Number of Solutions of Linear Congruence Systems

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Abstract

1 Introduction

We will consider the system of linear congruences,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \equiv b_1 \pmod{m} \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \equiv b_2 \pmod{m} \\
  & \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & \equiv b_n \pmod{m},
\end{align*}
\]

(1)

where \( m, n \) are positive integers, and all \( a_{ij}, b_i \) are integers. We are interested in finding an expression for the number of solutions to this system. Let \( A = (a_{ij}) \) denote the coefficient matrix. It is well known that this system has a unique solution if and only if \( \det(A) \) and \( m \) are relatively prime, see for example [8]. But, how many solutions do we have if \( \det(A) \) and \( m \) have a common divisor greater than 1? The authors became interested in this question when they were working on multidimensional \( p \)-adic monomial dynamical systems, see [7]. The problem was solved by Butson and Stewart [1], by rewriting the system into Smith normal form. The solvability of linear congruence systems and algorithms for finding a solutions have been of interest for many mathematicians and computer scientists over the years. Example of other contributions to the problem on solving systems of linear congruences can be found in [2, 3, 4, 6, 9].

Compared to Butson and Stewart we will use a more direct method. We use Gaussian elimination with successive reduction and the Chinese remainder theorem instead of the Smith normal form. We will find a different formula for the number of solutions than in [1]. The algorithm used in the proof of the
formula can with small changes also be used to find all incongruent solutions to the system, since we only have used elementary methods.

The paper is organized as follows: In Section 2 we give definitions and notations together with some theorems about solvability of systems of linear congruences. Section 3 is the main section in which we derive the formula for the number of solutions to homogeneous systems. Applications to inhomogeneous systems are mentioned in Section 4. In Appendix A we present an algorithm in pseudo code for calculating the number of solutions. The algorithm also gives us all the solutions. In Section 5 we discuss generalizations of our results.

2 Notations

Two solutions of the linear congruence system (1) are said to be incongruent modulo $m$ if they differ at least in one coordinate modulo $m$. We want to find the number of incongruent solutions modulo $m$. Let $A$ denote the coefficient matrix, $x$ the vector of the indeterminates, and $b$ the vector of the elements on the right hand side in the system, that is, $A = (a_{ij})$, $x = (x_1, x_2, \ldots, x_n)$ and $b = (b_1, b_2, \ldots, b_n)$. Then the system (1) can be written in matrix form as

$$Ax \equiv b \pmod{m}. \quad (2)$$

Let $\eta(A, b, m)$ denote the number of incongruent solutions modulo $m$ to the congruence (2).

Let $\text{adj}(A)$ denote the adjoint matrix of a square matrix $A$ of order $n$. It is known from linear algebra that $\text{adj}(A)$ has the following properties:

$$\det(\text{adj}(A)) = \det(A)^{n-1}$$

and

$$A \text{adj}(A) = \text{adj}(A) A = \det(A) I,$$

where $I$ is the identity matrix.

**Theorem 2.1.** Let $A$, $b$ and $m$ be as above. Then

$$\eta(A, b, m) \leq (\det(A), m)^n.$$  

If $(\det(A), m) = 1$, then $\eta(A, b, m) = 1$.

**Proof.** We multiply the congruence system $Ax \equiv b \pmod{m}$ by $\text{adj}(A)$ and get

$$\text{adj}(A) Ax \equiv \text{adj}(A) b \pmod{m} \iff \det(A) x \equiv \text{adj}(A) b \pmod{m}, \quad (3)$$

which is solvable if and only if $\det(A)$ divides all the elements in $\text{adj}(A)b$. If that is the case then this system has $(\det(A), m)^n$ different solutions. If that is not the case, then the original system has no solutions, since any solutions is also
solutions to the rewritten system \(\mathbf{A} \mathbf{x} = \mathbf{b}\). When we multiply by the matrix \(\text{adj}(\mathbf{A})\) it might happen that we introduce new solutions. This proves that the inequality 
\[ \eta(\mathbf{A}, \mathbf{b}, m) \leq \det(\mathbf{A}, m)^n \]
holds.

Assume that \(\det(\mathbf{A}, m) = 1\). From \(\mathbf{A} \mathbf{x} \equiv \mathbf{b} \pmod{m}\) it follows that
\[ \mathbf{x} \equiv \det(\mathbf{A})^{-1} \text{adj}(\mathbf{A}) \mathbf{b} \pmod{m} \]
is a solution to the original system. Hence \(\eta(\mathbf{A}, \mathbf{b}, m) = 1\).

\[\]  

**Theorem 2.2.** Let \(\mathbf{A}, \mathbf{b}\) and \(m\) be as above. Assume that \(m = m_1 \cdots m_k\), where the integers \(m_1, \ldots, m_k\) are pairwise relatively prime. Then
\[ \eta(\mathbf{A}, \mathbf{b}, m) = \eta(\mathbf{A}, \mathbf{b}, m_1) \cdots \eta(\mathbf{A}, \mathbf{b}, m_k). \]

Hence, \(\eta\) is multiplicative with respect to \(m\).

