Hypersensitivity to perturbation: An information-theoretical characterization of classical and quantum chaos

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Abstract

Hypersensitivity to perturbation is a criterion for chaos based on the question of how much information about a perturbing environment is needed to keep the entropy of a Hamiltonian system from increasing. In this paper we give a brief overview of our work on hypersensitivity to perturbation in classical and quantum systems.

1 Introduction

In both classical and quantum physics isolated systems can display unpredictable behavior, but the reasons for the unpredictability are quite different. In classical (Hamiltonian) mechanics unpredictability is a consequence of chaotic dynamics, or exponential sensitivity to initial conditions, which makes it impossible to predict the phase-space trajectory of a system to a certain accuracy from initial data given to the same accuracy. This unpredictability, which comes from not knowing the system’s initial conditions precisely, is measured by the Kolmogorov-Sinai (KS) entropy, which is the rate at which initial data must be supplied in order to continue predicting the coarse-grained phase-space trajectory [1]. In quantum mechanics there is no sensitivity to initial conditions in predicting the evolution of a state vector, because the unitary evolution of quantum mechanics preserves the inner product between state vectors. The absence of sensitivity to initial conditions

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seems to suggest that there is no quantum chaos. Yet quantum mechanics has an even more fundamental kind of unpredictability, which has nothing to do with dynamics: even if a system’s state vector is known precisely, the results of measurements are generally unpredictable.

To compare the unpredictability of classical and quantum dynamics, we first remove the usual sources of unpredictability from consideration and then introduce a new source of unpredictability that is the same in both classical and quantum dynamics. The first step is to focus in classical physics on the evolution of phase-space distributions, governed by the Liouville equation, instead of on phase-space trajectories, and to focus in quantum physics on the evolution of state vectors, governed by the Schrödinger equation. The Liouville equation preserves the overlap between distributions, so there is no sensitivity to initial conditions in predicting the evolution of a phase-space distribution. By shifting attention from phase-space trajectories to distributions, we remove lack of knowledge of initial conditions as a source of unpredictability. Moreover, by considering only Schrödinger evolution of state vectors, i.e., evolution uninterrupted by measurements, we eliminate the intrinsic randomness of quantum measurements as a source of unpredictability.

The conclusion that there is no chaos in quantum evolution is now seen to be too facile. Were things so simple, one would have to conclude that there is no chaos in classical Liouville evolution either. Having taken both classical and quantum unpredictability out of the picture, we introduce a new source of unpredictability to investigate chaos in the dynamics. We do this by adding to the system Hamiltonian, either classical or quantum mechanical, a stochastic perturbation. We measure the unpredictability introduced by the perturbation in terms of the increase of system entropy. By gathering information about the history of the perturbation, one can make the increase of system entropy smaller. To characterize the resistance of the system to predictability, we compare the information gathered about the perturbation with the entropy reduction that this information purchases. We say that a system is hypersensitive to perturbation if the perturbation information is much larger than the associated system-entropy reduction, and we regard hypersensitivity to perturbation as the signature of chaos in Liouville or Schrödinger evolution (see Sec. 2).

For classical systems we have shown that systems with chaotic dynamics display an exponential hypersensitivity to perturbation, in which the ratio of perturbation information to entropy reduction grows exponentially in time, with the exponential rate of growth given by the KS entropy. Thus, for classical systems, we have established that exponential hypersensitivity to perturbation characterizes chaos in Liouville evolution in a way that is exactly equivalent to the standard characterization of chaos in terms of the unpredictability of phase-space trajectories (see Sec. 3).

For a variety of quantum systems we have used numerical simulations to investigate hypersensitivity to perturbation. The simulations suggest that hypersensitivity to perturbation provides a characterization of chaos in quantum dynamics: quantum systems whose classical dynamics is chaotic display a quantum hypersensitivity to perturbation, which comes about because the perturbation generates state vectors that are nearly randomly distributed in the system Hilbert space, whereas quantum systems whose classical dynamics is not chaotic do not display hypersensitivity to perturbation (see Sec. 4).
2 Hypersensitivity to perturbation

Hypersensitivity to perturbation, in either classical or quantum mechanics, is defined in terms of information and entropy. The entropy $H$ of an isolated physical system (Gibbs entropy for a classical system, von Neumann entropy for a quantum system) does not change under Hamiltonian time evolution. If the time evolution of the system is perturbed through interaction with an incompletely known environment, however, averaging over the perturbation typically leads to an entropy increase $\Delta H_S$. Throughout this paper, we make the simplifying assumption that the interaction with the environment is equivalent to a stochastic perturbation of the Hamiltonian, a restriction we hope to be able to remove in the future. Conditions under which this assumption is valid are discussed in [8]. The increase of the system entropy can be limited to an amount $\Delta H_{tol}$, the tolerable entropy increase, by obtaining, from the environment, information about the perturbation. We denote by $\Delta I_{min}$ the minimum information about the perturbation needed, on the average, to keep the system entropy below the tolerable level $\Delta H_{tol}$. A formal definition of the quantities $\Delta H_S$, $\Delta H_{tol}$, and $\Delta I_{min}$ can be found in [5] for the classical case and in [8] for the quantum case.

