Analytic proof of the Sutherland conjecture

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Abstract

Using the integral representation of the inverse of the logarithmic derivative of the elliptic theta function, the spectrum of the Lax matrix for the 1D system of particles interacting via inverse sinh-squared potential is shown to be given by the asymptotic Bethe ansatz in the thermodynamic limit.

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The problem of the verification of the asymptotic Bethe ansatz method [1] still remains unsolved more than 30 years after its first presentation. The method consists in using only scattering data for the description of the integrable many-body systems in the thermodynamic limit. It is well known that if the system is integrable in the Yang-Baxter sense, the many-body scattering matrix is expressed via only two-particle phase shift but real structure of the wave functions might be rather complicated if the interaction is non-local. In particular, when the two-body potential is of the form $\sin^{-2}(\pi x/L)$, where $L$ is the size of the system, or $1/\sinh^2(x)$, the wave functions differ drastically from the linear combinations of the plane waves inherent for the Bethe ansatz. Despite the exact results in the thermodynamic limit available for $\sin^{-2}(\pi x/L)$ case are in complete coincidence with the asymptotic Bethe ansatz, the reason for this coincidence is still quite mysterious and it cannot be used as an argument to validate the method for the case of $1/\sinh^2 x$ pair potential which is more complicated from the mathematical viewpoint.

Due to the lack of a general approach to the problem, any particular exact results confirming the asymptotic Bethe ansatz are of interest. Some years ago, Sutherland [2] proposed one example of very good numerical coincidence of the asymptotic results with exact ones. It concerns the densities of the distribution of the eigenvalues of the $L$ matrix from the Lax pair [3] for the systems of particles interacting via $1/\sinh^2 x$ potential,

$$L_{jk} = p_j \delta_{jk} + (1 - \delta_{jk}) i \lambda \coth(x_j - x_k),$$

where $p_j = -i \partial/\partial x_j$ obey the canonical commutation relations $[x_j, p_k] = i \delta_{jk}$. The corresponding Hamilton operator reads

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \lambda^2 \sum_{j<k} \frac{1}{\sinh^2(x_j - x_k)}.$$

Asymptotically, if $x_1 \ll ... \ll x_N$, the particles have the momenta $k_1 < ... < k_N$ and the elements of the Lax matrix become c-numbers,

$$(L_{as})_{jk} = k_j \delta_{jk} + i \lambda \text{sgn}(j - k).$$

The asymptotic Bethe ansatz gives the asymptotic momenta as solutions to the equations

$$Lk_j = 2\pi I_j + \sum_{l \neq j}^{N} \tau(k_j - k_l),$$
where $\tau(k)$ is the two-body phase shift, $L$ is the total size of a system and $\{I_j\}$ are quantum numbers. In the classical limit $\lambda \to \infty$, for the ground state, (2) becomes an integral equation for large $N$ and $L$ [4],

$$2a = \int_{-A}^{A} dx' \gamma(x - x') \rho(x'),$$

where

$$\gamma(x) = \ln(1 + x^{-2}),$$

$\rho(x)$ is the density of the momentum distribution in the ground state, and $a = L/N$ is average nearest-neighbor spacing (or the lattice constant). The kernel of this integral equation is symmetric and positive definite. Thus it has unique solution at given $A$ [6]. The normalization condition

$$\int_{-A}^{A} \rho(x') dx' = 1$$

defines $A$ as a function of $a$. The distribution of the eigenvalues of the Lax matrix can be connected with the distribution of the momenta [2]. Indeed, it follows from (1) that the equation for the eigenvalues

$$\det(L_{as} - Iz) = \frac{1}{2} N \prod_{j=1}^{N} (k_j - z + i\lambda) + \prod_{j=1}^{N} (k_j - z - i\lambda)$$

$$= \prod_{s=1}^{N} (\omega_s - z) = 0$$

can be written as

$$(s + 1/2)\pi = \frac{1}{2i} \sum_{j=1}^{N} \ln \left[ \frac{k_j - \omega + i\lambda}{k_j - \omega - i\lambda} \right] = \sum_{j=1}^{N} \arctan \left[ \frac{\lambda}{k_j - \omega_s} \right].$$

