ABSTRACT

It has been shown that there is a sequential embedding structure in a $w_N$ string theory based on a linearized $W_N$ algebra. The $w_N$ string theory is obtained as a special realization of the $w_{N+1}$ string. The $w_\infty$ string theory is a universal string theory in this sense. We have also shown that there is a similar sequence for $N = 1$ string theory. The $N = 1 w_N$ string can be given as a special case of the $N = 1 w_{N+1}$ string. In addition, we show that the $w_3$ string theory is obtained as a special realization of the $N = 1 w_3$ string. We conjecture that the $w_N$ string can be obtained as a special $N = 1 w_N$ string for general $w_N$. If this is the case, $N = 1 w_\infty$ string theory is more universal since it includes both $N = 0$ and $N = 1 w_N$ string theories.

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1. Introduction

The symmetry breaking is one of the important concept in physics. On the unified theory viewpoint, it can be used to lead variety of theories from a unified theory which has higher symmetry. This idea has another advantage since the higher symmetry constrains the theory more strongly. If there exists a unique unified theory, it should have a maximum symmetry. It is a fascinating idea that there is such a universal theory from which all the possible theories can be lead.

In the string theory, it has been recently found that this kind of unification is possible.\cite{1,2} $N = 0$ ($N = 1$) string theory can be lead from a special $N = 1$ ($N = 2$) string theory. This is such a new kind of symmetry breaking that the physical subspace has less symmetry than the whole Hilbert space since some members of a multiplet are decoupled from the physical space as a BRST quartet. This result has been able to be extended to $N = 4$ string\cite{3} which is a maximal number of (linear) supersymmetries in two dimensions. This theory is a candidate of the universal theory which leads all the superstring theories.

There is another sequence of theories that is one of $W_N$ strings. While embedding of bosonic ($W_2$) string in the $W_3$ string has been found in Ref. \cite{4}, it is complicated due to the nonlinear nature of the $W_3$ algebra and is difficult to extend to general $W_N$ theories. In this paper, we consider linearized $w_N$ algebra, called linear $w_N$ algebra, and corresponding string theory which has been obtained in Ref. \cite{5}. This has an advantage that it is easy to investigate general structure of embeddings. The theory which has a maximum symmetry is the $w_\infty$ string theory and it is a universal theory of this sequence. Unfortunately the sequential embeddings of this case are not general. The $w_N$ string theory included in the $w_{N+1}$ string has the special form which is used to realize further symmetry breaking to the $w_{N-1}$ string. Only the bosonic ($w_2$) string theory is embedded in the general form.

We have two sequences of embeddings labeled by the number of supersymmetries and by the number of higher dimensional generators. The $w_\infty$ string theory
is not universal in a sense that it does not include the sequence of superstrings. We further consider the $N = 1 \ w_{N+1}$ string theory and show that this includes the $N = 1 \ w_N$ string. It is also shown that there is another embedding structure related these two sequences. The $w_3$ string can be obtained by a special $N = 1 \ w_3$ string. If this embedding structure holds for general $w_N$ string, $N = 1 \ w_\infty$ string theory is more universal which include a part of the sequence of supersymmetries.

The linearization of $W_N$ algebra is not unique and there is another linearized $w_N$ algebra[7] which we call the abelian $w_N$ algebra. This linearization is such a simple that all the higher dimensional generators commute. We can repeat all the investigations for another $w_N$ string theory based on the abelian $w_N$ algebra. In this case, the general $w_N$ string is obtained as a special $w_{N+1}$ string, which is different to the linear $w_N$ string. However, the geometrical meaning of the universal theory is not clear since the $N \to \infty$ limit of the abelian $w_N$ algebra is not the area preserving diffeomorphisms.

This paper is organized as follows.

We explain the linear $w_N$ algebra in §2. The string theory is constructed based on the linear $w_N$ algebra. The BRST operator is constructed for the $w_N$ string theory. For discussing the physical vertex operators, we define the invariant vacuum under the $sl(N, \mathbb{R})_c$ which is the finite subalgebra of the linear $w_N$ algebra.

In §3, we show that the $w_N$ string theory can be obtained by a special realization of the $w_{N+1}$ string. By a similarity transformation, the BRST operator of $w_{N+1}$ string is transformed into the sum of the BRST operators of the $w_N$ string and a topological theory. Since the cohomology of the topological theory is trivial, the cohomology of the $w_{N+1}$ string is the same with the one of the $w_N$ string. We also prove that amplitudes of $w_{N+1}$ string coincide with those of $w_N$ string.

The $N = 1 \ w_N$ string theory is introduced in §4, where the $N = 1 \ w_N$ algebra and the BRST operator are defined. We show, in §5, that there is a similar sequence to $N = 0$ for $N = 1 \ w_N$ string theory. The $N = 1 \ w_N$ string can be obtained by a special realization of $N = 1 \ w_{N+1}$ string.
In the above argument of the linear $w_N$ string theory, the $w_N$ string obtained by the special $w_{N+1}$ string is not the general one. In §6, it is shown that we can obtain the general $w_N$ string theory if we take another linearization of the $W_N$ algebra, the abelian $w_N$ algebra. We repeat the discussions in §2-5 for the abelian $w_N$ algebra. The general $w_N$ string is obtained as a special $w_{N+1}$ string while the algebra obtained by $N \to \infty$ limit is different from the area preserving diffeomorphisms.

In §7, we discuss another embedding of the string: the $w_N$ string can be obtained as a special $N = 1$ $w_N$ string. The explicit construction of embedding is given for the case of the $w_3$ string. If this embedding structure exists for general $w_N$, the $N = 1$ $w_{\infty}$ string is more universal theory than $w_{\infty}$ string.

§8 is devoted to the discussions.

2. $w_N$ string theory

In an analogous way to Ref. [1], it has been found that the bosonic ($W_2$) string theory can be obtained from the $W_3$ string theory by taking a special realization of $W_3$ algebra.\(^4\) It is natural to generalize this to string theories with higher symmetry, $W_N$ string theories. The general $W_N$ algebra is, however, too complicated to construct such a special realization. This complexity come from the fact that the $W_N$ algebra is nonlinear. In order to investigate the sequence of embeddings in $W_N$ string theories, it is better to use simpler algebra which can be constructed generally. In this section, we consider a linearized analog of the $W_N$ algebra which we call the linear $w_N$ algebra. It is explained that the linear $w_N$ algebra and the string theory constructed based on it.

