Fermion parity measurement and control in Majorana circuit quantum electrodynamics

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Combining superconducting qubits with mesoscopic devices that carry topological states of matter may lead to compact and improved qubit devices with properties useful for fault-tolerant quantum computation. Recently, a charge qubit device based on a topological superconductor circuit has been introduced where signatures of Majorana fermions could be detected spectroscopically in the transmon regime. This device stores quantum information in coherent superpositions of fermion parity states originating from the Majorana fermions, generating a highly isolated qubit whose coherence time could be greatly enhanced. We extended the conventional semi-classical method and obtained analytical derivations for strong transmon-photon coupling. The analytical challenge is rendered tractable via a formalism based on the WKB method that allows to extract the energy eigenstates of the qubit and its dipole matrix elements. Using this formalism, we study the effect of the Majorana fermions on the quantum electrodynamics of the device embedded within an optical cavity and develop protocols to initialise, control and measure the parity states. We show that, remarkably, the parity eigenvalue can be detected via dispersive shifts of the optical cavity in the strong coupling regime and its state can be coherently manipulated via a second order sideband transition.

INTRODUCTION

Advances in quantum information technology occurring over the past decade have given rise to a new generation of single qubit solid-state architectures which hold the promise of compatibility with existing electronics and fabrication techniques. Among present generation devices, superconducting circuit processors based on the transmon qubit [1–3] have shown great promise in terms of coherent control and measurement and constitute some of the leading designs in the effort to scale up these processors. These provide a unique opportunity to study fundamental quantum phenomena in an engineered macroscopic devices bearing two-level systems which are controlled by coherent microwave photons. Of particular interest now is the study of hybrid devices where a microscopic or a mesoscopic system is embedded within the superconducting circuit. The combined properties of the constituent devices can contribute to the optimization of the qubit, including the processes related to its preparation, manipulation and readout, to its coherence properties, and to its prospects for scaling up.

Recently it was suggested that integrating Majorana fermions [4, 5] into the superconducting circuit architecture [6–13] can potentially lead to improved qubits, with the ultimate goal being the realization of a topologically protected information storage and processing device [6, 7]. Such devices may also show reduced decoherence and may have better prospects for scalability. The recently introduced majorana-transmon (MT) device [13] sacrifices full topological protection by directly exploiting a weak interaction between two neighboring Majorana fermions, but gains a highly anharmonic spectrum, composed of well-separated nearly degenerate doublets. In order to facilitate both detection and control the circuit should be embedded within an optical cavity [14–19] where the strong interaction with the cavity field will provide the means for coherent qubit control and readout. In the proposed device the lowest doublet of states is analogous to the familiar ions hyperfine qubit [20] and does not couple directly to the cavity and the radiative envi-

FIG. 1. Majorana-Transmon circuit in a cavity. A topological superconductor (orange) bridges a Josephson junction nucleating Majorana fermions (yellow). Control gates (light blue and green) are used to drive the topological state and control the Majorana coupling. Together with its superconducting electrodes, this device is embedded inside a coplanar transmission line resonator. Microwave pulses on the input port allow logical gates realizations, while the output port can be used for homodyne or heterodyne detection of the fermion parity state.
environment owing to both a small matrix element and its small frequency. The consequence is an increased qubit lifetime and therefore potentially very high readout and control fidelities.

In this article we develop the hybrid circuit quantum electrodynamics of the MT system strongly coupled to a single mode electromagnetic field (see Fig. 1) and provide a scheme for readout and control. We dub this scheme Majorana-circuit QED (McQED). First, we use the WKB formalism to gain an analytical handle that allows further analysis, by providing semi-classical expressions for the eigenstates (Eq. 3) and spectrum (Eq. 4) of the device. We can then use these to provide analytical expressions for the dipole matrix elements (Eq. 5) demonstrating that the doublet forming the logical qubit is coupled to a higher doublet which can serve as a control. When the microwave transitions of the MT device are detuned from the cavity resonance we show that a dispersive interaction arises between the three degrees of freedom, namely the fermionic parity, the transmon oscillator, and the cavity photon. We use this regime to propose a scheme for measuring the state of the qubit, which is revealed via a fine-structure in the cavity dispersive frequency shifts that are, remarkably, sensitive to the different fermionic parities associated with the two levels. In addition, we discuss protocols for: i), qubit cooling; and, ii), implementing a single qubit rotation, demonstrating qubit control. Together, these provide the minimal ingredients required for establishing the relevance of the device for quantum information processing [21].

