Short Distance Properties from Large Distance Behaviour

Paul Mansfield and Marcos Sampaio
Department of Mathematical Sciences
University of Durham
South Road
Durham, DH1 3LE, England

and Jiannis Pachos
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307, USA.

P.R.W.Mansfield@durham.ac.uk, M.D.R.Sampaio@durham.ac.uk,
Pachos@ctpa02.mit.edu

Abstract
For slowly varying fields the vacuum functional of a quantum field theory may be expanded in terms of local functionals. This expansion satisfies its own form of the Schrödinger equation from which the expansion coefficients can be found. For scalar field theory in 1+1 dimensions we show that this approach correctly reproduces the short-distance properties as contained in the counter-terms. We also describe an approximate simplification that occurs for the Sine-Gordon and Sinh-Gordon vacuum functionals.
1 Introduction

Whilst asymptotic freedom has lead to an accurate determination of the Lagrangian of the Standard Model of particle physics from high energy experiments there are few analytical tools enabling us to calculate with that Lagrangian at low energies where the semi-classical approximation is no longer valid. For example, the eigenvalue problem for the Hamiltonian of Yang-Mills theory cannot be solved in a semi-classical expansion because the renormalisation group implies that the energy eigenvalues depend non-perturbatively on the coupling. Consequently the computation of the hadron spectrum can only be done numerically. In ordinary quantum mechanics there are many ways to tackle this problem which are not widely used in a field theory context, but which, if suitably generalised might allow non-perturbative methods to be developed for field theory. The oldest of these is the Schrödinger representation, (see [1]-[22] for applications to field theory, and [23]-[29] for applications to the Wheeler-de Witt equation).

In the Schrödinger representation the vacuum, $|E_0\rangle$, of a scalar quantum field theory is represented by the functional $\langle \phi | E_0 \rangle = \exp W[\phi]$, where $\langle \phi |$ is an eigenbra of the field operator $\hat{\phi}(x)$ at fixed time, belonging to eigenvalue $\phi(x)$. In general $W$ is non-local, but if $\phi(x)$ varies slowly on the scale of the inverse of the mass of the lightest particle, $m_0^{-1}$, it can be expanded in terms of local functionals, [30], for example in 1+1 dimensions

$$W = \int dx \sum B_{j_0..j_n} \phi(x)^{j_0} \phi'(x)^{j_1} \ldots \phi^{(n)}(x)^{j_n}. \tag{1}$$

The coefficients $B_{j_0..j_n}$ are constant, assuming translation invariance, and finite as the ultraviolet cut-off is removed, [2]. Particle structure is characterised by length scales smaller than $m_0^{-1}$, so this simplification in $W[\phi]$ does not appear useful, however, a knowledge of this local expansion is sufficient to reconstruct $W$ for arbitrary $\phi$, [31]. This is because if $W[\phi]$ is evaluated for a scaled field, $\phi_s(x) \equiv \phi(x/\sqrt{s})$, it extends to an analytic function of $s$ with cuts restricted to the negative real axis, so that Cauchy’s theorem can be used to relate the large-$s$ behaviour (when $\phi_s$ is slowly varying) to the $s = 1$ value:

$$W[\phi] = \lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_{|s| = \infty} ds \frac{1}{s - 1} e^{\lambda(s-1)} W[\phi_s]. \tag{2}$$

The exponential term removes the contribution of the cut as $\lambda \to \infty$. The vacuum functional satisfies the Schrödinger equation from which the coefficients $B_{j_0..j_n}$ can be found in principle, however care must be exercised because this equation depends explicitly on short-distance effects via the cut-off, whereas the local expansion is only valid for fields characterised by large length-scales, so we cannot simply substitute (1) into the Schrödinger equation and expect to be able to satisfactorily take the limit in which the cut-off is removed. However, we can again exploit Cauchy’s theorem to construct a version of the Schrödinger equation that acts directly on the local expansion by considering the effect of a scale-transformation on the cut-off, as well as on the field, [30]. The Hamiltonian with a scaled cut-off acting on the vacuum functional evaluated for the scaled field again extends to an analytic function with cuts on the negative real axis. This enables the limit in which the short-distance cut-off is taken to zero to be expressed in terms of large-distance behaviour described by the local expansion for $W[\phi]$. This leads to an infinite
set of algebraic equations for the coefficients $B_{j_0...j_n}$. By truncating the expansion the Schrödinger equation offers the possibility of solution beyond perturbation theory in the couplings, however, before this is attempted it is essential to show that this formulation is capable of reproducing the results that can be obtained using the standard approach of the semi-classical expansion and Feynman diagram perturbation theory. In particular, since the method consists of building states out of their large distance properties it is important to show that it gets right the short-distance behaviour as contained in the counter-terms of the Hamiltonian. The purpose of this paper is to demonstrate that this short-distance behaviour is correctly reproduced by our approach to the Schrödinger equation in which we build the vacuum state from its large-distance behaviour, and that, at least to low orders, the resulting local expansion coincides with the Feynman diagram calculation of the vacuum functional.

2 Semi-classical analysis of local expansion

The classical Hamiltonian of $\varphi^4$ theory is $\int dx \left( \frac{\hbar}{2} \left( \pi^2 + \varphi'^2 + m^2 \varphi^2 \right) + \varphi^4 \right)$ in 1+1 dimensions. In the Schrödinger representation the canonical momentum is represented by functional differentiation $\hat{\pi} = -i\hbar \delta/\delta \varphi(x)$, so the kinetic term leads to the product of two functional derivatives at the same point which we regulate by introducing a momentum cut-off $p^2 < 1/\epsilon$. The couplings must consequently be renormalised so that the Hamiltonian has a finite action on the vacuum. Writing the vacuum functional as $\exp (W[\varphi]/\hbar)$ gives the Schrödinger equation as $\lim_{\epsilon \downarrow 0} F_\epsilon[\varphi] = 0$ where

$$F_\epsilon[\varphi] = -\frac{\hbar}{2} \Delta_\epsilon W + \int dx \left( \frac{1}{2} \left( -\left( \frac{\delta W}{\delta \varphi} \right)^2 + \varphi'^2 + M^2(\epsilon) \varphi^2 \right) + \frac{9}{4!} \varphi^4 - \mathcal{E}(\epsilon) \right)$$

and

$$\Delta_\epsilon = \int dx \int_{p^2 < 1/\epsilon} \frac{dp}{2\pi} e^{ip(x-y)} \delta^2 \frac{\delta^2}{\delta \varphi(x)\delta \varphi(y)} = \int_{p^2 < 1/\epsilon} dp 2\pi \delta^2 \frac{\delta^2}{\delta \tilde{\varphi}(-p)\delta \tilde{\varphi}(p)},$$

where $\tilde{\varphi}(p) = \int dx \varphi(x) \exp(-ipx)$. In perturbation theory the only divergent diagrams with external legs are tadpoles, and these can be removed by normal ordering the Hamiltonian. This enables the $\epsilon$-dependence of the parameters to be calculated exactly as

$$M^2(\epsilon) = M^2 + \hbar \delta M^2 - \frac{\hbar^2}{4} \int_{p^2 < 1/\epsilon} dp \frac{1}{\sqrt{p^2 + M^2}},$$

$$\mathcal{E}(\epsilon) = \delta \mathcal{E} + \frac{\hbar}{2} \int_{p^2 < 1/\epsilon} dp \left( \sqrt{p^2 + M^2} + \frac{M^2(\epsilon) - M^2}{2\sqrt{p^2 + M^2}} \right) + \frac{g\hbar^2}{32} \left( \int \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M^2}} \right)^2$$

where $M^2, \delta M^2, \delta \mathcal{E}$ and $\mathcal{E}$ remain finite when the cut-off is removed. The ambiguity in the choice of counterterms represented by $\delta M^2$ and $\delta \mathcal{E}$ is resolved, as usual, by renormalisation conditions. We shall soon see that there is a natural way to do this in the present context. If $F_\epsilon[\varphi]$ is evaluated for a $\varphi$ whose Fourier transform is non-zero only for momenta less than $m_0$ it will reduce to a sum of local functionals of $\varphi$,

