Gδσδ Haar null sets without Gδ hulls in Zω

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Abstract. We show that in the non-locally-compact abelian Polish group Zω there exists a Π⁰₄ Haar null set that is not contained in any Π⁰₂ Haar null set. This partially answers a question of M. Elekes and Z. Vidnyánszky.

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1. Introduction and preliminaries

In [1] M. Elekes and Z. Vidnyánszky asked the following questions:

Question 1.1. Let G be a non-locally compact abelian Polish group. Does there exist an Fσ Haar null set that cannot be covered by a Gδ Haar null set?

Question 1.2. What is the least complexity of such a set? And in general, what is the least complexity of a Haar null set that cannot be covered by a Π₀⁸ Haar null set?

We partially answer this second question by constructing a Gδσδ Haar null subset of Zω that is not contained in any Gδ Haar null set. We follow the general structure of [1], but almost all parts of that proof used methods which construct Borel sets without giving an upper bound on the Borel class and we had to replace these by alternative approaches. Some of these approaches strongly use the properties of Zω, while others probably could work in a more general setting.

We use the notation \([m, n] = [m, n] \cap \mathbb{Z}\) for sets consisting of consecutive integers. Z₊ denotes the set of positive integers (and as usual, \(\mathbb{N}\) denotes the set of nonnegative integers). For \(n \in \omega\) let prₙ be the canonical projection \(prₙ : \mathbb{Z}^\omega \to \mathbb{Z}\), \(a \mapsto a(n)\).

We will use some notations related to sequences (i.e. function whose domain is either a natural number or ω). Let S be an arbitrary set. Let \(S^{<\omega} = \bigcup_{n \in \omega} S^n\) be the set of finite sequences of elements of S. For \(s \in S^{<\omega}\), \(|s|\) denotes the length of s; \(\emptyset\) denotes the sequence of length 0. If s and s’ are two sequences and s is finite

2010 Mathematics Subject Classification. Primary 03E15; Secondary 28C10, 22F99.
(s′ may be infinite), then s ∼ s′ denotes the sequence created by appending s′ after the elements of s. If x is a sequence of length at least n (usually x ∈ S^ω), then x | n denotes the sequence formed by the first n elements of x. For a sequence s ∈ S^≤ω, let [s] ⊆ S^ω be the set of sequences which have s as an initial segment, i.e. [s] = {x ∈ S^ω : x|n = s} = {s ∼ y : y ∈ S^ω}. For k ∈ ω, the lexicographic ordering on S^k is denoted by <_leκ and the minimum of a set X ⊆ S^k under this order is denoted by min X.

If S is endowed with the discrete topology, then ([s] : s ∈ S^≤ω) is a clopen base of the product topology of S^ω. We will use the following fact:

**Fact 1.3.** If S₁, S₂, . . . , Sₙ are countable sets endowed with the discrete topology, then every open set U ⊆ S₁^ω × S₂^ω × . . . × Sₙ^ω can be written as a disjoint union of the form

\[ U = \bigcup_{j \in \omega} \left( [s_1^{(j)}] \times [s_2^{(j)}] \times \ldots \times [s_k^{(j)}] \right) \]

where s_i^{(j)} ∈ S_i^≤ω for every 1 ≤ i ≤ k and j ∈ ω.

We omit the straightforward proof of this fact.

2. HAAR NULL SETS IN Z^ω

Let ν_k be the uniform probability measure on [0, k], that is, the (Borel probability) measure on Z defined by ν_k(X) = \(|X|/k+1|\).

If (a(n))ₙ∈ω is a sequence of positive integers, then let µ_a be the Borel probability measure on Z^ω defined as the product \(\otimes_{n \in \omega} ν_{a(n)}\). Clearly supp µ_a = \(\prod_{n \in \omega} [0, a(n)]\).

The following lemma characterizes the Borel Haar null subsets of Z^ω:

**Lemma 2.1.** A Borel subset B ⊆ Z^ω is Haar null if and only if there exists a sequence of positive integers (a(n))ₙ∈ω such that µ_a(B + x) = 0 for every x ∈ Z^ω. We call a sequence (a(n))ₙ∈ω satisfying this a witness sequence for B.

This lemma is motivated by [3, Theorem 4.1.], but that result works in a much more general setting and yields a slightly different statement. It would be possible to modify the proof of [3, Theorem 4.1.] to prove our lemma, but some technical complications make that path unpleasant. Instead, we give an unrelated proof which is relatively long, but uses only elementary ideas.

**Proof.** The “if” part of the statement is trivial. To prove the “only if” part, let µ be a witness measure for B with compact support (that is, a Borel probability measure µ with compact supp µ ⊆ Z^ω that satisfies µ(B + x) = 0 for every x ∈ Z^ω).

To simplify the calculations, choose a witness measure µ which is supported on the non-positive sequences (that is, supp µ ⊆ [−∞, 0]^ω). This is always possible, as we may replace µ by ν(X) = µ(X − ℓ) where ℓ ∈ Z^ω is the sequence ℓ(n) = max(pr_n(supp µ)).

