The Symmetry Group of Gaussian States in $L^2(\mathbb{R}^n)$

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Abstract  This is a continuation of the expository article [4] with some new remarks. Let $S_n$ denote the set of all Gaussian states in the complex Hilbert space $L^2(\mathbb{R}^n)$, $K_n$ the convex set of all momentum and position covariance matrices of order $2n$ in Gaussian states and let $G_n$ be the group of all unitary operators in $L^2(\mathbb{R}^n)$ conjugations by which leave $S_n$ invariant. Here we prove the following results. $K_n$ is a closed convex set for which a matrix $S$ is an extreme point if and only if $S = \frac{1}{2} L^T L$ for some $L$ in the symplectic group $Sp(2n, \mathbb{R})$. Every element in $K_n$ is of the form $\frac{1}{2}(L^T L + M^T M)$ for some $L, M$ in $Sp(2n, \mathbb{R})$. Every Gaussian state in $L^2(\mathbb{R}^n)$ can be purified to a Gaussian state in $L^2(\mathbb{R}^{2n})$. Any element $U$ in the group $G_n$ is of the form $U = \lambda W(\alpha) \Gamma(L)$ where $\lambda$ is a complex scalar of modulus unity, $\alpha \in \mathbb{C}^n$, $L \in Sp(2n, \mathbb{R})$, $W(\alpha)$ is the Weyl operator corresponding to $\alpha$ and $\Gamma(L)$ is a unitary operator which implements the Bogolioubov automorphism of the Lie algebra generated by the canonical momentum and position observables induced by the symplectic linear transformation $L$.

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1 Introduction

In [4] we defined a quantum Gaussian state in $L^2(\mathbb{R}^n)$ as a state in which every real linear combination of the canonical momentum and position observables $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ has a normal distribution on the real line. Such a state is uniquely determined by the expectation values of $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ and their covariance matrix of order $2n$. A real positive definite matrix $S$ of order $2n$ is the covariance matrix of the observables $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ if and only if the matrix inequality

$$2S - iJ \succeq 0 \quad (1.1)$$

holds where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad (1.2)$$

the right hand side being expressed in block notation with $0$ and $I$ being of order $n \times n$. We denote by $K_n$ the set of all possible covariance matrices of the momentum and position observables in Gaussian states so that

$$K_n = \{ S \mid S \text{ is a real symmetric matrix of order } 2n \text{ and } 2S - iJ \succeq 0 \} \quad (1.3)$$

Clearly, $K_n$ is a closed convex set. Here we shall show that $S$ is an extreme point of $K_n$ if and only if $S = \frac{1}{2} L^T L$ for some matrix $L$ in the real symplectic matrix group

$$Sp(2n, \mathbb{R}) = \{ L \mid L^T JL = J \} \quad (1.4)$$

with the superfix $T$ indicating transpose. Furthermore, it turns out that every element $S$ in $K_n$ can be expressed as

$$S = \frac{1}{2} (L^T L + M^T M)$$

for some $L, M \in Sp(2n, \mathbb{R})$. This, in turn implies that any Gaussian state in $L^2(\mathbb{R}^n)$ can be purified to a pure Gaussian state in $L^2(\mathbb{R}^{2n})$.

Let $\alpha \in (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{C}^n$, $L = ((\ell_{ij})) \in Sp(2n, \mathbb{R})$ and let $\alpha_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. Define a new set of momentum and position observables $p'_1, p'_2, \ldots, p'_n; q'_1, q'_2, \ldots, q'_n$ by

$$p'_i = \sum_{j=1}^{n} \{ \ell_{ij} (p_j - x_j) + \ell_{m+j} (q_j - y_j) \},$$

$$q'_i = \sum_{j=1}^{n} \{ \ell_{n+i} (p_j - x_j) + \ell_{n+i+j} (q_j - y_j) \},$$

for $1 \leq i \leq n$. Here one takes linear combinations and their respective closures to obtain $p'_i, q'_i$ as selfadjoint operator observables. Then $p'_1, p'_2, \ldots, p'_n; q'_1, q'_2, \ldots, q'_n$
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obey the canonical commutation relations and thanks to the Stone-von Neumann uniqueness theorem there exists a unitary operator $\Gamma(\alpha, L)$ satisfying

$$
p'_i = \Gamma(\alpha, L) p_i \Gamma(\alpha, L)^\dagger,
q'_j = \Gamma(\alpha, L) q_j \Gamma(\alpha, L)^\dagger
$$

for all $1 \leq i \leq n$. Furthermore, such a $\Gamma(\alpha, L)$ is unique upto a scalar multiple of modulus unity. The correspondence $(\alpha, L) \rightarrow \Gamma(\alpha, L)$ is a projective unitary and irreducible representation of the semidirect product group $\mathbb{C}^n \ltimes Sp(2n, \mathbb{R})$. Here any element $L$ of $Sp(2n, \mathbb{R})$ acts on $\mathbb{C}^n$ real-linearly preserving the imaginary part of the scalar product. The operator $\Gamma(\alpha, L)$ can be expressed as the product of

$$W(\alpha) = \Gamma(\alpha, I) \text{ and } \Gamma(L) = \Gamma(0, L).$$

Conjugations by $W(\alpha)$ implement translations of $p_j, q_j$ by scalars whereas conjugations by $\Gamma(L)$ implement symplectic linear transformations by elements of $Sp(2n, \mathbb{R})$, which are the so-called Bogolioubov automorphisms of canonical commutation relations. In the last section we show that every unitary operator $U$ in $L^2(\mathbb{R}^n)$, with the property that $U \rho U^\dagger$ is a Gaussian state whenever $\rho$ is a Gaussian state, has the form $U = \lambda W(\alpha) \Gamma(L)$ for some scalar $\lambda$ of modulus unity, a vector $\alpha$ in $\mathbb{C}^n$ and a matrix $L$ in the group $Sp(2n, \mathbb{R})$.

The following two natural problems that arise in the context of our note seem to be open. What is the most general unitary operator $U$ in $L^2(\mathbb{R}^n)$ with the property that whenever $|\psi\rangle$ is a pure Gaussian state so is $U|\psi\rangle$? Secondly, what is the most general trace-preserving and completely positive linear map $\Lambda$ on the ideal of trace-class operators on $L^2(\mathbb{R}^n)$ with the property that $\Lambda(\rho)$ is a Gaussian state whenever $\rho$ is a Gaussian state?

