Minimisation of Event Structures

Paolo Baldan
University of Padova, Italy
baldan@math.unipd.it

Alessandra Raffaetà
University Ca’ Foscari of Venice, Italy
raffaeta@unive.it

Abstract

Event structures are fundamental models in concurrency theory, providing a representation of events in computation and of their relations, notably concurrency, conflict and causality. In this paper we present a theory of minimisation for event structures. Working in a class of event structures that generalises many stable event structure models in the literature (e.g., prime, asymmetric, flow and bundle event structures), we study a notion of behaviour-preserving quotient, referred to as a folding, taking (hereditary) history preserving bisimilarity as a reference behavioural equivalence. We show that for any event structure a folding producing a uniquely determined minimal quotient always exists. We observe that each event structure can be seen as the folding of a prime event structure, and that all foldings between general event structures arise from foldings of (suitably defined) corresponding prime event structures. This gives a special relevance to foldings in the class of prime event structures, which are studied in detail. We identify folding conditions for prime and asymmetric event structures, and show that also prime event structures always admit a unique minimal quotient (while this is not the case for various other event structure models).

2012 ACM Subject Classification Theory of computation → Concurrency; Software and its engineering → Formal methods

Keywords and phrases Event structures, minimisation, history-preserving bisimilarity, behaviour preserving quotient

1 Introduction

When dealing with formal models of computational systems, a classical problem is that of minimisation, i.e., for a given system, define and possibly construct a compact version of the system which, very roughly speaking, exhibits the same behaviour as the original one, avoiding unnecessary duplications. The minimisation procedure depends on the notion of behaviour of interest and also on the expressive power of the formalism at hand, which determines its capability of describing succinctly some behaviour. One of the most classical examples is that of finite state automata, where one is typically interested in the accepted language. Given a deterministic finite state automaton, a uniquely determined minimal automaton accepting the same language can be constructed, e.g., as a quotient of the original automaton via a partition/refinement algorithm (see, e.g., [14]). Moving to non-deterministic finite automata, minimal automata become smaller, at the price of a computationally more expensive minimisation procedure and non-uniqueness of the minimal automaton [20].

In this paper we study the problem of minimisation for event structures, a fundamental model in concurrency theory [31, 32]. Event structures are a natural semantic model when one is interested in modelling the dynamics of a system by providing an explicit representation of the events in computations (occurrence of atomic actions) and of the relations between events, like causal dependencies, choices, possibility of parallel execution, i.e., in what is referred to as a true concurrent (non-interleaving) semantics. Prime event structures [22], probably the most widely used event structure model, capture dependencies between events in terms of causality and conflict. A number of variations of prime event structures have been
Minimisation of Event Structures

introduced in the literature. In this paper we will deal with asymmetric event structures [5], which generalise prime event structures with an asymmetric form of conflict which allows one to model concurrent readings and precedences between actions, and flow [8, 7] and bundle [18] event structures, which add the possibility of directly modelling disjunctive causes. Event structures have been used for defining a concurrent semantics of several formalisms, like Petri nets [22], graph rewriting systems [4, 3, 25] and process calculi [30, 29, 9]. Recent applications are in the field of weak memory models [23, 15, 11] and of process mining and differencing [12].

Behavioural equivalences, defined in a true concurrent setting, take into account not only the possibility of performing steps, but also the way in which such steps relate with each other. We will focus on hereditary history preserving (hhp-)bisimilarity [6], the finest equivalence in the true concurrent spectrum in [26], which, via the concept of open map, has been shown to arise as a canonical behavioural equivalence when considering partially ordered computations as observations [16].

The motivation for the present paper originally stems from some work on the analysis and comparison of business process models. The idea, advocated in [12, 1], is to use event structures as a foundation for representing, analysing and comparing process models. The processes, in their graphical presentation, should be understandable, as much as possible, by a human user, who should be able, e.g., to interpret the differences between two processes diagnosed by a comparison tool. For this aim it is important to avoid “redundancies” in the representation and thus to reduce the number of events, but clearly without altering the behaviour. The paper [2] explores the use of asymmetric and flow event structures and, for such models, it introduces some ad-hoc reduction techniques that allow one to merge sets of events without changing the true concurrent behaviour. A general notion of behaviour preserving quotient, referred to as a folding, is introduced over an abstract class of event structures, having asymmetric and flow event structures as subclasses. However, no general theory is developed. The paper focuses on a special class of foldings, the so-called elementary foldings, which can only merge a single set of events into one event, and these are studied separately on each specific subclass of event structures (asymmetric and flow event structures), providing only sufficient conditions ensuring that a function is a folding.

A general theory of behaviour preserving quotients for event structures is thus called for, settling some natural foundational questions. Is the notion of folding adequate, i.e., are all behaviour preserving quotients expressible in terms of foldings? Is there a minimal quotient in some suitably defined general class of event structures? What happens in specific subclasses? (for asymmetric and flow event structures the answer is known to be negative, but for prime event structures the question is open). Working in the specific subclasses of event structures, can we have a characterisation of general foldings, providing not only sufficient but also necessary conditions?

In this paper we address the above questions. We work in a general class of event structures based on the idea of family of posets in [24], sufficiently expressive to generalise most stable event structures models in the literature, including prime [22], asymmetric [5], flow [8] and bundle [18] event structures.

As a first step we study, in this general setting, the notion of folding, i.e., of behaviour preserving quotient. A folding is a surjective function that identifies some events while keeping the behaviour unchanged. Formally, it establishes a hhp-bisimilarity between the source and target event structure. Foldings can be characterised as open maps in the sense of [16]. Actually, it turns out that not all behaviour preserving quotients arise as a folding, but we show that for any behaviour preserving quotient, there is a folding that induces a
coarser equivalence, in a way that foldings properly capture all possible behaviour preserving quotients. Additionally, given two possible foldings of an event structure we show that it is always possible to “join” them. This allows to prove that for each event structure a maximally folded version, namely a uniquely determined minimal quotient always exists.

Relying on the order-theoretic properties of the set of configurations of event structures [24], and on the correspondence between prime event structures and domains [22], we derive that each event structure in the considered class arises as the folding of a canonical prime event structure. Moreover, all foldings between general event structures arise from foldings of the corresponding canonical prime event structures. Interestingly, this result can be derived from the characterisation of folding morphisms as open maps.

The results above give a special relevance to foldings in the class of prime event structures, which thus are studied in detail. We provide necessary and sufficient conditions characterising foldings for prime event structures. This allows establish a clear connection with the so-called abstraction morphisms, introduced in [10] for similar purposes. This characterisation of foldings can guide, at least in the case of finite structures, the construction of behaviour preserving quotients. Moreover we show that also prime event structures always admit a minimal quotient.

The fact that all event structures arise as foldings of prime event structures allows one to think of various brands of event structures in the literature, like asymmetric, flow, bundle event structures as more expressive models that allow for smaller realisations of a given behaviour, i.e., of smaller quotients. For all these classes, however, the uniqueness of the minimal quotient is lost. Despite the fact that foldings on wider classes of event structures can be studied on the corresponding canonical prime event structures, a direct approach can be theoretically interesting and it can lead more efficient minimisation procedures. In this paper, a characterisation of foldings is explicitly devised for asymmetric event structures.

Most results have a natural categorical interpretation, which is only hinted at in the paper. In order to keep the presentation simple, the categorical references are inserted in side remarks that can be safely skipped by the non-interested reader. This applies, in particular, to the possibility of viewing foldings as open maps in the sense of [16]. This correspondence, which in the present paper only surfaces, suggests the possibility of understanding and developing our results in a more abstract categorical setting. More details about this are provided in the appendices.

The rest of the paper is structured as follows. In § 2 we introduce the class of event structures we work with, hereditary history preserving bisimilarity and we discuss how various event structure models in the literature embed into the considered class. In § 3 we introduce and study the notion of folding, we prove the existence of a minimal quotient and we show the tight relation between general foldings and those on prime event structures. In § 4 we present folding criteria on prime and asymmetric event structures, and discuss the existence of minimal quotients. Finally, in § 5 we draw some conclusions, discuss connections with related literature and outline future work venues. An appendix contains all proofs and some additional technical results.

2 Event Structures and History Preserving Bisimilarity

In this section we define hereditary history-preserving bisimilarity, the reference behavioural equivalence in the paper. This is done for a abstract notion of event structure, introduced in [24], in a way that various stable event structure models in the literature can be seen as special subclasses. We will explicitly discuss prime [22], asymmetric [5], flow [8, 7] and
Minimisation of Event Structures

The family of posets compatible has an upper bound. A partial order on family of posets \( A \) where \( a \leq b \) if for all \( y \in Y \) and \( z \in Z \) it holds \( y \leq z \). When \( Y \) or \( Z \) are singletons, sometimes we replace them by their only element, writing, e.g., \( y \triangleright z \) for \( \{y\} \triangleright \{z\} \). The relation \( r \) is acyclic on \( Y \) if there is no \( \{y_0, y_1, \ldots, y_n\} \subseteq Y \) such that \( y_0 \triangleright y_1 \triangleright \ldots \triangleright y_n \triangleright y_0 \). Relation \( r \) is a partial order if it is reflexive, antisymmetric and transitive. Given a function \( f : X \rightarrow Y \) we will denote by \( f[x \mapsto y] : X \cup \{x\} \rightarrow Y \cup \{y\} \) the function defined by \( f[x \mapsto y][x] = y \) and \( f[x \mapsto y][z] = f(z) \) for \( z \in X \setminus \{x\} \). Note that the same notation can represent an update of \( f \), when \( x \in X \), or an extension of its domain, otherwise. For \( Z \subseteq X \), we denote by \( f|_Z : Z \rightarrow Y \) the restriction of \( f \) to \( Z \).

2.1 Event Structures

Following \([24, 27, 28, 2]\), we work on a class of event structures where configurations are given as a primitive notion. More precisely, we borrow the idea of family of posets from \([24]\).

Definition 1 (family of posets). A poset is a pair \((C, \preceq_C)\) where \( C \) is a set and \( \preceq_C \) is a partial order on \( C \). A poset will be often denoted simply as \( C \), leaving the partial order relation \( \preceq_C \) implicit. Given two posets \( C_1 \) and \( C_2 \) we say that \( C_1 \) is a prefix of \( C_2 \) and write \( C_1 \sqsubseteq C_2 \) if \( C_1 \subseteq C_2 \) and for all \( x_1 \in C_1, x_2 \in C_2 \), if \( x_2 \preceq_C x_1 \) then \( x_2 \in C_1 \) and \( x_2 \preceq_{C_1} x_1 \). A family of posets \( F \) is a prefix-closed set of finite posets i.e., a set of finite posets such that if \( C_2 \in F \) and \( C_1 \sqsubseteq C_2 \) then \( C_1 \in F \). We say that two posets \( C_1, C_2 \in F \) are compatible, written \( C_1 \sim C_2 \), if they have an upper bound, i.e., there is \( C \in F \) such that \( C_1, C_2 \sqsubseteq C \). The family of posets \( F \) is called coherent if each subset of \( F \) whose elements are pairwise compatible has an upper bound.

Posets \( C \) will be used to represent configurations, i.e., sets of events executed in a computation of an event structure. The order \( \preceq_C \) intuitively represents the order in which the events in \( C \) can occur. This motivates the prefix order that can be read as a computational extension: when \( C_1 \sqsubseteq C_2 \) we have that \( C_1 \preceq_C C_2 \), with events in \( C_1 \) ordered exactly as in \( C_2 \), and the new events in \( C_2 \setminus C_1 \) cannot precede events already in \( C_1 \). An example of family of posets can be found in Fig. 1 (left). Observe, for instance, that the configuration with set of events \( \{c\} \) is not a prefix of the one with set of events \( \{a, c\} \), since in the latter \( a \preceq c \).

An event structure is then defined simply as a coherent family of posets where events carry a label. Hereafter \( \Lambda \) denotes a fixed set of labels.

Definition 2 (event structure). A (poset) event structure is a tuple \( E = (E, \text{Conf}(E), \lambda) \) where \( E \) is a set of events, \( \text{Conf}(E) \) is a coherent family of posets such that \( E = \bigcup \text{Conf}(E) \).
and \( \lambda : E \to \Lambda \) is a labelling function. For a configuration \( C \in \text{Conf}(E) \) the order \( \leq_C \) is referred to as the local order.

In \cite{2} abstract event structures are defined as a collection of ordered configurations, without any further constraint. This is sufficient for giving some general definitions which are then studied in specific subclasses of event structures. Here, in order to develop a theory of foldings at the level of general event structures, we need to assume stronger properties, those of a family of posets from \cite{24} (e.g., the fact that Definition \cite{26} is well-given relies on this). This motivates the name poset event structure. Also note that, differently from what happens in other general concurrency models, like configuration structures \cite{25}, configurations are endowed explicitly with a partial order, which in turn intervenes in the definition of the prefix order between configurations. This will be essential to view asymmetric or flow event structures as subclasses. Since we only deal with poset event structures and their subclasses, we will often omit the qualification “poset” and refer to them just as event structures. Moreover, we will often identify an event structure \( E \) with the underlying set \( E \) of events and write, e.g., \( x \in E \) for \( x \in E \).

An isomorphism of configurations \( f : C \to C' \) is an isomorphism of posets that respects the labelling, namely for all \( x, y \in C \), we have \( \lambda(x) = \lambda(f(x)) \) and \( x \leq_C y \iff f(x) \leq_{C'} f(y) \). When configurations \( C, C' \) are isomorphic we write \( C \simeq C' \).

As mentioned above, the prefix order on configurations can be interpreted as computational extension. This will be later formalised by a notion of transition system over the set of configurations (see Definition \cite{3}).

Given an event \( x \) in a configuration \( C \) it will be useful to refer to the prefix of \( C \) including only those events that necessarily precede \( x \) in \( C \) (and \( x \) itself). This motivates the following definition.

\begin{definition}[history] Let \( E \) be an event structure, let \( C \in \text{Conf}(E) \) and let \( x \in C \). The history of \( x \) in \( C \) is defined as the set \( C[x] = \{ y \in C \mid y \leq_C x \} \) endowed with the restriction of \( \leq_C \) to \( C[x] \), i.e., \( \leq_{C[x]} = \leq_C \cap (C[x] \times C[x]) \). The set of histories in \( E \) is \( \text{Hist}(E) = \{ C[x] \mid C \in \text{Conf}(E) \land x \in C \} \). The set of histories of a specific event \( x \in E \) will be denoted by \( \text{Hist}(x) \).
\end{definition}

\subsection{Hereditary History Preserving Bisimilarity}

In order to define history preserving bisimilarity, it is convenient to have an explicit representation of the transitions between configurations.

\begin{definition}[transition system] Let \( E \) be an event structure. If \( C, C' \in \text{Conf}(E) \) with \( C \sqsubseteq C' \) we write \( C \xrightarrow{X} C' \) where \( X = C' \setminus C \).
\end{definition}

When \( X \) is a singleton, i.e., \( X = \{ x \} \), we will often write \( C \xrightarrow{x} C' \) instead of \( C \xrightarrow{\{ x \}} C' \). It is easy to see that in an event structure each configuration is reachable in the transition system from the empty one.

As it happens in the interleaving approach, a bisimulation between two event structures requires any event of an event structure to be simulated by an event of the other, with the same label. Additionally, the two events are required to have the same “causal history”.

\begin{definition}[(hereditary) history preserving bisimilarity] Let \( E, E' \) be event structures. A history preserving (hp-)bisimulation is a set \( R \) of triples \( (C, f, C') \), where \( C \in \text{Conf}(E) \), \( C' \in \text{Conf}(E') \) and \( f : C \to C' \) is an isomorphism of configurations, such that \((\emptyset, \emptyset, \emptyset) \in R \) and for all \((C_1, f, C'_1) \in R \)

Minimisation of Event Structures

1. for all \( C_1 \xrightarrow{x} C_2 \) there exists some \( C'_1 \xrightarrow{x'} C'_2 \) such that \((C_2, f[x \mapsto x'], C'_2) \in R\);
2. for all \( C'_1 \xrightarrow{x'} C'_2 \) there exists some \( C_1 \xrightarrow{x} C_2 \) such that \((C_2, f[x \mapsto x'], C'_2) \in R\).

