Integrability conditions at order 2 for homogeneous potentials of degree $-1$

Thierry Combot
IMCCE, 77 Avenue Denfert Rochereau 75014 Paris, France
E-mail: combot@imcce.fr

Received 29 March 2012, in final form 11 October 2012
Published 16 November 2012
Online at stacks.iop.org/Non/26/95

Abstract
We prove a meromorphic integrability condition at order 2 near a homothetic orbit for a meromorphic homogeneous potential of degree $-1$, which extends the Morales–Ramis conditions of order 1. Conversely, we prove that if this criterion is satisfied, then the Galois group of the second order variational equations is abelian and we compute explicitly the Galois group and Picard–Vessiot extension. Finally, we present an application corollary and some examples.

Mathematics Subject Classification: 37J30

((Some figures may appear in colour only in the online journal)

1. Introduction
In this paper we study dynamical systems of the form $\ddot{q} = \nabla V(q)$ where $V$ is a meromorphic homogeneous function of degree $-1$ in $q_1, \ldots, q_n$ and $n \in \mathbb{N}^*$. This system corresponds to a Hamiltonian system

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} - V(q) \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = p_i \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{\partial V}{\partial q_i} \quad (1)$$

This type of Hamiltonian system, corresponding to meromorphic homogeneous potentials, has already been studied significantly from the meromorphic integrability point of view, thanks to the Morales–Ramis–Simó theorems [14, 17]. Using these, Morales–Ramis found an explicit and simple integrability criterion given by theorem 2 (criterion of order 1). It is only necessary to find the solutions of an algebraic equation (the Darboux points) and then compute the eigenvalues of the Hessian matrices of the potential at these solutions. In the case of celestial mechanics (which are our primary concern due to the homogeneity degree $-1$), this algebraic equation corresponds to the equation of central configurations. As it is difficult to prove only the
finiteness of solutions to this equation \([1, 11, 20]\), to compute all of the solutions algebraically is often computationally too costly. Still, it is often possible to find one solution thanks to symmetry or a good parametrization of the potential. When we can only use, say, one central configuration, it can be interesting to find additional integrability constraints and to put them in a simple form so they can be easily checked.

In this article, our main interest will be the study of variational equations of order 2 and their Galois group to produce the additional integrability criterion of theorem 3 (criterion of order 2), and then the computation of the Galois group when the criterion is satisfied in theorem 4. We find in particular that it is only necessary to check that some third order derivatives of the potential vanish. The method we will use (holonomic computations \([8–10]\)) is not restricted to this single proof of theorem 3, but can also be used to study the case of a non-diagonalizable Hessian matrix in corollary 1 (producing a different proof from Duval and Maciejewski \([6]\)), and also other homogeneity degrees and variational equations of higher order (the main limitation would be computational power). Finally we prove corollary 2, which gives an idea of the usefulness of second order variational equations, and present some examples.

**Definition 1.** A point \(c \in \mathbb{C}^n\) is called a Darboux point of the dynamical system \(\ddot{q} = \nabla V(q)\) when it satisfies the equation

\[
\nabla V(c) = \alpha c
\]

The scalar \(\alpha \in \mathbb{C}\) is called the multiplier associated to \(c\). As \(\nabla V\) is homogeneous of degree \(-2\), we can always choose \(\alpha = 0, -1\); we will say that \(c\) is non-degenerate if \(\alpha \neq 0\). A homothetic orbit \(\Gamma\) associated to a Darboux point \(c\) with the multiplier \(\alpha\) is given by

\[
q(t) = \phi(t).c, \quad p(t) = \dot{\phi}(t).c, \quad \frac{1}{2} \dot{\phi}(t)^2 = -\frac{\alpha}{\phi(t)} + E
\]

with \(E \in \mathbb{C}^*\).

A homothetic orbit is an explicit particular solution of the dynamical system \((1)\), and it will be used for non-integrability proofs (as it is used in \([12, 13, 17]\)). In the following, we will only study non-degenerate Darboux points, and we always normalize the multiplier to \(-1\). For the associated homothetic orbit, we will always choose \(E = 1\), and thus

\[
\frac{1}{2} \dot{\phi}^2 = \frac{1}{\phi} + 1.
\]

Note that in the sequel, we will identify the dynamical system \((1)\) and the potential. For example, we will say that \(c\) is a Darboux point of \(V\). We will denote the ‘norm’ and scalar product by the expressions (see Craven \([5]\))

\[
|v|^2 = \sum_{i=1}^{n} v_i^2, \quad \langle v, w \rangle = \sum_{i=1}^{n} v_i w_i
\]
even for complex \(v, w\). We will say moreover that a matrix is orthonormal complex if its columns \(X_1, \ldots, X_n\) are such that

\[
\langle X_i, X_j \rangle = \sum_{k=1}^{n} (X_i)_k (X_j)_k = 0 \quad \forall i, j \quad |X_i|^2 = \sum_{k=1}^{n} (X_i)_k^2 = 1 \quad \forall i
\]
The following standard theorems provide a starting point for our method:

**Theorem 1 (Morales–Ramis–Simó [14] theorem 2).** Let us consider a symplectic analytical complex manifold \( M \) of dimension \( 2n \), with the Poisson bracket defined by the symplectic form, \( H \) a Hamiltonian analytic on \( M \) and \( \Gamma \subset M \) a particular (not a point) orbit. If \( H \) possesses a complete system of first integrals in involution, functionally independent and meromorphic on a neighbourhood of \( \Gamma \), then the identity component of the Galois group of variational equations is abelian at any order.

**Theorem 2 (Morales–Ramis [17] theorem 3).** Let \( V \) be a meromorphic homogeneous potential of degree \(-1\) and \( c \) a non-degenerate Darboux point. If we fix the multiplier of the Darboux point \( c \) to \(-1\), then the identity component of the Galois group of the first order variational equation of \( V \) near the homothetic orbit associated to \( c \) is abelian if and only if

\[
\text{Sp}(\nabla^2 V(c)) \subset \left\{ \frac{1}{2} (k - 1)(k + 2), \ k \in \mathbb{N} \right\}.
\]

The first and second order variational equations near a homothetic orbit are presented in sections 2.1 and 2.2. Note that theorem 1 is about an analytic Hamiltonian on a symplectic manifold but here we will only consider meromorphic homogeneous potentials of degree \(-1\), which correspond in particular to analytic Hamiltonians on a neighbourhood of the homothetic orbit \( \Gamma \). Theorem 3 in [17] is about any homogeneity degree but here in theorem 2 we only picked up the case of degree \(-1\). The first order condition is already known, computed by Yoshida [23] based on the classification of hypergeometric functions by Kimura [7], and also found in a more general way by Morales–Ramis in [15, 19]. It has been used many times in [12, 13, 16], particularly in the \( n \) body problem in the case of homogeneity degree \(-1\) in [18]. The Morales–Ramis–Simó theorem holds for variational equations at any order, and so here we want to study completely the second order, and give an integrability characterization at order 2. The main theorems of this article are the following

**Theorem 3.** Let \( V \) be a meromorphic homogeneous potential of degree \(-1\). Let \( c \) be a non-degenerate Darboux point of \( V \) with multiplier \(-1\). Assume that the Hessian \( \nabla^2 V(c) \) of \( V \) in \( c \) is diagonalizable. Let \( \lambda_1 \ldots \lambda_n \) denote its eigenvalues and \( X_1, \ldots, X_n \) the corresponding eigenvectors. Assume that the integrability conditions of Morales–Ramis 2 are satisfied, i.e.

\[
\lambda_i = \frac{1}{2} (n_i - 1)(n_i + 2), \quad n_i \in \mathbb{N}, \ i = 1 \ldots n
\]

We can build a set of indices \( J \subset \mathbb{N}^3 \), depending only on the \( n_i \), such that if \( V \) is meromorphically integrable, then

\[
\forall (i, j, k) \in J, \ D^3 V(c)(X_i, X_j, X_k) = 0
\]

The set of indices \( J \) is built in the following way: \((i, j, k) \in J \iff A_{n_i,n_j,n_k} = 0 \), where \( A \) is a 3 index table with values in \([0, 1]\), invariant by permutation, and given by

- For \( i, j, k \in \mathbb{N}^* \), \( A_{i,j,k} = 1 \) if and only if one of the following conditions are satisfied

\[
\begin{align*}
&i + j - k \geq 2 \\
&i - j + k \geq 2 \\
&-i + j + k \geq 2 \\
&i + j + k \mod 2 = 0 \\
&i - j + k \leq -3 \\
&i + j + k \mod 2 = 1 \\
or
\end{align*}
\]

or

\[
\begin{align*}
&-i + j + k \leq -3 \\
&i + j + k \mod 2 = 1
\end{align*}
\]
For $i = 0$, $j, k \in \mathbb{N}^*$, $A_{0,j,k} = 1$ if and only if $|j - k| \geq 2$

Theorem 4. Let $V$ be a meromorphic homogeneous potential of degree $-1$, $c$ a Darboux point of $V$ with multiplier $-1$. Assume that $\nabla^2 V(c)$, the Hessian of $V$ in $c$, is diagonalizable. If $V$ is meromorphically integrable, then the Galois group of the second order variational equation near the homothetic orbit associated to $c$ with $E = 1$ is always isomorphic to $(\mathbb{C}, +)$ (the additive group) and the Picard–Vessiot extension field is

$$
\mathbb{C}(\dot{\phi}, \ln(\dot{\phi} + \sqrt{2}) - \ln(\dot{\phi} - \sqrt{2})),
$$

except if one (or both) of the two following conditions are satisfied

- $D^3(V)(c)(v, v, v) \neq 0$ with $\nabla^2 V(c)v = -v, \ v \neq 0$
- $D^3(V)(c)(v, v, w) \neq 0$ with $\nabla^2 V(c)v = -v, \ \nabla^2 V(c)w = 0, \ v, w \neq 0$

and in this case the Galois group is $(\mathbb{C}^2, +)$ and the Picard–Vessiot field is

$$
\mathbb{C}(\dot{\phi}, \ln(\dot{\phi} + \sqrt{2}), \ln(\dot{\phi} - \sqrt{2})).
$$

The integrability constraint, when the system of second order variational equations is well written, can be found by computing a particular monodromy commutator; this is the approach of Maciejewski–Przybylska in [12]. Here the main difficulty is that this monodromy commutator depends on parameters, the eigenvalues of the Hessian matrix $\nabla^2 V(c)$. Indeed, the first order integrability condition from theorem 2 gives restrictions on such eigenvalues if the potential is meromorphically integrable, but still an infinite number of values remain possible. To compute this monodromy commutator, it is sufficient to study a particular 3 index sequence, and then to find its zero and non-zero entries. Of course it can be easily checked one by one (as done in section 3.1), but this is not enough. By chance, this 3 index sequence admits an explicit expression (12), but is not easy to find (and to prove). The property behind it is that the monodromy commutator is holonomic with respect to the eigenvalue parameters, and so it satisfies a 3 index linear recurrence with polynomial coefficients (12). These recurrences are found and proved using the creative telescoping approach with the holonomic package of Mathematica [10] in section 3.2. A closed form solution can then be guessed by the gfun Maple package [21], and then its validity checked. The non-nullity can thus be easily studied because this closed form expression is a hypergeometric sequence.

