An $SO(5)$ Symmetric Ladder

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Abstract

We construct an $SO(5)$ symmetric electron model on a two-chain ladder with purely local interactions on a rung. The ground state phase diagram of this model is determined in the strong-coupling limit. The relationship between the spin-gap magnon mode of the spin-gap insulator and the $\pi$ resonance mode of the $d$-wave pairing phase is discussed. We also present the exact ground state for an $SO(5)$ superspin model.
I. INTRODUCTION

The cuperate two-leg ladder materials are characterized by strong electronic correlation and a variety of competing ground states [1]. Numerical and analytic calculations for Hubbard [2] and \( t - J \) models [3] have shown that at half-filling, these two-leg ladder models exhibit a spin-gap insulating phase and that when the system is initially doped, \( d_{x^2-y^2} \)-like pairing and CDW correlation can become dominant. At higher doping the system is expected to behave as a one-dimensional Luttinger liquid [4]. Given the strongly interactive nature of these systems and the delicate balance between their competing ground states, one would like to have a more general framework for determining their properties. Recently it was suggested that one could capture some of the basic low energy physics with models having an \( SO(5) \) symmetry [5,6]. As we will discuss, there is in fact a natural way to construct an \( SO(5) \) symmetric model for a two-leg ladder which has only local interactions on a rung. In this work, we will show how to construct such a Hamiltonian and discuss its strong-coupling phase diagram and low lying collective excitations.

This work is partially motivated to use the ladder system as a theoretical laboratory to check some ideas of the \( SO(5) \) theory. Recently, two dimensional lattice electronic models with exact \( SO(5) \) symmetry have been constructed by three groups independently [7–9]. However, these models all involve long range interactions. The \( SO(5) \) symmetric ladder models constructed in this work involve only local interactions on the rung and are therefore much easier to visualize. There has been considerable progress in numerically checking the approximate \( SO(5) \) symmetry in the 2D \( t - J \) and Hubbard model [10,11]. The locally \( SO(5) \) symmetric ladder models constructed in this work are simple to implement numerically. By systematically varying the parameter away from the \( SO(5) \) symmetric point, one can trace the evolution of the \( SO(5) \) multiplet structure [11] and get a better sense of the nature of the approximate \( SO(5) \) symmetry. In this work, we find a continuous quantum phase transition from the spin gap Mott insulator phase to the \( d \) wave superconducting phase, and we show that the spin gap magnon mode of the Mott insulator evolves continuously.
into the $\pi$ resonance mode of the superconducting phase. These results also shed some light on the nature of the $\pi$ resonance mode in the 2D case. Perhaps one of the most important questions in the $SO(5)$ theory concerns the origin of such a symmetry in generic models. Recently, Shelton and Senechal studied the problem of two coupled 1D Tomonaga-Luttinger chains and concluded that approximate $SO(5)$ symmetry can emerge in the low energy limit of this model. Balents, Fisher and Lin have used a weak coupling RG method combined with abelian bosonization to show that a generic ladder model at half-filling flows to a manifold with $SO(5)$ symmetry (and in fact, to various phases characterized by higher symmetries). Arrigoni and one of us (WH) additionally included a next-nearest-neighbor hopping $t'$, which explicitly breaks the symmetry between the bonding and antibonding bands and thus the $SO(5)$ symmetry even in the non-interacting limit. In this case, the model still flows to an $SO(5)$ symmetric (at least up to order $O(t'/t)^2$) effective action, provided the $SO(5)$ symmetry is redefined by accounting for different single-particle renormalization factors for the fermions on the two bands of the ladder system. These results indicate that the $SO(5)$ symmetry can emerge as a result of RG flow, and it is therefore of interest to study the low-energy physics of fixed point Hamiltonians which have exact $SO(5)$ symmetry.

This paper is structured as follows. The formal construction of an $SO(5)$ symmetric two-leg ladder Hamiltonian will be described in Section II. Following this, Section III contains a discussion of the ground state phases in the strong coupling limit. The collective modes are discussed in Section IV, and Section V contains a summary of our results.

Before going into the technical details of the subsequent sections, we would like to conclude the introduction by explaining the basic idea. On a given rung of the ladder, there are two sites with 16 electronic states, as depicted in Fig. 1. These 16 states can be classified into 4 different groups, (the $E_0$, $E_1$, $E_2$ and $E_3$ groups in Fig. 1), each transforming as irreducible multiplets under $SO(5)$. If a local Hamiltonian on a given rung is $SO(5)$ symmetric, states within a given multiplet must have the same energy. However, simple visual inspection shows that the states in the $E_2$ and $E_3$ groups are already degenerate...
for generic interactions respecting spin invariance, particle-hole symmetry with respect to half-filling and symmetry under interchanging two sites. Therefore, only one condition is required to make the spin triplet magnon state and the two “pair states” in the $E_1$ group degenerate. This condition turns out to be $J = 4(U + V)$, relating the onsite interaction $U$, a near-neighbor interaction $V$, and an spin exchange interaction $J$ on a rung.

II. CONSTRUCTION OF $SO(5)$ SYMMETRIC LADDER MODELS

Rabello et al. [7] showed that the crucial step in constructing general exact $SO(5)$ symmetric models is to identify a 4 component fermion operator which transforms according to the fundamental spinor irreps of $SO(5)$, or its equivalent $Sp(4)$. The important point here is that the geometry of the 2-leg ladder makes it natural to group fermion operators on a rung into a 4-component spinor, which leads to a Hamiltonian with purely local interactions. In the following we will use $x, y, ..$ to denote the position of a rung on a ladder, and $c_\sigma(x)$ and $d_\sigma(x)$ to denote the spin up ($\sigma = 1$) and spin down ($\sigma = -1$) fermion destruction operators on the upper and lower chain, respectively. Our $SO(5)$ spinor operator is defined by

$$\Psi_\alpha(x) = \begin{pmatrix} c_\sigma(x) \\ d_\sigma^\dagger(x) \end{pmatrix}$$ (2.1)

for the even rungs with $(-1)^x = 1$, and

$$\Psi_\alpha(x) = \begin{pmatrix} d_\sigma(x) \\ c_\sigma^\dagger(x) \end{pmatrix}$$ (2.2)

for the odd rung with $(-1)^x = -1$. These spinor operators satisfy the canonical anticommutation relations:

$$\{\Psi^\dagger_\alpha(x), \Psi_\beta(y)\} = \delta(x - y)\delta_{\alpha\beta}$$ (2.3)

and

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = \{\Psi^\dagger_\alpha(x), \Psi^\dagger_\beta(y)\} = 0$$ (2.4)
Using these spinor operators and the Dirac $\Gamma$ matrices (for details see appendix A), we can construct a 5 dimensional $SO(5)$ superspin vector

$$n_a(x) = \frac{1}{2} \Psi^\dagger_\alpha(x) \Gamma^a_{\alpha\beta} \Psi_\beta(x)$$