RESULTS

Eigenstates of the Majorana-transmon device

The system we study consists of a Josephson junction capacitively coupled to a gate voltage source creating an offset charge $n_g$ on the superconducting electrode. A nano-wire which can support majorana fermions [22, 23] is placed along the junction. Another realization could be based on the recent discovery of Majorana fermions in a chain of magnetic impurities [24], where here the chain should cross a Josephson junction. The zero energy fermions composed of these Majorana fermions allow the relative number of cooper pairs, $n = -i\partial\phi$, to admit both integer and half-integer values [25]. Such a set up can be described by the model Hamiltonian

$$H = H_T \mathbf{1} + H_M \tau_x,$$

where $\tau_i$ ($i = x, y, z$) are Pauli matrices. The first term in $H_T$, containing the transmon Hamiltonian $H_T = -4E_C \partial^2 \phi - E_J \cos(\phi)$, has the eigenstates $|k\rangle_e = (f_k(\phi), 0)^T$, $|k\rangle_o = (0, g_k(\phi))^T$ which correspond to an even and odd fermionic parity. These wave functions obey the boundary conditions $f_k(\phi + 2\pi) = e^{-2\pi i n_y} f_k(\phi)$, $g_k(\phi + 2\pi) = e^{-2\pi i (n_y + 1/2)} g_k(\phi)$, which were chosen in such a way as to ensure the discreteness of the charge. Since the transmon operates in the regime $E_J/E_C \gg 1$, there is a close resemblance to an anharmonic oscillator which suggests a semi-classical derivation of its eigenvalues and eigenstates. The second term $H_M = E_M \cos(\phi/2)$ describes the interaction between the adjacent majorona fermions [19, 20, 27] and creates a condensate “parity flip” accompanying a single electron tunnelling process across the junction.

We proceed with the diagonalization of the hamiltonian and show that it can be approximately divided into independent bands, dominated by the transmon energy with a small splitting due to the majorana interaction. We define the overlap between the odd and even states due to $H_M$

$$m_{kk'} = E_M \int_{-\infty}^{\infty} \Psi_k(\phi) \Psi_{k'}(\phi) \cos(\phi/2) d\phi, \quad (1)$$

where we used the harmonic oscillator states $\Psi_k(\phi)$ as an approximation for $f_k(\phi)$ and $g_k(\phi)$, as described in the Methods section. The intra-band coupling is independent of the specific band and it is dominated by the interaction energy, $m_{kk} \simeq E_M$ to a leading order. The coupling between the bands decreases as $m_{kk'} \sim (E_C/E_J)^{|k-k'|/2}$, for even $|k-k'|$, and vanishes for odd $|k-k'|$. By neglecting terms of order $\sqrt{E_C/E_J}$ and higher, we can write the hamiltonian matrix as a sum of independent blocks $H = \bigoplus_{k=0}^{\infty} H^{(k)}$. In the $|k\rangle_e, |k\rangle_o$ basis these blocks take the form

$$H^{(k)} = \begin{pmatrix} \epsilon_k + t_k \cos(2\pi n_g) & E_M \\ E_M & \epsilon_k - t_k \cos(2\pi n_g) \end{pmatrix}, \quad (2)$$

where $\epsilon_k$ are the energies of the harmonic oscillator with first order anharmonic correction and $t_k$ is the transmon dispersion, obtained using the WKB approximation in the Methods section and defined in Eq. 30. The matrix can be diagonalized by a rotation around the $y$-axis $U = e^{i\eta_k \tau_y}$, $H^{(k)} \rightarrow H'^{(k)} = U H^{(k)} U^\dagger$, where $\eta_k = \frac{(-1)^{k+1}}{2} \text{atan} \left[ \frac{m_{kk}}{E_M (E_M - (-1)^{k+1} t_k \cos(2\pi n_g))} \right]$ and $\text{atan} x \equiv 2 \tan^{-1} \left[ \frac{y}{\sqrt{x^2 + y^2} + x} \right]$ is the quadrant dependent arctangent. The eigenvectors of Eq. 2 are

$$|k, -\rangle = \cos(\eta_k) |k\rangle_e + \sin(\eta_k) |k\rangle_o,$$

$$|k, +\rangle = -\sin(\eta_k) |k\rangle_e + \cos(\eta_k) |k\rangle_o, \quad (3)$$

and have corresponding eigenvalues

$$E_{k, \pm} = \epsilon_k \pm (-1)^{k} \sqrt{E_M^2 + t_k^2 \cos^2(2\pi n_g)}. \quad (4)$$

Some physical intuition can be gained by preparing the system in a given parity state. In addition to the quantum fluctuations of the relative number of cooper pairs in the transmon, a single electron will tunnel back and forth between the two superconducting electrodes. Such
a behaviour can be tuned by changing the offset charge \( n_0 \) or modifying the ratio \( E_J/E_C \).