Relation \( R \) is called a hereditary history preserving (hhp-)bisimulation if, in addition, it is downward closed, i.e., if \((C_1, f, C'_1) \in R \) and \( C_2 \subseteq C_1 \) then \((C_2, f|_{C_2}, f(C_2)) \in R\).

Observe that, in the definition above, an event must be simulated by an event with the same label. In fact, in the triple \((C \cup \{x\}, f[x \mapsto x'], C' \cup \{x'\}) \in R\), the second component \( f[x \mapsto x'] \) must be an isomorphism of configurations, i.e., of labelled posets, and thus it preserves labels. Hhp-bisimilarity has been shown to arise as a canonical behavioural equivalence on prime event structures, as an instance of a general notion defined in terms of the concept of open map, when considering partially ordered computations as observations [16].

2.3 Examples: Prime, Asymmetric, Flow and Bundle Event Structures

We next observe how different kinds of event structures, introduced for various purposes in the literature, can be naturally viewed as subclasses of the poset event structures in Definition 2. Verifying that the corresponding families of configurations satisfy the properties in the literature, can be naturally viewed as subclasses of the poset event structures in 2.3 Examples: Prime, Asymmetric, Flow and Bundle Event Structures.

Prime event structures. Prime event structures [22] are one of the simplest and most popular event structure models, where dependencies between events are captured in terms of causality and conflict.

- **Definition 6 (prime event structure).** A prime event structure (pes, for short) is a tuple \( P = (E, \leq, \#, \lambda) \), where \( E \) is a set of events, \( \leq \) and \( \# \) are binary relations on \( E \) called causality and conflict, respectively, and \( \lambda : E \rightarrow \Lambda \) is a labelling function, such that
  \begin{itemize}
    \item \( \leq \) is a partial order and \( \{x\} = \{y \in E \mid y \leq x\} \) is finite for all \( x \in E\);
    \item \( \# \) is irreflexive, symmetric and hereditary with respect to causality, i.e., for all \( x, y, z \in E\), if \( x \# y \) and \( y \leq z \) then \( x \# z \).
  \end{itemize}

  Configurations are sets of events without conflicts and closed with respect to causality. For later use, we also introduce a notation for the absence of conflicts, referred to as consistency.

- **Definition 7 (consistency, configuration).** Let \( P = (E, \leq, \#, \lambda) \) be a pes. We say that \( x, y \in E \) are consistent, written \( x \sim y \), when \( \neg(x \# y) \). A subset \( X \subseteq E \) is called consistent, written \( \sim X \), when its elements are pairwise consistent. A configuration of \( P \) is a finite set of events \( C \subseteq E \) such that (i) \( \sim C \) and (ii) for all \( x \in C \), \( \{x\} \subseteq C \).

  Some examples of pes can be found in Fig. 2. Causality is represented as a solid arrow, while conflict is represented as a dotted line. For instance, in \( P_0 \), event \( a_1 \) is a cause of \( b_1 \) and it is in conflict both with \( a_2 \) and \( b_3 \). Only direct causalities and non-inherited conflicts are
called causality is a tuple where both appear. This allows to represent faithfully the existence of precedences between actions and concurrent read accesses to a shared resource (intuitively, while readings can occur concurrently, destructive accesses can follow, but obviously not precede a reading).

Clearly pes can be seen as poset event structures. Given a pes $P = \langle E, \leq, \#, \lambda \rangle$ and its set of configurations $\text{Conf}(P)$, the local order of a configuration $C \in \text{Conf}(P)$ is $\leq_C = \leq \cap (C \times C)$, i.e., the restriction of the causality relation to $C$. The extension order turns out to be simply subset inclusion. In fact, given $C_1 \subseteq C_2$ clearly $\leq_{C_1} \subseteq \leq \cap (C_1 \times C_1)$ is the restriction to $C_1$ of $\leq_{C_2} = \leq \cap (C_2 \times C_2)$. Moreover, if $x_1 \in C_1$ and $x_2 \in C_2$, with $x_2 \leq x_2 x_1$, then necessarily $x_2 \in C_1$ since configurations are causally closed. As an example, the pes $P_2$ of Fig. 2 viewed as a poset event structure, can be found in Fig. 3.

Asymmetric event structures. Asymmetric event structures [5] are a generalisation of pes where conflict is allowed to be non-symmetric.

Definition 8 (asymmetric event structure). An asymmetric event structure (aes, for short) is a tuple $A = \langle E, \leq, \nearrow, \lambda \rangle$, where $E$ is a set of events, $\leq$ and $\nearrow$ are binary relations on $E$ called causality and asymmetric conflict, and $\lambda : E \rightarrow \Lambda$ is a labelling function, such that

- $\leq$ is a partial order and $[x] = \{y \in E \mid y \leq x\}$ is finite for all $x \in E$;
- $\nearrow$ satisfies, for all $x, y, z \in E$
  1. if $x < y$ then $x \nearrow y$;
  2. if $x \nearrow y$ and $y < z$ then $x \nearrow z$;
  3. $\nearrow$ is acyclic on $[x]$;
  4. $\nearrow$ is cyclic on $[x] \cup [y]$ then $x \nearrow y$.

In the graphical representation, asymmetric conflict is depicted as a dotted arrow. For instance, in the asymmetric event structure $A_0$ of Fig. 4 we have $a_{12} \nearrow b_{123}$. Again, only non inherited asymmetric conflicts are represented.

The asymmetric conflict relation has two natural interpretations, i.e., $x \nearrow y$ can be understood as (i) the occurrence of $y$ prevents $x$, or (ii) $x$ precedes $y$ in all computations where both appear. This allows to represent faithfully the existence of precedences between actions and concurrent read accesses to a shared resource (intuitively, while readings can occur concurrently, destructive accesses can follow, but obviously not precede a reading).

The interpretation of asymmetric conflict above should give some intuition for the conditions in Definition 8. Condition (1) naturally arises from interpretation (ii) above: when $x < y$ clearly $x$ precedes $y$ when both occur and thus $x \nearrow y$. Condition (2) is a form of hereditarity of asymmetric conflict along causality: if $x \nearrow y$ and $y < z$ then all runs where $x$ and $z$ appear, necessarily also include $y$, and $x$ precedes $y$ which in turn precedes $z$, hence $x \nearrow z$. Concerning (3) and (4), observe that events forming a cycle of asymmetric conflict cannot appear in the same run, since each event in the cycle should occur before itself in the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The configurations $\text{Conf}(P_2)$ of the pes $P_2$ in Fig. 2 viewed as poset event structures.}
\end{figure}
Minimisation of Event Structures

![Figure 4] Some asymmetric event structures

run. For instance, in the AES A₀ of Fig. 4, we have a₁ ↠ a₂ ↠ a₁, hence a₁ and a₂ cannot appear in the same computation. In this view, condition (3) corresponds to irreflexiveness of conflict in PESS, while condition (4) requires that binary symmetric conflict is explicitly represented by asymmetric conflict in both directions. Indeed, prime event structures can be identified with the subclass of AESs where ↠ is symmetric.

Configurations are again defined as causally closed and conflict-free sets of events.

**Definition 9 (AES configuration).** Let A = (E, ≤, ↠, λ) be an AES. A configuration of A is a finite set of events C ⊆ E such that (i) for any x ∈ C, [x] ⊆ C (causally closed) (ii) ↠ is acyclic on C (conflict-free).

Also AES can be seen as special poset event structures. Given an AES A = (E, ≤, ↠, λ) and its set of configurations Conf(A), the local order of a configuration C ∈ Conf(A) is ≤C = (↾ ↠ ↾ C)∗, i.e., the transitive closure of restriction of the asymmetric conflict to C. The prefix order on configurations is not simply set-inclusion: since a configuration C cannot be extended with an event which is prevented by some of the events already present in C. Hence for C₁, C₂ ∈ Conf(A) we have C₁ ⊍ C₂ iff C₁ ⊆ C₂ and for all x ∈ C₁, y ∈ C₂ \ C₁, ¬(y ↠ x). For instance, the configurations Conf(A₀) of A₀, ordered by prefix, can be obtained from Fig. 4 by replacing all occurrences of b₁₂ and b₁, by b₁₂₃. Note, e.g., that {b₁₂₃} ⊍ {a₁₂, b₁₂₃} since a₁₂ ↠ b₁₂₃.

**Flow event structures.** In some situations, it can be quite useful to have the possibility of modelling in a direct way the presence of multiple disjunctive and mutually exclusive causes for an event, something that is not possible in PESS and in AESs, where for each event there is a uniquely determined minimal set of causes. For instance, in a process calculus with non-deterministic choice “+” and sequential composition “;” in order to give a PES semantics to (a + b); c we are forced to use two different events to represent the execution of c, one for the execution of c after a and the other for the execution of c after b.

We briefly describe a model that overcomes this limitation, namely flow [8, 7] event structures.

**Definition 10 (flow event structure).** A flow event structure (FES) is a tuple (E, <, #, λ), where E is a set of events, < ⊆ E × E is an irreflexive relation called the flow relation, # ⊆ E × E is the symmetric conflict relation, and λ : E → Λ is a labelling function.

Causality is replaced by an irreflexive (in general non-transitive) flow relation <, intuitively representing immediate causal dependency. Moreover, conflict is no longer hereditary.

An event can have causes which are in conflict and these have a disjunctive interpretation, i.e., the event will be enabled by a maximal conflict-free subset of its causes. This is formalised by the notion of configuration.
Definition 11 (FES configuration). Let $F = (E, \prec, \#, \lambda)$ be an FES. A configuration of $F$ is a finite set of events $C \subseteq E$ such that (i) $\prec$ is acyclic on $C$, (ii) $\neg(x \# x')$ for all $x, x' \in C$ and (iii) for all $x \in C$ and $y \notin C$ with $y \prec x$, there exists $z \in C$ such that $y \# z$ and $z \prec x$.

Some examples of FESs can be found in Fig. 5. Relation $\prec$ is represented by a double headed solid arrow. For instance, consider the FES $F_1$. The set $C = \{a, d_01\}$ is a configuration. We have $b \prec d_01$ and $b \not\in C$, but this is fine since there is $a \in C$ such that $a \# b$ and $a \prec d_01$.

Under mild assumptions that exclude the presence of non-executable events (a condition referred to as fullness in [7]), FESs can be seen as poset event structures, by endowing each configuration $C$ with a local order arising as the reflexive and transitive closure of the restriction of the flow relation to $C$, i.e., $\leq_C = (\prec \cap (C \times C))^*$.

Bundle event structures. Bundle event structures [18, 19] are another event structure model that has been introduced in order to enable a direct representation of disjunctive causes, thus easing the definition of the semantics of the process description language LOTOS.

Definition 12 (bundle event structure). A bundle event structure is a triple $(E, \mapsto, \#)$, where $E$ is the denumerable set of events, $\# \subseteq E \times E$ is the (irreflexive) symmetric conflict relation and $\mapsto \subseteq 2^{\text{fin}}_E \times E$ is the bundle relation such that $X \times X \subseteq \#$.

Here a set of multiple disjunctive and mutually exclusive causes for an event is called a bundle set for the event, and comes into play as a primitive notion. The explicit representation of the bundles makes bundle event structures strictly less expressive than flow event structures. (see [19] for a wider discussion). On the other hand, bundle event structures offer the advantage of having a simpler theory. For instance, differently from what happens for flow event structures, non-executable events can be removed without affecting the behaviour of the event structure.

Configurations can be defined as conflict free sets of events that contain, for each event, a element from of each of its bundles. Formally, $C \subseteq E$ is a configuration if (i) the relation $\mapsto_C$ defined, for $x, y \in C$, by $x \mapsto_C y$ when $X \mapsto y$ and $x \in X$, is acyclic (ii) $\neg(x \# y)$ for all $x \in C$; (iii) $X \cap C \neq \emptyset$ for all $x \in C$. Endowing configurations with $\mapsto_C$ turns a bundle event structure into an event structure in the sense of Definition 2.

3 Foldings of Event Structures

In this section, we study a notion of folding, which is intended to formalise the intuition of a behaviour-preserving quotient for an event structure. We prove that there always exists a minimal quotient and we show that foldings between general poset event structures always arise, in a suitable formal sense, from foldings over prime event structures.
Minimisation of Event Structures

3.1 Morphisms and Foldings

We first endow event structures with a notion of morphism. Below, given two event structures \(E, E'\), a function \(f : E \to E'\) and a configuration \(C \in \text{Conf}(E)\), we write \(f(C)\) to refer to the configuration whose underlying set is \(\{f(x) \mid x \in C\}\), endowed with the order \(f(x) \leq f(y)\) iff \(x \leq y\).

Definition 13 (morphism). Let \(E, E'\) be event structures. A (strong) morphism \(f : E \to E'\) is a function between the underlying sets of events such that \(\lambda = \lambda' \circ f\) and for all configurations \(C \in \text{Conf}(E)\), the function \(f\) is injective on \(C\) and \(f(C) \in \text{Conf}(E')\).

Hereafter, the qualification “strong” will be omitted since this is the only kind of morphisms we deal with. It is motivated by the fact that normally morphisms on event structures are designed to represent simulations. If this were the purpose, then the requirement on preservation of configurations could have been weaker, i.e., we could have asked the order in the target configuration to be included in (not identical to) the image of the order of the source configuration (precisely, given a configuration \((C, \leq_C) \in \text{Conf}(E)\), then there exists \((C', \leq_{C'}) \in \text{Conf}(E')\) such that \(C' = f(C)\) and for all \(x, y \in C\), \(f(x) \leq_C f(y) \Rightarrow x \leq_C y\). Moreover, morphisms could have been partial. However, in our setting, for the objective of defining history-preserving quotients, the stronger notion works fine and simplifies the presentation.

Remark 14. The composition of morphisms is a morphism and the identity is a morphism. Hence the class of event structures and event structure morphisms form a category \(\text{ES}\).

Definition 15 (folding). Let \(E\) and \(E'\) be event structures. A folding is a morphism \(f : E \to E'\) such that the relation \(R_f = \{(C, f_C, f(C)) \mid C \in \text{Conf}(E)\}\) is a hhp-bisimulation.

In words, a folding is a function that “merges” some sets of events of an event structure into single event without altering the behaviour modulo hhp-bisimilarity. In [2] the notion of folding asks for the preservation of hp-bisimilarity, a weaker behavioural equivalence which is defined as hhp-bisimilarity but omitting the requirement of downward-closure. Note that, as far as the notion of folding is concerned, this makes no difference: \(R_f\) is downward-closed by definition, hence it is a hhp-bisimulation whenever it is a hp-bisimulation. Instead, taking hhp-bisimilarity as the reference equivalence appears to be the right choice for the development of the theory. E.g., it allows one to prove Lemma [B.6] that plays an important role for arguing about the adequateness of the notion of folding. Interestingly, foldings can be characterised as open maps in the sense of [16], by taking conflict free prime event structures as subcategory of observations. This is explicitly worked out in the appendix (Lemma [B.3]).

As an example, consider the \(\text{Pess}\) in Fig. 2 and the function \(f_{02} : P_0 \to P_2\) that maps events as suggested by the indices, i.e., \(f_{02}(a_1) = f_{02}(a_2) = a_{12}, f_{02}(b_1) = f_{02}(b_2) = b_{12}, f_{02}(b_3) = b_3\) and \(f_{02}(c) = c\). Then it is easy to see that \(f_{02}\) is a folding. Note that, instead, \(f_{01} : P_0 \to P_1\), again mapping events according to their indices, is not a folding. In fact, \(f_{01}(\{a_1\}) = \{a_{12}\} \xrightarrow{b_{12}} \{a_{12}, b_2\}\), but clearly there is no transition \(\{a_1\} \xrightarrow{b} \) with \(f_{01}(x) = b_2\), since the only counter-image of \(b_2\) in \(P_0\) is \(b_2\).

