The distribution of Pearson residuals in generalized linear models

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Abstract

In general, the distribution of residuals cannot be obtained explicitly. We give an asymptotic formula for the density of Pearson residuals in continuous generalized linear models corrected to order $n^{-1}$, where $n$ is the sample size. We define corrected Pearson residuals for these models that, to this order of approximation, have exactly the same distribution of the true Pearson residuals. Applications for important generalized linear models are provided and simulation results for a gamma model illustrate the usefulness of the corrected Pearson residuals.

Keywords: Exponential family; Generalized linear model; Pearson residual; Precision parameter

1 Introduction

The residuals carry important information concerning the appropriateness of assumptions that underlie statistical models, and thereby play an important role in checking model adequacy. They are used to identify discrepancies between models and data, so it is natural to base residuals on the contributions made by individual observations to measures of model fit. The use of residuals for assessing the adequacy of fitted regression models is nowadays commonplace due to the widespread availability of statistical software, many of which are capable of displaying residuals and diagnostic plots, at least for the more commonly used models. Beyond special models, relatively little is known about asymptotic properties of residuals in general regression models. There is a clear need to study second-order asymptotic properties of appropriate residuals to be used for diagnostic purposes in nonlinear regression models.

The unified theory of generalized linear models (GLMs), including a general algorithm for computing the maximum likelihood estimates (MLEs) is extremely important for
Analysis of real data. In these models, the random variables \( Y_1, \ldots, Y_n \) are assumed independent and each \( Y_i \) has a density function in the linear exponential family

\[
\pi(y; \theta_i, \phi) = \exp[\phi\{y\theta_i - b(\theta_i)\} + c(y, \phi)],
\]

where \( b(\cdot) \) and \( c(\cdot, \cdot) \) are known appropriate functions. We assume \( Y \) continuous and \( \pi \) a probability density function with respect to Lebesgue measure and that the precision parameter \( \phi = \sigma^{-2} \), \( \sigma^2 \) is the so-called dispersion parameter, is the same for all observations, although possibly unknown. We do not consider the discrete distributions in the form \( [1] \), such as Poisson, binomial and negative binomial. For two-parameter full exponential family distributions with canonical parameters \( \phi \) and \( \phi \theta \), the decomposition \( c(y, \phi) = \phi a(y) + d_1(y) + d_2(\phi) \) holds. The mean and variance of \( Y_i \) are, respectively, \( E(Y_i) = \mu_i = db(\theta_i)/d\theta_i \) and \( \text{Var}(Y_i) = \phi^{-1}V_i \), where \( V = d\mu/d\theta \) is the variance function. For gamma models, the dispersion parameter \( \sigma^2 \) is the reciprocal of the index, whereas for normal and inverse Gaussian models, \( \sigma^2 \) is the variance and \( \text{Var}(Y_i)/E(Y_i)^3 \), respectively. The parameter \( \theta = \int V^{-1}d\mu = q(\mu) \) is a known one-to-one function of \( \mu \). A linear exponential family is characterized by its variance function, which plays a key role in estimation.

A GLM is defined by the family of distributions \([1]\) and the systematic component \( g(\mu) = \eta = X\beta \), where \( g(\cdot) \) is a known one-to-one continuously twice-differentiable function, \( X \) is a specified \( n \times p \) model matrix of full rank \( p < n \) and \( \beta = (\beta_1, \ldots, \beta_p)^T \) is a set of unknown linear parameters to be estimated. Let \( \hat{\beta} \) be the MLE of \( \beta \).

Residuals in GLMs were first discussed by Pregibon (1981), though ostensibly concerned with logistic regression models, Williams (1984, 1987) and Pierce and Schafer (1986), McCullagh and Nelder (1989) provided a survey of GLMs with substantial attention to definition of residuals. Pearson residuals are the most commonly used measures of overall fit for GLMs and are defined by \( R_i = (Y_i - \hat{\mu}_i)/\hat{V}_i^{1/2} \), where \( \hat{\mu}_i \) and \( \hat{V}_i \) are respectively the fitted mean and fitted variance function of \( Y_i \). In this paper we consider only Pearson residuals appropriate to our particular asymptotic aims when the sample size \( n \to \infty \). Cordeiro (2004) obtained matrix formulae for the expectations, variances and covariances of these residuals and defined adjusted Pearson residuals having zero mean and unit variance to order \( n^{-1} \). Pearson residuals defined by Cordeiro (2004) are proportional to \( \sqrt{\phi} \), although we are considering here \( R_i \) as usual without the precision parameter \( \phi \). While Cordeiro’s adjusted Pearson residuals do correct the residuals for equal mean and variance, the distribution of these residuals is not equal to the distribution of the true Pearson residuals to order \( n^{-1} \).

Further, Cordeiro and Paula (1989) introduced the class of exponential family nonlinear models (EFNLMs) which extend the GLMs. Later, Wei (1998) gave a comprehensive introduction to these models. Recently, Simas and Cordeiro (2008) generalized Cordeiro’s (2004) results by obtaining matrix formulae of the \( O(n^{-1}) \) expectations, variances and covariances of Pearson residuals in EFNLMs.

In a general setup, the distribution of residuals usually differ from the distribution of the true residuals by terms of order \( n^{-1} \). Cox and Snell (1968) discussed a general definition of residuals, applicable to a wide range of models, and obtained useful expressions to this order for their first two moments. Loynes (1969) derived, under some regularity conditions, and again to order \( n^{-1} \), the asymptotic expansion for the density function of
Cox and Snell’s residuals, and then defined corrected residuals having the same distribution as the random variables which they are effectively estimating. In all but the simplest situations, the use of the results by Cox and Snell and Loynes will require a considerable amount of tedious algebra. Our chief goal is to obtain an explicit formula for the density of Pearson residuals to order \( n^{-1} \) which holds for all continuous GLMs.

In Section 2 we give a summary of key results from Loynes (1969) applied to Pearson residuals in GLMs. The density of Pearson residuals in these models corrected to order \( n^{-1} \) is presented in Section 3. We provide in Section 4 applications to some common models. In Section 5 we compare the corrected residuals with the adjusted residuals proposed by Cordeiro (2004). We present in Section 6 simulation studies to assess the adequacy of the approximations for a gamma model with log link. Some concluding remarks are given in Section 7. Finally, in the Appendix, we give a more rigorous proof of the general results discussed by Loynes (1969).

## 2 Conditional moments of Pearson residuals

The \( i \)th contribution for the score function from the observation \( Y_i \) follows from (1)

\[
U_r^{(i)} = \frac{\partial l_i}{\partial \beta_r} = \phi V_i^{-1/2} w_i^{1/2} (Y_i - \mu_i) x_{ir},
\]

where \( w = V^{-1/2} \mu_i^2 \) is the weight function and from now on the dashes indicate derivatives with respect to \( \eta \). Let \( \varepsilon_i = V_i^{-1/2} (Y_i - \mu_i) \) be the true Pearson residual corresponding to the Pearson residual \( R_i = V_i^{1/2} (Y_i - \hat{\mu}_i) \). Suppose we write the Pearson residual as \( R_i = \varepsilon_i + \delta_i \). We can write the following conditional moments given \( \varepsilon_i = x \) to order \( n^{-1} \) (Loynes, 1969)

\[
\text{Cov}(\hat{\beta}_r, \hat{\beta}_s | \varepsilon_i = x) = -\kappa^{rs},
\]

