PROOF OF THE TIJDEMAN-ZAGIER CONJECTURE VIA SLOPE IRRATIONALITY AND TERM COPRIMALITY

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Abstract. The Tijdeman-Zagier conjecture states no integer solution exists for \( A^X + B^Y = C^Z \) with positive integer bases and integer exponents greater than 2 unless \( \gcd(A, B, C) > 1 \). Any set of values that satisfy the conjecture correspond to a lattice point on a Cartesian graph which subtends a line in multi-dimensional space with the origin. Properties of the slopes of these lines in each plane are established as a function of coprimality of terms, such as irrationality, which enable us to explicitly prove the conjecture by contradiction.

1. Introduction

The Tijdeman-Zagier conjecture \([9,10,11]\), or more formally the generalized Fermat equation \([5]\) states that given

\[
A^X + B^Y = C^Z
\]

where \( A, B, C, X, Y, Z \) are positive integers, and exponents \( X, Y, Z \geq 3 \), then bases \( A, B, \) and \( C \) have a common factor. Some researchers refer to this conjecture as Beal’s conjecture \([24]\). There are considerable theoretical foundational advances in and around this topic \([11,30,40]\). Many exhaustive searches within limited domains have produced indications this conjecture may be correct \([14,24,31]\). More formal attempts to explicitly prove or counter-prove the conjecture abound in the literature but are often unable to entirely generalize due to difficulty in overcoming floating point arithmetic limits, circular logic, unsubstantiated claims, incomplete steps, or reliance on conjectures \([10,12,13,31]\). Many partial proofs are published in which limited domains, conditional upper bounds of the exponents, specific configuration of bases or exponents, or additional, relaxed, or modified constraints are applied in which the conjecture holds \([1,4,6–8,22,25,28,32–35]\).

Some researchers demonstrate or prove limited coprimality of the exponents \([19]\), properties of perfect powers and relationships to Pillai’s conjecture \([12]\), impossibility of solutions for specific bases \([27]\), influence of the parity of the exponents \([20]\), characterizations of related Diophantine equations \([29]\), relationship between the smallest base and the common factor \([39]\), and countless other insights.

To formally establish a rigorous and complete proof, we need to consider two complimentary conditions: 1) when \( \gcd(A, B, C) = 1 \) there is no integer solution to \( A^X + B^Y = C^Z \), and 2) if there is an integer solution, then \( \gcd(A, B, C) > 1 \). The approach we take is linked to the properties of slopes. An integer solution that satisfies the conjecture also marks a point \((A,B,C)\) that subtends a line through the origin on a 3 dimensional Cartesian graph. Being integers, this is a lattice point and thus the line has a rational slope in all 3 planes. Among other properties, it will be shown that if \( \gcd(A, B, C) = 1 \) with integer exponents, then one or more of the slopes of the subtended line is irrational and cannot pass through any non-trivial lattice points. Conversely it will be shown that if there exists a solution that satisfies the conjecture, the subtended line must pass through a non-trivial lattice point and must have rational slopes.

2. Details of the Proof

To establish the proof requires we first identify, substantiate, and then prove several preliminary properties:

- **Slopes of the Terms**: determine slopes of lines subtended by the origin and the lattice point \((A,B,C)\) that satisfy the terms of the conjecture (Theorem 2.1 on pages 2 to 3).
- **Coprimality of the Bases**: determine implications of 3-way coprimality on pairwise coprimality and implications of pairwise coprimality on 3-way coprimality (Theorems 2.2 to 2.4 on pages 3 to 5).

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Figure 1. Plot of \((A^X, B^Y, C^Z)\) given positive \(A, B, C, X, Y,\) and \(Z,\) where \(A^X + B^Y = C^Z,\)
\(A^X + B^Y \leq 10^{28},\) \(X \geq 4,\) \(Y, Z \geq 3.\)

- **Restrictions of the Exponents**: determine limits of the exponents related to coprimality of the bases and bounds of the conjecture (Theorem 2.5 on pages 5 to 6).
- **Reparameterization of the Terms**: determine equivalent functional forms of the terms and associated properties as related to coprimality of the terms (Theorems 2.6 to 2.13 on pages 6 to 17).
- **Impossibility of the Terms**: determine the relationship between term coprimality and slope irrationality, and between slope irrationality and solution impossibility (Theorem 2.14 on pages 17 to 21).
- **Requirement for Possibility of the Terms**: determine characteristics of \(\gcd(A, B, C)\) required for there to exist a solution given the properties of slopes and coprimality (Theorem 2.15 on pages 21 to 24).

Before articulating each of the underlying formal proofs, we establish two specific definitions to ensure consistency of interpretation.

1. **Reduced Form**: We define the bases of \(A^X, B^Y,\) and \(C^Z\) to be in reduced form, meaning that rather than let the bases be perfect powers, we define exponents \(X, Y,\) and \(Z\) such that the corresponding bases are not perfect powers. For example, \(8^5\) can be reduced to \(2^{15}\) and thus the base and exponent would be 2 and 15, respectively, not 8 and 5, respectively. Hence throughout this document, we assume all bases are reduced accordingly given that the reduced and non-reduced forms of bases raised to their corresponding exponents are equal.

2. **Resultant Function**: Without loss of generality, when establishing the impossibility of integer solutions, unless stated otherwise, we assume to start with integer values for \(A, B, X,\) and \(Y\) and then determine the impossibility of integers for \(C\) or \(Z.\) Given the commutative property of the equation, we hereafter base the determination of integrality of \(C\) and \(Z\) as a function of definite integrality of \(A, B, X,\) and \(Y,\) as doing so for any other combination of variables is a trivial generalization.

The following hereafter establishes the above objectives.

**2.1. Slopes of the Terms**. Although the exponents in \(A^X + B^Y = C^Z\) suggest a volumetric relationship between cubes and hypercubes, and given that the exponents cannot all be the same as it would violate Fermat’s last theorem [36, 38, 43], the expectation of apparent geometric interpretability is low. However every set of values that satisfy the conjecture correspond to a point on a Cartesian grid and subtend a line segment with the origin, which in turn means properties of the slopes of these line segments are directly related to the conjecture. Properties of these slopes form a crucial basis for the subsequent main proofs.

In the Cartesian plot of \(A^X \times B^Y \times C^Z,\) each point \((A^X, B^Y, C^Z)\) corresponds to a specific integer solution that satisfies the conjecture found from exhaustive search within a limited domain. See Figure 1. There exists
Figure 2. Line segment connecting the origin and point \((A^X, B^Y, C^Z)\) where \(A^X + B^Y = C^Z\) satisfying the conjecture from Figure 1.

Figure 3. Angles between the axes and the line segment subtended by the origin and point \((A^X, B^Y, C^Z)\) from Figure 1.

a unique line segment between each point and the origin. The line segment subtends a line segment in each of the three planes, and a set of corresponding angles in those planes made with the axes. See Figures 2 and 3.

**Theorem 2.1.** If the slope \(m\) of line \(y = mx\) is irrational, then the line does not go through any non-trivial lattice points.

**Proof.** Suppose there exists a line \(y = mx\), with irrational slope \(m\) that goes through lattice point \((X, Y)\). Then the slope can be calculated from the two known lattice points through which this line traverses, \((0, 0)\) and \((X, Y)\). Hence the slope is \(m = \frac{Y - 0}{X - 0}\). However, \(m = \frac{Y}{X}\) is the ratio of integers, thus contradicting \(m\) being irrational, hence a line with an irrational slope passing through the origin cannot pass through a non-trivial lattice point.

Since values that satisfy the conjecture are integer, by definition they correspond to a lattice point and thus the line segment between that lattice point and the origin will have a rational slope per Theorem 2.1. We next establish the properties of term coprimality and thereafter the relationship coprimality has on the slope irrationality. Thereafter we establish several other preliminary proofs relating to reparameterizations and non-standard binomial expansions before returning back to the connection between term coprimality, slope irrationality, and the conjecture proof.

### 2.2. Coprimality of the Bases

According to the conjecture, solutions only exist when \(\gcd(A, B, C) > 1\). Hence testing for impossibility when \(\gcd(A, B, C) = 1\) requires we establish the relationship between 3-way coprimality and the more stringent pairwise coprimality.
Theorem 2.2. Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if \( \gcd(A, B) = 1 \), then \( \gcd(A^X, C^Z) = \gcd(B^Y, C^Z) = 1 \).

{If \( A \) and \( B \) are coprime, then \( C^Z \) is pairwise coprime to \( A^X \) and \( B^Y \).}

Proof. Suppose \( A \) and \( B \) are coprime. Then \( A^X \) and \( B^Y \) are coprime, and we can define these terms based on their respective prime factors, namely

\[
\begin{align*}
A^X &= p_1 p_2 p_3 \cdots p_m \\
B^Y &= q_1 q_2 q_3 \cdots q_n
\end{align*}
\]

where \( p_i, q_j \) are prime, and \( p_i \neq q_j \), for all \( i, j \). Based on Equation (1), we can express \( C^Z \) as

\[
C^Z = p_1 p_2 p_3 \cdots p_m + q_1 q_2 q_3 \cdots q_n
\]

Without loss of generalization, suppose we choose \( p_i \). Thus

\[
\frac{C^Z}{p_i} = \frac{p_1 p_2 p_3 \cdots p_m + q_1 q_2 q_3 \cdots q_n}{p_i}
\]

The term \( p_1 p_2 p_3 \cdots p_m \) is an integer since by definition \( p_i \mid p_1 p_2 p_3 \cdots p_m \). However the term \( \frac{q_1 q_2 q_3 \cdots q_n}{p_i} \) cannot be simplified since \( p_i \nmid q_1 q_2 q_3 \cdots q_n \) and thus \( p_i \nmid (p_1 p_2 p_3 \cdots p_m + q_1 q_2 q_3 \cdots q_n) \). Hence by extension \( p_i \nmid C^Z \) and \( A^X \) must thus be coprime to \( C^Z \). By applying the same logic with \( q_j \), then \( B^Y \) must also be coprime to \( C^Z \). Therefore if \( A \) and \( B \) are coprime, then \( C^Z \) must be pairwise coprime to both \( A^X \) and \( B^Y \).

\[\square\]

Theorem 2.3. Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if \( \gcd(A, C) = 1 \) or \( \gcd(B, C) = 1 \) then \( \gcd(A^X, C^Z) = \gcd(B^Y, C^Z) = 1 \).

{If \( A \) or \( B \) is coprime to \( C \), then \( C^Z \) is pairwise coprime to \( A^X \) and \( B^Y \).}

Proof. Without loss of generalization, suppose \( A \) and \( C \) are coprime. Thus \( A^X \) and \( C^Z \) are coprime. We can define \( C^Z \) based on its prime factors, namely

\[
C^Z = r_1 r_2 r_3 \cdots r_s
\]

where \( r_k \) are primes. Based on Equations (2a) and (5), we can define \( B^Y \) based on the difference between \( C^Z \) and \( A^X \), namely

\[
B^Y = r_1 r_2 r_3 \cdots r_s - p_1 p_2 p_3 \cdots p_m
\]

We now take any prime factor of \( C^Z \) and divide both sides of Equation (6) by that prime factor. Without loss of generalization, suppose we choose \( r_k \). Thus

\[
\frac{B^Y}{r_k} = \frac{r_1 r_2 r_3 \cdots r_s - p_1 p_2 p_3 \cdots p_m}{r_k}
\]

The term \( r_1 r_2 r_3 \cdots r_s \) is an integer since by definition \( r_k \mid r_1 r_2 r_3 \cdots r_s \). However the term \( \frac{p_1 p_2 p_3 \cdots p_m}{r_k} \) cannot be simplified since \( r_k \nmid p_1 p_2 p_3 \cdots p_m \) and thus \( r_k \nmid (r_1 r_2 r_3 \cdots r_s - p_1 p_2 p_3 \cdots p_m) \). Hence by extension \( r_k \nmid B^Y \) and \( C^Z \) must thus be coprime to \( B^Y \). By applying the same logic with \( p_i \), then \( C^Z \) must also be coprime to \( A^X \). Therefore if either \( A \) or \( B \) is coprime to \( C \), then \( C^Z \) must be pairwise coprime to both \( A^X \) and \( B^Y \).

\[\square\]

Theorem 2.4. Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if \( \gcd(A, B, C) = 1 \) then \( \gcd(A^X, B^Y) = \gcd(A^X, C^Z) = \gcd(B^Y, C^Z) = 1 \).

{If \( A, B \) and \( C \) are 3-way coprime, then they are all pairwise coprime.}
Proof. We consider two scenarios when \( \gcd(A, B, C) = 1 \), namely: \( \gcd(A, B) > 1 \) and \( \gcd(A, C) > 1 \) (the later of which generalizes to \( \gcd(B, C) > 1 \)).