The $w_N$ algebra is generated by currents $w_i(z) \ (i = 0, \cdots, N - 2)$ which have dimension $i + 2$. The first current of them is the stress tensor $w_0(z) = T(z)$. A natural linearization of the $W_N$ algebra was obtained in Ref. [5] as

$$w_i(z)w_j(w) \sim \frac{2\delta_{i,j}\delta_{i,0}}{(z-w)^4} + \frac{(i+j+2)w_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial w_{i+j}(w)}{z-w},$$
for $i + j \leq N - 2$, for $i + j > N - 2$.

(2.1)

This algebra becomes $w_\infty$, area preserving diffeomorphisms, by taking the limit $N \to \infty$.

The $w_N$ string theory is constructed by using the linear $w_N$ algebra and fermionic ghosts $(b_i^{gh}(z), c_i^{gh}(z)), i = 0, \ldots, N - 2$. These ghost fields have dimensions $(i + 2, -i - 1)$ and satisfy the OPE

$$c_i^{gh}(z)b_j^{gh}(w) \sim \frac{\delta_{i,j}}{z - w}. \quad (2.2)$$

The Hilbert space is direct product of the representation space of the linear $w_N$ algebra and the ghost Fock space. The physical subspace is defined by BRST operator

$$Q_{\text{BRST}}^{(N)} = \oint dz \, \frac{N - 2}{2\pi i} \sum_{i=0}^{N-2} c_i^{gh}(w_i + \frac{1}{2} w_i^{gh}), \quad (2.3)$$

where the ghost's generators $w_i^{gh}(z)$ are defined by

$$w_i^{gh}(z) = \sum_{j=0}^{N-i-2} \left( -(i + j + 2)b_{i+j}^{gh} \partial c_j^{gh} - (j + 1)\partial b_{i+j}^{gh} c_j^{gh} \right). \quad (2.4)$$

This BRST operator is nilpotent if the matter system has the central charge

$$c(N) = \sum_{j=2}^{N} 2(6j^2 - 6j + 1)$$

$$= 2(N - 1)(2N^2 + 2N + 1). \quad (2.5)$$
For discussing the physical operators, let us introduce mode expansions as follows.

\[
w_i(z) = \sum_n w_{i,n} z^{-n-i-2},
\]

\[
b^gh_i(z) = \sum_n b^gh_{i,n} z^{-n-i-2},
\]

\[
c^gh_i(z) = \sum_n c^gh_{i,n} z^{-n+i+1}.
\]

By means of these mode expansion, the linear \(w_N\) algebra are written as

\[
[w_i,n, w_j,m] = \{(j+1)n - (i+1)m\}w_{i+j,n+m} + \frac{c}{12}n(n^2 - 1)\delta_{i,j}\delta_{i,0}\delta_{n+m,0},
\]

for \(i + j \leq N - 2\),

\[
[w_i,n, w_j,m] = 0,
\]

for \(i + j > N - 2\).

(2.6)

The linear \(w_N\) algebra has finite subalgebra which is generated by operators

\[
\bigoplus_{i=0}^{N-2} \{ \bigoplus_{n=-i-1}^{i+1} w_{i,n} \}.
\]

(2.7)

The physical state is obtained, up to picture changing explained later, as

\[
|\text{phys}\rangle = V_{mat}(0) |0\rangle_{mat} \otimes \prod_{i=0}^{N-2} \prod_{n=1}^{i+1} c^gh_{i,n} |0\rangle_{gh},
\]

(2.10)
or equivalently, the physical vertex operator has the form

\[
V_{\text{phys}}(z) = \prod_{i=0}^{N-2} \prod_{n=0}^{i} \partial^n c_i^h(z) V_{\text{mat}}(z). \tag{2.11}
\]

Here \(V_{\text{mat}}(z)\) is a primary field of the matter \(w_N\) algebra which must have the dimensions \(\frac{N}{6}(N^2 - 1)\).

Before closing this section, we should comment on the number of the \(w_N\) moduli. The moduli space of \(w_N\) algebra can be defined by the moduli space of flat \(sl(N, \mathbb{R})_c\) connections\(^{[6,5,7]}\)

\[
\mathcal{M}_{g,n} = \text{Hom}(\pi_1(\Sigma_{g,n}), SL(N, \mathbb{R})_c) / \sim
\]

for the genus \(g\) and \(n\) punctured surface. The dimension of this moduli space is obtained by an index theorem as

\[
dim \mathcal{M}_{g,n} = (N^2 - 1)(g - 1) + N(N - 1)n/2
\]

\[
= 3(g - 1) + n
\]

\[
+ 5(g - 1) + 2n
\]

\[
+ \cdots
\]

\[
+ (2N - 1)(g - 1) + (N - 1)n, \tag{2.12}
\]

where the number on the \(i\)-th line in the last equality is the number of the moduli comes from the spin \(i + 1\) gauge field.\(^{[8]}\)
3. \( w_N \subset w_{N+1} \)

In the previous section, we have constructed the \( w_N \) string theory based on the linear \( w_N \) algebra. Let us show, in this section, that this linear \( w_N \) string theory is interpreted as the special case of the linear \( w_{N+1} \) string theory.

