GENERALIZED COUNTEREXAMPLES TO THE SEIFERT CONJECTURE

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Abstract. Using the theory of plugs and the self-insertion construction due to the second author, we prove that a foliation of any codimension of any manifold can be modified in a real analytic or piecewise-linear fashion so that all minimal sets have codimension 1. In particular, the 3-sphere $S^3$ has a real analytic dynamical system such that all limit sets are 2-dimensional. We also prove that a 1-dimensional foliation of a manifold of dimension at least 3 can be modified in a piecewise-linear fashion so that there are no closed leaves but all minimal sets are 1-dimensional. These theorems provide new counterexamples to the Seifert conjecture, which asserts that every dynamical system on $S^3$ with no singular points has a periodic trajectory.

1. Introduction

In 1950, H. Seifert \cite{Sei} asked whether every dynamical system on the 3-sphere with no singular points has a periodic trajectory. The conjecture that the answer is yes became known as the Seifert conjecture. Seifert proved the conjecture for perturbations of the flow parallel to the Hopf fibration. In 1974, P. A. Schweitzer \cite{Sch1} found a $C^1$ counterexample to the Seifert conjecture, which was modified to a $C^2$ counterexample by J. Harrison \cite{Har}. In 1993, H. Hofer \cite{Hof} proved the conjecture for contact flows.

The main idea of this paper comes from the construction of a smooth counterexample to the Seifert conjecture due to the second author \cite{KuK}. Here, we outline more general consequences of that idea. In particular, we establish the following theorem:

**Theorem 1.** A foliation of any codimension of any manifold can be modified in an analytic or piecewise-linear fashion so that all minimal sets have codimension 1.

This theorem strengthens an earlier result of F. W. Wilson \cite{Wil} which for oriented 1-foliations, establishes minimal sets of codimension 2.

Theorem 1 is trivial for foliations which are themselves codimension 1. The problem of opening all compact leaves in codimension 1, which is a more natural question, has been partly settled by S. P. Novikov \cite{Nov} and P. A. Schweitzer \cite{Sch2}. Novikov proved that every $C^2$ codimension 1 foliation of $S^3$ has a closed leaf (which was later extended to continuous foliations by V. V. Solodov \cite{Sol} and G. Hector and U. Hirsch \cite{H-H}), while Schweitzer has shown that it is possible to modify any codimension 1 foliation in dimension 4 or higher in a $C^1$ fashion so that it has no compact leaf.

Here and throughout the paper, smooth means $C^\infty$ and analytic means real analytic or $C^\omega$. All manifolds are assumed to be paracompact and Hausdorff, and they are assumed...
to have no boundary unless explicitly stated otherwise. Also, dimension means covering
dimension, cohomological dimension, small inductive dimension, or large inductive dimen-
sion; they are all equal for topological spaces considered in this paper. In our context, we
can freely convert between an (oriented) 1-dimensional foliation and a dynamical system
with no singular points. Note that Theorem 1 implies the following theorem:

**Theorem 2.** The 3-sphere $S^3$ has an analytic dynamical system such that all limit sets are
2-dimensional. In particular, it has no circular trajectories.

Table 1 shows the best known continuity of various kinds of foliations of 3-manifolds. Entries with no citation are covered by Theorem 2 or its analogue for arbitrary 3-manifolds.

|                | not volume-preserving | volume-preserving |
|----------------|-----------------------|-------------------|
| discrete circles | $C^\infty$ [Wil]      | $C^\infty$ & PL [KuG] |
| 1-dim. minimal sets but no circles | $C^1$ [Sch], $C^2$ [Har], PL | $C^1$ [KuG] |
| 2-dim. minimal sets | $C^\infty$, PL | — |

**Table 1. Known foliations of 3-manifolds**

All counterexamples to the Seifert conjecture and its analogues described in this paper are based on constructions of aperiodic plugs. An (insertible, untwisted, attachable) plug, whose prototype was defined by Wilson, is an oriented, 1-dimensional foliation $F$ of a manifold with boundary, the Cartesian product $F \times I$ of an $(n-1)$-dimensional manifold $F$ and $I$. The foliation $F$ agrees with the trivial foliation in the $I$ direction on a neighborhood of $\partial (F \times I)$, if a leaf connects $(p,0)$ with $(q,1)$, then $p = q$, and there is a non-compact leaf containing some $(p,0)$. Here the base $F$ is an $(n-1)$-manifold that admits a bridge immersion in $\mathbb{R}^{n-1}$, an immersion that lifts to an embedding $\mathbb{R}^n$. A plug is aperiodic if it has no closed leaves. For example, the 3-dimensional Wilson plug on $A \times I$, where $A$ is an annulus, is not aperiodic, while the Schweitzer plug is an aperiodic plug on $pT \times I$, where $pT$ is a punctured torus. Although Wilson suggested the technique of inserting plugs to modify foliations, it was Schweitzer who first used plugs to break circular leaves, and it was his important observation that the base of a plug need only admit a bridge immersion, rather than an embedding, in order to be insertible. The results in this paper are based on an idea for constructing aperiodic plugs from a Wilson-type plug which breaks its own circular leaves by means of self-insertion.

The Schweitzer-Harrison $C^2$ aperiodic plug has two 1-dimensional minimal sets. In the PL category, we obtain an aperiodic plug with one 1-dimensional minimal set.

**Theorem 3.** A 1-foliation of a manifold of dimension at least 3 can be modified in a PL
fashion so there are no closed leaves but all minimal sets are 1-dimensional. Moreover, if
the manifold is closed, then there is an aperiodic PL modification with only one minimal
set, and the minimal set is 1-dimensional.

The authors would like to thank William P. Thurston for observing that the basic con-
struction is analytic and not merely smooth.
2. Preliminaries

This paper will consider four kinds of functions between manifolds: continuous, smooth (meaning \(C^\infty\)), analytic (meaning real analytic or \(C^\omega\)), and piecewise linear or PL. Each of these four classes of functions is a smoothness category.

Recall that many kinds of manifolds can be understood by gluing charts. A smooth manifold has smooth gluing maps, an analytic manifold has analytic gluing maps, and in general a manifold is said to be in a given smoothness category if its gluing maps are. We will need the following fundamental result [Mor, Gra]:

**Theorem 4** (Morrey, Grauert). Two analytic manifolds which are diffeomorphic are analytically diffeomorphic.

The Morrey-Grauert theorem has many formulations, all of which were rendered equivalent by Whitney [Whi] in work that predated the theorem itself: Given points \(p\) and \(q\) in \(M\), there exists an analytic function \(f\) such that \(f(p) \neq f(q)\). (This formulation is the closest to what Morrey and Grauert proved directly.) Every analytic manifold \(M\) admits an embedding in some \(\mathbb{R}^n\). Given a \(C^n\) function \(f\) on \(M\), there exists a sequence \(\{f_i\}\) of analytic functions on \(M\) such that \(f_i\) and its first \(n\) derivatives converge pointwise to those of \(f\). Another relevant result due to Whitney is the fact that every smooth manifold admits a smoothly compatible real analytic structure.

A \(k\)-dimensional foliation structure or \(k\)-foliation on an \(n\)-manifold \(M\) is an atlas of charts in \(\mathbb{R}^n\) that preserve the parallel \(k\)-plane foliation of \(\mathbb{R}^n\), which is a partition of \(\mathbb{R}^n\) into translates of flat \(\mathbb{R}^k \subset \mathbb{R}^n\). \(M\) is then a \(k\)-foliated manifold. The foliation structure is in a given category, such as smooth, if the gluing maps are simultaneously in the same category and preserve \(k\)-planes. An oriented foliation is a foliation structure whose gluing maps preserve the standard orientation of \(\mathbb{R}^k\) and its translates; however, they need not preserve the orientation of \(\mathbb{R}^n\).

Let \(M\) be a \(k\)-foliated manifold. Consider the topology on \(\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}\) which is the usual topology in the direction of the first factor \(\mathbb{R}^k\) and the discrete topology in the direction of the second factor \(\mathbb{R}^{n-k}\). This topology can be restricted to a topology on charts in \(\mathbb{R}^n\) and then pushed forward to a topology on \(M\) using the charts that describe the foliation of \(M\). The resulting topology is the leaf topology on \(M\) and it divides \(M\) into a disjoint union of connected \(k\)-manifolds which are the leaves of the foliation of \(M\). A foliation is often described in terms of its leaves. Also, a map between foliated manifolds is leaf-preserving if it is continuous in both the leaf topology and the usual topology. (Unless explicitly stated otherwise, all terms such as closed, connected, etc., will refer to the usual topology on \(M\) rather than the leaf topology.) A minimal set of a foliation is a set which is minimal among non-empty, compact subsets which are unions of leaves. A smooth or analytic foliation is also uniquely determined by the \(k\)-plane field (\(k\)-dimensional subbundle of the tangent bundle) parallel to it.