\[
\phi_s^{(i)}(x) = E(\hat{\beta}_s - \beta_s | \varepsilon_i = x) = B(\hat{\beta}_s) - \sum_{r=1}^p \kappa^{sr} U_r^{(i)}(x),
\]

where \(-\kappa^{sr}\) is the \((s, r)\)th element of the inverse information matrix \( K^{-1} \) for \( \beta \), \( B(\hat{\beta}_s) \) is the \( \mathcal{O}(n^{-1}) \) bias of \( \hat{\beta}_s \) and \( U_r^{(i)}(x) = E(U_r^{(i)} | \varepsilon_i = x) \) is the conditioned score function. The mean and variance of the asymptotic distribution of \( \delta_i \), given \( \varepsilon_i = x \), are to order \( n^{-1} \)

\[
\theta_x^{(i)} = E(\delta_i | \varepsilon_i = x) = \sum_{r=1}^p H_r^{(i)}(x) b_r^{(i)}(x) - \sum_{r,s=1}^p H_r^{(i)}(x) H_s^{(i)}(x) \kappa^{rs},
\]

\[
\text{Var}(\delta_i | \varepsilon_i = x) = \sum_{r,s=1}^p H_r^{(i)}(x) H_s^{(i)}(x) \kappa^{rs},
\]

where \( H_r^{(i)} = \partial \varepsilon_i / \partial \beta_r \), \( H_r^{(i)} = \partial^2 \varepsilon_i / \partial \beta_r \partial \beta_s \), \( H_r^{(i)}(x) = E(H_r^{(i)} | \varepsilon_i = x) \) and \( H_r^{(i)}(x) = E(H_r^{(i)} | \varepsilon_i = x) \). We obtain by simple differentiation

\[
H_r^{(i)} = \{-V_i^{-1/2} \mu_i' - \frac{1}{2} V_i^{-3/2} V_i^{(1)} \mu_i' (Y_i - \mu_i)\} x_{ir}
\]
and

\[ H^{(i)}_{rs} = \{-V_i^{-1/2} \mu_i'' + V_i^{-3/2} V_i^{(1)} \mu_i' + \frac{3}{4} V_i^{-5/2} V_i^{(1)^2} \mu_i' (Y_i - \mu_i) - \frac{1}{2} V_i^{-3/2} V_i^{(2)} \mu_i^2 (Y_i - \mu_i) - \frac{1}{2} V_i^{-3/2} V_i^{(1)} \mu_i' (Y_i - \mu_i)\} x_{ir} x_{is}. \]

Conditioning on \( \varepsilon_i = x \) leads to \( H^{(i)}_{ri}(x) = e_i(x)x_{ir} \) and \( H^{(i)}_{is}(x) = h_i(x)x_{ir}x_{is} \), where

\[ e_i(x) = -V_i^{-1/2} \mu_i' - \frac{1}{2} V_i^{-1} V_i^{(1)} \mu_i' x \quad (5) \]

and

\[ h_i(x) = -V_i^{-1/2} \mu_i'' + V_i^{-3/2} V_i^{(1)} \mu_i' + \frac{1}{4} \{(3V_i^{-2} V_i^{(1)^2} - 2V_i^{-1} V_i^{(2)}) \mu_i^2 - 2V_i^{-1} V_i^{(1)} \mu_i'\} x. \quad (6) \]

For canonical models \((\theta = \eta)\), \((5)\) and \((6)\) become

\[ e_i(x) = -V_i^{1/2} \text{ and } h_i(x) = \frac{1}{4} (V_i^{(1)^2} - 2V_i V_i^{(2)}) x. \]

Conditioning the score function \( U_r^{(i)} = \phi V_i^{-1/2} w_i^{1/2} (Y_i - \mu_i) x_{ir} \) on \( \varepsilon_i = x \), yields \( U_r^{(i)}(x) = \phi w_i^{1/2} x_{ir} x \), and then using \((2)\) we find

\[ b_s^{(i)}(x) = B(\hat{\beta}_s) + \phi w_i^{1/2} \tau_s^T K^{-1} X^T \gamma_i x, \]

where \( K^{-1} = \phi^{-1} (X^T W X)^{-1} \), \( W = \text{diag}(w_i) \) is the diagonal matrix of weights, \( \tau_s \) is a \( p \)-vector with the \( s \)th element equal to one and all other elements equal to zero and \( \gamma_i \) is an \( n \)-vector with one in the \( i \)th position and zeros elsewhere. Defining \( M = \{m_{si}\} = (X^T W X)^{-1} X^T \), we can easily verify that

\[ b_s^{(i)}(x) = B(\hat{\beta}_s) + w_i^{1/2} m_{si} x. \]

Cordeiro and McCullagh (1991) showed that the \( n^{-1} \) bias of \( \hat{\beta} \) is given by

\[ B(\hat{\beta}) = -(2\phi)^{-1} (X^T W X)^{-1} X^T Z_d F 1, \]

where \( F = \text{diag}\{V_i^{-1} \mu_i' \mu_i''\}, Z = \{z_{ij}\} = X (X^T W X)^{-1} X^T, Z_d = \text{diag}\{z_{ii}\} \) is a diagonal matrix with the diagonal elements of \( Z \) and 1 is an \( n \)-vector of ones. The asymptotic covariance matrix of the MLE \( \hat{\eta} \) of the linear predictor is simply \( \phi^{-1} Z \). We obtain

\[ \sum_{r=1}^n H_r^{(i)}(x) b_r^{(i)}(x) = e_i(x) \left\{ x w_i^{1/2} \sum_{r=1}^n m_{ri} x_{ir} + \sum_{r=1}^n B(\hat{\beta}_r) x_{ir} \right\} = e_i(x) \left\{ w_i^{1/2} z_{ii} x + B(\hat{\eta}) \right\}, \]

where \( B(\hat{\eta}) \) is the \( i \)th element of the \( \mathcal{O}(n^{-1}) \) bias \( B(\hat{\eta}) = -(2\phi)^{-1} ZZ_d F 1 \) of \( \hat{\eta} \). The bias expression depends on the model matrix, the variance function and the first two derivatives of the link function. Also,

\[ -\frac{1}{2} \sum_{r,s=1}^p H_{rs}^{(i)}(x) K^{rs} = -\frac{z_{ii}}{2}\phi h_i(x). \]
The conditional mean \( \theta_x^{(i)} \) from (3) is then a second-degree polynomial in \( x \) given by

\[
\theta_x^{(i)} = \{w_1^{1/2}z_i x + B(\hat{\eta}_i)\}e_i(x) + \frac{z_i}{2\phi} h_i(x),
\]

(7)

where \( e_i(x) \) and \( h_i(x) \) are obtained from (5) and (6).

We now compute the conditional variance \( \phi_x^{(i)} \). From (4) it follows

\[
\phi_x^{(i^2)} = \frac{z_i}{\phi} e_i(x)^2.
\]

Hence, \( \phi_x^{(i^2)} \) is also a second-degree polynomial in \( x \).