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2 Exponential vectors, Weyl operators, second quantization and the quantum Fourier transform

For any $z = (z_1, z_2, \ldots, z_n)^T$ in $\mathbb{C}^n$ define the associated exponential vector $e(z)$ in $L^2(\mathbb{R}^n)$ by

$$e(z)(x) = (2\pi)^{-n/4} \exp \sum_{j=1}^n (z_j x_j - \frac{1}{2} z_j^2 - \frac{1}{4} x_j^2).$$

Writing scalar products in the Dirac notation we have

$$\langle e(z) | e(z') \rangle = \exp \langle z | z' \rangle
= \exp \sum_{j=1}^n z_j z'_j.$$
\[ \Gamma(U)|e(z)\rangle = |e(Uz)\rangle \quad \forall \, z \in \mathbb{C}^n. \] (2.3)

The operator \( \Gamma(U) \) is called the second quantization of \( U \). For any two unitary matrices \( U, V \) in the unitary group \( \mathcal{U}(n) \) one has

\[ \Gamma(U)\Gamma(V) = \Gamma(UV). \]

The correspondence \( U \rightarrow \Gamma(U) \) is a strongly continuous unitary representation of the group \( \mathcal{U}(n) \) of all unitary matrices of order \( n \).

For any \( \alpha \in \mathbb{C}^n \) there is a unique unitary operator \( W(\alpha) \) in \( L^2(\mathbb{R}^n) \) satisfying

\[ W(\alpha)|e(z)\rangle = e^{-\frac{1}{2}\|\alpha\|^2 - \langle \alpha | z \rangle} |e(z + \alpha)\rangle \quad \forall \, z \in \mathbb{C}^n. \] (2.4)

For any \( \alpha, \beta \in \mathbb{C}^n \) one has

\[ W(\alpha)W(\beta) = e^{-i\text{Im} \langle \alpha | \beta \rangle} W(\alpha + \beta). \] (2.5)

The correspondence \( \alpha \rightarrow W(\alpha) \) is a projective unitary and irreducible representation of the additive group \( \mathbb{C}^n \). The operator \( W(\alpha) \) is called the Weyl operator associated with \( \alpha \). As a consequence of (2.5) it follows that the map \( t \rightarrow W(t\alpha) \), \( t \in \mathbb{R} \) is a strongly continuous one parameter unitary group admitting a selfadjoint Stone generator \( p(\alpha) \) such that

\[ W(t\alpha) = e^{-itp(\alpha)} \quad \forall \, \alpha \in \mathbb{C}^n. \] (2.6)

Writing \( e_j = (0,0,\ldots,0,1,0,\ldots,0)^T \) with 1 in the \( j \)-th position,

\[ p_j = 2^{\frac{1}{2}} p(e_j), \quad q_j = -2^{\frac{1}{2}} p(i e_j) \] (2.7)

\[ a_j = \frac{q_j + ip_j}{\sqrt{2}}, \quad a_j^\dagger = \frac{q_j - ip_j}{\sqrt{2}} \] (2.8)

one obtains a realization of the momentum and position observables \( p_j, q_j, 1 \leq i \leq n \) obeying the canonical commutation relations (CCR)

\[ [p_i, p_j] = 0, \quad [q_i, q_j] = 0, \quad [q_i, p_j] = i\delta_{ij}. \]

and the adjoint pairs \( a_j, a_j^\dagger \) of annihilation and creation operators satisfying

\[ [a_i, a_j] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij} \]

in appropriate domains. If we write

\[ p_j^x = 2^{\frac{1}{2}} p_j, \quad q_j^x = 2^{\frac{1}{2}} q_j \]

one obtains the canonical Schrödinger pairs of momentum and position observables in the form
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\[
\left(\rho_j^* \psi\right)(x) = \frac{1}{i} \frac{\partial \psi(x)}{\partial x_j}, \quad \left(\eta_j^* \psi\right)(x) = x_j\psi(x)
\]

in appropriate domains. We refer to [5] for more details.

We now introduce the symplectic group $Sp(2n, \mathbb{R})$ of real matrices of order $2n$ satisfying (1.4). Any element of this group is called a symplectic matrix. As described in [1], [4], for any symplectic matrix $L$ there exists a unitary operator $\Gamma(L)$ satisfying

\[
\Gamma(L) W(\alpha) \Gamma(L)^\dagger = W(\tilde{L} \alpha) \quad \forall \alpha \in \mathbb{C}^n
\]

where

\[
\begin{bmatrix}
\text{Re } L \alpha \\
\text{Im } L \alpha
\end{bmatrix} = L
\begin{bmatrix}
\text{Re } \alpha \\
\text{Im } \alpha
\end{bmatrix}.
\]

(2.9)

Whenever the symplectic matrix $L$ is also a real orthogonal matrix then $\tilde{L}$ is a unitary matrix and $\Gamma(L)$ coincides with the second quantization $\Gamma(\tilde{L})$ of $\tilde{L}$. Conversely, if $U$ is a unitary matrix of order $n, U_L$ is the matrix satisfying

\[
U_L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \text{Re } U(x + iy) \\ \text{Im } U(x + iy) \end{bmatrix}
\]

then $U_L$ is a symplectic and real orthogonal matrix of order $2n$ and $\Gamma(U_L) = \Gamma(U)$. Equations (2.9) and (2.10) imply that $\Gamma(L)$ implements the Bogolioubov automorphism determined by the symplectic matrix $L$ through conjugation.

For any state $\rho$ in $L^2(\mathbb{R}^n)$ its quantum Fourier transform $\hat{\rho}$ is defined to be the complex-valued function on $\mathbb{C}^n$ given by

\[
\hat{\rho}(\alpha) = \text{Tr } \rho W(\alpha), \quad \alpha \in \mathbb{C}^n.
\]

(2.11)

In [4] we have described a necessary and sufficient condition for a complex-valued function $f$ on $\mathbb{C}^n$ to be the quantum Fourier transform of a state in $L^2(\mathbb{R}^n)$. Here we shall briefly describe an inversion formula for reconstructing $\rho$ from $\hat{\rho}$. To this end we first observe that (2.11) is well defined whenever $\rho$ is any trace-class operator in $L^2(\mathbb{R})$. Denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively the ideals of trace-class and Hilbert-Schmidt operators in $L^2(\mathbb{R}^n)$. Then $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2$ is a Hilbert space with the inner product $\langle A | B \rangle = \text{Tr } A^* B$. There is a natural isomorphism between $\mathcal{F}_2$ and $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, which can, in turn, be identified with the Hilbert space of square integrable functions of two variables $x, y$ in $\mathbb{R}^n$. We denote this isomorphism by $\mathcal{I}$ so that $\mathcal{I}(A)(x, y)$ is a square integrable function of $(x, y)$ for any $A \in \mathcal{F}_2$ and

\[
\mathcal{I}(|e(u)| \langle e(\tilde{v})|)(x, y) = e(u)(x)e(\tilde{v})(y)
\]

