Abstract

A rational triangle has rational edge-lengths and area; a rational tetrahedron has rational faces and volume; either is Heronian when its edge-lengths are integer, and proper when its content is nonzero.

A variant proof is given, via complex number GCD, of the previously known result that any Heronian triangle may be embedded in the Cartesian lattice \( \mathbb{Z}^2 \); it is then shown that, for a proper triangle, such an embedding is unique modulo lattice isometry; finally the method is extended via quaternion GCD to tetrahedra in \( \mathbb{Z}^3 \), where uniqueness no longer obtains, and embeddings also exist which are unobtainable by this construction.

The requisite complex and quaternionic number theoretic background is summarised beforehand. Subsequent sections engage with subsidiary implementation issues: initial rational embedding, canonical reduction, exhaustive search for embeddings additional to those yielded via GCD; and illustrative numerical examples are provided.

A counter-example shows that this approach must fail in higher dimensional space. Finally alternative approaches by other authors are summarised.

Keywords: triangle, tetrahedron, lattice embedding, axial pose, Cayley-Menger determinant, quaternion arithmetic, Euclidean algorithm

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1 Introduction: Rational and Heronian Simplices

A rational triangle, tetrahedron, or simplex in \( n \)-space \( \mathbb{R}^n \) is a polytope with \( n + 1 \) vertices, and rational edge lengths, face areas, solid volumes, etc. When the edges have integer length, the simplex is Heronian; when the GCD (greatest common divisor) of those lengths equals unity, it is primitive (Heronian). When its \( n \)-dimensional content is nonzero, it is proper — we generally have only proper cases in mind, despite which much of the following may easily be seen to apply more widely. Obviously any rational simplex may be dilated to a Heronian simplex.

Our purpose here is to show that for \( n = 2, 3 \) the simplex in \( \mathbb{R}^n \) may be posed first axially in \( \mathbb{Q}^n \) (which is elementary), and finally embedded in the lattice \( \mathbb{Z}^n \); that is, congruently so that its vertices have Cartesian coordinates not just rational, but integer. For triangles the result is only about 10 years old [13]; we propose a construction based on complex-number GCD, and related to the approach of [5].

For tetrahedra the paradigm must be extended to embrace quaternion GCD, where commutativity and unique factorisation are no longer available: the correspondence
between the two arguments is illuminating, and we emphasise the analogies and contrasts by presenting the main theorems in sections 4, 8 in copycat format. The level of detail in their proofs might be felt excessive, to the point of obscuring the wood for the trees: it was motivated by a desire to avoid elementary blunders which have compromised numerous earlier versions.

Incidentally, an obscure but significant step in the argument involves first establishing that the preliminary rational pose (on which GCD is to act) has denominators with prime factors \( p \equiv 1 \pmod{4} \) only.

It is natural to enquire how these results might generalise to higher dimensions. There are currently several obstructions: the quaternion representation employed cannot be extended further; even if it could, the theorem in the form given here would no longer hold; and in any case, no examples of Heronian pentatopes are known to which any such theorem might be applicable.

Despite having spent much time contemplating this topic, this author still finds the connection revealed, between the number theoretic concept of GCD and the geometric concept of lattice embedding, both surprising and mysterious.

The contribution to this project from Warren D. Smith, in the form of copious literature references, enthusiastic criticism, and inexhaustible energy, is gratefully acknowledged; also some motivating ideas, particularly behind the proof of the embedding theorem, originated with Michael Reid.

2 Edges, Triangles, Tetrahedra and their Content

The Hero formula giving the area \( d \) of a triangle with edge lengths \( u, v, w \) as

\[
(4d)^2 = (u + v + w)(u + v - w)(u - v + w)(-u + v + w);
\]  

(1)

is case \( n = 2 \) of the Cayley-Menger formula for the content of a simplex in \( \mathbb{R}^n \), which deserves to be better known [12]:

\[
(4d)^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u^2 & v^2 \\ 1 & u^2 & 0 & w^2 \\ 1 & v^2 & w^2 & 0 \end{vmatrix}.
\]  

(2)

For \( n = 3 \) the volume \( e \) of the tetrahedron with edge lengths \( u, \ldots, z \) is given by

\[
2(12e)^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u^2 & v^2 \\ 1 & u^2 & 0 & w^2 \\ 1 & v^2 & w^2 & 0 \end{vmatrix};
\]  

(3)

Since these coefficients turn out to be even, \( (12e)^2 \) is again a homogeneous symmetric polynomial with integer coefficients, but now cubic in the edge squares.

To specify a particular free tetrahedron, a correspondence must be established between vertices and edges: denoting vertices \( P, Q, R, S \), edge-lengths will be specified by a hexad of integers presented in sequential order

\([QP, RP, RQ, SP, SQ, SR] \).

The amount of data output can be reduced by restriction to primitive cases, in which the GCD (greatest common divisor) of the edges equals unity; also by reduction
to canonical form, with vertices permuted so that the vector of edge-lengths represents the largest possible ‘number’, where individual components are regarded as ‘digits’. [A humbling exercise in algorithmic design is to sort such a hexad into canonical form efficiently, without first generating every vertex permutation: our best effort achieved a maximum 22 transpositions. And we are almost sure it actually works . . .]

Methods for constructing Heronian simplices are discussed in detail elsewhere [9], [2], [11].

Rational simplices are of interest as test-cases since many of their subsidiary elements are likewise rational — for triangles, the edges, altitudes, medians, centroid, orthocentre, incentre, circumcentre, etc. Less obvious is the rationality of the angular quantities

\[
\cot \frac{p}{2} = \frac{(s - w)s}{d}
\]

e tc., where \( p \) denotes the angle at vertex \( P \). Conway christens this property ”geodetic”: it implies that all the trigonometric functions of the angles are also rational.

