Presenting quantum Schur algebras as quotients of the quantized universal enveloping algebra of $\mathfrak{gl}_2$

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December 1, 2000

Abstract

We obtain a presentation of the quantum Schur algebras $S_v(2, d)$ by generators and relations. This presentation is compatible with the usual presentation of the quantized enveloping algebra $\mathbb{U} = U_v(\mathfrak{gl}_2)$. In the process we find new bases for $S_v(2, d)$. We also locate the $\mathbb{Z}[v, v^{-1}]$-form of the quantum Schur algebra within the presented algebra and show that it has a basis which is closely related to Lusztig’s basis of the $\mathbb{Z}[v, v^{-1}]$-form of $\mathbb{U}$.

1 Introduction

In [DG] we gave a description of the rational Schur algebra $S_\mathbb{Q}(2, d)$ in terms of generators and relations. This description is compatible with the usual presentation of the universal enveloping algebra $U(\mathfrak{gl}_2)$. We also described the integral Schur algebra $S_\mathbb{Z}(2, d)$ as a certain subalgebra of the rational version and an integral basis was exhibited. In this paper, we formulate and prove quantum versions of those results.

Consider the Drinfeld-Jimbo quantized enveloping algebra $\mathbb{U}$ corresponding to the Lie algebra $\mathfrak{gl}_2$. It has a natural two-dimensional module $E$, and thus the tensor product $E^\otimes d$ is also a module for $\mathbb{U}$. Let

$$\rho_d : \mathbb{U} \to \text{End}(E^\otimes d)$$

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be the corresponding representation. Amongst the various equivalent definitions of the quantum Schur algebra $S_v(2, d)$ is that it is precisely the image of the homomorphism $\rho_d$. (We shall not need to consider the other definitions in this work.) Since $S_v(2, d)$ is a homomorphic image of $U$, it is natural to ask for an efficient generating set of $\ker(\rho_d)$, thereby giving a presentation of $S_v(2, d)$.

In what follows, we obtain a precise answer to this question. Recall that $U$ is generated by elements $e, f, K_1^{\pm 1}, K_2^{\pm 1}$ subject to various well-known relations (see section 3). Now in the representation $\rho_d$, it is easy to see that $K_1 K_2 = v^d$, so we may use this relation to eliminate $K_2$ (or $K_1$) from the generating set for $S_v(2, d)$. Having done this, then our main result is that the only additional relation needed to give the desired presentation of $S_v(2, d)$ is the minimal polynomial of $K_1$ in $\text{End}(E^\otimes d)$.

Set $A = \mathbb{Z}[v, v^{-1}]$. The algebra $U$ contains a certain $A$-subalgebra $U_A$ which may be viewed as a quantum version of the integral form $U_{\mathbb{Z}}(\mathfrak{gl}_2)$ of the classical enveloping algebra. The algebra $U_A$, originally constructed by Lusztig, is generated by the $v$-divided powers of $e$ and $f$ along with $K_1^{\pm 1}$ and $K_2^{\pm 1}$. It has an $A$-basis which is a quantum analog of Kostant’s basis for the integral form of the classical enveloping algebra $U(\mathfrak{gl}_2)$. The image of the homomorphism $\rho_d$ upon restriction to $U_A$ gives us an “integral” Schur algebra $S_A(2, d)$, which can be used to define a version over any $A$-algebra. We show that the integral Schur algebra has a basis which is closely related to Lusztig’s basis of $U_A$.

Although the results of this paper are quantum versions of those appearing in [DG], the techniques are somewhat different, since some of the arguments given in that paper did not quantize directly. In particular, the treatment here of the degree zero part (generated by the images of $K_1^{\pm 1}$ and $K_2^{\pm 1}$) of $S_v(2, d)$ is totally different. Specifically, we exhibit an idempotent basis of this subalgebra, whereas in [DG] a PBW-type basis of the degree zero part was used. The idempotent basis is more amenable to computations and is precisely the kind of basis needed to handle the general Schur algebras $S(n, d)$ and their quantizations. Another difference between the results of this paper and the results of [DG] is that here we do not obtain analogues of the “restricted PBW-basis”, although we believe such results should be true in the quantum case.

Finally, it is clear that analogous results hold in general, for any $n$ and $d$, although one cannot expect to obtain such precise reduction formulas in the
general case as those given here and in [DG]. The authors expect to treat the general case in a later paper.

2 Statement of results

The main results of this paper are contained in the following theorems. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with fraction field $\mathbb{Q}(v)$. Each of the first three theorems gives a presentation of the quantum Schur algebra $S_v(2, d)$ in terms of generators and relations. The first result gives a presentation which is similar to that of $U_v(\mathfrak{sl}_2)$.

2.1 Theorem Over $\mathbb{Q}(v)$, the quantum Schur algebra $S_v(2, d)$ is isomorphic to the algebra generated by $e, f, K^\pm 1$ subject to the relations:

(a) $KK^{-1} = K^{-1}K = 1$, 
(b) $KeK^{-1} = v^2e$, $KfK^{-1} = v^{-2}f$, 
(c) $ef - fe = \frac{K - K^{-1}}{v - v^{-1}}$, 
(d) $(K - v^d)(K - v^{d-2}) \cdots (K - v^{-d+2})(K - v^{-d}) = 0$.

The next two results give presentations of $S_v(2, d)$ which are similar to that of $U_v(\mathfrak{gl}_2)$.

2.2 Theorem Over $\mathbb{Q}(v)$, the quantum Schur algebra $S_v(2, d)$ is isomorphic to the algebra generated by $e, f, K_1^\pm 1$ subject to the relations:

(a) $K_1K_1^{-1} = K_1^{-1}K_1 = 1$, 
(b) $K_1eK_1^{-1} = ve$, $K_1fK_1^{-1} = v^{-1}f$ 
(c) $ef - fe = \frac{v^{-d}K_1^2 - v^dK_1^{-2}}{v - v^{-1}}$, 
(d) $(K_1 - 1)(K_1 - v)(K_1 - v^2) \cdots (K_1 - v^d) = 0$.

By a change of variable ($K_2 = v^dK_1^{-1}$) we obtain another equivalent presentation of $S_v(2, d)$. 

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2.3 Theorem Over \(\mathbb{Q}(v)\), the quantum Schur algebra \(S_v(2, d)\) is isomorphic to the algebra generated by \(e, f, K_{2}^{\pm 1}\) subject to the relations:

(a) \(K_{2}K_{2}^{-1} = K_{2}^{-1}K_{2} = 1\),
(b) \(K_{2}eK_{2}^{-1} = v^{-1}e, \quad K_{2}fK_{2}^{-1} = vf\)
(c) \(ef - fe = \frac{v^{d}K_{2}^{-2} - v^{-d}K_{2}^{2}}{v - v^{-1}}\),
(d) \((K_{2} - 1)(K_{2} - v)(K_{2} - v^{2})\cdots(K_{2} - v^{d}) = 0\).

