TOPOLOGIES ON THE SET OF ALL SUBSPACES OF A BANACH SPACE AND RELATED QUESTIONS OF BANACH SPACE GEOMETRY
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1. Introduction.

For a Banach space $X$ we shall denote the set of all closed subspaces of $X$ by $G(X)$. In some kinds of problems it turned out to be useful to endow $G(X)$ with a topology. The main purpose of the present paper is to survey results on two the most common topologies on $G(X)$.

The organization of this paper is as follows. In section 2 we introduce some definitions and notation. In sections 3 and 4 we introduce two topologies on $G(X)$. Section 5 is devoted to the problem of comparison of these topologies. In section 6 we investigate the following general problem: How close should be the structure of the subspaces which are close with respect to the natural metrics, which generate introduced topologies? (It should be mentioned that both introduced topologies are metrizable.) In section 7 we survey those results on introduced topologies and related quantities which were not discussed in previous sections. Here we also try to describe known applications of introduced topologies and related quantities. This section is nothing more than guide to the literature. If $x$ is a vector of a Banach space $X$ and $A, D$ are subsets of $X$ then we shall denote the value $\inf_{a \in A} ||x - a||$ by $\text{dist}(x, A)$ and the value $\inf_{a \in A} \text{dist}(a, D)$ by $\text{dist}(A, D)$. The closed unit ball and the unit sphere of a Banach space $X$ are denoted by $B(X)$ and $S(X)$ respectively. For a subset $A$ of a Banach space $X$ by $A^\perp$, $\text{lin}(A)$, $\text{conv}(A)$ and $\text{cl}(A)$ we shall denote, respectively, the set $\{x^* \in X^* : (\forall x \in A)(x^*(x) = 0)\}$, the set of all finite linear combinations of vectors of $A$, the set of all convex combinations of vectors of $A$ and the closure of $A$ in the strong topology. For a subset $A$ of a dual Banach space $X^*$ we shall denote the set $\{x \in X : (\forall x^* \in A)(x^*(x) = 0)\}$ by $A^\perp$.

Let $Y$ and $Z$ be Banach spaces. For $1 \leq p \leq \infty$ we shall denote by $Y \oplus_p Z$ the Banach space of all pairs $(y, z), y \in Y, z \in Z$, with the norm $||(y, z)|| = (||y||^p + ||z||^p)^{1/p}$ (or $\max\{||y||, ||z||\}$, if $p = \infty$). It is clear that all these norms define the same topology. The corresponding topological vector space will be denoted by $Y \oplus Z$.

A closed linear subspace $Y$ of a Banach space $X$ is said to be a complemented subspace of $X$ if there is a bounded linear projection from $X$ onto $Y$ or, what is the same, if there exists a closed linear subspace $Z$ of $X$ such that every $x \in X$ can be in a unique manner represented in the form $x = y + z$, where $y \in Y$ and $z \in Z$. By $G_c(X)$ we shall denote the set of all complemented subspaces of $X$. For $Y \in G_c(X)$ we shall denote by $\lambda(Y, X)$ the value $\inf\{||P|| : P$ is a projection of $X$ onto $Y\}$. Banach space is said to be injective if its isomorphic embeddings into arbitrary Banach space have complemented images.

If $\{X_n\}_{n=1}^\infty$ is a sequence of Banach spaces we define the direct sum, of these spaces in the sense of $l_p, 1 \leq p < \infty$, namely $(\sum_{n=1}^\infty X_n)_p$, as the space of all sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ with $x_n \in X_n$ for all $n$, for which $||x|| = (\sum_{n=1}^\infty ||x_n||^p)^{1/p} < \infty$. Similarly, $(\sum_{n=1}^\infty X_n)_0$ denotes the direct sum of $\{X_n\}_{n=1}^\infty$ in the sense of $c_0$, i.e. the space of all sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ with $x_n \in X_n$ for all $n$, for which...
\[ \lim_n \|x_n\| = 0. \] The norm in this direct sum is taken as \( \|x\| = \max_n \|x_n\| \).

A sequence \( \{X_n\}_{n=1}^\infty \) of closed subspaces of a Banach space \( X \) is called a Schauder decomposition of \( X \) if every \( x \in X \) has a unique representation of the form \( x = \sum_{n=1}^\infty x_n \) with \( x_n \in X_n \) for every \( n \). In such case we write \( X = \bigoplus_{n=1}^\infty X_n \). Furthermore, if \( \{Y_n\}_{n=1}^\infty \) is a sequence of subspaces, \( Y_n \subset X_n \), then we shall denote \( \text{cl}(\text{lin}(\bigcup_{n=1}^\infty Y_n)) \) by \( \sum_{n=1}^\infty \oplus Y_n \). It is clear that in this case \( \{Y_n\}_{n=1}^\infty \) form a Schauder decomposition of \( \sum_{n=1}^\infty \oplus Y_n \).

Two Banach spaces \( Y \) and \( Z \) are called isomorphic if there exists an invertible operator from \( Y \) onto \( Z \). The Banach-Mazur distance between \( Y \) and \( Z \) is defined by \( d(Y,Z) = \inf \|T\|\|T^{-1}\| \), the infimum being taken over all invertible operators from \( Y \) onto \( Z \) (if \( Y \) is not isomorphic to \( Z \) we put \( d(Y,Z) = \infty \)). Let \( \{X_n\} \) and \( \{Y_n\} \) be two sequences of Banach spaces. We shall say that they are uniformly isomorphic if \( \sup_n d(X_n,Y_n) < \infty \).

The identity operator of a Banach space \( X \) is denoted by \( I_X \) (or simply by \( I \) if \( X \) is clear from the context.) For a mapping \( \varphi : A \to B \) by \( \text{im}\varphi \) we denote the set \( \{b \in B : (\exists a \in A) (b = \varphi a)\} \).

Let \( T \) be a linear mapping from a linear subspace \( L \) of a Banach space \( X \) into a Banach space \( Y \). Then \( L \) is called the domain of \( T \) and is denoted by \( D(T) \). The set \( \{x \in D(T) : Tx = 0\} \) is called the kernel of \( T \) and is denoted by \( \text{ker}T \). For Banach spaces \( X, Y \) by \( L(X,Y) \) we denote the space of all bounded linear operators from \( X \) into \( Y \), endowed with the norm \( \|T\| = \sup \{\|Tx\| : x \in B(X)\} \).

The least cardinal \( \alpha \) for which there exists a dense subset of \( X \) of cardinality \( \alpha \) is called the density character of \( X \) and is denoted by \( \text{dens}X \).

Referring to formula (3.5) we mean formula (5) from section 3.

3. Geometric opening.

3.1. Definition. Let \( Y, Z \in G(X) \). The geometric opening (or simply opening, sometimes gap) between \( Y \) and \( Z \) is defined to be

\[ \Theta(Y,Z) = \max \{\sup_{y \in S(Y)} \text{dist}(y,Z), \sup_{z \in S(Z)} \text{dist}(z,Y)\} \tag{1} \]

If the sphere of a subspace is empty (it happens when the subspace is \( \{0\} \)) then the corresponding supremum is set equal to zero.

3.2. Remark. This definition can be used for nonclosed subspaces as well. For nonclosed \( Y \) and \( Z \) we have \( \Theta(Y,Z) = \Theta(\text{cl}Y,\text{cl}Z) \).

3.3. For \( Y, Z \in G(X) \) we let

\[ \Theta_0(Y,Z) = \sup_{y \in S(Y)} \text{dist}(y,Z). \]

3.4. Properties of the geometric opening are described in the following theorem.

Theorem. Let \( X \) be a Banach space and let \( Y, Z \in G(X) \). Then

(a) \( 0 \leq \Theta_0(Y,Z) \leq 1 \) and \( 0 \leq \Theta(Y,Z) \leq 1 \).

(b) \( \Theta(Y,Z) = \Theta(Z,Y) \);

(c) \( \Theta(Y,Z) = 0 \Rightarrow Y = Z \);

(d) \( \Theta_0(Y,Z) = \Theta_0(Z^\perp,Y^\perp) \) and therefore

\[ \Theta(Y,Z) = \Theta(Y^\perp,Z^\perp). \tag{2} \]
(e) If at least one of the subspaces $Y$ and $Z$ is finite dimensional and $\Theta(Y, Z) < 1$ then both of them are finite dimensional and $\dim Y = \dim Z$.

(f) For every $Y_1, Y_2, Y_3 \in G(X)$ the following inequalities hold:

$$\Theta_0(Y_1, Y_3) \leq \Theta_0(Y_1, Y_2) + \Theta_0(Y_2, Y_3) + \Theta_0(Y_1, Y_2)\Theta_0(Y_2, Y_3);$$

$$\Theta(Y_1, Y_3) \leq \Theta(Y_1, Y_2) + \Theta(Y_2, Y_3) + \Theta(Y_1, Y_2)\Theta(Y_2, Y_3).$$

(3)

(Inequality (3) is called “weakened triangle inequality”).

(g) The function $d_g(Y, Z) = \log(1 + \Theta(Y, Z))$ is a metric on $G(X)$. The set $G(X)$ is a complete metric space with respect to this metric.

(h) If $X$ is a Hilbert space, then

$$\Theta(Y, Z) = ||P_Y - P_Z||,$$

where $P_Y$ and $P_Z$ are orthogonal projections onto $Y$ and $Z$ respectively.

Proof. Verification of (a), (b) and (c) is immediate. (d) Recall well-known formulas from the duality theory of Banach spaces. For $Z \in G(X)$, $y \in X$ and $y^* \in X^*$ we have

$$\text{dist}(y, Z) = \sup_{z^* \in S(Z^\perp)} |z^*(y)|;$$

$$\text{dist}(y^*, Z^\perp) = \sup_{z \in S(Z)} |y^*(z)|.$$

Therefore

$$\Theta_0(Y, Z) = \sup_{y \in S(Y)} \text{dist}(y, Z) =$$

$$= \sup_{y \in S(Y), z^* \in S(Z^\perp)} |z^*(y)| =$$

$$= \sup_{z^* \in S(Z^\perp)} \text{dist}(z^*, Y^\perp) = \Theta_0(Z^\perp, Y^\perp).$$

Statement (e) follows immediately from the following lemma.

3.5. Lemma. Let $Y, Z \in G(X)$, $Z$ be finite dimensional, $\dim Y \geq \dim Z$. Then there exists $y \in S(Y)$ such that $\text{dist}(y, Z) = 1$.

Proof. It is clear that we may suppose that $Y$ is finite dimensional and $\dim Y = \dim Z + 1$. Suppose first that $X$ is strictly convex i.e. that $||x_1 + x_2|| < ||x_1|| + ||x_2||$ for every linearly independent $x_1, x_2 \in X$. It is easy to see that in this case for every $x \in X$ there exists a unique element of best approximation of $x$ by elements of $Z$. Let us denote this element by $A(x)$. The mapping $A$ is, generally speaking, nonlinear but it is easy to verify that it is continuous and $A(-x) = -A(x)$. We shall use the following topological result.

3.6. Theorem. (K.Borsuk [Bor], see also [DKL]). Let $S^{n-1}$ be the unit sphere in $R^n$ and let $\varphi : S^{n-1} \to R^{n-1}$ be a continuous mapping, for which $\varphi(-x) = -\varphi(x)$. Then there exists a point $x \in S^{n-1}$ such that $\varphi(x) = 0$.

Since the unit sphere of $Y$ is homeomorphic to $S^{\dim Z}$ and $Z$ is homeomorphic to $R^{n-1}$ then this theorem is applicable in our situation and we can find $y \in S(Y)$ such that such that $0$ is the best approximation of $y$ by elements of $Z$ i.e. $\text{dist}(y, Z) = 1$. 3
Let us turn to the general case. We may assume that $X = \text{lin}(Y \cup Z)$ and therefore, $X$ is finite dimensional. Let $\{x_i\}_{i=1}^n$ be a basis of $X$ and $\{x_i^*\}$ be its biorthogonal functionals. It is easy to check that for every $k \in \mathbb{N}$ the norm
\[ \|x\|_k = (\|x\|^2 + (1/k) \sum_{i=1}^n (x_i^*(x))^2)^{1/2} \]
is strictly convex. Therefore for every $k \in \mathbb{N}$ we can find $y_k \in Y$ such that $\|y_k\|_k = 1$ and $\text{dist}_k(y_k, Z) = 1$. Since $\|y_k\| \leq \|y_k\|_k = 1$ then the sequence $\{y_k\}_{k=1}^\infty$ contains a convergent subsequence. It is easy to verify that its limit is the required vector. Lemma 3.5 and therefore statement (e) have been proved.

(f) It is clear that we need to prove only the first inequality. Let $\varepsilon > 0$ and $y_1 \in S(Y_1)$. Then for some $y_2 \in Y_2$ we have $\|y_1 - y_2\| < \Theta_0(Y_1, Y_2) + \varepsilon$, hence $\|y_2\| < 1 + \Theta_0(Y_1, Y_2) + \varepsilon$. For some $y_3 \in Y_3$ we have $\|y_2 - y_3\| < (\Theta_0(Y_2, Y_3) + \varepsilon)\|y_2\|$. Hence
\[ ||y_1 - y_3|| < ||y_1 - y_2|| + ||y_2 - y_3|| < \Theta_0(Y_1, Y_2) + \varepsilon + (\Theta_0(Y_2, Y_3) + \varepsilon)(1 + \Theta_0(Y_1, Y_2) + \varepsilon). \]
Taking supremum over $y_1 \in S(Y_1)$ and then infimum over $\varepsilon > 0$ we obtain the required inequality.

(g) By (a), (b) and (c) the only thing which we need to prove is the triangle inequality. Let $Y_1, Y_2, Y_3 \in G(X)$. By (f) we have
\[ d_g(Y_1, Y_3) = \log(1 + \Theta(Y_1, Y_3)) \leq \log(1 + \Theta(Y_1, Y_2) + \Theta(Y_2, Y_3) + \Theta(Y_1, Y_2)\Theta(Y_2, Y_3)) = \log((1 + \Theta(Y_1, Y_2))(1 + \Theta(Y_2, Y_3))) = \log(1 + \Theta(Y_1, Y_2)) + \log(1 + \Theta(Y_2, Y_3)) = d_g(Y_1, Y_2) + d_g(Y_2, Y_3). \]

Let $\{Y_n\}_{n=1}^\infty \subset G(X)$ be a Cauchy sequence with respect to $d_g$. We need to show that $\{Y_n\}$ is a convergent sequence. It is sufficient to prove that $\{Y_n\}$ contains a convergent subsequence. Therefore we may assume that $\Theta(Y_n, Y_{n+1}) < 2^{-n}$. Let us introduce subset $A \subset S(X)$ as a set of limits of all strongly convergent sequences $\{y_n\}_{n=1}^\infty$, for which $y_n \in S(Y_n)$ ($n \in \mathbb{N}$). Direct verification shows that $A$ is a unit sphere of some closed subspace $Y_0$ of $X$ and that $\lim_{n \to \infty} \Theta(Y_n, Y_0) = 0$. Hence $\lim_{n \to \infty} d_g(Y_n, Y_0) = 0$ and (g) is proved.

