Resonating delay equation

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Abstract – We propose here a delay differential equation that exhibits a new type of resonating oscillatory dynamics. The oscillatory transient dynamics appear and disappear as the delay is increased between zero to asymptotically large delay. The optimal height of the power spectrum of the dynamical trajectory is observed with the suitably tuned delay. This resonant behavior contrasts itself against the general behaviors where an increase of the delay parameter leads to the persistence of oscillations or more complex dynamics.

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Introduction. – Studies of time delays in dynamics have found rather intricate and complex behaviors. The time delays most commonly occur due to finite conduction and production times. They are intrinsic features of many control and interacting systems and have been studied in various fields including mathematics, biology, physics, engineering, and economics [1–14]. “Delay differential equations” are the main mathematical approaches and modeling tools for such systems.

Typically, delays induce instability of stable fixed points leading to oscillatory and more complex dynamics. Also, the complexity of dynamics generally increases with longer delays. For example, the Mackey-Glass equation [8], which was introduced to model the reproduction of the blood cells, shows the sequence of the monotonic convergence, transient oscillations, persistent oscillations, and chaotic dynamics as the delay parameter in the feedback function becomes longer. Understandings of the path to the complex behaviors of many systems with delays, including this model, have been gradually gained (e.g., [15]). There are, however, more to be investigated and explored, particularly with respect to the nature of time-dependent dynamical trajectories.

Against this background, we propose here a delay differential equation that exhibits a new type of dynamical behavior. Namely, the oscillatory transient dynamics appear and disappear as the delay is increased between zero to asymptotically large delay. We analyze the equation both mathematically and numerically to show that there is a resonance: the optimal height of the power spectrum of the dynamical trajectory is observed with the suitably tuned value of the delay parameter. This resonant behavior contrasts itself against the general behaviors induced by delay where an increase of delay leads to the persistence or increase of complex dynamics.

We emphasize that we are not investigating stability switching phenomena (e.g., [16]) with the delay as the bifurcation parameter. Indeed, in our analysis of the proposed model in this work, the asymptotic stability of the fixed point never changes with increasing delay. Changes are observed with the shapes of dynamical trajectories approaching the stable fixed point.

We end this paper with the discussion that there is an indication that similar transient resonant behaviors exist for more general types of equations.

Main equation and its properties. – The equation we propose is the following:

\[
\frac{dX(t)}{dt} + aX(t) = bX(t - \tau),
\]

where \( a \geq 0, b \geq 0, \tau \geq 0 \) are real parameters. When the parameter \( \tau \) is interpreted as a delay, we can consider this equation as a delay differential equation describing the dynamics of the variable \( X(t) \).

We investigate this equation both analytically and numerically to show that oscillatory transient dynamics appear and disappear as the value of delay increases.

Analysis of different cases. – Let us first consider the case where \( b = 0 \). With the initial condition \( X(t = 0) = X_0 \), the solution to the equation is given as

\[
X(t) = X_0 e^{-\frac{b}{a}t^2}.
\]

Thus, it is a trajectory with a Gaussian shape.

Next, the case where \( a = 0 \) is a special case of the much-studied Hayes’s equation, a first-order delay differential
equation with constant coefficients [4]. It is known that
the origin $X = 0$ is asymptotically stable only in the range
\[-\pi/2\tau < b < 0. \tag{3}\]

Now comes the general case with $a > 0, b > 0$. When
the delay $\tau = 0$, the solution $X(t) = 0$ is easily
obtained as
\[X(t) = \exp^{-\frac{1}{\tau^2} - \frac{i}{\tau}t} . \tag{4}\]
This is again a Gaussian with its peak at $b/a$.
To gain an insight when $\tau \neq 0$, we take the Fourier
transform of eq. (1),
\[i\omega \hat{X}(\omega) + i a \frac{d\hat{X}(\omega)}{d\omega} = -b \hat{X}(\omega)e^{i\omega\tau} , \tag{5}\]
where
\[\hat{X}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} X(t) dt . \tag{6}\]
The solution is again readily obtained with $C$ as the integra-
tion constant,
\[\hat{X}(\omega) = C \exp\left[ -\frac{1}{2a} \omega^2 + \frac{b}{\tau a} e^{i\omega\tau} \right] . \tag{7}\]
From this expression, we can infer when $\tau$ becomes very
large; the second term in the exponential approaches zero
leading to the Gaussian form. As we transform back, it is
again the Gaussian trajectory. Thus, with $\tau \to \infty$,
\[X(t) \to D e^{-\frac{1}{\tau^2} t^2} , \tag{8}\]
where $D$ is a constant.