$$F_\epsilon[\varphi] = \int dx \sum f_{j_0...j_n}(\epsilon) \varphi(x)^{j_0} \varphi'(x)^{j_1} ... \varphi^{(n)}(x)^{j_n}$$
The expansion functions, $\varphi^{j_0, j_1, \ldots}$, are related by partial integration so we can specify a linearly independent basis by insisting that the power of the highest derivative be at least two, and we will assume parity and $\varphi \rightarrow -\varphi$ invariance, restricting both the total number of $\varphi$ and the total number of derivatives in the expansion functions appearing in (1) and (7) to be even. It is important to note that (7) is not the same expression that would be obtained by acting with $\Delta_\varepsilon$ on the local expansion (1), because the former correctly includes differentiation with respect to the Fourier modes of $\varphi$ with momenta in the range $m_0^2 < p^2 < 1/\varepsilon$ absent from the second. Now if we scale the cut-off $(\Delta_\varepsilon W)[\varphi_\varepsilon]$ extends to an analytic function in the complex $s$-plane with singularities only on the negative real axis, because the former correctly includes differentiation with respect to the Fourier modes of $\varphi$ with momenta in the range $m_0^2 < p^2 < 1/\varepsilon$ absent from the second. Now if we scale the cut-off $(\Delta_\varepsilon W)[\varphi_\varepsilon]$ extends to an analytic function in the complex $s$-plane with singularities only on the negative real axis, the same is true of $M^2(\varepsilon)$ and $E(\varepsilon)$, and consequently of the coefficients of the linearly independent expansion functions, $f_{j_0, j_n}$ in (7), so the contour integral

$$I_{j_0, j_n}(\lambda) = \frac{1}{2\pi i} \int_{|s|=\infty} \frac{ds}{s} e^{\lambda s} \sqrt{\pi} \lambda s f_{j_0, j_n}(s\varepsilon)$$

(8)

can be calculated by collapsing the contour to a small circle about the origin and a contour along the cut on the negative real axis. The contribution from the circle about the origin is controlled by the small $\varepsilon$ behaviour of $f_{j_0, j_n}(\varepsilon)$. As $\varepsilon \rightarrow 0$ this vanishes due to the Schrödinger equation, and in perturbation theory the Feynman diagram expansion gives an asymptotic expansion of $f_{j_0, j_n}(\varepsilon)$ in positive powers of $\sqrt{\varepsilon}$. The inclusion of $\sqrt{\pi} \lambda s$ in (8) ensures that the contribution from the origin will be of order $1/\lambda$ rather than $1/\sqrt{\lambda}$. For large $|s|$ the scaled field $\varphi_s$ is slowly varying and the scaled cut-off $1/(s\varepsilon)$ is less than $m_0$ so $(\Delta_\varepsilon W)[\varphi_s]$ can now be calculated by acting with $\Delta_\varepsilon$ directly on the local expansion of $W$, (9). Furthermore as the real part of $\lambda$ tends to infinity the contribution from the cut tends to zero due to the $\exp(\lambda s)$ factor, (the contribution from that part of the cut for which $|s|$ is large is again given by the local expansion and seen to be suppressed as $\lambda \rightarrow \infty$). Thus the Schrödinger equation leads to an infinite set of algebraic equations

$$\lim_{\lambda \rightarrow \infty} I_{j_0, j_1, \ldots}(\lambda) = 0$$

where

$$I_0 = -\bar{E}(\lambda) - \hbar \frac{\sqrt{\lambda}}{\sqrt{\pi}} \left( B_2 + \frac{B_{0,2} \lambda}{3} + \frac{B_{0,2,0} \lambda^2}{10} + \ldots \right)$$

$$I_2 = \frac{\bar{M}^2(\lambda)}{2} - 2B_2^2 - \hbar \frac{\sqrt{\lambda}}{\sqrt{\pi}} \left( 6B_4 + \frac{B_{2,2} \lambda}{3} + \frac{B_{2,0,2} \lambda^2}{10} + \ldots \right)$$

$$I_4 = \frac{g}{4!} - 8B_2 B_4 - \hbar \frac{\sqrt{\lambda}}{\sqrt{\pi}} \left( 15B_6 + \frac{B_{4,2} \lambda}{3} + \frac{B_{4,0,2} \lambda^2}{4} + \ldots \right)$$

$$I_{0,2} = \frac{1}{2} - 4B_2 B_{0,2} - \hbar \frac{\sqrt{\lambda}}{\sqrt{\pi}} \left( B_{2,2} + 2B_{0,4} \lambda + \frac{4B_{2,0,2} \lambda^2}{3} + \ldots \right)$$

(9)

and

$$\bar{E}(\lambda) = \frac{1}{2\pi i} \int_{|s|=\infty} \frac{ds}{s} e^{\lambda s} \sqrt{\pi} \lambda s \bar{E}(s) = \sum_{n=0}^{\infty} \hbar^n \bar{E}(\lambda) h^n$$

(10)

$$\bar{M}^2(\lambda) = \frac{1}{2\pi i} \int_{|s|=\infty} \frac{ds}{s} e^{\lambda s} \sqrt{\pi} \lambda s \bar{M}^2(s) = M^2 + \hbar \bar{M}^2(\lambda) h$$

(11)
As the product $s\epsilon$ now plays the rôle of cut-off, rather than $\epsilon$ alone, we have taken $\epsilon$ to be finite and equal to unity. We will now choose renormalisation conditions. Note that the counter-terms only enter $I_0$ and $I_2$. If these are fixed then the above equations determine the coefficients $B_{j_0\ldots j_n}$ and the energy eigenvalue, $\mathcal{E}$, which are themselves finite as the cut-off is removed. Alternatively we could instead choose the values of two of these quantities, $B_2$ and $\mathcal{E}$ for example, and then think of the equations $I_0 = 0$ and $I_2 = 0$ as determining the counter-terms. So we will take $B_2 = -M/2$, which is its classical value, and $\mathcal{E} = 0$ as our renormalisation conditions. The advantage of imposing the renormalisation conditions on $\mathcal{E}$ and $B_2$ is that we are free to solve (9) for the remaining $B_{j_0\ldots j_n}$ without first computing the $\lambda$-dependence of the counter-terms which in a more general context can only be done in perturbation theory.

The equations (9) may be solved in the usual semi-classical approach in which we expand the coefficients as $B = \sum \hbar^n B^n$, by first ignoring the terms proportional to $\hbar$. Although the resulting equations are quadratic in the $B_{j_0\ldots j_n}$ they are readily solved by starting with the coefficients of local functions of the lowest dimension and number of $\varphi$, giving at tree-level

\[
W_{\text{tree}} = \int dx \left( -\frac{1}{2}\varphi^2 - \frac{1}{4}\varphi'^2 + \frac{1}{16}\varphi''^2 - \frac{1}{32}\varphi'''^2 + \frac{5}{256}\varphi''''^2 - \frac{1}{96}g\varphi^4 + \frac{1}{64}g\varphi^2\varphi'^2 \\
-\frac{1}{128}g\varphi^2\varphi''^2 + \frac{1}{256}g\varphi'^4 - \frac{5}{1024}g\varphi^2\varphi'^2 - \frac{3}{256}g\varphi\varphi'^2 - \frac{31}{1024}g\varphi^2\varphi'^2 \\
-\frac{7}{2048}g\varphi^2\varphi''^2 + \frac{41}{1024}g\varphi\varphi'^2 + \frac{75}{2048}g\varphi^2\varphi''^2 - \frac{93}{4096}g\varphi^4 + \ldots \right) \tag{12}
\]

where we have chosen our mass-scale so that $M = 1$. Particular tree-level coefficients that will be of use are

