Detecting laws in power subgroups

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ABSTRACT
A group law is said to be detectable in power subgroups if, for all coprime \( m \) and \( n \), a group \( G \) satisfies the law if and only if the power subgroups \( G^m \) and \( G^n \) both satisfy the law. We prove that for all positive integers \( c \), nilpotency of class at most \( c \) is detectable in power subgroups, as is the \( k \)-Engel law for \( k \) at most 4. In contrast, detectability in power subgroups fails for solvability of given derived length: for all coprime \( m \) and \( n \) we construct a finite group \( W \) such that \( W^m \) and \( W^n \) are metabelian but \( W \) has derived length 3. We analyse the complexity of the detectability of commutativity in power subgroups, in terms of finite presentations that encode a proof of the result.

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1. Introduction
This article studies the following broad question: what can be deduced about a group \( G \) by examining its power subgroups \( G^m = \langle g^m : g \in G \rangle \)? In particular, can one infer which laws \( G \) satisfies?

Let \( F_\infty = F \langle x_1, x_2, \ldots \rangle \) be the free group on the basis \( \{x_1, x_2, \ldots \} \). A law (or identity) is a word \( w \in F_\infty \), and we say a group \( G \) satisfies the law \( w \) if \( \varphi(w) = 1 \) for all homomorphisms \( \varphi : F_\infty \rightarrow G \). For notational convenience, when we require only variables \( x_1 \) and \( x_2 \) we will instead write \( x \) and \( y \). We can also think of a word \( w \) on \( k \) variables \( x_1, \ldots, x_k \) as a function \( w : G^{k \times \cdots \times G} \rightarrow G \), written \( w(g_1, \ldots, g_k) := \varphi(w) \) for a homomorphism \( \varphi : F_\infty \rightarrow G \) such that \( \varphi(x_i) = g_i \).

Laws give a common framework for defining various group properties; basic examples include commutativity (corresponding to the law \([x,y] \) ), having exponent \( m \) (the Burnside law \( x^m \) ), being metabelian (the law \( [[x_1, x_2], [x_3, x_4]] \) ), and nilpotency of class at most \( c \) (the law \( [[[\ldots[x_1, x_2], x_3], \ldots, x_c], x_{c+1}] \) ).

Definition 1.1. A group law \( w \) is detectable in power subgroups if, for all coprime \( m \) and \( n \), a group \( G \) satisfies \( w \) if and only if the power subgroups \( G^m \) and \( G^n \) both satisfy \( w \).

A subgroup of \( G \) will satisfy all the laws of \( G \), but in general it is possible even for coprime \( m \) and \( n \) that the power subgroups \( G^m \) and \( G^n \) satisfy a common law that \( G \) does not; for example, the holomorph \( G = \mathbb{Z}/7 \rtimes \mathbb{Z}/6 \) (where \( \mathbb{Z}/6 \cong \text{Aut}\mathbb{Z}/7 \) acts faithfully) was shown to have this property in [6, Example 8.2] (and is in fact the smallest such group). A concrete example of a
law that holds in \( G^2 \) and \( G^3 \) but not \( G \) is \([x^2, y^2]^3, y^3\). Another basic example is the holomorph \( G = \mathbb{Z}/9 \rtimes \mathbb{Z}/6 \), which does not satisfy the law \([x^2, x']\) although \( G^2 \) and \( G^3 \) do.

**Example 1.2.** The law \( x^r \) is detectable in power subgroups.

This basic example is immediate: for every \( g \in G \), if \((g^m)^r = 1\) and \((g^n)^r = 1\), then \( g^r = 1 \) as \( m \) and \( n \) are coprime.

A classical theme in group theory is the study of conditions that imply that a group is abelian. This was recently revived by Venkataraman in [10], where she proved that commutativity is detectable in power subgroups for finite groups. We can extend this to infinite groups using residual finiteness of metabelian groups (a theorem of P. Hall [7, 15.4.1]); it appears that this result is folklore. A. Olshanskii has informed us that it was discussed in the Group Theory seminar of the Moscow State University circa 1968.

In this article, we prove that this result generalizes to the nilpotent case:

**Corollary A1.** Let \( m \) and \( n \) be coprime and let \( c \geq 1 \). Then a group \( G \) is nilpotent of class at most \( c \) if and only if \( G^m \) and \( G^n \) are both nilpotent of class at most \( c \).