It is also interesting to observe that the greater expressiveness of \(\text{AES}\) allows one to obtain smaller quotients. For instance, while the \(\text{Pess}\) in Fig. 2 is minimal in the class of \(\text{Pess}\), if we view it as an \(\text{AES}\), it can be further reduced. In fact the obvious function from \(P_2\) to the \(\text{AES}\) \(A_0\) in Fig. 1 can be easily seen to be a folding.

Remark 16. The composition of foldings is a folding and the identity is a folding. We can consider a subcategory \(\text{ES}_f\) of \(\text{ES}\) with the same objects and foldings as morphisms (see Lemma [B.7] in the Appendix).
Again in the setting of $\text{aes}$, consider the structures in Fig. 4 and the functions $g_{12}: A_1 \rightarrow A_2$, and $g_{23}: A_2 \rightarrow A_3$, naturally induced by the indices. These can be seen to be foldings. The first one merges $c_1$, in conflict with $b$ and $c_2$ caused by $b$ to a single event $c_{12}$, in asymmetric conflict with $b$. The second one merges the two conflicting events $a_1$ and $a_2$ into a single one $a_{12}$. Their composition $g_{13} = g_{23} \circ g_{12}: A_1 \rightarrow A_3$ is again a folding.

Consider the FESS in Fig. 4. Again the obvious functions from $F_0$ to $F_1$ and $F_2$ can be seen to be foldings. Instead, seen as a PES, the event structure $F_0$ is minimal.

The next result shows that if we know that $f: E \rightarrow E'$ is a morphism, then half of the conditions needed to be a hps-bisimulation and thus folding, i.e., condition (1) in Definition 5 is automatically satisfied. This is used later in proofs whenever we need to show that some map is a folding.

Lemma 17 (from morphisms to foldings). Let $E$ and $E'$ be event structures and let $f: E \rightarrow E'$ be a morphism. If for all $C_1 \in \text{Conf}(E)$ and transition $f(C_1) \not\equiv x \rightarrow C_2'$ there exists $C_1 \not\equiv x \rightarrow C_2$ such that $f(C_2) = C_2'$ then $f$ is a folding.

A simple but crucial result shows that the target event structure for a folding is completely determined by the mapping on events. This allows us to view foldings as equivalences on the source event structures. We first define the quotient induced by a morphism.

Definition 18 (quotients from morphisms). Let $E$, $E'$ be event structures and let $f: E \rightarrow E'$ be a morphism. Let $\equiv_f$ be the equivalence relation on $E$ defined by $x \equiv_f y$ if $f(x) = f(y)$. We denote by $E_{\equiv_f}$ the event structure with configurations $\text{Conf}(E_{\equiv_f}) = \{[C]_{\equiv_f} \mid C \in \text{Conf}(E)\}$ where $[C]_{\equiv_f} = \{[x]_{\equiv_f} \mid x \in C\}$ is ordered by $[x]_{\equiv_f} \leq [y]_{\equiv_f}$ if $x \leq_C y$.

It is immediate to see that $E_{\equiv_f}$ is a well-defined event structure.

Lemma 19 (folding as equivalences). Let $E$, $E'$ be event structures and let $f: E \rightarrow E'$ be a morphism. If $f$ is a folding then $E_{\equiv_f}$ is isomorphic to $E'$.

The previous result allows us to identify foldings with the corresponding equivalences on the source event structures and motivates the following definition.

Definition 20 (folding equivalences). Let $E$ be an event structure. The set of folding equivalences over $E$ is $F\text{Eq}(E) = \{\equiv_f \mid f: E \rightarrow E' \text{ folding for some } E'\}$.

Hereafter, we will freely switch between the two views of foldings as morphisms or as equivalences, since each will be convenient for some purposes.

We next observe that given two foldings we can always take their “join”, providing a new folding that, roughly speaking, produces a smaller quotient than both the original ones.

Proposition 21 (joining foldings). Let $E$, $E'$, $E''$ be event structures and let $f': E \rightarrow E'$, $f'': E \rightarrow E''$ be foldings. Define $E'''$ as the quotient $E_{\equiv_{f'}}$ where $\equiv$ is the transitive closure of $\equiv_{f'} \cup \equiv_{f''}$. Then $g': E' \rightarrow E'''$ defined by $g'(x') = [x]_{\equiv}$ if $f'(x) = x'$ and $g'': E'' \rightarrow E'''$ defined by $g''(x'') = [x]_{\equiv}$ if $f''(x) = x''$ are foldings.

As an example, consider the PES in Fig. 2 and two morphisms $f_{30}: P_3 \rightarrow P_0$ and $f_{31}: P_3 \rightarrow P_1$. The way all events are mapped by $f_{30}$ and $f_{31}$ is naturally suggested by their labelling, apart for the $b_j$ for which we let $f_{30}(b_j) = b_1$ while $f_{31}(b_j) = b_j$. It can be seen that both are foldings. Their join, constructed as in Proposition 21, is $P_2$ with the folding morphisms $f_{02}: P_0 \rightarrow P_2$ and $f_{12}: P_1 \rightarrow P_2$. 
Minimisation of Event Structures

Remark 22. Proposition \[27\] is a consequence of the fact that the category \( \mathcal{ES} \) has pushouts of foldings. Indeed, \( \mathcal{E}'' \) as defined above is the pushout of \( f' \) and \( f'' \) (in \( \mathcal{ES} \) and also in \( \mathcal{ES}_f \)). It can be seen that, instead, \( \mathcal{ES} \) does not have all pushouts (see Fig. \[7\] in the Appendix for a counterexample).

When interpreted in the set of folding equivalences of an event structure, Proposition \[24\] has a clear meaning. Recall that the equivalences over some fixed set \( X \), ordered by inclusion, form a complete lattice, where the top element is the universal equivalence \( X \times X \) and the bottom is the identity on \( X \). Then Proposition \[24\] implies that \( \mathcal{FEq}(\mathcal{E}) \) is a sublattice of the lattice of equivalences. Actually, it can be shown that \( \mathcal{FEq}(\mathcal{E}) \) is itself a complete lattice. Therefore each event structure \( \mathcal{E} \) admits a maximally folded version.

Proposition 23 (lattice of foldings). Let \( \mathcal{E} \) be an event structure. Then \( \mathcal{FEq}(\mathcal{E}) \) is a sublattice of the complete lattice of equivalence relations over \( \mathcal{E} \).

Remark 24. The above result arises from a generalisation of Proposition \[23\] showing that, for any event structure \( \mathcal{E} \), each collection of foldings \( f_i : \mathcal{E} \to \mathcal{E}_i \), with \( i \in I \), admits a colimit in \( \mathcal{ES} \). Thus the coslice category \( (\mathcal{E} \downarrow \mathcal{ES}_f) \) has a terminal object, which is the maximally folded event structure.

It is natural to ask whether all behaviour preserving quotients correspond to foldings. Strictly speaking, the answer is negative. More precisely, there can be morphisms \( \mathcal{f} : \mathcal{E} \to \mathcal{E}' \) such that \( \mathcal{E}_{/\mathcal{f}} \) is hhp-bisimilar to \( \mathcal{E} \), but \( \mathcal{f} \) is not a folding. For an example, consider the \( \mathcal{PES} \) \( P_0 \) and \( P_1 \) in Fig. \[2\] and the morphism \( f_{01} : P_0 \to P_1 \) suggested by the indexing. We already observed this is not a folding, but \( P_{0/\mathcal{f}_{01}} \), which is isomorphic to \( P_1 \), is hhp-bisimilar to \( P_0 \).

However, we can show that for any behaviour preserving quotient, there is a folding that produces a coarser equivalence, and thus a smaller quotient. For instance, in the example discussed above, there is the folding \( f_{02} : P_0 \to P_2 \), that “produces” a smaller quotient.

This follows from the possibility of joining foldings (Proposition \[21\]) and the fact that a hhp-bisimulation can be always seen as an event structure, a result that generalises to our setting a property proved for \( \mathcal{PES} \) in \[6\].

Proposition 25 (foldings subsume behavioural quotients). Let \( \mathcal{E} \) be an event structure and let \( \mathcal{f} : \mathcal{E} \to \mathcal{E}' \) be a morphism such that \( \mathcal{E}_{/\mathcal{f}} \) is hhp-bisimilar to \( \mathcal{E} \). Then there exists a folding \( \mathcal{g} : \mathcal{E} \to \mathcal{E}'' \) such that \( \equiv_g \) is coarser than \( \equiv_f \).

3.2 Folding through Prime Event Structures

Here we observe that each event structure is the folding of some canonical \( \mathcal{PES} \). We then prove that, interestingly enough, all foldings between event structures arise from foldings of the corresponding canonical \( \mathcal{PES} \).

We start with the definition of the canonical \( \mathcal{PES} \) associated with an event structure.

Definition 26 (\( \mathcal{PES} \) for an event structure). Let \( \mathcal{E} \) be an event structure. Its canonical \( \mathcal{PES} \) is \( \mathcal{P}(\mathcal{E}) = (\mathcal{Hist}(\mathcal{E}), \subseteq, \#, \lambda') \) where \( \subseteq \) is prefix, \( \# \) is inconsistency, i.e., for \( H_1, H_2 \in \mathcal{Hist}(\mathcal{E}) \) we let \( H_1 \# H_2 \) if \( \neg(H_1 \sim H_2) \) and \( \lambda'(H) = \lambda(x) \) when \( H \in \mathcal{Hist}(x) \). Given a morphism \( \mathcal{f} : \mathcal{E} \to \mathcal{E}' \) we write \( \mathcal{P}(\mathcal{f}) : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}') \) for the morphism defined by \( \mathcal{P}(\mathcal{f})(H) = f(H) \).

It can be easily seen that the definition above is well-defined. In particular, \( \mathcal{P}(\mathcal{E}) \) is a well-defined \( \mathcal{PES} \) because, as proved in \[24\], a family of posets ordered by prefix is finitary coherent prime algebraic domain. Then the tight relation between this class of domains and
Pes highlighted in [31] allows one to conclude the proof. For instance, in Fig. 1 (right) one can find the canonical pes for the event structure on the left.

The canonical pes associated with an event structure can always be folded to the original event structure.

$\blacktriangleright$ Lemma 27 (unfolding event structures to pes’s). Let $E$ be an event structure. Define a function $\phi_E : \mathcal{P}(E) \to E$, for all $H \in \text{Hist}(E)$ by $\phi_E(H) = x$ if $H \in \text{Hist}(x)$ for $x \in E$. Then $\phi_E$ is a folding.

We next show that any morphism and any folding from a pes to an event structure $E$ factorises uniquely through the pes $\mathcal{P}(E)$ associated with $E$ (categorically, $\phi_E$ is cofree over $E$). This will be useful to relate foldings in $E$ with foldings in $\mathcal{P}(E)$.

$\blacktriangleright$ Lemma 28 (cofreeness of $\phi_E$). Let $E$ be an event structure, let $P'$ be a pes and let $f : P' \to E$ be an event structure morphism. Then there exists a unique morphism $g : P' \to \mathcal{P}(E)$ such that $f = \phi_E \circ g$.

Moreover, when $f$ is a folding then so is $g$.

$\blacktriangleright$ Remark 29. Lemma 28 means that the category PES of prime event structures is a coreflective subcategory of ES, i.e., $\mathbb{P} : \text{ES} \to \text{PES}$ can be seen as a functor, right adjoint to the inclusion $I : \text{PES} \to \text{ES}$. Moreover, $\mathbb{P}$ restricts to a functor on the subcategory of foldings, $\mathbb{P} : \text{ES}_f \to \text{PES}_f$, where an analogous result holds.

We conclude that all foldings between event structures arise from foldings of the associated pess. Given that PES is a coreflective subcategory of ES and foldings can be seen as open maps, this result (and also the fact that morphisms $\phi_E$ are foldings) can be derived from [16, Lemma 6]. The appendix gives more details on this point (and also reports a direct proof).

$\blacktriangleright$ Proposition 30 (folding through pess). Let $E, E'$ be event structures. For all morphisms $f : E \to E'$ consider $\mathbb{P}(f) : \mathcal{P}(E) \to \mathcal{P}(E')$ defined by $\mathbb{P}(f)(H) = f(H)$. Then $f$ is a folding iff $\mathbb{P}(f)$ is a folding.

4 Foldings for Prime and Asymmetric Event Structures

In this section we study foldings on specific subclasses of poset event structures, providing suitable characterisations. Motivated by the fact that foldings on general poset event structures always arise from foldings of the corresponding canonical pes we first and mainly focus on pess. Then we discuss how this can be extended to asymmetric event structures (and only give a hint to flow and bundle event structures). We will see that while pess admit a least folding, the other classes of event structures do not.

4.1 Folding Prime Event Structures

Since foldings are special morphisms, we first provide a characterisation of pess morphisms.

$\blacktriangleright$ Lemma 31 (pess morphisms). Let $P$ and $P'$ be pess and let $f : P \to P'$ be a function on the underlying sets of events. Then $f$ is a morphism iff for all $x, y \in P$
Minimisation of Event Structures

1. \( \lambda(f(x)) = \lambda(x) \);
2. \( f([x]) = [f(x)] \); namely (a) for all \( x' \in P' \), if \( x' \leq f(y) \) there exists \( x \in P \) such that \( x \leq y \) and \( f(x) = x' \); (b) if \( x \leq y \) then \( f(x) \leq f(y) \);
3. (a) if \( f(x) = f(y) \) and \( x \neq y \) then \( x \neq y \) and (b) if \( f(x) \neq f(y) \) then \( x \neq y \).

These are the standard conditions characterising (total) \( \text{PEs} \) morphisms (see, e.g., [31]), with the addition of condition (2b) that is imposed to ensure that configurations are mapped to isomorphic configurations, as required by the notion of (strong) morphism (Definition 13).

We know that not all \( \text{PEs} \) morphisms are foldings. We next identify some additional conditions characterising those morphisms which are foldings.

\[ \text{Proposition 32 (PEs foldings). Let } P \text{ and } P' \text{ be } \text{PEs} \text{ and let } f : P \to P' \text{ be a morphism. Then } f \text{ is a folding if and only if it is surjective and for all } X, Y \subseteq P, x, y \in P, y' \in P' \]

1. \( x \neq y \to f(x) \neq y' \);
2. \( \forall (X \cup \{x\}), \forall (Y \cup \{y\}), \forall (X \cup Y) \text{ and } f(x) = f(y) \text{ then there exists } z \in P \text{ such that } f(z) = f(x) \text{ and } \forall (X \cup Y \cup \{z\}) \).

The notion of folding on \( \text{PEs} \) turns out to be closely related to that of abstraction homomorphism for \( \text{PEs} \) introduced in [10] for similar purposes. More precisely, abstraction homomorphisms can be characterised as those \( \text{PEs} \) morphisms additionally satisfying condition 1 of Proposition 32 while they do not necessarily satisfy condition 2. Their more liberal definition is explained by the fact that they are designed to work on a subclass of structured \( \text{PEs} \) (see Lemma D.2 in the Appendix).

We finally show what the conditions characterising foldings look like when transferred to equivalences.

\[ \text{Corollary 33 (folding equivalences for } \text{PEs}). \text{ Let } P \text{ be a } \text{PEs and let } \equiv \text{ be an equivalence on } P. \text{ Then } \equiv \text{ is a folding equivalence in } \text{FEq}(P) \text{ iff for all } x, y \in P, \text{ if } x \equiv y \text{ then}
\]

1. \( \lambda(x) = \lambda(y) \);
2. \( [x]_\equiv = [y]_\equiv \);
3. \( x \neq y \).

Moreover, for all \( x, y \in P, X, Y \subseteq P \)

4. \( x \neq y \Rightarrow [x]_\equiv \neq [y]_\equiv \);
5. \( \forall (X \cup \{x\}), \forall (Y \cup \{y\}), \forall (X \cup Y) \text{ there exists } z \in [x]_\equiv \text{ such that } \forall (X \cup Y \cup \{z\}) \).

