On the genealogy of branching random walks and of directed polymers

BERNARD DERRIDA\textsuperscript{1,2} and PETER MOTTISHAW\textsuperscript{3}

\textsuperscript{1} Collège de France - 11 Place Marcelin Berthelot, 75005 Paris, France
\textsuperscript{2} LPS, École Normale Supérieure - 24 rue Lhomond, 75005 Paris, France
\textsuperscript{3} School of Physics and Astronomy, University of Edinburgh - James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, UK

received 22 July 2016; accepted in final form 31 August 2016
published online 22 September 2016

PACS 02.50.-r – Probability theory, stochastic processes, and statistics
PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion
PACS 65.60.+a – Thermal properties of amorphous solids and glasses: heat capacity, thermal expansion, etc.

Abstract – It is well known that the mean-field theory of directed polymers in a random medium exhibits replica symmetry breaking with a distribution of overlaps which consists of two delta functions. Here we show that the leading finite-size correction to this distribution of overlaps has a universal character which can be computed explicitly. Our results can also be interpreted as genealogical properties of branching Brownian motion or of branching random walks.

Copyright © EPLA, 2016

Introduction. – The study of branching Brownian motion and of branching random walks is central in the theory of probability \cite{1–5} and appears in several physical contexts \cite{6–11}. Here we focus on its relevance in the mean-field theory of directed polymers in a random medium \cite{12–15}. This mean-field version is an example of a disordered system which exhibits replica symmetry breaking in its low-temperature phase \cite{16–18}. In the models considered here only one step of replica symmetry breaking is required in the thermodynamic limit \cite{19}. Our motivation here is to determine how this broken symmetry is affected by finite-size fluctuations. In the present work we will show that the one-step replica symmetry breaking is smoothed in a universal way for which one can obtain an analytic expression.

To study the problem of directed polymers in its mean-field version, one may consider a binary tree of height $t$ as in fig. 1. On each edge $b$ of this tree there is a random energy $\epsilon_b$ distributed according to a given distribution $\rho(\epsilon)$. Then the possible configurations of the polymer are the $2^t$ paths of length $t$ connecting the top of the tree to its bottom. To each of these paths $i$ one associates an energy $X_i(t)$ which is simply the sum of the energies of all the bonds visited by this path,

$$X_i(t) = \sum_{b \in i} \epsilon_b.$$  

![Fig. 1: A binary tree of height $t$. On each edge there is a random energy $\epsilon_b$. The configurations of a directed polymer are the $2^t$ paths connecting the top of the tree to its bottom.](image)

The partition function is then given by

$$Z_t = \sum_{i=1}^{\mathcal{N}(t)} e^{-\beta X_i(t)},$$

where the sum is over the $\mathcal{N}(t) = 2^t$ configurations of the directed polymer, $\beta$ is the inverse temperature and the weight $W_i$ of a path is

$$W_i = \frac{e^{-\beta X_i(t)}}{Z_t}.$$  

(1)
For a general distribution $\rho(\epsilon)$ of the energies $\epsilon_b$, it is known [20] that there is a phase transition at $\beta_c$ given by the value of $\beta$ which minimizes the function $v(\beta)$ defined by

$$v(\beta) = \frac{g(\beta)}{\beta} + \log \frac{2}{\beta},$$

(2)

where

$$g(\beta) = \log \left[ \int \rho(\epsilon)e^{-\beta \epsilon}d\epsilon \right],$$

(3)

i.e., $\beta_c$ is the solution of

$$v'(\beta_c) = 0.$$  

(4)

It is also known that in the low-temperature phase ($\beta > \beta_c$), the extensive part of the entropy vanishes and that the partition function is dominated by the configurations whose difference of energy with the ground state is non-extensive.

In the directed polymer problem on a tree the overlap $q_{i,j}$ between two paths is simply the fraction of their length that the two paths share. So that, $q_{i,i} = 1$, while

$$q_{i,j} = \frac{r}{l}$$

if the two paths $i$ and $j$ have $r$ bonds in common. The distribution of overlaps $P(q)$ is then defined by

$$P(q) = \left\langle \sum_{i,j} W_i W_j \delta(q_{i,j} - q) \right\rangle,$$

(5)

where $\langle \cdot \rangle$ denotes an average over all the random energies $\epsilon_b$.

