Pitman sampling formula
and
an empirical study of choice behavior

Masato Hisakado
Nomura Holdings, Inc.
Otemachi 2-2-2, Chiyoda-ku Tokyo 100-8130, Japan

Fumiaki Sano† and Shintaro Mori‡
Department of Physics, Kitasato University
Kitasato 1-15-1, Sagamihara, Kanagawa 252-0373, Japan
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Abstract

This study discusses choice behavior using a voting model in which voters can obtain information from a finite number $r$ of previous voters. Voters vote for a candidate with a probability proportional to the previous vote ratio, which is visible to the voters. We obtain Pitman sampling formula as the equilibrium distribution of $r$ votes. We present the model as a process of making posts on a bulletin board system, 2ch.net, where users can choose one of many threads to create a post. We explore how this choice depends on the last $r$ posts and the distribution of $r$ posts across threads. We conclude that the posting process is described by our voting model for a small $r$, which might correspond to the time horizon of users response.

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I. INTRODUCTION

In physics, equilibrium state are comparatively well understood, while non-equilibrium ones continue to attract much attention [1–3]. The latter poses several interesting problems and clarifying and classifying the nature of non-equilibrium stationary state continue to be a central research theme [4]. In other disciplines, the non-equilibrium stationary state is referred as equilibrium state. The ecology literature highlights that the equilibrium state in a zero sum model is an important process [5]. In a zero sum model, the total number of individuals is constant. In economics, the equilibrium state in which companies survive competitive conditions are discussed [6, 7].

Ewens sampling formula is a one-parameter probability distribution on the set of all partitions of the integer [8]. Pitman sampling formula is a two-parameter extension of Ewens sampling formula [9]. The Chinese restaurant process [9] and a generalized Pólya urn [10, 11] are non-equilibrium stochastic processes which derives Pitman sampling formula. These processes permits new entries of individuals and the number of them increases. In a similar non-equilibrium process where the number of species or vertices increases, a power-law distributions was obtained [12–14].

We introduced a sequential voting model in previous papers [15]. At each time step $t$, one voter chooses one of two candidates. In addition, the $t$th voter can see $r$ previous votes, thus allowing access to public perception. When the voters vote for a candidate with the probability that is proportional to the previous refeerable votes and there are two candidates, the model can be termed Kirman’s ant colony model [16]. Beta-binomial distribution was derived as the equilibrium distribution of the refeerable $r$ votes in the stationary state of the process [17]. If we assume that voters can refer all votes, the process becomes a non-equilibrium one. The equilibrium distribution and the probability distribution in the non-equilibrium proceses are the same [15].

In this paper, we extend Kirman’s ant colony model when the number of candidates are more than two and not fixed [15, 16]. The model is a finite reference version of the generalized Pólya urn model [10, 11]. We derive Pitman sampling formula as an equilibrium distribution. As a comprehensive example of the model, we analyze time series data for posts on 2ch.net and electoral data for the Japanese House of Representatives. In the former case, votes and candidates in the voting model correspond to posts on bulletin boards and the bulletin
boards’ threads. When $r$ is small, the posting process is described by the voting model. For the parliament election data, the number of candidates is fixed. Using Pitmans sampling formula we compare the correlation between votes before and after the introduction of the small constituency system in 1993.

The remainder of this paper is organized as follows. Sec. II introduces a voting model and we derive Pitman sampling formula as equilibrium distribution of votes. Sec. III presents the characteristics of several parameters. Sec. IV studies the time series data for posts on 2ch.net using the voting model. Sec. V concludes. Appendix A studies the case in which the number of candidates is fixed case. Appendix B examines Japanese election data as a case where the number of candidates is fixed. Appendix C and Appendix D provide information on the data of 2ch.net and supplementary results of the data analysis.

II. MODEL

We examine choice behavior using a voting model with candidates $C_1, C_2, \ldots$. At time $t$, $C_j$ have $c_j(t)$ votes. In each time step, a voter votes for one candidate and the voting is sequential. Thus, at time $t$, the $t$th voter votes, after which the total number of votes is $t$. Voters are allowed to see $r$ previous votes for each candidate and thus, are aware of public perception. $r$ is a constant number. The candidates are allowed to both enter and exit. The voter votes for the new candidate $C_i$ with probability $(\theta + K_r \alpha)/(\theta + r)$, where $r$ is the number of referred votes and $K_r$ is the number of candidates who have more than one vote in the last $r$ votes. $\alpha$ and $\theta$ are parameters. If a candidate does not have more than one votes in the last $r$ votes, he/she exits. $i$ in $C_i$ is the number of candidates who have appeared in the past plus one. The number of candidate at $t = 1$ is one and there is only one candidate $C_1$.

In terms of the Chinese restaurant process or generalized Pólya urn, we describe the voting process as follows. At first, there is an urn with $\theta$ black balls in it. In each step, one ball is drawn from the urn and two balls are placed back in the urn. At turn one, a black ball is drawn and a ball of color 1 and the black ball are placed back in the urn. If the drawn ball is black, a ball of another color which does not appear in the past and the black ball are returned. If the drawn ball is not black, the ball is duplicated and two balls are placed back in the urn. The difference between the voting model and Chinese restaurant
process is that the voter refers to only recently added \( r \) balls and black balls.

![Parameter space diagram](image)

**FIG. 1.** Parameter space \( \alpha \) and \( \theta \). \( \alpha \) is the parameter that adjusts the entry probability of new candidates as per the number of candidates, \( K_r \). \( \theta \) is the parameter that controls the overall probability of adding a new candidate. The dotted line is not included in the parameter space. When \( \alpha < 0 \), the constraint \( \theta = \alpha K \) exists and \( K_r \) cannot exceed \( K \).

We demonstrate the parameter space in Fig. 1. \( \theta \) is the parameter that controls the overall probability of adding a new candidate. \( \alpha \) is the parameter that adjusts the entry probability of new candidates according to the number of candidates \( K_r = K \). When \( \alpha < 0 \), the constraint \( \theta = -\alpha K \) exists.

We consider the case in which voters are analog herders. If \( c_j(t) \geq 1 \), the transition is

\[
c_j(t) = k \rightarrow k + 1 : P_{j,k,t,t-r} = \frac{-\alpha + (k-l)}{\theta + r},
\]

where \( P_{j,k,t,t-r} \)s are the probabilities of the process. The number of votes for \( C_j \) at \( (t-r) \) is \( c_j(t-r) = l \). Hence, if \( (k-l) = 0 \), candidate \( C_j \) exits the system.
The process of a new candidate $C_j$ entering is

$$c_j(t) = 0 \rightarrow 1 : P_{j,k,t,t-r} = \frac{K_r\alpha + \theta}{\theta + r},$$

where the number of candidates who have more than 0 votes is $K_r$.

When $\alpha \geq 0$, from (1) and (2), constraints $\theta + \alpha > 0$ and $1 > \alpha \geq 0$ exist. (See Fig.1.)

