QUANTUM GRAVITY AS TOPOLOGICAL
QUANTUM FIELD THEORY

JOHN W. BARRETT

18 June 1995

ABSTRACT. The physics of quantum gravity is discussed within the framework of
topological quantum field theory. Some of the principles are illustrated with examples
taken from theories in which space-time is three dimensional.

CONTENTS

I. Introduction
II. A Combinatorial Problem
III. The Space of Quantum States
IV. Global Observables
V. Examples of TQFTs in Dimension Three
VI. Local Observables

I. INTRODUCTION

From the point of view of physics, a quantum theory of gravity is not a precisely
defined concept. There are two limits, $G \to 0$ and $\hbar \to 0$, which give particle
physics and general relativity, both of which are well understood. Apart from this,
there are few constraints. It is therefore necessary to make a series of assumptions
of a general nature. Thus from a mathematical point of view, one would like to
keep the general features of quantum theory on the one hand and the topological
framework of space-time on the other. Both of the features are embraced by the
precise notion of a topological quantum field theory. The aim of this paper is to
argue the thesis that the notion of a topological quantum field theory is sufficiently
broad and general that it is a proper framework for quantum gravity, and that it
encompasses many of the traditional approaches to the subject.

In this theory, the manifold is treated as an external, or unquantised object,
whereas the metric tensor is certainly regarded as quantum mechanical. An intu-
itive picture of this is given by the functional integral approach to quantum gravity,
where the idea is to integrate over the space of all metrics on a given manifold, sub-
ject to certain constraints associated with some given observables. In fact, one can
think of the axioms of TQFT as a way of encoding the desired properties of this
functional integral. Continuing with this intuitive picture, one can think of topo-
logical objects, such as manifolds, or particularly chosen submanifolds of a manifold

PACS 04.60.-m
as labels or names for quantum mechanical observables; the metrics and other fields being the spectra of these observables.

One of the main features of TQFT is that the way that diffeomorphisms of manifolds act on the algebraic objects of the theory is stated clearly at the outset. The consequences of this are surprisingly rich, and include direct analogues in this theory of the fact that the wavefunctions one can construct satisfy the equations known as momentum and Hamiltonian constraints in the canonical quantisation approach.

The most important reason for working with this level of abstraction is that there are examples. The most well-known examples are for space-time dimension three, including theories related to gravity. However examples exist in all dimensions, but none are known to be interesting, as far as quantum gravity is concerned, in dimension four. There is no prima facie reason why quantum gravity in dimension four may not exist within the framework of TQFT. This is despite the existence of propagating gravitational waves in four-dimensional general relativity, and the expectation that propagating gravitons play a role in quantum gravity. This phenomenon appears to require the use of infinite-dimensional state spaces or an effective substitute, which has some interesting consequences discussed in section V, which may require some restriction on the topologies considered. The subject of quantum cosmology is in many ways the forerunner of the present theory, the desiderata for the functional integral including the axioms of TQFT.

A lot of work on quantum gravity can be classified in two types. On the one hand there is the ‘particle physics + $\epsilon$’ approach, where one takes models of particle physics and perturbs them by adding new fields, more dimensions, extended structures or more elaborate quantisation schemes. In this approach, one hopes that the ‘correct’ model will emerge, and then it will become apparent what the physical and mathematical principles are with which to reformulate the theory on to a satisfactory conceptual footing. On the other hand there are the ‘fundamentalists’, who regard the revision of the basic notions of space, time, set theory, quantum mechanics or logic as a necessary preliminary before any proper progress can be made.

The problem with the later approach is that without the guide of some good examples, there is no reason to prefer any one modification of these basic notions to another one. The importance of the three-dimensional gravity and related TQFTs is that they can be viewed from either the fundamentalist or the particle physics standpoints. It shows at once that the mathematical formalism of topological quantum field theory is not empty and can deliver a theory in which metrics play a role, and on the other hand allows one to test some cherished assumptions about the nature of observables and the relationship of classical general relativity to quantum gravity.

II. A Combinatorial Problem

In a topological quantum field theory, the simplest case to consider is a compact oriented space-time manifold $M$, which is closed (without boundary). In this case, the TQFT defines a complex number, $Z(M)$, which is called the partition function of the manifold, due to the analogy with statistical mechanics. This number is to be thought of as the result of performing a functional integral over some fields, such as a metric tensor field, on $M$.
It is natural, and seemingly innocuous, to require that the partition functions of isomorphic manifolds are the same number. This is called an invariant of manifolds. This leads directly to a combinatorial problem to define a partition function for all the manifolds of a given dimension. A presentation of a manifold is a recipe for constructing it from a number of elementary pieces, such as balls or simplexes, by some combinatorial process which involves a finite amount of information about how the pieces are assembled. Probably the most well-known method is to present a manifold by a triangulation. In this method, the pieces are the simplexes and the information specifies which faces are glued to each other. The combinatorial problem is to give a computation of $Z(M)$ from the information specifying the presentation of $M$ in such a way that the same number is determined for any two different presentations of the same manifold.

This problem differs sharply in different dimensions. In dimension two, the closed manifolds are all isomorphic to a sphere with a number of handles attached. These are readily distinguished when a manifold is defined by a presentation by calculating the Euler number. Thus, $Z(M)$ is a recursive function of the Euler number.

In dimension three, there is no known list of invariants which specify a manifold uniquely up to isomorphism, and indeed it is not known even whether this is possible. Therefore there is no known limit to the complexity or subtlety of new invariants. The same remarks apply to simply-connected differentiable manifolds of dimension four. A new feature in dimension four is that inequivalent differentiable manifolds can be homeomorphic, for example the (non-compact) space $\mathbb{R}^4$ has a continuous family of inequivalent differential structures.

What are the implications of this zoo of differentiable manifolds? The main point is that the manifolds provide observables for the theory, and so a rich structure for the observables provides the framework for a non-trivial quantum theory. In this sense, the complexity of the theory of manifolds provides a worthy arena for the subject of quantum gravity.

A second point is that the character of these theories will be very different in different dimensions. Quantum gravity is also necessarily different in these different dimensions because general relativity is. General relativity in dimension two is trivial, because the action is just the Euler number. In dimension three the theory is not trivial, but the solutions are constant curvature metrics or connections. In dimension four, there are propagating gravitational waves.

III. The Space of Quantum States

In quantum mechanics, one usually has a space of quantum states associated to a given physical system. Often this space of states refers to a particular instant of time, which can be represented in general relativity by a space-like hypersurface. In topological quantum field theory, this vector space appears, as part of the definition, when the space-time manifold $M$ has a boundary. The general framework for a theory of dimension $d$ is as follows.

Every closed $(d-1)$-dimensional manifold $\Sigma$ has associated to it a vector space, $V(\Sigma)$, and every $d$-manifold $M$ determines a vector $Z(M)$ in the state space $V(\partial M)$ which is associated to its boundary.

This accords with the previous use of the notation $Z(M)$ for an invariant of a closed manifold, for in this case $V(\emptyset) = \mathbb{C}$, so $Z(M) \in \mathbb{C}$ when the boundary of $M$ is empty.
Hawking’s no-boundary proposal [1] for the wavefunction of the universe is an example of this formalism; there the state space of $S^3$ is defined to be the vector space of functions on the space of Riemannian metrics on $S^3$, and the wavefunction of the universe is defined to be the result of a functional integral on the four-dimensional ball,

$$Z(B^4) \in V(S^3).$$

The data in the TQFT, the state spaces and partition functions, have to satisfy a number of conditions, which were given by Atiyah [2]. These can be summarised as

1. The rules of quantum mechanics for composing amplitudes.
2. Functoriality, or the correct behaviour under diffeomorphisms of manifolds.