This means that the precise nilpotency class of \( G \) is the maximum of the precise nilpotency classes of \( G^m \) and \( G^n \). Fitting’s Theorem (see 2.1 below) readily implies a weak form of the “if” direction, namely that \( G \) is nilpotent of class at most \( 2c \), but it is much less obvious that the precise nilpotency class is preserved.

**Theorem A.** Let \( w \) be a group law such that every finitely generated group satisfying \( w \) is residually nilpotent, and let \( m \) and \( n \) be coprime integers. Then a group \( G \) satisfies \( w \) if and only if \( G^m \) and \( G^n \) both satisfy \( w \).

A topic with a rich history, dating to work of Burnside, is that of Engel laws. The \( k \)-Engel law is defined recursively by \( E_0(x, y) = x \) and \( E_{k+1}(x, y) = [E_k(x, y), y] \). For example, the 3-Engel law is \([[[x, y], y], y]\). Havas and Vaughan-Lee [4] proved local nilpotency for 4-Engel groups, so we have the following:

**Corollary A2.** Let \( m \) and \( n \) be coprime and let \( k \leq 4 \). A group \( G \) is \( k \)-Engel if and only if \( G^m \) and \( G^n \) are both \( k \)-Engel.

It is an open question whether a \( k \)-Engel group must be locally nilpotent for \( k \geq 5 \). Recently A. Juhasz and E. Rips have announced that this is not the case for sufficiently large \( k \).

In contrast to Corollary A1, solvability of a given derived length is not detectable in power subgroups; this fails immediately and in a strong sense as soon as we move beyond derived length one, that is, beyond abelian groups.

**Theorem B.** The metabelian law \([x_1, x_2], [x_3, x_4]\) is not detectable in power subgroups. Indeed, for every pair of coprime integers \( m, n > 1 \) there exists a finite group \( W \) such that \( W^m \) and \( W^n \) are both metabelian but \( W \) is of derived length 3.

This is yet another example of the chasm between nilpotency and solvability. Other properties that we lose when crossing from finitely generated nilpotent groups to finitely generated solvable groups include the following: residual finiteness, solvability of the word problem, polynomial growth, and finite presentability of the relatively free group.

As the free nilpotent group of class \( c \) is finitely presented, we know *a priori* that Corollary A 1 will be true for fixed \( m \) and \( n \) if and only if it is provable in a mechanical way, namely via a finite subpresentation of a canonical presentation for the universal group of rank \( c+1 \) with \( m \)-th and \( n \)-th power subgroups nilpotent of class \( c \). Since such a finite presentation ‘proving’ the theorem for those \( m \) and \( n \) exists, it is natural to ask what such a presentation looks like: what is the minimum number of relators needed, does that number depend on \( m \) and \( n \), and how must the specific relators change with \( m \) and \( n \)?
We analyse in detail the abelian case, where the answer to all of these questions is: surprisingly little.

**Theorem C.** Let \( m \) and \( n \) be coprime. The group \( G_{m,n} \) defined by the presentation

\[
\langle a, b \mid [a^m, b^n], [a^m, (ab)^m], [b^m, (ab)^m], [a^n, b^n], [a^n, (ab)^n], [b^n, (ab)^n] \rangle
\]

is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \).

The structure of this article is as follows, and reflects the structure of the introduction we have just given. We prove positive results, including Theorem A, in Section 2. We then prove the negative result Theorem B in Section 3. The proof of Theorem C follows in Section 4, and finally we record some open problems in Section 5. Detectability of laws admits an elegant phrasing in the language of varieties of groups (see [3 pp. 27–30] for this perspective), but this is not necessary for any of our proofs and is thus omitted in the present article.

## 2. Locally (residually) nilpotent laws are detectable

The starting point for this section is a desire to generalize the result that commutativity is detectable in power subgroups to the nilpotent case. For instance, can power subgroups detect whether a group is nilpotent of class at most 2? We are carried quite a way towards our goal by Fitting’s Theorem.

**Theorem 2.1** (Fitting, [7, 5.2.8]). Let \( M \) and \( N \) be normal nilpotent subgroups of a group \( G \). If \( c \) and \( d \) are the nilpotency classes of \( M \) and \( N \), then \( L = MN \) is nilpotent of class at most \( c + d \).

