Resolvent metrics and heat kernel estimates

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Abstract
Resolvent metrics are generalization of the resistance metric and provide unified treatment of heat kernel estimates of sub-Gaussian type under minimal conditions.

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1 Introduction

Heat propagation is not only interesting on its own, but reflects the very intrinsic structure of the space where it does take place. We gained such a new insight by Kigami’s resistance metric [10]. Unfortunately the use of resistance metric is applicable only on recurrent spaces. In this paper we eliminate this limitation and extend his notion to transient spaces, in particular for a class of transient graphs.

In the course of study of heat propagation the analogy between results on continuous and discrete spaces is utilized (see e.g. [5]) and switching between them become a powerful tool of the studies. That is why we think that it is useful if we tackle, in the present paper, the technically less involved random walk case and return to the measure metric space version in a forthcoming paper.

Kigami’s work and several other papers inspire the following questions. For any given measure (Dirichlet) space is there a ”good” metric in which:

- the elliptic Harnack inequality holds
- a parabolic Harnack inequality holds (in conjunction with heat kernel estimates)?

The presented results contribute to the answer of these questions. We introduce the resolvent metric which is direct generalization of the resistance metric and we make the following observations.

- Under the resolvent metric the volume doubling property implies the elliptic Harnack inequality.
- Under the resolvent metric volume doubling property turns to be equivalent to the parabolic Harnack inequality and two-sided heat kernel estimates in a fully local sense (c.f. [14]).

The main results are given in Theorem 5.1, 5.2, 5.3 and 6.1. The paper concludes with examples.
The main result of the paper can be summarized as follows. We consider a weighted graph \((\Gamma, \mu)\) and a random walk on it. We assume that for all \(\mu_{x,y} > 0\) we have \(P(x,y) \geq p_0 > 0\) uniformly. We construct the resolvent metric \(\rho\) and consider \(B_\rho(x,r)\), balls in \(\rho\), their volume \(V_\rho(x,r)\) and define the scaling function \(F(x,r) = (r^2 V_\rho(x,r))^{1/m}\) for a well chosen \(m\). Denote \(f(x,.) = F^{-1}(x,.)\) and \(\tilde{p}_n(x,y) = p_n(x,y) + p_{n+1}(x,y)\) the sum of the transition kernel.

**Definition 1.1** We define a set \(W_0\) of scaling functions, \(F : \Gamma \times [0, \infty] \to \mathbb{R}^+\): there is a \(C > 0\) such that for all \(x \in \Gamma, r > 0\)

\[
\frac{F(x,2r)}{F(x,r)} \leq C.
\]

**Theorem 1.1** Volume doubling holds \((V_\rho \in W_0)\) for \(\mu\) with respect to \(\rho\) if and only if there are \(C > c > 0, \beta > 1, \delta > 0\) and an \(F \in W_0\) such that for all \(x, y \in \Gamma\) and \(n > 0\)

\[
p_n(x,y) \leq \frac{C}{V_\rho(x,f(x,n))} \exp \left[ - \left( \frac{F(x,r)}{n} \right)^{\frac{1}{\beta-1}} \right]
\]

and for \(\rho(x,y) \leq \delta f(x,n)\)

\[
\tilde{p}_n(x,y) \geq \frac{C}{V_\rho(x,f(x,n))}
\]

hold.

## 2 Basic definitions

We consider \((\Gamma, \mu)\), weighted graph, \(\Gamma\) is a countable infinite set of vertexes and \(\mu_{x,y} = \mu_{y,x} \geq 0\) a symmetric weight. Edges are formed by the pairs for which \(\mu_{x,y} > 0\). We assume that the graph is connected. These weights define a measure on vertices:

\[
\mu(x) = \sum_{y \in \Gamma} \mu_{x,y}
\]

and on sets \(A \subset \Gamma\)

\[
\mu(A) = \sum_{z \in A} \mu(z).
\]
Due to the connectedness \( \mu(x) > 0 \) for all \( x \). It is natural to define the random walk on weighted graphs, which is a reversible Markov chain given by the one-step transition probabilities:

\[
P(x, y) = \frac{\mu_{x,y}}{\mu(x)}.
\]

In what follows we always assume the condition \((p_0)\): there is a constant \( p_0 > 0 \) such that for all \( x, y \) with \( \mu_{x,y} > 0 \)

\[
P(x, y) \geq p_0
\]

holds. One can define the transition operator \( P \) on \( c_0(\Gamma) \) functions \( Pf(x) = \sum P(x, y)f(y) \). The inner product for \( c_0(\Gamma, \mu) \) is defined by \( (f,g)_{c_0(\Gamma, \mu)} = \sum f(x)g(x) \mu(x) \).

If \( \rho \) is a metric, balls are defined with respect to it by

\[
\hat{B}_\rho(x, r) = \{ y : \rho(x, y) < r \}
\]

Denote \( B = B_\rho(x, r) \) the connected component of \( \hat{B}_\rho(x, r) \) containing \( x \). The volume of the connected part \( B \) is denote by \( V_\rho(x, r) = \mu(B_\rho(x, r)) \).

**Definition 2.1** We say that the volume doubling property, \((VD)_\rho\) holds if there is a \( C_\rho > 0 \) constant such that for all \( x \in \Gamma, r > 0 \)

\[
\frac{V_\rho(x, 2r)}{V_\rho(x, r)} \leq C_\rho.
\]

### 3 The resolvent metric

In case of recurrent spaces Kigami’s observation is that the effective resistance between two vertices \( R(x, y) \) is metric. The existence of the resistance metric has a particular consequence that, for any \( f \) in the domain of the Dirichlet form \( \mathcal{E} \)

\[
|f(x) - f(y)|^2 \leq R(x, y) \mathcal{E}(f, f).
\]

If the volume of balls \( V_R(x, r) \) with respect to the metric \( R \) satisfies the doubling condition the following important estimate holds

\[
R(x, B^*_R(x, r)) \asymp r.
\]

In particular

\[
R(x, B^*_R(x, r)) \geq cr
\]
while \( R(x, B^R_R(x, r)) \leq r \) is evident. One may recognize that (2) holds only for recurrent weighted graphs. This is a nice particular situation which has been successfully utilized in several papers to obtain heat kernel estimates and stability results ([6],[10],[11]). Almost the same proof which leads to (2) results the validity of the Einstein relation in the form:

\[
E_x(T_{B^R_R(x, r)}) \asymp r V_R(x, r)
\]

and that the elliptic Harnack inequality follows from the bounded covering condition. Having all that the heat kernel estimates and the parabolic Harnack inequality follows.

The crucial observations fail in the transient case, first of all (2) obviously does not hold, since \( R(x, B^R_R(x, r)) \to R_0 > 0 \) and the rate of convergence can be understand from the decay of \( R(B_R(x, r), B^R_R(x, 2r)) \).

