Approximation of the domain of attraction of an asymptotically stable fixed point of a first order analytical system of difference equations

E. Kaslik\textsuperscript{a} A.M. Balint\textsuperscript{b} S. Birauas\textsuperscript{a} St. Balint\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, West University of Timisoara, Bd V. Parvan No. 4, 1900 Timisoara, Romania
\textsuperscript{b}Department of Physics, West University of Timisoara, Bd V. Parvan No. 4, 1900 Timisoara, Romania

Abstract

In this paper a first order analytical system of difference equations is considered. For an asymptotically stable fixed point $x^0$ of the system a gradual approximation of the domain of attraction (DA) is presented in the case when the matrix of the linearized system in $x^0$ is a contraction. This technique is based on the gradual extension of the "embryo" of an analytic function of several variables. The analytic function is a Lyapunov function whose natural domain of analyticity is the DA and which satisfies an iterative functional equation. The equation permits to establish an "embryo" of the Lyapunov function and a first approximation of the DA. The "embryo" is used for the determination of a new "embryo" and a new part of the DA. In this way, computing new "embryos" and new domains, the DA is gradually approximated. Numerical examples are given for polynomial systems.

Key words: difference equations, fixed point, asymptotically stable, domain of attraction; AMS Subject Classification: 34.K.20

1 Introduction

We consider the system of difference equations

$$x_{k+1} = g(x_k) \quad k = 0, 1, 2...$$

(1)

where $g : \Omega \to \Omega$ is an analytic function defined on a domain $\Omega$ included in $\mathbb{R}^n$. 

Preprint submitted to Elsevier Science 29 March 2022
A point \( x^0 \in \Omega \) is a fixed point for the system (1) if \( x^0 \) satisfies

\[
x^0 = g(x^0)
\]  

(2)

The fixed point \( x^0 \) of (1) is "stable" provided that given any ball \( B(x^0, \varepsilon) = \{ x \in \Omega / \| x - x^0 \| < \varepsilon \} \), there is a ball \( B(x^0, \delta) = \{ x \in \Omega / \| x - x^0 \| < \delta \} \) such that if \( x \in B(x^0, \delta) \) then \( g^k(x) \in B(x^0, \varepsilon) \), for \( k = 0, 1, 2, ... \) [1].

If in addition there is a ball \( B(x^0, r) \) such that \( g^k(x) \to x^0 \) as \( k \to \infty \) for all \( x \in B(x^0, r) \) then the fixed point \( x^0 \) is "asymptotically stable" [1].

The domain of attraction \( DA(x^0) \) of the asymptotically stable fixed point \( x^0 \) is the set of initial states \( x \in \Omega \) from which the system converges to the fixed point itself i.e.

\[
DA(x^0) = \{ x \in \Omega | g^k(x) \xrightarrow{k \to \infty} x^0 \}
\]  

(3)

It is known that \( x^0 \) is a fixed point for system (1) if and only if \( 0 \in \mathbb{R}^n \) is a fixed point for the system

\[
y_{k+1} = f(y_k) \quad k = 0, 1, 2...
\]  

(4)

where \( f : \Omega - x^0 \to \Omega - x^0 \) is the analytic function defined by

\[
f(y) = g(y + x^0) - x^0 \quad \text{for} \; y \in \Omega - x^0
\]  

(5)

The fixed point \( x^0 \) of (1) is asymptotically stable if and only if the fixed point \( 0 \in \mathbb{R}^n \) of the system (4) is asymptotically stable.

The domain of attraction of \( x^0 \), \( DA(x^0) \) is related to the domain of attraction of 0, \( DA(0) \) by the equation

\[
DA(x^0) = DA(0) + x^0
\]  

(6)

For the above reason in the followings instead of the system (1) we will consider the system (4).

