Averaging Theory for Weakly Non-linear Oscillators

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Here I have first discussed how averaging theory can be an effective tool in solving weakly non-linear oscillators. Then I have applied the technique for a Van der Pol oscillator and extended the stability criterion of a Van der Pol oscillator for any integer $n$ (odd or even).

WEAKLY NONLINEAR OSCILLATORS

We consider systems of the form

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$$

where $0 \leq \epsilon \ll 1$ and $h(x, \dot{x})$ is an arbitrary smooth function.

Since such equations represent small perturbations of the linear oscillator $\ddot{x} + x = 0$, they are called weakly nonlinear oscillators. Some well known examples are the Van der Pol equation

$$\ddot{x} + x + \epsilon (x^2 - 1)\dot{x} = 0$$

and the Duffing equation

$$\ddot{x} + x + \epsilon x^3 = 0.$$

For the Van der Pol equation in $(x, \dot{x})$ phase space, if we choose an initial condition close to the origin and $\epsilon$ near 0, the trajectory is a slowly winding spiral; it takes many cycles for the amplitude to grow substantially. Eventually the trajectory asymptotes to an approximately circular limit cycle whose radius is close to 2.

AVERAGING THEORY

Introduction

Our system is

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad \text{where} \quad 0 \leq \epsilon \ll 1$$

Let

$$\dot{x} = y$$

$$\dot{y} = -x - \epsilon h(x, y)$$

When $\epsilon = 0$, the solutions are

$$x(t) = r \cos(t + \phi)$$

$$y(t) = -r \sin(t + \phi)$$

where $r, \phi$ are constant for simple harmonic oscillator on trajectories.

When $\epsilon \neq 0$, we expect a very slow drift of $r$ and $\phi$. They would evolve as expected; might approach a limit cycle. Our task at hand is to find the evolution equations of the amplitude and the phase given the effect of this nonlinear term $h(x, y)$. We can be definite that since $\epsilon$ is extremely small, the trajectories would be nearly circular and they would have periods of $2\pi$ approximately.

Now let

$$x(t) = r(t) \cos(t + \phi(t))$$

$$y(t) = -r(t) \sin(t + \phi(t))$$

Note here that although it might seem that the calculus done is wrong in computing $y$, but it is not so. Instead let it be a definition.

So view this as definition of $r(t)$ and $\phi(t)$ i.e.

$$r(t) = \sqrt{x^2(t) + y^2(t)}$$

$$\tan(t + \phi(t)) = -\frac{y(t)}{x(t)}$$

There are various ways one can exploit the fact, that oscillator is ‘close’ to a simple harmonic oscillator, to produce useful approximations. Here I will use the method of averaging theory to predict the period and radius of the limit cycle.
Calculating $\dot{r}$ and $\dot{\phi}$

Let us find equations for $\dot{r}, \dot{\phi}$:

\[
\begin{align*}
    r^2 &= x^2 + y^2 \\
    \Rightarrow \dot{r} &= x\dot{x} + y\dot{y} \\
    &= x(y) + y(-x - \epsilon h) \\
    &= -\epsilon y h \\
    &= -\epsilon h (-r \sin(t + \phi)) \\
\end{align*}
\]

\[
\Rightarrow \dot{r} = \epsilon h \sin(t + \phi)
\]

Main Concept of Averaging

Now what we have to do is exploit the separation of time scales of the evolution equations - **fast oscillation** versus **slow drift**. The main idea is to iron out the fast oscillations by averaging over one cycle of length $2\pi$ (1 oscillation) so that we can find out the explicit dependence of the evolution equations on $\epsilon$.

A **practical example** of the averaging theory would be our primary detector, eyes. All the things we see around us are made up of rapidly vibrating atoms. But our detectors average out this fast oscillation and we see only the slow scale dynamics explicitly.

Given $g(t)$, let us define the average over one cycle about the point $t$ as

\[
<g> (t) = \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} g(s) \, ds.
\]

So let us write out the equations for $\bar{r}$ and $\bar{\phi}$.

\[
\dot{\bar{r}} = <\epsilon h \sin(t + \phi)>_t \\
\dot{\bar{\phi}} = \left<\frac{\epsilon h}{r} \cos(t + \phi)\right>_t
\]

These equations are exact since R.H.S. depends on $r, \phi$, not $\bar{r}, \bar{\phi}$.

Now here you will notice a subtle difficulty, since we want a dynamical system with $\bar{r}$ and $\bar{\phi}$ so that we can compute the averages. We can get over this problem with an approximation. We well know that there is only a slight difference between $r$ and $\bar{r}$. [It is comparable to our height measured right at this instant versus our height averaged over the past. That is sort of the same thing with a slight error.] It is here that we are going to introduce an approximation.