It is well established [13,15,20] that in the low-temperature phase ($\beta > \beta_c$) and in the limit $t \to \infty$, this distribution $P(q)$ is the sum of two $\delta$-functions,

$$P(q) = \frac{\beta_c}{\beta} \delta(q) + \left(1 - \frac{\beta_c}{\beta}\right) \delta(q - 1).$$

(6)

In the low-temperature phase the distribution $P(q)$ of overlaps is non-trivial (in the sense that, in the large-$t$ limit, it does not reduce to a single $\delta$-function). This is one of the signatures of replica symmetry breaking [19]. The mean-field directed polymer problem is one of the simplest examples of such a broken symmetry: it exhibits a one-step broken replica symmetry since the distribution of overlaps $P(q)$ consists of a sum of two $\delta$-functions (6). In the present work we study the leading finite-size corrections to (6). We will show that for large $t$, $0 < q < 1$ and $\beta > \beta_c$,

$$P(q) \approx \frac{1}{\sqrt{\beta \epsilon_c}} \sqrt{\frac{1}{2\pi \epsilon_c \beta_v(\beta_c)}} \left(q(1-q)^{-\frac{3}{2}} \right)$$

and similarly when $\beta = \beta_c$

$$P(q) \approx \frac{1}{\sqrt{\beta_c \beta_v(\beta_c)}} \sqrt{\frac{1}{2\pi \beta_c \beta_v(\beta_c)}} q^{-\frac{3}{2}} (1-q)^{-\frac{3}{2}}.$$  

(7)

Note that for (7) and (8) to be valid we need $t$ large, $q$ arbitrary within the constraints $q t \gg 1$ and $(1 - q)t \gg 1$.

We have obtained expressions (7), (8) by making an approximation which consists in replacing the problem of the directed polymer on a tree by a generalized random energy model (GREM) [21–23]. The details of these calculations, which are too long to be given here, will be described in a forthcoming paper. In this letter we will only present an analytic argument leading to (7), (8) and numerical calculations which support these expressions. For a reason discussed below, we believe that the GREM approximation gives not only the right $t$- and $q$-dependence in (7), (8) but also the correct prefactors.

**Binary tree recursions.** — Before discussing our argument based on the GREM approximation we show how to determine numerically the distribution of overlaps $P(q)$ by recursion relations. For a given realization of the energies $\epsilon_b$ on the tree the partition function satisfies the following recursion:

$$Z_{t+1} = e^{-\beta \epsilon_1} Z_{t+1}^{(1)} + e^{-\beta \epsilon_2} Z_{t+1}^{(2)},$$

(9)

where $\epsilon_1$ and $\epsilon_2$ are distributed according to the density $\rho(\epsilon)$ and $Z_{t+1}^{(1)}$ and $Z_{t+1}^{(2)}$ are the partition functions of two independent copies of a tree of height $t$. Initially $Z_0 = 1$. If one introduces the generating function $G_t(x)$ defined by

$$G_t(x) = \langle \exp \left[-e^{-\beta x} Z_t\right]\rangle,$$

where the average is over all the random energies $\epsilon_b$, it is easy to see, using (9), that it satisfies

$$G_{t+1}(x) = \left[ \int \rho(\epsilon)de G_t(x + \epsilon) \right]^2 \equiv \langle (G_t(x + \epsilon))\rangle^2,$$

(10)

where $\langle \cdot \rangle_\epsilon$ denotes an average over a single random energy $\epsilon$. As $Z_0 = 1$ the initial condition is

$$G_0(x) = \exp \left[-e^{-\beta x}\right].$$

(11)

One can notice [13] that the recursion (10) on $G_t$ does not depend on the inverse temperature $\beta$. The temperature enters only in the initial condition (11).