**Scenario 1 of 2:** Suppose \( \gcd(A, B, C) = 1 \) while \( \gcd(A, B) > 1 \). Therefore \( A \) and \( B \) have a common factor. Thus we can express \( A^X \) and \( B^Y \) relative to their common factor, namely
\[
A^X = k \cdot p_1 p_2 p_3 \cdots p_m
\]
\[
B^Y = k \cdot q_1 q_2 q_3 \cdots q_n
\]
where integer \( k \) is the common factor, and \( p_i, q_j \) are prime, and \( p_i \neq q_j \), for all \( i, j \). Based on Equation (1), we can express \( C^Z \) as
\[
C^Z = k \cdot p_1 p_2 p_3 \cdots p_m + k \cdot q_1 q_2 q_3 \cdots q_n
\]

Per Equation (9a), \( k \) is a factor of \( C^Z \), just as it is a factor of \( A^X \) and \( B^Y \), thus \( \gcd(A, B, C) \neq 1 \), hence a contradiction. Thus when \( \gcd(A, B, C) = 1 \) we know \( \gcd(A, B) \neq 1 \) and thus \( k \) must be 1.

**Scenario 2 of 2:** Suppose \( \gcd(A, B, C) = 1 \) while \( \gcd(A, C) > 1 \). Therefore \( A \) and \( C \) have a common factor. Thus we can express \( A^X \) and \( C^Z \) relative to their common factor, namely
\[
A^X = k \cdot p_1 p_2 p_3 \cdots p_m
\]
\[
C^Z = k \cdot r_1 r_2 r_3 \cdots r_s
\]
where integer \( k \) is the common factor, and \( p_i, r_k \) are prime, and \( p_i \neq r_k \), for all \( i, k \). Based on Equation (1), we can express \( B^Y \) as
\[
B^Y = k \cdot r_1 r_2 r_3 \cdots r_s - k \cdot p_1 p_2 p_3 \cdots p_m
\]
\[
B^Y = k(r_1 r_2 r_3 \cdots r_s - p_1 p_2 p_3 \cdots p_m)
\]

Per Equation (11b), \( k \) is a factor of \( B^Y \), just as it is a factor of \( A^X \) and \( C^Z \), thus \( \gcd(A, B, C) \neq 1 \), hence a contradiction. Thus when \( \gcd(A, B, C) = 1 \) we know \( \gcd(A, C) \neq 1 \) and thus \( k \) must be 1. By extension and generalization, when \( \gcd(A, B, C) = 1 \) we know \( \gcd(B, C) \neq 1 \). \( \Box \)

Based on Theorems 2.2 to 2.4 if any pair of terms \( A, B, \) and \( C \) have no common factor, then all pairs of terms are coprime. Hence either all three terms share a common factor or they are all pairwise coprime. We thus formally conclude that if \( \gcd(A, B, C) = 1 \), then \( \gcd(A^X, B^Y) = \gcd(A^X, C^Z) = \gcd(B^Y, C^Z) = 1 \), and if \( \gcd(A, B) = 1 \) or \( \gcd(A, C) = 1 \) or \( \gcd(B, C) = 1 \), then \( \gcd(A, B, C) = 1 \).

### 2.3. Restrictions of the Exponents.

Trivial restrictions of the exponents are defined by the conjecture, namely integer and greater than 2. However, other restrictions apply such as, per Fermat’s last theorem, the exponents cannot be equal while greater than 2. More subtle restrictions also apply which will be required for the main proofs.

**Theorem 2.5.** Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), exponents \( X \) and \( Y \) cannot simultaneously be integer multiples of exponent \( Z \).

**Proof.** Suppose \( X \) and \( Y \) are simultaneously each integer multiples of \( Z \). Thus \( X = jZ \) and \( Y = kZ \) for positive integers \( j \) and \( k \). Therefore we can restate Equation (1) as
\[
(A^j)^Z + (B^k)^Z = C^Z
\]

Per Equation (12), we have 3 terms which are each raised to exponent \( Z \geq 3 \). According to Fermat’s last theorem [39, 40, 43], no integer solution exists when the terms share a common exponent greater than 2. Therefore \( X \) and \( Y \) cannot simultaneously be integer multiples of \( Z \). \( \Box \)

A trivially equivalent variation of Theorem 2.5 is that \( Z \) cannot simultaneously be a unit fraction of \( X \) and \( Y \). Given Theorem 2.5 there are only two possibilities:

1. Neither \( X \) or \( Y \) are multiples of \( Z \).
2. Only one of either \( X \) or \( Y \) is a multiple of \( Z \).
As such, given that at least one of the two exponents $X$ and $Y$ cannot be a multiple of $Z$, of the terms $A^X$ and $B^Y$, we therefore can arbitrarily choose $A^X$ to be the term whose exponent is not an integer multiple of exponent $Z$. Hence the following definition is used hereafter:

**Definition 2.1.** $X$ is not an integer multiple of $Z$.

Since per Theorem 2.5 at most only one of $X$ or $Y$ can be a multiple of $Z$ and given one can arbitrarily swap $A^X$ and $B^Y$, the arbitrary fixing hereafter of $A^X$ to be the term for which its exponent is not a multiple of $Z$ does not interfere with any of the characteristics or implications of the solution. Hence we hereafter define $A^X$ and $B^Y$ such that Definition 2.1 is maintained.

### 2.4. Reparameterization of the Terms

In exploring ways to leverage the binomial expansion and other equivalences, some researchers [2][3][15] explored reparameterizing one or more of the terms of $A^X + B^Y = C^Z$ so as to compare different sets of expansions. We broaden this idea to establish various irrationality conditions as related to coprimality of the terms, establish properties of the non-unique characteristics of key terms in the expansions, and showcase an exhaustive view to be leveraged when validating the conjecture.

The binomial expansion applied to the difference of perfect powers with different exponents is critical to mathematical research in general and to several proofs specifically later in this document. One feature of the binomial expansion in our application is the circumstance under which the upper limit of the sum is indeterminate [2][3] to be introduced in the following two theorems.

**Theorem 2.6.** If $p,q \neq 0$ and $v,w$ are real, then $p^v - q^w = (p + q)(p^{v-1} - q^{w-1}) - pq(p^{v-2} - q^{w-2})$.

*Proof.* Given non-zero $p$ and $q$, and real $v$ and $w$, suppose we can expand the difference $p^v - q^w$ as

\[
(13) \quad p^v - q^w = (p + q)(p^{v-1} - q^{w-1}) - pq(p^{v-2} - q^{w-2})
\]

Distributing $(p + q)$ on the right side of Equation (13) into $(p^{v-1} - q^{w-1})$ gives us $[p^v - pq^{w-1} + p^{v-1}q - q^w]$ and distributing $-pq$ into $(p^{v-2} - q^{w-2})$ gives us $[-p^{v-1}q + pq^{w-1}]$. Thus simplifying Equation (13) gives us

\[
(14a) \quad p^v - q^w = [p^v - pq^{w-1} + p^{v-1}q - q^w] + [-p^{v-1}q + pq^{w-1}]
\]

\[
(14b) \quad p^v - q^w = p^v + [pq^{w-1} - pq^{w-1}] + [p^{v-1}q - p^{v-1}q] - q^w
\]

\[
(14c) \quad p^v - q^w = p^v - q^w
\]

Thus the difference of powers can indeed be expanded per the above functional form accordingly. We also observe Equation (13) can be expressed in more compact notation, namely

\[
(15) \quad p^v - q^w = \sum_{i=0}^{1} (p + q)^{1-i}(-pq)^i(p^{v-1-i} - q^{w-1-i})
\]

We further observe in Equation (14b) of Theorem 2.6 that this expansion of the difference of two powers yields two other terms which are themselves differences of powers, namely $(p^{v-1} - q^{w-1})$ and $(p^{v-2} - q^{w-2})$. Each of these differences could likewise be expanded with the same functional form of Theorem 2.6. Recursively expanding the resulting terms of differences of powers leads to a more general form of Equation (15).

**Theorem 2.7.** If $p,q \neq 0$ and integer $n \geq 0$, then $p^v - q^w = \sum_{i=0}^{n} \binom{n}{i} (p + q)^{n-i}(-pq)^i(p^{v-n-i} - q^{w-n-i})$.

*General form of the expansion of the difference of two powers*
Proof. Suppose \( p, q \neq 0 \) and integer \( n \geq 0 \), and suppose

\[
(16) \quad p^v - q^w = \sum_{i=0}^{n} \binom{n}{i} (p+q)^{n-i}(-pq)^i(p^{v-n-i} - q^{w-n-i})
\]

Consider \( n = 0 \). The right side of Equation (16) reduces to \( p^v - q^w \), thus Equation (16) holds when \( n = 0 \). Consider \( n = 1 \). The right side of Equation (16) becomes

\[
(17a) \quad p^v - q^w = \binom{1}{0} (p+q)^{1-0}(-pq)^0(p^{v-1-0} - q^{w-1-0}) + \binom{1}{1} (p+q)^{1-1}(-pq)^1(p^{v-1-1} - q^{w-1-1})
\]

\[
(17b) \quad p^v - q^w = (p+q)(p^{v-1} - q^{w-1}) + (-pq)(p^{v-2} - q^{w-2})
\]

\[= p^v - q^w \]

The right side of Equation (17b) also reduces to \( p^v - q^w \). Hence Equation (16) holds for \( n = 0 \) and \( n = 1 \).

In generalizing, enumerating the terms of Equation (16) gives us

\[
(18) \quad p^v - q^w = \binom{n}{0} (p+q)^n(p^{v-n} - q^{w-n})
\]

\[+ \binom{n}{1} (p+q)^{n-1}(-pq)(p^{v-n-1} - q^{w-n-1})
\]

\[+ \binom{n}{2} (p+q)^{n-2}(-pq)^2(p^{v-n-2} - q^{w-n-2})
\]

\[+ \ldots
\]

\[+ \binom{n}{n-2} (p+q)^2(-pq)^{n-2}(p^{v-2n+2} - q^{w-2n+2})
\]

\[+ \binom{n}{n-1} (p+q)(-pq)^{n-1}(p^{v-2n+1} - q^{w-2n+1})
\]

\[+ \binom{n}{n} (-pq)^n(p^{v-2n} - q^{w-2n})
\]

Expanding each of the \( n+1 \) differences of powers \( (p^{v-n-i} - q^{w-n-i}) \) of Equation (18) per Theorem 2.6 gives us

\[
(19) \quad p^v - q^w = \binom{n}{0} (p+q)^n [(p+q)(p^{v-n-1} - q^{w-n-1}) - pq(p^{v-n-2} - q^{w-n-2})]
\]

\[+ \binom{n}{1} (p+q)^{n-1}(-pq)[(p+q)(p^{v-n-2} - q^{w-n-2}) - pq(p^{v-n-3} - q^{w-n-3})]
\]

\[+ \binom{n}{2} (p+q)^{n-2}(-pq)^2[(p+q)(p^{v-n-3} - q^{w-n-3}) - pq(p^{v-n-4} - q^{w-n-4})]
\]

\[+ \ldots
\]

\[+ \binom{n}{n-2} (p+q)^2(-pq)^{n-2}[(p+q)(p^{v-2n+1} - q^{w-2n+1}) - pq(p^{v-2n} - q^{w-2n})]
\]

\[+ \binom{n}{n-1} (p+q)(-pq)^{n-1}[(p+q)(p^{v-2n} - q^{w-2n}) - pq(p^{v-2n-1} - q^{w-2n-1})]
\]

\[+ \binom{n}{n} (-pq)^n[(p+q)(p^{v-2n-1} - q^{w-2n-1}) - pq(p^{v-2n-2} - q^{w-2n-2})]
\]
Distributing each of the \( \binom{n}{i} (p + q)^{n-i} (-pq)^i \) terms of Equation (19) into the corresponding bracketed terms then gives us

\[
\begin{align*}
\quad & p^v - q^w = \binom{n}{0} (p + q)^{n+1} (p^{v-n-1} - q^{w-n-1}) + \binom{n}{0} (p + q)^n (-pq)(p^{v-n-2} - q^{w-n-2}) \\
& + \left[ \binom{n}{1} + \binom{n}{0} \right] (p + q)^n (-pq)(p^{v-n-2} - q^{w-n-2}) \\
& + \left[ \binom{n}{2} + \binom{n}{1} \right] (p + q)^{n-1} (-pq)^2 (p^{v-n-3} - q^{w-n-3}) \\
& + \left[ \binom{n}{3} + \binom{n}{2} \right] (p + q)^{n-2} (-pq)^3 (p^{v-n-4} - q^{w-n-4}) \\
& + \cdots \\
& + \left[ \binom{n}{n-2} + \binom{n}{n-3} \right] (p + q)^3 (-pq)^{n-2} (p^{v-2n+1} - q^{w-2n+1}) \\
& + \left[ \binom{n}{n-1} + \binom{n}{n-2} \right] (p + q)^2 (-pq)^{n-1} (p^{v-2n} - q^{w-2n}) \\
& + \left[ \binom{n}{n} + \binom{n}{n-1} \right] (p + q)(-pq)^n (p^{v-2n-1} - q^{w-2n-1}) \\
& + \binom{n}{n} (-pq)n + 1 (p^{v-2n-2} - q^{w-2n-2}) 
\end{align*}
\]

