We provide an inhomogeneous solution concerning the dynamics of a real self-interacting scalar field minimally coupled to gravity in a region of the configuration space where it performs a slow rolling on a plateau of its potential. During the inhomogeneous de Sitter phase the scalar field dominant term is a function of the spatial coordinates only. This solution specialized nearby the FLRW model allows a classical origin for the inhomogeneous perturbations spectrum.

Keywords: inhomogeneous inflation, perturbaton spectrum

1. General Statements

A peculiar feature of the inflationary scenario consists of the violent expansion the Universe underwent during the de Sitter phase\(^1\),\(^2\) indeed via such a mechanism the inflationary model provides a satisfactory explanation of the so-called horizons and flatness paradoxes by stretching the inhomogeneities at a very large scale.\(^3\),\(^4\) However, when referred to a (homogeneous and isotropic) Friedmann–Lemaître–Robertson–Walker (FLRW) model\(^5\), the de Sitter phase of the inflationary scenario rules out the small inhomogeneous perturbations so strongly, that it makes them unable to become seeds for the later structures formation.\(^6\),\(^7\) This picture emerges sharply within the inflationary paradigm and it is at the ground level of the statements according to which the cosmological perturbations arise from the scalar field quantum fluctuations.\(^8\)

Though this argument is well settled down and results very attractive even because the predicted quantum spectrum of inhomogeneities takes the Harrison-Zeldovich form, nevertheless the question whether it is possible, in a more general context,
that classical inhomogeneities survive up to a level relevant for the origin of the actual Universe large scale structures remains open.

Indeed here we investigate the behaviour of an inhomogeneous cosmological model which undergoes a de Sitter phase and show how such a general scheme allows the scalar field to retain, at the end of the exponential expansion, a generic inhomogeneous term to leading order (for connected topic see [20]).

Thus our analysis provides relevant information either with respect to the morphology of an inhomogeneous inflationary model, either stating that the scalar field is characterized by an arbitrary spatial function which plays the role of its leading order.

The model we take into account refers to the coupled dynamics of an inhomogeneous cosmological model with a real self-interacting scalar field. The solution concerns the phase of evolution when the potential associated to the scalar field singles out a plateau and the Universe evolution is dominated by the effective cosmological constant associated with the energy level over the true vacuum state of the theory. We are in condition to neglect the contribution due to the ultra-relativistic matter because it would be relevant only for higher order terms and becomes more and more negligible as the exponential expansion develops (for a discussion of an inflationary scenario with relevant ultra-relativistic matter and different outcoming behaviour, see [13],[14]).

2. Inhomogeneous Inflationary Model

In a synchronous reference, the generic line element of a cosmological model takes the form (in units $c = h = 1$)

$$ds^2 = dt^2 - \gamma_{\alpha\beta}(t, x^\mu)dx^\alpha dx^\beta, \quad \alpha, \beta, \mu = 1, 2, 3$$

where $\gamma_{\alpha\beta}(t, x^\mu)$ is the three-dimensional metric tensor describing the geometry of the spatial slices. The Einstein equations in the presence of a self interacting scalar field $\{\phi(t, x^\mu), V(\phi)\}$ read explicitly

$$\frac{1}{2}\partial_t k^\alpha - \frac{k^\beta k^\alpha}{4} = \chi \left[-(\partial_t \phi)^2 + V(\phi)\right]$$

$$\frac{1}{2}(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}) = \chi (\partial_\alpha \phi \partial_t \phi)$$

$$\frac{1}{2\sqrt{\gamma}} \partial_t (\sqrt{\gamma} k^\beta_{\alpha}) + P^\beta_{\alpha} = \chi \left[\gamma^\beta_{\mu} \partial_\alpha \phi \partial_\mu \phi + V(\phi) \delta^\beta_{\alpha}\right].$$

(Einstein constant $\chi = 8\pi G$, $G$ being the Newton constant) where we used the notations

$$\gamma \equiv \det \gamma_{\alpha\beta}, \quad k_{\alpha\beta} \equiv \partial_t \gamma_{\alpha\beta}, \quad k^\beta_{\alpha} = \gamma^\beta_{\mu} k^\mu_{\alpha}.$$ 

The three-dimensional Ricci tensor $P^\beta_{\alpha} = \gamma^\beta_{\gamma} P_{\alpha\gamma}$ is constructed via the metric $\gamma_{\alpha\beta}$ which is also used to form the covariant derivative $(\ )_{;\alpha}$. 
The dynamics of the scalar field $\phi(t, x^\gamma)$ is described by the equation

$$\partial_t \phi + \frac{1}{2} k_\alpha^\gamma \partial_t \phi - \gamma^{\alpha\beta} \phi_{,\alpha;\beta} + \frac{dV}{d\phi} = 0,$$

coupled to the Einstein’s ones, with notation $\partial_t (\phantom{\phi}) \equiv \partial^2 (\phantom{\phi})_{tt}$.