Over one cycle,

\[
r = \bar{r} + \mathcal{O}(\epsilon)
\]

\[
\phi = \bar{\phi} + \mathcal{O}(\epsilon)
\]

Now we will replace $r, \phi$ by $\bar{r}, \bar{\phi}$ in the evolution equations which causes $\mathcal{O}(\epsilon^2)$ in the ODE’s. The beauty of this is that we get autonomous equations which we can analyse easily with phase plane methods. So the formal evolution equations are

\[
\dot{\bar{r}} = <\epsilon h \sin(t + \phi)> + \mathcal{O}(\epsilon^2)
\]

\[
\dot{\bar{\phi}} = \left<\frac{\epsilon h}{r} \cos(t + \phi)\right> + \mathcal{O}(\epsilon^2)
\]

Now since

\[
x^2 + y^2 = r^2 \text{ and } x = r \cos(t + \phi(t))
\]

we have,

\[
\dot{\bar{\phi}} = \frac{\epsilon h}{r} \cos(t + \phi)
\]

So we get slowly varying amplitude and phase, as expected, of the order $\epsilon$.

\[
\dot{\bar{r}} = \mathcal{O}(\epsilon), \dot{\bar{\phi}} = \mathcal{O}(\epsilon)
\]
We should treat $\bar{r}$ and $\bar{\phi}$ as constants while performing the averages.

**APPLICATION ON GENERAL VAN DER POL OSCILLATOR**

Now we shall apply this averaging technique to a general case of Van der Pol’s oscillator. The general Van der Pol equation is-

$$\ddot{x} + \epsilon \dot{x} (x^n - 1) = 0, \quad 0 \leq \epsilon \ll 1 \quad \text{‘}n\text{’ is a natural number}\quad \text{[this is even, or odd]}

I will try to show that when $n$ is odd, $r$ is going to grow exponentially and when $n$ is even, $r$ approaches a limit cycle.

I am going to use “$< >$” symbols which essentially means average over one cycle. We need to remember from previous discussions that $x = r \cos(t + \phi), y = -r \sin(t + \phi)$ and $\dot{x} = \dot{y}$. Let $(t + \bar{\phi})$ be replaced by $\theta$ in the following math.

$$h(x, \dot{x}) = (x^n - 1)\dot{x} = (x^n - 1)y$$

$$= (\bar{r}^n \cos^n (t + \bar{\phi}) - 1)(-\bar{r} \sin(t + \bar{\phi})) + O(\epsilon)$$

$$= -\bar{r}^{n+1} \cos^n \theta \sin \theta + \bar{r} \sin \theta + O(\epsilon)$$

$$\dot{r} = \epsilon \bar{r} \sin \theta + \bar{r} \dot{\theta} + O(\epsilon^2)$$

$$= \langle \epsilon \bar{r} \sin \theta - \bar{r}^{n+1} \cos^n \theta \sin \theta \rangle + O(\epsilon^2)$$

$$= \epsilon \bar{r} \langle \sin^2 \theta \rangle - \bar{r}^{n+1} \langle \cos^n \theta \sin^2 \theta \rangle + O(\epsilon^2)$$

$$= \frac{\epsilon \bar{r}}{2} - \bar{r}^{n+1} \langle \cos^n \theta \sin^2 \theta \rangle + O(\epsilon^2)$$

$$\dot{\phi} = \frac{\epsilon \bar{r}}{2} (t + \bar{\phi}) + O(\epsilon^2)$$

$$= \langle \epsilon \bar{r} (-\bar{r}^{n+1} \cos^n \theta \sin \theta + \bar{r} \sin \theta) \cos \theta \rangle + O(\epsilon^2)$$

$$= \langle \epsilon (-\bar{r}^{n+1} \cos^n \theta \sin \theta + \sin \theta \cos \theta) \rangle + O(\epsilon^2)$$

$$= \langle \epsilon \bar{r}^{n+1} \cos^n \theta \sin \theta \rangle + O(\epsilon^2)$$

$$= -\epsilon \bar{r}^{n+1} \langle \cos^n \theta \sin \theta \rangle + \epsilon \langle \sin \theta \cos \theta \rangle + O(\epsilon^2)$$

$$= O(\epsilon^2)$$

$: \text{both } \langle \cos^{n+1} \theta \sin \theta \rangle \text{ and } \langle \sin \theta \cos \theta \rangle \text{ are equal to 0.}$

