Decomposable graphs and definitions with no quantifier alternation

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Abstract

Let $D(G)$ be the minimum quantifier depth of a first order sentence $\Phi$ that defines a graph $G$ up to isomorphism. Let $D_0(G)$ be the version of $D(G)$ where we do not allow quantifier alternations in $\Phi$. Define $q_0(n)$ to be the minimum of $D_0(G)$ over all graphs $G$ of order $n$.

We prove that for all $n$ we have

$$\log^* n - \log^* \log^* n - 1 \leq q_0(n) \leq \log^* n + 22,$$

where $\log^* n$ is equal to the minimum number of iterations of the binary logarithm needed to bring $n$ to 1 or below. The upper bound is obtained by constructing special graphs with modular decomposition of very small depth.

Keywords: descriptive complexity of graphs, first order logic, Ehrenfeucht game on graphs, graph decompositions.

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1 Introduction

We are interested in defining a given graph $G$ in first order logic, being as succinct as possible. In order to state this problem formally, we have to specify what we mean by the terms defining, succinct, etc.

The vocabulary consists of the following symbols:

- variables ($x, y, y_1$, etc);
- the relations $=$ (equality) and $\sim$ (graph adjacency);
- the quantifiers $\forall$ (universality) and $\exists$ (existence);
- the usual Boolean connectives ($\lor, \land$, and $\neg$);
- parentheses (to indicate or change the precedence of operations).

These can be combined into first order formulas accordingly to the standard rules. The term first order means that the variables represent vertices so the quantifiers apply to vertices only. In this paper, a sentence is a first order formula without free variables. On the intuitive level it is perfectly clear what we mean when we say that a sentence $\Phi$ is true on a graph $G$. This is denoted by $G \models \Phi$; we write $G \not\models \Phi$ for its negation ($\Phi$ is false on $G$). We do not formalize these notions. A more detailed discussion can be found in e.g. [15, Section 1].

Of course, if $G \models \Phi$ and $H \cong G$ (i.e. $H$ is isomorphic to $G$), then $H \models \Phi$. On the other hand, for any graph $G$ it is possible to find a sentence $\Phi$ which defines $G$, that is, $G \models \Phi$ while $H \not\models \Phi$ for any $H \not\cong G$. Indeed, let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of $G$ and $E(G)$ be its edge set. The required sentence could read:

$$\Phi = \exists x_1 \ldots \exists x_n \ (\text{Distinct}(x_1, \ldots, x_n) \land \text{Adj}(x_1, \ldots, x_n))$$
$$\land \ \forall x_1 \ldots \forall x_{n+1} \neg\text{Distinct}(x_1, \ldots, x_{n+1}),$$

(1)

where, for the notational convenience, we use the following shorthands

$$\text{Distinct}(x_1, \ldots, x_k) = \bigwedge_{1 \leq i < j \leq k} \neg(x_i = x_j)$$

$$\text{Adj}(x_1, \ldots, x_n) = \bigwedge_{\{v_i,v_j\} \in E(G)} x_i \sim x_j \land \bigwedge_{\{v_i,v_j\} \not\in E(G)} \neg(x_i \sim x_j).$$

In other words, we first specify that there are $n$ distinct vertices, list the adjacencies and non-adjacencies between them, and then state that the total number of vertices is at most $n$.

A defining sentence $\Phi$ is not unique, so we are interested in finding one which is as succinct as possible. All natural succinctness measures of $\Phi$ are of interest:
• the length $L(\Phi)$ which is the total number of symbols in $\Phi$;

• the quantifier depth $D(\Phi)$ which is the maximum number of nested quantifiers in $\Phi$;

• the width $W(\Phi)$ which is the number of variables used in $\Phi$ (different occurrences of the same variable are not counted).

For example, for the sentence in (1) we have $L(\Phi) = \Theta(n^2)$ and $D(\Phi) = W(\Phi) = n + 1$. All three characteristics inherently arise in the analysis of the computational problem of checking if a $\Phi$ is true on a given graph, see e.g. Grädel [3]. They give us a small hierarchy of descriptive complexity measures for graphs: $L(G)$ (resp. $D(G)$, $W(G)$) is the minimum of $L(\Phi)$ (resp. $D(\Phi)$, $W(\Phi)$) over all sentences $\Phi$ defining $G$. These graph invariants will be referred to as the logical length, depth, and width of $G$. We have

$$W(G) \leq D(G) \leq L(G).$$

The former number is of relevance for graph isomorphism testing, see Cai, Fürer, and Immerman [4]. The parameters $W(G)$ and $D(G)$ admit a purely combinatorial characterization in terms of the Ehrenfeucht game, see [4, 15].

Here, we address the logical depth of graphs which was recently studied in Bohman et al [11, 9, 11, 12, 13, 16, 17]. We focus on the following general question: How do restrictions on logic affect the descriptive complexity of a graph? Call a sentence $\Phi$ $a$-alternating if it contains negations only in front of relation symbols and every sequence of nested quantifiers in $\Phi$ has at most a quantifier alternations, that is, the occurrences of $\forall \exists$ and $\exists \forall$. Let $D_a(G)$ denote the variant of $D(G)$ for $a$-alternating defining sentences. Clearly, for any integer $a \geq 0$ we have

$$D(G) \leq D_{a+1}(G) \leq D_a(G).$$

For example, the sentence in (11) has no alternations. Thus it shows that for any graph $G$ we have

$$D_0(G) \leq v(G) + 1,$$

where $v(G)$ denotes the number of vertices in $G$. This bound is in general best possible: for example, $D_0(K_n) = D(K_n) = n + 1$. In Kim et al [9] we proved that $D(G) = \log_2 n - \Theta(\log_2 n \log_2 n)$ and $D_0(G) \leq (2 + o(1)) \log_2 n$ for almost all graphs $G$ of order $n$.

In the above results, the functions $D(G)$ and $D_0(G)$ are the same or differ by at most a constant factor. However, they can be very far apart in general. In [11, Corollary 5.7] we demonstrated a superrecursive gap between $D(G)$ and $D_0(G)$: namely, we proved that for any total recursive function $f$ there is a graph $G$ with $f(D_0(G)) < D(G)$. This is not too surprising, since the logic of $0$-alternating sentences is very restrictive and provably weaker than the unbounded first order logic.
Whereas the problem of deciding if a first order sentence is satisfiable by some graph is unsolvable, it becomes solvable if restricted to 0-alternating sentences. The last result is due to Ramsey’s logical work [14] founding the combinatorial Ramsey theory (see Nešetřil [10, pp. 1336–1337] for historical comments on the relations between Ramsey theory and logic).

Given Ramsey’s decidability result, it is reasonable to concentrate on the first order definability with no quantifier alternation. As our main result here (Theorem 1), we determine the asymptotic behavior of the succinctness function $q_0(n)$, where for an integer $a \geq 0$ we define

$$q_a(n) = \min \{D_a(G) : G \text{ has order } n\}.$$  

Let $\log^* n$ be equal to the minimum number of iterations of the binary logarithm needed to bring $n$ to 1 or below.

**Theorem 1** For all $n$ we have

$$\log^* n - \log^* \log^* n - 1 \leq q_0(n) \leq \log^* n + 22.$$  

The estimates (3) are in sharp contrast to the result in [11, Corollary 9.1] which shows a superrecursive gap between

$$q(n) = \min \{D(G) : G \text{ has order } n\}$$

and $n$. Thus Theorem 1 besides being an interesting result on its own, implies that we cannot have $q_0(n) \leq f(q(n))$ for some total recursive $f$ and all $n$. This implies, again, a superrecursive gap between the graph invariants $D(G)$ and $D_0(G)$.

In [14, Theorem 7.1] a weaker bound $q_0(n) \leq 2 \log^* n + O(1)$ for an infinite sequence of values of $n$ is proved by inductively constructing large asymmetric trees and estimating $D_0(G)$ in terms of their (very small) radius. Here, our construction produces a graph of large order that has very short modular decomposition (as defined in Brandstädt, Le, and Spinrad [3, Section 1.5]), starting with small complement-connected graphs. It seems feasible that many other recursively defined constructions of graphs (see Borie, Parker, and Tovey [2] and Brandstädt, Le, and Spinrad [3, Section 11] for surveys) may lead to upper bounds on $q_0(n)$ compatible with (3). However, the proof of the upper bound in (3) required from us many delicate auxiliary lemmas, even though we chose a construction which is, in our opinion, most suitable for our purposes. So, a general theorem would probably be very messy and difficult to prove.

In [14, Theorem 9.3] we have shown that

$$\log^* n - \log^* \log^* n - 2 \leq q(n) \leq \log^* n + 4.$$
for infinitely many $n$. Combined with Theorem 1 and the obvious inequalities
\[ q_0(n) \geq q_a(n) \geq q(n), \quad \text{for any integer } a \geq 1, \]
this implies that for any fixed $a$ we have $q_a(n) = (1 + o(1)) \log^* n$ for infinitely many $n$. We do not even know if $q_1(n) = (1 + o(1)) \log^* n$ for all large $n$.

