Lattices in contact Lie groups and 5-dimensional contact solvmanifolds

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Abstract

We investigate the existence and properties of uniform lattices in Lie groups and use these results to prove that, in dimension 5, there are exactly seven connected and simply connected contact Lie groups with uniform lattices, all of which are solvable. Issues of symplectic boundaries are explored, as well. It is also shown that the special affine group has no uniform lattice.

1 Introduction

This paper investigates the geometry of compact contact manifolds that are uniformized by contact Lie groups, i.e., manifolds of the form \( \Gamma \setminus G \) for some Lie group \( G \) with a left invariant contact structure and uniform lattice \( \Gamma \subset G \). We re-examine Alexander’s criteria for existence of lattices on solvable Lie groups and apply them, along with some other well known tools. In particular, we restrict our attention to dimension five and describe which five-dimensional contact Lie groups admit uniform lattices. We prove that there are exactly seven connected and simply connected such Lie groups. Five of them are central extensions; the other two are semi-direct products. Furthermore, all seven are solvable. Let us remind that, in the symplectic counterpart, there are only 4 connected and simply connected Lie groups with a lattice, that can bear a left invariant symplectic form [19].

This paper is organized as follows. In Section 2, we give the preliminaries for the work ahead. This includes both a review of several classical results and some original results regarding the existence of lattices on certain Lie groups. Fundamental to this paper are Theorem 2.17 which describes all five-dimensional contact Lie algebras, and the list in Subsection 2.3.2 which delineates the Lie algebras of all the five-dimensional unimodular contact Lie groups. We also review some pertinent results of contact geometry on Lie groups.

In Section 3, the main theorem of the paper (Theorem 3.1) is stated as well as an immediate corollary. This theorem is proven in Section 4. A major yet technical aspect of this proof is the list of certain structures on the Lie algebras of the Lie groups in Subsection 2.3.2. For ease of reading, this list has been relegated to Appendix I (Section 6). In Section 4.2.2 we show that the special affine Lie group \( SL(n, \mathbb{R}) \ltimes \mathbb{R}^n \) has no uniform lattice (Theorem 4.1), although it may have a lattice.

Finally, Section 5 constructs compact symplectic \((2n+2)\)-manifolds whose boundaries are disconnected contact \((2n+1)\)-manifolds uniformized by contact Lie groups and hence, when \( n = 2 \), by the Lie groups of Theorem 3.1. This is a generalisation to all higher dimensions of a construction used in [12], to give counter-examples, when \( n = 1 \), to the question of E. Calabi as to whether symplectic compact manifolds with a boundary of contact type, admit a connected boundary, as it is the case for compact complex manifolds with strictly pseudo-convex boundary. The counterexamples in [12] encompass those by D. McDuff in [17].

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2 Preliminaries

2.1 Lattices on splittable solvable Lie groups

Recall that a lattice of a Lie group $G$ is a discrete subgroup $\Gamma$ such that the manifold $\Gamma \backslash G$ has a finite volume. If $\Gamma \backslash G$ is compact, then $\Gamma$ is called a uniform lattice. In this section, we present several results regarding the criteria for the existence of lattices on a specific category of solvable Lie groups, namely splittable solvable Lie groups. The general subject is a well-studied field, and much of the following material has been derived from Chapter 2 of Part I in [24], which is itself an exposition of classical results by Mostow ([21]), Auslander ([1], [2]) and Raghunathan ([24]). More details as well as more results on this topic can be found within these various sources.

Let $G$ be a simply-connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $N$ be the nilradical of $G$ with corresponding Lie algebra $\mathfrak{n}$, i.e., $N$ is the maximal nilpotent normal subgroup of $G$ so that $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$. This induces a short exact sequence

$$1 \to N \to G \to T \to 1,$$

where $T$ is the Abelian group given by $N \backslash G$. A Lie group $G$ is called splittable if this sequence splits, i.e., there is a homomorphism right inverse to the projection $G \to T$. It is straightforward to show that $G$ is splittable if and only if there is a homomorphism $b : T \to \text{Aut}(N)$ such that $G$ is isomorphic to the semi-direct product $N \rtimes_b T$. If $G$ is splittable and $b(t)$ is a semi-simple element of $\text{Aut}(N)$ for all $t \in T$, then $G$ is called semi-simple splittable. At the Lie algebra level, splittability of $G$ is equivalent to the existence of a homomorphism $\beta : T \to \text{der}(\mathfrak{n})$ such that $\mathfrak{g} = \mathfrak{n} + \beta(T)$. Note that in the above description $T$ has been identified with its Lie algebra. This convention will continue for the duration of this paper.

The main results of this section will apply to the category of splittable solvable Lie groups. For completeness, we begin with a number of classical results.

**Theorem 2.1** (Milnor [20]). If $G$ is a Lie group with a uniform lattice, then its Lie algebra is unimodular.

**Theorem 2.2.** A lattice on a solvable Lie group is a uniform lattice.

**Theorem 2.3.** Let $N$ be a simply-connected nilpotent Lie group with lattice $\Gamma$. Let $\ldots \subset N_2 \subset N_1 \subset N_0 = N$ be the decreasing central series of $N$. Then $\Gamma \cap N_j$ is a lattice of $N_j$ for all $j = 0, 1, 2, \ldots$

**Theorem 2.4.** A nilpotent Lie group $N$ has a lattice if and only if its Lie algebra $\mathfrak{n}$ has a $\mathbb{Q}$-algebra $\mathfrak{n}_\mathbb{Q}$, i.e., $\mathfrak{n}$ has a basis $e$ such that $[e, e] \subset \langle e \rangle \mathbb{Q}$.

Now we review some commonly known aspects of lattices on solvable Lie groups.

**Theorem 2.5** (Mostow [21]). Let $G$ be a simply-connected solvable Lie group with nilradical $N$ and projection $\pi : G \to T = N \backslash G$. If $\Gamma$ is a lattice of $G$, then $\Gamma \cap N$ is a lattice of $N$ and $\pi(\Gamma)$ is a lattice of $T$.

**Corollary 2.6.** Let $G = N \rtimes_b T$ be a simply-connected splittable solvable Lie group with nilradical $N$. Then any lattice of $G$ is isomorphic to the group $(\Gamma \cap N) \rtimes_b \Lambda$ for some lattice $\Lambda$ of $T$.

In [1], Auslander described criteria for the existence of a lattice on a general solvable Lie group. To this purpose, he invoked a concept due to Mal’tsev in [16] to which we refer here as a Mal’tsev splitting.

**Definition 2.7.** Let $G$ be a solvable Lie group. A Mal’tsev splitting of $G$ is an embedding $i : G \to M(G)$ as a normal subgroup into a simply-connected splittable solvable Lie group $M(G)$ with nilradical $U_G$ and corresponding representation $M(G) = U_G \rtimes \phi(T_G)$ such that

1. $M(G) = i(G)U_G$ and
2. $M(G) = i(G) \rtimes T_G$.

Every solvable Lie group has a unique Mal’tsev splitting, up to isomorphism (See [16]).

For a given Mal’tsev splitting $M(G) = U_G \rtimes \phi(T_G)$ of $G$, $i(N) \backslash M(G)$ is isomorphic to $(i(N) \backslash U_G) \rtimes T_G$. Let $p_1 : i(N) \backslash M(G) \to i(N) \backslash U_G$ and $p_2 : i(N) \backslash M(G) \to T_G$ be the resulting projections. Then, for each $j = 1, 2$, the restriction of $p_j$ to $i(G)/i(N)$, $q_j$, is an isomorphism.
**Theorem 2.8** (Auslander 1973). In a solvable simply-connected Lie group $G$ with nilradical $N$ and Mal’tsev splitting $M(G) = U_G \times_\phi T_G$, there exists a lattice if and only if

1. The nilradical $U_G$ of $M(G)$ has a $\mathbb{Q}$-form $U_G(\mathbb{Q})$ such that $N \cap U_G(\mathbb{Q})$ is a $\mathbb{Q}$-form of $N$, and

2. The subgroup $\Phi = q_1 \cdot q_2^{-1} (U_G(\mathbb{Q}) / N \cap U_G(\mathbb{Q})) \subset T_G$ contains a lattice subgroup $\Lambda$ such that the action $d\phi : T \to \text{Aut}(u_G)$, when restricted to $\Lambda$, can be represented by integer matrices with respect to some basis of $U_G(\mathbb{Q})$.