In what follows we will consider the three fundamental statements:

(i) the three metric tensor is taken in the general factorized form

$$\gamma_{\alpha\beta}(t, x^\mu) = \Gamma^2(t, x^\mu) \xi_{\alpha\beta}(x^\mu)$$

where $\xi_{\alpha\beta}$ is a generic symmetric three-tensor and therefore contains six arbitrary functions of the spatial coordinates, while $\Gamma$ is to be determined by the dynamics. The inverse metric reads

$$\gamma^{\alpha\beta}(t, x^\mu) = \frac{1}{\Gamma^2(t, x^\mu)} \xi^{\alpha\beta}(x^\mu), \quad \xi^{\alpha\nu} \xi_{\nu\beta} = \delta^\alpha_\beta;$$

(ii) the self interacting scalar field dynamics is described by a potential term which satisfies all the features of an inflationary one, i.e. a symmetry breaking configuration characterized by a relevant plateau region;

(iii) the inflationary solution is constructed under the assumptions

$$\frac{1}{2} (\partial_t \phi)^2 \ll V(\phi)$$

and neglecting all the spatial gradients.

Our analysis concerns the evolution of the cosmological model when the scalar field slow rolls on the plateau and the corresponding potential term is described as

$$V(\phi) = \Lambda_0 - \lambda U(\phi),$$

where $\Lambda_0$ behaves as an effective cosmological constant of the order $10^{15} - 10^{16}$ GeV and $\lambda (\ll 1)$ is a coupling constant associated to the perturbation $U(\phi)$.

Since the scalar field moves on a plateau almost flat, we infer that in the lowest order of approximation $\phi(t, x^\gamma) \sim \alpha(x^\gamma)$ (see below) and therefore the potential reduces to a space-dependent effective cosmological constant

$$\Lambda(x^\gamma) \equiv \Lambda_0 - \lambda U(\alpha(x^\gamma)).$$

In this scheme the $0 - 0$ and $\alpha - \beta$ components of the Einstein equations reduce, under condition (iii) and neglecting all the spatial gradients, to the simple ones

$$3 \partial_t \ln \Gamma + 3 (\partial_t \ln \Gamma)^2 = \chi \Lambda(x^\gamma)$$

and

$$(\partial_t \ln \Gamma) \delta_\beta^\gamma + 3 (\partial_t \ln \Gamma)^2 \delta_\beta^\gamma = \chi \Lambda(x^\gamma) \delta_\beta^\gamma,$$

respectively. A simultaneous solution for $\Gamma$ of both equations (10a) and (10b) takes the form

$$\Gamma(x^\gamma) = \Gamma_0(x^\gamma) \exp \left[ \frac{\chi \Lambda(x^\gamma)}{3} (t - t_0) \right],$$

respectively.
where $\Gamma_0(x^\gamma)$ is an integration function while $t_0$ a given initial instant of time for the inflationary scenario. Under the same assumptions and taking into account (11) for $\Gamma$, the scalar field equation (4) rewrites as

$$3H(x^\gamma)\partial_t\phi - \lambda W(\phi) = 0,$$

(12)

where we naturally defined

$$H(x^\gamma) = \partial_t \ln \Gamma = \sqrt{\chi \Lambda_0(x^\gamma)}, \quad W(\phi) = \frac{dU}{d\phi}.$$ 

(13)

We search a solution of the dynamical equation (12) in the form

$$\phi(t, x^\gamma) = \alpha(x^\gamma) + \beta(x^\gamma)(t - t_0).$$

(14)

Inserting expression (14) in (12) and considering it to the lowest order, we get the relation

$$3H\beta = \lambda W(\alpha), \quad W(\alpha) = \left. \frac{dU}{d\phi} \right|_{\phi=\alpha}.$$ 

(15)

This equation allows to express $\beta$ in terms of $\alpha$

$$\beta = \frac{\lambda W(\alpha)}{\sqrt{3\chi \Lambda_0 - \lambda U(\alpha)}}.$$ 

(16)

Of course the validity of solution (16) takes place in the limit

$$t - t_0 \ll \left| \frac{\alpha}{\beta} \right| = \left| \frac{\alpha}{W(\alpha)} \sqrt{\frac{\Lambda_0}{\lambda^2} - \frac{U(\alpha)}{\lambda}} \right|$$ 

(17)

where the ratio $\Lambda_0/\lambda^2$ takes in general very large values.

