STABILIZATION OF NONAUTONOMOUS PARABOLIC EQUATIONS BY A SINGLE MOVING ACTUATOR

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Abstract. It is shown that an internal control based on a moving indicator function is able to stabilize the state of parabolic equations evolving in rectangular domains. For proving the stabilizability result, we start with a control obtained from an oblique projection feedback based on a finite number of static actuators, then we used the continuity of the state when the control varies in a relaxation metric to construct a switching control where at each given instant of time only one of the static actuators is active, finally we construct the moving control by traveling between the static actuators.

Numerical computations are performed by a concatenation procedure following a receding horizon control approach. They confirm the stabilizing performance of the moving control.

1. Introduction. Stabilizability of controlled parabolic-like equations of the form
\[ \dot{y} + Ay + A_{rc}(t)y = u(t)\Phi(t), \quad y(0) = y_0, \quad t > 0, \] where the state evolves in a Hilbert space \( H \), that is, \( y(t) \in H \) for all \( t \geq 0 \), is investigated. The pair \((u, \Phi)\), with \( u(t) \in \mathbb{R} \) and \( \Phi(t) \in H \), with \( |\Phi(t)|_H = 1 \), is at our disposal. We shall look for a continuous function \( \Phi: [0, +\infty) \to H \), where \( \Phi(t) \) represents the actuator moving on a suitable compact subset \( \mathcal{S}^\mathbb{R}_H \subset \mathcal{S}_H \) of the unit sphere \( \mathcal{S}_H \) in \( H \), depending only on a finite set \( \mathcal{K}_c \subset C([0,1], \mathcal{S}_H) \) to be specified later on. Hereafter, \( A \) will be a symmetric diffusion-like linear operator and \( A_{rc} \) will be a reaction–convection-like linear operator. Under suitable assumptions to be specified later, in particular, under a stabilizability assumption by means of a finite number of static actuators the main result of this manuscript is the following.

Main Result. There exist a (signed) magnitude control function \( u = u(t) \) and a continuous moving actuator \( \Phi = \Phi(t) \) satisfying
\[
\begin{align*}
    u &\in L^2((0, +\infty), \mathbb{R}), \\
    \Phi(t) &\in \mathcal{S}_H^{\mathbb{R}} \quad \text{for} \quad t \geq 0, \\
    \dot{\Phi} &\in L^\infty((0, +\infty), H) \cap C([0, +\infty), H), \\
    \ddot{\Phi} &\in L^\infty((0, +\infty), H),
\end{align*}
\]
and constants $C \geq 1$ and $\mu > 0$, such that the solution of the system (1.1) satisfies
\[ |y(t)|_H \leq Ce^{-\mu t}|y_0|_H, \quad \text{for all} \quad t \geq 0, \quad (1.2) \]
and the mappings $y_0 \mapsto u(y_0)$ and $y_0 \mapsto \Phi(y_0)$ satisfy
\[ |u(y_0)|_{L^2(\mathbb{R}_0, \mathbb{R})} \leq C_0 |y_0|_H \quad \text{and} \quad \left| \Phi(y_0) \right|_{W^{1,\infty}(\mathbb{R}_0, H)} \leq C_1. \quad (1.3) \]

Furthermore, the tuple $(\mu, C_0, C_1)$ is independent of $y_0$.

In particular, the actuator $\Phi$ moves with a smooth trajectory, with continuous “velocity” $\dot{\Phi}$, which is meaningful from the applications/physical point of view.

The precise statement of Main Result is given in Corollary 3.2.

1.1. Example. As an illustration we consider a parabolic equation whose state evolves in $H = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, a regular bounded domain.
\[ \dot{y} - \nu \Delta y + ay + b \cdot \nabla y = u\hat{\omega}(c), \quad \mathcal{G} y|_{\Gamma} = 0, \quad y(0, \cdot) = y_0, \quad (1.4) \]
where $y = y(t, x) \in \mathbb{R}$, $y(t, \cdot) \in L^2(\Omega)$, $a = a(t, x) \in \mathbb{R}$, $b = b(t, x) \in \mathbb{R}^d$, and $\mathcal{G}$ denotes either Dirichlet or Neumann conditions on the boundary $\Gamma$ of $\Omega$, i.e. $\mathcal{G} y|_{\Gamma} = y(t, \bar{x})$ or $\mathcal{G} y|_{\Gamma} = n(\bar{x}) \cdot \nabla y(t, \bar{x})$, where $n(\bar{x})$ stands for the unit outward vector normal at $\bar{x} \in \Gamma$.

We shall apply the abstract Main Result to the more concrete system (1.4), after writing the later in the form (1.1). For this purpose it will be enough to take the operators $A = -\nu \Delta + 1$ and $A_{rc}(t) = (a(t, \cdot) - 1)1 + b(t, \cdot) \cdot \nabla$, and the actuator chosen as $\Phi(t) = \hat{\omega}(c(t))$, where $\hat{\omega}(c(t))$ denotes the normalized indicator function whose support is the rectangle $\omega(c(t))$. This rectangle $\omega(c(t)) := c(t) + \omega_0 \subset \Omega$ is the translation of a rectangular reference domain $\omega_0 \subset \mathbb{R}^d$, with $0 \in \omega_0$, and $|\omega_0| := \int_{\omega_0} 1 \, d\mathbb{R}^d$. Then
\[ \hat{\omega}(c(t))(x) := \begin{cases} |\omega_0|^{-\frac{1}{2}}, & \text{if} \quad x \in \omega(c(t)), \\ 0, & \text{if} \quad x \not\in \omega(c(t)) \end{cases} \quad \text{and} \quad \left| \hat{\omega}(c(t)) \right|_{L^2(\Omega)} = 1. \]

To simplify the exposition let us also assume that $0$ is the center of mass of $\omega_0$, so that we can simply say that $0$ is the center of $\omega_0$. Since $c(t) \in \omega(c(t))$, this justifies to call $c(t) \in \mathbb{R}^d$ the center of the actuator. Hence the motion of the actuator $\Phi(t) = \hat{\omega}(c(t))$ is described by the center of $\omega(c(t))$. See Figure 1, where we have taken $\omega_0 \subset \mathbb{R}^d$ as a small rectangular domain.

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** An internal moving actuator with support $\omega(c(t)) \subset \Omega$. 
The main result of this paper, when applied to (1.4), implies the following Theorem 1.1, concerning parabolic equations evolving in the rectangular domain

\[ \Omega := \bigotimes_{n=1}^{d} (0, L_n) \subset \mathbb{R}^d, \quad (1 - r)\Omega + \frac{r}{2} L := \bigotimes_{n=1}^{d} \left( \frac{r}{2} L_n, L_n - \frac{r}{2} L_n \right) \subset \Omega, \]  

(1.5a)

For any given \( r \in [0, 1] \) we further define the subsets

\[ r\Omega := \bigotimes_{n=1}^{d} (0, rL_n) \subset \Omega, \quad \omega_0 := (1 - r)\Omega - \frac{r}{2} L. \]  

(1.5b)

Observe that \( c + \omega_0 \subset \Omega \) if, and only if, \( c \in (1 - r)\Omega + \frac{r}{2} L \).

**Theorem 1.1.** Let \( \Omega \) be a bounded rectangular domain as in (1.5a), and let \( a \in L^\infty((0, +\infty) \times \Omega, \mathbb{R}) \) and \( b \in L^\infty((0, +\infty) \times \Omega, \mathbb{R}^d) \). Then for each sufficiently small \( r \in (0, 1) \), each initial state \( y_0 \in L^2(\Omega) \), and each initial actuator position \( c(0) = c_0 \in (1 - r)\Omega + \frac{r}{2} L \) with initial actuator velocity \( \dot{c}(0) = 0 \in \mathbb{R}^d \), there exists an actuator motion function \( c \) and a magnitude control function \( u \), with

\[ c(t) \in (1 - r)\Omega + \frac{r}{2} L, \quad \text{and} \quad u(t) \in \mathbb{R} \]

such that the solution of (1.4), with

\[ |\omega(c(t)) := c(t) + r\Omega - \frac{r}{2} L = \bigotimes_{n=1}^{d} \left( c_n(t) - \frac{r}{2} L_n, c_n(t) + \frac{r}{2} L_n \right) \]

satisfies

\[ |y(t, \cdot)|_{L^2(\Omega)} \leq C e^{-\mu t} |y_0|_{L^2(\Omega)}, \quad \text{for all} \quad t \geq 0, \]  

(1.6a)

with

\[ u \in L^2((0, +\infty), \mathbb{R}), \quad \dot{c} \in L^\infty((0, +\infty), \mathbb{R}^d), \quad \text{and} \quad \ddot{c} \in L^\infty((0, +\infty), \mathbb{R}^d). \]  

(1.6b)

Furthermore, the mapping \( y_0 \mapsto u(y_0) \) satisfies \( |u(y_0)|_{L^2((0, +\infty), \mathbb{R})} \leq C_u |y_0|_{L^2(\Omega)} \) and \( |\dot{c}|_{L^\infty([0, t], \mathbb{R}^d)} + |\ddot{c}|_{L^\infty([0, t], \mathbb{R}^d)} \leq C_c \). Above the constants \( C, C_u, C_c, \) and \( \mu > 0 \) are independent of \( y_0 \) and \( c_0 \).

The proof of Theorem 1.1 will be given in Section 4.

Besides the theoretical result we also discuss the numerical computation and implementation of a stabilizing control input based on a moving indicator function. Note that the control input \( u(t)\tilde{\omega}(c(t)) \) depends nonlinearly on the control functions \((u, c)\). In order to realize the geometrical constraint \( \omega(c(t)) \subset \Omega \), which can be obtained through constraints on the velocity \( \dot{c} \) and acceleration \( \ddot{c} \), it will be convenient to introduce a new auxiliary function

\[ \eta = \dot{c} + \varsigma \dot{c} + \epsilon c, \quad \text{for given} \quad \epsilon \geq 0, \quad \varsigma \geq 0. \]

We shall consequently consider system (1.4) in the extended form

\[ \begin{align*}
\dot{y} - \nu \Delta y + ay + b \cdot \nabla y &= u\tilde{\omega}(c), \quad G y|_\Gamma = 0, \quad y(0, \cdot) = y_0, \\
\dot{c} + \varsigma \dot{c} + \epsilon c &= \eta,
\end{align*} \]  

(1.7a)

(1.7b)

with proper constraints on the newly introduced additional control \( \eta \), in order to force the actuator to move in an appropriate way. Note that looking for \( c \) is equivalent to looking for \( \eta \), as soon as the initial actuator position \( c_0 \) is given. An
analogous extension argument is used in [5, 25, 28], with a first order ODE, \( \dot{c} + \epsilon c = \eta \) in order to deal with boundary control problems.

In order to compute the pairs \((u, \eta)\), the stabilization problem will be formulated as an infinite horizon optimal control problem (see (5.1)–(5.2)) whose solution will be a stabilizing pair \((u, \eta)\). To deal with the resulting infinite-horizon problem a receding horizon control framework will be employed. In this framework, a stabilizing moving control is constructed through the concatenation of solutions of open-loop problems defined on overlapping temporal domains covering \([0, \infty)\).

1.2. Related literature. Moving controls have been considered, for example, in [10, 20] where suitable moving Dirac delta functions are taken as actuators. In [20], both approximate controllability and exact null controllability results are proven for a semilinear 1D parabolic equation by means of two moving Dirac functions. Both Dirac delta functions and indicator functions are typical actuators in applications, see for instance [20] (cf. [20, Eqs. (1.2) and (1.3)]). Such actuators lead to lumped controls, which are essentially characterized by the temporal behavior only. Concerning again the terminology, in [20] the Dirac delta functions based controls are called point controls, and the indicator functions based controls are called average controls or zone controls. In [10] approximate controllability results for higher dimensional linear autonomous parabolic equations, by means of moving point controls and, more generally, with controls moving in a lower-dimensional submanifold, are presented. For semilinear 1D parabolic equations evolving in the spatial interval \((0, 1) \subset \mathbb{R}\), approximate controllability results have been derived in [19] by means of a single static average control \( u(t)1_{\omega}, \omega = (l_1, l_2) \subset (0, 1) \). The results are obtained under the condition that \( l_1 \pm l_2 \) are irrational numbers.

Concerning partial differential equations which are not of parabolic type we refer to [24], where controllability properties for 1D damped wave equations, under periodic boundary conditions, \( \Omega = \mathbb{T} \), are derived by means of a control based on a single moving point actuator \( u(t)\Phi(t) \). The actuator is either a Dirac delta \( \Phi(t) = \delta_{c(t)} \), see [24, Thm. 1.4], or a single moving function \( \Phi(t) = \phi_{c(t)} \in L^2(\mathbb{T}) \), see [24, Thm. 1.1] where we can also see that the function \( \phi_{c(t)} \) is required to have zero mean. We refer also to [9, 11, 23] where a moving average control is considered, but where the magnitude control function \( u = u(t, x) \) depends on both time and space variables. By means of such a moving control, in [11] the approximate controllability of higher dimensional damped wave equations is derived and, in [9] the inner null controllability of the one-dimensional wave equation is investigated theoretically and numerically. In [23] the null controllability is derived for a 1D coupled PDE-ODE system of FitzHugh–Nagumo type, again with a magnitude control function \( u = u(t, x) \) depending on both time and space variables. We recall that such systems are not null controllable by means of static average controls.

It is well known that observability properties and null controllability properties are related. In this respect we refer to the observability results in [18] for the autonomous higher dimensional case with point observations. We recall that often the tools used to derive controllability/observability results for autonomous systems are not appropriate or are not valid to deal with the nonautonomous case. See for example the solution representation in [18, Eq. (2.1)], and the discussion in [10, Sect. 6, §1].

Our result in Theorem 1.1 is of different nature, when compared to the ones mentioned above. Approximate and null controllability are properties concerning
the state $y(T)$ at a given time $T$. Instead, our goal in (1.6) is concerned with the asymptotic behavior of the state (as time goes to $+\infty$). Of course, if we have a control driving the state to $y(T) = 0$ at time $t = T$, then by switching the control off, for $t > T$, results in a stabilizing control. Thus exact controllability is a stronger property than stabilizability.

On the other hand for practical considerations controls driving the system to 0 at time $T$ may not be enough for applications, since, due to noise or computational error, the control may not drive the state exactly to the origin. If the latter is unstable and the control is nonetheless switched off then the state may diverge as time tends to infinity. Therefore, a control is still needed which stabilizes the state once it is close to the origin, or which keeps it in a small neighborhood of 0 which is proportional to the magnitude of noise and disturbances.

Moving indicator functions have also been considered in [13], where the goal is not the stabilizability of a given unstable free dynamics (as in this manuscript), but rather to speed up the stabilization and/or counteract the effect of external disturbances (sources). Though the nominal systems under study in [13] are stable parabolic equations, the proposed control design is interesting for applications.

1.3. On (lack of) stabilizability with $N$ static actuators. In this section we provide examples where, for a fixed $N$, there is no family of $N$ static actuators which are able to stabilize the system, no matter what the shape or placement of the actuators in the spatial domain are. This negative result can be seen as a motivation for our work in this manuscript, where we show that we can still stabilize the system if we are allowed to dynamically move a given indicator function as actuator.

Here we consider only the particular case of controlled autonomous systems as

$$
\dot{y} + Ay = \sum_{j=1}^{N} u_j \Psi_j, \quad y(0) = y_0, \quad t \geq 0,
$$

with $A = A^* \in \mathcal{L}(V, V')$, with a dense, continuous, and compact inclusion $V \subseteq H$, for suitable real separable Hilbert spaces $V$ and $H = H'$. Furthermore, we assume that, for a suitable constant $C > 0$, we have that

$$
\langle Ay, y \rangle_{V', V} \geq \lvert y \rvert_V^2 - C \lvert y \rvert_H^2, \quad \text{for all } y \in V,
$$

$$
y_0 \in H, \quad \{ \Psi_j \mid 1 \leq j \leq N \} \subset H, \quad \text{and } u \in L^2_{\text{loc}}((0, +\infty), \mathbb{R}^N).
$$

Finally, we assume that the symmetric operator $A$, with domain $\text{D}(A) = \{z \in H \mid Az \in H\}$, has a countable complete linearly independent system of eigenfunctions $\{ \tilde{e}_i \mid i = 1, 2, \ldots \}$, with corresponding eigenvalues $\tilde{\alpha}_i$ satisfying

$$
A\tilde{e}_i = \tilde{\alpha}_i \tilde{e}_i, \quad \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \leq \tilde{\alpha}_3 \leq \ldots, \quad \lim_{i \to +\infty} \tilde{\alpha}_i = +\infty.
$$

**Proposition 1.2.** Let $N \in \mathbb{N}_0$. If the first (smallest) eigenvalue $\tilde{\alpha}_1$ is nonpositive with multiplicity $N + 1$, then for each family $F = \{ \Psi_j \mid 1 \leq j \leq N \} \subset H$ of actuators we can find $y_0 \in H \setminus \{0\}$ such that $Ay_0 = \tilde{\alpha}_1 y_0$ and $y_0 \in F^\perp$, where $F^\perp$ is the orthogonal space to $F$, in $H$. In particular, for all $u \in L^2_{\text{loc}}((0, +\infty), \mathbb{R}^N)$ the weak solution $y$ of (1.8) satisfies $\lvert y(\cdot) \rvert_H \geq e^{-\tilde{\alpha}_1 t} \lvert y_0 \rvert_H \geq \lvert y_0 \rvert_H$.

