1. INTRODUCTION

Though commonly unrecognized, a superconducting BCS condensate consists of equal numbers of two-electron (2e) and two-hole (2h) Cooper pairs (CPs). A complete boson-fermion (statistical) model (CBFM), however, is able to depart from this perfect 2e-/2h-CP symmetry and yields [1] robustly higher $T_c$'s without abandoning electron-phonon dynamics mimicked by the BCS/Cooper model interaction $V_{k,k'}$ which is a nonzero negative constant $-V$, if and only if single-particle energies $\epsilon_k, \epsilon_{k'}$ are within an interval $[\max\{0, \mu - \hbar \omega_D\}, \mu + \hbar \omega_D]$ where $\mu$ is the electron chemical potential and $\omega_D$ is the Debye frequency. The CBFM is “complete” only in the sense that 2h-CPs are not ignored, and reduces to all the known statistical theories of superconductors (SCs), including the BCS-Bose “crossover” picture but goes considerably beyond it.

Boson-fermion (BF) models of SCs as a Bose-Einstein condensation (BEC) go back to the mid-1950’s [2-5], pre-dating even the BCS-Bogoliubov theory [6-8]. Although BCS theory only contemplates the presence of “Cooper correlations” of single-particle states, BF models [2-5, 9-17] posit the existence of actual bosonic CPs. Indeed, CPs appear to be universally accepted as the single most important ingredient of SCs, whether conventional or “exotic” and whether of low- or high-transition-temperatures $T_c$. In spite of their centrality, however, they are poorly understood. The fundamental drawback of early [2-5] BF models, which took 2e bosons as analogous to diatomic molecules in a classical atom-molecule gas mixture, is the notorious absence of an electron energy gap $\Delta(T)$. “Gapless” models cannot describe the superconducting state at all, although they are useful [16,17] in locating transition
temperatures if approached from above, i.e., $T > T_c$. Even so, we are not aware of any calculations with the early BF models attempting to reproduce any empirical $T_c$ values. The gap first began to appear in later BF models [9-14]. With two [12, 13] exceptions, however, all BF models neglect the effect of hole CPs accounted for on an equal footing with electron CPs, except the CBFM which consists of both bosonic CP species coexisting with unpaired electrons, in a ternary gas mixture. Unfortunately, no experiment has yet been performed, to our knowledge, that distinguishes between electron and hole CPs.

The “ordinary” CP problem [18] for two distinct interfermion interactions (the $\delta$-well [19, 20] or the Cooper/BCS model [6, 18] interactions) neglects the effect of 2h CPs treated on an equal footing with 2e [or, in general, two-particle (2p)] CPs. On the other hand, Green’s functions [21] can naturally deal with hole propagation and thus treat both 2e- and 2h-CPs [22, 23]. In addition to the generalized CP problem, a crucial result [12, 13] is that the BCS condensate consists of equal numbers of 2p and 2h CPs. This was already evident, though widely ignored, from the perfect symmetry about $\epsilon = \mu$ of the well-known Bogoliubov [24] $v^2(\epsilon)$ and $u^2(\epsilon)$ coefficients, where $\epsilon$ is the electron energy.

Here we show: a) how the crossover picture $T_c$s, defined self-consistently by both the gap and fermion-number equations, requires unphysically large couplings (at least for the Cooper/BCS model interaction in SCs) to differ significantly from the $T_c$ from ordinary BCS theory defined without the number equation since here the chemical potential is assumed equal to the Fermi energy; how although ignoring either 2h- or 2e-CPs in the CBFM b) one obtains the precise BCS gap equation for all temperatures $T$, but c) only half the $T = 0$ BCS condensation energy emerges. The gap equation gives $\Delta(T)$ as a function of coupling, from which $T_c$ is found as the solution of $\Delta(T_c) = 0$. The condensation energy is simply related to the ground-state energy of the many-fermion system, which in the case of BCS is a rigorous upper bound to the exact many-body value for the given Hamiltonian. Results (b) and (c) are also expected to hold for neutral-fermion superfluids (SFs)—such as liquid $^3$He [25, 26], neutron matter and trapped ultra-cold fermion atomic gases [27-38]—where the pair-forming two-fermion interaction of course differs from the Cooper/BCS one for SCs.