\[
B_{0,0\ldots,0,j_n=2}^1 = -\frac{1}{2} \left( \frac{1/2}{n} \right) \tag{13}
\]

and the coefficients $B_{1,1}^1, B_{2,2}^1, B_{2,0,2}^1, \ldots, B_{2,0\ldots,0,j_n=2}^1$. To simplify the following formulae we re-name some of the coefficients. Firstly let $B_2^1 \equiv b_0/2, B_{0,0\ldots,0,j_n=2}^1 \equiv b_n, n = 1,2,\ldots$ and $B_4^1 \equiv c_0/6, B_{2,2}^1, B_{2,0,2}^1, \ldots, B_{2,0\ldots,0,j_n=2}^1 \equiv c_n$ then the tree-level contribution to the equations $I_4 = 0, I_{2,2} = 0, I_{2,0,2} = 0, \ldots, I_{2,0\ldots,0,2} = 0$ can be written as

\[
b_0c_1 + b_1c_0 = 0 \\
b_0c_2 + b_1c_1 + b_2c_0 = 0 \\
b_0c_3 + b_1c_2 + b_2c_1 + b_3c_0 = 0 \\
\text{etc} \tag{14}
\]

which, in turn, may be expressed as the vanishing of each coefficient of $z$ in

\[
\left( \sum_{n=0}^\infty b_n z^n \right) \left( \sum_{m=0}^\infty c_m z^m \right) - b_0c_0 = 0, \tag{15}
\]

We can solve for the $c_n$ in (13) to give
\[
B_{2,0,\ldots,0,j_n=2}^1 = -\frac{g}{16} \begin{vmatrix}
B_{0,2}^1 & B_{0,0,2}^1 & \cdots & B_{0,0,\ldots,j_{n-1}=2}^1 & B_{0,0,\ldots,j_n=2}^1 \\
-1 & B_{0,2}^1 & \cdots & B_{0,0,\ldots,j_{n-2}=2}^1 & B_{0,0,\ldots,j_{n-1}=2}^1 \\
0 & -1 & \cdots & B_{0,0,\ldots,j_{n-3}=2}^1 & B_{0,0,\ldots,j_{n-2}=2}^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & B_{0,2}^1
\end{vmatrix}
\]  

(16)

We can find in a similar fashion the coefficients \(B_0^1 \equiv f_0/15, B_{1,0,\ldots,0,j_n=2}^1 \equiv f_n, n = 1\ldots\)
They are determined by the tree level equations \(I_0 = 0, I_{4,2} = 0, I_{4,0,2} = 0, \) etc. Redefining \(B_2^1 \equiv \frac{1}{3} b_0, B_4^1 = \frac{1}{8} c_0\) allows us to write those equations as

\[
\left(\sum_{n=0}^{\infty} c_n z^n\right)^2 + 2 \left(\sum_{m=0}^{\infty} b_m z^m\right) \left(\sum_{l=0}^{\infty} f_l z^l\right) - (c_0)^2 - 2 b_0 f_0 = 0
\]

(17)

where each coefficient of \(z\) must separately vanish. If we set \((\sum_{m=0}^{\infty} b_m z^m)^{-1} = \sum_{m=0}^{\infty} \beta_m z^m\)
as well as \((\sum_{n=0}^{\infty} c_n z^n)^2 = \sum_{n=0}^{\infty} \gamma_n z^n\) we can formally write, after substituting back the original values of \(b_0, c_0\) and \(f_0\),

\[
f_n = \frac{1}{2} \left((c_0)^2 + 2 b_0 f_0\right) \beta_n - \frac{1}{2} \sum_{k=0}^{n} \beta_k \gamma_{n-k} = -\frac{1}{2} \left(\frac{1}{27648} \beta_n + \sum_{k=0}^{n} \beta_k \gamma_{n-k}\right)
\]

(18)

\(n \geq 1\). Using the formulae for inversion and product of power series from the mathematical literature [32], we can calculate all the \(B_{1,0,\ldots,0,j_n=2}\)

The order-\(\hbar\) corrections are obtained by substituting the tree-level results into the previously ignored order-\(\hbar\) term in the Schrödinger equation and treating this as a perturbation to the classical equation. We want to use this to show that our large-distance expansion correctly gives the short-distance behaviour as contained in the divergent mass and energy subtractions \(\tilde{E}(\lambda)\) and \(\tilde{M}^2(\lambda)\), which occur only in \(I_0\) and \(I_2\). We first study \(I_2\). Using (13) we get the \(O(\hbar)\) expression

\[
I_{2}^\hbar(\lambda) = \frac{\tilde{M}^2(\lambda)^\hbar}{2} + 2 B_2^\hbar - g \sqrt{\frac{\lambda}{\pi}} \left(\frac{1}{16} - \frac{\lambda}{192} + \frac{\lambda^2}{1280} - \frac{5\lambda^3}{43008} + ..\right)
\]

(19)

This vanishes when \(\lambda \to \infty\), but we get a good approximation if we truncate the series and take \(\lambda\) as large as the truncation will allow, i.e. small enough for the first neglected term to be insignificant. Since \(I(\lambda)\) is of order \(1/\lambda\) for large \(\lambda\) the accuracy of this approximation is greatly improved if we perform a further contour integration, amounting to a re-summation of the series in \(\lambda\). Observe that substituting \(\lambda = 1/\sqrt{s}\) in \(I(\lambda)\) gives a function that is analytic in \(s\) with a cut on the negative real axis that we wish to evaluate as \(s\) tends to zero from real positive values, so we define

\[
\tilde{I}(\lambda) = \frac{1}{2\pi i} \int_{|s| = \infty} \frac{ds}{s} e^{s^2} \sqrt{\frac{\pi}{s^2}} \tilde{I}(s^{-1/2})
\]

(20)

for which \(\lim_{\lambda \to \infty} \tilde{I}(\lambda) = 0\). Thus

\[
\tilde{I}_0^\hbar(\lambda) = \frac{\delta M^2 + \tilde{M}^2(\lambda)^\hbar}{2} + 2 B_2^\hbar - g S(\lambda)
\]

(21)
where

\[
S(\lambda) = \frac{\sqrt{\lambda}}{\sqrt{\pi}} \left( \frac{1}{16 \Gamma(3/4)} - \frac{\lambda \sqrt{2} \Gamma(3/4)}{96 \pi} + \frac{\lambda^2}{960 \Gamma(3/4)} - \frac{\lambda^3 \sqrt{2} \Gamma(3/4)}{5376 \pi} + .. \right) \quad (22)
\]

The terms in \(S(\lambda)\) now decrease more rapidly than the corresponding terms in \(I(\lambda)\). Since \(I(1/\sqrt{s})\) behaves asymptotically as \(\sqrt{s}\) for small \(s\) this re-summation has the effect of eliminating the leading term so that \(\tilde{I}(\lambda)\) is now of order \(1/\lambda^2\). Further re-summations are only efficacious given a sufficient number of terms in the truncated series for the extra gamma-functions in the coefficients to be noticeable.

Figure 1: The mass subtraction

In fig. (1) we plot, the series \(S(\lambda)\) truncated to 13 terms, \(-\tilde{M}^2(\lambda)^h/(2g)\), their sum, and the limit of this sum as \(\lambda \to \infty\), (which we obtain exactly in the next section as \(-1/(8\pi) \simeq -0.0398\)). Clearly neither \(S(\lambda)\) nor \(-\tilde{M}^2(\lambda)^h/(2g)\) are constant for large \(\lambda\) but their sum is, to a good approximation for \(\lambda > 2\). This shows that our large-distance expansion correctly reproduces the short-distance effects encoded in \(M^2(e)^h\). The departure from this constant value for \(\lambda > 11\) is due to the error involved in truncating \(S(\lambda)\) to 13 terms. If we denote by \(S_n\) the series truncated to \(n\) terms minus \(\tilde{M}^2(\lambda)^h/(2g)\) then in fig. (2) we have shown \(S_n\) for \(n = 4, 5, 8, 9, 12, 13\).