The proof of Proposition 1.2 is given in the Appendix, Section A.1.

**Example 1.3.** Let $\beta_1$ be the smallest eigenvalue of the Laplacian $-\Delta$ defined in $L^2(\Omega)$, under either Dirichlet or Neumann boundary conditions. Then, for $\xi \geq \beta_1$ we have that the smallest eigenvalue of $-\Delta - \xi 1$ is $\beta_1 - \xi \leq 0$. Let $\beta_1 \leq \beta_2 \leq \ldots$
denote the increasing sequence of eigenvalues of $-\Delta$, $\beta_1 \to +\infty$, with corresponding orthonormal eigenfunctions $\tilde{e}_1, \tilde{e}_2, \ldots$. Then the eigenvalues $\alpha_j$ of the operator

$$\mathcal{A}y := -\Delta y - \xi y + \sum_{n=2}^{N+1} (\beta_1 - \beta_n)(y, \tilde{e}_n)_H \tilde{e}_n$$

are given by $\alpha_j = \beta_1 - \xi \leq 0$ for all $1 \leq j \leq N + 1$, and by $\alpha_j = \beta_j - \xi$ for all $j > N + 1$. The associated eigenfunctions are (or, can still be taken as) $\tilde{e}_j = \tilde{c}_j$, for all $j \in \mathbb{N}_0$. Indeed, we find:

If $j = 1$ or $j > N + 1$, \[ \mathcal{A}\tilde{c}_j = (-\Delta - \xi)(\tilde{c}_j) = (\beta_1 - \xi)\tilde{c}_j; \]

If $2 \leq j \leq N + 1$, \[ \mathcal{A}\tilde{c}_j = (-\Delta - \xi)(\tilde{c}_j) + \sum_{n=2}^{N+1} (\beta_1 - \beta_n)(\tilde{c}_j, \tilde{e}_n)_H \tilde{e}_n \]

Hence, by Proposition 1.2, system (1.8) is not stabilizable with $N$ static actuators.

Remark 1.4. The operator $y \mapsto \sum_{n=2}^{N+1} (\beta_1 - \beta_n)(y, \tilde{e}_n)_H \tilde{e}_n$ is in $\mathcal{L}(H) \subseteq \mathcal{L}(V, H)$ (cf. Assumption 2.3).

Remark 1.5. For concrete scalar parabolic equations, by stabilizability with $N$ static actuators we mean stabilizability with controls in the form $u = \sum_{j=1}^N u_j(t)\Phi_j(x)$ with a fixed family $\{\Phi_j \mid 1 \leq j \leq N\}$ of actuators. If we fix a control domain region $\Omega \subseteq \Omega$ and take controls as $v = 1_\omega u(t, x)$, then it is likely that we will be able to stabilize (1.8) (e.g., this is the case if $\mathcal{A}y = -\Delta y - \xi y$). By Proposition 1.2, if $v$ stabilizes (1.8) then the set $\{1_\omega u(t, x) \mid t \geq 0\}$ has to contain at least $N + 1$ linearly independent functions/profiles/actuators.

Next, for the sake of completeness, we present/recall also a positive result for stabilization with an appropriate single actuator $\Psi_1$.

Proposition 1.6. If all the nonpositive eigenvalues of $\mathcal{A}$ are simple, and if none of the corresponding eigenfunctions is orthogonal to $\Psi_1$, then system (1.8) is exponentially stabilizable.

The proof of Proposition 1.6 is given in the Appendix, Section A.1.
endowed with the norms $|(h,g)|_{X \times Y} := \left( |h|_X^2 + |g|_Y^2 \right)^{\frac{1}{2}}$, $|\tilde{h}|_{X \times Y} := |(\tilde{h},h)|_{X \times Y}$, and $|\tilde{h}|_{X+Y} := \inf_{(h,g) \in X \times Y} \{(h,g)_{X \times Y} \mid h = h + g\}$, respectively. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

For a given interval $I \subset \mathbb{R}$, we denote $W(I,X,Y) := \{f \in L^2(I,X) \mid \dot{f} \in L^2(I,Y)\}$, endowed with the norm $|f|_{W(I,X,Y)} := \left( \int_I |\dot{f}(t)|^2 \right)^{\frac{1}{2}}$. The space of continuous linear mappings from $X$ into $Y$ is denoted by $L(X,Y)$. In case $X = Y$ we write $L(X) := L(X,X)$.

The continuous dual of $X$ is denoted $X' := L(X,\mathbb{R})$. The adjoint of an operator $L \in L(X,Y)$ will be denoted $L^* \in L(Y',X')$.

The space of continuous functions from $X$ into $Y$ is denoted by $C(X,Y)$.

The orthogonal complement to a given subset $B \subset H$ of a Hilbert space $H$, with scalar product $(\cdot,\cdot)_H$, is denoted by $B^\perp := \{h \in H \mid (h,s)_H = 0 \text{ for all } s \in B\}$. Given two closed subspaces $F \subset H$ and $G \subset H$ of the Hilbert space given by $H = F \oplus G$, we denote by $P^F_G \in L(H,F)$ the oblique projection in $H$ onto $F$ along $G$. That is, writing $h \in H$ as $h = h_F + h_G$ with $(h_F,h_G) \in F \times G$, we have $P^F_G h := h_F$. The orthogonal projection in $H$ onto $F$ is denoted by $P_F \in L(H,F)$. Notice that $P_F = P^F_G$.

By $[a_1,\ldots,a_n]$ we denote a nonnegative function that increases in each of its nonnegative arguments $a_i \geq 0$, $1 \leq i \leq n$.

Finally, $C$, $C_i$, $i = 0, 1, \ldots$, stand for unessential positive constants.

2. Assumptions. The results will follow under general assumptions on the plant dynamics operators $A$ and $A_e$, and on a particular stabilizability assumption of (1.1) by means of controls based on a large enough finite number $M$ of suitable static actuators.

The Hilbert space $H$, in which system (1.1) is evolving in, will be set as a pivot space, that is, we identify, $H' = H$. Let $V$ be another Hilbert space with $V \subset H$.

Assumption 2.1. $A \in L(V,V')$ is symmetric and $(y,z) \mapsto \langle Ay,z \rangle_{V',V}$ is a complete scalar product in $V$.

From now on, we suppose that $V$ is endowed with the scalar product $(y,z)_V := \langle Ay,z \rangle_{V',V}$, which still makes $V$ a Hilbert space. Necessarily, $A : V \rightarrow V'$ is an isometry.

Assumption 2.2. The inclusion $V \subset H$ is dense, continuous, and compact.

Necessarily, we have that $\langle y,z \rangle_{V',V} = (y,z)_H$, for all $(y,z) \in H \times V$, and also that the operator $A$ is densely defined in $H$, with domain $D(A)$ satisfying $D(A) \xrightarrow{d.c} V \xleftarrow{d.c} H \xleftarrow{d.c} V' \xrightarrow{d.c} D(A)'$.

Further, $A$ has a compact inverse $A^{-1} : H \rightarrow H$, and we can find a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues $(\alpha_n)_{n \in \mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n \in \mathbb{N}_0}$:

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \rightarrow +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n. \quad (2.1)$$
We can define, for every $\zeta \in \mathbb{R}$, the fractional powers $A^\zeta$, of $A$, by
\[
y = \sum_{n=1}^{+\infty} y_n e_n, \quad A^\zeta y = A^\zeta \sum_{n=1}^{+\infty} y_n e_n := \sum_{n=1}^{+\infty} \alpha_n^\zeta y_n e_n,
\]
and the corresponding domains $D(A^\zeta) := \{ y \in H \mid A^\zeta y \in H \}$, and $D(A^{-\zeta}) := D(A^\zeta)'$. We have that $D(A^\zeta) \overset{d,\zeta}{\longrightarrow} D(A^\varphi)$, for all $\zeta > \zeta_1$, and we can see that $D(A^0) = H$, $D(A^1) = D(A)$, $D(A^{1/2}) = V$.

For the time-dependent operator we assume the following:

**Assumption 2.3.** For almost every $t > 0$ we have $A_{rc}(t) \in \mathcal{L}(V,H)$, and we have a uniform bound, that is, $|A_{rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(V,H))} := C_{rc} < +\infty$.

Finally, we will need the following norm squeezing property, by means of controls based on static actuators.

**Assumption 2.4.** There exist:
- a positive integer $M$, and positive real numbers $T > 0$ and $\theta \in (0,1)$,
- a linearly independent family $\{ \hat{\Phi}_j \mid j \in \{1,2,\ldots,M\} \} \subset H$ with $\hat{\Phi}_j|_H = 1$,
- a family of functions $\{ v_k \in \mathcal{L}(V,L^\infty((kT,kT+T),\mathbb{R}^M)) \mid k \in \mathbb{N} \}$, with $\sup_{k \in \mathbb{N}} |v_k|_{\mathcal{L}(V,L^\infty((kT,kT+T),\mathbb{R}^M))} \leq R$,

such that: for all $k \in \mathbb{N}$, the solution of
\[
\dot{y} + Ay + A_{rc}(t)y = \sum_{j=1}^{M} v_{k,j}(v)(t)\hat{\Phi}_j, \quad y(kT) = v, \quad t \in (kT,kT+T), \quad (2.2)
\]
satisfies
\[
|y(kT+T)|_V \leq \theta |v|_V, \quad \text{for all } v \in V. \quad (2.3)
\]

**Remark 2.5.** Assumptions 2.1–2.4 are satisfiable for parabolic equations as (1.4) evolving in bounded rectangular domains $\Omega \subset \mathbb{R}^d$. The satisfiability of such assumptions shall be revisited/proven later on, in Section 4, where we give the proof of Theorem 1.1, concerning standard parabolic equations.

**Remark 2.6.** Alternatively, in Assumption 2.3 we can take a reaction-convection term $A_{rc}(t) \in L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))$. The proof will however involve slightly different steps. Motivations and further details are given later in Section 4.4.

3. **Existence of a moving stabilizing control.** Hereafter $\mathcal{S}_H$ denotes the unit sphere in $H$,
\[
\mathcal{S}_H := \{ h \in H \mid |h|_H = 1 \},
\]
and for a given subset $R \subset C^2([0,1],\mathcal{S}_H)$ we denote
\[
\mathcal{S}_H^R := \{ f(s) \in \mathcal{S}_H \mid f \in R \text{ and } s \in [0,1] \}.
\]

Our main result reads as follows.

**Theorem 3.1.** Let Assumptions 2.1–2.4 hold true, let the $\hat{\Phi}_i$s, $1 \leq i \leq M$, be as in Assumption 2.4 with $M > 1$, and let $\mathcal{R}_i \in C^2([0,1],H)$ be functions satisfying,
for $1 \leq i \leq M-1$,
\[
\mathcal{R}_i(0) = \hat{\Phi}_i, \quad \mathcal{R}_i(1) = \hat{\Phi}_{i+1}, \quad \text{and } \mathcal{R}_i(s) \in \mathcal{S}_H \text{ for all } s \in [0,1].
\]
Let also $\mathcal{R} := \{\mathcal{R}_i \mid 1 \leq i \leq M - 1\}$. Then, there exist a magnitude control function $u$ and a continuously moving actuator $\Phi$ satisfying

$$ u \in L^2(\mathbb{R}_0, \mathbb{R}), \quad \Phi \in L^\infty(\mathbb{R}_0, \mathcal{H}), \quad \dot{\Phi} \in L^\infty(\mathbb{R}_0, \mathcal{H}), $$

$$ \Phi(0) = \tilde{\Phi}_1, \quad \dot{\Phi}(0) = 0, \quad \Phi(t) \in \mathcal{S}_H^\infty \text{ for } t \geq 0, $$

and constants $C \geq 1$ and $\mu > 0$, such that the solution of the system (1.1),

$$ \dot{y} + Ay + A_{rc}(t)y = u(t)\Phi(t), \quad y(0) = y_0 \in V, \quad t > 0, $$

satisfies (1.2),

$$ \|y(t)\|_V \leq C e^{-\mu t} |y_0|_V, \quad \text{for all } t \geq 0, $$

and the mapping $y_0 \mapsto (u(y_0), \Phi(y_0))$ satisfies

$$ |u(y_0)|_{L^2(\mathbb{R}_0, \mathbb{R})} \leq \mathcal{N}_0 |y_0|_V \quad \text{and} \quad \left|\dot{\Phi}(y_0)\right|_{W^{1,\infty}(\mathbb{R}_0, \mathcal{H})} \leq C_\Phi, $$

where the tuple $(\mu, C, \mathcal{N}_0, C_\Phi) \in (0, +\infty)^4$ is independent of $y_0$.

Note that Theorem 3.1 gives us stabilizability in the $V$-norm. The stabilizability in $H$-norm as stated in (1.2) follows as a consequence.

Note also that the paths $\mathcal{R}_i$ in Theorem 3.1 are arbitrarily fixed a priori.

**Corollary 3.2.** Let Assumptions 2.1–2.4 hold true and let $\Phi^* \in C^2([0, 1], \mathcal{H})$ satisfy

$$ \Phi^*(1) = \tilde{\Phi}_1, \quad \Phi^*(1) = 0, \quad \Phi^*(t) \in \mathcal{S}_H, \quad \text{for } t \in [0, 1]. $$

Let $\mathcal{R} = \{\mathcal{R}_i \mid 1 \leq i \leq M - 1\}$ be as in Theorem 3.1, and let $\mathcal{R}^* := \mathcal{R} \setminus \{\Phi^*\}$. Then, there exist a magnitude control function $u^*$ and a continuously moving actuator $\Phi^*$ satisfying

$$ u^* \in L^2(\mathbb{R}_0, \mathbb{R}), \quad \dot{\Phi}^* \in L^\infty(\mathbb{R}_0, \mathcal{H}), \quad \ddot{\Phi}^* \in L^\infty(\mathbb{R}_0, \mathcal{H}), $$

$$ \Phi^*|_{[0, 1]} = \Phi^*, \quad \Phi^*(t) \in \mathcal{S}_H^\infty \text{ for } t \geq 0, $$

and constants $C^* \geq 1$ and $\mu > 0$, such that the solution of the system (1.1),

$$ \dot{y} + Ay + A_{rc}(t)y = u^*(t)\Phi^*(t), \quad y(0) = y_0 \in H, \quad t > 0, $$

satisfies (1.2),

$$ \|y(t)\|_H \leq C^* e^{-\mu t} |y_0|_H, \quad \text{for all } t \geq 0, $$

and the mapping $y_0 \mapsto (u(y_0), \Phi^*(y_0))$ satisfies

$$ |u(y_0)|_{L^2(\mathbb{R}_0, \mathbb{R})} \leq \mathcal{N}_0^* |y_0|_H \quad \text{and} \quad \left|\ddot{\Phi}^*(y_0)\right|_{W^{1,\infty}(\mathbb{R}_0, \mathcal{H})} \leq C^*_\Phi, $$

with $C^*_\Phi = \max \left\{C_{\Phi}, \left|\Phi^*\right|_{W^{1,\infty}((0, 1), \mathcal{H})}\right\}$. Further, the tuple $(\mu, C^*, \mathcal{N}_0^*, C^*_\Phi) \in (0, +\infty)^4$ is independent (at $y_0, \Phi^*(0)$).

**Proof.** For $t \in [0, 1]$ we choose the control $u(t)\Phi^*(t)$ with $u = 0$. Using the smoothing property of parabolic-like equations (cf. [8, Lem. 2.4]), we arrive at a state $y(1) = y_1 \in V$, with

$$ \|y(1)\|_V \leq \overline{C}_{\mathcal{R}_1} |y(0)|_H. \quad (3.5) $$

In the time interval $\mathcal{R}_1$, we can find a control $u^1 \in L^2(\mathbb{R}_1, \mathbb{R})$ and a moving actuator $\Phi^1$ as in Theorem 3.1, with $\Phi^1(1) = \tilde{\Phi}_1$ and $\dot{\Phi}^1(1) = 0$, giving us

$$ \|y(t)\|_V \leq \overline{C}_{\mathcal{R}_1} e^{-\mu(t-1)} |y(1)|_V, \quad t \geq 1. \quad (3.6) $$
Indeed it is enough to consider a shift in time variable and use Theorem 3.1 to the function \( w(\tau) = y(1 + \tau) \), which solves the system

\[
\frac{d}{d\tau} w + \tilde{A}_{\text{rc}} w = u(\tau) \Phi(\tau), \quad w(0) = y_1, \quad \tau > 0,
\]

with \( \tilde{A}_{\text{rc}}(\tau) = A_{\text{rc}}(1 + \tau) \). Hence obtaining

\[
|w(\tau)|_V \leq \overline{C}_{[\tilde{A}_{\text{rc}},1]} e^{-\mu \tau} |w(0)|_V, \quad \tau \geq 0,
\]

with \( C_{\text{rc},1} = |\tilde{A}_{\text{rc}}|_{L^\infty(\mathbb{R}_0, L(V,H))} = |A_{\text{rc}}|_{L^\infty(\mathbb{R}_1, L(V,H))} \leq C_{\text{rc}} \) which implies (3.6), by taking for \( t \geq 1 \), \( u^1(t) = u(t-1) \) and \( \Phi^1(t) = \Phi(t-1) \).