2. Variational equations

2.1. First order variational equation

Let us first introduce the first order variational equation (see also [14, 17]). Near a homothetic orbit associated to a non-degenerate Darboux point $c$ with multiplier $-1$, the first order variational equation is

$$\ddot{X} = \frac{1}{\phi(t)^3} \nabla^2 V(c)X.$$

Note that using the Euler relation on the homogeneous function $V$, we obtain the relation $\nabla^2 V(c)c = 2c$, and thus the eigenvalue 2 always belongs to the spectrum of $\nabla^2 V(c)$. Let us now assume that $\nabla^2 V(c)$ is diagonalizable.

**Proposition 1 (Proved in Craven [5] theorems 1, 3).** Let $A \in M_n(\mathbb{C})$ be a complex symmetric matrix, meaning $A_{i,j} = A_{j,i}$. Assume that $A$ is diagonalizable. Then $A$ is diagonalizable in an orthonormal complex basis.
Thanks to this proposition, we can always make a symplectic variable change
\[ p \mapsto Pp \quad q \mapsto Pq \]
in the Hamiltonian \( H \) with \( P \) orthonormal complex such that the Hessian matrix of \( V \) is diagonal. As \( P \) is orthonormal complex, the Hamiltonian \( H(Pp, Pq) \) is still of the form (1). So in the following, as long as the Hessian matrix \( \nabla^2 V(c) \) is diagonalizable, we can after a linear variable change assume that \( \nabla^2 V(c) \) is diagonal, denoting it by \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \).

This small result is actually very important for the reduction of problems dealing with homogeneous potentials (additional similar properties can be found in Craven [5]). This is because the integrability status of a potential \( V \) is not changed after a rotation (even a complex one) nor dilatation. Such property allows a great simplification in classification in Maciejewski–Przybylska [12, 13]. The simplification at order 2 is even more important, because a simple criterion as in theorem 3 is not possible if the Hessian matrix is not diagonal. As we will see, the only hypothesis of diagonalizability is very weak, especially because the system is seldom integrable at order 1 if the Hessian matrix is not diagonalizable (the conditions on the spectrum are still necessary, but not sufficient).

In this setting, the first order variational equation \((VE_1)\) is now \( \ddot{X} = \frac{1}{\phi(t)^3} DX \).

The base field on which the differential Galois group \( G_1 \) of this equation should be computed in theorem 1 is the field \( \mathbb{C}(\phi, \dot{\phi}) = \mathbb{C}(\dot{\phi}) \) (because of the relation \( \dot{\phi}^2 / 2 = 1 / \phi + 1 \)). From now on we denote by \( K_1 \) the Picard–Vessiot field for \((VE_1)\). The criterion of theorem 2 is that the Galois group of \((VE_1)\) has a Galois group whose identity component is abelian (also called virtually abelian), then \( \lambda_i = \frac{1}{2}(n_i - 1)(n_i + 2), \quad n_i \in \mathbb{N}, \quad i = 1 \ldots n \).

In the following, we will often use the variable change \( \phi / \sqrt{2} \rightarrow t \) which transforms (one equation of) \((VE_1)\) to \( (t^2 - 1) \ddot{y} + 4t \dot{y} - (n_i - 1)(n_i + 2)y = 0 \).

Let us now study more closely the solutions of this equation. Replacing the parameter \( n_i \) by \( i \) in this last equation, a basis of solutions is given by \((P_i, Q_i)\) where \( P_i \) are polynomials and the functions \( Q_i \) can be written
\[ Q_i(t) = \frac{1}{\sqrt{2}} P_i(t) \int \frac{1}{(t^2 - 1)^2 P_i(t)^2} dt \]
choosing 0 for the start point of the integral, just to fix the definition. The polynomials \( P_i \) can be generated by the formula
\[ P_i(t) = \frac{1}{t^2 - 1} \left( t^{i-1} \frac{\partial^{i-1}}{\partial t^{i-1}} (t^2 - 1)^i \right) \]
which gives a normalization for the dominant coefficient of \( P_i \) that we will choose for now. The functions \( Q_i \) can be written
\[ Q_i(t) = \epsilon_i P_i(t) \arctanh \left( \frac{t}{i} \right) + \frac{W_i(t)}{t^2 - 1} \]
with \( W_i \) polynomials, and \( \epsilon_i \) is a real sequence, computed below. It follows that the differential Galois group \( G_1 \) of \((VE_1)\) is the additive group \( (\mathbb{C}, +) \) (except for \( i = 0 \).

**Lemma 5.** We have \( \epsilon_i = 4^{-i} i (i + 1) / i^2 \) for all \( i \in \mathbb{N}^* \).
Proof. The sequence $\epsilon_i$ can be computed thanks to the formula

$$\epsilon_i = \frac{1}{\int_C (t^2 - 1)^2 P_i(t)^2 \, dt}$$

with $C$ a circle around $-1, 1$ in the direct way (because $\epsilon_i$ is the term in front of $\text{arctanh}(1/t)$ which grows by 1 along $C$). Using the symmetry $t \mapsto -t$, we just need to compute the residue in $1$ for example. We have then

$$\epsilon_i = 2 \left. \frac{\partial}{\partial t} \left( \frac{1}{(t + 1)^2 P_i(t)^2} \right) \right|_{t=1}$$

knowing that 1 is never a root of $P_i$. Using a recurrence formula on the $P_i$(4)$^3 + 12 i^2 + 8 i) P_i + (-4 t i^2 - 14 ti - 12 t)P_{i+1} + (i + 3)P_{i+2} = 0$

we obtain

$$P_i(1) = 2^i (i + 1)! \left. \frac{\partial}{\partial t} P_i(t) \right|_{t=1} = 2^{i-2} i(i+3)(i+1)! \quad \epsilon_i = \frac{4^{-i} i(i+1)}{i!}$$

which gives us the lemma. \hfill $\Box$

So the functions $Q$ are multivalued except for $i = 0$, which is particular, because the Galois group is $\{Id\}$ instead of $(\mathbb{C}, +)$ and then all solutions are rational. Remark that the functions $P_i, Q_i$ have many interesting properties. They are linked to Legendre functions (after a variable change) and the polynomials $P_i$ form a family of orthogonal polynomials (related to Jacobi polynomials) for the weight $(t^2 - 1)$. The most important point to make about these functions is that they are holonomic. Indeed, in the following we will consider the system

$$\begin{cases}
(4n^3 + 12 i n^2 + 8 n) f_n(t) - (4 n i^2 + 14 i n + 12 t) f_{n+1}(t) \\
+ (n + 3) f_{n+2}(t), \quad (t^2 - 1) f''_n(t) + 4 t f'_n(t) - (n - 1) (n + 2) f_n(t)
\end{cases} \quad (2)$$

which vanishes for $f_i(t) = P_i$ and $f_j(t) = \epsilon_i^{-1} Q_i$.

2.2. Second order variational equation

The second order variational equation, when the Hessian matrix is diagonal, can be written as

$$\ddot{X} = \frac{1}{\phi(t)^3} DX + \frac{1}{2 \phi(t)^2} \left( \begin{array}{c} Y(t)^T T_1 Y(t) \\ \vdots \\ Y(t)^T T_n Y(t) \end{array} \right) \quad \ddot{Y} = \frac{1}{\phi(t)^3} DY \quad (3)$$

with $D$ diagonal, $T_i \in M_n(\mathbb{C})$, $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$. The matrix $D$ is the Hessian matrix of $V$ on the Darboux point $c$, and the matrices $T_i$, $i = 1 \cdots n$ are defined by $T_{i,j,k} = D^3(V)(c) \langle e_i, e_j, e_k \rangle$ where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$. A more detailed construction of the second variational equation and higher orders can be found in [2] and a formula is given in [14] p 860. This system is nonlinear; however, as explained in [2, 14], a linear variational equation can be constructed. We first produce a linear differential system whose solutions are linear combinations of $y_{i,j} = y_i y_j, i, j = 1 \ldots n$, called the 2nd symmetric power of equation $\ddot{Y} = \phi(t)^{-3}DY$. Equation (3) is then a linear differential system whose
Integrability conditions at order 2 for homogeneous potentials of degree $-1$

unknowns are $x_i, y_{i,j}, i, j = 1 \ldots n$. The matrix of this linear differential system is

$$
\begin{pmatrix}
\Delta_1 & \ldots & 0 & B_{1,1,1} & \ldots & B_{1,n,n} & B_{1,1,2} & \ldots & B_{1,n-1,n} \\
0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \Delta_n & B_{n,1,1} & \ldots & B_{n,n,n} & B_{n,1,2} & \ldots & B_{n,n-1,n} \\
0 & \ldots & 0 & \text{Sym}^2(\Delta_1) & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & \text{Sym}^2(\Delta_n) & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & \Delta_1 \otimes \Delta_2 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & \Delta_n-1 \otimes \Delta_n
\end{pmatrix}
$$

with

$$
\Delta_i = \begin{pmatrix} 0 & 1 \\ \lambda_i & \phi^5 \end{pmatrix}
$$

$$
B_{i,j,j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T_{i,j,j} & 0 & 0 \end{pmatrix}
B_{i,j,k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{T_{i,j,k}}{2\phi^4} & 0 & 0 \end{pmatrix}
$$

Lemma 6. Let $A_1, A_2, A_3$ be three block triangular matrices of the form

$$
A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_2 \end{pmatrix} \in \mathbb{C}(\phi) \quad A_2 = \begin{pmatrix} \alpha_1 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} \in \mathbb{C}(\phi) \quad A_3 = \begin{pmatrix} \alpha_1 & \beta_1 + \beta_2 \\ 0 & \alpha_2 \end{pmatrix}
$$

Assume that $\text{Gal}_{\text{dif}}(X = A_1 X)$ and $\text{Gal}_{\text{dif}}(X = A_2 X)$ are virtually abelian. Then $\text{Gal}_{\text{dif}}(X = A_3 X)$ is virtually abelian, and the Picard–Vessiot field is contained in the compositum of the Picard–Vessiot field of the two previous equations.

