SPT-3G secondary mirror geometry

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ABSTRACT

SPT-3G is a detector system for the 10m diameter South Pole Telescope, comprising 16,000 millimeter-wave bolometers. It is used for a deep Cosmic Microwave Background survey of the Southern sky. This paper describes the geometry of the secondary mirror, which is a section of a prolate spheroid, in several useful coordinate systems. There is application to off-axis mirrors in general. A geometric theorem is proven, relating to the Dragone condition: the intersection of a prolate spheroid and any plane is an ellipse; the lines connecting points on that ellipse to either focus compose a right circular cone; the central axes of the two cones from the two foci intersect outside the interior of the spheroid.

This paper describes the geometry of the SPT-3G (Benson et al. 2014; Pan et al. 2018) offset gregorian secondary mirror on the South Pole Telescope (Carlstrom et al. 2011), with respect to a global coordinate system centered on the primary mirror, and also the shape of the mirror in two coordinate systems local to the mirror itself (one referenced to the central ray, and one referenced to the edge of the mirror). This mirror has replaced the secondary mirror used in the first two generations of SPT receivers.

The telescope is oriented pointed at the horizon, toward the $+\hat{x}$ direction, so that the light from the star is moving in the $-x$ direction when intercepted by the primary mirror. The $\hat{y}$ axis points up, and the $\hat{z}$ axis is oriented according to the right-hand rule ($\hat{x}$ points right, $\hat{y}$ points up, and $\hat{z}$ points out of the page). The origin is at the vertex of the primary mirror. This coordinate system is the one used by Vertex. It differs from the one usually used in optics, by a rotation of $90^\circ$ about the $\hat{y}$ axis:

$$x_{\text{zemax}} = \hat{z}, \quad y_{\text{zemax}} = y, \quad z_{\text{zemax}} = -x.$$ 

While using this memo, it may be useful to consult Nils Halverson’s Optics Dimensional Control drawing: OpticsDimControl.pdf, and A. Stark’s program spt3gsecondary.c and its output, which can be found under http://spt.uchicago.edu/intweb/optics.
1. The Central Ray

The surface of the primary is defined by:

\[ x = \frac{y^2 + z^2}{4f_p} \]  

(1)

where \( f_p = 7000 \text{ mm} \) is the focal length of the primary mirror. Let \( y_c = 5300 \text{ mm} \). The central ray starts at \((+\infty, y_c, 0)\), and intercepts the primary mirror at

\[ \left( \frac{y_c^2}{4f_p}, y_c, 0 \right) = (1003.214286 \text{ mm}, 5300 \text{ mm}, 0) , \]

where it is reflected through a half-angle

\[ i_p = \tan(y_c/2f_p) = 20.735234^\circ , \]

and passes through the prime focus at

\[ F_1 = (f_p, 0, 0) = (7000 \text{ mm}, 0, 0) . \]

Between the primary and the secondary, the central ray satisfies the equation:

\[ y = \alpha(x - f_p) , \]  

(2)

where

\[ \alpha \equiv \frac{4f_p y_c}{y_c^2 - 4f_p^2} = -\tan(2i_p) = -0.8838068 , \]

is the tangent of the angle from the \(+\hat{x}\) axis to the central ray between the primary and the secondary mirrors.

The SPT3G secondary is a piece of a prolate spheroid. Let \( F_1 \) be the focal point at the prime focus of the primary, \( F_2 \) be the focus after reflection (the Gregorian focus), and \( Q \) be the point where the central ray strikes the surface and is reflected. In the drawing Optics Dimensional Control 6/18/14 by Nils Halverson, the spheroid is defined by a semi-major axis \( a = 1745 \text{ mm} \) and semi-minor axis \( b = 1644.556 \text{ mm} \). This immediately gives the eccentricity \( e = \sqrt{1 - (b/a)^2} = 0.334378 \), conic constant \( k = -e^2 = -0.111809 \), and end-cap radius of curvature \( R = a(1 + k) = 1549.89366 \text{ mm} \). The focal distance \( f_0 = \sqrt{a^2 - b^2} = ae = 583.4896 \text{ mm} \) is half the length of the line segment \( F_1F_2 \), and the vertex distance \( f_s = a - f_0 = 1161.5097 \text{ mm} \) is the length from vertex to focus. The surface can be expressed as:

\[ \frac{(x'' + f_0)^2}{a^2} + \frac{y''^2}{b^2} + \frac{z''^2}{b^2} = 1 . \]

(3)
Dimensional drawing for SPT-3G secondary mirror from Nils Halverson. Linear dimensions in millimeters.

The \((x'', y'', z'')\) coordinate system is defined with its origin at the \(F_1\) focus of the spheroid and the axis of the spheroid along the \(\hat{x}''\) direction. As shown in the Optics Dimensional Control diagram, the secondary can be put in place by rotating by angle \(\theta_s = 15.323°\) around the \(\hat{z}''\) axis then translating in \(\hat{x}\) by a distance \(f_p = 7000\) mm to place the \(F_1\) focus coincident with the prime focus:

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= \begin{pmatrix}
    \cos \theta_s & -\sin \theta_s & 0 \\
    \sin \theta_s & \cos \theta_s & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x'' \\
    y'' \\
    z''
\end{pmatrix}
+ \begin{pmatrix}
    f_p \\
    0 \\
    0
\end{pmatrix}.
\]

The inverse transform is:

\[
\begin{pmatrix}
    x'' \\
    y'' \\
    z''
\end{pmatrix}
= \begin{pmatrix}
    \cos \theta_s & \sin \theta_s & 0 \\
    -\sin \theta_s & \cos \theta_s & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x - f_p \\
    y \\
    z
\end{pmatrix}.
\]
Substituting Equation 5 into Equation 3 yields an equation for the secondary in \((x, y, z)\):

\[
\frac{(x - f_p) \cos \theta_s + y \sin \theta_s + f_0)^2}{a^2} + \frac{([x - f_p] \sin \theta_s - y \cos \theta_s)^2}{b^2} + \frac{z^2}{b^2} = 1. \tag{6}
\]

This equation can be solved for \(y\) as a function of \(x\) and \(z\):

\[
y = \frac{-\beta \gamma - \eta \lambda - \sqrt{(\beta \gamma + \eta \lambda)^2 - (\beta^2 + \eta^2)(\gamma^2 + \lambda^2 + \mu^2 - 1)}}{\beta^2 + \eta^2}, \tag{7}
\]

where

\[
\begin{align*}
\beta &= \sin \theta_s/a, \\
\gamma &= ([x - f_p] \cos \theta_s + f_0)/a, \\
\eta &= -\cos \theta_s/b, \\
\lambda &= ([x - f_p] \sin \theta_s)/b, \\
\mu &= z/b.
\end{align*}
\]

Substituting Equation 2 and \(z = 0\) into Equation 6 allows us to solve for \(x_{sc}\), the \(x\) coordinate of the intersection of the central ray with the secondary:

\[
x_{sc} = \frac{-\rho \sigma + \sqrt{\rho^2 \sigma^2 - (\rho^2 + \tau^2)(\sigma^2 - 1)}}{\rho^2 + \tau^2} + f_p = 7981.579 \, \text{mm}, \tag{8}
\]

where

\[
\begin{align*}
\rho &= (\cos \theta_s + \alpha \sin \theta_s)/a, \\
\sigma &= f_0/a = \varepsilon, \\
\tau &= (\sin \theta_s - \alpha \cos \theta_s)/b.
\end{align*}
\]

