Lower bounds for integration and recovery in $L_2$

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Abstract

Function values are, in some sense, “almost as good” as general linear information for $L_2$-approximation (optimal recovery, data assimilation) of functions from a reproducing kernel Hilbert space. This was recently proved by new upper bounds on the sampling numbers under the assumption that the singular values of the embedding of this Hilbert space into $L_2$ are square-summable. Here we mainly prove new lower bounds. In particular we prove that the sampling numbers behave worse than the approximation numbers for Sobolev spaces with small smoothness. Hence there can be a logarithmic gap also in the case where the singular numbers of the embedding are square-summable. We first prove new lower bounds for the integration problem, again for rather classical Sobolev spaces of periodic univariate functions.

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1 Introduction and Main Results

We always assume that \( H \) is a separable reproducing kernel Hilbert space (RKHS) on a domain \( D \) and that there is a measure \( \mu \) on \( D \) such that \( H \) is compactly embedded into \( L_2 = L_2(D, \mu) \). We study algorithms \( A_n \) for \( L_2 \)-approximation (or optimal recovery) of functions from \( H \) and consider the worst case error

\[
e(A_n) = \sup_{\|f\|_H \leq 1} \|f - A_n(f)\|_2.
\]

We study two kinds of information and algorithms: The algorithm \( A_n(f) = \sum_{i=1}^n L_i(f)g_i \), where \( g_i \in L_2 \), may use arbitrary linear functionals \( L_i \), while \( S_n(f) = \sum_{i=1}^n f(x_i)g_i \) can only use function values. The following question was recently studied in several papers: Are sampling algorithms \( S_n \) always, i.e., for any RKHS \( H \), almost as good as general algorithms \( A_n \)?

It is well known how to characterize the minimal worst case error \( e(A_n) \) for general algorithms \( A_n \). The answer is given by the approximation numbers

\[
a_n(H, L_2) = \inf_{A_n} e(A_n) = \sigma_{n+1},
\]

where \( \sigma_1 \geq \sigma_2 \geq \cdots \geq 0 \) are the singular values of the compact embedding of \( H \) into \( L_2 \). In addition to the approximation numbers (or linear widths) \( a_n(H, L_2) \) we also define the sampling numbers

\[
g_n(H, L_2) = \inf_{S_n} e(S_n),
\]

with algorithms that only use function values.

The lively history of upper bounds for the sampling numbers \( g_n \) for general RKHSs \( H \) was initiated by \[28\] and \[16\], where the authors assumed that the sequence \( \sigma = (\sigma_n)_{n \geq 1} \) of the singular values is in \( \ell_2 \). Under this condition it was proved in \[13\] that the polynomial order of the \( a_n \) and the \( g_n \) coincides (solving Open Problem 126 from \[21\]); but it was not clear whether a logarithmic gap is possible or not. Here we show that such a gap is possible and, in order to do so, prove new lower bounds. We discuss upper bounds on the sampling numbers \( g_n \) in Remark 2, this paper is mainly on lower bounds.

The lower bound

\[
g_n(H, L_2) \geq a_n(H, L_2)
\]
is trivial. Although there are several papers which improve upon this bound in the sense of tractability, see Remark 3, the authors only know one paper, namely [10], that contains results concerning a different asymptotic behavior of the $g_n$ and the $a_n$.

One way of obtaining nontrivial lower bounds for the numbers $g_n$ is to take a detour and prove lower bounds for the problem of numerical integration. This is the approach we will take in the present paper. For the necessary notation, let now $h \in L_2(D, \mu)$ with $\|h\|_2 = 1$ and consider the functional

$$\text{INT}_h(f) = \int_D f(x) h(x) \, d\mu(x)$$

on $H$. Let $Q_n(f) = \sum_{i=1}^n w_i f(x_i)$ with scalar $w_i$ be a quadrature formula to approximate the integral and let $e(Q_n) = \sup_{\|f\|_H \leq 1} \|\text{INT}_h(f) - Q_n(f)\|$ be the worst case error of $Q_n$. The minimal worst case error for $\text{INT}_h$ is then defined by

$$e_n(H, \text{INT}_h) = \inf_{Q_n} e(Q_n)$$

with the infimum taken over all quadrature formulas $Q_n$ using $n$ function values. Associating with a sampling algorithm $S_n(f) = \sum_{i=1}^n f(x_i) g_i$ the quadrature formula with $w_i = \text{INT}_h(g_i)$, we obtain $e(Q_n) \leq e(S_n)$ and conclude that

$$g_n(H, L_2) \geq e_n(H, \text{INT}_h). \quad (1)$$

Furthermore, it was observed in [9, Proposition 1] that lower bounds for the integration problem $\text{INT}_h$ are equivalent to showing that certain matrices involving the values of the reproducing kernel $K$ and the representer $h$ are positive semi-definite. To be more specific,

$$e_n(H, \text{INT}_h) \geq \|h\|^2_H - \alpha^{-1}$$

for a real parameter $\alpha > 0$ if, and only if,

$$\{K(x_j, x_k)\}_{j,k=1}^n \succeq \alpha \{h(x_j) h(x_k)\}_{j,k=1}^n$$

holds for every set of points $\{x_1, \ldots, x_n\} \subset D$. Here, if $A$ and $B$ are symmetric matrices, we mean by $A \succeq B$ that $A - B$ is positive semi-definite. This can be further combined with a recent result of [27], which shows that the entry-wise product (which is sometimes also called Schur product) of a positive semi-definite matrix with itself is not only positive semi-definite but also bounded.
from below by some rank-1 matrix (in the “⪰” ordering). Therefore, if \( K \) is a square of another reproducing kernel, the method of \([27]\) and \([9]\) can be applied to get new lower bounds for the integration problem.