Proof. Let us consider $(w_1(t), z_1(t)), (w_2(t), z_2(t))$ the general respective solution (so depending on several constants) of the differentials systems $X = A_1 X, \dot{X} = A_2 X$. As $\dot{z}_1 = \alpha_2 z_1$ (with the same $\alpha_2$), we can choose $z_1 = z_2$. By direct computation, we now find that $(w_1(t) + w_2(t), z_1(t))$ is the general solution of the differential system $X = A_3 X$. Thus the Picard–Vessiot field $K^{(3)}$ of $X = A_3 X$ is contained in the compositum of Picard–Vessiot fields $K^{(1)}, K^{(2)}$ of $X = A_1 X$ and $X = A_2 X$. As the Galois group action is normal on $K^{(1)}$, an automorphism $\sigma$ in $\text{Gal}_{\text{dif}}(X = A_3 X)$ acts on $K^{(3)}$ as its restriction on $K^{(1)}, K^{(2)}$. So the Galois group $\text{Gal}_{\text{dif}}(X = A_3 X)$ is virtually abelian. \hfill \Box

Lemma 7. Assume $(\triangledown E_1)$ has a virtually abelian Galois group. The Galois group of the second order variational equation (3) is virtually abelian if and only if for all $i, j, k = 1 \ldots n$ the systems (where the $T_{i,j,k}$ are the entries of the matrices $T_i$ of (3))

$$
\dot{x}_i = \frac{D_{i,i}}{\phi(t)^3} x_i + T_{i,j,k} y_j(t) y_k(t) \quad \dot{y} = \frac{1}{\phi(t)^3} D Y
$$

(5)
have a virtually abelian Galois group. The Picard–Vessiot field of the second order variational equation (3) is the compositum of the Picard–Vessiot fields of all the systems (5). Moreover, the Galois group and Picard–Vessiot extension depend only on the nullity or non-nullity of \( T_{i,j,k} \).

**Proof.** Assume that the Galois group of the second order variational equation (3) is virtually abelian. Let \( j \) be an integer in \( 1 \ldots n \). We now consider the subspace of solutions of equation (3) such that \( y_i^2 = 0 \), \( \forall i \neq j \). This corresponds to a subsystem of \((LV E_2)\), whose matrix is

\[
\begin{pmatrix}
\Delta_1 & \ldots & 0 & B_{1,j,j} \\
0 & \ldots & 0 & \ldots \\
0 & \ldots & \Delta_n & B_{n,j,j} \\
0 & \ldots & 0 & \text{Sym}^2(\Delta_j)
\end{pmatrix}
\]  

(6)

As it is a subsystem of \((LV E_2)\), the Picard–Vessiot field is a subfield of \( K_2 \), and the Galois group is virtually abelian. Picking the \( i \)th line of (6), the equation is of the form

\[
\begin{pmatrix}
\dot{x}_i \\
\ddot{x}_i
\end{pmatrix} = \Delta_i \begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} + B_{i,j,j} \begin{pmatrix} y_{j,j} \\ \dot{y}_{j,j} \end{pmatrix}.
\]

Making the variable change \( x_i \mapsto x_i/\alpha_i \) multiplies the \( T_{i,j,j} \) by \( \alpha_i \) (and thus the matrix \( B_{i,j,j} \)), and this variable change does not change the Galois group. So the Galois group of the differential system whose matrix is (6) depends only on the nullity or non-nullity of \( T_{i,j,j} \). Thus the Galois groups of the linear differential systems with matrices

\[
\begin{pmatrix}
\Delta_1 & \ldots & 0 & 0 & B_{1,j,j} & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \Delta_n & 0 & B_{n,j,j} & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & \text{Sym}^2(\Delta_1) & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & \text{Sym}^2(\Delta_n) & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & \Delta_1 \otimes \Delta_2 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & \Delta_{n-1} \otimes \Delta_n
\end{pmatrix}
\]  

(7)

are virtually abelian, and their Picard–Vessiot fields are subfields of \( K_2 \).

We now perform the same procedure for the terms in \( y_j y_k \) of (3). Let \( j, k \) be two distinct integers in \( 1 \ldots n \). We consider the solutions of equation (4) such that \( y_{i_1,i_2} = 0 \), \( \forall \{i_1, i_2\} \neq \{j, k\} \). This corresponds to a subsystem of the associated linear differential equation \((LV E_2)\), whose matrix is

\[
\begin{pmatrix}
\Delta_1 & \ldots & 0 & B_{1,j,k} \\
0 & \ldots & 0 & \ldots \\
0 & \ldots & \Delta_n & B_{n,j,k} \\
0 & \ldots & 0 & \Delta_j \otimes \Delta_k
\end{pmatrix}
\]  

(8)

As it is a subsystem of \((LV E_2)\), the Picard–Vessiot field is a subfield of \( K_2 \), and the Galois group is virtually abelian. Picking the \( i \)th line of (8), the equation is of the form

\[
\begin{pmatrix}
\dot{x}_i \\
\ddot{x}_i
\end{pmatrix} = \Delta_i \begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} + B_{i,j,k} \begin{pmatrix} y_{j,k} \\ \dot{y}_{j,k} \\ \ddot{y}_{j,k} \end{pmatrix}.
\]
Making the variable change $x_i \mapsto x_i/\alpha_i$ multiplies the $T_{i,j,k}$ by $\alpha_i$ (and thus the matrix $B_{i,j,k}$), and this variable change does not change the Galois group. So the Galois group of the differential system whose matrix is (8) depends only on the nullity or non-nullity of $T_{i,j,k}$.

Thus the Galois groups of the linear differential systems with matrices

$$
\begin{pmatrix}
\Delta_1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & B_{1,j,k} & 0 \\
0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\
0 & \ldots & \Delta_n & 0 & 0 & \ldots & 0 & \ldots & B_{n,j,k} & 0 \\
0 & \ldots & 0 & \text{Sym}^2(\Delta_1) & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & \text{Sym}^2(\Delta_n) & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & \Delta_1 \otimes \Delta_2 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & \Delta_{n-1} \otimes \Delta_n & 0 \\
\end{pmatrix}
$$

(9)

are virtually abelian, and their Picard–Vessiot fields are subfields of $K_2$. The matrix of the linear differential system associated to the nonlinear systems (5) is either a submatrix of (7) if $j = k$, or a submatrix of (9) if $j \neq k$, and thus their Galois groups are virtually abelian and their Picard–Vessiot fields are subfields of $K_2$.

Conversely, if the Galois group of the systems (5) are virtually abelian, then the differential equations whose matrices are (7) and (9) have virtual abelian Galois group. Thus, thanks to lemma 6, the Galois group of (3) is virtually abelian, and its Picard–Vessiot field is contained in the compositum of Picard–Vessiot fields of (7) and (9).

The Picard–Vessiot field of equation (3) is contained in the compositum of Picard–Vessiot fields of (5) and contains the Picard–Vessiot fields of all equations (5). Thus the Picard–Vessiot field of equation (3) is exactly the compositum of the Picard–Vessiot fields of (5).

The following lemma will have primary importance in the computation of monodromy. In fact, it will only be necessary to compute some sort of residue.

**Lemma 8.** We consider $F \in \mathbb{C}(z_1)[z_2]$ and

$$
f(t) = F\left(t, \text{arctanh}\left(\frac{1}{t}\right)\right)
$$

We consider the differential field and the Galois group

$$
K = \mathbb{C}\left(t, \text{arctanh}\left(\frac{1}{t}\right), \int f\, dt\right), \quad G = \text{Gal}_{\text{dif}}(K/\mathbb{C}(t))
$$

If $G$ is abelian, then

$$
\frac{\partial}{\partial \alpha} \lim_{t \to \infty} F\left(t, \text{arctanh}\left(\frac{1}{t}\right) + \alpha\right) = 0 \quad \forall \alpha \in \mathbb{C}
$$

(10)

**Proof.** First of all we recall that if the Galois group $G$ is abelian, then so is the monodromy group, because the monodromy group is always included inside the Galois group. At infinity, \text{arctanh}(1/t) is smooth, as

$$
\text{arctanh}\left(\frac{1}{t}\right) = \frac{1}{t} + \frac{1}{3t} + O(t^{-2})
$$

and thus $F(t, \text{arctanh}(1/t) + \alpha)$ has the following series expansion

$$
\int F\left(t, \text{arctanh}\left(\frac{1}{t}\right) + \alpha\right) \, dt = \sum_{n=0}^{\infty} a_n(\alpha) t^n + r(\alpha) \ln t
$$
Figure 1. Paths corresponding to monodromy elements \( \sigma_1, \sigma_2 \), and the Riemann surface associated to \( \text{arctanh}(1/t) \). The difference between two sheaves is \( 2i\pi \) and we see that \( \sigma_2 \), corresponding to monodromy around infinity, acts trivially on \( \text{arctanh}(1/t) \).

because the function \( \text{arctanh}(1/t) \) is smooth at infinity. We consider two paths, the eight path \( \sigma_1 \) around the singularities \(-1,1\) and the path \( \sigma_2 \) around both (figure 1). The monodromy element \( \sigma_1 \) fixes \( \text{arctanh} \), and

\[
\sigma_2 \left( \text{arctanh} \left( \frac{1}{t} \right) \right) = \text{arctanh} \left( \frac{1}{t} \right) + 2i\pi.
\]

We consider the commutator

\[
\sigma = \sigma_2^{-1} \sigma_1^{-1} = -\sigma_2 \sigma_1 = 2i\pi \quad \alpha \in 2i\pi \mathbb{Z}.
\]

If the monodromy group is abelian, then its derivative is \( \{\text{Id}\} \), and thus any commutator of elements of the monodromy group should act trivially on elements of \( K \). In particular, we should have \( \sigma(f) = f \). We have \( \sigma_1^{-1} \left( \frac{1}{t} \right) = F(t, \text{arctanh}(1/t) + \alpha) \) and \( \sigma_2 \left( \ln t \right) = \ln t + 2i\pi \).

We conclude that

\[
\sigma(f) = f + r(\alpha) - r(0).
\]

This \( r(\alpha) \) corresponds to the residue of \( F(t, \text{arctanh}(1/t) + \alpha) \) at infinity. The commutator \( \sigma \) should act trivially on \( f \). This is the case if and only if \( r(\alpha) - r(0) = 0 \), \( \forall \alpha \in 2i\pi \mathbb{Z} \). The function \( r \) is polynomial in \( \alpha \), then \( r(\alpha) - r(0) = 0 \) for all \( \alpha \) and so \( r(\alpha) \) is constant. This gives the formula (10).

2.4. Computation of terms of order 2

Let us now check that after a variable change, the \( D^3(V).(X_i, X_j, X_k) \) in the integrability condition of theorem 3 are equal to the \( T_{i,j,k} \) of equation (3) and lemma 7.

Proposition 2. Let \( V \) be a meromorphic homogeneous potential of degree \(-1\). Let \( c \in \mathbb{C}^n \) be a Darboux point with multiplier \(-1\). Assume that \( \nabla^2 V(c) \) is diagonalizable. We denote its eigenvectors \( X_1, \ldots, X_n \). Then the integrability constraints of theorem 3 do not depend on the choice of \( X_1, \ldots, X_n \). Moreover, if we make an orthonormal choice for the \( X_1, \ldots, X_n \), then coefficients \( T_{i,j,k} \) in equation (3) are equal to \( D^3(V). (X_i, X_j, X_k) \).
Proof. First of all, if we make an orthonormal choice (which is always possible thanks to proposition 1), we denote by \( P \) the associated orthonormal matrix. Then the potential \( W(q) = V(Pq) \) has a Darboux point in \( P^{-1} c \) and the corresponding Hessian matrix is diagonal.

