ON THE LOWER BOUND ESTIMATES OF SECTIONS OF
THE CANONICAL BUNDLES OVER A RIEMANN
SURFACE

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1. Introductions

Suppose $M$ is an $n$-dimensional Kähler manifold and $L$ is an ample line bundle over $M$. Let the Kähler form of $M$ be $\omega_g$ and the Hermitian metric of $L$ be $H$. We assume that $\omega_g$ is the curvature of $H$, that is, $\omega_g = \text{Ric}(H)$. The Kähler metric of $\omega_g$ is called a polarized Kähler metric on $M$.

Using $H$ and $\omega_g$, for any positive integer $m$, $H^0(M, L^m)$ becomes a Hermitian inner product space. We use the following notations: suppose that $S, T \in H^0(M, L^m)$. Let $\langle S, T \rangle_{H^m}$ be the pointwise inner product and

$$
(S, T) = \int_M \langle S, T \rangle_{H^m} \frac{\omega_g^n}{n!}
$$

be the inner product of $H^0(M, L^m)$. Let

$$
||S|| = \sqrt{\langle S, S \rangle_{H^m}}
$$

be the pointwise norm. In particular, $||S||(x)$ denotes the pointwise norm at $x \in M$. Let

$$
||S||_{L^2} = \sqrt{(S, S)}
$$

be the $L^2$-norm of $S$. 

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Let \( \{S_1, \cdots, S_{d(m)}\} \) be an orthonormal basis of \( H^0(M, L^m) \). The quantity (see [11])

\[
\sum_{i=1}^{d(m)} ||S_i||^2
\]  

(1.1)

plays an important role in Kähler-Einstein geometry and stability of complex manifolds. If \( m \) is sufficiently large, then there is a natural embedding \( \varphi_m : M \to CP^{d(m)-1} \) by \( x \mapsto [S_1(x), \cdots, S_{d(m)}(x)] \). This follows from the Kodaira’s embedding theorem. The metric \( \frac{1}{m}\varphi^*\omega_{FS} \) on \( M \) is called the Bergman metric. The following formula is an easy but key observation made by Tian [11] in his proof of the convergence of the Bergman metrics:

\[
\frac{1}{m}\partial\bar{\partial} \log \sum_{i=1}^{d(m)} ||S_i||^2 = \frac{1}{m}\varphi^*\omega_{FS} - \omega_g.
\]

A lot of work has been done by several authors [11], [9], [15], [2] and [8] on the estimates of (1.1). However, these works are concentrated on a single manifold. On the other hand, in studying the stability and Kähler-Einstein geometry of manifolds, we need to study the behavior of (1.1) for a family of manifolds. Tian’s work [12] on the Calabi Conjecture shows the importance and non-triviality of the problem of giving lower bound estimate of (1.1) for a family of complex surfaces. In the \( n \)-dimensional case, Tian [13] proved that (1.1) has a positive lower bound depends on the dimension \( n \), the upper bound of the Betti numbers, the positive upper and lower bound of the Ricci curvature and the \( L^n \) norm of the Riemannian sectional curvature. In general, (1.1) maybe a useful tool in studying algebraic fibrations over a compact Kähler manifold. The paper of interest to us are [5] and [6].

In this paper, we shall study the behavior of (1.1) on Riemann surfaces. Even in the case of Riemann surfaces, the problem of finding a uniform lower bound of (1.1) is nontrivial. In fact, by the counterexample in §3, we know that there is no uniform lower bound in general. We proved a partial uniform estimate in §4 which, I believe, is the “right” one in the sense that it gives all the information on stability of the Riemann surfaces.

We also consider the coordinate ring of Riemann surfaces. For smooth Riemann surfaces \( M \) of genus \( g \geq 2 \), it is well known that its coordinate ring is finitely generated. That is, there is a positive integer \( m_0 \), such that for any \( S \in H^0(M, K^{-m_0}_M) \) with \( m > m_0 \), we can find \( U_i \in H^0(M, K^{-m_0}_M), i \in I \) and \( T_i \in H^0(M, K^{m-m_0}_M), i \in I \) such that

\[
S = \sum_{i \in I} U_i T_i,
\]

where \( I \) is a finite set. In studying the behavior of Riemann surfaces near the boundary of the Teichmüller space, we need some uniform estimates. In this paper, we give a uniform estimate which will give us the information on singular Riemann surfaces.
The above setting is similar to that in the corona problem in complex analysis. The corona problem on the unit disk was studied by Carleson in [1]. Carleson’s result stimulates many ideas which proved to be useful for other problems. An extensive discussion of the Carleson’s corona theorem can be found in [4].

Modifying Wolff’s [14] proof of Carleson’s theorem together with the ∂-estimate, we give a uniform corona estimate in the last section of this paper as an application of Theorem 1.3. In order to obtain the result, we take special care to the points where the injective radius are small.

The organization of the paper is as follows: in §2, we give a lower bound of (1.1) in terms of the genus $g$ and the injective radius $\delta$ of $M$. In §3, we give a counterexample which shows that the lower bound must depend on $\delta$. In §4, we give the partial uniform estimate. That is, a lower bound of (1.1) at $x \in M$ depending only on the injective radius $\delta_x$ of $x$. In §5, we solve the uniform corona problem by the partial uniform estimate.

The main results of this paper are the following:

**Theorem 1.1.** Let $M$ be a Riemann surface of the genus $g \geq 2$. Let $K_M$ be the canonical line bundle of $M$ endowed with a Hermitian metric $H$. Let the curvature $\omega_g$ of $H$ be positive. $\omega_g$ gives a Kähler metric of $M$. Let the curvature $K$ of $\omega_g$ satisfy

$$-K_1 \leq K \leq K_2$$

for nonnegative constants $K_1, K_2 \geq 0$ and let $\delta'$ be the injective radius of $M$. Let

$$\delta = \min(\delta', \frac{1}{\sqrt{K_1 + K_2}}).$$

Then there is an absolute constant $C > 0$ such that for $m \geq 2$,

$$\sum_{i=1}^{d(m)} ||S_i||^2 \geq e^{-\frac{C g^3}{\delta^4}},$$

where $\{S_1, \cdots, S_{d(m)}\}$ is an orthonormal basis of $H^0(M, K_M^m)$.

**Theorem 1.2.** For any $\varepsilon > 0$ and $m \geq 2$, there is a Riemann surface $M$ of genus $g \geq 2$ with the constant Gauss curvature $(-1)$ such that

$$\inf_{x \in M} \sum_{i=1}^{d(m)} ||S_i||^2 \leq \varepsilon.$$

The theorem disproves the conjecture that the absolute lower bound exists.

When the injective radius of $M$ goes to zero, the first eigenvalue and the Sobolev constant will also go to zero. In this case, Theorem 1.1 gives no information. In the following theorem, we proved that (1.1) has a lower bound which depends only on the local information and is independent to
the injective radius of $M$. For this reason, we call the result partial uniform estimate.

**Theorem 1.3.** Let $M$ be a Riemann surface of genus $g \geq 2$ and constant curvature $(-1)$. Then there are absolute constants $m_0 > 0$ and $D > 0$ such that for any $m > m_0$ and any $x_0 \in M$, there is a section $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2} = 1$ such that

$$||S||_L^2 \geq \frac{\sqrt{m}}{D(1 + \frac{1}{\sqrt{m} \delta x_0} e^{-\pi \delta x_0})},$$

where $\delta x_0$ is the injective radius of $\delta x_0$.

On the coordinate ring $\oplus_{j=0}^{\infty} H^0(M, K^j_M)$, we have the following

**Theorem 1.4.** Let $M$ be a Riemann surface as above. Then there is an $m_0 > 0$ such that for any $m > m_0$ and $S \in H^0(M, K_M^m)$, there is a decomposition

$$S = \sum_{i=1}^{d} S_i$$

of $S_i \in H^0(M, K_M^{m_i})(i = 1, \cdots, d)$ such that

$$\|S_i\|_{L^2} \leq C(m, m_0, g) \|S\|_{L^2}$$
$$\|S_i\|_{L^\infty} \leq C(m, m_0, g) \|S\|_{L^\infty}$$

for $i = 1, \cdots, d$, and

$$S_i = T_i U_i$$

for a basis $U_1, \cdots, U_d$ of $H^0(M, K_M^{m_0})$ and $T_1, \cdots, T_d \in H^0(M, K_M^{m_0 - m_0})$.

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2. **A lower bound estimate**

Suppose that $M$ is a Riemann surface of genus $g \geq 2$. Let $K_M$ be the canonical line bundle over $M$ with the Hermitian metric $H$. We assume that the curvature $\omega_g$ of $H$ is positive and $\omega_g$ defines a Kähler metric of $M$.

Let $K$ be the Gauss curvature of the metric $\omega_g$. Let $K_1$ and $K_2$ be two nonnegative constants such that

$$-K_1 \leq K \leq K_2.$$  

(2.1)

Let $\delta'$ be the injective radius of $M$ with and

$$\delta = \min(\delta', \frac{1}{\sqrt{K_1 + K_2}}).$$

(2.2)

Let $x_0 \in M$ be a fixed point. Let $U$ be the open set

$$U = \{\text{dist}(x, x_0) < \delta\}.$$
It is well known that at each point of $U$ there is an isothermal coordinate. In the first part of this section, we prove that there is a holomorphic function $z$ on $U$ which gives the isothermal coordinate of $U$ with the required estimate.

Consider the equation

\[
\begin{align*}
\Delta h &= K/2 \\
h|_{\partial U} &= 0,
\end{align*}
\]

(2.3)

where $\Delta$ is the (complex) Laplacian of $M$. The solution $h$ exists and is unique. Let $ds^2$ be the Riemann metric of $U$. Then we have

**Lemma 2.1.** The metric $e^h ds^2$ on $U$ is a flat metric.

**Proof.** A straightforward computation using (2.3).

Since $U$ is an open set which is diffeomorphic to an open set in the Euclidean plane, we can assume that there are global frames on $U$. Let $\omega^1$ and $\omega^2$ be 1-forms on $U$ such that

\[
e^h ds^2 = \omega^1 + \omega^2.
\]

Let $\omega_{12}$ be the connection 1-form defined by

\[
\begin{align*}
d\omega_1 &= \omega_{12} \wedge \omega_2, \\
d\omega_2 &= -\omega_{12} \wedge \omega_1.
\end{align*}
\]

Then by Lemma 2.1, $d\omega_{12} = 0$. It follows that there is a real smooth function $\sigma$ on $U$ such that

\[
\omega_{12} = d\sigma.
\]

(2.4)

Let

\[
\xi = e^{i\sigma}(\omega_1 + i\omega_2).
\]

Then by (2.4) and (2.5), we have

\[
d\xi = 0.
\]

Thus there is a function $z$ on $U$ such that

\[
\xi = dz,
\]

and

\[
e^h ds^2 = dzd\bar{z}.
\]

(2.6)

Either $z$ or $\bar{z}$ will be holomorphic because it defines a conformal structure of $U$. Without losing generality, we assume that $z$ is holomorphic and at $x_0$, $z = 0$.

We have the following lemma:

**Lemma 2.2.** Let $\rho$ be the distance to the point $x_0$. $\rho(x) = \text{dist}(x,x_0)$. Then

\[
\frac{1}{3}\rho \leq |z| \leq 3\rho
\]

(2.7)

for $\rho < \delta$. 