There is no upper limit on the number of candidates. When $\alpha > 0$, the probability of a new entry increases with a rise in $K_r$. When $\alpha = 0$, the probability of a new entry is constant. When $\alpha < 0$, from (2), the constraint $\alpha K + \theta = 0$ exists. The probability of a new entry decreases with an increase in $K_r$ and $K$ is the upper limit of $K_r$. The candidates who have more than one vote do not exceed $K$. This is similar to the model that does not allow candidate entry, which is further discussed in Appendix A.

The distribution of $c_j(t)$ as the partition of integer $t$ is Pitman sampling formula in the generalized Pólya urn model [9, 11]. The process is a non-equilibrium one and the number of votes increases. We focus, not on the snap shot $c_j(t)$, but on the time series of state $c_j(t) - c_j(t-r)$. The process is an equilibrium one and the number of total votes is constant.

We consider a hopping rate among $(r+1)$ states $\hat{k}_j = k-l$, $\hat{k}_j = 0, 1, \cdots, r$. Here, we focus on the state. In each step of $t$, the vote at time $(t-r)$ is deleted and a new one is obtained. $\hat{k}_j$ is the number of votes that candidate $C_j$ obtained in the previous $r$ votes.

First, we consider the case $\hat{k}_j > 1$. The transition is

$$\hat{k}_j \rightarrow \hat{k}_j + 1 : P_{k_j,k_j+1,t} = \frac{r - \hat{k}_j - \alpha + \hat{k}_j}{r\theta + r - 1},$$

$$\hat{k}_j \rightarrow \hat{k}_j - 1 : P_{k_j,k_j-1,t} = \frac{\hat{k}_j(\theta + \alpha) + (r - \hat{k}_j - 1)}{\theta + r - 1},$$

$$\hat{k}_j \rightarrow \hat{k}_j : P_{k_j,k_j,t} = 1 - P_{k_j,k_j-1,t} - P_{k_j,k_j+1,t}.$$  \hspace{1cm} (3)

$P_{k_j,k_j \pm 1,t}$ and $P_{k_j,k_j,t}$ are the probabilities of the process. $P_{k_j,k_j \pm 1,t}$ is the product of the probabilities of an exit and entry.
We consider hopping from candidate $C_i$ to $C_j$.

$$
\hat{k}_i \rightarrow \hat{k}_i - 1, \hat{k}_j \rightarrow \hat{k}_j + 1: P_{\hat{k}_i \rightarrow \hat{k}_i - 1, \hat{k}_j \rightarrow \hat{k}_j + 1, t} = \frac{\hat{k}_i - \alpha + \hat{k}_j}{r \theta + r - 1},
$$

$$
\hat{k}_i - 1 \rightarrow \hat{k}_i, \hat{k}_j + 1 \rightarrow \hat{k}_j: P_{\hat{k}_i - 1 \rightarrow \hat{k}_i, \hat{k}_j + 1 \rightarrow \hat{k}_j, t} = \frac{\hat{k}_j + 1 - \alpha + \hat{k}_i - 1}{\theta + r - 1}.
$$

Here, we define $\mu_r(\hat{k}, t)$ as the distribution function of state $\hat{k}$ at time $t$. The number of all states is $(r + 1)$. Using the fact that the process is reversible, in the equilibrium, we have

$$
\frac{\mu_r(\hat{k}_i, \hat{k}_j, t)}{\mu_r(\hat{k}_i - 1, \hat{k}_j + 1, t)} = \frac{\hat{k}_j + 1 - \alpha + \hat{k}_i - 1}{\hat{k}_i - \alpha + \hat{k}_j}.
$$

(5)

We separate indexes $i$ and $j$ and obtain

$$
\frac{\mu^i_r(\hat{k}_i, t)}{\mu^i_r(\hat{k}_i - 1, t)} = \frac{-\alpha + \hat{k}_i - 1}{\hat{k}_i} c,
$$

$$
\frac{\mu^j_r(\hat{k}_j + 1, t)}{\mu^j_r(\hat{k}_j, t)} = \frac{-\alpha + \hat{k}_j}{\hat{k}_j + 1} c,
$$

(6)

where $c$ is a constant.

In the equilibrium, the number of candidates $\hat{k}_j > 0$ is $K_r$. We ignore candidates $\hat{k}_j = 0$ and change the number of candidates and votes from $C_j, \hat{k}_j$ to $\bar{C}_m, \bar{k}_m$, where $m = 1, \cdots, K_r$, wherein $\bar{k}_m > 0$.

We can write the distribution as

$$
\mu_r(\bar{a}, \infty) = \left(\frac{\theta + r - 1}{r}\right)^{-1} \prod_{m=1}^{K_r} \frac{(1 - \alpha)^{[\bar{k}_m]}}{\bar{k}_m!} \mu_r(\bar{a} = (1, 1, \cdots, 1), \infty),
$$

(7)

where $\bar{a} = (\bar{k}_1, \cdots, \bar{k}_{K_r})$ and $x^{[n]} = x(x + 1) \cdots (x + n - 1)$, which is the Pochhammer symbol.

Given (6), we can obtain the equilibrium condition between $\bar{a} = (1, 1, \cdots, 1) = 1$ and $\bar{a} = (0, 0, \cdots, 0) = 0$:

$$
\frac{\mu^i_r(\bar{a} = (1, \cdots, 1, 0, \cdots, 0), \infty)}{\mu^i_r(\bar{a} = (1, \cdots, 1, 0, \cdots, 0), \infty)} = \frac{\theta + (n - 1)\alpha}{n c},
$$

(8)
where $n = 1, \cdots, K_r$. Hence, we can get

$$\mu_r(\hat{a} = 1, \infty) = \prod_{m=1}^{K_r} \frac{\theta + (m-1)\alpha}{m} \mu_r(\hat{a} = 0, \infty) = \frac{(\theta)^{[K_r, \alpha]}}{K_r!},$$

where $x^{[\alpha]} = x(x + \alpha) \cdots (x + (n-1)\alpha)$. Therefore, we can write (7) as

$$\mu_r(\hat{a}, \infty) = \left(\frac{\theta}{K_r!}\right)^{-1} \prod_{j=1}^{r} \left(\frac{(1-\alpha)[j-1]}{j!}\right)^{a_j},$$

where $a_j$ is the number of candidates who have $j$ votes. Therefore, the number of candidates $\sum_{j=1}^{r} a_j = K_r$ and that of votes $\sum_{j=1}^{r} ja_j = r$ are related. Hereafter, we use a partition vector $\hat{a} = (a_1, \cdots, a_r)$.

We consider the partitions of integer $K_r$. To normalize, we add the following term of combination, $K_r!/a_1! \cdots a_r!$:

$$\mu_r(\hat{a}, \infty) = \frac{r!\theta^{[K_r, \alpha]}}{\theta^{[\alpha]}} \prod_{j=1}^{r} \left(\frac{1-\alpha}{j!}\right)^{a_j} \frac{1}{a_j!}.$$  

(11) is nothing but a Pitman sampling formula [9]. In the limit $\alpha = 0$, we can obtain Ewens sampling formula [8]:

$$\mu_r(\hat{a}, \infty) = \frac{r!}{\theta^{[\alpha]}} \prod_{j=1}^{r} \left(\frac{\theta}{j!}\right)^{a_j} \frac{1}{a_j!}.$$  

(12) Ewens sampling formula for the equilibrium process is presented in [6].

