Single-conflict colouring

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Abstract
Given a multigraph, suppose that each vertex is given a local assignment of \( k \) colours to its incident edges. We are interested in whether there is a choice of one local colour per vertex such that no edge has both of its local colours chosen. The least \( k \) for which this is always possible given any set of local assignments we call the \textit{single-conflict chromatic number} of the graph. This parameter is closely related to separation choosability and adaptable choosability. We show that single-conflict chromatic number of simple graphs embeddable on a surface of Euler genus \( g \) is \( O(g^{1/4} \log g) \) as \( g \to \infty \). This is sharp up to the logarithmic factor.

KEYWORDS
adaptable choosability, DP-colouring, graphs on surfaces, list colouring, single-conflict chromatic number
1 | INTRODUCTION

Dvořák and Postle [4] and Fraigniaud, Heinrich and Kosowski [7] independently defined the conflict $k$-colouring problem as follows. Given a (simple) graph $G = (V, E)$, each edge $uv \in E$ is assigned a list $K(u, v)$ of ordered pairs—called conflicts—of colours from $[k] = \{1, \ldots, k\}$. The question is whether $G$ admits a colouring $c : V \to [k]$ of the vertices so that no edge is in a conflict, that is, there is no edge $uv \in E$ and conflict $(\kappa_u, \kappa_v) \in K(u, v)$ such that $c(u) = \kappa_u$ and $c(v) = \kappa_v$. The authors in [4,7] also imposed further natural restrictions based on contrasting goals and perspectives, but here instead we only prescribe the maximum number $\mu$ of conflicts per edge.

In fact, this is equivalent to the “least nontrivially conflicting” version of the problem, with exactly one conflict per edge, provided we pass to a multigraph of maximum edge multiplicity $\mu$. (This motivates our first use of the letter $\mu$.) Let us be more precise. Let $G = (V, E)$ be a multigraph. For any positive integer $k$, a local $k$-partition of $G$ is a collection $\{\kappa_v\}_{v \in V}$ of maps of the form $\kappa_v : E(v) \to [k]$, where $E(v)$ denotes the set of edges incident to $v$. So each $\kappa_v$ is a partition of $E(v)$ into $k$ parts, and for each $e \in E(v)$ the colour $\kappa_v(e)$ can be thought of as the local colour of $v$ associated to $e$. For each edge $uv$ in the underlying simple graph we want the set $K(u, v)$ of conflicts to be composed of the pairs $(\kappa_u(e), \kappa_v(e))$, for every $e \in E$ with endpoints $u$ and $v$. Thus, given a local $k$-partition $\{\kappa_v\}$, we say $G$ is conflict $\{\kappa_v\}$-colourable if there is some colouring $c : V \to [k]$ of the vertices so that no edge $e = uv \in E$ has $c(u) = \kappa_u(e)$ and $c(v) = \kappa_v(e)$. The single-conflict chromatic number $\chi_s(G)$ of $G$ is the smallest $k$ such that $G$ is conflict $\{\kappa_v\}$-colourable for any local $k$-partition $\{\kappa_v\}_{v \in V}$.

As we discuss in Section 2, single-conflict chromatic number considerably strengthens upon two notable list colouring parameters, separation choosability (cf., [15]) and adaptable choosability (cf., [12]), and so its study could potentially yield new insights into these two parameters.

Before continuing, we give two easy but instructive examples. First, for a square integer $\mu$, consider two vertices with $\mu$ edges between them. Take the local $\sqrt{\mu}$-partition which lists all $\mu$ possible pairwise conflicts between the two vertices. So this is a $\mu$-edge planar multigraph with maximum degree and multiplicity both $\mu$ that has single-conflict chromatic number greater than $\sqrt{\mu}$. Second, for positive integers $a$ and $\mu$, consider $(a\mu)^a$ vertices, written $v_{i_1}, \ldots, i_a$, $i_1, \ldots, i_a \in [a\mu]$, each joined by $\mu$ edges to each of $a$ vertices $u_1, \ldots, u_a$. Take the local $(a\mu)$-partition where for each $j \in [a]$ and $i_1, \ldots, i_a \in [a\mu]$ the edges between $u_j$ and $v_{i_1}, \ldots, i_a$ include all conflicts having $i_j$ as a local colour for $u_j$ and some integer in $[\mu] \setminus [(j - 1)\mu]$ as a local colour for each $v_{i_1}, \ldots, i_a$. For any $i_1, \ldots, i_a \in [a\mu]$, if $u_j$ is coloured $i_j$ for all $j \in [a]$, there remains no possibility for colouring $v_{i_1}, \ldots, i_a$. Thus the multigraph formed from the complete bipartite graph $K_{a(a\mu)^a}$ by multiplying each edge by $\mu$ has single-conflict chromatic number greater than $a\mu$. (Note that by substituting $(a\mu)^a$ copies of the first example, one can boost this to $a\mu + \Omega(\sqrt{\mu})$). With $a = 2$ this yields a planar multigraph of maximum multiplicity $\mu$ that has single-conflict chromatic number greater than $2\mu$.

Besides introducing single-conflict chromatic number and setting down some of its basic behaviour, our main task in this paper is to treat it in a classic setting for chromatic graph theory. We prove the following.

1By relabelling, we alternatively may define the $\kappa_v$ as maps from $E(v)$ to $\mathbb{N}$ each image set of which contains at most $k$ elements, so not necessarily the same image for every $v$. 