Let $G = N \rtimes_b T$ be a simply-connected splittable solvable Lie group with nilradical $N$ and homomorphism $b : T \to \text{Aut}(N)$. This homomorphism $b$ can be decomposed into semisimple and nilpotent parts, $b = b_s \circ b_n$, such that

1. $b_n$ and $b_s$ commute,

2. $b_n(t)^k = \text{id}$ for some $k > 0$ and

3. each invariant subspace of the differential $db_s(t) \in \text{Aut}(n)$ of $b_s(t)$ at the unit of $T$ has an invariant complementary subspace for each $t \in T$.

Via identification of $T$ with its Lie algebra, the homomorphism $b$ induces homomorphisms $db : T \to \text{Aut}(n)$ and $\beta : T \to \text{der}(n)$. This notation will be used throughout this paper. Recall that $db(t)$ and $\beta(t)$ satisfy the relation $db(t) = \exp(\beta(t))$.

In [1], the Mal’tsev splitting of a general solvable Lie group is given as a certain subgroup within the Lie group of automorphisms of the nilradical. However, with a more specific category of solvable Lie group in mind (in this case, those that are splittable), a more specific description of the Mal’tsev splitting can be given. Define $\phi : T \to \text{Aut}(N \rtimes_{b_n} T)$ by

$$\phi(t)(h, t') = (b_s(t)(h), t')$$

for $h \in N$ and $t, t' \in T$. Then it is not difficult to show that the semidirect product

$$M(G) = (N \rtimes_{b_n} T) \rtimes_\phi T$$

is a Mal’tsev splitting for $G$, where the embedding $i : G \to M(G)$ of $G$ in $M(G)$, is given by $i(h, t) = ((h, t), t)$ for $h \in N$, $t \in T$ and

$$U(G) = N \rtimes_{b_n} T,$$

$$T(G) = T.$$ 

As the following theorem shows, this description of the Mal’tsev splitting streamlines Auslander’s criteria for the existence of a lattice considerably.

**Theorem 2.9.** Let $G = N \rtimes_b T$ be a simply connected splittable solvable Lie group, where $N$ is the nilradical of $G$ with Lie algebra $n$, $T$ an Abelian group, and $b : T \to \text{Aut}(N)$ a homomorphism with semisimple and nilpotent parts given by $b_s$ and $b_n$, respectively. Then $G$ contains a lattice if and only if there is a $\mathbb{Q}$-algebra $n_\mathbb{Q}$ of $n$ and a lattice $\Lambda$ of $T$ such that

1. $db_n(\Lambda) \subset \text{Aut}(n_\mathbb{Q})$; and

2. $db_s(\Lambda) \subset \text{Aut}(n_\mathbb{Q})$ and can be represented as a group of matrices with integer entries with respect to some basis of $n_\mathbb{Q}$.

**Proof.** Suppose that $G$ has a lattice. The first condition of Theorem 2.8 applied to the Mal’tsev splitting of $G$ given by Equation (1) implies that there is a $\mathbb{Q}$-form $\tilde{N}$ of $N \rtimes_{b_n} T$ such that $N_\mathbb{Q} = \tilde{N} \cap (N \times \{0\})$ is a $\mathbb{Q}$-form of $N \times \{0\}$, which we identify with $N$.

Let $\tilde{n}$ and $n_\mathbb{Q}$ be the Lie algebras of $\tilde{N}$ and $N_\mathbb{Q}$, respectively. Then $\tilde{n}$ is a $\mathbb{Q}$-algebra of $n \rtimes_{b_n} T$, where $\beta_n : T \to \text{der}(n)$ is the homomorphism of derivations induced by $db_n : T \to \text{Aut}(n)$, and $n_\mathbb{Q}$ is a $\mathbb{Q}$-algebra of $n$. Since $n_\mathbb{Q} = \tilde{n} \cap (n \rtimes_{\beta_n} \{0\})$, dimension-counting implies that $\tilde{n} = n_\mathbb{Q} \rtimes_{\beta_n} T_\mathbb{Q}$ for some $\mathbb{Q}$-algebra $T_\mathbb{Q}$ of $T$. In particular, $\beta_n(T_\mathbb{Q}) \subset \text{der}(n_\mathbb{Q})$, which implies that $db_n(T_\mathbb{Q}) \subset \text{Aut}(n_\mathbb{Q})$.

The subgroup $\Phi$ defined in the second condition of Theorem 2.8 is $T_\mathbb{Q}$. So, there is a lattice $\Lambda \subset T_\mathbb{Q}$ such that $db_n(\Lambda) \subset \text{Aut}(n)$ and all such transformations can be represented by elements in $\text{Sl}(n, \mathbb{Z})$ with respect to some basis of $n_\mathbb{Q}$. Clearly, $db_n(\Lambda) \subset \text{Aut}(n_\mathbb{Q})$ as well.

Conversely, suppose there is a $\mathbb{Q}$-algebra $n_\mathbb{Q}$ of $n$ and a lattice $\Lambda$ of $T$ such that
1. $db_n(\Lambda) \subset Aut(n_\mathbb{Q})$; and
2. $db_n(\Lambda) \subset Aut(n_\mathbb{Q})$ can be represented as a group of matrices with integer entries with respect to some basis of $n_\mathbb{Q}$.

Let $N_\mathbb{Q} = \{exp(qX) : q \in \mathbb{Q}, X \in n_\mathbb{Q}\}$ and $T_\mathbb{Q} = (\Lambda)_\mathbb{Q}$. These are $\mathbb{Q}$-forms of $N$ and $T$, respectively. By the first given condition, $db_n(T_\mathbb{Q}) \subset Aut(n_\mathbb{Q})$ so that $\beta_n(T_\mathbb{Q}) \subset der(n_\mathbb{Q})$, where $\beta_n : T \rightarrow der(n)$ is the homomorphism of derivations induced from $db_n$. In particular, $n_\mathbb{Q} \rtimes_{\beta_n} T_\mathbb{Q}$ is a $\mathbb{Q}$-algebra of $n \rtimes_{\beta_n} T$. Then $\bar{N} = \{exp(qX) : q \in \mathbb{Q}, X \in n_\mathbb{Q} \rtimes_{\beta_n} T_\mathbb{Q}\}$ is a $\mathbb{Q}$-form of $N \rtimes_{\beta_n} T$. Furthermore, $N_\mathbb{Q} = \bar{N} \cap (N \times \{0\})$ and is a $\mathbb{Q}$-form of $N \times \{0\}$. Thus, the first condition of Theorem 2.8 (using the Mal’tsev splitting given by Equation (1)) is satisfied. The second condition of Theorem 2.8 follows immediately from the second condition given above. Therefore, $G$ has a lattice.

We now list some corollaries to this theorem. The first two corollaries are immediate consequences of the theorem.

**Corollary 2.10.** Let $G = N \rtimes b T$ be a simply connected splittable solvable Lie group, where $N$ is the nilradical of $G$ with Lie algebra $n$, $T$ an Abelian group, and $b : T \rightarrow Aut(N)$ a homomorphism with semisimple and nilpotent parts given by $b_s$ and $b_n$, respectively. If $G$ has a lattice, then there are bases, $\nu$ of $n$ and $\nu$ of $T$, such that, for any $\epsilon \in \mathbb{L}$, the matrix representation of $db_\epsilon(\nu)$ in $Aut(n)$ with respect to $\nu$ is an integer matrix.

**Corollary 2.11.** Let $G = N \rtimes b T$ be a simply connected semi-simple splittable solvable Lie group, where $N$ is the nilradical of $G$ with Lie algebra $n$ of dimension $n$, $T$ an Abelian group, and $b : T \rightarrow Aut(N)$ a homomorphism. Then $G$ contains a lattice if and only if there are a $\mathbb{Q}$-algebra $n_\mathbb{Q}$ of $n$ with basis $\nu \subset n_\mathbb{Q}$ and a lattice $\Lambda$ of $T$ such that $db_\nu(\Lambda) \subset Aut(n_\mathbb{Q})$ and is a subgroup of integer matrices when represented as matrices with respect to $\nu$.

The following corollary is derived from Theorem 2.9 in conjunction with Theorem 2.5 and Theorem 2.3.

**Corollary 2.12.** Let $G = N \rtimes b T$ be a splittable solvable group with nilradical $N$, Abelian group $T$ and homomorphism $b : T \rightarrow Aut(N)$ with semi-simple and nilpotent components given by $b_s$ and $b_n$, respectively. Let $N_j$ be the $j^{th}$ central series subgroup of the nilradical $N$ with corresponding Lie algebras $n_j$ for $j = 0, 1, \ldots$, i.e., $n_0 = n$ and $n_j = [n_{j-1}, n]$ for $j = 1, 2, \ldots$. If $G$ has a lattice, then, there is a lattice $\Lambda$ of $T$ such that, for $j = 0, 1, \ldots$, there is a $\mathbb{Q}$-algebra $(n_j)^\mathbb{Q} \subset n_j$ with basis $\epsilon_{\mathbb{L}}$ such that

1. $[(db_n)|_{n_j}(\Lambda)]_{\mathbb{L}}$ is a subgroup of integer matrices, and
2. $[(db_n)|_{n_j}(\Lambda)]_{\mathbb{L}}$ is a subgroup of rational matrices.