III. FOUR REGIONS IN PARAMETER SPACE

In this section, we characterize four regions in the parameter space. Parameter $\theta$ denotes the correlations and parameter and $\alpha$ means competitive intense. In the upper-half plane, new comers increase with a rise in $\alpha$. In the lower-half plane, the bottom growing probability increases with a decrease in $\alpha$.

We consider an increase and decrease of $\hat{k}_j$. The probability of a decrease of candidate $j$ is $\hat{k}_j/r$ and that of an increase is $(-\alpha + \hat{k}_j)/(\theta + r - 1)$, as shown in (1). We consider the
FIG. 2. Four regions in parameter space. Average increase and decrease in regions.

condition in which the probability of an increase is larger than that of a decrease.

\[ \alpha \leq - \frac{k_j}{\theta - 1}. \]  \hspace{1cm} (13)
In the region, the number of votes increases on average.

We divide the parameter space into four regions, $I \sim IV$, as shown in Fig. 2(a). To clarify the regions, we define zone $U_{kj}$, where the probability of an increase in votes is larger than that of a decrease. $U_{kj}$ is defined in (13). $U_r$ is $\alpha \leq -(\theta - 1)$ and $U_0$ is $\alpha \leq 0$.

We define zone I, where $\alpha > 0$ and $\alpha < (1 - \theta)$, as $U_r$. In zone I, $U_r \supset U_{r-1} \supset \cdots \supset U_0$. We consider the case in which the parameter set $\mathbf{x} = (\theta, \alpha)$ is $\mathbf{x} \in U_l$ and $\mathbf{x} \notin U_{l-1}$. In this case, $k_j \geq l$ is the increase zone and $k_j < l$ the decrease zone. If a candidate has greater than $l$ vote, he/she can increase the number of votes and maintain his/her position. On the other hand, it is difficult to increase votes if the candidate has less than $l$ vote. We show the average trend in Fig 2.(b). In this region, the leader in the trapped zone is advantageous. On the other hand, it is competitive intense for new comers as $\alpha$ increases.

We define zone II: $\alpha > 0$ and $\alpha \geq (1 - \theta)$. In zone II, $\mathbf{x} \notin U_0, \cdots, U_r$. Further, it is difficult to increase the number of votes for every candidate and to be a stable leader. The zone becomes competitive intense with an increase in $\alpha$. In other words, it is possible to adjust the competitive intensity and protect new comers by adjusting $\alpha$, which denotes the number of new comers.

In the lower-half plane, there is a capacity limit and no new comer. We define zone III as $\alpha < 0$ and $\alpha > (1 - \theta)$. When $\alpha < 0$, $U_0 \supset U_1 \supset \cdots \supset U_r$. We consider the case in which the parameter set $\mathbf{x} = (\theta, \alpha)$ is $\mathbf{x} \in U_l$ and $\mathbf{x} \notin U_{l+1}$. In this case, $k_j > l$ is the decrease zone and $k_j \leq l$ the increase zone. It is easy to increase the number of votes to $\hat{k}_j = l$, but difficult to do so to greater than $\hat{k}_j = l + 1$. In this region, it is also difficult to be a stable leader.

We define zone IV as $\alpha < 0$ and $\alpha \leq (1 - \theta)$. In this zone, $\mathbf{x} \in U_0, \cdots , U_r$. It is easy to increase the votes for each candidate. In addition, it is competitive when the number of members is fixed.

Next, we consider Ewens sampling formula on the $\theta$ axis. When $\alpha = 0$ and $\theta = 1$, the probabilities of an increase and decrease become $\hat{k}_j/r$. In any $\hat{k}_j$, the probabilities of an increase and decrease are equal. Thus, the correlation becomes $\rho = 1/2$ (see Appendix A) and there is a uniform random permutation. The probability is proportional to the number of partitions.

When $\alpha = 0$ and $\theta < 1$, within the boundary of zone IV, if candidates can enter, the number of votes easily increases. In this zone, the correlation is high. When $\alpha = 0$ and
\( \theta > 1 \), in the boundary of zone II, it is difficult to increase the number of votes. Here, there is a low correlation. In summary, there are numerous candidates who have few votes.

IV. DATA ANALYSIS OF BULLETIN BOARD SYSTEM

In this section, we examine the data of posts on a bulletin board system (BBS), 2ch.net. 2ch.net is the largest BBS in Japan and covers a wide range of topics. Each bulletin board is separated by a field unit or a category, for example, news, food and culture, and net relations. Each category is further divided into genres, or boards, and each board contains numerous threads, which are segregated by topics that belong to the board. Writing and viewing boards is done on a thread. There are about 900 boards on 2ch.net. It is possible to make anonymous posts on all threads.

| No. | Board Name         | Obs. Period       | \( N \)  | \( T/N \) | S.D. | \( w_{\text{Max}} \) | Lifetime \( s_H \)[sec] |
|-----|--------------------|-------------------|--------|---------|-----|---------------------|------------------------|
| 1   | Business News      | Aug. 10, 2009–Dec. 31, 2009 | 8,248  | 140     | 290  | 7,707               | 7.5                    | 260.0                  |
| 2   | East Asia News     | Mar. 8, 2009–Aug. 5, 2009    | 8,225  | 388     | 1,022 | 27,966              | 7.5                    | 205.4                  |
| 3   | Live News          | Mar. 8, 2009–Aug. 5, 2009    | 15,307 | 53      | 333   | 30,443              | 2.0                    | 95.1                   |
| 4   | Music News         | Mar. 8, 2009–Aug. 5, 2009    | 23,000 | 332     | 1,123 | 78,388              | 2.8                    | 140.2                  |
| 5   | Breaking News      | Mar. 8, 2009–Dec.31, 2009    | 33,677 | 658     | 1,497 | 113,220             | 2.9                    | 94.5                   |
| 6   | Digital Camera     | Aug. 10, 2009–Dec. 31, 2009  | 835    | 527     | 1,530 | 33,494              | 251.4                  |
| 7   | Game               | Mar. 8, 2009–Aug. 10, 2009    | 1,371  | 241     | 286   | 2,043               | 132.6                  |
| 8   | Entertainment      | Aug. 10, 2009–Dec. 31, 2009  | 1,464  | 134     | 1188  | 30999               | 35.4                   |
| 9   | Int. Affairs       | Aug. 10, 2009–Dec. 31, 2009  | 688    | 233     | 241   | 1,000               | 1,123.8                |
| 10  | Press              | Aug. 10, 2009–Dec. 31, 2009  | 1,011  | 235     | 296   | 2,182               | 358.5                  |

We study the time series of posts on the following ten boards: business news, East Asia news, live news, music news, breaking news, digital camera, game, entertainment, international affairs, and press. We label them No. 1, No. 2, \cdots No. 10 boards, respectively. The first five boards fall under the news category. Each board has several hundred threads.
and managers maintain the number of threads by removing old ones and replacing them with new threads. The duration of a post on a thread is set to five days for the No. 4 and No. 5 boards. As a result, the lifetime of a thread is generally about few days. One cannot post more than 1,000 posts to a thread. Threads that can no longer be posted on are deleted from the thread list of boards and managers prepare a new sequential thread using the same thread title. The lifetime can be longer than the abovementioned duration, as the rule of postable duration applies to descendant threads with a new start date. We identify sequential descendant threads from a common ancestor thread as one thread. Table I summarizes the statistic of threads of the ten boards.

