Siegel Gauge in Vacuum String Field Theory

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We study the star algebra of ghost sector in vacuum string field theory (VSFT). We show that the star product of two states in the Siegel gauge is BRST exact if we take the BRST charge to be the one found in [1], and the BRST exact states are nil factors in the star algebra. By introducing a new star product defined on the states in the Siegel gauge, the equation of motion of VSFT is characterized as the projection condition with respect to this new product. We also comment on the comma form of string vertex in the ghost sector.

November 2001
1. Introduction

Physics of tachyon condensation in open string theory has been studied extensively in the framework of string field theories (see [4] for a review). One of the big challenges in this direction is to describe the final state of tachyon condensation and see how closed strings come out. Vacuum string field theory (VSFT) [3] was proposed as a possible candidate of the theory describing the end point of tachyon condensation. (See [4–14] for related papers.) The action of VSFT has the same form as the one of Witten’s bosonic open string field theory [15], but its BRST charge consists of purely ghost terms and has trivial cohomology which simply represents the absence of physical states of open strings after tachyon condensation.

A priori, there is no restriction on the form of BRST charge $Q$ in VSFT, except for the requirement that $Q$ should be a derivation of the star product of string fields. One possible form of $Q$ is given by

$$Q = c_0 + \sum_{n=1}^{\infty} f_n (c_{-n} + (-1)^n c_n)$$

(1.1)

where $f_n$ are arbitrary constants. However, if one wants to solve the equation of motion of VSFT in the Siegel gauge, the existence of the solution determines the coefficients $f_n$ uniquely [1]. In this paper, we show that $Q$ obtained in [1] is not tied to the specific form of the solution found in [1] but appears universally in the star product of two states in the Siegel gauge.

The key observation is that $Q$ in (1.1) satisfies

$$\{Q, b_0\} = 1.$$  

(1.2)

Therefore, the star algebra $\mathcal{A}$ of string fields splits into a direct sum of two parts as a vector space:

$$\mathcal{A} = \mathcal{A}_{b_0} \oplus \mathcal{A}_Q,$$

(1.3)

where

$$\mathcal{A}_{b_0} = \text{Ker}(b_0) = \text{Im}(b_0), \quad \mathcal{A}_Q = \text{Ker}(Q) = \text{Im}(Q),$$

(1.4)

i.e., $\mathcal{A}_{b_0}$ and $\mathcal{A}_Q$ are the space of states in the Siegel gauge and the space of BRST exact states, respectively. They are related with each other by multiplying $b_0$ or $Q$

$$\mathcal{A}_{b_0} = b_0(\mathcal{A}_Q), \quad \mathcal{A}_Q = Q(\mathcal{A}_{b_0}).$$

(1.5)
The natural question here is that what is the multiplication rule between $A_{b_0}$ and $A_Q$ under the star product of string fields? We will show that the star product of two states in the Siegel gauge is a BRST exact state, in other words

$$A_{b_0} \ast A_{b_0} \subset A_Q = Q(A_{b_0}),$$

and $A_Q$ acts as zero under the star product:

$$A_Q \ast A = A \ast A_Q = 0.$$

This paper is organized as follows: In section 2 we show the properties of string field algebra (1.6) and (1.7) by introducing a new set of variables which simplifies the string vertices. In section 3, we define a new star product on $A_{b_0}$ and discuss that the solution of the equation of motion of VSFT is given by the projection with respect to this new product. In section 4, we show that the 3-string vertex of ghost sector can be written as a comma form as in the case of matter sector. Section 5 is devoted to discussions.

Note added: Some of the results in this paper are discussed in [16] from the different viewpoint.

