Some Explicit Solutions of the Additive Deligne-Simpson Problem and Their Applications (Preprint)

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Abstract

In this paper we construct three infinite series and two extra triples (\(E_8\) and \(\hat{E}_8\)) of complex matrices \(B, C,\) and \(A = B + C\) of special spectral types associated to C. Simpson’s classification in [32] and P. Magyar, J. Weyman, A. Zelevinsky classification in [28]. This enables us to construct Fuchsian systems of differential equations which generalize the hypergeometric equation of Gauss-Riemann. In a sense, they are the closest relatives of the famous equation, because their triples of spectral flags have finitely many orbits for the diagonal action of the general linear group in the space of solutions. In all the cases except for \(E_8\), we also explicitly construct scalar products such that \(A, B,\) and \(C\) are self-adjoint with respect to them. In the context of Fuchsian systems, these scalar products become monodromy invariant complex symmetric bilinear forms in the spaces of solutions.

When the eigenvalues of \(A, B,\) and \(C\) are real, the matrices and the scalar products become real as well. We find inequalities on the eigenvalues of \(A, B,\) and \(C\) which make the scalar products positive-definite.

As proved by A. Klyachko, spectra of three hermitian (or real symmetric) matrices \(B, C,\) and \(A = B + C\) form a polyhedral convex cone in the space of triple spectra. He also gave a recursive algorithm to generate inequalities describing the cone. The inequalities we obtain describe non-recursively some faces of the Klyachko cone.

1 Introduction

Let \(V\) be a vector space over complex numbers such that \(\dim V = n\) where \(1 \leq n < \infty\). Let \(B, C\) be linear operators in \(V\) and let \(A = B + C\). We call the pair \(B, C\) irreducible if the operators do not preserve simultaneously any proper subspace of \(V\).

Let \(O_A\) be the adjoint orbit of \(A\) in \(\text{End} V\) under the \(\text{GL}(V)\) action. We call the triple \(A, B, C\) rigid, if any other triple \(B', C', A' = B' + C'\) such that \(A' \in O_A, B' \in O_B,\) and \(C' \in O_C\) is conjugate to the triple \(A, B, C\).

For a linear operator \(A \in \text{End} V\), we call the multiset of its eigenvalues the spectrum of \(A\). This means that each eigenvalue \(\lambda_i\) is taken with its multiplicity \(m_i\). Any ordering \(\lambda_1, \lambda_2, \ldots, \lambda_k\) of distinct eigenvalues of \(A\) allows us to represent the spectrum of \(A\) by a vector \(s(A) = (\underbrace{\lambda_1 \cdots \lambda_1}_{m_1 \text{ times}}, \underbrace{\lambda_2 \cdots \lambda_2}_{m_2 \text{ times}}, \ldots, \underbrace{\lambda_k \cdots \lambda_k}_{m_k \text{ times}}) \in \mathbb{C}^n\). For a diagonalizable operator \(A\), we call the par-
tion \((m_1, m_2, \ldots, m_k)\) of \(n\) the spectral type of \(A\). With slight abuse of terminology, we also call the spectral type of \(A\) any composition obtained by some ordering of \(\lambda_1, \ldots, \lambda_k\). We say that a vector \((x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) \in (\mathbb{C}^n)^3\) satisfies the trace condition if \(\sum_{i=1}^n x_i = \sum_{i=1}^n (y_i + z_i)\). Then \((s(A), s(B), s(C))\) belongs to the hyperplane in \((\mathbb{C}^n)^3\) defined by the trace condition. We call this hyperplane the space of triple spectra. Let \(\alpha = (m_1, m_2, \ldots, m_p), \beta = (n_1, n_2, \ldots, n_q), \gamma = (k_1, k_2, \ldots, k_r)\) (compositions of \(n\)) be the spectral types of \(A, B,\) and \(C\). Then \((s(A), s(B), s(C))\) lies in the part \(S(\alpha, \beta, \gamma) \subset \mathbb{C}^{3n}\) defined as follows. A vector \((x, y, z) \in \mathbb{C}^{3n}\) is in \(S(\alpha, \beta, \gamma)\) if \(x_1 = x_2 = \cdots = x_{m_1} \neq x_{m_1+1} = \cdots = x_{m_1+m_2} \neq \cdots\) and the same for \(y\) and \(z\).

Consider the following table of triples of spectral types.

| hypergeometric family | \((1, m - 1), (1^m), (1^m)\) | \(m \geq 2\) |
|-----------------------|-------------------------------|-----------------|
| even family           | \((m, m), (1, m - 1, m), (1^{2m})\) | \(m \geq 2\) |
| odd family            | \((m + 1, m), (1, m, m), (1^{2m+1})\) | \(m \geq 2\) |
| extra case            | \((4, 2), (2, 2, 2), (1^6)\)    |                 |

Here and later \((1^n)\) is a shorthand for \((1,1,\ldots,1)\). \(n\) times

**Theorem 1.1 (Simpson, Kostov)** Let \((\alpha, \beta, \gamma)\) be a triple of spectral types such that at least one of them is \((1^n)\). The following conditions are equivalent:

1. for a generic point \((x, y, z) \in S(\alpha, \beta, \gamma)\) there exists a rigid irreducible triple \((A = B + C, B, C)\) of diagonalizable operators such that \((s(A), s(B), s(C)) = (x, y, z)\);

2. \((\alpha, \beta, \gamma)\) is one of the triples in \((1.1)\).

**Remark 1.1** This theorem is an additive version of Theorem 4 from [32]. This version easily follows from Simpson’s results. A more elementary proof of Theorem 4 from [32] and a proof of Theorem [1.1] were given by V. Kostov in [13].

The first main result of this paper is that for each triple \((\alpha, \beta, \gamma)\) of spectral types from \((1.1)\) and a generic vector from \(S(\alpha, \beta, \gamma)\) we explicitly construct the corresponding triple \((A, B, C)\).

Recently there appeared algorithms to produce all rigid irreducible \(r\)-tuples of matrices \(M_1, \ldots, M_r\) such that \(M_1 + \cdots + M_r = 0\), see [4] and [3]. We use a different (less general, but more powerful for our particular purposes) tool: P. Magyar, J. Weyman, and A. Zelevinsky in [28] constructed all indecomposable triple partial flag varieties with finitely many orbits for the diagonal action of the general linear group. Their list \((1.4)\) is strikingly similar to the list \((1.1)\) of Simpson. It has just one more family: the \(E_8\)-family. A triple of spectral flags (for the definition, see Section 4) of the matrices \(A, B,\) and \(C\) constructed from Simpson’s list \((1.1)\) gives a representative of the open orbit of the corresponding triple flag variety from \((4.43)\).
Our $A$, $B$, and $C$ have the following common features. $B$ is block upper-triangular, $C$ is block lower-triangular. The block sizes of $B$ and $C$ are given by the compositions $\beta$ and $\gamma$ respectively. Each entry of $A$, $B$, and $C$ is a ratio of products of linear forms in the eigenvalues of $A$, $B$, and $C$. The coordinates of all eigenvectors of $A$, $B$, and $C$ are also ratios of products of linear forms in the eigenvalues. The linear forms are remarkably simple: all the coefficients are equal to either 1 or $-1$. As a corollary of our construction, we obtain the following.

**Theorem 1.2** For every composition $(\alpha, \beta, \gamma)$ from Simpson’s list [1,4], there exist open subsets $S''(\alpha, \beta, \gamma) \subset S'(\alpha, \beta, \gamma) \subset S(\alpha, \beta, \gamma)$ with the following properties.

1. Each of $S'(\alpha, \beta, \gamma)$ and $S''(\alpha, \beta, \gamma)$ is obtained from $S(\alpha, \beta, \gamma)$ by removing finitely many hyperplanes.

2. If $(s(A), s(B), s(C)) \in S'(\alpha, \beta, \gamma)$, then there exists a non-zero symmetric bilinear form on $V$ such that $A$, $B$, and $C$ are self-adjoint with respect to it.

3. If $(s(A), s(B), s(C)) \in S''(\alpha, \beta, \gamma)$, then there exists a non-degenerate symmetric bilinear form on $V$ such that $A$, $B$, and $C$ are self-adjoint with respect to it.

This theorem is proved case by case in Theorems [2.2, 2.3, 2.8, 2.11] for the bilinear forms given by the formulas (2.4), (2.14), (2.22), (2.27) correspondingly.

**Remark 1.2** The main virtue of this theorem is not the proof of existence of the objects, but an explicit construction of all of them.

In Simpson’s list (1.1), the last composition $\gamma$ is always $(1^n)$. Thus, the matrix $C$ has all eigenvalues distinct. Let $v_i$ be the eigenvector of $C$ corresponding to the eigenvalue $c_i$. If $C$ is self-adjoint with respect to a scalar product $\langle \cdot, \cdot \rangle$ on $V$, then $\langle v_i, v_j \rangle = \delta_{ij}$. If we manage to find $l_i$ such that the matrix $B$ becomes self-adjoint with respect to $\langle \cdot, \cdot \rangle$ as well, then $A$ is also self-adjoint with respect to $\langle \cdot, \cdot \rangle$ as $A = B + C$. We find the $l_i$ and it turns out that they are also ratios of products of linear forms in the eigenvalues of $A$, $B$, and $C$. And again all the coefficients of the forms are equal to either 1 or $-1$. The set of linear forms that appear in the $l_i$ includes the set of linear forms that appear in the matrix elements of $A$, $B$, and $C$ in the coordinates of their eigenvectors. The hyperplanes one has to remove from $S(\alpha, \beta, \gamma)$ to obtain $S''(\alpha, \beta, \gamma)$ of Theorem 1.3 are exactly the zero levels of the linear forms that appear in the $l_i$. The explicit formulas we find for the $l_i$ give explicit description of these hyperplanes.

Probably the most important applications of our explicit construction is to **Fuchsian systems** (see Section 1.2 for the definition). Let $z_1$, $z_2$, $z_3$ be distinct points of $\mathbb{CP}^1$. Consider the following system of differential equations

$$\frac{df}{dz} = \left[ \frac{B}{z - z_2} + \frac{C}{z - z_3} - \frac{A}{z - z_1} \right] f(z)$$

(1.2)

where $A = B + C$, $z \in \mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ and $f$ takes values in $V$. The matrices $A$, $B$, and $C$ are called the **residue matrices** of (1.2). Their eigenvalues are called **local exponents**. Real parts of the local exponents are the rates of growth of solutions of (1.2) expanded analytically towards the corresponding singularities (and restricted to sectors centered at the singularities). Thus, at each singularity the space of solutions stratifies into a flag. Local basis changes near each singularity turn this flag into a flag variety. The triple of flag varieties of the Gauss-Riemann equation has finitely many orbits for the action of the general linear group in the space of solutions. The Fuchsian systems constructed by means of our matrices exhaust the list of Fuchsian systems (with more than two singularities) having this property. In this sense, they are the simplest Fuchsian
systems possible and we expect their solutions to be interesting functions.

It was known to F. Klein that if the hypergeometric equation of Gauss-Riemann had real local exponents, then there existed a monodromy invariant hermitian form in the space of solutions. If the local exponents were generic, then the form became real as well. So do the matrices $A$, $B$, and $C$ themselves. Thus, $A$, $B$, and $C$ become matrices of real operators acting on the real space $V_R$ and self-adjoint with respect to the real symmetric bilinear form $\langle \ast, \ast \rangle_R$. Let us extend the form $\langle \ast, \ast \rangle_R$ to the hermitian form $(\ast, \ast)$ on $V$. This form gives rise to the monodromy invariant hermitian form in the space of solutions of (1.2). Once again, the forms are constructed explicitly. For the hypergeometric family, this result is not new. The Fuchsian systems from the hypergeometric family are equivalent to the generalized hypergeometric equations studied by F. Beukers and G. Heckman in [2]. Among other things, they construct the hermitian form. Also the generalized hypergeometric equations were studied in what became later known as the Okubo normal form by K. Okubo in [29]. For the generalized hypergeometric equations in the Okubo normal form, the monodromy invariant hermitian form was constructed by Y. Haraoka in [12].

As proved by A. Klyachko in [16], if a hermitian form is positive definite, then the spectra of hermitian matrices $B$, $C$ and $A = B + C$ form a polyhedral convex cone in the space of triple spectra. His proof contains a recursive algorithm to compute the inequalities describing the cone. We call this cone the Klyachko cone and we call the inequalities the Klyachko inequalities. Beukers and Heckman in [2] give explicitly the inequalities on the local exponents of the generalized hypergeometric equation which make the monodromy invariant hermitian form in the space of solutions positive definite. Thus, they describe non-recursively a non-trivial face of the Klyachko cone. We do the same for all the Fuchsian systems constructed from (1.1). Hence the second important application of our results is an explicit description of some interesting faces of the Klyachko cone. Beukers and Heckman in [2] use their criterion of positivity of the hermitian form to see when solutions of the generalized hypergeometric equations are algebraic functions. It is also needed to know the signature of the form for applications to number theory, see [2] and [5].

Let $\lambda$, $\mu$, and $\nu$ be highest weights of $GL(V)$. Let $V_\lambda$, $V_\mu$, and $V_\nu$ be the corresponding rational irreducible representations of $GL(V)$. Let $V_\lambda \otimes V_\mu = \sum_\nu c_{\lambda\mu}^\nu V_\nu$ be the decomposition of the tensor product of $V_\lambda$ and $V_\mu$ into the sum of irreducible representations. It follows from the results of A. Klyachko [16] combined with a refinement by A. Knutson and T. Tao [17] that the lattice points of the Klyachko cone are precisely the triples of weights $\lambda$, $\mu$, $\nu$ with non-zero Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$. Thus, our techniques allow us to explicitly describe some triples of highest weights with non-zero Littlewood-Richardson coefficients. In fact, for all the cases we consider, the Littlewood-Richardson coefficients are equal to 1.

The paper is organized as follows. In Section 2, we formulate main results of the paper. Namely, we list the triples $(A, B, C)$, the scalar products, and the Klyachko inequalities for all the partitions from Simpson’s list (1.1). In Section 3, theorems of Section 2 are proved and elaborated. For example, in subsection 3.4 we construct the residue matrices of the $m$-hypergeometric system as submatrices of the residue matrices of the $m$-even system.

Although the actual construction of the matrices relied heavily on the explicit description of representatives of the open orbits in [28], it turned out that once the answers were known, it was much simpler to prove them by inspection. We start using the results of [28] directly only in Section 4. In the Section, we construct the matrices $A$, $B$, and $C$ which give rise to the $E_8$-family.
of Magyar, Weyman, and Zelevinsky. We also prove the following theorem.

**Theorem 1.3** Let \((\alpha, \beta, \gamma)\) be a triple of spectral types from Theorem 1.1. If \((s(A), s(B), s(C))\) is a point of \(S''(\alpha, \beta, \gamma)\) then the triple \((A, B, C)\) is irreducible.

The proof of the Theorem for the hypergeometric family is on page 22, for all other families – on page 43.

**Remark 1.3** The genericity condition of Theorem 1.1 was somewhat a mysterious one. In general, it still is. A theorem of Katz (see [15]) excludes the coexistence of irreducible and reducible triples in rigid cases. If one deals with a reducible triple, then except for the “big” trace condition one also has a “small” trace condition coming from the reduced submatrices. Thus, people call generic spectra that stay away from all “small trace condition” hyperplanes possible (see Kostov’s papers). In our cases however, Theorem 1.3 gives an explicit meaning to the genericity condition: “generic” means “not in \(S''\”).

**Remark 1.4** If we take a different ordering of the eigenvalues of \(A, B,\) and \(C\), then the scalar products \(\langle \mathbf{v}_i, \mathbf{v}_i \rangle\) change. So does the set \(S'\), but the set \(S''\) does not.

Let \(A, B,\) and \(C\) be self-adjoint with respect to a non-zero symmetric bilinear form on \(V\). Let \((\alpha, \beta, \gamma)\) be a triple of spectral types from Theorem 1.1. Then the following corollary of Theorem 1.3 strengthens the third statement of Theorem 1.2.

**Corollary 1.1** If \((s(A), s(B), s(C))\) is a point of \(S''(\alpha, \beta, \gamma)\), then the form is unique up to a constant multiple.

**Proof** — If \((s(A), s(B), s(C))\) is a point of \(S''(\alpha, \beta, \gamma)\), then it follows from Theorem 1.3 that the triple \((A, B, C)\) is irreducible. If a triple \((A, B, C)\) is irreducible, then uniqueness of the form follows from Schur’s lemma. \(\square\)

In Section 5, we introduce the Berenstein-Zelevinsky triangles which provide a geometric version of the celebrated Littlewood-Richardson rule. For the \(E_8\) family, we do not have formulas for the hermitian form as nice as we have for other families. However, the Berenstein-Zelevinsky triangles enable us to compute the Klyachko inequalities for the \(E_8\)-family as well.

Section 6 contains no new results. In the section, we provide (very) basic facts about Fuchsian systems and raise questions we plan to answer in subsequent publications. In particular, we quote some results from [33, 11, 12] which are very similar to (but different from) ours.

Most of the proofs of the paper boil down to proofs of certain rational identities. These identities are collected in Section 7 (the Appendix).

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2 Main Results

2.1 Hypergeometric Family

Let us pick a vector \((a_1, a_2, \ldots, a_m, c_1, \ldots, c_m)\) from \(S((1, m - 1), (1^m))\). Recall that this means \(a_1 \neq a_2\), all \(b_i\) are distinct, all \(c_i\) are distinct, and the trace condition holds: \(a_1 + (m - 1)a_2 = \sum_{i=1}^{m}(b_i + c_i)\). Define the matrix elements of \(B\) and \(C\) as follows:

\[
B_{ij} = \begin{cases} 
0, & \text{if } i < j \\
b_i, & \text{if } i = j \\
b_i + c_{m+1-i} - a_2, & \text{if } i > j
\end{cases}
\]

\[
C_{ij} = \begin{cases} 
b_{m+1-i} - a_2, & \text{if } i < j \\
c_i, & \text{if } i = j \\
0, & \text{if } i > j
\end{cases}
\]

(2.3)

Here is an example with \(m = 5\).

Example 2.1

\[
B = \begin{bmatrix}
b_1 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 \\
b_2 & b_2 + c_4 - a_2 & b_2 + c_4 - a_2 & b_2 + c_4 - a_2 & b_2 + c_4 - a_2 \\
b_3 & b_3 + c_3 - a_2 & b_3 + c_3 - a_2 & b_3 + c_3 - a_2 & b_3 + c_3 - a_2 \\
b_4 & b_4 + c_2 - a_2 & b_4 + c_2 - a_2 & b_4 + c_2 - a_2 & b_4 + c_2 - a_2 \\
b_5 & b_5 + c_1 - a_2 & b_5 + c_1 - a_2 & b_5 + c_1 - a_2 & b_5 + c_1 - a_2 
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
c_5 & 0 & 0 & 0 & 0 \\
0 & c_4 & 0 & 0 & 0 \\
0 & 0 & c_3 & 0 & 0 \\
0 & 0 & 0 & c_2 & 0 \\
0 & 0 & 0 & 0 & c_1 
\end{bmatrix}
\]

It is clear that \(s(B) = \{b_1, \ldots, b_m\}\) and \(s(C) = \{c_1, \ldots, c_m\}\). Since all the \(b_i\) and all the \(c_i\) are distinct, \(B\) and \(C\) are diagonalizable.

Theorem 2.1 If \(B\) and \(C\) are given by (2.3), then \(A = B + C\) is diagonalizable and \(s(A) = \{a_1, a_2, a_3, \ldots, a_m\}\).

For \(i = 1, \ldots, m\), let \(v_i = (v_i^1, \ldots, v_i^{i-1}, 1, 0, \ldots, 0)\) be the eigenvector of \(B\) with the eigenvalue \(b_i\).

Lemma 2.1 For \(1 \leq j < i \leq m\), we have

\[
v_i^j = \frac{b_j + c_{m+1-j} - a_2}{b_i - b_j} \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k} - a_2}{b_i - b_{j+k}}
\]

Here and in the sequel, all empty products are understood to be equal to 1.
Here is an example with $m = 5$.

**Example 2.2**

\[ v_1 = (1, 0, 0, 0, 0) \]

\[ v_2 = \left( \frac{b_1 + c_2}{b_2 - b_1}, 1, 0, 0, 0 \right) \]

\[ v_3 = \left( \frac{(b_1 + c_5 - a_2)(b_4 + c_4 - a_2)}{(b_3 - b_1)(b_4 - b_2)}, \frac{b_2 + c_4 - a_2}{b_3 - b_2}, 1, 0, 0 \right) \]

\[ v_4 = \left( \frac{(b_1 + c_5 - a_2)(b_4 + c_4 - a_2)}{(b_4 - b_1)(b_4 - b_2)(b_4 - b_3)}, \frac{b_2 + c_4 - a_2}{(b_4 - b_2)(b_4 - b_3)}, b_3 - c_3, 1, 0 \right) \]

\[ v_5 = \left( \frac{(b_1 + c_2 - a_2)(b_5 + c_3 - a_2)(b_5 + c_4 - a_2)}{(b_5 - b_1)(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)}, \frac{(b_2 + c_3 - a_2)(b_5 + c_4 - a_2)}{(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)}, \frac{(b_3 + c_2 - a_2)(b_5 + c_3 - a_2)}{(b_5 - b_3)(b_5 - b_4)}, \frac{(b_4 + c_2 - a_2)(b_5 + c_3 - a_2)}{(b_5 - b_4)} \right) \]

We define a scalar product on $V$ by setting

\[
\langle v_i, v_j \rangle = \delta_{ij} \prod_{k=i+1}^{m} (b_i - b_k) \prod_{k=m+2-i}^{m} (b_i + c_k - a_2) \prod_{k=1}^{m+1-i} (b_i - b_k) \prod_{k=1}^{m+2-i} (b_i + c_k - a_2).
\]

(2.4)

Here is an example of the form with $m = 5$.