(h) We have
\[ ||P_Y - P_Z|| = \sup_{x \in S(X)} ||(P_Y - P_Z)x|| = \sup_{x \in S(X)} ||P_Y(I - P_Z)x - (I - P_Y)P_Zx||. \]
Hence
\[ ||P_Y - P_Z|| = \sup_{x \in S(X)} ((||P_Y(I - P_Z)x||^2 + ||(I - P_Y)P_Zx||^2)\^{1/2}. \quad (5) \]
Since $||P_Yu|| = \text{dist}(u, Y^\perp)$ and $||(I - P_Y)u|| = \text{dist}(u, Y)$ then we have $||P_Y - P_Z|| \leq \sup_{x \in S(X)} (\Theta_0(Z^\perp, Y^\perp)^2 ||(I - P_Z)x||^2 + \Theta_0(Z, Y)^2 ||P_Zx||^2)^{1/2}$. Therefore, using (2) we obtain
\[ ||P_Y - P_Z|| \leq \Theta(Y, Z). \]
On the other hand, taking supremum over $x \in S(Z)$ in (5), we obtain
\[ ||P_Y - P_Z|| \geq \Theta_0(Z, Y), \]
and taking supremum over $x \in S(Z^\perp)$ in (5), we obtain

$$||P_Y - P_Z|| \geq \Theta_0(Z^\perp, Y^\perp).$$

These inequalities together with statement (d) imply that

$$||P_Y - P_Z|| \geq \Theta(Y, Z).$$

Statement (h) is proved.

**Notes and remarks**

3.7. The notion of opening between subspaces of a Hilbert space was introduced by M.G.Krein and M.A.Krasnosel’skii [KK]. In [KK] the Hilbert space versions of parts (a)–(e) of Theorem 3.4 were proved. In [KK] opening was introduced by formula (1). The equality (4) appeared in the book due to N.I.Akhiezer and I.M.Glazman [AG]. The notion of geometric opening between subspaces of a Banach space was introduced by M.G.Krein, M.A.Krasnosel’skii and D.P.Milman in [KKM]. Parts (a)–(e) of Theorem 3.4 were proved in this paper. Later V.M.Tikhomirov [T] rediscovered part (e) of Theorem 3.4. His proof is more complicated. It should be noted that the authors of [KKM] did not use quantity $\Theta_0(Y, Z)$. This quantity was introduced and investigated by T.Kato in [K1].

The results of parts (f) and (g) of Theorem 3.4 are due to I.C.Gohberg and A.S.Markus [GM1].

Since the sets $U_{\varepsilon,Y} = \{Z \in G(X) : \Theta(Z, Y) < \varepsilon\}$ form a base of open sets of the metric space $(G(X), d_g)$ we shall say that its topology is generated by the geometric opening. The main reason for introduction of the notion of opening seems to be the following: this notion is an appropriate tool for generalization of the theory of Carleman-von Neumann of defect numbers of Hermitian operators onto the case of arbitrary operators in Hilbert and Banach spaces. Now we are going to present this generalization.

Let $H$ be a complex Hilbert space and $A$ be a linear mapping from a linear subspace $D(A) \subset H$ into $H$. The operator $A$ is called Hermitian if $D(A)$ is dense in $H$ and

$$(\forall x, y \in D(A))(\langle Ax, y \rangle = \langle x, Ay \rangle).$$

For every $\lambda \in \mathbb{C}$ by $N(\lambda, A)$ we shall denote the subspace $(\text{im}(A - \lambda I))^\perp$.

In [KK] and [KKM] the notion of opening was used to generalize the following well-known result (see e.g. [Na, §14]).

3.8. **Theorem.** Let $A$ be a Hermitian operator. Then for every complex number $\alpha$ from the open upper halfplane we have $\dim N(\alpha, A) = \dim N(i, A)$ and for every complex number $\beta$ from the open lower halfplane we have $\dim N(\beta, A) = \dim N(-i, A)$.

(Here $i = \sqrt{-1}$.) The numbers $\dim N(i, A)$ and $\dim N(-i, A)$ are called the defect numbers of $A$.

In order to formulate the Krein-Krasnosel’skii-Milman generalization of 3.8 we need the following notion. Let $X$ and $Y$ be complex Banach spaces. Let $A$ be a linear mapping from a linear subspace $D(A) \subset X$ into $Y$. A complex number $\alpha$ is called a point of regular type for $A$ if there exists a real number $c(\alpha) > 0$ such that

$$(\forall f \in D(A))(\| (A - \alpha I)f \| \geq c(\alpha)\| f \|).$$

For every $\alpha \in \mathbb{C}$ let us introduce the subspace $N(\alpha, A) = (\text{im}(A - \alpha I))^\perp \subset Y^*$. 

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It is easy to see that for every operator $A$ the set of points of regular type for $A$ is an open subset of $C$ and that complex numbers with nontrivial imaginary parts are points of regular type for every Hermitian operator. Therefore the following result due to M.G.Krein, M.A.Krasnoselskii and D.P.Milman [KKM, p. 111] is a generalization of 3.8.

**3.9. Theorem.** Let $G$ be a connected open subset of $C$ consisting of points of regular type for $A$. Then the numbers $\dim N(\lambda, A)$ are the same for all $\lambda \in G$.

Proof. It is sufficient to prove that for every point $\alpha \in G$ there exists a neighbourhood $W \subset G$ such that for every $\lambda \in W$ we have $\dim N(\lambda, A) = \dim N(\alpha, A)$.

Let us show that we may take $W = \{\lambda : |\lambda - \alpha| < c(\alpha)/4\} \cap G$. By definition we have
\[
\|(A - \alpha I)f\| \geq c(\alpha)\|f\|
\]
for every $f \in D(A)$. Hence for every $\lambda \in W$ and every $f \in D(A)$ we have
\[
\|(A - \lambda I)f\| \geq (3/4)c(\alpha)\|f\|;
\]
\[
\|(A - \lambda I)f - (A - \alpha I)f\| = |\lambda - \alpha|\|f\| < (1/3)\|(A - \lambda I)f\|;
\]
\[
\|(A - \lambda I)f - (A - \alpha I)f\| < (1/4)\|(A - \alpha I)f\|.
\]
Inequalities (6) and (7) imply that

$$\Theta(\text{im}(A - \lambda I), \text{im}(A - \alpha I)) \leq 1/3$$

Using Theorem 3.4 (d) we obtain

$$\Theta(N(\alpha, A), N(\lambda, A)) \leq 1/3.$$ 

By Theorem 3.4 (e) it follows that $\dim N(\alpha, A) = \dim N(\lambda, A)$. The theorem is proved.

**3.10.** Although opening $\Theta$ has some properties of metric (see parts (a), (b) and (c) of Theorem 3.4) for some Banach spaces it is not a metric on $G(X)$. More precisely, I.Gohberg and A.S.Markus [GM1] observed that $\Theta$ does not satisfy the triangle inequality for some Banach spaces.

**Example.** Let $X = l^2_1$ and let $0 < a < b \leq 1$. Let us introduce subspaces $Y_1, Y_2, Y_3 \in G(X)$ in the following way:

$Y_1 = \{(x, 0) : x \in \mathbb{R}\}$;

$Y_2 = \{(x, y) : y = ax, x \in \mathbb{R}\}$

$Y_3 = \{(x, y) : y = bx, x \in \mathbb{R}\}$

It is not hard to see that $\Theta(Y_1, Y_2) = a$, $\Theta(Y_1, Y_3) = b$ and $\Theta(Y_2, Y_3) = (b - a)/(1 + a)$. (The reader advised to draw the picture.) Since $a + (b - a)/(1 + a) = b - a(b - a)/(1 + a)$, and $a(b - a)/(1 + a) > 0$ then the triangle inequality is not satisfied.

This example shows that the equality in (3) may be attained for a triple of pairwise distinct subspaces.

**3.11.** In connection with example 3.10 and part (h) of Theorem 3.4 A.S.Markus proposed the following problem: Describe the class of Banach spaces for which the geometric opening satisfies the triangle inequality.
3.12. I. Gohberg and A. S. Markus [GM1] suggested to consider the following in some respects more convenient metric which generates the same topology on $G(X)$ as the geometric opening.

Let $X$ be a Banach space, $Y, Z \in G(X)$. The spherical opening between $Y$ and $Z$ is defined to be

$$\Omega(Y, Z) = \max\{ \sup_{y \in S(Y)} \text{dist}(y, S(Z)), \sup_{z \in S(Z)} \text{dist}(z, S(Y)) \}.$$ 

In the case when $Y$ or $Z$ is $\{0\}$, we let $\Omega(Y, Z) = \Theta(Y, Z)$.

The spherical opening satisfies the following conditions.

a) $\Omega$ is a metric on $G(X)$. (This statement immediately follows from the following observation: $\Omega(Y, Z)$ coincides with the Hausdorff distance between $S(Y)$ and $S(Z)$).

b) $\Theta(Y, Z) \leq \Omega(Y, Z) \leq 2\Theta(Y, Z)$.

c) $G(X)$ is complete with respect to $\Omega$. (This statement immediately follows from part (g) of Theorem 3.4.)

3.13. It is natural to mention one more modification of the geometric opening. This modification was introduced by R. Douady [Do] and V. I. Gurarii [Gu1].

Let $X$ be a Banach space, $Y, Z \in G(X)$. The ball opening between $Y$ and $Z$ is defined to be

$$\Lambda(Y, Z) = \max\{ \sup_{y \in S(Y)} \text{dist}(y, B(Z)), \sup_{z \in S(Z)} \text{dist}(z, B(Y)) \}.$$ 

In the case when $Y$ or $Z$ is $\{0\}$, we let $\Lambda(Y, Z) = \Theta(Y, Z)$.

It is easy to verify that the analogues of statements (a), (b), and (c) from 3.12 are valid for the ball opening. Furthermore, it satisfies the inequality

$$0 \leq \Lambda(Y, Z) \leq 1$$

and, if $X$ is a Hilbert space then

$$\Lambda(Y, Z) = \| P_Z - P_Y \|.$$ 

We shall sometimes use the quantities $\Lambda_0(Y, Z)$ and $\Omega_0(Y, Z)$ which are defined analogously to $\Theta_0(Y, Z)$.

3.14. It should be mentioned that the analogues of the duality formula (2) fail for the spherical and ball openings.

**Example.** Let $X = l_2^1$, $0 < a < 1$. Let $Y = \{(x, y) : y = ax, x \in \mathbb{R}\}$, $Z = \{(x, 0) : x \in \mathbb{R}\}$. We have $X^* = l_\infty^2$, $Y^\perp = \{(x, y) : y = -x/a, x \in \mathbb{R}\}$, $Z^\perp = \{(0, y) : y \in \mathbb{R}\}$.

It is easy to verify that

$$\Lambda(Y, Z) = \Omega(Y, Z) = 2a/(1 + a);$$

$$\Lambda(Y^\perp, Z^\perp) = \Omega(Y^\perp, Z^\perp) = a.$$ 

(The reader advised to draw a picture.) But for $0 < a < 1$ we have

$$2a/(1 + a) \neq a.$$
3.15. **Remark.** J.D.Newburgh [New1] introduced another metric on $G(X)$ which generates the same topology on $G(X)$ as the geometric opening does. This metric was investigated and compared with $\Theta$ by E.Berkson [Ber].

3.16. It is interesting to note that A.L.Brown [Br2] proved that the assertion of Theorem 3.6 can be easily deduced if we suppose that the assertion of Lemma 3.5 is true.

4. **Operator opening.**

4.1. Let $X$ be a Banach space. By $GL(X)$ we shall denote the group (with respect to composition) of all invertible linear operators on $X$. For $Y,Z \in G(X)$ let

$$r_0(Y,Z) = \inf\{||C-I|| : C \in GL(X), C(Y) = Z\},$$

if the set over which the infimum is taken is not empty, and $r_0(Y,Z) = 1$ otherwise.

**Definition.** The *operator opening* between $Y$ and $Z$ is defined by

$$r(Y,Z) = \max\{r_0(Y,Z), r_0(Z,Y)\}.$$ 

4.2. Properties of the operator opening are described in the following theorem.

**Theorem.** Let $X$ be a Banach space and let $Y,Z \in G(X)$.

(a) $0 \leq r_0(Y,Z) \leq 1$ and hence $0 \leq r(Y,Z) \leq 1$;

(b) $r(Y,Z) = r(Z,Y)$;

(c) $r_0(Y,Z) \geq \Theta_0(Y,Z)$ and hence $r(Y,Z) \geq \Theta(Y,Z)$;

(d) $r_0(Y,Z) < 1 \Rightarrow r_0(Z,Y) \leq r_0(Y,Z)/(1 - r_0(Y,Z))$;

(e) If $P_Y$ and $P_Z$ are projections with images $Y$ and $Z$ respectively, then $r(Y,Z) \leq ||P_Y - P_Z||$;

(f) $r(Y,Z) = 0 \iff Y = Z$.

(g) If $X$ is a Hilbert space then for every $Y,Z \in G(X)$ the following equality is valid

$$r(Y,Z) = \Theta(Y,Z).$$

(h) $r_0(Z^\perp,Y^\perp) \leq r_0(Y,Z)$.

(i) For every $Y_1,Y_2,Y_3 \in G(X)$ the following inequality is valid:

$$r_0(Y_1,Y_3) \leq r_0(Y_1,Y_2) + r_0(Y_2,Y_3) + r_0(Y_1,Y_2)r_0(Y_2,Y_3)$$

and hence

$$r(Y_1,Y_3) \leq r(Y_1,Y_2) + r(Y_2,Y_3) + r(Y_1,Y_2)r(Y_2,Y_3).$$

(The last inequality is called “weakened triangle inequality”).

(j) The function $d_{op}(Y,Z) = \log(1 + r(Y,Z))$ is a metric on $G(X)$. The set $G(X)$ is complete with respect to metric $d_{op}$.

Proof. (a) follows from the following observation: the set $\{C \in GL(X) : C(Y) = Z\}$ with every operator contains all its multiples.

(b) is evident.

(c) If $C_0 \in \{C \in GL(X) : C(Y) = Z\}$ then

$$||C_0 - I|| \geq \sup_{y \in S(Y)} ||y - C_0y|| \geq \sup_{y \in S(Y)} \text{dist}(y,Z).$$
This inequality implies (c).

(d) If $C \in GL(X)$ and $C(Y) = Z$ then $C^{-1} \in GL(X)$ and $C^{-1}(Z) = Y$. Furthermore, we have

$$\|C^{-1} - I\| \leq \|C - I\| \|C^{-1} - I\| \leq \|C - I\| \inf_{x \in S(X)} \|Cx\| \leq \|C - I\|/(1 - \|C - I\|).$$

Whence we have (d).

(e) The statement (e) is clear when $\|P_Z - P_Y\| \geq 1$. So we shall suppose that $\|P_Z - P_Y\| < 1$. In this case the operators $I - (P_Y - P_Z)$ and $I - (P_Z - P_Y)$ belong to $GL(X)$. Indeed, it is easy to verify that the operators $I + \sum_{n=1}^{\infty} (P_Y - P_Z)^n$ and $I + \sum_{n=1}^{\infty} (P_Z - P_Y)^n$ are their inverses. Therefore $(I - (P_Y - P_Z))X = X$ and hence $P_Y(I - P_Y + P_Z)X = P_YX$, therefore $P_YZ = Y$. Let us denote the operator $I - (P_Z - P_Y)$ by $C$. We have $C \in GL(X)$; $\|C - I\| = \|P_Z - P_Y\|$ and $C(Z) = (I - P_Z + P_Y)Z = Y$. By definition of $r_0(Z,Y)$ we have $r_0(Z,Y) \leq \|C - I\| = \|P_Y - P_Z\|$. In the same manner we can estimate $r_0(Y,Z)$. So (e) is proved.

Statement (f) immediately follows from (c) and the analogous statement about $\Theta$.