(2.5)

a 10 dimensional $SO(5)$ symmetry generator

$$L_{ab}(x) = -\frac{1}{2} \Psi^\dagger_\alpha(x) \Gamma^{ab}_{\alpha\beta} \Psi_\beta(x)$$

(2.6)

and a 1 dimensional $SO(5)$ scalar

$$\rho(x) = \frac{1}{2} \Psi^\dagger_\alpha(x) \Psi_\alpha(x)$$

(2.7)

on a given rung $x$. The local commutation relation between these operators are given by

$$[L_{ab}, L_{cd}] = -i(\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} - \delta_{cc} L_{ad})$$

(2.8)

$$[L_{ab}, n_c] = -i(\delta_{ac} n_b - \delta_{bc} n_a)$$

(2.9)

and

$$[L_{ab}, \rho] = 0$$

(2.10)

The superspin vector $n_a$ is related to the AF and SC operators by

$$n_1 = \frac{(\Delta^\dagger + \Delta)}{2} = \frac{1}{2}(-ic^\dagger \sigma_y d^\dagger + h.c.)$$

(2.11)

$$n_{2,3,4} = N_{x,y,z} = \frac{1}{2}(c^\dagger \bar{\sigma} c - d^\dagger \bar{\sigma} d)$$

(2.12)

$$n_5 = \frac{(\Delta^\dagger - \Delta)}{2i} = \frac{1}{2}(c^\dagger \sigma_y d^\dagger + h.c.)$$

(2.13)

where we supressed the spinor index on $c_\sigma$ and $d_\sigma$. The symmetry generators $L_{ab}$ are expressed in terms of the rung spin $S_\alpha = \frac{1}{2}(c^\dagger \sigma_\alpha c + d^\dagger \sigma_\alpha d)$, charge $Q = \frac{1}{2}(c^\dagger c + d^\dagger d - 2)$ and the $\pi_\alpha$ operators

$$\pi^\dagger_\alpha = -\frac{1}{2} c^\dagger \sigma_\alpha \sigma_y d^\dagger$$

(2.14)
with

\[
L_{ab} = \begin{pmatrix}
0 & \pi_1^+ + \pi_x & 0 \\
\pi_1^+ + \pi_y & -S_z & 0 \\
\pi_1^+ + \pi_z & S_y & -S_x & 0 \\
Q & \frac{1}{\sqrt{2}}(\pi_1^+ - \pi_x) & \frac{1}{\sqrt{2}}(\pi_1^+ - \pi_y) & \frac{1}{\sqrt{2}}(\pi_1^+ - \pi_z) & 0
\end{pmatrix}
\tag{2.15}
\]

The SO(5) singlet operator \( \rho = \frac{1}{2}(c^\dagger c - d^\dagger d + 2) \) has the physical interpretation of the charge-density-wave operator.

Having exhibited the local SO(5) operator algebra, we are now in a position to construct an SO(5) symmetric model. Let us first consider the problem of two sites on a given rung. As discussed in the introduction, there are 16 states which can be classified under SO(5) as

1) a spin singlet state on the rung

\[
|\Omega\rangle = \left(\begin{array}{c}
c_{\uparrow}^\dagger d_{\downarrow}^\dagger - c_{\downarrow}^\dagger d_{\uparrow}^\dagger \\
\end{array}\right) \frac{1}{\sqrt{2}}|0\rangle
\tag{2.16}
\]

which is also an SO(5) singlet \((L_{ab}|\Omega\rangle = 0)\), 2) an SO(5) vector quintet \(n_a|\Omega\rangle\) which contains a triplet of “magnon states” and a doublet consisting of a hole pair and its conjugate, 3) two SO(5) spinor quartets \(\Psi_\alpha|\Omega\rangle\) and \(\Psi_\alpha^\dagger|\Omega\rangle\), and finally 4) two additional SO(5) singlets of the form \(\Psi_\alpha R_{\alpha\beta}\Psi_\beta^\dagger|\Omega\rangle\) and \(\Psi_\alpha^\dagger R_{\alpha\beta}\Psi_\beta|\Omega\rangle\). The \(R\) matrix is an invariant tensor of the SO(5) algebra and it is defined in the Appendix A. These states are depicted in Fig. 1.

Let us first neglect the hopping within the rung, and consider the general spin and charge interaction Hamiltonian for the two sites:

\[
H_{\text{rung}} = U(n_{\alpha\uparrow} - \frac{1}{2})(n_{\alpha\downarrow} - \frac{1}{2}) + (a \to b)
+ V(n_a - 1)(n_b - 1) + JS_a S_b
\tag{2.17}
\]

In order for such a Hamiltonian to be SO(5) symmetric, we would require that the states within each of the four multiplets mentioned above be degenerate. As noted earlier, degeneracy within the \(\Psi_\alpha|\Omega\rangle, \Psi_\alpha^\dagger|\Omega\rangle\), multiplets is automatically ensured by spin rotation
invariance, the particle-hole symmetry and the symmetry under rung parity \((c_\sigma \rightarrow d_\sigma)\). The 
\(\Psi_\alpha R_{\alpha\beta} \Psi_\beta |\Omega\rangle\) and \(\Psi_\alpha^\dagger R_{\alpha\beta} \Psi_\beta^\dagger |\Omega\rangle\) states are already \(SO(5)\) singlets. This leaves only the \(n_a|\Omega\rangle\) quintet manifold. For the rung Hamiltonian, Eq (2.17), the triplet \(\text{“magnon states”}\) are seen to have energy \(J/4 - U/2\), while the doublet pair states have energy \(U/2 + V\). Therefore, these states will be degenerate if

\[ J = 4(U + V) \] (2.18)

Under this condition, the Hamiltonian (2.17) can be cast into the manifestly \(SO(5)\) symmetric form:

\[ H_{\text{rung}} = \frac{J}{4} \sum_{a<b} L_{ab}^2 + \left(\frac{J}{8} + \frac{U}{2}\right)(\Psi_\alpha^\dagger \Psi_\alpha - 2)^2 \] (2.19)

up to an additive constant \(3J/4 + U/2\). For this \(SO(5)\) symmetric form of the interaction, the above mentioned four multiplets have the energies \(E_0 = -\frac{7}{2}U - 3V\), \(E_1 = U/2 + V\), \(E_2 = 0\) and \(E_3 = U/2 - V\) respectively as indicated in Fig. 1.