We now turn to our results. It is by now well understood that (upper and lower) bounds on the sampling numbers \( g_n \) very much depend on whether the sequence \( \sigma \) of singular numbers is square summable or not. Equivalent conditions are that the embedding \( \text{id} \) of \( H \) into \( L^2 \) is a Hilbert-Schmidt operator or that the operator \( \text{id}^*\text{id} \) has a finite trace. Lower bounds in \([10]\) are proved only for the case \( \sigma \notin \ell_2 \), here we study both cases.

We begin with the assumption \( \sigma \in \ell_2 \), where we can work with classical Sobolev spaces \( H_\gamma \) of univariate periodic functions on \( D = [0, 1] \). Let \( \gamma = (\gamma_k)_{k \in \mathbb{Z}} \) be a bounded non-negative sequence and put \( e_j(x) = e^{2\pi i j x} \) for \( j \in \mathbb{Z} \). Then \( H_\gamma \) is the set of all 1-periodic functions given by

\[
f(x) = \sum_{j \in \mathbb{Z}} \alpha_j e_j(x), \quad x \in [0, 1],
\]

such that \( \alpha_j = 0 \) if \( \gamma_j = 0 \) and

\[
\| f \|_{H_\gamma}^2 = \sum_{j: \gamma_j \neq 0} \frac{\alpha_j^2}{\gamma_j^2} < \infty.
\]

It follows from the Cauchy-Schwarz inequality that the series in (2) is absolutely and uniformly convergent for \( \gamma \in \ell_2(\mathbb{Z}) \) and that the point evaluation functionals \( x \mapsto f(x) \) are continuous. Hence \( H_\gamma \) is a reproducing kernel Hilbert space of continuous 1-periodic functions with the kernel given by \( K(x, y) = \sum_{j \in \mathbb{Z}} \gamma_j^2 e_j(x - y) \) if \( \gamma \in \ell_2(\mathbb{Z}) \) and \( \alpha_j = \langle f, e_j \rangle_2 \) is the \( j \)-th Fourier coefficient of \( f \). The singular values \( \sigma \in \ell_2(\mathbb{N}) \) of \( H_\gamma \) in \( L^2 \) are given by the non-increasing rearrangement of \( \gamma \).

Surprisingly, we find that already for this classical example \( H_\gamma \), there is a gap of order \( \sqrt{\log n} \) between the sampling and the approximation numbers if we take \( \gamma_k \) of order \( |k|^{-1/2} \log^{-\beta} |k| \) for some \( \beta > 1/2 \). These spaces \( H_\gamma \) fall into the scale of function spaces of generalized (or logarithmic) smoothness, which can be traced back (at least) to the work of Lévy \([17]\). These spaces were since then studied intensively \([1, 5, 6, 18]\). Although different approximation quantities (like entropy numbers or approximation numbers) were studied in the frame of this scale of function spaces earlier, the possible gap
between approximation and sampling numbers in the Hilbert space setting went unnoticed so far.

We prove the logarithmic gap using (1) and lower bounds for the problem of approximating the integral $\int_0^1 f(x) \, dx$ for $f \in H_\gamma$. Note that we omit the $h$ in $\text{INT}_h$ in this special case $h = 1$. Of course these bounds are interesting by itself. Our first main result is the following, see Theorem 1.

Let $\mu \in \ell_1(\mathbb{Z})$ be a non-negative and non-zero sequence and let

$$
\gamma_\ell^2 = \sum_{j \in \mathbb{Z}} \mu_j \mu_{j+\ell}, \quad \ell \in \mathbb{Z}.
$$

Then we have for all $n \in \mathbb{N}_0$ that

$$
\epsilon_n(H_\gamma, \text{INT})^2 \geq \gamma_0^2 \left( 1 - \frac{n \gamma_0^2}{\|\gamma\|^2_2} \right).
$$

Using the spaces $H_\gamma$ and the above result one can finally answer the following question: Is there a Hilbert space $H$ such that the singular values of its embedding into $L_2$ are square summable and

$$
\lim_{n \to \infty} a_n(H, L_2)/g_n(H, L_2) = 0?
$$

Indeed, the result below shows that there is a gap of order $\sqrt{\log n}$ between sampling and approximation numbers for the spaces $H_\gamma$ with $\gamma$ just in $\ell_2$. In this case also the minimal worst case errors of uniform integration, which are a lower bound for the sampling numbers, actually have the same asymptotic behavior as the sampling numbers. The following is from Theorem 4.

Let $\beta > 1/2$. Then there exists $\gamma \in \ell_2(\mathbb{Z})$ such that

$$
a_n(H_\gamma, L_2) \asymp n^{-1/2} \log^{-\beta} n
$$

and

$$
e_n(H_\gamma, \text{INT}) \asymp g_n(H_\gamma, L_2) \asymp n^{-1/2} \log^{-\beta+1/2} n.
$$
The symbol \(\appropto\) means that the left hand side is bounded from above by a constant multiple of the right hand side for almost all (i.e., all except finitely many) \(n\) and vice versa; we use \(\lessapprox\) and \(\gtrapprox\) for the one-sided relations.

We also obtain a lower bound for arbitrary sequences of approximation numbers \((a_n) \in \ell_2(\mathbb{N}_0)\). This lower bound shows that, in general, one cannot expect the sampling numbers \(g_n\) to behave better than

\[
\max \left\{ a_n, \left( \frac{1}{n} \sum_{k \geq n} a_k^2 \right)^{1/2} \right\}.
\]

Note that this is conjectured to be also the worst possible behavior of the sampling numbers, see [13] and Remark 2. The following is from Theorem 2.

Let \(a \in \ell_2(\mathbb{N}_0)\) be non-increasing. Then there is some \(\gamma \in \ell_2(\mathbb{Z})\) such that \(a_n(H, L_2) = a_n\) for all \(n \in \mathbb{N}_0\) and

\[
g_n(H, L_2)^2 \geq \frac{1}{8n} \sum_{k \geq n} a_k(H, L_2)^2
\]

for infinitely many values of \(n\).

