Asymptotics of forward implied volatility

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Based on joint works with Patrick Roome (Imperial College London):

- The small-maturity Heston forward smile. *SIAM Journ on Fin. Math.* (4)1, 2013.
- Asymptotics of forward implied volatility. *Submitted, arxiv 1212.0779.*
- Large-maturity regimes of the Heston forward smile. *In progress.*
(Spot) implied volatility

- Asset price process: $(S_t = e^{X_t})_{t \geq 0}$, with $X_0 = 0$.
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:

$$C_{BS}(\tau, k, \sigma) := \mathbb{E}_0 \left( e^{X_\tau} - e^k \right)_+ = \mathcal{N}(d_+) - e^k \mathcal{N}(d_-),$$

$$d_{\pm} := -\frac{k}{\sigma \sqrt{\tau}} \pm \frac{1}{2} \sigma \sqrt{\tau}.$$

- Spot implied volatility $\sigma_\tau(k)$: the unique (non-negative) solution to

$$C_{observed}(\tau, k) = C_{BS}(\tau, k, \sigma_\tau(k)).$$

- Spot implied volatility: unit-free measure of option prices.
- However not available in closed form for most models.
Spot implied volatility ($\sigma_\tau(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Berestycki-Busca-Florent (2004): small-$\tau$ using PDE methods for diffusions.
- Henry-Labordère (2009): small-$\tau$ asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small-and large-$\tau$ using large deviations and saddlepoint methods.
- Lee (2003), Benaim-Friz (2009), Gulisashvili (2010-2012), De Marco-Jacquier-Hillairet (2013): $|k| \uparrow \infty$.
- Laurence-Gatheral-Hsu-Ouyang-Wang (2012): small-$\tau$ in local volatility models.
- Fouque et al.(2000-2011): perturbation techniques for slow and fast mean-reverting stochastic volatility models.
- Mijatović-Tankov (2012): small-$\tau$ for jump models.

Related works:

- Kim, Kunitomo, Osajima, Takahashi (1999-...): asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).
- Deuschel-Friz-Jacquier-Violante (CPAM 2014), De Marco-Friz (2014): small-noise expansions using Laplace method on Wiener space (Ben Arous-Bismut approach).

Note: from expansions of densities to implied volatility asymptotics is ‘automatic’ (Gao-Lee (2013)).
Forward implied volatility

- Fix $t > 0$: forward-starting date; $\tau > 0$: remaining maturity.
- Forward-start call option is a European call option with payoff
  \[
  \left( \frac{S_{t+\tau}}{S_t} - e^k \right)^+ = \left( e^{X_{t+\tau}} - X_t - e^k \right)^+, 
  \]
  and value today
  \[
  \mathbb{E}_0 \left( e^{X_{t+\tau}} - X_t - e^k \right)^+. 
  \]
- BSM model: its value today is simply worth $C_{BS}(\tau, k, \sigma)$ (stationary increments).
- Forward implied volatility $\sigma_{t,\tau}(k)$: the unique solution to
  \[
  C_{observed}(t, \tau, k) = C_{BS}(\tau, k, \sigma_{t,\tau}(k)). 
  \]
- Obviously, $\sigma_{0,\tau}(k) = \sigma_{\tau}(k)$.
- Alternative definition: $\left( S_{t+\tau} - S_t e^k \right)^+$. 

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Asymptotics of forward implied volatility
Motivation

Calibration:

- Forward-start options serve as natural hedging instruments for many exotic securities and it is therefore important for a model to be able to calibrate to liquid forward smiles.

Model Risk:

- Calibrate two different models to some observed spot implied volatility smiles: perfect calibration. Use these calibrated models to price some 'exotic' options, say barrier options: two different prices. One of the reasons: subtle dependence on the dynamics of implied volatility smiles.

