Bethe Equations "on the Wrong Side of Equator"

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Abstract

We analyse the famous Baxter’s $T−Q$ equations for $XXX$ ($XXZ$) spin chain and show that apart from its usual polynomial (trigonometric) solution, which provides the solution of Bethe-Ansatz equations, there exists also the second solution which should corresponds to Bethe-Ansatz beyond $N/2$. This second solution of Baxter’s equation plays essential role and together with the first one gives rise to all fusion relations.

1. Associated solutions of Bethe-Ansatz equations for $XXX$ - spin chains

The equations of Bethe-Ansatz in the case of $XXX$ - spin $1/2$ chain could be written in the following form:(see e.g. [2])

\[
\left( \frac{\lambda_j+i/2}{\lambda_j-i/2} \right)^N = \prod_{k \neq j}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} = -\prod_{k=1}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \quad (j = 1, 2, \ldots n), \quad (1)
\]
where $N$ - the length of the chain (total number of spins) and $n$ - the number of parameters $\lambda_j$, which describe the state vector.

The total spin of the eigenstate, described by $\lambda_j$ is equal to $\frac{N}{2} - n$, therefore only states with $n \leq N/2$ are meaningful. One can prove e.g. in the frameworks of QISM (see e.g. [3]), that if $n > N/2$, the corresponding Bethe vector vanishes.

Nevertheless, the solutions of (1) with $n$ beyond the equator $N/2$ do exist and moreover their consideration appears to be very useful.

In this section we shall prove the following.

**Theorem on extended Bethe-Ansatz for $XXX$ spin chain.**

For each solution of (1) with $n \leq N/2$ there exists the associated one-parametric solution with $n^* = N - n + 1 > N/2$.

Proof:

- Let us consider the set $\{\lambda_j\}$ which is the solution of (1) with $n \leq N/2$. This set defines the polynomial $Q(\lambda)$ whose roots are $\{\lambda_j\}$:

\[
Q(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j).
\]

The equations (1) could be represented in the following form:

\[
(\lambda_j - i/2)^N Q(\lambda_j + i) + (\lambda_j + i/2)^N Q(\lambda_j - i) = 0. \quad (j = 1, 2, \ldots n). \tag{3}
\]

wherefrom it follows that the polynomial of the degree $N + n$

\[
(\lambda - i/2)^N Q(\lambda + i) + (\lambda + i/2)^N Q(\lambda - i)
\]

vanishes at the roots of polynomial $Q(\lambda)$. For the case of the simple roots this statement implies the validity of the Baxter equation for $XXX$ spin chain [3, 4]:

\[
(\lambda - i/2)^N Q(\lambda + i) + (\lambda + i/2)^N Q(\lambda - i) = T(\lambda) Q(\lambda), \tag{5}
\]

where the polynomial $T(\lambda)$ of the degree $N$, is an eigenvalue of transfer matrix (the trace of monodromy matrix) for $XXX$ - model.

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1In more general situation of inhomogeneous XXZ spin chain this polynomial was introduced by Baxter [4].
Let us divide both sides of (5) on the product $Q(\lambda - i) Q(\lambda) Q(\lambda + i)$

$$\frac{T(\lambda)}{Q(\lambda + i) Q(\lambda - i)} = R(\lambda - i/2) + R(\lambda + i/2),$$  

(6)

where

$$R(\lambda) = \frac{\lambda^N}{Q(\lambda - i/2) Q(\lambda + i/2)}.$$  

(7)

The rational function $R(\lambda)$ can be presented in the following form:

$$R(\lambda) = \pi(\lambda) + \frac{q_-(\lambda)}{Q(\lambda - i/2)} + \frac{q_+(\lambda)}{Q(\lambda + i/2)},$$  

(8)

where $\pi(\lambda)$, $q_-(\lambda)$ and $q_+(\lambda)$ are polynomials, whose degrees satisfy:

$$\deg \pi(\lambda) = N - 2n,$$
$$\deg q_-(\lambda) < n,$$
$$\deg q_+(\lambda) < n.$$  

(9)

These inequalities will be used in the sequel.