Theorem 1. For some constant $C_1 > 0$, if $G$ is a multigraph of maximum multiplicity $\mu \geq 1$ that is embeddable on a surface of Euler genus $g$, then $\chi_m(G) \leq \max\{C_1, \sqrt{\mu} (g + 1)^{1/4}\log(\mu^2(g + 2)), 8\mu\}$.

Note that the $8\mu$ term cannot be lowered to $2\mu$ due to the second example above. We will see below that the other term is sharp up to at most a polylogarithmic factor.

Allow us to reiterate the $\mu = 1$ case, which may be interpreted as an analogue of Heawood’s classic formula for the chromatic number $\chi(G) = (l + 1)(l + 2)/2$.

Corollary 2. For some constant $C > 0$, if $G$ is a simple graph that is embeddable on a surface of Euler genus $g$, then $\chi(G) \leq C g^{1/4}\log(g + 2)$.

The $\mu = \Theta(\sqrt{g})$ case is also of special interest, in which case Theorem 1 implies a bound of the form $\chi_m(G) = O(\sqrt{g} \log g)$. Since the number of colours available per vertex is close to the maximum multiplicity and both are around $\sqrt{g}$, this result is evocative of Heawood’s bound itself. Naturally one could aspire towards an elimination of the logarithmic factor.

Conjecture 3. There exist $C, C' > 0$ such that, for any simple graph $G$ that is embeddable on a surface of Euler genus $g$, if every edge is assigned at most $Ck$ conflicts from $[k]^2$, then $G$ is conflict $k$-colourable, provided $k \geq C'\sqrt{g}$.

Note that $C < 1/2$ due to the second example above. In a previous version of the manuscript we incorrectly conjectured that $C$ is arbitrarily close to $1$.

Theorem 1 follows from the following perhaps more general result.

Theorem 4. For some constant $C_2 > 0$, if $G$ is a multigraph with $m \geq 3$ edges and maximum multiplicity $\mu \geq 1$, then $\chi_m(G) \leq C_2(\mu m)^{1/4}\log(\mu m)$.

We prove Theorems 1 and 4 in Section 4. The proof of Theorem 4 is partly probabilistic in nature. It relies on a stronger version (see Lemma 13) of the following simple bound.

Proposition 5. If $G$ is a multigraph of maximum degree $\Delta \geq 1$, then $\chi_m(G) \leq \lceil e(2\Delta - 1) \rceil$.

For completeness, we prove Proposition 5 in Section 3 by a standard application of the Lovász Local Lemma. This has the following strong yet still partial converse, also shown in Section 3.

Proposition 6. If $G$ is a multigraph of average degree $d \geq 3$, then $\chi_m(G) \geq \lceil \sqrt{d}/\log d \rceil$.

The last two assertions alone highlight a clear distinction between single-conflict chromatic number and, say, ordinary choosability, for which the behaviour of the complete graphs $K_{d+1}$ is linear in $d$, while that of the complete bipartite graphs $K_{d,d}$ is logarithmic in $d$ [5].

Notice that Proposition 6 helps provide a broad certificate of sharpness of Theorems 1 and 4 up to polylogarithmic factors. This is akin to the two-vertex example exhibited earlier.
In particular, consider the complete multigraph on $n$ vertices of uniform edge multiplicity $\mu$. It is a $\mu(n - 1)$-regular graph, so with $\mu \binom{n}{2}$ edges, that has $\Theta(n^2)$ Euler genus. By Proposition 6, the single-conflict chromatic number is $\Omega(\sqrt{\mu n/\log(\mu n)})$, and this is not far from the $O(\sqrt{\mu n \log(\mu n)})$ upper bound implied in both Theorems 1 and 4.

It may be challenging to eliminate the logarithmic factors in Theorems 1 and 4. Since we do not know the correct asymptotics in these results, we have made no effort to optimise the values of $C_1$ and $C_2$. On the other hand, we managed to avoid the logarithmic factors for separation and adaptable choosability (see Theorems 11 and 12). The simpler argument uses Proposition 5 directly (rather than needing Lemma 13), and we present it in Section 4 as a warm up to proving our main result.

One might wonder if degeneracy could be an alternative way to prove Theorem 1, at least in the $\mu = 1$ case. That was essentially Heawood’s original approach to bounding the chromatic number. As we will see in Section 2, density considerations have some use (see Lemma 7); however, a construction of Kostochka and Zhu [12] for the adaptable chromatic number shows that there are graphs of degeneracy $d$ which have single-conflict chromatic number greater than $d$. There might yet be some constant $C' > 0$ such that the single-conflict chromatic number of any $d$-degenerate graph on $n$ vertices is at most $C' \sqrt{d} \log n$ (which would imply the $\mu = 1$ case of Theorem 1), but we have not been able to prove this thus far. Theorem 4 implies an upper bound of $C(n d)^{1/4} \log(nd)$ in this situation.

For small $g$, it would be interesting to determine the optimal upper bound on $\chi_{sc}(G)$ over all multigraphs $G$ embeddable on a surface of Euler genus $g$ in terms of the maximum multiplicity $\mu$, particularly for the boundary cases $\mu = 1$ and $\mu \to \infty$. As we will indicate in Section 2 it is easy to verify that the extremal single-conflict chromatic number for simple planar graphs is 4, but we have not further investigated the precise values for $\mu = 1$ with, say, $g = 1, 2, 3$. For large $\mu$ (and fixed $g$) the second instructive example above and Theorem 1 together give a value asymptotically between $2\mu$ and $8\mu$, and it is tempting to narrow this range.

