Graph Fourier transforms on directed product graphs

Cheng Cheng, Yang Chen, Yeon Ju Lee and Qiyu Sun *

Abstract

Graph Fourier transform (GFT) is one of the fundamental tools in graph signal processing to decompose graph signals into different frequency components and to represent graph signals with strong correlation by different modes of variation effectively. The GFT on undirected graphs has been well studied and several approaches have been proposed to define GFTs on directed graphs. In this paper, based on the singular value decompositions of some graph Laplacians, we propose two GFTs on the Cartesian product graph of two directed graphs. We show that the proposed GFTs could represent spatial-temporal data sets on directed networks with strong correlation efficiently, and in the undirected graph setting they are essentially the joint GFT in the literature. In this paper, we also consider the bandlimiting procedure in frequency domains of the proposed GFTs, and demonstrate its performances to denoise the hourly temperature data sets collected at 32 weather stations in the region of Brest (France) and at 218 locations in the United States.

Keywords: Graph Fourier transform, singular value decomposition, directed product graphs.

1 Introduction

Data sets in many engineering applications are time-varying and pairwise interactions among agents of a network are not always mutual and equitable, such as the interaction data set on a social network and the temperature data set collected by a weather observation network. Those spatial-temporal data sets are usually modeled as graph signals residing on some directed product graphs [1]-[9].

Graph Fourier transform (GFT) is one of the fundamental tools to deal with spatial-temporal data sets [10]-[17]. The GFT on undirected graphs has been well studied and a conventional definition is based on the eigen-decomposition of the Laplacian on the graph [1],[18]-[25]. However, the above eigen-decomposition approach does not apply directly in the directed graph setting. In recent years, several approaches have been proposed to define GFTs on directed graphs.

The GFT should be designed to decompose graph signals into different frequency components and to efficiently represent them by different modes of variation [6,8,26,27]. The Jordan decomposition of the Laplacian has been widely used to define the GFT on directed graphs,
but the computational cost is high and Parseval’s identity may not hold [11, 12, 16, 20]. Several directed variations of signals along the graph structure have been proposed to define GFT on directed graphs [10, 11, 12, 28]. Based on the singular value decomposition (SVD) of the Laplacian on directed graphs, the authors of this paper introduced a GFT on directed graphs [17]. The SVD-based GFT in [17] has numerical stability and low computational cost, and on directed circulant graphs it is consistent with the classical discrete Fourier transform.

Let \( G_1 \) and \( G_2 \) be two directed graphs of orders \( N_1 \) and \( N_2 \). In this paper, we propose two GFTs \( F_\square \) and \( F_\otimes \) on the Cartesian product graph \( G_1 \square G_2 \) of two directed graphs \( G_1 \) and \( G_2 \), see Definitions 2.1 and 3.1. The proposed GFTs are based on the SVDs of the Laplacians on the Cartesian product graph \( G_1 \square G_2 \) and on directed graphs \( G_1 \) and \( G_2 \) respectively. We show that bandlimiting in the frequency domains of the proposed GFTs \( F_\square \) and \( F_\otimes \) provide good approximations to signals on the Cartesian product graph \( G_1 \square G_2 \) with strong spatial-temporal correlation, see Theorems 2.2 and 3.2. In this paper, we also show that the proposed GFTs \( F_\square \) and \( F_\otimes \) coincide only in the undirected graph setting, which become essentially the joint GFT in [2, 6, 7, 8, 26]. In this section, following the approach in [2, 6, 7], we introduce a GFT \( F_\square \) on the directed Cartesian product graph \( G \) and show that graph signals with strong spatial-temporal correlation may have their energy mainly concentrated on the low frequencies of the proposed GFT \( F_\square \), see Theorem 2.2.

Denote the adjacency, in-degree and (in-degree) Laplacian matrices of graphs \( G_l \) by \( A_l, D_l \) and \( L_l = D_l - A_l, l = 1, 2 \), respectively. One may verify that the adjacency and Laplacian matrices of the Cartesian product graph \( G \) are given by

\[
A_\square = A_1 \otimes I_{N_2} + I_{N_1} \otimes A_2
\]

and

\[
L_\square = L_1 \otimes I_{N_2} + I_{N_1} \otimes L_2.
\]  

A signal on the Cartesian product graph \( G \) is usually represented by a matrix \( X = [x_i]_{i \in V_1} \in \mathbb{R}^{N_1 \times N_2} \) and its vectorization \( x = \text{vec}(X) \), where for every \( i \in V_1 \), \( x_i \) is a graph signal on \( G_2 \). It could also be represented by a matrix \( Y = [y_j]_{j \in V_2} \) and its vectorization \( y = \text{vec}(Y) \), where for every \( j \in V_2 \), \( y_j \) is a graph signal on \( G_1 \). For our illustrative temporal-spatial (time-varying) scenario, \( x_i \) is the spatial signal at time \( i \in V_1 \) and \( y_j \) is the temporal signal at the vertex \( j \in V_2 \).
For the Laplacian $\mathbf{L}_\square$ on the directed Cartesian product graph $\mathcal{G}$, we take its SVD as follows,

$$\mathbf{L}_\square = \mathbf{U} \Sigma \mathbf{V}^T = \sum_{k=0}^{N-1} \sigma_k u_k v_k^T,$$

