Quadrature uncertainty and information entropy of quantum elliptical vortex states

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Abstract

We study the quadrature uncertainty of the quantum elliptical vortex state using the associated Wigner function. Deviations from the minimum uncertainty states were observed due to the absence of Gaussianity. We further observed that there exists an \textit{optimum value of ellipticity which gives rise to the maximum entanglement} of the two modes of the quantum elliptical vortex states. In our study of entropy, we noticed that with increasing vorticity, entropy increases for both the modes. A further increase in ellipticity reduces the entropy thereby resulting in a loss of information carrying capacity. We check the validity of the entropic inequality relations, namely the \textit{subadditivity} and the \textit{Araki–Lieb inequality}. The latter was satisfied only for a very small range of the ellipticity of the vortex, while the former seemed to be valid at all values.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Many of the fundamental concepts of quantum mechanics have been adapted from classical mechanics, albeit with some modifications. One such concept of particular importance is the phase space, which remains useful when applied to quantum mechanics. Similar to the probability density distribution functions in classical mechanics, quasi-probability distributions were introduced in quantum mechanics \cite{1}. Unlike the former, these can be negative which justifies the name quasi-probability distribution functions. Among them,
the Wigner function stands out because it is real, nonsingular and yields correct quantum-mechanical operator averages in terms of phase-space integrals and possesses positive definite marginal distributions [2]. The Wigner distribution function has come to play an ever increasing role in the description of both coherent and partially coherent beams and their passage through first-order systems [3]. Once the Wigner function is known, other properties of the system can be calculated from it.

Another concept, which quantum mechanics derives from classical mechanics, is that of entropy. It is a natural extension of the classical concept when dealing with quantum statistical mechanics. It is also a key concept in the field of quantum information theory. The entropy of quantum states, described in terms of density operators, replacing the classical probability theory, is defined by von Neumann entropy [4, 5]. It gives a measure of how much uncertainty there is in the state of a physical system and also gives an idea about the information carrying capability of a state, which is the aim of this study.

Optical vortices, possessing an orbital angular momentum, have been studied classically, by using a phase plate or a computer-generated hologram, for quite some time [6]. Agarwal et al gave the concept of the quantum optical vortex, which was circular in shape [7–9]. It was later generalized to the quantum elliptical vortex (QEV) [10]. In a recent work, it has been pointed out that photon subtraction from one of the spontaneous parametric down converted beams (idler) produces an elliptical vortex state [11]. It should be highlighted that the photon subtraction from spontaneous parametric down converted light has been realized experimentally as well [12, 13].

The paper is organized as follows. In section 2, we introduce the concept of QEV states in terms of basis vector states. We write the corresponding state for an elliptical vortex. We briefly discuss the Wigner distribution function for the generalized quantum vortex. Using the same function, we calculate the uncertainty products and discuss the results therein. We write down the corresponding reduced density matrices for QEV states in section 3 and study the entropy of the constituent modes. In section 4, we verify the validity of the entropic inequalities, namely subadditivity and the Araki–Lieb inequality for the QEV states. We conclude this paper after pointing out the significant results and directions for future work in section 5.

2. Uncertainties of the two modes of quantum elliptical vortex (QEV) states

Gaussian wave packets occupy a central place in studies involving wave packets of a quantum system. In the case of radiation fields, these packets play an important role as they are in fact minimum uncertainty states and describe both the coherent states [14] and the squeezed states [15]. The squeezed state $|\psi_i\rangle$ for the two-mode radiation field is defined as the direct product of the two squeezed mode states $|\psi_i\rangle_a$ and $|\psi_i\rangle_b$.

$$|\psi_i\rangle = |\psi_i\rangle_a |\psi_i\rangle_b = \exp[\zeta_i(a^\dagger^2 - a^2)]|0\rangle \exp[\zeta_i(b^\dagger^2 - b^2)]|0\rangle.$$  \hspace{1cm} (1)