2. THE COMPLETE BOSON-FERMION MODEL

The CBFM [12, 13] is described in $d$ dimensions by the Hamiltonian $H = H_0 + H_{\text{int}}$. The unperturbed Hamiltonian $H_0$ corresponds to a non-Fermi-liquid “normal” state, being an ideal (i.e., noninteracting) ternary gas mixture of unpaired fermions and both types of CPs namely, 2e and 2h. It is

$$H_0 = \sum_{k_1, s_1} \epsilon_{k_1} a^+_{k_1, s_1} a_{k_1, s_1} + \sum_K E_+(K) b^+_K b_K - \sum_K E_-(K) c^+_K c_K$$

where as before $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ is the CP center-of-mass momentum (CMM) wavevector while $\epsilon_{k_1} \equiv h^2 k_1^2 / 2m$ are the single-electron, and $E_\pm(K)$ the 2e-/2h-CP phenomenological, energies. Here $a^+_{k_1, s_1}$ ($a_{k_1, s_1}$) are creation (annihilation) operators for fermions
and similarly $b_k^+ (b_k)$ and $c_k^+ (c_k)$ for 2e- and 2h-CP bosons, respectively. Two-hole CPs are considered distinct and kinematically independent from 2e-CPs.

The interaction Hamiltonian $H_{int}$ (simplified by dropping all $K \neq 0$ terms, as is done in BCS theory in the full Hamiltonian but kept in the CBFM in $H_0$) consists of four distinct BF interaction vertices each with two-fermion/one-boson creation and/or annihilation operators. The vertices depict how unpaired electrons (subindex +) [or holes (subindex −)] combine to form the 2e- (and 2h-) CPs assumed in the $d$-dimensional system of size $L$, namely

$$
H_{int} = L^{-d/2} \sum_k f_+(k) \{ a_{k,\uparrow}^+ a_{-k,\downarrow}^+ b_0 + a_{-k,\downarrow}^- a_{k,\uparrow}^- b_0^+ \}
$$

$$
+ L^{-d/2} \sum_k f_-(k) \{ a_{k,\uparrow}^+ a_{-k,\downarrow}^+ c_0^+ + a_{-k,\downarrow}^- a_{k,\uparrow}^- c_0 \}
$$

(1)

where $k \equiv \frac{1}{2}(k_1 - k_2)$ is the relative wavevector of a CP. The interaction vertex form factors $f_{\pm}(k)$ in (1) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate of the two fermions. In Refs. [12, 13] they are taken as

$$
f_\pm(\epsilon) = \begin{cases} f & \text{if } \frac{1}{2} \left[ E_\pm(0) - \delta \epsilon \right] < \epsilon < \frac{1}{2} \left[ E_\pm(0) + \delta \epsilon \right] \\ 0 & \text{otherwise.} \end{cases}
$$

(2)

One then introduces the quantities $E_f$ and $\delta \epsilon$ as new phenomenological dynamical energy parameters (in addition to the positive BF vertex coupling parameter $f$) that replace the previous $E_{\pm}(0)$ parameters, through the definitions

$$
E_f \equiv \frac{1}{4} [E_+(0) + E_-(0)] \quad \text{and} \quad \delta \epsilon \equiv \frac{1}{2} [E_+(0) - E_-(0)]
$$

(3)

where $E_{\pm}(0)$ are the (empirically unknown) zero-CMM energies of the 2e- and 2h-CPs, respectively. Alternately, one has the two relations

$$
E_{\pm}(0) = 2E_f \pm \delta \epsilon.
$$

(4)

The quantity $E_f$ serves as a convenient energy scale; it is not to be confused with the Fermi energy $E_F = \frac{1}{2}mv_F^2 \equiv k_B T_F$ where $T_F$ is the Fermi temperature. The Fermi energy $E_F$ equals $\pi \hbar^2 n/m$ in 2D and $(\hbar^2/2m)(3\pi^2 n)^{2/3}$ in 3D, with $n$ the total number-density of charge-carrier electrons, while $E_f$ is the same with $n$ replaced by, say, $n_f$. The quantities $E_f$ and $E_F$ coincide only when perfect 2e/2h-CP symmetry holds, i.e. when $n = n_f$.