(2.2)

where $N = N_1 N_2$, $\mathbf{U} = [\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}]$ and $\mathbf{V} = [\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}]$ are orthogonal matrices, and the diagonal matrix $\Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{N-1})$ has singular values of the Laplacian $\mathbf{L}_\square$ deployed on the diagonal in a nondecreasing order, i.e.,

$$0 = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{N-1}. $$

The computational complexity to perform the SVD in (2.2) is $O(N^3)$ [29]. For the undirected graph setting, i.e., $\mathcal{G}_1$ and $\mathcal{G}_2$ are undirected graphs, the Laplacian matrices $\mathbf{L}_l, l = 1, 2$, are positive semi-definite and they have the following eigen-decomposition

$$\mathbf{L}_l = \sum_{i=0}^{N_l-1} \lambda_{i,l} \mathbf{w}_{i,l} \mathbf{w}_{i,l}^T, \quad l = 1, 2,$$

(2.3)

where $0 = \lambda_{l,0} \leq \ldots \leq \lambda_{l,N_l-1}$ are eigenvalues of $\mathbf{L}_l$, and $\mathbf{w}_{i,l}, 0 \leq i \leq N_l - 1$, form an orthonormal basis of $\mathbb{R}^{N_l}$. Therefore singular values of the Laplacian $\mathbf{L}_\square$ on the undirected Cartesian product graph $\mathcal{G}$ are the sum of eigenvalues of $\mathbf{L}_1$ and $\mathbf{L}_2$, and orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ are the same and consist of Kronecker products of eigenvectors of Laplacians $\mathbf{L}_1$ and $\mathbf{L}_2$, i.e.,

$$\mathbf{L}_\square = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (\lambda_{i,1} + \lambda_{j,2})(\mathbf{w}_{i,1} \otimes \mathbf{w}_{j,2})(\mathbf{w}_{i,1} \otimes \mathbf{w}_{j,2})^T$$

(2.4)

[6, 18, 30]. This implies that the computational complexity to perform the SVD (2.4) (and also the eigen-decomposition) of the Laplacian $\mathbf{L}_\square$ in the undirected graph setting is $O(N_1^3 + N_2^3)$ [29], instead of $O(N_1^3 N_2^3)$ in the general directed graph setting.

Based on the SVD (2.2) of the Laplacian matrix $\mathbf{L}_\square$, we can follow the approach in [17] to define the GFT on the directed Cartesian product graph $\mathcal{G}$.

**Definition 2.1.** Let $\mathcal{G}$ be the Cartesian product of directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, the Laplacian $\mathbf{L}_\square$ on $\mathcal{G}$ be given in (2.1), and orthogonal matrices $\mathbf{U}, \mathbf{V}$ of size $N \times N$ be as in (2.2). We define the graph Fourier transform $\mathcal{F}_\square : \mathbb{R}^N \mapsto \mathbb{R}^{N N}$ on $\mathcal{G}$ by

$$\mathcal{F}_\square \mathbf{x} := \frac{1}{2} \begin{pmatrix} (\mathbf{U} + \mathbf{V})^T \mathbf{x} \\ (\mathbf{U} - \mathbf{V})^T \mathbf{x} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (u_0 + v_0)^T x \\ (u_{N-1} + v_{N-1})^T x \\ (u_0 - v_0)^T x \\ \vdots \\ (u_{N-1} - v_{N-1})^T x \end{pmatrix},$$

(2.5)

where $\mathbf{x}$ is a graph signal on the Cartesian product graph $\mathcal{G}$. We also define the inverse graph Fourier transform $\mathcal{F}_\square^{-1} : \mathbb{R}^{2N} \mapsto \mathbb{R}^N$ by

$$\mathcal{F}_\square^{-1} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} := \frac{1}{2} \left( \mathbf{U}(\mathbf{z}_1 + \mathbf{z}_2) + \mathbf{V}(\mathbf{z}_1 - \mathbf{z}_2) \right)$$

$$= \frac{1}{2} \sum_{k=0}^{N-1} (z_{1,k} + z_{2,k}) u_k + (z_{1,k} - z_{2,k}) v_k$$

(2.6)

for $\mathbf{z}_l = [z_{l,0}, \ldots, z_{l,N_l-1}]^T \in \mathbb{R}^{N_l}, l = 1, 2$. 

3
For the proposed GFT $\mathcal{F}_\Box$ and inverse GFT $\mathcal{F}_\Box^{-1}$ in Definition 2.1, one may verify that
\[
\mathcal{F}_\Box^{-1} \mathcal{F}_\Box \mathbf{x} = \mathbf{x},
\]
where $\mathbf{x}$ is a signal on the Cartesian product graph $\mathcal{G}$. From the orthogonal properties of matrices $\mathbf{U}$ and $\mathbf{V}$ it follows that Parseval’s identity
\[
\| \mathcal{F}_\Box \mathbf{x} \|_2 = \| \mathbf{x} \|_2
\]
hold for all signals $\mathbf{x}$ on $\mathcal{G}$.

Following the terminology in [17], we may use singular values $\sigma_k, 0 \leq k \leq N - 1$, as frequencies of the proposed GFT $\mathcal{F}_\Box$, and $\mathbf{u}_k, \mathbf{v}_k, 0 \leq k \leq N - 1$, as the associated left/right frequency components. In the following theorem, we show that signals on the directed Cartesian product graph $\mathcal{G}$ with strong spatial-temporal correlation may have their energy mainly concentrated on the low frequencies of the proposed GFT $\mathcal{F}_\Box$, see Appendix A for the proof.