which can be simplified to

\[
\begin{align*}
\quad & p^v - q^w = \binom{n}{0} (p + q)^{n+1} (p^{v-n-1} - q^{w-n-1}) \\
& + \left[ \binom{n}{1} + \binom{n}{0} \right] (p + q)^n (-pq)(p^{v-n-2} - q^{w-n-2}) \\
& + \left[ \binom{n}{2} + \binom{n}{1} \right] (p + q)^{n-1} (-pq)^2 (p^{v-n-3} - q^{w-n-3}) \\
& + \left[ \binom{n}{3} + \binom{n}{2} \right] (p + q)^{n-2} (-pq)^3 (p^{v-n-4} - q^{w-n-4}) \\
& + \cdots \\
& + \left[ \binom{n}{n-2} + \binom{n}{n-3} \right] (p + q)^3 (-pq)^{n-2} (p^{v-2n+1} - q^{w-2n+1}) \\
& + \left[ \binom{n}{n-1} + \binom{n}{n-2} \right] (p + q)^2 (-pq)^{n-1} (p^{v-2n} - q^{w-2n}) \\
& + \left[ \binom{n}{n} + \binom{n}{n-1} \right] (p + q)(-pq)^n (p^{v-2n-1} - q^{w-2n-1}) \\
& + \binom{n}{n} (-pq)n + 1 (p^{v-2n-2} - q^{w-2n-2}) 
\end{align*}
\]
Pascal’s identity states that \( \binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1} \) for integer \( k \geq 1 \) and integer \( m \geq 0 \). Given this identity, each sum of the pairs of binomial coefficients in the brackets of Equation (21) simplifies to:

\[
p^v - q^w = \binom{n}{0} (p+q)^{n+1} (p^v - q^w - n - 1) \\
+ \binom{n+1}{1} (p+q)^n (-pq)(p^v - q^w - n - 2) \\
+ \binom{n+1}{2} (p+q)^{n-1} (-pq)^2 (p^v - q^w - n - 3) \\
+ \binom{n+1}{3} (p+q)^{n-2} (-pq)^3 (p^v - q^w - n - 4) \\
+ \ldots \\
+ \binom{n+1}{n-2} (p+q)^2 (-pq)^{n-2} (p^v - q^w - 2n + 1) \\
+ \binom{n+1}{n-1} (p+q) (-pq)^{n-1} (p^v - q^w - 2n) \\
+ \binom{n+1}{n} (-pq)^n (p^v - q^w - 2n - 1) \\
+ \binom{n+1}{n} (-pq)^{n+1} (p^v - q^w - 2n - 2)
\]

(22a)

\[
p^v - q^w = \sum_{i=0}^{n+1} \binom{n+1}{i} (p+q)^{n-i+1} (-pq)^i (p^v - q^w - n - i - 1)
\]

(22b)

The right side of Equation (16) and Equation (22b) both equal \( p^v - q^w \), thus the right side of these two equations are equal. Hence

\[
\sum_{i=0}^{n} \binom{n}{i} (p+q)^{n-i} (-pq)^i (p^v - q^w - n - i) = \sum_{i=0}^{n+1} \binom{n+1}{i} (p+q)^{n-i+1} (-pq)^i (p^v - q^w - n - i - 1)
\]

(23)

Therefore by induction, since \( p^v - q^w = \sum_{i=0}^{n} \binom{n}{i} (p+q)^{n-i} (-pq)^i (p^v - q^w - n - i) \) for \( n = 0 \) and \( n = 1 \) and per Equations (22b) and (23), this relation holds when \( n \) is replaced with \( n + 1 \). Hence this relation holds for all integers \( n \geq 0 \).

We observe an important property, per Equations (16), (22b), and (23), that \( n \) is indeterminate since \( p^v - q^w = \sum_{i=0}^{n} \binom{n}{i} (p+q)^{n-i} (-pq)^i (p^v - q^w - n - i) \) holds for every non-negative integer value of \( n \). Hence any non-negative integer value of \( n \) can be selected and the resulting expansion still applies, leading to different expansions that sum to identical outcomes.

Other preliminary properties required for the proof of the conjecture include the fact that each of the perfect powers \( A^X \), \( B^Y \) and \( C^Z \) can be expressed as a linear combination of an additive and multiplicative form of the bases of the other two terms. This property will reveal a variety of equivalences across various domains. We need to first establish a few basic principles.

**Theorem 2.8.** Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if there exists an integer solution, then there exists non-zero positive rational \( \alpha \) and \( \beta \) such that \( A^X = [(C + B)\alpha - CB\beta]^X \).

**Proof.** Given Equation (1), by definition \( A^X = C^Z - B^Y \). If there exist integer solutions that satisfy the conjecture then \( A \) and \( \sqrt[3]{C^Z - B^Y} \) must be integers. Suppose there exists non-zero rational \( \alpha \) and \( \beta \) such
that $A = (C + B)\alpha - CB\beta$, then these two expressions for $A$ are identical, namely

$$X = (C + B)\alpha - CB\beta \tag{26}$$

Solving for $\alpha$ when given any rational $\beta > 0$ gives us $\alpha = \frac{\sqrt[\beta]{\sqrt{C^2 - B^2} + CB\beta}}{C + B}$, which is positive since the numerator and denominator are each positive. Further, if $\beta$ is rational, then $\alpha$ must also be rational since $\sqrt[\beta]{\sqrt{C^2 - B^2}}$, $C$, and $B$ are each integer.

Solving instead for $\beta$ when given any arbitrary sufficiently large positive rational $\alpha$ gives us $\beta = \frac{\sqrt[\alpha]{\sqrt{C^2 - B^2} - (C + B)\alpha}}{-CB}$, which is positive given the numerator and denominator are both negative integers. Further, if $\alpha$ is rational, then $\beta$ must also be rational since $\sqrt[\alpha]{\sqrt{C^2 - B^2}}$, $C$, and $B$ are each integer. Hence there exist non-zero positive rational $\alpha$ and $\beta$ such that $A^X = C^Z - B^Y$ and $A^X = [(C+B)\alpha-CB\beta]^X$, when the terms of the conjecture are satisfied. $\square$

Without loss of generalization, it can be trivially shown that Theorem 2.8 establishes an alternate functional form in which $A^X = [(C+B)\alpha-CB\beta]^X$ also applies to $B^Y$ wherein there exists other non-zero positive rational $\alpha$ and $\beta$ such that $B^Y = [(C + A)\alpha - CA\beta]^Y$.

Suppose we arbitrarily let $\alpha = \sqrt[\beta]{\sqrt{C^2 - X - B^Y - X}}$ then per Theorem 2.8 we can solve for $\beta$ from $C^2 - B^Y = [(C + B)\alpha - CB\beta]^X$ which gives us $\beta = \frac{\sqrt[\alpha]{\sqrt{C^2 - B^2} - (C + B)\alpha}}{-CB}$. Likewise if we arbitrarily let $\beta = \sqrt[\alpha]{\sqrt{C^2 - 2X - B^Y - 2X}}$ then per Theorem 2.8 we can solve for $\alpha$ from $C^2 - B^Y = [(C + B)\alpha - CB\beta]^X$ which gives us $\alpha = \frac{\sqrt[\beta]{\sqrt{C^2 - B^2} + CB\beta}}{C + B}$. In either case, this yields set $\{\alpha, \beta\}$ that satisfies $C^2 - B^Y = [(C + B)\alpha - CB\beta]^X$ which shows these definitions of $\alpha$ and $\beta$ maintain $A^X$ as a perfect power of an integer based on $B$ and $C$, namely $A^X = [(C + B)\alpha - CB\beta]^X$ while satisfying $A^X = C^2 - B^Y$. Further, based on the indeterminacy of the upper bound in the binomial expansion of the difference of two perfect powers from Theorem 2.7 we can also find values of $\alpha$ and $\beta$ that are explicitly functions of $C$ and $B$.

**Theorem 2.9.** Given positive integers $A$, $B$, $C$, $X$, $Y$, $Z$, where $X, Y, Z \geq 3$ and $A^{X + B^Y} = C^Z$, values $\alpha = \sqrt[\beta]{\sqrt{C^2 - X - B^Y - X}}$, $\beta = \sqrt[\alpha]{\sqrt{C^2 - 2X - B^Y - 2X}}$, and real $M$ satisfy $C^2 - B^Y = [(C + B)M\alpha - CBM\beta]^X$.

**Proof.** Given Equation (1), we know that integer $A^X = C^2 - B^Y$ can be expanded with the general binomial expansion from Theorem 2.7 for the difference of perfect powers, namely

$$A^X = C^2 - B^Y = \sum_{i=0}^{n} \binom{n}{1} (C + B)^{n-i}(-CB)^i (C^{2n-i} - B^{Y-n}) \tag{25}$$

Per Theorem 2.7, since upper limit $n$ in Equation (25) is indeterminate, we can replace $n$ with any value such as $X$ while entirely preserving complete integrity of the terms, hence

$$A^X = C^2 - B^Y = \sum_{i=0}^{X} \binom{X}{i} (C + B)^{X-i}(-CB)^i (C^{2X-i} - B^{Y-X}) \tag{26}$$

Common to Equation (27) below

Furthermore, from Theorem 2.8 we know that $A = (C + B)\alpha - CB\beta$ for non-zero real $\alpha$ and $\beta$. Raising $A = (C + B)\alpha - CB\beta$ to $X$ and then expanding gives us

$$A^X = ((C + B)\alpha - CB\beta)^X \sum_{i=0}^{X} \binom{X}{i} ((C + B)\alpha)^{X-i}(-CB)^i \tag{27a}$$

$$A^X = ((C + B)\alpha - CB\beta)^X \sum_{i=0}^{X} \binom{X}{i} (C + B)^{X-i}(-CB)^i \alpha^{X-i} \beta^i \tag{27b}$$

Common to Equation (26) above
Equations (26) and (27b) have an identical number of terms, share the identical binomial coefficient, and share \((C + B)^{X-1}(-CB)^i\) for each value of \(i\). See the expansion in Table 1.

| Term \(i\) | Terms Common to Equations (26) and (27b) | Terms Unique to Equations (26) | Terms Unique to Equations (27b) |
|-------------|------------------------------------------|-------------------------------|---------------------------------|
| 0           | \(\binom{X}{0}(C + B)^X\)                | \(CZ^{-X} - BY^{-X}\)          | \(\alpha^X\)                   |
| 1           | \(\binom{X}{1}(C + B)^{X-1}(-CB)\)      | \(CZ^{-X-1} - BY^{-X-1}\)      | \(\alpha^{X-1}\beta\)         |
| 2           | \(\binom{X}{2}(C + B)^{X-2}(-CB)^2\)    | \(CZ^{-X-2} - BY^{-X-2}\)      | \(\alpha^{X-2}\beta^2\)       |
| ...         |                                          |                               |                                 |
| \(X - 1\)   | \(\binom{X}{X-1}(C + B)(-CB)^{X-1}\)   | \(CZ^{-2X+1} - BY^{-2X+1}\)    | \(\alpha \beta^{X-1}\)         |
| \(X\)       | \(\binom{X}{X}(-CB)^X\)                 | \(CZ^{-2X} - BY^{-2X}\)        | \(\beta^X\)                    |

Per Equations (26) and (27b) and Theorem 2.8, \(A^n\) equals both \(CZ^{-X} - BY^{-X}\) and \(((C + B)\alpha - CB\beta)^X\), thus the finite sums of Equations (26) and (27b) are equal. Each of the terms in the corresponding expansions have common components and unique components. Thus there exists a term-wise map between the unique components of the two sets of expansions.

When \(i = 0\), we observe in Table 1 that \(\alpha^X\) maps to \(CZ^{-X} - BY^{-X}\). Hence if \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X}}\) or more generically \(\alpha = \sqrt[2X]{|CZ^{-X} - BY^{-X}|}\), then per Theorem 2.8, the corresponding \(\beta\) is \(\beta = \sqrt[2X]{CZ^{-X} - BY^{-X}} - (C + B)\alpha\).

When \(i = X\), we observe in Table 1 that \(\beta^X\) maps to \(CZ^{-2X} - BY^{-2X}\). Hence if \(\beta = \sqrt[2X]{CZ^{-2X} - BY^{-2X}}\) or more generically \(\beta = \sqrt[2X]{|CZ^{-2X} - BY^{-2X}|}\), then per Theorem 2.8, the corresponding \(\alpha\) is \(\alpha = \sqrt[2X]{CZ^{-2X} - BY^{-2X}} + CB\beta\).