3 The density of Pearson residuals

A simple calculation from (11) gives the probability density function (pdf) of the true Pearson residual

\[
f_{e_i}(x) = \sqrt{V_i} \exp[\phi \{\sqrt{V_i} \theta_i x + \mu_i \theta_i - b(\theta_i)\} + c(\sqrt{V_i} x + \mu_i, \phi)],
\]

(9)

where \( \theta = q(\mu) \). Table 1 gives the densities of the true residuals for the normal, gamma and inverse Gaussian distributions, where \( \Gamma(\cdot) \) is the gamma function.

| Distribution      | Density in (1)                      | Density of the true residual \( (f_{e_i}(x)) \) |
|-------------------|-------------------------------------|-------------------------------------------------|
| Normal            | \( \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \) | \( \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right), x \in \mathbb{R} \) |
| Gamma             | \( \frac{(\phi x)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\phi x/\mu) \) | \( \frac{1}{\Gamma(\phi)} \exp\left(-\phi(1+x)\right), x > -1 \) |
| Inverse Gaussian  | \( \frac{1}{\sqrt{2\pi x^2}} \exp\left\{-\frac{\phi(x-\mu)^2}{2\mu^2 x}\right\} \) | \( \frac{\phi}{2\pi(\mu + x+1)^{1/2}} \exp\left(-\frac{\phi x^2}{2(\mu + x+1)}\right), x > \frac{-1}{\sqrt{\phi}} \) |

Throughout the following we assume that the standard regularity conditions of maximum likelihood theory are satisfied. The pdf of the Pearson residual \( R_i \) in continuous GLMs to order \( n^{-1} \) follows from Loynes (1969). See, also, equation (21) in the Appendix. We have

\[
f_{R_i}(x) = f_{e_i}(x) - \frac{d\{f_{e_i}(x)\theta_x^{(i)}\}}{dx} + \frac{1}{2} \frac{d^2\{f_{e_i}(x)\phi_x^{(i^2)}\}}{dx^2},
\]

(10)

where \( f_{e_i}(x), \theta_x^{(i)} \) and \( \phi_x^{(i^2)} \) come from (9), (7) and (8), respectively.

We now define corrected Pearson residuals for these models of the form \( R_i' = R_i + \rho_i(R_i) \), where \( \rho(\cdot) \) is a function of order \( \mathcal{O}(n^{-1}) \) constructed in order to produce the residual \( R_i' \) with the same distribution of \( e_i \) to order \( n^{-1} \). Loynes (1969) showed (see, also, the proof given in the Appendix) that if

\[
\rho_i(x) = -\theta_x^{(i)} + \frac{1}{2f_{e_i}(x)} \frac{d\{f_{e_i}(x)\phi_x^{(i^2)}\}}{dx},
\]

(11)
then \( f_{R'_i}(x) = f_{e_i}(x) \) holds to order \( n^{-1} \), i.e., the corrected residuals \( R'_i \) have the same distribution as the true residuals to this order of approximation. Combining (8) with (9) gives

\[
\frac{1}{2f_{e_i}(x)} \frac{d\{f_{e_i}(x)\phi_x^2\}}{dx} = \frac{z_{ii}}{\phi} e_i(x) \frac{d\phi}{dx} + \frac{z_{ii}}{2\phi} e_i(x)^2 \{ \phi \sqrt{V_i} \theta_i + \frac{d}{dx} c(\sqrt{V_i} x + \mu_i, \phi) \}. \tag{12}
\]

Using (11), (7) and (12), the correction function turns out to be

\[
\rho_i(x) = e_i(x) \left\{ -\frac{1}{2\phi} V_i^{-1} V_i^{(1)} \mu'_i z_{ii} - B(\tilde{\eta}_i) - w_i^{1/2} z_{ii} x \right\} - \frac{z_{ii}}{2\phi} h_i(x) + \frac{z_{ii}}{2\phi} e_i(x)^2 \left\{ \phi \sqrt{V_i} q(\mu_i) + \frac{d}{dx} c(\sqrt{V_i} x + \mu_i, \phi) \right\}. \tag{13}
\]

Direct substitution using (13) yields the corrected Pearson residuals \( R'_i \) for most models. The term \( \phi^{-1} z_{ii} \) in the above equation is just \( \text{Var}(\tilde{\eta}_i) \). Although there are several terms in (13), the correction term is simple to be applied to any continuous model since we need only to calculate \( e_i(x), h_i(x) \) and \( \frac{d}{dx} c(\sqrt{V_i} x + \mu_i, \phi) \) from (5), (6) and (1), the others terms being standard quantities in the theory of GLMs. More generally, the corrected residuals \( R'_i \) depend on the model only through the matrix \( X \), the precision parameter \( \phi \), the function \( c(\cdot, \cdot) \) and the variance and link functions with their first two derivatives.

The density of the true residual for the inverse Gaussian model given in Table 1 depends on the unknown mean \( \mu \). However, we can estimate this density using the general expression for the corrected MLE of \( \mu \), \( \hat{\mu} \), say, given by Cordeiro and McCullagh (1991, formula (4.4)). The resulting estimated density is identical to the true density except by terms of order less than \( n^{-1} \) and the results of Sections 3 and 4 could also be applied to this distribution. To prove this, let \( \hat{\mu} = \mu + c/n^2 \). Then, keeping only terms up to order \( n^{-2} \), we have

\[
\hat{\mu}^{1/2} = \sqrt{\hat{\mu}} \sqrt{1 + \frac{c}{n^2 \mu}} = \sqrt{\mu} \left( 1 + \frac{c}{2n^2 \mu} \right).
\]

Also,

\[
(\hat{\mu}^{1/2} x + 1)^{-3/2} = (\sqrt{\mu} x + 1)^{-3/2} \left\{ 1 - \frac{3 x c}{4n^2 \sqrt{\mu}(\sqrt{\mu} x + 1)} \right\}
\]

and

\[
\exp \left\{ \frac{-\phi x^2}{2(\hat{\mu}^{1/2} x + 1)} \right\} = \exp \left[ \frac{-\phi x^2}{2(\sqrt{\mu} x + 1)} \frac{1}{1 + \frac{4n^2 \sqrt{\mu}(\sqrt{\mu} x + 1)}{x c}} \right].
\]

Then,

\[
\exp \left\{ \frac{-\phi x^2}{2(\hat{\mu}^{1/2} x + 1)} \right\} = \exp \left\{ \frac{-\phi x^2}{2(\sqrt{\mu} x + 1)} \right\} \exp \left\{ \frac{\phi x^3 c}{4n^2 \sqrt{\mu}(\sqrt{\mu} x + 1)^2} \right\}.
\]

Hence,

\[
\sqrt{\phi} \frac{1}{2\pi (\hat{\mu}^{1/2} x + 1)^{3/2}} \exp \left\{ \frac{-\phi x^2}{2(\hat{\mu}^{1/2} x + 1)} \right\} = \sqrt{\frac{\phi}{2\pi}} (\sqrt{\mu} x + 1)^{-3/2} \left( 1 - \frac{c}{n^2 \mu} \right) \exp \left\{ \frac{-\phi x^2}{2(\sqrt{\mu} x + 1)} \right\} \exp \left\{ \frac{c x}{n^2 \mu} \right\}.
\]
where \( c_1 = \frac{3x\varepsilon}{4\sqrt{1 + \sqrt{x}} - 1} \) and \( c_2 = \frac{\phi x^3 \varepsilon}{4\sqrt{1 + \sqrt{x} + 1}} \). From this equation it is clear that the estimated density and the true density of \( \varepsilon \) are in agreement to order \( n^{-1} \).

### 4 Some special models

Formula (13) holds for all continuous GLMs including the models in common use: linear models, canonical models, normal models, gamma models and inverse Gaussian models.

We now compute the correction \( \rho_i(\cdot) \) in (13) for some important GLMs and obtain the corrected residuals \( R'_i = R_i + \rho_i(R_i) \). Table 2 and 3 give the quantities \( \mu', \mu'' \) and \( w \) for some useful link functions and \( q(\mu), V, w \) and \( \frac{d}{dx}c(\sqrt{V} x + \mu, \phi) \) for the normal, gamma and inverse Gaussian distributions, respectively.