- One metric that can be used to understand the dynamics of implied volatility smiles (Bergomi(2004) calls it a 'global measure' of the dynamics of implied volatilities) is to use the forward smile defined above.
Existing literature on forward smiles

- Glasserman and Wu (2011): different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility.
- Keller-Ressel (2011): when the forward-start date $t$ becomes large ($\tau$ fixed).
- Empirical results: Bergomi (2004), Bühler (2002), Gatheral (2006).
- Bompis-Hok (2013): expansion in local volatility models.
Today’s menu

- Asymptotics in time ($t$) / maturity ($\tau$) of forward implied volatility smiles.
- What will not be covered: connections with VIX / variance swaps.
Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

\[ \begin{align*}
    dX_t &= -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t, \quad X_0 = 0, \\
    d\sigma_t &= (a + b\sigma_t) dt + \xi dZ_t, \quad \sigma_0 = \sigma_0 > 0,
\end{align*} \]
Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

\[
\begin{align*}
\frac{dX_t}{dt} &= -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t, \quad X_0 = 0, \\
\frac{d\sigma_t}{dt} &= (a + b\sigma_t) dt + \xi dZ_t, \quad \sigma_0 = \sigma_0 > 0, 
\end{align*}
\]

Define \( X^{(t)}_\tau := X_{t+\tau} - X_t \), then

\[
\begin{align*}
\frac{dX^{(t)}_\tau}{dt} &= -\frac{1}{2} (\sigma^{(t)}_\tau)^2 d\tau + \sigma^{(t)}_\tau dW_\tau, \quad X^{(t)}_0 = 0, \\
\frac{d\sigma^{(t)}_\tau}{dt} &= (a + b\sigma^{(t)}_\tau) d\tau + \xi dZ_\tau, \quad \sigma^{(t)}_0 \sim \sigma_t, 
\end{align*}
\]
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    dX^{(t)}_\tau &= -\frac{1}{2} (\sigma^{(t)}_\tau)^2 d\tau + \sigma^{(t)}_\tau dW_\tau, \quad X^{(t)}_0 = 0, \\
    d\sigma^{(t)}_\tau &= (a + b \sigma^{(t)}_\tau) d\tau + \xi dZ_\tau, \quad \sigma^{(t)}_0 \sim \sigma_t,
\end{align*}
\]

**Pricing Fwd-start options:**

\[
    \mathbb{E}_0 (e^{X^{(t)}_\tau} - e^k)^+ = \mathbb{E}_0 \left\{ \mathbb{E}_t (e^{X^{(t)}_\tau} - e^k | \sigma_t)^+ \right\}
\]

**Problem:** Known expansions (as \( \tau \downarrow 0 \)) are NOT uniform in space.

*Easy case:* \( \sigma_t \) has compact support, e.g. finite-state Markov chain.
General results: framework

\((Y_\varepsilon)\): (general) stochastic process. Denote the re-normalised log moment generating function (lmgf) by \(\Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{uY_\varepsilon}{\varepsilon} \right) \right]\), for all \(u \in D_\varepsilon \subseteq \mathbb{R}\).

We require the following assumptions (Assumption OA) on the behaviour of \(\Lambda_\varepsilon\):

(i) **Expansion property:** \(\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + O(\varepsilon^3)\) holds for \(u \in D_0^o\) as \(\varepsilon \downarrow 0\);

(ii) **Differentiability:** The map \((\varepsilon, u) \mapsto \Lambda_\varepsilon(u)\) is of class \(C^\infty\) on \((0, \varepsilon_0) \times D_0^o\);

(iii) **Non-degenerate interior:** \(0 \in D_0^o\);

(iv) **Essential smoothness:** \(\Lambda_0\) is strictly convex and essentially smooth on \(D_0^o\);

(v) **Tail error control:** For any fixed \(p_r \in D_0^o \setminus \{0\}\),
   
   (a) \(\Re(\Lambda_\varepsilon(ip_i + p_r)) = \Re(\Lambda_0(ip_i + p_r)) + O(\varepsilon)\), for any \(p_i \in \mathbb{R}\);
   
   (b) \(L : \mathbb{R} \ni p_i \mapsto \Re(\Lambda_0(ip_i + p_r))\) has a unique maximum at zero and is bounded away from \(L(0)\) as \(|p_i|\) tends to infinity;
   
   (c) \(\Re(\Lambda_\varepsilon(ip_i + p_r) - \Lambda_0(ip_i + p_r)) \leq M\varepsilon\), for some \(M > 0\), for large \(|p_i|\) and small \(\varepsilon\).

