Disentangling mass and angle dependence in neutrino mixing

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Abstract. We show that it is possible to decompose the generator of mixing transformations for Dirac fields, in terms of a rotation and two ordinary Bogoliubov transformations, accounting for mass shift in the fermion fields. This result clarifies the nature of the mechanism which leads to the unitary inequivalence of the flavor and mass vacua.

1. Introduction

The phenomenon of neutrino mixing and oscillations [1, 2] is certainly one of the hottest topics in the physics beyond the Standard Model of elementary particle physics [3]. The study of particle mixing, in particular of neutrino mixing, has therefore attracted much attention in the past decades, especially in connection with the experimental observation of the mixing phenomenon. After Pontecorvo’s pioneer work, the theoretical basis of neutrino mixing has been studied in many details and the quantum field theory (QFT) formalism for mixed fields has been much developed [4]-[13]. The mixing phenomenon appears today to be a truly quantum field phenomenon, whose quantum mechanical approximation reproduces the Pontecorvo formalism and its results. Far from being a simple minded rotation among massive neutrinos, the field mixing appears to be the result of all the complex mathematical structure of QFT. The discovery of the unitary inequivalence between the massive neutrino vacuum and the flavored neutrino vacuum has displayed the condensate nature of the flavored vacuum. While the origin of the non vanishing neutrino masses remain a still open problem, the nature of the mixed, flavored vacuum seems to be clarified in many details. In the present report, by resorting to recent results [14], we analyze the apparently puzzling dependence of the mixing mechanism both, on the neutrino mass values and on the mixing angle. As well known, since Pontecorvo first formulation, in order to occur the neutrino mixing, neutrino masses need to be non zero and different among themselves, from one side; on the other side, the mixing angle must be, of course, non zero. The question then arises if any relation exists between the masses and the mixing angle, and which one is such a relation, if any. The result we obtain, to be presented in this report, is that a possibility exists to disentangle in the mixing transformation the dependence of the mixing generator on the angle form the one on the masses. The way this occurs shows the
crucial role played by the condensate of the flavored vacuum, thus reinforcing the QFT nature of the mixing phenomenon. Once the vacuum has acquired its condensate structure, it is no more invariant under rotation by the mixing angle \( \theta \).

Our analysis is limited to the case of two Dirac neutrinos. Extension to three neutrinos is in our plans. However, we have good reasons to believe that the present results are general, since our arguments are of algebraic nature. The results also extend to the mixing phenomenon of any particle, and are not limited to the case of Dirac neutrinos.

Since we will use them in our discussion, we need to introduce very few aspects and notations of the QFT mixing formalism, which is done in Section 2. In Section 3 the decomposition of the mixing generator is obtained in terms of elementary transformations and in Section 4 the energies of the various vacuum states associated to these transformations are considered. Section 5 is devoted to conclusions.

2. Fermion mixing

The mixing relations of two (Dirac) neutrino fields with definite flavors \( \nu_e, \nu_\mu \) are

\[
\nu_e(x) = \cos \theta \, \nu_1(x) + \sin \theta \, \nu_2(x) ; \quad \nu_\mu(x) = -\sin \theta \, \nu_1(x) + \cos \theta \, \nu_2(x) .
\]

\( \nu_1, \nu_2 \) are the (free) neutrino fields with definite masses \( m_1, m_2 \), respectively. \( \theta \) is the mixing angle. The fields \( \nu_1 \) and \( \nu_2 \) are expanded as

\[
\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{k,r} \left[ u_{k,i}^\tau(t) \alpha_{k,i}^\tau + u_{-k,i}^\tau(t) \beta_{-k,i}^\tau \right] e^{ikx}, \quad i = 1, 2 .
\]

where \( u_{k,i}^\tau(t) = e^{-i\omega_{k,ir} t} u_{k,i}^\tau \) and \( v_{k,i}^\tau(t) = e^{i\omega_{k,ir} t} v_{k,i}^\tau \), with \( \omega_{k,ir} = \sqrt{k^2 + m_i^2} \). The \( \alpha_{k,i}^\tau \) and the \( \beta_{k,i}^\tau \) are the annihilation operators for the vacuum state \( |0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2 \); \( \alpha_{k,i}^\tau |0\rangle_{1,2} = \beta_{k,i}^\tau |0\rangle_{1,2} = 0 \). The standard anticommutation relations are:

\[
\{ \nu_i^\alpha(x), \nu_j^\beta(y) \}_t = \delta^\beta(x - y) \delta_{\alpha\beta} \delta_{ij}, \quad \alpha, \beta = 1, \ldots, 4 ,
\]

\[
\{ \alpha_{k,i}^\tau, \alpha_{q,j}^\tau \} = \delta_{kq} \delta_{rs} \delta_{ij}; \quad \{ \beta_{k,i}^\tau, \beta_{q,j}^\tau \} = \delta_{kq} \delta_{rs} \delta_{ij}, \quad i,j = 1, 2 .
\]

Eqs.(1) can be recast as [4]:

\[
\nu_e^\alpha(x) = G_\theta^{-1}(t) \nu_1^\alpha(x) G_\theta(t) ; \quad \nu_\mu^\alpha(x) = G_\theta^{-1}(t) \nu_2^\alpha(x) G_\theta(t) ,
\]

where the generator \( G_\theta(t) \) is given by

\[
G_\theta(t) = \exp \left[ \theta \int d^4x \left( \nu_1^\alpha(x) \nu_2(x) - \nu_2^\alpha(x) \nu_1(x) \right) \right],
\]

\( G_\theta \) is an element of \( SU(2) \) [4]. It can be written as

\[
G_\theta(t) = \exp[\theta(S_+(t) - S_-(t))] \quad \text{with} \quad S_+(t) = S_-(t) \equiv \int d^4x \, \nu_1^\alpha(x) \nu_2(x).
\]

By introducing

\[
S_3 = \frac{1}{2} \int d^4x \left( \nu_1^\alpha(x) \nu_1(x) - \nu_2^\alpha(x) \nu_2(x) \right),
\]

the \( su(2) \) algebra is closed (at fixed \( t \)):

\[
[S_+(t), S_-(t)] = 2S_3, \quad [S_3, S_\pm(t)] = \pm S_\pm(t),
\]
By use of $G_\theta(t)$, the flavor fields can be expanded as:

$$\nu_\sigma(x) = \sum_{r=1,2} \int \frac{d^3 k}{(2\pi)^3} \left[ u_{k,i}^r(t) \alpha_{k,\sigma}^r(t) + v_{k,i}^r(t) \beta_{-k,\sigma}^r(t) \right] e^{ikx},$$  \hspace{1cm} (10)

with $(\sigma,i) = (e,1), (\mu,2)$. The flavor annihilation operators are defined as

$$\alpha_{k,\sigma}^r(t) \equiv G_\theta^{-1}(t) \alpha_{k,\sigma}^r G_\theta(t)$$

and

$$\beta_{-k,\sigma}^r(t) \equiv G_\theta^{-1}(t) \beta_{-k,\sigma}^r G_\theta(t).$$

For $k = (0,0,|k|)$, we have:

$$\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k^e(t) \alpha_{k,2}^r + e^r V_k(t) \beta_{-k,2}^r \right)$$  \hspace{1cm} (11)

$$\alpha_{k,\mu}^r(t) = \cos \theta \alpha_{k,2}^r - \sin \theta \left( U_k^\mu(t) \alpha_{k,1}^r - e^r V_k(t) \beta_{-k,1}^r \right)$$  \hspace{1cm} (12)

$$\beta_{-k,e}^r(t) = \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k^e(t) \beta_{-k,2}^r - e^r V_k(t) \alpha_{k,2}^r \right)$$  \hspace{1cm} (13)

$$\beta_{-k,\mu}^r(t) = \cos \theta \beta_{-k,2}^r - \sin \theta \left( U_k^\mu(t) \beta_{-k,1}^r + e^r V_k(t) \alpha_{k,1}^r \right)$$  \hspace{1cm} (14)

where $e^r = (-1)^{r+1}$ and

$$U_k(t) \equiv u_{k,2}^r(t) u_{k,1}^r(t) = v_{-k,1}^r(t) v_{-k,2}^r(t) = |U_k| e^{i(\omega_{k,2} - \omega_{k,1})t}$$  \hspace{1cm} (15)

$$V_k(t) \equiv e^r u_{k,1}^r(t) v_{-k,2}^r(t) = -e^r u_{k,2}^r(t) v_{-k,1}^r(t) = |V_k| e^{i(\omega_{k,2} + \omega_{k,1})t}$$  \hspace{1cm} (16)

$$|U_k| = \frac{|k|^2 + (\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}}, \quad |U_k|^2 + |V_k|^2 = 1.$$  \hspace{1cm} (17)

