A NOTE ON THE BUCHSBAUM-RIM MULTIPLICITY OF A PARAMETER MODULE

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Abstract. In this article we prove that the Buchsbaum-Rim multiplicity $e(F/N)$ of a parameter module $N$ in a free module $F = A^r$ is bounded above by the colength $\ell_A(F/N)$. Moreover, we prove that once the equality $\ell_A(F/N) = e(F/N)$ holds true for some parameter module $N$ in $F$, then the base ring $A$ is Cohen-Macaulay.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $F = A^r$ be a free module of rank $r > 0$, and let $M$ be a submodule of $F$ such that $F/M$ has finite length and $M \subseteq \mathfrak{m}F$.

In their article [5] from 1964 Buchsbaum and Rim introduced and studied a multiplicity associated to a submodule of finite colength in a free module. This multiplicity, which generalizes the notion of Hilbert–Samuel multiplicity for ideals, is nowadays called the Buchsbaum-Rim multiplicity. In more detail, it first turns out that the function

$$\lambda(n) := \ell_A(S_n(F)/R_n(M))$$

is eventually a polynomial of degree $d + r - 1$, where $S_A(F) = \bigoplus_{n \geq 0} S_n(F)$ is the symmetric algebra of $F$ and $R(M) = \bigoplus_{n \geq 0} R_n(M)$ is the image of the natural homomorphism from $S_A(M)$ to $S_A(F)$. The polynomial $P(n)$ corresponding to $\lambda(n)$ can then be written in the form

$$P(n) = \sum_{i=0}^{d+r-1} (-1)^i e_i \binom{n + d + r - 2 - i}{d + r - 1 - i}$$

with integer coefficients $e_i$. The Buchsbaum-Rim multiplicity of $M$ in $F$, denoted by $e(F/M)$, is now defined to be the coefficient $e_0$.

Buchsbaum and Rim also introduced in their article the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module $N$ in $F$ is said to be a parameter module in $F$ if the following three conditions are satisfied: (i) $F/N$ has finite length, (ii) $N \subseteq \mathfrak{m}F$,

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and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of $N$.

Buchsbaum and Rim utilized in their study the relationship between the Buchsbaum-Rim multiplicity and the Euler-Poincaré characteristic of a certain complex and proved the following:

**Theorem 1.1** (Buchsbaum-Rim [5, Corollary 4.5]). Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$. Then the following statements are equivalent:

1. $A$ is a Cohen-Macaulay local ring.
2. For any rank $r > 0$, the equality $\ell_A(F/N) = e(F/N)$ holds true for every parameter module $N$ in $F = A^r$.

Then it is natural to ask the following:

**Question 1.2.**

1. Does the inequality $\ell_A(F/N) \geq e(F/N)$ hold true for any parameter module $N$ in $F$?
2. Does the equality $\ell_A(F/N) = e(F/N)$ for some parameter module $N$ in $F$ imply that the ring $A$ is Cohen-Macaulay?

The purpose of this article is to give a complete answer to Question 1.2. Our results can be summarized as follows:

**Theorem 1.3.** Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$.

1. For any rank $r > 0$, the two inequalities $\ell_A(F/N) \geq e(F/N)$ and $\ell_A(A/I(N)) \geq e(F/N)$ always hold true for every parameter module $N$ in $F = A^r$, where $I(N)$ is the 0-th Fitting ideal of $F/N$.
2. The following statements are equivalent:
   i. $A$ is a Cohen-Macaulay local ring.
   ii. For some rank $r > 0$, there exists a parameter module $N$ in $F = A^r$ such that the equality $\ell_A(F/N) = e(F/N)$ holds true.
   iii. For some rank $r > 0$, there exists a parameter module $N$ in $F = A^r$ such that the equality $\ell_A(A/I(N)) = e(F/N)$ holds true.

When this is the case, the equality $\ell_A(F/N) = \ell_A(A/I(N)) = e(F/N)$ holds true for all parameter modules $N$ in $F = A^r$ of any rank $r > 0$.

Note that the equality $\ell_A(F/N) = \ell_A(A/I(N))$ is known by [1, 2.10].

