GENERALIZED POLYNOMIAL MODULES
OVER THE VIRASORO ALGEBRA

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Abstract. Let $B_r$ be the $(r + 1)$-dimensional quotient Lie algebra of the positive part of the Virasoro algebra $V$. Irreducible $B_r$-modules were used to construct irreducible Whittaker modules in a work of Mazorchuk and Zhao (2014) and irreducible weight modules with infinite dimensional weight spaces over $V$ in a work of Liu, Lu and Zhao (2015). In the present paper, we construct non-weight Virasoro modules $F(M, \Omega(\lambda, \beta))$ from irreducible $B_r$-modules $M$ and $(A, V)$-modules $\Omega(\lambda, \beta)$. We give necessary and sufficient conditions for the Virasoro module $F(M, \Omega(\lambda, \beta))$ to be irreducible. Using the weighting functor introduced by J. Nilsson, we also determine necessary and sufficient conditions for two $F(M, \Omega(\lambda, \beta))$ to be isomorphic.

1. Introduction

Let $V$ denote the complex Virasoro algebra, that is, the Lie algebra with a basis $\{c, d_i : i \in \mathbb{Z}\}$ and the Lie bracket defined as follows:

$$[d_i, d_j] = (j - i)d_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c; \quad [d_i, c] = 0, \quad \forall i, j \in \mathbb{Z}.$$ 

The Virasoro algebra $V$ is one of the most important algebras studied by physicists and mathematicians in the last few decades. It has a profound impact on mathematical and physical sciences; see [3,9,10,13,14]. The representation theory on the Virasoro algebra has attracted a lot of attention from mathematicians and physicists. The recent monograph [12] is a detailed survey of the classical part of the representation theory of $V$. There are two classical families of simple weight $V$-modules with all finite dimensional weight spaces: highest weight modules (completely described in [7]) and the so-called intermediate series modules. In [19] it is shown that these two families exhaust all simple weight modules with all finite dimensional weight spaces. In [21] it is even shown that the above modules exhaust all simple weight modules admitting a non-zero finite dimensional weight space. Very naturally, the next important task for the Virasoro algebra is to study simple weight modules with infinite dimensional weight spaces and non-weight simple modules.

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The first such examples of weight modules with infinite dimensional weight spaces were constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [30] in 1997, while the irreducibility of these tensor products was recently solved completely in [4]. Then Conley and Martin gave another class of such examples with four parameters in [5] in 2001. In [15], a large class of an irreducible Virasoro module with infinite dimensional weight spaces was constructed from $B_r$-modules, where $B_r$ (see Section 2) is an $(r + 1)$-dimensional quotient Lie algebra of the positive part of the Virasoro algebra; also see [17].

During the last decade, various families of non-weight simple Virasoro modules were studied. These include Whittaker modules, see [8, 16, 18, 20, 22, 25, 26, 29], $\mathbb{C}[d_0]$-free modules, see [23, 27, 28], highest-weight-like modules, see [11], and irreducible modules from Weyl modules, see [18]. In particular, all Whittaker modules and even more were described in a uniform way in [22], using irreducible $B_r$-modules. In [28], it was shown that if $M$ is a $V$-module which is a free $\mathbb{C}[d_0]$-module of rank 1, then $M \cong \Omega(\lambda, \beta)$ for some $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. One can refer to Example 2 for the definition of $\Omega(\lambda, \beta)$.

It is easy to see that an irreducible $V$-module is not a weight module if and only if it is $\mathbb{C}[d_0]$-free. So it is valuable to construct non-weight $V$-modules from $\Omega(\lambda, \beta)$. This is the motivation of the present paper.

Now we briefly describe the main results in the present paper. In Section 2, we first recall the definition of $(A, V)$-modules; see Section 3 in [6].

**Definition 2.1.** An $(A, V)$-module $M$ is a module both for the Lie algebra $V$ and the commutative associative algebra $A$ with compatible actions:

$$mx^{n+m}v = d_n x^{n}v - x^{n}d_nv, \quad cv = 0,$$

where $m, n \in \mathbb{Z}, v \in M$.

The following two examples are two interesting classes of $(A, V)$-modules.