Using the map between \(\alpha^X\) to \(CZ^{-X} - BY^{-X}\) based on \(i = 0\) and the map between \(\beta^X\) to \(CZ^{-2X} - BY^{-2X}\) based on \(i = X\), we can align to the other terms in the expansion of Equations (26) and (27b). When \(i = 1\), we have the terms \(CZ^{-X-1} - BY^{-X-1}\) and \(\alpha^{X-1}\beta\) (see Table 1). Substituting \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X}}\) and \(\beta = \sqrt[2X]{CZ^{-2X} - BY^{-2X}}\) for \(i = 1\), we have

\[
CZ^{-X-1} - BY^{-X-1} = \alpha^{X-1}\beta
\]

Equation (28b) holds in only trivial conditions. Hence \(\alpha\) and \(\beta\) cannot simultaneously equal \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X}}\) and \(\beta = \sqrt[2X]{CZ^{-2X} - BY^{-2X}}\). As such, there are only three possibilities regarding \(\alpha\) and \(\beta\) that ensure \(CZ^{-X} - BY^{-X} = [(C + B)\alpha - CB\beta]^X\): 

1. \(\alpha\) is arbitrarily defined and thus \(\beta = \sqrt[2X]{CZ^{-X} - BY^{-X} - (C + B)\alpha}\).
2. \(\beta\) is arbitrarily defined and thus \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X} + CB\beta}\).
3. \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X}}\) and \(\beta = \sqrt[2X]{CZ^{-2X} - BY^{-2X}}\) where one or both are scaled.

The first two cases, given \(\alpha\) or \(\beta\) and the other derived therefrom per Theorem 2.8, will satisfy \(CZ^{-X} - BY^{-X} = [(C + B)\alpha - CB\beta]^X\). In the third case, since \(\alpha = \sqrt[2X]{CZ^{-X} - BY^{-X}}\) and \(\beta = \sqrt[2X]{CZ^{-2X} - BY^{-2X}}\),...
do not simultaneously satisfy \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \), then scaling \( \alpha \) and \( \beta \) by \( M \) such that \( C^Z - B^Y = [(C + B)M\alpha - CBM\beta]^X \) will ensure equality, where

\[
(29a) \quad C^Z - B^Y = [(C + B)M\alpha - CBM\beta]^X
\]

\[
(29b) \quad \sqrt[\alpha]{C^Z - B^Y} = M[(C + B)\alpha - CB\beta]
\]

\[
(29c) \quad M = \frac{\sqrt[\alpha]{C^Z - B^Y}}{(C + B)\alpha - CB\beta}
\]

Since every set \( \{\alpha, \beta\} \) is unique, then per Equation \( (29c) \) there exists a unique \( M \) that satisfies \( C^Z - B^Y = [(C + B)M\alpha - CBM\beta]^X \) when \( \alpha = \sqrt[\alpha]{C^Z - B^Y} \) and \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) simultaneously.

Given that \( C^Z - B^Y \) and \( [(C + B)M\alpha - CBM\beta]^X \) are identical, their binomial expansions are structurally identical, and their sums are identical, then indeed \( \alpha = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) and \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) together ensure the equality of Equations \( (26) \) and \( (27b) \) hold for \( M \) as defined in Equation \( (29c) \).

**Characteristics of \( M, \alpha \) and \( \beta \) from Theorem 2.9**

The important feature of this is not the scalar but instead the characteristics of \( \alpha \) and \( \beta \) as defined above despite the scalar. Based on \( A^X + B^Y = C^Z \), we know \( A = \sqrt[\alpha]{C^Z - B^Y} \). The structural similarity between \( C^Z - B^Y \) in the formula for \( A \) and the expressions \( C^Z - X - B^Y - X \) and \( C^Z - 2^X - B^Y - 2^X \) in the formulas for \( \alpha \) and \( \beta \), respectively, is critical. This structural similarity will be explored and exploited later in this document.

We note \( \alpha \) and \( \beta \) could be defined differently from above and still maintain the equality of \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \), without scalar \( M \). However, if one defined \( \alpha \) or \( \beta \) differently than above, then either of these terms must be \( \beta = \frac{\sqrt[\alpha]{C^Z - B^Y} - (C + B)\alpha}{-CB} \) or \( \alpha = \frac{\sqrt[\alpha]{C^Z - B^Y} + CB\beta}{C + B} \) in order to satisfy \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \). Since any arbitrary \( \alpha \) corresponds to a unique \( \beta \), there are an infinite number of sets \( \{\alpha, \beta\} \) that satisfy this equation. In some conditions, there are no rational pairs among the infinite number of sets \( \{\alpha, \beta\} \), such as if \( \sqrt[\alpha]{C^Z - B^Y} \) is irrational. But per the above, there is only unique set \( \{\alpha, \beta\} \) when \( \alpha = \sqrt[\alpha]{C^Z - X - B^Y - X} \) and \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) when scalar \( M \) is applied accordingly.

We note all possible sets of \( \{\alpha, \beta\} \) map to \( \alpha = \sqrt[\alpha]{C^Z - X - B^Y - X} \) and \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) for scalar \( M \) since all sets must satisfy \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \).

**Theorem 2.10.** Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if there exists set \( \{A, B, C, X, Y, Z\} \) that does not satisfy these conditions, then \( \alpha \) or \( \beta \) that satisfies \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \) will be irrational.

**Proof.** From Theorem 2.9 we know for any and every possible value of \( \alpha \), that \( \beta = \frac{\sqrt[\alpha]{C^Z - B^Y} - (C + B)\alpha}{-CB} \). Without loss of generality, suppose \( B, C, X, Y, \) and \( Z \) are integer but there is no integer value of \( A \) that satisfies the conjecture, then \( A = \sqrt[\alpha]{C^Z - B^Y} \) must be irrational. Hence \( \beta \) is irrational given the irrational term \( \sqrt[\alpha]{C^Z - B^Y} \) in the numerator of \( \beta \).

We also know from Theorem 2.9 given any and every possible value of \( \beta \) that \( \alpha = \frac{C + B}{\sqrt[\alpha]{C^Z - B^Y} + CB\beta} \). Here too without loss of generality, if \( B, C, X, Y, \) and \( Z \) are integer but there is no integer value of \( A \) that satisfies the conjecture, then \( A = \sqrt[\alpha]{C^Z - B^Y} \) must be irrational. Hence \( \alpha \) is irrational given the irrational term \( \sqrt[\alpha]{C^Z - B^Y} \) in the numerator of \( \alpha \).

Hence when set \( \{A, B, C, X, Y, Z\} \) does not satisfy the conjecture, then the corresponding \( \alpha \) or \( \beta \) that satisfies \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \) is irrational. □

We note that the exclusion of scalar \( M \) from \( C^Z - B^Y = [(C + B)M\alpha - CBM\beta]^X \) in Theorem 2.10 (letting \( M = 1 \)) does not change the outcome in that if \( C^Z - B^Y \) is not a perfect power, then scaled or unscaled \( \alpha \) and \( \beta \) cannot change the irrationality of the root of \( C^Z - B^Y \). Since \( \alpha = \sqrt[\alpha]{C^Z - X - B^Y - X} \), \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \), and scalar \( M \) always together satisfy \( C^Z - B^Y = [(C + B)M\alpha - CBM\beta]^X \), then we can study the properties of \( \alpha = \sqrt[\alpha]{C^Z - X - B^Y - X} \) and \( \beta = \sqrt[\alpha]{C^Z - 2^X - B^Y - 2^X} \) as related to key
characteristics of the conjecture. To do so, we need to establish the implication of coprimality and irrationality as it relates to $\alpha$ and $\beta$.

**Theorem 2.11.** Given positive integers $A, B, C, X, Y, Z$, where $X,Y,Z \geq 3$ and $A^X + B^Y = C^Z$, if $\gcd(A, B, C) = 1$, then both $\alpha = \sqrt[3]{C^{Z-X} - B^{Y-X}}$ and $\beta = \sqrt[3]{C^{Z-2X} - B^{Y-2X}}$ are irrational.

**Proof.** Suppose $\alpha = \sqrt[3]{C^{Z-X} - B^{Y-X}}$ is rational, $A, B, C, X, Y$, and $Z$ are integers that satisfy the conjecture, and $\gcd(A, B, C) = 1$. Then we can express $\alpha$ as

\begin{equation}
\alpha^X = C^{Z-X} - B^{Y-X}
\end{equation}

\begin{equation}
\alpha^X = \frac{C^Z}{C^X} - \frac{B^Y}{B^X}
\end{equation}

\begin{equation}
\alpha^X = \frac{B^X C^Z - C^X B^Y}{B^X C^X}
\end{equation}

Since $\alpha$ is rational, then $\alpha$ can be expressed as the ratio of integers $p$ and $q$ such that

\begin{equation}
\frac{p^X}{q^X} = \frac{B^X C^Z - C^X B^Y}{B^X C^X}
\end{equation}

where $p^X = B^X C^Z - C^X B^Y$ and $q^X = B^X C^X$. We note that the denominator of Equation (31) is a perfect power of $X$ and thus we know that the numerator must also be a perfect power of $X$ for the $X^{th}$ root of their ratio to be rational. Hence the ratio of the perfect power of the numerator and the perfect power of the denominator, even after simplifying, must thus be rational.

Regardless of the parity of $B$ or $C$, per Equation (31), $p^X$ must be even as it is the difference of two odd numbers or the difference of two even numbers. Furthermore, since $p^X$ by definition is a perfect power of $X$ and given that it is even, then $p^X$ must be defined as $2^{iX}(f_1 f_2 \cdots f_{n_f})^X$ for positive integers $i, j, f_1, f_2, ..., f_{n_f}$, where $2^{iX}$ is the perfect power of the even component of $p^X$ and $(f_1 f_2 \cdots f_{n_f})^X$ is the perfect power of the remaining $n_f$ prime factors of $p^X$ where $f_1, f_2, ..., f_{n_f}$ are the remaining prime factors of $p^X$. Hence

\begin{equation}
\frac{p^X}{q^X} = \frac{2^{iX}(f_1 f_2 \cdots f_{n_f})^X}{BC} = \frac{B^X C^Z - C^X B^Y}{BC}
\end{equation}

$B$ and $C$ can also be expressed as a function of their prime factors, thus

\begin{equation}
\frac{p^X}{q^X} = \frac{2^{iX}j_1^{X-1}j_2^{X-2} \cdots f_{n_f}^{X-1}}{b_1 b_2 \cdots b_{n_b} c_1 c_2 \cdots c_{n_c}}
\end{equation}

where $b_1, b_2, ..., b_{n_b}$ and $c_1, c_2, ..., c_{n_c}$ are prime factors of $B$ and $C$ respectively. Based on the right side of Equation (33), the entire denominator $BC$ is fully subsumed by the numerator, and thus every one of the prime factors in the denominator equals one of the prime factors in the numerator. Thus after dividing, one or more of the exponents in the numerator reduces thereby canceling the entire denominator accordingly. For illustration, suppose $b_1 = b_2 = 2, b_{n_b} = f_1, c_1 = c_2 = f_2$ and $c_{n_c} = f_{n_f}$. As such, Equation (33) simplifies to

\begin{equation}
p^X = 2^{iX-2}j_1^{X-1}j_2^{X-2} \cdots f_{n_f}^{X-1}
\end{equation}

which has terms with exponents that are not multiples of $X$ and are thus not perfect powers of $X$. Therefore the $X^{th}$ root is irrational which contradicts the assumption that $\alpha = \frac{p}{q}$ is rational. We note that all factors in the denominator of Equation (33) cannot each be perfect powers of $X$ since the bases $B$ and $C$ are defined to be reduced. More generally, beyond the illustration, after simplifying one or more terms, Equation (34) will have an exponent that is not a multiple of $X$ and thus is irrational when taking the root accordingly.

Suppose $\beta = \sqrt[3]{C^{Z-2X} - B^{Y-2X}}$ is rational, $A, B, C, X, Y$, and $Z$ are integers, and $\gcd(A, B, C) = 1$. Then we can express $\beta$ as

\begin{equation}
\beta^X = C^{Z-2X} - B^{Y-2X}
\end{equation}

\begin{equation}
\beta^X = \frac{C^Z}{C^{2X}} - \frac{B^Y}{B^{2X}}
\end{equation}

\begin{equation}
\beta^X = \frac{B^{2X} C^Z - C^{2X} B^Y}{B^{2X} C^{2X}}
\end{equation}
Since $\beta$ is rational, then $\beta$ can be expressed as the ratio of integers $p$ and $q$ such that

$$
\frac{p^X}{q^X} = \frac{B^{2X}C^Z - C^{2X}B^Y}{B^{2X}C^{2X}}
$$

where $p^X = B^{2X}C^Z - C^{2X}B^Y$ and $q^X = B^{2X}C^{2X}$. We note that the denominator of Equation (36) is a perfect power of $X$ and thus we know that the numerator must also be a perfect power of $X$ for the $X^{th}$ root of their ratio to be rational. Hence the ratio of the perfect power of the numerator and the perfect power of the denominator, even after simplifying, must thus be rational.