2. Star Product in the Ghost Sector

2.1. Review of the 3-string Vertex

To study the structure of star algebra in the ghost sector, let us first review the properties of Neumann coefficients in the 3-string vertex [17–21,10]. The 3-string vertex in the ghost sector is written as

$$|V_3\rangle = \exp \left( \sum_{r,s=1}^{3} c^{(r)\dagger} \tilde{V}^{rs} b^{(s)\dagger} + c^{(r)\dagger} \tilde{v}^{rs} b^{(s)\dagger}_{0} \right) |+\rangle_{123}$$

where $|+\rangle$ is the oscillator vacuum of $b_n, c_n (n \geq 1)$ defined by

$$|+\rangle = c_0 |\rangle, \quad |\rangle = c_1 |0\rangle,$$

and $|0\rangle$ is the $SL(2, \mathbb{R})$ vacuum. The dagger is defined by $c^{\dagger}_n = c_{-n}$ and $b^{\dagger}_n = b_{-n}$, and the summation over $n(\geq 1)$ is suppressed in (2.1). The Neumann coefficients $\tilde{V}^{rs}_{nm}$ and $\tilde{v}^{rs}$ have the following twist properties:

$$C \tilde{V}^{rs} C = \tilde{V}^{sr}, \quad C \tilde{v}^{rs} = \tilde{v}^{sr},$$

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where $C_{nm} = (-1)^n \delta_{nm}$. They also satisfy the relations
\[
\sum_{r=1}^{3} \bar{V}^{rs} = C, \quad \sum_{t=1}^{3} \bar{V}^{rt} \bar{V}^{ts} = \delta^{rs}, \quad \sum_{r=1}^{3} \bar{v}^{rs} = 0, \quad \sum_{t=1}^{3} \bar{v}^{rt} \bar{v}^{ts} = -\bar{v}^{rs}.
\] (2.4)

It is convenient to introduce the notation
\[
\bar{X} = C \bar{V}^{rr}, \quad \bar{Y} = C \bar{V}^{r,r+1}, \quad \bar{Z} = C \bar{V}^{r,r-1},
\] (2.5)

and
\[
\bar{v}_0 = \bar{v}^{rr}, \quad \bar{v}_+ = \bar{v}^{r,r+1}, \quad \bar{v}_- = \bar{v}^{r,r-1}.
\] (2.6)

In terms of these variables, the relations (2.4) are written as
\[
\bar{X} + \bar{Y} + \bar{Z} = 1, \quad \bar{X} \bar{Y} + \bar{Y} \bar{Z} + \bar{Z} \bar{X} = 0, \quad \bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 = 1
\] (2.7)

and
\[
\bar{v}_+ = \frac{\bar{Z}}{\bar{X} - 1} \bar{v}_0, \quad \bar{v}_- = \frac{\bar{Y}}{\bar{X} - 1} \bar{v}_0.
\] (2.8)

2.2. Star Product of Coherent States in $A_{b_0}$

To know the property of star product on the space of Siegel gauge $A_{b_0}$, it is sufficient to compute the star product of coherent states constructed on the oscillator vacuum $|\alpha, \beta\rangle$:
\[
|\alpha, \beta\rangle = e^{\alpha C b^\dagger - c^\dagger C \beta} |\alpha, \beta\rangle \in A_{b_0},
\] (2.9)

since they form an (over-complete) basis of $A_{b_0}$. Here $\alpha$ and $\beta$ are grassmann parameters. The star product of two coherent states is calculated as
\[
|\alpha_1, \beta_1\rangle \star |\alpha_2, \beta_2\rangle = \exp \left( c^\dagger \mu^3 + \sum_{r=1,2} \alpha_r \mu^r - c^\dagger \bar{V}^3 \beta_r - \sum_{r,s=1,2} \alpha_r \bar{V}^{rs} \beta_s \right) |+\rangle,
\] (2.10)

where
\[
\mu^r = \bar{V}^r b^\dagger + \bar{v}^{r,3} b_0.
\] (2.11)