In fact, Theorem 1 holds also for digraphs, where instead of the adjacency relation $\sim$ we use the relation $x \mapsto y$ to denote that the ordered pair $(x, y)$ is an arc. For example, the digraph version of the lower bound in (3) reads as follows.

**Theorem 2** For any digraph $G$ on $n$ vertices we have
\[ D_0(G) \geq \log^* n - \log^* \log^* n - 1. \quad (4) \]

Let us see how these results are related. Take any graph $G$ and a 0-alternating sentence $\Phi$ defining it. Let the digraph $G'$ be obtained from $G$ by replacing each edge $\{x, y\} \in E(G)$ by a pair of arcs $(x, y)$ and $(y, x)$. Then the sentence
\[ (\forall x \neg (x \mapsto x)) \land (\forall x \forall y ((x \mapsto y) \land (y \mapsto x)) \lor (\neg (x \mapsto y) \land \neg (y \mapsto x))) \land \Phi' \]
defines $G'$, where $\Phi'$ is obtained from $\Phi$ by replacing each occurrence of $x \sim y$ by, for example, $x \mapsto y$. Thus
\[ D_0(G') \leq \max(2, D_0(G)) = D_0(G). \]
This shows that it is enough to prove the upper bound in Theorem 1 and the lower bound of Theorem 2.

2 Definitions

We denote $[m, n] = \{m, m + 1, \ldots, n\}$ and $[n] = [1, n]$. We define the tower-function by $\text{Tower}(0) = 1$ and $\text{Tower}(i) = 2^{\text{Tower}(i-1)}$ for each subsequent $i$. Note that $\log^*(\text{Tower}(i)) = i$. The notation $x \in^i X$ means $x \in X$ for odd $i$ and $x \notin X$ for even $i$. (The mnemonic rule to remember which is which is $\in^1 = \in$.) The abbreviation ‘iff’ means ‘if and only if.’ We do not allow infinite sentences nor infinite graphs (nor the degenerate graph with the empty vertex set).

We use the following graph notation: $\overline{G}$ is the complement of $G$; $G \sqcup H$ is the vertex-disjoint union of graphs $G$ and $H$; $G \subset H$ means that $G$ is isomorphic to an induced subgraph of $H$ (we will say that $G$ is embeddable into $H$). For graphs (resp. sets) $A$ and $B$ the relation $A \subset B$ does not exclude the case of isomorphism $A \cong B$ (resp. equality $A = B$).
We call $G$ complement-connected if both $G$ and $\overline{G}$ are connected. An inclusion-maximal complement-connected induced subgraph of $G$ will be called a complement-connected component of $G$ or, for brevity, cocomponent of $G$. Cocomponents have no common vertices and their vertex sets partition $V(G)$.

The decomposition of $G$, denoted by $\text{Dec} G$, is the set of all connected components of $G$ (this is a set of graphs, not just isomorphism types). Furthermore, given $i \geq 0$, we define the depth $i$ decomposition $\text{Dec}_i G$ of $G$ by

$$\text{Dec}_0 G = \text{Dec} G \quad \text{and} \quad \text{Dec}_{i+1} G = \bigcup_{F \in \text{Dec}_i G} \text{Dec} F.$$  

Note that $\text{Dec}_i G$ consists of connected graphs, and distinct vertices $x, y$ of an $F \in \text{Dec}_i G$ are adjacent in $F$ if and only if $\{x, y\} \in ^{i+1} E(G)$. Moreover,

$$P_i = \{V(F) : F \in \text{Dec}_i G\}$$

(5)

is a partition of $V(G)$ and $P_{i+1}$ refines $P_i$. The depth $i$ environment of a vertex $v \in V(G)$, denoted by $\text{Env}_i(v; G)$, is the graph $F$ in $\text{Dec}_i G$ containing $v$. If the underlying graph $G$ is clear from the context, we will usually write $\text{Env}_i(v)$.

We define the rank of a graph $G$, denoted by $\text{rk} G$, inductively as follows:

- If $G$ is complement-connected, then $\text{rk} G = 0$.
- If $G$ is connected but not complement-connected, then $\text{rk} G = \text{rk} G$.
- If $G$ is disconnected, then $\text{rk} G = 1 + \max \{\text{rk} F : F \in \text{Dec} G\}$.

Note that for connected graphs $\text{rk} G$ is equal to the smallest $k$ such that $P_{k+1} = P_k$ or, equivalently, such that $P_k$ consists of $V(F)$ for all cocomponents $F$ of $G$.

Let $G$ be a connected graph and let $k = \text{rk} G$. We call $G$ uniform if $\text{Dec}_{k-1} G$ contains no complement-connected graph, that is, every cocomponent appears in $\text{Dec}_k G$ and no earlier. We call $G$ inclusion-free if the following two conditions are true for every $0 \leq i \leq k$:

1. For any $K \in \text{Dec}_i G$, $\overline{K}$ contains no isomorphic connected components.
2. Of any two elements $K, M \in \text{Dec}_i G$ none is properly embeddable into the other, that is, either $K \cong M$ or none is an induced subgraph of the other.

Let us now describe the Ehrenfeucht game $\text{Ehr}_k(G, H)$ which will be our tool for studying the logical depth of graphs. The board consists of two vertex-disjoint graphs $G$ and $H$. There are $k$ rounds. The graphs $G, H$ and the number $k$ are known to both players, Spoiler and Duplicator (or he and she). In each round Spoiler selects one vertex in either $G$ or $H$; then Duplicator must choose a vertex
in the other graph. Let \( x_i \in V(G) \) and \( y_i \in V(H) \) denote the vertices selected by the players in the \( i \)-th round, irrespectively of who selected them. Duplicator wins the game if the componentwise correspondence between the ordered \( k \)-tuples \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) is a partial isomorphism from \( G \) to \( H \). Otherwise the winner is Spoiler. In the 0-alternation game Spoiler must play all the game in the same graph he selects in the first round.

Assume that \( G \not \cong H \). Let \( D(G, H) \) (resp. \( D_0(G, H) \)) denote the minimum of \( D(\Phi) \) over all (resp. 0-alternating) sentences \( \Phi \) that are true on one of the graphs and false on the other. The Ehrenfeucht theorem \([6]\) (see also Fraïssé \([7]\)) relates \( D(G, H) \) and the length of the Ehrenfeucht game on \( G \) and \( H \). We will use the following version of the theorem: \( D_0(G, H) \) is equal to the minimum \( k \) such that Spoiler has a winning strategy in the \( k \)-round 0-alternation Ehrenfeucht game on \( G \) and \( H \). We will also use the fact (see \([11, Proposition 3.6]\)) that 

\[
D_0(G) = \max \{ D_0(G, H) : H \not \cong G \}.
\]

We refer the Reader to \([15, Section 2]\) which contains a detailed discussion of the Ehrenfeucht game.

3 Proof of the Upper Bound in Theorem 1

3.1 Preliminaries

**Lemma 1** Every complement-connected graph \( G \) of order at least 5 has a vertex \( v \) such that \( G - v \) is still complement-connected.

**Proof.** Suppose that the claim is false. Take an arbitrary \( v \in V \), where \( V = V(G) \). This vertex does not work so assume that, for example, \( G - v \) is disconnected. Choose a proper partition \( V \setminus \{x\} = A_1 \cup A_2 \) such that no edge of \( G \) connects \( A_1 \) to \( A_2 \). Assume that \( |A_1| \geq |A_2| \). Since \( G \) is connected, the graph \( G_i = G[A_i \cup \{v\}] \) is connected, \( i = 1, 2 \). This implies that \( U_i \neq \emptyset \) for \( i = 1, 2 \), where

\[
U_i = \{u \in A_i \mid G_i - u \text{ is connected}\}.
\]

Let \( u \in U_1 \). The graph \( G - u \) is connected because any vertex of \( A_1 \setminus \{u\} \) can be connected (in \( G_1 - u \)) to \( v \) and then connected (in \( G_2 \)) to any vertex of \( A_2 \). Since \( G \) contains all edges between \( A_1 \) and \( A_2 \) (and \( |A_1| \geq 2 \)), the graph \( G - u - v \) is connected. Thus the only way that \( u \) can fail to satisfy the conclusion of the lemma is that \( v \) is adjacent (in \( G \)) to every other vertex except \( u \) (the vertex \( v \) cannot be adjacent to \( u \) too because \( G \) is complement-connected). The latter condition determines \( u \) uniquely and therefore \( U_1 = \{u\} \). If \( |A_2| \geq 2 \), then the same argument shows that \( U_2 \) should consist of the unique neighbor of \( u \) of \( v \), which is impossible.
Thus, $|A_2| = 1$ and hence $|A_1| \geq 3$. Let $w \in A_1$ be some neighbor of $u$ and let $z \in A_1 \setminus \{u, w\}$. Then $G_1 - z$ is still connected: $u$ is connected to $v$ via $w$ while any other vertex is directly adjacent to $v$. Hence, $z \in U_1$. This contradiction finishes the proof. 

Now we come to two strategic lemmas. The arguments of each lemma are listed in square brackets. This is convenient when we refer back to these results and, hopefully, makes the dependences between the lemmas easier to verify.

