Let $r > 0$ be a real number and $\mathcal{A}$ be the class of analytic functions defined in the disk $U(r) = \{ w \in \mathbb{C} : |w| < r \}$ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$. Let $(a_n)$, where $a_n \in \mathbb{C}$, $\forall n \geq 2$ be a sequence with

$$\lim_{n \to \infty} \sup |a_n|^{1/n} = r_f \geq 0,$$

(1)

where $r_f$ means the radius of convergence of the series $w + \sum_{n=2}^{\infty} a_n w^n = f(w) \in \mathcal{A}$. If $\lim_{n \to \infty} \sup |a_n|^{1/n} = 0$, then $r_f = +\infty$.

In 1999, Kanas and Wiśniowska [9] (also refer Goodman [7, 8], Rønning [15], and Ma and Minda [12]) proposed the idea of $\lambda$-uniform convexity denoted by $\lambda - UCV$. A function $f \in \mathcal{A}$ is said to be in $\lambda - UCV(\delta)$, the class of $\lambda$-uniformly Convex of order $\delta$ [3], iff

$$\text{Re} \left( 1 + \frac{wf'(w)}{f'(w)} \right) > \lambda \left| \frac{wf'(w)}{f'(w)} - 1 \right| + \delta, \lambda \geq 0, \delta \in [0, 1) \forall w \in U(r).$$

(2)

A function $f \in \mathcal{A}$ is said to be in $\lambda - \delta \mathcal{T}(\delta)$, the class of $\lambda$-starlike function of order $\delta$ [10], iff

$$\text{Re} \left( \frac{wf'(w)}{f'(w)} \right) > \lambda \left| \frac{wf'(w)}{f'(w)} - 1 \right| + \delta, \lambda \geq 0, \delta \in [0, 1) \forall w \in U(r).$$

(3)

Geometrically, the conditions (2) and (3) mean that for $f \in \lambda - UCV(\delta)$ and $f \in \lambda - \delta \mathcal{T}(\delta)$, the images of $U(r)$ under the functions $1 + wf''(w)/f'(w)$ and $wf'(w)/f'(w)$ are in the conic domain $\Omega_{\lambda,\delta}^A$ contained in the right half plane for which $1 \in \Omega_{\lambda,\delta}^A$ and $\partial \Omega_{\lambda,\delta}^A$ is the curve defined by the equation

$$\partial \Omega_{\lambda,\delta}^A = \{ \omega = u + iv : (u - \delta)^2 = \lambda^2 [(u - 1)^2 + v^2] \}, \lambda \geq 0.$$  

(4)

Moreover, $\Omega_{\lambda,\delta}^A$ is an elliptic region for $\lambda > 1$, parabolic for $\lambda = 1$, and hyperbolic for $0 < \lambda < 1$, and finally, $\Omega_{\lambda,\delta}^A$ is the whole right half plane.

The radius of $\lambda$-uniform convexity of order $\delta$ denoted by $r_{uc(f)}(\delta)$ and radius of $\lambda$-starlikeness of order $\delta$ denoted by $r_{st(f)}(\delta)$ are defined by
Let $a \in \mathbb{R}$ and $\alpha \in [0, 1)$. A function $f \in \mathcal{A}$ is said to be in $\mathcal{M}_\alpha(\delta)$, the class of $\alpha$-convex functions (Mocanu functions) of order $\delta$ [14, 16] iff
\[
\Re \left( (1 - \alpha) \frac{w f'(w)}{f(w)} + \alpha \left( 1 + \frac{w f''(w)}{f'(w)} \right) \right) > \delta, w \in \mathbb{U}(r), \ \delta \in (0, 1).
\]

The radius of $\alpha$-convexity (Mocanu functions) of order $\delta$ denoted by $\rho_{\alpha}(\delta)$ is defined by, for $0 \leq \delta < 1$,
\[
r_{\alpha}(\delta) = \sup \left\{ r \in (0, r_0) : \Re \left( (1 - \alpha) \frac{w f'(w)}{f(w)} + \alpha \left( 1 + \frac{w f''(w)}{f'(w)} \right) \right) > \delta, w \in \mathbb{U}(r) \right\}.
\]