Finally, the fourth corollary of Theorem 2.9 relates the existence of lattices and central extensions of certain symplectic Lie groups.

**Corollary 2.13.** Let $G'$ be the central extension of a semi-simple splittable symplectic Lie group $(G = N \rtimes b T, \omega)$. Let $g$, $n$, and $T$ be the corresponding Lie algebras of $G$, $N$ and $T$, respectively. Then $G'$ has a lattice if and only if there is a $\mathbb{Q}$-algebra of $n$ with basis $\epsilon$ and lattice $\Lambda$ of $T$ such that

1. $[db(\Lambda)]_\mathbb{L}$ is a subgroup of integer matrices, and
2. $\omega(X, Y) \in \mathbb{Q}$ for any $X, Y \in \mathbb{L} \cup \Lambda$.

**Proof:** Let $g'$ be the Lie algebra of $G'$ so that $g' = (n + \beta T) + \omega \mathbb{R}$. Let $\omega_1$ be the 2-form $\omega$ restricted on the ideal $n$ so that the nilradical of $G'$ is the central extension of $N$ by $\omega_1$. Also, define the homomorphism $\beta_\omega : T \rightarrow der(n + \omega_1 \mathbb{R})$ by $\beta_\omega(f)(X, x) = (\beta(f)(X), \omega(f, X))$, for $f \in T$, $X \in n$, and $x \in \mathbb{R}$. Then we may write $g'$ as a splittable solvable Lie group by $g' = (n + \omega_1 \mathbb{R}) + \beta_{\omega} T$. 
Note the following facts. First, a $\mathbb{Q}$-algebra of $n + \omega_i \mathbb{R}$ exists if and only if there is a $\mathbb{Q}$-algebra $n_\mathbb{Q}$ of $n$ such that $\omega(n_\mathbb{Q}, n_\mathbb{Q}) \subset \mathbb{Q}$. Second, by its construction and the semi-simplicity of $b$, the semi-simple and nilpotent components of $\beta_\omega$ are given by

\[
\begin{align*}
(\beta_\omega(f))_s (X, x) &= (\beta(f)(X), 0), \\
(\beta_\omega(f))_n (X, x) &= (0, \omega(f, X)).
\end{align*}
\]

The corollary then follows immediately from Theorem 2.9.

**Example** In [6], Benson and Gordon give two examples of solvable symplectic Lie groups with certain special properties. At the time, they were unable to determine the existence of lattices on either of these examples. More recently, Sawai and Yamada in [26] and Sawai in [25] generalized these examples to two families of solvable symplectic Lie groups with the same special properties and then used ad hoc methods to show the existence of lattices on these families of Lie groups.

Here we show how Theorem 2.9 can be systematically used to show the existence of a lattice on a given spittable solvable Lie group using the examples from [26]. Let $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 + a_2 - a_3 = 0$. Let $G$ be the simply-connected solvable Lie group corresponding to the Lie algebra $g = \langle A, X_1, X_2, X_3, Z_1, Z_2, Z_3 \rangle$ with Lie bracket identities given by

\[
\begin{align*}
[X_1, X_2] &= X_3, \\
[Z_1, Z_2] &= Z_3, \\
[A, X_j] &= a_j X_j, \\
[A, Z_j] &= -a_j Z_j \text{ for } j = 1, 2, 3.
\end{align*}
\]

Note that $g = (\mathcal{H}_3 \oplus \mathcal{H}_3) \times \beta \mathbb{R}$ where $\beta : \mathbb{R} \to \text{der} (\mathcal{H}_3 \oplus \mathcal{H}_3)$ is given by

\[
\exp(\beta(t)) = \begin{pmatrix}
e^{a_1 t} & e^{a_2 t} & e^{a_3 t} \\
e^{-a_1 t} & e^{-a_2 t} & e^{-a_3 t}
\end{pmatrix}.
\]

(Here $\mathcal{H}_3$ is the Lie algebra of the Heisenberg group $\text{Heis}^3$. See Subsection 2.2 for details).

Set $\lambda_j = e^{a_j}$ for $j = 1, 2, 3$. By Corollary 2.11 we see that $G$ has a lattice if and only if there is a $t_0 > 0$ such that the polynomial $f(x) = \Pi_{j=1}^3 (x - \lambda_j^{-t_0}) (x - \lambda_j^{-t_0})$, i.e., the characteristic polynomial of the above matrix, has integer coefficients. In particular, if $a_1, a_2 \in \mathbb{Z}$, then a lattice exists. This is the particular case for which Sawai and Yamada use ad hoc methods in [26] to show lattice existence. Similarly, these techniques can be used to show the existence of lattices on the family of Lie groups described in [25].

### 2.2 Heisenberg groups

Besides $\mathbb{R}^m$ under addition, the most encountered Lie group in the work below will be the Heisenberg groups in three and five dimensions. In general, the $(2n + 1)$-dimensional Heisenberg group $\text{Heis}^{2n+1}$ is the subgroup of $\text{Sl}(n + 2, \mathbb{R})$ given by

\[
\text{Heis}^{2n+1} = \left\{ \sigma = \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\},
\]

where the column vector $y^t$ is the transpose of the vector $y = (y_1, \ldots, y_n)$ and $I_n$ is the identity map of $\mathbb{R}^n$. Equivalently, $\text{Heis}^{2n+1}$ can be considered as the central extension of the symplectic Lie group $\mathbb{R}^{2n}$ under addition with the standard symplectic form $\omega_1$. Keeping this in mind, we define $\gamma : \mathbb{R}^{2n+1} \to \text{Heis}^{2n+1}$ by

\[
\gamma(z, x, y) = \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix}
\]

for $x, y \in \mathbb{R}^n$, $z \in \mathbb{R}$. 


The Lie algebra of $\mathcal{Heis}^{2n+1}$ is given by

$$\mathcal{H}_{2n+1} = \left\{ X = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b^t \\ 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{R}^n, c \in \mathbb{R} \right\}. $$

For $i, j \in \{1, \ldots, n+2\}$, let $e_{ij}$ be the $(n+2) \times (n+2)$ matrix, all of whose entries are zero except the $ij$-th entry which is equal to 1. We set $e_1 := e_{1,n+2}$, $e_k := e_{1,k}$ and $e_{n+k} := e_{k,n+2}$ for $k = 2, \ldots, n+1$. Then $\{e_1, \ldots, e_{2n+1}\}$ is a basis of $\mathcal{H}_{2n+1}$ with exactly $n$ nontrivial Lie brackets relations, namely, $[e_k, e_{n+k}] = e_1$ for all $k = 2, \ldots, n+1$. If we let $(e_1^*, \ldots, e_{2n+1}^*)$ stand for the dual basis of $(e_1, \ldots, e_{2n+1})$, then $e_1^*$ is a contact form on $\mathcal{H}_{2n+1}$. In terms of the original coordinates on $\mathcal{Heis}^{2n+1}$, the left invariant vector fields are given by $e_1^+ = \frac{\partial}{\partial z}$, $e_{k^+}^+ = \frac{\partial}{\partial y_k} + x_k \frac{\partial}{\partial z}$ for $k = 2, \ldots, n+1$. The left invariant contact form on $\mathcal{Heis}^{2n+1}$ corresponding to $e_1^*$ is $dz - \sum_{i=1}^{n} x_i dy_i$.

The exponential map $\exp : \mathcal{H}_{2n+1} \to \mathcal{Heis}^{2n+1}$ is a diffeomorphism, and we denote its inverse by $\ln$. Specifically, these mappings are given by

$$\exp \begin{pmatrix} 0 & a & c \\ 0 & 0 & b^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & c + \frac{1}{2}ab^t \\ 0 & 1 & b^t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \ln \begin{pmatrix} 1 & x & z \\ 0 & I_{2n} & y^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x & z - \frac{1}{2}xy^t \\ 0 & 0 & 0 \end{pmatrix}. $$

We consider the Lie algebra $\mathfrak{g} := \mathcal{H}_{2n+1} \oplus \mathbb{R}$, which is the direct sum of $\mathcal{H}_{2n+1}$ and $\mathbb{R}$. That is, $\mathcal{H}_{2n+1}$ is a subalgebra of $\mathfrak{g}$ and $\mathbb{R}$ is in its centre. The corresponding connected and simply connect Lie group is $\mathcal{Heis}^{2n+1} \times \mathbb{R}$.