#### Table 2: Values of \( \mu', \mu'' \) and \( w \) for some link functions.

| Link function | Formula | \( \mu' \) | \( \mu'' \) | \( w \) |
|---------------|---------|------------|------------|--------|
| Linear        | \( \mu = \eta \) | 1          | 0          | \( V^{-1} \) |
| Log           | \( \log(\mu) = \eta \) | \( \mu \) | \( \mu \) | \( \mu^2 V^{-1} \) |
| Reciprocal    | \( \mu^{-1} = \eta \) | \( -\mu^2 \) | 2\( \mu^3 \) | \( \mu^4 V^{-1} \) |
| Inverse of the square | \( \mu^{-2} = \eta \) | \( -\mu^3/2 \) | \( 3\mu^5/4 \) | \( \mu^6 V^{-1}/4 \) |

#### Table 3: Quantities \( q(\mu), V, w \) and \( \frac{d}{dx}c(\sqrt{V} x + \mu, \phi) \) for some models.

| Model        | \( q(\mu) \) | \( V \) | \( w \) | \( \frac{d}{dx}c(\sqrt{V} x + \mu, \phi) \) |
|--------------|---------------|--------|------|----------------------------------|
| Normal       | \( \mu \)    | 1      | \( \mu^2 \) | \(- (x + \mu) \phi \) |
| Gamma        | \(-1/\mu \)  | \( \mu^2 \) | \( \mu^{-2} \mu^2 \) | \( (\phi - 1)/(1 + x) \) |
| Inverse Gaussian | \(-1/(2\mu^2) \) | \( \mu^3 \) | \( \mu^{-3} \mu^2 \) | \(- \frac{3\mu^{3/2}}{2(\mu^{3/2} x + \mu)} + \frac{\phi \mu^{3/2}}{2(\mu^{3/2} x + \mu)^2} \) |

#### 4.1 Linear models

For linear models, \( \mu_i = \eta_i, \mu_i' = 1, \mu_i'' = 0, w_i = V_i^{-1} \) and \( B(\hat{\eta}_i) = 0 \). Then, \( e_i(x) = -V_i^{-1/2} - \frac{1}{2} V_i^{-1} \sqrt{V_i} \) and \( h_i(x) = V_i^{-3/2} \sqrt{V_i} + \frac{3}{4} V_i^{-2} \sqrt{V_i} x - \frac{1}{2} V_i^{-1} \sqrt{V_i} x \). Hence,

\[
\rho_i(x) = V_i^{-1} z_{ii} \left( 1 - \frac{V_i^{-1} \sqrt{V_i}}{8\phi} + \frac{V_i^{(2)}}{4\phi} + \frac{V_i^{-1/2} \sqrt{V_i}}{2} x \right) + \frac{z_{ii}}{2\phi} \left( V_i^{-1} + V_i^{-3/2} \sqrt{V_i} x + \frac{1}{4} V_i^{-2} (\sqrt{V_i})^2 x^2 \right) \left\{ \phi \sqrt{V_i} q(\mu_i) + \frac{d}{dx}c(\sqrt{V_i} x + \mu_i, \phi) \right\}.
\]
4.2 Canonical models

For canonical models, \( \eta_i = \theta_i \), \( w_i = V_i \), \( \mu_i' = V_i \) and \( \mu_i'' = V_i V_i^{(1)} \). Further, \( e_i(x) = -V_i^{1/2} - \frac{1}{2} V_i^{(1)} x \) and \( h_i(x) = \frac{1}{4} (V_i^{(1)^2} - 2V_i V_i^{(2)}) x \). Hence,

\[
\rho_i(x) = \left( V_i^{1/2} + \frac{V_i^{(1)}}{2} x \right) B(\hat{n}_i) + z_{ii} \left( \frac{V_i^{1/2} V_i^{(1)}}{2\phi} + V_i x + \frac{V_i^{(1)^2}}{8\phi} x + \frac{V_i V_i^{(2)}}{4\phi} x \right) + \frac{V_i^{1/2} V_i^{(1)}}{2} x^2 \\
+ \frac{z_{ii}}{2\phi} \left( V_i + V_i^{1/2} V_i^{(1)} x + \frac{1}{4} V_i^{(1)^2} x^2 \right) \left\{ \phi \sqrt{\frac{V_i V_i^{(1)}}{2}} \left( \frac{d}{dx} c(\sqrt{V_i} x + \mu_i, \phi) \right) \right\}.
\]

4.3 Normal models

For normal models, \( V_i = 1 \), \( w_i = \mu_i^2 \), \( c(x, \phi) = -1/2 \{ x^2 \phi + \log(2\pi/\phi) \} \), \( \frac{d}{dx} c(x + \mu, \phi) = -(x + \mu) \phi \), \( e_i(x) = -\mu_i' \) and \( h_i(x) = -\mu_i'' \). We have

\[
\rho_i(x) = B(\hat{n}_i) \mu_i' + \frac{\mu_i'' z_{ii}}{2\phi} + \frac{\mu_i^2 z_{ii}}{2} x.
\]

The normal linear model for which \( \mu = \theta \), \( e_i(x) = -1 \) and \( h_i(x) = 0 \) yields

\[
\rho_i(x) = z_{ii} x/2,
\]

and the corrected residuals follow as

\[
R_i' = R_i \left( 1 + \frac{z_{ii}}{2} \right).
\]

We can verify that \( \text{Var}(R_i') = 1 + \mathcal{O}(n^{-2}) \). A check of this expression can be obtained by considering the simplest case of independent and identically distributed observations. We have \( Z = n^{-1} 1 1^T, \ z_{ii} = n^{-1} \) and then

\[
R_i' = R_i \left( 1 + \frac{1}{2n} \right),
\]

which is identical to the result given in the example discussed by Loynes (1969).

4.4 Gamma models

For gamma models, \( V_i = \mu_i^2 \), \( w_i = \mu_i^{-2} \mu_i^2 \), \( c(x, \phi) = (\phi - 1) \log(x) + \phi \log(\phi) - \log(\Gamma(\phi)) \) and \( \frac{d}{dx} c(\mu x + \mu, \phi) = (\phi - 1)/(1 + x) \). We have \( e_i(x) = -\mu_i^{-1} \mu_i' - \mu_i^{-1} \mu_i'' x \) and \( h_i(x) = -\mu_i^{-1} \mu_i'' + 2\mu_i^{-2} \mu_i'' - \mu_i^{-1} \mu_i'' x + 2\mu_i^{-2} \mu_i'' x \). Then,

\[
\rho_i(x) = (1 + x) \left( \mu_i^{-1} \mu_i' B(\hat{n}_i) + \frac{\mu_i^{-1} \mu_i''}{2\phi} z_{ii} - \frac{\mu_i^{-2} \mu_i''}{2\phi} z_{ii} + \frac{\mu_i^{-2} \mu_i'' z_{ii}}{2} x \right).
\]
4.5 Inverse Gaussian models