**Lemma 1** (see [17]). If $a, b \in \mathbb{R}$ and $a > b > 0$, then
\[
\frac{a + w}{(b + w)^2} \leq \frac{a - |w|}{(b - |w|)^2}, \text{ for } |w| < b, w \in \mathbb{U} = \mathbb{U}(1).
\]

**2. Main Results**

**Theorem 2.** Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence with $c_n \cdot |e^{i \phi} \in \mathbb{C}, |c_n| \geq 1$ for $n \in \mathbb{N}\{0\}$, $r_0 = \inf \{c_n : n \in \mathbb{N}\{0\}\}$, and let $h(w)$ be an analytic function in $\mathbb{U}(r_0)$ with $|w| e^{i \phi} h'(|w| e^{i \phi}) \in \mathbb{R}$ and $|w| e^{i \phi} h''(|w| e^{i \phi}) \leq \mathcal{R}(h''(w))$, for $w \in \mathbb{U}(r_0)$. If the function $\mu : (0, r_0) \rightarrow \mathbb{R}$ defined by $\mu(r) = r e^{i \phi} h'(r e^{i \phi})$ is decreasing with respect to $r$ and $\mathcal{B}(w)$ is of the form (9) with $q_n \in \mathbb{N}\{0\}$ for $n \in \mathbb{N}\{0\}$, then the radius of $\lambda$-starlikeness of order $\delta$ of the function $\mathcal{B}(w)$ is $\rho_{\lambda}(\delta)$, the absolute value of the root of the equation $(1 + \lambda)\omega \mathcal{B}'(w) - (\lambda + \delta) \mathcal{B}(w) = 0$ having the smallest modulus and argument $\phi$.

**Proof.** By logarithmic differentiation, (9) becomes
\[
\frac{\omega \mathcal{B}'(w)}{\mathcal{B}(w)} = 1 + wh'(w) - \sum_{n=1}^{\infty} \frac{(w^n - 1)}{(1 - w)^n}.
\]

For $\omega \in \mathbb{U}$ and $k, n \in \mathbb{N}$,
\[
\mathcal{R} \left( \frac{w^n}{(1 - w)^k} \right) \leq \frac{|w^n|}{|1 - w|^k} \leq \frac{|w|^n}{(1 - |w|)^k} \leq \frac{|w|^n}{(1 - |w|)^k}.
\]
Since $|\omega/c_n| \leq 1$, (12) along with (13) implies
\[
\Re \left\{ \frac{\omega \mathcal{B}'(\omega)}{\mathcal{B}(\omega)} \right\} \geq 1 + \Re \left\{ \omega h'(\omega) \right\} - \sum_{n=1}^{\infty} \left| \frac{\omega/c_n}{1 - |\omega/c_n|} \right|^2 \geq 1 + |\omega| \mathcal{B}'(\omega)|/\mathcal{B}(\omega)'
\]
\[
\geq 1 + |\omega|^2 h'(\omega) + \sum_{n=1}^{\infty} \left| \frac{\omega/c_n}{1 - |\omega/c_n|} \right|^2 \geq 1 + |\omega|^2 \mathcal{B}'(\omega)|/\mathcal{B}(\omega)'
\]
\[
= |\omega|^2 \mathcal{B}'(\omega)|/\mathcal{B}(\omega)'.
\]
\[
\mathcal{B}_{\mathcal{B}}(\omega) = |\omega|^2 \mathcal{B}'(\omega)|/\mathcal{B}(\omega)'.
\]

Hence, the equation $(1 + \lambda)\mathcal{B}(\omega)\mathcal{B}'(\omega) - (\lambda + \delta)\mathcal{B}(\omega) = 0$ has a unique root in $(0, r_0)$, and this root is $r_{\mathcal{B}}(\omega)$. 

