1)}) S(m_{(1)2})
\]

\[
= b_M(1)(m) \otimes 1,
\]

and

\[
d_M((f \otimes m)_{(0)} \otimes (f \otimes m)_{(1)3}) = d_M(f_{(0)} \otimes m_{(0)} \otimes \beta(f_{(1)}) \alpha^{-1}(m_{(1)})
\]

\[
= f_{(0)} m_{(0)} \beta(f_{(1)}) \alpha^{-1}(m_{(1)})
\]

\[
= f(m_{(0)}) \alpha^{-1}(S(m_{(1)1}) m_{(1)2})
\]

\[
= d_M(f \otimes m)_{(0)} \otimes d_M(f \otimes m)_{(1)3}.
\]

It is straightforward to verify that

\[(id_M \otimes d_M)(b_M \otimes id_M) = id_M \text{ and } (d_M \otimes id_M)(id_M \otimes b_M) = id_M.\]

Similarly we can prove that \( \hat{M} \) is a right dual of \( M \). The proof is completed. \( \square \)

Now we are in a position to construct a coquasitriangular Turaev group algebra over \( G \), denoted by \( CT(H) \) such that the \( T \)-category \( \text{Corep}(CT(H)) \) of corepresentation of \( CT(H) \) is isomorphic to \( \mathcal{YD}(H) \) as braided \( T \)-categories.

For \( (\alpha, \beta) \in G \), the \( (\alpha, \beta) \)-component \( CT(H)_{\alpha, \beta} \) will be the diagonal crossed coproduct \( H^{op} \bowtie H(\alpha, \beta) \). Define multiplication by

\[m_{(\alpha, \beta)}(\cdot, \cdot) : H^{op} \bowtie H(\alpha, \beta) \otimes H^{op} \bowtie H(\gamma, \delta) \rightarrow H^{op} \bowtie H((\alpha, \beta) \ast (\gamma, \delta)),\]

\[\langle p \bowtie h \rangle \otimes \langle q \bowtie h' \rangle \rightarrow \langle p q \bowtie \delta(h) \delta(\alpha \delta^{-1}(h')) \rangle.\]  

\[ (4.1) \]

Then we have the following result.
Proposition 4.10. \(CT(H)\) becomes a Turaev G-algebra under the diagonal crossed coproduct and multiplication (4.1). The antipode is given by

\[
S_{(\alpha, \beta)} : H^{op} \Rightarrow H(\alpha, \beta) \longrightarrow H^{op} \Rightarrow H((\alpha, \beta)^{-1}),
\]

\[p \Rightarrow h \mapsto \sum_{i,j} h'^{-1}(p)S^{-1}(h') \Rightarrow \beta^{-1}(h_1)\beta^{-1} \alpha^{-1} S(h_1)\beta^{-1} \alpha^{-1} \beta(h_1).
\]

Proof. The multiplication is associative. For all \(f \Rightarrow h \in H^{op} \Rightarrow H(\alpha, \beta), p \Rightarrow h' \in H^{op} \Rightarrow H(\gamma, \delta), q \Rightarrow h'' \in H^{op} \Rightarrow H(\mu, \nu),\) we compute

\[
[(f \Rightarrow h)(p \Rightarrow h')](q \Rightarrow h'') = (pf \Rightarrow \delta(h)\alpha\delta^{-1}(h'))(q \Rightarrow h'')
\]

\[
= qp \Rightarrow \nu(h)\nu\delta^{-1}(h'')
\]

\[
= (f \Rightarrow h)(qp \Rightarrow \nu(h')\nu\delta^{-1}(h''))
\]

\[
= (f \Rightarrow h)[(p \Rightarrow h')(q \Rightarrow h'')],
\]

as claimed. Next we prove that \(m_{(\alpha, \beta), (\gamma, \delta)}\) is a coalgebra map. Indeed,

\[
m_{(\alpha, \beta), (\gamma, \delta)}((p \Rightarrow h_1) \otimes (q \Rightarrow h'_1)) \otimes m_{(\alpha, \beta), (\gamma, \delta)}((p \Rightarrow h_2) \otimes (q \Rightarrow h'_2))
\]