**Lemma 2** $[x, x', y, y', G, H, l]$ Consider the Ehrenfeucht game on graphs $G$ and $H$. Let $x, x' \in V(G)$, $y, y' \in V(H)$ and assume that the pairs $x, y$ and $x', y'$ were selected by the players in the same rounds. Furthermore, assume that all the following properties hold.

1. $Env_l(x) \neq Env_l(x')$.
2. $Env_l(y) = Env_l(y')$.
3. $V(Env_{l+1}(y)) \neq V(Env_l(y))$.

Then Spoiler can win in at most $l + 1$ rounds, playing all the time in $H$.

**Proof.** We proceed by induction on $l$. The induction step takes care of the base case $l = 0$ too. Observe that, for every $0 \leq i \leq l$, we have $V(Env_{i+1}(y)) \neq V(Env_i(y))$ so we do not have to worry about Assumption 3 when using induction.

Let $m \in [0, l]$ be the minimum number such that $x' \notin Env_m(x)$. If $m < l$, Spoiler wins in $m + 1 \leq l$ moves by induction. So suppose that $m = l$. Assume that $y$ and $y'$ are not adjacent in $Env_l(y)$ for otherwise Duplicator has already lost. By Assumption 3 the graph $Env_l(y)$ is connected but not complement-connected, so its diameter is at most 2. Spoiler selects any $y''$ adjacent to both $y$ and $y'$ in $Env_l(y)$. If Duplicator does not lose in this round, it means that her reply $x''$ lies outside $Env_{l-1}(x)$ (and that $l \geq 1$). We have $Env_{l-1}(x) \neq Env_{l-1}(x'')$ and $Env_{l-1}(y) = Env_{l-1}(y'')$. By the induction hypothesis applied to $[x, x'', y, y'', G, H, l - 1]$, Spoiler can win in at most $l$ extra moves. 

**Lemma 3** $[x_1, y_1, G, H, l]$ Suppose that $x_1 \in V(G)$ and $y_1 \in V(H)$ were selected in some round of the Ehrenfeucht game on $(G, H)$ so that there is an $l \geq 0$ satisfying the following Assumptions 1–3.

1. $G_1 = Env_l(x_1)$ is not isomorphic to $H_1 = Env_l(y_1)$.
2. $H_1$ is a uniform inclusion-free graph such that every cocomponent of $H_1$ has at most $c$ vertices.
3. For any \(i \geq 0\), no member \(A \in \text{Dec}_i H_1\) is embeddable as a proper subgraph into some \(B \in \text{Dec}_i G_1\).

Then Spoiler can win the game in at most \(k + c - 1\) extra moves, playing all the time inside \(H\), where \(k = rk H_1 + l\).

Proof. Suppose that it is Spoiler’s turn to move and, in addition to \(x_1\) and \(y_1\), we have the following configuration. Spoiler has already selected vertices \(y_2, \ldots, y_s \in V(H_1)\), Duplicator has selected \(x_2, \ldots, x_s \in V(G_1)\), and all of the following Properties 1–4 hold, where, for \(j \in [s]\), we let \(H_j = \text{Env}_{j+l-1}(y_j; H)\) and \(G_j = \text{Env}_{j+l-1}(x_j; G)\).

1. For \(i \in [2, s]\) we have \(y_i \in V(H_{i-1})\).
2. For \(i \in [2, s]\) we have \(x_i \in V(G_{i-1})\).
3. For every \(i \in [s]\) we have \(H_i \not\sim G_i\).
4. For every \(i \in [2, s]\) the vertices \(y_i\) and \(y_{i-1}\) belong to different components of \(\overline{H_{i-1}}\). (Note that \(y_i \in V(H_{i-1})\) by Property 1.)

Let us make a few remarks. Property 1 implies that
\[V(H_1) \supset \ldots \supset V(H_s),\]
and for \(1 \leq i \leq j \leq s\) we have \(y_j \in V(H_i)\). Likewise by Property 2,
\[V(G_1) \supset \ldots \supset V(G_s),\]
and for \(1 \leq i \leq j \leq s\) we have \(x_j \in V(G_i)\). Properties 1 and 4 imply that \(y_j \not\in V(H_i)\) for any \(1 \leq j < i \leq s\). We stated Properties 1–4 this way in order to reduce the number of checks needed to verify them. Also, note that we do not require that the vertices \(x_i\) satisfy the analog of Property 4.

The above properties determine all \(H\)-adjacencies between the vertices \(y_1, \ldots, y_s\). Indeed, take any \(1 \leq i < j \leq s\). By Properties 1 and 4, \(y_i\) and \(y_j\) belong to different components of \(\overline{H_i}\) so we have \(\{y_i, y_j\} \in E(H_i)\). This means that \(\{y_i, y_j\} \in E(H)\). In other words, the vertices \(y_i\) and \(y_j\) are adjacent in \(H\) if and only if \(i + l\) is odd.

If \(s = 1\), then Properties 1, 2, and 4 are vacuously true, while Property 3 is precisely Assumption 1 of the lemma.

We are going to show that Spoiler can either force the same situation after the next round (of course, with \(s\) increased by one) or win by making some extra moves.

Case 1. Suppose that \(s < k - l\).

As \(H_s \not\sim G_s\), Assumption 3 (for \(i = s - 1, A = H_s\), and \(B = G_s\)) implies that \(H_s \not\subset G_s\). By Assumption 2, the connected graph \(H_s \in \text{Dec}_{s-1} H_1\) is inclusion-free;
in particular, its complement does not contain two isomorphic components. Hence, there is a component $H_{s+1}$ of $H_s$ which is not isomorphic to any component of $G$. 

Suppose first that $y_s \notin V(H_{s+1})$. Spoiler chooses an arbitrary $y_{s+1} \in V(H_{s+1})$. Properties 1 and 4 hold automatically. Let $x_{s+1}$ be Duplicator’s reply. Assume that $x_{s+1}$ has the same adjacencies to the previously selected vertices as $y_{s+1}$ for otherwise Spoiler has already won having made $s \leq k - l - 1$ moves. (Note that we do not count $y_1$ as a move, here or later in the proof.) Suppose that $x_{s+1} \notin V(G_s)$, for otherwise Properties 2 and 3 hold and we are done.

Claim 1. We have $l \geq 1$ and $x_{s+1}$ does not belong to $Env_{l-1}(x_1; H)$.

Proof of Claim. First we argue that $x_{s+1} \notin V(G_1)$. Suppose that this is not true. In view of (i), take the largest $i \in [s-1]$ such that $x_{s+1} \in V(G_i)$. By the definition of $i$, $x_{s+1} \notin V(G_{i+1})$, the latter being the component of $G$ that contains $x_{i+1}$. Thus $\{x_{s+1}, x_{i+1}\} \in E(G_i)$. On the other hand, $x_{s+1}$ is not adjacent to $x_{i+1}$ in $G_i$ because $y_{s+1}$ is not adjacent to $y_{i+1}$ in $H_i$, a contradiction.

Next, we have $\{y_1, y_{s+1}\} \in ^{l+1}E(H)$, so $\{x_1, x_{s+1}\} \in ^{l+1}E(G)$. Since $x_{s+1} \notin V(G_1) = V(Env_l(x_1))$, we have $l \geq 1$. For any vertex $z \in V(Env_{l-1}(x_1)) \setminus V(Env_l(x_1))$ we have $\{x_1, z\} \in ^lE(G)$, so $x_{s+1} \notin V(Env_{l-1}(x_1))$, as required. 

At this point it is possible to argue that, if $s \geq 2$, then Duplicator has already lost. However, we still have to deal with the case $s = 1$ (when we have just $x_1$ and $x_2$). Since ruling out the case $s \geq 2$ would not make the proof shorter, we do not do this.

We have $V(Env_{l+1}(y_1)) \neq V(Env_l(y_1))$ because the latter set contains $y_{s+1}$ while the former does not (or because $rk H_1 \geq l + s - 1 \geq 1$). Hence, Lemma 2 applies to $[x_1, x_{s+1}, y_1, y_{s+1}, G, H, l - 1]$, and Spoiler can win the game in at most $l$ extra moves, having made at most $s + l \leq k - 1$ moves in total.

It remains to describe Spoiler’s strategy if $y_s \in V(H_{s+1})$, when Spoiler cannot just choose some $y_{s+1} \in V(H_{s+1})$ as this would violate Property 4. Here, Spoiler first selects some $y_{s+1} \in V(H_s) \setminus V(H_{s+1})$. (This set is non-empty since $s < rk H_1$.) Let Duplicator reply with $x_{s+1}$. If $x_{s+1} \notin V(G_s)$, then by the argument of Claim 1 we have that $l \geq 1$ and $Env_{l-1}(x_1) \neq Env_{l-1}(x_{s+1})$. Thus Spoiler can win in at most $l$ further moves by Lemma 2 having made at most $s + l \leq k - 1$ moves in total. Hence, let us assume that $x_{s+1} \in V(G_s)$. In this case, let us swap the vertices $y_s$ and $y_{s+1}$ as well as $x_s$ and $x_{s+1}$. It is clear that the new sequences $y_1, \ldots, y_{s+1}$ and $x_1, \ldots, x_{s+1}$ satisfy Properties 1–4. This completes the description of the case $s < k - l$.

