Quark and Lepton Masses from a $U(1) \times Z_2$ Flavor Symmetry

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Abstract

We show that solutions for the masses and mixings of the quarks and leptons based on a $U(1) \times Z_2$ horizontal symmetry are possible. The seesaw mechanism is shown to work consistently in the presence of the discrete symmetry. The discrete symmetry results in the phenomenologically useful suppressions of elements of the Yukawa matrices. The quark and lepton masses, the CKM mixing angles, and the neutrino mixing angles are accommodated at the order of magnitude level.

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I. INTRODUCTION

Experimental data on the quark and lepton sector masses and mixings may provide a clue to the nature of new physics beyond the Standard Model (SM). Masses and mixings are experimentally accessible, but as far as the SM is concerned, these parameters can be adjusted at will without destroying the consistency of the theory. Therefore any relationships between them must come from theoretical ideas beyond those already contained in the SM, and the experimental data can guide us in narrowing down the choices and freedom in these ideas.

The recent data on the mixing of neutrinos has rekindled interest in models of fermion masses and mixings since it supplements the existing data from the quark and charged lepton sectors. The neutrino observations have some intriguing features that one might hope to explain. First of all, the neutrinos are very light in comparison to the other fermions. This suggests that a heavy mass scale may be involved that is providing a small dimensionless number that is responsible for the small neutrino masses. Secondly, the atmospheric neutrino data [1] indicates that there is a large mixing angle involved. This is in contrast to the small mixing (Cabibbo-Kobayashi-Maskawa of CKM) angles of the quark sector. Since in grand unified theories (GUTs) the quarks and leptons are unified in representations of the larger gauge theory, this dichotomy of small CKM angles with large mixing in the lepton sector provides hints as to how the fermion masses might arise. In fact the quark and charged lepton sectors show large hierarchies of masses. This seems to indicate that there might be a flavor symmetry whose spontaneous breaking might result in the generation of naturally small contributions resulting in the hierarchical pattern of masses. One hope is that such a flavor symmetry can be implemented to understand the masses and mixings detailed above as well as the long-standing evidence for solar neutrino oscillations.

In this paper we show solutions to the quark and lepton masses and mixings based on a $U(1) \times Z_2$ Abelian flavor symmetry are possible. A particular solution (with nontrivial $Z_2$ charges) we detail is entirely consistent with all the phenomenological constraints, and one can implement the seesaw mechanism to explain the light neutrino masses.

The paper is organized as follows. In Section II we briefly review the approach of supersymmetric Abelian flavor (or horizontal) symmetries, and present the phenomenological requirements that must be satisfied in both the lepton and quark sectors. In Section III we discuss how a discrete symmetry can be used to overcome the constraints implied by a $U(1)$ symmetry, and show how the discrete symmetry can suppress an entry to the extent that it has no impact on the leading order predictions for the masses and mixing angles. In Sections IV and V we review how the light neutrino mass matrix is independent of the horizontal charges of the singlet neutrinos for the case where the symmetry is $U(1)$. In Section V we also generalize the derivation of the light neutrino mass matrix for the case where the horizontal symmetry is $U(1) \times Z_2$. In Section VI we present a solution for the quark sector that satisfies all the phenomenological requirements. Finally in Section VII we present our conclusions.
II. FLAVOR SYMMETRIES

The hierarchical structure of the fermion mass matrices strongly suggests that there is a spontaneously broken family symmetry responsible for the suppression of Yukawa couplings. In this paper we employ supersymmetric Abelian horizontal symmetries. These flavor symmetries allow the fermion mass and mixing hierarchies to be naturally generated from nonrenormalizable terms in the effective low-energy theory.

The idea is quite simple and easily implemented [2]. There is some field \( \Phi \) which is charged under a \( U(1) \) family symmetry, and without loss of generality, we can assume that its charge is -1. There are terms contributing to effective Yukawa couplings for the quarks,

\[
Q_i \overline{d}_j H_d \left( \frac{<\Phi>}{\Lambda_L} \right)^{m_{ij}} + Q_i \overline{u}_j H_u \left( \frac{<\Phi>}{\Lambda_L} \right)^{n_{ij}},
\]

and the integer exponents \( m_{ij} \) and \( n_{ij} \) are easily calculated in terms of the horizontal symmetry charges of the quark and Higgs fields. For example, if we choose to have the Higgs fields to be uncharged, then the exponent \( m_{ij} \) is just the sum of the horizontal charge of the fields \( Q_i \) and \( \overline{d}_j \). The hierarchy is generated from terms in the superpotential that carry integer charges \( m_{ij}, n_{ij} \geq 0 \). If we call the small breaking parameter \( \lambda \), then the generated terms for say the down quark Yukawa matrix will be of order \( \lambda^{m_{ij}} \). The holomorphy of the superpotential forbids terms from arising with \( m_{ij}, n_{ij} < 0 \). A nice analysis of the possible approaches to explaining the neutrino masses and mixings using \( U(1) \) symmetries only is given in Ref. [3]. In the remainder of this section we outline the experimental constraints that a solution employing the above idea must satisfy.

A. Phenomenological requirements for leptons

The first phenomenological constraints we consider involve the charged leptons whose masses are the most precisely measured parameters of the quark-lepton sector. For the experimental values for the masses, we require that

\[
\frac{m_\mu}{m_\tau} \sim \lambda^2, \quad \frac{m_e}{m_\mu} \sim \lambda^3,
\]

where the small parameter is identified as the Cabibbo angle, i.e. \( \lambda \sim 0.22 \).