\[
= \sum_{i,j} m_{(\alpha, \beta), (\gamma, \delta)}(p_1 \Rightarrow \beta(h_1)h_1aS^{-1}(h_1) \otimes q_1 \Rightarrow \delta(h_1)h'_1) \Rightarrow \gamma S^{-1}(h_1))
\]

\[
\otimes m_{(\alpha, \beta), (\gamma, \delta)}(h'^{-1}S^{-1}(h') \Rightarrow h_2 \otimes h'^{-1}q_2 \Rightarrow h'_2)
\]

\[
= \sum_{i,j} q_1p_1 \Rightarrow \delta\beta(h_1)\delta(h_1)\delta\alpha S^{-1}(h_1) \delta(h_1)\delta\alpha S^{-1}(h_1) \Rightarrow \gamma S^{-1}(h_1) \otimes h'^{-1}q_2 \Rightarrow h'_2
\]

\[
= \sum_{i,j} q_1p_1 \Rightarrow \delta\beta(h_1)\delta(h_1)\delta\alpha S^{-1}(h_1) \Rightarrow \gamma S^{-1}(h_1) \otimes h'^{-1}q_2 \Rightarrow h'_2
\]

\[
= (qp \Rightarrow \delta(h)\alpha\delta^{-1}(h')) \otimes (qp \Rightarrow \delta(h)\alpha\delta^{-1}(h'))_2
\]

\[
m_{(\alpha, \beta), (\gamma, \delta)}(p \Rightarrow h \otimes q \Rightarrow h')(1) \otimes m_{(\alpha, \beta), (\gamma, \delta)}(p \Rightarrow h \otimes q \Rightarrow h')_2,
\]

as required. Easy to see that \((\varepsilon \Rightarrow 1)_1 \otimes (\varepsilon \Rightarrow 1)_2 = \varepsilon \Rightarrow 1 \otimes \varepsilon \Rightarrow 1.
\]

We now check that \(S\) is the antipode of \(CT(H)\).

\[
S_{(\alpha, \beta)}((p \Rightarrow h_1)(p \Rightarrow h_2))
\]

\[
= \sum_{i,j} S_{(\alpha, \beta)}(p_1 \Rightarrow \beta(h_1)h_1aS^{-1}(h_1))(h'^{-1}p_2 \Rightarrow h'_2)
\]

\[
= \sum_{i,j} (h'^{-1}S^{-1}(p_1)S^{-1}(h') \Rightarrow \beta^{-1}(h_1)h_1S^{-1}(h_1) \Rightarrow \beta^{-1} \alpha^{-1} \beta S(h_1) \Rightarrow \beta^{-1} \alpha^{-1} \beta (h_1))(h'^{-1}p_2 \Rightarrow h'_2)
\]

\[
= \sum_{i,j} h'^{-1}p_2h'^{-1}S^{-1}(p_1)S^{-1}(h') \Rightarrow h_1 \alpha^{-1} s(h_1) \alpha^{-1} \beta S(h_1)h_2 \Rightarrow \alpha^{-1}(h_2)
\]

\[
= \sum_{i,j} h'^{-1}p_2h'^{-1}S^{-1}(p_1)S^{-1}(h') \Rightarrow h_1 \alpha^{-1} S(h_1) \alpha^{-1} \beta S(h_1)h_2 \Rightarrow \alpha^{-1}(h_2)
\]

\[
= \sum_{i,j} h'^{-1}p_2S^{-1}(p_1)S^{-1}(h') \Rightarrow h_1 \alpha^{-1} S(h_1) \alpha^{-1}(h_2)
\]

\[
= p(1)\varepsilon(h) \Rightarrow 1.
\]

Thus \(S_{(\alpha, \beta)} \ast id_{(\alpha, \beta)} = \varepsilon_{(\alpha, \beta)} \Rightarrow 1\). Similarly one can verify that \(id_{(\alpha, \beta)} \ast S_{(\alpha, \beta)} = \varepsilon_{(\alpha, \beta)} \Rightarrow 1\). \(S\) is the antipode of \(CT(H)\). The proof is completed. \(\Box\)
Proposition 4.11. Moreover $CT(H)$ is a crossed Turaev G-algebra with the crossing $\psi$ given by $\psi_{\alpha,\beta} : H^{op} \otimes H(\gamma, \delta) \rightarrow H^{op} \otimes H(\alpha, \beta) \cdot (\gamma, \delta) \cdot (\alpha, \beta)^{-1}$,

$p \otimes h \mapsto p \circ \alpha^{-1} \beta \triangleright \beta^{-1} \delta \alpha \delta^{-1}(h)$.

