Edge Universality of Sparse Random Matrices

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Abstract

We consider the statistics of the extreme eigenvalues of sparse random matrices, a class of random matrices that includes the normalized adjacency matrices of the Erdős-Rényi graph $G(N, p)$. Recently, it was shown in [40], up to an explicit random shift, the optimal rigidity of extreme eigenvalues holds, provided the averaged degree grows with the size of the graph, $pN > N^\epsilon$. We prove in the same regime, (i) Optimal rigidity holds for all eigenvalues with respect to an explicit random measure. (ii) Up to an explicit random shift, the fluctuations of the extreme eigenvalues are given the Tracy-Widom distribution.

1 Introduction

In this work we study the statistics of eigenvalues at the edge of the spectrum of sparse random matrices. A natural example is the adjacency matrix of the Erdős-Rényi graph $G(N, p)$, which is the random undirected graph on $N$ vertices in which each edge appears independently with probability $p$. Introduced in [13, 27], the Erdős-Rényi graph $G(N, p)$ has numerous applications in graph theory, network theory, mathematical physics and combinatorics. For further information, we refer the reader to the monographs [10, 34]. Many interesting properties of graphs are revealed by the eigenvalues and eigenvectors of their adjacency matrices.

The adjacency matrices of Erdős-Rényi graphs have typically $pN$ nonzero entries in each column and are sparse if $p \ll 1$. When $p$ is of constant order, the Erdős-Rényi matrix is essentially a Wigner matrix (up to a non-zero mean of the matrix entries). When $p \to 0$ as $N \to \infty$, the law of the matrix entries is highly concentrated at 0, and the Erdős-Rényi matrix can be viewed as a singular Wigner matrix. The singular nature of this ensemble can be expressed by the fact that the $k$-th moment of a matrix entry (in the scaling that the bulk of the eigenvalues lie in an interval of order 1) decays like ($k \geq 2$)

$$N^{-1}(pN)^{-(k-2)/2}. \quad (1.1)$$

When $p \ll 1$, this decay in $k$ is much slower than the $N^{-k/2}$ case of Wigner matrices, and is the main source of difficulties in studying sparse ensembles with random matrix methods.

The class of random matrices whose moments decay like (1.1) were introduced in the works [14, 16] as a natural generalization of the sparse Erdős-Rényi graph and encompass many other sparse ensembles. This is the class we study in this work.

The global statistics of the eigenvalues of the Erdős-Rényi graph are well understood. The empirical eigenvalue distribution converges to the semi-circle distribution provided $p \gg 1/N$, which follows from Wigner’s original proof. It was proven in [8,9,39,50], the spectral norm of normalized adjacency matrices
of Erdős-Rényi graphs converges to 2 if $p \gg \log N/N$. Some of their results also extend to nonhomogeneous Erdős-Rényi graphs. Later, sharp transition was identified in \cite{4,49}. It was proved that there exists a critical $b_\ast \approx 2.59$, if $p > b_\ast \log N/N$ then the spectral norm converges to 2, and if $p < b_\ast \log N/N$, then the extreme eigenvalues are determined by the largest degrees. For the global eigenvalue fluctuations, it was proven in \cite{44} that the linear statistics, after normalizing by $p^{1/2}$, converge to a Gaussian random variable.

Bulk universality of sparse random matrices (the statement that local statistics inside the bulk are asymptotically the same as those of Gaussian matrices) was proven in \cite{14,30} for $p \ge N^{\varepsilon-1}$ for any $\varepsilon > 0$. Universality for the edge statistics of sparse random matrices (the statement that the distribution of the extreme eigenvalues converge to the Tracy-Widom law) was more intricate. Edge universality for sparse random matrices was proven first in the regime $p \gg N^{-2/3}$ in \cite{14,41}. Later, it was observed in \cite{31} there is a transition in the behavior at $p = N^{-2/3}$. More precisely, it was proved for $N^{-2/9} \ll p \ll N^{-2/3}$, the extreme eigenvalues behave like

$$\mathcal{X} + N^{-2/3}\xi, \quad \mathcal{X} \sim \frac{\text{Gaussian}}{N\sqrt{p}}, \quad \xi \sim \text{Tracy-Widom law},$$

where $\mathcal{X}$ is related to the total number of edges.

In the regime $N^{-2/9} \ll p \ll N^{-2/3}$, the Gaussian term dominates and the leading behavior changes from the Tracy-Widom law to Gaussian at $p = N^{-2/3}$. This phenomenon that the leading behavior is Gaussian was extended down to the optimal scale $p \ge N^{\varepsilon-1}$ in \cite{29}. For the sparser regime when $(\log \log N)^3/N \ll p < b_\ast \log N/N$, it was proven in \cite{5}, the eigenvalues near the spectral edges form asymptotically a Poisson point process. There is however a nature question that \textit{what is the next order fluctuation term, and can we recover edge universality by subtracting all these higher order fluctuations?}

This question was partially settled in \cite{40} where it was proved that, for $N^{\varepsilon-1} \le p \ll N^{-2/3}$, there exists a sequence of explicit random correction terms, which capture higher (sub-leading) order fluctuations of extreme eigenvalues and after subtracting these explicit correction terms, the optimal rigidity of extreme eigenvalues holds. It was also explicitly conjectured in \cite{40} that up to this explicit random shift, the fluctuations of the extreme eigenvalues are given by the Tracy-Widom distribution. One main result of this paper proves this edge universality conjecture. As a consequence, the gaps between extreme eigenvalues of sparse random matrices with $p \ge N^{\varepsilon-1}$ are given asymptotically by the gaps of Airy point process.

Universality for the edge statistics of Wigner matrices was first established by the moment method \cite{46} under certain symmetry assumptions on the distribution of the matrix elements. The moment method was further developed in \cite{25,43} and \cite{45}. A different approach to edge universality for Wigner matrices based on the direct comparison with corresponding Gaussian ensembles was developed in \cite{24,47,48}.

For sparse random matrices, because of those random correction terms to the extreme eigenvalues, naive moment methods or direct comparison with Gaussian ensembles fail. Our proof of edge universality utilizes a three-step dynamical approach, which was originally developed to prove the bulk universality of Wigner matrices in a series of papers \cite{2,12,15,17-21,23,24,36,37}. This strategy is as follows: i) Establish a local semicircle law controlling the number of eigenvalues in windows of size $N^{\delta-1}$, where $\delta > 0$ is arbitrarily small. ii) Analyze the local ergodicity of Dyson Brownian motion to obtain universality after adding a small Gaussian noise to the ensemble. iii) A density argument comparing a general matrix to one with a small Gaussian component.