Let M(n) = − min(pr_n(supp µ)) (we use that supp µ is compact and hence its continuous image is also compact). It is clear that supp µ ⊆ \(\prod_{n \in \omega} [−M(n), 0]\).
Choose a sequence $N(n)$ of (large) positive integers such that $N(n) > 2M(n)$ (for every $n \in \omega$) and moreover

$$\prod_{n \in \omega} \left(1 - \frac{M(n)}{N(n) + 1}\right) > 0.$$ 

Let $\nu$ be the measure $\nu = \mu * \mu_N$. This is the convolution of Borel probability measures, hence itself a Borel probability measure on $\mathbb{Z}^\omega$. Applying the definition of convolution and Fubini’s theorem, then using that $\mu$ is a witness measure we can see that for every $x \in \mathbb{Z}^\omega$

$$\nu(B + x) = \int_{\mathbb{Z}^\omega} \mu(B + x - y) \, d\mu_N(y) = 0,$$

i.e. $\nu$ is also a witness measure for $B$. It is easy to see that $\text{supp} \, \nu \subseteq \prod_{n \in \omega} [-M(n), N(n)]$.

The measure $\nu$ is a “smoothed” variant of $\mu$, in fact if we ignore some “border zones” then the measure will be “uniform” on the “central zone” and this central zone will have positive measure. This will allow us to restrict $\nu$ to this central zone, normalize it and get a witness measure that is of the form $\mu_a$ for a witness sequence $a \in \mathbb{Z}^\omega_+$. 

More precisely, let us define the sequence $a(n) = N(n) - M(n)$ (this will be the “size” of the central zone). Our heuristic statement can be formalized as the following claim; after its proof we will be also able to finish the proof of Lemma 2.1 quickly.

**Claim 2.2.** The set $\text{supp} \, \mu_a = \prod_{n \in \omega} [0, a(n)] \cap \text{supp} \, \mu_a$ has positive $\nu$-measure. Moreover,

$$\mu_a(X) = \frac{\nu(X \cap \text{supp} \, \mu_a)}{\nu(\text{supp} \, \mu_a)}$$

holds for every $X \subseteq \mathbb{Z}^\omega$ (and these are defined for the same sets).

**Proof.** For every set $X \subseteq \text{supp} \, \mu_a$ one can apply Fubini’s theorem to get

$$\nu(X) = \int_{\mathbb{Z}^\omega} \mu_N(X - y) \, d\mu(y) = \int_{\text{supp} \, \mu} \mu_N(X - y) \, d\mu(y).$$

If $y \in \text{supp} \, \mu$ is arbitrary, then $-M(n) \leq y(n) \leq 0$ for every $n \in \omega$, hence

$$v_{N(n)}(W - y(n)) = v_{N(n)}(W)$$

for every $W \subseteq [0, a(n)]$ (using that $a(n) = N(n) - M(n)$ and $v_k$ is an uniform distribution). Moreover, notice that $\mu_N$ is the product of the measures $(v_{N(n)})_{n \in \omega}$ and the measure which assigns $\mu_N(P - y)$ to the set $P \subseteq \mathbb{Z}^\omega$ is the product of the measures which assign $v_{N(n)}(W - y(n))$ to the set $W \subseteq \mathbb{Z}$. Thus the equality $v_{N(n)}(W - y(n)) = v_{N(n)}(W)$, which holds for every $W \subseteq [0, a(n)]$, yields that these two product measures coincide when they are restricted to subsets of $\prod_{n \in \omega} [0, a(n)] = \text{supp} \, \mu_a$. If we apply this to $X \subseteq \text{supp} \, \mu_a$, then we get $\mu_N(X - y) = \mu_N(X)$ for every $y \in \text{supp} \, \mu$. Using this

$$\nu(X) = \int_{\text{supp} \, \mu} \mu_N(X) \, d\mu(y) = \mu_N(X).$$

(1)
For \( n \in \omega \) and \( W \subseteq \mathbb{Z} \) it is straightforward from the definitions that \( v_{a(n)}(W) = \int_{\mathbb{Z}} f_n(w) \, dv_{N(n)}(w) \) where
\[
f_n(w) = \begin{cases} \frac{N(n)+1}{a(n)+1} & \text{if } w \in [0,a(n)], \\ 0 & \text{otherwise}. \end{cases}
\]
From this it is clear that if \( X \subseteq \text{supp } \mu_a \), then
\[
(2) \quad \mu_a(X) = \lambda \cdot \mu_N(X) \text{ for the constant } \lambda = \prod_{n \in \omega} \frac{N(n)+1}{a(n)+1}.
\]
Here \( 1 < \lambda \) is trivial and \( \lambda < \infty \), because we assumed that \( \prod_{n \in \omega} (1 - \frac{M(n)}{N(n)+1}) > 0. \)

Considering the special case when \( X = \text{supp } \mu_a \), we get
\[
(3) \quad \lambda \cdot \mu_N(\text{supp } \mu_a) = \mu_a(\text{supp } \mu_a) = 1,
\]
Using (1), (2) and (3) we can see that \( \nu(X) = \mu_a(X) \cdot \nu(\text{supp } \mu_a) \) for every \( X \subseteq \mathbb{Z}^\omega \).

As we return to proving Lemma 2.1, we already know that for every \( x \in \mathbb{Z}^\omega \), \( \nu(B+x) = 0. \) This implies that for every \( x \in \mathbb{Z}^\omega \), \( \nu((B+x) \cap \text{supp } \mu_a) = 0 \) and hence (using the recently proved Claim 2.2) \( \mu_a(B+x) = 0. \) This means that \( a \) is indeed a witness sequence. \( \square \)

3. A function with a surprisingly thick graph

This theorem is the analogue of 1 Theorem 3.1 and the proof technique is also similar.

