APPROXIMATION OF THE LOWER OPERATOR IN NONLINEAR DIFFERENTIAL GAMES WITH NON-FIXED TIME

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Approximate properties of the lower operator in nonlinear differential games with non-fixed time are studied.

Abstract

The generalization of the Pontryagin’s second direct method [1–2] for nonlinear pursuit games led to the construction described by the operator $\widetilde{T}^t$, which is introduced in [3]. Operator’s construction in nonlinear differential games was developed in [4 - 18]. In particular, lower analogue of the operator $\widetilde{T}^t$ and its applications to study of qualitative structure of phase space of differential games of pursuit-evading were suggested [9]. Problems of approximation and simplified schemes for construction of operator $\widetilde{T}^t$ were studied in [7,10,13]. For the symmetry, $\widetilde{T}_t$ will be denoted the lower analogue of the operator $\widetilde{T}^t$.

In the present article we study approximation properties of the lower operator $\widetilde{T}_t$ for differential games of pursuit with non-fixed time.

Let us consider the differential game

$$\dot{z} = f(z,u,v),$$

where $z \in \mathbb{R}^d, u \in P, v \in Q, f : \mathbb{R}^d \times P \times Q \rightarrow \mathbb{R}^d$, $P$ and $Q$ are convex compact subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Along with the system (1) we also fix the set of $M$, $M \subset \mathbb{R}^d$, which is called terminal set.

We suppose that further the function $f$ holds the following conditions.

A. function $f : \mathbb{R}^d \times P \times Q \rightarrow \mathbb{R}^d$ is continuous and is locally Lipschitz type by $z$ (i.e. the function $f$ satisfies the Lipschitz condition on every compact set $D \subset \mathbb{R}^d$ with the constant $L_D$, depending on compact $D$).

B. There is the constant $C \leq 0$ such that for all $z \in \mathbb{R}^d, u \in P, v \in Q$ the inequality

$$| z \cdot f(z,u,v) | \leq C(1 + | z |^2)$$

holds.
C. The set \( f(z, u, Q) \) is convex for all \( z \in \mathbb{R}^d \), \( u \in P \).

Let \( X[\Delta] \) denote the set of all measurable functions \( a(\cdot) : \Delta \to X \). In the case of \( \Delta = [\alpha, \beta] \), we simply write \( X[\alpha, \beta] \). We call every function \( u(\cdot) \in P[\alpha, \beta] \) (respectively \( v(\cdot) \in Q[\alpha, \beta] \)) as admissible control of pursuer (respectively evader).

We denote by \( z(t, u(\cdot), v(\cdot), \xi) \) solution of equation (1), which corresponds to admissible controls \( u(t) \), \( v(t) \) and initial point \( \xi \).

**Definition 1.** Operator \( T_\varepsilon \) associates every set \( A \subset \mathbb{R}^d \) with the set \( T_\varepsilon A \) of all points \( \xi \subset \mathbb{R}^d \), such that there is admissible control pursuer \( u(\cdot) \in P[0, \varepsilon] \) for any admissible control of evader \( v(\cdot) \in Q[0, \varepsilon] \) the corresponding trajectory \( z(t, u(\cdot), v(\cdot), \xi) \) with the beginning at the point \( \xi \subset \mathbb{R}^d \) hits \( A \subset \mathbb{R}^d \) in time not greater than \( \varepsilon \), i.e. \( z(t_*) \in A \) for of certain \( t_* \in [0, \varepsilon] \).

By means of operations of association and intersection we can write the operator \( T_\varepsilon \) as follows:

\[
T_\varepsilon A = \bigcup_{u(\cdot)\in P[0,\varepsilon]} \bigcap_{v(\cdot)\in Q[0,\varepsilon]} \bigcup_{t_*\in [0,\varepsilon]} \left\{ \xi \subset \mathbb{R}^d \mid z(t_*, u(\cdot), v(\cdot), \xi) \in A \right\} .
\]

Let \( \omega = \{\tau_0, \tau_1, \tau_2, \ldots, \tau_n = t\} \) be partition of segment \([0, t]\) and \( \delta_i = \tau_i - \tau_{i-1} \), \( |\omega| = t \). We assume

\[
T_\omega M = T_{\delta_1} T_{\delta_2} T_{\delta_3} \ldots T_{\delta_n} M,
\]

where \( \delta_i = \tau_i - \tau_{i-1}, \quad i = 1, 2, \ldots, n \).