3 Representation of \( \mathbb{R}^2 \) and \( \mathbb{Z}^2 \) via Complex Numbers

With \( x, y \) real in \( \mathbb{R} \), denote a complex number in \( \mathbb{C} = \mathbb{R}[\iota] \) by

\[
Z = x + \iota y,
\]

its norm by

\[
\|Z\| \equiv x^2 + y^2,
\]

its conjugate by

\[
\overline{Z} \equiv x - \iota y.
\]

Any \( P = x + \iota y \) represents a finite point with Cartesian coordinate \((x, y)\). The representation is inhomogeneous: scalar multiples of \( P \) represent distinct points. The origin point is represented by \( X = 0 \). The distance squared between points \( P, Q \) equals \( \|P - Q\|^2 \).

Any unit \( U = \cos t + \iota \sin t \) represents rotation through angle \( t \) around the origin; non-unit \( U \) with \( \|U\| \neq 1 \) do not represent rotations. The identity rotation is represented by \( U = 1 \). Composition of rotations is represented by commutative complex product.

The transformation

\[
P \to UP \quad \text{where} \quad U = \cos t + \iota \sin t
\]

rotates the point \( P = x + \iota y \) through angle \( t \).

A (Gaussian) integer complex number lies in \( \mathbb{Z}[\iota] \), its components rational integers. A complex prime is either 2, a rational prime \( p \equiv -1 \) (mod 4), or of form \( x + \iota y \) where \( x^2 + y^2 = p \) with \( p \equiv 1 \) (mod 4). Factorisation is essentially unique modulo association — product with unit integers \( \pm 1, \pm \iota \).

Complex integers are a Euclidean domain: given \( X, Y \in \mathbb{Z}[\iota] \), their greatest common divisor \( \gcd(X, Y) \) is unique modulo association. Algorithmic details of GCD are discussed in section [6]. For further detail see [4] chap. 2.

**Assertion 1.** If \( Z \in \mathbb{Z}[\iota] \) is primitive, then \( \gcd(Z, \overline{Z}) = 1 \) (or associate).
Proof.
\[
\text{GCD}(x + i y, x - i y) = \text{GCD}(x + i y, 2x) = \text{GCD}(x + i y, 2i y),
\]
which divides the complex prime \(2 = 2 \text{GCD}(x, y)\); since one of \(x, y\) is odd, the GCD cannot be 2, so it must be (an associate of) unity.

Unit integers represent rotations through \(k\pi/2\) around the origin. The full group of lattice isometries is generated by these 4 rotations, reflections in axes (complex conjugacy etc), and translations (integer addition).

4 Lattice Embedding Theorem in \(\mathbb{R}^2\)

Assertion 2. Suppose all but one of a set \(\{Q\}\) of \(k + 1\) points already embedded in \(\mathbb{Z}^2\); the other point \(R\) rational in \(\mathbb{Q}^2\) with common denominator \(r\); and the squares of all their point-to-point distances in \(\mathbb{Z}\). Then rotation via complex product \(Q \rightarrow \overline{X}^2Q\) where \(X = \text{GCD}(R, r)\) embeds the entire set congruently in \(\mathbb{Z}^2\).

Given the first assertion, translating some point of the set to the origin and applying induction on the number \(k\) of points, immediately

Assertion 3. If a set of points lies in \(\mathbb{Q}^2\), and their squared distances lie in \(\mathbb{Z}\), then the entire set may be embedded congruently in \(\mathbb{Z}^2\).

And taking the set to be the vertices of a Heronian triangle in some rational pose (section 7), with edge-lengths themselves integer,

Assertion 4. Any Heronian triangle may be embedded congruently in \(\mathbb{Z}^2\).

Proof. of assertion 2 Let the points \(P, Q, \ldots\) lie in \(\mathbb{Z}^2\), except for \(R\) in \(\mathbb{Q}^2\) with Cartesian coordinate \((x/r, y/r)\) where \(r = \text{LCD}(x, y)\) (least common denominator). Represented by complex numbers, all are integer except for \(R\); now dilating the entire set by \(r\) yields also \(R'' = rR\) integer.

The first lattice point \(P\) of the set may be translated to the origin \(P = 0\); now if \(k = 1\) the result becomes trivial. Otherwise since all distances squared \(\|P - Q\|^2\) are integer, all dilated points have magnitude divisible by \(r^2\). Set \(X = \text{GCD}(R'', r)\), for which

\[
r = X\overline{X} = \|X\| = \overline{X}X.
\]
The rational point \(R\) is transformed by \(X\) to the lattice:

\[
X^2 \mid r^2 \mid \|R''\| = R'' \overline{R''},
\]
and by assertion 1 \(R'', \overline{R''}\) have common factor \(\text{GCD}(x, y)\), which is scalar and coprime to \(r\). Therefore \(X^2|R''\); and \(R\) is transformed to

\[
\overline{X}^2R''/r^2 = \overline{X}^2X^2R'/r^2 = R'
\]
for some integer \(R' = \overline{X}^2R/r \in \mathbb{Z}^2\).