So, we have the Gaussian trajectory with zero and
asymptotically large delay. What happens in between?
It turns out that oscillatory behaviors arise and disappear
as we increase the value of the delay $\tau$. We will show
these resonating phenomena together with numerical sim-
ulations of the equation.

Analysis of the oscillatory behavior. We have per-
formed the numerical simulation of eq. (1). Some rep-
resentative examples are shown in fig. 1 with parameter
choice so that the asymptotic stability is kept with any
value of the delay. With zero delay, the shape of the
dynamics is the Gaussian as derived in the previous subsec-
tion. With the increasing delay, the oscillatory behaviors
arise on top of the Gaussian trajectory. Further increase
changes the oscillatory shape into trains of pulses with
decreasing height at the delay interval. In the limit of
the long delay, the oscillation disappears. Note again that
the asymptotic stability of $X = 0$ does not change by the
increasing delay in this parameter set. This is the one
notable effect of having replaced the constant coefficient
by the linearly dependent one on $t$ in the second term of
eq (1); it is in contrast to the case of the much-studied
constant coefficient case, where the onset of the oscillation
leads to the destabilization by the increasing delay. It is
also different from the delay-induced transient oscillation

(DITO) [17,18]. The phenomena arise in coupled delay
differential equations exhibiting the prolonged duration of
oscillatory behaviors with increasing delay.
Let us now analyze the oscillatory behavior. Using
eq (6), we can compute the power spectrum as
\[S(\omega) = |\hat{X}(\omega)|^2 = \hat{X}(\omega)\hat{X}^*(\omega) =
C^2 \exp\left[ -\frac{1}{a} \omega^2 + \frac{2b}{\tau a} \cos \omega \tau \right] . \tag{9}\]
We have plotted this equation for the power spectrum
for the various delays. Results with the same parameter
setting as in fig. 1 are shown in fig. 2.

Also as shown, the peak of the power spectrum shows a
maximum height with the tuned value of the delay. This
indicates the resonance with the delay as a tuning param-
eter. We can analyze this by examining eq. (9). By taking
the derivative of (9), we see the maximum and minimum
points of the power spectrum occurs at $\omega^*$ satisfying
\[\omega^* = -b \sin \omega^* \tau . \tag{10}\]
The appearance and disappearance of the oscillatory
behavior correspond to that of the peaks in the power
spectrum.

They are given by the intersection points of the two
functions from both sides of this condition (fig. 3(A-i);
the value of $b$ is chosen to be different from figs. 1 and 2
so that the comparison is easier to be shown graphically).
The position of the first peak corresponds to the second

Fig. 1: Representative dynamics of the main equation (1) with
different values of the delays, $\tau$. The parameters are set at
$a = 0.15, b = 6.0$ with the initial interval condition as $X(t) =
0.1(-\tau \leq t \leq 0)$. The values of the delays $\tau$ are (A) 0, (B) 2,
(C) 4, (D) 7, (E) 10, (F) 20, (G) 50, (H) 80.
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Fig. 2: Representative power spectrums given by eq. (1) with different values of the delays \( \tau \). The parameters are the same as in fig. 1: \( a = 0.15, b = 6.0, C = 1 \) with the initial interval condition as \( X(t) = 0.1(\tau \leq t \leq 0) \). The values of the delays \( \tau \) are (A) 4, (B) 5, (C) 6, (D) 10, (E) 15, (F) 20.

We can numerically estimate this point, and obtain the height of the peak for various values of the delay, which is plotted in fig. 4. The resonance with the delay as the tuning parameter is clearly observed.

We can also infer the condition for the appearance of the oscillatory behavior. From above, the critical value \( \tau_c \) necessary for the existence of the power spectrum peaks are when the intersection point is the tangent point (fig. 3(B)). This gives the following conditions:

\[
\begin{align*}
\omega^* &= -b \sin \omega^* \tau_c, \quad 1 = -b \tau_c \cos \omega^* \tau_c, \\
\frac{\pi}{\tau_c} &< \omega^* < \frac{3\pi}{2\tau_c}. 
\end{align*}
\]  

If we set \( \lambda_c = b\tau_c \), the above condition leads to

\[
\begin{align*}
\frac{-1}{\lambda_c} &= \cos \sqrt{\lambda_c^2 - 1}, \\
\sqrt{\pi^2 + 1} < \lambda_c < \sqrt{\frac{9}{4}\pi^2 + 1}. 
\end{align*}
\]

We can obtain the value of \( \lambda_c \) numerically as \( \lambda_c = b\tau_c \approx 4.603 \).