Entropy and information acquire physical content in the presence of a heat reservoir at temperature $T$. If all energy in the form of heat is ultimately exchanged with the heat reservoir, then each bit of entropy, i.e., each bit of missing information about the system state, reduces by the amount $k_B T \ln 2$ the energy that can be extracted from the system in the form of useful work. The connection between acquired information and work is provided by Landauer’s principle [9, 10], according to which not only each bit of missing information, but also each bit of acquired information, has a free-energy cost of $k_B T \ln 2$. This cost, the Landauer erasure cost, is paid when the acquired information is erased. Acquired information can be quantified by algorithmic information [11, 12, 13, 14, 15].

We now define that a system is hypersensitive to perturbation if the information $\Delta I_{min}$ required to reduce the system entropy from $\Delta H_S$ to $\Delta H_{tol}$ is large compared to the entropy reduction $\Delta H_S - \Delta H_{tol}$, i.e.,

$$\frac{\Delta I_{min}}{\Delta H_S - \Delta H_{tol}} \gg 1.$$  (1)

The information $\Delta I_{min}$ purchases a reduction $\Delta H_S - \Delta H_{tol}$ in system entropy, which is equivalent to an increase in the useful work that can be extracted from the system; hypersensitivity to perturbation means that the Landauer erasure cost of the information is much larger than the increase in available work.

Hypersensitivity to perturbation means that the inequality (1) holds for almost all values of $\Delta H_{tol}$. The inequality (1) tends always to hold, however, for sufficiently small values of $\Delta H_{tol}$. The reason is that for these small values of $\Delta H_{tol}$, one is gathering enough information from the perturbing environment to track a particular system state whose entropy is nearly equal to the initial system entropy. In other words, one is essentially tracking a particular realization of the perturbation among all possible realizations. Thus, for small values of $\Delta H_{tol}$, the information $\Delta I_{min}$ becomes a property of the perturbation; it is the information needed to specify a particular realization of the perturbation. The important regime for assessing hypersensitivity to perturbation is where $\Delta H_{tol}$ is fairly close to $\Delta H_S$, and it is in this regime that one can hope that $\Delta I_{min}$ reveals something
about the system dynamics, rather than properties of the perturbation.

3 Classical chaos

In this section we do not aim for rigor; many statements in this section are without formal proof. Instead, our objective here is to extract the important ideas from the rigorous analysis given in [5] and to use them to develop a heuristic physical picture of why chaotic systems display exponential hypersensitivity to perturbation. For a simple illustration and a system where exact solutions exist, see [4]. This section is an abbreviated version of the discussion section of [5].

Consider a classical Hamiltonian system whose dynamics unfolds on a $2F$-dimensional phase space, and suppose that the system is perturbed by a stochastic Hamiltonian whose effect can be described as diffusion on phase space. Suppose that the system is globally chaotic with KS entropy $K$. For such a system a phase-space density is stretched and folded by the chaotic dynamics, developing exponentially fine structure as the dynamics proceeds. A simple picture is that the phase-space density stretches exponentially in half the phase-space dimensions and contracts exponentially in the other half of the dimensions.

The perturbation is characterized by a perturbation strength and by correlation cells. We can take the perturbation strength to be the typical distance (e.g., Euclidean distance with respect to some fixed set of canonical coordinates) that a phase-space point diffuses under the perturbation during an $e$-folding time, $F/K\ln 2$, in a typical contracting dimension. The perturbation becomes effective (in a sense defined precisely in Ref. [5]) when the phase-space density has roughly the same size in the contracting dimensions as the perturbation strength. Once the perturbation becomes effective, the effects of the diffusive perturbation and of the further exponential contraction roughly balance one another, leaving the average phase-space density with a constant size in the contracting dimensions.

The correlation cells are phase-space cells over which the effects of the perturbation are well correlated and between which the effects of the perturbation are essentially uncorrelated. We assume that all the correlation cells have approximately the same phase-space volume. We can get a rough idea of the effect of the perturbation by regarding the correlation cells as receiving independent perturbations. Moreover, the diffusive effects of the perturbation during an $e$-folding time $F/K\ln 2$ are compressed exponentially during the next such $e$-folding time; this means that once the perturbation becomes effective, the main effects of the perturbation at a particular time are due to the diffusion during the immediately preceding $e$-folding time.

Since a chaotic system cannot be shielded forever from the effects of the perturbation, we can choose the initial time $t = 0$ to be the time at which the perturbation is just becoming effective. We suppose that at $t = 0$ the unperturbed density is spread over $2^{-Kt_0}$ correlation cells, $t_0$ being the time when the unperturbed density occupies a single correlation cell. The essence of the KS entropy is that for large times $t$ the unperturbed density spreads over

$$\mathcal{R}(t) \sim 2^{K(t-t_0)}$$

(2)
correlation cells, in each of which it occupies roughly the same phase-space volume. The exponential increase of $R(t)$ continues until the unperturbed density is spread over essentially all the correlation cells. We can regard the unperturbed density as being made up of subdensities, one in each occupied correlation cell and all having roughly the same phase-space volume.