Note that $\mathcal{Heis}^{3} \times \mathbb{R}$ has a left invariant nondegenerate closed 2-form, hence defining a left invariant symplectic structure $\omega = dz \wedge dy + dt \wedge dx$, where $t$ is the coordinate in $\mathbb{R}$. It corresponds to the symplectic form $\omega_2 = e_1^* \wedge e_3^* + e_4^* \wedge e_2^*$, on $\mathcal{H}_3 \oplus \mathbb{R}$, where $\mathcal{H}_3 = \langle e_1, e_2, e_3 \rangle$ as above.

### 2.3 Five-dimensional contact Lie groups

A **contact Lie group** is a Lie group $G$ with dimension $2n + 1$ and left-invariant differential form $\eta$ such that $\eta \wedge d\eta^n \neq 0$. We set $\mathcal{H} = \ker \eta$. Then $\mathcal{H}$ is a left-invariant 2n-dimensional subbundle of $TG$. So, $\mathcal{H}$ induces a subspace of the Lie algebra $\mathfrak{g}$ of $G$, which we will also denote as $\mathcal{H}$. An element $X \in \mathfrak{g}$ is called **horizontal**, if $X \in \mathcal{H}$. A submanifold of $G$ is called **totally isotropic**, if its tangent space in $G$ is horizontal everywhere. A totally isotropic submanifold of (maximal) dimension $n$ is called a **Legendrian submanifold** of $G$.

**Lemma 2.14.** Let $(G, \eta)$ be a solvable contact Lie group with nilradical $N$. Let $\mathfrak{n}$ be the Lie algebra of $N$. Then $\mathfrak{n}$ is not contained in $\mathcal{H}$.

**Proof.** For any $X, Y \in \mathfrak{g}$, $d\eta(X, Y) = -\eta([X, Y])$. But, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. So, if $\mathfrak{n} \subset \mathcal{H}$, then $d\eta = 0$ on $\mathfrak{g}$. This proves the lemma. \qed

**Corollary 2.15.** Let $G = N \rtimes_\beta T$ be a simply-connected splittable solvable Lie group with nilradical $N$ and homomorphism $\beta : T \to \text{Aut}(N)$. Suppose $G$ has a contact structure given by $\eta \in \mathfrak{g}^*$, the dual space of the Lie algebra $\mathfrak{g}$ of $G$, and $\dim G = 2n + 1$. Then

1. $T \subset \mathcal{H}$ (as a Lie algebra in $\mathfrak{g} = \mathfrak{n} + \beta T$),

2. the subspace $\mathfrak{n} \cap \mathcal{H}$ has codimension 1 in $\mathfrak{n}$,

3. $\dim T \leq n$, $\dim \mathfrak{n} \geq n + 1$, and

4. for every nonzero $X \in T$, there is an $X' \in \mathfrak{n} \cap \mathcal{H}$ such that $d\eta(X, X') = 1$.

A Lie algebra $\mathfrak{g}$ is said to be **decomposable** if it is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Such a Lie algebra has a contact form if and only if $\mathfrak{g}_1$ has a contact form and $\mathfrak{g}_2$ an exact symplectic form, or vice versa.

**Lemma 2.16.** If a contact Lie algebra (resp. group) is unimodular, then it is necessarily nondecomposable.
If in [9], the first author classified the five-dimensional simply connected contact Lie groups (via their Lie algebras)

Proof. A decomposable Lie algebra \( g = g_1 \oplus g_2 \) is unimodular if and only if both \( g_1 \) and \( g_2 \) are both unimodular. If \( g \) had a contact form, then either \( g_1 \) or \( g_2 \) would have an exact symplectic form. But, due to the existence of a left invariant radiant vector field for the associated left invariant affine connection, a Lie group with a left invariant exact symplectic form cannot be unimodular (see [11]). This leads to a contradiction. \( \square \)

2.3.1 Five-dimensional solvable contact Lie algebras

In [9], the first author classified the five-dimensional simply connected contact Lie groups (via their Lie algebras) with the following theorem.

Theorem 2.17 (Diatta [9]). Let \( G \) be a five-dimensional Lie group with Lie algebra \( g \).

1. Suppose \( G \) is non-solvable. Then \( G \) is a contact Lie group if and only if \( g \) is one of the following Lie algebras:

   (a) \( \text{aff}(\mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \), \( \text{aff}(\mathbb{R}) \oplus \text{so}(3, \mathbb{R}) \) (decomposable cases) or

   (b) \( \text{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2 \) (non-decomposable case).

2. Suppose that \( G \) is solvable such that \( g \) is non-decomposable with trivial center \( Z(g) \). Then

   (a) If the derived ideal \( [g, g] \) has dimension three and is non-Abelian, then \( g \) is a contact Lie algebra.

   (b) If \( [g, g] \) has dimension four, then \( g \) is contact if and only if

      i. \( \dim(Z([g, g])) = 1 \) or

      ii. \( \dim(Z([g, g])) = 2 \) and there is a \( v \in g \) such that \( Z([g, g]) \) is not an eigenspace of \( ad_v \).

The first statement of this result taken with Lemma 2.1.6 implies that the only unimodular non-solvable five-dimensional contact Lie group is \( \text{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2 \). Furthermore, the second statement in conjunction with the list of five-dimensional solvable Lie algebras in [4] yields the list of all five-dimensional solvable contact Lie algebras, a total of 24 distinct nondecomposable Lie algebras and families of Lie algebras. Among these, exactly 12 are unimodular and listed below along with an example of a contact form \( \eta \). The label for each Lie algebra refers to that algebra’s position in the original list in [9] and will serve as the name of that Lie algebra (or corresponding simply connected Lie group) throughout this paper.

2.3.2 Five-dimensional unimodular solvable contact Lie algebras

Below is the list of unimodular solvable contact Lie algebras of dimension 5.

Central extensions

D1 \( [e_2, e_4] = e_1, [e_3, e_5] = e_1, \eta := e_1^* \). This is the Heisenberg Lie algebra \( \mathcal{H}_5 \). See Section 2.2.

D2 \( [e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, \eta := e_1^* \). This is the central extension \( b \ltimes_\omega \mathbb{R}e_1 \), where \( \omega = e_2^* \wedge e_4^* + e_3^* \wedge e_5^* \) and \( b = \mathcal{H}_3 \oplus \mathbb{R}e_4 \), as in 2.2.

D3 \( [e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, \eta = e_1^* \). This is the central extension \( b \ltimes_\omega \mathbb{R}e_1 \), where \( \omega = e_3^* \wedge e_4^* + e_2^* \wedge e_5^* \) and \( b = \text{span}(e_2, e_3, e_4, e_5) \) with Lie bracket \( [e_3, e_5] = e_2, [e_4, e_5] = e_3 \).

D5 \( [e_2, e_3] = e_1, [e_2, e_5] = e_2, [e_3, e_5] = e_3, e_4, e_5] = e_1, \eta = e_1^* \). This is \( b \ltimes_\omega \mathbb{R}e_1 \), where \( \omega = e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \) and \( b = \text{span}(e_2, e_3, e_4, e_5) \) with Lie bracket \( [e_2, e_5] = e_2, [e_3, e_5] = e_3 \).

D11 \( [e_2, e_3] = e_1, [e_2, e_5] = e_2, [e_3, e_5] = e_3, e_4, e_5] = -e_2, [e_4, e_5] = e_1, \eta = e_1^* \). Here \( g = b \ltimes_\omega \mathbb{R}e_1 \), where \( \omega = e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \) and \( b = \text{span}(e_2, e_3, e_4, e_5) \) with Lie bracket \( [e_2, e_5] = e_2, [e_3, e_5] = -e_2 \).

Semi-direct products

D4 \( [e_2, e_3] = e_1, [e_1, e_5] = (1 + p)e_1, [e_2, e_5] = e_2, [e_3, e_5] = pe_3, [e_4, e_5] = -2(p + 1)e_4, p \neq -1, \eta = e_1^* + e_4^* \). Here \( g \) is the semi-direct product \( (\mathcal{H}_3 \oplus \mathbb{R}e_5) \ltimes \mathbb{R}e_5 \) where \( \mathcal{H}_3 \oplus \mathbb{R}e_4 \) is as in Section 2.2.
Corollary 3.2. The following theorem indicates which of the simply-connected contact Lie groups in Theorem 2.17 have uniform lattices.

Five-dimensional contact Lie groups with uniform lattices

Theorem 3.1. Let $G$ be a five-dimensional connected and simply connected contact Lie group with a uniform lattice. Then $G$ satisfies one of the following statements.