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Proof. By the Gauss Lemma [3, page 8], the Riemann metric $ds^2$ can be written as

$$ds^2 = d\rho^2 + f^2(\rho, \theta)d\theta^2$$

for the polar coordinate $(\rho, \theta)$ where $f(\rho, \theta)$ is a smooth function satisfying

$$f(0, \theta) = 0, \; \frac{\partial f}{\partial \rho}(0, \theta) = 1,$$

and

$$\frac{\partial^2 f}{\partial \rho^2} = -Kf.$$

By the Hessian comparison theorem [10, page 4], we have

$$\Delta \rho \geq \frac{\sqrt{K_2}}{4} \cot \sqrt{K_2} \rho.$$

In particular, $\Delta \rho \geq 0$ on $U$. Noting that $\Delta$ is the complex Laplacian, we have

$$\Delta \rho = \frac{1}{2} |\nabla \rho|^2 + 2\rho \Delta \rho \geq \frac{1}{2}. \tag{2.8}$$

By (2.8), we have

$$\Delta (h + K_1 \rho^2) \geq K/2 + K_1/2 \geq 0,$$

$$\Delta (h - K_2 \rho^2) \leq K/2 - K_2/2 \leq 0.$$

By the maximal principle, we have

$$-1 \leq -K_2 \delta^2 \leq h|\partial U| - K_2 \rho^2 \leq h \leq h|\partial U| + K_1 \rho^2 \leq K_1 \delta^2 \leq 1. \tag{2.9}$$

Let $ds_1^2 = e^h ds^2$ denotes the flat metric. Then

$$e^{-1} ds^2 \leq ds_1^2 \leq eds^2.$$

By (2.8), $|z|$ is the distance to the point $x_0$ with respect to the metric $ds_1^2$. Thus by (2.9),

$$\frac{1}{3} \rho \leq e^{-1} \rho \leq |z| \leq e \rho \leq 3 \rho.$$

Proof of Theorem 1.1. Define a smooth function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\eta(t) = \begin{cases} 0 & t \geq 1, \\ 1 & 0 \leq t \leq \frac{1}{2}. \end{cases} \tag{2.10}$$

We assume that $|\eta'| \leq 4$ and $|\eta''| \leq 4$.

In the rest of this paper $C_1, C_2, \cdots$, are absolute constants, unless otherwise stated.

Let $\delta_1 = \frac{1}{4} \delta$. Define the smooth function $r$ on $M$ such that

$$r = \begin{cases} \eta\left(\frac{|z|}{\delta_1}\right) \log\left(\frac{|z|}{\delta_1}\right) & x \in U, \\ 0 & x \notin U. \end{cases} \tag{2.11}$$
r is well defined. For if \( x \in \partial U \), then \( \rho = \delta \). By the Lemma 2.2, \( |z| \geq \frac{4}{3} \delta_1 \) and thus \( r|_{\partial U} = 0 \) using either expression.

Note that if \( z \neq 0 \), then \( \Delta \log(|z|) = 0 \). Define the function \( \psi \) such that \( \psi = \Delta r \) for \( z \neq 0 \) and \( \psi = 0 \) for \( z = 0 \). We have

**Lemma 2.3.** There is a constant \( C_1 > 0 \) such that

\[
\begin{align*}
|\psi| & \leq \frac{C_1}{\delta_2}, \\
\int_M |\psi| & \leq C_1.
\end{align*}
\]

**Proof.** A straightforward computation gives

\[
\psi = \Delta r = \frac{1}{4} e^h \left( \frac{1}{\delta_1} \eta'' \log\left( \frac{|z|}{\delta_1} \right) + \eta' \frac{1}{\delta_1 |z|} \log\left( \frac{|z|}{\delta_1} \right) + 2 \eta' \frac{1}{\delta_1 |z|} \right)
\]

for \( \frac{\delta_1}{2} < z < \delta_1 \). Using (2.9), (2.10), we have the estimate

\[ |\psi| \leq C_2 \delta_2 \]

for some constant \( C_2 \). To get the estimate of the \( \int_M |\psi| \), we first see that by the volume comparison theorem [10, page 11],

\[
\text{vol}(U) \leq 2\pi \left( \frac{\cosh \sqrt{K_1} \delta_1 - 1}{K_1} \right).
\]

Since \( \sqrt{K_1} \delta_1 \leq 1 \), there is a constant \( C_3 \) such that

\[ \text{vol}(U) \leq C_3 \delta_2^2. \]

The lemma follows by setting \( C_1 = \max(C_2, C_2 C_3) \).

Let \( G(x, y) \) be the Green’s function of \( M \). That is,

\[
\begin{align*}
\Delta_x G(x, y) &= \frac{1}{4} (-\delta_x(y) + \frac{1}{\text{vol}(M)}), \\
\int_M G(x, y) dx &= 0,
\end{align*}
\]

where \( \Delta_x \) is the (complex) Laplacian with respect to \( x \) and \( \delta_x(\cdot) \) is the Dirac function. Let \( b \) be the function on \( M \) such that

\[
\begin{align*}
\Delta b &= K/4 + 1/2, \\
\int_M b &= 0.
\end{align*}
\]

Since the Kähler metric \( \omega_g \in -c_1(M) \), the above equation has a unique solution.

Let the function \( a : M \to \mathbb{R} \) defined by

\[
a = G(x, x_0) + \frac{1}{2\pi} \left( r - \frac{1}{\text{vol}(M)} \int_M r \right) + b,
\]

where \( x_0 \) is the fixed point of \( M \) and \( r \) is defined in (2.13). Then \( a \) is a smooth function on \( M \). We have

\[
\begin{align*}
\Delta a &= \frac{1}{4\text{vol}(M)} + \frac{1}{2\pi} \psi + K/4 + 1/2, \\
\int_M a &= 0.
\end{align*}
\]
Lemma 2.4. There is a constant $C_5$ such that
\[ |a| \leq \frac{C_5 g^2}{\delta^6}, \]
where $g$ is the genus of the Riemann surface $M$.

**Proof.** Let $\lambda_1$ be the first eigenvalue of $M$, then by the Poincare inequality, we have
\[ \lambda_1 \int_M a^2 \leq \int_M |\nabla a|^2. \]
Integration by parts using (2.17), we have
\[ \int_M |\nabla a|^2 \leq \int_M |a(\frac{1}{2\pi} \psi + \frac{1}{4 \text{vol}(M)} + \frac{K_1 + K_2}{4} + \frac{1}{2})|. \]
Let $g$ be the genus of $M$. By the Gauss-Bonnet Theorem, $\text{vol}(M) = 4\pi(g - 1)$. On the other hand, $K_1 + K_2 \leq \frac{1}{\delta^2}$. Let $a(x') = \max |a|$. By (2.17), (2.18) and Lemma 2.3
\[ \int_M a^2 \leq \frac{(C_1 + 6\pi)g}{\lambda_1 \delta^2} a(x'). \]
Consider a neighborhood $U'$ of $x'$ defined by
\[ U' = \{ x | \text{dist}(x, x') < \delta \}. \]
Let $z$ be the holomorphic function in Lemma 2.2 such that $z(x') = 0$. Let $U_1 = \{|z| < \frac{1}{3} \delta \}$.
Let $\tilde{\Delta} = \frac{\partial^2}{\partial z \partial \bar{z}}$ be the Euclidean Laplacian on $U_1$. Then by (2.9), (2.12) and (2.16), we have
\[ |\tilde{\Delta} a| \leq 3(2 + \frac{C_1 + 1}{\delta^2}). \]
It follows from an elementary fact that there is a constant $C_4$ such that
\[ a(x') \leq C_4(\log \frac{1}{\delta} + \frac{1}{\delta} (\int_{U_1} a^2(x))^{\frac{1}{2}}). \]
On the other hand, Cheeger’s inequality \[10\], page 91] gives
\[ \lambda_1 \geq \frac{1}{4g^2 \delta^2}. \]
Combining (2.19), (2.21) and (2.22), we have the required estimate. \qed

Let
\[ \varphi = -4\pi(G(x, x_0) + b). \]
Then for $x \neq x_0$, we have
\[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi \geq \left( - \frac{1}{2\text{vol}(M)} - \frac{K}{2} - 1 \right) \omega_g. \]
Lemma 2.5. There is a constant $C_6$ such that
\[ \varphi \leq \frac{C_6 g^3}{\delta^6}, \]
and for $|z| \leq \delta_1$,
\[ \varphi \geq -\frac{C_6 g^3}{\delta^6} + 2 \log |z|. \]

Proof. By (2.15),
\[ \varphi = -4\pi (a - \frac{1}{2\pi} (r - \frac{1}{\text{vol}(M)} \int_M r)) \]
The lemma follows from Lemma 2.4 and (2.11).

We need the following proposition from Demailly (see [11]):

Proposition 2.1. Suppose that $(M, g)$ is a complete Kähler manifold of complex dimension $n$, $L$ is a line bundle on $M$ with the Hermitian metric $h$, and $\varphi$ is a function on $M$, which can be approximated by a decreasing sequence of smooth functions $\{\varphi_l\}_{l \leq 1 < \infty}$. If
\[ <\partial\bar{\partial}\varphi_l + 2\pi \sqrt{-1} (\text{Ric}(h) + \text{Ric}(g)), v \wedge \overline{v} >_g \geq C||v||^2_g \]
for any tangent vector $v$ of type $(1, 0)$ at any point of $M$ and for each $l$, where $C > 0$ is a constant independent of $l$, and $<\cdot, \cdot>_g$ is the inner product induced by $g$, then for any $C^\infty$ $L$-valued $(0, 1)$-form $u_1$ on $M$ with $\overline{\partial}u_1 = 0$ and $\int_M ||u_1||^2 e^{-\varphi} dV_g$ finite, there exists a $C^\infty$ $L$-valued function $u$ on $M$ such that $\overline{\partial}u = u_1$ and
\[ \int_M ||u||^2 e^{-\varphi} dV_g \leq \frac{1}{C} \int_M ||u_1||^2 e^{-\varphi} dV_g, \]
where $dV_g$ is the volume form $g$ and the norm $||\cdot||$ is induced by $h$. The function $\varphi$ is called the weight function.