In the manifestly \(SO(5)\) symmetric Hamiltonian (2.19), the interactions are expressed in terms of tensor and scalar interactions. The readers may wonder why the vector interactions are missing. The reason is that there is a Fierz identity discussed in Appendix A, which relates the three channels of interactions, and only two of them are mutually independent.

Let us now consider the effect of the hopping within the rung. It is easy to see that the hopping term can be expressed in the manifestly \(SO(5)\) symmetric form:

\[ H_{t_\perp} = -2t_\perp (c_\sigma^\dagger d_\sigma + h.c) \]
\[ = t_\perp (\Psi_\alpha R^{\alpha\beta} \Psi_\beta + h.c.) \] (2.20)

This hopping term can split the degeneracy within the \(E_2\) and \(E_3\) manifold. The \(E_2\) manifold splits into anti-bonding and bonding states \((\Psi_\alpha \pm (R \Psi^\dagger)_\alpha)|\Omega\rangle\), with energies \(E_2^a = 2t_\perp\) and \(E_2^b = -2t_\perp\) respectively. \(H_{t_\perp}\) can in principle cause mixing between the \(E_0\) and the two states in the \(E_3\) manifolds. Since all three states are \(SO(5)\) singlets, their mixing does not violate \(SO(5)\) symmetry. Within degenerate state perturbation theory, the two \(E_3\) states
also form anti-bonding and bonding combinations \((\Psi_\alpha R_\alpha \Psi_\beta \pm \Psi_\alpha^\dagger R_\alpha \Psi_\beta^\dagger) )\), with energies 
\(E_3^a = U/2 - V - t_\perp^2/|2U + V|\) and \(E_3^b = U/2 - V + t_\perp^2/|2U + V|\) respectively.

So far the \(SO(5)\) symmetry is only realized on the two sites of a rung. The more non-trivial question is how the symmetry is realized when the hopping \(t_\parallel\) in the ladder direction is included. Remarkably, this hopping can also be expressed in manifestly \(SO(5)\) invariant form,

\[
H_{t_\parallel} = -2t_\parallel \sum_{<x,y>} (c_\sigma^\dagger(x)c_\sigma(y) + d_\sigma^\dagger(x)d_\sigma(y) + h.c)
= 2t_\parallel \sum_{<x,y>} (\Psi_\alpha^\dagger(x)R_\alpha^\beta \Psi_\beta(y) + h.c. ) \tag{2.21}
\]

It is important to point out that the alternating definition of the \(SO(5)\) spinors on the even \((2.1)\) and odd \((2.2)\) rungs makes it possible to express \(H_{t_\parallel}\) in manifestly \(SO(5)\) invariant form. This alternating definition was suggested to us by S. Rabello. Without such an alternating definition, the resulting Hamiltonian is still \(SO(5)\) symmetric, but the symmetry is not manifest. Our final \(SO(5)\) symmetric ladder Hamiltonian \(H\) is given by the sum of \((2.19)\), \((2.20)\) and \((2.21)\). For this Hamiltonian, all ten \(SO(5)\) generators

\[
L_{ab} = \sum_x L_{ab}(x) \tag{2.22}
\]

are exactly conserved,

\[
[H, L_{ab}] = 0 \tag{2.23}
\]

Notice that because of our alternating definitions \((2.1)\) and \((2.2)\) of the fermion operators, the \(\pi_\alpha\) operators have momentum \(\pi\) along the ladder, while the total charge and total spin operators are uniform. In the presence of a chemical potential term, \(H_\mu = -2\mu L_{15}\), the \(\pi_\alpha\) operators are exact eigen-operators

\[
[H + H_\mu, \pi_\alpha^\dagger] = -2\mu \pi_\alpha^\dagger \tag{2.24}
\]

so that the total Casimir charge of the ladder \(C = \sum_{x,a<b} L_{ab}^2\) is conserved.

\[
[H + H_\mu, C] = 0 \tag{2.25}
\]
Therefore, all states of the doped ladder are still labeled by their $SO(5)$ quantum numbers.

The $SO(5)$ symmetric ladder model we presented so far has only local interactions on the rungs. Obviously, one can generalize the model by including interaction between different rungs, for example one could write down $SO(5)$ invariant interactions having the form

$$
\begin{align*}
\sum_{x,y} V_1(x - y)n_a(x)n_a(y) + \sum_{x,y} V_2(x - y)L_{ab}(x)L_{ab}(y) \\
+ \sum_{x,y} V_0(x - y)(\rho(x) - 2)(\rho(y) - 2)
\end{align*}
$$

(2.26)

Here we shall restrict ourselves only to the analysis of models with local rung interactions, and defer the general analysis to future works. It is plausible that in the strong coupling limit, the local rung interaction dominates the physics.

### III. STRONG COUPLING PHASE DIAGRAM

In this section we discuss the phase diagram of the $SO(5)$ ladder Hamiltonian in the strong coupling limit. Setting $J = 4(U + V)$, the energies of the different rung manifolds are listed in Fig. 1. One can divide up the $U$-$V$ plane according to regions in which a given manifold lies lowest in energy and Fig. 2 shows such a plot. In the strong coupling regime, one can study a given sector of the $U$-$V$ plane, using the virtual hopping processes due to $H_{t\perp}$ and $H_{t\parallel}$ to resolve the degeneracies and determine the dynamics of the low lying excited states. In the following we examine the three different regions $E_0$, $E_3$, and $E_1$ shown in the $U$-$V$ phase diagram of Fig. 2.