**Theorem 3.1.** There exists a partial function \( f : \mathbb{Z}_+^\omega \times 2^\omega \rightarrow \mathbb{Z}^\omega \) which is \( \Sigma^0_2 \)-measurable (i.e. the preimage of every open set is a \( \Sigma^0_2 \) subset of \( \mathbb{Z}_+^\omega \times 2^\omega \)) and satisfies the following properties:

(I) \( (\forall (a,x) \in \mathbb{Z}_+^\omega \times 2^\omega)((a,x) \in \text{dom } f) \Rightarrow f(a,x) \in \text{supp } \mu_a \)

(II) \( (\forall a \in \mathbb{Z}_+^\omega)(\forall S \in \Pi^0_2(2^\omega \times \mathbb{Z}^\omega)) (\text{graph } (f_a) \subseteq S \Rightarrow (\exists x \in 2^\omega)(\mu_a(S_x) > 0)) \)

To prove this theorem we will need the following technical lemma:

**Lemma 3.2.** Suppose that \( X = S^\omega_1 \times S^\omega_2 \times \ldots \times S^\omega_k \) for some countable sets \( S_1, S_2, \ldots, S_k \) (this is a zero-dimensional Polish space if the \( S_i \)'s are endowed with discrete topology). If \( C \in \Sigma^0_2(\mathbb{Z}_+^\omega \times X \times \mathbb{Z}^\omega) \), then
\[
\{(a,x) \in \mathbb{Z}_+^\omega \times X : \mu_a(C_{a,x}) > 0\} \text{ is also a } \Sigma^0_2 \text{ set.}
\]
Proof: The $\Sigma_2^0$ set $C \subseteq X \times \mathbb{Z}^\omega$ can be written as $C = (X \times \mathbb{Z}^\omega) \setminus \bigcap_{U \in \mathcal{U}} U$ where $\mathcal{U} \subseteq \Sigma_2^0(X \times \mathbb{Z}^\omega)$ is a countable collection of open sets.

It is straightforward to check that
\[
\{(a, x) \in \mathbb{Z}^\omega_+ \times X : \mu_a(C_{a, x}) > 0\} = \bigcup_{U \in \mathcal{U}} \{(a, x) \in \mathbb{Z}^\omega_+ \times X : \mu_a(U_{a, x}) < 1\},
\]
hence it is enough to show that
\[
S_U = \{(a, x) \in \mathbb{Z}^\omega_+ \times X : \mu_a(U_{a, x}) < 1\}
\]
is a $\Sigma_2^0$ set for an arbitrary open subset $U$ of $X \times \mathbb{Z}^\omega$.

Applying [Fact 1.3] for the $(k + 2)$ countable sets $S_0 = \mathbb{Z}_+, S_1, S_2, \ldots, S_k$ and $S_{k+1} = \mathbb{Z}$ we can write the open set $U \subseteq \mathbb{Z}^\omega_+ \times X \times \mathbb{Z}^\omega$ as a disjoint union $U = \bigsqcup_{i \in \omega} (V(i) \times [s(i)])$ where $V(i) \subseteq \mathbb{Z}^\omega_+ \times X$ is a clopen set and $s(i) \in \mathbb{Z}^\omega$ (for all $i \in \omega$). Applying this decomposition it is easy to see that for $a \in \mathbb{Z}^\omega_+$ and $x \in X$
\[
\mu_a(U_{a, x}) = \mu_a\left(\bigcup_{i \in \omega} (V(i) \times [s(i)])_{a, x}\right) = \sum_{i \in \omega} \mu_a\left(V(i) \times [s(i)]_{a, x}\right) = \sum_{\substack{i \in \omega \\omega \in X \times \mathbb{Z}^\omega}} \mu_a([s(i)])
\]
We can make the inequality non-strict in (4) by using a countable union:
\[
S_U = \bigcup_{k \in \mathbb{Z}_+} \left\{(a, x) \in \mathbb{Z}^\omega_+ \times X : \mu_a(U_{a, x}) \leq 1 - \frac{1}{k}\right\}
\]
In this form we can substitute our infinite sum for $\mu_a(U_{a, x})$
\[
S_U = \bigcup_{k \in \mathbb{Z}_+} \left\{(a, x) \in \mathbb{Z}^\omega_+ \times X : \sum_{\substack{i \in \omega \\omega \in X \times \mathbb{Z}^\omega}} \mu_a([s(i)]) \leq 1 - \frac{1}{k}\right\}
\]
and then replace the infinite summation with finite summation and an intersection operation to get
\[
S_U = \bigcup_{k \in \mathbb{Z}_+} \bigcap_{K \in \omega} \left\{(a, x) \in \mathbb{Z}^\omega_+ \times X : \sum_{i \in K} \mu_a([s(i)]) \leq 1 - \frac{1}{k}\right\}
\]
It is clear that the map $\Phi^{(i)} : \mathbb{Z}^\omega_+ \to \mathbb{R}, a \mapsto \mu_a([s(i)])$ is continuous, as the value of $\Phi^{(i)}(a)$ depends only on the initial segment $a \downarrow |s(i)|$. As $V(i)$ is clopen, the map
\[
\Phi^{(i)} : \mathbb{Z}^\omega_+ \times X \to \mathbb{R}, \quad \Phi^{(i)}((a, x)) = \begin{cases} 
\mu_a([s(i)]) & \text{if } (a, x) \in V(i), \\
0 & \text{otherwise}
\end{cases}
\]
is also continuous, and hence the map $\Psi(K) = \sum_{i \in K} \Phi^{(i)}$ is also continuous. Notice that we can rewrite (5) as
\[
S_U = \bigcup_{k \in \mathbb{Z}_+} \bigcap_{K \in \omega} (\Psi(K))^{-1}\left(\left[0, 1 - \frac{1}{k}\right]\right)
\]
and this shows that $S_U$ is the countable union of intersections of continuous preimages of closed sets, hence $S_U$ is indeed a $\Sigma_2^0$ set. □
Proof of Theorem 3.1. Let $U \in \Sigma^0_2(2^\omega \times 2^\omega \times \mathbb{Z}^\omega)$ be universal for the $\Sigma^0_2$ subsets of $2^\omega \times 2^\omega$, that is, for every $B \in \Sigma^0_2(2^\omega \times \mathbb{Z}^\omega)$ there exists an $x \in 2^\omega$ such that $U_x = B$ (for the existence of such a set see \[2\] Theorem 22.3). The preimage of this set under the continuous map $(x, g) \mapsto (x, x, g)$ is the set