Remark 3. $\lambda \geq 0$ in Theorem 2 means that, if $\mathcal{B} \in \lambda - \mathcal{S}(\delta)$, then the image of $U(\mathcal{B})$ under the function $\mathcal{B}(\omega)\mathcal{B}'(\omega) / \mathcal{B}(\omega)$ is in a conic domain $\mathcal{D}_{\delta}$, contained in the right half plane for which $1 \in \mathcal{D}_{\delta}$ and $\mathcal{D}_{\delta}$ is the curve defined by equation (4).

In the following remarks, we deduce the radius of some special classes by specializing the parameters in Theorem 2.

Remark 4. Taking $\lambda \geq 0$, $\delta = 0$ in Theorem 2, we get $r_{\mathcal{B}}(\delta)$, the radius of $\lambda$-starlikeness of the function $\mathcal{B}(\omega)$, $r_{\mathcal{B}}(\delta)$ is the absolute value of the root of the equation $(1 + \lambda)\mathcal{B}(\omega)\mathcal{B}'(\omega) - \lambda\mathcal{B}(\omega) = 0$ having the smallest modulus and argument $\phi$.

Remark 5. Letting $\lambda = 0$, $0 \leq \delta < 1$ in Theorem 2, we get $r_{\mathcal{B}}(\delta)$, the radius of starlikeness of order $\delta$ of the function $\mathcal{B}(\omega)$, $r_{\mathcal{B}}(\delta)$ is the absolute value of the root of the equation $\mathcal{B}(\omega) - \delta\mathcal{B}(\omega) = 0$ having the smallest modulus and argument $\phi$.

In the following, we obtain the radius $r_{\mathcal{B}}(\omega)$ of $\lambda$-uniform convexity of order $\delta$ for $\mathcal{B}(\omega)$.

Theorem 6. Let $\{c_n\}_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence with $c_n = |c_n|e^{i\phi} \in \mathbb{C}$, $|c_n| \geq 1$ for $n \in \mathbb{N} \setminus \{0\}$, $r_0 = \inf \{|c_n|: n \in \mathbb{N} \setminus \{0\}\}$, and let $h(\omega)$ be an analytic function in $U(r_0)$ with $|\omega|e^{i\phi}h'(\omega) \in \mathbb{R}$, $|\omega|e^{i\phi}h''(\omega) \leq \Re \omega h'(\omega)$, and $|\omega|^2 e^{i\phi}h''(\omega) \leq \Re \omega h'(\omega)$ for $\omega \in U(r_0)$. If the function $\mu : (0, r_0) \rightarrow \mathbb{R}$ defined by $\mu(\omega) = re^{i\phi}h'(\omega)$ is increasing with respect to $r$ and $\mathcal{B}(\omega)$ is of the form (9) with $q_n \in \mathbb{N} \setminus \{0\}$ for $n \in \mathbb{N} \setminus \{0\}$; then, $\lambda$-uniform convexity of order $\delta$ of the function $\mathcal{B}(\omega)$ is $r_{\mathcal{B}}(\omega)$, the absolute value of the root of the equation $(1 + \lambda)\mathcal{B}(\omega)(\omega) + (1 - 2\lambda - \delta)\mathcal{B}'(\omega) = 0$ having the smallest modulus and argument $\phi$.

Proof. From (9),
\[
1 + \omega \mathcal{B}''(\omega) \mathcal{B}(\omega) = 2 + \omega h'(\omega) \mathcal{B}(\omega) - \sum_{n=1}^{\infty} \left( \frac{\omega/c_n}{1 - \omega/c_n} \right)^2
\]
\[
- \frac{1 - \omega h''(\omega) + \sum_{n=1}^{\infty} \left( (q_n - 1)\omega/c_n \right)^2}{1 + \omega h'(\omega) - \sum_{n=1}^{\infty} \left( (q_n - 1)\omega/c_n \right)^2} \frac{1}{1 - (\omega/c_n)}
\]
\[
= 1 + \frac{\omega \mathcal{B}'(\omega)}{\mathcal{B}(\omega) - \frac{1 - \omega^2 h''(\omega) + \sum_{n=1}^{\infty} \left( (q_n - 1)\omega/c_n \right)^2}{1 + \omega h'(\omega) - \sum_{n=1}^{\infty} \left( (q_n - 1)\omega/c_n \right)^2} \frac{1}{1 - (\omega/c_n)}.
\]
Using (14) and the inequality of Lemma 1, we have

\[ R \left( \frac{1 - w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{wB'(w)/B(w)} \right) \]

\[ \leq 1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) \]

\[ \frac{1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \]

\[ \left( \frac{wB''(w)/(w|e|^\theta)}{wB'(w)/(w|e|^\theta)} \right) \]