If \(M\) is an \(n\)-manifold with a \(k\)-foliation \(\mathcal{F}\) and a \(j\)-submanifold \(N\), then \(N\) and \(\mathcal{F}\) are transverse at a point \(p\) if there is a local equivalence that sends a neighborhood \(U\) of \(p\) into \(\mathbb{R}^n\), \(N \cap U\) into a \(j\)-plane, and \(\mathcal{F}|_U\) into parallel \(k\)-planes that intersect the \(j\)-plane in \((j+k-n)\)-planes. In particular, if \(N\) is transverse to a foliation, then \(N\) admits a tubular neighborhood; transversality is problematic without this condition. Note that transversality is defined relative to its smoothness category; for example, in the plane the curves \(y = x^3\) and \(y = 0\) are continuously but not smoothly transverse.
A flow on a metric space $A$ is a system of partial homeomorphisms of $A$ parametrized by time $t \in \mathbb{R}$. Specifically, let $U$ be an open subset of $\mathbb{R} \times A$ containing $\{0\} \times A$ such that $U \cap (\mathbb{R} \times \{p\})$ is connected for every $p$. A flow on $A$ is a continuous map $\Phi : U \to A$ such that $\Phi(0, p) = p$ and whenever $\Phi(t, p)$ and $\Phi(s, \Phi(t, p))$ are both defined, $\Phi(t + s, p)$ is also defined and equals $\Phi(s, \Phi(t, p))$. If $U = \mathbb{R} \times A$, then $\Phi$ is a dynamical system, or equivalently a topological group action of $\mathbb{R}$ on $A$. For example, a flow on a closed manifold is necessarily a dynamical system. If $\Phi$ is either a flow or a dynamical system, a trajectory of a point $p$ is the image of $\Phi((\mathbb{R} \times \{p\}) \cap U)$ in $A$. A point $p$ is a rest point if its trajectory consists only of $p$. A positive (negative) limit set of a trajectory of $p$ is the set of limit points of sequences $\{\Phi(t_n, p)\}$, such that $t_n \to \infty$ ($t_n \to -\infty$).

If $M$ is a manifold and $\Phi$ is smooth or analytic, then $\frac{d\Phi}{dt}|_{t=0}$ is the vector field of $\Phi$ and is in the same smoothness category as $\Phi$. Contrariwise, a standard integration theorem for differential equations says that a $C^1$ vector field can be integrated to produce a corresponding flow or dynamical system. If $\Phi$ has no rest points, the collection of all of its trajectories is the set of leaves of an oriented 1-dimensional foliation which is in the same smoothness category as $\Phi$. Contrariwise, if $F$ is an oriented, 1-dimensional foliation which is smooth or analytic, there exists a parallel non-vanishing vector field in this same category as $F$ which can be integrated to produce a compatible flow. For example, a smooth or analytic manifold always admits a Riemannian metric in the same category, and the unit vector field parallel to the foliation and pointing in the direction of its orientation suffices.

An oriented 1-foliation $F$ which is PL or continuous always has a parallel flow, by the following constructions: In the PL case, embed the manifold $M$ of $F$ in some $\mathbb{R}^n$, and define $\Phi(t, x)$ so that the oriented length of the leaf segment from $x$ to $\Phi(t, x)$ is $t$. For the continuous case choose a locally finite atlas of charts $\alpha_i : \mathbb{R}^n \to M$, and choose a partition of unity $\{f_i\}$ that refines $\{\alpha_i\}$. Let $s$ be a line segment in some leaf with endpoints $x$ and $y$ which is oriented from $x$ to $y$, and suppose that $s$ is entirely contained in every chart which intersects it. Then define $t$ by the formula

$$t = \sum_i \int_{\alpha_i^{-1}(s)} f_i$$

and define $y = \Phi(t, x)$. The function $\Phi$ then extends uniquely to a flow with no rest points.

Henceforth, we will loosely switch between flows, vector fields, and oriented 1-foliations, since they are almost the same geometrically.

If $M$ is a manifold with boundary, a foliation of $M$ is a foliation of the interior of $M$ which admits an extension to a foliation of an open manifold containing $M$; similarly, a flow on $M$ is a flow on the interior of $M$ which admits an extension to a flow on an open manifold containing $M$. If $F$ is a 1-foliation, then the parallel boundary of $F$ is the subset of $\partial M$ where $F$ is locally modelled by the foliation of upper half space by horizontal lines; similarly the transverse boundary is the subset of $\partial M$ where $F$ is locally modelled by the foliation by vertical lines. Note that $M$ can have boundary which is neither parallel nor transverse. A minimal set of a flow $\Phi$ on a manifold with boundary is a set $A$ such that the flow restricted to $A$ is a dynamical system, and $A$ is a minimal set of this dynamical system. A leaf may have boundary or endpoint(s). An infinite leaf is a non-compact leaf. A leaf that is neither closed nor infinite is a finite leaf.

A manifold with corners is a smooth or analytic manifold with piecewise smooth or analytic boundary. I.e., let $M$ be a compact $n$-manifold with boundary. If, for each point $p \in M$, there exists an $n$-dimensional PL submanifold $P$ of $\mathbb{R}^n$ with boundary which is
diffeomorphic to a neighborhood of $p$. For example, a manifold with boundary is a manifold with corners, and the Cartesian product of two manifolds with corners is a manifold with corners. Flows and foliations on manifolds with corners are defined in the same way as on manifolds with boundary, and the standard boundary types of parallel and transverse boundary can be extended to manifolds with corners. Let $N$ be a PL $(n-1)$-submanifold of $\mathbb{R}^{n-1}$. A 1-foliation $\mathcal{F}$ has corner separation at $p \in \partial M$ if $p$ is in neither parallel nor transverse boundary, but if $\mathcal{F}$ is locally equivalent at $p$ to $N \times [0, \infty)$ foliated by rays $\{x\} \times [0, \infty)$. In particular, $p$ belongs to the closure of the transverse boundary and the closure of the parallel boundary. Figure 1 gives an example of corner separation between parallel and transverse boundary.

Although the Morrey-Grauert theorem does not directly apply to a manifold with corners $M$, a manifold with corners is always contained in an open manifold, and the Morrey-Grauert theorem applied to this larger manifold implies that analytic functions on $M$ separate points and that $M$ admits an analytic embedding with the same geometry as any smooth embedding.

3. Plugs

Except where explicitly stated otherwise, the constructions in this section apply uniformly in each of the four smoothness categories. Typically, an object $O$ might involve a foliation, a gluing map, and an embedding, in which case $O$ is a smooth object (for example) if all three parts are in the smooth category.

A flow bordism is an oriented 1-foliation $\mathcal{P}$ of a connected, compact manifold $P$ with boundary or corners such that $\partial P$ is entirely transverse boundary, parallel boundary, or corner separation, and such that all leaves in the parallel boundary of $P$ are line segments. (A flow bordism is very similar to an oriented foliated cobordism. The principal differences are that a foliated cobordism may be a higher-dimensional foliation and it is usually considered without corner separation or parallel boundary.) If $\mathcal{P}$ is a flow bordism, let $F_-$ be the closure of the transverse boundary oriented inward, and similarly let $F_+$ be the closure of the transverse boundary oriented outward. The foliation $\mathcal{P}$ might in addition have one or both of the following properties:

(i): There exists an infinite leaf with an endpoint in $F_-$. 

Figure 1. Corner separation
(ii): There exists a manifold $F$ and homeomorphisms $\alpha_\pm : F \to F_\pm$ such that if $\alpha_+(p)$ and $\alpha_-(q)$ are endpoints of a leaf of $\mathcal{P}$, then $p = q$.

If $\mathcal{P}$ satisfies property (ii), it has \textit{matched ends}. The foliation $\mathcal{P}$ is a plug if it has properties (i) and (ii), but only a semi-plug if it has property (i) but not property (ii). It is an un-plug if it has property (ii) but not property (i). Table 2 gives a summary of these four definitions. The manifold $F_-$ is the entry region of $\mathcal{P}$, while $F_+$ is the exit region. If $\mathcal{P}$ has matched ends, then $F$ is the base of $\mathcal{P}$. The manifold $P$ is the support of $\mathcal{P}$. The entry stopped set $S_-$ of $\mathcal{P}$ is the set of points of $F_-$ which are endpoints of infinite leaves; the exit stopped set $S_+$ is defined similarly. If $\mathcal{P}$ has matched ends, the stopped set $S$ is defined as $\alpha_-^{-1}(S_-) = \alpha_+^{-1}(S_+)$. If $S$ has non-empty interior, then $\mathcal{P}$ stops content.

| Data | not (i) | semi-un-plug | un-plug |
|------|---------|--------------|---------|
| (i)  | semi-plug | plug         |         |

Table 2. Describing $\mathcal{P}$ based on its properties

Note that if $\mathcal{P}$ is a flow bordism with entry region $F_-$ and exit region $F_+$, then since $F_-$ and $F_+$ are the transverse boundaries together with some corner separation points, they possess trivially foliated neighborhoods. Specifically, if $F_\pm \times [0,1)$ is foliated by vertical fibers $\{p\} \times [0,1)$, there exists a leaf-preserving homeomorphism $\omega_\pm$ from an open neighborhood of $F_\pm$ to $F_\pm \times (0,1)$; moreover, we can take $\omega_\pm(p) = (p,0)$ for all $p \in F_\pm$.

\textbf{Lemma 5.} If $\mathcal{P}$ is a flow bordism with connected support $P$ and $\mathcal{P}$ has at least one circular or infinite leaf, then the entry stopped set $S_-$ (resp. the exit stopped set $S_+$) is non-empty if and only if $F_-$ (resp. $F_+$) is non-empty. In particular, if $\mathcal{P}$ has matched ends (and $F_- \cong F_+ \cong F$ is non-empty), then $\mathcal{P}$ is a plug.

