Einstein-Maxwell and Einstein-Proca theory from a modified gravitational action

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Abstract

A modified gravitational action is considered which involves the quantity $F_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$, where $\Gamma_\mu = \Gamma^\alpha_{\mu\alpha}$. Since $\Gamma_\mu$ transforms like a $U(1)$ gauge field under coordinate transformations terms such as $F_{\mu\nu} F^{\mu\nu}$ are invariant under coordinate transformations. If such a term is added to the usual gravitational action the resulting field equations, obtained from a Palatini variation, are the Einstein-Proca equations. The vector field can be coupled to point charges or to a complex scalar density of weight $ie$, where $e$ is the charge of the field. If this scalar density is taken to be $g^{-ie/2}$ and the overall factor of the scalar density Lagrangian takes on a particular value the resulting field equations are the Einstein-Maxwell equations.
Introduction

In the standard formulation of the general theory of relativity non-gravitational fields are not related to the geometry of space-time, but are fields that exist in space-time. In this paper I will show that both massive and massless geometrical vector fields can be obtained in the Palatini formalism from a modified gravitational action.

The contraction of the connection $\Gamma_\mu = \Gamma^\alpha_{\mu\alpha}$ is not a vector but has the following transformation law

$$\bar{\Gamma}_\mu = \partial x^\nu \Gamma_\nu + \frac{1}{e} \frac{\partial \chi}{\partial x^\mu} \left| \frac{\partial x}{\partial \bar{x}} \right|, \quad (1)$$

where $|\partial x/\partial \bar{x}|$ is the Jacobian of the transformation. It is interesting to compare this transformation law to that of a $U(1)$ gauge field $A_\mu$ under a combined gauge and coordinate transformation. Under a gauge transformation

$$A'_\mu = A_\mu + \frac{1}{e} \frac{\partial \chi}{\partial x^\mu}, \quad (2)$$

where $\chi$ is an arbitrary function and $e$ is the charge of the field. Performing a coordinate transformation gives

$$\bar{A}_\mu = \partial x^\nu A_\nu + \frac{1}{e} \frac{\partial \chi}{\partial x^\mu}. \quad (3)$$

This is the same as (1) if we take $\chi = \frac{1}{e} \ln |\partial x/\partial \bar{x}|$. Thus, under a coordinate transformation the field $\Gamma_\mu$ behaves like a $U(1)$ gauge field under a combined gauge and coordinate transformation. This implies that the quantity $F_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$ is a tensor under general coordinate transformations. In general relativity $\Gamma_\mu = \partial \ln (\sqrt{g})/\partial x^\mu$ and the associated field strength $F_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$ vanishes because $\Gamma_\mu$ is a pure gauge. It is possible however, that in modified theories this is not the case. In fact, I have shown [1] that $\Gamma_\mu$ is not necessarily a pure gauge in Born-Infeld-Einstein theory. In this paper I consider adding a term of the form $F_{\mu\nu} F^{\mu\nu}$ to the standard gravitational action and show that, under a Palatini variation

$$\bar{\Gamma}_\mu = \partial ln(\sqrt{g})/\partial x^\mu \Gamma_\mu + V_\mu, \quad (4)$$

where $V_\mu$ is an arbitrary vector. The field equation satisfied by $V_\mu$ is shown to be the Proca equation and the gravitational field equations are the Einstein-Proca equations.

Two types of sources for the vector field are considered. First it is shown that point charges can be coupled to the field. Then, following the formalism for Abelian gauge fields, it is shown that a complex scalar density of weight $ie$ can be coupled to the vector field. Here $e$ is the charge of the field that appears in the covariant derivative. There is a geometric quantity, namely $g^{-ie/2}$, that is a scalar density of weight $ie$. The Lagrangian for this scalar density is

$$L = -\frac{1}{4} \beta \sqrt{g} g^{-2} \nabla_\mu g \nabla^\mu g, \quad (5)$$

where $\beta$ is a constant. It is shown that if $\beta = -3/(4\kappa)$ the vector field is massless and we obtain the Einstein-Maxwell equations.
The Field Equations

The field equations follow from the variation of the action

\[ S = \int \left[ -\frac{1}{2\kappa} R - \frac{\alpha}{4} F^{\mu\nu} F_{\mu\nu} + L_M \right] \sqrt{g} \, d^4x , \]  

(6)

where

\[ R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\nu_{\mu\alpha} + \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\alpha\nu} - \Gamma^\beta_{\mu\nu} \Gamma^\alpha_{\beta\alpha} , \]  