Now we present a result for the case \(\sigma \not\in \ell_2(\mathbb{N})\). Here, it is already known from [10] that the convergence of the sampling numbers can be extremely slow. We improve upon [10] by providing a lower bound that holds for all \(n \in \mathbb{N}\) instead of only a thin sub-sequence. The following is from Theorem 6.

For illustration, one might imagine that \(a_n \approx n^{-1/2}\) and \(\tau_n \approx \log^{-1/2} n\).

Let \(a, \tau \in c_0(\mathbb{N}_0)\) be non-increasing with \(a \not\in \ell_2(\mathbb{N}_0)\). Then there is an example \((H, L_2)\) such that \(a_n(H, L_2) = a_n\) for all \(n \in \mathbb{N}_0\) and

\[
g_n(H, L_2) \geq \tau_n
\]

for almost all values of \(n \in \mathbb{N}_0\).

We note that also this lower bound already holds for some kind of integration problem, which is described in Section 4. Here is an open problem that also describes some of the progress of this paper.
Open Problem. Assume that, for some RKHS $H$,

$$a_n(H, L_2) \asymp n^{-r} \log^{-\beta} n,$$

where $(a_n)_n \in \ell_2(\mathbb{N}_0)$ (hence $r > 1/2$ or $r = 1/2$ and $\beta > 1/2$). Does it follow that

$$a_n(H, L_2) \asymp g_n(H, L_2)?$$

This was posed in [21] as a part of the Open Problem 140. We now know the answer “no” if $r = 1/2$ but do not know the answer if $r > 1/2$.

Remark 1 (Multivariate analogues). Theorem 4 answers the above open problem in the case $r = 1/2$. Nonetheless, it would be interesting to know whether a result like Theorem 4 is also true for Sobolev spaces with small smoothness on the $d$-dimensional sphere or torus for $d > 1$; see also [7, 25, 26]. We hope that our technique can be applied also for this purpose.

Remark 2 (Upper bounds). Upper bounds for particular spaces have a long history but it seems that [28] and [16] are the first papers that study upper bounds for general RKHSs. The history till 2012 can be found in [21].

This line of study was further developed in [13], where the authors showed that there are two absolute constants $c, C > 0$, such that for every RKHS $H$ and every $n \in \mathbb{N}$

$$g_n(H, L_2)^2 \leq \frac{C}{k_n} \sum_{j \geq k_n} a_j(H, L_2)^2$$

holds with $k_n \geq cn / \log(n + 1)$. It follows as a simple corollary, that in the case $\sigma \in \ell_2$ there cannot be a polynomial gap between the $a_n$ and the $g_n$. This solved an open problem from [10] that was also posed as Open Problem 126 in [21]. For the solution it was important to understand the geometry (and stochastics) of random sections of ellipsoids in high dimensional euclidean spaces, see [8]. We refer to [11, 14, 19, 24, 25] for improvements of (3) and further results. Let us also remark that one of the aims of our work is to study the optimality of (3) by providing appropriate lower bounds, cf. Theorem 2.

Remark 3 (Tractability and curse of dimensionality). Assume now that a whole sequence of Hilbert spaces $H_d$ is given; the functions $f$ from $H_d$ could be defined on $[0, 1]^d$. For some spaces we know that the curse of dimensionality is present, if only function values are allowed, while the problem is tractable for general linear information. This happens for certain (periodic and nonperiodic) Sobolev spaces where the singular values are in $\ell_2$, see
For the proof one again uses (1) together with lower bounds for the integration problem that follow from the technique of decomposable kernels, see [20]. A new technique to prove intractibility for integration problems is based on the method developed in [27], see [9]. This method is also used in the present paper, see the proof of Theorem 1.

Remark 4 (Randomized algorithms). In this paper we use the worst case setting for deterministic algorithms. We want to stress that results in the randomized setting are quite different. In particular, the results do not depend strongly on the assumption whether the singular values are in $\ell_2$ or not; see [11, 21, 29]. Together with the upper bound from [1], Theorem 4 gives an example where randomized algorithms achieve a better rate of convergence for $L_2$-approximation than deterministic algorithms.

## 2 Finite Trace - Lower Bounds

In this section we investigate the sampling numbers of the Sobolev spaces $H_\gamma$ of 1-periodic functions defined in the introduction. We start with a lower bound based on the results from [9].

**Theorem 1.** Let $\mu \in \ell_1(\mathbb{Z})$ be a non-negative sequence, $\mu \neq 0$, and let

$$\gamma_\ell^2 = \sum_{j \in \mathbb{Z}} \mu_j \mu_{j+\ell}$$

for $\ell \in \mathbb{Z}$. Then we have for all $n \in \mathbb{N}_0$ that

$$e_n(H_\gamma, \text{INT})^2 \geq \gamma_0^2 \left(1 - \frac{n\gamma_0^2}{\|\gamma\|_2^2}\right).$$

**Proof.** Recall that $e_k(x) = e^{2\pi ikx}$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. We define the kernels $M : [0, 1)^2 \to \mathbb{C}$ and $K : [0, 1)^2 \to \mathbb{R}$ by

$$M(x, y) = \sum_{k \in \mathbb{Z}} \mu_k e_k(x - y)$$

and

$$K(x, y) = |M(x, y)|^2 = \sum_{j, k \in \mathbb{Z}} \mu_j \mu_k e_{k-j}(x - y) = \sum_{\ell \in \mathbb{Z}} \gamma_\ell^2 e_{\ell}(x - y).$$
We observe that
\[
M(x, x)^2 = \left( \sum_{k \in \mathbb{Z}} \mu_k \right)^2 = \sum_{\ell \in \mathbb{Z}} \gamma_\ell^2 = \| \gamma \|_2^2
\]
for every \( x \in [0, 1) \). Furthermore, the representer of the integration functional \( f \mapsto \int_0^1 f(x) \, dx \) for \( f \in H_\gamma \) is given by \( h = \gamma_0^2 e_0 \), since for \( f \) as in (2) we have
\[
\int_0^1 f(x) \, dx = \alpha_0 = \langle \gamma_0 f, \gamma_0 e_0 \rangle_{H_\gamma} = \langle f, h \rangle_{H_\gamma}.
\]
In particular, the initial error satisfies
\[
e_0(H_\gamma, \text{INT})^2 = \| h \|_{H_\gamma}^2 = \gamma_0^2.
\]

