The weights of closed subgroups of a locally compact group

by Salvador Hernández, Karl H. Hofmann, and Sidney A. Morris

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1. Every infinite compact group contains an infinite closed metric subgroup.
2. For a locally compact group $G$ and $\aleph$ a cardinal satisfying $\aleph_0 \leq \aleph \leq w_0(G)$, where $w_0(G)$ is the local weight of $G$, there are either no infinite compact subgroups at all or there is a compact subgroup $N$ of $G$ with $w(N) = \aleph$.
3. For an infinite abelian group $G$ there exists a properly ascending family of locally quasiconvex group topologies on $G$, say, $(\tau_\aleph)_{\aleph_0 \leq \aleph \leq \text{card}(G)}$, such that $(G, \tau_\aleph) \overset{\text{eq}}{\cong} \hat{G}$.

Items (2) and (3) are shown in Section 5.
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Introduction

The weight $w(X)$ of a topological space $X$ is the smallest cardinal $\aleph$ for which
there is a basis $\mathcal{B}$ of the topology of $X$ such that $\text{card}(\mathcal{B}) = \aleph$. A compact group
$G$ is metric iff its weight $w(G)$ is countable, that is, $w(G) \leq \aleph_0$. (See e.g. [8],
A4.10 ff., notably, A1.16.) In particular, all compact Lie groups are metric. It is
not clear a priori that a compact group of uncountable weight contains an infinite
closed metric subgroup. Indeed, in Example 5.3(v) below we will show that the
precompact topological group defined on $\mathbb{Z}$ endowed with the Bohr-topology, which
it inherits from its universal almost periodic compactification, has no nonsingleton
metric subgroup, while its weight is the cardinality of the continuum.

However, we shall prove the following theorem which among other things will show
that every infinite compact group has an infinite metric subgroup.

Main Theorem. Let $G$ be a locally compact group of uncountable weight and
let $\aleph_0 \leq \aleph < w(G)$. Then $G$ has a closed subgroup $N$ with $w(N) = \aleph$.

In other words, for any infinite locally compact group $G$, the entire interval of
cardinals $[\aleph_0, w(G)]$ is occupied by the weights of closed subgroups of $G$.

If $G$ is compact and connected, we shall see that $[\aleph_0, w(G)]$ is filled even with
closed normal and indeed connected subgroups. It remains unsettled whether a
profinite group has, in this sense, enough normal closed subgroups.
We shall deal with a proof of the Main Theorem in a piecemeal way. For reasons of presenting a stepwise proof let us call $\mathcal{C}$ the class of all Hausdorff topological groups $G$ satisfying the following condition:

*for each infinite cardinal $\aleph \leq w(G)$ there is a closed subgroup $H$ of weight $\aleph$.*

Our Main Theorem says that all locally compact groups are contained in $\mathcal{C}$. We first aim to show that all compact groups are in $\mathcal{C}$ and we begin with compact abelian groups.

**1. Compact abelian groups**

For an abelian group $A$, let $\text{tor } A$ denote its torsion subgroup. The first portion of our first observation is a consequence of a more precise statement due to W. R. Scott [13]. Since we prove what we shall need in a shorter way (and quite differently) we present a proof which will also establish the second part. A divisible hull of an abelian group we are using can be constructed in a very special way by using the results of E. A. Walker in [14].

**Proposition 1.1.** Let $A$ be an uncountable abelian group and $\aleph_0 \leq \aleph < |A|$. Then $A$ contains a subgroup $B$ such that $(A : B) = \aleph$.

Moreover, If $(A : \text{tor } A)$ is at least $\aleph$ then $B$ may be picked so that $B$ is pure and $A/B$ is torsion free.

**Proof.** Let $D$ be a divisible hull of $A$ according to [8], Proposition A1.33. If $D = D_1 \oplus D_2$ is any direct decomposition, and $\text{pr}_1: D \to D_1$ is the projection onto the first summand of $D$, then $A/(A \cap D_2) \cong (A + D_2)/D_2 \cong \text{pr}_1(A)$. Now any subgroup of $D_1$ is a subgroup of the divisible hull of $A$ and therefore meets $A$ and thus $A \cap D_1 \subseteq \text{pr}_1(A)$ nontrivially (see [8], Proposition A1.33); therefore $D_1$ is a divisible hull of $\text{pr}_1(A)$. Hence either $\text{pr}_1(A)$ is finite or else $\text{card } D_1 = \text{card } \text{pr}_1(A) = \text{card } (A/(A \cap D_2))$ by [8], Proposition A1.33(i).

Since we control $\text{card } D_1$ by choosing $D_1$ appropriately, we aim to set $B = A \cap D_2$ and thereby prove our first assertion. We thus have to exclude the possibility that $\text{pr}_1(A)$ might turn out to be finite by an inappropriate choice of $D_1$. We now let $\text{tor } A$ denote the torsion subgroup of $A$. Then $\text{tor } D$ is a divisible hull of $\text{tor } A$, and $D \cong (\text{tor } D) \times (D/\text{tor } D)$ by [8], p. 657, Proposition A1.38. We now distinguish two cases:

(a) Case $\text{card } (\text{tor } A) = \text{card } A$. Since $A$ is uncountable, one of the $p$-primary components of $\text{tor } A$, as $p$ ranges through the countable set of primes, say $A(p)$, satisfies $\text{card } A(p) = \text{card } (\text{tor } A) = \text{card } A$. In particular $\text{card } A(p)$ is uncountable, that is, its $p$-rank $\text{card } A$ is uncountable and agrees with the $p$-rank of $D(p)$ (see...
[8], p. 656, Corollary A1.36(iii). In view of $D(p) \cong (\mathbb{Z}(p^{\infty})^{\text{card} A})$ by [8], p. 659, Theorem A1.42(iii), we find a direct summand $D_1$ of $D(p)$ of $p$-rank $\aleph$, giving us a direct summand of $\text{tor } D$ and thus yielding a direct sum decomposition $D = D_1 \oplus D_2$. Since the $p$-rank $\aleph$ of $D_1$ is infinite, and $D_1$ is the divisible hull of $\text{pr}_1(A)$ we know that $\text{pr}_1(A)$ cannot be finite, whence $\aleph = \text{card } D_1 = \text{card}(A/(A \cap D_2))$. Our first assertion then follows with $B = A \cap D_2$.