The collection of conditions which comes under (1) will become clear in the following discussion. It includes the condition that for the disjoint union of two closed $(d−1)$-manifolds $\Sigma_1$ and $\Sigma_2$,

$$V(\Sigma_1 \cup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2).$$

This accords with the normal rule in quantum mechanics that the state space for two systems isolated from each other is the tensor product space. In this topological theory, the stipulation that the systems are isolated from each other has been translated into the topological condition that $\Sigma_1$ and $\Sigma_2$ are disconnected.

It is worth spelling out condition (2) as its consequences are far-reaching. This condition is that the group of diffeomorphisms of a $(d−1)$-manifold $\Sigma$ is represented by linear transformations in $V(\Sigma)$, and that any diffeomorphism of $\Sigma = \partial M$ which extends over $M$ leaves the partition function $Z(M)$ invariant. The same conditions hold for diffeomorphisms between different manifolds.

Since any isotopy of $\partial M$, also called a small diffeomorphism, extends over any manifold $M$, an immediate consequence of this is that all partition functions are invariant under the action of isotopies. Translated into the language of quantum cosmology, the wavefunction of the universe satisfies the equations known as the momentum constraints.

An isotopy of $X$ is a diffeomorphism

$$X \times [0, 1] \rightarrow X \times [0, 1]$$

which preserves the ‘time’ parameter (so that the image of $(x, t)$ lies in $X \times \{t\}$) and leaves fixed all the points on one end, $(x, 0) \mapsto (x, 0)$. The mapping of the other end, $X \times \{1\}$, is the ‘small’ diffeomorphism of $X$. The reason that an isotopy of $\partial M$ extends over $M$ is that there is always a ‘collar’ neighbourhood of $\partial M$ diffeomorphic to the cylinder $\partial M \times [0, 1]$. The isotopy extends over the collar by definition, and extends to the other points of $M$ by the identity mapping.

If the boundary of a $d$-manifold $C$ is regarded as separated into two disjoint closed parts,

$$\partial C = \Sigma_1 \cup \Sigma_2,$$

then the vector $Z(C) \in V(\Sigma_1) \otimes V(\Sigma_2)$ can be regarded as a matrix of transition amplitudes

$$V(\Sigma_1) \rightarrow V(\Sigma_2).$$
The normal rules of quantum mechanics, (1), require that these multiply as matrices when the corresponding manifolds are glued together. A technicality to mention is that for this to work, the \((d - 1)\)-manifold \(\Sigma_1\) is \(\Sigma_1\) with the opposite orientation, and its state space is the dual space to that of \(\Sigma_1\). The bra and ket notation can be used to keep track of these.

Now, gluing an extra collar \(\partial M \times [0, 1]\) onto \(M\) does not change the topology. Therefore one has the linear equation

\[ Z(\partial M \times [0, 1])Z(M) = Z(M) \]

which shows that the vector \(Z(M)\) is a unit eigenvector of the matrix \(Z(\partial M \times [0, 1])\).

In the quantum cosmology, this equation is the statement that the wavefunction satisfies the Hamiltonian constraint. The transverse vector field which normally accompanies the Hamiltonian constraint is simply an infinitesimal version of the diffeomorphism

\[ M \cup (\partial M \times [0, 1]) \rightarrow M. \]

The Hamiltonian constraint is normally written

\[ HZ(M) = 0, \]

the operator \(H\) corresponding to an infinitesimal transverse deformation. The usual relationship between the Hamiltonian and the evolution operator is exponentiation, so one has the equation [3]

\[ e^{-H}Z(M) = Z(M). \]

From the analogy with canonical quantisation, one might expect one equation for every infinitesimal transverse deformation. They are however all the same, since any two transverse vector fields are related by an isotopy.

A further comparison with quantum cosmology shows some important points of physics. Although the no-boundary proposal is formulated for a particular manifold, \(B^4\), quantum cosmology allows the consideration of the functional integral on any manifold. One has therefore in mind a very specific construction in which \(V(\Sigma)\) is the vector space of functions on the space of Riemannian metrics on \(\Sigma\). Therefore one has right away a link with geometry and the necessary information to construct observables with a prescribed geometric meaning. This information is not present in the general framework of TQFT, but must be sought in the further structure which is provided by particular examples.

However it is remarkable that one can make a lot of progress using the general properties of TQFT alone, constructing particular wavefunctions and showing that they satisfy the momentum and Hamiltonian constraint equations. One might ask how this has been done, given that constructing solutions to these equations in the framework of canonical quantisation has been considered an important and apparently intractable problem. The answer is that the difficulty has been transformed into the combinatorial problem of constructing invariants. This problem is also difficult; but it is at least within the domain, and technology, of modern mathematics.

The primary problem is in fact the construction of invariants for closed manifolds, and the state spaces and partition functions for manifolds with boundary are usually obtained as a by-product.
Gluing two cylinders $\Sigma \times [0, 1]$ on top of each other, one obtains $\Sigma \times [0, 2]$, which is of course diffeomorphic to $\Sigma \times [0, 1]$. This translates into a matrix equation

$$Z(\Sigma \times [0, 1])Z(\Sigma \times [0, 1]) = Z(\Sigma \times [0, 1]).$$

In other words the cylinder partition function is a projection operator on the state space of $\Sigma$. It will be useful to use a shorthand notation for this projector,

$$P_\Sigma \equiv Z(\Sigma \times [0, 1]).$$

Apart from showing that the eigenvalues of $H$ are 0 and $\infty$ only, this raises the possibility of replacing each $V(\Sigma)$ in the theory with the image of the corresponding cylinder projector. From the point of view of the axiomatic framework of TQFT, which is all that has been introduced so far, one may as well do this. Since all partition functions lie in these images, the theory has the same properties and the same invariants. However, particular examples of TQFTs have further structure, for which the fact that the $V(\Sigma)$ is larger than the image of the cylinder projector is important.

An observable is represented by an operator in a state space, $V(\Sigma)$. The observable $O$ can be called a global observable if the operator acts non-trivially only in Image $P_\Sigma$, so that

$$OP_\Sigma = P_\Sigma O = O.$$

Otherwise it is called a local observable.

The argument given above that partition functions are invariant under the action of isotopies can be applied to the manifold $\Sigma \times [0, 1]$ itself to show that all elements of Image $P_\Sigma$ are invariant under isotopies. This means that global observables cannot refer to a particular subset of $\Sigma$ and they are indeed only sensitive to the global topology of $\Sigma$. By contrast, in particle physics most observables can be localised to one small region. Therefore the global observables are very limited and do not capture the full content of dynamics.

Global observables can be constructed using only the machinery introduced so far, as the examples described in the next section demonstrate, and allow for a relatively simple and self-contained discussion of quantum theory. Local observables will be discussed later.

### IV. Global Observables

The formalism which has been introduced contains some of the mathematical elements of quantum theory. A quantum theory proper must have a method of calculating probabilities, together with a statement of the physical objects or processes that these correspond to. To meet the first point, some projection operators will be defined, which enables a discussion of the relations between the probabilities which are determined by them. These operators are loosely referred to as observables, although the term more strictly refers to the physical object or process to which these are said to correspond.

If $C$ is a manifold with boundary $\Sigma \cup \bar{\Sigma}$, then $Z(C)$ is an operator

$$V(\Sigma) \to V(\Sigma)$$

which...
just as for the case $C = \Sigma \times [0, 1]$ considered previously. The simplest case to consider is when $Z(C)$ is a multiple of a projector, so let $C$ be this special type of manifold.

Suppose that, as in quantum cosmology, $M$ is a space-time manifold with boundary, and let $\partial M$ be identified, for the moment, with $\Sigma$. The observable labelled by $C$ acts on partition functions $Z(M)$ in $V(\Sigma)$ by gluing $C$ on to the boundary of $M$. It is a global observable because

$$C \cup_\Sigma (\Sigma \times [0, 1]) \cong C,$$

in which the cylinder can be glued to $C$ at either end.