From (14), (20), and (21),

\[ R \left( 1 + \left( \frac{wB''(w)}{wB'(w)} \right) \right) \]

\[ \geq 1 + R \left( \frac{wB'(w)/(B(w))}{1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \right) \]

Also, we have

\[ \left| \frac{wB''(w)}{B'(w)} \right| \leq \left| \frac{wB'(w)}{B(w)} \right| - \frac{1 - w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \]

\[ \leq \left| \frac{wB'(w)}{B(w)} \right| - \frac{1 - w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{1 + w^2 h''(w) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \]

\[ \leq 2 - \left| \frac{wB''(w)/(w|e|^\theta)}{wB'(w)/(w|e|^\theta)} \right| + \left| \frac{1 - w^2 e^{2\theta} h''(w|e|^\theta) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{1 + w^2 e^{2\theta} h''(w|e|^\theta) - \sum_{n=1}^{\infty} \left( (w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \right| \]

\[ \leq 2 - \left| \frac{wB''(w)/(w|e|^\theta)}{wB'(w)/(w|e|^\theta)} \right| + \left| \frac{1 - w^2 e^{2\theta} h''(w|e|^\theta) + \sum_{n=1}^{\infty} \left( (q_n)(w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) }{1 + w^2 e^{2\theta} h''(w|e|^\theta) - \sum_{n=1}^{\infty} \left( (w/c_n)^{4n+1} - (q_n - 1)(w/c_n)^{4n+2} \right) / (1 - (w/c_n)^2) } \right| \]

\[ = 2 - \left| \frac{wB''(w)/(w|e|^\theta)}{wB'(w)/(w|e|^\theta)} \right|. \]
From (22) and (23),

\[
\begin{align*}
\Re \left( 1 + \frac{wB''(w)}{B'(w)} - \lambda \frac{|wB''(w)|}{B'(w)} - \delta \right) \\
&\geq 1 + \frac{|w|e^{\delta \theta}B''(w)|w|e^{\delta \phi})}{B'(w)|w|e^{\delta \phi})} - \lambda \left( 2 - \frac{|w|e^{\delta \theta}B''(w)|w|e^{\delta \phi})}{B'(w)|w|e^{\delta \phi})} \right) - \delta \]
\end{align*}
\]

\[
= (1 + \lambda) \left( \frac{|w|e^{\delta \theta}B''(w)|w|e^{\delta \phi})}{B'(w)|w|e^{\delta \phi})} + (1 - \delta - 2\lambda), \right.
\]

where \( r \in (0, r_0) \). The function \( \psi : (0, r_0) \rightarrow \mathbb{R} \), defined by \( \psi(r) = (1 + \lambda)(r e^{\delta \theta}B''(w)|w|e^{\delta \phi}) + (1 - \delta - 2\lambda) \) is strictly decreasing; also, observe that \( \lim_{r \to 0} \psi(r) = 1 - \delta - 2\lambda > 0 \). Thus, it follows that the equation \((1 + \lambda)e^{\delta \theta}B''(w)|w|e^{\delta \phi}) + (1 - 2\lambda - \delta)B'(w)|w|e^{\delta \phi}) = 0\) has a unique root situated in \((0, r_0)\), and this root is \(r_0^1(\theta, \delta)\).

**Remark 7.** As \( \delta \in [0, 1) \) and \( 0 < \lambda < (1 - \delta)/2 \leq (1/2) \), we have \( \lambda \in [0, 1/2) \), which means that if \( B \in \lambda - \mathcal{A} \), then the image of \( U(r) \) under the function contained in the right half plane for which \( 1 + |(wB''(w))/B'(w))| \) is in hyperbolic domain \( \Omega^\lambda_\delta \) contained in the right half plane for which \( 1 \in \Omega^\lambda_\delta \) and \( \partial \Omega^\lambda_\delta \) is the curve defined by equation (4).

