Cosmic-ray current driven turbulence in shocks with efficient particle acceleration: the oblique, long-wavelength mode instability

A.M.Bykov\textsuperscript{1}\textsuperscript{*}, S.M.Osipov\textsuperscript{1}, D.C.Ellison\textsuperscript{2}
\textsuperscript{1}Ioffe Institute for Physics and Technology, 194021 St.Petersburg, Russia
\textsuperscript{2}Physics Department, North Carolina State University, Box 8202, Raleigh, NC 27695

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ABSTRACT
In order for diffusive shock acceleration (DSA) to accelerate particles to high energies, the energetic particles must be able to interact with magnetic turbulence over a broad wavelength range. The weakly anisotropic distribution of accelerated particles, i.e., cosmic rays (CRs), is believed capable of producing this turbulence in a symbiotic relationship where the magnetic turbulence required to accelerate the CRs is created by the accelerated CRs themselves. In efficient DSA, this wave-particle interaction can be strongly nonlinear where CRs modify the plasma flow and the specific mechanisms of magnetic field amplification. Resonant interactions have long been known to amplify magnetic fluctuations on the scale of the CR gyroradius, and Bell (2004) showed that the CR current can efficiently amplify magnetic fluctuations with scales smaller than the CR gyroradius. Here, we show with a multi-scale, quasi-linear analysis that the presence of turbulence with scales shorter than the CR gyroradius enhances the growth of modes with scales longer than the gyroradius, at least for particular polarizations. We use a mean-field approach to average the equation of motion and the induction equation over the ensemble of magnetic field oscillations accounting for the anisotropy of relativistic particles on the background plasma. We derive the response of the magnetized CR current on magnetic field fluctuations and show that, in the presence of short-scale, Bell-type turbulence, long wavelength modes are amplified. The polarization, helicity, and angular dependence of the growth rates are calculated for obliquely propagating modes for wavelengths both below and above the CR mean free path. The long-wavelength growth rates we estimate for typical supernova remnant parameters are sufficiently fast to suggest a fundamental increase in the maximum CR energy a given shock can produce.

Key words: radiation mechanisms: non-thermal—X-rays: ISM— (ISM:) supernova remnants—shock waves.

1 INTRODUCTION

Diffusive shock acceleration (DSA) is the most promising mechanism for producing superthermal and relativistic particles in a wide range of astrophysical objects.
ranging from the Earth bow shock to shocks in galaxy clusters (Blandford & Eichler 1987; Jones & Ellison 1991; Malkov & Drury 2001). While this mechanism is believed to be efficient and capable of producing cosmic rays (CRs) to energies well above $10^{15}$ eV in supernova remnants (SNRs), fast and efficient DSA demands that particles interact strongly with large amplitude magnetic fluctuations in the shock vicinity. The amplitude of the required MHD turbulence is substantially higher than the ambient MHD turbulence forcing a bootstrap scenario where the accelerated particles generate the turbulence required for their acceleration. Direct support for the modest self-generation of MHD turbulence has long been seen in heliospheric shocks (e.g., Gurnett 1985) and more recent observations of X-ray synchrotron radiation from several young SNRs provide indirect evidence for extreme super-adiabatic magnetic field amplification (MFA) associated with CR production and DSA (see, e.g., Vink & Laming 2003; Bamba et al. 2005; Uchihama et al. 2007; Vink 2008).

A critical aspect of MFA concerns the dynamic range of the self-generated turbulence. Since the maximum CR energy a given shock can produce is determined largely by the power in the longest wavelength turbulence – that turbulence which is needed to trap CRs with the largest gyroradii – the production of long-wavelength turbulence must be included in any full description of nonlinear DSA. The essential features of our calculation for the growth rate of turbulence with scales larger than the gyroradii of the generating CRs as these CRs interact with turbulence with scales shorter than their gyroradii, were presented for parallel propagating shocks in Bykov et al. (2009). Here, a more complete calculation is given which includes oblique shock geometry.

The situation where CRs interact with short-wavelength turbulence is just what is expected in a shock precursor where the CR current efficiently generates purely growing modes with wavelengths much shorter than the gyroradii of the CRs (i.e., Bell 2004). For wavelengths both below and above the CR mean free path, we perform a multi-scale, linear calculation that takes into account the polarization, helicity, and angular dependence of the growth rates for obliquely propagating modes. An important question for DSA and CR origin has always centered around the maximum particle energy a given shock can produce. For a shock of a given size, age, and magnetic field geometry, the maximum CR energy depends totally on the power in the longest wavelength turbulence. Our estimates suggest that the growth rates for long-wavelength modes are fast enough to allow a significant increase in CR energy.

The study of turbulence generation associated with CRs and DSA has a long history. Magneto-hydrodynamic (MHD) type wave amplification due to the resonant cosmic-ray streaming instability was studied in the context of galactic cosmic-ray origin and propagation since the 1960s (see, e.g. Kulsrud & Cesarsky 1971; Wentzel 1974; Achterberg 1981; Berezinskii et al. 1990; Zweibel 2003). It was proposed by Bell (1978) as a source of magnetic turbulence in the test particle DSA scenario, and nonlinear models of DSA including streaming instabilities and MFA were investigated by Amato & Biasi (2006; Vladimirov et al. 2008) and Reville et al. (2009). A Monte Carlo model of nonlinear DSA with MFA from resonant instabilities induced by accelerated particles, which also accounted for the effects of dissipation of turbulence upstream of a shock and the subsequent precursor plasma heating, was developed by Vladimirov et al. (2008). The Monte Carlo work showed that strong feedback effects between the plasma heating due to turbulence dissipation and particle injection are important for understanding the nonlinear nature of efficient DSA. While the resonant instability is arguably the simplest instability thus far studied, a full description is still beyond any single technique either analytic or computer simulation. Mixed techniques are required where analytical recipes to account for MFA, dissipation, and other effects are blended with simulations.

In addition to resonant instabilities, a number of non-resonant instabilities have been investigated for DSA. A non-resonant acoustic instability, where the cosmic-ray pressure gradient in the shock precursor amplifies compressional disturbances, was investigated by Dorfi & Drury (1985) and Drury & Falle (1986). Berezinskii (1986) and Chalov (1988) generalized this by accounting for a regular magnetic field. The obliquely propagating magnetosonic modes in the inhomogeneous precursors of cosmic-ray modified shocks were further examined by Zank et al. (1990), who also investigated the role of strong, intermediate, and weak cosmic-ray scattering regimes. The effects of the acoustic instability on the particle distribution were investigated by Kang et al. (1992) using a time-dependent numerical simulations of the diffusion-advection transport equations. The non-linear evolution of unstable acoustic waves in the precursors of cosmic-ray mediated shocks with large Mach numbers was shown by Chalov (2010) to result in the formation of a host of small, weak shock waves that could heat the thermal plasma and, therefore, change the parameters of the strong, large-scale shock. Other nonlinear work was done by Malkov & Diamond (2000) who they suggested that the development of unstable acoustic waves might result in shock-train formation that could speed up the acceleration rate and stimulate the inverse cascade of Alfvén waves generated by the accelerated particles. Recently, Beresnyak et al. (2009) proposed a model where the stochastic magnetic fields in the shock precursor are generated through a short-scale dynamo mechanism. In this model, the fluid velocity...
turbulence driving the dynamo is produced through interactions of the CR pressure gradient and density perturbations in the precursor.

A large amount of recent work has been stimulated by Bell’s discovery of a fast, non-resonant instability driven by the CR current (Bell 2004). In this instability, the CR current in the shock precursor drives purely growing, incompressible, electromagnetic modes. These modes have wavevectors along the magnetic field and have wavelengths much shorter than the CR gyroradius radius. Because of the short wavelengths, the CR current is essentially unmagnetized in this regime and induces a return current in the background plasma which is nearly parallel to the locally homogeneous magnetic field. This allows the approximation that the current generating CRs are only weakly perturbed by the magnetic fluctuations they create (we shall discuss this instability in some detail in section 3).

The focus of much of the analytical and numerical work on the Bell instability concerns the saturation level and the spectral properties of the instability (see e.g. Pelletier et al. 2006; Marcowith et al. 2006; Amato & Bland 2009; Luo & Melrose 2009; Vladimirov et al. 2009; Zweibel & Everett 2010). If the approximation that the CR current is fixed as an external parameter, scale fluctuations holds, nonlinear MHD-type simulations, where the CR current is only weakly affected by the short-scale fluctuations, which is certain to be nonlinear if DSA is efficient, places significant demands on plasma particle and fluid code simulations since they must cover a wide dynamic range to fully describe the problem from injection to maximum CR energy. These demands are particularly severe for nonrelativistic shocks with parameters typical of SNRs (see Vladimirov et al. 2008).

In the present paper we study the properties of long-wavelength, oblique modes in DSA for different assumptions on the CR scattering rate in background, short-scale turbulence. The long-wavelength growth rates we calculate depend on the CR current interacting with the short-wavelength turbulence generated by Bell’s instability, which has the fastest growth rate in the short wavelength regime and produces the mode polarizations we assume. While our calculation is linear in $\Delta B/B$, any application in a realistic DSA scenario would assume efficient acceleration with a non-negligible fraction of shock ram pressure being transformed to CRs. Our estimates given in Section 3 for young SNRs, suggest that, for some parameters at least, the production of long-wavelength turbulence in the forward shock precursors of young SNRs will be important. Since the shock structure depends critically on the efficiency of turbulence amplification, cascading, and dissipation, our work must be viewed as a step towards the development of a nonlinear model of DSA where all of the following are treated self-consistently: (1) efficient particle injection and acceleration occur; (2) particles of different energies participate differently in the instability generation and mag-

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1 In this context, hybrid means protons are treated as particles and electrons are modeled as a charge-neutralizing background fluid.
nette field amplification; (3) turbulence cascading and dissipation are accounted for; and (4) the nonlinear feedback of particles and fields on the bulk flow is included self-consistently. To our knowledge, the only techniques that are currently capable of including all of these coupled effects in a calculation suitable for modeling DSA in SNRs, even in parameterized form, are the semi-analytic methods of Blasi and co-workers (see Blasi et al. 2007, Caprioli et al. 2008, and references therein), and the Monte Carlo methods of Vladimirov and co-workers (see Vladimirov et al. 2009, and references therein). Before providing the details of our derivation we outline the steps in the next Section.