Here we introduce a notation where a point in space is represented by a vector extending from the origin of the \((x, y, z)\) coordinate system to that point. Of course, these vectors can be represented in other coordinate systems, even though they depend on the \((x, y, z)\) coordinate system for their definition. The central ray intersects the secondary at the point:

\[
\vec{S} = (x_{sc}, \alpha[x_{sc} - f_p], 0) = (7981.579 \, \text{mm}, -867.52618 \, \text{mm}, 0), \tag{9}
\]

and the Gregorian focus is the point:

\[
\vec{F}_2 = (f_p - 2f_0 \cos \theta_s, -2f_0 \sin \theta_s, 0) = (5874.5032 \, \text{mm}, -308.38675 \, \text{mm}, 0), \tag{10}
\]

in the \((x, y, z)\) coordinate system.
2. Dragone Relation

The distance from the Gregorian focus to point $S$ on the secondary is:

$$f_2 = \|\vec{F}_2 - \vec{S}\| = 2180.00 \text{ mm},$$  \hspace{1cm} (11)$$

while the distance from the prime focus to point $S$ on the secondary is:

$$f_1 = \|\vec{F}_1 - \vec{S}\| = 1310.00 \text{ mm},$$  \hspace{1cm} (12)$$

and the ratio of these distances is the magnification of the secondary:

$$M = -\frac{\|\vec{F}_2 - \vec{S}\|}{\|\vec{F}_1 - \vec{S}\|} = -\frac{f_2}{f_1} = -1.6641236,$$  \hspace{1cm} (13)$$

where by convention the magnification is negative for a Gregorian. The angle of incidence at the secondary is:

$$i_s = \frac{1}{2} \arccos \left[ \frac{(\vec{F}_2 - \vec{S}) \cdot (\vec{F}_1 - \vec{S})}{\|\vec{F}_2 - \vec{S}\| \|\vec{F}_1 - \vec{S}\|} \right] = 13.304411^\circ.$$  \hspace{1cm} (14)$$

The Dragone angle \cite{Dragone1982} is then:

$$\tan i_D \equiv (1 - M)\tan i_s + M\tan i_p = 0.$$  \hspace{1cm} (15)$$

We see that it is zero, since $\theta_s$ was chosen to yield that result.

3. The Secondary Mirror in Local Coordinates around the Central Ray

Define a new coordinate system $(x', y', z')$, where the origin is at $F_1$ like the $(x'', y'', z'')$ system, but rotated by angle $\theta_1$ about $\hat{y}''$ to place point $S$, the intersection of the central ray with the mirror surface, on the positive $\hat{z}'$ axis:

$$
\begin{pmatrix}
x'' \\
y'' \\
z''
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_1 & 0 & \sin \theta_1 \\
0 & 1 & 0 \\
-\sin \theta_1 & 0 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}, \hspace{1cm} (16)$$

and the inverse transform is

$$
\begin{pmatrix}
x'' \\
y'' \\
z''
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_1 & 0 & -\sin \theta_1 \\
0 & 1 & 0 \\
\sin \theta_1 & 0 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}. \hspace{1cm} (17)$$
Substituting Equation 17 into Equation 3 yields the equation of the spheroid in the \((x', y', z')\) system:

\[
\frac{(x' \cos \theta_1 - z' \sin \theta_1 + f_0)^2}{a^2} + \frac{y'^2}{b^2} + \frac{(x' \sin \theta_1 + z' \cos \theta_1)^2}{b^2} = 1.
\]

(18)

Setting \(x' = y' = 0\), and solving the resulting quadratic for \(z'\) gives the distance, \(f_1\), between \(F_1\) and \(S\):

\[
f_1 = \frac{b^2}{a} (1 + e \sin \theta_1)^{-1}.
\]

(19)

We see that \(\theta_1\) is the complementry angle of the “true anomaly” of the point \(S\). We can solve for \(\theta_1\):

\[
\sin \theta_1 = \left( \frac{b^2}{af_1} - 1 \right) e^{-1},
\]

(20)

\[
\theta_1 = 33.206532^\circ.
\]

(21)

Since \(f_2\) is the distance from \(Q\) to \(F_2\), from the properties of ellipses we have \(a = \frac{1}{2}(f_1 + f_2)\), \(b = \sqrt{\frac{1}{4}(f_1 + f_2)^2 - f_0^2}\), and we can express \(\theta_1\) in terms of \(f_0\), \(f_1\), and \(f_2\) only:

\[
\sin \theta_1 = \frac{f_2^2 - f_1^2 - 4f_0^2}{4f_0f_1}.
\]

(22)

This equation can also be derived by applying the law of cosines to the triangle \(F_1CF_2\).

In the \((x', y', z')\) system, the coordinates of point \(Q\) are \((0, 0, f_1)\). Substituting into Equation 17, the coordinates of \(Q\) in \((x'', y'', z'') = (−f_1 \sin \theta_1, 0, f_1 \cos \theta_1)\). Solve Equation 3 for \(z''\) as a function of \(x''\), and differentiate to obtain the slope of the ellipse at point \(C\):

\[
\tan \theta_2 = \frac{dz''}{dx''} = -\frac{b}{a^2} x'' \left(1 - \frac{x''^2}{a^2}\right)^{-1/2} = \frac{b}{a^2} (f_1 \sin \theta_1 + f_0) \left[1 - \left(\frac{f_1 \sin \theta_1 + f_0}{a^2}\right)^2\right]^{-1/2}
\]

(23)

Let \(i_s = \theta_2 - \theta_1\). Now we can define the \((x''', y''', z''')\) system, that has its origin at \(S\) in the center of the mirror, the \(\hat{x'''}-\hat{y'''}\) plane tangent to the spheroid, and the \(\hat{z'''}\) axis pointing into the mirror surface.

\[
\begin{pmatrix}
  x''' \\
  y''' \\
  z'''
\end{pmatrix}
= \begin{pmatrix}
  \cos i_s & 0 & \sin i_s \\
  0 & 1 & 0 \\
  -\sin i_s & 0 & \cos i_s
\end{pmatrix}
\begin{pmatrix}
  x' \\
  y' \\
  z' - f_1
\end{pmatrix},
\]

(24)

and the inverse transform is

\[
\begin{pmatrix}
  x'' \\
  y'' \\
  z''
\end{pmatrix}
= \begin{pmatrix}
  \cos i_s & 0 & -\sin i_s \\
  0 & 1 & 0 \\
  \sin i_s & 0 & \cos i_s
\end{pmatrix}
\begin{pmatrix}
  x''' \\
  y''' \\
  z'''
\end{pmatrix}
+ \begin{pmatrix}
  0 \\
  0 \\
  f_1
\end{pmatrix}.
\]

(25)
Note that $i_s$ is the angle of incidence at the secondary, since the incident ray lies along the $\hat{z}'$ axis, and $\hat{z}'''$ is normal to the surface. Applying the law of cosines to the triangle $F_1SF_2$, we have:
\[
4f_0^2 = f_1^2 + f_2^2 - 2f_1f_2\cos(2i_s).
\] (26)

We can now eliminate $a$, $b$, $e$, $f_0$, $\theta_1$, and $\theta_2$ in favor of $f_1$, $f_2$ and $i_s$ in all of the preceding equations. The shape of the mirror can therefore be described in terms of the two focal distances and the angle of incidence. Combine Equation 25 with Equation 26, substitute into Equation 3, and simplify:
\[
pz''' + (qx''' + 2r)z''' + y''' + sx''' = 0,
\] (27)

where
\[
p = 1 - e^2 \sin^2 \theta_2 = 0.94114858 \tag{28}
\]
\[
q = e^2 \sin (2 \theta_2) = 0.11165324 \tag{29}
\]
\[
r = f_1(\cos i_s + e \sin \theta_2) = f_2(\cos i_s - e \sin \theta_2) = 1592.6382 \text{ mm} \tag{30}
\]
\[
s = 1 - e^2 \cos^2 \theta_2 = 0.94704272 \tag{31}
\]

and
\[
f_0 = \frac{1}{2} \sqrt{f_1^2 + f_2^2 - 2f_1f_2\cos(2i_s)} = 583.49067 \text{ mm} \tag{32}
\]
\[
\theta_2 = i_s + \arcsin \left( \frac{f_2^2 - f_1^2 - 4f_0^2}{4f_0f_1} \right) = 46.51091^\circ \tag{33}
\]
\[
e = \frac{2f_0}{f_1 + f_2} = 0.334378, \tag{34}
\]

yielding the equation of the mirror surface sag in $(x'''', y'''', z'''')$ coordinates:
\[
z''' = \frac{1}{2p} \left[ \sqrt{(2r + qx''')^2 - 4p(y''' + sx''')} - (2r + qx''') \right]. \tag{35}
\]

This is the shape of the secondary mirror in a coordinate system centered on the intersection of the central ray with the surface, where the surface sag ($z'''$) is normal to the surface at that point. The value of $r$ has dimensions of length, and it is always positive (either of the expressions in equation 30 can be used, depending on the sign of $\theta_2$). It is the radius of curvature in the $\hat{y}'''$-$\hat{z}'''$ plane, but it is not the radius of curvature seen by the beam, as will be shown in Equation 36. The values of $p$ and $s$ are dimensionless and always between 0 and 1. The value of $q$ is dimensionless and always between $-1$ and 1; its sign is opposite that of the $x'''$ coordinate of the point $S$. Note that $z'''$ is everywhere negative, since $p$ and $s$ are positive in Equation 35. The $\hat{z}'''$ direction points into the mirror, and the $\hat{x}'''$-$\hat{y}'''$ plane...
is tangent to the mirror at point $S$, the origin of the $(x, y, z)$ system. The direction of $\hat{x}''$ is such that its dot product with the vector from $F_1$ to $F_2$ is positive.