**Theorem 2.2.** Let $\mathcal{G}$ be the Cartesian product of directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, $L_\Box$ be the Laplacian on $\mathcal{G}$, and $\mathbf{u}_k, \mathbf{v}_k, \sigma_k, 0 \leq k \leq N - 1$ be as in (2.2), where $N = N_1 N_2$, $N_1$ and $N_2$ are the orders of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. For a frequency bandwidth $M \in \{1, 2, \ldots, N\}$, define the low frequency component of a graph signal $\mathbf{x}$ on $\mathcal{G}$ with bandwidth $M$ by
\[
\mathbf{x}_{M, \Box} = \frac{1}{2} \sum_{k=0}^{M-1} (z_{1,k} + z_{2,k}) \mathbf{u}_k + (z_{1,k} - z_{2,k}) \mathbf{v}_k = \frac{1}{2} \sum_{k=0}^{M-1} (\mathbf{u}_k \mathbf{u}_k^T + \mathbf{v}_k \mathbf{v}_k^T) \mathbf{x},
\]
where $z_{1,k} = (\mathbf{u}_k + \mathbf{v}_k)^T \mathbf{x}/2$ and $z_{2,k} = (\mathbf{u}_k - \mathbf{v}_k)^T \mathbf{x}/2$, $0 \leq k \leq M - 1$. Then
\[
\| \mathbf{x} - \mathbf{x}_{M, \Box} \|_2 \leq \frac{1}{2\sigma_{M-1}} \left( \|L_\Box \mathbf{x} \|_2 + \|L_\Box^T \mathbf{x} \|_2 \right)
\]
\[
\leq \frac{1}{2\sigma_{M-1}} \left( \| (L_1 \otimes I_{N_2}) \mathbf{x} \|_2 + \| (L_1^T \otimes I_{N_2}) \mathbf{x} \|_2 + \| (I_{N_1} \otimes L_2) \mathbf{x} \|_2 + \| (I_{N_1} \otimes L_2^T) \mathbf{x} \|_2 \right),
\]
where $\sigma_{M-1}$ is the cut-off frequency of the bandlimiting procedure (2.9).

## 3 GFT on directed product graphs

Graph signals in some applications, such as time-varying signals, carry different correlation characteristics in different directions, and hence GFT in such scenario should be designed to reflect spectral characteristic for different directions [6, 8, 20, 27]. In this section, based on the SVDs of Laplacians on $\mathcal{G}_1$ and $\mathcal{G}_2$, we introduce another GFT $\mathcal{F}_\odot$ on the product graph $\mathcal{G}$, see Definition 3.1. Comparing with the GFT $\mathcal{F}_\Box$ in Definition 2.1, the new GFT $\mathcal{F}_\odot$ has lower computational complexity. On the other hand, they have similar performance to efficiently represent time-varying signals with strong correlation, see Theorem 3.2 and numerical demonstrations in Section 4. In this section, we also show that the proposed GFTs $\mathcal{F}_\odot$ and $\mathcal{F}_\Box$ coincide only in the undirected graph setting, see Theorem 3.3.

Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two directed graphs, and denote their Laplacians and orders by $L_l$ and $N_l, l = 1, 2$ respectively. For the Laplacian matrices $L_l, l = 1, 2$, we take their SVDs
\[
L_l = \mathbf{U}_l \Sigma_l \mathbf{V}_l^T = \sum_{i=0}^{N_l-1} \sigma_{l,i} \mathbf{u}_{l,i} \mathbf{v}_{l,i}^T,
\]
where \( \sigma_{l,i}, 0 \leq i \leq N_l - 1 \), are singular values of the Laplacian matrix \( L_l \) with a nondecreasing order, \( U_l = [u_{l,0}, \ldots, u_{l,N_l - 1}] \) and \( V_l = [v_{l,0}, \ldots, v_{l,N_l - 1}] \) are orthonormal matrices. Set

\[
U_\otimes = U_1 \otimes U_2 \quad \text{and} \quad V_\otimes = V_1 \otimes V_2.
\]

(3.2)

With the help of SVDs of Laplacians \( L_l, l = 1, 2 \), we propose the second GFT on the directed product graph \( G \) as follows.

**Definition 3.1.** Let directed graphs \( G_l, l \in \{1, 2\} \), have orders \( N_l \) and Laplacian matrices \( L_l, V_l \) be given as in (3.1), \( U_\otimes \) and \( V_\otimes \) be the orthogonal matrices in (3.2), and set \( N = N_1 N_2 \). Then we define the graph Fourier transform \( F_\otimes : \mathbb{R}^N \mapsto \mathbb{R}^{2N} \) and inverse graph Fourier transform \( F_\otimes^{-1} : \mathbb{R}^{2N} \mapsto \mathbb{R}^N \) on the product graph \( G \) by

\[
F_\otimes x := \frac{1}{2} \begin{pmatrix}
(U_\otimes + V_\otimes)^T x \\
(U_\otimes - V_\otimes)^T x
\end{pmatrix}
\]

(3.3)

and

\[
F_\otimes^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \frac{1}{2} (U_\otimes (z_1 + z_2) + V_\otimes (z_1 - z_2)),
\]

(3.4)

where \( x \in \mathbb{R}^N \) is a signal on the graph \( G \), and \( z_1, z_2 \) are vectors in \( \mathbb{R}^N \).