For edge universality, in the first step, a local semicircle law is not enough. For edge universality to be true, it is necessary that the extreme eigenvalues, up to an explicit (random) shift, fluctuate on scale $O(N^{-2/3})$. Such optimal rigidity estimate for extreme eigenvalues was obtained recently in \cite{40}. The proof is based on first constructing a higher order self-consistent equation for the Stieltjes transform of the empirical eigenvalue distributions, then computing the moments of the self-consistent equation by a recursive moment estimate. The rigidity estimates follow from a careful analysis of the recursive moment estimate.
This approach was first introduced in [41], where a deterministic higher order self-consistent equation was constructed and used to prove the optimal edge rigidity (with respect to a deterministically shifted edge) provided \( p \gg N^{-2/3} \). Later, a higher order self-consistent equation with one random correction term was constructed in [31], and used to prove the optimal edge rigidity (with respect to a randomly shifted edge) provided \( p \gg N^{-2/9} \). By including sufficiently many random correction terms, [40] proves an optimal edge rigidity down to the optimal scale \( p \gg N^{-1} \), and gives a full description of the randomly shifted edge. We revisit the recursive moment estimate for the random self-consistent equation introduced in [40]. By exploring a splitting phenomenon in the expansion, see Proposition 2.14, we improve the error in the recursive moment estimate, and obtain an optimal rigidity for all eigenvalues with respect to an explicit random measure. This is also crucial for the third step when we do the comparison.

For the second step, edge universality for ensembles with a small Gaussian noise was established in [38]. This work proves for wide classes of initial data, the edge statistics of Dyson Brownian motion coincides with Gaussian matrices. Moreover [38] finds the optimal time to equilibrium \( t \sim N^{-1/3} \) for sufficiently regular initial data.

In order to complete the three-step strategy, we need to compare sparse ensembles to Gaussian divisible ensembles, which is a sparse ensemble with a small Gaussian component. Similarly to [31,41], we can interpolate them by considering the Dyson matrix flow. The main challenge is to keep track of the change of the randomly shifted edge along the interpolation, and show the change of the Stieltjes transform over time is offset by the shift of the edge. The error term in the change of the Stieltjes transform around the randomly shifted edge has a \( (pN)^{-1/2} \) expansion. In [31,41], it was directly checked that the expansion vanishes up to the third order, with an error \( O((pN)^{-3/2}) \). A general principle of the cancellations up to arbitrary order is needed in order to solve the general case \( p \gg N^{-1} \). In this paper, we prove a version of such general cancellation principle. First we show the Stieltjes transform of sparse ensembles with a small Gaussian component satisfies a modified self-consistent equation, which can be used to precisely characterize the change of the randomly shifted edge along the interpolation. Next, we prove the change of the Stieltjes transform around the randomly shifted edge (up to negligible error) is given by the derivative of the modified self-consistent equation. Then a similar argument as in the first step, can be used to show its expectation is small up to arbitrary order.

It is worth to comparing the results with random \( d \)-regular graphs. Bulk universality of random \( d \)-regular graphs was proven in [6] for \( d \geq N^\varepsilon \) for any \( \varepsilon > 0 \). For edge statistics, those Gaussian fluctuations from degree fluctuations in Erdős–Rényi are absent in regular graphs. The eigenvalues of random regular graphs are more rigid than those of Erdős–Rényi graphs of the same average degree. We do not expect any shift of the spectral edge. It was proven that the law of the second largest eigenvalue (after shifting by 2 and proper normalization) converges to the Tracy-Widom distribution, for \( N^{-2/3} \ll d \ll N^{1/3} \) in [7]; for \( d \gg N^{2/3} \) in [28]; and for \( 1 \ll d \ll N^{1/3} \) in [32]. The edge universality was conjectured in [42] to be true down to \( d = 3 \). For fixed degree \( d \), even to show the concentration of extreme eigenvalues around the spectral edges \( \pm 2 \) requires significant work. It was first conjectured by Alon [3] and proven later in [11,26,33].

**Organization.** We define the model and present the main results in the rest of Section 1. In Section 2, we prove the optimal rigidity estimates for all eigenvalues with respect to an explicit random measure in the regime \( pN \geq N^\varepsilon \). In Section 3, we recall the results from [1,38] for edge universality of Gaussian divisible ensembles. In Section 4 we analyze the Stieltjes transform to compare a sparse ensemble to a Gaussian divisible ensemble and establish our results about Tracy-Widom fluctuations.

**Notations.** We use \( C \) to represent large universal constants, and \( c \) small universal constants, which may be different from line by line. Let \( Y \geq 0 \). We write \( X \lesssim Y \), \( Y \gtrsim X \) or \( X = O(Y) \) if there exists a constant \( C > 0 \), such that \( X \leq CY \). We write \( X \asymp Y \) or \( X = \Omega(Y) \) if there exists a constant \( C > 0 \) such that \( Y/C \leq X \leq Y/C \). We use \( \mathbb{C}_+ \) to represent the upper half plane. We denote \([a,b]=\{a,a+1,\ldots,b\}\).
For any index set \( m = \{m_1, m_2, \cdots, m_r\} \), we write
\[
\sum^* = \sum_{m_1, m_2, \cdots, m_r \in [1, N]} \sum_{m_1, m_2, \cdots, m_r \in [1, N]}
\]
For two index set \( m \) and \( v \), We denote \( v \cup m \) as \( vm \). If \( v = \{i\} \) has only one element, we simply write \( \{i\} \cup m \) as \( im \).

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1.1 Sparse random matrices
In this section we introduce the class of sparse random matrices that we consider. This class was introduced in [14, 16] and we repeat the discussion appearing there.

The Erdős-Rényi graph is the undirected random graph in which each edge appears with probability \( p \).