Let now \( n \in \mathbb{N} \) and fix \( x_1, \ldots, x_n \in [0, 1) \). Then the matrix \( (M(x_j, x_k))_{j,k=1}^n \) is positive semi-definite and, by [27] Theorem 1, also the matrix with entries
\[
K(x_j, x_k) - \frac{M(x_j, x_j)M(x_k, x_k)}{n} = K(x_j, x_k) - \frac{\| \gamma \|_2^2}{n \gamma_0^4} h(x_j)h(x_k)
\]
is positive semi-definite. By [9] Proposition 1, it follows that
\[
e_n(H_\gamma, \text{INT})^2 \geq \| h \|_{H_\gamma}^2 - \frac{n \gamma_0^4}{\| \gamma \|_2^2} = \gamma_0^2 - \frac{n \gamma_0^4}{\| \gamma \|_2^2}
\]
as claimed.

Theorem 1 shows that, if the sequence \( \gamma \) is given by (4), we need at least a constant multiple of \( \| \gamma \|_2^2 / \gamma_0^2 \) function values in order to reduce the initial error \( \gamma_0 \) by a constant multiple. It would be interesting to know whether this is true for all symmetric and non-increasing sequences \( \gamma \in \ell_2(\mathbb{Z}) \). This would simplify the remainder of this section. Indeed, assume for a moment that for each \( n \in \mathbb{N}_0 \) we could apply Theorem 1 to the sequence \( \tilde{\gamma} = \gamma_n 1_{\{-n, \ldots, n\}} + \gamma_1 1_{\mathbb{N}\setminus\{-n, \ldots, n\}} \). Then we obtain for every \( m \in \mathbb{N}_0 \) that
\[
e_m(H_\gamma, \text{INT})^2 \geq e_m(H_{\tilde{\gamma}}, \text{INT})^2 \geq \gamma_n^2 \left( 1 - \frac{m \gamma_n^2}{(2n + 1) \gamma_n^2 + 2 \sum_{k>n} \gamma_k^2} \right).
\]
This would imply that we need at least
\[
m(n) := \left\lceil n + \sum_{k>n} \gamma_k^2 / \gamma_n^2 \right\rceil
\]
function values to achieve a squared error smaller than $\gamma_n^2/2$. Thus we would get for all $m = m(n)$, $n \in \mathbb{N}_0$, that
\[
e_m(H_\gamma, \text{INT})^2 \geq \frac{\gamma_n^2}{2} \geq \frac{1}{2m} \sum_{k>n} \gamma_k^2 \geq \frac{1}{2m} \sum_{k>m} \gamma_k^2.
\]
In particular, this would imply Theorem 2. However, since Theorem 1 is only for sequences of the form (4), we need to do some additional work to get there. We approximate general sequences $\gamma$ by sequences of the form (4).

**Lemma 1.** Let $\gamma \in \ell_2(\mathbb{Z})$ be non-negative and non-increasing on $\mathbb{N}_0$ and let $\gamma_{-k} \geq \gamma_k$ for all $k \in \mathbb{N}$. For $r \in \mathbb{N}_0$, we put
\[
n(r) := \left\lceil \frac{\left(\sum_{j \geq r} \gamma_j^2\right)^2}{2 \sum_{j \geq r} \gamma_j^4} \right\rceil.
\]
Then we have
\[
e_n(r)(H_\gamma, \text{INT})^2 \geq \frac{1}{2(n(r) + 1)} \sum_{j \geq r} \gamma_j^2.
\]

**Proof.** For $k \in \mathbb{Z}$ we define
\[
\mu_k = \begin{cases} 
0 & \text{if } k < r, \\
\gamma_k^2 \left(\sum_{\ell=r}^{\infty} \gamma_{\ell}^2\right)^{-1/2} & \text{if } k \geq r.
\end{cases}
\]
We associate to $\mu = (\mu_k)_{k \in \mathbb{Z}}$ the sequence $\tilde{\gamma} = (\tilde{\gamma}_\ell)_{\ell \in \mathbb{Z}}$ by (4) and observe that
\[
\tilde{\gamma}_\ell^2 = \sum_{j=r}^{\infty} \gamma_j^2 \gamma_{\ell+j} \left(\sum_{u=r}^{\infty} \gamma_u^2\right)^{-1} \leq \gamma_\ell^2 \sum_{j=r}^{\infty} \gamma_j^2 \cdot \left(\sum_{u=r}^{\infty} \gamma_u^2\right)^{-1} = \gamma_\ell^2
\]
for $\ell \geq 0$ and
\[
\tilde{\gamma}_\ell^2 = \sum_{j \in \mathbb{Z}} \mu_{j} \mu_{j+\ell} = \sum_{k \in \mathbb{Z}} \mu_{k-\ell} \mu_k = \gamma_{-\ell}^2 \leq \gamma_\ell^2 \leq \gamma_{\ell}^2
\]
for $\ell < 0$. Therefore, we have $\tilde{\gamma}_\ell^2 \leq \gamma_\ell^2$ for all $\ell \in \mathbb{Z}$. By monotonicity and Theorem 1 we obtain for all $n \in \mathbb{N}_0$ that
\[
e_n(H_\gamma, \text{INT})^2 \geq e_n(H_{\tilde{\gamma}}, \text{INT})^2 \geq \tilde{\gamma}_0^2 \left(1 - \frac{n \tilde{\gamma}_0^2}{\|\tilde{\gamma}\|^2_2}\right).
\]
If we put
\[ n(r) := \left\lfloor \frac{\|\tilde{\gamma}\|_2^2}{2 \tilde{\gamma}_0^2} \right\rfloor, \]
we obtain
\[ e_{n(r)}(H_\gamma, \text{INT})^2 \geq \frac{\tilde{\gamma}_0^2}{2} \geq \frac{\|\tilde{\gamma}\|_2^2}{2(n(r) + 1)}. \]
It now only remains to observe that
\[ \tilde{\gamma}_0^2 = \sum_{j \in \mathbb{Z}} \mu_j^2 = \sum_{k \geq r} \gamma_k^2 \]
and
\[ \|\tilde{\gamma}\|_2^2 = \left( \sum_{j \in \mathbb{Z}} \mu_j \right)^2 = \sum_{k \geq r} \gamma_k^2. \]

From this we get some nice consequences.