By specializing the parameters in Theorem 6, we have

**Remark 8.** Substituting \( \delta = 0 \) and \( \lambda \in (0, 1/2) \) in Theorem 6, we get the radius \( r_0^\lambda(\theta) \) of \( \mathcal{A} \)-uniform convexity given by the absolute value of the root of the equation \((1 + \lambda)wB''(w) + (1 - 2\lambda)B'(w) = 0\) having the smallest modulus and argument \( \phi \).

**Remark 9** (see [17]). Taking \( \lambda = 0, 0 \leq \delta < 1 \) in Theorem 6, we get the radius \( r_0^\delta(\phi) \) of \( \mathcal{A} \)-convexity of order \( \delta \) given by the absolute value of the root of the equation \( wB''(w) + (1 - \delta)B'(w) = 0 \) having the smallest modulus and argument \( \phi \).

**Theorem 10.** Let \( \{c_n\}_{n \in \mathbb{N} / \{0\}} \) be a sequence with \( c_n = |c_n|e^{\delta \phi} \in \mathbb{C}, |c_n| \geq 1 \) for \( n \in \mathbb{N} / \{0\} \), \( r_0 = \inf \{|c_n|: n \in \mathbb{N} / \{0\}\} \), and let \( h(w) \) be an analytic function in \( U(r_0) \) with \( |w|e^{\delta \theta}h''(w)|w|e^{\delta \phi}) \in \mathbb{R} \), \( |w|e^{\delta \theta}h'(w)|w|e^{\delta \phi}) \leq \Re \{wh'(w))\}, \) and \( |w|e^{\delta \theta}h''(w)|w|e^{\delta \phi}) \leq \Re \{wh''(w))\}, \) and \( |w|^2e^{\delta \phi}h''(w)|w|e^{\delta \phi}) \) for \( w \in U(r_0) \). If the function \( \mu : (0, r_0) \rightarrow \mathbb{R} \) defined by \( \mu(r) = re^{\delta \theta}h'(w)|w|e^{\delta \phi}) \) is decreasing, the function \( \delta : (0, r_0) \rightarrow \mathbb{R} \) defined by \( \delta(r) = -r^2e^{\delta \theta}h''(w)|w|e^{\delta \phi}) \) is increasing with respect to \( r \), and \( B(w) \) is of the form (9) with \( q_n \in \mathbb{N} / \{0\} \) for \( n \in \mathbb{N} / \{0\} \) and \( \alpha \in [0, 1/2) \); then, the radius of \( \alpha \)-convexity of order \( \delta \) of the function \( B(w) \) is the smallest positive root of the equation \((1 + \alpha)e^{\delta \theta}B'(w)|w|e^{\delta \phi}) + \alpha e^{\delta \theta}B''(w)|w|e^{\delta \phi}) = \delta \) having the smallest modulus and argument \( \phi \).

**Proof.** Consider
for every $|w| < r_a$, and the equality holds for $w = |w|e^{i\phi}$. By the virtue of minimum principle for harmonic functions,

$$\inf_{|w|<r} \Re \{\mathcal{M}(\alpha, \mathcal{B}(w))\} = \mathcal{M}(\alpha, re^{i\phi}), \quad r \in (0, r_1),$$

(27)

Also, $\mathcal{M}(\alpha, re^{i\phi})$ is strictly decreasing; also, $\lim_{r \to 0} \mathcal{M}(\alpha, re^{i\phi}) = 1 > 0$ and $\lim_{r \to r_1} \mathcal{M}(\alpha, re^{i\phi}) = -\infty$. Hence, the equation $(1 - \alpha)(r^{\alpha} \mathcal{B}'(re^{i\phi})/\mathcal{B}(re^{i\phi})) + \alpha(1 + r^{\alpha} \mathcal{B}'(re^{i\phi})/\mathcal{B}(re^{i\phi})) = \delta$ has a unique root in $(0, r_0)$, and this root is $r_{\mathcal{B}(\alpha)}(\delta)$.