It is notationally convenient to replace the parameter \( p \) with \( q \) defined through
\[
pN = q^2, \quad q = \sqrt{pN}.
\]
We allow \( q \) to depend on \( N \). We denote by \( A \) the adjacency matrix of the Erdős-Rényi graph. \( A \) is an \( N \times N \) symmetric matrix whose entries \( A_{ij} \) above the main diagonal are independent and distributed according to
\[
A_{ij} = \begin{cases} 1 \text{ with probability } q^2/N \\ 0 \text{ with probability } 1 - q^2/N \end{cases}.
\]
We extract the mean of each entry and rescale the matrix so that the limiting eigenvalue distribution is roughly supported on \([-2, 2]\). We introduce the matrix \( H \) by
\[
H := \frac{A - q^2 |e\rangle \langle e|}{qN^{-1/2} - q^2 / N}
\]
where \( e \) is the unit vector
\[
e = (1, \ldots, 1)^T / \sqrt{N}.
\]
It is easy to check that the matrix elements of \( H \) (in the upper half triangle) have mean zero \( \mathbb{E}[h_{ij}] = 0 \), variance \( \mathbb{E}[h_{ij}^2] = 1/N \), and satisfy the moment bounds
\[
\mathbb{E}[h_{ij}^k] = \frac{1}{Nq^{k-2}} \left[ \left( 1 - \frac{q^2}{N} \right)^{-k/2+1} \left( 1 - \frac{q^2}{N} \right)^{k-1} + (-1)^k \left( \frac{q^2}{N} \right)^{k-1} \right] = \frac{\Omega(1)}{Nq^{k-2}},
\]
for \( k \geq 2 \). This motivates the following definition.

Definition 1.1 (Sparse random matrices). We assume that \( H = (h_{ij}) \) is an \( N \times N \) random matrix whose entries are real and independent up to the symmetry constraint \( h_{ij} = h_{ji} \). We further assume that \( (h_{ij}) \) satisfies \( \mathbb{E}[h_{ij}] = 0, \mathbb{E}[h_{ij}^2] = 1/N \) and that for any \( k \geq 2 \), the \( k \)-th cumulant of \( h_{ij} \) is given by

\[
\frac{(k - 1)C_k}{Nq^{k-2}}, \quad (1.2)
\]

where \( q = q(N) \) is the sparsity parameter, such that \( 0 < q \lesssim \sqrt{N} \). For \( C_k \) (which may depend on \( q, N \)) we make the following assumptions:
(1) $|C_k| \leq C_k$ for some constant $C_k > 0$.

(2) $C_4 \geq c$

**Remark 1.2.** By the definition, $C_2 = 1$. The lower bound, $C_4 \geq c$, ensures that the scaling by $q$ for the ensemble $H$ is “correct.”

We denote the eigenvalues of $H$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, corresponding eigenvectors $u_1, u_2, \ldots, u_N$, and the Green’s function of $H$ by

$$G(z) := (H - z)^{-1} = \sum_{\alpha=1}^{N} \frac{u_\alpha u_\alpha^*}{\lambda_\alpha - z}.$$ 

The Stieltjes transform of the empirical eigenvalue distribution is denoted by

$$m_N(z) := \frac{1}{N} \operatorname{Tr} G(z) = \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{\lambda_\alpha - z}.$$ 

**Definition 1.3** (overwhelming probability). We say an event $\Omega$ holds with overwhelming probability, if for any $D > 0$, $P(\Omega) \geq 1 - N^{-D}$ for $N \geq N(D)$ large enough.

**Definition 1.4** (Stochastic dominant). For $N$-dependent random (or deterministic) variables $A$ and $B$, we say $B$ stochastically dominate $A$, if for any $\varepsilon > 0$ and $D > 0$, then

$$P(A \geq \varepsilon B) \leq N^{-D}, \quad (1.3)$$

for $N \geq N(\varepsilon, D)$ large enough, and we write $A \prec B$ or $A = O_\prec(B)$.

### 1.2 Main Results

We first recall the edge rigidity estimates for the sparse random matrices from [40, Theorem 2.10].

**Theorem 1.5.** ([40, Theorem 2.10]) Let $H$ be as in Definition 1.1 with $N^\varepsilon \leq q \lesssim N^{1/2}$ and eigenvalues given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. There exists an explicit random measure $\tilde{\rho}$ supported on $[-\tilde{L}, \tilde{L}]$ (depending on certain averaged quantities of $h_{ij}$ as defined in Proposition 2.4). We have for any fixed index $k \geq 1$,

$$|\lambda_k - \tilde{L}| \prec \frac{1}{N^{2/3}}.$$ 

Analogous results hold for the smallest eigenvalues.

In the following Theorem, we improve the optimal edge rigidity results from [40, Theorem 2.10] to also include the bulk eigenvalues. The proof follows that of [40, Theorem 2.10] with some modifications. We give the proof in Section 2.3.

**Theorem 1.6.** Let $H$ be as in Definition 1.1 with $N^\varepsilon \leq q \lesssim N^{1/2}$ and eigenvalues given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. There exists an explicit random measure $\tilde{\rho}$ supported on $[-\tilde{L}, \tilde{L}]$ (depending on certain averaged quantities of $h_{ij}$ as defined in Proposition 2.4). We denote the classical eigenvalue locations of $\tilde{\rho}$ as $\gamma_1 > \gamma_2 > \cdots > \gamma_N$,

$$\frac{k - 1/2}{N} = \int_{\gamma_k}^{\gamma_k} \tilde{\rho}(x)dx, \quad 1 \leq k \leq N.$$ 

Then we have the following optimal rigidity estimates

$$|\lambda_k - \gamma_k| \prec \frac{1}{N^{2/3} \min\{k, N - k + 1\}^{1/3}}, \quad 1 \leq k \leq N. \quad (1.4)$$
The following theorem concerns the edge fluctuations for sparse random matrices as defined in Definition 1.1. We prove that the extreme eigenvalues have asymptotically Tracy-Widom fluctuation after subtracting the random edge location $\tilde{L}$ of $\tilde{\rho}$. As a consequence, the gaps between extreme eigenvalues are asymptotically the same as the gaps between the Airy point process.

**Theorem 1.7.** Let $H$ be as in Definition 1.1 with $N^c < q \lesssim N^{1/2}$ and eigenvalues given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. There exists an explicit random measure $\tilde{\rho}$ supported on $[-\tilde{L}, \tilde{L}]$ (depending on certain averaged quantities of $h_{ij}$ as defined in Proposition 2.4). Fix an integer $k \geq 1$, let $F : \mathbb{R}^k \to \mathbb{R}$ be a bounded test function. There is a universal constant $c > 0$ so that,

$$
\mathbb{E}_H[F(N^{2/3}(\lambda_1 - \tilde{L}), \ldots, N^{2/3}(\lambda_k - \tilde{L})] = \mathbb{E}_{GOE}[F(N^{2/3}(\mu_1 - 2), \ldots, N^{2/3}(\mu_k - 2))] + O(N^{-c}).
$$

(1.5)

The second expectation is with respect to a GOE matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$. Analogous results hold for the smallest eigenvalues.

2 Optimal Rigidity Estimates

In this section, we recall the main ingredients for the optimal edge rigidity estimates from [40], and improve it to the bulk eigenvalues.