After $t = 0$, when the perturbation becomes effective, the average density continues to spread exponentially in the expanding dimensions. As noted above, this spreading is not balanced by contraction in the other dimensions, so the phase-space volume occupied by the average density grows as $2^{Kt}$, leading to an entropy increase

$$\Delta H_S \sim \log_2(2^{Kt}) = Kt . \quad (3)$$

Just as the unperturbed density can be broken up into subdensities, so the average density can be broken up into average subdensities, one in each occupied correlation cell. Each average subdensity occupies a phase-space volume that is $2^{Kt}$ times as big as the volume occupied by an unperturbed subdensity.

The unperturbed density is embedded within the phase-space volume occupied by the average density and itself occupies a volume that is smaller by a factor of $2^{-Kt}$. We can picture a perturbed density crudely by imagining that in each occupied correlation cell the unperturbed subdensity is moved rigidly to some new position within the volume occupied by the average subdensity; the result is a perturbed subdensity. A perturbed density is made up of perturbed subdensities, one in each occupied correlation cell. All of the possible perturbed densities are produced by the perturbation with roughly the same probability.

Suppose now that we wish to hold the entropy increase to a tolerable amount $\Delta H_{tol}$. We must first describe what it means to specify the phase-space density at a level of resolution set by a tolerable entropy increase $\Delta H_{tol}$. An approximate description can be obtained in the following way. Take an occupied correlation cell, and divide the volume occupied by the average subdensity in that cell into $2^{\Delta H_S - \Delta H_{tol}}$ nonoverlapping volumes, all of the same size. Aggregate all the perturbed subdensities that lie predominantly within a particular one of these nonoverlapping volumes to produce a coarse-grained subdensity. There are $2^{\Delta H_S - \Delta H_{tol}}$ coarse-grained subdensities within each occupied correlation cell, each having a phase-space volume that is bigger than the volume occupied by a perturbed subdensity by a factor of

$$\frac{2^{Kt}}{2^{\Delta H_S - \Delta H_{tol}}} = 2^{\Delta H_{tol}} . \quad (4)$$

A coarse-grained density is made up by choosing a coarse-grained subdensity in each occupied correlation cell. A coarse-grained density occupies a phase-space volume that is bigger than the volume occupied by the unperturbed density by the factor $2^{\Delta H_{tol}}$ of Eq. (4) and hence represents an entropy increase

$$\log_2(2^{\Delta H_{tol}}) = \Delta H_{tol} . \quad (5)$$

Thus to specify the phase-space density at a level of resolution set by $\Delta H_{tol}$ means roughly to specify a coarse-grained density. The further entropy increase on averaging over the perturbation is given by

$$\log_2(2^{\Delta H_S - \Delta H_{tol}}) = \Delta H_S - \Delta H_{tol} . \quad (6)$$
What about the information $\Delta I_{\text{min}}$ required to hold the entropy increase to $\Delta H_{\text{tol}}$? Since there are $2^{\Delta H_S - \Delta H_{\text{tol}}}$ coarse-grained subdensities in an occupied correlation cell, each produced with roughly the same probability by the perturbation, it takes approximately $\Delta H_S - \Delta H_{\text{tol}}$ bits to specify a particular coarse-grained subdensity. To describe a coarse-grained density, one must specify a coarse-grained subdensity in each of the $\mathcal{R}(t)$ occupied correlation cells. Thus the information required to specify a coarse-grained density—and, hence, the information required to hold the entropy increase to $\Delta H_{\text{tol}}$—is given by

$$\Delta I_{\text{min}} \sim \mathcal{R}(t)(\Delta H_S - \Delta H_{\text{tol}}),$$

(7)

corresponding to there being a total of $(2^{\Delta H_S - \Delta H_{\text{tol}}})^{\mathcal{R}(t)}$ coarse-grained densities. The entropy increase (6) comes from counting the number of nonoverlapping coarse-grained densities that are required to fill the volume occupied by the average density, that number being $2^{\Delta H_S - \Delta H_{\text{tol}}}$. In contrast, the information $\Delta I_{\text{min}}$ comes from counting the exponentially greater number of ways of forming overlapping coarse-grained densities by choosing one of the $2^{\Delta H_S - \Delta H_{\text{tol}}}$ nonoverlapping coarse-grained subdensities in each of the $\mathcal{R}(t)$ correlation cells.

The picture developed in this section, summarized neatly in Eq. (7), requires that $\Delta H_{\text{tol}}$ be big enough that a coarse-grained subdensity is much larger than a perturbed subdensity, so that we can talk meaningfully about the perturbed subdensities that lie predominantly within a coarse-grained subdensity. If $\Delta H_{\text{tol}}$ becomes too small, Eq. (7) breaks down, and the information $\Delta I_{\text{min}}$, rather than reflecting a property of the chaotic dynamics as in Eq. (7), becomes essentially a property of the perturbation, reflecting a counting of the number of possible realizations of the perturbation.