Moreover, we have
\[
D^3(W)(e_i, e_j, e_k) = D^3(V)(Pe_i, Pe_j, Pe_k) = D^3(V)(X_i, X_j, X_k).
\]

Now, we verify that the criterion is well defined. We first consider the case where all eigenvalues of \( \nabla^2 V(c) \) are distinct. Then, up to multiplication by a non-zero constant, there is a unique choice of eigenvectors \( X_1, \ldots, X_n \). So this does not change the nullity or non-nullity of \( D^3(V)(X_i, X_j, X_k) \). Now, if there are multiple eigenvalues, there are an infinite number of choices for eigenvectors \( X \). We fix one. Assume that \( X_1, X_2 \) have the same eigenvalue. Since the nullity condition is associated only to the corresponding eigenvalues, if there is a nullity condition for some third order derivative involving \( X_1 \), it will be the same for \( X_2 \). Assume there is a condition
\[
D^3(V)(X_1, X_j, X_k) = 0 \quad D^3(V)(X_2, X_j, X_k) = 0.
\]

Then, for the vector \( \alpha X_1 + \beta X_2 \), we will also have \( D^3(V)(\alpha X_1 + \beta X_2, X_j, X_k) = 0 \) by multilinearity. Remark that even if \( j = 1, k = 1 \), it will still work. All basis changes can be written as such successive linear combinations. Thus the constraints of theorem 3 do not depend on the choice of \( X_1, \ldots, X_n \). □

3. Nonintegrability of second order variational equations

3.1. A first approach

Theorem 9. The second order variational equation (3) has a virtually abelian Galois group if and only if
\begin{itemize}
  \item \( D_{ij} = \frac{1}{2}(n_i - 1)(n_j + 2) \) with \( n_i \in \mathbb{N} \) (integrability condition of order 1)
  \item \( \forall i, j, k = 1 \cdots n, A_{n_i,n_j,n_k} = 0 \Rightarrow T_{i,j,k} = 0 \) where \( A \) is a three index table with values in \( \{0, 1\} \), invariant by permutation and whose first values are given by table 1.
\end{itemize}

One direct application of this theorem is the study of problems in celestial mechanics, for example [18, 22]. It is often unnecessary to know the table \( A \) for arbitrary high eigenvalues, except in some cases such as the open problem at the end of [13], which is, because of this, much more difficult.

Proof. Using lemma 7, the study of the Galois group of the second order variational equation (3) reduces to the study of equation (5). Let us look first at the expressions of functions \( Y_i \). The variable change \( \phi/\sqrt{2} \rightarrow t \) gives the equation
\[
(t^2 - 1) \ddot{y} + 4t \dot{y} - (i - 1)(i + 2)y = 0
\]
for \( Y_i(t) \). A basis for the solutions is given by \( (P_i, Q_i) \) as presented in section 2.1. We obtain for \( y(t) \) the following solution
\[
y(t) = C_1 P_i(t) + C_2 Q_i(t)
\]
\[
+ \int Y_j(t) Y_k(t) P_j(t)(t^2 - 1)^2 \, dt \, Q_i(t) - \int Y_j(t) Y_k(t) Q_j(t)(t^2 - 1)^2 \, dt \, P_i(t).
\]

Thus we need to study the monodromy of the nonhomogeneous part of the solution. Let us try to apply lemma 8. This theorem does not apply directly because there could be compensations
Table 1. Integrability table at order 2 for homogeneous potentials of degree $-1$. Only indices up to 7 are represented, which corresponds to the eigenvalue 27.

| $A_{0,i,j}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|---|---|
| 0           | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1           | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2           | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 2 |
| 3           | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 3 |
| 4           | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 4 |
| 5           | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 5 |
| 6           | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 6 |
| 7           | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 7 |

| $A_{1,i,j}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|---|---|
| 0           | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1           | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 2           | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 3           | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 3 |
| 4           | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 4 |
| 5           | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 5 |
| 6           | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 6 |
| 7           | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 7 |

| $A_{2,i,j}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|---|---|
| 0           | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1           | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 2           | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 |
| 3           | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 3 |
| 4           | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 5           | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 5 |
| 6           | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 6 |
| 7           | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 7 |

| $A_{3,i,j}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|---|---|
| 0           | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1           | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2           | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 3           | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 4           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 5           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 6           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 7           | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 7 |

| $A_{4,i,j}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|---|---|
| 0           | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1           | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2           | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 3           | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 3 |
| 4           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 5           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 6           | 0 | 0 | 1 | 0 | 1 | 0 | 4 | 0 |
| 7           | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 7 |

between the two integrals. But we can rewrite it as

\[
\int Q_j(t)Q_k(t)P_i(t)(t^2 - 1)^2 \, dt P_i(t) \int \frac{1}{(t^2 - 1)^2 P_i(t)^2} \, dt
\]

\[-\int Q_j(t)Q_k(t)P_i(t) \int \frac{1}{(t^2 - 1)^2 P_i(t)^2} \, dt (t^2 - 1)^2 \, dt P_i(t) \]

\[= \int Q_j(t)Q_k(t)P_i(t)(t^2 - 1)^2 \, dt \frac{1}{(t^2 - 1)^2 P_i(t)^2} \, dt P_i(t).\]

Then

\[\int Q_j(t)Q_k(t)P_i(t) \, dt\]

is in the Picard–Vessiot field of equation (5) (because $P_i$ is a polynomial and that the Picard–Vessiot field is stable by derivation). We also know that $Q_i$ is in the Picard–Vessiot field, and then by subtraction,

\[\int Q_j(t)Q_k(t)Q_i(t)(t^2 - 1)^2 \, dt\]

is in the Picard–Vessiot field of equation (5). We can now apply lemma 8 to this integral, and thus we only need to study the residue

\[S = \text{Res}_{t=\infty} (t^2 - 1)^2 (Q_j(t) + \epsilon_j \alpha P_j)(Q_j(t) + \epsilon_j \alpha P_j)(Q_k(t) + \epsilon_k \alpha P_k) \, dt.\]
The integrability conditions at order 2 only require that the identity component of the Galois group should be abelian. This does not say anything \textit{a priori} about the whole Galois group, except if it is connected. This is the case here. The Galois group at order 1 is \((\mathbb{C}, +)\) (or \(\{Id\}\)). Then at order 2, the Picard–Vessiot field will be of the form
\[
K = \mathbb{C}\left(t, \arctanh\left(\frac{1}{t}\right), \int f(t) \, dt\right) \quad \text{with} \quad f(t) \in \mathbb{C}\left(t, \arctanh\left(\frac{1}{t}\right)\right).
\]
We just add some integral in \(K\). Then the Galois group \(\text{Gal}_{\text{diff}}(K, \mathbb{C}(t))\) is still connected.

So, we write in the table \(A_{i,j,k} = 0\) if the constraint of lemma 8 is not satisfied. This criterion is \textit{a priori} a necessary criterion, but not sufficient, and thus we cannot conclude on the values of \(A_{i,j,k}\) if the constraint of lemma 8 is satisfied. So we then try to compute the integral using integration by parts
\[
\int Q_j(t)Q_k(t)P_i(t)(t^2 - 1)^2 \, dt \quad \text{or} \quad 
\int Q_j(t)Q_k(t)Q_l(t)(t^2 - 1)^2 \, dt P_i(t).
\]
Using the expressions of functions \(Q\), we need to integrate functions in \(\mathbb{C}[t, \arctanh(1/t)]\).

We make successive integrations by part, differentiating the term in \(\arctanh(1/t)\) of the highest degree, and finally we obtain an integral of a function in \(\mathbb{C}(t)\). This procedure could fail, but it works every time when the constraint of lemma 8 is satisfied. We can thus put \(A_{i,j,k} = 1\) (when \(A_{i,j,k} = 0\), the procedure fails because terms in \(\ln(t^2 - 1)\) appear). Using lemma 7, we know that it is necessary and sufficient that all equations
\[
\ddot{X} = \frac{1}{\phi(t)^3}d_1X + \frac{1}{2\phi(t)^2}Y_jY_k(t)
\]
have a virtually abelian Galois group for all non-zero \(T_{i,j,k}\) for the virtual abelianity of the Galois group of equation (3).

**Theorem 10.** The table \(A\) of theorem 9 has the following values

- For \(i, j, k \in \mathbb{N}^*\), \(A_{i,j,k} = 1\) if and only if one of the following conditions are satisfied

\[
\begin{align*}
  i + j - k &\geq 2 \\
  i - j + k &\geq 2 \\
  -i + j + k &\geq 2 \\
  i + j + k \mod 2 &= 0 \\
  i - j + k &\leq -3 \\
  i + j + k \mod 2 &= 1 \\
  i + j - k &\leq -3 \\
  i + j + k &\leq 2 \\
  i + j + k \mod 2 &= 1
\end{align*}
\]

- or

\[
\begin{align*}
  -i + j + k &\leq -3 \\
  i + j + k &\leq 2 \\
  i + j + k \mod 2 &= 1
\end{align*}
\]

- For \(i = 0, j, k \in \mathbb{N}^*\), \(A_{0,j,k} = 1\) if and only if \(|j - k| \geq 2\)

- For \(i = j = 0\), \(A_{i,j,k} = 1\).

Moreover, the table \(A\) is invariant by permutation of the indices \(i, j, k\).

This table is the direct generalization of the integrability table of [17] at order 2 for degree \(-1\). A similar process could be done for other homogeneity degrees, but in fact the degree \(-1\) is much simpler for three reasons:

- There is only one family in the Morales–Ramis table for degree \(-1\), and generically, there are two (and the complexity increases with the power three of the number of families).

- Some homogeneity degrees have very particular families, associated to the groups \(A_4, S_4, A_5\). This would produce exceedingly complicated computations. But this does not happen here.

- By studying only one homogeneity degree, we get one less parameter in the holonomic computations. This is important because computational cost usually increases exponentially with the number of parameters (at least).
3.2. General case

Proof of theorem 10. First part. We prove that the Galois group is not virtually abelian if the variational equation contains a term corresponding to indices such that $A_{i,j,k} = 0$. We will begin with the non-zero indices case (this is because the index 0 is very special, in particular the function $Q_0$ is not multivalued). We now need to compute the residues of lemma 8 for all indices and prove they are non-zero for $A_{i,j,k} = 0$. Knowing that $\epsilon_i \neq 0$ and $i \geq 1$, it comes down to the study of the sequence

$$S_{i,j,k} = \lim_{t \to \infty} (\epsilon_i^{-1} Q_i(t) + \alpha P_i)(\epsilon_j^{-1} Q_j(t) + \alpha P_j)(\epsilon_k^{-1} Q_k(t) + \alpha P_k)(t^2 - 1)^2.$$  

Moreover we know that the differential system we use vanishes for $P_n$ and $\epsilon_i^{-1} Q_i$, and thus is also vanishing for $\epsilon_i^{-1} Q_i + \alpha P_i$. Thanks to that, we will be able to find a recurrence on $S_{i,j,k}$ and also, it will not depend on $\alpha$.