Since we know the two focal distances $f_1$ and $f_2$, the thin lens formula gives the paraxial focal length $f$:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{2 \cos i_s}{r}, \quad (36)$$

using Equation 30 to substitute for $f_1$ and $f_2$. So $f = r/(2 \cos i_s) = 818.28089$ mm, and the central radius of curvature is $r/\cos i_s = 1636.5618$ mm.

4. Defining the Edge of the Secondary

Unlike the first SPT receiver and SPTpol, the edge of the secondary is not a stop of the optical system, and so the secondary need only be big enough not to significantly vignette; the exact figure of the edge doesn’t matter optically. For convenience in manufacturing, however, we’d like the edge of the secondary to lie in a plane. This can be accomplished by defining the edge as the intersection of the secondary spheroid with a right circular cone whose vertex is at the Gregorian focus $\bar{F}_2$ and whose axis is coincident with the central ray $\bar{F}_2 - \bar{Q}$. The half-angle of the cone is defined to be $\phi_2 = 23.5993^\circ$, and the angle between the rotation axis of the cone and the line between the focii is $\psi_2 = -30.1844^\circ$. These two angles, together with the major and minor axes of the spheroid ($a$ and $b$), fully define the mirror. The analysis is done in the Appendix below, and the results presented in Table 1. The calculations in the Appendix show that there are equivalent ways of defining the edge of the secondary: a right circular cone whose vertex is the prime focus, $F_1$, with half angle $\phi_1 = 38.8953^\circ$ and tilt $\psi_1 = -52.0529^\circ$ also intersects the spheroidal surface of the secondary in the same ellipse as the cone from $F_2$, although the axis of that cone does not lie on the central ray of the optics.

As a practical matter, the edge of the secondary mirror is chosen to be circular, the result of intersecting the prolate spheroid with a cylinder. The edge does not, therefore, lie in a plane. The edge of the primary is not illuminated by the detector optics, so the precise geometry does not matter.

5. Mirror Coordinates Relative to the Mirror Edge

Given the definition of the edge of the secondary, we can transform into a coordinate system centered on the ellipse that is the edge of the secondary and tilted so that the plane
of the mirror edge is the $x''' - y'''$ plane at $z''' = 0$:

\[
\begin{pmatrix}
x'''
\end{pmatrix}
= \begin{pmatrix}
\cos \theta_e & \sin \theta_e & 0 \\
0 & 0 & -1 \\
-\sin \theta_e & \cos \theta_e & 0
\end{pmatrix}
\begin{pmatrix}
x - c \\
y \\
z
\end{pmatrix}
+ \begin{pmatrix}
d \\
0 \\
0
\end{pmatrix}.
\]

(37)

The inverse transform is:

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
\cos \theta_e & 0 & -\sin \theta_e \\
0 & \cos \theta_e & 0 \\
-1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x''' - d \\
y'''
\end{pmatrix}
+ \begin{pmatrix}
c \\
0 \\
0
\end{pmatrix},
\]

(38)

where in this case $(x, y, z)$ is the coordinate system of Equation 41 and not the global coordinate system. The offsets $c$ and $d$ are given in Table 1, and $\theta_e = 47.9795^\circ$ is the angle $\theta$ defined in the Appendix ($\theta_e$, the slope of the mirror edge, differs from $\theta_2$ in Equation 33, the slope of the mirror surface at the central ray). This is the coordinate system needed to cut the mirror surface. In this coordinate system, the equation of the secondary mirror can be expressed as:

\[
z''' = \frac{P x''' + Q + a \sqrt{J y'''^2 + K x'''^2 + L x'''^2 + N}}{J}
\]

(39)

where

\[
c = \frac{(f_0 \cos \psi_2 - a \cos \phi_2)}{\sin \psi_2 \tan \theta_e} = 1961.81 \text{ mm}
\]

\[
d = b^2 c \cos \theta_e / (a^2 \sin^2 \theta_e + b^2 \cos^2 \theta_e) = 1227.92 \text{ mm}
\]

\[
J = -f_0^2 \cos^2 \theta_e - b^2 = -2857121.6 \text{ mm}^2
\]

\[
K = -b^2 = -2704564.4 \text{ mm}^2
\]

\[
L = 2 b^2 [d - c \cos \theta_e] = -4.614588 \times 10^8 \text{ mm}^3
\]

\[
N = b^2 \{b^2 + f_0^2 \cos^2 \theta_e - [d - c \cos \theta_e]^2\} = 7.7075858 \times 10^{12} \text{ mm}^4
\]

\[
P = f_0^2 \cos \theta_e \sin \theta_e = 169310.4 \text{ mm}^2
\]

\[
Q = -[f_0^2 d \cos \theta_e + b^2 c] \sin \theta_e = -4.1496374 \times 10^9 \text{ mm}^3.
\]

The $x'''$ direction is up, and the top of the mirror is $(x''', y''', z''') = (869.665 \text{ mm}, 0, 0)$. The point $(x''', y''', z''') = (0, 847.599, 0)$ is also on the mirror edge. The point of maximum depth in the secondary mirror is not below the center of the mirror (that is, $x''' = 0, y''' = 0$), but is slightly displaced:

\[
x_{\text{min}}''' = \frac{a^2 K L - L P^2 - P \sqrt{L^2 - 4 KN} P^2 + 4 a^2 K^2 N - a^2 K L^2}}{2 (K P^2 - a^2 K^2)}
\]

\[
= 14.2408 \text{ mm}
\]

\[
y_{\text{min}}''' = 0
\]

\[
z_{\text{min}}''' = -243.286 \text{ mm}
\]
is the lowest point on the mirror surface, about a half inch above the center.
6. Appendix: Theorems about Spheroids, Cones, and Planes

The coordinate systems and variables defined in this appendix are independent of the body of this memo above. Consider a prolate spheroid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1,$$

(41)

where $a > b > 0$. In the $(x, y, z)$ coordinate system, the focii are $F_1 = (+f_0, 0, 0)$ and $F_2 = (-f_0, 0, 0)$, where $f_0 \equiv \sqrt{a^2 - b^2}$. Cut the spheroid with an arbitrary plane, $y = (x - c)\tan \theta$, where $-90^\circ < \theta < 90^\circ$. By symmetry, this equation can describe any cutting of the spheroid except for the special cases of a plane parallel to the axis of the spheroid ($\theta = 0^\circ$ exactly) or a plane perpendicular to the axis of the spheroid ($\theta = 90^\circ$ exactly). These singular cases can easily be treated separately. Transform so that the plane becomes the $\hat{u} - \hat{z}$ plane, and translate in the $\hat{u}$ direction by an offset, $d$, to be determined below:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u - d \\ v \\ z \end{pmatrix} + \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix},$$

(42)

and the inverse transform is

$$\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x - c \\ y \\ z \end{pmatrix} + \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}.$$

(43)

Substitute Equation 42 into Equation 41, and set $v = 0$, (which we see from Equation 43 gives $y = (x - c)\tan \theta$) yielding:

$$\left[\frac{(u - d) \cos \theta + c}{a^2}\right]^2 + \frac{(u - d)^2 \sin^2 \theta}{b^2} + \frac{z^2}{b^2} = 1.$$

(44)

Let

$$g^2 \equiv a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

(45)

and

$$\gamma \equiv 1 - \frac{c^2 \sin^2 \theta}{g^2}.$$

(46)

Then, choosing the arbitrary translation in the $\hat{u}$ direction to be:

$$d \equiv \frac{b^2 c \cos \theta}{g^2},$$

(47)

Equation 44 can be written as

$$\frac{u^2}{a^2} + \frac{z^2}{\beta^2} = 1,$$

(48)
\[ \alpha^2 \equiv \frac{a^2 b^2}{g^2} \gamma \] \hspace{1cm} (49)

and

\[ \beta^2 \equiv b^2 \gamma. \] \hspace{1cm} (50)

Equation 48 shows that the intersection of the spheroid and plane is an ellipse, provided \( \gamma > 0 \) or \( c^2 < a^2 + b^2 \cot^2 \theta \) (i.e. the plane actually intersects the spheroid and doesn’t miss it). The intersection of a prolate spheroid and an arbitrary plane is an ellipse. (More generally, any closed figure that results from the intersection of a plane with an ellipsoid, paraboloid, or hyperboloid is an ellipse.)