(7)

\[ F_{\mu\nu} = \partial_\mu \Gamma^\nu_{\nu} - \partial_\nu \Gamma^\mu_{\nu} , \]  

(8)

\[ R = g^{\mu\nu} R_{\mu\nu} , \Gamma_\mu = \Gamma^\alpha_{\mu\alpha} , \kappa = 8\pi G , L_M \text{ is the matter Lagrangian and the connection is} \]  

\[ \text{taken to be symmetric. Here I will use a Palatini variation of the action, which treats} \]  

\[ g_{\mu\nu} \text{ and } \Gamma^\alpha_{\mu\nu} \text{ as independent variables. Varying the action with respect to} \]  

\[ g_{\mu\nu} \text{ gives} \]  

\[ G_{(\mu\nu)}(\Gamma) = -\alpha \kappa \left[ F_{\mu\alpha} F_{\nu}^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right] - \kappa T_{\mu\nu} , \]  

(9)

where \( G_{(\mu\nu)}(\Gamma) \) is the symmetric part of the Einstein tensor, which depends on \( \Gamma^\alpha_{\mu\nu} \), and \( T_{\mu\nu} \) is the energy-momentum tensor of the matter.

Varying the action with respect to \( \Gamma^\alpha_{\mu\nu} \) gives

\[ \nabla_\beta (\sqrt{g} g^{\mu\nu}) - \frac{1}{2} \left\{ \left[ \nabla_\beta (\sqrt{g} g^{\beta\mu}) + 2 \alpha \kappa \sqrt{g} \nabla_\beta F^{\beta\mu} \right] \delta^\nu_\alpha + \left[ \nabla_\beta (\sqrt{g} g^{\beta\nu}) + 2 \alpha \kappa \sqrt{g} \nabla_\beta F^{\beta\nu} \right] \delta^\mu_\alpha \right\} = 0 , \]  

(10)

where \( \nabla \) is the covariant derivative with respect to the Christoffel symbol. Contracting over \( \mu \) and \( \alpha \) gives

\[ \nabla_\beta (\sqrt{g} g^{\beta\mu}) = -\frac{10}{3} \alpha \kappa \sqrt{g} \nabla_\beta F^{\beta\mu} \]  

(11)

and substituting this into (10) gives

\[ \nabla_\alpha (\sqrt{g} g^{\mu\nu}) - \frac{1}{5} \left[ \delta^\nu_\alpha \nabla_\beta (\sqrt{g} g^{\beta\mu}) + \delta^\mu_\alpha \nabla_\beta (\sqrt{g} g^{\beta\nu}) \right] = 0 . \]  

(12)

Since the trace of the left hand side of (12) vanishes there are four too few equations and the system is under determined. Thus, we expect four arbitrary functions in the solution. Such a solution is given by

\[ \nabla_\alpha \left[ \sqrt{g} g^{\mu\nu} \right] = -\sqrt{g} \left[ \delta^\mu_\alpha V^\nu + \delta^\nu_\alpha V^\mu \right] , \]  

(13)

where \( V^\mu \) is an arbitrary vector. The connection that follows from this set of equations is given by [1]

\[ \Gamma^\alpha_{\mu\nu} = \left\{ \begin{array}{c} \alpha \\ \mu \nu \end{array} \right\} - \frac{1}{2} \left[ 3 g_{\mu\nu} V^\alpha - \delta^\alpha_\mu V^\nu - \delta^\alpha_\nu V^\mu \right] , \]  

(14)

where the first term on the right hand side is the Christoffel symbol. Thus,

\[ \Gamma^\mu_\mu = \frac{\partial \ln(\sqrt{g})}{\partial x^\mu} + V_\mu \]  

(15)
and \( F_{\mu\nu} = \hat{\nabla}_\mu V_\nu - \hat{\nabla}_\nu V_\mu \).

Substituting (13) into (11) gives
\[
\hat{\nabla}_\beta F^{\beta\mu} = \frac{3}{2\alpha \kappa} V^\mu ,
\]
so that \( V_\mu \) satisfies the Proca equation and has mass \( \sqrt{3/(2\alpha \kappa)} \). This implies that \( \Gamma_\mu \) satisfies the field equation
\[
\hat{\nabla}_\beta F^{\beta\mu} = \frac{3}{2\alpha \kappa} \left( \Gamma^\mu - \frac{\partial \ln(\sqrt{g})}{\partial x^\mu} \right) ,
\]
where \( F_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu \) is a tensor. Thus, \( \Gamma_\mu \) satisfies the Proca equation with a source. Note that even though \( \Gamma_\mu \) is a massive field the field equations are invariant under the transformation (1) since the source transforms in a similar way.