(2.12)

for all $u, v \in \mathbb{C}^n$, $\tilde{v}$ denoting $(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)$. From (2.4) and (2.11) we have

\[
(|e(u)| \langle e(\tilde{v})|)(\alpha) = \langle e(\tilde{v})| W(\alpha) | e(u)\rangle
\]

\[
= \exp\left\{-\frac{1}{2} \|\alpha\|^2 - \langle \alpha | u \rangle + \langle v | \alpha \rangle + \langle v | u \rangle \right\}.
\]
Substituting $\alpha = x + iy$ and using (2.1), the equation above, after some algebra, can be expressed as
\[
(|e(u)\rangle\langle e(v)|)^\wedge(x + iy) = (2\pi)^{n/2}e(u')|\sqrt{2}x\rangle e(v')|\sqrt{2}y\rangle
\] (2.13)
where
\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix}
= U
\begin{bmatrix}
  u \\
  v
\end{bmatrix},
\]
\[
U = 2^{-1/2}
\begin{bmatrix}
  -I & I \\
  il & il
\end{bmatrix},
\] (2.14)

Let $D_{\theta}, \theta > 0$ denote the unitary dilation operator in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ defined by
\[
(D_{\theta}f)(x, y) = \theta^n f(\theta x, \theta y).
\] (2.15)

Then (2.13) can be expressed as
\[
(|e(u)\rangle\langle e(v)|)^\wedge(x + iy) = \pi^{n/2} \left\{ D_{\sqrt{2}}\Gamma(U)e(u \otimes v) \right\}(x, y)
\]
where $\Gamma(U)$ is the second quantization operator in $L^2(\mathbb{R}^{2n})$ associated with the unitary matrix $U$ in (2.14) of order $2n$. Since exponential vectors are total and $D_{\sqrt{2}}$ and $\Gamma(U)$ are unitary we can express the quantum Fourier transform $\rho \rightarrow \hat{\rho}(x + iy)$ as
\[
\hat{\rho} = \pi^{n/2} D_{\sqrt{2}}\Gamma(U) \mathcal{F}(\rho).
\] (2.16)

In particular, $\hat{\rho}(x + iy)$ is a square integrable function of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and
\[
\rho = \pi^{-n/2} \mathcal{F}^{-1} \Gamma(U^\dagger) D_{2^{-1/2}} \hat{\rho}
\] (2.17)
is the required inversion formula for the quantum Fourier transform. It is a curious but an elementary fact that the eigenvalues of $U$ in (2.14) are all 12th roots of unity and hence the unitary operators $\Gamma(U)$ and $\Gamma(U^\dagger)$ appearing in (2.16) and (2.17) have their 12-th powers equal to identity. This may be viewed as a quantum analogue of the classical fact that the 4-th power of the unitary Fourier transform in $L^2(\mathbb{R}^n)$ is equal to identity.

### 3 Gaussian states and their covariance matrices

We begin by choosing and fixing the canonical momentum and position observables $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ as in equation (2.7) in terms of the Weyl operators. They obey the CCR. The closure of any real linear combination of the form $\sum_{j=1}^n (x_j p_j - y_j q_j)$ is selfadjoint and we denote the resulting observable by the same symbol. As
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in [4], for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $\alpha_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, the Weyl operator $W(\alpha)$ defined in Section 2 can be expressed as

$$W(\alpha) = e^{-i\sqrt{2} \sum_{j=1}^{n} (x_j p_j - y_j q_j)}.$$  \hspace{1cm} (3.1)

Sometimes it is useful to express $W(\alpha)$ in terms of the annihilation and creation operators defined by (2.8):

$$W(\alpha) = e^{\sum_{j=1}^{n} (\alpha_j a_j^\dagger - \bar{\alpha}_j a_j)}$$  \hspace{1cm} (3.2)

where the linear combination in the exponent is the closed version. A state $\rho$ in $L^2(\mathbb{R}^n)$ is said to be Gaussian if every observable of the form $\sum_{j=1}^{n} (x_j p_j - y_j q_j)$ has a normal distribution on the real line in the state $\rho$ for $x_j, y_j \in \mathbb{R}$. From [4] we have the following theorem.

**Theorem 1.** A state $\rho$ in $L^2(\mathbb{R}^n)$ is Gaussian if and only if its quantum Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(\alpha) = \text{Tr } \rho W(\alpha) = \exp \left\{ -i\sqrt{2} \left( \ell^T x - m^T y \right) - (x^T, y^T) S \left( \begin{array}{c} x \\ y \end{array} \right) \right\}$$  \hspace{1cm} (3.3)

for every $\alpha = x + iy$, $x, y \in \mathbb{R}^n$ where $\ell, m$ are vectors in $\mathbb{R}^n$ and $S$ is a real positive definite matrix of order $2n$ satisfying the matrix inequality $2S - iJ \succeq 0$, with $J$ as in (1.2).

**Proof.** We refer to the proof of Theorem 3.1 in [4]. \hfill \Box

We remark that $\ell, m$ and $S$ in (3.3) are defined by the equations

$$\ell^T x - m^T y = \text{Tr } \rho \sum_{j=1}^{n} (x_j p_j - y_j q_j)$$

$$\left( x^T, y^T \right) S \left( \begin{array}{c} x \\ y \end{array} \right) = \text{Tr } \rho X^2 - \left( \text{Tr } \rho X \right)^2 X = \sum_{j=1}^{n} (x_j p_j - y_j q_j).$$

It is clear that $\ell_j$ is the expectation value of $p_j$, $m_j$ is the expectation value of $q_j$ and $S$ is the covariance matrix of $p_1, p_2, \ldots, p_n; -q_1, -q_2, \ldots, -q_n$ in the state $\rho$ defined by (3.3). By a slight abuse of language we call $S$ the covariance matrix of the Gaussian state $\rho$. All such Gaussian covariance matrices constitute the convex set $K_n$ defined already in (1.3). We shall now investigate some properties of this convex set.