The remaining constraints on leptons involve the neutrino masses and mixings. The most interesting aspect of the neutrino data is that the atmospheric neutrino mixing appears to be large, perhaps even maximal. It is then hard to understand a hierarchical pattern for the neutrino masses, since large mixing should result when the neutrino masses are of roughly the same order of magnitude. The Super-Kamiokande data [4] suggest that

\[
\Delta m_{23}^2 \sim 2.2 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{23} \sim 1,
\]

where the subscripts indicate the generations of neutrinos involved in the mixing (here we assume the mixing is between \( \nu_\mu \) and \( \nu_\tau \)).

The solar neutrino flux can be explained by one of three distinct solutions. Two of these involve matter-enhanced oscillation (MSW), while the third involves vacuum oscillations
The two MSW solutions are differentiated by the size of the mixing angle, so one is usually called the small mixing angle (SMA) solution, and the other is called the large mixing angle (LMA) solution. The values required for the mixing parameters in each of these three cases are shown in the table below.

| Solution   | $\Delta m^2_{1x}$ [$eV^2$] | $\sin^2 2\theta_{1x}$ |
|------------|------------------------------|------------------------|
| MSW(SMA)   | $5 \times 10^{-6}$          | $6 \times 10^{-3}$     |
| MSW(LMA)   | $2 \times 10^{-5}$          | 0.8                    |
| VO         | $8 \times 10^{-11}$         | 0.8                    |

For example, consider the small mixing angle (SMA) solution of the solar neutrino problem. Combining the solar neutrino data with the atmospheric neutrino data, one requires then the following

$$\frac{\Delta m^2_{1x}}{\Delta m^2_{23}} \sim \lambda^4, \quad \sin \theta'_{12} \sim \lambda^2, \quad \sin \theta'_{23} \sim \lambda^0,$$

when the small parameter is taken to be the Cabibbo angle. As explained by Grossman, Nir, Shadmi [4] and Tanimoto [5], one can accommodate the phenomenological constraints on the neutrino masses and mixings as well as the charged lepton masses by postulating that there is a $U(1) \times Z_2$ horizontal symmetry. We show how such a solution can be obtained in the seesaw mechanism in Section V.

### B. Phenomenological requirements for quarks

Again taking the expansion parameter to be the Cabibbo angle, $\lambda = |V_{us}|$, then the experimental constraints [6]

$$|V_{us}| = 0.2196 \pm 0.0023, \quad |V_{cb}| = 0.0395 \pm 0.0017, \quad \frac{|V_{ub}|}{|V_{cb}|} = 0.08 \pm 0.02,$$

on the CKM matrix can be identified in terms of powers of $\lambda$ by the following

$$|V_{us}| \sim \lambda, \quad |V_{cb}| \sim \lambda^2, \quad |V_{ub}| \sim \lambda^3 - \lambda^4, \quad \frac{|V_{ub}|}{|V_{cb}|} \sim \lambda - \lambda^2.$$

The constraint on $|V_{ub}/V_{cb}|$ can be expressed in a stronger way at 90% confidence level as $0.25\lambda - 0.5\lambda$. One also has a constraint on the CKM elements from $B_d^0 - \overline{B}_d^0$ mixing [7],

$$|V_{tb} V_{td}| = 0.0084 \pm 0.0018,$$

which implies that

$$|V_{td}| \sim \lambda^3.$$

It has been argued that $|V_{ub}|$ is more accurately given as $\sim \lambda^4$ and taking it to be $\sim \lambda^3$ (as we will do) requires an unnatural cancellation. However, in our opinion, requiring $|V_{ub}| \sim \lambda^4$ is too restrictive for two reasons. Firstly, the fine-tuning required is much less if one uses
an expansion parameter $\lambda$ somewhat less than 0.22, say 0.18. Secondly, since there are four parameters in the CKM matrix we are trying to predict, it is not unnatural that one of these would appear mildly fine-tuned, given $\lambda \sim 1/5$.

One can appreciate the nature of the cancellation in terms of the unitarity of the CKM matrix. This constraint requires

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$

so to leading order in $\lambda$ one has the relation

$$V_{ub}^* + V_{td} + V_{cd}V_{cb}^* \simeq 0.$$  \hspace{1cm} (10)

Since $|V_{cd}| \sim \lambda$, $|V_{cb}| \sim \lambda^2$ and $|V_{td}| \sim \lambda^3$, unitarity implies that to leading order one might expect $|V_{ub}| \sim \lambda^3$ whereas the experimental data yields a value somewhat suppressed. One can show that with $U(1)$ or $Z_2$ as components of the horizontal symmetry, one can suppress $|V_{ub}|$ (or $|V_{td}|$) relative to $\lambda^3$ only by even powers of $\lambda$, so $|V_{ub}| \sim \lambda^4$ is not possible. The additional suppression one might desire to attribute to an additional power of $\lambda$ must in fact be resulting from a mild cancellation.

