Vector fields on $\mathfrak{osp}_{2m|2n}(\mathbb{C})$- and $\mathfrak{psp}_n(\mathbb{C})$-flag supermanifolds

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Abstract

The paper is devoted to a computation of the Lie superalgebras of holomorphic vector fields on isotropic flag supermanifolds of maximal type corresponding to the Lie superalgebras $\mathfrak{osp}_{2m|2n}(\mathbb{C})$ and $\mathfrak{psp}_n(\mathbb{C})$. The result is that under some restrictions on the flag type any holomorphic vector field is fundamental with respect to the natural action of the Lie superalgebras $\mathfrak{osp}_{2m|2n}(\mathbb{C})$ or $\mathfrak{psp}_n(\mathbb{C})$.

1 Introduction

Yu.I. Manin [Man] constructed four series of complex homogeneous supermanifolds that correspond to four series of linear Lie superalgebras: $\mathfrak{gl}_{m|n}(\mathbb{C})$, $\mathfrak{osp}_{m|2n}(\mathbb{C})$, $\mathfrak{psp}_n(\mathbb{C})$ and $\mathfrak{q}_n(\mathbb{C})$. (For definitions of these Lie superalgebras we refer to [Kac], see also Section 2.) These supermanifolds are super-analogues of classical flag manifolds. In this paper we calculate the Lie superalgebras of global holomorphic vector fields on isotropic flag supermanifolds of maximal type that correspond to the Lie superalgebras $\mathfrak{osp}_{2m|2n}(\mathbb{C})$ and $\mathfrak{psp}_n(\mathbb{C})$. We prove that under some restrictions on the flag type all such vector fields are fundamental with respect to the natural action of the corresponding Lie superalgebra. We use induction and a similar result for super-Grassmannians that was obtained in [OS2, OS3]. For isotropic flag supermanifolds and even for super-Grassmannians of non-maximal type an analogous result is not known so far. The Lie superalgebra of holomorphic vector fields on super-Grassmannians corresponding to the Lie superalgebras $\mathfrak{gl}_{m|n}(\mathbb{C})$ and $\mathfrak{q}_n(\mathbb{C})$ were studied in [OS1, O1] and a similar question in the case of flag supermanifolds were studied in [V1, V2].

The orthosymplectic Lie superalgebra $\mathfrak{osp}_{m|2n}(\mathbb{C})$ is the linear Lie superalgebra that annihilates a non-degenerate even symmetric bilinear form in

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The Lie superalgebra $\pi\mathfrak{sp}_n(\mathbb{C})$ is the linear Lie superalgebra that annihilates a non-degenerate odd skew-symmetric bilinear form in $\mathbb{C}^{n|n}$. (For a detailed description of these Lie superalgebras see Section 2.)

We denote by $\mathbf{F}_{k|l}$ the flag supermanifold of type $k|l$ in the vector superspace $\mathbb{C}^{m|n}$, see [Man] and also [V1]. Here $k = (k_0, \ldots, k_r)$ and $l = (l_0, \ldots, l_r)$ such that

$$0 \leq k_r \leq \ldots \leq k_0 = m, \quad 0 \leq l_r \ldots \leq l_0 = n, $$

$$0 < k_r + l_r < \ldots < k_0 + l_0 = m + n. \quad (1)$$

This flag supermanifold corresponds to the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. The number $r$ is called the length of $\mathbf{F}_{k|l}$. We denote by $\mathbf{F}^e_{k|l}$ and $\mathbf{F}^o_{k|l}$ the isotropic flag supermanifolds in $\mathbb{C}^{m|2n}$ corresponding to $\mathfrak{osp}_{m|2n}(\mathbb{C})$ and by $\mathbf{F}^v_{k|l}$, the isotropic flag supermanifold in $\mathbb{C}^{n|n}$ corresponding to $\pi\mathfrak{sp}_n(\mathbb{C})$. Here the subscripts $e$ and $o$ in $\mathbf{F}^e_{k|l}$ and $\mathbf{F}^o_{k|l}$ come from “even” and “odd”.

The idea of the proof is the following. For $r > 1$ the flag supermanifolds $\mathbf{F}^e_{k|l}$ and $\mathbf{F}^o_{k|l}$ are the total spaces of holomorphic superbundles with the base spaces that are isomorphic to the isotropic super-Grassmannians and with the fibers that are isomorphic to the flag supermanifold $\mathbf{F}_{k'|l'}$ of length $r - 1$. Here $k' = (k_1, \ldots, k_r)$ and $l' = (l_1, \ldots, l_r)$. Hence to obtain the result we can use induction and the results about Lie superalgebras of holomorphic vector fields on super-Grassmannians [OS2, O1] and on flag supermanifolds $\mathbf{F}^v_{k'|l'}$ from [V1].

We set $\mathfrak{pgl}_{m|n}(\mathbb{C}) := \mathfrak{gl}_{m|n}(\mathbb{C}) / \mathfrak{z}(\mathfrak{gl}_{m|n}(\mathbb{C}))$, where $\mathfrak{z}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ is the center of $\mathfrak{gl}_{m|n}(\mathbb{C})$. The main result of this paper was announced in [V4]. It is the following.

**Theorem 1.** Let $r > 1$.

1. Assume that $m = k_1$, $n = l_1$: $(k_i, l_i) \neq (k_{i-1}, 0)$, $(0, l_{i-1})$, $i \geq 2$, $k_1 \geq 1$, $l_1 \geq 1$ and $\mathfrak{v}(\mathbf{F}^e_{k|l'}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$. Then

$$\mathfrak{v}(\mathbf{F}^e_{k|l}) \simeq \mathfrak{osp}_{2m|2n}(\mathbb{C}).$$

2. Assume that $n = k_1 + l_1$: $(k_i, l_i) \neq (k_{i-1}, 0)$, $(0, l_{i-1})$, $i \geq 2$; $k_1 \geq 3$, $l_1 \geq 2$ and $\mathfrak{v}(\mathbf{F}^o_{k|l'}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$. Then

$$\mathfrak{v}(\mathbf{F}^o_{k|l}) \simeq \pi\mathfrak{sp}_n(\mathbb{C}).$$

The Lie superalgebras of holomorphic vector fields on the flag supermanifolds $\mathbf{F}_{k|l}$ and $I\mathbf{F}_{k|l}$ corresponding to $\mathfrak{gl}_{m|n}(\mathbb{C})$ and $\mathfrak{q}_{n}(\mathbb{C})$, respectively, were calculated in [V1, V2]. We obtained there the following result.
Theorem 2. A. Assume that $r > 1$ and that we have the following restrictions on the flag type:

$$(k_i, l_i) \neq (k_{i-1}, 0), (0, l_{i-1}), \ i \geq 2;$$

$$(k_{i-1}, k_i l_{i-1}, l_i) \neq (1, 0|l_{i-1}, l_{i-1} - 1), (1, 1|l_{i-1}, 1), \ i \geq 1;$$

$$(k_{i-1}, k_i l_{i-1}, l_i) \neq (k_i - 1|1, 0), (k_i, 1|1, 1), \ i \geq 1;$$

$k|l \neq (0, \ldots, 0|n, l_2, \ldots, l_r), k|l \neq (m, k_2, \ldots, k_r|0, \ldots, 0).$

Then

$$v(F_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

If $k|l = (0, \ldots, 0|n, l_2, \ldots, l_r)$ or $k|l = (m, k_2, \ldots, k_r|0, \ldots, 0)$, then

$$v(F_{k|l}) \simeq W_{mn} \oplus (\bigwedge (\xi_1, \ldots, \xi_{mn}) \otimes \mathfrak{pgl}_n(\mathbb{C})), $$

where $W_{mn} = \text{Der} \bigwedge (\xi_1, \ldots, \xi_{mn}).$

B. Assume that $r > 1$, then for any $k$

$$v(IIF_{k|l}) \simeq q_n(\mathbb{C})/\mathfrak{z}(q_n(\mathbb{C})), $$

where $\mathfrak{z}(q_n(\mathbb{C}))$ is the center of $q_n(\mathbb{C})$.

2 Examples of Lie supergroups and Lie superalgebras

A Lie supergroup is a group object in the category of supermanifolds. As in the classical Lie theory we can assign the Lie superalgebra to any Lie supergroup. For more information about Lie supergroups see for example [V5]. Further we will need a description of some classical Lie supergroups and their Lie superalgebras.

The general Lie supergroup $GL_{m|n}(\mathbb{C})$ is an open subsupermanifold in the superdomain

$$\text{Mat}_{m|n}(\mathbb{C}) = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \right\}. $$

Here we consider the elements of the matrices $X_{11} \in \text{Mat}_m(\mathbb{C})$ and $X_{22} \in \text{Mat}_n(\mathbb{C})$ as even coordinates of the superdomain $\text{Mat}_{m|n}(\mathbb{C})$ and the elements of the matrices $X_{12}, X_{21}$ as odd ones. The subsupermanifold $GL_{m|n}(\mathbb{C})$ is defined by the following equations

$$\det X_{11} \neq 0 \quad \text{and} \quad \det X_{22} \neq 0.$$
The multiplication in the Lie supergroup $GL_{m|n}(\mathbb{C})$ is given by the usual matrix multiplication.

The Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$ of $GL_{m|n}(\mathbb{C})$ has the following form:

$$\mathfrak{gl}_{m|n}(\mathbb{C}) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right\},$$

where $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ are matrices of complex numbers of size $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively. The even part $\mathfrak{gl}_{m|n}(\mathbb{C})_0$ of $\mathfrak{gl}_{m|n}(\mathbb{C})$ is determined by the equations $A_{12} = 0$, $A_{21} = 0$, and the odd part $\mathfrak{gl}_{m|n}(\mathbb{C})_1$ is given by $A_{11} = 0$, $A_{22} = 0$. The multiplication in this Lie superalgebra has the form:

$$[X,Y] = XY - (-1)^{p(X)p(Y)} YX,$$

where $X$, $Y$ are homogeneous elements in $\mathfrak{gl}_{m|n}(\mathbb{C})$ and $p(Z)$ is the parity of $Z$. The center $\mathfrak{z}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ of $\mathfrak{gl}_{m|n}(\mathbb{C})$ contains all matrices $\alpha E_{m+n}$, where $\alpha \in \mathbb{C}$ and $E_{m+n}$ is the identity matrix of size $m+n$. By definition we put $\mathfrak{pgl}_{m|n}(\mathbb{C}) := \mathfrak{gl}_{m|n}(\mathbb{C})/\mathfrak{z}(\mathfrak{gl}_{m|n}(\mathbb{C}))$.

Consider the following two classical Lie subsuperalgebras in $\mathfrak{gl}_{m|n}(\mathbb{C})$.

(1) The Lie superalgebra $\mathfrak{osp}_{m|2n}(\mathbb{C})$ is a Lie subsuperalgebra in $\mathfrak{gl}_{m|2n}(\mathbb{C})$ that annihilates a non-degenerate even symmetric bilinear form $\beta$ in $\mathbb{C}^{m|2n}$. The matrix $\Gamma$ of $\beta$ in the standard basis in $\mathbb{C}^{m|2n}$ for even and odd $m$ is given respectively by

$$\Gamma = \begin{pmatrix} 0 & E_s & 0 & 0 \\ E_s & 0 & 0 & 0 \\ 0 & 0 & 0 & E_n \\ 0 & 0 & -E_n & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & E_s & 0 & 0 \\ E_s & 0 & 0 & 0 \\ 0 & 0 & 0 & E_n \\ 0 & 0 & 0 & -E_n \end{pmatrix}. \quad (2)$$

Here $m = 2s$ or $m = 2s + 1$. Explicitly we have

$$\mathfrak{osp}_{2s|2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A_{11} & A_{12} & C_{11} & C_{12} \\ -A_{12}^T & C_{21} & C_{22} \\ -C_{22}^T & -C_{12}^T & B_{11} & B_{12} \\ C_{21}^T & C_{11}^T & B_{21} & -B_{11}^T \end{pmatrix} \right\}, \quad \begin{pmatrix} A_{21}^T = -A_{12}, \\ A_{12}^T = -A_{21}, \\ B_{12}^T = B_{12}, \\ B_{21}^T = B_{21} \end{pmatrix}, \quad (3)$$

and

$$\mathfrak{osp}_{2s+1|2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A_{11} & A_{12} & G_1 & C_{11} & C_{12} \\ A_{21} & -A_{21}^T & G_2 & C_{21} & C_{22} \\ -G_{22}^T & -G_{12}^T & 0 & G_3 & G_4 \\ -C_{22}^T & -C_{12}^T & -G_4^T & B_{11} & B_{12} \\ C_{21}^T & C_{11}^T & C_{12}^T & B_{21} & -B_{11}^T \end{pmatrix} \right\}, \quad \begin{pmatrix} A_{21} = -A_{21}, \\ A_{12} = -A_{12}, \\ B_{12} = B_{12}, \\ B_{21} = B_{21} \end{pmatrix}. \quad (4)$$
Here $A_{11}, B_{11}$ are square matrices of size $s$ and $n$, respectively. The center $\mathfrak{z}(\mathfrak{osp}_{m|2n}(\mathbb{C}))$ of $\mathfrak{osp}_{m|2n}(\mathbb{C})$ is trivial. The corresponding connected Lie supergroup we will denote by $\text{OSp}_{m|2n}(\mathbb{C})$. This is a subsupermanifold in $\text{GL}_{m|2n}(\mathbb{C})$ that is given by the following equation:

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}^{ST} \Gamma \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix} = \Gamma,
\]

where

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}^{ST} = \begin{pmatrix}
X_{11}^T & X_{21}^T \\
-X_{12}^T & X_{22}^T
\end{pmatrix},
\]

and $T$ is the usual transposition.

(2) The Lie superalgebra $\pi\mathfrak{sp}_n(\mathbb{C}) \subset \mathfrak{gl}_{n|n}(\mathbb{C})$ is a Lie subsuperalgebra in $\mathfrak{gl}_{n|n}(\mathbb{C})$ that annihilate a non-degenerate odd skew-symmetric bilinear form $\gamma$ in $\mathbb{C}^{n|n}$. The matrix $\Upsilon$ of $\gamma$ in the standard basis in $\mathbb{C}^{n|n}$ has the following form:

\[
\Upsilon = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}.
\]

Then we have

\[
\pi\mathfrak{sp}_n(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \right\}, \quad B^T = B, \quad C^T = -C,
\]

where $A$, $B$, $C$ are square matrices of size $n$. The center $\mathfrak{z}(\pi\mathfrak{sp}_n(\mathbb{C}))$ of $\pi\mathfrak{sp}_n(\mathbb{C})$ is trivial. We will denote the corresponding connected Lie supergroup by $\Pi\text{Sp}_n(\mathbb{C})$. This is a subsupermanifold in $\text{GL}_{n|n}(\mathbb{C})$ that is given by the following equation:

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}^{ST} \Upsilon \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix} = \Upsilon.
\]

3 Flag supermanifolds

An introduction to the theory of supermanifolds can be found in [BL, CCF, L, Man]. Throughout this paper we will be interested in the complex-analytic version of the theory. A complex-analytic supermanifold of dimension $p|q$ is a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a complex-analytic superdomain of dimension $p|q$, this is to a ringed space of the form $\mathcal{U} = (\mathcal{U}_0, \mathcal{F}_{\mathcal{U}_0} \otimes \mathbb{C} \wedge(q))$. Here $\mathcal{F}_{\mathcal{U}_0}$ is the sheaf of holomorphic functions on an open set $\mathcal{U}_0 \subset \mathbb{C}^p$ and $\wedge(q)$ is the Grassmann algebra with $q$ generators. We will denote a supermanifold by $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$, where $\mathcal{M}_0$ is the underlying
denote the Lie superalgebra of global holomorphic vector fields by \( V \). It is known that the Lie superalgebra \( V \) of the sheaf of holomorphic vector fields on a complex-analytic manifold and \( O_M \) is the structure sheaf of \( M \). The global sections of \( \mathcal{T} \) are a sheaf of Lie superalgebras with respect to the following multiplication

\[
[X, Y] = XY - (-1)^{p(X)p(Y)} YX.
\]

The global sections of \( \mathcal{T} \) are called holomorphic vector fields on \( M \). We will denote the Lie superalgebra of global holomorphic vector fields by \( V(M) \).

It is known that the Lie superalgebra \( V(M) \) is finite dimensional if \( M \) is compact. (A proof of this fact can be found for instance in [V1].)

\( gl_{m/n}(\mathbb{C}) \)-flag supermanifolds. We fix two sets of non-negative integers \( k = (k_0, \ldots, k_r) \) and \( l = (l_0, \ldots, l_r) \) such that (1) holds. Let us recall the definition of a flag supermanifold \( F_{k/l} \) in the superspace \( \mathbb{C}^{m|n} \), see [Man] and also [V1]. We use definitions and notations from [V1]. The only difference is that in this paper for convenience we denote by \( F_{k/l} \) the flag supermanifold of type \( k|l \), where \( k = (k_0, \ldots, k_r) \) and \( l = (l_0, \ldots, l_r) \) and we assume that \( k_0 = m \) and \( l_0 = n \). In [V1] this supermanifold was denoted by \( F_{m/n}^{k/l} \). The underlying space of the supermanifold \( F_{k/l} \) is the product \( F_k \times F_l \) of two flag manifolds of types \( k \) and \( l \), respectively.

Let us describe an atlas on \( F_{k/l} \), see [V1] for details. We fix two subsets \( I_{0} \subset \{1, \ldots, k_{s-1}\} \) and \( I_{1} \subset \{1, \ldots, l_{s-1}\} \) such that \( |I_{0}| = k_{s} \), and \( |I_{1}| = l_{s} \), where \( s = 1, \ldots, r \), and we put \( I_{s} := (I_{0}, I_{1}) \) and \( I := (I_{1}, \ldots, I_{r}) \). To any such \( I_{s} \) we assign the following \((k_{s-1} + l_{s-1}) \times (k_{s} + l_{s})\)-matrix

\[
Z_{I_{s}} = \begin{pmatrix} X_s & \Xi_s \\ H_s & Y_s \end{pmatrix}, \quad s = 1, \ldots, r. \tag{7}
\]

We assume that the matrices \( X_s = (x_{ij}^s) \) and \( Y_s = (y_{ij}^s) \) in (7) have size \((k_{s-1} \times k_s)\) and \((l_{s-1} \times l_s)\), respectively, and that \( Z_{I_{s}} \) contains the identity submatrix \( E_{k_{s-1}+l_{s-1}} \) of size \((k_s + l_s) \times (k_s + l_s)\) in the lines with numbers \( i \in I_{0} \) and \( k_{s-1} + i \), \( i \in I_{1} \). For example, if \( I_{0} = \{k_{s-1} - k_s + 1, \ldots, k_{s-1}\} \), \( I_{1} = \{l_{s-1} - l_s + 1, \ldots, l_{s-1}\} \), then the matrix \( Z_{I_{s}} \) has the following form:

\[
Z_{I_{s}} = \begin{pmatrix} X_s & \Xi_s \\ E_{k_s} & 0 \\ H_s & Y_s \\ 0 & E_{l_s} \end{pmatrix}.
\]

Here we denote by \( E_q \) the identity square matrix of size \( q \). To simplify notations we use the same letters \( X_s, Y_s, \Xi_s \) and \( H_s \) as in (7).
The matrices (7) determine the superdomain \( \mathcal{U}_I \) with even coordinates \( x^s_{ij} \) and \( y^s_{ij} \), and odd coordinates \( \xi^s_{ij} \) and \( \eta^s_{ij} \). The transition functions between two superdomains \( \mathcal{U}_I \) and \( \mathcal{U}_J \), where \( I = (I_s) \) and \( J = (J_s) \), are defined in the following way:

\[
Z_{J_1} = Z_{I_1} C^{-1}_{I_1,J_1}, \quad Z_{J_s} = C_{I_{s-1}J_{s-1}} Z_{I_s} C^{-1}_{I_s,J_s}, \quad s \geq 2.
\]

The matrix \( C_{I_1J_1} \) is an invertible submatrix in \( Z_{I_1} \) that consists of the lines with numbers \( i \in J_{00} \) and \( k_0 + i \), where \( i \in J_{11} \), and \( C_{I_sJ_s} \) is the invertible submatrix in \( C_{I_{s-1}J_{s-1}} Z_{I_s} \) that consists of the lines with numbers \( i \in J_{00} \) and \( k_{s-1} + i \), where \( i \in J_{11} \); see [V1] for details. Now the atlas on \( F_{k|l} \) is described. The supermanifold \( F_{k|l} \) is called the flag supermanifold of type \( k|l \). In case \( r = 1 \) this supermanifold is called the super-Grassmannian and sometimes it is denoted in the literature by \( \text{Gr}_{m|n,k|l} \).