1.1 | Probabilistic preliminaries

We make use of the following basic probabilistic tools. We refer the reader to the monograph of Molloy and Reed [16] for further details.

**The Chernoff Bound.** For any $0 \leq t \leq np$,

$$\mathbb{P}(|\text{Bin}(n, p) - np| > t) < 2\exp(-t^2/(3np)).$$

**The Lovász Local Lemma.** Consider a set $\mathcal{E}$ of (bad) events such that for each $A \in \mathcal{E}$

(i) $\mathbb{P}(A) \leq p < 1$, and

(ii) $A$ is mutually independent of a set of all but at most $d$ of the other events.

If $ep(d + 1) \leq 1$, then with positive probability none of the events in $\mathcal{E}$ occurs.
Consider a set \( \mathcal{E} = \{A_1, ..., A_n\} \) of (bad) events such that each \( A_i \) is mutually independent of \( \mathcal{E} - (\mathcal{D}_i \cup A_i) \), for some \( \mathcal{D}_i \subseteq \mathcal{E} \). If we have reals \( x_1, ..., x_n \in [0, 1) \) such that for each \( i \)

\[
\mathbb{P}(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j),
\]

then the probability that none of the events in \( \mathcal{E} \) occurs is at least \( \prod_i (1 - x_i) > 0 \).

## 2 | DEFINITIONS

In this section, we give some more definitions, one of single-conflict chromatic number, one of adaptable choosability and one of separation choosability. We also show how these three parameters are related, and give a few comments related to planar graphs.

First we give an alternative definition of single-conflict chromatic number, which may be insightful. Let \( G = (V, E) \) be a multigraph. Given a local \( k \)-partition \( \kappa_v \) of \( G \), we say \( G \) is conflict \( \kappa_v \)-orientable if there is some orientation of all edges of \( G \) such that for every vertex \( v \in V \), the set of local colours of \( v \) associated to the (oriented) edges leaving \( v \) does not contain all of \( [k] \). Then the single-conflict chromatic number \( \chi(G) \) of \( G \) is the least \( k \) such that \( G \) is conflict \( \kappa_v \)-orientable for any local \( k \)-partition \( \kappa_v \) of \( G \).

**Proof of equivalence.** Let \( G = (V, E) \) and fix a local \( k \)-partition \( \kappa_v \) of \( G \). It suffices to show that \( G \) is conflict \( \kappa_v \)-orientable if and only if it is conflict \( \kappa_v \)-colourable. If it has a conflict \( \kappa_v \)-orientation, then for every \( v \in V \) chooses a colour from \( [k] \) that is absent from the local colours of \( v \) associated to the edges leaving \( v \) to produce a conflict \( \kappa_v \)-colouring. If it has a conflict \( \kappa_v \)-colouring \( c \), then orient towards \( v \) all incident edges \( e \) such that \( \kappa_v(e) = c(v) \) to produce a conflict \( \kappa_v \)-orientation.

From this equivalence, the following proposition becomes plain.

**Proposition 7.** If there is an orientation of \( G \) such that every vertex has maximum outdegree less than \( k \), then \( \chi(G) \leq k \).

This implies \( \chi(G) \leq 1 + \max_{S \subseteq V} |E(G[S])/|S|\), compare, for example, [1, Lem. 3.1].

**Corollary 8.** If \( G \) is a planar graph, then \( \chi(G) \leq 4 \). If \( G \) is a triangle-free planar graph, then \( \chi(G) \leq 3 \). If \( G \) is a simple graph embeddable on a surface of Euler genus \( g > 0 \), then \( \chi(G) \leq H_g/2 + 1 \), where \( H_g \) is Heawood’s formula for Euler genus \( g \).

Recall that every \( k \)-degenerate graph has an orientation of maximum outdegree at most \( k \). So Proposition 7 cannot be improved in general, since there are \( k \)-degenerate graphs with adaptable chromatic number greater than \( k \) [12] (and, as we will shortly see, the same then is true of single-conflict chromatic number).

Next we discuss how single-conflict chromatic number is connected to two colouring parameters, both of which are weaker versions of list colouring, as introduced independently by Erdős, Rubin and Taylor [5] and by Vizing [17].
For completeness, we recall the classic definition. Let $G = (V, E)$ be a (multi)graph. For a positive integer $k$, a mapping $L : V \to \binom{\mathbb{Z}^+}{k}$ is called a $k$-list-assignment of $G$, and a colouring $c$ of $V$ is called an $L$-colouring if $c(v) \in L(v)$ for any $v \in V$. We say $G$ is $k$-choosable if there is a proper $L$-colouring of $G$ for any $k$-list-assignment $L$. The choosability $\chi(G)$ of $G$ is the least $k$ such that $G$ is $k$-choosable.

### 2.1 Adaptable choosability

The following list colouring parameter was proposed by Kostochka and Zhu [12]. Let $G = (V, E)$ be a multigraph. Given a labelling $\ell : E \to \mathbb{Z}^+$ of the edges, a (not necessarily proper) vertex colouring $c : V \to \mathbb{Z}^+$ is adapted to $\ell$ if for every edge $e = uv \in E$ not all of $c(u), c(v)$ and $\ell(e)$ are the same value. We say that $G$ is adaptably $k$-choosable if for any $k$-list-assignment $L$ and any labelling $\ell$ of the edges of $G$, there is an $L$-colouring of $G$ that is adapted to $\ell$. The adaptable choosability $\chi_a(G)$ of $G$ is the least $k$ such that $G$ is adaptably $k$-choosable. Every proper colouring is adapted to any labelling $\ell$, so $\chi(G) \geq \chi_a(G)$ always.

