Heat experiment design to estimate temperature dependent thermal properties

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Abstract. Experimental conditions are studied to optimize transient experiments for estimating temperature dependent thermal conductivity and volumetric heat capacity. A mathematical model of a specimen is the one-dimensional heat equation with boundary conditions of the second kind. Thermal properties are assumed to vary nonlinearly with temperature. Experimental conditions refer to the thermal loading scheme, sampling times and sensor location. A numerical model of experimental configurations is studied to elicit the optimal conditions. The numerical solution of the design problem is formulated on a regularization scheme with a stabilizer minimization without a regularization parameter. An explicit design criterion is used to reveal the optimal sensor location, heating duration and flux magnitude. Results obtained indicate that even the strongly nonlinear experimental design problem admits the aggregation of its solution and has a strictly defined optimal measurement scheme. Additional region of temperature measurements with allowable identification error is revealed.

1. Introduction
The estimation of thermal properties as a solution of an inverse problem is constantly of great interest in the theory of ill-posed problems. A very large number of investigations, combing experimental and mathematical activities, have been devoted to the reconstruction of the specific heat and thermal conductivity [2–5, 9-12, 16, 24]. The final trend in the development of heat inverse theory is a design of experiments [1, 6, 7, 13, 14, 18-21, 23]. The efforts in this direction are focused on the advance of numerical methods to determine the optimum conditions to conduct the experiment. The rules on the choice of various optimal experimental variables, including sensor locations, sampling times and heat loading are studied.

To design experiment the investigation of the state function sensitivity over the sought quantities is mainly executed. The sensitivity approach gives very important information on mathematical model features. However, in the framework of the sensitivity approach a number of theoretically and practically important questions remain unanswered.

The major open question is the existence of the structure of the design problem solution. The requirement of the structure seeking means that the typical classes of the problem solutions should be revealed. The basic feature of the structure is a similar behaviour of a certain set of solutions. By virtue of it, the respective change of the initial data will not vary the nature of the optimal design solution. Therefore, the absence of the structure knowledge limits the received solution only to a narrow particular field. On the other hand, the determination of the design solution structure indicates the typical cases of the identification error behaviour and specifies the ways of the design generalization. In total, the general solution of the design problem is divided into certain classes of the optimal experimental conditions, in the framework of which the best measurement scheme does not substantially vary.
The existence of the solution structure of the experimental design with a heat inverse problem was firstly proved in [19]. As it appears, the optimal estimation of the constant thermal properties can be expressed by the strictly determined cases of the identification error behaviour. Also, it was proved that the variations of the optimal sensor locations are not arbitrary, and the ones are determined by a certain set of dimensionless factors. The revealing of a typical behaviour of the identification error essentially specifies a situation with the definition of the concept of experimental design, because the required design solution depends on the sought parameters. The latter is now a serious obstacle to develop the mathematical theory of experimental design. A similar receiving structure allows us to bypass the above obstacle because the classes of experimental configurations, which generate certain weakly changing optimal conditions, are seeking.

The objective of the present investigation is to reveal the structure of the experimental design with the estimation of the temperature dependent thermal properties. The design approach [18] is used. This is based on the numerical analysis of the so-called guaranteed identification error. The approach developed takes the requirement of the stabilization of mathematical model parameters estimation into account. The sufficient restriction of the effect of initial data variations is ensured due to the special regularization scheme with the observation matching separately on each sample [15]. The motivation of the new method implementation is the existence of the conditions that expand maximally the number of sought quantities being formulated on a minimal volume sample [17, 20].

2. Problem formulation
Let us consider a widely used heat loading of a sample to estimate thermal properties of a solid body. This is a flash heat flux configuration, where one boundary has a pulse heat flux $Q$ with the time duration $\tau_p$ and other boundary has a thermal insulation. The investigation will be limited by the one-dimensional model to pay attention to the revealing of the basic features of the design problem solution. In this case the mathematical model is of the following form:

$$
\begin{align*}
    c(u) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ \lambda(u) \frac{\partial u}{\partial x} \right], & \quad 0 < x < \ell, 0 < t < T; \\
    u \big|_{x=0} = u_0(x), & \quad 0 < x < \ell; \\
    -\frac{\lambda}{\partial x} \bigg|_{x=0} = \frac{Q}{0, \tau_p < t \leq T} & \quad \frac{\lambda}{\partial x} \bigg|_{x=\ell} = 0, \quad 0 < t < T,
\end{align*}
$$

