Differential operators in exterior domain and application

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Abstract

The abstract elliptic and parabolic equations on exterior domain are considered. The equations have top-order variable coefficients. The separability properties of boundary value problems for elliptic equation and well-posedness of the Cauchy problem for parabolic equations are established. In application, the well-posedness of Wentzell-Robin type mixed problem for parabolic equation, Cauchy problem for anisotropic parabolic equations and system of parabolic equations are derived.

Key Words: differential-operator equations, exterior problems, semigroups of operators, Banach-valued function spaces, operator-valued Fourier multipliers, interpolation of Banach spaces

1. Introduction, notations and background

Boundary value problems (BVPs) for differential-operator equations (DOEs) have been studied extensively by many researchers (see [3, 5, 8-23, 26] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [13] and [26]. The maximal regularity properties for differential operator equations have been studied in [2], [8], [9] and [17-23] for instance. The main objective of the present paper is to discuss the exterior BVPs for the following DOE with variable coefficients

\[
\varepsilon au^{(2)}(x) + Au(x) + \varepsilon^{\frac{1}{2}}A_1 u^{(1)}(x) + A_0 u(x) = f(x), \quad x \in \sigma, \tag{1.1}
\]

\[
\sum_{i=0}^{\mu_1} \alpha_i \varepsilon^{\nu_i} u^{(i)}(0) = 0, \quad \sum_{i=0}^{\mu_2} \beta_i \varepsilon^{\nu_i} u^{(i)}(b) = 0,
\]

where \( \sigma \) is an exterior domain, i.e. \( \sigma = (-\infty, \infty) \setminus [0, b] \), \( a = a(x) \) is a complex-valued function, \( \varepsilon \) is a positive parameter, \( \nu_i = \frac{i}{2} + \frac{1}{2p} \), \( p \in (1, \infty) \); \( A = A(x) \), \( A_j = A_j(x) \) are linear operator functions in a Banach space \( E \), \( \alpha_i, \beta_i \) are complex numbers, \( \mu_k \in \{0, 1\} \).

In this paper, the \( E \)-valued \( L_p \)-separability properties of this problem is obtained. Especially, we prove that the corresponding differential operator is \( R \)-positive and also is a negative generator of the analytic semigroup.
Note that, the principal part of the corresponding differential operator is non selfadjoint. Nevertheless, the sharp uniform coercive estimates for the resolvent of corresponding differential operators are established. In section 6, nonlocal BVP for degenerate abstract elliptic equation considered in the moving domain. By using the maximal regularity properties of linear problem (1.1) we derive the existence and uniqueness of BVP for the following nonlinear degenerate abstract equation

\[ a(x)u^{(2)}(x) + B(x, u, u^{(1)}) u(x) = F(x, u, u^{(1)}) + f(x), \]  

in exterior domain, where \( a \) is a complex valued function, \( B \) and \( F \) are nonlinear operator in a Banach space \( E \).

Then, by using the separability properties of the elliptic problem (1.1), the \( L_p(\sigma_T; E) \) well-posedness is established for the following parabolic interior mixed problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \varepsilon a \frac{\partial^2 u}{\partial x^2} + A_0 u + A_1 \frac{\partial u}{\partial x} + A_0 u = f(t, x), & \quad t \in (0, T), \ x \in \sigma, \\
\sum_{i=0}^{\mu_1} \alpha_i \varepsilon^\nu \ u^{(i)}(t, 0) = 0, & \quad \sum_{i=0}^{\mu_2} \beta_i \varepsilon^\nu \ u^{(i)}(t, b) = 0, \\
u(0, x) = 0, & \quad x \in \sigma.
\end{aligned}
\]  

(1.3)

Here \( \sigma_T = \sigma \times (0, T), \ p = (p_1, p) \)

and \( L_p(\sigma_T; E) \) denotes the space of all \( E \)-valued \( p \)-summable functions with mixed norm i.e., the space of all \( E \)-valued measurable functions \( f \) defined on \( \sigma_T \) for which

\[
\|f\|_{L_p(\sigma_T; E)} = \left( \int_0^T \left( \int_\sigma \|f(t, x)\|^p_E \, dx \right)^{\frac{p}{p_1}} \, dt \right)^{\frac{1}{p}} < \infty.
\]

Moreover, let we choose \( E = L_2(0, 1) \) in (1.1) and \( A \) to be differential operator with generalized Wentzell-Robin boundary condition defined by

\[
D(A) = \left\{ u \in W^{2}_{p_1} (0, 1), \ B_j u = Au(j) + \sum_{i=0}^{1} \alpha_{ij} u^{(i)}(j), \ j = 0, 1 \right\},
\]

\[ Au = a_1 u^{(2)} + b_1 u^{(1)} + cu, \]

where \( \alpha_{ij} \) are complex numbers, \( a_1, b_1, c \) are complex-valued functions and \( u^{(0)}(x) = u(x) \). Then, we get the \( L_{p} (\Omega) \) - well-posedness of the following Wentzell-Robin type mixed problem for parabolic equation

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \varepsilon a \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial y} + cu = f(t, x, y), & \quad t \in (0, T), \ x, y \in \Omega, \\
u \left(\frac{\partial u}{\partial y}\right)|_{y=0} = 0, & \quad x \in \partial \Omega.
\end{aligned}
\]  

(1.4)
that are defined on $\Omega$ with the norm $\sum_{i=0}^{\mu_1} \alpha_i \varepsilon^{\nu_i} u_i^{(i)}(t, 0) = 0$, $\sum_{i=0}^{\mu_2} \beta_i \varepsilon^{\nu_i} u_i^{(i)}(t, b) = 0$,
\[ B_j u = 0, \quad j = 0, 1, \quad t \in (0, T), \quad x \in \sigma, \quad y \in (0, 1), \quad (1.5) \]

where $\tilde{p} = (\tilde{p}, 2)$, $\varepsilon$ is a small parameter and $\Omega = \sigma_T \times (0, 1)$.

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [41, 42] and the references therein. The maximal regularity properties of DOEs in Banach spaces were considered e.g. in [2, 4, 9, 16, 21-23, 25].

Let $L_p(\Omega; E)$ denote the space of strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

\[
\|f\|_{L_p} = \|f\|_{L_p(\Omega; E)} = \left( \int \|f(x)\|^p_{E} \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
\]

The Banach space $E$ is called an $UMD$-space if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{|x-y|^p} dy$ is bounded in $L_p(\mathbb{R}, E)$, $p \in (1, \infty)$ (see. e.g. [7]). $UMD$ spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p$, $q \in (1, \infty)$.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{C}$ be the set of the complex numbers and $S_{\varphi} = \{ \lambda \in \mathbb{C}, \mid \arg \lambda \mid \leq \varphi \} \cup \{0\}$, $0 \leq \varphi < \pi$.

Let $E_1$ and $E_2$ be two Banach spaces. $L(E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ into $E_2$. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\| (A + \lambda I)^{-1} \|_{L(E)} \leq M (1 + |\lambda|)^{-1}$ for any $\lambda \in S_{\varphi}$, $0 \leq \varphi < \pi$, where $I$ is the identity operator in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by $A_{\lambda}$. It is known [24, §1.15.1] that a positive operator $A$ has well-defined fractional powers $A^{\theta}$. Let $E(A^{\theta})$ denote the space $D(A^{\theta})$ with norm

\[
\|u\|_{E(A^{\theta})} = \left( \|u\|^p + \|A^{\theta} u\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \theta < \infty.
\]

Let $S(R^n; E)$ denote the Schwartz class, i.e., the space of all $E$-valued rapidly decreasing smooth functions on $R^n$ and $C(\Omega; E)$ denotes the space of all $E$-valued norm bounded functions on $\Omega$. Let $F$ denote the Fourier transformation. A function $\Psi \in C(R^n; L(E))$ is called Fourier multiplier in $L_{p, \gamma}(R^n; E)$ if the map

\[ u \to \Phi u = F^{-1} \Psi(\xi) Fu, \quad u \in S(R^n; E) \]

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is well defined and extends to a bounded linear operator in $L_p(R^n; E)$. The set of all multipliers in $L_p(R^n; E)$ will denoted by $M_p(E)$.

**Definition 1.1.** A Banach space $E$ is said to be a space satisfying multiplier condition with respect to $p \in (1, \infty)$ if for any $\Psi \in C^{(1)}(\mathbb{R}; L(E))$ the $R$-boundedness (see e.g. [9, § 4.1]) of the set
\[
\left\{ \xi^j\Psi^{(j)}(\xi) : \xi \in \mathbb{R}\setminus \{0\}, \ j = 0, 1 \right\}
\]
implies $\Psi \in M_p(E)$.

**Remark 1.1.** Note that if $E$ is $UMD$ space, then for example, by [25], [9], [11] this space satisfies the multiplier condition.

By $(E_1, E_2)_{\theta, p}, 0 < \theta < 1, 1 < p \leq \infty$ we will denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the $K$-method [24, §1.3.2].

The operator $A(x)$ is said to be $\varphi$-positive uniformly with respect to $x \in G$ in $E$ with bound $M > 0$ if $D(A(x))$ is independent of $x$, $D(A(x))$ is dense in $E$ and $\|A(x) + \lambda\|^{-1} \leq \frac{M}{|1 + |\lambda||}$ for all $\lambda \in S(\varphi), 0 \leq \varphi < \pi$, where $M$ is independent of $x$.

The $\varphi$-positive operator $A(x), x \in \sigma$ is said to be uniformly $R$-positive in a Banach space $E$ if there exists $\varphi \in [0, \pi)$ such that the set
\[
\left\{ A(x) (A(x) + \xi I)^{-1} : \xi \in S_\varphi \right\}
\]
is uniformly $R$-bounded, that is
\[
\sup_{x \in \sigma} R \left\{ \left\{ A(x) (A(x) + \xi I)^{-1} : \xi \in S_\varphi \right\} \right\} \leq M.
\]

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embedded into $E$. Let $\sigma$ be a domain in $\mathbb{R}$. Consider the Sobolev-Lions type space $W^m_p(\sigma; E_0, E)$ that consisting of all functions $u \in L_p(\sigma; E_0)$ that have generalized derivatives $u^{(m)} \in L_p(\sigma; E)$ with the norm
\[
\|u\|_{W^m_p} = \|u\|_{W^m_p(\sigma; E_0, E)} = \|u\|_{L_p(\sigma; E_0)} + \|u^{(m)}\|_{L_p(\sigma; E)} < \infty.
\]

The embedding theorems play a key role in the perturbation theory of DOEs. For estimating lower order derivatives we use following embedding theorems from [21]:

**Theorem A.** Assume the following conditions are satisfied:

1. $E$ is a Banach space satisfying the multiplier condition with respect to $p$;

2. $A$ is an $R$-positive operator in $E$, $\sigma \subset \mathbb{R}$;

3. $0 \leq j \leq m, 0 \leq \mu \leq 1 - \frac{m}{p}, 1 < p < \infty$; $h$ is a positive parameter that $0 < h < h_0 < \infty$;

4. There exists a bounded linear extension operator from $W_p^m(\sigma; E(A), E)$ to $W_p^m((-\infty, \infty); E(A), E)$. 

