Topological edge states in the spin 1 bilinear–biquadratic model

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Abstract
The spin 1 bilinear–biquadratic model
\[ H = \sum_{ij} \left[ \cos \phi S_i \cdot S_j + \sin \phi (S_i \cdot S_j)^2 \right] \]
on a square lattice in the region \(0 < \phi < \pi/4\) is studied in a fermion representation with a p-wave pairing Bardeen–Cooper–Schrieffer type of mean-field theory. Our results show there may exist a non-trivial gapped spin liquid with time-reversal symmetry spontaneously breaking. This exotic state manifests its topological nature by forming chiral states at the edges. To show this more clearly, we set up and solved a ribbon system. We got a gapless dispersion representing the edge modes beneath the bulk modes. The edge modes with nonzero longitudinal momentum \((k_x \neq 0)\) convect in opposite directions at the two edges, which leads to a twofold degeneracy, while the modes with zero longitudinal momentum \((k_x = 0)\) turn out to be Majorana fermion states. The edge spin correlation functions are found to decay following a power law with increasing distance. We also calculated the contribution of the edge modes to the specific heat and obtained a linear law at low temperatures.

(Some figures may appear in colour only in the online journal)

1. Introduction

In condensed matter physics, the Landau Fermi liquid theories and spontaneous symmetry breaking have been the basic principles that have accounted for a vast range of phenomena. For instance, the ground state of the two-dimensional spin \(S\) Heisenberg model on a square lattice possesses long range Néel order and Goldstone modes due to spontaneous spin-rotation symmetry breaking. People have been trying to look for more exotic states in spin systems for a long time. The seminal concept of the ‘resonating valence bond’ (RVB) spin liquid state was first proposed by Anderson [1]. Then, mostly due to its implication of the mechanism of high temperature superconductivity [2], this field has continued flourishing for more than two decades. As a new type of quantum matter, the spin liquid state itself is intriguing, since its properties have never been clarified before [3]. Various approaches have shown that quantum spin liquids may exist in the two-dimensional (2D) \(S = 1/2\) \(J_1-J_2\) model and the Heisenberg model on the Kagomé lattice. In these models, the quantum spin liquids are accessed (in principle) through appropriate frustrating interactions [4]. However, the nature of the quantum disordered ground state is still under debate. The RVB spin liquid state obviously goes beyond Landau’s theories, in which the quasi-particles of the Fermi liquid carry both spin and charge quantum numbers. Another exciting field in searching for quantum exotic states beyond Landau’s theory of spontaneous symmetry breaking is that of the quantum Hall (QH) and fractional quantum Hall (FQH) states. For these states, topological order plays an essential role [5, 6]. For the QH state, a quantized Hall conductance was measured, due to the formation of the Landau levels of a 2D electron gas at low temperatures and in strong magnetic field. One of the key features of the QH effect is the existence of chiral edge states around the system boundaries. Recently, involving both the quantized spin Hall effect [7, 8] and the quantized anomalous Hall effect [9], the topological insulator with gapless edge states has been one of the hot topics. The fundamental links between the above two fields have attracted much attention [10]. In this work, we aim to contribute to these interesting topics.

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We study the spin 1 bilinear–biquadratic model on a square lattice, whose Hamiltonian is written as
\[ H = \sum_{\langle ij \rangle} [\cos \phi \mathbf{S}_i \cdot \mathbf{S}_j + \sin \phi (\mathbf{S}_i \cdot \mathbf{S}_j)^2], \]  
where \( \mathbf{S}_i \) is a spin 1 operator. Its semiclassical version \((S \to \infty)\) on a bipartite lattice exhibits four ordered phases that are exactly divided by four \( SU(3) \) symmetric points with the model parameters \( \phi = \phi_0 = \pi/4, \pm \pi/2, -3\pi/4 \) [11, 12]. They are phases with ferromagnetic (FM), antiferromagnetic (AFM), ferroquadrupolar (FQ), and antiferroquadrupolar (AFQ) orders respectively. Whether these classically well-understood phases are stable in the quantum case and how the quantum model behaves constitute the interesting topics of current research. In one dimension (1D), many aspects have been revealed by extensive exploration [13–25], while in two dimensions (2D), a complete understanding of the system is still being anticipated. On a triangular lattice, many methods have revealed that the region \( \pi/4 < \phi \leq \pi/2 \) exhibits an AFQ order [26–28]. On the honeycomb lattice, a tensor renormalization group method showed that the AFQ order in the region \( \pi/4 < \phi \leq \pi/2 \) is destroyed by pure quantum fluctuations and there is a transition from the plaquette order to the AFM order [29]. A recent work proposed that a three-sublattice order exists at the \( SU(3) \) point \( \phi_0 = \pi/4 \) [30]. A quantum Monte Carlo simulation [31] revealed that the AFM phase is stable in the region \(-\pi/2 < \phi \leq 0\) on a square lattice, while for the region \(0 < \phi < \pi/4\) we lack evidence as regards whether the AFM order can survive or not.