**Example 1.** For each $\alpha, \beta \in \mathbb{C}$, there is a natural $(A, V)$-module structure on $A$ as follows:

$$d_n x^n = (n + \alpha + \beta m)x^{n+m}, \quad x^m x^n = x^{n+m}, \quad cx^n = 0.$$
We denote this module by $A(\alpha, \beta)$ which is called the intermediate series module. It is well known that, as a $V$-module $A(\alpha, \beta)$ is reducible if and only if $\alpha \in \mathbb{Z}$, and $\beta = 0$ or 1. Mathieu showed that any irreducible uniformly bounded weight module over $V$ is isomorphic to some irreducible sub-quotient of $A(\alpha, \beta)$; see [19].

**Example 2.** For $\lambda \in \mathbb{C} \setminus \{0\}, \beta \in \mathbb{C}$, denote by $\Omega(\lambda, \beta) = \mathbb{C}[t]$ the polynomial associative algebra over $\mathbb{C}$ in indeterminate $t$. In [18], a class of $(A, V)$-modules is defined on $\Omega(\lambda, \beta)$ by

$$cf(t) = 0, d_m f(t) = \lambda^m (t - \beta m) f(t - m),$$

$$x^m f(t) = \lambda^m f(t - m),$$

for all $m \in \mathbb{Z}, f(t) \in \mathbb{C}[t]$. From [18] we know that $\Omega(\lambda, \beta)$ is irreducible when it was restricted to $V$ if and only if $\beta \neq 0$. When $\beta = 0$, one can check that $\mathbb{C}[t] \Omega(\lambda, 0)$ is a proper Virasoro submodule of $\Omega(\lambda, 0)$. It was shown that if $M$ is a $V$-module which is a free $\mathbb{C}[d_0]$-module of rank 1, then $M \cong \Omega(\lambda, \beta)$ for some $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$; see [28].

For an $(A, V)$-module $M$, we define the following operator on $M$:

$$g(m) = x^{-m} d_m, \ m \in \mathbb{Z}.$$

We can check that

$$g(m) g(k) v - g(k) g(m) v = -k g(k) v + m g(m) v + (k - m) g(m + k) v,$$

$$g(m) x^n v - x^n g(m) v = n x^n v,$$

for any $m, k, n \in \mathbb{Z}, v \in M$.

By abuse of language, we denote $G$ by the Lie algebra with the basis $\{g(m) : m \in \mathbb{Z}\}$ and the Lie bracket defined as follows:

$$[g(m), g(k)] = -k g(k) + m g(m) + (k - m) g(m + k), \ \forall m, k \in \mathbb{Z}.$$

Let us define the notion of $(A, G)$-modules which appears naturally in the above construction.

**Definition 2.2.** An $(A, G)$-module $M$ is a module both for the Lie algebra $G$ and the commutative associative algebra $A$ with compatible actions:

$$g(m) x^n v - x^n g(m) v = n x^n v,$$

where $m, n \in \mathbb{Z}, v \in M$.

From (2.1) and (2.2), any $(A, V)$-module can be viewed as an $(A, G)$-module. Conversely, an $(A, G)$-module $M$ can also be viewed as an $(A, V)$-module via

$$d_m v = x^m g(m) v, \ cv = 0, \ \forall v \in M.$$

Let $M$ be a $G$-module and $W$ be an $(A, V)$-module. Since $W$ is also a $G$-module, considering the tensor product $M \otimes W$ of $G$-modules $M$ and $W$, there is a natural $(A, G)$-module structure on $M \otimes W$ as follows:

$$g(m) (v \otimes w) = v \otimes (g(m) w) + (g(m) v) \otimes w,$$

$$x^m (v \otimes w) = v \otimes (x^m w),$$

where $m \in \mathbb{Z}, v \in M, w \in W$. 

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From $d_m = x^m g(m)$, we know that the action of $V$ on $M \otimes W$ is
\begin{equation}
\label{2.6}
d_m(v \otimes w) = v \otimes (d_m w) + (g(m)v) \otimes x^m w,
\end{equation}
where $m \in \mathbb{Z}, v \in M, w \in W$.

Consequently, the formulas (2.5), (2.6) and (2.7) define an $(A, V)$-module structure on $M \otimes W$. We denote this $(A, V)$-module by $F(M, W)$.