where $u(x,t)$ is the temperature field, $c(u)$ is the specific heat, $\lambda(u)$ is the thermal conductivity, $\rho = \text{const}$ is the sample density, and $T \geq \tau_p$ is the upper bound of sampling times. It is also supposed that a sample $u^\delta$ corrupted with a noise $\varepsilon$

$$
    u^\delta = u(x_j, t_j) + \varepsilon(t_j), \quad j = 1, n, 0 < t_j \leq T
$$

deviates from its prototype $\tilde{u}$ (true state) by a known amount

$$
\left\{ \frac{1}{n} \sum_{j=1}^{n} \left[ u^\delta - \tilde{u}(x_j, t_j) \right]^2 \right\}^{1/2} \leq \delta
$$

The sample $u^\delta$ is the temperature observation, which is carried out by a single sensor. The minimal number of sensors aims the investigation of the heat experiment informativeness [17] with a limited observation volume. The sensor location $x_j$ will be considered on all sample length, $0 \leq x_j \leq \ell$. Concerning the noise of measurements it is proposed that we know only the norm $\delta$ of the $\varepsilon$.

This does not require any hypothesis on the error distribution, and also the expected value and covariance matrices are not specified. The estimator of the upper noise level $\delta$ may be derived in every experiment. The sample volume $n$ is large enough, $n \gg p$, to exclude the consideration of all worst point measurements [19] that takes place at a sample with a small volume $n = p$, where $p$ denotes the general number of unknowns.

It is required to seek both the optimal measurement scheme $\Xi = \{x_j, T\}$ and external impact capabilities $\{Q, \tau_p\}$ guaranteeing the reconstruction of the desired properties $\{c(u), \lambda(u)\}$ with the
minimal identification errors for the fixed non-zero level noise $\delta > 0$. The relative identification error is defined as

$$\mu_1 = \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{\tilde{c}(u) - c(u)}{c(u)} \right)^2 du,$$

$$\mu_2 = \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{\tilde{\lambda}(u) - \lambda(u)}{\lambda(u)} \right)^2 du,$$

(4)

where $u_{\text{min}}$ and $u_{\text{max}}$ are the known lower and upper temperature bounds, $\{\tilde{c}(u), \tilde{\lambda}(u)\}$ and $\{c(u), \lambda(u)\}$ denote the true values and stable inverse problem solution, respectively. Being based upon the relative errors (4), the design test is introduced in Section 3. Additional quantitative information about the true functions during the inverse problem solution is not used. We approximate the sought functions supposing only their sufficient smoothness. On this basis, the region of feasibility is expanded. Therefore, the inverse problem studied is the ill-posed problem and data variations will cause arbitrary numerical solution variations. To restrict these variations, we use sufficient regularization.

It is of importance to note that the present theory of experimental design with inverse problems does not take the necessity of the design solution regularization into account. In this connection, the next question is raised: if we know the optimal sensor location, then can the estimation method applied ensure the minimal identification error? In other words, the sensitivity analysis gives the conditions on the optimal experiment, but does not specify the way the optimal solution receiving. This leads to uncertainty in the optimal design solution concept. Below the determination of the optimal observation and the seeking of the best stable solution is unified in the frame of one notion.

Our main question is formulated as follows: how do the variations of the measurement scheme $\Xi$, external impact $\{Q, \tau_p\}$, noise level $\delta$ and the character of the sought properties $\{c(u), \lambda(u)\}$ effect on the magnitude of the identification error?

If the variations will indicate the existence of the typical error distribution, then the required structure of the design solution can be revealed and it is possible to connect the optimal sensor location $x_{(\text{opt})}$ to a certain experimental configuration.

3. Guaranteed identification error

Let us give a brief description of the technique that deals with the optimal experimental design in the sense of the guaranteed error. All necessary details of the approach formulation are discussed in [18, 19]. In this section we pay attention to the idea respect to the problem studied below.

3.1. Design tests

Essential features of the technique offered are the type and character of the design problem solution. They are formulated on the explicit form of the identification error generated by the worst measurement conditions.

**Definition 1.** A guaranteed identification error is a value of the worst accuracy of a stable sought quantity reconstruction obtained among all possible variants of the action in the maximum admissible error in the measurements at each observation.