There is a universal scaling factor associated with the renormalization group evolution of the CKM angles $|V_{ub}|$ and $|V_{cb}|$ from the high (grand unified) scale to the electroweak scale, but this scaling is not enough to change the expectations for the relevant exponents of $\lambda$. The mass ratios should satisfy

$$\frac{m_c}{m_t} \sim \lambda^4, \quad \frac{m_u}{m_c} \sim \lambda^4, \quad \frac{m_s}{m_b} \sim \lambda^3, \quad \frac{m_d}{m_s} \sim \lambda^2.$$  \hspace{1cm} (11)

To compare the predictions of flavor symmetries to these phenomenological constraints, one has to relate the CKM elements to the entries in the Yukawa matrices. The Yukawa matrices $U$ and $D$ can be diagonalized by biunitary transformations

$$U_{\text{diag}} = V^L_u U V^{R*}_u,$$

$$D_{\text{diag}} = V^L_d D V^{R*}_d.$$  \hspace{1cm} (12)\hspace{1cm} (13)

The CKM matrix is then given by

\[\text{Note:} a \simeq b\] indicates that $a$ and $b$ are equal to leading order in the small parameter $\lambda$, while we use $a \sim b$ to indicate that $a$ and $b$ are the same order in $\lambda$.

\[\text{Equation (10)}\] represents the familiar unitarity triangle, and the cancellation can be reinterpreted in terms of the size of the CP asymmetry angle $\beta$. Moreover, the unitarity triangle makes it clear how to interpret the three mixing angles and one CP-violating phase of the CKM matrix in terms of the four CKM elements in Eq. (11). The amount of CP-violation is proportional to the size of the triangle.

\[\text{In fact, the first (unsuccessful) solution in Section VI predicts } |V_{td}| \sim \lambda^5 \text{ and the relation } |V_{us}| \simeq |V_{ub}/V_{cb}|.\]
\[ V \equiv V_u^L V_d^{L†} . \] (14)

The left-handed transformation matrices \( V_u^L \) and \( V_d^{L} \) can be defined in terms of three successive rotations in the (2,3), (1,3) and (1,2) sectors. These rotation angles of the transformation matrices can be expressed in terms of the elements of the Yukawa matrices as follows [7,9]

\[ s_{12}^u = \frac{u_{12}}{\tilde{u}_{22}} + \frac{u_{11} u_{21}^*}{|\tilde{u}_{22}|^2} \left( u_{32} + u_{32}^* \tilde{u}_{22} \right) - \frac{u_{11} u_{31}^* (u_{23}^* + u_{32} u_{22}^*)}{|\tilde{u}_{22}|^2} , \] (15)

\[ s_{13}^u = u_{13} + u_{11} u_{31}^* + u_{12} \left( u_{32}^* + u_{22} u_{23} \right) + u_{11} u_{31}^* (u_{23} + u_{22} u_{32}^*) , \] (16)

\[ s_{23}^u = u_{23} + u_{22} u_{32}^* , \] (17)

where \( u_{ij} \) is the \( i,j \)th component of the up quark Yukawa matrix, \( U/(U)_{33} \), and \( \tilde{u}_{22} = u_{22} u_{33} - u_{23} u_{32} \). There are corresponding expressions for the \( s_{ij}^d \) in terms of the components of the down quark Yukawa matrix, \( D \) (which are slightly more complicated due to the fact that the (2,3) sector mixing in \( V_d^R \) might be of order one). Clearly contributions to the CKM matrix elements can come from a number of terms. We shall be interested in what follows in determining the leading order contribution(s) to the CKM angles and the fermion masses.

### III. TEXTURE ZEROS

The procedure of adopting a \( U(1) \) horizontal symmetry introduces nontrivial relationships between the parameters in the quark and lepton mass matrices. This results because previously undetermined entries in the matrices are described in terms of a few parameters. For example, the quark (up and down) mass matrices are described by nine parameters, namely the \( U(1) \) charges of the fields \( Q_L, \pi_R, \) and \( \eta_R \). Relationships between the masses and mixing angles are then obtained.

The motivation for including texture zeros in mass matrices was to derive more relationships between the masses and mixings. The earliest of these was the relationship between the Cabibbo angle and the first and second generation down quark masses, \( V_{us} \simeq \sqrt{m_d/m_s} \). The texture zeros responsible for this relation can be obtained in models where there is an additional discrete symmetry that forbids their occurrence, for example. Furthermore, these relationships might also include Clebsch-Gordon factors that allow one to obtain phenomenologically acceptable relationships: one of the earliest and most successful of these was the Georgi-Jarlskog model [10], which was shown later [11] to be successful in the case of electroweak scale supersymmetry (with the experimental data available at that time). The guiding principle for the case where the Yukawa matrices are symmetric is this: the mass hierarchy is of order \( \lambda^4 \) in the up quark matrices, and is of order \( \lambda^2 \) in the down quark matrices (c.f. Eq. (11)). So the dominant contribution to the Cabibbo matrix should come from the down quark matrices (If the down quark matrix is symmetric and the 1-1 component is suppressed, then one has the relation \( |V_{us}| \simeq \sqrt{m_d/m_s} \), while the dominant contribution to \( |V_{cb}| \sim \lambda^2 \) should come from the diagonalization of the up quark matrices. Furthermore the relation \( |V_{ub}/V_{cb}| \sim \sqrt{m_u/m_c} \) follow from suitable texture zero patterns [9]).

When the theory is supersymmetric, one can obtain zero entries in the mass matrices that are sometimes called holomorphic or supersymmetric zeros. They arise because the
superpotential must be holomorphic, so entries that would get a contribution from $\Phi^\dagger$ are absent. In terms of the mass matrices, this simply means that there are no entries with the small parameter $\lambda$ raised to a negative power. Rather entries, for which the quantum numbers would seem to require a negative exponent, are simply zero.