Denote by $V_+$ the Lie subalgebra of $V$ spanned by all $d_i$ with $i \geq 0$. For $r \in \mathbb{Z}_+$, denote by $V_+^{(r)}$ the Lie subalgebra of $V$ generated by all $d_i$, $i > r$, and by $B_r$ the quotient algebra $V_+ / V_+^{(r)}$. By $\bar{d}_i$ we denote the image of $d_i$ in $B_r$. Note that $B_r$ is a solvable Lie algebra of dimension $r + 1$.

Let $M$ be a $G$-module. Motivated by the notion of polynomial modules in [2], we suppose that the action of $g(m)$ on $M$ is defined by
\begin{equation}
\label{2.7}
g(m) = \sum_{i=0}^{r} \frac{m^{i+1} A_{i}}{(i+1)!},
\end{equation}
where $A_{i} \in \text{End}_{\mathbb{C}}(M)$ for $i : 0 \leq i \leq r$. From the Lie bracket (2.3) of $G$, we can check that
\[
[A_{i}, A_{j}] = (j - i)A_{i+j}, \forall i, j : 0 \leq i, j \leq r,
\]
where $A_{i+j} = 0$, when $i + j > r$. Thus $M$ can be viewed as a module over the Lie algebra $B_r$ via
\[
\bar{d}_i v = A_{i} v, \quad \forall v \in M.
\]

Conversely, for a module $M$ over $B_r$, we define the action of $G$ on $M$ by
\begin{equation}
\label{2.8}
g(m)v = \sum_{i=0}^{r} \frac{m^{i+1} \bar{d}_i v}{(i+1)!}, \quad \forall v \in M.
\end{equation}

**Lemma 2.3.** Let $M$ be a module over $B_r$. Then $M$ becomes a $G$-module under the action (2.8).

**Proof.** We can compute that
\[
g(m)g(k)v - g(k)g(m)v - mg(m)v + kg(k)v
\]
\[
= \left( \sum_{i=0}^{r} \frac{m^{i+1} A_{i}}{(i+1)!} \right) \left( \sum_{j=0}^{r} \frac{k^{j+1} A_{j}}{(j+1)!} \right) v
\]
\[
- \left( \sum_{i=0}^{r} \frac{k^{j+1} A_{i}}{(j+1)!} \right) \left( \sum_{i=0}^{r} \frac{m^{i+1} A_{i}}{(i+1)!} \right) v
\]
\[
+ \left( k \sum_{i=0}^{r} \frac{k^{i+1}}{(i+1)!} - m \sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!} \right) \bar{d}_i v
\]
\[
= \sum_{i,j=0}^{r} \frac{m^{i+1} k^{j+1}}{(i+1)! (j+1)!} \bar{d}_{i+j} v + \left( \frac{m^{i+1}}{(i+1)!} \right) \bar{d}_i v
\]

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Proposition 2.4. If \( M \) and \( N \) are modules over \( B_r \), then as \((A,V)\)-modules,
\[
F(M, N) \cong F(M \otimes N).
\]

Proof. For any \( v_1, v_2 \in M, w \in W, m, n \in \mathbb{Z} \), we have that
\[
d_m \left( v_1 \otimes (v_2 \otimes w) \right)
= v_1 \otimes d_m (v_2 \otimes w) + (g(m)v_1) \otimes (v_2 \otimes x^m w)
= v_1 \otimes (v_2 \otimes d_m w + g(m)v_2 \otimes x^m w) + (g(m)v_1) \otimes (v_2 \otimes x^m w)
= (v_1 \otimes v_2) \otimes d_m w + (g(m)(v_1 \otimes v_2)) \otimes x^m w
= d_m \left( (v_1 \otimes v_2) \otimes w \right),
\]
and
\[
x^m (v_1 \otimes (v_2 \otimes w)) = v_1 \otimes v_2 \otimes x^m w = x^m \left( (v_1 \otimes v_2) \otimes w \right).
\]
The proposition is proved.