Each truncation provides a good approximation to \(S(\lambda)\) up to a value of \(\lambda\) which is large enough for the highest order term to be a significant fraction of the whole. Taking this to be one per cent gives an estimate of \(S(\infty)\) with an error that ranges from three per cent (five terms) to half a per cent (13 terms).
To check that our large distance expansion correctly reproduces the energy subtraction we need the $O(\bar{h})$ part of $W[\varphi]$ that is quadratic in $\varphi$. We obtain this from the equations $I_0, I_0, I_0, 0, = 0, ..., $ having imposed the renormalisation condition $B_2 = -M/2$. We use the re-summation described earlier, truncate the series in $\lambda$ so that they include contributions from coefficients of functionals of $\varphi$ of dimension less than 26, and take $\lambda \approx \lambda$ so that the last included term is one per cent of the value of the truncated series. We also use Stieltje’s trick of halving the contribution of the last included term to improve the accuracy of the approximation [33]. This gives the estimate

\[
W_2^h = \frac{g}{1000} \int dx \left( 6.64\varphi'^2 - 6.02\varphi''^2 + 5.40\varphi'''^2 - 4.91\varphi''''^2 + 4.54\varphi^{(5)2} - 4.24\varphi^{(6)2} + 4.01\varphi^{(7)2} - 3.79\varphi^{(8)2} + 3.58\varphi^{(9)2} - 3.34\varphi^{(10)2} + .. \right) \quad (23)
\]
In the next section we obtain $W_2^h$ exactly. Rounding the exact results to three significant figures gives

$$W_2^h = \frac{g}{1000} \int dx \left( 6.63\varphi'^2 - 5.97\varphi''^2 + 5.33\varphi'''^2 - 4.84\varphi''''^2 + 4.45\varphi^{(5)2} \\
- 4.14\varphi^{(6)2} + 3.89\varphi^{(7)2} - 3.68\varphi^{(8)2} + 3.50\varphi^{(9)2} - 3.34\varphi^{(10)2} + \ldots \right)$$

which shows that our approximate results are good to a few per cent.

![Energy Subtraction](image)

**Figure 3:** The energy subtraction

Figure 3 shows the effect of substituting this estimate into the $O(h^2)$ contribution to $I_0$. The top curve, $A$, is the estimate of the re-summation of the series in $\lambda$, whilst the bottom curve, $B$, is the $O(h^2)$ contribution to the re-summation of $\mathcal{E}(\lambda)$ evaluated using (4) with $\delta M^2 = g/(4\pi)$. Neither of these curves appears to tend to a constant for large $\lambda$ whereas their sum, represented by the middle curve, $C$, provides a good approximation to a constant value for $\lambda$ larger than four until $\lambda$ is sufficiently large that the approximation of the infinite series by just ten terms breaks down. The straight line in the figure is the value 0.0052 which would be obtained by truncating the series at fifty terms using the expression for $W_2^h$ we find in the next section.

Having seen that our large distance expansion successfully reproduces the short-distance effects contained in the counter-terms of the Hamiltonian we turn to the one-loop evaluation of the $B_{j_0,\ldots,j_n}$ coefficients corresponding to higher numbers of fields. Begin with the coefficients of local functions containing four fields. Rounding the tree-level result to
three significant figures gives
\[ W_4^l = \frac{g}{1000} \int dx \left( -10.4 \varphi^4 + 15.6 \varphi^2 \varphi'^2 + 3.91 \varphi'^4 \right. \\
-7.81 \varphi^2 \varphi'^2 - 30.3 \varphi'^2 \varphi'' - 11.7 \varphi \varphi''' + 4.88 \varphi^2 \varphi''^2 - 22.7 \varphi''^4 \\
\left. +3.6 \varphi^2 \varphi''^2 + 40.0 \varphi \varphi''' \varphi'' - 3.42 \varphi^2 \varphi''^2 + \ldots \right) \] (25)

Estimating the \( O(h) \) contribution in the same way that we estimated \( W_2^h \) gives
\[ W_4^h = \frac{g^2}{10000} \int dx \left( 4.02 \varphi^4 - 20.0 \varphi^2 \varphi'^2 - 7.96 \varphi'^4 \right. \\
17.4 \varphi^2 \varphi'^2 + 83.8 \varphi'^2 \varphi'' + 37.6 \varphi \varphi''' - 15.6 \varphi^2 \varphi''^2 + 87.7 \varphi''^4 \\
\left. -129 \varphi^2 \varphi''^2 + -164 \varphi \varphi''' \varphi'' + 14.0 \varphi^2 \varphi''^2 + \ldots \right) \] (26)

There are two things to note. Firstly there is a proliferation of local functionals of the same dimension and number of \( \varphi \) as these increase. So, for example, there is a unique local functional with just two \( \varphi \) for any dimension, but there are two hundred and seven with twelve \( \varphi \) and dimension twelve. Secondly the ratio of the \( O(h) \) corrections to any two coefficients of functionals containing the same number of \( \varphi \) and the same dimension is approximately the same as the ratio of the tree-level values. For example the ratio of the \( O(h) \) coefficients of \( \varphi^2 \varphi'^2 \) and \( \varphi'^4 \) is \(-17.4/7.96 \simeq -2.19.. \) whereas the ratio of the corresponding tree-level values is exactly \(-2 \). Given that our estimate is probably only good to a few per cent it is not clear at this stage whether the one-loop ratios are exactly equal to the tree-level ratios, but we will investigate this with greater accuracy in the next section. We will now compare these results with those obtained by solving the Schrödinger equation without first expanding in terms of local functions.

3 Direct Semi-Classical Solution

It is straightforward to solve the Schrödinger equation \( \lim_{\epsilon \to 0} F_\epsilon[\varphi] = 0 \) without resorting to the local expansion, at least for low orders of an expansion in \( \varphi \) and \( h, \hbar \). This turns out to be remarkably efficient compared to the Feynman diagram expansion which we describe in the next section. Expand \( W[\varphi] \) as
\[ W[\varphi] = \sum_{n=1}^{\infty} \int dp_1 \ldots dp_{2n} \tilde{\varphi}(p_1) \ldots \tilde{\varphi}(p_{2n}) \Gamma_{2n}(p_1, \ldots, p_{2n}) \delta(p_1 + \ldots + p_{2n}) \] (27)

where the \( \Gamma \) are unknown functions. Then we can write \( \Delta_\epsilon W[\varphi] = \sum \Delta \Gamma_{2n} \) where \( \Delta \Gamma_{2n} \) is
\[ \int_{q^2 < 1/\epsilon} 2\pi dq \int dp_3 \ldots dp_{2n} 2(2n - 1) \tilde{\varphi}(p_3) \ldots \tilde{\varphi}(p_{2n}) \Gamma_{2n}(q, -q, p_3, \ldots, p_{2n}) \delta(p_3 + \ldots + p_{2n}) \] (28)
and
\[ \int dx \left( \frac{\delta W}{\delta \varphi} \right)^2 = \sum_{n,m} \Gamma_{2n} \circ \Gamma_{2m} \] (29)
where $\Gamma_{2n} \circ \Gamma_{2m}$ is

$$8nm\pi \int dp_2..dp_{2n}dk_2..dk_{2m} \varphi(p_2)\varphi(p_{2n})\varphi(k_2)\varphi(k_{2m})\Gamma_{2n}(-(p_2 + .. + p_{2n}), p_2, .., p_{2n}) \times \Gamma_{2m}(-(k_2 + .. + k_{2m}), k_2, .., k_{2m}) \delta(p_2 + .. + p_{2n} + k_2 + ..k_{2m})$$

(30)