In this paper, we show that a novel type of topological spin liquid might exist on a square lattice in the region \(0 < \phi < \pi/4\), in which a topological edge state circulates around the boundary of the system. Time-reversal symmetry is broken spontaneously for this non-trivial gapped spin liquid. We found that in a fermion representation the 2D spin liquid can be described very well by a (projected) spinless p-wave pairing Bardeen–Cooper–Schrieffer (BCS) type of Hamiltonian. We obtained the gapless dispersion of the edge modes. For a nonzero longitudinal momentum \( k_x \neq 0\), one edge mode splits into two half-modes that can exist individually, which could be termed \( \mathcal{L} \) (left) and \( \mathcal{R} \) (right) chiral modes, respectively, while for \( k_x = 0\), zero-momentum edge modes emerge; these turn out to be Majorana fermion states [32, 33].

The paper is organized as follows. In section 2 a brief introduction of the fermion representation is presented. In section 3 the mean-field theory is introduced and the solutions for a gapped chiral spin liquid are obtained. Then the corresponding edge state of a ribbon system is explored in section 4. And a summary is given in section 5.

2. The fermion representation with hard-core constraint

Firstly we introduce a fermion representation for quantum spin 1. Each spin has three eigenstates \( |m_i \rangle \) of \( S^z_i \) with the eigenvalues \( m_i = -1, 0, +1 \). We introduce three fermionic operators to generate three independent states \( i = \sqrt{-1} \),
\[ f^\dagger_{i,1}|0\rangle = \frac{1}{\sqrt{2}}(|m_i = -1\rangle + |m_i = 1\rangle), \]
\[ f^\dagger_{i,2}|0\rangle = |m_i = 0\rangle, \]
\[ f^\dagger_{i,3}|0\rangle = \frac{1}{\sqrt{2}}(|m_i = -1\rangle - |m_i = 1\rangle). \]

In terms of \( f \) operators, the spin operators can be expressed as
\[ S^x_i = i f^\dagger_{i,1} f_{i,2} - f^\dagger_{i,2} f_{i,1}, \]
\[ S^y_i = i f^\dagger_{i,1} f_{i,3} - f^\dagger_{i,3} f_{i,1}, \]
\[ S^z_i = f^\dagger_{i,1} f^\dagger_{i,2} f_{i,3} f^\dagger_{i,3}. \]

To restore the Hilbert space of spin 1, the hard-core constraint at each site must be imposed,
\[ \sum_{\mu=1}^3 f^\dagger_{i,\mu} f_{i,\mu} = 1. \]

In this way the Hamiltonian equation (1) is mapped to a frustrated \( SU(3) \) fermion model [34]
\[ H = -J_1 \sum_{\langle ij \rangle} F^\dagger_{ji} F_{ji} : -J_2 \sum_{ij} B^\dagger_{ji} B_{ji} + \sum_i \lambda_i \left( \sum_{\mu=1}^3 f^\dagger_{i,\mu} f_{i,\mu} - 1 \right), \]
where \( J_1 = \cos \phi > 0 \), \( J_2 = \cos \phi - \sin \phi > 0 \), denotes normal ordering of operators, the \( \lambda_i \) are the Lagrangian multipliers, and the bond operators are defined as
\[ F_{ji} = \sum_{\mu=1,2,3} f^\dagger_{j,\mu} f_{i,\mu}, \quad B_{ji} = \sum_{\mu=1,2,3} f^\dagger_{j,\mu} f_{i,\mu}. \]

