ASYMPTOTICS OF THE WEYL FUNCTION FOR
SCHRÖDINGER OPERATORS WITH MEASURE-VALUED
POTENTIALS

ANNEMARIE LUGER, GERALD TESCHL, AND TOBIAS WÖHRER

Abstract. We derive an asymptotic expansion for the Weyl function of a one-dimensional Schrödinger operator which generalizes the classical formula by Atkinson. Moreover, we show that the asymptotic formula can also be interpreted in the sense of distributions.

1. Introduction

The \( m \)-function or Weyl–Titchmarsh function introduced by Weyl in [30] plays a fundamental role in spectral theory for Sturm–Liouville operators. In particular, it is known that in the case of sufficiently "nice" potentials \( q \) all information about the spectral properties of self-adjoint realizations of the differential expression

\[
-\frac{d^2}{dx^2} + q(x),
\]

acting in \( L^2(0, \infty) \), are encoded in this function. In 1952 Marchenko proved (see [24, Theorem 2.2.1]) that the \( m \)-function corresponding to the Dirichlet boundary condition at \( x = 0 \) behaves asymptotically at infinity like the corresponding function of the unperturbed operator (corresponding to \( q \equiv 0 \)), that is,

\[
m(z) = -\sqrt{-z}(1 + o(1)),
\]

as \( z \to \infty \) in any nonreal sector in the open upper complex half-plane \( \mathbb{C}_+ \) (let us stress that the high-energy behavior of \( m \) can be deduced from the asymptotic behavior of the corresponding spectral function, see [23, Theorem II.4.3]). A simple proof of this formula was found by Levitan in [22] (a short self-contained proof of (1.2) can be found in, e.g., [29, Lemma 9.19]). Since then the high-energy asymptotics \( z \to \infty \) of the \( m \)-function received enormous attention over the past three decades as can be inferred, for instance from [2], [3], [4], [5], [6], [11], [12], [13], [14], [17], [18], [19], [20], [21], [26], [28] and the references therein.

Typically there are two directions which are of interest: If one assumes \( q \) to be smooth a full asymptotic expansion can be given. Otherwise, one tries to derive the leading asymptotic under minimal assumptions on \( q \). One of the key improvements in this latter direction is due to Atkinson [2] who showed

\[
m(z) = -\sqrt{-z} - \int_{(0,x_0)} e^{-2\sqrt{-y}q(y)}dy + o(z^{-1/2})
\]
for arbitrary \( x_0 \in (0, \infty) \). In particular, if 0 is a Lebesgue point of \( q \) this implies

\[
m(z) = -\sqrt{-z} - \frac{q(0)}{2\sqrt{-z}} + o(z^{-1/2}).
\] (1.4)

On the other hand, the case of a locally integrable potential does not cover the case where \( q \) is a single Dirac \( \delta \) one of the most popular toy models which can be found in any text book on quantum mechanics. Even though the case of delta potentials has a long tradition (see e.g. the monograph [1]) the case where \( q \) is replaced by an arbitrary measure got significant interest only recently and we refer to [7]–[9], [10], [25], [27] and the literature therein.

Our question in the present paper is to what extent (1.3) remains valid when \( q \) is replaced by a measure. Moreover, we will also show that (1.4) remains true when interpreted in the sense of distributions.

2. Schrödinger Operators with Measure-Valued Coefficients

Our main object are one-dimensional Schrödinger operators in the Hilbert space \( L^2(a, b) \), \(-\infty < a < b \leq \infty\), associated with the differential expressions

\[
\tau f = \left(-f' + \int f \, d\chi\right)',
\] (2.1)

where \( \chi \) is a locally finite signed Borel measure on \([a, b)\). In particular, we assume that \( \tau \) is regular at \( a \), that is, \( a \in \mathbb{R} \) and the total variation of \( \chi \) is finite near \( a \) (i.e., \( |\chi|([a, x_0]) < +\infty \) for every \( x_0 \in (a, b) \)).

The maximal domain of this differential expression is given as

\[
\mathcal{D}_\tau = \left\{ f \in AC_{loc}[a, b] \mid f', \tau f \in L^2(a, b) \right\},
\]

which leads to a jump condition for \( f'(x) \) at every point mass,

\[
f'(x+) - f'(x-) = \chi(\{x\}) f(x).
\] (2.2)

We fix \( f'(x) \) to be left continuous. At \( x = a \) the above condition has to be understood as the definition of the left limit.