Making use of the representation (8) for $R(\lambda)$ let us rewrite equation (6).

$$\frac{T(\lambda)}{Q(\lambda + i) Q(\lambda - i)} = \pi(\lambda - i/2) + \pi(\lambda + i/2) +$$
$$+ \frac{q_-(\lambda - i/2)}{Q(\lambda - i)} + \frac{q_+(\lambda - i/2)}{Q(\lambda)} + \frac{q_-(\lambda + i/2)}{Q(\lambda)} + \frac{q_+(\lambda + i/2)}{Q(\lambda + i)}.$$  

(10)

In the r.h.s. of (10) there are two terms with the denominator $Q(\lambda)$:

$$\frac{q_+(\lambda - i/2) + q_-(\lambda + i/2)}{Q(\lambda)}$$

which are absent in the l.h.s.. The degree of the nominator of this fraction according to (9) is less then degree of the denominator, therefore the two terms should cancel each other, hence

$$q_+(\lambda) = q(\lambda + i/2), \quad q_- = -q(\lambda - i/2).$$  

(11)
• With (11) the representation for \( R(\lambda) \) becomes

\[
R(\lambda) = \pi(\lambda) + \frac{q(\lambda + i/2)}{Q(\lambda + i/2)} - \frac{q(\lambda - i/2)}{Q(\lambda - i/2)}. \tag{12}
\]

The polynomial \( \pi(\lambda) \), as any other, also may be presented as the finite difference

\[
\pi(\lambda) = \rho(\lambda + i/2) - \rho(\lambda - i/2), \tag{13}
\]

where \( \rho(\lambda) \) is a polynomial of the degree \( N - 2n + 1 \). The explicit form of \( \rho(\lambda) \) one can obtain e.g. via binomial polynomials \( \binom{\lambda}{m} \), \( m = 0, 1, 2, \ldots \).

• Taking into account (13) we arrive at the following equation for our rational function \( R(\lambda) \):

\[
R(\lambda) \equiv \frac{\lambda^N}{Q(\lambda + i/2)Q(\lambda - i/2)} = \frac{P(\lambda + i/2)}{Q(\lambda + i/2)} - \frac{P(\lambda - i/2)}{Q(\lambda - i/2)}, \tag{14}
\]

where \( P(\lambda) \) is the last and most important polynomial of this theorem:

\[
P(\lambda) = \rho(\lambda)Q(\lambda) + q(\lambda) \tag{15}
\]

Counting the degree of \( P(\lambda) \) we obtain \( \text{deg} \ P(\lambda) = n^* = N + 1 - n \).

• Now we can get rid of the denominators in (14), and obtain the fundamental equation:

\[
P(\lambda + i/2)Q(\lambda - i/2) - P(\lambda - i/2)Q(\lambda + i/2) = \lambda^N. \tag{16}
\]

• This equation is invariant under substitution \( Q \to -P \), therefore the roots of the polynomial \( P(\lambda) \), which we denote as \( \{\lambda_j^*\} \) provide the solution of Bethe-Ansatz equations:

\[
\left( \frac{\lambda_j^* + i/2}{\lambda_j^* - i/2} \right)^N = \prod_{k \neq j}^{n^*} \frac{\lambda_j^* - \lambda_k^* + i}{\lambda_j^* - \lambda_k^* - i} \quad (j = 1, 2, \ldots n^*), \tag{17}
\]

as the roots of \( Q(\lambda) \) provide the solution of (1).
The polynomial $\rho(\lambda)$ in (13) is defined up to the arbitrary constant $\alpha$. This implies that the polynomial $P(\lambda)$, corresponding to $Q(\lambda)$ is actually one-parametric family:

$$P(\lambda, \alpha) = P(\lambda) + \alpha Q(\lambda),$$

with obvious agreement with (16).