\textbf{Proof:} Let $\Phi$ be a flow parallel to $\mathcal{P}$. If $p \in P$, then $L_+(p)$, the future longevity of $p$, is the supremum of $t$ such that $\Phi(t,p)$ is defined. Similarly, the past longevity $L_-(p)$ is the supremum of $t$ such that $\Phi(-t,p)$ is defined. Thus we have two functions $L_+: P \to [0,\infty]$, $L_- : P \to [0,\infty]$. If $L_+(p) < \infty$ or $L_-(p) < \infty$, then the leaf containing $p$ has an endpoint $q$ on $F_+$ or $F_-$, respectively. Investigating the points in a neighborhood of such a $q$, we see that two sets $L^+_\pm(\infty)$ are both closed. If $S_- = \emptyset$, then $F_- \cap L^+_\pm([0,\infty)) = F_-$, and the set $A$ consisting of leaves that intersect $F_-$ is closed. Then $P$ is the union of two disjoint non-empty closed sets, $A$ and $L^-_\pm(\infty) \cup L^+_\pm(\infty)$, which is a contradiction. Hence $S_- \neq \emptyset$ if $F_- \neq \emptyset$. Similarly, $S_+ \neq \emptyset$ if $F_+ \neq \emptyset$. \hfill $\square$

\textbf{Lemma 6.} If $\mathcal{P}$ is a semi-un-plug with support $P$ and entry region $F_-$, then there is a foliation isomorphism $\gamma : F_- \times I \to P$, where $F_- \times I$ is foliated by fibers $\{p\} \times I$. In particular, if $\mathcal{P}$ is an un-plug with base $F$, then there is a foliation isomorphism $\alpha : F \times I \to P$ which extends the maps $\alpha_\pm$.

\textbf{Proof:} Let $\Phi$ be a flow parallel to $\mathcal{P}$. Let $L_+$ be the future longevity as in the proof of Lemma 5. In each smoothness category other than the PL category, define $\gamma(p,t) = \Phi(p,t/L_+(p))$. The map $\gamma$ has all of the desired properties.

Unfortunately, the PL category is not closed under the arithmetic operation of division, so a more complicated argument is necessary. Let $U$ be the region in $F_- \times \mathbb{R}$ bounded by
$F_\times \{0\}$ and the graph of $L_+$ restricted to $F_-$. The map $(p,t) \mapsto \Phi(p,t)$ is a foliation isomorphism from $U$ to $P$; it remains to find a foliation isomorphism $\beta$ from $F_- \times I$ to $U$.

To construct $\beta$, let $T$ be a triangulation with vertex set $V$ of $F_-$ such that $L_+$ is linear on each simplex of $T$ and such that $T$ is the barycentric subdivision of another triangulation. Since $T$ is a barycentric subdivision, there is a map $o : V \to \{1, \ldots, n+1\}$, where $n$ is the dimension of $F_-$, which is bijective when restricted to the vertices of any single $n$-simplex of $T$. If $K$ is such a simplex, consider a triangulation $\{K_i\}$ of $K \times I$, such that the simplex $K_i$ has vertices $o^{-1}(\{1, \ldots, i\}) \times \{0\}$ and $o^{-1}(\{i, \ldots, n+1\}) \times \{1\}$. Then the map on vertices of $K \times I$ given by $(p,0) \mapsto (p,0)$ and $(p,1) \mapsto (L(p),1)$ extends to a PL foliation isomorphism from $K \times I$ to $(K \times R) \cap U$. These maps fit together for different simplices $K$ of $T$, yielding the desired map $\beta$.

The following construction due to Wilson [Wi] turns a semi-plug into a plug. If $P_1$ and $P_\epsilon$ are two flow bordisms such that the exit region of $P_1$ is the same as the entry region of $P_\epsilon$, their concatenation is a flow bordism obtained by identifying trivially foliated neighborhoods of this shared region. The mirror image $\overline{P}$ of $P$ is given by reversing the orientation of the leaves of $P$, which has the effect of switching the entry and exit regions. The mirror-image construction is the concatenation of $P$ with $\overline{P}$; it is easy to see that the result of this concatenation has matched ends. (Figure 2 shows a schematic picture of a mirror-image construction.)

The primary purpose of plugs is the operation of insertion. The geometric idea of insertion is illustrated in Figure 3. Given a foliation $X$ of an $n$-manifold $X$ and an $n$-dimensional plug $P$ with base $F$, we wish to find a leaf-preserving embedding of the trivially foliated $F \times I$ in $X$ and replace it with $P$. In full generality, this procedure raises three technical issues: insertibility, attachability, and twistedness.

An insertion map for a plug $P$ into a foliation $X$ is an embedding $\sigma : F \to X$ of the base of $P$ which is transverse to $X$. Such an insertion map can be extended to an embedding $\sigma : F \times I \to X$ which takes the fiber foliation of $F \times I$ to $X$. (It is convenient to denote the maps $F \to X$ and $F \times I \to X$ by the same $\sigma$ and this should not lead to a misunderstanding.) An $n$-dimensional plug $P$ is insertible if $F$ admits an embedding in $\mathbb{R}^n$ which is transverse to vertical lines. Such an embedding is equivalent to a bridge immersion of $F$ in $\mathbb{R}^{n-1}$, i.e., an immersion which lifts to an embedding of $F \times I$ in $\mathbb{R}^n$. For example, if $F$ is 2-dimensional, orientable, and has non-empty boundary, then $F$ necessarily admits a bridge immersion. Figure 4 shows a bridge immersion of a punctured torus $pT$ and a bridge immersion of a surface of genus three with three punctures. The corresponding embedding
of $pT \times I$ is one of the main steps in Schweitzer’s counterexample to the Seifert conjecture [Sch].

A plug is **attachable** if every leaf in the parallel boundary is finite. In general, if $M$ is a manifold with boundary, let $N_M$ denote an open neighborhood of $\partial M$ in $M$. The next step in plug insertion is to remove $\sigma((F \times I) - N_{F \times I})$ from $X$ and glue the open lip $\sigma(N_{F \times I})$ to the support $P$ of an attachable plug $P$ by a leaf-preserving homeomorphism $\alpha : N_{F \times I} \to N_P$, where $N_P$ is a neighborhood of $\partial P$. Moreover, the identification $\alpha$ should satisfy $\alpha(p, 0) = \alpha_-(p)$ and $\alpha(p, 1) = \alpha_+(p)$. A map $\alpha$ with these properties is an attaching map for $P$. Let $G$ be the parallel boundary of $P$, so that $\partial P = G \cup F_- \cup F_+$. Recall that all of the leaves in $G$ have two endpoints; this must also be true in a neighborhood $N_G$ of $G$. It follows by Lemma 6 that there exists a leaf-preserving homeomorphism $N_G \to N_{F \times I}$. This equivalence, together with the matched ends condition, ensures the existence of an attaching map for any attachable plug or un-plug.

Let $\sigma$ be an insertion map of a plug $P$ into a foliation $\mathcal{X}$ on a manifold $X$, and let $\tilde{\mathcal{X}}$ be the foliation on the manifold $\tilde{X}$ resulting from inserting $P$ into $\mathcal{X}$. The plug $P$ is **untwisted** if the attaching map $\alpha$ extends to a homeomorphism $F \times I \to P$. The manifolds $X$ and $\tilde{X}$ need not be homeomorphic, but they are if $P$ is untwisted. If $P$ is a smooth or PL plug, then it is assumed that the extension of $\alpha$ is smooth or PL just as $\alpha$ is. However,
if $\mathcal{P}$ is analytic, then an analytic extension of $\alpha$ is usually not possible. In this case, $\mathcal{P}$ is untwisted if $\alpha$ admits a smooth extension. The resulting manifolds $X$ and $\hat{X}$ have analytic structures which are a priori only smoothly diffeomorphic; however, the Morrey-Grauert theorem ensures that they are in fact analytically equivalent. All plugs in this paper are assumed to be untwisted unless explicitly stated otherwise.

A plug or a semi-plug is aperiodic if it has no circular leaves.

One of the main reasons to insert an aperiodic plug into another foliation is to break a circular leaf. Let $\mathcal{P}$ be a plug with stopped set $S$ and base $F$, and let $\sigma$ be an insertion map into a foliation $\mathcal{X}$. If $\sigma(S)$ intersects a circular leaf $l$ of $\mathcal{X}$, then the remnant of $l$ in $\hat{X}$ is an infinite leaf. On the other hand, if $l$ is disjoint from $\sigma(S)$ but intersects $\sigma(F)$, then there is a leaf $\hat{l}$ in $\hat{X}$ corresponding to $l$ with the same topology as $l$, although the geometry of $\hat{l}$ can differ from that of $l$.

The simplest interesting target for an insertion is an irrational foliation of the 3-sphere $S^3$. Given an irrational number $r \in \mathbb{R}$, the irrational foliation of slope $r$ is parallel to the vector field $\vec{V}$ on $\{ (x, y) \in \mathbb{C}^2 | |x|^2 + |y|^2 = 1 \}$ given by $\vec{V}(x, y) = ix + iry$.

The foliation has two circles, and the closure of any other leaf is a torus lying between these two circles. Two copies of any aperiodic plug can be inserted in small regions meeting these circles and breaking them [Sch1]. Thus, to find a counterexample to the Seifert conjecture, it suffices to construct an aperiodic plug. If the plug is smooth or analytic, the corresponding foliation of $S^3$ is respectively smooth or analytic.

4. Global stoppage

In 1935, K. Borsuk [Bor] gave an example of a fixed point free homeomorphism of an acyclic compact subset of $\mathbb{R}^3$; a solid cylinder with two narrowing and spiraling tunnels drilled out. His example can be easily modified to obtain a semi-plug on $D^2 \times I$ ($D^n$ is an $n$-cell) with two circular leaves whose entry and exit stopped sets $S_-$ and $S_+$ are disks. A similar construction on a solid torus (Borsuk’s example can be obtained by cutting or unwrapping the torus) was given in 1952 by F. B. Fuller [Ful]. The consequence of Fuller’s result that $D^{n-1} \times S^1$, $n \geq 3$, admits a dynamical system whose only minimal set is an $(n-2)$-torus $S^1 \times \cdots \times S^1$ contained in a $D^{n-1} \times \{p\}$ section motivated Wilson’s introduction of plugs, which he used to arrest flows globally.