In order to get a self-adjoint operator we look at the corresponding maximal operator associated with \( \tau \) in \( L^2(a, b) \) with the domain

\[
\text{dom} (T_{\text{max}}) = \{ f \in \mathcal{D}_\tau \mid f, \tau f \in L^2(a, b) \}.
\]

For \( f, g \in \text{dom} (T_{\text{max}}) \) we can define the Wronskian as usual

\[
W_x(f, g) = f(x)g'(x) - f'(x)g(x)
\] (2.3)

and one can verify the Lagrange identity

\[
\int_{[c, d]} (g \tau f - f \tau g) \, dx = W_d(f, g) - W_c(f, g)
\] (2.4)

where \( x, c, d \) include the interval endpoints as one-sided limits. In particular, the Wronskian is constant for two solutions of \( \tau u = zu \).

We say \( \tau \) is in the limit-circle (l.c.) case at \( b \) if all solutions of \( \tau u = zu \) are square integrable near \( b \) and we say that \( \tau \) is in the limit-point (l.p.) case at \( b \) otherwise.

To obtain a self-adjoint operator from \( T_{\text{max}} \) we will choose appropriate boundary conditions. First of all we will choose a Dirichlet boundary condition at \( a \).
Then, if $\tau$ is in the l.p. case at $b$, no further boundary condition is needed and the corresponding operator

$$\text{dom } (S) = \{ f \in \text{dom } (T_{\text{max}}) \mid f(a) = 0 \}$$

is a self-adjoint restriction of $T_{\text{max}}$. Otherwise, if $\tau$ is in the l.c. case at $b$, we need an additional boundary condition at $b$ in which case every restriction of $T_{\text{max}}$ with domain

$$\text{dom } (S) = \{ f \in \text{dom } (T_{\text{max}}) \mid f(a) = 0, W_b(f, w^*) = 0 \},$$

where $w \in \text{dom } (T_{\text{max}})$ satisfies $W_b(w, w^*) = 0$ and $W(h, w^*) \neq 0$ for some $h \in \text{dom } (T_{\text{max}})$, is a self-adjoint operator.

We refer to [10] for background and general theory.

### 3. Asymptotics for the Weyl function

In this section we will assume that the left endpoint $a$ is regular and without loss of generality we will assume $a = 0$. To simplify notation we denote

$$\chi(x) := \begin{cases} \chi([0, x)), & x \in (0, b), \\ 0, & x = 0. \end{cases}$$

In this case we have a basis of solutions $c(z, x)$, $s(z, x)$ of $\tau u = zu$ determined by the initial conditions

$$c(z, 0) = 1, \ c'(z, 0) = 0, \quad s(z, 0) = 0, \ s'(z, 0) = 1, \quad (3.1)$$

such that $W(c(z), s(z)) = 1$. Here and in what follows a prime will always denote a derivative with respect to the spatial coordinate $x$. They are given as the solutions of the following integral equations

$$c(z, x) = \cosh(\sqrt{-z}x) + \frac{1}{\sqrt{-z}} \int_{[0, x)} \sinh(\sqrt{-z}(x - y))c(z, y)d\chi(y), \quad (3.2)$$

$$s(z, x) = \frac{1}{\sqrt{-z}} \sinh(\sqrt{-z}x) + \frac{1}{\sqrt{-z}} \int_{[0, x)} \sinh(\sqrt{-z}(x - y))s(z, y)d\chi(y). \quad (3.3)$$

In fact, this can be verified using integration by parts, which also shows

$$c'(z, x) = \sqrt{-z} \sinh(\sqrt{-z}x) + \int_{[0, x)} \cosh(\sqrt{-z}(x - y))c(z, y)d\chi(y), \quad (3.4)$$

$$s'(z, x) = \cosh(\sqrt{-z}x) + \int_{[0, x)} \cosh(\sqrt{-z}(x - y))s(z, y)d\chi(y). \quad (3.5)$$

Here and in what follows $\sqrt{-z}$ will always denote the standard branch of the square root with branch cut along $(-\infty, 0)$.