$$U' = \{(x, g) \in 2^\omega \times \mathbb{Z}^\omega : (x, x, g) \in U\},$$

which is hence a $\Sigma^0_2$ set. Later we will use that that $U'_x = U_{x,x}$ for every $x \in 2^\omega$.

If we apply Lemma 3.2 for the $\Sigma^0_2$ set $C = \mathbb{Z}^\omega_+ \times U'$, then we can see that the set

$$P = \{(a, x) \in \mathbb{Z}^\omega_+ \times 2^\omega : \mu_a(U'_x) > 0\}$$

is also $\Sigma^0_2$.

Also notice that the set

$$S = \{(a, x, g) \in \mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega : g \in \text{supp } \mu_a\}$$

is equal to

$$\{(a, x, g) \in \mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega : (\forall n \in \omega)(0 \leq g(n) \leq a(n))\}$$

and hence trivially closed.

We can combine these to define the $\Sigma^0_2$ set

$$U'' = (\mathbb{Z}^\omega_+ \times U) \cap (P \times \mathbb{Z}^\omega) \cap S \quad (\subseteq \mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega).$$

(This $U''$ is the analog of the set denoted by $U'''$ in the proof of \[1\] Theorem 3.1.) For every $a \in \mathbb{Z}^\omega_+$ and $x \in 2^\omega$ it is straightforward to see that $\mu_a(U'_x) = \mu_a(U''_{a,x})$ and

$$U''_{a,x} = \begin{cases} U'_x \cap \text{supp } \mu_a & \text{if } \mu_a(U'_x) = \mu_a(U''_{a,x}) > 0, \\ \emptyset & \text{otherwise}. \end{cases}$$

In particular $P$ is the projection of $U''$ onto $\mathbb{Z}^\omega_+ \times 2^\omega$.

The proof of \[1\] Theorem 3.1] applied the ‘large section uniformization theorem’ \[2\] Theorem 18.6] and proved that the uniformizing function is a good choice for $f$. The following claim states that finding a suitable (i.e. $\Sigma^0_3$-measurable) uniformizing function is also enough in our case. For the sake of completeness we repeat this argument from \[1\].

Claim 3.3. If a partial function $f : \mathbb{Z}^\omega_+ \times 2^\omega \rightarrow \mathbb{Z}^\omega$ is $\Sigma^0_3$-measurable and satisfies that $\text{dom } (f) = P$, $\text{graph } (f) \subseteq U''$, then it satisfies properties (I) and (II) in Theorem 3.1.

Proof. Property (I) is clear because for all $(a, x) \in \text{dom } (f) = P$,

$$f(a, x) \in U'' = U'_x \cap \text{supp } \mu_a \subseteq \text{supp } \mu_a.$$

To prove Property (II), suppose for the contrary that there exists $a \in \mathbb{Z}^\omega_+$ and $S \in \Pi^0_3(2^\omega \times \mathbb{Z}^\omega)$ such that $\text{graph } (f) \subseteq S$ but for all $x \in 2^\omega$, $\mu_a(S_x) = 0$. The complements of $S$ is the $\Sigma^0_2$ set $B = (2^\omega \times \mathbb{Z}^\omega) \setminus S$. By the universality of $U$, there is an $x^* \in 2^\omega$ such that $U_{x^*} = B$. We know that for every $x \in 2^\omega$, $\mu_a(B_x) = 1 - \mu_a(S_x) = 1 > 0$, in particular $\mu_a(B_{x^*}) = \mu_a(U_{x^*,x^*}) = \mu_a(U'_{x^*}) > 0$. Therefore $(a, x^*) \in P = \text{dom } (f)$ and then $\text{graph } (f) \subseteq U''$ yields that

$$f(a, x^*) \in U'' \subseteq U'_x = U_{x^*,x^*} = B_{x^*}.$$
But we also supposed that graph $(f_a) \subset S$, and this yields $f(a, x^*) \in S_{x^*} = Z^\omega \setminus B_{x^*}$, a contradiction.