Next, defining

\[
\begin{align*}
\phi^c(t) &= 0 \quad \text{and} \quad \Phi^c(t) = \Phi^*(t), \quad \text{for} \quad t \in [0,1], \\
\phi^c(t) &= u^1(t) \quad \text{and} \quad \Phi^c(t) = \Phi^1(t), \quad \text{for} \quad t \geq 1,
\end{align*}
\]

we obtain, using (3.6) and (3.5),

\[
\begin{align*}
|y(t)|_H &\leq \overline{C}_{[\tilde{A}_{\text{rc}},1]} |y(t)|_V \leq \overline{C}_{[\tilde{A}_{\text{rc}},1]} e^{-\mu(t-1)} |y(1)|_V \\
&\leq \overline{C}_{[\tilde{A}_{\text{rc}},1]} e^{-\mu t} |y(0)|_H, \quad t \geq 1,
\end{align*}
\]

and (cf. [8, Lem. 2.2])

\[
|y(t)|_H \leq \overline{C}_{[\tilde{A}_{\text{rc}}]} |y(0)|_H \leq \overline{C}_{[\tilde{A}_{\text{rc}}]} e^{\mu t} |y(0)|_H \\
\leq \overline{C}_{[\tilde{A}_{\text{rc}},1]} e^{-\mu t} |y(0)|_H, \quad t \in [0,1].
\]

We can see that we can take \( C^\infty \) of the form \( \overline{C}_{[\tilde{A}_{\text{rc}},1]} |y(V,H)| \) in (3.4a).

Using \( \Phi^*(1) = \Phi^1(1) = \tilde{\Phi}_1 \) and \( \Phi^*(1) = \Phi^1(1) = 0 \), we can conclude that \( \Phi^c \in C^1([0, +\infty), H) \). Finally, by Theorem 3.1 we have \( \left| \tilde{\Phi}^1 \right|_{L^\infty(\mathbb{R}_1, H)} + \left| \tilde{\Phi}^1 \right|_{L^\infty(\mathbb{R}_1, H)} \leq C_{\Phi} \)
with \( C_{\Phi} \) independent of \( y_1 \) (and of \( \Phi^* \)).

We are going to use Assumption 2.4 together with a concatenation argument, and will prove that Theorem 3.1 is a corollary of the following result concerning the restriction of our system to the intervals

\[
I_k := (kT, kT + T), \quad \overline{T}_k := [kT, kT + T] \quad k \in \mathbb{N}. \tag{3.7}
\]

**Theorem 3.3.** Under Assumptions 2.1–2.4, there exist a magnitude control function \( u_k \) and a continuous moving actuator \( \Phi_k \) satisfying

\[
u \in L^2(I_k, \mathbb{R}), \quad \Phi_k \in L^\infty(I_k, H), \quad \Phi_k(t) \in \mathcal{S}_H \quad \text{for} \quad t \in I_k,
\]

\[
\Phi_k(kT) = \Phi_k(kT + T) = \tilde{\Phi}_1, \quad \Phi_k(kT) = \Phi_k(kT + T) = 0,
\]

such that the solution of the system

\[
\dot{y} + Ay + A_{\text{rc}}(t)y = u_k(t)\Phi_k(t), \quad y(kT) = v \in V, \tag{3.8}
\]

satisfies

\[
|y(kT + T)|_V \leq \frac{\mu + 1}{2} |v|_V, \tag{3.9a}
\]

and, the mapping \( v \mapsto (u_k(v), \Phi_k(v)) \) satisfies

\[
|u_k(v)|_{L^2(I_k, \mathbb{R})} \leq \mathcal{M}_1 |y_0|_V \quad \text{and} \quad \left| \Phi_k(y_0) \right|_{W^{1,\infty}(\mathbb{R}_0, H)} \leq C_1, \tag{3.9b}
\]

with the tuple \( (\mathcal{M}_1, C_1) \) independent of \( (v,k) \in V \times \mathbb{N} \).
Proof of Theorem 3.1. We consider the concatenation of controls $u_k$ given by Theorem 3.3 as follows
\[ u\Phi(y_0) = u_k\Phi_k(y(kT)), \quad \text{if } t \in I_k, \]
where the construction of $u$ is to be understood in a sequential manner: first we take $u\Phi(y_0)\mid_{I_0} = u_0\Phi_0(y(0T)) = u_0\Phi_0(y_0)$, then we consider the corresponding state $y(T)$ at final time $t = T$, which we then use to define $u\Phi(y_0)\mid_{I_1} = u_1\Phi_1(y(T))$, in this way, by concatenation, we have constructed a control on the interval $I_0 \cup I_1 = (0, 2T)$. Once we have constructed the control $u\Phi(y_0)\mid_{(0,kT)}$ on $(0, kT)$, we take $u\Phi(y_0)\mid_{I_k} = u_k\Phi_k(y(kT))$ and have a control defined for time $t \in (0, (k+1)T)$. Eventually we will have $u\Phi(y_0)$ defined in the entire time interval $\mathbb{R}_0$.

By (3.9a), we find that the solution associated to $u\Phi(y_0)$ satisfies
\[ |y(kT + T)|_V \leq \frac{2^{|k+1|}}{|k+1|} |y(kT)|_V \leq \frac{(k+1)}{|k+1|} |y(0)|_V, \tag{3.10} \]
that is, since $0 < \theta < 1$,
\[ |y(t)|_V \leq \frac{2^{|k+1|}}{|k+1|} |y(0)|_V, \quad \text{with } \theta := \frac{1}{r} \log \left(\frac{2^{|k+1|}}{|k+1|}\right) > 0. \tag{3.11} \]

Now, by a standard continuity argument (e.g., see [8, Lem. 2.3], recalling also that $W(I_k, D(A), H) \rightarrow C(I_k, V)$), we find that for $t \geq 0$,
\[ |y(t)|_V^2 \leq \bar{C}[T, C, \varepsilon] \left(|y(kT)|_V^2 + |u_k\Phi_k^2|_{L^2(I_k, H)}\right), \quad \text{with } k = \lfloor \frac{r}{2}\rfloor, \tag{3.12} \]
where $\lfloor r \rfloor$ denotes the integer satisfying $k \leq r < k + 1, r \in \mathbb{R}$. Since $|\Phi(t)|_H = 1$, it follows that, by using (3.10),
\[ |y(t)|_V^2 \leq \bar{C}[T, C, \varepsilon] \left(|y(kT)|_V^2 + |u_k\Phi_k^2|_{L^2(I_k, H)}\right) \leq \bar{C}[T, C, \varepsilon, \varepsilon_1] e^{-\pi kT} |y(0)|_V^2 \leq \bar{C}[T, C, \varepsilon, \varepsilon_1] e^{-\pi (t-kT)} e^{-\pi t} |y(0)|_V^2 \leq \bar{C}[T, C, \varepsilon_1] e^{-\pi t} |y(0)|_V^2, \]
because $e^{\pi (t-kT)} \leq e^{\pi T} = \bar{C}[T, \pi]$. Therefore, (3.2a) holds true.

It remains to show (3.2b). Due to (3.9b) and (3.10) we find
\[ |u(y_0)|_{L^2(\mathbb{R}_0, \mathbb{R})} = \sum_{k=0}^{\infty} |u_k(y(kT))|_{L^2(I_k, \mathbb{R})} \leq \mathfrak{M}_1 \sum_{k=0}^{\infty} |y(kT)|_V \leq \mathfrak{M}_1 |y(0)|_V \sum_{k=0}^{\infty} e^{-\pi kT} = \mathfrak{M}_1 \frac{1}{1-e^{-\frac{\pi}{T}}} |y_0|_V, \]
which gives us (3.2b), with $\mathfrak{M}_0 = \mathfrak{M}_1 \frac{1}{1-e^{-\frac{\pi}{T}}}$. \hfill \Box

Proof of Theorem 3.2. Let us fix an arbitrary $k \in \mathbb{N}$. By Assumption 2.4, we have that $\mathcal{V}_k(\nu)(t) = \sum_{j=1}^{M} v_{k,j}(\nu)(t)\Phi_j$, is a control function driving system (2.2) from $\nu \in V$ at time $t = kT$ to a state $y(kT + T)$ at time $t = kT + T$, with a norm squeezed by a factor $\theta \in (0, 1)$. The proof will follow by successive approximations of such control, hence we start by denoting $\mathcal{V}_k^0 = \mathcal{V}_k$, where the superscript underlines that $\mathcal{V}_k^0$ is our starting control. Since $k$ has been fixed, for simplicity we will omit the subscript $k$ in the control, $\mathcal{V}^0 = \mathcal{V}_k^0 = \mathcal{V}_k$.

\[ \mathcal{V}^0(\nu)(t) = \sum_{j=1}^{M} v_{j,0}(\nu)(t)\Phi_j, \quad v_{j,0}(\nu)(t) := v_{k,j}(\nu)(t), \quad (\nu, t) \in V \times I_k. \tag{3.13} \]
Let us consider our dynamical system (2.2) with a general external forcing \( f \),
\[
\dot{y} + Ay + A_{rc}y = f, \quad y(kT) = v \in V, \quad t \in I_k.
\] (3.14)
Denoting by \( y = \Psi_k(v, f) \), \( y(t) = \Psi_k(v, f)(t) \) the solution of (3.14). We can write
\[
|\Psi_k(v, V^0(t))(kT + T)|_V \leq \theta |v|_V
\] (3.15a)
where \( 0 \leq \theta < 1 \). We see that \( V^0(v) \) is a control based on the static actuators \( \Phi_j \), and recall that
\[
V^0(t) \in S_\Phi := \text{span}\{\Phi_j \mid 1 \leq j \leq M\} \subset H, \quad (3.15b)
\]
\[
v^0 = (v_1^0, v_2^0, \ldots, v_M^0) \in \mathcal{L}(V, L^\infty(I_k, \mathbb{R}^M)). \quad (3.15c)
\]
The proof is completed into 5 main steps in Sections 3.1–3.5, where we construct suitable approximations of \( V^0 \). \( V^0 \approx V^1 \approx V^2 \approx V^3 \approx V^4 \approx V^5 \), arriving at a moving control \( V^5 = V^5(v) \), taking values \( V^5(v)(t) \) in \( H \), with
\[
|\Psi_k(v, V^5(t))(kT + T)|_V \leq \frac{1+\theta}{\theta} |v|_V.
\]
That is, \( V^5(v) \) drives the system from \( v \in H \) at initial time \( t = kT \) to a state \( y(kT + T) \) at final time \( t = kT + T \) with a norm squeezed by a factor \( \frac{1+\theta}{\theta} \in (\theta, 1) \).

In each of remaining steps of the proof of Theorem 3.3 we will use a continuity argument for system (3.14). The main contents, in each step, are as follows.

\( \spadesuit \) Step 1: Taking auxiliary static actuators in \( D(A) \). In Section 3.1, we replace (i.e., approximate) our static actuators \( \Phi_j \in H \) by suitable static actuators \( \Phi_j \in D(A) \subset H \). In this way we obtain a control \( V^1(v)(t) = \sum_{j=1}^M v_j^0(t) \Phi_j \), taking values in \( S_\Phi := \text{span}\{\Phi_j \mid 1 \leq j \leq M\} \subset D(A) \). Taking actuators in \( D(A) \) is needed for technical reasons, which play a role in Step 2 of the proof.

\( \spadesuit \) Step 2: Piecewise constant static control in \( D(A) \). In Section 3.2, we approximate the control \( V^1 \), by a right-continuous piecewise constant control \( V^2(v) \) taking values \( V^2(v)(t) \) in the set \( \{s \Phi_j \mid 1 \leq j \leq M, \ -K \leq s \leq K\} \subset D(A) \) for a suitable constant \( K > 0 \), for all \( t \in I_k \). We underline that, in this key step we want to construct a switching control where at each instant of time \( t \geq 0 \) only one of the actuators is active, \( V^2(v)(t) = s \Phi_j \) for some \( j \in \{1, \ldots, M\} \).

\( \spadesuit \) Step 3: Piecewise constant static control in \( H \). Back to original actuators. In Section 3.3, we replace back the \( \Phi_j \)’s by the \( \Phi_j \)’s. In this way we arrive at a piecewise constant control \( V^3(v) \) defined by \( V^3(v)(t) := s \Phi_j \in H \) if \( V^2(v)(t) = s \Phi_j \).

\( \spadesuit \) Step 4: A piecewise constant static control with nondegenerate intervals of constancy. In Section 3.4 we construct a piecewise constant control \( V^4(v) \) taking values \( V^4(v)(t) \) in \( H \), where the lengths of the intervals of constancy are all larger than a suitable positive constant.

\( \spadesuit \) Step 5: A moving control in \( H \). In Section 3.5, we construct a moving control \( V^5 = V^5(v) = u(t) \Phi(t) \) which visits (several times) the positions of the static actuators \( \Phi_j \), spending a suitable amount of time at those positions, and travels, in \( H \), fast enough between those positions. In this way we obtain a moving control \( V^5 = V^5(v) \), taking values \( V^5(v)(t) \) in \( H \).
Steps 1 and 3 are needed only if some of our static actuators are in $H \setminus D(A)$. This will, in general, be the case for indicator functions $1_{\omega}, \omega \subset \Omega,$ for scalar parabolic equations evolving in bounded domains $\Omega \subset \mathbb{R}^d$.

The continuity arguments in Steps 1, 3, 4, and 5 are standard, namely the continuity of the solution of system (3.14) on the right-hand side as $y = \mathcal{Q}_k(v, \cdot) \in \mathcal{C}(L^2(I, H), C(T, V))$. The continuity argument in Step 2 is less standard, involving the continuity of the solution when the right-hand side varies in the so called relaxation metric, details will be given in Section 3.2.

### 3.1. A static control taking values in $D(A)$

Recall also that the solution of system (3.14) satisfies

$$\mathcal{Q}_k(v, f)(t) \leq D_Y \left( |v|^2_V + |f|^2_{L^2(I, H)} \right), \quad t \in I_k = (kT, kT + T), \quad (3.16)$$

with $D_Y = \mathcal{C}_{[T, C_{rc}]}$ independent of $k$, where $C_{rc}$ is defined in Assumption 2.3.

By Assumption 2.4, the actuators $\Phi_j$ are in the unit sphere $\mathfrak{S}_H$ of $H$. Then, from $D(A) \overset{\text{d}}{\subset} H$ we can choose a family $\{ \Phi_j \mid 1 \leq j \leq M \}$ such that

$$\Phi_j \in D(A) \cap \mathfrak{S}_H, \quad \left| \Phi_j - \hat{\Phi}_j \right|_H \leq \frac{1 + \theta - \theta}{10}, \quad 1 \leq j \leq M. \quad (3.17a)$$

The fact that the $\Phi_j$ can be taken in the unit sphere is a corollary of the following result, whose proof is given in Section A.2.

**Proposition 3.4.** Let $X \subset H$ be a vector space. Then, the density of $X \subset H$ implies the density of $X \cap \mathfrak{S}_H \subset \mathfrak{S}_H$.

Now we recall the control $\mathcal{V}^0$ in (3.13), and define a new control as

$$\mathcal{V}^1(v)(t) = \sum_{j=1}^{M} v_j^1(v)(t)\Phi_j, \quad v_j^1(v)(t) := v_0^0(v)(t), \quad (v, t) \in V \times I_k, \quad (3.18)$$

where we replace each actuator $\Phi_j \in H$ by the auxiliary actuator $\tilde{\Phi}_j \in D(A) \subset H$.