QED.

The theorem we just have proven may be illustrated by the concrete example of the set of polynomials $P$ and $Q$ for the case $N = 4$:

| Number | $S$ | $Q(\lambda)$ | $(2S + 1)iP(\lambda)$ | $T(\lambda)$ |
|--------|-----|---------------|------------------------|--------------|
| 1      | 0   | $\lambda^2 + \frac{1}{2}$ | $\lambda^2 + \frac{1}{2} \lambda + \alpha(\lambda^2 + \frac{1}{2})$ | $2\lambda^4 + 3\lambda^2 - \frac{3}{8}$ |
| 2      | 0   | $\lambda^2 - \frac{1}{2}$ | $\lambda^2 + \frac{1}{2} \lambda + \alpha(\lambda^2 - \frac{1}{2})$ | $2\lambda^4 + 3\lambda^2 + \frac{13}{8}$ |
| 3      | 1   | $\lambda - \frac{1}{2}$ | $\lambda^4 + \lambda^3 + \lambda^2 + \frac{5}{2} \lambda + \alpha(\lambda - \frac{1}{2})$ | $2\lambda^4 + \lambda^2 + 2\lambda + \frac{1}{16}$ |
| 4      | 1   | $\lambda + \frac{1}{2}$ | $\lambda^4 - \lambda^3 + \lambda^2 - \frac{5}{2} \lambda + \alpha(\lambda + \frac{1}{2})$ | $2\lambda^4 + \lambda^2 - 2\lambda + \frac{1}{16}$ |
| 5      | 1   | $\lambda$ | $\lambda^4 - \frac{1}{4} \lambda^2 - \frac{1}{16} + \alpha \lambda$ | $2\lambda^3 + \lambda^2 - \frac{7}{8}$ |
| 6      | 2   | 1 | $\lambda^6 + \frac{5}{6} \lambda^5 + \frac{5}{16} \lambda + \alpha$ | $2\lambda^4 - 3\lambda^2 + \frac{1}{8}$ |

Few comments are in order.

- The polynomial $Q(\lambda)$ is normalized in such a way that the coefficient at the highest degree is equal to 1. Comparing the highest degrees in (14), we obtain that the coefficient at the highest degree of $P(\lambda)$ is equal to $1/i(N - 2n + 1) = 1/i(2S + 1)$, where $S$ is the spin of Bethe state.

- Note, that the existence of one-parametric solution of Bethe equations "beyond equator" implies that these equations are not independent. We shall consider the consequences of this fact in a separate paper.

- Let us come back to the equation (6). Using the representation (14) for $R(\lambda)$, we obtain the following expression for eigenvalues of transfer matrix.

$$T(\lambda) = P(\lambda + i)Q(\lambda - i) - P(\lambda - i)Q(\lambda + i).$$ (19)
Combining (16), and (19) we easily obtain the equation:

\[(\lambda - i/2)^N P(\lambda + i) + (\lambda + i/2)^N P(\lambda - i) = T(\lambda)P(\lambda),\]  

(20)

similar to Baxter equation (3).

This means that \(P(\lambda)\) may be considered as the second independent solution of (3). The arbitrary linear combination of \(Q\) and \(P\) is the solution of Baxter equation as well.

Finally note that the polynomials \(Q(\lambda), P(\lambda)\), satisfying the relation (19) are completely different from the eigenvalues of operators \(Q\pm\), which have been constructed in the series of papers of Bazhanov, Lukyanov and Zamolodchikov [7]. In these papers authors considered the field theory analogues of some useful construction of lattice integrable models. The extension of their \(Q\pm\) operators for 6-vertex model [8] requires external magnetic field which spoils the rotational invariance of \(XXX\) model. We intend to give the detailed discussion of the properties of the solutions for Bethe Ansatz equations with magnetic field in subsequent publications. Also we intend to discuss the relation of our associated Bethe system (1) and (17) with similar construction of Krichiver, Lipan, Wiegmann and Zabrodin [9].