Theoretical research shows that the \( DA(0) \) and its boundary are complicated sets [2],[3],[4],[5],[6]. In most cases, they do not admit an explicit elementary representation. For this reason, different procedures are used for the approximation of the \( DA(0) \) with domain having a simpler shape. For example, in the case of the theorem 4.20 pg 170 [1] the domain which approximates the
$DA(0)$ is defined by a Lyapunov function $V$ built with the matrix $\partial_0 f$ of the linearized system in $0$. In this paper, we present a technique for the construction of a Lyapunov function $V$ in the case when the matrix $\partial_0 f$ is a contraction, i.e. $\|\partial_0 f\| < 1$. The Lyapunov function $V$ is built using the whole nonlinear system, not only the matrix $\partial_0 f$. $V$ is defined on the whole $DA(0)$, and more, the $DA(0)$ is the natural domain of analyticity of $V$. The formula which defines the Lyapunov function $V$ is used for determining an "embryo" of $V$ and a first approximation of $DA(0)$. The "embryo" is used for the determination of a new "embryo" and a new part of $DA(0)$. In this way, computing new "embryos" and new domains the $DA(0)$ is gradually approximated.

2 Theoretical results

Let be $f : \Omega \to \Omega$ an analytic function defined on a domain $\Omega \subset \mathbb{R}^n$ containing the origin $0 \in \mathbb{R}^n$.

Theorem 1 If the function $f$ satisfies the following conditions:

$$f(0) = 0$$  \hspace{1cm} (7)

$$\|\partial_0 f\| < 1$$  \hspace{1cm} (8)

then $0$ is an asymptotically stable fixed point. $DA(0)$ is an open subset of $\Omega$ and coincides with the natural domain of analyticity of the unique solution $V$ of the iterative first order functional equation

$$\begin{cases} 
V(f(x)) - V(x) = -\|x\|^2 \\
V(0) = 0
\end{cases}$$  \hspace{1cm} (9)

The function $V$ is positive on $DA(0)$ and $V(x) \xrightarrow{x \to x^0} +\infty$, for any $x^0 \in FrDA(0)$ ($FrDA(0)$ denotes the boundary of $DA(0)$).

Proof. Let be $\alpha$ such that $\|\partial_0 f\| < \alpha < 1$. By the continuity of $x \mapsto \|\partial_x f\|$ there exists a $\delta > 0$ such that $\|\partial_x f\| \leq \alpha$ for $\|x\| \leq \delta$. The mean value theorem gives

$$\|f(x') - f(x'')\| \leq \alpha \|x' - x''\|$$  \hspace{1cm} (10)

for any $x'$ and $x''$ in the ball $B(0, \delta)$. Therefore

$$\|f^k(x)\| \leq \alpha^k \|x\|$$  \hspace{1cm} (11)
for any \( x \) in the ball \( B(0, \delta) \) and \( k = 0, 1, 2, \ldots \). For a ball \( B(0, \varepsilon) \) we take \( \delta' = \min(\varepsilon, \delta) \) and the ball \( B(0, \delta') \). We have \( f^k(x) \in B(0, \delta') \) for any \( x \in B(0, \delta') \) and \( k = 0, 1, 2, \ldots \), which means that 0 is stable.

From (11) we obtain \( f^k(x) \xrightarrow{k \to \infty} 0 \) for any \( x \in B(0, \delta) \). We can conclude now that 0 is asymptotically stable.

In order to show that \( DA(0) \) is an open subset of \( \Omega \) we consider \( x' \) from \( DA(0) \) and \( k_0 > 0 \) such that \( \| f^{k_0}(x') \| < \frac{\delta}{3} \). Because \( f^{k_0} \) is a continuous function, there exists a ball \( B(x', \delta'') \) such that \( \| f^{k_0}(x) - f^{k_0}(x') \| < \frac{\delta}{3} \), for any \( x \in B(x', \delta'') \). Therefore, \( \| f^{k_0}(x) \| \leq \frac{2\delta}{3} < \delta \), for any \( x \in B(x', \delta'') \). It follows that \( x \in DA(0) \) and therefore, \( B(x', \delta'') \subset DA(0) \). This proves that \( DA(0) \) is an open subset of \( \Omega \).

Now we consider \( x \in DA(0) \) and the sequence \( \{ f^k(x) \}_{k \in \mathbb{N}} \). There exists \( k_x \) such that \( f^k(x) \in B(0, \delta) \), for any \( k \geq k_x \). Therefore, \( \| f^{k_x+k}(x) \| \leq \alpha^k \| f^{k_x}(x) \| \), for \( k = 0, 1, 2 \ldots \). It follows that the series \( \sum_{k=0}^{\infty} \| f^k(x) \|^2 \) is convergent for any \( x \in DA(0) \).