Let $L = K^m_M$ for $m \geq 2$. $H^m$ gives the positive Hermitian metric on $K^m_M$. Let $\omega_g$ be the Kähler form defined by the curvature of $H$ on $K_M$. Let $\varphi_l = \max(\varphi, -l)$ for $l \in \mathbb{Z}^+$, where $\varphi$ is defined in (2.23). Then by (2.24), we have
\[ <\partial\bar{\partial}\varphi_l + 2\pi \sqrt{-1} (\text{Ric}(H) + \text{Ric}(\omega_g)), v \wedge \overline{v} > \geq (m - 1 - \frac{1}{2\text{vol}(M)})||v||^2. \]

In order to prove Theorem 1.1, we need to proved that for any $m \geq 2$ and $x_0 \in M$, there is a section $S \in H^0(M, K^m_M)$ such that
\[ ||S||^2(x_0)/||S||^2_{L^2} \geq e^{-\frac{C_6 g^3}{\delta^6}}. \]
We will use Proposition 2.1 to construct such a section. Let $e^p$ be the local representation of the metric $H$. That is,

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} p = \omega_g.$$

Let $u_1 = \bar{\partial} \rho(\frac{m|z|}{\delta})e^{-m\frac{\partial p}{\partial z}(x_0)}(dz)^m$. Then $u_1 \in \Gamma(M, K^m_M)$. By (2.6), (2.7) and (2.10), we have

(2.26)  $||u_1||^2 \leq \frac{12m}{\delta^2} e^{m(p - 2Re \frac{\partial p}{\partial z}(x_0))}$

for $\frac{\delta}{\sqrt{m}} \leq |z| \leq \frac{\delta}{\sqrt{m}}$. Let $U_1 = \{x ||z| < \frac{1}{3}\delta\}$. Let $\tilde{\Delta}$ be the Euclidean Laplacian, by (2.9), we see that

$$|\tilde{\Delta} p| \leq 3$$
on $U_1$. Using the Possion formula we see that there is a constant $C_7$ such that

$$|\tilde{\nabla}^2 p(z)| \leq \frac{C_7}{\delta^2}$$

for $|z| \leq \frac{1}{2}\delta$. Thus

$$|p - 2Re \frac{\partial p}{\partial z}(x_0)z - p(x_0)| \leq \frac{C_7}{m}$$

for $|z| < \frac{\delta}{m}$. Using this estimate and (2.26), we have

$$||u_1||^2 \leq \frac{12m}{\delta^2} e^{C_7 e^{mp(x_0)}}$$

for $\frac{\delta}{\sqrt{m}} \leq |z| \leq \frac{\delta}{\sqrt{m}}$. Thus by Lemma 2.5 and (2.14),

$$\int_M ||u_1||^2 e^{-\varphi} = \int_{2\frac{\delta}{\sqrt{m}} \leq |z| \leq \frac{\delta}{\sqrt{m}}} ||u_1||^2 e^{-\varphi} \leq \frac{48C_3m}{\delta^2} e^{C_7 e^{mp(x_0)}}.$$

By Proposition 2.1 and (2.25), there is a $u \in \Gamma(M, K^m_M)$ such that $\bar{\partial} u = u_1$ and,

$$\int_M ||u||^2 e^{-\varphi} \leq \frac{1}{m - \frac{1}{2vol(M)}} \int_M ||u_1||^2 e^{-\varphi}.$$

Using Lemma 2.5 again, for $m \geq 2$, there is a $C_8$ such that

(2.27) $\int_M ||u||^2 \leq \frac{C_8}{\delta^2} e^{\frac{2C_7 e^{mp(x_0)}}{\delta^2} e^{mp(x_0)}}.$

On the other hand, we have

(2.28) $\int_M ||\rho(\frac{m|z|}{\delta})e^{-m\frac{\partial p}{\partial z}(x_0)}(dz)^m||^2 \leq e^{C_7} \int_{|z| \leq \frac{\delta}{\sqrt{m}}} e^{mp(x_0)} \leq e^{mp(x_0)} C_3 e^{C_7 \frac{\delta^2}{m}}.$

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Let $S = \rho(z) e^{-m \frac{\partial}{\partial z} (z^2)} (dz)^m - u$. Then $\overline{\partial} S = 0$. Since $\int_M e^{-\varphi} = +\infty$, $u(x_0) = 0$. In particular, $S \neq 0$. Using (2.27), (2.28), we have

$$||S||^2(x_0)/||S||^{2}_{L^2} \geq 1/ \left( 2C_3 e^{C_7 \delta^2/3m} + 2C_8 \delta^{2e} e^{-3/3}\right).$$

Thus for $m \geq 2$, there is a $C$ such that

$$||S||/||S||_{L^2} \geq \frac{C x^2}{3x}.$$

This completes the proof of Theorem 1.1.

3. A COUNTEREXAMPLE

In the last section, we give a lower bound estimate of (1.1) in terms of the injective radius of $M$. In this section, we give a counterexample that the uniform estimate is not true. More precisely, we are going to disapprove the following:

**Conjecture.** Let $K_M$ be the canonical line bundle of a Riemann surface $M$ of genus $g \geq 2$ and constant Gauss curvature $(-1)$. Then for $m$ sufficiently large, there is a number $C(m,g) > 0$, depending only on $m$ and $g$, such that for any orthonormal basis $S_1, \cdots, S_d$ of $H^0(M,K^m_M)$, we have

$$\inf \sum_{i=1}^{d} ||S_i||^2 \geq C(m,g).$$

In order to give the counterexample, we use the following Collar Theorem of Keen [7, p.264]:

**Theorem 3.1 (Keen).** Consider the region $T$ of $U$, the upper half plane, bounded by the curve $r = 1$, $r = e^l$, $\theta = \theta_0$ and $\theta = \pi - \theta_0$. Let $\gamma$ be a closed geodesic on $M$ with length $l$. Then there is a conformal isometric mapping $\varphi : T \rightarrow M$ such that $\varphi(iy) = \gamma$. The image $\varphi(T)$ of $T$ is called a collar. Then we can choose $\theta_0$ small enough such that the area of the Collar is at least $\frac{2}{\sqrt{5}}$.

The following theorem gives the counterexample and implies Theorem 1.2.

**Theorem 3.2.** For any $\varepsilon > 0$ and $m \geq 2$, there is a Riemann surface $M$ of constant curvature $(-1)$ and genus $g \geq 2$ such that there is a point $x_0 \in M$ satisfying

$$||S||(x_0) \leq \varepsilon$$

for any $S \in H^0(M, K^m_M)$ with $||S||_{L^2} = 1$.

The idea of the proof is that when the length of a closed geodesic line tends to zero, the collar will be longer and longer in order to keep the area of the collar having a lower bound. Topologically, a collar is a cylinder. By expanding the functions on the corollary using the Fourier series, we can find the suitable $x_0$ and the estimates. We begin by discussing some elementary properties of a collar.
Let $R > 0$ be a large real number. Let $(\rho, \theta) \in (-R, R) \times \mathbb{R}$. Let the group $\mathbb{Z}$ acting on the space $(-R, R) \times \mathbb{R}$ by

$$(n, \rho, \theta) \mapsto (\rho, \theta + n\delta)$$

for $n \in \mathbb{Z}$, where $\delta > 0$ satisfies

$$\delta \sinh R = \varepsilon_1 = \left(\frac{8}{\sqrt{5}}\right).$$

as in Theorem 3.1. Define the metric

$$ds^2 = d\rho^2 + (\cosh \rho)^2 d\theta^2$$

on $(-R, R) \times \mathbb{R}$ which descends to a metric on

$$C = (-R, R) \times \mathbb{R}/\mathbb{Z}.$$

The curvature of the metric is $(-1)$. Note that on $C$, $\rho$ is a global function but $\theta$ is only locally defined.

We call $C$ a collar of parameter $\delta$.

Let

$$(3.1) \quad \begin{cases} x = \theta \\ y = 2\arctan e^\rho - \frac{\pi}{2} \end{cases}.$$  

Define $z = x + iy$. Clearly $z$ is not a global function of $C$. But it defines a complex structure of $C$.

Let

$$(3.2) \quad w = e^{\frac{2\pi i}{\delta}(\theta + 2i(\arctan e^\rho - \frac{\pi}{4}))} = e^{\frac{2\pi i}{\delta}z}.$$  

Then $w$ is a global holomorphic function on $C$. Consequently

$$(3.3) \quad dz = \frac{\delta}{2\pi i} \frac{dw}{w}$$

is a global holomorphic 1-form on $C$.

Let $f$ be a holomorphic function on a neighborhood of $C$. Then $f$ is a period function on $[-R, R] \times \mathbb{R}$, satisfying

$$f(\rho, \theta \pm \delta) = f(\rho, \theta).$$

Let the Fourier expansion of $f(-R, \theta)$ and $f(R, \theta)$ be

$$f(-R, \theta) = \sum_{k=-\infty}^{+\infty} A_k e^{\frac{2\pi i}{\delta}k\theta};$$

$$f(R, \theta) = \sum_{k=-\infty}^{+\infty} B_k e^{\frac{2\pi i}{\delta}k\theta}.$$  

Define
\[ g_1 = \sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta} k \left( \frac{\pi}{4} - \arctan e^{-R} \right) w^k}, \]
\[ g_2 = B_0, \]
\[ g_3 = \sum_{k=-\infty}^{-1} B_k e^{-\frac{4\pi}{\delta} k \left( \frac{\pi}{4} - \arctan e^{-R} \right) w^k}, \]
(3.4)

where \( w \) is in (3.3). We have the following lemma:

**Lemma 3.1.** With the notations as above, \( g_1, g_2, g_3 \) are holomorphic functions on \( C \). Furthermore
\[ f = g_1 + g_2 + g_3. \]

**Proof.** \( g_2 \) is a constant. So it is automatically holomorphic. By equation (3.2), we have
\[ |w| \leq e^{-\frac{4\pi}{\delta} \left( \arctan e^\rho - \frac{\pi}{4} \right)}. \]
Thus we have
\[ |g_1| = \left| \sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta} k \left( \frac{\pi}{4} - \arctan e^{-R} \right) w^k} \right| \]
\[ \leq \sum_{k=1}^{\infty} |A_k| e^{-\frac{4\pi}{\delta} k \left( \arctan e^\rho - \arctan e^{-R} \right)} \]
\[ \leq \left( \sum_{k=1}^{\infty} |A_k|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} e^{-\frac{8\pi}{\delta} k \left( \arctan e^\rho - \arctan e^{-R} \right)} \right)^{1/2}. \]

By the Bessel inequality, we have
\[ \sum_{k=1}^{+\infty} |A_k|^2 \leq \frac{1}{\delta} \int_M |f(-R, \theta)|^2 d\theta. \]
Thus if \( \rho > -R \), the series is convergent absolutely. So \( g_1 \) defines a holomorphic function on \( \{-R < \rho < R\} \).

By the same argument, \( g_3 \) is also holomorphic.

In order to prove that
\[ f = g_1 + g_2 + g_3, \]
we just need to prove that on the set \( \{\rho = 0\} \), \( f = g_1 + g_2 + g_3 \). Define
\[ p_k(\rho) = \int_0^\delta f(\rho, \theta) e^{-\frac{4\pi}{\delta} \theta} d\theta \]
(3.5)
for \( k \in \mathbb{Z} \). Apparently
\[ p_k(R) = B_k \delta, \quad p_k(-R) = A_k \delta. \]
for $k \in \mathbb{Z}$. By the definition of $z$ in (3.1), we have
\[
\frac{\partial}{\partial z} = \frac{1}{2}(\sqrt{-1}\cosh \rho \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \theta}).
\]
Using the equation $\frac{\partial f}{\partial z} = 0$, from (3.5), we have
\[
p_k' = -\frac{2\pi k}{\delta \cosh \rho} p_k.
\]
Solving the above differential equation gives
\[
p_k(0) = A_k e^{-\frac{4\pi k}{\delta}(\frac{\pi}{4} - \arctan e^{-R})}, \quad k = 1, 2, \ldots,
\]
(3.6)
\[
p_0(0) = B_0,
\]
\[
p_k(0) = B_k e^{-\frac{4\pi k}{\delta}(\frac{\pi}{4} - \arctan e^R)}, \quad k = -1, -2, \ldots.
\]
From (3.4) and (3.6), we see that
\[
f|_{\rho=0} = (g_1 + g_2 + g_3)|_{\rho=0}.
\]
Thus
\[
f = g_1 + g_2 + g_3
\]
on $C$ because both sides are holomorphic functions.

Let $M$ be a Riemann surface of curvature $(-1)$ and genus $g \geq 2$. Assume that there is a closed geodesic $\gamma$ on $M$ such that $\text{length}(\gamma) = \delta > 0$. Assume that $\delta$ is small enough. Let $\theta$ be the arc length parameter and $\rho$ be the distance function to the geodesic. Then

**Lemma 3.2.** $(\rho, \theta)$ is the local coordinate system of $M$ as long as
\[
\sinh \rho \cdot \delta \leq \varepsilon_1 = \frac{8}{\sqrt{5}}.
\]

**Proof.** Note that the area of $\{R_0 < \rho < R_0\}$ is $\delta \sinh R_0$ for any $R_0 > 0$. The lemma follows from Theorem 5.1. □

Let $C = \{-R < \rho < R\}$, where $R$ satisfies $\delta \sinh R = \varepsilon_1$. Then $z = x + iy$ defines a complex structure of $C$ where $x$ and $y$ are in (3.1). We have

**Lemma 3.3.** Either $z = x + iy$ or $z = x - iy$ is holomorphic on $M$.

**Proof.** A straightforward computation gives
\[
ds^2 = (\cosh \rho)^2 dzd\overline{z}.
\]
Thus $z$ defines a conformal structure which is the same as the one on $M$. So either $z$ or $\overline{z}$ is holomorphic.