The boundary between the two kinds of behavior of $\Delta I_{\text{min}}$ is set roughly by the number $F$ of contracting phase-space dimensions. When $\Delta H_{\text{tol}}/F \gtrsim 1$, the characteristic scale of a coarse-grained subdensity in the contracting dimensions is a factor of

$$\left(2^{\Delta H_{\text{tol}}}ight)^{1/F} = 2^{\Delta H_{\text{tol}}/F} \gtrsim 2$$

(8)
larger than the characteristic size of a perturbed subdensity in the contracting dimensions. In this regime the picture developed in this section is at least approximately valid, because a coarse-grained subdensity can accommodate several perturbed subdensities in each contracting dimension. The information $\Delta I_{\text{min}}$ quantifies the effects of the perturbation on scales as big as or bigger than the finest scale set by the system dynamics. These effects, as quantified in $\Delta I_{\text{min}}$, tell us directly about the size of the exponentially fine structure created by the system dynamics. Thus $\Delta I_{\text{min}}$ becomes a property of the system dynamics, rather than a property of the perturbation.

In contrast, when $\Delta H_{\text{tol}}/F \lesssim 1$, we are required to keep track of the phase-space density on a very fine scale in the contracting dimensions, a scale smaller than the characteristic size of a perturbed subdensity in the contracting dimensions. Subdensities are considered to be distinct, even though they overlap substantially, provided that they differ by more than this very fine scale in the contracting dimensions. The information $\Delta I_{\text{min}}$ is the logarithm of the number of realizations of the perturbation which differ by more than this very fine scale in at least one correlation cell. The information becomes a property of the perturbation because it reports on the effects of the perturbation on scales finer than
the finest scale set by the system dynamics—i.e., scales that are, at the time of interest, irrelevant to the system dynamics.

We are now prepared to put in final form the exponential hypersensitivity to perturbation of systems with a positive KS entropy:

$$\frac{\Delta I_{\text{min}}}{\Delta H_S - \Delta H_{\text{tol}}} \sim R(t) \sim 2^{K(t-t_0)} \text{ for } \Delta H_{\text{tol}} \gtrsim F. \quad (9)$$

Once the chaotic dynamics renders the perturbation effective, this exponential hypersensitivity to perturbation is essentially independent of the form and strength of the perturbation. Its essence is that within each correlation cell there is a roughly even trade-off between entropy reduction and information, but for the entire phase-space density the trade-off is exponentially unfavorable because the density occupies an exponentially increasing number of correlation cells, in each of which it is perturbed independently.

What about systems with regular, or integrable dynamics? Though we expect no universal behavior for regular systems, we can get an idea of the possibilities from the heuristic description developed in this section. Hypersensitivity to perturbation requires, first, that the phase-space density develop structure on the scale of the strength of the perturbation, so that the perturbation becomes effective, and, second, that after the perturbation becomes effective, the phase-space density spread over many correlation cells.

For many regular systems there will be no hypersensitivity simply because the phase-space density does not develop fine enough structure. Regular dynamics can give rise to nonlinear shearing, however, in which case the density can develop structure on the scale of the strength of the perturbation and can spread over many correlation cells. In this situation, one expects the picture developed in this section to apply at least approximately: to hold the entropy increase to $\Delta H_{\text{tot}}$ requires giving $\Delta H_S - \Delta H_{\text{tot}}$ bits per occupied correlation cell; $\Delta I_{\text{min}}$ is related to $\Delta H_{\text{tot}}$ by Eq. (7), with $R(t)$ being the number of correlation cells occupied at time $t$. Thus regular systems can display hypersensitivity to perturbation if $R(t)$ becomes large (although this behavior could be eliminated by choosing correlation cells that are aligned with the nonlinear shearing produced by the system dynamics), but they cannot display exponential hypersensitivity to perturbation because the growth of $R(t)$ is slower than exponential.

A more direct way of stating this conclusion is to reiterate what we have explained in this section and shown in Ref. [5]: Exponential hypersensitivity to perturbation is equivalent to the spreading of phase-space densities over an exponentially increasing number of phase-space cells; such exponential spreading holds for chaotic, but not for regular systems and is quantified by a positive value of the Kolmogorov-Sinai entropy.