For instance, in Fig. 2 consider the equivalence \( \equiv_{01} \) over \( P_0 \) such that \( a_1 \equiv_{01} a_2 \). This produces \( P_3 \) as quotient. This is not a folding equivalence since condition (4) fails: a\( a_1 \neq b_2 \equiv_{01} \), but \( \neg(a_2 \neq b_2) \) and thus \( \neg([a_1]_\equiv \neq [b_2]_\equiv) \). Instead, the equivalence \( \equiv_0 \) over \( P_0 \) such that \( a_1 \equiv_0 a_2 \) and \( b_1 \equiv_0 b_2 \), producing \( P_2 \) as quotient, satisfies all five conditions.

When \( \text{PEs} \) are finite, the result above suggests a possible way of identifying foldings: one can pair candidate events to be folded on the basis of conditions (1)-(3) and then try to extend the sets with condition (4)-(5) when possible. The procedure can be inefficient due to the global flavor of the conditions. This will be further discussed in the conclusions.

We know from Proposition 21 that all event structures admit a “maximally folded” version. We next observe that the same result holds in the class of \( \text{PEs} \), i.e., that for each \( \text{PEs} \) there is a uniquely determined minimal quotient.

\[ \text{Lemma 34 (Joining foldings on } \text{PEs}). \text{ Let } P, P', P'' \text{ be } \text{PEs and let } f' : P \to P', f'' : P \to P'' \text{ be foldings. Define } E'' \text{ along with } g' : P' \to E'' \text{ and } g'' : P'' \to E'' \text{ as in Proposition 21. Then } E'' \text{ is a } \text{PEs}. \]

\[ \text{Remark 35. Lemma 34 is a consequence of the fact that the subcategory } \text{PEs}_{\text{f}} \text{ is a coreflective subcategory of } \text{ES}_{\text{f}} \text{ and thus it is closed under pushouts.} \]
4.2 Folding Asymmetric Event Structures

We know that foldings on all poset event structures arise from foldings on the corresponding canonical PESS. Still, for theoretical purposes and for efficiency reasons, a direct approach, not requiring the generation of the associated PESS, can be of interest. Here we explicitly discuss the case of asymmetric event structures. This generalises the results in [2] that identify conditions which are only sufficient and apply to a subclass of foldings (the so-called called elementary foldings, merging a single set of events). Note also that, despite the fact that in this paper we work in a slightly different framework, we continue to have that, as observed in [2], AESS (and also PESS) do not admit a unique minimal quotient in general.

We first characterise morphisms in the sense of Definition [19] on AESS.

Lemma 36 (AES morphisms). Let A and A' be AESS and let f : A → A' be a function on the underlying sets of events. Then f is a morphism if and only if for all x, y ∈ A, x ≠ y
1. λ(⟨f(x)⟩) = λ(x);
2. ⟨f([x])⟩ ⊆ f([x]);
3. (a) if f(x) ∨ y then x ∨ y and (b) if x ∨ y and ¬(y ∨ x) then f(x) ∨ f(y);
4. if f(y) = y then x ∨ y.

These are the standard conditions characterising (total) AES morphisms (see [5]), with the addition of (3b), needed in order to ensure that configurations are mapped to isomorphic configurations.

Proposition 37 (AES foldings). Let A and A' be AESS and let f : A → A' be a morphism. Then f is a folding if and only if it is surjective and for all X, Y ⊆ A, x, y ∈ A with x ∉ X, y ∉ Y
1. if f−1(y') ∨ x then y' ∨ f([x]);
2. if ¬(x ∨ X), ¬(y ∨ Y) and f(x) = f(y) then there exists z ∈ A such that f(z) = y and f(z) ∨ X, f(z) ∨ Y.
3. given H ∈ Hist(x), if ¬(H ∨ X), and H ⊆ H such that f(H) ∨ f(x) ∈ Hist(f(x)) there exists x₁ such that H₁ ∩ {x₁} ∈ Hist(x₁) and ¬(x₁ ∨ X).

We already observed that working in the class of AESS we can obtain smaller quotients than in the class of PESS (see, e.g., the hhp-bisimilar structures P₂ in Fig. 2 and A₀ in Fig. 4). However, not unexpectedly, the folding criteria for AESS are less elegant and more complex than those for PESS. In a practical use, the reference to histories could cause a loss of efficiency. Moreover, the uniqueness of the minimal quotient is lost. Consider for instance the AESS in Fig. 6. It can be seen that h₀₁ : A₀ → A₁ is a folding where the events c₁, caused by a and c₀ in conflict with a, are merged in a single event c₀₁ in asymmetric conflict with a. Similarly, h₀₂ : A₀ → A₂ is a folding obtained by merging c₀ and c₂. These are two minimal foldings that do not admit a join in the class of AESS. In fact, if we merge all three c-labelled events we obtain A₃, and it is easy to see that the function h₀₃ : A₀ → A₃ is not a folding. In fact, consider {a, b} ∈ Conf(A₀). Then h₀₃({a, b}) = {a, b} →₁₂, a transition that cannot be simulated in A₀. Indeed, it can be seen that the join of h₀₁ and h₀₂ is the event structure E in Fig. 4 (right), which cannot be represented as an AESS.

In passing, we note that also in the class of PESS the existence of minimal foldings is lost. In fact, consider Fig. 5. It can be easily seen that F₁ and F₂ are different minimal foldings of F₀. In particular, merging the three d-labelled events as in F₃ modifies the behaviour. In fact, in F₃, the event d₀₁₂ is not enabled in C = {a} since c ∼ d₀₁₂ and no event in C is in conflict with c. Instead, in F₀, the event d₁₂ is clearly enabled from {a}.

Existence of a unique minimal folding could be possibly recovered by strengthening the notion of folding and, in particular, by requiring that foldings preserve and reflect histories.
16 Minimisation of Event Structures

![Diagram of event structures](image)

Figure 6 Asymmetric event structures do not admit a minimal quotient

Note, however, that this would be against the spirit of our work where the notion of folding is not a choice. Rather, after having assumed hhp-bisimilarity as the reference behavioural equivalence, the notion of folding is essentially “determined” as a quotient (surjective function) that preserves the behaviour up to hhp-bisimilarity.

5 Conclusions

We studied the problem of minimisation for poset event structures, a class that encompasses many stable event structure models in the literature, taking hereditary history preserving bisimilarity as reference behavioural equivalence. We showed that a uniquely determined minimal quotient always exists for poset event structures and also in the subclass of prime event structures, while this is not the case for various models extending prime event structures. We showed that foldings between general poset event structures arise from foldings of corresponding canonical prime event structures. Finally, we provided a characterisation of foldings of prime event structures, and discussed how this could be generalised to other classes, developing explicitly the case of asymmetric event structures.

As underlined throughout the paper, our theory of folding has many connections with the literature on event structures. The idea of “unfolding” more expressive models to prime algebraic domains and prime event structures has been studied by many authors (e.g., in [24, 22, 27, 28, 8]). The same can be said for the idea of refining a single action into a complex computation (see, e.g., [26] and references therein). Instead, the problem of minimisation of event structures has received less attention. We already commented on the relation with the notion of abstraction homomorphisms for pess [10], which captures the idea of behaviour preserving abstraction in a subclass of structured pess. In some cases, given a Petri net or an event structure a special transition system can be extracted, on which minimisation is performed. In particular, in [21] the authors propose an encoding of safe Petri nets into causal automata, in a way that preserves hp-bisimilarity. The causal automata can be transformed into a standard labelled transition system, which in turn can be minimised. However, in this way, the correspondence with the original events is lost.

The notion of behaviour preserving function has been given an elegant abstract characterisation in terms of open maps [16]. In the paper, we mentioned the possibility of viewing our foldings as open maps and we observed that various results admit a categorical interpretation. This gives clear indications of the possibility of providing a general abstract view of the results in this paper, something which represents an interesting topic of future research.

The characterisation of foldings on prime (and asymmetric) event structures can be used as a basis to develop, at least in the case of finite structures, an algorithm for the definition of behaviour preserving quotients. The fact that conditions for folding refer to sets of events might make the minimisation procedure very inefficient. Determining suitable heuristics for the identification of folding sets and investigating the possibility of having more “local” conditions characterising foldings are interesting directions of future development.
Although not explicitly discussed in the paper, considering elementary foldings, i.e., foldings that just merge a single set of events, one can indeed determine some more efficient folding rules. This is essentially what is done for \textit{aess} and \textit{fess} in \cite{2}. However, restricting to elementary foldings is limitative, since it can be seen that general foldings cannot be always decomposed in terms of elementary ones (e.g., it can be seen that in Fig. \ref{fig:example}, the folding $f_{02} : P_0 \rightarrow P_2$ cannot be obtained as the composition of elementary foldings).

When dealing with possibly infinite event structures one could work on some finitary representation and try to devise reduction rules acting on the representation and inducing foldings on the corresponding event structure. Observe that working, e.g., on finite safe Petri nets, the minimisation procedure would be necessarily incomplete, given that \textit{hhp-bisimilarity} is known to be undecidable \cite{17}.

References

1. A. Armas-Cervantes, P. Baldan, M. Dumas, and L. García-Bañuelos. Diagnosing behavioral differences between business process models: An approach based on event structures. \textit{Information Systems}, 56:304–325, 2016.
2. A. Armas-Cervantes, P. Baldan, and L. García-Bañuelos. Reduction of event structures under history preserving bisimulation. \textit{Journal of Logical and Algebraic Methods in Programming}, 85(6):1110–1130, 2016.
3. P. Baldan. \textit{Modelling concurrent computations: from contextual Petri nets to graph grammars}. PhD thesis, University of Pisa, 2000.
4. P. Baldan, A. Corradini, H. Ehrig, M. Löwe, U. Montanari, and F. Rossi. Concurrent semantics of algebraic graph transformation systems. In G. Rozenberg, editor, \textit{Handbook of Graph Grammars and Computing by Graph Transformation}, volume III: Concurrency, pages 107–187. World Scientific, 1999.
5. P. Baldan, A. Corradini, and U. Montanari. Contextual Petri nets, asymmetric event structures and processes. \textit{Information and Computation}, 171(1):1–49, 2001.
6. M. A. Bednarczyk. Hereditary history preserving bisimulations or what is the power of the future perfect in program logics. Technical report, Polish Academy of Sciences, 1991.
7. G. Boudol. Flow Event Structures and Flow Nets. In \textit{Semantics of System of Concurrent Processes}, volume 469 of \textit{LNCS}, pages 62–95. Springer Verlag, 1990.
8. G. Boudol and I. Castellani. Permutation of transitions: an event structure semantics for CCS and SCCS. In \textit{Linear Time, Branching Time and Partial Order Semantics in Logics and Models for Concurrency}, volume 354 of \textit{LNCS}, pages 411–427. Springer Verlag, 1988.
9. R. Bruni, H. C. Melgratti, and U. Montanari. Event structure semantics for nominal calculi. In C. Baier and H. Hermanns, editors, \textit{CONCUR 2006}, volume 4137 of \textit{LNCS}, pages 295–309. Springer, 2006.
10. Ilaria Castellani. \textit{Bisimulations for Concurrency}. PhD thesis, University of Edinburgh, 1988.
11. S. Chakraborty and V. Vafeiadis. Grounding thin-air reads with event structures. \textit{PACMPL}, 3(POPL):70:1–70:28, 2019.
12. M. Dumas and L. García-Bañuelos. Process mining reloaded: Event structures as a unified representation of process models and event logs. In Raymond R. Devillers and Antti Valmari, editors, \textit{Petri Nets 2015}, volume 9115 of \textit{LNCS}, pages 33–48. Springer, 2015.
13. H. Herrlich and G. E. Strecker. Coreflective subcategories. \textit{Transactions of the American Mathematical Society}, 157:205–226, 1971.
14. John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman. \textit{Introduction to automata theory, languages, and computation - international (2nd ed.)}. Addison-Wesley, 2003.
15. A. Jeffrey and J. Riely. On thin air reads: Towards an event structures model of relaxed memory. In Martin Grohe, Eric Koskinen, and Natarajan Shankar, editors, \textit{LICS 2016}, pages 759–767. ACM, 2016.
A. Joyal, M. Nielsen, and G. Winskel. Bisimulation from open maps. Information and Computation, 127(2):164–185, 1996.

M. Jurdzinski, M. Nielsen, and J. Srba. Undecidability of domino games and hhp-bisimilarity. Information and Computation, 184(2):343–368, 2003.

R. Langerak. Bundle Event Structures: A Non-Interleaving Semantics for LOTOS. In 5th Intl. Conf. on Formal Description Techniques (FORTE’92), pages 331–346. North-Holland, 1992.

R. Langerak. Transformation and Semantics for LOTOS. PhD thesis, Department of Computer Science, University of Twente, 1992.

Albert R. Meyer and Larry J. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential space. In SWAT (FOCS), pages 125–129. IEEE Computer Society, 1972.

R. Langerak. History preserving process graphs. Draft available at http://theory.stanford.edu/~rvg/abstracts.html#hppg, 1996.

Rob van Glabbeek and Gordon D. Plotkin. Configuration structures, event structures and Petri nets. Theoretical Computer Science, 410(41):4111–4159, 2009.

G. Winskel. Event structure semantics for CCS and related languages. Technical Report DAIMI PB-159, University of Aarhus, 1983.

G. Winskel. Event Structures. In Petri Nets: Applications and Relationships to Other Models of Concurrency, volume 255 of LNCS, pages 325–392. Springer, 1987.

G. Winskel. Events, causality and symmetry. Computer Journal, 54(1):42–57, 2011.

Lemma A.1 (properties of histories). Let \( E \) be an event structure. Then

1. for all \( C \in \text{Conf}(E) \), we have \( C[x] \subseteq C \), hence \( C[x] \in \text{Conf}(E) \);
2. for all \( C_1, C_2 \in \text{Conf}(E) \), \( C_1 \subseteq C_2 \) iff for all \( x \in C_1 \), \( C_1[x] = C_2[x] \);
3. for all \( H_1, H_2 \in \text{Hist}(x) \), if \( H_1 \sim H_2 \) then \( H_1 = H_2 \);

Proof. 1. Immediate by the definition of \( C[x] \).
2. Let \( C_1, C_2 \in \text{Conf}(E) \) such that \( C_1 \subseteq C_2 \). For all \( x \in C_1 \) we have that

\[
C_2[x] = \{ y \in C_2 \mid y \leq_{C_2} x \} \\
= \{ y \in C_1 \mid y \leq_{C_1} x \} \quad \text{[since \( C_1 \subseteq C_2 \)]}
\]

\[
= C_1[x]
\]
Conversely, assume that for all \( x \in C_1 \) we have that \( C_1[x] = C_2[x] \). Then, since \( x \in C_1[x] \), for \( i \in \{1, 2\} \), clearly \( C_1 \subseteq C_2 \). Moreover, for all \( y \in C_1 \) and \( x \in C_2 \), if \( x \leq_{C_2} y \) then \( x \in C_2[y] \). Therefore, since by hypothesis \( C_1[y] = C_2[y] \), we have \( x \in C_1 \) and \( x \leq_{C_1} y \), as desired. Therefore, \( C_1 \subseteq C_2 \).

3. Let \( H_1, H_2 \in \text{Hist}(x) \) and assume that \( H_1 \sim H_2 \). This means that there exists \( C \in \text{Conf}(E) \) such that \( H_1, H_2 \subseteq C \). Therefore, by point (2), we have \( H_1 = H_1[x] = C[x] = H_2[x] = H_2 \).

\[ \square \]

Lemma A.2 (configurations are reachable). Let \( E \) be an event structure and let \( C \in \text{Conf}(E) \) be a configuration. Then \( \emptyset \rightarrow^\ast C \). More in detail, if \( x_1, x_2, \ldots, x_n \) is any linearisation of \( C \) compatible with \( \leq_{E} \) then, for all \( k \in \{1, \ldots, n\} \), \( \{x_1, \ldots, x_{k-1}\} \xrightarrow{x_k} \{x_1, \ldots, x_{k-1}, x_k\} \).