We will need their high-energy asymptotics as $\text{Im}(z) \to \infty$. 
Lemma 3.1. The function \( c(z, x) \) and its derivative \( c'(z, x) \) can be written as

\[
c(z, x) = \cosh(\sqrt{-z}x) + \frac{1}{2\sqrt{-z}} \sinh(\sqrt{-z}x)\chi(x)
+ \frac{e^{\sqrt{-z}x}}{4\sqrt{-z}} \left( \int_{[0,x]} e^{-2\sqrt{-z}y}d\chi(y) - \int_{(0,x]} e^{-2\sqrt{-z}(x-y)}d\chi(y) \right)
- \frac{e^{\sqrt{-z}x}}{z} E_1(z, x),
\]

\[
c'(z, x) = \sqrt{-z} \sinh(\sqrt{-z}x) + \frac{1}{2} \cosh(\sqrt{-z}x)\chi(x)
+ \frac{e^{\sqrt{-z}x}}{4z} \left( \int_{[0,x]} e^{-2\sqrt{-z}y}d\chi(y) + \int_{(0,x]} e^{-2\sqrt{-z}(x-y)}d\chi(y) \right)
+ \frac{e^{\sqrt{-z}x}}{\sqrt{-z}} E_2(z, x),
\]

with error functions \( E_j(z, x) \) satisfying \( |E_j(z, x)| \leq C|\chi|([0, x]) \) and

\[
E_j(z, x) = \frac{1}{8} \int_{(0,x]} (\chi(y) + \chi([0, x]))d\chi(y) + o(1), \quad j = 1, 2,
\]

as \( \text{Im}(z) \to +\infty \).

Similarly, the function \( s(z, x) \) and its derivative \( s'(z, x) \) can be written as

\[
\begin{align*}
s(z, x) &= \frac{1}{\sqrt{-z}} \sinh(\sqrt{-z}x) - \frac{1}{2z} \cosh(\sqrt{-z}x)\chi(x)
+ \frac{e^{\sqrt{-z}x}}{4z} \left( \int_{[0,x]} e^{-2\sqrt{-z}y}d\chi(y) + \int_{(0,x]} e^{-2\sqrt{-z}(x-y)}d\chi(y) \right)
+ \frac{e^{\sqrt{-z}x}}{\sqrt{-z}} E_3(z, x),
\end{align*}
\]

\[
\begin{align*}
s'(z, x) &= \cosh(\sqrt{-z}x) + \frac{1}{2\sqrt{-z}} \sinh(\sqrt{-z}x)\chi(x)
- \frac{e^{\sqrt{-z}x}}{4\sqrt{-z}} \left( \int_{[0,x]} e^{-2\sqrt{-z}y}d\chi(y) - \int_{(0,x]} e^{-2\sqrt{-z}(x-y)}d\chi(y) \right)
- \frac{e^{\sqrt{-z}x}}{z} E_4(z, x),
\end{align*}
\]

with error functions \( E_j(z, x) \) satisfying \( |E_j(z, x)| \leq C|\chi|([0, x]) \) and

\[
E_j(z, x) = \frac{1}{8} \int_{(0,x]} (\chi(y) - \chi([0, x]))d\chi(y) + o(1), \quad j = 3, 4,
\]

as \( \text{Im}(z) \to +\infty \).

Proof. Abbreviate \( k = \sqrt{-z} \) and note \( \text{Re}(k) \geq 0 \). First of all, considering the function \( \tilde{c}(z, x) = e^{-kx}c(z, x) \) we look at the corresponding integral equation

\[
\tilde{c}(z, x) = \frac{1 + e^{-2kx}}{2} + \int_{[0,x]} \frac{1 - e^{-2k(x-y)}}{2k} \tilde{c}(z, y)d\chi(y)
\]
from which it follows that there is a bounded solution satisfying
\[ |\hat{c}(z, x)| \leq \exp\left( |k|^{-1} |\chi|((0, x)) \right) \]
by using the usual iteration scheme (cf. [10, Theorem A.2]). Now we use bootstrapping and insert this information into our integral equation. First the integral equation for \( c(z, x) \) can be written as
\[ c(z, x) = \cosh(kx) + \frac{e^{kx}}{k} \tilde{E}_1(z, x), \tag{3.12} \]
with the error term
\[ \tilde{E}_1(z, x) = \int_{[0, x]} \frac{1 - e^{-2k(x-y)}}{2} \hat{c}(z, y) d\chi(y) \]
which is is locally uniformly bounded in \( x \) by the above estimate for \( \hat{c}(z, x) \). Reinserting (3.12) into the integral equation for \( c(z, x) \) leads to the desired representation of the solution \( c(z, x) \), where the error term
\[ E_1(z, x) = \int_{[0, x]} \frac{1 - e^{-2k(x-y)}}{2} \tilde{E}_1(z, y) d\chi(y) \]
is locally uniformly bounded in \( x \).