Lemma 11. The sequence $S_{i,j,k}$ satisfies the following recurrence relations

$$(1 + i + j - k)(k + 1)(i - j + k)S_{i,j,k} - j(-2 + i + j - k)(3 + i - j + k)S_{i,j-1,k+1} = 0$$

$$4(i - j - k)(1 + i + j - k)(k + 1)(k + 2)(i - j + k)(1 + i + j + k)S_{i,j,k} - (i - j - k - 3)(i + j - k - 2)(3 + i - j + k)(4 + i + j + k)S_{i,j,k+2} = 0$$

and all the recurrences produced permuting $i, j, k$ as well.

This recurrence relation can be proved automatically using the Mgfun package for Maple, the holonomic package for Mathematica [10], or even by hand using integration by parts and a formula between the derivative of $Q_i$ and $Q_j + \alpha Q_j$. Let us remark that these relations conserve the parity of $i + j + k$ which will allow us to treat both cases independently. Now we will actually prove much more than is required for proving theorem 10. We will find closed form solutions for the residue we need to compute.

Lemma 12. For $i, j, k \in \mathbb{N}^*$, we note

$$f(i, j, k) = \frac{2^{i+j+k}i!j!k!(1/2(d+1))\Gamma(\frac{d}{2})\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})\Gamma(\frac{c}{2})\Gamma(\frac{d+4}{2})}{\Gamma(\frac{a+3}{2})\Gamma(\frac{b+3}{2})\Gamma(\frac{c+3}{2})\Gamma(\frac{d+4}{2})}$$

with $a = -i + j + k$, $b = i - j + k$, $c = i + j - k$, $d = i + j + k$. We have then the following formula

$$\frac{\partial}{\partial \alpha} S_{i,j,k} = \begin{cases} \lim_{\epsilon \to 0} \frac{3}{4\pi} f(i + \epsilon, j + \epsilon, k + \epsilon) & \text{if } i + j + k \text{ mod } 2 = 1 \\ \lim_{\epsilon \to 0} \frac{1}{16 \Gamma(\epsilon)} f(i + \epsilon, j + \epsilon, k + \epsilon) & \text{if } i + j + k \text{ mod } 2 = 0. \end{cases}$$

The limit can be easily computed for all $i, j, k \in \mathbb{N}^*$ but there are no closed form expressions for the limit valid for all $i, j, k$. The limit depends in fact on the order of $i, j, k$. We choose such a complicated formula because of its generality. It holds in all cases and thus allows one to speed up the proof, avoiding repeating the same action three times, and also effectively shows the symmetry between the indices. With these formulas, it will be easy to prove theorem 10 because the cases $\partial_{\alpha} S_{i,j,k} = 0$ correspond to singular values of the $\Gamma$ functions in the denominator.
**Proof.**

The case $i + j + k \mod 2 = 1$. We begin by looking at $f$ for $i + j + k \mod 2 = 1$. This is the easy case, because when we replace $\epsilon$ by 0 in $f(i + \epsilon, j + \epsilon, k + \epsilon)$, the expression $f(i, j, k)$, $i, j, k \in \mathbb{N}^*$ is still meaningful if we assume $\Gamma(-n) = \infty$, $n \in \mathbb{N}$. Indeed, there can be at most one term of this kind and always in the denominator. The corresponding value of $S_{i,j,k}$ will be 0. We then check that this expression formally satisfies the recurrences (12). We now need to look at boundary values. First of all, we remark that one term in the relations (12) disappears for some specific $i, j, k$. We want to prove that a solution for $i + j + k \mod 2 = 1$ to the recurrences (12) is uniquely determined by its values on the axis $i = j = k$.

Let us look first at the case $i + j + k = 3 \mod 6$. Using the first relation of (12) (and its permutations), we can express $S_{i,j,k}$ in function of $S_{2n+1,2n+1,2n+1}$ with $i + j + k = 6n + 3$ (such a $n \in \mathbb{N}$ always exists) if

$$3 + i + j - k > 0 \quad 3 + i + j + k > 0 \quad 3 - i + j + k > 0$$

(12)

because in this case both terms appear in this first relation (and all its permutations). Indeed, we proceed step by step going closer to the central axis $i = j = k$ for which the condition (12) is clearly satisfied. If the condition (12) is not satisfied, we can also go closer to the central axis but we meet a singularity when one of the 3 quantities of (12) vanishes and this requires that $S_{i,j,k} = 0$ in this case. So if condition (12) is not satisfied, we get $S_{i,j,k} = 0$, and we can check it is compatible with our formula. Let us look now at the cases $i + j + k = 1, 5 \mod 6$.

We cannot go exactly to the axis because of that, but we can always go closely, for example to the cases $S_{2n+1,2n+1,2n+1}, S_{2n+1,2n+1,2n+3}$ (which always satisfy condition (12)). It is now just necessary to use the second relation of (12),

$$(n + 1)^2(6n + 5)S_{2n+1,2n+1,2n+1}$$

$$-8(2n - 1)^2(n + 2)n(3n + 1)S_{2n+1,2n+1,2n-1} = 0$$

$$(n + 2)^2(2n - 1)(6n + 7)S_{2n+1,2n+1,2n+3}$$

$$-8(2n + 1)^2(n + 1)^2(2n + 3)(3n + 2)S_{2n+1,2n+1,2n+1} = 0.$$

The coefficients of these relations are never zero for $n \in \mathbb{N}^*$, so we can always express $S_{2n+1,2n+1,2n-1}, S_{2n+1,2n+1,2n+3}$ in function of $S_{2n+1,2n+1,2n+1}$.

To conclude, we just need to check our formula on the axis $i = j = k$. Using again holonomic computation with the computer, we obtain the following relation

$$(n + 3)^3(3n + 4)(3n + 8)S_{n+2,n+2,n+2}$$

$$-6n^2(n + 1)^4(n + 2)^2(3n + 1)(3n + 5)S_{n,n,n} = 0.$$  

Our expression satisfies it and so it is only necessary to check the initial value. We have

$$\text{Res}_{t = 0} \left(\epsilon^{-1} Q_1(t) + \alpha P_1\right)(t^2 - 1)^2 = 8\alpha^2/5$$

and this fits our formula.

The case $i + j + k \mod 2 = 0$. We now look at the function $f$ for $i + j + k \mod 2 = 0$. This time, if we replace formally $\epsilon = 0$ in $f(i + \epsilon, j + \epsilon, k + \epsilon)\alpha$, we find a quotient of $\Gamma$ functions and in the numerator at most a term of the form $\Gamma(-n)$, $n \in \mathbb{N}$. We can still regularize the formula using the relation $\Gamma(n + 1) = n\Gamma(n)$. We obtain in particular that the limit

$$\lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon)} f(i + \epsilon, j + \epsilon, k + \epsilon)$$

is always finite. If there is no term of the form $\Gamma(-n)$, $n \in \mathbb{N}$ in the numerator of $f$, then the limit is zero. Using invariance by permutation, we will always assume in the following that...
In the case of a zero index, $i \geq j \geq k$, and so the only possible infinite term in the numerator is the term in $i + j - k$. We get then a zero limit for $k < i + j$, and for $k \geq i + j$, we can regularize the formula. We then check that the formula satisfies the recurrence. Now let us look at the boundary cases. This time we will try to go as far as possible away from the axis $i = j = k$. We want to prove that a solution for $i + j + k \mod 2 = 0$ to the recurrences (12) is uniquely determined by its values on $i = 1, j = 1$.

Using the first relation of (12) (and its permutations), we can express $S_{i,j,k}$ in the function of $S_{i-1,j,k}$ if $-2+i+j-k < 0$ because in that case both terms appear in the first relation of (12) (and all its permutations) and because it is satisfied for $i = 1$. If the condition $-2+i+j-k < 0$ is not satisfied, we also can go away from the axis $i = j = k$, but we meet a singularity when $-2+i+j-k = 0$ and this requires that $S_{i,j,k} = 0$ in this case. So if condition $-2+i+j-k < 0$ is not satisfied, we get $S_{i,j,k} = 0$, and we can check it is compatible with our formula. In the case $-1 + j - k < 0$, we can now try to reduce the index $j$ step by step using the relation

$$(2 + j - k)(k + 1)(1 - j + k)S_{i,j,k} - j(-1 + j - k)(4 - j + k)S_{i-1,j,k+1} = 0.$$ 

We know that $-1 + j - k$ never vanishes (it grows at each step), and $2 + j - k = 0$ is not possible because of the parity. So we can always express $S_{i,j,k}$ in function of $S_{i,j-1,k}$.

To conclude, one just need to compute $S_{1,1,2n+2}$, $n \in \mathbb{N}$. We prove the following formula

$$S_{1,1,2n+2} = -8\frac{16^6 \Gamma(n + 3/2)\Gamma(n - 1/2)\Gamma(n + 2)\Gamma(n + 1)}{\Gamma(n + 4)\Gamma(n + 5/2)\sqrt{\pi}}$$

using a two terms recurrence and the initial condition $S_{1,1,2} = 16/9\pi$. We eventually check that it fits our formula. \hfill \square

To conclude the proof of table A for non-zero indices, we now look at the expression of $\partial_0 S$ for $i + j + k \mod 2 = 1$, we see that it vanishes exactly when one of the three quantities

$$-i + j + k + 3 \quad -i + j + k + 3 \quad -i + j + k + 3$$

is non-positive. This accurately corresponds to the formulas of table A for $i + j + k \mod 2 = 1$.

In the case $i + j + k \mod 2 = 0$, the quantity

$$\lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon)} f(i + \epsilon, j + \epsilon, k + \epsilon)$$

vanishes if and only if all the numbers $-i + j + k$, $-i + j + k$, $-i + j + k$ are positive, which is equivalent to using the parity condition on

$$-i + j + k \geq 2 \quad -i + j + k \geq 2 \quad -i + j + k \geq 2$$

**The case of a zero index.** We now look at the case with at least one zero index. We can write

$$P_0(t) = \frac{1}{t^2 - 1} \quad Q_0(t) = \frac{1}{t^2 - 1} \quad Q_1(t) = \epsilon_i P_i(t) \arctanh \left( \frac{1}{t} \right) + \frac{W_i(t)}{t^2 - 1}$$

(the notation $P_0$ and $Q_0$ is here arbitrary because both are rational). We begin with the case where precisely one index is zero. We need to compute the residues

$$\text{Res}_{t=\infty} (\epsilon_i^{-1} Q_i(t) + \alpha P_i) (\epsilon_j^{-1} Q_j(t) + \alpha P_j) (t^2 - 1)$$

$$\text{Res}_{t=\infty} (\epsilon_i^{-1} Q_i(t) + \alpha P_i) (\epsilon_j^{-1} Q_j(t) + \alpha P_j) t (t^2 - 1).$$

These are polynomials of degree at most 2 in $\alpha$ but the coefficient in $\alpha^2$ is always zero because we take the residue of a polynomial at infinity. So one just needs to compute the residue in $\alpha$. 


