DOUBLE PRECISION COMPUTATION OF THE LOGISTIC MAP DEPENDS ON COMPUTATIONAL MODES OF THE FLOATING-POINT PROCESSING UNIT

MICHIRO YABUKI  
Department of Information Science, Meisei University, 2-1-1 Hodokubo  
Hino-shi, Tokyo 191-8506, Japan  
yabuki@is.meisei-u.ac.jp

TAKASHI TSUCHIYA*  
Department of Information Science, Meisei University, 2-1-1 Hodokubo  
Hino-shi, Tokyo 191-8506, Japan  
tsuchiya@is.meisei-u.ac.jp

Today’s most popular CPU can operate in two different computational modes for double precision computations. This fact is not very widely recognized among scientific computer users. The present paper reports the differences the modes bring about using the most thoroughly studied system in chaos theory, the logistic map. Distinct virtual periods due to finite precision come about depending on the computational modes for the parameter value corresponding to fully developed chaos. For other chaotic regime various virtual periods emerge depending on the computational modes and the mathematical expressions of the map. Differences in the bifurcation diagrams due to the modes and the expressions are surveyed exhaustively. A quantity to measure those differences is defined and calculated.

Keywords: double precision, logistic map, bifurcation diagram, virtual periodicity, Floating-point Processing Unit.

1. Introduction

Most numerical calculations are currently done in double precision(DP), whose accuracy is approximately 16 decimal digits is well known among the scientific computer users. The following fact, however, might not be so well recognized; today’s most popular Central Processing Unit(CPU) by Intel Corporation(as well as that by Advanced Micro Devices, Inc.) has the Floating-point Processing Unit(FPU) which can operate in two different computational modes; one is the DP formalized by IEEE 754 with 64 bits [ANS/IEEE754-1985, 1985] and the other is called the extended precision mode utilizing 80 bits. This difference in computational modes may yield a slight discrepancy in the final results. As is easily expected, the discrepancy is so small that it can be ignored as long as the absolute tolerance is set to be larger than $10^{-16}$ in usual calculations where convergent results are assumed.

However, for chaotic systems in which a small difference in the initial states is extended to the size of the system itself [Ott, 2002], the discrepancy stated above can possibly be manifested macroscopically. In this paper we report that the differences in computational modes actually bring about the differences

*Professor Emeritus
in the final results using the simplest and most frequently studied chaos generator, the logistic map. If we
denote the system variable by \( x \), an adjustable external parameter by \( a \) and count the number of iterations
by \( n \), the logistic map is commonly expressed as

\[
x_{n+1} = ax_n(1 - x_n).
\]  

(1)

It is readily seen that the right-hand-side of Eq. (1) can be expressed in several different forms, all of
which are, of course, mathematically equivalent. We show, in this report, that there are cases the different
expressions of the logistic map in computer programs give different macroscopic outputs depending on the
two computational modes of FPU.

Periodic orbits should not be observed when a system is in chaotic regime, but it is already well known
that the results of numerical calculations of the logistic map (1) in DP which has a finite precision eventually
fall into virtual periodic orbits, though their periods are very long [Keller & Wiese, 2007] [Wang et al.,
2004] [Wagner, 1992]. We here show further that the periodicity of the orbits thus produced depends on the
computational modes of FPU and on the expressions of the logistic map used in the computer program.

The bifurcation diagram is commonly drawn in order to grasp dynamical behaviors of the system
under consideration for a wide range of external parameter values. We show that the differences in the
computational modes of FPU and the expressions in the computer program bring about differences in bi-
furcation diagrams. Furthermore, we propose a measure that can quantitatively distinguish the differences,
and actually calculate the measure.

Although the system treated in this paper as an example is only the logistic map, our most general aim
of this paper is to make the following proposal; in reports of numerical investigations of chaotic systems
explicit mentions of mathematical expression used in the program and the computational mode of FPU
should be standardized in order to make the reports maximally reproducible.

In the next section we explain about the two computational modes of FPU under the condition that
the precision is set at DP. In the third section we present six possible expressions in computer programs
for the logistic map Eq. (1). In section 4 it is reported that periodicity that inevitably emerges as an
artifact when the logistic map in chaotic regime is computed with DP. The point is that the period is
strongly dependent on the computational modes of FPU and mathematical expressions of the map in the
program used. In section 5 we exhaustively discuss the difference between the bifurcation diagrams when
the bifurcation diagrams are computed using different modes of FPU and the map expressions. In the final
section we come back to the proposal stated above.