Proof. First of all $\psi_{\alpha,\beta}$ is bijective and for all $p \in H, h \in H$,

\[
\psi_{\alpha,\beta}(p \otimes h) = (p \circ \alpha^{-1} \beta \triangleright \beta^{-1} \delta \alpha \delta^{-1}(h))_1 \otimes (p \circ \alpha^{-1} \beta \triangleright \beta^{-1} \delta \alpha \delta^{-1}(h))_2
\]

\[
= \sum_{ij} p_1 \circ \alpha^{-1} \beta \triangleright \beta^{-1} \delta(h) \beta^{-1} \delta \alpha \delta^{-1}(h_1) \beta^{-1} \delta \alpha \delta^{-1}(h_2) \otimes h'(p_2 \circ \alpha^{-1} \beta) \triangleright \beta^{-1} \delta \alpha \delta^{-1}(h_2)
\]

\[
= \sum_{ij} \psi_{\alpha,\beta}(p_1 \otimes \delta(h_1) S^{-1}(h_1)) \otimes \psi_{\alpha,\beta}(h' p_2 \triangleright h)
\]

\[
= \psi_{\alpha,\beta}((p \otimes h)_1) \otimes \psi_{\alpha,\beta}((p \otimes h)_2).
\]

Thus $\psi_{\alpha,\beta}$ is a coalgebra isomorphism. And

(i) $\psi$ is multiplicative, since for $h \in H(\mu, \nu)$

\[
\psi_{\alpha,\beta}(\psi_{\alpha,\beta}(p \otimes h) = \psi_{\alpha,\beta}(p \circ \gamma^{-1} \delta \triangleright \delta^{-1} \nu \gamma \nu^{-1}(h))
\]

\[
= p \circ \gamma^{-1} \delta \alpha \delta^{-1}(h) \triangleright \beta^{-1} \nu \gamma \nu^{-1}(h)
\]

\[
= \psi(\alpha^{-1} \gamma \nu)(p \otimes h)
\]

\[
= \psi_{\alpha,\beta}(p \otimes h).
\]

Obviously $\psi_{\alpha,\beta}(CT(\alpha, \beta)) = id(\alpha, \beta)$.

(ii) For $p, q \in H$ and $h \in H(\gamma, \delta), h' \in H(\mu, \nu),

\[
\psi_{\alpha,\beta}(p \otimes h) \psi_{\alpha,\beta}(q \otimes h') = (p \circ \alpha^{-1} \beta \triangleright \beta^{-1} \delta \alpha \delta^{-1}(h)) (q \circ \alpha^{-1} \beta \triangleright \beta^{-1} \nu \gamma \nu^{-1}(h'))
\]

\[
= qp \circ \alpha^{-1} \beta \triangleright \beta^{-1} \nu \gamma \nu^{-1}(h) \triangleright \beta^{-1} \nu \gamma \nu^{-1}(h')
\]

\[
= \psi_{\alpha,\beta}(qp \otimes (h) \otimes \gamma \nu \nu^{-1}(h'))
\]

\[
= \psi_{\alpha,\beta}((p \otimes h)(q \otimes h')).
\]

(iii) $\psi_{\alpha,\beta}(\varepsilon \otimes 1) = \varepsilon \otimes 1$.