**Remark 11** (see [17]). Taking $\alpha = 0$ in Theorem 10, we get the radius $r_{\mathcal{B}(\alpha)}(\delta)$ of starlikeness of order $\delta$, given by the absolute value of the root of the equation $w\mathcal{B}'(w) - \delta\mathcal{B}(w) = 0$, having the smallest modulus and argument $\phi$.

**Remark 12** (see [17]). Taking $\alpha = 1$, in Theorem 6, we get the radius $r_{\mathcal{B}(\alpha)}(\delta)$ of convexity of order $\delta$, given by the absolute value of the root of the equation $w\mathcal{B}''(w) + (1 - \delta)\mathcal{B}'(w) = 0$, having the smallest modulus and argument $\phi$.

In the following remark, we discuss the radius of $\lambda$-starlikeness, $\lambda$-uniform convexity, and $\alpha$-convexity of order $\Gamma$ for the function $\Gamma$.

**Remark 13.** Let $h(w) = \gamma w$ where $\gamma$ is the Euler-Mascheroni constant [5], and let $q_n = 1, c_n = -n, n \in \mathbb{N}$, and $\phi = 0$. Then,

$$\mathcal{B}(w) = \frac{1}{\Gamma(w)} = we^{\gamma w} \sum_{n=1}^{\infty} \left(1 - \frac{w}{n}\right)e^{\gamma wn}. \quad (28)$$

We now have $w^h(w) = \gamma w, w^h^\prime w(w) = \gamma w^2$, and it is easy to verify $\Re\{w\mathcal{B}(w)\} \geq |\gamma w|, w \in \mathbb{U}$ with equality iff $w \in \mathbb{R}$ and $|w\mathcal{B}(w)| = |\gamma w|^2, w \in \mathbb{U}$. The conditions of Theorems 2, 6, and 10 are satisfied.

By Theorem 2, the radius $r_{\mathcal{B}(\alpha)}(\delta)$ of $\lambda$-starlikeness of order $\delta$ of the function $1/\Gamma(w)$ is the modulus of the biggest negative root of the equation $(\Gamma(w)/\Gamma'(w) + (\lambda + \delta)/(1 + \lambda)) = 0$. Numerical approach gives $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(0)) = 0.504083$, $r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(0)) = 0.416321$, $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(1/2)) = 0.358071, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.304311, \text{and } r_{\mathcal{B}(\alpha)}^0(1/\Gamma(1/2)) = 0.180823$.

By Theorem 6, the radius $r_{\mathcal{B}(\alpha)}^0(\delta)$ of $\lambda$-uniform convexity of order $\delta$ of the function $1/\Gamma(w)$ is the modulus of the biggest negative root of the equation

$$\frac{w\Gamma''(w)}{\Gamma'(w)} + \frac{2w\Gamma'(w)}{\Gamma(w)} + \frac{1 - 2\lambda - \delta}{1 + \lambda} = 0. \quad (29)$$

Numerical approach gives $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(0)) = 0.266701, r_{\mathcal{B}(\alpha)}^0(1/\Gamma(1/2)) = 0.190771, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(0)) = 0.125966, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.108467, \text{and } r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.166153$. By Theorem 10, the radius $r_{\mathcal{B}(\alpha)}^0(\delta)$ of $\alpha$-convexity of order $\delta$ for the function $1/\Gamma(w)$ is the modulus of the biggest negative root of the equation

$$\alpha \left(1 + \frac{w\Gamma''(w)}{\Gamma'(w)} + \frac{1 - \alpha}{\Gamma(w)} = \delta. \quad (30)$$

Numerical approach gives $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(0)) = 0.504083, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(0)) = 0.266701, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.258289, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/3)) = 0.269676, \text{and } r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.234978$.

In the following remark, we give an example which shows that the Theorems 2, 6, and 10 work even if $w\mathcal{B}(w)$ is not starliken. That is, the example given in the following remark shows that the hypotheses of Theorems 2, 6, and 10 are free from the hypothesis of Theorem 3 in [13], proved by Merkes et al.