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Then the embedding \( D^j W^m_p (\sigma; E(A), E) \subset L_p \left( \sigma; E \left( A^{1 - \frac{i}{m} - \mu} \right) \right) \) is continuous. Moreover, for \( u \in W^m_p (\sigma; E(A), E) \) the following estimate holds
\[
\left\| u^{(j)} \right\|_{L_p \left( \sigma; E \left( A^{1 - \frac{i}{m} - \mu} \right) \right)} \leq h^\mu \left\| u \right\|_{W^m_p (\sigma; E(A), E)} + h^{- (1 - \mu)} \left\| u \right\|_{L_p (\sigma; E)} .
\]

Consider the DOE with variable coefficients on \((0, \infty)\)
\[
\varepsilon a(x) u^{\nu\nu} (x) + A(x) u(x) + \sum_{i=0}^{1} \varepsilon^i A_i (x) u^{(i)} (x) + \lambda u(x) = f(x), \quad (1.6)
\]
where \( a(\cdot) \) is a real-valued function, \( \varepsilon \) is a positive parameter, \( A(\cdot) \) and \( A_j (\cdot) \) are linear operator functions in a Banach space \( E \), \( \lambda \) is a complex parameter.

Let \( \omega_1 = \omega_1 (x) \), \( \omega_2 = \omega_2 (x) \) be roots of the equation \( a(x) \omega^2 + 1 = 0 \).

From [21] we obtain

**Theorem A.2.** Suppose the following conditions are satisfied:
(1) \( E \) is a Banach space satisfying the multiplier condition with respect to \( p \in (1, \infty) \);
(2) \( A(x) \) is an \( R \)-positive operator in \( E \) for \( \varphi \in [0, \pi) \) uniformly with respect to \( x \in [0, 1] \) and \( A(x) A^{-1} (x_0) \in C \left( (\infty, \infty) ; L (E) \right) \) for a.e. \( x_0 \in (\infty, \infty) \);
(3) for any \( \delta > 0 \) there is a positive \( C(\delta) \) such that
\[
\left\| A_1 (x) u \right\| \leq \delta \left\| u \right\|_{\left( E(A), E \right)} + C(\delta) \left\| u \right\|
\]
for \( u \in \left( E(A), E \right) \), \( \delta \in (0, \infty) \) and \( \left\| A_0 (x) u \right\| \leq \delta \left\| A u \right\|_{E} + C(\delta) \left\| u \right\| \) for \( u \in D(A) \);
(4) \( a \in C_b (\infty, \infty) \) and \( \text{Re} \omega_k \neq 0 \) and \( \omega_k \in S(\varphi) \) for \( \lambda \in S(\varphi), 0 \leq \varphi < \pi, k = 1, 2 \) a.e. \( x \in \mathbb{R} \).

Then problem (1.6) has a unique solution \( u \in W^2_p (\mathbb{R}; E(A), E) \) for \( f \in L_p (\mathbb{R}; E) \). Moreover, for \( |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \) the following uniform coercive estimate holds
\[
\sum_{i=0}^{2} \left| \lambda \right|^{- \frac{i}{2}} \varepsilon^i \left\| u^{(i)} \right\|_{L_p (\mathbb{R}; E)} + \left\| A u \right\|_{L_p (\mathbb{R}; E)} \leq C \left\| f \right\|_{L_p (\mathbb{R}; E)} .
\]

Consider the nonhomogenous BVP for DOE with constant coefficients on half plane
\[
\varepsilon a u^{\nu\nu} (x) + A u(x) + \lambda u(x) = f(x), \quad x \in (0, \infty), \quad (1.7)
\]
where \( \nu \in (E(A), E) \), \( \nu \in (0, \infty) \), \( a \) is a complex number, \( \varepsilon \) is a positive parameter, \( \nu_1 = \frac{1}{2} + \frac{1}{2} p \), \( A \) is a linear operator in a Banach space \( E \), \( \lambda \) is a complex parameter, \( \alpha_i \) are complex numbers and \( \nu \in \{0, 1\}, \alpha_i \neq 0 \).
Let $\omega_1, \omega_2$ be roots of equation $a\omega^2 + 1 = 0$.

From [22] we obtain.

**Theorem A.** Suppose the following conditions are satisfied:

1. $E$ is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$;
2. $A$ is an $R$-positive operator in $E$ for $\varphi \in [0, \pi)$;
3. $a$ is a complex number such that $\text{Re}\omega_k \neq 0$ and $\frac{1}{\omega_k} \in S(\varphi)$ for $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, $k = 1, 2$.

Then problem (1.7) has a unique solution $u \in W^2_p(0, \infty; E(A), E)$ for $f \in L_p(0, \infty; E)$. Moreover, for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$ the following uniform coercive estimate holds

$$
\sum_{i=0}^2 |\lambda|^{1 - \frac{i}{2}} \varepsilon^\frac{i}{2} \|u^{(i)}\|_{L_p(0, \infty; E)} + \|Au\|_{L_p(0, \infty; E)} \leq C \|f\|_{L_p(0, \infty; E)} + \|\varphi\|_{(E(A), E)^p, p}.
$$

Consider the nonlocal BVP for DOE with constant coefficients

$$
\varepsilon a u^{(2)}(x) + Au(x) + \lambda u(x) = f(x),
$$

where $f_k \in (E(A), E)_{\mu_k + 1, p}$, $A$ is a linear operator in a Banach space $E$, $\varepsilon$ is a positive parameter, $\nu_i = \frac{1}{2} + \frac{1}{2p}$, $\lambda$ is a complex parameter, $a$, $\alpha_{k_i}$, $\beta_{k_i}$ are complex numbers and $\mu_k \in \{0, 1\}$.

From [20] we obtain.

**Theorem A.** Suppose the following conditions are satisfied:

1. $E$ is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$;
2. $A$ is an $R$-positive operator in $E$ for $\varphi \in [0, \pi)$;
3. $a$ is a complex number such that $\text{Re}\omega_k \neq 0$ and $\frac{1}{\omega_k} \in S(\varphi)$ for $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, $k = 1, 2$;
4. $\alpha_k = \alpha_{k_{\nu_k}} \neq 0$, $\beta_k = \beta_{k_{\nu_k}} \neq 0$, $\eta = (-1)^{\nu_1} \alpha_1 \beta_2 - (-1)^{\nu_2} \alpha_2 \beta_1 \neq 0$, $a > 0$.

Then problem (1.8) has a unique solution $u \in W^2_p(0, 1; E(A), E)$ for $f \in L_p(0, 1; E)$ and $f_k \in (E(A), E)_{\mu_k + 1, p}$. Moreover, for $|\arg \lambda| \leq \varphi$ and sufficiently large $|\lambda|$ the following uniform coercive estimate holds

$$
\sum_{i=0}^2 |\lambda|^{1 - \frac{i}{2}} \varepsilon^\frac{i}{2} \|u^{(i)}\|_{L_p(0, 1; E)} + \|Au\|_{L_p(0, 1; E)} \leq C \left[ \|f\|_{L_p(0, 1; E)} + \sum_{k=1}^2 \|f_k\|_{(E(A), E)_{\mu_k + 1, p}} \right].
$$

By virtue Lions Petree trace theorem (see of [24, §1.8.2]) we obtain
\textbf{Theorem A5.} Assume \( m \) and \( j \) are integers, \( 0 \leq j \leq m - 1, \), \( \theta_j = \frac{m+1}{pm} \), \( p \in (1, \infty) \); \( \varepsilon \in (0, 1) \) is a parameter, \( x_0 \in [0, b] \). Then, the linear transformation \( u \rightarrow u^{(j)}(x_0) \) is bounded from \( W^m_p(0, b; E_0, E) \) onto \( (E_0, E)_{\theta_j, p} \) and the following inequality holds
\[
\varepsilon^{\theta_j} \left\| u^{(j)}(x_0) \right\|_{(E_0, E)_{\theta_j, p}} \leq C \left( \left\| \varepsilon u^{(m)} \right\|_{L_{p, \gamma}(0, b; E)} + \| u \|_{L_{p, \gamma}(0, b; E_0)} \right).
\]

2. Abstract equation with variable coefficients

Consider the exterior BVP for differential-operator equation with variable coefficients
\[
Lu = \varepsilon au^{(2)} + Au + \sum_{i=0}^{1} \varepsilon^{\frac{i}{\gamma}} A_i u^{(i)} + \lambda u = f, \quad (2.1)
\]
\[
L_1 u = \sum_{i=0}^{\mu_1} \alpha_i \varepsilon^{\gamma_i} u^{(i)}(0) = 0, \quad L_2 u = \sum_{i=0}^{\mu_2} \beta_i \varepsilon^{\gamma_i} u^{(i)}(b) = 0, \quad (2.2)
\]
where \( a = a(x) \) is a complex-valued function, \( \varepsilon \) is a positive parameter, \( \nu_i = \frac{i}{\gamma} + \frac{1}{pm}, \) \( u = u(x) \), \( f = f(x) \), \( x \in \sigma \) are \( E \)-valued unknown and date functions; \( A = A(x) \) and \( A_j = A_j(x) \) are linear operator functions in a Banach space \( E \), \( \lambda \) is a complex parameter, \( \alpha_i, \beta_i \) are complex numbers, \( \mu_k \in \{0, 1\} \) and \( \sigma = \mathbb{R} \setminus [0, b] \).

A function \( u \in W^2_p(\sigma; E(A), E) \) satisfying the equation (2.1) a.e. on \( \sigma \) is said to be the solution of the equation (2.1) on \( \sigma \).

Consider the problem (2.1)–(2.2). Let \( X = L_p(\sigma; E) \) and \( Y = W^2_p(\sigma; E(A), E) \).

Let \( \omega_1 = \omega_1(x) \), \( \omega_2 = \omega_2(x) \) be roots of equation \( a(x) \omega^2 + 1 = 0 \).