2 OUTLINE OF THE MODEL

Our goal is to obtain the growth rates for the cosmic-ray current driven, long-wave instability in a shock precursor containing short-scale fluctuations. The terms “long” and “short” are relative to the minimum CR gyroradius and the first step in our derivation is to obtain the equations of the plasma and magnetic field dynamics averaged over the short-scale fluctuations.

In Section 3 we describe the properties of the short-scale CR-current driven modes produced by Bell’s instability (Bell 2004, 2005). We discuss the polarization properties of the modes, the helicity they induce in the background plasma, and emphasize how the mode polarization is important for the long-wavelength dynamics.

The equation of motion and the magnetic induction equation, both for the background plasma and both averaged over the short-scale modes that are needed to study the long-wavelength growth rates, are discussed in Section 4. The detailed derivations of these averaged equations are given in the Appendix. In Section 5 of the Appendix we obtain the averaged magnetic induction equation. In Section 6 we average the momentum equation. We generalized the mean field method developed in dynamo theory (see Blackman & Field 2002, Brandenburg & Subramanian 2002, Brandenburg 2003) to average the equations accounting for the specific effects of the CRs. The distinctive feature of the derivation is that the CR current, which is responsible for the instability, is fully accounted for. The CR current produces specific polarizations of the modes and this influences the ratio of their kinetic and magnetic energy densities. This, in turn, modifies the mode correlation functions that determine the turbulent transport coefficients in the mean field approach. The correlation functions are presented in Section 7 of the Appendix.

To get the linear dispersion relation for the long-wavelength fluctuations, one needs to know the response of the CR current to the short-scale fluctuating magnetic field, that is, to the Bell turbulence. This is an important point. For solely the Bell instability, the response of the CR current to the short-scale fluctuations can be ignored. In the long-wavelength regime, however, the CR current changes in response to the magnetic field perturbations imposed on the plasma system and these changes need to be accounted for. Here, we calculate the CR response using a kinetic equation for the CR distribution function with a collision operator that describes the CR scattering by magnetic fluctuations. The details of the CR current response derivation are presented in Section A of the Appendix.

Finally, the growth rates we obtain for long-wavelength fluctuations in plasma systems with a CR current are presented in Section 6. Two distinct long-wavelength regimes are discussed. The first regime is where the growing modes, at some position in the shock precursor, have wavelengths between the gyroradius and the mean free path of the lowest energy CRs present at that precursor position. We call this regime “intermediate” and these modes are complementary to those seen in mean-field dynamo theory except now the modes are modified by the presence of the CR current. These long-wavelength modes are produced by the anisotropic CR distribution in the presence of strong, small-scale fluctuations. The second regime is where the growing modes have wavelengths longer than the CR mean free path. We refer to this regime as “hydrodynamical.” We note that in the case of very strong CR scattering, that is when the Bohm limit is obtained and the CR mean free path equals the gyroradius, the intermediate regime has no dynamic range and only the hydrodynamical regime produces the long-wavelength instability.

3 SHORT-SCALE DYNAMICS AND BELL’S INSTABILITY

In MHD-type flows with cosmic rays imbedded in a background plasma, the momentum equation of the background plasma, including the Lorentz force, is given by

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} \right) = -\nabla P + \frac{1}{c}(\mathbf{j} \times \mathbf{B}) + c(n_{i} - n_{e})\mathbf{E} + \nu \Delta \mathbf{u},$$

where \( \mathbf{u} \) and \( \mathbf{j} \) are the bulk velocity and the electric current of the background plasma, respectively. The viscosity \( \nu \) is due to Coulomb collisions (or due to plasma oscillations on scales much less than the CR gyroradii) and \( \nu \) is typically small.
for the effects discussed here. We assume quasi-neutrality for the whole system consisting of background plasma ions of number density \(n_i\), electrons of number density \(n_e\), and cosmic rays of number density \(n_{cr}\). For simplicity we consider cosmic-ray protons only such that \(n_i + n_{cr} = n_e\). Both the background electric current \(\mathbf{j}\) and the electric current of accelerated particles \(\mathbf{J}^{\text{cr}}\) are the sources of magnetic fields in Maxwell’s equations, where the displacement current can be omitted for the slow MHD-type processes under consideration. Furthermore, we assume ideal plasma conductivity in Ohm’s law:

\[
\mathbf{\text{rot}} \mathbf{B} = \frac{4\pi}{c} (\mathbf{j} + \mathbf{J}^{\text{cr}}), \quad \mathbf{E} = -\frac{1}{c} (\mathbf{u} \times \mathbf{B}).
\] (2)

Then the induction equation is given by

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \nu_m \nabla \times \mathbf{B},
\] (3)

where \(\nu_m\) is the magnetic diffusivity due to the Coulomb collisions or MHD plasma oscillations.

To study instabilities in flows with initially quasi-homogeneous magnetic fields, the global current neutralization condition should be fulfilled in Eq. (2). Substituting Eq. (2) in Eq. (3), we obtain the momentum equation in the following form (Bell 2004)

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u} \right) = -\nabla P + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{1}{c} (\mathbf{J}^{\text{cr}} - \epsilon n_{cr} \mathbf{u}) \times \mathbf{B} + \nu \Delta \mathbf{u}.
\] (4)

In this equation, the electric current of accelerated particles \(\mathbf{J}^{\text{cr}}\) is an external current in the momentum equation of the background plasma. The current is governed by sources of energetic particles and by local electromagnetic fields. The current \(\mathbf{J}^{\text{cr}}\) initiates a compensating return current in the background plasma. Bell (2004, 2005) discovered that the system is unstable against linear perturbations that are \(\propto \exp(\gamma t + i k \mathbf{r})\) and the return current drives nearly purely growing modes. Here, \(\gamma\) is the linear growth rate. The wavenumbers of growing modes must satisfy the condition \(kr_{g0} > 1\), where \(r_{g0} = c\rho_0/(\epsilon B)\) is the gyroradius of an accelerated particle of momentum \(p_0\). All particles with \(p > p_0\) contribute to mode growth.

In a cold plasma with sound speed \(a_s\), well below the Alfvén velocity \(v_a\), the linear growth rate obtained by Bell (2004) depends only on the wavevector projection \(k_z\) on the local mean magnetic field, i.e.,

\[
\gamma \approx \gamma_{\text{max}} k_z / k,
\] (5)

where

\[
\gamma_{\text{max}} = v_a \sqrt{k_1 |k| - k^2}
\] (6)
is the growth rate for the modes propagating along the mean field and

\[
k_1 = \frac{4\pi \sqrt{\gamma}}{c^2 \langle B \rangle}.
\] (7)

Here, the bar means the CR current and magnetic field are averaged over fluctuations with scales below \(r_{g0}\).

According to the linear analysis of Bell (2005), the wavenumber of a growing mode must satisfy the condition \(r_{g0}^{-1} < k < k_1\). Therefore, the instability growth rate is higher than the Alfvén frequency \(\nu_a k\). Note that this condition for mode growth \(r_{g0} k_1 > 1\), together with Eq. (7), implies that the anisotropy of the relativistic particle distribution, \(\delta_{\text{cr}}\), exceeds the ratio of the mean magnetic field energy density to the energetic particle energy density \(E_{\text{cr}}\). That is, \(\delta_{\text{cr}} > B^2/(4\pi E_{\text{cr}})\), where the CR current is given by \(J^{\text{cr}} \approx \delta_{\text{cr}} e n_{cr} c\).

The polarization of the growing mode, given by

\[
b_x = i \frac{k_z}{|k_1|} b_y,
\] (8)
is important and we use the fact that the kinetic energy density in the growing mode dominates over the magnetic energy density to get simplified mean field equations. The linear relation between the amplitude of the growing magnetic field \(b(k)\) and the velocity \(v(k)\) of the perturbations is given by

\[
\gamma b(k) = i B_0 k_z v(k).
\] (9)

This yields

\[
|v(k)|^2 \approx \frac{1}{4\pi \rho} \frac{k_1}{|k|} |b(k)|^2,
\] (10)

provided that the kinetic energy density in the growing mode dominates over the magnetic energy density because \(k_1 > k_z\). This is in contrast to Alfvén modes where the energy densities are equal. In Section B (namely, Eq. (152)), the mode polarizations are used to express the pair correlation functions for the fluctuating fields.