This situation can be generalised. The closed $(d - 1)$-manifold $\Sigma$ need only be one component of the boundary of $M$. More generally, it could be any embedded submanifold of $M$, called a hypersurface, such that cutting $M$ along $\Sigma$ produces a manifold $M_\Sigma$ with two extra boundary components, $\Sigma$ and $\overline{\Sigma}$. The state space for $M_\Sigma$ is

$$V(\Sigma) \otimes V(\overline{\Sigma}) \otimes V(\partial M) \cong \text{End}(V(\Sigma)) \otimes V(\partial M),$$

End($V$) being the set of linear operators on $V$. The observable labelled by $C$ also acts in the space $V(\Sigma)$. Gluing $C$ onto $M_\Sigma$ on both boundary components $\Sigma$ and $\overline{\Sigma}$ produces a manifold

$$M_{\Sigma}C = M_\Sigma \cup_{\Sigma \cup \overline{\Sigma}} C$$

which is $M$ with $C$ sewn into it. The gluing rules give

$$Z(M_{\Sigma}C) = \text{tr}_{V(\Sigma)} Z(M_\Sigma) Z(C) \in V(\partial M),$$

taking the trace over $V(\Sigma)$. In the language of physics, $C$ operates on a subsystem which is coupled to other systems.

One of the shifts in thinking about quantum theory which is involved is that some of the considerations are only valid for a particular quantum state. Here, the observable labelled by $C$ makes a difference by replacing the state vector $Z(M)$ with $Z(M_{\Sigma}C)$. However this is not defined as the result of applying a linear operation to the space $V(\partial M)$. If $M'$ is another manifold with the same boundary, $\partial M$, then the action of $C$ on $Z(M')$ is not defined because one does not know, in a general situation, how $\Sigma$ should be inserted in $M'$. In the special situation where $\Sigma$ lies in $M$ as a level hypersurface of a collar neighbourhood of the boundary $\partial M$ (i.e., parallel to the boundary), then the usual rule is recovered,

$$Z(M_{\Sigma}C) = Z(C)Z(M),$$

with $Z(C)$ acting as a linear map on the vector $Z(M)$. This is essentially the same as when $\Sigma = \partial M$.

How many of these subsystems are there? If $\Sigma'$ is a second embedded hypersurface in $M$, then there is an equivalent observable on $\Sigma'$ if there is a diffeomorphism

$$f: M \to M, \quad f|_{\partial M} = \text{identity}$$

which takes $\Sigma$ to $\Sigma'$. Then inserting $C$ at $\Sigma$ is equivalent to inserting $C$ at $\Sigma'$, provided $\partial C$ is identified with $\Sigma' \cup \overline{\Sigma}$ using $f$. 


The question of the number of inequivalent embeddings of a given \((d-1)\)-manifold in \(M\) gets rapidly complicated as the dimension increases through \(d = 2, 3, 4\). In dimension two, there is only one way to embed a circle in \(S^2\), and a finite number of ways of embedding it in a closed surface, enumerated by the topologies of the pieces obtained by removing the circle. In dimension three, there is still only one way to embed \(S^2\) in \(S^3\), but an infinite number of ways to embed a torus - taking a tube around any knot, for example. However, any surface embeds in \(S^3\) in a unique way if the complement is required to be two handlebodies. In dimension four, it is not even known if the embedding of \(S^3\) in \(S^4\) is unique. This is known as the Schönflies conjecture.

In the usual quantum mechanics, operators are conveniently regarded as being defined at one particular instant of time, with a sequence of interactions being represented by the multiplication of the corresponding operators in the order given by the causal structure, or time coordinate. Therefore, the essential topological feature of the usual quantum mechanics is the time ordering of observables. This has been regarded as a difficult feature of quantum gravity, since the causal structure of a space-time is a feature of the classical metrics, which arise as the spectra of quantum observables, and is therefore not available for the overall structure of the quantum theory. Another puzzling feature, on physical grounds, is the postulate that the observables should form an algebra. It may not make sense to consider two interactions which happen in one time order as happening in the opposite time order, yet in an algebra the product is always defined in both orders.

In a TQFT, this information is specified by choosing a particular hypersurface \(\Sigma \subset M\), together with some further information concerning the observable in \(\Sigma\). This is identifying some topological objects (here \(\partial C\)) on \(\Sigma\). Thus specifying a place in the manifold replaces specifying a time ordering in the usual theory.

If \(M = \Sigma \times [0, 1]\), and the hypersurfaces considered are just

\[
\Sigma \times \{t\} \subset \Sigma \times [0, 1]
\]

for different values of \(t\), then the normal time ordering is apparent in the TQFT. The gluing rules of the TQFT imply that, in this situation, the observables are indeed multiplied as operators in the corresponding order. The multiplication of an ordered set of operators is merely a particular case of the dynamics of a topological theory. These conclusions are also justified for a general manifold \(M\) if all observables considered are in a collar neighbourhood of the boundary.

However in the general situation, the linear time ordering for the observables may not exist, and in general it may not make sense to swap the places at which two different observables are defined. For this more general notion of quantum theory, it is necessary to discuss the probabilities that the theory determines directly. An example of a theory in which this more general discussion makes sense will follow.

A. Hermitian theories

For this discussion one wants to have Hermitian inner products, for the same reason as in quantum mechanics, that the square of a vector should be a probability. A Hermitian theory is one in which each state space has a Hermitian inner product, which, considered as an (antilinear) map

\[
V(\Sigma) \to V(\Sigma)
\]
QUANTUM GRAVITY AS TOPOLOGICAL QUANTUM FIELD THEORY

9

takes any partition function \( Z(M) \) to \( Z(\overline{M}) \), for \( \partial M = \Sigma \). In the bra-ket notation, \( Z(M) \) is written \( |M\rangle \) and \( Z(\overline{M}) \) is written \( \langle M| \). In the Hermitian theory, \( \langle N|M \rangle \) is the Hermitian inner product of \( Z(N) \) with \( Z(M) \). Therefore for the double of any manifold, obtained by gluing two copies across the boundary, \( Z(M \cup \partial M \overline{M}) = \langle M|M \rangle \geq 0 \).

Also, if \( P \) is a closed manifold, then \( Z(\overline{P}) \in \mathbb{C} \) is the complex conjugate of \( Z(P) \). This is a special case of the more general fact that for the manifold \( C \), the matrix \( Z(C) \) is equal to the Hermitian adjoint \( Z^+(C) \) (To show this, one has to use the natural assignments of Hermitian inner products to the dual of a state space and the tensor product of two state spaces.).

**B. Probabilities**

A discussion of probabilities is essential to give the theory some meaning as a theory of physics. It is worth starting with a discussion of standard quantum mechanics in a single state space. Let \( \pi_1 \) be an orthogonal projector in the inner product space \( V(\partial M) \). Then the usual discussion of quantum mechanics is to associate \( \pi_1 \) to a proposition \( X \). If one wishes to assert \( X \) as a fact, then a state vector, say \( |M\rangle \), is simply replaced by the vector \( \pi_1 |M\rangle \) for the purposes of future discourse.