**Theorem 2.** Let \( a \in \ell_2(\mathbb{N}_0) \) be non-negative and non-increasing. If we put \( \gamma = (..., a_3, a_1, a_0, a_2, a_4, ...) \), we have \( a_n(H_\gamma, L_2) = a_n \) for all \( n \in \mathbb{N}_0 \) and
\[ g_n(H_\gamma, L_2)^2 \geq \frac{1}{8n} \sum_{k \geq n} a_k(H_\gamma, L_2)^2 \]
for infinitely many values of \( n \).

**Proof.** For \( r \in \mathbb{N}_0 \), let again \( n(r) \) be defined by (5). We distinguish two cases. In the first case, we assume \( n(r) \geq 2r \) for infinitely many \( r \in \mathbb{N} \). For these values of \( r \), we get from Lemma 1 that
\[ g_{n(r)}(H_\gamma, L_2)^2 \geq e_{n(r)}(H_\gamma, \text{INT})^2 \geq \frac{1}{2(n(r) + 1)} \sum_{j \geq r} \gamma_j^2 \geq \frac{1}{4n(r)} \sum_{j \geq r} a_{2j}^2 \]
\[ \geq \frac{1}{8n(r)} \sum_{j \geq r} (a_{2j}^2 + a_{2j+1}^2) \geq \frac{1}{8n(r)} \sum_{k \geq n(r)} a_k^2 \]
and we are done because in this case, the sequence \((n(r))_{r \in \mathbb{N}}\) is unbounded.
In the second case, we assume \( n(r) \leq 2r \) for infinitely many \( r \in \mathbb{N} \). This means that
\[
2r \geq \left( \frac{\sum_{j \geq r} \gamma_j^2}{2 \sum_{j \geq r} \gamma_j^4} \right) \geq \left( \frac{\sum_{j \geq r} \gamma_j^2}{2 \gamma_r^2} \right)
\]
and thus
\[
2r \geq \frac{\sum_{j \geq r} \gamma_j^2}{4 \gamma_r^2}.
\]
Here we estimate
\[
g_{2r}(H_\gamma, L_2)^2 \geq a_{2r}(H_\gamma, L_2)^2 = \gamma_r^2 \geq \frac{1}{8r} \sum_{j \geq r} \gamma_j^2 \geq \frac{1}{16r} \sum_{k \geq 2r} a_k^2
\]
to obtain the desired statement. \( \square \)

A lower bound of this type can be obtained for all \( n \in \mathbb{N} \) (instead of just infinitely many) if the sequence of singular values has some additional regularity. For example, we get the following.

**Theorem 3.** Let \( a \in \ell_2(\mathbb{N}_0) \) be non-negative, non-increasing and assume that there is a constant \( b > 0 \) such that \( a_{2n} \geq ba_n \) for all \( n \in \mathbb{N}_0 \). If we put \( \gamma = (..., a_3, a_1, a_0, a_2, a_4, ...) \), we have \( a_n(H_\gamma, L_2) = a_n \) for all \( n \in \mathbb{N}_0 \) and
\[
g_n(H_\gamma, L_2)^2 \geq e_n(H_\gamma, \text{INT})^2 \geq \frac{1}{n} \sum_{k \geq n} a_k^2(H_\gamma, L_2)^2.
\]

This theorem is an immediate consequence of the following lemma.

**Lemma 2.** Let \( \gamma \in \ell_2(\mathbb{Z}) \) be non-negative, non-increasing on \( \mathbb{N}_0 \) and let \( \gamma_{-k} \geq \gamma_k \) for all \( k \in \mathbb{N} \). If there is a constant \( b > 0 \) such that \( \gamma_{2k} \geq b \gamma_k \) for all \( k \in \mathbb{N} \), then we have
\[
e_n(H_\gamma, \text{INT})^2 \geq \frac{1}{n} \sum_{j \geq n} \gamma_j^2.
\]

**Proof.** We use Lemma \( \Box \) and observe that
\[
n(r) \geq \frac{\left( \sum_{j \geq r} \gamma_j^2 \right)^2}{2 \sum_{j \geq r} \gamma_j^4} - 1 \geq \frac{\sum_{j \geq r} \gamma_j^2}{2 \gamma_r^2} - 1 \geq \frac{r \gamma_{2r}^2}{2 \gamma_r^2} - 1 \geq \frac{b^2 r}{2} - 1.
\]
In particular, \( n(r) \to \infty \) and \( n(r) \geq 1 \) for \( r \geq r_0 \). On the other hand, we have

\[
\sum_{j \geq 2r} \gamma_j^4 \geq \sum_{j \geq r} \gamma_{2j}^4 \geq b^4 \sum_{j \geq r} \gamma_j^4
\]

and thus for all \( r \geq r_0 \) that

\[
n(2r) \leq \left( \frac{\sum_{j \geq 2r} \gamma_j^2}{2 \sum_{j \geq 2r} \gamma_j^4} \right)^2 \leq \left( \frac{\sum_{j \geq r} \gamma_j^2}{2b^4 \sum_{j \geq r} \gamma_j^4} \right)^2 \leq \frac{2}{b^4} n(r).
\]