We note here that the cosmic-ray current has only a weak response to the short-scale fluctuations, while its response to fluctuations with scales longer than \(r_{g0}\) is much larger. Therefore, the CR current variations can be neglected when averaging the dynamic equations over fluctuations produced by the Bell instability and we perform this averaging in the next Section. For the long-wavelength fluctuations, the current variations must be considered and this averaging is done in Section A of the Appendix.
The averaged equations of large-scale dynamics

Here we average the momentum equation of the background plasma and the induction equation over the short-scale fluctuations produced by Bell’s instability to obtain the mean field dynamics equation. Since the linear growth rate is fast for the Bell mode with the wavevector along the local mean magnetic field, the bulk velocity of the background plasma \( \mathbf{u}(r, t) \) and \( \mathbf{B}(r, t) \) both experience rapid, incoherent fluctuations on scales that are small compared to \( r_{g0} \), the gyroradius of a relativistic particle. The unstable modes with wavevectors along the mean magnetic field are nearly incompressible making the averaging of the mass continuity equation straightforward. The time and spatial scales separation relations are defined by \( J^E = J^\parallel + J^\perp \), \( \mathbf{B} = \mathbf{B} + \mathbf{b} \) and \( \mathbf{u} = \mathbf{V} + \mathbf{v} \), where \( J^\parallel \), \( \mathbf{B} \), and \( \mathbf{V} \) are the averaged electric current of accelerated particles, the averaged magnetic field, and the background plasma bulk velocity, while \( \mathbf{v} \) and \( \mathbf{b} \) are the short-scale bulk velocity and magnetic field. We can now average over the ensemble of short-scale fluctuations, placing these averages in angular brackets, and obtain the averaged momentum equation Eq. (11):

\[
\frac{\partial \mathbf{V}}{\partial t} + \langle (\mathbf{v} \nabla) \mathbf{V} \rangle = -\langle (\mathbf{v} \nabla) \mathbf{v} \rangle - \frac{1}{4\pi \rho} \langle \nabla \times (\nabla \times \mathbf{b}) \rangle - \frac{1}{\rho} \nabla P - \frac{1}{c^2} \langle \nabla^2 - e \sigma_{ec} \nabla \mathbf{V} \rangle + \frac{1}{4\pi \rho} \langle (\nabla \times \mathbf{B}) \times \mathbf{B} \rangle, \tag{11}
\]

and the averaged equation of magnetic induction

\[
\frac{\partial \mathbf{B}}{\partial t} = c \nabla \times \mathbf{V} + \nabla \times (\nabla \times \mathbf{B}) + \nu_{m} \Delta \mathbf{B}. \tag{12}
\]

Here \( \mathbf{V} = (\mathbf{v} \times \mathbf{b}) \) is the average turbulent electromotive force. The coordinate system is defined relative to the unperturbed magnetic field \( \mathbf{e}_z = \mathbf{B}_0 / B_0 \).

5 The long-wavelength cosmic-ray current driven modes

In the presence of a cosmic-ray current, Bell’s instability results in the fast growth of short-scale modes with wavelengths shorter than the gyroradius of the cosmic-ray particles. However, as we showed above when we obtained the mean field dynamical equations averaged over the ensemble of short-scale motions, the strong short-scale turbulence influences the plasma dynamics on scales larger than the CR gyroradii producing the turbulence. Equations (11) and (12) are designed to be applied to the dynamics of modes with scales larger than \( r_{g0} \), i.e., CR particles are magnetized on these scales.

In diffusive shock acceleration with strong scattering, the particle mean free path, \( \Lambda \), is often taken to be \( \Lambda = \eta r_{g0} \), where \( \eta \geq 1 \) and \( \eta = 1 \) is the Bohm limit. If \( \eta > 1 \) there are two different regimes for the large-scale dynamics. The first regime, discussed next, corresponds to \( \eta^{-1} < kr_{g0} < 1 \), where the CR particles have gyroradii small compared to the turbulence scales and can be considered magnetized. The second regime is for \( kr_{g0} < \eta^{-1} \), where the particle gyroradii are large compared to the turbulence wavelengths. These modes are driven by the transverse current of anisotropic, magnetized cosmic rays and are discussed in Section 5.2.

5.1 Long-wavelength current driven modes in the intermediate regime \( \eta^{-1} < kr_{g0} < 1 \)

The dynamic equations averaged over the short-scale fluctuations, that we obtained in Sections 4 and 5 can now be used to study the long-wavelength modes. We start with the linearized equation of motion Eq. (11) and the induction equation for the mean field Eq. (12). Denoting the small departures of physical values from their unperturbed magnitudes by \( \delta \), and performing the analysis in the rest frame of the unperturbed upstream plasma so that \( \mathbf{v} = 0 = \mathbf{B} \), we obtain:

\[
\frac{\partial \delta \mathbf{V}}{\partial t} = -\frac{1}{\rho} \nabla P - \frac{1}{c^2} (\delta j^e_z \mathbf{e}_z + \delta \mathbf{b}) - \frac{1}{c^2} \delta \mathbf{V} \times \nabla \times \mathbf{B} + \frac{1}{4\pi \rho} \delta \nabla \times (\nabla \times \mathbf{B}) + \nu_{m} \Delta \delta \mathbf{B}, \tag{13}
\]

where \( \delta j^e_z \) is the momentum flux of accelerated particles, \( \delta \mathbf{b} \) is the averaged magnetic field, and \( \nu_{m} \) is the magnetic viscosity.

\[
\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\delta \mathbf{V} \times \mathbf{B}_0) + \frac{2}{\rho c} \alpha_t \nabla \times \delta j^e_z \mathbf{e}_z + \frac{1}{2\rho c} \alpha_t \nabla \times (\delta j^e_y \mathbf{e}_y + \delta j^x \mathbf{e}_x) + \frac{3}{2\rho c} \alpha_t \nabla \times (g' \delta b_x \mathbf{e}_y + g' \delta b_y \mathbf{e}_x), \tag{14}
\]

From Eqs. (13-14), and using Eq. (A10), one obtains...
the dispersion relation in the form:

\[ \omega^2 \mp \omega i k_0 \frac{\alpha}{4 \pi \rho} \left[ \frac{1}{2} A(x_0) + \frac{3}{2} \right] - k^2 v_a^2 \pm \]

\[ \pm k_0 v_a^2 \left( 1 + \frac{\kappa_s}{B_0} \right) [A(x_0) - 1] = 0, \tag{15} \]

where \( k_0 = 4 \pi q/c \), \( v_a = B_0/\sqrt{4 \pi \rho} \), the function \( A(x_0) \) is defined by Eq. (A9) in Section A1, and \( \kappa_s \) is a turbulent transport coefficient in the averaged equation of motion and is defined in Eq. (173). The ± signs give the modes with the opposite circular polarizations in the dispersion relation.

To estimate the coefficients in Eq. (15) we assume \((b^2) \approx B_0^2\) and introduce a dimensionless parameter for the amplitude of the Bell turbulence \( N_B = \sqrt{(b^2)/B_0}\). The turbulence correlation time \( \tau_{cor} \) is estimated as the short-scale mode vortex turnover time, while the amplitude of the turbulent velocity is \( \sqrt{\langle \nu^2 \rangle} \). Then, the turbulent mixing length is defined as \( \tau_{cor} \sqrt{\langle \nu^2 \rangle} \approx \tau_{cor} \sqrt{\xi(b^2)/(4 \pi \rho)} = 2 \pi \xi/k_0 \) where \( \xi \) is a dimensionless parameter characterizing the turbulence mixing length. In our numerical estimations below, we assume \( \xi \sim 5 \) for the short-scale turbulence produced by Bell’s instability.

According to Bell (2004), the maximum growth rate of the instability is at the wavelength \( \lambda = 4 \pi/k_0 \). However, due to the nonlinear dynamics of the Bell instability, the maximum of the turbulent energy density may be at somewhat larger wavelengths if the initial spectrum of the fluctuations is a declining function of the wavenumber \( k \) (e.g., Zirakashvili et al. 2008; Riquelme & Spitkovsky 2009). We specify the scalings of the turbulent kinetic coefficients with the dimensionless parameter \( \xi \) as \( k_0 \alpha_\xi/(4 \pi \rho) \approx 2 \pi \sqrt{\xi N_B} v_a \), \( \kappa_s/B_0 \approx \pi N_B \), and assume that the minimum momentum of accelerated protons is \( p_0 = m c \). With these values, the solution to the dispersion equation Eq. (15) is

\[ \omega = \frac{1}{2} (-d \pm \sqrt{d^2 - 4c}), \tag{16} \]

with

\[ d = \mp ik_0 \frac{\alpha}{4 \pi \rho} \left[ \frac{1}{2} A(x_0) + \frac{3}{2} \right], \tag{17} \]

and

\[ c = -k^2 v_a^2 \pm k_0 v_a^2 \left( 1 + \frac{\kappa_s}{B_0} \right) [A(x_0) - 1]. \tag{18} \]

Below, we present simple estimations for frequencies, growing-rates and polarizations for the fastest growing modes. As has long been known, the standard resonant instability (e.g., Blandford & Eichler 1987; Kulsrud 2003) operates in the intermediate magnetized regime \( \eta^{-1} < k r_g \eta < 1 \) in the lack of strong, short-scale turbulence. This resonant effect in the dispersion equation (15) is dominated by the imaginary part of the current response function \( A(x_0) \) defined by Eqs. (A9, A12). The response function was calculated for a power-law momentum distribution of CRs of index \( \alpha \) as defined in Eq. (A2) Modes with different circular polarizations that are distinguished by the sign \( \mp \) in Eqs. (A9, A12) have the same growth rate and with no short-scale turbulence (i.e., \( \alpha = 0 \) and \( \kappa_s = 0 \)), the frequencies of the two circularly polarized modes are determined by

\[ \frac{\omega r_g}{v_a} \approx \pm (1 + i) \sqrt{\frac{3 \pi}{8} \left( \frac{1}{\alpha - 2} - \frac{1}{\alpha} \right) \frac{k_0 r_g}{v_a} (k r_g)^{\alpha - 2}/2}. \tag{19} \]

This growth rate is plotted as the dashed curve in Fig. 1 for \( k_0 r_g = 100 \) and \( \alpha = 4 \).