The grand potential $\Omega$ for the full $H = H_0 + H_{int}$ is then constructed via

$$
\Omega(T, L^d, \mu, N_0, M_0) = -k_B T \ln \left[ \text{Tr} e^{-\beta (H - \mu \hat{N})} \right]
$$

(5)

where “Tr” stands for “trace.” Following the Bogoliubov recipe [39], one sets $b_0^+$, $b_0$ equal to $\sqrt{N_0}$ and $c_0^+$, $c_0$ equal to $\sqrt{M_0}$ in (1), where $N_0$ is the $T$-dependent number
of zero-CMM 2e-CPs and $M_0$ the same for 2h-CPs. This allows exact diagonalization, through a Bogoliubov transformation, giving \[40\]

\[
\frac{\Omega}{L^d} = \int_0^\infty d\epsilon N(\epsilon)[\epsilon - \mu - E(\epsilon)] - 2k_B T \int_{0}^{\infty} d\epsilon N(\epsilon) \ln\{1 + \exp[-\beta E(\epsilon)]\} + [E_+(0) - 2\mu]n_0 + k_B T \int_{0^+}^{\infty} d\epsilon M(\epsilon) \ln\{1 - \exp[-\beta\{E_+(0) + \epsilon - 2\mu]\} \] + [2\mu - E_-(0)]m_0 + k_B T \int_{0^+}^{\infty} d\epsilon M(\epsilon) \ln\{1 - \exp[-\beta\{2\mu - E_-(0) + \epsilon]\} \] \tag{6}
\]

where $N(\epsilon)$ and $M(\epsilon)$ are respectively the electronic and bosonic density of states, $E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^2(\epsilon)}$ where $\Delta(\epsilon) \equiv \sqrt{n_0 f_+(\epsilon) + \sqrt{m_0} f_-(\epsilon)}$, with $n_0(T) \equiv N_0(T)/L^d$ and $m_0(T) \equiv M_0(T)/L^d$ being the 2e-CP and 2h-CP number densities, respectively, of BE-condensed bosons. Minimizing (6) with respect to $N_0$ and $M_0$, while simultaneously fixing the total number $N$ of electrons by introducing the electron chemical potential $\mu$, namely

\[
\frac{\partial \Omega}{\partial N_0} = 0, \quad \frac{\partial \Omega}{\partial M_0} = 0, \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = -N \tag{7}
\]

specifies an equilibrium state of the system with volume $L^d$ and temperature $T$. Here $N$ evidently includes both paired and unpaired CP electrons. The diagonalization of the CBFM $H$ is exact, unlike with the BCS $H$, so that the CBFM goes beyond mean-field theory. Some algebra then leads \[40\] to the three coupled integral Eqs. (7)-(9) of Ref. [12]. Self-consistent (at worst, numerical) solution of these three coupled equations then yields the three thermodynamic variables of the CBFM

\[
n_0(T, n, \mu), \quad m_0(T, n, \mu), \quad \text{and} \quad \mu(T, n). \tag{8}
\]

Fig. 1 displays the three BE condensed phases—labeled $s+$, $s-$ and $ss$—along with the normal phase $n$, that emerge \[13\] from the CBFM.

Vastly more general, the CBFM contains \[1\] the key equations of all five distinct statistical theories as special cases; these range from BCS to BEC theories, which are thereby unified by the CBFM. Perfect 2e/2h CP symmetry signifies equal numbers of 2e- and 2h-CPs, more specifically, $n_B(T) = m_B(T)$ as well as $n_0(T) = m_0(T)$. With (4) this implies that $E_f$ coincides with $\mu$, and the CBFM then reduces to the gap and number equations [viz., (11) and (12) below] of the BCS-Bose crossover picture with the Cooper/BCS model interaction—if its parameters $V$ and $\hbar \omega_D$ are identified with the BF interaction Hamiltonian $H_{int}$ parameters $f^2/2\delta$ and $\delta$, respectively. The crossover picture for unknowns $\Delta(T)$ and $\mu(T)$ is now supplemented by the central relation

\[
\Delta(T) = f\sqrt{n_0(T)} = f\sqrt{m_0(T)}. \tag{9}
\]