**Definition 2.** \( \tilde{T}_t = \bigcup_{|\omega|=t} T_\omega M \).

The operator \( \tilde{T}_t \) is called the lower operator of nonlinear differential games pursuit with non-fixed time.

In what follows, we shall assume that the boundary of \( M \) (\( \partial M \)) is compact. We denote by \( D_* \) the set of all points of \( \xi \in \mathbb{R}^d \), of which it is possible to achieve the set \( \partial M \) (the boundary of \( M \)) at the appropriate admissible controls \( u(\cdot) \) and \( v(\cdot) \) for a time not exceeding \( \theta \). Let \( D = D_* + H \) and constants is the quantity that can depend only on the function \( f \), sets \( P, Q, D \) and we shall suppose \( t \leq \theta \).

Condition B guarantees boundedness of the set \( D \) [14]. We assume
$K = \max\{|f(z,u,v)| \ z \in D, u \in P, v \in Q\}$ and $L_1$ is the constant Lipshitz of $f$ on the set $D$.

Let operator $\overline{T}_\varepsilon$ differs from the operator $T_\varepsilon$ in that in Definition 1 only constant controls $u(\cdot) = u \in P$ are taken instead of arbitrary admissible controls $u(\cdot) \in P[0,\varepsilon]$.

Let $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n = t\}$ be partition of segment $[0,t]$.

$$\overline{T}_\omega M = T_{\delta_1}T_{\delta_2}...T_{\delta_n},$$

where $\delta_i = \tau_i - \tau_{i-1}$, $i = 1, 2, ...n$.

**Definition 3.** $\overline{T}_t M = \bigcup_{|\omega|=t} \overline{T}_\omega M$.

For completeness, we present some well-known properties of the operator $\overline{T}_t$.

**Theorem 1 [15].** If $M$ is an open subset of $\mathbb{R}^d$, then

$$\overline{T}_t M = T_t M$$

We note that for arbitrary family $A_\alpha$ the following inclusion

$$\bigcup_\alpha T_{\varepsilon} A_\alpha \subset T_{\varepsilon} \bigcup_\alpha A_\alpha$$

is valid.

**Lemma 1 [10].** Let $A_\alpha \subset \mathbb{R}^d$ non-decreasing direction of open sets. Then following equality holds

$$\bigcup_\alpha T_{\varepsilon} A_\alpha = T_{\varepsilon} \bigcup_\alpha A_\alpha.$$

**Lemma 2 [10].** Let $\omega_k$ be infinitely reducing sequence of partitions of the segment $[0,t]$ i.e. $\omega_k \subset \omega_{k+1}$ , $|\omega_k| = t, \text{max} |\tau_i^k - \tau_{i-1}^k| \to 0$ for $k \to \infty$. Then the following equality holds

$$\overline{T}_t M = \bigcup_{k \geq 1} \overline{T}_{\omega_k} M$$

for open set $M \subset \mathbb{R}^d$.

A simplified schemes for constructing of alternating integral were proposed in [10,13].
For nonlinear differential games the problem of working out a simplified schemes for the construction of the operator $\tilde{T}_tM$ is relevant.

Consider the following operator

$$\Theta_{\varepsilon}B = \bigcup_{u \in P} \bigcap_{v \in Q \cap [0, \varepsilon]} \bigcup_{0 \leq t_* \leq \varepsilon} \{ \xi \in \mathbb{R}^d \mid z(\varepsilon, u, v, \xi) = \xi + t_*f(\xi, u, v) \in B \}. $$

The definition of the operator $\tilde{\Theta}_t$ is similar to the definition of the operator $\tilde{T}_t$.