**Proof.** Since \(m_i | m\) for each \(i\), any solution to \(\mathbf{A} \mathbf{x} \equiv \mathbf{b} \pmod{m}\) is also a solution to every \(\mathbf{A} \mathbf{x} \equiv \mathbf{b} \pmod{m_i}\). Assume that \(\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})\) is a solution to the congruence system \(\mathbf{A} \mathbf{x} \equiv \mathbf{b} \pmod{m_i}\), for \(i = 1, 2, \ldots, k\). Then for any \(k\)-tuple \((\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)\) of solutions of the systems modulo \(m_i\) we construct a solution \(\mathbf{x} = (x_1, x_2, \ldots, x_n)\) modulo \(m\), by defining \(x_i\) to be the unique solution, due to the Chinese Remainder Theorem, modulo \(m\), to the system
\[ \begin{cases} x \equiv x_{i1} \pmod{m_1} \\ x \equiv x_{2i} \pmod{m_2} \\ \vdots \\ x \equiv x_{ki} \pmod{m_k}. \end{cases} \]

Since there are \(\eta(\mathbf{A}, \mathbf{b}, m_1) \cdots \eta(\mathbf{A}, \mathbf{b}, m_k)\) possible such \(k\)-tuples of solutions modulo \(m_i\), the theorem follows (two different \(k\)-tuples can not generate the same solution modulo \(m\)).

\[\]  

### 3 Homogeneous Systems

According to Theorem 2.2 we only have to consider congruence systems modulo a prime power. So, from now on \(m = p^k\), where \(p\) is a prime number and \(k\) a positive integer. Let \(d = \gcd(\mathbf{A}, p^k)\) be the greatest common divisor of all the elements in the matrix \(\mathbf{A}\) and the integer \(p^k\), that is,
\[ (\mathbf{A}, p^k) = (a_{11}, a_{12}, \ldots, a_{nn}, p^k). \]
Hence, \(d = p^e\) for some non-negative integer \(e \leq k\).

**Lemma 3.1.** Let \(d = \gcd(\mathbf{A}, p^k) = p^e\) and let \(\mathbf{R} = \mathbf{A}/d\). Then
\[ \eta(\mathbf{A}, \mathbf{0}, p^k) = d^n \eta(\mathbf{R}, \mathbf{0}, p^{k-e}) = p^{en} \eta(\mathbf{R}, \mathbf{0}, p^{k-e}). \]
Note that \(\eta(\mathbf{R}, \mathbf{0}, 1) = 1\).
Proof. Let \( q = p^k/d = p^{k-\epsilon} \). Assume that \( y \in \mathbb{Z}_q^n \) is a solution to the congruence system \( Rx \equiv 0 \) (mod \( p^k/d \)). Then it follows from the definition of \( R \) that \( y \) is also a solution to the original system \( Ax \equiv 0 \) (mod \( p^k \)). This is also true for all elements on the form

\[
x = y + qv = y + p^{k-\epsilon}v, \quad \text{where } v \in \mathbb{Z}_d^n,
\]

since

\[
A(y + qv) \equiv \frac{p^k}{d} Av \equiv p^k Rv \equiv 0 \pmod{p^k}
\]

and all the elements in \( R = A/d \) are integers. The components of the vector \( qv \) are all non-negative integers less than \( qd = p^k \). Hence, two vectors of the form \( v \) are incongruent modulo \( p^k \).

Next, we prove that different solutions modulo \( q \) are lifted to different solutions modulo \( p^k \). If \( x_1 = y_1 + qv_1 \) and \( x_2 = y_2 + qv_2 \) are two solutions to \( Ax \equiv 0 \pmod{p^k} \) where \( y_1 \) and \( y_2 \) are solutions to \( Ry \equiv 0 \pmod{q} \), then

\[
x_1 - x_2 = y_1 - y_2 + q(v_1 - v_2) \equiv y_1 - y_2 \pmod{q}.
\]

Hence, if \( x_1 = x_2 \), then \( y_1 \) and \( y_2 \) is congruent modulo \( q \). It therefore follows that

\[
\eta(A, 0, p^k) \geq d^n \eta(R, 0, p^k/d).
\]