**Note:** (i)-(iv) are Gårtner-Ellis assumptions for large deviations:

\[
\mathbb{P}(Y_\varepsilon \in A) \sim \exp \left( -\frac{1}{\varepsilon} \inf \{\Lambda^*(x) : x \in A\} \right),
\]

for \(A \subset \mathbb{R}\), as \(\varepsilon \downarrow 0\), \(\Lambda^*\): dual of \(\Lambda_0\).
Main result: Option price asymptotics

Theorem (J-Roome, 2013)

Let \((Y_\varepsilon)\) satisfy OA and \(f : \mathbb{R}_+ \to \mathbb{R}_+\) be a function satisfying \(f(\varepsilon)\varepsilon = c + \mathcal{O}(\varepsilon)\), for some constant \(c \geq 0\) as \(\varepsilon \downarrow 0\). For \(k > \Lambda'_0(c)\), as \(\varepsilon \downarrow 0\),

\[
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{kf(\varepsilon)} \right)^+ = \psi(k, c, \varepsilon) \exp \left\{ -\frac{\Lambda^*(k)}{\varepsilon} + kf(\varepsilon) \right\} \left[ 1 + \alpha_1(k, c)\varepsilon + \mathcal{O}(\varepsilon^2) \right],
\]

where \(\psi(k, c, \varepsilon) \equiv \alpha(k, c) \left( c\sqrt{\varepsilon} \mathbb{1}_{\{c > 0\}} + \varepsilon^{3/2} f(\varepsilon) \mathbb{1}_{\{c = 0\}} \right)\), and \(\Lambda^* : \mathbb{R} \to \mathbb{R}_+\) is the Fenchel-Legendre transform of \(\Lambda_0\):

\[
\Lambda^*(k) := \sup_{u \in \mathcal{D}_0} \{uk - \Lambda_0(u)\}, \quad \text{for all } k \in \mathbb{R}.
\]

We shall denote \(u^*(k)\) the corresponding saddlepoint: \(\Lambda^*(k) = u^*(k)k - \Lambda_0(u^*(k))\).

Analogous results hold for Put options when \(k < \Lambda'_0(c)\) and for covered calls when \(k \in (\Lambda'_0(0), \Lambda'_0(c))\).
Application I: forward-start large-maturity

Recall the fwd-start process: \( X^{(t)}_{\tau} := X_{t+\tau} - X_t \). Let \( (Y_\varepsilon) := (\varepsilon X^{(t)}_{\tau/\varepsilon}) \) and \( f(\varepsilon) \equiv 1/\varepsilon \):

\[
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{k f(\varepsilon)} \right)^+ = \mathbb{E} \left( e^{X^{(t)}_{\tau/\varepsilon}} - e^{k/\varepsilon} \right)^+ 
\]

Corollary (Large-maturity, \( t \geq 0 \))

If \( (\tau^{-1} X^{(t)}_{\tau})_{\tau>0} \) satisfies OA with \( \varepsilon = \tau^{-1} \) and \( 1 \in \mathcal{D}_0^0 \), then for \( k > \Lambda'_0(1) \), as \( \tau \uparrow \infty \):

\[
\mathbb{E}_0 \left( e^{X^{(t)}_{\tau}} - e^{k \tau} \right)^+ = \frac{e^{-\tau(\Lambda^*(k) - k) + \Lambda_1(u^*(k))} \tau^{-1/2}}{u^*(k)(u^*(k) - 1) \sqrt{2\pi \Lambda'_0(u^*(k))}} \left[ 1 + \frac{\alpha_1(k)}{\tau} + O \left( \frac{1}{\tau^2} \right) \right]
\]

Corollary: \( (\tau^{-1} X^{(t)}_{\tau})_{\tau} \) satisfies a LDP with speed \( \tau^{-1} \) and rate function \( k \mapsto \Lambda^*(k) - k \).