The main result of this section is the following:

\textbf{Theorem 2.1.} Assume the following conditions are satisfied:

Suppose the following conditions are satisfied:

(1) \( E \) is a Banach space satisfying the multiplier condition with respect to \( p \in (1, \infty) \);

(2) \( A(x) \) is an \( R \)-positive operator in \( E \) for \( \varphi \in [0, \pi] \) uniformly with respect to \( x \in [0, 1] \) and \( A(x) A^{-1}(x_0) \in C(\tilde{\sigma}; L(E)) \) for \( x_0 \in (0, 1) \);

(3) for any \( \delta > 0 \) there is a positive \( C(\delta) \) such that
\[
\left\| A_1(x) u \right\| \leq \delta \left\| u \right\|_{L_{p, \gamma}(E(A), E)_{\frac{1}{2}, \infty}} + C(\delta) \left\| u \right\| \text{ for } u \in (E(A), E)_{\frac{1}{2}, \infty}
\]
and
\[
\left\| A_0(x) u \right\| \leq \delta \left\| Au \right\|_E + C(\delta) \left\| u \right\|
\]
for \( u \in D(A) \);

(4) \( a \in C(\tilde{\sigma}), \text{ Re} \omega_k \neq 0 \) and \( \frac{1}{\omega_k} \in S(\varphi) \) for \( \lambda \in S(\varphi), 0 \leq \varphi < \pi, \)
\( k = 1, 2 \). a.e. \( x \in \sigma \).

Then problem (2.1) – (2.2) has a unique solution \( u \in W^2_p(\sigma; E(A), E) \) for \( f \in L_p(\sigma; E) \). Moreover, for \( |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \) the following uniform coercive estimate holds

\[
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\]
\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{j}{p}} \|u^{(i)}\|_{L^{p}(\sigma; E)} + \|Au\|_{L^{p}(\sigma; E)} \leq C \|f\|_{L^{p}(\sigma; E)}. \tag{2.3}
\]

**Proof.** First of all, we will show the uniqueness of solution. Let \(G_1, G_2, \ldots, G_n, \ldots\) be regions in \(\mathbb{R}\) and \(\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots\) correspond to a partition of unit on \(\sigma\), which functions \(\varphi_j\) are smooth functions on \(\mathbb{R}\), \(\text{supp} \varphi_j \subset G_j\) and \(\sum_{j=1}^{\infty} \varphi_j(x) = 1\) for \(x \in \sigma\). Then for all \(u \in Y\) we have \(u(x) = \sum_{j=1}^{\infty} u_j(x)\), where \(u_j(x) = u(x) \varphi_j(x)\).

Let \(u \in Y\) be a solution of (2.1) – (2.2). Then from (2.1) – (2.2) we obtain

\[
(L + \lambda) u_j = \varepsilon a(u_j^{(2)}(x)) + (A + \lambda) u_j(x) = f_j(x), \tag{2.4}
\]

where

\[
f_j = f \varphi_j + \varepsilon a \left( 2u^{(1)} \varphi_j^{(1)} + u \varphi_j^{(2)} \right) + \varepsilon \frac{1}{2} \varphi_j^{(1)} A1 u,
\]

\[
L_1 u_j = \tau_1, \quad L_2 u_j = \tau_2, \quad j = 1, 2, \ldots, \infty, \quad \tau_1 = \alpha_1 u(0) \varphi_j(0), \quad \tau_2 = \alpha_1 u(b) \varphi_j(b).
\]

By Lemma \(A_5\), \(\tau_1, \tau_2 \in (E(A), E)_{\frac{1}{2}, p}\). By freezing coefficients in (2.4) we obtain that

\[
\varepsilon a(x_{0j}) u_j^{(2)}(x) + (A(x_{0j}) + \lambda) u_j(x) = f_j(x), \tag{2.6}
\]

\[
L_1 u_j = \tau_1, \quad L_2 u_j = \tau_2, \quad j = 1, 2, \ldots, \infty,
\]

where

\[
f_j = f_j + [A(x_{0j}) - A(x)] u_j - \varepsilon [a(x) - a(x_{0j})] u_j^{(2)}.
\]

Since functions \(u_j(x)\) have compact supports, by extending \(u_j(x)\) on the outsides of \(\text{supp} \varphi_j\) we obtain BVPs for DOE with constant coefficients

\[
\varepsilon a(x_{0j}) u_j^{(2)}(x) + (A(x_{0j}) + \lambda) u_j = f_j, \quad L_1 u_j = \tau_1, \quad L_2 u_j = \tau_2. \tag{2.8}
\]

Since \(a\) is uniformly bounded on \(\sigma\) for all small \(\rho > 0\) there is a large \(r_0 > 0\) such that \(|a(x) - a(\pm \infty)| \leq \delta\) for all \(|x| \geq r_0\). Let

\[
G_0 = (-\infty, \infty) \setminus O_{r_0}(0), \quad O_{r_0}(0) = \{x \in \sigma, \ |x| \leq r_0\}.
\]

Cover \(O_{r_0}(0)\) by finitely many intervals \(G_j = O_{r_j}(x_{0j})\) such that

\[
|a(x) - a(x_{0j})| \leq \delta \text{ for } |x - x_{0j}| \leq r_j, j = 1, 2, \ldots,
\]

8
Define coefficients of local operators, i.e.

\[
a^0 (x) = \begin{cases} 
    a (x), & x \notin O_{r_0} (0) \\
    a \left( r_0^2 \frac{\| x \|}{| x |^2} \right), & x \in O_{r_0} (0)
\end{cases}
\]

\[
a^j (x) = \begin{cases} 
    a (x), & x \in O_{r_j} (x_{0j}) \\
    a \left( x_{0j} + r_0^2 \frac{x - x_{0j}}{| x - x_{0j} |^2} \right), & x \notin O_{r_j} (x_{0j})
\end{cases}
\]

and

\[
A^0 (x) A^{-1} (x_{0j}) = \begin{cases} 
    A (x) A^{-1} (x_{0j}), & x \notin O_{r_0} (0) \\
    A \left( r_0^2 \frac{\| x \|}{| x |^2} \right) A^{-1} (x_{0j}), & x \in O_{r_0} (0)
\end{cases}
\]

\[
A^j (x) A^{-1} (x_{0j}) = \begin{cases} 
    A (x) A^{-1} (x_{0j}), & x \in O_{r_j} (x_{0j}) \\
    A \left( x_{0j} + r_0^2 \frac{x - x_{0j}}{| x - x_{0j} |^2} \right) A^{-1} (x_{0j}), & y \notin O_{r_j} (x_{0j})
\end{cases}
\]

for each \( j = 1, 2, \ldots \). Then, for all \( x \in \sigma \) and \( j = 0, 1, 2, \ldots \) we get

\[ |a^j (x) - a^j (x_{0j})| \leq \delta \text{ and } \| A^j (x) A^{-1} (x_{0j}) - A^j (x_{0j}) A^{-1} (x_{0j}) \|_{B(E)} < \delta. \]

Let \( \varphi_j \) such that \( 0, b \in \text{ supp } \varphi_j \). Then by virtue of Theorem A4 we obtain that problem (2.8) has a unique solution \( u_j \) and the coercive uniform estimates hold

\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{r}{2}} \varepsilon^{\frac{r}{2}} \left\| u_j^{(i)} \right\|_{G_j, p} + \| A u_j \|_{G_j, p} \leq C \| F_j \|_{G_j, p} + \sum_{k=1}^{2} \| \kappa_k \|_{E_p},
\]

where, \( \| \cdot \|_{G_j, p} \) denote \( E \)-valued \( L_p \)-norms on \( G_j \) and \( E_p = (E(A), E) \). Then by using Theorems A1 and A6 we obtain from the above estimate the following

\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{r}{2}} \varepsilon^{\frac{r}{2}} \left\| u_j^{(i)} \right\|_{G_j, p} + \| A u_j \|_{G_j, p} \leq C \| F_j \|_{G_j, p}.
\]

Let \( \varphi_j \) such that \( 0, 1 \in \text{ supp } \varphi_j \). Hence, \( \kappa_k = 0 \). Then in a similar way, Theorem A2 and Theorem A3 imply the same estimates

\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{r}{2}} \varepsilon^{\frac{r}{2}} \left\| u_j^{(i)} \right\|_{G_j, p} + \| A u_j \|_{G_j, p} \leq C \| F_j \|_{G_j, p}
\]

for domains \( G_j \) adjoin the boundary point \( 0 \) and \( b \). Hence, using properties of the smoothness of coefficients of equations (2.5), (2.7) and choosing diameters of \( \text{ supp } \varphi_j \) sufficiently small, we get

\[
\| F_j \|_{G_j, p} \leq \delta \| u_j \|_{W^2(G_j, E(A), E)} + C (\delta) \| f_j \|_{G_j, p},
\]

where \( \delta \) is a sufficiently small positive number and \( C (\delta) \) is a continuous function. Consequently, from (2.9)-(2.11) by using Theorem A1 we get

\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{r}{2}} \varepsilon^{\frac{r}{2}} \left\| u_j^{(i)} \right\|_{G_j, p} + \| A u_j \|_{G_j, p} \leq C (\delta),
\]
Choosing $\delta < 1$ from the above inequality we have

$$
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{i} \left\| u^{(i)} \right\|_{G_{j},p} + \| Au \|_{G_{j},p} \leq C \left[ \| f \|_{G_{j},p} + \| u \|_{G_{j},p} \right], \quad j = 1, 2, \ldots.
$$

(2.12)

Then using the equality $u(x) = \sum_{j=1}^{\infty} u_{j}(x)$ and the estimate (2.12) for $u \in Y$ we have

$$
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{i} \left\| u^{(i)} \right\|_{p} + \| Au \|_{p} \leq C \left[ \| (L + \lambda) u \|_{p} + \| u \|_{p} \right].
$$

(2.13)

Let $u \in Y$ be solution of problem (2.1) - (2.2). Then for $|\arg \lambda| \leq \phi$ we have

$$
\| u \|_{X} = \| (L + \lambda) u - Lu \|_{X} \leq \frac{1}{\lambda} \left[ \| (L + \lambda) u \|_{X} + \| u \|_{Y} \right].
$$

(2.14)

Then by Theorem A1, by virtue of (2.12) and (2.14) for sufficiently large $|\lambda|$ we have

$$
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{i} \left\| u^{(i)} \right\|_{X} + \| Au \|_{X} \leq C \| (L + \lambda) u \|_{X}.
$$

(2.15)