In the \((u, v, z)\) coordinate system, the center of the ellipse is \( C = (0, 0, 0) \), and the extrema of the ellipse are \( A = (\pm abg^{-1} \sqrt{\gamma}, 0, 0) \), and \( B = (0, 0, \pm b \sqrt{\gamma}) \); the focii are \( F_1 = ([f_0 - c] \cos \theta + d, -[f_0 - c] \sin \theta, 0) \) and \( F_2 = (-[f_0 + c] \cos \theta + d, -[f_0 + c] \sin \theta, 0) \).

In the \((x, y, z)\) coordinate system, the center of the ellipse is \( C = (-d \cos \theta + c, -d \sin \theta, 0) \), and the extrema of the ellipse are

\[ A = ([\pm abg^{-1} \sqrt{\gamma} - d] \cos \theta + c, [\pm abg^{-1} \sqrt{\gamma} - d] \sin \theta, 0), \] \hspace{1cm} (51)

and

\[ B = (-d \cos \theta + c, -d \sin \theta, \pm b \sqrt{\gamma}). \] \hspace{1cm} (52)

Consider a right circular cone whose vertex is \( F_1 \), and whose axis passes through the point \( Q_1 \). The points on cone satisfy the equation:

\[ (\vec{P} - \vec{F}_1) \cdot (\vec{Q}_1 - \vec{F}_1) = \| \vec{P} - \vec{F}_1 \| \| \vec{Q}_1 - \vec{F}_1 \| \cos \phi_1, \] \hspace{1cm} (53)

where \( P \) is any point on the cone, \( \phi_1 \) is the opening half-angle of the cone, and \( Q_1 \) is a point in the \( x-y \) plane that defines the axis of the cone. Without loss of generality, we take \( 0 < \phi_1 < 90^\circ \), so \( \cos \phi_1 > 0 \).

Define a coordinate system translated so that the origin is at \( F_1 \): \( x' = x - f_0 \), \( y' = y \), \( z' = z \), and let the angle \( \psi_1 \) be the angle between the unit vector in the \( x' \) direction and the axis of the cone, \( \vec{Q}_1 - \vec{F}_1 \):

\[ x' \cdot (\vec{Q}_1 - \vec{F}_1) = \| \vec{Q}_1 - \vec{F}_1 \| \cos \psi_1. \] \hspace{1cm} (54)

Without loss of generality, we take we take \( -90^\circ < \psi_1 < 90^\circ \), so \( \cos \psi_1 > 0 \).

Then Equation 53 becomes:

\[ x' \cos \psi_1 + y' \sin \psi_1 = \sqrt{x'^2 + y'^2 + z'^2} \cos \phi_1. \] \hspace{1cm} (55)
Solve Equation 55 for \( z^2 \), substitute into Equation 41, and solve for \( y \) to yield:

\[
y = \frac{(\pm f_0 \cos \phi_1 - a \cos \psi_1) x + a f_0 \cos \psi_1 \mp a^2 \cos \phi_1}{a \sin \psi_1}.
\]  

(56)

We see that

\[
\tan \theta = \frac{\pm e \cos \phi_1 - \cos \psi_1}{\sin \psi_1},
\]  

(57)

and

\[
c = \pm \frac{a \cos \phi_1 - f_0 \cos \psi_1}{\sin \psi_1 \tan \theta},
\]  

(58)

where \( e \equiv f_0/a \) is the eccentricity of the spheroid, describe the two planes

\[
y = (x - c) \tan \theta
\]  

(59)

whose intersection with the spheroid yields the same ellipses as the intersection of the spheroid with the two nappes of the cone that has a vertex at \( F_1 \), is tilted by angle \( \psi_1 \) with respect to the \( \hat{x} \) axis, and has a half opening angle of \( \phi_1 \). Choosing the + signs in Equations 57 and 58 yields the nappe of the cone to the left, where the points on its axis satisfy \( x < f_0 \). Note that \( \psi_1 \) and \( \theta \) have opposite signs for the nappe to the right, and usually have opposite signs for the nappe on the left unless the spheroid is sufficiently eccentric \((e > \cos \psi_1 / \cos \phi_1)\). The intersection of a prolate spheroid with a right circular cone, whose vertex is one of the focii of the spheroid, is a planar figure, specifically an ellipse. That is not true in general of a cylinder whose axis passes through the focus — the intersection of that cylinder with the spheroid will not lie in a plane — because the intersection of the cylinder with a circular cone sharing the same axis is a circle, not an ellipse, and we know that the intersection between the cone and the spheroid is in general elliptical.

By symmetry, this result applies equally well to a cone whose vertex is at \( F_2 \), but if we define the tilt of such a cone with respect to the positive \( \hat{x} \) axis:

\[
\hat{x}'' \cdot (\vec{Q}_2 - \vec{F}_2) = ||\vec{Q}_2 - \vec{F}_2|| \cos \psi_2.
\]  

(60)

where \( x'' = x + f_0 \), that breaks the symmetry and we get slightly different equations for planes:

\[
\tan \theta = \frac{\mp e \cos \phi_2 - \cos \psi_2}{\sin \psi_2},
\]  

(61)

and

\[
c = \pm \frac{a \cos \phi_2 + f_0 \cos \psi_2}{\sin \psi_2 \tan \theta},
\]  

(62)

where like before, we take \( 0 < \phi_2 < 90^\circ \) and \(-90^\circ < \psi_2 < 90^\circ\). Choosing the upper signs in Equations 61 and 62 yields the nappe to the left (points on the axis satisfy \( x < -f_0 \)).
Equations 57 and 58 can be solved to yield $\phi_1$ and $\psi_1$ as a function of $\theta$ and $c$:

$$\tan \psi_1 = \frac{b^2 \cot \theta}{c f_0 - a^2}, \quad (63)$$

and

$$\cos \phi_1 = \left| \frac{\tan \theta \sin \psi_1 + \cos \psi_1}{e} \right|, \quad (64)$$

while Equations 61 and 62 can be similarly be inverted to yield:

$$\tan \psi_2 = \frac{-b^2 \cot \theta}{c f_0 + a^2}, \quad (65)$$

and

$$\cos \phi_2 = \left| \frac{\tan \theta \sin \psi_2 + \cos \psi_2}{e} \right|. \quad (66)$$

For any plane defined by $\theta$ and $c$ that cuts the spheroid, there are right circular cones from each focus defined by $\phi_1$, $\psi_1$, $\phi_2$, and $\psi_2$ that generate the same ellipse as the plane. Any arbitrary plane cutting a prolate spheroid results in an ellipse. The set of lines connecting points on that ellipse to the focii of the spheroid comprise two right circular cones whose vertices lie on the foci.

The points $Q_1$ and $Q_2$ cannot in general be coincident with the center of the ellipse, point $C$. When an ellipse is generated by cutting a cone with a plane, the center of the ellipse does not fall on the axis of the cone, except in the singular case where the ellipse is a circle.