2. Fusion relations for transfer matrices

As was emphasised in [4, 9], the fundamental equation

\[P(\lambda + i/2)Q(\lambda - i/2) - P(\lambda - i/2)Q(\lambda + i/2) = \lambda^N,\]  

(21)

implies the existence of the class of functional relations known as fusion relations for transfer matrices (see e.g. [10]).

Now we have shown that the fundamental relations (21) follows from Bethe-Ansatz equations, therefore these fusion relations also arise due to Bethe-Ansatz!

Let us consider the details of the connection of (21) and fusion relations.

2 see also [8]

3 authors use a special parameter \(\nu\) which plays the role of the magnetic field
First of all let us define the functions $T_s(\lambda)$ as follows:

$$T_s(\lambda) = P(\lambda + i(s + \frac{1}{2}))Q(\lambda - i(s + \frac{1}{2})) - P(\lambda - i(s + \frac{1}{2}))Q(\lambda + i(s + \frac{1}{2})).$$ \hspace{1cm} (22)$$

The parameter $s$ may be considered as spin in the auxiliary space and therefore may take integer or half integer, but generally speaking the r.h.s. in (22) is well defined for arbitrary complex $s$.

From this definition immediately follows the equation:

$$T_{-s-1}(\lambda) = -T_s(\lambda),$$ \hspace{1cm} (23)$$

and for particular values of $s$ we have:

$$T_{1/2}(\lambda) = T(\lambda), \quad T_{-1/2}(\lambda) = 0,$$

$$T_{-1}(\lambda) = -T_0(\lambda) = -\lambda^N \hspace{1cm} (24)$$

For the sake of brevity we shall use the following notation:

$$\Delta(a, b) \equiv P(a)Q(b) - P(b)Q(a) \hspace{1cm} (25)$$

The function $\Delta(a, b)$ changes sign while $a \to b$, what implies the identity:

$$\Delta(a, b)Q(c) + \Delta(b, c)Q(a) + \Delta(c, a)Q(b) = 0. \hspace{1cm} (26)$$

Making use of the definitions (22) and (25) we can rewrite the last equation as follows:

$$T_{s_1}(\lambda + i(s_2 - s_3)/3)Q(\lambda + 2i(s_3 - s_2)/3) +$$

$$+ T_{s_2}(\lambda + i(s_3 - s_1)/3)Q(\lambda + 2i(s_1 - s_3)/3) +$$

$$+ T_{s_3}(\lambda + i(s_1 - s_2)/3)Q(\lambda + 2i(s_2 - s_1)/3) = 0,$$

$$s_1 + s_2 + s_3 + 3/2 = 0. \hspace{1cm} (27)$$

Apparently this equation may be considered as generalization of $T - Q$ equation \([3]\).

Another simple identity:

$$\Delta(a, b)\Delta(c, d) - \Delta(a, c)\Delta(b, d) + \Delta(a, d)\Delta(b, c) = 0 \hspace{1cm} (28)$$
leads to the following quadratic relations:

\[ T_{s_1}(\lambda - i(s_1 + 1/2))T_{s_3-s_2-1/2}(\lambda - i(s_2 + s_3 + 1)) - \\
-T_{s_2}(\lambda - i(s_2 + 1/2))T_{s_3-s_1-1/2}(\lambda - i(s_1 + s_3 + 1)) + \\
+T_{s_3}(\lambda - i(s_3 + 1/2))T_{s_2-s_1-1/2}(\lambda - i(s_1 + s_2 + 1)) = 0, \]

(29)

For \( s_2 = -1 \), \( s_3 = 0 \), the last equation due to (23) and (24) may be written as famous fusion relations:

\[ T_s(\lambda - i(s + 1/2))T(\lambda) = \\
= (\lambda + i/2)^N T_{s-1/2}(\lambda - i(s + 1)) + (\lambda - i/2)^N T_{s+1/2}(\lambda - i) s, \]

(30)

where \( T_s(\lambda) \) is the eigenvalue of transfer matrix of quantum spin 1/2 and auxiliary spin \( s \).