Consider the operator $O_{\varepsilon}$ in $X$ generated by problem (2.1) - (2.2), i.e.,

$$
D(O_{\varepsilon}) = W^{2}_{p} (\sigma; E(A), E, L_{1}, L_{2}),
$$

$$
O_{\varepsilon} u = \varepsilon a_{j}^{(1)}(x) + A_{j}(x) + \sum_{i=0}^{1} \varepsilon^{\frac{i}{2}} A_{j}^{(i)} u^{(i)}.
$$

The estimate (2.15) implies that the problem (2.1) - (2.2) has only a unique solution and the operator $O + \lambda$ has an invertible operator in its rank space. We need to show that this rank space coincides with the space $X$. We consider the smooth functions $g_{j} = g_{j}(x)$ with respect to the partition of the unique $\varphi_{j} = \varphi_{j}(x)$ on $\sigma$ that equal one on $\text{supp} \varphi_{j}$, where $\text{supp} g_{j} \subseteq G_{j}$ and $|g_{j}(x)| < 1$. Let us construct the function $u_{j}$ for all $j$, that are defined on $\Omega_{j} = \sigma \cap G_{j}$ and satisfying the problem (2.1) - (2.2). The problem (2.1) - (2.2) can be expressed as

$$
\varepsilon a_{j}(x_{0j}) u_{j}^{(2)} + (A_{j}(x_{0j}) + \lambda) u_{j} = g_{j} \{ F_{j} + [A(x_{0j}) - A(x)] u_{j} \}
$$

(2.16)
Consider the \( L_p(G_j;E) \) – realization of the above local operators \( O_{j\lambda\varepsilon} = O_{j\varepsilon} + \lambda \) defined as

\[
D(O_{j\lambda\varepsilon}) = W_p^2(G_j;E(A),E,L_1,L_2),
\]

\[
O_{j\lambda\varepsilon}u = \varepsilon a(x_{0j})u^{(2)}(x) + (A(x_{0j}) + \lambda)u(x).
\]

By virtue of Theorem A_1, for \( f \in L_p(G_j;E) \), \(|\arg \lambda| \leq \varphi\) and sufficiently large \(|\lambda|\) we have

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^\frac{i}{2} \left\| \frac{d^i}{dx^i} O_{j\lambda\varepsilon}^{-1} f \right\|_p + \left\| AO_{j\lambda}^{-1} f \right\|_p \leq C \|f\|_p.
\]  

(2.17)

Extending \( u_j \) zero on the outside of \( \text{supp} \varphi_j \) and passing substitutions \( u_j = O_{j\lambda\varepsilon}^{-1} v_j \) in (2.17), obtain equations with respect to \( v_j \).

\[
v_j = K_{j\lambda\varepsilon} v_j + g_j f, \quad j = 1, 2, ..., N.
\]  

(2.18)

By virtue of Theorem A_1 and estimate (2.17), in view of the smoothness of the coefficients of the expression \( K_{j\lambda} \), for sufficiently large \(|\lambda|\) we have \( \|K_{j\lambda}\| < \delta \), where \( \delta \) is sufficiently small. Consequently, equations (2.18) have unique solutions \( v_j = [I - K_{j\lambda\varepsilon}]^{-1} g_j f \). Moreover,

\[
\|v_j\|_X = \left\| [I - K_{j\lambda\varepsilon}]^{-1} g_j f \right\|_X \leq \|f\|_X.
\]

Whence, \( [I - K_{j\lambda\varepsilon}]^{-1} g_j \) are bounded linear operators from \( X \) to \( L_p(G_j;E) \). Thus, we obtain that

\[
u_j = U_{j\lambda\varepsilon} f = O_{j\lambda\varepsilon}^{-1} [I - K_{j\lambda\varepsilon}]^{-1} g_j f
\]

are solutions of (2.18). Consider the linear operator \((U + \lambda)\) in \( X \) such that

\[
(U + \lambda) f = \sum_{j=1}^{\infty} \varphi_j (y) U_{j\lambda\varepsilon} f.
\]

It is clear from the constructions \( U_{j\lambda\varepsilon} \) and the estimate (2.17) that operators \( U_{j\lambda\varepsilon} \) are bounded linear from \( X \) to \( Y \) and

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^\frac{i}{2} \left\| \frac{d^i}{dx^i} U_{j\lambda\varepsilon}^{-1} f \right\|_X + \left\| AU_{j\lambda\varepsilon}^{-1} f \right\|_X \leq C \|f\|_X.
\]  

(2.19)

Therefore, \((U + \lambda)\) is a bounded linear operator from \( L_p \) to \( L_p \). Let \( O \) denote the operator in \( X \) generated by BVP (2.1) – (2.2). Then act of \((O + \lambda)\) to \( u = \sum_{j=1}^{\infty} \varphi_j U_{j\lambda\varepsilon} f \) gives \((O + \lambda) u = f + \sum_{j=1}^{\infty} \Phi_{j\lambda\varepsilon} f\), where \( \Phi_{j\lambda\varepsilon} \) are a linear combination of \( U_{j\lambda\varepsilon} \) and \( \frac{d}{dx} U_{j\lambda\varepsilon} \). By virtue of embedding Theorem A_1, the
estimate (2.19) and from the expression $\Phi_{j\lambda}$ we obtain that operators $\Phi_{j\lambda}$ are bounded linear from $X$ to $L_p(G_j; E)$ and $\|\Phi_{j\lambda}\| < 1$. Therefore, there exists a bounded linear invertible operator $\left(I + \sum_{j=1}^{\infty} \Phi_{j\lambda}\right)^{-1}$. So, we obtain that the BVP (2.1) - (2.2) for $f \in X$ has a unique solution

$$u(x) = (O_\varepsilon + \lambda)^{-1} f = (U_\varepsilon + \lambda)^{-1} \left(I + \sum_{j=1}^{\infty} \Phi_{j\lambda}\right) f = (2.20)$$

Then by using the above representation and by using Theorem A1 we obtain the estimate (2.3).

**Result 2.1.** Theorem 2.1 implies that the differential operator $O_\varepsilon$ has a resolvent $(O_\varepsilon + \lambda)^{-1}$ for $|\arg \lambda| \leq \varphi$, and the uniform estimate holds

$$\sum_{i=0}^{\mu_1} |\lambda|^{-\frac{i}{p}} \varepsilon \left\| \frac{d^i}{d \varepsilon^i} (O_\varepsilon + \lambda)^{-1} \right\|_{L(\sigma; L)} + \left\| A (O_\varepsilon + \lambda)^{-1} \right\|_{L(X)} \leq C.$$

**3. R-positive properties of the abstract differential operator**

Result 2.1 implies that the operator $O$ is positive in $L_p(\sigma; E)$. In the following theorem we prove that this operator is $R$-positive of the operator $O$ in $L_p(\sigma; E)$.

**Theorem 3.1.** Let all condition of Theorem 2.1 be satisfied. Then the operator $O$ is $R$-positive in $L_p(\sigma; E)$.

**Proof.** Consider first of all the problem with constant coefficients

$$\varepsilon a u^{(2)}(x) + Au(x) + \lambda u(x) = f(x), \ x \in \sigma, \quad (3.1)$$

$$\sum_{i=0}^{\mu_1} \varepsilon^{\nu_i} \alpha_i u^{(i)}(0) = 0, \sum_{i=0}^{\mu_2} \varepsilon^{\nu_i} \beta_i u^{(i)}(1) = 0, \quad (3.2)$$

where $a$ is a complex number, $A$ is a linear operator in a Banach space $E$, $\lambda$ is a complex parameter, $\varepsilon$ is a positive parameter, $\nu_i = \frac{1}{\varepsilon} + \frac{1}{2p}$, $\alpha_i, \beta_i$ are complex numbers, $\mu_1, \mu_2 \in \{0, 1\}$.

Consider the operator $O_0$ in $X$ generated by problem (3.1) - (3.2) for $\lambda = 0$, i.e.

$$D(O_0) = W^2_p(\sigma; E(A), E, L_1, L_2), \ O_0u = \varepsilon au^{(2)} + Au.$$

Since $A$ is a positive operator in $E$, then in view of [9, Lemma 2.6] there exists semigroups $U_{j\lambda}(x) = e^{\frac{x}{\varepsilon} \omega_1 A_1^\ast}$ for $\text{Re} \omega_1 < 0$, $U_{j\lambda}(x) = e^{-\frac{x}{\varepsilon} (b-\varepsilon) \omega_2 A_2^\ast}$ for...
Re $\omega_2 > 0$ that are holomorphic for $x > 0$ and strongly continuous for $x \geq 0$. By using a technique similar to that applied in [26, Lemma 5.3.2/1], we obtain that for $f \in D(\sigma; E(A))$ the solution of the equation (3.1) is represented as

$$
u(x) = \sum_{j=1}^{2} U_{j\lambda}(x) g_k + \int_{\sigma} U_{\lambda}(x-y) f(y) dy, \ g_k \in E,$$

(3.3)

where

$$U_{\lambda}(x-y) = \begin{cases} -A^\frac{-\lambda}{2} U_{\varepsilon 1\lambda}(x-y), & x \geq y, \\ A^\frac{-\lambda}{2} U_{\varepsilon 2\lambda}(y-x), & x \leq y. \end{cases}$$

By taking into account the boundary conditions (3.2), we obtain the following equation with respect to $g_1, g_2$

$$\sum_{k=1}^{2} L_k(U_{\varepsilon j\lambda}) g_k = L_k(\Phi), \ j = 1, 2, \ \Phi = \int_{\sigma} U_{\varepsilon 0\lambda}(x-y) f(y) dy.$$ 

By solving the above system and substituting it into (3.3) we obtain the representation of the solution for problem (3.1) – (3.2):

$$\nu(x) = |O_0 + \lambda|^{-1} f = \int_{\sigma} G_\varepsilon(\lambda, x, y) f(y) dy,$$

(3.4)

$$G_\varepsilon(\lambda, x, y) = \sum_{k=1}^{2} \sum_{j=1}^{2} A^\frac{-\lambda}{2} B_{kj}(\lambda) U_{j\lambda}(x) \tilde{U}_{kj\lambda}(x-y) + U_{\lambda}(x-y),$$

where $B_{kj}(\lambda)$ are are uniformly bounded operators in $E$ and

$$\tilde{U}_{kj\lambda}(x-y) = \begin{cases} b_{kj} U_{\varepsilon k\lambda}(x-y), & x \geq y, \\ \beta_{kj} U_{\varepsilon k\lambda}(y-x), & x \leq y, \ b_{kj}, \ \beta_{kj} \in \mathbb{C}. \end{cases}$$