We observe adaptable choosability is at most the single-conflict chromatic number.

**Observation 9.** For any multigraph $G, \chi(G) \geq \chi_a(G)$.

**Proof.** Fix $G = (V, E)$ and let $k = \chi_a(G)$. Let $L$ be a $k$-list-assignment and let $\ell$ be a labelling of the edges of $G$. For each $v \in V$, locally colour each edge $e$ incident to $v$ with colour $a$ if $a \in L(v)$ and $\ell(e) = a$. This yields a local $k$-partition $\{\kappa_v\}$ (as mentioned in Section 1, it is not important that the image of each map $\kappa_v$ is equal to $[k]$, the image of each $\kappa_v$ can be different sets of $k$ elements for each vertex $v$). By the choice of $k$ there must be a conflict $\{\kappa_v\}$-colouring. It follows from our definition of $\{\kappa_v\}$ that this corresponds to an $L$-colouring that is adapted to $\ell$. \qed

We remark that adaptable choosability is in turn a strengthening of the adaptable chromatic number (for which the list assignment always takes all lists equal) and Hell and Zhu [10] have exhibited planar graphs with adaptable chromatic number at least 4. So the single-conflict chromatic number is also exactly 4 for such graphs.

### 2.2 Separation choosability

The following list colouring parameter was proposed by Kratochvíl, Tuza and Voigt [15]. Let $G = (V, E)$ be a graph. We say a $k$-list-assignment $L$ has maximum separation if $|L(u) \cap L(v)| \leq 1$ for every edge $uv$ of $G$. We say $G$ is separation $k$-choosable if there is a proper $L$-colouring of $G$ for any $k$-list-assignment $L$ that has maximum separation. The separation choosability $\chi_{sep}(G)$ of $G$ is the least $k$ such that $G$ is separation $k$-choosable. Since the choosability $\chi(G)$ of $G$ omits any separation requirement on the lists, $\chi(G) \geq \chi_{sep}(G)$ always.

Let us see that separation choosability is at most adaptable choosability. This observation was made earlier [6], but we include it here for cohesion.
Observation 10. For any simple graph $G$, $\chi(a)(G) \geq \chi_{\text{sep}}(G)$.

Proof. Fix $G = (V, E)$ and let $k = \chi(a)(G)$. Let $L$ be a $k$-list-assignment of maximum separation. Let $\ell$ be a labelling defined for each $uv \in E$ by taking $\ell(uv)$ as the unique element of $L(u) \cap L(v)$ if it is nonempty, and arbitrary otherwise. By the choice of $k$, there is guaranteed to be an $L$-colouring $c$ that is adapted to $\ell$. Due to the maximum separation property of $L$ and the definition of $\ell$, the colouring $c$ must be proper. \[\square\]

Single-conflict chromatic number is a direct strengthening of separation choosability, in the same way that “DP-colouring” is a strengthening of choosability [4].

Alternative proof that $\chi_{\text{sep}}(G) \leq \chi''(G)$ for any simple graph $G$. Fix $G = (V, E)$ and let $k = \chi''(G)$. Let $L$ be a $k$-list-assignment of maximum separation. Let $\{\kappa_v\}$ be a local $k$-partition of $G$ defined as follows. For each edge $e = uv \in E$, if $i$ is the unique colour in $L(u) \cap L(v)$, then let $\kappa_u(e) = i$ and $\kappa_v(e) = i$. By the choice of $k$, there is guaranteed to be a conflict $\{\kappa_v\}$-colouring $c$. Due to the maximum separation property of $L$ and the definition of $\{\kappa_v\}$, the colouring $c$ is proper. \[\square\]

We remark that Kratochvíl, Tuza and Voigt [14] proved that $\chi_{\text{sep}}(K_n) \sim \sqrt{n}$ as $n \to \infty$ by the use of affine planes. This is enough to certify sharpness of our Theorems 1 and 4 each up to a logarithmic factor (and Proposition 5 up to a constant factor) for simple graphs.

We also note that Škrekovski [18] conjectured that every planar graph has separation choosability at most 3, but this remains open to the best of our knowledge. If true, it would imply that separation choosability and adaptable choosability can be distinct for some planar graphs.

3 | DEGREE

In this section, we for completeness give the proofs of Propositions 5 and 6. These results closely relate single-conflict chromatic number to the maximum and average degrees, respectively, of the multigraph.

The following proof is analogous to proofs for separation and adaptable choosability [12,14].

Proof of Proposition 5. Let $G = (V, E)$ be a multigraph of maximum degree $\Delta$ and fix $k = \lceil \sqrt{e(2\Delta - 1)} \rceil$. Let $\{\kappa_v\}$ be a local $k$-partition of $G$. Consider a random colouring $c : V \to [k]$ where each vertex is given an independent uniform choice. For each edge $e = uv \in E$, let $A_e$ be the event that $c(u) = \kappa_u(e)$ and $c(v) = \kappa_v(e)$. For all $e \in E$, $P(A_e) = 1/k^2$ and $A_e$ is mutually independent of all but at most $2\Delta - 2$ other events $A_f$. Observe that $c$ is a conflict $\{\kappa_v\}$-colouring if and only if all the events $A_e$ do not occur. The Lovász Local Lemma guarantees with positive probability a conflict $\{\kappa_v\}$-colouring if $e(2\Delta - 1)/k^2 < 1$, which follows from the choice of $k$. \[\square\]

Note that the bound $\sqrt{e(2\Delta - 1)}$ in Proposition 5 can be slightly improved to $2\sqrt{\Delta}$ using the Local Cut Lemma [3, Theorem 3.1] instead of the Lovász Local Lemma, using the same set of bad events. We have deliberately chosen to present the simpler, weaker bound.