To compute the desired estimate for the error term \( E_1(z, x) \) we insert (3.12) into the definition of \( \tilde{E}_1(z, x) \), which leads to
\[ \tilde{E}_1(z, x) = \begin{cases} \frac{1}{2} (\chi(x) + \chi(\{0\})) + O(\frac{1}{x^2}), & x > 0, \\ 0, & x = 0, \end{cases} \]
by the dominated convergence theorem, where the estimate is locally uniform in \( x \) as \( \text{Im}(z) \to +\infty \). Now inserting this estimate into the definition of \( E_1(z, x) \) and applying the dominated convergence theorem again, leads to the desired estimate for the error term \( E_1(z, x) \).

Similarly, considering \( \tilde{s}(z, x) = ke^{-kx}s(z, x) \) we look at the corresponding integral equation
\[ \tilde{s}(z, x) = \frac{1 - e^{-2kx}}{2} + \int_{[0, x]} \frac{1 - e^{-2k(x-y)}}{2k} \tilde{s}(z, y) d\chi(y) \]
and conclude that there is a bounded solution satisfying
\[ |\tilde{s}(z, x)| \leq \exp\left( |k|^{-1} |\chi|((0, x)) \right). \]
The rest follows as before. \( \square \)

Next we recall the Weyl function \( m(z) \) defined such that
\[ u(z, x) = c(z, x) + m(z)s(z, x), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.13} \]
is square integrable near \( b \) and satisfies the boundary condition of our operator at \( b \) (if there is one). Following, the original approach of Weyl we recall the Weyl circles with center, radius given by
\[ q(z, x_0) = - \frac{W_{x_0}(c(z, \cdot), s(z, \cdot)^*)}{W_{x_0}(s(z, \cdot), s(z, \cdot)^*)}, \quad r(z, x_0) = \frac{1}{|W_{x_0}(s(z, \cdot), s(z, \cdot)^*)|}. \tag{3.14} \]
with \( x_0 \in [0, b) \), respectively. By construction the solutions \( c(z, x) + m s(z, x) \) with \( m \) on the Weyl circle are precisely the ones which satisfy a real boundary condition at \( x_0 \):

\[
\frac{c'(z, x_0) + ms'(z, x_0)}{c(z, x_0) + ms(z, x_0)} \in \mathbb{R} \cup \{\infty\}.
\]

(3.15)

Taking \( x_0 \not\to b \) these circles are nested and hence converge to a circle (limit circle case) or to a point (limit point). In the first case, the points on the circle correspond to the Weyl functions corresponding to different self-adjoint realizations and in the second case the point corresponds to the unique Weyl function of the unique self-adjoint realization.

Moreover, for \( \text{Im}(z) > 0 \), those where the quotient in (3.15) is in the upper, lower half-plane are those for which \( m \) is in the interior, exterior of the Weyl circle, respectively. Hence, if we find an \( m \) in the interior, the distance between \( m \) and \( m(z) \) can be at most \( 2r(z, x_0) \). This is precisely the idea (due to [2]) of the following lemma:

**Lemma 3.2.** For every \( x_0 \in (0, b) \) we have

\[
m(z) = -\frac{c(z, x_0) \sqrt{-z} + c'(z, x_0)}{s'(z, x_0) + \sqrt{-z}s(z, x_0)} + O(z e^{-2\sqrt{-z}x_0}),
\]

(3.16)
as \( \text{Im}(z) \to +\infty \), where the error depends only on the total variation \( |\chi|(0, x_0)\).