To prove Theorem 3.1 it is sufficient to show that there exists a partial function $f : Z^\omega \times 2^\omega \to Z^\omega$ is $\Sigma^0_3$-measurable and satisfies that $\text{dom}(f) = P$ and $\text{graph}(f) \subseteq U''$. We will use the following result, its proof can be found in [2, Theorem 22.21].

**Theorem 3.4.** Let $X$ be a Polish space and $A \in \Sigma^0_\xi(X)$. If $\xi > 1$, then there is a system $\{A^s : s \in N^{<\omega}\}$ which is a Lusin scheme (that is, $A^{s_n} \subseteq A^s$ and $A^{s_n} \cap A^{s_m} = \emptyset$ if $s, t \in N^{<\omega}$, $i, j \in N$ and $i \neq j$) and also satisfies the following properties:

1. $A^s \in \Delta^\emptyset_\xi$ for every $s \in N^{<\omega} \setminus \{\emptyset\}$,
2. $A^\emptyset = A$ and $A^s = \bigcup_n A^{s_n}$ for $s \in N^{<\omega}$ and $n \in N$,
3. if $x \in N^\omega$ and $A^{x_k} \neq \emptyset$ for every $k \in \omega$ then $\bigcap_k A^{x_k}$ is a singleton

Moreover, if we fix a metric on $X$ (that is compatible with the Polish topology), then we can make sure that $\text{diam}(A^s) < 2^{-|s|}$ for every $s \in N^{<\omega} \setminus \{\emptyset\}$.

Remarks: The proof of [2, Theorem 18.6] used [2, Theorem 13.9] which does not state the Borel class of the sets $A^s$, we replaced it with this stronger result. We will not need the full strenght of Property (i), we will only use that $A^s \in \Sigma^0_\xi$ for every $s \in N^{<\omega}$ (Property (i) does not consider the case when $s = \emptyset$, but $A^\emptyset = A \in \Sigma^0_\xi$ follows from Property (ii)).

If we apply this theorem for the Polish space $Z^\omega_+ \times 2^\omega \times Z^\omega$ and the $\Sigma^0_3$ set $U''$ in it, then it yields a Lusin scheme $\{A^s : s \in N^{<\omega}\}$ satisfying properties (i)-(iii).

This theorem only states that the ("deeper" layers of) the Lusin scheme consists of small sets, but we will want to fit the sets in the Lusin scheme into small, nice (in fact, clopen) sets. The following claim states that this is possible:

**Claim 3.5.** There exists a Lusin scheme $\{B^s : s \in N^{<\omega}\}$ which satisfies (i)-(iii) from the previous theorem (for $X = Z^\omega_+ \times 2^\omega \times Z^\omega$, $A = U''$ and $\xi = 2$), a function $\varphi : N^{<\omega} \to N^{<\omega}$ and another function $t : N^{<\omega} \to Z^{<\omega}$ such that they have the following properties:

1. $B^s \subseteq A^{t(s)}$ for every $s \in N^\omega$,
2. for every $s \in N^{<\omega}$, $|s| = |\varphi(s)| = |t(s)|$,
3. for every $s \in N^{<\omega}$, $B^s \subseteq Z_+^\omega \times 2^\omega \times [t(s)]$.

**Proof.** Fix an arbitrary bijection $\psi : N \times Z \to N$. The construction will be recursive; for the initial step let $B^\emptyset = A^\emptyset = U''$, $\varphi(\emptyset) = \emptyset$ and $t(\emptyset) = \emptyset$.

Suppose that we already constructed $B^s$, $\varphi(s)$ and $t(s)$ for some $s \in N^{<\omega}$. For every $n \in N$ and $z \in Z$, let

$$B^{s_n}(n, z) = A^{t(s) - n} \cap (Z^\omega_+ \times 2^\omega \times [t(s) \cap z]),$$

$$\varphi(s \cup \psi(n, z)) = \varphi(s) \cup n \quad \text{and} \quad t(s \cup \psi(n, z)) = t(s) \cup z.$$
It is clear that \( \{ B^s \} \) is a Lusin scheme which satisfies (i), (ii), (iv), (v) and (vi). For Property (iii), if some \( x \in \mathbb{N}^{<\omega} \) satisfies that \( B^x \neq \emptyset \) for every \( k \in \omega \), then
\[
\bigcap_{k \in \omega} B^x(k) = \bigcap_{k \in \omega} A^x(x(k)) \cap \bigcap_{k \in \omega} (\mathbb{Z}_+^2 \times 2^\omega \times [t(x \mid k)]) ,
\]
and it is easy to see that that here \( \bigcap_{k \in \omega} A^x(x(k)) \) is a singleton and
\[
\bigcap_{k \in \omega} (\mathbb{Z}_+^2 \times 2^\omega \times [t(x \mid k)]) = \mathbb{Z}_+^2 \times 2^\omega \times \{ y \}
\]
for the \( y \in \mathbb{Z}_+^2 \) that satisfies \( y \mid k = t(x \mid k) \). If we constructed \( \{ A^s \} \) in a way that the diameters are convergent to 0, then it is easy to check that this singleton set will fall into the set \( \mathbb{Z}_+^2 \times 2^\omega \times \{ y \} \). The second part of (iii) immediately follows from the same Property of \( \{ A^s : s \in \mathbb{N}^{<\omega} \} \).