3. Gapped spin liquid with time-reversal symmetry breaking

3.1. Bond operator mean-field theory

To find the ground state properties of this spin 1 system, we take the mean-field approximation by introducing two order parameters for the bond operators \( B^\dagger_{ji} \) and \( F_{ji} \),
\[ \langle F_{i+y,x} \rangle = \langle F_{i,y+x} \rangle = F, \]
\[ -i \langle B_{i+y,x} \rangle = B_x = B^\dagger_{y+i} = B_y = B^\dagger_{x+i}, \]
where \( F, B, \eta_x, \) and \( \eta_y \) are real and to be determined self-consistently. Here we have taken a uniform phase factor in \( F \), which turns out to be negligible when the mean-field equations are established. Two phase factors are kept for the \( B \) field, and we will see that the final results only rely on the phase difference \( \Delta \eta = \eta_y - \eta_x \). Under this prescription, the
The effective Hamiltonian reads
\[ H_{\text{eff}} = \sum_{i,\mu} \lambda f_{i,\mu}^\dagger f_{i,\mu} + \sum_{\langle ij \rangle,\mu} \left( J_{ij,\mu} T_{ij} f_{i,\mu}^\dagger + \text{h.c.} \right) + \sum_{\langle ij \rangle,\mu} \left( J_{ij,\mu} P_{ij} f_{i,\mu}^\dagger + \text{h.c.} \right) - \lambda N = 2N_A J_1 F^2 + 2N_A J_2 B^2, \]

where \( T_{ij} = -2J_1 (F_{ij}), \) \( P_{ij} = -2J_2 (B_{ij}), \) and \( N_A \) is the total number of lattice sites. The hard-core constraint will be imposed on an average level by minimizing the free energy. After performing the Fourier transformation, we arrive at a complex p-wave-like pairing of independent flavors of fermions [36],
\[ H_{\text{eff}} = \frac{1}{2} \sum_{k,\mu} \Phi_{\mu}(k) M(k) \Phi_{\mu}(k) + \epsilon_0, \]
where the sum over momentum \( k \) is carried out in the first Brillouin zone (1st BZ), the spinor \( \Phi_{\mu}^\dagger(k) = (f_{k,\mu}^\dagger, f_{k,\mu}^\dagger), \) the \( 2 \times 2 \) Hermition matrix
\[ M(k) = d(k) \cdot \sigma , \]
and the coefficients satisfy
\[ |\alpha_k|^2 = \frac{1}{2} \left[ 1 + \frac{d_\alpha(k)}{\omega(k)} \right], \]
\[ |\beta_k|^2 = \frac{1}{2} \left[ 1 - \frac{d_\beta(k)}{\omega(k)} \right], \]
\[ 2a_k \beta_k = \frac{d_\beta(k) + i d_\gamma(k)}{\omega(k)} . \]

If one chooses a real and even \( \nu_k (\nu_k^\dagger = \nu_k) \), then \( \alpha_k \) is complex and odd, and vice versa. The spectrum is
\[ \omega(k) = |d(k)| = \sqrt{d_\mu^2(k) + d_\gamma^2(k) + d_\beta^2(k)}. \]
The p-wave paired ground state of the bulk system reads [36]

\[ |\Omega_0\rangle = \sum_{\mathbf{k}, \mu} \langle \mathbf{k} | \Phi_{\mu}^{+}(\mathbf{k}) \mathbf{n}(\mathbf{k}) \times \partial_{\mathbf{k}} \mathbf{n}(\mathbf{k}) |0\rangle, \]

where the prime on the product indicates that each distinct pair (\( \mathbf{k}, -\mathbf{k} \)) is to be taken once. This ground state exhibits a non-trivial topological property that can be signified by the Chern number of the spinless \( SU(3) \) fermions. For each flavor of spinless fermions, the Chern number is defined by [36, 39]

\[ C = \frac{1}{4\pi} \int_{\Omega} d^{2}k |\mathbf{n}(\mathbf{k}) \cdot \partial_{\mathbf{k}} \mathbf{n}(\mathbf{k})| \]

where \( \Omega \) means the volume of the first Brillouin zone, \( \mathbf{n}(\mathbf{k}) \) is defined as \( \mathbf{n}(\mathbf{k}) = \frac{\partial |\mathbf{k}\rangle}{\partial |\mathbf{k}\rangle} \). By substituting equation (17) in equation (34), we get

\[ C = \frac{\sin(\Delta\eta)}{4\pi} \left( \frac{2J_{z}B}{\lambda} \right)^{2} \int_{\Omega} d^{2}k \frac{\epsilon_{\mathbf{k}}}{(\omega(\mathbf{k})/\lambda)^{3}}. \]

