Blow-up analysis involving isothermal coordinates on the boundary of compact Riemann surface

Yunyan Yang
Department of Mathematics, Renmin University of China, Beijing 100872, China

Jie Zhou
Department of Mathematics, Tsinghua University, Beijing 100084, China

Abstract
Using the method of blow-up analysis, we obtain two sharp Trudinger-Moser inequalities on a compact Riemann surface with smooth boundary, as well as the existence of the corresponding extremals. This generalizes early results of Chang-Yang [7] and the first named author [32], and complements Fontana’s inequality of two dimensions [15]. The blow-up analysis in the current paper is far more elaborate than that of [32], and particularly clarifies several ambiguous points there. In precise, we prove the existence of isothermal coordinate systems near the boundary, the existence and uniform estimates of the Green function with the Neumann boundary condition. Also our analysis can be applied to the Kazdan-Warner problem and the Chern-Simons Higgs problem on compact Riemann surfaces with smooth boundaries.

Key words: isothermal coordinate system, Trudinger-Moser inequality, blow-up analysis

2010 MSC: 58J05, 58J32

1. Introduction

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \), \( W^{1,2}_0(\Omega) \) be the completion of all smooth functions with compact support under the norm

\[
||u||_{W^{1,2}_0(\Omega)} = \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.
\]

It was proved by Yudovich [37], Pohozaev [26], Peetre [25], Trudinger [29] and Moser [22] that

\[
\sup_{u \in W^{1,2}_0(\Omega), ||u||_{W^{1,2}_0(\Omega)} \leq 1} \int_\Omega \exp(\gamma u^2) \, dx < +\infty, \quad \forall \gamma \leq 4\pi;
\]

moreover, if \( \gamma > 4\pi \), then the above supremum is infinity. In literature, such kind of inequalities are known as Trudinger-Moser inequalities. Concerning all smooth functions with mean value

Email addresses: yunyanyang@ruc.edu.cn (Yunyan Yang), zhoujie2014@mails.ucas.ac.cn (Jie Zhou)
Preprint submitted to ***  September 22, 2020
zero instead of boundary value zero, Chang and Yang [3] obtained by using their isoperimetric inequality that

$$\sup_{u \in W^{1,2}(\Omega), \|\nabla u\|_{L^2} \leq 1, \int u \, dx = 0} \int_{\Omega} \exp(\gamma u^2) \, dx < +\infty, \quad \forall \gamma \leq 2\pi. \quad (2)$$

Analogous to (1), the supremum in (2) is infinity for any \( \gamma > 2\pi \). This inequality was applied by Chang and Yang to the Nirenberg problem with the Neumann boundary condition.

Now we consider \((\Sigma, g)\), a closed Riemann surface, i.e. a compact Riemann surface without boundary, and let \(W^{1,2}(\Sigma, g)\) be the usual Sobolev space. Representing a function by the Riesz potential of its gradient and using a manifold version of Adams’ potential estimate [1], L. Fontana obtained far more than (3) in his elegant paper [15]. Later, via a method of blow-up analysis, Li [18] proved that the supremum in (3) can be attained for all \( \gamma \leq 4\pi \).

In view of (2), one would naturally expect (3) for compact Riemann surfaces with smooth boundaries. Indeed, in the case that \((\Sigma, g)\) is a compact Riemann surface with smooth boundary \(\partial \Sigma\), following the approach of Li [18], the first named author [32] extended (2) as below:

$$\sup_{u \in W^{1,2}(\Sigma, g), \|\nabla u\|_{L^2} \leq 1, \int u \, dv = 0} \int_{\Sigma} \exp(\gamma u^2) \, dv < +\infty, \quad \gamma \leq 2\pi. \quad (4)$$

$$\sup_{u \in W^{1,2}(\Sigma, g), \|\nabla u\|_{L^2} \leq 1, \int u \, dv = 0} \int_{\Sigma} \exp(\gamma u^2) \, dv < +\infty, \quad \gamma \leq 2\pi. \quad (5)$$

Furthermore, both supremums can be attained for all \( \gamma \leq 2\pi \), but they are infinite when \( \gamma > 2\pi \).
Secondly we prove that $c_k u_k$ converges to some Green function $G_{s_0}$ weakly in $W^{1,q}(\Sigma, g)$ for any $1 < q < 2$, strongly in $L^q(\Sigma, g)$ with $s < 2q/(2 - q)$, and in $C^1_{\text{loc}}(\Sigma \setminus \{x_0\})$, where $G_{s_0}$ is a distributional solution of the equation

$$
\begin{cases}
\Delta_g G_{s_0} = \delta_{s_0} - \frac{\partial}{\partial \text{Area}(\Sigma)} & \text{in } \Sigma \\
\partial G_{s_0}/\partial n = 0 & \text{on } \partial \Sigma \\
\int_{\Sigma} G_{s_0} dv_g = 0.
\end{cases}
$$

Based on elliptic estimates in the isothermal coordinate system near $x_0$, $G_{s_0}$ can be locally decomposed as

$$
G_{s_0}(x) = -\frac{1}{\pi} \log \text{dist}_g(x_0, x) + A_{s_0} + O(\text{dist}_g(x_0, x)).
$$

Thirdly, using the capacity estimate introduced by Li [18], we derive

$$
\sup_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} |\nabla u|^2 dv_g \leq 1, \int_{\Sigma} udv_g = 0} \int_{\Sigma} \exp(2\pi u^2) dv_g \leq \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi A_{s_0}).
$$

Finally we construct a function sequence $\phi_k \in W^{1,2}(\Sigma, g)$ with $\int_{\Sigma} \phi_k dv_g = 0$ and $\int_{\Sigma} |\nabla_g \phi_k|^2 dv_g = 1$ satisfying

$$
\int_{\Sigma} \exp(2\pi \phi_k^2) dv_g > \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi A_{s_0}),
$$

provided that $k$ is chosen sufficiently large. The contradiction between (8) and (9) implies that $c_k$ must be bounded. Then applying elliptic estimates to (6), one has up to a subsequence, $u_k \to u_0$ in $C^1(\Sigma)$ as $k \to \infty$ and thus the supremum in (4) can be attained by $u_0$.

Checking the proof in [32], we found at least three key points that should have been seriously treated there. The first one is the claimed existence of isothermal coordinate system on the boundary $\partial \Sigma$, which is very important in the subsequent blow-up analysis; The second one is the way of finding a constant $C$ depending only on $(\Sigma, g)$ and $q < 2$ such that

$$
\int_{\Sigma} |\nabla_g (c_k u_k)|^q dv_g \leq C,
$$

which leads to the convergence of $c_k u_k$ to $G_{s_0}$; The third one is the decomposition of $G_{s_0}$ with the form (7).

Our goals are twofold. One is to clarify the above three concerns. Specifically, we employ Riemann mapping theorems to construct isothermal coordinate systems near the boundary $\partial \Sigma$; To prove (10), we first construct a Green function with the Neumann boundary condition, and then use the Green representation formula; The decomposition of $G_{s_0}$ will be based on elliptic estimates in an isothermal coordinate system. The other one is to improve Theorems 1.1 and 1.2 in [32]. To describe this improvement, we define a space of functions by

$$
\mathcal{H} = \left\{ u \in W^{1,2}(\Sigma, g) : \int_{\Sigma} udv_g = 0 \right\},
$$

the first eigenvalue of the Laplace-Beltrami operator with respect to the Neumann boundary condition by

$$
\lambda_N(\Sigma) = \inf_{u \in \mathcal{H}, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 dv_g}{\int_{\Sigma} u^2 dv_g}.
$$
and a Sobolev norm on $\mathcal{H}$ in the case $\alpha < \lambda_N(\Sigma)$ by

$$
\|u\|_{1,\alpha} = \left( \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \right)^{1/2}.
$$

(13)

Our main result reads as follows:

**Theorem 1.** Let $(\Sigma, g)$ be a compact Riemann surface with smooth boundary $\partial \Sigma$. Then for any $\alpha < \lambda_N(\Sigma)$, there holds

$$
\sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp(\gamma u^2) dv_g < +\infty, \quad \forall \gamma \leq 2\pi,
$$

(14)

where $\mathcal{H}$, $\lambda_N(\Sigma)$ and $\| \cdot \|_{1,\alpha}$ are defined as in (11), (12) and (13) respectively. Moreover, the above supremum is infinity for any $\gamma > 2\pi$. Furthermore, for any fixed $\alpha < \lambda_N(\Sigma)$ and $\gamma \leq 2\pi$, the supremum in (14) can be attained by some $u^* \in \mathcal{H} \cap C^1(\Sigma)$ with $\|u^*\|_{1,\alpha} = 1$.

Similarly we have the following:

**Theorem 2.** Let $(\Sigma, g)$ be a compact Riemann surface with smooth boundary $\partial \Sigma$. Then for any real number $\tau > 0$, there holds

$$
\sup_{u \in W^{1,2}(\Sigma, g), \|u\|_{1,\tau} \leq 1} \int_{\Sigma} \exp(\gamma u^2) dv_g < +\infty, \quad \forall \gamma \leq 2\pi,
$$

(15)

where $\|u\|_{1,\tau} = \left( \int_{\Sigma} (|\nabla u|^2 + \tau u^2) dv_g \right)^{1/2}$. Moreover, the above supremum is infinity for any $\gamma > 2\pi$. Furthermore, for all real numbers $\tau > 0$ and $\gamma \leq 2\pi$, the supremum in (15) can be attained by some $u_0 \in \mathcal{H} \cap C^1(\Sigma)$ with $\|u_0\|_{1,\tau} = 1$.

Theorems 1 and 2 are complements of [7, 15, 18, 35, 36]. We remark that the inequality (14) involving the norm $\| \cdot \|_{1,\alpha}$ was motivated by [28], while the inequality (15) involving the norm $\| \cdot \|_{1,\tau}$ was motivated by Adimurthi-Yang [3] and do Ó-Yang [13]. Although the method of blow-up analysis is now standard, the technique is far more delicate than the existing related works [12, 2, 18, 32]. Our technique can certainly be used in the study of Trudinger-Moser inequalities on boundaries [20, 19, 33, 34, 21, 23], as well as in the Chern-Simons Higgs problem with Neumann boundary condition [10, 11, 17, 30, 31], and other related problems [6, 9, 8, 38, 39].

As far as the inequality itself is concerned, (15) is apparently weaker than (14), but unexpectedly they are equivalent. Motivated by [24], we have the following:

**Theorem 3.** Let $(\Sigma, g)$ be a compact Riemann surface with smooth boundary $\partial \Sigma$ and $\lambda_N(\Sigma)$ be defined as in (12). Given any $0 \leq \alpha < \lambda_N(\Sigma)$ and any $\tau > 0$. Assume that (15) holds for all $\gamma < 2\pi$. Then the inequality

$$
\sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp(2\pi u^2) dv_g < +\infty
$$

(16)

is equivalent to

$$
\sup_{u \in W^{1,2}(\Sigma, g), \|u\|_{1,\tau} \leq 1} \int_{\Sigma} \exp(2\pi u^2) dv_g < +\infty,
$$

(17)

where $\mathcal{H}$, $\| \cdot \|_{1,\alpha}$ and $\| \cdot \|_{1,\tau}$ are the same as in Theorems 1 and 2 respectively.
Throughout this paper, sequence and subsequence are not distinguished, and various constants are often denoted by the same \( C \). The remaining part of this paper is organized as follows: In Section \ref{existence} we prove the existence of isothermal coordinate system around any point on the boundary \( \partial \Sigma \); in Section \ref{Green} we construct a Green function with the Neumann boundary condition and give its uniform estimates; Theorems \ref{main} will be proved in Sections \ref{existence} and \ref{Green} respectively.

2. Isothermal coordinate systems near the boundary

In this section, we prove existence of isothermal coordinate systems near the boundary. This is based on the classical existence result near inner points of Riemann surface and Riemann mapping theorems involving the boundary. From now on, we always denote

\[ B^+_x = \{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < r, \ y_2 > 0 \}, \quad \mathbb{R}^+ = \{ y = (y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0 \} \]

and the closure of a set \( E \) by \( \overline{E} \).

**Lemma 4.** Let \((\Sigma, g)\) be a compact Riemann surface with smooth boundary \( \partial \Sigma \). For any fixed point \( x \in \partial \Sigma \), there exist a number \( \delta > 0 \) and an isothermal coordinate system \((\overline{U}_x, \psi_x; (y_1, y_2))\) near \( x \) such that \( \psi_x(x) = (0, 0) \), \( \overline{U}_x \subset \Sigma \) is a neighborhood of \( x \), \( \psi_x(U_x) = \overline{B}_1^+ \) and \( \psi_x(U_x \cap \partial \Sigma) = \overline{B}_1^+ \cap \partial \mathbb{R}^2 \). In this coordinate system, there exists a function \( f \in C^1(\overline{B}_1^+, \mathbb{R}) \) such that for all \( y = (y_1, y_2) \in \overline{B}_1^+ \), the metric \( g \) can be written as

\[ g = \exp (2f(y))(dy_1^2 + dy_2^2). \]

Suppose that \( \nu \) is an unit outward vector field defined on \( \psi_x^{-1}(\overline{B}_1^+ \cap \partial \mathbb{R}^2) \subset \partial \Sigma \). For any \( p \in \psi_x^{-1}(\overline{B}_1^+ \cap \partial \mathbb{R}^2) \), if we write \( y = \psi_x(p) \), then

\[ (\nu_x)_*(\nu(p)) = \exp(-f(y))\partial/\partial y_2. \]

**Proof.** We divide the construction into several steps.

**Step 1.** There exists a neighborhood \( \overline{U}_1 \) of \( x \), a domain \( \Omega_1 \subset \mathbb{R}^2 \) verifying that \( \partial \Omega_1 \) is smooth except for two corners, and a homeomorphism \( \psi_1 : \overline{U}_1 \to \Omega_1 \) such that \( \psi_1(x) = (0, 0) \) and \( \psi_1(\overline{U}_1 \cap \partial \Sigma) = \Gamma_1 \subset \partial \Omega_1 \). In the coordinate system \((\overline{U}_1, \psi; [x_1, x_2])\), the metric \( g \) can be written as

\[ g = \exp(2f_1(x_1, x_2))(dx_1^2 + dx_2^2) \]

for all \( (x_1, x_2) \in \overline{U}_1 \), where \( f_1 \) is a smooth function with \( f_1(0, 0) = 0 \). Denote \( \nu_1 = (\psi_1)_*(\nu) \). Then \( \nu_1 = \exp(-f_1(x_1, x_2))\nu_0 \), where \( \nu_0 \) is the unit outward vector field on \( \partial \Omega_3 \).

Indeed, since \((\Sigma, g)\) is a compact Riemann surface with smooth boundary \( \partial \Sigma \), we understand that there exists another compact Riemann surface \((\Sigma^*, g^*)\) with smooth boundary \( \partial \Sigma^* \) such that \( \Sigma \subset \Sigma^*, \text{dist}_g(\Sigma, \partial \Sigma^*) > 0 \) and \( g^* = g \) on \( \Sigma \). Note that \( x \) is an inner point of \( \Sigma^* \). By \([5]\), there exist \( U \subset \Sigma^* \), a neighborhood of \( x \), and a diffeomorphism \( \psi_1 : U \to \mathbb{B}_r \subset \mathbb{R}^3 \) with \( \psi_1(x) = (0, 0) \) such that the metric \( g \) reads as

\[ g = \exp(2f_1(x_1, x_2))(dx_1^2 + dx_2^2) \quad (18) \]

where \( f_1 \) is a smooth function with \( f_1(0, 0) = 0 \). Denote \( U_1 = U \cap \Sigma \) and \( \Omega_1 = \psi_1(U_1) \). To finish this step, it suffices to estimate \( \nu_1 \). Write \( \nu_1 = a_1 \partial/\partial x_1 + a_2 \partial/\partial x_2 \). Then

\[ 1 = |\nu|^2 = \exp(2f_1(x_1, x_2))(a_1^2 + a_2^2). \quad (19) \]
which immediately leads to the representation of \( v_1 \).

Step 2. Replace \( \Omega_1 \) by a smooth domain \( \Omega_2 \subset \Omega_1 \) verifying that \( \phi_1(x) = (0, 0) \) is an inner point of a smooth curve \( \Gamma_2 \subset \partial \Omega_2 \).

Step 3. \( \overline{\Omega_2} \) is conformal to a unit disc \( \overline{D} \subset \mathbb{R}^2 \). In fact, according to the Riemann mapping theorem [2], there exists a conformal map \( \psi_2 : \Omega_2 \to D \) denoted by \( w = \psi_2(z) \) with \( \psi_2(0, 0) = (0, -1) \), where \( z = x_1 + i x_2 \). By (27), Theorem 3.5), \( \psi_2 \) extends to a map in \( C^1(\overline{\Omega_2}, D) \); moreover, \( \psi_2(z) \neq 0 \) for all \( z \in \overline{\Omega_2} \). Here and in the sequel we slightly abuse some notations. In particular we identify \( z \in \mathbb{C} \) with \( (x_1, x_2) \in \mathbb{R}^2 \), and so on.