The action of the mixing generator on the vacuum $|0\rangle_{1,2}$ is non-trivial; at finite volume $V$ we have:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t) |0\rangle_{1,2}.$$  \hspace{1cm} (18)

In the infinite volume limit one obtains [4]

$$\lim_{V \to \infty} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{V} e^{i(\omega_{k,2} - \omega_{k,1})t} = 0.$$  \hspace{1cm} (19)

Eq.(19) expresses the unitary inequivalence between the flavor and the mass representations and shows the non-trivial nature of the mixing transformations (1) resulting in the condensate structure of the flavor vacuum [4, 13].

The form of the flavor vacuum (at $t = 0$) is the following one:

$$|0\rangle_{e,\mu} = \prod_{k,r} \left[ (1 - \sin^2 \theta |V_k|^2) - e^r \sin \theta \cos \theta |V_k| ((\alpha_{k,1}^r \beta_{-k,2}^r + \alpha_{k,2}^r \beta_{-k,1}^r) +$$

$$+ e^r \sin^2 \theta |V_k||U_k| ((\alpha_{k,1}^r \beta_{-k,1}^r - \alpha_{k,2}^r \beta_{-k,2}^r) + \sin^2 \theta |V_k|^2 (\alpha_{k,1}^r \beta_{-k,2}^r + \beta_{-k,1}^r)) |0\rangle_{1,2}$$  \hspace{1cm} (20)

from which it is evident the condensate nature made of particle-antiparticle pairs with same or different masses.

The condensation density of the flavor vacuum is given by

$$e^r (0(t)|\alpha_{k,i}^r \alpha_{k,i}^r|0(t))_{e,\mu} = e^r (0(t)|\beta_{-k,i}^r \beta_{-k,i}^r|0(t))_{e,\mu} = \sin^2 \theta |V_k|^2,$$  \hspace{1cm} (21)

with the same result for antiparticles. Note that the $|V_k|^2$ has a maximum at $\sqrt{m_1 m_2}$ and $|V_k|^2 \approx \frac{(m_1 - m_2)^2}{4|k|^2}$ for $|k| \gg \sqrt{m_1 m_2}$.

Finally, let us consider the explicit expression of the mixing generator, obtained by use of the expressions for the free fields $\nu_i$:

$$S_{k}^{i,r} = \left( U_k^i \alpha_{k,1}^r \alpha_{k,2}^r - e^r V_k \beta_{-k,1}^r \beta_{-k,2}^r + e^r V_k \beta_{-k,1}^r \alpha_{k,2}^r + U_k \beta_{-k,1}^r \beta_{-k,2}^r \right)$$  \hspace{1cm} (22)

where we defined $S_{\pm} = \sum_k S_{k}^{i,r}$, with $S_{+}^{i} = S_{-}^{i} \equiv \sum_r S_{k}^{i,r}$.
3. Decomposition of the mixing generator

The mixing generator $G_\theta(t)$, Eq.(6), is a function of $m_1$, $m_2$, and $\theta$. Our purpose is now to disentangle the mass dependence of $G_\theta(t)$ from its dependence on the rotation angle\(^1\). In order to do that, let us define [14]:

$$R(\theta) = \exp\left\{ \theta \sum_{k,r} \left[ \left( \alpha^r_{k,1} \alpha^r_{k,2} + \beta^r_{-k,1} \beta^r_{-k,2} \right) e^{i\psi_k} - \left( \alpha^r_{k,2} \alpha^r_{k,1} + \beta^r_{-k,2} \beta^r_{-k,1} \right) e^{-i\psi_k} \right] \right\}, \quad (23)$$

$$B_i(\Theta_i) = \exp\left\{ \sum_{k,r} \Theta_{k,i} e^r \left[ \alpha^r_{k,i} \beta_{-k,i} e^{-i\phi_k} - \beta^r_{-k,i} \alpha_{k,i} e^{i\phi_k} \right] \right\}, \quad i = 1, 2. \quad (24)$$