The location of $Q_1$ could be chosen to lie anywhere on the line $y = (x - f_0) \tan \psi_1$, while the location of $Q_2$ could be chosen to lie anywhere on $y = (x + f_0) \tan \psi_2$. From Equations 63 and 65, we see that these two lines intersect at the point

$$Q_0 = \left( \frac{a^2}{c}, \frac{-b^2}{c \tan \theta}, 0 \right), \quad (67)$$

and this point can serve as both $Q_1$ and $Q_2$, but in general this point does not lie on the spheroid. $Q_0$ is, of course, the intersection of the axes of the two cones, and $c$ is the intercept of the plane with the $\hat{x}$ axis and $-c \tan \theta$ is the intercept of the plane with the $\hat{y}$ axis. $Q_0$ lies on the spheroid if $c^2 = a^2 + b^2 \cot^2 \theta$, but that implies that the plane is tangent to the spheroid, $\gamma = 0$ in Equation 46, and the ellipse has shrunk to a point — not interesting or useful. The singular on-axis case, where $\psi_1 = \psi_2 = 0$ and the ellipse is actually a circle, allows $Q_0 = (a, 0, 0)$. In the general case, point $Q_0$ lies outside the spheroid.

The points $Q_1$ and $Q_2$ could also lie in the plane of the ellipse or on the surface of the spheroid. The intersections of the cone axes with the line $y = (x - c) \tan \theta$, in the plane of
the ellipse, in \((x, y, z)\) coordinates, are:

\[ Q_{1e} = \left( \frac{c \tan \theta - f_0 \tan \psi_1}{\tan \theta - \tan \psi_1}, \frac{[c - f_0] \tan \theta \tan \psi_1}{\tan \theta - \tan \psi_1}, 0 \right), \] (68)

\[ Q_{2e} = \left( \frac{c \tan \theta + f_0 \tan \psi_2}{\tan \theta - \tan \psi_2}, \frac{[c + f_0] \tan \theta \tan \psi_2}{\tan \theta - \tan \psi_2}, 0 \right). \] (69)

The intersections of the cone axes with the spheroid are:

\[ Q_{1s} = \left( \frac{a^2 f_0 \tan^2 \psi_1 \pm ab^2 \sec \psi_1}{a^2 \tan^2 \psi_1 + b^2}, \frac{-b^2 f_0 \pm ab^2 \sec \psi_1}{a^2 \tan \psi_1 + b^2 \cot \psi_1}, 0 \right), \] (70)

\[ Q_{2s} = \left( \frac{-a^2 f_0 \tan^2 \psi_2 \pm ab^2 \sec \psi_2}{a^2 \tan^2 \psi_2 + b^2}, \frac{+b^2 f_0 \pm ab^2 \sec \psi_2}{a^2 \tan \psi_2 + b^2 \cot \psi_2}, 0 \right). \] (71)

Choose the + sign for the nappe to the right.

If the cone represents a bundle of light rays coming from \(F_2\) and striking the spheroidal mirror, then \(Q_2 - F_2^r\) is the central ray, and it strikes the mirror at point \(Q_{2s}\). The light rays in the cone will strike the mirror and reflect, converging in a different cone onto \(F_1\). An ellipse that is the intersection of a spheroid and a right circular cone whose vertex is one focus of the spheroid can also be described as the intersection of the spheroid with a different right circular cone whose vertex is the other focus. This second cone will have a different opening angle \(\phi_1\) and tilt \(\psi_1\). Also, the axes of the two cones do not, in general, intersect at the spheroidal surface, so the points \(Q_{1s}\) and \(Q_{2s}\) will be different. The central beam coming from \(F_2\) will strike the mirror and be reflected before it gets to \(Q_0\). This is important for optics, because if the axis of one cone is the central ray of the beam, that ray will not reflect onto the axis of the other cone, causing a skewness in the beam that can only be corrected by an appropriate choice of angles in subsequent reflections, i.e. the Dragone condition.
| Variable | Coordinate System |
|----------|-------------------|
| $a$ | semimajor axis of spheroid | 1745.000 |
| $b$ | semiminor axis of spheroid | 1644.556 |
| $f_0$ | focal distance of spheroid | 583.490 |
| $\theta$ | angle of plane | 47.9795° |
| $c$ | $x$ intercept of plane | 1961.810 |
| $d$ | $u$ coordinate offset | 1227.920 |
| $g$ | $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/2}$ | 1700.726 |
| $\gamma$ | $1 - c^2 \sin^2 \theta / g^2$ | 0.265634 |
| $\alpha$ | semimajor axis ellipse | 869.665 |
| $\beta$ | semiminor axis ellipse | 847.599 |
| $\phi_1$ | half-angle cone 1 | 38.8953° |
| $\psi_1$ | tilt cone 1 | -52.0529° |
| $\phi_2$ | half-angle cone 2 | 23.5993° |
| $\psi_2$ | tilt cone 2 | -30.1844° |
| $A_t$ | top of ellipse | 1721.997 | -266.148 | 0 | s |
| $A_t$ | bottom of ellipse | 8168.367 | 44.177 | 0 | p |
| $A_b$ | bottom of ellipse | 557.696 | -1558.305 | 0 | s |
| $A_b$ | bottom of ellipse | 7386.922 | -1509.725 | 0 | p |
| $B$ | extrema of ellipse | 1139.847 | -912.226 | ±847.599 | s |
| $B$ | extrema of ellipse | 7777.645 | -732.774 | ±847.599 | p |
| $Q_0$ | intersection of axes | 1552.150 | -1242.2 | 0 | s |
| $Q_0$ | intersection of axes | 8262.490 | -942.058 | 0 | p |
| $Q_{1e}$ | axis 1 intercept ellipse | 1222.935 | -820.0142 | 0 | s |
| $Q_{1e}$ | axis 1 intercept ellipse | 7833.411 | -621.883 | 0 | p |
| $Q_{1s}$ | axis 1 intercept spheroid | 1374.021 | -1013.765 | 0 | s |
| $Q_{1s}$ | axis 1 intercept spheroid | 8030.327 | -768.820 | 0 | p |
| $Q_{2e}$ | axis 2 intercept ellipse | 1086.548 | -971.378 | 0 | s |
| $Q_{2e}$ | axis 2 intercept ellipse | 7741.872 | -803.908 | 0 | p |
| $Q_{2s}$ | axis 2 intercept spheroid | 1300.929 | -1096.073 | 0 | s |
| $Q_{2s}$ | = point $S$ in Equation 9 | 7981.584 | -867.517 | 0 | p |

Parameters of SPT3G Secondary Mirror: Variables refer to definitions in the Appendix. All dimensions in millimeters. The spheroid is the mirror surface, the ellipse is the mirror edge. Subscript 1 refers to the cone between the prime focus and the secondary. Subscript 2 refers to the cone between the secondary and the Gregorian focus. Coordinate system “s” has its origin halfway between the two focii and the $x$ axis is the line between the two focii. Coordinate system “p” is the global coordinate system with the origin at the vertex of the primary.
/* file spt3gsecondary.c */
/* Calculate spt coordinate values from memo */
/* A. Stark */
/* 7–4–14 */
/* compile with command
cc spt3gsecondary.c -lm -o spt3gsecondary */

#include <math.h>
#include <stdio.h>

double fp;
double theta_s;
double xpe, ype;
double theta_e;
double theta_r;
double theta_c;
double theta_1, theta_2;
double a, b, fs, f_0;
double rtod;
double i_p;
double eslope;
double xppe, yppe, zppe;
double xdaggere, ydaggere, zdaggere;

main()
{

double norm(), dot();
void stoptopprimary();
void ptou(), utop(), utopp(), ppou(), pptop(), ptopp();
void ppptop(), ptoppppp();
void daggertou();
double uonellipse(), pponellipse(), pponcone();
double daggeronellipse(), daggeroncone();
double quadp(), quadm();

double yc, xc;
double r2, k, e;
double alpha, beta, gamma, nu, lambda, mu;
double rho, sigma, tau;
double F1x, F1y, F1z;
double Scx, Scy, Scz;
double F2x, F2y, F2z;
double f_1, f_2;
double M;
double i_s, i_D;
double iota;
double temp;
double xpptop, xppbot, ypptop, yppbot;
double xdaggertop, x daggerbot, ydaggertop, y daggerbot;
double omega;
double upsilon;
double ae, be;
double J, K, L, N, P, Q;
double p, q, r, s;
double fparaxial;

double x, y, z;
double xp, yp, zp;
double xpp, ypp, zpp;
double x dagger, y dagger, z dagger;
double xppp, yppp, zppp;
double xpppp, ypppp, zpppp;
double xppppmin, yppppmin, zppppmin;
double Tslope, Bslope;
double Atemp, Btemp, Ctemp;

rtod = 45.0 / atan(1.0); /* convert radians to degrees */

yc = 5300.0; /* vertical offset of primary */
fp = 7000.0; /* focal length of primary */

i_p = atan(yc/(2*fp))*rtod; /* angle of incidence at primary */
printf("\n i_p = %12.8gdeg\t angle of incidence at primary \n", i_p);