Remark 2.5. From Proposition 2.4, it is very hard to determine the irreducibility of \( F(M, W) \) for any irreducible \((A,V)\)-module \( W \), since it is a difficult problem to discuss the irreducibility of the tensor product of two \( B_r \)-modules. Moreover, up to now there is no explicit classification of all irreducible \((A,V)\)-modules. However, it is still valuable to research the structure of \( F(M, W) \) for some interesting \((A,V)\)-modules.
Let $M$ be a module over $B_r$. From Example 1, (2.6) and (2.8), the action of $\mathcal{V}$ on $F(M, A(\alpha, \beta))$ is defined by

$$d_m(v \otimes x^n) = \left((n + \alpha + \beta m)v + \sum_{i=0}^{r} \frac{m^{i+1}d_i v}{(i+1)!}\right) \otimes x^{n+m},$$

where $m, n \in \mathbb{Z}, v \in M$. Clearly $F(M, A(\alpha, \beta))$ is a weight module over $\mathcal{V}$, which is isomorphic to a module $\mathcal{N}(M, \alpha)$ defined in [15] where $M$ is modified by the action of $d_0$. Thus our Virasoro modules $F(M, W)$ are generalizations of the modules $\mathcal{N}(M, \alpha)$ defined in [15].

If $M$ is a finite dimensional irreducible $B_r$-module, then by Lie's Theorem, $M = \mathbb{C}v, d_i M = 0$ for any $i \in \mathbb{N}$ and $d_0 v = \gamma v$ for some $\gamma \in \mathbb{C}$. We denote this $B_r$-module by $M_\gamma$. Clearly

$$F(M_\gamma, A(\alpha, \beta)) \cong A(\alpha, \beta + \gamma).$$

The following theorem was given by Liu-Lu-Zhao; see Theorem 4 and Theorem 5 in [15].

**Theorem 2.6.** The following statements hold:

(a) Let $M$ be an irreducible $B_r$-module. Then the Virasoro module $F(M, A(\alpha, \beta))$ is reducible if and only if $M \cong M_\gamma$ and $\alpha \in \mathbb{Z}, \beta + \gamma = 0$ or 1.

(b) Let $M, M'$ be an infinite dimensional irreducible module over $B_r$, and $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$. Then $F(M, A(\alpha, \beta)) \cong F(M', A(\alpha', \beta'))$ if and only if $M \cong M'$, $\alpha - \alpha' \in \mathbb{Z}$ and $\beta = \beta'$.

### 3. Irreducibility and the isomorphism classes of $F(M, \Omega(\lambda, \beta))$

In this section, we will determine the irreducibility and the isomorphism classes of the Virasoro modules $F(M, \Omega(\lambda, \beta))$ for any irreducible $B_r$-module $M$.

#### 3.1. Irreducibility of $F(M, \Omega(\lambda, \beta))$

From (2.6) and (2.8), the $\mathcal{V}$-module structure on $F(M, \Omega(\lambda, \beta))$ is given by

$$d_m(v \otimes f(t)) = v \otimes \lambda^m(t - m) f(t - m) + \sum_{i=0}^{r} \frac{m^{i+1}d_i v}{(i+1)!} \otimes \lambda^m f(t - m).$$

**Lemma 3.1.** Let $M$ be an irreducible module over $B_r$. Then either $\bar{d}_r M = 0$ or the action of $\bar{d}_r$ on $M$ is bijective.

**Proof.** It is straightforward to check that $\bar{d}_r M$ and $\text{ann}_M(\bar{d}_r) = \{v \in M|\bar{d}_r v = 0\}$ are submodules of $V$. Then the lemma follows from the simplicity of $M$. \qed

For any $n \in \mathbb{Z}_+, m \in \mathbb{Z}$, let $h_0^m = 1$ and

$$h_n^m = \prod_{j=m+1}^{m+n} (t - j), \forall \ n > 0.$$

It is clear that $\{h_n^m | n \in \mathbb{Z}_+\}$ forms a basis of $\Omega(\lambda, \beta)$ for any $m \in \mathbb{Z}$. Moreover

$$h_n^m - h_n^{m+1} = nh_{n-1}^{m+1}.$$  \hspace{1cm} (3.2)

From the definition of $\Omega(\lambda, \beta)$, we have that

$$d_m h_k^m = \lambda^m(t - m) h_{k+m}, \quad x^m h_k^m = \lambda^m h_{k+m},$$

for any $m, k \in \mathbb{Z}, n \in \mathbb{Z}_+$.  \hspace{1cm} (3.3)
\textbf{Theorem 3.2.} Let $M$ be an irreducible module over $\mathcal{B}_r$, $\lambda \in \mathbb{C} \setminus \{0\}$. Then the Virasoro module $F(M, \Omega(\lambda, \beta))$ is reducible if and only if $M \cong M_\beta$.