Consider the general critical bosonic \((w_2)\) string theory represented by the stress tensor \( T(z) \) with \( c = 26 \). By adding bosonic matter fields \((\beta_i(z), \gamma_i(z))\), \( i = 1, \cdots, N-1 \), we can construct generators of the \( w_{N+1} \) algebra \( \{w_i(z)\}, \, i = 0, \cdots, N-1 \) as

\[
w_i(z) = \delta_{i,0} T + i \beta_i + \sum_{j=1}^{N-i-1} \left( - (i + j + 2) \beta_{i+j} \partial \gamma_j - (j+1) \partial \beta_{i+j} \gamma_j \right). \tag{3.1}
\]

Here the additional fields have dimensions \((i + 2, -i - 1)\) and satisfy the OPE

\[
\gamma_i(z) \beta_j(w) \sim \frac{\delta_{i,j}}{z-w}. \tag{3.2}
\]

This realization can be interpreted as being constructed by the \( w_N \) string defined by generators \( \{\tilde{w}_i\}, \, i = 0, \cdots, N-2 \) and \((\beta_{N-1}(z), \gamma_{N-1}(z))\), where \( \tilde{w}_i \) is obtained by removing \((\beta_{N-1}(z), \gamma_{N-1}(z))\) in (3.1). However, this \( w_N \) string is the special one realized by means of \( T(z) \) and \((\beta_i(z), \gamma_i(z)), \, i = 1, \cdots, N-2 \). Only the \( w_2 \) (bosonic) string theory is included as a whole. By this construction, therefore, we can only show that the linear \( w_{N+1} \) string is including special realizations of the linear \( w_i \) string \((i = 3, \cdots, N)\) and the general \( w_2 \) string.

Since the central charge of this matter system is equal to the critical value \( c(N+1) \), the \( w_{N+1} \) string theory is obtained by adding the fermionic ghosts \((b_i^{gh}(z), c_i^{gh}(z))\) \( i = 1, \cdots, N-1 \) as explained in §2. We can prove the coincidence of the cohomology of this special realization of \( w_{N+1} \) string and of the \( w_N \) string by giving a similarity transformation \([9]\) defined by

\[
R = \frac{1}{N-1} \oint \frac{dz}{2\pi i} \sum_{i=0}^{N-2} c_i^{gh} \left[ (N+1)b_{N-1}^{gh} \partial \gamma_{N-i-1} + (N-i) \partial b_{N-1}^{gh} \gamma_{N-i-1} \right]. \tag{3.3}
\]
The BRST operator of the $w_{N+1}$ string $Q^{(N+1)}_{BRST}$ is transformed by this similarity transformation into the sum of the BRST operators of the $w_N$ string $Q^{(N)}_{BRST}$ and a topological theory $Q_{top}$.

$$e^R Q^{(N+1)}_{BRST} e^{-R} = Q^{(N)}_{BRST} + Q_{top}, \quad (3.4)$$

where

$$Q_{top} = \oint \frac{dz}{2\pi i} (N - 1)c_{N-1}^{gh}\beta_{N-1}. \quad (3.5)$$

Therefore the physical states of the $w_{N+1}$ string are given by the tensor product of the physical states of the $w_N$ string and the topological theory. The further discussions hereafter are given in the transformed theory. The results can be inversely transformed if one want to get those in the original form.

The cohomology of the topological theory is trivial since the topological BRST operator has the bilinear form and thus all the fields $(\beta_{N-1}(z), \gamma_{N-1}(z))$ and $(b_{N-1}^{gh}(z), c_{N-1}^{gh}(z))$ make the quartet and are decoupled from the physical subspace. Only the physical state in the topological sector is the physical vacuum defined later. To explain this further, we note that the $sl(N+1, \mathbb{R})_c$ vacuum split into the direct product of the $sl(N, \mathbb{R})_c$ vacuum in the $w_N$ sector and the vacuum in the topological sector. The former is defined in eq. (2.9) and the latter is obtained by

$$\beta_{N-1,n} |0\rangle_{top} = 0, \quad \text{for } n \geq -N,$$
$$\gamma_{N-1,n} |0\rangle_{top} = 0, \quad \text{for } n > N,$$
$$b_{N-1,n}^{gh} |0\rangle_{top} = 0, \quad \text{for } n \geq -N,$$
$$c_{N-1,n}^{gh} |0\rangle_{top} = 0, \quad \text{for } n > N, \quad (3.6)$$

where we introduce the mode expansion of $(\beta_{N-1}(z), \gamma_{N-1}(z))$ as

$$\beta_{N-1}(z) = \sum_{n} \beta_{N-1,n} z^{-n-N-1}$$
\begin{equation}
\gamma_{N-1}(z) = \sum_{n} \gamma_{N-1,n} z^{-n+N}.
\end{equation}

In order to define the physical vacuum, we must bosonize the matter fields \((\beta_{N-1}(z), \gamma_{N-1}(z))\) in a similar way to the bosonic ghost fields in the superstring theory\(^{[10]}\)\(^{*}\):

\begin{align*}
\beta_{N-1}(z) &= e^{-\phi_{N-1}(z)} \eta_{N-1}(z), \\
\gamma_{N-1}(z) &= -\partial \xi_{N-1}(z) e^{\phi_{N-1}(z)}.
\end{align*}

The mode expansions of these fields is defined by

\begin{align*}
i \partial \phi_{N-1}(z) &= \sum_{n} \phi_{N-1,n} z^{-n-1}, \\
\eta_{N-1}(z) &= \sum_{n} \eta_{N-1,n} z^{-n-1}, \\
\xi_{N-1}(z) &= \sum_{n} \xi_{N-1,n} z^{-n}.
\end{align*}

The topological vacuum (3.6) is defined by these bosonized fields as

\begin{align*}
\phi_{N-1,n} |0\rangle_{\text{top}} &= 0, \quad \text{for } n \geq 0, \\
\eta_{N-1,n} |0\rangle_{\text{top}} &= 0, \quad \text{for } n \geq 0, \\
\xi_{N-1,n} |0\rangle_{\text{top}} &= 0, \quad \text{for } n > 0.
\end{align*}

The physical vacuum of the topological sector is defined by

\begin{equation}
|0\rangle_{\text{top}}^{\text{phys}} = \prod_{n=0}^{N-1} \partial^{n} e_{N-1}(0) e^{-N\phi_{N-1}(0)} |0\rangle_{\text{top}}.
\end{equation}

Therefore the physical vertex operator of the \(w_{N+1}\) string is obtained by

\begin{equation}
V_{\text{phys}}^{(N+1)}(z) = \prod_{n=0}^{N-1} \partial^{n} e_{N-1}(z) e^{-N\phi_{N-1}(z)} V_{\text{phys}}^{(N)}(z),
\end{equation}

where \(V_{\text{phys}}^{(N)}(z)\) is the physical vertex operator in the \(w_{N}\) string.