If we require this product state $|\alpha_1, \beta_1\rangle \star |\alpha_2, \beta_2\rangle$ to be BRST closed (which implies BRST exact), the coefficient $f_n$ in (1.1) is determined uniquely. The BRST transform of this product state is
\[
Q \left( |\alpha_1, \beta_1\rangle \star |\alpha_2, \beta_2\rangle \right) = \left( c^\dagger f - c^\dagger \{ Q, \mu^3 \} - \sum_{r=1,2} \alpha_r \{ Q, \mu^r \} \right) |\alpha_1, \beta_1\rangle \star |\alpha_2, \beta_2\rangle.
\] (2.12)
Therefore we obtain the condition for \( f_n \)

\[
f - \{Q, \mu^3\} = \{Q, \mu^1\} = \{Q, \mu^2\} = 0, \tag{2.13}
\]

which is written as

\[
(1 - \tilde{X})f - \tilde{v}_0 = \tilde{v}_- + \tilde{Y} f = \tilde{v}_+ + \tilde{Z} f = 0. \tag{2.14}
\]

Although this seems to be an over-determined system, thanks to the identity (2.8) we have a solution for \( f_n \)

\[
f = \frac{1}{1 - \tilde{X}} \tilde{v}_0. \tag{2.15}
\]

This is exactly the one obtained in [1,10]. We conclude that the star product of two states in \( \mathcal{A}_{b_0} \) gives a state in \( \mathcal{A}_Q \), i.e., \( \mathcal{A}_{b_0} \ast \mathcal{A}_{b_0} \subset \mathcal{A}_Q \) for this particular choice of \( Q \).

Note that since this \( f \) is twist even

\[
Cf = f, \tag{2.16}
\]

\( Q \) can be written as

\[
Q = c_0 + (c + c^\dagger) f. \tag{2.17}
\]

2.3. 3-string Vertex on New Basis

The result in the previous subsection can be clearly understood by rewriting the 3-string vertex in new variables. Notice that \( f \) in (2.15) appears in the relation of Neumann coefficients as

\[
\tilde{v}^{rs} = (\delta^{rs} - \tilde{V}^{rs}) f. \tag{2.18}
\]

This expression of \( \tilde{v}^{rs} \) makes it clear that it is an eigenvector of \( \tilde{V}^{rs} \) with eigenvalue \( -1 \) (see (2.4)). Plugging this into (2.1), \( |V_3\rangle \) becomes

\[
|V_3\rangle = \exp \left( \sum_{r,s=1}^3 c^{(r)\dagger} \tilde{V}^{rs} (b^{(s)\dagger} - f b^{(s)}_0) + \sum_{r=1}^3 c^{(r)\dagger} f b^{(r)}_0 \right) |+\rangle_{123}. \tag{2.19}
\]

Since the last term is diagonal in the index \( r \) of 3-string, it can be absorbed by introducing a new vacuum \( |\tilde{+}\rangle \) by

\[
|\tilde{+}\rangle = e^{c^\dagger f b_0} |+\rangle = U |+\rangle. \tag{2.20}
\]

Here \( U \) is the operator defined by

\[
U = e^{(c + c^\dagger) f b_0}. \tag{2.21}
\]
This is a unitary operator in the sense of both hermitian and BPZ conjugation:

\[
U^\dagger = \text{bpz}(U) = U^{-1}. \tag{2.22}
\]

This unitary transformation \(U\) naturally accounts for the appearance of \(Q\) in the star product of Siegel gauge states. Actually, \(Q\) is the \(U\)-transform of \(c_0\)

\[
Q = Uc_0U^{-1} = c_0 + (c + c^\dagger)f. \tag{2.23}
\]

Also the shift of \(b_n\) by \(f_n b_0\) in the first term of (2.19) can be understood as the effect of \(U\)-transformation:

\[
\tilde{b}_n \equiv Ub_nU^{-1} = b_n - f_n b_0. \tag{2.24}
\]

Under this unitary transformation, \(b_0, c_n\) and \(|-\rangle\) remain intact

\[
Ub_0U^{-1} = b_0, \quad Uc_nU^{-1} = c_n, \quad U|-\rangle = |-\rangle. \tag{2.25}
\]