Regardless of the parity of $B$ or $C$, per Equation (36), $p^X$ must be even as it is the difference of two odd numbers or the difference of two even numbers. Furthermore, since $p^X$ by definition is a perfect power of $X$ and given that it is even, then $p^X$ must be defined as $2^X (f_1 f_2 \cdots f_{n_f})^X$ for positive integers $i, j, f_1, f_2, ..., f_{n_f}$ where $2^X$ is the perfect power of the even component of $p^X$ and $(f_1 f_2 \cdots f_{n_f})^X$ is the perfect power of the remaining $n_f$ prime factors of $p^X$ where $f_1, f_2, ..., f_{n_f}$ are the remaining prime factors of $p^X$. Hence

$$
\frac{p^X}{q^X} = \frac{2^X (f_1 f_2 \cdots f_{n_f})^X}{B^{2X}C^Z - C^{2X}B^Y} = \frac{B^{2X}C^Z - C^{2X}B^Y}{B^{2X}C^{2X}}
$$

$B^2$ and $C^2$ can also be expressed as a function of their prime factors, thus

$$
\frac{p^X}{q^X} = \frac{2^X f_1^X f_2^X \cdots f_{n_f}^X}{b_1^2 b_2^2 \cdots b_{n_b}^2 c_1^2 c_2^2 \cdots c_{n_c}^2}
$$

where $b_1, b_2, ..., b_{n_b}$ and $c_1, c_2, ..., c_{n_c}$ are prime factors of $B$ and $C$ respectively. Based on the right side of Equation (37), the entire denominator $B^{2X}C^{2X}$ is fully subsumed by the numerator, and thus every one of the prime factors in the denominator equals one of the prime factors in the numerator. Thus after dividing, one or more of the exponents in the numerator reduces thereby canceling the entire denominator accordingly. For illustration, suppose $b_1 = b_2 = 2, b_{n_b} = f_1, c_1 = c_2 = f_2$ and $c_{n_c} = f_{n_f}$. As such, Equation (38) simplifies to

$$
\frac{p^X}{q^X} = \frac{2^X f_1^X f_2^X \cdots f_{n_f}^X}{f_1^X f_2^X \cdots f_{n_f}^X} = \frac{1}{2^{X-4} f_1^{X-2} f_2^{X-4} \cdots f_{n_f}^{X-2}}
$$

which has terms with exponents that are not multiples of $X$ and are thus not perfect powers of $X$. Therefore the $X^{th}$ root is irrational which contradicts the assumption that $\alpha = \frac{p}{q}$ is rational. We note that all factors in the denominator of Equation (38) cannot each be perfect powers of $X$ since the bases $B$ and $C$ are defined to be reduced. More generally, beyond the illustration, after simplifying one or more terms, Equation (39) will have an exponent that is not a multiple of $X$ and thus is irrational when taking the root accordingly.

Since the definition of a rational number is the ratio of two integers in which the ratio is reduced, given that $\gcd(B, C) = 1$, then both $\alpha = \sqrt[2X]{C^{Z-X} - B^{Y-X}}$ and $\beta = \sqrt[2X]{C^{Z-2X} - B^{Y-2X}}$ are irrational.

Theorem 2.11 establishes that if $\gcd(A, B, C) = 1$ then $\alpha = \sqrt[2X]{C^{Z-X} - B^{Y-X}}$ and $\beta = \sqrt[2X]{C^{Z-2X} - B^{Y-2X}}$ are irrational. We now establish the reverse, such that if $\alpha = \sqrt[2X]{C^{Z-X} - B^{Y-X}}$ and $\beta = \sqrt[2X]{C^{Z-2X} - B^{Y-2X}}$ are rational then $\gcd(A, B, C) > 1$.

**Theorem 2.12.** Given positive integers $A, B, C, X, Y, Z$, where $X, Y, Z \geq 3$ and $A^X + B^Y = C^Z$, if $\sqrt[2X]{C^{Z-X} - B^{Y-X}}$ or $\sqrt[2X]{C^{Z-2X} - B^{Y-2X}}$ are rational, then $\gcd(A, B, C) > 1$.

**Proof.** Suppose $\alpha = \sqrt[2X]{C^{Z-X} - B^{Y-X}}$ is rational and $A, B, C, X, Y, Z$ are integers that satisfy the conjecture. Then we can express $\alpha$ as

$$
\alpha^X = C^{Z-X} - B^{Y-X}
$$

$$
\alpha^X = C^Z - B^Y
$$

$$
\alpha^X = \frac{B^X C^Z - C^X B^Y}{B^X C^X}
$$

Since $\alpha$ is rational, then $\alpha$ can be expressed as the ratio of integers $p$ and $q$ such that

$$
\frac{p^X}{q^X} = \frac{B^X C^Z - C^X B^Y}{B^X C^X}
$$
where \( p^X = B^X C^Z - C^X B^Y \) and \( q^X = B^X C^X \). We note that the denominator of Equation (41) is a perfect power of \( X \) and thus we know that the numerator must also be a perfect power of \( X \) for the \( X^{th} \) root of their ratio to be rational. Hence the ratio of the perfect power of the numerator and the perfect power of the denominator, even after simplifying, must thus be rational.

Suppose \( \gcd(A, B, C) = k \) where integer \( k \geq 2 \). Thus \( A = ak \), \( B = bk \), and \( C = ck \) for pairwise coprime integers \( a, b, \) and \( c \). We can express Equation (41) with the common term, namely

\[
(42a) \quad \frac{p^X}{q^X} = \frac{(kb)^X (kc)^Z - (kc)^X (kb)^Y}{(kb)^X (kc)^X}
\]

\[
(42b) \quad \frac{p^X}{q^X} = \frac{k^X + z b^X c^Z - k^X + y c^X b^Y}{k^{2X} b^X c^X}
\]

\[
(42c) \quad \frac{p^X}{q^X} = \frac{k^Z - z x b^X c^Z - k^Y - x c^X b^Y}{b^X c^X}
\]

\[
(42d) \quad \frac{p^X}{q^X} = \frac{k^{\min(Z-X,Y-X)} (z b^X c^Z - k^Y - z x c^X b^Y)}{b^X c^X}
\]

Regardless of the parity of \( b, c, \) or \( k \) per Equations (42c) and (42d), \( p^X \) must be even as it is the difference of two odd numbers or the difference of two even numbers. Furthermore, since \( p^X \) by definition is a perfect power of \( X \) and given that it is even, then \( p^X \) must be defined as \( 2^{iX} k^{\min(Z-X,Y-X)} (f_1 f_2 \cdots f_n) j^X \) for positive integers \( i, j, f_1, f_2, \ldots, f_n \) where \( 2^{iX} \) is the perfect power of the even component of \( p^X \), \( k^{\min(Z-X,Y-X)} \) is the common factor based on \( \gcd(A, B, C) \), and \( (f_1 f_2 \cdots f_n) j^X \) is the perfect power of the remaining \( n_f \) prime factors of \( p^X \) where \( f_1, f_2, \ldots, f_n \) are the remaining prime factors of \( p^X \). Hence

\[
(43) \quad \frac{p^X}{q} = \frac{2^{iX} k^{\min(Z-X,Y-X)} (f_1 f_2 \cdots f_n) j^X}{bc} = \frac{k^Z - z x b^X c^Z - k^Y - x c^X b^Y}{bc}
\]

Both \( b \) and \( c \) can also be expressed as a function of their prime factors, thus

\[
(44) \quad \frac{p^X}{q} = \frac{2^{iX} k^{\min(Z-X,Y-X)} (f_1 f_2 \cdots f_n) j^X}{b_1 b_2 \cdots b_n c_1 c_2 \cdots c_{n_c}}
\]

where \( b_1, b_2, \ldots, b_n \) and \( c_1, c_2, \ldots, c_{n_c} \) are prime factors of \( b \) and \( c \) respectively. Based on the right side of Equation (43), the entire denominator \( bc \) is fully subsumed by the numerator, and thus every one of the prime factors in the denominator equals one of the prime factors in the numerator. Thus after dividing, one or more of the exponents in the numerator reduces thereby canceling the entire denominator accordingly. For illustration, suppose \( b_1 = b_2 = 2, b_n = f_1, c_1 = c_2 = f_2 \) and \( c_{n_c} = f_{n_f} \). As such, Equation (44) simplifies to

\[
(45) \quad \frac{p^X}{q} = \frac{2^{iX-4} k^{\min(Z-X,Y-X)} (f_1 f_2 \cdots f_n) j^X}{1^{X-2} f_1^{X-4} \cdots f_{n_f}^{X-2}}
\]

which has terms with exponents that are not multiples of \( X \) and are thus not perfect powers of \( X \). If \( k = 1 \), then the \( X^{th} \) root is irrational which contradicts the assumption that \( \alpha = \frac{p}{q} \) is rational. However if \( k > 1 \) such that it is a composite of the factors that are not individually perfect powers of \( X \), then the resulting expression is a perfect power of \( X \). Hence when \( k = 1 \), then per Theorem 2.11 \( \alpha = \sqrt[2]{C^{Z-X} - B^{Y-X}} \) is irrational. However if \( \alpha = \sqrt[2]{C^{Z-X} - B^{Y-X}} \) is rational, then \( k \neq 1 \) and thus \( \gcd(A, B, C) \neq 1 \).

Suppose \( \beta = \sqrt[2]{C^{Z-2X} - B^{Y-2X}} \) is rational and \( A, B, C, X, Y, \) and \( Z \) are integers that satisfy the conjecture. Then we can express \( \beta \) as

\[
(46a) \quad \beta X = C^{Z-2X} - B^{Y-2X}
\]

\[
(46b) \quad \beta X = \frac{C^Z}{C^2 X} - \frac{B^Y}{B^2 X}
\]

\[
(46c) \quad \beta X = \frac{B^{2X} C^Z - C^{2X} B^Y}{B^{2X} C^{2X}}
\]

Since \( \beta \) is rational, then \( \beta \) can be expressed as the ratio of integers \( p \) and \( q \) such that

\[
(47) \quad \frac{p^X}{q^X} = \frac{B^{2X} C^Z - C^{2X} B^Y}{B^{2X} C^{2X}}
\]
where \( p^X = B^{2X}C^Z - C^{2X}B^Y \) and \( q^X = B^{2X}C^{2X} \). We note that the denominator of Equation (47) is a perfect power of \( X \) and thus we know that the numerator must also be a perfect power of \( X \) for the \( X^{th} \) root of their ratio to be rational. Hence the ratio of the perfect power of the numerator and the perfect power of the denominator, even after simplifying, must thus be rational.