\(^{*}\) Here we invert the role of \((\eta_{N-1}(z), \partial \xi_{N-1}(z))\) comparing with the bosonic ghosts. This bosonization is convenient for this case as becoming clear later.
In the remaining part of this section, we discuss that the amplitudes of the $w_{N+1}$ string coincide with those of the $w_N$ string. This is nontrivial since the rules of calculating the amplitudes are different for two theories. For calculating the amplitudes, we need to introduce the picture changing operator $Z_{N-1}(z)$ as in the superstring theory.\[^{10}\] It is obtained by

$$
Z_{N-1}(z) = \{Q^{(N+1)}_{BRST}, \xi_{N-1}(z)\} \\
= \{Q_{top}, \xi_{N-1}(z)\} \\
= (N - 1)e^{\frac{gh}{N-1}}N_{-1}(z)e^{-\phi_{N-1}(z)}. 
$$

The inverse picture changing operator, which is needed to calculate the amplitudes, is easily found as

$$
Z_{N-1}^{-1}(z) = \frac{1}{N - 1}e^{\phi_{N-1}(z)}b_{N-1}^{gh}(z),
$$

which satisfies

$$
\lim_{z \to w} Z_{N-1}^{-1}(z)Z_{N-1}(w) = 1, \\
[Q^{(N+1)}_{BRST}, Z_{N-1}^{-1}(z)] = 0. 
$$

If we assume that the rules of calculating the amplitudes of the $w_N$ string are given, (the holomorphic part of) the amplitudes of the $w_{N+1}$ string can be defined by using the inverse picture changing operator as

$$
\mathcal{A}^{(N+1)}_{g,n} = \int \prod_{i=1}^{3g-3+n} dm_i \left\langle \mathcal{O} \prod_{i=1}^{(2N+1)(g-1)+Nn} Z_{N-1}^{-1}(z_i) \prod_{j=1}^{n} V^{(N+1)}_{phys}(z_j) \right\rangle_{N+1} 
$$

where we denote the operators needed to insert for calculating the amplitudes of the $w_N$ string as $\mathcal{O}$. The number of the inserted $Z_{N-1}^{-1}$ is the same as the number of the spin $N + 1$ moduli explained in §2. We can interpret this insertion as the
result of the integration of these moduli. One can easily see that this is factorized
to the product of the amplitudes of the $w_N$ string and of the topological theory. It
coincides the amplitudes of the $w_N$ string since the part of the topological theory
are one always up to sign comes from ordering of the fermionic operators.

$$A^{(N+1)}_{g,n} = \int \prod_{i=1}^{3g-3+n} dm_i \left\langle O \prod_{j=1}^{n} V^{(N)}_{\text{phys}}(z_j) \right\rangle_N \times$$

$$\left\langle \prod_{i=1}^{(2N+1)(g-1)+Nn} (e^{\phi_{N-1}(z_i)} b^{gh}_{N-1}(z_i)) \prod_{j=1}^{n} \partial^n c^{gh}_{N-1}(z_j) e^{-N\phi_{N-1}(z_j)} \right\rangle_{\text{top}}$$

$$= \int \prod_{i=1}^{3g-3+n} dm_i \left\langle O \prod_{j=1}^{n} V^{(N)}_{\text{phys}}(z_j) \right\rangle_N$$

$$= A^{(N)}_{g,n}, \quad (3.17)$$

where it is noted that the similarity transformation (3.3) acts not only on the
topological sector but also on the $w_N$ sector. One can see however that it does not
affect the $w_N$ amplitudes due to the (anomalous) $\beta_i^\gamma_i$ number conservation.

Therefore it is completed the proof that $w_N$ string theory can be obtained by
a special realization of $w_{N+1}$ string. We have shown that there is a sequence of the
$w_N$ string theory. $w_N$ string theory can be interpreted as a special $w_{N+1}$ string
theory. We can consider the $w_\infty$ string theory as a limit, which has a maximum
symmetry in this sequence and is a universal theory in this sense.
4. $N = 1 \ w_N$ string theory

We have shown that the $w_\infty$ string theory is a universal theory which includes the $w_N$ string theories as a special case. The $w_\infty$ string, however, does not include the superstring theories. Next we consider the $N = 1 \ w_N$ string theory and the sequential embeddings including $N = 1$ superstring.

We define the $N = 1 \ w_N$ string theory in this section. $N = 1$ extension of the linear $w_N$ algebra is generated by bosonic generators $\{w_i(z)\}$ and fermionic generators $\{v_i(z)\}, \ i = 0, \cdots, N - 2$. Here we denote that $w_0(z) = T(z)$ and $v_0(z) = G(z)$. The dimensions of the bosonic (fermionic) generators are $i + 2 \ (i + 3/2)$. The OPE relations of the $N = 1$ linear $w_N$ algebra is given by

$w_i(z)w_j(w) \sim \frac{6\delta_{ij}\delta_{i0}}{(z-w)^4} + \frac{(i+j+2)w_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial w_{i+j}(w)}{z-w},$

$w_i(z)v_j(w) \sim \frac{(i+j+\frac{3}{2})v_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial v_{i+j}(w)}{z-w},$

$v_i(z)v_j(w) \sim \frac{2c\delta_{i,j}\delta_{i0}}{(z-w)^3} + \frac{2w_{i+j}(w)}{z-w}, \quad \text{for} \ i+j \leq N - 2,$

$w_i(z)w_j(w) \sim w_i(z)v_j(w) \sim v_i(z)v_j(w) \sim 0, \quad \text{for} \ i+j > N - 2. \quad (4.1)$