For the GFT \( F_\otimes \) just defined, we may use pairs \((\sigma_{l,i}, \sigma_{l,j})\) of singular values of Laplacians \( L_1 \) and \( L_2 \) as frequency pairs of the proposed GFT, and \( u_{l,i} \otimes u_{l,j} \) and \( v_{l,i} \otimes v_{l,j} \), \( 0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1 \), as the associated left/right frequency components. The computational complexity to evaluate the left/right frequency components of the GFT \( F_\otimes \) is \( O(N_1^3 + N_2^3) \) [29], c.f. \( O(N_1^3 + N_2^3) \) to evaluate the left/right frequency components \( u_k, v_k, 0 \leq k \leq N_1 N_2 - 1 \), of the GFT \( \mathcal{F}_\otimes \) (2.5) in the undirected graph setting, and \( O(N_1^3 N_2^3) \) to evaluate them in general directed graph setting, see (2.4) and (2.2) and also numerical simulations in Section 4.

By the orthogonality of the matrices \( U_l, V_l, l = 1, 2 \), one may verify that

\[
\|F_\otimes x\|_2 = \|x\|_2
\]

(3.5)

and

\[
F_\otimes^{-1} F_\otimes x = x
\]

(3.6)

hold for all signals \( x \) on the product graph \( G \). Similar to the conclusion in Theorem 2.2 we can show that bandlimiting in the frequency domain of the GFT \( F_\otimes \) provides good approximations to graph signals with strong spatial-temporal correlation, see Appendix B for the proof.

**Theorem 3.2.** Let \( G \) be the Cartesian product of directed graphs \( G_1 \) and \( G_2 \), \( \sigma_{l,i}, u_{l,i}, v_{l,i}, 0 \leq i \leq N_l - 1, l = 1, 2 \), be as in (3.1), and \( \mu_k, 0 \leq k \leq N - 1 \), be the ascending order of \( \sigma_{1,i} + \sigma_{2,j} \), \( 0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1 \), where \( N = N_1 N_2 \). For a frequency bandwidth \( 1 \leq M \leq N \) of the GFT \( \mathcal{G}_\otimes \) in (3.3), define the low frequency component of a graph signal \( x \) on \( G \) with bandwidth \( M \) by

\[
x_{M,\otimes} = \frac{1}{2} \sum_{(i,j) \in S_M} (u_{1,i} \otimes u_{2,j})(u_{1,i} \otimes u_{2,j})^T x
\]

\[+ (v_{1,i} \otimes v_{2,j})(v_{1,i} \otimes v_{2,j})^T x,
\]

(3.7)

where \( S_M \) contains all pairs \((i,j)\) with \( \sigma_{1,i} + \sigma_{2,j} \) being some \( \mu_k \), \( 0 \leq k \leq M - 1 \). Then

\[
\|x - x_{M,\otimes}\|_2 \leq \frac{1}{2 \mu_{M-1}} \left( \|(L_1 \otimes I_{N_2})x\|_2 + \|(I_{N_1} \otimes L_2)x\|_2 + \|(I_{N_1} \otimes L_2^T)x\|_2 \right),
\]

(3.8)

where \( \mu_{M-1} \) is the cut-off frequency of the bandlimiting procedure (3.7).
For a graph signal $X = [x_i]_{i \in V} \in \mathbb{R}^N$ or its vectorization $x = \text{vec}(X)$ on the product graph $\mathcal{G}$, using the mixed Kronecker matrix-vector product property, we can rewrite its GFT $F_\otimes x$ as follows:

$$F_\otimes x = \frac{1}{2} \begin{pmatrix} \text{vec}(U_1^T X U_1 + V_1^T X V_1) \\ \text{vec}(U_2^T X U_1 - V_2^T X V_1) \end{pmatrix}.$$  \hspace{1cm} (3.9)

Thus just as taking classical discrete Fourier transform of two-dimensional signals by directions, we can implement the GFT $F_\otimes x$ in the direction of the graph $\mathcal{G}_1$ and then of the graph $\mathcal{G}_2$, or vice versa, see Algorithm 3.1.

**Algorithm 3.1** Algorithm to implement the GFT $F_\otimes$

**Input:** Graph signal $X$.

**Steps:**
1) Do $Y_1 = XU_1$ and $\hat{Y}_1 = XV_1$;
2) Do $Y_2 = U_2^T Y_1$ and $\hat{Y}_2 = V_2^T Y_1$;
3) Do $\hat{X}_1 = (Y_2 + \hat{Y}_2)/2$ and $\hat{X}_2 = (Y_2 - \hat{Y}_2)/2$.

**Outputs:** The first component $\hat{X}_1$ and the second component $\hat{X}_2$ of the GFT $F_\otimes \text{vec}(X)$.

Similarly, we have

$$F_\otimes^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} (U_2(Z_1 + Z_2)U_1^T + V_2(Z_1 - Z_2)V_1^T)$$

for $z_1, z_2 \in \mathbb{R}^N$, where $Z_1 = \text{vec}^{-1}(z_1)$ and $Z_2 = \text{vec}^{-1}(z_2)$, see Algorithm 3.2 for the implementation.