**Example 2.3**

\[
\langle v_1, v_1 \rangle = \frac{(b_1 - b_2)(b_1 - b_3)(b_1 - b_4)(b_1 - b_5)}{(b_1 + c_1 - a_2)(b_1 + c_2 - a_2)(b_1 + c_3 - a_2)(b_1 + c_4 - a_2)(b_1 + c_5 - a_2)}
\]

\[
\langle v_2, v_2 \rangle = \frac{(b_2 - b_3)(b_2 - b_4)(b_2 - b_5)(b_2 + c_2 - a_2)}{(b_2 - b_1)(b_2 + c_1 - a_2)(b_2 + c_2 - a_2)(b_2 + c_3 - a_2)(b_2 + c_4 - a_2)(b_2 + c_5 - a_2)}
\]

\[
\langle v_3, v_3 \rangle = \frac{(b_3 - b_4)(b_3 - b_5)(b_3 + c_3 - a_2)(b_3 + c_4 - a_2)(b_3 + c_5 - a_2)}{(b_3 - b_1)(b_3 - b_2)(b_3 + c_1 - a_2)(b_3 + c_2 - a_2)(b_3 + c_3 - a_2)(b_3 + c_4 - a_2)}
\]

\[
\langle v_4, v_4 \rangle = \frac{(b_4 - b_5)(b_4 + c_4 - a_2)(b_4 + c_5 - a_2)}{(b_4 - b_1)(b_4 - b_2)(b_4 - b_3)(b_4 + c_1 - a_2)(b_4 + c_2 - a_2)(b_4 + c_3 - a_2)}
\]

\[
\langle v_5, v_5 \rangle = \frac{(b_5 + c_2 - a_2)(b_5 + c_3 - a_2)(b_5 + c_4 - a_2)(b_5 + c_5 - a_2)}{(b_5 - b_1)(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)(b_5 + c_1 - a_2)}
\]

Let $S'((1, m - 1), (1^m), (1^m))$ be obtained from $S((1, m - 1), (1^m), (1^m))$ by removing the hyperplanes which are zero levels of the linear forms in the denominators of $\langle v_i, v_i \rangle$. Let $S''((1, m - 1), (1^m), (1^m))$ be obtained from $S'((1, m - 1), (1^m), (1^m))$ by removing the hyperplanes which are zero levels of the linear forms in the numerators of $\langle v_i, v_i \rangle$. It is clear that if $(s(A), s(B), s(C))$ lies in $S'((1, m - 1), (1^m), (1^m))$, then the form $[2.4]$ is well-defined. It is clear that if $(s(A), s(B), s(C))$ lies in $S''((1, m - 1), (1^m), (1^m))$, then the form $[2.4]$ is non-degenerate.

**Theorem 2.2** The operators $A$, $B$, and $C$ are self-adjoint with respect to the scalar product $[2.4]$.

Now suppose that the eigenvalues of the matrices $A$, $B$, and $C$ are real numbers. Let $b_1 > b_2 > \cdots > b_m$ and $c_1 > c_2 > \cdots > c_m$. We call a real symmetric bilinear (or hermitian) form *sign-definite* if it is either positive-definite or negative-definite.
Theorem 2.3 Let the form $\langle \ast, \ast \rangle$ be defined by (2.4). Then it is sign-definite precisely in the following two situations:

$$
\begin{array}{c|c|c}
 b_m + c_1 & a_2 & b_m + c_2 \\
 b_{m-1} + c_2 & a_2 & b_{m-1} + c_3 \\
 \vdots & \vdots & \vdots \\
 b_2 + c_{m-1} & a_2 & b_2 + c_m \\
 b_1 + c_m & a_2 & b_1 + c_m \\
\end{array}
\begin{array}{c|c|c}
 b_1 + c_{m-1} & a_2 & b_1 + c_m \\
 b_2 + c_{m-2} & a_2 & b_2 + c_{m-1} \\
 \vdots & \vdots & \vdots \\
 b_{m-1} + c_1 & a_2 & b_{m-1} + c_2 \\
 a_2 & b_m + c_1 & \\
\end{array}
$$

(2.5)

If the form $\epsilon \langle \ast, \ast \rangle$ is positive-definite for $\epsilon = \pm 1$, then $\epsilon = \text{sign}(a_1 - a_2)$. If the inequalities of the first column hold, then $a_1 > a_2$. If the inequalities of the second column hold, then $a_1 < a_2$.

2.2 Even Family

Let us pick a vector $(a_1 \cdots, a_1, a_2, \cdots, a_2, b_1, b_2, \cdots, b_2, b_3, \cdots, b_3, c_1, \cdots, c_{2m})$ from $S((m, m), (1, m - 1, m), (1^{2m}))$. Recall that this means $a_1 \neq a_2$, all $b_i$ are distinct, all $c_j$ are distinct, and the trace condition holds: $ma_1 + ma_2 = b_1 + (m - 1)b_2 + mb_3 + \sum_{i=1}^{2m} c_i$. Let us set up the following notation:

$$
p_{jk} = c_i + b_j - a_k, \quad q_{ij} = c_i + c_j + b_2 + b_3 - a_1 - a_2.
$$

(2.6)

We now define the matrices $B$ and $C$ by setting

$$
\begin{array}{cccc}
1 & m-1 & m \\
1 & b_1 & B_{1,1+j} & B_{1,m+j} \\
\vdots & \vdots & \vdots & \vdots \\
m-1 & 0 & b_2 I_{m-1} & B_{1+i,m+j} \\
m & 0 & 0 & b_3 I_m \\
\end{array}
$$

, where

$$
B_{1,1+j} = (-1)^{m+1-j} \frac{\prod_{k=j+1}^{m} q_{k,2m-j}}{\prod_{k=m+1}^{2m-1-j} (c_k - c_{2m-j})} \quad (1 \leq j \leq m - 1);
$$

(2.7)
\[ B_{1,m+j} = (-1)^{m-j} p_{m+1-j}^{31} \prod_{k=m+1}^{2m-1} q_{m+1-j,k} \prod_{k=1}^{m-j} (c_k - c_{m+1-j}) \quad (1 \leq j \leq m); \]  
(2.8)

\[ B_{1+i,m+j} = (-1)^{m-j} p_{m+1-j}^{31} \prod_{k=m+1}^{2m-1} q_{k,2m-i} \prod_{k=1}^{2m-1} q_{m+1-j,k} \prod_{k=1}^{m-j} (c_{2m-i} - c_k) \prod_{k=1}^{m-j} (c_k - c_{m+1-j}) \]  
(2.9)

\[ (1 \leq i \leq m-1, \ 1 \leq j \leq m); \]

\[
\begin{array}{|c|c|c|}
\hline
& 1 & m-1 & m \\
\hline
1 & c_{2m} & 0 & 0 \\
\hline
m-1 & C_{1+i,1} & \begin{array}{c}
\vdots \\
0 \\
\vdots \\
c_{m+1}
\end{array} & 0 \\
\hline
m & C_{m+i,1} & \begin{array}{c}
\vdots \\
0 \\
\vdots \\
c_1
\end{array} & C_{m+i,1+j} \\
\hline
\end{array}
\]

, where

\[ C_{1+i,1} = - \prod_{k=1}^{i} q_{k,2m-i} \prod_{k=2m+1-i}^{2m-1} (c_{2m-i} - c_k) \quad (1 \leq i \leq m-1); \]  
(2.10)

\[ C_{m+i,1} = - p_{m+1-i}^{32} \prod_{k=m+1}^{2m+1-i} q_{m+1-i,k} \prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k) \quad (1 \leq i \leq m); \]  
(2.11)

\[ C_{m+i,1+j} = (-1)^{m+1-j} p_{m+1-i}^{32} \prod_{k=m+2-i}^{m} q_{k,2m-j} \prod_{k=m+1-j}^{m} q_{m+1-j,k} \prod_{k=1}^{m-j} (c_{m+1-j} - c_k) \prod_{k=1}^{m-j} (c_k - c_{m+1-j}) \]  
(2.12)

\[ (1 \leq i \leq m \ 1 \leq j \leq m-1). \]

Here is an example with \( m = 3 \).
Example 2.4

\[
\begin{pmatrix}
\begin{array}{c|ccc}
& b_1 & -\frac{q_{35} q_{35}}{c_4 - c_3} & q_{34} \\
\hline
0 & b_2 & 0 & 0 \\
0 & 0 & b_2 & 0 \\
0 & 0 & 0 & b_3 \\
0 & 0 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\begin{array}{llllll}
& c_6 & 0 & 0 & 0 & 0 \\
\hline
-\frac{q_{15} q_{34}}{c_4 - c_3} & c_5 & 0 & 0 & 0 \\
-\frac{q_{32} q_{24}}{c_2 - c_3} & c_4 & 0 & 0 & 0 \\
-\frac{p_{32} q_{31} q_{35}}{c_1 - c_2} & \frac{p_{31} q_{31} q_{34} q_{35}}{(c_1 - c_2)(c_2 - c_3)} & \frac{p_{31} q_{31} q_{34}}{c_1 - c_2} & \frac{p_{31} q_{31}}{c_1 - c_2} & \frac{p_{31}}{c_1 - c_2} & 0 \\
\end{array}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
\begin{array}{llllll}
& c_6 & 0 & 0 & 0 & 0 \\
\hline
-\frac{q_{15} q_{34}}{c_4 - c_3} & c_5 & 0 & 0 & 0 \\
-\frac{q_{32} q_{24}}{c_2 - c_3} & c_4 & 0 & 0 & 0 \\
-\frac{p_{32} q_{31} q_{35}}{c_1 - c_2} & \frac{p_{31} q_{31} q_{34} q_{35}}{(c_1 - c_2)(c_2 - c_3)} & \frac{p_{31} q_{31} q_{34}}{c_1 - c_2} & \frac{p_{31} q_{31}}{c_1 - c_2} & \frac{p_{31}}{c_1 - c_2} & 0 \\
\end{array}
\end{pmatrix}
\]

It is clear that \( B \) and \( C \) are diagonalizable and that their spectra are 
\[ s(B) = \{b_1, b_2, \ldots, b_2, b_3, \ldots, b_3, b_3\}, \quad s(C) = \{c_1, \ldots, c_{2m}\}. \]

**Theorem 2.4** If \( B \) and \( C \) are as above, then \( A = B + C \) is diagonalizable and 
\[ s(A) = \{a_1, \ldots, a_1, a_2, \ldots, a_2\}. \]

For \( i = 1, \ldots, 2m \), let \( v_i = (0, \ldots, 0, 1, v_i^{2m+2-i}, \ldots, v_i^{2m}) \) be the eigenvector of the matrix \( C \) with the eigenvalue \( c_i \).

**Lemma 2.2**

1. For \( 1 \leq i \leq m \), we have \( v_i^j = 0 \) for all \( 2m + 1 - i < j \).

2. For \( 1 \leq i \leq m - 1 \), we have \( v_{m+i}^j = 0 \) when \( m + 1 - i < j \leq m \) and
   \[
   v_{m+i}^{m+j} = (-1)^i \frac{p_{32}^{m+1-j}}{(c_{m+1-j} - c_{m+i}) \prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k) \prod_{k=m+1}^{m-1+i} (c_k - c_{m+i})} \prod_{k=m+1-j}^{m-1+i} (q_{k,m+i}^{m+1-j})
   \]
   for \( 1 \leq j \leq m \).
3. For $1 \leq j \leq m - 1$ we have 

$$v_{2m}^{1+j} = \prod_{k=1}^{j} q_{k,2m-j} \prod_{k=2m+1-j}^{2m} (c_{2m-j} - c_k)$$

and for $1 \leq j \leq m$ we have 

$$v_{2m}^{m+j} = (-1)^{m+1} \prod_{k=m+1-j}^{m} q_{k,2m-j} \prod_{k=m+1}^{m-j} (c_k - c_{2m}) \prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k)$$

(2.13)

Here is an example with $m = 3$ ($e_1, \cdots, e_{2m}$ is the standard basis of $V$).

**Example 2.5**

$v_1 = e_6, \ v_2 = e_5, \ v_3 = e_4,$

$v_4 = \left(0, 0, 1, -\frac{p_3^2}{c_3-c_4}, -\frac{p_2^2 q_24}{(c_2-c_3)(c_2-c_4)}, -\frac{p_1^2 q_15 q_{34}}{(c_1-c_2)(c_1-c_3)(c_1-c_4)} \right),$

$v_5 = \left(0, 1, 0, -\frac{p_3^2 q_{25}}{(c_3-c_5)(c_4-c_6)}, -\frac{p_2^2 q_{24} q_{25}}{(c_2-c_3)(c_2-c_5)(c_4-c_6)}, -\frac{p_1^2 q_{14} q_{25} q_{35}}{(c_1-c_2)(c_1-c_3)(c_1-c_5)(c_4-c_6)} \right),$

$v_6 = \left(1, -\frac{q_{15}}{c_5-c_6}, -\frac{q_{14} q_{24}}{(c_4-c_5)(c_4-c_6)}, -\frac{p_3^2 q_{24} q_{26}}{(c_3-c_4)(c_4-c_5)(c_5-c_6)}, -\frac{p_2^2 q_{24} q_{16} q_{36}}{(c_2-c_3)(c_2-c_4)(c_4-c_5)(c_5-c_6)} \right).$

We define a scalar product on $V$ by setting 

$$\langle v_i, v_j \rangle = \delta_{ij} \prod_{k=1}^{i-1} (c_i - c_k) \prod_{k=i+1}^{2m} (c_i - c_k) \prod_{k=1}^{2m-i} q_{ik} \prod_{k=i+1}^{m} q_{ik} \times \begin{cases} \frac{p_1^3}{p_i^2} q_{16}, & \text{if } i \leq m; \\ \frac{p_i^3}{p_1^2} q_{16}, & \text{if } i > m. \end{cases}$$

(2.14)

Here is an example with $m = 3$.

**Example 2.6**

$$\langle v_1, v_1 \rangle = \frac{(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6)}{c_2 - c_1} \times \frac{p_1^3}{p_1^2},$$

$$\langle v_2, v_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_5)(c_2 - c_6)}{c_2 - c_1} \times \frac{p_3^3}{p_2^2},$$

$$\langle v_3, v_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)}{c_3 - c_1} \times \frac{p_3^3}{p_3^2},$$

$$\langle v_4, v_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)}{c_4 - c_1} \times \frac{p_4^3}{p_4^2},$$

$$\langle v_5, v_5 \rangle = \frac{c_5 - c_6}{c_5 - c_1} \times \frac{p_5^3}{p_5^2},$$

$$\langle v_6, v_6 \rangle = \frac{1}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)} \times \frac{p_6^3}{p_6^2}.$$
The sets $S'((m, m), (1, m - 1, m), (1^{2m}))$ and $S''((m, m), (1, m - 1, m), (1^{2m}))$ are constructed from (2.14) similarly to the hypergeometric case (see page 7) and have the same properties.

**Theorem 2.5** The operators $A$, $B$, and $C$ are self-adjoint with respect to the scalar product (2.14).

Now suppose that the eigenvalues of the matrices $A$, $B$, and $C$ are real numbers. Let $a_1 > a_2$ and $c_1 > c_2 > \cdots > c_{2m}$.

**Theorem 2.6** The form $(\ast, \ast)$ defined by (2.14) is sign-definite precisely in the following six situations:

| Condition 1 | Condition 2 | Condition 3 |
|-------------|-------------|-------------|
| $b_1 > b_3 > b_2$ | $b_1 > b_2 > b_3$ | $b_2 > b_1 > b_3$ |
| $p_{m-1}^{31} > 0 > p_{m}^{31}$ | $0 > p_{m}^{31}$ | $0 > p_{m+1}^{31}$ |
| $p_{2m-1}^{32} > 0 > p_{2m}^{32}$ | $p_{m}^{32} > 0 > p_{m+1}^{32}$ | $q_{1,2m-1} > 0 > q_{1,2m}$ |
| $q_{1,2m-2} > 0 > q_{1,2m-1}$ | $q_{2,2m-2} > 0 > q_{2,2m-1}$ | $q_{2,2m-1} > 0 > q_{3,2m-1}$ |
| $q_{3,2m-4} > 0 > q_{3,2m-3}$ | $q_{m-1,m+1} > 0 > q_{m-1,m+2}$ | $q_{m-1,m+2} > 0 > q_{m,m+2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $q_{m-1,m} > 0 > q_{m-1,m+1}$ | $q_{m-1,m} > 0 > q_{m,m+1}$ | $q_{m,m+1} > 0$ |

| Condition 4 | Condition 5 | Condition 6 |
|-------------|-------------|-------------|
| $b_2 > b_3 > b_1$ | $b_3 > b_2 > b_1$ | $b_3 > b_1 > b_2$ |
| $p_{m+1}^{31} > 0 > p_{m}^{31}$ | $p_{m}^{31} > 0 > p_{m+1}^{31}$ | $p_{m}^{31} > 0 > p_{m+1}^{31}$ |
| $p_{m+2}^{32} > 0 > p_{m}^{32}$ | $p_{m}^{32} > 0 > p_{m+2}^{32}$ | $p_{m}^{32} > 0 > p_{m+2}^{32}$ |
| $q_{2,2m} > 0 > q_{3,2m}$ | $q_{2,2m-1} > 0 > q_{3,2m-1}$ | $q_{2,2m-1} > 0 > q_{1,2m}$ |
| $q_{3,2m-1} > 0 > q_{4,2m-1}$ | $q_{4,2m-2} > 0 > q_{5,2m-2}$ | $q_{3,2m-1} > 0 > q_{1,2m}$ |
| $q_{4,2m-2} > 0 > q_{5,2m-2}$ | $\vdots$ | $q_{2,2m-2} > 0 > q_{2,2m-1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $q_{m,m+2} > 0 > q_{m+1,m+2}$ | $q_{m,m+2} > 0 > q_{m,m+2}$ | $q_{m,m+1} > 0 > q_{m-1,m+2}$ |
| $q_{m,m+1} > 0$ | $q_{m,m+1} > 0$ | $0 > q_{m,m+1}$ |

If the form $\epsilon(\ast, \ast)$ is positive-definite for $\epsilon = \pm 1$, then $\epsilon = \text{sign}((b_1 - b_2)(b_1 - b_3))$. In each case, the inequalities between $b_1$, $b_2$, and $b_3$ are implied by other inequalities.

### 2.3 Odd Family

Let us pick a vector $(\underbrace{a_1, \ldots, a_1}_{m+1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{m \text{ times}}, b_1, \underbrace{b_2, \ldots, b_2}_{m \text{ times}}, \underbrace{b_3, \ldots, b_3}_{m \text{ times}}, c_1, \ldots, c_{2m+1})$ from $S((m + 1, m), (1, m, m), (1^{2m+1}))$. Recall that this means $a_1 \neq a_2$, all $b_i$ are distinct, all $c_j$ are distinct, and the trace condition holds: $(m + 1)a_1 + ma_2 = b_1 + mb_2 + mb_3 + \sum_{i=1}^{2m+1} c_i$.

We now define the matrices $B$ and $C$ by setting
\[
B = \begin{bmatrix}
1 & b_1 & B_{1,1+j} & B_{1,m+1+j} \\
m & 0 & b_2 \text{Id}_m & B_{1+i,m+1+j} \\
m & 0 & 0 & b_3 \text{Id}_m
\end{bmatrix}, \text{where}
\]

\[
B_{1,1+j} = (-1)^{m-j} p_{2m+1-j}^{21} \prod_{k=j+1}^{m} q_{k,2m+1-j}^{m} \prod_{k=m+1}^{2m-j} (c_k - c_{2m+1-j}) (1 \leq j \leq m); \quad (2.15)
\]

\[
B_{1,m+1+j} = (-1)^{m-j} p_{m+1-j}^{31} \prod_{k=m+1+1}^{2m} q_{m+1-j,k}^{2m} \prod_{k=m+1}^{m-j} (c_k - c_{m+1-j}) (1 \leq j \leq m); \quad (2.16)
\]

\[
B_{1+i,m+1+j} = (-1)^{m-j} p_{m+1-j}^{31} \prod_{k=2m+2-i}^{i} q_{k,2m+1-i}^{i} \prod_{k=m+1+1}^{2m} q_{m+1-j,k}^{m} \prod_{k=2m+1-i}^{m-j} (c_{2m+1-i} - c_k) \prod_{k=1}^{m-j} (c_k - c_{m+1-j}) (1 \leq i \leq m, \quad 1 \leq j \leq m); \quad (2.17)
\]
Here is an example with $m = 3$. 

\[
C = \begin{pmatrix}
1 & c_{2m+1} & 0 & 0 \\
0 & C_{1+i,1} & 0 & 0 \\
0 & 0 & c_{m+1} & 0 \\
c_{m+1+i,1} & C_{m+1+i,1+j} & c_m & 0 \\
0 & 0 & 0 & c_1
\end{pmatrix},
\]

where

\[
C_{1+i,1} = -\frac{\prod_{k=1}^{i} q_{k,2m+1-i}}{\prod_{k=2m+2-i}^{2m} (c_{2m+1-i} - c_k)} \quad (1 \leq i \leq m); \quad (2.18)
\]

\[
C_{m+1+i,1} = -\frac{\prod_{k=m+1}^{m+i} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k)} \quad (1 \leq i \leq m); \quad (2.19)
\]

\[
C_{m+1+i,1+j} = (-1)^{m-j} p_{2m+1-j}^{21} \frac{\prod_{k=m+1}^{m+i} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k)} \frac{\prod_{k=m+1}^{m} q_{k,2m+1-j}}{\prod_{k=m+2-i}^{m} (c_k - c_{2m+1-j})} \quad (1 \leq i \leq m, \quad 1 \leq j \leq m). \quad (2.20)
\]
Example 2.7

\[
B = \begin{bmatrix}
0 & b_2 & 0 & 0 & 0 & b_2 & 0 & b_2 & 0 & b_2 & 0 \\
0 & 0 & b_2 & 0 & p_2 & 0 & 0 & 0 & p_2 & 0 & 0 \\
0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_3 \\
b_1 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & &
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
c_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{16} & c_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{15} q_{35} c_5 & 0 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{14} q_{24} q_{35} c_4 & 0 & 0 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{34} c_2 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{24} q_{25} c_2 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{14} q_{15} q_{34} c_1 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{14} q_{15} q_{24} q_{35} c_1 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{14} q_{15} q_{24} q_{36} c_1 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 \\
-q_{14} q_{15} q_{36} c_1 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 \\
-q_{34} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 \\
& & & & & & & & & & &
\end{bmatrix}
\]

It is clear that \(B\) and \(C\) are diagonalizable and that their spectra are \(s(B) = \{b_1, b_2, b_2, \ldots, b_2, b_2, b_3, \ldots, b_3\}\), \(s(C) = \{c_1, c_2, \ldots, c_{2m+1}\}\).