Now let’s consider the effect of strong, short-scale Bell turbulence on the resonant instability when \( \alpha_\xi \) and \( \kappa_s \) are nonzero. The main contribution to the dispersion equation (16) at \( \xi \sim 5 \) is due to the coefficient \( d \) in Eq. (17). Since the response function \( A(x_0) \approx 1 \) for \( k r_g < 1 \), the growth rate can be approximated as

\[ \omega = i 4 \pi \sqrt{\xi N_B} v_a k. \tag{20} \]

For \( k r_g < 1 \), only the mode with polarization \( \delta b = \delta b(e_x + ie_y) \) is growing while for the case \( k r_g > 1 \), Bell’s instability amplifies the other mode with polarization \( \delta b = \delta b(e_x - ie_y) \). The ratio of the kinetic energy density to the magnetic energy density in the growing mode can be estimated from

\[ |\delta \nabla(k)|^2 \sim \frac{1}{4 \pi \rho_0} \left( \frac{3 k_0 r_g (1 + \pi N_B)}{64 \pi \sqrt{\xi N_B}} \right)^2 |\delta b(k)|^2. \tag{21} \]

It should be noted that the helicity of the unstable, long-wavelength mode studied above is opposite that of the short-scale Bell mode. This provides, in principle at least, the possibility of balancing the global helicity of the system by combining short and long-wavelength modes. Care must be taken however, since recent numerical simulations show a high saturation amplitude of the Bell mode making a nonlinear analysis necessary to address the helicity balance issue. The estimations given above are valid in the intermediate regime and provide simple analytical approximations to the growth rates shown in Fig. 1 for \( k r_g \lesssim 1 \). To turn to the hydrodynamical regime \( k r_g < \eta^{-1} \) (considered in Section 5.2), one should just change \((1 + \kappa_\xi/B_0)\) to \( \kappa_s/B_0 \) in Eq. (18), so that

\[ c = -k^2 v_a^2 \pm k_0 v_a^2 \kappa_s [A(x_0) - 1]/B_0, \tag{22} \]

and then substitute this in Eq. (16).

The coefficient given by Eq. (22) (or more exactly the imaginary part of Eq. A11) dominates the dispersion equation Eq. (16) if \( x_0 \ll 1 \) and \( \eta \) is finite. Because of the ± sign in the imaginary part of Eq. (A11), both circular polarizations will be growing with the same growth rate given
by
\[ \gamma \approx \sqrt{\frac{\pi N_G}{2}} \sqrt{\frac{kk_0}{\eta}} v_a. \] (23)

The transition from the regime described by Eq. (20) to that described by Eq. (22) takes place at
\[ x_0 \sim \frac{1}{\eta} \frac{k_0 r_{g_0}}{32 \pi \xi \sqrt{N_B}}. \] (24)

5.2 Long-wavelength current driven modes in the hydrodynamical regime \((kr_{g_0} < \eta^{-1})\)

The ponderomotive force \((\langle \mathbf{j} \cdot \mathbf{v} \rangle \times \mathbf{b})/(\epsilon \rho)\) in the mean-field momentum equation of the background plasma Eq. (25), is due to the momentum exchange between the background plasma and cosmic rays. Contrary to the short-wavelength regime, the cosmic-ray current response on the magnetic fluctuations is essential in the long-wavelength regime, \(kr_{g_0} < 1\), and results in a non-negligible ponderomotive force. If the perturbation wavelength is longer than the CR mean free path and taking into account Eq. (2), the hydrodynamic approximation can also be applied to the cosmic-ray dynamics. Then, the momentum density \(\mathbf{P}^{(r)}\) and the momentum density flux tensor \(\Pi^{(r)}_{\alpha \beta}\) of the CR-fluid, defined as
\[ \Pi^{(r)}_{\alpha \beta} = \int v_\alpha p_\beta f(r, p, t) d^3p = \rho_c u^{(r)}_{\alpha} u^{(r)}_{\beta} + P_c \delta_{\alpha \beta}, \] (25)
can be approximately derived in closed form. Here \(f(r, p, t)\) is the CR distribution function. Then, the CR momentum equation, derived as a moment of the kinetic equation, takes the form
\[ \frac{\partial \Pi^{(r)}_{\alpha \beta}}{\partial t} + \nabla \cdot \Pi^{(r)}_{\alpha \beta} = \frac{1}{c} \left( [\mathbf{j}^{(r)} \times \mathbf{B}] - c \mathbf{n}_c \mathbf{E} \right)_{\alpha \beta}. \] (26)

Equation (26), averaged over the fluctuations with scales below the CR mean free path and taking into account Eq. (2), has the form
\[ \left\langle \frac{\partial \Pi^{(r)}_{\alpha \beta}}{\partial t} + \nabla \cdot \Pi^{(r)}_{\alpha \beta} \right\rangle = \frac{1}{c} \left( [\mathbf{j}^{(r)} - c \mathbf{n}_c \mathbf{v}] \times \mathbf{B} + [\mathbf{j}^{(r)} - c \mathbf{n}_c \mathbf{v}] \times \mathbf{b} \right)_{\alpha \beta}. \] (27)

The CR distribution in Eq. (28) is nearly isotropic for scales larger than the CR mean free path \(kr_{g_0} < \eta^{-1}\). Then, the isotropic cosmic-ray pressure dominates in \(\Pi^{(r)}_{\alpha \beta}\) and using Eq. (28) one may write a simplified Eq. (28) in the form
\[ \frac{\partial \delta \mathbf{V}}{\partial t} = - \frac{1}{\rho} \nabla (\delta P + \delta P_{cr}) + \frac{1}{4 \pi \rho} (\langle \mathbf{j}^{(r)} - c \mathbf{n}_c \mathbf{v} \rangle \times \mathbf{B})_{\alpha} + \frac{\kappa_t}{\epsilon \rho} \left( \left| \delta j^{(r)} - g' \delta b \right| e_y - \left| \delta j^{(r)} - g' \delta b \right| e_x \right). \] (28)

In the rest frame of the unperturbed shock precursor, the velocity has only the perturbed component \(\mathbf{V} = \delta \mathbf{V}\). The term \(\sim c n_c \delta \mathbf{V}\) is small compared with \(g' \delta \mathbf{b}\) if \(v_{pb}/v_a < 1\), and it is omitted in the dispersion relation.

To study the dispersion properties of the CR current driven modes in the long-wavelength regime, we use the mean-field dynamic equations averaged over short-scale fluctuations, Eqs. (27) and (28), and the mass continuity equation:
\[ \frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{V}) = 0. \] (29)

The equation of state for the background plasma is assumed to be adiabatic, i.e.,
\[ \frac{\partial P}{\partial t} + (\mathbf{V} \cdot \nabla) P + \gamma P \nabla \nabla = 0, \] (30)
where \(\gamma\) is the adiabatic index. Using Eq. (29) and Eq. (30), one obtains
\[ \nabla \delta P = a_0^2 \nabla \delta \rho, \] (31)
where \(a_0 = \sqrt{\gamma P_0/\rho_0}\) is the sound speed of the background plasma.

To get the dispersion equation one must substitute the expressions \(\delta \mathbf{b} = \mathbf{B} - \mathbf{B}_0, \delta \mathbf{V} = \mathbf{V}, \delta j = \mathbf{J}^{(r)} - \mathbf{J}\) into Eqs. (27) and (28). Then, using Eqs. (29) and (31), by neglecting the terms that are small at \(x < x_0\) (where \(x_0\) is defined by Eq. (24)), we finally get the linear dispersion relation for the perturbations with wavelengths larger than the mean free path of the cosmic-ray particle \(\Lambda\), i.e.,
\[ \langle \omega^2 - k^2 \delta \rho - i k \delta \delta \mathbf{b}_0 \rangle \{\mathbf{J}^{(r)} - \langle \mathbf{J}^{(r)} \rangle\} = 0, \] (32)
where we assumed \(k_x = 0, k_y = k_z\), and
\[ W_0 = g' \frac{\kappa_t}{\epsilon \rho_0} \frac{1}{\eta} D_1(k_||, k_\perp), \]
\[ W_1 = g' \frac{\kappa_t}{\epsilon \rho_0} D_1(k_||, k_\perp) - 1, \]
\[ W_2 = g' \frac{\kappa_t}{\epsilon \rho_0} D_1(k_||, k_\perp) - 1 - g' \frac{B_0}{\epsilon \rho_0 \eta^2} D_2(k_||, k_\perp), \]
\[ W_3 = g' \frac{\kappa_t}{\epsilon \rho_0} \frac{1}{\eta} D_1(k_||, k_\perp) + D_2(k_||, k_\perp) + \frac{g' B_0}{\epsilon \rho_0 \eta^2} D_2(k_||, k_\perp), \]
\[ W_4 = g' \frac{B_0}{\epsilon \rho_0 \eta^2} D_3(k_||, k_\perp), \]
\[ W_5 = g' \frac{B_0}{\epsilon \rho_0 \eta^2} D_3(k_||, k_\perp). \] (33)

The angular dependence of the coefficients \(W_i\) for \(i = 0, \ldots,\)

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The coefficients $W_i$ are well above the standard resonant growth rates calculated for the model with short-scale turbulence. All of the coefficients have different asymptotic behaviors at $\eta \gg 1$. The coefficients $W_0$, $W_3$, and $W_4$ scale $\propto \eta^{-1}$, while all of the others are $\propto \eta^{-2}$. It is important to note, however, that contrary to $W_3$, $W_4 \propto k_0$ and $W_0 \propto k^2$.

The growth rates depend on the mode propagation angle and we first consider parallel propagating modes. As discussed above, for modes propagating parallel to the initial magnetic field, the growth rate is

$$\gamma_0(k) \approx \sqrt{\frac{\pi N_B}{2}} \sqrt{\frac{k k_0}{\eta}} v_a,$$

and these modes have the fastest growth rates for the Bohm diffusion regime with $\eta \sim 1$. In Fig. 1 we illustrate the effect of short-scale turbulence on the long-wavelength instability for a particular set of parameters. The dashed and dotted curves show the result without short-scale turbulence. The two curves show two growing short-scale modes, two other modes in Bell’s dispersion equation are not growing and aren’t shown. The departure of the solid and dot-dashed curves from the linear dependence (evident flattening) at small $k$ is due to the transition to the hydrodynamical regime when $k r_{g0} < \eta^{-1}$. It is clearly seen in Fig. 1 that the resonant growth rates calculated for the model with short-scale turbulence are well above the standard resonant growth rates shown by the dashed line at $k r_{g0} < 1$.