In this paper we want to introduce another concept that we will call a texture zero by flavor suppression. The horizontal symmetry we are considering here does not by itself give us any information on the order one coefficients of the entries in the mass matrices. When certain entries are suppressed because of the discrete symmetry, it can result that the entry is sufficiently suppressed that it does not affect the leading order of the phenomenology. Equivalently this entry could be replaced with an exact zero, and the leading order expectation for the masses and mixing angles would remain the same. Consider a simple $2 \times 2$ example of quark mass matrices where there is a horizontal $U(1)$ symmetry, and where the phenomenological constraints (listed below) are motivated by the (2,3) sector of the quark mass matrices. We require that the mixing angle be $\sim \lambda^2$ and the quark mass ratios satisfy $m_c/m_t \sim \lambda^4$ and $m_s/m_b \sim \lambda^2$. This can be obtained by assuming the charges $Q_L : 2, 0$, $\bar{u}_R : 2, 0$, $\bar{d}_R : 0, 0$:

$$U \sim \begin{pmatrix} \lambda^4 & \lambda^2 \\ \lambda^2 & \lambda^0 \end{pmatrix}, \quad D \sim \begin{pmatrix} \lambda^2 & \lambda^0 \\ \lambda^0 & \lambda^0 \end{pmatrix}.$$  \hspace{1cm} (18)

This is a unique solution of $U(1)$ charges, and thereby determines already relationships between the mixings and masses of the first generation. The procedure for determining the exponents of $\lambda$ in a model with a $U(1)$ solution, clearly leads to the relations between exponents,

$$n_{ii} + n_{jj} = n_{ij} + n_{ji},$$  \hspace{1cm} (19)

for all $i, j = 1, 2, 3$.

Turning now to the case of a $U(1) \times Z_2$ symmetry, suppose the charges are $Q_L : (2, 1), (0, 1)$, $\bar{u}_R : (1, 0), (0, 1)$, $\bar{d}_R : (0, 1), (0, 1)$, and assume a common symmetry breaking parameter $\lambda$. We still achieve the hierarchies for $U$ and $D$ shown in Eq. (18). So there are additional charge assignments that can be made that satisfy the phenomenological constraints.

However there is something more that adding a discrete symmetry can do. Notice that in Eq. (18), the mixing $V_{cb}$ arises from contributions from diagonalizing $U$ and from diagonalizing $D$, since both of these contributions are of order $\lambda^2$. We can however find stronger relationships if we can suppress one of these contributions. For example, if the mixing contribution from the (2,3) block of the up quark matrix is suppressed and the $(D)_{22}$ entry is suppressed, then one has that the mixing angle is the same power of $\lambda$ as the square root of the mass ratio $|V_{cb}| \sim \sqrt{m_c/m_t}$. (If the up quark matrix is known to be exactly symmetric, then one even knows that the order one coefficient of the leading contributions in $\lambda$ is the same, $|V_{cb}| \simeq \sqrt{m_c/m_t}$.) Just this kind of suppression of the element $(U)_{23}$ can be obtained by employing a discrete symmetry. So if the exponent of $\lambda$ in $(U)_{23}$ is sufficiently large that it plays no role in determining the phenomenological predictions (to leading order), then we say it is a texture zero. Returning to our example, take the $U(1) \times Z_2$ charges to be $Q_L : (3, 1), (0, 0)$, $\bar{u}_R : (1, 1), (0, 0)$, $\bar{d}_R : (1, 0), (0, 1)$. Then one obtains the Yukawa matrices
\[ U \sim \begin{pmatrix} \lambda^4 & \lambda^4 \\ \lambda^2 & \lambda^0 \end{pmatrix}, \quad D \sim \begin{pmatrix} \lambda^5 & \lambda^3 \\ \lambda^1 & \lambda^1 \end{pmatrix}, \tag{20} \]

for which the phenomenological predictions in terms of powers of \( \lambda \) are the same, but the mixing comes at leading order from diagonalizing \( D \). We see that the relations, Eq. (19), need not necessarily hold if one adds a \( Z_2 \) symmetry. Since it is the off-diagonal entries that are suppressed in \( U \), we can define the texture pattern in the following schematic way,

\[ U \sim \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad D \sim \begin{pmatrix} 0 & X \\ X & X \end{pmatrix}, \tag{21} \]

where \( X \) denotes a phenomenologically relevant entry, and the 0 entries are suppressed sufficiently that the exponent that appears there is irrelevant. Even though the entries in the first column of the up and down quark matrices are not suppressed by the discrete symmetry, we denote these as zeros because they do not contribute at leading order to either the phenomenologically relevant (left-handed) mixing angles or the quark masses. We leave the categorization of what patterns of texture zeros one can employ in \( 3 \times 3 \) matrices to a future work \[12\].

### IV. NEUTRINO MASSES

Assume that the lepton fields have charges under a \( U(1) \) family symmetry

\[ \begin{align*}
\mathbf{\bar{e}}_{R1} & \quad \mathbf{\bar{e}}_{R2} & \quad \mathbf{\bar{e}}_{R3} & \quad \ell_{L1} & \quad \ell_{L2} & \quad \ell_{L3} & \quad \mathbf{\bar{\nu}}_{R1} & \quad \mathbf{\bar{\nu}}_{R2} & \quad \mathbf{\bar{\nu}}_{R3} \\
\mathbf{E}_1 & \quad \mathbf{E}_2 & \quad \mathbf{E}_3 & \quad \mathbf{L}_1 & \quad \mathbf{L}_2 & \quad \mathbf{L}_3 & \quad \mathbf{N}_1 & \quad \mathbf{N}_2 & \quad \mathbf{N}_3
\end{align*} \tag{22} \]

We assume here that the quantum numbers satisfy the hierarchies \( E_1 \geq E_2 \geq E_3 \geq 0 \), \( L_1 \geq L_2 \geq L_3 \geq 0 \), and \( N_1 \geq N_2 \geq N_3 \geq 0 \).