2.1 Local law for sparse random graphs

In this section, we recall the following entrywise local semicircle law for sparse random matrices from [16]. We also collect some estimates following from the local law, which will be used in the rest of this paper.

**Theorem 2.1.** ([16, Theorem 2.8]) Let $H$ be as in Definition 1.1. Let $b > 0$ be any large constant. Then uniformly for any $z = E + i\eta$ such that $-b \leq E \leq b$ and $0 < \eta \leq b$, we have

$$
\max_{i,j} |G_{ij}(z) - \delta_{ij}m_{sc}(z)| \prec \left( \frac{1}{q} + \sqrt{\frac{\text{Im} m_{sc}(z)}{N\eta}} + \frac{1}{N\eta} \right),
$$

(2.1)

where $m_{sc}(z)$ is the Stieltjes transform of the semi-circle distribution.

As an easy consequence of Theorem 2.1, we have that the eigenvectors of $H$ are completely delocalized: with overwhelming probability, uniformly for $1 \leq \alpha \leq N$,

$$
\|u_\alpha\|_\infty \prec \frac{1}{\sqrt{N}}.
$$

(2.2)

The following estimates utilizing Ward identity and delocalization of eigenvectors (2.2) will be used repeated in the rest of the paper

$$
\sum_{j=1}^{N} |G_{ij}(z)|^2 = \frac{\text{Im}[G_{ij}(z)]}{\text{Im}[z]} = \frac{1}{\text{Im}[z]} \text{Im} \left[ \sum_{\alpha=1}^{N} \frac{u_\alpha^2(i)}{\lambda_\alpha - z} \right] = \frac{1}{N} \sum_{\alpha=1}^{N} \frac{w_\alpha^2(i)}{|\lambda_\alpha - z|^2} \prec \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{|\lambda_\alpha - z|^2} = \frac{\text{Im}[m_N(z)]}{\text{Im}[z]}.
$$

(2.3)

2.2 Higher order self-consistent equation

In this section, we recall the higher order self-consistent equations for sparse random matrices, and some useful estimates on the equilibrium measure and its Stieltjes transform from [31,40]. We recall from [40],
the random equilibrium measure $\tilde{\rho}$ in Theorem 1.5 and its Stieltjes transform $\tilde{m}$
\[\tilde{m}(z) = \int \frac{\tilde{\rho}(x) dx}{x - z}, \quad z \in \mathbb{C}_+.\]
Both $\tilde{\rho}$ and $\tilde{m}$ are random and depend on $N$ and certain averaged quantities of $h_{ij}$. They are characterized by a random polynomial $P(z, m)$,
\[P(z, m) = 0, \quad z \in \mathbb{C}_+.\] (2.4)
Explicitly, $P(z, m)$ is given by
\[P(z, m) = 1 + zm + m^2 + Q(m),\] (2.5)
where
\[Q(m) = a_2m^2 + a_4m^4 + a_6m^6 + \cdots + a_{2L}m^{2L},\] (2.6)
is an even polynomial of $m$, with degree $2L$ (where $L$ is a sufficiently large integer, which will be chosen later). The coefficients $a_{2\ell}$ for $1 \leq \ell \leq L$ are explicit random polynomials in the variables $h_{ij}$. To construct them, we need to introduce some notations.

**Definition 2.2. (Weighted forest).** By a weighted forest we mean a finite simple graph which is a union of trees: $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}), W(\mathcal{F})) = (V, E, W)$. Here $V$ is a finite set of vertices, $E$ is a finite set of edges, and each edge $e \in E$ connects $\{\alpha_e, \beta_e\} \in V$. $W$ is a set of edge weights: each edge $e \in E$ is associated with a positive odd integer $s_e \in \mathbb{N}$. We denote the number of connected components of $\mathcal{F}$ as $\theta(\mathcal{F})$.

We remark that by definition a weighted forest can have arbitrary degrees and weights. But given the total sum of weights $\sum_e (s_e + 1) = 2\ell$, there are only finite number of weighted forests. We treat the vertices as indeterminate. We will later assign them to be numbers in $[1, N]$, and sum over them.

For any $1 \leq \ell \leq L$, the coefficient $a_{2\ell}$ is a linear combination of terms (with bounded coefficients) in the following form
\[w(\mathcal{F}) = \sum_{x_1, x_2, \cdots, x_{|V(\mathcal{F})|}} \frac{1}{N^{\theta(\mathcal{F})}} \prod_{e \in E(\mathcal{F})} w(h_{\alpha_e, \beta_e}; s_e), \quad w(h; s) := h^{s+1} - \frac{1(s = 1)}{N},\] (2.7)
where $\mathcal{F}$ is a weighted forest as in Definition 2.2, $x_1, x_2, \cdots, x_{|V(\mathcal{F})|}$ enumerate the vertices of $\mathcal{F}$ and weights $s_e$ satisfy $\sum_e (s_e + 1) = 2\ell$. In (2.7), we slightly misuse the notations. The summation means we assign each vertex a distinct value in $[1, N]$, $x_1 \neq x_2 \neq \cdots \neq x_{|V(\mathcal{F})|} \in [1, N]$, and sum over all possible assignments. The summation can be viewed as a sum over all the possible embeddings of $\mathcal{F}$ to the complete graph on $N$ vertices $\{1, 2, \cdots, N\}$.

**Example 2.3.** If the graph $\mathcal{F}$ consists of a single edge, with weight $s = 1$, then (2.7) simplifies to
\[w(\mathcal{F}) = \sum_{1 \leq i \neq j \leq N} \frac{1}{N} \left( h_{ij}^2 - \frac{1}{N} \right).\]
Later one can see from the proof, we will have $a_2 = \sum_{ij} (h_{ij}^2 - 1/N)/N$.

It has been proven in [40], with high probability the random coefficients $a_{2\ell}$ are small and satisfy $|a_{2\ell}| \ll 1/q$. We remark that this bound is not sharp. From the expression of $a_2$ as in Example 2.3, it is not hard to check that $|a_2| \ll 1/(q\sqrt{N})$. As long as $a_2$ all go to zero as $N$ goes to infinite, we can view (2.4) as a small perturbation of the equation $1 + zm_{sc}(z) + m_{sc}(z)^2 = 0$, where $m_{sc}(z)$ is the Stieltjes transform of the semi-circle distribution. By a perturbation argument, the solution $\tilde{m}(z)$ of $P(z, \tilde{m}(z)) = 0$ defines a holomorphic function from the upper half plane $\mathbb{C}_+$ to itself. It turns out that it is the Stieltjes transform of a probability measure $\tilde{\rho}$. The following proposition from [31, Proposition 2.5] collects some properties of $\tilde{m}(z)$ and the measure $\tilde{\rho}$. 