When \( t = 0 \), we recover Jacquier, Keller-Ressel, Mijatović (2013), which intuitively makes sense.
Application II: forward-start diagonal small-maturity

Recall the fwd-start process: \( X_{t+\tau}^{(t)} := X_{t+\tau} - X_t \). Let \((Y_\varepsilon) := (X_{\varepsilon\tau}^{(\varepsilon t)})\) and \(f(\varepsilon) \equiv 1:\)

\[
\mathbb{E} \left( e^{Y_\varepsilon f(\varepsilon)} - e^{k f(\varepsilon)} \right)^+ = \mathbb{E} \left( e^{X_{\varepsilon\tau}^{(\varepsilon t)}} - e^k \right)^+
\]

**Corollary (Diagonal small-maturity, \( t, \tau > 0 \))**

If \( (X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon > 0} \) satisfies OA, then the following holds for \( k > \Lambda_0'(0) \), as \( \varepsilon \downarrow 0 \):

\[
\mathbb{E}_0 \left( e^{X_{\varepsilon\tau}^{(\varepsilon t)}} - e^k \right)^+ = \frac{\exp \left\{ - \frac{\Lambda^*(k)}{\varepsilon} + k + \Lambda_1(u^*(k)) \right\} \varepsilon^{3/2}}{u^*(k)^2 \sqrt{2\pi \Lambda_0''(u^*(k))}} [1 + \alpha_1(k)\varepsilon + O(\varepsilon^2)] .
\]

**Corollary:** \( (X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon > 0} \) satisfies a LDP with speed \( \varepsilon \) and rate function \( \Lambda^* \).
Corollary I: large-maturity forward smile

Corollary (Large-maturity forward smile asymptotics)

If \((\tau^{-1}X_{\tau}(t))_{\tau>0}\) satisfies OA with \(\varepsilon = \tau^{-1}\) and \(\Lambda_0(1) = 0\) with \(1 \in D_0^o\), then for all \(k \in \mathbb{R}\), as \(\tau \uparrow \infty\):

\[
\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k, t) + \frac{v_1^\infty(k, t)}{\tau} + \frac{v_2^\infty(k, t)}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right),
\]

where \(v_0^\infty(\cdot, t)\), \(v_1^\infty(\cdot, t)\) and \(v_2^\infty(\cdot, t)\) are continuous functions on \(\mathbb{R}\).

- If \(S = e^X\) is a true martingale, then \(\Lambda_0(1) = 0\).
- For \(t = 0\) (spot smiles), we recover Jacquier, Keller-Ressel, Mijatović (2013) in the case of affine stochastic volatility models with jumps.
Corollary II: diagonal small-maturity forward smile

Corollary (Diagonal small-maturity forward smile asymptotics)

If $(X_{\varepsilon t}^{\tau})_{\varepsilon > 0}$ satisfies OA and $\Lambda_0'(0) = 0$, then for all $k \in \mathbb{R}$, as $\varepsilon \downarrow 0$,

$$
\sigma_{\varepsilon t, \varepsilon \tau}^2(k) = v_0(k, t, \tau) + v_1(k, t, \tau)\varepsilon + v_2(k, t, \tau)\varepsilon^2 + O(\varepsilon^3),
$$

where $v_0(\cdot, t, \tau)$, $v_1(\cdot, t, \tau)$ and $v_2(\cdot, t, \tau)$ are continuous functions on $\mathbb{R}$.