Hence, there is a constant \( C \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) we find some \( r \in \mathbb{N} \) with \( n \leq n(r) \leq Cn \). We get

\[
e_n(H_\gamma, \text{INT})^2 \geq e_{n(r)}(H_\gamma, \text{INT})^2 \geq \frac{1}{2(n(r) + 1)} \sum_{j \geq r} \gamma_j^2 \geq \frac{1}{4Cn} \sum_{j \geq 4Cn/b^2} \gamma_j^2.
\]

Now, choosing \( t \in \mathbb{N} \) with \( 2^t \geq 4C/b^2 \), we continue

\[
\sum_{j \geq 4Cn/b^2} \gamma_j^2 \geq \sum_{j \geq n} \gamma_{2^t j}^2 \geq b^{2t} \sum_{j \geq n} \gamma_j^2
\]

and obtain the statement.

\[\square\]

### 3 Finite Trace - Upper Bounds

We now complement the lower bound of Lemma \[4\] with an appropriate upper bound. Let us recall that the lower bounds of Lemma \[2\] were based on a new technique from \[9\] and \[27\]. Compared to that, the upper bounds of Proposition \[1\] and Theorem \[4\] are based on a classical approximation scheme using the Dirichlet kernel, see, e.g., \[23\] Theorem 3.3).

Here, we approximate \( f \) by

\[
S_n(f) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(x_j^n) D_n(\cdot - x_j^n),
\]

where \( x_j^n = \frac{j}{2n+1} \) and \( D_n(x) = \sum_{|l| \leq n} e_l(x) \) is the Dirichlet kernel of degree \( n \). The integral of \( f \) is approximated by the midpoint rule

\[
Q_n(f) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(x_j^n).
\]

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Note that $Q_n(f)$ is the integral of $S_n(f)$.

**Lemma 3.** Let $f = \sum_{m \in \mathbb{Z}} \alpha_m e_m$ be a 1-periodic function pointwise represented by its Fourier series. Then, for every $n \in \mathbb{N}$,

$$
\|f - S_n(f)\|_2^2 = \sum_{|j| > n} \alpha_j^2 + \sum_{|k| \leq n} \left| \sum_{\theta \in \mathbb{Z} \setminus \{0\}} \alpha_{k+\theta(2n+1)} \right|^2.
$$

**Proof.** We calculate the $k$-th Fourier coefficient of $S_n(f)$ by

$$
\int_0^1 S_n(f)(x) e_k(x) \, dx = \frac{1}{2n+1} \sum_{j=0}^{2n} f(x_j^n) \sum_{l=-n}^{n} e_{-l}(x_j^n) \int_0^1 e_{l-k}(x) \, dx.
$$

The last expression vanishes if $|k| > n$ and for $|k| \leq n$, using the Fourier expansion of $f$ at $x_j^n$, it is equal to

$$
\sum_{m \in \mathbb{Z}} \alpha_m \cdot \frac{1}{2n+1} \sum_{j=0}^{2n} e_m(x_j^n) e_{-k}(x_j^n) = \sum_{\theta \in \mathbb{Z}} \alpha_{k+\theta(2n+1)}.
$$

If we make use of the partial sum operator $T_n(f) = \sum_{|k| \leq n} \alpha_k e_k$, the calculation above shows that $S_n(T_n(f)) = T_n(f)$ and we can compute

$$
\|f - S_n(f)\|_2^2 = \|f - T_n(f) + S_n(T_n(f) - f)\|_2^2
= \|f - T_n(f)\|_2^2 + \|S_n(T_n(f) - f)\|_2^2
= \sum_{|j| > n} \alpha_j^2 + \sum_{|k| \leq n} \left| \sum_{\theta \in \mathbb{Z} \setminus \{0\}} \alpha_{k+\theta(2n+1)} \right|^2
$$

to obtain the desired identity. \qed

From this, one obtains the following general upper bound.

**Proposition 1.** Let $\gamma \in \ell_2(\mathbb{Z})$ be symmetric and non-increasing on $\mathbb{N}_0$. Then, for all $n \in \mathbb{N}$, we have

$$
e(Q_n, H_\gamma, \text{INT}) \leq e(S_n, H_\gamma, L_2) \leq 2 \max \left\{ \gamma_{n+1}, \left( \frac{1}{n} \sum_{k > n} \gamma_k^2 \right)^{1/2} \right\}.
$$
Proof. The first inequality is clear from $Q_n(f) = \text{INT}(S_n(f))$ since

$$|\text{INT}(f) - Q_n(f)| = |\text{INT}(f - S_n(f))| \leq \|f - S_n(f)\|_2.$$ 

Regarding the second inequality, Lemma 3 yields for $f = \sum_{m \in \mathbb{Z}} \alpha_m e_m$ with $\|f\|_{H_{\gamma}} \leq 1$ that

$$\|f - S_n(f)\|_2^2 = \sum_{|j| > n} |\alpha_j|^2 + \sum_{|k| \leq n} \left| \sum_{\theta \in \mathbb{Z} \setminus \{0\}} \alpha_{k+\theta(2n+1)} \right|^2 \leq \sum_{|j| > n} \frac{|\gamma_j|^2}{\gamma_j^2} \cdot \sum_{|k| \leq n} \left( \sum_{\theta \in \mathbb{Z} \setminus \{0\}} \frac{\alpha_{k+\theta(2n+1)}}{\gamma_{k+\theta(2n+1)}^2} \cdot \sum_{\varphi \in \mathbb{Z} \setminus \{0\}} \gamma_{k+\varphi(2n+1)}^2 \right) \leq \gamma_{n+1}^2 + \max_{|k| \leq n} \frac{\gamma_{k+\varphi(2n+1)}^2}{\gamma_{k+\varphi(2n+1)}^2} \cdot \sum_{|k| \leq n} \frac{\alpha_{k+\theta(2n+1)}}{\gamma_{k+\theta(2n+1)}^2} \leq \gamma_{n+1}^2 + \frac{1}{n} \sum_{k > n} \gamma_k^2,$$

and thus the stated inequality follows. □

Recall that Theorem 2 gives a lower bound for the recovery problem that matches the upper bound of Proposition 1 up to a constant factor for infinitely many values of $n \in \mathbb{N}$. In this sense, the results show that there is no significant improvement over the recovery method $S_n$ for any of the Sobolev spaces $H_{\gamma}$. If $\gamma$ has some additional regularity, we can even say a little more. Then both $S_n$ and $Q_n$ are optimal for all $n \in \mathbb{N}$ up to constants.