Let be \( V = V(x) \) the function defined by

\[
V(x) = \sum_{k=0}^{\infty} \| f^k(x) \|^2 \quad \text{for} \ x \in DA(0)
\] (12)

The above function defined on \( DA(0) \) is analytical, positive and satisfies (9). In order to show that the function \( V \) defined by (12) is the unique function which satisfies (9) we consider \( V'' = V'(x) \) satisfying (9) and we denote by \( V'' \) the difference \( V'' = V - V' \). It is easy to see that \( V''(f(x)) - V''(x) = 0 \), for any \( x \in DA(0) \). Therefore, we have \( V''(x) = V''(f^k(x)) \) for any \( x \in DA(0) \) and any \( k = 0, 1, 2 \ldots \). It follows that \( V''(x) = \lim_{k \to \infty} V''(f^k(x)) = 0 \) for any \( x \in DA(0) \). In other words, \( V(x) = V'(x) \), for any \( x \in DA(0) \), so \( V \) defined by (12) is the unique function which satisfies (9).

In order to show that \( V(x) \xrightarrow{x \to x^0} \infty \) for any \( x^0 \in FrDA(0) \) we consider \( x^0 \in FrDA(0) \) and \( r > 0 \) such that \( \| f^k(x^0) \| > r \), for any \( k = 0, 1, 2 \ldots \). For an arbitrary positive number \( N > 0 \) we consider the first natural number \( k_1 \) which satisfies \( k_1 \geq \frac{2N}{r^2} + 1 \). Let be \( r_1 > 0 \) such that \( \| f^k(x) \| \geq \frac{\delta}{\sqrt{2}} \), for any \( k = 1, 2, \ldots, k_1 \) and \( x \in B(x^0, r_1) \). For any \( x \in B(x^0, r_1) \cap DA(0) \) we have \( \sum_{k=0}^{k_1} \| f^k(x) \|^2 > N \). Therefore, \( V(x) \xrightarrow{x \to x^0} \infty \). \( \square \)

**Remark 2** Newton’s method for solving systems of \( n \) nonlinear equations with \( n \) unknowns always leads to a system of difference equations which satisfies the conditions of Theorem 1.
We consider the expansion of $V$ in $0$:

$$V(x_1, x_2, ..., x_n) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1j_2...j_n} x_1^{j_1} x_2^{j_2} ... x_n^{j_n}$$

(13)

The domain of convergence of the series (13) is the set of those $x \in \Omega$ which have the property that the series (13) is absolutely convergent in a neighborhood of $x$ [7]. We denote by $D^0$ this domain. Actually, the domain of convergence $D^0$ coincides with the interior of the set of all points $x^0$ in which the series (13) is absolutely convergent [7].

**Theorem 3** (Cauchy-Hadamard see [7]) A point $x$ belongs to $D^0$ if and only if

$$\lim_{m \to \infty} \sqrt{\sum_{|j|=m} |B_{j_1j_2...j_n} x_1^{j_1} x_2^{j_2} ... x_n^{j_n}|} < 1$$

(14)

By the previous theorem we find a part $D^0_p$ of $DA(0)$.

In practice, we can compute the coefficients $B_{j_1j_2...j_n}$ up to a finite degree $p$. This degree $p$ has to be big enough for assuring that the domain $D^0_p$ given by

$$D^0_p = \{ x \in \Omega / \sqrt{\sum_{|j|=p} |B_{j_1j_2...j_n} x_1^{j_1} x_2^{j_2} ... x_n^{j_n}|} < 1 \}$$

approximates the region of convergence $D^0$ of the series of $V$ and the ”embryo”

$$V^0_p(x_1, x_2, ..., x_n) = \sum_{m=2}^{p} \sum_{|j|=m} B_{j_1j_2...j_n} x_1^{j_1} x_2^{j_2} ... x_n^{j_n}$$

(15)

approximates $V$ with accuracy.