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Proposition 2.4. There exists an algebraic function \( \tilde{m} : \mathbb{C}_+ \to \mathbb{C}_+ \), which depends on the coefficients \( a_2, a_4, \cdots, a_{2L} \) of \( Q \), such that the following holds:

1. \( \tilde{m} \) is the solution of the polynomial equation, \( P(z, \tilde{m}(z)) = 0 \).

2. \( \tilde{m} \) is the Stieltjes transform of a symmetric probability measure \( \tilde{\rho} \), with \( \text{supp} \tilde{\rho} = [-\tilde{L}, \tilde{L}] \), where \( \tilde{L} \) depends smoothly on the coefficients of \( Q \), and its derivatives with respect to the coefficients of \( Q \) are uniformly bounded. Moreover, \( \tilde{\rho} \) is strictly positive on \(( -\tilde{L}, \tilde{L}) \) and has square root behavior at the edge.

3. We have the following estimate on the imaginary part of \( \tilde{m} \),
   \[
   \text{Im}[\tilde{m}(E + i\eta)] \simeq \begin{cases} 
   \sqrt{\kappa + \eta}, & \text{if } E \in [-\tilde{L}, \tilde{L}], \\
   \eta/\sqrt{\kappa + \eta}, & \text{if } E \notin [-\tilde{L}, \tilde{L}],
   \end{cases} 
   \]
   and
   \[
   |\partial_2 P(z, \tilde{m}(z))| \simeq \sqrt{\kappa + \eta}, \quad \partial_2^2 P(z, \tilde{m}(z)) = 1 + O(1/q),
   \]
   where \( \kappa = \text{dist}(\text{Re}[z], \{-\tilde{L}, \tilde{L}\}) \).

Remark 2.5. Since the coefficients of the polynomial \( Q \) are random, the edge location \( \tilde{L} \) is also random, depending on certain averaged quantities of \( h_{ij} \). We can write \( \tilde{L} \) as its mean plus its fluctuation \( \tilde{L} = L + \Delta L \), where \( L = \mathbb{E}[\tilde{L}] \). Then \( L \) is close to 2 and has a \( 1/q \) expansion: \( L = 2 + 6C_4/q^2 + (120C_6 - 81C_4^2)/q^4 + \cdots \). The leading fluctuation of \( \tilde{L} \) is given by \( \sum_{ij}(h_{ij}^2 - 1/N)/N \) which is of size \( 1/(\sqrt{Nq}) \).

2.3 Optimal Rigidity Estimates

In this section we compute higher order moments of the self-consistent equation \( P(z, m_N(z)) \). This gives us a recursive moment estimate for the Stieltjes transform \( m_N(z) \). The rigidity estimates Theorem 1.6 follow from a careful analysis of the recursive moment estimate and an iteration argument.

We only analyze the behavior of the Stieltjes transform \( m_N(z) \) with \( z \) close to the right edge \( \tilde{L} \) (or bounded away from \(-\tilde{L}\)). The case that \( z \) is close to the left edge can be analyzed in the same way. Fix a small constant \( \varepsilon > 0 \), we define the shifted spectral domain \( \mathcal{D} \) as

\[
\mathcal{D} = \{ \kappa + i\eta : |\kappa| \leq 3/2, 0 < \eta \leq 1, N\eta/\sqrt{\kappa + \eta} \geq N^{\varepsilon} \}. \tag{2.9}
\]

Proposition 2.6. Let \( H \) be as in Definition 1.1 with \( N^{\varepsilon} \leq q \leq N^{1/2} \). There exist a finite number of random correction terms \( a_2, a_4, \cdots, a_{2L} \), and polynomial \( P \) as in (2.6). For \( z = \tilde{L} + w \) where \( \tilde{L} \) is from Proposition 2.4 and \( w = \kappa + i\eta \in \mathcal{D} \), let \( m_N(z) \) be the Stieltjes transform of the empirical eigenvalue density of \( H \), then we have

\[
\mathbb{E}[|P(z, m_N(z))|^{2r}] \prec \mathbb{E}[\Phi_r(w)], \tag{2.10}
\]

where \( \Phi_r(w) \) is defined as

\[
\Phi_r(w) := \sum_{s=1}^{2r} \left( \frac{\text{Im}[m_N(z)]}{N\eta} \right)^s |P(z, m_N(z))|^{2r-s} \]
\[+ \sum_{s=1}^{2r} 1/N\eta \left( \frac{\text{Im}[m_N(z)]|\partial_2 P(z, m_N(z))|}{N\eta} + \frac{1}{N} \right)^s |P(z, m_N(z))|^{2r-s-1}.
\]

Proposition 2.6 improves [40, Proposition 3.1]. In the control parameter \( \Phi_r \) (2.11), comparing with [40, Proposition 3.1] we no longer have the term

\[
\sum_{s=1}^{2r} \frac{\left[ \text{Im}[m_N(z)] \right]}{N\eta} \left( \frac{\text{Im}[m_N(z)]|\partial_2 P(z, m_N(z))|}{N\eta} + \frac{1}{N} \right)^s |P(z, m_N(z))|^{2r-s-1}.
\]

\[ (2.12) \]
Inside the bulk of the spectral, namely $\kappa = \Omega(1)$, we have $\text{Im}[m_N(z)] = \Omega(1)$. The error (2.12) is $\sqrt{N}\eta$ bigger than the second term on the righthand side of (2.11). Getting rid of the error (2.12) is crucial to the optimal rigidity estimates for eigenvalues inside the bulk of the spectral.

In this proof we write, for simplicity of notation, $z = \bar{L} + w$ with $w = \kappa + i\eta \in \mathcal{D}$, $P = P(z,m_N(z))$, $G = G(z)$, $m_N = m_N(z)$, $P' = \partial_2 P(z,m_N(z))$, $\partial_{ij} = \partial_{h_{ij}}$, $\partial_{ij} G = \partial_{ij}(G(z))$, $\partial^2_{ij} m_N = \partial_{ij}(m_N(z))$. We also denote $D_{ij} G = (\partial_{ij} G)(z)$ and $D_{ij} m_N = (\partial_{ij} m_N)(z)$, where the derivatives do not hit $z$.

A central object in our proof is the following notion of a polynomial in the entries of the Green’s function.