In several previous works resolvents are used with success to analyze transient walks and diffusions. The simplest resolvent is the following:

\[
\sum_{n=0}^{\infty} n^m P_n(x, y).
\]

It is clear that it is monotonically increasing in \( m \) and may be infinite if \( P_n \) decays polynomially. In probabilistic terms one may consider this sum as the average visit time of \( y \) by the increasing family of independent walkers which has \( n^m \) members at time \( n \). In independent walkers we mean here that on a given site some new walkers ”born” (according to the expansion of the family tree) and start independent walk. In what follows we need a modified version of the resolvent which provides nice correspondence to the power of the Laplace operator while it has basically the same properties. We fix an \( m \in \mathbb{N} \) which will be specified later and reserved as the parameter of the resolvent.

In [9] we started the utilization of polyharmonic functions, Green function as well as Green operators (or resolvents). Now we follow this direction and find a new metric for non strongly recurrent graphs (weakly recurrent and transient) which posses nice features.

Denote \( P \) the transition operator on \( l_1(\Gamma) \). \( P f(x) = \sum_{y \sim x} P(x, y)f(y) \).

**Definition 3.1** The Laplace operator is defined as \( \Delta = P - I \). The Dirichlet form corresponding to the Laplace operator is given by

\[
\mathcal{E}(f, g) = \mathcal{E}_1(f, g) = (-\Delta f, g) = ((I - P)f, g) = \frac{1}{2} \sum_{x,y} (f(x) - f(y))(g(x) - g(y)) \mu_{x,y}
\]
Definition 3.2 For $A, B \subset \Gamma, A \cap B = \emptyset$ we define the resistance

$$R(A, B) = \inf_{f \in c_0} [\mathcal{E}(f, f) : f|_A = 1, f|_B = 0]^{-1}$$

Let $A \subset \Gamma, \Delta^A = P^A - I, (\Delta^A)^m = (-1)^m (I - P^A)^m$ the $m$-th iteration of the Laplace operator for $m \geq 1$ integer. If $A = \Gamma$ we drop it from the notation.

Let us recall, that $\lambda(A) = \inf_{f \neq 0} \frac{\mathcal{E}^A(f, f)}{\|f\|^2} = \|f\|^2_{\mathcal{P}(\Gamma, \mu)}$.

The domain of the Dirichlet form on $\Gamma$ is defined by $\mathcal{F}_m = \mathcal{F}^A(\mathcal{E}_m) = \{f \in l_2(\Gamma, \mu), \mathcal{E}^A_m(f, f) > 0\}$, where the bilinear form $\mathcal{E}^A_m$ is defined as

$$\mathcal{E}^A_m(f, g) = ((I - P^A)^m f, g)_{l_2(A, \mu)}.$$

The quasi resolvent metric on $A$ is defined as

$$R^A_m(x, y) = \sup_f \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}^A_m(f, f)} : f(x) \neq f(y), f \in \mathcal{F}_m \right\}.$$ 

Note that $R^A_m$ is decreasing in $A$ since $\mathcal{E}^A_m$ is increasing by definition, consequently $R_m$ is existing.

That has the equivalent forms

$$R_m(x, y) = \sup_f \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}_m(f, f)} : f(x) \neq f(y), f \in \mathcal{F}_m \right\}. \quad (3)$$

and

$$R^{-1}_m(A, B) = \inf \{ \mathcal{E}_m(f, f) : f \in \mathcal{F}_m, \mathcal{E}_m(f, f) > 0, f|_A(x) = 1, f|_B = 0 \}.$$ 

The former one can be seen using $g = \frac{f}{\sqrt{\mathcal{E}_m(f, f)}}$.

Lemma 3.1 1. For any $f \in \mathcal{F}(\mathcal{E}_m)$

$$|f(x) - f(y)|^2 \leq R_m(x, y) \mathcal{E}_m(f, f). \quad (4)$$

2. If $\Gamma$ is connected and $A \cap B = \emptyset$ then

$$0 < R^{-1}_m(A, B) < \infty.$$
Proof. The statements follow from the definition. ■

Lemma 3.2 If \( A \subset B \subset D \subset \Gamma \) then

\[
R_m(A, B^c) \leq R_m(A, D^c)
\]  (5)

Proof. By definition for \( D^c \subset B^c \)

\[
R^{-1}_m(A, B^c) = \inf \{ \mathcal{E}_m(f, f) : f|_A(x) = 1, f|_{B^c} = 0 \}
\]

\[
\geq \inf \{ \mathcal{E}_m(f, f) : f|_A(x) = 1, f|_{D^c} = 0 \}
\]

\[
= R^{-1}_m(A, D^c).
\]

Lemma 3.3 \( R_m(x, y) \) is a quasi metric:

for any \( x, y \in \Gamma \),

\[
R_m(x, y) = R_m(y, x),
\]

\[
R_m(x, y) = 0 \text{ if and only if } x = y
\]

\[
R_m(x, y) \leq 2(R_m(x, z) + R_m(z, y)).
\]  (8)

Proof. The first statement ensured by the definition. For the second see the end of the proof of [12] Proposition 3.1. The weak triangular inequality can be see as follows:

\[
R_m(x, y) = \sup_g \{ |g(x) - g(y)|^2 : 0 < \mathcal{E}_m(g, g) \leq 1 \}
\]

\[
\leq \sup_g \{ 2|g(x) - g(z)|^2 + 2|g(z) - g(y)|^2 : 0 < \mathcal{E}_m(g, g) \leq 1 \}
\]

\[
\leq \sup_g \{ 2|g(x) - g(z)|^2 : 0 < \mathcal{E}_m(g, g) \leq 1 \}
\]

\[
+ \sup_g \{ 2|g(z) - g(y)|^2 : 0 < \mathcal{E}_m(g, g) \leq 1 \}
\]

\[
= 2(R_m(x, z) + R_m(z, y))
\]

The next result of Mac’ıas and Segovia is essential in our work.

Theorem 3.1 ([13]) If \( X \) is a non-empty set and \( d \) is a quasisymmetric with constant \( K \) :

\[
d(x, y) \leq K(d(x, z) + d(z, y))
\]

then, there is a metric \( \rho \), such that

\[
d^p(x, y) \lesssim \rho(x, y)
\]

with \( p = \frac{1}{1 + \log_2 K} \).
Corollary 3.1 There is a metric $\rho$ and $C > 1 > c > 0$ such that for all $x, y \in \Gamma$

$$cp^2(x, y) \leq R_m(x, y) \leq C\rho^2(x, y) \quad (9)$$

Based on this theorem we define balls with respect to $\rho$: $\tilde{B}_\rho(x, r) = \{y: \rho(x, y) < r\}$. As in the case of the resistance metric it can be that the balls are not connected. Let $B = B(x, r) = B_\rho(x, r) \subset \tilde{B}_\rho(x, r)$ be the connected subset of $\tilde{B}_\rho(x, r)$ containing $x$. With the same slight abuse of notation we shall use $B_R$ for the sets (balls) with respect to the quasi-metric $R_m$. One can immediately observe that $(\Gamma, \rho, \mu)$ satisfies volume doubling if and only if $(\Gamma, R_m, \mu)$ does. In addition if the bounded covering property holds with respect to one of $R_m$ or $\rho$ it holds for the other as well and it follows from volume doubling (c.f. [14]).