For constructing the string theory, we now need both fermionic and bosonic ghosts $(b_i^{gh}(z), c_i^{gh}(z))$ and $(\beta_i^{gh}(z), \gamma_i^{gh}(z)), \ i = 0, \cdots, N - 2$. The dimensions of these ghost fields are $(i + 2, -i - 1)$ and $(i + 3/2, -i - 1/2)$. The BRST operator is constructed as

$$Q_{BRST}^{N=1(N)} = \oint \frac{dz}{2\pi i} \sum_{i=0}^{N-2} \left( c_i^{gh}(w_i + \frac{1}{2}w_i^{gh}) - \gamma_i^{gh}(v_i + \frac{1}{2}v_i^{gh}) \right), \quad (4.2)$$

where

$$w_i^{gh}(z) = \sum_{j=0}^{N-i-2} \left( -(i+j+2)b_{i+j}^{gh}\partial c_j^{gh} - (j+1)\partial b_{i+j}^{gh}c_j^{gh} \right).$$
\[ v_{i}^{gh}(z) = \sum_{j=0}^{N-i-2} \left( 2b_{i+j}^{gh} \gamma_{j}^{gh} - (i + j + \frac{3}{2}) \beta_{i+j}^{gh} \partial \gamma_{j}^{gh} - (j + 1) \partial \beta_{i+j}^{gh} \gamma_{j}^{gh} \right). \]

(4.3)

The critical central charge is now

\[ c^{N=1}(N) = \sum_{j=2}^{N} \left\{ 2(6j^2 - 6j + 1) - 2(6j - \frac{1}{2})^2 - 6(j - \frac{1}{2}) + 1 \right\} \]
\[ = 3(N-1)(2N+1). \]

(4.4)

The number of super moduli is \[^{[8]}\]

\[ (\# \text{ of super moduli})_{g,n} = 2(g - 1) + n \]
\[ + 4(g - 1) + 2n \]
\[ + \cdots \]
\[ + 2(N - 1)(g - 1) + (N - 1)n \]
\[ = N(N - 1)(g - 1) + N(N - 1)n/2. \]

(4.5)

Here we assume that all the punctures are NS states, which is sufficient for later investigations.
5. $N = 1 \ w_N \subset N = 1 \ w_{N+1}$

In this section, we show that there is a similar sequential embedding structure to the $w_N$ string in the $N = 1$ case. For obtaining a special realization of the $N = 1$ linear $w_{N+1}$ algebra, we must add the matter fields $(\beta_i(z), \gamma_i(z))$ and $(b_i(z), c_i(z)), i = 1, \cdots, N - 1$ to the general critical $N = 1$ superstring theory. Here these additional fields have dimensions $(i + 2, -i - 1)$ and $(i + 3/2, -i - 1/2)$ and the superstring theory is represented by $N = 1$ super conformal generators $(T(z), G(z))$ with $c = 15$. A special realization of the $N = 1$ linear $w_{N+1}$ algebra is given by

$$
\begin{align*}
\omega_i(z) &= \delta_{i,0}T + i\beta_i \\
&\quad + \sum_{j=1}^{N-1} \left( -(i + j + 2)\beta_{i+j}\partial\gamma_j - (j + 1)\partial\beta_{i+j}\gamma_j - (i + j + \frac{3}{2})b_{i+j}\partial c_j - (j + \frac{1}{2})\partial b_{i+j}c_j \right), \\
\nu_i(z) &= \delta_{i,0}G + ib_i \\
&\quad + \sum_{j=1}^{N-1} \left( 2\beta_{i+j}c_j - (i + j + \frac{3}{2})b_{i+j}\partial\gamma_j - (j + 1)\partial b_{i+j}\gamma_j \right).
\end{align*}
$$

The BRST operator is transformed into the sum of the BRST operators of the $N = 1 \ w_N$ string and the topological theory by the similarity transformations:

$$
 e^R Q_{BRST}^{N=1(N+1)} e^{-R} = Q_{BRST}^{N=1(N)} + Q_{top},
$$

where

$$
Q_{top} = \oint \frac{dz}{2\pi i} \left( (N - 1)c_{N-1}^{gh}\beta_{N-1} - (N - 1)\gamma_{N-1}^{gh}b_{N-1} \right).
$$

The generator of this similarity transformation is

$$
R = \frac{1}{N - 1} \oint \frac{dz}{2\pi i} \sum_{i=0}^{N-2} \left[ c_i^{gh} \left( (N + 1)b_{N-1}^{gh}\partial\gamma_{N-i-1} + (N - i)\partial b_{N-1}^{gh}\gamma_{N-i-1} \right) \right].
$$
\[-(N + \frac{1}{2})\beta_{N-1}^g c_{N-1-i-1} - (N - i - \frac{1}{2})\partial \beta_{N-1}^g c_{N-1-i-1} \]
\[\gamma_i^g \left(2b_{N-1}^g c_{N-1-i-1} \right. \]
\[+ (N + \frac{1}{2})\beta_{N-1}^g \partial \gamma_{N-i-1} + (N - i)\partial \beta_{N-1}^g \gamma_{N-i-1} \bigg] \]

Therefore the cohomology of the $N = 1 \ w_{N+1}$ string coincides with that of the
$N = 1 \ w_N$ string.

The coincidence of the amplitudes is easily shown by considering the transformed theory. We bosonize $(\beta_{N-1}^g(z), \gamma_{N-1}^g(z))$ as

\[\gamma_{N-1}^g(z) = e^{\phi_{N-1}^g(z)} \eta_{N-1}^g(z), \quad \beta_{N-1}^g(z) = \partial \xi_{N-1}^g(z) e^{-\phi_{N-1}^g(z)}. \quad (5.5)\]

By means of these bosonized fields, the physical vertex operators in the canonical
picture can be factorized to the physical vertex of the $N = 1 \ w_N$ string and of the
topological theory.

\[V_{\text{phys}}^{(N+1)N=1}(z) = V_{\text{phys}}^{(N)N=1}(z) \left( \prod_{n=0}^{N-1} \partial^n c_{N-1}(z) e^{-N\phi_{N-1}^g(z)} \right) \left( \prod_{n=0}^{N-1} \partial^n c_{N-1}^g(z) e^{-N\phi_{N-1}(z)} \right) \]
\[\equiv V_{\text{phys}}^{(N)N=1}(z)V^{\text{top}}(z). \quad (5.6)\]

Thus the amplitudes of the $N = 1 \ w_{N+1}$ string become the product of those of the
$N = 1 \ w_N$ string and of the topological theory.