Moreover,

\[
\frac{c(z, x)\sqrt{-z} + c'(z, x)}{s'(z, x) + \sqrt{-z}s(z, x)} = \sqrt{-z} \frac{1 + \frac{1}{\sqrt{-z}} \int_{[0, x]} \tilde{c}(z, y)d\chi(y)}{1 + \frac{1}{\sqrt{-z}} \int_{[0, x]} \tilde{s}(z, y)d\chi(y)}
\]

(3.17)

where \( \tilde{c}(z, x) = e^{-\sqrt{-z}x}c(z, x) \) and \( \tilde{s}(z, x) = \sqrt{-z}e^{-\sqrt{-z}x}s(z, x) \).

**Proof.** For \( \text{Im}(z) > 0 \) it follows that the solution defined via the initial condition

\[
v(z, x_0) = 1, \quad v'(z, x_0) = -\sqrt{-z}
\]

with \( x_0 \in (0, b) \) corresponds to a point in the interior of the Weyl circle. Indeed we have

\[
\frac{v'(z, 0)}{v(z, 0)} = \frac{W_0(c(z, \cdot), v(z, \cdot))}{W_0(v(z, \cdot), s(z, \cdot))}
\]

and the constancy of the Wronskian implies

\[
\frac{v'(z, 0)}{v(z, 0)} = \frac{W_{x_0}(c(z, \cdot), v(z, \cdot))}{W_{x_0}(v(z, \cdot), s(z, \cdot))} = -\frac{c(z, x_0)\sqrt{-z} + c'(z, x_0)}{s'(z, x_0) + \sqrt{-z}s(z, x_0)}.
\]

(3.18)

Now an easy computation shows that

\[
\frac{c'(z, x_0) + s'(z, x_0)\frac{v'(z, 0)}{v(z, 0)}}{c(z, x_0) + s(z, x_0)\frac{v'(z, 0)}{v(z, 0)}} = -\sqrt{-z} \in \mathbb{C}_+.
\]

Hence the point \( \frac{v'(z, 0)}{v(z, 0)} \) lies in the interior of the Weyl circle by the considerations prior to this lemma. As the same is true for the Weyl function \( m(z) \) of our problem, we obtain

\[
m(z) = \frac{v'(z, 0)}{v(z, 0)} + O(r(z, x_0)) = \frac{v'(z, 0)}{v(z, 0)} + O(z e^{-2\sqrt{-z}x_0})
\]

as \( \text{Im}(z) \to \infty \), where we have used Lemma 3.1 for the second identity.

The last part is a straightforward calculation using (3.2)–(3.4). □
Combining this lemma with Lemma 3.1 gives our main result:

**Theorem 3.3.** For every $x_0 \in (0, b)$ the Weyl m-function has the asymptotic behavior

$$m(z) = -\sqrt{z} - \int_{[0, x_0]} e^{-2\sqrt{y}}d\chi(y) + o(z^{-1/2})$$

as $\text{Im}(z) \to \infty$. Moreover, the error satisfies an estimate of the type $|o(z^{-1/2})| \leq C|z|^{-1/2}$, where the constant depends only on the total variation $|\chi((0, x_0))|.$

**Proof.** By inserting Lemma 3.1 into the identity (3.18) a long but straightforward computation shows that

$$m(z) = -k - I_1 - \int_{[0, x_0]} e^{-2\sqrt{y}z}d\chi(y)$$

as $\text{Im}(z) \to +\infty$, where we abbreviated $k = \sqrt{-z}$ and $I_1(z) = \int_{[0, x_0]} e^{-2ky}d\chi(y)$.

Inserting the estimates for the error terms $E_j(z) = E_j(z, x_0)$ of Lemma 3.1 as well as the estimate

$$I_1(z) = \chi(0) + o(1)$$

as $\text{Im}(z) \to +\infty$, finally proves the theorem. □

**Remark 3.4.** (i). For an arbitrary left endpoint $a$ equation (3.19) reads

$$m(z) = -k - \int_{[a, x_0]} e^{-2\sqrt{z}(y-a)}d\chi(y) + o(z^{-1/2}).$$

(ii). Of course one can iterate this procedure to get further terms in the above expansion. For example using one more step one obtains:

$$m(z) = -k - \int_{[0, x_0]} e^{-2ky}d\chi(y)$$

$$- \frac{1}{2k} \int_{[0, x_0]} \left(1 - e^{-2k(x_0-y)}\right) \int_{[0, y]} e^{-2kr}d\chi(r) \right)d\chi(y)$$

$$+ \frac{1}{2k} \int_{[0, x_0]} \left(1 - e^{-2ky}\right)d\chi(y) \int_{[0, x_0]} E(y)d\chi(y) + O(k^{-2}).$$

