An exact BPS wall solution
in Five Dimensional Supergravity

Masato Arai \textsuperscript{a} \textsuperscript{1,2}, Shigeo Fujita \textsuperscript{b} \textsuperscript{3}, Masashi Naganuma \textsuperscript{b} \textsuperscript{4},
and Norisuke Sakai \textsuperscript{b} \textsuperscript{5}

\textsuperscript{a Institute of Physics, AS CR, 182 21, Praha 8, Czech Republic

\textsuperscript{b Department of Physics, Tokyo Institute of Technology
Tokyo 152-8551, JAPAN

Abstract

In five-dimensional supergravity, an exact solution of BPS wall is
found for a gravitational deformation of the massive Eguchi-Hanson non-
linear sigma model. The warp factor decreases for both infinities of the
extra dimension. Thin wall limit gives the Randall-Sundrum model with-
out fine-tuning of input parameters. We also obtain wall solutions with
warp factors which are flat or increasing in one side, by varying a defor-
mation parameter of the potential.

1 Introduction

One of the most interesting models in the brane-world scenario is given by Randall and
Sundrum, where the localization of four-dimensional graviton \cite{RS} has been obtained by a
spacetime metric containing a warp factor $e^{2U(y)}$ which decreases exponentially for both
infinities of the extra dimension $y \to \pm\infty$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2U(y)}\eta_{mn}dx^m dx^n + dy^2, \quad (1)$$

where $\mu, \nu = 0,..,4$, $m, n = 0, 1, 3, 4$ and $y \equiv x^2$. They had to introduce both a bulk
cosmological constant and a boundary cosmological constant, which have to be fine-tuned
each other.

This scenario is based on the assumption of the existence of delta functional domain
wall. Thus, it would be nice to obtain the domain wall as a classical solution in some field

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\textsuperscript{2}e-mail address: arai@fzu.cz
\textsuperscript{3}e-mail address: fujita@th.phys.titech.ac.jp
\textsuperscript{4}e-mail address: naganuma@th.phys.titech.ac.jp
\textsuperscript{5}e-mail address: nsakai@th.phys.titech.ac.jp
theory in a phenomenological point of view. Studies of domain wall solutions in gauged supergravity theories in five dimensions revealed that hypermultiplets are needed \cite{2} to obtain warp factors decreasing for both infinities $y \to \pm \infty$ (infra-red (IR) fixed points in AdS/CFT correspondence \cite{3}). It has been shown that the target space of hypermultiplets in five-dimensional supergravity theory must be quaternionic Kähler (QK) manifolds \cite{4}. Further, in order to obtain domain wall solution in massive QK nonlinear sigma models (NLSM) in supergravity theories have been studied using mostly homogeneous target manifolds. Unfortunately, supersymmetric (SUSY) vacua in homogeneous target manifolds are not truly IR critical points, but can only be saddle points with some IR directions \cite{5}, \cite{6}. Inhomogeneous manifolds and a wall solution have also been constructed \cite{7}, \cite{8}. However, these manifolds do not allow a limit of weak gravitational coupling.

The purpose of this paper is to present an exact BPS domain wall solution in five-dimensional supergravity coupled with hypermultiplets (and vector multiplets). Our strategy to construct the model is to deform the NLSM in SUSY theory having domain wall solution to the model with gravity. Massive hyper-Kähler NSLMs without gravity in four dimensions have been constructed in harmonic superspace as well as in $\mathcal{N} = 1$ superfield formulation \cite{9}, and have yielded the domain wall solution for the Eguchi-Hanson (EH) manifold \cite{10}. Inspired by this solution, we deform this model into five-dimensional supergravity model and we consider the BPS domain wall solution. We also discuss a limit of weak gravitational coupling. This paper is based on our paper \cite{11} where complete analysis and references are found.

2 Bosonic action of our model in 5D Supergravity

To find a gravitational deformation of the NLSM with EH target manifold, we use the recently obtained off-shell formulation of five-dimensional supergravity (tensor calculus) \cite{12}, \cite{13} combined with the quotient method via a vector multiplet without kinetic term and the massive deformation (central charge extension). \footnote{We adopt the conventions of Ref. \cite{12} except the sign of our metric $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1,+1)$. This induces a change of Dirac matrices and the form of SUSY transformation of fermion.} We start with the system of a Weyl multiplet, three hypermultiplets and two $U(1)$ vector multiplets. One of the two vector multiplets has no kinetic term and plays the role of a Lagrange multiplier for hypermultiplets to obtain a curved target manifold. The other vector multiplet serves to give mass terms for hypermultiplets.