The first estimate of the $DA$ will be $D^0_p$.

In order to extend the first estimate $D^0_p$, we expand $V$ in a point $x^0$ close to $FrD^0_p$ in which $|V^0_p(x^0)|$ is still small. That is because, according to Theorem 1, the points $x$ close to $FrD^0_p$ for which $|V^0_p(x)|$ is extremely high are close to $FrDA(0)$.

To find the expansion of $V$ in $x^0$ close to $FrD^0_p$, we will compute the expansion in $x^0$ of the ”embryo” $V^0_p$ of $V$. We obtain:
\[ V_p^1(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{p} \sum_{|j|=m} \frac{\partial^m V_0^p}{\partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_n^{j_n}} (x_1^0 - x_1^0)^{j_1} \ldots (x_n - x_n^0)^{j_n} / m! = \]
\begin{equation}
\sum_{m=0}^{p} \sum_{|j|=m} B_{1, j_2 \ldots j_n}^1 (x_1 - x_1^0)^{j_1} (x_2 - x_2^0)^{j_2} \ldots (x_n - x_n^0)^{j_n} \tag{16}
\end{equation}

We consider the set

\[ D_p^1 = \{ x \in \Omega / \sqrt{\sum_{|j|=p} |B_{1, j_2 \ldots j_n} (x_1 - x_1^0)^{j_1} (x_2 - x_2^0)^{j_2} \ldots (x_n - x_n^0)^{j_n}| < 1} \}

which provides a new part \( D_p^1 \) of \( DA(0) \).

So, \( D_p^0 \cup D_p^1 \) gives a larger estimate of \( DA(0) \). We can continue this procedure for a few steps, till the values \( |V_p^k| \) become extremely large and we obtain the estimate \( D_p^0 \cup D_p^1 \cup \ldots \cup D_p^k \) of \( DA(0) \).

3 Numerical results

3.1 Systems with known domains of attraction

In this subsection, we will present some examples of systems of one, two or three difference equations, for which we can compute easily the \( DA \). We will apply our technique to these examples, and we will show how the real domains of attraction are gradually approximated. These examples are meant to validate our procedure.

The computations were made using a program written in Mathematica 4, Wolfram Research on an Intel Pentium III PC (2Ghz, 512MB of RAM). The data for the estimations (the degree up to which the approximation is made, the necessary timing for the estimations) are displayed in Table 1. In our figures, the thick black line represents the true boundary of the domain of attraction, the dark grey set denotes the first estimate \( D_p^0 \) of \( DA \) and the further estimates \( D_p^k \) of \( DA \) with \( k \geq 1 \) are colored in light grey.
**Example 1**

We consider the following one dimensional difference equation:

\[ x_{n+1} = \frac{1}{2}x_n - x_n^2 + 2x_n^3 - 4x_n^4 \]  \hspace{1cm} (17)

This difference equation has two fixed points: \( x = 0 \) and \( x = -0.271845 \). The fixed point \( x = 0 \) is asymptotically stable, while the other fixed point is unstable. It can be proved (by the staircase method) that \( DA(0) \) is the interval \((-0.271845, 0.653564)\).

We applied the procedure described above to obtain an estimation of the \( DA(0) \).

- After the first step, we obtained \( D^0 = (-0.27184, 0.27184) \).
- After the second step applied in \( x^1 = 0.2718 \), we obtained \( D^1 = (0.01345, 0.53015) \).
- After the third step applied in \( x^2 = 0.5 \), we obtained \( D^2 = (0.38378, 0.61622) \).
- After the forth step applied in \( x^3 = 0.61 \), we obtained \( D^3 = (0.59785, 0.622175) \).

Therefore, the estimate of \( DA(0) \) obtained after four steps is \((-0.27184, 0.622175)\).