Both $\Delta(T)$ and $n_0(T)$ and $m_0(T)$ are the familiar “half-bell-shaped” order-parameter curves. These are zero above a certain critical temperature $T_c$, rising monotonically upon cooling (lowering $T$) to maximum values $\Delta(0)$, $n_0(0)$ and $m_0(0)$ at $T = 0$. The energy gap $\Delta(T)$ is the order parameter describing the superconducting (or
superfluid) condensed state, while \( n_0(T) \) and \( m_0(T) \) are the BEC order parameters depicting the macroscopic occupation that arises below \( T_c \) in a BE condensate. This \( \Delta(T) \) is precisely the BCS energy gap if the boson-fermion coupling \( f \) is made to correspond to \( \sqrt{2V/\hbar\omega_D} \). Note that the BCS and BE \( T_c \)s are the same. Writing (9) for \( T = 0 \), and dividing this into (9) gives the much simpler \( f \)-independent relation involving order parameters normalized in the interval \([0, 1]\)

\[
\frac{\Delta(T)}{\Delta(0)} = \frac{\sqrt{n_0(T)/n_0(0)}}{\sqrt{m_0(T)/m_0(0)}} = \frac{\sqrt{m_0(T)/m_0(0)}}{\sqrt{n_0(T)/n_0(0)}} \xrightarrow{T \to 0} 1. \tag{10}
\]

The first equality, apparently first obtained in Ref. [9], simply relates the two heretofore unrelated “half-bell-shaped” order parameters of the BCS and the BEC theories. The second equality [12, 13] implies that a BCS condensate is precisely a BE condensate of equal numbers of 2e- and 2h-CPs. Since (10) is independent of the particular two-fermion dynamics of the problem, it can be expected to hold for either SCs and SFs.

3. BCS-BOSE CROSSOVER THEORY

The crossover theory (defined by two simultaneous equations, the gap and number equations) was introduced by many authors beginning in 1967 with Friedel and
co-authors [41]; for a review see Ref. [42]. The critical temperature $T_c$ is defined by $\Delta(T_c) = 0$, and is to be determined self-consistently with $\mu(T_c)$. The two equations to be solved, in 2D for the Cooper/BCS model interaction, are [43]

$$1 = \lambda \int_0^{\hbar \omega_D} dx \frac{\tanh x}{x} \left( \text{if } \mu > \hbar \omega_D \right); \quad 1 = \lambda \int_{-\mu(T_c)/2k_BT_c}^{\hbar \omega_D} dx \frac{\tanh x}{2x}, \quad \left( \text{if } \mu < \hbar \omega_D \right)$$

(11)

$$\int_0^\infty \frac{d\epsilon}{\exp[\epsilon - \mu(T_c)/2k_BT_c] + 1} = 1.$$  \quad \text{ (12)}

The last integral can be done analytically, and leaves

$$\mu(T_c) = k_B T_c \ln(e^{E_F/k_BT_c} - 1).$$  \quad \text{ (13)}

The $\mu(T_c)$ is then eliminated (numerically) from (11) to give $T_c$ as a function of $\lambda$. Using $\hbar \omega_D/E_F = 0.05$ as a typical value for cuprates, increasing $\lambda$ makes $\mu(T_c)$ decrease from its weak-coupling (where $T_c \to 0$) value of $E_F$ down to $\hbar \omega_D$ when $\lambda \approx 56$, an unphysically large value. Fig. 2 displays $T_c$ (in units of $T_F$) as function of $\lambda$.