**Theorem 7** (Wilson). *Every smooth $n$-manifold of Euler characteristic zero or non-compact has a smooth dynamical system with a discrete set of minimal sets. Each of the minimal sets is an $(n-2)$-torus, and every trajectory originates (resp. limits) on one of these tori.*

Wilson’s theorem implies that an analogue to the Seifert conjecture for higher dimensional spheres of odd dimension is false. Although Wilson claimed Theorem 7 in the smooth category, the construction is actually analytic. Also, with slight modifications, Wilson’s construction is valid in a PL setting.

Wilson constructed a content-stopping, mirror-image plug and inserted many small copies of the plug into the foliation to break the leaves into small pieces. More specifically, he proved that if $\mathcal{F}$ is an oriented 1-foliation of an $n$-manifold $M$, $n \geq 3$, and $\mathcal{U}$ is open cover of $M$, then $\mathcal{X}$ can be modified to an oriented 1-foliation with every leaf contained in a star of an element of $\mathcal{U}$. The following result of [K-R1] is therefore a special case of Theorem 7.
Theorem 8. For every \( \epsilon > 0 \), there exists an oriented 1-foliation of \( \mathbb{R}^3 \) with all leaves of diameter less than \( \epsilon \).

To establish an aperiodic version of Theorem 7 and 8 in dimension 3, it suffices to use an aperiodic plug that stops content, for instance, a variation of Schweitzer’s plug [KR] or a plug obtained by breaking the circles of Wilson’s periodic plug with any aperiodic plug. Sections 6, 7, and 8 give constructions of analytic and PL aperiodic plugs such that each has a unique minimal set and it has codimension 1. These plugs yield stronger versions of Theorems 7 and 8.

Theorem 9. If \( M \) is a continuous, \( C^r \), \( C^\infty \), \( C^\omega \), or PL manifold of dimension \( \geq 3 \) admitting an oriented 1-foliation in the same smoothness category, and \( U \) is an open cover of \( M \), then there exists an aperiodic oriented 1-foliation of \( M \) in the same smoothness category, such that each leaf is contained in an element of \( U \), and whose minimal sets have codimension one.

Theorem 10. Let \( M \) be a PL manifold of dimension \( n \geq 3 \), \( 1 \leq k \leq n - 1 \), and let \( U \) be an open cover of \( M \). A PL 1-foliation of \( M \) can be modified in a PL fashion so that each leaf is contained in an element of \( U \), there are no circular leaves, and all minimal sets are \( k \)-dimensional.

The stopped set of each of the plugs constructed in this paper contains an arc. If the manifold \( M \) is closed, then the foliation specified in Wilson’s theorem has finitely many minimal sets. A plug can be “weaved” through the foliation so the arc in the stopped set meets a dense leaf in each of the minimal sets. Hence for closed manifolds, we get the following variations of the above theorems:

Theorem 11. If \( M \) is a continuous, \( C^r \), \( C^\infty \), \( C^\omega \), or PL closed manifold of dimension \( \geq 3 \) admitting an oriented 1-foliation in the same smoothness category, then there exists an aperiodic oriented 1-foliation of \( M \) in the same smoothness category, with exactly one minimal set, and the minimal set has codimension one.

Theorem 12. Let \( M \) be a closed PL manifold of dimension \( n \geq 3 \) and \( 1 \leq k \leq n - 1 \). A PL 1-foliation of \( M \) can be modified in a PL fashion so that there are no circular leaves, and there is exactly one minimal set which is \( k \)-dimensional.

5. Self-insertion

As a warm-up to the important technique of self-insertion, we first define and analyze partial insertion of plugs.

Let \( P \) be an \( n \)-dimensional plug with base \( F \) and support \( P \). Let \( \alpha : N_{F \times I} \to N_P \) be an attaching map for \( P \). Let \( X \) be an \( n \)-manifold with an oriented 1-foliation \( \mathcal{X} \). Let \( D \) be a closed domain in \( F \) whose boundary in \( F \) (in the point-set topology sense of boundary) is a properly embedded \((n-2)\)-manifold \( A \) transverse to \( \partial F \). It may happen that either \( A \) or \( D \) is not connected. The domain \( D \) is, in particular, a manifold with corners. A partial insertion map of \( P \) is an embedding \( \sigma : D \to X \) such that \( \sigma(A) \subset \partial X \), but \( \sigma(D - A) \) is disjoint from \( \partial X \). The map \( \sigma \) can be extended to an embedding of \( D \times I \) which takes vertical fibers to leaves. Define \( \hat{X} \) to be \( X - \sigma((D \times I) - N_{F \times I}) \) glued to \( P \) by the composition \( \sigma \circ \alpha^{-1} \) where it is defined. Define \( \hat{\mathcal{X}} \) by gluing \( \mathcal{X} \) to \( \mathcal{P} \) in the same way.
Figure 5 shows an example of a partial insertion of a plug with base $D^2$. The partial insertion creates transverse boundary (resembling a ledge and an overhang); unlike with a complete insertion, some leaves in the result have endpoints. By the matched ends condition, all leaves of the partial insertion either have both endpoints or neither endpoint at the transverse boundary. A leaf $\hat{X}$ from the depths of $X$ might enter $P$ and never return to $X$, but only by approaching some minimal set of $P$ and never by terminating.

Let $P, A$, and $D$ be as before. A self-insertion map for $P$ is an embedding $\sigma : D \to P$ such that the image of $\sigma$ is transverse to $P$ and disjoint from $F_{\pm}$, $\sigma(A) \subset \partial P$, $\sigma(D - A)$ is disjoint from $\partial P$, $\sigma(A)$ avoids leaves with endpoints in $\alpha_-(A)$, and $\sigma(A)$ does not intersect any leaf twice. Choose an attaching map $\alpha : N_F \times I \to NP$ such that the image of $\sigma$ is disjoint from the closure of $\alpha((D \times I) \cap N_F \times I)$. The map $\sigma$ can be extended to an embedding of $D \times I$ which is also disjoint from the closure of $\alpha((D \times I) \cap N_F \times I)$ and from $F_{\pm}$. Let $\hat{P}$ be the space $P - \sigma((D \times I) - N_F \times I)$ identified to itself by $\sigma \circ \alpha^{-1}$ where it is defined. The self-insertion of $P$ at $\sigma$ is the foliation $\hat{P}$ on $\hat{P}$ obtained by the same gluing.

Figure 7 shows an example of self-insertion of an un-plug with base $I$. Keeping the above notation, the surfaces $\alpha_-(D - \partial(A))$ and $\alpha_+(D - \partial(A))$, which are on the boundary of $P$, are in the interior of $\hat{P}$. These are the internal entry and exit regions, respectively, of $\hat{P}$. Similarly, $F_+ - \alpha_- (D - \partial(A))$ and $F_- - \alpha_+ (D - \partial(A))$ are the external entry and exit regions of $\hat{P}$. Many leaves of $\hat{P}$ cross the internal entry and exit regions once or many times. The points where a leaf $l$ does so are the transition points of $l$, or the entry and exit points. In addition, $l$ may begin with an external entry and/or end with an external exit. Progressing along a leaf $l$ of $\hat{P}$ in the positive direction, we encounter a history of transition points. In general $l$ consists of segments of leaves of $P$ separated by transition points. If $p \in l$ is an entry (exit), the leaf of $P$ preceding $p$ is interrupted (interrupting) leaf $p$, while the leaf following $p$ is the interrupting (interrupted) leaf.

In the self-insertion in Figure 7a, a leaf has the following history of transition points:

(external) entry, entry, entry, exit, exit, (external) exit.

In the self-insertion in Figure 7b, a leaf entering the bottom has the history

(external) entry, entry, entry, entry, entry, ...

In a given history, an entry and an exit are matched if the exit follows the entry and they satisfy the following inductive rule: They are matched if they are adjacent, and otherwise
they are matched if all entries and exits between them are matched to each other. For example, if a leaf in a hypothetical self-insertion has the history given in Figure 6, then the entries and exit are matched as indicated. The proof of the following lemma is an instructive exercise for the reader; it can also be found in [KuK]. Note that the lemma would not hold for a hypothetical definition of self-insertion of a semi-plug.

Figure 6. Matched transitions

Lemma 13. If two transition points are matched, they have the same interrupted leaf and they are the endpoints of the same interrupting leaf.

The definition of matched transition points and Lemma 13 suggest an interpretation of the geometry of a self-insertion in terms of a recursive algorithm. The following procedure follows a leaf of \( \hat{P} \) by following segments of leaves of \( P \). The input to the procedure is a base point \( p \in F \); the procedure follows the corresponding leaf \( \hat{l} \) of \( \hat{P} \).

\[\text{Procedure } \text{followleaf}(p)\]

1.: Let \( l \) be the leaf of \( P \) that begins at \( \alpha_-(p) \).
2.: Follow \( l \) until some \( \sigma(q) \) or \( \alpha_+(p) \) is reached.
3.: If \( \alpha_+(p) \) is reached, quit.
4.: If \( \sigma(q) \) is reached, then do \( \text{followleaf}(q) \) and go to step 2.

In computer programming terms, a recursive procedure such as \( \text{followleaf} \) is implemented by means of a pushdown stack: When \( \text{followleaf} \) is called, its argument is pushed onto the top of the stack. It is removed when that particular instantiation of \( \text{followleaf} \) terminates. Given a point \( p \in \hat{P} \), the stack of \( q \) is the sequence of points \( E(p), E(\sigma(E(p))), E(\sigma(E(\sigma(E(p))))) \), ..., where \( q = E(p) \) is a point in \( F \) such that at least one of \( \alpha_+(q) \) and \( \alpha_-(q) \) is an endpoint of the leaf of \( P \) containing \( p \). The function \( E \) is not defined on all of \( P \); the stack at \( q \) extends as long as \( E \) is defined and may therefore be either a finite or infinite sequence. It is easy to check that as \( q \) moves along its leaf, its stack changes in the same way as the stack used to execute \( \text{followleaf} \).
Inspection of followleaf demonstrates that in a finite leaf, all transition points are matched. In particular, the two endpoints are matched to each other.