In the present article we consider the problem of approximation of the operator $\tilde{T}_t$ by means of iteration of operator $\Theta_{\varepsilon}$ and its application to the problem of pursuit.

**Lemma 3.** There is a positive number $L$ such that the following inclusions

$$\overline{T}_\varepsilon(A \ast 2L\varepsilon^2H) \subset \Theta_{\varepsilon}(A \ast L\varepsilon^2H) \subset \overline{T}_\varepsilon A$$

hold.

Proof. The first we prove the left-side of the inclusion (2). Let $\xi \in \overline{T}_\varepsilon(A \ast 2L\varepsilon^2H)$. Then, there exists an admissible control of the pursuer $u \in P$ such that for any admissible control evader $v(\cdot) \in Q[0, \varepsilon]$, there is $t_* \in [0, \varepsilon]$ for trajectory $z(t_*, u, v(\cdot), \xi)$ corresponding to controls $u \in P$, $v(\cdot) \in Q$ and the initial point $\xi \in \mathbb{R}^d$ the following inclusion $z(t_*, u, v(\cdot), \xi) \in A \ast 2L\varepsilon^2H$ holds. i.e.

$$z(t_*, u, v(\cdot), \xi) = \xi + \int_0^{t_*} f(z(t), u, v(t)) dt + 2L\varepsilon^2H \in A. $$

By virtue of the condition A for arbitrary controls $u \in P$, $v(\cdot) \in Q$ and the initial point $\xi \in \mathbb{R}^d$ we have the relation

$$| f(z(t), u, v(t)) - f(\xi, u, v(t)) | \leq L_1 | z(t) - \xi |. $$

On the other hand,

$$| z(t, u, v(\cdot), \xi) - \xi | \leq K\varepsilon, t \in [0, \varepsilon].$$

Hence, using the inequality (4), we obtain

$$| f(z(t), u, v(t)) - f(\xi, u, v(t)) | \leq L\varepsilon,$$

where $L = L_1K$. 
Now we prove that for any \( v(\cdot) \in Q[0, \varepsilon] \) there is a constant control \( v \in Q \) for which the equality

\[
\xi + t_* f(\xi, u, v) = \xi + \int_0^{t_*} f(\xi, u, v(t))dt \quad (6)
\]

is fulfilled.

By virtue of the condition C, the set \( f(\xi, u, Q) \) is convex for any \( u \in P \). Therefore

\[
\int_0^{t_*} f(\xi, u, v(t))dt \in t_* f(\xi, u, Q).
\]

It follows that there is a \( v \in Q \) such that

\[
\int_0^{t_*} f(\xi, u, v(t))dt = t_* f(\xi, u, v).
\]

Therefore, for any \( v(\cdot) \in Q[0, \varepsilon] \) there is a constant control \( v \in Q \) for which equality

\[
\xi + \int_0^{t_*} f(\xi, u, v(t))dt = \xi + t_* f(\xi, u, v)
\]

holds.

Applying inequality (5) to the right side of equality (6) we have

\[
\xi + t_* f(\xi, u, v) \in \xi + \int_0^{t_*} f(z(t), u, v(t))dt + L\varepsilon^2 H.
\]

Hence, using the condition (3) we obtain

\[
\xi + t_* f(\xi, u, v) + L\varepsilon^2 H \subset \xi + \int_0^{t_*} a(t) f(z(t), u, v(t))dt + 2L\varepsilon^2 H \subset A.
\]

Consequently,

\[
\xi \in \Theta_\varepsilon(A_\#L\varepsilon^2 H).
\]

Similarly, the right side of the turn proved (2).