For indeterminates \(X, X^{-1}\) satisfying \(XX^{-1} = X^{-1}X = 1\) and any \(t \in \mathbb{N}\) we formally set

\[
\begin{bmatrix} X \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{Xv^{-s+1} - X^{-1}v^{s-1}}{v^{s} - v^{-s}}.
\]

This expression will make sense if \(X\) is replaced by any invertible element of a \(\mathbb{Q}(v)\)-algebra. The next result describes the \(A\)-form \(S_{A}(2, d)\) in terms of the above generators, and gives an \(A\)-basis for the algebra. In this we can take \(S_v(2, d)\) to be given by either presentation 2.2 or 2.3, but we always assume that \(K_{1}\) and \(K_{2}\) are related by the condition \(K_{1}K_{2} = v^{d}\).

2.4 Theorem The integral Schur algebra \(S_{A}(2, d)\) is isomorphic to the \(A\)-subalgebra of \(S_v(2, d)\) generated by

\[
e^{(m)} := \frac{e^{m}}{[m]!}, \quad f^{(m)} := \frac{f^{m}}{[m]!} \quad (m \in \mathbb{N}), \quad K_{1}^{\pm 1}.
\]

The preceding statement is true when \(K_{1}\) is replaced by \(K_{2}\). Moreover, an \(A\)-basis for \(S_{A}(2, d)\) is the set consisting of all

\[
e^{(a)} \begin{bmatrix} K_{1} \\ b_{1} \\ b_{2} \end{bmatrix} f^{(c)}
\]

such that the natural numbers \(a, b_{1}, b_{2}, c\) are constrained by the conditions \(a + b_{1} + c \leq d, \quad b_{1} + b_{2} = d\). Another such basis consists of all

\[
f^{(a)} \begin{bmatrix} K_{1} \\ b_{1} \\ b_{2} \end{bmatrix} e^{(c)}
\]

such that the natural numbers \(a, b_{1}, b_{2}, c\) satisfy the constraints \(a + b_{2} + c \leq d, \quad b_{1} + b_{2} = d\).
2.5 **Theorem** In $S_A(2, d)$ we have the following reduction formulas for all $s \geq 1$ and all $a, b_1, b_2, c \in \mathbb{N}$ with $b_1 + b_2 = d$:

\[
\begin{align*}
\text{(a)} & \quad e^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f^{(c)} = \\
& \quad \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k - 1 \\ s - 1 \end{bmatrix} \begin{bmatrix} b_1 + k \\ k \end{bmatrix} e^{(a-k)} \begin{bmatrix} K_1 \\ b_1 + k \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - k \end{bmatrix} f^{(c-k)} \\
\text{(b)} & \quad f^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} e^{(c)} = \\
& \quad \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k - 1 \\ s - 1 \end{bmatrix} \begin{bmatrix} b_2 + k \\ k \end{bmatrix} f^{(a-k)} \begin{bmatrix} K_1 \\ b_1 - k \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 + k \end{bmatrix} e^{(c-k)}
\end{align*}
\]

where $s = a + b_1 + c - d$ in (a) and $s = a + b_2 + c - d$ in (b).

The next theorem, the quantum analogue of [DG, Thm. 2.4], provides yet another kind of basis for $S_A(2, d)$. In this case we will deduce it as a direct consequence of Theorem 2.4, while in [DG] the analogue of the idempotent basis in Theorem 2.4 was not needed. We remark that the analogues of the integral idempotent bases in Theorem 2.4 above do hold in the classical situation.

2.6 **Theorem** The set

\[
\left\{ e^{(a)} \begin{bmatrix} K_1 \\ b \end{bmatrix} f^{(c)} \mid a, b, c \in \mathbb{N}, a + b + c \leq d \right\}
\]

is an $\mathcal{A}$-basis for $S_A(2, d)$. Another such basis is given by the set

\[
\left\{ f^{(a)} \begin{bmatrix} K_2 \\ b \end{bmatrix} e^{(c)} \mid a, b, c \in \mathbb{N}, a + b + c \leq d \right\}.
\]
3 Quantized enveloping algebras

3.1 The Drinfeld-Jimbo quantized enveloping algebra \( U = U_v(g\mathfrak{l}_2) \) is defined to be the \( \mathbb{Q}(v) \)-algebra with generators \( e, f, K_1, K_1^{-1}, K_2, K_2^{-1} \) and relations

(a) \( K_1K_2 = K_2K_1 \),
(b) \( K_iK_i^{-1} = K_i^{-1}K_i = 1 \) \((i = 1, 2)\),
(c) \( K_1eK_1^{-1} = ve, \; K_1fK_1^{-1} = v^{-1}f \),
(d) \( K_2eK_2^{-1} = v^{-1}e, \; K_2fK_2^{-1} = vf \),
(e) \( ef - fe = \frac{K_1K_2^{-1} - K_1^{-1}K_2}{v - v^{-1}} \).

Let \( U^+ \) (respectively \( U^- \)) be the subalgebra generated by \( e \) (respectively \( f \)) and let \( U^0 \) be the subalgebra generated by \( K_1^\pm, K_2^\pm \). There are \( \mathbb{Q}(v) \)-vector space isomorphisms

\( U \cong U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+ \)

and moreover it is well-known that the algebra \( U \) has PBW-type bases \( \{ e^aK_1^{b_1}K_2^{b_2}f^c \} \) and \( \{ f^aK_1^{b_1}K_2^{b_2}e^c \} \) where \( a, c \in \mathbb{N} \) and \( b_1, b_2 \in \mathbb{Z} \).

In the algebra \( U \), set \( K = K_1K_2^{-1} \), and define \( U_v(\mathfrak{sl}_2) \) to be the subalgebra of \( U \) generated by \( e, f, \) and \( K \). The familiar relations that these elements satisfy are easily deducible from (a) - (e).

3.2 For \( r, s \in \mathbb{Z} \) define

(a) \[ [r] = \frac{v^r - v^{-r}}{v - v^{-1}} \]
(b) \[ \left[ \begin{array}{c} r \\ s \end{array} \right] = \frac{[r][r-1]\cdots[r-s+1]}{[1][2]\cdots[s]} \]

These satisfy the well-known identities

(c) \[ [r + s] = v^{-s}[r] + v^r[s] \]
(d) \[ \left[ \begin{array}{c} r + 1 \\ s \end{array} \right] = v^{-s}\left[ \begin{array}{c} r \\ s \end{array} \right] + v^{r-s+1}\left[ \begin{array}{c} r \\ s - 1 \end{array} \right] \]
For every \( m, t \in \mathbb{N} \), \( c \in \mathbb{Z} \) and any element \( X \) an \( \mathbb{Q}(v) \)-algebra define

\[(e) \quad [m]! = [m][m-1] \cdots [1]\]
\[(f) \quad X^{(m)} = \frac{X^m}{[m]!}\]
\[(g) \quad \left[ \begin{array}{c} X; c \\ t \end{array} \right] = \begin{cases} \prod_{i=1}^{t} \frac{X^{v^{c-i+1}} - X^{-1}v^{-c+i-1}}{v^i - v^{-i}} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases} \quad (X \text{ invertible})\]

We note that \( \left[ X \right] \), as defined in section 2, coincides with the element \( \left[ X; 0 \right] \).