In the lepton sector, the effective Yukawa couplings come from nonrenormalizable terms, giving

\[ \ell_{Li} \mathbf{\bar{e}}_{Rj} H_d \lambda^{p_{ij}} + \frac{1}{M_R} \ell_{Li} \ell_{Lj} H_u H_u \lambda^{q_{ij}}, \tag{23} \]

where \( M_R \) is the relevant high mass scale at which the light neutrino masses are generated by the effective (nonrenormalizable) operator in the second term. There are two cases we can consider: (1) the mechanism that gives rise to the light neutrino masses generates all possible contributions to the nonrenormalizable terms \( \ell_{Li} \ell_{Lj} H_u H_u \). In this case the exponent \( q_{ij} \) that makes the second term an invariant under the horizontal symmetry (before symmetry breaking) is just \( q_{ij} = L_i + L_j \). So the light neutrino mass matrix is simply

\[ m_{\nu} \sim \begin{pmatrix} \lambda^{2L_1} & \lambda^{L_1+L_2} & \lambda^{L_1+L_3} \\ \lambda^{L_1+L_2} & \lambda^{2L_2} & \lambda^{L_2+L_3} \\ \lambda^{L_1+L_3} & \lambda^{L_2+L_3} & \lambda^{2L_3} \end{pmatrix} \frac{v_2}{\Lambda_L}, \tag{24} \]

where \( v_2 \) is the electroweak scale vev of \( H_u \) (and \( v_1 \) is the vev of \( H_d \)). (2) The horizontal symmetry can play a role in the generation of the second term in which case it is not
necessarily the case that the exponent $q_{ij}$ is given by a naive inspection of the charges $L_i$ and $L_j$. The seesaw mechanism for the generation of the light neutrino masses is such an example, and we explore it further in the next section. In particular we show that with the appropriate assignment of heavy neutrino charges, $N_i$, one can enhance the $(m_\nu)_{33}$ entry of the light neutrino mass matrix.

V. NEUTRINO SEESAW

The neutrino seesaw mechanism is a popular way to explain the lightness of the observed neutrino masses. It follows naturally from the group theory structure of the Standard Model (SM). We have left-handed neutrino doublets in the SM we can supplement by adding a right-handed neutrino singlet. In fact we have three generations, so the resulting masses will involve mass matrices. The neutrino doublets can pair up with the singlets to form a Dirac mass matrix, $m_D$, coming from the breakdown of the electroweak symmetry. The neutrino singlets can pair up with themselves to form a $3 \times 3$ Majorana matrix, $M_N$. This mass matrix is expected to be superheavy; it is not generated by the electroweak symmetry breaking. The group theory dictates that the neutrino doublets cannot pair up with each other. So the result is a $6 \times 6$ mass matrix of the form

$$
\begin{pmatrix}
0 & m_D^T \\
m_D & M_N
\end{pmatrix},
$$

and upon diagonalization of the $3 \times 3$ light neutrino mass matrix is given by the seesaw formula

$$
m_\nu = m_D (M_N)^{-1} m_D^T.
$$

In the rest of this section we will explore the implications of assuming the mass matrix entries are governed by an Abelian horizontal symmetry. A simple result emerges for the case where the symmetry is $U(1)$, and some interesting enhancements (or suppressions) can occur when the symmetry is extended to $U(1) \times Z_2$ which have some phenomenological usefulness.

A. $U(1)$

Given lepton doublet charges $L_i$ and right-handed neutrino charges $N_i$ one has the following pattern for the neutrino Dirac mass matrix

$$
m_D \sim \begin{pmatrix}
\lambda^{L_1+N_1} & \lambda^{L_1+N_2} & \lambda^{L_1+N_3} \\
\lambda^{L_2+N_1} & \lambda^{L_2+N_2} & \lambda^{L_2+N_3} \\
\lambda^{L_3+N_1} & \lambda^{L_3+N_2} & \lambda^{L_3+N_3}
\end{pmatrix} v_2,
$$

and the following pattern for the Majorana mass matrix

$$
M_N \sim \begin{pmatrix}
\lambda^{2N_1} & \lambda^{N_1+N_2} & \lambda^{N_1+N_3} \\
\lambda^{N_1+N_2} & \lambda^{2N_2} & \lambda^{N_2+N_3} \\
\lambda^{N_1+N_3} & \lambda^{N_2+N_3} & \lambda^{2N_3}
\end{pmatrix} \Lambda_L.
$$
Then one obtains the same form for the light neutrino mass matrix via the see-saw formula
\[ m_\nu = m_D (M_N)^{-1} m_D^T \]
that was presented in the previous section in Eq. (24).

It is easy to see that if one wants to have a hierarchy in light neutrino masses \( m_\nu / m_\nu \sim \lambda^2 \), and large mixing in the (2,3) generation in the lepton sector, one cannot rely on a \( U(1) \) symmetry alone. From Eq. (24) one sees that \( L_2 = L_3 + 1 \) is required to give the proper mass ratio. This then immediately prevents the large mixing from arising in the neutrino sector, because the mixing is necessarily of order \( \lambda \). However we must still examine the diagonalization of the charged lepton matrix. Let the \( U(1) \) charges of the charged lepton singlets be \( E_i \), then the charged lepton matrix is
\[
m_{\ell \pm} \sim \begin{pmatrix}
\lambda^{L_1+E_1} & \lambda^{L_1+E_2} & \lambda^{L_1+E_3} \\
\lambda^{L_2+E_1} & \lambda^{L_2+E_2} & \lambda^{L_2+E_3} \\
\lambda^{L_3+E_1} & \lambda^{L_3+E_2} & \lambda^{L_3+E_3}
\end{pmatrix} v_1.
\]