Also each lattice point \(Q\) is transformed by \(X\) to the lattice:

\[
r^2 \mid \|rQ - R''\| = (rQ - R'')(rQ - R'')
\]
which is transformed to the equal

\[
\|\overline{X}^2rQ - \overline{X}^2R''\|^2 = (\overline{X}^2rQ - \overline{X}^2R'')(\overline{X}^2rQ - \overline{X}^2R'')/r^2;
\]

4
hence
\[ r^4 \mid (X^2rQ - X^2R'')(X^2rQ - X^2R''). \]
Now for \( r \in \mathbb{Z}, Y \in \mathbb{Z}[i] \) we have \( r \mid Y \) iff \( r \mid Y; \) so via unique factorisation
\[ r^2 \mid X^2rQ - X^2R''; \quad \text{also} \quad r^2 \mid X^2R'', \]
and adding, finally
\[ r^2 \mid rX^2Q. \]
Therefore \( Q \) is transformed to \( Q' = \frac{X^2R}{r} \in \mathbb{Z}^2 \).

The construction is in fact reversible, although this seems difficult to establish directly. The following argument relies heavily on unique factorisation.

**Assertion 5.** Lattice embedding of a proper Heronian triangle is unique modulo lattice isometry (association, integer addition, conjugacy).

**Proof.** Suppose \( P, Q, R \) and \( P, S, T \) are lattice embeddings into \( \mathbb{Z}[i] \) from vertices of the same free triangle, translated and reflected to share the origin \( P = 0 \) and to have anticlockwise sense.

Let \( Q' = Q/GCD(Q, S), S' = S/GCD(Q, S) \), and set \( U = S/Q = S'/Q' \); the isometry \( U \) rotates \( P, Q, R \) to \( P, S, T \). Since \( \|Q\| = \|S\| \) we have \( \|Q'\| = \|S'\| = q \) say.

If \( q = 1 \) then \( U \in \mathbb{Z}[i] \) is a lattice isometry, so the embeddings are essentially equivalent. Otherwise, each rational prime factor of \( q \) splits into conjugate primes; so \( Q' = \frac{S'}{q} \) and \( U = S'/Q' = S'^2/q, \) and similarly \( U = T'^2/q. \) But now \( S' = \pm T' \), so the triangle was after all improper. \( \square \)

## 5 Representation of \( \mathbb{R}^3 \) and \( \mathbb{Z}^3 \) via Quaternions

With \( s, p, q, r \) real in \( \mathbb{R}, \)
\[ X = s + pi + qj + rk \]
denotes a quaternion in \( \mathbb{H} = \mathbb{R}[i, j, k]; \)
\[ \|X\| \equiv s^2 + p^2 + q^2 + r^2 \]
denotes its norm;
\[ \overline{X} \equiv s - pi - qj - rk \]
denotes its conjugate.

In general \( X \neq 0 \) represents the finite point at distances \( p/s, q/s, r/s \) from the Cartesian coordinate planes. The representation is homogeneous: any nonzero scalar multiple of \( X \) represents the same point. The origin point is represented by \( X = 1 \) (or any nonzero scalar). A pure quaternion with scalar component \( s = 0 \) represents a point at projective infinity. The distance squared between points \( P, Q \) equals \( \|P - Q\| \) only when both have been co-normalised, with scalar parts unity.

Dually \( X \neq 0 \) represents rotation through angle \( t \) around a line meeting the origin, where \( s = \cot t/2 \) and the direction cosines (of angles made by the line with the coordinate planes) are \( p, q, r. \) The representation is homogeneous: any nonzero scalar multiple of \( X \) represents the same rotation. The identity rotation is represented by \( X = 1 \) (or any nonzero scalar). Composition of rotations is represented by quaternion
product, which is in general non-commutative, unless its arguments differ only in their scalar component.

In vector terms, given 3-vector \( u \), unit 3-vector \( v \), and angle \( t \), the transformation (group-theoretic conjugation)

\[ P \rightarrow XPX \quad \text{where} \quad X = \cot t/2 + v \]

rotates the point \( P = 1 + u \) through angle \( t \) about axis \( v \).

Interpreted as rotations, quaternions are analogous to conventional polar or contravariant vectors; as points, to axial or covariant vectors: and this duality accounts for their varying normalisations. Again, transformation by a rotation is the only situation in which it is legitimate to multiply a point.

Symmetries of the Cartesian lattice are generated by translations (integer additions), reflection in the origin (conjugacy), and 24 rotational fixing the unit cube via transformations:

\[
\begin{align*}
1 & \quad \text{(identity) } \times 1; \\
i, j, k & \quad \text{ (} \pi \text{ around axis) } \times 3; \\
(1 \pm i)/\sqrt{2} & \quad \text{ (} \pi/2 \text{ around axis) } \times 6; \\
(i \pm j)/\sqrt{2} & \quad \text{ (} \pi/2 \text{ around edge perpendicular) } \times 6; \\
(1 \pm i \pm j \pm k)/2 & \quad \text{ (} 2\pi/3 \text{ around diagonal) } \times 8. \\
\end{align*}
\]

A (Lipschitzian) integer quaternion \( \in \mathbb{Z}[i,j,k] \) has rational integer components. It is primitive when the GCD of its components equals unity. Rational points are representable by integer quaternions, lattice points by integers with unit scalar component; and conversely.

Quaternion number theory is complicated by non-commutativity of multiplication and non-unique factorisation. A prime \( X \in \mathbb{Z}[i,j,k] \) has prime norm \( \|X\| = p \), while \( X \notin \mathbb{Z} \) itself; an integer quaternion may factorise into primes in several essentially distinct ways. Factorisation is discussed in detail in \[4\] chap. 5.

Integer quaternions are almost a Euclidean domain: via \[5\] Theorem 3, provided at least one of \( Y, Z \in \mathbb{Z}[i,j,k] \) has odd norm, their left- and right-hand greatest common divisors

\[ \text{GCD}_L(Y,Z), \quad \text{GCD}_R(Y,Z) \]

are guaranteed to exist — though in general distinct — and are unique modulo association — product with units \( \pm 1, \pm i, \pm j, \pm k \).