**Definition 2.7.** Let $R = R(\{x_{st}\}_{s,t=1}^r, y)$ be a monomial in the $r^2 + 1$ abstract variables $\{x_{st}\}_{s,t=1}^r, y$. We denote its degree by $\deg(R)$. For $i \in \mathbb{N}^r$, we define its evaluation on the Green’s function and the Stieltjes transform by

$$R_i = R(\{G_{i,s_t}\}_{s,t=1}^r, m),$$

(2.13)

and say that $R_i$ is a monomial in the Green’s function entries $\{G_{i,s_t}\}_{s,t=1}^r$. We denote the number of off-diagonal entries of $R_i$ as $\chi(R_i)$.

In [40], the terms with one off-diagonal Green’s function are bounded using the Ward identity (2.3), which leads to an error in the form (2.12). Instead of using the Ward identity, we estimate such terms using the $L_2$ norm of the Green’s function, i.e. $\|G\|_2 \leq 1/\eta$, which leads to the following estimate:

**Proposition 2.8.** Adopt the assumptions of Proposition 2.6. Given a weighted forest $\mathcal{F}$ with vertex set $V(\mathcal{F}) = ijm$, where $m = \{m_1, m_2, \cdots, m_r\}$. Then for any monomial $R_{ijm}$ as in Definition 2.7 and nonnegative integers $p = \{p_e\}_{e \in E(\mathcal{F})}$ such that $p_e \geq s_e$, we have

$$\frac{1}{N^{r+2}} \sum_{ijm}^* \mathbb{E} \left[ G_{ij} R_{ijm} \left( \prod_{e \in E(\mathcal{F})} \partial^{p_e-s_e}_{\alpha_e \beta_e} \right) (P^{r-1} P^r) \right] \prec \mathbb{E}[\Phi_r],$$

(2.14)

and for any choice of $m \in [1, N]$ (possibly not distinct), it holds

$$\left( \frac{1}{N} + \frac{\text{Im}[m_N]}{N\eta} \right) \left( \prod_{e \in E(\mathcal{F})} \partial^{p_e-s_e}_{\alpha_e \beta_e} \right) (P^{r-1} P^r) \prec \Phi_r,$$

(2.15)

where $\Phi_r$ is as defined in (2.11).

Using Proposition 2.8 as input, in the following, we give a proof of Proposition 2.6 following [40, Proposition 3.1]. We postpone the proof of Proposition 2.8 and some estimates to next section. The proof uses the cumulant expansion to compute the expectations. It turns out, all the terms we will get in the expansion are in the following form

$$\frac{1}{q^a} \times \frac{1}{N^{r+2}} \sum_{m}^* \mathbb{E} \left[ R_m \left( \prod_{e \in E(\mathcal{F})} \partial^{p_e-s_e}_{\alpha_e \beta_e} \right) (P^{r-1} P^r) \right],$$

(2.16)

where $\mathcal{F}$ is a weighted forest with vertex set $V(\mathcal{F}) = m = \{m_1, m_2, \cdots, m_r\}$, $R$ is a monomial as in Definition 2.7, $p = \{p_e\}_{e \in E(\mathcal{F})}$ are nonnegative integers, and $a \geq 0$ is the order parameter. Since the second factor in (2.16) can be trivially bounded by $O_<(1)$, the whole expression can be bounded by $O_<(1/q^a)$. For terms with order at least $M$, we will trivially bound them by $O_<(1/q^M)$.

**Proof of Proposition 2.6.** We divide the proof of Proposition 2.6 into three steps.

**Step 1 (eliminate off-diagonal Green’s function terms)** The starting point is the following identity,

$$1 + zm_N(z) = \sum_{ij} h_{ij} G_{ij}(z).$$
Using the cumulant expansion, we can write the moment of \( P(z, m_N(z)) \) as,

\[
\mathbb{E}[|P(z, m_N(z))|^2] = \mathbb{E} \left[ (1 + zm_N(z)) P^{r-1} \bar{P} \right] + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}]
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \sum_{ij} h_{ij} G_{ij} P^{r-1} \bar{P} \right] + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}]
\]

\[
= \frac{1}{N} \sum_{p=1}^{M} \sum_{ij} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \mathbb{E}[\partial^p_{ij} (G_{ij} P^{r-1} \bar{P})] + O_{\prec} \left( \frac{1}{q^{2M}} \right) + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}]
\]

\[
= \frac{1}{N} \sum_{ij} \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[\partial^p_{ij} G_{ij} \partial^{p-s}_{ij} (P^{r-1} \bar{P})] + O_{\prec} \left( \mathbb{E}[\Phi_r] \right) + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}],
\]

where \( M \) is large enough such that \( 1/q^M \ll \Phi_r \). For the last line in (2.17), we will prove later in Proposition 2.14, \( \partial^1_{ij} G_{ij} = D^1_{ij} G_{ij} + O_{\prec}(\text{Im}[m_N]/Nq) \ll 1 \). We can replace \( \partial^1_{ij} \) by \( D^1_{ij} \) and error can be bounded by \( \mathbb{E}[\Phi_r] \). Thus combining with (2.15), we can bound the terms with \( i = j \) in (2.17) by \( \mathbb{E}[\Phi_r] \)

\[
\frac{1}{N} \sum_{i=j} \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[\partial^p_{ij} G_{ij} \partial^{p-s}_{ij} (P^{r-1} \bar{P})] \ll \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{1}{N^2} \mathbb{E}[|\partial^p_{ij} (P^{r-1} \bar{P})|] \ll \mathbb{E}[\Phi_r].
\]  

(2.18)

With (2.18), we can rewrite the summation on the righthand side of (2.17) as

\[
\frac{1}{N} \sum_{i=j} \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[\partial^p_{ij} G_{ij} \partial^{p-s}_{ij} (P^{r-1} \bar{P})] + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}] + O_{\prec} \left( \mathbb{E}[\Phi_r] \right). \tag{2.19}
\]

The derivatives \( D^1_{ij} G_{ij} \) is a sum of terms in the form \( G_{ii}^c G_{jj}^c G_{ij}^c \). Thanks to Proposition 2.8, terms with \( c \geq 1 \) is bounded by \( O_{\prec}(\mathbb{E}[\Phi_r]) \), and

\[
D^1_{ij} G_{ij} = -1(s \text{ is odd})! G_{ii}^c G_{jj}^c G_{ij}^c + \{\text{terms with off-diagonal entries}\}.
\]