(g) By (e) we have $r(Y,Z) \leq \|P_Y - P_Z\|$, where $P_Y$ and $P_Z$ are the orthogonal projections onto $Y$ and $Z$ respectively. By Theorem 3.4 (h) $\|P_Y - P_Z\| = \Theta(Y,Z)$. So we have $r(Y,Z) \leq \Theta(Y,Z)$. Comparing this inequality with (c) we obtain the desired inequality.

(h) The assertion immediately follows from the following observation: if $C \in GL(X)$ and $C(Y) = Z$ then $C^* \in GL(X^*)$ and $C^*(Z^\perp) = Y^\perp$.

(i) The assertion is clear if $r_0(Y_1, Y_2) = 1$ or $r_0(Y_2, Y_3) = 1$. So we shall suppose that it is not the case.

Let $\varepsilon > 0$ be arbitrary positive number. Let $C_1 \in GL(X)$ be such that $C_1(Y_1) = Y_2$ and $\|C_1 - I\| < r_0(Y_1, Y_2) + \varepsilon$ and let $C_2 \in GL(X)$ be such that $C_2(Y_2) = Y_3$ and $\|C_2 - I\| < r_0(Y_2, Y_3) + \varepsilon$. Then $C_2C_1 \in GL(X)$, $C_2C_1(Y_1) = Y_3$ and

$$\|C_2C_1 - I\| = \|(C_2 - I)(C_1 - I) + (C_1 - 1) + (C_2 - I)\| \leq \|C_2 - I\|\|C_1 - I\| +
\|C_1 - I\| + \|C_2 - I\| <
(r_0(Y_2, Y_3) + \varepsilon)(r_0(Y_1, Y_2) + \varepsilon) + r_0(Y_1, Y_2) + \varepsilon + r_0(Y_2, Y_3) + \varepsilon.$$

Since $\varepsilon$ is arbitrary then the required inequality follows.

(j) Since we already proved statements (a), (b) and (f) we need to verify the triangle inequality and completeness only.

The triangle inequality for $d_{op}$ follows from (i) (see the proof of Theorem 3.4 (g)).

Let sequence $\{Y_n\}_{n=1}^{\infty} \subset G(X)$ be such that

$$\lim_{m,n \to \infty} d_{op}(Y_n, Y_m) = 0.$$

Since it is enough to prove that $\{Y_n\}$ contains a convergent subsequence, we may assume that

$$r_0(Y_n, Y_{n+1}) \leq 2^{-n-1}.$$

Therefore for every $n \in \mathbb{N}$ there exists an operator $C_n \in GL(X)$ such that $C_n(Y_n) = Y_{n+1}$ and $\|C_n - I\| < 2^{-n}$. It is easy to verify that this condition implies the convergence
of the sequence $T_n = \prod_{k=1}^{n} C_k \ (n \in \mathbb{N})$ in the uniform topology. Let us denote its limit by $T_0$. It is easy to verify that $T_0 \in GL(X)$. Let $Y_0 = T_0(Y_1)$. It is easy to verify that $r_0(Y_k, Y_0) \to 0$ when $k \to \infty$ and, hence, $\lim_{k \to \infty} d_{op}(Y_k, Y_0) = 0$. We finished the proof of Theorem 4.2.

4.3. Remark. It is clear that the sets

$$V_{\varepsilon, Y} = \{Z \in G(X) : r(Y, Z) < \varepsilon\}, \ Y \in G(X), \varepsilon > 0$$

form a base of the topology corresponding to the metric $d_{op}$. So it is reasonable to say that this topology is induced by the operator opening.

Notes and remarks

4.4. Part (g) of Theorem 4.2 implies that in the Hilbert space case the operator opening is a metric on $G(X)$. However in general case it is not so: the triangle inequality may fail for it. Indeed, let $X = \ell^2_1$ and $Y_1, Y_2, Y_3 \in G(X)$ be subspaces introduced in example 3.10. It can be verified that in this case $r(Y_k, Y_j) = \Theta(Y_k, Y_j) \ (k, j = 1, 2, 3)$ and so, the triangle inequality is not satisfied.

4.5. The operator opening was introduced by J.L.Massera and J.J.Schäffer [MS1, p. 563]. This concept appeared in a natural way in their investigations of linear differential equations in Banach spaces. Analogous concept somewhat later was introduced by A.L.Garkavi [Ga]. He used this concept in the theory of best approximation in Banach spaces.

4.6. All statements of Theorem 4.2 except statement (h) can be found in [Ber], statement (e) was earlier proved by B.Sz-Nagy [Sz1], [Sz2, p. 132]. Some of them seems to be known to J.L.Massera and J.J.Schäffer [MS1] (see p. 563). Statement (h) I added since it is a natural analogue for the corresponding statement about $\Theta$. It seems that the example from 4.4 was known to J.L.Massera and J.J.Schäffer (see [MS2], §13), but it seems that it was not published anywhere.

5. Comparison of the topologies induced by geometric and operator openings of subspaces.

5.1. When geometric and operator openings were introduced the problem of the comparison of the topologies induced by them arose in a natural way. By Theorem 4.2(c) the topology induced by the operator opening majorize the topology induced by the geometric opening. From Theorem 4.2(g) it follows that in Hilbert space these topologies coincide. In the general case the following version of this result is valid.

5.2. Theorem. Geometric and operator openings induce the same topology on $G_c(X)$.

This theorem immediately follows from the next result.

5.3. Proposition. Let $X$ be a Banach space and $Y \in G_c(X)$. If $Z \in G(X)$ is such that $\Omega(Z, Y) < 1/\lambda(Y, X)$ then $Z \in G_c(X)$ and

$$r_0(Y, Z) \leq \lambda(Y, X)\Omega(Z, Y)(1 + \lambda(Y, X) - \Omega(Z, Y)\lambda(Y, X))/(1 - \Omega(Z, Y)\lambda(Y, X))$$

Proof. We shall use the following geometrical concept. Let $Y, Z \in G(X)$. The number

$$\delta(Z, Y) = \text{dist}(S(Z), Y)$$

is called the inclination of $Z$ to $Y$. 

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Let $Y \cap Z = \{0\}$. It is easy to verify that $\delta(Z,Y) > 0$ if and only if the subspace $\text{lin}(Z \cup Y) \subset X$ is closed. Moreover, if $\delta(Z,Y) > 0$ then there exists a projection from $\text{lin}(Z \cup Y)$ onto $Z$, whose kernel is $Y$ and whose norm is equal to $1/\delta(Z,Y)$.

By definition of $\lambda(Y,X)$ it follows that for every $\varepsilon > 0$ there exists a projection $P_{Y,\varepsilon} : X \rightarrow Y$ such that $||P_{Y,\varepsilon}|| < \lambda(Y,X) + \varepsilon$. Therefore for $U_\varepsilon = \ker P_{Y,\varepsilon}$ we have $\delta(Y,U_\varepsilon) > 1/(\lambda(Y,X) + \varepsilon)$. Using the definition of $\Omega$ we obtain $\delta(Z,U_\varepsilon) > \delta(Y,U_\varepsilon) - \Omega(Y,Z) > (1 - \Omega(Z,Y)(\lambda(Y,X) + \varepsilon))/(\lambda(Y,X) + \varepsilon)$. So, if $\varepsilon$ is small enough then $\delta(Z,U_\varepsilon) > 0$. Let us show that if $\varepsilon$ is small enough then $\text{lin}(Z \cup U_\varepsilon) = X$. In fact, let us suppose that it is not the case. Since we may assume without loss of generality that $\text{lin}(Z \cup U_\varepsilon)$ is closed, then we can find $x \in S(X)$ such that $\text{dist}(x,\text{lin}(Z \cup U_\varepsilon)) > 1 - \varepsilon$. Furthermore, we have $x = y + u$, where $y \in Y$, $u \in U_\varepsilon$ and $||y|| < \lambda(Y,X) + \varepsilon$. Therefore there exists $z \in Z$ such that $||z - y|| < (\lambda(Y,X) + \varepsilon)(\Omega(Z,Y) + \varepsilon)$.

Hence $1 - \varepsilon < \text{dist}(x,\text{lin}(Z \cup U_\varepsilon)) \leq ||x - (z + u)|| < (\lambda(Y,X) + \varepsilon)(\Omega(Z,Y) + \varepsilon)$.

It is clear that for $\varepsilon$ small enough this inequality is false. So we may assume that there exists a projection of $X$ onto $Z$ whose norm is not greater than $(\lambda(Y,X) + \varepsilon)/(1 - \Omega(Z,Y)(\lambda(Y,X) + \varepsilon))$ and whose kernel is $U_\varepsilon$. We shall denote this projection by $P_{Z,\varepsilon}$. Using Theorem 4.2(e) we obtain

$$r(Y,Z) \leq \inf_{\varepsilon} ||P_{Z,\varepsilon} - P_{Y,\varepsilon}||.$$ 

We have

$$||P_{Z,\varepsilon} - P_{Y,\varepsilon}|| = \sup_{x \in S(X)} ||P_{Z,\varepsilon}x - P_{Y,\varepsilon}x|| = \sup_{x \in S(X)} ||(P_{Z,\varepsilon} - I)P_{Y,\varepsilon}x|| \leq \sup_{x \in S(X)} \inf_{z \in Z} ||(P_{Z,\varepsilon} - I)(P_{Y,\varepsilon}x - z)|| \leq (||P_{Z,\varepsilon}|| + 1)||P_{Y,\varepsilon}||\Omega(Y,Z).$$

Taking infimum over $\varepsilon > 0$ we obtain the required inequality. The proposition is proved.

5.4. V.I.Gurarii and A.S.Markus [GuM] proved that in general topologies induced on $G(X)$ by geometric and operator openings are different. Here are their arguments. Let $K$ be a complemented subspace of a Banach space $Y$ and let $L$ be an uncomplemented subspace of a Banach space $Z$ and let $K$ and $L$ be isomorphic. V.I.Gurarii and A.S.Markus proved that in such a case the topologies induced on $G(Y \oplus Z)$ by geometric and operator openings are different.

In fact, let $T : L \rightarrow K$ be an isomorphism. Let us introduce the following family of subspaces of $Y \oplus Z$:

$$L(\lambda) = \{(\lambda Tx, x) : x \in L \} (\lambda \in \textbf{R}, \lambda > 0).$$

It is easy to verify that $\lim_{\lambda \rightarrow 0} \Theta(L(\lambda), L) = 0$. Since $L$ is uncomplemented then in order to prove that $\lim_{\lambda \rightarrow 0} r(L(\lambda), L) \neq 0$ it is sufficient to verify that $L(\lambda)$ ($\lambda > 0$) are complemented. Let us do that. Let $P : Y \rightarrow K$ be a projection onto $K$. Let $P_{\lambda} : Y \oplus Z \rightarrow Y \oplus Z$ be defined by the equality $P_{\lambda}(y,z) = (Py, \lambda^{-1}T^{-1}Py)$. It can be directly verified that $P_{\lambda}$ is a continuous projection onto $L(\lambda)$.

5.5. The theory of complemented and uncomplemented subspaces is highly developed now. Comparing results of this theory [BDGJN], [B], [LT1], [Pel2], [Ros] with the arguments of 5.4 we can deduce the distinction of the topologies induced by geometric
and operator openings for the most part of common Banach spaces (but with such exceptions as $c_0, l_\infty$, Hilbert spaces).

Below we shall describe another method of proving the distinction of the topologies induced by geometric and operator openings. This method works in the cases $X = c_0, l_\infty$. But the following problem posed by V.I.Gurarii and A.S.Markus [GuM] remains unsolved.

**Problem.** Does there exist a Banach space which is nonisomorphic to a Hilbert space but is such that the topologies induced on $G(X)$ by geometric and operator openings coincide?

At the moment it is known that these topologies are different for Banach spaces which are not “almost Hilbert” (in the sense described below).

Recall some definitions. Let us denote by $r_i(t)$ ($i \in \mathbb{N}$, $t \in [0, 1]$) the Rademacher functions. A Banach space $X$ is said to have type $p$ (1 $\leq p \leq 2$) if for some constant $T_p(X) < \infty$ and for every finite set $\{x_i\}_{i=1}^n$ of vectors of $X$ the following inequality takes place

$$\left( \int_0^1 \left| \sum_{i=1}^n r_i(t)x_i \right|^2 dt \right)^{1/2} \leq T_p(X) \left( \sum_{i=1}^n ||x_i||^p \right)^{1/p}.$$  

A Banach space $X$ is said to have cotype $q$ (2 $\leq q < \infty$) if for some constant $C_q(X) < \infty$ and every finite set $\{x_i\}_{i=1}^n$ of vectors of $X$ the following inequality takes place:

$$C_q(X) \left( \int_0^1 \left| \sum_{i=1}^n r_i(t)x_i \right|^2 dt \right)^{1/2} \geq \left( \sum_{i=1}^n ||x_i||^q \right)^{1/q}.$$  

Let $p(X) = \sup\{p : X has type p\}$ and $q(X) = \inf\{q : X has cotype q\}$. It is known that for infinite compact $K$ we have $p(C(K)) = 1$ and $q(C(K)) = \infty$. If a measure $\mu$ is distinct from the atomic measure with finite set of atoms and $1 \leq r < \infty$ then $p(L_r(\mu)) = \min\{r, 2\}$ and $q(L_r(\mu)) = \max\{r, 2\}$. S.Kwapień [Kw] proved that if a Banach space has type 2 and cotype 2 then it is isomorphic to a Hilbert space. This result allows us to consider a Banach space $X$ with $q(X) = p(X) = 2$ as an “almost Hilbert” one.

Extensive information on type and cotype (including results mentioned above) may be found in [LT2], [MiS] and [Pis1].

**5.6 Theorem.** If a Banach space $X$ is such that $p(X) \neq 2$ or $q(X) \neq 2$, then the topology induced on $G(X)$ by the operator opening is strictly stronger than the topology induced by the geometric opening.

We divide the proof of the theorem into two parts. The first, Proposition 5.7, is a “pasting together” of infinite-dimensional subspaces $\{W_j\}_{j=0}^\infty$ for which the following conditions are satisfied:

$$\lim_{j \to \infty} r(W_0, W_j) \neq 0, \quad \lim_{j \to \infty} \Theta(W_0, W_j) = 0.$$  

from finite-dimensional “pieces”. The second, Proposition 5.8, is a construction of suitable finite-dimensional “pieces” in spaces satisfying the condition of the theorem.

**5.7. Proposition.** Let a Banach space $X$ be such that for some $\gamma > 1$, every $\varepsilon > 0$ and every $Y \in G(X)$ of finite codimension in $X$ there exist finite-dimensional
subspaces $Z_1$ and $Z_2$ in $G(Y)$ such that $\Theta(Z_1, Z_2) < \varepsilon$ and $d(Z_1, Z_2) > \gamma$. Then the topology induced by the operator opening on $G(X)$ is strictly stronger than the topology induced by the geometric opening on $G(X)$.

Proof. We construct a sequence of subspaces of $X$ that satisfy conditions (1) and (2). Let the numbers $\{\delta_i\}_{i=1}^\infty$ be such that $\delta_i > 0$ and $\Delta = \Pi_{i=1}^\infty (1 + 2\delta_i) < \infty$. We take $Y_1 = X, \varepsilon_1 = 1$. According to the condition of the proposition, finite dimensional $Z_1^1, Z_2^1 \subset Y_1$ with $\Theta(Z_1^1, Z_2^1) < \varepsilon_1, d(Z_1^1, Z_2^1) > \gamma$ can be found. We consider the $\delta_1$-net $\{y_i^{(1)}\}_{i=1}^{n(1)}$ on $S(\cup_{i=1}^1 Z_i^1)$ and functionals $\{y_i^{*}\}_{i=1}^{n(1)} \subset S(X^*)$ such that $y_i^*(y_i) = 1$. We let $Y_2 = \{y_i^{*}\}_{i=1}^{n(1)}$ (intersection of the kernels of functionals $y_i^*$), $\varepsilon_2 = 1/2$. For them in turn we find finite-dimensional $Z_1^2, Z_2^2 \subset Y_2$ such that $\Theta(Z_1^2, Z_2^2) < \varepsilon_2, d(Z_1^2, Z_2^2) > \gamma$.