If, however, one wants to discuss the relative merits of \( X \) and not \( X \), which is assigned the projector \( 1 - \pi_1 \), then the probabilities

\[
p_1 = \frac{\langle M|\pi_1|M \rangle}{\langle M|M \rangle} \quad \text{and} \quad p_\bar{1} = \frac{\langle M|1 - \pi_1|M \rangle}{\langle M|M \rangle}
\]

are used. It is obviously essential that these numbers are not negative, and add up to 1. Assuming the denominator is not zero, this is because the numerators are both the square of a vector, which implies \( p_1, p_\bar{1} \geq 0 \), and because of the condition

\[
\langle M|M \rangle = \langle M|\pi_1|M \rangle + \langle M|1 - \pi_1|M \rangle.
\]

which implies

\[
1 = p_1 + p_\bar{1}.
\]

This is the most elementary example of a consistency condition for the probabilities, which is satisfied automatically, as \( \pi_1 \) is a projector.

Now suppose \( \pi_2 \) is a second projector, which need not commute with \( \pi_1 \), and define \( p_2, p_\bar{2} \) in the same way using the state \( |M\rangle \). According to Griffiths \[4\], the numbers

\[
p_{12} = \frac{\langle M|\pi_1\pi_2\pi_1|M \rangle}{\langle M|M \rangle} \quad \quad p_{\bar{1}\bar{2}} = \frac{\langle M|\pi_1(1 - \pi_2)\pi_1|M \rangle}{\langle M|M \rangle}
\]

\[
p_{\bar{1}2} = \frac{\langle M|(1 - \pi_1)\pi_2(1 - \pi_1)|M \rangle}{\langle M|M \rangle} \quad \quad p_{2\bar{2}} = \frac{\langle M|(1 - \pi_2)(1 - \pi_1)(1 - \pi_2)|M \rangle}{\langle M|M \rangle}
\]

determine a joint probability distribution on the four-point space \( \{12, \bar{1}2, \bar{1}\bar{2}, \bar{2}\bar{2}\} = \{1, \bar{1}\} \times \{2, \bar{2}\} \) if a consistency condition is satisfied. This is the condition

\[
p_{12} + p_{\bar{1}\bar{2}} = 1.
\]
The three other analogous relations for $p_2$, $p_1$, $p_1$ follow automatically. These conditions are the conditions for the compatibility of the marginal distributions for \{1, 1\} and \{2, 2\} determined by $p_1$ and $p_2$ with this joint probability distribution. As explained by Omnès [5], this condition, and its generalisation to more than two propositions, is a necessary and sufficient prerequisite for consistent reasoning in a quantum theory.

The interpretation of $p_{12}$ in quantum theory is that it represents the probability of the proposition $\pi_1$ followed by $\pi_2$ at a later time. The joint probability distribution is used to discuss the correct inferences which can be drawn connecting different propositions at different times. These are the implications which hold with probability one. Quantum theory is a language whose elements refer to the many different propositions which may be considered and whose content is the many relations between them. This is the sense in which the consistency conditions are essential; the individual probability distributions have a very limited scope of application on their own.

Omnès argues that all true statements about quantum systems fall into the class of correct implications which can be made using this probability calculus. This argument is not provable in a mathematical sense, but has an experimental status, like Church’s thesis on computability: all situations which have been analysed have indeed followed this pattern. It is fairly easy to demonstrate, on the other hand, that the failure of the consistency conditions signals the inapplicability of Boolean logic and the possibility of logical paradoxes. It is also the essential feature of quantum interference phenomena.

An important point to note is that the consistency condition depends on the quantum state. Indeed, requiring it to hold for all states in the state space would force the projectors $\pi_1$ and $\pi_2$ to commute, which, if required universally, would rule out all quantum phenomena. The fact that Boolean logic applies for some quantum states in some situations where the observables do not commute is a relatively recent discovery. It invalidates the earlier assumption that a modification of Boolean logic is necessary to discuss any quantum phenomena at all. This analysis extends to probabilities which can be calculated in a Hermitian topological quantum field theory. As explained above, the operator $Z(C)$ can be considered to act in the vector space $V(\Sigma)$ if the boundary of $C$ is identified appropriately with $\Sigma$. Equally it makes sense to act with a linear combination of such operators, even if the geometrical interpretation is less clear. Thus if $Z(C)$ is not a projector, some polynomials in $Z(C)$ may be. Let us continue with the case mentioned above, where $Z(C)$ is a multiple of a projector, which will hold in the specific example to be introduced below. The projector is

$$\pi = \lambda Z(C), \quad \lambda \neq 0 \in \mathbb{C},$$

acting in the space $V(\Sigma)$, for the hypersurface $\Sigma \subset M$, which need no longer be the boundary of $M$ or parallel to it. The complementary observable is

$$1 - \pi = 1 - \lambda Z(C).$$

Accordingly, the two vectors obtained from the action of these two operators are

$$|\pi M\rangle = \lambda |M, C\rangle \quad \text{and} \quad |(1 - \pi)M\rangle = |M\rangle - \lambda |M, C\rangle.$$
The notation $|\pi M\rangle$ has been chosen to emphasise that $\pi$ does not act in $V(\partial M)$; it is only in the special case where $\Sigma$ is parallel to $\partial M$ that one can write

$$|\pi M\rangle = \pi|M\rangle.$$ 

The probabilities are defined as the normalised square of these vectors

$$p = \frac{\langle \pi M|\pi M \rangle}{\langle M|M \rangle} \quad \text{and} \quad \bar{p} = \frac{\langle (1-\pi)M|(1-\pi)M \rangle}{\langle M|M \rangle}.$$

There is a consistency condition for these probabilities, namely that $p + \bar{p} = 1$. This can be written in the form

$$\langle \pi M|\pi M \rangle = \frac{1}{2}\left(\langle \pi M|M \rangle + \langle M|\pi M \rangle\right),$$

or as

$$\text{Re}\langle (1-\pi)M|\pi M \rangle = 0.$$

The later form has a state space interpretation which is that the two vectors are orthogonal in the real-linear form given by the real part of the Hermitian inner product. The condition is analogous to the Griffiths consistency condition for ordinary quantum mechanics, with the non-trivial topology of the manifold playing the role of the interposition of the projector $\pi_2$.

This formalism can be extended to consider the joint probability distributions determined by a number of projectors each acting in a different place in the manifold. Several consistency conditions will arise. These conditions are that the joint probability distributions are compatible with each of the marginal distributions, which are calculated in the same manner from the quantum theory.

The formalism has the property that if a number of observables act in parallel hypersurfaces in a submanifold isomorphic to $\Sigma \times [0,1]$, then the calculation for global observables will reduce to the usual quantum mechanical multiplication of projectors. In particular, it is possible to show that repeating the same proposition is reliable. Let $X_t$ be a proposition which is represented by the projector $\pi$ acting in the state space associated to $\Sigma \times \{t\}$. Then

$$p(X_t \cap X_s) = p(X_t) = p(X_s),$$

due to the fact that $\pi P_\Sigma \pi = P_\Sigma \pi = \pi P_\Sigma$. This means that

$$p(X_t|X_s) = \frac{p(X_t \cap X_s)}{p(X_s)} = 1,$$

and according to Omnès interpretive rule for quantum mechanics, the logical implication

$$X_s \iff X_t$$

therefore holds.
C. Semantics

The previous discussion can be completed by describing a framework for a semantics. This will allow some general conclusions to be drawn about the scope of the theory.

What remains is to discuss the correspondence between the physical objects or processes, and the mathematical entities consisting of certain specific projection operators acting in certain state spaces. This is a difficult point in all discussions of quantum gravity.

In more conventional physics, the semantics is constructed from the specification that there are some physical objects which behave in the manner of classical deterministic dynamics. The classical example is ‘the particle is at a position \( x \in [a, b] \). Let us call this proposition \( X \). From this statement one can construct, in particle physics theories, a projector in the particular Hilbert space of interest. These projectors, for different subsets of \( \mathbb{R} \), together with the analogues for momentum observables, form a privileged subset of the set of all projectors in the Hilbert space.