#### A. The $E_0$ Spin-Gap $d$-Wave Phase

In the region $E_0$ bounded by $V = -2U$ for negative values of $U$ and $V = -U$ for positive values of $U$, the singlet rung state $|\Omega\rangle$ (see Fig. 3a) is lowest in energy and the system is expected to be in a spin-gap insulating ground state. In the strong-coupling limit this state is simply a product of rung singlets. The spin-gap corresponds to the energy to create a
magnon triplet, as illustrated in Fig. 3b, in which a rung singlet is replaced by a magnon triplet from the $E_1$ manifold. This costs an energy

$$\Delta_{sg} = \left(\frac{U}{2} + V\right) + \left(\frac{7U}{2} + 3V\right) = 4(U + V) = J \quad (3.1)$$

If we were to add two holes to this phase, the lowest energy state occurs when the two holes are placed on the same rung as illustrated in Fig. 3c. In this case, the excitation energy is again $\Delta_{sg} = J$, as expected for an $SO(5)$ symmetric system. Alternatively, one could imagine adding two holes by placing each one on a separate rung creating two $E_2 = 0$ states. However, this would cost an energy $2(\frac{7}{2}U + 3V)$, because of the two singlet rung states that are destroyed, and this is a larger cost in energy than placing the holes on the same rung throughout the entire $E_0$ region of Fig. 2. Thus the doped holes will form rung pairs and we expect that the doped system will exhibit power law pairing and CDW correlations in the $E_0$ region. If one defines bonding and antibonding rung orbitals

$$b^\dagger_\sigma(x) = \frac{c^\dagger_\sigma(x) + d^\dagger_\sigma(x)}{\sqrt{2}} \quad a^\dagger_\sigma(x) = \frac{c^\dagger_\sigma(x) - d^\dagger_\sigma(x)}{\sqrt{2}} \quad (3.2)$$

then the singlet rung state has the form

$$\left| \frac{c^\dagger_\uparrow d^\dagger_\downarrow - c^\dagger_\downarrow d^\dagger_\uparrow}{\sqrt{2}} \right| 0 \rangle = \left| \frac{b^\dagger_\uparrow b^\dagger_\downarrow - a^\dagger_\uparrow a^\dagger_\downarrow}{\sqrt{2}} \right| 0 \rangle \quad (3.3)$$

Thus in the $E_0$-region, the hole pairs go into a “$d$-wave” like state in which the amplitudes of the singlet pair in the bonding and antibonding orbitals have opposite signs.

Both the magnon and the hole pair can propagate coherently along the ladder leading to an energy dispersion in $q_x$. In strong coupling we can calculate their dispersion relations to second order in $t_\parallel$ as follows. A magnon excitation on rung $x$ can hop to rung $x + 1$ by going through an $E_1$ intermediate state. If $|\psi_x\rangle$ is a state with the magnon on site $x$, then the second order virtual hopping process has a matrix element

$$\langle \psi_{x+1}|H_{t_\parallel}\frac{1}{E_0 - H_{\text{rung}}}H_{t_\parallel}|\psi_x\rangle = \frac{t^2_{\parallel}}{3U + 2V} \quad (3.4)$$

and the magnon dispersion is
There is also a $t_\parallel^2$ shift in the zero point energy when a magnon is created. Taking these virtual processes into account gives the complete magnon dispersion to second order in $t_\parallel^2$:

$$\omega_q = J \left(1 - \frac{2t_\parallel^2}{3U + 2V(2V/2 + 3V)}\right) + \frac{2t_\parallel^2}{3U + 2V}\cos q_x \quad (3.6)$$

It is straightforward to carry out a similar calculation for the hole pair dispersion and one finds that only the sign of the $\cos q_x$ term in Eq. (3.6) is changed. Thus, as expected for an SO(5) symmetric ladder, the magnon dispersion about $q_x = \pi$ is identical to the hole pair dispersion about $q_x = 0$.

If two hole pairs (or two magnons) are added, one finds that they repel each other if they are located on neighboring rungs. This repulsion simply reflects the reduction in the zero point fluctuation energy from the background fermions when the two hole pairs are adjacent. This gives rise to an effective near neighbor repulsion $V^* = 4t_\parallel^2/(3U + 2V)$. Therefore, in the $E_0$ region of the $U-V$ phase diagram the doped system behaves as a dilute one-dimensional boson gas with a repulsive near-neighbor interaction, and is expected show quasi-long-ranged-order in the superconducting correlation function. The transition from the spin gap Mott insulator to the superconductor occurs when the chemical potential is given by

$$2|\mu| = 2\mu_c = \Delta_{sg} \quad (3.7)$$

and such a transition is expected to be second order.

**B. The $E_3$ Spin-gap $s$-wave and CDW Phase**

In the $E_3$-region of the $U-V$ strong coupling phase diagram (Fig. 2) the singlet states $\Psi_{\alpha}^\dagger R_{\alpha\beta} \Psi_\beta^\dagger |\Omega\rangle$ and $\Psi_{\alpha}^\dagger R_{\alpha\beta} \Psi_\beta |\Omega\rangle$ are coupled by second order kinetic energy processes. In this regime, the problem can be mapped onto an effective Ising model in a magnetic field. We
identify $\Psi^\dagger_\alpha R_{\alpha\beta} \Psi^\dagger_\beta |\Omega\rangle$ with the Ising state $|\sigma^z(x) = 1\rangle$ and $\Psi_\alpha R_{\alpha\beta} \Psi_\beta |\Omega\rangle$ with the Ising state $|\sigma^z(x) = -1\rangle$. On a rung we have

$$\langle \sigma^z(x) = 1 | H_{t\perp} \frac{1}{E_0 - H_0} H_{t\perp} | \sigma^z(x) = -1 \rangle = \frac{t_{\perp}^2}{2U + V} \quad (3.8)$$

and between two near neighbor rungs

$$\langle \sigma^z(x + 1) = 1 | H_{t\parallel} \frac{1}{E_0 - H_0} H_{t\parallel} | \sigma^z(x) = -1 \rangle = \frac{t_{\parallel}^2}{U/2 - V} \quad (3.9)$$