Step 4. \( D \) is conformal to a half plane. Let \( q \notin \psi_2(\Gamma_2) \) be fixed. Then via a Möbius transformation \( \zeta = h(w) \), the set \( D \setminus \{q\} \) can be mapped into the upper half plane \( \mathbb{R}^{2+} \) with \( h(\psi_2(\phi_1(x))) = (0, 0) \). Define a function by

\[
\varphi(z) = f_1(z) - \log |h'(w)| \psi_2'(z)|
\]

and a dilation \( \tau : \mathbb{R}^{2+} \to \mathbb{R}^{2+} \) by \( y = \tau(\zeta) = \exp(\varphi(0, 0)) \zeta \). Thus

\[
dy = \exp(\varphi(0, 0))h'(w)|\psi_2'(z)|dz.
\]

Set \( \psi_x = \tau \circ h \circ \psi_2 \circ \phi_1 \). Choose \( \delta > 0 \) sufficiently small so that \( \psi_x^{-1}(\partial \mathbb{B}_\delta^+ \cap \partial \mathbb{R}^{2+}) \subset \psi_2^{-1}(\Gamma_2) \).

Step 5. \( (\psi_x^{-1}(\overline{\mathbb{B}_\delta}), \psi_x; \{y_1, y_2\}) \) is an isothermal coordinate system near \( x \in \partial \Sigma \) as we required. Indeed, since \( \phi_1, \psi_2, h \) and \( \tau \) are all conformal maps, we conclude that \( \psi_x \) is also a conformal map. This together with (19), (20) and (21) leads to the representation of the metric \( g \) as

\[
g = \frac{\exp(2f_1(z))}{|h'(w)|^2 \psi_2'(z)^2} \exp(-2\varphi(0, 0))|dy|^2
\]

\[
= \exp(2f(y))(dy_1^2 + dy_2^2)
\]

for all \( y \in \overline{\mathbb{B}_\delta^+} \), where \( f(y) = \varphi(z) - \varphi(0, 0) \) and \( z = \psi_x^{-1}(h^{-1}(\exp(-\varphi(0, 0))y)) \). By the above definitions of \( h \) and \( \psi_2 \), we have that \( y = (0, 0) \) if and only if \( z = (0, 0) \). Thus \( f(0, 0) = 0 \). Moreover, we can assume \( (\psi_x)_*(y)(p) = b_2(y) \partial/\partial y_2 \) for any \( p \in \psi_x^{-1}(\overline{\mathbb{B}_\delta^+}) \cap \partial \Sigma \) with \( y = \psi_x(p) \). Similar to (19), we calculate \( b_2(y) = \exp(-f(y)) \). Clearly \( f \in C^0(\overline{\mathbb{B}_\delta^+}) \). Further application of (27), Theorem 3.6) implies that \( f \) is smooth on \( \overline{\mathbb{B}_\delta^+} \). This ends the proof of the lemma.  

3. The Green function with the Neumann boundary condition

In this section, we concern the Green function on \((\Sigma, g)\) with the Neumann boundary condition, whose construction is based on the method of (Aubin [4], Chapter 4). For its uniform estimate, we use elliptic estimates as Aubin did in ([4], Chapter 4), and as Druet, Robert, Wei did in [14]. To begin with, we need the following:

**Lemma 5.** Let \((\Sigma, g)\) be a compact Riemann surface with smooth boundary \( \partial \Sigma \). If \( f \in \mathcal{L}^2(\Sigma, g) \) satisfies \( \int_\Sigma f dv_\Sigma = 0 \), then there exists a unique weak solution of

\[
\begin{aligned}
\Delta_g u &= f & \text{in} & \Sigma \\
\partial u/\partial n &= 0 & \text{on} & \partial \Sigma \\
\int_\Sigma u dv_\Sigma &= 0,
\end{aligned}
\]

(22)
or equivalently there exists a \( u \in \mathcal{H} \) defined by (11) satisfies
\[
\int_{\Sigma} \nabla u \varphi \, dv_g = \int_{\Sigma} f \varphi \, dv_g, \quad \forall \varphi \in C^1(\Sigma). \tag{23}
\]
Moreover there exists some constant \( C \) depending only on \((\Sigma, g)\) such that
\[
|u|_{W^{2,2}(\Sigma, g)} \leq C \|f\|_{L^2(\Sigma, g)}. \tag{24}
\]
If further \( f \in C^0(\Sigma) \) for some \( 0 \leq \alpha < 1 \), then \( u \in C^{2,\alpha}(\Sigma) \).

**Proof.** The uniqueness is obvious. To see this, we let \( u_1 \) and \( u_2 \) be two weak solutions of (22) and \( u^* = u_1 - u_2 \). Since \( C^1(\Sigma) \) is dense in \( W^{1,2}(\Sigma, g) \), it follows from (23) that
\[
\int_{\Sigma} \nabla u^* \nabla v \, dv_g = 0, \quad \forall v \in W^{1,2}(\Sigma, g).
\]
Choosing \( v = u^* \) in the above equality, we conclude \( u^* \equiv 0 \) since \( u^* \in \mathcal{H} \).

The Existence of weak solution of (22) is based on a direct method of variation. Let us consider the functional
\[
J(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 \, dv_g - \int_{\Sigma} f u \, dv_g.
\]
For any \( u \in \mathcal{H} \), we have by the Hölder inequality and the Poincare inequality
\[
\int_{\Sigma} f u \, dv_g \leq C \left( \int_{\Sigma} f^2 \, dv_g \right)^{1/2} \left( \int_{\Sigma} |\nabla u|^2 \, dv_g \right)^{1/2},
\]
which implies that \( J \) has a lower bound on \( \mathcal{H} \). Now we take a sequence of functions \( u_j \in \mathcal{H} \) satisfying \( J(u_j) \to \inf_{u \in \mathcal{H}} J(u) \). One can easily see that \( u_j \) is bounded in \( \mathcal{H} \). Thus one can assume up to a subsequence, \( u_j \) converges to some \( u_0 \in \mathcal{H} \) weakly in \( W^{1,2}(\Sigma, g) \), strongly in \( L^q(\Sigma, g) \) for any \( q > 1 \) and almost everywhere in \( \Sigma \). Clearly \( u_0 \in \mathcal{H} \) and
\[
J(u_0) \leq \lim_{j \to \infty} J(u_j) = \inf_{u \in \mathcal{H}} J(u).
\]
Hence \( u_0 \) is a minimizer of \( J \) on \( \mathcal{H} \) and satisfies the Euler-Lagrange equation (23).

We now prove (24). Since \( (\Sigma, g) \) is compact, we have by the standard \( W^{2,2} \)-estimate (see for example [4], Theorem 3.54) that
\[
\|u_0\|_{W^{2,2}(\Sigma, g)} \leq C (\|u_0\|_{L^2(\Sigma, g)} + \|f\|_{L^2(\Sigma, g)}) \tag{25}
\]
for some constant \( C \) depending only on \((\Sigma, g)\). Noting that \( u_0 \) is a unique solution of (22), we have by the definition of distributional solution and the Hölder inequality that
\[
\int_{\Sigma} |\nabla u_0|^2 \, dv_g = \int_{\Sigma} f u_0 \, dv_g \leq \|u_0\|_{L^2(\Sigma, g)} \|f\|_{L^2(\Sigma, g)}.
\]
This together with the Poincare inequality leads to
\[
\|u_0\|_{L^2(\Sigma, g)} \leq C \|f\|_{L^2(\Sigma, g)}. \tag{26}
\]
Inserting (26) into (25), we conclude (24), as desired.

Finally, if \( f \in C^0(\Sigma) \), then we have \( u \in C^{2,\alpha}(\Sigma) \) by using Lemma 4 and the classical Schauder estimate ([16], Theorem 6.6). \( \square \)

An analog of ([4], Theorems 4.13 and 4.17) reads as follows.
Lemma 6. There exists a unique Green function \( G(x, \cdot) \in L^1(\Sigma, g) \) satisfying

\[
\begin{aligned}
\Delta_{\Sigma, g} G(x, y) &= \delta_{\Sigma}(y) - \frac{1}{\text{Area}(\Sigma)} \quad \text{in} \quad \Sigma \\
\frac{\partial}{\partial \nu} G(x, y) &= 0 \quad \text{on} \quad \partial \Sigma \\
\int_{\Sigma} G(x, y) dv_{g, y} &= 0
\end{aligned}
\]

(27)

in the distributional sense, or equivalently for any \( \varphi \in C^1(\bar{\Sigma}) \) with \( \partial \varphi / \partial \nu = 0 \) on \( \partial \Sigma \), there holds

\[
\int_{\Sigma} G(x, y) \Delta_{\Sigma} \varphi(y) dv_{g, y} = \varphi(x) - \overline{\varphi},
\]

(28)

where \( \overline{\varphi} = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \varphi dv_{g} \). Moreover, for all \( x, y \in \Sigma \) with \( x \neq y \), \( G(x, y) = G(y, x) \), and there exists some constant \( C \) depending only on \( (\Sigma, g) \) such that

\[
|G(x, y)| \leq C(1 + |\log \text{dist}_g(x, y)|), \quad |\nabla_{g, y} G(x, y)| \leq C(\text{dist}_g(x, y))^{-1},
\]

(29)

where \( \text{dist}_g(x, y) \) denotes the geodesic distance between \( x \) and \( y \).

Proof. Part I. Uniqueness of the Green function. If \( G_1(x, y) \) and \( G_2(x, y) \) are two Green functions satisfying (28), then we set \( h(y) = G_1(x, y) - G_2(x, y) \). By Lemma 5 for any \( f \in C^\alpha(\Sigma) \), \( 0 < \alpha < 1 \), there exists a unique \( \varphi \in C^2(\bar{\Sigma}) \) such that \( \Delta_{\Sigma} \varphi = f - \overline{f} \), \( \partial \varphi / \partial \nu = 0 \) on \( \partial \Sigma \), and \( \int_{\Sigma} \varphi dv_{g} = 0 \). Hence (28) implies that

\[
\int_{\Sigma} h f dv_{g} = \int_{\Sigma} h \Delta_{\Sigma} \varphi dv_{g} = 0.
\]

This together with the facts \( h \in L^1(\Sigma, g) \) and \( C^\alpha(\bar{\Sigma}) \) is dense in \( L^1(\Sigma, g) \) leads to \( h \equiv 0 \).

Part II. Existence of the Green function.

Case 1. \( x \) is an inner point of \( \Sigma \). We follow the line of (4), Theorem 4.13. Let \( i(x) \) be the injectivity radius of \( x \), and \( \phi(r) \) be a decreasing function, which is equal to 1 in a neighborhood of zero, and to zero for \( r > i(x)/8 \). Define

\[
H(x, y) = -\frac{1}{2\pi} \phi(\text{dist}_g(x, y)) \log \text{dist}_g(x, y),
\]

\[
\Gamma(x, y) = \Gamma_1(x, y) = \Delta_{\Sigma} H(x, y), \quad \Gamma_{i+1}(x, y) = \int_{\Sigma} \Gamma_i(x, z) \Gamma_i(z, y) dv_{g, z} \quad \text{for} \ i = 1, 2,
\]

and set

\[
G(x, y) = H(x, y) + \sum_{i=1}^{2} \int_{\Sigma} \Gamma_i(x, z) H(z, y) dv_{g, z} + F(x, y),
\]

where \( F(x, y) \) satisfies

\[
\begin{aligned}
\Delta_{\Sigma} F(x, y) &= -\Gamma_3(x, y) - \frac{1}{\text{Area}(\Sigma)} \quad \text{in} \quad \Sigma \\
\frac{\partial}{\partial \nu} F(x, y) &= 0 \quad \text{on} \quad \partial \Sigma
\end{aligned}
\]

and

\[
\int_{\Sigma} F(x, y) dv_{g, y} = -\int_{\Sigma} H(x, y) dv_{g, y} - \sum_{i=1}^{2} \int_{\Sigma} \left( \int_{\Sigma} \Gamma_i(x, z) H(z, y) dv_{g, z} \right) dv_{g, y}.
\]
Such an \( F(x, y) \) exists in view of Lemma \([S]\). It can be easily checked that \( G(x, \cdot) \) satisfies \([AB]\).

Case 2. \( x \in \partial \Sigma \). By Lemma \([B]\) we choose an isothermal coordinate system \((U_x, \psi_x; |z_1, z_2|)\) near \( x \) such that \( \psi_x : U_x \rightarrow \mathbb{H}^n_0 \) for some \( r_0 > 0 \). In this coordinate system, the metric \( g \) can be written as
\[
g = \exp(2f(z))(dz_1^2 + dz_2^2);
\]
mOREOVER \( \Delta_y = -\exp(2f(z))\Delta_{\mathbb{H}^2} \). Let \( \phi : [0, \infty) \rightarrow [0, \infty) \) be a decreasing function such that \( \phi \equiv 1 \) on \([0, r_0/2]\) and \( \phi \equiv 0 \) on \([r_0, \infty)\). Set
\[
H(x, y) = \begin{cases}
\frac{1}{2}\phi(|z|) \log |z|, & y = \psi_x^{-1}(z) \in \psi_x^{-1}(\mathbb{H}^n_0) \\
0, & y \in \Sigma \setminus \psi_x^{-1}(\mathbb{H}^n_0).
\end{cases}
\]

One can check that
\[
\begin{align*}
\Delta_y \phi \mathcal{H}(x, y) &= \delta_y(y) - \eta(x, y) & \text{in } & \Sigma \\
\frac{\partial}{\partial y} \mathcal{H}(x, y) &= 0 & \text{on } & \partial \Sigma
\end{align*}
\]
in the distributional sense \([28]\), where
\[
\eta(x, y) = \begin{cases}
\exp(-2f(z))\Delta_{\mathbb{H}^2} \left(-\frac{1}{2}\phi(|z|) \log |z|\right), & y = \psi_x^{-1}(z) \in \psi_x^{-1}(\mathbb{H}^n_0) \\
0, & y \in \Sigma \setminus \psi_x^{-1}(\mathbb{H}^n_0).
\end{cases}
\]

According to Lemma \([S]\) one can find a unique \( F(x, y) \) satisfying
\[
\begin{align*}
\Delta_y \phi \mathcal{H}(x, y) &= \delta_y(y) - \phi \mathcal{H}(x, y) - \frac{1}{\text{Area}(\Sigma)} & \text{in } & \Sigma \\
\frac{\partial}{\partial y} \mathcal{H}(x, y) &= 0 & \text{on } & \partial \Sigma \\
\int_{\Sigma} \mathcal{H}(x, y) dv_{g, \phi} &= -\int_{\Sigma} H(x, y) dv_{g, \phi}.
\end{align*}
\]

We set \( G(x, y) = H(x, y) + F(x, y) \) for all \( y \in \Sigma \). Then \( G(x, \cdot) \) is a distributional solution of \([29]\).