The proof of our Theorem 1.3 will be completed in section 4. Section 2 is of a preliminary character. In that section we will recall the definition and some basic facts about the generalized Koszul complex. In order to prove Theorem 1.3 we will investigate in section 3 the higher Euler-Poincaré characteristics of the generalized Koszul complex and show that they are non-negative. Finally, in section 4, we will obtain Theorem 1.3 as a corollary of a more general result (Theorem 1.1).

2. Preliminaries

In this section we will recall the definition and some basic facts about the generalized Koszul complex introduced in [3, 8] (for more details, see also [7, Appendix A.2.6]).

Let $A$ be a commutative Noetherian ring, and let $n \geq r > 0$ be integers. Let $a = (a_{ij})$ be an $r \times n$ matrix over $A$, and let $I_r(a)$ denote the ideal generated by
the maximal minors of \( a \). Let \( F \) and \( G \) be free modules with bases \( \{ f_1, \ldots, f_r \} \) and \( \{ c_1, \ldots, c_n \} \), respectively. Let \( S \) be the symmetric algebra of \( F \), and let \( S^t \) be the \( t \)-th symmetric power of \( F \). Let \( \wedge \) be the exterior algebra of \( G \), and let \( \wedge^t \) be the \( t \)-th exterior power of \( G \). Associated with the \( i \)-th row \( [a_{i1} \cdots a_{in}] \) of \( a \), there is a differentiation homomorphism \( \delta_i : \wedge \rightarrow \wedge \) given by

\[
\delta_i(f_{j1} \wedge \cdots \wedge f_{jp}) = \sum_{k=1}^{p} (-1)^{k-1} a_{ijk} f_{j1} \wedge \cdots \wedge \widehat{f_{jk}} \wedge \cdots \wedge f_{jp}.
\]

Let \( f_i : S \rightarrow S \) and \( f_i^{-1} : S \rightarrow S \) denote the multiplication and division maps by \( f_i \), respectively, i.e.,

\[
f_i^{-1}(f_1^{\mu_1} \cdots f_i^{\mu_i} \cdots f_r^{\mu_r}) = \begin{cases} f_1^{\mu_1} \cdots f_i^{\mu_i-1} \cdots f_r^{\mu_r} & (\mu_i > 0) \\ 0 & (\mu_i = 0). \end{cases}
\]

Then the generalized Koszul complex \( K_\bullet(a; t) \) associated to a matrix \( a \) and an integer \( t \) is the complex

\[
K_\bullet(a; t) : \cdots \rightarrow K_{p+1}(a; t) \xrightarrow{d_{p+1}} K_p(a; t) \xrightarrow{d_p} K_{p-1}(a; t) \rightarrow \cdots
\]

defined by

\[
K_p(a; t) = \begin{cases} \wedge^{r-p-1} \otimes_A S_{p-t-1} & (p \geq t+1) \\ \wedge^p \otimes_A S_{t-p} & (p \leq t) \end{cases}
\]

and

\[
d_{p+1} = \begin{cases} \sum_{j=1}^{r} \delta_j \otimes f_j^{-1} & (p > t) \\ \delta_{p+1} \cdots \delta_1 \otimes 1 & (p = t) \\ \sum_{j=1}^{r} \delta_j \otimes f_j & (p < t). \end{cases}
\]

The generalized Koszul complex \( K_\bullet(a; t) \) is a free complex of \( A \)-modules. We note that it is of length \( n-r+1 \) when \( -1 \leq t \leq n-r+1 \). Also recall that \( K_\bullet(a; t) \) coincides with the ordinary Koszul complex for any \( t \) in the case \( r = 1 \), whereas \( K_\bullet(a; 0) \) is the Eagon-Northcott complex and \( K_\bullet(a; 1) \) is the Buchsbaum-Rim complex. Moreover, the generalized Koszul complex has the following important properties (see [8, 10] and [7, Appendix A2.6]):

**Proposition 2.1.** Let \( a \) be an \( r \times n \) matrix over \( A \) with \( n \geq r > 0 \). Then

1. [8, Theorem 1] For any \( t, p \in \mathbb{Z} \), \( I_t(a) H_p(K_\bullet(a; t)) = (0) \).
2. [7, Theorem A2.10] If the grade of \( I_t(a) \) is at least \( n-r+1 \), then \( K_\bullet(a; t) \) is acyclic for all \( -1 \leq t \leq n-r+1 \). Furthermore, if \( a \) is a generic matrix, then \( K_\bullet(a; t) \) is acyclic for all \( t \geq -1 \).