The total number of posts on each board is about 0.2–22 million. The average number of posts in each thread is several hundreds and the standard deviation is large. The maximum number of posts $w_{Max}$ is 50–100 times larger than the average. The average lifetime of a thread on news boards is several days, which is derived from the strict rule defining the period within which a post can be made on threads. There is no strict rule for the remaining five boards and the average lifetimes are considerably longer than those of the news boards.

![FIG. 3. Scatter plot for post data $(n(t), s(t))$ for the No. 5 and No. 8 boards. Each dot corresponds to a post.](image)

We label threads by $n \in \{1, \ldots, N\}$ and $N$ is the total number of threads which appears in the board. We describe the $t$th post in the board at $s[sec]$ by the thread number $n$ and $s$ as $(n(t), s(t)), t = 1, \ldots, T$. We measure post time $s$ by setting the first post time on the board as 0 seconds. We present the scatter plot of the time series post data $(n(t), s(t)), t = 1, \ldots, T$.
in the \((n, s)\) plane of the No. 5 and No. 8 board in Fig. 3. Because the threads have a
strict finite lifetime of five days in the No. 5 board, the plot shows a narrow strip pattern.
Some threads have a longer lifetime because they have a long family tree from ancestors to
descendants. No. 8 board shows a wide strip pattern, which indicates that the number of
threads becomes considerably large.

A. Correlation function \(C(\tau)\) for equilibrium \(r\)

\[
\text{FIG. 4. Posts, threads, and a board. A board is represented as a rectangle with rounded corners. Posts are shown as balls and labeled as } t. \text{ There are } K_r = 6 \text{ threads with non-zero posts. } n \text{ shows the number of threads and the posts to a thread is put on it. The "21st post refers to the previous 10 posts, } t = 11, 12, \cdots, 10. \text{ The thread numbers are } n = 5, 9, 11, 16, 18, 10 \text{ and appear } (k_5, k_9, k_{11}, K_{16}, k_{18}, k_{10}) = (3, 1, 2, 1, 1) \text{ times, respectively. The multiplicities } a \text{ are } a_1 = 3, a_2 = 2 \text{ and } a_3 = 1. \text{ } a_1 + a_2 + a_3 = K_r = 6 \text{ and } 1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 = 10 \text{ holds. The probability of a post on a thread with two posts is } 2 \cdot \frac{2-a}{\theta+10}. \text{ The probability of a thread not appearing in ten posts is } \frac{\theta+6-a}{\theta+10}.\]

We identify threads and posts as candidates and votes in the voting model in Fig. 4. As
previously shown, when voting is done with reference to the previous \(r\) votes, the stationary
distribution of votes of \(r\) consecutive voting obeys Pitman sampling formula in (11). In this
case, the correlation between \(n(t)\) and \(n(t - \tau)\) for voting lag \(\tau\) does not decay for \(\tau < r\)
as the distribution is stationary in \(r\) consecutive votes. In the range \(\tau > r\), \(C(\tau)\) dumps.
If the post on the board is described by the voting model with reference \( r \), \( C(\tau) \) should demonstrate the feature. We adopt the expectation value of the coincidence of \( n(t) \) and \( n(t - \tau) \) as the correlation between \( n(t) \) and \( n(t - \tau) \),

\[
C(\tau) \equiv E(\delta_{n(t), n(t-\tau)}).
\]

We assume \( C(\tau) \) does not depend on \( t \) and estimate it using time series data \( \{n(t)\}, t = 1, \ldots, T \) as

\[
C(\tau) = \frac{1}{T - \tau} \sum_{t=1}^{T-\tau} \delta_{n(t), n(t+\tau)}.
\]

**FIG. 5.** Plot of \( C(\tau) \) vs. \( \tau \). The left (right) panel shows the results for the first (remaining) five boards. \( \tau \) on the x axis indicates the voting lag and \( C(\tau) \) denotes the auto-correlation function between \( n(t) \) and \( n(t + \tau) \).

Fig.5 illustrates the semi-logarithmic plot \( C(\tau) \) vs. \( \tau \). The properties of \( C(\tau) \) of a voting model with finite \( r \) is summarized in [18]. We see a plateau structure in which \( C(\tau) \) does not decrease with \( \tau \) for the first five news boards in the left panel. For the No. 4 and No. 5 boards, \( C(\tau) \) is almost constant for \( \tau \leq \tau_c \approx 80 \). As for other boards, \( C(\tau) \) decreases for \( \tau \geq \tau_c = 20 \) for No. 1 and No. 2 boards and for the smaller values of \( \tau_c \sim 5 \) for No. 3 board. In these news boards, \( C(\tau) \) are almost constant among posts within \( \tau_c \). On the other hand, \( C(\tau) \) decreases with \( \tau \) for the latter boards in the right panel with the exception of No.8 board. As for the No. 8 board, \( C(\tau) \) is large for large \( \tau \) and gradually decreases, indicating that the board has special features.
We apply the voting model to the posts on 2ch.net in the boards for the news category with $r < \tau_c$. We adopt $r = 80$ for the No. 4 and No. 5 boards, $r = 20$ for the No. 1 and No. 2 boards, and $r = 5$ for the No. 3 boards. To interpret $\tau_c$, we highlight the response time of board users. We believe a user needs several minutes to respond to posts. Next, the post should be random for short time interval and the probability of a post on a thread is roughly estimated as the post ratio in the previous posts. In the last column of Table I we indicate the time horizon $s_H[sec]$ for $\tau_c$, which is defined as the mean duration between posts multiplied by $\tau_c$, as

$$s_H \equiv \frac{s(T) - s(1)}{T - 1} \cdot \tau_c.$$  

(14)

$s_H$ is about 1.5 minute to 4 minutes, which is possibly the requisite time duration to respond to posts.