Under the assumption $\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + O(\varepsilon^3)$, as $\varepsilon \downarrow 0$, $v_0$, $v_1$, and $v_2$ depend on the derivatives of $\Lambda_0$, $\Lambda_1$ and $\Lambda_2$ evaluated at $u^*(k)$. When $t = 0$ (spot smiles), we recover Forde-Jacquier-Lee (2012), Gao-Lee (2013), Berestycki-Busca-Florent (2004).
Examples

- \( S \): exponential Lévy model. Stationary increment property implies \( \sigma_{t, \tau} \) does not depend on \( t \).
- \( S \): time-changed exponential Lévy model.
- Stochastic volatility models; Schöbel-Zhu: \( \text{d}\sqrt{V_t} = \kappa(\theta - \sqrt{V_t})\text{d}t + \xi \text{d}Z_t \).
- Heston (affine stochastic volatility) model:

\[
\begin{align*}
\text{d}X_t &= -\frac{1}{2} V_t \text{d}t + \sqrt{V_t} \text{d}W_t, \\
\text{d}V_t &= \kappa (\theta - V_t) \text{d}t + \xi \sqrt{V_t} \text{d}Z_t, \\
\text{d}\langle W, Z \rangle_t &= \rho \text{d}t,
\end{align*}
\]

with \( \kappa > 0, \xi > 0, \theta > 0 \) and \( |\rho| < 1 \).
Heston diagonal small-maturity

• We can compare spot and forward (diagonal) small-maturity smiles:

\[
\sigma_{\varepsilon t,\varepsilon \tau}(0) = \sigma_{0,\varepsilon \tau}(0) - \frac{\varepsilon t}{8\sqrt{v}} (\xi^2 + 4\kappa(v - \theta)) + O(\varepsilon^2),
\]

\[
\partial^2_k \sigma_{\varepsilon t,\varepsilon \tau}(0) = \partial^2_k \sigma_{0,\varepsilon \tau}(0) + \frac{\xi^2 t}{4\tau v^{3/2}} + O(\varepsilon).
\]

• At zeroth order in \(\varepsilon\) the wings of the forward smile increase to arbitrarily high levels with decreasing maturity.

• Bühler (2002): ‘Heston implied forward volatility: short skew becomes U-shaped, which is inconsistent with observations.’
**Figure:** In (a) circles, squares and diamonds represent the zeroth, first- and second-order asymptotics respectively and triangles represent the true forward smile using Fourier inversion. In (b): differences between the true forward smile and the asymptotic. We use $t = 1/2$ and $\tau = 1/12$ and the Heston parameters $\nu = 0.07$, $\theta = 0.07$, $\kappa = 1$, $\xi = 0.34$, $\rho = -0.8$.

**Copyright:** Numerics performed with the IPython-based Zanadu interface provided by Zeliade Systems.
Heston small-maturity asymptotics: problem overview

Consider now fixed $t > 0$, and $\tau \downarrow 0$. The framework above does not apply.
Heston small-maturity asymptotics: problem overview

Consider now fixed $t > 0$, and $\tau \downarrow 0$. The framework above does not apply.

Rescaled log mgf: $\Lambda_{\tau}(t)(u, a) := a \log \mathbb{E}\left(e^{uX_{\tau}(t)/a}\right)$.

**Lemma**

If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda_{\tau}(t)(u, h(\tau)) = 0$, for $|u| < a/\sqrt{\beta_t}$ and $\infty$ otherwise;

Define $\Lambda(u) := \lim_{\tau \downarrow 0} \Lambda_{\tau}(t)(u)$ on $D_\Lambda$, and its Fenchel-Legendre transform $\Lambda^*(k) := \sup\{uk - \Lambda(u), u \in D_\Lambda\}$.

**Corollary**

$\Lambda^*(k) = |k|/\sqrt{\beta_t}$, for all $k \in \mathbb{R}$.