One can easily see that \(|-\rangle\) and \(|\tilde{+}\rangle\) behave as a doublet under \(b_0\) and \(Q\)

\[
b_0|\tilde{+}\rangle = |-\rangle, \quad Q|-\rangle = |\tilde{+}\rangle, \tag{2.26}
\]

and the inner product of these states is the same as the original vacua \(|\pm\rangle\):

\[
\langle -|\tilde{+}\rangle = \langle \tilde{+}| -\rangle = 1, \quad \langle \tilde{+}|\tilde{+}\rangle = \langle -| -\rangle = 0. \tag{2.27}
\]

Finally, \(|V_3\rangle\) in this new basis becomes

\[
|V_3\rangle = \exp \left( \sum_{r,s=1}^{3} c^{(r)\dagger} \tilde{V}^{rs} \tilde{b}^{(r)\dagger} \right) |\tilde{+}\rangle_{123}. \tag{2.28}
\]

We can also rewrite the reflector \(\langle R|\) in this new basis. \(\langle R|\) in the original variable is given by

\[
\langle R| = 12\langle +| \delta(b_0^{(1)} - b_0^{(2)}) \exp \left( -c^{(1)} C b^{(2)} - c^{(2)} C b^{(1)} \right). \tag{2.29}
\]

In terms of the new vacuum \(|\tilde{+}\rangle\), the zero mode part of reflector is rewritten as

\[
12\langle +| \delta(b_0^{(1)} - b_0^{(2)}) = 12\langle \tilde{+}| \exp \left( c^{(1)} f b_0^{(1)} + c^{(2)} f b_0^{(2)} \right) \delta(b_0^{(1)} - b_0^{(2)})
\]

\[
= 12\langle \tilde{+}| \delta(b_0^{(1)} - b_0^{(2)}) \exp \left( c^{(1)} f b_0^{(2)} + c^{(2)} f b_0^{(1)} \right). \tag{2.30}
\]

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Therefore, the reflector in the new basis has the same form as the original one:

$$\langle R | = 12 \langle \tilde{\Theta} | \delta (b_0^{(1)} - b_0^{(2)}) \exp \left( -c^{(1)} C \tilde{b}^{(2)} - c^{(2)} C \tilde{b}^{(1)} \right).$$

(2.31)

In summary, the 3-string vertex and reflector in this new basis have the same form as the original one except for the absence of $b_0$ in the exponent in $|V_3\rangle$. Combining this observation with the fact that $A_{b_0}$ and $A_Q$ can be written as

$$A_{b_0} = \text{Span} \left\{ c_{n_1}^\dagger \cdots c_{n_k}^\dagger b_{m_1}^\dagger \cdots b_{m_l}^\dagger | - \right\},$$

$$A_Q = \text{Span} \left\{ c_{n_1}^\dagger \cdots c_{n_k}^\dagger \tilde{b}_{m_1}^\dagger \cdots \tilde{b}_{m_l}^\dagger | \tilde{\Theta} \right\},$$

we arrive at the same conclusion $A_{b_0} \star A_{b_0} \subset A_Q$. Because of the inner product structure (2.27) and the absence of $b_0$ in (2.28), all the elements in $A_Q$ behave as zero in the star algebra

$$QA \star B = B \star QA = 0 \quad \forall A, B.$$  \hspace{1cm} (2.33)

For example, $c_0 |I\rangle$ belongs to $A_Q$ and hence decouples from the star product [22,3,10]. Note that the “identity string field $|I\rangle$” is not the identity of star product [10] since $|I\rangle \star A_Q = 0$.