As opposed to the transmon where the dependence on \( n_0 \) is exponentially suppressed in the form of a charge dispersion \([11]\), the MT eigenstates are extremely susceptible to changes in \( n_0 \). This dependence leads to parity based interference effect in the dipole transitions which can be controlled via \( n_0 \). This can be exploited by coupling the system to a photon cavity as we now proceed to discuss.

**Generalized Jaynes-Cummings Model**

We now couple the MT to a single mode of a quantized electromagnetic field operating in the microwave range, confined within a cavity consisting of a transmission line resonator \([17, 28]\). The full quantum description of this system is given by a generalized Jaynes-Cummings (JC) hamiltonian \([1, 14]\) (\( h = 1 \))

\[
H_{JC} = (H_T + \omega_c a^\dagger a) \mathbb{1} + H_M \tau_s + \hat{G} (a^\dagger + a),
\]

where \( \omega_c \) is the frequency of the photons created (annihilated) by the operator \( a^\dagger \) (\( a \)). The interaction between the MT and the cavity is achieved via the dipole coupling \( \hat{G} = g (i\partial_x - n_0) \mathbb{1} \), where \( g = 2e\beta \xi_{rms} d \) is the dipole coupling strength, \( d \) is the distance between the superconducting electrodes, \( \xi_{rms} \) is the rms field at the ground state of the resonator, and \( \beta \) is the ratio between the gate capacitance and the capacitance of the junction.

We employ the rotating wave approximation (RWA) by keeping in the interaction term only the transitions that include excitation of the MT with a de-excitation of the cavity, and vice-versa. In the harmonic oscillator approximation, the parity of the wave functions ensures that the dipole transitions are non zero only between neighboring bands. Projecting Eq. (5) on the eigenstates given in Eq. (3) the hamiltonian takes the form

\[
H_{JC} = \sum_{k,s} E_{k,s} |k,s\rangle \langle k,s| + \omega_c a^\dagger a
+ \left( \sum_{kss'} \hat{G}_{k,s;k+1,s'} |k,s\rangle \langle k+1,s'|a^\dagger + \text{h.c.} \right),
\]

where \( \hat{G}_{k,s;k+1,s'} = |k,s\rangle \langle k+1,s'| \) and \( s, s' = \pm \) designate the parity of the state according to Eq. (3). We proceed to analyze the hamiltonian, Eq. (6), in the combined qubit–cavity basis \( |k, s; N\rangle \), where \( N = |a^\dagger a\rangle \) is the number of photons in the cavity. The states \( |0, s; 0\rangle \) are completely isolated from transitions to other levels, and within the harmonic approximation, transitions between the two states \( |0, +; 0\rangle \equiv |0, -; 0\rangle \) are also neglected. This creates a two-level system we call the “qubit”, given by \( H_q = \text{diag} (E_{0,-}, E_{0,+}) \). For a non-zero photon number in the cavity the qubit states are dressed, to first order in the dipole coupling, only with the first excited transmon level \( |1, s; N\rangle \) (\( N \geq 1 \)):

\[
H_N = \begin{pmatrix}
E_{1,+;N-1} & 0 & -\sqrt{N}G_x^* & \sqrt{N}G_o^* \\
0 & E_{1,-;N-1} & \sqrt{N}G_o^* & \sqrt{N}G_x^* \\
-\sqrt{N}G_x & \sqrt{N}G_o & E_{0,-;N} & 0 \\
\sqrt{N}G_o & \sqrt{N}G_x & 0 & E_{0,+;N}
\end{pmatrix},
\]

where \( E_{k,s;N} = E_{k,s} + N \omega_c \). The relevant dipole matrix elements are

\[
G_o = G_T \cos(\eta_1 - \eta_0), \quad G_x = G_T \sin(\eta_1 - \eta_0),
\]

where \( G_T \) is the dipole transition associated with the transmon

\[
G_T = ig \int_{-\infty}^{\infty} \Psi_0(\varphi) \Psi_1(\varphi) d\varphi = ig \left( \frac{E_I}{32E_C} \right)^{1/4}.
\]

These results are compared against numerics in Fig. 2.