**Part III. Uniform estimate.**

We first prove that there exists some constant \( C \) depending only on \((\Sigma, g)\) such that for all \( x \in \Sigma \), there holds
\[
||G(x, \cdot)||_{L^2(\Sigma, g)} \leq C. \tag{30}
\]

To see this, for any \( w \in C^2(\Sigma) \), we conclude from Lemma \([S]\) that the equation
\[
\begin{align*}
\Delta_x u &= w - \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} w dv_g & \text{in } & \Sigma \\
\frac{\partial u}{\partial n} &= 0 & \text{on } & \partial \Sigma \\
\int_{\Sigma} u dv_g &= 0
\end{align*}
\]
has a unique solution \( u \in C^2(\Sigma) \). Combining \([27]\) and \([24]\), we obtain
\[
\int_{\Sigma} G(x, y)w(y) dv_{g, \phi} = \int_{\Sigma} G(x, y)\Delta_x w(y) dv_{g, \phi} \leq ||u||_{L^2(\Sigma, g)} \leq C||w||_{L^2(\Sigma, g)},
\]
where \( C \) is a constant depending only on \((\Sigma, g)\). This together with the density of \( C^2(\Sigma) \) in \( L^2(\Sigma, g) \) implies \(\text{(30)}\).
Given any fixed \( x_0 \in \partial \Sigma \). Take an isothermal coordinate system \((\overline{U}_{x_0}, \psi_{x_0}; \{z_1, z_2\})\) near \( x_0 \) such that \( \psi(x_0) = (0, 0), \psi_{x_0}(\overline{U}_{x_0}) = \overline{B}^r_{r_0} \) and \( \psi_{x_0}^{-1}(\overline{B}^r_{r_0} \cap \partial \mathbb{R}^{2+}) = \overline{U}_{x_0} \cap \partial \Sigma \) for some \( r_0 > 0 \), and the metric \( g = \exp(2f(z))(dz_1^2 + dz_2^2) \) with \( f \in C^2(\overline{B}^r_{r_0}) \) and \( f(0) = 0 \). For any \( x \in \psi_{x_0}^{-1}(\overline{B}^r_{r_0}) \), we define

\[
G^*_x(z) = \begin{cases} \mathcal{G}(x, \psi_{x_0}^{-1}(z_1, z_2)), & z = (z_1, z_2) \in \overline{B}^r_{r_0} \\ \mathcal{G}(x, \psi_{x_0}^{-1}(z_1, -z_2)), & z = (z_1, z_2) \in \mathbb{B}_{r_0} \setminus \overline{B}^r_{r_0}. \end{cases}
\]

Then \( G^*_x \) is a distributional solution of

\[
-\Delta_{\Sigma} G^*_x(z) = \delta_{z_0}(z) + \delta_{z'_0}(z) - \frac{\exp(2f'(z))}{\text{Area}(\Sigma)} \quad \text{in} \quad \mathbb{B}_{r_0}, \tag{31}
\]

where \( z_0 = \psi_{x_0}(x) = (z_0,1, z_0,2), z'_0 = (z_0,1, -z_0,2) \) and

\[
f'(z) = \begin{cases} f(z_1, z_2), & z_2 \geq 0 \\ f(z_1, -z_2), & z_2 < 0. \end{cases}
\]

Denote

\[
F^*_x(z) = G^*_x(z) + \frac{1}{2\pi} \log |z - z_0| + \frac{1}{2\pi} \log |z - z'_0|. \tag{32}
\]

It follows from (31) that \( F^*_x \) satisfies

\[
\Delta_{\Sigma} F^*_x(z) = \frac{\exp(2f'(z))}{\text{Area}(\Sigma)} \quad \text{in} \quad \mathbb{B}_{r_0} \tag{33}
\]

in the distributional sense. By (30), we have \( ||F^*_x(\cdot)||_{L^2(\mathbb{B}_{r_0})} \leq C \) for some constant \( C \) depending only on \((\Sigma, g)\) and \( r_0 \). Then applying \( W^{2,2} \)-estimate to (33), we can see that \( F^*_x \) is bounded in \( W^{2,2}(\mathbb{B}_{2r_0/3}) \) uniformly with respect to \( x \in \psi_{x_0}^{-1}(\overline{B}^r_{r_0/2}) \). Further elliptic estimate leads to

\[
||F^*_x(\cdot)||_{C^1(\overline{B}_{r_0/4})} \leq C
\]

for some constant \( C \) depending only on \((\Sigma, g)\) and \( r_0 \). This together with (32) gives

\[
|G^*_x(z)| \leq C(1 + \log |z - z_0| + \log |z - z'_0|) \leq C(1 + \log |z - z_0|)
\]

and

\[
|\nabla_{\Sigma} G^*_x(z)| \leq C(|z - z_0|^{-1} + |z - z'_0|^{-1}) \leq C|z - z_0|^{-1}
\]

for all \( z \in \overline{B}^r_{r_0/4} \setminus \{z_0\} \), since \( |z - z_0| < |z - z'_0| \). Therefore there exists some constant \( C \) depending only on \((\Sigma, g)\) and \( r_0 \) such that

\[
|\mathcal{G}(x, y)| \leq C(1 + |\log \dist_y(x, y)|), \quad |\nabla_y \mathcal{G}(x, y)| \leq C(\dist_y(x, y))^{-1} \tag{34}
\]

for all \( x \in \psi_{x_0}^{-1}(\overline{B}^r_{r_0}) \) and \( y \in \psi_{x_0}^{-1}(\overline{B}^r_{r_0/2}) \) with \( y \neq x \). Now for any fixed \( x \in \psi_{x_0}^{-1}(\overline{B}^r_{r_0/8}) \), in view of (30), we have by applying elliptic estimate to (27) that

\[
||\mathcal{G}(x, \cdot)||_{C^1(\overline{\psi_{x_0}^{-1}(B_{r_0/8})})} \leq C
\]
for some constant \( C \) depending only on \((\Sigma, g)\) and \( r_0 \). This implies (34) already holds for all \( x \in \psi_{r_0}(\Sigma_{\Sigma_0/8}) \) and all \( y \in \Sigma \) with \( y \neq x \). Since \( \partial \Sigma \) is compact, one can find a real number \( r_1 > 0 \) and a constant \( C \) depending only on \((\Sigma, g)\) and \( r_1 \) such that

\[
(34) \text{ holds for all } x \in \Sigma_{r_1} = \{ x \in \Sigma : \text{dist}_g(x, \partial \Sigma) \leq r_1 \} \quad \text{and } y \in \Sigma \text{ with } y \neq x.
\]

If \( x_0 \) is an inner point of \( \Sigma \), we take an isothermal coordinate system \((U_{x_0}, \psi_{x_0} ; \{ z_1, z_2 \})\) near \( x_0 \) such that \( \psi_{x_0}(x_0) = (0, 0), U_{x_0} \subset \Sigma \setminus \Sigma_{1/2}, \psi_{x_0}(U_{x_0}) = B_{r_0} \), and the metric \( g = \exp(2f(z))d\xi_1^2 + d\xi_2^2 \) with \( f(0, 0) = 0 \). For any \( x \in \psi_{r_0}^{-1}(B_{r_0/8}) \), we define \( G_r(z) = G(x, \psi_{r_0}^{-1}(z)) \) for \( z \in B_{r_0} \). Denote \( z_0 = \psi_{x_0}(x) = (z_{0,1}, z_{0,2}) \). Then \( G_r(z) \) is a distributional solution of

\[
-\Delta_{g_r} G_r(z) = \delta_{z_0}(z) - \frac{\exp(2f(z))}{\text{Area}(\Sigma)} \quad \text{in } B_{r_0}.
\]

As a consequence

\[
-\Delta_{g_r} \left( G_r(z) + \frac{1}{2\pi} \log |z| \right) = \frac{\exp(2f(z))}{\text{Area}(\Sigma)} \quad \text{in } B_{r_0}
\]

in the distributional sense. In view of (35), applying elliptic estimate to (36), we conclude \( G_r(z) + \frac{1}{2\pi} \log |z| \) is bounded in \( C^1(B_{r_0/2}) \) uniformly in \( x \in \psi_{r_0}^{-1}(B_{r_0/8}) \), and thus

\[
\left\| G(x, \cdot) + \frac{1}{2\pi} \log \text{dist}(x, \cdot) \right\|_{C^1(B_{r_0/2})} \leq C
\]

for all \( x \in \psi_{r_0}^{-1}(B_{r_0/8}) \), where \( C \) is a constant depending only on \((\Sigma, g)\), \( \psi_{x_0} \) and \( r_0 \). In addition, we have by applying elliptic estimate to (37) that \( \| G(x, \cdot) \|_{C^1(B_{r_0/2})} \leq C \) for all \( x \in \psi_{r_0}^{-1}(B_{r_0/8}) \). This together with (37) implies that (34) holds for some constant \( C \) depending only on \((\Sigma, g)\), \( \psi_{x_0} \) and \( r_0 \), and for all \( x \in \psi_{r_0}^{-1}(B_{r_0/8}) \). As a result, in view of the compactness of \( \Sigma \setminus \Sigma_{r_1} \), we conclude that there exists some constant \( C \), depending only on \((\Sigma, g)\) and \( r_1 \), such that

\[
(34) \text{ holds for all } x \in \Sigma \setminus \Sigma_{r_1} \text{ and } y \in \Sigma \text{ with } y \neq x.
\]

Combining (35) and (38), we conclude (29), as desired.

**Part IV. Symmetry.** We shall prove that \( G(x, y) = G(y, x) \) for all \((x, y) \in \Sigma \times \Sigma \) with \( x \neq y \).

For any \( f \in C^0(\Sigma) \), we set

\[
F(x) = \int_{\Sigma} G(y, x)(f(y) - \overline{f}) \, dv_{g, y}.
\]

where \( \overline{f} = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} f \, dv_g \). In view of (29), we have \( G \in L^1(\Sigma \times \Sigma) \). Hence we obtain by the Fubini theorem

\[
\overline{F} = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \left( \int_{\Sigma} G(y, x) \, dv_{g, y} \right) (f(y) - \overline{f}) \, dv_{g, x} = 0.
\]

By Lemma 5 there exists a unique \( \varphi \in C^{2,\alpha}(\overline{\Sigma}) \) satisfying

\[
\left\{ \begin{array}{ll}
\Delta_g \varphi = f - \overline{f} & \text{in } \Sigma \\
\partial \varphi / \partial n = 0 & \text{on } \partial \Sigma \\
\varphi = 0.
\end{array} \right.
\]
We now claim that
\[ F(x) = \varphi(x) \] for all \( x \in \overline{\Sigma} \). (42)

By (29) and the Lebesgue dominated convergence theorem, one can easily see that \( F \) is continuous on \( \overline{\Sigma} \). For any \( h \in C^0(\overline{\Sigma}) \), there exists a unique \( \psi \in C^{2,\alpha}(\overline{\Sigma}) \) such that
\[
\begin{align*}
\Delta_y \psi &= h - \overline{h} \quad \text{in } \Sigma \\
\partial \psi / \partial n &= 0 \quad \text{on } \partial \Sigma \\
\psi &= 0.
\end{align*}
\] (43)

By (39), (40), (41), (43) and the Fubini theorem, we calculate
\[
\int_{\Sigma} F h dv_g = \int_{\Sigma} F(h - \overline{h}) dv_g.
\]
\[
= \int_{\Sigma} \left( \int_{\Sigma} G(y, x)(f(y) - \overline{f}) dv_{g,y} \right) \Delta_y \varphi(x) dv_{g,x}
\]
\[
= \int_{\Sigma} \left( \int_{\Sigma} G(y, x) \Delta_y \varphi(x) dv_{g,x} \right) (f(y) - \overline{f}) dv_{g,y}
\]
\[
= \int_{\Sigma} \psi(y) \Delta_y \varphi(y) dv_{g,y}
\]
\[
= \int_{\Sigma} (h - \overline{h}) \varphi(y) dv_{g,y}
\]
\[
= \int_{\Sigma} h \varphi dv_g.
\]

Noting that \( h \in C^0(\overline{\Sigma}) \) is arbitrary, \( F \in C^0(\overline{\Sigma}) \) and \( \varphi \in C^{2,\alpha}(\overline{\Sigma}) \), we conclude (42).

It follows from (29) and (42) that
\[
\int_{\Sigma} G(y, x)(f(y) - \overline{f}) dv_{g,y} = \int_{\Sigma} G(x, y) \Delta_y \varphi(y) dv_{g,y} = \int_{\Sigma} G(x, y)(f(y) - \overline{f}) dv_{g,y}.
\]
As a consequence
\[
\int_{\Sigma} (G(y, x) - G(x, y))(f(y) - \overline{f}) dv_{g,y} = 0. \tag{44}
\]

Denote \( \mu(x) = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} G(y, x) dv_{g,y} \). Clearly \( \mu \in C^0(\overline{\Sigma}) \) because of (29). Since \( f \in C^0(\overline{\Sigma}) \) is arbitrary, we conclude from (44),
\[
G(y, x) - G(x, y) = \mu(x) \quad \text{for a.e. } y \in \Sigma.
\]

Integrating both sides of the above equation with respect to \( x \), we have by the Fubini theorem
\[
\mu(y) = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} G(x, y) dv_{g,x} = -\frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \mu(x) dv_{g,x} = 0,
\]
which implies that \( \mu \equiv 0 \) on \( \overline{\Sigma} \), and whence \( G(y, x) = G(x, y) \) for a.e. \( (x, y) \in \Sigma \times \Sigma \). Since \( G(x, \cdot) \in C^1(\overline{\Sigma} \setminus \{x\}) \) due to (29), we have \( G(\cdot, x) \in C^1(\overline{\Sigma} \setminus \{x\}) \). Therefore \( G(x, y) \) is continuous for all \( (x, y) \in \overline{\Sigma} \times \Sigma \) with \( x \neq y \), and this gives the symmetry of \( G(\cdot, \cdot) \). \qed
4. Proof of Theorem 1

In this section, we shall prove Theorem 1 by using the method of blow-up analysis. Pioneer works related to this topic are due to Ding-Jost-Li-wang [12], Adimurthi-Struwe [2], and Li [18]. Here, in our situation, blow-up happens on the boundary $\partial \Sigma$. This brings new difficulties compared with the previous situation [18, 36]. In particular, we use the Green representation formula of $c_k u_k$ to obtain the boundedness of $\|\nabla (c_k u_k)\|_{L^q(\Sigma)}$ for any $1 < q < 2$, which is the key step in the study of the convergence of $c_k u_k$ (see Lemma 19 below). It should be mentioned that our blow-up analysis and decomposition of certain Green function depend on the existence of isothermal coordinate system near the boundary $\partial \Sigma$.

Since the proof is very long, we sketch it as follows: In the subsection 4.1, let $\gamma^* \ast$ be the best constant for the inequality (14), which will be explicitly defined by (49) below. Then $\gamma^* \ast$ must be $2\pi$. In the subsections 4.2 and 4.3, there exists a smooth maximizer for any subcritical Trudinger-Moser functional. If blow-up happens (the maximizers are not uniformly bounded), by a process of blow-up analysis on a sequence of maximizers, we obtain an accurate estimate on the supremum in (14). In the subsection 4.4, we construct a sequence of admissible functions to show that the supremum in (14) is strictly greater than that we obtained in the subsection 4.3. This implies that no blow-up happens in the subsection 4.3. Thus elliptic estimate leads to the attainability of the supremum in (14) for $\gamma = 2\pi$.

4.1. The best constant

Let $\mathcal{H}$, $\lambda_N(\Sigma)$ and $\|\cdot\|_{1,\alpha}$ be defined as in (11)-(13) respectively. We first have

**Lemma 7.** For any $\alpha < \lambda_N(\Sigma)$, there exists some constant $\gamma_0 > 0$ such that

$$\sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp(\gamma_0 u^2) dv_g < +\infty.$$  

**Proof.** Since $\alpha < \lambda_N(\Sigma)$, we have for any $u$ with $\|u\|_{1,\alpha} \leq 1$,

$$1 \geq \int_{\Sigma} |\nabla_g u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \geq \left(1 - \frac{\alpha}{\lambda_N(\Sigma)}\right) \int_{\Sigma} |\nabla_g u|^2 dv_g. \quad (45)$$

If $x$ is an inner point of $\Sigma$, we choose an isothermal coordinate system $(U_x, \psi_x; \{y_1, y_2\})$ around $x$, where $U_x \subset \Sigma$ is a neighborhood of $x$ and $\psi_x : U_x \to \Omega_x \subset \mathbb{R}^2$ is a diffeomorphism. In this coordinate system, the metric $g = \exp(2f(y))(dy_1^2 + dy_2^2)$, where $f \in C^1(\Omega_x)$. As a consequence, we have by (45) that

$$\int_{\Omega_x} |\nabla_{\mathbb{R}^2}(u \circ \psi_x)|^2 dy = \int_{U_x} |\nabla_g u|^2 dv_g \leq \lambda_N(\Sigma)/(\lambda_N(\Sigma) - \alpha) \quad (46)$$

and that

$$\int_{\Omega_x} (u \circ \psi_x)^2 dy \leq \left(\max_{\Omega_x} \exp(2f)\right) \int_{U_x} u^2 dv_g \leq \frac{\max_{\Omega_x} \exp(2f)}{\lambda_N(\Sigma) - \alpha}. \quad (47)$$

Combining (46), (47) and Chang-Yang’s result (2), we conclude that there must be two constants $\gamma_x < 2\pi$ and $C_x > 0$ satisfying

$$\int_{\Omega_x} \exp(\gamma_x(u \circ \psi_x)^2) dy \leq C_x.$$

13
It then follows that
\[
\int_{U_x} \exp(\gamma x^2) dv_g \leq C_x \max_{\Omega_x} \exp(-2f).
\] (48)

In the case \( x \in \partial \Sigma \), the estimate (48) still holds for some constant \( C_x \) due to Lemma 7 and Chang-Yang’s result (2). Since \((\Sigma, g)\) is compact, we can choose \( \ell \) sets \( \{U_{x_i}\}_{i=1}^{\ell} \) satisfying \( \cup_{i=1}^{\ell} U_{x_i} \supset \Sigma \), where \( U_{x_i} \) is given as above. We immediately get the desired result. \( \square \)

In view of Lemma 7, we let
\[
\gamma^* = \sup \left\{ \gamma : \sup_{u \in H, \|u\|_{1,2} \leq 1} \int_{\Sigma} \exp(\gamma u^2) dv_g < +\infty \right\}.
\] (49)

**Lemma 8.** There holds \( \gamma^* \leq 2\pi \).

**Proof.** Recall the Moser function sequence (22)
\[
M_k(y, r) = \begin{cases} \sqrt{\log^k \frac{4\pi}{|y|}} & \text{when } |y| \leq rk^{-1/4} \\ \sqrt{\frac{1}{\pi \log^k \frac{4\pi}{r}}} \log \frac{4\pi}{|y|} & \text{when } rk^{-1/4} < |y| \leq r \\ 0 & \text{when } |y| > r \end{cases}
\] (50)
for all \( y \in \mathbb{R}^2, r > 0 \) and \( k \in \mathbb{N} \). It can be checked that
\[
\int_{\mathbb{R}^2} |\nabla M_k(y, r)|^2 dy = 1,
\] (51)
\[
\int_{\mathbb{R}^2} M_k(y, r) dy = o_k(1) + o_r(1),
\] (52)
that
\[
\int_{\mathbb{R}^2} M_k^2(y, r) dy = o_k(1) + o_r(1),
\] (53)
where \( o_k(1) \to 0 \) as \( k \to \infty \), \( o_r(1) \to 0 \) as \( r \to 0 \), and that
\[
\int_{\mathbb{R}^2} \exp(\gamma M_k^2(y, r)) dy \geq \int_{|y| \leq 1/4} \exp(\gamma M_k^2(y, r)) dy = \frac{\pi \gamma^2 k^{\gamma - 4}}{2}.
\] (54)

Now we fix a point \( p \in \partial \Sigma \) and choose an isothermal coordinate system \((\overline{U}_p, \psi; (y_1, y_2))\) near \( p \), where \( \overline{U}_p = \psi^{-1}(\mathbb{B}^2) \subset \Sigma \) for some \( r > 0 \). In this coordinate system, the metric
\[
g = \exp(2f(y)) (dy_1^2 + dy_2^2),
\]
where \( f \in C^1(\overline{\mathbb{B}^2}) \) with \( f(0, 0) = 0 \). Define a sequence of functions
\[
\tilde{M}_k(x, r) = \begin{cases} M_k(\psi(x), r) & \text{if } x \in \overline{U}_p \\ 0 & \text{if } x \in \Sigma \setminus \overline{U}_p. \end{cases}
\]

14
In view of (50), we have that
\[
\int_\Sigma |\nabla \tilde{M}_k|^2 \, dv_\Sigma = 1, \quad \int_\Sigma \tilde{M}_kdv_\Sigma = o_\epsilon(1) + o_\epsilon(1), \quad \int_\Sigma \tilde{M}_k^2 \, dv_\Sigma = o_\epsilon(1) + o_\epsilon(1).
\]
Let
\[
Q_k = \tilde{M}_k - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \tilde{M}_kdv_\Sigma.
\]
It follows that \(Q_k \in \mathcal{H}\) and \(\|Q_k\|_{1,0} = 1 + o_\epsilon(1) + o_\epsilon(1)\). This together with (54) implies
\[
\int_\Sigma \exp(\gamma \|Q_k\|^2_{1,0}) \, dv_\Sigma \geq (1 + o_\epsilon(1)) \frac{\exp(\pi^2/(2k^2))}{15}.
\]
Therefore if \(\gamma > 2\pi\), then we have by choosing sufficiently small \(r\) and passing to the limit \(k \to \infty\),
\[
\int_\Sigma \exp(\gamma \|Q_k\|^2_{1,0}) \, dv_\Sigma \to +\infty.
\]
This leads to \(\gamma^* \leq 2\pi\).