In the $E_3$-region, the energy denominators in the expressions are negative so that the $t_{\perp}^2$ term favors the formation of the $s$-wave like rung singlet

$$\Psi^\dagger_\alpha R_{\alpha\beta} \Psi^\dagger_\beta |\Omega\rangle + \Psi_\alpha R_{\alpha\beta} \Psi_\beta |\Omega\rangle = \frac{b^\dagger \uparrow b^\dagger \downarrow + a^\dagger \uparrow a^\dagger \downarrow}{\sqrt{2}} |0\rangle \quad (3.10)$$

Here $b^\dagger$ and $a^\dagger$ are the bonding and antibonding creation operators of Eq (3.2). On the other hand, the $t_{\parallel}^2$ process favors a staggered charge density wave state. Combining equations (3.7) and (3.8) we can write an effective Ising-like Hamiltonian for the $E_3$-region in the form

$$\mathcal{H}_3 = \sum_x \left(-h \sigma^z(x) + K_3 (\sigma_{x+1}^z \sigma^z(x) - 1)\right) \quad (3.11)$$

with $h = t_{\perp}^2 / |2U + V|$ and $K_3 = 2t_{\parallel}^2 / |U/2 - V|$. The ground state of $\mathcal{H}_3$ is known to have an Ising-like phase transition for $h = K_3$. For $h < K_3$, the half-filled $SO(5)$ ladder will be in a CDW phase corresponding to one of the two degenerate states illustrated in Fig. 4a and Fig. 4b. For $h > K_3$, the system will be disordered. In this region, the half-filled $SO(5)$ ladder will be in a spin-gap insulating phase. For $t_{\parallel} = t_{\perp}$, the $h = K_3$ dividing line corresponds to $V = -\frac{2}{3}U$. When holes are doped into the disordered region, they will tend to go onto a rung forming $s$-wave like, Eq (3.10), pairing correlations.

C. The $E_1$ $SO(5)$ Superspin Phase

Here the Hilbert space per rung is restricted to the "superspin" quintet manifold $n_a(x) |\Omega\rangle$. In this case, each rung is either occupied by a triplet magnon or a doublet "pair"
state. The effective Hamiltonian in the quintet manifold is easily determined using second order perturbation theory:

\[ H_1 = K_1 \sum_{<x,y>} L_{ab}(x)L_{ab}(y) \]  \hspace{1cm} (3.12)

where \( K_1 = t^2/|U/2 + V| \). This model can be viewed as the \( SO(5) \) generalization of the spin one Heisenberg chain. Therefore, we would expect to find many properties simply from this analogy, for example, a ground state with a finite excitation gap, and short ranged correlations, etc. One useful model of the spin one Heisenberg chain is the AKLT model for which an exact ground state is known. In this section we shall construct an \( SO(5) \) generalization of the AKLT model and present its exact ground state.

We begin by considering two neighboring rungs \( x \) and \( y \). The wave function for the two superspins defined on the two rungs can be decomposed as

\[ 5 \times 5 = 1 + 10 + 14 \]  \hspace{1cm} (3.13)

i.e. the product wave function can transform like an \( SO(5) \) singlet, an \( SO(5) \) antisymmetric tensor or an \( SO(5) \) symmetric traceless tensor. Therefore, we can defined a complete set of bond projection operators \( P_1(xy), P_{10}(xy) \) and \( P_{14}(xy) \) onto these subspaces, satisfying:

\[ 1 = P_1(xy) + P_{10}(xy) + P_{14}(xy) \]  \hspace{1cm} (3.14)

The \( SO(5) \) generalization of the AKLT model is then given by

\[ \tilde{H}_1 = 2K_1 \sum_{<x,y>} P_{14}(xy) \]  \hspace{1cm} (3.15)

Let us first see how this Hamiltonian can be expressed in terms of the \( SO(5) \) superspin exchange operators. We start by defining

\[ \mathcal{L}_{ab} = L_{ab}(x) + L_{ab}(y) \]  \hspace{1cm} (3.16)

where \( \mathcal{L}_{ab} \) measures the total \( SO(5) \) generator on the bond \( <xy> \). Squaring this equation and noticing that the Casimir charge \( C_5 = \sum_{a<b} L_{ab}^2(x) \) for the 5 irreps on a given rung is \( C_5 = 4 \), we obtain
\[
\sum_{a<b} L_{ab}(x)L_{ab}(y) = \frac{1}{2} \sum_{a<b} \mathcal{L}_{ab}^2 - 4
\]  
(3.17)

The operator \( \sum_{a<b} \mathcal{L}_{ab}^2 \) is the total Casimir charge for the bond \( <xy> \), therefore, it can be expressed as

\[
\sum_{a<b} \mathcal{L}_{ab}^2 = C_1 P_1(xy) + C_{10} P_{10}(xy) + C_{14} P_{14}(xy)
\]  
(3.18)

where \( C_1 = 0 \), \( C_{10} = 6 \) and \( C_{14} = 10 \) are the Casimir charge for the 1, 10 and 14 irreps respectively. Therefore, we obtain

\[
\sum_{a<b} L_{ab}(x)L_{ab}(y) = -4P_1(xy) - P_{10}(xy) + P_{14}(xy)
\]  
(3.19)

Squaring this equation again gives

\[
(\sum_{a<b} L_{ab}(x)L_{ab}(y))^2 = 16P_1(xy) + P_{10}(xy) + P_{14}(xy)
\]  
(3.20)

where we used the property of the projection operators \( P_i P_j = \delta_{ij} P_i \). Equations (3.14), (3.19) and (3.20) finally allows us to express the \( P_{14}(xy) \) operator as

\[
P_{14}(xy) = \frac{1}{10}(L_{ab}(x)L_{ab}(y))^2 + \frac{1}{2}(L_{ab}(x)L_{ab}(y)) + \frac{2}{5}
\]  
(3.21)

Inserting this equation into (3.15) then gives the desired expression for the \( SO(5) \) generalization of the AKLT Hamiltonian.