Integrability conditions at order 2 for homogeneous potentials of degree $-1$

We expand, suppress the polynomial terms, and divide by $\epsilon_i \epsilon_j - 1$ (which never vanishes) and we get

$$S^1_{i,j} = \text{Res } \text{arctanh} \left( \frac{1}{t} \right) P_i P_j (t^2 - 1)$$
$$S^2_{i,j} = \text{Res } \text{arctanh} \left( \frac{1}{t} \right) P_i P_j t (t^2 - 1).$$

One just needs to prove that either $S^1_{i,j}$ or $S^2_{i,j}$ is not zero for $i \in \mathbb{N}^*$, $j = i, i+1, i-1$ (the condition on the indices of table $A$ such that $A = 0$ corresponds here to $-1 \leq i - j \leq 1$).

Using the parity on $t$ of the polynomials $P_i$, we find that only $S^1_{i,i}$, $S^2_{i,i}$ can be non-zero. These sequences can be easily computed for finding a recurrence with Mgfun and then a closed form

$$S^1_{i,i} = -2 \frac{4^i \Gamma(i+1)^2}{i(i+1)(2i+1)i}$$
$$S^2_{i,i+1} = -4 \frac{4^i \Gamma(i+1)^2}{(2i+1)(2i+3)}.$$

These expressions do not vanish.

3.3. Integrability in the cases where $A_{i,j,k} = 1$

Second part: we now prove that if all the non-zero terms of second order variational equation correspond only to cases such that $A_{i,j,k} = 1$, then the Galois group is abelian. We use the following lemma

**Lemma 13.** We consider

$$F(t) = \sum_{i=0}^3 H_i(t) \text{arctanh} \left( \frac{1}{t} \right)^i$$

with $H_0, \ldots, H_3 \in \mathbb{C}[t]$. If the conditions of lemma 8 are satisfied, then

- If $\text{Res }_{t=\infty} F(t) = 0$, then $\int F \, dt \in \mathbb{C}[t, \text{arctanh}(1/t)]$
- If $\text{Res }_{t=\infty} F(t) \neq 0$, then $\int F \, dt \in \mathbb{C}[t, \text{arctanh}(1/t), \ln(t^2 - 1)]$

**Proof.** We proceed using integration by parts. We look at the term of the highest degree and we differentiate $\text{arctanh}(1/t)^3$. Posing

$$J(t) = \int_{-1}^t H_3(s) \, ds$$

we get $J(1) = 0$ using the condition in $a^2$ of lemma 8 and making a series expansion at infinity of $\text{arctanh}(1/t)$. Then $(t^2 - 1)$ divide the polynomial $J$. After integration by parts, we get a term of the form

$$R(t) \text{arctanh} \left( \frac{1}{t} \right)^2$$

with $R$ a polynomial. Let us try another integration by parts. We get the term

$$\frac{2}{t^2 - 1} \int_{-1}^t \frac{3J(t)}{t^2 - 1} = H_2(t) dt \text{arctanh} \left( \frac{1}{t} \right).$$

We want this term to be written $Z(t) \text{arctanh}(1/t)$ with $Z$ a polynomial (with a good choice of integration constant). We just need that

$$\int_{-1}^1 \frac{3J(t)}{t^2 - 1} = H_2(t) dt = 0.$$
Let us look now at the coefficient in $\alpha$ of the residue (10). We know it is equal to zero.

$$[\alpha] \text{Res} \int t, \text{arctanh} \left( \frac{1}{t} \right) + \alpha = \text{Res} \int t, 3H_3(t)\text{arctanh} \left( \frac{1}{t} \right)^2 + 2H_2(t)\text{arctanh} \left( \frac{1}{t} \right).$$

which gives using an integration by parts (we can see the residue as an integration along a small circle around infinity)

$$\text{Res} \int t, \frac{6J(t)}{t^2 - 1} \text{arctanh} \left( \frac{1}{t} \right) + 2H_2(t)\text{arctanh} \left( \frac{1}{t} \right).$$

Using the Taylor expansion of $\text{arctanh}\left(\frac{1}{t}\right)$ at infinity, we get

$$\frac{1}{2} \int_{-1}^{1} \frac{6J(t)}{t^2 - 1} + 2H_2(t) \, dt = 0.$$

This is exactly our condition (10). So the last remaining integral to compute is of the form

$$\int Z(t)\text{arctanh} \left( \frac{1}{t} \right) \, dt \in \mathbb{C} \left[ t, \text{arctanh} \left( \frac{1}{t} \right), \ln \left( t^2 - 1 \right) \right]$$

which can be proved using an integration by parts. Now let us look closer at the possible terms in $\ln(t^2 - 1)$. Assume there exists a term $\ln(t^2 - 1)$ in $\int F \, dt$. We will have

$$\int F \, dt = Z_3(t)\text{arctanh} \left( \frac{1}{t} \right)^3 + \ldots + Z_0(t) + r \ln(t^2 - 1)$$

with $Z_3, \ldots, Z_0$ polynomials and $r$ is a constant because $\ln(t^2 - 1)$ does not appear in $F$. A function in $\mathbb{C}[t, \text{arctanh}(1/t)]$ is meromorphic near infinity. Differentiating this expression will give

$$F = g' + \frac{rt}{t^2 - 1}$$

with $g$ a meromorphic function on a neighbourhood of infinity. Then the residue of $F$ at infinity equals to $r$. So, if this residue is zero, there will be no $\ln(t^2 - 1)$ terms. □

The integrals to compute for the solutions of the second order variational method are the following

$$\int (t^2 - 1)^2 Q_i(t)Q_j(t)Q_k(t) \, dt \quad \int (t^2 - 1)^2 P_i(t)Q_j(t)Q_k(t) \, dt$$

$$\int (t^2 - 1)^2 P_i(t)P_j(t)Q_k(t) \, dt \quad \int (t^2 - 1)^2 P_i(t)P_j(t)P_k(t) \, dt.$$ (13)

(14)

They all are of the form given by lemma 8. We already know that the third one and the last one satisfy the condition of lemma 8, so they all belong to

$$\mathbb{C} \left[ t, \text{arctanh} \left( \frac{1}{t} \right), \ln \left( t^2 - 1 \right) \right].$$

For the first one, we use lemma 12. This proves that the residue is constant for $i, j, k \in \mathbb{N}^*$. For the case $i = 0$, we only need to look at the residues

$$S^1_{i,j} = \text{Res} \int t, \text{arctanh} \left( \frac{1}{t} \right) P_i P_j (t^2 - 1) \quad S^2_{i,j} = \text{Res} \int t, \text{arctanh} \left( \frac{1}{t} \right) P_i P_j t (t^2 - 1).$$

Making a Taylor expansion at infinity, we obtain the formulas

$$S^1_{i,j} = \int_{-1}^{1} P_i P_j (t^2 - 1) dt \quad S^2_{i,j} = \int_{-1}^{1} P_i P_j t (t^2 - 1) dt.$$
Moreover we have the property that the family of polynomials $P_i$ is orthogonal for the weight $(t^2 - 1)$ (due to the orthogonality relations for the Legendre polynomials). So $S^1_{i,j} = 0$ for $i \neq j$ (and we have already computed the case $i = j$ in the previous section). For $S^2_{i,j}$ we use the orthogonal property on $P_i$ and $P_j t$. If $i \geq j + 2$, we have that $P_i$ and $P_j t$ are orthogonal and so $S^2_{i,j} = 0$. By symmetry, it is also the case for $j \geq i + 2$. These correspond exactly to the cases $A_{0,i,j} = 1$. Finally, in the case $i = j = 0$, the first integral of (13) can be directly computed using one integration by parts (including the case $i = j = k = 0$).

We now look at the second integral of (13). The coefficient in $\alpha^2$ is automatically 0 because, by expanding the formula, we take the residue of a polynomial. We now want to compute the coefficient in $\alpha$ of the residue. This gives (using a Taylor expansion at infinity)

$$[\alpha] \operatorname{Res}_{t=\infty} (t^2 - 1)^2 P_i(t) Q_j(t) Q_k(t) = \frac{1}{2} \epsilon_i \epsilon_j \epsilon_k \int_{-1}^{1} P_i(t) P_j(t) P_k(t) (t^2 - 1)^2 dt$$

$$= [\alpha^2] \epsilon_i^{-1} \operatorname{Res}_{t=\infty} (t^2 - 1)^2 Q_i(t) Q_j(t) Q_k(t)$$

which equals to zero because it corresponds to the condition in $\alpha^2$ for the first integral of (13).

So the residue condition of lemma 8 is also satisfied, and then thanks to lemma 13, it also belongs to

$$\mathbb{C} \left[ t, \arctanh \left( \frac{1}{t} \right), \ln (t^2 - 1) \right].$$

Eventually, lemma 7 and theorems 9, 10 imply theorem 3.

### 3.4. Study of the Galois group in the integrable case

We now prove theorem 4, analysing more precisely the Galois group in the case where the integrability conditions of lemma 8 are satisfied. We will see that in fact the Galois group almost never grows, and the Galois group can be in fact precisely computed thanks to lemma 13.

**Proof of theorem 4.** In the integrable case, the first order variational equation $(\mathcal{V} E_1)$ involves the functions $P_i$, $Q_i$ which are in $\mathbb{C}[t, \arctanh(1/t)]$ (after variable change). The only univaluated function $Q_i$ is the function $Q_0$, but the eigenvalue 2 is always in the spectrum, and so the Galois group is always at least $(\mathbb{C}, +)$. At order 2, using lemma 13, we know that the solutions are in $\mathbb{C}[t, \arctanh(1/t), \ln(t^2 - 1)]$ and that the logarithmic term $\ln(t^2 - 1)$ can appear only if $S_{i,j,k}$ is a non-zero constant (which is independent of $\alpha$ because we assumed that the second order variational equation has a virtually abelian Galois group). Let us prove that

$$S_{i,j,k} \mid_{\alpha=0} = 0 \quad \forall i, j, k \in \mathbb{N}^*.$$  

(15)

We only need to use the recurrence (12) for $S_{i,j,k}$. To prove that this sequence is zero, we then only need to prove it vanishes on the boundary, and here it comes down to the cases $i = j = k = 1$ and $i = j = k = 2$ (as in the proof of theorem 10). We have that $S_{1,1,1} = 8\alpha^2/5$ and $S_{1,1,2} = 16\alpha/9$, and so vanish for $\alpha = 0$. This proves relation (15). Let us look now at the case where one of the indices is zero. We need to study

$$S^1_{i,j} = \operatorname{Res}_{t=\infty} (t^2 - 1) Q_i(t) Q_j(t) \quad S^2_{i,j} = \operatorname{Res}_{t=\infty} (t^2 - 1) t Q_i(t) Q_j(t).$$

We also prove they vanish for $\alpha = 0$ using recurrence. If two indices are zero, then we need to study

$$S^1_{i,j} = \operatorname{Res}_{t=\infty} Q_i(t) \quad S^2_{i,j} = \operatorname{Res}_{t=\infty} t Q_i(t) \quad S^3_{i,j} = \operatorname{Res}_{t=\infty} t^2 Q_i(t).$$
All these sequences are zero except for \( i = 1 \) for which \( S_3^1 = -2/3 \). Eventually, in the case where all indices are zero, we need to compute the following integrals
\[
\int \frac{1}{t^2 - 1} \, dt \quad \int \frac{t}{t^2 - 1} \, dt \quad \int \frac{t^2}{t^2 - 1} \, dt \quad \int \frac{t^3}{t^2 - 1} \, dt.
\]
The second and the fourth integral have a term in \( \ln(t^2 - 1) \). This gives theorem 4.