Without losing generality, we assume that $z$ is holomorphic. Fixing $m \geq 2$. Let $S \in H^0(M, K_M^m)$. We choose an $x_0 \in M$ as follows: let $\rho_0 < 0$ be the number such that
\[
\delta(\cosh \rho_0)^2 = \varepsilon_1.
\]
Then $\rho_0 \to -\infty$, $\rho_0 + R \to +\infty$ as $\delta \to 0$.

By (3.3), we see that $(dz)^m \in H^0(C, K_M^m)$. Furthermore $(dz)^m \neq 0$ on $C$. Thus for any $S \in H^0(M, K_M^m)$, there is a holomorphic function $f$ on $C$ such that

$$S|_C = f(dz)^m.$$ 

Let

$$f = g_1 + g_2 + g_3,$$

where $g_1, g_2, g_3$ are defined in Lemma 3.1. Let

(3.7) $S_i = g_i(dz)^m, \quad i = 1, 2, 3.$

**Lemma 3.4.** With the notations as above, let $x_0 = (\rho_0, 0)$. Then

$$\lim_{\delta \to 0} \frac{||S_i||^2(x_0)}{||S_i||^2_{L^2(C)}} = 0$$

for $i = 1, 2, 3$.

**Proof.** By (3.2), we have

$$w(x_0) = e^{-\frac{4\pi}{\delta}(\arctan e^{\rho_0} - \frac{\pi}{4})}.$$ 

Thus

(3.8)

$$||S_1||^2(x_0) = \frac{1}{(\cosh \rho_0)^{2m}} \sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta} k(\arctan e^{\rho_0} - \arctan e^{-R}) + \frac{2\pi k}{\delta}} \sum_{k=1}^{\infty} e^{-\frac{8\pi}{\delta} k(\arctan e^{\rho_0} - \arctan e^{-R})} \cdot \sum_{k=1}^{\infty} e^{-\frac{4\pi k}{\delta \cosh(\rho_0 - 1)}}.$$ 

We assume that $\delta$ is so small that

$$\frac{4\pi}{\delta \cosh(\rho_0 - 1)} \geq \frac{\mu}{\sqrt{\delta}} > 0,$$

where $\mu$ is an absolute constant. In addition, assume that $e^{-\frac{4\pi}{\delta}} < \frac{1}{2}$. Then

(3.9)

$$\sum_{k=1}^{\infty} e^{-\frac{4\pi k}{\delta \cosh(\rho_0 - 1)}} \leq \sum_{k=1}^{\infty} e^{-\frac{\mu k}{\sqrt{\delta}}} \leq 2e^{-\frac{\mu}{\sqrt{\delta}}}.$$ 

On the other hand
\begin{equation}
||S_1||_2^2(C) = \delta \int_{-R}^R \frac{1}{(\cosh \rho)^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})} d\rho.
\end{equation}

Assuming \( \rho_0 - 1 + R > 1 \) and \( \cosh R < 2 \sinh R \), we have
\begin{equation}
||S_1||_2^2(C)
\geq \delta \int_{-R}^{\rho_0-1} \frac{1}{(\cosh \rho)^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})} d\rho
\geq \frac{\delta}{(\frac{s}{8})^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})}.
\end{equation}

By (3.8), (3.9) and (3.10), we have
\begin{equation}
\frac{||S_1||_2^2(x_0)}{||S_1||_2^2(C)} \leq \frac{2^{2m-1} \varepsilon_1^{2m-1}}{\delta^{2m} e^{-\frac{s\pi}{8}}} \to 0
\end{equation}
for \( \delta \to 0 \).

The idea for estimating \( S_2 \) and \( S_3 \) are almost the same. By (3.7), we have
\begin{equation}
||S_2||_2^2(x_0) = \frac{1}{(\cosh \rho_0)^{2m}} |B_0|^2,
\end{equation}
\begin{equation}
||S_2||_2^2(L_2^2(C)) = \delta \int_{-R}^R \frac{1}{(\cosh \rho)^{2m-1}} |B_0|^2 d\rho.
\end{equation}
If \( \delta \to 0 \), then \( R \to +\infty \). Thus if \( R \) is large enough, we have
\begin{equation}
\int_{-R}^R \frac{1}{(\cosh \rho)^{2m-1}} d\rho \geq \mu_1 > 0,
\end{equation}
where \( \mu_1 \) is a constant only depending on \( m \). Thus
\begin{equation}
\frac{||S_2||_2^2(x_0)}{||S_2||_2^2(L_2^2(C))} = \frac{1}{\mu_1 (\cosh \rho_0)^{2m}} = \frac{1}{\mu_1 \varepsilon_1^{2m}} \to 0
\end{equation}
for \( \delta \to 0 \).

For \( S_3 \), we have
\begin{equation}
||S_3||_2^2(x_0) = \frac{1}{(\cosh \rho_0)^{2m}} \sum_{k=-\infty}^{-1} B_k e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})} \sum_{k=-\infty}^{-1} B_k e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})}.
\end{equation}
\begin{equation}
\leq \sum_{k=-\infty}^{-1} \left| B_k \right|^2 e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})} \sum_{k=-\infty}^{-1} e^{-\frac{s\pi k}{8} (\arctan e^{\rho_0} - \arctan e^{-R})}.
\end{equation}
Since \( \rho_0 \to -\infty \), we can assume
\begin{equation}
\arctan e^{\rho_0} < \frac{\pi}{8}.
\end{equation}
Thus
\begin{equation}
\sum_{k=-\infty}^{-1} e^{-\frac{8\pi}{\pi}\left(\arctan e^{\rho_0} - \frac{\pi}{4}\right)} \leq \sum_{k=-\infty}^{-1} e^{\frac{e^2}{2}}k \leq 2e^{-\frac{e^2}{2}}
\end{equation}
for $e^{-\frac{e}{2}} < \frac{1}{2}$. On the other hand,
\begin{equation}
||S_3||^2_{L^2(C)} = \delta \int_{-R}^{R} \frac{1}{(\cosh \rho)^{2m-1}} \sum_{k=-\infty}^{-1} |B_k|^2 e^{-\frac{8\pi k}{\pi} (\arctan e^{\rho} - \arctan e^{R})} d\rho
\end{equation}
\begin{equation}
\geq \delta \int_{0}^{R} \frac{1}{(\cosh R)^{2m-1}} \sum_{k=-\infty}^{-1} |B_k|^2 e^{-\frac{8\pi k}{\pi} (\arctan e^{R})} d\rho
\end{equation}
\begin{equation}
\geq \frac{\delta R}{(\cosh R)^{2m-1}} \sum_{k=-\infty}^{-1} |B_k|^2 e^{-\frac{8\pi k}{\pi} (\arctan e^{R})}
\end{equation}
By (3.14), (3.15) and (3.16), we have
\begin{equation}
||S_3||^2(x_0) \leq \frac{2e^{-\frac{e^2}{2}}}{\delta R} (\cosh R)^{2m-1} \leq \frac{2^{2m}e^{-\frac{e^2}{2}}}{\delta^{2m}} \to 0.
\end{equation}
Thus for any $\varepsilon > 0$ and $m \geq 2$, from (3.11), (3.13) and (3.17), we can find $M$ such that there is a closed geodesic with the length sufficiently small and an $x_0 \in M$ such that
\begin{equation}
\frac{||S_i||^2(x_0)}{||S_i||^2_{L^2(C)}} \leq \varepsilon
\end{equation}
for $i = 1, 2, 3$.
One can check that
\begin{equation}
(S_i, S_j)_{L^2(C)} = 0.
\end{equation}
Thus for any $S \in H^0(M, K^m_M)$ with $||S||_{L^2(M)}$ and $x_0 \in M$,
\begin{equation}
||S||^2(x_0) \leq \frac{||S||^2(x_0)}{||S||^2_{L^2(C)}} \leq \frac{3\sum_{i=1}^{3} ||S_i||^2(x_0)}{\sum_{i=1}^{3} ||S_i||^2_{L^2(C)}} \leq 3\varepsilon.
\end{equation}
Theorem 3.2 is proved.

4. Partial uniform estimates
Let $M$ be a Riemann surface of genus $g$ and constant curvature $(-1)$. In this section, we prove that there is a (positive) lower bound of (1.1) depending only on the injective radius of the point. More precisely, for $m$ large enough, for any $x \in M$, there is a section $S \in H^0(M, K^m_M)$ such that $||S||_{L^2} = 1$ and $||S||(x) \geq C(\delta_x)$ where $C(\delta_x)$ is a positive constant depending only on $\delta_x$ and $\delta_x$ is the injective radius at $x$. 
Note that in the result the lower bound doesn’t depend on the injective radius of $M$, which will go to zero as $M$ approaches the boundary of the Teichmüller space.

We use all the notations of in §3 about the collars and the functions on them. The following proposition is a corollary of the collar theorem:

**Proposition 4.1.** Let $M$ be a Riemann surface of genus $g \geq 2$ and constant curvature $(-1)$. Let $\gamma_1, \cdots, \gamma_s$ be the closed geodesics on $M$ such that
\[ \text{length}(\gamma_i) \leq \frac{1}{1000}, \quad 1 \leq i \leq s. \]

Let $C_{\gamma_i}(1 \leq i \leq s)$ be the corresponding collars embedded in $M$ (Theorem [2.1]). Then for any $x \in M \setminus \bigcup_{i=1}^{s} C_{\gamma_i}$, there is an absolute constant $\varepsilon_2 > 0$ such that
\[ \delta_x \geq \varepsilon_2. \]

**Proof.** Let $\delta = \frac{1}{1000}$. Let $x \in M \setminus \bigcup_{i=1}^{s} C_{\gamma_i}$ and $\text{inj}(x) \geq \delta_x > 0$. Then there is a point $y \in M$ such that there are two geodesics $l_1$ and $l_2$ connecting $x$ and $y$ but $l_1$ and $l_2$ are not homotopic to each other. If $\delta_x > 1$, the theorem has been proved. Otherwise, let $\gamma'$ be the shortest closed curve homotopic to the closed curve $l_1^{-1}l_2$ in
\[ D' = M \setminus \bigcup_{\text{length}(\gamma_i)<2\delta} C_{\gamma_i}(R_i - 2), \]
where
\[ C_{\gamma_i}(R_i - 2) = \{ x | \text{dist}(x, \gamma_i) < R_i - 2 \}. \]

Since $D'$ is a compact set. If $\gamma'$ doesn’t touch any of the boundary $\partial C_{\gamma_i}(R_i - 2)$ for any $i$, then $\gamma'$ must be a closed geodesic and by the definition, we have $\text{length}(\gamma) \geq \frac{1}{1000}$ and thus $\delta_x \geq \frac{1}{2000}$. Otherwise either $\delta_x > 1$ or $\gamma' \subset C_{\gamma_i}(R_i - 1) \setminus C_{\gamma_i}(R_i - 2)$ for some $i$. In the latter case, since $\gamma'$ is not homotopic to zero, we see that
\[ \text{length}(\gamma') \geq \text{length}(\gamma_i) \cosh(R_i - 2) \geq \frac{1}{18} \varepsilon_1 \]
(remember $\text{length}(\gamma_i) \sinh R_i = \varepsilon_1$). Thus
\[ \delta_x \geq \frac{1}{2} \text{length}(\gamma') \geq \frac{1}{36} \varepsilon_1 \geq \varepsilon_2 \]
for $\varepsilon_2 = \frac{1}{36} \varepsilon_1$.