Clearly, no convexity argument holds here.
Theorem (J-Roome, 2012)

Let \( t > 0 \). In the Heston model, the following expansion holds for all \( k \in \mathbb{R}^* \) as \( \tau \downarrow 0 \):

\[
\mathbb{E} \left( e^{X^{(t)}_{\tau}} - e^k \right)^+ = \left( 1 - e^k \right) \mathbb{1}_{\{k < 0\}} \\
+ \exp \left( -\frac{\Lambda^*(k)}{\sqrt{\tau}} + \frac{c_0(k)}{\tau^{1/4}} + c_1(k) + k \right) \tau^{7/8} - \frac{\kappa \theta}{2\sigma^2} c_2(k) \left[ 1 + c_3(k) \tau^{1/4} + o(\tau^{1/4}) \right].
\]

Corollary: \( (X^{(t)}_{\tau})_{\tau \geq 0} \) satisfies a LDP with speed \( \sqrt{\tau} \) and rate function \( \Lambda^* \) as \( \tau \downarrow 0 \).

Compare with (see Forde-Jacquier-Lee (2012)), when \( t = 0 \):

- \[
\mathbb{E} \left( e^{X^{(0)}_{\tau}} - e^k \right)^+ = \left( 1 - e^k \right) \mathbb{1}_{\{k < 0\}} + \exp \left( \frac{\Lambda^*(k)}{\tau} \right) \tau^{3/2} c_2(k) \left( 1 + \mathcal{O}(\tau) \right),
\]

- \( (X^{(0)}_{\tau})_{\tau \geq 0} \) satisfies a LDP with speed \( \tau \) and good rate function \( \Lambda^* \) as \( \tau \downarrow 0 \).
Small-maturity smile

Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$
\sigma_{t, \tau}^2(k) = \begin{cases} 
\frac{\nu_0(k, t)}{\tau^{1/2}} + \frac{\nu_1(k, t)}{\tau^{1/4}} + o\left(\frac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\
\frac{v_0(k, t)}{\tau^{1/2}} + \frac{v_1(k, t)}{\tau^{1/4}} + v_2(k, t) + v_3(k, t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2.
\end{cases}
$$

Compare with the $t = 0$ case: $\sigma_{0, \tau}^2(k) = \sigma_0^2(k) + a(k)\tau + o(\tau)$, when $k \neq 0$. 
Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$
\sigma_{t,\tau}^2(k) = \begin{cases}
\frac{v_0(k, t)}{\tau^{1/2}} + \frac{v_1(k, t)}{\tau^{1/4}} + o\left(\frac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\
\frac{v_0(k, t)}{\tau^{1/2}} + \frac{v_1(k, t)}{\tau^{1/4}} + v_2(k, t) + v_3(k, t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2.
\end{cases}
$$

Compare with the $t = 0$ case: $\sigma_{0,\tau}^2(k) = \sigma_0^2(k) + a(k)\tau + o(\tau)$, when $k \neq 0$.

At-the-money case $k = 0$, $t > 0$.

As $\tau \downarrow 0$,

$$
\sigma_{t,\tau}(0) = \begin{cases}
\Delta_0(t) + o(1), & \text{if } 4\kappa\theta \leq \xi^2, \\
\Delta_0(t) + \Delta_1(t)\tau + o(\tau), & \text{if } 4\kappa\theta > \xi^2.
\end{cases}
$$
**Figure:** Here $t = 1$ and $\tau = 1/12$. In (a) circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively and backwards triangles represent the true forward smile using Fourier inversion. In (b) we plot the errors.
Sketch of proof: large deviations analysis
Step 1: Find the speed of convergence

Rescaled log mgf: \( \Lambda^{(t)}_\tau(u, a) := a \log \mathbb{E} \left( e^{uX^{(t)}_\tau} / a \right) \).

Lemma

Let \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) continuous with \( \lim_{\tau \downarrow 0} h(\tau) = 0 \) and \( a > 0 \). Then

(i) If \( h(\tau) \equiv a\sqrt{\tau} \) then \( \lim_{\tau \downarrow 0} \Lambda^{(t)}_\tau(u, h(\tau)) = 0 \), for \( |u| < a/\sqrt{\beta_t} \) and \( \infty \) otherwise;
(ii) if \( \sqrt{\tau} / h(\tau) \uparrow \infty \) then \( \lim_{\tau \downarrow 0} \Lambda^{(t)}_\tau(u, h(\tau)) = 0 \), for \( u = 0 \) and \( \infty \) otherwise;
(iii) if \( \sqrt{\tau} / h(\tau) \downarrow 0 \) then \( \lim_{\tau \downarrow 0} \Lambda^{(t)}_\tau(u, h(\tau)) = 0 \), for all \( u \in \mathbb{R} \).