As demonstrated in the figure, the addition of anharmonic corrections to \( G_T \) further improves the agreement with numerics, so one can replace \( G_T \rightarrow G_T^{\text{ah}} \) in Eq. (8), where

\[
G_T^{\text{ah}} = G_T \left( 1 - \sqrt{\frac{E_C}{32E_J}} + \frac{15E_C}{256E_J} \right).
\]

**Spectroscopy and parity state detection**

By coupling the MT to a high-Q superconducting resonator and tuning it to the strong dispersive regime \([28, 33]\) it would be possible to, (1), probe the telling features of the spectrum \([13]\) and to verify its dependence on the
parameters $n_g$ and $E_J$ that was discussed in the previous sections; and, (2), extend the applicability of the dispersive readout to perform a dispersive quantum state measurement of the parity doublet qubit. Experimentally these would involve a homodyne measurement setup for detecting changes in the complex amplitude of a coherent microwave tone which is transmitted at the resonance of the cavity, while spectroscopic pulses are driving the MT resonances.

Spectroscopic measurement of the qubit’s transition frequency relies on the existence of a dispersive cavity frequency shift which itself depends on the state of the qubit [24]. When a spectroscopic pulse excites the qubit the transition through the cavity resonance is diminished, indicating the transition frequency. The pulse is emitted from a dedicated transmission line terminating in the vicinity of the qubit [35]. The hamiltonian describing the qubit drive is

$$H_{D}^{\text{dir}} = 2\epsilon c \mathcal{E}_{\text{dir}}(t) \mathbf{L}_c \sum_{s} \left( \langle 0, s | i \partial_{\phi} | 1, s \rangle | 0, s \rangle \langle 1, s | + h.c. \right)$$

(11)

where $\mathcal{E}_{\text{dir}}(t)$ is the strength of the electric field at the vicinity of the transmon and $\mathbf{L}_c$ is the cavity identity operator. When the qubit is coupled to the cavity in the strongly dispersive regime, the final states are approximately the dressed states. Usually in the strong coupling regime the qubit is driven via the cavity in which case the strength of the drive will depend on the degree of mixing of the excited MT state $| 1, s; 0 \rangle$, dressed by the cavity-qubit interaction given by Eq. (7), with the bare photon state $| 1, s; 0 \rangle$. The drive Hamiltonian can be written approximately in terms of the itinerant electric field at the port of the resonator with the drive amplitude $\xi(t)$

$$H_{D}^{\text{cav}} = \xi(t) \sum_{s} \left( \frac{g_{s}}{\Delta} | 0, s; 0 \rangle \langle 1, s; 0 | + h.c. \right)$$

(12)

where $\Delta = \omega_c - (E_{1,+} - E_{0,-})$ is the cavity-qubit detuning. To first order in $| g_{o,x} |$ this scaling does not depend on $E_M$ and is similar to the transmon-cavity case. The general dependence on the small parameters $| g_{o,x} | / \Delta$ and $E_M / \Delta$ is different with higher order terms depending on $E_M$. Therefore in the first order this form of the drive would evidently lead to a qualitatively similar spectroscopic pattern of the transition strengths on $n_g$.

Returning to the system hamiltonian, Eq. (4), and assuming that $E_M$ dominates over the transmon dispersion then the closed part of the Jaynes-Cummings-Majorana hamiltonian (i.e., without drive) in the $\ell$th excitation sector can be diagonalized to give $H_{\ell} = \omega_{\ell} - \sigma_s \sqrt{\Delta^2} + \mathbf{g}_{\ell}^2 + h_M$ with $h_M = E_M^2 + E_M \tau \sigma_z \sqrt{\Delta^2} + 4 \mathbf{g}_{\ell}^2 \ell$, where $\tau_{\ell}$ and $\sigma_z$ operate in the parity ($s = \pm$) and the dressed transmon ($k = 0, 1$) degrees of freedom respectively. In the dispersive regime we expand in the small parameters $| g_{o,x} | / \Delta$, $E_M / \Delta$ and substitute $\ell \rightarrow a^\dagger a + \frac{1+\epsilon_\ell}{2}$ which gives us the the effective