Note that, by Assumption 2.4, for $v^1 := (v_1^1, v_2^1, \ldots, v_M^1)$, we find

$$\sup_{t \in I_k} \left| v^1(v)(t) \right|_{L^M} = \sup_{t \in I_k} \left| v^0(v)(t) \right|_{L^M} \leq \mathfrak{K} |v|_V. \quad (3.19)$$

Let us denote $d^1 := \mathcal{Q}_k(v, \mathcal{V}^1(v)) - \mathcal{Q}_k(v, \mathcal{V}^0(v))$, which satisfies

$$d^1 + Ad^1 + A_{rc}d^1 = \mathcal{V}^1(v) - \mathcal{V}^0(v), \quad \text{for} \ t \in I_k, \quad d^1(kT) = 0. \quad (3.20)$$

By (3.16) and (3.17), we find that, since $M \geq 1$,

$$\left| d^1(t) \right|_V \leq D^\frac{1}{2} \left( \mathcal{V}^1(v) - \mathcal{V}^0(v) \right) \in \mathcal{C}(L^2(I, H), \mathcal{C}(T, V)) \leq D^\frac{1 + \theta}{10} T^{-\frac{1}{2}} M^{-1} \left| v^0(v) \right|_{L^2(I, H)} \leq D^\frac{1 + \theta}{10} T^{-\frac{1}{2}} M^{-1} \left| v^0(v) \right|_{L^2(I, H)} \leq D^\frac{1 + \theta}{10} T^{-\frac{1}{2}} M^{-1} \left| v^0(v) \right|_{L^\infty(I, H)}.$$

Thus, using Assumption 2.4,

$$\left| \mathcal{Q}_k(v, \mathcal{V}^1(v))(t) - \mathcal{Q}_k(v, \mathcal{V}^0(v))(t) \right|_V \leq \frac{1 + \theta}{10} \left| v \right|_V, \quad \text{for all} \ t \in T_k. \quad (3.21)$$
3.2. A piecewise constant static control taking values in \( D(A) \). Let us denote the closed unit ball in \( D(A) \) by \( \overline{B}_{D(A)} \). Recall the control \( \mathcal{V}^1(\mathbf{v})(t) \), defined in (3.18), taking values in \( A \mathcal{F} = \text{span}\{ \overline{\Phi}_j \mid 1 \leq j \leq M \} \). We will prove that the solution of (3.14) varies continuously in \( C(T, V) \) when the external forcing varies continuously in the so called (weak) relaxation metric (cf. [14, Ch. 3])

\[
\mathcal{D}_k^w(f, g) := \sup_{t \in T} \left| \int_{kT}^{(k+1)T} f(s) - g(s) \, ds \right|_{D(A)}, \quad \{f, g \} \subset L^{\infty}(I_k, \mathcal{S}_A \cap K \overline{B}_{D(A)}),
\]

for a given \( K > 0 \). Hence we will approximate \( \mathcal{V}^1(\mathbf{v}) \) by a piecewise constant control taking values in \( \mathcal{V}^2(\mathbf{v}) \) in such a metric. We underline here that \( f \) and \( g \) above are functions taking their values in the bounded subset \( \mathcal{S}_A \cap K \overline{B}_{D(A)} \) of the finite dimensional subspace \( \mathcal{S}_A \subset D(A) \).

As the reference [14] shows, such continuity is known in control theory of ordinary differential equations. It has also been used to derive (approximate) controllability results for partial differential equations, see for example [1, Sect. 12.3], [2, Sect. 6.3], [26, Sect. 9], [27, Sect. 3.2.2].

We follow a variation of the procedure in [14, Ch. 3], which allows us to construct a piecewise constant control taking values in \( \{ s \overline{\Phi}_j \mid 1 \leq j \leq M, -K \leq s \leq K \} \), for a suitable fixed \( K > 0 \), see (3.38). The fact that the control takes its values in a subset of the cone \( \{ r \overline{\Phi}_j \mid 1 \leq j \leq M, r \in \mathbb{R} \} \) will be important in Section 3.5. With respect to this, we would like to refer also to [32, Lemma 3.5], for a different approximation involving piecewise constant controls, but where the control is allowed to take values which are not necessarily in the cone above.

In order to construct a piecewise constant control, we start with a partition of the time interval \( I_k \) into \( N \) subintervals of constant size \( \frac{T}{N} \),

\[
I_{k,n} := (kT + (n - 1)\frac{T}{N}, kT + n\frac{T}{N}), \quad 1 \leq n \leq N,
\]

and we denote \( I_{k,n} := [kT + (n - 1)\frac{T}{N}, kT + n\frac{T}{N}] \). We are going to construct a piecewise constant control on each of the subintervals \( I_{k,n} \) with exactly \( 2M \) subintervals of constancy (possibly with vanishing length) where each of the \( M \) actuators \( \overline{\Phi}_j \), \( 1 \leq j \leq M \) will be active in exactly two of such intervals.

We start by defining the nonnegative constant

\[
\Sigma_n(\mathbf{v}) := \frac{N}{T} \sum_{m=1}^{M} \left| \int_{I_{k,n}} v^1_m(\mathbf{v})(t) \, dt \right|_R.
\]

Observe that

\[
\Sigma_n(\mathbf{v}) \leq \frac{N}{T} \int_{I_{k,n}} \sum_{m=1}^{M} |v^1_m(\mathbf{v})(t)|_R \, dt \leq \frac{N}{T} M \int_{I_{k,n}} |v^1(\mathbf{v})(t)|_{RM} \, dt
\]

and, by (3.19), it follows that

\[
\Sigma_n(\mathbf{v}) \leq M \mathcal{R} |\mathbf{v}|_V.
\]

To simplify the exposition we denote

\[
\begin{align*}
\tilde{\Phi}_{2M+1-j} & := -\overline{\Phi}_j, \\
v^1_{2M+1-j}(\mathbf{v})(t) & := v^1_j(\mathbf{v})(t), \quad 1 \leq j \leq M.
\end{align*}
\]

Next, we consider the cases \( \Sigma_n(\mathbf{v}) \neq 0 \) and \( \Sigma_n(\mathbf{v}) = 0 \) separately.
• The case $\Sigma_n(v) \neq 0$. We rewrite our control $\mathcal{V}^1(v)$, as

$$\mathcal{V}^1(v)(t) = \sum_{j=1}^{M} v_j^1(v)(t) \Phi_j = \sum_{j=1}^{M} v_j^1(v)(t) \Sigma_n(v) \Phi_j = \sum_{j=1}^{2M} v_j^1(v)(t) \Sigma_n(v) \Phi_j.$$ 

Let us denote

$$l_{k,n,j} = l_{k,n,2M+1-j} := \frac{1}{\Sigma_n(v)} \int_{I_{k,n}} v_j^1(v)(t) \, dt, \quad 1 \leq j \leq M.$$ (3.26a)

We define a piecewise constant control in each interval $I_{k,n} \subset I_k$, where the lengths of the intervals of constancy are given by

$$|l_{k,n,j}| = |l_{k,n,j}|_R.$$ (3.26b)

Observe that

$$\sum_{j=1}^{2M} |l_{k,n,j}| = 2 \frac{1}{\Sigma_n(v)} \sum_{j=1}^{M} \left| \int_{I_{k,n}} v_j^1(v)(t) \, dt \right|_R = \frac{T}{N}.$$ (3.26c)

Note also that some of the lengths may vanish.

Next, we denote the switching time instants $t_{k,n,j} = t_{k,n,j}^{[N]}$ as follows

$$t_{k,n,0} := kT + (n-1) \frac{T}{N},$$ (3.27a)

$$t_{k,n,j} := kT + (n-1) \frac{T}{N} + j \frac{T}{MN}, \quad 1 \leq j \leq 2M.$$ (3.27b)

In particular, we have $t_{k,n,2M} = kT + n \frac{T}{N}$.

We define

$$I_{k,n,j} := [t_{k,n,j-1}, t_{k,n,j}), \quad 1 \leq j \leq 2M.$$ (3.28a)

$$\mathcal{V}^{[N]}(v)(t) = \text{sign}(l_{k,n,j}) \Sigma_n(v) \Phi_j, \quad \text{if} \quad t \in I_{k,n,j}, \quad 1 \leq j \leq 2M.$$ (3.28b)

• The case $\Sigma_n(v) = 0$. We define $\mathcal{V}^{[N]}(v)(t) = 0$, for all $t \in I_{k,n}$. Which we can still rewrite as a piecewise constant control as follows.

Firstly we define

$$t_{k,n,0} := kT + (n-1) \frac{T}{N}, \quad t_{k,n,j} := kT + (n-1) \frac{T}{N} + j \frac{T}{MN}, \quad 1 \leq j \leq 2M,$$ (3.29)

and, then we set analogously to (3.28),

$$I_{k,n,j} := [t_{k,n,j-1}, t_{k,n,j}) = [kT + (n-1) \frac{T}{N} + j \frac{T}{MN}], \quad 1 \leq j \leq 2M.$$ (3.30a)

$$\mathcal{V}^{[N]}(v)(t) = \text{sign}(l_{k,n,j}) \Sigma_n(v) \Phi_j = 0 \Phi_j, \quad \text{if} \quad t \in I_{k,n,j}, \quad 1 \leq j \leq 2M.$$ (3.30b)

In either case we obtain a piecewise constant control in the entire interval $I_k$.

Observe that $\mathcal{V}^{[N]}(v)(t)$ tells us that we activate the actuators $\Phi_j$ in each interval $I_{k,n}$ in the order

$$\Phi_1 \rightarrow \Phi_2 \rightarrow \cdots \rightarrow \Phi_{M-1} \rightarrow \Phi_M \rightarrow \Phi_{M+1} \rightarrow \Phi_{M+2} \rightarrow \cdots \rightarrow \Phi_{2M-1} \rightarrow \Phi_{2M},$$

which is the same, by (3.25), as the cycle

$$\Phi_1 \rightarrow \Phi_2 \rightarrow \cdots \rightarrow \Phi_{M-1} \rightarrow \Phi_M \rightarrow \Phi_{M-1} \rightarrow \cdots \rightarrow \Phi_2 \rightarrow \Phi_1.$$ (3.31)

Some actuators may be active in degenerate intervals of length zero. The actuators are activated with the same input of constant magnitude $\text{sign}(l_{k,n,j}) \Sigma_n(v)$. 

Next, we show that $\mathcal{V}_{[N]}(\mathbf{v})(t)$ approaches $\mathcal{V}^1(\mathbf{v})(t)$ in the relaxation metric (3.22). We set
\[ I_{[N]}(t) := \int_{kT}^{t} (\mathcal{V}_{[N]}(\mathbf{v})(s) - \mathcal{V}^1(\mathbf{v})(s)) \, ds. \]
Then
\[ D_{I_k}^{\text{rxx}}(\mathcal{V}_{[N]}(\mathbf{v}), \mathcal{V}^1(\mathbf{v})) = \sup_{t \in I_k} \left| I_{[N]}(t) \right|_{D(A)}. \]

We show now that $I_{[N]}$ vanishes at the extrema of the intervals $I_{k,n}$. Clearly
\[ I_{[N]}(kT) = 0. \] (3.32)

Further, if we assume that $I_{[N]}(kT + n T_N) = 0$ for a given $0 \leq n \leq N - 1$, then:
• if $\Sigma_n(\mathbf{v}) \neq 0$ we obtain
  \[ I_{[N]}(kT + (n + 1) T_N) = \int_{I_{k,n}} (\mathcal{V}_{[N]}(\mathbf{v})(s) - \mathcal{V}^1(\mathbf{v})(s)) \, ds \]
  \[ = \sum_{j=1}^{2M} |l_{k,n,j}| \Sigma_n(\mathbf{v}) \Phi_j - \int_{I_{k,n}} \mathcal{V}^1(\mathbf{v})(s) \, ds \]
  \[ = \sum_{j=1}^{2M} \frac{1}{2} \int_{I_{k,n}} v_j^1(\mathbf{v})(s) \, ds \Phi_j - \int_{I_{k,n}} \mathcal{V}^1(\mathbf{v})(s) \, ds = 0. \]

• if $\Sigma_n(\mathbf{v}) = 0$ we obtain
  \[ I_{[N]}(kT + (n + 1) T_N) = \int_{I_{k,n}} (0 - \mathcal{V}^1(\mathbf{v})(s)) \, ds = 0. \]

Therefore, in either case we have that
\[ I_{[N]}(kT + n T_N) = 0 \implies I_{[N]}(kT + (n + 1) T_N) = 0, \quad 0 \leq n \leq N - 1. \] (3.33)

From (3.32) and (3.33), by induction we can conclude that
\[ I_{[N]}(kT + n T_N) = 0, \quad \text{for all } n \in \{0, 1, 2, \ldots, N\}. \] (3.34)

Now for an arbitrary $t \in I_{k,n}$, we find
\[ \left| I_{[N]}(t) \right|_{D(A)} \leq T_N \left( \sup_{s \in I_{k,n}} \left| \mathcal{V}_{[N]}(\mathbf{v})(s) \right|_{D(A)} + \sup_{s \in I_{k,n}} \left| \mathcal{V}^1(\mathbf{v})(s) \right|_{D(A)} \right) \]
\[ \leq T_N \left( \Sigma_n(\mathbf{v}) \sup_{1 \leq j \leq M} \left| \Phi_j \right|_{D(A)} + \sup_{s \in I_k} \left| v_j^1(\mathbf{v})(s) \right|_{D(M)} \sup_{1 \leq j \leq M} \left| \vec{\Phi}_j \right|_{D(A)} \right) \]
and by (3.24) and Assumption 2.4,
\[ \left| I_{[N]}(t) \right|_{D(A)} \leq T_N \left( M \| \mathbf{r} |_{V} + \| \mathbf{r} |_{V} \right) \sup_{1 \leq j \leq M} \left| \vec{\Phi}_j \right|_{D(A)} \]
\[ \leq T_N (M + 1) \left\| \mathbf{r} \right\|_{V}, \quad t \in I_k, \quad \left\| \mathbf{r} \right\|_{V} := \sup_{1 \leq j \leq M} \left| \Phi_j \right|_{D(A)}. \] (3.35)

Next we show the continuity of the solution when the right-hand side control varies in the relaxation metric. Let $d_N := \mathcal{V}_{[N]}(\mathbf{v}) - \mathcal{V}^1(\mathbf{v})$, and observe that $d_N$ satisfies (3.14), as
\[ d_N + Ad_N + A_e d_N = \mathcal{V}_{[N]}(\mathbf{v}) - \mathcal{V}^1(\mathbf{v}), \quad d_N(kT) = 0. \]
With \( z_N := d_N - \bar{I}_{[N]} \), we see that 
\[
\dot{z}_N = \dot{d}_N - \dot{\bar{I}}_{[N]} = -Ad_N - A_{rc}d_N,
\]
which implies
\[
\dot{z}_N + A_d z_N + A_{rc} z_N = -A \bar{I}_{[N]} - A_{rc} \bar{I}_{[N]},
\]
and also, by (3.34),
\[
z_N(kT + n \frac{T}{N}) = d_N(kT + n \frac{T}{N}), \quad 0 \leq n \leq N. \tag{3.36}
\]
Therefore, for \( z_N = \mathcal{Y}_k(0, -A \bar{I}_{[N]} - A_{rc} \bar{I}_{[N]}) \) we obtain, see (3.16),
\[
|z_N(t)|_V^2 \leq D_Y \left| A \bar{I}_{[N]} - A_{rc} \bar{I}_{[N]} \right|_{L^2(I_k, H)}^2, \quad t \in \bar{T}_k.
\]
By standard computations we find, for all \( t \in \bar{T}_k \),
\[
|z_N(t)|_V^2 \leq D_Y \left( |A \bar{I}_{[N]}|_{L^2(I_k, H)} + |A_{rc} \bar{I}_{[N]}|_{L^2(I_k, H)} \right)^2
\]
\[
\leq D_Y \left( |\bar{I}_{[N]}|_{L^2(I_k, D(A))} + C_{rc} |I_{[N]}|_{L^2(I_k, V)} \right)^2
\]
\[
= D_Y (1 + C_{rc} |I_{[N]}|_{L([I_k, D(A)], V)})^2 T \left| I_{[N]} \right|_{L^\infty(I_k, D(A))}^2,
\]
and using (3.35),
\[
|z_N(t)|_V \leq \frac{1}{N} D_Y (1 + C_{rc} |I|_{L([D(A)], V)}) T \left( \frac{2}{3} (M + 1) \frac{\left\| \bar{\Phi} \right\| \mathcal{R} |v|_V \right).
\]
Now we can take \( N \) large enough, namely
\[
N = \tilde{N} \geq D_Y \left( 1 + C_{rc} |I|_{L([D(A)], V)}) T \left( \frac{2}{3} (M + 1) \frac{\left\| \bar{\Phi} \right\| \mathcal{R} |v|_V \right)
\]
\[
in order to obtain \( |z_\tilde{N}(t)|_V \leq \frac{1-\theta}{10} |v|_V. \) Then, we set
\[
\mathcal{Y}^2(v)(t) := \mathcal{Y}_{[N]}(v)(t) = \text{sign}(I_{k,n,j}) \Sigma_n(v) \bar{\Phi}_j, \quad \text{if} \ t \in I_{k,n,j}, 1 \leq j \leq 2M, \tag{3.38}
\]
where the intervals \( I_{k,n,j} \) are defined as in (3.28) and (3.30),
\[
I_{k,n,j} = I^{[\tilde{N}]}_{k,n,j} = [t^{[\tilde{N}]}_{k,n,j-1}, t^{[\tilde{N}]}_{k,n,j}) = [t_{k,n,j-1}, t_{k,n,j}). \tag{3.39}
\]
We find, using (3.35),
\[
\left| \mathcal{Y}_k(v, \mathcal{Y}^2(v))(t) - \mathcal{Y}_k(v, \mathcal{Y}^1(v))(t) \right|_V = \left| d_{\tilde{N}}(t) \right|_V \leq \left| z_{\tilde{N}}(t) \right|_V + \left| \bar{I}_{[N]}(t) \right|_V
\]
\[
\leq \frac{1-\theta}{10} |v|_V + T \frac{\left\| \bar{\Phi} \right\| \mathcal{R} |v|_V, \quad \text{for all} \ t \in \bar{T}_k, \tag{3.40a}
\]
and, using (3.36),
\[
\left| \mathcal{Y}_k(v, \mathcal{Y}^2(v))(kT + T) - \mathcal{Y}_k(v, \mathcal{Y}^1(v))(kT + T) \right|_V
\]
\[
= \left| z_{\tilde{N}}(kT + T) \right|_V \leq \frac{1-\theta}{10} |v|_V. \tag{3.40b}
\]