Now to calculate the distribution $P(q)$ of overlaps, it is convenient to introduce the probability $Q_t(r)$ that two paths $i,j$ have an overlap such that $t q_{i,j} \geq r$ (obviously $Q_t(0) = 1$). We are going to show that

$$Q_t(r) = \frac{2r}{\beta} \int_{-\infty}^{\infty} dx H_t^{(1-r)}(x),$$

(12)

where $H_t^{(1-r)}(x)$ is obtained by the following recursion for $t \geq s$:

$$H_t^{(1-s)}(x) = \langle G_t(x + \epsilon)\rangle \langle H_t^{(s)}(x + \epsilon)\rangle,$$

(13)

with the initial condition

$$H_s^{(s)}(x) = e^{-\beta \beta_c} \frac{d}{dx} \left( e^{\beta \beta_c} \frac{d}{dx} G_s(x) \right).$$

(14)
It is clear from the definition of $Q_t(r)$ that

$$\text{Prob} \left( q = \frac{r}{t} \right) = Q_t(r) - Q_t(r+1) \quad (15)$$

(with the convention that $Q_t(0) = 0$) allowing one to determine $P(q)$ by

$$P(q) dq = P(q) \cdot \frac{1}{t} = \text{Prob} \left( q = \frac{r}{t} \right) \quad (16)$$

since increasing $r$ by 1 changes $dq$ by $1/t$.

To justify (12)–(14) one can consider the following modified partition function:

$$\Xi_t^{(r)}(\phi_1, \cdots, \phi_{2^r}) = \sum_{j=1}^{2^r} e^{\beta j - \beta X_j(r)} Z_{t-r}^{(j)},$$

where the $X_j(r)$ are the $2^r$ energies of a tree of height $r$ and the $Z_{t-r}^{(j)}$ are $2^r$ independent partition functions. It is easy to see that

$$Q_t(r) = \left\langle \sum_{j=1}^{2^r} \left[ e^{-\beta X_j(r)} Z_{t-r}^{(j)} \right]^2 \right\rangle = -\frac{1}{\beta^2} \sum_{j=1}^{2^r} \frac{\partial}{\partial \phi_j} \left( e^{-\beta \phi_j} \times \frac{\partial}{\partial \phi_j} \left( \log \Xi_t^{(r)}(\phi_1, \cdots, \phi_{2^r}) \right) \right)|_{\phi_1 = \cdots = \phi_{2^r} = 0}. \quad (17)$$

This expression shows that the calculation can be limited to the case where only one $\phi_1$, say $\phi_1$, is non-zero. Then it is easy to check that $\mathcal{H}_t^{(s)}$ defined by

$$\mathcal{H}_t^{(s)}(x, \phi_1) = \left\langle \exp \left[ -e^{-\beta x} \Xi_t^{(t-s)}(\phi_1, 0 \cdots 0) \right] \right\rangle \quad (18)$$

satisfies for $t \geq s$ the following recursion:

$$\mathcal{H}_t^{(s)}(x, \phi_1) = \langle G_t(x + \epsilon) \rangle_x \langle \mathcal{H}_t^{(s)}(x + \epsilon, \phi_1) \rangle_{\epsilon} \quad (19)$$

with the initial condition

$$\mathcal{H}_s^{(s)}(x, \phi_1) = G_s(x - \phi_1). \quad (20)$$

Then (using the identity $\log x = \int_0^\infty (e^{-u} - e^{-ux}) du/u$) one can see that

$$\left\langle \log \Xi_t^{(r)}(\phi_1, 0 \cdots 0) \right\rangle = \int_0^\infty e^{-u} - \left\langle \exp \left[ -u \Xi_t^{(r)} \right] \right\rangle du = \beta \int [G_0(x) - \mathcal{H}_t^{(t-r)}(x, \phi_1)] dx. \quad (21)$$

The last step to complete the derivation of (12)–(14) is to take two derivatives with respect to $\phi_1$ in (19)–(21).

