Accelerating the convergence rate of distributed consensus algorithm based on label propagation algorithm

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Abstract. In order to accelerate the convergence rate of distributed consensus problem under complex topology, in the paper, the distributed consensus algorithm based label propagation algorithm was proposed. Firstly, we composed the complex topology into two layer of topologies by label propagation algorithm, the first layer of topology was consist of a few small communities, and every small community was considered as a node of the second layer of topology. The consensus firstly was reached in the first layer topology, then the consensus was reached in the second layer topology. In the paper, the convergence performance of the algorithm was proved. The analysis and simulation on convergence rate were done, the results show the convergence rate of the algorithm was higher than that of the usually first-order distributed consensus algorithm.

1. Introduction
In the past years, the distributed consensus problem attracted much attention, the convergence rate and convergence performance of the distributed consensus algorithm were one of the most important problem. In a general way, the convergence rate of the distributed consensus algorithm become slow with expansion of the communication topology, in order to improving the convergence rate, a large number of different methods were developed [1-12]. The methods can be mainly categorized into four kinds according to their characteristics. The first kind of methods utilized weighted matrix to accelerate the convergence rate, such as the algorithms in [2,3]. The second kind of methods utilized the past information of the adjacent nodes, such as the algorithms in [4,5]. The third kind of methods was based on broadcast gossip algorithms, such as the algorithms in [6,7]. The last kind of methods optimized network topology to accelerate the convergence rate of the distributed consensus algorithm, such as the algorithms in [8,9].

In the past years, the community detection algorithms were proposed, in the algorithms, the complex networks can be divided into many small communities. By community composition, the distributed consensus problem can be converted to the multi-layers distributed consensus problem. In [13,14], the hierarchical consensus algorithm based on the community detection was applied to accelerate the convergence rate of average consensus problem, but the average consensus can't be reached in the hierarchical consensus algorithms. In [15], the community detection algorithm based on label propagation (abbreviated to LPA) was proposed, the LPA was a simple and near linear time algorithm, so the algorithm had wide application in the community detection, but the stability of the LPA was bad. In [16], the LPA with damping factor was proposed, the LPA become stable.

In order to accelerate the convergence rate of the distributed consensus problem, in the paper, a distributed consensus algorithm based on community detection was proposed. In the paper, firstly, we
composed the large scale network into two layer of network topologies based on the LPA. In the first layer of topology, the nodes in the same community had the same label, the labels of nodes in the different communities were different. The average consensus of every communities firstly was reached, if a node in a community can communicate with nodes in the other communities, the node was considered as a node in the second layer of topology, then, the distributed consensus problem was solved by two steps. In the first step, the topology was divided into different communities based on the LPA, the average consensus of every community was computed online, then, the average consensus in the second layer of topology was computed. In the paper, the simulation, analysis and comparison were done.

2. Distributed consensus and label propagation algorithm

We introduce some notations and concepts about label propagation algorithm and the distributed consensus problem. The community detection can compose a large-scale network into many small communities, the label propagation algorithm was a simple and near linear time algorithm, so the algorithm had wide application in the community detection.

For a graph \( (V, E) \), where \( V \) is a set of vertices, and \( E \) is a set of edges. For a graph \( (V, E) \) with \( n \) node, Olfati-Saber propose the first-order distributed consensus algorithm in [1] can be written as:

\[
x_i(k+1) = x_i(k) - \varepsilon \sum_{j \in N_i} (x_j(k) - x_i(k))
\]  

(1)

where \( N_i \) is adjacent nodes of node \( i \). The collective dynamics of the group of agents can be written as:

\[
x(k+1) = (I_n - \varepsilon L)x(k)
\]  

(2)

In the label propagation algorithm (LPA), firstly every node of communication topology was labeled with different numbers, secondly, every node updated the label of oneself based on the labels of neighbor nodes in every step. Of all labels of neighbor nodes, the node select the label that the number of the label was the most as the new label of oneself, step by step, the nodes with the same label belong to the same community. The label propagation algorithm includes the synchronization algorithm and the asynchronous algorithm. In step \( k \), the synchronization algorithm utilized the labels of adjacent nodes in step \( k \) to update the label of oneself, the asynchronous algorithm utilized the labels of adjacent nodes in step \( k \) and step \( k - 1 \) to update the label of oneself, and the stability of the asynchronous algorithm was better than that of the synchronization algorithm.