(iii). We want to emphasize that in contradistinction to [2] our approach is more direct and avoids the use of Riccati equations for the Weyl function. In addition to being simpler this approach also retains a good control over the error with respect to the total variation of $\chi$. This will turn out crucial for our following application which states that (1.4) continues to hold in the sense of distributions.

**Theorem 3.5.** Denote by $m(z, t)$ the Weyl function associated with our operator restricted to the interval $(t, b)$ with a Dirichlet boundary condition at $t \in [a, b)$ and keeping the boundary condition at $b$ (if any) fixed. Then for any test function $\phi \in C^\infty([a, b)$ we have

$$\int_a^b m(z, t)\phi(t)dt = -\sqrt{-z} \int_a^b \phi(t)dt - \frac{1}{2\sqrt{-z}} \int_a^b \phi(s)d\chi(s) + o(z^{-1/2}).$$
Proof. All we have to do is multiply (3.19) with \( \phi \) and integrate with respect to \( t \). By our bound on the error term we can integrate the error term using dominated convergence and the rest follows by Fubini:

\[
\int_a^b m(z, t)\phi(t)dt = -\sqrt{-z}\Phi_0 - \int_{\mathbb{R}^2} \phi(t)\mathbf{1}_{[t, t+\varepsilon]}(s)e^{-2\sqrt{-z}(s-t)}d\chi(y)dt + o(z^{-1/2})
\]

\[
= -\sqrt{-z}\Phi_0 - \int_{\mathbb{R}^2} \phi(t)\mathbf{1}_{[t-\varepsilon, t]}(s)e^{2\sqrt{-z}(t-s)}dt d\chi(s) + o(z^{-1/2})
\]

\[
= -\sqrt{-z}\Phi_0 - \frac{1}{2\sqrt{-z}} \int_a^b \phi(s)d\chi(s) + o(z^{-1/2}),
\]

where we have abbreviated \( \Phi_0 = \int_a^b \phi(t)dt \) and \( \mathbf{1}_\Omega \) denotes the indicator function of a set \( \Omega \). Moreover, in the last step we have used

\[
\int_{t-\varepsilon}^t \phi(t)e^{2\sqrt{-z}(t-s)}dt = \frac{1}{2\sqrt{-z}}\phi(s) + O(z^{-1}),
\]

which follows from a simple integration by parts. \( \square \)

Finally, we look at the example

**Example 3.6.** Denote by \( \delta_0 \) a single Dirac delta measure located at \( x = 0 \) and set

\[
\chi = \alpha\delta_0, \quad \alpha \in \mathbb{R}.
\] (3.21)

In this case the solution \( u(z, x) \) from (3.13) is given as \( u(z, x) = e^{-\sqrt{-z}x} \) and thus \( u(z, 0) = 1 \) and \( u'(z, 0) = u'(z, 0-) = -\sqrt{-z} - \alpha \) implying

\[
m(z, 0) = -\sqrt{-z} - \alpha
\] (3.22)

in agreement with (3.19).

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References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2005.

[2] F. V. Atkinson, *On the location of Weyl circles*, Proc. Roy. Soc. Edinburgh **88A**, 345–356 (1981).

[3] C. Bennewitz, *A note on the Titchmarsh-Weyl m-function*, Argonne Nat. Lab. preprint, ANL-87-26, Vol. 2, 1988, 105–111.

[4] C. Bennewitz, *Spectral asymptotics for Sturm–Liouville equations*, Proc. London Math. Soc. (3) **59**, no. 2, 294–338 (1989).

[5] A. Boutet de Monvel and V. Marchenko, *Asymptotic formulas for spectral and Weyl functions of Sturm-Liouville operators with smooth potentials*, in *New Results in Operator Theory and its Applications*, I. Gohberg and Yu. Lubich (eds.), Operator Theory, Advances and Applications, Vol. 98, Birkhäuser, Basel, 1997, pp. 102–117.