Proof. Immediate consequence of the prefix-closedness of the family of configurations. \[ \square \]

B Proofs for Section 3 (Foldings of Event Structures)

Lemma B.1 (foldings are closed under composition). Let \( E, E', E'' \) be event structures and let \( f : E \rightarrow E' \) and \( f' : E' \rightarrow E'' \) be foldings. Then \( f' \circ f : E \rightarrow E'' \) is a folding.

Proof. We rely on the characterisation of foldings provided in Lemma 17. Let \( C_1 \in \text{Conf}(E) \) and assume that \( f'(f(C_1)) \xrightarrow{x'} C_2'' \). Since \( f(C_1) \in \text{Conf}(E') \) and \( f' \) is a folding, there exists \( x' \) such that \( f(C_1) \xrightarrow{x'} C_2' \) with \( f'(x') = x'' \) and \( f'(C_2') = C_2'' \). In turn, since \( f \) is a folding, from \( f(C_1) \xrightarrow{\ell} C_2' \), we derive the existence of a transition \( C_1 \xrightarrow{\ell} C_2 \) with \( f(x) = x' \) and \( f(C_2) = C_2' \). Therefore \( f'(f(x)) = x'' \) and \( f'(f(C_2)) = C_2'' \), as desired. \[ \square \]

Lemma 17 (from morphisms to foldings). Let \( E \) and \( E' \) be event structures and let \( f : E \rightarrow E' \) be a morphism. If for all \( C_1 \in \text{Conf}(E) \) and transition \( f(C_1) \xrightarrow{x} C_2 \) there exists \( C_1 \xrightarrow{\ell} C_2 \) such that \( f(C_2) = C_2'' \) then \( f \) is a folding.

Proof. We have to show that \( R_f = \{(C, f(C)) \mid C \in \text{Conf}(E) \} \) satisfies conditions (1) and (2) of Definition 5. Condition (2) is in the hypotheses. Concerning (1), let \( C_1 \in \text{Conf}(E) \) and consider a transition \( C_1 \xrightarrow{\ell} C_2 \). Then by definition of morphism, \( f(C_1) \in \text{Conf}(E) \) and isomorphic to \( C_i \), for \( i \in \{1, 2\} \). Therefore \( f(C_1) \xrightarrow{\ell} f(C_2) \). \[ \square \]

Relying on Lemma 17 we can derive that foldings arise as open maps in the sense of 16.

Definition B.2 (open map). Let \( M \) be a category and let \( C \) be a subcategory of \( M \). A morphism \( f : M \rightarrow M' \) is \( C \)-open if for all morphisms \( e : C \rightarrow C' \) and commuting square\[ \begin{array}{ccc} C & \xrightarrow{e} & E \\ \downarrow^{e''} & & \downarrow^{f} \\ C' & \xrightarrow{e'} & E' \end{array} \]
there exists a morphism \( e'' : C' \rightarrow E \) such that the two triangles commute.

Let \( \text{Pom} \) denote the subcategory of \( \text{ES} \) having conflict-free \( \text{PESs} \) as objects and injective morphisms as arrows. Then Lemma 17 leads to show that foldings are \( \text{Pom} \)-open morphisms in \( \text{ES} \), generalising to our setting a result proved for prime event structures in 10.

Lemma B.3 (foldings as open maps). Let \( E, E' \) be event structures and let \( f : E \rightarrow E' \) be a morphism. Then \( f \) is a folding if and only if \( f \) is \( \text{Pom} \)-open.
20 Minimisation of Event Structures

**Proof.** Let \( f \) be a folding. In order to prove that \( f \) is a Pom-open map, assume to have a commuting square as in Definition B.2 Since \( C \) is a conflict-free prime event structures, its set of events, ordered by causality, which abusing the notation, we still denote by \( C \) is a configuration. Since \( e \) is a morphism \( e(C) \in \text{Conf}(E) \) and \( e(C) \simeq C \), and thus \( f(e(C)) \in \text{Conf}(E') \) and \( f(e(C)) \simeq C \). Similarly, \( e'(C') \in \text{Conf}(E') \) and \( e'(C') \simeq C' \). Finally observe that \( e(C) \subseteq C' \). Thus \( e'(e(C)) = f(e(C)) \subseteq e'(C') \), meaning that \( f(e(C)) \xrightarrow{X'} f(e'(C')) \) for a suitable \( X' \). By definition of folding, there must be a transition \( c \in C \xrightarrow{X} D \) such that \( f(D) = e'(C') \). Therefore, we can define \( e'' : C'' \rightarrow E \) as follows: for all \( x' \in C' \), let \( e''(x') \) be the unique \( y \in D \) such that \( f(y) = e'(x') \).

Conversely, assume that \( f \) is an Pom-open map. We show that \( f \) satisfies the condition of Lemma 17 Let \( C_1 \in \text{Conf}(E) \) and consider a transition \( f(C_1) \xrightarrow{c} C_2' \). If we view configurations \( C_1, C_2' \) as pomsets, then we can build the following commuting square

\[
\begin{array}{ccc}
C_1 & \rightarrow & E \\
\downarrow f_{|C_1} & & \downarrow f \\
C_2' & \rightarrow & E'
\end{array}
\]

By the fact that \( f \) is open, we get the morphism \( e'' \), and it is immediate to see that \( C_1 \xrightarrow{\approx} e''(C_2') \) is the desired transition that completes the proof. □

**Lemma 19** (folding as equivalences). Let \( E, E' \) be event structures and let \( f : E \rightarrow E' \) be a morphism. If \( f \) is a folding then \( E_{/f_{|E}} \) is isomorphic to \( E' \).

**Proof.** Consider the function \( g : E_{/f_{|E}} \rightarrow E' \) defined by \( g([x]_{/f_{|E}}) = f(x) \). It is well defined, since all elements in \([x]_{/f_{|E}}\) have the same \( f \)-image, and clearly injective. Moreover, it is also surjective. In fact, if \( x' \in E' \) then there exists \( C' \in \text{Conf}(E') \) such that \( x' \in C' \). By Lemma A.2 configuration \( C' \) is reachable from the empty one, and thus, since \( f \) is an hp-bisimulation, there exists \( C \in \text{Conf}(E) \) such that \( C' = f(C) \). Therefore there is \( e \in C \) such that \( f(x) = x' \) and thus \( g([x]_{/f_{|E}}) = x' \).

Finally, observe that by definition, for all configuration \( C' \in \text{Conf}(E_{/f_{|E}}) \), we have \( g(C') \simeq C' \), hence we conclude. □

**Lemma B.4** (factorising morphisms). Let \( E, E', E'' \) be event structures and let \( f : E'' \rightarrow E' \) be a morphism and \( h : E'' \rightarrow E \) be a folding. Let \( g : E \rightarrow E' \) be a function such that \( f = g \circ h \).

\[
\begin{array}{ccc}
E & \xrightarrow{g} & E' \\
\downarrow h & & \downarrow f \\
E'' & \rightarrow & E'
\end{array}
\]

Then \( g \) is a morphism. Moreover, if \( f \) is a folding then \( g \) is.

**Proof.** Let us show that \( g \) is a morphism. For all \( C \in \text{Conf}(E) \), since \( h \) is a folding, there exists \( C'' \in \text{Conf}(E'') \) such that \( h(C'') = C \) and \( C'' \simeq C \). Since \( f \) is a morphism \( f(C'') \in \text{Conf}(E') \). Therefore \( g(C) = g(h(C'')) = f(C'') \), as desired.

Let assume now that \( g \) is a folding. Let \( C_1 \in \text{Conf}(E) \) and suppose that there is a transition \( g(C_1) \xrightarrow{c} C_2' \). Since \( h \) is a folding, there is a configuration \( C'' \in \text{Conf}(E'') \) such that \( C_1 = h(C'') \). Therefore \( f(C''_1) = g(h(C'')) = g(C_1) \xrightarrow{c} C_2' \). Since \( f \) is a folding there is a transition \( C''_1 \xrightarrow{c''} C''_2 \) with \( f(C''_2) = C_2 ' \). Therefore \( h(C''_2) = C_1 \xrightarrow{h(c'')} g(h(C''_2)) \) with \( g(h(C''_2)) = f(C_2) = C_2' \), as desired. □
Proposition 21 (joining foldings). Let $E, E', E''$ be event structures and let $f' : E \to E'$, $f'' : E \to E''$ be foldings. Define $E'''$ as the quotient $E_{f''} = \equiv$ is the transitive closure of $\equiv_f \cup \equiv_{f''}$. Then $g' : E' \to E'''$ defined by $g'(x') = [x]_\equiv$ if $f'(x) = x'$ and $g'' : E'' \to E'''$ defined by $g''(x'') = [x]_\equiv$ if $f''(x) = x''$ are foldings.

Proof. We actually show that the construction described in the statement produces the pushout in the category ES and also in ESr. Consider the diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{f'} & E'' \\
\downarrow{g'} & & \downarrow{g''} \\
E & \xrightarrow{f''} & E''
\end{array}
\]

Observe that $E'''$, with functions $g'$ and $g''$ is the pushout in Set, as it easily follows recalling that $f'$ and $f''$ are surjective. Another immediate observation is that the set of configurations of $E'''$ can be written

\[
\text{Conf}(E''') = \{ g'(f'(C)) \mid C \in \text{Conf}(E) \} = \{ g''(f''(C)) \mid C \in \text{Conf}(E) \}
\]

We prove that $g'$ is a folding. In fact

- $g'$ is a morphism.

For all $C' \in \text{Conf}(E')$, since $f'$ is a folding, there is $C \in \text{Conf}(E)$ such that $f'(C) = C'$. Therefore $g'(C') = g'(f'(C)) \in \text{Conf}(E''')$, by construction. Moreover, $g'$ is injective on $C'$. In fact, take $x', y' \in C'$, with $g'(x') = g'(y')$. Since $C' = f'(C)$, there are $x, y \in C$ such that $f'(x) = x'$ and $f'(y) = y'$. Therefore, $g'(f'(x)) = g'(f'(y))$, and thus, by the properties of pushouts, $f''(x) = f''(y)$. Since $f''$ is a folding, thus a morphism, this implies $x = y$ and thus $x' = f'(x) = f'(y) = y'$, as desired.

- $g'$ is a folding.

Let $C'_1 \in \text{Conf}(E')$ and assume that $f'(C'_1) \xrightarrow{f''} D''_2$. By (1) we know that there is $D_2 \in \text{Conf}(E)$ such that $D''_2 = g'(f''(D_2))$ and $D_2 \cong D''_2$. Therefore, there is $D_1 \subseteq D_2$ such that $f'(g'(D_1)) = g'(C'_1)$ and

\[
D_1 \xrightarrow{f''} D_2.
\]

Define $D_1' = f'(D_1) \in \text{Conf}(E')$. Now, since $f'$ is a folding and $C'_1 \in \text{Conf}(E_1)$, there is also $C_1 \in \text{Conf}(E)$ such that $f'(C_1) = C'_1$. Recall that $g'(D_1) = g'(f'(D_1)) = g'(C'_1)$, hence, by pushout properties, it must be $f''(C_1) = f''(D_1)$. From (2), since $f''$ is a folding, we deduce $f''(C_1) \xrightarrow{f''} D''_2$, with $f''(C_2) = D''_2$. And, using again the fact that $f''$ is a folding, this implies $C_1 \xrightarrow{f''} C_2$, with $f''(C_2) = D''_2 = f''(D_2)$.

Now, we use the fact that $f'$ is a folding, and derive that $C'_1 = f'(C_1) \xrightarrow{f'(x_1)} f'(C_2)$. If we call $C'_2 = f'(C_2)$, we have that $g'(C'_2) = g'(D'_2)$, as desired, since $f''(C_2) = f''(D_2)$.

In the same way, one concludes that also $g''$ is a folding.

Given any other $E_1$ with morphisms $g'_1 : E' \to E_1$ and $g''_1 : E'' \to E_1$ such that $g'_1 \circ f' = g''_1 \circ f''$, we show that there exists a unique morphism $h : E''' \to E_1$ that makes the diagram commute.
Consider the unique map \( h : E'' \to E_1 \) making the diagram commute in \( \text{Set} \). Since \( g'_1 \) is a folding and \( g'_1 \) is a morphism, by Lemma B.4, also \( h \) is a morphism. This proves that \( E'' \) is a pushout in \( \text{ES} \).

By the same result, if \( g'_1 \) is a folding, also the mediating morphism \( h \) is. This means that the same construction produces a pushout in \( \text{ES}_f \).

As a counterexample to the existence of pushouts in \( \text{ES} \) for general morphisms, consider the obvious mappings \( f_{45} : P_4 \to P_5 \) and \( f_{46} : P_4 \to P_6 \) in Fig. 7.

**Lemma B.5** (multi-colimit). Let \( E \) be an event structure. Each collection of foldings \( f_i : E \to E_i \) with \( i \in I \) has a colimit in \( \text{ES} \). Therefore the coslice category \( (E \downarrow \text{ES}_f) \) has a terminal object.

**Proof.** When \( I \) is finite, the proof proceeds by straightforward induction on \( I \), using Proposition 21. If instead \( I \) is infinite, let \( E' \) be the colimit of the \( f_i \)'s in \( \text{Set} \).

\[
\begin{array}{c}
E' \\
\downarrow g_i \quad \downarrow g_i' \quad \downarrow g_i'' \quad \downarrow g_i''' \\
E_1 \\
\end{array}
\]

with configurations \( \text{Conf}(E') = \{ g_i(f_i(C)) \mid C \in \text{Conf}(E) \} \). The proof of the fact that the \( g_i \)'s are foldings then proceeds as in Proposition 21. The only delicate point is the following. Given configurations \( C, C' \in \text{Conf}(E) \), define \( C_i = f(C) \) and \( C'_i = f(C') \in \text{Conf}(E_i) \). If \( g_i(C_i) = g_i(C'_i) \), then it is not necessarily the case that \( f_j(C) = f_j(C') \) for some \( j \in I \). However, since configurations are finite, there is a finite subset \( J \subseteq I \) such that, if \( E_J \) is the colimit of \( \{ f_j \mid j \in J \} \) and \( f_J : E \to E_J \) the corresponding folding, whose existence is proved in the first part, then \( f_J(C) = f_J(C') \). Exploiting this fact, we can conclude exactly as in Proposition 21.

**Proposition 23** (lattice of foldings). Let \( E \) be an event structure. Then \( \text{FEq}(E) \) is a sublattice of the complete lattice of equivalence relations over \( E \).

**Proof.** Immediate corollary of Lemma B.5
Lemma B.6 (hhp-bisimulation as an event structure). Let $E', E''$ be event structures and let $R$ be a hhp-bisimulation between them. Then there exists a (prime) event structure $E_R$ and two foldings $\pi': E_R \rightarrow E'$ and $\pi'': E_R \rightarrow E''$.

Proof. Let $E', E''$ be event structures and let $R$ be a hhp-bisimulation between them. Define $E_R$ as follows. Events are histories related by $R$, namely the triples $\{(H', f, H'') \mid H' \in Hist(E')\}$, labelled by $\lambda_E(H', f, H'') = \lambda_R(x')$ when $H' \in Hist(x')$. For each $(C', f, C'') \in R$, define

$$C_I = \{(C'[x'], f_{C'[x']}, C''[f(x')]) \mid x \in C'\}$$

ordered by pointwise inclusion, i.e., $(H'_1, f_1, H''_1) \subseteq (H'_2, f_2, H''_2)$ if $f_1 \subseteq f_2$, and thus $H'_1 \subseteq H''_1$, $H'_2 \subseteq H''_2$. The set of configurations of $E_R$ is $Conf(E_R) = \{C_R \mid C \in Conf(E)\}$.