However, this will only tell us, for instance, that if the power subgroups are nilpotent of class at most 2, then our group of interest is nilpotent of class at most 4. We quickly lay some foundations towards proving the general Theorem A, then see in Theorem 2.4 how we can reduce the bound of \( 2c \) to \( c \), as in Corollary A1. By proving the general theorem, we will also be able to conclude that certain Engel laws are detectable.

**Notation.** We write conjugation \( g^h = h^{-1}gh \) and commutator \( [g, h] = g^{-1}h^{-1}gh \).

An easily proved property of power subgroups is the following:

**Lemma 2.2.** Let \( \varphi : G \to Q \) be a surjective group homomorphism and \( m \) an integer. Then \( \varphi(G^m) = Q^m \) and \( G^m \leq \varphi^{-1}(Q^m) \).

Recall a well-known fact about torsion groups, which we will apply several times.

**Proposition 2.3** ([7, 5.4.11]). A finitely generated solvable torsion group is finite.

**Theorem 2.4.** Let \( G \) be a finitely generated residually nilpotent group and let \( m \) and \( n \) be coprime. If \( G^m \) and \( G^n \) both satisfy a law \( w \), then \( G \) satisfies \( w \).

**Proof.** Suppose for the sake of contradiction that there is a homomorphism \( \varphi : F_\infty \to G \) with \( \varphi(w) \neq 1 \). Since \( G \) is finitely generated and residually nilpotent, it is residually finite [7, 5.4.17], so there is a map \( q : G \to Q \) for some finite nilpotent group \( Q \) such that \( q(\varphi(w)) \neq 1 \). The group \( Q \) is the direct product of its Sylow subgroups [7, 5.2.4]. We compose \( q \) with a projection onto a Sylow subgroup in which \( q(\varphi(w)) \) has non-trivial image, to get \( q_p : G \to Q_p \). Without loss of generality, the prime \( p \) does not divide \( m \) so that \( Q_p^m = Q_p \) as every element \( g \in Q_p \) has order coprime to \( m \) and thus is contained in the cyclic subgroup generated by \( g^m \). This gives a contradiction, as \( Q_p^m = q_p(G^m) \) (Lemma 2.2), and \( G^m \) satisfies the law \( w \).
Remark 2.5. Note that a group which is the product of two locally residually nilpotent normal subgroups need not be locally residually nilpotent [1] (in contrast to the locally nilpotent case, which is the Hirsch–Plotkin Theorem [7, 12.1.2]), so $G^m$ and $G^n$ satisfying $w$ does not immediately imply that $G$ is locally residually nilpotent. Nonetheless, one can actually prove that if all groups satisfying $w$ are locally residually nilpotent, then groups satisfying $w$ are in fact locally nilpotent [8]. Here, we instead prove the locally residually nilpotent version directly, as the argument is shorter.

Proof of Theorem A. Suppose that $G^m$ and $G^n$ satisfy $w$. A group satisfies a law if and only if all its finitely generated subgroups satisfy that law, and if $G_0 \leq G$ then $G_0^m \leq G^m$ so $G_0^m$ satisfies $w$, and likewise for $G_0^n$, so we can assume without loss of generality that $G$ is finitely generated.

Consider the group $G/G^m$, which is finitely generated and of exponent $m$. By the second isomorphism theorem

\[ G/G^m \cong G^n/(G^m \cap G^n) \]

and so $G/G^m$ satisfies $w$, being a quotient of $G^m$, so it is residually nilpotent. Thus $G/G^m$ is a finitely generated, residually finite group of exponent $m$. By Zelmanov’s solution of the Restricted Burnside Problem [11, 12], there is a bound on the orders of finite quotients of $G/G^m$, so residual finiteness forces $G/G^m$ to be finite itself. Hence the subgroup $G^m \triangleleft G$ is of finite index, so it is finitely generated and thus residually nilpotent. Similarly, $G^n$ is residually nilpotent.

As $G$ has a residually finite subgroup of finite index, it is residually finite itself. If $q : G \rightarrow Q$ is any finite quotient, then $Q$ is the normal product of $q(G^m)$ and $q(G^n)$, which are finite groups satisfying $w$, hence are residually nilpotent and thus nilpotent. By Fitting’s Theorem we conclude that all finite quotients of $G$ are nilpotent, and thus $G$ is residually nilpotent. The result now follows from Theorem 2.4.