Definition 3.3 The graph $\Gamma$ with metric $\sigma$ satisfies the bounded covering condition if there is an integer $M > 0$ such that for all $x \in \Gamma$, $r > 0$ the ball $B_\sigma(x, 2r)$ can be covered at most $M$ balls of radius $r$.

Definition 3.4 The $m$-resolvent is defined for an integer $m > 0$ as follows. Let $Q_m(n) = \binom{n+m-1}{m-1}$, $A \subset \Gamma$ finite set and for $x, y \in A$

$$G^A_m(x, y) = \sum_n Q_m(n) P_n(x, y)$$

the corresponding Green kernel is $g^A_m(x, y) = \frac{1}{\mu(y)} G^A_m(x, y)$.

The Green operators $G^A$ defined as usual. It is worth to observe immediately, that for $m = 0$ $G^A_m = I^A$ and for $m = 1$ $G^A_m = G^A$, the usual Green operator. For infinite $A$ the resolvent operator may be unbounded and the Green function is $\infty$. For finite sets due to the transience of the Markov chain with Dirichlet boundary, these objects are well-defined.

Lemma 3.4 The Dirichlet Green kernel $g^A_m(x, y)$ for finite $A \subset \Gamma$ is a reproducing kernel with respect to $\mathcal{E}_m$.

Proof. Let $u \in \mathcal{F}_m$, $u|_{A^c} = 0$

$$\mathcal{E}_m \left( g^A_m, u \right) = \left( (-\Delta)^m G^A_m \frac{1}{\mu(\cdot)}, u \right)$$

$$= \left( \delta_x \frac{1}{\mu(\cdot)}, u \right)$$

$$= u(x).$$

The next corollary is immediate.
Corollary 3.2
\[ \mathcal{E}_m \left( g_m^A (x,.) , g_m^A (x,.) \right) = g_m^A (x,x) \]

Lemma 3.5 The minimal value in the definition of \( R_m (x, A^c) \) of \( \mathcal{E}_m (f,f) \) is taken by \( g (y) = \frac{1}{g_m^A (x,x)} g_m^A (x,y) \) and
\[ R_m (x, A^c) = g_m^A (x,x) \] (10)

**Proof.** Let \( h \) be an other function with \( h (x) = 1 \ h |_{A^c} = 0 \) then \( d = h - g, h = d + g \)
\[ \mathcal{E}_m (h,h) = \mathcal{E}_m (g,g) + \mathcal{E}_m (d,d) + 2 \mathcal{E}_m (d,g) \]
but \( \mathcal{E}_m (d,d) \geq 0 \) while \( \mathcal{E}_m (d,g) = cd (x) = 0. \)

**Lemma 3.6** Assume \((\Gamma, \rho)\) has the bounded covering property (or \((VD) \rho \) ), then
\[ R_m (x, B^c_{\rho} (x,r)) < cr^2 \]

**Proof.** From (9) and (5) we have that there is a \( c > 0 \) such for \( s = cr^2, B_R (x,s) \subset B_{\rho} (x,r) \)
\[ R_m (x, B_{\rho}^c (x,r)) \geq R_m (x, B_R^c (x,s)) \] .

If
\[ R_m (x, B^c_{\rho} (x,s)) \geq cs \] (11)
we are ready. Now we prove (11) following the steps of [12]. Let \( B = B_R (x,s), y, z \in B_R (x,s) \) and \( R_m (y,z) < \lambda s, \lambda \leq 1 \). Let us fix a \( c \) and a \( z \in B \) with \( c_1 s < R_m (x,z) < s \). We consider \( 0 \leq q_z (y) = \frac{g_m^B (x,y)}{g_m^B (x,x)} \leq 1 \) \( m \)-
harmonic function on \( D = B \setminus \{ z \} \) with \( q_z (x) = 1, q_z (z) = 0 \). By definition and the reproducing property of the Green kernel
\[ \mathcal{E}_m (q_z,q_z) = q_z = \frac{1}{R_m^B (x,z)} \leq \frac{1}{R_m (x,z)} \]
where \( R_m^B (x,z) \) denotes the resolvent metric within \( B \), while
\[ |q_z (y)|^2 = |q_z (y) - q_z (z)|^2 \leq R_m (x,y) \mathcal{E}_m (q,q) \leq \frac{CR_m (y,z)}{R_m (x,z)} \leq \frac{C (\lambda s)^2}{(c_1 s)^2} < \frac{1}{2} \]
if \( \lambda = \lambda_1 \) is chosen enough small. Note that volume doubling implies bounded covering of \( B_R (x,s) \setminus B_R (x,c_1 s) \). Let \( B_R (z_i, \lambda_1 s) \) the set of covering sets (via the covering with smaller \( B_{\rho} \) balls: \( B_{\rho} (z_i, cr) \subset B_R (z_i, \lambda_1 s) \) balls with
some extra increase of the covering number), $i = 1, ..., K$. Denote $q(y) = \min_i q_i(y)$ and $q = 2 \left( q - \frac{1}{2} \right)^+ 1_{B_\rho(x,r)}$. We have that $q(x) = 1$ and $q(y) = 0$ on $B^c_R(x,s)$. Finally we obtain that

$$R_m^{-1}(x, B^c_R(x,s)) \leq E_m(q, q) \leq \sum_i E_m(q_z, q_z) \leq \frac{4M}{\min_i R_m(x,z_i)} \leq \frac{4M}{c_1 s}.$$

\[ \square \]

**Corollary 3.3**

$$R_m(x, B^c_R(x,r)) \propto cr^2 \quad (12)$$

**Proof.** The lower estimate was given above, the upper one is almost immediate. We chose $S = Cr^2$ so that $B_R(x,S) \supset B_\rho(x,r)$. Let $y \in \partial B_R(x,S)$ and apply (5) for $\{y\} \subset B^c_R(x,S) = R_m(x, B^c_R(x,S)) \leq R_m(x,y) = S = Cr^2$.

\[ \square \]

## 4 The tail distribution of the exit time

This section contains two key results. One establishes an estimate similar to the Einstein relation, the other presents the estimate of the tail distribution of the exit time. The novelty in the approach is that in the lack of the usual Einstein relation all the arguments should be accommodated to the $m$–resolvent.