When $n$ is Odd

Let $n = 2m + 1$.

$$\langle \cos^{2m+1} \theta \sin^2 \theta \rangle$$

$$= \int_0^{2\pi} \cos^{2m+1} \theta \sin^2 \theta d\theta$$

[Now let $t = \sin \theta$, then $dt = \cos \theta d\theta$]

$$= \int_0^1 (1 - t^2)^m t^2 dt$$

$$= 0.$$

So for odd $n$,

$$\dot{\bar{r}} = \frac{\epsilon \bar{r}}{2} + O(\epsilon^2)$$

If we neglect $O(\epsilon^2)$ error, then

$$\frac{d\bar{r}}{dt} = \frac{\epsilon \bar{r}}{2}$$

$$\Rightarrow \frac{d\bar{r}}{\bar{r}} = \frac{\epsilon}{2} dt$$

$$\Rightarrow \log \bar{r} = \frac{\epsilon}{2} t + k \quad [\text{where } k \text{ is any constant}]$$

$$\therefore \bar{r} = \rho \exp \left( \frac{\epsilon}{2} t \right) \quad [\text{where } \rho \text{ is a constant of integration}].$$

So the final approximate solution for odd $n$ is

$$x = \rho \exp \left( \frac{\epsilon}{2} t \right) \cos(t + \phi).$$

Therefore we assert that $x$ is going to blow up given sufficiently long time and the phase is going to change negligibly.
When $n$ is Even

We can be sure, that since $\cos^n \theta \sin^2 \theta \geq 0$ throughout, $\langle \cos^n \theta \sin^2 \theta \rangle$ must be equal to some positive quantity say $k^2$.

$$\dot{r} = \frac{\epsilon \bar{r}^n}{2} - \epsilon \bar{r}^{n+1} k^2$$

$$= \frac{\epsilon \bar{r}}{2} (1 - 2\bar{r}^n k^2)$$

So $\bar{r}$ has one real positive root $\left(\frac{1}{2\pi^2}\right)^{\frac{1}{n}}$. Whenever $\bar{r}$ is greater than $\left(\frac{1}{2\pi^2}\right)^{\frac{1}{n}}$, $\dot{\bar{r}}$ is less than 0 and $\bar{r}$ continues to shrink till it reaches $\left(\frac{1}{2\pi^2}\right)^{\frac{1}{n}}$. And whenever $\bar{r}$ is less than $\left(\frac{1}{2\pi^2}\right)^{\frac{1}{n}}$, $\dot{\bar{r}}$ is greater than 0 and $\bar{r}$ continues to grow till it reaches $\left(\frac{1}{2\pi^2}\right)^{\frac{1}{n}}$. So this means that the oscillator is going to approach a limit cycle of amplitude 2. This is the beauty of the problem when $n$ is even. Now let’s do a problem based on this.

**Example:** Let us consider a Van der Pol oscillator

$$\ddot{x} + x + \epsilon \dot{x} (x^2 - 1) = 0, \quad 0 \leq \epsilon \ll 1$$

$$h(x, \dot{x}) = (x^2 - 1) \dot{x} = (x^2 - 1)y$$

$$= (\bar{r}^2 \cos^3 (t + \bar{\phi}) - 1)(-\bar{r} \sin (t + \bar{\phi})) + O(\epsilon)$$

$$\dot{\bar{r}} = \epsilon h \sin (t + \bar{\phi}) > +O(\epsilon^2)$$

$$= \langle \epsilon (\bar{r} \sin \theta - \bar{r}^3 \cos^n \theta \sin \theta) \sin \theta \rangle + O(\epsilon^2)$$

$$= \epsilon \bar{r} \langle \sin^2 \theta \rangle - \epsilon \bar{r}^3 \langle \cos^n \theta \sin^2 \theta \rangle + O(\epsilon^2)$$

$$= \frac{\epsilon \bar{r}}{2} - \frac{\epsilon \bar{r}^3}{8} + O(\epsilon^2)$$

$$= \frac{\epsilon \bar{r}}{8} (4 - \bar{r}^2) + O(\epsilon^2)$$

From the diagram it is clear that as $t$ tends to $\infty$, value of $\bar{r}$ tends to 2. It means that after sufficiently long time, the Van der Pol Oscillator settles down to a limit cycle of amplitude 2.

$$\bar{\phi} \sim O(\epsilon^2)$$ [proved earlier in the general case] So $\phi$ changes on super slow time-scale. Period of the Van der Pol oscillator $= 2\pi + O(\epsilon^2)$.

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