Therefore, the leading terms in (2.17) are those which do not contain any off-diagonal Green’s function terms,

\[
\frac{1}{N} \sum_{i=j} \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[\partial^p_{ij} G_{ij} \partial^{p-s}_{ij} (P^{r-1} \bar{P})] + O_{\prec} \left( \mathbb{E}[\Phi_r] \right) + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}]
\]

\[
= -\frac{1}{N} \sum_{p=1}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ G_{ii}^c G_{jj}^c \partial^p_{ij} (P^{r-1} \bar{P}) \right] + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}] + O_{\prec}(\mathbb{E}[\Phi_r])
\]

\[
= -\frac{1}{N} \sum_{p=2}^{M} \sum_{s=0}^{p} \frac{C_{p+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ G_{ii}^c G_{jj}^c \partial^p_{ij} (P^{r-1} \bar{P}) \right] + \mathbb{E}[(m_N^2 + Q) P^{r-1} \bar{P}] + O_{\prec}(\mathbb{E}[\Phi_r]). \tag{2.20}
\]

Here in the last line, we used that the term corresponding to \( p = 1, s = 1 \)

\[
-\frac{1}{N} \sum_{ij} C_N \mathbb{E} \left[ G_{ij} G_{jj} (P^{r-1} \bar{P}) \right] = -\mathbb{E}[m^2 P^{r-1} \bar{P}] + O \left( \frac{1}{N} \mathbb{E}[(P^{2r-1})^2] \right)
\]

\[
= -\mathbb{E}[m^2 P^{r-1} \bar{P}] + O(\mathbb{E}[\Phi_r]),
\]

which cancels with \( \mathbb{E}[m^2 P^{r-1} \bar{P}] \) in the second term.
We remark that for any given \( p \), the term on the righthand side of (2.20) is in the form (up to some bounded multiplicative factor)

\[
\frac{1}{N^{r+1}} \sum_{2 \leq p \leq M} \sum_{m} E \left[ R_m \prod_{e \in E(F)} \frac{C_{p}+1s_e!}{q^{p-1}} \left( \frac{p_e}{s_e} \right)^{\partial_{\alpha, \beta}^{s_e} (P^{r-1} \hat{P}^r)} \right],
\]

(2.21)

where the weighted forest \( F \) as in Definition 2.2 is a single edge \( e = \{m_1, m_2\} \) with vertex set \( m = \{m_1, m_2\} \). The edge \( e \) has weight \( s_e \). The monomial \( R_m \) as in (2.7) has no off-diagonal entries, i.e. \( \chi(R_m) = 0 \), and total degree \( \deg(R_m) = s_e + 1 \).

**Step 2 (replace diagonal Green’s function entries by \( m_N \))**

For the diagonal terms \( G_{ii}^{\frac{\gamma_i}{\beta_i}} G_{jj}^{\frac{\gamma_j}{\beta_j}} \) in (2.20) we can replace them by diagonal terms with different indices using the following proposition.

**Proposition 2.9.** Adopt the assumptions of Proposition 2.6. Given a weighted forest \( F \) with vertex set \( V(F) = im \), where \( m = \{m_1, m_2, \cdots, m_r\} \). Then for any monomial \( R_m \) as in Definition 2.7 with no off-diagonal Green’s function entries, i.e. \( \chi(R_m) = 0 \), and integers \( p = \{p_e\}_{e \in E(F)} \) with \( p_e - s_e \geq 0 \), we have

\[
\frac{1}{N^{r+1}} \sum_{im}^s E \left[ G_{ii}^{\gamma_i} R_m V \right] = \frac{1}{N^{r+1}} \sum_{im}^s E \left[ G_{ii}^{\gamma_i} m_N R_m V + \Omega_1 + \Omega_2 + O_{\prec} (E[\Phi_i]) \right],
\]

(2.22)

\[
V = \left( \prod_{e \in E(F)} \partial_{\alpha, \beta}^{p_e - s_e} \right) (P^{r-1} \hat{P}^r),
\]

where the \( \Omega_1, \Omega_2 \) are given by

\[
\Omega_1 = -\frac{1}{N^{r+2}} \sum_{p=2, s \text{ odd}}^M \sum_{i,j,k,m} \frac{C_{p}+1s!}{Nq^{p-1}} \left( \frac{p}{s} \right) \sum_{i,j,k} E[ \Phi_{i,j,k}^{\frac{\gamma_i}{\beta_i}} G_{jj}^{\frac{\gamma_j}{\beta_j}} G_{kk}^{\frac{\gamma_k}{\beta_k}} R_m \partial_{ijkl}^{p-e} V] \right],
\]

(2.23)

\[
\Omega_2 = \frac{1}{N^{r+1}} \sum_{p=2, s \text{ odd}}^M \sum_{i,j,k,m} \frac{C_{p}+1s!}{Nq^{p-1}} \left( \frac{p}{s} \right) s! \left( \frac{\gamma_i}{\beta_i} - 1 \right) \sum_{i,j,k,m} E[ m_N G_{ii}^{\gamma_i+\frac{\gamma_j}{\beta_j}} G_{jj}^{\frac{\gamma_j}{\beta_j}} G_{kk}^{\frac{\gamma_k}{\beta_k}} R_m \partial_{ijkl}^{p-e} V] \right].
\]

Comparing these terms \( \Omega_1, \Omega_2 \) with (2.22), since \( p \geq 2 \) they are of order at least 1 (recall from (2.16)).

**Remark 2.10.** These terms \( \Omega_1, \Omega_2 \) in (2.23) are in the same form as in (2.22). With given \( p \), the term in \( \Omega_1 \) is associated with a weighted forest \( F_1 \), which is from \( F \) by adding vertices \( j, k \) and an edge \( \{j, k\} \) with weight \( s \). In total \( \Omega_1 \) has \( r + 2 \) vertices. \( \Omega_2 \) is associated with a weighted forest \( F_2 \), which is from \( F \) by adding one vertex \( i \) and an edge \( \{i, k\} \) with weight \( s \). In total \( \Omega_2 \) has \( r + 1 \) vertices. For given \( s \), both \( \Omega_1, \Omega_2 \) have an extra derivative \( \partial^p \) and the total number of diagonal Green’s function entries increases by \( s + 1 \).
Thanks to Proposition 2.9, we can replace one copy of $G_{ij}$ in the last line of (2.20) by $m_N$:

$$-\frac{1}{N} \sum_{p=2}^{M} \sum_{s \text{ odd}} \sum_{ij} c_{p+1} \frac{s!}{N q^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ F_{ij}^{s+1} F_{jj}^{s+1} \partial_{ij}^{p-s} (P^{r-1} \tilde{P}^r) \right]$$