In [LU], Lusztig investigates many relations which hold in quantized enveloping algebras. Those which we will need are contained in the following Lemma.

3.3 Lemma For any \( c, n \in \mathbb{Z} \) and \( m, t, t' \in \mathbb{N} \) the following identities hold in \( U \):

\[(a) \quad K_1^n e K_1^{-n} = v^n e, \quad K_2^n e K_2^{-n} = v^{-n} e,\]
\[(b) \quad K_1^n f K_1^{-n} = v^{-n} f, \quad K_2^n f K_2^{-n} = v^n f,\]
\[(c) \quad \left[ K_1; c \right]_t e \left[ K_1; c+1 \right]_t = e \left[ K_1; c+1 \right]_t, \quad \left[ K_1; c \right]_t f \left[ K_1; c-1 \right]_t = f \left[ K_1; c-1 \right]_t,\]
\[(d) \quad \left[ K_2; c \right]_t e \left[ K_2; c-1 \right]_t = e \left[ K_2; c-1 \right]_t, \quad \left[ K_2; c \right]_t f \left[ K_2; c+1 \right]_t = f \left[ K_2; c+1 \right]_t,\]
\[(e) \quad f^{(m)} e = e f^{(m)} - \left[ K_1 K_2^{-1}; m-1 \right]_1 f^{(m-1)},\]
\[(f) \quad f e^{(m)} = e^{(m)} f - e^{(m-1)} \left[ K_1 K_2^{-1}; m-1 \right]_1,\]
\[(g) \quad \left[ K_i; c+1 \right]_t = v^{t+1} \left[ K_i; c \right]_{t+1} + v^{-c} K_i^{-1} \left[ K_i; c \right]_t,\]
\[(h) \quad \left[ K_i \right]_t \left[ K_i; -t \right]_{t'} = \left[ K_i \right]_t \left[ t + t' \right] \left[ K_i \right]_{t+t'},\]
\[(i) \quad \left[ K_i; c \right]_t = \sum_{j=0}^{t} v^{c(t-j)} \left[ c \right]_j K_i^{-j} \left[ K_i \right]_{t-j} \quad (c \geq 0).\]
Proof. All of these identities are special cases of those appearing on pages 269-270 of Lu. □

4 The algebra $B_d$

In this section we define a homomorphic image of $U$ which will turn out to be isomorphic to the quantum Schur algebra $S_v(2, d)$.

4.1 Let $d$ be a fixed nonnegative integer and define $B_d$ to be the $\mathbb{Q}(v)$-algebra generated by $e, f, K_1^{\pm 1}, K_2^{\pm 1}$ subject to relations (3.1(a)-(e), along with the additional relations

(a) \[ K_1K_2 = v^d \]
(b) \[ (K_1 - 1)(K_1 - v)\cdots(K_1 - v^d) = 0 \]

Note that relation (b) can be replaced by the equivalent relation

(c) \[ (K_1^{-1} - 1)(K_1^{-1} - v^{-1})\cdots(K_1^{-1} - v^{-d}) = 0 \]

and in the presence of relation (a) it can also be replaced by either

(d) \[ (K_2 - 1)(K_2 - v)\cdots(K_2 - v^d) = 0, \quad \text{or} \]
(e) \[ (K_2^{-1} - 1)(K_2^{-1} - v^{-1})\cdots(K_2^{-1} - v^{-d}) = 0. \]

The defining relations of $B_d$ are invariant if $e$ and $f$ are interchanged along with $K_1$ and $K_2$. These interchanges therefore induce an automorphism of $B_d$. We shall often make use of this property, which we will call symmetry, in the sequel.

Let $B^0_d$ be the subalgebra of $B_d$ generated by $K_1^{\pm 1}, K_2^{\pm 1}$. It follows from relations (3.1(a)-(e)) that $\dim(B^0_d) = d + 1$.

4.2 Lemma In the algebra $B^0_d$ we have

\[
\begin{bmatrix}
  K_1 \\
  d + 1
\end{bmatrix} = 0 = \begin{bmatrix}
  K_2 \\
  d + 1
\end{bmatrix}, \quad \begin{bmatrix}
  K_1^{-1} \\
  d + 1
\end{bmatrix} = 0 = \begin{bmatrix}
  K_2^{-1} \\
  d + 1
\end{bmatrix}.
\]
Proof. Using definition 3.2 we have that
\[
\begin{bmatrix}
K_1 \\
\vdots \\
K_1 
\end{bmatrix}
\begin{bmatrix}
d + 1 \\
\vdots \\
d + 1 
\end{bmatrix}
= \prod_{i=1}^{d+1} \frac{K_1 v^{1-i} - K_1^{-1} v^{i-1}}{v^i - v^{-i}}
\]
\[
= \prod_{i=1}^{d+1} \frac{(K_1^{-1} v^{1-i})(K_2 - v^{2(i-1)})}{v^i - v^{-i}}
\]
\[
= \prod_{i=1}^{d+1} \frac{(K_1^{-1} v^{1-i})(K_1 - v^{(i-1)})(K_1 + v^{(i-1)})}{v^i - v^{-i}}
\]
\[
= 0 \text{ by relation 4.1(b)}. 
\]
The other equalities follow in a similar manner using 4.1(c)-(e).
\[\square\]

More generally we have

4.3 Lemma. In the algebra \( \mathcal{B}_d^0 \) we have
\[
\begin{bmatrix}
K_1 \\
b_1 \\
K_2 \\
b_2 
\end{bmatrix}
= 0
\]
whenever \( b_1 + b_2 = d + 1 \).

Proof.
\[
\begin{bmatrix}
K_1 \\
b_1 \\
K_2 \\
b_2 
\end{bmatrix}
\begin{bmatrix}
K_1 \\
b_1 \\
K_2 \\
b_2 
\end{bmatrix}
= \prod_{i=1}^{b_1} \frac{K_1 v^{1-i} - K_1^{-1} v^{i-1}}{v^i - v^{-i}} \prod_{i=1}^{b_2} \frac{K_2 v^{1-i} - K_2^{-1} v^{i-1}}{v^i - v^{-i}}
\]
\[
= \prod_{i=1}^{b_1} \frac{K_1 v^{1-i} - K_1^{-1} v^{i-1}}{v^i - v^{-i}} \prod_{i=1}^{b_2} \frac{K_2 v^{1-i} - K_2^{-1} v^{i-1}}{v^i - v^{-i}}
\]
\[
= (-1)^{b_2} \begin{bmatrix}
d + 1 \\
b_2 
\end{bmatrix}
\begin{bmatrix}
K_1 \\
d + 1 
\end{bmatrix}
\]
\[
= 0 \text{ by 4.1(b)}. 
\]
\[\square\]