The idea suggests the examining of not every inverse problem solution, but only those which satisfy the conditions of sufficient stabilization. Among all relevant stable solutions it is necessary to minimize the one with the worst identification error. This specifies substantially the concept of the experimental design with ill-posed problems, because the given initial data and observation can lead to uncertainty in the design solution if a method without regularization or with poor regularization is applied. Also the notion implies the noise model as the upper bound of measurement errors. A similar model conveys a wide class of noise sources and allows one to analyze many practically important cases.

To specify the experimental design test in terms of the guaranteed identification error we introduce the following criterion.

**Definition 2.** R-optimal observation design is a measurement scheme that minimizes a root mean square norm of a guaranteed identification error.

In this case the desired test is

$$\mu_{\text{rms}} = \left( \sum_{k} \mu_k^2 \right)^{1/2},$$
where \( \mu_k \) denotes the errors (4) of the stable solution. Again, we will not consider every arbitrary inverse problem solution and will require sufficient regularization of the design solution as the necessary condition. The desired optimal measurement scheme \( \Xi^{(R)} \) should minimize the rms-error,

\[
\Xi^{(R)} = \text{Arg} \{ \min_{\mu} (\mu_{\text{rms}}) \}.
\]

Consequently, the value \( \mu^{(R)} = \min_{\Xi} (\mu_{\text{rms}}) \) is the R-optimal guaranteed error.

In addition to the rms-estimator, other error norms of the inverse problem solution can be considered. In particular, it is the absolute-error estimator. By analogy with the previous case, the relevant measurement scheme \( \Xi^{(M)} \) is said to be M-optimal.

**Definition 3.** M-optimal observation design is a measurement scheme that minimizes an absolute norm of a guaranteed identification error.

The one is based upon the error estimator

\[
\mu_{\text{max}} = \max_k |\mu_k|.
\]

Its minimization is the purpose of the experimental design. Accordingly, the min-max solution \( \Xi^{(M)} = \text{Arg} \{ \min_{\mu} (\mu_{\text{max}}) \} \) is the essential of M-optimal design and \( \mu^{(M)} = \min_{\Xi} (\mu_{\text{max}}) \) is the M-optimal guaranteed error.

The criteria introduced are the typical error estimators in the theory of approximation. However, they differ from the conventional design tests [1, 7, 13, 14, 21, 23]. The difference is the transition from the implicit to the explicit form of the identification error. Also the difference is regarded with the model of the measurement errors. The postulation of a noise distribution and their covariance matrices are not required. The sufficient volume of statistical information for the existence of the sought solution is a limited norm of noise, \( \delta < \infty \). Thereby, the optimal experimental conditions are derived with the most general assumptions on the noise, including fixed errors. In the mean time, additional data including a noise distribution law may be used to increase the estimation precision.

Summarizing, we note that the solutions obtained below is only one of the possible applications of the approach [18]. By virtue of its generality, no limitations are imposed on the mathematical models used. Further generalizations, both on the kinds of boundary conditions and sought quantities, as well as the choice of other measurement scheme and test criteria may be accomplished after the studying of basic experimental design regularities. In particular, other measurement schemes and boundary conditions of the first and third kinds can be investigated.

### 3.2. Planning procedure

Following [18] the experimental design problem is divided into four subproblems. Generally, they consist in a minimization of a stabilizing functional on a set whose elements differ from actual sought quantities at most by fixed value, which should satisfy the observation matching equations. The idea is formulated on the regularization scheme proposed in [15]. Its main advantage is the absence of the regularization parameter. This is attained due to a direct minimization of a stabilizing functional.