**B. Estimation of parameters $\theta, \alpha$**

We use time series data $\{n(t)\}, t = 1, \cdots, T$, $n \in \{1, \ldots, N\}$ and estimate the model parameters $\theta$ and $\alpha$ using the maximum likelihood principle. In the model, the probability of a post on a thread that appears in the past $r$ posts with $\hat{k}$ times is defined as in (1). The probability of a post to $a_{\hat{k}}$ threads with $\hat{k}$ is

$$P_{Existing}(\hat{k}) = a_{\hat{k}} \cdot \frac{\hat{k} - \alpha}{\theta + r}.$$  

(15)

The probability of a post on a new thread that does not appear in the past $r$ posts depends on the number of thread $K_r$ in the $r$ past and is defined in (2):

$$P_{New}(K_r) = \frac{K_r\alpha + \theta}{\theta + r}.$$  

(16)

For $t \in [1 \times 10^4, T - r]$, we choose $t_n, n = 1, \cdots, S$ randomly and study the following $r + 1$ sequence, $n(t_n), n(t_n + 1), \cdots, n(t_n + r - 1)$. We estimate the number of threads $K_r$ and the number of threads with $\hat{k}$ posts $a_{\hat{k}}$ in the $r$ posts. $\sum_{\hat{k}} a_{\hat{k}} = K_r$ holds. If thread $n(t_n + r)$ does not exist in the $K_r$ threads, the likelihood is $P_{New}(K_r)$. If it exists and thread $n(t_n + r)$ appears $\hat{k}$ times, the likelihood is $P_{Existing}(\hat{k})$. The likelihood for $S$ sample is then estimated by the products of these likelihood for all $s = 1, \cdots, S$. We adopt $S = 2 \times 10^5$. In addition,
we fit the parameters using the maximum likelihood principle for the distribution of the partitions of \( a_k \) with the Pitman sampling formula in (11).

TABLE II. Fitting results of \( \theta \) and \( \alpha \) for probabilistic rules in (15) and (16). We use the maximum likelihood principle and the sample number \( S \) is \( 2 \times 10^5 \). We show the estimates for the No. 1 and No. 2 boards with \( r = 20 \), No. 3 board with \( r = 5 \), and No. 4 and No. 5 boards with \( r = 80 \) in the third and fourth columns. For \( r = 5 \) and \( r = 20 \), we show the goodness-of-fit results using the Pitman sampling formula in (11) in the fifth and sixth columns. We adopt the same samples for the two fittings. The standard error (S.E.) in the last digit of the estimate is provided in parentheses.

| No. | \( r \) | \( \theta \) (S.E.) | \( \alpha \) (S.E.) | \( \theta \) (S.E.) | \( \alpha \) (S.E.) |
|-----|-------|----------------|----------------|----------------|----------------|
| 1   | 20    | 8.8(2)         | 0.37(1)        | 8.7(0)         | 0.390(2)       |
| 2   | 20    | 2.2(1)         | 0.42(1)        | 2.0(0)         | 0.418(2)       |
| 3   | 5     | 1.7(0)         | 0.560(4)       | 1.3(0)         | 0.623(3)       |
| 4   | 80    | 10.0(3)        | 0.35(1)        | NA             | NA             |
| 5   | 80    | 11.9(4)        | 0.28(1)        | NA             | NA             |

The estimated values for the parameters are summarized in Table II. We adopted \( r = 20 \) for the No. 1 and No. 2 boards, \( r = 5 \) for the No. 3 board, and \( r = 80 \) for the No. 4 and No. 5 boards by the correlation analysis. The standard errors are estimated using the square root of the minus eigenvalue for the Hessian of log-likelihood. For \( r = 80 \), we only show the results by fitting with probabilistic rules.

To verify the probabilistic rules, we directly estimate \( P_{Existing}(\hat{k}) \) and \( P_{New}(K_r) \). We calculate the number of threads with post times \( \hat{k} \) for \( r \) posts and denote it as \( N(\hat{k}) \). In addition, we count the number of times a post is made on an exiting thread with posts \( \hat{k} \) and denote it as \( N_{post}(\hat{k}) \). The estimator for \( P_{Existing}(\hat{k}) \) is

\[
\hat{P}_{Existing}(\hat{k}) = \frac{N_{Post}(\hat{k})}{N(\hat{k})}.
\]  

Likewise, we count the number of threads \( K_r \) and the number of times a post is made on a new thread when the number of threads is \( K_r \). We denote them as \( N_{K_r}(K_r) \) and \( N_{New}(K_r) \). The estimator for \( P_{New}(K_r) \) is then denoted as

\[
\hat{P}_{New}(K_r) = \frac{N_{New}(K_r)}{N_{K_r}(K_r)}.
\]
FIG. 6. Plots of $P_{\text{Existing}}(\hat{k})$ vs. $\hat{k}$ and $P_{\text{New}}(K_r)$ vs. $K_r$. The symbols denote the estimated results using $\hat{P}_{\text{Existing}}(\hat{k})$ in (17) and $\hat{P}_{K_r}(K_r)$ in (18). The lines denote the plots of (15) and (16) with fitted parameters for $\theta$ and $\alpha$ in Table II.

Fig. 6 presents the estimates for $\hat{P}_{\text{Existing}}(\hat{k})$ and $\hat{P}_{\text{new}}(K_r)$. We also plot $P_{\text{Existing}}(\hat{k})$ and $P_{\text{New}}(K_r)$ in (15) and (16) with fitted values for $\theta$ and $\alpha$ in Table II. The estimated results for the maximum likelihood fit well with the results from estimators $\hat{P}_{\text{Existing}}(\hat{k})$ and $\hat{P}_{\text{New}}(K_r)$. The parameters are in zone II, which were introduced in the previous section. In this zone, it is difficult for a leader to appear.

C. Distribution of $K_r$ and $\hat{k}$

We compare (11) and the probability mass function for $K_r$ using the fitted parameters in Table III and those of an empirical distribution. The probability mass function $P_r(K_r)$ for the number of candidates $K_r$ with reference $r$ is given as

$$P_r(K_r) = \frac{\theta^{[r,\alpha]}}{\theta^{[r]}} c(r, K_r, \alpha) \alpha^{-K_r}, \quad (19)$$

where $c(r, K_r, \alpha)$ is the generalized Stirling number or the C-numbers [9]. As for the probability mass function for the post times $\hat{k}$, we calculate the number of posts $\hat{k}_{1st} \geq \hat{k}_{2nd} \geq \hat{k}_{3rd}$ for the most popular three threads in $r$ posts in addition to all post times $\hat{k}$ for all threads. We plot the results in Fig. 7.
FIG. 7. Plots of distribution of post times $\hat{k}$ and number of threads $K_r$ in $r$ posts for data with symbols and for (11) and (19). We sort $\hat{k}$ in descending order and select the largest three values $\hat{k}_1 \geq \hat{k}_2 \geq \hat{k}_3$. The symbols □, ◦, and △ denote the empirical distribution of $\hat{k}_1$, $\hat{k}_2$, and $\hat{k}_3$, respectively. ♦ and ▽ show the distribution of all $\hat{k}$ and $K_r$. The lines indicate the plots of $P_r(K_r)$ in (19) and the distribution of ordered $\hat{k}_j$ and $\hat{k}$, which are calculated using Pitman's sampling formula in eq.(11). We adopt $r = 20(80)$ for the No. 1 and No. 2 (No. 4 and No. 5) boards. The parameters for (11) and (19) are presented in Table II. We adopt the second set for $r = 20$ and the first set for $r = 80$.