3. Reduced Star Product on $A_{b_0}$

Although the star product does not close on the space of Siegel gauge states $A_{b_0}$, we can define a new product on $A_{b_0}$ using the isomorphism $A_Q \cong A_{b_0} = b_0(A_Q)$. We define a new product $\star_{b_0}$ on $A_{b_0}$ as

$$\star_{b_0} : A_{b_0} \times A_{b_0} \to A_{b_0},$$

$$(A, B) \mapsto b_0(A \star B) \equiv A \star_{b_0} B,$$

which we will call “reduced star product”. The original star product $\star$ and the reduced product $\star_{b_0}$ are related by

$$A \star_{b_0} B = b_0(A \star B), \quad A \star B = Q(A \star_{b_0} B).$$  \hspace{1cm} (3.2)

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1 Assuming $\langle A | B \rangle = \langle I | A \star B \rangle$, the authors of [10] argued that eq. (2.33) should be relaxed to $

\langle C | QA \star B \rangle = 0$ for Fock space states $A, B, C$.  

6
The first equation is the definition of \( \star_{b_0} \). The second equation can be shown as

\[
Q(A \star_{b_0} B) = Qb_0(A \star B) = (1 - b_0Q)(A \star B) = A \star B.
\]  

(3.3)

In the last step, we used the fact \( A \star B \in \mathcal{A}_Q \). This reduced product is (at least formally) associative since this product is written in terms of \( \tilde{V}^{rs} \) as in the original star product. We can consistently extend this reduced product to the whole space by simply setting the elements of \( \mathcal{A}_Q \) to be zero under this product:

\[
QA \star_{b_0} B = B \star_{b_0} QA = 0 \quad \forall A, B.
\]  

(3.4)

Using this product \( \star_{b_0} \), we can define the wedge-like state \( |n\rangle_w \in \mathcal{A}_{b_0} \) as in [22]

\[
|n\rangle_w = (|\rangle - \rangle)^{n-1}_{b_0},
\]  

(3.5)

and the sliver-like state as the limit of wedge-like state

\[
|S_-\rangle = \lim_{n \to \infty} |n\rangle_w.
\]  

(3.6)

In the same way as the original sliver state, this sliver-like state turns out to be a projection with respect to the reduced star product

\[
|S_-\rangle = |S_-\rangle \star_{b_0} |S_-\rangle.
\]  

(3.7)

Due to the relation (3.2), \(-|S_-\rangle\) gives a solution of the equation of motion of VSFT:

\[
Q|S_-\rangle = Q(|S_-\rangle \star_{b_0} |S_-\rangle) = |S_-\rangle \star |S_-\rangle.
\]  

(3.8)

In the oscillator representation, \( |S_-\rangle \) is given by [1114]

\[
|S_-\rangle = \mathcal{N}_{\infty}e^{c\dagger C\tilde{T}b^\dagger}|\rangle
\]  

(3.9)

where \( \mathcal{N}_{\infty} \) is a normalization factor and \( \tilde{T} \) is defined by

\[
\tilde{T} = \frac{1}{2\tilde{X}}\left[1 + \tilde{X} - \sqrt{(1 - \tilde{X})(1 + 3\tilde{X})}\right].
\]  

(3.10)

In general, there is a one-to-one correspondence between the solution of the equation of motion of VSFT and the projection with respect to \( \star_{b_0} \) under the map of \( \star \) and \( \star_{b_0} \) (3.2). As a special projection, there is the identity of \( \star_{b_0} \) given by

\[
|I_-\rangle = \mathcal{N}_1e^{c\dagger Cb^\dagger}|\rangle.
\]  

(3.11)
4. Comma Vertex in the Ghost Sector

As in the matter sector, we can rewrite the 3-string vertex of ghost sector in the comma form $[23,9]$. This form of vertex is useful to clarify the left-right split structure of the string field algebra.