This can be construed as saying that $\hat{P}$ has matched ends. Unfortunately, $\hat{P}$ is not even a flow bordism, because the top and bottom of $\hat{P}$ have stair-steps and are not transverse to $\hat{P}$. However, without disturbing the transition point scheme, it is possible to approximate the top and bottom of $\hat{P}$ by surfaces $\hat{F}_+$ and $\hat{F}_-$ that are transverse to $\hat{P}$. Let $\hat{P}$ be the manifold that results from cutting along these surfaces, and let $\hat{P}$, the self-inserted plug, be the restriction of $\hat{P}$ to $\hat{P}$.

As Figure 7b demonstrates, a leaf can have unmatched transition points. Indeed, by Lemma 13, if a transition point (whether internal or external) is the endpoint of an infinite leaf of $\mathcal{P}$, then it cannot be matched in $\hat{P}$. Its leaf in $\hat{P}$ (and hence in $\hat{P}$) is therefore either infinite or circular, and by Lemma 5, $\hat{P}$ is a plug. In particular, if $p \in F$ is in the stopped set of $\mathcal{P}$ and $\alpha_-(p)$, then it lies in an infinite leaf of $\hat{P}$. Thus, a self-insertion cannot close infinite leaves of this type. On the other hand, Figure 7 also demonstrates that a self-insertion can create new circular leaves; these necessarily have unmatched transition points.

Finally, since a self-insertion changes the base of a plug, it may happen that the self-insertion of an insertible plug is not insertible. It may also happen that the self-insertion of an untwisted plug is twisted.

The following example shows that a self-insertion may cause the boundary of a plug to disappear completely. Let $Q$ be a plug with support $D^2 \times I$ such that the transverse boundary consists of the top and bottom of the cylinder, and the side of the cylinder is the parallel boundary. A self-insertion of $Q$ using an immersion of the entire disk $D^2$ results in a plug with a boundaryless support (e.g., $S^2 \times S^1$).

6. AN ANALYTIC PLUG

This section contains a construction, similar to that of KuK, of a two-component self-insertion of a plug $W$ which is a concatenation of two semi-plugs $W_s$ and $\overline{W}_s$, each a mirror image of the other.
Parametrize $S^1$ by $\theta$ with $0 \leq \theta < 10$. Let be $F = [-1, 1] \times S^1$ ($F$ is an annulus), and the support of $W_s$ be the cylinder $W_s = F \times [-1, 1]$. The coordinates of $W_s$ are $r$, $\theta$, and $z$, similar to the cylindrical coordinates in $\mathbb{R}^3$. The support of $\overline{W}_s$ is also $F \times [-1, 1]$, but it will be denoted $\overline{W}_s$ to avoid confusion; let $W = W_s \cup \overline{W}_s$.

The semi-plugs $W_s$ and $\overline{W}_s$ are generated by the vector fields
\[
\tilde{W}_s(r, \theta, z) = \frac{\partial}{\partial \theta} + (r^2 + z^6) \frac{\partial}{\partial z},
\]
and
\[
\overline{W}_s(r, \theta, z) = -\frac{\partial}{\partial \theta} + (r^2 + z^6) \frac{\partial}{\partial z}.
\]

The concatenation identifies the top $F \times \{1\}$ of $W_s$ with the bottom $F \times \{-1\}$ of $\overline{W}_s$; the map $\alpha_- : F \to W$ sends $F$ onto the bottom of $W$ by $\alpha_-(p) = (p, -1) \in W_s$. The vector fields parallel to $W_s$ and $\overline{W}_s$ are oriented in the positive $z$ direction. Since $W_s$ and $\overline{W}_s$ are trivially foliated in the respective neighborhoods of $F \times \{1\}$ and $F \times \{-1\}$ and they are analytic, the plug $W$ is analytic.

Let $T = \{(r, \theta, z) \in W_s | r = 0, z = 0\}$ and $\overline{T} = \{(r, \theta, z) \in \overline{W}_s | r = 0, z = 0\}$ be the circles of $W_s$ and $\overline{W}_s$.

We define a self-insertion whose domain $D \subset F$ consists of two components $D_s$ and $\overline{D}_s$. The self-insertion map $\sigma : D \to W$ divides into a map $\sigma : D_s \to W_s$ given by the formula
\[
\sigma(r, \theta) = (r - \frac{1}{4} r^2 - 2(\theta - 2)^2, 6, 2 - \theta),
\]
where $D_s$ is the set of all points $(r, \theta)$ such that $(r - \frac{1}{4} r^2 - 2(\theta - 2)^2, 6, 2 - \theta)$ lies in $W_s$, and a map $\sigma : \overline{D}_s \to \overline{W}_s$ similarly defined by
\[
\sigma(r, \theta) = (r - \frac{1}{4} r^2 - 2(\theta - 8)^2, 4, \theta - 8),
\]
where $\overline{D}_s$ is the set of all points $(r, \theta)$ such that $(r - \frac{1}{4} r^2 - 2(\theta - 8)^2, 4, \theta - 8)$ lies in $\overline{W}_s$.

To check that the equations for $\sigma$ form a valid self-insertion, we first note that $D_s$ and $\overline{D}_s$ are disjoint, because if $(r, \theta) \in D_s$, then $1 < \theta < 3$, while if $(r, \theta) \in \overline{D}_s$, then $7 < \theta < 9$. Each leaf of $\partial W$ intersects $\text{Im}(\sigma)$ at most once, because the slope $\frac{dz}{d\theta}$ of any leaf in $\partial W$ is bigger than 1 and is positive in $W_s$. The variable $\theta$ would have to increase by 8 for a leaf to connected $\sigma(D_s)$ with $\sigma(D_s)$ and by 10 for it to connect either component with itself, but this is impossible. Finally, the condition that $\sigma$ avoid leaves in $\partial W$ that begin in its domain is automatically satisfied, because such leaves have $r = 1$ while $\text{Im}(\sigma)$ only intersects $\partial W$ at $r = -1$. The self-inserted plug $\tilde{W}$ given by $\sigma$ is pictured in Figure [3].

Observe that $\sigma(0, 2) \in T$ and that $\sigma(0, 8) \in \overline{T}$ and that $(0, 2)$ and $(0, 8)$ are both in the stopped set of $W$. In other words, the self-insertion breaks both circles of $W$. Moreover, if $(r, \theta, z) = \sigma(r', \theta')$, then $r \leq r'$, with equality occurring only for two points of $F$; one point is sent to the circle $T$ by $\sigma$ and the other is sent to $\overline{T}$. This is the important radius inequality for self-insertions.

A point $p \in \tilde{W}$ may be considered as a point in $W$. To make a distinction between the leaves of $\tilde{W}$ and $W$ containing $p$, the leaves are denoted by $\tilde{l}$ and $l$.

**Theorem 14.** The self-inserted plug $\tilde{W}$ has no circular leaves.

**Proof:** Let $p_1, p_2, \ldots$ be the stack of some point in $\tilde{W}$. By the radius inequality, the points have strictly decreasing radii $r_1 > r_2 > \ldots$. If some $p_n$ is either $(0, 2)$ or $(0, 8)$, then
the stack ends at $p_n$, because $\sigma(p_n)$ lies on the circle $T$ or $\bar{T}$ and $E(\sigma(p_n))$ is undefined. At all other points of $D_s \cup \bar{D}_s$, the radius inequality is a strict inequality.

Suppose that $\tilde{l}$ is a circular leaf of $\tilde{W}$ and let $p$ be a point that varies along $\tilde{l}$. As $p$ goes all the way around $\tilde{l}$, the stack of $p$ either grows, shrinks, or stays at the same level. If it grows or shrinks, then it must be infinite, and moreover it must be eventually periodic, which is impossible by the radius inequality. If it stays at the same level, then for some $q \in l$ the stack reaches a minimum height; the segment of $\tilde{l}$ containing $q$ must therefore correspond to some leaf $l$ of $W$ which is interrupted but does not interrupt other leaves.

The leaf $l$ must be a circle, and must therefore be one of the two circles $\bar{T}$ and $\tilde{T}$. However, the transition points on these circles lead to the entry points $(0, 2)$ and $(0, 8)$ which lie in the stopped set of $W$ and cannot be matched. In conclusion, all avenues for a circular leaf in $\tilde{W}$ lead to contradiction.

**Lemma 15.** If $\tilde{l}$ is a leaf of $\tilde{W}$ whose radii avoid the interval $(-\epsilon, +\epsilon)$ for some $\epsilon > 0$, then $\tilde{l}$ is a finite leaf.

**Proof:** Let $r_1 > r_2 > \ldots$ be the radii of the stack of some point on such a leaf $\tilde{l}$. By hypothesis the points of the stack are at least $\epsilon$ away from the critical points $(0, 2)$ and $(0, 8)$. Therefore there exists a $C$ such that $r_n > C + r_{n+1}$, which in turn implies that the height of the stack is bounded by $2/C$. Moreover, all leaves of $W$ that intersect the image of $\sigma$ infinitely many times have $r = 0$, so there exists an $N$ such that all component leaves of $\tilde{l}$ intersect $\sigma$ at most $N$ times. Modelling the behavior of $\tilde{l}$ by FOLLOLEAF, each instantiation of FOLLOLEAF can only call FOLLOLEAF at most $N$ times, and the recursion can only extend to a depth of $2/C$. Therefore there are at most $N^{2/C}$ internal calls, which means that $\tilde{l}$ has at most $N^{2/C}$ internal entry points, or $2N^{2/C}$ transition points in total. This implies that $\tilde{l}$ is finite, since none of the component leaves is infinite.