For inverse Gaussian models, \( V_i = \mu_i^3, w_i = \mu_i^{-3} \mu_i'^2, c(x, \phi) = (1/2)\log\{\phi/(2\pi x^3)\} - \phi/(2x) \) and 
\[
\frac{d}{dx}c(\mu^{3/2}x + \mu, \phi) = -\frac{3\mu^{3/2}}{2(\mu^{3/2}x + \mu)^2} + \frac{6\phi x}{2(\mu^{3/2}x + \mu)^2}.
\]
Further, \( e_i(x) = -\mu_i^{-3/2}\mu_i' - \frac{3}{4}\mu_i^{-1} \mu_i'^2 x \) and 
\[
h_i(x) = -\mu_i^{-3/2}\mu_i'^2 + 3\mu_i^{-1} \mu_i'^2 x - \frac{3}{2}\mu_i^{-1} \mu_i'^2 x. \]
Then,
\[
\rho_i(x) = \left( \mu_i^{-3/2}\mu_i' + \frac{3\mu_i'^2}{2\mu_i} \right) B(\hat{\eta}_i) + \frac{3\mu_i'^2}{2\mu_i} \frac{\mu_i'^2 z_i^2}{2\phi} + \frac{3\mu_i'^2}{2\mu_i} \mu_i'^2 + \frac{3\mu_i'^2}{2\mu_i} x^2
\]
\[
+ \frac{\mu_i'^2}{2\phi} \left( \frac{9\mu_i'^2}{4} - \frac{\phi}{(\mu_i'^2 x + \mu_i)} + \frac{\phi}{\mu_i'^2} \right) \right). \]

5 Expansion for Cordeiro’s adjusted residual

We now obtain the density function of the adjusted Pearson residuals proposed by Cordeiro (2004). He gave simple expressions to order \( n^{-1} \) for the mean and variance of the Pearson residual \( R_i \) in GLMs, namely \( E(R_i) = m_i/n + O(n^{-2}) \) and \( \text{Var}(R_i) = \sigma^2 + v_i/n + O(n^{-2}) \), where
\[
m_i/n = -\frac{\sigma^2}{2} \gamma_i(I - H) J z \quad \text{and} \quad v_i/n = \frac{\sigma^2}{2} \gamma_i(Q H J - T) z,
\]
\( I \) is the identity matrix of order \( n \), \( H = W^{1/2} X (X^T W X)^{-1} X^T W^{1/2} \) is the projection matrix, \( J, Q \) and \( T \) are diagonal matrices given by \( J = \text{diag}\{V_i^{-1/2}\mu_i'\}, Q = \text{diag}\{V_i^{-1/2}V_i^{(1)}\}, \)
\( T = \text{diag}\{2\phi w_i + V_i^{(2)} + V_i^{-1}V_i^{(1)}\\mu_i'^2\} \), \( z = (z_1, \ldots, z_m)^T \) is an \( n \)-vector with the diagonal elements of \( Z = X (X^T W X)^{-1} X^T \), and \( \gamma_i \) was defined in Section 2. Cordeiro’s (2004) adjusted residuals are
\[
R_i^* = \frac{R_i - \hat{m}_i/n}{\left(\frac{\sigma^2}{2} + \hat{v}_i/n\right)^{1/2}}. \tag{14}
\]
Expanding \( (\sigma^2 + \hat{v}_i/n)^{-1/2} \) as \( \sigma^{-1}(1 - \frac{\hat{v}_i}{2n\sigma^2} + \ldots) \) yields to order \( n^{-1} \)
\[
R_i^* = \sigma^{-1} \left\{ \left( 1 - \frac{\hat{v}_i}{2n\sigma^2} \right) R_i - \frac{\hat{m}_i}{n} \right\}. \]
Since \( \hat{m}_i = m_i + O_p(n^{-1/2}) \) and \( \hat{v}_i = v_i + O_p(n^{-1/2}) \), we can write \( R_i^* \) equivalently to order \( n^{-1} \) as
\[
R_i^* = \sigma^{-1} \left\{ R_i - n^{-1} \left( m_i + \frac{v_i R_i}{2\sigma^2} \right) \right\}, \tag{15}
\]
which implies trivially that \( E(R_i^*) = 0 + O(n^{-3/2}) \) and \( \text{Var}(R_i^*) = 1 + O(n^{-3/2}) \). Then, the adjusted residuals \([14]\) have zero mean and unit variance to order \( n^{-1} \).

Let \( S_i = \{R_i - n^{-1}(m_i + \frac{v_i R_i}{2\sigma^2})\} \). Since \( R_i = O_p(1) \), the cumulative distribution function (cdf) of \( S_i \), \( F_{S_i}(x) \) say, can be obtained from \([15]\) to order \( n^{-1} \) following the approach developed by Cordeiro and Ferrari (1998, Section 2)
\[
F_{S_i}(x) = F_{R_i}(x) + \frac{1}{n} \left( m_i + \frac{v_i x}{2\sigma^2} \right) f_{R_i}(x). \tag{16}
\]
Differentiation of (10) with respect to $x$, and replacing $f_{R_i}(x)$ by its asymptotic expansion in (10), yields the density of $S_i$ to the same order

$$f_{S_i}(x) = f_\varepsilon(x) - \frac{d}{dx}\{\theta_x f_\varepsilon(x)\} + \frac{1}{n}\left\{ \frac{v_i x}{2\sigma^2} \frac{df_\varepsilon(x)}{dx} + \frac{v_i}{2\sigma^2} f_\varepsilon(x) \right\}.$$  

(17)

The density function of $R^*_i$ is $f_{R^*_i}(x) = \sigma f_{S_i}(\sigma x)$, where $f_{S_i}(\sigma x)$ comes from (17) with $\sigma x$ replacing $x$. The sum of the second and third terms in (17) are expressed as $\frac{d}{dx}\{\mu_i(x)f_\varepsilon(x)\}$. Since $m_i/n, v_i/n, \theta_x^{(i)}$, and $\phi_x^{(i)^2}$ are all quantities of order $O(n^{-1})$, the terms on the right hand side of (17), except $f_\varepsilon(x)$, are of this order and then the densities $f_{R_i}(x)$ and $f_\varepsilon(x)$ differ by terms of order $O(n^{-1})$. However, we showed in Section 3, that the densities $f_{R_i}(x)$ and $f_\varepsilon(x)$ are equal to this order. Thus, the distribution of the corrected residuals $R^*_i$, even in small samples, is closer to the distribution of the true Pearson residuals than the distribution of the adjusted residuals $R^*_i$.

A simple expansion for the density $f_{R^*_i}(x)$ of the adjusted residuals $R^*_i$ to order $n^{-1}$ for the normal model with any link function is given by

$$f_{R^*_i}(x) = e^{-\frac{x^2}{2}} \left( 1 + a_0 - a_1 x - a_2 x^2 \right),$$

where the constant terms

$$a_0 = \frac{3\mu_x^2 z_i}{2} + \frac{v_i}{2n\sigma^2}, \quad a_1 = \frac{m_i}{\sigma n} - \frac{\mu_x^2 B(\eta_i)}{\sigma} - \frac{2\mu_x^2 z_i}{2}$$

and $a_2 = \frac{v_i}{2n\sigma^2} - \frac{3\mu_x^2 z_i}{2}$

that depend on the model are all of order $O(n^{-1})$.

6 Simulation results

We present some simulation results for studying the finite-sample distributions of the Pearson $R_i$, corrected $R'_i$, adjusted $R^*_i$ and the true $\varepsilon_i$ residual. We use a gamma model with log link

$$\log \mu = \beta_0 + \beta_1 x_1 + \beta_2 x_2,$$

where the true values of the parameters were taken as $\beta_0 = 1/2, \beta_1 = 1, \beta_2 = -1$ and $\phi = 4$. The explanatory variables $x_1$ and $x_2$ were generated from the uniform $U(0,1)$ distribution for $n = 20$ and their values were held constant throughout the simulations. The number of Monte Carlo replications was set at 10,000 and all simulations were performed using the statistical software R.