The picture changing operators for calculating the topological amplitudes are

\[Z_{N-1}(z) \equiv \{Q_{\text{top}}, \xi_{N-1}(z)\} = (N - 1)c_{N-1}^g(z) e^{-\phi_{N-1}(z)}, \quad (5.7)\]

for matter bosonic fields and

\[Z_{N-1}^g(z) \equiv \{Q_{\text{top}}, \xi_{N-1}^g(z)\} = (N - 1)c_{N-1}^{gh}(z) b_{N-1}(z), \quad (5.8)\]
for ghost bosonic fields. The inverse picture changing operators are easily found:

\[
Z^{-1}_{N-1}(z) = \frac{1}{N-1} e^{\phi_{N-1}(z)} c_{N-1}^{gh}(z), \quad Z_{N-1}^{gh-1}(z) = \frac{1}{N-1} c_{N-1}(z) e^{-\phi_{N-1}^{gh}(z)}.
\]

The topological amplitudes are given by

\[
A^\text{top}_{g,n} = \left\langle \prod_{i=1}^{(2N+1)(g-1)+Nn} Z^{-1}_{N-1}(z_i) \prod_{i=1}^{2N(g-1)+Nn} Z_{N-1}^{gh}(z_i) \prod_{k=1}^{n} V^\text{top}(z_k) \right\rangle = 1,
\]

(5.10)

up to sign comes from interchanging the order of the fermionic operators. The \(Z^{-1}_{N-1} (Z_{N-1}^{gh})\) insertions come from the spin \(N + 1\) moduli (the spin \(N + 1/2\) supermoduli) integrals. Therefore the amplitudes of the \(N = 1\) \(w_{N+1}\) string coincide with those of the \(N = 1\) \(w_N\) string.

6. Another linearized \(w_N\) algebra

So far we have shown that \((N = 1)\) \(w_N\) string theory can be obtained as a special realization of the \((N = 1)\) \(w_{N+1}\) string. However, this \(w_N\) string is not general but a special as mentioned in §3 and §5. Only the \(w_2\) (bosonic) string theory is included in the general form. One may consider that this is insufficient and the general \(w_N\) string theory should be obtained by a special realization of the \(w_{N+1}\) string. In this section, we explain that this is possible by considering another linearization of the nonlinear \(W_N\) algebra.

The linearization of the \(W_N\) algebra is not unique and there is another linearized \(w_N\) algebra which was defined in Ref. [7]:

\[
w_i(z)w_j(w) \sim 0, \quad \text{for } i, j = 1, \ldots, N - 2,
\]

(6.1)

and the OPE related to \(w_0(z) = T(z)\) is the same with that for the linear \(w_N\) algebra. This can be obtained from the linear \(w_N\) algebra by a group contraction. We call this linearized \(w_N\) algebra as abelian \(w_N\) algebra.
To obtain the BRST operator, we have to replace the ghost generators in (2.4) to

\[ w^g_i(z) = -(i + 2)b^g_i \partial c^g_0 - \partial b^g_i c^g_0, \quad \text{for } i = 1, \cdots, N - 2. \]  

(6.2)

The general $w_N$ string theory is represented by the generators \{\tilde{w}_i\}, $i = 1, \cdots, N - 2$, with the critical central charge $c(N)$ obtained in (4.4). A special realization of the abelian $w_{N+1}$ algebra is constructed by the matter system composed of the above general theory with $w_N$ symmetry and the bosonic fields $(\beta_{N-1}(z), \gamma_{N-1}(z))$, which have dimensions $(N + 1, -N)$. Generators of the $w_{N+1}$ symmetry \{\tilde{w}_i\}, $i = 0, \cdots, N - 1$ is defined by

\[ w_i(z) = \tilde{w}_i + \delta_{i,0} \left( - (N + 1)\beta_{N-1} \partial \gamma_{N-1} - N\partial \beta_{N-1} \gamma_{N-1} \right), \quad \text{for } i = 0, \cdots, N - 2 \]

\[ w_{N-1}(z) = (N - 1)\beta_{N-1}. \]  

(6.3)

The BRST operator of this $w_{N+1}$ string theory is transformed into the same form (3.4) by the similarity transformation generated by

\[ R^{\text{abelian}} = \frac{1}{N - 1} \oint \frac{dz}{2\pi i} c^g_0 \left( (N + 1)\beta^g_{N-1} \partial \gamma_{N-1} + N\partial b^g_{N-1} \gamma_{N-1} \right). \]  

(6.4)

The remaining proofs are the same with those of the linear $w_N$ cases.

The $N = 1$ abelian $w_N$ algebra is similarly obtained by replacing the r.h.s. of (4.1) to zero in the case of $i \neq 0, j \neq 0$. The ghost generators in BRST operator are now

\[ w^g_i(z) = -(i + 2)b^g_i \partial c^g_0 - \partial b^g_i c^g_0 - (i + \frac{3}{2})\beta^g_i \partial \gamma^g_0 - \frac{1}{2} \partial \beta^g_i \gamma^g_0, \]

\[ v^g_i(z) = 2b^g_i \gamma^g_0 - (i + \frac{3}{2})\beta^g_i \partial c^g_0 - \partial \beta^g_i c^g_0, \]  

(6.5)

for $i = 1, \cdots, N - 2$.  