Recall that we denote by \( \mathfrak{gl}_{m|n}(\mathbb{C}) \) the general Lie superalgebra of the superspace \( \mathbb{C}^{m|n} \) and by \( \text{GL}_{m|n}(\mathbb{C}) \) the connected Lie supergroup of the Lie superalgebra \( \mathfrak{gl}_{m|n}(\mathbb{C}) \). In [Man] an action of \( \text{GL}_{m|n}(\mathbb{C}) \) on \( F_{k|l} \) is defined. Let us recall this definition in our notations and in our atlas. Let

\[
L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}
\]

be a coordinate matrix of the Lie supergroup \( \text{GL}_{m|n}(\mathbb{C}) \). Then the action of \( \text{GL}_{m|n}(\mathbb{C}) \) on \( F_{k|l} \) in our coordinates is given by the following formulas:

\[
(L, (Z_{I_1}, \ldots, Z_{I_s})) \longmapsto (\tilde{Z}_{J_1}, \ldots, \tilde{Z}_{J_s}),
\]

where \( \tilde{Z}_{J_1} = LZ_{I_1} C^{-1}_{I_1,J_1} \) and \( \tilde{Z}_{J_s} = C_{s-1} Z_{I_s} C^{-1}_{I_s,J_s} \). Here \( C_1 \) is the invertible submatrix in \( LZ_{I_1} \) that consists of the lines with numbers \( i \in J_{00} \) and \( m+i \), where \( i \in J_{11} \), and \( C_s \), where \( s \geq 2 \), is the invertible submatrix in \( C_{s-1} Z_{I_s} \) that consists of the lines with numbers \( i \in J_{00} \) and \( k_{s-1} + i \), where \( i \in J_{1s} \). This Lie supergroup action induces the following Lie superalgebra homomorphism

\[
\mu : \mathfrak{gl}_{m|n}(\mathbb{C}) \rightarrow \mathfrak{v}(F_{k|l}).
\]

**osp_{m|2n}(\mathbb{C})- and \( \pi \mathfrak{sp}_{n}(\mathbb{C}) \)-flag supermanifolds.** We can assign to the Lie superalgebras \( \mathfrak{osp}_{m|2n}(\mathbb{C}) \) and \( \pi \mathfrak{sp}_{n}(\mathbb{C}) \) isotropic flag supermanifolds \( F^e_{k|l} \) and \( F^o_{k|l} \), respectively. The underlying spaces of these supermanifolds are the manifolds of isotropic flags with respect to the form \( \beta \) and \( \gamma \), see Section 2. In case \( r = 1 \) these supermanifolds are called isotropic super-Grassmannians. These supermanifolds were studied in [OS2, O1]. Let us describe the supermanifolds \( F^e_{k|l} \) and \( F^o_{k|l} \) using charts and local coordinates.
The supermanifold $F_{e\ell}$, where $l_0$ is even, is a subsupermanifold of $F_{k\ell}$ that is given in local coordinates (7) by the following equation:

$$
\left( \begin{array}{cc}
X_1 & \Xi_1 \\
H_1 & Y_1
\end{array} \right)^{ST} \Gamma \left( \begin{array}{cc}
X_1 & \Xi_1 \\
H_1 & Y_1
\end{array} \right) = 0,
$$

where $\Gamma$ is as in (2) and $ST$ is the supertransposition, see Section 2. In case $m = 2k_1$ or $m = 2k_1 + 1$ and $n = l_1$, we say that the supermanifold $F_{e\ell}$ has maximal type. The subsupermanifold $F_{o\ell}$ of the supermanifold $F_{k\ell}$ is given in the local coordinates (7) by the following equation:

$$
\left( \begin{array}{cc}
X_1 & \Xi_1 \\
H_1 & Y_1
\end{array} \right)^{ST} \Upsilon \left( \begin{array}{cc}
X_1 & \Xi_1 \\
H_1 & Y_1
\end{array} \right) = 0,
$$

where $\Upsilon$ is as (5). In case $n = k_1 + l_1$, we say that the supermanifold $F_{o\ell}$ has maximal type.

There are transitive actions $\mu_e$ and $\mu_o$ of the Lie supergroups $OSp_{m|2n}(\mathbb{C})$ and $IIIsp_n(\mathbb{C})$ on $F_{e\ell}$ or $F_{o\ell}$, respectively. It is given by (9), if we replace $L$ by a coordinate matrix of the Lie supergroup $OSp_{m|2n}(\mathbb{C})$ or $IIIsp_n(\mathbb{C})$, respectively. This action induces the homomorphism of Lie superalgebras

$$
\mu_e : osp_{m|2n}(\mathbb{C}) \to \mathfrak{v}(F_{e\ell}) \quad \text{and} \quad \mu_o : IIIsp_n(\mathbb{C}) \to \mathfrak{v}(F_{o\ell}).
$$

## 4 Vector fields on a superbundle

For computation of the Lie superalgebra of holomorphic vector fields on isotropic flag supermanifolds we will use the following fact. For $r > 1$ the isotropic flag supermanifold $F_{e\ell}$ is a superbundle with the base space that is isomorphic to the isotropic super-Grassmannian $F_{k_0,k_1|l_0,l_1}$ and with the fiber that is isomorphic to $F_{k'|l'}$. Similarly, for $r > 1$ the isotropic flag supermanifold $F_{o\ell}$ is a superbundle with the base spaces $F_{k_0,k_1|l_0,l_1}$ and the fiber $F_{k'|l'}$. (A similar statement holds for $F_{k\ell}$, see [V1] for details.) In local coordinates that we introduced the bundle projection say $\pi$ is given by $(Z_1, Z_2, \ldots, Z_n) \mapsto (Z_1)$. Moreover, from Formulas (9) we can deduce that the projection $\pi$ is equivariant with respect to the action of the Lie supergroups $OSp_{m|2n}(\mathbb{C})$ and $IIIsp_n(\mathbb{C})$ on $F_{e\ell}$ and $F_{o\ell}$, respectively.

We will need some facts about vector fields on superbundles.

**Definition 1.** Let $\pi = (\pi_0, \pi^*) : \mathcal{M} \to \mathcal{N}$ be a morphism of holomorphic supermanifolds. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called **projectible** with respect to $\pi$, if there exists a vector field $v_1 \in \mathfrak{v}(\mathcal{N})$ such that $\pi^*(v_1(f)) = v(\pi^*(f))$ for all $f \in \mathcal{O}_\mathcal{N}$. In this case we say that $v$ is **projected** to $v_1$. 
Denote the Lie superalgebra of projectible vector fields by $\mathfrak{v}(\mathcal{M})$. Assume that $\pi = (\pi_0, \pi^*) : \mathcal{M} \to \mathcal{N}$ is the projection of a holomorphic superbundle. Then the homomorphism $\pi^* : \mathcal{O}_\mathcal{N} \to \pi_*(\mathcal{O}_\mathcal{M})$ is injective and any projectible vector field $v$ is projected into a unique vector field $v_1 = \mathcal{P}(v)$. Hence we have the following homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{N}), \quad v \mapsto v_1.$$  

A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called vertical, if $\mathcal{P}(v) = 0$. The Lie subalgebra $\text{Ker} \mathcal{P}$ is an ideal in $\mathfrak{v}(\mathcal{M})$.

If $S$ is a supermanifold, then the global sections $\mathcal{O}_S(S_0)$ of the structure sheaf $\mathcal{O}_S$ are also called holomorphic functions on $S$.

**Proposition 1.** [B] Let $p : \mathcal{M} \to \mathcal{B}$ be the projection of a superbundle with fiber $S = (S_0, \mathcal{O}_S)$. Assume that $\mathcal{O}_S(S_0) = \mathbb{C}$, then any global holomorphic vector field on $\mathcal{M}$ is projectible.

Let $\pi : \mathcal{M} \to \mathcal{B}$ be a superbundle with the fiber $S$. We denote by $\mathcal{W}$ the sheaf of vertical holomorphic vector fields. It is a sheaf of the base space $\mathcal{B}_0$ of $\mathcal{B}$. More precisely, to any open set $U \subset \mathcal{B}_0$ we assign the set of all vertical vector fields on the supermanifold $(\pi_0^{-1}(U), \mathcal{O}_\mathcal{M})$.

**Proposition 2.** [V6] Assume that the base space $S_0$ of the fiber $S$ is compact. Then $\mathcal{W}$ is a locally free sheaf of $\mathcal{O}_\mathcal{B}$-modules and $\dim \mathcal{W} = \dim \mathfrak{v}(S)$.