**Proposition 1 (Williamson’s normal form [11]).** Let $A$ be any real strictly positive definite matrix of order $2n$. Then there exists a unique diagonal matrix $D$ of order $n$
with diagonal entries \( d_1 \geq d_2 \geq \cdots \geq d_n > 0 \) and a symplectic matrix \( M \) in \( \text{Sp}(2n, \mathbb{R}) \) such that

\[
A = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M. \tag{3.4}
\]

**Proof.** Define

\[
B = A^{1/2} JA^{1/2}
\]

where \( J \) is given by (1.2). Then \( B \) is a real skew symmetric matrix of full rank. Hence its eigenvalues, inclusive of multiplicity, can be arranged as \( \pm id_1, \pm id_2, \ldots, \pm id_n \) where \( d_1 \geq d_2 \geq \cdots \geq d_n > 0 \). Define \( D = \text{diag}(d_1, d_2, \ldots, d_n) \), i.e., the diagonal matrix with \( d_i \) as the \( ii \)-th entry for \( 1 \leq i \leq n \). Then there exists a real orthogonal matrix \( \Gamma \) of order \( 2n \) such that

\[
\Gamma^T B \Gamma = \begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}.
\]

Define

\[
L = A^{1/2} \Gamma \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix}.
\]

Then \( L^T J L = J \) and

\[
L^T J L = J
\]

and

\[
L^T J L L = J
\]

Putting \( M = (L^{-1})^T \) we obtain (3.4).

To prove the uniqueness of \( D \), suppose \( D' = \text{diag}(d'_1, d'_2, \ldots, d'_n) \) with \( d'_1 \geq d'_2 \geq \cdots \geq d'_n > 0 \) and \( M' \) is another symplectic matrix of order \( 2n \) such that

\[
A = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M = M'^T \begin{bmatrix} D' & 0 \\ 0 & D' \end{bmatrix} M'.
\]

Putting \( N = MM'^{-1} \) we get a symplectic \( N \) such that

\[
N^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} N = \begin{bmatrix} D' & 0 \\ 0 & D' \end{bmatrix}.
\]

Substituting \( N^T = J N^{-1} J^{-1} \) we get

\[
N^{-1} \begin{bmatrix} D & 0 \\ -D & 0 \end{bmatrix} N = \begin{bmatrix} D' & 0 \\ 0 & D' \end{bmatrix}.
\]

Identifying the eigenvalues on both sides we get \( D = D' \). \( \Box \)

**Theorem 2.** A real positive definite matrix \( S \) is in \( K_n \) if and only if there exists a diagonal matrix \( D = \text{diag}(d_1, d_2, \ldots, d_n) \) with \( d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2} \) and a symplectic matrix \( M \in \text{Sp}(2n, \mathbb{R}) \) such that

\[
A = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M.
\]
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\[ S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M. \quad (3.5) \]

In particular,

\[ \det S = \prod_{i} d_j^2 \geq 4^{-n}. \quad (3.6) \]

**Proof.** Let $S$ be a real strictly positive definite matrix in $K_n$. From (1.3) we have $S \geq \frac{i}{2}J$ and therefore, for any $L \in Sp(2n, \mathbb{R})$,

\[ L^T S L \geq \frac{i}{2}J. \quad (3.7) \]

Using Proposition 1 choose $L$ so that

\[ L^T S L = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \]

where $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. Now (3.7) implies

\[ \begin{bmatrix} D & \frac{i}{2}I \\ -\frac{i}{2}I & D \end{bmatrix} \geq 0. \]

The minor of second order in the left hand side arising from the $jj$, $jn+j$, $n+j$, $n+jn+j$ entries is $d_j^2 - \frac{1}{2} \geq 0$. Choosing $L = M^{-1}$ we obtain (3.5) and (3.6). Now we drop the assumption of strict positive definiteness on $S$. From the definition of $K_n$ in (1.3) it follows that for any $S \in K_n$ one has $S + \epsilon I \in K_n$ for every $\epsilon > 0$. Since $S + \epsilon I$ is strictly positive definite $\det S + \epsilon I \geq 4^{-n} \forall \epsilon > 0$. Letting $\epsilon \to 0$ we see that (3.6) holds and $S$ is strictly positive definite.

To prove the converse, consider an arbitrary diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$. Clearly

\[ 2 \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0, \]

and hence for any $M \in Sp(2n, \mathbb{R})$

\[ 2M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0. \]

In other words,

\[ M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M \in K_n \quad M \in Sp(2n, \mathbb{R}). \]

Finally, the uniqueness of the parameters $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$ in the theorem is a consequence of Proposition 1. \qed
We now prove an elementary lemma on diagonal matrices before the statement of our next result on the convex set $K_n$.

**Lemma 1.** Let $D \geq I$ be a positive diagonal matrix of order $n$. Then there exist positive diagonal matrices $D_1, D_2$ such that

$$D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2}(D_1^{-1} + D_2^{-1}).$$

**Proof.** We write $D_2 = D_1 X$ and solve for $D_1$ and $X$ so that

$$2D = D_1 (I + X) = D_1^{-1} (I + X^{-1}),$$

$D_1$ and $X$ being diagonal. Eliminating $D_1$ we get the equation

$$(I + X)(I + X^{-1}) = 4D^2$$

which reduces to the quadratic equation

$$X^2 + (2 - 4D^2)X + I = 0.$$

Solving for $X$ we do get a positive diagonal matrix solution

$$X = I + 2(D^2 - 1) + 2D(D^2 - I)^{1/2}.$$

Writing

$$D_1 = 2D(I + X)^{-1}, \quad D_2 = D_1 X$$

we get $D_1, D_2$ satisfying the required property. \(\square\)

**Theorem 3.** A real positive definite matrix $S$ of order $2n$ belongs to $K_n$ if and only if there exist symplectic matrices $L, M$ such that

$$S = \frac{1}{4}(L^T L + M^T M).$$

Furthermore, $S$ is an extreme point of $K_n$ if and only if $S = \frac{1}{2}L^T L$ for some symplectic matrix $L$.

**Proof.** Let $S \in K_n$. By Theorem 2 we express $S$ as

$$S = N^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} N \quad (3.8)$$

where $N$ is symplectic and $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$. Thus $2D \geq I$ and by Lemma 2 there exist diagonal matrices $D_1 > 0, D_2 > 0$ such that

$$2D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2}(D_1^{-1} + D_2^{-1}).$$

We rewrite (3.8) as
The Symmetry Group of Gaussian States in $L^2(\mathbb{R}^n)$

\[ S = \frac{1}{4} N^T \left( \begin{bmatrix} D_1 & 0 \\ 0 & D_1^{-1} \end{bmatrix} + \begin{bmatrix} D_2 & 0 \\ 0 & D_2^{-1} \end{bmatrix} \right) N. \]

Putting
\[ L = \begin{bmatrix} D_1^{1/2} & 0 \\ 0 & D_1^{-1/2} \end{bmatrix}, \quad M = \begin{bmatrix} D_2^{1/2} & 0 \\ 0 & D_2^{-1/2} \end{bmatrix} \]
we have
\[ S = \frac{1}{4} (L^T L + M^T M). \]

Since \( \begin{bmatrix} D_1^{1/2} & 0 \\ 0 & D_1^{-1/2} \end{bmatrix}, \ i = 1, 2 \) are symplectic it follows that $L$ and $M$ are symplectic. This proves the only if part of the first half of the theorem.