Here $a$ and $b$ are the regular bosonic annihilation operators for the two modes and $\zeta_i$ are the squeezing parameters. Invoking the disentangling theorem [15], the exponential for one mode in equation (1) can be expressed as

$$\exp[\zeta_i(a^\dagger^2 - a^2)] = \exp \left[ \frac{\xi_i}{2} a^\dagger^2 \right] \exp \left\{ -\ln[cosh(2\zeta_i)] \left( a^\dagger a + \frac{1}{2} \right) \right\} \exp \left[ \frac{\xi_i}{2} a^2 \right],$$

$$\xi_i = \tanh(2\zeta_i),$$ \hspace{1cm} (2)
which is a product of exponentials. Using the fact that $a$ and $b$ acting on their respective vacuum states give zero, we can write the squeezed state for mode $a$ as

$$|\psi\rangle_0 = \frac{1}{\sqrt{\cosh(2\xi)}} \exp\left[\frac{\xi t}{2} d^2\right]|0\rangle.$$  

(3)

Following a similar approach, we can also write the corresponding expression for $|\psi\rangle_b$. The vortex state can now be written as

$$|\psi_{\text{QEV}}\rangle = A(n_1a^\dagger - in_2b^\dagger)^m|\psi\rangle,$$

(4)

where $A$ is the normalization constant, $|\psi\rangle$ represents the two squeezed mode vacuum. Moreover, as the two modes are entangled, our results show that the QEV state is not a ground state but an excited state unlike the Gaussian beam. Using the fact that

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is obtained as

$$\psi_{\text{QEV}}(x, y) \sim (n_1x - in_2y)^m \exp\left[-\frac{1}{2}\left\{\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2\right\}\right].$$

(5)

If we put $n_1 = n_2 = 1$, $\zeta = \zeta = \zeta$ (real), then it reduces to the circular vortex state in a Gaussian beam. Using $\eta = 1/\sqrt{2\sigma}$, the normalized spatial distribution of the QEV state [10] is obtained as

$$\Psi_{\text{QEV}}(x, y) = \frac{2^{m-1}}{\sigma_x \sigma_y \Gamma(m + \frac{1}{2})} \left\{\sqrt{x^2} \pm i \sqrt{2\sigma_x}\right\}^m \exp\left[-\frac{1}{2}\left\{\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2\right\}\right].$$

(6)

where $\sigma_i = \exp(2\zeta_i)$. As is clear from equation (6), the QEV states are non-Gaussian in structure.

We use $\sigma_x = \sqrt{3}\sigma_x$ or equivalently $\zeta = \frac{\ln5}{4} + \zeta_x$ arbitrarily and change the variables to a new set of scaled ones defined as $X_1 = \frac{x}{\sigma_x}, Y_1 = \frac{y}{\sigma_y}, X_2 = \frac{x}{\sigma_x}, Y_2 = \frac{y}{\sigma_y}$. $P_{X_1} = \frac{\sigma_x}{\sqrt{2}}p_x, P_{Y_1} = \frac{\sigma_y}{\sqrt{2}}p_y, P_{X_2} = \frac{\sigma_y}{\sqrt{2}}p_x, P_{Y_2} = \frac{\sigma_x}{\sqrt{2}}p_y$ [10]. Following the treatments of [16], the four-dimensional Wigner function for the state $\Psi_{\text{QEV}}(x, y)$ is obtained in a compact fashion as

$$W(x, y, p_x, p_y) = K \exp\left[-\left(X_1^2 + Y_1^2 + P_{X_1}^2 + P_{Y_1}^2\right)\right] L_m^{-1/2} \left[\frac{(P_{X_1} + P_{Y_1} - X_2 - Y_2)^2}{\sigma_x^2 + \sigma_y^2}\right],$$

(7)

where $K = \frac{2^{m-1}m!}{\pi^{3/4} \Gamma(m + \frac{1}{2})} \left[-2\left(\sigma_x^2 + \sigma_y^2\right)\right]^m$ and $L_m^{-1/2}$ is the associated Laguerre polynomial.