By definition we have $\mathcal{W}(\mathcal{B}_0) = \text{Ker} \mathcal{P}$. In [V1] we described the corresponding to $\mathcal{W}$ graded sheaf $\tilde{\mathcal{W}}$. It is defined in the following way:

$$\tilde{\mathcal{W}} = \bigoplus_{p \geq 0} \tilde{\mathcal{W}}_p, \quad \tilde{\mathcal{W}}_p = \mathcal{W}_p / \mathcal{W}_{p+1}. \quad (12)$$

Here $\mathcal{W}_p = \mathcal{J}^p\mathcal{W}$ and $\mathcal{J}$ is the sheaf of ideals in $\mathcal{O}_\mathcal{B}$ generated by odd elements. Clearly, $\tilde{\mathcal{W}}$ is the $\mathbb{Z}$-graded sheaf of $\mathcal{F}_{\mathcal{B}_0}$-modules, where $\mathcal{F}_{\mathcal{B}_0}$ is the structure sheaf of the underlying space $\mathcal{B}_0$. By Proposition 2 we get the following result.

**Proposition 3.** Assume that the base space $S_0$ of the fiber $S$ is compact. Then $\tilde{\mathcal{W}}_0$ is a locally free sheaf of $\mathcal{F}_{\mathcal{B}_0}$-modules and any fiber of the corresponding vector bundle is isomorphic to $\mathfrak{v}(S)$.

Denote by $\mathcal{W}_0$ the vector bundle corresponding to the locally free sheaf $\tilde{\mathcal{W}}_0$. To calculate the Lie superalgebra of holomorphic vector fields on isotropic flag supermanifolds we will use Proposition 1 and the following result.

**Theorem 3.** [V3] Consider the flag supermanifold $\mathcal{M} = \mathcal{F}_{k|l}$. Assume that

$$(k|l) \neq (m, \ldots, m, k_{s+2}, \ldots, k_r|l_1, \ldots, l_s, 0, \ldots, 0),$$

$$(k|l) \neq (k_1, \ldots, k_s, 0, \ldots, 0|n, \ldots, n, l_{s+2}, \ldots, l_r), \quad (13)$$
for any \( s \geq 0 \). Then \( \mathcal{O}_M(\mathcal{M}_0) = \mathbb{C} \). Otherwise \( \mathcal{O}_M(\mathcal{M}_0) = \wedge(mn) \), where \( \wedge(mn) \) is the Grassmann algebra with \( mn \) generators. \( \square \)

5 The Borel-Weil-Bott Theorem

Further we will use the Borel-Weil-Bott Theorem to compute the vector space of global sections of locally free sheaves. Details about the Borel-Weil-Bott Theorem can be found for example in [A]. Recall that this theorem is used to compute cohomology groups with values in a homogeneous holomorphic bundle over a product of classical or isotropic flag manifolds. For completeness we formulate this theorem here adapting to our notations. First of all we need to describe the underlying space of an isotropic super-Grassmannian of maximal type.

Consider the isotropic super-Grassmannian of maximal type \( F_{e,2s,s}^{c,2|2n,n} \), where \( m = 2s \) or \( 2s + 1 \). Denote by \( G_{2s}^e \cong \text{SO}_{2s}(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C}) \) the underlying space of the Lie supergroup \( \text{OSp}_{2s|2n}(\mathbb{C}) \) and by \( P_{2s}^e \) the parabolic subgroup in \( G_{2s}^e \) that contains all matrices in the following form:

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 \\
C_1 & (A_1^T)^{-1} & 0 & 0 \\
0 & 0 & A_2 & 0 \\
0 & 0 & C_2 & (A_2^T)^{-1}
\end{pmatrix},
\]

where \( A_1 \in \text{GL}_s(\mathbb{C}) \) and \( A_2 \in \text{GL}_n(\mathbb{C}) \). We also denote by \( R_{2s}^e \) the reductive part of \( P_{2s}^e \). Clearly, \( R_{2s}^e \cong \text{GL}_s(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \). The underlying manifold of \( F_{2s,s}^{c,2|2n,n} \) is isomorphic to \( G_{2s}^e/P_{2s}^e \). We see that it is a product of two isotropic Grassmannians.

Further, the underlying manifold of \( F_{2s+1,s}^{c,2|2n,n} \) is isomorphic to the homogeneous space \( G_{2s+1}^e/P_{2s+1}^e \), where \( G_{2s+1}^e \cong \text{SO}_{2s+1}(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C}) \) is the underlying space of \( \text{OSp}_{2s+1|2n}(\mathbb{C})_{\text{red}} \) and \( P_{2s+1}^e \) is the parabolic subgroup in \( G_{2s+1}^e \) that contains all matrices in the following form:

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 & 0 \\
C_1 & (A_1^T)^{-1} & G & 0 & 0 \\
H & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & A_2 & 0 \\
0 & 0 & 0 & C_2 & (A_2^T)^{-1}
\end{pmatrix}.
\]

Here \( A_1 \in \text{GL}_s(\mathbb{C}) \) and \( A_2 \in \text{GL}_n(\mathbb{C}) \). The reductive part \( R_{2s+1}^e \) of \( P_{2s+1}^e \) has the form \( R_{2s+1}^e \cong \text{GL}_s(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \).
πsp\(_n\)(C)-super-Grassmannians of maximal type. The underlying manifold of the isotropic super-Grassmannian of maximal type \(F^o_{n,s|n,t}\), where \(s + t = n\), is the usual Grassmannian \(F\), see [Man, OS3]. It is isomorphic to \(G^o/P^o\), where \(G^o \simeq \text{GL}_n(C)\) is the underlying space of \(\Pi \text{Sp}_n(C)\) and \(P^o\) is the parabolic subgroup in \(G^o\) that contains all matrices in the following form:

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 \\
C_1 & B_1 & 0 & 0 \\
0 & 0 & (A_1^T)^{-1} & D_1 \\
0 & 0 & 0 & (B_1^T)^{-1}
\end{pmatrix}, \tag{16}
\]

where \(A_1 \in \text{GL}_t(C)\) and \(B_1 \in \text{GL}_s(C)\). The reductive part of \(P^o\) has the form \(R^o \simeq \text{GL}_t(C) \times \text{GL}_s(C)\).

To use the Borel-Weil-Bott Theorem we need to fix Cartan subalgebras and root systems.

**Cartan subalgebras and root systems.** In the Lie algebra \(\text{osp}_{2n|2n}(C)\) we fix the following Cartan subalgebra:

\[
t(\text{osp}_{2n|2n}(C)) = t_0(\text{so}_{2s}(C)) \oplus t_1(\text{sp}_{2t}(C)),
\]

where

\[
t_0(\text{so}_{2s}(C)) = \{\text{diag}(\mu_1, \ldots, \mu_s, -\mu_1, \ldots, -\mu_s)\},
\]

\[
t_1(\text{sp}_{2t}(C)) = \{\text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)\}.
\]

We fix the following system of positive roots:

\[
\Delta^+(\text{osp}_{2n|2n}(C)) = \Delta_1^+(\text{so}_{2s}(C)) \cup \Delta_2^+(\text{sp}_{2t}(C)),
\]

where

\[
\Delta_1^+(\text{so}_{2s}(C)) = \{\mu_i - \mu_j, \mu_i + \mu_j, \, i < j\},
\]

\[
\Delta_2^+(\text{sp}_{2t}(C)) = \{\lambda_p - \lambda_q, \, p < q, \lambda_p + \lambda_q, \, p \leq q\},
\]

and the following system of simple roots

\[
\Phi(\text{osp}_{2n|2n}(C)) = \Phi_1(\text{so}_{2s}(C)) \cup \Phi_2(\text{sp}_{2t}(C)),
\]

where

\[
\Phi_1(\text{so}_{2s}(C)) = \{\alpha_1, \ldots, \alpha_s\}, \quad \alpha_i = \mu_i - \mu_{i+1}, \, i = 1, \ldots, s-1, \alpha_s = \mu_{s-1} + \mu_s,
\]

\[
\Phi_2(\text{sp}_{2t}(C)) = \{\beta_1, \ldots, \beta_n\}, \quad \beta_j = \lambda_j - \lambda_{j+1}, \, j = 1, \ldots, n-1, \beta_n = 2\lambda_n.
\]

In \(\text{osp}_{2s+1|2n}(C) = \text{so}_{2s+1}(C) \oplus \text{sp}_{2n}(C)\) we fix the following Cartan subalgebra:

\[
t(\text{osp}_{2s+1|2n}(C)) = t_0(\text{so}_{2s+1}(C)) \oplus t_1(\text{sp}_{2n}(C)),
\]
where
\[ t_0(\mathfrak{so}_{2s+1}(\mathbb{C})) = \{\text{diag}(\mu_1, \ldots, \mu_s, 0, -\mu_1, \ldots, -\mu_s)\}, \]
\[ t_1(\mathfrak{sp}_{2n}(\mathbb{C})) = \{\text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)\}. \]

We fix the following system of positive roots:
\[ \Delta^+(\mathfrak{osp}_{2s+1|2n}(\mathbb{C})_0) = \Delta^+_1(\mathfrak{so}_{2s+1}(\mathbb{C})) \cup \Delta^+_2(\mathfrak{sp}_{2n}(\mathbb{C})), \]
where
\[ \Delta^+_1(\mathfrak{so}_{2s+1}(\mathbb{C})_0) = \{\mu_i - \mu_j, \mu_i + \mu_j, i < j, \mu_i, \}, \]
\[ \Delta^+_2(\mathfrak{sp}_{2n}(\mathbb{C})) = \{\lambda_p - \lambda_q, p < q, \lambda_p + \lambda_q, p \leq q\}, \]
and the following system of simple roots
\[ \Phi(\mathfrak{osp}_{2s+1|2n}(\mathbb{C})_0) = \Phi_1(\mathfrak{so}_{2s+1}(\mathbb{C})) \cup \Phi_2(\mathfrak{sp}_{2n}(\mathbb{C})), \]
where
\[ \Phi_1(\mathfrak{osp}_{2s+1}(\mathbb{C})) = \{\alpha_1, \ldots, \alpha_s\}, \alpha_i = \mu_i - \mu_{i+1}, i = 1, \ldots, s - 1, \alpha_s = \mu_s, \]
\[ \Phi_2(\mathfrak{sp}_{2n}(\mathbb{C})) = \{\beta_1, \ldots, \beta_n\}, \beta_j = \lambda_j - \lambda_{j+1}, j = 1, \ldots, n - 1, \beta_n = 2\lambda_n. \]