We will use the Lusin scheme \( \{ B^s \} \) to construct the uniformizing function \( f \). For each \( (a, x) \in \mathbb{Z}_+^2 \times 2^\omega \) let \( B_a^s = (B^s)_{a,x} \). It is straightforward to check that for every \( (a, x) \in \mathbb{Z}_+^2 \times 2^\omega \), the system \( \{ B^s_a : s \in \mathbb{N}^{<\omega} \} \) satisfies (i)-(iii) for \( B_{a,x} \).

For each \( (a, x) \in \mathcal{P} \), let \( T_{a,x} = \{ s \in \mathbb{N}^{<\omega} : \mu_{a}(B^s_{a,x}) > 0 \} \). It follows from \( B^\emptyset = U'' \) the definition of \( P \) that \( \emptyset \in T_{a,x} \). It is straightforward to check that \( T_{a,x} \) is a pruned tree on \( \mathbb{N} \) (that is, if \( s \in T_{a,x} \), then \( s \prec n \in T_{a,x} \) for some \( n \in \mathbb{N} \)). These allow us to define \( b_{a,x} \in \mathbb{N}^{<\omega} \) as the leftmost branch of \( T_{a,x} \). As \( \{ B^{s}_{a,x} \}_{s \in \mathbb{N}^{<\omega}} \) satisfies Property (iii), the intersection \( \bigcap_{k} B^{s}_{b_{a,x} \mid k} \) is a singleton, say \( \{ f(a, x) \} \). This defines our uniformizing function. It is clear from the definition that \( \text{dom}(f) = P \) and \( \text{graph}(f) \subset U'' \), it remains to show that it is \( \Sigma^0_3 \)-measurable.

Let
\[
d(g, h) = \inf \{ 2^{-k} : g \mid k = h \mid k \}
\]
be the usual metric on \( \mathbb{Z}^\omega = \mathbb{Z}^\omega \) (this is compatible with the Polish topology of \( \mathbb{Z}^\omega \)). We will define a sequence of \( \Sigma^0_3 \)-measurable functions \( f_k : P \to \mathbb{Z}^\omega \) such that \( f_k \to f \) uniformly. (This idea from the hints for [2, Exercise 24.8], where it is attributed to K.)

The definition of \( f \) means that \( f(a, x) \in B^{s}_{b_{a,x} \mid k} \) for every \( k \in \omega \), and hence \( f(a, x) \in [t(b_{a,x}) \mid k] \) by Property (vi) of Claim 3.5. It is clear from the definition of the metric \( d \) that \( [t(b_{a,x}) \mid k] \) is the open ball of radius \( 2^{-k+1} \) around \( f(a, x) \) (we use that \( |t(b_{a,x}) \mid k| = k \) by Property (v)). This means that if we enforce that the sequence \( f_k(a, x) \in \mathbb{Z}^\omega \) starts with \( t(b_{a,x}) \mid k \in \mathbb{Z}^k \) then this will guarantee that \( f_k \to f \) uniformly.

Let us define \( f_k(a,x) \) as \( f_k(a, x) = t(b_{a,x} \mid k) \prec (0, 0, \ldots) \) (i.e. continue the finite starting sequence with infinitely many zeroes). This choice yields that for every \( k \in \omega, a \in \mathbb{Z}_+^\omega \) and \( x \in 2^\omega \), \( f_k(a, x) \in D \) for the countable set
\[
D = \{ s \prec (0, 0, \ldots) : s \in \mathbb{Z}^{<\omega} \} = \{ g \in \mathbb{Z}^\omega : (\exists N_g \in \omega)(\forall n > N_g) g(n) = 0 \}.
\]

Fix an arbitrary \( k \in \omega \). We will prove that \( f_k \) is \( \Sigma^0_3 \)-measurable, that is, \( f_k^{-1}(V) \in \Sigma^0_3(\mathbb{Z}_+^2 \times 2^\omega) \) for every open set \( V \subseteq \mathbb{Z}^2 \). As \( f_k^{-1}(V) = \bigcup_{g \in D \cap V} f_k^{-1}(\{ g \}) \) and \( \Sigma^0_3 \) is closed under countable unions, it is enough to prove that \( f_k^{-1}(\{ g \}) \in \Sigma^0_3(\mathbb{Z}_+^2 \times 2^\omega) \) for every \( k \in \omega \) and \( g \in D \).
We may assume that \( g = (g \upharpoonright k) \cap (0, 0, \ldots) \), because otherwise \( f_k^{-1}(\{g\}) = \emptyset \in \Sigma^0_3 \) follows immediately from the definition of \( f_k \). Using this assumption

\[
f_k^{-1}(\{g\}) = \{(a, x) \in P : t(b_{a,x} \mid k) = g \upharpoonright k\}.
\]