(iv) $\psi_{\alpha,\beta}(S_{\gamma, \delta}(p \otimes h) = \sum_{ij} \psi_{\alpha,\beta}(h' S^{-1}(p) S^{-1}(h')) \otimes \delta^{-1}(h_1) \delta^{-1}(S(h)) \delta^{-1}(\gamma^{-1}(h_2))$

\[
= \sum_{ij} (h' \otimes h_1) \otimes \delta^{-1}(h_1) \delta^{-1}(S(h)) \delta^{-1}( \gamma^{-1}(h_2))
\]

\[
= \sum_{ij} (h' \otimes h_1) \otimes \delta^{-1}(h_1) \delta^{-1}(S(h)) \delta^{-1}( \gamma^{-1}(h_2))
\]

\[
= \sum_{ij} \psi_{\alpha,\beta}(h' \otimes h_1) \otimes \delta^{-1}(h_1) \delta^{-1}(S(h)) \delta^{-1}( \gamma^{-1}(h_2))
\]

\[
= \psi_{\alpha,\beta}(S(p \otimes h) \otimes S^{-1}(h')) \triangleright \beta^{-1} \delta^{-1} \alpha \delta^{-1} \beta^{-1} \gamma^{-1} S(h) \delta^{-1} \gamma^{-1} \delta(h_2)
\]

\[
= \psi_{\alpha,\beta}(h') \triangleright \beta^{-1} \delta^{-1} \alpha \delta^{-1} \beta^{-1} \gamma^{-1} S(h) \delta^{-1} \gamma^{-1} \delta(h_2)
\]

\[
= \psi_{\alpha,\beta}(h).
\]
The proof is completed.

**Proposition 4.12.** CT(H) is coquasitriangular with the structure

\[
\sigma_{(a, b), (\gamma, \delta)}(p \mapsto h, \psi \mapsto h') = p(\delta^{-1}(h'))q(1)\epsilon(h).
\]

**Proof.** For all \(f, p, q \in H', h \in H(\alpha, \beta), h' \in H(\gamma, \delta), h'' \in H(\mu, \nu),\)

For (TCT1):

\[
\sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}((f \mapsto h)(p \mapsto h'), (q \mapsto h'')) = \sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}(pf \mapsto \delta(h)\delta\alpha\delta^{-1}(h'), (q \mapsto h''))
\]

\[
= pf(\nu^{-1}(h''))q(1)\epsilon(hh')
\]

\[
= p(\nu^{-1}(h''))f(\nu^{-1}(h''))q(1)\epsilon(hh'),
\]

and

\[
\sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}(f \mapsto h, (q \mapsto h''))\sigma_{(\gamma, \delta), (\mu, \nu)}(p \mapsto h', (q \mapsto h''))
\]

\[
= \sum_{i,j} \sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}(f \mapsto h, h'qph' \mapsto h''_{i,j})\sigma_{(\gamma, \delta), (\mu, \nu)}(p \mapsto h', q_{1} \mapsto \nu(h_{1})h''_{i,j}\mu S^{-1}(h_{1}))
\]

\[
= \sum_{i,j} f(\nu^{-1}(h''_{i,j}))(h''_{i,j})(1)q_{2}(1)h''_{i,j}(1)\epsilon(h)p(h_{1}\nu^{-1}(h''_{i,j})\nu^{-1}\mu S^{-1}(h_{1}))
\]

\[
= f(\nu^{-1}(h''_{i,j}))p(\nu^{-1}(h''_{i,j}))\epsilon(hh'q(1)).
\]

For (TCT2):

\[
\sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}((f \mapsto h, (p \mapsto h'), (q \mapsto h'')) = \sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}((f \mapsto h, (p \mapsto h'), (q \mapsto h''))
\]

\[
= \sum_{i,j} \sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}(f \mapsto h, (p \mapsto h'), (q \mapsto h''))
\]

\[
= \sum_{i,j} f(\nu^{-1}(h''_{i,j}))(h''_{i,j})(1)q_{2}(1)h''_{i,j}(1)\epsilon(h)p(h_{1}\nu^{-1}(h''_{i,j})\nu^{-1}\mu S^{-1}(h_{1}))
\]

\[
= f(\nu^{-1}(h''_{i,j}))p(\nu^{-1}(h''_{i,j}))\epsilon(hh'q(1)).
\]