In \( \pi\mathfrak{sp}_n(\mathbb{C})_0 = \mathfrak{gl}_n(\mathbb{C}) \) we fix the following Cartan subalgebra
\[ t(\pi\mathfrak{sp}_n(\mathbb{C})_0) = \{\text{diag}(\mu_1, \ldots, \mu_n)\}, \]
the following system of positive roots:
\[ \Delta^+(\pi\mathfrak{sp}_n(\mathbb{C})_0) = \{\mu_i - \mu_j, i < j\} \]
and the following system of simple roots
\[ \Phi(\pi\mathfrak{sp}_n(\mathbb{C})_0) = \{\alpha_1, \ldots, \alpha_{n-1}\}, \alpha_i = \mu_i - \mu_{i+1}. \]

Let \( \mathfrak{g} = \mathfrak{osp}_{m|2n}(\mathbb{C}) \) or \( \pi\mathfrak{sp}_n(\mathbb{C}) \). Denote by \( t(\mathfrak{g}_0)^\ast(\mathbb{R}) \) the real subspace in \( t(\mathfrak{g}_0)^\ast \) spanned by \( (\mu_j, \lambda_i) \) or by \( (\mu_j) \). Consider in \( t(\mathfrak{g}_0)^\ast(\mathbb{R}) \) the scalar product \( (, ) \) such that the vectors \( (\mu_j, \lambda_i) \) or \( (\mu_j) \) form an orthonormal basis. An element \( \gamma \in t(\mathfrak{g}_0)^\ast(\mathbb{R}) \) is called \textit{dominant} if \( (\gamma, \alpha) \geq 0 \) for all positive roots \( \alpha \in \Delta^+(\mathfrak{g}_0) \).

Let \( G = G^\circ_m \), where \( m = 2s \) or \( 2s + 1 \), or \( G^\circ \). In other words, \( G = \text{OSp}_{m|2n}(\mathbb{C})_{\text{red}} \), where \( m = 2s \) or \( 2s + 1 \), or \( \text{IISp}_n(\mathbb{C})_{\text{red}} \). We also put \( P = P^\circ_m \), where \( m = 2s \) or \( 2s + 1 \), or \( P^\circ \), and \( R = R^\circ_m \), where \( m = 2s \) or \( 2s + 1 \), or \( R^\circ \). Let \( E_\varphi \rightarrow M = G/P \) be a homogeneous holomorphic vector bundle
corresponding to a representation $\varphi$ of $P$ in the fiber $E = (E_\varphi)_P$ of $E_\varphi$ at the point $P$. (An introduction to the theory of homogeneous holomorphic vector bundles can be found in [A].) We denote by $E_\varphi$ the sheaf of holomorphic sections of $E_\varphi$.

**Theorem 4.** [Borel-Weil-Bott]. Assume that the representation $\varphi : P \to \text{GL}(E)$ is completely reducible and $\lambda_1, ..., \lambda_s$ are highest weights of $\varphi|_R$. Then the $G$-module $H^0(G/P, E_\varphi)$ is isomorphic to the sum of irreducible $G$-modules with highest weights $\lambda_{i_1}, ..., \lambda_{i_t}$, where $\lambda_{i_a}$ are dominant highest weights.

## 6 Vector fields on super-Grassmannians

Let us repeat briefly main definitions from the theory of homogeneous supermanifolds. An action of a Lie supergroup $\mathcal{G}$ on a supermanifold $M$ is called *transitive* if the underlying action of the Lie group $\mathcal{G}_0$ is transitive on the underlying space $M_0$ of $M$ and the corresponding action of the Lie superalgebra is also transitive, see [V3] for details. A supermanifold $M$ is called *homogeneous* if it possesses a transitive action of a certain Lie supergroup.

From Formulas (9) we can deduced that the flag supermanifolds $F_{k|l}$, $F^o_{k|l}$ and $F^e_{k|l}$ are homogeneous with respect to the action (9) of the Lie supergroups $\mathcal{G} = \text{GL}_m(n)(\mathbb{C})$, $\text{OSp}_{m|2n}(\mathbb{C})$ and $\text{IISp}_n(\mathbb{C})$, respectively. Indeed, the underlying action of $\mathcal{G}_0$ is transitive, because it is just the standard action of $\mathcal{G}_0$ on a (isotropic) flag manifold. The corresponding action of the Lie superalgebra is also transitive. For instance this can be explicitly verified using our local coordinates, see also [V3]. Further, the actions of Lie supergroups $\text{OSp}_{m|2n}(\mathbb{C})$ and $\text{IISp}_n(\mathbb{C})$ induce the following Lie superalgebra actions

$$\mu^e : \text{osp}_{m|2n}(\mathbb{C}) \to \mathfrak{v}(F^e_{k|l}) \quad \text{and} \quad \mu^o : \pi\text{sp}_n(\mathbb{C}) \to \mathfrak{v}(F^o_{k|l}),$$

respectively. We will prove that with some exceptional cases $\mu^e$ and $\mu^o$ are isomorphisms.

The Lie superalgebras of holomorphic vector fields on isotropic super-Grassmannians corresponding to $\text{OSp}_{m|2n}(\mathbb{C})$ and $\text{IISp}_n(\mathbb{C})$ were calculated in [OS2, OS3].

**Theorem 5.** Let $r = 1$.

1. Assume that $m = 2k_1$ and $n = 2l_1$, i.e the super-Grassmannian $F^e_{m,k_1|n,l_1}$ is of maximal type. If $k_1 \geq 1$ and $l_1 \geq 1$, then the homomorphism

$$\mu^e : \text{osp}_{m|n}(\mathbb{C}) \to \mathfrak{v}(F^e_{m,k_1|n,l_1})$$
is in fact an isomorphism. Moreover, if \( k_1 \geq 2 \) and \( l_1 \geq 1 \) the super-Grassmannian \( F_{m-1,k_1-1|n,l_1}^e \) is isomorphic to a connected component of the super-Grassmannian \( F_{m,k_1|n,l_1}^e \).

2. Assume that \( n = k_1 + l_1 \), i.e. the super-Grassmannian \( F_{n,k_1|n,l_1}^o \) is of maximal type. If \( k_1 \geq 3 \), \( l_1 \geq 2 \), then the homomorphism

\[
\mu^o : \pi \mathfrak{sp}_n(\mathbb{C}) \to \mathfrak{v}(F_{n,k_1|n,l_1}^o).
\]

is in fact an isomorphism.

The Lie superalgebra of holomorphic vector fields on the super-Grassmannian \( F_{m,k_1|n,l_1} \) corresponding to \( \mathfrak{gl}_{m|n}(\mathbb{C}) \) was computed in [Bun, OS1, O2, Ser], see also [V1]. The case of flag supermanifolds corresponding to \( \mathfrak{gl}_{m|n}(\mathbb{C}) \) was studied in [V1]. The case of super-Grassmannian corresponding to \( \mathfrak{q}_n(\mathbb{C}) \) can be found in [O1], and the case of flag supermanifolds of this type can be found in [V2].

7 Vector fields on isotropic flag supermanifolds

Assume that \( r > 1 \). From now on we consider isotropic supermanifolds of maximal type. As we have seen in Section 4 the isotropic flag supermanifold \( M^e := F_{k|l}^e \) (or \( M^o := F_{k|l}^o \)) is a superbundle. We denote by \( B^e \) and \( S^e \) (or by \( B^o \) and \( S^o \)) its base space and its fiber, respectively. In other words we put

\[
M^e = F_{k|l}^e, \quad B^e = F_{k_0,k_1|l_0,l_1}^e \quad \text{and} \quad S^e = F_{k'|l'}
\]

for \( \mathfrak{osp}_{2m|2n}(\mathbb{C}) \)-flag supermanifolds, and

\[
M^o = F_{k|l}^o, \quad B^o = F_{k_0,k_1|l_0,l_1}^o \quad \text{and} \quad S^o = F_{k'|l'}
\]

for \( \pi \mathfrak{sp}_n(\mathbb{C}) \)-flag supermanifolds. Note that in both cases the fiber is a usual \( \mathfrak{gl}_{k_1|l_1}(\mathbb{C}) \)-flag supermanifold.

7.1 Plan of the proof

Consider the case of \( \mathfrak{osp}_{2m|2n}(\mathbb{C}) \)-flag supermanifolds. Assume that we do not have non-constant holomorphic functions on the fiber \( S^e \), i.e. \( O_{S^e}(S^e_0) = \mathbb{C} \). Then by Proposition 1, the projection \( M^e \to B^e \) determines the homomorphism of Lie superalgebras \( \mathfrak{p}^e : \mathfrak{v}(M^e) \to \mathfrak{v}(B^e) \). This projection is \( \mathfrak{OSP}_{2m|2n}(\mathbb{C}) \)-equivariant. Hence for the Lie superalgebra homomorphisms

\[
\mu^e : \mathfrak{osp}_{2m|2n}(\mathbb{C}) \to \mathfrak{v}(M^e) \quad \text{and} \quad \mu_B^e : \mathfrak{osp}_{2m|2n}(\mathbb{C}) \to \mathfrak{v}(B^e)
\]
we have $\mu^e_{\mathcal{B}} = \mathcal{P}^e \circ \mu^e$. Assuming conditions of Theorem 5, the homomorphisms $\mu^e_{\mathcal{B}}$ and hence the homomorphism $\mathcal{P}^e$ is surjective. In case if $\mathcal{P}^e$ is also injective we have

$$\mu^e = (\mathcal{P}^e)^{-1} \circ \mu^e_{\mathcal{B}}$$

(17)
is surjective and therefore $\nu(\mathcal{M}^e) \simeq \mathfrak{osp}_{2m|2n}(\mathbb{C})$. Hence our goal is to prove that $\mathcal{P}^e$ is injective. For $\pi\mathfrak{sp}_n(\mathbb{C})$-flag supermanifolds the idea is similar.