**Theorem 2.7** Let \(B\) and \(C\) be as above and let \(A = B + C\). Then \(A\) is diagonalizable and \(s(A) = \{a_1, \ldots, a_1 a_2, \ldots, a_2\}\).

For \(i = 1, \ldots, 2m+1\), let \(v_i = (0, \ldots, 0, 1, v_i^{2m+3-j}, \ldots, v_i^{2m+1})\) be the eigenvector of the matrix \(C\) with the eigenvalue \(c_i\).

**Lemma 2.3** 1. For \(1 \leq i \leq m\), we have \(v_i^j = 0\) for all \(2m + 2 - i < j\).
2. For $1 \leq i \leq m$, we have $v^j_{m+i} = 0$ when $2m + 2 - i < j \leq m + 1$ and 
\[
v^m_{m+i} = \frac{(-1)^i m! q^j_{m+i}}{c_{m+1-j} - c_{m+i}} \prod_{k=m+1}^{m+j} (c_{m+1-j} - k) \prod_{k=m+2}^{m+j} (c_{m+1-j} - k) \prod_{k=m+1}^{m+j} (c_k - c_{m+i})
\]
for $1 \leq j \leq m$.

3. For $1 \leq i \leq m$, we have $v^{1+i}_{2m+1} = \frac{\prod_{k=1}^{i} q_{2m+1-i}}{m+1} (c_{2m+1-i} - c_k)$ and for $1 \leq j \leq m$, we have 
\[
v^{m+1+j}_{2m+1} = \frac{(-1)^m m! q^{1+i}_{2m+1}}{c_{m+1-j} - c_{2m+1}} \prod_{k=m+1}^{m+1+j} (c_{m+1-j} - k) \prod_{k=m+2}^{m+1+j} (c_{m+1-j} - k)
\]
and 
\[
\prod_{k=m+1}^{m+1+j} (c_k - c_{2m+1}) \prod_{k=m+1}^{m+1+j} (c_{m+1-j} - c_k)
\]

(2.21)

Here is an example with $m = 3$.

**Example 2.8**

$v_1 = e_7, \ v_2 = e_6, \ v_3 = e_5,$

$v_4 = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0 \\ -\frac{p^2_{21}}{c_3 - c_1} \\ -\frac{p^2_{21}}{c_2 - c_1} (c_2 - c_4) \\ -\frac{p^2_{21}}{c_1 - c_3} (c_1 - c_3) (c_1 - c_4) \\ \end{pmatrix},$

$v_5 = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0 \\ -\frac{p^2_{21}}{c_3 - c_1} (c_3 - c_5) (c_4 - c_5) \\ -\frac{p^2_{21}}{c_2 - c_1} (c_2 - c_5) (c_2 - c_5) (c_4 - c_5) \\ -\frac{p^2_{21}}{c_1 - c_5} (c_1 - c_5) (c_1 - c_5) (c_4 - c_5) \\ \end{pmatrix},$

$v_6 = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0 \\ -\frac{p^2_{21}}{c_3 - c_1} (c_3 - c_5) (c_4 - c_6) (c_5 - c_6) \\ -\frac{p^2_{21}}{c_2 - c_1} (c_2 - c_5) (c_2 - c_6) (c_4 - c_6) (c_5 - c_6) \\ -\frac{p^2_{21}}{c_1 - c_5} (c_1 - c_5) (c_1 - c_6) (c_4 - c_6) (c_5 - c_6) \\ \end{pmatrix},$

$v_7 = \begin{pmatrix} 1, 0, 0, 0, 0, 0, 0 \\ \frac{q_{16}}{c_3 - c_1} (c_3 - c_7) (c_4 - c_7) (c_5 - c_7) (c_6 - c_7) \\ \frac{q_{16}}{c_2 - c_1} (c_2 - c_7) (c_2 - c_7) (c_4 - c_7) (c_5 - c_7) (c_6 - c_7) \\ \frac{q_{16}}{c_1 - c_7} (c_1 - c_7) (c_1 - c_7) (c_4 - c_7) (c_5 - c_7) (c_6 - c_7) \\ \end{pmatrix}.$

We define a scalar product on $V$ by setting 
\[
\langle v_i, v_j \rangle = \delta_{ij} \frac{2m+1}{\prod_{k=1}^{2m+1} (c_i - c_k)} \prod_{k=2m+2}^{2m+1} q_{ik} \times \begin{cases} \frac{p^2_{31}}{p^2_{i1}}, & \text{if } i \leq m; \\ \prod_{k=1}^{2m+1-i} q_{ik} \times \frac{p^3_{31}}{p^3_{i1}} \frac{p^{21}_{31}}{p^{21}_{i1}}, & \text{if } i > m. \end{cases}
\]

(2.22)

Here is an example with $m = 3$. 

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Example 2.9

\[ \langle v_1, v_1 \rangle = (c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6)(c_1 - c_7) \times \frac{p_1^{31}}{P_1} \times q_1 \]

\[ \langle v_2, v_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_5)(c_2 - c_6)}{c_2 - c_1} \times \frac{p_2^{31}}{P_2} \times q_2 \]

\[ \langle v_3, v_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)}{(c_3 - c_1)(c_3 - c_2)} \times \frac{p_3^{31}}{P_3} \times q_3 \]

\[ \langle v_4, v_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)}{(c_4 - c_1)(c_4 - c_2)} \times \frac{p_4^{31}}{P_4} \times q_4 \]

\[ \langle v_5, v_5 \rangle = \frac{(c_5 - c_6)}{(c_5 - c_1)(c_5 - c_2)(c_5 - c_3)(c_5 - c_4)} \times \frac{p_5^{31}}{P_5} \times q_5 \]

\[ \langle v_6, v_6 \rangle = \frac{c_6 - c_7}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)} \times \frac{p_6^{31}}{P_6} \times q_6 \]

\[ \langle v_7, v_7 \rangle = \frac{1}{(c_7 - c_1)(c_7 - c_2)(c_7 - c_3)(c_7 - c_4)(c_7 - c_5)(c_7 - c_6)} \times \frac{p_7^{31}}{P_7} \times q_7 \]

The sets \( S'( (m + 1, m), (1, m, m) \), \( (1, m, m) \) \) and \( S'' ( (m + 1, m), (1, m, m) \), \( (1^{2m+1}) \) \) are constructed from \( 2.22 \) similarly to the hypergeometric case (see page 5) and have the same properties.

**Theorem 2.8** The operators \( A, B, \) and \( C \) are self-adjoint with respect to the scalar product \( 2.22 \).

Now suppose that the eigenvalues of the matrices \( A, B, \) and \( C \) are real numbers. Let \( b_2 > b_3 \) and \( c_1 > c_2 > \cdots > c_{2m+1} \).

**Theorem 2.9** Under the condition \( b_2 > b_3 \), the form \( \langle *, * \rangle \) defined by \( 2.22 \) is sign-definite precisely in the following three situations:

| \( b_1 > b_2 > b_3 \) | \( b_2 > b_1 > b_3 \) | \( b_2 > b_3 > b_1 \) |
|--------------------------|--------------------------|--------------------------|
| \( 0 > p_1^{31} \) | \( 0 > p_2^{31} \) | \( 0 > p_1^{31} \) |
| \( p_2^{21} > 0 > p_2^{21} \) | \( p_2^{21} > 0 > p_2^{21} \) | \( p_2^{21} > 0 > p_2^{21} \) |
| \( q_{1,2m} > 0 > q_{1,2m+1} \) | \( q_{1,2m+1} > 0 > q_{2,2m+1} \) | \( q_{2,2m+1} > 0 > q_{3,2m+1} \) |
| \( q_{2,2m-1} > 0 > q_{2,2m} \) | \( q_{2,2m} > 0 > q_{3,2m} \) | \( q_{3,2m} > 0 > q_{4,2m} \) |
| \( q_{3,2m-2} > 0 > q_{3,2m-1} \) | \( \vdots \) | \( \vdots \) |
| \( q_{m, m+1} > 0 > q_{m, m+2} \) | \( \vdots \) | \( \vdots \) |

If the form \( \epsilon \langle *, * \rangle \) is positive-definite for \( \epsilon = \pm 1 \), then \( \epsilon = \text{sign}((b_1 - b_2)(b_1 - b_3)) \). In each case, the inequalities between \( b_1 \) and \( b_2 \) or \( b_3 \) are implied by other inequalities.

**2.4 Extra Case**

Let us pick a vector \( (a_1, a_1, a_1, a_2, a_2, b_1, b_1, b_2, b_2, b_3, b_3, c_1, c_1, c_2, c_3, c_4, c_5, c_6) \) from \( S((4, 2), (2, 2, 2), (1^6)) \). Recall that this means \( a_1 \neq a_2 \), all \( b_i \) are distinct, all \( c_j \) are distinct, and the trace condition holds:
$4a_1 + 2a_2 = 2b_1 + 2b_2 + 2b_3 + \sum_{i=1}^{6} c_i$. Let us set up the following notation:

$$p_{ij} = b_i + c_j - a_1, \quad q_{ijk} = 2c_i + 2c_j + 2c_k - \frac{1}{2} \sum_{i=1}^{6} c_i.$$  \hfill (2.23)

We now define the matrices $B$ and $C$ by setting

$$B = \begin{bmatrix}
    b_1 & 0 & -\frac{p_{14} q_{245}}{c_3-c_4} & p_{16} & -\frac{p_{16} q_{245}}{c_1-c_2} & p_{16} \\
    0 & b_1 & \frac{p_{13} q_{235} q_{246}}{(c_3-c_4)(c_3-c_6)} & -\frac{p_{15} q_{236}}{c_3-c_6} & -\frac{p_{15} q_{236} q_{246}}{(c_1-c_2)(c_3-c_6)} & -\frac{p_{15} q_{236} q_{246}}{c_3-c_6} \\
    0 & 0 & b_2 & 0 & -\frac{p_{24} q_{246}}{c_1-c_2} & p_{24} \\
    0 & 0 & 0 & b_2 & -\frac{p_{24} q_{246} q_{246}}{(c_1-c_2)(c_3-c_4)} & p_{23} q_{246} \\
    0 & 0 & 0 & 0 & b_3 & 0 \\
    0 & 0 & 0 & 0 & 0 & b_3
\end{bmatrix}$$  \hfill (2.24)

and

$$C = \begin{bmatrix}
    c_6 & 0 & 0 & 0 & 0 & 0 \\
    0 & c_5 & 0 & 0 & 0 & 0 \\
    -\frac{p_{23} q_{235} q_{246}}{c_3-c_6} & -p_{24} & c_4 & 0 & 0 & 0 \\
    -\frac{p_{23} q_{235} q_{246}}{(c_3-c_4)(c_5-c_6)} & -\frac{p_{23} q_{235}}{c_3-c_4} & 0 & c_3 & 0 & 0 \\
    -\frac{p_{23} q_{235} q_{246}}{(c_1-c_2)(c_3-c_6)} & -p_{32} & \frac{p_{32} q_{235}}{c_1-c_4} & -p_{32} & c_2 & 0 \\
    -\frac{p_{31} q_{235} q_{246}}{(c_1-c_2)(c_3-c_4)} & \frac{p_{31} q_{235} q_{246}}{c_1-c_2} & \frac{p_{31} q_{235} q_{246}}{(c_1-c_2)(c_3-c_4)} & -\frac{p_{31} q_{235} q_{246}}{c_1-c_2} & 0 & c_1
\end{bmatrix}.$$  \hfill

It is clear that $B$ and $C$ are diagonalizable and that their spectra are $s(B) = \{b_1, b_1, b_2, b_2, b_3, b_3\}$, $s(C) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$.

**Theorem 2.10** For $B$ and $C$ as above, let $A = B + C$. Then $A$ is diagonalizable and 
$s(A) = \{a_1, a_1, a_1, a_2, a_2, a_2\}$.

**Lemma 2.4** The following are the eigenvectors of the matrix $C$ ($v_i$ corresponds to the eigenvalue $c_i$):
\[ \mathbf{v}_1 = (0, 0, 0, 0, 0, 1) \]
\[ \mathbf{v}_2 = (0, 0, 0, 0, 1, 0) \]
\[ \mathbf{v}_3 = \left(0, 0, 1, \frac{p_{32}}{q_{32}}, \frac{p_{31}}{q_{31}}, \frac{p_{36}}{q_{36}} \right) \]
\[ \mathbf{v}_4 = \left(0, 0, 1, 0, \frac{p_{32} q_{235}}{(c_2 - c_4)(c_1 - c_4)}, \frac{p_{31} q_{246}}{(c_1 - c_2)(c_1 - c_4)(c_3 - c_4)} \right) \]
\[ \mathbf{v}_5 = \left(0, 1, \frac{p_{32}}{c_4 - c_5}, \frac{p_{23} q_{245}}{(c_3 - c_4)(c_3 - c_5)}, \frac{p_{25} p_{32} q_{234}}{(c_2 - c_5)(c_3 - c_5)(c_4 - c_5)}, \frac{p_{25} p_{31} q_{245} q_{256}}{(c_1 - c_2)(c_1 - c_5)(c_3 - c_5)(c_4 - c_5)} \right) \]
\[ \mathbf{v}_6 = \left(1, 0, \frac{p_{24} q_{235}}{(c_4 - c_5)(c_3 - c_5)}, \frac{p_{26} p_{25} q_{231} q_{235}}{(c_2 - c_6)(c_3 - c_6)(c_4 - c_6)(c_5 - c_6)} \right) \]

Let us define a scalar product on \( V \) by setting \( \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \) for \( i \neq j \) and setting

\[ \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = -\frac{(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6)}{p_{31} p_{32} p_{31}} \]
\[ \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_6)(c_3 - c_5)}{(c_2 - c_1)p_{12} p_{22} p_{32}} \times q_{134} q_{125} q_{234} q_{145} q_{146} q_{156} \]
\[ \langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)}{(c_3 - c_1)p_{13} p_{23}} \times q_{124} q_{125} q_{126} q_{145} q_{146} q_{156} \]
\[ \langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)}{(c_4 - c_1)(c_4 - c_2)(c_4 - c_3)p_{44}} \times q_{123} q_{124} q_{125} q_{134} q_{135} q_{145} q_{146} \]
\[ \langle \mathbf{v}_5, \mathbf{v}_5 \rangle = \frac{(c_5 - c_6)p_{55} p_{35}}{(c_5 - c_1)(c_5 - c_2)(c_5 - c_3)(c_5 - c_4)p_{15}} \times q_{123} q_{124} q_{125} q_{134} q_{135} q_{145} q_{146} \]
\[ \langle \mathbf{v}_6, \mathbf{v}_6 \rangle = \frac{p_{66}}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)p_{16}} \times q_{123} q_{124} q_{125} q_{134} q_{135} q_{145} q_{146} \]

The sets \( S'((4, 2), (2, 2, 2), (1^6)) \) and \( S'((4, 2), (2, 2, 2), (1^6)) \) are constructed from (2.25) similarly to the hypergeometric case (see page 19) and have the same properties.

**Theorem 2.11** The operators \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \) are self-adjoint with respect to the scalar product (2.25).

Now suppose that the eigenvalues of the matrices \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \) are real numbers. Let \( b_1 > b_2 > b_3 \) and \( c_1 > c_2 > \cdots > c_6 \).

**Theorem 2.12** The form \( \langle *, * \rangle \) defined by (2.25) is sign-definite precisely in the following two situations:

| b_1 + c_4 > a_1 > b_1 + c_5 | b_3 + c_2 > a_1 > b_3 + c_3 |
| b_2 + c_2 > a_1 > b_2 + c_3 | b_2 + c_4 > a_1 > b_2 + c_5 |
| a_1 > b_1 + c_1 | b_1 + c_6 > a_1 |
| c_1 + c_4 + c_5 > c_2 + c_1 + c_6 | c_1 + c_4 + c_5 > c_2 + c_3 + c_6 |
| c_1 + c_3 + c_6 > c_2 + c_4 + c_5 | c_1 + c_3 + c_6 > c_2 + c_4 + c_5 |
| c_2 + c_3 + c_5 > c_1 + c_4 + c_6 | c_2 + c_3 + c_5 > c_1 + c_4 + c_6 |
If the form $\epsilon \langle \ast, \ast \rangle$ is positive-definite for $\epsilon = \pm 1$, then $\epsilon = \text{sign}(a_1 - a_2)$. If the inequalities of the first column hold, then $a_1 > a_2$. If the inequalities of the second column hold, then $a_1 < a_2$.

## 3 Proofs and More Results

In this section we prove theorems from Section 2. In the process, some new results are obtained.

The following simple observation is helpful in this section. If we replace a triple $(A = B + C, B, C)$ by the triple

\[
\tilde{A} = k A + \theta \text{Id}, \\
\tilde{B} = k B + \phi \text{Id}, \\
\tilde{C} = k C + (\theta - \phi) \text{Id}
\]

(3.26)

then we still have $\tilde{A} = \tilde{B} + \tilde{C}$. This transformation changes neither irreducibility nor rigidity of the triple. If, say, $v$ is an eigenvector of $B$ with the eigenvalue $b$, then $v$ is an eigenvector of $\tilde{B}$ with the eigenvalue $kb + \phi$. If $A, B,$ and $C$ were self-adjoint with respect to a symmetric bilinear form, then $\tilde{A}, \tilde{B},$ and $\tilde{C}$ are self-adjoint with respect to the form as well.

### 3.1 Hypergeometric Family

An affine transformation (3.26) with $k = 1$, $\theta = -a_2$, and $\phi = -a_2/2$ normalizes $A$ to $\tilde{A}$ such that the eigenvalue of $\tilde{A}$ of multiplicity $m - 1$ is 0. So, without loss of generality, we can assume that $a_2 = 0$. Now let us prove Theorem 2.1.

**Proof of Theorem 2.1** — Consider the matrix $A$ (with $a_2 = 0$). Here is an example with $m = 5$.

\[
A = \begin{bmatrix}
    b_1 + c_5 & b_1 + c_5 & b_1 + c_5 & b_1 + c_5 & b_1 + c_5 \\
    b_2 + c_4 & b_2 + c_4 & b_2 + c_4 & b_2 + c_4 & b_2 + c_4 \\
    b_3 + c_3 & b_3 + c_3 & b_3 + c_3 & b_3 + c_3 & b_3 + c_3 \\
    b_4 + c_2 & b_4 + c_2 & b_4 + c_2 & b_4 + c_2 & b_4 + c_2 \\
    b_5 + c_1 & b_5 + c_1 & b_5 + c_1 & b_5 + c_1 & b_5 + c_1
\end{bmatrix}
\]

Now $A$ has rank 1, and its image is the linear span of the vector $i = (b_1 + c_m, b_2 + c_{m-1}, \ldots, b_m + c_1)$. $A i = a_1 i \neq 0$. Thus, $A$ is diagonalizable, and $s(A) = \{ \sum_{i=1}^{m} (b_i + c_i), 0, 0, \ldots, 0 \}$. Thus, before the normalizing affine transformation we had $s(A) = \{ a_1, a_2, \ldots, a_2 \}$.

In our normalized version,

\[
B_{ij} = \begin{cases}
    0, & \text{if } i < j \\
    b_i, & \text{if } i = j \\
    b_i + c_{m+1-i}, & \text{if } i > j
\end{cases}, \quad C_{ij} = \begin{cases}
    0, & \text{if } i > j \\
    c_{m+1-i}, & \text{if } i = j \\
    b_i + c_{m+1-j}, & \text{if } i < j
\end{cases}
\]

(3.27)

We are ready to prove Lemma 2.1 that is to show that for every $i = 1, \ldots, m$ the vector $v_i = (v_i^1, \ldots, v_i^{i-1}, 1, 0, \ldots, 0)$ with

\[
v_j^i = \frac{b_i + c_{m+1-j}}{b_i - b_j} \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k}}{b_i - b_j+k} \quad (j = 1, \ldots, i-1)
\]

(3.28)
is an eigenvector of $B$ with the eigenvalue $b_i$.