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**FIG. 3. Qubit initialization and coherent control in the strong-dispersive regime.** (a) Qubit initialization: Numerical calculation of the four levels population during the thinking process. A microwave pump with amplitude $\xi_{p} / h = 10^{-3} \text{ GHz}$ and frequency $\omega / 2\pi = 5.34 \text{ GHz}$ is admitted to the system with a preliminary temperature of 0.03 K. The decay rates for parity-conserving transitions are:

$$\gamma_0 = \gamma | \cos(\eta_1 - \eta_0)|^2$$

and sub-

$$\eta_0 = \gamma | \sin(\eta_1 - \eta_0)|^2$$

where $\gamma / 2\pi = 10^{-3} \text{ GHz}$ (Fermi-

$$\xi_0 = \gamma | \sin(\eta_1 - \eta_0)|^2$$

and detuning $\gamma_0 = \gamma | \cos(\eta_1 - \eta_0)|^2$ (neglecting the small energy difference between the two transitions). For parity-flipping transitions $\gamma_0 = \gamma | \sin(\eta_1 - \eta_0)|^2$, where $\gamma / 2\pi = 10^{-3} \text{ GHz}$ (Fermi-Golden-Rule/Classical approximation). For $E_M = 0.025 E_C$, $E_J = 25 E_C$, $E_C / h = 0.4 \text{ GHz}$, $n_g = 0$, $g / \Delta = 0.3$ (b) Coherent control: The system prepared in its ground state and driven via two coherent microwave tones using a rectangular pulse with amplitude $\xi / h = 8 \times 10^{-4} \text{ GHz}$ and detuning $\delta = E_M$, resulting in transfer between the lower levels with a minimal higher level population. (c) Level diagram: The qubit-cavity interaction (left) produces a shifted resonance frequency $\omega' = \omega_c + \chi_M$ accompanied by a parity based shift $\pm \chi_M$. A process of coherent control (middle) in the double $\Lambda$ system, is preceded by the cooling process (right).
Hamiltonian

\[ H_{\text{eff}} = \left( \frac{\omega_c}{2} - \frac{\Delta}{2} - E_M \tau_z \right) \sigma_z + \frac{1 + \sigma_z}{2} \left( -\chi_T + \chi_M \tau_z \right) + a^\dagger a \left( \omega_c - \chi_T \sigma_z + \chi_M \sigma_z \tau_z \right). \]  

(13)

The dispersive shift \( \chi_T = \left| G_T \right|^2 / \Delta \), characteristic to the transmon, is accompanied by an additional higher order shift \( \chi_M = 2E_M |G_\alpha|^2 / \Delta^2 \) which couples dispersively the photon, transmon and parity degrees of freedom. Focusing on the \( k = 0 \) sector, we obtain the hamiltonian for the qubit with the parity dependent dispersive interaction

\[ H_{\text{eff},q} = E_M \tau_z + a^\dagger a \left( \omega_c + \chi_T - \chi_M \tau_z \right). \]  

(14)

We realize that the cavity resonance frequency of \( 0 \to 1 \) photon transition is dependent on the parity of the ground state. This interaction, which scales as \( \chi_M \), can be used in a homodyne measurement setup to determine which of the two ground states the MT occupies [34, 36–39], opening the way to using this pair of states as a qubit.

The signal-to-noise ratio of such a readout scheme depends also on the strength of the decoherence processes, the scale of \( E_M \) and the quality factor of the cavity.

**Qubit initialization and control**

In this section we provide protocols for qubit initialization (cooling) and control. We take the logical qubit to be the two-level system \( H_q \) (which lacks direct dipole couplings between its two states), and the lowest dressed doublet \( |\Gamma, s; 0 \rangle \) of \( H_1 \) as the control, forming together a double \( \Lambda \) system.

Due to the small energy splitting of the two-level system forming the qubit, the equilibrium thermal state of the system generically mixes the two levels. To prepare the qubit in a pure state a cooling procedure should initially be performed. This involves an external drive operating at the frequency \( \omega_{ss'} \), where \( \omega_{ss'} \) corresponds to \( |\Gamma, s; 0 \rangle \leftrightarrow |0, s'; 0 \rangle \) transitions. Photonic decay channels exist between the control states and the qubit states. The system then relaxes to the ground state as depicted in Fig. 3.