Expanding the $\Gamma$ in powers of $\hbar$ as $\Gamma_{2n} = \sum h^n \Gamma^h_{2n}$ and ignoring order-$\hbar$ terms in the Schrödinger equation gives the tree-level result

$$\Gamma^1_2 \circ \Gamma^1_2 = 2\Gamma^1_2 \circ \Gamma^1_4 + 2\Gamma^1_2 \circ \Gamma^1_6 + \Gamma^1_4 \circ \Gamma^1_4 + .. = \int dx \left( \varphi^2 + M^2\varphi^2 + \frac{g}{12}\varphi^4 \right)$$

(31)

The term quadratic in $\varphi$ is

$$\Gamma^1_2 \circ \Gamma^1_2 = \int dp 8\pi (\Gamma^1_2(p, -p))^2 \varphi(p)\varphi(-p) = \int \frac{dp}{2\pi}(\varphi^2 + M^2)\varphi(p)\varphi(-p)$$

(32)

so if we take the negative root for normalisability of the vacuum functional we get

$$\Gamma^1_2 = -\sqrt{p^2 + M^2} \equiv -\frac{\omega(p)}{4\pi}.$$ 

(33)

Using

$$\Gamma^1_2 \circ \Gamma^1_{2n} = -\int dp_1..dp_{2n} \varphi(p_1)\varphi(p_{2n}) \left( \sum_{i=1}^{2n} \omega(p_i) \right) \Gamma_{2n}(p_1, .., p_{2n}) \delta(p_1 + .. + p_{2n})$$

(34)

in

$$\Gamma^1_2 \circ \Gamma^1_4 = \frac{g}{4!} \int dx \varphi^4.$$ 

(35)

gives, for $p_1 + .. + p_4 = 0$,

$$\Gamma^1_4(p_1, .., p_4) = -\frac{g}{(2\pi)^3(4!)(\omega(p_1) + .. + \omega(p_4))}.$$ 

(36)

The terms of higher order in $\varphi$, for which there are no contributions from the potential give $\sum_{n+m=\text{const}} \Gamma^1_2n \circ \Gamma^1_2m = 0$ which can be solved recursively as

$$\Gamma^1_{2r}(p_1, .., p_{2r}) = \frac{4\pi}{\sum_{r=2}^{n} \omega(p_1) \sum_{n=2}^{r-1} n(r + 1 - n) S \left\{ \Gamma^1_{2n}(-(p_2 + .. + p_{2n}), p_2, .., p_{2n}) \right\} \times \Gamma^1_{2(r+1-n)}(-(p_{2n+2} + .. + p_{2(r+1-n)}), p_{2n+2}, .., p_{2(r+1-n)})}.$$ 

(37)

where $S$ symmetrises the momenta. Expanding $\Gamma^1_2$ and $\Gamma^1_4$ in positive powers of the momenta reproduces (12) as it should since no re-summation is involved in either approach to the tree-level result.

The order-$\hbar$ contribution to the Schrödinger equation is

$$\sum_{n,m} \Gamma^h_{2n} \circ \Gamma^h_{2m} + \sum_{n} \Delta \Gamma^1_{2n} + \int dx \left( 2\mathcal{E}^h - (M^2)^h \varphi^2 \right) = 0.$$ 

(38)
The term quadratic in $\varphi$ gives the limit as $\epsilon \to 0$ of
\[
\frac{\delta M^2 + M^2(\epsilon)^h}{4\pi} - 2\omega(p)\Gamma_2^h(p, -p) + \frac{g}{32\pi^2} \int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} \frac{dq}{\omega(q) + \omega(p)} = 0 \quad (39)
\]
which can be solved for $p \neq 0$ as, [4],
\[
\Gamma_2^h(p, -p) = \frac{g}{32\pi^2} \int_0^\infty dq \left( \frac{1}{\omega(q)(\omega(q) + \omega(p))} \right) - \frac{\delta M^2}{8\pi\omega(p)} \quad (40)
\]
and for $p = 0$ we get $\Gamma_2^h = g/(32\pi^2 M) - \delta M^2/(8\pi M)$. The renormalisation condition that fixes $B_2$ at its classical value requires that $\Gamma_2^h(0, 0) = 0$, which determines $\delta M^2 = g/(4\pi)$. Setting $p = 0$ in (39) and taking the limit $\epsilon \to 0$ is meant to yield the same as taking the limit $\lambda \to \infty$ of $\tilde{I}^h_0(\lambda)$ in (21) when we identify $\Gamma_2^h(0, 0) = B_2/(2\pi)$. This gives the value $-g/(8\pi)$ quoted earlier that agrees well with the large $\lambda$ behaviour of $S(\lambda) - M^2(\lambda)^h/2$.

More particularly $S(\lambda)$ should be obtained from the large $\epsilon$ expansion of
\[
H(\epsilon) = \frac{1}{16\pi} \int_{q^2 < 1/\epsilon} dq \left( \frac{1}{\omega(q)(\omega(q) + \omega(p))} \right) \quad (42)
\]
by applying the two contour integral re-summations, giving
\[
-\frac{\lambda}{4\pi} \int_{|s| = \infty} d\tilde{s} \tilde{s}^{-3/4} e^{\lambda^2 \tilde{s}} \int_{|s| = \infty} ds s^{-1/2} e^{s/\sqrt{\tilde{s}}} H(s) \quad (43)
\]
which does in fact coincide with (22). Since the large $\lambda$ behaviour corresponds to small $\epsilon$ we can use (22) to investigate this. Thus for small $\epsilon$
\[
\int_{0}^{1/\sqrt{\epsilon}} dq \left( \frac{1}{\omega(q)(\omega(q) + M)} \right) = \sqrt{M^{-1} + \epsilon} - \sqrt{\epsilon} \approx \sqrt{M} - \sqrt{\epsilon} + \frac{\epsilon}{2\sqrt{M}} - \frac{\epsilon^2}{8\sqrt{M^3}} + .. \quad (44)
\]
which leads to the power law corrections to the large-$\lambda$ behaviour described earlier. Expanding (41) in positive powers of $p^2$ leads to the exact results for the $W_2^h$ quoted earlier
\[
\int \frac{dp}{2\pi} \Gamma_2^h(p, -p)\tilde{\varphi}(p)\tilde{\varphi}(-p) = -\frac{g}{\pi} \int dx \left( \frac{\varphi'^2}{48} + \frac{3\varphi'^4}{160} + \frac{15\varphi'^6}{896} + \frac{35\varphi'^8}{2304} + \frac{315\varphi'^{10}}{22528} + \frac{693\varphi'^{12}}{53248} + \frac{1001\varphi'^{14}}{81920} + \frac{6435\varphi'^{16}}{557056} + \frac{109395\varphi'^{18}}{9961472} + \frac{230945\varphi'^{20}}{22020096} + .. \right) \quad (45)
\]

The $O(h)$ contribution to the part of $W[\varphi]$ that is quartic in $\varphi$ is obtained from
\[
2\Gamma_2^1 \circ \Gamma_4^h + 2\Gamma_4^1 \circ \Gamma_2^h + \Delta \Gamma_6^1 = 0 \quad (46)
\]
which leads to

\[
\Gamma_4^h(p_1,\ldots,p_4) = \frac{g^2}{(2\pi)^34!\pi \sum_1^1 \omega(p_i)} \left\{ \int_0^\infty \frac{dq}{2\omega(q) + \sum_1^1 \omega(p_i)} \left( -\frac{1}{\omega(q)(\omega(q) + \omega(p_1))} \right) \right. \\
+ \frac{3}{(\omega(q) + \omega(p_1) + \omega(p_2) + \omega(q + p_1 + p_2))(\omega(q) + \omega(p_3) + \omega(p_4) + \omega(-q + p_3 + p_4))} \\
\left. + \frac{1}{2\omega(p_1) \sum_1^1 \omega(p_i)} \right\} 
\]  

(47)