(b) Case $\text{card}(A/\text{tor } A) = \text{card } A$. Then the (torsion free) rank of $D$ is $\text{card } A$ (see [8], p. 656, Corollary A1.36(iii)). By the structure theorem of divisible groups (see e.g. [8], Theorem A1.42) and elementary cardinal arithmetic, we can write $D = D_1 \oplus D_2$ with a torsionfree subgroup $D_1$ of cardinality $\aleph$. Then $\text{pr}_1(A) \subseteq D_1$ cannot be finite, and as in the first case, we let $B = A \cap D_2$ and have $\aleph = \text{card } D_1 = \text{card } A/B$ as in our first assertion.

It remains to inspect the case that $\text{card}(A/\text{tor } A) \geq \aleph$. Then we may assume $\text{tor } A \subseteq D_2$ and $D_1$ torsion free; but then $\text{tor } A \subseteq A \cap D_2 = B$ whence $A/B$ is torsion free.

The second part of the preceding proposition is also a consequence of Theorem 4 of [14].

**Corollary 1.2.** Every uncountable abelian group has a proper subgroup of index $\aleph_0$. □

(A different, but likewise not entirely trivial proof by Hewitt and Ross is found in [6], p. 227.)

As usual, for a topological group $G$, the identity component of $G$ will be denoted $G_0$.

**Corollary 1.3.** Let $G$ be an infinite compact abelian group and assume $\aleph_0 \leq \aleph < w(G)$. Then $G$ has a closed subgroup $M$ with $w(M) = \aleph$.

Moreover, if $w(G_0) \geq \aleph$, then $M$ may be chosen to be connected. □

**Proof.** By the Annihilator Mechanism ([8], Theorem 7.64), for a subgroup $H$ of $G$ and its annihilator $H^\perp$ in $\hat{G}$, one has $\hat{H} \cong \hat{G}/H^\perp$ and thus $w(H) = \text{card } \hat{H} = \text{card } \hat{G}/H^\perp$. Since card $\hat{G} = w(G)$ we have card $\hat{G} > \aleph$.

Moreover, $H$ is connected iff $\hat{G}/H^\perp$ is torsion free. (See [8], Corollary 8.5.) The Corollary is therefore equivalent to the Proposition 1.1 above. □
Corollary 1.4. Let $\aleph$ be an infinite cardinal and let $G$ be a compact connected group with $w(G) > \aleph$. Then $G$ has a compact connected abelian subgroup $T$ with $w(T) = \aleph$.

Proof. Let $T$ be a maximal compact connected abelian subgroup of $G$. Then $w(T) = w(G)$ by [8], Theorem 9.36(vi). Then the assertion follows from Corollary 1.3.

Let us see what we have in the case of $\aleph = \aleph_0$. A compact abelian group $A$ is metric iff $\hat{A}$ is countable ([8], Theorem 7.76). Thus Corollary 1.3 trivially implies

Corollary 1.5. Every compact abelian group has an infinite closed metric subgroup.

A result by Efim Zelmanov says:

Theorem 1.6. An infinite compact group contains an infinite abelian subgroup.

Proof. See [15] or [12], p. 162.

This together with Corollary 1.5, implies

Corollary 1.7. Every compact group contains an infinite metric compact abelian subgroup.

While this corollary answers the question whether infinite compact groups have infinite compact metric subgroup in the affirmative, we should keep in mind, that Zelmanov’s Theorem in itself is not a simple matter. It therefore appears worthwhile to pursue the question further.

2. Connectivity versus total disconnectivity in compact groups

Lemma 2.1. Let $G$ be an arbitrary compact group. Then the following conclusions hold:

(i) $G$ and $G_0 \times G/G_0$ are homeomorphic. In particular, if $G$ is infinite, then

$$w(G) = \max\{w(G_0), w(G/G_0)\}.$$ 

(ii) There is a profinite subgroup $D$ of $G$ such that $G = G_0D$ and $G_0 \cap D$ is normal in $G$ and central in $G_0$. If $w(G_0) < w(G)$, then $w(G) = w(G/G_0) = w(D)$.

(iii) If $G$ is profinite, then $G$ and $(\mathbb{Z}/2\mathbb{Z})^w(G)$ are homeomorphic.

Proof. For (i) see [8], 10.38.
Regarding (ii), see [8], 9.41 and note $G/G_0 \cong D/(D \cap G_0)$.

For (iii), see [8], 10.40.

After Lemma 2.1 the question whether a compact group $G$ is in $C$ splits into two cases:

Case 1. If $w(G_0) = w(G)$, one may assume that $G$ is connected.

Case 2. If $w(G_0) < w(G)$, one may assume that $G$ is profinite.

We recall from Corollary 1.4 that for a compact connected group $G$ the set of all infinite cardinals $\leq w(G)$ is filled with the set of all infinite cardinals representing the weights of closed connected abelian subgroups. In the following we amplify this observation.