For (TCT3):

\[
\sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}((f \mapsto h, (p \mapsto h'), (q \mapsto h'))(\mu S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= \sum_{i,j} \sigma_{(a, b), (\gamma, \delta), (\mu, \nu)}(f_{1} \mapsto \beta(h_{1})h_{1}\alpha S^{-1}(h_{1}), p_{1} \mapsto \delta(h_{1})h''_{i,j}\gamma S^{-1}(h_{1}))(h'p_{2}h' \mapsto h''_{i,j})\psi_{(\gamma, \delta), (\mu, \nu)}(f_{2} \mapsto h_{2})
\]

\[
= \sum_{i,j} f_{1}(h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(p_{1}(1)h'p_{2}h' \mapsto h''_{i,j})\psi_{(\gamma, \delta), (\mu, \nu)}(f_{2} \mapsto h_{2})
\]

\[
= \sum_{i,j} f_{1}(h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))p_{1}(1)h'p_{2}h' \mapsto h''_{i,j}(f_{2} \mapsto hh')
\]

\[
= \sum_{i,j} f_{1}(h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(f_{2} \mapsto h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= \sum_{i,j} f_{1}(h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(f_{2} \mapsto h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= \sum_{i,j} f_{1}(h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(f_{2} \mapsto h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= f_{1}(\delta^{-1}(h'_{i,j}))(f_{2} \mapsto h_{1}\delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= f_{2}(\delta^{-1}(h'_{i,j}))(f_{1} \mapsto \delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]

\[
= f_{2}(\delta^{-1}(h'_{i,j}))(f_{1} \mapsto \delta^{-1}(h'_{i,j})\delta^{-1}\gamma S^{-1}(h_{1}))(h_{1})(\nu h' \mapsto h_{2})
\]
and
\[
(f \triangleright h)_1(p \triangleright h')_2 = \sum_{i,j} (f_1 \triangleright \beta(h_1i)h_1jA^j(h_1i)(p_1 \triangleright \delta(h_1i)h_1j)\sigma(a_\beta, b_\gamma)(h_1f_2h_1i \triangleright h_2, h_1p_2h_1i \triangleright h'_2)
\]
\[
= \sum_{i,j} p_1f_1 \triangleright \delta\beta(h_1i)h_1j\Delta A^j(h_1i)\Delta\alpha\delta^{-1}(h_1i)\Delta\alpha\delta^{-1}(h_1f_2h_1i)(\delta^{-1}(h_2')f_2h_1i(\delta^{-1}(h_2'))e(h_2h_1p_2h_1i(1))
\]
\[
= pf_1 \triangleright \delta\beta\delta^{-1}(h_1i)h_1j\Delta\alpha\delta^{-1}(h_1f_2h_1i(\delta^{-1}(h_2'))f_2(\delta^{-1}(h_2'))).
\]

For (TC5):
\[
\sigma(a_\beta)(\gamma, \delta)(p \triangleright h'), \psi(a_\beta)(q \triangleright h'')
\]
\[
= \sigma(a_\beta)(\gamma, \delta)(p \triangleright \alpha^{-1}\beta \triangleright \alpha^{-1}\beta \Delta\alpha\delta^{-1}(h')q \triangleright \alpha^{-1}\beta \triangleright \alpha^{-1}\beta \Delta\alpha\delta^{-1}(h''))
\]
\[
= p(v^{-1}(h''))q(1)(\delta(h'))
\]
\[
= \sigma(\gamma, \delta, \mu, \nu)(p \triangleright h', q \triangleright h'').
\]
The proof is completed. □

By the arguments after Proposition 4.3 we obtain the main result:

**Theorem 4.13.** Corep(CT(H)) and \(\mathcal{YD}(H)\) are isomorphic as braided T-categories over \(G\).

**Example 4.14.** Let \(\pi\) be a group, then we have a group algebra \(k(\pi)\). It is well known that the group \(\text{Aut}_{\text{Hopf}}(k(\pi))\) of Hopf automorphisms of \(k(\pi)\) is equal to the group \(\text{Aut}(\pi)\) of automorphisms of \(\pi\). Let \(\alpha, \beta \in \text{Aut}(\pi)\). An \((\alpha, \beta)\) Yetti-Drinfel’d module is a left \(\pi\)-module \(M\) with a decomposition \(M = \bigoplus_{\alpha, \beta} M_{\alpha, \beta}\), where \(M_{\alpha, \beta} = \{a \in M | m(0) \otimes m(1) = m(\alpha \otimes a)\} \).