Case 2. Suppose that $s = k - l$ (or that $s = 1$ and $k = l$).

This means that $H_s$ is a cocomponent of $H_1$ (and thus has at most $c$ vertices). Spoiler selects all vertices in $V(H_s) \setminus \{y_s\}$. We claim that Duplicator has lost by now.
Indeed, if Duplicator replies all the time inside $G_s$, then she has lost because $G_s \not\supset H_s$ by Assumption 3 and Property 3. Otherwise, her response to the whole set $V(H_s)$ cannot be complement-connected because it contains both a vertex outside of $G_s$ and the vertex $x_s \in V(G_s)$. Thus Spoiler wins, having made at most $s-1+c-1 \leq k+c-1$ further moves.

3.2 Finishing the Proof

**Lemma 4 (Main Lemma)** Let $G$ be a connected uniform inclusion-free graph. Let $c \geq 5$ and suppose that every cocomponent of $G$ has at most $c$ vertices. Then $D_0(G) \leq \text{rk}_G + c + 1$.

**Proof.** Let $k = \text{rk}_G$. Since the case of $k = 0$ is trivial (namely we have $D_0(G) \leq v(G) + 1 \leq c + 1$ by (2)), we assume that $k \geq 1$.

Fix a graph $H \not\cong G$. We will design a strategy allowing Spoiler to win the 0-alternation Ehrenfeucht game on $(G, H)$ in at most the required number of moves. There are a few cases to consider.

**Case 1.** $H$ has a cocomponent $C$ non-embeddable into any cocomponent of $G$.

If $C$ has no more than $c$ vertices, Spoiler selects all vertices of $C$. Otherwise he selects $c + 1$ vertices spanning a complement-connected subgraph in $C$ which is possible by Lemma 1 (since $c \geq 5$). If Duplicator’s response $A$ is within a cocomponent of $G$, then $C \not\cong A$ by the assumption. Otherwise $A$ is not complement-connected and Duplicator loses anyway.

**Case 2.** There are an $l \in [0, k]$ and an $A \in \text{Dec}_l G$ properly embeddable into some $B \in \text{Dec}_l H$, and not Case 1.

Let $H_0$ be a copy of $A$ in $B$. Fix an arbitrary vertex $y_0 \in V(B) \setminus V(H_0)$. Note that since we are not in Case 1, the connected graph $B$ cannot be a cocomponent of $H$ by Property 2 in the definition of an inclusion-free graph. Hence

$$V(Env_l(y_0; H)) \neq V(Env_{l+1}(y_0; H)).$$

Let $Z = V(B) \setminus V(H_0)$. We will need the following routine claim, whose proof uses (7) and the connectedness of $H_0$.

**Claim 2.** For any $m \geq 0$ and $y \in V(H_0)$ we have

$$Env_m(y; H_0) = Env_{m+l}(y; H - Z).$$

**Proof of Claim.** It is enough to prove the case $m = 0$ only, because the remaining cases would follow by a straightforward induction on $m$. Since $H_0$ is connected, the
claim for \( m = 0 \) amounts to proving that \( H_0 = Env_1(y; H - Z) \). The latter identity is precisely the case \( s = l \) of

\[
Env_s(y; H - Z) = Env_s(y; H) - Z, \quad \text{for any } s \in [0, l]. \tag{8}
\]

We prove \( \text{(8)} \) by induction on \( s \). the case \( s = 0 \) being routine to check. Let \( s \in [l] \). By \( \text{(7)} \) the complement of \( Env_{s-1}(y; H) \) has at least two components, one of which, namely \( Env_s(y; H) \), contains \( V(H_0) \). In order to prove \( \text{(8)} \) by induction on \( s \) we have to show that \( F = Env_s(y; H) - Z \) is still connected. If \( s = l \), then this is true because \( F = H_0 \). If \( s < l \), then \( F \) is connected because it contain a spanning complete bipartite graph with one part being \( V_1 = V(Env_{s+1}(y; H)) \setminus Z \). (This bipartite graph is not degenerate: \( V_1 \supset V(H_0) \neq \emptyset \) while \( V_1 \neq V(F) \) by \( \text{(7)} \).)

Spoiler plays in \( H \). At the first move he selects \( y_0 \). Denote Duplicator’s response in \( G \) by \( x_0 \) and set \( G_0 = Env_0(x_0) \). There are two alternatives to consider.

**Subcase 2.1.** \( G_0 \not\cong H_0 \).

Suppose first that \( l < k \). Since \( G_0 \) and \( H_0 \) are non-isomorphic copies of elements of \( \text{Dec}_l G \) and \( G \) is inclusion-free, Spoiler is able to make his next choice \( y_1 \) in some \( H_1 \in \text{Dec}_0 \overline{G}_0 \) with no isomorphic graph in \( \text{Dec} \overline{G}_0 \). Denote Duplicator’s response by \( x_1 \).

If \( x_1 \not\in V(G_0) \), then Lemma \( \text{2} \) applies to \([x_0, x_1, y_0, y_1, G, H, l]\) in view of \( \text{(7)} \). Thus Spoiler can win by using at most \( l + 3 \leq k + 2 \) moves in total. So, assume that \( x_1 \in V(G_0) \). Lemma \( \text{3} \) applies to \([x_1, y_1, G, H - Z, l + 1]\) in view of Claim \( \text{2} \) (For example, Assumption 3 is satisfied because \( G \) is uniform inclusion-free and \( \text{Dec}_l G \) contains both \( G_0 \) and an isomorphic copy of \( H_0 \).) Thus Spoiler can win in at most \( 2 + (k + c - 1) \) moves in total, as desired.

It remains to consider the case \( l = k \). Spoiler selects all vertices of \( H_0 \). There are at most \( c \) of them because \( H_0 \) is isomorphic to a cocomponent of \( G \). If Duplicator’s replies lie in \( V(G_0) \), she has already lost in view of \( G_0 \not\cong H_0 \) (which holds since \( G \) is inclusion-free). Otherwise, Duplicator’s reply to \( V(H_0) \) contains both a vertex outside \( G_0 \) and the vertex \( x_0 \in V(G_0) \), so it cannot be complement-connected, and she loses. So, Spoiler wins having made at most \( c \) moves in total.

**Subcase 2.2.** \( G_0 \cong H_0 \).

Though the graphs are isomorphic, the crucial fact is that \( G_0 \), unlike \( H_0 \), contains a selected vertex. By the definition of an inclusion-free graph, every automorphism of \( G_0 \cong H_0 \) takes each cocomponent onto itself. Therefore all isomorphisms between \( G_0 \) and \( H_0 \) match cocomponents of these graphs in the same way. Let \( Y \) be the \( H_0 \)-counterpart of the cocomponent \( X = Env_{k-1}(x_0; G_0) \) with respect to this matching. In the second round Spoiler selects an arbitrary \( y_1 \) in \( Y \). Denote Duplicator’s answer by \( x_1 \).
Suppose first that \( x_1 \in X \). Spoiler selects all vertices of \( Y \setminus \{y_1\} \). At least one of Duplicator’s replies lies outside \( V(X) \) for otherwise she has already lost having chosen some vertex in \( X \) twice. But then Duplicator’s reply to \( Y \) cannot be complement-connected. In any case Spoiler wins, having made at most \( c + 1 \) moves in total.

If \( x_1 \in V(G_0) \setminus X \), then there is an \( m \leq k - l \) such that \( \text{Env}_m(x_1; G_0) \) and \( \text{Env}_m(y_1; H_0) \) are non-isomorphic. By Claim \( \ref{claim:non-isomorphic} \) Spoiler can apply the strategy of Lemma \( \ref{lemma:complement-connected} \) to \([x_1, y_1, G, H - Z, l + m]\), winning in at most \( 2 + (k + c - 1) \) moves. If \( x_1 \not\in V(G_0) \), then Spoiler wins by Lemma \( \ref{lemma:complement-connected} \) applied to \([x_0, x_1, y_0, y_1, G, H, l]\), having made at most \( 2 + l + 1 < k + c + 1 \) moves in total.

**Case 3.** \( H \) has a component \( H_0 \) isomorphic to \( G \), and not Cases 1–2.

Spoiler plays in \( H \). In the first round he selects a vertex \( y_0 \) outside \( H_0 \) and further plays exactly as in Subcase 2.2 with \( G_0 = G \).

**Case 4.** Neither of Cases 1–3.

Spoiler plays in \( G_0 = G \). His first move \( x_0 \) is arbitrary. Denote Duplicator’s response in \( H \) by \( y_0 \) and set \( H_0 = \text{Env}_0(y_0) \). Since we are not in Cases 1–3, \( G_0 \not\subset H_0 \). As \( G_0 \) is inclusion-free, \( G_0 \) has a connected component \( G_1 \) with no isomorphic component in \( H_0 \).

If \( x_0 \not\in V(G_1) \), then Spoiler just selects any vertex \( x_1 \in V(G_1) \). Let Duplicator respond with \( y_1 \). Assume that \( y_1 \in V(H_0) \), for otherwise Duplicator has already lost: \( \{y_0, y_1\} \not\in E(H) \) while \( \{x_0, x_1\} \in E(G) \).