It remains to prove that all the solutions of the original congruence system \( Ax \equiv 0 \pmod{p^k} \) can be written on the form \( \mathbf{4} \) where the corresponding vector \( y \) is a solution to \( Rx \equiv 0 \pmod{p^k/d} \). Assume that \( x \in \mathbb{Z}_{p^k}^n \) is a solution to \( Ax \equiv 0 \pmod{p^k} \). By the division algorithm, applied on each component of \( x \), there are vectors \( y \in \mathbb{Z}_q^n \) and \( v \in \mathbb{Z}_d^n \) such that \( x = y + qv \). Then

\[
0 \equiv Ax \equiv Ay + \frac{p^k}{d} Av \equiv dRy + p^k Rv \equiv p^\epsilon Ry \pmod{p^k},
\]

or equivalent

\[
Ry \equiv 0 \pmod{p^{k-\epsilon}}.
\]

This proves the theorem. \( \square \)

Let \( A_n = A \) and \( p^l_n = p^l/d_n \), where \( l_{n+1} = k \) and \( d_n = p^{\tau_n} = (A_n, p^k) \). Set \( R_n = A_n/d_n \). If \( \epsilon_n = k \), then \( A_n \equiv 0 \pmod{p^k} \) and

\[
\eta(A_n, 0, p^k) = p^{kn},
\]

since any vector in \( \mathbb{Z}_{p^k}^{n_n} \) is a solution to \( A_n x \equiv 0 \pmod{p^k} \).

Assume that \( \epsilon_n < k \). Then \( (R_n, p) = 1 \) and therefore we can find an element in \( R_n \) that is relatively prime to \( p \), say \( r_{11} \) after a possible rearrangement of rows and columns. Note that \( r_{11} \) is then invertible modulo \( p^l_n \). By Gaussian elimination we can rewrite

\[
R_n x \equiv 0 \pmod{p^{l_n}} \quad \text{(5)}
\]
to get the equivalent system

$$\begin{align*}
  r_{11} x_1 + r_{12} x_2 + r_{13} x_3 + \cdots + r_{1n} x_n &\equiv 0 \pmod{p^n} \\
  a'_{22} x_2 + a'_{23} x_3 + \cdots + a'_{2n} x_n &\equiv 0 \pmod{p^n} \\
  &\vdots \\
  a'_{n2} x_2 + a'_{n3} x_3 + \cdots + a'_{nn} x_n &\equiv 0 \pmod{p^n},
\end{align*}$$

(6)

where

$$a'_{ij} \equiv r_{ij} - r_{11}^{-1} r_{1j} \pmod{p^n}$$

for $i, j = 2, 3, \ldots, n$. Note that we do not change the number of solutions of the system (5) since the two systems are equivalent because Gaussian transform is invertible. Hence, the two systems (5) and (6) have equally many solutions modulo $p^n$ (any solution to one of the systems is also a solution to the other).

Let $A_{n-1} = (a'_{ij})_{2 \leq i, j \leq n}$ and $x' = (x_2, x_3, \ldots, x_n)$, see (6). Note that the matrix $A_{n-1}$ depends on $A_n$ and the choice of $r_{11}$ in $R_n$. Since $r_{11}$ is invertible modulo $p^n$, any solution to the congruence system

$$A_{n-1} x' \equiv 0 \pmod{p^n}$$

(7)

can be extended with respect to $x_1$ in an unique way to a solution to (6). Hence, the number of solution to (7) is equal to the number of solutions to (5). This proves that

$$\eta(A_n, 0, p^k) = d_n^p \eta(R_n, 0, p^k) = d_n^p \eta(A_{n-1}, 0, p^k),$$

(8)

according to Lemma 3.1. Further, we have that

$$\det(A_n) = d_n^p \det(R_n),$$

(9)

since $A_n = d_n R_n$. From (5) and (6) it follows that

$$\det(R_n) \equiv (-1)^{1+1} r_{11} \det(A_{n-1}) \pmod{p^n},$$

and therefore is

$$(\det(R_n), p^j) = (\det(A_{n-1}), p^j),$$

(10)

since $(r_{11}, p) = 1$.

**Remark.** Note that if we choose an integer to represent $r_{11}^{-1}$ which is a multiplicative inverse modulo $p^j$ to $r_{11}$, where $j \geq l_n$, then (10) can be reformulated as

$$(\det(R_n), p^i) = (\det(A_{n-1}), p^i),$$

(11)

for all integers $i$ such that $1 \leq i \leq j$.