The definition of \( b_{a,x} \) as the leftmost branch of the pruned tree \( T_{a,x} \) means that

\[
b_{a,x} \upharpoonright k = \min(\mathbb{N}^k \cap T_{a,x}) = \min\{s \in \mathbb{N}^k : \mu_a(B_{a,x}^s) > 0\}
\]

where we consider \( \mathbb{N}^k \) ordered lexicographically. Thus a pair \((a, x) \in P\) satisfies \( t(b_{a,x} \mid k) = g \upharpoonright k \) if and only if

\[
(\exists s \in \mathbb{N}^k)(t(s) = g \mid k \land \mu_a(B_{a,x}^s) > 0 \land (\exists s' \in \mathbb{N}^k)(s' < s \land \mu_a(B_{a,x}^{s'}) > 0))
\]

Thus

\[
f_k^{-1}(\{g\}) = P \cap \bigcup_{s \in \mathbb{N}^k, t(s) = g \mid k} \left( P^s \setminus \left( \bigcup_{s' \in \mathbb{N}^k, s' < s} P^{s'} \right) \right)
\]

where we define \( P^s \) as \( P^s = \{(a, x) \in \mathbb{Z}^{\omega} \times 2^{\omega} : \mu_a(B_{a,x}^s) > 0\} \). As we noted earlier, properties (i) and (ii) in Theorem 3.4 imply that \( B^s \in \Sigma^0_2 \) for every \( s \in \mathbb{N}^{<\omega} \). This allows us to apply Lemma 3.2 and conclude that \( P^s \) is also a \( \Sigma^0_2 \) set for every \( s \in \mathbb{N}^{<\omega} \). As a corollary of this, \( f_k^{-1}(\{g\}) \) is indeed a \( \Sigma^0_3 \) set (we already used that \( P \in \Sigma^0_2 \), but this also follows from the fact that \( P = P^0 \)).

This proved that \( f_k \) is indeed a \( \Sigma^0_3 \)-measurable function for every \( k \in \omega \). \cite[Theorem 24.3]{2} and part i) of \cite[Exercise 24.3]{2} basically state that if \( \xi \geq 2 \), then an uniform limit of \( \Sigma^0_\xi \)-measurable functions between separable metric spaces is also \( \Sigma^0_\xi \)-measurable. This means that \( f \) is also \( \Sigma^0_3 \)-measurable and this concludes our proof (using Claim 3.3).

\[\Box\]

### 4. The Main Result

Now we are ready to prove our main result, which is stated in the following theorem.

**Theorem 4.1.** In the (non-locally-compact abelian Polish) group \( \mathbb{Z}^\omega \) there exists a \( \Pi^1_1 \) Haar null set \( E \) that is not contained in any \( \Pi^1_2 \) Haar null set.

**Proof:** First we will construct some simple functions which will be useful in our proof.

Let us define the function

\[
\theta : \mathbb{Z}_+ \times \{0, 1\} \times \mathbb{Z} \to \mathbb{Z}, \quad \theta(n, b, z) = (n - 1)(n + 4) + b(n + 2) + z.
\]

Elementary calculations show that \( \theta(n, 1, 0) = \theta(n, 0, 0) + (n + 2), \theta(n + 1, 0, 0) = \theta(n, 1, 0) + (n + 2) \) and hence if we restrict \( \theta \) to the set

\[
T = \{(n, b, z) : n \in \mathbb{Z}_+, b \in \{0, 1\}, z \in [0, n + 1]\},
\]

then \( \theta(T) = \mathbb{N} \) is the set of nonnegative integers, and the restricted function \( \theta \mid T \) is an order preserving bijection (when \( T \) is ordered lexicographically and \( \mathbb{N} \) has its usual ordering). Let \( \iota : \mathbb{N} \to T \) be the inverse of this restriction.
We can let $\theta$ act elementwise on sequences of length $\omega$, that is, we can define
\[
t : \mathbb{Z}_+ \times 2^\omega \times \mathbb{Z}_+ \to \mathbb{Z}_+ , \quad (t(a, x, g))(k) = \theta(a(k), x(k), g(k)) \text{ for all } k \in \omega.
\]

Analogously, we may also let $\iota$ act elementwise on sequences of length $\omega$ to get a function $\iota : \mathbb{N}_+^\omega \to T^\omega$. It is clear that both $t$ and $\iota$ are continuous (in fact, Lipschitz).

With a slight abuse of notation let us indentify $T^\omega$ and the set
\[
T = \{(a, x, g) \in \mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega : (\forall k)(g(k) \in [0, a(k) + 1])\}
\]
($T^\omega$ contains sequences of triples, $T$ contains triples of sequences, the natural map between them is a homeomorphism). As $\iota$ is the inverse of a restriction of $\theta$, the same holds for $\iota$ and $t$: every $(a, x, g) \in T$ satisfies $i(t(a, x, g)) = (a, x, g)$ and every $s \in \mathbb{N}_+^\omega$ satisfies $t(\iota(s)) = s$.