1. $G$ is the central extension of a solvable symplectic Lie group with a lattice that extends to $G$. In particular, $G$ is one of the following groups:

   i. $\text{Heis}^3 = \mathbb{R}^4 \times \omega_1 \mathbb{R}$, where $\omega_1$ is the standard symplectic form on $\mathbb{R}^4$,

   ii. $(\text{Heis}^3 \times \mathbb{R}) \times \omega_2 \mathbb{R}$, where $\omega_2$ is the symplectic form on $\text{Heis}^3 \times \mathbb{R}$, or

   iii. $B_j \times \omega_j \mathbb{R}$ ($j = 3, 4, 5$), where $\omega_j$ is the symplectic form on $B_j = \mathbb{R}^3 \times F_j \mathbb{R}$ with $F_j : \mathbb{R} \to \text{Gl}(3, \mathbb{R})$ defined by the matrices

      \[
      i. \quad F_3(t) = \begin{pmatrix} 1 & -t & \frac{1}{2}t^2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}, \\
      ii. \quad F_4(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
      iii. \quad F_5(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.
      \]

2. $G$ is a solvable semi-direct product and is one of the following groups:

   i. $\mathbb{R}^3 \times_{b_1} \mathbb{R}^2$, where $b_1$ is given by $b_1(s, t) = \begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{s+t} \end{pmatrix}$.

   ii. $\mathbb{R}^3 \times_{b_2} \mathbb{R}^2$, where $b_2$ is given by $b_2(s, t) = \begin{pmatrix} e^{2s} & 0 & 0 \\ 0 & e^{-s} \cos(t) & -e^{-s} \sin(t) \\ 0 & e^{-s} \sin(t) & e^{-s} \cos(t) \end{pmatrix}$.

Corollary 3.2. Let $X$ be a compact five-dimensional contact manifold uniformized by a five-dimensional contact Lie group $G$. Then $G$ is solvable.
4 Proof of Theorem 3.1

We will prove Theorem 3.1 by showing first that the stated Lie groups have lattices and second that the rest of the five-dimensional contact Lie groups given by Theorem 2.17 and the list in Subsection 2.3.2 do not. As stated before, each solvable Lie group (or Lie algebra) will be referred to by its label in the list, e.g. D2, D13. For ease of reading, we have relegated several technical results to appendices at the end of the paper. In Appendix I (Section 6), the reader will find a description of the nilradical of each solvable Lie algebra in the list in Subsection 2.3.2 as well as matrix representations of $db$ and $\beta$ for the splitting $n \nrightarrow T$.

4.1 Positive cases

The groups listed in Theorem 3.1 are the simply connected Lie groups with Lie algebras D1, D2, D3, D5, D11, D18, and D20, respectively.

Claim 1: The Lie groups with Lie algebras D1, D2 and D3 have lattices.

Proof of Claim 1: The Lie algebras D1, D2 and D3 are all nilpotent. By Theorem 2.4 each of these Lie groups has a lattice if and only if its Lie algebra also has a $\mathbb{Q}$-algebra, i.e., there is a basis on which all the coefficients of all of the bracket relations are in $\mathbb{Q}$. The bases of the Lie algebras for these Lie groups as given in Appendix I (Section 2.3.2) all satisfy this property. Thus, groups with Lie algebras D1, D2 and D3 have lattices, and Claim 1 is proven.

Claim 2: The Lie groups with Lie algebras D5 and D11 have lattices.

Proof of Claim 2: Both D5 and D11 are central extensions over semi-simple splittable solvable symplectic Lie groups. We will use Corollary 2.13 to prove the existence of a lattice on each Lie group.

The Lie group D5 is the central extension over the symplectic Lie group $(H, \omega)$ with Lie algebra $\mathfrak{h} = \langle e_2, e_3, e_4, e_5 \rangle_\mathbb{R}$ given by structure equations $[e_2, e_3] = e_2$ and $[e_3, e_5] = -e_3$ and symplectic form $\omega = e_2^* \wedge e_3^* + e_4^* \wedge e_5^*$. That is, its Lie algebra is of the form $\mathfrak{h} = \mathbb{R}^3 + \beta \mathbb{R}$ where $\beta : \mathbb{R} \rightarrow GL(\mathbb{R}^3)$ is given with respect to $\{e_2, e_3, e_4\}$ by

$$\beta(t) = \begin{pmatrix}
-t & 0 & 0 \\
0 & t & 0 \\
0 & 0 & 0
\end{pmatrix}$$

so that

$$exp \beta(t) = \begin{pmatrix}
e^{-t} & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & 1
\end{pmatrix}.$$ 

Choose $t_0 > 0$ such that $m_0 = e^{t_0} + e^{-t_0} \in \mathbb{Z}$, and choose $r, s > 0$ so that $rs (e^{t_0} - e^{-t_0}) \in \mathbb{Q}$. Let $X$ be the basis of $\mathbb{R}^3$ given by

$$X_2 = re_2 + se_3, \\
X_3 = re^{t_0}e_2 + se^{-t_0}e_3, \\
X_4 = e_4.$$ 

Then, with respect to $X$,

$$[exp \beta(t_0)]_X = \begin{pmatrix} 0 & -1 & 0 \\
1 & m_0 & 0 \\
0 & 0 & 1 \end{pmatrix}.$$ 

Furthermore, $\omega(X_2, X_3) = rs (e^{t_0} - e^{-t_0}) \in \mathbb{Q}$, and $\omega(X_4, e_5) = 1$. By Corollary 2.13 the Lie group D5 has a lattice.

The Lie group D11 is the central extension of the semi-simple splittable symplectic Lie group $(K, \zeta)$, the Lie algebra of which $\mathfrak{k} = \langle e_2, \ldots, e_5 \rangle_\mathbb{R}$ has a symplectic form $\zeta$ given by $\zeta = e_2^* \wedge e_3^* + e_4^* \wedge e_5^*$ and has the form...
of \( \mathfrak{f} = \mathbb{R}^3 + \gamma \mathbb{R} \) where \( \gamma : \mathbb{R} \to \mathfrak{gl}(3, \mathbb{R}) \) is given with respect to the basis \( \{ e_2, e_3, e_4 \} \) as
\[
\gamma(te_5) = \begin{pmatrix}
0 & t & 0 \\
-t & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
so that
\[
\exp \gamma(te_5) = \begin{pmatrix}
\cos(t) & -\sin(t) & 0 \\
\sin(t) & \cos(t) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Choose \( k_0 \) to be any positive integer and \( q_0 \) to be any positive rational number. Set \( t_0 = \frac{k_0}{2} \pi \), and define a basis \( \vec{X} \) of \( \mathbb{R}^3 \) by
\[
X_2 = e_2, \ X_3 = e_3, \ X_4 = q_0\pi^{-1}e_4.
\]
Then, with respect to \( \vec{X} \), \( \exp \gamma(t_0e_5) \in \text{SL}(3, \mathbb{Z}) \). Furthermore, \( \zeta(X_2, X_3) = 1 \) and \( \zeta(X_4, t_0e_5) = \frac{k_0q_0}{2} \in \mathbb{Q} \).

By Corollary 2.13 the Lie group \( D_{11} \) has a lattice. Claim 2 is thus proven.

**Claim 3:** The Lie groups with Lie algebras \( D_{18} \) and \( D_{20} \) have lattices.

**Proof of Claim 3:** Each of these groups is semi-simple splittable. So, we will use Corollary 2.11 to show that both of these groups have lattices.

**\( D_{18} \):** is of the form \( \mathbb{R}^3 \rtimes_b \mathbb{R}^2 \), where \( b : \mathbb{R}^2 \to GL(\mathbb{R}^3) \) is given by
\[
b(s, t) = \begin{pmatrix}
e^{-s} & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & e^{s+t}
\end{pmatrix}.
\]
We need only to show that there is a lattice \( \Lambda \subset \mathbb{R}^2 \) such that \( b(\Lambda) \) is represented by a group of integer matrices with respect to some basis of \( \mathbb{R}^3 \).

Let \( T_1 = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & -5 \\
0 & 1 & 6
\end{pmatrix} \) and \( T_2 = \begin{pmatrix}
-4 & -4 & -3 \\
-4 & 16 & 11 \\
-4 & -3 & -2
\end{pmatrix} \). The characteristic polynomial of \( T_1 \) is \( f_1(X) = X^3 - 6X^2 + 5X - 1 \) and that of \( T_2 \) is \( f_2(X) = X^3 - 10X^2 + 17X - 1 \). A little calculation determines that both of these polynomials have the following approximate roots:
\[
f_1 : \quad 0.3080, \ 0.6431, \ 5.0489 \\
f_2 : \quad 0.0610, \ 2.0882, \ 7.8509.
\]

This implies that both \( T_1 \) and \( T_2 \) have exactly three distinct eigenvalues and are thus diagonalizable.