Furthermore, we have

**Lemma 9.** There holds \(\gamma^* = 2\pi\).

**Proof.** In view of Lemma 9 we only need to show \(\gamma^*\) cannot be strictly less than \(2\pi\). By the definition of \(\gamma^*\) (see (49) above), there exists a function sequence \((w_j) \subset \mathcal{H}\) with \(\|w_j\|_{1,0} \leq 1\) such that

\[
\int_\Sigma \exp(\gamma^* + 1/j) w_j^2 \, dv_\Sigma \to +\infty
\]

as \(j \to \infty\). Clearly, there exists some \(w \in \mathcal{H}\) with \(\|w\|_{1,0} \leq 1\) such that \(w_j \rightharpoonup w\) weakly in \(W^{1,2}(\Sigma, g)\), \(w_j \to w\) strongly in \(L^q(\Sigma)\) for any \(q > 0\) and \(w_j \to w\) almost everywhere in \(\Sigma\). We now claim \(w \equiv 0\) in \(\Sigma\). Supposing the contrary, we would have
\[
\|w_j - w\|_{1,0}^2 = 1 - \|w\|_{1,0}^2 \leq 1 - \frac{1}{2} \|w\|_{1,0}^2
\]
for sufficiently large \(j\). For any \(\epsilon > 0\), one has by using the Young inequality, \(ab \leq \epsilon a^2 + b^2/(4\epsilon)\), and the Hölder inequality that
\[
\int_\Sigma \exp((\gamma^* + 1/j) w_j^2) \, dv_\Sigma \leq C \left( \int_\Sigma \exp \left( (\gamma^* + 1/j)(1 + 2\epsilon) \|w_j - w\|_{1,0}^2 \frac{(w_j - w)^2}{\|w_j - w\|_{1,0}^2} \right) \, dv_\Sigma \right)^{1/2}
\]
for some constant \(C\) depending only on \(\epsilon\) and \(w\). Taking \(\epsilon\) such that \(1 + 2\epsilon = (1 - \|w\|_{1,0}^2/3)/(1 - \|w\|_{1,0}^2/2)\), we have
\[
(\gamma^* + 1/j)(1 + 2\epsilon) \|w_j - w\|_{1,0}^2 \leq (1 - \|w\|_{1,0}^2/3)(\gamma^* + 1/j) \leq (1 - \|w\|_{1,0}^2/4)^{\gamma^*},
\]
provided that \(j \geq j_0\) for sufficiently large \(j_0\). As a consequence
\[
\int_\Sigma \exp((\gamma^* + 1/j) w_j^2) \, dv_\Sigma \leq \int_\Sigma \exp \left( (1 - \|w\|_{1,0}^2/4) \frac{(w_j - w)^2}{\|w_j - w\|_{1,0}^2} \right) \, dv_\Sigma \leq C
\]
for some constant $C$. This contradicts (55) and confirms our claim $w \equiv 0$.

Suppose that $\gamma' < 2\pi$. Similarly as in the proof of Lemma 7, for any $x \in \Sigma$, we choose an isothermal coordinate system $(U_x, \psi_x, \{y_1, y_2\})$, where $\psi_x : U_x \to \Omega_x \subset \mathbb{R}^2$ is a diffeomorphism. In such a coordinate system, the metric $g = \exp(2f(y))(dy_1^2 + dy_2^2)$, where $f \in C^1(\Omega_x)$ with $f(0, 0) = 0$. By the above consideration, $w_j$ converges to 0 strongly in $L^q(\Sigma)$ for any $q > 0$. It follows that

$$
\int_{\Omega_x} w_j \circ \psi_x^{-1} dy = o_j(1)
$$

and

$$
\int_{\Omega_x} |\nabla_x(w_j \circ \psi_x^{-1})|^2 dy \leq 1 + o_j(1).
$$

Hence for any $\gamma, \gamma' < \gamma < 2\pi$, we have by Chang-Yang’s result (2),

$$
\int_{\Omega_x} \exp(\gamma( w_j \circ \psi_x)^2) dy \leq C.
$$

Similarly we have

$$
\int_{U_x} \exp(\gamma w_j^2) dv_g \leq C.
$$

Since $\Sigma$ is compact, by choosing finitely many isothermal coordinate systems covering $\Sigma$, we conclude

$$
\int_{\Sigma} \exp(\gamma w_j^2) dv_g \leq C
$$

for some constant $C$ depending only on $(\Sigma, g), \gamma'$ and $\gamma$. This contradicts (55) and concludes that $\gamma'$ must be $2\pi$. \hfill \Box

4.2. Existence of extremals for subcritical Trudinger-Moser functionals

Using a direct method of variation, we can prove the attainability of the supremum in (14) in the case $\gamma < 2\pi$. In particular, we have the following:

**Lemma 10.** For any $k \in \mathbb{N}$, there exists a $u_k \in \mathcal{H} \cap C^1(\Sigma)$ with $\|u_k\|_{1,\alpha} = 1$ such that

$$
\int_{\Sigma} \exp((2\pi - 1/k)u_k^2) dv_g = \sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} = 1} \int_{\Sigma} \exp((2\pi - 1/k)u^2) dv_g.
$$

Moreover $u_k$ satisfies the following Euler-Lagrange equation

$$
\begin{cases}
\Delta_g u_k - au_k = \frac{1}{4} u_k \exp(\gamma_k u_k^2) - \frac{\mu_k}{\nu} & \text{in } \Sigma \\
\partial u_k / \partial \nu = 0 & \text{on } \partial \Sigma \\
\lambda_k = \int_{\Sigma} u_k^2 \exp(\gamma_k u_k^2) dv_g \\
\mu_k = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} u_k \exp(\gamma_k u_k^2) dv_g \\
\gamma_k = 2\pi - 1/k,
\end{cases}
$$

(56)

where $\nu$ denotes the unit outward vector field on $\partial \Sigma$. 16
Proof. The proof is based on a direct variational method. Take a function sequence \((u_j) \subset \mathcal{H}\) satisfying \(\|u_j\|_{1,\alpha} \leq 1\) and
\[
\lim_{j \to \infty} \int_{\Sigma} \exp((2\pi - 1/k)u_j^2) dv_g = \sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp((2\pi - 1/k)u^2) dv_g. \tag{57}
\]
Since \(\alpha < \lambda_0(\Sigma)\), \((u_j) \subset \mathcal{H}\) is bounded in \(W^{1,2}(\Sigma)\). Hence one can find \(u_k \in \mathcal{H}\) such that \(u_j\) converges to \(u_k\) weakly in \(W^{1,2}(\Sigma, g)\), strongly in \(L^q(\Sigma)\) for all \(q > 0\), and almost everywhere in \(\Sigma\). It then follows that \(\|u_k\|_{1,\alpha} \leq 1\). By Lemma 9, \(\exp((2\pi - 1/k)u_j^2)\) is bounded in \(L^1(\Sigma)\) for some \(s > 1\), and thus \(\exp((2\pi - 1/k)u_j^2)\) converges to \(\exp((2\pi - 1/k)u_k^2)\) in \(L^1(\Sigma)\). This together with (57) leads to
\[
\int_{\Sigma} \exp((2\pi - 1/k)u_k^2) dv_g = \sup_{u \in \mathcal{H}, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp((2\pi - 1/k)u^2) dv_g. \tag{58}
\]
Now we show \(\|u_k\|_{1,\alpha} = 1\). Suppose \(\|u_k\|_{1,\alpha} < 1\). Then
\[
\int_{\Sigma} \exp((2\pi - 1/k)u_k^2)/\|u_k\|_{1,\alpha}^2 dv_g > \int_{\Sigma} \exp((2\pi - 1/k)u_k^2) dv_g.
\]
This contradicts (58) and implies that \(\|u_k\|_{1,\alpha} = 1\). A simple calculation shows the Euler-Lagrange equation of \(u_k\) is (56). Applying elliptic estimates to (56), we have \(u_k \in C^1(\Sigma)\). \(\square\)

4.3. Blow-up analysis on the boundary

Let \(c_k = \max_{\Sigma}|u_k|\). If \(c_k\) is bounded, then applying elliptic estimates to (56), we conclude that there exists some \(u^* \in \mathcal{H}\) with \(\|u^*\|_{1,\alpha} = 1\) such that \(u_k \to u^*\) in \(C^1(\Sigma)\). Clearly \(u^*\) is the desired extremal function. In the following, noting that \(-u_k\) is also a solution of (56), we assume without loss of generality that
\[
c_k = u_k(x_k) = \max_{\Sigma} |u_k| \to +\infty \tag{59}
\]
and
\[
x_k \to x_0 \in \partial \Sigma \tag{60}
\]
as \(k \to \infty\). Since \(u_k\) is bounded in \(W^{1,2}(\Sigma, g)\), we assume up to a subsequence \(u_k\) converges to \(u_0 \in \mathcal{H}\) weakly in \(W^{1,2}(\Sigma, g)\), strongly in \(L^q(\Sigma, g)\) for all \(q > 0\), and almost everywhere in \(\Sigma\).

Lemma 11. \(u_0 \equiv 0\), \(x_0 \in \partial \Sigma\) and \(\|\nabla u_k\|^2 dv_g \to \delta_{x_0}\) in the sense of measure.

Proof. Firstly we prove \(u_0 \equiv 0\). Suppose not. There holds
\[
\|u_k - u_0\|_{1,\alpha}^2 = 1 - \|u_0\|_{1,\alpha}^2 + o_k(1) \leq 1 - \frac{1}{2}\|u_0\|_{1,\alpha}^2
\]
for sufficiently large \(k\). By the inequality \((a + b)^2 \leq (1 + e)a^2 + (1 + 1/e)b^2\), the Hölder inequality, and Lemma 3 we have that \(\exp(\gamma_k u_k^2)\) is bounded in \(L^q(\Sigma, g)\) for some \(q > 1\). Then applying elliptic estimates to (56), we conclude that \(u_k\) is uniformly bounded in \(\Sigma\), which contradicts (59).

Hence \(u_0 \equiv 0\).

Secondly, in view of (60), we show \(x_0 \in \partial \Sigma\). Suppose \(x_0\) is an inner point of \(\Sigma\). Choose an isothermal coordinate system \((U_{x_0}, \psi_{x_0}, (y_1, y_2))\) around \(x_0\) such that \(\psi_{x_0}(U_{x_0}) = B_{x_0} \subset \mathbb{R}^2\). In this
coordinate system, the metric $g$ can be written as $g = \exp(2f(y))(dy_1^2 + dy_2^2)$, where $f \in C^1(\overline{B_0})$ with $f(0,0) = 0$. Take a cut-off function $\phi \in C_0^\infty(\overline{B_0})$ satisfying $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $\overline{B_{r_0}/2}$.

One has $\phi(u_k \circ \psi_{\phi_{k}}^{-1}) \in W^{1,2}_0(\overline{B_0})$ and $\|\nabla_g(\phi(u_k \circ \psi_{\phi_{k}}^{-1}))\|_2^2 \leq 1 + o_k(1)$. Thus Moser's inequality \ref{eq:moser} leads to

$$
\int_{B_{r_0/2}} \exp\left(2\pi q(u_k \circ \psi_{\phi_{k}}^{-1})^2\right) dx \leq \int_{B_0} \exp\left(2\pi q\phi^2(u_k \circ \psi_{\phi_{k}}^{-1})^2\right) dx \leq C
$$

for some $q > 1$ and constant $C$. This immediately implies that $\lambda_k^{-1}(u_k \exp(y_k u_k^2) - \lambda_k^{-1}\mu_k$ is bounded in $L^q(\psi_{\phi_{k}}^{-1}(\overline{B_{r_0/2}}))$ for some $1 < q' < q$. Applying elliptic estimates to \ref{eq:elliptic}, we conclude that $u_k$ is uniformly bounded in $\psi_{\phi_{k}}^{-1}(\overline{B_{r_0}/4}))$, contradicting \ref{eq:claim}. Therefore $x_0 \in \partial \Sigma$. As for the final assertion, we first claim the following

$$
\lim_{k \to \infty} \lim_{r \to 0} \int_{B_k(x_0)} |\nabla_g u_k|^2 dv_g = 1,
$$

where $B_k(x_0) \subset \Sigma$ denotes the geodesic ball centered at $x_0$ with radius $r$. For otherwise, there exist $a < 1$, $r > 0$ and $k_0 > 0$ such that

$$
\int_{B_k(x_0)} |\nabla_g u_k|^2 dv_g \leq a, \quad \forall k \geq k_0.
$$

Then similarly as we derive $x_0 \in \partial \Sigma$, we conclude that $u_k$ is uniformly bounded in $B_{r/2}(x_0)$, which contradicts \ref{eq:claim} again. Hence \ref{eq:claim} holds. For any $\varphi \in C^0(\Sigma)$, we have

$$
\lim_{k \to \infty} \int_{\Sigma} \varphi |\nabla_g u_k|^2 dv_g = \lim_{k \to \infty} \left( \int_{\Sigma \setminus B_k(x_0)} \varphi |\nabla_g u_k|^2 dv_g + \int_{B_k(x_0)} \varphi |\nabla_g u_k|^2 dv_g \right) = \varphi(x_0),
$$

which is the desired result. \hfill \Box

Let $x_0$ be given as in Lemma \ref{lemma:claim}. From now on until the end of this section, we use the isothermal coordinate system

$$
\left(\overline{U_0}, \psi_{\phi_{k}}^{-1}, \{y_1, y_2\}\right), \quad \psi_{\phi_{k}}(U_0) = \mathbb{B}_0^+, \quad \psi_{\phi_{k}}(\overline{U_0} \cap \partial \Sigma) = \partial \mathbb{B}_0^+ \cap \partial \mathbb{R}^{2+},
$$

$$
\psi_{\phi_{k}}(x_0) = (0); \quad g = \exp(2f(y))(dy_1^2 + dy_2^2), \quad f(0,0) = 0, \quad f \in C^1(\overline{B_0});
$$

moreover, in this coordinate system, the unit outward vector field $\nu$ on the boundary $\partial \Sigma$ can be written as $\nu = \exp(-f(y))\partial / \partial y_2$. For any $u \in C^1(\overline{\Sigma})$, the normal derivative $\partial u / \partial \nu$ on the boundary $\partial \Sigma$ can be represented by

$$
\frac{\partial u}{\partial \nu} = \exp(-f(y)) \frac{\partial}{\partial y_2} (u \circ \psi_{\phi_{k}}^{-1}).
$$

For simplicity we write

$$
f_k = \lambda_k^{-1} u_k \exp(y_k u_k^2) - \lambda_k^{-1}\mu_k + au_k,
$$

where $\alpha, u_k, \mu_k, y_k$ and $\lambda_k$ are defined as in \ref{eq:claim}. We set

$$
u_k(y) = \begin{cases} 
  u_k \circ \psi_{\phi_{k}}^{-1}(y) & \text{if } y \in \mathbb{B}_0^+
  \\
  u_k \circ \psi_{\phi_{k}}^{-1}(y_1, -y_2) & \text{if } y \in \overline{\mathbb{B}_0^{-}}
\end{cases}
$$

\ref{eq:nu_k}}
and

\[ f_k^*(y) = \begin{cases} \exp(2f(y))(f_k \circ \psi_{\alpha_k}^{-1})(y) & \text{if } y \in \mathbb{B}^+ \setminus \mathbb{B}_0^+ \\ \exp(2f(y_1, -y_2))(f_k \circ \psi_{\alpha_k}^{-1})(y_1, -y_2) & \text{if } y \in \mathbb{B}_0^+, \end{cases} \]

where \( \mathbb{B}_0^+ = \mathbb{B}_0 \setminus \mathbb{B}_0^+ \). Define two function sequences \( \phi_k : \mathbb{B} \to \mathbb{R} \) and \( \eta_k : \mathbb{B} \to \mathbb{R} \) for any fixed \( R > 0 \) by

\[ \phi_k(z) = \frac{u_k'(\tilde{x}_k + r_k z)}{c_k}, \quad \eta_k(z) = c_k(u_k'(\tilde{x}_k + r_k z) - c_k), \]

where \( \tilde{x}_k = \psi_{\alpha_k}(x_k) \), \( c_k \) is defined as in (59) and \( r_k > 0 \) satisfies

\[ r_k^2 = \frac{\lambda_k}{c_k} \exp(-\gamma_k c_k^2). \]

Lemma 12. \( \phi_k \) and \( \eta_k \) are distributional solutions of

\[ - \Delta_\mathbb{R}^2 \phi_k(z) = \frac{r_k^2}{c_k} f_k^*(\tilde{x}_k + r_k z) \quad \text{in} \quad \mathbb{B}_R \]

and

\[ - \Delta_\mathbb{R}^2 \eta_k(z) = r_k^2 c_k f_k^*(\tilde{x}_k + r_k z) \quad \text{in} \quad \mathbb{B}_R \]

respectively.