Let at first, to show that the set $\Phi = \{G_\varepsilon(\lambda, x, y) : \lambda \in S(\varphi)\}$ is uniformly $R$-bounded. By using the generalized Minkowskii’s, Young inequalities and by using of the holomorphic semigroups estimates [9] we have the uniform estimate

$$\|G_\varepsilon(\lambda, x, y) f\|_{X} \leq C \sum_{k=1}^{2} \sum_{j=1}^{2} \left\{ \left\|A^\frac{-\lambda}{2}\right\| \left\|B_{kj}(\lambda)\right\| \left\|\tilde{U}_{kj\lambda}(x) f\right\|_{X} + \right.$$

$$\left. \left\|U_{\lambda}(x-y) f\right\|_{X} \right\} \leq C \|f\|_{X}.$$ 

Due to $R$-positivity of $A$, uniform boundedness of operators $B_{kj}(\lambda)$ and in view of the Kahane’s contraction principle and from the product properties of the collection of $R$-bounded operators [9, Lemma 3.5, Proposition 3.4] we get that the sets

$$b_{kj}(\lambda, x, y) = \left\{ B_{kj}(\lambda) A^\frac{-\lambda}{2} U_{j\lambda}(x) [U_{\varepsilon k\lambda}(1-y) + U_{\varepsilon k\lambda}(y)] : \lambda \in S(\varphi) \right\},$$

$$b_0(\lambda, x, y) = \{ U_{\varepsilon 0\lambda}(x-y) : \lambda \in S(\varphi) \}$$
are uniformly $R$-bounded. Then by using the Kahane’s contraction principle, product and additional properties of the collection of $R$-bounded operators and in view of $R$-boundedness of the sets $b_{kj}$, $b_0$, for all $u_1, u_2, \ldots, u_\mu \in F$, $\lambda_1, \lambda_2, \ldots, \lambda_\mu \in S(\varphi)$, and independent symmetric $\{ -1, 1 \}$-valued random variables $r_i(y)$, $i = 1, 2, \ldots, \mu$, $\mu \in N$ we have the uniform estimate

$$
\int_\Omega \left\| \sum_{i=1}^\mu r_i(y) G_\epsilon (\lambda_i, x, y) u_i \right\|_X d\tau \leq C \left\{ \sum_{k,j=1}^2 \int_\Omega \left\| \sum_{i=1}^\mu r_i(y) b_{kj} (\lambda_i, x, y) u_i \right\|_X d\tau \right\} + \int_\Omega \left\| \sum_{i=1}^\mu r_i(y) b_0 (\lambda_i, x, y) u_i \right\|_X d\tau, \beta < 0.
$$

This implies that

$$
R \{ G_\epsilon (\lambda, x, y) : \lambda \in S_\varphi \} \leq Ce^{\beta|\lambda|^2|x-y|}, \beta < 0, x, y \in (0, b).
$$

By applying the $R$-bondedness property of kernel operators (see e.g. the Proposition 4.12 in [9]) and due to density of $D(\sigma; E(A))$ in $X$ (see e.g. [14, § 2.2]) we get that the operator $O_0$ is uniformly $R$-positive in $X$. From the representation (3.4) of solution of problem (3.1) – (3.2) it is easy to see that the operator $(O_0 + \lambda)^{-1}$ can be expressed as a linear combination of operators $O_{j\lambda}^{-1}$ like $(O_0 + \lambda)^{-1}$. Then, in view the representation (3.4) and by virtue of Kahane’s contraction principle, product and additional properties of the collection of $R$-bounded operators we obtain that the operator $O_0$ is $R$-positive in $L_p(\sigma; E)$.

Now, consider the problem (2.1) – (2.2). By virtue of (2.20) from Theorem 2.1 we obtain that for $f \in L_p(\sigma; E)$ the BVP (2.1) – (2.2) have a unique solution expressing in the form

$$
u(x) = (O_\epsilon + \lambda)^{-1} f = \sum_{j=1}^\infty \varphi_j O^{-1}_{\epsilon j \lambda} (I - K_{\epsilon j \lambda})^{-1} g_j \left( I + \sum_{j=1}^\infty \Phi_{\epsilon j \lambda} \right)^{-1} f, \quad (3.5)
$$

where $O_{\epsilon j \lambda} = O_{\epsilon j} + \lambda$ are local operators generated by BVPs with constant coefficients of type (2.16) and $K_{\epsilon j \lambda}$, $\Phi_{\epsilon j \lambda}$ are uniformly bounded operators defined in the proof of the Theorem 2.1. By virtue of the first part of this theorem, the operators $O_j$ are $R$-positive in $L_p(G_j; E)$. Then by using the representation (3.5) and by virtue of Kahane’s contraction principle, product and additional properties of the collection of $R$-bounded operators (see e.g. [9] Lemma 3.5, Proposition 3.4) we obtain the assertion.

4. Abstract Cauchy problem for parabolic equation on exterior domain

Consider the following mixed problem for parabolic DOE equation with parameter
\[
\frac{\partial u}{\partial t} + \varepsilon a(x) \frac{\partial^2 u}{\partial x^2} + [A(x) + d] u + \varepsilon^2 A_1(x) \frac{\partial u}{\partial x} + A_0(x) u = f(t, x),
\]

\[
\sum_{i=0}^{\mu_1} \varepsilon^{\nu_i} \alpha_i u_x^{(i)}(t, 0) = 0, \quad \sum_{i=0}^{\mu_2} \varepsilon^{\nu_i} \beta_i u_x^{(i)}(t, b) = 0, \quad (4.1)
\]

where \( \sigma = (-\infty, \infty) \setminus [0, 1] \), \( \alpha_i, \beta_i \) are complex numbers, \( \varepsilon \) is a positive parameter, \( \nu_i = \frac{1}{2} + \frac{1}{2p} \), \( d \) is a positive number, \( \mu_k \in \{0, 1\} \), \( A(\cdot) \) and \( A_j(\cdot) \) are linear operator functions in a Banach space \( E \).

The problem (4.1) can be express as the following Cauchy problem

\[
\frac{du}{dt} + (O_\varepsilon + d) u = f(t), \quad u(0) = 0, \quad (4.2)
\]

where \( O_\varepsilon \) denote the operator generated by (2.1) – (2.2). The Theorem 3.1 implies that the operator \( O_\varepsilon \) is \( R \)-positive in \( X = L_p(\sigma; E) \). By virtue of [24, §1.14], the operator \( O_\varepsilon \) is a generator of an analytic semigroup in \( X \).

For \( p = (p, p_1), \Delta_T = (0, T) \times \sigma, L_p(\Delta_T; E) \) will be denoted the space of all \( E \)-valued \( p \)-summable functions with mixed norm (see e.g. [6]), i.e., the space of all measurable functions \( f \) defined on \( \Delta_T \) for which

\[
\|f\|_{L_p(\Delta_T)} = \left( \int_0^T \left( \int_{\sigma} f(t, x)^p \, dx \right)^{\frac{p}{p_1}} \, dt \right)^{\frac{p_1}{p}} < \infty.
\]

Analogously, \( W^{2,1}_p(\sigma_T, E(\sigma); E) \) denotes the Sobolev space with corresponding mixed norm (see [6] for scalar case).

**Theorem 4.1.** Let the conditions of Theorem 2.1 hold for \( \varphi > \frac{\alpha}{2} \). Then for all \( f \in L_p(\sigma_T; E) \) and sufficiently large \( d > 0 \), problem (4.1) has a unique solution belonging to \( W^{1,2}_p(\sigma_T; E(\sigma); E) \) and the following coercive estimate holds

\[
\left\| \frac{\partial u}{\partial t} \right\|_{L_p(\sigma_T; E)} + \left\| \varepsilon \frac{\partial^2 u}{\partial x^2} \right\|_{L_p(\sigma_T; E)} + \|Au\|_{L_p(\sigma_T; E)} \leq C \|f\|_{L_p(\sigma_T; E)}.
\]

**Proof.** The problem (4.1) can be express as the following Cauchy problem

\[
\frac{du}{dt} + (O_\varepsilon + d) u = f(t), \quad u(0) = 0,
\]

where \( O_\varepsilon \) denote the operator generated by (2.1) – (2.2). The Theorem 3.1 implies that the operator \( O_\varepsilon \) is \( R \)-positive in \( X = L_p(\sigma; E) \).

Then applying [9, Theorem 4.4] we obtain that for \( f \in L_{p_1}(0, T; X) \) and sufficiently large \( d > 0 \), problem (4.1) has a unique solution belonging to \( W^{1,2}_{p_1}(0, T; D(O) \cdot X) \) and the following estimate holds

\[
\left\| \frac{du}{dt} \right\|_{L_{p_1}(0, T; X)} + \|(O_\varepsilon + d) u\|_{L_{p_1}(0, T; X)} \leq C \|f\|_{L_{p_1}(0, T; X)}.
\]

Since \( L_{p_1}(0, T; X) = L_p(\sigma_T; E) \), by Theorem 2.1 we have

\[
\|(O_\varepsilon + d) u\|_{X} = \|u\|_{W^{2,1}_p(\sigma_T; E(\sigma); E)}.
\]

These relations and the above estimate prove the hypothesis to be true.
5. Elliptic DOE on the moving domain

Consider the BVP on the exterior moving domain $\sigma (s) = \mathbb{R} / [0, b (s)]:$

\begin{equation}
au^{(2)} (x) + Au (x) + A_1 u^{(1)} (x) + A_0 u (x) = f (x), \ x \in \sigma,
\end{equation}

\begin{equation}
\sum_{i=0}^{\mu_1} \alpha_i u^{(i)} (0) = 0, \sum_{i=0}^{\mu_2} \beta_i u^{(i)} (b (s)) = 0,
\end{equation}

where $\alpha_i, \beta_i$ are complex numbers, $a$ is a complex valued function; $A = A (x)$ and $A_j = A_j (x)$ are linear operators in a Banach space $E,$ the end point $b (s)$ depend on the parameter $s$ and $b(s)$ is a positive continues function on compact domain $\Delta \subset \mathbb{R}, \mu_k \in \{0, 1\}$.