If $\eta > 1$, scattering is less efficient than the Bohm limit and the maximum growth rate occurs for obliquely propagating modes. The non-parallel propagating modes are driven by both the cosmic-ray current and the cosmic-ray pressure gradient (i.e., the diffusive part of the cosmic ray current). An analysis of the relative contributions of the corresponding terms in $W_3$, defined in Eq. (33) using Eq. (A19), shows that, at the maximum growth rate, the relative contribution of the cosmic-ray pressure gradient to that of the cosmic-ray current in the shock precursor is $\propto \pi N_B$. Therefore, the cosmic-ray current contribution is larger if the short-scale Bell turbulence is strong. In Fig. 2 the angular dependence of the hydrodynamical, long-wavelength modes is illustrated for a finite temperature plasma with the parameter $\beta = \alpha_0^2 / v_a^2 = 1.0$. One can see that, in contrast to the short-scale Bell instability, the modes propagating along the unperturbed magnetic field are growing while the modes propagating in the opposite direction are damped. It is also clear in Fig. 2 that the fastest growing modes are propagating nearly perpendicular to the unperturbed field $B_0$ for $\eta = 10$, a typical result for $\eta \gg 1$.

To get the propagation angle $\theta_{\text{max}}$ of the mode of maximum growth for $\eta > 1$, one needs to find the maximum of the expression $k_0 W_3(k_0 / k = \cos \theta)$, i.e.,

$$\cos \theta_{\text{max}} = 1/\eta,$$

and the maximum growth rate at $\eta \gg 1$ is determined by

$$\gamma(k) \approx \sqrt{\frac{\pi N_B}{4}} \sqrt{k k_0 v_a}.$$

Figure 1. The figure shows growth rates of the parallel propagating modes as a function of the wavenumber to illustrate the effect of short-scale turbulence on the long-wavelength instability. Equation (15) was solved numerically to generate the curves. The model parameters are $k r_{g0} = 100$ and $\alpha = 4.0$. The solid and dot-dashed curves are simulated for two modes in the model with short-scale turbulence of $\xi = 5$ and $\eta = 10$ to demonstrate the behavior of the modes in the intermediate regime. For comparison, the dashed and dotted curves are calculated for the model without the short-scale turbulence, i.e., with $N_B = 0$ and $\eta \to \infty$ (c.f. Bell 2004).
the validity condition in the form \( \sqrt{\frac{\pi N_B}{4}} \sqrt{\frac{k_0}{k}} M_a^{-1} \ll 1 \) (where \( M_a \) is the Alfvén Mach number of the shock).

Consider the polarization of a mode propagating along the direction giving the maximum growth rate for \( \eta \gg 1 \) (i.e., \( k_\perp \gg k_\parallel \)), see Fig. 2. The amplitude of the magnetic field along the initial magnetic field, \( \delta b_\parallel \), in this mode, exceeds the transverse magnetic field perturbations \( \delta b_\perp \). The maximum velocity component \( \delta V_y \) is in the plane determined by the wave vector and the initial magnetic field

\[
\delta V_y = \frac{\omega}{v_a k_\perp \sqrt{4\pi \rho_0}} \approx - \frac{\omega}{v_a k_\parallel} \delta b_y
\]

and the energy density of the mode is dominated by the kinetic energy

\[
|\delta V_y(k)|^2 \approx \frac{1}{4\pi \rho_0} \frac{\pi N_B}{4} \frac{k_0}{k} |\delta b_y(k)|^2,
\]

since \( k_0 \gg k \), and we assumed \( |k_\perp| \approx k \).

The modes are compressible and from the continuity equation one can write

\[
\frac{\delta \rho}{\rho_0} = \frac{k_\perp \delta V_y + k_\parallel \delta V_z}{\omega} \approx k_\perp \frac{\delta V_y}{\omega}.
\]

Then, using the induction equation given above, the equation \( \nabla \delta B = 0 \), \( k_\perp > k_\parallel \), and \( \delta V_y \gg \delta V_z \), one can estimate \( \frac{\delta \rho}{\rho_0} \approx \frac{\delta \rho}{\rho_0} \approx \frac{\delta \rho}{\rho_0} \approx \frac{\delta \rho}{\rho_0} \approx \frac{\delta \rho}{\rho_0} \).

The complex frequency \( \omega \) is dominated by the growth rate \( \gamma(k) \) in the regime under consideration. The analysis above was performed using the turbulent kinetic coefficients determined by Bell’s turbulence that are valid if

\[
\frac{v_a}{\nu_{\perp}} > k_0 r_{\perp} \frac{v_a}{c},
\]

where \( v_{\perp} \) is the thermal ion velocity. A thorough discussion of the effects of a hot plasma on the short-scale modes was done by Zweibel & Everett (2010).

We emphasize that the growth rates obtained here account for both the CR current and the CR pressure gradient in the presence of short-scale Bell turbulence. In the presence of the short-scale fluctuations, the momentum exchange between the CRs and the flow in the hydrodynamical regime, \( kr_{\perp} \ll \eta^{-1} \), results in a ponderomotive force proportional to the CR current (the last terms in the mean-field momentum equation). As a result, there exist transverse growing modes with wavevectors along the initial magnetic field with growth rates that are proportional to the turbulent coefficient \( \kappa_t \) defined in Appendix D. In the opposite regime with \( \eta \gg 1 \), the fastest growth rates are for the modes having wavevectors nearly transverse to the initial magnetic field. These modes are compressible. If the short-scale Bell turbulence is absent (i.e., when the turbulent coefficients vanish), unstable acoustic modes are produced, as has been studied earlier by Chalov (1988).

6 DISCUSSION AND CONCLUSIONS

Collisionless shocks are complex phenomena where a number of relaxation processes are involved that redistribute the bulk ram kinetic energy into individual superthermal particles and magnetic fields. Strong astrophysical shocks can transfer a sizable fraction of the ram energy from thermal particles to extremely relativistic ones and the great challenge for modeling these shocks comes from this wide dynamic range. Furthermore, if the particles have a wide range in momentum, the self-generated magnetic turbulence must have a correspondingly wide range in wavelengths. Thus far, no exact treatment of this process with plasma simulations or other methods has been possible and approximate techniques must be used.

In this paper, we derive a mechanism for long-wavelength magnetic fluctuation growth in the presence of...
a cosmic-ray current and the short-scale magnetic turbulence produced by Bell’s instability. We use the anisotropic cosmic-ray distribution, interacting with strong Bell-like turbulence with scales below the CR gyroradii, to calculate the growth rate of fluctuations with wavelengths longer than the CR gyroradius or mean free path. As we have emphasized, the power in the longest wavelength turbulence is critical for determining the highest energy CRs a given shock can produce.

The algebra needed to describe the mechanism is long but, schematically, this is what happens: The $\mathbf{J}^\times \times \mathbf{B}/c$ force from the CR current drives the Bell short-scale instability at scales below the CR gyroradius. Strong, short-scale turbulence is produced if the $\mathbf{J}^\times \times \mathbf{B}/c$ force is large enough to dominate the magnetic field tension in the momentum equation. The short-scale turbulence influences the large-scale dynamics through the ponderomotive forces imposed on the plasma by the turbulence and the CR current. To derive the mean ponderomotive force one must average the momentum equation over the ensemble of short-scale fluctuations. When this is done, it is seen that there is a contribution to the mean ponderomotive force (Eqs. D5, D8, D9). In the intermediate regime, to first order in the amplitude of the long-wavelength ponderomotive force, we are able to average the mean electromotive force in the averaged magnetic induction equation (11). The CR contributions are proportional to the turbulent transport coefficient $\alpha_t$. The amplification of long-wavelength magnetic fields in this regime is reminiscent of the large-scale magnetic field dynamo model that is widely discussed in the literature (see e.g. Brandenburg & Subramanian, 2004, for a review).

The fastest growing long-wavelength mode in the intermediate regime has non-zero helicity with a sign opposite to the short-wavelength Bell modes. In the hydrodynamical regime ($kr_0 < \eta^{-1}$), the two modes with similar growth rates (the solid and dot-dashed curves in Fig. 1) dominating the regime $kr_0 < \eta^{-1}$, also have opposite helicity. While it is not possible to draw exact conclusions for the properties of the strong turbulence with our linear analysis, the growth rates of the long-wavelength turbulence we obtain are comparable to those of Bell’s instability for a reasonable range of parameters (compare the maxima of the solid and dashed curves in Fig. 1). This suggests that it may be possible to overcome a fundamental problem and balance the overall helicity with these two instabilities, but non-linear analysis is needed to address the issue.

The nonlinear, long-wavelength instability requires an accounting of the CR response to the short-scale turbulence. Here, we have exploited the fact that the process of long-scale turbulence growth in the presence of short-scale motions and magnetic field fluctuations, has some analogy with the dynamo theory reviewed recently by Brandenburg (2009). The key ingredient in our model, however, is the anisotropic CR distribution.