Proof. In view of (66), \( u_k' \in W^{1,2}(\mathbb{B}_0) \cap C^1(\overline{\mathbb{B}_0}) \). Since \( \partial u_k / \partial y = 0 \) on \( \partial \Sigma \), in view of (64), we have \( \partial u_k' / \partial y_2 = 0 \) on \( \partial \mathbb{B}_0^+ \cap \partial \mathbb{R}^{2\ast} \). We claim that \( u_k' \) is a distributional solution of the equation

\[ - \Delta_\mathbb{R}^2 u_k' = f_k^* \quad \text{in} \quad \mathbb{B}_0. \]

To see this, for any \( \varphi \in C_0^\infty(\mathbb{B}_0) \), we obtain

\[- \int_{\mathbb{B}_0} u_k' \Delta_\mathbb{R}^2 \varphi dy = \int_{\mathbb{B}_0} \nabla_\mathbb{R}^2 u_k' \nabla_\mathbb{R}^2 \varphi dy = \int_{\mathbb{B}_0^+} \nabla_\mathbb{R}^2 u_k' \nabla_\mathbb{R}^2 \varphi dy + \int_{\mathbb{B}_0^\ast \setminus \mathbb{B}_0^+} \nabla_\mathbb{R}^2 u_k' \nabla_\mathbb{R}^2 \varphi dy = \frac{\partial u_k'}{\partial y_2} dy_2 - \int_{\mathbb{B}_0^+} (\Delta_\mathbb{R}^2 u_k') \varphi dy + \frac{\partial u_k'}{\partial y_2} dy_2 - \int_{\mathbb{B}_0^\ast \setminus \mathbb{B}_0^+} (\Delta_\mathbb{R}^2 u_k') \varphi dy = \int_{\mathbb{B}_0} f_k^* \varphi dy,\]

which concludes that \( u_k' \) satisfies (70) in the distributional sense.

We next prove that \( \phi_k \) is a distributional solution of (66). Let \( R > 0 \) be fixed. For any \( \psi \in C_0^\infty(\mathbb{B}_R) \), we denote \( \tilde{\psi}(y) = \psi(\tilde{x}_k + r_k z) \). Obviously \( \tilde{\psi} \in C_0^\infty(\mathbb{B}_{R_0}(\tilde{x}_k)) \). Since \( u_k' \) is a distributional
solution of (70), it then follows that
\[
\int_{B^2_r} \phi_k(z) \Delta_{B^2} \psi(z) dz = \int_{B^2_r(\tilde{x}_k)} \frac{u^*_k(y)}{c_k} \Delta_{B^2} \psi \left( \frac{y - \tilde{x}_k}{r_k} \right) dy = \int_{B^2_r(\tilde{x}_k)} \frac{f_k^*(y)}{c_k} \psi(y) dy = \int_{B^2_r} \frac{f^*_k(\tilde{x}_k + r_k z)}{c_k} \psi(z) |z|^2 dz.
\]
Hence \( \phi_k \) satisfies (68) in the distributional sense.

In the same way, it can be proved that \( \eta_k \) is a distributional solution of (69). \( \Box \)

**Lemma 13.** For any \( \nu < 2\pi \), there holds \( r_k \exp(\nu c^2_k) \) converges to 0 as \( k \to \infty \).

**Proof.** Using the Hölder inequality, the fact \( u_k \to 0 \) strongly in \( L^q(\Sigma, g) \) for any \( q > 0 \), and Lemma 9 we have for any \( \nu < 2\pi \),
\[
\int_{B^2} u_k^\nu \exp(\nu c^2_k) dv_g = o_k(1).
\]
This together with (67) and the definition of \( A_k \) (see (56) above) gives the desired result. \( \Box \)

**Lemma 14.** Let \( \phi_k \) and \( \eta_k \) be defined as in (66). Then \( \phi_k \to \phi_0 \) and \( \eta_k \to \eta_0 \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \), where \( \phi_0(z) \equiv 1 \) and \( \eta_0(z) = -\frac{1}{2\pi} \log(1 + \frac{\pi}{2} |z|^2) \), \( \forall z \in \mathbb{R}^2 \).

**Proof.** The proof is based on the elliptic estimates on (68) and (69). We omit the details but refer the reader to [18, 35]. \( \Box \)

**Lemma 15.** Let \( \tilde{x}_k = (y_{1,k}, y_{2,k}) \). Then \( y_{2,k}/r_k \to 0 \) as \( k \to \infty \).

**Proof.** Without loss of generality, we assume as \( k \to \infty \),
\[
y_{2,k}/r_k \to \ell \quad (71)
\]
for some \( \ell > 0 \). Noting that under the change of variable \( y = \tilde{x}_k + r_k z \), the set \( B_{B^2_r(\tilde{x}_k)} \cap \mathbb{R}^{2+} = \{ y = (y_1, y_2) \in B_{B^2_r(\tilde{x}_k)} : y_2 > 0 \} \) is mapped onto \( B_{\ell,k} = \{ z = (z_1, z_2) \in \mathbb{B}_k : z_2 > -y_{2,k}/r_k = -\ell(1 + o_k(1)) \} \), we calculate by noticing (71),
\[
1 = \int_{\mathbb{R}^2} \frac{1}{A_k} u^\nu_k \exp(\gamma_k u^\nu_k) dv_g
\geq \int_{B_{\ell,k} \cap \mathbb{R}^{2+}} \frac{1}{A_k} u^\nu_k \exp(\gamma_k u^\nu_k) \exp(2\nu(y)) dy = (1 + o_k(1)) \int_{B_{\ell,k}} \phi_k^\nu(z) \exp(\gamma_k(1 + \phi_k(z)) \eta_k(z)) dz = (1 + o_k(1)) \int_{B_{\ell,k} \cap \{ z_2 > -\ell(1 + o_k(1)) \}} \exp(4\pi \eta_0(z)) dz.
\]
Note that
\[ \int_{\Omega_k} \exp(4\pi\eta_0)dz > \int_{\mathbb{R}^2} \exp(4\pi\eta_0)dz = 1. \]
By passing to the limit \( k \to \infty \) and then \( R \to \infty \) in (72), we get a contradiction. This ends the proof of the lemma. \( \square \)

Let \( \tilde{x}_{0,k} = (y_{1,k}, 0) \). We define two function sequences modified from (66) by
\[
\phi_{1,k}(z) = \frac{u_k^*(\tilde{x}_{0,k} + rz)}{c_k}, \quad \eta_{1,k}(z) = c_k(u_k^*(\tilde{x}_{0,k} + rz) - c_k)
\]
for \( z \in \Omega_k = \{ z \in \mathbb{R}^2 : \tilde{x}_{0,k} + rz \in B_{r_k} \} \).

**Lemma 16.** \( \phi_{1,k} \to \phi_0 \) and \( \eta_{1,k} \to \eta_0 \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \), where \( \phi_0 \) and \( \eta_0 \) are given as in Lemma 14.

**Proof.** Note that
\[
\tilde{x}_{0,k} + rz = \tilde{x}_k + r_k \left( z + \frac{\tilde{x}_{0,k} - \tilde{x}_k}{r_k} \right) = \tilde{x}_k + r_k (z + (0, y_{2,k}/r_k)).
\]
Then the lemma follows from Lemmas 14 and 15. \( \square \)

For any \( 0 < \beta < 1 \), let \( u_{k,\beta} = \min\{u_k, \beta c_k\} \). Similar to [18], we shall show
\[
\lim_{k \to \infty} \int_{\Sigma} |\nabla g u_{k,\beta}|^2 dv_g = \beta. \quad (74)
\]
To this end, since \( \partial u_k/\partial \nu = 0 \) on \( \partial \Sigma \), we have by the divergence theorem, Lemmas 14 and 15
\[
\int_{\Sigma} |\nabla g u_{k,\beta}|^2 dv_g = \int_{\Sigma} u_{k,\beta} \Delta g u_{k,\beta} dv_g
\]
\[
= \int_{\Sigma} u_{k,\beta} \frac{1}{A_k} \Delta g u_{k,\beta} dv_g + o_k(1)
\]
\[
\geq (1 + o_k(1)) \int_{B_{r_k}(\tilde{x}_k) \cap B_{r_k}^c} \frac{\beta}{A_k} c_k u_k^* \exp(\gamma_k u_k^2) dy + o_k(1)
\]
\[
= (\beta + o_k(1)) \int_{B_{r_k}(\tilde{x}_k) \cap B_{r_k}^c} \exp(4\pi\eta_0(z))dz + o_k(1)
\]
\[
= \beta \int_{B_{r_k}(\tilde{x}_k) \cap B_{r_k}^c} \exp(4\pi\eta_0(z))dz + o_k(1).
\]
Letting \( k \to \infty \) first, and then \( R \to \infty \), we have
\[
\liminf_{k \to \infty} \int_{\Sigma} |\nabla g u_{k,\beta}|^2 dv_g \geq \beta. \quad (75)
\]
In the same way, we estimate
\[
\int_{\Sigma} |\nabla g (u_k - u_{k,\beta})|^2 dv_g = \int_{\Sigma} (u_k - u_{k,\beta})^* \Delta g u_k dv_g
\]
\[
\geq (1 - \beta) \int_{B_{r_k}(\tilde{x}_k) \cap B_{r_k}^c} \exp(4\pi\eta_0(z))dz + o_k(1).
\]
Then we get an analog of (75), namely

$$\liminf_{k \to \infty} \int_{\Sigma} |\nabla g(u_k - u_{k,\beta})|^2 \, dv_g \geq 1 - \beta.$$  

(76)

Hence the equality

$$\int_{\Sigma} |\nabla u_{k,\beta}|^2 \, dv_g + \int_{\Sigma} |\nabla g(u_k - u_{k,\beta})|^2 \, dv_g = \int_{\Sigma} |\nabla u_k|^2 \, dv_g = 1 + \alpha \int_{\Sigma} u_k^2 \, dv_g$$

together with (74), (76) and Lemma 11 implies (74).

**Lemma 17.** Under the assumption $c_k \to \infty$, there holds

$$\sup_{u \in H, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp(2\pi u^2) \, dv_g = \text{Area}(\Sigma) + \lim_{k \to \infty} \frac{\lambda_k}{c_k^2}.$$  

(77)

As a consequence,

$$c_k/\lambda_k \to 0 \quad \text{as} \quad k \to \infty.$$  

(78)

**Proof.** Note that

$$\lim_{k \to \infty} \int_{\Sigma} \exp(\gamma u_k^2) \, dv_g = \sup_{u \in H, \|u\|_1 \leq 1} \int_{\Sigma} \exp(2\pi u^2) \, dv_g.$$  

(79)

Given any $0 < \beta < 1$. On one hand, we have by (74),

$$\int_{\Sigma} \exp(\gamma u_k^2) \, dv_g \geq \int_{u_k > \beta k} \frac{u_k^2}{c_k^2} \exp(\gamma u_k^2) \, dv_g + \int_{u_k \leq \beta k} \exp(\gamma u_k^2) \, dv_g $$

$$= \frac{\lambda_k}{c_k^2} - \int_{u_k \leq \beta k} \frac{u_k^2}{c_k^2} \exp(\gamma u_k^2) \, dv_g + \int_{u_k \leq \beta k} \exp(\gamma u_k^2) \, dv_g $$

$$= \frac{\lambda_k}{c_k^2} + \text{Area}(\Sigma) + o_k(1).$$  

(80)

On the other hand, we also obtain by using (74),

$$\int_{\Sigma} \exp(\gamma u_k^2) \, dv_g \leq \int_{u_k > \beta k} \frac{u_k^2}{\beta^2 c_k^2} \exp(\gamma u_k^2) \, dv_g + \int_{u_k \leq \beta k} \exp(\gamma u_k^2) \, dv_g $$

$$\leq \frac{1}{\beta^2} \frac{\lambda_k}{c_k^2} + \text{Area}(\Sigma) + o_k(1).$$  

(81)

Combining (79)-(81), we get (77) by passing to the limit $k \to \infty$ first, and then $\beta \to 1$.

For the second assertion, we suppose the contrary, there exists some constant $\varrho > 0$ such that up to a subsequence, $c_k/\lambda_k \geq \varrho$. Hence $\lambda_k/c_k^2 \leq 1/(\varrho c_k)$, which together with (77) leads to

$$\sup_{u \in H, \|u\|_{1,\alpha} \leq 1} \int_{\Sigma} \exp(2\pi u^2) \, dv_g = \text{Area}(\Sigma),$$

which is impossible. Therefore (78) holds. □
Lemma 18. For any \( \varphi \in C^2(\Sigma) \), there holds

\[
\int_{\Sigma} \varphi \frac{c_k}{\lambda_k} |u_k| \exp(\gamma_k u_k^2) dv = \varphi(x_0) + o_k(1), \quad \int_{\Sigma} \varphi \frac{c_k}{\lambda_k} u_k \exp(\gamma_k u_k^2) dv = \varphi(x_0) + o_k(1). \tag{82}
\]

Proof. We only prove the first equality of (82), since the proof of the second one is the same.

Let \((U_{y_0},\psi_{y_0};(y_1,y_2))\) be the isothermal coordinate system around \(x_0\) given by (62) and (63). We calculate by using Lemma 16 that

\[
\int_{\{y < \beta \}} \varphi \frac{c_k}{\lambda_k} |u_k| \exp(\gamma_k u_k^2) dv \leq C \left(1 - \int_{\mathbb{R}^n} \frac{u_k^2}{\lambda_k} \exp(\gamma_k u_k^2) \exp(2\gamma) dy\right)
\]

\[
= C \left(1 - \int_{\mathbb{R}^n} \exp(4\pi\eta_0) dy\right)
\]

\[
= \varphi(x_0) + o_k(1)
\]

Also we have for any fixed \(0 < \beta < 1\),

\[
\int_{\{u > \beta \}} \varphi \frac{c_k}{\lambda_k} |u_k| \exp(\gamma_k u_k^2) dv \leq C \frac{c_k}{\lambda_k} \int_{\Sigma} \frac{|u_k|}{\lambda_k} \exp(\gamma_k u_k^2) dv = o_k(1),
\]

where \(C\) is a constant depending only on \(\beta\) and \(\max \varphi\). Finally we estimate by (74) and Lemma 11 that

\[
\int_{u < \beta |e_k|} \varphi \frac{c_k}{\lambda_k} |u_k| \exp(\gamma_k u_k^2) dv \leq C \frac{c_k}{\lambda_k} \int_{\Sigma} |u_k| \exp(\gamma_k u_k^2) dv = o_k(1),
\]

where we used \(c_k/\lambda_k = o_k(1)\), which is a consequence of Lemma 17. Combining the above three estimates, we conclude (82).