2. Computational Modes of FPU

IEEE 754-2008 formulates floating-point number for single precision as 32 bits, double precision(DP) as
64 bits and quadratic precision as 128 bits [ANSI/IEEE754-2008, 2008]. DP is most prevalently used for
numerical calculations today. In DP, 1 bit is allocated for the sign, 11 bits for the exponent and 52 bits (53
bits if one includes the so-called hidden bit which is a leading bit of normalized mantissa not actually stored
in the datum) for the mantissa. After each arithmetic operation, rounding is done. The rounding modes are
also formulated by IEEE 754-2008 as follows [ANSI/IEEE754-1985, 1985], namely, round-to-even, round-
toward-zero, round-up and round-down. Since most current FPU’s default round-to-even rounding mode
and most programmers take that mode for granted, we restrict ourselves only to this rounding mode.

Even if one restricts the rounding mode to round-to-even, there are two DP computational modes in
today’s popular FPU’s as mentioned in section 1. In one mode, 53 bits are used for mantissa both for the
left-hand-side and the right-hand-side of each arithmetical expression in the program, while in the other
mode only the left-hand-side value has 53-bit-mantissa length and for the right-hand-side value 64-bit-
mantissa length is utilized. Since our main interest in this paper is in the difference between these two
modes, we devise the following notation to distinguish the mode, namely,

\[
[\alpha : \beta]
\]

(2)

where \( \alpha \) represents the left-hand-side mantissa bit-length and \( \beta \) the right-hand-side mantissa bit-length.
Hence, we say there are \([53 : 53]\) mode and \([53 : 64]\) mode for DP calculations. As far as the present authors
have surveyed, each of the currently popular compilers defaults one of the two modes as:
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As we can easily imagine, there are situations that produce different results depending on the mode we employ; for example (in this paper we use C language to represent a part of a computer program)

```c
double x, y, z;
x = 10.0;
y = 2.718281810;
z = x / (y * y);
printf("%21.16e\n", z);
```

yields 1.3533528507465618e+00 when the mode [53 : 53] is used, and 1.3533528507465620e+00 for the mode [53 : 64] giving rise to the difference $2 \times 10^{-16}$ in decimal digit. For such cases as this example, the difference can easily be ignored by setting the tolerance larger than $10^{-15}$. But things are not that simple for chaotic systems such as the logistic map in chaotic regime because each iterate of the map can be different depending on its mathematical expression, as will be presented in the following sections.

3. Different Expressions for the Logistic Map

The logistic map which is most frequently written in the form given in Eq. (1) can be expressed in computer programs in the following six distinct forms

\[
\begin{align*}
x_{n+1} &= (ax_n)(1.0 - x_n) \quad (L1) \\
x_{n+1} &= a(x_n(1.0 - x_n)) \quad (L2) \\
x_{n+1} &= (a - ax_n)x_n \quad (L3) \\
x_{n+1} &= a(x_n - x_n)x_n \quad (L4) \\
x_{n+1} &= ax_n - (ax_n)x_n \quad (L5) \\
x_{n+1} &= ax_n - a(x_n)x_n \quad (L6)
\end{align*}
\]

all of which are, of course, equivalent mathematically. Here we restrict ourselves to the expressions each of which can be written in one program sentence, in other words, without using an extra variable representing any term in the expressions above.

We are not the first to study expression dependence as well as machine dependence of the computational results of chaotic systems. Colonna calculated iterations up to 80 steps for the forward-discretized (which means using the Euler method of discretization) logistic differential equation, which is well known to be essentially equivalent (conjugate) to the logistic map, on two different IBM machines using five different expressions [Colonna, 1993]. He reported that 10 results are incompatible to one another even at the 80th iteration for the case that corresponds to our results for $a = 4.0$. Our studies here however, is more specific, namely the effects of the computational modes [53 : 53] and [53 : 64], and more thorough, namely six different expressions (L1)~(L6) over millions of iterations if necessary.