And by substituting the numerical mean-field solutions at zero temperature in, we obtain the simplified result,

\[ C = \pm 1, \]

for \( 0 < \phi < \pi/4 \). Thus the total Chern number of this topological state is \( C = \pm 3 \) due to the symmetry for the fermions of different flavors. The bulk system’s non-trivial ground state can be labeled with this Chern number.

### 3.2. The ground state and the Chern number

The p-wave paired ground state of the bulk system reads [36]

\[ |\Omega_0\rangle = \sum_{\mathbf{k}, \mu} \langle \mathbf{k} | \Phi_{\mu}^{+}(\mathbf{k}) \mathbf{n}(\mathbf{k}) \times \partial_{\mathbf{k}} \mathbf{n}(\mathbf{k}) |0\rangle, \]

where the prime on the product indicates that each distinct pair (\( \mathbf{k}, -\mathbf{k} \)) is to be taken once. This ground state exhibits a non-trivial topological property that can be signified by the Chern number of the spinless \( SU(3) \) fermions. For each flavor of spinless fermions, the Chern number is defined by [36, 39]

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### 4. Spin edge states

#### 4.1. Edge modes of the ribbon system

To demonstrate the spin edge states explicitly, we set up a ribbon (or ladders) system with a pair of open edges in \( \hat{y} \) direction and keep a periodic boundary condition along the \( \hat{x} \) axis (figure 3). Noticing that \( k_{y} \) is still a good quantum number, we start with a Hamiltonian with \( L_{\text{max}} \) legs,

\[ H_{\text{eff}} = \sum_{k_{x} \geq 0, \mu} \Phi_{\mu}^{+}(k_{x}) M(k_{x}) \Phi_{\mu}(k_{x}) + \varepsilon_{0}, \]

\[ \Phi_{\mu}(k_{x}) = \{f_{(k_{x}, 1), \mu}, \ldots, f_{(k_{x}, L_{\text{max}}), \mu}, f_{(-k_{x}, 1), \mu}, \ldots, f_{(-k_{x}, L_{\text{max}}), \mu}\}. \]
The matrices $M(k)$ are too large to be presented here. One can easily solve the Hamiltonian numerically. The resulting diagonalized Hamiltonian can be written in the form

$$H_{\text{eff}} = \sum_{k_x \geq 0, \mu} \Psi_{\mu}^\dagger(k_x) P(k_x) \Psi_{\mu}(k_x) + \varepsilon_0,$$

(39)

$$P(k_x) = \text{diag}[\omega(k_x,1), \ldots, \omega(k_x,L_{\text{max}}); -\omega(k_x,1), \ldots, -\omega(k_x,L_{\text{max}})].$$

(40)

and reads

$$H_{\text{eff}} = \sum_{k_x \geq 0, \mu, i} \alpha(k_x,i) \left[ \gamma^\dagger_{(k_x,i),\mu} \gamma_{(k_x,i),\mu} + \gamma^\dagger_{(-k_x,i),\mu} \gamma_{(-k_x,i),\mu} \right] - \sum_{k_x \geq 0, \mu, i} \alpha(k_x,i) + \varepsilon_0.$$

(42)

We choose the canonical operator $\gamma^\dagger_{(k_x,1),\mu}$ (with subscript $i = 1$) to denote the edge excitations above the ground state and $\omega(k_x,1)$ as the edge excitation energy. The rest of the modes are bulk modes. The new ground state $|\Omega^0_{\text{ef}}\rangle$ is quite different from that in equation (33). $|\Omega^0_{\text{ef}}\rangle$ itself contains the chirality of the edges and should satisfy the condition

$$\gamma_{(k_x,1),\mu} |\Omega^0_{\text{ef}}\rangle = 0, \quad \forall k_x, i, \mu.$$

(43)

We will specify the ground state numerically later when evaluating some quantities. The lowest energy mode for each $k_x$ could be collected as

$$H_{\text{lowest}} = \sum_{k_x \geq 0, \mu} \alpha(k_x,1)$$

$$\times \left[ \gamma^\dagger_{(k_x,1),\mu} \gamma_{(k_x,1),\mu} + \gamma^\dagger_{(-k_x,1),\mu} \gamma_{(-k_x,1),\mu} \right].$$

(44)

It seems that this effective Hamiltonian denotes the edge states. However, this is not necessarily the case. The lowest energy mode for each $k_x$ is too large to be presented.
contributes zero energy to the system, and can be rewritten as

\[ E_k = 0 \]

It is in fact a Majorana fermion state \[32\], since the mode that this zero-momentum mode manifests itself at both edges.