Introducing a new set of oscillators which annihilate $|S_−\rangle$
\[ t = \frac{1}{\sqrt{1-T^2}}(b - C\tilde{T}b^\dagger), \quad u = \frac{1}{\sqrt{1-T^2}}(c + C\tilde{T}c^\dagger), \quad (4.1) \]
and making use of the formulas $[23]$
\[ e^{c^\dagger A b^\dagger + c^\dagger M b + b^\dagger M c + c B b} \]
\[ = \det[e^M(1-M)^{-1} A b^\dagger e^{-c^\dagger \log(1-M)b} e^{-b^\dagger \log(1-M)c} e^{c B (1-M)^{-1} b}], \quad (4.2) \]
\[ e^{c^\dagger A (b+\lambda)} = e^{c^\dagger (e^\lambda - 1)\lambda} e^{c^\dagger A b}, \]
the 3-string vertex constructed on $|S_+\rangle = c_0|S_-\rangle$ is found to be
\[ |V_3\rangle = \exp\left(\sum_{r,s=1}^3 u^{(r)\dagger} \hat{V}^{rs} t^{(s)\dagger} + u^{(r)\dagger} \hat{v}^{rs} b_0^{(s)}\right)|S_+\rangle_{123}. \quad (4.3) \]
Here the transformed Neumann coefficients are given by
\[ C\tilde{V} = (1 - C\tilde{V}\tilde{T})^{-1}(C\tilde{V} - \tilde{T}) = \begin{pmatrix} 0 & \tilde{L} & \tilde{R} \\ \tilde{R} & 0 & \tilde{L} \\ \tilde{L} & \tilde{R} & 0 \end{pmatrix}, \quad (4.4) \]
and
\[ \hat{v} = \frac{\sqrt{1-T^2}}{1 - \hat{V} CT}\tilde{v} = \frac{1 + \hat{V} C\tilde{T} \tilde{v}}{\sqrt{1-T^2}}, \quad (4.5) \]
\[ \tilde{L} \quad \text{and} \quad \tilde{R} \quad \text{are orthogonal projections given by} \]
\[ \tilde{L} = \frac{\tilde{Y} + \tilde{T}\tilde{Z}}{(1-\tilde{X})(1+\tilde{T})}, \quad \tilde{R} = \frac{\tilde{Z} + \tilde{T}\tilde{Y}}{(1-\tilde{X})(1+\tilde{T})}. \quad (4.6) \]
The relation $(4.3)$ can be written explicitly as
\[ \hat{v}_0 = \frac{1}{\sqrt{1-T^2}}(\tilde{v}_0 + \tilde{T}\tilde{R}\tilde{v}_0 + \tilde{T}\tilde{L}\tilde{v}_0), \]
\[ \hat{v}_+ = \frac{1}{\sqrt{1-T^2}}(\tilde{v}_+ + \tilde{T}\tilde{R}\tilde{v}_0 + \tilde{T}\tilde{L}\tilde{v}_-) = -\tilde{R}\tilde{v}_0, \quad (4.7) \]
\[ \hat{v}_- = \frac{1}{\sqrt{1-T^2}}(\tilde{v}_- + \tilde{T}\tilde{R}\tilde{v}_+ + \tilde{T}\tilde{L}\tilde{v}_0) = -\tilde{L}\tilde{v}_0. \]
These transformed Neumann coefficients satisfy the same relation as the original one:

\[
\begin{align*}
\sum_{r=1}^{3} \hat{V}^{rs} &= C, \\
\sum_{t=1}^{3} \hat{V}^{rt} \hat{V}^{ts} &= \delta^{rs}, \\
\sum_{r=1}^{3} \hat{v}^{rs} &= 0, \\
\sum_{t=1}^{3} \hat{V}^{rt} \hat{v}^{ts} &= -\hat{v}^{rs}, \\
\hat{v}^{rs} &= (\delta^{rs} - \hat{V}^{rs}) \hat{v}_{0}.
\end{align*}
\] (4.8)

Note that the role of \(f\) in the original picture is played by \(\hat{v}_{0}\) in this transformed picture. We can use the same trick as in section 2.3 to eliminate \(b_{0}\) in the exponent of \(|V_{3}\rangle\) by performing a unitary transformation

\[
U = e^{(u + u^\dagger) \hat{v}_0 b_0}.
\] (4.9)