If \( x_0 \in V(G_1) \), then Spoiler selects any vertex \( x_1 \in V(G_0) \setminus V(G_1) \). (The latter set is non-empty since \( k \geq 1 \).) Let Duplicator respond with \( y_1 \). As before we can assume that \( y_1 \in V(H_0) \). Now, let us swap \( x_0 \) and \( x_1 \) as well as \( y_0 \) and \( y_1 \).

What we have achieved in both cases is that \( G_1 \not\cong H_1 \), where \( H_1 = \text{Env}_1(y_1; H) \). Also, \( G_1 \) is a uniform inclusion-free graph of rank \( k - 1 \). Lemma \( \ref{lemma:uniform-inclusion-free} \) applies to \([y_1, x_1, H, G, 1]\). (For example, Assumption 3 of the lemma holds because we are not in Cases 1–2.) This shows that Spoiler can win the 0-alternation game in at most \( 2 + (k + c - 1) = k + c + 1 \) moves. This completes the proof of Lemma \( \ref{lemma:uniform-inclusion-free} \).

**Proof of the upper bound in Theorem** \( \ref{theorem:upper-bound} \) Fix an integer \( c \) so that there are \( 4c + 4 \) pairwise non-embeddable into each other complement-connected graphs

\[
H_{i,j}, \quad c \leq i \leq 2c, \ 1 \leq j \leq 4,
\]

such that \( H_{i,j} \) has order \( i \). The existence of \( c \) can be easily deduced by choosing each \( H_{i,j} \) uniformly at random from all graphs of order \( i \), independently from the other graphs. Indeed, for any \( c \leq i \leq f \leq 2c \) with \( (i, j) \neq (f, g) \) the probability
that $H_{ij}$ is embeddable into $H_{fg}$ is at most
\[ \frac{f!}{(f-i)!} 2^{-\binom{i}{2}} \]
while the probability of $H_{i,j}$ not being complement-connected is at most
\[ \frac{1}{2} \sum_{h=1}^{i-1} \binom{i}{h} 2^{-h(i-h)+1}, \]
where the factor $\frac{1}{2}$ accounts for the fact that each vertex partition is counted twice.

Hence, by looking at the expected number of ‘bad’ events, we conclude that if
\[ 16 \sum_{c \leq i \leq f \leq 2c} \frac{f!}{(f-i)!} 2^{-\binom{i}{2}} + 4 \times \frac{1}{2} \sum_{i=c}^{2c-1} \sum_{h=1}^{i-1} \binom{i}{h} 2^{-h(i-h)+1} < 1, \quad (9) \]
then the required graphs exist. The exact-arithmetic calculation with Mathematica shows that $c = 10$ works in (9). (This value of $c$ can perhaps be improved with more work.)

We define, inductively on $i$, a family $R_i$ of graphs, starting with
\[ R_0 = \{ H_{c,1}, H_{c,2}, H_{c,3}, H_{c,4} \}. \]
Assume that $R_{i-1}$ is already specified. Given a non-empty subset $S \subset R_{i-1}$, we define the graph
\[ G_{i,S} = \bigcup_{G \in S} G, \]
or, in words, $G_{i,S}$ is the complement of the vertex-disjoint union of the graphs in $S$. We let
\[ R_i = \{ G_{i,S} : |S| = |R_{i-1}|/2 \}, \]
where we view $R_i$ as the set of isomorphism types of graphs. It is proved in Claim 3 below that the graphs $G_{i,S}$ are pairwise non-isomorphic. (In particular, this implies by induction on $i$ that $|R_i|$ is even because $\binom{2m}{m}$ is even for any integer $m \geq 1$.) Let
\[ r_i = |R_i|. \]

Let us list some properties of these graphs.

**Claim 3.**

1. For any $S \subset R_{i-1}$ with $|S| \geq 2$, $G_{i,S}$ is a connected inclusion-free uniform graph of rank $i$.

2. For any $S, T \subset R_{i-1}$ with $S \not\subset T$, the graph $G_{i,S}$ is not embeddable into $G_{i,T}$. 
Proof of Claim. We prove all claims by induction on $i$, the case $i = 1$ directly following from the definition of $R_0$. Let $i \geq 2$.

First, we verify Property 1, assuming that Properties 1–3 hold for all smaller values of $i$. Since $|S| \geq 2$, $G_{i,S}$ is connected. The components of $G_{i,S}$ belong to $R_{i-1}$, each being isomorphic to $G_{i-1,S'}$ for some $S' \subset R_{i-2}$. From Property 3 and the initial value $r_0 = 4$, it is easy to deduce that $|S'| = r_{i-2}/2 \geq 2$. By the inductive Property 1, all components of $G_{i,S}$ are uniform of rank $i - 1$, so $G_{i,S}$ is uniform of rank $i$.

Next, let us verify that $G_{i,S}$ is an inclusion-free graph. For any $j \in [i]$ all elements of $Dec_j G_{i,S}$ belong to $R_{i-j}$; by induction, each is inclusion-free. Let us show that none of these graphs is properly embeddable into another. Assume that $j < i$ for otherwise the claim follow from the definition of $R_0$. Take any two non-isomorphic $G_{i-j,S'}, G_{i-j,S''} \in R_{i-j}$. We have $S' \not\subset S''$ because $S' \neq S''$ and $|S'| = |S''| = r_{i-j-1}/2$. By induction (Property 2), we conclude that $G_{i-j,S'} \not\subset G_{i-j,S''}$, giving the stated. Since $G_{i,S}$ is connected, it remains to observe that $G_{i,S}$ has no two isomorphic components, which follows from Property 2 again. Thus $G_{i,S}$ is indeed inclusion-free. We have completely finished the inductive step for Property 1.

Let us turn to Property 2. All components of $\overline{G_{i,S}}$ and $\overline{G_{i,T}}$ belong to $R_{i-1}$. Take any $H \in S \setminus T$. The graph $H \in R_{i-1}$ appears as a component in $G_{i,S}$. By induction (Property 2) and the definition of $R_{i-1}$, $H$ cannot be embedded into any component of $\overline{G_{i,S}}$. Thus $G_{i,S} \not\subset G_{i,T}$, as required. Property 3 follows from Property 2 which implies that the graphs $G_{i,S}$, for $S \subset R_{i-1}$, are pairwise non-isomorphic. \[ \]

All graphs in $R_i$ have the same order which we denote by $n_i$. We have $n_0 = c$ and, for $i \geq 1$,$n_i = n_{i-1}r_{i-1}/2$. We have $v(G_{i,S}) = |S|n_{i-1}$. If we denote $m_i = r_i/2$, then we have $m_0 = 2$ and $m_1 = 3$. Thus for $i \geq 1$ we have

$$m_{i+1} = \frac{1}{2}r_{i+1} = \frac{1}{2} \left( \frac{r_i}{r_i/2} \right) = \frac{1}{2} \left( \frac{2m_i}{m_i} \right) \geq 2^{m_i}.$$ 

We conclude that $m_i > \text{Tower}(i)$ for all $i \geq 0$ and thus

$$n_i \geq m_{i-1} > \text{Tower}(i - 1).$$ \[ 10 \]

At this point we are able to prove the required upper bound on $q_0(n)$ for an infinite sequence of $n$, namely,

$$\ldots, n_{i-1}, 2n_{i-1}, 3n_{i-1}, \ldots, m_{i-1}n_{i-1} = n_{i+1}, 2n_{i+1}, \ldots$$ \[ 11 \]
Indeed, by Lemma 4 for every \( 2 \leq s \leq m_i \) and an \( s \)-set \( S \subset R_{i-1} \), we have

\[
q_0(sn_{i-1}) \leq D_0(G_{i,S}) \leq i + c + 1.
\]

Also, we have \( i \leq \log^* n_{i-1} + 1 \) by (10). Thus

\[
q_0(sn_{i-1}) \leq \log^*(n_{i-1}) + c + 2 \leq \log^*(sn_{i-1}) + 12.
\]

It now remains to fill in the gaps in (11). We need some auxiliary notions and claims first. We define the operation of a \textit{cocomponent replacement} as follows. Suppose that \( A \) is a cocomponent of a graph \( G \) and \( B \) is a complement-connected graph. The result of the \textit{replacement of \( A \) with \( B \) in \( G \)} is the graph \( G' \) with \( V(G') = (V(G) \setminus V(A)) \cup V(B) \) such that \( G'[V(B)] = B, G' - B = G - A \), and every vertex in \( B \) is adjacent to a vertex \( v \) outside \( B \) in \( G' \) if and only if every vertex in \( A \) is adjacent to \( v \) in \( G \). (Here, we assume that \( V(G) \cap V(B) = \emptyset \), and we use the fact that any two vertices inside a cocomponent have the same adjacency pattern to the rest of the graph.)

**Claim 4.** Let \( G \) be a uniform inclusion-free graph of rank \( i \) with all cocomponents being isomorphic to one of \( H_{c,l} \) with \( 1 \leq l \leq 4 \). Let \( G' \) be obtained from \( G \) by replacing each cocomponent \( A \cong H_{c,l} \) with some \( H_{j,l} \), where \( j \in [c, 2c] \) may depend on \( A \). Then \( G' \) is a uniform inclusion-free graph of rank \( i \).