**Proposition 2.2.** Let $\aleph$ be an infinite cardinal such that $\aleph < w(G)$ for a compact connected group $G$. Then $G$ contains a closed connected and normal subgroup $N$ such that $w(N) = \aleph$.

**Proof.** Following the Levi-Mal’cev Structure Theorem for Compact Connected Groups ([8], Theorem 9.24) we have $G = G' Z_0(G)$ where the algebraic commutator subgroup $G'$ is a characteristic compact connected semisimple subgroup and the identity component of the center $Z_0(G)$ is a characteristic compact connected abelian subgroup.

Case 1. $w(G') \leq \aleph$. Then $w(G) = w(Z_0(G))$, and by Corollary 1.3, $Z_0(G)$ contains a connected closed subgroup $N$ of weight $\aleph$; since it is central, it is normal.

Case 2. $w(G') > \aleph$. If we find a compact connected normal subgroup $N$ of $G'$, we are done, since the normalizer of $N$ contains both $G'$ and the central subgroup $Z_0(G)$, hence all of $G = G' Z_0(G)$. Thus it is no loss of generality to assume that $G = G'$ is a compact connected semisimple group.

Case 2a. $G = \prod_{j \in J} G_j$ for a family of compact connected (simple) Lie groups. Then $w(G_j) = \aleph_0$, and $\aleph < w(G) = \max\{\aleph_0, \text{card } J\}$ (see e.g. [8], EA4.3.). Since $\aleph$ is infinite and smaller than $w(G)$, we have $w(G) = \text{card } J$. Then we find a subset $I \subseteq J$ such that $\text{card } I = \aleph$, and set $N = \prod_{i \in I} G_i$. Then $w(N) = \text{card } N = \aleph$.

Case 2b. By the Sandwich Theorem for Semisimple Compact Connected Groups ([8], 9.20) there is a family of simply connected compact simple Lie groups $S_j$ with center $Z(S_j)$ and there are surjective morphisms

$$
\prod_{j \in J} S_j \xrightarrow{f} G \xrightarrow{q} \prod_{j \in J} S_j/Z(S_j)
$$

such that $q f$ is the product $\prod_{j \in J} p_j$ of the quotient morphisms $p_j: S_j \to S_j/Z(S_j)$.

Now both products $\prod_{j \in J} S_j$ and $\prod_{j \in J} S_j/Z(S_j)$ have the same weight $\text{card } J$.
which agrees with the weight of the sandwiched group $G$. Define $I$ as in Case 2a and set $N = f(\prod_{i \in I} S_i)$ and note that $q(N) = \prod_{i \in I} S_i/Z(S_i)$. Hence $N$ is sandwiched between two products with weight $\text{card } I = \aleph$ and hence has weight $\aleph$. This proves the existence of the asserted $N$ in the last case. $\square$

\section{The generating degree}

Now we have to reach beyond connectivity, all the while still staying within the class of compact groups.

We refer to a cardinal invariant for compact groups $G$ which is one of several alternatives to the weight $w(G)$, namely, the so called \textit{generating degree} $s(G)$ (see [8], Definition 12.15). The definition relies on the Suitable Set Theorem, loc. cit. Theorem 12.11, which in turn invokes the so called Countable Layer Theorem (see [7] or [8], Theorem 9.91). Indeed recall that in a compact group $G$ a subset $S$ is called \textit{suitable} iff it does not contain 1, is closed and discrete in $G \setminus \{1\}$, and satisfies $G = \langle S \rangle$. The Suitable Set Theorem asserts, that every compact group $G$ has a suitable set. A suitable set is called \textit{special} iff its cardinality is minimal among all suitable subsets of $G$. The cardinality $s(G)$ of one, hence every special suitable set is called the \textit{generating degree} of $G$.

The relevance of the generating degree in our context is the following

\textbf{Proposition 3.1.} \textit{Let $G$ be a profinite group with uncountable weight. Then $w(G) = s(G)$.}

\textbf{Proof.} By Proposition 12.28 of [8], for an infinite profinite, that is, compact totally disconnected group we have

$$w(G) = \max\{\aleph_0, s(G)\}.$$ 

This implies the assertion immediately in the case of $w(G) > \aleph_0$. \hfill $\square$

The next step, namely, proving that every profinite group is in $C$ will be facilitated by a lemma on suitable sets for which all ingredients are contained in [8].

\textbf{Lemma 3.2.} \textit{(a) Let $S$ be any suitable set of a compact group $G$. Then $\text{card } S \leq w(G)$.}

\textit{(b) If $G$ is profinite and $S$ is an infinite suitable subset of $G$, then $\text{card } S = w(G)$.}

\textbf{Proof.} (a) Let $B$ be a basis of the topology of $H$ of cardinality $w(H)$. Since $S$ is discrete in $G \setminus \{1\}$, for every element $x \in S$ there is an element $U(x) \in B$
with $U(x) \cap X = \{x\}$. Then $x \mapsto U(x): S \to \mathcal{B}$ is an injective function and thus $\text{card } S \leq \text{card } \mathcal{B} = w(H)$. (Cf. [8], p. 620, proof of 12.16.)

(b) Assume that $G$ is profinite and $w(G)$ is uncountable. Then $w(G) = s(G) \leq \text{card } S$ by Proposition 3.1 and the definition of $s(G)$. From this and (a), $w(G) = \text{card } S$ follows. Now assume that $S$ is infinite. Then either $S$ is uncountable or $\text{card } S = \aleph_0$. In the first case, $w(G)$ is uncountable by (a) and $w(G) = \text{card } S$ holds. If, on the other hand, $\text{card } S = \aleph_0$ and $w(X) = \aleph_0$, then $\text{card } S = w(X)$ as well. \qed

The significance of this lemma is that

for a profinite group, infinite suitable subsets all have the same cardinality, namely, the weight of the group.