Comparison with the predictions (7), (8). – The dashed lines in figs. 2 and 3 show the rescaled distributions $t^{3/2} P(q)$ vs. $r/t$ obtained by a numerical integration of (10)–(16) for $t = 50, 150$ and 450 and a $\rho(\epsilon)$ of the form

$$\rho(\epsilon) = \rho(\epsilon + 1) + (1 - \rho) \delta(\epsilon).$$

We choose $p = (2 - \sqrt{3})/4 \approx 0.669873$. For this choice $\beta_k = 2 \log(2 + \sqrt{3}) \approx 2.63392$. $v(\beta_k) = 1/2$. $v(\beta_k) = 0$. $\beta_k v(\beta_k) = 1/4$. (The reason for chosing this value of $p$ is that $\beta_k$ has an exact expression. For other choices of $p$ one can determine $\beta_k$ by solving numerically the equation $v'(\beta_k) = 0$. Doing so we obtained results very similar to the ones shown in figs. 2 and 3.) As $t$ increases the results of the numerical integration of the recursion relations (10)–(16) seem to converge to the predictions of (7), (8) represented by the continuous curves.

The GREM approximation. – Let us now explain how (7), (8) can be derived by an argument based on a GREM approximation. The approximation consists in

40005-p3
replacing the tree of height \( t \) of fig. 1 by a “renormalized” tree of height \( \tau = t/n \) with at each level \( 2^n \) branches and with, on each bond \( b \) of this renormalized tree, an energy \( E_b \) which is the sum of \( n \) independent energies \( \epsilon \) distributed according to \( \rho(\epsilon) \). Varying \( n \) interpolates between the original binary tree \((n = 1)\) and a random energy model \((n = t)\) [24]. (We repeated the numerical calculations shown in figs. 2 and 3 in the case \( n = 2 \) and, for large enough \( t \), the numerical results were indistinguishable on the figure from those for \( n = 1 \). As (7), (8) (obtained as we will see by taking first the limit \( n \rightarrow \infty \) with \( \tau \) fixed and then the limit \( \tau \rightarrow \infty \)) agree so well with the numerical data of figs. 2 and 3, we believe that all values of \( n \), as long as \( \tau \) remains large, give the same results (7), (8) in the limit \( t \rightarrow \infty \).

In the GREM approximation, \( \tau \) is fixed and we take the limit of large \( n \). Then we take the large \( \tau \) limit at the end.

In the low-temperature phase (that is for \( \beta \geq \beta_c \) and at each level of the GREM, only the energies sufficiently close to the minimal energy (at distance of order \( \sqrt{n} \) when \( \beta \geq \beta_c \)) contribute. We are going to show that at level \( p \) from the top of the GREM tree of height \( \tau \), these energies, close to this minimal energy, are well described by the points of a Poisson process with density \( \tilde{\rho}_p(E) \) given by

\[
\tilde{\rho}_p(E) \simeq e^{\beta v_p(E)} F_p(E - E_p),
\]

where

\[
F_p(E) = \left( \frac{B}{\pi} \right)^{\frac{\beta}{2}} \int_0^\infty \cdots \int_0^\infty dx_{p-1} \cdots dx_1 \times \exp \left[ -B \left( E - x_{p-1} \right)^2 + (x_{p-1} - x_{p-2})^2 \right. \\
\left. + \cdots (x_2 - x_1)^2 + x_1^2 \right]
\]

and

\[
E_p = -p \mu v(\beta_c); \quad B = \left( 2n \beta_c v''(\beta_c) \right)^{-1}.
\]

We will need below the following two large-\( p \) asymptotics:

\[
F_p(0) = \sqrt{\frac{B}{\pi p}}^{\beta/2},
\]

\[
\int_0^\infty F_p(E) dE \simeq \frac{2^{p+1}}{4^p p!} \left( \frac{2 \beta}{\pi p} \right)^{\beta/2} \simeq \frac{1}{\sqrt{\pi}} p^{-1/2}
\]

(25), (26), valid for all \( p \geq 1 \), are exact [25], (see appendix B of [23]).