Here the $B_i(\Theta_i)$, $i = 1, 2$, are the generators of Bogoliubov transformations and $R(\theta)$ is the rotation generator. Since $[B_1(\Theta_1), B_2(\Theta_2)] = 0$, we may also define $B(\Theta_1, \Theta_2) \equiv B_2(\Theta_2)B_1(\Theta_1)$ and then calculate how ladder operators transform under $B(\Theta_1, \Theta_2)$:

$$\tilde{\alpha}^r_{k,i} \equiv B^{-1}(\Theta_1, \Theta_2) \alpha^r_{k,i} B(\Theta_1, \Theta_2) = \cos \Theta_{k,i} \alpha^r_{k,i} - \sin \Theta_{k,i} \beta^r_{-k,i}$$

$$\tilde{\beta}^r_{-k,i} \equiv B^{-1}(\Theta_1, \Theta_2) \beta^r_{-k,i} B(\Theta_1, \Theta_2) = \cos \Theta_{k,i} \beta^r_{-k,i} + \sin \Theta_{k,i} \alpha^r_{k,i} \quad (25)$$

On the other hand, the action of the rotation generator on the annihilation operators gives:

$$R(\theta)^{-1} \alpha^r_{k,1} R(\theta) = \cos \theta \alpha^r_{k,1} + e^{i\psi_k} \sin \theta \alpha^r_{k,2}$$

$$R(\theta)^{-1} \alpha^r_{k,2} R(\theta) = \cos \theta \alpha^r_{k,2} - e^{-i\psi_k} \sin \theta \alpha^r_{k,1} \quad (26)$$

and similar ones for the $\beta^r_{k,i}$.

The action of the above transformations on the vacuum for the free fields $\nu_1$, $\nu_2$ is given by

$$\langle 0 \rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2) |0\rangle_{1,2} = \prod_{i=1,2} \prod_{k,r} \left[ \cos \Theta_{k,i} + e^{i\phi_k} \sin \Theta_{k,i} \alpha^r_{k,i} \beta^r_{-k,i} \right] |0\rangle_{1,2} \quad (29)$$

$$R^{-1}(\theta) |0\rangle_{1,2} = |0\rangle_{1,2}. \quad (30)$$

Define now $\tilde{R} \equiv \tilde{R}(\theta, \Theta_1, \Theta_2) = B^{-1}(\Theta_1, \Theta_2) R^{-1}(\theta) B(\Theta_1, \Theta_2)$, which can be written as

$$\tilde{R} = \exp \left\{ \theta \sum_{k,r} \left[ \left( \tilde{\alpha}^r_{k,1} \tilde{\alpha}^r_{k,2} + \tilde{\beta}^r_{-k,1} \tilde{\beta}^r_{-k,2} \right) e^{i\psi_k} - \left( \tilde{\alpha}^r_{k,2} \tilde{\alpha}^r_{k,1} + \tilde{\beta}^r_{-k,2} \tilde{\beta}^r_{-k,1} \right) e^{-i\psi_k} \right] \right\}, \quad (31)$$

Thereby, using Eqs.(25), (26) and imposing the following constraint

$$U_k(t) = e^{i\psi_k} \cos(\Theta_{k,1} - \Theta_{k,2}) \quad (32)$$

$$V_k(t) = e^{\left(\Theta_{k,1} + \Theta_{k,2}\right) \frac{\psi_k}{2}} \sin(\Theta_{k,1} - \Theta_{k,2}) \quad (33)$$

we obtain

$$\tilde{R} = \exp \left\{ \sum_r \left( U_k^* \alpha_{k,1} \alpha_{k,2} - e^r V_k^* \beta_{-k,1} \alpha_{k,2} + e^r V_k \alpha_{k,1} \beta_{-k,2} + U_k \beta_{-k,1} \beta_{-k,2} \right) \right\} \quad (34)$$