An immediate consequence of the preceding lemma is that
\[
\begin{bmatrix}
K_1 \\
b_1 \\
K_2 \\
b_2 
\end{bmatrix}
= 0
\]
whenever \( b_1 + b_2 \geq d + 1 \). When \( b_1 + b_2 \leq d \), these elements are non-zero, but we shall see that the most important case is \( b_1 + b_2 = d \).
4.4 Lemma Suppose that $b_1 + b_2 = d$ and $t \in \mathbb{N}$. Then for $i = 1, 2$ the following identities hold in $\mathcal{B}_d^0$:

(a) $K_i \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix} = v^{b_i} \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix}$

(b) $\begin{bmatrix} K_i; c \\ t \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} b_i + c \\ t \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix}$

Proof. The second equality of the lemma follows immediately from the first using definition 3.2(g), and so we only need to prove (a). Consider the case $i = 1$. We have

$$(K_1 - v^{b_1}) \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix}$$

$$= (K_1 - v^{b_1}) \prod_{i=1}^{b_1} \frac{K_1 v^{1-i} - K_1^{-1} v^{i-1}}{v^i - v^{-i}} \prod_{i=1}^{b_2} \frac{K_2 v^{1-i} - K_2^{-1} v^{i-1}}{v^i - v^{-i}}$$

$$= (-1)^{b_2} (K_1 - v^{b_1}) \prod_{i=1}^{d+1} \frac{K_1 v^{1-i} - K_1^{-1} v^{i-1}}{v^i - v^{-i}}$$

$$= (-1)^{b_2} (K_1 - v^{b_1}) \prod_{i=1}^{d+1} \frac{v^{1-i} K_1^{-1} (K_1^2 - v^{2(i-1)})}{v^i - v^{-i}}$$

$$= 0 \text{ by } \text{3.3(3)}. $$

This proves identity (a) for $i = 1$. The case $i = 2$ follows from symmetry. $\square$

The following result establishes the structure of the algebra $\mathcal{B}_d^0$; it will later be crucial in determining the structure of the entire algebra $\mathcal{B}_d$.

4.5 Theorem In the algebra $\mathcal{B}_d^0$, the set

$$\left\{ \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} \mid b_1 + b_2 = d \right\}$$

is a basis of mutually orthogonal idempotents whose sum is the identity.

Proof. By Lemma 4.4 we have that

$$\left( \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix} \right)^2 = \begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ b_1 & b_2 \end{bmatrix}$$
and so each $\begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix}$ is an idempotent. Now suppose that $b_1 + b_2 = b'_1 + b'_2 = d$ and $b_1 \neq b'_1$. Then either $b_1 + b'_2 \geq d + 1$ or $b'_1 + b_2 \geq d + 1$ and so orthogonality follows by Lemma 4.3. Thus in the algebra $B_d^2$ (which has dimension $d + 1$), we have a set of $d + 1$ distinct mutually orthogonal idempotents. It follows that these must form a basis and that their sum is the identity. □

These idempotents have pleasant commutation relations with the elements $e$ and $f$, given in the next lemma.

4.6 Lemma Suppose that $b_1 + b_2 = d$ and $a \in \mathbb{N}$. Then in the algebra $B_d$

(a) $\begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix} e^a = \begin{cases} e^a \begin{bmatrix} K_1 \\ b_1 - a \\ K_2 \\ b_2 + a \end{bmatrix} & \text{if } b_1 \geq a \\ 0 & \text{if } b_1 < a \end{cases}$

(b) $e^a \begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix} = \begin{cases} e^a \begin{bmatrix} K_1 \\ b_1 + a \\ K_2 \\ b_2 - a \end{bmatrix} & \text{if } b_2 \leq a \\ 0 & \text{if } b_2 > a \end{cases}$

(c) $f^a \begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix} = \begin{cases} f^a \begin{bmatrix} K_1 \\ b_1 - a \\ K_2 \\ b_2 + a \end{bmatrix} & \text{if } b_1 \geq a \\ 0 & \text{if } b_1 < a \end{cases}$

(d) $\begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix} f^a = \begin{cases} f^a \begin{bmatrix} K_1 \\ b_1 + a \\ K_2 \\ b_2 - a \end{bmatrix} & \text{if } b_2 \leq a \\ 0 & \text{if } b_2 > a \end{cases}$

Proof. Each of the relations is similar to prove, so we will only verify the first one. By 3.3(i) and 3.3(j) we have

$\begin{bmatrix} K_1 \\ b_1 \\ K_2 \\ b_2 \end{bmatrix} e^a = e^a \begin{bmatrix} K_1; a \\ b_1 \\ K_2; -a \\ b_2 \end{bmatrix}$
and thus
\[
\begin{bmatrix}
  b_2 + a \\
  a
\end{bmatrix}^{-1} \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} e^a \begin{bmatrix}
  K_2 \\
  a
\end{bmatrix} = \begin{bmatrix}
  b_2 + a \\
  a
\end{bmatrix}^{-1} e^a \begin{bmatrix}
  K_1; a \\
  b_1 \\
  K_2; -a \\
  b_2
\end{bmatrix} \begin{bmatrix}
  K_2 \\
  a
\end{bmatrix}.
\]

Now using 3.3(d) and 3.3(h), this equality becomes
\[
\begin{bmatrix}
  b_2 + a \\
  a
\end{bmatrix}^{-1} \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} \begin{bmatrix}
  K_2; a \\
  b_1
\end{bmatrix} e^a = \begin{bmatrix}
  b_2 + a \\
  a
\end{bmatrix}^{-1} e^a \begin{bmatrix}
  K_1; a \\
  b_1 \\
  K_2 \\
  b_2 + a
\end{bmatrix}.
\]

Transforming this further using 3.3(i) and 4.4(b) yields
\[
\begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} e^a = \begin{bmatrix}
  b_2 + a \\
  a
\end{bmatrix}^{-1} e^a \left( \sum_{j=0}^{b_1} u^{a(b_1-j)} \begin{bmatrix}
  a \\
  j
\end{bmatrix} \begin{bmatrix}
  K_1 \\
  b_1 - j
\end{bmatrix} \begin{bmatrix}
  K_2 \\
  b_2 + a
\end{bmatrix} \right).
\]

Since \( \begin{bmatrix}
  K_1 \\
  b_1 - j \\
  K_2 \\
  b_2 + a
\end{bmatrix} = 0 \) whenever \( j < a \) and \( \begin{bmatrix}
  a \\
  j
\end{bmatrix} = 0 \) whenever \( j > a \), the only possible non-zero term on the right side of the equality corresponds to the case \( j = a \). Consequently, \( \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} e^a = 0 \) whenever \( b_1 < a \) and if \( b_1 \geq a \) then
\[
\begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} e^a = u^{a(b_1-a)} K_1^{-a} \begin{bmatrix}
  K_1 \\
  b_1 - a
\end{bmatrix} \begin{bmatrix}
  K_2 \\
  b_2 + a
\end{bmatrix}.
\]

The remaining part of claim (a) of the lemma now follows using 4.4(b). \( \square \)

4.7 Remark An immediate consequence of the preceding lemma is that
\[
e^a \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} = 0 = \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} f^a \text{ whenever } a + b_1 \geq d + 1
\]
and
\[
f^a \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} = 0 = \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} e^a \text{ whenever } a + b_2 \geq d + 1.
\]

In particular, this implies that both \( e \) and \( f \) are nilpotent of index \( d + 1 \).