As we can see, the fitting results are good. The distributions of $\hat{k}_{1st}$, $\hat{k}_{2nd}$, $\hat{k}_{3rd}$, $\hat{k}$, and $K_r$ are well described by Pitman's sampling formula (11) and $P_r(K_r)$ with fitted parameters for $\theta, \alpha$ in Table II. We present the results for the statistical test using Kolmogorov–Smirnov (KS) statistics in Appendix C.
D. Distribution of total votes $c_n(T)$

In this subsection, we discuss the distribution of total votes $c_n(T)$ for thread $n$, where $c_n(t) = \sum_{t'=1}^{t} \delta_{n(t'),n}$. Fig. 8 shows the semi-logarithmic plot of the cumulative distribution $P(k) \equiv P(c_n(T) \geq k)$ vs. $k$. The left (right) panel depicts the results for the first (remaining) five boards. The dotted line denotes the cumulative distribution of the log-normal distribution with the same mean and variance. As is clearly shown, the results show good fits. In the previous subsection, we confirm that the distribution of posts in the four boards obey the equilibrium Pitman sampling formula. The difference between the equilibrium and non-equilibrium suggests that the posting process for a large $r$ is not described by the voting process.

![FIG. 8. Plot of $P(k) \equiv P(c_n(T) \geq k)$ vs. $k$. The left (right) panel presents the results for the first (remaining) five boards. The mean and standard deviations of $c_j$ for the first five boards are (3.93, 1.41), (4.79, 1.48), (3.03, 1.10), (4.76, 1.45), and (5.47, 1.47).](image)

In the remaining boards, we confirm that the distribution of the number of posts on the No. 8 board obey the power-law distribution and the power-law index is 1.65, $P(k) \sim k^{-(1.65-1)}$. We have seen that the board has a long memory in Fig 5. Further, we can confirm that the probability of a post is proportional to the number of posts and that of a new thread is proportional to the number of threads for a large $r$ in Fig 9. As $r$ increases, the latter dependence disappears and the process is described by the Yule process 12. As the power-law exponent is less than two, the fitness model for evolving networks might be a
FIG. 9. Plot of $P_{Existing}(\hat{k})$ vs. $P_{New}(K_r)$ for No. 8 boards. The symbols show the estimated results using $\hat{P}_{Existing}(\hat{k})$ in (17) and $\hat{P}_{K_r}(K_r)$ in (18). The lines denote the plots of (15) and (16) better candidate to describe the posting process [19, 20].

V. CONCLUDING REMARKS

In this study, we discuss choice behavior using a voting model comprising voters and candidates. Voters vote for a candidate with the probability that is proportional to the previous votes ratio, which is visible to the voters. In addition, voters can obtain information from a finite number $r$ of latest previous voters.

In the large $t$ limit, the system is equilibrated and the partition of $r$ votes obeys Pitman sampling formula. Kirman’s ant colony model is the special case and corresponds to the number of states, $K = 2$. The equilibrium probability distribution and the non-equilibrium probability distribution for $t = r$ are the same. We propose the voting model for the posting process of a BBS, 2ch.net, where users can select one of many threads to make a post. We explore how this choice depends on the last $r$ posts and the $r$ posts are distributed to boards.

We conclude that the posting data in the news category is described by the voting model. The equilibrium time or time horizon $s_H$ is about 1.5-4 minutes. Up to this time horizon, the probability of posting a thread is proportional to the ratio of posts on the thread.

When the number of candidate $K$ is fixed at $\theta = -K\alpha$ for $\alpha < 0$, we show the Dirichlet
multinomial distribution reduces to Pitman’s sampling formula in Appendix A. In Appendix B as its application, we examine the parliament election data for Japan. We ignore the inhomogeneities of the candidates in the elections. The model has only one parameter $\theta$. We estimate the correlation strength between votes as $1/(\theta+1)$ and show that the correlation between votes becomes stronger after the introduction of the small constituency system in 1993.

Recently, a wide variety of social systems, including election votes, citations of scientific papers, rating dynamics on E commerce, and social tagging system, have been extensively studied [21–25] using simple probabilistic models. We hope our study provide a new perspectives from equilibrium viewpoint in non-equilibrium system.

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Appendix A: Fixed number of candidates case

We model the voting of $K$ candidates, $C_1 \cdots C_K$. At time $t$, $C_j$ have $c_j(t)$ votes. In this appendix, we consider the case in which the number of candidates $K$ is fixed, that is, no new entry is allowed. In each time step, one voter votes for one candidate; the voting is sequential. Hence, at time $t$, the $t$th voter votes, after which the total number of votes is $t$.

Voters are allowed to see $r$ previous votes for each candidate and thus, are aware of public perception. $r$ is a constant number. We consider the case in which all voters vote for the candidate with a probability proportional to the previous votes ratio, which is visible to the voters.

The transition is

$$c_j(t) = k \rightarrow k + 1 : P_{j,k,t,t-r} = \frac{q_j(1-\rho) + (k-l)}{1-\rho + r} = \frac{\beta_j + (k-l)}{\theta + r},$$

where $c_j(t-r) = l$, $\rho$ is the correlation coefficient and $q_j$ is the initial constant of the $j$th candidate [17]. $\rho$ is the correlation of the beta binomial model. The constraint $\sum_{j=1}^{K} q_j = 1$ exists. We define $\theta = (1-\rho)/\rho$ and $\beta_j = q_j(1-\rho)/\rho$. $P_{j,k,t,t-r}$ is the probabilities of the process. The voting ratio for $C_j$ at $t-r$ is $c_j(t-r) = l$. We consider the case $\beta_j \geq 0$ from $P_{j,k,t,t-r} > 0$ and the constraint $\sum_j \beta_j = \theta$. When $\beta_j = \beta$, the constraint becomes $\beta K = \theta$.

We consider the hopping rate among $(r+1)$ states $\hat{k}_j = k - l$, $\hat{k}_j = 0, 1, \cdots, r$. In each step of $t$, the vote at time $(t-r)$ is deleted and a new vote is obtained. $\hat{k}$ is the number of votes candidate $C_j$ obtained in the latest $r$ votes. In case $K = 2$, the model becomes Kirmans ant colony model [16]. The dynamic evolution of the process is given by

$$\hat{k}_j \rightarrow \hat{k}_j + 1 : P_{\hat{k}_j,\hat{k}_j+1,t} = \frac{r - \hat{k}_j}{r} \frac{\beta_j + \hat{k}_j}{\theta + r - 1},$$

$$\hat{k}_j \rightarrow \hat{k}_j - 1 : P_{\hat{k}_j,\hat{k}_j-1,t} = \frac{\hat{k}_j}{r} \frac{(\theta - \beta_j) + (r - 1 - \hat{k}_j)}{\theta + r - 1},$$

$$\hat{k}_j \rightarrow \hat{k}_j : P_{\hat{k}_j,\hat{k}_j,t} = 1 - P_{\hat{k},\hat{k}-1,t} - P_{\hat{k},\hat{k}+1,t}.$$
We consider hopping from candidate $C_i$ to $C_j$.

\[
\hat{k}_i \rightarrow \hat{k}_i - 1, \hat{k}_j \rightarrow \hat{k}_j + 1 : P_{\hat{k}_i \rightarrow \hat{k}_i - 1, \hat{k}_j \rightarrow \hat{k}_j + 1, t} = \frac{\hat{k}_i \beta_j + \hat{k}_j}{r \theta + r - 1},
\]

\[
\hat{k}_i - 1 \rightarrow \hat{k}_i, \hat{k}_j + 1 \rightarrow \hat{k}_j : P_{\hat{k}_i - 1 \rightarrow \hat{k}_i, \hat{k}_j + 1 \rightarrow \hat{k}_j, t} = \frac{\hat{k}_j + 1 \beta_i + \hat{k}_i - 1}{r \theta + r - 1}.
\]