The convergence of \(c_k u_k\) away from \(x_0\) can be described as

Lemma 19. For any \(1 < q < 2\), \(c_k u_k\) converges to \(G_{a,x_0}\) weakly in \(W^{1,q}(\Sigma)\), strongly in \(L^s(\Sigma)\) with \(s < 2q/(2-q)\), and in \(C^1_\text{loc}(\Sigma \setminus \{x_0\})\) as \(k \to \infty\), where \(G_{a,x_0}\) satisfies

\[
\begin{align*}
\Delta_k G_{a,x_0} - \alpha G_{a,x_0} = \delta_{x_0} - \frac{1}{\lambda_k} & \quad \text{in } \Sigma \\
\frac{\partial}{\partial n} G_{a,x_0} = 0 & \quad \text{on } \partial \Sigma \\
\int_{B(x_0,\rho)} G_{a,x_0} dv = 0
\end{align*} \tag{83}
\]

in the distributional sense.

Proof. In view of (56), \(c_k u_k\) is a solution of

\[
\Delta_k (c_k u_k) - \alpha c_k u_k = f_k \equiv \frac{1}{\lambda_k} c_k u_k \exp(\gamma_k u_k^2) - \frac{c_k c_k}{\lambda_k} \quad \text{in } \Sigma. \tag{84}
\]
Integrating both sides of (84) and recalling Lemma 18 we conclude that
\[ c_k u_k / A_k \to 1 / \text{Area}(\Sigma) \quad \text{as} \quad k \to \infty, \]  
and that \( f_k \) is bounded in \( L^1(\Sigma, g) \). We claim that \( c_k u_k \) is also bounded in \( L^1(\Sigma, g) \). Suppose not. Let \( v_k = c_k u_k / \| c_k u_k \|_1 \), where \( \| \cdot \|_1 \) denotes the \( L^1(\Sigma, g) \) norm. Then \( \| v_k \|_1 = 1 \) and satisfies
\[ \Delta_k v_k = \alpha v_k + f_k / \| c_k u_k \|_1 \quad \text{in} \quad \Sigma. \]  
By the Green representation formula (Lemma 6),
\[ v_k(x) = \int_{\Sigma} G(x,y) \Delta_k v_k(y) dv_{g,\Sigma}. \]
Recalling (29), we have for any \( 1 < q < 2 \) by using the H\ölder inequality and the Fubini theorem,
\[ \int_{\Sigma} |\nabla v_k|^q dv_g \leq \int_{\Sigma} \left( \int_{\Sigma} |\nabla G(x,y)\Delta_k v_k(y)| dv_{g,\Sigma} \right)^{q/2} \left( \int_{\Sigma} |\Delta_k v_k(y)| dv_{g,\Sigma} \right)^{q/2} dv_{g,\Sigma} \leq C. \]  
This together with the Poincare inequality implies that \( v_k \) is bounded in \( W^{1,q}(\Sigma, g) \). Then up to a subsequence, we assume \( v_k \) converges to \( v_0 \) weakly in \( W^{1,q}(\Sigma, g) \), strongly in \( L^1(\Sigma, g) \) with \( s < 2q/(2-q) \), and almost everywhere in \( \Sigma \). As a consequence, \( \| v_0 \|_1 = 1 \) and \( v_0 \) is a distributional solution of
\[ \Delta_k v_0 - \alpha v_0 = 0 \quad \text{in} \quad \Sigma, \]  
where we have used (86) and \( f_k / \| c_k u_k \|_1 \to 0 \) in \( L^1(\Sigma, g) \) as \( k \to \infty \). Since \( \alpha < A_k(\Sigma) \), it follows from (88) that \( v_0 \equiv 0 \) in \( \Sigma \), which contradicts \( \| v_0 \|_1 = 1 \). Hence we conclude our claim \( \| c_k u_k \|_1 \leq C \). Then coming back to (84), we see that \( \Delta_k (c_k u_k) \) is bounded in \( L^1(\Sigma, g) \). In the same way as (87), we obtain
\[ \int_{\Sigma} |\nabla G(c_k u_k)|^q dv_k \leq C. \]  
Hence \( c_k u_k \) is bounded in \( W^{1,q}(\Sigma, g) \). There exists some \( G_{\alpha,x_0} \) such that \( c_k u_k \) converges to \( G_{\alpha,x_0} \) weakly in \( W^{1,q}(\Sigma, g) \), strongly in \( L^1(\Sigma, g) \) for any \( s < 2q/(2-q) \), and almost everywhere in \( \Sigma \). In view of Lemma 18 \( G_{\alpha,x_0} \) satisfies (83) in the distributional sense.

It follows from (78), (85), Lemmas 9 and 11 that for any \( \Omega' \subset\subset \Sigma \setminus \{x_0\} \), there exists some \( p > 2 \) such that \( f_k \) is bounded in \( L^p(\Omega') \). Applying elliptic estimates to (84), we have \( c_k u_k \to G_{\alpha,x_0} \) in \( C^1(\overline{\Omega'}) \) for any \( \Omega'' \subset \subset \Omega' \). This ends the proof of the lemma. \( \square \)

The function \( G_{\alpha,x_0} \) can be decomposed near \( x_0 \) as below.

**Lemma 20.** In the isothermal coordinate system \( (62) \) around \( x_0 \), the function \( G_{\alpha,x_0} \) can be written as the form
\[ G_{\alpha,x_0} \circ \psi_{x_0}^{-1}(y) = -\frac{1}{\pi} \log |y| + h(y), \quad \forall y \in \overline{B_{x_0}} \setminus \{0\}, \]  
where \( h \in C^1(\overline{B_{x_0}}) \).
Proof. In the isothermal coordinate system (62) near \( x_0 \), we set
\[
G^*_0(x_0, y) = \begin{cases} 
G_{α,0} \circ ϕ^{-1}_0(y_1, y_2) & \text{if } y_2 ≥ 0 \\
G_{α,0} \circ ϕ^{-1}_0(y_1, -y_2) & \text{if } y_2 < 0.
\end{cases}
\]

It follows from (83) and the fact \( ∂G^*_0 / ∂y_2 = 0 \) on \( ∂B^*_0 \cap ∂]\(R^2+\) that \( G^*_0 \) satisfies
\[
- Δ g^*G^*_0 - α exp(2f)G^*_0 = 2δ_0 - exp(2f) / Area(Σ) \quad \text{in } B^*_0
\]
in the distributional sense. Namely, for any \( ϕ \in C^1_0(\mathbb{R}^n) \), there holds
\[
- \int_{B^*_0} (Δ g^*G^*_0) dy - α \int_{B^*_0} ϕ exp(2f)G^*_0 dy = 2φ(0) - \int_{B^*_0} ϕ exp(2f) / Area(Σ) dy.
\]
Noting also
\[
Δ g^* log |y| = 2πδ_0 \quad \text{in } B^*_0,
\]
we obtain by subtracting (91) from (90) that
\[
- Δ g^* \left( G^*_0 + \frac{1}{|x|} log |y| \right) = α exp(2f)G^*_0 - exp(2f) / Area(Σ) \quad \text{in } B^*_0.
\]
Then (89) follows immediately from elliptic estimates on (92). \( \square \)

Let \( \bar{x}_{0,k} \) and \( r_0 \) be given as in (75) and (62) respectively. For any real numbers \( R > 0 \) and \( 0 < s < r_0 \), we denote
\[
T^*_s = B^*_s(\bar{x}_{0,k}) \setminus B^*_{R|s|}(\bar{x}_{0,k}), \quad Γ^s = ∂B^*_s(\bar{x}_{0,k}) \setminus ∂R^{2+}, \quad Γ^R = ∂B^*_{R,s}(\bar{x}_{0,k}) \setminus ∂R^{2+}
\]
and
\[
m_{s,k} = \sup_{Γ^s} u_s \circ \psi^{-1}_{s,k}, \quad i_{s,k} = \inf_{Γ^s} u_s \circ ϕ^{-1}_{s,k}.
\]
In view of Lemmas 16, 19 and 20 there holds
\[
\begin{cases}
m_{s,k} = \frac{-\frac{1}{s} log |s^{-1}\bar{x}_{0,k} + s ξ + (0)|}{r_0} \quad & (93) \\
i_{s,k} = c_k + \frac{-\frac{1}{s} log |s^{-1}\bar{x}_{0,k} + s ξ + (0)|}{r_0}
\end{cases}
\]
where \( ξ_{s}(s) → 0 \) as \( s → 0^+ \). Define a sequence of function sets
\[
\mathscr{K} = \left\{ u ∈ W^{1,2}(T^*_s) : u|_{Γ^s} = m_{s,k}, \quad u|_{Γ^R} = i_{s,k} \right\}.
\]
Since \( i_{s,k} > m_{s,k} \) for sufficiently large \( k \), the Poincare inequality holds on \( \mathscr{K} \). By a direct method of variation, it then follows that
\[
\inf_{u ∈ \mathscr{K}} \int_{T^*_s} |∇ u|^2 dy
\]
can be attained by the harmonic function
\[
h_k(y) = \frac{m_{s,k}(log |y - \bar{x}_{0,k}| - log(R_{s,k})) + i_{s,k}(log s - log |y - \bar{x}_{0,k}|)}{log s - log(R_{s,k})}.
\]
As a consequence
\[ \int_{T_1} |\nabla G(x)|^2 \, dy = \frac{\pi(m_{x,k} - i_{R_k})^2}{\log s - \log(Rr_k)} \]  
(95)

Define a sequence of functions
\[ \tilde{u}_k(y) = \max\{m_{x,k}, \min\{u_{k,y} \circ \psi_{\xi_{\alpha}}^{-1}(y), i_{R_k}\}\} \quad y \in \mathbb{R}^e. \]

One can see that \( \tilde{u}_k \) belongs to \( \mathcal{K}_k \) and that
\[ \int_{T_2} |\nabla \tilde{u}_k|^2 \, dy \leq \int_{T_2} |\nabla (u_k \circ \psi_{\xi_{\alpha}}^{-1})|^2 \, dy \]
\[ = \int_{\psi_{\xi_{\alpha}}^{-1}(T_1)} |\nabla u_k|^2 \, dv_g \]
\[ = 1 + \alpha \int_{\Sigma} u_k^2 \, dv_g - \int_{\Sigma(\psi_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}})))} |\nabla u_k|^2 \, dv_g \]
\[ - \int_{\tilde{G}_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}}))} |\nabla (u_k \circ \psi_{\xi_{\alpha}}^{-1})|^2 \, dy. \]  
(96)

Combining (94), (95) and (96), we have
\[ \frac{\pi(m_{x,k} - i_{R_k})^2}{\log s - \log(Rr_k)} \leq 1 + \alpha \int_{\Sigma} u_k^2 \, dv_g - \int_{\Sigma(\psi_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}})))} |\nabla u_k|^2 \, dv_g \]
\[ - \int_{\tilde{G}_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}}))} |\nabla (u_k \circ \psi_{\xi_{\alpha}}^{-1})|^2 \, dy. \]  
(97)

It follows from (93) that
\[ \frac{\pi(m_{x,k} - i_{R_k})^2}{\log s - \log(Rr_k)} = \frac{2\pi \gamma_k^2 - 2 \log(1 + \frac{1}{4} R^2) + 4 \log s - 4 \pi h(0) + o(1)}{\gamma_k c_k^2 + 2 \log s - 2 \log R - \log \frac{1}{p}}. \]  
(98)

Let \( \nu \) be the unit outward vector on \( \partial \psi_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}}(\tilde{G}_{\xi_{\alpha}}))) \). We write \( \nu = v^1 \partial/\partial y_1 + v^2 \partial/\partial y_2 \). Then there holds on \( \partial \psi_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}}(\tilde{G}_{\xi_{\alpha}})) \) \( \partial \Sigma \),
\[ \frac{\partial G_{\alpha_{\xi_{\alpha}}} \, dv}{\partial \nu} = v^1 \frac{\partial}{\partial y_1}(G_{\alpha_{\xi_{\alpha}}} \circ \psi_{\xi_{\alpha}}^{-1}) + v^2 \frac{\partial}{\partial y_2}(G_{\alpha_{\xi_{\alpha}}} \circ \psi_{\xi_{\alpha}}^{-1}), \]
\[ 1 = |\nu|^2 = \exp(2 f(y))((v^1)^2 + (v^2)^2), \]
\[ d\sigma_g = \exp(f(y)) \, d\sigma_0. \]

and
\[ \int_{\partial \psi_{\xi_{\alpha}}^{-1}(\tilde{G}_{\xi_{\alpha}}(\tilde{G}_{\xi_{\alpha}}))) \, d\sigma_g = \int_{\partial G_{\alpha_{\xi_{\alpha}}} \circ \psi_{\xi_{\alpha}}^{-1}} (G_{\alpha_{\xi_{\alpha}}} \circ \psi_{\xi_{\alpha}}^{-1}) \frac{\partial}{\partial \nu_0}(G_{\alpha_{\xi_{\alpha}}} \circ \psi_{\xi_{\alpha}}^{-1}) \exp(f) \, d\sigma_0 \]
\[ = \int_0^\alpha \left( \frac{- \log s}{\pi} + h(s \cos t, s \sin t) \right) \left( - \frac{1}{\pi s} + \frac{\partial h}{\partial s} \right) s \, dt + o_1(1) \]
\[ = - \frac{1}{\pi} \log s + h(0) + o_1(1) + o_2(1), \]
where \( \nu = (\nu^1, \nu^2) \) is a normal vector field on \( \partial B_1^+(\tilde{\gamma}_0, k) \setminus \partial B^2 \) and \( \nu_0 \) denotes its Euclidean arc length element. Thus
\[
\int_{\Sigma|\phi_0^{-1}(\partial B^1(|\tilde{\gamma}_0|))} |\nabla_x G_{\alpha, \nu_0}|^2 dv_x = \int_{\Sigma|\phi_0^{-1}(\partial B^1(|\tilde{\gamma}_0|))} G_{\alpha, \nu_0} \Delta_x G_{\alpha, \nu_0} dv_x
\]
\[
+ \int_{\Sigma|\phi_0^{-1}(\partial B^1(|\tilde{\gamma}_0|))} G_{\alpha, \nu_0} \frac{\partial G_{\alpha, \nu_0}}{\partial \nu} dv_x
\]
\[
= \alpha \int_{\Sigma} G_{\alpha, \nu_0}^2 dv_x - \frac{1}{\pi} \log s + A_0 + o_2(1) + o_3(1).
\]
This together with Lemma[12] leads to
\[
\int_{\Sigma|\phi_0^{-1}(\partial B^1(|\tilde{\gamma}_0|))} |\nabla_x u_k|^2 dv_x = \frac{1}{c_k^2} \left( \int_{\Sigma|\phi_0^{-1}(\partial B^1(|\tilde{\gamma}_0|))} |\nabla_x G_{\alpha, \nu_0}|^2 dv_x + o_2(1) \right)
\]
\[
= \frac{1}{c_k^2} \left( \alpha \int_{\Sigma} G_{\alpha, \nu_0}^2 dv_x - \frac{1}{\pi} \log s + h(0) + o_2(1) + o_3(1) \right).
\]
By Lemma[16] we obtain
\[
\int_{\phi_0^{-1}(B^1(|\tilde{\gamma}_0|))} |\nabla_x u_k|^2 dv_x = \int_{\phi_0^{-1}(B^1(|\tilde{\gamma}_0|))} |\nabla_x (u_k \circ \psi_0^{-1})|^2 dv
\]
\[
= \frac{1}{c_k^2} \left( \frac{1}{2\pi} \log(1 + \frac{\pi}{2} R^2) - \frac{1}{2\pi} + o_2(1) + o_3(1) \right),
\]
where \( o_2(1) \to 0 \) as \( R \to \infty \). Combining (97), (98), (99), (100) and passing to the limit \( k \to \infty \) firstly, then \( R \to \infty \) and finally \( s \to 0 \), we calculate
\[
\limsup_{k \to \infty} \frac{A_k}{c_k} \leq \frac{\pi}{2} \exp(1 + 2\pi h(0)),
\]
which together with (77) and (79) leads to
\[
\sup_{\alpha \in H, ||\phi_0||_{1, \alpha} \leq 1} \int_{\Sigma} \exp(2\pi \alpha^2) dv_x = \lim_{k \to \infty} \int_{\Sigma} \exp(\gamma_k u_k^2) dv_x \leq \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)).
\]

4.4. Test function computation

We shall construct a sequence of functions \( \phi_k \in \mathcal{H} \) with \( ||\phi_k||_{1, \alpha} = 1 \) such that
\[
\int_{\Sigma} \exp(2\pi \phi_k^2) dv_x > \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)).
\]
The contradiction between (102) and (101) implies that (59) can not hold. Then applying elliptic estimates to (56), we finish the proof of Theorem[1].