Proof of Corollary A2. The 4-Engel law was shown to be locally nilpotent by Havas and Vaughan-Lee [4]. This also implies the (previously known) $k \leq 3$ cases, as it is clear from the definition

\[ E_{k+1}(x, y) = [E_k(x, y), y] \]

that a $k$-Engel group is also $(k + 1)$-Engel.

For a survey on Engel groups, the reader is referred to [9].

Remark 2.6. One can prove that the class of virtually nilpotent groups is detectable in power subgroups [3, Corollary 2.25, p. 34], using the fact that the nilpotent radical is a characteristic subgroup.

3. Derived length is not detectable

In this section we show by explicit example that one cannot extend the above results for the nilpotent case to the solvable case. Of course, a group with solvable coprime power subgroups is itself solvable: the class of solvable groups is closed under extensions. The point is that we do not have precise control over derived length like we did for nilpotency class.

The following construction was suggested by A. Olshanskii, generalizing a construction specific to $m = 2$ and $n = 3$.

Proof of Theorem B. Let $p$ be a prime dividing $m$ and $q$ a prime dividing $n$; it will suffice to prove that $W^p \geq W^m$ and $W^q \geq W^n$ are metabelian. We let

\[ H = \left\{ \begin{pmatrix} ap + 1 & c \\ 0 & bp + 1 \end{pmatrix} \right\} \leq \text{GL}_2(\mathbb{Z}/p^2), \]

a matrix group of order $p^4$. The product

\[ \begin{pmatrix} ap + 1 & c \\ 0 & bp + 1 \end{pmatrix} \begin{pmatrix} dp + 1 & c' \\ 0 & bp + 1 \end{pmatrix} = \begin{pmatrix} (a + dp + 1)(ap + 1)c' + (b'p + 1)c \\ 0 & (b + b'p + 1)c \end{pmatrix} \]
from which we see that the cyclic subgroup \( Z \) generated by \( \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \) is central, that \( H \) is not abelian, and that \( H/Z \cong (\mathbb{Z}/p)^3 \). Thus in particular \([H, H] = Z\).

Let \( R = (\mathbb{Z}/p^2)^\infty \) and let \( N = R \times \cdots \times R \). We make \( H \) act on \( N \) diagonally, with the usual left-multiplication on each \( R \) factor, and we let \( \mathbb{Z}/q \) act by coordinate shift \((r_1, r_2, ..., r_q) \rightarrow (r_q, r_1, ..., r_{q-1})\). These actions commute, so we can define
\[
W = N \rtimes (H \times \mathbb{Z}/q).
\]

First, we show that \( W_p \) is metabelian. By Lemma 2.2 we see that, since \( H/Z \cong (\mathbb{Z}/p)^3 \), we have \( H^p = Z \) and hence \( W_p \leq N \rtimes (\mathbb{Z} \times \mathbb{Z}/q) \). This group is manifestly an abelian-by-abelian group.

Now we consider \( W_q = N \rtimes H \) (we have \( \leq \) by Lemma 2.2 and \( \geq \) follows from \( N \) and \( H \) being \( p \)-groups). Recall the following:

**Lemma 3.1.** Let \( G = N \rtimes K \). Then the derived subgroup \( G' = (N'[N, K]) \rtimes K' \).

One can prove the lemma by verifying that every commutator in \( G \) lies in the subgroup generated by \( N' \), \([N, K]\) and \( K'\), and then noting that the action of \( K \) on \( N \) restricts to an action on \( N'[N, K] \). In the present case, we see that each factor \([R, H]\) is generated by the elements
\[
\begin{pmatrix} ap + 1 \\ 0 \end{pmatrix} \begin{pmatrix} c \\ bp + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ap + cy \\ bp \end{pmatrix}
\]
so that \([N, H]\) is \( q \) copies of \( \left\{ \begin{pmatrix} x \\ py \end{pmatrix} \right\} \), on which \([H, H] = Z \) acts trivially, thus \((N \rtimes H)'\) is abelian, that is, \( W_q \) is metabelian.

Finally, \([W, W] \cap N\) certainly contains an isomorphic copy of \( R \) with the usual \( H \)-action. For instance, for all \( r \in R \), the commutator of the generator \( t \) of \( \mathbb{Z}/q \) and \((r, 0, ..., 0) \in C(t) = (r, -r, 0, ..., 0) \). Since \( Z \leq [W, W] \) does not act trivially on this copy of \( R \), we see that \([W, W] \) is not abelian.

\[ \square \]

4. A concise presentation proving the abelian case

We can formulate detectability of commutativity using an infinitely presented group having the appropriate universal property.