It is easy to see that $Conf(E_R)$ is well-defined. Prefix-closedness of $Conf(E_R)$ follows from the fact that $R$ is downward-closed by definition of hhp-bisimulation. It can be seen that $E_R$ is actually a prime event structure, with causality defined by $(H'_1, f_1, H''_1) \leq (H'_2, f_2, H''_2)$ if $H'_1 \subseteq H'_2$ and $f_1 \subseteq f_2$, and conflict defined by $(H'_1, f_1, H''_1) \#(H'_2, f_2, H''_2)$ if there is no $(C', f, C'') \in R$ such that $H'_1, H''_1 \subseteq C'$ and $f_1, f_2 \subseteq f$.

Consider two configurations $C_{f_1}, C_{f_2} \in Conf(E_R)$, arising from the triples $(C_i', f_i, C_i'') \in R$, for $i \in \{1, 2\}$. Then it holds that

$$C_{f_1} \subseteq C_{f_2}$$

iff $C_{f_1} \subseteq C_{f_2}$

iff for all $x' \in C_1'$, $(C_1'[x'], f_{(C_1'[x'])}, C''_1[f(x')]) \in C_{f_2}$

iff for all $x' \in C_1'$, $C_1'[x'] = C_2'[x']$ and $f_1(x') = f_2(x')$

iff $C_1' \subseteq C_2'$ and $f_1 \subseteq f_2$.

We can now define $\pi': E_R \rightarrow E'$ as $\pi'(H', f, H'') = x'$ if $H' \in Hist((x'))$ and, similarly, $\pi'': E_R \rightarrow E'$ as $\pi''(H', f, H'') = x''$ if $H'' \in Hist((x''))$.

Then $\pi'$ and $\pi''$ are well-defined morphisms and they are foldings. We prove this for $\pi'$ (for $\pi''$ the proof is completely analogous).

- $\pi'$ is a morphism.

This is immediately by observing that for any configuration $C_f \in Conf(E_R)$, arising from the triple $(C', f, C'') \in R$, then we have $\pi'(C_f) = C'$. Note that, concerning the local order, for $x', y' \in C'$ we have $(C'[x'], f_{C'[x']}, C''[f(x')]) \leq_{C_f} (C'[y'], f_{C'[y']}, C''[f(y')])$ if inclusion holds pointwise iff $x' \subseteq C' y'$, which means $\pi'(C'[x']) = x' \subseteq C'$ $y' = \pi'(C'[y'])$.

- $\pi$ is a folding.

In fact, for any configuration $C_f \in Conf(E_R)$, arising from the triple $(C', f, C'') \in R$, if $\pi'(C_f) = C' \xrightarrow{f} D'$ then, since $R$ is an hhp-bisimulation, there is $C'' \xrightarrow{g} D''$ with $(C'', g, D'') \in R$ with $g = f[x'] \rightarrow x''$. Hence, if we let $H' = D'[x']$, we have that $C_f \xrightarrow{(H', g, D'[H'])} C_g$ and $\pi'(C_g) = D'$, as desired.

Proposition 25 (foldings subsume behavioural quotients). Let $E$ be an event structure and let $f: E \rightarrow E'$ be a morphism such that $E_{/f}$ is hhp-similar to $E$. Then there exists a folding $g: E \rightarrow E''$ such that $\equiv_g$ is coarser than $\equiv_f$.

Proof. Let $R$ be a hhp-bisimulation between $E$ and $E_{/f}$. Consider the event structure $E_R$ and the foldings $\pi: E_R \rightarrow E$ and $\pi': E_R \rightarrow E_{/f'}$, given by Lemma B.6. By Proposition 21 we can close the diagram as follows:
24 Minimisation of Event Structures

[Diagram]

and both $g$ and $g'$ are foldings. Then $E'' = E_{/=g} = (E_{/=1})_{/=g'}$, and we conclude. ▷

Lemma B.7 (configurations of the canonical $\text{PES}$). Let $E$ be an event structure. Then $\text{Conf}(E)$ and $\text{Conf}(\text{PES})$ seen as partial orders, ordered by prefix, are isomorphic.

More in detail, for all $C \in \text{Conf}(E)$ it holds $\text{hs}(C) = \{C[x] \mid x \in C\}$, with inclusion as local order, is in $\text{Conf}(\text{PES})$. Moreover $C \equiv \text{hs}(C)$ and $\text{hs} : \text{Conf}(E) \to \text{Conf}(\text{PES})$ is a poset isomorphism.

Its inverse is as follows. For $D \in \text{Conf}(\text{PES})$ consider $\text{fl}(D) = \bigcup D$. Then, for each $x \in \text{fl}(D)$ there exists a unique $H_x \in D$ such that $H_x \in \text{Hist}(x)$. Define the order $\leq_{\text{fl}(D)}$, for $x, y \in \text{fl}(D)$, by $x \leq_{\text{fl}(D)} y$ iff $x \in H_y$. Then $\text{fl}(D) \in \text{Conf}(E)$ and $\text{fl}(D) \simeq D$ as posets.

Proof. Let $C \in \text{Conf}(E)$ and let us show that $\text{hs}(C) = \{C[x] \mid x \in C\}$, with inclusion as local order, is in $\text{Conf}(\text{PES})$. First, note that $\text{hs}(C)$ is consistent by construction, since $C[x] \subseteq C$ for all $x \in C$. Moreover, it is causally closed. In fact, if $H \subseteq C[x]$ for some $H \in \text{Hist}(E)$, then, if $H \in \text{Hist}(y)$, by Lemma A.12 we have $H = C[x][y] = C[y] \in \text{hs}(C)$.

Moreover, $\text{hs}(C)$ is isomorphic to $C$, the isomorphism established by the mapping $C[x] \mapsto x$. It is clearly bijective. Moreover, for all $x_1, x_2 \in C$ it holds that $C[x_1] \subseteq C[x_2]$ iff $x_1 \in C[x_2]$ and thus $x_1 \leq_{C} x_2$.

Let us show that $\text{hs} : \text{Conf}(E) \to \text{Conf}(\text{PES})$ is a poset isomorphism. It is injective. In fact, if $\text{hs}(C_1) = \text{hs}(C_2)$ then clearly $C_1$ and $C_2$ contain the same events. Moreover, $\leq_{C_1} = \leq_{C_2}$ and thus the two configurations coincide. Otherwise, there would be $x, y \in C_1$ such that $x \leq_{C_1} y$ and $\neg(x \leq_{C_2} y)$, or conversely $\neg(x \leq_{C_1} y)$ and $x \leq_{C_2} y$. Assume, without loss of generality, that we are in the first case. Then $x \in C_1[y]$ and $x \notin C_2[y]$, and thus $\text{hs}(C_1) \neq \text{hs}(C_2)$ contradicting the hypotheses. Moreover, it preserves and reflects the prefix order, i.e., given $C_1, C_2 \in \text{Conf}(E)$ we have $C_1 \subseteq C_2$ iff $\text{hs}(C_1) \subseteq \text{hs}(C_2)$ as it immediately follows from Lemma A.12.

We conclude, by showing that it is also surjective. Consider any configuration $D \in \text{Conf}(\text{PES})$. Since $D$ has no conflicts, its elements are pairwise compatible. Therefore, by coherence of the class of configurations, there exists $C \in \text{Conf}(E)$ such that $H \subseteq C$ for all $H \in D$. Let $\text{fl}(D) = \bigcup D$. Then, for each $x \in \text{fl}(D)$ there exists a unique $H_x \in D$ such that $H_x \in \text{Hist}(x)$, since by Lemma A.12 different histories of the same event are not compatible. Define the order $\leq_{\text{fl}(D)}$, for $x, y \in \text{fl}(D)$, by $x \leq_{\text{fl}(D)} y$ iff $x \in H_y$. It is easy to check that $\text{fl}(D) \subseteq C$, and thus by prefix closedness of $\text{Conf}(E)$, we have $\text{fl}(D) \in \text{Conf}(E)$. It is now immediate to see that $\text{hs}(\text{fl}(D)) = D$, thus we conclude. ▷

Lemma 27 (unfolding event structures to $\text{PES}$’s). Let $E$ be an event structure. Define a function $\phi_E : \text{P}(E) \to E$, for all $H \in \text{Hist}(E)$ by $\phi_E(H) = x$ if $H \in \text{Hist}(x)$ for $x \in E$. Then $\phi_E$ is a folding.

Proof. This results can be derived from the characterisation of foldings as $\text{Pom}$-open maps (Lemma B.3), the fact that $\text{PES}$ is a coreflective subcategory of $\text{ES}$ (Lemma 28) and then using Lemma 6(ii)].

Explicitly, the fact that $\phi_E$ is a morphism immediately follows from the observation that $\phi_E(D) = \text{fl}(D)$. Then by Lemma B.7 we have $D \simeq \phi_E(D)$, as desired.
In order to conclude that it is a folding we show that given \( D_1 \in \text{Conf}(\mathbb{P}(E)) \), if \( \phi_E(D_1) \xrightarrow{H} C_2 \) then \( D_1 \xrightarrow{H} D_2 \) with \( \phi_E(D_2) = C_2 \). Let \( C_1 = \phi_E(D_1) \) and assume \( C_1 \xrightarrow{H} C_2 \). By definition of transition (Definition 3), we have \( C_1 \subseteq C_2 \). Let \( H_x = C_2[x] \). By definition of \( \mathbb{P}(E) \), the causes \( \{H_x\} = \{H_x[y] \mid y \in H_x\} \). For all \( y \in H_x \setminus \{x\} \), clearly \( y \in C_1 \). Moreover \( H_x[y] = C_2[y] = C_1[y] \). Therefore, by Lemma A.1(2), \( H_x[y] \in D_1 \). We thus conclude that \( D_1 \xrightarrow{H} D_2 \) and moreover \( \phi_E(D_2) \simeq C_2 \). For the last statement, the only thing to observe is that the image of the causes of \( H_x \) are exactly the causes of \( x \). Indeed we have, for all \( H \in D_2 \), say \( H \in \text{Hist}(y) \), that \( H \subseteq H_x \) iff \( y \in H_x \) iff \( y \leq C_2 \) \( x \), as desired.

**Lemma 28** (cofreeness of \( \phi_E \)). Let \( E \) be an event structure, let \( \mathbb{P}' \) be a PES and let \( f : \mathbb{P}' \to \mathbb{P}(E) \) be an event structure morphism. Then there exists a unique morphism \( g : \mathbb{P}' \to \mathbb{P}(E) \) such that \( f = \phi_E \circ g \).

\[
\begin{array}{c}
\mathbb{P}(E) \xrightarrow{\phi_E} E \\
\mathbb{P}' \xrightarrow{f} \mathbb{P}(E)
\end{array}
\]

Moreover, when \( f \) is a folding then so is \( g \).

**Proof.** The function \( g \) can be defined, for all \( x' \in \mathbb{P}' \) as \( g(x') = f([x']) \)

Note that this is a well-defined morphism. First observe that \( g(x') \in \text{Hist}(E) \), hence it is an event in \( \mathbb{P}(E) \). In fact, for all \( x' \in \mathbb{P}' \), since \( f \) is a morphism and \( [x'] \in \text{Conf}(\mathbb{P}') \), \( f([x']) \in \text{Conf}(E) \), and \( f([x']) \simeq [x'] \), therefore \( g(x') = f([x']) = f([x'])[f(x')] \in \text{Hist}(E) \). Moreover, the reasoning above shows that \( g(x') \in \text{Hist}(f(x')) \). Therefore, if \( g(x') = g(y') \) then \( f(x') = f(y') \). This fact, recalling that \( f \) is injective on configurations, implies that also \( g \) is. Finally, for all \( C' \in \text{Conf}(\mathbb{P}') \), since \( f \) is a morphism, \( f(C') \in \text{Conf}(E) \) and \( f(C') \simeq C' \). Therefore its \( g \)-image is

\[
g(C') = \{g(x') \mid x' \in C'\}
= \{f([x']) \mid x' \in C'\}
= \{f([x'])[f(x')] \mid x' \in C'\} \quad \text{[Since morphisms preserve prefix order]}
= \{f(C')[f(x')] \mid x' \in C'\}
= \text{hs}(f(C'))
\]

Hence, by Lemma B.7, \( g(C') = \text{hs}(f(C')) \in \text{Conf}(\mathbb{P}(E)) \) and \( \text{hs}(f(C')) \simeq C' \), as desired.

For the second part, assume that \( f \) is a folding and let us show that also \( g \) is. We use the characterisation in Lemma 17. Let \( C_1' \in \text{Conf}(\mathbb{P}') \) and assume that \( g(C_1') \xrightarrow{H} D_2 \). Since \( \phi_E \) is a morphism, this implies that \( f(C_1') = \phi_E(g(C_1')) \xrightarrow{\phi_E(H)} \phi_E(D_2) \). Since \( f \) is a folding, by Lemma 17, there exists a transition \( C_1' \xrightarrow{x'} C_2' \) such that \( f(C_2') = \phi_E(C_2) \). Observe that this implies \( f(x') = \phi_E(H) \) and more generally \( f([x']) = \phi_E([H]) \), but since \( \phi_E([H]) = H \)

\[
f([x']) = H.
\]

We only need to show that \( g(C_2') = D_2 \). This is an immediate consequence of the fact that \( g(C_2') = g(C_1') \cup \{g(x')\} = D_1 \cup \{H\} = D_2 \), as desired.

\[\blacksquare\]
Proposition 30 (folding through \textit{PES}). Let $E, E'$ be event structures. For all morphisms $f : E \rightarrow E'$ consider $P(f) : P(E) \rightarrow P(E')$ defined by $P(f)(H) = f(H)$. Then $f$ is a folding iff $P(f)$ is a folding.

Proof. This can be derived from the characterisation of foldings as \textit{Pom}-open maps (Lemma [B.3]), the fact that \textit{PES} is a coreflective subcategory of \textit{ES} (Lemma [28]) and then using [16, Lemma 6(iii)].

Explicitly, let $E, E'$ be event structures, let $f : E \rightarrow E'$ be a morphism and consider the commuting diagram

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\phi_E & & \phi_{E'} \\
P(E) & \xrightarrow{P(f)} & P(E')
\end{array}
$$

If $f$ is a folding then $f \circ \phi_E : P(E) \rightarrow E'$ is a composition of foldings and thus, by Remark [16], it is a folding. In turn, by Lemma [28] this implies that $P(f)$ is a folding.

Conversely, if $P(f)$ is a folding, then $\phi_E \circ P(f) : P(E) \rightarrow E'$ is a composition of foldings and thus, by Remark [16] it is a folding. In turn, by Lemma [B.4] this implies that $f$ is a folding.

C Some Properties of Morphisms and Foldings

In this section, we define some relations between the events of an event structure, based on the way in which such events occur in configurations. They can be used to prove general properties of morphisms of event structures, that then can be instantiated on specific subclasses.

Definition C.1 (precedence). Let $E$ be an event structure. The precedence as the relation $\cdot \subseteq E \times E$, defined for $x, y \in E$ by $x \bowtie y$ if for all $C \in \text{Conf}(E)$ such that $x, y \in C$ it holds $x \prec_C y$. We say that $E$ has global precedence if for $x, y \in E$, if $x, y \in C$ and $x \prec_C y$ then $x \bowtie y$.

In words, $x \bowtie y$ whenever in each computation where $x, y$ occur necessarily $x$ occurs before $y$. The precedence relation is useful also to define a notion of semantic conflict. Observe that for any configuration $C$ the precedences expressed by $\bowtie$ are always respected by $\preceq_C$, i.e., $\bowtie \subseteq \preceq_C$. When the event structure has global precedence, the precedence relation is sufficient to completely characterise the local order of configuration, i.e., for all configurations $C$ it holds that $\bowtie_C = (\bowtie_C)^*$.

Closely connected, we can introduce a notion of semantic conflict.