A discontinuous branch of $\arctan$ with values in $[0, \pi]$ is used here. In the thermodynamic limit, the eigenvalues $\{\omega\}$ are distributed with the density $\sigma(\omega)$: $N\sigma(\omega)d\omega$ gives the number of $\{\omega\}$ in the interval $(\omega, \omega + d\omega)$. Hence

$$\frac{ds}{d\omega} = N\sigma(\omega).$$

Differentiating the above relation with respect to $\omega$ and taking classical limit gives

$$\sigma(\omega) = \frac{1}{2\pi} \int_{-A}^{A} \frac{dx \rho(x)}{(x - \omega)^2 + 1/4}$$

(5)
after rescaling of variables [2]. One can see that in the classical limit the density $\sigma(\omega)$ can be calculated via the solution of the integral equation of the asymptotic Bethe ansatz method (3).

On the other hand, in this limit the particles take their equilibrium positions at $x_j = ja$ in the ground state with $k_j = 0$. The form of the Lax matrix becomes simpler,

$$L_{jk} = (1 - \delta_{jk}) i \lambda \coth[a(j - k)],$$

and the distribution of its eigenvalues can be calculated directly since (6) is of the Toeplitz form. Its eigenvectors are the plane waves, and after imposing periodic boundary condition (i.e. regularization of the determinant) and taking thermodynamic limit $N \to \infty$ one could introduce the variable $\phi = 2\pi s/N$ defining the continuous distribution

$$\omega(\phi) = \omega_s = \omega_{N\phi/2\pi}.$$

The result can be written upon rescaling $\omega \to 2\lambda \omega$ in the form [2]

$$\omega(\phi) = -\frac{\theta_1'(\phi/2)}{2\theta_1(\phi/2)},$$

where $\theta_1(x)$ is the standard theta function

$$\theta_1(x) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n + 1)x$$

with the nome $q = e^{-a}$. The density of the eigenvalues is given now by the formula

$$\sigma(\omega) = \frac{1}{2\pi} \frac{d\phi}{d\omega},$$

where the derivative is calculated through the relation

$$\frac{d\omega}{d\phi} = -\frac{1}{4} \left[ \frac{\theta_1'(\phi/2)}{\theta_1(\phi/2)} \right]' = -\frac{K^2}{\pi^2} \left[ \frac{K - E}{K} - \frac{1}{\sin^2(K\phi/\pi)} \right].$$

Thus one gets two representations for the density of the eigenvalues of the classical Lax matrix, one exact (formulae (7-9)) and one obtained by using the asymptotic Bethe ansatz method (formulae (3-5)). If it is true, they should coincide. The main difficulty in verifying this fact is that there is no chance to find analytic solution to the integral equation (3). In [2], Sutherland found good coincidence of both expressions by solving
this equation numerically with high accuracy. However, analytic solution of the problem has not been found.

In what follows, we propose a construction which uses analytic properties of the elliptic functions and provides the desired proof. Let us introduce the notation $\chi_r = \Re e \chi$, $\chi_i = \Im m \chi$ for any complex $\chi$. Consider at first the problem of explicit construction of the inverse function $\phi(\omega)$ such that

$$\phi(\omega(\lambda)) \equiv \lambda.$$  

It is clear that it is no longer holomorphic in the $\omega$-plane. Indeed, on the lines $\phi = \phi_r \pm ia$ one finds $\omega(\phi) = \omega_r \pm i/2$ due to the quasiperiodicity property

$$\omega(\phi + 2ia) = \omega(\phi) + i, \quad \omega(\phi + 2\pi) = \omega(\phi).$$