\(^1\) A first (partial) attempt in doing this is contained in Ref.[4].
which indeed coincides with the expression of the mixing generator $G_\theta(t)$, cf. Eq.(22), provided we make the following identifications:

$$
\psi_k = (\omega_{k,1} - \omega_{k,2}) t \\
\phi_{k,i} = 2 \omega_{k,i} t \\
\Theta_{k,i} = \frac{1}{2} \cot^{-1} \left( \frac{|k|}{m_i} \right)
$$

In definitive, we have shown that it is possible to decompose the mixing generator $G_\theta$ in the following way (for $t = 0$)

$$
G(\theta, m_1, m_2) = B^{-1}_0(\Theta_1, \Theta_2) R(\theta) B(\Theta_1, \Theta_2)
$$

i.e., we have been able to write $G$ as a product of operators depending only on the masses or on the mixing angle. From the above decomposition, it is quite evident that the non-trivial nature of the mixing generator arises as a consequence of the non-commutativity of the rotation generator with the generator of Bogoliubov transformation(s).

From Eq.(35), we see that $\Theta_{k,i}$ are functions of the masses and the momentum only. Thus we can regard the generator $B(\Theta_1, \Theta_2)$, where the momentum has been integrated out, as dependent on the mass parameters, i.e. as $B(m_1, m_2)$. Then we rewrite Eq.(39) as

$$
G_t(\theta, m_1, m_2) = B^{-1}_t(m_1, m_2) R_t(\theta) B_t(m_1, m_2)
$$

where we have included now the dependence on time as an index $t$.

Finally, let us consider the flavor vacuum (for $t = 0$), which can be written as

$$
|0\rangle_{e,\mu} \equiv G^{-1} |0\rangle_{1,2} = |0\rangle_{1,2} + \left[ B(m_1, m_2) , R^{-1}(\theta) \right] |\tilde{0}\rangle_{1,2}
$$

where we see as the condensate structure arises as a consequence of the non vanishing commutator $[B, R^{-1}]$.

4. Vacuum energies

Having shown that the mixing generator can be written as $G = B^{-1} R B$, we use now such a decomposition to investigate the nature of the flavor vacuum, starting from $|0\rangle_{1,2}$ and acting on it with the above Bogoliubov transformations and rotation generators. In the following table we report the vacuum expectation values of the (unordered) Hamiltonian on the various vacua obtained by acting step-by-step with the above generators.

| $\langle H_{k,1} + H_{k,2} \rangle$ | State |
|---------------------------------|-------|
| $-(\omega_{k,1} + \omega_{k,2})$ | $|0\rangle_1 \otimes |0\rangle_2 \equiv |0\rangle_{1,2}$ |
| $-(k + k)$ | $B^{-1}(m_1, m_2) |0\rangle_{1,2} \equiv |\tilde{0}\rangle_{1,2}$ |
| $-k(2 \cos^2 \theta + \left( \frac{\omega_{k,1}}{\omega_{k,2}} + \frac{\omega_{k,2}}{\omega_{k,1}} \right) \sin^2 \theta)$ | $R^{-1}(\theta) B^{-1}(m_1, m_2) |0\rangle_{1,2} = R^{-1}(\theta) |\tilde{0}\rangle_{1,2}$ |
| $-(\omega_{k,1} + \omega_{k,2})(1 - 2 \sin^2 \theta \sin^2 (\Theta_{k,1} - \Theta_{k,2}))$ | $B(m_1, m_2) R^{-1}(\theta) B^{-1}(m_1, m_2) |0\rangle_{1,2} \equiv |0\rangle_{e,\mu}$ |

From the results shown in the table we can understand the steps that lead to the non-equivalence between the two vacua. Starting from the vacuum for the free fields $|0\rangle_{1,2}$, the first Bogoliubov
transformation(s) acts as a mass shift that brings the two species of neutrinos to have a massless vacuum. The second transformation that acts is a rotation, then it follows the inverse Bogoliubov transformation which however cannot put back masses at their original position, since in the meanwhile they have been “rotated” by \( R \). This mechanism is essentially due to the non vanishing commutator \([B, R^{-1}]\).