\textit{Proof.} If $M$ is finite dimensional, then $M \cong M_\gamma$ for some $\gamma \in \mathbb{C}$. In this case, we have that

$$d_m(v \otimes f(t)) = v \otimes (t - m(\beta - \gamma))f(t - m),$$

for any $f(t) \in \Omega(\lambda, \beta)$. Hence

$$F(M_\gamma, \Omega(\lambda, \beta)) \cong \Omega(\lambda, \beta).$$

Thus $F(M_\gamma, \Omega(\lambda, \beta))$ is reducible if and only if $\beta - \gamma = 0$, i.e., $\beta = \gamma$.

Next we assume that $M$ is infinite dimensional. By Lemma 3.1, there exists an $r_1 \in \mathbb{N}$ with $r_1 \leq r$ such that the action of $\bar{d}_{r_1}$ on $M$ is injective and $\bar{d}_i M = 0$ for $i > r_1$. We may simply assume that $r_1 = r$.

Let $N$ be a non-zero submodule of $F(M, \Omega(\lambda, \beta))$. Since

$$F(M, \Omega(\lambda, \beta)) \cong F(M, \Omega(1, \beta)),$$

see Theorem 3.6 below, we may further assume that $\lambda = 1$.

\textbf{Claim 1.} If $u = \sum_{n=0}^l v_n \otimes h_0^n$ is a non-zero element in $N$ with $v_l \neq 0$, then

$$\sum_{n=0}^l (\bar{d}_r^2 v_n) \otimes h_m^n \in N \text{ for any } m \in \mathbb{Z}.$$

By (3.3) and (3.3),

$$d_m(v_n \otimes h_0^n) = v_n \otimes (t - m\beta) h_m^n + (g(m)v_n) \otimes h_m^n.$$

Consequently

$$d_k d_{m-k}(v_n \otimes h_0^n) = d_k (v_n \otimes (t - (m-k)\beta) h_{m-k}^n + g(m-k)v_n \otimes h_{m-k}^n) = v_n \otimes (t - k\beta)(t - (m-k)\beta - k) h_m^n + g(m-k)v_n \otimes (t - k\beta) h_m^n$$

$$+ g(k)v_n \otimes (t - (m-k)\beta - k) h_m^n + g(k)g(m-k)v_n \otimes h_m^n.$$

Since $k$ is an arbitrary integer, considering the coefficient of $k^{2r+2}$ in $d_k d_{m-k}u$, we obtain that

$$\sum_{n=0}^l (\bar{d}_r^2 v_n) \otimes h_m^n \in N \text{ for } m \in \mathbb{Z}.$$

Then Claim 1 follows.

\textbf{Claim 2.} If $u = \sum_{n=0}^l v_n \otimes h_0^n$ is a non-zero element in $N$, then $(\bar{d}_r^2 v_l) \otimes 1 \in N$.

Consequently $(\bar{d}_r^2 v_l) \otimes \Omega(\lambda, \beta) \subset N$.

Note that $h_m^k \in \mathbb{C}[t]$ can be expressed as a polynomial in $m$ with degree $k$, and the highest term is a constant (i.e., a degree 0 polynomial in $t$). From Claim 1,

$$u(m) := \sum_{n=0}^l (\bar{d}_r^2 v_n) \otimes h_m^n \in N,$$

for any $m \in \mathbb{Z}$. Considering the coefficient of $m^l$ in $u(m)$, we know that $(\bar{d}_r^2 v_l) \otimes 1 \in N$. From $d_0((\bar{d}_r^2 v_l) \otimes 1) = (\bar{d}_r^2 v_l) \otimes t$, we have that $(\bar{d}_r^2 v_l) \otimes \Omega(\lambda, \beta) \subset N$. Then Claim 2 follows.

Since the action of $\bar{d}_r$ on $M$ is injective, $(\bar{d}_r^2 v_l) \neq 0$. From Claim 2, there exists a non-zero element $v \in M$ such that $v \otimes \Omega(\lambda, \beta) \subset N$.