For brevity we will use the following notations:

- $E_m(A|x) = E_x(Q_{m+1}(T_A) | X_0 = x)$
- $\overline{E_m}(A) = \max_{x \in A} E_m(A|x)$
- $E_m(x,r) = E_m(x,r) = E_m(B_\rho(x,r)|x)$. We will use the particular notation for $m = 0$,
- $E_\rho(x,r) = E_0(B_\rho(x,r)|x)$ is the usual mean exit time.

**Lemma 4.1** For a set $A \subset \Gamma, x \in A$, there is a $C_0 > 1$ such that

$$\mathbb{P}_x(T_A < n) \leq 1 - \frac{E_m(A)}{CE_m(A)} + \frac{Cn^m}{E_m(A)}$$

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Proof.

\[ T_A \leq 2n + I(T_A > n)T_A \circ \Theta_n, \]
\[ T_A^m \leq 2^m ((2n)^m + I(T_A > n)T_A^m \circ \Theta_n), \]

where \( \Theta_n \) is the time shift operator. From the strong Markov property one obtains with \( C = 2^\lceil m \rceil \)

\[ E_m(A) \leq C^2 n^m + C\mathbb{E}_x(I(T_A > n)\mathbb{E}_{X_n}(T_A^m)) \]
\[ \frac{E_m(A)}{C^2 E_m(A)} \leq \frac{Cn^m}{C^2} + \mathbb{P}_x(T_A > n) \]

and the statement follows. \( \blacksquare \)

Let us recall here that under \((VD)_\rho\) the scaling function \( H(x, r) = r^2 V_\rho(x, r) \) has nice regularity properties.

**Corollary 4.1** If \((VD)_\rho\) holds then there is a \( c_0 \) such that if \( n = \left( \frac{1}{2} C_0^{-2} E_m(x, r) \right)^{1/m} \)

\[ \mathbb{P}_x(T_{B_\rho(x, r)} \geq n) \geq c_0. \]  

Here \( C_0 \) is given by the Lemma 4.1.

**Theorem 4.1** If \((\Gamma, \rho)\) satisfies \((VD)_\rho\) then, for \( B = B_\rho(x, r) \)

\[ \mathbb{P}_x(T_{B_\rho(x, r)} < n) \leq C \exp \left( -ck_m(x, n, r) \right) \]

where \( k = k_m(x, n, r) > 1 \) is the maximal integer for which

\[ \frac{n^m}{k} \leq q \min_{y \in B_\rho(x, r)} E_m(B_\rho(y, \frac{r}{k})), \]

where \( q \) is a small constant (to be specified later).

**Definition 4.1** Let us define \( \beta_m \) as the smallest possible exponent for which

\[ \frac{R^2 V_\rho(x, R)}{r^2 V_\rho(x, r)} \leq C \left( \frac{R}{r} \right)^{\beta_m}, \]

and observe that (16) equivalent to \((VD)_\rho\).

**Remark 4.1** There are several further transcript of (14). In the simplest case if \( r^2 V_\rho(x, r) \asymp r^\beta, B = B_\rho(x, r) \) one has

\[ \mathbb{P}_x(T_B < n) \leq C \exp \left( -c \left( \frac{r^\beta}{n^m} \right)^{\frac{1}{\beta - 1}} \right). \]
Remark 4.2 From (17) one can see that the estimate is weaker as $m$ increases. If a lower estimate of the same form and magnitude is aimed, $m$ should be chosen as small as possible. However it should be recognized, that the increase of $m$ not only increase the upper bound but the probability on the left hand side of (14).

Proof of Theorem 4.1. The proof follows the old, nice idea of [4] (see also [1] Lemma 3.14). The only modification is that we use the very rough estimate:

$$ T_{B_{c}(x,r)}^{m} \geq \sum_{i=1}^{k} \tau_{i}^{m} $$

where $\tau_{i}$ is the exit time of $\partial B_{c}(\xi_{i}, \frac{r}{k})$, $\xi_{i} = X_{\tau_{i}} - 1$ and $k \geq 1$ will be chosen later. From Lemma 4.1 we have that with $t = \frac{k}{2}$

$$ P(\tau < t) \leq p + at^{m} \quad (18) $$

where $p \in \left[\frac{1}{2}, 1 - \varepsilon\right]$ and $a = \frac{2^{m}}{E_{m}(x, \frac{r}{k})}$. Let $\eta$ be such that $P(\tau < t) = (p + at^{m}) \land 1$. The relation (18) can be rewritten as

$$ P(\tau^{m} < s) \leq p + as $$

$$ \mathbb{E}(\exp(-\lambda \tau^{m})) \leq \mathbb{E}(\exp(-\lambda \eta^{m})) \leq p + a\lambda^{-1}. $$

From that point the proof can be finished as in [1].

4.1 The Einstein relation

The relation between the mean exit time of a ball, its volume and resistance is regarded as a key tool to obtain heat kernel estimates. In this section we obtain the corresponding relation with respect to the distance $\rho$ assuming only volume doubling and existence of the $m \geq 0$ integer. More precisely we show the following statements.

Theorem 4.2 If $(\Gamma, \mu, \rho)$ satisfies $(VD)_{\rho}$ then, satisfies $(ER)_{\rho}$:

$$ E_{\rho}(x, 2r) \asymp [R_{m}(x, B^{c})\nu_{\rho}(x, 2r)]^{1/m} \quad (19) $$

with $B = B_{\rho}(x, 2r)$, $E_{\rho}(x, r) = \mathbb{E}_{m}(B_{\rho}(x, r) \mid x)$.

Theorem 4.2 will follow from the next statement and from the tail estimate (26) of the exit time.
Theorem 4.3 If $\mu$ satisfies $(VD)_\rho$ then, $(ER)_m$:

$$E_m(x, 2r) \asymp R_m(x, 2r) \rho(x, 2r)$$ (20)

holds, where $B = B_{\rho}(x, r)$. $E_m(x, r) = \mathbb{E}_m(B|x), R_m(x, 2r) = R_m(x, B_{\rho}^c(x, 2r))$

Let us recall, that $\mathbb{E}_m(B|x) = \mathbb{E}(Q_{m+1}(T_B) | X_0 = x)$ and Lemma 8.4 from[14].

The first lemma is elementary.

Lemma 4.2 Let $B = B_{\rho}(x, r), T = T_B$ then

$$\mathbb{E}_x(Q_{m+1}(T)) \asymp \mathbb{E}_x(T^m).$$

Proof. Let $T = T_{B_{\rho}(x, r)}$. Assume that $r$ is large enough to ensure $T > 2m$ i.e. $B_d(x, 2m + 1) \subset B_{\rho}(x, r)$ and obtain

$$\frac{(2T)^m}{m!} \geq \frac{(T + m)^m}{m!} \geq Q_{m+1}(T) \geq \frac{(T - m)^m}{m!} \geq c \left( \frac{T - m}{m} \right)^m \geq \left( \frac{T}{2m} \right)^m.$$ (21)

For small values the inequality follows by adjusting the constants. ■

Of course the statement holds for arbitrary finite set as well.