We consider the $\delta_2$-net $\{y_i^{(2)}\}_{i=1}^{n(2)}$ of $S(\cup_{i=1}^2 (Z_1^1 \cup Z_2^1))$ and functionals $\{y_i^{*}\}_{i=1}^{n(2)}$ such that $y_i^*(y_i) = 1$. Further, we take $Y_3 = \{y_i^{*}\}_{i=1}^{n(2)}$ such that $y_i^*(y_i) = 1$. We shall prove that $\Theta(Z_1^1, Z_2^1) < \varepsilon_3, d(Z_1^2, Z_2^2) > \gamma$. Repeating the arguments from [LT1, p. 4] we can prove that spaces $Z^i$ form a Schauder decomposition of $V$. Let $\pi_{k,n}, k \leq n$, be operators on $V$ defined by the equations $\pi_{k,n}(\sum_{i=1}^k z^i) = \sum_{i=k}^n z^i$. By arguments analogous to those in [LT1, p. 5], we can show that the operators $\pi_{k,n}$ are bounded and $\sup_{k,n} \|\pi_{k,n}\| \leq 2\Delta < \infty$. Therefore $[z] = \sup_{k,n} \|\pi_{k,n}z\|$ is an equivalent norm on $V$. We extend it onto the whole $X$ using the following construction due to A. Pelczynski [Pel1] (Proposition 1). Let $B_1$ be the unit ball of $V$ in the new norm and let $\alpha > 0$ be such that $(\alpha B(X) \cap V) \subset B_1$, where $B(X)$ is the unit ball of $X$ in the original norm. Let us introduce new norm on $X$ as a Minkowski functional of $\text{cl}(\text{conv}(\alpha B(X) \cup B_1))$. It is easy to verify that this norm coincides with $[\cdot]$ on $V$. So we may denote it also by $[\cdot]$. It is easy to verify that this norm coincides with the original on $Z^i, i = 1, 2, \ldots$, and in addition has the following property:

\[
(\forall m(2) > m(1) > n(1) > n(2))(\forall (z^i)_{i=n(2)}^{m(2)}) (\forall (z^i)_{i=n(1)}^{m(1)}) \left( \sum_{i=n(1)}^{m(1)} z^i \right) \leq \left( \sum_{i=n(2)}^{m(2)} z^i \right).
\]

It is clear that it is sufficient to prove the distinction of the topologies for $X$ with the norm $[\cdot]$. The desired sequence of subspaces will be the following:

\[
W_0 = \sum_{i=1}^\infty \oplus Z^i_1; \quad W_j = (\sum_{i\neq j} \oplus Z^i_1) \oplus Z^j_2.
\]

In order to prove (2) it is sufficient to show that $\Theta(W_j, W_0) < \varepsilon_j$. Let $z \in S(W_j)$, $z = z_2^j + \sum_{i \neq j} z_1^i$. By (3) we have $[z_2^j] \leq 1$. Using the fact that a new norm coincides on $Z^j$ with the original, and consequently the inequality $\Theta(Z^1_1, Z^2_1) < \varepsilon_j$ is preserved, we find a vector $z_1^j \in Z^j_1$ such that $[z_1^j - z_2^j] < \varepsilon_j$. For the vector $\bar{z} = \sum_{i=1}^\infty z_i^1$ we have $\bar{z} \in W_0, [\bar{z} - z] < \varepsilon_j$. Analogous arguments can be carried out also for $\bar{z} \in S(W_0)$. Thus (2) is proved.

We shall prove (1) by contradiction. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence of bounded linear operators on $X$ that satisfy the conditions:

I) $[I_X - \phi_j] \to 0$;
II) \( \phi_j(W_j) = W_0. \)

We introduce operator \( \tau_j : Z^j_2 \to Z^j_1 \) as the restriction of operator \( \pi_{j,j} \phi_j \) to the space \( Z^j_2. \) For \( z^j_2 \in Z^j_2 \) we have \( \phi_j(z^j_2) = \tau_j z^j_2 + \sum_{i \neq j} w_i^j \) for some \( w_i^j \in Z^j_1. \) Consequently, by (2) we have

\[
[z^j_2 - \tau_j z^j_2] \leq [(z^j_2 - \tau_j z^j_2) + \sum_{i \neq j} w_i^j] = [z^j_2 - \phi_j z^j_2] \leq [z^j_2][I_X - \phi_j].
\]

We obtain \( [z^j_2 - \tau_j z^j_2] \to 0 \) uniformly over \( z^j_2 \in S(Z^j_2). \) Therefore, starting from some \( j, \)
the operators \( \tau_j \) are isomorphisms of \( Z^j_2 \) into \( Z^j_1, \)
and, in addition \( [\tau_j][\tau_j^{-1}] \to 1 \) when \( j \to \infty. \) This contradicts inequality \( d(Z^j_1, Z^j_2) > \gamma > 1, \) which is preserved also in the new norm, since it coincides on \( Z^j \) with the original. The proposition is proved.

5.8. Proposition. If a Banach space \( X \) is such that \( q(X) \neq 2 \) or \( p(X) \neq 2, \) then for every \( \gamma > 1 \) and every \( \varepsilon > 0 \) there exist finite dimensional subspaces \( Z_1, Z_2 \in G(X) \) such that \( d(Z_1, Z_2) > \gamma \) and \( \Theta(Z_1, Z_2) \leq \varepsilon. \)

In the proof of this proposition we shall use the following construction. Let \( X \) be a
Banach space and let \( Y \in G(X). \) Let us denote by \( \vartheta \) the quotient mapping \( X \to X/Y. \) In the space \( X \oplus_1 (X/Y) \) we introduce the following subspaces \( G_0 = Y \oplus_1 (X/Y) \) and \( G_0 = \{(x, \vartheta x) : x \in X \}, \) \( 1 > \varepsilon > 0. \)

5.9. Lemma. The equalities \( d(G_0, X) \leq (1 + \varepsilon)/\varepsilon \) and \( \Theta(G_0, G_0) \leq \varepsilon \) take place.

Proof. The first inequality follows from the fact that the operator \( \tau : X \to G_0, \)
defined by the equation \( \tau x = (\varepsilon x, \vartheta x), \) satisfies inequalities \( ||\tau|| \leq 1 + \varepsilon, ||\tau^{-1}|| \leq 1/\varepsilon. \)
We shall prove the second inequality. Let \( u \in G_0, u = (\varepsilon x, \vartheta x). \) Then \( ||u|| = \varepsilon ||x|| + ||\vartheta x||. \) By the fact that \( \vartheta \) is a quotient mapping, for every \( \delta > 0 \) we can find \( y_\delta \in \text{Ker} \vartheta = Y \) such that \( ||x - y_\delta|| < ||\vartheta x|| + \delta. \) We introduce a vector \( v_\delta(u) = (\varepsilon y_\delta, \vartheta x) \in G_0. \) For \( v_\delta(u) \) we have \( ||u - v_\delta(u)|| < (||\vartheta x|| + \delta)\varepsilon. \) Therefore,

\[
\sup_{u \in S(G_0)} \text{dist}(u, G_0) \leq \sup_{\delta > 0} \{\inf_{\delta > 0} ([||u - v_\delta(u)||/||u||]) : u \in G_0, u \neq 0\} \leq \sup_{\delta > 0} \{\inf_{\delta > 0} [\varepsilon (||\vartheta x|| + \delta)/(\varepsilon ||x|| + ||\vartheta x||)] : u = (\varepsilon x, \vartheta x) \neq 0\} \leq \varepsilon.
\]

Let \( v \in G_0, v = (y, z). \) We take \( x_\delta \in X \) such that \( z = \vartheta x_\delta \) and \( ||x_\delta|| \leq ||z|| + \delta. \) We introduce vectors \( u_\delta(v) = (y + \varepsilon x_\delta, z) \in G_0. \) We have

\[
\sup_{v \in S(G_0)} \text{dist}(v, G_0) \leq \sup_{\delta > 0} \{\inf_{\delta > 0} ([||u - u_\delta(v)||/||v||]) : v \in G_0, v \neq 0\} \leq \sup_{\delta > 0} \{\inf_{\delta > 0} [\varepsilon (||z|| + \delta)/(||y|| + ||z||)] : 0 \neq v = (y, z) \in G_0\} \leq \varepsilon.
\]

Lemma 5.9 is proved.

5.10. Proof of Proposition 5.8. Let \( 1 < p = p(X) < 2, q = p/(p - 1). \) In [Ros, p. 286] it was proved that there exists a sequence of spaces \( \{X_n\}, \text{dim} X_n = n, \) that satisfy the following conditions:

1) \( X_n \) is a subspace of \( l^m_q \) for some \( m = m(n) \in \mathbb{N}. \)
2) The subspaces \( X_n^\perp \subset l^m_p \) are such that for any sequence \( \{Y_n\} \) of subspaces of \( l_p \) which are uniformly isomorphic to \( \{X_n^\perp\} \) we have

\[
\lim_{n \to \infty} \lambda(Y_n, l_p) = \infty.
\]
3) Spaces \( \{X^*_n\} \) are uniformly isomorphic to some subspaces \( \{W_n\} \) of \( l_p \).

We introduce the spaces

\[
L_n = l_p^{m(n)} \oplus_1 (l_p^{m(n)}/(X^*_n)),
\]

where sequences \( \{X_n\}_{n=1}^{\infty} \) and \( \{m(n)\}_{n=1}^{\infty} \) are such that conditions 1, 2 and 3 are satisfied. We note that the second term in (4) is isometric to \( X^*_n \), and therefore from condition 3 and the Maurey-Pisier theorem [MiS, p. 85] it follows that a sequence of subspaces \( M_n \subset X, n = 1, 2, \ldots \) can be found such that \( \sup_n d(M_n, L_n) < \infty \). Therefore it is sufficient to construct the desired \( Z_1 \) and \( Z_2 \) in some of \( L_n \). After using Lemma 5.9 for \( X = l_p^{m(n)}, Y = X^*_n, \) we find \( G_0^n, G_\epsilon^n \subset L_n \) such that \( \Theta(G_0^n, G_\epsilon^n) \leq \varepsilon; d(l_p^{m(n)}, G_\epsilon^n) \leq (1 + \varepsilon)/\varepsilon \). From condition 2 and the equation \( G_0^n = X^*_n \oplus_1 (l_p^{m(n)}/(X^*_n)) \) we conclude that \( d(l_p^{m(n)}, G_0^n) \to \infty \) when \( n \to \infty \). Since \( d(G_0^n, G_\epsilon^n) \geq d(l_p^{m(n)}, G_0^n)/d(l_p^{m(n)}, G_\epsilon^n) \), then by choosing \( n \) sufficiently large, we get \( d(G_0^n, G_\epsilon^n) > \gamma \). Therefore for \( Z_1 \) and \( Z_2 \) we can take, respectively, \( G_\epsilon^n \) and \( G_0^n \).

In the case \( p(X) = 1 \) we use the fact that the above-constructed \( Z_1 \) and \( Z_2 \) can be embedded in \( X \) with the help of M.I.Kadets’ theorem [Kad1] (see also [MiS, p. 50]).

We consider now the case \( q = q(X) > 2 \). We shall use the fact that for \( p = q/(q - 1) \) \( (p = 1 \text{ in the case } q = \infty) \) we have the following assertion [MiS, pp. 21, 23]: \( \alpha > 0 \) can be found such that for each \( n \in \mathbb{N} \) in \( l_q^n \), a subspace \( Y_n \) can be found with \( \dim Y_n = [\alpha n] \) (integral part) and \( d(Y_n, l_2^{[\alpha n]}) < 2 \). In the same time, if subspaces \( Z_n \subset l_q^n \) are such that \( \dim Z_n/n^{2/q} \to \infty \) when \( n \to \infty \) \( (\dim Z_n/\ln(n) \to \infty \text{ in the case } q = \infty) \), then

\[
d(Z_n, l_2^{[\alpha n]}) \to \infty, \text{ when } n \to \infty,
\]

where \( t(n) = \dim Z_n \). From the first statement it follows that subspaces \( X_n \) of codimension \( [\alpha n] \) can be found in \( l_q^n \) such that \( d(l_q^n/X_n, l_2^{[\alpha n]}) < 2 \).

We consider spaces \( L_n = l_q^n \oplus_1 (l_q^n/X_n) \). Using the Maurey-Pisier theorem [MiS, p. 85] and Dvoretzky’s theorem [MiS, p. 24], we get that subspaces \( M_n \) can be found in \( X \) such that \( \sup_n d(L_n, M_n) < \infty \). Therefore it is sufficient to construct the desired \( Z_1 \) and \( Z_2 \) in some of \( L_n \). For this we use the lemma with \( X = l_q^n, Y = X_n \). We obtain spaces \( G_\epsilon^n, G_0^n \subset L_n \) for which \( d(G_\epsilon^n, l_q^n) \leq (1 + \varepsilon)/\varepsilon, \Theta(G_\epsilon^n, G_0^n) \leq \varepsilon \). Since \( G_0^n = X_n \oplus_1 (l_q^n/X_n) \) has a subspace \( l_q^n/X_n \) for which \( d(l_q^n/X_n, l_2^{[\alpha n]}) < 2 \), then from (5) it follows that \( d(G_0^n, l_q^n) \to \infty \) when \( n \to \infty \). We argue further in the same way as in the first case. The proposition is proved.

5.11. The theorem follows from Propositions 5.7 and 5.8 since, as it is easy to see, if \( Y \subset X \) and \( \dim Y < \infty \), then \( p(X) = p(Y) \) and \( q(X) = q(Y) \).

Notes and remarks.

5.12. Theorem 5.2 is due to E.Berkson [Ber]. Theorem 5.6 and Propositions 5.7 and 5.8 are due to the author [O7]. After publication of [O7] the author learned that Lemma 5.9 was already known to A.Douady [D] (see pp. 15–16). In [O4] the “localized” version of 5.4 was developed. Using this version it can be proved that in order to solve in negative the Gurarii-Markus problem from 5.5 it is sufficient to prove the following inversion of B.Maurey’s theorem from [Ma]: If a Banach space \( X \) does not have type 2 then there exists a sequence of finite dimensional subspaces \( X(n) \subset X \) such that \( \sup_n d(X(n), l_2^{\dim(X(n))}) < \infty \) and \( \sup_n \lambda(X(n), X) = \infty \).
6. Community of properties of subspaces which are close with respect to the opening.

6.1. Subspaces which are close with respect to the operator opening are close with respect to the Banach-Mazur distance.

**Proposition.** If \( r_0(Y, Z) < 1 \), then

\[
d(Y, Z) \leq \frac{(1 + r_0(Y, Z))}{(1 - r_0(Y, Z))}.
\]

Indeed, let \( T \in GL(X) \) be such that \( T(Y) = Z \) and \( ||T - I|| \leq r_0(Y, Z) + \varepsilon < 1 \). Then \( T^{-1} = I + \sum_{i=1}^{\infty} (I - T)^i \). Therefore

\[
||T^{-1}|| \leq 1 + \sum_{i=1}^{\infty} (r_0(Y, Z) + \varepsilon)^i = 1/(1 - r_0(Y, Z) - \varepsilon).
\]

Whence it follows the desired inequality.