### 3. Higher Euler-Poincaré Characteristics

In this section we will investigate higher Euler-Poincaré characteristics of a generalized Koszul complex.

Throughout this section, let \( (A, \mathfrak{m}) \) be a Noetherian local ring of dimension \( d > 0 \). Let \( F = A^r \) be a free module of rank \( r > 0 \) with a basis \( \{ f_1, \ldots, f_r \} \). Let \( M \) be a submodule of \( F \) generated by \( c_1, c_2, \ldots, c_n \), where \( n = \mu_A(M) \) is the minimal number of generators of \( M \). Writing \( c_j = c_{ij} f_1 + \cdots + c_{rj} f_r \) for some \( c_{ij} \in A \), we have an \( r \times n \) matrix \( (c_{ij}) \) associated to \( M \). We call this matrix the matrix of \( M \) and denote it by \( \tilde{M} \). Let \( I(M) = \text{Fitt}_0(F/M) \) be the 0-th Fitting ideal of \( F/M \). We assume that \( F/M \) has finite length and \( M \subseteq \mathfrak{m} F \). Then \( I(M) \) is an \( \mathfrak{m} \)-primary ideal, because \( \sqrt{I(M)} = \sqrt{\text{Ann}_A(F/M)} \). Hence each homology
module $H_p(K_\bullet(\widetilde{M}; t))$ has finite length by Proposition 2.11. So the Euler-Poincaré characteristics of $K_\bullet(\widetilde{M}; t)$ can be defined as follows:

**Definition 3.1.** For any integer $q \geq 0$, we set
\[
\chi_q(K_\bullet(\widetilde{M}; t)) := \sum_{p \geq q} (-1)^{p-q} \ell_A(H_p(K_\bullet(\widetilde{M}; t)))
\]
and call it the $q$-th partial Euler-Poincaré characteristic of $K_\bullet(\widetilde{M}; t)$. When $q = 0$, we simply write $\chi(K_\bullet(\widetilde{M}; t))$ for $\chi_0(K_\bullet(\widetilde{M}; t))$ and call it the Euler-Poincaré characteristic of $K_\bullet(\widetilde{M}; t)$.

Buchsbaum and Rim studied in [5] the Euler-Poincaré characteristic of the Buchsbaum-Rim complex in analogy with the Euler-Poincaré characteristic of the ordinary Koszul complex in the case of usual multiplicities. In 1985 Kirby investigated in [9] Euler-Poincaré characteristics of the complex $K_\bullet(\widetilde{M}; t)$ for all $t$ and proved the following:

**Theorem 3.2** (Buchsbaum-Rim, Kirby). For any integer $t \in \mathbb{Z}$, we have
\[
\chi(K_\bullet(\widetilde{M}; t)) = \begin{cases} 
  e(F/M) & (n = d + r - 1), \\
  0 & (n > d + r - 1),
\end{cases}
\]
where $n = \mu_A(M)$ is the minimal number of generators of $M$. In particular, $\chi(K_\bullet(\widetilde{M}; t)) \geq 0$ for all $t \in \mathbb{Z}$.