The $0 - \alpha$ component (2b) of the Einstein equations remains to be solved. In view of (11) and (14) through (16) this provides the relation

$$-2\sqrt{\frac{\chi}{3}} \partial_\gamma \left( \sqrt{\Lambda} \right) = \chi(\partial_\gamma \alpha) \beta = \sqrt{\frac{\chi}{3\lambda}} \lambda \partial_\gamma U$$ 

(18)

or, simplifying easily,

$$\partial_\gamma (\Lambda + \lambda U) = 0,$$

(19)

which is reduced to an identity by (9) for $\Lambda(x^\gamma)$.

The validity of the obtained inflationary solution is guaranteed by considering that all the spatial gradients, either of the three-metric field either of the scalar one, behave as $\Gamma^{-2}$ and therefore decay exponentially.

If we take into account the coordinate characteristic lengths $L$ and $l$ for the inhomogeneity scales regarding the functions $\Gamma_0$ and $\xi_{\alpha\beta}$, i.e.

$$\partial_\gamma \Gamma_0 \sim \frac{\Gamma_0}{L}, \quad \partial_\gamma \xi_{\alpha\beta} \sim \frac{\xi_{\alpha\beta}}{l},$$ 

(20)
respectively, then negligibility of the spatial gradients at the initial instant \( t_0 \) leads to the inequalities for the physical quantities

\[
\Gamma_0 l = l_{\text{phys}} \gg H^{-1}, \quad (21a)
\]
\[
\Gamma_0 L = L_{\text{phys}} \gg H^{-1}. \quad (21b)
\]

These conditions state that all the inhomogeneities have to be much greater than the physical horizon \( H^{-1} \).

The assumption made on the negligibility of the spatial gradients at the beginning of the inflation is required (as well known) by the existence of the de Sitter phase itself; however, spatial gradients having a passive dynamical role allow to deal with a fully inhomogeneous solution. This feature simply means that space point dynamically decouple to leading order.

The analysis is completed by stressing that the condition (7a) becomes

\[
W^2(\alpha) \ll \chi \left( \frac{\Lambda}{\lambda} \right)^2, \quad (22)
\]
or equivalently by (9)

\[
\lambda^2 W^2(\alpha) \ll \chi (\Lambda_0 - \lambda U(\alpha))^2 \quad (23)
\]
which, neglecting all terms in \( \lambda^2 \), simply states that the dominant contribution in \( \Lambda(x^\gamma) \) is provided by \( \Lambda_0 \), i.e.

\[
\lambda U(\alpha) \ll \Lambda_0, \quad (24)
\]
whereas (22) is always naturally satisfied. By other words, we get the only important restriction on the spatial function \( \alpha(x^\gamma) \) which reads

\[
|\alpha| \ll |U^{-1}(\Lambda_0/\lambda)|. \quad (25)
\]

In order to get a satisfactory exponential expansion able to overcome the SCM shortcomings, we require that in each space point the condition

\[
H(t_f - t_i) \sim O(10^2) \quad (26)
\]
holds, where \( t_i \) and \( t_f \) denote the instants when the de Sitter phase starts and ends, respectively. We may take \( t_i \equiv t_0 \) and \( t_f \) must satisfy the inequality

\[
t_f \ll t^* \equiv t_0 + \left| \frac{\alpha}{\beta} \right|. \quad (27)
\]

Hence we have

\[
H(t_f - t_i) \ll H(t^* - t_0) = H \left| \frac{\alpha}{\beta} \right|, \quad (28)
\]
or equivalently

\[
H(t_f - t_i) \ll \frac{\Lambda_0}{\lambda} \left| \frac{\alpha}{W(\alpha)} \right|. \quad (29)
\]
where we made use of (16). Being $\Lambda_0/\lambda$ a very large quantity, no serious restrictions appear for the e-folding of the model.

A fundamental feature of our analysis relies on the very general nature of the obtained solution; in fact, once satisfied all the dynamical equations, still eight arbitrary spatial functions remain, i.e. six for $\xi_{\alpha\beta}(x^\gamma)$, and then $\Gamma_0(x^\gamma)$, $\alpha(x^\gamma)$.

However, taking into account the possibility to choose an arbitrary gauge via the set of the spatial coordinates, we have to kill three degrees of freedom; hence five physically arbitrary functions finally remain: four corresponding to gravity degrees of freedom and one related to the scalar field.

This picture corresponds exactly to the allowance of specifying a generic Cauchy problem for the gravitational field, on a spatial non-singular hypersurface, nevertheless one degree of freedom of the scalar field is lost against the full generality.