/* F1 is vector to prime focus */
F1x = 7000.0; F1y = 0.0; F1z = 0.0;
printf("\n F1 = %12.8gmm, %12.8gmm, %12.8gmm\t prime focus ",
F1x,F1y,F1z);

/* slope of central ray between primary and secondary */
alpha = (4.0*fp*yyc)/(yyc*yyc - 4.0*fp*fp);
printf("\n alpha = %12.8g\t slope of central ray", alpha);

/* Major and minor axes are defining values for SPT3G secondary */
a = 1745.;
b = 1644.556;
/* radius of curvature and conic constant of secondary */
/* these values , chosen here , define the shape of the secondary */
e = sqrt(1.0 - b*b/(a*a));
k = -1.0*e*e;
r2 = a*(1.0+k);
/*
r2 = 1549.89366;
k = -0.111809;
e = sqrt(-k);
*/
printf("\n r2 = %12.8gmm\t secondary radius of curvature",r2);
printf("\n k = %12.8g\t conic constant of secondary",k);
printf("\n e = %12.8g\t eccentricity of secondary",e);

/* semi-major, semi-minor, focal distance of secondary */
/*
a = r2/(k+1.0);
b = r2/sqrt(k+1.0);
*/
f_0 =   sqrt(a*a-b*b);
fs = a - f_0;
printf("\n a = %12.8gmm\t major axis of secondary spheroid",a);
printf("\n b = %12.8g mm\t minor axis of secondary spheroid\n",b);
printf("\nf_0 = %12.8g mm\t focal distance 0 of spheroid\n",f_0);
printf("\nfs = %12.8g mm\t vertex to f_1 distance of spheroid\n",fs);

/∗ rotation of secondary axis ∗/
/∗ this value is chosen so that Dragone angle (calculated below) is zero ∗/
theta_s = 15.323;
printf("\n theta_s = %12.8g deg\t rotation secondary axis\n",theta_s);

/∗ solve for intersection of central ray with secondary ∗/
rho = (cos(theta_s/rtod)+alpha*sin(theta_s/rtod))/a;
sigma = (a−fs)/a;
tau = (sin(theta_s/rtod)−alpha*cos(theta_s/rtod))/b;

xc = fp + (−rho*sigma+sqrt(rho*rho*sigma*sigma−(rho*rho+tau*tau)*(sigma*sigma−1.0)))/(rho*rho+tau*tau);

Scx = xc; Scy = alpha*(xc−fp); Scz = 0.0;
printf("\n C = %12.8g mm, %12.8g mm, %12.8g mm\t\n" "central ray at secondary", Scx, Scy, Scz);

/∗ F2 is vector to Gregorian focus ∗/
F2x = fp−2.0*(a−fs)*cos(theta_s/rtod);
F2y = −2.0*(a−fs)*sin(theta_s/rtod);
F2z = 0.0;
printf("\n F2 = %12.8g mm, %12.8g mm, %12.8g mm\t Gregorian focus\n", F2x, F2y, F2z);

/∗ distance from Gregorian focus to secondary along central ray ∗/
 f_2 = norm(F2x−Scx,F2y−Scy,F2z−Scz);
 printf("\n f_2 = %12.8g mm\t Gregorian focus to secondary\n",f_2);
f_2 = 2180.0;

/∗ distance from prime focus to secondary along central ray ∗/
f_1 = norm(F1x-Scx,F1y-Scy,F1z-Scz);
printf("\n f_1 = %12.8g mm prime focus to secondary\n", f_1);
f_1 = 1310.0;

/* magnification of secondary, is negative for Gregorians */
M = -f_2 / f_1;
printf("\n M = %12.8g magnification\n", M);

/* solve for angle of incidence of central ray at secondary */
i_s = 0.5 * acos(dot(F2x-Scx,F2y-Scy,F2z-Scz,
 F1x-Scx,F1y-Scy,F1z-Scz)
/(f_2*f_1))*rtod;
printf("\n i_s = %12.8g deg angle of"
  " incidence at secondary \n", i_s);

printf("\n f_0 = %12.8g \n", f_0);

/*
 f_0 = 0.5*sqrt(f_1*f_1+f_2*f_2-2.0*f_1*f_2*cos(2.0*i_s/rtod));
 printf(" f_0 second time = %12.8g \n", f_0);
*/

/* solve for Dragone angle */
i_D = atan( (1.0 - M)*tan(i_s/rtod)+M*tan(i_p/rtod)) * rtod;
printf("\n i_D = %12.3g deg Dragone angle\n", i_D);

/* angle that rotates dagger coordinate system to p coordinate system */
theta_r = -30.1844;
printf("\n theta_r = %12.8g deg\n ", theta_r);

theta_r = (-1.0)* (theta_s + 2.0*i_p - 2.0*i_s);
printf("\n alternate theta_r = %12.8g deg\n ", theta_r);

theta_1 = rtod * asin((((b*b)/(a*f_1) - 1.0)/e));
printf("\n theta_1 = %12.8g deg rotate pp to p\n ", theta_1);

theta_1 = rtod *
\[
\begin{align*}
\text{asinh} &\left(\frac{f_2 f_2 - f_1 f_1 - 4.0 f_0 f_0}{4.0 f_0 f_1}\right) \\
\text{printf} &\left(\text{"n theta}_1 = %12.8g\text{deg\t rotate pp to p }\right)\text{", theta}_1) \\
\text{temp} &\text{ = } f_1 \ast \sin(\theta_1/\text{rtod}) + f_0; \\
\theta_2 &\text{ = } \text{rtod} \ast \text{atan}(\text{b*temp}/(a*a*sqrt(1.0 - temp*temp/(a*a)))); \\
\text{printf} &\left(\text{"n theta}_2 = %12.8g\text{deg }\right)\text{", theta}_2) \\
\text{e} &\text{ = } 2f_0/(f_1+f_2); \\
\text{printf} &\left(\text{"n e = %12.8g\t eccentricity }\right)\text{", e);} \\
\text{p} &\text{ = } 1.0 - \text{e*e} \ast \sin(\theta_2/\text{rtod}) \ast \sin(\theta_2/\text{rtod}); \\
\text{printf} &\left(\text{"n p = %12.8g }\right)\text{", p);} \\
\text{q} &\text{ = } \text{e*e} \ast \sin(2.0*\theta_2/\text{rtod}); \\
\text{printf} &\left(\text{"n q = %12.8g }\right)\text{", q}); \\
\text{r} &\text{ = } f_1*(\cos(\text{i._s/rtod})+\text{e*sin(}\theta_2/\text{rtod})); \\
\text{printf} &\left(\text{"n r = %12.8g }\right)\text{", r}); \\
\text{r} &\text{ = } f_2*(\cos(\text{i._s/rtod}) - \text{e*sin(}\theta_2/\text{rtod})); \\
\text{printf} &\left(\text{"n r = %12.8g }\right)\text{", r}); \\
\text{s} &\text{ = } 1.0 - \text{e*e} \ast \cos(\theta_2/\text{rtod}) \ast \cos(\theta_2/\text{rtod}); \\
\text{printf} &\left(\text{"n s = %12.8g }\right)\text{", s}); \\
\text{fparaxial} &\text{ = } r / (2.0 \ast \text{cos(}\text{i._s/rtod})); \\
\text{printf} &\left(\text{"n fparaxial} = %12.8g\t central radius} = %12.8g\n\right)\text{", fparaxial}, 2.0*fparaxial); \\
\text{printf} &\left(\text{"n Is S on ellipse? %12.8g }\right), \text{uonellipse(Scx,Scy,Scz)}); \\
\text{theta}_c &\text{ = } 23.5993; \\
\text{printf} &\left(\text{"n theta}_c = %12.8g\text{deg\t defined"}\right) \\
\text{\" half angle of cone }\right)\text{", theta}_c); \\
\text{eslope} &\text{ = } (-1.0*f_0*\text{cos(}\theta_c) \\
\text{ \-a*cos(}\text{theta}_r/\text{rtod}))/\text{(a*sin(}\text{theta}_r/\text{rtod}));} \\
\text{printf} &\left(\text{"n e slope Equation 44 = %12.8g }\right)\text{", eslope});
\[ \theta_e = \text{rtod} \times \text{atan}(\text{eslope}); \]
\[
\text{printf}("\t \theta_e = %12.8g \text{deg} \t \text{angle of ellipse}"\n\t \text{in dagger } \text{n", } \theta_e); \]