Remark 4.3 From the definitions, the Theorem 4.1, $(VD)_\rho$ and $(ER)_\rho$ it is immediate that

$$\frac{n^m}{k + 1} \geq q \min_{y \in B_{\rho}(x, r)} E_m(B_{\rho}(y, \frac{r}{k})) \geq cq \min_{y \in B_{\rho}(x, r)} E_m(B_{\rho}(y, r)) k^{-\beta_m},$$ (22)

$$\begin{aligned}
(k + 1)^{\beta_m - 1} & \geq \frac{E_m(B_{\rho}(y, r))}{n^m} \\
& \geq c \left( \frac{E_m(B)}{n^m} \right)^{\frac{1}{\beta_m - 1}} .
\end{aligned}$$ (23)

where $B = B(x, r)$, which yields

$$\mathbb{P}_x(T_B < n) \leq C \exp \left( -c \left( \frac{E_m(B)}{n^m} \right)^{\frac{1}{\beta_m - 1}} \right)$$ (25)

$$\mathbb{P}_x(T_B < n) \leq C \exp \left( -c \left( \frac{H(x, r)}{n^m} \right)^{\frac{1}{\beta_m - 1}} \right)$$ (26)
Lemma 4.3 (Feynmann-Kacc formula, c.f. [9] or [14]) Let \( f \) be a function on \( \Gamma, A \subset \Gamma, \lambda > 0 \) satisfying

\[
\Delta f - \lambda f = 0 \text{ in } B.
\]

Then for any \( x \in A, \omega = (1 + \lambda)^{-1}, T = T_A \)

\[
f(x) = \mathbb{E}_x \left[ \omega^T f(X_T) \right]
\]

and for any \( m \geq 0 \)

\[
\left( \sum \omega^{n+m}Q_m(n)P_n^A f \right)(x) = \mathbb{E}_x \left( Q_{m+1}(T) \omega^{T+m}f(X_T) \right).
\] (27)

Corollary 4.2 If we choose \( f \equiv 1, \lambda = 0 \) we have from (27) that

\[
E_m(A|x) = \mathbb{E}_x \left( Q_{m+1}(T_A) \right) = \sum_{y \in B} G_m^A(x,y).
\] (28)

Lemma 4.4 For any \( A \subset \Gamma, x \in A \)

\[
E_m(A|x) \leq g_m^A(x,x) \mu(A).
\] (29)

Proof. The proof follows from (28).

Corollary 4.3

\[
E_m(A|x) \leq R_m(x,A^c) \mu(A)
\]

and in particular for \( x \in \Gamma, r > 0 \)

\[
E_m(x,r) \leq Cr^2V_p(x,r)
\]

Both statements direct consequence of (29).

Proof. (of Theorem 4.3). The upper estimate is provided by Lemma 4.4 for the lower estimate we apply the proof of Proposition 4.2 in [12]. Denote \( B = B_\rho(x,2r) \). We start with (1): If \( f \in \mathcal{F}(E_m) \)

\[
|f(x) - f(y)|^2 \leq R_m(x,y)E_m(f,f)
\]

in particular let \( g(z) = g_m^B(x,z) \) and \( z \in B_\rho(x,\delta r) \) then

\[
|g(x) - g(z)|^2 \leq R_m(x,z)E_m(g,g).
\]

From reproducing property of \( g_m^B(x,z) \) we have that \( E_m(g,g) = R_m(x,2r) = g_m^B(x,x) \geq cr^2 \)

\[
|g(x) - g(z)|^2 \leq C\delta^2r^2 = C\delta^2 \leq C\delta^2g(x)^2.
\]
We can choose $\delta$ such that $C\delta^2 \leq 2$, and we obtain from $g(z) \leq g(x)$, that for $z \in B_\rho(x, \delta r)$

$$g^B_m(x, z) \geq \frac{1}{2} g^B_m(x, x).$$

Now we finish immediately using the definition and $(V D)_\rho$.

$$E_m (x, 2r) = \sum_{y \in B} g^B_m(x, y) \mu(y) \geq \sum_{z \in B_\rho(x, \delta r)} g^B_m(x, z) \mu(z) \geq \frac{1}{2} g^B_m(x, x) \text{Var}_\rho (x, \delta r) \geq cR_m \left(B_\rho(x, r), B_\rho(x, 2r)\right) \text{Var}_\rho (x, 2r),$$

where the last step follows from (10) and $(V D)_\rho$. ■

After the above preparations the proof of Theorem 4.2 is immediate from Theorem 4.3 and the next Lemma.

**Lemma 4.5**

$$E_\rho (x, r) \asymp E_m(x, r)^{1/m}$$

**Proof.** Let $B = B_\rho(x, r), T = T_B$ From the Jensen inequality we obtain that for $m \geq 1$

$$E_m (x, r) = \mathbb{E}_x \left(Q_{m+1} \left(T_{B_\rho(x, r)}\right)\right) \asymp \mathbb{E}_x (T^m) \geq [E_\rho (x, r)]^m.$$

For the opposite estimate denote $E = E_m(x, r)$ and

$$E_\rho (x, r) = \sum_{n=0}^{2c_0 E^{1/m}} P(T > n) \geq \sum_{n=0}^{2c_0 E^{1/m}} P(T > n) \geq c_0 E^{1/m} P(T_B > 2c_0 E^{1/m}).$$

Now we use Theorem 4.1, in particular (26)

$$\mathbb{P}_x (T < n) \leq C \exp \left(-c \left(\frac{H(x, r)}{n^m}\right)^{\frac{1}{m-1}}\right)$$

(30)

$$\mathbb{P}_x (T \geq n) = 1 - \mathbb{P}_x (T < n) \geq 1 - C \exp \left(-c \left(\frac{H(x, r)}{n^m}\right)^{\frac{1}{m-1}}\right) \geq 1/2$$

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if we chose \( n^m \geq H(x,r) \) and \( c_0 \) such that \( \log C - c \left( \frac{1}{2\alpha} \right)^{\frac{1}{\beta_m}} = 1/2 \) i.e. \( c_0 = \frac{1}{2} \left( \frac{c}{\log C - 1/2} \right)^{\beta_m^{-1}} \) the proof is complete. ■

**Proof of Theorem (4.2).** The statement is immediate from Lemma 4.2 4.5 and Theorem 4.3 ■

5 Heat kernel estimates

In this section we show that \((V\ell D)_{\rho}\) implies the off-diagonal upper and near diagonal lower estimates. The proofs are adaptation of the ones developed in the works \([6],[12]\) all, on resistance forms in case of recurrent (or strongly recurrent) spaces, graphs.