[However provided \( q \mid Q \) and \( q \in \mathbb{Z} \), then via \[6\] Theorem 2 there do still exist integers \( X,W \) such that \( q = X\overline{X} \) and \( Q = WX \); but only when \( q \) is odd can \( X = \text{GCD}_L(Q,q) \) be guaranteed unique. Analogous observations apply on the right-hand side.]

### 6 Algorithms for GCD

Define rounding by \( \langle x \rangle \equiv \lfloor x + 1/2 \rfloor \in \mathbb{Z} \), the integer nearest to \( x \in \mathbb{R} \); and for quaternions

\[ \langle X \rangle = \langle s + p i + q j + r k \rangle = \langle s \rangle + \langle p \rangle i + \langle q \rangle j + \langle r \rangle k; \]
then left and right remainder functions are defined by

\[ X \mod_L Y \equiv X - Y(\langle Y X/\|Y\rangle) \]
\[ X \mod_R Y \equiv X - (X\langle Y/\|Y\rangle)Y \]

where \( Y/\|Y\| = Y^{-1} \) equals the commutative multiplicative inverse of \( Y \).

Now the Euclidean algorithm can be employed, enhanced to catch those cases where no GCD exists:

\[
\text{GCD}_L(Y, Z) \leftarrow \text{while } Z \neq 0 \text{ do} \\
\quad X \leftarrow Y; \; Y \leftarrow Z; \; Z \leftarrow X \mod_L Y; \\
\quad \text{if } \|Z\| \geq \|Y\| \text{ then ABORT } \text{fi od;} \\
\text{return } Y;
\]

For GCD\(_R(Y, Z)\), substitute mod\(_R\) for mod\(_L\) above.

This procedure is guaranteed to terminate only provided at least one of \( \|Y\| \) or \( \|Z\| \) is odd, see [9] Theorem 3. For example when \( Y = 2, Z = 1 + i + j + k \) (both with norm 4), the distinct primes \( 1 + i, 1 + j, 1 + k \) all divide both, but no product of all three (with norm 8) can possibly divide either; so no common divisor \( X \) can be divisible by every divisor of both.

GCD in \( \mathbb{C} \) follows lines similar to GCD in \( \mathbb{H} \), though simplified by commutativity, just two components, and existence for all arguments.

7 Rational Embedding of Heronian Tetrahedra

**Assertion 6.** In order that a simplex in \( \mathbb{R}^n \) be rationally embeddable in \( \mathbb{Q}^n \), it suffices that the edge-lengths have rational squares, and there exists some flag (incrementally ascending sequence of vertex subsets) all having rational content.

The result extends in a natural fashion to polytopes with more than \( n + 1 \) vertices, see [3] Prop. 3.3; here we are principally concerned only with the case \( n = 3 \).

**Proof.** Given an arbitrary free tetrahedron with edge lengths \( u, v, w, x, y, z \) to be posed in \( \mathbb{Q}^3 \), its vertices may be posed in axial subspaces of increasing dimension thus:

\[ P = 1, \quad Q = 1 + Q_1 i, \quad R = 1 + R_1 i + R_2 j, \quad S = 1 + S_1 i + S_2 j + S_3 k. \]

Elimination of the 6 equations \( \|P - Q\| = u \) etc. displays the vertex coordinates as rational functions of edge length \( u \) of \( PQ \), face area \( d \) of \( PQR \), volume \( e \) of \( PQRS \), and the squares of other edge lengths:

\[
Q_1 = u, \\
R_1 = (v^2 - w^2 + Q_1^2)/2Q_1, \\
R_2 = 2d/u, \\
S_1 = (x^2 - y^2 + Q_1^2)/2Q_1, \\
S_2 = (x^2 - z^2 + R_1^2 + R_2^2 - 2S_1R_1)/2R_2, \\
S_3 = 3e/d.
\]
Each vertex will then be rescaled homogeneously as a primitive integer quaternion, with scalar component the LCD of the reduced rational coordinate components, for submission to the lattice embedding procedure in section 8. [The expressions are only valid for length $u$ and area $d$ nonzero: a more elaborate algorithm is required to cope with all improper cases.]

The denominators of these rationals possess an obscure property, unexpectedly necessary to permit the use of GCD in construction of lattice embedding transformations. 

**Assertion 7.** Each coordinate component of each vertex of a Heronian tetrahedron in axial pose has denominator the product of primes $p \equiv 1 (\mod 4)$. 

**Proof.** Borrowing a technique from [3], the coordinate components are expressed as homogeneous rationals in terms of cotangent half-angles: at triangular vertices in $\mathbb{R}^2$, and at dihedral edges in $\mathbb{R}^3$. These cotangents are rationals $f'/f'' \in \mathbb{Q}$ with $\text{GCD}(f', f'') = 1$; when substituted into the coordinates, they give rise to denominators with factors of form $f'^2 + f''^2$.

A theorem from elementary number theory due to Euler [7] guarantees that $f'^2 + f''^2$ factorises into primes $p = 4k + 1$, possibly with a single extra factor 2 when both $f'$, $f''$ are odd. It then remains only to check that these surplus 2's cancel with the numerator in each component.