**Lemma 16.** If $\tilde{l}$ is a leaf of $\tilde{W}$ with entry point at $r = 0$, then all subsequent transition points are matched and the positive limit set of $\tilde{l}$ contains the (formerly circular) leaf $\bar{T}$.
Then the numbers that with \( C \) the relation and for \( n \) minimal set it suffices to follow the leaf \( \tilde{\theta} \). Then there is only one minimal set and it contains \( \tilde{\theta} \) when \( r = 0 \) contains the leaf \( \tilde{T} \).

A similar argument shows that the negative limit set of a leaf of \( \tilde{W} \) with exit point at \( r = 0 \) contains the leaf \( \tilde{T} \).

**Theorem 17.** The plug \( \tilde{W} \) has a unique non-trivial minimal set and it is 2-dimensional.

**Proof:** By Lemmas 15 and 16, the closure of any infinite leaf contains the leaf \( \tilde{T} \). Therefore there is only one minimal set and it contains \( \tilde{T} \). Since \( \sigma(0, 2) \in T \), to understand the minimal set it suffices to follow the leaf \( \tilde{l} \) of \( \tilde{W} \) starting at \( (0, 2, -1) = \alpha_-(0, 2) \).

Let \( A(x) \) be the antiderivative of \( 1/(1 + x^6) \) with \( A(0) = 0 \); in particular,

\[
\int_{\infty}^{-\infty} \frac{1}{1 + x^6} dx = 2A(\infty)
\]

by abuse of notation. Solving the differential equation defined by \( \tilde{W} \), the leaves of \( W_s \) are given by the equation

\[
\theta = -\frac{1}{5z^5} + C
\]

when \( r = 0 \), and by

\[
\theta = r^{-5/3} A(r^{-1/3} z) + C
\]

when \( r \neq 0 \). If \( l \) is the leaf of \( W_s \) starting at \( (0, 2, -1) \), then \( l \) is given by the former equation with \( C = 9/5 \). For each integer \( n \geq 0 \), \( l \) intersects the surface \( \theta = 6 \) with

\[
z = z_n = -(21 + 50n)^{-1/5},
\]

and for \( n \) large enough, these points also lie in \( \sigma(D_s) \). For such \( n \), choose \((r_n, \theta_n) \in F \) so that

\[
\sigma(r_n, \theta_n) = (0, 6, z_n).
\]

Then the numbers \( \theta_n = 2 - z_n \to 2 \) as \( n \to \infty \), while \( r_n \approx 2z_n^2 \) as \( n \to \infty \) by inspection of the relation

\[
r_n - \frac{1}{4} r_n^2 - 2z_n^2 = 0.
\]

Let \( l_n \) be the leaf of \( W_s \) beginning at \((r_n, \theta_n, -1) \). By Lemma 16 in the leaf \( \tilde{l} \), each leaf \( l_n \) concatenated with its mirror image interrupts the leaf \( l \). Let \((r_n, \theta'_n, 1) \) be the other endpoint of \( l_n \) in \( W_s \); \( \theta'_n \) is given by

\[
\theta'_n = \theta_n + 2r_n^{-5/3} A(r_n^{-1/3}).
\]

Combining several identities and approximations,

\[
\theta'_n \approx 2 + 21/3(21 + 50n)^{2/3} A(\infty)
\]

for \( n \) large. Thus, \( \theta'_n \to \infty \), but \( \theta'_{n+1} - \theta'_n \to 0 \) as \( n \to \infty \). It follows that \( \{\theta'_n \mod 10\} \) is a dense subset of the circle \( \mathbb{R}/10\mathbb{Z} \), and that the circle \( K \) of all points \((0, \theta, 1) \in W_s \) is in the closure of \( \tilde{l} \). Following the leaves of \( W_s \) and \( \tilde{W_s} \), the circle \( K \) sweeps out the entire annulus \( N \) between \( T \) and \( \tilde{T} \). The surface \( N \), except where it is cut by \( \text{Im}(\sigma) \), is therefore in the closure of \( \tilde{l} \) in \( \tilde{W} \), which demonstrates that the minimal set of \( \tilde{W} \) is 2-dimensional. \( \square \)
Finally, the following proposition was pointed out to the authors by É. Ghys, who credits S. Matsumoto [Ghy]. It slightly simplifies Section 4, since it means that it is not necessary to insert $\tilde{W}$ into another content-stopping plug.

**Proposition 18.** The plug $\tilde{W}$ stops content.

**Proof:** The set of points $(r, 6, z) = \sigma(r', \theta) \subset W_s$ with $r' \leq 0$ is the region in the $\theta = 6$ plane with $r \leq -2z^2$. In this region, the slope of a leaf of $W$ satisfies

$$\frac{dz}{d\theta} = r^2 + z^6 \leq \sqrt{2}r^2.$$

If $l$ is a leaf of $W$ at radius $r$ with $0 > r > -\frac{1}{20}$, then in the region $r \leq -2z^2$, the successive intersections of $l$ with $\Im(\sigma)$ are less than $10\sqrt{2}r^2 < 10r$ apart. But the region $r \leq -2z^2$ has width $\sqrt{2r} > \frac{1}{10}$ in the $z$ direction, so it follows that the leaf in $\tilde{l}$ in $\tilde{W}$ containing segments of $l$ eventually meets an internal entry point with interrupting radius $r'$ such that $0 \geq r' > r$. All entry points that $l$ meets at radius $r' > 0$ are matched by Lemma 16, but an entry at radius $0$ cannot be matched, and an entry at radius $0 > r' > r$ begins a new leaf $l'$ of $W$ with the same properties as $l$. Therefore the stack of $\tilde{l}$ grows indefinitely and $\tilde{l}$ is an infinite leaf. In conclusion, all points of $\tilde{F}$ with radius $0 > r > -\frac{1}{20}$ are in the stopped set of $\tilde{W}$.

Proposition 18 applies equally well to any conceivable variation of the formulas defining the plug $\tilde{W}$, with the conclusion that no such analytic (or even $C^1$) plug can preserve volume given by a volume form. On the other hand, Proposition 18 is valid for some but not all PL self-insertions. Nevertheless, it seems impossible to satisfy the radius inequality and preserve volume also. Reference [KuG] gives a version of the Schweitzer plug which preserves volume. Like Schweitzer’s example, it is $C^1$.

### 7. Higher dimensions

We first consider oriented 1-foliations in $n$ dimensions with $n > 3$.

Let $T^{n-2}$ be an $(n-2)$-dimensional torus, parametrized by coordinates $\theta_1, \theta_2, \ldots, \theta_{n-2}$ with period 10. Let

$$\tilde{\theta} = \sum_i k_i \frac{\partial}{\partial \theta_i}$$

be a vector field, where the coefficients $\{k_i\}$ are between 0 and 1 and are linearly independent over $\mathbb{Q}$, the rational numbers. All leaves of the foliation parallel to $\tilde{\theta}$ are dense. Parametrize the manifold $W_{s,n} = [-1, 1] \times T^{n-2} \times [-1, 1]$ by the coordinates $r, \theta_1, \ldots, \theta_{n-2}, z$, and define the vector field $\tilde{W}_{s,n}$ on $W_{s,n}$ by the formula

$$\tilde{W}_{s,n} = \tilde{\theta} + (r^2 + z^6) \frac{\partial}{\partial z}.$$

The foliation $\mathcal{W}_{s,n}$ parallel to $\tilde{W}_{s,n}$ is a semi-plug, and the mirror-image construction yields a plug $\mathcal{W}_n$ with base $I \times T^{n-2}$. The plug $\mathcal{W}_n$ is aperiodic and is similar to a construction of Wilson which settled the higher-dimensional Seifert conjecture for all manifolds with Euler characteristic 0. However, its minimal sets are at most $(n-2)$-dimensional; to achieve $(n-1)$-dimensional minimal sets we will perform a self-insertion.
Let $a = (2, 2, \ldots, 2)$ in the parametrization by the $\theta_i$'s, and let $b = (8, 8, \ldots, 8)$. Consider a self-insertion with two components, one component going into $W_{s,n}$ and the other going into $\overline{W}_{s,n}$. The formula for the part of $\sigma$ that maps into $W_{s,n}$ is

$$\sigma(r, \theta_1, \ldots, \theta_{n-2}) = (r - \frac{1}{4}r^2 - 2 \sum_{1}^{n-2} (\theta_i - 2)^2, 6 \ldots, 6, 2 - \theta_1 + \sum_{2}^{n-2} (\theta_i - 2)),$$

while the formula that maps into $\overline{W}_{s,n}$ is

$$\sigma(r, \theta_1, \ldots, \theta_{n-2}) = (r - \frac{1}{4}r^2 - 2 \sum_{1}^{n-2} (\theta_i - 8)^2, 4 \ldots, 4, \sum_{2}^{n-2} (\theta_i - 8)).$$

Let $\tilde{W}_n$ be the self-inserted plug.

The self-insertion clearly satisfies the radius inequality with respect to the variable $r$. The base after self-insertion is $I \times T^{n-2}$ with the boundary components connected by two boundary handles, which admits a bridge immersion in $\mathbb{R}^{n-1}$ as illustrated in Figure 10 in the case $n = 4$. Following the argument for three dimensions, if $\tilde{l}$ is the leaf of $W_n$ beginning at $\alpha_-(0, 2, 2, \ldots, 2)$, then $\tilde{l}$ is in the minimal set of $\tilde{W}_n$. In $\tilde{I}$, the leaf $\tilde{l}$ is interrupted by a sequence of leaves $l_n$ with radius $r_n \to 0$. In the $\theta_i$ directions, these leaves progress in the $\theta_i$ directions by an amount that goes to infinity as $n \to \infty$, but at a rate that goes to 0. It follows that the entire surface with $r = 0$ and $z = 1$ in $W_{s,n}$ is in the closure of the leaves $l_n$, which in turn forces the minimal set of $\tilde{W}_n$ to have codimension 1.