Expanding in positive powers of the momenta \(p_1,\ldots,p_4\) and integrating over \(q\) numerically, (using MAPLE), leads to

\[
W_4^h = \frac{g^2}{10000} \int dx \left( 3.973\varphi^4 - 19.45\varphi^2\varphi'^2 - 7.961\varphi'^4 \\
16.27\varphi^2\varphi''^2 + 85.78\varphi^2\varphi''^2 + 33.66\varphi\varphi'^4 - 14.15\varphi^2\varphi''^2 + .. \right). 
\]  

(48)

From this it is clear that our previous estimate was quite good, but that the observation that the ratios \(\rho_{0,4},\rho_{2,0,2} = B_{0,4}^h/gB_{0,4}^h\) are the same for coefficients of functionals of the same dimension and number of fields is only approximate, since

\[
\rho_4 = -0.03814 
\]  

(49)

\[
\rho_{2,2} = -0.1245 
\]  

(50)

\[
\rho_{0,4}, \rho_{2,0,2} = -0.2038, -0.2082 
\]  

(51)

\[
\rho_{0,2,2}, \rho_{1,0,3}, \rho_{2,0,0,2} = -0.2834, -0.2872, -0.2898. 
\]  

(52)

These ratios may be explained by observing that the dominant contribution to (47) comes from the last term in the braces, which itself originates in the mass renormalisation. \(h\delta M^2\) was fixed by imposing \(B_2^h = 0\), but if instead we had taken \(h\delta M^2 = 0\) then this term would have been absent. The effect of this choice on the ratios can be calculated using dimensional analysis and gives

\[
\rho_4 = -0.00165 
\]  

(53)

\[
\rho_{2,2} = -0.00513 
\]  

(54)

\[
\rho_{0,4}, \rho_{2,0,2} = -0.00486, -0.00926 
\]  

(55)

\[
\rho_{0,2,2}, \rho_{1,0,3}, \rho_{2,0,0,2} = -0.00488, -0.00868, -0.0113. 
\]  

(56)

The advantage of this choice is that the one-loop corrections to the coefficients \(B_{0,4},..\) for functionals containing four fields are now significantly smaller. The same is true for the coefficients corresponding to two fields, with the exception of \(B_2\). This suggests that a more effective choice of renormalisation condition which would reduce the size of the one-loop corrections, would be to fix \(B_4\) at its classical value, rather than \(B_2\).
4 Sinh-Gordon Model

From the standpoint of perturbation theory $\varphi^4$-theory is ‘close’ to a theory that is quite special, namely the Sinh-Gordon theory which has an infinite number of conserved quantities that imply the absence of particle production, it is interesting to calculate the vacuum functional for this case. The potential of the Sinh-Gordon theory may be taken to be

$$V = \frac{M^2 + \hbar \delta M^2}{\beta^2} \cosh(\beta \varphi) \exp \left( -\frac{\beta^2}{4\pi} \int_0^{1/\sqrt{\epsilon}} \frac{dp}{\omega(p)} \right) = (M^2 + \hbar \delta M^2) \left( \frac{1}{\beta^2} + \frac{\varphi^2}{2} + \frac{\beta^4 \varphi^4}{4!} + \frac{\beta^6 \varphi^6}{6!} + \ldots \right) \left( 1 - \frac{\beta^2}{4\pi} \int_0^{1/\sqrt{\epsilon}} \frac{dp}{\omega(p)} + \ldots \right) \quad (57)$$

Apart from the replacement $g \to M^2/\beta^2$ the Sinh-Gordon potential leads to the same expressions for the tree-level values of $\Gamma_1^1$, $\Gamma_1^4$, and the one-loop result $\bar{\Gamma}_4^1$. The tree-level $\Gamma_6^1$ is modified by the $\varphi^6$ term in the potential

$$\Gamma_6^1 \to \Gamma_6^1 - \frac{\beta^4 M^2}{6!(2\pi)^5 \sum_i \omega(p_i)} \quad (58)$$

this, together with the $\delta M^2 \varphi^4$ term in $V$ modifies the one-loop value $\bar{\Gamma}_4^1$

$$\bar{\Gamma}_4^1 \to \bar{\Gamma}_4^1 - \frac{\beta^4 M^2}{(2\pi)^4! \sum_i \omega(p_i)} \left( \int_0^\infty dq \left( \frac{1}{2\omega(q) + \sum_i \omega(p_i)} - \frac{1}{2\omega(q)} \right) + \frac{1}{2} \right) \quad (59)$$

so that for the sinh-Gordon model

$$W_4^h = \frac{\beta^4}{\pi} \int dx \left( \frac{\varphi^4}{384} + \frac{5\pi - 22}{1280} \varphi^2 \varphi'^2 - \frac{2275\pi - 8952}{860160} \varphi'^4 + \frac{651\pi - 2768}{172032} \varphi^2 \varphi''^2 - \frac{1041705\pi - 4243072}{41287680} \varphi'^2 \varphi''^2 - \frac{689535\pi - 2920448}{82575360} \varphi \varphi'^3 + \frac{13905\pi - 58624}{3932160} \varphi^2 \varphi''' + \ldots \right)$$

$$\approx \frac{\beta^4}{10000} \int dx \left( 8.2893 \varphi^4 - 15.647 \varphi^2 \varphi'^2 - 6.6791 \varphi'^4 + 13.3743 \varphi^2 \varphi''^2 + 74.818 \varphi'^2 \varphi''^2 + 29.0731 \varphi \varphi'^3 - 12.0941 \varphi^2 \varphi''' + \ldots \right) \quad (60)$$

The ratios of the one-loop coefficients to their tree-level values for this model are

$$\rho_4 = -0.07958 \quad (61)$$

$$\rho_{2,2} = -0.1001 \quad (62)$$

$$\rho_{0,4} = -0.1710, \quad \rho_{2,0,2} = -0.1712 \quad (63)$$

$$\rho_{0,2,2} = -0.2471, \quad \rho_{1,0,3} = -0.2481, \quad \rho_{2,0,0,2} = -0.2477. \quad (64)$$

Note that again the ratios are approximately the same for coefficients of functionals of the same number of fields and dimension, however this cannot be explained away as simply
the effect of mass or coupling renormalisation. If we had taken $\delta M^2 = 0$ we would have obtained

$$\rho_4 = -0.1194$$

(65)

$$\rho_{2,2} = -0.06035$$

(66)

$$\rho_{0,4} = -0.05162$$

(67)

$$\rho_{0,2,2} = -0.04915$$

(68)

which are approximately constant for coefficients of functionals of the same number of fields and dimension. These coefficients would, with a change of sign, apply to the Sine-Gordon model as well.

5 Feynman Diagram Expansion

We will now describe how the results of the previous section are obtained within the more conventional approach to field theory based on Feynman diagrams. There are several ways to represent the vacuum functional as a functional integral, the most convenient for our purposes is the representation, $\Psi[\phi] = \int D\phi e^{-\mathcal{S}_E}[\phi] + \int d^4x \dot{\phi}$

(69)

where $\phi$ vanishes on the surface $t = 0$, which is the boundary of the Euclidean space-time $t < 0$ in which the field $\phi$ lives. Therefore, $W[\phi]$ is a sum of connected Euclidean Feynman Diagrams in which $\phi$ is the source for $\dot{\phi}$ on this boundary, the only major difference from the usual Feynman diagrams encountered in free space is that the propagator vanishes when either of its arguments lies on the boundary. Such a propagator can be obtained using the method of images as

$$G(x, y) = G_0(x, y) - G_0(x, y) = G_0(x, y) - G_0(x, y)$$

(70)

where $x = (t, x)$, $\bar{x} = (-t, x)$, similarly for $y$ and $\bar{y}$, and $G_0(x, y)$ is the free space propagator. Developing the Feynman diagram expansion yields the tree level diagrams up to $\varphi^6$ which are shown in figure (4).