So consider the $\mathbb{R}^3$ vertex $S$ above, which subsumes all other cases. Denote by $k = 2c/u$ the height of face $PSQ$; at vertex $P$ we find $g$ in terms of $S$ etc.

$$
\cos P = S_1/x, \quad \sin P = k/x, \\
g = \cot P/2 = (1 + \cos P)/\sin P = (x + S_1)/k;
$$

and similarly $h$ at dihedral $U$ along edge $PQ$

$$
\cos U = S_2/k, \quad \sin U = S_3/k, \\
h = \cot U/2 = (1 + \cos U)/\sin U = (2c/u + S_2)/S_3;
$$

then inverting to get $S$ in terms of $g, h$ etc.

$$
S = 1 + x \cos P i + x \sin P \cos U j + x \sin P \sin U k
$$

$$
= 1 + (g^2 - 1)/(g^2 + 1) \cdot x i \\
+ 2g(h^2 - 1)/(g^2 + 1)(h^2 + 1) \cdot x j \\
+ 4gh/(g^2 + 1)(h^2 + 1) \cdot x k
$$

$$
= 1 + (g'^2 - g''^2)/(g'^2 + g''^2) \cdot x i \\
+ 2g'g''(h'^2 - h''^2)/(g'^2 + g''^2)(h'^2 + h''^2) \cdot x j \\
+ 4g'g''h'h''/(g'^2 + g''^2)(h'^2 + h''^2) \cdot x k,
$$

where $g = g'/g''$ and $h = h'/h''$ are positive reduced fractions. If it is even, $(f'^2 + f''^2) \equiv 2 (\mod 4)$ and $(f'^2 - f''^2)$ is also even; so all 2's cancel from the denominators as required. $\Box$

8 Lattice Embedding Theorem in $\mathbb{R}^3$

**Assertion 8.** Suppose all but one of a set $\{Q\}$ of $k + 1$ points already embedded in $\mathbb{Z}^3$; the other point $S$ rational in $\mathbb{Q}^3$ with common denominator $s$; and the squares of all their point-to-point distances in $\mathbb{Z}$. Then rotation via quaternion transformation $Q \rightarrow X Q X$ where $X = \text{GCD}_L(S, s)$ embeds the entire set congruently in $\mathbb{Z}^3$. 

8
Given the first assertion, translating some point of the set to the origin, and applying induction on the number $k$ of points, immediately

**Assertion 9.** If a set of points lies in $\mathbb{Q}^3$, and their squared distances lie in $\mathbb{Z}$, then the entire set may be embedded congruently in $\mathbb{Z}^3$.

And taking the set to be the vertices of a Heronian tetrahedron in some rational pose (section 7) with edge-lengths themselves integer,

**Assertion 10.** Any Heronian tetrahedron may be embedded congruently in $\mathbb{Z}^3$.

**Proof.** of assertion 9. Let the points $P, Q, \ldots$ lie in $\mathbb{Z}^3$, except for $S$ in $\mathbb{Q}^3$ with Cartesian coordinate $(x/s, y/s, z/s)$ where $s = \text{LCD}(x, y, z)$. Represented by integer quaternions, every point is primitive; and each has scalar component unity, excepting $S = s + xi + yj + zk$. We suppose that $s$ is odd, to be justified under assertion 7.

The first lattice point $P$ of the set may be translated to the origin $P = 1$; now if $k = 1$ the result becomes trivial. Otherwise since all distances squared are integer, $s^2 \mid \|P - Q\|$, so $s^2 \mid \|S - sP\|$. By section 5 there exists $X = \text{GCD}_L(S, s)$, for which $s = XX = \|X\| = XXT$.

Now the rational point $S$ is transformed by $X$ to the lattice:

$$X = \text{GCD}_L(S, s) = \text{GCD}_L(2s - S, s) = \text{GCD}_L(\overline{S}, s) = \text{GCD}_R(S, s),$$

so

$$\text{GCD}_L(S, s) = X \quad \text{and} \quad \text{GCD}_R(S, s) = \overline{X}. $$

Since $\|S\| \geq \|s\|$, the prime factorisation of $S$ is no shorter than that of $s$; the extra factors $S'$ must occur between $X$ and $\overline{X}$ (for some factorisation), so $S = XS\overline{X}$. That is, $S$ is transformed to

$$\overline{X}SX = \overline{X}XS\overline{X} = s^2S',$$

where $S'$ is integer. The scalar component of each side equals $s^2$, so $S'$ has unit scalar, and $S' \in \mathbb{Z}^3$.

Also each lattice point $Q$ is transformed by $X$ to the lattice:

Note that $sQ - S = -(sQ - S)$ is pure, so

$$s^2 \mid \|sQ - S\| = -(sQ - S)^2$$

which is transformed to (scaled by $s^2$)

$$s^4 \mid \|\overline{X}sQX - \overline{X}SX\| = -(\overline{X}sQX - \overline{X}SX)^2;$$

so

$$s^2 \mid \overline{X}sQX - \overline{X}SX, \quad s^2 \mid \overline{X}SX, \quad s^2 \mid s\overline{X}QX$$

via commutativity of scalars, and adding. That is, $Q$ is transformed to $\overline{X}QX = sQ'$, where $Q'$ is integer with unit scalar as before, and $Q' \in \mathbb{Z}^3$. □

In contrast to dimension 2, permuting vertices may yield embeddings which are essentially distinct, in the sense defined in section 4. Consider the penultimate stage of the embedding procedure, where $P, Q, R \in \mathbb{Z}^3$ and only $S$ remains to be located: since $P, Q, R$ are essentially unique via assertion 9 at most 4 distinct configurations are possible, depending upon which vertex was initially specified as $S$. Therefore
Assertion 11. The number of essentially distinct embeddings of a Heronian tetrahedron into $\mathbb{Z}^3$, as constructed via the GCD procedure, is at most 4.

However, it will emerge in section [2] that there are embeddings which are not constructible in this fashion. Whether there is a bound on the total number of distinct embeddings is not known: modulo lattice isometry, it would have to be at least 36.