The Hamiltonian (3.15) has an exact ground state. In general, the ground state can be expressed as

\[
|\psi_0\rangle = \sum_{a_1,..,a_N} \psi_0(a_1,..,a_N)n_{a_1}(x_1)\ldots n_{a_N}(x_N)|\Omega\rangle
\]  
(3.22)

for a ladder with \( N \) rungs. \( \psi_0(a_1,..,a_N) \) is the corresponding ground state wave function in the superspin vector basis. It is easy to see that the exact ground state wave function for the \( SO(5) \) AKLT Hamiltonian (3.15) for a period ladder is given by

\[
\psi_0(a_1,a_2,..,a_N) = Tr(\Gamma^{a_1} \Gamma^{a_2} \ldots \Gamma^{a_N})
\]  
(3.23)
This follows from the following property of the Dirac Γ matrices (for more details, see appendix A):

\[ \Gamma^a \Gamma^b = 2\delta^{ab} + 2i\Gamma^{ab} \]  

(3.24)

From this equation we see that the product of two Γ matrices involves no symmetric traceless components. Therefore, the wave function \( \psi_0(a_1, ..., a_i, a_{i+1}, ..., a_N) \) viewed as a 5×5 matrix in \( a_i \) and \( a_{i+1} \) with all other indices fixed has no symmetric traceless components. This means that the \( 14 \) irreps on bond \( < x_i x_{i+1} > \) are absent, which implies that \( |\psi_0\rangle \) is annihilated by the projector Hamiltonian (3.15). Since the Hamiltonian (3.15) is positive definite, we can conclude that \( |\psi_0\rangle \) is indeed the exact ground state.

For a ladder with open boundary conditions, the corresponding ground state wave function is given by

\[ \psi_0(a_1, a_2, ..., a_N) = \Gamma^{a_1}\Gamma^{a_2}...\Gamma^{a_N} \]  

(3.25)

This implies 4 edge states at each end of the ladder, giving rise to a 16 fold ground state degeneracy.

To our knowledge, this is the first exact solution to a problem of interacting magnon and Cooper pairs. The ground state described by (3.22) and (3.23) is translationally invariant in the ladder direction and is an \( SO(5) \) singlet state. It has short ranged antiferromagnetic and superconducting order along the ladder. Because it is a resonating state, it is hard to draw a simple picture for this state. The spin part of this wave function can be basically visualized as resonating between the states depicted in Fig. 5a and Fig. 5b.

**IV. COLLECTIVE MODES**

As discussed in Section IIIA in the \( E_0 \)-phase the magnon dispersion relation about \( q_x = \pi \) is the same as the one hole pair dispersion relation around \( q_x = 0 \). Here we examine what happens when the system becomes superconducting upon doping with a finite concentration of hole pairs.
The rung operator $L_{15}(x)$ is equal to

$$L_{15}(x) = \frac{1}{2}(N_e(x) - 2) \quad (4.1)$$

with

$$N_e(x) = \sum_s \left( c^\dagger_s(x)c_s(x) + d^\dagger_s(x)d_s(x) \right) \quad (4.2)$$

Thus $Q = \sum_x L_{15}(x)$ counts the number of pairs relative to half-filling and in the presence of a chemical potential $\mu$, as discussed in Section II, one adds the term

$$H_\mu = -2\mu Q \quad (4.3)$$

to the ladder Hamiltonian $H$. In the $E_0$-phase, as $\mu$ becomes increasingly negative, the $E_1$ quintet splits with the $\Delta|\Omega\rangle$ mode linearly decreasing its energy, the $\Delta^{\dagger}|\Omega\rangle$ mode linearly increasing its energy and the $\vec{N}|\Omega\rangle$ magnon modes remaining constant in energy, as illustrated in Fig. 6a. When $2|\mu|$ becomes greater than the spin gap $\Delta_{sg}$, the ladder becomes doped with a finite density of hole pairs. As discussed in Section III, these hole pairs behave in strong coupling like a dilute hard core boson system with a near neighbor repulsion. In one dimension, hard core bosons can be treated as spinless fermions. Therefore, one can imagine that this band is filled with spinless fermions up to some Fermi energy, and $2|\mu|$ can be physically identified with this Fermi energy. The physical origin of the increase in the energy to add an extra hole pair is due to the hard-core repulsion with the hole pairs in the condensate.

On the other hand, the magnon band has a band minimum at momentum $\pi$ and it is completely empty. A naive argument would suggest that one could insert a magnon simply by putting it at the band minimum which would cost energy $\Delta_{sg}$. However, this argument neglects the repulsive interaction between the magnon and the hole pairs in the condensate. In an $SO(5)$ model, the interaction between the magnon and the hole pairs are the same as the mutual interaction between the hole pairs, therefore, the energy of the magnon is the same as the energy of a hole pair, which is $2|\mu|$. Thus we arrive at a simple physical
interpretation of the energy of the magnon in the superconducting state: \textit{the magnon energy} 2|\mu| \textit{in the superconducting state is the sum of two contributions, the rest energy } \Delta_{sg} \textit{to create a magnon and the interaction energy between the magnon and the hole pair in the condensate.} Therefore, the energy of the spin triplet momentum \pi excitation is \Delta_{sg} for |\mu| < \mu_c \textit{and } 2|\mu| \textit{for } |\mu| > \mu_c, \textit{see (Fig. 6b).}

This physical interpretation was based upon a strong-coupling picture, but a more general result can be obtained as follows. According to Eq. (2.24) for an SO(5) ladder the \pi operators

\[[H + H_\mu, \pi_\alpha] = 2\mu\pi_\alpha \] (4.4)

which add a pair and generate an exact excited state with momentum (\pi, \pi) and \(S = 1\).

Now, suppose we were to start with the ground state of \(H\) with charge \(Q\), \(|\psi_0(Q)\rangle\), and add a hole pair to obtain the ground state of the electron system with charge \(Q - 1\), \(|\psi_0(Q - 1)\rangle\). The energy cost to insert the hole pair is given by the difference in the ground state energy of \(H\) for \(Q - 1\) and \(Q\) electron pairs, \(2|\mu| = E_0(Q - 1) - E_0(Q)\). On the other hand, we can act on \(|\psi_0(Q - 1)\rangle\) with the \(\pi_\alpha^\dagger\) operator and rotate the added hole pair into a magnon of the \(Q\) electron pair system. For the SO(5) ladder, this rotation costs no energy, so we see that the energy for inserting a magnon into the \(Q\) electron pair system is \(2|\mu|\). Furthermore, one sees that the \(\pi\)-mode is just the natural continuation of the spin gap magnon mode [17]. This corresponds to the idea of the \(\pi\)-mode in the two dimensional \(t - J\) model originally proposed by Demler \textit{et al} [6,12] and studied in numerical calculation by Meixner et. al. [10] and Eder et. al. [11].