**Algorithm 3.2** Algorithm to implement the inverse GFT $F_\otimes^{-1}$

**Inputs:** $z_1, z_2 \in \mathbb{R}^N$.

**Inverse vectorization:** $Z_1 = \text{vec}^{-1}(z_1)$ and $Z_2 = \text{vec}^{-1}(z_2)$.

**Steps:**
1) Do $W_1 = (Z_1 + Z_2)U_1^T$ and $\tilde{W}_1 = (Z_1 - Z_2)V_1^T$;
2) Do $W_2 = U_2 W_1$ and $\tilde{W}_2 = V_2 W_1$;
3) Do $X = (W_2 + \tilde{W}_2)/2$.

**Output:** $x = \text{vec}(X) = F_\otimes^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

In the undirected graph setting, we obtain from (2.3) that $U, V$ in (2.2) and $U_\otimes, V_\otimes$ in (3.2) can be chosen to be the same, i.e., $U = V = U_\otimes = V_\otimes$. Therefore

$$F_\boxtimes x = F_\otimes x = \begin{pmatrix} U_\otimes^T x \\ 0_N \end{pmatrix}.$$ \hspace{1cm} (3.10)

hold for all signals $x$ on the Cartesian product of two undirected graphs. We remark that in the undirected graph setting, $U_\otimes^T x$ is used in (2.6) to define the joint GFT of a graph signal $x$ on the product graph.

In the following theorem, we show that the proposed GFTs $F_\boxtimes$ and $F_\otimes$ coincide only in the undirected graph setting, see Appendix C for the proof.

**Theorem 3.3.** Let $F_\boxtimes$ and $F_\otimes$ be the GFTs on the Cartesian product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are not edgeless graphs. If $F_\boxtimes = F_\otimes$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are undirected graphs.
4 Numerical simulations

In this section, we first consider the hourly temperature data set measured in Celsius collected at 32 weather stations in the region of Brest (France) in January 2014, published by French national meteorological service [4]. We represent the above temperature data set by matrices $X_d, 1 \leq d \leq 31$, as signals on the Cartesian product graph $T \square S$, where $T$ is the unweighted directed line graph with 24 vertices and $S$ is the directed graph with 32 locations of weather observation stations as vertices and edges constructed by the 5 nearest neighboring stations in physical distances, and weights on the edges are randomly chosen in $[0, 1]$. In this section, we demonstrate the performances of the proposed GFTs $F_{\Box}$ and $F_{\otimes}$ by bandlimiting the first $M$-frequencies of the noisy temperature data set

$$\hat{X}_d = X_d + \eta_d, \quad 1 \leq d \leq 31, \tag{4.1}$$

where $\eta_d$ are additive random noises with entries being i.i.d. drawn on $[-c, c]$ with $c \in [0, 8]$. All experiments are implemented on a Macbook pro (2.3 GHz quad-core Intel Core i7 and 32 GB memory) by Matlab R2020b.

For the unweighted directed line graph $T$ of order $N_1 = 24$, weighted directed graph $S$ of order $N_2 = 32$ and their Cartesian product graph $T \square S$ of order 768 = $24 \times 32$, we take $\sigma_k, u_k, v_k, 0 \leq k \leq 767$, as in (2.2), and $\sigma_{l,i}, u_{l,i}, v_{l,i}, 0 \leq i \leq N_l - 1, l = 1, 2$. Inspired by the eigen-decomposition in (2.4) and the coincidence (3.10) of GFTs in the undirected graph setting, we arrange frequencies of the GFT $F_{\Box}$ in the ascending order $0 = \sigma_0 \leq \ldots \leq \mu_767$, and frequency pairs $(\sigma_{l,i}, \sigma_{2,j})$ of the GFT $F_{\otimes}$ in the ascending order of $\sigma_{l,i} + \sigma_{2,j}, 0 \leq i \leq 23, 0 \leq j \leq 31$, which are represented by $0 = \mu_0 \leq \ldots \leq \mu_767$. The time to find the left/right frequency components $u_k, v_k, 0 \leq k \leq 767$, of the GFT $F_{\Box}$ and the ones $u_{1,i}, v_{1,i} \otimes v_{2,j}, 0 \leq i \leq 23, 0 \leq j \leq 31$, of the GFT $F_{\otimes}$ are 0.0861 and 0.0189 seconds respectively. This confirms that the GFT $F_{\otimes}$ has lower computational complexity than the GFT $F_{\Box}$ does. Our numerical simulations also show that $0 \leq \sigma_k, \mu_k \leq 13.3206$ and $\sigma_k \leq \mu_k \leq 0.4047, 0 \leq k \leq 767$, see Figure [2]. Therefore, the proposed GFTs $F_{\Box}$ and $F_{\otimes}$ may have similar frequency information.