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A special realization of the $N = 1$ abelian $w_{N+1}$ algebra is obtained by adding the matter fields $(\beta_{N-1}(z), \gamma_{N-1}(z))$ and $(b_{N-1}(z), c_{N-1}(z))$ to the system of the general critical $N = 1$ abelian $w_N$ string. The dimensions of the additional fields are $(N + 1, -N)$ and $(N + 1/2, -N + 1/2)$. Then the special realization of the $N = 1$ abelian $w_{N+1}$ algebra is

$$w_i(z) = \tilde{w}_i + \delta_{i,0} \left( - (N + 1) \beta_{N-1} \partial \gamma_{N-1} - \partial \beta_{N-1} \gamma_{N-1} \right),$$

$$v_i(z) = \tilde{v}_i + \delta_{i,0} \left( 2 \beta_{N-1} \partial \gamma_{N-1} - (N + \frac{1}{2}) b_{N-1} \partial \gamma_{N-1} - N \partial c_{N-1} \gamma_{N-1} \right),$$

for $i = 0, \cdots, N - 2$

$$w_{N-1}(z) = (N - 1) \beta_{N-1},$$

$$v_{N-1}(z) = (N - 1) b_{N-1}, \quad (6.6)$$

where $(\tilde{w}_i(z), \tilde{v}_i(z)), i = 0, \cdots, N - 2$ are generators of abelian $w_N$ algebra. The transformed BRST operator has the same form with (5.2) if we replace the generator of the similarity transformation with

$$R^{\text{abelian}} = \frac{1}{N - 1} \oint \frac{dz}{2\pi i} \left[ c_{N-1}^{gh} (N + 1) b_{N-1}^{gh} \partial \gamma_{N-1} + N \partial c_{N-1}^{gh} \gamma_{N-1} \right.$$

$$- (N + \frac{1}{2}) \beta_{N-1}^{gh} \partial c_{N-1} - (N - \frac{1}{2}) \partial \beta_{N-1}^{gh} c_{N-1}$$

$$- \gamma_{0}^{gh} \left( 2 b_{N-1}^{gh} c_{N-1} + (N + \frac{1}{2}) \beta_{N-1}^{gh} \partial \gamma_{N-1} + N \partial \beta_{N-1}^{gh} \gamma_{N-1} \right) \right].$$

$$\quad (6.7)$$

We have shown that one can construct another $w_N$ string theory based on the abelian $w_N$ algebra. There is also embedding structure in the abelian $w_N$ strings. The general abelian $(N = 1)$ $w_N$ string is obtained as a special realization of the abelian $(N = 1)$ $w_{N+1}$ string.
7. $w_N \subset N = 1 w_N$

We can consider another embedding structure since the $w_N$ algebra can be embedded also in the $N = 1 w_N$ algebra. In this section, we show that the $w_3$ string theory can be actually obtained as a special realization of the $N = 1 w_3$ string.\[ \star \]

A realization of the $N = 1 w_3$ algebra is obtained by using the $w_3$ algebra generated by $(\tilde{w}_0(z), \tilde{w}_1(z))$ and the fermionic fields $(b_0(z), c_0(z))$ and $(b_1(z), c_1(z))$ with dimensions $(3/2, -1/2)$ and $(5/2, -3/2)$. Generators of $N = 1 w_3$ algebra is given by

\[
\begin{align*}
w_0(z) &= \tilde{w}_0 - \frac{3}{2} b_0 \partial c_0 - \frac{1}{2} \partial b_0 c_0 - \frac{5}{2} b_1 \partial c_1 - \frac{3}{2} \partial b_1 c_1 + \frac{1}{2} \partial^2 (c_0 \partial c_0), \\
w_1(z) &= \tilde{w}_1 - 2 c_0 \partial c_0 \tilde{w}_1 - \frac{5}{2} b_1 \partial c_0 + \frac{1}{2} c_0 \partial b_1, \\
v_0(z) &= b_0 + c_0 (\tilde{w}_0 - \frac{5}{2} b_1 \partial c_1 - \frac{3}{2} \partial b_1 c_1 - b_0 \partial c_0) + 7 \partial^2 c_0 - \frac{9}{8} c_0 \partial c_0 \partial^2 c_0, \\
v_1(z) &= b_1 + 2 c_0 \tilde{w}_1 + \frac{5}{2} c_0 \partial c_0 b_1. \quad (7.1)
\end{align*}
\]

The BRST operator can be transformed by the similarity transformation generated by

\[
R = \int \frac{dz}{2 \pi i} \left[ \begin{array}{c} g^h_0 \\ g^h_1 \\ g^h_0 \\ g^h_1 \\ g^h_0 \\ g^h_1 \end{array} \right] \left( -\frac{3}{2} \beta_0^g \partial c_0 - \frac{1}{2} \partial \beta_0^g c_0 - \frac{5}{2} \beta_1^g \partial c_1 - \frac{3}{2} \partial \beta_1^g c_1 - \frac{1}{2} \beta_0^g c_0 \partial c_0 \\
- \frac{3}{2} \partial (\nu_1^g c_0 c_1) - \nu_1^g c_0 \partial c_1 - \frac{35}{24} \beta_1^g c_0 \partial c_0 \partial c_1 - \frac{7}{8} \partial \beta_1^g c_0 \partial c_0 c_1 - \frac{5}{8} \beta_1 c_0 \partial^2 c_0 c_1 \\
+ c_1^g \left( \frac{5}{2} \beta_1^g \partial c_0 + \frac{1}{2} \partial \beta_1^g c_0 \right) \\
+ g_0^g \left( - b_0^g c_0 + \frac{1}{4} \beta_0^g c_0 \partial c_0 + \frac{5}{2} \partial (\beta_1^g c_0 c_1) - \frac{1}{2} \partial \beta_1^g c_0 c_1 + \frac{3}{4} b_1^g c_0 \partial c_0 c_1 \right) \\
+ g_1^g \left( - 2 b_1^g c_0 + \frac{5}{4} \beta_1^g c_0 \partial c_0 \right) + \frac{5}{4} c_0 \partial c_0 b_1 c_1 \right) \quad (7.2)
\]

* We should note that the linear and the abelian $w_N$ algebras are identical for $w_3$. 