**Remark 3.5.** The chosen cycle (3.31) is not unique. For example, we could adapt the proof for the cycle \( \bar{\Phi}_1 \to \bar{\Phi}_2 \to \bar{\Phi}_{M-1} \to \bar{\Phi}_M \to \bar{\Phi}_1. \)
3.3. **A piecewise constant static control taking values in \( H \).** To simplify the exposition we denote

\[
\bar{\Phi}_{2M+1-j} := \bar{\Phi}_j, \quad 1 \leq j \leq M. \tag{3.41}
\]

Recall that \( V^2(v) \) takes its values \( V^2(v)(t) \) in the set \( \{\pm \Sigma_n(v) \bar{\Phi}_j\} \subset D(A) \), for \( t \in T_k \). We define a new piecewise constant control \( V^3(v)(t) \), taking its values in the set \( \{\pm \Sigma_n(v) \bar{\Phi}_j\} \subset H \), by setting for \( t \in I_{k,n,j} \),

\[
V^3(v)(t) := \text{sign}(l_{k,n,j}) \Sigma_n(v) \bar{\Phi}_j, \quad \text{for } V^2(v)(t) = \text{sign}(l_{k,n,j}) \Sigma_n(v) \bar{\Phi}_j. \tag{3.42}
\]

Using (3.24), we can see that the corresponding solutions satisfy

\[
\left| \mathcal{G}_k(v, V^3(v))(t) - \mathcal{G}_k(v, V^2(v))(t) \right|_V^2 \leq D_Y \left| V^3(v) - V^2(v) \right|_{L^2(T_k, H)}^2 \leq D_Y T \max_{1 \leq j \leq M} \left| \bar{\Phi}_j - \Phi_j \right|_H^2 \leq D_Y TM^2 \mathcal{R}^2 \sup_{1 \leq j \leq M} \left| \bar{\Phi}_j - \Phi_j \right|_H^2.
\]

and by (3.17),

\[
\left| \mathcal{G}_k(v, V^3(v))(t) - \mathcal{G}_k(v, V^2(v))(t) \right|_V \leq \frac{1 - \theta}{10} |v|_V, \quad t \in T_k. \tag{3.43a}
\]

Note also that

\[
\left| V^3(v)(t) \right|_V \leq \sup_{1 \leq n \leq N} \Sigma_n(v) \leq M \mathcal{R} |v|_V, \quad t \in T_k, \tag{3.43b}
\]

\[
V^3(v)(kT + (n - 1) \frac{T}{N}) \in \{\pm \Sigma_n(v) \bar{\Phi}_1\}, \quad 1 \leq n \leq \hat{N}. \tag{3.43c}
\]

Observe that \( V^2 \) switches between the actuators \( \bar{\Phi}_j \) as described in (3.31), and hence \( V^3 \) switches between the actuators \( \bar{\Phi}_j \) as in the analogous cycle

\[
\bar{\Phi}_1 \rightarrow \bar{\Phi}_2 \rightarrow \cdots \rightarrow \bar{\Phi}_{M-1} \rightarrow \bar{\Phi}_M \rightarrow \bar{\Phi}_{M-1} \rightarrow \cdots \rightarrow \bar{\Phi}_2 \rightarrow \bar{\Phi}_1, \tag{3.44}
\]

which, due to (3.25), results in

\[
\bar{\Phi}_1 \rightarrow \bar{\Phi}_2 \rightarrow \cdots \rightarrow \bar{\Phi}_{M-1} \rightarrow \bar{\Phi}_M \rightarrow \bar{\Phi}_{M+1} \rightarrow \bar{\Phi}_{M+2} \rightarrow \cdots \rightarrow \bar{\Phi}_{2M-1} \rightarrow \bar{\Phi}_{2M}. \]

3.4. **A control with no degenerate intervals of constancy.** By construction, the length \( |l_{k,n,j}| \) vanishes if \( v^1_j(v) \) has zero average on \( I_{k,n} \), see (3.26a). It will be convenient, also to simplify the exposition in Section 3.5 below, that all the intervals of constancy have a length larger than a suitable positive constant.

Recall the switching time instants \( t_{k,n,j} \) on each interval \( T_{k,n} \subset T_k \), see (3.39),

\[
t_{k,n,0} \leq t_{k,n,1} \leq t_{k,n,2} \leq \cdots \leq t_{k,n,2M-1} \leq t_{k,n,2M}, \quad 1 \leq n \leq \hat{N}. \tag{3.45a}
\]

and recall that

\[
t_{k,n-1} := kT + (n - 1) \frac{T}{N} = t_{k,n,0} \quad \text{and} \quad t_{k,n} = kT + n \frac{T}{N} = t_{k,n,2M}. \tag{3.45b}
\]

To guarantee nondegenerate intervals of constancy we will define new switching time instants satisfying

\[
t_{k,n,0}^* < t_{k,n,1}^* < t_{k,n,2}^* < \cdots < t_{k,n,2M-1}^* < t_{k,n,2M}^* \tag{3.46a}
\]

with

\[
t_{k,n,0}^* := t_{k,n-1} \quad \text{and} \quad t_{k,n,2M}^* := t_{k,n}. \tag{3.46b}
\]
To do so we fix a positive number $\varepsilon > 0$, and define
\[
\ell^\varepsilon_{k,n,j} = t_{k,n} - \vartheta_{\varepsilon}(t_{k,n,j} - t_{k,n} + \frac{j+1}{2} \varepsilon), \quad 0 \leq j \leq 2M, 
\]  
with
\[
\vartheta_{\varepsilon} := \frac{T}{T + N(2M+1)M\varepsilon}.
\]
Now we show that, indeed, the sequence (3.47) satisfies (3.46). We find
\[
\varepsilon > 0.
\]
To do so we fix a positive number $\varepsilon > 0$, and define
\[
\ell^\varepsilon_{k,n,0} = t_{k,n} - \vartheta_{\varepsilon}0 = t_{k,n} - \vartheta_{\varepsilon}(t_{k,n} + \frac{T}{N} + (2M + 1)M\varepsilon) = t_{k,n} + \frac{T}{N} = t_{k,n},
\]
\[
\ell^\varepsilon_{k,n,2M} = t_{k,n} - \vartheta_{\varepsilon}(t_{k,n-1} + \frac{T}{N} + (2M + 1)M\varepsilon) = t_{k,n} - \frac{T}{N} = t_{k,n},
\]
\[
\ell^\varepsilon_{k,n,j} - \ell^\varepsilon_{k,n,j-1} = \vartheta_{\varepsilon}(t_{k,n,j} - t_{k,n,j-1} + (\frac{j+1}{2} - \frac{j}{2} (j-1)\varepsilon), \quad 1 \leq j \leq 2M.
\]
From (3.48) we see that (3.46) is satisfied.

Next we define the piecewise constant control, for time $t \in I_k$, as follows
\[
V_{\varepsilon}(v)(t) := \text{sign}(l_{k,n,j})(\Sigma_n(v)\tilde{\Phi}_j), \quad \text{if} \quad t \in [\ell^\varepsilon_{k,n,j-1}, \ell^\varepsilon_{k,n,j}), \quad 1 \leq n \leq M, \quad 1 \leq j \leq 2M,
\]
where the intervals of constancy have a positive minimum length, see (3.48),
\[
\min_{1 \leq j \leq 2M} \{\ell^\varepsilon_{k,n,j-1} - \ell^\varepsilon_{k,n,j-1}\} \geq \varepsilon \vartheta_{\varepsilon} > 0.
\]
Observe that, from (3.38) and (3.42), we have that
\[
V^\varepsilon(v)(t) = \text{sign}(l_{k,n,j})(\Sigma_n(v)\tilde{\Phi}_j), \quad \text{if} \quad t \in [t_{k,n,j-1}, t_{k,n,j}).
\]
Note that as $\varepsilon \to 0$ we have $\ell^\varepsilon_{k,n,j} \to t_{k,n,j}$. Now we show that we also have $V_{\varepsilon}(v)(t) \to V_{\varepsilon}(v)(t)$ in $L^2(I_k, H)$, as $\varepsilon \to 0$.

**Proposition 3.6.** Let $[a, b] \subset \mathbb{R}$ be a nonempty interval, $a < b$, let $X$ be a Banach space, and let $K$ be positive integer. Let us be given a finite sequence in $X$
\[
\phi_j \in X, \quad 1 \leq j \leq K,
\]
and two finite sequences in $[a, b]$,
\[
a = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{K-1} \leq \tau_K = b \quad \text{and} \quad a = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{K-1} \leq \sigma_K = b.
\]
Then, for the following two functions defined for $t \in (a, b)$ by
\[
f_r(t) := \phi_j, \quad \text{if} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad f_\sigma(t) := \phi_j, \quad \text{if} \quad t \in [\sigma_{j-1}, \sigma_j),
\]
we have the estimate
\[
|f_r - f_\sigma|_{L^2((a,b), X)} \leq K^{\frac{1}{2}}R^{\frac{1}{2}}X,
\]
with $R := \max_{0 \leq j \leq K} |\tau_j - \sigma_j|$, and $X := \max_{1 \leq i, j \leq K} |\phi_j - \phi_i|_X$.

The proof of Proposition 3.6 is given in Section A.3.

From Proposition 3.6 it follows that
\[
|V_{\varepsilon}(v) - V^\varepsilon(v)|_{L^2(I_k, H)} \leq (2M\tilde{N})^{\frac{1}{2}} \max_{0 \leq j \leq 2M} \left|\ell^\varepsilon_{k,n,j} - \ell^\varepsilon_{k,n,j-1}\right|_{L^\infty, \mathbb{R}} \max_{1 \leq i, j \leq M} \left|\Sigma_n(v)\tilde{\Phi}_j - \Sigma_n(v)\tilde{\Phi}_i\right|_H \leq 2(2M\tilde{N})^{\frac{1}{2}} \max_{1 \leq n \leq \tilde{N}} \Sigma_n(v) \max_{0 \leq j \leq 2M} \left|\ell^\varepsilon_{k,n,j} - \ell^\varepsilon_{k,n,j-1}\right|_{L^\infty, \mathbb{R}}.
Recalling (3.24), we arrive at
\[ |V_ε(v) - V^3(v)|_{L^2(I_k, H)} \leq (2M)^{\frac{3}{2}} \sqrt{N} \| \mathcal{R}v \|_V \max_{0 \leq j \leq 2M} |t^ε_{k,n,j} - t_{k,n,j}|_R. \]  
(3.52)

Next, from (3.47) we find that
\[ |t^ε_{k,n,j} - t_{k,n,j}|_R = |t_{k,n,j} - t_{k,n-1} + \varepsilon \varepsilon (t_{k,n,j} - t_{k,n-1} + \frac{j+1}{2} \varepsilon)|_R \\
= |(\varepsilon \varepsilon - 1)(t_{k,n,j} - t_{k,n-1} + \frac{j+1}{2} \varepsilon)|_R \\
\leq (1 - \varepsilon \varepsilon) \frac{T}{N} + (2M + 1) M \varepsilon \varepsilon =: \Theta(\varepsilon). \]  
(3.53)

By combining (3.52) and (3.53), we obtain that
\[ |V_ε(v) - V^3(v)|_{L^2(I_k, H)} \leq (2M)^{\frac{3}{2}} \sqrt{N} \| \mathcal{R}v \|_V \Theta(\varepsilon)^{\frac{1}{2}}, \]  
(3.54)

and by using (3.16) it follows that
\[ |\mathcal{G}_k(v, V_ε(v))(t) - \mathcal{G}_k(v, V^3(v))(t)|_V \leq (8M^3 D \sqrt{N}) \frac{1}{2} \| \mathcal{R}v \|_V, \quad t \in T_k. \]  
(3.55)

From (3.47) we also find
\[ 1 - \varepsilon \varepsilon = \frac{N(1 + 1 + 1) M \varepsilon \varepsilon}{T + N(2M + 1) M \varepsilon \varepsilon}, \quad \varepsilon \varepsilon = \frac{T}{T + N(2M + 1) M \varepsilon \varepsilon}, \]  
and
\[ 1 - \varepsilon \varepsilon \to 0, \quad \varepsilon \varepsilon \to 0, \quad \text{and} \quad \Theta(\varepsilon) \to 0, \quad \text{as} \quad \varepsilon \to 0. \]

Therefore, there exists \( \varepsilon \) small enough, so that
\[ \Theta(\varepsilon) \leq (8M^3 D \sqrt{N})^{-1} \sqrt{N} \frac{1}{4} (\frac{1 - \theta}{10})^2. \]  
(3.56)

Now we set the control
\[ V^4(v) = V^c(v), \quad t \in I_k, \]  
(3.57a)

with nondegenerate intervals of constancy (cf. (3.50)),
\[ \min_{1 \leq j \leq 2M} \{ t^ε_{k,n,j} - t^ε_{k,n,j-1} \} \geq \varepsilon \varepsilon > 0. \]  
(3.57b)

From (3.55), and (3.56), it follows that
\[ |\mathcal{G}_k(v, V^4(v))(t) - \mathcal{G}_k(v, V^3(v))(t)|_V \leq \frac{1 - \theta}{10} |v|_V, \quad t \in T_k. \]  
(3.58)

3.5. A continuously moving control taking values in \( H \). We will travel in \( H \) between the static actuators \( \Phi_i \), following the cycle (3.44).

For traveling we fix a set of roads, in the unit sphere \( \mathcal{S}_H \), connecting the static actuators, as follows:
\[ \mathcal{R}_j : C^p([0, 1], \mathcal{S}_H), \quad 1 \leq j \leq M - 1, \quad p \in \mathbb{N}, \]  
(3.59a)

with \( \mathcal{R}_j(0) = \hat{\Phi}_j \) and \( \mathcal{R}_j(1) = \hat{\Phi}_{j+1} \),
\[ \mathcal{R}_M(s) = \hat{\Phi}_M, \quad s \in [0, 1], \]  
(3.59c)

\[ \mathcal{R}_{M-j}(s) = \mathcal{R}_{M-j}(1 - s), \quad 1 \leq j \leq M - 1. \]  
(3.59d)

Note that by (3.41), we also have
\[ \mathcal{R}_{M-j}(0) = \mathcal{R}_{M-j}(1) = \hat{\Phi}_{M-j+1} = \hat{\Phi}_{M+j}, \]  
(3.59e)

\[ \mathcal{R}_{M-j}(1) = \mathcal{R}_{M-j}(0) = \hat{\Phi}_{M-j} = \hat{\Phi}_{M+j+1}. \]  
(3.59f)
We also introduce the scalar function

\[ r^\xi,\zeta_{k,n,j}(t) := \frac{\xi + t^\xi_{k,n,j} - t}{2\xi} \text{sign}(l_{k,n,j}) + \frac{\xi - t^\xi_{k,n,j} + t}{2\xi} \text{sign}(l_{k,n,j+1}), \quad (3.60a) \]

with, recall (3.57),

\[ \xi \in (0, \frac{\epsilon_1}{2}). \quad (3.60b) \]

Then we define a moving control \( \mathcal{V}_\xi(v) \), for \( t \in T_k \) as follows:

\[ \mathcal{V}_\xi(v)(t) := \text{sign}(l_{k,n,1}) \Sigma_n(v) \Phi_1, \quad (3.61a) \]

if \( t \in [kT + (n - 1)\frac{T}{N}, t^\xi_{k,n,1} - \xi] \), \( 1 \leq n \leq \hat{N} \).

\[ \mathcal{V}_\xi(v)(t) := \text{sign}(l_{k,n,2M}) \Sigma_n(v) \Phi_1, \quad (3.61b) \]

if \( t \in [t^\xi_{k,n,2M-1} + \xi, kT + n\frac{T}{N}] \), \( 1 \leq n \leq \hat{N} \).

\[ \mathcal{V}_\xi(v)(t) := \text{sign}(l_{k,n,j}) \Sigma_n(v) \Phi_j, \quad (3.61c) \]

if \( t \in [t^\xi_{k,n,j-1} + \xi, t^\xi_{k,n,j} - \xi], \quad 1 \leq n \leq \hat{N}, \quad 2 \leq j \leq 2M - 1. \)

\[ \mathcal{V}_\xi(v)(t) := r^\xi_{k,n,j}(t) \Sigma_n(v) \Phi_j \left( \frac{\xi - t^\xi_{k,n,j} + t}{2\xi} \right), \quad (3.61d) \]

if \( t \in [t^\xi_{k,n,j} - \xi, t^\xi_{k,n,j} + \xi], \quad 1 \leq n \leq \hat{N}, \quad 1 \leq j \leq 2M - 1. \)

Observe that \( \mathcal{V}_\xi \) differs from \( \mathcal{V}^4 \) only in the intervals \( (t^\xi_{k,n,j} - \xi, t^\xi_{k,n,j} + \xi), \) \( 1 \leq n \leq \hat{N}, \quad 1 \leq j \leq 2M - 1, \) when we travel from the static actuator \( \Phi_j \) to the static actuator \( \Phi_{j+1} \). These are exactly \( \hat{N}(2M - 1) \) intervals, where each has length \( 2\xi \).