Previous nonlinear work on the short-scale Bell instability using MHD simulations (e.g., Bell 2004; Zirakashvili & Ptuskin 2008; Zirakashvili et al. 2008) assumed a fixed CR current as an external parameter. These models studied the spectral evolution in the short-scale range as well as the transformation of the turbulence through the subshock. The evolution of the Bell modes downstream from the shock was addressed by Pelletier et al. (2004); Marcowith et al. (2006). In DSA, the shock precursor containing the CR current has a scale length $l_f(p) \approx c\Lambda(p)/v_t$. Therefore, the growing modes are advected through the instability region on a time scale

$$\tau_d(p) \approx c\Lambda(p)/(3v_t^2) .$$

(42)

To amplify a mode of wavenumber $k$ by a factor of a few, one needs the growth rate to satisfy

$$\gamma(k) \cdot \tau_d > 1.$$

(43)

The growth rates can be estimated for a particle diffu-
gating at the angle \( \cos \theta \) is given by

\[
\Lambda(p) = \eta r_g(p) = 3 \cdot 10^{12} \eta \left( \frac{B_0}{1 \mu G} \right)^{-1} \left( \frac{p}{m_p c} \right) \text{ cm.}
\]

Consider the energy density, \( E_{cr} \), for a power-law CR distribution as expected in test-particle DSA. This is the same as given by Eq. (A2) only here we normalize the CR number density to \( n_{cr} \). Then, if the power-law index is \( \alpha \), the CR energy density can be written as

\[
E_{cr} = c \rho_0 n_{cr} \Phi(\alpha),
\]

where

\[
\Phi(\alpha) = \begin{cases} 
\alpha - 3 & \text{for } \alpha > 4 \\
\alpha - 4 & \text{for } \alpha = 4 \\
\ln \left( \frac{p_m}{p_0} \right) & \text{for } \alpha < 4,
\end{cases}
\]

and \( p_m \) is the maximum CR momentum and \( p_0 \) is the minimum. It is convenient to write

\[
E_{cr} = \epsilon_{cr} \frac{n_p m_p v_s^2}{2},
\]

where \( \epsilon_{cr} \) is the fraction of the far upstream shock ram pressure transferred to the cosmic-ray energy density. Note that \( \epsilon_{cr} \) varies with precursor position and with the minimum CR momentum, \( p_0 \). Then the maximum growth rate achievable in the model, for a mode of wavenumber \( k = 2\pi/\Lambda \) propagating at the angle \( \cos \theta \approx \eta \gamma^{-2} \) to the initial magnetic field, is given by

\[
\gamma_{\max} \tau_0 \approx 1.7 \left( \frac{n N_B}{10} \right)^{1/2} \left( \frac{\Phi}{10} \right)^{-1/2} \times \left( \frac{\epsilon_{cr}}{0.1} \right)^{1/2} \left( \frac{v_s}{0.01c} \right)^{-1/2}.
\]

Equation (48) demonstrates that, for parameters typical of young SNRs, and for acceleration efficiencies that are high enough (i.e., \( \epsilon_{cr} \geq 0.1 \)), long-wavelength fluctuations can be strongly amplified. The validity requirement given by Eq. (11) is fulfilled for most of the conditions expected in young galactic SNRs.

In DSA, particle spectra far upstream in the shock precursor close to the escape boundary can be approximated by Eq. (A2) where the minimal momentum \( p_0 \) can be just a few times smaller than \( p_m \). In contrast, just upstream from the subshock, the minimal momentum is much lower, i.e., \( p_0 \sim m_p c. \) Therefore, for \( p_m \geq 10^5 m_p c \), the particle mean free path \( \Lambda \lesssim 10^{18} \) cm and fluctuations with wavelengths larger than \( \Lambda \) would grow with a time scale on the order of 1,000 yr for a shock of velocity \( v_s \sim 0.01c \).

Besides increasing the maximum CR proton energy a shock can produce, the presence of strong, long-wavelength magnetic field fluctuations can affect synchrotron emission from relativistic electrons (see, e.g., Vink & Laming 2003; Bamba et al. 2003; Uchiyama et al. 2007; Vink 2008). Temporal, spatial, spectral, and polarization features seen in synchrotron emission from the shells of young supernova remnants will all be modified to some extent by long-wavelength turbulence. The observation of these features can provide unique information on the properties of the long-wavelength fluctuations (see, e.g., Bykov et al. 2008, 2009). In case of \( \eta > 1 \), and for the quasi-parallel shocks considered here, the growing long-wavelength magnetic field fluctuations are propagating obliquely to the initial magnetic field. This means they must be accompanied by plasma density fluctuations that are interacting with the shock front, an effect that potentially can be studied through the optical line observations of SNRs (see, e.g., Raymond et al. 2010).

The interpretation of optical observations might also be influenced, at least for the quasi-parallel shocks we studied here. For \( \eta > 1 \), the growing, long-wavelength magnetic field fluctuations are propagating obliquely to the initial magnetic field and they must be accompanied by plasma density fluctuations. These density fluctuations might produce observable features in optical line observations of SNRs (see, e.g., Raymond et al. 2010).

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APPENDIX A: LINEAR RESPONSE OF THE COSMIC RAY CURRENT

A small perturbation $\delta B$ imposed on the local mean magnetic field in the form of a plane monochromatic wave results in a linear response of the cosmic-ray current $\delta J^{cr}$. General expressions for the linear response of a background plasma at rest with an arbitrary particle distribution, were presented by [Krall & Trivelpiece (1973)]. Kinetic equations will be used here to derive the linear response of the cosmic-ray current. For the unperturbed CR distribution, we consider distributions that are typical for the diffusive shock acceleration model (e.g., Blandford & Eichler 1987). In this case, the momentum distribution of accelerated particles allows a local approximation to a weakly inhomogeneous (on scales of the order of the cosmic-ray gyroradii) distribution given by

$$f_0(p) = \frac{n_0}{4\pi} N(p) \left(1 + \frac{3v_s p_s}{c p}\right),$$  \hspace{1cm} (A1)

where $v_s$ is the drift velocity of the accelerated particles along the mean magnetic field $B_0$ relative to the background plasma ions, and $N(p)$ is the isotropic part of the cosmic-ray particle momentum distribution of the power-law form

$$N(p) = \frac{(\alpha - 3)p_0^{\alpha-3}}{p^{\alpha}}, \quad p_0 \leq p \leq p_m, \quad \alpha > 3,$$  \hspace{1cm} (A2)

normalized as $\int_{p_0}^{p_m} N(p)p^2 dp = 1$.

Actually, in DSA the particle distribution is inhomogeneous on the scale $l_f \approx \Delta c/v_s$, which is much larger than the particle gyroradius in the nonrelativistic shocks considered here. Thus, in our local analysis, the minimum momentum of the particle distribution $p_0$ in Eq. (A2) is position dependent. In the far upstream region, $p_0$ approaches $p_m$ making the distribution given by Eq. (A2) rather narrow. In contrast, at the shock front $p_0 \approx m_v c$ and the distribution is much broader.
The particle distribution function of energetic particles, \( f(x, p, \theta, \phi, t) \), disturbed by a magnetic field, satisfies the kinetic equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + eE \cdot \frac{\partial f}{\partial p} - \frac{e}{c} (B_0 + b) \cdot \vec{\partial} f = I[f]. \tag{A3}
\]

where \( E \) and \( b \) are the field amplitudes of the imposed MHD disturbance, and \( \vec{\partial} \) is the momentum rotation operator, defined by

\[
\vec{\partial} = p \times \frac{\partial}{\partial p}. \tag{A4}
\]

### A1 Perturbations propagating along the initial magnetic field

Let \( \delta f \) be the perturbation of the distribution function due to imposed harmonic perturbations of electromagnetic fields \( \delta b \) and \( E \sim \exp(\mathbf{k} \cdot \mathbf{r} - \omega t) \) propagating along the initial magnetic field direction \( \mathbf{k} = k_x = B_0/B_0 \). We introduce the CR scattering rate by magnetic fluctuations, \( \nu \), assuming the simplest form for the CR collision operator in Eq. (A3), i.e., \( I[f] = -\nu (\delta f - \delta f_{iso}) \), where \( \delta f_{iso} \) is the isotropic part of the disturbed distribution function. Then, \( \delta f \) satisfies

\[
-i(\omega - kv_x)\delta f + e\left(\mathbf{v} \times \mathbf{B}_0\right) \frac{\partial \delta f}{\partial p} + c^2 \nu \delta f =
\]

\[
= -e \left( E + \frac{\mathbf{v} \times \delta b}{c} \right) \frac{\partial f_0}{\partial p} + c^2 \nu \delta f. \tag{A5}
\]

In the case of parallel propagation, it is convenient to distinguish between two circular polarization modes \( \delta b = \delta b (e_x \pm ie_y) \). Then, the CR current response from Eq. (A5) takes the form

\[
\delta J^c \approx \pm \frac{3}{4} \frac{e}{\omega} \int_0^\infty \frac{v_0}{1 + x t \pm i \eta} N(p) p^2 dp,
\]

where \( x = -\frac{k c p}{e B_0}, a = \frac{\nu}{\Omega} = \frac{\epsilon}{\eta} B_0/c, \) and \( \Omega = \frac{e B_0}{p} \). From Maxwell's equations one obtains \( E = E (e_x \pm ie_y) = \pm i \frac{\omega}{k c} \delta b (e_x \pm ie_y) \). We neglected \( E \) in the right hand side of Eq. (A5) assuming \( v_{ph}/v_x \ll 1 \). Then using \( J^c = e N c v s \) from Eq. (A1), an integration over time of Eq. (A5) yields

\[
\delta J^c \approx g' \delta b \int_0^\infty \sigma(p) N(p) p^2 dp, \tag{A7}
\]

\[
\sigma = \frac{3}{2x} + \frac{3}{8x} \left( 1 - \frac{1}{x^2} + \left( \frac{x}{a} \right)^2 \right) \Psi_1 - \frac{3a}{2x^3} \Psi_2 \mp
\]

\[
\mp i \left\{ \frac{3}{4x} \left( 1 - \frac{1}{x^2} + \left( \frac{x}{a} \right)^2 \right) \Psi_2 - \frac{3a}{2x^3} + \frac{3a}{2x^3} \Psi_1 \right\},
\]

\[
\Psi_1(x) = \ln \left[ \frac{(x + 1)^2 + a^2}{(x - 1)^2 + a^2} \right],
\]

\[
\Psi_2(x) = arctg \left( \frac{x + 1}{a} \right) + arctg \left( \frac{x - 1}{a} \right). \tag{A8}
\]