In each of the 10,000 replications, we fitted the model and computed the MLE $\hat{\beta}$ and fitted mean $\hat{\mu}$, the Pearson residuals $R_i$, the corrected function $\rho(\cdot)$ and the corrected residuals $R'_i$. Further, we calculated their expected values and variances from the expressions given by Cordeiro (2004) to obtain the adjusted residuals $R^*_i$. Finally, we calculated the true residuals $\varepsilon_i$. Tables 4 and 5 give the sample means, variances, skewness and kurtosis of the residuals $R_i$, $R'_i$, $R^*_i$ and $\varepsilon_i$, respectively, out of 10,000 values.
The corrected residuals $R'_i$ should agree with the true Pearson residuals rather than to the normal distribution. A good agreement with the normal distribution happens when these figures are, on average, close to 0, 1, 0 and 3, respectively.

The figures in Tables 4 and 5 show that the distribution of all residuals for the gamma model are positively skewed. All four cumulants of the corrected Pearson residuals $R'_i$ are generally closer to the corresponding cumulants of the true residuals $\varepsilon_i$ than those of the other residuals. The adjusted residuals $R'_*i$ have cumulants much closer to the cumulants of a standard normal distribution as claimed by Cordeiro (2004). Further, the distribution of the corrected residuals is generally closer to the distribution of the true residuals than the distribution of the Pearson residuals. In short, the correction $\rho(.)$ appears to be effective even when the sample size is small.

In Table 6 we give the values of the Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) (see, for instance, Anderson and Darling, 1952; Thode, 2002, Section 5.1.4) distances between the empirical distribution of each set of the 10,000 uncorrected $R_i$ and corrected $R'_i$ residuals for $i = 1, \ldots, 20$, and the estimated distribution of the true residuals. The estimated distribution here is the shifted gamma distribution with dispersion parameter $\phi$ taken to be the sample average of the estimated dispersion parameters at each step of the Monte Carlo simulation. In Table 7, we follow the same procedure for Table 6, but we now examine if the uncorrected $R_i$ and corrected $R'_i$ residuals follow the empirical distribution of the true residual $\varepsilon_i$. We then calculated both K-S and A-D distances between the empirical distributions of both (uncorrected and corrected) residuals and the empirical distribution of the true residuals $\varepsilon_i$.

We see from Tables 6 and 7 that the distribution of the corrected residuals is closer to the distribution of the true residuals than the distribution of the uncorrected residuals. Furthermore, the distances for the corrected residuals are substantially smaller than the distances for the uncorrected ones. These facts show that, when the model is well-specified, our correction works very well for the set of the corrected residuals.

We conclude the study providing an application of the corrected residuals to assess the adequacy of the above gamma model. We could expect that under a well-specified model, the distribution of the corrected residuals will follow approximately the distribution of the true residuals. However, even though it is common to compare the distribution of the Pearson residuals with the normal distribution, it is not clear that this approximation should be good in small samples. Therefore, we compare the empirical distribution of the corrected residuals with the distribution of the true residuals and the distribution of the uncorrected residuals with the normal distribution. For doing this, we use a QQ-Plot which displays a quantile-quantile plot of the sample quantiles of the corrected and uncorrected residuals versus theoretical quantiles from the estimated distribution of the true residuals and the normal distribution with mean zero and variance $\hat{\phi}^{-1}$, respectively. If the distribution of the corrected residuals is well approximated by the distribution of the true residuals, the plot will be close to linear. Therefore, we expect that a QQPlot of the Studentized corrected residuals versus the estimated distribution of the true residuals should be closer to the diagonal line than that QQPlot of the uncorrected residuals against the normal $N(0, \hat{\phi}^{-1})$ distribution. Moreover, we also consider the QQPlot of the adjusted residuals suggested by Cordeiro (2004) against the theoretical quantiles of a standard normal distribution.

Figure 1 gives two QQPlots, one for the vector of the 10,000 ordered uncorrected
Table 4: Mean and variance of uncorrected, corrected, adjusted and true residuals.

|    | Mean | Variance |
|----|------|----------|
|    | Mean | Variance |
| i  |     |          |
| 1  | 0.013 | 0.006 | 0.011 | 0.004 | 0.234 | 0.255 | 1.059 | 0.257 |
| 2  | -0.010 | -0.006 | 0.007 | 0.001 | 0.183 | 0.232 | 1.112 | 0.255 |
| 3  | 0.002 | -0.002 | -0.004 | -0.002 | 0.220 | 0.248 | 1.040 | 0.254 |
| 4  | 0.006 | 0.003 | 0.010 | 0.005 | 0.208 | 0.241 | 1.051 | 0.253 |
| 5  | 0.015 | 0.004 | 0.005 | 0.002 | 0.237 | 0.247 | 1.006 | 0.249 |
| 6  | -0.003 | -0.005 | 0.003 | -0.001 | 0.188 | 0.229 | 1.043 | 0.245 |
| 7  | -0.002 | -0.006 | -0.008 | -0.005 | 0.201 | 0.237 | 1.038 | 0.244 |
| 8  | -0.012 | -0.009 | 0.001 | -0.001 | 0.180 | 0.230 | 1.107 | 0.258 |
| 9  | 0.000 | -0.001 | 0.008 | 0.000 | 0.201 | 0.244 | 1.087 | 0.253 |
| 10 | -0.005 | -0.010 | -0.014 | -0.010 | 0.207 | 0.235 | 0.999 | 0.236 |
| 11 | -0.000 | 0.001 | 0.010 | 0.002 | 0.201 | 0.244 | 1.079 | 0.254 |
| 12 | 0.010 | -0.001 | -0.006 | -0.000 | 0.243 | 0.252 | 1.022 | 0.259 |
| 13 | -0.009 | -0.012 | -0.016 | -0.009 | 0.199 | 0.230 | 1.002 | 0.239 |
| 14 | -0.003 | -0.001 | 0.019 | 0.005 | 0.176 | 0.227 | 1.116 | 0.248 |
| 15 | 0.012 | 0.005 | 0.010 | 0.004 | 0.221 | 0.243 | 1.017 | 0.252 |
| 16 | -0.017 | -0.017 | -0.014 | -0.007 | 0.174 | 0.225 | 1.105 | 0.249 |
| 17 | 0.004 | -0.004 | -0.009 | -0.004 | 0.221 | 0.241 | 1.022 | 0.246 |
| 18 | 0.001 | -0.004 | -0.005 | -0.004 | 0.214 | 0.240 | 1.020 | 0.246 |
| 19 | 0.000 | -0.004 | -0.006 | -0.002 | 0.215 | 0.239 | 1.019 | 0.249 |
| 20 | -0.003 | -0.004 | -0.003 | -0.002 | 0.196 | 0.230 | 1.008 | 0.240 |

The figures show that even for a well-specified model, the plot for the uncorrected residuals is very distant from the diagonal line when compared with the plot for the corrected residuals. The adjusted residuals given in Figure 2 provides an improvement in regard to the uncorrected residuals, but the plot is also distant from the diagonal line when compared to the corrected residuals. Therefore, the corrected residuals have a good behavior that leads to the right conclusion, i.e., that the model is well-specified. We thus recommend the corrected residuals to build up QQPlots.
Table 5: Skewness and kurtosis of uncorrected, corrected, adjusted and true residuals.