Thus

\[ |\mathcal{V}_\xi(v) - \mathcal{V}^4(v)|^2_{L^2(I_k, H)} \leq 2\xi \hat{N}(2M - 1) \max_{1 \leq n \leq \hat{N}} \left\{ \left| r^\xi_{k,n,j}(t) \right| \Sigma_n(v)^2 \left| \Phi_j \right|^2_H \right\}. \]

Since \( \left| r^\xi_{k,n,j}(t) \right| \leq 1 \) and \( \left| \Phi_j \right|^2_H = 1 \), using (3.24), we arrive at

\[ |\mathcal{V}_\xi(v) - \mathcal{V}^4(v)|^2_{L^2(I_k, H)} \leq 2\xi \hat{N}(2M - 1)M^2R^2 |v|^2_V. \]

Recalling (3.16), we obtain

\[ |\mathcal{Q}_k(v, \mathcal{V}^4(v))(t) - \mathcal{Q}_k(v, \mathcal{V}^4(v))(t)|^2 \leq D_Y \left| \mathcal{V}_\xi(v) - \mathcal{V}^4(v) \right|^2_{L^2(I_k, H)} \]

\[ \leq D_Y 2\xi \hat{N}(2M - 1)M^2R^2 |v|^2_V, \quad t \in I_k. \quad (3.62) \]

Now choosing small enough \( \xi \), namely

\[ \xi = \xi := \min \left\{ \frac{\epsilon_2}{4}, \left( D_Y 2\xi \hat{N}(2M - 1)M^2R^2 \right)^{-1} \left( \frac{1 - \theta}{10} \right)^2 \right\}, \quad (3.63) \]

and setting

\[ \mathcal{V}^5(v)(t) := \mathcal{V}_\xi(v)(t), \quad t \in I_k, \quad (3.64) \]

we find

\[ |\mathcal{Q}_k(v, \mathcal{V}^5(v))(t) - \mathcal{Q}_k(v, \mathcal{V}^4(v))(t)|_V \leq \frac{1 - \theta}{10} |v|_V. \quad (3.65) \]

Finally, note that \( \mathcal{V}^5(v) \) is a moving control of the form

\[ \mathcal{V}^5(v)(t) := u(t) \Phi(t), \quad t \in I_k, \quad \text{with} \quad |u(t)|_H \leq M \hat{N} |v|_V, \quad \Phi(t) \in \mathcal{S}_H, \quad (3.66a) \]
for suitable \( u \in L^\infty(\bar{T}_k, \mathbb{R}) \) and \( \Phi \in C(\bar{T}_k, \mathcal{S}_H) \). Furthermore, by choosing \( p \geq 0 \) in (3.59) we can obtain a regular motion of the actuator. Namely, if we have

\[
\frac{d^q}{ds^q} |_{s=0} \mathcal{R}_j = 0 = \frac{d^q}{ds^q} |_{s=1} \mathcal{R}_j, \quad 1 \leq q \leq p,
\]

then

\[
\Phi \in C^p(\bar{T}_k, \mathcal{S}_H)
\]

and

\[
\max_{\tau \in \bar{T}_k} \frac{d^p}{ds^p} |_{s=\tau} \Phi \leq \left( \frac{1}{2\xi} \right)^p \max_{1 \leq j \leq M, s_0 \in [0,1]} \frac{d^p}{ds^p} |_{s=s_0} \mathcal{R}_j \quad H.
\]

In particular, we have that

\[
|\Phi|_{C^p(\bar{T}_k, \mathcal{S}_H)} \leq \left( \frac{1}{2\xi} \right)^p \max_{1 \leq j \leq M} |\mathcal{R}_j|_{C^p([0,1], H)}.
\]

Conclusion of the proof of Theorem 3.3. By using (3.21), (3.40), (3.43), (3.58), and (3.65), together with the triangle inequality, we arrive at

\[
|\Phi|_{C^p(\bar{T}_k, \mathcal{S}_H)} \leq \left( \frac{1}{2\xi} \right)^p \max_{1 \leq j \leq M} |\mathcal{R}_j|_{C^p([0,1], H)}.
\]

Remark 3.7. Observe that the actuators \( \Phi_j \) in (3.17), the integer \( \hat{N} \) in (3.37), the parameter \( \hat{\varepsilon} \) in (3.56), and the parameter \( \xi \) in (3.63), were all chosen independently of \( k \in \mathbb{N} \). Furthermore, from (3.66) we can also see that \( |\Phi|_{C^p(\bar{T}_k, \mathcal{S}_H)} \) is bounded by a constant independent of \( k \in \mathbb{N} \). Again from (3.66), by recalling Assumption 2.4 we also have \(|u|_{L^\infty(I_k, H)} \leq M \Re |v|_V = M \Re |y(kT)|_V \) with the product \( M \Re \) independent of \( k \in \mathbb{N} \).

4. Proof of Theorem 1.1. We start by writing (1.4) as

\[
\dot{y} + Ay + A_{rc}y = u \tilde{1}_\omega(c), \quad y(0) = y_0, \quad t > 0,
\]

with \( A := -\nu \Delta + 1 \) and \( A_{rc}z = A_{rc}(t)z := (a(t, \cdot) - 1)z + b(t, \cdot) \cdot \nabla z \).

It is not hard to check that Assumptions 2.1, 2.2, and 2.3, are satisfied by \( A \in \mathcal{L}(V, V') \) and \( A_{rc} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H)) \), namely with \( V = H^1_0(\Omega) \) in the case of Dirichlet boundary conditions and with \( V = H^1(\Omega) \) in the case of Neumann boundary conditions (see, e.g., [29, Sect.5.1]).

4.1. Satisfiability of Assumption 2.4. Assumption 2.4 follows from the results in [29, Thm. 4.5] (when applied to linear equations), from which we know that

\[
\dot{y} + Ay + A_{rc}y = P_{U_M}^{E_{\tilde{1}}}(A_{rc}y - \lambda y), \quad y(0) = y_0, \quad t > 0,
\]

is a stable system, for arbitrary \( \lambda > 0 \) and a suitable oblique projection \( P_{U_M}^{E_{\tilde{1}}} \). Namely, its solution satisfies,

\[
|y(t)|_V \leq C e^{-\mu (t-s)} |y(s)|_V, \quad t \geq s \geq 0,
\]

with suitable \( C \geq 1 \) and \( \mu \in (0, \lambda] \), both independent of \( (t, s, y(s)) \). Actually in [29, Thm. 4.5] only the case \( s = 0 \) is mentioned, however by a time shift argument \( t = s + \tau, \quad w(\tau) = y(s + \tau), \quad A_{rc}(\tau) = A_{rc}(s + \tau) \), we can rewrite (4.2) as

\[
\frac{d}{d\tau} w + Aw + \tilde{A}_{rc}w = P_{U_M}^{E_{\tilde{1}}}(\tilde{A}_{rc}w - \lambda w), \quad w(0) = y(s), \quad \tau > 0,
\]
and the results in [29, Thm. 4.5] give us
\[
|w(\tau)|_V \leq C_s e^{-\mu \tau} |y(s)|_V, \quad \tau \geq 0,
\]
which is equivalent to (4.3). The constant \( C_s \) is of the form \( \overline{C} \left[ \mathcal{A}_e \right]_{L^\infty(\mathcal{B}_0, \mathcal{L}(V,H))} \), \( \mathcal{A}_e \) and so \( C_s \leq C_0 \), that is we can take \( C \) independent of \( s \) in (4.3).

This stability result in [29, Thm. 4.5] holds for an arbitrary \( \lambda > 0 \) and large enough \( M \), where \( U_M = \text{span}\{1_{\omega_j} | 1 \leq j \leq M\} \) is the span of suitable indicator functions supported in small rectangles \( \omega_j \subset \Omega \). The operator \( P_{U_M} \) is the oblique projection in \( L^2(\Omega) \) onto \( U_M \) along an auxiliary space \( E_M^+ \), where \( E_M \) is the span of a suitable set of eigenfunctions of the diffusion \( A \) defined in \( L^2(\Omega) \), where \( \Omega \) is a bounded rectangular domain. For precise definitions of suitable \( U_M \) and \( E_M \) we refer to [22, Sect. 4.8.1] and [29, Sect. 2.2]. We refer also to [30] for a different feedback. Furthermore, for such choice we have
\[
\sup_{M \geq 1} \left| P_{U_M}^E \right|_{\mathcal{L}(H)} =: \|P\| < +\infty. \quad (4.4)
\]

Observe that \( U_M \) is the range of \( P_{U_M}^E \), hence our control is of the form
\[
P_{U_M}^E (A_{ec}y - \lambda y) = \sum_{j=1}^M \tilde{v}_j(t) 1_{\omega_j} = \sum_{j=1}^M v_j(t) \tilde{\theta}_j =: \mathbf{v}(t).
\]

In particular, from (4.3), for any given \( \theta \in (0,1) \) we have that, for all \( k \in \mathbb{N} \),
\[
|y(kT + T)|_V \leq C e^{-\mu T} |y(kT)|_V \leq \theta |y(kT)|_V, \quad \text{if} \quad T \geq \mu^{-1} \log \left( \frac{C}{\theta} \right).
\]

To prove that Assumption 2.4 is satisfied, it remains to show that the \( v_j(t) \) are appropriately essentially bounded. From (4.3) and \( P_{U_M}^E = P_{U_M}^E P_{E_M} \), we find
\[
|\mathbf{v}(t)|_H \leq \|P\| \left| P_{E_M} (A_{ec}y - \lambda y) \right|_H \\
\leq \left(|A_{ec}|_{L^\infty(\mathcal{B}_0,\mathcal{L}(V,H))} + \lambda |1|_{\mathcal{L}(V,H)} \right) \|P\| |y(t)|_V \\
\leq \left(C_{ec} + \lambda |1|_{\mathcal{L}(V,H)} \right) \|P\| C e^{-\mu(t-kT)} |y(kT)|_V, \quad t \geq kT, \quad (4.5)
\]
which implies that
\[
|\mathbf{v}(t)|_H \leq \mathcal{R}_0 |y(kT)|_V, \quad t \in \mathcal{T}_k = [kT, kT + T], \quad (4.6)
\]
with \( \mathcal{R}_0 = \left(C_{ec} + \lambda |1|_{\mathcal{L}(V,H)} \right) \|P\| C \) independent of \( k \).

Therefore, Assumption 2.4 holds for \( M \) large enough, and with \( T = \mu^{-1} \log \left( \frac{C}{\theta} \right) \) and \( \mathcal{R} = \mathcal{R}_0 \) as above.

**Remark 4.1.** Let \( R > 0 \). From [29, Thm. 4.5], for semilinear systems, we have that there are \( M \) actuators which are able to stabilize the system provided the norm of the initial state satisfies \( |y(0)|_V \leq R \). Since in this manuscript our system is linear, we can conclude that the same number of actuators are able to stabilize the system for every initial condition.
4.2. Illustration of a path for the moving actuator. Let \( \Omega = \prod_{n=1}^{d} (0, L_n) \) and \( \omega_0 = \frac{\rho}{2}(\Omega - \frac{L}{2}) \) with \( \rho \in (0, 1) \) (cf. Theorem 1.1). We consider \( M = S^d \) static actuators \( I_{\omega_j} = I_{\omega(c_m(j))} \) with centers \( c^j = c^m(j) \) as in [29, Sect. 2.2], illustrated in Figure 2. Namely,
\[
c^j = c^m(j) = \left( \frac{(2m(j) - 1)L_1}{2^S}, \frac{(2m(j) - 1)L_2}{2^S}, \ldots, \frac{(2m(j) - 1)L_d}{2^S} \right),
\]
where \( m : \{1, 2, \ldots, S^d\} \to \{1, 2, \ldots, S\}^d \) is a bijection. We may order the actuators and consider the corresponding cycle, for example, as illustrated in the figure for the case \( S = 3, d = 2 \), we start at the first actuator in the bottom-left corner, going up until the \( M \)th actuator (here, \( M = 9 \)) in the top-right corner and returning back down to the bottom-left corner. In this way we are considering roads, see (3.65),

\[
S_1 = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure1.png}
\end{array}
\quad S_2 = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure2.png}
\end{array}
\quad S_3 = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure3.png}
\end{array}
\]

\textbf{Figure 2.} Supports of the static actuators. Case \( \Omega \subset \mathbb{R}^2 \).

Note that for such roads, we may take
\[
c_j(0) = c^j, \quad c_j(1) = c^{j+1}, \quad \dot{c}_j(0) = \dot{c}_j(1) = \ddot{c}_j(0) = \ddot{c}_j(1) = 0.
\]

(4.9b)

Note that for such a cycle, we may take
\[
c_j(s) = c^j + \phi(s)(c^{j+1} - c^j), \quad 1 \leq j \leq M,
\]
where \( \phi \in C^2([0, 1], [0, 1]) \) is increasing and satisfies the relations \( \phi(0) = 0, \phi(1) = 1, \) and \( \dot{\phi}(0) = \dot{\phi}(1) = \phi(0) = \phi(1) = 0. \) Note that we have that for each \( t \geq 0 \)
\[
\omega(c(t)) = c^j(s) + \omega_0 \quad \text{for some} \quad (j, s) \in \{1, 2, \ldots, M\} \times [0, 1].
\]

(4.9c)

That is, \( \omega(c(t)) \) is a translation of the rectangle \( \omega_0 \), as in Theorem 1.1 with \( r = \frac{L}{2} \).

Observe also that, with \( c(s) = \zeta + \phi(s)(\tau - \zeta) \), we have that \( \omega(c(s)) \subset \Omega \) if \( \omega(\zeta) \subset \Omega \) and \( \omega(\tau) \subset \Omega \). Indeed \( c(s) = (1 - \phi(s))\zeta + \phi(s)\tau \) is a convex combination between \( \zeta \) and \( \tau \), which allows us to conclude that
\[
\omega(c(s)) = c(s) + \omega_0 = \{(1 - \phi(s))\zeta + \phi(s)\tau | b \in \omega_0\} \subset \Omega,
\]
because \( \Omega \) is convex.

Finally, we recall that we will need to travel fast enough, say in time \( 2\tilde{\zeta} < 1 \) between the centers \( c^j \) and \( c^{j+1} \). In this case we find that
\[
\left| c_j\left(\frac{1}{2^S}\right) \right|_{C^m([0, 2\tilde{\zeta}], \mathbb{R}^d)} \leq \left(\frac{1}{2^S}\right)^m |\phi(\cdot)|_{C^m([0, 1], \mathbb{R})}, \quad m \in \{0, 1, 2\},
\]
(4.10)

Recall also that a suitable traveling time \( 2\tilde{\zeta} \) can be chosen bounded from below and independent of \( y_0 \); see (3.63) and (3.66).
Remark 4.2. With actuators (indicator functions) located in the supports as in (4.8) we have the stabilizability of the system for \( M = S^d \) large enough, as shown in [29], by using an oblique projection based feedback. Of course we can start with any set of \( M \) indicator functions, whose supports are equal up to translations and which allow us to stabilize the system, and afterwards construct, as above, the roads between those \( M \) indicator functions. For other results concerning stabilizability by means of a finite number of actuators, namely, based on Riccati feedbacks, we refer the reader to [8, 21]. Finally, note that if we were not particularly interested in indicator functions as actuators, we could also start with any set \( M \) stabilizing actuators, whose shapes are equal up to translations. The support/shape of the static actuators may also differ by rotations. In this case slightly different roads are needed, where the moving actuator support/shape rotates during the travel.