Theorem 2.1 implies the following:

**Proposition 5.1.** Assume the Condition 2.1 hold for $b = b (s)$. Then, problem (5.1) has a unique solution $u \in W^{2,p} ([0, b); (E, A, E))$ for $f \in L^p (0, b; E)$ and sufficiently $d > 0$. Moreover, the following coercive uniform estimate holds

\begin{equation}
\left\| u^2 \right\|_{L^p (0, b; E)} + \left\| Au \right\|_{L^p (0, b; E)} \leq C \left\| f \right\|_{L^p (0, b; E)}.
\end{equation}

**Proof.** Under the substitution $\tau = xb^{-1} (s)$ the problem (5.1) reduced to the following BVP in fixed domain $(0, 1):$

\begin{equation}
b^{-2} (s) \tilde{a} (\tau) \tilde{u}^{(2)} + \tilde{A} (\tau) \tilde{u} + \sum_{i=0}^{1} b^{-i} (s) \tilde{A}_i (\tau) \tilde{u}^{(i)} (\tau) = \tilde{f} (\tau), \ \tau \in (0, 1),
\end{equation}

\begin{equation}
\sum_{i=0}^{1} b^{-i} (s) \alpha_i \tilde{u}^{(i)} (0) = 0, \sum_{i=0}^{1} b^{-i} (s) \beta_i \tilde{u}^{(i)} (1) = 0,
\end{equation}

where

\begin{equation}
\tilde{u} (\tau) = u (\tau b^{-1}), \ \tilde{a}_k (\tau) = a_k (\tau b^{-1}), \ \tilde{A} (\tau) = A ((\tau b^{-1})).
\end{equation}

Then, by virtue of Theorem 2.1 we obtain the required assertion.

6. Nonlinear abstract elliptic problem in exterior domain

Consider the following nonlinear elliptic problem

\begin{equation}
a (x) u^{(2)} (x) + B (x, u, u^{(1)}) u = F (x, u, u^{(1)}) + f (x), \ x \in \sigma,
\end{equation}

\begin{equation}
L_1 u = \sum_{i=0}^{\mu_1} \alpha_i u^{(i)} (0) = 0, \ L_2 u = \sum_{i=0}^{\mu_2} \beta_i u^{(i)} (b) = 0,
\end{equation}

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where \( g \) is \( E \)-valued given function, \( a \) is a complex valued function, \( \alpha_i, \beta_i \) are complex numbers, \( \mu_k \in \{0,1\}, \sigma = \mathbb{R} \setminus [0,b] \).

In this section we will prove the existence and uniqueness of maximal regular solution for the nonlinear problem (6.1) \-(6.2).\ Let

\[
U = (u_0,u_1), \quad X = L_p(\sigma;E), \quad Y = W^2_p(\sigma;E(A),E),
\]

\[
E_i = (E(A),E)_{\theta_i,p}, \quad \theta_i = \frac{i + \frac{d}{2}}{2}, \quad X_0 = \prod_{i=0}^1 E_i,
\]

**Remark 6.1.** By using J. Lions-I. Petree result (see e.g [24, § 1.8.]) we obtain that the embedding \( D^2 Y \in E_i \) is continuous and there is a constant \( C_1 \) such that for \( w \in Y, W = \{w_i\}, w_i = D^i w(\cdot), i = 0,1, \)

\[
\|u\|_{\infty,X_0} = \prod_{i=0}^1 \|D^i w\|_{C(\sigma,E_i)} = \sup_{x \in [0,b]} \prod_{i=0}^1 \|D^i w(x)\|_{E_i} \leq C_1 \|w\|_Y.
\]

For \( r > 0 \) denote by \( O_r \) the closed ball in \( X_0 \) of radios \( r \), i.e.

\[
O_r = \{u \in X_0, \|u\|_{X_0} \leq r\}.
\]

Consider the linear problem,

\[
Lu = a(x) w^{(2)}(x) + (A(x) + d) w(x) = g(x), \quad (6.3)
\]

\[
L_kw = 0, \quad k = 1,2,
\]

where \( A(x) \) is a linear operator in a Banach space \( E \) for \( x \in \sigma \), \( L_k \) are boundary conditions defined by (6.1) and \( d > 0 \).

Assume \( E \) is a UMD space and \( A(x) \) is uniformly \( R \)-positive in \( E \), \( A(0) A^{-1}(y_0) = A(a) A^{-1}(y_0) \). By virtue Theorem 2.1 and Proposition 5.1, problem (6.3) has a unique solution \( w \in Y \) for all \( g \in X \) and for sufficiently large \( d > 0 \). Moreover, the following coercive estimate holds

\[
\|w\|_Y \leq M \|g\|_X,
\]

where the constant \( C_0 \) do not depend on \( f \in X \) and \( b \in (0,b_0) \).

Let \( \omega_1 = \omega_1(x), \omega_2 = \omega_2(x) \) be roots of equation \( a(x) \omega^2 + 1 = 0 \).

**Condition 6.1.** Assume the following satisfied:

1. \( a \in C(\bar{\sigma}), \ \text{Re} \omega_k \neq 0 \) and \( \frac{1}{\omega_k} \in S(\varphi) \) for \( \lambda \in S(\varphi), \ 0 \leq \varphi < \pi, \)
   \( k = 1,2 \), a.e. \( x \in \sigma \);

2. \( E \) is an UMD space, \( p \in (1,\infty) \);

3. \( F : \sigma \times X_0 \to E \) is a measurable function for each \( u_i \in E_i, i = 0,1 \) and \( F(x,U) \in X \). Moreover, for each \( r > 0 \) there exists the positive functions \( h_k(x) \) such that

\[
\|F(x,U)\|_E \leq h_1(x) \|U\|_{X_0},
\]

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\[ \|F(x, U) - F(x, \bar{U})\|_E \leq h_2(x) \|U - \bar{U}\|_{X_0}, \]

where \( h_k \in L_p(\sigma) \) with

\[ \|h_k\|_{L_p(\sigma)} < M^{-1}, \ k = 1, 2; \]

and \( U = \{u_0, u_1\}, \bar{U} = \{\bar{u}_0, \bar{u}_1\}, u_i, \bar{u}_i \in E_i \) and \( U, \bar{U} \in O_r. \)

(4) there exist \( \Phi_i \in E_i, \) such that the operator \( B(x, \Phi) \) for \( \Phi = \{\Phi_i\} \) is \( R \)-positive in \( E \) uniformly with respect to \( x \in [0, b]; B(x, \Phi) B^{-1}(x, \Phi) \in C(\bar{\sigma}; L(E)); B(x, 0) = A(x) ; \)

(5) \( B(x, U) \) for \( x \in (0, a) \) is a uniform positive operator in a Banach space \( E, \)

where domain definition \( D(B(x, U)) \) does not depend on \( x, U \) and \( B: \sigma \times X_0 \to L(E(A), E) \) is continuous. Moreover, for each \( r > 0 \) there is a positive constant \( L(r) \) such that

\[ \|B(x, U) - B(x, \bar{U})\| \leq L(r) \|U - \bar{U}\|_{X_0} \|Av\|_E \text{ for } x \in \sigma, U, \bar{U} \in O_r \]

and \( v \in D(B(x, U)). \)

**Theorem 6.1.** Assume the Condition 6.1 holds. Then, there exist a radius \( 0 < r \leq r_0 \) and \( \delta > 0 \) such that for each \( f \in L_p(\sigma; E) \) with \( \|f\|_{L_p(x, E)} \leq \delta \) there is a unique solution \( u \in W_p^2((\sigma; E(A), E) \) of the problem (6.1) – (6.2) with \( \|u\|_{W_p^2(\sigma; E(A), E)} \leq r. \)

**Proof.** We want to solve the problem (6.1) – (6.2) locally by means of maximal regularity of the linear problem (6.3) via the contraction mapping theorem. For this purpose, let \( w \) be a solution of the linear problem (6.3). Consider a ball

\[ O_r = \{v \in Y, \ L_k(v - w) = 0, \ \|v - w\|_Y \leq r\}. \]

Let \( w \in Y \) be a solution of the problem (6.3) and

\[ W = \{w, w^{(1)}\}. \]

Given \( v \in B_r \) solve the linear problem

\[ a(x) u^{(2)}(x) + B(x, 0) u(x) + du = F(x, v) + [B(x, 0) - B(x, V)] v(x) + f(x), \ L_k u = 0, \ k = 1, 2, \]

where

\[ V = \left( v, v^{(1)} \right), \ v \in Y. \]

Consider the function

\[ g(x) = [B(x, 0) - B(x, V)] v(x) + F(x, V) + f(x). \]

Let first of all, we show that \( g \in X \) and \( \|g\|_X \leq M^{-1} r \) for \( v \in Y, \ \|v\|_Y \leq r. \)

Indeed, by Remark 6.1 \( V \in C(\bar{\sigma}; X_0), \) one has

\[ B(x, 0) - B(x, V) \in C(\bar{\sigma}; L(E(A), E)). \]
Hence, by assumption (3), \( g \) is measurable and
\[
\|g(x)\|_E \leq L(r) \|V\|_{X_0} + h_1(x) \|V\|_{X_0} + \|f(x)\|_E
\]
for a.e. \( x \in \sigma \). Then, by using the Remark 6.1 and by chosing \( \delta \) we obtain
\[
\|g\|_X \leq rL(r) \|v\|_X + r\|h_1\|_{L_p} + \|f\|_X \leq r^2L(r) + r\|h_1\|_{L_p} + \delta \leq M^{-1}r.
\]
Define a map \( Q \) on \( O_r \) by
\[
Qv = w,
\]
where \( w \) is a solution of the problem (6.3) with \( g \) defined by (6.4). We want to show that \( Q(B_r) \subset B_r \) and that \( Q \) is a contraction operator in \( Y \) provided \( \delta \) is sufficiently small, and \( r \) is chosen properly. For this aim, by using maximal regularity properties of the problem (6.3) we have
\[
\|Qv - w\|_Y = \|u - w\|_Y \leq M \\{\|F(x, V) - F(x, 0)\|_X + \|B(x, 0) - B(x, V)\|_X\}.
\]
By assumption (3) for \( v \in O_r \) we get
\[
\|F(x, V) - F(x, 0)\|_X \leq h_2\|L_\sigma\| \|V\|_{X_0}.
\]
By assumptions (4), (5) and Remark 6.2, for \( v \in O_r \) and \( W = (w, w^{(1)}) \), \( w \in Y \) we have
\[
\|\|B(x, 0)v - B(x, V)\|_X\| \leq \sup_{x \in \sigma} \left\{\|\|B(x, 0) - B(x, W)\|_X\|_Y v\|\}_{L(X_0, X)}
\]
\[
+ \|\|B(x, W) - B(x, V)\|_X v\|\}_{L(X_0, X)} \|v\|_Y \leq L(r) \left[\|W\|_{X_0} A_{\sigma} v\|_X + \|v - w\|_{X_0} \right] \|v - w\|_Y + \|w\|_Y \leq
\]
\[
l(r) \left[\|W\|_{X_0} \|v\|_Y + C_1 \|v - w\|_Y \right] + l(r) \|w\|_Y \right\).
\]
By chosing \( r \) and \( b \in (0 b_0] \) so that \( \|w\|_Y < \delta_a \) by assumptions (3)-(5) we obtain from the above inequalities
\[
\|Qv - w\|_Y \leq r + r^2l(r) \|W\|_{X_0} + r^2l(r) C_1 + rL(r) \|w\|_Y < r.
\]
That is the operator \( Q \) maps \( B_r \) into itself, i.e.
\[
Q(B_r) \subset B_r.
\]
Let \( u_1 = Q(v_1) \) and \( u_2 = Q(v_2) \). Then \( u_1 - u_2 \) is a solution of the problem
\[
a(x)u^{(2)}(x) + A(x)u(x) + du = F(x, v_1) -
\]
\[
F(x, v_1) + [B(x, v_2) - B(x, 0)] [v_1(x) - v_2(x)] - [B(x, v_1) - B(x, v_2)] v_1(x), \quad L_k u = 0, \quad k = 1, 2.
\]
In a similar way, by using the assumption (5) we obtain
\[ \|u_1 - u_2\|_Y \leq C_0 \left\{ rL (r) \|v_1 - v_2\|_X + L (r) \|v_1 - v_2\|_Y \right\} \]
\[ + \|h_2\|_{L_p} \|v_1 - v_2\|_Y \right\} \leq C_0 \left[ 2rL (r) + \|h_2\|_{L_p} \right] \|v_1 - v_2\|_Y . \]