6.2. The following proposition may be considered as an inversion of statement from 6.1.

**Proposition.** Let \( Y \) and \( Z \) be isomorphic Banach spaces. Then for every \( \varepsilon > 0 \) there exists a Banach space \( X \) and isometric embeddings of \( Y \) and \( Z \) into \( X \) such that for their images (which we still denote by \( Y \) and \( Z \)) the following inequalities are satisfied:

\[
r_0(Y, Z) \leq d(Y, Z) + \varepsilon - 1; \tag{1}
\]

\[
\Omega(Y, Z) \leq d(Y, Z) + \varepsilon - 1 \tag{2}
\]

**Proof.** Let \( U : Y \to Z \) be an isomorphism such that \( ||U|| = 1 \) and \( ||U^{-1}|| \leq d(Y, Z) + \varepsilon \). On the direct sum \( Y \oplus Z \) we introduce the following seminorm:

\[
p(y, z) = \max\{|z + Uy|, \sup\{|z^*(z) + U^*z^*(y)\langle U^*z^* ||: z^* \in S(Z^*)\}|\}.
\]

This seminorm generates a norm on the quotient of \( Y \oplus Z \) by the zero space of this seminorm. We shall denote the completion of the corresponding normed space by \( X \). The spaces \( Y \) and \( Z \) isometrically embed into \( X \) in a natural way.

Let us introduce an operator \( T \) on \( Y \oplus Z \) by the equality:

\[
T(y, z) = (-U^{-1}z, 2z + Uy).
\]

We have

\[
p((T - I)(y, z)) = p(-U^{-1}z - y, z + Uy) = \\
\max\{|(z + Uy) + U(-U^{-1}z - y)|, \sup\{|z^*(z + Uy) + U^*z^*(-U^{-1}z - y)\langle ||U^*z^* ||: z^* \in S(Z^*)\}| \leq ||z + Uy|| \sup\{|1 - 1/||U^*z^* ||: z^* \in S(Z^*)\} \leq \\
p(y, z)(||U^*||^{-1} - 1) \leq p(y, z)(d(Y, Z) + \varepsilon - 1).
\]

This inequality implies that \( T \) induces continuous operator on \( X \) for which \( T(Y) = Z \) and \( ||T - I|| \leq d(Y, Z) + \varepsilon - 1 \). Thus, we have inequality (1).

In order to prove inequality (2) it is sufficient to verify that for every \( y \in S(Y) \) we have \( p(y, -Uy/||Uy||) \leq d(Y, Z) + \varepsilon - 1 \). We have

\[
p(y, -Uy/||Uy||) =
\]

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max\{\|Uy\|(1/\|Uy\|-1); \sup\{\|z^*(Uy)/\|Uy\| + U^*z^*(y)/\|U^*z^*\|\} : z^* \in S(Z^*)\} \leq \\
max\{\|U^{-1}\| - 1\}; \sup\{\|z^*(Uy)/\|U^*z^*\| - 1/\|Uy\|\} : z^* \in S(Z^*)\} \leq \\
d(Y, Z) + \varepsilon - 1.

The proposition is proved.

6.3. By Propositions 6.1 and 6.2 the class of properties which are common for subspaces which are close with respect to the operator opening coincides with the class of properties which are common for spaces which are close with respect to the Banach-Mazur distance. The situation with the analogous problem for the geometric opening turns out to be quite different. No smallness of the \(\Theta(Y, Z)\) implies that \(Y\) and \(Z\) are isomorphic. This assertion may be deduced from Lemma 5.9. Indeed, it is well-known that for \(X = l_1\) there exists a subspace \(Y \in G(X)\) such that \(X/Y\) is isometric to \(l_2\) [LT1, p. 108]. Let \(G_\varepsilon\) (\(\varepsilon > 0\)) and \(G_0\) be the subspaces introduced before Lemma 5.9. By Lemma 5.9 all \(G_\varepsilon\) (\(\varepsilon > 0\)) are isomorphic to \(l_1\) and \(\Theta(G_\varepsilon, G_0) \to 0\) when \(\varepsilon \to 0\). But the space \(G_0 = Y \oplus l_2\) is not isomorphic to \(l_1\) [LT1, p. 54]. This arguments prove that the set of all subspaces in \(G(X)\) which are isomorphic to the given one \((Y \oplus l_2\) in the example) may be even non-open in \(G(X)\) in the topology induced by the geometric opening.

6.4. In this connection it seems natural to introduce the following definitions.

By property we shall mean a subclass in the class of all Banach spaces. When \(X\) satisfies property \(P\) we shall write \(X \in P\).

A property \(P\) will be called open if for every Banach space \(X\) the subset \(P \cap G(X) \subset G(X)\) is open in the topology induced by the opening \(\Theta\).

A property \(P\) is called stable if there exists a number \(\alpha > 0\) such that for every Banach space \(X\) and every \(Y, Z \in G(X)\), if \(Y \in P\) and \(\Theta(Y, Z) < \alpha\), then \(Z \in P\). The least upper bound of numbers \(\alpha\) for which this statement is true is denoted by \(s(P)\) and is called the stability exponent of \(P\).

A property \(P\) will be called extendedly stable if there exists a number \(\alpha > 0\) such that for every Banach space \(X\) and every \(Y, Z \in G(X)\), if \(Y \in P\) and \(\Theta_0(Z, Y) < \alpha\), then \(Z \in P\). The least upper bound of numbers \(\alpha\) for which this statement is true is denoted by \(es(P)\) and is called the extended stability exponent of \(P\).

6.5. Remark. It is clear that stable properties are open and extendedly stable properties are stable.

6.6. Proposition. If property \(P\) is open, then for every \(Y \in P\) there exists a number \(\alpha > 0\) such that for every isometric embedding \(U : Y \to X\) and every \(Z \in G(X)\), if \(\Theta(Z, UY) < \alpha\), then \(Z \in P\).

Proof. Let us suppose the contrary. Then there exists a space \(Y \in P\), spaces \(X(i)\) (\(i \in \mathbb{N}\)), isometric embeddings \(U_i : Y \to X(i)\) and subspaces \(Z(i) \in G(X(i))\) such that \(\Theta(U_i Y, Z(i)) \to 0\) when \(i \to \infty\).

Let us introduce a space \(X = (\sum_{i=1}^\infty X(i))_1\) and a subspace \(K \subset X\) consisting of those sequences \(\{x(i)\}_{i=1}^\infty\) for which \(x(i) \in \text{im} U_i\) and \(\sum_{i=1}^\infty U_i^{-1} x(i) = 0\). Then the restrictions of the quotient mapping \(\varphi : X \to X/K\) to the subspaces \(X(i)\) are isometries. For every \(i, j \in \mathbb{N}\) we have \(\varphi U_i Y = \varphi U_j Y\). Let us denote this subspace of \(X/K\) by \(V\). We have \(V \in P\), \(\varphi(Z(i)) \notin P\) and \(\lim_{i \to \infty} \Theta(\varphi(Z(i)), V) = 0\). This contradicts to the fact that \(P\) is open. The proposition is proved.
6.7. Proposition. Stable properties are isomorphic invariants.
Proof. Let $P$ be a stable property and $s(P)$ be its stability exponent. Using formula (2) with $\varepsilon = s(P)/2$ we obtain that if $d(Y, Z) < s(P)/2 + 1$ and $Y \in P$ then $Z \in P$.

Let $Y \in P$, $Z$ be isomorphic to $Y$ and $T : Y \to Z$ be an isomorphism. Let us introduce the following family of equivalent norms on $Y$:

$$||y||_t = (1 - t)||y||_Y + t||Ty||_Z, \ t \in [0, 1].$$

It is clear that $Y(0)$ is isometric to $Y$, $Y(1)$ is isometric to $Z$, and the function $t \to d(Y, Y(t))$ is continuous. Hence we can find a set of points $t_1, t_2, \ldots, t_n \in [0, 1]$ such that

$$d(Y, Y(t_1)) < s(P)/2 + 1; \ d(Y(t_n), Z) < s(P)/2 + 1;$$

$$d(Y(t_k), Y(t_{k+1})) < s(P)/2 + 1, \ (k = 1, 2, \ldots, n - 1).$$

By this and the observation made at the begining of the proof it follows that $Z \in P$. The proposition is proved.

6.8. An open property need not be an isomorphic invariant.

Example. $P = \{ X : (\exists x_1, x_2 \in B(X))(||x_1 + x_2|| > 3/2) \ & (||x_1 - x_2|| > 3/2) \}$. It is not hard to show that this property is open. On the other hand, by the parallelogram identity it follows that $l_2 \notin P$. In the same time, the space $l_1^2 \oplus_1 l_2$ is isomorphic to $l_2$ and $l_1^2 \oplus_1 l_2 \notin P$.

6.9. Set-theoretic operations (union, intersection, complement) may be introduced for properties in a natural way. It is not hard to verify the following assertions:

Proposition. a) The intersection of finite collection of open properties is open. The union of every class of open properties is open.

b) Let $\{ P_\alpha : \alpha \in A \}$ be some class of stable properties and $\inf_{\alpha \in A} s(P_\alpha) > 0$, then $\cap_{\alpha \in A} P_\alpha$ and $\cup_{\alpha \in A} P_\alpha$ are stable properties. Furthermore, $s(\cap_{\alpha \in A} P_\alpha) > \inf_{\alpha \in A} s(P_\alpha)$ and $s(\cup_{\alpha \in A} P_\alpha) > \inf_{\alpha \in A} s(P_\alpha)$. Analogous assertion is valid for extendedly stable properties.

c) The complement of the stable property is stable.

As a rule we shall not formulate consequences of our results which can be obtained by immediate application of this proposition.

6.10. Example. By Theorem 3.4 (e) it follows that for every natural number $n$ the class of all $n$-dimensional subspaces is stable and its stability exponent is 1. By Lemma 3.5 the class of the spaces whose dimension is not greater than $n$ is extendedly stable. It is clear that its complement is not extendedly stable.

6.11. The following result is an immediate consequence of Lemma 5.9 and Proposition 6.7.

Proposition. a) Let property $P$ be such that for some Banach spaces $X$ and $W$ the following conditions are satisfied:

1) $X \in P$.

2) $X$ can be decomposed into a direct sum $X = Y \oplus Z$.

3) $W$ contains a subspace $W_0$, which is isomorphic to $Y$ and such that the quotient $W/W_0$ is isomorphic to $Z$.

4) $W$ is not isomorphic to any space from $P$.

Then $P$ is not open.
b) If there exists $X \in P$ such that for some $Y \in G(X)$ the space $Y \oplus_1 (X/Y)$ is not in $P$, then $P$ is not stable.

c) If $P$ is such that for some Banach spaces $X$ and $W$ conditions 1–3 of (a) are satisfied and $W$ contains a subspace which is not in $P$, then $P$ is not extendedly stable.

d) If there exists $X \in P$ such that for some $Y \in G(X)$ the space $Y \oplus_1 (X/Y)$ contains a subspace which is not in $P$, then $P$ is not extendedly stable.

6.12. Part (a) of proposition 6.11 shows that if property $P$ is an isomorphic invariant and is such that for $Y, Z \in P$ we have $Y \oplus Z \in P$ then the negative solution of the so-called “three space problem” is a sufficient condition for $P$ to be non-open.

Recall that the “three space problem” for property $P$ is the following problem: Let $X$ be a Banach space such that for some $Y \in G(X)$ we have $Y \in P$ and $X/Y \in P$. Does it follow that $X \in P$?

This problem was investigated for different properties by many authors. In the present context results of the negative character are of interest. Such results can be found in [CG1], [CG2], [C], [ELP], [JR], [KP], [Lu], [O8], [OP].

It is natural to ask the following question: Let property $P$ be non-open. Is it true that the “three space problem” for $P$ has the negative answer? In order to avoid trivial situations we shall additionally suppose that $P$ is closed with respect to formation of direct sums and is isomorphic invariant.

It turned out that even under this restriction the answer to the posed question is negative. The corresponding example is given by the introduced in 5.5 class of “almost Hilbert” spaces. In [ELP] it was proved that the “three space problem” for this class has positive solution. On the other hand we shall see later that this class is not open.

It should be noted that by the known results on type and cotype [Pis2] it follows that the class of “almost Hilbert” spaces does not satisfy the condition of part (b) of Proposition 6.11.

6.13. Example. Proposition 5.3 immediately implies that the class of injective Banach spaces is open. Let us consider $X = l_\infty$ and let $Y \in G(X)$ be isometric to $l_2$. By part (b) of Proposition 6.11 it follows that the class of injective spaces is not stable.

6.14. Now we turn to another method of establishing unstability of classes of Banach spaces. This method is quite general: every unstable class satisfies its conditions. But sometimes it is not clear how to apply this method to concrete classes.

Let $Y$ and $Z$ be Banach spaces and let $T : S(Y) \to S(Z)$ and $D : S(Y^*) \to S(Z^*)$ be surjective mappings. Let us introduce on the algebraic sum $Y \oplus Z$ the following seminorm:

$$p(y, z) = \sup\{|y^*(y) - (Dy^*)(z)| : y \in S(Y^*)\}.$$  

Seminorm $p$ generates norm on the quotient of $Y \oplus Z$ by the zero-space of $p$. We denote the completion of this normed space by $X$.

By properties of $D$ it follows that $Y$ and $Z$ are isometric with their natural images in $X$. (This images we shall still denote by $Y$ and $Z$.) We have

$$\Omega_0(Y, Z) = \sup\{\text{dist}(y, S(Z)) : y \in S(Y)\} \leq \sup\{||y - Ty|| : y \in S(Y)\} =$$

$$\sup\{|y^*(y) - (Dy^*)(Ty)| : y \in S(Y), y^* \in S(Y^*)\}.$$
\[ \Omega_0(Z,Y) = \sup \{ \text{dist}(z, S(Y)) : z \in S(Z) \} \leq \sup \{ \| y - y^* \| : y \in T^{-1}z, z \in S(Z) \} = \sup \{ \| y - Ty \| : y \in S(Y) \}. \]

Hence \( \Omega(Y, Z) \) is not greater than the following quantity.

\[ \sup \{ \| y^*(y) - (DY^*)(Ty) \| : y \in S(Y), y^* \in S(Y^*) \}. \] (3)

Let us introduce the quantity \( k(Y, Z) \) as the infimum of quantities (3) over all surjective mappings \( T : S(Y) \to S(Z) \) and \( D : S(Y^*) \to S(Z^*) \). We have

\[ \inf_{X,U,V} \Omega(UY, VZ) \leq k(Y, Z), \] (4)

where the infimum is taken over all Banach spaces \( X \) containing isometric copies of \( Y \) and \( Z \) and over all isometric embeddings \( U : Y \to X \) and \( V : Z \to X \). It turns out [O9] that for certain \( a > 0 \) (e.g., we may let \( a = 1/20 \)) we have

\[ \inf_{X,U,V} \Omega(UY, VZ) \geq ak(Y, Z). \] (5)

Using inequalities (4) and (5) and Proposition 6.6 we obtain:

**Proposition.** (a) Property \( P \) is non-open if and only if for some \( Y \in P \) and some Banach spaces \( Z_n \notin P \) \((n \in \mathbb{N})\) we have

\[ \lim_{n \to \infty} k(Y, Z_n) = 0. \]

(b) Property \( P \) is unstable if and only if for some \( Y_n \in P \) and some Banach spaces \( Z_n \notin P \) \((n \in \mathbb{N})\) we have

\[ \lim_{n \to \infty} k(Y_n, Z_n) = 0. \]

6.15. M.I.Kadets [Kad2] applied construction from 6.14 to \( Y = l_2 \) and \( Z = l_p \), where \( 1 < p < 2 \). Considering mappings \( T : l_2 \to l_p \) and \( D : l_2 \to l_q \) (where \( q \) is such that \( 1/q + 1/p = 1 \)) defined by the equations

\[ T(\{ x_i \}_{i=1}^\infty) = \{|x_i|^{2/p}\text{sign}(x_i)\}_{i=1}^\infty; \]
\[ D(\{ x_i \}_{i=1}^\infty) = \{|x_i|^{2/q}\text{sign}(x_i)\}_{i=1}^\infty; \]

he obtained the following estimate:

\[ k(l_2, l_p) \leq 2(2/p - 1). \] (6)

By proposition 6.14 (a) this inequality implies that the class of spaces isomorphic to \( l_2 \) is non-open.