The results obtained thus far show that the sets
\[
\left\{ e^{(a)} \begin{bmatrix}
  K_1 \\
  b_1 \\
  K_2 \\
  b_2
\end{bmatrix} f^{(c)} \mid a, c \leq d, b_1 + b_2 = d \right\}
\]
and
\[ \left\{ f(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f(c) \mid a, c \leq d, b_1 + b_2 = d \right\} \]
are spanning sets for the algebra \( B_d \). We will henceforth refer to elements of these spanning sets simply as \textit{monomials}. These sets, however, are not bases. To establish this fact, we need some terminology. Define the \textit{fake degree} of a monomial \( e(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f(c) \) (respectively \( f(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} e(c) \)) to be \( a + b_1 + c \) (respectively \( a + b_2 + c \)) and, in both cases, define its \textit{height} to be \( a + c \).

4.8 \textbf{Theorem} Suppose that \( b_1 + b_2 = d \). Then in the algebra \( B_d \) all monomials of the form \( e(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f(c) \) (respectively \( f(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} e(c) \)) of fake degree \( d + 1 \) are expressible as \( \mathbb{Q}(v) \)-linear combinations of monomials of the same form of strictly smaller fake degree and height.

\textit{Proof.} By symmetry it is enough to prove the claim for monomials of the form \( e(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f(c) \). We use induction on height. The base case here is height one since there are no height zero monomials of fake degree \( d + 1 \). Consider a monomial of fake degree \( d + 1 \) and height \( s \geq 1 \). Suppose that \( a \geq c \). Then \( a \geq 1 \) and by Lemma \[ \text{Lemma 4.6} \] we have
\[ e(a-1) \begin{bmatrix} K_1 \\ b_1 + 1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - 1 \end{bmatrix} e f^{(c)} = e(a) \begin{bmatrix} K_1 \\ b_1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f^{(c)} \]
and so the claim will follow once we show that \( e(a-1) \begin{bmatrix} K_1 \\ b_1 + 1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - 1 \end{bmatrix} e f^{(c)} \) can be expressed in the desired form. But, by \[ \text{Lemma 3.3} \] and \[ \text{Lemma 4.3} \] we have
\[ e(a-1) \begin{bmatrix} K_1 \\ b_1 + 1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - 1 \end{bmatrix} e f^{(c)} = e(a-1) \begin{bmatrix} K_1 \\ b_1 + 1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - 1 \end{bmatrix} f^{(c-1)} \]
and
\[ + |b_1 - b_2 + c - 1| e(a-1) \begin{bmatrix} K_1 \\ b_1 + 1 \\ \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 - 1 \end{bmatrix} f^{(c-1)} \]
where the second term on the right hand side is zero if \( c = 0 \). However, in general, the second term is a monomial of fake degree \( d \) and height \( s - 2 \), which is of the desired form. To analyze the first term, we need to use the fact that if \( M \) is a monomial of fake degree \( d' \) and height \( s' \), then \( M e \) is an \( \mathbb{Q}(v) \)-linear combination of monomials of fake degree at most \( d' + 1 \) (this claim can be verified using 3.3(e) and 4.4(b)). Now by induction,

\[
e^{(a-1)} \left[ \begin{array}{c} K_1 \\ b_1 + 1 \end{array} \right] \left[ \begin{array}{c} K_2 \\ b_2 - 1 \end{array} \right] f^{(c)} \text{ is expressible as an } \mathbb{Q}(v) \text{-linear combination of monomials of fake degree at most } d \text{ and height at most } s - 2.
\]

Thus, by our claim, \( e^{(a-1)} \left[ \begin{array}{c} K_1 \\ b_1 + 1 \end{array} \right] \left[ \begin{array}{c} K_2 \\ b_2 - 1 \end{array} \right] f^{(c)} e \) is expressible as a linear combination of terms of fake degree at most \( d \) and height at most \( s - 1 \). This completes the proof in the case \( a \geq c \); the case \( a \leq c \) is similar and is omitted. \( \square \)

5 Identifications

In this section we show that \( \mathcal{B}_d \) is isomorphic to \( S_v(2, d) \).

5.1 Let \( E \) be an \( \mathbb{Q}(v) \)-module with basis \( \{ e_1, e_2 \} \). There is a canonical representation \( \rho : U \rightarrow \text{End}_{\mathbb{Q}(v)}(E) \) defined by:

\[
e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_1 \mapsto \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}.
\]

Since \( U \) is a bialgebra, we obtain a representation

\[
\rho_d : U \rightarrow \text{End}_{\mathbb{Q}(v)}(E^\otimes d)
\]

in the \( d \)th tensor power of \( E \), for every \( d \in \mathbb{N} \). Specifically, \( \rho_d = \rho^d \circ \Delta^{d-1} \), where \( \Delta^{d-1} : U \rightarrow U^\otimes d \) is the iterated comultiplication map. As stated earlier, \( S_v(2, d) \) is the image of the homomorphism \( \rho_d \).

5.2 Lemma Write \( \overline{X} \) for the image of \( X \in U \) under the representation \( \rho_d : U \rightarrow \text{End}_{\mathbb{Q}(v)}(E^\otimes d) \). Then we have the identities

(a) \[
\overline{K_1} \overline{K_2} = v^d
\]

(b) \[
(\overline{K_1} - 1)(\overline{K_1} - v) \cdots (\overline{K_1} - v^d) = 0
\]

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Proof. We have $K_1 K_2 = \rho_d(K_1 K_2) = \rho_1(K_1 K_2)^{\otimes d}$ since $\Delta(K_i) = K_i \otimes K_i$. But by 5.1,

$$\rho_1(K_1 K_2) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} = v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = v 1_V$$

and so $\rho_1(K_1 K_2)^{\otimes d} = v^d 1_V \otimes 1_V$, which proves claim (a). The case $d = 1$ for claim (b) follows immediately since $\rho_1(K_1) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$. Now $\rho_d(K_1) = K_1^{\otimes d}$ is a diagonal matrix whose entries are $1, v, \ldots, v^d$ (not counting multiplicities). The claim follows.

Henceforth we shall omit the bar, writing the image of $X$ simply as $X$ instead of $\overline{X}$.

5.3 Theorem The Schur algebra $S_v(2, d)$ is isomorphic as a $\mathbb{Q}(v)$-algebra to $B_d$. Moreover, the set of all monomials $e^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f^{(c)} (a + b_1 + c \leq d)$ is a basis over $\mathbb{Q}(v)$. Similarly, another such basis consists of all the monomials $f^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} e^{(c)} (a + b_2 + c \leq d)$.