To justify (22)–(24) we note that, for large \( n \), the energy \( E \) on each bond of the GREM is the sum of \( n \) random variables \( \epsilon \). The distribution of this energy \( E \) is characterized by a large deviation function \( f(\epsilon) \),

\[
P(E = n\epsilon) dE \sim \frac{1}{\sqrt{n \pi f''(\epsilon)}} e^{-n f(\epsilon)} dE,
\]

which can be written in a parametric form (\( \epsilon = -g'(\beta) \) and \( f(\epsilon) = -g(\beta) + \beta g'(\beta) \)) in terms of the generating function \( g(\beta) \) defined in (2) (see the appendix). For \( 2^n \) independent such energies, the minimal one is close to an energy \( E_1 = n\epsilon_c \) such that

\[
\log 2 - f(\epsilon_c) = 0,
\]

(i.e. \( \epsilon_c = -g'(\beta_c) \) where \( \epsilon_c \) is solution of \( v'(\beta_c) = 0 \)).

Near this energy, in the range \( |E - E_1| \lesssim O(\sqrt{n}) \), one can approximate \( P(E) \) by

\[
P(E) \simeq \sqrt{\frac{B}{\pi}} e^{\beta v(\epsilon_c) - B E^2}.
\]

(28) (see the appendix). Then one can see that expressions (22)–(24) follow simply from the recursion

\[
\tilde{\rho}_{p+1}(E) = \int_{E_{p}-A}^{\infty} P(E - E') \tilde{\rho}_p(E') dE'
\]

with the initial condition

\[
\tilde{\rho}_1(E) = P(E),
\]

where \( A \) is a cut-off \( 1 \ll A \ll \sqrt{n} \) (for example \( A = n^{1/2} \)).

Conditioned on this fluctuation (of having a minimal energy \( E_p + y \) at level \( p \) from the top) the minimal energy \( E_p + y \) takes a value of order \( \log n \) lower than its typical value. A similar idea was already used to understand the fluctuations of the position or the genealogy of branching random walks with selection [26, 27] and corrections to the position of a traveling wave [28].

To be quantitative let \( \psi(y) \) be the probability of finding, at level \( p \) from the top of the tree, a minimal energy \( E_p + y \). One has

\[
\psi(y) = \tilde{\rho}_p(E_p + y) \exp \left[ -\int_{-\infty}^y \tilde{\rho}_p(E_p + y') dy' \right]
\]

(30) (because we anticipate that \( y \sim -\log n \), one can drop the exponential and evaluate (22) using the fact that \( F_p(y) \simeq F_p(0) \)).

Conditioned on this fluctuation (of having a minimal energy \( E_p + y \) at level \( p \) from the top) the energies at the bottom of the tree are the points of a Poisson process whose density is

\[
\tilde{\rho}^{\text{total}}(E) = \tilde{\rho}_{\tau-p}(E - E_p - y) + \tilde{\rho}_p(E)
\]

\[
\sim e^{\beta v(\epsilon)} \left( e^{-\beta y} F_{\tau-p}(E - E_r + y) + F_r(E - E_r) \right),
\]

(31)
where the first term represents the subtree generated by the minimal energy $E_p + y$ at level $\tau - p$ and the second term represents the rest of the tree.

Then averaging over all realizations of the Poisson process and on all the paths gives for the probability that two paths (of the subtree) chosen according to their Boltzmann weights merge at level $p$

$$\text{Pro} \left( q = \frac{p}{\tau} \middle| y \right) = \int_0^\infty udv \int dE' \int dE''
\times \bar{\rho}_{\tau-p}(E' - E_p - y) \bar{\rho}_{\tau-p}(E'' - E_p - y) e^{-\beta(E' + E'')}
\times \exp \left[ -u \left( e^{-\beta E'} + e^{-\beta E''} \right) \right]
+ \int dE (e^{-\beta E} - 1) \bar{\rho}_{\text{total}}(E),$$

(32)

where we have used the identity $\frac{X^2}{(X + Y)^2} = \int_0^\infty udv X e^{-u(X + Y)}$.

To complete the derivation of (7), (8) one has to discuss separately the two cases $\beta > \beta_c$ and $\beta = \beta_c$.