Using analogous estimates the present author [O1] proved that for every \( 1 < p < \infty \) the class of spaces isomorphic to \( l_p \) is non-open.

6.16. **Corollary.** The class of “almost Hilbert” spaces is non-open.

Indeed, since for \( r < 2 \) we have \( p(l_r) = r < 2 \) (see 5.5) it follows that for every \( r < 2 \) the space \( l_r \) is not “almost Hilbert”. Comparing this assertion with Proposition 6.14 and estimate (6) we obtain the required result.
6.17. Let us turn to stable properties. We are going to describe a method of finding
extendedly stable properties.
Let $\Gamma$ be a set, $l_1(\Gamma)$ be the corresponding Banach space. (I.e. the space of functions
$f : \Gamma \to \mathbb{R}$ with countable support, denoted by $\text{supp} f$ and such that $\sum_{\gamma \in \text{supp} f} |f(\gamma)| < \infty$. The norm on $l_1(\Gamma)$ is defined as $||f|| = \sum_{\gamma \in \text{supp} f} |f(\gamma)|$.) Let $X$ be a Banach space.
By $l_\infty(\Gamma, X)$ we denote the space of functions $x : \Gamma \to X$ such that $\sup_{\gamma \in \Gamma} ||x(\gamma)||_X < \infty$, with the norm $||x|| = \sup_{\gamma \in \Gamma} ||x(\gamma)||_X$.
Let $A$ be a subset of the unit sphere of $l_1(\Gamma)$. For every $a \in A$ we introduce a linear
operator from $l_\infty(\Gamma, X)$ into $X$ defined in the following way:
$$x \to \sum_{\gamma \in \Gamma} a(\gamma)x(\gamma).$$
This operator will be also denoted by $a$. It is clear that the norm of this operator equals 1.

6.18. Definition. By the index of $A$ in $X$ we mean the supremum $h(X, A)$ of those
$\delta$ for which there exists $x \in S(l_\infty(\Gamma, X))$ such that $\inf_{a \in A} ||a(x)|| \geq \delta$.

Many common properties of Banach spaces can be described in terms of introduced
indices.

6.19. Example. Recall the characterization of reflexivity due to R.C.James [J1]
and D.P.Milman-V.D.Milman [MM].
Let $X$ be a Banach space. The following assertions are equivalent:
a) $X$ is nonreflexive.
b) For some $\varepsilon > 0$ there exists a sequence $\{x_i\}_{i=1}^\infty \subset B(X)$ such that
$$\forall n \in \mathbb{N} \exists x \in S(l_\infty(\Gamma, X)) \text{ such that } \inf_{a \in A} ||a(x)|| \geq \varepsilon$$
    \hspace{1cm} (7)
c) For every $1 > \varepsilon > 0$ there exists a sequence $\{x_i\}_{i=1}^\infty \subset B(X)$ such that (7) is
    satisfied.
    In order to restate these results in terms of introduced indices let us define $R \subset S(l_1)$
as a set of all vectors of the form
$$(a_1, \ldots, a_n, -a_{n+1}, \ldots, -a_m, 0, \ldots),$$
where $n < m$ are arbitrary natural numbers and numbers $a_i$ ($i = 1, \ldots, m$) are such
that $a_i \geq 0$, $\sum_{i=1}^n a_i = 1/2$; $\sum_{i=n+1}^m a_i = 1/2$.
It is easy to see that the formulated above characterization of reflexivity can be
restated as following:
A Banach space $X$ is reflexive if and only if $h(X, R) = 0$. If $X$ is nonreflexive, then
$h(X, R) \geq 1/2$.

6.20. Example. Let $\alpha$ be some uncountable cardinal. M. G. Krein, M. A. Kras-
noselskii and D. P. Milman [KKM, p. 98] proved that the following conditions are
equivalent:
(a) $\text{dens} X \geq \alpha$.
(b) For some $\varepsilon > 0$ the unit ball $B(X)$ contains a subset of cardinality $\alpha$ such that
the distance between each two elements of this subset is not less than $\varepsilon$. 
(c) For every $0 < \varepsilon < 1$ the unit ball $B(X)$ contains a subset of cardinality \( \alpha \) such that the distance between each two elements of this subset is not less than \( \varepsilon \).

This result also may be reformulated in terms of introduced indices. Let set \( \Gamma \) be such that \( \text{card}\Gamma = \alpha \). Let us denote by \( D(\alpha) \) the subset of \( S(l_1(\Gamma)) \) consisting of all vectors with two-point support for which one of the values is \( 1/2 \) and the other is \((-1/2)\). It is easy to see that the mentioned characterization of spaces with density character not less than \( \alpha \) can be reformulated in the following way.

The density character of a Banach space \( X \) is less than \( \alpha \) if and only if \( h(X, D(\alpha)) = 0 \).

6.21. Proposition. Let \( X \) be a Banach space, \( Y, Z \in G(X) \). Then for every set \( \Gamma \) and every \( A \subset S(l_1(\Gamma)) \) the following inequalities take place:

\[
\Lambda_0(Y, Z) \geq h(Y, A) - h(Z, A) \quad (8)
\]

\[
\Theta_0(Y, Z)(1 + h(Z, A)) \geq h(Y, A) - h(Z, A) \quad (9)
\]

Proof of (8). Let \( \varepsilon > 0 \) be arbitrary. Let vector \( y \in S(l_\infty(\Gamma, Y)) \) be such that \( \inf_{a \in A} ||a(y)|| \geq h(Y, A) - \varepsilon \). Using definition of \( \Lambda_0 \) we find \( z \in B(l_\infty(\Gamma, Z)) \) such that

\[
(\forall \gamma \in \Gamma)(||z(\gamma) - y(\gamma)|| < \Lambda_0(Y, Z) + \varepsilon).
\]

Therefore \( \inf_{a \in A} ||a(z)|| \geq h(Y, A) - \Lambda_0(Y, Z) - 2\varepsilon \). Hence \( h(Z, A) \geq h(Y, A) - \Lambda_0(Y, Z) \).

Proof of (9). Let \( \varepsilon > 0 \) be arbitrary. Let vector \( y \in S(l_\infty(\Gamma, Y)) \) be such that \( \inf_{a \in A} ||a(y)|| \geq h(Y, A) - \varepsilon \). Using definition of \( \Theta_0 \) we find \( z \in l_\infty(\Gamma, Z) \) such that

\[
(\forall \gamma \in \Gamma)(||z(\gamma) - y(\gamma)|| < \Theta_0(Y, Z) + \varepsilon).
\]

Therefore

\[
\inf_{a \in A} ||a(z)|| \geq h(Y, A) - \Theta_0(Y, Z) - 2\varepsilon.
\]

On the other hand

\[
||z|| \leq \sup_{\gamma} ||y(\gamma)|| + \Theta_0(Y, Z) + \varepsilon.
\]

Hence

\[
h(Z, A) \geq (h(Y, A) - \Theta_0(Y, Z))/(1 + \Theta_0(Y, Z))
\]

It is easy to see that this inequality is equivalent to (9). The proposition is proved.

Applying this proposition to examples 6.19 and 6.20 we obtain the following consequences.

6.22. Corollary. Let \( P \) be a class of reflexive spaces. Then \( P \) is extendedly stable and \( es(P) \geq 1/2 \).

6.23. Corollary. Let \( \alpha \) be uncountable ordinal and let \( P \) be a class of Banach spaces whose density character is less than \( \alpha \). Then \( P \) is extendedly stable and \( es(P) \geq 1/2 \).

6.24. It turns out that many known isomorphic invariants can be described in the same manner as properties in examples 6.19 and 6.20. In this connection it is natural to introduce the following definition.
**Definition.** Class $P$ of Banach spaces is said to be $l_1$-property, if there exist a set $\Gamma$ and a subset $A \subset S(l_1(\Gamma))$ such that for some positive $\delta > 0$ the following assertions are equivalent:

(a) $X \notin P$.
(b) $h(X, A) > 0$.
(c) $h(X, A) \geq \delta$.

By Proposition 6.21 every $l_1$-property is extendedly stable and extended stability exponent is not less than $\delta$.

**6.25. Definition.** Let $P$ be an $l_1$-property. The supremum of those $\delta$ for which there exist a set $\Gamma$ and a subset $A \subset S(l_1(\Gamma))$ such that the conditions (a), (b) and (c) of definition 6.24 are satisfied is called the $l_1$-exponent of $P$ and is denoted by $e_1(P)$.

It is clear that $es(P) \geq e_1(P)$.

**6.26.** It seems that $e_1(P)$ can be less than $es(P)$ (see e.g. example 6.27.4 below). But now I haven’t proof of this assertion.

**6.27.** Here is the list of known $l_1$-properties and the estimates for their $l_1$-exponents and extended stability exponents.

**6.27.1.** Reflexivity. By 6.19 it follows that reflexivity is an $l_1$-property and that its $l_1$-exponent is not less than $1/2$. The precise values of the $l_1$-exponent and the extended stability exponent of reflexivity seems to be unknown.

**6.27.2.** Let $\alpha$ be an uncountable cardinal. The class of all Banach spaces, whose density character is less than $\alpha$ is an $l_1$-property by the result of M. G. Krein, M. A. Krasnoselskii and D. P. Milman mentioned in 6.20, and its $l_1$-exponent is not less than $1/2$. The $l_1$-exponents and the extended stability exponents of those classes for which $\alpha$ is not greater than the cardinality of continuum are equal to $1/2$ (see 6.43). For those $\alpha$ which are greater than the cardinality of continuum the values of $l_1$-exponents and extended stability exponents seem to be unknown.

**6.27.3.** The class of all finite dimensional spaces. One can show that this class is an $l_1$-property by consideration of the subset $A = S(l_1^{n+1})$. Using notion of Auerbach system (see [LT1, p. 16]) it can be shown that the $l_1$-exponent of the class of spaces whose dimension is not greater than $n$, is not less than $1/(n + 1)$. It is known that this estimate is not precise. This assertion can be derived, for example, from the results on the estimates of Banach-Mazur distances between $l_1^n$ and arbitrary $n$-dimensional space (see [Sz] and [SzT]). Even the order (with respect to $n$) of these $l_1$-exponents seems to be unknown. By Theorem 3.4(e) the extended stability exponents of these classes are equal to $1$.

**6.27.4.** Let $n \in \mathbb{N}$. The class of all finite dimensional Banach spaces, whose dimension is not greater than $n$ is an $l_1$-property. It can be shown by consideration of $A = S(l_1^{n+1})$. Using notion of Auerbach system (see [LT1, p. 16]) it can be shown that the $l_1$-exponent of the class of spaces whose dimension is not greater than $n$, is not less than $1/(n + 1)$. It is known that this estimate is not precise. This assertion can be derived, for example, from the results on the estimates of Banach-Mazur distances between $l_1^n$ and arbitrary $n$-dimensional space (see [Sz] and [SzT]). Even the order (with respect to $n$) of these $l_1$-exponents seems to be unknown. By Theorem 3.4(e) the extended stability exponents of these classes are equal to $1$.

**6.27.5.** $B$-convexity. 

**Definition.** A Banach space $X$ is said to be $B$-convex if

$$\lim_{n \to \infty} \inf \{d(X_n, l_1^n) : X_n \text{ is an } n \text{-dimensional subspace of } X\} = \infty.$$
It was shown by D.P.Giesy [Gi] (Lemmas I.4 and I.6), see also [LT2, p. 62], that for every non-$B$-convex Banach space $X$, every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a subspace $X_n \subset X$ such that $d(X_n, l^n_1) < 1 + \varepsilon$.

Let $A \subset S(l_1)$ be the set of all sequences, which for some $n \in \mathbb{N}$ have the following form:

$$(0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots),$$

where $a_1$ is preceded by $n(n - 1)/2$ zeros.

It is clear that the mentioned result due to Giesy can be reformulated in the following way.

Let $X$ be a Banach space. The following three assertions are equivalent.

(a) $X$ is non-$B$-convex.
(b) $h(X, A) > 0$.
(c) $h(X, A) = 1$.

Hence $B$-convexity is an $l_1$-property and its $l_1$-exponent and its extended stability exponent are equal to 1.

6.27.6. Super-reflexivity.

**Definition.** A Banach space $X$ is said to be **finitely representable** in a Banach space $Y$ if for each finite dimensional subspace $X_n$ of $X$ and each positive number $\varepsilon$ there exists a subspace $Y_n$ of $Y$ such that $d(X_n, Y_n) < 1 + \varepsilon$. A Banach space $X$ is said to be **super-reflexive** if every Banach space which is finitely representable in $X$ is reflexive.

The class of super-reflexive spaces has many equivalent descriptions. For our purposes the most interesting is the following one (see [Be2, p. 236, 237, 270]).

For a Banach space $X$ the following properties are equivalent.

(a) $X$ is not super-reflexive.
(b) For some $\varepsilon > 0$ and every $n \in \mathbb{N}$ there is a sequence $x^n_1, \ldots, x^n_n$ in the unit ball of $X$ such that for every $k \in \mathbb{N}, 1 \leq k \leq n$,

$$\text{dist}(\text{conv}(x^n_1, \ldots, x^n_k), \text{conv}(x^n_{k+1}, \ldots, x^n_n)) > \varepsilon. \quad (10)$$

(c) For every $0 < \varepsilon < 2$ and every $n \in \mathbb{N}$ there is a sequence $x^n_1, \ldots, x^n_n$ in the unit ball of $X$ such that inequality (10) is satisfied for every $k \in \mathbb{N}, 1 \leq k \leq n$.

Let $A \subset S(l_1)$ be the set of all sequences, which for some $n \in \mathbb{N}$ have the following form:

$$(0, \ldots, 0, a_1, \ldots, a_n, 0, \ldots),$$

where for some $k \in \mathbb{N}, 1 \leq k \leq n$, we have $a_i \leq 0$ for $i \leq k$ and $a_i \geq 0$ for $i \geq k + 1$; $\sum_{i=1}^{k} a_i = -1/2$ and $a_1$ is preceded by $n(n - 1)/2$ zeros.

It is clear that the mentioned characterization of super-reflexivity can be reformulated in the following way.

Let $X$ be a Banach space. The following three assertions are equivalent.

(a) $X$ is not super-reflexive.
(b) $h(X, A) > 0$.
(c) $h(X, A) = 1$.