**Proof of Lemma 2.1** — Remembering the definition (2.27) of $B$, we need to show the following equality for all $j < i$:

$$(b_i - b_j)v_i^j = (b_i + c_{m+1-j})\sum_{k=j+1}^i b_k^{i-k}.$$  \hspace{1cm} \text{(3.29)}

This identity becomes obvious once we rewrite (3.28) as

$$v_i^j = \prod_{k=1}^{i-j} \frac{b_i + c_{m+1-j-k}}{b_i - b_{j+k}}.$$  \hspace{1cm} \text{(3.30)}

and use telescoping.

Now we are ready to prove Theorem 2.2, that is to show that the operators $A$, $B$, and $C$ are self-adjoint with respect to the scalar product (2.4).

**Proof of Theorem 2.2** — The operator $B$ is self-adjoint with respect to the scalar product by construction. To show that $A$ is self-adjoint, we have to show that $\langle A v_i, v_j \rangle = \langle v_i, A v_j \rangle$. As we have seen, $A$ has a one-dimensional image spent by the vector $i = (b_1 + c_m, b_2 + c_{m-1}, \cdots, b_m + c_1)$. Namely, for any vector $k = (k_1, k_2, \cdots, k_m)$, $\langle A v_i, v_j \rangle = (\sum_{i=1}^m k_i) i$. In particular, $A v_i = \langle \sum_{i=1}^m v_i^j \rangle i$.

In view of (3.29), we have

$$A v_i = \left(\prod_{k=1}^{i-1} \frac{b_i + c_{m+1-k}}{b_i - b_k}\right) i.$$  \hspace{1cm} \text{(3.31)}

It will be convenient to introduce the following notation:

$$s_i = \prod_{k=1}^{i-1} \frac{b_i + c_{m+1-k}}{b_i - b_k}; \quad x_i = (b_i + c_{m+1-i}) \prod_{k=1}^{m-i} \frac{b_i + c_{m+1-i-k}}{b_i - b_i+k}.$$  \hspace{1cm} \text{(3.32)}

Then $A v_i = s_i i$ and (2.4) can be rewritten as $\langle v_i, v_i \rangle = s_i / x_i$. Now the desired equality $\langle A v_i, v_j \rangle = s_i x_j (\langle v_j, v_j \rangle = s_i x_i x_i) / x_j = s_i s_j = (\langle v_i, A v_j \rangle)$ becomes a consequence of the following lemma.

**Lemma 3.1** \(\sum_{i=1}^m x_i v_i = i\).

**Proof** — We have $v_i = (v_i^1, \cdots, v_i^{i-1}, 1, 0 \cdots, 0)$. Thus, to prove the lemma we have to prove that the identity $\sum_{j=i}^m x_j v_j^i = b_i + c_{m+1-i}$ holds for $1 \leq i \leq m$. This is equivalent to $x_i = b_i + c_{m+1-i} - \sum_{k=1}^{m-i} x_{i+k} v_{i+k}^i$. This identity after minor simplification becomes

$$\sum_{j=i}^m \prod_{k=1}^{m-j} \frac{b_j + c_k}{b_j - b_k} = 1.$$  \hspace{1cm} \text{(3.33)}

Let us set $n = m - i + 1; \quad x_1 = b_1, \quad x_2 = b_{i+1}, \cdots, x_n = b_m; \quad y_1 = -c_1, \quad y_2 = -c_2, \cdots, y_{n-1} = -c_{m-i}$. This change of variables transforms the last identity into identity (7.62) which we prove in the Appendix.
Now we prove Theorem 1.3 for the hypergeometric family. That is, we show that if the vector $(s(A), s(B), s(C))$ lies in $S''((1, m - 1), (1^m), (1^a))$, then the triple $(A, B, C)$ is irreducible.

**Proof of Theorem 1.3** — Suppose that the triple preserves a non-trivial subspace of $V$. Then this subspace is spanned by some of the eigenvectors $v_j$ of $B$. But $A v_j = \sum s_j x_j v_i$. If $(s(A), s(B), s(C))$ lies in $S''((1, m - 1), (1^m), (1^a))$, then all the coefficients $s_j x_j$ are non-zero, so $A$ does not preserve any such proper subspace. Thus, the triple $(A, B, C)$ is irreducible. □

Let us prove Theorem 2.3, that is determine the inequalities on the real spectra of $A$, $B$, and $C$ which make the form (2.4) sign-definite. Recall that we are working with the normalized version $C_2.3$ that this subspace is spanned by some of the eigenvectors $v$. Proof of Theorem 2.3 — It immediately follows from Theorem 2.3 that

$$\text{sign}((v_i, v_j)) = (-1)^{i-1} \text{sign} \left( \prod_{j=1}^{m} (b_i + c_j) \right).$$

Construct an $m \times m$ matrix $T$ where $T_{i,j} = b_i + c_j$. Notice that $T_{i,j} > T_{i,j+1}$ and $T_{i,j} > T_{i+1,j}$ for all $i$ and $j$. Then $\text{sign}((v_i, v_j)) = (-1)^{i-1} \times (-1)^{\#(j: T_{i,j} \leq 0)}$. Thus, to keep the sign constant, $\# \{ j : T_{i,j} < 0 \}$ must differ from $\# \{ j : T_{i+1,j} < 0 \}$ by an odd number for all $m$ rows of $T$. This gives us only two possibilities: either $T_{i,m-i} > 0 > T_{i,m+1-i}$, or $T_{i,m+1-i} > 0 > T_{i,m+2-i}$. Here is a picture which illustrates the two situations for $m = 5$. The line separates $T_{i,j} > 0$ from $T_{i,j} < 0$.

![Matrix Diagram](image_url)

The sum of $T_{i,j}$ along the non-main diagonal of $T$ equals $a_1$. Thus, $T_{i,m-i} > 0 > T_{i,m+1-i}$ forces $a_1 < 0$ and $T_{i,m+1-i} > 0 > T_{i,m+2-i}$ forces $a_1 > 0$. □

**Remark 3.1** Let the eigenvalues of $A$, $B$, and $C$ be real numbers, and let the form (2.4) be positive-definite. Then in the basis $e_i = v_i / \sqrt{(v_i, v_i)}$, the form becomes standard ($\langle e_i, e_j \rangle = \delta_{ij}$). Let $A$, $B$, and $C$ be the matrices $A$, $B$, and $C$ in the basis $e_1, e_2, \cdots, e_m$. Then for $i, j = 1, 2, \cdots, m$,

$$\tilde{A}_{ij} = \sqrt{T_{i,j}s_is_j},$$

$$\tilde{B}_{ij} = \delta_{ij} b_i,$$

$$\tilde{C}_{ij} = \tilde{A}_{ij} - \tilde{B}_{ij}.$$

### 3.2 Even Family

In order to make proofs simpler, let us normalize $A$, $B$, and $C$ so that they become traceless and $a_1 = 1, a_2 = -1$. The affine transformation (3.26) with $k = 2/(a_1 - a_2), \theta = -(a_1 + a_2)/2$, and $\phi = -(b_1 + (m - 1)b_2 + m b_3)/(2m)$ does the job.

Let us prove Theorem 2.4. In our normalized version, we have to prove that $A$ is diagonalizable and that $s(A) = \{1, \cdots, 1, -1, \cdots, -1\}$. 

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Proof of Theorem 2.4 — First, let us prove that $A^2 = \textbf{Id}$. For that, we have to prove the following eleven identities.

1. The identity $\sum_{i=1}^{2m} A_{1i} A_{1i} = 1$ with the help of identity (7.62) can be reduced to the identity

$$((m-1)b_2 + mb_3 + c_1 + c_2 + \cdots + c_{2m-1})^2 = \sum_{i=1}^{m-1} \prod_{j=1}^{m-1-i} (b_2 + b_3 + c_j + c_{m+i})$$

$$+ \sum_{i=1}^{m} \left( \frac{m-1}{i} \right) \prod_{j=1}^{m-1-i} (b_2 + b_3 + c_j + c_{m+i}) \prod_{j=1}^{m-i} (c_j - c_i).$$

For $1 \leq i \leq m-1$, let us set $x_i = c_{m+i} + b_2$. For $1 \leq i \leq m$, let us set $y_i = c_i + b_3$. Then the last identity becomes

$$\left( \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} y_i \right)^2 = \sum_{i=1}^{m-1} \prod_{j=1}^{m-i} (x_i + y_j) + \sum_{i=1}^{m} \prod_{j=1}^{m-i} (y_i - y_j) \quad (3.33)$$

Introducing $x_m = x_{m+1} = 0$, we can rewrite the last term in (3.33) as

$$\sum_{i=1}^{m} y_i^2 \prod_{j=1}^{m-i} (y_i - y_j) = \sum_{i=1}^{m} \prod_{j=1}^{m-i} (y_i - y_j).$$

Now we can prove identity (3.33) with the help of identity (7.64) from the Appendix.

2. For $1 \leq i \leq m-1$, the identity $\sum_{i=1}^{2m} A_{1i} A_{1i+i} = 0$ after some simplification becomes

$$\sum_{j=1}^{m} \frac{(b_3 + c_j)^2 - 1}{m} \prod_{k=1}^{2m-1} \frac{q_{j,k}}{c_j - c_k} = \sum_{j=1}^{m} (c_j + b_3) + \sum_{j=m+1}^{2m-1} (c_j + b_2).$$

For $1 \leq j \leq m$, set $x_j = b_3 + c_j$. For $1 \leq j \leq m-1$ and $j \neq m-i$, set $y_j = -(b_2 + c_{m+i})$.

The identity

$$\sum_{j=1}^{m} \prod_{k=1}^{m} \frac{q_{j,k}}{c_j - c_k} = 0$$
is equivalent to identity (7.61) from the Appendix. Now the identity to prove becomes

\[
\sum_{j=1}^{m} (b_3 + c_j)^2 \frac{\prod_{k=m+1}^{2m-1} q_{j,k}}{\prod_{k=1}^{m} (c_j - c_k)} = \sum_{j=1}^{m} (c_j + b_3) + \sum_{j=m+1}^{2m-1} (c_j + b_2).
\] (3.34)

Introducing \( y_{m-i} = y_m = 0 \), we reduce the last identity to identity (7.63) from the Appendix.

3. For \( 1 \leq j \leq m \), the identity \( \sum_{l=1}^{2m} A_{1,l} A_{l,m+j} = 0 \) reduces to identity (7.63) from the Appendix.

4. For \( 1 \leq i \leq m - 1 \), the identity \( \sum_{l=1}^{2m} A_{1+i,l} A_{l,1} = 0 \) reduces to identity (7.63) from the Appendix.

5. Let \( 1 \leq i, j \leq m - 1 \), \( i \neq j \). The identity \( \sum_{l=1}^{2m} A_{1+i,l} A_{l,1+j} = 0 \) reduces to identity (7.62) from the Appendix.

6. For \( 1 \leq i \leq m - 1 \), the identity \( \sum_{l=1}^{2m} A_{1+i,l} A_{l,1+i} = 1 \) after some simplification becomes

\[
\sum_{j=1}^{2m} p_{j}^{31} p_{j}^{32} \prod_{k=1}^{m} q_{k,i} \prod_{k=m+1}^{2m-1} q_{j,k} \prod_{k=1}^{m} (c_j - c_k) = 1 - (b_2 + c_i)^2 + \prod_{k=1}^{m} q_{k,i} \prod_{k=m+1}^{2m-1} (c_i - c_k).
\]

Recall that in the normalized version \( p_{j}^{31} = c_i + b_3 - 1 \) and \( p_{j}^{32} = c_i + b_3 + 1 \). For \( 1 \leq i \leq m \), let us set \( x_i = b_3 + c_i \). For \( 1 \leq i \leq m - 1 \), let us set \( y_i = -b_2 - c_{m+i} \). The above identity splits into two homogeneous identities: one of degree 0 and the other of 2 (in \( x_i \) and \( y_j \)). The first is equivalent to identity (7.63) from the Appendix. The second is equivalent to identity (7.66) from the Appendix.

7. For \( 1 \leq i \leq m - 1 \) and \( 1 \leq j \leq m \), the identity \( \sum_{l=1}^{2m} A_{1+i,l} A_{l,m+j} = 0 \) reduces to the trivial identity \( q_{2m-1,m+1-i} q_{2m-1,m+1-j} - 1 = 0 \).

8. For \( 1 \leq i \leq m \), the identity \( \sum_{l=1}^{2m} A_{m+j,l} A_{l,1} = 0 \) reduces to the identity

\[
\sum_{j=1}^{m-1} \prod_{k=m+1}^{2m-1} q_{k,2m-j} \prod_{k=m+1}^{2m-1} (c_{2m-j} - c_k) = -b_1 - b_3 - c_{m+1-i} - c_{2m}.
\]

The latter follows from identity (7.63) of the Appendix and from the fact that the normalized \( B \) and \( C \) are traceless.

9. Let \( 1 \leq i \leq m \) and \( 1 \leq j \leq m - 1 \). The identity \( \sum_{l=1}^{2m} A_{m+j,l} A_{l,1+i} = 0 \) reduces to the trivial identity \( q_{2m-j,m+1-i} q_{2m-j,m+1-j} - 1 = 0 \).

10. Let \( 1 \leq i \neq j \leq m \). The identity \( \sum_{l=1}^{2m} A_{m+j,l} A_{l,m+i} = 0 \) reduces to identity (7.62) from the Appendix.
11. Let $1 \leq i \leq m$. To prove that $\sum_{i=1}^{2m} A_{m+i,1} A_{1,m+i} = 1$, we set $x_1 = b_3 + c_1$, $x_2 = b_3 + c_2, \ldots, x_m = b_3 + c_m; \ y_1 = b_2 + c_{m+1}, y_2 = b_2 + c_{m+2}, \ldots, y_{m-1} = b_2 + c_{2m-1}$. This reduces the identity in question to identity (7.62) of the Appendix.

Now we are ready to prove that if $(s(A), s(B), s(C))$ is a point of $S''((m, m), (1, m-1, m), (12m))$, then $s(A) = \{1, \ldots, 1, -1, \ldots, -1\}$. We know that $A^2 = \text{Id}$. Thus, $A$ is diagonalizable and the eigenvalues of $A$ are $1$ and $-1$. For $1 \leq i \leq m$, let us set

$$a^+_i = (A + \text{Id}) e_{m+i},$$

$$a^-_i = (A - \text{Id}) e_{m+i}. \tag{3.35}$$

Then $(A - \text{Id}) a^+_i = (A - \text{Id}) (A + \text{Id}) e_{m+i} = 0$ and $(A + \text{Id}) a^-_i = (A + \text{Id}) (A - \text{Id}) e_{m+i} = 0$. If we take a look at the matrix $A$, we see that the condition $(s(A), s(B), s(C)) \in S''((m, m), (1, m-1, m), (12m))$ guarantees that the vectors $(a^+_i)_{i=1,\ldots,m}$ are linearly independent as well as the vectors $(a^-_i)_{i=1,\ldots,m}$. (If this condition is violated, then $a^+_i$ and $a^-_i$ are not necessarily linearly independent. For example, if $p_i^{31} = 0$, then $a^-_i = 0$.)

Let us prove Lemma 2.2 that computes the coordinates $v^j_i$ of the eigenvectors $v_i$ of the matrix $C$.

**Proof of Lemma 2.2** — The only non-trivial part of the lemma is formula (2.13). A direct computation gives

$$v^m_{2m} = \frac{1}{c_{m+1-i} - c_{2m}} \left[ -C_{m+i,1} + \sum_{j=1}^{m-1} \frac{C_{m+i,j+1} C_{j+1,1}}{c_{2m-j} - c_{2m}} \right],$$

where $C_{j+1,1}$ is given by (2.10), $C_{m+i,1}$ is given by (2.11), and $C_{m+i,j+1}$ is given by (2.12), see page 25. Comparing the formula for $v^m_{2m}$ given by (2.13) to the right hand side of the last formula, we obtain an identity which reduces to identity (7.62) from the Appendix.

The following lemma expresses the vectors $e_i$ of the standard basis in terms of the eigenvectors $v_j$ of the matrix $C$.

**Lemma 3.2**

1. $e_1 = v_{2m} - \sum_{i=1}^{m-1} \frac{\prod_{k=1}^{m-i} q_k, m+i}{\prod_{k=m+1+i}^{2m} (c_{m+i} - c_k)} v_{m+i} - \sum_{i=1}^{m} \frac{\prod_{k=1}^{2m-i} q_{ik}}{\prod_{k=1+i}^{2m} (c_i - c_k)} v_i$, \n
2. $e_{1+i} = v_{2m-i} + \sum_{j=1}^{m} (-1)^{m+1-i} \frac{p^3_j}{\prod_{k=m+1+i}^{2m-j} q_{j,k}} v_{j} - \sum_{k=1+j}^{m} \frac{\prod_{k=m+1+i}^{2m-1-i} q_{k,2m-i}}{\prod_{k=m+i}^{2m} (c_k - c_{2m-i})} v_j$, \n
3. $e_{m+i} = v_{m+i-1}$ for $i = 1, 2, \ldots, m$.

**Proof** — For $1 \leq i \leq m$, let

$$c^i = \sum_{i=1}^{m} \frac{\prod_{k=1+i}^{2m} q_{ik}}{\prod_{k=1+i}^{2m} (c_i - c_k)} \tag{3.36}$$
be the \( i \)-th coordinate of \( e_1 \) in the basis \( v_1 \). The formula \((3.36)\) is the only nontrivial part of the lemma. To prove it we have to show that

\[
e_i^j + v_{2m+1-i}^{2m+1-i} - \sum_{k=m+1}^{2m-1} v_k^{2m+1-i} v_{2m+1-k} = 0, \quad \text{where } i = 1, 2, \ldots, m.
\] (3.37)

Let us use the formulas for \( v_j^i \) given in Lemma 3.2 and the following change of variables: for a fixed \( 1 \leq i \leq m \), set \( x_0 = c_1 + b_3 \), \( x_1 = c_{m+1} + b_3 \), \( x_2 = c_{m+2} + b_3 \), \ldots, \( x_m = c_{2m+3} + b_3 \), \( y_1 = -b_2 - c_1 \), \( y_2 = -b_2 - c_2 \), \ldots, \( y_{i-1} = -b_2 - c_{i-1} \), \( y_i = -b_2 - c_{i+1} \), \ldots, \( y_{m-1} = -b_2 - c_m \). This reduces \((3.37)\) to identity \((3.61)\) from Appendix.

To prove Theorem 2.4, we have to compute eigenvectors of the matrix \( B \) (see page 8 for its description). Let us recall that \( B \) is diagonalizable. Let us call \( w_1 \) the eigenvector of \( B \) corresponding to the eigenvalue \( b_1 \), \( w_2, \ldots, w_m \) the eigenvectors of \( B \) corresponding to the eigenvalue \( b_2 \), and \( w_{m+1}, \ldots, w_{2m} \) the eigenvectors of \( B \) corresponding to the eigenvalue \( b_3 \).

**Lemma 3.3**

1. We have \( w_1 = e_1 \).

2. For \( 1 \leq i \leq m - 1 \), we have \( w_{1+i} = -\frac{B_1 + w_2 - b_3}{b_1 - b_2} e_1 + e_{1+i} \).

3. For \( 1 \leq i \leq m \), we have \( w_{m+i} = x^{m+i} e_1 - \sum_{j=1}^{m-1} \frac{B_{j+1,m+i}}{b_2 - b_3} e_{j+1} + e_{m+i} \), where

\[
x^{m+i} = \frac{(-1)^{m+1-i} p_{m+1-i}^{31} (b_1 + b_2 + c_{m+1-i} + c_{2m})}{(b_1 - b_3)(b_2 - b_3)} \prod_{k=m+i}^{2m-1} q_{m+1-i,k} \prod_{k=1}^{m-i} (c_k - c_{m+1-i}).
\] (3.38)

Here is an example with \( m = 3 \).

**Example 3.1**

\[
w_1 = (1, 0, 0, 0, 0, 0),
\]
\[
w_2 = \left( \frac{q_{25}}{(c_4 - c_5)(b_1 - b_2)}, 1, 0, 0, 0, 0 \right),
\]
\[
w_3 = \left( -\frac{q_{24}}{b_1 - b_2}, 0, 1, 0, 0, 0 \right),
\]
\[
w_4 = \left( -\frac{(b_1 + b_2 + c_5 + c_6)p_{31}^{24} q_{25}}{(c_4 - c_5)(b_1 - b_2)(c_1 - c_2)(c_2 - c_3)}, -\frac{p_{31}^{24} q_{25}}{(b_2 - b_3)(c_4 - c_5)}, 1, 0, 0 \right),
\]
\[
w_5 = \left( \frac{(b_1 + b_2 + c_5 + c_6)p_{31}^{24} q_{25}}{(b_1 - b_2)(b_2 - b_3)(c_1 - c_2)}, -\frac{p_{31}^{24} q_{25}}{(b_2 - b_3)(c_4 - c_5)}, 0, 1, 0 \right),
\]
\[
w_6 = \left( -\frac{(b_1 + b_2 + c_5 + c_6)p_{31}^{24} q_{25}}{(b_1 - b_2)(b_2 - b_3)}, -\frac{p_{31}^{24} q_{25}}{(b_2 - b_3)(c_4 - c_5)}, 0, 1 \right).
\]

**Proof of Lemma 3.3** — All the formulas in Lemma 3.3 are immediate except for \((3.38)\). A direct computation gives

\[
(b_1 - b_3)x^{m+i} + \sum_{j=1}^{m-1} (-1)^{m-j} \frac{m}{2m-1-j} \prod_{k=m+1}^{2m-1} q_{2m-j} B_{j+1,m+i} \frac{(-1)^{m-i} p_{m+1-i}^{31}}{(b_2 - b_3) \prod_{k=m+i}^{2m-1} (c_k - c_{m+1-i})} = 0,
\]
where $B_{1+j,m+i}$ is given by (2.3). To prove (3.38), we have to show that the formula we derive for $x^{m+i}$ from the equation above equals the formula for $x^{m+i}$ from (3.38). This boils down to a proof of identity (7.62) from the Appendix.

\[\square\]

Lemma 3.4 The following is the matrix of the scalar product (2.14) in the standard basis $e_1, \cdots, e_{2m}$. 