3. XXX\(_{s_q}\) - model

Now let us consider inverse situation when quantum spin is \( s_q \), while auxiliary spin 1/2. This situation corresponds to the XXX\(_{s_q}\) spin chain. The above discussion could be easily generalized for this case.

Indeed, the Bethe ansatz equations have the following form: (see e.g. [4])

\[ \left( \frac{\lambda_j + i s_q}{\lambda_j - i s_q} \right)^N = \prod_{k \neq j}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} = - \prod_{k=1}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \quad (j = 1, 2, \ldots n), \]

(31)

where the notations are the same as in [1].

Now the set of meaningful solutions \( \{\lambda_j\} \) are those for \( n \leq s_q N \). The eigenvalues of the transfer matrix is given by:

\[ T_{1/2,s_q}(\lambda) = (\lambda + i s_q)^N \prod_{j=1}^{n} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + (\lambda - i s_q)^N \prod_{j=1}^{n} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j}, \]

(32)

while \( T - Q \) Baxter equations look like:

\[ (\lambda - is_q)^N Q_{s_q}(\lambda + i) + (\lambda + is_q)^N Q_{s_q}(\lambda - i) = T_{1/2,s_q}(\lambda; s_q) Q_{s_q}(\lambda), \]

(33)

To simplify further consideration we shall limit ourself with the case \( s_q = 3/2 \).
As we did in the case $s_q = 1/2$ we divide both sides of (33) on the product $Q(\lambda - i)Q(\lambda)Q(\lambda + i)$. But now trying to represent r.h.s. as a finite difference we meet an obstacle do to different shift of spectral parameter in numerators and denominators of the fractions. To overcome this difficulty we have to multiply both sides to the additional multipliers $(\lambda + i/2)^N(\lambda - i/2)^N$. (In general case the number of this auxiliary multipliers is $2^{s_q-1}$):

$$
\frac{T(\lambda)(\lambda + i/2)^N(\lambda - i/2)^N}{Q(\lambda + i)Q(\lambda - i)} = R(\lambda - i/2) + R(\lambda + i/2),
$$

(34)

where

$$
R(\lambda) = \frac{(\lambda - i)^N\lambda^N(\lambda + i)^N}{Q(\lambda - i/2)Q(\lambda + i/2)}.
$$

Further steps are the same as above and finally we arrive at the following fundamental relation:

$$
P_\frac{1}{2}(\lambda + \frac{i}{2})Q_\frac{1}{2}(\lambda - \frac{i}{2}) - \frac{1}{2}P_\frac{1}{2}(\lambda - \frac{i}{2})Q_\frac{1}{2}(\lambda + \frac{i}{2}) = (\lambda - i)^N\lambda^N(\lambda + i)^N,
$$

(36)

and expression for eigenvalues of transfer matrix $T_{\frac{1}{2}\frac{1}{2}}(\lambda)$:

$$
T_{\frac{1}{2}\frac{1}{2}}(\lambda)(\lambda + \frac{i}{2})^N(\lambda - \frac{i}{2})^N = P_\frac{1}{2}(\lambda + i)Q_\frac{1}{2}(\lambda - i) - \frac{1}{2}P_\frac{1}{2}(\lambda - i)Q_\frac{1}{2}(\lambda + i).
$$

(37)