### 4. GAP EQUATION

Curiously, the standard procedure in all SC and SF theories of many-fermions is to ignore 2h-CPs altogether. Indeed, the BCS gap equation for all $T$ can be derived without them. Neglecting in (6) all terms containing $m_0(T)$, $E_-(0)$ and $f_-(\epsilon)$ leaves an $\Omega(T, L^d, \mu, N_0)$ defining an incomplete BFM. Minimizing it over $N_0$ (for fixed total electron number $N$) requires that

$$\partial \Omega/\partial N_0 = 0 \quad \text{or} \quad \partial \Omega/\partial n_0 = 0,$$

which becomes

$$\int_0^\infty d\epsilon N(\epsilon) \left[ -1 + \frac{2\exp{-\beta E(\epsilon)}}{1 + \exp{-\beta E(\epsilon)}} \right] \frac{dE(\epsilon)}{dn_0} + [E_+(0) - 2\mu] = 0$$

or

$$2[E_+(0) - 2\mu] = f^2 \int_{E_f}^{E_f+\delta \epsilon} d\epsilon N(\epsilon) \frac{1}{E(\epsilon)} \tanh \frac{1}{2} \beta E(\epsilon).$$  \quad \text{ (14)}

Using (4) yields precisely the BCS gap equation for all $T$, namely

$$1 = \lambda \int_0^{\hbar \omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2(T)}} \tanh \frac{1}{2} \beta \sqrt{\xi^2 + \Delta^2(T)}$$

(15)

where $\xi \equiv \epsilon - \mu$, provided one picks $E_f = \mu$, since $\lambda \equiv f^2 N(0)/2\delta \epsilon$ while $\delta \epsilon = \hbar \omega_D$. The companion number equation follows from the last equation of (7) and will thus be

$$n = n_f(T) + 2n_B(T)$$  \quad \text{ (16)}

where $n_f(T)$ is the number density of unpaired electrons

$$n_f(T) \equiv \int_0^\infty d\epsilon N(\epsilon) \left[ 1 - \frac{\epsilon - \mu}{E(\epsilon)} \tanh \frac{1}{2} \beta E(\epsilon) \right]$$

(17)
Figure 2. Critical SC temperatures $T_c$ in units of $T_F$ for the BCS-Bose crossover theory (full curve), the BCS value from the exact implicit equation (Ref. [21], p. 447) $1 = \lambda \int_0^{\hbar \omega_D / 2 k_B T_c} dx x^{-1} \tanh x$ (upper dashed curve) and its weak-coupling solution $T_c \approx 1.134 \hbar \omega_D \exp(-1/\lambda)$ (lower dashed curve). The dot-dashed “appendage” signals a breakdown in the BCS/Cooper interaction model when $\mu(T_c)$ turns negative, as the Fermi surface at $\mu$ then washes out. The value of $\lambda = 1/2$ marked is the maximum possible value allowed [45] for this interaction model just short of lattice instability.

while the number density of composite bosons, both with $K = 0$ and with $K > 0$, is

$$n_B(T) \equiv n_0(T) + n_{B+}(T); \quad n_{B+}(T) \equiv \int_{0+}^{\infty} d\varepsilon M(\varepsilon) \frac{1}{e^{\beta(\varepsilon - 2\mu)} - 1}.$$  (18)

Note that the number equation $n = n_f(T) + 2n_B(T)$ differs from (12) of the previous section which follows from $n = n_f(T)$ only.

Similarly, ignoring 2e-CPs and keeping only 2h-CPs leads to $\Omega(T, L^d, \mu, M_0)$ which to minimize over $M_0$ requires that $\partial \Omega / \partial M_0 = \partial \Omega / \partial m_0 = 0$ and, since $E(\xi) \equiv E(-\xi)$. This again leads to (15) but now with the companion number equation

$$m = n_f(T) - 2m_B(T)$$  (19)

with the same previous $n_f(T)$ and where

$$m_B(T) \equiv m_0(T) + m_{B+}(T); \quad m_{B+}(T) \equiv \int_{0+}^{\infty} d\varepsilon M(\varepsilon) \frac{1}{e^{\beta(2\mu - E_-(0) + \varepsilon)} - 1}.$$  (20)
However, ignoring either 2e- or 2h-CPs does not give the entire BCS ground-state energy, as we now show.