If the sought quantities \( \{c(u), \lambda(u)\} \) belong to the twice differentiable space, then the above-mentioned subproblems are formulated as follows. The first one is a direct problem (1) with the known initial-boundary conditions and preset magnitudes \( \{\bar{c}(u), \bar{\lambda}(u)\} \). The second subproblem is the minimization of the functional

\[
\Omega[c, \lambda] = W_{11} \int_{u_{\text{min}}}^{u_{\text{max}}} c^2 \, du + W_{12} \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{dc}{du} \right)^2 \, du + W_{13} \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{d^2 c}{du^2} \right)^2 \, du
\]

\[
+ W_{21} \int_{u_{\text{min}}}^{u_{\text{max}}} \lambda^2 \, du + W_{22} \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{d\lambda}{du} \right)^2 \, du + W_{23} \int_{u_{\text{min}}}^{u_{\text{max}}} \left( \frac{d^2 \lambda}{du^2} \right)^2 \, du
\]

with the observation matching conditions

\[
\sqrt{\frac{1}{n} \sum_{j=1}^{n} [\bar{u}(x_j, t_j) - u(x_j, t_j) \pm \delta] ^2} \leq \delta,
\]

(5)
where $\Omega$ denotes the stabilizing functional [22], $\{W_{i,j}\}_{i,j=1,2,3}$ are the weighting coefficients, and $u(x,t)$ denotes the current temperature field determined with the model coefficients $c(u) = \tilde{c}(u)(1 - \mu_1), \lambda(u) = \tilde{\lambda}(u)(1 - \mu_2)$. The weighting coefficients are necessary due to the relative size of the functions and their derivatives.

The third subproblem is the computing of the $\text{rms}$- or $\text{max}$- design test, when the worst solution $\{\hat{c}, \hat{\lambda}\}$ among all variants of the restriction (6) is selected. The forth subproblem is the minimization of the design test.

The matching conditions (6) express the observation model (2) according to Definition 1. The idea of the guaranteed error requires the displacing of the random noise $\varepsilon$ by the upper noise level $\delta$. In this case all probable variants of the worse observation should be considered. The large volume of the sample reduces the observation to two kinds of worst samples. This means that all combinations of point measurements, as it was considered in [19, 20] for the case of a small volume sample, are not required to be examined.

Also according to Definition 1, the design problem studied should ensure the stable behaviour of the sought solution. Therefore, the conditions of inverse problem regularization should be satisfied. To fulfil these conditions the approach (5) and (6) is the seeking the design solution, which provides for the restriction of a domain of admissible solutions and ensures the coordination with measurement errors.

It should be specially emphasized here that the idea developed is based on two requirements of the solution regularization. One of them is a restriction of a domain of admissible solutions. The other one is a matching of the inverse problem solution with noisy data separately at each sample. A computerized inversion without the sufficient restrictions results in large errors despite stability of obtained solutions. Similarly, the poor matching causes large identification errors, even with sufficient restrictions. If the identification method does not ensure both the necessary matching and sufficient restriction, then it is impossible to derive the satisfactory estimation since a specific, as a rule, small noise level. Choice of the relevant stabilizer and matching norm guarantees the satisfactory inverse problem solution even with the intense measurement noise [15, 16]. Formulation (5), (6) satisfies the above conditions.

In addition, taking the existence of the non-unique roots of the observation matching equations into account, it is of importance that the offered design approach is based upon a stabilizer minimization. This condition ensures the necessary root filtration. Below it will be demonstrated how this rule improves the solution precision even in the case of large measurement errors. Thus the optimal solution obtained in the framework of Definition 1 is the minimized identification error under the worst observation conditions, when all regularities of the inverse problem are taken into account.

The regularization proposed is oriented to the standard mathematical programming methods [8]. The penalty function for the finite-dimensional basis $X_{c,\lambda} = \{x_{k,\lambda}\}_{k=1,p}$ of the sought functions approximation is introduced as follows:

$$P(\chi) = \Omega[X_{c,\lambda}] + \sum_{i=1,2} \Pi_i \max(0, \gamma_i[X_{c,\lambda}]),$$

where $\{x_{k,\lambda}\}_{k=1,p}$ is the parameterization vector, $p$ denotes the general number of the sought parameters, $\Pi_i$ are the penalty coefficients and $\gamma_i$ denote the residual of the matching condition

$$\gamma_1 = \frac{1}{n} \sum_{j=1}^{n} \left[ u(x_j, t_j) - u(x_j, t_j) \pm \delta \right]^2 - \delta, \quad 0 < t_j \leq \tau_p,$$

$$\gamma_2 = \frac{1}{n} \sum_{j=1}^{n} \left[ u(x_j, t_j) - u(x_j, t_j) \pm \delta \right]^2 - \delta, \quad \tau_p < t_j \leq T.$$  