Let $1 \leq M \leq 768$. Applying the bandlimiting procedure of the first $M$-frequencies of the GFTs $F_{\Box}$ and $F_{\otimes}$ to the noisy temperature data set $\hat{X}_d$ in [4.1], we obtain

$$\hat{X}_{d,M,\Box} = \text{vec}^{-1} \left( \frac{1}{2} \sum_{k=0}^{M-1} (u_k^T u_k^T + v_k^T v_k^T) \text{vec}(\hat{X}_d) \right), \tag{4.2}$$
Figure 2: Plotted in red dotted line and blue solid line are the frequencies $\sigma_k$ and $\mu_k$, $0 \leq k \leq 767$, associated with the GFTs $\mathcal{F}_{\Box}$ and $\mathcal{F}_{\otimes}$ scaled at the left $y$-axis respectively. Plotted in lime green are the differences $\mu_k - \sigma_k$, $0 \leq k \leq 767$, scaled at the right $y$-axis.

and

$$\hat{X}_{d,M,\otimes} = \frac{1}{2} \sum_{(i,j) \in S_M} (u_{2,j}^T \tilde{X}_{d} u_{1,i}) u_{2,j} u_{1,i}^T + (v_{2,j}^T \tilde{X}_{d} v_{1,i}) v_{2,j} v_{1,i}^T,$$

\( (4.3) \)

where $S_M$ contains all pairs $(i, j)$ with $\sigma_{1,i} + \sigma_{2,j}$ being some $\mu_k$, $0 \leq k \leq M - 1$, one of the first $M$-frequencies in the frequency domain of the GFT $\mathcal{F}_{\otimes}$. Shown in Figure 3 are the GFTs of the temperature data set $X_1$ on January 1st, 2014 and its bandlimiting approximations $\hat{X}_{1,M,\Box}$ and $\hat{X}_{1,M,\otimes}$ of the noisy temperature data set $\tilde{X}$ in the frequency domain of the GFTs $\mathcal{F}_{\Box}$ and $\mathcal{F}_{\otimes}$, where $M = 32$, $c = 4$, $\|X_1\|_F = 286.6332$, $\|\hat{X}_{1,M,\Box} - X_1\|_F = 24.1248$ and $\|\hat{X}_{1,M,\otimes} - X_1\|_F = 23.7247$. This shows that the hourly temperature data set $X_1$ has about 91.583% and 91.723% energy concentrated on the first 32 out of total 768 (about 4.167%) frequencies of the GFTs $\mathcal{F}_{\Box}$ and $\mathcal{F}_{\otimes}$ respectively.

Define the input signal-to-noise ratio (ISNR) and the bandlimiting signal-to-noise ratio (SNR) by

$$\text{ISNR}(c) = -20 \log_{10} \frac{\| \tilde{X} - X \|_F}{\| X \|_F}$$

and

$$\text{SNR}(c, M) = -20 \log_{10} \frac{\| \hat{X} - X \|_F}{\| X \|_F},$$

where $X$ is the original temperature data $X_d$, $1 \leq d \leq 31$, $\tilde{X}$ is the noisy temperature data in (4.1), $\hat{X}$ is the bandlimited temperature data in (4.2) or (4.3). Denote the SNR obtained by (4.2) and (4.3) by $\text{SNR.IV2}$ and $\text{SNR.IV3}$ respectively. Shown in Tables 1 and 2 are the denoising performances of the proposed GFTs for different noise levels $c$ and bandlimiting frequency bandwidths $M$, where the ISNR, $\text{SNR.IV2}$ and $\text{SNR.IV3}$ are taken over the average of 100 trials per day and over 31 days. From Table 1 we observe that the proposed GFTs $\mathcal{F}_{\Box}$ and $\mathcal{F}_{\otimes}$ have similar good performance on denoising the noisy temperature data sets collected in the region of Brest, and from Table 2 that the SNR has slow change for larger frequency bandwidth $M \geq 24$ (about 3.125% of the total numbers of frequencies). The possible reasons for the second observation could be that the temperature data set in the region of Brest (France) has strong correlation for different hours and locations, and energy of the original data set is mainly concentrated on the low frequencies of the proposed GFTs, see Figure 1. This demonstrates that the proposed GFTs $\mathcal{F}_{\Box}$ and $\mathcal{F}_{\otimes}$ could be used to decompose graph signals on product graphs into different frequency components and represent those signals with strong correlation efficiently in the frequency domain, cf. Theorems 2.2 and 3.2 and see also Figure 3.
Figure 3: Plotted on the top left and right are the first component $(U + V)^T x_1/2$ and the second component $(U - V)^T x_1/2$ of the GFT $\mathcal{F}_\square x_1$ of the signal $x_1$ respectively, where $U, V$ are the orthogonal matrices in (2.2) of $L_\square$ and $x_1 = \text{vec}(X_1)$ is the vectorization of the temperature data set on January 1st, 2014. On the middle left and right are the first component $(U \otimes V + V \otimes U)^T x_1/2$ and the second component $(U \otimes V - V \otimes U)^T x_1/2$ of the GFT $\mathcal{F}_\otimes x_1$ respectively, where $U, V$ are the orthogonal matrices in (3.2). On the bottom left and right are the snapshots of bandlimiting approximations $\hat{X}_{1,M,\square}$ and $\hat{X}_{1,M,\otimes}$ of the noisy temperature data set $\tilde{X}_1$ at noon on January 1st, 2014 respectively, where $c = 4$ and $M = 32$. 