To see that (9) are the gravitational field equations associated with the Proca field it is necessary to express them in terms of the Einstein tensor \( \tilde{G}_{\mu\nu} \), which is defined in terms of the Christoffel symbol. The relationship between the Ricci tensors is given by
\[
R_{\mu\nu}(\Gamma) = \tilde{R}_{\mu\nu} - \frac{3}{2} \left[ V_\mu V_\nu - \frac{1}{4} g_{\mu\nu} \left( 2 \tilde{\nabla}_\alpha V^\alpha + V^\alpha V_\alpha \right) \right] .
\]
From (16) it is easy to see that \( \tilde{\nabla}_\alpha V^\alpha = 0 \). Thus, the Einstein field equations are given by
\[
\tilde{G}_{\mu\nu} = -\alpha \kappa \left[ F_{\mu\alpha} F^{\alpha}_\nu - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + \frac{3}{2\alpha \kappa} (V_\mu V_\nu - \frac{1}{2} g_{\mu\nu} V^\alpha V_\alpha) \right] - \kappa T_{\mu\nu}
\]
and are the field equations for a Proca field plus any additional matter that contributes to \( T_{\mu\nu} \). Note that if \( \alpha \sim 1 \) the mass of the vector field is of order of the Planck mass.

It is interesting to note that the transformation law of the full connection
\[
\tilde{\Gamma}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \bar{x}^\lambda} \frac{\partial x^\sigma}{\partial \bar{x}^\alpha} \frac{\partial x^\kappa}{\partial \bar{x}^\beta} \Gamma^\lambda_{\sigma\kappa} + \frac{\partial}{\partial \bar{x}^\alpha} \left[ \frac{\partial x^\sigma}{\partial \bar{x}^\beta} \right] \frac{\partial x^\mu}{\partial \bar{x}^\alpha}
\]
is identical with the transformation law for the Yang-Mills potential
\[
(A_{\mu})_i^j = \frac{\partial x^\nu}{\partial \bar{x}^\mu} S_i^j (A_{\nu})_m^l (S^{-1})_k^m - \frac{i}{e} \frac{\partial S^i}{\partial x^\mu} (S^{-1})_k^m .
\]
if we take
\[ S^i_{\ k} = \frac{\partial x^i}{\partial \bar{x}^k} \] (24)
and
\[ \Gamma^l_{\ alpha k} = i e (A_{\alpha})^l_{\ k} \] (25)
Note that both Latin and Greek indices are space-time indices and run from 0 to 3. The field strength tensor is proportional to the Riemann tensor, but the action is taken to be proportional to \( R \), not to the Riemann tensor squared (see [2] for a more detailed discussion on the Yang-Mills fields associated with diffeomorphisms). Thus, we can think of general relativity as the gauge theory associated with diffeomorphisms (see [3] for alternative approaches to gauge theories of gravity).

**Sources of the vector field**

I will first consider coupling point particles to the vector field. The simplest coupling is given by
\[ L_c = -\alpha \sqrt{g} \Gamma^\mu J^\mu \] (26)
where \( J^\mu \) is the conserved current associated with the source. For a point particle with “charge” \( e \)
\[ J^\mu(x^\alpha) = \frac{e}{\sqrt{g}} \int \frac{dx^\mu(p)}{dp} \delta(x^\alpha - x^\alpha(p)) dp \] (27)
where \( p \) is a parameterization of the particle's world line. Of course, the particles that produce \( J^\mu \) must also appear in \( L_M \).

At first sight it may appear that there is a problem with this Lagrangian since \( \Gamma^\mu \) is not a vector. An analogous situation occurs in electrodynamics where the interaction Lagrangian \( \sqrt{g} A_\mu J^\mu \) appears not to be gauge invariant. However, if \( \bar{\nabla}_\mu J^\mu = 0 \) the Lagrangian only changes by a total derivative under a gauge transformation. Now, under a coordinate transformation
\[ \bar{\Gamma}^\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \Gamma^\nu + \frac{\partial}{\partial \bar{x}^\mu} \ln \left| \frac{\partial x}{\partial \bar{x}} \right| \] (28)
where \( |\partial x/\partial \bar{x}| \) is the Jacobian of the transformation. This is analogous to a gauge transformation and it is easy to see that the Lagrangian \( L_c \) only changes by a total derivative if \( \bar{\nabla}_\mu J^\mu = 0 \).