Hence super-reflexivity is an $l_1$-property and its $l_1$-exponent and its extended stability exponent are equal to 1.
6.27.7. Class of spaces which do not contain isomorphic copies of $l_1$. R.C. James [J2] (see also [LT1, p. 97]) proved that if a Banach space $X$ contains a subspace isomorphic to $l_1$, then, for every $\varepsilon > 0$ there exists a subspace $Y \subset X$ for which
\[ d(Y, l_1) < 1 + \varepsilon. \]

Let $A$ be the set of all finitely non-zero vectors from $S(l_1)$. It is clear that mentioned result due to James can be reformulated in the following way.

Let $X$ be a Banach space. The following three assertions are equivalent.

(a) $X$ contains an isomorphic copy of $l_1$.
(b) $h(X, A) > 0$.
(c) $h(X, A) = 1$.

Hence the class of spaces which do not contain isomorphic copies of $l_1$ is an $l_1$-property and its $l_1$-exponent and its extended stability exponent are equal to 1.

6.27.8. Alternate-signs Banach-Saks property.

Definition. A Banach space $X$ is said to have alternate-signs Banach-Saks property (ABS) if every bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ contains a subsequence $\{x_{n(i)}\}_{i=1}^\infty$ such that the alternate-signs Cesaro means $n^{-1} \sum_{k=1}^n (-1)^k x_{n(k)}$ are convergent in the strong topology.

B. Beauzamy [Be1, p. 362] (see also [BL]) obtained the following characterization of ABS. Let $X$ be a Banach space. Then the following assertions are equivalent.

(a) $X \notin$ ABS.
(b) There exist $\varepsilon > 0$ and a sequence $\{x_n\} \subset B(X)$ such that for all $k \in \mathbb{N}$, if $k \leq n(1) \leq \ldots \leq n(k)$ ($n(i) \in \mathbb{N}$), then for all scalars $c_1, \ldots, c_k$,
\[ || \sum_{i=1}^k c_i x_{n(i)} || \geq \varepsilon || \sum_{i=1}^k |c_i| \]  \tag{11}

(c) For every $0 < \varepsilon < 1$, there exists a sequence $\{x_n\} \subset B(X)$ such that for all $k \in \mathbb{N}$, if $k \leq n(1) \leq \ldots \leq n(k)$ ($n(i) \in \mathbb{N}$), then for all scalars $c_1, \ldots, c_k$, inequality (11) is satisfied.

Let $A \subset S(l_1)$ be the set consisting of all finitely non-zero vectors satisfying the following conditions. If $\text{supp} a = \{n(1), \ldots, n(k)\}$ and $n(1) \leq \ldots \leq n(k)$, then:

6.27.9. Banach-Saks property.

Definition. A Banach space $X$ is said to have Banach-Saks property (BS) if every bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ contains a subsequence $\{x_{n(i)}\}_{i=1}^\infty$ such that the Cesaro means $n^{-1} \sum_{k=1}^n x_{n(k)}$ are convergent in the strong topology.

Let $A \subset S(l_1)$ be the set of all finitely non-zero vectors satisfying the following conditions. If $\text{supp} a = \{n(1), \ldots, n(k)\}$ and $n(1) \leq \ldots \leq n(k)$, then:
(a) \( k \leq n(1) \).
(b) There exists \( j \leq k \) such that \( a_{n(1)}, \ldots, a_{n(j)} \geq 0 \); \( a_{n(j+1)}, \ldots, a_{n(k)} \leq 0 \) and 
\[ \sum_{i=1}^{j} a_{n(i)} = - \sum_{i=j+1}^{k} a_{n(i)} \leq 1/2. \]

Using the descriptions of reflexivity (6.19), \( \text{ABS} \) (6.27.8) it is not hard to prove that the following statements are equivalent.

(a) \( X \not\in \text{BS} \).
(b) \( h(X, A) > 0 \).
(c) \( h(X, A) \geq 1/2 \).

Hence \( \text{BS} \) is an \( l_1 \)-property and its \( l_1 \)-exponent and its extended stability exponent are not less than 1/2. Their precise values seem to be unknown.

**6.28.** The list of \( l_1 \)-properties presented above is by no means complete. Later (6.38) we shall describe methods of constructing new \( l_1 \)-properties from the known ones.

It should be mentioned that every two \( l_1 \)-properties from 6.27 are different. Some of the corresponding verifications are quite nontrivial (see books [Be2], [BL], [Du] and references in 6.40).

**6.29.** Now we shall describe two ways of obtaining new stable properties from the known ones.

Let \( P \) be some property of Banach spaces. The class of all Banach spaces for which \( X^{**}/X \in P \) is denoted by \( P^{co} \). J. Alvarez, T. Alvarez and M. Gonzalez [AAG] proved the following general result.

**Theorem.** If property \( P \) is extendedly stable then \( P^{co} \) is also extendedly stable and \( es(P^{co}) \geq es(P)/2 \). The analogous statement is valid for stable properties. If property \( P \) is an isomorphic invariant and is open then \( P^{co} \) is also open.

In order to prove this theorem we need the following result due to M. Valdivia. (Having in mind further applications of it we shall prove it in slightly more general form than is needed now.)

**6.30. Proposition.** Let \( C \) be a convex set in a Banach space \( X \) and let \( y^{**} \in w^{*} - \text{cl}(C) \subset X^{**} \) and \( \|y^{**} - x\| < 1 \) for some \( x \in X \). Then there exists \( y \in C \) such that \( \|y^{**} - y\| < 2 \).

Proof. Let real number \( \delta \) be such that \( \|y^{**} - x\| \leq \delta < 1 \), i.e. \( y^{**} - x \in \delta B(X^{**}) \).

Then \( x = (x - y^{**}) + y^{**} \in \delta B(X^{**}) + w^{*} - \text{cl}(C) \subset w^{*} - \text{cl}(\delta B(X) + C) \). Hence \( x \in w - \text{cl}(\delta B(X) + C) \).

Since \( \delta B(X) + C \) is convex, then we can find \( u \in \delta B(X) \) and \( y \in C \) such that \( \|x - (u + y)\| < 1 - \delta \). Then \( \|y - x\| < \|u\| + 1 - \delta \leq 1 \). Hence \( \|y^{**} - y\| \leq \|y^{**} - x\| + \|x - y\| < 2 \). Proposition is proved.

**6.31.** Let \( X \) be arbitrary Banach space and \( Y \in G(X) \). Let us denote by \( Q \) the quotient mapping \( Q : X^{**} \to X^{**}/X \). Applying Proposition 6.30 to \( C = Y \) we obtain the following statement.

**Corollary.** The subspace \( Q(Y^{\perp\perp}) \) belongs to \( G(X^{**}/X) \) and is isomorphic to \( Y^{**}/Y \).

**6.32.** Each of the statements of Theorem 6.29 follows from the comparison of corollary 6.31 and the following lemma. (We use notation of 6.31).

**Lemma.** For arbitrary Banach space \( X \) and arbitrary subspaces \( Y, Z \in G(X) \) we have \( \Theta_0(Q(Z^{\perp\perp}), Q(Y^{\perp\perp})) \leq 2\Theta_0(Z, Y) \).

Proof. Let \( \varepsilon > 0 \) be arbitrary. Let \( \xi \in S(Q(Z^{\perp\perp})) \). Then we can find \( z^{**} \in Z^{\perp\perp} \) and \( x \in X \) such that \( Qz^{**} = \xi \) and \( \|z^{**} - x\| < 1 + \varepsilon \). By Proposition 6.30 we can find
It is clear that every $z \in Z$ such that $\|z^* - z\| < 2(1 + \varepsilon)$. Since $\Theta_0(Z^\perp, Y^\perp) = \Theta_0(Z, Y)$ (by Theorem 3.4(d)), then we can find $y^* \in Y^\perp$ such that $\|z^* - z - y^*\| < 2(1 + \varepsilon)\Theta_0(Z, Y)$. Hence $\|\xi - Qy^*\| < 2(1 + \varepsilon)\Theta_0(Z, Y)$. Since $\varepsilon > 0$ is arbitrary, we obtain the desired inequality. Lemma is proved.

6.33. Definition. Let $P$ be a property of Banach spaces. Class $Q$ is said to be the preproperty of $P$ (Notation: $Q = \text{pre}(P)$) if

$$(X \in Q) \iff (X^* \in P).$$

6.34. Proposition. If $P$ is stable property then pre$P$ is also stable. If $P$ is extendedly stable property then pre$P$ is also extendedly stable. Moreover we have $s(\text{pre}(P)) \geq s(P)$ and $es(\text{pre}(P)) \geq es(P)$.

Proof. Let $X$ be a Banach space and let $Y, Z \in G(X)$ be such that $Y \in \text{pre}(P)$ and $\Theta_0(Z, Y) < es(P)$. Let us prove that $Z \in \text{pre}(P)$. In order to do that let us consider space $X_1 = X \oplus_2 Y$ and let $Y_t(t > 0)$ be the following family of subspaces:

$$Y_t = \{(y, ty) : y \in Y\}.$$

It is clear that every $Y_t$ is isomorphic (even isometric) to $Y$ and $\lim_{t \to 0} \Theta(Y, Y_t) = 0$. Therefore choosing $t$ small enough we obtain $\Theta_0(Z, Y_t) < es(P)$. On the other hand we have $Z \cap Y_t = 0$ and $\delta(Z, Y_t) > 0$. Let $X_2 = \text{lin}(Z \cup Y_t)$. By remark from 5.3 $X_2$ is a closed subspace of $X$ and $Y_t$ is a complemented subspace of it. So we have:

(a) $Y_t^\perp$ is isomorphic to $Z^*$.
(b) $Z^\perp$ is isomorphic to $(Y_t)^*$.

By duality formula (Theorem 3.4(d)) we have

(c) $\Theta_0(Y_t^\perp, Z^\perp) < es(P)$.

Since $(Y_t)^* \in P$ and $P$ is isomorphic invariant by Proposition 6.7, we obtain that $Z^* \in P$. So $Z \in \text{pre}(P)$. Proposition is proved.

6.35. Remark. Analogous statement can be proved for open properties which are isomorphic invariants. But the known proof of this statement is rather long (see 7.12).

6.36. The following conjecture seems to be excessively audacious but at the moment I don’t no counterexamples.

Conjecture. For every cardinal $\alpha$ the intersection of every extendedly stable property with the set of Banach spaces with density character less than $\alpha$ is an $l_1$-property.

We need to consider intersections here in order to avoid such trivial counterexamples as the class of all Banach spaces.

It should be noted that stable properties need not be $l_1$-properties. (It follows because the complement of stable property is stable and the complement of $l_1$-property is not an $l_1$-property.)

6.37. In [O9] the present author obtained several results in support of this conjecture. In this connection the following definition is useful.

Definition. Class $P$ of Banach spaces is called a regular $l_1$-property if there exist a real number $\delta > 0$, set $\Gamma$ and a subset $A \subset S(l_1(\Gamma))$ satisfying the conditions:

1. The set $A$ consists of finitely non-zero vectors.
2. If $a_0 \in A$, then $A$ contains all vectors $a \in S(l_1(\Gamma))$ for which

$$(\forall \gamma \in \Gamma)(\text{sign} a_0(\gamma) = \text{sign} a(\gamma)).$$
3. For a Banach space \( X \) the following conditions are equivalent:

(a) \( X \not\in P \).
(b) \( h(X, A) > 0 \).
(c) \( h(X, A) \geq \delta \).

Supremum of those \( \delta > 0 \) for which there exist \( \Gamma \) and \( A \subset S(l_1(\Gamma)) \) such that conditions 1–3 are satisfied is called the regular exponent of \( P \).

6.38. In [O9] the following results were proved.

(a) Properties listed in 6.27 are regular \( l_1 \)-properties.
(b) The union of every set of regular \( l_1 \)-properties with uniformly bounded away from zero regular exponents is a regular \( l_1 \)-property.
(c) If \( P \) is regular \( l_1 \)-property then \( \text{pre}(P) \) and \( P^{\text{co}} \) are regular \( l_1 \)-properties.

I do not know examples of \( l_1 \)-properties which are not regular. In particular I do not know whether any intersection of regular \( l_1 \)-properties with uniformly bounded away from zero regular \( l_1 \)-exponents is a regular \( l_1 \)-property.

Notes and remarks

6.39. Let us give some historical comments.

Proposition 6.1 and the part of Proposition 6.2 which concerns the operator opening seems to be new. The second part of proposition 6.2 is taken from [O1]. It was also announced in [Fr]. Observation 6.3 is due to A. Douady [D]. Construction of 6.14 is a straightforward generalization of M.I. Kadets’ construction [Kad2]. Approach of 6.17 was introduced by the author [O2], [O3], [O4]. Proposition 6.21 is taken from [O4] (see also [O3]).

Corollary 6.22 is a slight strengthening of the following result due to A.L. Brown [Br1]: Class of reflexive spaces is stable with the stability exponent no less than \( \sqrt{2} - 1 \).

Corollary 6.23 was proved in [KKM].

The main idea of the description of super-reflexivity, which was mentioned in 6.27.6 is due to R.C. James [J2]. Necesssary additions was made in [J3], [SS] and [JS]. Many other result on super-reflexive spaces can be found in [Be2] and [Du].

Papers [AAG] and [Ja2] contains somewhat weaker versions of Proposition 6.34 with more complicated proofs. The present proof is due to the author.

It turns out that the paper [O6] which is relevant to the topic of this chapter is “almost free” of new results and is only of historical interest. This paper contains proofs of Proposition 6.11 and of stability of quasireflexivity.

6.40. I would like to add the following information concerning the statement of 6.28 about the distinction between \( l_1 \)-properties. The direct sum \( (\sum_{n=1}^{\infty} \oplus l_1^n) \) is an example of reflexive non-\( B \)-convex space.

Examples of non-reflexive and hence non-super-reflexive \( B \)-convex spaces was constructed by R.C. James [J4], see also [JL].

It is known [BrS] that \( c_0 \in ABS \). Hence \( ABS \) does not imply any of the properties: reflexivity, \( B \)-convexity, super-reflexivity, \( BS \).

A. Baernstein [Bae] constructed reflexive space without \( BS \).

6.41. In connection with definition 6.18 and Proposition 6.21 it is natural to propose the problem of description of the sets

\[ \sigma(A) = \{ h(X, A) : X \text{ running through the class of all Banach spaces} \} \]
for different sets $A \subset S(l_1(\Gamma))$ and to calculate $h(X, A)$ for classical Banach spaces.

In addition to results mentioned in 6.19, 6.20 and 6.27 I know results of the mentioned type only for the set $A \subset S(l_1)$ described in 6.27.3 and its uncountable analogues mentioned in 6.27.2. It is known that $h(l_p, A) = 2^{1/p}/2$ for $1 \leq p < \infty$ [BRR] (see also [Ko1] and [WW], p. 91). The book [WW] contains also results on evaluation of $h(L_p(\mu), A)$, where $\mu$ is arbitrary measure.