Definition C.2 (conflict). Let $E$ be an event structure. The conflict is relation $\# \subseteq 2^E$, defined for a finite $X \subseteq E$ by $X \# Y$ if there is no $C \in \text{Conf}(E)$ such that $X \subseteq C$. When $\{x, y\}$ we often write $x \# y$.

We observe that conflict and precedence are strictly related. In particular, binary conflict can be characterised in terms of precedence.

Proposition C.3 (precedence vs conflict). Let $E$ be an event structure. Then

- for $X \subseteq E$, if $\bowtie_X$ is cyclic then $\# X$.
- for $x, y \in E$, we have $x \# y$ iff $\prec x \prec y$.
Proof. Let \( X \subseteq E \). If \( \cap_X \) is cyclic, i.e., there are \( x_1, \ldots, x_n \in X \) such that \( x_1 \lhd x_2 \lhd \ldots \lhd x_n \lhd x_1 \) then the events \( x_1, \ldots, x_n \) and thus \( X \) can never occur together in the same computation, i.e., there cannot be \( C \in \text{Conf}(E) \) such that \( X \subseteq C \). In fact, otherwise, we should have \( \cap_C \subseteq \subseteq C \), contradicting the fact that \( \subseteq_C \) is a partial order. In words, each of the events \( x_i \) should occur before the others, which is impossible.

In particular, if \( x \neq y \) then \( x, y \) can never be in the same computation, hence trivially \( x \lhd y \) and \( y \lhd x \), and observe that also the converse holds.

Morphism on event structures can be shown to enjoy interesting properties with respect to the semantic relations.

Lemma C.4 (morphism properties). Let \( E, E' \) be event structures and let \( f : E \to E' \) be a morphism. Then for all \( x, y \in E \)

1. if \( f(x) \lhd f(y) \) then \( x \lhd y \);
2. if \( f(x) = f(y) \) then \( x \lhd y \), hence by duality \( x \neq y \).

Moreover, if \( E, E' \) have global precedence, then

3. if \( x \lhd y \) and \( \lnot(y \lhd x) \) then \( f(x) \lhd f(y) \);

Proof. Let \( x, y \in E \)

1. Assume \( f(x) \lhd f(y) \). Let \( C \in \text{Conf}(E) \) be a configuration such that \( x, y \in C \). Then \( f(x), f(y) \in f(C) \) and \( C \in \text{Conf}(E') \). Since \( f(x) \lhd f(y) \) we have that \( f(x) \lhd f(y) \) and thus, since \( f \) is a morphism, \( x \lhd y \). Since this holds for any configuration, we conclude \( x \lhd y \).

2. Assume \( f(x) = f(y) \). Since \( f \) is injective on configurations, there cannot be \( C \in \text{Conf}(E) \) such that \( x, y \in C \). Therefore, trivially \( x \lhd y \) (and \( y \lhd x \), whence \( x \neq y \)).

3. If \( E, E' \) have global precedence, \( f \) is a folding and \( x \lhd y \) and \( \lnot(y \lhd x) \) then \( \lnot(x \neq y) \) and thus there is some configuration \( C \in \text{Conf}(E) \) such that \( x, y \in C \). Since \( E \) has global precedence, \( x \lhd y \). Now \( f(x), f(y) \in f(C) \) which is in \( \text{Conf}(E') \). Therefore \( f(x) \lhd f(y) \). Again, since \( E' \) has global precedence, \( f(x) \lhd f(y) \), as desired.

Proofs for Section 4 (Foldings for Prime and Asymmetric Event Structures)

Lemma 31 (pes morphisms). Let \( P \) and \( P' \) be pess and let \( f : P \to P' \) be a function on the underlying sets of events. Then \( f \) is a morphism iff for all \( x, y \in P \)

1. \( \lambda(f(x)) = \lambda(x) \);
2. \( f([x]) = [f(x)] \); namely (a) for all \( x \in P \), if \( x' \leq f(y) \) there exists \( x \in P \) such that \( x \leq y \) and \( f(x) = x' \) (b) if \( x \leq y \) then \( f(x) \leq f(y) \);
3. (a) if \( f(x) = f(y) \) and \( x \neq y \) then \( x \neq y \) and (b) if \( f(x) \neq f(y) \) then \( x \neq y \).

Proof. First observe that pess have global precedence and \( x \lhd y \) iff \( x \leq y \) or \( x \neq y \).

Now, assume that \( f \) is a morphism. Then property (1) holds by definition. Property (2) follows from the fact that \( [x] \in \text{Conf}(P) \). Hence \( f([x]) \in \text{Conf}(P') \) and \( f([x]) \preceq [x] \), which implies \( f([x]) = [f(x)] \).

Concerning condition (3b), observe that from Lemma C.4, instantiated with the notion of \( \lhd \) for pess, we get

\[
\begin{align*}
&f(x) \leq f(y) \text{ or } f(x) \neq f(y) \text{ implies } x \leq y \text{ or } x \neq y.
\end{align*}
\]
In particular, if \( f(x) \# f(y) \) then \( x \leq y \) or \( x \# y \) and, since conflict is symmetric, we also have \( y \leq x \) or \( y \# x \). It is now easy to see that only the second possibility \( x \# y \) can hold true, which is the desired conclusion. Property (3a) immediately derives from Lemma C.4(2).

Conversely, assume that \( f \) satisfies conditions (1)-(3) above. Given a configuration \( C \in \text{Conf}(P) \), by conditions (2a) and (3b), \( f(C) \) is a configuration in \( P' \). By condition (3a), \( f \) is injective on \( C \). This, together with condition (2b), implies that \( C \cong f(C) \). □

\textbf{Proposition 32 (pes foldings).} Let \( P \) and \( P' \) be \textit{pes} and let \( f : P \to P' \) be a morphism. Then \( f \) is a folding if and only if it is surjective and for all \( X, Y \subseteq P \), \( x, y \in P \), \( y' \in P' \)

1. if \( x \# y \) then \( f(x) \# y' \);
2. if \( ^\wedge(X \cup \{x\}), ^\wedge(Y \cup \{y\}), ^\wedge(X \cup Y) \) and \( f(x) = f(y) \) then there exists \( z \in P \) such that \( f(z) = f(x) \) and \( ^\wedge(X \cup Y \cup \{z\}) \).

\textbf{Proof.} Let \( f : P \to P' \) be a folding. Let us first observe that \( f \) is surjective. Take \( x' \in P' \).

Since \( \{x'\} \in \text{Conf}(P') \), we have that \( \emptyset \xrightarrow{\{x'\}} \{x'\} \). Since \( f \) is a folding, there must be \( C \in \text{Conf}(P) \) such that \( f(C) = \{x'\} \), and thus there is \( x \in C \) such that \( f(x) = x' \), as desired.

We next show that properties (1) and (2) hold.

1. We prove the contranominal, namely that if \( f(x) \sim y \) then there is \( y \in P \) such that \( f(y) = y' \) and \( x \sim y \). Assume that \( f(x) \sim y' \). We distinguish two possibilities:

   a. If \( y' \leq f(x) \) then, by Lemma 31.2a, there exists \( y \leq x \) such that \( f(y) = y' \). Hence \( x \sim y \), as desired.

   b. Assume that, instead, \( \neg(y' \leq f(x)) \). Therefore, if we let \( C' = \{f(x)\} \cup \{y'\} \) and \( X' = C' \setminus \{f(x)\} \)

   \[
   f(x) \xrightarrow{X'} C'
   \]

   By Lemma 31.3, we have that \( f(\{x\}) = \{f(x)\} \). Therefore, since \( f \) is a folding, there must be a transition \( \{x\} \xrightarrow{X} C \) with \( f(C) = C' \). This means that there exists \( y \in C \) such that \( f(y) \in C' \) and, since \( x \in C \), necessarily \( x \sim y \), as desired.

2. Assume that \( ^\wedge(X \cup \{x\}), ^\wedge(Y \cup \{y\}), ^\wedge(X \cup Y) \) and \( f(x) = f(y) \). Define \( C = \{X \cup Y\} \in \text{Conf}(P) \). We distinguish two cases.

   a. If \( x \in C \) then we can simply take \( z = x \), since clearly \( ^\wedge(X \cup Y \cup \{x\}) \).

   b. Assume now that \( x \notin C \). Clearly \( f(x) \notin f(C) \). Moreover, \( ^\wedge(f(C) \cup \{f(x)\}) \). In fact, by Lemma 31.3, if for some \( w \in C \) it were \( f(w) \# f(x) = f(y) \) we would have \( w \# x \)

   and \( w \# y \), contradicting either \( ^\wedge(X \cup \{x\}) \) or \( ^\wedge(Y \cup \{y\}) \).

   Therefore \( f(C) \xrightarrow{X} f(C) \cup \{f(x)\} \) with \( X' = f(f(x)) \setminus f(C) \). Since \( f \) is a folding, this implies that \( C \xrightarrow{X} D \) with \( f(D) = f(C) \cup \{f(x)\} \) and \( D \cong f(C) \cup \{f(x)\} \).

   Therefore there exists \( z \in D \) such that \( f(z) = f(x) \). Since \( X \cup Y \subseteq D \), we have that \( ^\wedge(X \cup Y \cup \{z\}) \), as desired.

For the converse implication, assume that \( f \) is a surjective morphisms satisfying conditions (1) and (2). We have to prove that it is a folding.

Let \( C_1 \in \text{Conf}(P) \) and assume that \( f(C_1) \xrightarrow{X} C_0 \). If \( C_1 = \emptyset \), take any \( x \in P \) such that \( f(x) = x' \), which exists by surjectivity. By Lemma 31.2b we have \( f(\{x\}) = \{x'\} \), and thus \( \{x\} = \{x'\} \). This means that \( C_1 = \emptyset \xrightarrow{X} \{x\} \), and we conclude.
Otherwise, if $C_1 \neq \emptyset$, since for all $y \in C_1$ it holds that $f(y) \sim x'$, by condition (1), there exists some element $x_y \in P$ such that $x_y \sim y$ and $f(x_y) = x$. Note that necessarily $\sim (x_y \leq y)$, otherwise, by Lemma [31][2] we would have $x' = f(x_y) \leq f(y)$, which is not the case.

Since $C_1$ is finite and consistent, an inductive argument based on condition (2), allows to derive the existence of $x$ such that $f(x) = x'$ and $\sim (C_1 \cup \{x\})$. Moreover, as argued above for the $x_y$s, it is not the case that $x \leq y$ for some $y \in C_1$. Therefore there is a transition $C_1 \xrightarrow{X} C_1 \cup \{x\}$

where $X = [x] \setminus C_1$.

We argue that $X = \{x\}$ and thus we conclude. In fact, assume that there is some $z \in X \setminus \{x\}$. Since $f$ is a morphism $f(z) \leq f(x) = x'$. Now, since there is the transition $f(C_1) \xrightarrow{x'}$, all causes of $x'$ must be in $f(C_1)$. Note that, since $f$ is a morphism, by Lemma [31][2], we have $[x'] = [f(x)] = f([x])$. Therefore, there must exist $z_1 \in C_1$ such that $f(z_1) = f(z)$. However, since $z, z_1 \in C_1 \cup \{x\} \setminus \{x\}$ which is a configuration in $Conf(P)$, and $f$ is injective on configurations, we get $z = z_1 \in C_1$, contradicting the hypothesis.

Given a PES $P$ and an event $x \in P$ let us define $\lfloor x \rfloor = [x] \setminus \{x\}$, $\lceil x \rceil = \{y \mid y \in P \land x < y\}$, and $\text{conc}(x) = \{y \mid y \in P \land \neg (x \leq y \lor y \leq x \land x \# y)\}$.

**Definition D.1 (abstraction homomorphisms [10]).** Let $P, P'$ be PESS. An abstraction morphism is a function $f : P \rightarrow P'$ such that for all $x, y \in P$

1. $\lambda'(f(x)) = \lambda(x)$;
2. $f([x]) = [f(x)]$;
3. $f([x]) = [f(x)]$;
4. $\text{conc}(f(x)) = \text{conc}(f(x))$

**Lemma D.2 (foldings vs abstraction homomorphisms).** Let $P, P'$ be PES and let $f : P \rightarrow P'$ be a function. Then $f$ is an abstraction morphism iff $f$ is a PES morphism additionally satisfying condition (1) of Lemma [32].

**Proof.** Let $f$ be an abstraction homomorphism. We first prove conditions [1]-[3] of Lemma [31]. The first condition is already in Definition D.1. Condition (2), is immediately implied by Definition D.1(1). Concerning condition (3), let $x, y \in P$ such that $f(x) = f(y)$ and $x \neq y$. Observe that we cannot have $x < y$, otherwise by Definition D.1(2), we would have $f(x) < f(y)$. Dually, it cannot be $y < x$. Moreover, it cannot be $x \in \text{conc}(y)$, otherwise Definition D.1(3) would be violated. Therefore, necessarily $x \# y$. The validity of condition (3) is proved analogously.

We finally show that $f$ satisfies also condition [1] of Lemma [32]. Let $x \in P, y \in P'$ such that $\neg (f(x) \# y')$ and we show that $\neg (x \# y)$ for some $y \in P$ such that $f(y) = y'$. We distinguish various possibilities:

- If $f(x) = y'$, we simply take $y = x$.
- If $y' < f(x)$, by Definition D.1(2), there exists $y \in P$ with $y < x$ such that $f(y) = y'$, and we conclude.
- If $f(x) < y'$, by Definition D.1(3), there exists $y \in P$ with $x < y$ such that $f(y) = y'$, and we conclude.
- If none of the above holds, necessarily $y' \in \text{conc}(f(x))$, and thus by Definition D.1(4), there exists $y \in P$ with $y \in \text{conc}(x)$ such that $f(y) = y'$, and we conclude.
We prove that conditions (1)-(4) of Definition D.1 hold. As above, the first conditions is already in Lemma 31. The second condition, namely $f([x]) = [f(x)]$, immediately follows from Lemma 31(1), i.e., $f([x]) = [f(x)]$. In fact, we only need to observe that for all $y < [x]$, $f(y) \neq f(x)$, otherwise, by Lemma 31(2a) we would have $x \# y$.

Concerning (3), i.e., for $x \in P$, $f([x]) = [f(x)]$ let us prove separately the two inclusions.

- $\subseteq$: Let $y' \in f([x])$, i.e., $y' = f(y)$ for some $y \in [x]$. Since $x < y$, by Lemma 31(2b), $f(x) < f(y)$ and thus $y' = f(y) \in [f(x)]$, as desired.

- $\supseteq$: Let $y' \in [f(x)]$, i.e., $f(x) < y'$. Then, for all $y \in f^{-1}(y')$, since $f(x) < y' = f(y)$, by Lemma 31(2a), there is $z < y$ such that $f(z) = f(x)$. Hence either $z = x$ and thus $x < y$ or $z \neq x$, hence, by Lemma 31(3a), $x \# z$ and thus $x \# y$.

It cannot be that $x \# f^{-1}(y')$, otherwise, by Lemma 32(1), we would have $x \# y$, which is not the case. Therefore there must exists $y \in f^{-1}(y')$ such that $x < y$. Therefore $y' = f(y) \in f([x])$.

Let us finally prove condition (4), i.e., for $x \in P$, $f(\text{conc}(x)) = \text{conc}(f(x))$. Again, we prove separately the two inclusions.

- $\subseteq$: Let $y' \in f(\text{conc}(x))$, i.e., $y' = f(y)$ for some $y \in \text{conc}(x)$. By Lemma 31(2b) and Lemma 31(3a), it must be $y' = f(y) \in \text{conc}(f(x))$, as desired.

- $\supseteq$: Let $y' \in \text{conc}(f(x))$. Since $\neg(f(x) \# y')$, by Lemma 32(1), we deduce that $\neg(x \# f^{-1}(y'))$. Take any $y \in f^{-1}(y')$ such that $\neg(x \# y)$. Now observe that it cannot be $x < y$ or $y < x$, otherwise, by Lemma 31(2a), $f(x)$ and $y'$ would be ordered in the same way, contradicting $y' \in \text{conc}(f(x))$. It cannot be $x = y$ either, otherwise $y' = f(y) = f(x)$, again contradicting $y' \in \text{conc}(f(x))$.