**Remark 1.** The computation of the sequence \( S_{i,j,k}|_{\alpha = 0} \) makes no sense when \( S_{i,j,k} \) depend on \( \alpha \). Indeed \( \alpha \) corresponds to the multivaluation of functions \( Q \). If \( S_{i,j,k}|_{\alpha = 0} = 0 \) and \( S_{i,j,k} \) depend on \( \alpha \), then when we replace \( \alpha \) by \( \alpha + 1 \) (by changing the convention for the definition of functions \( Q \)), we get \( S_{i,j,k}|_{\alpha = 0} \neq 0 \). So in fact this vanishing term only corresponds to a convention taken for the \( Q_i \). Still the convention is well chosen, because it allows one to study \( S_{i,j,k}|_{\alpha = 0} \) without making a distinction between integrable cases and non integrable cases (in particular, \( S_{i,j,k}|_{\alpha = 0} \) is almost always zero for all values, and this property allows a much faster proof than in previous sections).

4. Applications

4.1. The nondiagonalizable case

We now look at the diagonalizability hypothesis of theorem 3. We do not make a complete analysis of the non-diagonalizable case at order 2 because it does not seem possible to make an efficient reduction in this case to produce a nice criterion, and this case is very rare in practice (still in this case, the theorem 3 gives integrability constraints on a subsystem, but this constraint is not a priori optimal).

Using our method of analysis for the second order variational equation, let us now reprove the results of Duval and Maciejewski on the non-diagonalizable case.

**Corollary 1 (See a different proof of Duval and Maciejewski in [6]).** We consider the equation
\[
\ddot{X}(t) = \frac{1}{\phi(t)^3} AX(t)
\]
with \( A \) a matrix in Jordan form. Then this equation has a virtually abelian Galois group if and only if

- \( A_{i,i} = \frac{1}{2}(n_i - 1)(n_i + 2) \) with \( n_i \in \mathbb{N} \)
- \( A \) is diagonal except maybe for eigenvalue \(-1\) for which the Jordan blocks should be at most of size 2.

We see that the diagonalizability is not a strong hypothesis as, if the first order variational equation has a virtually abelian Galois group, there is only one possibility for which the Hessian matrix can be non-diagonalizable, and only for the eigenvalue \(-1\).

**Proof.** In the non-diagonalizable case, the nonhomogeneous part of the variational equation corresponds to terms outside the diagonal in the matrix \( A \). For a Jordan block of size 2, the equation (16) can be rewritten
\[
\ddot{X} = \frac{1}{\phi(t)^3} d_i X + \frac{1}{\phi(t)^3} Y \quad \ddot{Y} = \frac{1}{\phi(t)^3} d_i Y
\]
because a Jordan block has the same eigenvalue on the diagonal. For the case with a bigger Jordan block, we would obtain even stronger conditions because then the equation (16) would
be a subsystem of equation (17). In our case (except for eigenvalue $-1$), this analysis is not necessary. We compute the solutions and we prove that the following function should be in the Picard–Vessiot field.

$$\int (t^2 - 1)Q_i(t)^2 \, dt.$$  

With lemma 8 and lemma 13, we know it is enough to study the sequence

$$S_i = \text{Res}_{t=\infty} (t^2 - 1)(Q_i(t) + \epsilon_i \alpha P_i)^2.$$  

The interesting term is in $\alpha$, because the coefficient in $\alpha^2$ is always zero. The Mgfun package give us a recurrence and then a closed form for this residue

$$S_i = -2 \frac{4i i! (i)}{2(i + 1)(i + 1)}$$  

which is never zero for $i \in \mathbb{N}^*$. The case $i = 0$ requires the analysis of a Jordan block of size 3 (a larger Jordan block size would still have this system as a subsystem). This gives the equations

$$\begin{align*}
1/2(t^2 - 1)\ddot{X}_1 + 2t \dot{X}_1 + X_1 &= 0 \\
1/2(t^2 - 1)\ddot{X}_2 + 2t \dot{X}_2 + X_2 &= X_1 \\
1/2(t^2 - 1)\ddot{X}_3 + 2t \dot{X}_3 + X_3 &= X_2.
\end{align*}$$  

(18)

Again, we compute the solution and we compute a solution for $X_3$

$$X_3(t) = (t^2 - 1)^{-1} \int \int 2t \arctanh \left( \frac{1}{t} \right) + \ln(t^2 - 1) \frac{dr \, dt}{t^2 - 1}.$$  

Thus the following integral belongs to the Picard–Vessiot field

$$\int \frac{2t \arctanh \left( \frac{1}{t} \right) + \ln(t^2 - 1)}{t^2 - 1} \, dt = \ln(t - 1)(2 \ln 2 - \ln(t + 1)) - 2 \text{dilog} \left( \frac{t + 1}{2} \right).$$  

All the terms are in $\mathbb{C}[t, \arctanh(\frac{1}{t}), \ln(t^2 - 1)]$ except one, the dilogarithmic term

$$\text{dilog} \left( \frac{t + 1}{2} \right) = \int \frac{\ln(t + 1) - \ln 2}{1 - t} \, dt.$$  

With the same idea as in lemma 8, we prove that this term has a non-commutative monodromy because of the following residue at $t = 1$ (also proved in [2])

$$\text{Res}_{t=1} \frac{\ln(t + 1) - \ln 2 + \alpha}{1 - t} = -\alpha$$  

which depends explicitly on $\alpha$. To conclude, we notice that the Galois group of equation (18) is always connected because we only take integrations recursively (no algebraic functions are involved) and because the Galois group of the first equation of (18) is always connected (it is $\{Id\}$). 

\[\square\]

\[4.2.\, A\, useful\, corollary\]

Corollary 2. Let $V$ be a meromorphic homogeneous potential of degree $-1$ in dimension $n$, $c$ a Darboux point of $V$ with multiplier $-1$. Let $\lambda_i, \ i = 1 \ldots n$ denote the eigenvalues of $\nabla^2 V(c)$ with $\lambda_1 = 2$ (the eigenvalue 2 always appears in the spectrum). Assume $\nabla^2 V(c)$ is diagonalizable and

$$\{\lambda_2, \ldots, \lambda_n\} = \{(2k - 1)(k + 1), \ k \in B\} \quad \text{with}$$  

$$B \subset \mathbb{N}, \ \max(B) \leq \max(2 \min(B) - 1, 0).$$  

(19)
Then the variational equation at order 2 near the homothetic orbit associated to \( c \) has a virtually abelian Galois group.

**Remark 2.** In practice, this corollary tells us that if the eigenvalues of the Hessian matrix all have an even index and are sufficiently close to each other, then the system is always integrable at order 2 without any additional conditions. Moreover, this theorem is in some sense ‘optimal’, because if it is not satisfied, then there will be strong additional integrability constraints (of codimension at least 1) for integrability. It also allows one to have a strong intuition about what will be the easy and the hard cases in proving the non-integrability of a particular problem depending on parameters. In practice, eigenvalues of odd index lead to strong at integrability conditions at order 2, and thus only cases where the eigenvalues of the Hessian matrix all have an even index lead to integrable cases at order 2. So typically, in dimension \( d \) with \( k \) Darboux points, the number of resistant cases (to prove non-integrability) is divided by \( 2^{(d-1)k} \).

**Proof.** We have the Euler relation due to homogeneity and the Darboux point condition

\[
\sum_{i=1}^{n} q_i \frac{\partial}{\partial q_i} V = -V \quad \frac{\partial}{\partial q_i} V(c) = -c_i. \tag{20}
\]

Differentiating the Euler relation in \( q_j \), we obtain

\[
\sum_{i=1}^{n} q_i \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} V = -2 \frac{\partial}{\partial q_j} V.
\]

With the Darboux point relation, this implies that \( c \) is an eigenvector with eigenvalue 2. Let us note \( X_1 = c, X_2, \ldots, X_n \) a basis of eigenvectors of \( \nabla^2 V(c) \). We will first prove that

\[
D^3(V)(c). (X_1, X_a, X_b) = 0 \quad \forall a \neq b
\]

We differentiate the Euler relation (20) two times and evaluate on \( c \):

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (X_a)_j c_i \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} V(c) = -3 \lambda_a (X_a)_k \quad \forall k
\]

with \( \lambda_a \) the eigenvalue associated to \( X_a \) and using the fact that \( X_a \) is an eigenvector of \( \nabla^2 V(c) \).

We then multiply each line with index \( k \) by \( (X_b)_k \) and we sum over the \( k \)

\[
\sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (X_b)_k (X_a)_j c_i \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} V(c) = -3 \lambda_a (X_a|X_b) = 0 \tag{21}
\]

thanks to orthogonality. This is the expression of \( D^3(V)(c). (X_1, X_a, X_b) \). Let us now use theorem 3. We first remark that all invoked indices of table \( A \) are even. Moreover, in the case of three even indices \( i, j, k \) with \( \max(i, j, k) \leq 2 \min(i, j, k) - 2 \), we have

\[
A_{i,j,i}, A_{j,j,j}, A_{k,k,k}, A_{i,j,j}, A_{i,k,k}, A_{i,j,i}, A_{i,j,k}, A_{k,i,i}, A_{k,j,j}, A_{i,j,k} = 1.
\]

We also have \( A_{0,0,0} = 1 \). So if the eigenvalues of \( \nabla^2 V(c) \) satisfy

\[
\text{Sp}(\nabla^2 V(c)) = \{(2k - 1)(k + 1), \quad k \in \mathbb{N}_0 \}
\]

with \( \mathbb{B} \subset \mathbb{N}, \quad \max(\mathbb{B}) \leq \max(2 \min(\mathbb{B}) - 1, 0) \)
Integrability conditions at order 2 for homogeneous potentials of degree $-1$

then the system is integrable at order 2. Still, knowing that the eigenvalue 2 always appears in the spectrum of $\nabla^2 V(c)$, this imply that $\min(B) = 1$, and thus that $\tilde{B} = \{1\}$. So this criterion is useless. We need to ‘remove’ this eigenvalue 2.