**Proposition 4.1.** The detectability of commutativity in power subgroups is equivalent to the fact that, for all coprime \( m \) and \( n \), the group defined by the infinite presentation
\[
Q = \langle a, b \mid [u^m, v^m], [u^n, v^n] \forall u, v \in F(a, b) \rangle
\]
is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \).

**Proof.** If the group \( G \) defined by \( Q \) is non-abelian then it is a counterexample. Suppose \( G \) is abelian, and let \( H \) be a group with \( H^m, H^n \) abelian. Then for every pair of elements \( g, h \in H \) there is a homomorphism from \( G \) to \( (g, h) \leq H \) defined by \( a \rightarrow g, b \rightarrow h \), since all the relators have trivial image, and thus \( g \) and \( h \) commute. Therefore \( H \) is abelian. Moreover, as all relators in the presentation of \( Q \) are a consequence of the commutativity of the two generators, \( G \) is in fact isomorphic to the free abelian group on 2 generators, \( \mathbb{Z} \times \mathbb{Z} \).

Thus for coprime \( m \) and \( n \), the word \([a, b] \) is expressible in the free group as a product of conjugates of terms of the form \([u^m, v^m]\) and \([u^n, v^n]\). For a treatment of this phenomenon in generality, see [3, § 2.5, p. 36 ff.].

At this point, it is natural to ask how many different such terms are needed to encode such a proof; Theorem C gives 6 specific terms that suffice. After proving this theorem, we became aware of another proof [5] that groups with abelian power subgroups are abelian from which one can extract a 2-generator 6-relator presentation which defines \( \mathbb{Z} \times \mathbb{Z} \), just as in Theorem C.
However, our proof has the advantage of uniformity in the words from the "verbal" subgroup used, whereas in the other proof the length of the words grows quadratically with \( m \) and \( n \).

The proof proceeds by first showing that the commutator \([a, b]\) is central; once we know this, the proof that it is trivial is very short. Rather than prove that \( G_{m,n} \) (defined via the presentation of Theorem C) is nilpotent of class 2 directly, we instead prove the stronger result that the group \( \Gamma \) (defined below), an extension of \( G_{m,n} \), is nilpotent of class 2. This group is moreover a common extension of all the \( G_{m,n} \), so we see our introducing \( \Gamma \) as abstracting away \( m \) and \( n \) from the proof. (We will of course prove later that each \( G_{m,n} \cong \mathbb{Z} \times \mathbb{Z} \), and so technically \( \mathbb{Z} \times \mathbb{Z} \) itself is also a common extension \emph{a posteriori}, but we are constructing a group which is \emph{a priori} a common extension.)

**Definition 4.2.** Let the group \( \Gamma \) be defined by the presentation

\[
\langle a, b, x, y, z \mid \begin{array}{c}
[a, x], \ [b, y], \ [ab, z], \ [x, y], \ [x, z], \ [y, z], \ [ax, by], \ [ax, abz], \ [by, abz] \end{array} \rangle.
\]

**Lemma 4.3.** The group \( \Gamma \) is an extension of \( G_{m,n} \), with the quotient map sending \( a \mapsto a \) and \( b \mapsto b \).

**Proof.** As \( m \) and \( n \) are coprime, there exist integers \( p \) and \( q \) such that \( pm - qn = 1 \), that is, \( pm = qn + 1 \). Define a map \( \Gamma \to G_{m,n} \) by \( a \mapsto a, b \mapsto b, x \mapsto a^q y, y \mapsto b^p \) and \( z \mapsto (ab)^{qn} \). This is easily checked to be well defined, as every defining relator for \( \Gamma \) is mapped to a relator of the form \([u^k, v^l] \) for some \( u \) and \( v \) with \([u, v] \) a defining relator of \( G_{m,n} \) and \( k, l \in \mathbb{Z} \).

**Proposition 4.4.** The subgroup \( \langle a, b \rangle \leq \Gamma \) is nilpotent of class 2.

**Remark 4.5.** The group \( \Gamma \) itself is nilpotent of class 2, with \( [\Gamma, \Gamma] \cong \mathbb{Z} \). However, we confine ourselves here to proving Proposition 4.4, which is all that is required for Theorem C.