Using the above lemma, we know that outside the collars whose shortest closed geodesics are small, the injective radius has a lower bound and the weight function in Proposition [2.3] can be constructed in the ordinary way. If $x \in C_{\gamma_i}$ for some $i$, we are going to construct the weight functions having the compact support within $C_{\gamma_i}$. For this reason, let’s first assume that $C_{\delta}$ is a collar with $\delta < \frac{1}{1000}$ and do some analysis on it.
Let’s fix some notations: there are absolute constants $\varepsilon_3, \varepsilon_4 > 0$ such that
\begin{equation}
\varepsilon_3 < \delta \cosh R, \delta \cosh(R+4), \delta \sinh(R+4), \delta e^{R+4} < \varepsilon_4.
\end{equation}

Let $(\rho, \theta)$ be the local coordinate of the collar $C = C_\delta$ as in §3. Then
\begin{equation}
w = e^{\frac{2\pi i}{4} \theta - \frac{4\pi}{4} (\arctan e^\rho - \frac{\pi}{4})}
\end{equation}
is the holomorphic function on $C_\delta$. Let $x_0$ and $p_0$ be the points on $C_\delta$ such that the local coordinate of $x_0$ and $p_0$ can be represented as: $x_0 = (\rho_0, 0)$ for $R - 4 > \rho_0 \geq 0$ and $p_0 = (R-1,0)$. The function $w$ at $x_0$ and $p_0$ has the values
\begin{align*}
w_0 &= e^{-\frac{4\pi}{5} (\arctan e^{\rho_0} - \frac{\pi}{4})}, \\
w_{p_0} &= e^{-\frac{4\pi}{5} (\arctan e^{R-1} - \frac{\pi}{4})}
\end{align*}
at $x_0$ and $p_0$ respectively. Let
\begin{equation}
\alpha = 2 \arctan \frac{e^{\rho_0}}{\pi},
\end{equation}
and define the functions $\varphi_1$, $\varphi_2$ and $\varphi_3$ on $C_\delta$ to be
\begin{align*}
\varphi_1 &= \log \left| \frac{w}{w_0} - 1 \right|, \\
\varphi_2 &= \log \left| \frac{w}{w_{p_0}} - 1 \right|, \\
\varphi_3 &= \varphi_1 - \alpha \varphi_2.
\end{align*}

The Riemann metric on $C_\delta$ can be represented as
\begin{equation}
ds^2 = d\rho^2 + (\cosh \rho)^2 d\theta^2.
\end{equation}

Let the injective radius at $x_0$, $p_0$ and $x$ be $\delta_{x_0}$, $\delta_{p_0}$ and $\delta_x$. Then we have an absolute constant $\varepsilon_5 > 0$ such that
\begin{equation}
\begin{aligned}
\frac{1}{2} \delta \cosh \rho_0 &> \delta_{x_0} > \varepsilon_5 \delta \cosh \rho_0, \\
\frac{1}{2} \varepsilon_1 &> \delta_{p_0} > \varepsilon_5, \\
\frac{1}{2} \delta \cosh \rho &> \delta_x > \varepsilon_5 \delta \cosh \rho.
\end{aligned}
\end{equation}

We establish some elementary properties of the function $\varphi_3$. Let $d = d(x)$ be the distance function to the point $x_0$. Then we have

**Lemma 4.1.** With the notations as above, there are constants $C_9, C_{10} > 0$ such that
\begin{equation}
\varphi_3 \leq C_9,
\end{equation}
for $-R+1 \leq \rho \leq R-2$ and
\begin{equation}
\varphi_3 \geq \log d(x) - \frac{4\pi}{\delta e^\rho} - C_{10}
\end{equation}
for $d(x) \leq \delta_{x_0}$. 

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Proof. By (4.2) and (4.3), we have
\[
\frac{w}{w_0} = e^{\frac{2\pi i}{\delta}(\arctan e^\rho - \arctan e^{\rho_0})},
\]
(4.10)
\[
\frac{w}{w_{p_0}} = e^{\frac{2\pi i}{\delta}(\arctan e^\rho - \arctan e^{R-1})}.
\]
From (4.10), we have
\[
\begin{cases}
\log |\frac{w}{w_0} - 1| \leq \log 2 & \rho_0 \leq \rho \leq R - 2, \\
\log |\frac{1 - \frac{w}{w_0}}{w_{p_0}}| \leq 0 & -R + 1 \leq \rho \leq \rho_0, \\
\log |\frac{w}{w_{p_0}}| \geq 0 & -R + 1 \leq \rho \leq R - 1, \\
\log |\frac{1 - \frac{w}{w_{p_0}}}{w}| \geq \log(1 - e^{-\frac{2\pi}{\epsilon 3}}) & -R + 1 \leq \rho \leq R - 2.
\end{cases}
\]
(4.11)
If \(\rho_0 \leq \rho \leq R - 2\), then we can write
\[
\varphi_3 = \log |\frac{w}{w_0} - 1| - \alpha \log |\frac{w}{w_{p_0}}| - \alpha \log |\frac{1 - \frac{w}{w_{p_0}}}{w}|.
\]
By (4.11), we have
(4.12) \(\varphi_3 \leq \log 2 - \log(1 - e^{-\frac{2\pi}{\epsilon 3}})\).
If \(-R + 1 \leq \rho \leq \rho_0\). We can write
(4.13) \(\varphi_3 = \log |\frac{w}{w_0}| - \alpha \log |\frac{w}{w_{p_0}}| + \log |\frac{1 - \frac{w}{w_0}}{w}| - \alpha \log |\frac{1 - \frac{w}{w_{p_0}}}{w}|\).
Using (4.11), we have
(4.14) \[\log |\frac{w}{w_0}| - \alpha \log |\frac{w}{w_{p_0}}| = -\frac{8}{\delta}(\frac{\pi}{2} - \arctan e^{\rho_0}) \arctan e^\rho + \frac{8}{\delta} \arctan e^{\rho_0}(\frac{\pi}{2} - \arctan e^{R-1}).\]
Thus by (4.1) and (4.7),
(4.15) \[\frac{-4\epsilon 5}{\delta x_0} \leq \log |\frac{w}{w_0}| - \alpha \log |\frac{w}{w_{p_0}}| \leq \frac{4\pi}{\epsilon 3}.\]
By (4.11), (4.13) and (4.15), we have
(4.16) \[\varphi_3 \leq \frac{4\pi}{\epsilon 3} + \log 2 - \log(1 - e^{-\frac{2\pi}{\epsilon 3}}).\]
Thus by (4.12) and (4.16), we have \(\varphi_3 \leq C_9\) for
\[C_9 = \frac{4\pi}{\epsilon 3} + \log 2 - \log(1 - e^{-\frac{2\pi}{\epsilon 3}}).\]
Let's now assume that \(d(x) < \delta x_0\). Then by the triangle inequality we have
(4.17) \[
\begin{cases}
|\rho - \rho_0| + |\theta| \cosh \rho_0 \geq d(x) & 0 \leq \theta \cosh \rho_0 < d(x), \\
|\rho - \rho_0| + |\theta - \delta \cosh \rho_0| \geq d(x) & \delta \cosh \rho_0 - d(x) < \theta \cosh \rho_0 < \delta \cosh \rho_0.
\end{cases}
\]
Without losing generality we assume that $0 \leq \theta \leq d(x)$ and $|\rho - \rho_0| + |\theta| \cosh \rho_0 \geq d(x)$, then

$$|w_{w_0} - 1| \geq e^{-\frac{4\pi}{\delta}(\arctan e^{\rho} - \arctan e^{\rho_0})} \sin \frac{2\pi}{\delta} \theta \geq \frac{\pi}{4} e^{-\pi} d(x).$$

(4.18)

On the other hand, if $|\rho - \rho_0| \geq \frac{1}{2} d(x)$, then

$$|w_{w_0} - 1| \geq e^{-\frac{2\pi}{\delta} d}.$$

(4.19)

By (4.18) and (4.19), there is a constant $C_{11} > 0$ such that

$$\log |w_{w_0} - 1| \geq \log d - C_{11}.$$

(4.20)

We also have

$$\log |w_{w_0} - 1| \leq \log 2 + \frac{4\pi}{\delta e^{\rho}}$$

(4.21)

for $d \leq \delta_{x_0}$. By (4.20) and (4.21), from (4.5)

$$\varphi_3 \geq \log d - C_{11} - (\log 2 + \frac{4\pi}{\delta e^{\rho}}),$$

This completes the proof of Lemma 4.1.

Lemma 4.2. There is a constant $C_{12} > 0$ such that

$$|\varphi_3| \leq C_{12}, \quad |\nabla \varphi_3| \leq C_{12}$$

for $R - 3 < |\rho| < R - 2$.

Proof. By the above lemma, we see that

$$\varphi_3 \leq C_9$$

for the constant $C_9$. Thus we just need to prove the lower bound of $\varphi_3$ and the bound for the derivative of $\varphi_3$.

If $R - 3 < \rho < R - 2$, by (4.10), we have

$$|w_{w_0}| \leq e^{-\frac{2\pi}{\delta}}.$$

(4.22)

Thus

$$\varphi_1 \geq \log(1 - e^{-\frac{2\pi}{\delta}}).$$

(4.23)

Also we have

$$|w_{w_{p_0}}| \leq e^{\frac{4\pi}{\delta}}$$

(4.24)

for $R - 3 < \rho < R - 2$. Thus

$$\varphi_3 \geq \log(1 - e^{-\frac{2\pi}{\delta}}) - \log(1 + e^{\frac{4\pi}{\delta}}).$$

(4.25)

If $R - 3 < -\rho < R - 2$, then by (4.14), we have

$$\log |w_{w_0}| - \alpha \log |w_{w_{p_0}}| \geq -\frac{8}{\delta} (\frac{\pi}{2} - \arctan e^{\rho_0}) \arctan e^{-(R-3)} \geq -\frac{2}{\epsilon_2}.$$
By (4.13), we have

\[ \varphi_3 \geq -\frac{8}{\varepsilon_3} + \log(1 - e^{-\frac{\pi^2}{2}T}) - \log 2. \]  

Combining (4.25) and (4.26), we get the lower bound of \( \varphi_3 \). Next let’s consider \( \nabla \varphi_3 \). Obviously

\[ |\nabla \varphi_3| \leq |\nabla \varphi_1| + |\nabla \varphi_2|. \]

Thus we just need to estimate \( |\nabla \varphi_1| \) and \( |\nabla \varphi_2| \). By (4.6), the Riemann metric under the coordinate \( w \) can be written as

\[ ds^2 = \delta^2(cosh \rho)^2 \frac{1}{|w - w_p|^2} dwd\bar{w}. \]

Thus

\[ |\nabla \varphi_1|^2 = \frac{4\pi^2|w|^2}{\delta^2(cosh \rho)^2} \cdot \frac{1}{|w - w_0|^2}, \]

\[ |\nabla \varphi_2|^2 = \frac{4\pi^2|w|^2}{\delta^2(cosh \rho)^2} \cdot \frac{1}{|w - w_{p0}|^2}. \]

Using the same elementary estimates as above, we get,

\[
\begin{align*}
|\frac{u_0}{w}| &\geq e^{\frac{2\pi}{\varepsilon_1}} R - 3 < \rho < R - 2, \\
|\frac{u_p}{w}| &\leq e^{\frac{2\pi}{2\varepsilon_1}} R - 3 < -\rho < R - 2, \\
|\frac{u_{p0}}{w}| &\leq e^{\frac{2\pi}{\varepsilon_1}} R - 3 < \rho < R - 2, \\
|\frac{w_{p0}}{w}| &\leq e^{\frac{2\pi}{2\varepsilon_1}} R - 3 < -\rho < R - 2.
\end{align*}
\]

Using these results, we get the bound for the gradient of \( \varphi_1 \) and \( \varphi_2 \). This completes the proof of the lemma.