(i) is the only non-trivial zero limit and \( \mathcal{D}_\Lambda = (-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t}) \).

Define \( \Lambda(u) := \lim_{\tau \downarrow 0} \Lambda^{(t)}_\tau(u) \) on \( \mathcal{D}_\Lambda \), and its Fenchel-Legendre transform \( \Lambda^*(k) := \sup \{ uk - \Lambda(u), u \in \mathcal{D}_\Lambda \} \).

Lemma

\( \Lambda^*(k) = |k|/\sqrt{\beta_t} \), for all \( k \in \mathbb{R} \).
Step 2: Weak convergence of (rescaled) measure

Consider the saddlepoint equation (\(\star\)): \(\partial_u \Lambda^{(t)}_\tau(u^*_\tau(k)) = k\).

Lemma

For any \(k \neq 0\), \(\tau > 0\), \((\star)\) admits a unique solution \(u^*_\tau(k)\), and
\(u^*_\tau(k) = a_0(k) + a_1(k)\tau^{1/4} + a_2(k)\tau^{1/2} + a_3(k)\tau^{3/4} + O(\tau) \in D^o_\Lambda\), as \(\tau \downarrow 0\).

For small \(\tau\), introduce the time-dependent change of measure

\[
\frac{dQ_{k,\tau}}{dP} := \exp \left( \frac{u^*_\tau(k)X^{(t)}(t)}{\sqrt{\tau}} - \frac{\Lambda^{(t)}_\tau(u^*_\tau(k))}{\sqrt{\tau}} \right).
\]

Define \(Z_{\tau,k} := (X^{(t)}(t) - k)/\tau^{1/8}\) and \(\Phi_{\tau,k}(u) := \mathbb{E}^{Q_{k,\tau}}(e^{iuZ_{\tau,k}})\).

Lemma

The following expansion holds for all \(k \neq 0\) as \(\tau \downarrow 0\):

\[
\Phi_{\tau,k}(u) = e^{-\frac{1}{2} \zeta^2(k)u^2} \left[ 1 + \phi_1(k, u)\tau^{1/8} + \phi_2(k, u)\tau^{1/4} + O(\tau^{3/8}) \right]. \tag{1}
\]

Corollary: \(Z_{\tau,k}\) converges weakly to \(\mathcal{N}(0, \zeta(k)^2)\) under \(Q_{k,\tau}\).
Step 3: Wrapping up

\[
E \left[ e^{X^{(t)}_\tau} - e^k \right]^+ = E^{Q_{k,\tau}} \left[ \frac{dQ_{k,\tau}}{dP} \left\{ e^{X^{(t)}_\tau} - e^k \right\}^+ \right] = e \frac{\Lambda^{(t)}_\tau(u^*_\tau)}{\sqrt{\tau}} E^{Q_{k,\tau}} \left[ e^{-\frac{u^*_\tau X^{(t)}_\tau}{\sqrt{\tau}}} \left\{ e^{X^{(t)}_\tau} - e^k \right\}^+ \right] = e^{-\frac{k u^*_\tau - \Lambda^{(t)}_\tau(u^*_\tau)}{\sqrt{\tau}}} e^k E^{Q_{k,\tau}} \left[ e^{-\frac{u^*_\tau Z_{\tau,k}}{\tau^{3/8}}} \left( e^{Z_{\tau,k} \tau^{1/8}} - 1 \right)^+ \right].
\]