Suppose \( \gcd(A, B, C) = k \) where integer \( k \geq 2 \). Thus \( A = ak \), \( B = bk \), and \( C = ck \) for pairwise coprime integers \( a, b, \) and \( c \). We can express Equation (47) with the common term, namely

\[
\frac{p^X}{q^X} = \frac{(kb)^2X(kc)^Z - (kc)^2X(kb)^Y}{(kb)^2X(kc)^2X}
\]

\[
\frac{p^X}{q^X} = \frac{k^2X + ZbXCZ - k2X + YcXbY}{k^{4X}b^XcX}
\]

\[
\frac{p^X}{q^X} = \frac{kZ - 2XbXCZ - kY - 2XcXbY}{b^YcX}
\]

\[
\frac{p^X}{q^X} = \frac{k^{\min(Z-2X,Y-2X)bXCZ} - k^{Y-min(Z-2X,Y-2X)cXbY}}{b^YcX}
\]

Regardless of the parity of \( b, c, \) or \( k \) per Equations (48c) and (48d), \( p^X \) must be even as it is the difference of two odd numbers or the difference of two even numbers. Furthermore, since \( p^X \) by definition is a perfect power of \( X \) and given that it is even, then \( p^X \) must be defined as \( 2^{\alpha}k^{\min(Z-2X,Y-2X)}(f_1 f_2 \cdots f_n)^X \) for positive integers \( i, j, f_1, f_2, \ldots, f_n \) where \( 2^{\alpha}X \) is the perfect power of the even component of \( p^X \), \( k^{\min(Z-2X,Y-2X)} \) is the common factor based on \( \gcd(A, B, C) \), and \( (f_1 f_2 \cdots f_n)^X \) is the perfect power of the remaining \( n_f \) prime factors of \( p^X \) where \( f_1, f_2, \ldots, f_n \) are the remaining prime factors of \( p^X \).

\[
\frac{p^X}{q} = 2^{\alpha}k^{\min(Z-2X,Y-2X)}(f_1 f_2 \cdots f_n)^X
\]

Both \( b \) and \( c \) can also be expressed as a function of their prime factors, thus

\[
\frac{p^X}{q} = \frac{2^{\alpha}k^{\min(Z-2X,Y-2X)}(f_1 f_2 \cdots f_n)^X}{b1b2\cdots b_{n_i}c1c2\cdots c_{n_c}}
\]

where \( b_1, b_2, \ldots, b_{n_i} \) and \( c_1, c_2, \ldots, c_{n_c} \) are prime factors of \( b \) and \( c \) respectively. Based on the right side of Equation (49), the entire denominator \( bc \) is fully subsumed by the numerator, and thus every one of the prime factors in the denominator equals one of the prime factors in the numerator. Thus after dividing, one or more of the exponents in the numerator reduces thereby canceling the entire denominator accordingly. For illustration, suppose \( b_1 = b_2 = 2, b_{n_i} = f_1, c_1 = c_2 = f_2 \) and \( c_{n_c} = f_{n_f} \). As such, Equation (50) simplifies to

\[
p^X = 2^{\alpha-4}k^{\min(Z-2X,Y-2X)}(f_1^{X-2} f_2^{X-4} \cdots f_{n_f}^{X-2})
\]

which has terms with exponents that are not multiples of \( X \) and are thus not perfect powers of \( X \). If \( k = 1 \), then the \( X^{th} \) root is irrational which contradicts the assumption that \( \alpha = \frac{p}{q} \) is rational. However if \( k > 1 \) such that it is a composite of the factors that are not individually perfect powers of \( X \), then the resulting expression is a perfect power of \( X \). Hence when \( k = 1 \), then per Theorem 2.11 \( \beta = \sqrt[3]{C^{Z-2X} - B^{-2X}} \) is irrational. However if \( \beta = \sqrt[3]{C^{Z-2X} - B^{-2X}} \) is rational, then \( k \neq 1 \) and thus \( \gcd(A, B, C) \neq 1 \).

Thus if either or both \( \sqrt[3]{C^{Z-2X} - B^{-Y}} \) or \( \sqrt[3]{C^{Z-2X} - B^{-2X}} \) are rational, then \( \gcd(A, B, C) > 1 \). \( \square \)

The values of \( \alpha = \sqrt{C^{Z-X} - B^{-Y}} \) and \( \beta = \sqrt{C^{Z-2X} - B^{-2X}} \) have critical properties as related to coprimality and other relationships with integer solutions that satisfy the conjecture. We know from Theorem 2.11 that if \( \gcd(A, B, C) = 1 \), then \( \alpha = \sqrt{C^{Z-X} - B^{-Y}} \) and \( \beta = \sqrt{C^{Z-2X} - B^{-2X}} \) are both irrational. Even though all feasible values of \( \alpha \) and \( \beta \) map to this pair given Theorem 2.9 we can still consider other feasible values of \( \alpha \) and \( \beta \) as related to coprimality.

**Theorem 2.13.** Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if \( \gcd(A, B, C) = 1 \), then any value of \( \alpha \) or \( \beta \) that satisfies \( C^Z - B^Y = [(C + B)\alpha - CB\beta]^X \) must be irrational or indeterminate.
Proof. Given \( \sqrt[3]{C^2 - B^2} = (C + B) \alpha - CB \beta \) per Theorem 2.8. If we suppose \( \sqrt[3]{C^2 - B^2} \) is irrational, then \((C + B) \alpha - CB \beta \) is irrational. With C and B integer, if \( \alpha \) and \( \beta \) were rational, then \((C + B) \alpha - CB \beta \) is composed solely of rational terms and thus must be rational, which contradicts the assumption of irrationality. Hence \( \alpha \) or \( \beta \) must be irrational. Further, per Theorem 2.11, \( \alpha \) and \( \beta \) are irrational if \( \sqrt[3]{C^2 - B^2} \) is irrational and \( \gcd(A, B, C) = 1 \). Thus here too \( \alpha \) or \( \beta \) must be irrational.

If we suppose instead \( \sqrt[3]{C^2 - B^2} \) is rational, then \((C + B) \alpha - CB \beta \) is rational. Given \( \gcd(A, B, C) = 1 \), per Theorem 2.11, we know both \( \alpha = \sqrt[3]{C^{2x} - B^{2y} - x} \) and \( \beta = \sqrt[3]{C^{2x} - B^{2y} - 2x} \) are irrational.

Suppose instead \( \alpha \) is to be defined as any real other than \( \sqrt[3]{C^2 - B^2} \). As such, per Theorem 2.8, \( \beta \) is derived by \( \beta = \frac{\sqrt[3]{C^2 - B^2} - (C + B) \alpha}{CB} \). Hence substituting into Equation (24) gives us

\[
\begin{align*}
(52a) & \quad \sqrt[3]{C^2 - B^2} = (C + B) \alpha - CB \beta \\
(52b) & \quad \sqrt[3]{C^2 - B^2} = (C + B) \alpha - CB \frac{\sqrt[3]{C^2 - B^2} - (C + B) \alpha}{CB} \\
(52c) & \quad \sqrt[3]{C^2 - B^2} = (C + B) \alpha + \sqrt[3]{C^2 - B^2} - (C + B) \alpha \\
(52d) & \quad \sqrt[3]{C^2 - B^2} = \sqrt[3]{C^2 - B^2}
\end{align*}
\]

Regardless of the value selected for \( \alpha \), both \( \alpha \) and \( \beta \) fall out and thus both are indeterminate when \( \gcd(A, B, C) = 1 \).

Suppose instead \( \beta \) is to be defined as any real other than \( \sqrt[3]{C^{2x} - B^{2y} - 2x} \). As such, per Theorem 2.8, \( \alpha \) is derived by \( \alpha = \frac{\sqrt[3]{C^2 - B^2} + CB \beta}{C + B} \). Hence substituting into Equation (24) gives us

\[
\begin{align*}
(53a) & \quad \sqrt[3]{C^2 - B^2} = (C + B) \alpha - CB \beta \\
(53b) & \quad \sqrt[3]{C^2 - B^2} = (C + B) \sqrt[3]{C^2 - B^2} + CB \beta - CB \beta \\
(53c) & \quad \sqrt[3]{C^2 - B^2} = \sqrt[3]{C^2 - B^2} + CB \beta - CB \beta \\
(53d) & \quad \sqrt[3]{C^2 - B^2} = \sqrt[3]{C^2 - B^2}
\end{align*}
\]

Regardless of the value selected for \( \beta \), both \( \alpha \) and \( \beta \) fall out and thus both are indeterminate when \( \gcd(A, B, C) = 1 \).

Hence if \( \gcd(A, B, C) = 1 \), then both of the terms \( \alpha = \sqrt[3]{C^{2x} - B^{2y} - x} \) and \( \beta = \sqrt[3]{C^{2x} - B^{2y} - 2x} \) must be irrational, while any other random value of \( \alpha \) or \( \beta \) that satisfies \( C^2 - B^2 = [(C + B) \alpha - CB \beta] \) must be irrational or indeterminate.

\[\square\]

2.5. Impossibility of the Terms. Having established that pairwise coprimality is a definite byproduct when \( \gcd(A, B, C) = 1 \) (Theorems 2.2 to 2.4), there is a unique reparametrization of the bases of \( A^X + B^Y = C^Z \) whose rationality is tied to coprimality of terms (Theorems 2.9 to 2.12), and that a line through the origin with an irrational slope does not go through any non-trivial lattice points (Theorem 2.1), we now delve into proving the conjecture under two mutually exclusive conditions:

1. geometric implications when \( \gcd(A, B, C) = 1 \)
2. geometric implications when \( \gcd(A, B, C) > 1 \)

These steps lead to a critical contradiction which demonstrates the impossibility of the existence of counter-examples due to fundamental features of the conjecture.

As implied by Catalan’s conjecture and proven by Mihăilescu [27], no integer solutions exist when \( A, B, \) or \( C \) equals 1, regardless of coprimality. Hence we consider the situation in which \( A, B, C \geq 2 \). Given the configuration of the conjecture, \( A, B, C, X, Y, \) and \( Z \) are positive integers, and thus \( A^X, B^Y, \) and \( C^Z \) are also integers. A set of values that satisfy the conjecture can be plotted on a Cartesian coordinate grid with
axes $A^X$, $B^Y$, and $C^Z$. See Figure 2. Based on Equation (1) the line passing through the origin and the point $(A^X, B^Y, A^X + B^Y)$ can be expressed based on the angles in relation to the axes (see Figure 3), namely

\[
\theta_{CZBY} = \tan^{-1} \frac{A^X + B^Y}{B^Y} \tag{54a}
\]

\[
\theta_{CZAX} = \tan^{-1} \frac{A^X + B^Y}{A^X} \tag{54b}
\]

\[
\theta_{BYAX} = \tan^{-1} \frac{B^Y}{A^X} \tag{54c}
\]

where $\theta_{CZBY}$ is the angle subtended between the $B^Y$ axis to the line through the origin and the given point in the $C^Z \times B^Y$ plane, $\theta_{CZAX}$ is the angle subtended between the $A^X$ axis to the line through the origin to the given point in the $C^Z \times A^X$ plane, and $\theta_{BYAX}$ is the angle subtended between the $B^Y$ axis to the line through the origin to the given point in the $B^Y \times A^X$ plane.

The line subtended in each plane based on the origin and the given point $(A^X, B^Y, A^X + B^Y)$ has slopes that by definition are identical to the arguments in the corresponding tangent functions in Equations (54a) to (54c). In each case, the numerator and denominator are integers, and thus the corresponding ratios (and therefore slopes) are rational. Given that this line passes through the origin and has a rational slope in all three planes, then we conclude the infinitely long line passes through infinitely many lattice points, namely at specific integer multiples of the slopes.

Based on the conjecture the given lattice point $(A^X, B^Y, A^X + B^Y)$ relative to axes $A^X$, $B^Y$, and $C^Z$ corresponds to a lattice point $(A, B, \sqrt[2]{A^X + B^Y})$ in a scatter plot based on axes $A$, $B$, and $C$. See Figure 4. The conjecture states there is no integer solution to $A^X + B^Y = C^Z$ that simultaneously satisfies all the conditions if $\gcd(A, B, C) = 1$. Thus from a geometric perspective, this means that if $\gcd(A, B, C) = 1$, then the line in the scatter plot based on axes $A$, $B$, and $C$ in Figure 4 could never go through a non-trivial lattice point since $A$, $B$, and $C$ could not all be integer simultaneously. Conversely, if the corresponding line in the scatter plot based on axes $A$, $B$, and $C$ in Figure 4 does go through a non-trivial lattice point, then based on the conjecture we know $\gcd(A, B, C) > 1$. Hence to validate the conjecture, we need to test the relationship between $A^X + B^Y = C^Z$, the lattice points in both graphs, and the slopes subtended by the origin and these lattice points.

**Theorem 2.14.** Given positive integers $A$, $B$, $C$, $X$, $Y$, $Z$, where $X, Y, Z \geq 3$ and $A^X + B^Y = C^Z$, if $\gcd(A, B, C) = 1$, then there is no integer solution.
Figure 5. Angles between the axes and the line segment subtended by the origin and a single point A, B, and C from Figure 1.

Proof. The line through the origin and point \((A, B, \sqrt{A^X + B^Y})\) can be expressed based on the angles in relation to the axes. See Figure 5. These angles are

\[
\theta_{CB} = \tan^{-1} \frac{\sqrt{A^X + B^Y}}{B} \quad \tag{55a}
\]

\[
\theta_{CA} = \tan^{-1} \frac{\sqrt{A^X + B^Y}}{A} \quad \tag{55b}
\]

\[
\theta_{BA} = \tan^{-1} \frac{B}{A} \quad \tag{55c}
\]

where \(\theta_{CB}\) is the angle subtended between the \(B\) axis to the line through the origin and the given point in the \(C \times B\) plane, \(\theta_{CA}\) is the angle subtended between the \(A\) axis to the line through the origin to the given point in the \(C \times A\) plane, and \(\theta_{BA}\) is the angle subtended between the \(B\) axis to the line through the origin to the given point in the \(B \times A\) plane. See Figure 5.

The line that corresponds to \(\theta_{CB}\) in Equation (55a) has slope \(m_{CB} = \frac{\sqrt{A^X + B^Y}}{B}\) in the \(C \times B\) plane, and the line that corresponds to \(\theta_{CA}\) in Equation (55b) has slope \(m_{CA} = \frac{\sqrt{A^X + B^Y}}{A}\) in the \(C \times A\) plane. These two slopes are different than the slope of the line that corresponds to \(\theta_{BA}\) in Equation (55c) which is \(m_{BA} = \frac{B}{A}\) in the \(B \times A\) plane since this latter slope is merely the ratio of two integers whereas the numerator of the first two are \(\sqrt{A^X + B^Y}\) which may not be rational.