The relevant mixing comes from the right hand side of this matrix, \( \lambda^{L_2+E_3}/\lambda^{L_3+E_3} \). So the mixing parameter here is also order \( \lambda \), since \( L_2 = L_3 + 1 \). Therefore the atmospheric neutrino mixing is necessarily of the order \( \sin \theta_{23}^\nu \sim \lambda \) in contradiction to the order one mixing observed (c.f. Eq. (3))

\section*{B. \( U(1) \times Z_2 \)}

We consider next the changes that can occur when a discrete symmetry is utilized. This avenue has been exploited to understand the large mixing in the atmospheric neutrino oscillations together with a hierarchy in the neutrino masses, \( m_\nu / m_\nu << 1 \). It can also lead to an enhancement in the production of a matter/antimatter symmetry in the early universe \( [13] \), if the heavy neutrinos are assumed to be decaying asymmetrically. In the rest of this section we explain in detail how to implement the discrete symmetry with the seesaw mechanism so that the \( m_\nu / m_\nu \sim \lambda^2 \), and large mixing results.

Take the following \( U(1) \times Z_2 \) charges for the lepton fields
\[
\begin{pmatrix}
\ell_{R1} \\
\ell_{R2} \\
\ell_{R3} \\
\ell_{L1} \\
\ell_{L2} \\
\ell_{L3} \\
\nu_{R1} \\
\nu_{R2} \\
\nu_{R3}
\end{pmatrix}
\begin{pmatrix}
(E_1, 0) \\
(E_2, 0) \\
(E_3, 0) \\
(L_1, 0) \\
(L_2, 0) \\
(L_3 - 1, 1) \\
(N_1, 0) \\
(N_2, 0) \\
(N_3 - 1, 1)
\end{pmatrix}
\]

Assume the symmetry breaking is characterized by the single expansion parameter \( \lambda \). The formulas given above for the heavy neutrino mass matrix, \( M_N \), the neutrino Dirac mass matrix, \( m_D \), and the resulting light neutrino mass matrix, \( m_\nu \) are modified. With the above assignments one finds that
\[
M_N \sim \begin{pmatrix}
\lambda^{2N_1} & \lambda^{N_1+N_2} & \lambda^{N_1+N_3} \\
\lambda^{N_1+N_2} & \lambda^{2N_2} & \lambda^{N_2+N_3} \\
\lambda^{N_1+N_3} & \lambda^{N_2+N_3} & \lambda^{2N_3-2}
\end{pmatrix} \Lambda_L,
\]
so that
\[
(M_N)^{-1} \sim \begin{pmatrix}
\lambda^{-2N_1} & \lambda^{-N_1-N_2} & \lambda^{-N_1-N_3+2} \\
\lambda^{-N_1-N_2} & \lambda^{-2N_2} & \lambda^{-N_2-N_3+2} \\
\lambda^{-N_1-N_3+2} & \lambda^{-N_2-N_3+2} & \lambda^{-2N_3+2}
\end{pmatrix} \Lambda_L^{-1}.
\]
So the effect of the discrete symmetry in our case is to enhance the 3-3 entry of the $M_N$ matrix, and thereby alter the results for the third row and the third column on the inverse matrix, $(M_N)^{-1}$. With our charge assignments, one also has an enhanced entry in the 3-3 component of the neutrino Dirac mass matrix,

$$m_D \sim \begin{pmatrix} 
\lambda^{L_1+N_1} & \lambda^{L_1+N_2} & \lambda^{L_1+N_3} \\
\lambda^{L_2+N_1} & \lambda^{L_2+N_2} & \lambda^{L_2+N_3} \\
\lambda^{L_3+N_1} & \lambda^{L_3+N_2} & \lambda^{L_3+N_3-2}
\end{pmatrix} v_2,$$

(33)

It is easy to see then that the light neutrino mass matrix $m_\nu$ is modified so that only the 3-3 entry is enhanced as desired,

$$m_\nu \sim \begin{pmatrix} 
\lambda^{2L_1} & \lambda^{L_1+L_2} & \lambda^{L_1+L_3} \\
\lambda^{L_1+L_2} & \lambda^{2L_2} & \lambda^{L_2+L_3} \\
\lambda^{L_1+L_3} & \lambda^{L_2+L_3} & \lambda^{2L_3-2}
\end{pmatrix} \frac{v_2^2}{\Lambda L}.$$

(34)

A phenomenologically viable solution has been presented in Ref. [4]: taking $L_1 = 3$, $L_2 = L_3 = 1$, $E_1 = 5$, $E_2 = 4$, and $E_3 = 2$ yields mass matrices of the form

$$m_\nu \sim \begin{pmatrix} 
\lambda^6 & \lambda^4 & \lambda^4 \\
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda^4 & \lambda^2 & 1
\end{pmatrix} v_2^2, \quad m_\ell \sim \begin{pmatrix} 
\lambda^8 & \lambda^7 & \lambda^5 \\
\lambda^6 & \lambda^5 & \lambda^3 \\
\lambda^6 & \lambda^5 & \lambda^3
\end{pmatrix} v_1,$$

(35)

which give the correct orders of magnitude for the small mixing angle MSW solution

$$\frac{\Delta m_{12}^2}{\Delta m_{23}^2} \sim \lambda^4, \quad \sin \theta_{12}^\nu \sim \lambda^2, \quad \sin \theta_{23}^\nu \sim \lambda^0,$$

(36)

and the correct orders of magnitude for the charged lepton mass ratios, Eq. (2). It also gives $\sin \theta_{13}^\nu \sim \lambda^2$. In fact it has been shown [4] that one can obtain a VO solution as well by a proper quantum number assignment to the lepton fields.