After integrating out a part of the auxiliary fields by their on-shell conditions in the off-shell supergravity action \cite{13}, we obtain the bosonic part of the action for our model

$$e^{-1} \mathcal{L} = -\frac{1}{2\kappa^2}R - \frac{1}{4} \left( \partial_{\mu} W^0_{\nu} - \partial_{\nu} W^0_{\mu} \right) \left( \partial^{\mu} W^{0\nu} - \partial^{\nu} W^{0\mu} \right)$$

$$- \nabla^A i^A_j d_{\beta\alpha} \nabla_a A^i_{\alpha} - \kappa^2 [A^i_{\beta} d_{\beta\alpha} \nabla_a A^j_{\alpha}]^2$$

$$- \left[ -A_i^{\beta} d_{\gamma\alpha} (g_0 M^0 t_0 + M^1 t_1)^2 A_i^{\beta} - \frac{\kappa^2}{12} (g_0 M^0)^2 (2 A_i^{\alpha} d_{\gamma\alpha} (t_0)^2 A_i^{\gamma})^2 \right],$$

$$\nabla_{\mu} A^i_{\alpha} = \partial_{\mu} A^i_{\alpha} - (g_0 W^0_{\mu} t_0 + W^1_{\mu} t_1)^{\alpha} A^{\beta}_{j}, \quad A^i_{\alpha} \equiv \epsilon^{ij} A^{\beta}_{j} \rho_{\beta\alpha} = -(A^i_{\alpha})^*,$$

where $d_{\alpha\beta} = \text{diag}(1,1,-1,-1,-1)$, $\kappa$ is the five-dimensional gravitational coupling, $A^i_{\alpha}$, $i = 1, 2$, $\alpha = 1, \ldots, 6$ are the scalars in hypermultiplets, and $W^0_{\mu}$ ($W^1_{\mu}$), $M^0$ ($M^1$)
and $t_0$ ($t_1$) are vector fields, scalar fields and generators of the $U(1)$ vector multiplets with (without) a kinetic term. The gauge coupling of $W_0^\mu$ is denoted by $g_0$. Another gauge coupling $g_1$ is absorbed into a normalization of $W_1^\mu$ in order to drop the kinetic term by taking $g_1 \to \infty$. Hypermultiplet scalars are subject to two kinds of constraints

$$\mathcal{A}^2 = \mathcal{A}_i^\beta d_\beta^\alpha \mathcal{A}_i^\alpha = -2\kappa^{-2}, \quad \frac{1}{g_1^2} \mathcal{A}_i^{ij} = 2\mathcal{A}_i^\alpha d_\alpha^\gamma (t_1)^\gamma_\beta \mathcal{A}_j^\beta = 0. \quad (3)$$

The first constraint comes from the gauge fixing of dilatation, and make target space of hypermultiplets to be a non-compact version of quaternionic projective space, combined with the gauge fixing of $SU(2)_R$ symmetry. The second constraint is required by the on-shell condition of auxiliary fields of the $U(1)$ vector multiplet without kinetic term, and corresponds to the constraint for the EH target space in the limit of $\kappa \to 0$.

The third line of (2) is a scalar potential. The scalar $M^0$ is fixed as $(M^0)^2 = \frac{1}{2}\kappa^{-2}$ from the requirement of canonical normalizations of the Einstein-Hilbert term and the kinetic term of the gravi-photon $W_0^\mu$ for Poincaré supergravity. The scalar $M^1$ without kinetic term is a Lagrange multiplier, and is found to be

$$M^1 = \frac{-\mathcal{A}_i^\alpha (t_0 t_1)_{\alpha}^\beta \mathcal{A}_i^\beta}{\mathcal{A}_i^\gamma d_\gamma^\alpha (t_1)^{2\alpha}_\beta \mathcal{A}_j^\beta} g_0 M^0. \quad (4)$$