**Theorem 3.2.** Let $A$ be a square matrix of order $n$ with integer elements, $p$ a prime number and $k$ a positive integer. Set $A_n = A$. Let $A_i$ denote the matrix consisting of the $i$ last rows and columns of $A$ after the $(n-i)$th step of
Gaussian elimination including a possible reordering of rows or columns, where 
\( i = n - 1, \ldots, 3, 2 \). Further, for \( i = n, \ldots, 3, 2 \) define \( d_i, e_i \) and \( l_i \) recursively by

\[
p_i = \frac{p_i+1}{d_i} \quad \text{and} \quad d_i = p_i = (A_i, p_i+1)
\]

with \( l_{n+1} = k \). Hence, \( l_i = l_{i+1} - e_i \). Then

\[
\eta(A, 0, p^k) = (\det(A), d_2d_3^2 \cdots d_n^{n-1}p^k).
\]

Proof. We prove this by induction over \( n \). First assume that \( n = 2 \) and set \( R_2 = A_2/d_2 \), where \( d_2 = (A_2, p^3) < p^3 \). Note that \( k = l_3 \) in this case. The congruence system \( R_2x = 0 \pmod{p^2} \) is given by

\[
\begin{align*}
 r_{11}x_1 + r_{12}x_2 &\equiv 0 \pmod{p^2} \\
r_{21}x_1 + r_{22}x_2 &\equiv 0 \pmod{p^2},
\end{align*}
\]

(12)

where \( p^2 = p^3/d_2 \) and

\[
R_2 = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}
\]

is an integer matrix. Since \( (R_2, p) = (r_{11}, r_{12}, r_{21}, r_{22}, p) = 1 \) one of the matrix entries must be relatively prime to \( p \). Assume that it is \( r_{11} \). Let \( r_{11}^{-1} \) denote the multiplicative inverse of \( r_{11} \) modulo \( p^2 \). From the first congruence in the system (12) we have

\[
x_1 \equiv -r_{11}^{-1}r_{12}x_2 \pmod{p^2}.
\]

Hence, \( x_1 \) is uniquely determined by \( x_2 \pmod{p^2} \). By putting this expression for \( x_1 \) in the second congruence in the system (12) we get

\[
(r_{22} - r_{11}^{-1}r_{12}r_{21})x_2 \equiv 0 \pmod{p^2},
\]

or equivalent

\[
(r_{11}r_{22} - r_{12}r_{21})x_2 \equiv 0 \pmod{p^2}.
\]

The coefficient for \( x_2 \) is \( \det(R_2) \) and from the theory of linear congruences we know that this equation has \( (\det(R_2), p^2) \) incongruent solutions modulo \( p^2 \). We conclude that \( \eta(R_2, 0, p^2) = (\det(R_2), p^2) \). Hence

\[
\eta(A_2, 0, p^3) = d_2^2(\det(R_2), p^2)
\]

\[
= (d_2^2 \det(R_2), d_2^2p^2) = (\det(A_2), p^{2e_2}p^{l_3-e_2}) = (\det(A_2), d_2p^{l_3}).
\]

If \( d_2 = p^{l_3} \), then \( A_2 \equiv 0 \pmod{p^{l_3}} \) and \( \eta(A_2, 0, p^k) = p^{2l_3} \). Since \( d_2p^{l_3} = p^{2l_3} \) it follows that the equality

\[
\eta(A_2, 0, p^{l_3}) = (\det(A_2), d_2p^{l_3})
\]
also holds in the case when \((A_2, p^{l_3}) = p^{l_3}\). By repeating the procedure with Gaussian elimination described before the theorem we get a sequence \(A_n, A_{n-1}, \ldots, A_2\) of matrices. Assume that
\[
\eta(A_m, 0, p^{l_{m+1}}) = (\det(A_m), d_2^2 \cdots d_m^{m-1} p^{l_{m+1}})
\]
when \(m = n-1, \ldots, 2\). Let \(R_n = A_n/d_n\), where \(d_n = p^{r_n} = (A_n, p^{l_{n+1}})\). The exponent of the prime power
\[
d_2^2 \cdots d_{n-1}^{n-2} p^{l_n}
\]
is
\[
(l_3 - l_2) + 2(l_4 - l_3) + \cdots + (n-2) (l_n - l_{n-1}) + l_n
\]
\[
= -l_2 - l_3 - \cdots - l_{n-1} + (n-1) l_n \leq nl_{n+1} = nk,
\]
since \(l_n \leq l_{n+1}\) and all \(l_i\) are non-negative integers. Hence, by choosing an integer which is an multiplicative inverse to \(r_{11}\) modulo \(p^{nk}\) the equality \([13]\) will be fulfilled when \(p^k = d_2^2 \cdots d_{n-1}^{n-2} p^{l_n}\). Then it follows from \([5], [9], [11]\) and \([13]\) that
\[
\eta(A, 0, p^k) = \eta(A_n, 0, p^{n+1})
\]
\[
= d_n^p \eta(A_{n-1}, 0, p^{l_n})
\]
\[
= d_n^p (\det(A_{n-1}), d_2^2 \cdots d_{n-1}^{n-2} p^{l_n})
\]
\[
= d_n^p (\det(R_n), d_2^2 \cdots d_{n-1}^{n-2} p^{l_n})
\]
\[
= (d_n^p \det(R_n), d_2^2 \cdots d_{n-1}^{n-2} d_n^{n-1} p^{l_n})
\]
\[
= (\det(A_n), d_2^2 \cdots d_{n-1}^{n-2} d_n^{n-1} p^{l_{n+1}})
\]
\[
= (\det(A), d_2^2 \cdots d_{n-1}^{n-2} d_n^{n-1} p^k),
\]
since \(k = l_{n+1} = l_n + c_n\).