Oteo and Ros also compared the expressions essentially equivalent to our (L1) and (L5) in their paper concerning errors when the logistic map is calculated with double precision (DP) [Oteo & J., 2007]. For $a = 4.0$ they reported that the difference in the mathematical expressions brings about the exponential growth of the distance between the two orbits even for the same initial conditions, namely after 50-60 iterations the difference becomes as large as the size of the system itself. Our concern here is again more thorough; we consider possible six different mathematical expressions as well as two different computational modes in DP calculation not only for $a = 4.0$ but also for other values of $a$. 

For some special values of $a$, especially for $a = 4.0$, six expressions above are grouped into only two. As is well known for $a = 4.0$ the logistic map generates so-called fully developed chaos, hence very many numerical calculations are done for this parameter value to show typical results of the chaotic map. The value 4.0 can be realized as an error-free number on any FPU’s, hence multiplication by 4.0 does not produce further errors in mantissa of the result. As the proof is given in the Appendix, the six expressions of the logistic map are grouped as

- Group A: (L1), (L2), (L3)
- Group B: (L4), (L5), (L6)

for $a = 4.0$.

The point we here make is that the value $a = 4.0$ is quite particular as far as numerical results are concerned, since for the values $2.0 < a < 4.0$ this grouping is no longer valid. Thus one must consider six different expressions separately for the region. We show that six different results emerge for six expressions even for the value of $a$ that is smaller than for 4.0 by only 1 bit in the next section.

### 4. Virtual Periodicity

In chaotic regime all the periodic orbits become unstable, hence they should not be observed experimentally or numerically. However, when computed on computers, chaotic orbits inevitably fall into virtual periodic orbits due to restricted finite precision. For the logistic map the virtual periodicity due to double precision (DP) has already been investigated \[ Keller \& Wiese, 2007 \] \[ Wang \textit{et al.}, 2004 \] \[ Wagner, 1992 \]. Keller and Wiese reported virtual periods for $a = 3.90$ and $a = 3.99$, and Wang \textit{et al.} found the period length 5638349 of the most frequently observed period for $a = 4.0$. Wagner confirmed the same period length as that of Wang \textit{et al.}, when he used Weitek, Mips and Sparc machines but found a new length 86058417 as the most frequently observed when calculation was done on VAX machines. All the authors of \[ Keller \& Wiese, 2007 \] \[ Wang \textit{et al.}, 2004 \] \[ Wagner, 1992 \] however did not mention the computational modes or the map expressions in their programs.

Results of our own calculations of virtual periods for $a = 4.0$ are tabulated in Table 1. For $a = 4.0$ only two expressions are needed to be compared as pointed out at the end of the previous section, we used the expression (L1) as the representative of group A and (L5) for group B. One thousand initial conditions ($x_0$) that evenly distributed in $0.000000 \leq x_0 \leq 0.999999$ which simulates the unit interval. Table 1 lists the top 5 virtual-orbit frequencies when 1000 orbits are calculated for possible 4 computational conditions. From our results we can identify the period 5638349 that was reported in \[ Wang \textit{et al.}, 2004 \] and \[ Wagner, 1992 \] had been obtained when computational mode [53:53] and the map expression of group A were used.

| Group A [53:53] | Group A [53:64] | Group B [53:53] | Group B [53:64] |
|-----------------|-----------------|-----------------|-----------------|
| period frequency | period frequency | period frequency | period frequency |
| 5638349         | 678             | (zero)          | 588             | 9458152         | 296             | 33525897         | 745             |
| (zero)          | 173             | 15784521        | 409             | 21739953        | 273             | 3210244          | 144             |
| 14632801        | 89              | 1122211         | 3               | 17503666        | 217             | 17354121         | 56              |
| 2441806         | 25              | (zero)          | 166             | (zero)          | 52              |
| 2625633         | 20              | 2857100         | 47              | 1176817         | 3               |

Period (zero) indicates that the value of the iterates falls into 0.

The results of the frequencies vary slightly depending on the initial conditions.
If the value of $a$ is different from 4.0, six expressions of the logistic map can no longer be grouped into two. We have hence 12 different virtual periods as tabulated in Table 2 for the value of the parameter $a$ whose mantissa is only 1 bit smaller than 4.0, i.e., 3.9999999999999995 in decimal or 0x400fffffffffffff in hexadecimal. This value is the largest possible value smaller than 4.0 that can be expressed in DP. One thousand orbits were also surveyed and the most frequent virtual period is tabulated in Table 2 for each category. Notice that no two entries become identical in Table 2, which means that once the parameter value of $a$ is smaller than 4.0 even by just one bit, the virtual periodic phenomena become quite complicated. Furthermore, no entries in Table 2 become identical to any of those in Table 1.