Here we introduce two two-component complex fields $\phi_1$ and $\phi_2$ to parametrize $\mathcal{A}_i^\alpha$ by a matrix with $i = 1, 2$ as rows and $\alpha = 1, \ldots, 6$ as columns

$$\mathcal{A}_i^\alpha \equiv \frac{1}{\kappa} \tilde{A}^{-1/2} \begin{pmatrix} 1 & 0 & \kappa \phi_1 & -\kappa \phi_2^* \\ 0 & 1 & \kappa \phi_2 & \kappa \phi_1^* \end{pmatrix} \quad (5)$$

satisfying the first constraint in (3) by taking $\tilde{A} = 1 - \kappa^2 (|\phi_1|^2 + |\phi_2|^2)$. In this basis, we can choose two $U(1)$ generators as $t_1^{\alpha}_\beta = diag(ia\alpha, -ia\alpha, i, i, -i, -i)$ and $t_0^{\alpha}_\beta = diag(ia\alpha, -ia\alpha, -i, i, i, -i)$, where $\alpha$ and $a$ are real parameters. The parameter $\alpha$ in $t_1$ makes target manifold inhomogeneous generally through the second constraint in (3), and a special case of $\alpha = 1$ corresponds to a homogeneous manifold of $SU(2,1)/U(2)$ [14]. Here we define $\alpha \equiv \kappa^2 \Lambda^2$, where $\Lambda$ is a real parameter of unit mass dimension.

In order to solve constraints (3) in terms of (5), we introduce the spherical coordinates as

$$\phi_1^1 = g(r) \cos(\frac{\theta}{2}) \exp(\frac{i}{2} (\Psi + \Phi)), \quad \phi_2^1 = g(r) \sin(\frac{\theta}{2}) \exp(\frac{i}{2} (\Psi - \Phi)), \quad \phi_2^1 = f(r) \sin(\frac{\theta}{2}) \exp(-\frac{i}{2} (\Psi - \Phi)), \quad \phi_2^2 = -f(r) \cos(\frac{\theta}{2}) \exp(-\frac{i}{2} (\Psi + \Phi)). \quad (6)$$
Here we set
\[ f(r)^2 = \frac{1}{2}(\Lambda^3 + \sqrt{4r^2 + \Lambda^6}), \quad g(r)^2 = \frac{1}{2}(\Lambda^3 + \sqrt{4r^2 + \Lambda^6}) \quad (7) \]
in order to satisfy (3).

Substituting the solution (6) into the action, it can be described by independent variables. The target metric of the kinetic term is found to be a quaternionic extension of the EH metric [15], [16]. Since the metric is Einstein, the Weyl tensor is anti-selfdual and the scalar curvature is negative \( R = -24\kappa^2 \), it is locally a quaternionic manifold [4] for any values of \( \kappa \neq 0 \).

The scalar potential part in (2) becomes a function of fields \( r, \theta \) and depends on the parameter \( a \) and the gravitational coupling \( \kappa \), i.e. \( V = V(r, \theta, a, \kappa) \). Here we do not show the explicit form but only show the plot of the potential. Fig. 1-(a) shows 3D plot of the potential and it is found there exist two vacua at \((r, \theta) = (0, 0), (0, \pi)\) as local minima. These two vacua become saddle points with an unstable direction along \( r \) for \( \kappa^2 \Lambda^3 > 3/4 \) for \( a = 0 \). Fig. 1-(b) shows a typical unstable behavior of potential at \( \kappa^2 \Lambda^3 = 9 \), which is close to \( \kappa^2 \Lambda^3 = 1 \), where the target space of hypermultiplets becomes a homogeneous space of \( SU(2,1)/U(2) \). For \( a \neq 0 \), potential takes different values at these two vacua.

### 3 BPS equation and the solution

Instead of solving Einstein equations directly, we solve BPS equations to obtain a classical solution conserving a half of SUSY. Since we consider bosonic configurations, we need to examine the on-shell SUSY transformation of gravitino and hyperino [12]

\[
\delta \varepsilon_i^\mu = D_\mu \varepsilon_i^\mu - \frac{\kappa^2}{6} M_0 \gamma_i^\mu \varepsilon_j^j, \quad (8)
\]

\[
\delta \varepsilon^\alpha = -D_\mu A_\mu^\gamma \varepsilon^\gamma - (M^1 t_1 + g_0 M^0 t_0)^\alpha \beta \varepsilon_j^j + \frac{\kappa^2}{2} A_j^\gamma M_0 \gamma_j^\kappa \varepsilon^k, \quad (9)
\]

where

\[
D_\mu \varepsilon_i^\mu = \left( \partial_\mu - \frac{1}{4} \gamma_{ab} \omega_{\mu}^{ab} \right) \varepsilon_i^\mu - \kappa^2 V_i^\gamma i \varepsilon_j^j, \quad (10)
\]