The following proposition summarizes the technical results of this section.

**Proposition 4.2.** Suppose \( M \) is a compact Riemann surface of genus \( g \geq 2 \) and constant curvature \((-1)\). Then for any \( x_0 \in M \), there is a function \( \varphi = \varphi_{x_0} \) such that \( \varphi \) is smooth on \( M \setminus \{x\} \) and

1. In a neighborhood \( U_x \) of \( x \), \( \varphi \) can be written as

\[ \varphi = 2 \log d(x) + \psi, \]

where \( \psi \) is a smooth function on \( U_x \). Consequently,

\[ \int_{U_x} e^{-\varphi} = +\infty. \]

2. There is a constant \( C_{13} \) such that

\[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi \geq -C_{13} \omega_g \]

on \( M \setminus \{x\} \), where \( \omega_g \) is the Kähler form of \( M \);
3. \( \varphi \) satisfies
\begin{equation}
\varphi \leq C_{13} \tag{4.30}
\end{equation}
on \( M \) and
\begin{equation}
\varphi \geq 2 \log d(x) - \frac{2\pi}{\delta x_0} - C_{13} \tag{4.31}
\end{equation}
for \( d(x) \leq \delta x_0 \).

**Proof.** Let \( \gamma_1, \cdots, \gamma_s \) be the closed geodesics such that \( \text{length}(\gamma_i) < \frac{1}{1000} \). Let \( C_{\gamma_i} \) be the corresponding collars. Let
\[ C_{\gamma_i}(R_i - 4) = \{ x | \text{dist}(x, \gamma_i) \leq R_i - 4 \} \quad i = 1, \cdots s. \]
For any \( x_0 \in M \), if \( x_0 \in C_{\gamma_i}(R_i - 4) \) for some \( i \), then let \( C_\delta = C_{\gamma_i} \) and define \( \varphi = \varphi_{x_0} : M \to \mathbb{R} \) as follows
\begin{equation}
\varphi = \begin{cases} 
2 \eta (\rho - (R - 3)) \varphi_3 & \rho \geq 0 \quad \text{and} \quad x \in C_\delta, \\
2 \eta (-\rho - (R - 3)) \varphi_3 & \rho < 0 \quad \text{and} \quad x \in C_\delta, \\
0 & \text{otherwise},
\end{cases} \tag{4.32}
\end{equation}
where the function \( \eta \) is defined in (2.10). By Lemma 4.1, Lemma 4.2 and the fact that \( \varphi_3 \) is harmonic on \( C_\delta \setminus \{x_0\} \setminus \{p_0\} \), it is easy to check that the function \( \varphi \) satisfies all the assertions in the proposition. On the other hand, if \( x_0 \notin C_{\gamma_i}(R_i - 4) \) for any \( 1 \leq i \leq s \), then by Proposition 4.1, \( \delta x_0 \geq \varepsilon_2 \). The argument becomes quiet standard: define
\begin{equation}
\varphi = 2 \eta (\frac{d(x)}{\varepsilon_2}) \log (\frac{d(x)}{\varepsilon_2}) \tag{4.33}
\end{equation}
Then we can prove that \( \varphi \) satisfies all the requirements by using the same method as in Lemma 2.3.

\[ \square \]

**Theorem 4.1.** Let \( M \) be a Riemann surface of genus \( g \geq 2 \) and constant curvature \((-1)\). Then there are absolute constants \( m_0 > 0 \) and \( D > 0 \) such that for any \( m > m_0 \) and any \( x_0 \in M \), there is a section \( S \in H^0(M, K^m_M) \) with \( ||S||_{L^2} = 1 \) such that
\begin{equation}
||S||(x_0) \geq \frac{\sqrt{m}}{D(1 + \frac{1}{\sqrt{m} \delta x_0} e^{\varepsilon_0})}. \tag{4.34}
\end{equation}

**Proof.** Let \( x_0 \in M \) and \( U_{x_0} = \{ x | \text{dist}(x, x_0) < \delta x_0 \} \). Let \( z_1 \) be the holomorphic function on \( U_{x_0} \) such that the hermitian metric can be represented as
\[ ds^2 = \frac{1}{(1 - \frac{1}{4}|z_1|^2)^2} dz_1 d\overline{z}_1. \]
For \( m > 0 \) large enough, let
\[ u_1 = \overline{\partial} (\eta (\frac{2|z_1|}{\delta x_0^{23}})) (dz_1)^m. \]
Then $\overline{\partial} u_1 = 0$ and 
$$||\overline{\partial} u_1||^2 \leq \frac{16}{\delta x_0} (1 - \frac{1}{4}|z_1|^2)^{m+1}$$
for $\frac{1}{4}\delta x_0 \leq \frac{1}{2}|z_1| \leq \frac{1}{2}\delta x_0$. Thus there is a $C_{14} > 0$ such that
$$\int_M ||\overline{\partial} u_1||^2 e^{-\varphi x_0} \leq \frac{1}{m} \cdot \frac{C_{14}}{\delta^4 x_0} e^{\frac{2x}{x_0}}.$$  

Let $m_0 = C_{13} + 2$. By Proposition 2.1, for $m > m_0$, we can find a $u \in \Gamma(M, K^m_M)$ such that $\overline{\partial} u = u_1$ with

(4.35) $$\int_M ||u||^2 e^{-\varphi x_0} \leq \frac{1}{m(m-C_{13}-1)} \cdot \frac{C_{14}}{\delta^4 x_0} e^{\frac{2x}{x_0}}.$$  

Let $S = \eta(\frac{2|x_1|}{x_0})(dz_1)^m - u$. Then $\overline{\partial} S = 0$. Thus $S$ is an element of $H^0(M, K^m_M)$. Furthermore, since $\int_M e^{-\varphi x_0} = +\infty$, we must have $u(x_0) = 0$. So

(4.36) $$||S||(x_0) = 1.$$  

On the other hand,

(4.37) $$||S||^2_{L^2} \leq 2(\int_M ||u||^2 + \int_M ||\eta(dz_1)^m||^2).$$  

By (4.36) and (4.37), we have

We also have

(4.38) $$\int_M ||\eta(dz_1)^m||^2 \leq \frac{\pi}{m}.$$  

The theorem follows from (4.36), (4.37) and (4.38).

5. THE UNIFORM CORONA PROBLEM

Let $M$ be a Riemann surface of genus $g \geq 2$. It is well known that the coordinate ring $\bigoplus_{m=0}^{\infty} H^0(M, K^m_M)$ is finitely generated. That is, there is an $m_0 > 0$ such that for any $m > 0$ and $S \in H^0(M, K^m_M)$, $S$ can be represented by

(5.1) $$S = \sum_{i=1}^{d} U_i T_i,$$

where $U_i \in H^0(M, K^{m_0}_M)$ and $T_i \in H^0(M, K^{m-m_0}_M)$ for $i = 1, \ldots, d = \dim H^0(M, K^m_M)$. Finding a suitable set of $\{T_i\}_{i=1}^{d}$ is called the corona problem(cf. [4]).

We need to consider the case where $M$ approaches to the boundary of the moduli space in the Teichmüller theory. So in addition to the existence
of $U_i$ and $T_i$, we need some uniform estimates. In this section, we give the uniform estimate for the corona problem on Riemann surfaces.

**Theorem 5.1.** Let $M$ be a Riemann surface of genus $g$ and constant curvature $(-1)$. Then there is an $m_0 > 0$ such that for any $m > m_0$ and $S \in H^0(M, K^m_M)$, there is a decomposition

$$S = \sum_{i=1}^{d} S_i$$

of $S_i \in H^0(M, K^m_M)(i = 1, \cdots, d)$ such that

$$||S_i||_{L^2} \leq C(m, m_0, g)||S||_{L^2}$$

$$||S_i||_{L^\infty} \leq C(m, m_0, g)||S||_{L^\infty}$$

for $i = 1, \cdots, d$, and

$$S_i = T_i U_i$$

for a basis $U_1, \cdots, U_d$ of $H^0(M, K^{m_0}_M)$ and $T_1, \cdots, T_d \in H^0(M, K^{m-m_0}_M)$.

**Remark 5.1.** An estimate on $T_i(i = 1, \cdots, d)$ alone is not expected because of the counterexample in §3, where $\sum ||U_i||^2$ can be arbitrarily small.

Throughout this section, we will use the notation $D_1, D_2, \cdots$ to denote the constants depending only on $m_0$ and the genus $g$. We also use $A \lesssim B$ to mean that there is a positive constant $C = C(m_0, g)$, depending only on $m_0$ and $g$ such that $A \leq CB$. Likewise, we use $A \gtrsim B$ to denote the fact $A \geq CB$ for some constant $C = C(m_0, g)$.

The idea of the proof is that, if the injective radius of $M$ is greater than an absolute constant, then $\sum ||U_i||^2$ has a lower bound by an absolute positive constant. In this case, we can solve the corona problem exactly using the method in §4. So we just need to prove the theorem in the case where $inj(M)$ is arbitrarily small. By the collar theorem, we know that in this case, there are finite many collars $C_{\delta_1}, \cdots, C_{\delta_s}$ (with max $\delta_i$ small) embedded in $M$ and they do not intersect each other. By Proposition 4.1, outside the collars, the injective radius has an absolute lower bound. Special care must be taken for the sections over these collars. In order to take care of the collars to get the estimates, we first fix a collar $C_\delta$ embedded in $M$ with the parameter $\delta$ small. We will use all the notations about collars in §3. For any $R > 0$, let

$$C_\delta(R) = \{x||\rho| \leq \tilde{R}\}.$$ 

In particular, $C_\delta = C_\delta(R)$ with $\delta \sinh R = \varepsilon_1$.

We choose and fix a number $m_0 > 0$ such that $K^{m_0}_M$ is very ample. Let $\tilde{\eta}$ be the cut-off function of $M$ defined as

$$\tilde{\eta} = \begin{cases} 
\eta(\rho - (R - 1)) & \rho \geq 0 \text{ and } x \in C_\delta, \\
\eta(-\rho - (R - 1)) & \rho < 0 \text{ and } x \in C_\delta, \\
0 & \text{otherwise,}
\end{cases}$$

(5.3)
where the function $\eta$ is defined in (2.10). Let
\begin{equation}
(5.4)
  u_1 = \frac{1}{\sqrt{\delta}} \tilde{\eta}(dz)^{m_0}
\end{equation}
be a section of $K_{M}^{m_0}$ over $M$ using this cut-off function, where $dz$ is defined in (3.3). We can check that
\[
\frac{\partial u_1}{\partial \rho} = 0 \quad \text{if} \quad |\rho| > R
\]
Thus
\[
\int_{M} \frac{1}{\sqrt{\delta}} \tilde{\eta}(dz)^{m_0} \leq \frac{8}{(m_0 - 1)(\cosh (R-1))^{2m_0-1}}.
\]
By Proposition 2.1, there is a section $u$ of $K_{M}^{m_0}$ such that
\[
\nabla u = \overline{\partial} u
\]
with
\[
\int_{M} |u|^2 \leq \frac{1}{m_0 - 1} \int_{M} \frac{1}{\sqrt{\delta}} \tilde{\eta}(dz)^{m_0} \leq \frac{8}{(m_0 - 1)(\cosh (R-1))^{2m_0-1}}.
\]
By using (4.1), we see that
\[
(5.5)
  \int_{M} |u|^2 \leq \sim \delta^{2m_0-1}.
\]
Let
\begin{equation}
(5.6)
  U' = u_1 - u
\end{equation}
Then $\overline{\partial} U' = 0$.