Such a statement is made in a particular context, which the usual theory presupposes. This is that there are also a number of macroscopic reference bodies, whose existence delineates the classical ‘space’, \( \mathbb{R} \), and that the particle is in fact correlated with these macroscopic bodies. This context is a matter of physical fact, rather than mathematical formalism. For example, a particle emitted by radioactive decay of a nuclear state has a position variable which is in fact correlated with the atom which has emitted it. The atom is, in turn part of a lump of solid matter, which is fixed in a laboratory. This laboratory contains macroscopic measuring devices which serve to define the interval \([a, b]\). Finally, the description of the particle by its wavefunction can be tested by a measurement interaction which alters the positions of the macroscopic bodies.

The existence of the macroscopic bodies, and the initial conditions, is asserted as fact, say by a collection of statements \( Y \). This has a different status to the proposition \( X \), in that \( Y \) is asserted as fact, and the possibility of \( \neg Y \) is not entertained, or even defined. On other hand, \( X \) and \( \neg X \) are both defined and ascribed probabilities which satisfy a consistency condition when they are used. This consistency condition ensures that \( X \) and \( \neg X \) relate to each other in the way that is usually meant. The future discourse may introduce further propositions of this type, and the truth of \( X \), \( \neg X \), or combinations of these with other propositions is a matter of calculation and logical inference. The facts \( Y \) on the other hand are simply a prerequisite for the debate and their truth is not open to discussion.

The theory of a single particle is normally written without the need to include macroscopic reference bodies explicitly in the quantum mechanical formalism. The correct procedure, as ever, is to include only the least amount of information about these other bodies in the description of the particle. This information is precisely the commonly agreed meaning of the privileged sets of projectors. Other projectors in Hilbert space, such as those specifying momenta, can be introduced, but only in as far as they have a particular specified relation to the reference bodies. In this way, the rather featureless geometry of Hilbert space is replaced by the geometry of the phase space of a particle.

Quantum theory does supply an interpretation for the situation where sets of projectors and a quantum state fail to satisfy the Griffiths consistency condition. For example, in the double slit experiment, the proposition \( X \) could be that the particle

particle has gone through one of the slits. As is well known, the probability sum rules in this experiment are not satisfied, so that the probabilities \( p(X) \) and \( p(\text{not } X) \) calculated from quantum theory do not sum to 1 when normalised with the probability calculated for the double slit experiment where the particle is free to travel through either slit. This logical paradox is resolved by the necessity to insert a measuring device in front of one of the slits to test the truth of \( X \) or \( \text{not } X \). Therefore, the only situations where \( X \) or \( \text{not } X \) have any logical consequences are where the physical facts \( Y' \) actually differ from the asserted facts \( Y \) for the double slit experiment without the determination of \( X \) or \( \text{not } X \), according to the presence or absence of the measuring probe. In other words, the theory asserts that a discrepancy in the consistency condition is due to the difference in the asserted facts \( Y \) or \( Y' \), and this is due to an actual physical cause. In established quantum physics, this cause can be described within the quantum formalism, and consistency is restored.

There is an analogy in Newtonian mechanics. Suppose that the known forces on a body do not account for its acceleration. Then according to Newton’s laws, there must be another force on the body which is due to the action of another physical object which has not yet been included in the formalism. Including all bodies which act does eventually resolve the discrepancy.

Finally, conventional quantum mechanics allows one to assert any prior facts \( Y \) which are not absolutely contradictory. This means that the quantum state vector is not precisely zero. In that case, the state vector can be normalised by dividing by its norm, which is the same as normalising the probability measure to give a total measure one. This means one can add any fact to \( Y \) asserted by the action of a projector, as long as the resulting state vector is still non-zero. This wide choice of quantum state appears to rule out any explanation for any objective facts, common to all observers, within conventional quantum theory.

The semantics for quantum gravity is necessarily more abstract. In the preceding discussion of global observables, the various choices of manifold \( M \), hypersurfaces \( \Sigma \) and observables, labelled by \( C \), need to be assigned a physical counterpart. By deliberate construction, all three of these are topological objects. Therefore, I would like to propose a general principle that the topology of the physical situation is reflected in the topology which enters into the mathematical description of the state vector and the observables. This means that the projectors which occur in the theory have a very particular construction in terms of the topology of manifolds. The topological construction singles out a privileged set of projectors, which plays the role that the set of position and momentum projectors enjoys in particle physics.

The way in which this principle is applied will depend on the particular model, and the ideas at present are rather vague. The global observables are rather limited, and a more thorough investigation would start with the properties of local observables. However, some general remarks can be made.

Following the ideas of quantum cosmology, one could propose that the manifold \( M \) represents both space-time and the initial conditions. By analogy with the quantum mechanics of a particle, the choice of \( M \), and thus the initial state vector, \( |M\rangle \), is one of the asserted facts \( Y \), and one is free to choose any \( M \) so long as \( |M\rangle \neq 0 \). As the global observables change the topology of \( M \), this confirms that they are truly global in the space-time sense, as opposed to having a localised effect in a small region.

The introduction of propositions whose projectors satisfy consistency conditions
has immediate implications for the boundary of $M$. As noted above, one form of the condition for one observable is that the two vectors in $V(\partial M)$ are orthogonal. This feature persists in the consistency conditions for several observables, with the number of mutually orthogonal vectors increasing with the number of consistent observables. This means that unless the vectors are zero, the dimension of $V(\partial M)$ has to be sufficiently large to support the observables. In particular a closed manifold $M$ has little use. It also implies that the observables which are required must have a certain dynamical connection with the boundary $\partial M$.

One interpretation of these conditions is that $\partial M$ represents the ‘present instant of time’ and that the vectors in $V(\partial M)$ which are determined by various choices of observables represent a memory of past facts. At least if the vectors are orthogonal in the Hermitian inner product, a stronger condition, then they are in principle distinguishable in a deterministic way by operations in $V(\partial M)$. Such a dynamical connection with the boundary is also required for a theory of *measurements*, where facts are correlated with memory devices.

These general remarks on the framework for a physical interpretation of the theory show that it is reasonable to construct the semantics for quantum gravity in a fashion which is analogous to, but more abstract than, the standard quantum mechanics for particles. The idea is to replace the classical world (or in more sophisticated treatments, the approximately classical world) of rigid bodies in space-time with some prior notions of topology. In a sense, one can say that quantum gravity is, amongst other things, a ‘quantum theory without large objects’. This means that the description of a small object should not ultimately rest on the existence of large objects such as rigid bodies to construct the semantics for the requisite observables.

Some more ambitious goals are sometimes mooted for quantum gravity, often under the name of quantum cosmology. One such project is to use the extra information in the initial state vector, namely its norm

$$\langle M|M \rangle,$$

as a prior probability for the ‘occurrence’ of $M$. The idea is to take a ‘one world’ view in which every possible fact or potentiality does occur in the one universe with some frequency or probability. The problem with this is that the event space (in the sense of probability theory) is not clear, with propositions such as ‘space-time is not $M$’ having no clear translation into the quantum formalism. A second problem with this project is that the quantum formalism appears to treat the observer, namely the person or object which is entertaining the propositions, as outside of the quantum system described by the TQFT. This type of observer dependency may be described by a framework in which quantum states of the usual kind are relative to observers, but some type of abstract absolute quantum theory is retained as an overall description [6, 7]. However such a framework has not been developed in the context of topological quantum field theory.

An even more ambitious project is to assert the existence of common or objective facts. This requires not only the selection of some universal initial state but also some restriction on the allowed propositions, with say, the consistency condition allowing only a single set of propositions [8]. At the present stage it is too early to say whether these doctrines have a precise formulation within the framework which is discussed here.
D. An example

A simple example can be presented in dimension three (corresponding to two-dimensional ‘space’, $3=2+1$). This example is for a theory which satisfies a condition on connected sums. The connected sum of two connected manifolds, $M$ and $N$, is obtained by removing a ball $B^3$ from each, and joining $M$ and $N$ across the extra boundary components $S^2$ so created on each one. The balls can be taken to be small neighbourhoods of some arbitrarily chosen interior point on each. The connected sum is denoted $M \# N$.