**Proof of Claim.** The partitions \( P_0, \ldots, P_i \) defined in (3) are completely determined by the vertex sets of the cocomponents and the adjacencies between them. This shows that \( G' \) is uniform of rank \( i \). Let us check that \( G' \) is inclusion-free.

Let \( 0 \leq j \leq i \), \( K' \in \text{Dec}_j G' \), and \( C_1', C_2' \) be some distinct components of the complement of \( K' \). Suppose on the contrary that a bijection \( f' : V(C_1') \rightarrow V(C_2') \) establishes an isomorphism between \( C_1' \) and \( C_2' \). The isomorphism \( f' \) induces a correspondence \( g' \) between the cocomponents of \( C_1' \) and \( C_2' \).

The definition of component replacement allows us to point the corresponding \( K \in \text{Dec}_i G, C_1, C_2 \in \text{Dec} \overline{K} \), and \( g \). Since \( C_1 \not\cong C_2 \), there is a cocomponent \( X_1 \) of \( C_1 \) such that the cocomponent \( X_2 = g(X_1) \) is not isomorphic to \( X_1 \). It means that, if \( X_1 \cong H_{c,l_1} \) and \( X_2 \cong H_{c,l_2} \), then \( l_1 \neq l_2 \). But in \( G' \) these are replaced by \( X_1' \cong H_{j,l_1} \) and \( X_2' \cong H_{j,l_2} \), which are still non-isomorphic since \( l_1 \neq l_2 \). This contradicts the assumption that \( f' \) is an isomorphism. Thus \( G' \) satisfies Property 1 of the definition of an inclusion-free graph. The other property in the definition can be checked similarly.

If \( n \leq 2c = 20 \), then the upper bound (3) follows from the trivial inequality \( q_0(n) \leq n + 1 \). So assume that \( n > 2c = 2m_0 \). Choose the integer \( i \) satisfying \( 2n_i \leq n < 2n_{i+1} \). Since \( n_{i+1} = n_i m_i \), let \( s \in [2, 2m_i - 1] \) satisfy \( sn_i \leq n < (s + 1)n_i \). Pick any \( s \)-set \( S \subset R_i \) and let \( G = G_{i+1,S} \). We have \( v(G) = sn_i \leq n \) and, by Claim (3) the graph \( G \) is inclusion-free and uniform of rank \( i + 1 \).
Let \( f : \text{Dec}_{i+1} G \to [c, 2c] \) be some function. We construct a new graph \( G_f \) by replacing every cocomponent \( A \) of \( G \) by a copy of \( H_{f(A), j} \), where \( j \) is defined by \( A \cong H_{c,j} \). If \( f \) is the constant function assuming the value \( 2c \), then \( v(G_f) = 2v(G) > n \). Hence there is some choice of \( f \) such that \( v(G_f) = n \). By Claims 3 and 4, the graph \( G_f \) is a uniform inclusion-free graph of rank \( i+1 \). By Lemma 4, we have \( D_0(G_f) \leq i + 2c + 2 \). On the other hand, \( n \geq n_i > \text{Tower}(i-1) \), that is, \( \log^* n \geq i \).

This finishes the proof of the upper bound in (3).

4 Lower bound: Proof of Theorem 2

From now on we will be dealing with digraphs.

Given a first order formula \( \Phi \) in which the negation sign occurs only in front of atomic subformulas, let the \textit{alternation number} of \( \Phi \), denoted by \( \text{alt}(\Phi) \), be the maximum number of quantifier alternations, i.e. the occurrences of \( \exists \forall \) and \( \forall \exists \), in a sequence of nested quantifiers of \( \Phi \). For a non-negative integer \( a \), we denote

\[
\Lambda_a = \{ \Phi : \text{alt}(\Phi) \leq a \}.
\]

We also define \( \Lambda_{1/2} \) to be the class of formulas \( \Phi \) with \( \text{alt}(\Phi) \leq 1 \) such that any sequence of nested quantifiers of \( \Phi \) starts with \( \exists \) or has no quantifier alternation. Note that \( \Lambda_0 \subset \Lambda_{1/2} \subset \Lambda_1 \subset \Lambda_2 \subset \ldots \).

Now we somewhat extend our notation. Let \( F \) be some class of first order formulas. If a digraph \( G \) has a defining sentence in \( F \), let \( D_F(G) \) (resp. \( L_F(G) \)) denote the minimum quantifier rank (resp. length) of a such sentence; otherwise, we let \( D_F(G) = L_F(G) = \infty \). The \textit{succinctness function} is defined as

\[
q_F(n) = \min \{ D_F(G) : v(G) = n \}.
\]

Whenever the index \( F \) is omitted, it is supposed that \( F \) is the class of all first order formulas. We also simplify notation by \( D_a(G) = D_{\Lambda_a}(G) \) and similarly with \( L_a(G) \) and \( q_a(n) \). Clearly,

\[
q(n) \leq \ldots \leq q_2(n) \leq q_1(n) \leq q_{1/2}(n) \leq q_0(n).
\]

**Lemma 5** For every \( a \in \{0, 1/2, 1, 2, 3, \ldots \} \) and any digraph \( G \) we have

\[
L_a(G) < \text{Tower}(D_a(G) + \log^* D_a(G) + 2).
\]

An analog of this lemma for \( L(G) \) and \( D(G) \) appears in [11, Theorem 10.1]. However, the proof of Lemma 5 we give below is not just an easy adaptation of the proof in
because the restrictions on the class of formulas do not allow to run the same argument directly. Moreover, if \( a = 1/2 \), there appears another obstacle — the class of formulas \( \Lambda_{1/2} \) is not closed with respect to negation.

Lemma 5 is proved in the next section in a stronger form since the argument is presentable more naturally in a more general situation. Here, let us show how Lemma 5 implies Theorem 2.

Given \( n \), denote \( k = q_0(n) \) and fix a digraph \( G \) on \( n \) vertices such that \( D_0(G) = k \). By Lemma 5, \( G \) is definable by a 0-alternating sentence \( \Phi \) of length less than \( \text{Tower}(k + \log^* k + 2) \). First, we convert \( \Phi \) to an equivalent prenex \( \exists^* \forall^* \)-sentence \( \Psi \), i.e. of form (12). This can be easily done as follows. By renaming variables, ensure that each variable is quantified exactly once. Let the existential (resp. universal) quantifiers appear with variables \( x_1, \ldots, x_l \) (resp. \( y_1, \ldots, y_m \)) in this order as we scan \( \Phi \) from left to right. To obtain the required sentence \( \Psi \) simply ‘pull’ all quantifiers at front:

\[
\Psi = \exists x_1 \ldots \exists x_l \forall y_1 \ldots \forall y_m \text{ (quantifier-free part) (12)}
\]

The obtained sentence \( \Psi \) is equivalent to \( \Phi \) since the latter does not contain an \( \exists \)-quantifier in the range of a \( \forall \)-quantifier. Also, this reduction does not increase the total number of quantifiers. Therefore, as a rather rough estimate, we have \( D(\Psi) \leq L(\Phi) \).

It is well known and easy to see that, if a sentence of the form (12) is true on some structure \( H \), then it is true on some structure of order at most \( l \leq D(\Psi) \). (Indeed, fix any satisfying assignment for \( x_1, \ldots, x_l \) and take the substructure of \( H \) induced by the corresponding vertices.) Since the defining sentence \( \Psi \) is true only on \( G \), we have

\[
n \leq D(\Psi) \leq L(\Phi) < \text{Tower}(k + \log^* k + 2).
\]

This implies that

\[
\log^* n \leq k + \log^* k + 1. \tag{13}
\]

Suppose on the contrary to Theorem 2 that \( k \leq \log^* n - \log^* \log^* n - 2 \). Then \( \log^* k \leq \log^* \log^* n \) and (13) implies that

\[
\log^* n \leq (\log^* n - \log^* \log^* n - 2) + \log^* \log^* n + 1,
\]

which is a contradiction, proving Theorem 2.

Note that identically the same argument works for \( a = 1/2 \) as well, giving that

\[
q_{1/2}(n) \geq \log^* n - \log^* \log^* n - 1 \quad \text{for all } n. \tag{14}
\]
5  Length vs. depth for restricted classes of defining sentences

Writing $A(x_1, \ldots, x_s)$, we mean that $x_1, \ldots, x_s$ are all free variables of $A$. We allow $s = 0$ which means that $A$ is a sentence. A formula $A(x_1, \ldots, x_s)$ of quantifier rank $k - s$ is normal if

- all negations occurring in $A$ stay only in front of atomic subformulas,
- $A$ has occurrences of variables $x_1, \ldots, x_k$ only,
- every sequence of nested quantifiers of $A$ has length $k - s$ and quantifies the variables $x_{s+1}, \ldots, x_k$ exactly in this order.

A simple inductive syntactic argument shows that any $A(x_1, \ldots, x_s)$ has an equivalent normal formula $A'(x_1, \ldots, x_s)$ of the same quantifier rank. Such a formula $A'$ will be called a normal form of $A$.