If \(d_i = p^c = (A_i, p^{l_{i+1}}) = p^{l_{i+1}}\) for some \(i\), then \(l_i = 0\) and we will consider the system \(R_i x \equiv 0\) (mod 1), which have exactly one solution, namely \(x = 0\). Further, \(d_j = 1\) and \(l_j = 0\) for all \(j < i\). The number of solutions of the system \(A_i x \equiv 0\) (mod \(p^{l_{i+1}}\)) is
\[
(p^{l_{i+1}})^i = d_i^{l_i} = d_i^{l_i-1} p^{c_i} = d_i^{l_i-1} p^{l_{i+1}}.
\]
Further, we also have that \(p^{l_{i+1}} | \det(A_i)\). Since \(d_2^2 \cdots d_{i-1}^{i-2} = 1\) it follows that
\[
\eta(A_i, 0, p^{l_{i+1}}) = d_2^2 \cdots d_{i-1}^{i-1} p^{l_{i+1}}.
\]
We have proved the theorem. \(\square\)
Corollary 3.3. Assume that $p^l$ divide $\det(A)$ exactly. If $l \leq k$, then

$$\eta(A, 0, p^k) = p^l.$$ 

The determinant of a matrix gives an upper bound for how many solutions there can exist, and therefore the number of solutions will not increase if we fix the matrix and choose $k$ larger than $l$. If $A$ is not the zero matrix and $\det(A) = 0$, then we have to find all $d_2, d_3, \ldots, d_n$ to determine the number of solutions.

Example 3.1. Let

$$A = \begin{pmatrix} 3 & 6 & 0 \\ 2 & 5 & 1 \\ 6 & 1 & 9 \end{pmatrix}.$$ 

Then $\det(A) = 60 = 2^2 \cdot 3 \cdot 5$. Hence, the number of solutions of $Ax \equiv 0 \pmod{2^k}$, for $k \geq 2$ is 4 and for modulo 2 we have 2 or 4 solutions. In this case we get two solutions. For modulo $3^k$ or $5^k$ the number of solutions is always 3 and 5, respectively. Note that for all other prime numbers $p$ the system has exactly one solution modulo $p^k$.

Example 3.2. Let

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

Then $Ax \equiv 0 \pmod{3}$ has nine solutions.

Corollary 3.4. Let $d_1 = p^{e_1} = (A_1, p^{l_1})$. Then

$$\eta(A_n, 0, p^k) = d_1 d_2^2 \cdots d_n^n = p^\varepsilon,$$

where $\varepsilon = e_1 + 2e_2 + \cdots + ne_n$.

Proof. If we complete the Gaussian elimination we get the system

$$\begin{cases} r_{11} x_1 + r_{12} x_2 \equiv 0 \pmod{p^{l_1}} \\ a_{22}' x_2 \equiv 0 \pmod{p^{l_1}}, \end{cases}$$

where $(r_{11}, p) = 1$ and $A_1 = a_{22}'$. Then $\det(A_1) = a_{22}^\prime$ and the number of solutions to the system is $d_1 = (a_{22}^\prime, p^{l_1})$. From (8) it follows that

$$\eta(A_n, 0, p^k) = d_2^2 d_3^3 \cdots d_n^n \eta(A_1, 0, p^{l_1}) = d_1 d_2^2 d_3^3 \cdots d_n^n = p^{e_1} (p^{e_2})^2 (p^{e_3})^3 \cdots (p^{e_n})^n = p^{e_1 + 2e_2 + \cdots + ne_n},$$

and by that the corollary. $\Box$
Corollary 3.5. Let $A$ be a square integer matrix of order $n$, and $m$ a positive integer. Assume that $m = \prod_{i=1}^{N} p_i^{k_i}$, where $p_i$ are different primes. Then

$$\eta(A, 0, m) = \prod_{i=1}^{N} p_i^{\varepsilon_i}$$

where

$$\varepsilon_i = e_{i,1} + 2e_{i,2} + \cdots + ne_{i,n}$$

is the exponent of the prime power given in Corollary 3.4 when we consider the system $A x \equiv 0 \pmod{p^{k_i}}$.