Proof. By the preceding lemma we see that the surjection $\rho_d : U \to S_v(2, d)$ factors through $B_d$, giving the commutative diagram

$$\begin{array}{ccc}
U & \longrightarrow & S_v(2, d) \\
\downarrow & & \uparrow \\
B_d & \longrightarrow & B_d
\end{array}$$

in which all arrows are surjections. By Theorem 4.8, the set of all monomials of the form $e^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} f^{(c)}$ satisfying $a + b_1 + c \leq d$ spans the algebra $B_d$. This spanning set is in one-to-one correspondence with the set of all monomials in 4 commuting variables of total degree $d$ (set one of the variables equal to 1 to get the correspondence). It is well known that this number is the dimension of $S_v(2, d)$. Hence the map $B_d \to S_v(2, d)$ must be an isomorphism and the spanning set is a basis. A similar argument establishes that the set of all $f^{(a)} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix} \begin{bmatrix} K_2 \\ b_2 \end{bmatrix} e^{(c)}$ satisfying $a + b_2 + c \leq d$ is also a basis of $B_d$. This completes the proof. □
5.4 Theorems 2.2 and 2.3 follow immediately from the previous result by using the relation $K_1 K_2 = v^d$ to remove either $K_1$ or $K_2$ from the generating set. However, the results of the preceding theorem are not sufficient to establish Theorem 2.4 since we only know that the coefficients in the reduction formulas are elements of $\mathbb{Q}(v)$, not necessarily of $\mathcal{A}$. In the next section we indeed show that these coefficients all lie in $\mathcal{A}$. The process by which we prove this does not rely on Theorem 5.3 and so the forthcoming results give alternative proofs of Theorems 2.2 and 2.3 as well as a proof of 2.4.

We now prove Theorem 2.1. In the algebra $S_v(2, d)$, set $K = v^{-d} K_1^2$. Using Lemma 4.4 and Theorem 4.5 we obtain the equality

$$K_1^2 = v^d K = \sum_{b_1 + b_2 = d} K_{b_1} K_{b_2} = \left( \sum_{b_1 + b_2 = d} v^{b_1} K_{b_1} K_{b_2} \right)^2.$$ 

Thus, $e$, $f$, and $K^\pm 1$ comprise another generating set for $S_v(2, d)$. Relations (a)-(c) of Theorem 2.1 are easily seen to be equivalent to the corresponding parts of Theorem 2.2. Relation (d) of these theorems are also equivalent. To see this, note that

$$\prod_{i=0}^{d} (K - v^{d-2i}) = v^{-d^2 - d} \prod_{i=0}^{d} (K_1^2 - v^{2(d-i)})$$

$$= v^{-d^2 - d} \prod_{i=0}^{d} (K_1 - v^{(d-i)})(K_1 + v^{(d-i)}).$$

But by Theorem 2.2 this expression is equal to zero. This completes the proof of Theorem 2.1.

6 The integral reduction

In this section we prove Theorems 2.4, 2.5, and 2.6. For typeographic convenience, we will throughout this section use the abbreviation $K_{b_1, b_2}$ in place of the more cumbersome notation $[K_{b_1)} [K_{b_2)]$. 

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6.1 Theorem In the algebra $B_d$ we have the equality

$$e^{(a)} K_{b_1, b_2} f^{(c)} = \sum_{k=1}^{\min(a, c)} (-1)^{k-1} \begin{bmatrix} b_1 + k \\ k \end{bmatrix} e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}$$

for all $a, b_1, b_2, c \in \mathbb{N}$ satisfying $a + b_1 + c = d + 1$ and $b_1 + b_2 = d$. Similarly we have the equality

$$f^{(a)} K_{b_1, b_2} e^{(c)} = \sum_{k=1}^{\min(a, c)} (-1)^{k-1} \begin{bmatrix} b_2 + k \\ k \end{bmatrix} f^{(a-k)} K_{b_1-k, b_2+k} e^{(c-k)}$$

for all $a, b_1, b_2, c \in \mathbb{N}$ satisfying $a + b_2 + c = d + 1$ and $b_1 + b_2 = d$.

Proof. By symmetry it suffices to prove the first reduction formula only. Suppose first that $c = 0$. Then the claim follows from the remark following Lemma 4.6. If $c > 0$ we will prove the desired result by induction on $a$. The base case here is $a = 0$, which also follows from the remark mentioned above. The identity to be proved can be rewritten in the form

$$0 = \sum_{k=0}^{\min(a, c)} (-1)^k \begin{bmatrix} b_1 + k \\ k \end{bmatrix} e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}.$$  

Suppose this equation is satisfied for some fixed quadruple $a, b_1, b_2, c$ satisfying the conditions $a + b_1 + c = d + 1$, $b_1 + b_2 = d$ and $c \geq 0$. From this we will derive the result for the case $a + 1, b_1 - 1, b_2 + 1, c$. The idea is to multiply (a) on the right by $e$ and commute it all the way to the left. Using Lemma 3.3 we obtain

$$0 = \sum_{k=0}^{\min(a, c)} (-1)^k \begin{bmatrix} b_2 + k \\ k \end{bmatrix} f^{(a-k)} K_{b_1-k, b_2+k} e^{(c-k)} \times \left( e f^{(c-k)} - \begin{bmatrix} K_1 K_2^{-1} \\ 1 \end{bmatrix} f^{(c-k-1)} \right).$$

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Then by Lemmas 4.4 and 4.6 this equation becomes

\[
0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b_1 + k}{k} e^{(a-k)}e^{K_{b_1+k-1,b_2-k+1}f(c-k)}
\]

which is equivalent to the identity

\[
0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b_1 + k}{k} [a - k + 1]e^{(a-k+1)}K_{b_1+k-1,b_2-k+1}f(c-k)
\]

\[
+ \sum_{k=1}^{1+\min(a,c)} (-1)^{k-1} \binom{b_1 + k - 1}{k - 1} [a - b_1 - k + 1]
\]

\[
\times e^{(a-k+1)}K_{b_1+k-1,b_2-k+1}f(c-k).
\]

(The equality \( b_1 - b_2 + k + c - 1 = -(a - b_1 - k) \) follows from the hypotheses \( a + b_1 + c = d + 1 \) and \( b_1 + b_2 = d \).) Writing \( m \) for \( \min(a, c) \) we obtain

\[
(b) \quad 0 = \sum_{k=1}^{m} Re^{(a-k+1)}K_{b_1+k-1,b_2-k+1}f(c-k)
\]

\[
+ [a + 1]e^{(a+1)}K_{b_1,b_2}f(c) + (-1)^m[a - b_1 - m]e^{(a-m)}K_{b_1+m,b_2-m}f(c-m-1)
\]

where

\[
R = \binom{b_1 + k}{k} [a - k + 1] - \binom{b_1 + k - 1}{k - 1} [a - b_1 - k + 1]
\]

\[
= [a + 1] \binom{b_1 + k - 1}{k}.
\]