Finally, complex numbers turn out to have been something of a waste of space all along:

Assertion 12. Embedding in $\mathbb{Z}^2$ is a special case of embedding in $\mathbb{Z}^3$.

Proof. The claim is equivalent to the following. Given a rational point $R = 1 + x/r i + y/r j$ in $\mathbb{Q}^2$, the quaternion GCD rotates it to some lattice point remaining in $\mathbb{Z}^2$; that is, $X = \text{GCD}_L(r + x i + y j, r) = f + g k$ for some $f, g \in \mathbb{Z}$. To show this,

$$\text{GCD}_L(r + x i + y j, r) = \text{GCD}_L(r, (r \mod r) i + (y \mod r) j)$$

remainder modulo $r$;

$$= \text{GCD}_L(r, x \mod r + (y \mod r) k)$$

via right product of second argument by $-i$ (units are coprime);

$$= \text{GCD}(r, x \mod r + \iota y \mod r)$$

setting $k \equiv \iota$, its polar complex equivalent;

$$= f + \iota g,$$

say. \qed

Notice how (polar) rotations $\cos t + \iota \sin t \in \mathbb{C}$ correspond to $\cos t + \sin t \cdot k \in \mathbb{H}$; whereas (axial) points $x + \iota y \in \mathbb{C}$ correspond to $1 + x i + y j \in \mathbb{H}$.

9 Essentially Distinct Embeddings; Canonical Reduction

Here we consider only tetrahedra in $\mathbb{R}^3$; generalisation to simplices in $\mathbb{R}^n$ is a matter of routine, were it ever required.

While there are infinitely many lattice embeddings of any Heronian tetrahedron, factoring out translations by fixing one vertex at the origin reduces this to a finite number. [For technical reasons however, it proves more convenient instead to further translate so that all coordinates are positive, with each component zero for at least one vertex.]

Many of these fall into sets of (in general) 48 lattice isomorphs of one another: to eliminate these trivial variations from consideration, define the weak canonical form to be the earliest isomorph, modulo lattice symmetries, in numerical order — where coordinate components taken in the natural ordering are regarded as individual ‘digits’ in a 12-digit ‘number’.

However even with this refinement embeddings may proliferate, particularly (and paradoxically) for symmetric cases. They may be further reduced by a potential factor 4 via permutation of vertices; so define the strong canonical form to be the earliest
weak form modulo vertex permutation. [The relationship between input edge lengths and vertices is disrupted, but can straightforwardly be recovered when required.]

The symmetry types occurring in Heronian tetrahedra are of interest in this connection. The table records number of isomorphs for each group, along with the corresponding pattern of equal edges, and a (permuted) example where applicable.

| Symmetry Type        | Number | Pattern | Example                  |
|----------------------|--------|---------|--------------------------|
| scalene              | 1      | [u, v, w, x, y, z] | [117, 84, 51, 80, 53, 52] |
| semi-isosceles       | 2      | [u, v, v, x, x, z] | [680, 680, 208, 15, 185, 185] |
| isosceles            | 4      | [u, v, v, v, v, z] | [1073, 1073, 990, 896, 1073, 1073] |
| semi-isohedral       | 2      | [u, v, w, w, v, z] | [990, 901, 793, 793, 901, 308] |
| isohedral            | 4      | [u, v, w, w, v, u] | [203, 195, 148, 148, 195, 203] |
| isohedral-isosceles  | 8      | [u, v, v, v, v, u] | NONE |
| equilateral          | 6      | [u, u, u, x, x, x] | NONE |
| regular              | 24     | [u, u, u, u, u, u] | NONE |

The last two cases above are trivially impossible, since a unit equilateral triangle has irrational area $\sqrt{3}/2$; the 8-symmetric case is dispatched at sect. 3 case 2(ii).

The first Heronian triangle for which the lattice embedding is not also an axial pose is

$$[30, 29, 5];$$

the first tetrahedron with some non-axial embedding, also the first without an axial embedding, is

$$[160, 153, 25, 120, 56, 39];$$

the first with some embedding not constructible via GCD from any axial pose is

$$[888, 875, 533, 533, 875, 888];$$

the first with 2,3,4 distinct embeddings via GCD are respectively

$$[1073, 975, 448, 495, 952, 840],$$
$$[1360, 1092, 548, 975, 865, 663],$$
$$[45100, 43911, 6929, 34476, 40544, 36975].$$

The particularly prolix case

$$[8484, 6625, 6409, 6409, 6625, 8484],$$

yields 9(36) strongly(weakly) distinct embeddings, versus just 1(4) via GCD.

10 Exhaustive Search for Lattice Embeddings

The quaternion GCD-based construction establishes that some lattice embedding exists for every Heronian tetrahedron case, and is furthermore fast — far more time is spent in reduction to canonical form than in construction. GCD embeddings have been computed for every $\mathbb{R}^3$ case with diameter $\leq 100,000$. [Indeed in 4 cases, new embeddings were found which had been overlooked — to considerable embarrassment on the part of its implementor — by the allegedly complete search described below.]

We shall need the parameterisation generating all primitive solutions of the Diophantine equation $x^2 + y^2 + z^2 = w^2$ from the identity

$$(p^2 + q^2 - u^2 - v^2)^2 + (2pu + 2qv)^2 + (2pv - 2qu)^2 = (p^2 + q^2 + u^2 + v^2)^2. \quad (4)$$

11
[In [13], where it is credited to Mordell, this same idea is employed in an entirely
different capacity — see end of section [13].]

