Symplectic matrices with predetermined left eigenvalues

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Abstract

We prove that given four arbitrary quaternion numbers of norm 1 there always exists a $2 \times 2$ symplectic matrix for which those numbers are left eigenvalues. The proof is constructive. An application to the LS category of Lie groups is given.

Key words: quaternion, left eigenvalue, symplectic group, LS category

2000 MSC: 15A33, 11R52, 55M30

1. Introduction

Left eigenvalues of quaternionic matrices are only partially understood. While their existence is guaranteed by a result from Wood [11], many usual properties of right eigenvalues are no longer valid in this context, see Zhang’s paper [10] for a detailed account. In particular, a matrix may have infinite left eigenvalues (belonging to different similarity classes), as has been proved by Huang and So [5]. By using this result, the authors characterized in [7] the symplectic $2 \times 2$ matrices which have an infinite spectrum. In the present paper we prove that given four arbitrary quaternions of norm 1 there always exists a matrix in $Sp(2)$ for which those quaternions are left eigenvalues. The proof is constructive. This non-trivial result is of interest for the computation of the so-called LS category, as we explain in the last section of the paper.

*Partially supported by FEDER and MICINN Spain Research Project MTM2008-05861
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2. Left eigenvalues of symplectic matrices

Let $Sp(2)$ be the Lie group of $2 \times 2$ symplectic matrices, that is quaternionic matrices such that $AA^* = I$, where $A^* = A'$ is the conjugate transpose.

By definition, a quaternion $\sigma$ is a left eigenvalue of the matrix $A$ if there exists a vector $v \in \mathbb{H}^2$, $v \neq 0$, such that $Av = \sigma v$; equivalently, $A - \sigma I$ is not invertible.

It is easy to prove that the left eigenvalues of a symplectic matrix must have norm 1.

**Remark 2.1.** $\mathbb{H}^2$ will be always considered as a right quaternionic vector space, endowed with the product $\langle v, w \rangle = v^*w$.

The following Theorem was proven by the authors in [7], using the results of Huang and So in [5].

**Theorem 2.2.** The only $2 \times 2$ symplectic matrices with an infinite number of left eigenvalues are those of the form

$$L_q \circ R_\theta = \begin{bmatrix} q \cos \theta & -q \sin \theta \\ q \sin \theta & q \cos \theta \end{bmatrix}, \quad q \in \mathbb{H}, |q| = 1, \quad \theta \in \mathbb{R}, \sin \theta \neq 0.$$

Any other symplectic matrix has one or two left eigenvalues.

The matrix $L_q \circ R_\theta$ above corresponds to the composition of a real rotation $R_\theta \neq \pm \text{id}$ with a left translation $L_q$, $|q| = 1$. We need to characterize its eigenvalues.

**Lemma 2.3.** Let the matrix $A = L_q \circ R_\theta$ be as in Theorem 2.2 and let $\sigma \in \mathbb{H}$ be a quaternion. The following conditions are equivalent:

1. $\sigma$ is a left eigenvalue of $A$;
2. $\sigma = q(\cos \theta + \sin \theta \cdot \omega)$ with $\omega \in \langle i, j, k \rangle_\mathbb{R}, |\omega| = 1$;
3. $|\sigma| = 1$ and $\Re(q\sigma) = \cos \theta$;
4. $\bar{q}\sigma$ is conjugate to $\cos \theta + i \sin \theta$.

**Proof.** Part 2 can be checked by a direct computation from the definition or by using the results in [5]; part 3 follows because $\bar{q}\sigma - \cos \theta$ has not real part; finally part 4 is proved from the fact that two quaternions are conjugate if and only if they have the same norm and the same real part. \hfill $\Box$
We also need the following elementary result.

**Lemma 2.4.** Let $M \in M(n+1, \mathbb{R})$ be a real matrix with rows $\sigma_1, \ldots, \sigma_{n+1}$ and let $w \in \mathbb{R}^{n+1}$, $w \neq 0$. Suppose that $M$ has maximal rank $n+1$ and that its rows have euclidean norm 1. Then,

$$|M \cdot w| < \sqrt{n+1} |w|.$$

**Proof.** Let the matrix $M = (m_{ij})$ and let $w = \sum_{j=1}^{n+1} w_j e_j$ where $e_j$ are the vectors of the canonical basis. Then

$$|M \cdot w| = \left| \sum_{i=1}^{n+1} (\sum_{j=1}^{n+1} m_{ij} w_j) e_i \right| = \left[ \sum_{i=1}^{n+1} (\sum_{j=1}^{n+1} w_j m_{ij})^2 \right]^{1/2} = \left[ \sum_{i=1}^{n+1} \langle w, \sigma_i \rangle^2 \right]^{1/2},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^{n+1}$. Moreover, for any $1 \leq i \leq n+1$ we have

$$\langle w, \sigma_i \rangle = |w| |\sigma_i| \cos \langle w, \sigma_i \rangle = |w| \cos \langle w, \sigma_i \rangle \leq |w|$$

and equality implies that $w$ is a multiple of $\sigma_i$. Since the rows $\sigma_i$ are $\mathbb{R}$-independent by hypothesis, the vector $w \neq 0$ can not be in the direction of the $n+1$ rows at the same time, so $\langle w, \sigma_i \rangle < |w|$ for some $i$. Then, from (1),

$$|M \cdot w| < (n+1)|w|^2)^{1/2} = \sqrt{n+1} |w|.$$

Next theorem is the main result of this paper.