\textbf{Claim 3.} If $v \otimes \Omega(\lambda) \subset N$, then $(\bar{d}_l v) \otimes \Omega(\lambda) \subset N$ for any $l \in \mathbb{Z}_+$. From $d_m(v \otimes h_0^k) \subset N$, then $(\bar{d}_l v) \otimes \Omega(\lambda) \subset N$ for any $l \in \mathbb{Z}_+$.
It is easy to see that \( \dim(\Omega) = N \). Therefore \( F(M, \Omega(\lambda, \beta)) \) is irreducible when \( M \) is infinite dimensional.

3.2. Isomorphism of \( F(V, \Omega(\lambda, \beta)) \). We will first recall the weighting functor introduced in [24].

For \( a \in \mathbb{C} \), let \( I_a \) be the maximal ideal of \( \mathbb{C}[\text{d}_0] \) generated by \( \text{d}_0 - a \). For a \( \mathcal{V} \)-module \( M \) and \( n \in \mathbb{Z} \), let
\[
M_n := M/I_nM, \quad \mathfrak{W}(M) := \oplus_{n \in \mathbb{Z}} (M_n \otimes x^n).
\]
By Proposition 8 in [24], we have the following construction.

**Proposition 3.3.** The vector space \( \mathfrak{W}(M) \) becomes a weight \( \mathcal{V} \)-module under the following action:
\[
(3.4) \quad \text{d}_m \cdot ((v + I_nM) \otimes x^n) := (\text{d}_m v + I_{n+m}M) \otimes x^{n+m}.
\]
We first establish the following useful lemma.

**Lemma 3.4.** We have \( \mathfrak{W}(\Omega(\lambda, \beta)) \cong A(0, 1 - \beta) \).

**Proof.** It is easy to see that \( \dim(\Omega(\lambda, \beta)/I_n(\Omega(\lambda, \beta))) = 1 \) for any \( n \in \mathbb{Z} \). Let
\[
v_n = 1 + I_n(\Omega(\lambda, \beta)) \in \Omega(\lambda, \beta)/I_n(\Omega(\lambda, \beta)).
\]
We see that
\[
\text{d}_m v_n = \lambda^m(t - m\beta) + I_{m+n}(\Omega(\lambda, \beta))
= \lambda^m(m + n - m\beta) + I_{m+n}(\Omega(\lambda, \beta))
= \lambda^m(n + (1 - \beta))v_{m+n}.
\]
Set \( w_n = \lambda^n v_n \). Then \( \text{d}_m w_n = (n + m(1 - \beta))w_{m+n} \). Thus the lemma follows.

We know that \( \Omega(\lambda, 1) \) is irreducible, and \( A(0, 0) \) is reducible as \( \mathcal{V} \)-modules. Thus the weighting functor \( \mathfrak{W} \) does not map irreducible modules to irreducible modules.

**Proposition 3.5.** As Virasoro modules, we have the isomorphism
\[
\mathfrak{W}(F(M, \Omega(\lambda, \beta))) \cong F(M, A(0, 1 - \beta)).
\]

**Proof.** Note that \( F(M, \Omega(\lambda, \beta)) = M \otimes \Omega(\lambda, \beta) \). For any \( n \in \mathbb{Z} \), using (3.1), we have
\[
I_n(F(M, \Omega(\lambda, \beta))) = I_n(M \otimes \Omega(\lambda, \beta)) = M \otimes I_n(\Omega(\lambda, \beta)).
\]
We can easily deduce that
\[
\mathfrak{W}(F(M, \Omega(\lambda, \beta))) \cong F(M, A(0, 1 - \beta)).
\]

Combining Proposition 3.5 with Theorem 2.6 we obtain the following isomorphism criterion.

**Theorem 3.6.** Let \( M, M' \) be two infinite dimensional irreducible \( \mathcal{B}_r \)-modules, \( \lambda, \lambda', \beta, \beta' \in \mathbb{C} \). Then \( F(M, \Omega(\lambda, \beta)) \cong F(M', \Omega(\lambda', \beta')) \) if and only if \( M \cong M', \beta = \beta' \).

**Remark 3.7.** From Theorem 3.2 we can construct irreducible non-weight Virasoro modules from irreducible \( \mathcal{B}_r \)-modules. All irreducible modules over \( \mathcal{B}_1 \) were classified in [1], while all irreducible modules over \( \mathcal{B}_2 \) were classified in [22]. The classification of irreducible modules over \( \mathcal{B}_r \) remains open, for any \( r > 2 \).
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