Furthermore, since \( T_1T_2 = T_2T_1 \), they are simultaneously diagonalizable, i.e., there is a \( \Psi \in GL(3, \mathbb{R}) \) such that, for \( j = 1, 2, \)
\[
\Psi^{-1}T_j\Psi = \begin{pmatrix} \alpha_j & 0 & 0 \\
0 & \beta_j & 0 \\
0 & 0 & (\alpha_j\beta_j)^{-1}
\end{pmatrix}.
\]
It is unclear the order in which the eigenvalues of each transformation appear in these simultaneous diagonalizations. However, using the approximations above, it is easy to determine that \( \frac{\ln x_1}{\ln y_2} \neq \frac{\ln y_1}{\ln y_2} \) where \( x_j \) and \( y_j \) are any two distinct eigenvalues of \( T_j \) for \( j = 1, 2, \). So, \( f_1 := (\ln \alpha_1, \ln \beta_1) \) and \( f_2 := (\ln \alpha_2, \ln \beta_2) \) with \( \alpha_1, \ldots, \beta_2 \) determined by the simultaneous diagonalization of \( T_1 \) and \( T_2 \), is a basis of \( \mathbb{R}^2 \). Now set \( \Lambda = \langle f_1, f_2 \rangle \). Then \( \Lambda \) is a lattice of \( \mathbb{R}^2 \) satisfying the desired property.

**\( D_{20} \):** is of the form \( \mathbb{R}^3 \rtimes_b \mathbb{R}^2 \), where \( b : \mathbb{R}^2 \to \mathbb{R}^3 \) is given by
\[
b(s, t) = \begin{pmatrix} e^{2s} & 0 & 0 \\
0 & e^{-s}\cos(t) & -e^{-s}\sin(t) \\
0 & e^{-s}\sin(t) & e^{-s}\cos(t)
\end{pmatrix}.
\]
Again, we need to construct a lattice on $\mathbb{R}^2$ such that its image via $b$ is representable by integer matrices on $\mathbb{R}^2$.

Let $U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 0 & 1 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

The characteristic polynomial for $U_1$ is $X^3 - 3X^2 + 2X - 1$, which has exactly one real root (approximately 2.3247) and two complex roots. Similarly, The characteristic polynomial for $U_2$ is $X^3 + X^2 - 1$, which also has exactly one real root (approximately 0.7549) and two complex roots. Furthermore, $U_1U_2 = U_2U_1$ so that there is a matrix $\Psi \in GL(3, \mathbb{R})$ such that, for $j = 1, 2$,

$$\Psi U_j \Psi^{-1} = \begin{pmatrix} \alpha_j^2 & 0 & 0 \\ 0 & \alpha_j^{-1} \cos(\beta_j) & -\alpha_j^{-1} \sin(\beta_j) \\ 0 & \alpha_j^{-1} \sin(\beta_j) & \alpha_j^{-1} \cos(\beta_j) \end{pmatrix},$$

where $\alpha_1 \approx \sqrt{2.3247}$, $\beta_1 \approx 1.0300$, $\alpha_2 \approx \sqrt{0.7549}$ and $\beta_2 \approx 2.4378$. Furthermore,

$$\frac{\ln \alpha_1}{\ln \alpha_2} \approx -3.0000, \quad \frac{\beta_1}{\beta_2} \approx 1.4589,$$

so that the vectors $f_1 := (\ln \alpha_1, \beta_1)$, $f_2 := (\ln \alpha_2, \beta_2)$ are linearly independent and thus generate a lattice $\Lambda := \langle f_1, f_2 \rangle \in \mathbb{R}^2$ on $\mathbb{R}^2$, satisfying the above claim.

### 4.2 Negative cases

#### 4.2.1 Solvable

We now show that the rest of the solvable contact Lie groups of five dimensions do not have lattices. There are only two classes of such Lie groups remaining from the list in Subsection 4.3.2, namely, those whose nilradical is $\mathcal{H}eis^3 \times \mathbb{R}$ (D4, D8, D10, D13) and one whose nilradical is a semidirect product $\mathbb{R}^3 \rtimes_f \mathbb{R}$ (D15).

**Case 1:** Nilradical $N = \mathcal{H}eis^3 \times \mathbb{R}$

This case involves the semi-direct products D4, D8, D10 and D13. The Lie algebra of the nilradical is given as $n = \langle e_1, e_2, e_3 \rangle_\mathbb{R} \oplus \langle e_4 \rangle_\mathbb{R} = \mathcal{H}_3 \oplus \mathbb{R}$. Appendix I lists the corresponding homomorphisms $\beta : T \rightarrow \text{der}(n)$ and $db : T \rightarrow \text{Aut}(n)$ of each group. For each of these cases, the latter homomorphism $db : \mathbb{R} \rightarrow \text{Aut}(\mathcal{H}_3 \oplus \mathbb{R})$ has the form, with respect to the basis $\{e_1, ..., e_4\}$

$$db(te_5) = \begin{pmatrix} \alpha^t & 0 \\ 0 & B_t \end{pmatrix}$$

for some $\alpha > 0$ and $B_t \in GL(3, \mathbb{R})$ so that the semi-simple part of $db(t)$, $db_s(t)$ is of the same form

$$db_s(te_5) = \begin{pmatrix} \alpha^t & 0 \\ 0 & \tilde{B}_t \end{pmatrix}$$

where $\tilde{B}_t \in GL(3, \mathbb{R})$ is the semi-simple part of the matrix $B_t$. It is important to note that $\alpha^t \det(\tilde{B}_t) = 1$.

Additionally, since $n_1 = [\mathcal{H}_3 \oplus \mathbb{R}, \mathcal{H}_3 \oplus \mathbb{R}] = \langle e_1 \rangle_\mathbb{R}$, $db(t)$ restricted to $n_1$ is simply multiplication by $\alpha^t$.

Suppose there is a lattice on one of these groups. Then there is a $t_0 > 0$ and a basis $e$ of $\mathcal{H}_3 \oplus \mathbb{R}$ such that

1. the matrix $A_{t_0} = [db_s(t_0e_5)]_e \in GL(4, \mathbb{Z})$ (by Theorem 2.9), and
2. $\alpha^{t_0} \in \mathbb{Z}$ (by Corollary 2.12).

In particular, the characteristic polynomial $F_{t_0}(X)$ of $A_{t_0}$ would have integer coefficients and be divisible by $(X - \alpha^{t_0})$. In fact, the polynomial given by $\frac{F_{t_0}(X)}{X - \alpha^{t_0}}$ is the characteristic polynomial of the matrix $\tilde{B}_{t_0}$, an polynomial with integer coefficients by Gauss’ Polynomial Lemma. This implies that $\det(\tilde{B}_{t_0}) \in \mathbb{Z}$. That is, $1 = \alpha^{t_0} \det(\tilde{B}_{t_0})$ with both $\alpha^{t_0} \in \mathbb{Z}$ and $\det(\tilde{B}_{t_0}) \in \mathbb{Z}$. Therefore, $1 = \alpha^{t_0} = \det(\tilde{B}_{t_0})$, which means that $t_0 = 0$, a contradiction. Therefore, none of these groups have lattices.

**Case 2:** Nilradical $N = \mathbb{R}^3 \rtimes_f \mathbb{R}$
The last solvable case to be settled is that of Lie group **D15**. The Lie algebra of the nilradical is of the form
\[ n = \langle e_1, ..., e_3 \rangle + \phi \langle a_4 \rangle \] where \( \phi : \mathbb{R} \to \mathbb{R}^3 \) is given with respect to the basis \( \{ e_1, ..., e_3 \} \) by the matrix
\[
\phi(se_4) = \begin{pmatrix}
0 & -s & 0 \\
0 & 0 & -s \\
0 & 0 & 0
\end{pmatrix}
\]
for any \( s \in \mathbb{R} \). In particular, the lower central series of the nilradical of **D15** have Lie algebras given by:
\[
\begin{align*}
n_0 &= \langle e_1, ..., e_4 \rangle \\
n_1 &= \langle e_1, e_2 \rangle \\
n_2 &= \langle 0 \rangle.
\end{align*}
\]

The homomorphism \( b : \mathbb{R} \to Aut(N) \) for which the structure of **D15** is given by \( N \rtimes_b \mathbb{R} \) has derivative given with respect to \( e = \{ e_1, ..., e_4 \} \) by the matrix
\[
[db(te_5)]_{ei} = \begin{pmatrix}
e^{-\frac{4}{3}t} & 0 & 0 & 0 \\
0 & e^{\frac{1}{3}t} & 0 & 0 \\
0 & 0 & e^{\frac{1}{2}t} & 0 \\
0 & 0 & 0 & e^{-t}
\end{pmatrix}.
\]