**Lemma 4.** The following inclusions

\[
\Theta_\varepsilon(A_\#L\delta^2(1 + L_1\varepsilon)H) + L\delta^2 H \subset \Theta_\varepsilon A \quad (7)
\]
\[
T_\varepsilon(A\ast L\delta^2(1 + L_1\varepsilon)H) + L\delta^2H \subset T_\varepsilon A
\] 
(8)

hold. Proof. Let \( \eta \) be an arbitrary element from the left part of the inclusion (7). Then there is \( \xi \in \Theta_\varepsilon(A\ast L\delta^2(1 + L_1\varepsilon)H) \) such that

\[
| \eta - \xi | \leq L\delta^2.
\] 
(9)

By virtue of condition A, we have

\[
| f(\xi, u, v) - f(\eta, u, v) | \leq L_1 | \eta - \xi |.
\]

From inequality (9) we get

\[
| f(\xi, u, v) - f(\eta, u, v) | \leq L_1 L\delta^2.
\] 
(10)

Consider the sum \( \eta + t_\ast f(\eta, u, v) \). Using inequality (9) and (10) we have

\[
\eta + t_\ast f(\eta, u, v) \in \xi + L\delta^2H + t_\ast(f(\eta, u, v) + L_1 L\delta^2H) \subset \xi + t_\ast f(\xi, u, v) + L\delta^2(1 + L_1\varepsilon).
\]

Now, considering that \( \xi \in \Theta_\varepsilon(A\ast L\delta^2(1 + L_1\varepsilon)H) \) we come to the inclusion \( \eta + t_\ast f(\eta, u, v) \in A \). this implies \( \eta \in \Theta_\varepsilon(M) \). This was to be proved. Similarly, the inclusion (8) will be proved. Lemma 4 is proved.

Further, we consider only uniform partitions of the segments \([0, t]\). Let \( \omega_n = \{0, \varepsilon, 2\varepsilon, \ldots, n\varepsilon = t\} \), where \( \varepsilon = \frac{t}{n} \). Let \( \Gamma(n, \varepsilon) = L\varepsilon^2 \sum_{k=1}^{n} (1 + L_1\varepsilon)^{k-1} \). We assume

\[
\Theta_{2\varepsilon}A = \Theta_\varepsilon \Theta_{\varepsilon}A, \Theta_{k\varepsilon}A = \Theta_\varepsilon \Theta_{(k-1)\varepsilon}A, \quad \Theta_{\omega_n}A = \Theta_{n\varepsilon}A.
\]

Note that the notation \( T_{k\varepsilon} \) is entered in the same way as \( \Theta_{k\varepsilon} \)

**Theorem 2.** The following inclusions

\[
T_{\omega_n}(M\ast 2\Gamma(n, \varepsilon)H) \subset \Theta_{\omega_n}(M\ast \Gamma(n, \varepsilon)H) \subset T_{\omega_n}(M)
\] 
(11)

hold.

Proof. We prove the right side of inclusions (11). Let \( \omega_n = \{0, \varepsilon, 2\varepsilon = t\} \), where \( \varepsilon = \frac{t}{2} \). From Lemma 3 it follows that

\[
\Theta_\varepsilon(M\ast L\varepsilon^2H) \subset T_\varepsilon M.
\]
Now using the inclusion (7) we have

\[ \Theta_2(\mathcal{M} \ast \Gamma(2, \varepsilon) H) \subset \Theta_\varepsilon(\Theta_\varepsilon(\Theta_\varepsilon(\mathcal{M} \ast \Gamma(1, \varepsilon) H)) \ast L \varepsilon^2 H). \]

Applying Lemma 3 to the right-hand side of this inclusion, we arrive at the following relation

\[ \Theta_2(\mathcal{M} \ast \Gamma(2, \varepsilon) H) \subset \overline{T}_\varepsilon(\Theta_\varepsilon(\mathcal{M} \ast L \varepsilon^2 H)). \]

Using again Lemma 3, we obtain

\[ \Theta_2(\mathcal{M} \ast \Gamma(2, \varepsilon) H) \subset \overline{T}_\varepsilon \overline{T}_\varepsilon M = \overline{T}_2 M. \]

Suppose

\[ \Theta_\varepsilon(\mathcal{M} \ast \Gamma(p, \varepsilon) H) \subset \overline{T}_\varepsilon M. \]  

(12)