4 Inhomogeneous Systems

Theorem 4.1. Let $p$ be a prime number, $k$ a positive integer, $A_n$ a square integer matrix of order $n$, and $b_n$ an integer vector of length $n$. The inhomogeneous linear system

$$A_n x \equiv b_n \pmod{p^k} \quad (14)$$

is solvable if and only if $d_n | b_n$ and $d_i | b_i$ for all $i = n-1, \ldots, 2, 1$, where the integers $d_i$ are given in similar way as in Theorem 3.2 and the vectors $b_i$ is the last $i$ elements in the right hand side of the system after the $j$th step of the Gaussian elimination. Moreover, if the system is solvable then

$$\eta(A_n, b_n, p^k) = \eta(A_n, 0, p^k) = p^\varepsilon,$$

where $\varepsilon$ is the exponent given by Corollary 3.4.

Proof. The system (14) is solvable when $d_n$ divide each component of $b$. Let $R_n = A_n/d_n$ and $r_n = b_n/d_n$. Reduce the system in the same way as described for (6). We get the inhomogeneous system

$$A_{n-1} x \equiv b_{n-1} \pmod{p^{l_n}},$$

with $n-1$ unknowns. In order for this system to have solutions $d_{n-1} = (A_{n-1}, p^{l_n})$ must divide all the components of $b_{n-1}$. If we continue in this way we get that (14) is solvable if and only if $d_n | b$ and $d_i | b_i$ for all $i = n-1, \ldots, 2, 1$. The number of solutions, if they exists, are the same as in homogeneous case—the backward substitution result in the same number of solutions in each step.

In the following example we will use the algorithm described in Appendix A.

Example 4.1. Study the linear congruence system

$$\begin{align*}
123x_1 + 152x_2 + 28x_3 + 22x_4 + 144x_5 &\equiv 193 \\
38x_1 + 189x_2 + 127x_3 + 171x_4 + 141x_5 &\equiv 2 \\
132x_1 + 232x_2 + 215x_3 + 22x_4 &\equiv 96 \pmod{243}, \\
155x_1 + 30x_2 + 178x_3 + 142x_4 + 127x_5 &\equiv 198 \\
194x_1 + 171x_2 + 16x_3 + 24x_4 + 98x_5 &\equiv 162
\end{align*} \quad (15)$$

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where $243 = 3^5$. Then $l_6 = 5$. Let

$$A_5 = \begin{pmatrix} 123 & 152 & 28 & 22 & 144 \\ 38 & 189 & 127 & 171 & 141 \\ 132 & 232 & 215 & 22 & 0 \\ 155 & 30 & 178 & 142 & 127 \\ 194 & 171 & 16 & 24 & 98 \end{pmatrix} \quad \text{and} \quad b_5 = \begin{pmatrix} 193 \\ 2 \\ 155 \\ 30 \\ 171 \end{pmatrix}.$$

Then $3^2$ is the largest power of 3 which divide $\det(A_5) = -134741218779$. Hence, the number of solutions of the congruence system can not exceed 9. We have that $d_5 = (A_5, 3^5) = 1$, and therefore is $e_5 = 0$, $l_5 = l_6 - e_5 = 5$ and $R_5 = A_5$. Since $d_5$ divides all elements in $b_5$ we can continue. Next, we interchange the first and second row. Hence, $r_{11} = 38$ and $r_{11}^{-1} = 32$ modulo $3^5$. Note that $r_{11}^{-1} = 200673618026$ modulo $p^k = 3^5$. After the first step in the Gaussian elimination we get the matrices

$$A_4 = \begin{pmatrix} 38 & 189 & 127 & 171 & 141 \\ 0 & 71 & 7 & 76 & 180 \\ 0 & 151 & 68 & 157 & 9 \\ 0 & 84 & 114 & 52 & 121 \\ 0 & 63 & 135 & 123 & 56 \end{pmatrix} \quad \text{and} \quad b_4 = \begin{pmatrix} 2 \\ 97 \\ 153 \\ 241 \\ 139 \end{pmatrix}.$$

Hence,

$$A_4 = \begin{pmatrix} 71 & 7 & 76 & 180 \\ 151 & 68 & 157 & 9 \\ 84 & 114 & 52 & 121 \\ 63 & 135 & 123 & 56 \end{pmatrix} \quad \text{and} \quad b_4 = \begin{pmatrix} 97 \\ 153 \\ 241 \\ 139 \end{pmatrix}.$$

We have that $d_4 = (A_4, 3^5) = 1$, and therefore is $e_4 = 0$, $l_4 = l_5 - e_4 = 5$ and $R_5 = A_5$. Since $d_4$ divides all elements in $b_4$ we can continue. Further, we do not have to interchange any rows or columns this time since $r_{11} = 71$ is relatively prime to 3. Next step in the Gaussian elimination gives us the matrices

$$A_3 = \begin{pmatrix} 36 & 122 & 54 \\ 27 & 10 & 175 \\ 9 & 213 & 218 \end{pmatrix} \quad \text{and} \quad b_3 = \begin{pmatrix} 22 \\ 181 \\ 94 \end{pmatrix}.$$