We will also use [Theorem 3.1] let $f$ be a $\Sigma^0_3$-measurable partial function $f : \mathbb{Z}^\omega_+ \times 2^\omega \to \mathbb{Z}^\omega$ which satisfies properties (I) and (II) in [Theorem 3.1]

Now we are able to define $E$ as
\[
E = t(\text{graph } (f)) = \{(a, x, g) : (a, x, g) \in \text{graph } (f)\}.
\]

**Claim 4.2.** $E$ is a $\Pi^0_4$ set.

**Proof.** First we show that graph $(f)$ is a $\Pi^0_3$ subset of $\mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega$. Let $\mathcal{B}$ be a countable basis of the topology of $\mathbb{Z}^\omega$, then clearly $(a, x, g) \in \text{graph } (f)$ if and only if $f(a, x) \in U \iff g \in U$ for every $U \in \mathcal{B}$. Therefore
\[
\text{graph } (f) = \bigcap_{U \in \mathcal{B}} \left( (f^{-1}(U) \times \mathbb{Z}^\omega) \cap (\mathbb{Z}^\omega_+ \times 2^\omega \times U) \right) \cup \left( ((\mathbb{Z}^\omega_+ \times 2^\omega) \setminus f^{-1}(U)) \times \mathbb{Z}^\omega \cap (\mathbb{Z}^\omega_+ \times 2^\omega \times (\mathbb{Z}^\omega \setminus U)) \right)
\]
and in this form it is easy to see that graph $(f)$ is indeed a $\Pi^0_3$ set.

Notice that if we apply first the definition of $T$ and then Property (I) of [Theorem 3.1] then we get
\[
T \supset \{(a, x, g) \in \mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega : g \in \text{supp } \mu_0 \} \supset \text{graph } (f).
\]
This implies that graph $(f) = i(t(\text{graph } (f))) = i(E)$ and hence $E = i^{-1}(\text{graph } (f))$.

As graph $(f) \subset T$ is a $\Pi^0_3$ subset of $\mathbb{Z}^\omega_+ \times 2^\omega \times \mathbb{Z}^\omega$, it is also a $\Pi^0_3$ subset of $T$. This means that its preimage under the continuous function $i : \mathbb{N}_+^\omega \to T$ is a $\Pi^0_3$ subset of $\mathbb{N}_+^\omega$. As $\mathbb{N}_+^\omega$ is a closed subset of $\mathbb{Z}^\omega$, this yields that $E = i^{-1}(\text{graph } (f))$ is indeed a $\Pi^0_3$ set in $\mathbb{Z}^\omega$.

**Claim 4.3.** $E$ is Haar null.

**Proof.** We will show that $a_0 = (1, 1, \ldots)$ is a witness sequence for this fact. Notice that $\mu_{a_0}$ is just the usual coinflip measure with $\text{supp } (\mu_{a_0}) = \{0, 1\}^\omega \subset \mathbb{Z}^\omega$. It is clearly sufficient to show that $|(E + r) \cap \{0, 1\}^\omega| \leq 1$ for all $r \in \mathbb{Z}^\omega$. This is equivalent to saying that if $e, e' \in E$ and $e \neq e'$, then $|e(k) - e'(k)| \geq 2$ for at least $k \in \omega$.

Fix arbitrary $e, e' \in E$ with $e \neq e'$. By the definition of $E$ there are $a, a' \in \mathbb{Z}^\omega_+$ and $x, x' \in 2^\omega$ such that $e = t(a, x, f(a, x))$ and $e' = t(a', x', f(a', x'))$. As we assumed
that \( e \neq e' \), we can find a \( k \in \omega \) where \( (a(k), x(k)) \neq (a'(k), x'(k)) \). Without loss of generality, we may assume that \( (a(k), x(k)) < (a'(k), x'(k)) \) lexicographically. By Property (I) of Theorem 3.1 we know that \( f(a, x) \in \text{supp}(\mu_a) = \prod_{k \in \omega} [0, a(k)] \), hence \( 0 \leq f(a, x)(k) \leq a(k) \) and analogously \( 0 \leq f(a', x')(k) \leq a'(k) \). Straightforward and elementary calculations (using these bounds and the definition of \( t \) and \( \theta \)) show that 
\[ e(k) + 2 \leq e'(k) \]
both in the case when \( a(k) < a'(k) \) and in the case when \( a(k) = a'(k) \) (and hence \( x(k) < x'(k) \), i.e. \( x(k) = 0 \) and \( x'(k) = 1 \)). These allow us to conclude that \( E \) is indeed Haar null.

\[ \square \]

**Claim 4.4.** There is no Haar null set \( H \in \Pi_0^0(\mathbb{Z}^\omega) \) containing \( E \).

**Proof.** Suppose that \( H \in \Pi_0^0(\mathbb{Z}^\omega) \) is such a set. The by [Lemma 2.1](#) there exists a sequence \( a \in \mathbb{Z}_+^\omega \) such that \( \mu_a(H + r) = 0 \) for every \( r \in \mathbb{Z}^\omega \). As \( t \) is continuous, the section map
\[
t_a : 2^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}^\omega, \quad (x, g) \mapsto t(a, x, g)
\]
is also continuous, hence \( S = t_a^{-1}(H) \subseteq 2^\omega \times \mathbb{Z}^\omega \) is a \( \Pi_0^0 \) set.

It is easy to check that \( \text{graph}(f_a) \subset S \), and therefore by Property (II) of Theorem 3.1 there exists an \( x^* \in 2^\omega \) such that \( \mu_a(S_{x^*}) > 0 \). By the definition of \( S \), \( t(a, x^*, S_{x^*}) \subset t_a(S) \subset H \). But \( g \mapsto t(a, x^*, g) \) is a translation, so a translate of \( H \) contains \( S_{x^*} \), but \( S_{x^*} \) has positive \( \mu_a \)-measure and hence this contradicts that \( a \) is a witness sequence for \( H \).

\[ \square \]

This concludes the proof.

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