\[
D_\mu A_i^\alpha = \partial_\mu A_i^\alpha + \kappa^2 V_{\mu i}^\beta \varepsilon_j^j - W_{\mu i}^{\alpha \beta} A_i^\beta, \quad (11)
\]

\[
\gamma_0^{ij} = 2 A_i^\alpha d_\gamma (g_0 t_0)^\gamma \varepsilon_j^j, \quad V_{\mu i}^\beta = -A_i^\alpha d_\gamma \varepsilon_j^j \varepsilon^\mu . \quad (12)
\]

Let us require vanishing of the SUSY variation of gravitino and hyperino to preserve four SUSY specified by

\[
\gamma^\nu \varepsilon_i^\nu (y) = \iota \tau_3 \varepsilon_i^\nu (y), \quad (13)
\]

where \( \tau_3 \) is one of the Pauli matrix. Substituting this condition (13) and the metric ansatz (11) into (8) and (9), we obtain BPS equation. We can solve it in the spherical coordinate (6). The wall solution interpolating between the two vacua \((r, \theta) = (0, 0), (0, \pi)\) is obtained from (9)

\[
r = 0, \quad \cos \theta = \tanh \left( 2g_0 M^0 (y - y_0) \right), \quad \Phi = \varphi_0, \quad (14)
\]

\[
p = 0, \quad \sin \theta = \tanh \left( 2g_0 M^0 (y_0 - y) \right), \quad \Phi = \varphi_0. \quad (15)
\]
with $\Psi$ undetermined, and $y_0$ and $\varphi_0$ are constants. Here we take the boundary condition $r = 0$ at $y = -\infty$. Using (14), we obtain the BPS solution of the warp factor and the Killing spinor from (8)

$$U(y) = -\frac{\kappa^2 \Lambda^3}{3(1 - \kappa^2 \Lambda^3)} \left[ \ln \left\{ \cosh \left( 2g_0 M^0 (y - y_0) \right) \right\} + 2a g_0 M^0 (y - y_0) \right],$$

$$\varepsilon^i(y) \equiv e^{U(y)/2} \tilde{\varepsilon}^i, \quad \gamma^y \tilde{\varepsilon}^i = i \tau_3 \varepsilon^i,$$

where $\tilde{\varepsilon}^i$ is a constant spinor.

The warp factor $e^{2U(y)}$ of this solution decreases exponentially for both infinities $y \to \pm \infty$ for $|a| < 1$ (see Fig. 2) similarly to the case of the bulk AdS space. The case of $|a| = 1$ becomes the wall solutions interpolating between AdS and flat Minkowski vacua. On the other hand, warp factor increases exponentially either one of the infinities for $|a| > 1$. Following the AdS/CFT conjecture, a vacuum reached by a decreasing (increasing) warp factor corresponds to IR (UV) fixed point of a four-dimensional field theory [8]. Our BPS wall solutions interpolate two IR fixed points for $|a| < 1$. The wall solutions for $|a| > 1$ interpolate one IR and one UV fixed points which cannot realize the warped extra dimension, but should be related to a Renormalization Group (RG) flow. The family of our BPS solutions contains a parameter $a$ interpolating between three classes of field theories: one with two IR fixed points ($|a| < 1$), another with one IR and one UV fixed point ($|a| > 1$), and one with one IR fixed point and flat space ($|a| = 1$). We find it remarkable that a single family of models can realize all these possibilities as we change a parameter.

We can obtain a thin wall limit by taking $g_0 M^0 \to \infty$ and $\Lambda \to 0$ with $g_0 M^0 \Lambda^3$, $\kappa$, and $a$ fixed. Substituting the solutions (14) and (15) to the Lagrangian of hypermultiplets and taking the thin wall limit, we obtain for $y_0 = 0$

$$-\frac{1}{2\kappa^2} R + e^{-1} L_{kin} + e^{-1} L_{pot} \to -\frac{1}{2\kappa^2} R - \Lambda_c^\pm(y) - T_w \delta(y),$$

where $T_w$ is a cosmological constant (wall tension) $\Lambda_c^+$, $\Lambda_c^-$ is bulk cosmological constant for $y < 0 (y > 0)$ as

$$T_w = 4(g_0 M^0 \Lambda^3), \quad \Lambda_c^\pm = -\frac{8\kappa^2 (g_0 M^0 \Lambda^3)^2}{3}(1 \pm a^2).$$