We see the coefficients \( U_\omega \) excitation with increasing size of the system, the zero-momentum energy value is so small that one can hardly discern it from the machine precision. We may denote the zero-momentum modes as

\[ R_{\mu k} \equiv \gamma(k, 1, \mu) \]

where the coefficients \( U \) and \( V \) are real (in contrast to the coefficients \( u_k \) and \( v_k \) for the periodic boundary in the previous section) and are depicted in figures 5(a) and (b). With increasing size of the system, the zero-momentum edge excitation \( \epsilon_{(0, 1)} \) goes to zero rapidly. Beyond \( L_{\text{max}} = 50 \), its energy value is so small that one can hardly discern it from the machine precision. We may denote the zero-momentum modes as

\[ E_{(0, 1), \mu} = \sum_{j=1}^{L_{\text{max}}} [U_{(0, j), \mu} f_{(0, j), \mu} + V_{(0, j), \mu} f_{(0, j), \mu}^\dagger] \]

where the real coefficients are depicted in figure 5(c). We see that this zero-momentum mode manifests itself at both edges. It is in fact a Majorana fermion state \[32\], since the mode contributes zero energy to the system, and can be rewritten as

\[ E_{\mu} E_{\mu} = 1 - 2 \mathcal{M}_\mu^R \mathcal{M}_\mu^L \]

with two Majorana fermions

\[ \mathcal{M}_\mu^R = \sum_{j=1}^{L_{\text{max}}} \frac{U_{(0, j)} + V_{(0, j)} f_{(0, j), \mu} + f_{(0, j), \mu}^\dagger}{2}, \]

\[ \mathcal{M}_\mu^L = \sum_{j=1}^{L_{\text{max}}} \frac{U_{(0, j)} - V_{(0, j)} f_{(0, j), \mu} + f_{(0, j), \mu}^\dagger}{2}, \]

localizing at the two opposite edges. All of the above coefficients satisfy the relations numerically for large enough \( L_{\text{max}}(k_x > 0) \):

\[ \sum_{j=1}^{L_{\text{max}}} [U^2_{(+k_x, j)} + V^2_{(+k_x, j)}] = 1, \]

\[ U_{(k_x, j)} = V_{(k_x, j)}, \quad U_{(-k_x, j)} = -V_{(-k_x, j)}, \]

\[ U_{(-k_x, j)} = -U_{(k_x, L_{\text{max}}-j+1)}, \]

\[ V_{(-k_x, j)} = V_{(-k_x, L_{\text{max}}-j+1)}, \]

\[ U_{(-k_x, j)} U_{(k_x, j)} = V_{(-k_x, j)} V_{(k_x, j)} = 0. \]

But notice that each mode possesses \( U(1) \) symmetry, so the values may be changed according to the symmetry transformation.

### 4.3. Edge spin correlation functions

Although the edge states of the ribbon are clear to see in the fermion representation, they are still elusive from the point of view of the spin language. In order to show the properties
of the spin edge state, we measure the spin correlations and thermodynamic quantities, such as the specific heat, contributed by the edge.