Using equation (7), we determine the uncertainty in $x, y, p_x$ and $p_y$. We study the uncertainties as a function of $\sigma_x$ for different vorticities $m$. It is seen from figure 1 that $\Delta x$ increases monotonically with increasing $\sigma_x$, whereas $\Delta p_x$ starts from infinity and falls to a much lower value with increasing $\sigma_x$. This is expected as $\Delta x$ varies linearly with $\sigma_x$ and $\Delta p_x$ varies as $1/(\Delta x)$ and hence $1/(\sigma_x)$. On the other hand, $\Delta y$ increases nonlinearly with $\sigma_x$ (figure 2). This is mainly because of the parametrization that we used for our calculations where $\Delta y$ varies as $\sqrt{3}\sigma_y$. $\Delta p_y$ on the other hand, starts from infinity and falls as $1/\sqrt{\sigma_y}$, thus preserving the Heisenberg uncertainty relation. It is further observed that at a particular value of $\sigma_x$ all the uncertainties are same. It is noticed from figure 3 that it is no longer the minimum uncertainty state because the QEV state is not a ground state but an excited state unlike the two squeezed mode vacuum. Moreover, as the two modes are entangled, our results show that when the uncertainty product of one mode is decreased, the other one is increased. Hence the
uncertainty product $\Delta x \Delta p_x$ decays to $1/\sqrt{2}$. $\Delta y \Delta p_y$ has an almost complementary nature to that of the former. It starts from around $1/\sqrt{2}$ and gradually increases to reach saturation at a much higher value. The maximum value attained by $\Delta x \Delta p_x$ is different than that attained by $\Delta y \Delta p_y$. The same holds true for the minimum values as well. However, it is observed that neither of the uncertainty products violates the uncertainty principle. In figure 4, we study the sum of the uncertainty products, $\Delta x \Delta p_x + \Delta y \Delta p_y$.

3. Reduced state for each mode and information entropy

In this section, we calculate the reduced density matrices of the two modes and calculate the corresponding entropies. We start with determining the expression for the density matrix of the QEV state $|\psi_{QEV}\rangle$. Using equations (3) and (4), one can write down the vortex state in an expanded form in the Fock state basis as
\[ |\psi_{QEV}\rangle = A (\eta_x a^\dagger \eta_y b^\dagger)^m |\psi\rangle_a |\psi\rangle_b \]
\[ = \frac{A}{\sqrt{\cosh(2\zeta_x) \cosh(2\zeta_y)}} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (-i\eta_x)^k (\eta_x^{m-k}) \exp \left( \frac{\xi_x}{2} a^\dagger^2 \right) \eta_x^{m-k} \exp \left( \frac{\xi_y}{2} b^\dagger^2 \right) |0\rangle_a |m-k\rangle_a |m-k\rangle_b. \]  

The density matrix \( \rho \) of the vortex state is
\[ \rho = |\psi_{QEV}\rangle \langle \psi_{QEV}|. \]  

Using the properties of the creation operators \( a^\dagger \) and \( b^\dagger \) and their conjugates, we can write down equation (9) in the following form:
\[ \rho = \frac{A^2}{\cosh(2\zeta_x) \cosh(2\zeta_y)} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (\eta_x^{2(m-k)}) (\eta_y^{2k}) \exp \left( \frac{\xi_x}{2} a^\dagger^2 \right) \exp \left( \frac{\xi_y}{2} b^\dagger^2 \right) |m-k\rangle_a |m-k\rangle_b |k\rangle_b |k\rangle_a. \]  

Figure 3. Uncertainty products of both the modes with \( \sigma_x \).