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into the form:

\[ e^{-R} Q_{BRST}^{(3)N=1} e^R = Q_{BRST}^{(3)} + Q_{\text{top}}, \]  (7.3)

where

\[ Q_{\text{top}} = - \oint \frac{dz}{2\pi i} (\gamma_0^{gh} b_0 + \gamma_1^{gh} b_1). \]  (7.4)

Thus the cohomology of \( N = 1 \) \( w_3 \) string is equivalent to the \( w_3 \) string. We note that the total stress tensor \( w_0^{\text{tot}}(z) = w_0 + w_0^{gh} \), which has unusual total derivative term, is transformed into the usual form \(^{†}\):

\[ e^R w_0^{\text{tot}} e^{-R} = \tilde{w}_0 - \frac{3}{2} b_0 \partial c_0 - \frac{1}{2} \partial b_0 c_0 - \frac{5}{2} b_1 \partial c_1 - \frac{3}{2} \partial b_1 c_1 \]
\[ - 2b_0^{gh} \partial c_0^{gh} - 2b_0^{gh} \partial c_0^{gh} - 3b_1^{gh} \partial c_1^{gh} - 2\tilde{b}_1^{gh} c_1^{gh} \]
\[ - \frac{3}{2} \beta_0^{gh} \partial \gamma_0^{gh} - \frac{1}{2} \beta_0^{gh} \partial \gamma_0^{gh} - \frac{5}{2} \beta_1^{gh} \partial \gamma_1^{gh} - \frac{3}{2} \beta_1^{gh} \gamma_1^{gh}. \]  (7.5)

The physical vertex operators in the transformed theory are

\[ V_{\text{phy}}^{(3)N=1} = V_{\text{phy}}^{(3)}(c_0(z)e^{-\phi_0^{gh}(z)})(c_1(z)\partial c_1(z)e^{-2\phi_1^{gh}(z)}). \]  (7.6)

The topological amplitudes can be calculated by using the picture changing operators

\[ Z_i^{gh}(z) \equiv \{Q_{\text{top}}, \xi_i^{gh}(z)\} = e^{\phi_i^{gh}(z)} b_i(z), \]
\[ Z_i^{gh} -1(z) = c_i(z)e^{-\phi_i^{gh}(z)}, \]  (7.7)

where \( i = 0, 1 \). One can easily show the amplitudes of the \( N = 1 \) \( w_3 \) string coincide with those of the \( w_3 \) string in a similar way to the previous cases.

Here we conjecture that the \( w_N \) string is obtained as a special realization of the \( N = 1 \) \( w_N \) string for general \( w_N \). If this is indeed the case, the \( N = 1 \) \( w_\infty \) string is more universal than the \( w_\infty \) string since it includes both sequences of \( N = 0 \) and \( N = 1 \) \( w_N \) string theories. Such an investigation remains to the future study.

\(^{†}\) The total stress tensors in the previous cases, which have no unusual term, are invariant under the similarity transformations.

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8. Discussions

We have constructed the $w_N$ string theory based on the linear $w_N$ algebra. The linear $w_N$ algebra naturally tends to $w_\infty$, area preserving diffeomorphisms, by taking the limit $N \to \infty$. We have shown that there is sequential embedding structure: the $w_N$ string theory can be obtained by a special realization of the $w_{N+1}$ string. This $w_N$ string theory is, however, not general one but it further gives $w_{N-1}$ string theory etc. Only the $w_2$ (bosonic) string is included as a whole. The $w_\infty$ string theory is a universal string theory in this sequence. It has been also shown that there is a similar sequence for $N=1$ string theories. We conjecture that this sequence includes the $N=0$ string sequence by explicit proof in the case of $w_3$ string theory. The $N=1$ $w_\infty$ string theory is therefore more universal.

We have also shown that the above discussions can be repeated even if we replace the linear $w_N$ algebra to the abelian $w_N$ algebra given by a contraction. In this case, the general $w_N$ string theory can be obtained from the $w_{N+1}$ string, while $N \to \infty$ limit does not give the usual $w_\infty$ algebra. It gives a contraction of the area preserving diffeomorphisms.

From the results obtained in this paper, we can expect that the $N=4$ $w_\infty$ string theory is a candidate of the most universal string theory since $N=4$ supersymmetry is the maximum in two dimensions and $w_\infty$ is also the maximum extensions by using the higher spin generators. There is a fascinating possibility that all the string theories can be obtained as a special realization of the $N=4$ $w_\infty$ string theory.

It is interesting to study a geometrical picture of the symmetry breaking considered in this paper. A realization of the string theory is considered to be corresponding to a geometry of the target space. We can ask what is the background geometry corresponding to the special realization which is equivalent to the string based on a smaller symmetry. We hope to discuss this issue elsewhere.
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REFERENCES

1. N. Berkovits and C. Vafa, *On the Uniqueness of String Theory*, preprint HUTP-93/A031, KCL-TH-93-13, hep-th/9310170 (1993).

2. J.M. Figueroa-O’Farrill, *On the Universal String Theory*, preprint QMW-PH-93-29, hep-th/9310200 (1993); *The Mechanism behind the Embeddings of String Theories*, preprint QMW-PH-93-30, hep-th/9312033.
   H. Ishikawa and M. Kato, *Note on N=0 string as N=1 string*, preprint UT-Komaba/93-23, hep-th/9311139.
   N. Ohta and J.L. Petersen, *N=1 from N=2 Superstring*, preprint NBI-HE-93-76, hep-th/9312187.

3. F. Bastianelli, N. Ohta and J.L. Petersen, *Toward the Universal Theory of Strings*, preprint NORDITA-94/6 P, hep-th/9402042.

4. N. Berkovits, M. Freeman and P. West, *A W-String Realization of the Bosonic String*, preprint KCL-TH-93-15, hep-th/9312013.

5. K. Li, *Linear $w_N$ Gravity*, preprint CALT-68-1724 (1991).

6. E. Witten, *Surprises with Topological Field Theories*, in the proceedings of Strings ’90 ed. by R. Arnowitt et al. (World Scientific) (1991).

7. S. Hosono, *Int. J. Mod. Phys.* A7 (1992) 5193.

8. E. Martinec, *Nucl. Phys.* B281 (1987) 157.
9. H. Ishikawa and M. Kato, *Note on N=0 string as N=1 string*, preprint UT-Komaba/93-23, [hep-th/9311139](http://arxiv.org/abs/hep-th/9311139).

10. D. Friedan, E. Martinec and S. Schenker, *Nucl. Phys.* **B271** (1986) 93.