4 Quantum chaos

4.1 Distribution of vectors in Hilbert space

The simplifying restriction on the interaction with the environment made in Sec. 2 means, for the quantum case, that the interaction with the environment is equivalent to a stochastic unitary time evolution. Given this assumption, we can proceed as follows. At a given
time, we describe the result of the perturbed time evolution by a list \( \mathcal{L} = (|\psi_1\rangle, \ldots, |\psi_N\rangle) \) of \( N \) vectors in \( D \)-dimensional Hilbert space, with probabilities \( q_1, \ldots, q_N \), each vector in the list corresponding to a particular realization of the perturbation, which we call a *perturbation history*. Averaging over the perturbation leads to a system density operator

\[
\hat{\rho}_S = \sum_{j=1}^{N} q_j |\psi_j\rangle\langle\psi_j| ,
\]

with entropy

\[
\Delta H_S = -\text{tr}\left(\hat{\rho}_S \log_2 \hat{\rho}_S\right) .
\]

Consider the class of measurements on the environment whose outcomes partition the list \( \mathcal{L} \) into \( R \) groups labeled by \( r = 1, \ldots, R \). We denote by \( N_r \) the number of vectors in the \( r \)th group (\( \sum_{r=1}^{R} N_r = N \)). The \( N_r \) vectors in the \( r \)th group and their probabilities are denoted by \( |\psi_{r1}\rangle, \ldots, |\psi_{rN_r}\rangle \) and \( q_{r1}, \ldots, q_{rN_r} \), respectively. The measurement outcome \( r \), occurring with probability

\[
p_r = \sum_{i=1}^{N_r} q_{ri} ,
\]

indicates that the system state is in the \( r \)th group. The system state conditional on the measurement outcome \( r \) is described by the density operator

\[
\hat{\rho}_r = p_r^{-1} \sum_{i=1}^{N_r} q_{ri} |\psi_{ri}\rangle\langle\psi_{ri}| .
\]

We define the conditional system entropy

\[
\Delta H_r = -\text{tr}\left(\hat{\rho}_r \log_2 \hat{\rho}_r\right) ,
\]

the average conditional entropy

\[
\Delta H = \sum_r p_r \Delta H_r ,
\]

and the average information

\[
\Delta I = -\sum_r p_r \log_2 p_r .
\]

We now describe nearly optimal measurements, i.e., nearly optimal groupings, for which \( \Delta I \) is a close approximation to \( \Delta I_{\text{min}} \), the minimum information about the environment, on the average, to keep the system entropy below a given tolerable entropy \( \Delta H_{\text{tol}} \), as described in Sec. 2. Given \( \Delta H_{\text{tol}} \), we want to partition the list of vectors \( \mathcal{L} \) into groups so as to minimize the information \( \Delta I \) without violating the condition \( \Delta H \leq \Delta H_{\text{tol}} \). To minimize \( \Delta I \), it is clearly favorable to make the groups as large as possible. Furthermore, to reduce the contribution to \( \Delta H \) of a group containing a given number of vectors, it is favorable to choose vectors that are as close together as possible in Hilbert space. Here the distance between two vectors \( |\psi_1\rangle \) and \( |\psi_2\rangle \) can be quantified in terms of the Hilbert-space angle \( \phi \)

\[
\phi = \cos^{-1}\left(\langle\psi_1|\psi_2\rangle\right) .
\]
Consequently, to find a nearly optimal grouping, we choose an arbitrary resolution angle $\phi$ ($0 \leq \phi \leq \pi/2$) and group together vectors that are less than an angle $\phi$ apart. More precisely, groups are formed in the following way. Starting with the first vector, $|\psi_1\rangle$, in the list $L$, the first group is formed of $|\psi_1\rangle$ and all vectors in $L$ that are within an angle $\phi$ of $|\psi_1\rangle$. The same procedure is repeated with the remaining vectors to form the second group, then the third group, continuing until no ungrouped vectors are left. This grouping of vectors corresponds to a partial averaging over the perturbations. To describe a vector at resolution level $\phi$ amounts to averaging over those details of the perturbation that do not change the final vector by more than an angle $\phi$.

For each resolution angle $\phi$, the grouping procedure described above defines an average conditional entropy $\Delta H \equiv \Delta H(\phi)$ and an average information $\Delta I \equiv \Delta I(\phi)$. If we choose, for a given $\phi$, the tolerable entropy $\Delta H_{\text{tol}} = \Delta H(\phi)$, then to a good approximation, the information $\Delta I_{\text{min}}$ is given by $\Delta I_{\text{min}} \simeq \Delta I(\phi)$. By determining the entropy $\Delta H(\phi)$ and the information $\Delta I(\phi)$ as functions of the resolution angle $\phi$, there emerges a rather detailed picture of how the vectors are distributed in Hilbert space. If $\Delta I(\phi)$ is plotted as a function of $\Delta H(\phi)$ by eliminating the angle $\phi$, one obtains a good approximation to the functional relationship between $\Delta I_{\text{min}}$ and $\Delta H_{\text{tol}}$.

As a further characterization of our list of vectors, we calculate the distribution $g(\phi)$ of Hilbert-space angles $\phi = \cos^{-1}(|\langle \psi | \psi' \rangle|)$ between all pairs of vectors $|\psi\rangle$ and $|\psi'\rangle$. For vectors distributed randomly in $D$-dimensional Hilbert space, the distribution function $g(\phi)$ is given by

$$g(\phi) = 2(D - 1)(\sin \phi)^{2D-3} \cos \phi.$$  \hspace{1cm} (18)

The maximum of this $g(\phi)$ is located at $\phi = \arccos\left(\frac{2}{D-1}\right)$; for large-dimensional Hilbert spaces, $g(\phi)$ is very strongly peaked near the maximum, which is located at $\phi \approx \pi/2 - 1/\sqrt{2D}$, very near $\pi/2$.