It is well known that a gapped spin system exhibits an exponentially decaying spin correlation in the bulk. Of all the spin correlations for the ribbon system, the one at the edge is of great interest to us. We choose the right edge of the ribbon spin correlations to measure the spin correlations in the ground state, (figure 3) to measure the spin correlations in the ground state,

\[ C_{\text{edge}}(i, i + r) \equiv C_{\text{edge}}(r) = \langle S_i^{(L, \text{max})} S_{i+r}^{(L, \text{max})} \rangle. \] (57)

Now we need to find the ground state \(|\Omega_0\rangle\). The edge modes in \(H'_{\text{edge}}\) could be singled out and serve as a quasi-1D effective Hamiltonian,

\[ H'_{\text{eff}} = H'_{\text{bulk}} + H'_{\text{edge}}, \]
\[ H'_{\text{edge}} = \sum_{0 \leq k_x \leq k_T', \mu} \alpha_{ij}^{\text{edge}} Y_{(k_x, 1), \mu} Y_{(k_x, 1), \mu} + Y_{(-k_x, 1), \mu} Y_{(-k_x, 1), \mu}, \] (59)

which could be utilized to evaluate quantities along the edges. Near \(k_x \approx 0\), the edge modes behave linearly: \(\alpha_{ij}^{\text{edge}} \approx 2JzB|k_x|\) [33] (please see the lowest thick line in figure 4).

Since the bulk and edge modes are independent, one can write the ground state in a separable form:

\[ |\Omega_0\rangle = |\Omega_{0, \text{bulk}}\rangle \otimes |\Omega_{0, \text{edge}}\rangle, \] (60)

where \(|\Omega_{0, \text{bulk}}\rangle\) and \(|\Omega_{0, \text{edge}}\rangle\) are the lowest energy states of \(H'_{\text{bulk}}\) and \(H'_{\text{edge}}\) respectively. We have

\[ |\Omega_{0, \text{edge}}\rangle = \prod_{0 \leq k_x \leq k_T', \mu} [U(k_x, j) - V(k_x, j)] \prod_{(-k_x, j), \mu} 0 \], \] (61)

where the prime on the product indicates that each distinct pair \((k_x, -k_x)\) is to be taken once. One can easily verify that \(\gamma(\mp k_x, 1), \mu |\Omega_{0, \text{edge}}\rangle = 0\). Since we are considering the quantities along the edge, the edge correlation function equation (57) can be evaluated approximately just by \(|\Omega_{0, \text{edge}}\rangle\). First, one can work out

\[ \langle f_{(k_x, \text{max}), \mu}^\dagger f_{(-k_x, \text{max}), \mu} \rangle = -U(k_x, \text{max}) V(k_x, \text{max}), \]
\[ \langle f_{(k_x, \text{max}), \mu}^\dagger f_{(-k_x, \text{max}), \mu} \rangle = V^2(k_x, \text{max}). \] (62)

![Figure 6.](image-url)
Then the correlation function is deduced as

\[
C_{\text{edge}}^\phi (r) = 2 \left[ \frac{1}{N_A} \sum_{0 \leq k_z \leq k_T} e^{i k_T r} \left( \langle \phi_{(k_z, L_{\text{max}}), \mu}^i | \phi_{(-k_z, L_{\text{max}}), \mu}^i \rangle \right)^2 \right] + 2 \left[ \frac{1}{N_A} \sum_{0 \leq k_z \leq k_T} e^{i k_T r} \left( \langle \phi_{(k_z, L_{\text{max}}), \mu}^i | \phi_{(k_z, L_{\text{max}}), \mu} \rangle \right)^2 \right]
\]

\[
+ 2 \left[ \frac{1}{N_A} \sum_{0 \leq k_z \leq k_T} e^{i k_T r} U_{(k_z, L_{\text{max}})} V_{(k_z, L_{\text{max}})} \right]^2 + 2 \left[ \frac{1}{N_A} \sum_{0 \leq k_z \leq k_T} e^{i k_T r} V_{(k_z, L_{\text{max}})}^2 \right]^2.
\]

(63)

The results show a general power law

\[
C_{\text{edge}}^\phi (r) \approx \frac{\alpha}{r^\delta},
\]

(64)

In practice, we fit the numerical data using the formula

\[
\ln C_{\text{edge}}^\phi (r) \approx \ln \alpha - \delta \ln r
\]

(65)

instead. At the model parameter \( \phi = 0.435 \, 847 \), we obtain \( \alpha \approx 0.015 \, 6407 \) and \( \delta \approx 2.174 \, 343 \) (see figure 6(a)). For other model parameters, the results are not very different (figures 6(b) and (c)). So we see that the edge spin correlations decay like a power law along the edge and exclude the possibility of exponential decay behavior in the main region of model parameters.