It is instructive to take note of the following remarks which are pertinent to this context:

**Remark a.** The universal monothetic and the universal solenoidal compact groups $G$ have generating degree $s(G) = 1$, density $d(G) = \aleph_0$ (i.e. they are separable), and weight $w(G) = 2^{\aleph_0}$. (For the concept of density see e.g. [8], p. 620, Definition 12.15.)

**Remark b.** If $H$ is a precompact group whose Weil completion $G$ is profinite and if $H$ has an infinite relatively compact suitable subset, then $\text{card } H \geq w(H)$.

**Proof.** Assume that $S$ is an infinite relatively compact suitable subset of $H$. Then by [8], p. 616, Lemma 12.4, $S$ is a suitable subset of $G$ and by Lemma 3.2 it follows that $\text{card } H \geq \text{card } S = w(G) = w(H)$. \qed

**Remark c.** (i) For every compact group $G$ and infinite cardinal number $\aleph$, the inequalities $\aleph < w(G) \leq 2^{\aleph}$ imply $d(G) < w(G)$.

(ii) Let $G$ be a compact group of weight $\aleph_1$. Then $G$ contains countable dense subgroups. If $G$ is profinite, then none of these contains an infinite relatively compact suitable set.

**Proof.** (i) This follows from the equation $d(G) = \log w(G)$ valid for any compact group $G$, see [1].

(ii) Let $G$ be a compact group of weight $\aleph_1$. Then by (i) it has a countable dense subgroup $H$. Suppose that $G$ is profinite and $H$ has an infinite relatively compact suitable subset $S$. Then by Remark b we would have $\aleph_0 = \text{card } H \geq w(H) = w(G) = \aleph_1$, a contradiction. \qed
Now we show that every profinite group of uncountable weight is in $\mathcal{C}$.

**Lemma 3.3.** Let $G$ be a profinite group of uncountable weight and let $\aleph < w(G)$ be an infinite cardinal. Then there is a closed subgroup $H$ such that $w(H) = \aleph$.

**Proof.** Let $T$ be a suitable subset of $G$ with $\text{card } T = w(G)$ according to Proposition 3.1. Then $T$ contains a subset $S$ of cardinality $\aleph$. We set $H = \langle S \rangle$. Now $S$ is discrete in $H \setminus \{1\}$ since $T$ is discrete in $G \setminus \{1\}$. Hence $S$ is an infinite suitable subset of the profinite group $H$. Hence, by Lemma 3.1(b), $w(H) = \text{card}(S) = \aleph$ follows. \hfill $\square$

**Corollary 3.4.** An infinite profinite group contains an infinite compact metric subgroup.

**Proof.** Let $G$ be an infinite metric group. If $w(G) = \aleph_0$ then $G$ itself is metric. If $G$ has uncountable weight, then we apply Lemma 3.3 with $\aleph = \aleph_0$. \hfill $\square$

Now we are ready to prove that every compact group is in $\mathcal{C}$ which is the main portion of the following result:

**Theorem 3.5.** Let $\aleph$ be an infinite cardinal and $G$ a compact group such that $\aleph < w(G)$. Then there is a closed subgroup $H$ such that $w(H) = \aleph$, and if $G$ is connected, $H$ may be chosen normal and connected.

**Proof.** Let $G_0$ denote the identity component of $G$. The case that $w(G) = w(G_0)$ is handled in Proposition 2.2. So we assume $w(G_0) < w(G)$. By Lemma 2.1 we may assume that $G$ is totally disconnected, that is, profinite. Then Lemma 3.3 proves the assertion of the theorem. \hfill $\square$

In particular, we have the following conclusion:

**Corollary 3.6.** Every infinite compact group contains an infinite closed metric subgroup. \hfill $\square$

Recall that by Corollary 1.7 we know that we find even a closed abelian metric subgroup. For a compact connected group $G$, Corollary 1.4 shows that for any infinite cardinal $\aleph \leq w(G)$ there is in fact a closed connected abelian subgroup of weight $\aleph$.

**Problem.** If $G$ is a compact group and $\aleph$ is an infinite cardinal $\leq w(G)$. Is there a normal closed subgroup $H$ such that $w(H) = \aleph$?
The answer is affirmative if it is affirmative for profinite groups. A profinite group has weight $\aleph_1$ iff its set of open-closed normal subgroups has cardinality $\aleph_1$. This observation points into the direction of an affirmative answer to the problem.

4. The weights of closed subgroups of a locally compact group

Now we finish the proof of the Main Theorem by showing that every locally compact group is in $C$. A first step is the following:

**Lemma 4.1.** Every locally compact pro-Lie group belongs to $C$.

**Proof.** Let $N$ be a compact normal subgroup of $G$ such that $G/N$ is a Lie group. Then $w(G/N) = \aleph_0$. We conclude $w(G) = w(N)$. We apply Theorem A to $N$ and obtain a closed subgroup of $N$ of weight $\aleph_1$. $\square$

In Montgomery’s and Zippin’s classic [10] we note Lemma 2.3.1 on p. 54 and the Theorem on p. 175 and thus find

**Lemma 4.2.** Every locally compact group contains an open pro-Lie group. $\square$

**Lemma 4.3.** Let $H$ be an infinite open subgroup of a topological Hausdorff group $G$. Let $\mathcal{X}$ be a set of cosets $Hg$ of $G$ modulo $H$. Then $w(\langle \bigcup \mathcal{X} \rangle) = \max\{w(H), \text{card}(\mathcal{X})\}$.