Figure 4. Sum of the uncertainty products with \( \sigma_x \).
To calculate the entropy for each mode, we need to obtain the corresponding reduced density matrices, which is defined by \( \rho_a \equiv \text{Tr}_b(\rho) \). In this context, we should mention that the reduced state is a mixed state even though the two-mode state of equation (1) is a pure state. Tracing out the \( b \) mode, we can write down \( \rho_a \) as:

\[
\rho_a = \frac{A_x^2}{\cosh(2\zeta_x)\cosh(2\zeta_y)} \sum_{k=0}^{m} \frac{m!^2}{k!(m-k)!} \eta_x^{2(m-k)} \eta_y^{2k} \exp \left( \frac{\xi_x}{2} a^2 \right) \langle m-k \rangle \langle m-k \rangle \|k\rangle \langle k\|
\]

where \( A_x \) is the normalization constant for mode \( a \). Solving the term in the square bracket and rearranging we obtain:

\[
\rho_a = \frac{A_x^2}{\cosh(2\zeta_x)\cosh(2\zeta_y)} \sum_{k=0}^{m} \frac{m!^2}{k!(m-k)!} \eta_x^{2(m-k)} \eta_y^{2k} \exp \left( \frac{\xi_x}{2} a^2 \right) \langle m-k \rangle \langle m-k \rangle \|k\rangle \langle k\|
\]

where \( \sum_{k=0}^{m} \frac{m!^2}{k!(m-k)!} \eta_x^{2(m-k)} \eta_y^{2k} \exp \left( \frac{\xi_x}{2} a^2 \right) \langle m-k \rangle \langle m-k \rangle \|k\rangle \langle k\| = \rho \)

On rearranging the terms, we finally arrive at the following form:

\[
\text{Tr} \rho_a = \sum_{k=0}^{m} C_k^a
\]

Using these coefficients, we calculate the von Neumann entropy of mode \( a \), following the treatment of [8] as follows:

\[
S_a = - \sum_{k=0}^{m} C_k^a \log_2 C_k^a
\]

where \( C_k^a \) stands for the coefficients in (14). The logarithm is taken to base 2 as is the norm for information entropy.
Following a similar approach, we can determine the von Neuman entropy for the $b$ mode. Repeating the procedures of equations (11) and (12), with the only difference being that we trace out the $a$ mode instead of $b$, we can write the reduced density matrix $\rho_b \equiv \text{Tr}_a(\rho)$ as

$$\rho_b = \frac{A_y^2}{\cosh(2\xi_x)\cosh(2\xi_y)} \sum_{k=0}^{m} \frac{m^{2k}}{k!(m-k)!} \eta_x^{2(m-k)}\eta_y^{2k} \exp \left( \frac{\xi_x^2}{2}b^2 \right) \exp \left( \frac{\xi_y^2}{2}b^2 \right) \sum_{k'=0}^{m} C_{k'}^b \frac{m^{2k'}}{(m-k')!(m-k)!} \xi_x^{(m-k)2} \xi_y^{2k'}$$

where $A_x$ is the corresponding normalization constant for mode $b$. Using equation (16), we determine the diagonal elements of the reduced density matrix for the $b$ mode as follows:

$$\text{Tr}\rho_b = \sum_{k=0}^{m} C_{k}^b = \sum_{k=0}^{m} \frac{A_y^2}{\cosh(2\xi_x)\cosh(2\xi_y)} \eta_x^{2(m-k)}\eta_y^{2k} \frac{m^{2k}}{(m-k)!k!} \sum_{k'=0}^{m} C_{k'}^b \frac{m^{2k'}}{(m-k')!(m-k)!} \xi_x^{(m-k)2} \xi_y^{2k'} F_1 \left[ \frac{m-k+1}{2}, \frac{m-k+2}{2}, 1, \xi_x^2 \right].$$

The corresponding entropy, $S_b$, can be calculated using equation (17) and the equivalent form of equation (15) by replacing $a$ with $b$.

We study the corresponding entropies with respect to $\eta_x$. We use $\sigma_x = 3$, $\sigma_y = 5$ in our calculations for generating the graphs. We use the same parametrizations for $\xi_x$ and $\xi_y$ as in section 2, where the subscript $i$ stands for $x$ and $y$. We choose $\eta_y = 1/(\sqrt{2}\eta_x)$ arbitrarily. The normalization constant is evaluated for the two modes as

$$A_y^2 = \frac{\gamma^{2-4i} a^{2(m-2k)}(1+\sigma_x)(1+\sigma_y)^1k!(m-k)!}{(1+\sigma_x^2)(1+\sigma_y^2)^k!(m-k)!},$$

where the subscript $i$ is used to denote the modes $a$ and $b$. $F_1$ represents the corresponding Hypergeometric function as expressed in equation (12) for mode $a$ and in equation (16) for mode $b$.