For convenience, we introduce the function \( A(x_0) \) defined as

\[
A(x_0) = \int_0^\infty \sigma(p) N(p) p^2 dp,
\]

where \( x_0 = \frac{k c p_0}{e B_0} \) giving:

\[
\delta J^c = g' A(x_0) \delta b. \tag{A10}
\]

Then, in the long-wavelength limit \( x_0 \ll 1 \), \( A(x_0) \) takes the form

\[
A(x_0) \approx \frac{1}{1 + 1/\eta^2} \left( 1 \mp \frac{i \eta}{\eta} \right). \tag{A11}
\]

In the intermediate limit, \( \eta \rightarrow \infty \), Eq. (A8) has a form equivalent to that of Bell (2004), i.e.,

\[
\sigma = \frac{3}{2x^3} + \frac{3}{4x} \left( 1 - \frac{1}{x^2} \right) \ln \frac{x + 1}{x - 1} \mp
\]

\[
\mp i \frac{3}{4x} \left( 1 - \frac{1}{x^2} \right) \Theta (|x| - 1), \tag{A12}
\]

where \( \Theta(x) \) is the Heaviside function. For the important case of a test particle distribution of shock accelerated CRs with \( \alpha = 4 \), Eq. (A10) reduces to that of Bell (2004) and then

\[
A(x_0) = \int_{p_0}^\infty \sigma(p) N(p) p^2 dp =
\]

\[
= \frac{3}{8} \left( 1 + \frac{1}{x_0^2} \right) - \frac{3}{16} x_0 \left( \frac{x_0 + 1}{x_0 - 1} \right)^2 \ln \frac{x_0 + 1}{x_0 - 1} \mp
\]

\[
\mp \frac{3}{16} x_0 \left( 1 - \frac{1}{x_0^2} \right), \ \text{for} \ x_0 > 1
\]

\[
\mp 1, \ \text{for} \ x_0 < 1. \tag{A13}
\]

In the short-wavelength limit, \( x_0 \gg 1 \) and \( A(x_0) \) has the same asymptotic behavior as for finite \( \eta \) as is clearly seen in Fig. (A1).

### A2 Long-wavelength CR current and pressure responses to oblique magnetic perturbations

In the case of parallel propagating perturbations, the simple approximations for the linear response of the CR current
were obtained in both the intermediate and the hydrodynamical regimes. In the case of oblique perturbations, the general equations were obtained in the form of infinite sequences of Bessel functions with cyclotron and Cherenkov resonant denominators (see e.g., Krall & Trivelpiece 1973). Our main aim in this paper is to address the long-wavelength dynamics of the system with cosmic rays. If the wavelength exceeds the particle mean free path (the long-wavelength limit), or in the other words if $kc/\nu$ is small, one can get simple analytic expressions for both the cosmic-ray current and the cosmic-ray pressure responses that can be used to obtain the wave dispersion equations. Note here that, contrary to longitudinal propagating perturbations, the oblique perturbations induce a cosmic-ray pressure response. The linear response function satisfies

$$\frac{\partial \delta f}{\partial t} + \nu \frac{\partial \delta f}{\partial r} - \frac{ec}{\varepsilon} B_0 \cdot \nabla f = 0.$$  
(A14)

In the long-wavelength limit, one can neglect terms of the order of $kc/\nu \ll 1$. Now, we define a coordinate system with the z-axis along the unperturbed initial magnetic field, $B_0$, the polar angle $\theta$, and the azimuthal angle $\varphi$ in the $x-y$ plane. Thus, $k_\parallel = k \cos \theta$ and $k_\perp = k \sin \theta$. To obtain a solution to Eq. (A14), we neglect the electric field $E$ on the right-hand-side, assuming $\nu \ll v_s$, and also assume the low frequency perturbation $\omega \ll ck$. Then, the solution to Eq. (A14) can be presented as a superposition of spherical harmonics in $\theta$ and $\varphi$ with coefficients that depend on the particle energy.

$$\frac{ec}{\varepsilon} b \cdot \nabla f_0 = S_1 \sin \theta \cos \varphi + S_2 \sin \varphi \sin \varphi,$$

$$S_1 = \frac{e}{c} \delta b_ \parallel 3v_e \eta \mu N(p),$$

$$S_2 = \frac{e}{c} \delta b_ \perp 3v_e \eta \mu N(p).$$  
(A15)

Then, presenting the angular dependence of the distribution function as

$$\delta f(p) = A_0(p) + A_1(p) \cos \theta + A_2(p) \sin \theta \cos \varphi + A_3(p) \sin \theta \sin \varphi,$$  
(A16)

one obtains, in the limit $ck/\nu \ll 1$, the four equations for the coefficients:

$$ik_\parallel c A_0(p) + \frac{1}{\eta} \Omega A_1(p) = 0,$$

$$ik_\perp c A_0(p) + \frac{1}{\eta} \Omega A_2(p) - \Omega A_3(p) = S_1,$$

$$ik_\parallel c A_0(p) + \Omega A_2(p) + \frac{1}{\eta} \Omega A_3(p) = S_2.$$  
(A17)

Using the above, the linear responses of the cosmic-ray current and pressure can be obtained from

$$\delta J^\nu = \epsilon \int v \delta f(p) d^3p,$$

$$\delta P_{cr} = \frac{1}{3} \int \epsilon v^2 \delta f(p) d^3p.$$  
(A18)

Substituting the solutions of Eqs. (A17), into (A16) and (A18), and performing the integration over the particle momentum $d^3p$, we obtain finally the cosmic-ray current and pressure response on the imposed oblique magnetic fluctuations:

$$\delta J^\nu = \frac{4\pi}{3} \epsilon c \int_0^\infty \{A_1(p)e_\parallel + A_2(p)e_\perp + A_3(p)e_\varphi\} p^2 dp =$$

$$= g' \frac{k}{k^2 + \frac{1}{\eta^2}} \left\{k_\parallel \delta b + \frac{1}{\eta} k \times \delta b\right\},$$

$$\delta P_{cr} = \frac{4\pi}{3} \epsilon \int_0^\infty \frac{A_0(p)p^3 dp}{3} =$$

$$= \frac{iB_0}{c} g' \frac{e_\parallel}{k^2 + \frac{1}{\eta^2}} \left\{k_\parallel \delta b + \frac{1}{\eta} k \times \delta b\right\}.$$  
(A19)

We note that in the case of the short-wavelength limit, the procedure described above cannot be applied because $kc > 1$. In this case, one should treat cyclotron and Cherenkov resonances.

**APPENDIX B: CORRELATIONS IN BELL MODES**

To obtain the mean field dynamic equations in a closed form one must express the mean electromotive force $\delta E$, $(\langle \nabla \varepsilon \rangle \nu)$, $(\langle b \nu \rangle b)$, and other terms through the correlators of the short-scale Bell fluctuations $(\langle v_\alpha(k) b_\beta(k) \rangle)$ and $(\langle b_\alpha^*(k) b_\beta(k) \rangle)$. Since the maximal growth rates of the Bell modes are along the local mean magnetic field, it is convenient to present the correlations in a coordinate system with the z-axis along $e_z = \mathbf{B}/|\mathbf{B}|$, where $e_\alpha, e_\alpha', e_\beta$ are the unit vectors in the plane transverse to $e_z$. Doing this we obtain:

$$\langle b_\alpha^*(k) b_\beta(k) \rangle = \frac{2}{\mathbf{B}^2(k_{\beta}))} \times$$

$$\times \delta(k_{\beta}) \delta(k_{\alpha'}) \left( \begin{array}{ccc} 1 & -i & 0 \\ -i & \frac{k_{\alpha'}}{|k_{\alpha'}|} & 0 \\ 0 & 0 & 1 \end{array} \right).$$  
(B1)
Figure A1. The dependence of the dimensionless complex response function $A(x_0)$ on $x_0 = kr_0$. The imaginary parts $\Im A(x_0)$ are shown as the dotted curves, while $1 - \Re A(x_0)$ are the solid curves. Left panel: the hydrodynamic case with the dimensionless particle mean free path $\eta = 2$ Right panel: the limit of Bell (2004) corresponding to $\eta \to \infty$.

\[ \langle v_α^*(k) v_β(k) \rangle = \frac{1}{4\pi |k|^3} \int \frac{k_1}{|k|^3} \langle b^*(k) b(k) \rangle \]  

\[ \langle v_α^*(k) b_β(k) \rangle = \frac{1}{\sqrt{4\pi |k|^3}} \int \frac{k_1}{|k|^3} \langle b^*(k) b(k) \rangle \times \delta(k_{x'}) \delta(k_{y'}) \left( \begin{array}{ccc} \frac{1}{|k|^3} & 1 & 0 \\ -1 & \frac{1}{|k|^3} & 0 \\ 0 & 0 & \frac{1}{|k|^3} \end{array} \right) \]  

\[ \langle b_α^*(k) v_β(k) \rangle = -\langle v_α^*(k) b_β(k) \rangle \]  