Thus \( Q \) is a strict contraction. Eventually, the contraction mapping principle implies a unique fixed point of \( Q \) in \( O_r \) which is the unique strong solution
\[ u \in Y = W^2_p (\sigma; E (A), E) . \]

### 7. Exterior BVP for elliptic equations

The regularity property of BVP for elliptic equations were studied e.g. in \[1\], \[9\], \[26\]. Let \( \Omega = \sigma \times G \), where \( \sigma = \mathbb{R} \setminus [0, b] \), \( G \subset R^n \), \( n \geq 2 \) is a bounded domain with \((n - 1)\)-dimensional boundary \( \partial G \). Let us consider the following BVP for elliptic equation with parameter
\[ Lu = \varepsilon a (x) D^2_x u (x, y) + \sum_{|\alpha| \leq 2m} b_{\alpha} (x) a_{\alpha} (y) D^\alpha_y u (x, y) + \]
\[ + \sum_{i=0}^{1} \sum_{|\beta| \leq \mu_i} a_{i\alpha} (x, y) D^\beta_x D^\alpha_y u (x, y) + du (x, y) = f, \ x \in \sigma, \ y \in G, \quad (7.1) \]
\[ \sum_{i=0}^{1} \sum_{|\beta| \leq \mu_i} a_{i\alpha} (x, y) D^\beta_x D^\alpha_y u (x, y) + \sum_{i=0}^{\eta_2} \varepsilon^{\nu_i} \beta_i u^{(i)} (0, y) = 0, \ \text{for a.e.} \ y \in G, \quad (7.2) \]
\[ B_j u = \sum_{|\beta| \leq m_j} b_{j\beta} (y) D^\beta_y u (x, y) = 0, \ x \in \sigma, \ y \in \partial \Omega, \ j = 1, 2, ..., m, \quad (7.3) \]
where \( \eta_k \in \{0, 1\} \), \( \alpha_i, \beta_i \) are complex numbers, \( \varepsilon \) is a positive parameter, \( \nu_i = \frac{1}{2} + \frac{1}{d} \), \( d > 0 \),
\[ D^k_x = \frac{\partial^k}{\partial x^k}, \ D^k_j = -i \frac{\partial^k}{\partial y^j}, \ D^\alpha_y = (D_1, ..., D_n), \ y = (y_1, ..., y_n) \]
and \( a, a_\alpha, b_\alpha, a_{i\alpha}, b_{j\beta} \) are the complex valued functions, \( \mu_i < 2m \). Let \( p = (p_1, p) \).

Let \( \xi' = (\xi_1, \xi_2, ..., \xi_{n-1}) \in R^{n-1} \), \( \alpha' = (\alpha_1, \alpha_2, ..., \alpha_{n-1}) \in Z^n \)
and
\[ A (y_0, \xi', D_y) = \sum_{|\alpha'| + j \leq 2m} a_{\alpha'} (y_0) \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_{n-1}^{\alpha_{n-1}} D^j_y \text{ for } y_0 \in \bar{G} \]
\[ B_j (y_0, \xi', D_y) = \sum_{|\beta'| + j \leq m_j} b_{j\beta'} (y_0) \xi_1^{\beta_1} \xi_2^{\beta_2} ... \xi_{n-1}^{\beta_{n-1}} D^j_y \text{ for } y_0 \in \partial G. \]
Let $Q$ denote the differential operator in $L^p_{\varphi}(\Omega)$ generated by BVP (7.1) – (7.3).

**Theorem 5.1.** Let the following conditions be satisfied:

1. $a \in C(\varphi)$, $\Re \omega_k \neq 0$ and $\frac{\omega_k}{\varphi_k} \in S(\varphi)$ for $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, $k = 1, 2$, a.e. $x \in \sigma$, $b_\alpha \in C(\sigma)$, $a_\alpha \in C(\Omega)$ for each $|\alpha| = 2m$ and $a_\alpha \in L^\infty(\Omega)$ for each $|\alpha| < 2m$;

2. $b_{j\beta} \in C^{2m-\eta_j}(\partial \Omega)$ for each $j$, $\beta$ and $m_j < 2m$, $\sum_{j=1}^m b_{j\beta}(y)\sigma_j \neq 0$, for $|\beta| = m_j, y \in \partial G$, where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in R^n$ is a normal to $\partial G$;

3. for $y \in \bar{\Omega}$, $\xi \in R^n$, $\lambda \in S(\varphi_y)$, $|\xi| + |\lambda| \neq 0$ let $\lambda+ \sum_{|\alpha|=2m} a_\alpha (y)\xi^\alpha \neq 0$;

4. for each $y_0 \in \partial \Omega$ local BVP in local coordinates corresponding to $y_0$

$$\lambda + A(y_0, \xi', D_y) \vartheta(y) = 0,$$

has a unique solution $\vartheta \in C_0(\mathbb{R}_+)$ for all $h = (h_1, h_2, ..., h_n) \in C^n$ and for $\xi' \in R^{n-1}$.

Then:

(a) problem (7.1) – (7.3) has a unique solution $u \in W^{2, m}_{p,y}(\Omega)$ for $f \in L^p_{\varphi}(\Omega)$ and sufficiently large $d > 0$. Moreover, the uniform coercive estimate holds

$$\varepsilon \|D^2 u\|_{L^p_{\varphi}(\Omega)} + \sum_{|\alpha| \leq 2m} \|D^\alpha u\|_{L^p_{\varphi}(\Omega)} \leq C \|f\|_{L^p_{\varphi}(\Omega)} ;$$

(b) the operator $Q$ is $R$-positive in $L^p_{\varphi}(\Omega)$.

**Proof.** Let us consider operators $A$ and $A_i (x)$ in $E = L_{p_i}(G)$ that are defined by the equalities

$$D(A) = \{u \in W^{2m}_{p_i}(G) ; B_j u = 0, j = 1, 2, ..., m \} , \quad Au = \sum_{|\alpha| \leq 2m} a_\alpha (y) D^\alpha_y u (y) ,$$

$$A_i u = \sum_{|\beta| \leq \mu_i} a_{i\beta} (x, y) D^\beta_y u (y) , i = 0, 1.$$

Then the problem (7.1) – (7.3) can be rewritten as the problem (2.1) – (2.2), where $u(x) = u(x,.)$, $f(x) = f(x,.)$, $x \in \sigma$ are the functions with values in $E = L_{p_i}(G)$. By virtue of [2, Theorem 4.5.2] the space $E = L_{p_i}(G)$, $p_i \in (1, \infty)$ satisfies the multiplier condition. By virtue of [9, Theorem 8.2] operator $A + \mu$ for sufficiently large $\mu > 0$ is $R$-positive in $L_{p_i}$. Moreover, (1) and (2) implies the (3) condition of Theorem 2.1, i.e., conditions (1)–(3) of Theorem 2.1 are fulfilled. It is known that the embedding $W^{2m}_{p_i}(G) \subset L_{p_i}(G)$ is compact (see e.g. [24, § 3, Theorem 3. 2. 5]). Using interpolation properties of Sobolev spaces [24, § 4] we obtain that the condition (4) of Theorem 2.1 is satisfied. Hence, all hypotheses of Theorem 2.1 are valid and the assertion of (a) holds. Then the Theorem 3.1 implies the assertion (b).
Consider the Cauchy problem for the system of parabolic equation of arbitrary number

\[
\frac{\partial u_j}{\partial t} + \varepsilon a (x) \frac{\partial^2 u_j}{\partial x^2} + [a_j (x) + d] u + \sum_{i=0}^{1} b_{ij} (x) \frac{\partial u_j}{\partial x} + du = f (t, x),
\]

\[
\sum_{i=0}^{\mu_1} \varepsilon^{\nu_i} \alpha_i \frac{\partial u_j}{\partial x} (t, 0) = 0, \quad \sum_{i=0}^{\mu_2} \varepsilon^{\nu_i} \beta_i \frac{\partial u_j}{\partial x} (t, 1) = 0,
\]

where \(a (x), a_j (x), b_{ij} (x)\) are complex valued functions, \(\alpha_i, \beta_i\) are complex numbers, \(\varepsilon\) is a small positive parameter, \(\nu_i, \beta_i\) are complex numbers, \(\varepsilon = \frac{1}{2} + \frac{1}{2p}\), \(d\) is a positive number, \(\mu_k \in \{0, 1\}\), \(\sigma = \mathbb{R} \setminus [0, 1]\).