For $\beta > \beta_c$ only energies $E$ at distances $|E - E_\tau| = O(\log n) \ll \sqrt{n}$ contribute and one can approximate the expression (31) of $\bar{\rho}_{\text{total}}(E)$ by

$$\bar{\rho}_{\text{total}}(E) \sim e^{\beta_c(E - E_\tau)} \left[ e^{-\beta_c y} F_p(0) + F_t(0) \right]$$

where $E_\tau = -n\tau v(\beta_c) = -tv(\beta_c)$. Using (32) this leads to

$$\text{Pro} \left( q = \frac{p}{\tau} \middle| y \right) = \frac{\beta_c}{\beta} \left( \frac{F_{\tau-p}(0) e^{-\beta_c y}}{F_{\tau-p}(0) e^{-\beta_c y} + F_t(0)} \right)^2.$$

On the other hand, for $\beta = \beta_c$, the integrals over energies in (32) are dominated by the range $E - E_\tau = O(\sqrt{n})$ and one gets

$$\text{Pro} \left( q = \frac{p}{\tau} \middle| y \right) = \frac{1}{\beta} \left( \frac{F_{\tau-p}(0) e^{-\beta_c y}}{F_t(0)} \right)^2 \left( \frac{\tau}{p(\tau - p)} \right)^\frac{1}{2},$$

(33)

and for $\beta = \beta_c$

$$\text{Pro} \left( q = \frac{p}{\tau} \right) = \frac{1}{\beta_c} \left( \frac{F_{\tau-p}(0) \int_0^\infty F_p(E) dE}{\int_0^\infty F_t(E) dE} \right)^\frac{1}{2} \left( \frac{\tau}{p(\tau - p)} \right)^\frac{1}{2},$$

(34)

where we have used the value (24) of $B$ and the asymptotics (25), (26). The last step to obtain (7), (8) is to replace $\tau$ by $\frac{\tau}{p}$ and $p$ by $\tau$ in the above expressions (33), (34) and the fact that

$$\text{Pro} \left( q = \frac{p}{\tau} \right) = \int_q^{q+\frac{1}{2}} P(q') dq' \approx \frac{1}{\tau} P(q).$$

The definition of $P(q)$ given in (5) can be extended to overlaps of more than two paths, and the approach given above can be used to compute these overlaps. We will describe this in the follow-up publication to this one.

**Branching Brownian motion.** It is straightforward to repeat the argument in the case of branching Brownian motion. A difference is that the number $N(t)$ of end points at the bottom of the tree fluctuates but if the $X_i(t)$ are the positions of these end points of this branching Brownian motion, the analysis is essentially the same. If we take a branching Brownian motion characterized by a branching rate 1 (i.e. which has a probability $dt$ of branching during each infinitesimal time interval $dt$) and which diffuses according to $\langle \text{d}x^2 \rangle = 2dt$, the only changes are that $G(x)$ and $H(x)$ become continuous functions of $t$ with the evolution equations (10), (13) replaced by

$$\frac{dG}{dt} = \frac{d^2G}{dx^2} + G^2 - G$$

which is the Fisher-KPP equation [1] and

$$\frac{dH}{dt} = \frac{d^2H}{dx^2} + (2G - 1)H.$$

The velocity $v(\beta)$ in (2) is then replaced by

$$v(\beta) = \frac{\beta + 1}{\beta}.$$

With this choice, one has $\beta_c = 1, v(\beta_c) = 2, v''(\beta_c) = 2$ and we expect (7), (8) to hold for two end points of branching Brownian motion sampled according to (1) have their most recent common ancestor at time $dt$.

The distributions (7), (8) are characteristic of the large $t$ properties of the edge of the branching Brownian motion [15,29–35]. If instead of choosing the end points sampled by (1), we take any pair of points close to the leftmost particle, for example the two leftmost points of the branching Brownian motion, or the 5th and the 7th leftmost particles, expression (7) would hold (up to the prefactor $\beta_c / \beta$ which would not be there).