**Proof.** Abbreviate $\langle \bigcup \mathcal{X} \rangle$ by $K$. Then $w(H) \leq w(K)$ and $\text{card}(\mathcal{X}) \leq w(K)$ and thus

$$\max\{\text{card}(\mathcal{X}), w(H)\} \leq w(K).$$

In order to prove the reverse inequality, let $D$ be a dense subset of $H$ of cardinality $d(H)$. For each finite tuple $\mathcal{F} = (Hg_1, Hg_2, \ldots, Hg_n) \in \mathcal{X}^n$, $n \in \mathbb{N}$, the set $P_{\mathcal{F}} \overset{\text{def}}{=} Hg_1Hg_2\cdots Hg_n$ has a dense subset $\Delta_{\mathcal{F}} \overset{\text{def}}{=} Dg_1Dg_2\cdots Dg_n$ of cardinality $d(H)$ with the density of $H$. Then $K = \bigcup_{\mathcal{F}} P_{\mathcal{F}}$, with the union extended over the set of finite $n$-tuples $\mathcal{F}$, has a dense subset $\Delta = \bigcup_{\mathcal{F}} \Delta_{\mathcal{F}}$ whose cardinality is $\leq \text{card} \cdot d(H) \leq \max\{\text{card}(\mathcal{X}), w(H)\}$ (see [8], p. 620, Proposition 6.20 and its proof). It follows that

$$w(K) \leq \text{card}(\Delta)\cdot w(H) \leq \max\{\text{card}(\mathcal{X}), w(H)\}.$$ 

Now (1) and (2) together prove the assertion.$\square$
The following conclusion now completes the proof of the Main Theorem:

**Corollary 4.4.** Every locally compact group belongs to $C$.

**Proof.** If $\aleph = w(G) = \aleph_0$ there is nothing to prove. Assume $\aleph_0 \leq \aleph < w(G)$; we have to find a closed subgroup such that $w(K) = \aleph$. By Lemma 4.2, let $H$ be an open pro-Lie subgroup of $G$. Now $w(G) = \max\{w(H), \text{card}(G/H)\}$. If $w(G) = w(H)$, we find $K$ by Lemma 4.1. If $w(G) = \text{card}(G/H)$ then we pick a subset $X$ of $G/H$ of cardinality $\aleph$. Then $K = \langle \bigcup X \rangle$ has weight $\text{card} X = \aleph$ by Lemma 4.3. \qed

5. An application of the Main Theorem

Let $(G, \tau)$ be an arbitrary topological abelian group. We shall continue to write abelian groups additively, unless specified otherwise such as in the case of the multiplicative circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. A character on $(G, \tau)$ is a continuous morphism $\chi: G \to T = \mathbb{R}/\mathbb{Z}$. The pointwise sum of two characters is again a character, and the set $\hat{G}$ of all characters is a group with pointwise multiplication as the composition law. If $\hat{G}$ is equipped with the compact open topology $\hat{\tau}$, it becomes a topological group $(\hat{G}, \hat{\tau})$ which is called the dual group of $(G, \tau)$.

**Definition 5.1** (Varopoulos) Let $G$ and $H$ be two abelian groups then we say that $G$ and $H$ are in duality if and only if there is a $\mathbb{Z}$-bilinear function

$$\langle \cdot, \cdot \rangle: G \times H \to T = \mathbb{R}/\mathbb{Z}$$

such that

$$(\forall 0 \neq g \in G)(\exists h \in H) \langle g, h \rangle \neq 0_{\hat{T}}$$

and

$$(\forall 0 \neq h \in H)(\exists g \in G) \langle g, h \rangle \neq 0_{\hat{T}}.$$

**Definition 5.2** (Varopoulos) Assume that $G$ and $H$ are in duality. Then a topology $\tau$ on $G$ is compatible with the duality if $(G, \tau)^{\sim} = H$.

Typically, if $G$ is a locally compact abelian group and $\hat{G}$ its Pontryagin dual, that is, its character group, then $G$ and $\hat{G}$ are in duality, but there are group topologies $\tau$ on $G$ which are in general coarser than the given locally compact topology on $G$ but which are nevertheless compatible with this duality. One of the best known is the so-called Bohr topology $\tau^+$ on $G$ which we discuss in the next example for the sake of completeness and because it plays a role in our subsequent discussion.
which, at least in the case that $G$ is discrete, provides a substantial cardinality of topologies on $G$ which are compatible with the Pontryagin duality of $G$.

**Example 5.3.** Let $G = (A, \tau)$ be a locally compact abelian group and let $\eta: G \to G^\alpha$ be the Bohr compactification morphism and let $\tau^+$ be the pull-back topology on $A$ (that is, the topology which makes $\eta$ an embedding $\epsilon: G^+ \to G^\alpha$). Equivalently, $\tau^+$ is the topology of pointwise convergence when $A$ is considered as the space of characters of $\hat{G}$ via Pontryagin duality. We write $G^+$ for the topological group $(A, \tau^+)$. Then

(i) If $(a_n)_{n \in \mathbb{N}}$ is a sequence of $A$, and $a \in A$, then $a = \tau$-lim$_n a_n$ iff $a = \tau^+$-lim$_n a_n$. If $\tau$ is the discrete topology, then $(a_n)_{n \in \mathbb{N}}$ converges w.r.t. $\tau^+$ iff it is eventually constant.

(ii) $\tau^+$ is compatible with the duality between $A$ and $\hat{G}$, that is, $\hat{G}^+ = \hat{G}$. Also, $\hat{G}^+ = (G^\alpha)^\wedge$.