In the case of $\mathfrak{osp}_{2m-1|2n}(\mathbb{C})$-flag supermanifolds a similar argument does not work since $\mu^e_{\mathcal{B}} : \mathfrak{osp}_{2m-1|2n}(\mathbb{C}) \to \nu(\mathcal{B}^e)$ is not surjective, see Theorem 5. This case we will consider in a separate paper.

7.2 Vector bundles $W^e_0$ and $W^o_0$

In Section 4 we defined the locally free sheaf $\tilde{W}_0$ for any superbundle $\mathcal{M}$. Denote by $\tilde{W}^e_0$ and $\tilde{W}^o_0$ the locally free sheaves corresponding to the superbundles $\mathcal{F}^e_{k,l}$ and $\mathcal{F}^o_{k,l}$, respectively. We denote also by $W^e_0$ and $W^o_0$ the corresponding to $\tilde{W}^e_0$ and $\tilde{W}^o_0$ vector bundles over $\mathcal{B}^e_0 = G^e_m/F^e_m$ and $\mathcal{B}^o_0 = G^o/P^o$, respectively. Our goal now is to compute the vector space of global sections of $W^e_0$ and $W^o_0$ using Theorem 4. Note that the vector bundles $W^e_0$ and $W^o_0$ are homogeneous because the sheaves $\tilde{W}^e_0$ and $\tilde{W}^o_0$ possess the natural actions of the Lie groups $\text{OSp}_{m|2n}(\mathbb{C})_{\text{red}}$ and $\Pi\mathfrak{sp}_n(\mathbb{C})_{\text{red}}$, respectively. Let us compute the corresponding to $W^e_0$ and $W^o_0$ representations of $P^e_m$ and $P^o$.

7.3 Representations of $P^e_{2m}$ and $P^o$

Consider the local chart on the super-Grassmannian $\mathcal{B}^e$ of maximal type corresponding to

$$I^e_{10} = \{m - k_1 + 1, \ldots, m\} \quad \text{and} \quad I^e_{11} = \{n - l_1 + 1, \ldots, n\},$$

(18)

where $m = 2k_1$ and $n = 2l_1$, and the local chart on the super-Grassmannian $\mathcal{B}^o$ of maximal type corresponding to

$$I^o_{10} = \{n - k_1 + 1, \ldots, m\} \quad \text{and} \quad I^o_{11} = \{1, \ldots, l_1\},$$

(19)

where $k_1 + l_1 = n$. We put $I^e_i = (I^e_{10}, I^e_{11})$ and $I^o_i = (I^o_{10}, I^o_{11})$. The coordinate matrices $Z^e_{I^e_i}$ and $Z^o_{I^o_i}$ in this cases have the following form, respectively.

1. $Z^e_{I^e_i} = \begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ -\Xi_1 & Y_1 \\ 0 & E_{l_1} \end{pmatrix}$, 2. $Z^o_{I^o_i} = \begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ 0 & E_{l_1} \\ H_1 & -X_1 \end{pmatrix}$

(20)
1. Here $X_1, Y_1$ are matrices of size $k_1 \times k_1$ and $l_1 \times l_1$, respectively, that contain even coordinates, and $\Xi_1$ is a matrix of size $k_1 \times l_1$, that contains odd coordinates. Moreover we have $X_1^t = -X_1$ and $Y_1^t = Y_1$. 2. $X_1$ is a matrix of size $l_1 \times k_1$ that contains even coordinates, and $\Xi_1, H_1$ are matrices of size $l_1 \times l_1$ and $k_1 \times k_1$, respectively, that contain odd coordinates. Moreover we have $\Xi_1^t = \Xi_1, H_1^t = -H_1$.

Denote by $x^e$ the point in $\mathcal{B}_0^e$ defined by the following equations

$$X_1 = 0, \quad Y_1 = 0, \quad \Xi_1 = 0,$$

and by $x^o$ the point in $\mathcal{B}_0^o$ defined by the following equations

$$X_1 = 0, \quad \Xi_1 = 0, \quad H_1 = 0.$$

It is easy to see that the Lie groups $P_{2m}^e$ and $P^o$, see (14) and (16), are stabilizers of $x^e$ and $x^o$, respectively. Recall that we denoted by $R_{2m}^e$ and $R^o$ the reductive parts of $P_{2m}^e$ and $P^o$, respectively.

Let us compute the representation $\psi^e$ and $\psi^o$ of $P_{2m}^e$ and $P^o$ in the fibers $(W_0^e)_{P_{2m}^e}$ and $(W_0^o)_{P^o}$, respectively. We identify $(W_0^e)_{P_{2m}^e}$ (or $(W_0^o)_{P^o}$) with the Lie superalgebra of holomorphic vector fields $v(S^e)$ (or $v(S^o)$), see Proposition 3. Let us choose an atlas on $\mathcal{M}^e$ and $\mathcal{M}^o$ defined by $I_t^e$ and $I_t^o$, as above, and by certain $I_s^e, s = 2, \ldots, r$ and $I_s^o, s = 2, \ldots, r$. In notations (20) the Lie groups $P_{2m}^e$ and $P^o$ act at $x^e$ and $x^o$ in the chart on the super-Grassmannians $\mathcal{B}^e$ and $\mathcal{B}^o$ defined by $Z_{I_t^e}$ and $Z_{I_t^o}$ in the following way:

$$1. \begin{pmatrix} A_1 & 0 & 0 & 0 \\ C_1 & (A_1^T)^{-1} & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & C_2 & (A_2^T)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & E_{k_1} & 0 \\ 0 & 0 & 0 & E_{l_1} \\ E_{k_1} & 0 & 0 & 0 \\ 0 & 0 & E_{l_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & (A_1^T)^{-1} & 0 \\ 0 & 0 & 0 & (A_2^T)^{-1} \end{pmatrix}.$$

$$2. \begin{pmatrix} A_1 & 0 & 0 & 0 \\ C_1 & B_1 & 0 & 0 \\ 0 & 0 & (A_1^T)^{-1} & D_1 \\ 0 & 0 & (B_1^T)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & E_{k_1} & 0 \\ 0 & 0 & 0 & E_{l_1} \\ E_{k_1} & 0 & 0 & 0 \\ 0 & 0 & E_{l_1} & 0 \end{pmatrix} = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & (A_1^{-1})^T & 0 & 0 \end{pmatrix}.$$

(Note that a chart on $\mathcal{B}_e$ is defined by $Z_{I_t^e}$, and a chart on the whole flag supermanifold $\mathcal{M}^e$ is defined by $Z_{I_t^e}$, where $s = 1, \ldots, r$. The same holds for $\mathcal{M}^o$.) Further, for $Z_{I_t^e}$ and $Z_{I_t^o}$, we have

$$1. \begin{pmatrix} (A_1^T)^{-1} & 0 \\ 0 & (A_2^T)^{-1} \end{pmatrix} \begin{pmatrix} X_2 & \Xi_2 \\ H_2 & Y_2 \end{pmatrix} = \begin{pmatrix} (A_1^T)^{-1}X_2 & (A_1^T)^{-1}\Xi_2 \\ (A_2^T)^{-1}H_2 & (A_2^T)^{-1}Y_2 \end{pmatrix},$$

$$2. \begin{pmatrix} B_1 & 0 \\ 0 & (A_1^T)^{-1} \end{pmatrix} \begin{pmatrix} X_2 & \Xi_2 \\ H_2 & Y_2 \end{pmatrix} = \begin{pmatrix} B_1X_2 & B_1\Xi_2 \\ (A_1^T)^{-1}H_2 & (A_1^T)^{-1}Y_2 \end{pmatrix}. \quad (21)$$
Note that the local coordinates of $Z_{l^2_s}$, $s \geq 2$, (or $Z_{l^2_1}$, $s \geq 2$) can be interpreted as local coordinates on the fiber $S^e$ (or $S^o$) of the superbundle $\mathcal{M}^e$ (or $\mathcal{M}^o$). Hence to obtain the actions of $P_{2m}^e$ and $P^o$ we use (21) and modify $Z_{l^2_s}$ and $Z_{I^2_1}$, $s \geq 3$, accordingly. We see that the nilradicals of $P_{2m}^e$ and $P^o$ act trivially on $S^e$ and $S^o$. Further, the action of the reductive parts $R_{2m}^e$ and $R^o$ coincide with the restriction

1. of $R_{2m}^e$ on the subgroup $\text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C})$ that acts on $S^e$ as the composition of the standard action of $\text{GL}_{k_1l_1}(\mathbb{C})_{\text{red}}$, see (9), and

\[ \text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C}) \to \text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C}), \ (A_1, A_2) \mapsto ((A_1^T)^{-1}, (A_2^T)^{-1}); \]

2. of $R^o$ on the subgroup $\text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C})$ that acts on $S^o$ as a composition of the standard action of $\text{GL}_{k_1l_1}(\mathbb{C})_{\text{red}}$, see (9), and

\[ \text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C}) \to \text{GL}_{k_1}(\mathbb{C}) \times \text{GL}_{l_1}(\mathbb{C}), \ (B_1, A_1) \mapsto (B_1, (A_1^T)^{-1}). \]