Now the last term in (b) is zero if \( m = c \). Otherwise \( m = a < c \) and the term takes the form

\[
(-1)^a \binom{b_1 + a}{a} [-b_1]K_{b_1+a,b_2-a}f(c-a-1)
\]

\[
= (-1)^{a+1}[a + 1] \binom{b_1 + a}{a + 1} K_{b_1+a,b_2-a}f(c-a-1).
\]
This shows that all the terms in equation (B) have a common factor of \([a + 1]\). Putting these terms together and dividing by \([a + 1]\) we obtain the equality

\[
0 = \sum_{k=0}^{M} (-1)^k \binom{b_1 + k - 1}{k} e^{(a-k+1)} K_{b_1+k-1,b_2-k+1} f(c-k)
\]

where \(M = m = \min(a + 1, c)\) in the case \(m = c\) and \(M = m + 1 = a + 1 = \min(a + 1, c)\) otherwise. In either case \(M = \min(a + 1, c)\) and the induction is complete. \(\square\)

6.2 Theorem Suppose \(a, b_1, b_2, c \in \mathbb{N}\) with \(b_1 + b_2 = d\). Then if \(s = a + b_1 + c - d > 0\) we have the equality

\[
e^{(a)} K_{b_1,b_2} f(c) = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b_1 + k}{k} e^{(a-k)} K_{b_1+k,b_2-k} f(c-k)
\]

and if \(s = a + b_2 + c - d\) we have the equality

\[
f^{(a)} K_{b_1,b_2} e^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b_2 + k}{k} f^{(a-k)} K_{b_1-b-k,b_2+k} e^{(c-k)}.
\]

Proof. By symmetry we need only verify the first equality. We proceed by induction on \(s\). The case \(s = 1\) is the content of Theorem 6.1. Let \(a, b_1, b_2, c\) be given such that \(a + b_1 + c - d = s + 1\) and \(b_1 + b_2 = d\). If \(c < s\) then \(a + b_1 \geq d + 1\) and so by Lemma 4.6 we have

\[
e^{(a)} K_{b_1,b_2} f(c) = 0.
\]

By a similar argument one sees that this also holds if \(a < c\). Hence we may assume that both \(a\) and \(c\) are \(\geq s\). It is enough to prove the result for the case \(a \geq c \geq s\) since the other case is similar.
Thus \( a \geq 1 \) and we have by induction
\[
e^{(a)} K_{b_1, b_2} f^{(c)} = \frac{e}{a} e^{(a-1)} K_{b_1, b_2} f^{(c)}
\]
\[
= \frac{e}{a} \sum_{k=s}^{\min(a-1, c)} (-1)^{k-s} \left[ \frac{k-1}{s-1} \right] \left[ \frac{b_1 + k}{k} \right] e^{(a-1-k)} K_{b_1+k, b_2-k} f^{(c-k)}
\]
\[
= \frac{1}{a} \sum_{k=s}^{\min(a-1, c)} (-1)^{k-s} \left[ \frac{k-1}{s-1} \right] \left[ \frac{b_1 + k}{k} \right] \left[ a - k \right] e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}
\]
\[
= \frac{a-s}{a} \left[ \frac{b_1 + s}{s} \right]\left[ a - s \right] e^{(a-s-k)} K_{b_1+s+k, b_2-s+k} f^{(c-s-k)}
\]
\[
- \frac{1}{a} \sum_{k=s+1}^{\min(a-1, c)} (-1)^{k-(s+1)} \left[ \frac{k-1}{s-1} \right] \left[ \frac{b_1 + k}{k} \right] \left[ a - k \right] e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}
\]
\[
= \frac{a-s}{a} \left[ \frac{b_1 + s}{s} \right]\left[ a - s \right] e^{(a-k)} K_{b_1+k, b_2+k} f^{(c-k)}
\]
\[
- \frac{1}{a} \sum_{k=s+1}^{\min(a-1, c)} (-1)^{k-(s+1)} \left[ \frac{k-1}{s-1} \right] \left[ \frac{b_1 + k}{k} \right] \left[ a - k \right] e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}
\]

Now the second term in the last equality above can be taken from \( s+1 \) to \( c \) since \( \min(a-1, c) \) is different from \( c \) only if \( a = c \), in which case the additional term in the sum will be zero (the factor \([a-k] \) is zero when \( k = c = a \)). Putting the two sums together and using the identity
\[
\frac{a-s}{a} \left[ \frac{b_1 + s}{s} \right]\left[ a - s \right] - \frac{a-k}{a} \left[ \frac{k-1}{s-1} \right] \left[ \frac{b_1 + k}{k} \right] = \left[ \frac{k-1}{s} \right] \left[ b_1 + k \right] \left[ a - k \right] e^{(a-k)} K_{b_1+k, b_2-k} f^{(c-k)}.
\]
we obtain

\[
e^{(a)}K_{b_1,b_2}f^{(c)} = \sum_{k=s+1}^{c} (-1)^{k-(s+1)} \binom{k-1}{s} \binom{b_1+k}{k} e^{(a-k)}K_{b_1+k,b_2-k}f^{(c-k)}
\]

and this completes the induction. □

6.3 Theorem 6.2 combines with the isomorphism \(B_d \cong S_v(2, d)\) to prove Theorem 2.5.

We now prove Theorem 2.4. First recall the representation

\[(a)\]  \[\rho_d : \mathcal{U} \to \text{End}_{\mathbb{Q}(v)}(E^{\otimes d})\]

of section 5.1. Let \(\mathcal{U}_A\) be the subalgebra of \(\mathcal{U}\) generated by all \(e^{(m)}, f^{(m)} (m \in \mathbb{N})\) and \(K_i^{\pm 1}\). The map in (3) restricts to give a map \(\mathcal{U}_A \to \text{End}_{\mathbb{Q}(v)}(E^{\otimes d})\). Let \(E_A\) be the \(A\)-submodule of \(E\) spanned by the canonical basis elements \(e_1\) and \(e_2\). It is clear that \(E_A\) is stable under the action of \(\mathcal{U}_A\) and hence so is \(E^{\otimes d}_A\). Thus the image of the above map is contain in \(\text{End}_A(E^{\otimes d}_A)\). Consequently we have a representation

\[(b)\]  \[\rho_d^A : \mathcal{U}_A \to \text{End}_A(E^{\otimes d}_A)\]

and by a result of Du [Du] (see also [Gr]) the algebra \(S_A(2, d)\) is precisely the image of this representation.