Building from Equations (55a) and (55b), let \(m_{CB}\) and \(m_{CA}\) be the slopes of the lines through the origin and the given point in the \(C \times B\) and \(C \times A\) planes, respectively. Thus we have

\[
m_{CB} = \frac{\sqrt{A^X + B^Y}}{B} \quad \tag{56a}
\]

\[
m_{CA} = \frac{\sqrt{A^X + B^Y}}{A} \quad \tag{56b}
\]

\[
m_{CB} = \left(\frac{A^X}{B^2} + \frac{B^Y}{A^2} - z\right)^{\frac{1}{2}} \quad \tag{56c}
\]

\[
m_{CA} = \left(\frac{A^X - z + \frac{B^Y}{A^2}}{A^2}\right)^{\frac{1}{2}}
\]

Per Theorem 2.1 if a line through the origin has an irrational slope, then that line does not pass through any non-trivial lattice points. Relative to the conjecture, a line in 3 dimensions that passes through the origin also passes through a point equal to the integer bases which satisfy the terms of the conjecture. Since the solution that satisfies the conjecture is integer and must be a lattice point, then the corresponding lines must have rational slopes. Hence if there exist integer solutions, then slopes \(m_{CB}\) and \(m_{CA}\) are rational. If the slopes are irrational, then there is no integer solution. We must now consider three mutually exclusive scenarios in relation to slopes \(m_{CB}\) and \(m_{CA}\).
Scenario 1 of 2: \( A^X \neq B^Y \). Suppose \( A^X \neq B^Y \). Dividing both terms by \( B^Z \) gives us \( \frac{A^X}{B^Z} \neq \frac{B^Y - Z}{A^Z} \).

Likewise, dividing both terms of \( A^X \neq B^Y \) by \( A^Z \) gives us \( \frac{A^{X-Z}}{A^Z} \neq \frac{B^Y - A^Z}{A^Z} \). Using this relationship, \( m_{CB} \) and \( m_{CA} \) from Equation (56c) become

\[
\begin{align*}
\text{(57a)} & \quad m_{CB} = \left( \frac{A^X}{B^Z} \right)^{\frac{1}{Z}} \left( 1 + \frac{B^Y - Z}{A^X} \right)^{\frac{1}{Z}} \\
\text{(57b)} & \quad m_{CA} = \left( \frac{A^X}{A^{X-Z}} \right)^{\frac{1}{Z}} \left( 1 + \frac{B^Y}{A^{X-Z}} \right)^{\frac{1}{Z}} 
\end{align*}
\]

Both \( m_{CB} \) and \( m_{CA} \) must be rational for there to exist an integer solution that satisfies the conjecture. Based on Equation (57b), there are two cases that can ensure both \( m_{CB} \) and \( m_{CA} \) are rational:

1. The term \( \left( 1 + \frac{B^Y}{A^X} \right)^{\frac{1}{Z}} \) from Equation (57b) is rational, therefore both terms \( \frac{A^X}{B^Z} \) and \( A^\frac{X}{Z} - 1 \) must be rational so their respective products are rational.

2. The term \( \left( 1 + \frac{B^Y}{A^X} \right)^{\frac{1}{Z}} \) from Equation (57b) is irrational, therefore both terms \( \frac{A^X}{B^Z} \) and \( A^\frac{X}{Z} - 1 \) are irrational. However the irrationality of these two terms are canceled with the irrationality of the denominator of \( \left( 1 + \frac{B^Y}{A^X} \right)^{\frac{1}{Z}} \) so their respective products are rational.

Case 1 of 2: the terms of \( m_{CB} \) and \( m_{CA} \) are rational

Starting with the first case, assume \( \left( 1 + \frac{B^Y}{A^X} \right)^{\frac{1}{Z}} \) in Equation (57b) is rational and thus \( \frac{A^X}{B^Z} \) and \( A^\frac{X}{Z} - 1 \) are rational. Since \( A \) is reduced (not a perfect power), \( \text{gcd}(A, B) = 1 \), and \( A \) is raised to exponent \( \frac{X}{Z} \) and \( X \) \( \frac{X}{Z} - 1 \), respectively, these terms are rational only if \( X \) is an integer multiple of \( Z \). However, per Definition 2.1 on page 6, \( X \) is not an integer multiple of \( Z \), therefore the requirements to ensure \( m_{CB} \) and \( m_{CA} \) are rational cannot be met.

Case 2 of 2: the terms of \( m_{CB} \) and \( m_{CA} \) are irrational

Considering the second case, assume \( \left( 1 + \frac{B^Y}{A^X} \right)^{\frac{1}{Z}} \) in Equation (57b) is irrational, and thus \( \frac{A^X}{B^Z} \) and \( A^\frac{X}{Z} - 1 \) must also be irrational such that they cancel the irrationality with multiplication. Since slopes \( m_{CB} \) and \( m_{CA} \) in Equation (57b) are rational with irrational terms, both Equations (57a) and (57b) must be rational. We can re-express Equation (56a) equivalently as

\[
\begin{align*}
\text{(58)} & \quad m_{CB} = \frac{C}{\sqrt{C^Z - A^X}} \\
m_{CA} = \frac{C}{\sqrt{C^Z - B^Y}}
\end{align*}
\]

in which the denominators must be rational. Per Theorem 2.8 the denominators in Equation (58) can be reparameterized and thus Equation (58) becomes

\[
\begin{align*}
\text{(59)} & \quad m_{CB} = \frac{C}{(C + A)M_4 \alpha_{CA} - CAM_4 \beta_{CA}} \\
m_{CA} = \frac{C}{(C + B)M_4 \alpha_{CB} - CBM_4 \beta_{CB}}
\end{align*}
\]

where \( \alpha_{CA}, \beta_{CA}, \alpha_{CB}, \text{ and } \beta_{CB} \) are positive rational numbers and \( M_4 \) and \( M_4 \) are positive scalars. Per Theorems 2.8 and 2.9 \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \text{ and } \beta_{CB} \) can be defined as

\[
\begin{align*}
\text{(60a)} & \quad \alpha_{CA} = \sqrt{C^Z - A^X} \\
\text{(60b)} & \quad \beta_{CA} = \sqrt{C^Z - A^Y - 2A^X}
\end{align*}
\]

Per Theorem 2.11 \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \text{ and } \beta_{CB} \) as defined in Equations (60a) and (60b) are irrational when \( \text{gcd}(A, B, C) = 1 \). Thus \( m_{CB} \) and \( m_{CA} \) in Equation (59) are irrational.
Therefore as consequences of both cases 1 and 2, slopes \( m_{CB} \) and \( m_{CA} \) must be irrational when \( A^X \neq B^Y \) and \( \gcd(A, B, C) = 1 \).

As a side note, an alternate way to define \( \alpha_{CA}, \beta_{CA}, \) and \( \beta_{CB} \) is to select any real value for \( \alpha_{CA} \) and \( \alpha_{CB} \) and then derive \( \beta_{CA} \) and \( \beta_{CB} \) or select any real value for \( \beta_{CA} \) and \( \beta_{CB} \) and then derive \( \alpha_{CA} \) and \( \alpha_{CB} \), per Theorem 2.9. However per Theorem 2.13 when \( \gcd(A, B, C) = 1 \), the derived values of \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \) and \( \beta_{CB} \) are either irrational or indeterminate. Thus \( m_{CB} \) and \( m_{CA} \) in Equation (59) are irrational or indeterminate. If the slopes \( m_{CB} \) and \( m_{CA} \) were rational, then \( \alpha \) and \( \beta \) would both be determinate, thus a contradiction. Hence here too the requirements to ensure \( m_{CB} \) and \( m_{CA} \) are rational cannot be met when \( \gcd(A, B, C) = 1 \).

**Scenario 2 of 2: \( A^X = B^Y \).** Suppose \( A^X = B^Y \). Given the bases are reduced, we conclude that \( A = B \). Per this theorem, \( \gcd(A, B) = 1 \) and thus \( A \neq B \), hence this scenario is impossible.

Since the scenarios are mutually exclusive and exhaustive given \( \gcd(A, B) = 1 \), and given that in each scenario, the slopes \( m_{CB} \) and \( m_{CA} \) are irrational, then the line that goes through the origin and through the integer solution must also have irrational slopes. However, as already proven in Theorem 2.1, lines through the origin with irrational slopes cannot go through any non-trivial lattice points. Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if \( \gcd(A, B, C) = 1 \), then slopes \( m_{CB} \) and \( m_{CA} \) are irrational and thus there is no integer solution.

We know that each grid point \((A, B, C)\) subtends a line through the origin and that point, whereby that point is supposed to be an integer solution that satisfies the conjecture. We also know that each grid point corresponds to a set of slopes. Further, we know from Theorem 2.14 that a line through the origin with an irrational slope does not pass through any non-trivial lattice points. Since both \( m_{CB} \) and \( m_{CA} \) are irrational when \( \gcd(A, B, C) = 1 \), then their corresponding lines fail to go through any non-trivial lattice points, and thus these slopes mean there are no corresponding integer solutions for \( A, B, \) and \( C \). Hence there is no integer solution satisfying the conjecture when \( \gcd(A, B, C) = 1 \).

### 2.6. Requirement for Possibility of the Terms.

Having established that the slopes of a line through the origin and the lattice point \((A, B, C)\) are irrational when \( \gcd(A, B, C) = 1 \) translates to no integer solutions to the conjecture and the non-existence of a non-trivial lattice point. We now consider the reverse; if there is an integer solution that satisfies the conjecture, then \( \gcd(A, B, C) \) must be greater than 1 translates to the existence of a non-trivial lattice point through which the line passes.

**Theorem 2.15.** Given positive integers \( A, B, C, X, Y, Z \), where \( X, Y, Z \geq 3 \) and \( A^X + B^Y = C^Z \), if there are integer solutions, then \( \gcd(A, B, C) > 1 \).

**Proof.** Given the configuration of the conjecture, \( A, B, C, X, Y, \) and \( Z \) are positive integers, and thus \( A^X, B^Y, \) and \( C^Z \) are also integers. A set of values that satisfy the conjecture correspond to a point on a scatter plot with axes \( A, B, \) and \( C \). See Figure 1. Based on Equation (1) the line passing through the origin and the point \((A, B, \sqrt[A]{A^X+B^Y})\) can be expressed based on its slopes in the three planes (see Equations (55a), (55b) and (55c), and Figure 5), namely

\[
\begin{align*}
(61a) & \quad m_{CB} = \frac{\sqrt[A]{A^X+B^Y}}{B} = \frac{C}{B} \quad & \quad m_{CA} = \frac{\sqrt[A]{A^X+B^Y}}{A} = \frac{C}{A} \\
(61b) & \quad m_{CB} = \sqrt[B]{A^X+B^Y} = \frac{C}{B} \quad & \quad m_{CA} = \sqrt[A]{A^X+B^Y} = \frac{C}{A} \\
\end{align*}
\]

where \( m_{CB} \) and \( m_{CA} \) are the slopes of the lines through the origin and the given point in the \( C \times B \) and \( C \times A \) planes, respectively. We can likewise define \( m_{BA} \) as the slope of the lines through the origin and the given point in the \( A \times B \) plane based on Equation (55c), namely

\[
m_{BA} = \frac{B}{A}
\]
We know from Theorem 2.14 that $m_{CB}$ and $m_{CA}$ cannot be rational if $\gcd(A, B, C) = 1$. Suppose $\gcd(A, B, C) = k$ where $k \geq 2$. Thus with integer $k$ common to $A$, $B$, and $C$, we can express Equation (1) with the common term, namely

$$a^x k^x + b^y k^y = c^z k^z$$

where $a$, $b$, and $c$ are positive coprime integer factors of $A$, $B$, and $C$ respectively, and where $A = ak$, $B = bk$, $C = ck$, and where $A^x = a^x k^x$, $B^y = b^y k^y$, and $C^z = c^z k^z$. We can thus express the slopes from Equations (61b) and (62) with the common term, namely

$$m_{CB} = \sqrt[\frac{x}{2}]{\frac{a^x k^x + b^y k^y}{bk^2}}$$

$$m_{CA} = \sqrt[\frac{y}{2}]{\frac{a^x k^x + b^y k^y}{ak^2}}$$

$$m_{BA} = \frac{bk}{ak} = b/a$$

We observe that the common term $k$ cancels from $m_{BA}$ in Equation (64b) and thus $m_{BA}$ is rational regardless of the common term. We can simplify Equation (64a) as

$$m_{CB} = \left(\frac{(ak)^x}{(bk)^z} + (bk)^{y-z}\right)^{\frac{1}{2}}$$

$$m_{CA} = \left((ak)^x z - (bk)^y\right)^{\frac{1}{2}}$$

Before applying Newton’s generalized binomial expansion to slopes $m_{CB}$ and $m_{CA}$ in Equation (65), we must first consider two mutually exclusive scenarios:

**Scenario 1 of 2: $A^X \neq B^Y$.** Suppose $A^X \neq B^Y$. As such $\frac{A^X}{B^Z} \neq B^{-YZ}$ and $A^{X-Z} \neq B^Y$ and thus $\frac{(ak)^X}{(bk)^Z} \neq (bk)^{y-z}$ and $(ak)^{x-z} \neq (bk)^y$. Therefore we can re-express $m_{CB}$ and $m_{CA}$ in Equation (65) as

$$m_{CB} = \left(\frac{(ak)^x}{(bk)^z}\right)^{\frac{1}{2}} \left[1 + \frac{(bk)^{y-z}}{(ak)^{x-z}}\right]^{\frac{1}{2}}$$

$$m_{CA} = \left((ak)^x z - (bk)^y\right)^{\frac{1}{2}}$$

Given there are integer solutions that satisfy the conjecture, then $m_{CB}$ and $m_{CA}$ must be rational. Based on Equation (66b), there are two cases that can ensure both $m_{CB}$ and $m_{CA}$ are rational:

1. The term $\left(1 + \frac{(bk)^y}{(ak)^z}\right)^{\frac{1}{2}}$ from Equation (66b) is rational, therefore both terms $\frac{(ak)^{\frac{x}{z}}}{bk}$ and $(ak)^{\frac{x-1}{z}}$ must be rational so their respective products are rational.