**Example 2**

In [1], the following system of difference equations is considered:

\[
\begin{align*}
x_{n+1} &= -y_n(1 - x_n^2 - y_n^2) \\
y_{n+1} &= -x_n(1 - x_n^2 - y_n^2)
\end{align*}
\]  \hspace{1cm} (18)

It is easy to see that \( (x, y) = (0, 0) \) is an asymptotically stable fixed point for the system (18), and the matrix of the linearized system in \((0, 0)\) satisfies the conditions from Theorem 1. There are two more fixed points of system (18) (which are unstable) which are represented by the gray points in Fig 1.1-1.2. The \( DA(0) \) is the set:

\[
DA(0) = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 < 3\}
\]

Using the technique described above, after 2 steps of gradual extension of the "embryo", we find the estimate of \( DA(0) \) presented in Figure 1.1. If we apply the second step for several points close to \( FrD^0 \), we obtain the estimate of \( DA(0) \) presented in Figure 1.2.
Example 3

The following decoupled system of difference equations is considered:

\[
\begin{aligned}
    x_{n+1} &= 4x_n^3 \\
y_{n+1} &= 9y_n^3
\end{aligned}
\]  

(19)

It is clear that \((0, 0)\) is an asymptotically stable fixed point for the system (19) and 

\[DA(0) = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right)\]

Applying the above procedure, after only one step, we obtain the whole \(DA(0)\).
3.2 Systems for which we don’t know the domain of attraction

In this subsection, some systems of difference equations are presented for which we don’t know the $DA(0)$. For these examples, we will apply the procedure presented above.

**Example 4**

We consider the difference equation:

$$x_{n+1} = -x_n^2 - 2x_n^3 - 4x_n^4 - 8x_n^5$$  \hspace{1cm} (20)

This equation has only one fixed point, namely $x = 0$. We obtained

- $D^0 = (-0.442585, 0.442585)$, after the first step.
- $D^1 = (-0.673088, -0.212082)$, after the second step applied in $x^1 = -0.44258$.

The estimate of the $DA(0)$ obtained after two steps is $(-0.673088, 0.442585)$.

**Example 5**

The following system is considered:

$$\begin{align*}
    x_{n+1} &= -\frac{1}{2}x_n + x_n y_n \\
    y_{n+1} &= -\frac{1}{2}y_n + x_n y_n
\end{align*}$$  \hspace{1cm} (21)

This system of difference equations has besides the asymptotically stable fixed point $(0, 0)$, an unstable fixed point represented in the following figure by a black point. After two steps, we can obtain the following estimate of the $DA(0)$:
Example 6

The following system of three difference equations is considered:

\[
\begin{align*}
x_{n+1} &= \frac{1}{2} x_n y_n + \frac{1}{4} x_n z_n + \frac{1}{3} x_n^2 y_n + \frac{1}{12} x_n^2 z_n - \frac{1}{3} x_n y_n^2 - \frac{1}{12} x_n z_n^2 - \frac{1}{12} x_n y_n z_n \\
y_{n+1} &= -\frac{1}{2} x_n y_n + \frac{1}{2} y_n z_n - \frac{1}{3} x_n^2 y_n + \frac{1}{3} x_n y_n^2 + \frac{1}{3} y_n z_n^2 - \frac{1}{3} y_n z_n^2 + \frac{1}{6} x_n y_n z_n \\
z_{n+1} &= -\frac{1}{2} y_n z_n - \frac{1}{4} x_n z_n - \frac{1}{12} x_n^2 z_n - \frac{1}{3} y_n^2 z_n + \frac{1}{12} x_n z_n^2 + \frac{1}{3} y_n z_n^2 - \frac{1}{12} x_n y_n z_n
\end{align*}
\]

The estimate of the $DA$ of the asymptotically stable fixed point $(0, 0, 0)$ obtained after one step is presented in Fig. 4.
Table 1. Numerical data

| example | order of approximation \( p \) | timing for 1\(^{st} \) step | timing for 2\(^{nd} \) step |
|---------|-------------------------------|-----------------------------|-----------------------------|
| 1       | 4096                          | 9.3 h                       | 32.5 h                      |
| 2       | 164                           | 10 min                      | 24.1 h                      |
| 3       | 500                           | 12.4 min                    | -                           |
| 4       | 625                           | 1.2 h                       | 12.4 h                      |
| 5       | 256                           | 16.8 min                    | 45.2 h                      |
| 6       | 54                            | 4.6 h                       | -                           |

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