Here, we define \( \mu_r(\hat{k}, t) \) as a distribution function of the state \( \hat{k} \) at time \( t \). The number of all states is \( (r + 1) \). Given that the process is reversible, we have

\[
\frac{\mu_r(\hat{k}_i, \hat{k}_j, t)}{\mu_r(\hat{k}_i - 1, \hat{k}_j + 1, t)} = \frac{\hat{k}_j + 1 \beta_i + \hat{k}_i - 1}{\hat{k}_i \beta_j + \hat{k}_j}.
\] (A2)

We can separate indexes \( i \) and \( j \) and obtain

\[
\frac{\mu_r(i, \hat{k}_i, t)}{\mu_r(i - 1, \hat{k}_i + 1, t)} = \frac{\beta_i + \hat{k}_i - 1}{\hat{k}_i} c, \]

\[
\frac{\mu_r(j, \hat{k}_j, t)}{\mu_r(j - 1, \hat{k}_j + 1, t)} = \frac{\beta_j + \hat{k}_j}{\hat{k}_j + 1} c, \] (A3)

where \( c \) is a constant. Using (A3) sequentially, in the limit \( t \to \infty \), we can obtain the equilibrium distribution, which can be written as

\[
\mu_r(\hat{a}, \infty) = \left( \frac{\theta + r - 1}{r} \right)^{-1} K \prod_{j=1}^{\infty} \left( \frac{\beta_i + \hat{k}_i - 1}{\hat{k}_i} \right), \] (A4)

where \( \hat{a} = (\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_K) \). This distribution is written as

\[
\mu_r(\hat{a}, \infty) = \frac{r!}{\theta^r} \prod_{i=1}^{\infty} \frac{\beta_i^{[\hat{k}_i]}}{\hat{k}_i!}, \] (A5)

where \( x^{[n]} = x(x + 1) \cdots (x + n - 1) \). This is the Dirichlet multinomial distribution.

Here, we set \( \beta_j = \beta \). The relation \( \beta = -\alpha \) exists, where \( \alpha \) is the parameter used in the main text. We write (A4) as

\[
\mu_r(\hat{a}, \infty) = \left( \frac{\theta + r - 1}{r} \right)^{-1} r \prod_{j=1}^{r} \left( \beta + j - 1 \right)^{a_j}, \] (A6)
where $a_j$ is the number of candidates whom $j$ voters voted for and $\hat{a} = (a_1, \ldots, a_r)$. Hence, the relations $\sum_{i=1}^r a_i = K_r < K$ and $\sum_{i=1}^r ia_i = r$ exist. Here, we define $K_r$ as the number of candidates who have more than one vote. $(K - K_r)$ candidates have no vote.

We consider the partitions of integer $K_r$. To normalize, we add the term of combination: $K! / a_1! \cdots a_r! (K - K_r)!$. We obtain

$$
\mu_r(\hat{a}, \infty) = \frac{K!}{a_1! \cdots a_r! (K - K_r)!} \left( \theta + r - 1 \right)^{-1} \prod_{j=1}^r \left( \beta + j - 1 \right)^{a_j} \left( \frac{\theta}{\beta} \right)^{\frac{1}{a_j}} \prod_{j=1}^r \left( \frac{1 + \beta}{j!} \right)^{a_j} a_j!
$$

where $x^{[n: -\beta]} = x(x - \beta) \cdots (x - (n - 1)\beta)$. We use the relation $\theta = K\beta$. (A7) is nothing but the Pitman sampling formula [9].

In the limit $\beta \to 0$, $K \to \infty$, subject to a fixed $\theta = \beta K$, we can obtain Ewens sampling formula. In this case, the sum of probabilities that a candidate who has zero votes can obtain one vote is $\theta / (\theta + r)$. This is the case of $\alpha = 0$ in Section 2.

Appendix B: Data analysis of election data

In this appendix, we study the distribution of vote shares in the elections of Japans House of Representatives. Previously, we proposed a mechanical model that is based on a voting model and certain assumptions about the inhomogenities of candidates [26]. We describe the vote shares of candidates of a political party as the mixture of the votes of fixed supporters of the political party in the region and the votes of floating voters which obeys Dirichlet distribution. The former one becomes a source of inhomogeneity of the system. Here, we neglect the inhomogenities and treat all candidates equally.

1. Election data

We study the data from the 28th general election in 1958 to the 47th general election in 2014 and find a change in the election system. A middle constituency system was adopted prior to the 40th election, where the number of winners in each district is between three and five in general. In the case of the correction of congress seats, there are districts with 2–6
congress seats, which are rare cases. Following the 40th election in 1993, a small constituency system was installed and only one person was elected from each district. In the analysis of election data, we used the dataset \[27\] that records the elections results in several small regions in each electoral district. We separate the election data between before and after the introduction of a small constituency system. Next, we classify data on the basis of the number of congress seats \(W\) and candidates \(K\). Table III shows the sample number case \((W, K)\). Hereafter, we only study the case \((W, K)\) in which the sample number is more than \(10^3\). For \(W = 1\) small constituency system, we study \(K = 3, 4, 5\). For \(2 \leq W \leq 5\) middle constituency system, we examine \(K = 4, 5, 6, 7\) for \(W = 3\), \(K = 6, 7, 8, 9\) for \(W = 4\), and \(K = 6, 7, 8, 9, 10\) for \(W = 5\), respectively.

TABLE III. Number of samples in case \((K, W)\). Election data are classified by the number of congress seats \(W\) and that of candidates \(K\) in the electoral districts. The second column presents the results for the elections under the small constituency system. The third, fourth, and fifth columns provide the results for \(W = 3, 4, 5\) for the elections under the middle constituency system.

| \(K\) | \(W\) | 1   | 3   | 4   | 5   |
|-------|-------|-----|-----|-----|-----|
| 2     | 814   | NA  | NA  | NA  |
| 3     | 9,868 | NA  | NA  | NA  |
| 4     | 5,650 | 1,050 | NA  | NA  |
| 5     | 1,762 | 4,413 | 846 | NA  |
| 6     | 428   | 3,422 | 3,638 | 1,062 |
| 7     | 115   | 1,652 | 4,104 | 4,613 |
| 8     | 18    | 562  | 2,850 | 5,358 |
| 9     | 6     | 227  | 1,112 | 4,511 |
| 10    | NA    | 65   | 497  | 2,369 |
| 11    | NA    | 81   | 148  | 796  |

2. Parameter estimation

In the elections, the number of candidates \(K\) is fixed and the model parameters \(\theta, \alpha\) should satisfy \(\alpha < 0\) and \(\theta = -K\alpha\). We estimate the model parameter \(\theta\) using the maximum likelihood principle. As for \(r\), we adopt \(r = 50, 100\) and the Dirichlet limit \(r \to \infty\). For a finite \(r\), we transform the vote share \(v_i, i = 1, \cdots, K\) into votes \(\hat{k}_i, i = 1, \cdots, K\) as \(\hat{k}_i = \lfloor r \cdot v_i \rfloor\). Here, \(\lfloor x \rfloor\) is the floor function. As for the treatment of fractions \(r \cdot v_i - \hat{k}_i\), we distribute the remaining votes \(r - \sum \hat{k}_i\) to \(\hat{k}_i\) with the largest fractions.
The results are presented in Table IV. The estimated parameters $\theta$ for $(W, K)$ in the three cases are almost the same. The parameters are in zone III, which is introduced in Section III. In this region, it is difficult to be the stable leader. As the number of winners $W$ increases, the correlation decreases for the same number of candidates $K$. This means that the correlation strengthened following the introduction of the small constituency system in 1993.