Varying the action with respect to \( g_{\mu\nu} \) gives (9) and varying the action with respect to \( \Gamma^\alpha_{\ \mu\nu} \) gives (10) with
\[ \bar{\nabla}_\beta F^{\beta\mu} \rightarrow \bar{\nabla}_\beta F^{\beta\mu} - J^\mu \] (29)
One interesting property of this choice of Lagrangian is that the current \( J^\mu \) does not enter into the equation defining the connection, so that (14) is still valid. The field \( F^{\mu\nu} \)
now satisfies the equation
\[ \tilde{\nabla}_\beta F^{\beta \mu} = \frac{3}{2\alpha \kappa} V^\mu + J^\mu \]  
which is the Proca equation with source \( J^\mu \). This equation tells us that \( \tilde{\nabla}_\mu V^\mu = 0 \) and that (21) holds. The field equations therefore correspond to a Proca equation with source minimally coupled to gravity.

Next consider coupling a complex scalar density to the vector field. In a \( U(1) \) gauge theory consisting of a complex scalar field \( \phi \) coupled to a gauge field \( A_\mu \) the fields transform as
\[ \tilde{\phi} = e^{-i\chi} \phi, \]  
\[ \tilde{A}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} A_\nu + \frac{1}{e} \frac{\partial \chi}{\partial \tilde{x}^\mu}, \]  
under a combined gauge and coordinate transformation, the covariant derivative of \( \phi \) is given by
\[ D_\mu \phi = (\partial_\mu + ieA_\mu)\phi, \]  
and the Lagrangian is given by
\[ L = \frac{1}{2} D_\mu \phi D^\mu \phi^* - \frac{1}{2} m^2 \phi^* \phi - \frac{1}{4} F^{\mu \nu} F_{\mu \nu}, \]  
where
\[ D_\mu \phi^* = (\partial_\mu - ieA_\mu) \phi^*. \]  
In the theory presented here \( A_\mu \) will be replaced by \( \Gamma_\mu \). From (28) and (32) we see that
\[ \chi = e \ln \left| \frac{\partial x}{\partial \tilde{x}} \right| \]  
and from (42) we see that
\[ \tilde{\phi} = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{-ie} \phi. \]  
Thus, \( \phi \) is a scalar density of complex weight \( ie \). With \( D_\mu \phi \) defined as
\[ D_\mu = (\partial_\mu + ie\Gamma_\mu) \phi \]  
it is easy to show that
\[ \tilde{D}_\mu \tilde{\phi} = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{-ie} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} D_\alpha \phi. \]  
In fact, \( D_\mu \phi \) is just the covariant derivative of the tensor density \( \phi \) with respect to the connection \( \Gamma_\mu^\alpha \) and I will write \( \nabla_\mu \phi \) instead of \( D_\mu \phi \). The Lagrangian (34) is therefore a scalar under coordinate transformations.
The action for the theory will therefore be given by (6) with \( L_M \) given by the scalar density terms in (34). Varying the action with respect to \( \phi^* \) gives
\[
\nabla_\mu \left[ \sqrt{g} \nabla^\mu \phi \right] - m^2 \phi = 0 .
\]
(40)

Using
\[
\nabla_\mu \sqrt{g} = -\sqrt{g} V_\mu
\]
gives the field equation
\[
\nabla_\mu \nabla^\mu \phi = V_\mu \nabla^\mu \phi - m^2 \phi = 0 .
\]
(42)
The field equations that follow from varying the action with respect to \( g_{\mu\nu} \) are
\[
G_{\mu\nu} = -\alpha \kappa \left[ F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right] - \kappa \left[ \nabla_{(\mu} \phi \nabla_{\nu)} \phi^* - \frac{1}{2} g_{\mu\nu} \left( \nabla_\alpha \phi \nabla^\alpha \phi^* + m^2 \phi^* \phi \right) \right].
\]
(43)