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Table 1: The average ISNR and bandlimiting SNR for fixed frequency bandwidth $M = 32$ and different noise levels $c$.

| $c$ | ISNR  | SNR.IV2 | SNR.IV3 |
|-----|-------|---------|---------|
| 1   | 23.2701 | 17.8334 | 17.9590 |
| 2   | 17.2473 | 17.6570 | 17.7780 |
| 3   | 13.7260 | 17.3822 | 17.4981 |
| 4   | 11.2296 | 17.0294 | 17.1400 |
| 5   | 9.2902  | 16.6332 | 16.7344 |
| 6   | 7.7021  | 16.1838 | 16.2836 |
| 7   | 6.3647  | 15.7238 | 15.8187 |
| 8   | 5.2108  | 15.2610 | 15.3483 |

Table 2: The average bandlimiting SNR for different frequency bandwidths $M$ and fixed noise levels $c = 0$ and 4, where the average ISNR is 11.2272 for $c = 4$.

| $M$ | $c=0$ | $c=4$ | $c=0$ | $c=4$ |
|-----|-------|-------|-------|-------|
|     | SNR.IV2 | SNR.IV3 | SNR.IV2 | SNR.IV3 |
| 16  | 16.3048 | 16.6723 | 16.0119 | 16.3485 |
| 24  | 17.4866 | 17.4409 | 16.8893 | 16.8667 |
| 32  | 17.8944 | 18.0213 | 17.0320 | 17.1419 |
| 48  | 18.2314 | 18.2260 | 16.8823 | 16.8886 |
| 64  | 18.6767 | 19.3312 | 16.8299 | 17.2950 |
| 128 | 20.5466 | 20.5481 | 16.4607 | 16.4767 |
| 256 | 23.0639 | 23.6230 | 15.0905 | 15.2524 |

We also do the simulations to implement the denoising procedures (4.2) and (4.3) on the hourly temperature data set collected at 218 locations in the United States on August 1st, 2010 [8, 17]. Similar to the temperature data set in France, the data set can be modeled as signals on the Cartesian product graph of order $24 \times 218$ (about 6.8125 times the order $24 \times 32$ of the Cartesian product graph to model temperature data set in France). Our experiments show that the time spent on finding the left/right frequency components of the GFTs $\mathcal{F}_\square$ and $\mathcal{F}_\otimes$ are 24.3662 and 0.294226 seconds respectively, which are about 283.00 and 15.57 times more than the time spent on finding frequency components when dealing with the temperature data set in France. This reaffirms numerically that the GFT $\mathcal{F}_\otimes$ has much lower computational complexity than the GFT $\mathcal{F}_\square$ does for the directed product graph of a large order. For different noise levels $c$ and frequency bandwidths $M$, our simulations indicate that the proposed GFTs have similar performance on denoising the U.S. temperature data set to the one on denoising the temperature data set in the region of Brest (France).

A Proof of Theorem 2.2

By (2.2), we have

$$\|L_\square x\|_2^2 = x^T \Sigma^2 V^T = \sum_{k=0}^{N-1} \sigma_k^2 (v_k^T x)^2$$

$$\geq \sigma_{M-1}^2 \sum_{k=M}^{N-1} (v_k^T x)^2 \quad \text{(A.1)}$$
\[
\|L^T_2 x\|_2^2 = x^T U \Sigma^2 U^T x \geq \sigma_{M-1}^2 \sum_{k=M}^{N-1} (u_k^T x)^2. \tag{A.2}
\]

From (2.7) and (2.9), it follows that
\[
\|x - x_{M,\square}\|_2 = 1 \frac{1}{2} \left( \sum_{k=M}^{N-1} (u_k^T x)^2 \right)^{1/2} + \frac{1}{2} \left( \sum_{k=M}^{N-1} (v_k^T x)^2 \right)^{1/2}.
\]

This together with (2.1), (A.1) and (A.2) completes the proof.

**B Proof of Theorem 3.2**

By (3.1), we have
\[
\|(L_1 \otimes I_{N_2})x\|_2^2 = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \sigma_{1,i}^2 ((v_{1,i} \otimes v_{2,j})^T x)^2
\]
and
\[
\|(I_{N_1} \otimes L_2)x\|_2^2 = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \sigma_{2,j}^2 ((v_{1,i} \otimes v_{2,j})^T x)^2.
\]

This implies that
\[
\left( \|(L_1 \otimes I_{N_2})x\|_2 + \|(I_{N_1} \otimes L_2)x\|_2 \right)^2 \geq \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (\sigma_{1,i} + \sigma_{2,j})^2 ((v_{1,i} \otimes v_{2,j})^T x)^2 \geq \mu_{M-1}^2 \sum_{(i,j) \not\in S_M} ((v_{1,i} \otimes v_{2,j})^T x)^2. \tag{B.1}
\]

Similarly, we obtain from (3.1) that
\[
\left( \|(L_1^T \otimes I_{N_2})x\|_2 + \|(I_{N_1} \otimes L_2^T)x\|_2 \right)^2 \geq \mu_{M-1}^2 \sum_{(i,j) \not\in S_M} ((u_{1,i} \otimes u_{2,j})^T x)^2. \tag{B.2}
\]

From (3.6) and (3.7) it follows that
\[
\|x - x_{M,\square}\|_2 \leq \left( \sum_{(i,j) \not\in S_M} ((u_{1,i} \otimes u_{2,j})^T x)^2 \right)^{1/2} + \left( \sum_{(i,j) \not\in S_M} ((v_{1,i} \otimes v_{2,j})^T x)^2 \right)^{1/2}. \tag{B.3}
\]