In addition, J. Elton and E. Odell [EO] (see also [Di, p. 241]) proved that $h(X, A)$ for certain subspaces $X$ also $\in K_01$ and $\in WW$, p. 91). The book $WW$ contains also results on evaluation of $h(l_1, A)$ for different sets $A$. In addition to results mentioned in 6.19, 6.20 and 6.27 I know results of the mentioned type only for the set $A \subset S(l_1)$. Let us estimate $\Lambda(Y, Z)$. It is also clear that the space $Y$ is separable and $\text{dens} Y = \text{dens} Z$. This question was solved in negative in [O1], where a Banach space $X$ was constructed such that for certain subspaces $Y, Z \in G(X)$ we have $\Lambda(Y, Z) \leq 2\sqrt{2} - 2$, $Z$ is separable and $\text{dens} Y$ equals to the power of continuum. Here we reproduce this example. Let $a = \sqrt{2} - 1$. Consider the algebraic sum

$$X = c_0([0, 1]) \oplus (\sum_{i=1}^{\infty} \oplus C(0, 1))_1 \oplus (\sum_{j=1}^{\infty} \oplus C(0, 1))_1.$$  

We endow it with the norm:

$$\|(h_0, (h_i)_{i=1}^{\infty}, (g_j)_{j=1}^{\infty})\| =$$

$$\max(\sum_{i=0}^{\infty} ||h_i - (1/2)g_{i+1}||, \sum_{j=1}^{\infty} ||g_j - (1/2)h_j||),$$

where all the norms on the right-hand side are suprema of the modulus on $[0,1]$. It is clear that the space

$$Y = \{(h_0, (h_i)_{i=1}^{\infty}, (0)_{j=1}^{\infty})\}$$

is isometric to $c_0([0, 1]) \oplus (\sum_{i=1}^{\infty} \oplus C(0, 1))_1$ and the space

$$Z = \{(0, (0)_{i=1}^{\infty}, (g_j)_{j=1}^{\infty})\}$$

is isometric to $(\sum_{j=1}^{\infty} \oplus C(0, 1))_1$. It is also clear that $Z$ is separable and $\text{dens} Y$ equals to the power of continuum. Let us estimate $\Lambda(Y, Z)$. To do this it suffices to estimate $\text{dist}(y_i, B(Z)) (i \in N \cup \{0\})$ and $\text{dist}(z_j, B(Y)) (j \in N)$ for vectors $y_i \in S(Y)$ of the form

$$y_0 = (h_0, (0)_{i=1}^{\infty}, (0)_{j=1}^{\infty})$$

and

$$y_i = (0, (0, \ldots, 0, h_i, 0, \ldots), (0)_{j=1}^{\infty}),$$

where $h_i$ is at the $i$-th position, and

$$z_j = (0, (0)_{i=1}^{\infty}, (0, \ldots, 0, g_j, 0, \ldots)),$$

where $g_j$ is at the $j$-th position. This is done in the same manner for all such vectors except for vectors of the form $y_0$.  

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We verify the estimate for \( y_1 \). Consider the vector

\[
f = (0, (0)_{i=1}^\infty, (0, ah_1, 0, \ldots)) \in B(Z).
\]

We have

\[
\text{dist}(y_1, B(Z)) \leq \|y_1 + f\| = \max\{\|(1 - a^2)h_1\|, \|ah_1\| + \|ah_1\|\} = \max\{(1 - a^2), 2a\} = 2\sqrt{2} - 2.
\]

Let us now verify the estimate for \( y_0 \). Let \( \varepsilon > 0 \). We introduce the set \( A = \{x: h_0(x) \geq \varepsilon\} \). This set is finite. Let the function \( w_\varepsilon : A \to \mathbb{R} \) be defined as \( w_\varepsilon(x) = h_0(x) \) and extend it as a continuous function to \([0,1]\) in such a way that \( \sup\{w_\varepsilon(x): x \in [0,1]\} = 1 \). Then for every \( b \in [0,1] \) we have

\[
\|h_0 - bw_\varepsilon\| = \max\{1 - b, b + \varepsilon\}
\]

(12)

Let \( z = (0, (0)_{i=1}^\infty, (aw_0, 0, \ldots)) \). We have \( \|z\| \leq a \) and \( \text{dist}(y_0, B(Z)) \leq \|y_0 + z\| = \max\{\|h_0 - a^2w_0\|, \|aw_0\|\} = \max\{1 - a^2, a^2 + a, a\} = 2\sqrt{2} - 2 \).

**6.43. Remark.** Inequality (12) implies that for subspaces \( c_0([0,1]) \subset l_\infty([0,1]) \) and \( C(0,1) \subset l_\infty([0,1]) \) we have

\[
\Lambda_0(c_0([0,1]), C(0,1)) = 1/2.
\]

(13)

Therefore the estimates of extended stability exponents in 6.27.2 are precise when \( \alpha \) is not greater then the cardinality of continuum.

Equality (13) was independently noticed by A.N.Plichko, who used it in the theory of biorthogonal systems [Pl].

**6.44.** The author proved [O1] that if \( Y \) and \( Z \) are subspaces of a Banach space with an extended unconditionally monotone basis (see definition in [S2], §17), then \( \Theta(Y, Z) < 1 \) implies \( \text{dens}Y = \text{dens}Z \).

V.I.Gurarii [Gu2] proved that if we suppose that \( X \) is uniformly convex (=uniformly rotund) and \( \delta \) is its modulus of convexity (see necessary definitions in [Da] or [Di]), then \( \Theta(Y, Z) < 1/2 + \delta(1/7)/2 \) implies \( \text{dens}Y = \text{dens}Z \).

**6.45.** The examples from 6.42 has predecessor (see [Le] and [S3, p. 271]).

**6.46.** Starting from [KKM] the following dual version of the problem to which the present section is devoted is considered: How close should be the structure of the quotients over the subspaces which are close with respect to the geometric opening?

**Definition.** A property \( P \) is called *co-open* if for every Banach space \( X \) the set of those \( Y \in G(X) \) for which \( X/Y \in P \) is open in the topology induced by \( \Theta \).

A property \( P \) is called *costable* if there exists a number \( \alpha > 0 \) such that for every Banach space \( X \) and every \( Y, Z \in G(X) \), if \( X/Y \in P \) and \( \Theta(Y, Z) < \alpha \), then \( X/Z \in P \). The least upper bound of numbers \( \alpha \) for which this statement is true is denoted by \( c(P) \).

A property \( P \) is called *extendedly costable* if there exists a number \( \alpha > 0 \) such that for every Banach space \( X \) and every \( Y, Z \in G(X) \), if \( X/Y \in P \) and \( \Theta(Y, Z) < \alpha \), then \( X/Z \in P \). The least upper bound of numbers \( \alpha \) for which this statement is true is denoted by \( ec(P) \).
It turns out that the concept of costability coincides with the concept of stability and the concept of extended costability coincides with the concept of extended stability. Moreover, we have \( s(P) = c(P) \) and \( \text{es}(P) = \text{ec}(P) \) for every property \( P \). This assertion can be proved using arguments of Proposition 6.34 and “co-analogue” of Proposition 6.2. Somewhat weakened version of this statement was proved by J. Alvarez, T. Alvarez and M. Gonzalez [AAG]. Some other predecessors of this result can be found in [O2].

If we restrict ourselves by consideration of isomorphic invariants then the concept of open property would coincide with the concept of co-open property. The known proof of this result is rather complicated (see [Ja2]).

6.47. Proposition 6.11 (a) implies that if property \( P \) is (a) open; (b) isomorphic invariant; (c) closed with respect to formation of direct sums; then the solution of the three space problem for \( P \) is positive. In particular it is so for stable properties which are closed with respect to formation of direct sums. Here it should be noted that the three space problems for properties listed in 6.27 (with obvious exception of 6.27.4) were already solved. Here is the list of corresponding references. Reflexivity: [KS, p. 575]; \( B \)-convexity [Gi, p. 103] (see also [ELP], [R]); super-reflexivity: [ELP] (see also [H, p. 86], [R], [Sc]), class of spaces, which do not contain isomorphic copies of \( l_1 \): [R] (see also [AGO]), alternate-signs Banach-Saks property: [O3]; Banach-Saks property: [GR] (see also [O3]).

6.48. The papers [Ja2], [O1], [O3] (see also [O6]) deals with the following question: for what Banach spaces \( Z \) classes \( P_Z = \{ X : X \text{ is isomorphic to } Z \} \) are open and what is the least upper bound of \( \alpha > 0 \) for which the assertion of Proposition 6.6 is true if we let \( P = P_Z \) and \( Y = Z \)? The most interesting problem seems to be the following: is it true that every \( Z \) for which \( P_Z \) is open is either injective or isomorphic to \( c_0 \) or \( l_1(\Gamma) \)? The fact that \( P_Z \) is open when \( Z \) is injective can be easily derived from the Proposition 5.3.

6.49. There is another survey devoted to the same topic as section 6 of the present survey. It is the survey due to J. Alvarez and T. Alvarez [AA]. Unfortunately because of certain confusion in different concepts of stability, some statements of this survey (Theorem 12, Observation 15) are incorrect.

6.50. I am sure that the results and the methods of papers [DJL], [DL], [Fa], [J5] and [PX] will be useful in further investigations of \( l_1 \)-properties.

7. Additional remarks on the topologies on the set of all subspaces of a Banach space and their applications

7.1. Basic facts about geometric opening are discussed in well-known course of I.M.Glazman and Yu.I.Lyubich [GL].

7.2. Openings found applications in the theory of best approximation in Banach spaces. Recall that a subspace \( Y \) of a Banach space \( X \) is called a Chebyshev subspace if for every \( x \in X \) there exists unique \( y \in Y \) such that \( ||x - y|| = \text{dist}(x, Y) \). Subspace \( Y \) is called almost-Chebyshev if the condition above is satisfied for all \( x \) except some set of the first category. It is known [Ga] that \( c_0 \) does not contain almost-Chebyshev proper infinite dimensional subspaces. A.L.Garkavi [Ga] proved that if a separable Banach space \( X \) contains a reflexive infinite dimensional subspace then it contains an almost-Chebyshev proper infinite dimensional subspace. He also proved that a separa-
ble dual Banach space contains almost-Chebyshev proper infinite dimensional subspace. The proof relies heavily on Baire category argument for metric space \((G(X), d_\text{op})\). Using Corollary 6.22, A.L.Brown [Br1] showed that the first result can be proved using category argument for the metric space \((G(X), d_\rho)\).

Some other applications of openings in the theory of best approximation can be found in [S3].

7.3. V.P.Fonf [Fo] used geometric opening in his investigations of polyhedral Banach spaces. Recall that a Banach space is called \textit{polyhedral} if the unit ball of each of its finite dimensional subspaces is a polytope. V.P.Fonf [Fo] proved that every infinite dimensional polyhedral Banach space contains an isomorphic copy of \(c_0\).

7.4. The topology induced by the opening on the set of all subspaces of a Hilbert space was used by R.G.Douglas and C.M.Pearcy [DP] in investigations of the lattice of invariant subspaces of bounded linear operators in Hilbert space.

7.5. Theorem 3.4 (e) has many applications in constructions of “nicely bounded” biorthogonal systems (see [Da, p. 93], [Pel3], [Te]).

7.6. Geometric opening is a natural tool in some questions of the theory of bases (see [S1], [S2]).

7.7. The topology on \(G(X)\) induced by the geometric opening appeared in a natural way in investigations of infinite dimensional analytic manifolds [D], operator function equations [Ja1], [KT], [Man], [Sl].

7.8. The geometric opening and quantity \(\Theta_0\) turned out to be important tool in the creation of the theory of Fredholm operators in Banach spaces [GK], [K1]. They are repeatedly used in the theory of operators in Banach spaces (see, in particular, [AZ], [CPY], [GM2], [Go], [GoM], [MeS], [Sob], [Va2], [Va3]), Fredholm and semi-Fredholm complexes of Banach spaces (see [AV], [A], [Do], [F], [FS], [Va1]).

7.9. Let \(X\) and \(Y\) be Banach spaces. If we have a topology on \(G(X \oplus Y)\) then, considering graphs we obtain a topology on the set of all closed linear operators with domains in \(X\) into \(Y\). The topology obtained in such a way from the topology induced by the geometric opening turned out to be very useful. Many important characteristics of closed linear operators are stable with respect to this topology. This circle of problems was investigated by J.D.Newburgh [New1], [New2], H.O.Cordes and J.-P.Labrousse [CL], G.Neubauer [N1], [N2], T.Kato [K2], J.-P.Labrousse [L1], [L2]. Some of these results are presented in the book due to K.R.Partasarathy [Par]. If we restrict ourselves by consideration of self-conjugate operators in a Hilbert space then this topology coincides with the topology of uniform resolvent convergence (see [RS], chapter VIII).

It should be mentioned that some “unbounded” analogues of known results fails for this topology (see [L3] and forthcoming book due to J.-P.Labrousse). In the case of unbounded closed densely defined operators in Hilbert space the situation sometimes can be saved by the use of another metric (see [L3], [LM] and forthcoming book due to J.-P.Labrousse). This metric goes back to C.Davis [Dav]. (This metric is defined only on the subset of \(G(H \oplus H)\) consisting of graphs of closed densely defined operators on the separable Hilbert space \(H\).)

7.10. Let \(X, Y\) be Banach spaces and \(T \in L(X, Y)\). The \textit{minimum modulus} of \(T\) is
defined by
\[ \gamma(T) = \sup\{c \geq 0 : (\forall x \in X)((||Tx|| \geq c \cdot \text{dist}(x, \ker T))} \).

(We refer to [K2] for basic properties of \( \gamma \).

A.S. Markus [M1] proved the following result.

**Theorem.** Let \( T, S \in L(X, Y) \). Then

(a) \( \Theta_0(\ker S, \ker T) \leq \gamma(T)^{-1}||S - T|| \).

(b) \( \Theta_0(\text{im} S, \text{im} T) \leq \gamma(S)^{-1}||S - T|| \).

(c) If \( \Theta(\ker S, \ker T) < 1/2 \), then

\[ |\gamma(S) - \gamma(T)| \leq 3||S - T||(1 - 2\Theta(\ker S, \ker T))^{-1}. \]

(d) If \( \Theta(\text{im} S, \text{im} T) < 1/2 \), then

\[ |\gamma(S) - \gamma(T)| \leq 3||S - T||(1 - 2\Theta(\text{im} S, \text{im} T))^{-1}. \]

7.11. R. Janz [Ja2] proved the following statement. Let \( D \) be a metric space and \( T : D \to G(X) \) be a mapping continuous in the topology induced by the geometric opening. Then there exist Banach spaces \( Z_1 \) and \( Z_2 \) and continuous mappings \( R_1 : D \to L(Z_1, X) \) and \( R_2 : D \to L(X, Z_2) \) such that for every \( d \in D \) we have

(a) \( \text{im}(R_1(d)) = T(d) \).

(b) \( \ker(R_2(d)) = T(d) \).

(c) \( ||R_1(d)|| \leq 1, ||R_2(d)|| \leq 1 \).

(d) \( \gamma(R_1) \geq 1/10, \gamma(R_2) \geq 1/10 \).

7.12. Applying 7.11 to the identity mapping \( T : G(X) \to G(X) \) and using 7.10 it is not hard to prove assertions formulated in 6.35 and 6.46.

7.13. Let \( X \) be a linear space with two norms and \( Y, Z \) be linear subspaces of \( X \). By \( \Theta_1(Y, Z) \) and \( \Theta_2(Y, Z) \) we shall denote the geometric openings in the sense of the first and the second norms respectively. (Here we use the fact that the geometric opening is well-defined for non-closed subspaces of non-complete normed spaces.

A.S. Markus [M2] proved the following statement: if for some \( c > 0 \) and every linear subspaces \( Y, Z \subset X \) we have

\[ \Theta_1(Y, Z) \geq c\Theta_2(Y, Z), \]

then the norms are equivalent.

7.14. Some properties of the metric space \( (G(H), d_g) \), where \( H \) is the Hilbert space were considered in [I1], [I2], [Lo1], [Lo2], [Rod].

7.15. H. Porta and L. Recht [PR] proved that for every Banach space \( X \) there exists a mapping \( \phi : G_c(X) \to G_c(X) \) such that for every \( Y \in G(X) \) the space \( \phi(Y) \) is a complement of \( Y \) and \( \phi \) is continuous with respect to the \( d_{op} \).

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