Constraining the integer parameters $p, q, u, v$
by

$$
1 \leq p \leq \lfloor \sqrt{w} \rfloor,
$$

$$
\lceil \sqrt{\max(0, w/2 - p^2)} \rceil \leq q \leq \lfloor \sqrt{w - p^2} \rfloor,
$$

$$
0 \leq u \leq \lfloor \sqrt{\min(p^2 + q^2, w - p^2 - q^2)} \rfloor,
$$

$$
v = \lfloor \sqrt{w - p^2 - q^2 - u^2} \rfloor,
$$

before verifying $p^2 + q^2 + u^2 + v^2 = w$ will ensure that $x \leq y \leq z$; though some solutions
may be imprimitive or duplicated. The complete solution requires merging scaled-up solutions for every divisor of $w$.

Now the following algorithm can be designed to enumerate the set of all essentially
distinct embeddings of a given case:

1. Find some rational axial pose for vertices $P, Q, R, S$ via section 7 noting that
altitude of $S$ has ambiguous sign;

2. Generate complete solutions to $x^2 + y^2 + z^2 = u^2, v^2$ resp. via (4);

3. With vertex $P$ at origin, scan all possible lattice locations for $Q, R$, retaining
partial embeddings where length $QR^2 = w^2$;

4. Solve over-determined homogeneous linear equations (via null space) for the
quaternion $X$ rotating face $PQR$ from axial pose to lattice embedding;

5. If $X$ also rotates (either) $S$ to the lattice, reduce the rotated pose $P'Q'R'S'$ to
canonical form and add to result set.

A Magma implementation of this algorithm has currently found many more $\mathbb{Z}^3$
embeddings for tetrahedron diameter $\leq 20,000$, despite evidently remaining in need
of more thorough development! A version employing assertion 12 was employed (after
much ill-tempered adjustment of dimension) to verify uniqueness in $\mathbb{Z}^2$ for the first
1000 triangles.

11 Worked Numerical Examples

The following illustrate the complete procedure for computing a canonical lattice em-
bedding of a Heronian tetrahedron via quaternion GCD applied to a rational axial pose.
Throughout this section the quaternion $s + p i + q j + r k$ will be denoted $[s, p, q, r]$.
Input data comprise six edge-lengths.

The first example involves the scalene case

$$
[2431, 2375, 1044, 2296, 2175, 1479].
$$

Apply the currently selected permutation on vertices, say $PQRS \rightarrow QRPS$, and via
section 7 construct the corresponding rational pose $P, Q, R, S =$

$$
[1, 0, 0, 0], [1, 1044, 0, 0], [29, 18876, 67925, 0], [13, 22620, 8613, 14616];
$$

notice the odd scalar component (Cartesian LCD) in $R, S$. Since (inevitably) $P, Q$
already lie in the lattice, via section 8 relocate $R$ by setting

$$
X = \text{GCD}_L(R, 29) = [-5, 0, 0, 2]
$$

12
and transforming all four vertices by $X$ to $P, Q, R, S =$

\[ [1, 0, 0, 0], [1, 756, 720, 0], [1, -1144, 2145, 0], [13, 10440, 21837, 14616]; \]

then repeat to relocate $S$ by

\[ X = \text{GCD}_L(S, 13) = [-2, 2, 2, 1] \]

transforming to $P, Q, R, S =$

\[ [1, 0, 0, 0], [1, 396, 864, 432], [1, 396, -561, 2332], [1, 1740, 783, 1044], \]

with every vertex now in the lattice. Finally translate to the positive octant; scan lattice symmetries and vertex permutations as per section 9, finding the canonical isomorph

\[ [1, 0, 0, 396], [1, 561, 2332, 0], [1, 1344, 1288, 1740], [1, 1425, 1900, 396]. \]

The second example involves the same case, but permuting $PQRS \to PQSR$. Now the sequence follows rational pose

\[ [1, 0, 0, 0], [1, 2431, 0, 0], [13, 17248, 24360, 0], [17, 36575, 13680, 10260], \]

by $X = [0, -3, -2, 0]$ to

\[ [1, 0, 0, 0], [1, 935, 2244, 0], [1, 2240, 504, 0], [17, 26695, 28500, -10260], \]

by $X = [0, 3, -2, -2]$ to

\[ [1, 0, 0, 0], [1, -1529, -1848, 396], [1, -224, -1848, -1344], [1, -665, -2280, 0], \]

reducing canonically to

\[ [1, 0, 0, 396], [1, 224, 1848, 1740], [1, 665, 2280, 396], [1, 1529, 1848, 0]; \]

different from the previous result and so representing an essentially distinct embedding. [In scalene cases there is no difference between strong and weak.] For this case all 24 permutations of vertices yield one of these two results.

Nonetheless, tedious exhaustive search (section 10) eventually discovers one more embedding, distinct from both above:

\[ [1, 0, 0, 0], [1, 224, 672, 2184], [1, 665, 1824, 1368], [1, 1529, 1716, 792], \]

showing that not every $\mathbb{Z}^3$ embedding is related via GCD to an axial pose.

The third example employs isohedral case

\[ [8484, 6625, 6409, 6409, 6625, 8484]. \]

This embedding might be thought to lack variety: whatever the vertex permutation, any axial pose, say

\[ [1, 0, 0, 0], [1, 6625, 0, 0], [5, 28224, 31668, 0], [5, 4901, 22932, 21840] \]

which happens to transform in a single shot by $X = [0, -1, -2, 0]$ to

\[ [1, 0, 0, 0], [1, -3975, 5300, 0], [1, 1680, 8316, 0], [1, 3081, 3536, -4368], \]
always yields the same strongly canonical
\[ [1, 0, 0, 1401], [1, 0, 3016, 7056], [1, 0, 8316, 3081], [1, 4368, 4780, 0]. \]
Invoking only weak canonicity reduplicates this as 4 vertex-permut ed variations. More
noteworthy are the results of exhaustive search in this case, yielding in total 9(36)
distinct embeddings.
An intriguing feature of isohedral cases is the sparseness of their rotations: it
appears that such \( X \neq 1 \) have just two nonzero components, which furthermore are
very small — eg. \( X = -(i + 2j) \) above. For all 11 isohedral cases with diameter
\( \leq 20,000 \), there is some vertex permutation for which the rotations have height at
most 4.