5. CONDENSATION ENERGY

The \( T = 0 \) condensation energy per unit volume according to the CBFM (i.e., with both 2e- and 2h-CPs) is

\[
\frac{E_s - E_n}{L^d} = \frac{\Omega_s(T = 0) - \Omega_n(T = 0)}{L^d}
\]

since for any \( T \) the Helmholtz free energy \( F = \Omega + \mu N = E - TS \), with \( S \) the entropy, and \( \mu \) is the same for either superconducting \( s \) or normal \( n \) phases with internal energies \( E_s \) and \( E_n \), respectively. In the normal phase \( n_0 = 0, m_0 = 0 \) so that \( \Delta(T = 0) = 0 \) for all \( T \geq 0 \), and (6) reduces to

\[
\Omega_n(T = 0) = \int_{-\mu}^{\mu} d\xi N(\xi) [\xi - \mu - |\xi - \mu|] = 2 \int_{-\mu}^{\mu} d\xi N(\xi) \xi.
\]

For the superconducting phase at \( T = 0 \), and when \( n_0(T) = m_0(T) \) and \( n_B(T) = m_B(T) \) hold, one deduces from (4) and (6) that \( \mu = E_f \). Letting \( \Delta(T = 0) \equiv \Delta \) in (6) and putting \( \delta \varepsilon \equiv h \omega_D \) while using (4) gives for the superconducting phase

\[
\Omega_s(T = 0) = 2h\omega_D n_0(0) + \int_{-\mu}^{\mu} d\xi N(\xi) \left( \xi - \sqrt{\xi^2 + \Delta^2} \right) - 2 \int_{-\mu}^{\mu} d\xi N(\xi) \sqrt{\xi^2 + \Delta^2}.
\]

Subtracting (22) from (23) and putting \( N(\xi) \approx N(0) \), the density of electronic states at the Fermi surface, one is left with

\[
\frac{E_s - E_n}{L^d} = 2h\omega_D n_0(0) + 2N(0) \int_{-\mu}^{\mu} d\xi \left( \xi - \sqrt{\xi^2 + \Delta^2} \right)
\]

Employing Eq. (2), p. 158 of Ref. [44] the integral becomes

\[
\frac{(h\omega_D)^2}{2} - \frac{1}{2} h\omega_D \sqrt{(h\omega_D)^2 + \Delta^2} + \frac{1}{2} \Delta^2 \ln \frac{\Delta}{h\omega_D + \sqrt{(h\omega_D)^2 + \Delta^2}} \rightarrow \frac{1}{2} \Delta^2 \ln \left( \frac{\Delta}{2h\omega_D} \right) - \frac{1}{4} \Delta^2 - \frac{1}{16} \left( \frac{\Delta^4}{(h\omega_D)^2} \right) + \frac{\Delta^6}{[h\omega_D]^4}.
\]

Using (9) for \( T = 0 \) and weak coupling \( f \rightarrow 0 \) implies that \( \Delta = f \sqrt{n_0(0)} = f \sqrt{m_0(0)} \rightarrow 0 \) so that (24) yields the expansion

\[
\frac{E_s - E_n}{L^d} \rightarrow 2h\omega_D n_0(0) + 2N(0) \left[ \frac{1}{2} \Delta^2 \ln \left( \frac{\Delta}{2h\omega_D} \right) - \frac{1}{4} \Delta^2 - \frac{1}{16} \left( \frac{\Delta^4}{(h\omega_D)^2} \right) + \frac{\Delta^6}{[h\omega_D]^4} \right].
\]
Given that for small $\lambda$

$$\Delta = \frac{\hbar \omega_D}{\sinh(1/\lambda)} \quad \xrightarrow{\lambda \to 0} \quad 2\hbar \omega_D \exp(-1/\lambda)$$

(27)

the log term in (26) is just

$$\ln \left( \frac{\Delta}{2\hbar \omega_D} \right) = -\frac{2\hbar \omega_D}{f^2 N(0)}$$

(28)

since

$$\lambda \equiv V N(0) = \frac{f^2 N(0)}{2\hbar \omega_D}$$

(29)

so that (26) finally simplifies to

$$\frac{E_s - E_n}{L^d} \quad \xrightarrow{\Delta \to 0} \quad -\frac{1}{2} N(0) \Delta^2 \left[ 1 - \frac{1}{4} \left( \frac{\Delta}{\hbar \omega_D} \right)^2 + O \left( \frac{\Delta}{\hbar \omega_D} \right)^4 \right]$$