Figure 4: Tree Diagrams up to $\varphi^6$

The Feynman rules in the coordinate space for the diagram (b), for instance, yield

$$g s f \int d^4x_1...d^4x_4 d^2v \varphi(x_1)\varphi(x_2) \varphi(x_3) \varphi(x_4) \frac{\partial G(x_1, v)}{\partial t_1} \frac{\partial G(x_2, v)}{\partial t_2} \frac{\partial G(x_3, v)}{\partial t_3} \frac{\partial G(x_4, v)}{\partial t_4}$$

(71)
where \( s_f \) is the symmetry factor associated to the diagram, the times \( t_1, \ldots, t_4 \) are all set to zero, and the time-like component of \( v \) is integrated over negative values only. When a propagator ends on the boundary it is equal to twice the free space propagator, so that in momentum space this is proportional to

\[
\int dp_1 \cdots dp_4 dq_1 \cdots dq_4 \bar{\phi}(p_1) \cdots \bar{\phi}(p_4) \delta(p_1 + \cdots + p_4) \delta(q_1 + \cdots + q_4) \prod_{j=1,4} \frac{q_j}{p_j^2 + q_j^2 + M^2}
\] (72)

The momenta \( q_1, \ldots, q_4 \) are the time-like components which, when integrated over, give rise to \( \delta \)-functions that impose the vanishing of \( t_1, \ldots, t_4 \) in the configuration space representation (71). In the momentum-space representation these integrals are readily performed as contour integrals leading to our previous expression (36) for \( \Gamma_4 \). Diagram (a) gives

\[
\int \frac{2}{(2\pi)^2} dp dq \, \bar{\phi}(p) \bar{\phi}(-p) \frac{q^2}{p^2 + q^2 + M^2}
\] (73)

The integral over \( q \) is divergent. Formally we can write it as

\[
\int \frac{dp}{2\pi} \, \phi(p) \bar{\phi}(-p) \left( \delta(0) - \sqrt{p^2 + M^2} \right)
\] (74)

The origin of \( \delta(0) \) is in the construction of the original path integral representation of \( W \), (30). Rather than the functional integral (39), we should start with

\[
\Psi[\varphi] = \langle \varphi | 0 \rangle = \langle D | e^{i \int \pi \varphi \, dx} | 0 \rangle
\] (75)

(where the state \( \langle D \) satisfies \( \langle D | \varphi = 0 \) ). As operators \( \pi = \hat{\pi} \) , but in the passage from (43) to the functional integral \( \pi \) is represented by \( \hat{\pi} \) plus terms coming from the time derivative acting on the \( T \)-ordering because the functional integral represents \( T \)-ordered products. This leads to terms like \( \int dx \varphi^2 \delta(0) \), because this is local it may be cancelled by an equal but opposite counterterm, which amounts to simply discarding the divergence. Alternatively, and perhaps more satisfactorily, we can deal with this divergence by placing the source terms \( \pi \varphi \) not at \( t = 0 \) but at small, distinct times \( t_i \) and finally taking the limit \( t_i \to 0 \). This leads to an extra factor of the form \( e^{i q \epsilon} \) in (73) which regularizes the divergence and enables the integral to be done by closing the \( q \)-contour in the upper half-plane, yielding a finite result in limit as \( \epsilon \to 0 \), so that we end up with our earlier expression (33).

When neither end of a propagator is on the boundary, \( t = 0 \), the image charge breaks energy conservation leading to more complicated expressions. However for the tree-level diagram (c) these may be combined into an expression proportional to

\[
\int \left( \prod_{i=1}^6 dp_i dq_i \, \bar{\varphi}(q_i) \, p_i \right) \frac{\delta(p_1 + \cdots + p_6) \delta(q_1 + \cdots + q_6)}{(p_1 + p_2 + p_3)^2 + (q_1 + q_2 + q_3)^2 + M^2}
\] (76)

which, after integration over the \( q_i \) gives the same as (37).

The diagrams represented in figure (5) enable us to calculate the \( O(\hbar) \) correction for the coefficients of 2 and 4 fields.
The symmetry factors $s$ we have

This is divergent due to the factor $G$ in powers of the momenta $p$ before, diagram (e) is the counterterm diagram associated to the bubble appearing in (d). In momentum space (c) gives

From fig. (a) above we can work out the first order correction to $B_2$ as

This is divergent due to the factor $G(z, z)$. Since the propagator does not touch the boundary, it has an energy non-conserving contribution from the image represented by the exponential in

where $t$ is the time-like coordinate of $z$. We regulate the integral by restricting the space-like component of momentum, $p^2 < 1/\epsilon$, just as we did for the Laplacian in the Hamiltonian. The divergence is then cancelled by the counter-term represented by fig. (b), which is due to the order-\(\hbar\) terms in $M^2(\epsilon)$, (f). Taken together these diagrams give $\Gamma_2^\hbar$ as in (11).

The first order correction $\Gamma_2^\hbar$ is given by the sum of diagrams (c), (d) and (e). As before, diagram (e) is the counterterm diagram associated to the bubble appearing in (d).

In momentum space (c) gives

where $l = p_3 + p_4 + k$. whereas, for the sum of the other two diagrams, (with $\delta M^2 = 0$), we have

The symmetry factors $s_f^{(c)}$ and $s_f^{(d)}$ are respectively equal to $1/16$ and $1/12$. If we expand in powers of the momenta $p_1, \ldots, p_4$ the loop integration over $k$ can be done to reproduce the results for $W_4^\hbar$ given by (18).

The extra $\phi^6$ term in the potential of the Sinh-Gordon model generates the diagram fig. (f). Its analytic expression after subtracting the counterterm reads

$$\beta^4 s_f \frac{1}{(2\pi)^3} \int dk dp_1 \ldots dp_4 \frac{\tilde{\phi}(p_1) \ldots \tilde{\phi}(p_4) \delta(p_1 + \ldots + p_4) }{\omega(k)(\omega(p_4) + \omega(k)) \sum_1^4 \omega(p_i)(\sum_1^4 \omega(p_i) + 2\omega(k))} .$$

$$\Gamma_2^\hbar = \frac{\beta^4}{(2\pi)^3} \int dk dp_1 \ldots dp_4 \frac{\tilde{\phi}(p_1) \ldots \tilde{\phi}(p_4) \delta(p_1 + \ldots + p_4) }{\omega(k)(\omega(p_4) + \omega(k)) \sum_1^4 \omega(p_i)(\sum_1^4 \omega(p_i) + 2\omega(k))} .$$

$$\Gamma_2^\hbar = \frac{\beta^4}{(2\pi)^3} \int dk dp_1 \ldots dp_4 \frac{\tilde{\phi}(p_1) \ldots \tilde{\phi}(p_4) \delta(p_1 + \ldots + p_4) }{\omega(k)(\omega(p_4) + \omega(k)) \sum_1^4 \omega(p_i)(\sum_1^4 \omega(p_i) + 2\omega(k))} .$$
where $s_f$ for this diagram is $1/96$. This gives the modifications to the $\varphi^4$ results described earlier.