Varying the action with respect to \( \Gamma^\alpha_{\mu\nu} \) gives (10) with
\[
\nabla_\beta F^{\beta\mu} \to \nabla_\beta F^{\beta\mu} - J^\mu ,
\]
(44)

where
\[
J^\mu = \frac{ie}{2\alpha} \left[ \phi \nabla_\mu \phi^* - \phi^* \nabla_\mu \phi \right] .
\]
(45)
The field \( F^{\mu\nu} \) satisfies the equation
\[
\nabla_\beta F^{\beta\mu} = \frac{3}{2\alpha \kappa} V^\mu + J^\mu ,
\]
(46)
which is the Proca equation with source \( J^\mu \). From (42) and (45) it is easy to see that
\[
\nabla_\mu J^\mu = V_\mu J^\mu ,
\]
(47)
which implies that
\[
\nabla_\mu J^\mu = 0 .
\]
(48)

Equations (46) and (48) give
\[
\nabla_\mu V^\mu = 0 .
\]
(49)

This together with (20) and (43) gives
\[
\tilde{G}_{\mu\nu} = -\alpha \kappa \left[ F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right] + \frac{3}{2\alpha \kappa} (V_\nu V_\mu - \frac{1}{2} g_{\mu\nu} V_\alpha V_\alpha)
\]
\[-\kappa \left[ \nabla_{(\mu} \phi \nabla_{\nu)} \phi^* + \frac{3}{2} g_{\mu\nu} \left( \nabla_\alpha \phi \nabla^\alpha \phi^* + m^2 \phi^* \phi \right) \right].
\]
(50)

It is interesting to note that
\[
\nabla_\mu \phi = \left( \tilde{\nabla}_\mu + ie V_\mu \right) \phi \equiv \tilde{D}_\mu \phi ,
\]
(51)
where \( \tilde{D}_\mu \) is the standard covariant derivative operator with respect to the Christoffel symbol and that equation (42) can be written as
\[
\tilde{D}_\mu \tilde{D}^\mu \phi - m^2 \phi = 0 .
\]
(52)
Thus, the above set of equations represents Einstein gravity minimally coupled to a massive vector field \( V_\mu \) and a complex scalar density field \( \phi \).
A geometric scalar density and the Einstein-Maxwell equations

A simple geometric quantity that is a scalar density of weight \( ie \) is \( g^{-ie/2} \). In this case the scalar density is not an independent variable and the field equations will differ from the ones given in the previous section. The scalar density Lagrangian can be obtained from (34) by letting \( \phi \rightarrow g^{-ie/2} \) and is given by

\[
L = -\frac{1}{4} \beta \sqrt{g} \left[ g^{-2} \nabla_\mu g \nabla_\nu g \right],
\]

where I have introduced an overall constant \( \beta \). I have also taken \( m = 0 \) for simplicity since it only corresponds to a cosmological constant. The variation with respect to the connection gives (14) as before and (46) with \( J^\mu \) given by

\[
J^\mu = -\frac{\beta}{\alpha} \nabla^\mu g.
\]

Now from (13) one can show that

\[
\nabla_\alpha g = -2 g V_\alpha.
\]

The vector field equation becomes

\[
\tilde{\nabla}_\beta F^{\beta \mu} = \left[ \frac{3}{2\alpha \kappa} + \frac{2\beta}{\alpha} \right] V^\mu.
\]

Variation of the action with respect to \( g_{\mu\nu} \) gives

\[
\tilde{G}_{\mu\nu} = -\alpha \kappa \left[ F_{\mu\alpha} F^{\alpha \nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha \beta} F_{\alpha \beta} \right] - \left( 2\kappa \beta + \frac{3}{2} \right) \left[ V_\mu V_\nu - \frac{1}{2} g_{\mu\nu} V_\alpha V_\alpha \right].
\]

Thus, if \( \beta = -3/(4\kappa) \) the resulting equations are the Einstein-Maxwell equations. Point charge sources for the electromagnetic field can be included by adding the Lagrangian (26) with \( J^\mu \) given by (27). Of course, the Lagrangian for the point sources also has to be included and will contribute a source term to the Einstein Field equations.

Conclusion

A term proportional to \( F^{\mu\nu} F_{\mu\nu} \), where \( F_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu \) and \( \Gamma_\mu = \Gamma_\mu^{\alpha} \), was added to the standard gravitational Lagrangian. Since the field \( \Gamma_\mu \) transforms like a \( U(1) \) gauge field under coordinate transformations the field strength \( F_{\mu\nu} \) is a tensor. The field equations obtained from a Palatini variation are the Einstein-Proca equations. It was shown that the vector field could be coupled to point charges or to a complex scalar density of weight \( ie \). If the scalar density is taken to be \( g^{-ie/2} \) and the overall factor of the scalar density Lagrangian takes on a particular value the resulting equations were shown to be the Einstein-Maxwell equations.
Acknowledgements

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