Assume that

\[ \mathfrak{v}(S^e) \simeq \mathfrak{v}(S^o) \simeq \mathfrak{gl}_{k_1l_1}(\mathbb{C})/(E_{k_1+l_1}) = \left\{ \begin{pmatrix} Z_1 & T_1 \\ T_2 & Z_2 \end{pmatrix} + <E_{k_1+l_1}> \right\}, \]

where $Z_1 \in \mathfrak{gl}_{k_1}(\mathbb{C})$ and $Z_2 \in \mathfrak{gl}_{l_1}(\mathbb{C})$. Then

1. the representation $\psi^e$ of $R_{2m}^e$ on $\mathfrak{v}(S^e)$ is determined by

\[ \begin{pmatrix} (A_1^T)^{-1} & 0 \\ 0 & (A_2^T)^{-1} \end{pmatrix} \left( \begin{pmatrix} Z_1 & T_1 \\ T_2 & Z_2 \end{pmatrix} + <E_{k_1+l_1}> \right) \begin{pmatrix} A_1^T & 0 \\ 0 & A_2^T \end{pmatrix} = \left( \begin{pmatrix} (A_1^T)^{-1}Z_1A_1^T & (A_1^T)^{-1}T_1A_1^T \\ (A_2^T)^{-1}T_1A_1^T & (A_2^T)^{-1}Z_2A_2^T \end{pmatrix} + <E_{k_1+l_1}> \right); \]

where $A_1 \in \text{GL}_{k_1}(\mathbb{C})$, $A_2 \in \text{GL}_{l_1}(\mathbb{C})$;

2. the representation $\psi^o$ of $R^o$ on $\mathfrak{v}(S^o)$ is determined by

\[ \begin{pmatrix} B_1 & 0 \\ 0 & (A_1^T)^{-1} \end{pmatrix} \left( \begin{pmatrix} Z_1 & T_1 \\ T_2 & Z_2 \end{pmatrix} + <E_{k_1+l_1}> \right) \begin{pmatrix} B_1^{-1} & 0 \\ 0 & A_1^T \end{pmatrix} = \left( \begin{pmatrix} B_1Z_1B_1^{-1} & B_1T_1A_1^T \\ (A_1^T)^{-1}T_1B_1^{-1} & (A_1^T)^{-1}Z_2A_1^T \end{pmatrix} + <E_{k_1+l_1}> \right), \]

where $B_1 \in \text{GL}_{k_1}(\mathbb{C})$, $A_1 \in \text{GL}_{l_1}(\mathbb{C})$. 

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Denote by $\rho_1$ and $\rho_2$ the standard representations of $GL_{k_1}(C)$ and $GL_{l_1}(C)$ on $C^{k_1}$ and $C^{l_1}$, respectively, and by $Ad_1$ and $Ad_2$ the adjoint representations of $GL_{k_1}(C)$ and $GL_{l_1}(C)$ on $sl_{k_1}(C)$ and $sl_{l_1}(C)$, respectively. We also denote by 1 the one dimensional trivial representation of $GL_{k_1}(C) \times GL_{l_1}(C)$. We have proved the following lemma.

**Lemma 1.** The representations $\psi^e$ and $\psi^o$ of $P^e_{2m}$ and $P^o$ in the fibers $(W^e_0)_{P^e_{2m}}$ and $(W^o_0)_{P^o}$, respectively, are completely reducible. If $v(S) \simeq gl_{k_1+l_1}(C)/(E_{k_1+l_1})$, then

1.

$$\psi^e|_{R^e_{2m}} = \begin{cases} Ad_1 + Ad_2 + \rho_1 \otimes \rho_2 + \rho_1 \otimes \rho_2^* + 1 & \text{for } k_1, l_1 > 0, \\
Ad_1 & \text{for } k_1 > 0, l_1 = 0, \\
Ad_2 & \text{for } k_1 = 0, l_1 > 0. \end{cases}$$ (22)

2.

$$\psi^o|_{R^o} = \begin{cases} Ad_1 + Ad_2 + \rho_1 \otimes \rho_2 + \rho_1^* \otimes \rho_2^* + 1 & \text{for } k_1, l_1 > 0, \\
Ad_1 & \text{for } k_1 > 0, l_1 = 0, \\
Ad_2 & \text{for } k_1 = 0, l_1 > 0. \end{cases}$$ (23)

Further we will use the charts $U^e$ and $U^o$ on $F^e_{k|l}$ and $F^o_{k|l}$ defined by $I^e_s = I^e_{s\bar{0}} \cup I^e_{s1}$ and $I^o_s = I^o_{s\bar{0}} \cup I^o_{s1}$, where $I^e_{1\bar{1}}$ and $I^o_{1\bar{1}}$ are as above, and

$$I^e_{s\bar{0}} = I^e_{s\bar{0}} = \{k_{s-1} - k_s + 1, \ldots, k_{s-1}\}, \quad I^o_{s\bar{1}} = I^o_{s\bar{1}} = \{l_{s-1} - l_s + 1, \ldots, l_{s-1}\}$$

for $s \geq 2$. The coordinate matrices of these charts have the following form

$$Z_{I^e_s} = Z_{I^o_s} = \begin{pmatrix} X_s & \Xi_s \\
E_s & 0 \\
H_s & Y_s \\
0 & E_{l_s} \end{pmatrix}, \quad s = 2, \ldots, r,$$

where again the local coordinates are $X_s = (x^e_{ij})$, $Y_s = (y^o_{ij})$, $\Xi_s = (\xi^e_{ij})$ and $H_s = (h^o_{ij})$. We denote by $U^e_{B^e}$ and by $U^o_{B^o}$ the corresponding charts on $B^e$ and $B^o$, respectively. In other words, $U^e_{B^e}$ and $U^o_{B^o}$ are given by (20).

The proofs of the following two lemmas are similar to the proof of Lemma 2 and Lemma 3 in [V1].

**Lemma 2.** The following vector fields 1. $\frac{\partial}{\partial \xi_{ij}}$, 2. $\frac{\partial}{\partial \xi_{ij}}$, $\frac{\partial}{\partial \eta_{ij}}$ are fundamental. That is they are induced by the natural actions of Lie supergroups $OSp_{2m|2n}(C)$ and $II Sp_{m}(C)$ on $F^e_{k|l}$ and $F^o_{k|l}$, respectively.

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Lemma 3. Assume that $\text{Ker } \mathcal{P} \neq \{0\}$. Then $\dim \mathcal{W}_c^e(\mathcal{B}_0) > \dim \mathcal{W}_c^e(\mathcal{B}_0)$ and $\dim \mathcal{W}_c^o(\mathcal{B}_0) > \dim \mathcal{W}_c^o(\mathcal{B}_0)$.

Here $\mathcal{W}_c^e$ and $\mathcal{W}_c^o$ are locally free sheaves as in Section 4 corresponding to superbundles $\mathbf{F}_k^e$ and $\mathbf{F}_k^o$, respectively.

Now we need the following lemma.

Lemma 4. Assume that $\mathcal{O}_{S^e}(\mathcal{S}_0^e) \simeq \mathbb{C}$, $\mathcal{O}_{S^o}(\mathcal{S}_0^o) \simeq \mathbb{C}$, $\psi(S^e) \simeq \mathfrak{pgl}_{k_1,l_1}(\mathbb{C})$ and $\psi(S^o) \simeq \mathfrak{pgl}_{k_1,l_1}(\mathbb{C})$. Then

$$\mathcal{W}_0(\mathcal{B}_0^o) \simeq \begin{cases} \mathbb{C}, & k_1 > 2, l_1 \geq 1; \\ \mathbb{C} \oplus \mathfrak{r}_1, & k_1 = 2, l_1 \geq 1; \\ \mathbb{C} \oplus \mathfrak{r}_2, & k_1 = 1, l_1 > 1, \end{cases}$$

where $\mathfrak{r}_1$ is the $\mathfrak{so}_4(\mathbb{C}) \oplus \mathfrak{sp}_n(\mathbb{C})$-module with the highest weight $\mu_1 - \mu_2$, $\mathfrak{r}_2$ is the $\mathfrak{sp}_{2l_1}(\mathbb{C})_0$-module with the highest weight $\lambda_1$ and $\mathbb{C}$ is the trivial $\mathfrak{so}_m(\mathbb{C}) \oplus \mathfrak{sp}_n(\mathbb{C})$-module that corresponds to the highest weight 0.

Further we have

$$\mathcal{W}_0(\mathcal{B}_0^o) \simeq \begin{cases} \mathbb{C}, & k_1 > 1, l_1 \geq 1; \\ \mathbb{C} \oplus \mathfrak{r}_1, & k_1 = 1, l_1 \geq 1; \\ \mathbb{C} \oplus \mathfrak{r}_2, & k_1 > 1, l_1 = 1, \end{cases}$$

where $\mathfrak{r}_1$ is the $\mathfrak{sl}_n(\mathbb{C})$-module with the highest weight $-\mu_1 - \mu_{l_1+1}$, $\mathfrak{r}_2$ is the $\mathfrak{sl}_n(\mathbb{C})$-module with the highest weight $\mu_1 + \mu_2$, $\mathbb{C}$ is the trivial $\mathfrak{sl}_n(\mathbb{C})$-module that corresponds to the highest weight 0.