Recall that by $F$ we denote a class of first order formulas. Given $F$, the class of sentences (i.e. closed formulas) in $F$ of quantifier rank $k$ is denoted by $F^k$. We call $F$ regular if

- $F$ is closed under subformulas and renaming of bound variables,
- with each $A(x_1, \ldots, x_s)$ in $F$, the class $F$ contains a normal form of $A$,
- for any $k \geq 1$, $F^k$ has the pattern set $P^k \subseteq \{\forall, \exists\}^k$ such that a normal sentence $A$ belongs to $F^k$ iff every sequence of nested quantifiers of $A$ belongs to $P^k$. (By the normality of $A$ all quantifier sequences have the same length.)

**Theorem 3** Suppose that $F$ is regular and $G$ is definable in $F$. Then

$$L_F(G) < \text{Tower } (D_F(G) + \log^* D_F(G) + 2).$$

Note that Theorem 3 generalizes Lemma 5 because the classes $\Lambda_a$ are regular.

When we write $\bar{z}$, we will mean an $s$-tuple $(z_1, \ldots, z_s)$. If $\bar{u} \in V(G)^s$, we write $G, \bar{u} \models A(\bar{x})$ if $A(\bar{x})$ is true on $G$ with each $x_i$ assigned the respective $u_i$. Notation $\models A(\bar{x})$ will mean that $A(\bar{x})$ is true on all digraphs with $s$ designated vertices.

A formula $A(\bar{x})$ in $F$ is called an $F$-description of $(G, \bar{u})$ if

- $G, \bar{u} \models A(\bar{x})$, and
- for every $B(\bar{x}) \in F$ such that $G, \bar{u} \models B(\bar{x})$, we have $\models A(\bar{x}) \Rightarrow B(\bar{x})$, where $X \Rightarrow Y$ is a shorthand for $(\neg X) \lor Y$. 
The next proposition will be useful.

**Lemma 6** Suppose that $G$ is definable in $F$. Let $A$ be a sentence in $F$. Then $A$ defines $G$ iff $A$ is an $F$-description of $G$.

**Proof.** Suppose that $A$ defines $G$. Then $G \models A$. Let $B \in F$ satisfy $G \models B$. We have to show that $H \models A \Rightarrow B$ for any $H$. If $H \not\models A$, we are done immediately. If $H \models A$, then $H \cong G$ and $H \models B$, as required.

For the other direction, suppose that $A$ is an $F$-description of $G$. We have to show that $H \not\models A$ for any $H \not\models G$. Fix a sentence $B \in F$ defining $G$. Since $H \not\models B$ and $\models A \Rightarrow B$, we conclude that $H \not\models A$, as required. □

Let $G$ and $H$ be digraphs, $\bar{u} \in V(G)^s$, and $\bar{v} \in V(H)^s$. We write $G, \bar{u} \equiv H, \bar{v} \pmod{F}$ if, for any $A(\bar{x})$ in $F$, we have $G, \bar{u} \models A(\bar{x})$ exactly when $H, \bar{v} \models A(\bar{x})$.

**Lemma 7** Suppose that $G, \bar{u} \equiv H, \bar{v} \pmod{F}$ and let $A(\bar{x}) \in F$. Then $A$ is an $F$-description of $(G, \bar{u})$ iff it is an $F$-description of $(H, \bar{v})$.

**Proof.** As $A$ is in $F$, we have $G, \bar{u} \models A(\bar{x})$ iff $H, \bar{v} \models A(\bar{x})$. Let $B(\bar{x}) \in F$. Again $G, \bar{u} \models B(\bar{x})$ iff $H, \bar{v} \models B(\bar{x})$. It follows that $G, \bar{u} \models B(\bar{x})$ implies $\models A(\bar{x}) \Rightarrow B(\bar{x})$ iff $H, \bar{v} \models B(\bar{x})$ implies $\models A(\bar{x}) \Rightarrow B(\bar{x})$. □

Furthermore, we define

$$(G, \bar{u}) \pmod{F} = \{ (H, \bar{v}) : G, \bar{u} \equiv H, \bar{v} \pmod{F} \}.$$

Let $0 \leq s \leq k$. The class of formulas in $F$ with $s$ free variables and quantifier rank $k-s$ is denoted by $F_{k-s}$. In particular, $F_{k,0} = F_k$. We define

$$E(F_{k,s}) = \{ (G, \bar{u}) \pmod{F_{k,s}} : G \text{ is a digraph, } \bar{u} \in V(G)^s \}.$$

We will also use the following notation. Given $P^k \subseteq \{\forall, \exists\}^k$ and $\sigma \in \{\forall, \exists\}^s$, let

$$P_{\sigma}^{k,s} = \{ \rho \in \{\forall, \exists\}^{k-s} : \sigma \rho \in P^k \}.$$

Furthermore, given a regular $F$ with pattern set $P^k$, let $F_{\sigma}^{k,s}$ consist of the normal formulas in $F_{k,s}$ whose sequences of nested quantifiers are in $P_{\sigma}^{k,s}$. We say that a formula $A(\bar{x}) \in F_{\sigma}^{k,s}$ describes a class $\alpha \in E(F_{\sigma}^{k,s})$ if $A(\bar{x})$ is an $F_{\sigma}^{k,s}$-description of some $(G, \bar{u}) \in \alpha$. By Lemma 7, this definition does not depend on the particular choice of a representative $(G, \bar{u})$ of $\alpha$, and the word *some* in the definition can be replaced with *every*. 
Definitions with no quantifier alternation

Proof of Theorem. For each $\alpha \in E(F^{k,s}_\sigma)$ we will construct a formula $A_\alpha(\vec{x}) \in F^{k,s}_\sigma$ describing $\alpha$. We will use induction on $k - s$. Afterwards we will estimate the length of the obtained $A_\alpha$ and show how this implies Theorem 3.

We start with $s = k$. Let $\sigma \in \{\forall, \exists\}^k$. Assume that $\sigma \in P^k$, for otherwise $F^{k,k}_\sigma$ is empty and there is nothing to do. For any such $\sigma$, $F^{k,k}_\sigma = F^{k,k}$ is exactly the class of all quantifier-free formulas in $F$ over the set of variables $\{x_1, \ldots, x_k\}$. Clearly, $(G, \vec{u}) \equiv (H, \vec{v}) (\mod F^{k,k}_\sigma)$ iff the componentwise correspondence between $\vec{u}$ and $\vec{v}$ gives a partial isomorphism. So, any given class $\alpha \in E(F^{k,s}_\sigma)$ can be described as follows. Pick any representative $(G, u_1, \ldots, u_k)$ of $\alpha$ and let $A_\alpha(x_1, \ldots, x_k)$ be the conjunction of all atomic formulas $x_i \leftrightarrow x_j$ for $(u_i, u_j)$ in $G$, all negations $\neg(x_i \leftrightarrow x_j)$ for $(u_i, u_j)$ not in $G$, all $x_i = x_j$ for identical $u_i, u_j$, and all $\neg(x_i = x_j)$ for distinct $u_i, u_j$. Clearly, $H, \vec{v} \models A_\alpha(\vec{x})$ iff $(H, \vec{v}) \in \alpha$. It follows that $A_\alpha$ indeed describes $\alpha$. Note that $L(A_\alpha) \leq 18k^2$.

Assume now that $0 \leq s < k$ and that for any $\tau \in \{\forall, \exists\}^{s+1}$ with $F^{k,s+1}_\tau \neq \emptyset$ and $\beta \in E(F^{k,s+1}_\tau)$ we have a formula $A_\beta(\vec{x}, x_{s+1}) \in F^{k,s+1}_\tau$ describing $\beta$. Given a digraph $G$, an $s$-tuple of vertices $\vec{u} \in V(G)^s$, and a non-empty class of formulas $F$, we set

$$S(G, \vec{u}; F) = \{(G, \vec{u}, u) \mod F : u \in V(G)\}.$$ We also set $S(G, \vec{u}; \emptyset) = \emptyset$. We will write $A \equiv A'$ if formulas $A$ and $A'$ are literally identical. Let $\sigma \in \{\forall, \exists\}^s$ and $\alpha \in E(F^{k,s}_\sigma)$. To construct $A_\alpha(\vec{x})$, we fix $(G, \vec{u})$ being an arbitrary representative of $\alpha$ and put

$$A_\alpha(\vec{x}) \equiv \bigwedge_{\beta \in S(G, \vec{u}; F^{k,s+1}_{\vec{x}})} \exists x_{s+1} A_\beta(\vec{x}, x_{s+1}) \land \forall x_{s+1} \bigvee_{\beta \in S(G, \vec{u}; F^{k,s+1}_{\vec{x}})} A_\beta(\vec{x}, x_{s+1}).$$

Claim 5. $A_\alpha(\vec{x}) \in F^{k,s}_\sigma$.

Proof of Claim. This follows from the assumption that $A_\beta(\vec{x}, x_{s+1}) \in F^{k,s+1}_{\vec{x}}$ for $\beta \in S(G, \vec{u}; F^{k,s+1}_{\vec{x}})$.