**Theorem 2.5.** Let $\sigma_1, \ldots, \sigma_4$ be four quaternions with norm 1. Then there exists a matrix $A \in Sp(2)$ for which those quaternions are left eigenvalues.
Proof. Accordingly to Theorem 2.2 the matrix must be of the form $A = L_q \circ R_{\theta}$, $|q| = 1$, $\sin \theta \neq 0$.

Now, part (3) of Lemma 2.3 means that, in order to find $A$, we have to fix a possible $\cos \theta \neq \pm 1$ –the exact value of $\cos \theta$ will be determined later–, and then to solve the system of linear equations

$$\Re(q\sigma_m) = \cos \theta, \quad m = 1, \ldots, 4. \quad (2)$$

Moreover, the solution $q$ must verify $|q| = 1$.

Let us write

$$q = t + xi + yj + zk, \quad t, x, y, z \in \mathbb{R},$$

and analogously

$$\sigma_m = t_m + x_m i + y_m j + z_m k, \quad m = 1, \ldots, 4.$$

Then system (2) can be written as

$$\begin{pmatrix} t_1 & x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ t_4 & x_4 & y_4 & z_4 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \cos \theta \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which we abbreviate as

$$M \cdot q = \cos \theta \cdot u. \quad (3)$$

If $0 < \text{rank} M < 4$ we take $\cos \theta = 0$, so system (3) is homogeneous. The set of solutions is a non trivial vector space, hence it contains at least one solution $q$ with norm $|q| = 1$. Then a matrix having $\sigma_1, \ldots, \sigma_4$ as eigenvalues would be, for instance, $A = L_q \circ R_{\pi/2}$.

On the other hand, if the matrix $M$ has maximal rank 4, it is invertible and the unique solution of (3) is given by

$$q = \cos \theta \cdot M^{-1} \cdot u. \quad (4)$$

By Lemma 2.4 above we have

$$2 = |u| = |M \cdot M^{-1} \cdot u| < \sqrt{4} |M^{-1} \cdot u|.$$
hence $|M^{-1}u| > 1$ and so we can choose $\theta$ such that

$$0 < |\cos \theta| = \frac{1}{|M^{-1}u|} < 1$$

which implies

$$|q| = |\cos \theta||M^{-1}u| = 1.$$

With that angle $\theta$ and the solution $q$ in $(4)$ we have obtained a matrix $A = L_q \circ R_\theta$ having $\sigma_m$, $m = 1, \ldots, 4$ among its eigenvalues. 

**Example 2.6.** The four quaternions $1, i, j, k$ give rise to the system

$$\text{id} \cdot q = \cos \theta \cdot u,$$

with unitary solutions $q = \pm (1/2)u$ (we are using (4) and (5)). Then, the only two symplectic matrices having those four numbers as left eigenvalues are

$$\frac{1}{4} \left( \begin{array}{cc} u & -\sqrt{3}u \\ \sqrt{3}u & u \end{array} \right)$$

and

$$\frac{1}{4} \left( \begin{array}{cc} u & \sqrt{3}u \\ -\sqrt{3}u & u \end{array} \right),$$

where $u = 1 + i + j + k$.

3. Application to LS category

The Lusternik-Schnirelmann category of a topological space $X$, denoted by $\text{cat } X$, is the minimum number of open sets (minus one), contractible in $X$, which are needed to cover $X$. This homotopical invariant has been widely studied and has many applications which go from the calculus of variations to robotics, see [1, 2, 6]. The computation of the LS category of Lie groups and homogeneous spaces is a central problem in this field, where many questions are still unanswered. For instance, for the symplectic groups the only known results are $\text{cat } Sp(1) = 1$, $\text{cat } Sp(2) = 3$ and $\text{cat } Sp(3) = 5$ [3].

There is a standard technique which has been successfully applied in the complex setting, for instance to the unitary group $U(n)$ [9] and to the symmetric spaces $U(2n)/Sp(n)$ and $U(n)/O(n)$ [8]. It consists in considering, for a given complex number $z$ with $|z| = 1$, the set $\Omega(z)$ of unitary matrices
A such that $A - zI$ is invertible. It turns out that this set is contractible. So, since a unitary $n \times n$ matrix can not have simultaneously $n + 1$ different eigenvalues, it is possible to cover the cited spaces by $n + 1$ contractible open sets $\Omega(z_1), \ldots, \Omega(z_{n+1})$, showing that they have category $\leq n$ (that $n$ is also a lower bound can be proved with homological methods).

In the quaternionic setting, we must consider left eigenvalues. If $\sigma \in \mathbb{H}$, the open set

$$\Omega(\sigma) = \{ A \in Sp(2) : A - \sigma I \text{ is invertible} \}$$

is contractible, for instance by means of the Cayley contraction

$$A_t = \frac{(1 + t)A - (1 - t)\sigma I}{(1 + t)I - (1 - t)\bar{\sigma}A}, \quad t \in [0, 1]$$

(see [4] for a general discussion). Hence our main result in this paper (Theorem 2.5) implies that four contractible sets of the type $\Omega(\sigma)$ will never cover $Sp(2)$, despite the fact that $\text{cat} Sp(2) = 3$.

**Remark 3.1.** We observe (cf. Lemma 2.3) that all the eigenvalues $\sigma$ of the two matrices in Example 2.6 verify $\Re(\sigma) = \pm 1/2$. Then if we take $\sigma_5 = (i + j)/\sqrt{2}$, we have $\Re(\sigma_5) = \pm 1/\sqrt{2}$, hence the two matrices belong to $\Omega(\sigma_5)$. So five open sets associated to eigenvalues do suffice to cover the group.

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