It is standard to deduce a diagonal lower estimate from (13), (see also in \([14]\) Theorem 6.2).

**Proposition 5.1** If \((V\ell D)_{\rho}\) holds there is a \( c > 0 \) such that for all \( n > 0 \)

\[
p_{2n}(x,x) \geq \frac{c}{\nu(x, f(x,n))}, \tag{DLE}
\]

where \( f(x,.) \) is the inverse of \( F(x,.) \) in the second variable.

**Remark 5.1** If we consider the example, \( V_{\rho}(x,r) \propto r^{\alpha} \) then \( H(x,r) = r^2 V_{\rho}(x,r) \propto r^{\alpha+2} \), \( F(x,r) \propto r^{\frac{\alpha+2}{\alpha}} \), \( f(x,n) \propto n^{\frac{\alpha}{\alpha+2}} \)

\[
p_{2n}(x,x) \geq cn^{-\frac{\alpha}{\alpha+2}}. \]

Thus

\[
\sum_n n^m p_n(x,x) = \infty.
\]

**Remark 5.2** One may object that \( m \left( 1 - \frac{\alpha}{\alpha+2} \right) > -1 \) holds for all \( m \), that seemingly contradicts to the initial argument, which indicates that \( m \) had to be chosen enough large. Let us notice, that if \( m \) is not enough large then the whole above argument is meaningless, the resolvent operator \( G_m \) is bounded and we can not subtract the needed asymptotic information from it. Among others the notion \( R_m(x,y) \) similarly to the usual resistance metric \( R(x,y) \) in the transient case is meaningless, furthermore the key observations \([1]\) and \([4]\) are not in our possession.
5.1 Estimates of higher order time derivatives of the heat kernel and relation to $E_m$

A fairly simple but powerful method is developed in the mentioned works (see also [7]). The key observation is the following (see for the simplified proof [10] Theorem 10.4). Without any further assumption for any finite set $A \subset \Gamma$

$$p_n(x, x) \leq \frac{2R(x, A)}{n} + \frac{\sqrt{2}}{\mu(A)},$$

where $R(x, A) = \sup_{y \in A} R(x, y)$. We have the following version of the statement.

**Proposition 5.2** There is a $C > 0$ such that, for any finite set $A \subset \Gamma$ and $x \in A$, $n > 0$ integer

$$p_n(x, x) \leq C \left( \frac{R_m(x, A)}{n^m} + \frac{1}{\mu(A)} \right),$$

where $R_m(x, A) = \sup_{y \in A} R_m(x, y)$.

Before we prove the statement we show how one can obtain the diagonal upper bound from it.

**Theorem 5.1** Assume $(VD)_\rho$ then

$$p_n(x, x) \leq \frac{C}{V_\rho(x, f(x, n))},$$

where $f(x, n)$ is the inverse of $F(x, r)$ in the second variable, furthermore

$$p_n(x, y) \leq \frac{C}{\sqrt{V_\rho(x, f(x, n))} V_\rho(y, f(y, n))}.$$  \hspace{1cm} \hspace{1cm} (32)

**Proof.** Let $A = B = B_\rho(x, r)$ and choose $r$ to have $\frac{C\pi_m(x,B)}{n^m} = \frac{C'}{V_\rho(x, r)}$ and $n = [C r^2 V_\rho(x, r)]^{1/m}$ then

$$p_n(x, x) \leq \frac{C}{V_\rho(x, r)},$$

$$p_n(x, x) \leq \frac{C}{V_\rho(x, f(x, n))}.$$  \hspace{1cm} \hspace{1cm} (33)

This shows the statement for even $n$, for odd $n$ it can be seen together with (32) as in [11]. \[\blacksquare\]
**Definition 5.1** Let us introduce the time differential operator and its iterations for \( k \geq 1, n \geq 0 
)

\[
(Df)_n = f_{n+2} - f_n \\
(D^k f)_n = (D (D^{k-1} f))_n \\
f_n^{(k)} = (D^k f)_n.
\]

**Lemma 5.1** Let \( f_n (y) = p_n (x, y) \) then there is a \( C > 0 \) such that for all \( k \geq 0, n > 0, x \in \Gamma \)

\[ (-1)^k f_n^{(k)} (x) \leq \frac{1}{(2n)^k} f_{2n} (x) \tag{33} \]

**Proof.** From the spectral decomposition of \( p_{2n} (x, x) \) we know that \( h_{2n}^{(k)} = (-1)^k f_n^{(k)} (x) \geq 0 \) for all \( k \geq 0 \) and the same implies that the map \( n \rightarrow (-1) \left( h_{2n+2}^{(k-1)} - h_{2n}^{(k-1)} \right) (x) \) non-decreasing. We show the statement by induction using a slightly stronger statement. Assume it holds for all \( 0 \leq i < k \),

\[ h_{3n}^{(i)} \leq \frac{1}{(4 \lfloor S_i n \rfloor)} f_{4n-4[S_i n]} (x), \]

where \( c_i = 2^{-(i+2)} \) , \( S_k = S_{k-1} + c_k \), and note that for \( i = 0 \) the assumption holds.

\[ h_{4n}^{(k)} = \left[ h_{4n-1}^{(k-1)} - h_{4n+2}^{(k-1)} \right] \]

\[ \leq \frac{1}{4 \lfloor c_k n \rfloor} \sum_{i=0}^{4[S_k n]} \left[ h_{4n-2i}^{(k-1)} - h_{4n+2-2i}^{(k-1)} \right] \]

\[ \leq \frac{1}{4 \lfloor c_k n \rfloor} \left[ h_{4n-4[S_k n]}^{(k-1)} - h_{2n+2}^{(k-1)} \right] \]

\[ \leq \frac{1}{4 \lfloor c_k n \rfloor} h_{4n-4[S_k n]}^{(k-1)} \]

now by induction, if \( m = n - \lfloor c_k n \rfloor \)

\[ h_{4n}^{(k)} \leq \frac{1}{4m} h_{4m}^{(k-1)} \leq \frac{1}{4m \ (4 \lfloor S_{k-1} m \rfloor)^{k-1} f_{4m-4[S_{k-1} m]} (x)} \]

Let us recall that \( f_{2k} (x) \) is non-increasing in \( k \) and find that

\[ 4m - 4 \lfloor S_{k-1} m \rfloor = 4 \left( n - \lfloor c_k n \rfloor \right) - 4 \lfloor S_{k-1} (n - \lfloor c_k n \rfloor) \rfloor \]

\[ \geq 4 \left( n - \lfloor c_k n \rfloor \right) - 4 \lfloor S_{k-1} n \rfloor \]

\[ \geq 4n - 4 \left( \lfloor c_k n \rfloor + \lfloor S_{k-1} n \rfloor \right) \]

\[ \geq 4n - 4 \lfloor (S_{k-1} + c_k) n \rfloor \]

\[ = 4n - 4 \lfloor S_k n \rfloor \]
which leads to the needed inequality.