**Lemma 5.1.** Let $U''$ be a holomorphic section of $K_{M}^{m_0}$ on $C_\delta(R)$ such that $(U'', u_1)_{C_\delta(R-2)} = 0$. That is,
\[
(5.7)
  \int_{C_\delta(R-2)} U'' < 0
\]
where $u_1$ is the section defined in (5.4). Then there is an absolute constant $\varepsilon_6 > 0$ such that
\[
|U''| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|+m_0(R-|\rho|)}} ||U''||_{L^2(C_\delta)},
\]
and
\[
(5.9)
  ||\nabla U''|| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|+(m_0+1)(R-|\rho|)}} ||U''||_{L^2(C_\delta)}.
\]
We use the notation in § 8. Let
\[
(8)
  \begin{align*}
  w_1 &= e^{2s+\pi/3} - 4s \arctan e^\rho - \arctan e^{-(R-2)}, \\
  w_2 &= e^{-2s+\pi/3} + 4s \arctan e^\rho - \arctan e^{R-2}.
  \end{align*}
\]
Let
\[
U'' = (g_1(w_1) + a + g_3(w_2))(dz)^{m_0}
\]
be the decomposition similar to that in (3.4) where \(g_1(w_1)\) and \(g_2(w_2)\) are \(\text{holomorphic}\) functions of \(w_1\) and \(w_2\) respectively, \(g_1(0) = g_2(0) = 0\), \(a\) is a constant, and \(dz\) is defined in (3.3).

Using (5.7), we see that \(a = 0\). By the Schwartz Lemma, we have
\[
|g_1(w_1)| \leq |w_1| \max_{|w_1|=1} |g_1(w_1)|.
\]
(5.10)
At each point of \(|w_1| = 1\) or \(\rho = R - 2\), by the collar theorem, there is an absolute lower bound for the injective radius. Thus by the Cauchy integral formula, we have
\[
\max_{|w_1|=1} |g_1(dz)^{m_0}| \leq |g_1(dz)^{m_0}|_{L^2(C_\delta)}.
\]
(5.11)
Using (5.10) and (5.11), we have
\[
||g_1(dz)^{m_0}|| \leq e^{-4\pi \rho \frac{1}{\sqrt{\delta}} (\text{arctan e}^\rho \text{arctan e}^{-(R-2)}) \left( \frac{\cosh R}{\cosh \rho} \right)^{m_0} ||g_1(dz)^{m_0}||_{L^2(C_\delta)}.}
\]
(5.12)
Using the same argument as above, we get (5.9).

**Lemma 5.2.** With the notations as above, there is a constant \(r > 2\), depending only on \(m_0\) and the genus \(g\), such that \(U' \neq 0\) on \(C_\delta(R - r)\), where \(U'\) is defined in (5.6).

**Proof.** By (5.6), we know that \(u\) is holomorphic on \(C_\delta(R - 2)\). Let
\[
u = u' + \alpha u_1
\]
(5.17)
be the decomposition of \(u\) such that \((u', u_1)_{C_\delta(R - 2)} = 0\) and \(\alpha\) is a constant. Then
\[
\int_M ||u||^2 \geq \int_{C_\delta(R - 2)} ||u'||^2 + \int_{C_\delta(R - 2)} ||\alpha||^2 ||u_1||^2.
\]
(5.18)
In particular
\[
\int_{C_\delta(R-2)} ||u||^2 \geq ||\alpha||^2 \int_{-(R-2)}^{R-2} \frac{1}{(\cosh \rho)^{2m_0-1}} d\rho.
\]
By (5.5) and (5.19), we have
\[
|\alpha| \leq \sim \delta^{m_0-\frac{1}{2}}.
\]
On the other hand, by Lemma 5.1,
\[
||u'|| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \delta^{m_0-\frac{1}{2}}.
\]
Thus by (5.4), (5.20) and (5.21), there are constants $D_1$ and $D_2$ such that
\[
||U'(x)|| \geq \frac{1}{\sqrt{\delta_i} (\cosh \rho)^{m_0}} (1 - D_1 \delta^{m_0-\frac{1}{2}})
\]
\[
- D_2 e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \delta^{m_0-\frac{1}{2}}
\]
for $|\rho| < R - 3$. If $r$ is large enough, then $||U'(x)|| > 0$ for $|\rho| < R - r$. In particular $U' \neq 0$ on $C_\delta(R-r)$. This completes the proof of the lemma.

Now we assume the general case. Let $C_{\delta_1}, \cdots, C_{\delta_s}$ be the collars of parameters $\delta_1, \cdots, \delta_s$ respectively embedded into $M$. We assume that $C_{\delta_i}$, $(i = 1, \cdots, s)$ do not intersect each other. By Lemma 5.2 there are $R_i$, $(i = 1, \cdots, s)$ such that we can find $U_i' \in H^0(M, K^{m_0}_M)$, $(i = 1, \cdots, s)$ with $U_i'|_{C_{\delta_i}(R_i)} \neq 0$. Let $\rho_i, dz_i, (i = 1, \cdots, s)$ be defined in §3 corresponding to $C_{\delta_i}(i = 1, \cdots, s)$, respectively. Let
\[
\eta_i = \begin{cases} 
\eta(\rho_i - (R_i - 1)) & \rho_i \geq 0 \quad \text{and} \quad x \in C_{\delta_i} \\
\eta(-\rho_i - (R_i - 1)) & \rho_i \leq 0 \quad \text{and} \quad x \in C_{\delta_i} \\
0 & \text{otherwise}
\end{cases}
\]
be the cut-off functions for $i = 1, \cdots, s$. Assume that max $\delta_i$ is small enough. Then by Lemma 5.2, $U_1', \cdots, U_s' \in H^0(M, K^{m_0}_M)$ have the following properties
1. $||U'|| \geq \frac{1}{\sqrt{\delta_i} e^{-m_0|\rho_i|}}$ on $C_{\delta_i}(R_i)$ for $i = 1, \cdots, s$ (by (5.22));
2. There are decompositions $U_i'|_{C_{\delta_i}(R_i)} = \alpha_i v_{1i} + v_{2i}$ with
\[
v_{1i} = \frac{1}{\sqrt{\delta_i}} \eta_i(dz_i)^{m_0}, \quad 1 \leq i \leq s
\]
and
\[
(v_{1i}, v_{2i})_{C_{\delta_i}(R_i)} = 0, \quad 1 \leq i \leq s,
\]
where $\alpha_i (i = 1, \cdots, s)$ are a constant such that for $1 \leq i \leq s$,
\[
1 \leq \sim \alpha_i \leq \sim 1;
\]
3. By (5.4) and (5.5),

\[
\int_{M \setminus C_{\delta_i}(R_i)} ||U'_i||^2 \leq \delta_i^{2m_0-1}
\]

for \(i = 1, \cdots, s\).

We have the following lemma:

**Lemma 5.3.** With the notations as above, there are holomorphic sections \(U_1, \cdots, U_s \in H^0(M, K^m_{M})\) such that

1. \(||U_i|| \geq \frac{1}{\sqrt{\delta_i}} e^{-m_0|\rho_i|} \) on \(C_{\delta_i}(R_i)\);
2. \((U_i|_{C_{\delta_j}(R_j)}, v_{1j})_{C_{\delta_j}(R_j)} = 0 \) for \(i \neq j, 1 \leq i, j \leq s\);
3. \(\int_{M \setminus C_{\delta_i}(R_i)} ||U'_i||^2 \leq \delta_i^{2m_0-1}\).

**Proof.** Let

\[
\beta_{ij} = (U'_i, v_{1j})_{C_{\delta_j}(R_j)}, \quad 1 \leq i, j \leq s.
\]

Then if \(i \neq j\) we have

\[
|\beta_{ij}| \leq \frac{1}{\sqrt{\delta_i}} e^{-m_0|\rho_i|}
\]

by (5.25) and the definition of \(v_{1j}\), \(j = 1, \cdots, s\) in (5.23). We also have

\[
1 \geq \beta_{ii} \geq 1
\]

by (5.24).

Let \(B = (\beta_{ij})_{s \times s}\) be the matrix of \((\beta_{ij})\) for \(1 \leq i, j \leq s\). Then by (5.26), (5.27), \(B\) is an invertible matrix, when max \(\delta_i\) is small enough. Let \(A = B^{-1}\) be the inverse matrix and let \(A = (\alpha_{ij})_{s \times s}\). Define

\[
U_i = \sum_{j=1}^{s} \alpha_{ij} U'_j, \quad 1 \leq i \leq s.
\]

Then \(U_i(i = 1, \cdots, s)\) satisfies all the requirements in the lemma by the fact that

\[
|\alpha_{ij}| \leq \delta_i^{m_0-\frac{1}{2}} \quad i \neq j,
\]

\[
1 \geq \alpha_{ii} \geq 1 \quad i = j.
\]

Let \(U'_{s+1}, \cdots, U'_d\) be an orthonormal basis of the space \(span\{U_1, \cdots, U_s\}^\perp\). Assume that

\[
(U'_i, U'_j) = 0
\]

for \(s < i \leq d, 1 \leq j \leq s\). Let

\[
U_i = U'_i - \sum_{j=1}^{s} \gamma_{ij} U'_j, \quad s < i \leq d,
\]

29
where
\begin{equation}
\gamma_{ik} = \frac{1}{(U_k, v_{1k})C_{\delta_k}(R_k)}(U_i^t, v_{1k})C_{\delta_k}(R_k), \quad 1 \leq k \leq s, s < i \leq d.
\end{equation}

Then we have
\begin{equation}
(U_i|_{C_{\delta_j}(R_j)}, v_{1j})C_{\delta_j}(R_j) = 0
\end{equation}
for \( s < i \leq d \) and \( 1 \leq j \leq s \).

We have the following lemma:

**Lemma 5.4.** If \( x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j - 2) \), then
\begin{equation}
\sum_{i=1}^{d} ||U_i||^2 \geq 1.
\end{equation}

**Proof.** Let
\begin{equation}
l_{ij} = (U_i, U_j)
\end{equation}
for \( 1 \leq i, j \leq d \). If \( 1 \leq i, j \leq s \), we have
\begin{equation}
(U_i, U_j) \leq \max \delta_i \quad i \neq j,
1 \geq (U_i, U_j) \geq 1 \quad i = j
\end{equation}
by (5.26), (5.27) and lemma 5.3.