1. $\langle e_1, e_1 \rangle = (b_1 - b_2)(b_1 - b_3)$

2. $\langle e_1, e_{1+i} \rangle = (b_1 - b_3) \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)}$ for $i = 1, 2, \cdots, m - 1$

3. $\langle e_1, e_{m+i} \rangle = -p_{m+1-i}^{31} \frac{\prod_{k=m+1}^{m-i} q_{m+1-i,k}}{\prod_{k=1}^{m-j} (c_{m+1-i} - c_k)}$ for $i = 1, 2, \cdots, m$

4. $\langle e_{1+i}, e_{1+j} \rangle = \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \left\{ \frac{1 - (b_2 - b_3) \sum_{r=1}^{m} c_r}{\prod_{k=1}^{2m} q_{r,k}} \prod_{k=m+1}^{2m} (c_r - c_k) \right\}$

for $i, j = 1, 2, \cdots, m - 1$, $i \neq j$

5. $\langle e_{1+i}, e_{1+i} \rangle = \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \left\{ \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} - (b_2 - b_3) c_{2m-i} \right\}$

\[\frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} - (b_2 - b_3) \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \sum_{r=1}^{m} c_r \prod_{k=m+1}^{2m} q_{r,k} \prod_{k=m+1}^{2m} (c_r - c_k) \right\}$

6. $\langle e_{1+i}, e_{m+j} \rangle = \frac{p_{m+1-j}^{31}}{q_{m+1-j,2m-1}^{2m-1-i}} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \frac{\prod_{k=1}^{m-j} q_{m+1-j,k}}{\prod_{k=m+1}^{2m-1-i} (c_{m+1-j} - c_k)} \frac{\prod_{k=1}^{m} q_{m+1-j,k}}{\prod_{k=m+1}^{2m-1-j}}$ for $i = 1, 2, \cdots, m - 1$ and $j = 1, 2, \cdots, m$
7. \((e_{m+j}, e_{m+i}) = \delta_{ij} \frac{p_{m+1-i}^{31} \prod_{k=m+2-i}^{2m} (c_{m+1-i} - c_k) \prod_{k=m+1}^{2m} q_{m+1-i,k}}{p_{m+1-i}^{2m+1} \prod_{k=1}^{m-1} (c_{m+1-i} - c_k) \prod_{k=1}^{m-1+i} q_{m+1-i,k}}\) for \(i, j = 1, 2, \ldots, m\)

Proof —

1. We obtain by direct computation

   \[
   \langle e_1, e_1 \rangle = \sum_{i=1}^{2m} p_i^{31} p_i^{32} \prod_{k=1}^{2m-1} (c_{2m-i} - c_k) \prod_{k=m+1}^{2m} q_{m+1-i,k}.
   \]

   For \(1 \leq i \leq 2m\) let us set \(x_i = c_i + (b_2 + b_3)/2\). Identities \(7.68\), \(7.69\), and \(7.70\) from the Appendix finish the proof.

2. A direct computation gives

   \[
   \langle e_1, e_{1+i} \rangle = -p_{2m-i}^{31} p_{2m-i}^{32} \prod_{k=1}^{2m-1} (c_{2m-i} - c_k) \prod_{k=m+1}^{2m} q_{m+1-i,k} \sum_{r=1}^{m} p_r^{31} p_r^{32} \prod_{k=1}^{2m-i} (c_r - c_{2m-i}) \prod_{k=m+1}^{2m} (c_{2m-i} - c_k).
   \]

   After canceling out common multiples, we have to prove that

   \[
   \sum_{r=1}^{m} p_r^{31} p_r^{32} \prod_{k=1}^{2m-i} (c_r - c_{2m-i}) \prod_{k=m+1}^{2m} (c_{2m-i} - c_k) = -p_{2m-i}^{31} p_{2m-i}^{32} \prod_{k=m+1}^{2m} q_{m+1-i,k} \prod_{k=1}^{2m} (c_r - c_{2m-i}) - (b_1 - b_3).
   \]

   Let us set \(x_r = c_r + (b_2 + b_3)/2, y_r = c_{m+r} + (b_2 + b_3)/2\). Now identities \(7.65\), \(7.71\), and \(7.72\) from the Appendix finish the proof.

3. This formula is proved by direct computation.

4. A direct computation gives

   \[
   \langle e_{1+i}, e_{1+j} \rangle = \sum_{r=1}^{m} p_r^{31} p_r^{32} \prod_{k=1}^{2m-1} (c_{2m-i} - c_k) \prod_{k=m+1}^{2m} q_{m+1-i,k} \prod_{k=1}^{2m-j} (c_{2m-j} - c_k) \prod_{k=m+1}^{2m} \frac{(c_r - c_k)}{q_{r,k} \prod_{k=1}^{2m-j} (c_r - c_k)}.
   \]

   Let us set \(x_1 = c_1 + (b_2 + b_3)/2, x_2 = c_2 + (b_2 + b_3)/2, \ldots, x_m = c_m + (b_2 + b_3)/2; y_1 = c_{m+1} + (b_2 + b_3)/2, y_2 = c_{m+2} + (b_2 + b_3)/2, \ldots, y_m = c_{m+m} + (b_2 + b_3)/2\). Identities \(7.61\) and \(7.62\) from the Appendix finish the proof.
5. A direct computation gives

\[
\langle \{ e_{1+i}, e_{1+i} \rangle = \left\{ \prod_{k=1+i}^{m} q_{2m-i,k} \prod_{k=m+1}^{m} q_{2m-i,k} \right\} \sum_{r=1}^{m} p_{2m-i}^{31} p_{2m-i}^{32} \times \left( \prod_{k=1+i}^{2m} q_{2m-i,k} \prod_{k=m+1}^{2m} q_{2m-i,k} \right) \times \prod_{k=1}^{m} q_{r,k} \prod_{k=1}^{m} (c_r-c_k) + \prod_{k=1+k}^{m} q_{2m-i,k} \prod_{k=m+1}^{2m-i} (c_{2m-i}-c_k) \right\}.
\]

Let us set \( x_r = c_r + (b_2 + b_3)/2 \), \( y_r = c_m + r + (b_2 + b_3)/2 \) for \( r = 1, 2, \ldots, m \). Identities (7.62) and (7.63) from Appendix finish the proof.

6. Proved by direct computation.

7. Proved by direct computation.

Now it is time to prove Theorem 2.5, that is prove that the matrices \( A, B, \) and \( C \) are self-adjoint with respect to the scalar product (2.14). \( C \) is self-adjoint with respect to the scalar product by construction. The space \( V \) splits into the direct sum \( V = V_{b_1} \oplus V_{b_2} \oplus V_{b_3} \) of the spectral subspaces of \( B \). If the subspaces \( V_{b_1}, V_{b_2}, \) and \( V_{b_3} \) are mutually orthogonal with respect to the scalar product (2.14), then \( B \) is self-adjoint with respect to it as well. Then \( A \) is also self-adjoint, as \( A = B + C \). Proof of the following lemma finishes the proof of Theorem 2.5.

Lemma 3.5 The subspaces \( V_{b_1}, V_{b_2}, \) and \( V_{b_3} \) are mutually orthogonal with respect to the scalar product (2.14).

Proof — We use the formulas of Lemma 3.3 to express the eigenvectors \( w_i \) of the matrix \( B \) in terms of the standard basis \( \{ e_1, e_2, \ldots, e_{2m} \} \). Then we use the formulas of Lemma 3.4 to expand \( (e_i, e_j) \).

1. \[
\langle w_i, w_{1+i} \rangle = \left\langle e_1, e_{1+i} - \frac{1}{b_1 - b_2} \prod_{k=1+i}^{m} q_{k,2m-i} \prod_{k=m+1}^{m} (c_{2m-i}-c_k) \right\rangle = \left\langle e_1, e_{1+i} \right\rangle - \frac{1}{b_1 - b_2} \prod_{k=1+i}^{m} q_{k,2m-i} \prod_{k=m+1}^{m} (c_{2m-i}-c_k) \right\rangle
\]

2. The identity \( \langle w_1, w_{m+i} \rangle = 0 \) \( (i = 1, 2, \ldots, m) \) reduces to identity (7.63) from the Appendix.

3. Lemma 3.3 gives us

\[
w_{1+i} = e_{1+i} - \frac{1}{b_1 - b_2} \prod_{k=1+i}^{m} q_{k,2m-i} \prod_{k=m+1}^{m} (c_{2m-i}-c_k) \]

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We know that $e_1 = w_1$ and that $(w_1, w_{m+i}) = 0$. Thus, $(w_{1+i}, w_{m+j}) = (e_{1+i}, w_{m+j})$. The identity $(e_{1+i}, w_{m+j}) = 0$ after expansion and some simplification becomes the following identity:

$$
1 = \frac{\prod_{r=1}^{m-1} \prod_{k=1}^{m-1} q_{2m-r,k}}{\prod_{k=m+1}^{m} q_{2m-r,k}} \frac{\prod_{s=1}^{m-1} q_{2m-s,i} (c_{s} - c_{2m-i})}{\prod_{k=m+1}^{m} q_{2m-s,i} (c_{s} - c_{2m-i})} = \frac{2m!}{(m-1)!} \frac{\prod_{r=1}^{m-1} q_{r,k}}{\prod_{k=m+1}^{m} q_{r,k}} \frac{(c_{s} - c_{k})}{(c_{s} - c_{k})} \frac{q_{m+i,1} q_{m+i,2}}{q_{m+1-j,2m}} \frac{q_{m+1-j,2m}}{q_{m+1-j,2m}}.
$$

For $i = 1, 2, \cdots, m$, let us set $x_i = c_i + (b_2 + b_3)/2$ and $y_i = c_{m+i} + (b_2 + b_3)/2$. Let us write $c_s = x_s - y_{2m-i} + y_{2m-i} - (b_2 + b_3)/2$ and $c_{2m-i} = y_{2m-i} - (b_2 + b_3)/2$. Now identities (7.61), (7.65), and (7.73) of the Appendix finish the proof.

Let us prove Theorem 2.6, that is determine the inequalities on the real spectra of A, B, and C which make the form (2.14) sign-definite. It is an assumption of Theorem 2.6 that $c_1 > c_2 > \cdots > c_{2m}$. The assumption $a_1 > a_2$ of the Theorem is satisfied automatically, because in our normalized version $a_1 = 1$ and $a_2 = -1$.

**Proof of Theorem 2.7** — It is immediately clear from Theorem 2.7 that

$$
\text{sign}((v_i, v_i)) = (-1)^{i-1} \text{sign}(p_{i}^{31} p_{i}^{32}) \text{sign} \left( \prod_{j=1}^{m} q_{i,j} \right).
$$

Let $Q$ be a $2m \times 2m$ array such that $Q_{i,j} = q_{i,j}$ for $i \neq j$ and $Q_{i,i}$ are not defined. Then $Q_{i,j} > Q_{i,j+1}$ and $Q_{i,j} > Q_{i+1,j}$ for all $i$ and $j$ such that neither of the array elements involved belongs to the main diagonal. Let $P$ be a $2m \times 2m$ matrix such that $P_{i,1} = p_{i}^{32}$ and $P_{2,j} = p_{j}^{31}$. The fact $a_1 > a_2$ implies $p_{1}^{32} > p_{1}^{31}$. Then sign($v_i, v_i)$) = $(-1)^{i-1}$ if $(j; Q_{i,j} < 0)$ or $(j; P_{i,j} < 0)$. In order to keep sign($v_i, v_i$) constant, the number of negative elements in the $i$ row of the arrays $Q$ and $P$ must differ from the number of negative elements in the $i + 1$ row by an odd number. This and the fact that $Q_{i,j} = Q_{j,i}$ leaves room for the following six configurations. The first is given by the inequalities:

$$
\begin{align*}
& p_{m-1}^{31} > 0 > p_{m}^{31} \quad q_{1,2m-2} > 0 > q_{1,2m-1} \\
& p_{2m-1}^{31} > 0 > p_{2m}^{31} \quad q_{2,2m-3} > 0 > q_{2,2m-2} \\
& \vdots \quad q_{m-1,m} > 0 > q_{m-1,m+1} \\
\end{align*}
$$

We have $q_{i,2m-i} < 0$ for $i = 1, 2, \cdots m - 1$. Let us sum up these inequalities with $p_{2m}^{31} < 0$ and $p_{m}^{31} < 0$. Recalling that $C$ is traceless, we obtain $(m - 1)b_2 + (m + 1)b_3 < 0$. Recalling that $B$ is traceless, we obtain $b_1 > b_3$. We have $p_{m-1}^{31} > 0$ and $p_{2m-1}^{31} > 0$. Thus, $-p_{m-1}^{31} - p_{2m-1}^{31} < 0$. We also have $q_{m-1,2m-m} < q_{m-1,m+1} < 0$ for $m > 2$ and $q_{m-1,2m-1} = q_{m-1,m+1} < 0$ for $m = 2$ because $c_i < c_j$ for $i > j$. Thus, we have $-p_{m-1}^{31} - p_{2m-1}^{31} + q_{m-1,2m-1} < 0$. This gives us $b_3 > b_2$. So, we have $b_1 > b_3 > b_2$. Here is a picture illustrating the case of $m = 3$. The line separates positive elements from negative.
The second configuration is given by the following inequalities.

\[
\begin{align*}
p_{31}^1 & > 0 > p_{2}^{31} \\
p_{m+1}^3 & > 0 > p_{m+2}^3 \\
p_{32}^m & > 0 > q_{3,2m} \\
q_{3,2m-1} & > 0 > q_{4,2m-1} \\
q_{4,2m-2} & > 0 > q_{5,2m-2} \\
& \vdots \\
q_{m,m+2} & > 0 > q_{m+1,m+2}
\end{align*}
\]

We have \( p_{2}^{31} < 0 \) and \( p_{m+2}^3 < 0 \). Thus, we have \(-p_{31}^{31} - p_{32}^m > 0\). We also have \( q_{2,m+2} > q_{m,m+2} > 0 \) for \( m > 2 \) and \( q_{2,m+2} = q_{m,m+2} > 0 \) for \( m = 2 \). Thus, \( q_{2,m+2} - p_{31}^{31} - p_{32}^m > 0 \). This implies \( b_2 > b_3 \). We have \( q_{i,2m+2-i} > 0 \) for \( i = 2, 3, \ldots, m \). Summing up these inequalities with \( p_{31}^{31} > 0 \) and \( p_{m+1}^3 > 0 \), we obtain \((m-1)b_2 + (m+1)b_3 > 0\). Thus, \( b_3 > b_1 \). So, we have \( b_2 > b_3 > b_1 \).

Here is the picture illustrating the case of \( m = 3 \).

The third configuration is given by the following inequalities.

\[
\begin{align*}
0 & > p_{1}^{31} \\
q_{1,2m-1} & > 0 > q_{1,2m} \\
q_{2,2m-2} & > 0 > q_{2,2m-1} \\
p_{m}^3 & > 0 > p_{m+1}^3 \\
q_{m-1,m+1} & > 0 > q_{m-1,m+2} \\
0 & > q_{m,m+1}
\end{align*}
\]

The inequalities \( p_{m+1}^3 < 0 \) and \( p_{1}^{31} < 0 \) imply the inequality \( 2b_3 + c_1 + c_{m+1} < 0 \). The inequality \( q_{1,2m-1} > 0 \) implies the inequality \( q_{1,m+1} > 0 \) because \( c_{m+1} > c_{2m-1} \) for \( m > 2 \) and \( c_{m+1} = c_{2m-1} \) for \( m = 2 \). Now, the inequalities \( b_2 + b_3 + c_1 + c_{m+1} > 0 \) and \(-2b_3 - c_1 - c_{m+1} > 0\) imply the inequality \( b_2 - b_3 > 0 \). So, \( b_2 > b_3 \). Let us sum up the inequalities \( q_{i,2m+1-i} < 0 \) for \( i = 1, 2, \ldots, m \). The sum of all the \( c_i \) is equal to zero. Thus we obtain \( mb_2 + mb_3 < 0 \) which is equivalent to \( b_2 - b_1 < 0 \). This gives us \( b_1 > b_2 > b_3 \). Here is the picture illustrating the case of \( m = 3 \).
The fourth configuration is given by the following inequalities.

\[ p_{31}^m > 0 > p_{31}^{m+1}, \quad q_{1,2m} > 0 > q_{2,2m}, \quad q_{2,2m-1} > 0 > q_{3,2m-1} \]

\[ p_{2m}^{32} > 0 \quad \vdots \quad q_{m-1,m+2} > 0 > q_{m,m+2}, \quad q_{m,m+1} > 0 \]

We have \( q_{m,2m} < q_{m,m+2} < 0 \) for \( m > 2 \) and \( q_{m,2m} = q_{m,m+2} < 0 \) for \( m = 2 \). We also have \(-p_{2m}^{32} < 0\) and \(-p_{m}^{31} < 0\). Summing up these inequalities, we obtain \( b_3 > b_2 \). We also have \( q_{i,2m+1-i} > 0 \) for \( i = 1, 2, \ldots, m \). Summing up these inequalities, we obtain \( b_2 > b_1 \). This gives us \( b_3 > b_2 > b_1 \). Here is the picture illustrating the case of \( m = 3 \).

The fifth configuration is given by the following inequalities.

\[ 0 > p_{1}^{31}, \quad q_{1,2m} > 0 > q_{2,2m}, \quad q_{2,2m-1} > 0 > q_{3,2m-1} \]

\[ p_{m}^{32} > 0 > p_{m+1}^{32} \quad \vdots \quad q_{m-1,m+2} > 0 > q_{m,m+2}, \quad q_{m,m+1} > 0 \]

We have \( q_{i,2m+2-i} < 0 \) for \( i = 2, 3, \ldots, m \). Summing up these inequalities with \( p_{1}^{31} < 0 \) and \( p_{m+1}^{32} < 0 \), we obtain \((m-1)b_2 + (m+1)b_3 < 0\). The last is equivalent to \( b_1 > b_3 \). We also have \( q_{i,2m+1-i} > 0 \) for \( i = 1, 2, \ldots, m \). Summing up these inequalities, we obtain \( mb_2 + mb_3 > 0 \) which is equivalent to \( b_2 > b_1 \). This gives us \( b_2 > b_1 > b_3 \). Here is the picture illustrating the case of \( m = 3 \).

The last configuration possible is given by the following inequalities.

\[ p_{m}^{31} > 0 > p_{m+1}^{31}, \quad q_{1,2m-1} > 0 > q_{1,2m}, \quad q_{2,2m-2} > 0 > q_{2,2m-1} \]

\[ p_{2m}^{32} > 0 \quad \vdots \quad q_{m-1,m+1} > 0 > q_{m-1,m+2}, \quad 0 > q_{m,m+1} \]
We have \( q_{i,2m+1-i} < 0 \) for \( i = 1, 2, \cdots, m \). Summing up these inequalities, we obtain \( mb_2 + mb_3 < 0 \) which is equivalent to \( b_1 > b_2 \). We also have \( q_{i,2m-i} > 0 \) for \( i = 1, 2, \cdots, m-1 \). Summing up these inequalities with \( p_{2m}^{32} > 0 \) and \( p_{m}^{31} > 0 \), we obtain \( (m-1)b_2 + (m+1)b_3 > 0 \) which is equivalent to \( b_3 > b_1 \). This gives us \( b_3 > b_1 > b_2 \). Here is the picture illustrating the case of \( m = 3 \).

So, in these six cases the form \( \langle *, * \rangle \) is sign-definite. Lemma 3.4 gives \( (e_1, e_1) = (b_1 - b_2)(b_1 - b_3) \). Thus, \( \text{sign}(\langle *, * \rangle) = \text{sign}((b_1 - b_2)(b_1 - b_3)) \).

3.3 Odd Family

For the hypergeometric, odd, and even family, let us call the objects \( \{ V; A = B + C, B, C; \langle *, * \rangle \} \) where \( (A, B, C) \) is a rigid irreducible triple of matrices of the corresponding spectral types and \( \langle *, * \rangle \) is the non-degenerate scalar product such that \( A, B, \) and \( C \) are self-adjoint with respect to it, \( m \)-hypergeometric module, \( m \)-even module, and \( m \)-odd module. Let us denote these objects as \( HGM_m, EM_m, \) and \( OM_m \). The reason for calling these objects modules comes from the theory of quiver representations and will not be explained in this paper.

It is possible to prove Theorems 2.7, 2.8, Lemma 2.3, etc. in the same fashion as for the even family. But we choose a different approach. We show that by means of violating the “generic eigenvalues” condition it is possible to construct \( OM_{m-1} \) as a submodule and as a factor module of \( EM_m \). Then all the formulas follow from the corresponding formulas for the even family.

Let \( V \) be the same \( 2m \)-dimensional linear space as in the previous subsection and let \( e_1, \cdots, e_{2m} \) be the standard basis of \( V \). Let \( A, B, \) and \( C \) be the matrices from the previous subsection, too. Fix an integer \( i \) such that \( 1 \leq i \leq m \). Let \( V_i^* \) be the subspace of \( V \) spanned by the vectors \( e_1, e_2, \cdots, e_{2m-i}, e_{2m+1-i}, e_{2m+2-i}, \cdots, e_{2m} \). It follows from the formulas of Lemma 2.3 that \( V_i^* \) is spanned by \( v_1, v_2, \cdots, v_{i-1}, \hat{v}_i, v_{i+1}, \cdots, v_{2m} \) (hence the notation). Then the following lemma follows at once from the formulas for \( A, B, \) and \( C \) of Subsection 2.3.