**Conclusion.** In the past, the understanding of finite-size corrections and of the contribution of fluctuations in systems with one replica symmetry breaking has been a puzzling question [36–38]. In the present work we have computed, without recourse to the replica trick, the finite-size corrections of the distribution of overlaps of the mean-field version of the directed polymer. Our results extend those obtained last year [39] on the random energy model.
to a more complex case. Although the random energy model and the directed polymer have exactly the same distribution of overlaps (6) in the thermodynamic limit, and are both representative examples of systems with one broken symmetry of replica, their finite-size corrections are quite different: here they are of order $t^{-\frac{1}{2}}$ while for the random energy model they are of order $t^{-1}$. Trying to see whether our result (7), (8) could be recovered by the replica approach is in our opinion an interesting open question.

Also determining the genealogies of directed polymers in a non–mean-field case, in particular in 1+1 dimension, where major progresses have been done recently [40–44], also determining the genealogies of directed polymers in a non–mean-field case, in particular in 1+1 dimension, where major progresses have been done recently [40–44], is the random energy model they are of order $\log 2 + o(1)$, $\log 2 + o(1)$ will be published in a forthcoming work.

***

We thank Eric Brunet and Bastien Mallein for useful discussions. We would also like to thank the Higgs Centre and the Institute for Condensed Matter and Complex Systems at the University of Edinburgh for their kind hospitality.

Appendix

In this appendix we explain the relations between the large deviation function (27), the generating function (3) and the velocity (2). If $E$ is the sum of $n$ random variables $\epsilon$ distributed according to a distribution $\rho(\epsilon)$ one has

$$\mathbb{E} e^{-\beta E} = \rho(\beta)$$

with $g(\beta)$ given by (3). As

$$\mathbb{E} e^{-\beta E} = \int P(E) e^{-\beta E} dE$$

one has for large $n$

$$P(E) \simeq \sqrt{\frac{n f''(\frac{E}{n})}{2\pi}} \exp \left[ -nf\left(\frac{E}{n}\right) \right]$$

with

$$g(\beta) = \max_{\epsilon} \left[ -\beta \epsilon - f(\epsilon) \right].$$

(A.1)

If one knows $g(\beta)$ one can obtain $f(\epsilon)$ in a parametric form,

$$\epsilon = -g'(\beta); \quad f(\epsilon) = -g(\beta) + \beta g'(\beta)$$

(A.2)

by inverting the Legendre transform (A.1). This implies in particular that

$$f'(\epsilon) = -\beta; \quad f''(\epsilon) = \frac{1}{g''(\beta)}$$

(A.3)

In order to determine the solution $\epsilon_0$ of (28)

$$\log 2 - f(\epsilon) = 0$$

one has to find the value $\beta_\epsilon$ such that

$$\log 2 + g(\beta_\epsilon) - \beta_\epsilon g'(\beta_\epsilon) = 0,$$

i.e., the solution of

$$v'(\beta_\epsilon) = 0,$$

where $v(\beta)$ is defined as in (2)

$$v(\beta) = \frac{g(\beta) + \log 2}{\beta}.$$

Therefore, near $\epsilon_0$, one has

$$f(\epsilon) = \log 2 + f'(\epsilon_0)(\epsilon - \epsilon_0) + \frac{1}{2} f''(\epsilon_0)(\epsilon - \epsilon_0)^2 + \cdots$$

$$= \log 2 - \beta_\epsilon (\epsilon - \epsilon_0) + \frac{1}{2} \beta_\epsilon^2 v''(\beta_\epsilon) (\epsilon - \epsilon_0)^2 + \cdots$$

and replacing $\epsilon$ by $E/n$ and $\epsilon_0$ by $E_t/n$ one gets (29) and (24) using the fact that $g''(\beta_\epsilon) = \beta_\epsilon v''(\beta_\epsilon)$.