We shall prove the validity of the following relation

\[ \Theta_{(p+1)\varepsilon}(\mathcal{M} \ast \Gamma(p + 1, \varepsilon) H) \subset \overline{T}_{(p+1)\varepsilon} M. \]  

(13)

Let us consider the set

\[ \Theta_{(p+1)\varepsilon}(\mathcal{M} \ast \Gamma(p + 1, \varepsilon) H) = \Theta_\varepsilon \Theta_\varepsilon(\Theta_\varepsilon(\mathcal{M} \ast \Gamma(p, \varepsilon) H) \ast L \varepsilon^2 (1 + L_1 \varepsilon)^p H). \]

Applying Lemma 4 to the right side of this inclusion p-times we have

\[ \Theta_{(p+1)\varepsilon}(\mathcal{M} \ast \Gamma(p + 1, \varepsilon) H) \subset \Theta_\varepsilon(\Theta_\varepsilon(\mathcal{M} \ast \Gamma(p, \varepsilon) H) \ast L \varepsilon^2 H). \]

By virtue of Lemma 3 one obtains

\[ \Theta_{(p+1)\varepsilon}(\mathcal{M} \ast \Gamma(p + 1, \varepsilon) H) \subset \overline{T}_\varepsilon \Theta_\varepsilon(\mathcal{M} \ast \Gamma(p, \varepsilon) H). \]

Now due to the inclusion (12) we have

\[ \Theta_{(p+1)\varepsilon}(\mathcal{M} \ast \Gamma(p + 1, \varepsilon) H) \subset \overline{T}_\varepsilon \overline{T}_\varepsilon M = \overline{T}_{(p+1)\varepsilon}. \]

This implies the inclusion

\[ \Theta_{n\varepsilon}(\mathcal{M} \ast \Gamma(n, \varepsilon) H) \subset \overline{T}_{n\varepsilon} M, \]
is valid for any $n \in \mathbb{N}$. Consequently, $\Theta_{\omega_n}(M \ast \Gamma(n, \varepsilon) H) \subset \overline{T_{\omega_n} M}$. Similarly of that, the left side of the inclusion (11) will established. Theorem 2 is proved.

**Theorem 3.** The following equality holds

$$T_t M = \bigcup_{\delta > 0} \Theta_t(M \ast \delta H),$$

for open $M, M \subset \mathbb{R}^d$.

Proof. Consider the quantity $\Gamma(\varepsilon) = L \varepsilon^2 \sum_{k=1}^{n} (1 + L_1 \varepsilon)^k$. It is not difficult to see that $\Gamma(\varepsilon) \leq \varepsilon L(e^{L_1 \varepsilon} - 1)$. We choose $\varepsilon$ such that $\Gamma(\varepsilon) \leq \varepsilon L(e^{L_1 \theta} - 1) < \delta$, i.e. $\varepsilon < \frac{\delta}{L(e^{L_1 \theta} - 1)}$. By virtue of this, inclusion (11) implies

$$\overline{T_{\omega_n} (M \ast \delta H)} \subset \Theta_{\omega_n}(M \ast \delta H) \subset \overline{T_{\omega_n} M}.$$ 

Passing to the union over all $\omega_n$ in these relations by term, we obtain

$$\overline{T_t (M \ast \delta H)} \subset \Theta_t(M \ast \delta H) \subset T_t M.$$ 

Turning to the union over all $\delta > 0$ in these inclusions, we arrive to the following inclusions

$$\bigcup_{\delta > 0} \overline{T_t (M \ast \delta H)} \subset \bigcup_{\delta > 0} \Theta_t(M \ast \delta H) \subset \bigcup_{\delta > 0} T_t M.$$ 

It follows, by Lemmas 1 and 2, we have

$$\overline{T_t M} = \bigcup_{\delta > 0} \Theta_t(M \ast \delta H).$$ 

Theorem 3 is proved.

Theorems 1 and Theorems 3 imply

**Corollary.** The following equality holds

$$T_t M = \bigcup_{\delta > 0} \Theta_t(M \ast \delta H),$$

for open $M, M \subset \mathbb{R}^d$. 
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