The condition that the theory satisfies for the example to work is

$$Z(S^3)Z(M \# N) = Z(M) \otimes Z(N).$$

The scalar $Z(S^3)$ multiplying $Z(M \# N)$ can be thought of as correcting for the two balls which are removed from $M$ and $N$. This condition is satisfied for the Turaev-Viro model discussed in the next section, for example.

The observables are obtained by the operation of surgery. Suppose that $b$ is a band $S^1 \times [0,1]$ embedded in the 3-manifold $M$. The band can be thought of as lying in a small neighbourhood of the circle $S^1 \times \{0\}$. This circle has a neighbourhood in $M$ which is a small solid tube with topology $S^1 \times B^2$. The band gives a preferred way of identifying the boundary of this tube with a standard torus $S^1 \times S^1$, at least up to isotopy, which is what counts. Surgery is the operation of removing the tube and re-inserting it after swapping the variables

$$(x, y) \mapsto (y, x)$$

on its boundary, $S^1 \times S^1$. The new manifold obtained from $M$ by surgery on $b$ is denoted

$surgery(M, b)$.

The other end of the band, $S^1 \times \{1\}$, can be taken to lie on the boundary of the tube. After surgery, this curve is contractible, whereas before the surgery it may not be.

The observable is constructed from a surface (2-manifold) $\Sigma$ and a band $b \subset \Sigma$. The band $b$ can also be embedded in the middle of the cylinder $\Sigma \times [0,1]$, say $\overline{b} = b \times \{1/2\} \subset \Sigma \times [0,1]$. Now let

$$C = surgery(\Sigma \times [0,1], \overline{b}).$$

With the above condition on connected sums, one can show that $Z(C)$ is a multiple of a projector $\pi$, in fact,

$$Z(C)^2 = \frac{Z(S^2 \times S^1)}{Z(S^3)} Z(C).$$

This is because if $C$ is glued to a copy of itself by one end, then the band $\overline{b}_1$ in one copy can be slid up the cylinder until it lies parallel to the other band $\overline{b}_2$. Then after the surgery on $\overline{b}_2$, $\overline{b}_1$ can be slid to a band which lies in a small ball neighbourhood of one point, an ‘unknot’. Finally, surgery on the band $\overline{b}_1$, now in the small ball, gives the connected sum of $C$ with $S^2 \times S^1$. 

Following the above formalism for observables, suppose that $M$ is a 3-manifold and $\Sigma \subset M$. Then $C$ can be inserted in $M$ at $\Sigma$ to give $M_{\Sigma}C$. However there is an isomorphism

$$M_{\Sigma}C \cong \text{surgery}(M, b),$$

and the later does not depend on the choice of $\Sigma$. Therefore the consistency condition is satisfied if a very simple topological condition is satisfied. If $b$ can be deformed to lie in the boundary of $M$, then one has

$$\langle \pi M | \pi M \rangle = \langle M | \pi M \rangle,$$

sometimes called the strong consistency condition. This follows from the same argument that shows that $Z(C)$ is a multiple of a projector.

There are an infinite number of inequivalent such observables, but typically only a few of them will satisfy this consistency condition. Picking a surface $\Sigma \subset M$, one can see that there will be some bands in $\Sigma$ which will satisfy the condition, but others which may not, unless of course $\Sigma$ is parallel to the boundary of $M$, when all will be consistent. This situation is analogous to the situation in quantum mechanics, where for intermediate times in a Griffiths history, the typical observable will not be consistent with the rest of the history, but some special choices will be.

This verifies the existence of quantum interference phenomena in this example.

V. Examples of TQFTs in dimension three

The main examples of TQFTs were preceded by some theories which have a similar construction but turn out to be defined only for some manifolds. This is for the usual reason in quantum field theory, that some infinite sums fail to converge. I shall describe these theories first, before turning to the mathematically watertight TQFTs, mostly based on quantum groups.

A. Three-dimensional gravity

A state-sum model for a partition function is familiar from statistical mechanics, where it is defined by summing the Boltzmann weights, given by the exponentials of an energy, over a set of states or configurations for a system. The key features are that the states are determined by variables which are distributed over a lattice, and the weights are the product of factors defined locally on each lattice site.

The main ingredients for a state-sum model defining quantum gravity in dimension three were assembled by Ponzano and Regge[9]. The partition function is defined by a sum which can be thought of as a discrete version of a path integral. The variables are on the edges of a triangulation of the 3-manifold, somewhat similar to a lattice gauge theory. However, the theory differs from the lattice gauge theory in that a change of triangulation (for example a refinement) in the interior of the manifold is supposed to give exactly the same partition function. This is an expression of the topological invariance.

The variables are summed over an infinite set of values, and in general this sum will diverge. Ponzano and Regge suggested a regularisation scheme which clearly gives a finite answer for the simplest cases, but the properties of this regularisation have not been developed in general. There are some triangulations of some 3-manifolds for which the regularisation is not needed, namely those for which the boundary is ‘sufficiently large’ that the fixed boundary data place constraints on the interior variables appearing in the state sum, which constrain these to a finite range.
of values. Therefore, without needing to consider the properties of a regularisation, one can say that the theory is partly defined.

The variable associated to each edge ranges over the set of representations of the Lie group SU(2). This set can be identified with the non-negative integers. In the analogy with the path integral, these integers are regarded as lengths for these edges, thus defining a metric tensor on the manifold, which is flat on each simplex. This affords the most direct link between topological theories in three dimensions, and quantum gravity. As remarked earlier, it depends on some extra properties which this model has, in addition to the general properties investigated in the previous sections.

There is a Boltzmann weight for each tetrahedron, given by the value of the $6j$-symbol at the six representations of SU(2) on the edges of the tetrahedron. Ponzano and Regge gave arguments for a remarkable formula for an asymptotic limit for this Boltzmann weight, as the integers on the edges approach infinity. The limit is that the $6j$-symbol tends to the exponential of the Einstein action for the metric [9, 10, 11, 12], as given by Regge calculus [13]. There considerations were all heuristic, and it is still an open problem to provide a precise statement of the asymptotic limit, and proofs.

Witten provided a definition of quantum gravity in three dimensions starting from the Einstein action and the path integral [14]. The evaluation of the path integral boils down to a formula involving the Ray-Singer analytic torsion, a formula which is only finite in some circumstances. Some explicit calculations have been done by Carlip and Cosgrove[15]. The first circumstantial piece of evidence that this theory is the same as Ponzano-Regge is that the manifolds for which this is well-defined, such as handlebodies, are precisely those for which the Regge-Ponzano partition function can be calculated via a finite sum. The fact that both theories have a construction (approximately or exactly) from the Einstein action is the second piece of circumstantial evidence. The third piece of evidence is that both theories can be regarded, heuristically, as a limit of a well-defined theory, described below, for which there is a theorem asserting that the two theories are equivalent.

An argument which would settle the issue once and for all would be a regularisation of the Regge-Ponzano theory which gave the combinatorial definition in terms of the Reidemeister torsion. The equivalence would then rest on the equality of Reidemeister and Ray-Singer torsion. However such an argument has yet to be carried through.