The final expression of the comma vertex is

\[
|V_{3}\rangle = \exp \left( \sum_{r,s=1}^{3} u^{(r)\dagger} \hat{V}^{rs} \tilde{t}^{(s)\dagger} \right) |\tilde{S}_{+}\rangle_{123}
\]

\[
= \exp \left( \sum_{r=1}^{3} u^{(r)\dagger} C \tilde{L} \tilde{t}^{(r+1)\dagger} + u^{(r)\dagger} C \tilde{R} \tilde{t}^{(r-1)} \right) |\tilde{S}_{+}\rangle_{123},
\] (4.10)

where \(|\tilde{S}_{+}\rangle = U|S_{+}\rangle\) and \(\tilde{t} = UtU^{-1}\). This form of vertex explains the appearance of projections \(\tilde{L}, \tilde{R}\) in the calculation of star product of coherent states constructed over the sliver-like state \([10]\).

The BRST charge \(Q\) determined from this form of vertex is

\[
Q = U c_{0} U^{-1} = c_{0} + (u + u^\dagger) \hat{v}_0 = c_{0} + (c + c^\dagger) \sqrt{\frac{1 + \tilde{T}}{1 - \tilde{T}}} \hat{v}_0.
\] (4.11)

Therefore, \(f\) and \(\hat{v}_0\) should be related by

\[
f = \sqrt{\frac{1 + \tilde{T}}{1 - \tilde{T}}} \hat{v}_0.
\] (4.12)

From (4.7), one can see that this expression of \(f\) agrees with (2.13).
5. Discussions

In this paper, we studied the structure of star algebra in the ghost sector which is summarized by (1.6) and (1.7). However, there are some subtleties of our result as shown in the following examples.

First we discuss the value of classical action in VSFT. If we write the action of VSFT in a form

$$S = \int \frac{1}{2} A \star QA + \frac{1}{3} A \star A \star A,$$  \hspace{1cm} (5.1)

the action for the solution $A_0$ satisfying $QA_0 + A_0 \star A_0 = 0$ becomes

$$S_0 = \int \frac{1}{6} A_0 \star QA_0.$$  \hspace{1cm} (5.2)

It seems naively that this action vanishes identically due to the property (1.7) and cannot reproduce the tension of $D$-branes. This problem is related to the subtlety of the identity string field in the ghost sector. Since there is no identity of star product which defines the integral $\int$, eq.(5.2) should be understood as

$$S_0 = \frac{1}{6} \langle A_0 | QA_0 \rangle.$$  \hspace{1cm} (5.3)

Note that the BPZ inner product between $A_{b_0}$ and $A_Q$ does not vanish identically.

The second example of subtleties is the gauge invariance in VSFT. Using (3.2), we find that the gauge transformation of string field is BRST exact

$$\delta A = Q\epsilon + [A, \epsilon]_\star = Q(\epsilon + [A, \epsilon]_{\star b_0}).$$  \hspace{1cm} (5.4)

If we naively use the relation between $\star$ and $\star_{b_0}$, the BRST charge around a solution $A_0$ of VSFT also becomes $Q$-exact

$$Q_B\Psi = QA_0 \star \Psi - (-1)^\Psi \Psi \star A_0$$
$$= Q(\Psi + A_0 \star_{b_0} \Psi - (-1)^\Psi \Psi \star_{b_0} A_0).$$ \hspace{1cm} (5.5)

However, the true BRST charge around the $D25$-brane background, i.e., the one constructed from the matter and ghost energy momentum tensors, does not satisfy $Q_B\Psi \in A_Q$ for a generic $\Psi$.

These examples tell us that we should carefully treat the anomalous behavior of star algebra associated with the infinite dimensional nature of Neumann coefficients such as associativity anomaly $[22]$, twist anomaly $[13]$ and the singularity of sliver $[14]$, which can be easily missed in the naive computation.

Acknowledgments

I would like to thank I. Kishimoto for valuable discussions. I am also grateful to Prof. Y. Nambu for his continuous encouragement.
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