We have that $d_3 = (A_3, 3^4) = 1$, and therefore is $e_3 = 0$, $l_3 = l_4 - e_3 = 5$ and $R_3 = A_3$. Since $d_3$ divides all elements in $b_3$ we can continue. Further, we have to interchange the first and second column. Then $r_{11} = 122$. Next step in the Gaussian elimination gives us the matrices

$$A_2 = \begin{pmatrix} 36 & 67 \\ 225 & 56 \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 227 \\ 199 \end{pmatrix}.$$

We have that $d_2 = (A_2, 3^4) = 1$, and therefore is $e_2 = 0$, $l_2 = l_3 - e_2 = 5$ and $R_2 = A_2$. Since $d_2$ divides all elements in $b_2$ we can continue. Further, we have
to interchange the first and second column. Then \( r_{11} = 67 \). Next step in the Gaussian elimination gives us the “matrices”

\[
A_1 = 126 \quad \text{and} \quad b_1 = 216.
\]

If we put these results together we get the matrices

\[
\begin{pmatrix}
38 & 189 & 171 & 141 & 127 \\
0 & 71 & 76 & 180 & 7 \\
0 & 0 & 122 & 54 & 36 \\
0 & 0 & 0 & 67 & 36 \\
0 & 0 & 0 & 0 & 126
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 \\
97 \\
22 \\
227 \\
216
\end{pmatrix},
\]

which corresponds to the linear congruence system

\[
\begin{align*}
38y_1 + 189y_2 + 171y_3 + 141y_4 + 127y_5 & \equiv 2 \pmod{3^5} \\
71y_2 + 76y_3 + 180y_4 + 7y_5 & \equiv 97 \pmod{3^5} \\
122y_3 + 54y_4 + 36y_5 & \equiv 22 \pmod{3^5} \\
67y_4 + 36y_5 & \equiv 227 \pmod{3^5} \\
14y_5 & \equiv 24 \pmod{3^5},
\end{align*}
\]

where

\[
\begin{pmatrix}
38^{-1} & 32 \pmod{3^5} \\
71^{-1} & 89 \pmod{3^5} \\
122^{-1} & 2 \pmod{3^5} \\
67^{-1} & 214 \pmod{3^5} \\
14^{-1} & 2 \pmod{3^5}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
x_1 = y\sigma(1) = y_1 \\
x_2 = y\sigma(2) = y_2 \\
x_3 = y\sigma(3) = y_5 \\
x_4 = y\sigma(4) = y_3 \\
x_5 = y\sigma(5) = y_4
\end{pmatrix}
\]

Where

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 5 & 3 & 4
\end{pmatrix}
\]

denotes the permutations of the columns done in the computation. Since \( d_1 = (A_1, 3^5) = (126, 3^5) = 3^2 \) we have that \( e_1 = 2 \) and \( l_1 = l_2 - e_1 = 3 \), and therefore is the number of solutions

\[
\eta(A, b, 3^5) = 3^{3+2+3+3+5} = 3^{2+0+0+0+0} = 9.
\]

Backward substitution gives the following. First we have that

\[
x_3 = y_5 \equiv 2 \cdot 24 \equiv 21 \pmod{3^3}.
\]

We lift the result to \( \mathbb{Z}_{243} \), that is,

\[
x_3 = y_5 = 21 + 3^3 h = 21 + 3^3 h, \quad h \in \mathbb{Z}_{d_1} = \mathbb{Z}_9.
\]

Hence, \( x_3 \in \{ 21, 48, 75, 102, 129, 156, 183, 210, 237 \} \). Since 36 is divisible by 9, neither \( y_4 \) or \( y_3 \) depends on \( y_4 \), which follows from

\[
36 \cdot (21 + 3^3 h) \equiv 36 \cdot 21 \equiv 27 \pmod{3^5}.
\]
We get that \( x_4 = y_3 = 179 \) and \( x_5 = y_4 = 32 \). The two first congruences in the system \((16)\) gives us the nine solutions:

\[
\begin{align*}
(43, 127, 21, 179, 32) & \quad (151, 73, 48, 179, 32) & \quad (16, 19, 75, 179, 32) \\
(124, 208, 102, 179, 32) & \quad (2154, 129, 179, 32) & \quad (97, 100, 156, 179, 32) \\
(205, 46, 183, 179, 32) & \quad (70, 235, 210, 179, 32) & \quad (178, 181, 237, 179, 32)
\end{align*}
\]

We have solve the system \((15)\).