B. Topological quantum field theories

The Ponzano-Regge theory requires very little modification to determine a bona fide TQFT. The Lie group SU(2) is replaced by the quantised enveloping algebra of the Lie algebra sl(2) at a root of unity. This manoeuvre ensures that there is a finite set of representations for which a completeness relation still holds. At a stroke, the infinite sums which plague three-dimensional gravity are replaced by finite sums and the theory satisfies all of Atiyah’s axioms for a TQFT. This is the theory of Turaev and Viro[16]. The theory was subsequently generalised to other quantised enveloping Lie algebras (quantum groups) [17, 18]. Although the formalism generalises easily enough, the non-trivial part of this work is to check that these other quantum groups do satisfy all of the relevant conditions. There are also models which apparently have nothing to do with quantum groups [19].

The path integral theory which is equivalent was described by Witten [20], and is
the theory determined by the action for three-dimensional gravity with a cosmological constant. With the appropriate choices of signature, this action is the difference of two independent Chern-Simons actions with group SU(2). This means, at least on the heuristic level of path integrals, that the partition function is the modulus square of the partition function for the Chern-Simons theory which Witten famously associated with knot theory [21]. This later theory was defined rigorously by Reshetikhin and Turaev [22] by combinatorial methods which were suggested by the properties of Witten’s theory. Therefore the modulus square of the Reshetikhin-Turaev partition function can be taken to be a definition of three-dimensional gravity with a cosmological constant. Roberts[23] proved that this is equal to the Turaev-Viro partition function, thus providing the final link to interpret the Turaev-Viro theory as a physical model.

The question of the corresponding physical interpretation of the TQFTs based on other quantum groups remains open.

The limiting procedure referred to above is to take the quantum parameter $q \to 1$ in the Turaev-Viro theory, or equivalently, the level $k \to \infty$ in the Chern-Simons theory. This corresponds to taking the cosmological constant $\Lambda \to 0$ in the Einstein action. It is not known whether such limits exist. However, it would be expected that $q \to 1$ leads to the Ponzano-Regge theory on the one hand, and $\Lambda \to 0$ to the three-dimensional gravity on the other hand.

C. Relevance to four dimensions

It is interesting to speculate on whether the experience with three-dimensional theories tells us anything about four dimensions. In a TQFT,

$$Z(S^1 \times \Sigma) = \text{tr}(P_\Sigma) = \dim(\text{Image } P_\Sigma),$$

a well-defined number. Therefore the space of ‘propagating modes’ is finite-dimensional. If, on the other hand, this space is infinite-dimensional, as is widely believed to be the case for quantum gravity, then $Z(S^1 \times \Sigma)$ is infinite and the theory is not completely definable. This argument is less clear-cut than it may seem, as we know little about the space of propagating modes in quantum gravity. When local observables are included in the theory, as considered in the next section, the theory has a sequence of similar such spaces of finite but arbitrarily large dimension, where the dimension increases with the resolution of the measurements. It is possible that such a construction is an effective substitute for an infinite-dimensional state space.

The other factor which has to be kept in mind is the different nature of topology in three and four dimensions. The manifold $S^1 \times \Sigma$ has an infinite fundamental group, which one can conjecture is the reason that its partition function is not defined in the infinite-dimensional case. Indeed this is the case in three-dimensional gravity[14]. The topology of closed 3-manifolds is dominated by the fundamental group, as there are no known simply-connected 3-manifolds other than $S^3$. Therefore the theory is drastically restricted if it is limited to invariants of simply-connected 3-manifolds or 3-manifolds with finite fundamental group. This is not such a restriction in four dimensions, where the diversity of the simply-connected manifolds is the fundamental issue of four-dimensional topology.

The three-dimensional gravity theory exists at all because it is defined for certain manifolds $M$ with boundary. These are where each element of the fundamental group is generated by a curve lying in the boundary of $M$. It is interesting to note that the consistency conditions in section IV C and the example in section IV D
also led to restrictions of a somewhat similar nature on the relation of \( M \) to its boundary.

VI. Local observables

A desirable feature for a theory of physics is localisation, the ability to decompose space into subsets, which may have boundary points in common, to discuss the physics of each subset separately, and the interaction between them due to the common boundary points. Localisation is a fundamental property of the standard theories of dynamics, the theory of fields being a local explanation for action at a distance.

For example, Newtonian gravity was originally formulated as a theory with an inverse-square law force, but can be reformulated using Poisson’s equation for the potential. This equation can be solved in a region \( A \) of space providing the potential is given on the boundary of \( A \). The boundary data is all that one needs to know about the sources in the rest of space. If \( B \) is a second region of space, disjoint from \( A \) except on the boundary, it is clear that the boundary data for \( B \) has to be equal to the boundary data for \( A \) for the points which coincide.

Now consider the case of topological quantum field theory. Let \( \Sigma \) be a closed \((d-1)\)-manifold, which can be thought of as ‘space’. Consider a decomposition

\[
\Sigma = A \cup B,
\]

where the manifolds \( A \) and \( B \) have common boundary \( A \cap B \). An observable which uses \( A \) and \( B \) as data is necessarily a local observable; as explained above, a global observable has an action which is invariant under isotopies of \( \Sigma \), and these isotopies do not preserve the decomposition into \( A \) and \( B \). An example of such an observable might be

There is a particle in \( A \).

Let this observable be represented by the projector \( \mathcal{O} \) in \( V(\Sigma) \). Since \( \mathcal{O} \) is not global, it ‘sees’ the kernel

\[
\ker(P_\Sigma) = \{ v : P_\Sigma(v) = 0 \},
\]

the part of \( V(\Sigma) \) that is redundant from the point of view of the Atiyah axioms. This is because \((n-1)\)-manifolds with boundary, such as \( A \) and \( B \), are not part of Atiyah’s scheme.

A concrete example of these constructions can be given in the three-dimensional state-sum models. The state space for a surface is constructed with the aid of an arbitrary triangulation, which we can assume is such that \( A \) and \( B \) are the union of complete triangles. Then there are state spaces \( V(A) \) and \( V(B) \), and \( V(\Sigma) \) is obtained as a certain subspace of \( V(A) \otimes V(B) \). This subspace is determined by a condition that certain boundary data common to \( A \) and \( B \) are equal. These are the labels on edges.

It is worth contrasting the operator \( \mathcal{O} \) with \( \mathcal{G} = P_\Sigma \mathcal{O} P_\Sigma \), which is a global observable, and thus ‘gauge invariant’ in the terminology borrowed from Yang-Mills theory. If \( \mathcal{O} \) were global then these would be equal. However in the case of a local observable these operators are not the same. For example, the norm of \( \mathcal{G} | M \rangle \) is not necessarily equal to \( \mathcal{O} | M \rangle \), as \( \mathcal{O} \) and \( P_\Sigma \) do not commute.
The operator $G$ corresponds to an observable constructed from the topological data given by identifying the decomposition $A \cup B$ with the middle $\Sigma \times \{0\}$ of the cylinder $\Sigma \times [-1, 1]$. This is diffeomorphic to the same construction with $A$ and $B$ replaced by their images under an isotopy of $\Sigma$, $A'$ and $B'$. Therefore the global observable cannot distinguish between $A$ and $A'$ or $B$ and $B'$. This accords with the fact that global observables commute with isotopies. The fact that $G$ is not a projector is related to this. One cannot prove, as was done in section IV B for global observables, that repeating the proposition $O$ at different times, which is the same as repeating $G$, is reliable, in the sense that the proposition at one time implies the same proposition at a later time. Indeed, it does not follow logically that

There is a particle in $A \iff$ There is a particle in $A'$,

since $A$ and $A'$ need not coincide.

The logical approach to quantum mechanics requires that observables are associated with particular sets, such as $A$, and not their equivalence classes under diffeomorphism or isotopy. This is because the theory is centred on the logical relation between propositions, which is governed by the relations between the corresponding sets. A topological relation such as ‘$A$ intersects $B$’ makes sense whereas there is no such relation between equivalence classes. The difficulty with the reliability of $O$ is that the TQFT has not registered any one particular identification of $\Sigma \times \{t_1\}$ with $\Sigma \times \{t_2\}$.