To investigate if a quantum map shows hypersensitivity to perturbation, we use the following numerical method. We first compute a list of vectors corresponding to different perturbation histories. Then, for about 50 values of the angle $\phi$ ranging from 0 to $\pi/2$, we group the vectors in the nearly optimal way described above. Finally, for each grouping and for each chosen angle $\phi$, we compute the information $\Delta I(\phi)$ and the entropy $\Delta H(\phi)$. In addition, we compute the angles between all pairs of vectors in the list and plot them as a histogram approximating the distribution function $g(\phi)$.

### 4.2 A typical numerical result

In this section, we present a typical numerical result for the quantum kicked top taken from [8], where more details can be found. We look at the time evolution of an initial Hilbert-space vector $|\psi_0\rangle$ at discrete times $nT$. After $n$ time steps, the unperturbed vector is given by

$$|\psi_n\rangle = \hat{T}^n|\psi_0\rangle ,$$  \hspace{1cm} (19)

where $\hat{T}$ is the unitary Floquet operator [17, 18]

$$\hat{T} = e^{-i(k/2J_z)J_z^2}e^{-i\pi J_x/2} ,$$  \hspace{1cm} (20)
and where $\hat{\mathbf{J}} = \hbar(\hat{J}_x, \hat{J}_y, \hat{J}_z)$ is the angular momentum vector for a spin-$J$ particle evolving in $(2J + 1)$-dimensional Hilbert space.

Depending on the initial condition, the classical map corresponding to the Floquet operator (20) displays regular as well as chaotic behavior [18]. Following [13], we choose initial Hilbert-space vectors for the quantum evolution that correspond to classical initial conditions located in regular and chaotic regions of the classical dynamics, respectively. For this purpose, we use coherent states [20, 21, 22]. In this section, we consider two initial states. The first one is a coherent state centered in a regular region of the classical dynamics; we refer to it as the regular initial state. The second one, referred to as the chaotic initial state, is a coherent state centered in a chaotic region of the classical dynamics.

The perturbation is modeled as an additional rotation by a small random angle about the $z$ axis. The system state after $n$ perturbed steps is thus given by

$$|\psi_n\rangle = \hat{T}(l_n) \cdots \hat{T}(l_1) |\psi_0\rangle,$$

where $\hat{T}(l_m) = e^{-igl_m\hat{J}_z} \hat{T}$, with $l_m = \pm 1$, is the unperturbed Floquet operator (20) followed by an additional rotation about the $z$ axis by an angle $l_m g = \pm g$, the parameter $g$ being the perturbation strength. There are $2^n$ different perturbation histories obtained by applying every possible sequence of perturbed unitary evolution operators $\hat{T}(-1)$ and $\hat{T}(+1)$ for $n$ steps. We have applied the method described in Sec. 4.1 to find numerically a nearly optimal grouping of the list $\mathcal{L}$ of $2^n$ vectors generated by all perturbation histories.

Figure 1 shows results for spin $J = 511.5$ and a total number of $2^{12} = 4096$ vectors after $n = 12$ perturbed steps [8]. We used a twist parameter $k = 3$ and perturbation strength $g = 0.003$. For Fig. 1(a), the chaotic initial state was used. The distribution of Hilbert-space angles, $g(\phi)$, is concentrated at large angles; i.e., most pairs of vectors are far apart from each other. The information $\Delta I$ needed to track a perturbed vector at resolution level $\phi$ is 12 bits at small angles, where each group contains only one vector. At $\phi \approx \pi/16$ the information suddenly drops to 11 bits, which is the information needed to specify one pair of vectors out of $2^{11}$ pairs, the two vectors in each pair being generated by perturbation sequences that differ only at the first step. The sudden drop of the information to 10 bits at $\phi \approx \pi/8$ similarly indicates the existence of $2^{10}$ quartets of vectors, generated by perturbation sequences differing only in the first two steps. Figure 1(a) suggests that, apart from the organization into pairs and quartets, there is not much structure in the distribution of vectors for a chaotic initial state. The $2^{10}$ quartets seem to be rather uniformly distributed in a $n_d = 46$-dimensional Hilbert space (see [8] for a definition of the number of explored Hilbert-space dimensions, $n_d$).