Following the initialization to the pure state \( |0, -; 0 \rangle \), single qubit quantum gates can be performed on the qubit. To demonstrate a simple gate operation we focus here on a population flip, taking \( |0, -; 0 \rangle \to |0, +; 0 \rangle \). The trick is to use a two-tone microwave photon drive operating in the frequencies \( \omega_{+-} - \delta \) and \( \omega_{-+} - \delta \) sharing the same detuning \( \delta \), fixed between the control doublet levels. This results in a coherent population transfer between the states of the qubit, see Fig. 3. The transition is a coherent evolution where a dynamical phase is accumulated, which can be described as a combination of \( R_x \) and \( R_z \) rotations.

**DISCUSSION**

In summary, we have extended the semi-classical analysis of the transmon to obtain the optical properties of an embedded topological superconductor. The analytical results obtained using these methods allowed us to explore the potential use of the fermionic parity as part of the coherent system potentially realising a high-coherence qubit. The anharmonic level structure and selective coupling to the cavity via the transmon excited states are favourable properties for quantum information processing. In this vein we have shown how the qubit can be optically initialised and controlled. We see that the zero energy modes arising from the topological properties have the potential of altering the properties of the hosting superconducting circuit. Remarkably, the quantum information hidden in the parity state can still be accessed via its imprint in the dispersive shifts of the cavity manifesting that all the main ingredients for a qubit realisation exist in this design.

**METHODS**

In this section we provide an asymptotic solution, based on the WKB method [40–42], to the equation

\[ H_T f(\varphi) = E f(\varphi) \]  

with the boundary condition \( f(\varphi + 2\pi) = e^{i\theta} f(\varphi), (\theta \in \mathbb{R}), \)

\[ f''(\varphi) + \left( \frac{E}{4E_C} + \frac{E_J}{4E_C} \cos(\varphi) \right) f(\varphi) = 0. \]  

(15)

We are interested in the transmon regime \( E_J / E_C \gg 1 \) where the fluctuations of \( \varphi \) are mostly localized around \( \varphi = 0 \) and the energy has a small deviation \( \delta E_k \) from the harmonic oscillator values: \( E_k = -E_J + \sqrt{8E_CE_J} (k + 1/2) + \delta E_k \), where \( k = 0, 1, 2, \ldots \) and \( \delta E_k \ll \sqrt{8E_CE_J} \). Inserting this expression into Eq. (15) and rearranging we get

\[ f''(\varphi) + \sqrt{\frac{E_J}{2E_C}} \left( \nu + 1 - \sqrt{\frac{E_J}{2E_C}} \sin^2(\varphi/2) \right) f(\varphi) = 0, \]  

(16)

where \( \nu = k + \delta E_k / \sqrt{8E_CE_J} \). As \( E_J \) increases \( \nu \) approaches an integer value.

In deriving the solution we focus for convenience on the domain \( -\pi < \varphi < \pi \), which contains two barriers centered at \( \varphi = \pm \pi \). To describe tunnelling through the barriers, the wave function should be a linear combination of two independent functions, one exponentially increasing and one exponentially decaying. For these we assume the form \( \phi_{\pm}(\varphi) = A_{\pm}(\varphi) e^{\mp S(\varphi)} \), where \( S(\varphi) \) is the action through the barrier, for which we take the ansatz \( S(\varphi) = \sqrt{\frac{2E_J}{E_C}} \cos(\varphi/2) \). Inserting \( \phi_{\pm}(\varphi) \) into Eq. (16) and neglecting terms of order \( \sim \sqrt{E_C/E_J} \), we obtain a
first order equation

\[ A_\pm (\varphi) \cos (\varphi / 2) \mp (2\nu + 1) + 4A_\pm' (\varphi) \sin (\varphi / 2) = 0, \]  

which is readily solved to find the two independent solutions

\[ \phi_\pm (\varphi) = \frac{\tan (\varphi / 4)^{\pm(\nu+1)/2}}{\sqrt{\sin (\varphi / 2)}} e^{\pm \sqrt{2EJ/E_C} \cos (\varphi / 2)}. \]  

These are valid mainly close to \( \varphi = \pm \pi \). Close to \( \varphi = 0 \) we can rewrite Eq. (16) approximately as

\[ \frac{d^2 f(z)}{dz^2} + \left( \nu + \frac{1}{2} - \frac{z^2}{4} \right) f(z) = 0, \]  

where \( z = (EJ/2E_C)^{1/4} \). Eq. (19) is the Weber equation, which has two independent solutions \( D_\nu(z) \) and \( D_{-\nu+1}(iz) \), the Parabolic Cylinder Functions (following the notation of Abramowitz and Stegun [43]). For positive integer \( \nu \) this equation simply describes the harmonic oscillator.