Let \(p = (p, p_1), \Delta_+ = (0, T) \times \sigma\) and \(L_p (\Delta_T) = \mathbb{L}_p (\Delta_T; \mathbb{C})\) will be denoted the space of all complex-valued functions with mixed norm i.e., the space of all measurable functions \(f\) defined on \(\Delta_T\) for which

\[
\|f\|_{L_p, \gamma (\Delta_+)} = \left( \int_0^T \left( \int_\sigma |f (t, x)|^p \, dx \right)^{\frac{1}{p}} \, dt \right)^{\frac{1}{\gamma}} < \infty.
\]

Analogously, \(W^2_p (\Delta_T)\) denotes the Sobolev space with corresponding mixed norm (see e.g. [6]).

Let \(E = l_q\) and \(A (x) = [\delta_{ij} a_i (x)], A_1 (x) = [b_{ij} (x)]\) are diagonal matrices in \(l_q\), where \(i, j = 1, 2, \ldots N\), \(\delta_{ij} = 1\) for \(i = j\) and \(\delta_{ij} = 0\) and

\[
l_q (A) = \left\{ u \in l_q, ||u||_{l_q (A)} = ||Au||_{l_q} = \right\}
\]

\[
\left( \sum_{j=1}^{N} \left| (Au)_j \right|^q \right)^{\frac{1}{q}} = \left( \sum_{j=1}^{N} \left| a_j u_j \right|^q \right)^{\frac{1}{q}} < \infty,
\]

\[
u = \{ u_j \}, \quad Au = \left\{ \sum_{j=1}^{N} a_j u_j \right\}, \quad j = 1, 2, \ldots N.
\]

**Condition 8.1.** Assume the following conditions are satisfied:

1. \(a \in C (\bar{\sigma}), Re \omega_k \neq 0\) and \(\lambda_k \in S (\varphi)\) for \(\lambda \in S (\varphi)\), \(k = 1, 2, a_j \in C (\bar{\sigma})\) and \(a_j (x) \in S (\varphi)\), \(x \in \sigma, 0 \leq \varphi < \pi\);
(2) \( b_{ij} \in L_\infty(0,1), |b_{ij}(x)| \leq C |a_j^{-\frac{p}{2}\delta_i}(x)| \) for \( 0 < \delta_i < 1 - \frac{1}{2} \) and a.e. \( x \in \sigma; \)

(5) \( p, q \in (1,\infty) \) and \( \prod_{j=1}^{N} |a_j(x)| \leq \infty \) for a.e. \( x \in \sigma. \)

Let

\[
f(x) = \{f_j(x)\}_{1}^{N}, \quad u = \{u_j(x)\}_{1}^{N}.
\]

**Theorem 8.1.** Assume Condition 8.1 are satisfied. Then for \( f(x) \in L_p(\Delta_+;l_q) \) and for sufficiently large \( d \) problem (8.1) has a unique solution \( u \) that belongs to the space \( W^{1,2}_p(\Delta_+;l_q(A),l_q) \) and the following coercive estimate holds

\[
\left[ \int_{\Delta_+} \left( \sum_{j=1}^{N} \left| \frac{\partial u_j}{\partial t} \right|^q \right) dx \right]^\frac{1}{q} + \left[ \int_{\Delta_+} \left( \sum_{j=1}^{N} \left| \frac{\partial^2 u_j}{\partial x^2} \right|^q \right) dx \right]^\frac{1}{q} \leq C \left[ \int_{\Delta_+} \left| \sum_{i=1}^{N} f_i(x) \right|^q dx \right]^\frac{1}{q}.
\]

**Proof.** Let first all of, we suppose \( N < \infty \). Then \( \det A(x) = \prod_{j=1}^{N} a_j(x). \)

It is easy to see that

\[
B(\lambda) = \lambda(A + \lambda)^{-1} = \frac{\lambda}{D(\lambda)} [A_{ji}(\lambda)], \quad i, j = 1, 2, ...N,
\]

where \( D(\lambda) = \prod_{j=1}^{N} (a_j(x) + \lambda)^{-1}, A_{ji}(\lambda) \) are entries of the corresponding adjoint matrix of \( A + \lambda I \). By using the (1) assumption it is clear to see that the matrix \( A \) generates a positive operator in \( l_q \). For all \( u_1, u_2, ..., u_\mu \in l_q, \lambda_1, \lambda_2, ..., \lambda_\mu \in \mathbb{C} \) and independent symmetric \( \{-1,1\} \)-valued random variables \( r_k(y), k = 1, 2, ..., \mu, \mu \in \mathbb{N} \) we have

\[
\int_{\Omega} \left\| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right\|_{l_q}^q dy \leq C \left\{ \int_{\Omega} \left[ \sum_{j=1}^{N} \left( \sum_{k=1}^{\mu} \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) r_k(y) u_{ki} \right)^q \right] dy \right. \\
\left. \sup_{k,i} \sum_{j=1}^{N} \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) \left| \int_{\Omega} \left[ \sum_{k=1}^{\mu} r_k(y) u_{kj} \right]^q dy \right. \right\}.
\]
Since $A$ is symmetric and positive definite, we have

$$
\sup_{k,i} \sum_{j=1}^{N} \left| \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) \right|^q \leq C. \quad (8.3)
$$

From (8.2) and (8.3) we get

$$
\int_{0}^{1} \left\| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right\|_{l_q}^q \, dy \leq C \int_{0}^{1} \left\| \sum_{k=1}^{\mu} r_k(y) u_k \right\|_{l_q}^q \, dy,
$$
i.e., the operator $A$ is $R$-positive in $l_q$.

Let $N = \infty$, then we define determinant of infinite dimensional matrix $A$ as:

$$
det A = \lim_{n \to \infty} \prod_{j=1}^{n} a_j < \infty.
$$

The resolvent set $R(A)$ of the infinite dimensional matrix $A$ is defined as:

$$
R(A) = \left\{ \lambda \in \mathbb{C}, \lim_{n \to \infty} \prod_{j=1}^{n} (a_j + \lambda)^{-1} < \infty \right\}.
$$

In a similar way we obtain that

$$
B(\lambda) = \lambda (A + \lambda)^{-1} = \frac{\lambda}{D(\lambda)} [A_{ji}(\lambda)] , \quad i, j = 1, 2, ..., N,
$$

where $D(\lambda) = \lim_{n \to \infty} \prod_{j=1}^{n} (a_j + \lambda)$ and $A_{ji}(\lambda)$ are entries of the corresponding adjoint matrix of $A + \lambda$. By reasoning as the above and by taking limit when $n \to \infty$ we obtain that the matrix $A$ generates $R$-positive operator in $l_q$ also for $N = \infty$. From the Theorem 3.1 we obtain that problem (8.1) has a unique solution $u \in W^{1,2}(\Delta_{+}; l_q(A), l_q)$ for $f \in L_p(\Delta_{+}; l_q)$ and the following uniform estimate holds

$$
\left\| \frac{\partial u}{\partial t} \right\|_{L_p(\Delta T; l_q)} + \left\| \varepsilon \frac{\partial^2 u}{\partial x^2} \right\|_{L_p(\Delta T; l_q)} + \left\| Au \right\|_{L_p(\Delta_{+}; E)} \leq C \left\| f \right\|_{L_p(\Delta T; E)}.
$$

From the above estimate we obtain the assertion.

9. Wentzell-Robin type mixed problem for parabolic equation in exterior domain

Consider the problem

$$
\frac{\partial u}{\partial t} + d \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial y} + cu = f(t, x, y), \quad (9.1)
$$
possessing the derivatives \( \partial u \) mixed norm, i.e., \( \| u \|_{W^1_p(\Omega)} \), will denote the space of all \( \tilde{p} \)-summable scalar-valued functions with mixed norm. Analogously, \( W^{2,1}_p(\Omega) \) denotes the Sobolev space with corresponding mixed norm, i.e., \( W^{2,1}_p(\Omega) \) denotes the space of all functions \( u \in L_{\tilde{p}}(\Omega) \) possessing the derivatives \( \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial t \partial x^2}, \frac{\partial^2 u}{\partial t \partial y^2} \in L_{\tilde{p}}(\Omega) \) with the norm

\[
\| u \|_{W^{2,1}_p(\Omega)} = \| u \|_{L_{\tilde{p}}(\Omega)} + \left\| \frac{\partial u}{\partial t} \right\|_{L_{\tilde{p}}(\Omega)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_{\tilde{p}}(\Omega)} + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L_{\tilde{p}}(\Omega)}.
\]

**Condition 9.1** Assume:

1. \( a(t, x, y) \in C(\sigma), y \in (0, 1) \) and \( t \in (0, T) \), \( \Re \omega_k \neq 0 \) and \( \frac{\lambda}{\omega_k} \in S(\varphi) \) for, \( x \in \sigma, \lambda \in S(\varphi), k = 1, 2, p_k \in (1, \infty) \);
2. \( a_1(t, x, .) \in W^1_{\infty}(0, 1) \), \( a_1(t, x, .) \geq \delta > 0, b_1(t, x, .), c(t, x, .) \in L_{\infty}(0, b) \) for a.e. \( x \in \sigma, t \in (0, T) \);

In this section, we present the following result:

**Theorem 9.1.** Suppose the Condition 9.1 hold. Then, for \( f \in L_{\tilde{p}}(\Omega; E) \), problem (9.1) – (9.3) has a unique solution \( u \) belonging to \( W^{2,1}_p(\Omega; E) \) and the following coercive estimate holds

\[
\left\| \frac{\partial u}{\partial t} \right\|_{L_{\tilde{p}}(\Omega; E)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_{\tilde{p}}(\Omega)} + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L_{\tilde{p}}(\Omega)} + \| Au \|_{L_{\tilde{p}}(\sigma; E)} \leq C \| f \|_{L_{\tilde{p}}(\Omega; E)}.
\]

**Proof.** Let \( E = L_2(0, 1) \). It is known [10] that \( L_2(0, 1) \) is an \( UMD \) space. Consider the operator \( A \) defined by

\[
D(A) = W^2_{\tilde{p}}(0, 1; B_2 u = 0), \quad Au = a_1 \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial y} + cu.
\]

Therefore, the problem (9.1) – (9.3) can be rewritten in the form of (4.1), where \( u(x) = u(x, .), f(x) = f(x, .) \) are functions with values in \( E = L_2(0, 1) \).

By virtue of [30, 31] the operator \( A \) generates analytic semigroup in \( L_2(0, 1) \). Then in view of the Hill-Yosida theorem (see e.g. [28, § 1.13]) this operator is sectorial in \( L_2(0, 1) \). Since all uniform bounded set in Hilbers sapace is an \( R \)-bounded (see [10]), i.e. we get that the operator \( A \) is \( R \)-sectorial in \( L_2(0, 1) \). Then from Theorem 4.1 we obtain the assertion.

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