The illustrative example for the case $s_q = 3/2, N = 2$:

| Number | $S$ | $Q(\lambda)$ | $(2S+1)iP(\lambda)$ | $T(\lambda)$ |
|--------|-----|--------------|---------------------|-------------|
| 1      | 0   | $\lambda^4 + \frac{1}{4}\lambda$ | $2\lambda^2 + \frac{1}{2}i$ |             |
| 2      | 1   | $\lambda^4 + \frac{1}{4}\lambda + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$ | $2\lambda^2 + \frac{1}{2}$ |             |
| 3      | 2   | $\lambda^4 + \frac{1}{4}\lambda^3 + \frac{1}{8}\lambda^2 + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \alpha$ | $2\lambda^2 + \frac{1}{2}$ |             |
| 4      | 3   | $\lambda^4 + \frac{1}{4}\lambda^3 + \frac{1}{8}\lambda^2 + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \alpha$ | $2\lambda^2 - \frac{1}{2} - \frac{1}{2}$ |             |

In conclusion of this section we formulate the second theorem, which generalizes the first one.

**Theorem 2:** For each solution of equations (31) with $n \leq sN$, there exists the one-parametric associated solution with $n^* = 2sN - n + 1 > sN$.

Note that with fundamental relation of the type (36) for arbitrary $s_q$ we can obtain the rational analogues of all fusion relations considered in [10].
4. Trigonometric case - XXZ spin chain

There we shall consider the Bethe-Ansatz ”beyond the equator” for XXZ spin chain. The general ideas of this generalization are the same, as in the first section.

We shall use the Baxter’s parametrization (see e.g. [3]) for spectral $\phi$ and crossing $\eta$ parameters. In these notations $T - Q$ Baxter equation looks like:

$$T(\phi)Q(\phi) = \sin^N(\phi + \eta)Q(\phi - 2\eta) + \sin^N(\phi - \eta)Q(\phi + 2\eta).$$

Usual q-parameter of the XXZ model is defined by $q = e^{2i\eta}$.

Recall that

$$Q(\phi) = \prod_{j=1}^{n} \sin(\phi - \phi_j)$$

is now trigonometric polynomial of the degree $n \leq N/2$, where a set $\{\phi_j\}$ substitutes the set of $\{\lambda_j\}$ in [1], all other notations was introduced in the first section.

Eigenvalues of the transfer matrix $T(\phi)$ are also trigonometric polynomial of the degree $N$. Instead of the rational function $R(\lambda)$ we now have meromorphic function:

$$R(\phi) = \frac{\sin^N \phi}{Q(\phi - \eta)Q(\phi + \eta)}.$$  

The analogue of the decomposition on the primitive fractions in trigonometric case is the decomposition of (14) on to the primitive functions $1/\sin(\phi - \phi_j \pm \eta)$ for odd $N$ and $\cot(\phi - \phi_j \pm \eta)$ for even $N$.

Making use of such expansion and taking into account Bethe Ansatz we obtain the trigonometric analogue of representation (12):

$$R(\phi) = \pi(\phi) + \frac{q(\phi + \eta)}{Q(\phi + \eta)} - \frac{q(\phi - \eta)}{Q(\phi - \eta)},$$

where $\pi(\phi)$ is the trigonometric polynomial of the degree $N - 2n$, while $\deg q(\phi) < n$.

Now the construction of the $P(\phi)$ which is the analogue of $P(\lambda)$ is reduced to the construction of the trigonometric polynomial $\rho(\phi)$ satisfying

$$\rho(\phi + \eta) - \rho(\phi - \eta) \equiv \pi(\phi).$$

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In the present paper we shall consider the case of $q$ - parameter is not the root of unity i.e. $\eta$ is not the rational part of $\pi$.

In this case $\sin(k\eta) \neq 0$, $k \in \mathbb{Z}$ and so we can use the following simple formulas:

$$
\sin(k\phi) = \frac{\cos(k(\phi - \eta)) - \cos(k(\phi + \eta))}{\sin(k\eta)},
$$

$$
\cos(k\phi) = \frac{\sin(k(\phi + \eta)) - \sin(k(\phi - \eta))}{\sin(k\eta)},
$$

(43)

For odd $N$, the degree of $\pi(\phi)$ is also odd and it may be decomposed in the harmonics $\cos(k\phi), \sin(k\phi)$, $k \neq 0$.