The penalty function (7) reduces the variational problem (5) and (6) to the non-constraint programming problem of the vector $X_{c,\lambda}$. This vector is to be used in the formulation of the minimization problem. The known effective algorithms [8] can be applied to solve this problem.
4. Numerical analysis of guaranteed error

Let us study the guaranteed identification error behaviour being formulated on the numerical simulation. In this case it is necessary to specify the probe functions \( \{ \tilde{c}(u), \tilde{\lambda}(u) \} \) and to reconstruct the sought properties \( \{ \hat{c}, \hat{\lambda} \} \) for a number of the magnitude \( \delta \) and sensor location variations at the segment \( 0 \leq x_s \leq \ell \) with the fixed initial and boundary conditions. The sought thermal properties are approximated by cubic B-spline functions. Their nodes

\[
\chi_k = c(u_k), \, u_{\min} \leq u_k \leq u_{\max}, \, k = 1, p_c
\]

\[
\chi_{k+p_c} = \lambda(u_k), \, u_{\min} \leq u_k \leq u_{\max}, \, k = 1, p_{\lambda}
\]

constitute the parameterization vector \( \chi_{c, \lambda} \). Being grounded on these nodes, the current coefficients of the heat equation are approximated and the direct problem is solved. The number of nodes \( p_{c, \lambda} \) is specified to ensure sufficient approximation precision. For the given \( \delta \) and \( x_s \), the penalty function (7) is minimized. Four cases of the observation matching conditions (6) are computed in accordance to the number of the residuals (8). The variant of the worst identification error among these four solutions is used to calculate the \( \text{rms} \)-test. The seeking of its minimum is the final purpose of the simulation.

The spline approximation of the sought quantities gives rise to the significant complexities of the numerical solution. The basic difficulty is the expansion of a region of feasibility. As two unknown functions are reconstructed at once, and a single sensor sample is used, it is of doubtless interest to propose that the method can ensure the realization of the optimal design. Again, the optimal sensor location does not guarantee the minimal identification error, if a numerical method without a sufficient regularization is applied.

The numerical analysis limits a degree of generality of the received results. Its application does not allow the distribution of the received results to the general theoretical cases. However, the received results will reflect the general properties of the design and specify practical ways of the optimum planning of the thermal experiment.

Consider a number of the test functions. They are as follows:

\[
\tilde{c}(u) = 5000 + \left( \frac{u}{95} \right)^3 - \left( \frac{u}{95} \right)^4, \quad \tilde{\lambda}(u) = \frac{1}{8} \left[ 100 - \left( \frac{u}{50} \right)^2 - \left( \frac{u}{100} \right)^3 + \left( \frac{u}{130} \right)^4 \right], \quad (9)
\]

\[
\tilde{c}(u) = 1000 + 25 u - \left( \frac{u}{25} \right)^3 + 0.1 \left( \frac{u}{75} \right)^5, \quad \tilde{\lambda}(u) = 10 + 0.05 u - \left( \frac{u}{100} \right)^2 + \left( \frac{u}{450} \right)^6, \quad (10)
\]

\[
\tilde{c}(u) = 3000 - 0.08 (u - 100)^2 + 53 (u - 100)^3 - 15 (u - 100)^4 + 238 (u - 100)^5 - 235 (u - 100)^6 + 138 (u - 100)^7 - 36 (u - 100)^8,
\]

\[
\tilde{\lambda}(u) = 70 - 26 (u - 100)^2 + 176 (u - 100)^3 - 5 (u - 100)^4 + 79 (u - 100)^5 - 78 (u - 100)^6 - 46 (u - 100)^7 - 12 (u - 100)^8, \quad (11)
\]

The functions (9) express the typical unimodal kind of the thermal properties. Also, their theoretically important feature is the existence of intervals both low and high thermal conductivity at the estimation segment \([u_{\min}, u_{\max}]\). The functions (10) depict the typical character of the temperature dependence of the thermal properties. The later has the strongly marked as maximal and minimal values. The functions (11) express the case of the abnormal variations of the sought properties.

The direct problem (1) was solved by the method of finite differences with the number of nodes \( N_u = 200, N_t = 400 \). Other initial data are \( Q = 20000 \text{ W/m}^2 \), \( \tau_p = 500 \text{ s} \), \( T = 1500 \text{ s} \), \( u_0 = 0 \), \( \rho = 1 \text{ kg/m}^3 \), \( p_c = p_{\lambda} = 10 \), \( n = 100 \). The stabilizer weighting coefficients are \( W_{11} = W_{12} = W_{21} = W_{22} = 0 \), \( W_{13} = W_{23} = 10^{10} \).