The following proof is analogous to that in [13] or in [2].
Proof of Proposition 6. Let $G = (V, E)$ be a multigraph of average degree $d = 2m/n$, where $n = |V|$ and $m = |E|$. Let $k = \lceil \sqrt{d/\log d} \rceil$ and consider a random local $k$-partition $\{ \kappa_v \}$ of $G$ where, for each edge $e = uv \in E$, the pair $(\kappa_u(e), \kappa_v(e))$ is independently, uniformly chosen from pairs in $[k]^2$. For any fixed $c : V \to [k]$, $c$ is a conflict $\{ \kappa_v \}$-colouring with probability $(1 - 1/k^2)^m$. By the union bound and Markov's inequality, the probability that $G$ is conflict $\{ \kappa_v \}$-colourable is at most $k^m (1 - 1/k^2)^m \leq \exp(-m/k^2)$. Since $G$ has average degree $d$, we have by the choice of $k$ that $k^2 \log k = d \log d < d/2 = m/n$. This implies $n \log k - m/k^2 < 0$ and so $k^m \exp(-m/k^2) < 1$. We have thus shown that with positive probability there is a local $k$-partition $\{ \kappa_v \}$ for which $G$ is not conflict $\{ \kappa_v \}$-colourable.

We remark that since $(1 + o(1)) \log d \leq \text{ch}(K_{d,d}) \leq \text{ch}(K_{d,d}) \leq (1 + o(1)) \log d$ as $d \to \infty \ [5,8]$, Proposition 6 implies that the ratio between single-conflict chromatic number and choosability or adaptable choosability or separation choosability can be arbitrarily large even for bipartite graphs.

4 PROOF OF THEOREM 1

As a warm up to the main proof, we show the following result, an adaptable choosability analogue of Theorem 4.

Theorem 11. If $G$ is a multigraph with $m \geq 2^{16}$ edges and maximum multiplicity $\mu \geq 1$, then $\text{ch}(G) \leq 2^{11/4} \sqrt{\mu}(\mu m)^{1/4}$.

The proof of Theorem 11 can be viewed as a simplified version of the proof of Theorem 4. Afterwards, we show how the following result, an adaptable choosability analogue of Theorem 1, is a consequence of Theorem 11. (At the same time, we also show how Theorem 4 implies Theorem 1).

Theorem 12. For some constant $C_3 > 0$, if $G$ is a multigraph of maximum multiplicity $\mu \geq 1$ that is embeddable on a surface of Euler genus $g$, then $\text{ch}(G) \leq C_3 \sqrt{\mu}(g + 1)^{1/4}$.

Theorems 11 and 12 imply the same bounds for separation choosability, and both are sharp up to the choice of $C_3$ due to the complete graphs with uniform edge multiplicity $\mu \ [14]$. Let us mention that the question of whether graphs of Euler genus $g$ have adaptable chromatic and choice numbers at most of order $g^{1/4}$ was first raised in December of 2007 during the Graph Theory 2007 meeting in Fredericia, Denmark.

Proof of Theorem 11. Let $G = (V, E)$ be a multigraph with $|E| = m$ and maximum multiplicity $\mu$. Let $k = 2^{11/4} \sqrt{\mu}(\mu m)^{1/4}$, let $L$ be a $k$-list-assignment, and consider any labelling $\ell$ of the edges of $G$. We want to prove that there is an $L$-colouring of $G$ that is adapted to $\ell$. We can assume that $G$ is connected (or else we consider each component separately), and in particular $G$ has $n \leq m + 1$ vertices.
Let \( X = \bigcup_{v \in V} L(v) \), and let \( X_1 \subseteq X \) be chosen uniformly at random. Set \( X_2 = X \setminus X_1 \). For any \( i \in \{1, 2\} \) and \( v \in V \), \( L(v) \cap X_i \) is binomially distributed with parameter \( 1/2 \). The Chernoff Bound implies that \( |L(v) \cap X_i| \leq k/4 \) with probability at most \( \exp(-k/24) \leq \frac{1}{2m} < \frac{1}{2n^2} \), where the first inequality uses \( m \geq 2^{16} \). By a union bound, there is a bipartition \( X = X_1 \cup X_2 \) such that \( |L(v) \cap X_i| \geq k/4 \) for any \( i \in \{1, 2\} \) and \( v \in V \).