We find that our BPS solution for $a = 0$ automatically satisfies the fine-tuning condition $\sqrt{-\Lambda_c} = \frac{\kappa}{\sqrt{6}} T_w$ of the Randall-Sundrum model between $T_w$ and $\Lambda_c$, as a result of combined
dynamics of scalar field and gravity. In terms of the asymptotic linear exponent $c$ of the warp factor $U \sim -c|y - y_0|$, $c \equiv 2\kappa^2(g_0 M^0 \Lambda^3)/3$ for $|y - y_0| \to \infty$, the wall tension $T_w = 24c/(4\kappa^2)$, and cosmological constant $\Lambda_c = -24c^2/(4\kappa^2)$ satisfy precisely the same relation as in Ref. [1] (with $M_p^3 \equiv (4\kappa^2)^{-1}$). Therefore we have realized the single-wall Randall-Sundrum model as a thin-wall limit of our solution of the coupled scalar-gravity theory, instead of an artificial boundary cosmological constant put at an orbifold point.

Finally, we discuss the properties of our model and solution in the weak gravity limit, which is defined by taking the limit of $\kappa \to 0$ with $g_0 M^0 \equiv \bar{M}$ held fixed. In the limit, we find that the action (2) in terms of (5) reduces to the five-dimensional version of the massive NLSM with the EH target metric in the basis in Ref. [17]. The wall solution for $\kappa = 0$ is the five-dimensional version of the kink solution in Ref. [9]. Their solution is exactly identical to our solution (14) obtained for finite $\kappa$. It is very interesting that BPS solution for the hypermultiplet in the global SUSY model coincides with that in the corresponding supergravity. This mysterious coincidence has also appeared in the analytic solution in a four-dimensional $\mathcal{N} = 1$ supergravity model [18].

References

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690, [hep-th/9906064].
[2] R. Kallosh and A. Linde, JHEP 0002 (2000) 005, [hep-th/0001071]; K. Behrndt, C. Herrmann, J. Louis and S. Thomas, JHEP 0101 (2001) 011, [hep-th/0008112].
[3] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, [hep-th/9711200]; O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. 323 (2000) 183, [hep-th/9912001].
[4] J. Bagger and E. Witten, Nucl. Phys. B222 (1983) 1.
[5] D.V. Alekseevsky, V. Cortés, C. Devchand and A. Van Proeyen, Comm. Math. Phys. 238 (2003) 525, [hep-th/0109094].
[6] K. Behrndt and M. Cvetic, Phys. Rev. D65 (2002) 126007, [hep-th/0201272].
[7] K. Behrndt and G. Dall’Agata, Nucl. Phys. B627 (2002) 357, [hep-th/0112136].
[8] L. Anguelova and C.I. Lazaroiu, JHEP 0209 (2002) 053, [hep-th/0208154].
[9] M. Arai, M. Naganuma, M. Nitta and N. Sakai, Nucl. Phys. B652 (2003) 35, [hep-th/0211103].
[10] T. Eguchi and A.J. Hanson, Phys. Lett. 74B (1978) 249; Ann. Phys. 120 (1979) 82.
[11] M. Arai, S. Fujita, M. Naganuma and N. Sakai, Phys. Lett. B556 (2003) 192, [hep-th/0212175].
[12] T. Kugo and K. Ohashi, Prog. Theor. Phys. 105 (2001) 323, [hep-ph/0010288]; T. Fujita and K. Ohashi, Prog. Theor. Phys. 106 (2001) 221, [hep-th/0104130].
[13] T. Fujita, T. Kugo and K. Ohashi, Prog. Theor. Phys. 106 (2001) 671, [hep-th/0106051].
[14] P. Breitenlohner and M.F. Sohnius, *Nucl. Phys.* **B187** (1981) 409.

[15] K. Galicki, *Nucl. Phys.* **B271** (1986) 402.

[16] E. Ivanov and G. Valent, *Nucl. Phys.* **B576** (2000) 543, [hep-th/0001165].

[17] T.L. Curtright and D.Z. Freedman, *Phys. Lett.* **90B** (1980) 71.

[18] M. Eto, N. Maru, N. Sakai and T. Sakata, *Phys. Lett.* **B553** (2003) 87, [hep-th/0208127].