If \( 1 \leq i \leq s, s < j \leq d \), we have
\begin{equation}
(U_i, U_j) = -\sum_{k=1}^{s} \gamma_{jk}(U_i, U_k)
\end{equation}
using (5.28). By the definition of \( \gamma_{jk} \) in (5.29), we have
\begin{equation}
|\gamma_{jk}| \leq \max \delta_j.
\end{equation}

Thus
\begin{equation}
||(U_i, U_j)|| \leq \max \delta_j
\end{equation}
for \( 1 \leq i \leq s \) and \( s < j \leq d \). Finally, if \( s < i, j \leq d \), then
\begin{equation}
|(U_i, U_j)| \leq \max \delta_i \quad i \neq j,
1 \geq (U_i, U_j) \geq 1 \quad i = j
\end{equation}
by (5.28) and (5.31). Using (5.30), (5.32) and (5.33), we have
\begin{equation}
|l_{ij}| \leq \max \delta_i \quad i \neq j,
1 \geq |l_{ij}| \geq 1 \quad i = j.
\end{equation}

Let \((m_{ij})_{d\times d}\) be the matrix such that
\begin{equation}
\sum_{i=1}^{d} \sum_{t=1}^{d} m_{ij}m_{tk}l_{it} = \delta_{jk}
\end{equation}
for \( 1 \leq i, j, k \leq d \).
for $1 \leq j, k \leq d$. We can choose $m_{ij}$ such that
\begin{equation}
|m_{ij}| \lesssim \max_{i \neq j} \delta_i, \quad 1 \geq m_{ij} \gtrsim 1 \quad i = j.
\end{equation}

It is easy to check that $\sum_{j=1}^{d} m_{ij} U_j$ for $1 \leq i \leq d$ forms an orthonormal basis of $H^0(M, K^m_M)$. Thus we have
\begin{equation}
\sum_{i=1}^{d} \left| \sum_{j=1}^{d} m_{ij} U_j \right|^2 \gtrsim 1
\end{equation}
by Theorem 5.3. The lemma thus follows from (5.34) and the fact that $\max \delta_i$ is small.

We summarize the results up to now in the following

**Proposition 5.1.** Let $U_1, \ldots, U_d$ be sections of $H^0(M, K^m_M)$ as above. Then
\begin{equation}
||U_i|| + ||\nabla U_i|| \lesssim 1 \quad x \notin \cup_{j=1}^{s} C_{\delta_j}(R_j - 2),
\end{equation}
\begin{equation}
||U_i|| + ||\nabla U_i|| \lesssim e^{-(m_0+1)(R_j - |\rho_j|)} \quad x \in C_{\delta_j}(R_j - 2), 1 \leq j \leq s,
\end{equation}
for $i > s$ and
\begin{equation}
||U_i|| + ||\nabla U_i|| \lesssim \delta_i^{m_0-\frac{1}{2}} \quad x \notin \cup_{j=1}^{s} C_{\delta_j}(R_j - 2),
\end{equation}
\begin{equation}
||U_i|| + ||\nabla U_i|| \lesssim \frac{1}{\sqrt{\delta_i e^{m_0 |\rho_i|}}} \quad x \in C_{\delta_i}(R_i - 2),
\end{equation}
\begin{equation}
||U_i|| + ||\nabla U_i|| \lesssim \delta_i^{m_0-\frac{1}{2}} e^{-(m_0+1)(R_j - |\rho_j|)} \quad x \in C_{\delta_j}(R_j - 2), j \neq i.
\end{equation}
for $1 \leq i \leq s$. Furthermore,
\begin{equation}
\sum_{k=1}^{d} ||U_i||^2 \gtrsim 1 \quad x \notin \bigcup_{i=1}^{s} C_{\delta_i}(R_i - 2).
\end{equation}

**Proof.** If $x \notin \cup_{j=1}^{s} C_{\delta_j}(R_j)$, then $\delta_x$ has a uniform lower bound. (5.35) follows from the Cauchy integral formula. (5.36) is a corollary of Lemma 5.1. (5.37) follows from (5.3) and the Cauchy integral formula. (5.38) follows from a straightforward computation. (5.39) follows from Lemma 5.1 and (5.25). Finally, (5.40) is just a restatement of the conclusion of Lemma 5.4.

Define an inner product $<,>$ in the coordinate ring $\bigoplus_{m=0}^{\infty} H^0(M, K^m_M)$. Let $S_1 \in H^0(M, K^m_M)$ and $S_2 \in H^0(M, K^m_M)$. Suppose that $m \geq m_1$. We define a section of $K^m_{M} - m_1$ as follows: Let $x \in M$ and $U_x$ is a local trivialization of $K_M$. Let $S_1|U_x = S'| S'' \in \Gamma(U_x, K^{m_1}_{M})$ and $S'' \in \Gamma(U_x, K^{m_1}_{M})$. Then

$S_3|U_x = S' < S'', S_2 >_{H^{m_1}}$, where $<,>$ is the pointwise inner product.
Proof of Theorem 5.1. We modify the method of Wolff's [14] of solving the corona problem on the unit disk. First we construct a $C^\infty$ solution. Let $S \in H^0(M, K_M^m)$ for a fixed $m > m_0$. Let

$$
\begin{aligned}
\begin{cases}
  b_k = \eta_k \frac{S}{U_k} + (1 - \sum_{j=1}^s \eta_j) \frac{<S, U_k>}{\sum_{j=1}^d ||U_j||^2} & 1 \leq k \leq s, \\
  b_k = (1 - \sum_{j=1}^s \eta_j) \frac{<S, U_k>}{\sum_{j=1}^d ||U_j||^2} & k > s.
\end{cases}
\end{aligned}
$$

Here $b_k (1 \leq k \leq s)$ is well defined because of Lemma 5.2. We can check that

$$
S = \sum_{k=1}^d U_k b_k.
$$

If $x \notin \cup_{j=1}^s C_{\delta_j} (R_j - 2)$, then by Lemma 5.4,

$$
\begin{aligned}
\begin{cases}
  ||b_k|| \sim (\sqrt{\delta_k} (\cosh \rho_k)^{m_0} + 1) ||S|| & 1 \leq k \leq s, \\
  ||b_k|| \sim ||S|| & k > s.
\end{cases}
\end{aligned}
$$

By (5.41), we have

$$
\begin{aligned}
\bar{\partial} b_k &= \bar{\partial} \eta_k \frac{S}{U_k} - \bar{\partial} \sum_{j=1}^s \eta_j \frac{<S, U_k>}{\sum_{j=1}^d ||U_j||^2} + (1 - \sum_{j=1}^s \eta_j) \frac{<S, \nabla U_k>}{\sum_{j=1}^d ||U_j||^2} \\
&- (1 - \sum_{j=1}^s \eta_j) \frac{<S, U_k> \sum_{j=1}^d <U_j, \nabla U_j>}{(\sum_{j=1}^d ||U_j||^2)^2} & 1 \leq k \leq s, \\
\bar{\partial} b_k &= -\bar{\partial} \sum_{j=1}^s \eta_j \frac{<S, U_k>}{\sum_{j=1}^d ||U_j||^2} + (1 - \sum_{j=1}^s \eta_j) \frac{<S, \nabla U_k>}{\sum_{j=1}^d ||U_j||^2} \\
&- (1 - \sum_{j=1}^s \eta_j) \frac{<S, U_k> \sum_{j=1}^d <U_j, \nabla U_j>}{(\sum_{j=1}^d ||U_j||^2)^2} & k > s.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\begin{cases}
  ||\bar{\partial} b_k|| \sim \frac{m_0}{\delta_k} ||S|| & 1 \leq k \leq s, \\
  ||\bar{\partial} b_k|| \sim ||S|| & k > s.
\end{cases}
\end{aligned}
$$

Let

$$
\begin{aligned}
c_{ik} &= \frac{<\bar{\partial} b_k, U_i>}{\sum_{j=1}^d ||U_j||^2}.
\end{aligned}
$$

for $i \neq k$ and $1 \leq i, k \leq d$. Then by (5.42) and (5.44), we have

$$
\begin{aligned}
\begin{cases}
  ||c_{ik}|| \sim \frac{1}{\delta_k^{m_0/2}} ||S|| & 1 \leq k \leq s, \\
  ||c_{ik}|| \sim ||S|| & k > s.
\end{cases}
\end{aligned}
$$

Let’s consider the equations

$$
\bar{\partial} b_{ik} = c_{ik}
$$
for \( i \neq k \) and \( 1 \leq i, k \leq d \). By Proposition 2.1, the solutions exist and we may assume that
\[
\|b_{ik}\|_{L^2} \leq \|c_{ik}\|_{L^2}
\]
for \( i \neq k, 1 \leq i, k \leq d \). Thus by (5.46), we have
\[
(5.47) \quad \begin{cases}
\|b_{ik}\|_{L^2} \leq \frac{1}{\delta_k} \|S\|_{L^2} & 1 \leq k \leq s, \\
\|b_{ik}\|_{L^2} \leq \|S\|_{L^2} & s < k \leq d.
\end{cases}
\]

Let
\[
T_i = b_i + \sum_{k=1}^{d} (b_{ik} - b_{ki}) U_k.
\]

One can check that \( \partial T_i = 0 \) and
\[
S = \sum_{i=1}^{d} T_i U_i.
\]

In order to prove the theorem, we need to estimate \( \|T_i U_i\| \) for \( 1 \leq i \leq d \).

By (5.47) and Proposition 5.1, we have
\[
(5.48) \quad \|T_i U_i\|_{L^2(M \cup \cup_{j=1}^{s} C_{\delta_j}(R_j))} \leq \|S\|_{L^2}.
\]

By the Cauchy integral formula and Proposition 4.1
\[
(5.49) \quad \|T_i U_i\|(x) \leq \|S\|_{L^2}
\]
for \( 1 \leq i \leq d \) and \( x \notin \cup_{j=1}^{s} C_{\delta_j}(R_j) \).

Let
\[
E_i(\rho) = \{ x \in C_{\delta_i}(R_i) | |\rho_i(x)| \geq \rho \}, \quad 1 \leq i \leq s
\]
for positive number \( \rho > 0 \). By (5.47) and Proposition 5.1 again, we have
\[
(5.50) \quad \begin{cases}
\|T_i U_i - \eta_i S\|_{L^2(E_i(\rho) - E_i(\rho + 2))} \leq e^{-(R_i - \rho)} \|S\|_{L^2} & 1 \leq i \leq s, \\
\|T_i U_i\|_{L^2(E_i(\rho) - E_i(\rho + 2))} \leq e^{-(R_i - \rho)} \|S\|_{L^2} & i > s.
\end{cases}
\]

By the Cauchy formula,
\[
(5.51) \quad \begin{cases}
\|T_i U_i - \eta_i S\|(x) \leq \frac{1}{\delta_x} e^{-(R_i - \rho)} \|S\|_{L^2} & 1 \leq i \leq s, \\
\|T_i U_i\|(x) \leq \frac{1}{\delta_x} e^{-(R_i - \rho)} \|S\|_{L^2} & i > s.
\end{cases}
\]
for any \( x \in E_i(\rho + \frac{1}{2}) - E_i(\rho + 1) \). Since
\[
\delta_x \geq \varepsilon_5 \delta_j (\cosh \rho_j)
\]
by (4.7), we have
\[
(5.52) \quad \begin{cases}
\|T_i U_i - \eta_i S\|(x) \leq \|S\|_{L^2} & 1 \leq i \leq s, \\
\|T_i U_i\|(x) \leq \|S\|_{L^2} & i > s
\end{cases}
\]
for any \( x \in E_i(\rho + \frac{1}{2}) - E_i(\rho + 1) \) and any \( |\rho_j| < R_j - 3, j = 1, \cdots s \). Thus if \( \|S\|_{L^2} = 1 \), then by (5.49) and (5.52),
\[
\|T_i U_i\|_{L^2} \leq \sim 1
\]
for $1 \leq i \leq d$. If $||S||_{L^\infty} = 1$, then

$$||T_i U_i||_{L^\infty} \leq 1$$

for $1 \leq i \leq d$. These results give the inequality (5.2).

\[ \square \]

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