**Theorem 4.** Let $\gamma \in \ell_2(\mathbb{Z})$ be symmetric and non-increasing on $\mathbb{N}_0$ and assume that there is a constant $b > 0$ such that $\gamma_{2n} \geq b \gamma_n$ for all $n \in \mathbb{N}_0$. Then

$$e_n(H_{\gamma}, \text{INT})^2 \preceq g_n(H_{\gamma}, L_2)^2 \asymp \frac{1}{n} \sum_{j > n} \gamma_j^2.$$

In particular, if we have $\gamma_k = k^{-1/2} \log^{-\beta} k$ for some $\beta > 1/2$ and all $k \geq k_0$, then

$$a_n(H_{\gamma}, L_2) \asymp n^{-1/2} \log^{-\beta} n$$
and
\[ e_n(H_\gamma, \text{INT}) \asymp g_n(H_\gamma, L_2) \asymp n^{-1/2} \log^{-\beta+1/2} n. \]

**Proof.** The lower bound is already stated in Lemma 2. On the other hand, Proposition 1 yields the upper bound
\[ e_n(H_\gamma, \text{INT})^2 \leq g_n(H_\gamma, L_2)^2 \leq 4 \max \left\{ \gamma_{m+1}^2, \frac{1}{m} \sum_{k>m} \gamma_k^2 \right\} \]
with \( m = \lfloor \frac{n-1}{2} \rfloor \). Because of the regularity assumption for \( \gamma \), both \( \gamma_{m+1}^2 \) and \( \frac{1}{m} \sum_{k>m} \gamma_k^2 \) are dominated by \( \frac{1}{n} \sum_{j>n} \gamma_j^2 \). \( \square \)

**Remark 5.** By Theorem 4 we have a gap of order \( \sqrt{\log(n)} \) between sampling and approximation numbers if \( \gamma_n \asymp n^{-1/2} \log^{-\beta} n \) with \( \beta > 1/2 \). If we consider \( \beta = 1/2 \) and add a double logarithm of order bigger than \( 1/2 \), we obtain a gap of order \( \sqrt{\log(n) \cdot \log(\log(n))} \), and so forth. This is in a sharp contrast to the situation where \( \gamma_n \asymp n^{-s} \) with \( s > 1/2 \). In that case, it is known (cf. [15, Theorems 2.1.2 and 2.2.1]) that
\[ e_n(H_\gamma, \text{INT}) \asymp g_n(H_\gamma, L_2) \asymp a_n(H_\gamma, L_2) \asymp n^{-s}. \]

**Remark 6.** In the case of small smoothness, where we observe the logarithmic gap between the sampling and approximation numbers, an upper bound like in Proposition 1 can also be proven for a piecewise constant approximation. This was done in an earlier version of this manuscript; see also [3, Section 2 of Chapter 12]. Following the advice of a referee, we replaced this approach by the more general approach of Dirichlet approximation, which works for any sequence \( \gamma \).

## 4 Infinite Trace

In this section, we consider \( \sigma \notin \ell_2(\mathbb{N}) \) and want to show that there exists a reproducing kernel Hilbert space \( H \) whose singular values in \( L_2 \) are given by \( \sigma \) and whose sampling numbers \( g_n(H, L_2) \) show an arbitrarily bad behavior. We cannot use the spaces \( H_\gamma \) from the previous sections in this case since those are not reproducing kernel Hilbert spaces any more. We need different examples. Here, we consider (real) sequence spaces \( H \subset \ell_2(\mathbb{N}) \), which are
reproducing kernel Hilbert spaces on the domain $D = \mathbb{N}$. An integration problem $\text{INT}_h$ for $h \in \ell^2$ therefore takes the form

$$\text{INT}_h(f) = \sum_{j=1}^{\infty} h_j f_j.$$ 

In [10], Hinrichs, Novak and Vybíral proved the following.

Lemma 4 ([10]). Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ such that $\sum_{j=1}^{\infty} \sigma_j^2 = \infty$ and let $n_0 \in \mathbb{N}$ and $\varepsilon > 0$. Then there is some $m \in \mathbb{N}$ and a Hilbert space $H \subset \mathbb{R}^m$ as well as some $h \in \mathbb{R}^m$ with $\|h\|_2 = 1$ such that $a_n(H, \ell^m_2) = \sigma_{n+1}$ for all $n < m$ and

$$e_{n_0}(H, \text{INT}_h) \geq (1 - \varepsilon)\sigma_1.$$ 

Proof. We note that in [10, Theorem 1] only $\sigma_1 = 1$ is considered but this is just a matter of scaling. Moreover, it is only written that $a_n(H, \ell^m_2) = \sigma_{n+1}$ for $n = n_0$ and that

$$g_{n_0}(H, \ell^m_2) \geq (1 - \varepsilon)\sigma_1.$$ 

But a second look quickly shows that the authors prove precisely the stated lemma for $h = e_1$ and $e_1$ as in [10]. \qed

From this, they concluded the following.

Theorem 5 ([10]). Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ such that $\sum_{j=1}^{\infty} \sigma_j^2 = \infty$ and $\tau_0 \geq \tau_1 \geq \ldots \geq 0$ such that $\lim_{n \to \infty} \tau_n = 0$. Then there is a Hilbert space $H \subset \ell_2(\mathbb{N})$ such that $a_n(H, \ell_2) = \sigma_{n+1}$ for all $n \in \mathbb{N}_0$ and

$$g_n(H, L_2) \geq \tau_n$$

for infinitely many values of $n \in \mathbb{N}$.