3. Coleman–Weinberg Model

Let us specify our solution in the case of the Coleman and Weinberg (CW) zero-temperature potential

$$V(\phi) = \frac{B\sigma^4}{2} + B\phi^4 \left[ \ln \left( \frac{\phi^2}{\sigma^2} \right) - \frac{1}{2} \right]$$

(30)

where $B \approx 10^{-3}$ is connected to the fundamental constants of the theory, while $\sigma \approx 2 \cdot 10^{15}$GeV gives the energy associated with the symmetry breaking process.

In the region $|\phi| \ll |\sigma|$ the potential (30) approaches a plateau behaviour profile similar to (8) and acquires the form

$$V(\phi) \approx \frac{B\sigma^4}{2} - \frac{\lambda}{4} \phi^4,$$

$\lambda \approx 80B \approx 0.1$. (31)

This is effectively reducible to (8) by

$$\Lambda_0 = \frac{B\sigma^4}{2}, \quad U(\phi) = \frac{\phi^4}{4}, \quad W(\phi) = \phi^3,$$

(32)

and the relations (16) and (22) rewrite

$$\beta = \frac{\lambda\alpha^3}{3H},$$

(33a)

$$\alpha^3 \left( \alpha + \sqrt{\frac{8}{3\chi}} \right) \ll \frac{\Lambda_0}{\lambda} \approx \frac{\sigma^4}{160},$$

(33b)

respectively. The inequality in (33a) is equivalent to fulfil the initial assumption

$$\Lambda_0 \gg \lambda U(\alpha) \approx \frac{\lambda}{4} \alpha^4,$$

(34)

like as in (24).

The restriction (24) reflects over the free function $\alpha$ as

$$|\alpha| \ll \sqrt[4]{\frac{\Lambda_0}{\lambda}} \approx \sigma.$$  

(35)
4. Towards FLRW Universe

The conditions (21) state that the validity of the inhomogeneous inflationary scenario discussed in the previous Section requires the inhomogeneous scales to be out of the horizon when inflation starts. The situation is different when treating the small perturbations to the FLRW case; in fact, the negligibility of the spatial curvature corresponds to require the radius of curvature of the universe to be much greater than the physical horizon, the inhomogeneous terms being small in amplitude. To this end, let us consider the three-metric

\[ \gamma_{\alpha\beta} = \Gamma(t, \varphi^\mu)^2 \left[ h_{\alpha\beta} + (t - t_0)\delta\theta_{\alpha\beta}(\varphi^\mu) \right], \]

(36)

where \( h_{\alpha\beta} \) denotes the FLRW spatial part of the three metric (\( \{\varphi^\mu\} \) are the three usual angular coordinates) and \( \delta\theta_{\alpha\beta} \) denote a small inhomogeneous perturbation.

The Einstein equations (2) coupled to the scalar field dynamics (4) on the plateau (31) admit, to leading order in the inhomogeneities, the solution

\[ \Gamma = \Gamma_0 e^{H(t-t_0)}, \quad H = H_0 - \frac{\delta\theta}{6}, \]

(37a)

\[ \phi = \alpha_0 \left[ 1 + \frac{\lambda\alpha_0^2}{3H_0}(t - t_0) \right] + \frac{\delta\theta}{3\chi} \left[ 1 + \frac{\lambda\alpha_0^2}{H_0}(t - t_0) \right], \]

(37b)

\[ \delta\theta_{\alpha\beta} = \frac{\delta\theta}{3} h_{\alpha\beta}, \quad H_0 = \sigma^2 \sqrt{\frac{\chi B}{6}}, \]

(37c)

where \( t_0 \) and \( \Gamma_0 \) are constants. The solution (37) holds and provides the correct e-folding of order \( \mathcal{O}(10^2) \) when the following inequalities take place

\[ t - t_0 \ll \frac{3H_0}{\lambda\alpha_0^2}, \]

(38a)

\[ \alpha_0 \ll \mathcal{O} \left( \frac{1}{10\sqrt{\frac{\Lambda_0}{\lambda}}} \right), \]

(38b)

\[ R_{\text{curv}} = \frac{\Gamma_0}{\sqrt{\mathcal{K}}} \gg H_0^{-1}, \]

(38c)

\[ \Gamma_0 l = l_{\text{phys}} \gg H_0^{-1}\delta, \quad l_{\text{phys}} \gg \frac{\delta}{\sqrt{\lambda\alpha_0^2}}. \]

(38d)

in (38c), \( \mathcal{K} \) is the signature of the spatial curvature, while \( \delta (\ll H_0/100) \) and \( l \) in (38c) denote the characteristic amplitude and length, respectively, of the arbitrary function \( \delta\theta \) which is the trace of the tensor \( \delta\theta_{\alpha\beta} \). The inequality (38a) ensures that the dominant term of the scalar field remains the time-independent one during the de Sitter phase; inequality (38b) allows for an e-folding of order \( \mathcal{O}(10^2) \); finally, equations (38c) and (38d) provide the negligibility of the spatial gradients in the Einstein and scalar field equations, respectively. When inflation starts, in agreement with (38c) and (38d), the inhomogeneous scales can be inside the physical horizon \( H_0^{-1} \).