We have studied the entropies of the two modes as functions of $\eta_x$ in figures 5 and 6 for different $m$, the vorticity. We observed that the entropy increases with increasing vorticity. It
is also observed that the peaks occurred at almost a fixed value for each mode for all values of $m$, although it occurred for different values of $\eta_x$ for modes $a$ and $b$. This signifies that there exists an optimum value of $\eta_x$ for which maximum entanglement can be achieved for the QEV state. In other words, an optimum level of ellipticity exists for which we can obtain the maximum entanglement. We use the word entanglement as information entropy is a measure of the degree of entanglement of the constituent states which make up the system of states.

A further look at figures 5 and 6 reveals another very interesting fact. The maximum entropy occurs not at the value of $\eta_x = 1$, but at some other value. It is to be noted that $\eta_x = 1$ corresponds to the circular vortex, which has been a topic of study for a long time. This observation leads us to conclude that the elliptical vortex has more entropy than the circular vortex which means more information transfer is possible by elliptical vortices, choosing corresponding ellipticity, than the circular vortex. This observation further emphasizes the importance of the need to study elliptical vortices.

If the ellipticity is further increased, entropy falls off exponentially. Thus, we can conclude that increasing the elliptic nature of the vortex beyond a certain value would lead to disentanglement and hence loss of information carrying capacity.

4. Entropic inequalities and their validity for QEV states

In the previous section, we calculated the reduced density matrices for the two modes using the partial trace operation and also calculated the corresponding entropies. In this section, we check the validity of the entropic uncertainty relations for the QEV states.

If two systems $a$ and $b$ have a joint quantum state $\rho_{ab}$, then the entropy of the combined states is expected to satisfy the following inequalities [4]:

$$S_{ab} \leq S(a) + S(b)$$  \hspace{1cm} (19)

$$S_{ab} \geq |S(a) - S(b)|.$$  \hspace{1cm} (20)

We adapt these equations to suit our system where these are equally valid as the subsystems of the QEV state, $\rho_a$ and $\rho_b$ are the distinct quantum states, though they are correlated. So we expect the subadditivity, equation (19), to hold with the inequality. The second equation,
i.e. equation (20), is the Araki–Lieb inequality. It is generally satisfied for the von Neumann entropy. For a pure state, it signifies that the entropy is cancelled only by an equal amount of entropy. But for the mixed states, $S_{ab} > 0$, so the entropies of the subsystems do not cancel each other completely. However, it is generally expected to be fulfilled for the mixed states as well. This inequality also has some important implications for the index of correlation [18]. The index of correlation, $I_c$, is a measure of the information content of the correlation between the components of an $N$-component system. For a two-component system, it can be expressed in a simplified form:

$$ I_c = S_a + S_b - S_{ab}, \quad (21) $$

where $S_{ab}$ stands for the entropy of the combined system. If the two-component system is in a pure state, it is shown that the maximum correlation occurs [19, 20]. The Araki–Lieb inequality serves an important purpose here by limiting the maximum value that $I_c$ can take (equation (21)), thereby limiting the information content of the correlation of the components.

To verify these inequalities, we need to determine the entropy of the combined system, that is, both the modes taken together. In section 3, we have written the density matrix of the QEV state in equation (10). We use it to calculate the von Neumann entropy of the state. Proceeding as we did to determine the trace of the reduced density matrices, we can find the trace of the entire system as follows:

$$ \text{Tr} \rho = \sum_{k=0}^{m} C_k^{ab} = \frac{A^2}{\cosh(2\xi_\alpha) \cosh(2\xi_\beta)} \sum_{k=0}^{m} \frac{m!^2}{(m-k)!k!} \eta_x^{2(m-k)} \eta_y^{2k}. \quad (22) $$