Table 2. The virtual periods and their frequencies for two computational modes and six expressions ($a = 0x400fffffffffffff$ in hexadecimal).

| expression | [53:53] period | frequency | [53:64] period | frequency |
|------------|----------------|-----------|----------------|-----------|
| L1         | 144666122      | 973       | 27919860       | 943       |
| L2         | 133933248      | 922       | 34082242       | 999       |
| L3         | 22875200       | 805       | 14808396       | 900       |
| L4         | 9726075        | 894       | 46556242       | 776       |
| L5         | 9440450        | 845       | 62494847       | 719       |
| L6         | 26643051       | 966       | 25600176       | 879       |

Although emergence of the virtual periods is quite complicated, they are very long—even the shortest is of the order of $10^6$—and are seldom observed in usual computer experiments. But we found that in more familiar results such as the bifurcation diagram the effects of the six different expressions due to the difference in the computational modes become apparent for $a < 4.0$, as will be presented in the next two sections.

5. Difference in Bifurcation Diagrams

For a continuous range of the external parameter $a$, the bifurcation diagrams are most commonly drawn to grasp dynamical behaviors of the system. In this section, in order to show that the bifurcation diagram is actually dependent on the expressions (L1) ~ (L6) and the computational modes of FPU, namely, [53 : 53] or [53 : 64], we pay attention to the difference between two bifurcation diagrams calculated with different expressions and computational modes. Since there are as many as $6!(2!)^4 = 15$ ways to select two expressions from the expressions (L1) ~ (L6), and each pair must be computed with the computational mode [53 : 53] and [53 : 64], we first use the expression (L1) and (L5) as representatives for our exposition below. These two are the most frequently employed expressions in literatures because when one expresses Eq.(1) directly in program language it becomes (L1), and (L5) is obtained when the expanded form of Eq.(1), i.e., $ax - ax^2$ is directly programed. After the explanation, though, results for all the 30 combinations are presented.

The bifurcation diagram is drawn as follows: a range of interest for the external parameter $a$, specifically $a_{min} \leq a \leq a_{max}$ is divided evenly by a positive integer $M$, for each value of $a$ the iterated values of the map starting from the initial value $x_0$ are plotted. In most studies behaviors of the system after sufficiently many iterations attract attentions, hence the values of the first $I_I$ iterations are not plotted and ensuing $(I_I - I_d)$ iterations are plotted. Here $I_I$ is the total number of iterations for each value of $a$. Although seldom shown in literatures, it is possible to observe transient behaviors of the map if one plots all the iterations by putting $I_d = 0$.

Now, let us begin the exposition of our investigation using (L1) and (L5). It is quite natural to guess that bifurcation diagrams are the same for all the mathematically equivalent expressions (L1) ~ (L6), as long as the parameters $a_{min}$, $a_{max}$, $M$, $x_0$, $I_d$ and $I_I$ are all unaltered. However, we show that the difference
emerges depending on the expression and the mode of computation. Furthermore, we show that it is possible to evaluate the difference quantitatively.

To make our discussion transparent we use the variable $x$ when we run the expression (L1) with $[53 : 53]$ mode, namely,

$$x_{n+1} = (ax_n)(1.0 - x_n),$$

and we use $y$ for the expression (L5) with $[53 : 53]$ mode as

$$y_{n+1} = ay_n - (ay_n)y_n.$$  

We employ the upper-case letters $X$ and $Y$ for (L1) with $[53 : 64]$ mode as

$$X_{n+1} = (aX_n)(1.0 - X_n),$$

and (L5) with $[53 : 64]$ as

$$Y_{n+1} = aY_n - (aY_n)Y_n.$$ 

Our first results are presented in Figs.1 and 2, where the parameter values are fixed as $a_{\text{min}} = 3.0$, $a_{\text{max}} = 4.0$, $M = 1000$, $x_0 = 0.1$, $I_d = 0$ and $I_t = 100$, in other words, the first 100 iterations starting from $x_0 = 0.1$ are plotted for 1000 evenly separated values of $a$ from 3.0 to 4.0. The bifurcation diagram for Eq.(3) is shown in Fig.1(a), Eq.(4) in Fig.1(b), Eq.(5) in Fig.2(a) and Eq.(6) in Fig.2(b). To our naked eyes all the four diagrams appear to be similar, but comparing Fig.1(c) with Fig.2(c) one immediately sees that there really is a difference. In Fig.1(c) the difference

$$z_n = x_n - y_n$$

and in Fig.2(c) the difference

$$Z_n = X_n - Y_n$$

is plotted, respectively. The bifurcation diagram in Fig.1(a) is different from that in Fig.1(b) in chaotic regime, whereas the bifurcation diagrams in Fig.2(a) and Fig.2(b) are the same at least up to $I_t = 100$.