The physical implications on the density perturbation spectrum of such a nearly homogeneous model rely on the dominant behaviour of the potential term over
the energy density $\rho_\phi$ associated to the scalar field during the de Sitter phase and therefore
\[ \Delta \equiv \left| \frac{\delta \rho_\phi}{\rho_\phi} \right| \sim \left| \frac{d \ln V}{d \phi} \delta \phi \right| \simeq \frac{\lambda}{\Lambda_0} W(\alpha_0) \delta \alpha, \] (39)
where $\delta \alpha = \delta \theta/(3 \chi)$ for our scalar field solution (37). In particular, in the CW case (39) reduces to
\[ \Delta_{CW} \simeq \frac{5 \theta}{\sigma^4 \alpha_0^3} \frac{\delta \theta}{\chi}. \] (40)

However, to get an information about the problem of computing the physically relevant perturbations after the scales re-entry in the horizon, we have to deal with the gauge invariant quantity $\zeta$ (62) which has the form
\[ \zeta = \frac{\delta \rho}{\rho + p} \simeq \frac{\delta \theta}{W(\alpha_0)} \frac{\Lambda_0}{\lambda} \] (41)
when the perturbations leave the horizon and $\rho + p = (\partial_t \phi)^2$; in the CW case it reads as
\[ \zeta_{CW} = \frac{\sigma^4}{160} \frac{\delta \theta}{\alpha_0^3}. \] (42)

Since $\zeta$ remains constant during the super-horizon evolution of the perturbations, then at the re-entry to the causal scale in the matter-dominated era, we get $\zeta_{MD} \sim \delta \rho/\rho \sim \zeta_{CW}$.

By restoring physical units and assuming $\alpha_0 \lesssim 10^{-4} \sigma/\sqrt{hc}$ in agreement with (36), then it is required $\delta \alpha/\alpha_0 \lesssim 10^{-2}$ in order to obtain perturbations $\delta \rho/\rho \sim 10^{-4}$ at the horizon re-entry during the matter-dominated age.

Hence the expression (42) explains how the perturbation spectrum after the de Sitter phase can still arise from classical inhomogeneous terms. Indeed, the function $\delta \theta(\phi)$ is an arbitrary one and can be chosen for it a Harrison–Zeldovich spectrum by assigning its Fourier transform as
\[ |\delta \alpha(k)|^2 \propto \frac{\text{const.}}{k^3}; \] (43)
such a spectrum has to hold for $k \ll \frac{\Gamma_m}{H_0^\gamma}$.

Thus, the pre-inflationary inhomogeneities of the scalar field remain almost of the same amplitude during the de Sitter phase as a consequence of the linear form of the scalar field solution (14). Hence we get that the Harrison–Zeldovich spectrum can be a pre-inflationary picture of the density perturbations and it survives to the de Sitter phase, becoming a classical seed for structure formation. The existence of such a classical spectrum is not related with the quantum fluctuations of the scalar field whose effect is an independent contribution to the classical one.
5. Concluding Remarks

The merit of our analysis relies on having provided a dynamical framework within which classical inhomogeneous perturbations to a real scalar field minimally coupled with gravity can survive even after that the de Sitter expansion of the universe stretched the geometry; the key feature underlying this result consists (i) of constructing an inhomogeneous model for which the leading order of the scalar field is provided by a spatial function and then (ii) of showing how the very general case contains as a limit a model close to the FLRW one.

It is relevant to remark that the metric tensor (36) seems of the same form as the one considered in 14, however in the present paper the function \( \eta(t) \) appearing in the previous work is linear in time and does not decay exponentially. The different behaviour relies on the negligibility of the matter with respect to the scalar field which is at the ground of the present analysis. We are here assuming the dynamics of \( \eta(t) \) to be driven by the scalar field alone, instead of by the ultra-relativistic matter. This situation corresponds to an initial conditions for which the scalar field dominates over the ultra-relativistic matter when inflation starts and this is the reason for the resulting different issues of the two analyses.

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