Let \( A \) be the set of vertices of degree at least \( \sqrt{2m} \) in \( G \) and let \( B = V \setminus A \). Since \( |E| = m \), \( A \) has most \( 2m/\sqrt{2m} = \sqrt{2m/\mu} \) vertices, and thus \( G[A] \) has maximum degree at most \( \mu \sqrt{2m/\mu} = \sqrt{2\mu m} \). By definition, \( G[B] \) also has maximum degree at most \( \sqrt{2\mu m} \). We remove all the colours of \( X_1 \) from \( L(v) \) for each \( v \in A \), and all the colours of \( X_2 \) from \( L(v) \) for each \( v \in B \). After this operation, each list has at least \( k/4 \) colours left. Since \( k/4 = \frac{\sqrt{2}}{2e\sqrt{2\mu m}} \), it follows from Proposition 5 that \( G[A] \) has an \( L \)-colouring adapted to \( \ell \) using only colours from \( X_2 \) while \( G[B] \) has an \( L \)-colouring adapted to \( \ell \) using only colours from \( X_1 \). Since \( X_1 \) and \( X_2 \) are disjoint, we obtain an \( L \)-colouring of \( G \) adapted to \( \ell \), as desired.

Let us now see that Theorems 1 and 12 follow from Theorems 4 and 11, respectively.

**Proofs of Theorems 1 and 12.** Assume for a contradiction that there is a counterexample \( G \) to Theorem 1 or 12. Take \( G \) in such way that \( g \) is minimised, and subject to this the number \( n \) of vertices of \( G \) is minimised. We can assume that \( G \) is connected (or else we consider each component separately). Let \( \tilde{G} \) be the simple graph underlying \( G \). By the minimality of \( g \), \( \tilde{G} \) has no embedding on a surface of smaller Euler genus, and thus has a cellular embedding on a surface \( \Sigma \) of Euler genus \( g \). It follows from Euler’s Formula that \( \tilde{G} \) has \( \tilde{m} \leq 3n + 3g - 6 \) edges, and so \( G \) has \( m \leq \mu(3n + 3g - 6) \) edges. Let \( k = \max\{[C_1 \sqrt{\mu} (g + 1)^{1/4} \log(\mu^2(g + 2))], 8\mu\} \) (for Theorem 1) or \( k = [C_3 \sqrt{\mu} (g + 1)^{1/4}] \) (for Theorem 12), and assume that each vertex has \( k \) local colours. If \( G \) has a vertex \( v \) of degree less than \( k \), then remove \( v \). By the minimality of \( n \), we can colour \( G - v \) and then find a suitable colour for \( v \) (since \( v \) has at least \( k \) local colours and fewer than \( k \) neighbours in \( G \)). Thus, we can assume that \( G \) has minimum degree at least \( k \), and thus at least \( \frac{1}{2} nk \) edges. Consequently, \( nk/2 \leq \mu(3n + 3g - 6) \).

For Theorem 1, since \( k/(2\mu) \geq 4 \), we have \( n \leq 3g - 6 \) and \( m \leq \mu(12g - 24) \). It then follows from Theorem 4 and a large enough choice of constant that \( G \) has single-conflict chromatic number strictly smaller than \( k \), which is a contradiction.

For Theorem 12, observe that not only \( G \), but also \( \tilde{G} \) has minimum degree at least \( k \). Thus \( nk/2 \leq \tilde{m} \leq 3n + 3g - 6 \). For a large enough choice of constant \( C_3 \), \( k \geq 8 \) and thus \( n \leq 3g - 6 \) and \( m \leq \mu(12g - 24) \). It then follows from Theorem 11 and a large enough choice of constant that \( G \) has single-conflict chromatic number strictly smaller than \( k \), which is a contradiction.

To prove Theorem 4, we require the following slightly technical result.

**Lemma 13.** For any \( d \geq 2^{23} \), let \( G = (V, E) \) be a multigraph with a vertex partition \( V = A \cup B \) such that
(i) the induced submultigraph $G[A]$ has maximum degree at most $d$, 
(ii) all vertices in $A$ have maximum degree at most $d^2$ in $G$, and 
(iii) all vertices in $B$ have maximum degree at most $d$ in $G$.

There is a constant $C > 0$ such that for any local $k$-partition $\{\kappa_i\}$ of $G$, where $k \geq C\sqrt{d} \log d$, there is a colouring $c : A \to [k]$ such that $c$ is a conflict $\{\kappa_i\}$-colouring of $G[A]$ and no vertex $x \in B$ has more than $\sqrt{d}$ incident edges $e = xy$, $y \in A$, such that $c(y) = \kappa_y(e)$.

**Proof of Theorem 4.** Let $G = (V, E)$ be a multigraph with $m$ edges and maximum multiplicity $\mu$. Let $A$ be the set of vertices of degree at least $\mu m / 2$ in $G$ and let $B = V \setminus A$. Since $|E| = m$, $A$ has at most $2m / \sqrt{2\mu m} = \sqrt{2m / \mu}$ vertices, and thus $G[A]$ has maximum degree at most $\mu \sqrt{2m / \mu} = \sqrt{2\mu m}$. It follows from the definition of $B$ that $G[B]$ also has maximum degree at most $\sqrt{2\mu m}$. Note that $\chi_\infty(G)$ is trivially at most $m$. So by a large enough fixed choice of $C_2$ we may assume $m$ is large enough so that the conditions of Lemma 13 are satisfied with $d = \sqrt{2\mu m}$. Let $C > 0$ be the constant associated to the corresponding application of Lemma 13. Let $k$ be an integer at least $\max\{C\sqrt{d} \log d, \lceil \sqrt{e(2d - 1)} \rceil + \sqrt{d}\}$ and let $\{\kappa_i\}$ be a local $k$-partition of $G$. It follows from an application of Lemma 13 that there is a conflict $\{\kappa_i\}$-colouring $c$ of $G[A]$. It remains to colour $B$ in such a way that it is compatible with $c$.