APPENDIX C: MEAN-FIELD INDUCTION EQUATION

To get the mean field equations we apply an averaging procedure that is widely used in the mean field dynamo theory (Blackman & Field 2002; Brandenburg & Subramanian 2002; Brandenburg 2005). The distinctive feature of our model is the presence of the cosmic-ray current and we shall derive the cosmic-ray current effect on the mean field dynamics and unstable modes. The electromotive force, $\overline{E}$, satisfies the equation

\[ \frac{\partial \overline{E}}{\partial t} = \left\langle \frac{\partial \nu}{\partial t} \times b \right\rangle + \langle \nu \times \frac{\partial b}{\partial t} \rangle. \]  

To treat $\partial \overline{E}/\partial t$ in Eq. (C1) we obtain from Eq. (B1) the equation for short-scale variations of the bulk velocity, $\nu$, including second-order correlations:

\[ \frac{\partial \nu}{\partial t} = -\frac{1}{c_p} (j^\nu - en_{\nu^2} \nabla) \times b \]  

\[ + \frac{en_{\nu^2}}{c_p} (\nu \times B) - (\nabla \nu) v - (\nabla v) \nabla + \frac{1}{4\pi \rho} (\nabla \times b) \times B + \frac{1}{4\pi \rho} (\nabla \times (\nabla \times b)) - (\nabla v) v + (\nu \nabla) v + \frac{en_{\nu^2}}{c_p} (\nu \times b) - \frac{en_{\nu^2}}{c_p} (\nu \times b) + \nu \Delta \nu, \]  

where $\overline{E}$ is the averaged cosmic-ray current. In Eq. (C2), the short-scale density fluctuations were omitted because the fastest growing modes are the incompressible modes with wave vectors along the local mean magnetic field. The short-scale fluctuations of the cosmic-ray current were also neglected being small for $kcp_{\rho}/(cB_0) > 1$ as shown in Appendix A.

The fluctuating part of the magnetic field, $b$, satisfies the equation

\[ \frac{\partial b}{\partial t} = \nabla \times (\nu \times B) + \nabla \times (\nabla \nu \times B) + \nabla \times (\nu \times B) + \nu \nabla \Delta B. \]  

Substituting Eqs. (C2) and (C3) into Eqs. (C1), (D1), and (D2), one obtains:

\[ \frac{\partial \overline{E}}{\partial t} = -\frac{1}{c_p} \left(\langle j^\nu - en_{\nu^2} \nabla \rangle \times b \right) + \langle \nu \times \left(\nabla \times \nu \times B \right) + \nabla \nabla \left(\nabla \nu \times B \right) \times B \rangle \]  

The last term, $\overline{E}/\tau_{\nu^2}$, in Eq. (C4) approximates the time relaxation of triple correlations with the time scale $\tau_{\nu^2}$ (see, e.g., Brandenburg & Subramanian 2005). The correlation time is typically expected to be about the turnover time of the turbulence. For the sake of simplicity we use
the same relaxation time $\tau_{cor}$ for all of the triple correlations. A more rigorous analysis that distinguishes the correlation times of different triple correlations is beyond of the scope of this paper and will be done separately. The time derivative, $\partial \mathbf{E} / \partial t$, in Eq. (C4) suppresses the mean-field variations on time scales below $\tau_{cor}$ and is analogous to the Faraday displacement current in Maxwell’s equations (e.g., Brandenburg 2009). This term can be omitted in the analysis of the long-wavelength modes of frequencies $\omega \tau_{cor} \ll 1$ in the close analogy with the well-known MHD approximation.

For Bell’s instability, the terms in Eq. (C4) containing the short-scale magnetic field correlators Eq. (B5) and Eq. (B1) (apart from a term with the cosmic-ray current) are $k_x/k_{x'} \gg 1$ times smaller than the velocity correlator Eq. (B2). With this approximation, the equation for the long-wavelength perturbations

$$\nabla \cdot \mathbf{V} \psi_{cr,0} = \frac{1}{\rho c} (\mathbf{B}^2) \left( \frac{2}{2} \nabla \cdot \mathbf{V} \psi_{cr,0} \right) \mathbf{e}_{x',0} +$$

and $\nabla \cdot \mathbf{V} \psi_{cr,0} + $ (C5)

Then, the mean field induction equation yields

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{V} \times \mathbf{B} \right) +$$

(\mathbf{B}^2) \tau_{cor} \mathbf{V} \times (\mathbf{V} \cdot \mathbf{e}_{x',0} +$$

(C7)

The long-wavelength perturbations $\mathbf{J}_{x'}^w$ in Eqs. (C5) and (D5) were derived in a coordinate system with the axis $\mathbf{e}_{x'} = \mathbf{B}/|\mathbf{B}|$ along the local mean magnetic field. To make a linear analysis of the dispersion relations it is convenient to transform the vector coordinates to the laboratory system where the $z$-axis is along the unperturbed magnetic field and the shock normal $\mathbf{e}_z = \mathbf{B}_0/B_0$. To first order in the small amplitudes of the current and field perturbations, the transformation yields:

$$\mathbf{e}_{x'} \approx (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_{x'} \approx -g' \delta b_x + (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_{x'} \approx -g' \delta b_y + (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_y \,.$$ (C8)

(\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_y \approx -g' \delta b_y + (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_y \,.$$ (C9)

APPENDIX D: THE AVERAGED MOMENTUM EQUATION

To get the equation of motion averaged over the short-scale fluctuations in a closed form, we derive $\langle (\mathbf{v} \nabla) \mathbf{v} \rangle$ and $\langle (\mathbf{b} \nabla) \mathbf{b} \rangle$ in Eq. (11) for large-scale motions using the mean-field approximation as it was described in Appendix C. This yields:

$$\frac{\partial}{\partial t} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle = \left( \frac{\partial (\mathbf{v} \nabla) \mathbf{v}}{\partial t} \right) + \langle (\mathbf{v} \nabla \mathbf{v}) \rangle \mathbf{v} \rangle,$$ (D1)

$$\frac{\partial}{\partial t} \langle (\mathbf{b} \nabla) \mathbf{b} \rangle = \left( \frac{\partial (\mathbf{b} \nabla) \mathbf{b}}{\partial t} \right) + \langle (\mathbf{b} \nabla \mathbf{b}) \rangle \mathbf{b} \rangle,$$ (D2)

$$\frac{\partial}{\partial t} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle + \frac{\tau_{cor}}{2} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle =$$

$$= - \frac{1}{c \rho} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle \mathbf{b} \rangle +$$

$$\frac{1}{4 \pi \rho} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle \mathbf{b} \rangle +$$

$$\frac{\tau_{cor}}{2} \langle (\mathbf{v} \nabla) \mathbf{v} \rangle \mathbf{b} \rangle +$$

$$\mathbf{e}_{x'} \approx -g' \delta b_x + (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_{x'} \approx -g' \delta b_y + (\mathbf{J}_{x'}^w - \mathbf{e}_{x'} \nabla \mathbf{J}_{x'}^w) \mathbf{e}_y \,.$$ (C10)

$$\frac{\partial}{\partial t} \langle (\mathbf{b} \nabla) \mathbf{b} \rangle + \frac{\tau_{cor}}{2} \langle (\mathbf{b} \nabla) \mathbf{b} \rangle =$$

Then, the equation of motion in the mean field approxi-
The equation is

\[
\frac{\partial \mathbf{V}}{\partial t} + (\nabla \mathbf{V}) \cdot \mathbf{V} = -\frac{1}{\rho} \nabla P - \frac{1}{c\rho}(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V}) \times \mathbf{B} + \frac{1}{4\pi \rho}((\nabla \times \mathbf{B}) \times \mathbf{B}) + \frac{\kappa_t}{c\rho}(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V})_x e_y' - (\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V}_x) e_x' + \frac{\zeta_t}{c\rho}(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V})_y e_x' - (\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V}_y) e_y' \]

\[
- \frac{\partial (\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V})_x}{\partial x'} e_y' - (\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V}_x) e_x' + \frac{\tau_{\text{cor}}}{2}\left(\frac{\partial \mathbf{V}}{\partial y'} \cdot e_x' - \frac{\partial \mathbf{V}}{\partial x'} \cdot e_y'\right) + \frac{\tau_{\text{cor}}}{2} \left(\frac{\partial \mathbf{V}}{\partial x'^2} + \frac{\partial \mathbf{V}}{\partial y'^2}\right) + \nu \nabla^2 \mathbf{V} - \frac{1}{c\rho}(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V}) \times \mathbf{b}) \ldots,
\]

(D5)

where the turbulent transport coefficients

\[
\kappa_t = \tau_{\text{cor}} \int_0^{\infty} dk'_z \frac{1}{\sqrt{4\pi \rho}} \sqrt{k_1 |k_{z'|}} \langle b^2(k_{z'}) \rangle,
\]

(D6)

and

\[
\zeta_t = \tau_{\text{cor}} \int_0^{\infty} dk'_z \frac{1}{\sqrt{4\pi \rho}} \frac{1}{\sqrt{|k_{z'|}}} \langle b^2(k_{z'}) \rangle
\]

(D7)

are expressed through the correlation time \( \tau_{\text{cor}} \).

As in Appendix C we transform the vector coordinates to the laboratory system where the z-axis is along the unperturbed magnetic field and the shock normal \( \mathbf{e}_z = \mathbf{B}_0^1 / \mathbf{B}_0 ^1 \).

Then, to first-order in the small amplitudes of the current and field perturbations the transformation yields

\[
(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V})_x e_y' \approx [-g' \delta b_x + (\delta j_x^{\text{cr}} - e_{\text{cr}} \delta \mathbf{V}_x)] e_y',
\]

(D8)

\[
(\mathbf{j}^{\text{cr}} - e_{\text{cr}} \mathbf{V})_y e_x' \approx [-g' \delta b_y + (\delta j_y^{\text{cr}} - e_{\text{cr}} \delta \mathbf{V}_y)] e_x',
\]

(D9)

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