Without introducing more theoretical input (e.g. grand unified theories) to relate the quark and lepton charges, we cannot say more about which of the solutions is the correct one. We have shown here that the lepton sector phenomenology and the neutrino seesaw mechanism is consistent with a $U(1) \times Z_2$ symmetry, and in fact a hierarchy in the neutrino parameters $m_\nu^\mu << m_\nu^e$ requires the extra $Z_2$. Furthermore we have shown in detail how to implement the neutrino mass enhancement in the seesaw mechanism. In the next section, we show that the quark sector phenomenological constraints also admit a solution consistent with Eqs. (6) and (11), and with a $U(1) \times Z_2$ symmetry.

VI. A $U(1) \times Z_2$ SOLUTION

Our solution to the quark Yukawa matrices is based upon the Elwood-Irges-Ramond (EIR) solution [14] obtained with a $U(1)$ flavor symmetry. Here we show that one can impose a texture pattern by choosing quark fields to be charged under the new $Z_2$ symmetry. EIR obtained the Yukawa matrices

\begin{align*}
\end{align*}
The EIR solution was obtained by the $U(1)$ charges (after adding appropriate overall constants to each field which don’t affect the hierarchy pattern)

\[
U \sim \begin{pmatrix}
\lambda^8 & \lambda^5 & \lambda^3 \\
\lambda^7 & \lambda^4 & \lambda^2 \\
\lambda^5 & \lambda^2 & \lambda^0
\end{pmatrix}, \quad D \sim \begin{pmatrix}
\lambda^4 & \lambda^3 & \lambda^3 \\
\lambda^3 & \lambda^2 & \lambda^2 \\
\lambda^1 & \lambda^0 & \lambda^0
\end{pmatrix}.
\] (37)

The CKM elements can be expressed in terms of the Yukawa matrix elements,

\[
|V_{us}| = \left( \frac{d_{12}}{d_{22}} - \frac{d_{13}d_{32}}{d_{22}} \right) - \left( \frac{u_{12}}{u_{22}} - \frac{u_{13}u_{32}}{u_{22}} \right),
\] (38)

\[
|V_{cb}| = d_{23} + d_{22}d_{32}^* - u_{23},
\] (39)

\[
|V_{ub}| = (d_{13} + d_{12}d_{32}^* - u_{13}) - \left( \frac{u_{12}}{u_{22}} - \frac{u_{13}u_{32}}{u_{22}} \right) (d_{23} + d_{22}d_{32}^* - u_{23}),
\] (40)

\[
|V_{td}| = -(d_{13} + d_{12}d_{32}^* - u_{13}) + \left( \frac{d_{12}}{d_{22}} - \frac{d_{13}d_{32}}{d_{22}} \right) (d_{23} + d_{22}d_{32}^* - u_{23}),
\] (41)

where it is understood that there are possible phases associated with each term on the right hand sides of the equations. From Eq. (40), one sees that $|V_{ub}|$ is receiving contributions in the EIR solution of order $\lambda^3$ from both $u_{13}$ and $d_{13}$, as well as from the final term

\[
\left( \frac{u_{12}}{u_{22}} - \frac{u_{13}u_{32}}{u_{22}} \right) V_{cb}.
\] (42)

Then the experimental value for $|V_{ub}|$ must arise from a partial cancellation of these three contributions.

Tanimoto showed that a solution involving a $U(1) \times Z_2$ symmetry is not possible if the Cabibbo mixing, $|V_{us}|$, arises from the diagonalization of the down quark Yukawa matrix. The Cabibbo mixing in our scheme comes from diagonalizing the up quark matrix $U$. Our first attempt has the following assignments for the quantum numbers of the quark fields

\[
\begin{align*}
& i = 1 \quad 2 \quad 3 \\
& Q_L : (4, 0) \quad (2, 1) \quad (0, 1) \\
& \pi_R : (4, 0) \quad (1, 0) \quad (0, 1) \\
& d_R : (0, 0) \quad (0, 1) \quad (0, 1)
\end{align*}
\]

which is easily related to the EIR $U(1)$ assignment above by replacing the $Z_2$ charge with $+1$ for $Q_L$ and with $-1$ for $\pi_R$ and $d_R$. One can always substitute $0 \leftrightarrow 1$ for $Z_2$ charges without affecting the results. We obtain the following Yukawa matrices for the up and down quarks
\[
\begin{align*}
U &\sim \begin{pmatrix} \lambda^8 & \lambda^5 & \lambda^5 \\ \lambda^7 & \lambda^4 & \lambda^2 \\ \lambda^5 & \lambda^2 & \lambda^0 \end{pmatrix}, & D &\sim \begin{pmatrix} \lambda^4 & \lambda^5 & \lambda^5 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^1 & \lambda^0 & \lambda^0 \end{pmatrix}.
\end{align*}
\tag{43}
\]

The mass matrices can be obtained by multiplying these Yukawa matrices by the relevant vev \((v_1 \text{ for } D \text{ and } v_2 \text{ for } U)\). The intergenerational hierarchy, \(m_b/m_t \sim \lambda^3\), is then accounted for either by \(\tan \beta = v_2/v_1\), and/or by increasing the \(U(1)\) charges for \(d_R\). It is straightforward to check that these matrices have the texture pattern

\[
\begin{align*}
U &\sim \begin{pmatrix} X & X & 0 \\ X & X & X \\ 0 & X & X \end{pmatrix}, & D &\sim \begin{pmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{pmatrix}.
\end{align*}
\tag{44}
\]