We prove Proposition 4.4 in a sequence of lemmas. It will be convenient to know that the symmetry in \( a \) and \( b \) of \( G_{m,n} \) extends to \( \Gamma \).

**Lemma 4.6.** Let \( \varphi : a \mapsto b \mapsto a, x \mapsto y \mapsto x, z \mapsto z^a \). Then \( \varphi \) is an automorphism of \( \Gamma \).

**Proof.** Since the above also defines an automorphism of the free group \( F(a, b, x, y, z) \), it suffices to check that \( \varphi \) is a well-defined group homomorphism. To verify this we now show that the images of the relators are trivial, in the cases where this is not immediate. Note that since \([ab, z] = 1\), we have \( z^a = z^{b^{-1}} \).

\[
\begin{align*}
\varphi([ab, z]) &= [ba, z^a] = [ab, z^a] = 1 \\
\varphi([x, z]) &= [y, z^a] = [y, z^{b^{-1}}] = [y, z]^{b^{-1}} = 1 \\
\varphi([y, z]) &= [x, z^a] = [x, z]^a = 1 \\
\varphi([ax, abz]) &= [by, baza] = [by, baza] = [by, zab]^{b^{-1}} = [by, abz]^{b^{-1}} = 1 \\
\varphi([by, abz]) &= [ax, baz^a] = [ax, baz]^a = [ax, abz]^a = 1
\end{align*}
\]

**Lemma 4.7.** The commutator \([ab, yx] = 1\) in the group \( \Gamma \).

**Proof.** In light of Lemma 4.6, we can instead prove \([ba, xy] = 1\) as follows:

\[
\begin{align*}
axybabz &= abxz(ab)y \\
= (ax)(by)(abz) &= abx(ab)zy \\
= (abz)(ax)(by) &= abaxbyz \\
= abzxaby &= abaxybz.
\end{align*}
\]

After cancelling on the left and right, we have \( xyb = baxy \).
Lemma 4.8. The commutator \([a, bzy] = 1\) in the group \(\Gamma\).

Proof. We have
\[
\begin{align*}
a(bzy) &= (abz)(by)b^{-1} = by(ab)z^{-1} \\
&= \frac{by(ab)z^{-1}}{C_0} \\
&= \frac{by(ab)b^{-1}}{C_0} \quad \text{Lemma 4.8} \\
&= \frac{byz(ab)b^{-1}}{C_0} \\
&= \frac{(bzy)a}{C_0}.
\end{align*}
\]

Lemma 4.9. The commutator \([b, zax] = 1\) in the group \(\Gamma\).

Proof. This follows from Lemma 4.8 by symmetry, as we have \(\varphi([b, zax]) = [a, z^a by] = [a, z^{b^{-1}} by] = [a, bzy] = 1\).

In the following computations, we will frequently use the basic fact that if \([g, hk] = 1\) then \(g^h = g^{k^{-1}}\).

Lemma 4.10. The commutator \([b, z^{-1}x] = 1\) in the group \(\Gamma\).

Proof. Note first that since \(abz\) and \(a\) both commute with \(ax\), we have \([ax, bz] = 1\) and thus \((ax)^b = (ax)^{z^{-1}}\). Now
\[
\begin{align*}
abxb^{-1} &= (ab)(xy)y^{-1}b^{-1} = (ax)^b \\
&= \frac{(ax)^b}{C_0} \quad \text{Lemmas 4.7} \\
&= \frac{(ax)^{z^{-1}}}{C_0} \\
&= \frac{(ab)^{z^{-1}}(b^{-1}x)^{z^{-1}}}{C_0} \\
&= \frac{abzb^{-1}x^{-1}}{C_0} \\
&= \frac{abzb^{-1}z^{-1}x}{C_0}.
\end{align*}
\]
Left-multiplying both sides by \(z^{-1}b^{-1}a^{-1}\) gives \(z^{-1}xb^{-1} = b^{-1}z^{-1}x\).

Lemma 4.11. The commutator \([b, [a, z]] = 1\) in the group \(\Gamma\).

Proof. By Lemma 4.9 we have \(zax \in C_\Gamma(b)\), the centralizer of \(b\), and by Lemma 4.10 we have \(z^{-1}x \in C_\Gamma(b)\). The centralizer thus contains \(z^{-1}xax = xax\), and thus also
\[
[xax, x^{-1}z] = [a, z]
\]
since \(x\) is central in \(a, x, z\).

We are now equipped to prove Proposition 4.4.