![Figure 10. Bridge immersion of $I \times T^2$](image)

In the general case, the goal is to open the leaves of a $k$-dimensional foliation of an $n$-manifold so that the closure of any such leaf is at least $(n-1)$-dimensional. If $k = n-1$, there is almost nothing to prove. To eliminate the possibility of a minimal set of codimension 0 (the whole manifold), choose a circle or properly embedded line transverse to the foliation and put in a Reeb structure along this curve.

The interesting case is $1 < k < n-1$. Following Schweitzer [Sch], consider the manifold $W_{n-k+1} \times S^k$ foliated by leaves $l \times S^k$, where $l$ is a leaf of $W_{n-k+1}$ and $S^k$ is the $k$-sphere.
The techniques of insertion and global stoppage readily generalize to such a foliation, and it is easy to check that the only minimal set has codimension 1.

8. The PL case

The PL construction mostly uses the same geometric ideas as the analytic (and therefore smooth) case, but they are realized somewhat differently because PL foliations do not have well-behaved parallel vector fields. The construction has one new detail, a vertical annulus of circles, which gives the minimal set high dimension in a different way.

Let $H$ be a compact manifold, usually with boundary or corners. Let $f : H \times [a, b] \to H \times [a, b]$ be a PL homeomorphism and let $0 < l < 1$ be a real number. Let $L$ be the foliation of $H \times [a, b] \times [0, 1]$ such that, for fixed $p \in H$ and $z \in \mathbb{R}$, the set $\{(p, z + lx, x) | x \in [0, 1], z + lx \in [a, b]\}$ is a leaf. Orient all such leaves from $H \times [a, b] \times \{0\}$ to $H \times [a, b] \times \{1\}$.

The slanted suspension of $f$ with slant $l$ is defined as $Z$, the manifold $H \times [a, b] \times [0, 1]$ with $(f(p, z), 0)$ identified with $(p, z, 1)$, together with the foliation $\mathcal{Z}$ induced from $L$. The slanted suspension $Z$ is a PL foliation by construction, and moreover is a flow bordism with entry and exit regions $H \times S^1$.

Figure 11. PL vertical collar

Let $H = [-1, 1]$ and $[a, b] = [-2, 2]$. Let $f : [-1, 1] \times [-2, 2] \to [-1, 1] \times [-2, 2]$ be a PL homeomorphism which is the identity at $\partial([-1, 1] \times [-2, 2])$, such that $f(0, 0) = (0, -1)$, $f(0, 1) = (0, 0)$, and such that $f$ is linear on the two triangles and two trapezoids illustrated in Figure 11. It is easy to check that the slanted suspension of $f$ with slant $l = 1$ is a semi-plug with an annulus of circular leaves corresponding to the line segment from $(0, -1)$ to $(0, 0)$. It is convenient to take a 20-fold covering of the slanted suspension, to obtain a semi-plug $W_{PL,s}$ with support $W_{PL,s} = [-1, 1] \times [0, 20] \times [-2, 2]$ (where 0=20) is parametrized by $r, \theta$, and $z$ with the suspension direction $\theta$. The semi-plug $W_{PL,s}$ is similar to the analytic semi-plug $W_s$, except that it has an annulus of circular leaves instead of one circular leaf. Let $\overline{W}_{PL,s}$ (with support $\overline{W}_{PL,s}$) be a the mirror image of $W_{PL,s}$ obtained by changing $z$ to $-z$. Let $W_{PL}$ (with support $W_{PL}$) be the plug obtained from $W_{PL,s}$ by the mirror-image construction, the concatenation of $W_{PL,s}$ and $\overline{W}_{PL,s}$.

Although $W_{PL}$ has two annuli of circles, $T$ and $\overline{T}$, it is still possible to break all of them with a self-insertion, since the stopped set is also a circle. The radius inequality preserving self-insertion $\sigma : D \to W$ is defined on two parts of $D$: $D_s$ containing the segment with endpoints $(0, 9)$, $(0, 10)$ which is mapped linearly onto the whole segment of $T$ at $\theta = 20,$
and $D_s$ containing the segment with endpoints $(0, 14), (0, 15)$ which is mapped linearly onto the whole segment of $T$ at $\theta = 5$. (Figure 12 shows the $D_s$ part of $\sigma$.)

Its geometry is more or less the same as that in the analytic case, since the leaves of $W_{PL}$ are at constant $r$ and the self-insertion satisfies the radius inequality. Lemma 13, slightly modified Lemma 16, and Theorem 17 hold for $\tilde{W}_{PL}$. The internal entries of a point at $r = 0$ have radii that converge to $r = 0$, and their leaves converge to the annuli of circles of $W_{PL}$. Therefore the remnants of these annuli after self-insertion are contained in the minimal set, and the minimal set is still 2-dimensional.

The construction for higher-dimensional foliations also apply to the PL case without modification.

9. Symbolic Dynamics

Although most aperiodic self-inserted Wilson-type plugs have 2-dimensional minimal sets, a carefully chosen self-insertion may result in a 1-dimensional minimal set. Such a self-insertion has interesting symbolic dynamics which can be described explicitly. The self-insertion is easiest to define in the continuous category, but with yet more care it is also possible in the PL category.

**Theorem 19.** There exists an aperiodic PL plug with 1-dimensional minimal sets.

**Proof:** First, we consider a self-insertion that only breaks one circle. The other circle is the unique minimal set, but the leaf containing the broken circle is in a 1-dimensional invariant set which resembles the minimal set in the final construction.

Let $R = [-1, \frac{3}{2}] \times [-3, \frac{3}{2}]$ be a rectangle parametrized by $r$ and $z$, and let $f : R \to R$ be the PL homeomorphism illustrated in Figure 13. The map $f$ is linear in each of the regions delineated by the four rays $z = -r \leq 0, z = -r \geq 0, z = \frac{r}{2} \geq 0, \text{ and } z = 2r \leq 0$. In the upper region, $f(r, z) = (r, 2z - \frac{3}{2})$; in the lower region, $f(r, z) = (r, \frac{3}{2} - \frac{3}{2})$, and in the two side regions, $f$ matches the unique linear transformation which makes it continuous along the four rays. Note that $f(0, 0) = (0, -\frac{3}{2})$, and for any other point $(r, z) \in R$, the difference between $z$ and the $z$-coordinate of $f(r, z)$ is greater than $-\frac{3}{2}$. Let $V_s$ be the slanted suspension of $f$ with slant $\frac{3}{2}$, and let $V_s$ be its support. More precisely, $V_s$ is obtained from $[-1, \frac{3}{2}] \times [0, 1] \times [-3, \frac{3}{2}]$ by identifying $(r, 0, \bar{z})$ with $(r, 1, z)$, where $(\bar{r}, \bar{z}) = f(r, z)$. We use the coordinates $(r, \theta, z)$, and the suspension is in the direction of $\theta$. Each leaf of $V_s$ is the
union of segments \( \{(r, \theta, z + \frac{3}{2} \theta) | r \in [-1, \frac{5}{3}], z \in \mathbb{R}, \theta \in [0, 1], z + \frac{3}{2} \theta \in [-2, 2]\} \). The semi-plug \( \mathcal{V}_s \) is another PL analogue of the analytic semi-plug \( \mathcal{W}_s \). Like \( \mathcal{W}_j \), it has one circular leaf \( T \). Note that \( T \) passes through the point \((0, 0, -\frac{3}{2}) = (0, 1, 0)\).

Let \( \overline{\mathcal{V}}_s \) be the mirror image of \( \mathcal{V}_j \), and let \( \mathcal{T} \) be the circular leaf of \( \overline{\mathcal{V}}_s \). Let \( \mathcal{V} \) be the concatenation of the two mirror-image semi-plugs \( \mathcal{V}_s \) and \( \overline{\mathcal{V}}_s \). The support \( V \) of \( \mathcal{V} \) is obtained by identifying the top of \( \mathcal{V}_s \) with the bottom of \( \overline{\mathcal{V}}_s \). Let \( F \) be the base of \( \mathcal{V} \).

Let \( B = \{(r, \theta)| r \geq 0, \frac{7}{12} \leq \theta - \frac{1}{3} \leq \frac{5}{4}\} \). We define a self-insertion map \( \sigma : D \to \mathcal{V}_s \subset V \) on an appropriate \( D \subset F \) containing \( B \). The image of \( \sigma \) lies in the \( \theta = 1 \) section of \( \mathcal{V}_s \), and \( \sigma \) is described explicitly on \( B \) by:

\[
g(r, \theta) = \left(\frac{1}{2}r, 9\theta - \frac{7}{4}r - 3\right) = (x, y),
\]

\[
h(x, y) = \begin{cases} 
(2x - 2y, 1, \frac{3}{2}) & \text{if } y \geq \frac{1}{2}x \\
(x, 1, y) & \text{if } -x < y < \frac{1}{2}x, \\
(2x + y, 1, -x) & \text{if } y \leq -x
\end{cases}
\]

and \( \sigma(r, \theta) = h \circ g \). On \( D - B \), \( \sigma \) tapers in the \( z \) direction to have \( \text{Im}(\sigma) \cap \partial \mathcal{V}_s \) small enough so that leaves of \( \partial \mathcal{V}_s \) intersect \( \text{Im}(\sigma) \) at most once, and none of the leaves that do intersect have an endpoint in \( D \). Figure 14 shows the shape of \( \text{Im}(\sigma) \) in the section of \( \mathcal{V}_s \) with \( \theta = 1 \). The formulas demonstrate that \( \sigma \) satisfies the radius inequality on \( B \). Outside \( B \) the radius inequality is easy to achieve. Let \( \tilde{\mathcal{V}} \) be \( \mathcal{V} \) self-inserted by \( \sigma \). Let \( \tilde{T} \) be the leaf of \( \tilde{\mathcal{V}} \) containing segments of \( T \). By Lemma 14, \( \sigma|_B \) determines the geometry of the leaf \( \tilde{T} \).