We can completely satisfy the solution in the domain \(-\pi < \varphi < \pi\) using the three functions

\[ f_L(\varphi) = A_L \phi_+(\varphi) + B_L \phi_-(\varphi), \]
\[ f_M(\varphi) = A_M \Psi(\varphi) + B_M \Omega(\varphi), \]
\[ f_R(\varphi) = A_R \phi_+(\varphi) + B_R \phi_-(\varphi), \]  

where \( \Psi(\varphi) = D_\nu(z) \) and \( \Omega(\varphi) = D_{-\nu+1}(iz) \) (for simplicity we omit the band index \( \nu \) from the basis functions). Here \( f_L \) (\( f_R \)) is the solution to the left (right) of \( \varphi = 0 \) and \( f_M \) is valid in the region close to \( \varphi = 0 \). In order to impose the boundary condition on Eq. (20) we need to represent the coefficients of \( f_R \) as a linear combination of the coefficients of \( f_L \). This is achieved by comparing \( \phi_\pm(\varphi) \) to \( \Psi(\varphi) \) and \( \Omega(\varphi) \) in their common region of validity.

Using an asymptotic approximation for the parabolic cylinder functions [43], for any \( \nu \) and \( z \gg 1 \) we obtain the form

\[ \Psi(\varphi) \simeq \left( \frac{EJ}{2E_C} \right)^{\nu/2} e^{-\sqrt{8EJ/E_C} \varphi^2}, \]
\[ \Omega(\varphi) \simeq e^{-i\frac{\pi}{2}(\nu+1)} \left( \frac{EJ}{2E_C} \right)^{(\nu+1)/2} \left( \frac{EJ}{2E_C} \right)^{-\nu} e^{-\sqrt{8EJ/E_C} \varphi^2}. \]  

In addition, Eq. (18) takes the approximate form near \( \varphi = 0 \),

\[ \phi_+(\varphi) \simeq 2^{-2(\nu+\frac{1}{2})} e^{\sqrt{8EJ/E_C} \varphi^2} e^{-\sqrt{8EJ/E_C} \varphi^2}, \]
\[ \phi_-(\varphi) \simeq 2^{2(\nu+\frac{1}{2})} e^{-\sqrt{8EJ/E_C} \varphi^2} e^{-\sqrt{8EJ/E_C} \varphi^2}. \]  

Writing Eq. (22) in terms of Eq. (21) for \( \varphi \gtrless 0 \), we get

\[ \phi_+(\varphi) = 2^{-(2\nu+\frac{1}{2})} \left( \frac{EJ}{2E_C} \right)^{-\frac{\nu}{2}} e^{\frac{\nu+1}{\nu} \sqrt{8EJ/E_C} \varphi}, \]
\[ \phi_-(\varphi) = 2^{(2\nu+\frac{3}{2})} e^{i\frac{\pi}{2}(\nu+1)} \left( \frac{EJ}{2E_C} \right)^{-\frac{\nu-1}{\nu} \sqrt{8EJ/E_C} \varphi}. \]  

Using the above, \( f_R(\varphi) \) can be expressed approximately using \( \Psi(\varphi) \), \( \Omega(\varphi) \) for \( \varphi \gtrless 0 \). A similar method can be applied in the region \( \varphi \lesssim 0 \), but the approximations in Eq. (21) cannot be used directly since they are valid only for \( \varphi > 0 \). Instead we use the identities [43]

\[ \Psi(-\varphi) = e^{i\pi \nu} \Psi(\varphi) - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\frac{i\pi}{2}(\nu-1)} \Omega(\varphi), \]
\[ \Omega(-\varphi) = e^{i\pi (\nu+1)} \Omega(\varphi) + \frac{\sqrt{2\pi}}{\Gamma(\nu+1)} e^{i\frac{\pi}{2} \nu} \Psi(\varphi). \]  