Note that \( db(te_5) \) is semi-simple so that **D15** is semi-simple splittable. Furthermore, restricted to the second lower central series subgroup \( n_1 \), \( db(te_5) \) is given with respect to the basis \( e^1 = \{ e_1, e_2 \} \) by the matrix
\[
[db(te_5)]_{e_1} = \begin{pmatrix}
(e^{t_0})^{-1/4} \\
(e^{t_0})^{1/4}
\end{pmatrix}.
\]

Since **D15** is semi-splitable, Corollary 2.12 implies that a necessary condition for this group to have a lattice is the existence of a lattice \( \Lambda \) of \( \mathbb{R} (= \langle e_5 \rangle) \) and, for each \( j = 0, 1, 2, ..., \) a basis \( e^j \) of the \( j \)-th central series \( n_j \) such that \( [db]_{n_j}(\Lambda)_{ei} \) is a subgroup of integer matrices. That is, there would be a \( t_0 > 0 \) such that the matrices \( [db(te_5)]_{ei} \) and \( [db(te_5)]_{e_1} \) are each conjugate to integer matrices, and hence their characteristic polynomials \( F_0 \) and \( F_1 \), respectively, would have integer coefficients. In addition, Gauss’ Lemma implies that the quotient of the two polynomials \( F_0/F_1 \) would also be a polynomial with integer coefficients. However this would imply that \( F_0(0)/F_1(0) = (e^{t_0})^{-1/4} \) would be a non-unit positive integer dividing 1, a contradiction. Thus, **D15** does not have any lattices.

### 4.2.2 Non-solvable: the general case of \( \mathbb{R}^n \rtimes Sl(n, \mathbb{R}) \).

According to Theorem 2.17 the only unimodular non-solvable contact Lie group of dimension five is the group \( \mathbb{R}^2 \rtimes Sl(2, \mathbb{R}) \) of special affine transformations of the plane. We obtain the following more general result stating the nonexistence of uniform lattices in \( \mathbb{R}^n \rtimes Sl(n, \mathbb{R}) \), for every \( n \geq 2 \). Recall from [2] that \( \mathbb{R}^n \rtimes Sl(n, \mathbb{R}) \) is a contact Lie group. The contact structure on on the Lie algebra \( \mathbb{R}^n \rtimes sl(n, \mathbb{R}) \) of \( \mathbb{R}^n \rtimes Sl(n, \mathbb{R}) \) is constructed by considering it as the subalgebra \( \mathbb{R}^n \rtimes sl(n, \mathbb{R}) = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \), where \( A \in sl(n, \mathbb{R}) \) and \( v \in \mathbb{R}^n \) of the Lie algebra \( gl(n+1, \mathbb{R}) \) of \( (n+1) \times (n+1) \) real matrices. The \( (n+1) \times (n+1) \) matrices \( e_{i,j} \), all of whose entries are zero except the \( ij \)-th one which is equal to 1, form a basis of \( gl(n+1, \mathbb{R}) \). Let by \( \{ e^*_{i,j} \} \) be the corresponding dual basis. Then, \( \eta := \sum_{i=1}^{n} e^*_{i,i+1} \) is a contact form on \( \mathbb{R}^n \rtimes sl(n, \mathbb{R}) \), with Reeb vector \( \xi := \frac{1}{n} \sum_{i=1}^{n} e_{i,i+1} \). Now, we have the following.

**Theorem 4.1.** For \( n \geq 2 \), the group of special affine transformations of \( \mathbb{R}^n \), \( \mathbb{R}^n \rtimes Sl(n, \mathbb{R}) \), has no uniform lattice.
Proof: Let $G := \mathbb{R}^n \rtimes Sl(n, \mathbb{R})$ and suppose $\Gamma$ is a lattice of $G$. The radical of $G$ is the subgroup $\mathbb{R}^n \times \{I\}$. Then $\Gamma' = \Gamma \cap \mathbb{R}^n \times \{I\}$ is a lattice of $\mathbb{R}^n \times \{I\}$ (Corollary 1.8 on p. 107 of [23]). Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be such that $(v_1, I) \ldots (v_n, I)$ generate $\Gamma'$. Let $A \in Sl(n, \mathbb{R})$ and $w \in \mathbb{R}^n$ such that $(w, A) \in \Gamma$. Then, for $j = 1, \ldots, n$, 

$$(w, A)(v_j, I)(w, A)^{-1} = (Av_j + w, A)(-A^{-1}w, A^{-1}) = (Av_j, I) \in \Gamma.$$ 

Hence the set $M_\Gamma$ given by 

$$M_\Gamma = \{ A \in Sl(n, \mathbb{R}) : (w, A) \in \Gamma \text{ for some } w \in \mathbb{R}^n \}$$

preserves the lattice $\Gamma'$ on $\mathbb{R}^n$. In particular, by the change of basis $v_j \mapsto e_j$ for $j = 1, \ldots, n$, we can assume that $M_\Gamma \subset Sl(n, \mathbb{Z})$. Now, it is known that $Sl(n, \mathbb{Z})$ is a lattice of $Sl(n, \mathbb{R})$ but not a uniform lattice ([8]). In other words, there is a sequence $\{\gamma_j\} \subset Sl(n, \mathbb{R})$ such that its projection in $Sl(n, \mathbb{Z}) \setminus Sl(n, \mathbb{R})$ has no convergent subsequences. So, its projection in $M_\Gamma \setminus Sl(n, \mathbb{R})$ also has no convergent subsequences, which means that the sequence $\{0, \gamma_j\} \subset \Gamma \setminus \mathbb{R}^n \times Sl(n, \mathbb{R})$ has no convergent subsequences. Therefore, $\Gamma \setminus \mathbb{R}^n \times Sl(n, \mathbb{R})$ is not compact. Since $\Gamma$ was assumed to be an arbitrary lattice of $\mathbb{R}^n \rtimes Sl(n, \mathbb{R})$, $\mathbb{R}^n \rtimes Sl(n, \mathbb{R})$ has no uniform lattices.

5 Symplectic manifolds with a disconnected boundary

The question whether symplectic compact manifolds with a boundary of contact type, admit a connected boundary, as it is the case for compact complex manifolds with strictly pseudo-convex boundary, was raised up by E. Calabi. In [12], H. Geiges used some 3-dimensional contact Lie groups to construct symplectic manifolds of dimension 4 which are counterexamples to such a question. Geiges’ counterexamples contain those constructed by Dusa McDuff in [17]. By the time Geiges gave his examples in 3D, not so many examples of contact Lie groups with lattices were known. The constructions in [12] and the proof can be easily generalised to any odd dimension to unimodular contact Lie groups admitting a uniform lattice.

Theorem 5.1. Suppose $(G, \eta^+)$ is a connected contact Lie group of dimension $(2n + 1)$, with a uniform lattice $\Gamma$. Then the manifold $M := G/\Gamma \times I$ is a compact symplectic manifold with a disconnected boundary $\partial M$ of contact type, where $I := [0, 1]$.

Proof. The contact form $\eta^+$ on $G$ descends to a contact form on $\Gamma \setminus G$, which we denote by $\bar{\eta}$. Let $\bar{\xi}$ stand for the corresponding Reeb vector field. On $M$, consider the differential exact 2-form $\Omega := d(s\bar{\eta})$ and the vector field $X = \xi + s \frac{\partial}{\partial s}$, where $s$ is the $I$-coordinate. Let us check that $\Omega$ is nondegenerate on $M$. Obviously, if $\bar{\alpha}$ and $\bar{\beta}$ are two differential 1-forms, then $(\bar{\alpha} \wedge \bar{\beta})^i = 0$ for all $i \geq 2$. Applying this remark to $\bar{\alpha} = ds$ and $\bar{\beta} = \bar{\eta}$ and using the fact that $(d\bar{\eta})^{n+1} = 0$, we have

$$\Omega^{n+1} = \sum_{i=0}^{n+1} C^i_{n+1} s^{n+1-i} (ds \wedge \bar{\eta})^i \wedge (d\bar{\eta})^{n+1-i} = (n+1) s^n ds \wedge \bar{\eta} \wedge (d\bar{\eta})^n \neq 0.$$

The above is true whenever $s \neq 0$. Now let us prove that $X$ is a Liouville vector field for $\Omega$, i.e. $L_X \Omega = \Omega$. First, we have

$$i_X \Omega = i_{\bar{\xi}}(ds \wedge \bar{\eta} + s \bar{\eta}) + s i_s (ds \wedge \bar{\eta} + s \bar{\eta}) = -ds + s \bar{\eta}.$$

So, the Lie derivative of $\Omega$ on the direction of $X$ comes to the following.

$$L_X \Omega = i_X d\Omega + d i_X \Omega = d i_X \Omega = d(s\bar{\eta}) = \Omega.$$

Now it is readily seen that $X$ points outward on $\partial M$. \[\square\]
6 Appendix I: List of nilradicals of the unimodular contact Lie algebras of dimension 5

The following is a list of all of the unimodular Lie algebras among those in the first author’s list of solvable contact Lie groups in five dimensions from \([9]\). Their Lie brackets, in a basis \((e_1, e_2, e_3, e_4, e_5)\), are given in Section 2.3.2. Each of the corresponding Lie groups will be of the form \(N \rtimes_b T\), where \(N\) is the nilradical, \(T\) is an Abelian group and \(b : T \rightarrow Aut(N)\) a homomorphism. For each of these, the Lie algebra \(\mathfrak{n}\) of the nilradical \(N\) of the simply-connected Lie group corresponding to each Lie algebra is provided as well as the Abelian group \(T\).