REFERENCES

[1] McKean H. P., Commun. Pure Appl. Math., 28 (1975) 323.
[2] Bramson M., Mem. Am. Math. Soc., 44, No. 285 (1983).
[3] Lalley S. P. and Sellke T., Ann. Probab., 15 (1987) 1052.
[4] Shi Z., Branching Random Walks (Springer International Publishing) 2016.
[5] Bovier Anton, Gaussian Processes on Trees (Cambridge University Press) 2016.
[6] Krapivsky P. L. and Majumdar S. N., Phys. Rev. Lett., 85 (2000) 5492.
[7] Majumdar S. N. and Krapivsky P. L., Phys. Rev. E, 62 (2000) 7735.
[8] Majumdar S. N. and Krapivsky P. L., Phys. Rev. E, 65 (2002) 036127.
[9] Montanari A., Eur. Phys. J. B, 18 (2000) 121.
[10] Munier S., Phys. Rep., 473 (2009) 1.
[11] Ramola K., Majumdar S. N. and Schehr G., Phys. Rev. E, 91 (2015) 042131.
[12] Halfin-Healy T. and Zhang Y. C., Phys. Rep., 254 (1995) 215.
[13] Derrida B. and SPOHN H., J. Stat. Phys., 51 (1988) 817.
[14] Hu Y. and Shi Z., Ann. Probab., 37 (2009) 794.
[15] Arguin L. P., Bovier A. and Kistler N., Commun. Pure Appl. Math., 64 (2011) 1647.
[16] Mezard M. and Parisi G., J. Phys. I, 1 (1991) 809.
[17] Mezard M. and Parisi G., J. Phys. A: Math. Gen., 23 (1990) L1229.
On the genealogy of branching random walks and of directed polymers

[18] PARISI G., J. Phys., 51 (1990) 1595.
[19] MEZARD M., PARISI G. and VIRASORO M. A., Spin Glass Theory and Beyond, Lect. Notes Phys., Vol. 9 (World Scientific) 1987.
[20] DERRIDA B., Physica A, 163 (1990) 71.
[21] DERRIDA B., J. Phys. Lett., 46 (1985) 401.
[22] DERRIDA B. and GARDNER E., J. Phys. C: Solid State Phys., 19 (1986) 2253.
[23] COOK J. and DERRIDA B., J. Stat. Phys., 63 (1991) 505.
[24] SCHMIDT M. A. and KISTLER N., Electron. Commun. Probab., 20 (2015) 1.
[25] BRUNET E., private communication.
[26] BRUNET E., DERRIDA B., MUELLER A. H. and MUNIER S., Phys. Rev. E, 73 (2006) 056126.
[27] BRUNET E., DERRIDA B., MUELLER A. H. and MUNIER S., Phys. Rev. E, 76 (2007) 041104.
[28] MUELLER A. H. and MUNIER S., Phys. Rev. E, 90 (2014) 042143.
[29] AIDEKON E., BERESTYCKI J., BRUNET E. and SHI Z., Probab. Theory Relat. Fields, 157 (2013) 405.
[30] ARGuin L. P., BOVIER A. and KISTLER N., Probab. Theory Relat. Fields, 157 (2013) 535.
[31] BRUNET E. and DERRIDA B., EPL, 87 (2009) 60010.
[32] BRUNET E. and DERRIDA B., J. Stat. Phys., 143 (2011) 420.
[33] MALLEIN B., Stoch. Processes Appl., 125 (2015) 3958.
[34] MALLEIN B., arXiv:1606.01748 (2016).
[35] DEAN D. S. and MAJUMDAR S. N., Phys. Rev. E, 64 (2001) 046121.
[36] NIEUWENHUIZEN T. M., J. Phys. I, 6 (1996) 109.
[37] FERRERO M. E., PARISI G. and RANIERI P., J. Phys. A: Math. Gen., 29 (1996) L569.
[38] CAMPELLONE M., PARISI G. and VIRASORO M. A., J. Stat. Phys., 138 (2010) 29.
[39] DERRIDA B. and MOTTISHAW P., J. Stat. Mech.: Theory Exp. (2015) P01021.
[40] AMIR G., CORWIN I. and QUASTEL J., Commun. Pure Appl. Math., 64 (2011) 466.
[41] DOTSENO V., EPL, 90 (2010) 20003.
[42] SASAMOTO T. and SPOHN H., Phys. Rev. Lett., 104 (2010) 230602.
[43] JOHANNSSON K., Commun. Math. Phys., 209 (2000) 437.
[44] CALABRESE P. and LE DOUSSAL P., Phys. Rev. Lett., 106 (2011) 250603.