(iii) If $\sigma$ is any group topology on $A$ such that $\tau^+ \subseteq \sigma \subseteq \tau$, then $\sigma$ is compatible with the duality between $A$ and $\hat{G}$.

(iv) $w(G) = w(\hat{G})$, $w(G^+) = w(G^\alpha) = \text{card}(\hat{G})$.

(v) If $G$ has no nonsingleton compact subgroups, such as $G = \mathbb{Z}$ or $G = \mathbb{R}$, then $G^+$ has no nonsingleton metric subgroups.

**Proof.** For easy reference we provide proofs.

(i) The fact that $G$ and $G^+$ have the same converging sequences is based on two implications of which “$a = \tau$-lim$_n a_n$ implies $a = \tau^+$-lim$_n a_n$” is immediate from the continuity of $\eta$ and the definition of $\tau^+$. The other implication has fascinated several authors (see e.g. [3], [4], [9], [11]); the following argument was credited by Reid to originate from Varopoulos in the 1960s but may have been around before: Let $a = \tau^+$-lim$_n a_n$. We consider $G$ as the character group of $\hat{G}$. For any $f \in L^1(\hat{G})$ we calculate the value of the Fourier transform $\hat{f} \in C_0(G)$ at $a_n$ as

$$\hat{f}(a_n) = \int_{\hat{G}} f(\chi) \exp(-i\langle\chi, a_n\rangle) d\chi$$

By the definition of $\tau^+$ as topology of pointwise convergence, the sequence of functions $f(\bullet) \exp(-i\langle\bullet, a_n\rangle)$ is dominated by $|f|$ since the absolute value of the exponential is 1, and it converges pointwise to $f(\bullet) \exp(-i\langle\bullet, a\rangle)$. Hence by the Lebesgue Dominated Convergence Theorem we have

$$\lim_n \hat{f}(a_n) = \hat{f}(a).$$
Since the algebra of Fourier transforms \( \hat{f} \) is uniformly dense in \( C_0(G) \), this implies
\[ a = \tau-\lim_n a_n. \]

(ii) Let \( \chi: G^+ \to \mathbb{T} \) be a (continuous) character. Then there is a unique character \( \overline{\chi}: G^\alpha \to \mathbb{T} \) such that \( \overline{\chi} \circ \eta = \chi \) by the definition of \( \tau^+ \). However, if \( \phi: G \to \mathbb{T} \) is a character, then there is a unique character \( \phi^\alpha: G^\alpha \to \mathbb{T} \) such that \( \phi^\alpha \circ \eta = \phi \) by the universal property of the Bohr compactification. Hence there is a bijective correspondence \( \beta: \hat{G} \to (G^\alpha)^\sim \). Then \( \chi \mapsto \beta^{-1}(\overline{\chi}) : \hat{G} \to \hat{G} \) is a bijection such that \( \beta^{-1}(\overline{\chi}) \circ \eta = \chi \). The following commutative diagram may help
\[
\begin{array}{c}
G^+ \\
\downarrow \text{id}_A \\
G \\
\downarrow \eta \\
G^\alpha \\
\uparrow \text{id}_{G^\alpha}
\end{array}
\]

Thus a character \( A \to \mathbb{T} \) is \( \tau^+ \)-continuous iff it is \( \tau \)-continuous.

(iii) Every character \( \chi \in \hat{G}, \chi: G = (A, \tau) \to \mathbb{T} \) is \( \tau^+ \)-continuous by (ii). Then it is \( \sigma \)-continuous since \( \tau^+ \subseteq \sigma \). Conversely, let \( \chi: (A, \sigma) \to \mathbb{T} \) be a \( \sigma \)-continuous character, then it is \( \tau \)-continuous, since \( \sigma \subseteq \tau \). Hence a character \( A \to \mathbb{T} \) is \( \sigma \)-continuous iff it is \( \tau \) continuous.

(iv) For the equalities \( w(G) = w(\hat{G}) \) and \( w(G^\alpha) = \text{card}(G^\alpha)^\sim \) see e.g. [8], Theorem 7.76(i) and (ii), p. 364. Since \( G^+ \) has a dense homeomorphic image in \( G^\alpha \) we have \( w(G^+) = w(G^\alpha) \). Now \( (G^\alpha)^\sim = \hat{G} \) by (ii) above. This proves the assertion.

(v) Let \( H^+ \) be a first countable nonsingleton subgroup of \( G^+ \). Then \( H^+ \) is precompact since \( G^+ \) is precompact, and its topology is determined by its convergent sequences since it is first countable. On the other hand, by (i) above, it has the same convergent sequences as the subgroup \( H \) which has the same underlying group as \( H^+ \) but whose topology is the one induced by the topology of \( G \). Hence \( H^+ = H \) as topological groups. Since \( H^+ \) is precompact, the locally compact group \( \overline{H} \) is precompact and thus is compact. Now assume that \( G \) is a locally compact abelian group without compact nondegenerate subgroups, say, \( G = \mathbb{Z} \) or \( G = \mathbb{R} \). In those cases \( G^+ \) cannot have any nonsingleton metric subgroup.

By its very definition, the Bohr topology \( \tau^+ \) is a precompact topology. Indeed, a topological group \( G \) is said to be precompact or totally bounded if for any neighborhood \( U \) of the neutral element in \( G \), there is a subset \( F \subseteq G \) with \( \text{card}(F) < \aleph_0 \) such that \( FU = G \). Next we need to generalize the concept of total boundedness:

**Definition 5.4** Let \( \aleph \) be a cardinal number. A topological group \( G \) is said to be \( \aleph \)-bounded when for every neighborhood \( U \) of the neutral element in \( G \), there is a subset \( S \subseteq G \) with \( \text{card}(S) < \aleph \) such that \( SU = G \). 

\[ \square \]
According to this definition, a group $G$ is totally bounded iff it is $\aleph_0$-bounded. It is uniformly Lindelöf if it is $\aleph_1$-bounded.