| i | $R_i$ | $R'_i$ | $R^*_i$ | $\varepsilon_i$ | $R_i$ | $R'_i$ | $R^*_i$ | $\varepsilon_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.837 | 0.943 | 0.626 | 1.005 | 3.798 | 4.105 | 2.967 | 4.468 |
| 2 | 0.586 | 0.822 | 0.494 | 0.986 | 3.205 | 3.780 | 2.805 | 4.399 |
| 3 | 0.824 | 0.973 | 0.605 | 1.080 | 3.898 | 4.350 | 3.020 | 4.859 |
| 4 | 0.703 | 0.863 | 0.550 | 0.979 | 3.417 | 3.825 | 2.882 | 4.395 |
| 5 | 0.876 | 0.942 | 0.626 | 0.964 | 4.040 | 4.232 | 3.012 | 4.275 |
| 6 | 0.628 | 0.829 | 0.523 | 0.987 | 3.278 | 3.772 | 2.823 | 4.387 |
| 7 | 0.715 | 0.901 | 0.548 | 0.960 | 3.548 | 4.052 | 2.920 | 4.317 |
| 8 | 0.611 | 0.864 | 0.500 | 1.068 | 3.318 | 3.984 | 2.865 | 4.813 |
| 9 | 0.711 | 0.923 | 0.557 | 1.017 | 3.561 | 4.162 | 2.911 | 4.667 |
| 10 | 0.809 | 0.965 | 0.628 | 1.018 | 3.904 | 4.387 | 3.061 | 4.811 |
| 11 | 0.727 | 0.936 | 0.556 | 1.018 | 3.590 | 4.176 | 2.920 | 4.532 |
| 12 | 0.939 | 1.001 | 0.659 | 1.077 | 4.361 | 4.560 | 3.076 | 4.929 |
| 13 | 0.746 | 0.907 | 0.603 | 0.938 | 3.607 | 4.052 | 3.006 | 4.106 |
| 14 | 0.553 | 0.801 | 0.474 | 0.939 | 3.150 | 3.709 | 2.820 | 4.254 |
| 15 | 0.808 | 0.928 | 0.606 | 1.048 | 3.813 | 4.154 | 3.033 | 4.737 |
| 16 | 0.593 | 0.851 | 0.506 | 1.006 | 3.246 | 3.833 | 2.859 | 4.510 |
| 17 | 0.793 | 0.910 | 0.606 | 0.958 | 3.727 | 4.058 | 2.994 | 4.202 |
| 18 | 0.783 | 0.923 | 0.610 | 0.992 | 3.686 | 4.078 | 2.977 | 4.411 |
| 19 | 0.776 | 0.904 | 0.603 | 0.963 | 3.687 | 4.060 | 2.993 | 4.292 |
| 20 | 0.715 | 0.888 | 0.569 | 0.963 | 3.532 | 4.004 | 2.960 | 4.346 |

7 Conclusion

Using the results given in Loynes (1969), we calculate the $O(n^{-1})$ distribution of the Pearson residuals in GLMs (see, for instance, McCullagh and Nelder, 1989). It is important to mention that the distribution of residuals in regression models are typically unknown, and therefore all inference regarding these residuals are done by asymptotic assumptions which may not hold in small or moderate sample sizes. Then we can use this knowledge to define corrected Pearson residuals in these models in such a way that the corrected residuals will have, to order $O(n^{-1})$, the same distribution of the true Pearson residuals, which is known. The corrected residuals have practical applicability for all continuous GLMs. We simulate a gamma model with log link to conclude the superiority of the corrected Pearson residuals $R'_i$ over the uncorrected residuals $R_i$ and also over the adjusted residuals suggested by Cordeiro (2004) with regard to the approximation to the reference distribution, which for the corrected and uncorrected residuals was the distribution of the true residuals and for the adjusted residuals was the standard normal distribution. The paper is concluded with an application of the corrected residuals to assess the adequacy of the model.
Table 6: One-sample K-S and A-D statistics for uncorrected and corrected residuals.

| i   | K-S stat. for $R_i$ | A-D stat. for $R_i$ | K-S stat. for $R'_i$ | A-D stat. for $R'_i$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| All | 0.0232          | 300.42          | 0.0036          | 6.7641          |
| 1   | 0.0230          | 7.5944          | 0.0103          | 1.7875          |
| 2   | 0.0317          | 30.7031         | 0.0074          | 1.2504          |
| 3   | 0.0208          | 7.9810          | 0.0077          | 1.0710          |
| 4   | 0.0287          | 17.5666         | 0.0100          | 1.2283          |
| 5   | 0.0216          | 8.9498          | 0.0098          | 1.1761          |
| 6   | 0.0307          | 28.3464         | 0.0074          | 0.8353          |
| 7   | 0.0273          | 17.8230         | 0.0088          | 1.0719          |
| 8   | 0.0311          | 34.7206         | 0.0109          | 1.7666          |
| 9   | 0.0277          | 19.3796         | 0.0081          | 0.9986          |
| 10  | 0.0244          | 12.6919         | 0.0123          | 1.9530          |
| 11  | 0.0306          | 19.5087         | 0.0089          | 0.6709          |
| 12  | 0.0167          | 3.6271          | 0.0106          | 2.0631          |
| 13  | 0.0208          | 15.1071         | 0.0107          | 2.2356          |
| 14  | 0.0401          | 49.8411         | 0.0117          | 1.6905          |
| 15  | 0.0277          | 16.1354         | 0.0150          | 1.8709          |
| 16  | 0.0360          | 43.4746         | 0.0155          | 2.5022          |
| 17  | 0.0186          | 7.3023          | 0.0082          | 1.0216          |
| 18  | 0.0235          | 9.8595          | 0.0068          | 0.6481          |
| 19  | 0.0172          | 7.6480          | 0.0085          | 0.8816          |
| 20  | 0.0282          | 21.7356         | 0.0072          | 0.7146          |

Appendix

Suppose we write the residual $R$ in terms of the true residual $\varepsilon$ as $R = \varepsilon + \delta$, where $\varepsilon$ and $\delta$ are absolutely continuous random variables with respect to Lebesgue measure and $\delta$ is of order $O_p(n^{-1})$. Our goal is to define a corrected residual $R'$ having the same density of $\varepsilon$ to order $n^{-1}$. Initially, we have

$$E(e^{isR}) = E(e^{is\varepsilon}E(e^{is\delta} | \varepsilon))$$

and

$$\frac{\partial^k}{\partial s^k} E(e^{is\delta} | \varepsilon) \bigg|_{s=0} = i^k E(\delta^k | \varepsilon).$$

Expanding $E(e^{is\delta} | \varepsilon)$ in a Taylor series around $s = 0$ gives

$$E(e^{is\delta} | \varepsilon) = 1 + (is)E(\delta | \varepsilon) + \frac{(is)^2}{2} E(\delta^2 | \varepsilon) + \cdots.$$

Let $\theta_x = E(\delta | \varepsilon = x)$ and $\phi^2_x = \text{Var}(\delta | \varepsilon = x)$. Thus,

$$E\left\{e^{is\varepsilon}E(e^{is\delta} | \varepsilon)\right\} = \int_{-\infty}^{\infty} e^{isx} \left\{ 1 + (is)\theta_x + \frac{(is)^2}{2} (\phi^2_x + \theta^2_x) + \cdots \right\} f_\varepsilon(x) dx,$$

where $f_\varepsilon(\cdot)$ is the density function of $\varepsilon$. By using formulae (25) and (26) from Cox and Snell (1968) with $\varepsilon = 0$, it is possible to conclude that $E(\delta)$ and $\text{Var}(\delta)$ (and thus $E(\delta^2)$)
Table 7: Two-sample K-S and A-D statistics for uncorrected and corrected residuals.