It is understandable that there are no differences in the period-doubling region ($3.0 \leq a \ll 3.57$), but remarkable results are for the chaotic region ($3.57 \lesssim a \leq 4.0$), namely $z_n$ spreads chaotically (Fig.1(c)) whereas $Z_n$ remains 0 (Fig.2(c)). Chaotic behavior of $z_n$ for this region can be understood that for the mode $[53 : 53]$ the orbit calculated with (L1) becomes different from that with (L5) within 100 iterations, however, the fact that $Z_n$ stays at 0 reflects the fact that for the mode $[53 : 64]$ despite the fact that both $X_n$ and $Y_n$ in Eq.(5) dance chaotically the values of them coincide completely within the double precision(DP) up to 100 iterations. Figs.1(c) and 2(c) show clearly that the computational mode does affect chaotic orbits.

\footnote{If one carefully looks at Fig.2(c) some scattered points near $a \lesssim 4.0$ are to be recognized.}
Fig. 2. The bifurcation diagrams for the computational mode [53 : 64] with the same parameter values as those of Fig.1 (a) The bifurcation diagram for the expression (L1). (b) The bifurcation diagram for (L5). (c) The bifurcation diagram for the difference Eq. (8) where \( X_n \) stands for (L1) and \( Y_n \) for (L5).

Up to this point the total number of iterations \( I_t \) is restricted to 100. Now let us increase \( I_t \) as \( I_t = 1000, 3000 \) and 10000, and see what happens to \( z_n \) (Fig.3) and to \( Z_n \) (Fig.4). If we compare Fig.3(b) with Fig.4(b), we see that \( z_n \) is still different from \( Z_n \). However, from Fig.3(c) and Fig.4(c) we cannot recognize any conspicuous differences. The observational results stated above (independence of the initial value \( x_0 \) has been confirmed), that for the mode [53 : 53] the difference due to expressions (L1) and (L5) disappears by \( I_t = 1000 \), but for the mode [53 : 64] there exist certain values of \( a \) for which \( X_n \) and \( Y_n \) take the same value even after 1000 iterations. For \( I_t = 10000 \), however, the bifurcation diagram for \( Z_n \) cannot be distinguished from that for \( z_n \) (Figs.3(c) and 4(c)).

Fig. 3. The bifurcation diagrams for the computational mode [53 : 53] for the difference Eq. (7) for expressions (L1) and (L5). (a) \( I_t = 1000 \). (b) \( I_t = 3000 \). (c) \( I_t = 10000 \). Other parameter values are the same as those used for Fig.1.

Fig. 4. The bifurcation diagrams for the computational mode [53 : 64] for the difference Eq. (8) for expressions (L1) and (L5). (a) \( I_t = 1000 \). (b) \( I_t = 3000 \). (c) \( I_t = 10000 \). Other parameter values are the same as those used for Fig.1.

In the following we try to quantify the difference between the bifurcation diagrams for \( z_n \) and \( Z_n \). Since the difference becomes 0 for the periodic region, we consider only the chaotic region putting \( a_{\text{min}} = 3.5699456718 \) which is the available value \cite{Sprott:2003} for the accumulation point of the period doubling or the Feigenbaum point and \( a_{\text{max}} = 4.0 \).