For a topological abelian group $\Gamma$ and a cardinal $\aleph$ we let $\mathcal{K}_\aleph(\Gamma)$ denote the set of all compact subsets $K \subseteq \Gamma$ with $w(K) < \aleph$.

**Lemma 5.5.** Let $G$ be the character group of an abelian topological group $\Gamma$. Let $\aleph$ be a cardinal $\leq w(\Gamma)$, and let $G$ have the topology of uniform convergence on compact subsets $K \in \mathcal{K}_\aleph(\Gamma)$. Then $G$ is $\aleph$-bounded.

**Proof.** From Theorem 3.4 in [2] we have the following information due to Ferrer and Hernández who deal with the following set-up:

Let $X$ be a set, let $M$ be a metrizable space, and let $Y$ be a subset of $M^X$ that is equipped with some bornology $B$ consisting of pointwise relatively compact sets. Denote by $\mu_B$ the uniformity on $X$ defined as $\sup\{\mu_F : F \in B\}$. The concept of an $\aleph$-bounded topological group generalizes rather immediately to that of an $\aleph$-bounded uniform space. Now we have

**The $\aleph$-Boundedness Theorem.** If $\aleph$ is a cardinal such that $w(M) < \aleph$, then the following statements are equivalent:

(i) $(\forall F \in B) \ w(F) < \aleph$.

(ii) $(X, \mu_B)$ is $\aleph$-bounded.

Now we apply this result with $\Gamma = X$, $\mathbb{T} = M$, $G = Y$, $\mathcal{K}_\aleph(\Gamma) = B$, and, finally, with the topology of uniform convergence on sets of $\mathcal{K}_\aleph(\Gamma)$ being the uniform topology of $\mu_B$. We have $w(M) = w(\mathbb{T}) = \aleph_0 \leq \aleph$. Hence the implication (i) $\Rightarrow$ (ii) of the $\aleph$-Boundedness Theorem yields the assertion of the Lemma.

Let $G$ be a topological group, the local weight (or character) $w_0(G)$ is the smallest among the cardinals of neighbourhood bases at the neutral element. We say that a collection $\{K_i : i \in I\}$ is a compact cover of $G$ when $\bigcup_{i \in I} K_i = G$ and $K_i$ is compact for all $i \in I$. The compact covering number $\kappa(G)$ of $G$ is defined as the smallest of the cardinals of the members of the set of compact covers of $G$.

If $G$ is any locally compact group, let $H$ be an almost connected open subgroup and let $C$ be a maximal compact subgroup of $H$. Then $G$ is homeomorphic to $\mathbb{R}^n \times C \times G/H$ and $w_0(G) = \max\{\aleph_0, w(C)\}$ and $\kappa(G) = \text{card} \ G/H$. If $G$ is abelian and $\Gamma = \hat{G}$, then $w_0(\Gamma) = \kappa(G)$ and $\kappa(\Gamma) = w_0(G)$. In this sense, $w_0$ and $\kappa$ are "dual" cardinals.

In view of our Main Theorem we may summarize:

**The Local Weight Lemma for Locally Compact Groups.** For a locally compact nondiscrete group $G$ select any almost connected open subgroup $H$ and any maximal compact subgroup $C$ of $H$. Then
(1) \( w_0(G) = \max\{\aleph_0, w(C)\} \),
(2) \( w(G) = \max\{w_0(G), \text{card}(G/H)\} \).
(3) If \( \aleph_0 \leq \aleph \leq w_0(G) \), then either \( G \) contains no infinite compact subgroups or else there is a compact subgroup of weight \( \aleph \). 

Consider now a locally compact abelian group \( G \) with dual group \( \Gamma \). If \( w_0(\Gamma) \) is uncountable, then \( \Gamma \) contains a compact subgroup \( C_\Gamma \) with weight \( w(C_\Gamma) = w_0(\Gamma) \). Now let \( \aleph \leq \kappa(G) = w_0(\Gamma) \) be any infinite cardinal. Consider \( G \) as the character group of the locally compact group \( \Gamma \).

Write \( \tau_\aleph \) for the topology of uniform convergence on the sets \( K \in \mathcal{K}_\aleph(\Gamma) \). We note that by definition,

all topological abelian groups \( (G, \tau_\aleph), 0 \leq \alpha \in \Omega \) are locally quasiconvex.

Let \( \Omega \) the initial set of ordinals \( 0 < 1 < \cdots \leq \xi \) such that \( \{ \aleph_\alpha : \alpha \in \Omega \} \) is the interval \( [\aleph_0, w_0(\Gamma)] \) of cardinals.1

It is in the proof of the following lemma that we use the principal result of this paper.

Main Lemma 5.6. For \( \alpha < \xi \) in \( \Omega \), the following conclusions hold

(i) \( \mathcal{K}_\aleph \subsetneq \mathcal{K}_{\aleph+1} \),
(ii) \( \tau_\aleph \not\subseteq \tau_{\aleph+1} \).

Proof. (i) By the Main Theorem (indeed by Theorem 3.5), the compact group \( C_\Gamma \) contains a subgroup \( K \) of of weight \( w(K) = \aleph_\alpha \). What is relevant for the proof is the fact that \( \Gamma \) contains a compact subspace \( K \) of weight \( \aleph_\alpha \). Thus \( K \in \mathcal{K}_{\aleph+1} \setminus \mathcal{K}_\alpha \). The inclusion \( \mathcal{K}_{\aleph_\alpha}(\Gamma) \subseteq \mathcal{K}_{\aleph_\alpha+1}(\Gamma) \) is trivial.