Final steps, take the Fourier transform:

\[
\mathcal{F} \left( e^{-\frac{u^*_\tau Z_{\tau,k}}{\tau^{3/8}}} \left( e^{Z_{\tau,k} \tau^{1/8}} - 1 \right)^+ \right)(u) = C_{k,\tau}(u)
\]

use Parseval’s identity (or so):

\[
E \left[ e^{-\frac{u^*_\tau Z_{\tau,k}}{\tau^{3/8}}} \left( e^{Z_{\tau,k} \tau^{1/8}} - 1 \right)^+ \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi_{\tau,k}(u) \overline{C_{k,\tau}(u)} \, du,
\]

‘conclude’ using (1) and a control of the tails

\[
\left| \int_{|u|>1/\sqrt{\varepsilon}} \Phi_{\tau,k}(u) \overline{C_{\varepsilon,k}(u)} \, du \right| = O(e^{-\gamma/\varepsilon}).
\]
Getting some intuition: Gärtner-Ellis theorem

Rescaled cumulant generating function (\( h \) continuous with \( \lim_{\tau \downarrow 0} h(\tau) = 0 \)):

\[
\Lambda_{\tau}(u, h) := h(\tau) \log \mathbb{E} \left( S_{\tau}^{u/h(\tau)} \right) = h(\tau) \log \mathbb{E} \left( e^{(u/h(\tau)) X_{\tau}} \right), \quad u \in D_{\tau,h} \subset \mathbb{R}.
\]

**Theorem (Gärtner-Ellis)**

If \( \Lambda(u) := \lim_{\tau \downarrow 0} \Lambda_{\tau}(u, h) \) exists in \( \mathbb{R} \) for \( u \in D_0 \), and \( \Lambda \) strictly convex and differentiable on \( D_0^0 \), with \( \lim_{u \in \partial D_0} \Lambda'(u) = +\infty \), then \( (X_{\tau})_{\tau > 0} \) satisfies a large deviations principle (LDP) (with speed \( h(\tau) \)) as \( \tau \downarrow 0 \):

\[
P(X_{\tau} \in A) \sim \exp \left( - \frac{1}{h(\tau)} \inf \{ \Lambda^*(x) : x \in A \} \right), \quad A \subset \mathbb{R}.
\]

**Lemma**: For diffusions (Black-Scholes, stochastic volatility,...), exp-Lévy, \( h(\tau) \equiv \tau \).
Getting some intuition: Gärtner-Ellis theorem

Rescaled cumulant generating function ($h$ continuous with $\lim_{\tau \downarrow 0} h(\tau) = 0$):

$$\Lambda_{\tau}(u, h) := h(\tau) \log \mathbb{E} \left( S_{\tau}^{u/h(\tau)} \right) = h(\tau) \log \mathbb{E} \left( e^{(u/h(\tau))X_{\tau}} \right), \quad u \in D_{\tau,h} \subset \mathbb{R}.$$ 

Theorem (Gärtner-Ellis)

If $\Lambda(u) := \lim_{\tau \downarrow 0} \Lambda_{\tau}(u, h)$ exists in $\mathbb{R}$ for $u \in D_0$, and $\Lambda$ strictly convex and differentiable on $D_0^o$, with $\lim_{u \in \partial D_0} \Lambda'(u) = +\infty$, then $(X_{\tau})_{\tau > 0}$ satisfies a large deviations principle (LDP) (with speed $h(\tau)$) as $\tau \downarrow 0$:

$$\mathbb{P}(X_{\tau} \in A) \sim \exp \left( -\frac{1}{h(\tau)} \inf \{ \Lambda^*(x) : x \in A \} \right), \quad A \subset \mathbb{R}.$$ 

Lemma: For diffusions (Black-Scholes, stochastic volatility,...), exp-Lévy, $h(\tau) \equiv \tau$.

Forward problem: $S$: Heston model, and $\Lambda_{\tau}^{(t)}$ the rescaled forward cgf, then

Lemma

If $h(\tau) \equiv \sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda_{\tau}^{(t)}(u, h(\tau)) = 0$, for $u \in (\underline{u}, \overline{u})$ and $\infty$ otherwise;