4.4. Specific heat contributed by the edge modes

Now we turn to the specific heat. At low temperatures, the bulk states contribute little to the specific heat due to the existence of a bulk gap, while the gapless edge modes give the main contribution. By adopting equation (58) as the effective Hamiltonian, one can work out the contribution of the edge modes to the specific heat which behaves linearly in temperature \( T \):

\[
\frac{C_V}{N_A k_B} = \int_0^{E_m} \left( \frac{E}{2k_B T} \right)^2 \cosh^{-2} \left( \frac{E}{2k_B T} \right) \rho(E) \, dE
\]

\[
\approx \frac{\pi k_B}{8J_2 B} T,
\]

(66)

where we have released the upper limit of the integral for simplicity \( E_m \to \infty \) and the density of states is

\[
\rho(E) = \frac{1}{N_A} \sum_{k_z, \mu} \delta \left( E - \omega_{(k_z, \mu)} \right) \approx \frac{3}{2\pi J_2 B}
\]

(67)

5. Summary

To briefly summarize, we showed a possible gapped chiral spin liquid in the 2D square bilinear–biquadratic system in the region of \( 0 < \phi < \pi/4 \). As a consequence, the time-reversal symmetry breaks spontaneously and an interesting topological ground state is revealed. We numerically analyzed the resulting spin edge states for a ribbon system in detail. This method may be applied to other related systems to specify a spin liquid state. We found \( L \) (left) and \( R \) (right) chiral edge modes for nonzero longitudinal momentum \( k_z \neq 0 \) and a zero-momentum edge mode for \( k_z = 0 \). The power-law decay of the edge spin correlation function and the contribution of the non-trivial spin edge state to the specific heat at low temperatures are found. In future work, the properties of low energy excitations would be of great interest.

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Appendix. Classification of topological states using \( Z_2 \) topological invariants

For each flavor of fermions (omit the flavor index \( \mu \) in equation (14)), the effective Hamiltonian is

\[
H_{\text{eff}} = \frac{1}{2} \sum_k \Phi^\dagger(k) M(k) \Phi(k),
\]

(68)

with \( M(k) \) defined in equation (16). From the results in [38], the \( 2 \times 2 \) Pauli matrices can be divided into two groups—the even matrix \( \sigma_z \) and odd matrices, \( \sigma_x \) and \( \sigma_y \). Because the coefficients of odd matrices are zero at four high symmetry points of the square lattice in momentum space, we can focus solely on the coefficients of the even matrix, \( d_i(k) = \lambda - 2J_1 F \cos k_x + \cos k_y \). The four \( Z_2 \) topological invariants are defined as

\[
\zeta_k = 1 - \Theta(d_i(k)),
\]

(69)

where \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) if \( x < 0 \).

Hence, for points \( (0, 0), (0, \pi), (\pi, 0), (\pi, \pi) \), the four \( Z_2 \) topological invariants are explicitly given by

\[
\zeta_{k=(0,0)} = \Theta[\lambda - 4J_1 F], \quad \zeta_{k=(\pi,0)} = \Theta[\lambda], \quad \zeta_{k=(\sigma,\pi)} = \Theta[\lambda + 2J_1 F].
\]

(70)

For \( k = (\pi, \pi), k = (0, \pi), k = (\pi, 0) \), we have a trivial result as follows:

\[
\zeta_{k=(\pi, \pi)} = \zeta_{k=(0, \pi)} = \zeta_{k=(\pi, 0)} = 0;
\]

(71)

and for \( k = (0, 0) \), we have

\[
\zeta_{k=(0,0)} = \Theta(\lambda - 4J_1 F).
\]

(72)

Thus we identify two distinct topological states: the topological state with trivial topological invariants

\[
\zeta_{k=(0,0)} = \zeta_{k=(\pi, \pi)} = \zeta_{k=(0, \pi)} = \zeta_{k=(\pi, 0)} = 0
\]

(73)
for $\lambda > 4J_1F$ and the topological state with non-trivial topological invariants

$$\zeta_k = (\pi, \pi) = 0, \quad \zeta_k = (0, \pi) = 1$$

(74)

for $\lambda < 4J_1F$. And in the topological spin liquid state in this paper, we find a special fermion parity pattern: even fermion parity at $k = (\pi, \pi)$, $k = (0, \pi)$, and $k = (\pi, 0)$ and odd fermion parity at $k = (0, 0)$.

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