The last statement holds for the higher Euler-Poincaré characteristics, too:

**Theorem 3.3.** For any $q \geq 0$ and any $t \geq -1$, we have
\[
\chi_q(K_\bullet(\widetilde{M}; t)) \geq 0.
\]

**Proof.** We use ideas from [6]. Let $\widetilde{M} = (c_{ij}) \in \text{Mat}_{r \times n}(A)$ be the matrix of $M$, and let $X = (X_{ij})$ be the generic matrix of the same size $r \times n$. Let $A[X] = A[X_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n]$ be a polynomial ring over $A$, and let $B = A[X]_{(m,n)}$. We will consider the ring $A$ as a $B$-algebra via the substitution homomorphism $\phi : B \to A ; X_{ij} \mapsto c_{ij}$. Let
\[
b = \text{Ker} \phi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)B.
\]
We note here that $K_\bullet(X; t) \otimes_B A \cong K_\bullet(\widetilde{M}; t)$, because the generalized Koszul complex is compatible with the base change. Let $C_t(X) := H_0(K_\bullet(X; t))$. By Proposition 2.12, the complex $K_\bullet(X; t)$ is a $B$-free resolution of the $B$-module $C_t(X)$ for any $t \geq -1$. By tensoring with $A$ and taking the homology, we have that
\[
H_p(K_\bullet(\widetilde{M}; t)) \cong H_p(K_\bullet(X; t) \otimes_B A) \cong \text{Tor}^B_p(C_t(X), A)
\]
for all $p \geq 0$. On the other hand, since the ideal $b$ in $B$ is generated by a regular sequence of length $rn$, the ordinary Koszul complex $K_\bullet(b)$ associated to the sequence $b$ is a $B$-free resolution of $A$. Hence, by tensoring with $C_t(X)$, we can compute the Tor as follows:
\[
\text{Tor}^B_p(C_t(X), A) \cong H_p(K_\bullet(b) \otimes_B C_t(X)).
\]
Therefore, for any $p \geq 0$, we have
\[
H_p(K_\bullet(\widetilde{M}; t)) \cong H_p(K_\bullet(b) \otimes_B C_t(X)).
\]
It follows that for any \( t \geq -1 \) and any \( q \geq 0 \) we have the equality
\[
\chi_q(M(t; t)) = \chi_q(M(b) \otimes_B C_t(X)).
\]
Here the right hand side is non-negative by Serre’s Theorem ([12, Ch. IV, Appendix II]). Therefore \( \chi_q(M(t; t)) \geq 0 \).

4. Proof of Theorem 4.1

Theorem 4.1 will be a consequence of the following more general result:

**Theorem 4.1.** Let \( (A, \mathfrak{m}) \) be a Noetherian local ring of dimension \( d > 0 \).

1. For any rank \( r > 0 \), the inequality \( \ell_A(H_0(K_\bullet(M(N; t))) \geq e(F/N) \) holds true for any integer \( t \geq -1 \) and any parameter module \( N \) in \( F = A^r \).

2. The following statements are equivalent:
   (i) \( A \) is a Cohen-Macaulay local ring.
   (ii) For some rank \( r > 0 \), there exists an integer \( -1 \leq t \leq d \) and a parameter module \( N \) in \( F = A^r \) such that the equality \( \ell_A(H_0(K_\bullet(M(N; t))) = e(F/N) \) holds true.

When this is the case, the equality \( \ell_A(H_0(K_\bullet(M(N; t))) = e(F/N) \) holds true for any integer \( -1 \leq t \leq d \) and any parameter module \( N \) in \( F = A^r \) of any rank \( r > 0 \).

**Proof.** (1): Let \( N \) be a parameter module in \( F = A^r \), and let \( t \geq -1 \). By Theorem 3.2 we obtain that
\[
e(F/N) = \chi(M(N; t)) = \ell_A(H_0(K_\bullet(M(N; t) - \chi_0(K_\bullet(M(N; t)).
\]

Since \( \chi_0(K_\bullet(M(N; t)) \geq 0 \) by Theorem 3.3, the desired inequality follows.

2: Assume that \( A \) is Cohen-Macaulay. Let \( N \) be any parameter module in \( F = A^r \) of any rank \( r > 0 \). Let \( n = \mu_A(N) = d + r - 1 \). Then grade \( I(N) = \text{ht} I(N) = d = n - r + 1 \). Hence, by Proposition 2.12, \( K_\bullet(N; t) \) is acyclic for all \( -1 \leq t \leq n - r + 1 = d \). Therefore, by Theorem 3.2 we have \( e(F/N) = \chi(K_\bullet(N; t)) = \ell_A(H_0(K_\bullet(M(N; t))). \) This proves the implication (i) \( \Rightarrow \) (ii) and also the last assertion.