To proceed, we use the isothermal coordinate system \( (U_{x_0}, \psi_{x_0}, (y_1, y_2)) \), which is defined as in (62), and let
\[
\tilde{\phi}_k(y) = \begin{cases} 
\frac{c + \frac{1}{c} \left( \frac{1}{2\pi} \log(1 + \frac{\pi}{2} k|y|^2) + B \right)}{\frac{1}{c} (G_{\alpha, \nu_0} \circ \psi_0^{-1}(y) - \eta(y) \beta(y))} & \text{when } |y| \leq \frac{\log k}{k} \\
\frac{1}{c} (G_{\alpha, \nu_0} \circ \psi_0^{-1}(y) - \eta(y) \beta(y)) & \text{when } \frac{\log k}{k} < |y| < \frac{2\log k}{k}
\end{cases}
\]
where $\beta(y) = G_{n,0} \circ \psi_0^{-1}(y) + \frac{1}{2} \log |y| - h(0)$, $\eta(y) = \eta(|y|)$ is a radially symmetric function satisfying $\eta \in C^1_c(\mathbb{B}_{k^{-1} \log k}, \eta \equiv 1 \text{ in } \mathbb{B}_{k^{-1} \log k}$, $\|\nabla g \eta\|_{L^\infty} = O(\frac{1}{\log k})$, $B$ and $c$ are constants depending only on $k$ to be determined later. Define

$$\phi_k = \begin{cases} \tilde{\phi}_k \circ \psi_0 & \text{on } \psi_0^{-1}(\mathbb{B}_{k^{-1} \log k}^+), \\ \frac{G_{n,0}}{c} & \text{on } \Sigma \setminus \psi_0^{-1}(\mathbb{B}_{k^{-1} \log k}^+). \end{cases} \quad (103)$$

On $\partial \mathbb{B}_{k^{-1} \log k}^+ \setminus \mathbb{R}^2_+$, we let

$$c + \frac{1}{c} \left(-\frac{1}{2\pi} \log(1 + \frac{\pi}{2}(\log k)^2) + B\right) = \frac{1}{c} \left(-\frac{1}{\pi} \log(k^{-1} \log k) + h(0)\right), \quad (104)$$

which leads to $\phi_k \in W^{1,2}(\Sigma, g)$. It follows from (104) that

$$2\pi c^2 = 2 \log k - 2\pi B + 2\pi h(0) + \log \frac{\pi}{2} + O\left(\frac{1}{(\log k)^2}\right). \quad (105)$$

Clearly we calculate

$$\int_{\psi_0^{-1}(\mathbb{B}_{k^{-1} \log k}^+)} |\nabla \tilde{\phi}|^2 dv_g = \int_{\mathbb{B}_{k^{-1} \log k}^+} |\nabla \tilde{\phi}|^2 dy = \int_{\mathbb{B}_{k^{-1} \log k}^+} \frac{1}{4c^2} \int_{\mathbb{B}_{k^{-1} \log k}^+} \left|z\right|^2 \left(1 + \frac{\left|z\right|^2}{c^2}\right) dz = \frac{1}{2\pi c^2} \left(2 \log(\log k) + \log \frac{\pi}{2} - 1 + O\left(\frac{1}{(\log k)^2}\right)\right). \quad (106)$$

Denoting $T^+_k = \mathbb{B}_{k^{-1} \log k}^+ \setminus \mathbb{B}_{k^{-1} \log k}$, we have

$$\int_{\psi_0^{-1}(T^+_k)} |\nabla \tilde{\phi}|^2 dv_g = \int_{T^+_k} |\nabla \tilde{\phi}|^2 dy = \int_{T^+_k} \frac{1}{c^2} \left|\nabla_{\mathbb{R}^2}(G_{n,0} \circ \psi_0^{-1})\right|^2 dy + \int_{T^+_k} \frac{1}{c^2} \left|\nabla_{\mathbb{R}^2}(\eta\beta)\right|^2 dy$$

$$- \int_{T^+_k} \frac{2}{c^2} \nabla_{\mathbb{R}^2}(G_{n,0} \circ \psi_0^{-1}) \nabla_{\mathbb{R}^2}(\eta\beta) dy = \frac{1}{c^2} \left(2 \log 2 + O\left(\frac{1}{(\log k)^2}\right)\right). \quad (107)$$

Writing $\tilde{G} = G_{n,0} \circ \psi_0^{-1}$ and $\nu = v^1 \partial/\partial y_1 + v^2 \partial/\partial y_2$, we get on $\psi_0^{-1}(\mathbb{B}_{k^{-1} \log k}^+) \cap \partial \Sigma$,

$$\frac{\partial G_{n,0}}{\partial v} = v^1 \frac{\partial \tilde{G}}{\partial y_1} + v^2 \frac{\partial \tilde{G}}{\partial y_2} = \exp(-f) \frac{\partial \tilde{G}}{\partial y_0},$$

where $v_0 = \psi_0^{-1}(\nu)/|\psi_0^{-1}(\nu)|$ is the unit outward vector field on $\partial \mathbb{B}_{k^{-1} \log k}^+ \setminus \partial \mathbb{R}^2_+$. Moreover $d\sigma_g = \exp(f) d\sigma_0$, where $d\sigma_0$ is the Euclidean arc-length element of $\partial \mathbb{B}_{k^{-1} \log k}^+ \setminus \partial \mathbb{R}^2_+$. It then
follows that
\[
\int_{\omega_0^{-1}(\mathbb{R}^n)} G_{a_0} \frac{\partial G_{a_0}}{\partial y} d\sigma_y = \int_{\omega_0^{-1}(\mathbb{R}^n)} \tilde{G} \frac{\partial \tilde{G}}{\partial y_0} d\sigma_0
\]
\[
= \int_{\omega_0^{-1}(\mathbb{R}^n)} \left( -\frac{1}{\pi} \log |y| + h(0) + O(|y|) \right)
\times \left( -\frac{1}{\pi |y|} + O(1) \right) d\sigma_0
\]
\[
= \frac{\log 2}{\pi} + \frac{1}{\pi} \log \left( \frac{\log k}{k} \right) - h(0) + O \left( \frac{1}{(\log k)^2} \right).
\]
This together with
\[
\int_{\Sigma} \phi^2 d\nu = \int_{\Sigma} \phi^2 + O \left( \frac{1}{(\log k)^2} \right)
\]
and
\[
\int_{\Sigma} \phi d\nu = -\int_{\phi^{-1}(\mathbb{R}^n)} \phi d\nu = O \left( \frac{1}{(\log k)^2} \right)
\]
leads to
\[
\int_{\Sigma} \phi \frac{\partial \phi}{\partial y} d\nu = \int_{\Sigma} \phi \frac{\partial G_{a_0}}{\partial y} d\nu
\]
\[
= \frac{1}{c^2} \int_{\partial \Sigma} \phi \frac{\partial G_{a_0}}{\partial y} d\sigma_y
+ \frac{1}{c^2} \int_{\Sigma} \phi \frac{\partial \phi}{\partial y} d\nu
\]
\[
= \frac{1}{c^2} \left\{ \int_{\omega_0^{-1}(\mathbb{R}^n)} \phi \frac{\partial \phi}{\partial y} d\nu + \alpha \int_{\Sigma} \phi^2 d\nu + \frac{1}{\text{Area}(\Sigma)} \int_{\omega_0^{-1}(\mathbb{R}^n)} \phi d\nu \right\}
\]
\[
= \frac{1}{c^2} \left\{ -\frac{1}{\pi} \log \left( \frac{\log k}{k} \right) \frac{\log 2}{\pi} \right. + \frac{1}{\pi} \log \left( \frac{\log k}{k} \right) - h(0) + \alpha \int_{\Sigma} \phi^2 d\nu + O \left( \frac{1}{(\log k)^2} \right) \right\}.
\] (108)

Combining (106), (107) and (108), we conclude
\[
\int_{\Sigma} |\nabla \phi|^2 d\nu = \frac{1}{c^2} \left\{ \log k \frac{\log k}{\pi} + h(0) + \frac{1}{2\pi} \log \frac{\pi}{2} - \frac{1}{2\pi} + \alpha \int_{\Sigma} \phi^2 d\nu + O \left( \frac{1}{(\log k)^2} \right) \right\}. \] (109)

Also one can compute
\[
\bar{\phi} = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \phi d\nu = \frac{1}{c} O \left( \frac{1}{(\log k)^2} \right)
\]
and
\[ \int_{\Sigma} (\phi_k - \overline{\phi}_k)^2 dv_g = \frac{1}{c^2} \left( \int_{\Sigma} G_{\alpha, \beta}^2 dv_g + O\left( \frac{1}{(\log k)^2} \right) \right). \]

This together with (109) gives
\[ \|\phi_k - \overline{\phi}_k\|^2_{L^2, \alpha} = \int_{\Sigma} |\nabla g(\phi_k - \overline{\phi}_k)|^2 dv_g - \alpha \int_{\Sigma} (\phi_k - \overline{\phi}_k)^2 dv_g \]
\[ = \frac{1}{c^2} \left( \frac{\log k}{\pi} + h(0) + \frac{1}{2\pi} \log \frac{\pi}{2} - \frac{1}{2\pi} + O\left( \frac{1}{(\log k)^2} \right) \right). \quad (110) \]

Now we set
\[ \|\phi_k - \overline{\phi}_k\|_{L^2, \alpha} = 1. \quad (111) \]

It follows from (110) and (111) that
\[ c^2 = \frac{\log k}{\pi} + h(0) + \frac{1}{2\pi} \log \frac{\pi}{2} - \frac{1}{2\pi} + O\left( \frac{1}{(\log k)^2} \right). \quad (112) \]

Inserting (112) into (105), we obtain
\[ B = \frac{1}{2\pi} + O\left( \frac{1}{(\log k)^2} \right). \quad (113) \]

In view of (103), (112) and (113), there holds
\[ \int_{\phi_k^{-1}(B_{l+1}^{+})} \exp(2\pi(\phi_k - \overline{\phi}_k)^2) dv_g = \int_{B_{l+1}^{+}} \exp \left( 2\pi (\phi_k(y) - \overline{\phi}_k)^2 + 2f(y) \right) dy \]
\[ = (1 + O((\log k)^{-2})) \int_{B_{l+1}^{+}} \exp(2\pi \overline{\phi}_k^2) dy \]
\[ \geq (1 + O((\log k)^{-2})) \int_{B_{l+1}^{+}} \exp \left( 2\pi c^2 - 2\log(1 + \frac{\pi}{2} |z|^2) \right. \]
\[ \left. + 4\pi B \right) \frac{1}{k^2} dz \]
\[ = (1 + O((\log k)^{-2})) \frac{\pi}{2} \exp(1 + 2\pi h(0)), \]

and
\[ \int_{\Sigma\setminus\phi_k^{-1}(B_{l+1}^{+})} \exp(2\pi(\phi_k - \overline{\phi}_k)^2) dv_g \geq \int_{\Sigma\setminus\phi_k^{-1}(B_{l+1}^{+})} (1 + 2\pi(\phi_k - \overline{\phi}_k)^2) dv_g \]
\[ = \text{Area}(\Sigma) + \frac{2\pi}{c^2} \int_{\Sigma} G_{\alpha, \beta}^2 dv_g + O((\log k)^{-2}). \]

Therefore
\[ \int_{\Sigma} \exp(2\pi(\phi_k - \overline{\phi}_k)^2) dv_g \geq \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)) + \frac{2\pi}{c^2} \int_{\Sigma} G_{\alpha, \beta}^2 dv_g + O((\log k)^{-2}). \quad (114) \]
Since \((\log k)^{-2} = o(e^{-2})\), we have by (114) that
\[
\int \Sigma \exp(2\pi (\phi_k - \overline{\phi_k})^2)dv_g > \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)) \]
for sufficiently large \(k\). Therefore \(\phi_k - \overline{\phi_k} \in \mathcal{H}\) satisfies (102) provided that \(k\) is chosen sufficiently large, and thus the proof of Theorem 1 is completely finished.

5. Proof of Theorem 2

In this section, we shall prove Theorem 2 by using the same method of proving Theorem 1. We only give its outline but emphasize their differences.

5.1. The best constant
Let \(\tau > 0\) be a fixed positive real number, \(u\) be any function in \(W^{1,2}(\Sigma, g)\), \(\|u\|_{1,\tau}\) be defined as in (15) and \(\overline{\pi} = \frac{1}{\text{Area}(\Sigma)} \int_\Sigma udv_g\). By the H"older inequality,
\[
\|u\|^2 \leq \frac{1}{\text{Area}(\Sigma)} \int_\Sigma u^2dv_g \leq \frac{\|u\|_{1,\tau}}{\tau \text{Area}(\Sigma)} \tag{115}
\]
Hence, if \(\|u\|_{1,\tau} \leq 1\), then \(\int_\Sigma |\nabla g(u - \overline{u})|^2dv_g \leq 1\), and the Young inequality together with (115) implies that for any \(\epsilon > 0\), there holds a constant \(C\) depending only on \((\Sigma, g), \alpha\) and \(\epsilon\) such that
\[
\int_\Sigma \exp(\alpha u^2)dv_g \leq C \left( \int_\Sigma \exp(\alpha(1 + \epsilon)(u - \overline{u})^2)dv_g \right)^{1/(1+\epsilon)} \tag{116}
\]
Define
\[
\alpha^* = \sup \left\{ \alpha \mid \sup_{u \in W^{1,2}(\Sigma, g), \|u\|_{1,\tau} \leq 1} \int_\Sigma \exp(\alpha u^2)dv_g < \infty \right\}.
\]
It follows from (115) and Lemma 9 that
\[
\alpha^* \geq 2\pi \tag{117}
\]
Let \(M_k\) be defined as in (50). Then we have
\[
\|M_k\|_{1,\tau, r}^2 = \int_{B_r} (|\nabla g M_k|^2 + \tau M_k^2)dy = 1 + o_\tau(1) + o_r(1),
\]
where \(o_r(1) \to 0\) as \(r \to 0\). For any \(\gamma > 2\pi\), there holds
\[
\int_{B_r} \exp(\gamma M_k^2/\|M_k\|_{1,\tau, r}^2)dy \geq \int_{B_{r^{-1/4}}} \exp(\gamma(1 + o_\tau(1) + o_r(1))M_k^2)dy
\]
\[
= \exp\left(\gamma(1 + o_\tau(1) + o_r(1))\frac{\log k}{4\pi}\right) \frac{\pi}{2} r^{2k^{-1/2}}
\]
\[
= \frac{\pi}{2} r^{2k^{1/2}(1 + o_\tau(1) + o_r(1))^{-1}}.
\]
Let \((U_{a_0}, \psi_{a_0}; \{y_1, y_2\})\) be the isothermal coordinate system around \(x_0 \in \partial \Sigma\), and the metric \(g\) can be written as \(g = \exp(2f(y))(dy_1^2 + dy_2^2)\). Define a sequence of functions \(M_k = M_{\nu \cdot \psi_{a_0}}\). Then we have

\[
\|\tilde{M}_k\|_{1,2} = \int_\Sigma |(\nabla g \tilde{M}_k)^2 + \tau \tilde{M}_k|^2 dv_g = 1 + o(1) + o_r(1).
\]

It follows that for any fixed \(\gamma > 2\pi\), if \(r > 0\) is chosen sufficiently small,

\[
\int_\Sigma \exp(\gamma \tilde{M}^2_k / \|M_k\|_{1,2}^2)dv_g \geq \int_{U_r} \exp(\gamma \tilde{M}^2_k / \|M_k\|_{1,2}^2)dv_g
\]

\[
\geq \int_{B_r^{+\xi}} \exp(\gamma(1 + \alpha(1) + o_r(1))M^2_k)\exp(2f)dy
\]

\[
= (1 + o(1))\pi r^2 k^{2(1+o(1))} \to +\infty
\]

as \(k \to \infty\). This leads to \(\alpha^* \leq 2\pi\), which together with (117) implies that \(\alpha^* = 2\pi\).

5.2. The existence of extremals for the supremums in (15)

By a direct method of variation, for any \(k \in \mathbb{N}\), there exists a nonnegative function \(u_k\) with \(\|u_k\|_{1,2} = 1\) such that

\[
\int_\Sigma \exp(\gamma \lambda_k u_k^2)dv_g = \sup_{\|u\|_{1,2} \leq 1} \int_\Sigma \exp(\gamma u^2)dv_g,
\]

where \(\gamma_k = 2\pi - 1/k\). One can easily check that \(u_k\) satisfies the Euler-Lagrange equation

\[
\begin{cases}
\Delta u_k + \tau u_k = \frac{1}{4} u_k \exp(\gamma \lambda_k u_k^2) & \text{in } \Sigma \\
u_k > 0 & \text{in } \Sigma \\
\frac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial \Sigma \\
\lambda_k = \int_\Sigma u_k^2 \exp(\gamma \lambda_k u_k^2)dv_g.
\end{cases}
\] (118)

With no loss of generality, we assume \(c_k = u_k(x_k) = \max_{\Sigma} u_k \to +\infty\) and \(x_k \to x_0 \in \Sigma\) as \(k \to \infty\). Then as in Lemma 11, we have \(x_0 \in \partial \Sigma\), \(u_k\) converges to 0 weakly in \(W^{1,2}(\Sigma, g)\), strongly in \(L^q(\Sigma)\) for any \(q > 1\), and \(|\nabla \psi_{a_0}|dv_g \to \delta_{x_0}\) in the sense of measure.