The situation is clarified by Fig. 1, where we plot, for sample value of the parameters, the (absolute values of) expectation values of \( H_{k,1} \) and \( H_{k,2} \) for the above vacua. The arrows indicate the “way” from \(|0\rangle_{1,2} \) to \(|0\rangle_{e,\mu} \), making clear the origin of the energy gap between these two vacua.

In the following figures, we consider other values of the parameters, showing various limits in which the effects of the condensate tends to be less important and eventually to disappear.

**Figure 1.** Plot of vacuum expectation values of \( H_1 \) and \( H_2 \) for the states of Table 1. Sample values of parameters are chosen for \( \theta = \pi/4, k = 80, m_1 = 20, m_2 = 150 \). We put \( A = |0\rangle_{1,2}, B = |\tilde{0}\rangle_{1,2}, C = R^{-1}(\theta)|\tilde{0}\rangle_{1,2}, \) and \( D = |0\rangle_{e,\mu} \).

**Figure 2.** Plot of vacuum expectation values of \( H_1 \) and \( H_2 \) for the states of Table 1. Sample values of parameters are chosen for \( \theta = \pi/4, k = 300, m_1 = 20, m_2 = 150 \).
Figure 3. Plot of vacuum expectation values of $H_1$ and $H_2$ for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/10$, $k = 80$, $m_1 = 20$, $m_2 = 150$.

Figure 4. Plot of vacuum expectation values of $H_1$ and $H_2$ for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/40$, $k = 80$, $m_1 = 20$, $m_2 = 150$.

5. Conclusions

We have shown that it is possible to decompose the generator of flavor mixing transformations for two Dirac fields in terms of a rotation, parameterized only by the mixing angle $\theta$, and two Bogoliubov transformations, depending only on the masses $m_1$ and $m_2$. This result allows for a better understanding of the mixing phenomenon and the associated inequivalence among the mass and flavor representations (Hilbert spaces), in terms of the non-commutativity of the two transformations (Bogoliubov and rotation).

It is interesting to note that the Bogoliubov transformations appearing in the above decomposition are responsible for mass shift in the fermion fields, and are of the same type of those used in Ref.[15], where the generation of masses and mixing was studied in the context of dynamical symmetry breaking.

Finally, the energy gap between the flavor and mass vacua can be associated to the entropy of the flavor vacuum. Such an issue will be discussed in detail in a separate publication [14].
Figure 5. Plot of vacuum expectation values of $H_1$ and $H_2$ for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/4$, $k = 0$, $m_1 = 20$, $m_2 = 150$.

References
1. Bilenky S M and Pontecorvo B 1978 Phys. Rept. 41 225
2. Giunti C and Kim C W 2007 Fundamentals of Neutrino Physics and Astrophysics, (Oxford University Press)
3. Cheng T and Li L 1989 Gauge Theory of Elementary Particle Physics (Clarendon Press)
4. Blasone M and Vitiello G 1995 Ann. Phys. (N.Y.) 244 283
5. Blasone M, Henning P A and Vitiello G 1996 In La Thuile 1996, Results and perspectives in particle physics 139 [hep-ph/9605335].
6. Binger M and Ji C R 1999 Phys. Rev. D 60 056005
7. Blasone M, Henning P A and Vitiello G 1999 Phys. Lett. B 451 140
8. Blasone M, Jizba P and Vitiello G 2001 Phys. Lett. B 517 471
9. Blasone M, Capolupo A and Vitiello G 2002 Phys. Rev. D 66, 025033
10. Beuthe M 2003 Phys. Rept. 375 105
11. Blasone M, Di Mauro M and Vitiello G 2011 Phys. Lett. B 697 238
12. Blasone M 2011 J. Phys. Conf. Ser. 306 012037
13. Blasone M, Jizba P and Vitiello G 2011 Quantum Field Theory and its Macroscopic Manifestations - (World Scientific & ICP)
14. Blasone M, Gargiulo M V and Vitiello G 2015, in preparation
15. Blasone M, Jizba P, Lambiase G and Mavromatos N E 2014 J.Phys.Conf.Ser. 538 012003.