Thus, in the case of an infinite trace, there are no reasonable upper bounds for the sampling numbers in terms of the approximation numbers. At least there are none that hold for (almost) all values of $n$. For example, it may happen that $a_n(H, L_2) = n^{-1/2}$ for all $n \in \mathbb{N}$, but $g_n(H, L_2) \geq \log^{-1/2} n$ for infinitely many values of $n \in \mathbb{N}$.

But the theorem still leaves us with some hope. On the one hand, it leaves room for upper bounds on the sampling numbers that hold for infinitely many values of $n \in \mathbb{N}$. On the other hand, there might still be upper bounds for the simpler problem of computing an integral. In both cases, the hope is not justified. We prove the following.
Theorem 6. Let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq 0 \) such that \( \sum_{j=1}^{\infty} \sigma_j^2 = \infty \) and \( \tau_0 \geq \tau_1 \geq \ldots \geq 0 \) such that \( \lim_{n \to \infty} \tau_n = 0 \). Then there is a Hilbert space \( H \subset \ell_2(\mathbb{N}) \) such that \( a_n(H,\ell_2) = \sigma_{n+1} \) for all \( n \in \mathbb{N}_0 \) and some \( h \in \ell_2 \) with \( \|h\|_2 = 1 \) such that
\[
g_n(H,\ell_2) \geq \epsilon_n(H,\text{INT}_h) \geq \tau_n
\]
for almost all values of \( n \in \mathbb{N} \).

For the proof, we first add a slight modification of Lemma 4.

Lemma 5. Let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq 0 \) such that \( \sum_{j=1}^{\infty} \sigma_j^2 = \infty \) and let \( n_0 \in \mathbb{N} \) and \( \varepsilon > 0 \). Then there is a Hilbert space \( H \subset \ell_2(\mathbb{N}) \) as well as some \( h \in \ell_2 \) with \( \|h\|_2 = 1 \) such that \( a_n(H,\ell_2) = \sigma_{n+1} \) for all \( n \in \mathbb{N}_0 \) and
\[
e_n(H,\text{INT}_h) \geq (1 - \varepsilon)\sigma_1.
\]

Proof. We take the Hilbert space \( H_1 \subset \mathbb{R}^m \) and \( h \in \mathbb{R}^m \subset \ell_2 \) from Lemma 4 and a Hilbert space \( H_2 \) that is contained in the orthogonal complement of \( \mathbb{R}^m \) in \( \ell_2 \) and has singular values \( (\sigma_k)_{k>m} \). We choose \( H \) as the direct sum of \( H_1 \) and \( H_2 \) in \( \ell_2 \). Then the approximation numbers of \( H \) in \( \ell_2 \) are given by \( \sigma \). Moreover, the lower bound of Lemma 4 extends to \( e_n(H,\text{INT}_h) \), since the unit ball of \( H \) is larger than the unit ball of \( H_1 \) and since the additional point evaluations \( f \mapsto f_k \) for \( k > m \) that are gained from replacing \( \ell_2^m \) by \( \ell_2 \) are equal to the zero functional on \( H_1 \).

Proof of Theorem 6. First, we partition \( \mathbb{N} \) into index sets \( I_j, j \in \mathbb{N}, \) such that \( I_j \) starts with \( 2j - 1 \) and such that the square sum of \( \sigma \) over each index set \( I_j \) is still infinite. For \( I_j \), in addition to the odd index \( 2j - 1 \), we take every other of the even indices which have not been used yet for \( I_i \) with \( i < j \). Namely,
\[
I_j = \{2j - 1\} \cup \{k \in \mathbb{N} : k \equiv 2^j \mod 2^{j+1}\}.
\]

Then, because of monotonicity,
\[
\sum_{k \in I_j} \sigma_k^2 \geq \sum_{l=1}^{\infty} \sigma_{2j+1}^2 \geq \frac{1}{2j+1} \sum_{k=2j+1}^{\infty} \sigma_k^2 = \infty.
\]

Secondly, we choose natural numbers \( n_0 < n_1 < n_2 < \ldots \) such that we have for all \( j \in \mathbb{N} \) that
\[
\tau(n_{j-1}) \leq 2^{-j/2} \frac{\sigma_{2j-1}}{2}.
\]
Then, by Lemma 5, there is an example \((H_j, \ell_2(I_j))\) and some \(h_j \in \ell_2(I_j)\) with \(\|h_j\|_2 = 1\) such that the sequence of singular numbers is given by \((\sigma_k)_{k \in I_j}\) and

\[
e_n(H_j, \text{INT}_{h_j}) \geq \frac{\sigma_{2j-1}}{2}.
\]

We define \(H \subset \ell_2(\mathbb{N})\) as the direct sum of the spaces \(H_j\). Namely, \(H\) contains all \(f \in \ell_2(\mathbb{N})\) for which \((f_k)_{k \in I_j} \in H_j\) for all \(j \in \mathbb{N}\) and for which

\[
\|f\|_H := \left(\sum_{j \in \mathbb{N}} \left\| (f_k)_{k \in I_j} \|_2^2 H_j \right\|^2 \right)^{1/2}
\]

is finite. Then the sequence of singular numbers of \(H\) in \(\ell_2(\mathbb{N})\) is the sequence \(\sigma\). We put

\[
h := \sum_{j=1}^{\infty} 2^{-j/2} h_j,
\]

which satisfies \(\|h\|_2 = 1\). Let \(n \geq n_0\) and choose \(j \in \mathbb{N}\) such that \(n_{j-1} \leq n < n_j\). Then

\[
e_n(H, \text{INT}_h) \geq e_n(H_j, \text{INT}_{h_j}) = e_n(H_j, 2^{-j/2} \text{INT}_{h_j}) = 2^{-j/2} e_n(H_j, \text{INT}_{h_j}) \geq 2^{-j/2} \frac{\sigma_{2j-1}}{2} \geq \tau(n_{j-1}) \geq \tau_n,
\]

as claimed. \(\Box\)

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