(ii) From (i) it follows that \( \tau_{\aleph_\alpha} \subseteq \tau_{\aleph_\alpha+1} \). From Lemma 5.5 we know that \( (G, \tau_{\aleph_\alpha}) \) is \( \aleph_\alpha \)-bounded and that \( (G, \tau_{\aleph_\alpha+1}) \) is \( \aleph_{\alpha+1} \)-bounded. Suppose, by way of contradiction that the topologies \( \tau_{\aleph_\alpha} \) and \( \tau_{\aleph_\alpha+1} \) were equal. Then the implication (ii) \( \Rightarrow \) (i) of the \( \aleph \)-Boundedness Theorem would imply that every compact subset \( K \subseteq \Gamma \) of weight \( w(K) < \aleph_{\alpha+1} \) would have weight \( w(K) < \aleph_\alpha \). This would contradict (i) above, and this contradiction shows that \( \tau_{\aleph_{\alpha+1}} \) is strictly finer than \( \tau_{\aleph_\alpha} \). 

The Main Lemma 5.6 establishes the essential result of this section:

Theorem 5.7. Let \( (G, \tau) \) be a locally compact abelian group which has uncountable compact covering number \( \kappa(G) \). Then

(i) for each cardinal \( \aleph \) with \( \aleph_0 \leq \aleph \leq \kappa(G) \) there is a locally quasiconvex group topology \( \tau_\aleph \) on \( G \) such that \( (G, \tau_\aleph) \hat{=} \hat{G} \), and

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(ii) for \( \aleph_0 \leq \kappa \leq \kappa(G) \) one has \( \tau_\kappa \subsetneq \tau_{\kappa'} \subseteq \tau \).

Thus, if \([\aleph_0, \kappa(G)]\) denotes the full interval of infinite cardinals up to the cardinality of \( G \), then Theorem 4.7 provides card\([\aleph_0, \kappa(G)]\)-many locally quasiconvex group topologies on the abelian group \( G \) all of which are coarser than the original topology of \( G \) and yield the (locally compact abelian) group \( \hat{G} \) as character group. This means that all topologies \( \tau_\kappa \) have the same compact subsets as \( \tau \).

**Corollary 5.8.** Let \( G \) be an uncountable abelian group. Then

(i) for each cardinal \( \aleph \) with \( \aleph_0 \leq \aleph \leq \text{card}(G) \) there is a locally quasiconvex group topology \( \tau_\aleph \) on \( G \) such that \((G, \tau_\aleph)^\sim = \hat{G} \), and

(ii) for \( \aleph_0 \leq \aleph < \aleph' \leq \text{card}(G) \) one has \( \tau_\aleph \subsetneq \tau_{\aleph'} \).

Again, if \([\aleph_0, \text{card}(G)]\) denotes the full interval of infinite cardinals up to the cardinality of \( G \), then Corollary 5.8 provides card\([\aleph_0, \text{card}(G)]\)-many locally quasiconvex group topologies \( \tau \) on the abelian group \( G \) all of which yield the (compact abelian) group \( \hat{G} \) as character group of \((G, \tau)\).

**References**

[1] Comfort, W. W., *Topological Groups*, in: K. Kunen and J. E. Vaughan, Eds., North-Holland, Amsterdam, 1984, Chapter 24, 1143–1263.

[2] Ferrer, Maria V., Hernández, Salvador, *Dual topologies on non-abelian groups*, Submitted to the Proceedings of the conference “Algebra meets Topology,” Barcelona, July 2010.

[3] Flor, P., *Zur Bohr-Konvergenz der Folgen*, Math. Scand. 23 (1968), 169–170.

[4] Glicksberg, I., *Uniform boundedness for groups*, Canadian J. Math. 14 (1962), 269–276.

[5] Hernández, Salvador, *Questions raised at the conference “Algebra meets Topology,”* Barcelona, July 2010. hernande@mat.uji.es

[6] Hewitt, E., and K. A. Ross, “Abstract Harmonic Analysis I,” Grundlehren 115 Springer Verlag, Berlin etc., 1963.

[7] Hofmann, K. H., and S. A. Morris, *A structure theorem on compact groups*, Math. Proc. Camb. Phil. Soc. 130 (2001), 409–426.

[8] —, “The Structure of Compact Groups”, de Gruyter, Berlin, 2nd Edition 2006, xvii+858 pp.

[9] Leptin, H., *Abelsche Gruppen mit kompakten Charaktergruppen und Dualitätstheorie gewisser linear topologischer abelscher Gruppen*, Abh. Math. Sem. Univ. Hamburg 19 (1955), 244-263.
[10] Montgomery, D., and Zippin, L. “Topological transformation groups,” Robert E. Krieger Publishing Co., 1955. xi+282 pp. 2nd edition 1966, xi+289 pp.

[11] Reid, G. A., *On sequential convergence in groups*, Math. Z. **102** (1967), 225–235.

[12] Ribes, L., and P. Zaleskii, “Profinite Groups,” Springer-Verlag, Berlin etc., 2000, xiv+435 pp. 2nd edition 2010, xvi+464 pp.

[13] Scott, W. R., *The number of subgroups of given index in a nonenumerable group*, Proc. Amer. Math. Soc. **5** (1954), 19–22.

[14] Walker, E. A., *Subdirect sums and infinite abelian groups*, Pac. J. Math. **9** (1959), 287–291.

[15] Zelmanov, E. I., *On periodic compact groups*, Israel J. of Math. **77** (1992), 83–95.

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