First we divide \((a_{\text{max}} - a_{\text{min}})\) by \( M \) and for each value of \( a \) we prepare evenly distributed \( P \) initial
values of $x_0$. The total iteration number is $I_t$. If the absolute value $|z_n|$ or $|Z_n|$ exceeds a certain threshold value $\epsilon$, it can grow exponentially afterwards, so the corresponding point $(a, x_0)$ can be judged to give a chaotic orbit. If the total number of points thus determined to produce chaotic orbits up until $I_t$ is denoted by $S(I_t)$, then

$$R(I_t) = \frac{S(I_t)}{MP} \quad (9)$$

represents the ratio of the chaotic orbits up until $I_t$ iterations. In Fig. 5(a) the ratio $R(I_t)$ is plotted against $I_t$ for the mode [53 : 53]. The number of initial points is $P = 1000$ so that the region $0.000999 \leq x_0 \leq 0.999000$ divided by $P$ generate 1000 orbits. In Fig. 5(b), $R(I_t)$ is plotted against $I_t$ for the [53 : 64] mode. The threshold value $\epsilon$ is set at $10^{-6}$ for both graphs. Note that these two graphs are different: for the [53 : 53] mode, a sharp transition occurs for $I_t < 10^2$, whereas for [53 : 64] mode a far gentler transition takes place whose mid-point is seen for $I_t > 10^3$.

Fig. 5. The graph of the ratio of chaotic orbits $R(I_t)$ defined by Eq.(9) against $I_t$ calculated for the difference between expressions (L1) and (L5). (a) The mode [53 : 53]. (b) The mode [53 : 64].

Fig. 6. The ratio of chaotic orbits $R(I_t)$ for all the 15 pairs selected from 6 different mathematical expressions against $I_t$. (a) The mode [53 : 53]. (b) The mode [53 : 64]. Each curve respectively represents the pair (L3)(L6), (L2)(L6), (L3)(L5), (L1)(L6), (L4)(L6), (L4)(L5), (L2)(L5), (L1)(L5), (L3)(L4), (L5)(L6), (L1)(L3), (L1)(L4), (L2)(L3), (L2)(L4) and (L1)(L2) from left to right.

Until now, for clarity of our explanation, we have restricted $x_n$ and $X_n$ to represent the expression (L1), and $y_n$ and $Y_n$ are for (L5), but the ratio $R(I_t)$ defined by Eq.(9) can be used for any pair of expressions selected from the possible six expressions (L1)∼(L6). Figs. 6(a) and (b) show the results of $R(I_t)$ calculated...
for all the 15 possible pairs; (a) for the [53 : 53] mode and (b) for the [53 : 64] mode. All the parametric conditions are the same as those for Figs. 5(a) and (b).

Remarkable difference is that for the mode [53 : 53] all the graphs for the 15 pairs are superposed into one, whereas for the mode [53 : 64] each of the 15 transition graphs emerges at separate $I_t$ values with almost the same slope which is much less steep than that of the transition curve in Fig. 6(a). As far as the chaotic-orbit ratio $R(I_t)$ is concerned, it is independent of the expression used in the computer program for the mode [53 : 53], whereas it is sensitively dependent on the expression for the [53 : 64] mode.

6. Conclusions

We first attracted attention of the computer users in scientific fields to the fact that currently used FPU’s have two computational modes to handle the floating-point numbers in double precision (DP). We showed that those two modes actually give rise to macroscopically observable differences using the familiar logistic map as an example.

In papers reporting results of numerical calculations, what kind of computer was used, programming language or codes themselves are seldom included, let alone the computational mode examined in this report. One of the most important objects of scientific papers, the present authors believe, must be reproducibility of the results reported therein. Since the fact that there are situations that the computational modes of FPU in DP produce differences is made clear by this work, the present authors propose that in papers reporting numerical results, computational mode and expressions used should be mentioned explicitly for further reproduction of the results. This proposal does not at all urge every author to inspect his system to find out which computational mode it utilizes. All he should do, we propose, is to include a line that mentions what compiler was used on what machine in his numerical research. Then, if the compiler is one of the popular ones, readers can see its computational mode as indicated in section 2.

Appendix

In this appendix, we prove that for $a = 4.0$ the expressions (L1)–(L6) can be grouped into two groups as stated at the end of section 3.

We assume that for all expressions the exponents of the resulting floating-point numbers become identical, hence the two expressions are concluded to be equal when the mantissae coincide. We premise the following (A1), (A2), and (A3) as formulas for our proof. A floating-point number $x$ is defined as

$$x = 2^e m(x)$$

where $e$ is the exponent for the base 2 and $m(x)$ is the mantissa of $x$.