For each vertex $x \in B$, remove from $G$ any edge $f$ incident to $x$ if there exists some incident edge $e = xy$, $y \in A$, such that $c(y) = \kappa_y(e)$ and $\kappa_x(f) = \kappa_x(e)$. We also (locally) remove each of the colours associated to the edges we removed. By one of the properties of $c$ guaranteed by Lemma 13, this process removes at most $\sqrt{d}$ of the colours incident to each vertex in $B$. By arbitrarily deleting any excess local colours as well as any of the incident edges with those colours, then relabelling colours, we are left with a local $k'$-partition $\{\kappa_i\}'$ of a submultigraph of $G[B]$ with maximum degree at most $d$, where $k' = \lceil \sqrt{e(2d - 1)} \rceil$. By Proposition 5, this submultigraph admits a conflict $\{\kappa_i\}'$-colouring $c'$. The colour and edge removal process we performed ensures that, by reversing the relabelling, $c'$ corresponds to a conflict $\{\kappa_i\}$-colouring of $G[B]$ that combines with $c$ to produce a conflict $\{\kappa_i\}$-colouring of all of $G$. \hfill \square

It remains only to prove Lemma 13. This is done with an application of the General Local Lemma (see Section 1.1).

**Proof of Lemma 13.** Let $k = [C\sqrt{d} \log d]$ where $C$ is some constant large enough to guarantee certain properties as specified later in the proof. Let $\{\kappa_i\}$ be a local $k$-partition of $G$.

We must do a pruning operation before proceeding—in fact, this is the crucial step in the proof. By taking $C$ large enough, we may assume for each $v \in A$ and each $i \in [k]$ that the number of edges in $\kappa_{v^{-1}}(i)$ with its other endpoint also in $A$ is at most $\sqrt{d}$. (We summarily remove all edges associated to every colour not satisfying the property,
and since the maximum degree of $G[A]$ is at most $d$ this removes at most $\sqrt{d}$ of the colours around each vertex in $A$.

Let $p = 2^{-4}/\sqrt{d}$. Consider a random selection of colours where each of the $|V|k$ local colours is selected according to an independent Bernoulli trial of probability $p$. With an eye to applying the General Local Lemma, let us define three types of (bad) events.

I. For a vertex $x \in A$, none of the colours around $x$ is selected.
II. For an edge $e = xy \in E$ with $x, y \in A$, $\kappa_x(e)$ and $\kappa_y(e)$ are both selected.
III. For a vertex $x \in B$, there are more than $\sqrt{d}$ edges $e = xy, y \in A$, for which $\kappa_y(e)$ is selected.

If we obtain a selection for which none of the above events occurs, then we are done. This is because the deselection of a colour does not introduce any new event of Type II or III. So we can arbitrarily deselect all but one of the colours around each vertex, and the remaining selection induces the desired colouring $c$, thanks to the fact that no events of Type II or III hold.

For each $x \in A$, the probability of a Type I event is $\mathbb{P}(\text{Bin}(k, p) = 0) = (1 - p)^k \leq \exp(-pk) \leq \exp(-2^{-4}C \log d) < 2^{-8}/d$ if $C$ is chosen large enough. For each edge $e = xy \in E$, the probability of a Type II event is $p^2 = 2^{-8}/d$. For each vertex $x \in B$, the probability of a Type III event is at most $\mathbb{P}(\text{Bin}(d, p) > \sqrt{d}) \leq \mathbb{P}(|\text{Bin}(d, p) - dp| > \sqrt{3} \cdot 2^{-4} \sqrt{d}) < 2\exp(-2^{-4} \sqrt{d})$ by the Chernoff Bound.

The choice to generate the random colouring according to independent Bernoulli trials rather than a uniform colour per vertex (as in Proposition 5) is important for us in establishing the following bounds on dependence between bad events, especially for Type III events. Each Type I event is mutually independent of all but at most $d$ events of Type I, at most $d^2$ events of Type II, and at most $d^2$ events of Type III. Each Type II event is mutually independent of all but at most $2$ events of Type I, at most $2d - 1$ events of Type II, and at most $2d^2$ events of Type III. By the pruning operation we did at the beginning, each Type III event is mutually independent of all but at most $d$ events of Type I, at most $d^{3/2}$ events of Type II, and at most $d^{3/2}$ events of Type III. (To be more explicit, each Type III event is determined by up to $d$ independent Bernoulli random variables, each of which corresponds to a local colour of a neighbour. Thanks to the pruning, the number of Type II events, say, that also use this randomness is at most $d^{3/2}$. The Type III event is mutually independent of all other Type II events).

We associate weight $x_i = 2^{-7}/d$ to each event $i$ of Type I or II, and weight $x_i = 2\exp(-2^{-6} \sqrt{d})$ to each event $i$ of Type III. By the considerations above, the General Local Lemma guarantees the desired selection of colours with positive probability, provided the following three inequalities hold (where we repeatedly used that $\exp(-x - x^2) \leq 1 - x \leq \exp(-x)$ if $0 < x < 0.69$):

$$
1/2 \leq \exp \left( -\frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{2d^2}{\exp(\sqrt{d}/2^2)} - \frac{4d^2}{\exp(\sqrt{d}/2^2)} \right),
$$