[6] A.A. Danielyan and B.M. Levitan, *On the asymptotic behavior of the Weyl–Titchmarsh m-function*, Math. USSR Izv. **36**, 487–496 (1991).

[7] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Supersymmetry and Schrödinger-type operators with distributional matrix-valued potentials*, arXiv:1206.4966.

[8] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials*, Opuscula Math. **33**, 467–563 (2013).

[9] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Inverse spectral theory for Sturm–Liouville operators with distributional potentials*, Preprint.
[10] J. Eckhardt and G. Teschl, Sturm–Liouville operators with measure-valued coefficients, J. Anal. Math. 120, 151–224 (2013).

[11] W. N. Everitt, On a property of the m-coefficient of a second-order linear differential equation, J. London Math. Soc. (2), 4, 443–457 (1972).

[12] W. N. Everitt and S. G. Halvorsen, On the asymptotic form of the Titchmarsh-Weyl m-coefficient. Appl. Anal. 8, 153–169 (1978).

[13] W. N. Everitt, D. B. Hinton, and J. K. Shaw, The asymptotic form of the Titchmarsh-Weyl coefficient for Dirac systems, J. London Math. Soc. (2), 27, 465–476 (1983).

[14] B. J. Harris, On the Titchmarsh-Weyl m-function, Proc. Roy. Soc. Edinburgh 95A, 223–237 (1983).

[15] B. J. Harris, The asymptotic form of the Titchmarsh-Weyl m-function, J. London Math. Soc. (2), 30, 110–118 (1984).

[16] B. J. Harris, The asymptotic form of the Titchmarsh-Weyl m-function associated with a second order differential equation with locally integrable coefficient, Proc. Roy. Soc. Edinburgh 102A, 243–251 (1986).

[17] B. J. Harris, An exact method for the calculation of certain Titchmarsh-Weyl m-functions, Proc. Roy. Soc. Edinburgh 106A, 137–142 (1987).

[18] E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, 1969.

[19] D. B. Hinton, M. Klaus, and J. K. Shaw, Series representation and asymptotics for Titchmarsh-Weyl m-functions, Diff. Integral Eqs. 2, 419–429 (1989).

[20] H. G. Kaper and M. M. Kwong, Asymptotics of the Titchmarsh-Weyl m-coefficient for integrable potentials, Proc. Roy. Soc. Edinburgh 103A, 347–358 (1986).

[21] H. G. Kaper and M. M. Kwong, Asymptotics of the Titchmarsh-Weyl m-coefficient for integrable potentials, II, in Differential Equations and Mathematical Physics, I. W. Knowles and Y. Saito (eds.), Lecture Notes in Mathematics, Vol. 1285, Springer, Berlin, 1987, pp. 222–229.

[22] B. M. Levitan, A remark on one theorem of V. A. Marchenko, Trudy Moskov. Mat. Otech. 1, 421–422 (1952) [In Russian]

[23] B. M. Levitan and I. S. Sargsjan, Sturm–Liouville and Dirac Operators (Russian), Nauka, Moscow, 1988.

[24] V. A. Marchenko, Some questions in the theory of one-dimensional second-order linear differential operators. I, Trudy Moskov. Mat. Otech. 1, 327–420 (1952) [In Russian]; Amer. Math. Soc. Transl. (2) 101, 1–104 (1973).

[25] A. S. Pechentsov, Trace of a difference of singular Sturm–Liouville operators with a potential containing Dirac δ-functions, Russ. J. Math. Phys. 20, 230–238 (2013).

[26] A. Rybkin, On the trace approach to the inverse scattering problem in dimension one, SIAM J. Math. Anal. 32, 1248–1264 (2001).

[27] A. M. Savchuk and A. A. Shkalikov, Trace Formula for Sturm–Liouville operators with singular potentials, Math. Notes 69, 387–400 (2001).

[28] B. Simon, A new approach to inverse spectral theory, I. Fundamental formalism, Ann. of Math. 150, 1029–1057 (1999).

[29] G. Teschl, Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators, 2nd ed., Graduate Studies in Math., Amer. Math. Soc., Vol. 157, RI, 2014.

[30] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklung willkürlicher Funktionen, Math. Ann. 68, 220–269 (1910).