\[ \iota = (b \times b \times \cos(\theta_c / \text{rtod}))/ (a \times \sin(\theta_r / \text{rtod})); \]
\[
\text{printf}("\n \iota \ \text{Equation45} = %12.8g \text{ mm \n", } \iota); \]

\[ \text{Tslope} = \tan((\theta_r + \theta_c) / \text{rtod}); \]
\[ \text{Bslope} = \tan((\theta_r - \theta_c) / \text{rtod}); \]
\[
\text{printf}("\n \text{Tslope} = %12.8g \n", \text{Tslope}); \]
\[ \text{printf}("\n \text{Bslope} = %12.8g \n", \text{Bslope}); \]
\[
A_{temp} = a \times a \times \text{Tslope} \times \text{Tslope} + b \times b; \]
\[
B_{temp} = -2.0 \times b \times b \times f_0; \]
\[ C_{temp} = f_0 \times f_0 \times b \times b - a \times a \times b \times b; \]
\[
\text{if (quadp}(A_{temp}, B_{temp}, C_{temp}) > \text{quadm}(A_{temp}, B_{temp}, C_{temp}) \}) \{ \]
\[
\quad \text{xdaggertop=} \text{quadp}(A_{temp}, B_{temp}, C_{temp}); \]
\[
\quad \text{printf}(" \text{xdaggertop positive} "); \]
\[
\} \text{ else } \{ \]
\[
\quad \text{xdaggertop=} \text{quadm}(A_{temp}, B_{temp}, C_{temp}); \]
\[
\quad \text{printf}(" \text{xdaggertop negative} "); \]
\[
\}
\]
\[ \text{ydaggertop} = \text{Tslope} \times \text{xdaggertop}; \]
\[
\text{printf}("\n \text{xdaggertop, ydaggertop=} %12.8g \text{ mm, } %12.8g \text{ mm }", \text{xdaggertop, ydaggertop}); \]
\[
\text{printf}("\n \text{Is top on ellipse pp? } %12.8g \n", \text{daggeronellipse}(\text{xdaggertop, ydaggertop, 0.0})); \]
\[
\text{printf}("\n \text{Is top on cone pp? } %12.8g \n", \text{daggeroncone}(\text{xdaggertop, ydaggertop, 0.0})); \]

\[ A_{temp} = a \times a \times \text{Bslope} \times \text{Bslope} + b \times b; \]
\[
\text{if (quadp}(A_{temp}, B_{temp}, C_{temp}) \times \text{Bslope} < \]
\[
\quad \text{quadm}(A_{temp}, B_{temp}, C_{temp}) \times \text{Bslope}) \{ \]
\[
\quad \text{xdaggerbot=} \text{quadp}(A_{temp}, B_{temp}, C_{temp}); \]
\[
\quad \text{printf}(" \text{xxpbot positive} "); \]
\[
\} \text{ else } \{ \]
\[
\quad \text{xdaggerbot=} \text{quadm}(A_{temp}, B_{temp}, C_{temp}); \]
\[
\quad \text{printf}(" \text{xxpbot negative} "); \]
ydaggerbot = Bslope*xdaggerbot;
printf("\n xdaggerbot , ydaggerbot= %12.8g mm, %12.8g mm ",
        xdaggerbot , ydaggerbot);
printf("\n Is bot on ellipse dagger? %12.8g ",
        daggeronellipse(xdaggerbot , ydaggerbot , 0.0));
printf("\n Is bot on cone dagger? %12.8g ",
        daggeroncone(xdaggerbot , ydaggerbot , 0.0));

/*
 xdaggett0p = (a*b*b/cos((theta_c-theta_r)/rtod) - b*b*f_0)
    /(a*a*Tslope*Tslope + b*b);
 ydaggett0p = Tslope*xpptop;
 printf("\n alternate xdaggett0p , ydaggett0p=%12.8g mm, %12.8g mm", 
        xdaggett0p , ydaggett0p);
 xdaggetbot = (a*b*b/cos((-1.0*theta_c-theta_r)/rtod)
    - b*b*f_0)/(a*a*Bslope*Bslope + b*b);
 ydaggebot = Bslope*xdaggerbot;
 printf("\n alternate xppbot, yppbot= %12.8g mm, %12.8g mm ",
        xdaggett0p , ydaggett0p);
*/
temp = xdaggett0p-xdaggerbot;
ae = temp*temp;
temp = ydaggett0p-ydaggerbot;
ae += temp*temp;
ae = sqrt(ae)/2.0;
printf("\n ae= %12.8g mm ", ae);

xdaggere = (xdaggett0p+xdaggerbot)/2.0;
ydaggere = (ydaggett0p+ydaggerbot)/2.0;
zdaggere = b*sqrt(1.0 - (xdaggere-f_0)*(xdaggere-f_0)/(a*a)
    - ydaggere*ydaggere/(b*b));
printf("\n xdaggere, ydaggere, zdaggere="
    "%12.8g mm, %12.8g mm, %12.8g mm\t\n", 
    xdaggere , ydaggere ,zdaggere);
printf("\n Is daggere on ellipse dagger? %12.8g ",
        daggeronellipse(xdaggere , ydaggere ,zdaggere));
printf("\n Is dagger on cone dagger? %12.8g \n", 
daggeroncone(xdagger, ydagger, zdagger));

eslope = (ydaggertop−ydaggerbot)/(xdaggertop−xdaggerbot);
printf("\n e slope= %12.8g ", eslope);

theta_e = rtod∗atan(eslope);
printf("\t theta_e = %12.8gdeg\t angle of ellipse in dagger \n", 
theta_e);

printf("\n theta_e+theta_s = %12.8gdeg", theta_e + theta_s);

iota = ydaggertop−eslope∗xdaggertop;
printf("\n iota = %12.8g mm", iota);
printf("\n ydaggerbot = %12.8g mm", xdaggerbot∗eslope+iota);

daggertou(xdaggertop, ydaggertop, 0.0, &x, &y, &z);
printf("\n top of secondary edge = %12.8gmm, %12.8gmm, %12.8gmm", 
 x, y, z);
printf("\n Is top on ellipse? %12.8g \n", uonellipse(x, y, z));

daggertou(xdaggerbot, ydaggerbot, 0.0, &x, &y, &z);
printf("\n bottom of secondary edge = %12.8gmm, %12.8gmm, %12.8gmm", 
 x, y, z);
printf("\n Is bottom on ellipse? %12.8g \n", uonellipse(x, y, z));

daggertou(xdagger, ydagger, zdagger, &x, &y, &z);
printf("\n center of secondary edge = 
 "
 "%12.8gmm, %12.8gmm, %12.8gmm\t \n", x, y, z);

printf("\n Is center edge on ellipse? %12.8g \n", uonellipse(x, y, z));

printf("\nInput mirror coordinate xpppp: ");
scanf("%lg", &xpppp);
printf(" Input mirror coordinate ypppp: ");
scanf("%lg",&ypppp);

J = -1.0*(f_0*f_0*cos(theta_e/rtod)*cos(theta_e/rtod)+b*b);
printf("\n J = %12.8g mm\t\n", J);

K = -1.0*b*b;
printf("\n K = %12.8g mm\t\n", K);

temp = ydaggere*sin(theta_e/rtod)+(xdaggere-f_0)*cos(theta_e/rtod);
L = -2.0*b*b*temp;
printf("\n L = %12.8g mm\t\n", L);

N = b*b*(b*b+f_0*f_0*cos(theta_e/rtod)*
    cos(theta_e/rtod)-temp*temp);
printf("\n N = %12.8g mm\t\n", N);