Since \( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0 \)
multiplication by $L^T$ on the left and $L$ on the right shows that $L^T L - iJ \geq 0$ for any symplectic $L$. Hence $\frac{1}{2} L^T L \in K_n \forall L \in Sp(2n, \mathbb{R})$. Since $K_n$ is convex, $\frac{1}{4} (L^T L + M^T M) \in K_n$, completing the proof of the first part.

The first part also shows that for an element $S$ of $K_n$ to be extremal it is necessary that $S = \frac{1}{2} L^T L$ for some symplectic $L$. To prove sufficiency, suppose there exist $L \in Sp(2n, \mathbb{R}), S_1, S_2 \in K_n$ such that
\[ \frac{1}{2} L^T L = \frac{1}{2} (S_1 + S_2). \]

By the first part of the theorem there exist $L_j \in Sp(2n, \mathbb{R})$ such that
\[ L^T L = \frac{1}{4} \sum_{j=1}^{d} L_j^T L_j \quad (3.9) \]

where $S_1 = \frac{1}{4} (L_1^T L_1 + L_2^T L_2), S_2 = \frac{1}{4} (L_3^T L_3 + L_4^T L_4)$. Left multiplication by $(L^T)^{-1}$ and right multiplication by $L^{-1}$ on both sides of (3.9) yields
\[ I = \frac{1}{4} \sum_{j=1}^{d} M_j \quad (3.10) \]

where
\[ M_j = (L_j^T)^{-1} L_j^T L_j L^{-1}. \]

Each $M_j$ is symplectic and positive definite. Multiplying by $J$ on both sides of (3.10) we get
\[
J = \frac{1}{4} \sum_{j=1}^{4} M_j J
= \frac{1}{4} \sum_{j=1}^{4} M_j M_j^{-1}
= \frac{1}{4} \sum_{j=1}^{4} M_j^{-1}.
\]

Thus
\[
I = \frac{1}{4} \sum_{j=1}^{4} M_j = \frac{1}{4} \sum_{j=1}^{4} M_j^{-1} = \frac{1}{4} \sum_{j=1}^{4} \frac{1}{2} (M_j + M_j^{-1}),
\]
which implies
\[
\sum_{j=1}^{4} \left( M_j^{1/2} - M_j^{-1/2} \right)^2 = 0,
\]
or
\[
M_j = I \quad 1 \leq j \leq 4
\]
Thus
\[
L_j^T L_j = L_j^T L \quad \forall j
\]
and \(S_1 = S_2\). This completes the proof of sufficiency. \(\square\)

**Corollary 1.** Let \(S_1, S_2\) be extreme points of \(K_n\) satisfying the inequality \(S_1 \geq S_2\). Then \(S_1 = S_2\).

**Proof.** By Theorem 3 there exist \(L_i \in Sp(2n, \mathbb{R})\) such that \(S_i = \frac{1}{2} L_i^T L_i, i = 1, 2\). Note that \(M = L_2 L_1^{-1}\) is symplectic and the fact that \(S_1 \geq S_2\) can be expressed as \(M^T M \leq I\). Thus the eigenvalues of \(M^T M\) lie in the interval \((0, 1]\) but their product is equal to \((\det M)^2 = 1\). This is possible only if all the eigenvalues are unity, i.e., \(M^T M = I\). This at once implies \(L_1^T L_1 = L_2^T L_2\). \(\square\)

Using the Williamson’s normal form of the covariance matrix and the transformation properties of Gaussian states in Section 3 of [4] we shall now derive a formula for the density operator of a general Gaussian state. As in [4] denote by \(\rho_\ell(\ell, m, S)\) the Gaussian state in \(L_2^2(\mathbb{R}^n)\) with the quantum Fourier transform
\[
\rho_\ell(\ell, m, S)^\wedge(z) = \exp \left(-i \sqrt{2} (\ell^T x - m^T y) - (x^T y^T) S \begin{pmatrix} x \\ y \end{pmatrix} \right), z = x + iy
\]
where \(\ell, m \in \mathbb{R}^n\) and \(S\) has the Williamson’s normal form
\[
S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M
\]
with \(M \in Sp(2n, \mathbb{R}), D = \text{diag}(d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}\). From Corollary 3.3 of [4] we have
Theorem 4. Let $\rho_\varepsilon(\ell, m, S)$ be the Gaussian state in $L^2(\mathbb{R}^2)$ with mean momentum and position vectors $\ell, m$ respectively and covariance matrix $S$ with Williamson’s normal form

$$S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M, \quad M \in Sp(2n, \mathbb{R}),$$

$$D = \text{diag}(d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \cdots \geq d_m > d_{m+1} = \cdots = d_n = \frac{1}{2}, \quad d_j = \frac{1}{2} \coth \frac{1}{2} s_j, \quad 1 \leq j \leq m, \quad s_j > 0.$$ 

Then

$$\rho_\varepsilon(\ell, m, S) = W\left(\frac{m + i\ell}{\sqrt{2}}\right)^{\top} \rho_\varepsilon(\ell, m) W\left(\frac{m + i\ell}{\sqrt{2}}\right) = \rho_\varepsilon(0, 0, S)$$

and Corollary 3.5 of [4] implies

$$\rho_\varepsilon(0, 0, S) = \Gamma(M)^{-1} \rho_\varepsilon(0, 0, \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}) \Gamma(M).$$

Since $\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ is a diagonal covariance matrix

$$\rho_\varepsilon(0, 0, \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}) = \bigotimes_{j=1}^n \rho_\varepsilon(0, 0, d_j I_2)$$

where the $j$-th component in the right hand side is the Gaussian state in $L^2(\mathbb{R})$ with means 0 and covariance matrix $d_j I_2, I_2$ denoting the identity matrix of order 2. If $d_j = \frac{1}{2}$ we have

$$\rho_\varepsilon(0, 0, \frac{1}{2} I_2) = |\langle 0 | 0 \rangle| \text{ in } L^2(\mathbb{R}).$$

If $d_j > 1/2$, writing $d_j = \frac{1}{2} \coth \frac{1}{2} s_j$, one has

$$\rho_\varepsilon(0, 0, d_j I_2) = (1 - e^{-s_j}) e^{-s_j a^a} = 2 \sinh \frac{1}{2} s_j e^{-\frac{1}{2} s_j (p^2 + q^2)} \text{ in } L^2(\mathbb{R})$$

with $a, a^a, p, q$ denoting the operator of annihilation, creation, momentum and position respectively in $L^2(\mathbb{R})$. We now identify $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R})^\otimes n$ and combine the reductions done above to conclude the following:

Theorem 4. Let $\rho_\varepsilon(\ell, m, S)$ be the Gaussian state in $L^2(\mathbb{R}^n)$ with mean momentum and position vectors $\ell, m$ respectively and covariance matrix $S$ with Williamson’s normal form

$$S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M, \quad M \in Sp(2n, \mathbb{R}),$$

$$D = \text{diag}(d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \cdots \geq d_m > d_{m+1} = \cdots = d_n = \frac{1}{2}, \quad d_j = \frac{1}{2} \coth \frac{1}{2} s_j, \quad 1 \leq j \leq m, \quad s_j > 0.$$ 

Then

$$\rho_\varepsilon(\ell, m, S) = W\left(\frac{m + i\ell}{\sqrt{2}}\right)^{\top} \Gamma(M)^{-1} \prod_{j=1}^m (1 - e^{-s_j}) \times \left(\rho_\varepsilon(0, 0, S)\right)^{\otimes n} \Gamma(M) W\left(\frac{m + i\ell}{\sqrt{2}}\right)^{-1} \quad (3.11)$$

where $W(\cdot)$ denotes Weyl operator, $\Gamma(M)$ is the unitary operator implementing the Bogolioubov automorphism of CCR corresponding to the symplectic linear trans-
formation \( M \) and \( |e(0)\rangle \) denotes the exponential vector corresponding to 0 in any copy of \( L^2(\mathbb{R}) \).

**Proof.** Immediate from the discussion preceding the statement of the theorem. \( \Box \)

**Corollary 2.** The wave function of the most general pure Gaussian state in \( L^2(\mathbb{R}^n) \) is of the form

\[
|\psi\rangle = W(\alpha) \Gamma(U) |e_{\lambda_1}\rangle |e_{\lambda_2}\rangle \cdots |e_{\lambda_n}\rangle
\]

where

\[
e_{\lambda}(x) = (2\pi)^{-1/4} \lambda^{-1/2} \exp(-4^{-1} \lambda^{-2} x^2), \quad x \in \mathbb{R}, \lambda > 0,
\]

\( \alpha \in \mathbb{C}^n, U \) is a unitary matrix of order \( n \), \( W(\alpha) \) is the Weyl operator associated with \( \alpha \), \( \Gamma(U) \) is the second quantization unitary operator associated with \( U \) and \( \lambda_j, 1 \leq j \leq n \) are positive scalars.

**Proof.** Since the number operator \( a^\dagger a \) has spectrum \( \{0, 1, 2, \ldots\} \) it follows from Theorem 4 that \( \rho_g(\ell, m, S) \) is pure if and only if \( m = 0 \) in (3.11). This implies that the corresponding wave function \( |\psi\rangle \) can be expressed as

\[
|\psi\rangle = W(\alpha) \Gamma(M)^{-1} |e(0)\rangle \otimes^n
\]

(3.12)

where \( M \in Sp(2n, \mathbb{R}) \) and \( \alpha = \frac{m+i\ell}{\sqrt{2}} \). The covariance matrix of this pure Gaussian state is \( \frac{1}{2} M^T M \). The symplectic matrix \( M \) has the decomposition

\[
M = V_1 \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} V_2
\]

where \( V_1 \) and \( V_2 \) are real orthogonal as well as symplectic and \( D \) is a positive diagonal matrix of order \( n \). Thus

\[
M^T M = V_2^T \begin{bmatrix} D^2 & 0 \\ 0 & D^{-2} \end{bmatrix} V_2 = N^T N
\]

where

\[
N = \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} V_2.
\]

Since the covariance matrix of \( |\psi\rangle \) in (3.12) can also be written as \( \frac{1}{2} N^T N \), modulo a scalar multiple of modulus unity \( |\psi\rangle \) can also be expressed as

\[
|\psi\rangle = W(\alpha) \Gamma(V_2)^{-1} \Gamma \left( \begin{bmatrix} D^{-1} & 0 \\ 0 & D \end{bmatrix} \right) |e(0)\rangle \otimes^n.
\]

(3.13)

If \( U \) is the complex unitary matrix of order \( n \) satisfying
The Symmetry Group of Gaussian States in $L^2(\mathbb{R}^n)$

$$U(x + iy) = x' + iy', \quad \begin{bmatrix} x' \\ iy' \end{bmatrix} = V_T^2 \begin{bmatrix} x \\ y \end{bmatrix} \quad \forall \ x, y \in \mathbb{R}^n$$

and $D^{-1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ we can express (3.13) as

$$|\psi\rangle = W(\alpha) \Gamma(U) \{ \bigotimes_{j=1}^n \Gamma\left( \begin{bmatrix} \lambda_j \\ 0 \\ 0 \end{bmatrix} \right) |e(0)\rangle \}$$

$$= W(\alpha) \Gamma(U) |e_{\lambda_1}\rangle |e_{\lambda_2}\rangle \cdots |e_{\lambda_n}\rangle$$

where we have identified $L^2(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$.

We conclude this section with a result on the purification of Gaussian states. □

**Theorem 5.** Let $\rho$ be a mixed Gaussian state in $L^2(\mathbb{R}^n)$. Then there exists a pure Gaussian state $|\psi\rangle$ in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ such that

$$\rho = \text{Tr}_2 U |\psi\rangle \langle \psi| U^\dagger$$

for some unitary operator $U$ in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ with $\text{Tr}_2$ denoting the relative trace over the second copy of $L^2(\mathbb{R}^n)$.

**Proof.** First we remark that by a Gaussian state in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ we mean it by the canonical identification of this product Hilbert space with $L^2(\mathbb{R}^{2n})$. Let $\rho = \rho_g(\ell, m, S)$ where by Theorem 3 we can express

$$S = \frac{1}{4} (L_1^2 L_1 + L_2^2 L_2), \quad L_1, L_2 \in \text{Sp}(2n, \mathbb{R}).$$

Now consider the pure Gaussian states,

$$|\psi_{L_i}\rangle = \Gamma(L_i)^{-1} |e(0)\rangle, \quad i = 1, 2$$

in $L^2(\mathbb{R}^n)$ and the second quantization unitary operator $\Gamma_0$ satisfying

$$\Gamma_0 e(u \oplus v) = e\left( \frac{u + v}{\sqrt{2}} \oplus \frac{u - v}{\sqrt{2}} \right) \quad \forall \ u, v \in \mathbb{C}^n$$

in $L^2(\mathbb{R}^{2n})$ identified with $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, so that

$$e(u \oplus v) = e(u) \otimes e(v).$$

Then by Proposition 3.11 of [4] we have

$$\text{Tr}_2 \Gamma_0 (|\psi_{L_1}\rangle \langle \psi_{L_1}| \otimes |\psi_{L_2}\rangle \langle \psi_{L_2}|) \Gamma_0^\dagger = \rho_g(0, 0, S).$$