We know that $c$ is always an eigenvector with eigenvalue 2, and that all the possible conditions linked to this eigenvector are of the form $D^3(V)(c). (X_1, X_a, X_b) = 0$. These are automatically satisfied for $a \neq b$. For $a = b$, we have that $A_{2,i}i = 1$ for all $i \neq 1$, so the only possible problem would be if $X_a$ has the eigenvalue 0, but in this particular case, we also get with equation (21)

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (X_a)_k (X_a)_j c_i \frac{\partial}{\partial q_i \partial q_j \partial q_k} V(c) = -3\lambda_a \langle X_a | X_a \rangle = 0$$

because $\lambda_a = 0$. So the possible integrability conditions involving the eigenvector $X_1 = c$ are always satisfied, and thus we can remove one time the eigenvalue $\lambda_1 = 2$ from the spectrum, and this gives exactly the condition (19).

□

4.3. Some examples

Let us consider the 3 body problem on a line with masses $(1, m, 1)$. The corresponding Hamiltonian is the following

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2m} + \frac{p_3^2}{2} + \frac{m}{q_1 - q_2} + \frac{m}{q_2 - q_3} + \frac{1}{q_1 - q_3}.$$  

After the variable change $(p_2, q_2) \rightarrow (p_2/\sqrt{m}, q_2/\sqrt{m})$, the Hamiltonian becomes

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} \frac{m}{q_1 - q_2/\sqrt{m}} + \frac{m}{q_2/\sqrt{m} - q_3} + \frac{1}{q_1 - q_3}$$

and so corresponds to a homogeneous potential of degree $-1$. Let us study this potential using our integrability table.

**Proposition 3.** We consider the potential

$$V = \frac{m}{q_1 - q_2/\sqrt{m}} + \frac{m}{q_2/\sqrt{m} - q_3} + \frac{1}{q_1 - q_3}$$

with $m \in \mathbb{C}$ and

$$c = \left( \frac{1}{2} (8m + 2)^{1/3}, 0, -\frac{1}{2} (8m + 2)^{1/3} \right)$$

Then $c$ is a Darboux point of $V$ with multiplier $-1$ and the identity component of the Galois group of the second order variational equation is abelian if and only if

$$\frac{8(m + 2)}{4m + 1} \in \left\{ \frac{1}{2} (k - 1)(k + 2), \ k \in \mathbb{N} \right\}$$

In this case, the Galois group is then always $(\mathbb{C}, +)$.

**Proof.** We first check that $c$ is a Darboux point of $V$ with multiplier $-1$. We then compute the Hessian matrix of $V$ at $c$ and we find the following spectrum

$$\left\{ 0, 2, \frac{8(m + 2)}{4m + 1} \right\}$$

...
and so the condition at order 1 is exactly the condition (22). Looking at the table $A$ of theorem 3, we have $A_{k,k,k} = 1$ if $k$ even and $A_{k,k,k} = 0$ if $k$ odd. As proved in corollary 2, the eigenvalue 2 can be discarded because anyway the corresponding integrability conditions will be satisfied. Let us look at integrability conditions of theorem 3. The eigenvalue 0 has for eigenvector $X_1 = (1, \sqrt{m}, 1)$, and the potential is invariant by translation along this vector. Thus all corresponding derivatives of $V$ will vanish. The only possible integrability constraint can come from the cases $A_{k,k,k} = 0$ corresponding to $k$ odd. We find that
\[ D^3V(c). (X_3, X_3, X_3) = 0 \quad \text{with} \quad X_3 = (-\sqrt{m}, 2, -\sqrt{m}) \]
and so the constraint is always satisfied. Moreover, using theorem 4, we find that the Galois group is always $(\mathbb{C}, +)$. □

This case is an unfortunate one because variational equations of order 2 give nothing more than order 1, and order 3 would be necessary to conclude using solely this Darboux point. The problem is still tractable if one uses the other Darboux points. Still, some 'resistant' potentials require in all cases higher variational equations.

**Proposition 4.** Let us consider the potential
\[ V(r, \theta) = r^{-1} \left( (1 + a) - 2ae^{i\theta} + ae^{2i\theta} \right) \quad a \in \mathbb{C} \]
where $r, \theta$ correspond to polar coordinates in the plane. If $V$ is meromorphically integrable, then
\[ -2a - 1 \in \{(2k - 1)(k + 1) : k \in \mathbb{N}\} \]
and in this case, the Galois group of the second order variational equation near the homothetic orbit associated to the unique Darboux point is always $(\mathbb{C}, +)$.

**Proof.** Let us first look for the Darboux points. After computation, we find only one Darboux point which is $c = (1, 0)$ and has multiplier $-1$. The spectrum of the Hessian matrix at $c$ is given by $[2, -2a - 1]$. The integrability condition of order 1 gives that
\[ -2a - 1 = \frac{1}{2}(k - 1)(k + 2) \quad k \in \mathbb{N} \]
The only condition of order 2 can come from the eigenvalue $-2a - 1$, whose eigenvector is $X = (0, 1)$. Looking at the table $A$ of theorem 3, we have $A_{k,k,k} = 1$ if $k$ even and $A_{k,k,k} = 0$ if $k$ odd. For $k$ even we have no conditions at all, and for $k$ odd, we obtain the condition
\[ D^3(V)(c). (X, X, X) = -6a = 0. \]
For $k$ odd, the only possible case is then $a = 0$, but this implies that $-2a - 1 = -1$ which corresponds to $k = 0$ (even). So the case $k$ odd is never possible. We compute the Galois group, and we find that for $k$ even it is always $(\mathbb{C}, +)$, including the case $k = 0$ because the condition of theorem 4 is satisfied (we have $D^3(V)(c). (X, X, X) = 0$). □

**Remark 3.** Note that in this second example, there is a square root in the potential. As explained in Combot [4], the potential $V$ is in fact well defined on the complex manifold $S = \{(q_1, q_2, r) : r^2 = q_1^2 + q_2^2, \quad q \neq 0\}$ (instead of $\mathbb{C}^n$ as in the whole article). On this complex manifold $S$, $V$ is meromorphic (and even rational), and the corresponding Hamiltonian is holomorphic on $\mathbb{C}^2 \times S$. We can apply the most general version of the Morales–Ramis–Simó theorem 1, using the orbit ($c$ is a Darboux point)
\[ \Gamma = \{ p = \dot{\phi}.c, \quad q = \phi.c, \quad r = |c| \phi, \quad \frac{1}{2} \phi = 1/\phi + 1, \phi \neq 0 \} \subset \mathbb{C}^2 \times S. \]
the Hamiltonian $H$ being holomorphic on a neighbourhood of $\Gamma$. Thus theorems 3, 4 subsequently apply. Note that, in this second example, the second order integrability condition was useful but not sufficient to entirely solve the problem. This is because there still remains infinitely many values of the parameter $a$ for which the system might be integrable (as expected, half of these cases were removed, but an infinity still remains). As there is just one Darboux point, a study of the third variational equation is necessary to conclude (we have done this in [3]).

Acknowledgments

I wish to thank Christoph Koutschan for the computations of the recurrences (12) using his own Mathematica implementation [10]. I wish to also thank Jacques-Arthur Weil for his valuable comments on this article. The author is financially supported by University Paris Diderot.

References

[1] Albouy A and Kaloshin V 2012 Finiteness of central configurations of five bodies in the plane Ann. Math. 176 535–88
[2] Aparicio-Monforte A and Weil J A 2011 A reduction method for higher order variational equations of Hamiltonian systems Symmetries and Related Topics in Differential and Difference Equations: Jairo Charris Seminar 2009, Escuela de Matematicas. Universidad Sergio Arboleda, Bogotá, Colombia (Contemporary Mathematics vol 549) (Providence, RI: American Mathematical Society)
[3] Combot T and Koutschan C 2012 Third order integrability conditions for homogeneous potentials of degree 1 J. Math. Phys. 53 at press
[4] Combot T 2012 A note on algebraic potentials and morales-ramis theory, arXiv:1209.4747
[5] Craven B D 1969 Complex symmetric matrices J. Aust. Math. Soc. 10 341–54
[6] Duval G and Maciejewski A J 2009 Jordan obstruction to the integrability of Hamiltonian systems with homogeneous potentials (Obstruction de Jordan à l’intégrabilité de systèmes Hamiltoniens avec potentiel homogène) Ann. l’inst. Fourier 59 2839–90
[7] Kimura T 1970 On Riemann’s equations which are solvable by quadratures Funkcialaj Ekvacioj 12 269–81
[8] Koutschan C 2009 Advanced applications of the holonomic systems approach PhD Thesis RISC, Johannes Kepler University, Linz, Austria
[9] Koutschan C 2010 A fast approach to creative telescoping Math. Comput. Sci. 4 259–66
[10] Koutschan C 2010 Holonomic Functions (User’s Guide) Technical Report 10-01, RISC Report Series, Johannes Kepler University Linz http:/www.risc.jku.at/research/combinat/HolonomicFunctions/
[11] Lee T L and Santoprete M 2009 Central configurations of the five-body problem with equal masses Celest. Mech. Dyn. Astron. 104 369–81
[12] Maciejewski A J and Przybylska M 2004 All meromorphically integrable 2D Hamiltonian systems with homogeneous potential of degree 3 Phys. Lett. A 327 461–73
[13] Maciejewski A J and Przybylska M 2005 Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential J. Math. Phys. 46 062901
[14] Morales-Ruiz J J, Ramis J P and Simó C 2007 Integrability of Hamiltonian systems and differential Galois groups of higher variational equations Ann. Sci. l’Ecole Normale Supérieure 40 845–84
[15] Morales-Ruiz J J and Ramis J P 2001 Galoisian obstructions to integrability of Hamiltonian systems Methods Appl. Anal. 8 33–96
[16] Morales-Ruiz J J and Ramis J P 2001 Galoisian obstructions to integrability of Hamiltonian systems: II. Methods Appl. Anal. 8 97–112
[17] Morales-Ruiz J J and Ramis J P 2001 A note on the non-integrability of some Hamiltonian systems with a homogeneous potential Methods Appl. Anal. 8 113–20
[18] Morales-Ruiz J J and Simón S 2009 On the meromorphic non-integrability of some N-body problems Discrete Cont. Dyn. Syst. (DCDS-A) 24 1225–73
[19] Morales Ruiz J J 1999 Differential galois theory and non-integrability of Hamiltonian systems Progress in Mathematics (Boston, MA: Birkhauser) p 179
[20] Pina E and Lonngi P 2010 Central configurations for the planar newtonian four-body problem *Cele. Mech. Dyn. Astron.* **100** 73–93

[21] Salvy B and Zimmermann P 1994 Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable *ACM Trans. Math. Softw.* **20** 163–77

[22] Tsygvintsev A 2001 The meromorphic non-integrability of the three-body problem *J. Reine Angew. Math. (Crelles J.)* **2001** 127–49

[23] Yoshida H 1987 A criterion for the non-existence of an additional integral in Hamiltonian systems with a homogeneous potential *Physica D* **29** 128–42