Let \( \mathcal{E}_1, \mathcal{E}_2, \ldots \) be a sequence of closed disks in \( B \) such that

\[
\mathcal{E}_n = \{(r, \theta) \in B| 4 \cdot 2^{-n} - 2 \cdot 4^{-n} \leq r \leq 4 \cdot 2^{-n} + 2 \cdot 4^{-n}\}.
\]

Let \( \mathcal{L}_n \) be the tube of leaves of \( \mathcal{V} \) with endpoints in \( \mathcal{E}_n \). The intersection of each \( \mathcal{L}_n \) with the \( \theta = 1 \) section is also shown in Figure 15 along with the regions \( \sigma(\mathcal{E}_n) \). The figure suggests, and a computation shows, that \( \sigma^{-1}(\mathcal{L}_n) \subset \bigcup_k \mathcal{E}_k \) for every \( n \). Let \( \mathcal{L}_{0,n} = \mathcal{E}_n \) and let \( \mathcal{L}_{0,n} = \mathcal{L}_n \). The set \( \bigcup \sigma^{-1}(\mathcal{L}_n) \) is also a union of disjoint disks, which we denote \( \mathcal{E}_{1,n}, \mathcal{E}_{1,2}, \ldots \). Let \( \mathcal{L}_{1,n} \) be the tube of leaves of \( \mathcal{V} \) which begins at \( \mathcal{E}_{1,n} \). In general, for each \( k \), let \( \mathcal{E}_{k+1,1}, \mathcal{E}_{k+1,2}, \ldots \) be a sequence of disjoint closed disks whose union is \( \sigma^{-1}(\bigcup_n \mathcal{L}_{k,n}) \), and let \( \mathcal{L}_{k+1,n} \) be the tube of leaves beginning at \( \mathcal{E}_{k+1,n} \). Another computation shows that the diameter of \( \mathcal{E}_{k,n} \) goes to zero as \( k \) goes to infinity, irrespective of the behavior of \( n \) as a function of \( k \). Moreover, Figure 15 shows that there are infinitely many (and in particular more than one) disks \( \mathcal{E}_{k+1,n} \) in a given \( \mathcal{E}_{k,n} \). Therefore the closure of the intersection

\[
\bigcap_k \bigcup_n \mathcal{E}_{k,n}
\]
Figure 14. $\sigma$ restricted to $A$ is a Cantor set $C$.

Figure 15. A 1-dimensional minimal set

In the leaf $\tilde{T}$, the leaf $T$ is interrupted by the leaf $l$ beginning at $(0, \frac{1}{3}, -3)$. Since the leaf $l$ is infinite, the leaf $\tilde{T}$ never returns to $T$. The leaf $l$, in turn, is interrupted by an infinite sequence of internal entry points, and the $n$th such point lies in $\sigma(D_n)$. It follows that the closure of all internal entry points, which is a cross-section of the closure of $\tilde{T}$, is the Cantor set $C$. In particular, the closure of $\tilde{T}$ is 1-dimensional.

The main extra step in the full construction is to take the double-cover $V'$ of $V$. Similarly, let $V'_s$, $\overline{V}'_s$, and $F'$ be double-covers of $V_s$, $\overline{V}_s$, and $F$. Let $\pi: V' \to V$ be the covering map, let $\beta: V' \to V'$ be the non-trivial deck translation (also use $\pi$ and $\beta$ for the covering $F'$ of $F$), and let $\gamma: V' \to V'$ be the mirror-image involution which switches $V'_s$ and $\overline{V}'_s$.

Let $\sigma_1: D_1 \to V'$ be a lift of $\sigma: D \to V'$, where the disk $D_1$ is a lift of the disk $D$, let $D_2 = \beta(D_1)$, and let $\sigma_2: D_2 \to V'$ be given by

$$\sigma_2 = \beta \circ \gamma \circ \sigma_1 \circ \beta.$$ 

Let $D' = D_1 \cup D_2$, and define the self-insertion map $\sigma': D' \to V'$ to be both $\sigma_1$ and $\sigma_2$. The entire system of disks $\{E_{k,n}\}$ and tubes $\{L_{k,n}\}$ has a lift $E_{1,k,n}$ and $L_{1,k,n}$ and
another lift $E_{2,k,n}$ and $L_{2,k,n}$. Moreover, each $L_{i,k,n}$ intersects $\text{Im}(\sigma')$ in some collection of disks $\{E_{j,k+1,m}\}$, where $j$ and $m$ may both vary. So the closure of $\tilde{T}$ still has a Cantor set cross-section and is still 1-dimensional, but since the self-insertion at $\sigma'$ breaks both circles $T$ and $\tilde{T}$, the closure of $\tilde{T}$ is in this case the minimal set.

The construction of Theorem 19 gives rise to interesting symbolic dynamics. Consider the sequence of pairs of integers $(j, n)$ such that the $i$th internal entry of the leaf $\tilde{T}$ after the interruption of $T$ is in the disk $E_{j,0,n}$. The sequence runs:

$$(1, 2), (1, 1), (1, 1), (2, 1), (2, 1), (1, 4), (1, 2), (1, 1), (2, 1), (2, 1), (1, 3),$$

$$(1, 2), (1, 1), (1, 1), (2, 1), (2, 1), (1, 2), (1, 1), (1, 1), (2, 1), (2, 1), (1, 1), \ldots$$

The sequence is generated by calling FOLLOWDISKS$(1, \infty)$, where the recursive procedure FOLLOWDISKS is defined as:

- Procedure FALLOWDISKS$(j,n)$
  1. Print $(j, n)$ if $n$ is finite.
  2. Do the following three steps with $i = 1, 2$:
     3. For each $k$ from 1 to $n - 2$, do FALLOWDISKS$(i,k)$ if $j + k$ is odd.
     4. Do FALLOWDISKS$(i,n - 1)$ twice.
     5. For each $k$ from $n - 2$ to 1, do FALLOWDISKS$(i,k)$ if $j + k$ is even.
  6. Quit.

Finally, note that there are many different constructions that resemble the construction of Theorem 19, and in general they have similar but different symbolic dynamics. For example, where FALLOWDISKS says “Do FALLOWDISKS$(i,n - 1)$ twice,” its analogue for some other self-inserted plug may call itself once or three times or any other integer.

The above PL construction yielding a 1-dimensional minimal set easily generalizes to $n$-dimensional manifolds, $n \geq 3$: for any $k = 1, 2, \ldots, n - 2$, there is an aperiodic PL plug with one minimal set which is of dimension $k$. 
References

[Bor] K. Borsuk, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. 24 (1935), 51-58.

[Ful] F. B. Fuller, *Note on trajectories in a solid torus*, Ann. of Math. 56 (1952), 438-439.

[Gra] H. Grauert, *On Levi’s problem and the imbedding of real-analytic manifolds*, Ann. of Math. 68 (1958), 460-472.

[Ghy] É. Ghys, *Construction de champs de vecteurs sans orbite périodique*, Séminaire BOURBAKI 785 (1993-94).

[Har] J. Harrison, *C^2 counterexamples to the Seifert conjecture*, Topology 27 (1988), 249-278.

[H-H] G. Hector and U. Hirsch, *Introduction to the geometry of foliations. Part B*, Aspects of Math. E3, Friedr. Vieweg & Sohn, Braunschweig (1987).

[Hof] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. 114 (1993), 515-563.

[KuG] G. Kuperberg, *A volume-preserving counterexample to the Seifert conjecture*, to appear in Comment. Math. Helvetici.

[KuK] K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, Ann. of Math., 140 (1994), 723-732.

[K-R1] K. Kuperberg and C. Reed, *A rest point free dynamical system on \(\mathbb{R}^3\) with uniformly bounded trajectories*, Fund. Math. 114 (1981), 229-234.

[K-R2] K. M. Kuperberg and C. S. Reed, *A dynamical system on \(\mathbb{R}^3\) with uniformly bounded trajectories and no compact trajectories*, Proc. Amer. Math. Soc. 106 (1989), 1095-1097.

[Mor] C. B. Morrey, *The analytic embedding of abstract real-analytic manifolds*, Ann. of Math. 68 (1958), 159-201.

[Nov] S. P. Novikov, *The topology of foliations*, Trudy Moscov. Mat. Obšč. 14 (1965), 248-278.

[Sch1] P. A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. 100 (1974), 386-400.

[Sch2] P. A. Schweitzer, *Codimension one foliations without compact leaves*, Comment. Math. Helvetici 70 (1995) 171-209.

[Sei] H. Seifert, *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Amer. Math. Soc. 1 (1950), 287-302.

[Sma] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747-817.

[Sol] V. V. Solodov, *Components of topological foliations*, Mat. Sb. (N.S.) 119(161) (1982), 340-354.

[Tam] I. Tamura, *Topology of foliations: An Introduction*, Translations Math. Monographs 97 (1992).

[Whi] H. Whitney, *Differentiable manifolds*, Ann. of Math. 37 (1936), 645-680.

[Wil] F. W. Wilson, *On the minimal sets of non-singular vector fields*, Ann. of Math. 84 (1966), 529-536.

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