Proof of Proposition 4.4. Starting by applying Lemma 4.7, we see
\[
\begin{align*}
ab &= (yx)ab(yx)^{-1} = a^{bx}b^{-a} \\
&= \frac{a^{bx}b^{-a}}{C_0} \quad \text{Lemmas 4.8 and 4.9} \\
yax^{-1}bx^{-1} &= (a^{bx}b^{-a})^{z^{-1}} \\
&= \frac{a^{bx}b^{-a}z^{-1}}{C_0} \\
yay^{-1}xbx^{-1} &= a^{bx}b^{-a}z^{-1} \\
a^{-1}b^{-1}b^{-1} &= a^{-1}b^{-1}b^{-1} \quad \text{LHS ab and z commute} \\
&= \frac{a^{bx}b^{-a}}{C_0} \quad \text{Lemma 4.11}
\end{align*}
\]
Thus \(b^a = (a^b)^{-1}ab = b^{-1}a^{-1}bab = b^{-1}b^a b\). Since \(b\) commutes with \(b^a\), it commutes with \(b^{-1}b^a = [b, a] = [a, b^{-1}]\). Applying \(\varphi\), we see that also \([a, [a, b]] = 1\). Thus the subgroup \(\langle a, b \rangle \leq \Gamma\) is nilpotent of class 2.

Proof of Theorem C. As \([a, b]\) is central in \(\langle a, b \rangle \leq \Gamma\) (Proposition 4.4) and \(\Gamma\) is an extension of \(G_{m,n}\) (Lemma 4.3), it follows that \([a, b]\) is central in \(G_{m,n}\). Thus \([a^n, b^m] = [a, b]^{nm}\) and
\[ [a^n, b^m] = [a, b]^{mn}. \] Since \( m \) and \( n \) are coprime, so are \( m^2 \) and \( n^2 \). Now coprime powers of \([a, b]\) are both trivial, so \( [a, b] = 1 \).

5. Open problems

While we have a surprisingly and uniformly small presentation proving the abelian case, we still do not know what is the minimum number of relators needed.

**Question 1.** For which coprime \( m \) and \( n \) is there a 4-relator presentation for \( \mathbb{Z} \times \mathbb{Z} \) that encodes a proof that \( G^m \) and \( G^n \) being abelian imply that \( G \) is abelian, that is, when does the infinite presentation \( Q \) (see Proposition 4.1) have a 4-relator subpresentation defining \( \mathbb{Z} \times \mathbb{Z} \)? In particular, when is

\[ \Delta_{m,n} = \langle a, b \mid [a^m, b^m], [a^n, b^n], [(ab)^m, (ba)^m], [(ab)^n, (ba)^n] \rangle \]

isomorphic to \( \mathbb{Z} \times \mathbb{Z} \)?

The van Kampen diagram in Figure 1 proves that the question has a positive answer for \( (m, n) = (2, 3) \). A van Kampen diagram per se is purely topological; the geometry of the drawing has been chosen such that corners generally delimit the 4 subwords \( u^{-1}, v^{-1}, u \) and \( v \) in a commutator \( [u, v] \).

Computational experiments using GAP [2] provided some evidence for Question 1 having a positive answer; in particular, we have verified this for \( m = 2 \) and odd \( n < 50 \), but could not answer the question either way for \( (m, n) = (3, 4) \).

**Question 2.** Determine the analogous complexity for the nilpotency law

\[ \nu_c = [[[\ldots[x_1, x_2], x_3], \ldots, x_c], x_{c+1}] \].

It seems that the following classification problem would require substantial progress.

**Question 3.** Which laws are detectable in power subgroups?

The difficulty is exemplified by the fact that the 4-Engel law is detectable, but it has been claimed that not all \( k \)-Engel laws imply the essential local nilpotency that we used. We thus ask in particular:

**Question 4.** For which \( k \) is the \( k \)-Engel law detectable in power subgroups?

To summarize our knowledge at this time, \( x^m \) is detectable in power subgroups, as is every locally (residually) nilpotent law (for example, the 4-Engel law \([x, y, y, y, y] \) or a nilpotency law...
such as $[[x_1, x_2], x_3]$). On the other hand, $[[x_1, x_2], [x_3, x_4]]$ is not detectable, and neither are some assorted laws for which detectability also fails in finite groups, such as $[[x^2, y^3], y^3]$ and $[x^2, x^y]$.

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