The inset in Fig. 1(a) shows the approximate functional dependence of the information needed about the perturbation, $\Delta I_{\text{min}}$, on the tolerable entropy $\Delta H_{\text{tol}}$, based on the data points $\Delta I(\phi)$ and $\Delta H(\phi)$. There is an initial sharp drop of the information, reflecting the grouping of the vectors into pairs and quartets. Then there is a roughly linear decrease of the information over a wide range of $\Delta H_{\text{tol}}$ values, followed by a final drop with increasing slope down to zero at the maximum value of the tolerable entropy, $\Delta H_{\text{tol}} = \Delta H_S$. The large slope of the curve near $\Delta H_{\text{tol}} = \Delta H_S$ can be regarded as a signature of hypersensitivity to perturbation. The linear regime at intermediate values of $\Delta H_{\text{tol}}$ is due to the
finite size of the sample of vectors: in this regime the entropy $\Delta H_r$ of the $r$th group is limited by $\log_2 N_r$, the logarithm of the number of vectors in the group.

Figure 1(b) shows data for $2^{12}$ vectors after 12 perturbed steps in the regular case. The distribution of perturbed vectors starting from the regular initial state is completely different from the chaotic initial condition of Fig. 1(a). The angle distribution $g(\phi)$ is conspicuously nonrandom: it is concentrated at angles smaller than roughly $\pi/4$, and there is a regular structure of peaks and valleys. Accordingly, the information drops rapidly with the angle $\phi$. The number of explored dimensions is $n_d = 2$, which agrees with results of Peres [19] that show that the quantum evolution in a regular region of the kicked top is essentially confined to a 2-dimensional subspace. The $\Delta I_{\text{min}}$ vs. $\Delta H_{\text{tot}}$ curve in the inset bears little resemblance to the chaotic case. Summarizing, one can say that, in the regular case, the vectors do not get far apart in Hilbert space, explore only few dimensions, and do not explore them randomly.

To obtain better numerical evidence for hypersensitivity in the chaotic case and for the absence of it in the regular case would require much larger samples of vectors, a possibility that is ruled out by restrictions on computer memory and time. The hypothesis most strongly supported by our data is the random character of the distribution of vectors in the chaotic case. In the following section we show that randomness in the distribution of perturbed vectors implies hypersensitivity to perturbation.

4.3 Discussion

Guided by our numerical results we now present an analysis of hypersensitivity to perturbation for quantum systems based on the conjecture that, for chaotic systems, Hilbert space is explored randomly by the perturbed vectors. We consider a Hamiltonian quantum system whose classical phase-space dynamics is chaotic and assume the system is perturbed by a stochastic Hamiltonian that classically gives rise to diffusion on phase space. We suppose that at time $t = 0$ the system’s state vector has a Wigner distribution that is localized on phase space. We further assume that at $t = 0$ the perturbation is just becoming effective in the classical sense described in Sec. 3.

Our numerical analyses [6, 7, 8] suggest the following picture. For times $t > 0$, the entropy $\Delta H_S$ of the average density operator $\hat{\rho}_S$ (10) increases linearly with time. This is in accordance with an essentially classical argument given by Zurek and Paz [23]. Denoting the proportionality constant by $\kappa$, we have

$$\Delta H_S \simeq \kappa t. \quad (22)$$

Since the von Neumann entropy of a density operator is bounded by the logarithm of the dimension of Hilbert space, it follows that the realizations of the perturbation—i.e., the state vectors that result from the different perturbation histories—explore at least a number

$$D(t) \equiv 2^{\Delta H_S} \simeq 2^{\kappa t} \quad (23)$$

of Hilbert-space dimensions, which increases exponentially. Our main conjecture now is that these dimensions are explored quasi-randomly, i.e., that the realizations of the perturbation at time $t$ are distributed essentially like random vectors in a $D(t)$-dimensional Hilbert space.
Starting from this main conjecture, we will now derive an estimate of the information $\Delta I_{\text{min}}$ needed to keep the system-entropy increase below the tolerable amount $\Delta H_{\text{tol}}$. Following the discussion on grouping vectors in Sec. 4.1, a tolerable entropy increase $\Delta H_{\text{tol}}$ corresponds to gathering the realizations of the perturbation into Hilbert-space spheres of radius $\phi$. The state vectors in each such sphere fill it randomly (since the perturbation is diffusive, there are plenty of vectors), so the entropy of their density operator—which is the tolerable entropy—is

$$
\Delta H_{\text{tol}} = -\left(1 - \frac{D-1}{D} \sin^2 \phi \right) \log_2 \left(1 - \frac{D-1}{D} \sin^2 \phi \right) - \frac{D-1}{D} \sin^2 \phi \log_2 \left(\frac{\sin^2 \phi}{D}\right)
$$

(Eq. (B6) of [7]). The number of spheres of radius $\phi$ in $D$-dimensional Hilbert space is $(\sin \phi)^{-2(D-1)}$ (Eq. (5.1) of [7]), so the information needed to specify a particular sphere is

$$
\Delta I_{\text{min}} \simeq \Delta \tilde{I}_{\text{min}} \equiv \log_2 \left((\sin \phi)^{-2(D-1)}\right) = -(D-1) \log_2(\sin^2 \phi).
$$

(25)

The information $\Delta \tilde{I}_{\text{min}}$ consistently underestimates the actual value of $\Delta I_{\text{min}}$, which comes from an optimal grouping of the random vectors; the reason is that the perfect grouping into nonoverlapping spheres of uniform size assumed by Eq. (25) does not exist.