![Diagram](image)

Figure 6: 6 point interaction for the Sinh-Gordon Model

6 Reconstructing the Vacuum Functional

![Graph](image)

Figure 7: Tree-level vacuum functional: $\Gamma_2^1(p, -p)$

We will now use the above results for the $\varphi^2$ part of $W[\varphi]$ to illustrate how the vacuum functional can be reconstructed from its large distance expansion. The tree-level contribution has been treated in detail in \[30\]. Applying formula (2) to $\int dp \tilde{\varphi}(p) \tilde{\varphi}(-p) \Gamma_2(p, -p)$ amounts to the expression

$$
\Gamma_2(p, -p) = \lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_{|s|=\infty} ds \frac{e^{\lambda(s-1)}}{s-1} \sqrt{s} \Gamma_2(p/\sqrt{s}, -p/\sqrt{s})
$$

(82)
Since \(|s|\) is large on the contour we can use the local expansion \(\Gamma_2(p,-p) = \sum_0^\infty a_n p^{2n}\). Shifting \(s\) we get

\[
\lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_{|s|=\infty} \frac{ds}{s} e^{\lambda s} \sqrt{s+1} \sum_0^\infty a_n \frac{p^{2n}}{(s+1)^n} \equiv \lim_{\lambda \to \infty} S(p, \lambda) \tag{83}
\]

Expanding the \((s+1)\) factors in powers of \(1/s\) enables the integral to be done, yielding a power series in \(\lambda\). For example, at tree-level

\[
\frac{1}{2\pi i} \int_{|s|=\infty} \frac{ds}{s-1} e^{\lambda(s-1)} \sqrt{p^2 + s} = \sum_0^\infty \frac{(-)^{n+1} \lambda^{n-1/2} (1+p^2)^n}{n!(2n-1)\sqrt{\pi}}. \tag{84}
\]

This series converges for all positive \(\lambda\). We get an approximation by truncating the expansion by including terms up to and including \(\lambda^{N-1/2}\), say. This requires a knowledge of the local expansion only up to terms in \((\varphi^{(N)}(x))^2\). To demonstrate this approximation we have plotted in fig. (7) the series (84) truncated at \(N = 12\), \(S_{12}\). The value of \(\lambda\) is chosen so that the last term included is one per cent of the value of the series. We have also plotted \(\sqrt{p^2 + 1}\), and the expansion of \(\sqrt{p^2 + 1}\) in powers of \(p^2\), \(C\) truncated to fourteen terms. The full series fails to converge for \(p^2 > 1\), and this is reflected in the fact that the truncated series ceases to be a good approximation for \(p^2 > 1\). However, \(S_{12}\), which is a resummation of this series is a very good approximation for a much larger range of momenta.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure8.png}
\caption{One-loop vacuum functional: \(\Gamma_2^h(p,-p)\)}
\end{figure}

The one-loop correction, \(\Gamma_2^h\), may be treated in the same way. In fig. (8) we have plotted \(\text{arcsinh}(p)/p\). The small \(p\) expansion, \(C\) is again only good for \(p^2 < 1\). Our
approximation that re-sums this series, $S_{12}$ provides a good approximation only over a slightly larger range. The accuracy of this approximation is greatly improved by further re-summations, just as we did for the Laplacian. Let us define the re-summation operator acting on a function of $\lambda$ and $p$ to be

$$R \cdot S(\lambda, p) \equiv \frac{1}{2\pi i} \int_{|s|=\infty} ds \frac{1}{s} e^{\lambda s} S(s^{-1/2}, p).$$

(85)

Then the curve $R_{12}$ shown in fig. (8) results from applying $R$ twice to $S_{12}$, and provides a good approximation to arcsinh($p$)/$p$ for values of $p$ up to about $p = 5$. Since the effect of applying $R^p$ to a term in $S_{12}$ that is proportional to $\lambda^n$ is simply to divide it by $\Gamma(n/2 + 1)\Gamma(n/4 + 1)\Gamma(n/2p + 1)$, further applications would have no significant effect when we take just twelve terms in the expansion.

7 Conclusions

The purpose of this paper has been to test an approach to quantum field theory in which states are constructed in the Schrödinger representation from their large-distance behaviour. The vacuum functional is expanded as a series of local functionals. Truncating this series at some convenient order leads to an approximation scheme. For scalar $\varphi^4$ theory in 1+1-dimensions we compared the results obtained by solving the form of the Schrödinger equation that applies in this approach with those obtained from a semiclassical expansion, and from the Feynman diagram expansion of the wave-functional. We have found that the expansion coefficients agree to within a few per cent when we truncate these series at about ten terms. Also we found that the known counterterms that contain information about short-distance effects are correctly reproduced by our large-distance expansion to a similar accuracy.

We also found a curious simplification that occurs in the vacuum functional of the Sinh-Gordon and Sine-Gordon models. The ratios of coefficients of the one-loop corrections to the coefficients of local functionals containing four fields to their tree-level values are approximately the same for functionals of the same dimension.

8 Acknowledgements

M. Sampaio acknowledges a grant from CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil, and J. Pachos acknowledges a studentship from the University of Durham.

References

[1] Hatfield, B. Quantum Field Theory of Particle and Strings, Addison Wesley, 1992, ISBN 0-201-11-11982X.

[2] Symanzik, K. Nucl. Phys. B190[FS3] (1983) 1.
[3]  Symanzik, K. Schrodinger Representation in Renormalizable Quantum Field Theory, Les Houches 1982, Proceedings, Recent Advances In Field Theory and Statistical Mechanics.

[4]  Luscher, M., Schrodinger Representation in Quantum Field Theory, Nucl. Phys. B254 (1985) 52-57.

[5]  Chan, H.S., Nucl. Phys. B278 (1986) 721.

[6]  Lüscher, M., Narayanan, R., Weisz, P. and Wolff, U., Nucl. Phys. B384 (1992) 168.

[7]  K. Jansen, C. Liu, M. Luscher, H. Simma, S Sint, R. Sommer, P. Weisz, U. Wolff, Phys. Lett. B372 (1996) 275.

[8]  S. Sint, Nucl. Phys. B451 (1995) 416.

[9]  S. Sint, R. Sommer, Nucl. Phys. B465 (1996) 71.

[10]  Jackiw, R., Analysis on Infinite Dimensional Manifolds: Schrodinger Representation for Quantized Fields, Brazil Summer School 1989:78-143.

[11]  McAvity, D.M. and Osborn, H. Nucl. Phys. B394 (1993) 728.

[12]  Yee, J. H., Schrodinger Picture Representation of Quantum Field Theory, Mt. Sorak Symposium 1991:210-271.

[13]  Floreanini, R., Applications of Schrodinger Picture in Quantum Field Theory, Luc Vinet (Montreal U.), MIT-CTP-1517, Sept. 1987.

[14]  C. Kiefer, Phys. Rev. D 45 (1992) 2044.

[15]  C. Kiefer and Andreas Wipf, Ann. Phys. 236 (1994) 241.

[16]  Feynman, R. P., Nucl. Phys. B188 (1981) 479.

[17]  Greensite, J., Nucl. Phys. B158 (1979) 469, Nucl. Phys. B166 (1980) 113.

[18]  Kawamura, M., Maeda, K., Sakamoto, M., KOBE-TH-96-02 hep-th/9607176.

[19]  P. Mansfield, Nucl. Phys. B418 (1994) 113.

[20]  G.V. Dunne, R. Jackiw, C.A. Trugenberger, Chern-Simons Theory in the Schrodinger Representation, Ann. Phys. 194:197,1989.

[21]  A.V. Ramallo, Two Dimensional Chiral Gauge Theories in the Schrodinger Representation, Int. J. Mod. Phys. A5:153, 1990.

[22]  K. Heck, Some Considerations on the Problem of Renormalization of Quantum Field Theory in the Schrodinger Representation. In German, Heidelberg Univ. - HD-THEP 82-04.

[23]  T. Horiguchi, KIFR-94-01, KIFR-94-03, KIFR-95-02, KIFR-96-01, KIFR-96-03.
[24] T. Horiguchi, Nuovo. Cim. 111B (1996) 49,85,293.

[25] T. Horiguchi, K. Maeda, M. Sakamoto, Phys.Lett. B344 (1994) 105.

[26] Kowalski-Glikman, J., Meissner, K.A., Phys.Lett. B376 (1996) 48.

[27] Kowalski-Glikman, J. gr-qc/9511014.

[28] Blaut, A., Kowalski-Glikman, J. gr-qc/9607004.

[29] Maeda, K., Sakamoto, M., Phys.Rev. D54 (1996) 1500.

[30] Mansfield, P., Phys. Lett. B358 (1995) 287.

[31] Mansfield, P., Phys. Lett. B365 (1996) 207.

[32] Jeffrey, A., *Handbook of Mathematical Formulas and Integrals*, Academic Press. (1995).

[33] Copson, E., *Asymptotic Series*, Cambridge University Press, 1965.

[34] Coleman, S., Phys. Rev. D11 (1975) 2088.