3. Distributed consensus algorithm based on LPA

In the paper, the graph \( (V, E) \) was divided into two layer of topology \( (V', E') \) and \( (V'', E'') \). The first layer of topology were \( G(V', E') \), there were \( m \) independent communities, i.e. \( G(V_1, E_1) \ldots G(V_m, E_m) \), and \( V_1 \ldots V_m \subset V' = V, E_1 \ldots E_m \subset E', V_i \cap V_k = \emptyset, E_i \cap E_k = \emptyset, i,k = 1 \ldots m \). In step \( k \), firstly, the consensus was computed among nodes of the different community, i.e. the consensus in the second layer of topology \( G(V'', E'') \). In the second layer of topology \( G(V'', E'') \), if a node can exchange information with the nodes in the different community, the corresponding links was considered as 1, the link of two nodes in the same communities was consider as 0, the distributed consensus algorithm can be written as:

\[
x(k+1) = (I_n - \varepsilon L'')x(k)
\]  

\( L'' = \{l_{ij} \} \)

\[ l_{ij} = \begin{cases} 
1 & (i,j) \notin V_k \ k = 1 \ldots m \\
0 & \text{other} 
\end{cases}
\]

\[
l_{ij} = \sum_{j \in V_k} l_{ij}
\]

(3)

Where \( L'' \) was the Laplacian matrix of the second layer of topology \( G(V'', E'') \). In the following, the consensus among nodes in the same community was computed, the distributed consensus can be written by:

\[
x'(k+1) = (I' - \varepsilon L')x'(k+1)
\]

(4)
Where $x'(k+1)$ is a state vector of nodes in the $r$ community, and $r = 1 \cdots m$. $L'$ is the corresponding Laplacian matrix of the graph $G(V_r,E_r)$, the collective dynamics equation in the first layer of topology can be written as:

$$x(k) = (I_n - \epsilon L')x(k) \quad L' = \text{diag}\{L_1', \cdots, L_m'\} \quad L = L' + L''$$  \hspace{1cm} (5)

Where $L$ was the Laplacian matrix of the topology $G(V,E)$, the distributed consensus algorithm based on LPA can be completed by two steps, in the first step, the consensus problem was computed according to the equation (4), i.e. the consensus in the every community was computed independently. In the second step, the consensus among communities was computed according to the equation (3). The collective consensus algorithm can be written as:

$$x(k) = (I_n - \epsilon L'')(I_n - \epsilon L')x(k)$$  \hspace{1cm} (6)

In the following, the convergence performance of the algorithm was proven.

**Theorem 1.** For the distributed consensus algorithm in equation (6), when $G(V,E)$ was connected, and $\rho(I - \epsilon L - \epsilon^2 L'L'') < 1$, then an average consensus for an undirected connected graph $G(V,E)$ can be reached asymptotically, i.e. $\lim_{k \to \infty} x(k) = \frac{1}{n} 1^T x(0)$.

**Proof.** In step $k$, we can obtain the following equation according to equation (6),

$$x(k+1) = (I_n - \epsilon L')(I_n - \epsilon L'')x(k) = (I_n - \epsilon L' - \epsilon L'' + \epsilon^2 L'L'')x(k),$$

obviously, $L', L''$ were symmetric positive semi-definite matrixes, every column sum and every row sum of the matrix $L'$ and $L''$ are zero, i.e., $1^T L' = 0, L' 1 = 0$, $1^T L'' = 0, L'' 1 = 0$. If $\rho(I - \epsilon L - \epsilon^2 L'L'') < 1$, then we can get,

$$\lim_{k \to \infty} x(k+1) = \lim_{k \to \infty} (I - \epsilon L - \epsilon^2 L'L'')x(k) = \lim_{k \to \infty} \frac{1}{n} 1^T (I - \epsilon L - \epsilon^2 L'L'')^{k+1} x(0) = \frac{1}{n} 1^T x(0).$$

4. Simulation

In order to verify the performance and convergence rate of the distributed consensus based on LPA proposed in the paper, simulation results were present in the paper. There were 100 sensors scattered
Figure 2. Comparison of convergence performance of two algorithm randomly in 100 square meter, there were only communication links among sensors within 20 meters, the communication topology was as shown in the Figure 1.

We assumed the step size $\varepsilon = 0.02$ (second), the state value of every node was respectively a number between 1 and 100. After running 1000 Monte-Carlo simulations, the simulating results were as shown in the Figure 2. Figure 2 show the convergence rate of the distributed consensus algorithm based on LPA was higher than that of the first-order distributed consensus algorithm in [1], and the distributed consensus algorithm based on LPA can reach an average consensus.

5. Conclusion
By analysis and simulation in the above section, the convergence rate of the distributed consensus algorithm based on LPA proposed in the paper was faster than the convergence rate of the first-order distributed consensus algorithm in [1], and the proposed algorithm can reach an average consensus.

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