2. The term $\left(1 + \frac{(bk)^y}{(ak)^z}\right)^{\frac{1}{2}}$ from Equation (66b) is irrational, therefore both terms $\frac{(ak)^{\frac{x}{z}}}{bk}$ and $(ak)^{\frac{x-1}{z}}$ are irrational. However the irrationality of these two terms are canceled with the irrationality of the denominator of $\left(1 + \frac{(bk)^y}{(ak)^z}\right)^{\frac{1}{2}}$ so their respective products are rational.

**Case 1 of 2: the terms of $m_{CB}$ and $m_{CA}$ are rational**

Starting with the first case, assume $\left(1 + \frac{(bk)^y}{(ak)^z}\right)^{\frac{1}{2}}$ in Equation (66b) is rational and thus $\frac{(ak)^{\frac{x}{z}}}{bk}$ and $(ak)^{\frac{x-1}{z}}$ are rational. Applying the binomial expansion to Equation (66b) gives us

$$m_{CB} = \frac{(ak)^{\frac{x}{z}}}{bk} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \left(\frac{bk}{(ak)^z}\right)^i$$

$$m_{CA} = (ak)^{\frac{x-1}{z}} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \left(\frac{bk}{(ak)^z}\right)^i$$
The binomial coefficient \( \binom{\frac{k}{i}}{j} \) and ratio \( \frac{(bk)^{Y_i}}{(ak)^{X_i}} \) in both formulas in Equation (67) are rational for all \( i \), as are their products, regardless of the value of \( k \). Hence we consider the terms \( \frac{(ak)^{\frac{k}{b}}}{bk} \) and \( (ak)^{\frac{k}{b} - 1} \) in Equation (67). Since integers \( A \) and \( ak \) are reduced (not perfect powers), there are only three possibilities that ensure they are rational:

1. \( a \) is a perfect power of \( Z \), or
2. \( X \) is an integer multiple of \( Z \), or
3. \( k > 1 \) such that \( ak \) is a perfect power of \( Z \).

Suppose \( a \) is a perfect power of \( Z \). Since \( A = ak \) cannot be a perfect power of \( Z \) given that \( A \) is reduced, then \( k \) must be greater than 1 so that when factored from composite \( A \), the remaining factors of \( A \) are a perfect power of \( Z \). Hence \( k > 1 \) and thus \( \gcd(A, B, C) > 1 \).

Suppose instead \( X \) is an integer multiple of \( Z \), hence \( X = iZ \) for some positive integer \( i \). However, per Definition 2.1, on page 8, \( X \) is not an integer multiple of \( Z \). Thus if \( k = 1 \), the term \( (ak)^{\frac{k}{b}} = A^{\frac{k}{b}} = a^{\frac{k}{b}} \) is not a perfect power and thus \( m_{CB} \) and \( m_{CA} \) are irrational, contradicting their given rationality. If instead \( k \) is a multiple of \( a \) such as \( k = a^j \) for some positive integer \( j \), then \( (ak)^{\frac{k}{b}} = a^{\frac{k}{b} + j} \) for which \( X + j \) could be a multiple of \( Z \). Thus \( m_{CB} \) and \( m_{CA} \) are rational only for some values \( k > 1 \) and thus \( \gcd(A, B, C) > 1 \).

**Case 2 of 2: the terms of \( m_{CB} \) and \( m_{CA} \) are irrational**

Considering the second case, assume \( \left( 1 + \frac{(bk)^{Y_i}}{(ak)^{X_i}} \right)^{\frac{k}{b}} \) in Equation (66b) is irrational, and thus \( \frac{(ak)^{\frac{k}{b}}}{bk} \) and \( (ak)^{\frac{k}{b} - 1} \) must also be irrational such that they cancel the irrationality with multiplication. Since slopes \( m_{CB} \) and \( m_{CA} \) in Equation (66b) are rational with irrational terms, both slopes in Equation (64a) must be rational.

We can re-express slopes \( m_{CB} \) and \( m_{CA} \) from Equation (61a) as

\[
\begin{align*}
m_{CB} &= \frac{C}{\sqrt{C^Z - AX^Y}} \\
m_{CA} &= \frac{C}{\sqrt{C^Z - BY^X}}
\end{align*}
\]

in which the denominators must both be rational. Per Theorem 2.8, the denominators in Equation (68) can be reparameterized and thus Equation (68) becomes

\[
\begin{align*}
m_{CB} &= \frac{C}{(C + A)M_1 \alpha_{CA} - CAM_2 \beta_{CA}} \\
m_{CA} &= \frac{C}{(C + B)M_2 \alpha_{CB} - CBM_2 \beta_{CB}}
\end{align*}
\]

where \( \alpha_{CA}, \beta_{CA}, \alpha_{CB}, \beta_{CB} \) are positive rational numbers and \( M_1 \) and \( M_2 \) are positive scalars. Per Theorems 2.8 and 2.9, \( \alpha_{CA}, \beta_{CA}, \alpha_{CB}, \beta_{CB} \) are defined as

\[
\begin{align*}
\alpha_{CA} &= \sqrt[2]{C^Z - AX^Y} \\
\beta_{CB} &= \sqrt[2]{C^Z - Y^X}
\end{align*}
\]

Per Theorem 2.11, \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \beta_{CB} \) as defined in Equations (70a) and (70b) are irrational when \( \gcd(A, B, C) = k = 1 \). However, since \( m_{CB} \) and \( m_{CA} \) in Equations (70a) and (70b) are rational, we need to consider the common factor in the bases.

Per Theorem 2.12, \( \sqrt[2]{C^Z - AX^Y} \) and \( \sqrt[2]{C^Z - AX^Y} \) are rational only if \( \gcd(A, B, C) > 1 \). By extension, \( \sqrt[2]{C^Z - BY^X} \) and \( \sqrt[2]{C^Z - BY^X} \) are also rational only if \( \gcd(A, B, C) > 1 \).

Since slopes \( m_{CB} \) and \( m_{CA} \) in Equation (69) are rational, then their denominators are rational, and thus \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \beta_{CB} \) must be rational. Per the definitions of \( \alpha_{CA}, \alpha_{CB}, \beta_{CA}, \beta_{CB} \) in Equations (70a) and (70b) and given Theorem 2.12 in which these terms can only be rational when \( \gcd(A, B, C) > 1 \), we conclude \( k \) must be greater than 1 and thus \( \gcd(A, B, C) > 1 \).

Therefore as consequences of both cases 1 and 2, for slopes \( m_{CB} \) and \( m_{CA} \) to be rational when \( A^X \neq B^Y \) requires \( \gcd(A, B, C) > 1 \), and thus \( A, B, \) and \( C \) must share a common factor greater than 1.

**Scenario 2 of 2: \( A^X = B^Y \).** If \( A^X = B^Y \), then \( \gcd(A, B) = k > 1 \). Hence by definition, \( k \) is a factor of \( A^X + B^Y \) and of \( C^Z \), and thus \( A, B, \) and \( C \) must share a common factor greater than 1.
Given positive integers $A$, $B$, $C$, $X$, $Y$, $Z$, where $X, Y, Z \geq 3$ and $A^X + B^Y = C^Z$, when there exist integer solutions, two scenarios show common factor $k$ must be greater than 1 to ensure slopes $m_{CB}$ and $m_{CA}$ are rational. We know that each grid point $(A, B, C)$ subtends a line through the origin and that point, whereby that point is supposed to be an integer solution that satisfies the conjecture. We also know that each grid point corresponds to a set of slopes. Further, we know from Theorem 2.1 that a line through the origin with an irrational slope does not pass through any non-trivial lattice points. Since both $m_{CB}$ and $m_{CA}$ are rational only for some common factor $k > 1$, then $\gcd(A, B, C) = k$ is required. We know $m_{BA}$ is always rational but since $m_{CB}$ and $m_{CA}$ can be rational only when $\gcd(A, B, C) > 1$ for only certain common factors, then we know the lines go through non-trivial lattice points, and thus these slopes mean there can be integer solutions for $A$, $B$, and $C$. Hence there can be integer solutions satisfying the conjecture only when $\gcd(A, B, C) > 1$.

\[ \tag{3. Conclusion} \]

Every set of values that satisfy the Tijdeman-Zagier conjecture corresponds to a lattice point on a multi-dimensional Cartesian grid. Together with the origin this point defines a line in multi-dimensional space. This line requires a rational slope in order for it to pass through a non-trivial lattice point. Hence the core of the various proofs contained herein center on the irrationality of the slope based on the coprimality of the terms. Several key steps were required to establish this relationship and then support the proof.

Theorems 2.2 to 2.4 establish that within the relation $A^X + B^Y = C^Z$ if any pair of terms, $A$, $B$, and $C$ is coprime, then all 3 terms must be coprime, and if all 3 terms are coprime, then each pair of terms must likewise be coprime. Likewise, Theorem 2.5 establishes a similarly restrictive relationship between the exponents, namely that exponents $X$ and $Y$ cannot be integer multiples or unit fractions of exponent $Z$ and that $Z$ cannot be an integer multiple of $X$ or $Y$.

Theorems 2.6 and 2.7 establish that the difference of powers can be factored and expanded based on an arbitrary and indeterminate upper limit.

Theorems 2.8 and 2.9 establish that $A^X$ could be parameterized as a linear combination of $C + B$ and $CB$, with two parameters. Theorems 2.10, 2.11, and 2.13 establish that when these parameters are irrational there can be no integer solution satisfying the conjecture and if $\gcd(A, B, C) = 1$, then these parameters must be irrational. Theorem 2.12 establishes that if these two parameters are rational, then $\gcd(A, B, C) > 1$.

The relationships between coprimality of terms and irrationality of the parameters (Theorems 2.8 to 2.13) are critical to the slopes that are core to the remaining theorems. It is shown that the slopes are functions of these parameters and thus the irrationality properties of the parameters translate to irrationality conditions for the slopes.

Theorem 2.14 establishes that a line with an irrational slope that passes through the origin will not pass through any non-trivial lattice points. This simple, subtle theorem is critical to the proof since the link between irrationality of slope and non-integer solutions is key to relating the outcomes to coprimality of terms. The logic of the proof is that integer solutions which satisfy the conjecture can be expressed only with an arbitrary and indeterminate upper limit.

Theorem 2.14 establishes that when $\gcd(A, B, C) = 1$, the slopes are irrational. Thus if the slopes are irrational, then the line that is equivalent to the integer solution does not pass through non-trivial lattice points, hence there is no integer solution. Theorem 2.15 establishes the reverse, namely that the slopes of the corresponding lines can only be rational when $\gcd(A, B, C) > 1$, and that integer solutions satisfying the conjecture fall on the lines with rational slopes.

Any proof of the Tijdeman-Zagier conjecture requires four conditions be satisfied:

- $A$, $B$, $C$, $X$, $Y$, and $Z$ are positive integers.
- $X, Y, Z \geq 3$
- $A^X + B^Y = C^Z$
- $\gcd(A, B, C) = 1$

Since the set of values that satisfy the conjecture is directly a function of rationality of slopes, we have demonstrated the explicit linkage between the coprimality aspect of the conjecture, the integer requirement of the framework, and properties of slopes of lines through the origin. Via contradiction these theorems prove
the four conditions cannot be simultaneously met. Given the fully exhaustive and mutual exclusivity of the theorems, the totality of the conjecture is thus proven.

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