Proof. 1. We compute the vector space of global sections of $\mathcal{W}_0^e$ using Theorem 4. The representation $\psi^e$ of the Lie group $P_{2m}^e$ was computed in Lemma 1. It follows that the highest weights of $\psi^e$ have the following form:

- $\mu_1 - \mu_{k_1}, \mu_1 - \lambda_{l_1}, \lambda_1 - \mu_{k_1}, \lambda_1 - \lambda_{l_1}, 0$ for $k_1 > 1, l_1 > 1$;
- $\mu_1 - \lambda_{l_1}, \lambda_1 - \mu_1, \lambda_1 - \lambda_{l_1}, 0$ for $k_1 = 1, l_1 > 1$;
- $\mu_1 - \mu_{k_1}, \mu_1 - \lambda_1, \lambda_1 - \mu_{k_1}, 0$ for $k_1 > 1, l_1 = 1$;
- $\mu_1 - \lambda_1, \lambda_1 - \mu_1, 0$ for $k_1 = 1, l_1 = 1$.

(Note that we have $k_1 > 0$ and $l_1 > 0$, since otherwise $m = 0$ or $n = 0$.) Therefore the dominant weights of $\psi^e$ are:

- 0, if $k_1 > 2, l_1 \geq 1$;
- 0, $\mu_1 - \mu_{k_1}$, if $k_1 = 2, l_1 \geq 1$;
- 0, $\lambda_1 - \mu_1$, if $k_1 = 1, l_1 > 1$;
• 0, $\lambda_1 - \mu_1$, if $k_1 = 1, l_1 = 1$. This case we will not consider further since if $k_1 = 1, l_1 = 1$ we have $\mathcal{O}_{S^r}(S_0^0) \neq \mathbb{C}$, see Theorem 3.

By the Borel-Weil-Bott Theorem we get the result.

2. Again we use the Borel-Weil-Bott Theorem to compute the vector space of global sections of $W_0^r$. The representation $\psi^r$ of the Lie group $P^r$ was computed in Lemma 1. It follows that the highest weights of $\psi^r$ have the following form:

• $\mu_l + \mu_1$, $-\mu_n - \mu_1$, $0$ for $k_1 > 1, l_1 > 1$;
• $\mu_1 + \mu_1$, $-\mu_n - \mu_1$, $0$ for $k_1 = 1, l_1 > 1$;
• $\mu_2 - \mu_n$, $\mu_2 + \mu_1$, $0$ for $k_1 > 1, l_1 = 1$;
• $\mu_1 + \mu_2$, $-\mu_2 - \mu_1$, $0$ for $k_1 = 1, l_1 = 1$.

(Note that by definition of $F_{k,l}^e$ we have $k_1 > 0$ and $l_1 > 0$. Indeed, if for example $k_1 = 0$, then $l_1 = n$ and $F_{k,l}^e$ is isomorphic to a point.) Therefore the dominant weights of $\psi^r$ are:

• 0, if $k_1 > 1, l_1 > 1$;
• 0, $-\mu_l$, if $k_1 = 1, l_1 > 1$;
• 0, $\mu_1$, if $k_1 > 1, l_1 = 1$;

The case $k_1 = 1$ and $l_1 = 1$ we will not consider further since in this case we have $\mathcal{O}_{S^r}(S_0^0) \neq \mathbb{C}$, see Theorem 3. By the Borel-Weil-Bott Theorem we get the result. □

7.4 Main results

Now we are ready to prove the following two theorems.

**Theorem 6.** Assume that $r > 1$, $m = k_1$ and $n = l_1$. If $\mathcal{O}_{S^r}(S_0^0) \simeq \mathbb{C}$, $v(F_{2m,k_1|2n,l_1}^e) \simeq \mathfrak{osp}_{2m|2n}(\mathbb{C})$ and $v(S^e) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$, then $v(F_{k_1}^e) \simeq \mathfrak{osp}_{2m|2n}(\mathbb{C})$.

**Proof.** Consider the super-stabilizer $P_{2m}^e \subset \text{OSp}_{2m|2n}(\mathbb{C})$ of $x^e$. It contains all super-matrices of the following form:

$$
\begin{pmatrix}
A_1 & 0 & C_{11} & 0 \\
C_1 & (A_1^T)^{-1} & C_{21} & C_{22} \\
-C_{22}^T & 0 & A_2 & 0 \\
C_{21}^T & C_{11}^T & C_2 & (A_2^T)^{-1}
\end{pmatrix},
$$

(24)
where the size of the matrices is as in (14). Denote by \( \mathcal{L} \) the Lie sub-supergroup in \( \mathcal{P}_{2m} \) defined by the following submatrix:

\[
\begin{pmatrix}
(A_1^T)^{-1} & C_{22} \\
C_{12} & (A_2^T)^{-1}
\end{pmatrix}
\]

We see that \( \mathcal{L} \simeq \text{GL}_{k_1|l_1}(\mathbb{C}) \). And if we replace \((A_1^T)^{-1}\) by \( W_1 \in \text{GL}_{k_1}(\mathbb{C}) \) and \((A_2^T)^{-1}\) by \( W_2 \in \text{GL}_{l_1}(\mathbb{C}) \) we will see that \( \mathcal{L} \) acts \((W_0^e)_{x^e} \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})\) in the standard way. In other words, \((W_0^e)_{x^e} \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})\) is isomorphic to the adjoint \( l \)-module, where \( l \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C}) \) is the Lie superalgebra of \( \mathcal{L} \).

Now we repeat the argument used in [V1]. Let \( \pi : W^e \to \tilde{W}_0^e = W^e/W^e_{(1)} \) be the natural map and \( \pi_{x^e} : W^e \to (W_0^e)_{x^e} \) be the composition of \( \pi \) and of the evaluation map at the point \( x^e \). We have the following commutative diagram:

\[
\begin{array}{c}
W^e(B_0) \xrightarrow{[X,\cdot]} W^e(B_0) \\
\pi_{x^e} \downarrow \quad \pi_{x^e} \downarrow \\
(W_0^e)_{x^e} \xrightarrow{[X,\cdot]} (W_0^e)_{x^e}
\end{array}
\]

where \( X \in l \). (Note that the vector space \( W^e(B_0) \) is an ideal in \( \mathfrak{v}(\mathcal{M}^e) \) and in particular it is invariant with respect to the action of \( \mathcal{L} \).) Denote by \( V \) the image \( \pi_{x^e}(W^e(B_0)) \). From the commutativity of this diagram it follows that \( V \subset (W_0^e)_{x^e} \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \) is invariant with respect to the adjoint representation of \( \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \). Therefore, \( V \) is an ideal in \( \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \).

Analyzing ideals in \( \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \), we see that \( V \subset \text{Im}(\gamma) \), where \( \gamma : \tilde{W}_0^e(B_0) \to (W_0^e)_{x^e} \) is the evaluation map, never coincides with non-trivial ideals. Hence, \( V = \{0\} \). In other words, we proved that all sections of \( \pi(W^e(B_0)) \) are equal to 0 at the point \( x^e \). Since \( W_0^e \) is a homogeneous bundle, we get that sections from \( \pi(W^e(B_0)) \) are equal to 0 at any point. Therefore, we have \( \pi(W^e(B_0)) = \{0\} \) and \( W^e(B_0)_{(1)} \simeq W^e(B_0)_{(1)} \). From Lemma 3, it follows that \( \text{Ker} \mathcal{P} = \{0\} \).

We have proved the following theorem.

**Theorem 7.** Assume that \( r > 1 \), \( m = k_1 \), \( n = l_1 \); the conditions (13) hold; \( k_1 \geq 1 \), \( l_1 \geq 1 \) and \( \mathfrak{v}(\mathcal{F}_{k'|r}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \). Then \( \mathfrak{v}(\mathcal{F}_{k|l}) \simeq \mathfrak{osp}_{2m|2n}(\mathbb{C}) \).

**Theorem 8.** Assume that \( r > 1 \) and \( n = k_1 + l_1 \). If \( \mathcal{O}_{S^o}(S^o_0) \simeq \mathbb{C} \), \( \mathfrak{v}(\mathcal{F}_{n,k_1|l_1}) \simeq \pi \mathfrak{sp}_n(\mathbb{C}) \) and \( \mathfrak{v}(S^o) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \), then \( \mathfrak{v}(\mathcal{F}_{k|l}^o) \simeq \pi \mathfrak{sp}_n(\mathbb{C}) \).

**Proof.** Consider the super-stabilizer \( \mathcal{P}^o \subset \mathbb{I} \mathfrak{S}p_n(\mathbb{C}) \) of \( x^o \). It contains all
super-matrices of the following form:

\[
\begin{pmatrix}
A_1 & 0 & 0 & C_{12} \\
C_1 & B_1 & C_{12}^T & C_{22} \\
D_{11} & D_{12} & (A_1^T)^{-1} & -C_1 \\
-D_{12}^T & 0 & 0 & (B_1^T)^{-1}
\end{pmatrix}
\]

where the size of the matrices is as in (16). Denote by \( \mathcal{L} \) the Lie subsupergroup in \( \mathcal{P} \) defined by the following coordinate matrix.

\[
\begin{pmatrix}
B_1 & C_{22} \\
D_{12} & (B_1^T)^{-1}
\end{pmatrix}
\]

We see that \( \mathcal{L} \simeq \text{GL}_{k_1|l_1}(\mathbb{C}) \). And if we replace \( (B_1^T)^{-1} \) by \( W_1 \in \text{GL}_{l_1}(\mathbb{C}) \) we will see that \( \mathcal{L} \) acts \( (W_0^0)_x \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \) in the standard way. In other words, \( (W_0^0)_x \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \) is isomorphic to the adjoint \( \mathfrak{l} \)-module, where \( \mathfrak{l} \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C}) \) is the Lie superalgebra of \( \mathcal{L} \). The rest of the proof is similar to the proof of Theorem 6. \( \square \)

We have proved the following theorem.

**Theorem 9.** Assume that \( n = k_1 + l_1 \); the conditions (13) hold; \( \mathfrak{v}(\mathbf{F}_{k_1|l_1}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}) \) and \( k_1 \geq 3, l_1 \geq 2 \). Then \( \mathfrak{v}(\mathbf{F}_{k_1|l_1}) \simeq \mathfrak{psp}_n(\mathbb{C}) \). \( \square \)

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