It remains to show the implication (ii) \( \Rightarrow \) (i). Assume that there exist integers \( r > 0 \), \( -1 \leq t \leq d \), and a parameter module \( N \) in \( F = A^r \) such that \( \ell_A(H_0(K_\bullet(M(N; t))) = e(F/N) \). Arguing as in the proof of Theorem 3.3 and using the same notation, we get
\[
\chi_1(K_\bullet(b) \otimes_B C_t(X)) = \chi_0(K_\bullet(M(N; t)) = \ell_A(H_0(K_\bullet(M(N; t))) - e(F/N) = 0.
\]

We observe here that \( \sqrt{\text{Ann}_B C_t(X)} = \sqrt{I_t(X)} \) (see [14, Lemma 2.7]). Thus \( \text{dim}_B C_t(X) = \text{dim} B/I_t(X) = d + (n + 1)(r - 1) = rn \) (see [2] (5.12), Corollary). Therefore \( b \) is a parameter ideal of \( C_t(X) \). Hence we have the equality
\[
\ell_B(C_t(X)/bC_t(X)) - e(b; C_t(X)) = \chi_1(K_\bullet(b) \otimes_B C_t(X)) = 0,
\]
where \( e(b; C_t(X)) \) is the multiplicity of the module \( C_t(X) \) with respect to an ideal \( b \). Since \( \ell_B(C_t(X)/bC_t(X)) = e(b; C_t(X)) \), this implies that \( C_t(X) \) is a Cohen-Macaulay \( B \)-module. On the other hand, \( \text{pd}_B C_t(X) = d \), because the complex
$K_\bullet(X; t)$ is a minimal $B$-free resolution of $C_t(X)$ of length $n - r + 1 = d$. Hence, by the Auslander-Buchsbaum formula, we have

\begin{align*}
  d + rn &= pd_B C_t(X) + \text{depth}_B C_t(X) \\
  &= \text{depth} B \\
  &\leq \dim B \\
  &= d + rn.
\end{align*}

Thus depth $B = \dim B$ so that $B$ is Cohen-Macaulay. Therefore $A$ is also a Cohen-Macaulay local ring. □

Taking $t = 0, 1$ in Theorem 4.1 now readily gives Theorem 1.3.

We want to close this article with a question. For that, let us first recall the notion of a Buchsbaum local ring, which was introduced by Stückrad and Vogel (for more details on Buchsbaum rings, we refer the reader to [13]). Let $A$ be a Noetherian local ring. Then $A$ is said to be a Buchsbaum ring if the difference

\[ \ell_A(A/Q) - e(A/Q) \]

between the colength and multiplicity of a parameter ideal $Q$ in $A$ is independent of the choice of $Q$. This difference, which is an invariant of a Buchsbaum ring $A$, is denoted by $I(A)$. The ring $A$ is Cohen-Macaulay if and only if it is Buchsbaum and $I(A) = 0$. In this sense, the notion of a Buchsbaum ring is a natural generalization of that of a Cohen-Macaulay ring. In Theorem 4.1, the inequality $\ell_A(F/N) \geq e(F/N)$, for any parameter module $N$ in $F$, is an analogue of the well-known inequality $\ell_A(A/Q) \geq e(A/Q)$ for any parameter ideal $Q$ in $A$. Also, the characterization of the Cohen-Macaulay property of $A$ based on the equality $\ell_A(F/N) = e(F/N)$ generalizes the usual one using parameter ideals. With these remarks in mind, it is natural to ask the following question:

**Question 4.2.** Let $F$ be a fixed free module of rank $r > 0$. Is it then true that the difference

\[ \ell_A(F/N) - e(F/N) \]

between the colength and multiplicity of a parameter module $N$ in $F$ is independent of the choice of $N$ if the ring $A$ is Buchsbaum?

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