A fundamental result in [Du] (applied to the \(\mathfrak{gl}_2\) case) is that \(\mathcal{U}_A\) is a free \(A\)-module with basis

\[(c)\]  \[\{e^{(a)}K_1^{\delta_1}K_2^{\delta_2}K_{b_1,b_2}f^{(c)} \mid a, b_1, b_2, c \in \mathbb{N}, \ \delta_i \in \{0, 1\}\}\]

Hence \(S_A(2, d)\) is spanned over \(A\) by the images of the basis elements of (3). By Lemma [4.4] and Theorem [1.5] the elements \(K_1^{\delta_1}K_2^{\delta_2}K_{b_1,b_2}\) with \(\delta_i \in \{0, 1\}\) and \(b_1, b_2\) arbitrary are \(A\)-linear combinations of those \(K_{b_1,b_2}\) with \(b_1+b_2 = d\). Therefore \(S_A(2, d)\) is spanned by all \(e^{(a)}K_{b_1,b_2}f^{(c)}\) with \(b_1+b_2 = d\). But by Theorem 6.2 we see that the set of all such terms satisfying the condition \(a+b_1+c \leq d\) is a spanning set for \(S_A(2, d)\) over \(A\). Being linearly independent over \(\mathbb{Q}(v)\), this set is also linearly independent over \(A\). By symmetry, the set \(f^{(a)}K_{b_1,b_2}e^{(c)}\) with \(b_1+b_2 = d\) and \(a+b_2+c \leq d\) is linearly independent over \(A\). This completes the proof of Theorem 2.4.
It remains to prove Theorem 2.6. By Lemma 4.4 and Theorem 4.5 it follows that

\[
\begin{bmatrix} K_1 \\ b \end{bmatrix} = \begin{bmatrix} K_1 \\ b \end{bmatrix} \cdot 1 = \sum_{b_1+b_2=d} K_{b_1,b_2} \begin{bmatrix} b_1 \\ b \end{bmatrix} = \sum_{b_1+b_2=d} \begin{bmatrix} b_1 \\ b \end{bmatrix} K_{b_1,b_2}
\]

and similarly we have

\[
\begin{bmatrix} K_2 \\ b \end{bmatrix} = \sum_{b_1+b_2=d} \begin{bmatrix} b_2 \\ b \end{bmatrix} K_{b_1,b_2}.
\]

Moreover, the matrix of coefficients in these equations is, with respect to an appropriate ordering of its rows and columns, triangular with 1’s on the main diagonal. So these equations can be inverted over \( \mathcal{A} \) to obtain formulas expressing each \( K_{b_1,b_2} \) as an \( \mathcal{A} \)-linear combination of the idempotents \( K_{b_1,b_2} \) as \( b \) ranges from 0 to \( d \). Thus it follows that the sets

\[
\{1, \begin{bmatrix} K_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} K_1 \\ d \end{bmatrix}\} \quad \text{and} \quad \{1, \begin{bmatrix} K_2 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} K_2 \\ d \end{bmatrix}\}
\]

are spanning sets (over \( \mathcal{A} \)) for the algebra \( S^0_{\mathcal{A}}(n,d) \). Since it is already known that the rank of this free \( \mathcal{A} \)-module is \( d + 1 \) and since the ring \( \mathcal{A} \) is commutative, it follows that the two sets are in fact \( \mathcal{A} \)-bases for \( S^0_{\mathcal{A}}(n,d) \).

6.4 Lemma Let \( a, b, c \) be nonnegative integers satisfying \( a + b + c > d \). Then the elements \( e^{(a)} \begin{bmatrix} K_1 \\ b \end{bmatrix} f^{(c)} \) and \( f^{(a)} \begin{bmatrix} K_2 \\ b \end{bmatrix} e^{(c)} \) of degree \( a + b + c \) are each expressible as \( \mathcal{A} \)-linear combinations of elements of the same form but of degree not exceeding \( d \).

Proof. From the remarks preceding the lemma we know that \( \begin{bmatrix} K_1 \\ b \end{bmatrix} \) and \( \begin{bmatrix} K_2 \\ b \end{bmatrix} \) are expressible as \( \mathcal{A} \)-linear combinations of the idempotents \( K_{b_1,b_2} \) (\( b_1 + b_2 = d \)). Thus the elements \( e^{(a)} \begin{bmatrix} K_1 \\ b \end{bmatrix} f^{(c)} \) (resp., \( f^{(a)} \begin{bmatrix} K_2 \\ b \end{bmatrix} e^{(c)} \)) are expressible as \( \mathcal{A} \)-linear combinations of elements of the form

\[
e^{(a)} K_{b_1,b_2} f^{(c)} \quad \text{(resp.,} \quad f^{(a)} K_{b_1,b_2} e^{(c)}\text{)}
\]
where $b_1 + b_2 = d$ and where the fake degree $a + b_1 + c$ (resp., $a + b_2 + c$) is strictly greater than $d$. By Theorem 2.3 it follows that each element of the form (4) above is expressible as an $\mathcal{A}$-linear combination of elements of the same form
\[(b) \quad e^{(a')}K_{u_1,u_2}f^{(c')} \quad \text{(resp., } f^{(a')}K_{u_1,u_2}e^{(c')}\text{)} \]
where $a' < a$, $c' < c$, $u_1 + u_2 = d$, and $a' + u_1 + c' \leq d$ (resp., $a' + u_2 + c' \leq d$). Now by expressing $K_{u_1,u_2}$ as an $\mathcal{A}$-linear combination of elements of the form $\begin{bmatrix} K_1 \\ b' \end{bmatrix}$, (resp., $\begin{bmatrix} K_2 \\ b' \end{bmatrix}$) where $0 \leq b' \leq d$ we obtain (via left and right multiplication by appropriate elements) formulas expressing each of the elements of the form (4) in terms of $\mathcal{A}$-linear combinations of elements of the form
\[(c) \quad e^{(a')} \begin{bmatrix} K_1 \\ b' \end{bmatrix} f^{(c')} \quad \text{(resp., } f^{(a')} \begin{bmatrix} K_2 \\ b' \end{bmatrix} e^{(c')}\text{)} \]
If an element of this form satisfies the constraint $a' + b' + c' \leq d$ then we leave it be, but for those elements which do not satisfy this constraint we repeat the entire process given above, replacing the element by an $\mathcal{A}$-linear combination of elements of the same form, in which for each element the degree of $e$ is strictly smaller, as is the degree of $f$. After repeating the process finitely many times we obtain the desired result. \[\square\]

Now we can prove Theorem 2.6. By Theorem 2.4 we know that the set
\[B = \{ e^{(a)}K_{b_1,b_2}f^{(c)} \mid a + b_1 + c \leq d, \ b_1 + b_2 = d \} \]
is an $\mathcal{A}$-basis for $S_\mathcal{A}(2, d)$. Now consider the set
\[B' = \left\{ e^{(a)} \begin{bmatrix} K_1 \\ b \end{bmatrix} f^{(c)} \mid a + b + c \leq d \right\}. \]
The sets $B$ and $B'$ have the same cardinality and the ring $\mathcal{A}$ is commutative, thus it will follow that $B'$ is a basis once we can show that it spans.

We know that $S_\mathcal{A}(2, d)$ is spanned by elements of the form $e^{(a)} \begin{bmatrix} K_1 \\ b \end{bmatrix} f^{(c)}$ since $S_\mathcal{A} = S^+_\mathcal{A}S_\mathcal{A}^0S^-_\mathcal{A}$. By the above lemma we know that each such element not satisfying the constraint $a + b + c \leq d$ is expressible as an $\mathcal{A}$-linear combination of elements which do satisfy that constraint. It follows that the set $B'$ is a spanning set for $S_\mathcal{A}(2, d)$. This proves the first part of Theorem 2.6. The second part of the theorem follows by symmetry.
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