| i  | K-S stat. for $R_i$ | A-D stat. for $R_i$ | K-S stat. for $R_i'$ | A-D stat. for $R_i'$ |
|----|----------------------|----------------------|----------------------|----------------------|
| All| 0.0283               | 44.193               | 0.0041               | 9.5703               |
| 1  | 0.0246               | 10.5690              | 0.0086               | 0.5191               |
| 2  | 0.0356               | 44.3716              | 0.0125               | 1.1887               |
| 3  | 0.0273               | 15.8880              | 0.0100               | 0.8209               |
| 4  | 0.0331               | 22.3655              | 0.0079               | 0.9083               |
| 5  | 0.0227               | 9.41018              | 0.0070               | 0.4727               |
| 6  | 0.0355               | 33.9614              | 0.0094               | 1.5002               |
| 7  | 0.0348               | 23.8043              | 0.0077               | 0.5297               |
| 8  | 0.0394               | 54.3072              | 0.0126               | 1.3377               |
| 9  | 0.0325               | 27.3448              | 0.0071               | 0.5715               |
| 10 | 0.0270               | 14.1363              | 0.0065               | 0.2252               |
| 11 | 0.0336               | 28.1285              | 0.0100               | 0.8706               |
| 12 | 0.0218               | 9.17441              | 0.0102               | 0.4995               |
| 13 | 0.0342               | 22.1684              | 0.0112               | 1.2824               |
| 14 | 0.0426               | 56.0126              | 0.0132               | 2.5277               |
| 15 | 0.0281               | 15.7758              | 0.0121               | 1.1146               |
| 16 | 0.0444               | 57.0837              | 0.0109               | 1.9487               |
| 17 | 0.0255               | 14.0805              | 0.0081               | 0.5073               |
| 18 | 0.0310               | 16.4867              | 0.0089               | 0.4553               |
| 19 | 0.0282               | 16.7720              | 0.0094               | 1.0549               |
| 20 | 0.0303               | 22.7455              | 0.0087               | 0.8158               |

are of order $O(n^{-1})$ and, in the same way, that the higher moments of $\delta$ are of order $o(n^{-1})$. In a similar manner, we can show that $E(\delta \mid \varepsilon = x)$ and $\text{Var}(\delta \mid \varepsilon = x)$ are also of order $O(n^{-1})$, and that the higher-order conditional moments are of order $o(n^{-1})$. Then, we can rewrite equation (18) as

$$E\{e^{is\delta} E(e^{i\delta} \mid \varepsilon)\} = \int_{-\infty}^{\infty} e^{isx} \left\{1 + (is)\theta_x + \frac{(is)^2}{2}\phi_x^2\right\} f_\varepsilon(x) dx + o(n^{-1}).$$

(19)

Note that we can express the integral on the right side of (19) as a sum of three integrals. Then, integration by parts, one time for the integral containing $\theta_x$ on the integrand and two times for the integral containing $\phi_x^2$ on the integrand, yields the following formula

$$E(e^{isR}) = \int_{-\infty}^{\infty} e^{isx} \left[f_\varepsilon(x) - \frac{d\{f_\varepsilon(x)\theta_x\}}{dx} + \frac{1}{2} \frac{d^2\{f_\varepsilon(x)\phi_x^2\}}{dx^2}\right] dx + o(n^{-1}).$$

(20)

The uniqueness theorem for characteristic functions yields the density of $R$ to order $n^{-1}$

$$f_R(x) = f_\varepsilon(x) - \frac{d\{f_\varepsilon(x)\theta_x\}}{dx} + \frac{1}{2} \frac{d^2\{f_\varepsilon(x)\phi_x^2\}}{dx^2} + o(n^{-1}).$$

(21)

Equation (21) is identical to formula (5) in Loynes (1969).
Further, we now define corrected residuals of the form $R' = R + \rho(R)$, where $\rho(\cdot)$ is a function of order $\mathcal{O}(n^{-1})$ used to recover the distribution of $\varepsilon$. We may proceed as above, noting that $E\{\rho(R) \mid R = x\} = \rho(x)$, to obtain the density of $R'$ to order $n^{-1}$

$$f_{R'}(x) = f_R(x) - \frac{d}{dx}\{\rho(x)f_R(x)\}.$$ 

Since the quantities $\rho(x), \theta_x$ and $\phi_x^2$ are all of order $\mathcal{O}(n^{-1})$, we have that $\frac{d}{dx}\{\rho(x)f_R(x)\} = \frac{d}{dx}\{\rho(x)f_\varepsilon(x)\}$ to this order. Therefore, the densities of $R$ and $\varepsilon$ will be the same to order $n^{-1}$ if

$$\frac{d}{dx}\{\rho(x)f_\varepsilon(x)\} = -\frac{d}{dx}\{f_\varepsilon(x)\theta_x\} + \frac{1}{2} \frac{d^2}{dx^2}\{f_\varepsilon(x)\phi_x\}.$$ 

Integration gives

$$\rho(x) = -\theta_x + \frac{1}{2f_\varepsilon(x)} \frac{d}{dx}\{f_\varepsilon(x)\phi_x\}. \quad (22)$$

Equation (22) is identical to equation (6) given by Loynes (1969) and it is clear from the proof that the support of $\varepsilon$ does not need to be the entire line and we can have proper intervals as support. We should note that the assumptions needed can be made weaker if we require that an expansion of the Taylor polynomial of order two with a remainder term (for instance, Lagrange remainder) can be done instead of the complete series.

We could also prove Loynes’ (1969) results by using the equivalence of (3c) and (4c), together with (5) and (6) of Cox and Reid (1987) and appropriate regularity conditions. The idea to this approach is as follows: consider in equation (3c) of Cox and Reid (1987) $X_0 = \varepsilon$, $X_1 = n^{1/2} \delta$ and $X_2 = 0$. This means that we are writing $Y_n$ as $Y_n = \varepsilon + \delta + \mathcal{O}_p(n^{-3/2})$, where $\varepsilon$ and $\delta$ are of orders $\mathcal{O}_p(1)$ and $\mathcal{O}_p(n^{-1})$, respectively. Then, from (4c), (5) and (6) of Cox and Reid (1987), we can write de cdf of $Y_n$ as

$$G_n(y) = F_0(y) - E(\varepsilon \mid \varepsilon = y)f_0(y) + \frac{1}{2} \frac{\partial}{\partial y}\{E(\varepsilon^2 \mid \varepsilon = x)f_0(y)\} + \mathcal{O}(n^{-3/2}),$$

where $F_0(\cdot)$ and $f_0(\cdot)$ are the cdf and pdf of $\varepsilon$, respectively. The expression above implies equation (21). We can also obtain the expansion for $R + \rho(R)$ from the equivalence of (3c) and (4c) of Cox and Reid (1987) by setting $X_0 = R, X_1 = 0$ and $X_2 = \rho(R)$. The rest of the proof is identical to the one given before. Note also, that for this proof $\varepsilon$ does not need to have a support in the entire line since this is not an assumption in the usual regularity conditions.

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Figure 1: QQPlots for the Pearson and corrected residuals

QQPlot of the uncorrected residuals

Theoretical quantiles of a normal distribution

QQPlot of the corrected residuals

Theoretical quantiles of a shifted gamma distribution

Figure 1: QQPlots for the Pearson and corrected residuals
Figure 2: QQPlot for the adjusted residuals