We refer to this type of suppression as a 3-texture zero solution, since three entries are suppressed by the charge assignments in the discrete symmetry. Referring back to Eq. (44), one can see that the dominant contribution to \(|V_{ub}|\) comes only from the third term and is of order \(\lambda^3\). This solution was motivated by the desire\(^4\) to derive that \(|V_{ub}|\) is proportional to \(|V_{cb}|\); this can be achieved if the first term in parentheses in Eq. (40) is suppressed, and this requires a texture zero in the \((1,3)\) position of both \(U\) and \(D\). Consequently one only has a contribution from the final term \((u_{13} \sim \lambda^5\) and \(d_{13} \sim \lambda^5\)). But then \(|V_{td}| \sim \lambda^5\) is inconsistent with Eq. (40), and unitarity (Eq. (10)) requires the relation

\[
|V_{us}| \simeq \frac{|V_{ub}|}{|V_{cb}|},
\tag{45}
\]

which is also not supported by the experimental data, Eq. (5). Clearly the 3-texture zero pattern in Eq. (44) is too restrictive.

We can relax the problematic constraint, Eq. (45), by removing the texture zero in the \((1,3)\) position of the up quark matrix. Consider the following texture zero pattern

\[
\begin{align*}
U &\sim \begin{pmatrix} 0 & X & X \\ X & X & X \\ X & X & X \end{pmatrix}, & D &\sim \begin{pmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{pmatrix}.
\end{align*}
\tag{46}
\]

from

\[
\begin{align*}
i & = 1 \quad 2 \quad 3 \\
Q_L : & \ (4,0) \quad (2,1) \quad (0,1) \\
\pi_R : & \ (6,1) \quad (2,0) \quad (0,0) \\
\overline{d}_R : & \ (0,0) \quad (0,1) \quad (0,1)
\end{align*}
\]

\(^4\)Since the experimental data for \(|V_{cb}| = a\lambda^2\) already requires the order one coefficient to be less than one, \(a \simeq 0.6\) \(\mathbb{[5]}\), it is more likely that this final term (which includes another order one coefficient we will call \(b\)) will give agreement with the experimental data, \(|V_{ub}| = ab\lambda^3\) with \(b\) somewhat smaller than one. Furthermore it is then consistent with the experimental data on \(|V_{ub}/V_{cb}| = 0.08 \pm 0.02 = b\lambda\) for \(b \simeq 0.4\). So the combination of order one coefficients satisfy the required relation, \(ab \simeq \lambda\).
We obtain the following Yukawa matrices for the up and down quarks

\[
U \sim \begin{pmatrix}
\lambda^{11} & \lambda^6 & \lambda^4 \\
\lambda^8 & \lambda^5 & \lambda^3 \\
\lambda^6 & \lambda^3 & \lambda^1
\end{pmatrix}, \quad D \sim \begin{pmatrix}
\lambda^4 & \lambda^5 & \lambda^2 \\
\lambda^3 & \lambda^2 & \lambda^0 \\
\lambda^1 & \lambda^0 & \lambda^0
\end{pmatrix}.
\] (47)

As is clear from the texture pattern in Eq. (46), the Cabibbo angle is arising in the up quark matrix \(U\). However one avoids the uncomfortable relation \(|V_{us}| \sim \sqrt{m_u/m_c}\) because the matrix is not symmetric. All phenomenological constraints are satisfied by this solution with \(|V_{ub}|\) receiving contributions of order \(\lambda^3\) from only the \(u_{13}\) term and the last term in Eq. (40). The matrices in Eq. (33) and (47) show that nontrivial \(Z_2\) charges can be assigned to the quark and lepton fields, and that all phenomenological requirements can be met.

VII. CONCLUSION

We have shown that one can explain all the masses and mixings of the quarks and leptons with a \(U(1) \times Z_2\) symmetry. The phenomenological requirements can be met if the Cabibbo mixing in the two light generations is generated in the up quark mass matrix. This runs counter to the bias of assuming that the Cabibbo mixing is coming from the down quark mass matrix so that the relation \(|V_{us}| \sim \sqrt{m_d/m_s}\) is obtained. This prejudice should be abandoned in the context of these Abelian flavor symmetries, because the resulting Yukawa matrices need not be symmetric. and \(m_u/m_c \sim \lambda^4\) is a reasonable hierarchy even with the leading contribution to the Cabibbo angle \(|V_{us}|\) coming from the up quark matrix.

The advantages of employing the \(U(1) \times Z_2\) as a flavor symmetry are the following:

- One can understand a mass hierarchy \(m_{\nu_u}/m_{\nu_c} \sim \lambda^2\) and large atmospheric neutrino mixing \(\sin \theta_{23}' \sim 1\), without invoking unnatural cancellations.
- The discrete symmetry can be implemented consistently with the neutrino seesaw mechanism to give the necessary neutrino mass hierarchy.
- The source for CKM mixing angles in terms of the original parameters in the Yukawa matrices is reduced via the presence of texture zeros. For example \(|V_{ub}|\) arises from a single contribution in Eq. (37), since the other contributions are suppressed by a flavor suppression. The EIR model has the Cabibbo angle, \(|V_{us}| \sim \lambda\), arising at leading order from both the up and down quark matrices (c.f. Eqs. (38) and (37)). Our solution suppresses the contribution from the down quark Yukawa matrix, and thus the leading contribution arises entirely in the up quark matrix.
- The discrete \(Z_2\) symmetry can enhance the lepton asymmetry generated by the decay of heavy right-handed neutrinos \([13]\). This can be exploited to straightforwardly explain the baryon asymmetry of the Universe.

A discrete flavor symmetry offers some attractive features for generating phenomenologically reasonable models. We leave to future work the question of whether these models can be embedded in a larger theoretical structure.
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