Multiplication by 4.0 does not alter the mantissa, thus we have

$$m(x) = m(4.0x). \quad \text{(A1)}$$

The mantissa of the product of two floating-point numbers $x$ and $y$ is given, in general, as the product of the mantissae of $x$ and $y$, therefore we have

$$m(xy) = m(m(x)m(y)). \quad \text{(A2)}$$

Subtraction of two floating-point numbers $x$ and $y$ is done for the mantissae after equalizing the two exponents. If the differences of the two exponents are the same, the resulting mantissae are also the same. Hence, we have

$$m(4.0x - 4.0y) = m(x - y), \quad \text{(A3)}$$

First we prove that (L1)=(L2)=(L3). The expression (L1) is $(4.0x)(1.0-x)$, whose mantissa can be written from (A2) and (A1) as

$$m((4.0x)(1.0-x)) = m(m(4.0x)m(1.0-x)) = m(m(x)m(1.0-x)).$$
The expression (L2) is \(4.0(x(1.0 - x))\), whose mantissa can be written from (A1) and (A2) as
\[
m(4.0(x(1.0 - x))) = m(x(1.0 - x)) = m(m(x)m(1.0 - x)).
\]

Therefore, we have proved that (L1)=(L2). The expression (L3) is \((4.0 - 4.0)x\), whose mantissa can be written from (A2) and (A3) as
\[
m((4.0 - 4.0)x) = m(m(4.0 - 4.0)x) = m(m(1.0 - x)m(x)) = m(m(x)m(1.0 - x)).
\]

Since the order of multiplication does not matter, we have
\[
m((4.0 - 4.0)x) = m(m(x)m(1.0 - x)).
\]

Therefore, for \(a = 4.0\) we have proved that (L1)=(L2)=(L3).

Next, we prove that (L4)=(L5)=(L6). The expression (L4) is \(4.0(x - xx)\), whose mantissa can be written from (A1) as
\[
m(4.0(x - xx)) = m(x - xx).
\]

The expression (L6) is \(4.0x - 4.0(xx)\), whose mantissa can be written from (A1) as
\[
m(4.0x - 4.0(xx)) = m(x - xx).
\]

Hence, we have proved (L4)=(L6) first. The expression (L5) is \(4.0x - (4.0)x\), whose first term is identical with that of (L6). Therefore, if the second terms of (L5) and (L6) are identical, we can conclude that (L5)=(L6). The mantissa of the second term of (L5) can be written from (A2) and (A1) as
\[
m((4.0)x) = m(m(4.0)xm(x)) = m(m(x)m(x)).
\]

The mantissa of the second term of (L6) can be written from (A1) and (A2) as
\[
m(4.0(xx)) = m(xx) = m(m(x)m(x)).
\]

Hence we see that the second terms of (L5) and (L6) are identical. We have proved that (L4)=(L5)=(L6).

References

ANSI/IEEE754-1985 [1985] “IEEE standard for binary floating-point arithmetic,” Tech. rep., The Institute of Electrical and Electronics Engineers, Inc, East 47th Street, New York.

ANSI/IEEE754-2008 [2008] “IEEE standard for binary floating-point arithmetic,” Tech. rep., The Institute of Electrical and Electronics Engineers, Inc, East 47th Street, New York.

Colonna, J.-F. [1993] “The subjectivity of computers,” CACM 36, 15–18.

Keller, J. & Wiese, H. [2007] “Period lengths of chaotic pseudo-random number generators,” CNIS ’07 Proceedings of the Fourth IASTED International Conference on Communication, Network and Information Security, 2007 (Berkeley, California, USA), pp. 7–11.

Oteo, J. A. & J., R. [2007] “Double precision errors in the logistic map: Statistical study and dynamical interpretation,” Phys. Rev. E 76, 8pp.

Ott, E. [2002] Chaos in Dynamical Systems, 2nd ed. (Cambridge University Press, United Kingdom).

Sprott, J. C. [2003] “One-dimensional Maps,” Chaos and Time-Series Analysis (Oxford University Press), p. 27.

Wagner, N. R. [1992] “The logistic lattice in random number generation,” Proceedings of the Thirtieth Annual Allerton Conference on Communications, Control, and Computing (Monticello, Illinois, USA), pp. 922–931.

Wang, S., Liu, W., Lu, H., Kuang, J. & Hu, G. [2004] “Periodicity of chaotic trajectories in realizations of finite computer precisions and its implication in chaos communications,” International Journal of Modern Physics B 18, 2617–2622.