(1)
\[
\frac{1}{2} \leq \left(1 - \frac{1}{2d}\right)^2 \exp\left(-\frac{1}{2^6} - \frac{1}{2^{13}d} - \frac{4d^2}{\exp\left(\frac{\sqrt{d}}{2^9}\right)} - \frac{8d^2}{\exp\left(\frac{\sqrt{d}}{2^5}\right)}\right),
\]

(2)

\[
-\frac{\sqrt{d}}{2^4} + \frac{\sqrt{d}}{2^6} \leq -\frac{1}{2^7} - \frac{1}{2^{14}d} - \frac{\sqrt{d}}{2^7} - \frac{1}{2^{14}\sqrt{d}} - \frac{2d^{3/2}}{\exp\left(\frac{\sqrt{d}}{2^6}\right)} - \frac{4d^{3/2}}{\exp\left(\frac{\sqrt{d}}{2^5}\right)}.
\]

(3)

It is straightforward to check that \(d \geq 2^{23}\) suffices. \(\square\)

The above proof can be straightforwardly adapted for the same upper bound (with a larger constant \(C\)) on a stronger type of single-conflict chromatic number where additionally we must assign \(\Omega(\log d)\) distinct colours per vertex instead of just one. What this then directly implies is that, for any simple graph \(G\) that is embeddable on a surface of Euler genus \(g\), the single-conflict chromatic number is \(O(g^{1/4}(\log g)^{3/4})\) even if we allow \(O(\log g)\) conflicts per edge and demand \(\Omega(\log d)\) distinct colours per vertex.

4.1 Notes added

In a version of this study first circulated on arXiv (arXiv:1803.10962v1), we called \(\chi_{\leftrightarrow}\) the least conflict choosability and denoted it instead by \(\text{ch}_{\leftrightarrow}\). Upon the suggestion of a referee, we have reformulated our terminology to better place it amongst the extant colouring notions.

It transpires that single-conflict chromatic number is also naturally related to the classic problem of finding independent transversals in vertex-partitioned graphs. Specifically, given \(G\) with a local \(k\)-partition \(\{\kappa_v\}\), one can define its cover graph \(H\) and an associated vertex partition as follows. The vertex set \(V(H)\) consists of all pairs \((v, i)\) with \(v \in V(G)\) and \(i \in [k]\). The edge set \(E(H)\) includes \((v, i)(v', i')\) if there is an edge \(e = vv' \in E(G)\) such that \(\kappa_v(e) = i\) and \(\kappa_{v'}(e) = i'\). The parts of \(H\) are defined according to \(V(G)\), that is, for each \(v \in V(G)\) the vertices \((v, i), i \in [k]\), are all in one part. Then \(G\) is conflict \(\{\kappa_v\}\)-colourable if and only if \(H\) contains an independent set that is transversal to the partition of \(H\). Thus one may convert between results on single-conflict chromatic number and on independent transversals. For instance, one may recast Proposition 5 as a result about independent transversals subject to some average degree condition. See \cite{11} for more discussion on this perspective and related references.

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REFERENCES
1. N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), no. 2, 125–134.
2. A. Bernshteyn, The asymptotic behavior of the correspondence chromatic number, Discrete Math. 339 (2016), no. 11, 2680–2692.
3. A. Bernshteyn, The local cut lemma, European J. Combin. 63 (2017), 95–114.
4. Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, J. Combin. Theory Ser. B 129 (2017), 38–54.
5. P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, CA, 1979), Congress. No. XXVI Utilitas Math., 1980, pp. 125–157.
6. L. Esperet, R. J. Kang, and S. Thomassé, Separation choosability and dense bipartite induced subgraphs, Combin. Probab. Comput. 28 (2019), no. 5, 720–732.
7. P. Fraigniaud, M. Heinrich, and A. Kosowski, Local conflict coloring, IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9–11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA, 2016, pp. 625–634.
8. Z. Füredi, A. Kostochka, and M. Kumbhat, Choosability with separation of complete multipartite graphs and hypergraphs, J. Graph Theory 76 (2014), no. 2, 129–137.
9. P. J. Heawood, Map colour theorem, Q. J. Math. 24 (1890), 332–338.
10. P. Hell and X. Zhu, On the adaptable chromatic number of graphs, European J. Combin. 29 (2008), no. 4, 912–921.
11. R. J. Kang and T. Kelly, Colourings, transversals and local sparsity, arXiv e-prints, arXiv:2003.05233, Mar. 2020.
12. A. V. Kostochka and X. Zhu, Adapted list coloring of graphs and hypergraphs, SIAM J. Discrete Math. 22 (2008), no. 1, 398–408.
13. D. Král, O. Pangrác, and H.-J. Voss, A note on group colorings, J. Graph Theory 50 (2005), no. 2, 123–129.
14. J. Kratochvíl, Z. Tuza, and M. Voigt, Brooks-type theorems for choosability with separation, J. Graph Theory 27 (1998), no. 1, 43–49.
15. J. Kratochvíl, Z. Tuza, and M. Voigt, Complexity of choosing subsets from color sets, Discrete Math. 191 (1998), no. 1–3, 139–148.
16. M. Molloy and B. Reed, Graph colouring and the probabilistic method, Algorithms and Combinatorics, vol. 23, Springer-Verlag, Berlin, 2002.
17. V. G. Vizing, Coloring the vertices of a graph in prescribed colors, Metody Diskret. Anal. v Teorii Kodov i Shem. 101 (1976), no. 29, pp. 3–10.
18. R. Škrekovski, A note on choosability with separation for planar graphs, Ars Combin. 58 (2001), 169–174.

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