The reconstruction results are shown in figures 1 – 3. They indicate the novel features of the identification error behaviour.
Figure 1. Identification error distribution for the case (9), (10): 1) $\delta=0.1$; 2) $\delta=0.35$; 3) $\delta=1$; 4) $\delta=5$

Figure 2. Measurement error effect on the optimal estimation for the case (9), (10)

Figure 3. $R$-optimal thermal properties (9), (10): 1) $\delta=0$; 2) $\delta=1$; 3) $\delta=10$; 4) $\delta=20$; 5) $\delta=30$; 6) $\delta=50$
Firstly, there are two areas of the minimal identification error, see figure 1. The global minimum of the guaranteed error $\mu^{(R)}$ comes to the boundary with the heat flux. It is the commonly known fact. The used approach [18] indicates that the location of the global minimum $\mu^{(R)}$ is retained during the noise level growing. The estimation of the thermal dependent properties has the tolerable accuracy even for large noise level, see figure 2. The largest identification errors take place on the boundaries of the approximation segment, see figure 3.

Secondly, the observation with one sensor ensures the simultaneous reconstruction of two functions of the parabolic differential equation with the noisy data. The regularization can guarantee the small sensitivity of the identification error $\mu^{(R)}$ to the sensor location. As a result, the deviations of the sensor location from its optimal position in the limit of 20% of the specimen length do not substantially vary the identification error.

Thirdly, the second minimum of the identification error takes place on the isolate boundary. As it appears, the temperature measurements on the isolated boundary have the allowable informativeness to reconstruct the thermal conductivity and volumetric heat capacity simultaneously.

There exists a strongly marked region of worst temperature measurements. These give rise to large but finite identification errors. In contrast, the case with constant thermal properties [20] results in the infinite growth of the identification error if the sample is obtained in the region of the worst temperature measurements. The latter appears already in the case of small noise levels and quickly grows. However, the regularization scheme (5) and (6) limits the infinite growth of the identification error. It is of importance that the approach develops responses both where and how the optimal estimation can be determined.

Thus the numerical investigation of the initial data variations indicates the existence of a certain solution structure. One is expressed as the typical distribution of the identification error. Their characters for small and large measurement errors, as well as for the optimal and worst sensor locations, are elicited.

5. Conclusions

The inverse problem for estimating the thermal properties of heat conductors is investigated to determine the optimal experimental configuration. The explicit design test criterion is used to construct the planning procedure. A numerical model of heat experimental configurations has been studied to determine the regularities in the design problem solution.

The main outcome is the revealing of the design problem solution structure. The latter determines the typical identification error behaviour with a wide set of initial data. For the heat loading with a heat flash flux the solution structure is represented in the form of two regions of minimal identification errors. The global minimum is determined on the sample boundary with the heat flux. There are conditions when the temperature measurements on the insulated boundary give the allowable information to reconstruct the thermal conductivity and volumetric heat capacity simultaneously.

It is indicated that the satisfactory estimating of the thermal properties can be realized with one sensor sample. Optimal experiments ensure the satisfactory identification of a wide branch of measurement errors. The optimal solution demonstrates that the initial data variations do not change the character of the solution structure. At the same time, the sensor locations of the worst temperature measurements are elicited. The relevant sample guarantees the satisfactory identification only with a small noise level.

Nomenclature

- $c(u)$ – specific heat,
- $n$ – number of measurements,
- $p$ – general number of unknowns,
- $u$ – temperature field,
- $u^s$ – sample,
- $x_s$ – sensor location,
- $T$ – upper bound of measurement time,
- $Q$ – boundary heat flux,
- $\delta$ – upper bound of measurement noise,
\( \varepsilon \) – measurement noise,
\( \tau \) – time duration,
\( \lambda(u) \) – thermal conductivity,
\( \mu \) – relative identification error,
\( \mu_{\text{rms}} \) – root mean square estimator of guaranteed identification error,
\( \mu^{(R)} \) – \( R \)-optimal design test,
\( \Xi \) – measurement scheme,
\( \Omega \) – stabilizer.

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