(CBFM). (30)

By contrast, the original BCS expression from Eq. (2.42) of Ref. [6] is

$$\frac{E_s - E_n}{L^d} = N(0)(\hbar \omega_D)^2 \left[ 1 - \sqrt{1 + (\Delta/\hbar \omega_D)^2} \right]$$

(BCS) (31)

which on expansion leaves

$$\frac{E_s - E_n}{L^d} \quad \xrightarrow{\lambda \to 0} \quad -\frac{1}{2} N(0) \Delta^2 \left[ 1 - \frac{1}{4} \left( \frac{\Delta}{\hbar \omega_D} \right)^2 + O \left( \frac{\Delta}{\hbar \omega_D} \right)^4 \right]$$

(BCS). (32)

Thus, the CBFM condensation energy (30), and consequently its ground-state energy, is lower (or larger in magnitude) than the BCS result (32). Therefore, the CBFM satisfies a prime expectation of any theory that improves upon BCS, which being based on a trial wave function gives a ground-state energy which is a rigorous upper bound to the exact energy associated with the BCS Hamiltonian ground state. Consequently, there is no a priori reason why the CBFM is limited to weak coupling, at least for all $\lambda \leq 1/2$ [45].

What happens on ignoring either $2e$- or $2h$-CPs, as seems to be common practice in theories of SCs and SFs? Starting from (6) for $T = 0$, and following a similar procedure to arrive at (23) but without $2h$-CPs such that $f_- = 0$, $m_0(0) = 0$ and $n_0(0) = \Delta^2/f^2$, one gets

$$\left[ \frac{\Omega_s(T = 0)}{L^d} \right]_+ = \hbar \omega_D n_0(0) + 2 \int_{-\mu}^0 d\xi N(\xi) \xi + N(0) \int_0^{\hbar \omega_D} d\xi \left( \xi - \sqrt{\xi^2 + \Delta^2} \right).$$

(33)

Subtracting (22) from (33) gives

$$\left[ \frac{E_s - E_n}{L^d} \right]_+ = \hbar \omega_D n_0(0) + N(0) \int_0^{\hbar \omega_D} d\xi \left( \xi - \sqrt{\xi^2 + \Delta^2} \right)$$

(34)
which is just one half the full CBFM result (24). Furthermore, if \( [(E_s - E_n)/L^d]_+ \) is the contribution from 2h-CPs alone, assuming now that \( f_+ = 0 \) and \( n_0(0) = 0 \) we eventually arrive at precisely rhs of (34) but with \( m_0(0) = \Delta^2/f^2 \) in place of \( n_0(0) \). Hence

\[
\left[ \frac{E_s - E_n}{L^d} \right]_+ = \left[ \frac{E_s - E_n}{L^d} \right]_- \xrightarrow{\lambda \to 0} -\frac{1}{4} N(0) \Delta^2 \left[ 1 + \frac{1}{4} \left( \frac{\Delta}{\hbar \omega_D} \right)^2 + O \left( \frac{\Delta}{\hbar \omega_D} \right)^4 \right]
\]

which again is just one half the full CBFM condensation energy (30) that in leading order in \( \Delta \) was found to be the full BCS condensation energy.

Including both 2e- and 2h-CPs gave similarly striking conclusions on generalizing [22, 23] the ordinary [18] CP problem from unrealistic infinite-lifetime pairs to the physically expected finite-lifetime ones.

6. CONCLUSIONS

The recent “complete boson-fermion model” (CBFM) contains as a special case the BCS-Bose crossover theory which, at least for the Cooper/BCS model interaction, predicts virtually the same \( T_c \)s to well beyond physically unreasonable values of coupling than the allegedly less general BCS theory where the number equation is replaced by the assumption that \( \mu = E_F \).

The CBFM reveals that, while the BCS gap equation for all temperatures follows rigorously without either electron or hole pairs, the resulting \( T = 0 \) condensation energy is only one half the entire BCS value. In view of this, if BEC is at all relevant in SCs and SFs taken as many-fermion systems where pairing into bosons undoubtedly occurs, two-hole CPs cannot and must not be ignored.

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