\[ h_{4n}^{(k)} \leq \frac{1}{(4 \lfloor S_k n \rfloor)} f_{4n-4[S_k n]} (x). \]

Finally observing that \( S_k = \sum_{i=0}^{k} 2^{-i+2} \) we have that \( S_k < \frac{1}{2} \) and we obtain (33). ■

**Lemma 5.2** Again, if \( f_n (y) = p_n (x, y) \), then

\[ E_m (f_n, f_n) = (I - P)^{m} f_n, f_n \]

Proof. Let \( A \subset \Gamma \) be a finite set and choose \( y^* \) so that

\[ p_{2n} (x, y^*) := \min_{y \in A} p_{2n} (x, y) \]

\[ p_{2n} (x, y^*) \sum_{z \in A} \mu (z) \leq \sum_{z \in A} p_{2n} (x, z) \mu (z) \leq \sum_{z \in \Gamma} P_{2n} (x, z) \leq 1, \]

and

\[ p_{2n} (x, y^*) \leq \frac{1}{\mu (A)} \]

follows. Let us denote \( f_n (y) = p_n (x, y) \). By elementary estimates we have that

\[ \frac{1}{2} p_{2n}^2 (x, x) \leq p_{2n}^2 (x, y^*) + |p_{2n} (x, x) - p_{2n} (x, y^*)|^2 \]

\[ \leq \frac{1}{\mu^2 (A)} + \overline{R} (x, A) E_m (f_{2n}, f_{2n}) \]

\[ \leq \frac{1}{\mu^2 (A)} + \overline{R} (x, A) \frac{C}{n^m} p_{2n} (x, x), \]
where in the last step Lemma 5.1 is used. Solving this for \( p_{2n}(x,x) \) we obtain

\[
p_{2n}(x,x) \leq C_1 \frac{\mathcal{R}_m(x,A)}{\ell^m} + \left( \frac{2}{\mu^2(A)} + C_2 \frac{\mathcal{R}_m^2(x,A)}{\ell^{2m}} \right)^{1/2}
\]

(34)

\[
\leq C \left( \frac{\mathcal{R}_m(x,A)}{n^m} + \frac{1}{\mu(A)} \right).
\]

(35)

The proof is finished by noting that \( p_{2n+1}(x,x) \leq p_{2n}(x,x) \).

5.2 The off-diagonal upper estimate

The off-diagonal estimate can be easily obtained from the diagonal one.

**Theorem 5.2** Assume \((p_0),(VD)_\rho\) and \((DUE)_F\) then

\[
p_n(x,y) \leq \frac{C}{V(x,f(x,n))} \exp(-ck(x,n,r))
\]

\[
\leq \frac{C}{\nu(x,f(x,n))} \exp\left(-c\left(\frac{F_\rho(x,d(x,y))}{n}\right)^\frac{1}{m-1}\right).
\]

The proof is word by word the same as for Theorem 8.5 in [14] or an alternative proof is combination of Theorem 8.6 and 8.10 in [14].

5.3 Lower estimates

The next task is to show the Near Diagonal Lower Estimate \((NDLE)_F\):

There are \( \delta \) and \( c > 0 \) such that, for all \( x \in \Gamma, r > 0, y \in B(x,r), n > 0 \) if \( \rho(x,y) \leq \delta f(x,n) \) then,

\[
\tilde{p}_n(x,y) \geq \frac{c}{\nu(x,f(x,n))}.
\]

**Theorem 5.3** If \((\Gamma,\mu)\) satisfies \((VD)_\rho\) and \((DUE)_F\) then, \((NDLE)_F\) holds.

**Proof.** First we prove

\[
p_{2n}(x,y) \geq \frac{c}{\nu(x,f(x,n))},
\]

for \( x,y \in \Gamma \) satisfying \( d(x,y) \equiv 0 \mod 2 \). Let us choose \( r \) such that \( n = F_\rho(x,r) = [r^2 \nu(x,r)]^{1/m} \) and denote \( f_{2n}(y) = p_{2n}(x,y) \), then

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\[ |f_{2n}(x) - f_{2n}(y)|^2 \leq R_m(x, y)E_m(f, f). \]

By Lemma 5.2 we have that
\[ |f_{2n}(x) - f_{2n}(y)|^2 \leq R_m(x, y)E_m(f_{2n}, f_{2n}) = R_m(x, y)(-1)^m f_{4n}^{(m)} \]
and by Lemma 5.1 and the diagonal upper estimate
\[ |f_{2n}(x) - f_{2n}(y)|^2 \leq R_m(x, y)E_m(f_{2n}, f_{2n}) = R_m(x, y)(-1)^m f_{4n}^{(m)} \]

if \( \delta \) is small enough. The above inequality means that
\[ p_{2n}(x, y) \geq \frac{1}{2}p_{2n}(x, x) \geq \frac{c}{V_\rho(x, r)}. \]

Finally
\[ p_{2n+1}(x, y) = p(x, z)\mu(z)p_{2n}(z, y) \geq p_0p_{2n}(z, y) \geq \frac{c}{V_\rho(x, r)}. \]

\section{6 Stability}

In this section we show that \((VD)_\rho\) implies the parabolic Harnack inequality via the two-sided estimates. It is an interesting by-product that in our scenarios the volume doubling property implies the elliptic Harnack inequality.

It is already known that \((UE_F)\) and \((NDLE_F)\) are equivalent to the \(F\)-parabolic Harnack inequality (c.f., [14]) where \(F\) is properly regular function and both imply \((V D)_\rho\).

\textbf{Definition 6.1} The function class \(W_1\) is defined as follows. \(F \in W_1\) if there are \(\beta \geq \beta' > 1, C \geq c > 0\) such that for all \(R > r > 0, x \in \Gamma, y \in B(x, R)\),
\[ c \left( \frac{R}{r} \right)^{\beta'} \leq \frac{F(x, R)}{F(y, r)} \leq C \left( \frac{R}{r} \right)^\beta. \quad (36) \]
Definition 6.2 We say that $(PH)_F$, the parabolic Harnack inequality holds for a weighted graph $(\Gamma, \mu)$ with respect to a function $F \in W_1$ if there is a constant $C > 0$ such that for any $x \in \Gamma, R, k > 0$ and any solution $u \geq 0$ of the heat equation
\[ \partial_t u = \Delta u \]
on $\mathcal{D} = [k, k+F(x, R)] \times B(x, 2R)$, the following is true. On smaller cylinders defined by $\mathcal{D}^- = [k + \frac{1}{4}F(x, R), k + \frac{1}{2}F(x, R)] \times B(x, R)$, and $\mathcal{D}^+ = [k + \frac{3}{4}F(x, R), k + F(x, R)] \times B(x, R)$, and taking $(n_-, x_-) \in \mathcal{D}^-$, $(n_+, x_+) \in \mathcal{D}^+$,
\[ d(x_-, x_+) \leq n_+ - n_- \]the inequality
\[ u_{n_-}(x_-) \leq C \tilde{u}_{n_+}(x_+) \]
holds, where the short notation $\tilde{u}_n = u_n + u_{n+1}$ was used.