Lemma 3.6 If \( p_i^{32} = 0 \), then \( V_i^* \) is invariant with respect to \( A, B, \) and \( C \).

Thus, it makes sense to consider the restrictions of \( A, B, \) and \( C \) to \( V_i^* \) and call them \( A_i^*, B_i^*, \) and \( C_i^* \). We will also call \( \langle *, * \rangle_i^* \) the form \( \langle *, * \rangle \) restricted to \( V_i^* \). Note that \( p_i^{32} = 0 \) forces \( q_{ij} = p_{ij}^{21} \) and \( c_i - c_j = -p_j^{32} \).

Theorem 3.1 If \( (s(A), s(B), s(C)) \) is a generic point of the intersection of \( S((m, m), (1, m - 1, m), (12m)) \) with the hyperplane given by the equation \( p_i^{32} = 0 \) for a fixed \( 1 \leq i \leq m \), then \( \{ V_i^*; A_i^*, B_i^*, C_i^*; \langle *, * \rangle_i^* \} \) is \( OM_{m-1} \).

Here is an example of the matrices \( B_i^* \) and \( C_i^* \) obtained from the matrices \( B \) and \( C \) of Example 2.4 by setting \( p_2^{32} = 0 \) and restricting them to \( V_2^* \).
Example 3.2

\[
B^s_2 = \begin{bmatrix}
  b_1 & \frac{-p_{31}^2 q_{35}}{c_4-c_5} & \frac{-p_{31}^2 q_{34} q_{35}}{p_{31}^2 (c_1-c_3)} & p_{31}^3 \\
  0 & b_2 & 0 & \frac{-p_{31}^2 q_{15} q_{34}}{p_{31}^2 (c_1-c_3)} & p_{31}^3 \\
  0 & 0 & b_2 & \frac{-p_{31}^2 q_{14} q_{35}}{p_{31}^2 (c_1-c_3)(c_4-c_5)} & p_{31}^3 \frac{q_{14} q_{35}}{c_4-c_5} \\
  0 & 0 & 0 & b_3 & 0 \\
  0 & 0 & 0 & 0 & b_3
\end{bmatrix}
\]

\[
C^s_2 = \begin{bmatrix}
  c_6 & 0 & 0 & 0 & 0 \\
  -q_{15} & c_5 & 0 & 0 & 0 \\
  -\frac{q_{31}^2 q_{14}}{c_4-c_5} & 0 & c_4 & 0 & 0 \\
  -p_{31}^2 & \frac{-p_{31}^2 p_{21}^2}{c_4-c_5} & p_{33} & c_3 & 0 \\
  -q_{14} q_{15} & \frac{-p_{31}^2 q_{14} q_{35}}{(c_1-c_3)(c_4-c_5)} & \frac{q_{14} q_{35}}{c_1-c_3} & 0 & c_1
\end{bmatrix}
\]

This way $OM_{m-1}$ is constructed as a submodule of $EM_m$. It is also possible to construct $OM_{m-1}$ as a factor module of $EM_m$. Let us fix $i$ such that $1 \leq i \leq m$. The following lemma follows at once from the formulas for $A$, $B$, and $C$ of Subsection 2.2.

Lemma 3.7 If $p_{31}^3 = 0$, then $A$, $B$, and $C$ preserve the one-dimensional subspace of $V$ spanned by the vector $e_{2m+1-i}$.

Thus, it makes sense to consider $A$, $B$, and $C$ acting on the factor space $V_i^f$. Let us call these new operators $A_i^f$, $B_i^f$, and $C_i^f$. If $p_{31}^3 = 0$, then $\langle v_i, v_i \rangle = 0$. Thus, the form $\langle *, * \rangle_{i}^f$ induced on $V_i^f$ by the form $\langle *, * \rangle$ on $V$ is well defined. Note that $p_{31}^3 = 0$ forces $q_ij = p_{22}^2$ and $c_i - c_j = -p_{31}^3$.

Theorem 3.2 If $(s(A), s(B), s(C))$ is a generic point of the intersection of $S((m,m),(1,m-1,m),(1^{2m}))$ with the hyperplane given by the equation $p_{31}^3 = 0$ for a fixed $1 \leq i \leq m$, then $\{V_i^f; A_i^f, B_i^f, C_i^f; \langle *, * \rangle_i^f \}$ is $OM_{m-1}$.

Here is an example of the matrices $B_i^f$ and $C_i^f$ obtained from the matrices $B$ and $C$ of Example 2.4 by setting $p_{21}^3 = 0$ and passing to the factor space $V_i^f$. **
Example 3.3

\[
\begin{bmatrix}
  b_1 & -\frac{p_{32}^2 p_{14}}{c_4 c_5} & q_{34} & -\frac{q_{34} q_{35}}{c_1 c_3} & p_{31}^3 \\
  0 & b_2 & 0 & -\frac{q_{15} q_{34}}{c_1 c_3} & p_{31}^3 \\
  0 & 0 & b_2 & -\frac{p_{32}^2 q_{14} q_{35}}{(c_1 - c_3)(c_4 - c_5)} & p_{31}^3 \\
  0 & 0 & 0 & b_3 & 0 \\
  0 & 0 & 0 & 0 & b_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
  c_6 & 0 & 0 & 0 & 0 \\
  -q_{15} & c_5 & 0 & 0 & 0 \\
  -\frac{p_{32}^2 q_{14}}{c_4 c_5} & 0 & c_4 & 0 & 0 \\
  -p_{32}^3 & -\frac{p_{32}^3 p_{32}^2}{c_4 c_5} & p_{32}^3 & c_3 & 0 \\
  -\frac{p_{32}^2 q_{14} q_{15}}{p_{31}^3 (c_1 - c_3)} & -\frac{p_{32}^2 q_{14} q_{35}}{p_{31}^3 (c_1 - c_3)(c_4 - c_5)} & \frac{p_{32}^3 q_{15} q_{34}}{p_{31}^3 (c_1 - c_3)} & 0 & c_1
\end{bmatrix}
\]

We first prove Theorem 3.1 and then we derive all the proofs for the odd family from what we already know about the even family. Let us prove Theorem 3.1.

**Proof of Theorem 3.1** — It is clear that \(B_2\) is diagonalizable and that \(s(B_2) = \{b_1, b_2, \ldots, b_3, \ldots, b_3\}\).

It is clear that \(C_2\) is diagonalizable and that \(s(C_2) = \{c_1, c_2, \ldots, c_{m-1}, v_i, \hat{c}_i, c_i, c_{i+1}, \ldots, c_{2m}\}\). In view of Lemma 3.6 and Theorem 2.4, it is clear that \(A_i^e\) is diagonalizable as well. In the notations of Subsection 3.2, \(A\) has eigenvectors \(a_j^+\) corresponding to the eigenvalue \(a_1\) (normalized to 1). Vectors \(a_j^-\) are eigenvectors of \(A\) corresponding to the eigenvalue \(a_2\) (normalized to \(-1\)). Once we set \(p_1^{32} = 0\), all the eigenvectors of \(A\) belong to \(V_i^{2\epsilon}\) except for \(a_m^{m+1-i}\) and the proof follows immediately.

To finish the rest of the proofs for the odd family, we just have to say that all the formulas for \(OM_m\) in this paper were obtained from the formulas for \(EM_m\) by setting \(p_{m+1}^{32} = 0\) and renumbering the remaining \(c_1, c_2, \ldots, c_m, c_{m+2}, \ldots, c_{2m+2}\) as \(c_1, c_2, \ldots, c_{2m+1}\).

**Remark 3.2** In exactly the same fashion, we can construct \(EM_m\) as a factor module of \(OM_m\) by setting either \(p_i^{31} = 0\) for a fixed \(1 \leq i \leq m\) or \(p_i^{21} = 0\) for a fixed \(m+1 \leq i \leq 2m\). Also similarly, one can show that setting \(b_1 + c_{m+1-i} = a_2 = 0\) for a fixed \(1 \leq i \leq m\) creates \(HGM_m\) as a submodule of \(HGM_m\).
3.4 $HGM_m$ from $EM_m$

In this subsection we show that there exist $HGM_m$ and $HGM_{m+1}$ which naturally “live inside” $EM_m$ with roles of $A$ and $B$ interchanged.

Let us consider $EM_m$ constructed in Subsection 2.2. Let $V_1$ be the subspace of $V$ spanned by $e_1$. Let $V_2$ be the subspace of $V$ spanned by the vectors $e_2, e_3, \cdots, e_m$. Finally, let $V_3$ be the subspace of $V$ spanned by the vectors $e_{m+1}, e_{m+2}, \cdots, e_{2m}$. Let $\Gamma : V_1 \oplus V_2 \rightarrow V$ be an inclusion. Let $P' : V \rightarrow V_1 \oplus V_2$ be the projection along $V_3$. Set $A' = P' (-B) \Gamma', B' = P' (-A) \Gamma$, and $C' = P' C \Gamma'$.

**Theorem 3.3** If $(s(A), s(B), s(C))$ is a generic point of $S''((m, m), (1, m - 1, m), (1^{2m}))$, then \{V_1 \oplus V_2; A', B', C'\} is $HGM_m$.

Let $\Gamma'' : V_1 \oplus V_3 \rightarrow V$ be an inclusion. Let $P'' : V \rightarrow V_1 \oplus V_3$ be the projection along $V_2$. Set $A'' = P'' (-B) \Gamma'', B'' = P'' (-A) \Gamma''$, and $C'' = P'' C \Gamma''$.

**Theorem 3.4** If $(s(A), s(B), s(C))$ is a generic point of $S''((m, m), (1, m - 1, m), (1^{2m}))$, then \{V_1 \oplus V_3; A'', B'', C''\} is $HGM_{m+1}$.

The following lemma is a simple exercise in linear algebra.

**Lemma 3.8**

$$
\begin{vmatrix}
\alpha_1 & \beta_1 & \beta_2 & \cdots & \beta_{m-1} \\
\gamma_1 & \alpha_2 & 0 & \cdots & 0 \\
\gamma_2 & 0 & \alpha_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{m-1} & 0 & 0 & \cdots & \alpha_m
\end{vmatrix} = \prod_{i=1}^{m} \alpha_i - \sum_{j=1}^{m-1} \beta_j \gamma_j \prod_{k=2}^{m} \alpha_k.
$$

**Proof of Theorem 3.3** — It is clear that $A' = B' + C'$. It is clear that $A'$ is diagonalizable and that $s(A') = (-b_1, -b_2, \cdots, -b_2)$. It is also clear that $C'$ is diagonalizable and that $s(C_m) = (c_{m+1}, c_{m+2}, \cdots, c_{2m})$. Let us prove that $B'$ is diagonalizable and $s(B') = (b_3 + c_1 - a_1 - a_2, b_3 + c_2 - a_1 - a_2, \cdots, b_3 + c_m - a_1 - a_2)$. Let us consider the matrix $B'$. Application of Lemma 3.8 immediately yields the following formula for the characteristic polynomial of $B'$:

$$
|B' - x \text{Id}| = (-1)^m \left(b_1 + c_{2m} + x\right) \prod_{k=1}^{m-1} \left(b_2 + c_{m+k} + x\right) + \\
\sum_{i=1}^{m-1} \frac{\prod_{k=1}^{m} q_{i,k,2m-i}}{\prod_{k=m+1}^{m} (c_{2m-i} - c_k)} \prod_{k=2}^{m} \left(b_2 + c_{2m+1-k} + x\right) \right).
$$

(3.39)

where $p_{i,j}$ and $q_{i,j}$ are as in (2.6).

Let us prove that $x_j = b_3 + c_j - a_1 - a_2$ where $j = 1, 2, \cdots, m$ are the roots of the polynomial. For that, we have to prove the following identity:
\[
\sum_{i=1}^{m-1} \prod_{k \neq m+1-i}^{m} \frac{q_{k,2m-i}}{2m-1} = a_1 + a_2 - b_1 - b_3 - c_j - c_{2m}.
\]

Identity (7.63) from the Appendix reduces this identity to the trace identity. The matrix \( B' \) is diagonalizable because it has all eigenvalues distinct. If \( (s(A), s(B), s(C)) \) is a generic point of \( S''((m, m), (1, m - 1, m), (1^2 m)) \), then \( (s(A'), s(B'), s(C')) \) is a generic point of \( S''((1, m - 1), (1^m), (1^m)) \). Then it follows from Theorems 1.2 and 1.3 and Corollary 1.1 that there exists the unique non-trivial non-degenerate symmetric bilinear form \( (\cdot, \cdot)' \) such that \( A', B', \) and \( C' \) are self-adjoint with respect to it.

**Proof of Theorem 3.4** — It is clear that \( A'' \) is diagonalizable and that \( s(A'') = (-b_1, -b_3, \ldots, -b_3) \).

It is also clear that \( C'' \) is diagonalizable and that \( s(C'') = (c_1, c_2, \ldots, c_m, c_{2m}) \). Let us prove that \( B'' \) is diagonalizable and that \( s(B'') = (-a_1, -a_2, b_2 + c_{m+1} - a_1 - a_2, b_2 + c_{m+2} - a_1 - a_2, \ldots, b_2 + c_{2m-1} - a_1 - a_2) \). Let us consider the matrix \( B'' \). Application of Lemma 3.3 immediately yields the following formula for the characteristic polynomial of \( B'' \):

\[
|B'' - x \text{Id}| = (-1)^{m+1} \left[ (b_1 + c_{2m} + x) \prod_{k=1}^{m} (b_3 + c_{m+1-k} + x) + \sum_{i=1}^{m} p_{m+1-i}^{31} p_{m+1-i}^{32} \prod_{k=1}^{2m-1} q_{k,m+1-i} \prod_{k \neq m+1-i}^{m} (c_{m+1-i} - c_k) \prod_{k=1}^{m} (b_3 + c_{m+1-k} + x) \right] =
\]

\[
(-1)^{m+1} \left[ (b_1 + c_{2m} + x) \prod_{k=1}^{m} (b_3 + c_k + x) + \sum_{i=1}^{m} p_{i}^{31} p_{i}^{32} \prod_{k=1}^{2m-1} q_{k,i} \prod_{k \neq i}^{m} (c_i - c_k) \prod_{k=1}^{m} (b_3 + c_k + x) \right].
\]

(3.40)

Let us prove that \(-a_1\) is a root of this polynomial. For that, we have to show

\[
\sum_{i=1}^{m} p_{i}^{32} \prod_{k=m+1}^{m} q_{i,k} \prod_{k \neq i}^{m} (c_i - c_k) = a_1 - b_1 - c_{2m}.
\]

Let us set \( x_1 = p_{1}^{32}, x_2 = p_{2}^{32}, \ldots, x_m = p_{m}^{32}, y_1 = -p_{m+1}^{31}, y_2 = -p_{m+2}^{31}, \ldots, y_{m-1} = -p_{2m-1}^{31} \). Then \( q_{i,k} = x_i - y_k, c_i - c_k = x_i - x_k \), and the formula to prove becomes equivalent to identity (7.63) from the Appendix. The proof that \(-a_2\) is a root of the polynomial is exactly the same.

Let us fix \( 1 \leq j \leq m - 1 \). The same identity (7.63) proves that \( b_2 + c_{m+j} - a_1 - a_2 \) is a root of the polynomial. The rest of the proof of this theorem is the same as the corresponding part of the proof of Theorem 3.3.

**Remark 3.3** Similarly, one can construct \( HGM_{m+1} \) “inside” \( OM_m \).
3.5 Extra case of Simpson

Consider the following vectors.

\[ w_1 = (1, 0, 0, 0, 0, 0), \]
\[ w_2 = (0, 1, 0, 0, 0, 0), \]
\[ w_3 = \left( \frac{P_{16} Q_{245}}{(b_1 - b_2)(c_3 - c_4)}, \frac{P_{15} Q_{243} Q_{246}}{(b_1 - b_2)(c_3 - c_4)}, 1, 0, 0, 0 \right), \]
\[ w_4 = \left( -\frac{P_{16} Q_{236}}{b_1 - b_2}, \frac{P_{15} Q_{236}}{b_1 - b_2}(c_3 - c_4), 0, 1, 0, 0 \right), \]
\[ w_5 = \left( -\frac{P_{16} Q_{245}}{(b_1 - b_3)(b_2 - b_3)(c_1 - c_2)}, \frac{P_{15} Q_{236} Q_{246}}{(b_1 - b_3)(b_2 - b_3)(c_1 - c_2)}, \frac{P_{24} Q_{236}}{b_1 - b_3}(c_1 - c_2), 1, 0 \right), \]
\[ w_6 = \left( -\frac{P_{16} Q_{245}}{(b_1 - b_3)(b_2 - b_3)}, \frac{P_{15} Q_{235} Q_{243}}{(b_1 - b_3)(b_2 - b_3)(c_5 - c_6)}, \frac{P_{24} Q_{235}}{b_2 - b_3}(c_5 - c_6), 0, 1 \right). \]

(3.41)

Theorem 2.10 and Lemma 2.4 are proved by direct computation as well as the following two lemmas.

**Lemma 3.9** Let \( B \) be as in (2.24). Then \( w_1 \) and \( w_2 \) are eigenvectors of \( B \) with the eigenvalue \( b_1 \), \( w_3 \) and \( w_4 \) are eigenvectors of \( B \) with the eigenvalue \( b_2 \), and \( w_5 \) and \( w_6 \) are eigenvectors of \( B \) with the eigenvalue \( b_3 \).

**Lemma 3.10** Let \( \langle *, * \rangle \) be defined by (2.23). Let \( V_{b_1} \) be the subspace of \( V \) spanned by \( w_1 \) and \( w_2 \). Let \( V_{b_2} \) be the subspace of \( V \) spanned by \( w_3 \) and \( w_4 \). Let \( V_{b_3} \) be the subspace of \( V \) spanned by \( w_5 \) and \( w_6 \). Then \( V_{b_1}, V_{b_2}, \) and \( V_{b_3} \) are mutually orthogonal with respect to \( \langle *, * \rangle \).

Theorem 2.11 follows from Lemmas 2.4 and 3.10. Finally, Theorem 2.12 can be proved similarly to Theorems 2.3 and 2.6.

4 Indecomposable Triple Flag Varieties with Finitely Many Orbits

Let \( i \in \{1, 2, 3\} \). For a triple of flags \( \emptyset = V_0^i \subset V_1^i \subset V_2^i \subset \cdots \subset V_{k_i - 1}^i \subset V_{k_i}^i = V \), we call the dimension vector in the jump coordinates the vector \( (\dim(V_{b_1}^i/V_{b_1 - 1}^i), \dim(V_{b_2}^i/V_{b_2 - 1}^i), \cdots, \dim(V_{k_i}^i/V_{k_i - 1}^i)) \). We say that this triple of flags is in a standard form, if \( V \) is given a basis \( z_1, \cdots, z_n \) with the following property: for the flag \( \emptyset = V_0^3 \subset V_1^3 \subset V_2^3 \subset \cdots \subset V_{k_3}^3 \subset V_{k_3 - 1}^3 \subset V_{k_3}^3 = V \), the subspace \( V_j^3 \) of dimension \( d_j^3 \) is spanned by the first \( d_j^3 \) basis vectors \( z_1, z_2, \cdots \); for the flag \( \emptyset = V_0^3 \subset V_1^3 \subset V_2^3 \subset \cdots \subset V_{k_3}^3 \subset V_{k_3}^3 = V \), the subspace \( V_j^3 \) of dimension \( d_j^3 \) is spanned by the last \( d_j^3 \) basis vectors \( z_n, z_{n-1}, \cdots \).

P. Magyar, J. Weyman, and A. Zelevinsky classify in [28] all indecomposable triple partial flag varieties with finitely many orbits of the diagonal action of the general linear group. They work over an algebraically closed field. \( \mathbb{C} \) is enough for our purposes. Among other results, they give the dimension vectors in the jump coordinates as well as explicit representatives of the open orbit in
the standard form. For every element of their list, the first flag turns to be just a subspace \( V_1^1 \subset V \). It also turns out that this subspace is spanned by vectors \( a_i \) such that all their coordinates in the standard basis \( z_1, \ldots, z_n \) are equal to either 0 or 1. Their list is given on page \[40\].

**Remark 4.1** Our definition of a standard form for a triple of flags is weaker than that of Magyar, Weyman, and Zelevinsky (includes more triple flags).

Recall that we proved Theorem 1.3 only for the hypergeometric family so far. Now we use the results of Magyar, Weyman and Zelevinsky to prove the counterparts of this result for all other families of Simpson. Let us begin with the even family. Recall that we work with the normalized matrices \( A, B, \) and \( C \). This means that they are traceless and the eigenvalues of \( A \) are 1 and \(-1\).

Let \( Z \) be the following matrix

\[
Z = \begin{bmatrix}
1 & m-1 & m \\
1 & 0 & 0 \\
m-1 & Z_{1+i,1+j} & 0 \\
m & 0 & Z_{m+i,m+j}
\end{bmatrix},
\]

where

\[
Z_{1+i,1+j} = \begin{cases}
0 & \text{if } i < j \\
1 & \text{if } i = j \\
\prod_{k=1+j}^{i} q_{k,2m-i} & \text{if } i > j,
\end{cases}
\]

and

\[
Z_{m+i,m+j} = \begin{cases}
0 & \text{if } i < j \\
1 & \text{if } i = j \\
\frac{\prod_{k=m+i}^{m-1} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m+i} (c_{m+1-i}-c_k)} & \text{if } i > j.
\end{cases}
\]

Note that \( Z \) is lower-triangular with all the diagonal elements equal to 1.