Therefore, $y \in \text{conc}(x)$ and thus $y' = f(y) \in f(\text{conc}(x))$, as desired.

For instance, consider the pess $P_7$ and $P_8$ in Fig. 8. It can be seen that obvious function $f_{78} : P_7 \rightarrow P_8$ is an abstraction homomorphism but not a folding. Indeed, consider the configuration $\{b_0, a_1\}$. Then the step $f_{78}(\{b_0, a_1\}) \overset{\text{\textsuperscript{\text{eq}}}}{\rightarrow} \{b_{01}, a_{01}, c_{01}\}$ cannot be simulated by $\{b_0, a_1\}$.

**Corollary 33** (folding equivalences for pess). Let $P$ be a pess and let $\equiv$ be an equivalence on $P$. Then $\equiv$ is a folding equivalence in $\text{FEq}(P)$ iff for all $x, y \in P$, if $x \equiv y$ then
1. $\lambda(x) = \lambda(y)$;
2. $[[x]]_\equiv = [[y]]_\equiv$;
3. $x \# y$.

Moreover, for all $x, y \in P$, $X, Y \subseteq P$
4. if $x \# y \equiv \rightarrow [x]_\equiv \# [y]_\equiv$;
5. if $\cap(X \cup \{x\})$, $\cap(Y \cup \{y\})$, $\cap(X \cup Y)$ there exists $z \in [x]_\equiv$ such that $\cap(X \cup Y \cup \{z\})$. 

![Figure 8](image-url) Abstraction homomorphisms vs folding morphisms.
Proof. Let \( P \) be a pess and let \( \equiv \) be a folding equivalence. This means that there exists a folding \( f : P \to P' \) such that \( \equiv \) and \( \equiv_f \) coincide. By Lemma \ref{lem:19} we know that \( P_{/\equiv_f} \) is isomorphic to \( P' \). Therefore using Lemma \ref{lem:31} and Proposition \ref{prop:32} we immediately get the validities of properties (1)-(5).

Conversely, assume that \( \equiv \) satisfies properties (1)-(5) above. Define a \( \text{PES} \) \( P' \) as follows.

\[ E' = E/f; \]
\[ [x]_\equiv \leq'_t [y]_\equiv \text{ if } [x]_\equiv \leq_\equiv [y]_\equiv; \]
\[ [x]_\equiv \#'_t [y]_\equiv \text{ if } [x]_\equiv \#_\equiv [y]_\equiv; \]
\[ \lambda([x]_\equiv) = \lambda(x). \]

Observe that \( P' \) is a well-defined \( \text{PES} \). A simple key observation is that

\[ [x]_\equiv \leq'_t [y]_\equiv \leq'_t [z]_\equiv \implies \exists x' \in [x]_\equiv, y' \in [y]_\equiv, z' \in [z]_\equiv \text{ such that } x' \leq y'. \quad (4) \]

In fact, since \([y]_\equiv \leq'_t [z]_\equiv\), by definition we have the existence of \( y' \in [y]_\equiv \) and \( z' \in [z]_\equiv \) such that \( y' \leq z' \). Moreover, since \([x]_\equiv \leq'_t [y]_\equiv\), by definition we have the existence of \( x'' \in [x]_\equiv \) and \( y'' \in [y]_\equiv \) such that \( x'' \leq y'' \). Since \( y' \equiv y'' \), by condition \( (2) \), \([y]'_\equiv = [y'']_\equiv\). Hence from \( x'' \leq y'' \) we deduce the existence of \( x' \leq y' \) with \( x' \in [x]_\equiv \) as desired.

Using \( (4) \), we can immediately inherit the partial order properties of \( \leq'_t \) and irreducibility and hereditarity of \( \#'_t \) from the analogous properties of \#.

If we define a function \( f : P \to P' \) as \( f(x) = [x]_\equiv \), it is now easy to show that it satisfies properties \( (1)-(3) \) in Lemma \ref{lem:31} and \( (1),(2) \) in Proposition \ref{prop:32} hence it is a folding and we conclude.

\[ \blacktriangleright \text{Lemma 34} \text{ (joining foldings on pess's).} \text{ Let } P, P', P'' \text{ be pess and let } f' : P \to P', f'' : P \to P'' \text{ be foldings. Define } E''' \text{ along with } g' : P' \to E''' \text{ and } g'' : P'' \to E''' \text{ as in Proposition } \ref{prop:27}. \text{ Then } E''' \text{ is a pess.} \]

Proof. The result can be proved by using the fact that, by Lemma \ref{lem:28} \( \text{PES}_f \) is a coreflective category of \( \text{ES}_f \), hence it is closed under pushout as proved in \cite[Corollary 1]{13}.

Explicitly, the fact that \( \text{Pr}(g') : P' \to \text{Pr}(E''') \) and \( \text{Pr}(g'') : P'' \to \text{Pr}(E''') \) are foldings derive from Proposition \ref{prop:30}. Now observe that, since \( \text{In order to show that this actually provide a pushout in } \text{PES}, \) consider two morphisms \( g'_1 \) and \( g'_2 \) as in the diagram below, such that \( g'_1 \circ f' = g'_2 \circ f'' \):

\[ \begin{array}{cccc}
\text{P} & \xrightarrow{f'} & \text{P}' & \xrightarrow{f''} & \text{P}'' \\
\phi_{E'''}(h) & \downarrow & \phi_{E'''}(g') & \downarrow & \phi_{E'''}(g'') \\
\text{Pr}(E''') & \xrightarrow{\text{Pr}(g')} & \text{Pr}(E''') & \xrightarrow{\text{Pr}(g'')} & \text{Pr}(E''')
\end{array} \]

Since \( E''' \) is a pushout and \( \text{Pr}(g') \circ f' = \text{Pr}(g'') \circ f'' \), there is a unique morphism \( h : E''' \to \text{Pr}(E''') \), making the diagram commute. Now, observe that \( \phi_{E'''} \circ h : E'''' \to E''' \) can be used in the diagram below as mediating morphisms:
Now, also the identity works as mediating morphisms we deduce that \( h \circ \phi_{E'''} = id_{E'''} \), which implies that \( \phi_{E'''} \) is injective. Since it is a folding, it is also surjective, and therefore it is an isomorphism, as desired.

Lemma 36 (aes morphisms). Let \( A \) and \( A' \) be \( \text{aes} \) and let \( f : A \rightarrow A' \) be a function on the underlying sets of events. Then \( f \) is a morphism if and only if for all \( x, y \in A, x \neq y \)
1. \( \lambda(f(x)) = \lambda(x) \);
2. \( [f(x)] \subseteq f([x]) \);
3. (a) if \( f(x) \not\succ y \) then \( x \not\succ y \) and (b) if \( x \not\succ y \) and \( \neg(y \not\prec x) \) then \( f(x) \not\succ f(y) \);
4. if \( f(x) = f(y) \) then \( x \not\succ y \).

Proof. Let \( f : A \rightarrow A' \) be a morphism. Just observe that \( \text{pess} \) have global precedence and \( x \wedge y \) if \( x \not\succ y \). Condition (1) is obviously true. Property (2) follows by observing that, for all \( x \in A \), since \( [x] \in \text{Conf}(A) \) and \( f \) is a morphism, then \( f([x]) \in \text{Conf}(A) \). Since configurations are causally closed we deduce that \( [f(x)] \subseteq f([x]) \). The validity of properties (3) and (4) is given directly by Lemma C.4.

Conversely, assume that \( f : A \rightarrow A' \) enjoys properties (1)-(4). Let \( C \in \text{Conf}(A) \) be a configuration. Function \( f \) is injective on \( C \) since, otherwise, if there are \( x, y \in C \) such that \( f(x) = f(y) \) and \( x \neq y \), we would get \( x \not\succ y \not\succ x \), contradicting acyclicity of \( \not\succ \) in \( C \). Observe that \( f(C) \) is a configuration. In fact, \( \not\succ \) is acyclic in \( f(C) \) since \( C \) is and, by (3b), cycles are reflected by \( f \). In addition, \( f(C) \) is causally closed by (2), since \( C \) is. Finally, note that \( C \simeq f(C) \). In fact, for all \( x, y \in C \), if \( x \not\succ y \), since \( \neg(y \not\prec x) \), by (3b), we get \( f(x) \not\succ f(y) \). Conversely, if \( f(x) \not\succ f(y) \) then \( x \not\succ y \), by (3a).

Proposition 37 (aes foldings). Let \( A \) and \( A' \) be \( \text{aes} \) and let \( f : A \rightarrow A' \) be a morphism. Then \( f \) is a folding if and only if it is surjective and for all \( X, Y \subseteq A, x, y \in A \) with \( x \notin X, y \notin Y, y' \in A' \)
1. if \( f^{-1}(y') \not\succ x \) then \( y' \not\succ X \);
2. if \( \neg(x \not\prec X), \neg(y \not\prec Y), \wedge(X \cup Y) \) and \( f(x) = f(y) \) then there exists \( z \in A \) such that \( f(z) = f(x) \) and \( \neg(z \not\prec X \cup Y) \);
3. given \( H \in \text{Hist}(x) \), \( \neg(H \not\prec X) \), and \( H_1 \subseteq H \) such that \( f(H_1) \cup \{f(x)\} \in \text{Hist}(f(x)) \) there exists \( x_1 \) such that \( H_1 \cup \{x_1\} \in \text{Hist}(x_1) \) and \( \neg(x_1 \not\prec X) \).

Proof. Let \( f : A \rightarrow A' \) be a folding. Surjectivity of \( f \) can be proved exactly as in Proposition 32. We show that properties (1)-(3) hold.

1. We prove the contronomin, namely that if \( \neg((y' \not\prec X \cup Y)) \) then there is \( y \in A \) such that \( f(y) = y' \) and \( \neg(y \not\prec x) \). Let \( H = [x] \in \text{Conf}(A) \) and assume that \( \neg((y' \not\prec X)) \).
   Since \( f \) is a morphism \( H' = f(H) \in \text{Hist}(f(x)) \). Observe that we can safely assume that \( y' \notin H' \). In fact, otherwise, since \( \neg((y' \not\prec H')) \), the only possibility would be \( y' = f(x) \).
and thus we could take \( y = x \) since \( \neg(x \not\rightarrow x) \), as desired. Using the fact that \( \neg(y' \not\rightarrow^3 H') \) and \( y \notin H' \), if we let \( C' = H' \cup [y'] \) and \( Y' = C' \setminus H' \)

\[
H' \xrightarrow{Y'} C' 
\]

(5)

Therefore, since \( f \) is a folding, there must be a transition \( H \xrightarrow{X} C \) with \( f(C) = C' \). This means that there exists \( y \in X \) such that \( f(y) = y' \) and since \( H = [x] \), necessarily \( \neg(y \not\rightarrow x) \), as desired.

2. Assume that \( x \notin X \), \( y \notin Y \) \( \neg(x \not\rightarrow^3 X \), \( \neg(y \not\rightarrow^3 Y \), \( \wedge(X \cup Y) \) and \( f(x) = f(y) \). Define \( C = [X \cup Y] \in \text{Conf}(A) \). We show that \( x \notin C \). In fact, \( x \notin [X] \) since \( x \notin X \) and \( \neg(x \not\rightarrow^3 X \), and, for analogous reasons, \( y \notin [Y] \). Now, if \( x = y \) we are done. Otherwise, we can prove that \( x \notin [Y] \) and conclude. In fact, assume by contradiction that \( x \in [Y] \), i.e., \( x \leq w \) for some \( w \in Y \). Since \( f(x) = f(y) \) and \( x \neq y \), we deduce, by Lemma 36(4), that \( y \not\rightarrow x \). Recalling \( x \leq w \), by inheritance of asymmetric conflict, we get \( y \not\rightarrow^3 Y \), contradicting the hypotheses.

Since \( x \notin C \), we have \( f(x) \notin f(C) \). Moreover, if we let \( y' = f(x) = f(y) \), we have \( \neg(y' \not\rightarrow^3 f(C)) \). Otherwise, by Lemma 36(3), we would deduce \( x \not\rightarrow^3 X \) or \( y \not\rightarrow^3 Y \), contradicting the hypotheses.

Therefore \( f(C) \xrightarrow{X} f(C) \cup \{f(x)\} \) with \( X' = f([f(x)]) \setminus f(C) \). Since \( f \) is a folding, this implies that \( X' \rightarrow D \) with \( f(D) = f(C) \cup \{f(x)\} \) and \( D = f(C) \cup \{f(x)\} \). Therefore there exists \( z \in D \) such that \( f(z) = f(x) \). Therefore \( \neg(z \not\rightarrow^3 C) \). Hence, recalling \( C = [X] \cup [Y] \), we have \( \neg(z \not\rightarrow^3 X \cup Y) \), as desired.

3. Take \( H \in \text{Hist}(x) \) with \( \neg(H \not\rightarrow^3 X) \) and \( H_1 \subseteq H \) such that \( f(H_1) \cup \{f(x)\} \in \text{Hist}(f(x)) \), hence \( f(H_1) \xrightarrow{f(x)} f(H_1) \cup \{f(x)\} \). Consider \( C = H_1 \cup [X] \). Since \( H_1 \cup \{x\} \subseteq H \) and \( \neg(H \not\rightarrow^3 X) \), we have \( \neg(H_1 \cup \{x\} \not\rightarrow^3 [X]) \). Hence, by Lemma 36(3), \( \neg([f(H_1 \cup \{x\})] \not\rightarrow^3 f([X])) \). Therefore \( f(H_1 \cup [X]) \rightarrow f(H_1) \cup f([X]) \xrightarrow{f(x)} C'_1 \), and since \( f \) is a folding \( H_1 \cup [X] \xrightarrow{f(x)} C'_1 \), with \( f(x_1) = f(x) \) and clearly (given that the transition exists, \( x_1 \not\rightarrow^3 X \), as desired.

For the converse implication, assume that \( f \) is a surjective morphism satisfying conditions (1)-(3). We have to prove that it is a folding.

Let \( C_1 \in \text{Conf}(A) \) and assume that \( f(C_1) \xrightarrow{X} C'_2 \). When \( C_1 = \emptyset \) we argue as in Proposition 32. Otherwise, if \( C_1 \neq \emptyset \), for all \( y \in C_1 \) it holds \( [y] \subseteq C_1 \) and thus \( \neg(x' \not\rightarrow^3 f([y])) \). Thus, by condition (1), there exists some element \( x_y \in A \) such that \( f(x_y) = x' \) and \( \neg(x_y \not\rightarrow y) \). Note that necessarily \( x_y \neq y \).

Since \( C_1 \) is finite and consistent, an inductive argument based on condition (2), allows to derive the existence of \( x \) such that \( f(x) = x' \) and \( \neg(x \not\rightarrow^3 C_1) \). Therefore there is a transition

\[
C_1 \xrightarrow{X} C_2
\]

where \( C_2 = C_1 \cup [x] \) and \( X = [x] \setminus C_1 \).

Let \( H = C_2[x] \). By definition of history, if \( \neg(H \not\rightarrow^3 C_2 \setminus H) \). Let \( H'_1 = f(C_1)[x'] \setminus \{x'\} \) and let \( H_1 \) its \( f \)-counterimage in \( C_1 \). We have \( H_1 \subseteq H \), \( x' = f(x) \notin f(H_1) \) and \( f(H_1) \cup \{f(x)\} \in \text{Hist}(f(x)) \). Then, by condition (3), there exists \( x_1 \) such that \( H_1 \cup \{x_1\} \in \text{Hist}(x_1) \) and \( \neg(x_1 \not\rightarrow^3 C_2 \setminus H) \), hence \( \neg(x_1 \not\rightarrow^3 C_1 \setminus H_1) \). This implies \( C_1 \xrightarrow{X} C_1 \cup \{x_1\} \), as desired. \(\blacksquare\)