**Remark 14.** Let $h(w) = w^2/(w^2 - 1)$ with $\phi = 0$, and let $q_n = 1, c_n = n$. Clearly, $w\mathcal{B}(w)$ is not starliken. Then, we have $w^h(w) = (-2w^2/(w^2 - 1))$ and $w^{h\prime h}(w) = ((2w^2 + 6w^4)/((w^2 - 1)^3))$. Also, $\Re\{w^h(w)\} \geq (2|w|^2)/((|w|^2 - 1)^2)$ and $|w^{h\prime h}(w)| \geq (2|w|^2 + 6|w|^4)/(1 - |w|^2)^3$, $w \in \mathbb{U}$ with equality iff $w \in \mathbb{R}$.

By Theorem 2, the radius $r_{\mathcal{B}(\alpha)}^1(\delta)$ of $\lambda$-starlikeness of order $\delta$ of the function

$$\mathcal{B}_1(w) = we^{w^2/(w^2 - 1)} \sum_{n=1}^{\infty} \left(1 - \frac{w}{n}\right)e^{w^2n} \quad (31)$$

is the smallest positive root of the equation $1 - (2r^2/(1 - r^2)^3) - \sum_{n=1}^{\infty} (r^2/(n - r)) - (\lambda + \delta)/(1 + \lambda) = 0$. Numerical approach gives $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(0)) = r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(0)) = 0.325887, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.241843, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.201282, \text{and } r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.274465$.

By Theorem 6, the radius $r_{\mathcal{B}(\alpha)}^0(\delta)$ of $\lambda$-uniform convexity of order $\delta$ is the smallest positive root of the equation

$$1 - \frac{2r^2}{(1 - r^2)^3} - \sum_{n=1}^{\infty} r^2 \left((1 - r^2)^3\right) + \sum_{n=1}^{\infty} (r^2/(n - r)^2) \left((1 - r^2)^3\right) - \sum_{n=1}^{\infty} r^2/(n - r)^2 \left((1 - r^2)^3\right) + \sum_{n=1}^{\infty} (1 - \delta - 2\lambda)/(1 + \lambda) = 0. \quad (32)$$

Numerical approach gives $r_{\mathcal{B}(\alpha)}^0(1/\Gamma(0)) = 0.242015, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.187093, r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.108455, \text{and } r_{\mathcal{B}(\alpha)}^{1/2}(1/\Gamma(1/2)) = 0.126439$. 


By Theorem 10, the radius $r^α_{\mathcal{A}_1}(δ)$ of $α$-convexity of order $δ$ is the smallest positive root of the equation

$$1 - \frac{2r^2}{(1 - r^2)^2} - \sum_{n=1}^{\infty} \frac{r^2}{n(n - r)} + \alpha \left(1 - \frac{1 + \left(2r^2 + 6r^4\right)/(1 - r^2)^3 + \sum_{n=1}^{\infty} (r^2((n - r)^2))}{1 - \left(2r^2/(1 - r^2)^2\right)} - \sum_{n=1}^{\infty} (r^2/(n(n - r)))\right) = δ.$$  

(33)

Numerical approach gives

- $r^0_{\mathcal{A}_1}(0) = r^*_{\mathcal{A}_1} = 0.426948$,
- $r^1_{\mathcal{A}_1}(0) = r^*_{\mathcal{A}_1} = 0.242015$,
- $r^3_{\mathcal{A}_1}(1/2) = r^*_{\mathcal{A}_1}(1/2) = 0.325887$,
- $r^{1/4}_{\mathcal{A}_1}(1/2) = r^*_{\mathcal{A}_1}(1/2) = 0.187093$,
- $r^{1/2}_{\mathcal{A}_1}(1/2) = r^*_{\mathcal{A}_1}(1/2) = 0.222952$,
- $r^{1/4}_{\mathcal{A}_1}(1/4) = 0.139653$, and
- $r^{1/4}_{\mathcal{A}_1}(1/2) = 0.254242$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

There is no conflict of interest regarding the publication of this article.

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