The transformations \(\beta\) and \(db\) are matrix representations (with respect to the given basis of \(\mathfrak{n}\)) of the corresponding homomorphisms \(\beta : T \rightarrow der(\mathfrak{n})\) and \(db(x) = exp(\beta(x)) : T \rightarrow Aut(\mathfrak{n})\) (for \(x \in T\)) induced from the semidirect product \(N \rtimes_b T\).

D1 \(n = < e_1, \ldots, e_5 > = H_5, T = (0)\),

D2 \(n = ( < e_1, e_3, e_4 > \oplus < e_2 >) + df < e_5 > = (H_3 \oplus \mathbb{R}) + df \mathbb{R}, T = (0)\) where

\[
    ad(e_5) = \begin{pmatrix}
        0 & 0 & 0 & -1 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & -1 & 0 & 0
    \end{pmatrix}, \quad df(te_5) = \begin{pmatrix}
        1 & \frac{1}{2}t^2 & 0 & -t \\
        0 & 1 & 0 & 0 \\
        0 & 0 & 1 & 0 \\
        0 & -t & 0 & 1
    \end{pmatrix}.
\]

D3 \(n = ( < e_1, e_3, e_4 > \oplus < e_2 >) + df < e_5 > = (H_3 \oplus \mathbb{R}) + df \mathbb{R}, T = (0)\) where

\[
    ad(e_5) = \begin{pmatrix}
        0 & 0 & 0 & -1 \\
        0 & 0 & -1 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & -1 & 0 & 0
    \end{pmatrix}, \quad df(te_5) = \begin{pmatrix}
        1 & \frac{1}{2}t^2 - \frac{1}{2}t^3 & -t \\
        0 & 1 & -t & 0 \\
        0 & 0 & 1 & 0 \\
        0 & -t & 0 & 1
    \end{pmatrix}.
\]

D4 \(n = < e_1, e_2, e_3 > \oplus < e_4 > = H_3 \oplus \mathbb{R}, T = \mathbb{R}e_5\),

\[
    \beta(e_5) = \begin{pmatrix}
        -(p+1) & 0 & 0 & 0 \\
        0 & -1 & 0 & 0 \\
        0 & 0 & -p & 0 \\
        0 & 0 & 2(p+1) & 0
    \end{pmatrix}, \quad db(te_5) = \begin{pmatrix}
        e^{-(p+1)t} & 0 & 0 & 0 \\
        0 & e^{-t} & 0 & 0 \\
        0 & 0 & e^{-pt} & 0 \\
        0 & 0 & 0 & e^{2(p+1)t}
    \end{pmatrix}.
\]

D5 \(n = < e_1, e_2, e_3 > \oplus < e_4 > = H_3 \oplus \mathbb{R}, T = \mathbb{R}e_5\),

\[
    \beta(e_5) = \begin{pmatrix}
        0 & 0 & 0 & -1 \\
        0 & -1 & 0 & 0 \\
        0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 0
    \end{pmatrix}, \quad db(te_5) = \begin{pmatrix}
        1 & 0 & 0 & -t \\
        0 & e^{-t} & 0 & 0 \\
        0 & 0 & e^t & 0 \\
        0 & 0 & 0 & 1
    \end{pmatrix}.
\]

D8 \(n = < e_1, e_2, e_3 > \oplus < e_4 > = H_3 \oplus \mathbb{R}, T = \mathbb{R}e_5\),

\[
    \beta(e_5) = \begin{pmatrix}
        -2 & 0 & 0 & 0 \\
        0 & -1 & 0 & 0 \\
        0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 4
    \end{pmatrix}, \quad db(te_5) = \begin{pmatrix}
        e^{-2t} & 0 & 0 & 0 \\
        0 & e^{-t} & 0 & 0 \\
        0 & -te^{-t} & e^{-t} & 0 \\
        0 & 0 & 0 & e^{4t}
    \end{pmatrix}.
\]

D10 \(n = < e_1, e_2, e_3 > \oplus < e_4 > = H_3 \oplus \mathbb{R}, T = \mathbb{R}e_5\),

\[
    \beta(e_5) = \begin{pmatrix}
        -2p & 0 & 0 & 0 \\
        0 & -p & 1 & 0 \\
        0 & 0 & -p & 0 \\
        0 & 0 & 0 & 4p
    \end{pmatrix}, \quad db(te_5) = \begin{pmatrix}
        e^{-2pt} & 0 & 0 & 0 \\
        0 & e^{-pt} \cos(-t) & -e^{-pt} \sin(-t) & 0 \\
        0 & e^{-pt} \sin(-t) & e^{-pt} \cos(-t) & 0 \\
        0 & 0 & 0 & e^{4pt}
    \end{pmatrix}.
\]
D11  \( n = < e_1, e_2, e_3 > \oplus < e_4 > = \mathcal{H}_3 \oplus \mathbb{R}, \ T = \mathbb{R}e_5, \)

\[
\beta(e_5) = \begin{pmatrix}
0 & 0 & 0 & \pm 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \ db(te_5) = \begin{pmatrix}
1 & 0 & 0 & \pm t \\
0 & \cos(t) & -\sin(t) & 0 \\
0 & \sin(t) & \cos(t) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

D13  \( n = < e_1, e_2, e_3 > \oplus < e_4 > = \mathcal{H}_3 \oplus \mathbb{R}, T = \mathbb{R}e_5, \)

\[
\beta(e_5) = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{3}{2} & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1
\end{pmatrix}, \ db(te_5) = \begin{pmatrix}
e^{\frac{1}{2}t} & 0 & 0 & 0 \\
0 & e^{\frac{3}{2}t} & 0 & 0 \\
0 & 0 & e^{-t} & 0 \\
0 & 0 & -te^{-t} & e^{-t}
\end{pmatrix}.
\]

D15  \( n = < e_1, \ldots, e_4 > = < e_1, e_2, e_3 > + f_*, < e_4 >, \) with \( f_*(e_4) = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}, T = \mathbb{R}e_5, \)

\[
\beta(e_5) = \begin{pmatrix}
-\frac{2}{t} & 0 & 0 & 0 \\
0 & \frac{1}{t} & 0 & 0 \\
0 & 0 & \frac{4}{t} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \ db(te_5) = \begin{pmatrix}
e^{-\frac{2}{t}t} & 0 & 0 & 0 \\
0 & e^{\frac{1}{t}t} & 0 & 0 \\
0 & 0 & e^{\frac{4}{t}t} & 0 \\
0 & 0 & 0 & e^{-t}
\end{pmatrix}.
\]

D18  \( n = < e_1, e_2, e_3 >, \) with

\[
\beta(se_4 + te_5) = \begin{pmatrix}
-s & 0 & 0 \\
0 & -t & 0 \\
0 & 0 & s + t
\end{pmatrix}, \ db(se_4 + te_5) = \begin{pmatrix}
e^{-s} & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & e^{s+t}
\end{pmatrix}.
\]

D20  \( n = < e_1, e_2, e_3 >, \) with

\[
\beta(se_4 + te_5) = \begin{pmatrix}
2s & 0 & 0 \\
0 & -s & -t \\
0 & t & -s
\end{pmatrix}, \ db(se_4 + te_5) = \begin{pmatrix}
e^{2s} & 0 & 0 \\
0 & e^{-s}\cos(t) & -e^{-s}\sin(t) \\
0 & e^{-s}\sin(t) & e^{-s}\cos(t)
\end{pmatrix}.
\]

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