P = f_0*f_0*cos(theta_e/rtod)*sin(theta_e/rtod);
printf("\n P = %12.8g mm\t\n", P);

Q = a*a*ydaggere*cos(theta_e/rtod)
    -b*b*(xdaggere-f_0)*sin(theta_e/rtod);
printf("\n Q = %12.8g mm\t\n", Q);

zpppp = (P*xpppp+Q+a*sqrt(J*ypppp*ypppp
    +K*xpppp*xpppp+L*xpppp+N))/J;
printf(" mirror sag zpppp = %12.8g\t\n", zpppp);

xppppmin = (a*a*K*L-L*P*P-P*sqrt((L*L-4.0*K*N)*P*P+
    4.0*a*a*K*K*N-a*a*K*L*L))/(2.0*(K*P*P-a*a*K*K));

xpppp = xppppmin;
ypppp = 0.0;

zpppp = (P*xpppp+Q+a*sqrt(J*ypppp*ypppp
    +K*xpppp*xpppp+L*xpppp+N))/J;
printf(" lowest point on mirror=
    "%12.8gmm, %12.8gmm, %12.8gmm\t\n", xpppp,ypppp,zpppp);
uotp(7981.579, -867.52618, 0, &x, &y, &z);
pptopppp(x, y, z, &xpppp, &ypppp, &zpppp);
printf(" central ray at mirror=

\%12.8gmm, \%12.8gmm, \%12.8gmm\t\n",
        xpppp, ypppp, zpppp);

printf("\\n");
}

double norm(x, y, z) /* norm of a vector */
double x, y, z;
{
    return (sqrt(x*x+y*y+z*z));
}

double dot(x1, y1, z1, x2, y2, z2) /* dot product of two vectors */
double x1, y1, z1, x2, y2, z2;
{
    return (x1*x2+y1*y2+z1*z2);
}

double uonellipse(double xonel, double yonel, double zonel)
/* Is the point on the spheroid in unprimed coords? */
{
    double temponel, sumonel;

    temponel = ((xonel-fp)*cos(theta_s/rtod)
            +yonel*sin(theta_s/rtod)+f_0)/a ;
    sumonel = temponel*temponel;
    temponel = ((xonel-fp)*sin(theta_s/rtod)
            -yonel*cos(theta_s/rtod))/b;
    sumonel += temponel*temponel;
    temponel = zonel/b;
    sumonel += temponel*temponel;
    return (sumonel);
}
double pponellipse(double xonel, double yonel, double zonel)  
/* Is the point on the spheroid in double prime coords? */  
{  
double temponel, sumonel;  

    temponel = (xonel+f_0)/a;  
    sumonel = temponel*temponel;  
    temponel = yonel/b;  
    sumonel += temponel*temponel;  
    temponel = zonel/b;  
    sumonel += temponel*temponel;  
    return(sumonel);  
}  

double daggeronellipse(double xonel, double yonel, double zonel)  
/* Is the point on the spheroid in dagger coords? */  
{  
double temponel, sumonel;  

    temponel = (xonel-f_0)/a;  
    sumonel = temponel*temponel;  
    temponel = yonel/b;  
    sumonel += temponel*temponel;  
    temponel = zonel/b;  
    sumonel += temponel*temponel;  
    return(sumonel);  
}  

double daggeroncone(double xonco, double yonco, double zonco)  
/* Is the point on the cone in dagger coords? */  
{  
double temponco;  

    temponco = sqrt(xonco*xonco+yonco*yonco+zonco*zonco)  
        *cos(theta_c/rtod);  
    return((xonco*cos(theta_r/rtod)+yonco  
            *sin(theta_r/rtod))/temponco);
void ptou(double xp, double yp, double zp,
    double *x, double *y, double *z)
/* convert primed to unprimed coordinates */
{
    *x = cos(2.0*i_p/rtod)*(xp)+sin(2.0*i_p/rtod)*yp+fp;
    *y = (-1.0)*sin(2.0*i_p/rtod)*(xp)+cos(2.0*i_p/rtod)*yp;
    *z = zp;
}

void ptopp(double xp, double yp, double zp,
    double *xpp, double *ypp, double *zpp)
/* convert primed to double primed coordinates */
{
    *xpp = cos(theta_1/rtod)*(xp) - sin(theta_1/rtod)*zp;
    *ypp = yp;
    *zpp = sin(theta_1/rtod)*(xp)+cos(theta_1/rtod)*zp;
}

void pptou(double xpp, double ypp, double zpp,
    double *x, double *y, double *z)
/* convert double primed to unprimed coordinates */
{
    *x = cos(theta_s/rtod)*(xpp)-sin(theta_s/rtod)*ypp+fp;
    *y = sin(theta_s/rtod)*(xpp)+cos(theta_s/rtod)*ypp;
    *z = zpp;
}

void daggertou(double xdagger, double ydagger,
    double zdagger, double *x, double *y, double *z)
/* convert double primed to unprimed coordinates */
{
    *x = cos(theta_s/rtod)*(xdagger - 2.0*f_0)
        -sin(theta_s/rtod)*ydagger +fp;
    *y = sin(theta_s/rtod)*(xdagger - 2.0*f_0)
        +cos(theta_s/rtod)*ydagger;
    *z = zdagger;
void pptop(double xpp, double ypp, double zpp,
        double *xp, double *yp, double *zp)
/* convert double primed to primed coordinates */
{    
    *xp = cos(theta_1/rtod)*(xpp)+sin(theta_1/rtod)*zpp;
    *yp = ypp;
    *zp = (-1.0)*sin(theta_1/rtod)*(xpp)+cos(theta_1/rtod)*zpp;
}

void utopp(double x, double y, double z,
        double *xpp, double *ypp, double *zpp)
/* convert unprimed to double primed coordinates */
{    
    *xpp = cos(theta_s/rtod)*(x-fp) + sin(theta_s/rtod)*y;
    *ypp = (-1.0)*sin(theta_s/rtod)*(x-fp)+cos(theta_s/rtod)*y;
    *zpp = z;
}

void utop(double x, double y, double z,
        double *xp, double *yp, double *zp)
/* convert unprimed to primed coordinates */
{    
    *xp = cos(2.0*i_p/rtod)*(x-fp) - sin(2.0*i_p/rtod)*y;
    *yp = sin(2.0*i_p/rtod)*(x-fp)+cos(2.0*i_p/rtod)*y;
    *zp = z;
}

void pptopppe(double xpp, double ypp, double zpp,
        double *x, double *y, double *z)
/* convert double primed to quadruple primed coordinates */
{    
    *x = cos(theta_e/rtod)*(xpp-xppe)+sin(theta_e/rtod)*(ypp-yppe);
    *y = (-1.0)*zpp;
    *z = (-1.0)*sin(theta_e/rtod)*(xpp-xppe)+cos(theta_e/rtod)*(ypp-yppe);
}
void stoptopprimary(double xs, double ys, double zs,
    double *x, double *y, double *z)
/* project points at the stop onto primary */
{
    *y = 2.0*fp*ys*(xs−fp−sqrt((xs−fp)*(xs−fp)
        +ys*ys+zs*zs))/(ys*ys+zs*zs);
    *x = (*y)*(xs−fp)/ys+fp;
    *z = (*y)*zs/ys;
}

void ppptop(double xppp, double yppp, double zppp,
    double *xp, double *yp, double *zp)
/* convert mirror coordinates to primed coordinates */
{
    *xp=cos(theta_e/rtod)*xppp−sin(theta_e/rtod)*zppp+xpe;
    *yp=sin(theta_e/rtod)*xppp+cos(theta_e/rtod)*zppp+ype;
    *zp=−yppp;
}

double quadp(double A, double B, double C)
{
    double surd;
    surd = B*B−4.0*A*C;
    if (surd < 0) {
        printf(“\nbad surd in quadp\n”);
        return(0.0);
    } else {
        return((-B-sqrt(surd))/(2.0*A));
    }
}

double quadm(double A, double B, double C)
{
    double surd;
    surd = B*B−4.0*A*C;
    if (surd < 0) {

printf("\nbad surd in quad\n");
return (0.0);
}
else {
    return ((-B-sqrt(surd))/(2.0*A));
}
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