Remark 6.1 At present it is not clear how the $\beta' > 1$ condition follows from the assumptions, while we expect it holds if $\mu$ is $(VD)_\rho$.

Definition 6.3 We say that the elliptic Harnack inequality, $(H)$ holds with respect to $\mu$, and $\rho$ if there is a $C > 0$ such that for all $x \in \Gamma, r > 0$ if $h$ is harmonic on $B_\rho(x, 2r)$:
\[ (I - P) h (x) = 0 \text{ for } x \in B(x, 2r) \]
then
\[ \max_{z \in B(x, r)} h(z) \leq C \min_{y \in B(x, r)} h(y). \]

Theorem 6.1 Assume that $(\Gamma, \mu)$ satisfies $(p_0)$, then the following statements are equivalent.

1. $(VD)_\rho$, and $F \in W_1$
2. $(UE)_F$ and $(NDLE)_F$ hold for an $F \in W_1$,
3. $(PH)_F$ holds for $F \in W_1$ with respect to $\rho$-balls.

Proof. The proof can be reproduced from the one of [14] Theorem 12.1.
**Theorem 6.2** Assume that \((\Gamma, \mu)\) satisfies \((p_0)\), then \((VD)_\rho\) and \(F \in W_1\) implies \((H)\).

**Proof.** It is evident that \((PH)_F\) implies \((H)\) and from Theorem 6.1 we know that \((VD)_\rho\) and \(F \in W_1\) implies \((PH)_F\).

**Definition 6.4** Two weighted graphs \(\Gamma\) with \(\mu\) and \(\Gamma'\) with \(\mu'\) are roughly isometric (or quasi isometric) with respect to the metrics \(d, d'\) (c.f. [8, Definition 5.9]) if there is a map \(\phi\) from \(\Gamma\) to \(\Gamma'\) such that there are \(a, b, c, M > 0\) for which

\[
\frac{1}{a} d(x, y) - b \leq d'(\phi(x), \phi(y)) \leq ad(x, y) + b
\]  

for all \(x, y \in \Gamma\),

\[
d'(\phi(\Gamma), y') \leq M
\]  

for all \(y' \in \Gamma'\) and

\[
\frac{1}{c} \mu(x) \leq \mu'(\phi(x)) \leq c\mu(x)
\]  

for all \(x \in \Gamma\).

**Theorem 6.3** The \(F\)-parabolic Harnack inequality is rough isometry invariant with respect to \(\rho\) and \(\rho'\).

**Proof.** We know that \((VD)_\rho\) on \((\Gamma, \mu)\) if and only if \((VD)_{\rho'}\) on \((\Gamma, \mu)\). The equivalence of \((VD)_{\rho}\) and \((PH)_F\) on both graphs are given by Theorem 6.1 and the statement follows.

**6.1 Comments**

Kigami in [10] constructed a metric which is quasisymmetric to \(RV_R(x, d(x, y)) + RV_R(x, d(y, x))\). This procedure can be applied to \(\rho^2(x, y)V_\rho(x, \rho(x, y)) + \rho^2(x, y)V_\rho(y, \rho(x, y))\) as well. All the conditions are satisfied to obtain a new metric \(\sigma\) which is quasi symmetric to \(\rho\). We know that \((VD)_\rho\) implies \((VD)_\sigma\) and by Kigami’s result [10] we have

\[
p_n(x, y) \leq \frac{C}{V_\sigma(x, g^{-1}(n))} \exp \left( -c \left( \frac{\sigma(x, y)}{n} \right)^{\frac{1}{\beta - 1}} \right)
\]

\[
\tilde{p}_n(x, y) \geq \frac{c}{V_\sigma(x, g^{-1}(n))}.
\]
where \( g^{-1}(n) \) the inverse of \( g(r) = r^\alpha V_\sigma(x, r) \) and \( \alpha \) is the exponent determined by construction of \( \sigma \) from \( \rho \). The lower estimate and parabolic Harnack inequality follows for \( \sigma \) as well. It should be noted here, that with the introduction of the second new metric the dependence from \( x \) in the exponential term is eliminated and \( F(x, r) \) replaced by \( g(r) \).

7 Examples

There are several examples of fractals and fractal-like graph possessing two-sided heat kernel estimates (see [2]) and examples on which direction dependence destroy it [8]. The major contribution of resistance metric in showing heat kernel estimates is emphasized in [6] and [10]. The same arguments apply to our work.

**Example 7.1** The graphs in Example 4, [6], are strongly recurrent. There, \( \beta > \alpha \) i.e. strong recurrence is assumed, that is not needed anymore. For instance the high dimensional graphical Sierpinski carpet, can be handled by our method.

Kigami constructed \( \mathcal{G} \) a family of fractal structures in [10]. The structures can be discretized and get the so called graphical Sierpinski gaskets, graphs. The typical structures are recurrent but it is easy to lift up them and get transient graphs. In [2] Barlow has Proposition 5 as follows.

**Proposition 7.1** Let \( \alpha \geq 1 \) and a graph \((\Gamma, \mu) \in \mathcal{G}\) which satisfies \( V(x, r) \simeq r^\alpha \) and \( E(x, r) \simeq r^\beta \) with respect to the shortest path metric, furthermore the graph is very strongly recurrent (see the definition there). Let \( \lambda > 0 \). Then there is a weighted graph \((\tilde{\Gamma}, \tilde{\mu})\) such that \( V(x, r) \simeq r^{\alpha + \lambda} \) but \( E(x, r) \simeq r^\beta \) and satisfies the elliptic Harnack inequality.

The graph \((\tilde{\Gamma}, \tilde{\mu})\) is product of \((\Gamma, \mu)\) and an ultrametric space. The choice of \( \lambda > 0 \) can ensure that \( \beta < \alpha + \lambda \). From Proposition 3 of [2] we know that such graphs are transient hence the resistance metric is not applicable, while the resolvent metric does if volume doubling holds for it. Polynomial volume growth in the resolvent metric follow from the asymptotic spherical symmetry of the Green kernel and from polynomial volume growth in the original metric.
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