Using Eq. (25) to eliminate $\phi$ from Eq. (24) gives an expression for $\Delta H_{\text{tol}}$ as a function of $\Delta \tilde{I}_{\text{min}}$,

$$
\Delta H_{\text{tol}} = -\left(1 - \frac{D-1}{D} 2^{-\Delta \tilde{I}_{\text{min}}/(D-1)}\right) \log_2 \left(1 - \frac{D-1}{D} 2^{-\Delta \tilde{I}_{\text{min}}/(D-1)}\right)
$$

$$
-\frac{D-1}{D} 2^{-\Delta \tilde{I}_{\text{min}}/(D-1)} \log_2 \left(\frac{2^{-\Delta \tilde{I}_{\text{min}}/(D-1)}}{D}\right),
$$

(26)

from which $D$ could be eliminated in favor of $\Delta H_S$ by invoking Eq. (23). The behavior of $\Delta \tilde{I}_{\text{min}}$ as a function of $\Delta H_{\text{tol}}$ expressed in Eq. (26) is the universal behavior that we conjecture for chaotic systems, except for when $\Delta H_{\text{tol}}$ is so close to $\Delta H_S$ that $\Delta \tilde{I}_{\text{min}} \lesssim 1$, as the spheres approximation used above breaks down for angles $\phi$ for which Hilbert space can accommodate only one sphere. Since $\Delta H_{\text{tol}}$ increases and $\Delta \tilde{I}_{\text{min}}$ decreases with $\phi$, $\Delta \tilde{I}_{\text{min}}$ increases as $\Delta H_{\text{tol}}$ decreases from its maximum value of $\Delta H_S$.

To gain more insight into Eq. (26), we calculate the derivative

$$
\frac{d \Delta \tilde{I}_{\text{min}}}{d \Delta H_{\text{tol}}} = -\frac{D}{\sin^2 \phi \ln(1 + D \cot^2 \phi)},
$$

(27)

which is the marginal tradeoff between between information and entropy. For $\phi$ near $\pi/2$, so that $\epsilon = \pi/2 - \phi \ll 1$, the information becomes $\Delta \tilde{I}_{\text{min}} = (D-1) \epsilon^2 / \ln 2$, and the derivative (27) can be written as

$$
\frac{d \Delta \tilde{I}_{\text{min}}}{d \Delta H_{\text{tol}}} \simeq -\frac{D}{\ln(1 + D \epsilon^2)} = -\frac{D}{\ln \left(1 + \frac{D}{D-1} \Delta \tilde{I}_{\text{min}} \ln 2\right)}.
$$

(28)
For $\Delta I_{\text{min}} \gtrsim 1$, i.e., when Eq. (26) is valid, the size of the derivative (28) is determined by $D(t) = 2^{\Delta H_S} \simeq 2^{2t}$, with a slowly varying logarithmic correction. This behavior, characterized by the typical slope $D(t)$, gives an exponential hypersensitivity to perturbation, with the classical number of correlation cells, $R(t)$, roughly replaced by the number of explored Hilbert-space dimensions, $D(t)$.

It is a remarkable fact that the concept of perturbation cell or perturbation correlation length (see Sec. 3) did not enter this quantum-mechanical discussion. Indeed, our numerical results suggest that our main conjecture holds for a single correlation cell, i.e., for a perturbation that is correlated over all of the relevant portion of phase space. That we find this behavior indicates that we are dealing with an intrinsically quantum-mechanical phenomenon. What seems to be happening is the following. For tolerable entropies $\Delta H_{\text{tol}} \gtrsim F$, where $2F$ is the dimension of classical phase space as in Sec. 3, we can regard a single-cell perturbation as perturbing a classical system into a set of nonoverlapping densities. In a quantum analysis these nonoverlapping densities can be crudely identified with orthogonal state vectors. The single-cell quantum perturbation, in conjunction with the chaotic quantum dynamics, seems to be able to produce arbitrary linear superpositions of these orthogonal vectors, a freedom not available to the classical system. The result is a much bigger set of possible realizations of the perturbation.

5 Conclusion

This paper compares and contrasts hypersensitivity to perturbation in classical and quantum dynamics. Although hypersensitivity provides a characterization of chaos that is common to both classical and quantum dynamics, the mechanisms for hypersensitivity are different classically and quantum mechanically. The classical mechanism has to do with the information needed to specify the phase-space distributions produced by the perturbation—this is classical information—whereas the quantum mechanism has to do with the information needed to specify the random state vectors produced by the perturbation—this is quantum information because it relies on the superposition principle of quantum mechanics. Captured in a slogan, the difference is this: a stochastic perturbation applied to a classical chaotic system generates classical information, whereas a stochastic perturbation applied to a quantum system generates quantum information.

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Figure 1: Results characterizing the distribution of Hilbert-space vectors for the perturbed kicked top. (a) Chaotic case, (b) regular case. For details see the text.
Figure 1