4.3. Conclusion of proof of Theorem 1.1. Let us fix \( y_0 \in H \) and \( c_0 \in \mathbb{R}^M \) with \( \omega(c_0) \subset \Omega \), and let \( c^1 \) be the center of the static actuator \( 1_{\omega_1} \). For \( \hat{c}(t) := c_0 + t^2(2-t)^2(c^1 - c_0) \) we have that
\[
\hat{c}(0) = c_0, \quad \hat{c}(1) = c^1, \quad \hat{c}(0) = \hat{c}(1) = 0, \quad \hat{c} \in L^\infty((0,1), \mathbb{R}^M).
\]
We proceed as in Corollary 3.2 by taking the actuator path \( \Phi^*(t) = \hat{1}_{\omega(c(t))} \), for time \( t \in [0,1] \), and the path illustrated in section 4.2 for time \( t \geq 1 \) where we use Theorem 3.1. Note that \( \Phi^*(1) = \hat{1}_{\omega(c(1))} = \hat{1}_{\omega_1} \) and \( \Phi^*(1) = 0 \).

Observe also that \( |\hat{c}|_{W^{1,\infty}((0,1), \mathbb{R}^M)} \leq |\hat{\varphi}|_{W^{1,\infty}((0,1), \mathbb{R}^M)} |c^1 - c_0|_{\mathbb{R}^M} \leq C_3 \) with \( \varphi(t) := t^2(2-t)^2 \), where \( C_3 \) can be taken independent of \( c_0 \) because \( \Omega \) is bounded. Therefore, we have that \( |\hat{c}|_{W^{1,\infty}(\mathbb{R}_0, \mathbb{R}^M)} \leq \max\{C_3, |\hat{c}|_{W^{1,\infty}(\mathbb{R}_1, \mathbb{R}^M)} \} \leq C_4 \), with \( C_4 \) independent of \((y_0, c_0)\), because \( |\hat{c}|_{W^{1,\infty}(\mathbb{R}_1, \mathbb{R}^M)} \) is independent of \( y(1) \) (cf. (4.10)), hence independent of \( y(0) \).

4.4. A remark on Assumption 2.3 and weak solutions. Instead of a reaction-convection operator \( A_{rc} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H)) \), we can also take an operator \( A_{rc} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, V')) \) which is the case for a convection term as \( \nabla \cdot (by) \) under homogeneous Dirichlet boundary conditions, with \( b \in L^\infty(\mathbb{R}_0, \mathbb{R}^d) \). For the latter case we can repeat the procedure and prove the stabilizability result in the \( H \)-norm. That is, we must work with weak solutions \( y \in \mathcal{C}(\mathbb{R}_0, H) \) instead of strong solutions \( y \in \mathcal{C}(\mathbb{R}_0, V) \). In particular we would just need to replace \( V \) by \( H \) in Assumption (2.4) and in (3.16). Recall that, with \( \bar{C}_{rc} = |A_{rc}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))} \) and \( \bar{D}_Y = \mathcal{C}[T, \bar{C}_{rc}] \), we will have (cf. [25, Lem. 2.2], recalling that \( \mathcal{C}(\mathcal{T}_k, H) \hookrightarrow W(I_k, V, V') \))

\[
|\mathcal{G}_{k}(u, f(t))|_H^2 \leq \bar{D}_Y \left( |u|_H^2 + |f|_{L^2(I_k, V')}^2 \right), \quad t \in I_k = (kT, kT + T).
\]

Concerning parabolic equations we can see that instead of (4.5) we would obtain
\[
|\mathbf{v}(t)|_{V'} \leq \|P\| \left( |P_{E_M}|_{\mathcal{L}(V', H)} |A_{rc}|_{\mathcal{L}(H, V')} + \lambda \|y\|_H \right)
\]
\[
\leq \left( |P_{E_M}|_{\mathcal{L}(V', H)} |A_{rc}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))} + \lambda \right) \|P\| \|y(t)\|_H
\]
\[
\leq \left( |P_{E_M}|_{\mathcal{L}(V', H)} C_{rc} + \lambda \right) \|P\| Ce^{-\mu(t-kT)} |y(kT)|_H, \quad t \geq kT, \quad (4.11)
\]
which implies that
\[
|\mathbf{v}(t)|_{V'} \leq \mathcal{R}_0 |y(kT)|_H, \quad t \in \mathcal{T}_k = [kT, kT + T]. \quad (4.12)
\]
Such inequality implies that the control is essentially bounded as required in Assumption 2.4.

Such weak solutions are also defined for $A_{rc} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H))$, but we cannot show that the control remains essentially bounded in the case we only know that $|y(t)|_H$ remains bounded. For that we would need to bound (4.5) by $|y(kT)|_H$ instead of $|y(kT)|_V$, but this seems to be not possible in general.

5. Numerical simulations. The construction in Sections 3 and 4, for a fixed set of roads $\mathcal{R}_j$ connecting a suitable set of static actuators, see (4.9), will likely lead us to an actuator which is moving very fast in such roads. In applications, this is likely not the “best” motion for the actuator, sometimes it could be better to stay longer in a particular region or it could be better to leave the roads $\mathcal{R}_j$ in order to cover other regions of $\Omega$. Therefore, we are going to compute the center $c = c(t)$ of the moving actuator and the control magnitude $u = u(t)$ using tools from optimal control. For the position/center $c$ of the actuator we choose a dynamics based on the equation of motion $\ddot{c} = a$ from physics. We consider the acceleration as $a = -\varsigma c - \varepsilon c + \eta$, for given constants $\varsigma$ and $\varepsilon$, where $\eta = \eta(t)$ can be seen as a control force/acceleration at our disposal. Note that such dynamics is more realistic than a first order dynamics as $\dot{c} = -\varepsilon c + \varsigma$, with $\varsigma$ being a control on the velocity $\dot{c}$. Indeed, it is more realistic to apply a force/acceleration to a moving object than to instantaneously impose/change its velocity.

5.1. Computation of a stabilizing single actuator based receding horizon control. We deal with system (1.7), where now we will consider $(y, c)$ as the state of the system and $(u, \eta)$ as the control. To compute the force $\eta$ and the magnitude $u$, we formulate the following infinite-horizon optimal control problem defined by minimizing the performance index function defined by

$$ J_\infty(u, \eta : (y_0, c_0, 0)) := \frac{1}{2} \int_0^\infty |\nabla y(t, \cdot)|^2_{L^2(\Omega, \mathbb{R}^d)} + \beta |u(t)|^2 \, dt. $$

That is, we define the infinite-horizon optimization problem

$$ \inf_{(u, \eta) \in L^2(\mathbb{R}_0, \mathbb{R}) \times L^2_{loc}(\mathbb{R}_0, \mathbb{R}^d)} J_\infty(u, \eta : (y_0, c_0, 0)) $$

subject to

$$\begin{align*}
\dot{y} - \nu \Delta y + ay + b \cdot \nabla y &= u 1_{\omega(c)}, \\
\dot{c} + \varsigma \dot{c} + \varepsilon c &= \eta, \\
y(0, \cdot) &= y_0, \\
c(0) &= c_0, \\
\dot{c}(0) &= 0,
\end{align*}$$

as well as to the constraints

$$\begin{align*}
c &\in \mathcal{C} := \{ y \in C(\mathbb{R}_0, \mathbb{R}^d) \mid \omega(y(s)) \subset \Omega, \text{ for } s \in \mathbb{R}_0 \}, \\
\eta &\in \mathcal{X} := \{ \kappa \in L^2_{loc}(\mathbb{R}_0, \mathbb{R}^d) \mid \|\kappa(s)\| \leq K \text{ for a.e. } s \in \mathbb{R}_0 \},
\end{align*}$$

where $K = (K_1, K_2, \ldots, K_d) \in \mathbb{R}^d$ is a vector with coordinates $K_i > 0$, for all $1 \leq i \leq d$, and where by $\|\kappa(s)\| \leq K$ we mean that $|\kappa_i(s)| \leq K_i$ for all $1 \leq i \leq d$. For tackling this infinite-horizon problem we employ a receding horizon framework. This framework relies on successively solving finite-horizon open-loop problems on bounded time-intervals as follows. Let us fix $T > 0$ and let an initial vector of the
form $I_0 := (t_0, y_0, c_0, c_0^0) \in \mathbb{R}_+ \times L^2(\Omega) \times \mathbb{R}^d \times \mathbb{R}^d$ be given. We define the time interval $I_{t_0} := (t_0, t_0 + T)$, and the finite-horizon cost functional

$$J_T(u, \eta; I_0) := \frac{1}{2} \int_{t_0}^{t_0 + T} |\nabla y(t, \cdot)|_{L^2(\Omega, \mathbb{R}^d)}^2 + \beta |u(t)|^2 \, dt,$$

and introduce the finite-horizon optimization problem

$$\min_{(u, \eta) \in L^2(I_{t_0}, \mathbb{R}^{1+d})} J_T(u, \eta; I_0) \tag{5.3a}$$

subjected to the dynamical constraints

$$\begin{align*}
\dot{y} - \nu \Delta y + ay + b \cdot \nabla y &= u_1 \omega(c), \quad y|_{\Gamma} = 0, \\
\dot{\epsilon} + \zeta \dot{\epsilon} + \epsilon \dot{c} &= \eta, \\
\epsilon(t_0, \cdot) &= y_0, \quad c(t_0) = c_0, \quad \dot{\epsilon}(t_0) = c_0^0,
\end{align*} \tag{5.3b}$$

in the time interval $I_{t_0}$, as well as to the constraints

$$\begin{align*}
\{ c \in C_{t_0,T} := \{ g \in C(I_{t_0}, \mathbb{R}^d) \mid \omega(g(s)) \subset \Omega \text{ for } s \in I_{t_0} \} \\
\{ \eta \in X_{t_0,T} := \{ \kappa \in L^2(I_{t_0}, \mathbb{R}^d) \mid ||\kappa(s)|| \leq K \text{ for a.e. } s \in I_{t_0} \} \}.
\end{align*} \tag{5.3c}$$

The steps of the RHC are described in Algorithm 1, where we use the subset

$$\mathbb{R}^d_{[\omega]} := \{ c \in \mathbb{R}^d \mid \omega(c) \in \Omega \}.$$

### Algorithm 1 Receding Horizon Algorithm

**Require:** The prediction horizon $T > 0$, the sampling time $\delta < T$, and an initial vector $\mathbb{I}_\infty = (y_0, c_0) \in H \times \mathbb{R}^d_{[\omega]}$.

**Ensure:** The suboptimal RHC pair $(u_{rh}, \eta_{rh})$.

1. Set $t_0 = 0$ and $\mathcal{I}_0 = (t_0, y_0, c_0, 0)$;
2. Find the solution $(y^\tau_T(\cdot; \mathcal{I}_0), u^\tau_T(\cdot; \mathcal{I}_0), c^\tau_T(\cdot; \mathcal{I}_0), \eta^\tau_T(\cdot; \mathcal{I}_0))$ over the time horizon $I_{t_0}$ by solving the open-loop problem (5.3);
3. For all $\tau \in [t_0, t_0 + \delta)$, set $u_{rh}(\tau) = u^\tau_T(\tau; \mathcal{I}_0)$ and $\eta_{rh}(\tau) = \eta^\tau_T(\tau; \mathcal{I}_0)$;
4. Update: $t_0 \leftarrow t_0 + \delta$;
5. Update: $\mathcal{I}_0 \leftarrow (t_0, y^\tau_T(t_0; \mathcal{I}_0), c^\tau_T(t_0; \mathcal{I}_0), \dot{c}^\tau_T(t_0; \mathcal{I}_0))$;
6. Go to Step 2.

#### 5.2. Numerical discretization and implementation.
Here we report on numerical experiments related to Algorithm 1. These experiments confirm the capability of the moving control computed by Algorithm 1. The examples deal with one-dimensional controlled systems of the form (1.4) defined on $\Omega := (0, 1)$ which are exponentially unstable without control. We compare the performance of one single moving control with finitely many static actuators. Throughout, the spatial discretization was done by the standard Galerkin method using piecewise linear and continuous basis functions with mesh-size $h = 0.0025$. Moreover, for temporal discretization we used the Crank–Nicolson/Adams–Bashforth scheme [16] with step-size $t_{step}$. In this scheme, the implicit Crank–Nicolson scheme is used except for the nonlinear term $u_1 \omega(c)$ and convection term $b \cdot \nabla y$ which are treated with the explicit Adams–Bashforth scheme. To deal with the open-loop problems (5.3a), we considered the reduced formulation of the problem with respect to the independent variables $(\eta, u)$. The state constraints $C_{t_0,T}$ were treated using the Moreau–Yosida [17] regularization with parameter $\mu = 10^{-5}$. The box constraints $|u(t)| \leq K$.
were handled using projection. We used the projected Barzilai–Borwein gradient method \cite{4,7,12} equipped with a nonmonotone line search strategy. Further, we terminated the algorithm as the $L^2$-norm of the projected gradient for the reduced problem was smaller than $10^{-4}$ times of the norm of the projected gradient for the initial iterate.

For the case with static actuators, we choose the indicator functions $1_{\omega_i}$ with the placements

$$
\omega_i := \left( \frac{1}{2M}(2i - 1) - \frac{r}{2}, \frac{1}{2M}(2i - 1) + \frac{r}{2} \right) \quad \text{for } i = 1, \ldots, M, \tag{5.4}
$$

where $r > 0$ and $M \in \mathbb{N}$. This is motivated by the stabilizability results given in \cite[Thm. 4.4]{31} and \cite[Sect. 4.8.1]{22}. Further, for every $t \geq 0$, the moving actuator $1_{\omega(c(t))}$ is described by

$$
\omega(c(t)) := (c(t) - \frac{r}{2}, c(t) + \frac{r}{2}).
$$

In the case of the static actuators, we employed the receding horizon framework given in \cite[Alg. 1]{3} for the choice of $| \cdot |_\ast = | \cdot |_{\ell_2}$ with control cost parameter $\beta$.

**Example 5.1.** In this example, we set (cf. (5.2b)–(5.2c))

$$
\nu = 0.1, \quad \varsigma = 1, \quad \epsilon = 0,
$$

$$
a(t, x) = -3 - 2|\sin(t + x)|, \quad b(t, x) = |\cos(t + x)|, \quad K = 500.
$$

Further, we chose the initial conditions

$$
y_0(x) := \sin(\pi x), \quad (c_0, c_0^1) := (0.5, 0).
$$

For all actuators, moving and fixed ones, we chose $r = 0.04$. Thus the support of every actuator covers 4% of the whole of domain. We chose $T = 1.25$ and $\delta = 0.5$ for RHCs.

Figures 3 and 4 correspond to the choices $\beta = 0.1$ and $\beta = 0.5$. Figures 3(a) and 4(a) illustrate the evolution of the $L^2(\Omega)$-norm for the states corresponding to uncontrolled system, one single moving actuator, and fixed actuators ($M = 1, \ldots, 5$) for $t_{\text{step}} = 0.001$. The black dotted line in both figures corresponds to the uncontrolled state. It shows that the uncontrolled state is exponentially unstable. For both cases $\beta = 0.1$ and $\beta = 0.5$, we can see that the moving control obtained by Algorithm 1 is stabilizing and its stabilization rate is smaller than the one corresponding to one single static actuator ($M = 1$), and comparable to the cases $M = 2, 3, 4$.

Further, by comparing Figures 3(a) and 4(a), we can infer that, as we could expect, $\beta = 0.1$ leads to a faster stabilization compared to the case $\beta = 0.5$. As can be seen from Figure 4(a), it is not clear for the case $\beta = 0.5$ that one fixed actuator is asymptotically stabilizing. Moreover, for $M = 4, 5$ we have better stabilization results compared to the single moving control. Figures 3(b) and 4(b) illustrate the time evolution of the control domain $\omega(c_{rh}(t))$. Figures 3(c) and 4(c) show the evolution of the absolute value of the magnitude control $u_{rh}$ and Figures 3(d) and 4(d) demonstrate the evolution of the force $\eta_{rh}(t)$.

**Example 5.2.** Here, as an illustration of Proposition 1.2 we consider an example which is not stabilizable with $N = 5$ static actuators. A moving control steers, however, the system to zero. We take an operator as in Example 1.3 and consider the parabolic-like dynamics as

$$
\frac{\partial y}{\partial t} + Ay = \sum_{j=1}^{5} u_j 1_{\omega_j}, \quad y_0(x) = (1 - P^e_{\varphi})c_{io}
$$

(5.5a)
with
\[ Ay := -\nu \Delta y - 5y + \nu \pi^2 \sum_{m=2}^{6} (1 - m^2)(y, \hat{e}_m)_{L^2(\Omega)} \hat{e}_m \]  
(5.5b)
where \( \nu = 0.1 \), \( \hat{e}_m(x) = 2^j \sin(m\pi x) \), and
\[ i_0 := \min \left\{ i \mid 1 \leq i \leq 6 \text{ and } |{(1 - P_G)\hat{e}_i}_{L^2(\Omega)}| = \max_{1 \leq j \leq 6} |{(1 - P_G)\hat{e}_j}_{L^2(\Omega)}| \right\}, \quad (5.5c)\]
with \( \mathcal{G} = \text{span}\{1_{\omega_j} \mid 1 \leq j \leq 5\} \) and \( \mathcal{E}_6 = \text{span}\{\hat{e}_i \mid 1 \leq i \leq 6\} \). Note that the (Dirichlet) eigenvalues of \( -\nu \Delta \) are given by \( \beta_j = \nu \pi^2 j^2 \) and the first 6 eigenvalues of \( A \) are given by \( \bar{\alpha}_j = \beta_1 - 5 = \nu \pi^2 - 5 < 0 \), for \( 1 \leq j \leq 6 \).