For \( 1 \leq i \leq 2m \), let \( z_i = Ze_i \). The matrix \( Z \) is non-degenerate, so \( z_i \) is a basis of \( V \). Consider the following flags: \( V_1^2 \subset V_2^2 \subset V \) and \( V_1^3 \subset V_2^3 \subset \cdots \subset V_{2m-1}^3 \subset V \) where \( V_1^2 \) is spanned by \( z_1, V_2^2 \) is spanned by \( z_1, z_2, \ldots, z_m \), and \( V_3^3 \) is spanned by \( z_{2m}, z_{2m-1}, \ldots, z_{2m+1-i} \). They are the second and the third flags of the even family with the dimension vector \((m, m), (1, m-1, m), (1^{2m})\) in (4.43).
| Hypergeometric Family | \( a_k = z_1 + z_{k+1} \) \((1 \leq k \leq m - 1)\) |
|----------------------|-----------------------------------------------|
| \((m - 1, 1), (1^m), (1^m)\) | \( a_1 = z_1 + z_2 + \cdots + z_m \) |
| \((1, m - 1), (1^m), (1^m)\) |

| Even Family | \( a_k = z_1 + z_{k+1} + z_{2m+1-k} \) \((1 \leq k \leq m - 1)\) |
|-------------|-------------------------------------------------------------|
| \((m, m), (1, m - 1, m), (1^{2m})\) | \( a_m = z_1 + z_{m+1} \) |
| \(a_1 = z_1 + z_2\) |
| \(a_k = z_1 + z_{k+1} + z_{2m+2-k} \) \((2 \leq k \leq m)\) |
| \((m, m), (m - 1, 1, m), (1^{2m})\) | \( a_k = z_k + z_{2m-k} + z_{2m} \) \((1 \leq k \leq m - 1)\) |
| \(a_m = z_m + z_{2m}\) |
| \((m, m), (m, m - 1, 1), (1^{2m})\) | \( a_k = z_k + z_{m+1} + z_{2m+1-k} \) \((1 \leq k \leq m - 1)\) |
| \(a_1 = z_{m+1}, a_{m+1} = z_1 + z_{m+2}\) |
| \(a_k = z_k + z_{m+1} + z_{2m+2-k} \) \((2 \leq k \leq m)\) |

| Odd Family | \( a_k = z_1 + z + k + 1 + z_{2m+2-k} \) \((1 \leq k \leq m)\) |
|-------------|----------------------------------------------------------------|
| \((m, m + 1), (1, m, m), (1^{2m+1})\) | \( a_1 = z_1 + z_2, a_{m+1} = z_1 + z_{m+2}, a_k = z_k + z_{m+1} + z_{2m+2-k} \) \((1 \leq k \leq m)\) |
| \(a_k = z_k + z_{m+1} + z_{2m+3-k} \) \((2 \leq k \leq m)\) |
| \((m + 1, m), (m, m, m), (1^{2m+1})\) | \( a_1 = z_1 + z_m+1, a_{m+1} = z_{m+1} + z_{m+2}, a_k = z_k + z_{m+1} + z_{2m+3-k} \) \((2 \leq k \leq m)\) |
| \((m, m + 1), (m, m, 1), (1^{2m+1})\) | \( a_1 = z_1 + z_{2m+1}, a_{m+1} = z_{m+1} + z_{2m+1}, a_k = z_k + z_{2m+2-k} + z_{2m+1} \) \((2 \leq k \leq m)\) |

| E₈ Family | \( a_1 = z_1 + z_2 + z_3 + z_6, a_2 = z_1 + z_4 + z_5 \) |
|------------|----------------------------------------------------------------|
| \((2, 4), (2, 2, 2), (1, 1, 1, 1, 1)\) | \( a_1 = z_1 + z_5, a_2 = z_2 + z_3, a_3 = z_2 + z_5 + z_6, a_4 = z_4 + z_5 \) |
| \((4, 2), (2, 2, 2), (1, 1, 1, 1, 1)\) |

| E₆ Family | \( a_1 = z_1 + z_2 + z_3, a_2 = z_1 + z_6, a_3 = z_2 + z_4 + z_5 \) |
|------------|----------------------------------------------------------------|
| \((3, 3), (2, 2, 2), (2, 1, 1, 1)\) | \( a_1 = z_1 + z_2 + z_3, a_2 = z_1 + z_6, a_3 = z_2 + z_4 + z_5 \) |
| \((3, 3), (2, 2, 2), (1, 2, 1, 1)\) | \( a_1 = z_1 + z_2 + z_4 + z_6, a_3 = z_1 + z_3, a_3 = z_1 + z_5 \) |
| \((3, 3), (2, 2, 2), (1, 1, 2, 1)\) | \( a_1 = z_1 + z_5 + z_6, a_2 = z_2 + z_4 + z_6, a_3 = z_1 + z_3, a_3 = z_1 + z_5 \) |
| \((3, 3), (2, 2, 2), (1, 1, 1, 2)\) | \( a_1 = z_1 + z_4 + z_6, a_2 = z_2 + z_4 + z_5, a_3 = z_2 + z_3 \) |
Here is an example of the matrix $Z$ with $m = 3$.

**Example 4.1**

$$Z = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & \frac{q_{24}}{c_{1}-c_{3}} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{q_{24}}{c_{2}-c_{3}} & 1 \\
0 & 0 & 0 & \frac{q_{14}+q_{15}}{(c_{1}-c_{2})(c_{2}-c_{3})} & \frac{q_{15}}{c_{1}-c_{2}} & 1
\end{bmatrix}$$

**Definition 4.1** Let $A$ be a diagonalizable complex linear operator with the spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $V_{\lambda_i}$ be the eigenspace of $A$ corresponding to the eigenvalue $\lambda_i$. We will call the flag $V_{\lambda_i} \subset V_{\lambda_i} \oplus V_{\lambda_2} \subset \cdots \subset V$ the spectral flag of $A$ corresponding to the ordering $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of its spectrum.

If $(m_1, m_2, \ldots, m_k)$ are the multiplicities of the spectrum of $A$ from the above definition, then the dimension vector in the jump coordinates of its spectral flag is also $(m_1, m_2, \ldots, m_k)$.

If we take another look at the eigenvectors of $B$ (Lemma 3.3) and at the eigenvectors of $C$ (Lemma 2.2), we see that the spectral flags of these matrices are exactly the second and the third flags of the Magyar, Weyman, Zelevinsky triple $((m, m), (1, m - 1, m), (12m))$.

**Lemma 4.1** The subspace $V^1_1$ spanned by the vectors $a_1, \ldots, a_m$ (from the $((m, m), (1, m - 1, m), (12m))$ entry in (1.4)) is the spectral subspace of the matrix $A$ corresponding to the eigenvalue $-1$.

**Proof** — In order to prove $(A + \text{Id}) a_i = 0$ for $1 \leq i \leq m$, we have to prove the following identities.

1. The first identity says that the first component of $(A + \text{Id}) a_i$ is zero.

   $$b_1 + c_{2m} + 1 + \sum_{j=i}^{m-1} Z_{1+j,1+i} B_{1,1+j} + \sum_{m+1-i}^{m} Z_{m+j,2m+1-i} B_{1,m+j} = 0 \quad (4.44)$$

2. The second identity says that the components 2 through $m$ of $(A + \text{Id}) a_i$ are zero.

   $$C_{1+j,1} + (c_{2m-j} + b_2 + 1)Z_{1+j,1+i} + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i} B_{1+j,m+k} = 0 \quad (4.45)$$

3. The third identity says that the components $m+1$ through $2m$ of $(A + \text{Id}) a_i$ are zero.

   $$C_{m+j,1} + (c_{m+1-j} + b_3 + 1)Z_{m+j,2m+1-i} + \sum_{k=i}^{m-1} Z_{1+k,1+i} C_{m+j,1+k} = 0 \quad (4.46)$$
Recall that the matrix elements of the matrices $B$ and $C$ are given by formulas (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) on pages 3 – 4. Then the first identity becomes

$$b_1 + c_{2m} + 1 + \sum_{j=i}^{m-1} \frac{\prod_{k=1+i}^{m} q_{k,2m-j}}{\prod_{k=m+1-i}^{2m-i} (c_{m-j} - c_k)} + \sum_{j=m+1-i}^{m} \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{k=m+1-j}^{m} (c_{m-j} - c_k)} = 0. \quad (4.47)$$

In our normalized version, $p_{m+1-j}^{31} = c_{m+1-j} + b_3 - 1$. Thus, (4.47) splits into two identities of homogeneous degrees 0 and 1. The part of degree 0 is

$$\sum_{j=m+1-i}^{m} \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{k=m+1-j}^{m} (c_{m-j} - c_k)} = 1.$$

This identity is equivalent to (7.62) from Appendix. The part of degree 1 is

$$b_1 + c_{2m} + \sum_{j=i}^{m-1} \frac{\prod_{k=1+i}^{m} q_{k,2m-j}}{\prod_{k=m+1-i}^{2m-i} (c_{m-j} - c_k)} + \sum_{j=m+1-i}^{m} (b_3 + c_{m+1-j}) \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{k=m+1-j}^{m} (c_{m-j} - c_k)} = 0.$$

This one is proved similarly to identity (3.34) on page 24 with the help of identity (7.63) from the Appendix applied separately to each of the sums.

To prove the second identity (4.45), let us recall that

$$Z_{1+j,1+i} = \begin{cases} 
0 & \text{if } j < i \\
1 & \text{if } j = i \\
\frac{\prod_{k=1+i}^{j} q_{k,2m-j}}{\prod_{k=m+1-i}^{2m-i} (c_{m-j} - c_k)} & \text{if } j > i
\end{cases}.$$

Thus, the second identity (4.45) splits into three different identities. In case $j < i$, we have

$$C_{1+j,1} + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i} B_{1+j,m+k} = 0.$$

This identity, after cancelling out common multiples, splits into a sum of two identities: one of degree $−1$ and the other of degree 0. The first reduces to identity (7.64) and the second reduces to identity (7.62) from the Appendix.

In case $i = j$, we have $C_{1+j,1} + (c_{2m-j} + b_2 + 1) + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i} B_{1+j,m+k} = 0$. This identity splits into two parts of degree 0 and 1. The part of degree 0 reduces to identity (7.65) from the Appendix. The part of degree 1 is a sum of two identities: one is equivalent to (7.62) and the other is equivalent to (7.63) from the Appendix. The case $j > i$, after cancelling out common multiples, becomes equivalent to the case $j = i$.  

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To prove the third identity (4.47), recall that

\[
Z_{m+j, 2m+1-i} = \begin{cases} 
0 & \text{if } i + j < m + 1 \\
1 & \text{if } i + j = 1 \\
\prod_{k=2m+1-i}^{m+1-j} \prod_{k=m+2-j}^{m-1+j} q^{m+1-j-c_k} & \text{if } i + j > m + 1
\end{cases}.
\]

Thus, the third identity splits into three different identities. In case \(i + j < m + 1\), after cancelling out common multiples, (4.47) becomes equivalent to identity (7.62) from the Appendix. In cases \(i + j = m + 1\) and \(i + j > m + 1\), identity (7.63) from the Appendix does the job.

Finally, it is clear from (4.48) and (4.42) that for \(1 \leq i \leq m\), the vectors \(a_i\) are linearly independent. \(\square\)

Now let us get back to the proof of Theorem 1.3.

**Proof of Theorem 1.3** — For a triple of partitions \(S''(\alpha, \beta, \gamma)\) from Simpson’s list (4.4), we want to prove that if \((s(A), s(B), s(C))\) is a point of \(S''(\alpha, \beta, \gamma)\), then the triple \((A, B, C)\) is irreducible. We have proved it already for the hypergeometric family in subsection 5.1. To prove it for the even family, let us recall that if \((s(A), s(B), s(C))\) is a point of \(S''(\alpha, \beta, \gamma)\), then the scalar product (2.14) is well-defined and non-degenerate. Assume that the triple of matrices \((A, B, C)\) is reducible. Then they all preserve a non-trivial subspace \(V'\). This subspace is spanned by some eigenvectors of \(C\). Let \(V''\) be the subspace of \(V\) spanned by the complementary eigenvectors of \(C\). But all eigenvectors of \(C\) are orthogonal to each other with respect to (2.14). Thus, the space \(V\) splits into the orthogonal direct sum \(V' \oplus V''\). Thus, the matrices \(A\) and \(B\) preserve the subspace \(V''\) as well. So, no matter how we introduce linear orders on the spectra of \(A\), \(B\), and \(C\), the corresponding triple of flags will decompose. However, if \((s(A), s(B), s(C))\) is a point of \(S''((m, m), (1, m - 1, m), (1^2m))\), then as follows from Lemma 4.1 and the preceding discussion, the spectral flags of the matrices \(A\), \(B\), and \(C\) give the Magyar, Weyman, Zelevinsky representative of the open orbit of the corresponding triple flag variety. According to Magyar, Weyman, and Zelevinsky, this triple of flags is indecomposable. Thus, the assumption that the triple \((A, B, C)\) is reducible cannot be true. This proves Theorem 1.3 in the even case.

The \(Z\)-matrix for the odd family triple \(((m + 1, m), (1, m, m), (1^{2m+1}))\) can be obtained from the \(Z\)-matrix for the even family triple \(((m + 1, m + 1), (1, m, m + 1), (1^{2m+2}))\) by restricting the latter to \(V_{m+1}^*\) as in the proof of Theorem 1.3 on page 35. The rest of the argument is the same. Finally, let us give the \(Z\)-matrix for the extra case of Simpson (or, more precisely, for the triple of compositions \(((4, 2), (2, 2, 2), (1^6))\) of the \(E_8\)-family from (4.43)).

\[
Z = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{q_{145}}{c_5 - c_6} & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{q_{245}}{c_3 - c_4} & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{q_{236}}{c_1 - c_2} \\
\end{bmatrix}.
\]

(4.48)
The only family of Magyar, Weyman, Zelevinsky which does not appear in the list of Simpson, is the $E_8$-family. We now construct the matrices $A = B + C, B, C$ such that their spectral flags form the Magyar, Weyman, Zelevinsky representative for the open orbit of the triple flag variety of dimension $((3, 3), (2, 2, 2), (1, 1, 1, 1, 2))$. This time the standard basis $e_i$ and the $z$-basis of Magyar, Weyman, and Zelevinsky $z_i$ coincide ($z_i = e_i$).

Let $(a_1, a_1, a_2, a_2, b_1, b_1, b_2, b_2, b_3, c_1, c_2, c_3, c_4, c_5, c_5) \in S((3, 3), (2, 2, 2), (1, 1, 1, 1, 2))$.

Let $B = \begin{bmatrix}
    b_1 & 0 & 0 & a_1 + a_2 & -a_1 - a_2 + \\
    0 & b_1 & a_1 - b_1 - c_5 & -b_1 - b_3 & b_1 + b_3 + \\
    0 & 0 & b_2 & -c_1 - c_5 & c_1 + c_5
\end{bmatrix}$.

(4.49)
Products of irreducible representations into sums of irreducibles:

\[ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \]

There exists a famous combinatorial algorithm to compute the Littlewood-Richardson coefficient called the Klyachko cone. The lattice points of the Klyachko cone are exactly the triples of highest weights with non-zero Littlewood-Richardson coefficients.

5 Connections with the Littlewood-Richardson rule

An irreducible rational representation of \( GL(n, \mathbb{C}) \) is uniquely determined by its highest weight \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) where \( \lambda_i \) are integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). We can decompose tensor products of irreducible representations into sums of irreducibles:

\[ V_\lambda \otimes V_\mu = \sum_\nu c_{\lambda \mu}^\nu V_\nu. \] (5.51)

The number \( c_{\lambda \mu}^\nu \) of copies of \( V_\nu \) in \( V_\lambda \otimes V_\mu \) is called the Littlewood-Richardson coefficient. There exists a famous combinatorial algorithm to compute the Littlewood-Richardson coefficient called the Littlewood-Richardson rule (see [9] for more information). It follows from the results of A. Klyachko [16] combined with a refinement by A. Knutson and T. Tao [17], that the lattice points of the Klyachko cone are exactly the triples of highest weights with non-zero Littlewood-Richardson coefficients.
coefficients (see also [8] for a nice survey). The question whether all the lattice points of the Klyachko cone were such triples was raised in [36] under the name of the saturation conjecture. The conjecture was proved by Knutson and Tao in [17]. Some of the Klyachko inequalities describing the Klyachko cone are redundant. Knutson, Tao, and Woodward give in [18] the set of necessary inequalities for the Klyachko cone. H. Derksen and J. Weyman in [6] give a proof of the saturation conjecture different from that of Knutson and Tao. They use methods of the theory of quiver representations, developing further ideas of A. Schofield from [31]. Moreover, in [7], Derksen and Weyman give description of all the faces of the Klyachko cone of arbitrary dimension. However, all these results involve recursive computations.

The inequalities of Theorem 2.3, Theorem 2.6, Theorem 2.9, and Theorem 2.12 give non-recursive description of some faces of the Klyachko cone. Thus, integral solutions to these inequalities give non-recursive description of some triples of highest weights with \( c^r_{\nu \lambda \mu} \neq 0 \).

Let us show a different way to derive these inequalities and also show that the corresponding \( c^r_{\nu \lambda \mu} = 1 \). For that, we use the Berenstein-Zelevinsky triangle. It was invented in [1] as a geometric version of the Littlewood-Richardson rule. A variation of the BZ-triangle was used in [17] under the name of a honeycomb tinkertoy. A different variation of the triangle was used in [10] under the name of a web-function to examine relations of the Littlewood-Richardson coefficients with a Quantum-Yang-Baxter-Type equation.

Consider the following graph.

![Berenstein-Zelevinsky triangle](image)

This is the Berenstein-Zelevinsky triangle for \( sl_r \). In order to define it formally, it is convenient to use the baricentric coordinates in \( \mathbb{R}^2 \). Namely, we represent a point in \( \mathbb{R}^2 \) by a triple \((\alpha, \beta, \gamma)\) such that \( \alpha + \beta + \gamma = 0 \). The \( r \)-th Berenstein-Zelevinsky triangle \( BZ_r \) is the set of points in \( \mathbb{R}^2 \) with baricentric coordinates \((\alpha, \beta, \gamma)\), such that

1. \( 0 < \beta < -\alpha < r + 1 \);
2. the numbers $2\alpha$, $2\beta$, and $2\gamma$ are integers;

3. at least one $\alpha$, $\beta$, or $\gamma$ is not integer.

Every integer point $(a, b, c)$, $a + b + c = 0$, with $0 < b < -a < r + 1$ has six neighbors in $\text{BZ}_r$ that form vertices of the following hexagon:

$$A = (a, b - \frac{1}{2}, c + \frac{1}{2}) \quad B = (a - \frac{1}{2}, b, c + \frac{1}{2})$$

$$F = (a + \frac{1}{2}, b - \frac{1}{2}, c) \quad (a, b, c) \quad C = (a - \frac{1}{2}, b + \frac{1}{2}, c)$$

$$E = (a + \frac{1}{2}, b, c - \frac{1}{2}) \quad D = (a, b + \frac{1}{2}, c - \frac{1}{2})$$

**Definition 5.1** A function $f : \text{BZ}_r \rightarrow \{0, 1, 2, \ldots \}$ is called a $\text{BZ-filling}$ if for any hexagon as above we have $f(A) + f(B) = f(D) + f(E)$, $f(B) + f(C) = f(E) + f(F)$, and $f(C) + f(D) = f(F) + f(A)$ (the last condition follows from the first two). We call this the hexagon condition.

Let $\lambda = \sum_{i=1}^r l_i \omega_i$, $\mu = \sum_{i=1}^r m_i \omega_i$, and $\nu = \sum_{i=1}^r n_i \omega_i$, where the $\omega_i$ are the **fundamental weights** of $\text{sl}(r + 1, \mathbb{C})$. Let us assign $l_i$, $m_i$, and $n_i$ to the boundary segments of the $\text{BZ}_r$ as shown on the picture on page 46. Note that $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$, $n_i = \nu_i - \nu_{i+1}$.

**Definition 5.2** A filling $f$ of $\text{BZ}_r$ satisfies boundary conditions if for any boundary segment with vertices $A$, $B$, and a nonnegative integer value $v$ assigned to the segment, $f(A) + f(B) = v$.

**Theorem 5.1 (Berenstein, Zelevinsky)** Let $\lambda = \sum_{i=1}^r l_i \omega_i$, $\mu = \sum_{i=1}^r m_i \omega_i$, and $\nu = \sum_{i=1}^r n_i \omega_i$ be dominant weights of $\text{sl}(r + 1, \mathbb{C})$. Then $c^\nu_{\lambda\mu} = \# \{ \text{fillings of $\text{BZ}_r$ satisfying the boundary conditions} \}$.

Let us use the BZ-triangle for a different proof of Theorem 2.3 and also to show that the corresponding Littlewood-Richardson coefficient is equal to one.