The triangular relation between the \(\pi\) resonance, the magnon, and the Cooper pair can be illustrated by the SO(5) representation theory discussed by Eder et. al [11]. The general traceless symmetric tensor irreps of SO(5) are characterized by three intergers \((S_z, Q, \nu)\), where \(\nu\) is related to the Casimir charge by \(L^2_{ab} = \nu(\nu + 3)\). This class of eigenstates of any SO(5) model can therefore be represented by an SO(5) pyramid, labeled by the Cartesian coordinates \((S_z, Q, \nu)\). If we take an \(S_z = 0\) slice of the SO(5) pyramid, and plot the resulting
energy diagram, we obtain Fig. 7. Each box in Fig. 7 denotes the collection of all eigenstates with the same $SO(5)$ quantum numbers. The superconducting states lie on the ridge of the pyramid. In the canonical ensemble, all states with the same $\nu$ are strictly degenerate, but the lowest energy states with $\Delta \nu = 1$ are spaced by $2|\mu| = E_0(Q - 1) - E_0(Q)$. Starting from the ground state $|\psi_0(Q)\rangle$, one can create a magnon using the $N_\alpha$ operator, or by first adding a hole pair using the $\Delta$ operator, and then acting on the resulting $|\psi_0(Q - 1)\rangle$ state with the $\pi_\alpha^\dagger$ operator. (See Fig. 7). In $SO(5)$ symmetric models, this triangle closes exactly, and the magnon mode energy is therefore predicted to be exactly $2|\mu|$.

V. CONCLUSION

We have found that a two-leg ladder with a rung interaction characterized by an onsite $U$ interaction, a rung near neighbor $V$ interaction, and a rung exchange interaction $J$ can have $SO(5)$ symmetry if $J = 4(U+V)$. Furthermore, in the $E_0$ regime, the strong-coupling ground state is a spin-gap insulator. In this half-filled state the equal time rung magnetization and rung pair field correlations are identical. In addition, and of particular importance, the dispersion relation of a magnon rung excitation with $q_x$ measured from $\pi$ is identical to the rung hole pair dispersion measured from $q_x = 0$. We have also seen that when the chemical potential is increased such that $2|\mu|$ exceeds the spin gap $\Delta_{sg}$, $d$-wave-like hole pairs form a dilute hard core bose gas with a near neighbor repulsion. The spin gap magnon mode of the Mott insulator evolves continuously into the $\pi$ resonance mode of the superconductor.

There are also other ground states in the $U - V$ phase diagram such as the $E_3$ regime which can have a CDW state or a spin gap insulating phase which when doped has $s$-wave hole pairs. In addition, the $E_1$-phase at half-filling corresponds to an $SO(5)$-like Heisenberg model with ground state gaps analogous to the $S = 1$ Heisenberg model.

A key question remains regarding the relationship of this $SO(5)$ ladder to the more standard Hubbard or $t - J$ ladders. Physically if we want $J$ and $U$ to be positive, this requires a negative rung interaction $V$. Furthermore $|V|$ must nearly balance $U$ in order for the system
to be in the physical interesting regime in which $J/t < 1$. Therefore, the standard ladder models are not exactly $SO(5)$ symmetric in the sense defined in this paper. However, at half-filling, it is likely that standard ladder models flow towards a rung singlet ground state in the strong coupling limit. In this work we showed that such a state is not only a total spin singlet, but also an $SO(5)$ singlet. Therefore, we would expect the static correlation to be approximately $SO(5)$ symmetric. Recent results by Shelton and Senechal \cite{13}, Balents, Fisher, and Lin \cite{14} and Arrigoni and Hanke \cite{15} show that the generic interaction parameters of the ladder model tend to flow towards the $SO(5)$ symmetric manifold under RG. However, their results were obtained in the weak coupling regime. Clearly, there remain various questions such as whether one will have a sufficient renormalization flow to approach the $SO(5)$ regime when the physical $U$ is of order the bandwidth. The $SO(5)$ symmetric model studied in this work offer a reference point around which departures from the $SO(5)$ symmetric point can be studied systematically. It would be desirable to develop a numerical RG analysis to study the flow around the $SO(5)$ symmetric point in the strong coupling limit.

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APPENDIX A: DIRAC Γ MATRICES AND FIERZ IDENTITY

The general method introduced by Rabello et. al to construct $SO(5)$ symmetric models uses the five Dirac $Γ$ matrices $Γ_a$ ($a = 1, ..., 5$) which satisfy the Clifford algebra,

$$\{Γ^a, Γ^b\} = 2δ^{ab} \quad (A1)$$

Rabello et. al introduced the following explicit representation which is naturally adapted for discussing the unification of AF and dSC order parameters,

$$Γ^1 = \begin{pmatrix} 0 & -iσ_y \\ iσ_y & 0 \end{pmatrix}, \quad Γ^{(2,3,4)} = \begin{pmatrix} 0 & 0 \\ 0 & t\bar{σ} \end{pmatrix}, \quad Γ^5 = \begin{pmatrix} 0 & σ_y \\ σ_y & 0 \end{pmatrix} \quad (A2)$$