In this case the equations (43) solve the problem (42) and $\rho(\phi)$ is the trigonometric polynomial of the degree $N - 2n$. The polynomial:

$$
P(\phi) \equiv \rho(\phi)Q(\phi) + q(\phi),
$$

(44)

is the second solution of (38). Its degree is $N - n$.

Apparently its decomposition:

$$
P(\phi) = \text{const} \prod_{j=1}^{n^*} \sin(\phi - \phi^*_j),
$$

(45)

where $n^* = N - n$ gives the solution for trigonometric Bethe-Ansatz equation

$$
\left( \frac{\sin(\phi_j + \eta)}{\sin(\phi_j - \eta)} \right)^N = \prod_{k \neq j}^{n^*} \frac{\sin(\phi_j - \phi_k + 2\eta)}{\sin(\phi_j - \phi_k - 2\eta)} \quad (j = 1, 2, \ldots n^*)
$$

(46)

"beyond equator".

For even $N$, the polynomial $\pi(\phi)$ has the zero harmonic and therefore the solution of (42) acquires term with linear (nonperiodic) dependence of $\phi$.

As the result we have the following

**Theorem on the associated solution of Baxter equation for XXZ spin chain**

For odd length of spin chain $N$ the equation (38) has the couple of associated solutions which are the trigonometrical polynomials of the degrees $n \leq N/2$ and $n^* = N - n > N/2$. 

11
For even length $N$ one solution is the trigonometrical polynomial of the degrees $n$ while the second has the form (44), where $\rho(\phi)$ contains the linear (nonperiodic) dependence of $\phi$.

For the construction of fusion relation in the case of $XXZ$ - model it is sufficient to use two main ingredients - the analogues of eqs. (21) and (22). The first one can be extracted from representation for $R(\phi)$:

$$P(\phi + \eta)Q(\phi - \eta) - P(\phi - \eta)Q(\phi + \eta) = \sin^N \phi. \quad (47)$$

The second may be written as follows:

$$T_s(\phi) = P(\phi + (2s + 1)\eta)Q(\phi - (2s + 1)\eta) - P(\phi - (2s + 1)\eta)Q(\phi + (2s + 1)\eta). \quad (48)$$

Therefore all the results of Sections 2 and 3 holds true for generic $XXZ$ spin chain.

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**References**

[1] Bethe H., *Zeitschrift fur Physik* **71** (1931) 205-226.

[2] Faddeev L.D., *Les-Houches lectures* How Algebraic Bethe Ansatz works for integrable model. (1995) 1-59. [hep-th/9605187](http://arxiv.org/abs/hep-th/9605187)

[3] Faddeev L.D., Takhtajan L.A., *Zap. nauch. semin. LOMI* **109**. Leningrad, “Nauka”, (1981) 134-178.

[4] Baxter R.J., *Studies in Applied Mathematics* **L** (1971) 51-69.

[5] Baxter R.J., *Ann. Phys. (N.Y.)* **70** (1972) 193-228.

[6] Baxter R.J., *Ann. Phys. (N.Y.)* **76** (1973) 1-24, 25-47, 48-71.
[7] Bazhanov, V.V., Lukyanov, S.L. and Zamolodchikov, A.B., Commun. Math. Phys. 177, (1996) 381-398, 190, (1997) 247-278; ”Integrable Structure of Conformal Field Theory III. The Yang-Baxter Relation”, hep-th/9805008.

[8] Antonov, A., Feigin, B., ”Quantum Group Representations and Baxter Equation”, hep-th/9603105.

[9] Krichiver, I., Lipan, O., Wiegmann, P., and Zabrodin, A., Commun. Math. Phys. 188, (1997) 267-304.

[10] Kirillov, A.N. and Reshetikhin, N.Yu., J. Phys. A20 (1987) 1565-1585.