One has also

$$\omega_r(\pm ia) = \omega_r(2\pi \pm ia) = 0.$$

The derivative $\frac{d\omega}{d\phi}$ is double periodic with periods $(2\pi, 2ia)$ and has double pole in the fundamental domain $0 \leq \Re e \phi < 2\pi, \quad -a \leq \Im m \phi < a.$

Therefore it must have just two zeros giving two extremal points of $\omega(\phi)$: one minimum of $\omega_r$ located at $\phi_{\text{min}} + ia$ and one maximum located at $2\pi - \phi_{\text{min}} + ia$. Both these extrema are considered with respect to line $\Im m \phi = a$, and $\phi_{\text{min}} \in (0, 2\pi)$. Let us denote $\Omega_0 = \omega_r(2\pi - \phi_{\text{min}} + ia)$. Then it is evident that the function $\phi(\omega)$ should have two cuts in the $\omega$-plane represented by the segments $-\Omega_0 \leq \omega_r \leq \Omega_0$, $\omega_i = 1/2$ and $-\Omega_0 \leq \omega_r \leq \Omega_0$, $\omega_i = -1/2$. Following Haldane [5], let us express $\phi$ as a Cauchy integral over the contour along the image of the boundary of the fundamental domain and use the symmetry properties of $\omega(\phi)$. We skip these rather long but in fact simple considerations. Only integral over the finite interval remains and after integrating by parts we obtain

$$\phi(\omega) = \int_{-\Omega_0}^{\Omega_0} dx \rho_0(x) \frac{1}{i} \ln \frac{\omega - x - i/2}{\omega - x + i/2}. \quad (10)$$

The still unknown function $\rho_0(x)$ is normalized due to the properties of the function $\phi$: $\omega(\phi + 2\pi) = \omega(\phi)$ and the integral representation (10),

$$\int_{-\Omega_0}^{\Omega_0} \rho_0(x) dx = 1. \quad (11)$$
On the other hand, we know that $\phi_i(\omega \pm i/2) = \pm ia$ for all real $\omega$ in the interval $-\Omega_0 \leq \omega \leq \Omega_0$. This gives an integral equation for the function $\rho_0(x)$ entering (10) of the form quite similar to (3),

$$\Im m\phi(\omega + i/2) = a = \frac{1}{2} \int_{-\Omega_0}^{\Omega_0} dx \rho_0(x) \ln(1 + (\omega - x)^{-2}).$$

(12)

Note also that the same equation can be obtained with the use of quasiperiodicity property $\omega(\phi + 2ia) = \omega(\phi) + i$ and the representation (10).

The equations (3) and (12) become completely identical if one puts $A = \Omega_0$, i.e. the meaning of the parameter $A$ is that it defines the maximal value of $\omega_r(\phi)$ on the segment $0 \leq \phi_r \leq 2\pi$, $\phi_i = ia$ due to the uniqueness of the solution of (3) mentioned above.

It is straightforward now to verify by differentiating (10) with respect to $\omega$ that the derivative $\frac{d\omega}{d\phi}$ has the integral representation

$$\left(\frac{d\omega}{d\phi}\right)^{-1} = \int_{-\Omega_0}^{\Omega_0} dx \rho_0(x) \frac{dx}{(\omega - x)^2 + 1/4}.$$  (13)

Comparing both sets of formulas (3-5) and (11-13), one can easily see that the expressions for the spectral density of the Lax matrix in the classical limit coincide after identification $\rho(x) = \rho_0(x)$. This completes the proof.

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References

1. B. Sutherland. J.Math.Phys. 12, 251 (1971)

2. B. Sutherland. Phys. Rev. Lett. 75, 1248 (1995)

3. F. Calogero, C. Marchioro and O. Ragnisco. Lett. Nuovo Cim. 13, 383 (1975)

4. B. Sutherland, R.A. Römer and B.S. Shastry. Phys. Rev. Lett. 73, 2154 (1994)

5. F.D.M. Haldane (unpublished)

6. E.T. Whittaker and G.N. Watson. A Course of Modern Analysis, Cambridge at the University Press, 1927