**Proof of Theorem 2.3** — Let us assume that $a_1 < a_2$. Consider $\text{BZ}_r$ for the hypergeometric case ($r = m - 1$). For that, we have to switch from $gl(n, \mathbb{C})$ to $sl(n, \mathbb{C})$. Let us set $\tilde{A} = A - \frac{1}{r+1} \text{tr}(A) \text{Id}$, $\tilde{B} = B - \frac{1}{r+1} \text{tr}(B) \text{Id}$, and $\tilde{C} = C - \frac{1}{r+1} \text{tr}(C) \text{Id}$. Then $s(\tilde{A}) = \{\tilde{a}, \tilde{a}, \ldots, \tilde{a}, -r \tilde{a}\}$. Let us call the eigenvalues of $\tilde{B}$ and $\tilde{C}$ $\tilde{b}_i$ and $\tilde{c}_j$. We have $\sum_{j=1}^{r+1} \tilde{b}_j = \sum_{j=1}^{r+1} \tilde{c}_j = 0$. Recall that $l_i = \tilde{b}_i - \tilde{b}_{i+1}$; $m_i = \tilde{c}_i - \tilde{c}_{i+1}$; $n_1 = n_2 = \cdots = n_{r-1} = 0$, and $n_r = (r + 1)a$. 47
If a hexagon has two zeros on a side, then the nonnegativity of BZ-fillings and the hexagon condition force two zeros on the opposite side. In the hypergeometric case, this mechanism reduces the BZ-triangle to a strip. Let us put a variable $x$ in an unfilled vertex of the strip. Then the filling is expressed in terms of $x$, and the boundary conditions. Also the $l_r$-boundary condition gives a linear equation on $x$. 
In the notations of (4.49), (4.50), let $A$ to compute the inequalities on the eigenvalues of the corresponding Hermitian form. However, the BZ-triangle enables us to see that $x$ is obtained from $-\tilde{b}_2 - \tilde{c}_{n-1} + \tilde{a}$, and $b_1 + \tilde{b}_2 + \tilde{c}_{n-1} + \tilde{c}_n + (n-2)\tilde{a}$. All the numbers in the middle part of the strip except for the atmost right one $(-\tilde{b}_n - \tilde{c}_1 + \tilde{a})$ do not produce generating inequalities for the same reason. The numbers on the lower part of the strip together with the last middle number produce the following generating inequalities:

$$
\begin{align*}
\tilde{b}_1 + \tilde{c}_{m-1} &> \tilde{b}_1 + \tilde{c}_m \\
\tilde{b}_2 + \tilde{c}_{m-2} &> \tilde{b}_2 + \tilde{c}_{m-1} \\
\vdots &> \tilde{a} \\
\tilde{b}_{m-1} + \tilde{c}_1 &> \tilde{b}_{m-1} + \tilde{c}_2 \\
&> \tilde{b}_m + \tilde{c}_1.
\end{align*}
$$

(5.52)

Switching back to $gl(n, \mathbb{C})$ proves the theorem in this case. The case $a_1 > a_2$ is obtained from the case $a_1 < a_2$ in the following way. Let us multiply $A$, $B$, and $C$ by $-1$. Then let us renumber $b_i$ and $c_j$ so that $b_1 > b_2 > \cdots > b_m$ and $c_1 > c_2 > \cdots > c_m$ again.

One can similarly prove Theorems 2.6, 2.9, 2.12 and also show that the corresponding Littlewood-Richardson coefficients are equal to one.

In the $E_8$ case of the Magyar, Weyman, Zelevinsky list (4.43), we do not have an explicit criterion for positivity of the corresponding Hermitian form. However, the BZ-triangle enables us to compute the inequalities on the eigenvalues of $A$, $B$, and $C$ which make the form sign-definite. In the notations of (4.49), (4.50), let $a_1 > a_2$ and $b_1 > b_2 > b_3$. Then the form is sign-definite precisely in the following situations.
The first set of inequalities forces $c_1 > c_2 > c_3 > c_4 > c_5$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1)$. The second set of inequalities forces $c_1 > c_2 > c_3 > c_5 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1)$. The third set of inequalities forces $c_1 > c_2 > c_5 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1)$. The forth set of inequalities forces $c_1 > c_5 > c_2 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1)$. The last set of inequalities forces $c_5 > c_1 > c_2 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1)$. Thus, all the members of the $E_8$-family from $[4.43]$ can be constructed this way with the help of the corresponding eigenvectors of $A$, $B$, and $C$.

6 Fuchsian systems, Fuchsian equations, Okubo normal forms, and the list of Haraoka-Yokoyama.

Let us consider a system of linear differential equations on a $\mathbb{C}^n$-valued function $f$ on $\mathbb{C}P^1$. 

| $-a_1 - a_2 + b_1 + b_3 + c_1 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_3 + c_2 + c_5$ |
| $-a_1 - a_2 + b_1 + b_2 + c_3 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_2 + c_4 + c_5$ |
| $a_1 - a_2 - c_1 + c_2 - c_3 + c_4$ | $> 0$ | $a_1 - a_2 - c_1 - c_2 + c_3 + c_4$ |
| $0$ | $> -a_2 + b_2 + c_5$ |

| $-a_1 - a_2 + b_1 + b_3 + c_2 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_3 + c_3 + c_5$ |
| $-a_2 + b_2 + c_5$ | $> 0$ | $-a_2 + b_2 + c_4$ |
| $-a_1 + a_2 + c_1 - c_2 + c_3 - c_4$ | $> 0$ | $0$ |
| $0$ | $> -a_1 - a_2 + b_2 + b_3 + c_1 + c_5$ |
| $0$ | $> -a_1 + b_1 + c_4$ |

| $-a_1 - a_2 + b_2 + b_3 + c_1 + c_5$ | $> 0$ | $-a_1 - a_2 + b_2 + b_3 + c_2 + c_5$ |
| $-a_1 - a_2 + b_1 + b_2 + c_3 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_2 + c_4 + c_5$ |
| $a_1 - a_2 - c_1 + c_2 - c_3 + c_4$ | $> 0$ | $0$ |
| $-a_1 + b_1 + c_5$ | $> 0$ |

| $-a_1 - a_2 + b_1 + b_3 + c_2 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_3 + c_3 + c_5$ |
| $-a_1 + b_2 + c_1$ | $> 0$ | $-a_1 + b_2 + c_5$ |
| $-a_1 - a_2 + b_1 + b_2 + c_4 + c_5$ | $> 0$ | $0$ |
| $-a_2 + b_3 + c_1$ | $> 0$ |
| $0$ | $> a_1 - a_2 - c_1 + c_2 - c_3 + c_4$ |

| $-a_1 - a_2 + b_1 + b_3 + c_3 + c_5$ | $> 0$ | $-a_1 - a_2 + b_1 + b_3 + c_4 + c_5$ |
| $-a_1 - a_2 + b_2 + b_3 + c_1 + c_5$ | $> 0$ | $-a_1 - a_2 + b_2 + b_3 + c_2 + c_5$ |
| $a_1 + a_2 + c_1 + c_2 - c_3 - c_4$ | $> 0$ | $0$ |
| $-a_1 + a_2 + c_1 - c_2 + c_3 - c_4$ | $> 0$ |
| $-a_1 + b_2 + c_5$ | $> 0$ |
\[ df = \omega f, \quad (6.54) \]

where \( \omega \) is a \((n \times n)\) matrix-valued 1-differential form on \( \mathbb{CP}^1 \). Let the form be holomorphic everywhere on \( \mathbb{CP}^1 \) except for a finite set of points \( D = \{ z_1, z_2, \ldots z_k \} \). Let us consider a solution of (6.54) restricted to a sectorial neighborhood centered at any \( z_i \in D \). If any such solution has polynomial growth when it approaches \( z_i \) within any such sector, then the system (6.54) is called linear regular. If \( \omega \) has only first order poles at \( D \), then the system is called Fuchsian. Any Fuchsian system is linear regular, but there exist linear regular systems which are not Fuchsian (for more detailed treatment, see [3] or [33]).

An \( n \) order Fuchsian equation is a linear differential equation

\[ f^n(z) + q_1(z)f^{n-1}(z) + \cdots + q_n(z)f(z) = 0 \quad (6.55) \]

such that its coefficients \( q_j(z) \) have a finite set of poles \( D = \{ z_1, z_2, \ldots z_k \} \) and in a small neighborhood of a pole \( z_i \) the coefficients of (6.55) have the form

\[ q_j(z) = \frac{r_j(z)}{(z - z_i)^j}, \quad j = 1, \ldots, n, \quad (6.56) \]

where the \( r_j(z) \) are holomorphic functions. Solutions of Fuchsian equations have polynomial growth when continued analytically towards a pole. This distinguishes Fuchsian differential equations from all other linear differential equations on \( \mathbb{CP}^1 \). Thus, for linear differential equations the notions “Fuchsian” and “linear regular” coincide.

The matrix \( R_i = \text{Res}_{z = z_i} \omega(z) \) is called the residue of a linear regular system at \( z_i \). By Cauchy residue theorem, \( \sum_{i=1}^{k} R_i = 0 \).

**Theorem 6.1** (see [3]) Any Fuchsian system has the standard form

\[ \frac{df}{dz} = \sum_{i=1}^{k} \frac{R_i}{z - z_i} f(z). \quad (6.57) \]

**Theorem 6.2** (see [3]) For any Fuchsian equation on the Riemann sphere, it is possible to construct a Fuchsian system with the same singular points and the same monodromy. Conversely, it is not true.

**Remark 6.1** Thus, the notion of a residue matrix makes sense for a Fuchsian equation as well as for a Fuchsian system.

To study Fuchsian differential equations, K. Okubo invented what became later known as the Okubo normal form of a Fuchsian equation. In [29], he proves that any Fuchsian equation can be written in the following form:

\[ (t \text{Id} - B) \frac{dx}{dt} = Ax, \quad (6.58) \]

where \( t \) is a complex variable, \( x \in \mathbb{C}^n \) is an unknown vector, \( \text{Id} \) is the identity matrix of order \( n \), \( B \) is a constant diagonal \( n \times n \) matrix, and \( A \) is a constant \( n \times n \) matrix. Let

\[ B = \text{diag}(z_{11}, \ldots, z_{1n_1}, z_{21}, \ldots, z_{2n_2}, \ldots, z_{k1}, \ldots, z_{kn_k}), \quad (6.59) \]
such that $z_i \neq z_j$ for $i \neq j$, $n_1 + n_2 + \cdots + n_k = n$, and $n_1 \geq n_2 \geq \cdots \geq n_k$. The partition $(n_1, n_2, \cdots, n_k)$ of $n$ endows $A$ with the block decomposition $A = (A_{ij})_{1 \leq i,j \leq d}$. Let us call $\Lambda_i$ the set of eigenvalues of $A_{ii}$ and let us call $\Lambda_\infty$ the set of eigenvalues of $A$. Then $z_1, z_2, \cdots, z_k$ and $\infty$ are the singular points of (6.58). At the point $z_i$, (6.58) has $n_i$ non-holomorphic solutions with local exponents $\lambda_j \in \Lambda_i$. At $\infty$, (6.58) has local exponents $\lambda_j \in \Lambda_\infty$.

In [35], T. Yokoyama used Okubo theory to classify the spectral types of rigid irreducible Fuchsian equations. For such, all $A_{ii}$ are diagonalizable as well as $A$ itself. Quoting the result of Yokoyama, we will not give the spectral types of $A_{ii}$ and $A$ the way he does. Instead, we will list spectral types of the residue matrices (which are diagonalizable, too).

| I   | $(m - 1, 1), (1^m), (1^m)$          | $m \geq 2$ |
|-----|------------------------------------|------------|
| II  | $(m, 1^m), (m, 1^m), (m, m - 1, 1)$ | $m \geq 2$ |
| III | $(m, 1^{m+1}), (m + 1, 1^m), (m, m, 1)$ | $m \geq 2$ |
| IV  | $(4, 1, 1), (2, 1, 1, 1), (2, 2, 2)$ |            |
| I*  | $(m - 1, 1), \cdots, (m - 1, 1)$  | $m \geq 2$ |
| I*  |                                          | $m$ times   |
| II* | $(m, 1^m), (m + 1, 1^{m-1}), (2m - 1, 1), (m, m)$ | $m \geq 2$ |
| III*| $(m + 1, 1^m), (m + 1, 1^m), (2m, 1), (m + 1, m)$ | $m \geq 2$ |
| IV* | $(4, 1, 1), (4, 1, 1), (4, 1, 1), (4, 2)$ |            |

Y. Haraoka explicitly constructed the equations of the above spectral types in [11]. In [12], he explored the solutions of these equations: computed their monodromies, found monodromy invariant forms in their spaces of solutions, etc. It turns out that the solutions of these equations are important hypergeometric functions. It also turns out that the Fuchsian systems constructed in our paper are closely related to Yokoyama-Haraoka equations: sometimes the $A$ matrices are just the same! We think it is interesting to understand the nature of this relation, find solutions to our systems, and their monodromies. We plan to do it in a subsequent publication.

7 Appendix

In this section we collect the identities needed for the proofs in the previous sections.
For $k < n - 1$,

$$
\sum_{i=1}^{n} \frac{n}{\prod_{j \neq i}^{j=1} (x_i - y_j)} = 0 \quad (7.61)
$$

$$
\sum_{i=1}^{n} \frac{n}{\prod_{j \neq i}^{j=1} (x_i - x_j)} = 1 \quad (7.62)
$$

$$
\sum_{i=1}^{n} \frac{n}{\prod_{j \neq i}^{j=1} (x_i - y_j)} = \sum_{i=1}^{n} (x_i - y_i) \quad (7.63)
$$

$$
\sum_{i=1}^{n} \frac{n}{\prod_{j \neq i}^{j=1} (x_i - x_j)} = \sum_{i=1}^{n} \frac{n}{\prod_{j \neq i}^{j=1} (x_i - x_j)} \quad (7.64)
$$

For $1 \leq i \leq m - 1$,

$$
\sum_{j=1}^{m} \frac{m}{\prod_{k \neq j}^{j=1} (x_j - x_k)} \frac{m}{\prod_{k \neq j}^{j=1} (y_i - y_k)} = 1 \quad (7.65)
$$

$$
\frac{m}{\prod_{k \neq j}^{j=1} (y_i - y_k)} + \frac{m}{\prod_{k \neq j}^{j=1} (x_j - x_k)} \sum_{j=1}^{m} \frac{x_j^2}{(y_i - x_j)^2} \frac{m}{\prod_{k \neq j}^{j=1} (x_j - x_k)} = 1 \quad (7.66)
$$

For $1 \leq i \leq m$,

$$
\frac{m}{\prod_{k \neq i}^{k=1} (x_i - x_k)} + \frac{m}{\prod_{k \neq i}^{k=1} (y_i - y_k)} + \frac{m}{\prod_{k \neq i}^{k=1} (y_i + x_k)} = 1 \quad (7.67)
$$

\[
\sum_{i=1}^{n} \frac{(x_i + x_1)(x_i + x_2) \cdots (x_i + x_n)}{(x_i - x_1)(x_i - x_2) \cdots (x_i - x_n)} = \begin{cases} 
0, & \text{if } n \text{ is even} \\
1, & \text{if } n \text{ is odd} 
\end{cases} \quad (7.68)
\]

\[
\sum_{i=1}^{n} \frac{x_i(x_i + x_1)(x_i + x_2) \cdots (x_i + x_n)}{(x_i - x_1)(x_i - x_2) \cdots (x_i - x_n)} = x_1 + x_2 + \cdots + x_n \quad (7.69)
\]

\[
\sum_{i=1}^{n} \frac{x_i^2(x_i + x_1)(x_i + x_2) \cdots (x_i + x_n)}{(x_i - x_1)(x_i - x_2) \cdots (x_i - x_n)} = (x_1 + x_2 + \cdots + x_n)^2 \quad (7.70)
\]
\[
\frac{\prod_{k=1}^{m} (y_j + y_k)}{\prod_{k=1}^{m} (y_j - x_k)} + \sum_{r=1}^{m} \frac{m}{x_r - y_j} = 1 \quad \text{for} \quad j = 1, 2, \ldots, m - 1 \quad (7.71)
\]

\[
\frac{y_j^2}{\prod_{k=1}^{m} (y_j - x_k)} + \sum_{r=1}^{m} \frac{x_r^2}{x_r - y_j} = \sum_{r=1}^{m} (x_r + y_r) \quad \text{for} \quad j = 1, 2, \ldots, m - 1 \quad (7.72)
\]

For \( i = 1, 2, \ldots, m - 1 \) and \( j = 1, 2, \ldots, m, \)
\[
\frac{m - 1}{\prod_{k=1}^{m} (y_r - y_k)} \sum_{s=1}^{m} \frac{1}{x_s + y_i} = \frac{m}{\prod_{k=1}^{m} (x_r - y_k)} + \frac{m}{\prod_{k=1}^{m} (x_j - y_k)} = 1 \quad (7.73)
\]

All these identities have the following features: the left hand side \( L(x, y) \) is a rational homogeneous function in \( x_i \) and \( y_j \). All the denominators of \( L(x, y) \) are products of linear forms \( \alpha \) of the form \( (x_i \pm x_j), (y_i \pm y_j), \) or \( (x_i \pm y_j) \). The power of every such form in any denominator is 1. The right hand sides \( R(x, y) \) are constants or homogeneous polynomials in \( x_i \) and \( y_j \) of degree 1 or 2.

The first step to prove such an identity is to prove that \( L(x, y) \) is in fact a polynomial. For that, it is enough to prove that \( \alpha L(x, y) |_{\alpha = 0} = 0 \) for every form \( \alpha \) from any denominator of the identity. For all the identities except for \( (7.73) \), the restriction of \( \alpha L(x, y) \) to the hyperplane \( \alpha = 0 \) turns to be a sum of just two terms with equal absolute values and opposite signs. For example, consider identity \( (6.2) \). Let us fix \( p \) and \( q \) such that \( 1 \leq p < q \leq n \). Consider
\[
(x_p - x_q) \sum_{i=1}^{n} \prod_{j=1}^{n} \frac{(x_i - y_j)}{(x_i - x_j)}
\]
restricted to the hyperplane \( x_p = x_q \). The restriction equals
\[
\prod_{j=1}^{n} \frac{(x_p - y_j)}{(x_p - x_j)} - \prod_{j=1}^{n} \frac{(x_q - y_j)}{(x_q - x_j)} = 0.
\]

For identity \( (7.73) \), the same technique works for all the forms in the denominators except for \( \alpha = x_p + x_m \) where \( 1 \leq p \leq m - 1 \). If \( \alpha = x_p + x_m \), then the restriction of \( \alpha L(x, y) \) to the hyperplane \( \alpha = 0 \) is
\[
- \sum_{r=1}^{m-1} \frac{m-1}{\prod_{k=1}^{m} (y_r - y_k)} \sum_{s=1}^{m} \frac{m}{x_s - y_p} + \frac{m}{\prod_{k=1}^{m} (x_p + y_k)} - \sum_{r=1}^{m-1} \frac{m-1}{\prod_{k=1}^{m} (y_r - y_k)} \sum_{s=1}^{m} \frac{m}{x_s - y_p} + \frac{m}{\prod_{k=1}^{m} (x_p + y_k)}.
\]
The fact that this restriction equals zero is equivalent to the identity
\[
\sum_{r=1}^{m} \prod_{k \neq p}^{m} (y_r + x_k) \prod_{k \neq r}^{m} (x_p - y_k) = 1
\]
which is similar to identity (7.65), but different from it. In order to prove (7.74), it is convenient to rewrite it as
\[
\sum_{r=1}^{m} \prod_{k \neq p}^{m} (y_r + x_k) / \prod_{k \neq r}^{m} (x_p - y_r) - \prod_{k \neq p}^{m} (x_p + x_k) / \prod_{k \neq r}^{m} (x_p - y_k) = 0
\]
and use the same technique over again.

The second step in the proofs is to show that a polynomial \(L(x, y)\) equals the corresponding polynomial \(R(x, y)\). Let us, for example, consider (7.70). In this case, \(L(x)\) and \(R(x)\) are symmetric homogeneous polynomials in \(x\) of degree 2. The space of such polynomials is two-dimensional. It is spanned by \(s_2 = x_1^2 + \cdots + x_n^2\) and \(s_1 = x_1 + \cdots + x_n\). To prove that \(L(x) \equiv R(x)\), we have to find two linearly independent functionals \(f_1\) and \(f_2\) on this space such that \(f_i(L) = f_i(R)\) for \(i = 1, 2\). We will treat the cases \(n = 2k\) and \(n = 2k+1\) separately.

Let \(n = 2k\). Let \(p_1 = (-k, -k+1, \ldots, -1, 1, 2, \ldots, k)\) and \(p_2 = (-k+1, -k+2, \ldots, -1, 1, 2, \ldots, k+1)\). For a symmetric homogeneous polynomial \(s\) of degree 2, let \(f_i(s) = s(p_i)\), where \(i = 1, 2\). Then
\[
\begin{vmatrix}
  f_1(s_2) & f_1(s_1^2) \\
  f_2(s_2) & f_2(s_1^2)
\end{vmatrix}
= \begin{vmatrix}
  \frac{(2k+1)(k+1)k}{3} & 0 \\
  \frac{(2k-1)k(k-1)}{3} + k^2 + (k+1)^2 & (2k+1)^2
\end{vmatrix}
= \frac{(2k+1)^3(k+1)}{3} \neq 0.
\]
Thus, \(f_1\) and \(f_2\) are linearly independent. We have \(f_1(L) = L(p_1) = 0 = R(p_1) = f_1(R)\) and \(f_2(L) = L(p_2) = (2k+1)^2 = R(p_2) = f_2(R)\). This finishes the proof for \(n = 2k\). For \(n = 2k+1\), take \(p_1 = (-k, -k+1, \ldots, k)\) and \(p_2 = (-k+1, -k+2, \ldots, k+1)\). The rest of the proof is analogous to the case \(n = 2k\). Proofs of other identities are finished similarly.

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