Claim 6. $G, \vec{u} \models A_\alpha(\vec{x})$.

Proof of Claim. Let us show first that all conjunctions over $\beta \in S(G, \vec{u}; F^{k,s+1}_{\vec{x}})$ are satisfied. Each such $\beta$ is of the form $(G, \vec{u}, u_\beta) \mod F^{k,s+1}_{\vec{x}}$ for some $u_\beta \in V(G)$. By assumption, $G, \vec{u}, u_\beta \models A_\beta(\vec{x}, x_{s+1})$ and hence $G, \vec{u} \models \exists x_{s+1} A_\beta(\vec{x}, x_{s+1}).$

It remains to show that the universal member of the conjunction is also satisfied. Consider an arbitrary $u \in V(G)$. Let $\beta_u = (G, \vec{u}, u) \mod F^{k,s+1}_{\vec{x}}$. By assumption, $G, \vec{u}, u \models A_{\beta_u}(\vec{x}, x_{s+1})$ and hence the disjunction is always true.

\footnote{Here $A_{\alpha}$ has the same form as the Hintikka formula in [23] page 18. Curiously, in a similar context in [11] Lemma 3.4 we use another generic defining formula borrowed from [13] Theorem 2.3.2], which is not usable now because $F$ may be not closed with respect to negation.}
Claim 7. We have \( \models A_\alpha(\bar{x}) \Rightarrow B(\bar{x}) \) for any \( B(\bar{x}) \in F_{\sigma}^{k,s} \) such that
\[
G, \bar{u} \models B(\bar{x}).
\] (15)

Proof of Claim. Given a class of formulas \( F \), let \( F|_{\exists} \) (resp. \( F|_{\forall} \)) denote the class of those formulas in \( F \) having form \( \exists x(\ldots) \) (resp. \( \forall x(\ldots) \)). First, we settle two special cases of the claim.

Case 1. \( B \in F_{\sigma}^{k,s}|_{\exists} \)

Let \( B \models \exists x_{s+1} C(\bar{x}, x_{s+1}) \). Note that \( C(\bar{x}, x_{s+1}) \in F_{\sigma}^{k,s+1} \). Assume that \( H, \bar{v} \models A_\alpha(\bar{x}) \). We have to verify that \( H, \bar{v} \models B(\bar{x}) \). By (15) we can choose a vertex \( u \in V(G) \) such that \( G, \bar{u}, u \models C(\bar{x}, x_{s+1}) \). Let
\[
\beta = (G, \bar{u}, u) \text{ mod } F_{\sigma}^{k,s+1}.
\]
We have \( H, \bar{v} \models \exists x_{s+1} A_\beta(\bar{x}, x_{s+1}) \) and hence \( H, \bar{v}, v \models A_\beta(\bar{x}, x_{s+1}) \) for some \( v \in V(H) \). Since we have assumed that \( A_\beta(\bar{x}, x_{s+1}) \) is an \( F_{\sigma}^{k,s+1} \)-description of \( \beta \), we have \( H, \bar{v}, v \models C(\bar{x}, x_{s+1}) \) and hence \( H, \bar{v} \models B(\bar{x}) \) as needed.

Case 2. \( B \in F_{\sigma}^{k,s}|_{\forall} \)

Let \( B \models \forall x_{s+1} C(\bar{x}, x_{s+1}) \). Note that \( C(\bar{x}, x_{s+1}) \in F_{\sigma}^{k,s+1} \). Assume that \( H, \bar{v} \models A_\alpha(\bar{x}) \). It follows that for every \( v \in V(H) \) there is a \( \beta_v \in S(G, \bar{u}; F_{\sigma}^{k,s+1}) \) such that \( H, \bar{v}, v \models A_{\beta_v}(\bar{x}, x_{s+1}) \). By (15) we have \( G, \bar{u}, u \models C(\bar{x}, x_{s+1}) \) for all \( u \in V(G) \). Let \( u_v \) be such that
\[
\beta_v = (G, \bar{u}, u_v) \text{ mod } F_{\sigma}^{k,s+1}.
\]
We have \( G, \bar{u}, u_v \models C(\bar{x}, x_{s+1}) \), and, by our assumption that \( A_{\beta_v} \) describes \( \beta_v \), we have \( H, \bar{v}, v \models C(\bar{x}, x_{s+1}) \). Since \( v \) is arbitrary, we conclude that \( H, \bar{v} \models B(\bar{x}) \), finishing the proof of Case 2.

Finally, take an arbitrary \( B(\bar{x}) \in F_{\sigma}^{k,s} \). Since \( B \) is normal (and \( s < k \)), it is equivalent to a DNF formula \( \lor_i (\land_j B_{i,j}) \) with all \( B_{i,j} \) belonging to \( F_{\sigma}^{k,s}|_{\exists} \cup F_{\sigma}^{k,s}|_{\forall} \).

This can be routinely shown by induction on \( L(B) \). For example, if \( B = B_1 \land B_2 \), where, by induction, \( B_h \) is equivalent to \( \lor_i (\land_j B_{i,j}) \), \( h = 1, 2 \), then we can take \( \lor_i (\land_j B_{i,j}) \land (\land_j B_{i,j}) \) for \( B \).

Since \( G, \bar{u} \models B(\bar{x}) \), we have \( G, \bar{u} \models B_{i_0,j}(\bar{x}) \) for some \( i_0 \) and all \( j \). From Cases 1–2 it follows that \( H, \bar{v} \models B_{i_0,j}(\bar{x}) \) for all \( j \) whenever \( H, \bar{v} \models A_\alpha(\bar{x}) \). This means that \( H, \bar{v} \models B(\bar{x}) \) whenever \( H, \bar{v} \models A_\alpha(\bar{x}) \), as required.

Let us now estimate the length of the constructed formulas. The estimates are similar to those in [11] Theorem 10.1]. Our bound will depend on \( k \) and \( s \) only, so we define
\[
l(k,s) = \max_{\tau \in \{\forall,\exists\}^s} \max \{ L(A_\beta) : \beta \in E(F_{\tau}^{k,s}) \}.
\]
Let $f(k, s) = |Ehrv(k, s)|$, where $Ehrv(k, s) = E(FO^{k,s})$ with $FO$ being the class of all first order formulas. (The elements of $Ehrv(k, s)$ are called digraph Ehrenfeucht values, see [15].) The function $f(k, s)$ is an upper bound on $|E(F^{k,s}_\tau)|$ for any $\tau \in \{\forall, \exists\}^*$. The number of Ehrenfeucht values for (unoriented) graphs was estimated in [15, Theorem 2.2.1]. The obvious modifications of the proofs from [15] give the following bounds for digraphs:

$$f(k, k) \leq 4k^2,$$

$$f(k, s) \leq 2f(k, s+1).$$

We already know that $l(k, k) \leq 18k^2$. Also, the analysis of our construction shows that for $0 \leq s < k$ we have

$$l(k, s) \leq 2f(k, s+1)(l(k, s+1) + 9). \quad (16)$$

Let $k \geq 2$. Set $g(x) = 2 \cdot 2^x(x + 9)$. A simple inductive argument shows that

$$f(k, s) \leq 2g(k-s)(18k^2) \quad \text{and} \quad l(k, s) \leq g(k-s)(18k^2).$$

Define the two-parameter function $\text{Tower}(i, x)$ inductively on $i$ by $\text{Tower}(0, x) = x$ and $\text{Tower}(i+1, x) = 2^{\text{Tower}(i, x)}$ for $i \geq 1$. This is a generalization of the old function: $\text{Tower}(i, 1) = \text{Tower}(i)$. One can prove by induction on $i$ that for any $x \geq 5$ and $i \geq 1$ we have

$$g^{(i)}(x) < \text{Tower}(i + 1, x)/2. \quad (17)$$

Indeed, it is easy to check the validity of (17) for $i = 1$, while for $i \geq 2$ we have

$$g^{(i)}(x) < g(\text{Tower}(i, x)/2) < 2^{\text{Tower}(i, x)-1} = \text{Tower}(i + 1, x)/2. \quad (18)$$

If $k \geq 12$, then $18k^2 < 2^k$ and by (17) we have

$$l(k, 0) \leq g^{(k)}(18k^2) < \text{Tower}(k + 1, 18k^2)/2 < \text{Tower}(k + \log^* k + 2) \quad (19)$$

Also, $18 \cdot 11^2 < \text{Tower}(4)/2$ and, similarly to (18), we have $g^{(k)}(18k^2) < \text{Tower}(k + 4)/2$ for $k \leq 11$. Thus (19) holds for $k \in [3, 11]$ too. For $k = 2$ one can still prove (19) using (16) and the sharper initial estimates $f(2, 2) = 10$ and $l(2, 2) \leq 24$.

To finish the proof of Theorem 3 let $k = DF(G) \geq D(G) \geq 2$ and $\alpha = G \mod F^{k,0}$. Since $G$ is definable in $F^{k,0}$, the sentence $A_\alpha$ defines $G$ by Lemma 6. By (19),

$$L_F(G) \leq L(A_\alpha) \leq l(k, 0) < \text{Tower}(k + \log^* k + 2),$$

completing the proof. \hfill \blacksquare
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