Here $\bar{σ} = (σ_x, σ_y, σ_z)$ are the usual Pauli matrices and $t\bar{σ}$ denotes their transposition. These five $Γ_a$ matrices form the 5 dimensional vector irreps of $SO(5)$. Their commutators

$$Γ^{ab} = -\frac{i}{2} [Γ^a, Γ^b] \quad (A3)$$

define the 10 dimensional antisymmetric tensor irreps of $SO(5)$. In the above representation, the 10 $Γ^{ab}$'s are given explicitly by

$$Γ^{15} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Γ^{(i+1)(j+1)} = ε_{ijk} \begin{pmatrix} σ_k & 0 \\ 0 & -tσ_k \end{pmatrix} \quad (i, j = 1, 2, 3)$$

$$Γ^{(2,3,4)1} = \begin{pmatrix} 0 & -\bar{σ}σ_y \\ -σ_y\bar{σ} & 0 \end{pmatrix} = σ_y \begin{pmatrix} 0 & t\bar{σ} \\ -\bar{σ} & 0 \end{pmatrix}$$

$$Γ^{(2,3,4)5} = \begin{pmatrix} 0 & -i\bar{σ}σ_y \\ iσ_y\bar{σ} & 0 \end{pmatrix} = iσ_y \begin{pmatrix} 0 & t\bar{σ} \\ -\bar{σ} & 0 \end{pmatrix}$$

These $Γ$ matrices satisfy the following commutation relations:

$$[Γ^{ab}, Γ^c] = 2i(δ_{ac}Γ^b - δ_{bc}Γ^a) \quad (A4)$$

$$[Γ^{ab}, Γ^{cd}] = 2i(δ_{ac}Γ^{bd} + δ_{bd}Γ^{ac} - δ_{ad}Γ^{bc} - δ_{bc}Γ^{ad}) \quad (A5)$$
A very important property of the $SO(5)$ Lie algebra is the pseudo-reality of its spinor representation. This means that there exists a matrix $R$ with the following properties:

$$R^2 = -1, \quad R^\dagger = R^{-1} = {}^tR = -R \quad \text{(A6)}$$

$$R \Gamma^a R = -{}^t \Gamma^a, \quad R \Gamma^{ab} R = {}^t \Gamma^{ab} \quad \text{(A7)}$$

The relations $R \Gamma^{ab} R^{-1} = -(\Gamma^{ab})^*$ indicate that the spinor representation is real, and the antisymmetric nature of the matrix $R$ indicates that it is pseudo-real. The $R$ matrix plays a role similar to that of $\epsilon_{\alpha\beta}$ in $SO(3)$. In our representation, the $R$ matrix takes the form

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{(A8)}$$

The sixteen $\Gamma^A = 1, \Gamma^a, \Gamma^{ab}$ matrices form a complete basis in the space of $4 \times 4$ Hermitian matrices. This basis is orthonormal by virtue of the trace operation:

$$Tr(\Gamma^A \Gamma^B) = 4 \delta^{AB} \quad \text{(A9)}$$

Therefore, any $4 \times 4$ Hermitian matrix $M_{\alpha\beta}$ can be expanded as

$$M_{\alpha\beta} = \sum_A \lambda_A \Gamma^A_{\alpha\beta} \quad \text{(A10)}$$

with

$$\lambda_A = \frac{1}{4} Tr(M \Gamma^A) \quad \text{(A11)}$$

This observation can be used to derive a series of Fierz identities, relating interactions in the scalar, vector and tensor channels. For example, the fermion bilinear $\Psi^\dagger_\alpha(x) \Psi_\beta(y)$ can be expanded as

$$\Psi^\dagger_\alpha(x) \Psi_\beta(x) = \frac{1}{4} \sum_A (\Psi^\dagger_\alpha(x) \Gamma^A \Psi(y)) \Gamma^A_{\alpha\beta} \quad \text{(A12)}$$

Using this Fierz identity, one can show that the scalar, vector and the tensor interactions on the same rung are not independent of each other. They are instead related by the following equation

$$(\Psi^\dagger \Psi - 2)^2 = 4 - \frac{1}{5}(\Psi^\dagger \Gamma^a \Psi)^2 - \frac{1}{5}(\Psi^\dagger \Gamma^{ab} \Psi)^2 \quad \text{(A13)}$$
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these quasi-partical pair excitations varies as the doping and near half-filling is small compared to the collective nagnon mode which we are discussing.
FIG. 1. The 16 states of a rung are laid out in their $SO(5)$ multiplets. Here the two lines in the kets represent the two sites of a rung which can be in four states (empty, one electron spin up, spin down, or two electrons). The energies of the multiples for $J = 4(U + V)$ are also listed.
FIG. 2. The half-filled strong-coupling $U - V$ phase diagram showing the regions in which the single rung manifolds indicated have the lowest energy. As discussed in Section III, the $E_0$ manifold is a spin-gap phase which has $d_{x^2-y^2}$-like power law pair field correlations when it is doped. The $E_3$ manifold is divided into a CDW phase and a spin-gapped phase which exhibits $s$-wave like pairing correlations upon doping. The $E_1$ regime is an $SO(5)$ generalization of the Heisenberg spin one chain.
FIG. 3. Schematic illustration of (a) the $E_0$ ground state, (b) a spin $S = 1$ magnon, (c) a hole pair. The ellipses signify a singlet state.

FIG. 4. Illustration of the two-degenerate CDW $E_3$ ground states which can occur when $K_3 > h$.

FIG. 5. Schematic of two local configurations between which the system resonates in the $E_1$ superspin phase.
FIG. 6. (a) The $SO(5)$ vector modes versus $\mu$. Here $\Delta^+ |\Omega\rangle$ and $\Delta^- |\Omega\rangle$ add or remove a pair, while $\vec{N} |\Omega\rangle$ represents the three $S = 1$ magnon modes. $\Delta_{sg} = J$ is the spin gap. (b) The $q = (\pi, \pi)$ magnon versus $\mu$. These modes remain constant until $\mu$ exceeds $\mu_c = \Delta_{sg}/2$ and a finite pair hole density is formed. When this happens, the energy of the magnon-$\pi$ mode increases in energy as $2|\mu|$. 

27
FIG. 7. An $S_z = 0$ slice of the $SO(5)$ pyramid. Here $Q = (N_e - N)/2$ is half of the electron charge measured from half-filling, and $\nu$ is related to the $SO(5)$ Casimir charge. Each box contains many states with the same $SO(5)$ quantum numbers. The ground states of a given charge sector lie within the edge of this triangle. One can create a magnetic excitation of a charge $Q$ state either by the $N_\alpha$ operator, or by first inserting a hole pair using the $\Delta$ operator and then “rotate” the resulting state by the $\pi_\alpha^+$ operator. This triangular relationship is indicated on the figure.