Equation (22) is used to determine the entropy of the combined system, i.e. $S_{ab}$. We use the same parametrization as in the previous section to study them in figures 7–9. It is observed that $S_a + S_b$ starts from zero and increases to reach saturation or a plateau region and falls off to zero. The joint entropy $S_{ab}$, on the other hand, increases with increasing $\eta_x$ until it reaches a maximum. Then it starts decreasing gradually to reach zero. $|S_a - S_b|$ exhibits a strikingly different nature from the other two. It has a couple of peaks with the same maximum
value, while the other two attain the maximum value only once. $|S_a - S_b|$ starts from zero and increases to attain the maximum. After which it falls off to zero before rising again to the same maximum value for a different value of $\eta_x$. Then it gradually falls off to zero like $S_a + S_b$ and $S_{ab}$. The occurrence of the zero at the middle can be explained very easily. It is clear from figures 5 and 6 that both $S_a$ and $S_b$ attain the same value for that particular value of $\eta_x$ due to which $|S_a - S_b|$ goes to zero.

A further look at figures 7–9 enables us to infer that the entropies satisfy the subadditivity, equation (19), for all values of $\eta_x$ but not with an equality as is expected for correlated systems. This signifies that equation (19) holds for QEV states. This was also expected as the entropy for an entangled state should be less than that of the summation of the entropies of the constituent systems. On the other hand, the Araki–Lieb inequality, equation (20), was satisfied only in a very small range of values of $\eta_x$. For all other values, it is violated. We can argue that, since the vortex is bounded by a sharply peaked Gaussian distribution, it is in this region that equation
(20) is violated as verified by Keitel and Wodkiewicz [21]. More importantly, it is not fulfilled completely mainly because the correlation is present between the two subsystems. As the subsystems A and B are entangled, the Araki–Lieb inequality is not fully fulfilled. Since \( \eta_x \) controls the ellipticity of the vortex, it is observed that the Araki–Lieb inequality is valid only for a very short range of ellipticity. We thus state that the subsystems of the QEV state exhibit an optimum level of entanglement only in a limited range of the ellipticity of the QEV state where both the inequalities hold together. As the ellipticity increases, \( S_{\text{ent}} \) also increases and attains a maximum value where the inequality holds, but as the ellipticity increases further the combined entropy falls off and the inequality is violated. This has some direct consequences which should be further investigated from a quantum information theoretical point of view to achieve the maximum entanglement for this class of states.

5. Conclusion

In this paper, we have calculated the uncertainty products using the Wigner function of the QEV state. We noticed that the uncertainty product \( \Delta x \Delta p_x \) attains a minimum value of \( 1/\sqrt{2} \). It has a maximum value of about 1.25. \( \Delta y \Delta p_y \), on the other hand, has an initial value of about \( 1/\sqrt{2} \). It starts increasing gradually and saturates at nearly 1.2. It can be argued that the presence of the vortex modifies the characteristics of this state and hence it is no longer the minimum uncertainty state which it would have been if the vortex was not present.

We have studied the von Neumann entropy of the QEV states in terms of basis vector states. We found that the entropy was raised considerably with the increase in the vorticity of the states. It was noticed that the peaks for both the modes occur at the different values of \( \eta_x \), where \( \eta_x \) is a measure of the ellipticity of the vortex. But the peak value of the entropies for the two modes remained the same. It was further observed that there exists an optimum value of ellipticity which gives rise to the maximum entanglement of the two modes of the QEV states. A further increase in ellipticity reduces the entropy thereby resulting in a loss of information carrying capacity.

We checked and verified the entropic inequalities. We observed that the strong subadditivity was satisfied for all conditions. This is expected for any entangled state, as the combined entropy for such state would be always less than that of the constituent systems taken together. We noticed that the Araki–Lieb inequality, an indicator of the degree of entanglement, was violated in all the regions, except in a very narrow region of ellipticity values.

Our results serve as a pointer for further investigation and studies of quantum elliptical vortices as a means of information transport. The study of decoherence in this system needs to be pursued in order to find out the robustness of this correlated system. An investigation of quantum discord in this case will also be exciting from both the fundamental and application points of view.

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