12 Heronian Pentatopes in \( \mathbb{R}^4 \)
Consider the free pentatope \( PQRST \) specified by squared edge-lengths
\[
\begin{array}{cccccccccccc}
\text{edge} & \text{QP} & \text{RP} & \text{RQ} & \text{SP} & \text{SQ} & \text{SR} & \text{TP} & \text{TQ} & \text{TR} & \text{TS} \\
\text{square length} & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 1.
\end{array}
\]
\( PQRST \) can be rationally posed in \( \mathbb{Q}^4 \), with projective coordinates for its vertices:
\[
\begin{align*}
P &= [1, 0, 0, 0, 0], \\
Q &= [1, 1, 0, 0, 0], \\
R &= [1, 0, 1, 1, 0], \\
S &= [1, 0, 1, 0, 1], \\
T &= [3, 0, 1, 2, 2].
\end{align*}
\]
Taking the determinant shows \( PQRST \) is proper, with volume \( 1/24 \).
However,

\textbf{Assertion 13.} The simplex \( PQRST \) above has no lattice embedding in \( \mathbb{Z}^4 \).
\textit{Proof.} We may choose lattice axes so that \( T \) lies at the origin
\[
T = [1, 0, 0, 0, 0].
\]
Since \( TS^2 = TR^2 = TP^2 = 1 \) we may choose
\[
S = [1, 0, 0, 0, 1], \quad R = [1, 0, 0, 1, 0], \quad P = [1, 1, 0, 0, 0].
\]
Since \( QP^2 = 1 \), and the volume is nonzero, the vacant column must be filled by
choosing
\[
Q = [1, 0, 1, 0, 0].
\]
Now \( RQ^2 = 2 \), contradicting \( RQ^2 = 3 \) in the specification. \( \square \)

The point of all this is that assertions\[8, 9\] require only that a polytope be rationally
embedded, and have edge lengths squaring to integers, in order to be embeddable in \( \mathbb{Z}^2 \)
or \( \mathbb{Z}^3 \) [constraints garbled in the final paragraph of \cite{13}]. The counter-example above
demonstrates that these constraints are insufficient to ensure embeddability in \( \mathbb{Z}^4 \);
it follows that any theorem concerning Heronian embeddability in higher dimensions
must impose stronger conditions.
That’s always assuming that somebody can manage to find anything to embed.
Sascha Kurz \cite{10} recently completed the enumeration of primitive Heronian simplices
for diameter \( \leq 600,000 \). He reports 41563542 triangles, 2526 tetrahedra, and no
pentatope at all — nought, zero, zilch.
13 Remarks on Alternative Approaches

During the course of an interesting but in places unspecific paper covering many of the same topics as here, the proof of [8] Theorem 4.4 develops a construction for $\mathbb{Z}^2$ triangle embeddings similar in spirit to our proof of assertion[4]; however, there are significant differences.

In the first place no GCD is explicitly invoked: instead the LCD $r$ is factored into rational primes $p$, then by induction each factor is dispatched via a new rotation $X$ — obtained in essentially the same manner as above. From a computational viewpoint, this decomposition is both more elaborate and slower.

When it comes to dealing with the set however, instead of each point dispatched in turn via a new rotation, every point is (partially) embedded via the same $X$. Computationally speaking, what is lost on the swings — the number of factors of $q$ — is (perhaps) regained on the roundabouts — the number $k$ of points.

So we should enquire whether — in $\mathbb{R}^3$ especially — our induction on $k$ might also be aggregated, winning on both rides. Instead of transforming by $X = \text{GCD}_L(Q, q)$ for each point $Q$ in turn, consider just setting $X = \text{GCD}_L(S, s)$ for the final point $S$ of the rational pose, in the hope that $X$ will also embed every other point.

Clearly this could work only if all previous $q \mid s$. But the situation is actually rather worse: it turns out that, after rotation and reduction, the scalar $Q'$ remains $\text{GCD}(s/q, q)$ rather than unity. So rotated points may miss the lattice unless, for each scalar LCD $q$ and prime factor $p$, either $p$ divides $q$ to the same power as $p$ divides $s$, or $p$ does not divide $q$ at all.

And in practice cases do frequently occur where, besides denominators involving various powers of the same prime, the rational pose has no LCD equalling their LCM: for example (for various permutations of the vertices) the early case

\[[160, 153, 25, 120, 56, 39]\]

yields some axial poses with scalars $q = 1, 1, 25, 65$. Tough luck!

Turning to $\mathbb{R}^3$, the paper provides a partial solution to the existence problem: [8] Theorem 4.6 asserts that computation has established the existence of suitable rotation matrices, provided all LCDs are products only of primes $p \leq 37$. Unfortunately no details are given: one conjectures that the method involves some variation of the ‘modularisation’ employed elsewhere in attacking divisibility of volume, see section 2.

A complete contrast to the complex approach to embedding in $\mathbb{R}^2$ is provided by [13], where the problem is reformulated in terms of a quadratic Diophantine equation, for which the complete parametric solution [14] is known. It seems improbable that an analogous attack could easily be mounted in $\mathbb{R}^3$: corresponding parameterisations for simultaneous quadratic and cubic Diophantine equations are (as far as we are aware) currently unknown, and quite possibly do not exist.

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