5 Discussion

The main motivation to investigate the problem was to find a formula, as simple as possible, for the number of solutions of a linear congruence system.

We made the choice to use as elementary methods as possible. Mainly because the problem is of an elementary nature but also because of the interest from applications. It is possible to lift both the problem and its solution into the context of free modules, see [5], over the ring \( \mathbb{Z}/m\mathbb{Z} \). The derivation will then be almost identical but with different vocabulary.

By small changes it is also possible to solve system of congruence equations with different modulus. We construct a system modulo the least common multiple of all the moduli, that are equivalent to the original system. This is the same technique that is mentioned in [1].

A Algorithm

Let \( p \) be a prime number, \( n \) and \( k \) positive integers, \( A = (a_{ij})_{n \times n} \) an integer matrix, and \( b = (b_i)_{n \times 1} \) a vector with integer entries. The following algorithm determines if the congruence

\[
Ax \equiv b \pmod{p^k}
\]

is solvable, and in that case it finds the number of solutions, denoted \( \eta \), and the set \( X \) of all solutions.

1. [Initialization] Set \( l_{n+1} \leftarrow k \), \( s \leftarrow n \) and \( t \leftarrow 1 \).
2. [Factor] Set \( ds \leftarrow \det(A), p^{s+1} \Rightarrow p^s \) and \( ls \leftarrow l_{s+1} - es \).
3. [Solvable?] If \( d_s \not| b_i \) for some \( i = t, \ldots, n \), then stop and return “Not solvable”.
4. [Cancel factor] Set \( a_{ij} \leftarrow a_{ij}/ds \) and \( b_i \leftarrow b_i/ds \) for all \( i, j = t, \ldots, n \).
5. [Zero matrix?] If \( l_s = 0 \), then set \( e_i \leftarrow 0 \) and \( l_i \leftarrow 0 \) for all \( i = s + 1, \ldots, n \) and go step 6.
6. [Pivoting] Find an element among \( a_{ij} \) for \( i, j = t, \ldots, n \) which is relative prime to \( p \), and perform, if necessary, an interchange of rows or columns so that the element is the \( t \)th element in the main diagonal of \( A \).
7. [Gaussian elimination] For \( i, j = t + 1, \ldots, n \) set \( a_{ij} \leftarrow 0 \),

\[
a_{ij} \leftarrow a_{ij} - a_{it}^{-1}a_{ii}a_{tj} \mod{p^k} \quad \text{and} \quad b_i \leftarrow b_i - a_{it}^{-1}a_{ii}b_i \mod{p^k}.
\]
The arithmetic is done modulo $p^{nk}$ to make sure that the formula in step 9 give the correct number of solutions.

8. [Done?] If $s > 2$, then set $s ← s - 1$ and $t ← t + 1$, and go to step 2.

9. [Number of solutions] Set $η ← p^{e_1 + 2e_2 + \cdots + ne_n}$.

To determine the solutions the algorithm can be continued in the following way.

1. [Initialization] Set $s ← 2$ and $t ← n - 1$.

2. [Introduce solution set] Let $X$ be the set of all vectors $(\cdot, \cdot, \cdot, y_n)$, where $y_n$ is one of the $p^{e_1}$ solutions of $a_{nn}y ≡ b_n (\text{mod } p^{l_1})$. If $l_1 \neq l_2$, then for each $(\cdot, \cdot, \cdot, y) \in X$ add to $X$ all vectors $(\cdot, \cdot, \cdot, y + ip^{l_1})$, for $i = 0, \ldots, p^{e_1} - 1$.

3. [Backward substitution] For each $y = (\cdot, \cdot, \cdot, y_{t+1}, \ldots, y_n) ∈ X$ set

   $$y_t ← a_{tt}^{-1} \left( b_t - \sum_{j=t+1}^{n} a_{tj}y_j \right) \mod p^{l_s}$$

and store $y_t$ at position $t$ in $y$. Note that $(a_{tt}, p) = 1$.

4. [Lifting] If $l_s \neq l_{s+1}$, then for each $y = (\cdot, \cdot, \cdot, y_t, \ldots, y_n) ∈ X$ add to $X$ all vectors $(\cdot, \cdot, \cdot, y_t + i_1p^{l_s}, \ldots, y_n + i_np^{l_s})$, where $i_1, \ldots, i_n = 0, \ldots, p^{e_s} - 1$.

5. [Done?] If $t > 1$, then set $s ← s + 1$ and $t ← t - 1$, and go to step 3.

6. [Rearrange solution] Let $σ ∈ S_n$ be the permutation which summarize the interchanges of columns. Replace each $(y_1, \ldots, y_n) ∈ X$ with $(x_1, \ldots, x_n)$, where $x_i = y_{σ(i)}$.

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