In an isothermal coordinate system \((U_{a_0}, \psi_{a_0}; \{y_1, y_2\})\) around \(x_0, \psi_{a_0}(U_{a_0}) = B_{r_0}\), the metric \(g\) can be written as \(g = \exp(2f(y))(dy_1^2 + dy_2^2)\) with \(f \in C^1(\Sigma)\) and \(f(0,0) = 0\); moreover, the unit outward vector field \(\nu\) on the boundary \(\partial \Sigma\) can be written as \(\nu = \exp(-f(y))\partial/\partial y_2\). For any \(u \in C^1(\Sigma)\), the normal derivative \(\partial u / \partial \nu\) can be represented by

\[
\frac{\partial u}{\partial \nu} = \exp(-f(y)) \frac{\partial}{\partial y_2}(u \circ \psi_{a_0}^{-1}).
\]

Denote \(\tilde{x}_k = \psi_{a_0}(x_k) = (y_{1,k}, y_{2,k})\) and \(\tilde{x}_{0,k} = (y_{1,k}, 0)\). Let \(r_k > 0\) satisfy

\[
r_k^2 = \frac{\lambda_k}{c_k^2} \exp(-\gamma_k c_k^2).
\]

32
Using the same argument in the proof of Lemmas 17 and 18, we have as $k \to \infty,$
\[ c_k(u_k \circ \psi_{-1}(\tilde{x}_0) + r_k^\cdot) - c_k \to -\frac{1}{2\pi} \log(1 + \frac{\pi}{2} |\tilde{\cdot}|^2) \] in $C^1_{\text{loc}}(\mathbb{R}^2 \cup \partial \mathbb{R}^2)$.

Similar to Lemma 18, we also have that for any $\varphi \in C^2(\tilde{\Sigma}),$ there holds
\[ \int_{\Sigma} \varphi \frac{c_k}{A_k} u_k \exp(\gamma_k u_k^2)dv_g = \varphi(x_0) + o_k(1). \]
In particular,
\[ \frac{1}{A_k} c_k\|u_k\|_2 \exp(\gamma_k u_k^2) \leq C \] (119)
and in the sense of measure
\[ \frac{1}{A_k} c_k u_k \exp(\gamma_k u_k^2) \to \delta_{x_0}. \]

In view of (118), there holds
\[ \Delta_k(c_k u_k) + \tau(c_k u_k) = \frac{1}{A_k} c_k u_k \exp(\gamma_k u_k^2) \] in $\Sigma$. (120)

Integrating both sides of (120), we have by noticing (119), $u_k > 0$ in $\Sigma$ and $\partial u_k / \partial \nu = 0$ on $\partial \Sigma$ that
\[ \int_{\Sigma} c_k u_k dv_g \leq C \] (121)
and
\[ \int_{\Sigma} |\Delta_k(c_k u_k)| dv_g \leq C. \]

Let
\[ w_k = c_k u_k - \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} c_k u_k dv_g. \]

Then we obtain by using the Green representation formula,
\[ w_k(x) = \int_{\Sigma} G(x, y)\Delta_y w_k(y)dv_g(y), \]
where $G(\cdot, \cdot)$ is defined as in Lemma 6. An obvious analog of (87) reads $\|\nabla w_k\|_{L^2(\Sigma, g)} \leq C$ for all $1 < q < 2$. Hence $\|\nabla G(c_k u_k)\|_{L^2(\Sigma, g)} \leq C$ for all $1 < q < 2$. This together with (121) implies that $c_k u_k$ is bounded in $W^{1,q}(\Sigma, g)$ for any $1 < q < 2$. Similar to Lemma 19, $c_k u_k$ converges to $G_{\tau, x_0}$ weakly in $W^{1,q}(\Sigma, g)$, strongly in $L^s(\Sigma, g)$ with $s < 2q/(2 - q)$, and in $C^1_{\text{loc}}(\tilde{\Sigma} \setminus \{x_0\})$ as $k \to \infty,$ where $G_{\tau, x_0}$ satisfies in the distributional sense
\[ \left\{ \begin{array}{ll}
\Delta_s G_{\tau, x_0} + \tau G_{\tau, x_0} = \delta_{x_0} & \text{in } \Sigma \\
\frac{\partial}{\partial \nu} G_{\tau, x_0} = 0 & \text{on } \partial \Sigma \\
\int_{\Sigma} G_{\tau, x_0} dv_g = 0.
\end{array} \right. \]

Similar to Lemma 20, in the isothermal coordinate system $(U_{x_0}, \psi_{x_0}; [y_1, y_2])$ near $x_0,$ we have
\[ G_{\tau, x_0} \circ \psi_{x_0}^{-1}(y) = -\frac{1}{3\pi} \log |y| + h(y), \]
where $h \in C^1(B + r_0)$. Then repeating the argument of deriving (101), we obtain

$$\sup_{||u||_{1,\tau} \leq 1} \int_{\Sigma} \exp(2\pi u^2)dv_g = \lim_{k \to \infty} \int_{\Sigma} \exp(\gamma_k u^2)dv_g \leq \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)).$$

(122)

Let $\phi_k$ be defined as in (103). We first require $\phi_k \in W^{1,2}(\Sigma, g)$. In particular, (104) and thus (105) hold. A straightforward calculation shows

$$\int_{\Sigma} \left( |\nabla g \phi_k|^2 + \tau \phi_k^2 \right) dv_g = \frac{1}{c^2} \left\{ \log k \pi + h(0) + \frac{1}{2\pi} \log \frac{\pi}{2} - \frac{1}{2\pi} + O \left( \frac{1}{(\log k)^2} \right) \right\}. $$

We further require

$$||\phi_k||_{1,\tau}^2 = \int_{\Sigma} (|\nabla g \phi_k|^2 + \tau \phi_k^2)dv_g = 1.$$

It then follows that (112) and (113) hold. As a consequence, we calculate as before

$$\int_{\Sigma} \exp(2\pi \phi_k^2)dv_g \geq \text{Area}(\Sigma) + \frac{\pi}{2} \exp(1 + 2\pi h(0)), $$

(123)

provided that $k$ is sufficiently large.

The contradiction between (122) and (123) implies that $c_k$ must be bounded, and thus the supremum in (15) can be attained for $\gamma = 2\pi$. This ends the proof of Theorem 2.

6. Proof of Theorem 3

Proof of Theorem 3. Let $0 \leq \alpha < \lambda_N(\Sigma)$ and $\tau > 0$ be two fixed real numbers. The inequality (16) implies the inequality (17). Suppose (16) holds. To derive (17), let $\tau$ be a positive real number and $u \neq \text{null}$ be any function in $W^{1,2}(\Sigma, g)$ satisfying

$$||u||_{1,\tau}^2 = \int_{\Sigma} (|\nabla g u|^2 + \tau u^2)dv_g \leq 1. $$

(124)

By the Young inequality, one has for any $\epsilon > 0$,

$$u^2 \leq (1 + \epsilon)(u - \overline{u})^2 + (1 + \frac{1}{4\epsilon})\overline{u}^2, $$

(125)

where

$$\overline{u} = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} u dv_g. $$

(126)

Since $\int_{\Sigma} |\nabla g u|^2 dv_g \leq 1 - \tau \int_{\Sigma} u^2 dv_g$ by (124), we have

$$||u - \overline{m}||_{1,\alpha}^2 = \int_{\Sigma} |\nabla g u|^2 dv_g - \alpha \int_{\Sigma} (u - \overline{u})^2 dv_g \leq 1 - \tau \int_{\Sigma} u^2 dv_g - \alpha \int_{\Sigma} (u - \overline{u})^2 dv_g. $$

Thus $||u - \overline{m}||_{1,\alpha}^2 < 1$ for $0 \leq \alpha < \lambda_N(\Sigma)$. As a consequence, we can choose $\epsilon > 0$ verifying $1 + \epsilon = 1/||u - \overline{m}||_{1,\alpha}^2$. This leads to

$$\frac{1}{\epsilon} = \frac{1 - \tau \int_{\Sigma} u^2 dv_g - \alpha \int_{\Sigma} (u - \overline{u})^2 dv_g}{\tau \int_{\Sigma} u^2 dv_g} \leq \frac{1}{\tau \int_{\Sigma} u^2 dv_g}. $$

(127)
Combining (126) and (127), we get

$$\frac{1}{\gamma} \leq \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} u^2 dv_g$$

and thus

$$\frac{1}{4\pi} \gamma^2 \leq \frac{1}{4\pi \text{Area}(\Sigma)}.$$  \hspace{1cm} (129)

In view of (125), (128) and (129), we obtain

$$\exp(2\pi u^2) \leq \exp\left(2\pi \left(\frac{1}{1+\frac{1}{4\pi}}\right)\right) \leq C \exp\left(2\pi \left(\frac{1}{1+\frac{1}{4\pi}}\right)\right)$$

for some uniform constant $C$. Hence by applying (14) of Theorem 1, we conclude

$$\int_{\Sigma} \exp(2\pi u^2) dv_g \leq C$$

for some uniform constant $C$. Therefore (17) follows immediately.

**The inequality (17) implies the inequality (16).**

Assume that (17) holds. To prove (16), we use the method of blow-up analysis. Suppose that (16) does not hold. By (15) for any $\gamma < 2\pi$, we let $u_k$ be as in Lemma 10. Then we must have

$$\lim_{k \to \infty} \int_{\Sigma} \exp(\gamma_k u_k^2) dv_g = +\infty.$$  \hspace{1cm} (130)

As before we assume with no loss of generality, $c_k = u_k(x_k) = \max_{\Sigma} |u_k|$ and $x_k \to x_0$ as $k \to \infty$. Then the assumption (130) implies that $c_k \to +\infty$ as $k \to \infty$. By Lemma 19, $c_k u_k$ converges to $G_{\alpha,\lambda}$ strongly in $L^2(\Sigma, g)$. Since $|u_k|_{1,\tau} = 1$, we have

$$\begin{align*}
\gamma_k u_k^2 &= \gamma_k \frac{u_k^2}{|u_k|_{1,\tau}^2} |u_k|_{1,\tau}^2 \\
&= \gamma_k \frac{u_k^2}{|u_k|_{1,\tau}^2} \left( |u_k|_{1,\tau}^2 + \alpha \int_{\Sigma} u_k^2 dv_g + \tau \int_{\Sigma} u_k^2 dv_g g \right) \\
&= \gamma_k \frac{u_k^2}{|u_k|_{1,\tau}^2} + \frac{\alpha \gamma_k}{|u_k|_{1,\tau}^2} \int_{\Sigma} u_k^2 dv_g + \frac{\gamma_k}{|u_k|_{1,\tau}^2} \int_{\Sigma} u_k^2 dv_g,
\end{align*}$$

$$\gamma_k = 2\pi - 1/k, \quad |u_k|_{1,\tau} = 1 + o_k(1), \quad \alpha < \lambda S(\Sigma)$$

and

$$u_k^2 \int_{\Sigma} u_k^2 dv_g \leq \int_{\Sigma} c_k^2 u_k^2 dv_g = \int_{\Sigma} G_{\alpha,\lambda}^2 dv_g + o_k(1),$$

we conclude

$$\exp(\gamma_k u_k^2) \leq C \exp(\gamma_k u_k^2/|u_k|_{1,\tau}^2)$$

for some uniform constant $C$. It follows from (17) that

$$\lim_{k \to \infty} \int_{\Sigma} \exp(\gamma_k u_k^2) dv_g \leq C \lim_{k \to \infty} \int_{\Sigma} \exp(\gamma_k u_k^2/|u_k|_{1,\tau}^2) dv_g \leq C$$

for some constant $C$. This contradicts (130) and leads to (16) immediately.  \hfill \Box

**Acknowledgement.** This work is partly supported by the National Science Foundation of China (Grant No. 11761131002).
References

[1] D. Adams, A sharp inequality of J. Moser for higher order derivatives, Ann. Math. 128 (1988) 385-398.
[2] A. Adimurthi, M. Struwe, Global compactness properties of semilinear elliptic equation with critical exponential growth, J. Funct. Anal. 175 (2000) 125-167.
[3] Adimurthi, Y. Yang, An interpolation of Hardy inequality and Trudinger-Moser inequality in $\mathbb{R}^N$ and its applications, Int. Math. Res. Notices 13 (2010) 2394-2426.
[4] T. Aubin, Nonlinear analysis on manifolds, Springer, 1982.
[5] L. Bers, Riemann surfaces, Courant Institute Lecture Notes, 1957-58.
[6] D. Bonheure, E. Serra, M. Tarallo, Symmetry of extremal functions in Moser-Trudinger inequalities and a Hénon type problem in dimension two, Adv. Differential Equations 13 (2008) 105-138.
[7] A. Chang, P. Yang, Conformal deformation of metrics on $S^2$, J. Differential Geometry 27 (1988) 259-296.
[8] S. Deng, New solutions for critical Neumann problems in $\mathbb{R}^2$, Adv. Nonlinear Anal. 8 (2019) 615-644.
[9] S. Deng, M. Musso, Critical points of the Trudinger-Moser trace functional with high energy levels, Ann. Inst. H. Poincare Anal. Non Lineaire 32 (2015) 59-95.
[10] W. Ding, J. Jost, J. Li, G. Wang, An analysis of the two vortex case in the Chern-Simons Higgs model, Calc. Var. Partial Differential Equation 7 (1998) 87-97.
[11] W. Ding, J. Jost, J. Li, G. Wang, Self-duality equations for Ginzburg-Landau and Seiberg-Witten type functionals with 6th order potentials, Comm. Math. Phys. 217 (2001) 383-407.
[12] W. Ding, J. Jost, J. Li, G. Wang, The differential equation $\Delta u = 8\pi - 8\pi \nu^2$ on a compact Riemann surface, Asian J. Math. 1 (1997) 230-248.
[13] J. M. do Ó, Y. Yang, A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary, Ann. Global Anal. Geom. 38 (2010) 317-334.
[14] O. Druet, P. Robert, J. Wei, The Lin-Ni’s problem for mean convex domains, Mem. Amer. Math. Soc. 218 (2012) 1027.
[15] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv. 68 (1993) 415-454.
[16] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order, Springer, 2001.
[17] X. Lan, J. Li, Asymptotic behavior of the Chern-Simons Higgs 6th theory, Comm. Partial Differential Equations 32 (2007) 1473-1492.
[18] Y. Li, Moser-Trudinger inequality on compact Riemannian manifolds of dimension two, J. Partial Differ. Equ. 14 (2001) 163-192.
[19] Y. Li, P. Liu, Moser-Trudinger inequality on the boundary of compact Riemannian surface, Math. Z. 250 (2005) 363-386.
[20] P. Liu, A Moser-Trudinger type inequality and blow up analysis on compact Riemannian surface, Doctoral thesis, Max-Plank Institute, Germany, 2005.
[21] G. Lu, Y. Yang, A sharpened Moser-Pohozaev-Trudinger inequality with mean value zero in $\mathbb{R}^2$, Nonlinear Anal. 70 (2009) 2992-3001.
[22] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1971) 1077-1092.
[23] Q. Ngô, V. Nguyen, An improved Moser-Trudinger inequality involving the first non-zero Neumann eigenvalue with mean value zero in $\mathbb{R}^2$, arXiv: 1702.08883.
[24] V. Nguyen, A sharp Adams inequality in dimension four and its extremal functions, arXiv: 1701.08249.
[25] J. Peetre, Espaces d’interpolation et théoreme de Soboleff, Ann. Inst. Fourier (Grenoble) 16 (1966) 279-317.
[26] S. Pohozaev, The Sobolev embedding in the special case $pl = n$, Proceedings of the technical scientific conference on advances of scientific research 1964-1965, Mathematics sections, 158-170, Moscov. Energet. Inst., Moscow, 1965.
[27] C. Pommerenke, Boundary behavior of conformal maps, Springer, 1992.
[28] C. Tintarev, Trudinger-Moser inequality with remainder terms, J. Funct. Anal. 266 (2014) 55-66.
[29] N. Trudinger, On embeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473-484.
[30] M. Wang, The self-dual Chern-Simons Higgs equation on a compact Riemann surface with boundary, Internat. J. Math. 21 (2010) 67-76.
[31] M. Wang, The asymptotic behavior of Chern-Simons Higgs model on a compact Riemann surface with boundary, Acta Math. Sin. (Engl. Ser.) 28 (2012) 145-170.
[32] Y. Yang, Extremal functions for Moser-Trudinger inequalities on 2-dimensional compact Riemannian manifolds with boundary, Internat. J. Math. 17 (2006) 315-330.
[33] Y. Yang, Moser-Trudinger trace inequalities on a compact Riemannian surface with boundary, Pacific J. Math. 227 (2006) 177-200.
[34] Y. Yang, A sharp form of trace Moser-Trudinger inequality on compact Riemannian surface with boundary, Math. Z. 255 (2007) 373-392.
[35] Y. Yang, A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface, Trans. Amer. Math. Soc. 359 (2007) 5761-5776.
[36] Y. Yang, Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two, J. Differential Equations 258 (2015) 3161-3193.
[37] V. Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, Sov. Math. Dokl. 2 (1961) 746-749.
[38] T. Zhang, C. Zhou, Asymptotical behaviors for Neumann boundary problem with singular data, Acta Math. Sin. (Engl. Ser.) 35 (2019) 463-480.
[39] X. Zhu, Solutions for Toda system on Riemann surface with boundary, Acta Math. Sin. (Engl. Ser.) 27 (2011) 1501-1520.