Furthermore, though we cannot guarantee stabilizability for an arbitrary a priori given actuator trajectory, we also make the following experiment. We start by fixing a trajectory \( c(t) = \tilde{c}(t) \) for the moving actuator \( 1_{\tilde{c}(t)}, \tilde{w}(t) = (\tilde{c}(t) - \frac{r}{2}, \tilde{c}(t) + \frac{r}{2}) \) and determine a magnitude control \( u_{rh} = \tilde{u}_{rh} \) by RHC. For the trajectory we choose \( \tilde{c}(t) := \frac{1}{2} - \frac{1}{2} (1 - r) \sin(\omega t) \), with \( \omega = 1 \) and \( r = 0.05 \).

We choose \( r = 0.05 \) for each static and moving actuator and set
\[ \nu = 0.1, \quad \varsigma = 1, \quad \epsilon = 0, \]
\[ t_{\text{step}} = 0.01, \quad \beta = 0.001, \quad K = 50, \]
and \( T = 2 \) and \( \delta = 0.5 \) for the receding horizon algorithms.
As expected, from Proposition 1.2, we cannot stabilize the corresponding solution to zero with the five static actuators \( \{1_{\omega_j} \mid 1 \leq j \leq 5 \} \). This is confirmed in Figure 5(a), where the curve corresponding to the uncontrolled state overlaps that corresponding to the controlled state with 5 static actuators. This is also expected due to the choice of the initial state, which is inspired by the proof of Proposition 1.2.

On the other hand, from Figure 5(a), we see that single moving controls are able to stabilize the system both with the actuator trajectory \( c_{rh} \) obtained by Algorithm 1 and with the a priori given actuator trajectory \( \hat{c} \). These trajectories are depicted in Figure 5(b). The fastest stabilization is obtained with the computed trajectory \( c_{rh} \). Figure 5(c) depicts the evolution of the magnitude \( |\eta_{rh}| \), and Figure 5(d) depicts the evolution of the forcing \( \eta_{rh} \) corresponding to \( c_{rh} \).

We would like to mention that the RHC computations are cheaper for an a priori given actuator trajectory \( \hat{c} = \hat{c}(t) \). However, we do not know, in general, for which trajectories stabilizability of the system holds.

Finally, we would like to underline that though, in the simulations, the 5 static actuators were taken as the indicator functions \( 1_{\omega_j} \) as in (5.4), for a different placement of those 5 actuators we can repeat the steps above, namely, on the choice a suitable initial state, leading us to a similar unstable behavior of the norm of the corresponding controlled state.
6. Final remarks. Concerning the computation of the RHC, note that Examples 5.1 and 5.2 show that a single moving actuator is able to stabilize unstable systems, including situations when fixed actuators fail to. These examples were set in dimension 1. In the future we plan to consider higher dimensional cases. This will require extra care due to the movement of the actuator within the spatial grid of the partial differential equation. We also plan to investigate the asymptotic stability of the RHC and the effect of the constraints.

Concerning the velocity of the moving actuator, note that Proposition 1.2 shows that moving the actuator is necessary to achieve stability (for the entire general class of systems considered). Our theoretical constructions shows that the ability of moving the actuators fast enough is a sufficient condition for stabilizability, recall (3.63) and (3.66d). These two facts do not imply that we cannot achieve stability by nonstatic arbitrarily slow motion. Proving or refuting the stabilizability under arbitrarily small positive velocity is not the scope of the manuscript, but this could be an interesting question for future work. Another related question concerns the upper bound $K > 0$ for the magnitude of $\eta$ for the forcing $\eta$ in (5.2). By fixing the dynamics for the motion, for example, $\ddot{c} + \kappa \dot{c} + \epsilon c = \eta$ as in (5.2), the velocity is determined by the control “acceleration” $\eta$ at our disposal. From practical and physical points of view, it would be interesting to investigate whether we can stabilize the system for arbitrary $K > 0$. 

Figure 5. Example 5.2.
Appendix.

A.1. Proofs of Propositions 1.2 and 1.6. Assume the notation from Section 1.3. Let $\mathcal{E}_{N+1} := \text{span } E_{N+1}$, with $E_{N+1} := \{\tilde{e}_j \mid 1 \leq j \leq N+1\}$ be the eigenspace associated with $\bar{\alpha}_1$, hence $Au = \bar{\alpha}_1 v$ for all $v \in \mathcal{E}_{N+1}$.

Let $\mathcal{F} := \text{span}\{\Psi_j \mid 1 \leq j \leq N\}$ and let $P_X \in \mathcal{L}(H,X)$ be the orthogonal projection in $H$ onto a given closed subspace $X \subset H$.

We start with the following auxiliary result.

**Lemma A.1.** Let the smallest eigenvalue satisfy $\bar{\alpha}_1 \leq 0$ and have multiplicity $N + 1 \geq 2$, then there exists an associated eigenfunction $\bar{\varphi} \in \mathcal{E}_{N+1} \cap \mathcal{F}$.\hfill \Box

**Proof.** Let $\mathcal{G} := P_{\mathcal{E}_{N+1}} \mathcal{F} \subset \mathcal{E}_{N+1}$. We have that $\{\tilde{e}_i - P_{\mathcal{G}} \tilde{e}_i \mid 1 \leq i \leq N+1\} \neq \{0\}$, because otherwise we would have that $E_{N+1} \subset \mathcal{G}$, which is impossible due to $\dim \mathcal{E}_{N+1} = N + 1 > N \geq \dim \mathcal{F} \geq \dim \mathcal{G}$. Therefore, $\bar{\varphi} := \tilde{e}_{i_0} - P_{\mathcal{G}} \tilde{e}_{i_0} \neq 0$ for some $i_0 \in \{1,2,\ldots,N+1\}$. Clearly $\bar{\varphi} \in \mathcal{E}_{N+1} \cap \mathcal{G}$. Further, for every $h \in \mathcal{F}$ we find that $(\bar{\varphi}, h)_H = (\bar{\varphi}, P_{\mathcal{E}_{N+1}} h)_H = (\bar{\varphi}, P_{\mathcal{G}} P_{\mathcal{E}_{N+1}} h)_H = 0$, because $P_{\mathcal{E}_{N+1}} h \in P_{\mathcal{E}_{N+1}} \mathcal{F} = \mathcal{G}$.\hfill \Box

**Proof of Proposition 1.2.** We take an eigenfunction as given by Lemma A.1 for the initial condition, $y_0 := \bar{\varphi}$ satisfying $(\bar{\varphi}, \Psi_j)_H = 0$, for all $1 \leq j \leq N$. By decomposing the solution into orthogonal components $y = q + Q$, with $q \in \text{span}\{\bar{\varphi}\}$ and $Q \in \{\bar{\varphi}\}^\perp$, we obtain

\begin{align}
\dot{q} + Aq &= 0, & q(0) &= \bar{\varphi}, \quad (A.1a) \\
\dot{Q} + AQ &= \sum_{j=1}^N u_j \Psi_j, & Q(0) &= 0. \quad (A.1b)
\end{align}

Observe that the dynamics of the component $q$ is independent of $u$, and such component is then given by $q(t) = e^{-\bar{\alpha}_1 t} \bar{\varphi}$, $t > 0$. Now, for the norm of the entire state we find

$$|y(t)|_H^2 = |q(t)|_H^2 + |Q(t)|_H^2 \geq e^{-2\bar{\alpha}_1 t} |\bar{\varphi}|_H^2 \geq |\bar{\varphi}|_H^2 > 0, \quad t > 0,$$

for every control $u$.\hfill \Box

**Proof of Proposition 1.6.** Let $j_0 := \min\{j \in \mathbb{N} \mid \bar{\alpha}_j > 0\}$. If $j_0 = 1$ then the free dynamics is exponentially stable. If $j_0 > 1$, then we consider the dynamics onto the linear span of the first $j_0 - 1$ eigenfunctions

$$\mathcal{E}_{j_0-1} := \text{span}\{\tilde{e}_j \mid 1 \leq j \leq j_0 - 1\}, \quad q(t) := P_{\mathcal{E}_{j_0-1}} y(t) \in \mathcal{E}_{j_0-1}.$$  

We decompose the system as

\begin{align}
\dot{q} + Aq &= u P_{\mathcal{E}_{j_0-1}} \Psi_1, & q(0) &= P_{\mathcal{E}_{j_0-1}} y_0, \quad (A.2a) \\
\dot{Q} + AQ &= u P_{\mathcal{E}_{j_0-1}}^\perp \Psi_1, & Q(0) &= P_{\mathcal{E}_{j_0-1}}^\perp y_0. \quad (A.2b)
\end{align}

with $Q = y - q = P_{\mathcal{E}_{j_0-1}}^\perp y$. Next we prove that the finite dimensional system (A.2a) is null controllable. Writing

\begin{align}
q &= \sum_{k=1}^{j_0-1} q_k \tilde{e}_k, & \bar{q} &= \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{j_0-1} \end{bmatrix}
\end{align}
we obtain the system
\[ \dot{\overline{q}} = A\overline{q} + Bu, \] (A.3)
with \( A \in \mathbb{R}^{(j_0-1)\times(j_0-1)} \) and \( B \in \mathbb{R}^{(j_0-1)\times1} \) as follows
\[
A = \text{diag}(\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{j_0-1}) \quad \text{and} \quad B = \begin{bmatrix}
(\overline{c}_1, \Psi) \\
(\overline{c}_2, \Psi) \\
\vdots \\
(\overline{c}_{j_0-1}, \Psi)
\end{bmatrix}.
\]
The matrix \( A \) is diagonal with entries \( A_{(i,i)} = \overline{\alpha}_i \).
For any given \( T > 0 \), we have that system (A.3) is controllable at time \( T \). Indeed, this follows from Kalman rank condition (see, e.g., [34, Sect 1.3, Thm. 1.2]), because we have that
\[
\det([A \mid B]) = \left( \bigotimes_{k=1}^{j_0-1} (\overline{c}_k, \Psi) \right) \det \mathcal{V}
\]
where
\[
[A \mid B] := [B \ AB \ldots \ A^{j_0-2}B]
\]
and \( \mathcal{V} \) is the Vandermonde matrix whose entries are
\[
\mathcal{V}_{(i,j)} = \overline{\alpha}_i^{j-1}.
\]
Hence
\[
\det([A \mid B]) = \left( \bigotimes_{k=1}^{j_0-1} (\overline{c}_k, \Psi) \right) \prod_{1 \leq i < j \leq j_0-1} (\overline{\alpha}_j - \overline{\alpha}_i) \neq 0.
\]
Therefore, we can choose a control \( u \) such that \( \overline{q}(T) = 0 \), which implies \( q(T) = 0 \).
Then, we take the concatenated control defined as: \( u^c(t) \) if \( t \in [0, T) \), and \( u^c(t) = 0 \) for \( t > T \). For time \( t \geq T \) we have that \( q(t) = 0 \) and
\[
|y(t)|_H^2 = |Q(t)|_H^2 = e^{-\overline{\alpha}_{j_0}(t-T)} Q(T) |_H, \quad t \geq T.
\]
Since, by definition of \( j_0 \), we have that \( \overline{\alpha}_{j_0} > 0 \), it follows that \( |y(t)|_H^2 \) converges exponentially to zero, as \( t \to +\infty \). Hence, \( u^c \) is a stabilizing (open-loop) control. \( \square \)

The proofs of Propositions 1.2 and 1.6 above are quite intuitive for the considered system. Shorter proofs would follow from applications of the Fattorini criterium for stability that we find for example in [6]. At this point we would like to refer the reader also to the work by Hautus in [15, Sect. 3] for further comments on controllability and stabilizability of more general finite-dimensional autonomous systems (including both continuous-time and discrete-time cases). See also [33, Sect. 6.5] for a version of the Hautus test for the observability of linear abstract infinite-dimensional systems (here recall also the duality between observability and controllability).

A.2. Proof of Proposition 3.4. Let us fix an arbitrary \( \delta > 0 \) and let \( h \in \mathcal{S}_H \) be in the unit ball of \( H \). Since \( X \) is dense in \( H \), we can choose \( \overline{h} \in X \setminus \{0\} \) such that \( |\overline{h} - h|_H \leq \frac{\delta}{2} \). Now, for \( \overline{\overline{h}} := |\overline{\overline{h}}|^{-1} \overline{h} \in \mathcal{S}_H \) we find
\[
|h - \overline{\overline{h}}|_H \leq |\overline{\overline{h}}|^{-1} (|\overline{h} - \overline{h}|_H + |\overline{h} - h|_H) \leq |\overline{\overline{h}}|^{-1} - 1 |\overline{h}|_H + \frac{\delta}{2}
\]
\[
= |\overline{\overline{h}}|^{-1} |\overline{h}|_H - |\overline{\overline{h}}|^{-1} |h|_H + \frac{\delta}{2} = |h|_H - |\overline{h}|_H + \frac{\delta}{2} \leq |h - \overline{h}|_H + \frac{\delta}{2} = \delta.
\]
Hence we can conclude that \( X \cap \mathcal{S}_H \) is dense in \( \mathcal{S}_H \). \( \square \)
A.3. Proof of Proposition 3.6. We start by defining

$$\tilde{t}_j := \max\{\tau_j, \sigma_j\}, \quad \underline{t}_j := \min\{\tau_j, \sigma_j\}, \quad 0 \leq j \leq K,$$

and by writing, with $g := f_{\tau} - f_{\sigma}$,

$$|f_{\tau} - f_{\sigma}|^2_{L^2((a,b), X)} = \sum_{j=1}^{K} \int_{\underline{t}_j}^{\tilde{t}_j} |f_{\tau}(t) - f_{\sigma}(t)|^2_X dt = \sum_{j=1}^{K} \int_{\underline{t}_j}^{\tilde{t}_j} |g(t)|^2_X dt. \quad (A.4)$$

We proceed by Induction. Firstly, we find that

$$\int_a^{\tilde{t}_1} |g(t)|^2_X dt = \int_{t_0}^{\tilde{t}_1} |g(t)|^2_X dt = \int_{\underline{t}_1}^{\tilde{t}_1} |g(t)|^2_X dt \leq \mathcal{R}\mathcal{X}^2, \quad (A.5)$$

where in the last inequality we used the fact that $g(t) = f_{\tau}(t) - f_{\sigma}(t) = \phi_1 - \phi_1 = 0$ for $t \in [t_0, \underline{t}_1)$.

Next, we assume that for a given $i \in \{1, 2, \ldots, K - 1\}$ we have

$$\int_a^{\tilde{t}_i} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2. \quad (\ast)$$

Then we obtain

$$\int_a^{\tilde{t}_{i+1}} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_i}^{\tilde{t}_{i+1}} |g(t)|^2_X dt,$$

which implies that

$$\int_a^{\tilde{t}_{i+1}} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_i}^{\tilde{t}_{i+1}} |g(t)|^2_X dt + \int_{\underline{t}_{i+1}}^{\tilde{t}_{i+1}} |g(t)|^2_X dt, \quad \text{if } \tilde{t}_i < \underline{t}_{i+1} \leq \tilde{t}_{i+1},$$

and

$$\int_a^{\tilde{t}_{i+1}} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_{i+1}}^{\tilde{t}_{i+1}} |g(t)|^2_X dt, \quad \text{if } \underline{t}_{i+1} \leq \tilde{t}_i \leq \tilde{t}_{i+1},$$

Observe that, if $\tilde{t}_i < \underline{t}_{i+1}$, then $g(t) = f_{\tau}(t) - f_{\sigma}(t) = \phi_{i+1} - \phi_{i+1} = 0$, for $t \in (\tilde{t}_i, \underline{t}_{i+1}) \subseteq (\tau_i, \tau_{i+1}) \Delta (\sigma_i, \sigma_{i+1})$. Therefore, in either case we have

$$\int_a^{\tilde{t}_{i+1}} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2 + \int_{\underline{t}_{i+1}}^{\tilde{t}_{i+1}} |g(t)|^2_X dt \leq i\mathcal{R}\mathcal{X}^2 + (i + 1)\mathcal{R}\mathcal{X}^2 = (i + 1)\mathcal{R}\mathcal{X}^2. \quad (\Psi)$$

Hence, assumption $(\ast)$ implies $(\Psi)$, which together with $(A.5)$ imply, by Induction,

$$\int_a^{\tilde{t}_j} |g(t)|^2_X dt \leq j\mathcal{R}\mathcal{X}^2, \quad \text{for all } j \in \{1, 2, \ldots, K\}. \quad (A.6)$$

In particular, since $\tilde{t}_K = b$, for $j = K$ we obtain $|f_{\tau} - f_{\sigma}|_{L^2((a,b), X)} \leq K^{\frac{1}{2}}\mathcal{R}^{\frac{1}{2}}\mathcal{X}^2$. □

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