Combining (B.1), (B.2) and (B.3) establishes the desired estimate in (3.8).
C Proof of Theorem 3.3

For \( l = 1, 2 \), let \( U_l \) and \( V_l \) be orthogonal matrices in the SVD (3.1) of Laplacians \( L_l \) on the graphs \( G_l \), and write \( U_l = [u_l(i,j)]_{0 \leq i,j \leq N_l-1} \), \( V_l = [v_l(i,j)]_{0 \leq i,j \leq N_l-1} \) and \( L_l = [a_l(i,j)]_{0 \leq i,j \leq N_l-1} \). By the assumption on GFTs \( F_\square \) and \( F_\odot \), the Laplacian \( L_\square \) in (2.1) has the following decomposition

\[
L_\square = U_\odot \Sigma V_\odot^T, \tag{C.1}
\]

where \( U_\odot \) and \( V_\odot \) are given in (3.2) and \( \Sigma \) is a diagonal matrix with nonnegative diagonal entries which are not necessarily in a nondecreasing order. Let \( \delta(i,j) \), \( 0 \leq i, j \leq N_1 - 1 \), be the Kronecker delta and write \( \Sigma = \text{diag}(\Sigma_0, \ldots, \Sigma_{N_1-1}) \), where \( \Sigma_i \), \( 0 \leq i \leq N_1 - 1 \), are diagonal matrices of size \( N_2 \). By (2.1) and (C.1), we have

\[
\sum_{k=0}^{N_1-1} u_1(i,k)v_1(j,k) U_2 \Sigma_k V_2^T = a_1(i,j) I_{N_2} + \delta(i,j) L_2, \quad 0 \leq i, j \leq N_1 - 1. \tag{C.2}
\]

Let \( \mathbb{R}^{N_2 \times N_2} \) be the Hilbert space of the real matrices of size \( N_2 \times N_2 \) with the inner product of two matrices \( A = [a(i,j)]_{0 \leq i,j \leq N_2-1} \) and \( B = [b(i,j)]_{0 \leq i,j \leq N_2-1} \) defined by

\[
\langle A, B \rangle = \sum_{i,j=0}^{N_2-1} a(i,j)b(i,j).
\]

Write

\[
U_2 \Sigma_k V_2^T = b_k I_{N_2} + c_k L_2 + W_k, \quad 0 \leq k \leq N_1 - 1, \tag{C.3}
\]

where \( b_k, c_k \in \mathbb{R} \) and \( W_k, 0 \leq k \leq N_1 - 1 \), are orthogonal to the linear subspace of \( \mathbb{R}^{N_2 \times N_2} \) spanned by \( I_{N_2} \) and \( L_2 \). By (C.2) and (C.3), we have

\[
\sum_{k=0}^{N_1-1} u_1(i,k)v_1(j,k) W_k = O_{N_2}, \quad 0 \leq i, j \leq N_1 - 1. \tag{C.4}
\]

This together with the orthogonal properties of matrices \( U_1 \) and \( V_1 \) implies that

\[
W_k = O_{N_2}, \quad 0 \leq k \leq N_1 - 1. \tag{C.4}
\]

By the non-edgeless assumption on the graph \( G_2 \), the unit matrix \( I_{N_2} \) and the Laplacian \( L_2 \) on the graph \( G_2 \) are linearly independent in \( \mathbb{R}^{N_2 \times N_2} \). Therefore combining (C.2), (C.3) and (C.4), we obtain

\[
\sum_{k=0}^{N_1-1} u_1(i,k)v_1(j,k)b_k = a_1(i,j) \tag{C.5}
\]

and

\[
\sum_{k=0}^{N_1-1} u_1(i,k)v_1(j,k)c_k = \delta(i,j), \tag{C.6}
\]

where \( 0 \leq i, j \leq N_1 - 1 \). Let \( B \) and \( C \) be the diagonal matrices with diagonal entries \( b_k \) and \( c_k, 0 \leq k \leq N_1 - 1 \). Then we can rewrite (C.5) and (C.6) in the following matrix formulation:

\[
U_1 B V_1^T = L_1 \quad \text{and} \quad U_1 C V_1^T = I_{N_1}. \tag{C.7}
\]

This implies that \( L_1 = U_1 B C^{-1} U_1^T \) is symmetric. Hence the graph \( G_1 \) is undirected.
By the non-edgeless assumption on the graphs \( G_1 \), the unit matrix \( I_{N_1} \) and the Laplacian \( L_1 \) on the graph \( G_1 \) are linearly independent. Then we conclude from (C.7) that the diagonal matrix \( BC^{-1} \) is not a multiple of the identity matrix \( I_{N_1} \), which in turn implies that there exist \( 0 \leq k_1 \neq k_2 \leq N_1 - 1 \) such that \((b_{k_1}, c_{k_1})\) and \((b_{k_2}, c_{k_2})\) are linearly independent in \( \mathbb{R}^2 \). Hence there are two diagonal matrices \( B \) and \( C \) by (C.2), (C.3) and (C.4) such that

\[
U_2 \tilde{B} V_T^2 = L_2 \quad \text{and} \quad U_2 \tilde{C} V_T^2 = I_{N_2}.
\]

Therefore \( L_2 = U_2 \tilde{B}(\tilde{C})^{-1} U_T^2 \) is symmetric. This completes the proof that the graph \( G_2 \) is undirected.

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