If $\alpha = \frac{m + i\ell}{\sqrt{2}}$ we have
\[ W(\alpha) \rho_0(0,0,S) W(\alpha)^\dagger = \rho_0(\ell,m,S). \]

Putting
\[ U = (W(\alpha) \otimes I) \Gamma_0 (\Gamma(L_1)^{-1} \otimes \Gamma(L_2)^{-1}) \]
we get
\[ \rho_g(\ell,m,S) = \text{Tr}_2 U \left| e(0) \otimes e(0) \right\rangle \langle e(0) \otimes e(0) | U^\dagger \]
where \( |e(0)\rangle \) is the exponential vector in \( L^2(\mathbb{R}^n) \).

4 The symmetry group of the set of Gaussian states

Let \( S_n \) denote the set of all Gaussian states in \( L^2(\mathbb{R}^n) \). We say that a unitary operator \( U \) in \( L^2(\mathbb{R}^n) \) is a Gaussian symmetry if, for any \( \rho \in S_n \), the state \( U \rho U^\dagger \) is also in \( S_n \). All such Gaussian symmetries constitute a group \( G_n \). If \( \alpha \in \mathbb{C}^n \) and \( L \in \text{Sp}(2n, \mathbb{R}) \) then the associated Weyl operator \( W(\alpha) \) and the unitary operator \( \Gamma(L) \) implementing the Bogolioubov automorphism of CCR corresponding to \( L \) are in \( G_n \). (See Corollary 3.5 in [4].) The aim of this section is to show that any element \( U \) in \( G_n \) is of the form \( \lambda W(\alpha) \Gamma(L) \beta \) where \( \lambda \) is a complex scalar of modulus unity, \( \alpha \in \mathbb{C}^n \) and \( L \in \text{Sp}(2n, \mathbb{R}) \). This settles a question raised in [4].

We begin with a result on a special Gaussian state.

**Theorem 6.** Let \( s_1 > s_2 > \cdots > s_n > 0 \) be irrational numbers which are linearly independent over the field \( \mathbb{Q} \) of rationals and let
\[ \rho_s = \rho_0(0,0,S) = \prod_{j=1}^n (1 - e^{-s_j}) e^{-\sum_{j=1}^n s_j a_j^\dagger a_j} \]
be the Gaussian state in \( L^2(\mathbb{R}^n) \) with zero position and momentum mean vectors and covariance matrix
\[ S = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \quad D = \text{diag}(d_1, d_2, \ldots, d_n) \]
with \( d_j = \frac{1}{2} \coth \frac{s_j}{2} \). Then a unitary operator \( U \) in \( L^2(\mathbb{R}^n) \) has the property that \( U \rho_s U^\dagger \) is a Gaussian state if and only if, for some \( \alpha \in \mathbb{C}^n \), \( L \in \text{Sp}(2n, \mathbb{R}) \) and a complex-valued function \( \beta \) of modulus unity on \( \mathbb{Z}_+^n \),
\[ U = W(\alpha) \Gamma(L) \beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n) \quad (4.1) \]
where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \).

**Proof.** Sufficiency is immediate from Corollary 3.3 and Corollary 3.5 of [4]. To prove necessity assume that
\[ U \rho_s U^\dagger = \rho_0(\ell,m,S') \quad (4.2) \]
Since $a^j a$ in $L^2(\mathbb{R})$ has spectrum $\mathbb{Z}_+$ and each eigenvalue $k$ has multiplicity one, it follows that the selfadjoint positive operator $\sum_{j=1}^n s_j a^j a_j$, being a sum of commuting self adjoint operators $s_j a^j a_j$, $1 \leq j \leq n$ has spectrum $\mathbb{Z}_+^+$ and each eigenvalue $k$ has multiplicity one \cite{2}.

Since $\rho_s$ and $U \rho_s U^{-1}$ have the same set of eigenvalues and same multiplicities it follows from Theorem 4 that
\[
U \rho_s U^{-1} = W(z) \Gamma(M)^{-1} \rho_t \Gamma(M) W(z)^{-1}
\] (4.3)
where $z \in \mathbb{C}^n$, $M \in Sp(2n, \mathbb{R})$, $t = (t_1, t_2, \ldots, t_n)^T$ and
\[
\rho_t = \prod_{j=1}^n (1 - e^{-t_j}) e^{-\sum_{j=1}^n t_j a^j a_j}.
\]

Since the maximum eigenvalues of $\rho_s$ and $\rho_t$ are same it follows that
\[
\prod_{j=1}^n (1 - e^{-t_j}) = \prod_{j=1}^n (1 - e^{-s_j}).
\]
Since the spectra of $\rho_s$ and $\rho_t$ are same it follows that
\[
\left\{ \sum_{j=1}^n s_j k_j \mid k_j \in \mathbb{Z}_+ \quad \forall \ j \right\} = \left\{ \sum_{j=1}^n t_j k_j \mid k_j \in \mathbb{Z}_+ \quad \forall \ j \right\}.
\]
Choosing $k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$ with 1 in the $k$-th position we conclude the existence of matrices $A, B$ of order $n \times n$ and entries in $\mathbb{Z}_+$ such that
\[
t = A s, \quad s = B t
\]
so that $B A s = s$. The rationally linear independence of the $s_j$’s implies $BA = I$. This is possible only if $A$ and $B = A^{-1}$ are both permutation matrices.

Putting $V = \Gamma(M) W(z)^T U$ we have from (4.3)
\[
V \rho_s = \rho_t V.
\]
Denote by $|k\rangle$ the vector satisfying
\[
a^j a_j |k\rangle = k_j |k\rangle
\]
where $|k\rangle = |k_1\rangle |k_2\rangle \cdots |k_n\rangle$. Then
\[
V \rho_s |k\rangle = \prod_{j=1}^n (1 - e^{-s_j}) e^{-\sum_{j=1}^n s_j k_j} V |k\rangle
= \rho_t V |k\rangle, \quad k \in \mathbb{Z}_+^n.
\]
Thus $V |k\rangle$ is an eigenvector for $\rho_t$ corresponding to the eigenvalue

$$\prod_{j=1}^{n}(1 - e^{-s_j}) e^{-t^T b^T k} = \prod_{j=1}^{n}(1 - e^{-s_j}) e^{-t^T T B T k} = \prod_{j=1}^{n}(1 - e^{-s_j}) e^{-t^T T A k}.$$ 

Hence there exists a scalar $\beta(k)$ of modulus unity such that

$$V |k\rangle = \beta(k) |A k\rangle = \Gamma(A) \beta(a_1^+, a_2^+, \ldots, a_n^+) \forall k \in \mathbb{Z}^n_+.$$ 

where $\Gamma(A)$ is the second quantization of the permutation unitary matrix $A$ acting in $\mathbb{C}^n$. Thus

$$U = W(z) \Gamma(M)^T \Gamma(A) \beta(a_1^+, a_2^+, \ldots, a_n^+),$$

which completes the proof.  