For this reason, observables referring to specific regions cannot be gauge invariant. Gauge invariance is a necessary feature of observables in Yang-Mills theories, but one has to bear in mind that general relativity is not a Yang-Mills gauge theory. It may seem puzzling that particular sets are singled out in the theory, when the theory is explicitly invariant under the action of diffeomorphisms in an appropriate way. At one level, this can be seen as a book-keeping device, the need to name one particular set $A$, so that it can be referred to explicitly by other related propositions. However there is a somewhat more physical explanation.

In general relativity, the set of all space-time events is a manifold. Therefore it is clear that specifying a particular property for the space-time events specifies a particular subset of this manifold. Diffeomorphisms of the space-time manifold do not play a basic role in the physical interpretation of the theory. Even the points of space-time which are ‘unobserved’ are determined by the unique collection of geometries of paths which lead to them from a fixed base-point [24]. However, when the space-time events are represented, either mathematically as points in a coordinate chart, or say in a computer, then there is an arbitrariness in the representation, in general, which is accounted for by the action of diffeomorphisms. The simplest and clearest interpretation of TQFT is that these features of general relativity continue to hold, namely, that the points of the manifold, perhaps together with some other data, each have an unambiguous physical meaning. This can be weakened somewhat, in that one does not require an interpretation for each and every point on the manifold, but for the elements of the relevant combinatorial presentation of the manifold, which is of a finite nature. This point reinforces the topological principle which was described in section IV C.

Since diffeomorphisms of $\Sigma$ act in $V(\Sigma)$, one can formulate a condition that the observable $O$ depends only on the topology of $A$ and $B$, in the same way that global observables depend only on the topology of $\Sigma$. 
This is that if $h$ is an isotopy $\Sigma \times [0,1] \to \Sigma \times [0,1]$, for which the sets $A \times [0,1]$ and $B \times [0,1]$ are invariant (for example $h(a,t) \in A \times \{t\}$), then

$$h\mathcal{O} = \mathcal{O}h = \mathcal{O}.$$

Suppose $P_\Sigma(A,B)$ is a projector in $V(\Sigma)$ whose image consists of vectors each fixed by every such isotopy $h$, and whose kernel is invariant by every such $h$. Then observables which satisfy

$$\mathcal{O}P_\Sigma(A,B) = P_\Sigma(A,B)\mathcal{O} = \mathcal{O}$$

satisfy this criterion. This can be described as saying that they are global relative to $\{A,B\}$.

If the theory provides such a projector, then it is natural to associate it with the $d$-manifold $\Sigma \times [0,1]$ equipped with the decomposition into the two subsets $A \times [0,1]$ and $B \times [0,1]$. Clearly if this projector is used for ‘time evolution’ instead of $P_\Sigma$, then one can prove that the observable $\mathcal{O}$ is reliable for measurements repeated in time. Intuitively, this is because specifying the subset $A \times [0,1]$ provides the required identification of the set $A$ at different times.

The picture which emerges is that the more structure on $\Sigma$ which is fixed, the larger the state space becomes ($\text{Image } P_\Sigma \subset \text{Image } P_\Sigma(A,B)$), and the correspondingly smaller the symmetry group becomes. The specification of a particular metric tensor, for example, can be regarded as a limiting case of this procedure. This can be considered in the context of the Turaev-Viro model. Any given metric tensor can be approximated to any given accuracy by picking a sufficiently fine triangulation and assigning a length to each edge [25].

A systematic investigation of the local observables goes beyond the general framework of topological quantum field theory. The set $A \times [0,1]$ is an example of a manifold with corners. The boundary is

$$\partial A \times [0,1] = (A \times \{0,1\}) \cup (\partial A \times [0,1]),$$

which is a union of $(d-1)$-manifolds with boundary. The boundary of these is the closed $(d-2)$-manifold $\partial A \times \{0,1\}$, the corners of this manifold. More generally, one can have corners in all dimensions down to zero, the basic examples being the cube or the simplex. A set of axioms for topological field theory which includes manifolds with corners should reduce to Atiyah’s axioms for the special cases where the corners are empty. The notion of an $n$-category[26] provides a framework for the general formalism, although at present it is not clear exactly which type of $n$-category is most useful. In any event, the state-sum models provide explicit examples of theories with corners.

References

[1] J.B. Hartle and S.W. Hawking, *Wave function of the universe*, Phys. Rev. D28, 2960–2975, (1983).
[2] M.F. Atiyah, *Topological quantum field theories*, Publ. Math. IHES 68, 175-186, (1989).
[3] M.F. Atiyah, *The geometry and physics of knots*, (Cambridge University Press, Cambridge, 1990).
[4] R.B. Griffiths, *Consistent histories and the interpretation of quantum mechanics*, J. Stat. Phys. 36, 219–272, (1984).
[5] R. Omnès, *From Hilbert space to common sense*, Ann. Phys. **201**, 354–447, (1990).
[6] H. Everett, “Relative State” formulation of quantum mechanics, Rev. Mod. Phys. **29**, 454–462, (1957).
[7] C. Rovelli, *On quantum mechanics*, hep-th/9403015, (1994).
[8] F. Dowker and A. Kent, *Properties of consistent sets*, hep-th/9409037, (1994).
[9] G. Ponzano and T. Regge, Semiclassical limit of Racah coefficients, Spectroscopic and group theoretical methods in physics (F. Bloch, ed.), pp. 1-58, (North-Holland, Amsterdam, 1968).
[10] T. Regge, General relativity without coordinates, Nuovo Cimento **19**, 558-571, (1961).
[11] K. Schulten and R.G. Gordon, *Semiclassical approximations to 3j- and 6j-coefficients for quantum-mechanical coupling of angular momenta*, J. Math. Phys. **16**, 1971–1988, (1975).
[12] J.W.Barrett and T.J. Foxon, Semiclassical limits of simpicial quantum gravity, Class. Quant. Grav. **11**, 543–556, (1994).
[13] E. Witten, Topology-changing amplitudes in 2+1 dimensional gravity, Nuclear Physics **B323**, 113–140, (1989).
[14] S. Carlip and R. Cosgrove, Topology change in (2+1)-dimensional gravity, gr-qc/9406006, (1994).
[15] V.G. Turaev and O.Y. Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, Topology **31**, 865–902, (1992).
[16] J.W. Barrett and B.W. Westbury, Invariants of piecewise-linear 3-manifolds, hep-th/ 9311155, Trans. Amer. Math. Soc., to appear (1996).
[17] V.G. Turaev, Quantum invariants of knots and 3-manifolds, De Gruyter studies in mathematics, vol. 18, (de Gruyter, Berlin, 1994).
[18] J.W. Barrett and B.W. Westbury, The equality of 3-manifold invariants, hep-th/9406019, Math. Proc. Cam. Phil. Soc., to appear (1995).
[19] E. Witten, 2+1 dimensional gravity as an exactly soluble system, Nuc. Phys. **B311**, 46–78, (1988).
[20] E.Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. **121**, 351–399, (1989).
[21] N. Reshetikhin, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. **103**, 547–597, (1991).
[22] J.D. Roberts, Skein theory and Turaev-Viro invariants, Topology, to appear (1995).
[23] J.W. Barrett, Holonomy and path structures in general relativity and Yang-Mills theory, Int. J. Th. Phys. **30**, 1171–1216, (1991).
[24] J.W. Barrett and P.E. Parker, Smooth limits of piecewise-linear approximations, J. Approx. Theory **76**, 107–122, (1994).
[25] R. Street, The algebra of oriented simplexes, J. Pure Appl. Alg. **49**, 283–335, (1987).