If \(\alpha, \beta, \gamma, \delta \in \text{Aut}(\pi), M \in k(\pi)\mathcal{YD}(\pi, \gamma, \delta), N \in k(\pi)\mathcal{YD}(\pi, \gamma, \delta)\) then \(M \otimes N \in k(\pi)\mathcal{YD}(\pi, \gamma, \delta)\) with action \(a \cdot (m \otimes n) = a \cdot m \otimes a \cdot n\) for all \(a \in \pi, m \in M, n \in N\), and decomposition \(M \otimes N = \bigoplus_{\alpha, \beta} (\bigoplus_{\gamma, \delta} M_{\alpha, \beta} \otimes N_{\gamma, \delta})\).

If \(\alpha, \beta \in \text{Aut}(\pi)\) and \(N \in k(\pi)\mathcal{YD}(\pi, \gamma, \delta)\) then \((\alpha, \beta)N = N\) as vector space with action \(a \cdot n = \alpha^{-1}\beta(a) \cdot n\) for all \(a \in \pi, n \in N\), and decomposition \((\alpha, \beta)N = \bigoplus_{\gamma, \delta} N_{\gamma, \delta}\).

With the above notations, the braiding \(c_{MN} : M \otimes N \rightarrow MN \otimes M\) acts on homogeneous elements \(m \in M_{\alpha}, n \in N_{\beta}\) as \(c_{MN}(m \otimes n) = \alpha^{-1}(a) \cdot n \otimes m(\alpha)\). Therefore \(M_{\alpha} \otimes N_{\beta}\) is sent to \(N_{\beta} \otimes M_{\alpha}\).

Now assume that \(M \in k(\pi)\mathcal{YD}(\pi, \gamma, \delta)\) is finite dimensional. Since \(S = S^{-1}\) for \(k(\pi)\), we have \(M^* \cong M\), and for all \(a \in \pi, m \in M, f \in M^*\), \((a \cdot f)(m) = f(a^{-1} \cdot m)\) with decomposition \(M^* = \bigoplus_{\alpha, \beta} (M_{\alpha^{-1}, \beta} \otimes \beta^{-1}_\alpha)\).

Let \(\pi\) be a finite group and \([p_\pi, \otimes]\) the dual of \(k(\pi)\). For \(\alpha, \beta \in \text{Aut}(\pi)\), the component \(CT(k(\pi))(\alpha, \beta) = k(\pi)^{\alpha, \beta} \cong k(\pi)\) with comultiplication
\[
\tilde{\Delta}(p_\pi \triangleright d) = \sum_{a \in \pi} p_\pi \triangleright (\beta(b) \Delta(a^{-1}) \otimes p_\pi \triangleright d),
\]
for all \(a, d \in \pi\). Furthermore for \(a \in k(\pi)^{\alpha, \beta} \) and \(b \in k(\pi)^{\gamma, \delta}\),
\[
(p_\pi \triangleright a)(p_\pi \triangleright b) = \delta_{\alpha, \beta} p_\pi \triangleright (a)(\alpha \Delta(b^{-1})),
\]
\[
1_{CT(k(\pi))}(a \otimes d) = \sum_{\pi} p_\pi \otimes 1,
\]
\[
\psi(l_{\alpha, \beta})(p_\pi \triangleright d) = p_{\beta^{-1}(\alpha)} \otimes \beta^{-1} \Delta(a^{-1})(d),
\]
\[
S_{l_{\alpha, \beta}}(p_\pi \triangleright a) = p_l \triangleright \beta^{-1}(c) \beta^{-1}(\alpha^{-1})(a) \beta^{-1}(\alpha^{-1})(c^{-1}) \beta^{-1}(\alpha^{-1})(c^{-1}) \beta^{-1}(\alpha^{-1})(c^{-1}),
\]
\[
\sigma(a_\beta, b_\gamma)(\gamma, \delta)(p_\pi \triangleright a, p_\pi \triangleright b) = \delta_{l_{\alpha, \beta}}(\gamma, \delta)(d, d).
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