**Theorem 7.** A unitary operator $U$ in $L^2(\mathbb{R}^n)$ is a Gaussian symmetry if and only if there exist a scalar $\lambda$ of modulus unity, a vector $\alpha$ in $\mathbb{C}^n$ and a symplectic matrix $L \in Sp(2n, \mathbb{R})$ such that

$$U = \lambda W(\alpha) \Gamma(L)$$

where $W(\alpha)$ is the Weyl operator associated with $\alpha$ and $\Gamma(L)$ is a unitary operator implementing the Bogolioubov automorphism of CCR corresponding to $L$.

**Proof.** The if part is already contained in Corollary 3.3 and Corollary 3.5 of [4]. In order to prove the only if part we may, in view of Theorem 6, assume that $U = \beta(a_1^+, a_2^+, \ldots, a_n^+)$ where $\beta$ is a function of modulus unity on $\mathbb{Z}^n_+$. If such a $U$ is a Gaussian symmetry then, for any pure Gaussian state $|\psi\rangle$, $U |\psi\rangle$ is also a pure Gaussian state. We choose $|\psi\rangle = e^{-\frac{1}{2} \|u\|^2} e(\alpha^+) = W(u) |e(0)\rangle$ where $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{C}^n$ with $u_j \neq 0 \forall j$. By our assumption

$$|\psi'\rangle = e^{-\frac{1}{2} \|u\|^2} \beta(a_1^+, a_2^+, a_3^+, \ldots, a_n^+) e(\alpha^+)$$

is also a pure Gaussian state. By Corollary 2 $\exists \alpha \in \mathbb{C}^n$, a unitary matrix $A$ of order $n$ and $\lambda_j > 0$, $1 \leq j \leq n$ such that

$$|\psi'\rangle = W(\alpha) \Gamma(A) |e_{\lambda_1}\rangle |e_{\lambda_2}\rangle \cdots |e_{\lambda_n}\rangle.$$ 

Using (4.4) and (4.5) we shall evaluate the function $f(z) = \langle \psi | e(z) \rangle$ in two different ways. From (4.4) we have
Thus

where $|k_1 k_2 \cdots k_n| = |k_1| |k_2| \cdots |k_n|$ and $|e(z)| = \sum_{k \in \mathbb{Z}_+} \frac{e^k}{\sqrt{k!}} |k|$ for $z \in \mathbb{C}$.

Since $|\beta(k)| = 1$, (4.6) implies

$$|f(z)| \leq \exp \left\{ -\frac{1}{2} ||u||^2 + \sum_{j=1}^{n} |u_j| |z_j| \right\}.$$  \hspace{1cm} (4.7)

From the definition of $|e_\lambda\rangle$ in Corollary [2] and the exponential vector $|e(z)\rangle$ in $L^2(\mathbb{R})$ one has

$$\langle e_\lambda | e(z) \rangle = \sqrt{\frac{2\lambda}{1+\lambda^2}} \exp \frac{1}{2} \left( \frac{\lambda^2 - 1}{\lambda^2 + 1} \right) z^2, \quad \lambda > 0, \; z \in \mathbb{C}. $$

This together with (4.5) implies

$$f(z) = \langle e_{\lambda_1} \otimes e_{\lambda_2} \cdots \otimes e_{\lambda_n} | \Gamma(A^{-1}) W(-\alpha) e(z) \rangle$$

$$= e^{\langle \alpha | z \rangle - \frac{1}{2} ||\alpha||^2} \langle e_{\lambda_1} \otimes e_{\lambda_2} \cdots \otimes e_{\lambda_n} | e(A^{-1}(z + \alpha)) \rangle$$

which is a nonzero scalar multiple of the exponential of a polynomial of degree 2 in $z_1, z_2, \ldots, z_n$ except when all the $\lambda_j$’s are equal to unity. This would contradict the inequality (4.6) except when $\lambda_j = 1 \; \forall \; j$. Thus $\lambda_j = 1 \; \forall \; j$ and (4.5) reduces to

$$|\psi'\rangle = W(\alpha) \Gamma(A) |e(0)\rangle$$

$$= e^{-\frac{1}{2} ||\alpha||^2} |e(\alpha)\rangle.$$ 

Now (4.4) implies

$$\beta(a_1^1 a_2^2 \cdots a_n^a | e(u)\rangle$$

$$= e^{\frac{1}{2} ||u||^2 - ||\alpha||^2} |e(\alpha)\rangle,$$

or

$$\sum_{k \in \mathbb{Z}_+^n} \frac{u_1^k \cdot u_2^k \cdots u_n^k}{\sqrt{k_1! \cdots k_n!}} \beta(k_1 k_2 \cdots k_n | k_1 k_2 \cdots k_n) \rangle$$

$$= e^{\frac{1}{2} ||u||^2 - ||\alpha||^2} \sum_{\alpha_1^1 \alpha_2^2 \cdots \alpha_n^n} \frac{\alpha_1^1 \alpha_2^2 \cdots \alpha_n^n}{\sqrt{k_1! \cdots k_n!}} |k_1 k_2 \cdots k_n\rangle.$$ 

Thus
\[ \beta(k_1, k_2, \ldots, k_n) = e^{\frac{i}{2}(||a||^2 - ||\alpha||^2)} \left( \frac{\alpha_1}{u_1} \right)^{k_1} \cdots \left( \frac{\alpha_n}{u_n} \right)^{k_n}. \]

Since \(|\beta(k)| = 1 \) and \(u_j \neq 0 \forall j\) it follows that \(|\frac{\alpha_j}{u_j}| = 1 \) and

\[ \beta(k) = e^{i \sum_{j=1}^{n} \theta_j k_j} \quad \forall k \in \mathbb{Z}_n^+ \]

where \(\theta_j\)'s are real. Thus \(\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n) = \Gamma(D)\), the second quantization of the diagonal unitary matrix \(D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})\). This completes the proof. \(\square\)

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