Writing \( \phi_\pm(\varphi) \) in the region \( \varphi \lesssim 0 \) as \( \phi_\pm(-|\varphi|) \):

\[ \phi_+(\varphi) = e^{i\pi \nu} 2^{-2(\nu+\frac{1}{2})} \left( \frac{EJ}{2E_C} \right)^{-\frac{\nu}{2}} e^{\frac{\nu+1}{\nu} \sqrt{8EJ/E_C} \varphi}, \]
\[ \phi_-(\varphi) = e^{-i\frac{\pi}{2}(\nu-1)} 2^{(2\nu+\frac{3}{2})} \left( \frac{EJ}{2E_C} \right)^{-\frac{\nu-1}{\nu} \sqrt{8EJ/E_C} \varphi}. \]  

Using Eq. (24) we can now represent also \( f_L(\varphi) \) using \( \Psi(\varphi) \), \( \Omega(\varphi) \). By comparing the coefficients of this basis in \( f_R(\varphi) \) to the coefficients in \( f_L(\varphi) \) we acquire the connection matrix

\[ \begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} e^{2i\pi \nu} & \frac{A_L}{B_L} \end{pmatrix} \equiv C \begin{pmatrix} A_L \\ B_L \end{pmatrix}, \]  

where we neglected the wave function terms containing \( e^{-\sqrt{8EJ/E_C} \varphi} \). The factor \( \zeta \) is

\[ \zeta = 2^{-(4\nu+2)} \left( \frac{EJ}{2E_C} \right)^{-\frac{\nu}{2} - \frac{1}{2}} e^{\frac{\nu+1}{\nu} \sqrt{8EJ/E_C} \varphi}. \]  

Next we construct a matrix which represents the boundary condition via the coefficients by using the property of Eq. (18): \( \phi_\pm(\varphi + 2\pi) = e^{i\pi \nu} \phi_\mp(\varphi) \). Together with the required boundary condition we obtain

\[ \begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} 0 & e^{-i(\nu \varphi - \theta)} \\ e^{i(\pi \nu \theta)} & 0 \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix} \equiv \mathbb{B} \begin{pmatrix} A_L \\ B_L \end{pmatrix}. \]  

Combining Eq. (26) with Eq. (28) gives us a system of equations for the coefficients, which has a non trivial solution only when \( \det(\mathbb{C} - \mathbb{B}) = 0 \). Using the approximation \( \nu \approx k \) (everywhere except in \( \Gamma(\nu - 1) \) which we deal with separately below) the condition can be written as

\[ \zeta = 2 \cos(\theta). \]  

In order to retrieve the value of \( \delta E_k \) we next use the identity \( \Gamma(-\nu)\Gamma(\nu + 1) = -\frac{\pi}{\sin(\pi \nu)} \) [43] to get, expanding to first order in \( \delta E_k \) in the denominator,

\[ \Gamma(-\nu) \simeq (-1)^{k+1} \frac{\sqrt{8E_CE_J}}{\Gamma(\nu + 1) \delta E_k}. \]
plugging this expression into Eq. [27] we obtain the tight-binding like spectrum $\delta E_k = t_k \cos(\theta)$ with the “tunnelling amplitude” $t_k$ defined as

$$ t_k = (-1)^{k+1} \frac{2^{4(k+1)} E_C}{k!} \sqrt{\left( \frac{2}{2E_C} \right)^{\frac{1}{2} + \frac{n}{4}} e^{-\sqrt{\frac{2}{E_C}}} . $$

(30)

The functions described in Eq. [20] outline the entire solution in the region $-\pi < \varphi < \pi$. Although we only found the coefficients of $f_L(\varphi)$ and $f_R(\varphi)$, a similar process can be applied to find the connections to $f_M(\varphi)$. In keeping with the spirit of our approximation we argue that in cases where the tunnelling process between the barriers is negligible, $f_M(\varphi)$ which need not satisfy the relevant boundary condition, can be used as the wave function of the transmon to a good approximation. Finally, since $\Omega(\varphi)$ is a bounded function in the region $-\pi < \varphi < \pi$ and the ratio $B_M/A_M \sim e^{-\sqrt{8E_J/E_C}}$, we can approximate $f_M(\varphi) \simeq \Psi_k(\varphi)$, where $\Psi_k(\varphi)$ is the $k$th harmonic oscillator wave function. Using this wave function, Eqs. [1] and [2], forming the basis for the rest of the analysis, are now justified.

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