To check the fit of the Pitman sampling formula (11) for $\theta = -K\alpha$, we plot the distribution of $\hat{k}_i$ for $r = 100$ and $i \leq 5$ for the 12 cases in Fig. 10. The middle constituency system $W \neq 1$ results show a good fit. On the other hand, this is not the case for the small constituency system, $W = 1$. This is because we neglected the inhomegeieties of the system which affects the voting results, particularly in the small constituency system.

Appendix C: Goodness-of-fit test for 2ch.net data

Given the observed dataset and Pitman sampling formula, we test whether the empirical data are truly drawn from the formula. A standard approach is to perform a goodness-of-fit test, which generates a p-value that quantifies the plausibility of the hypothesis. To adopt the procedure, it is necessary to introduce a measure of distance between the distribution of empirical data and Pitman sampling formula. This distance is compared with distance measurements for comparable synthetic datasets drawn from the Pitman sampling formula and the p-value is defined as the fraction of the synthetic distances that are larger than the empirical distance [28]. If the p-value is rather small, one can reject the Pitman sampling formula as a plausible fit to the data.

Pitman sampling formula provides a probability for each decomposition $\hat{a} = (a_1, a_2, \cdots, a_r)$ of integer $r$ as $\sum_j j a_j = r$. $K_r$ is defined as $K_r = \sum_j a_j$. We adopt the Kolmogorov–Smirnov (KS) statistics as a measure quantifying the distance between Pitman sampling formula and empirical data. We explain the procedure below, which is based on the goodness-of-fit test of power-laws [28].

- First, we randomly choose $S$ sequences of length $r$ from post data $\{n(t)\}, t = 1, \cdots, T$. We then randomly draw $S$ integers $t_n, n = 1, \cdots, S$ from $\{10^1, \cdots, T-r\}$ and choose $S$ sequences as $(n(t_n), n(t_n + 1), \cdots, n(t_n + r - 1))$. 

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FIG. 10. Plots of distribution of $c_k$ for $k \leq 5$ and $r = 100$. We choose three cases with the largest sample numbers for $W \in \{1, 3, 4, 5\}$.
We calculate the decomposition $\hat{a}_n$ for the sequence $(n(t_n), n(t_n+1), \cdots, n(t_n+r-1)$ and obtain the empirical distribution of the decomposition of $r$. We then use the maximum likelihood principle to estimate $(\theta, \alpha)$. In addition, we estimate the KS statistics for this fit. We define the KS statistics as the maximum distance between...
the cumulative distribution functions (CDFs) of the data and fitted model:

\[ D = \max |S(\hat{a}) - P(\hat{a})|. \]

Here, \( P_r(\hat{a}) \) is the CDF of Pitman sampling formula that best fits the data and \( S(\hat{a}) \) is the CDF of the data. In the calculation of CDFs, we order \( \hat{a} \) according to size.

- We generate Pitman sampling formula for the distributed synthetic datasets with parameters \( \theta, \alpha \) equal to those of the distribution that best fits the above data. The sample number is \( S \). We fit synthetic data to Pitman sampling formula and calculate the KS statistics. We repeat the procedure 2,500 times and obtain the same number of KS statistics. Then, we estimate the 90% point \( KS_{90\%} \) of the KS statistics distribution.

In general, we can count the fraction of the synthetic samples whose KS value is larger than the value for the empirical data and treat it as the p-value for the empirical data. Instead, we calculate the ratio \( KS/KS_{90\%} \), where \( KS \) in the numerator is the KS statistics for the empirical data. If the ratio is greater than one, the empirical data p-value is less than 10%. Thus, we can reject the hypothesis that the empirical data obeys Pitman sampling formula. Undoubtedly, even if the p-value is large, it does not guarantee that Pitman sampling formula is the correct distribution of the data. Some other model may prove a better fit to the data. In addition, if the sample size \( S \) is too small, the p-value could become large. We avoid the latter by adopting \( S = 3 \times 10^4 \).

In addition, we estimate ratio \( KS/KS_{90\%} \) because one can use the value as the proxy for the difference between Pitman sampling formula and empirical data.

We estimate ratio \( KS/KS_{90\%} \) for the first five boards. The number of partitions \( \hat{a} \) of \( r \) for \( r = 3 \) is 3, \( 3 = 1+1+1 = 2+1+3 \). We use two parameters as the fit and there is no degree of freedom. For \( r = 4 \), there are five partitions, \( 4 = 1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4 \). The degree of freedom that fits is \( 5 - 2 - 1 = 2 \). Here, we adopt a \( r \) that is greater than four as \( r \in \{4, 5, 6, 7, 8, 9, 10, 11, 14, 17, 20\} \). As \( r \) increases, the number of partitions rapidly increases. For \( r = 20 \), there are 627 passions and the remaining degree of freedom is 624.

We illustrate the results in Fig. 11. We can see that the ratio is less than one for the No. 1 board with \( r = 5, 6, 7, 8, 9, 11 \) and the No. 5 board with \( r = 4 \). For the large \( r \) for these two boards and other boards, Pitman’s sampling formula is rejected.
FIG. 11. Plot of $KS/KS_{90\%}$ vs. $r$ for the first five boards, No.1, 2, 3, 4, and 5. $r \in \{4, 5, 6, 7, 8, 9, 10, 11, 14, 17, 20\}$.

Appendix D: Additional information about BBS 2ch.net

In this appendix, we provide supplementary information about 2ch.net and the dataset studied here. 2ch.net is a collection of multiple BBS and was founded in May 1999. According to a NetRating survey, in 2009, the number of 2ch.net users was 11.7 million.

To obtain the post data for 2ch.net, we chose ten boards for several genres, as shown in Table I. The first five board genres are news and the other five belong to various genres. The program processes the HTML files and extracts the number of threads on the boards. All threads have a 10-digit ID, date of post, and ID of the user who made the post. The IDs are randomly assigned by 2ch.net to maintain user anonymity. If the thread is in sequence with a previous thread, we record the ancestors thread ID.

The data and R scripts to generate the figures presented in this study are available online: https://202.24.143.74/2ch. The data for No. $k$ boards are denoted by $k$.csv and comprise $(s(t), n(t), id(t)), t = 1, \cdots, T$ in three columns.