Thus, if we fix the parameter \( b \), the oscillation appears for a delay longer than this critical delay:

\[
\tau_c = \frac{1}{b} < \tau. 
\]

Also, if we increase the delay further, there are more peaks to appear as there are more roots to satisfy (11). Asymptotically, components of all frequency signals are mixed, leading to suppression of the oscillatory behavior.

Discussion. – In this paper, we proposed a simple delay differential equation that exhibits oscillatory resonance. Resonance with both stochasticity and delay have previously been investigated as “delayed stochastic resonance” [19]. There, the tuned combination of strength of noise and the amount of delay led to more regular oscillatory patterns. Our equation here is simpler as it induces resonance only with the delay. We can consider equations of the more general form,

\[
\frac{dX(t)}{dt} + aX(t) = f(X(t - \tau)),
\]

where the function \( f \) can be a non-linear function. Our preliminary simulation results show that if we set \( f \) in the
The parameters are \(a = 0.15, b = 2.0\) with the initial interval condition as \(X(t) = 0.1(-\tau \leq t \leq 0)\). The critical value of the delay is \(\tau_c = \frac{\lambda_c b}{b} \approx 2.3017\). The values of the delays \(\tau\) are (A) 4.802, (B) 2.3017, (C) 2.002.

form of the “negative feedback function” and the “mixed feedback function” [8,20,21], we can observe transient oscillations and chaotic dynamics, respectively. They are also suppressed with asymptotically larger delay. More investigations, however, are needed to understand the nature of these behaviors with the general equation.

REFERENCES

[1] An der Heiden U., J. Math. Biol., 8 (1979) 345.
[2] Bellman R. and Cooke K., Differential-Difference Equations (Academic Press, New York) 1963.
[3] Cabrera J. L. and Milton J. G., Phys. Rev. Lett., 89 (2002) 158702.
[4] Hayes N. D., J. London Math. Soc., 25 (1950) 226.
[5] Insperger T., IEEE Trans. Control Syst. Technol., 14 (2007) 974.
[6] Küchler U. and Mensch B., Stoch. Stoch. Rep., 40 (1992) 23.
[7] Longtin A. and Milton J. G., Biol. Cybern., 61 (1989) 51.
[8] Mackey M. C. and Glass L., Science, 197 (1977) 287.
[9] Milton J. G., Cabrera J. L., Ohira T., Tajima S., Tonosaki Y., Zurich W. W. and Campbell S. A., Chaos, 19 (2009) 026110.
[10] Ohira T. and Yamane T., Phys. Rev. E, 61 (2000) 1247.
[11] Smith H., An Introduction to Delay Differential Equations with Applications to the Life Sciences (Springer, New York) 2010.
[12] Štěpán G., Retarded Dynamical Systems: Stability and Characteristic Functions (Wiley & Sons, New York) 1989.
[13] Štěpán G. and Insperger T., Ann. Rev. Control, 30 (2006) 159.
[14] Szydlowski M. and Krawiec A., J. Nonlinear Math. Phys., 8 (2010) 266.
[15] Taylor S. R. and Campbell S. A., Phys. Rev. E, 75 (2007) 046215.
[16] Yan X., Liu F. and Zhang C., Nonlinear Dyn., 99 (2020) 2011.
[17] Milton J. G., Naie P., Chan C. and Campbell S. A., Math. Model. Nat. Phenom., 5 (2010) 125.
[18] Pakdaman K., Grotta-Ragazzo C. and Malta C. P., Phys. Rev. E, 58 (1998) 3623.
[19] Ohira T. and Sato Y., Phys. Rev. Lett., 82 (1999) 2811.
[20] Glass L., Beuter A. and Larocque D., Math. Biosci., 90 (1988) 111.
[21] Glass L. and Mackey M. C., From Clocks to Chaos: The Rhythms of Life (Princeton University Press, Princeton) 1988.