$$= -\frac{1}{N} \sum_{p=2}^{M} \sum_{s \text{ odd}} \sum_{ij} c_{p+1} \frac{s!}{N q^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ F_{ij}^{s+1} m_N G_{jj}^{s+1} \partial_{ij}^{p-s} (P^{r-1} \tilde{P}^r) \right] + O_e(\mathbb{E}[\mathcal{F}_r])$$

$$+ \frac{1}{N^2} \sum_{s, s' \text{ odd}, p, p'=2}^{M} c_{p+1} \frac{s!}{N q^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ (G_{ij}^{s+1} G_{jj}^{s+1} G_{kk}^{s+1} G_{\ell \ell}^{s+1}) \partial_{ik}^{p-s} \partial_{ij}^{p-s} (P^{r-1} \tilde{P}^r) \right]$$

$$- \frac{1}{N} \sum_{s, s' \text{ odd}, p, p'=2}^{M} c_{p+1} \frac{s!}{N q^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ (m_N G_{ij}^{s+1} G_{jj}^{s+1} G_{kk}^{s+1} G_{\ell \ell}^{s+1}) \partial_{ik}^{p-s} \partial_{ij}^{p-s} (P^{r-1} \tilde{P}^r) \right]$$

$$= -\frac{1}{N} \sum_{p=2}^{M} \sum_{s \text{ odd}} \sum_{ij} c_{p+1} \frac{s!}{N q^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ m_N^{s+1} \partial_{ij}^{p-s} (P^{r-1} \tilde{P}^r) \right],$$

with higher order terms (which have at least one more copies of $1/q$) as linear combination of terms (with bounded coefficients) in the form

$$\frac{1}{N^r} \sum_{2 \leq p_1, \cdots, p_r \in \mathcal{F}} \sum_{m \leq M} R_m \prod_{e \in E(\mathcal{F})} \frac{c_{p_1} \cdots c_{p_r} \left( \frac{p_e}{s_e} \right)^{s_e}}{q^{p_e-1}} \partial_{\alpha_e \beta_e}^{p_e-s_e} (P^{r-1} \tilde{P}^r),$$

where $\mathcal{F}$ is a weighted forest as in Definition 2.2 with vertex set $m = \{m_1, m_2, \cdots, m_r\}$; the monomial $R_m$ has no off-diagonal entries, i.e. $\chi(R_m) = 0$ and $\deg(R_m) = \sum_{e \in E(\mathcal{F})} (s_e + 1)$.

We can repeat Step 2 for these higher order terms (2.26). If a term has order bigger than $M$, we can trivially bound it by $1/q^M \prec \Phi$, as in (2.16). The final expression is a linear combinations (with bounded coefficients) of terms in the form:

$$\mathbb{E}[m_N^{2\ell} L_{\mathcal{F}}(P^{r-1} \tilde{P}^r)],$$

$$L_{\mathcal{F}} = \frac{1}{N^{\mid V(\mathcal{F}) \mid}} \sum_{x_1, \cdots, x_{\mid V(\mathcal{F}) \mid} \in E(\mathcal{F})} \prod_{e \in E(\mathcal{F})} \frac{c_{p_1} \cdots c_{p_r} \left( \frac{p_e}{s_e} \right)^{s_e}}{q^{p_e-1}} \partial_{\alpha_e \beta_e}^{p_e-s_e},$$

where $\mathcal{F}$ is a forest as in Definition 2.2, $x_1, x_2, \cdots, x_{\mid V(\mathcal{F}) \mid}$ enumerare the vertices of $\mathcal{F}$. Moreover, all the weights $s_e$ are odd positive integers, and the total weights satisfies $\sum_e (s_e + 1) = 2\ell$. The above discussion leads to the following claim.

**Claim 2.11.** Under the assumptions of Proposition 2.6, with an error $O_e(\mathbb{E}[\mathcal{F}_r])$, the first term on the righthand side of (2.20) is a linear combinations of terms in the form,

$$\mathbb{E}[m_N^{2\ell} L_{\mathcal{F}}(P^{r-1} \tilde{P}^r)],$$

where $\mathcal{F}$ is a forest as in Definition 2.2, and $2\ell = \sum_{e \in E(\mathcal{F})} (s_e + 1)$ and $L_{\mathcal{F}}$ is as defined in (2.27).
Step 3 (rewrite differential operators as an expectation) Finally by the cumulant expansion we have,

\[ \mathbb{E}[h_{ij}^{s+1}m_N^{2q}(P^{r-1}\tilde{P}^r)] = \sum_{p=1}^{M} \frac{C_p+1}{Nq^{p-1}} \mathbb{E}[\partial_{ij}^p(h_{ij}^s m_N^{2q}P^{r-1}\tilde{P}^r)] + O_\prec(\mathbb{E}[\Phi^r]) \]

\[ = \sum_{p=1}^{M} \frac{C_p+1}{Nq^{p-1}} \left( \frac{p}{s', s''} \right) \mathbb{E}[\partial_{ij}^p(h_{ij}^s)\partial_{ij}^{p-s'}(m_N^{2q})\partial_{ij}^{p-s''}(P^{r-1}\tilde{P}^r)] + O_\prec(\mathbb{E}[\Phi^r]). \]

If \( s'' \geq 1 \), then we have \( |\partial_{ij}^{s''}(m_N^{2q})| < \text{Im}[m_N/N\eta] \) from (2.47) in Proposition 2.14. Then it follows

\[ \partial_{ij}^p(h_{ij}^s)\partial_{ij}^{p-s'}(m_N^{2q})\partial_{ij}^{p-s''}(P^{r-1}\tilde{P}^r) < \frac{\text{Im}[m_N]}{N\eta} \partial_{ij}^{p-s'-s''}(P^{r-1}\tilde{P}^r) < \Phi^r, \]

where we used (2.15) in the last inequality. If \( s'' = 0 \) and \( s' < s \), then \( \partial_{ij}^{s'}(h_{ij}^s) = s(s-1)\cdots(s-s'+1)h_{ij}^{s-s'} \), we can do another cumulant expansion

\[ \mathbb{E}[\partial_{ij}^p(h_{ij}^s)\partial_{ij}^{p-s'}(P^{r-1}\tilde{P}^r)] \]

\[ = \sum_{p'=1}^{M} \frac{C_{p'}+1}{Nq^{p'-1}} \left( \frac{s'}{s-s'+1} \right) \mathbb{E}[\partial_{ij}^{p'}(h_{ij}^{s-s'+1}\partial_{ij}^{p-s'}(P^{r-1}\tilde{P}^r))] + O_\prec \left( \frac{1}{qM} \right) < \mathbb{E}[\Phi^r], \]

where we used (2.15) in the last inequality. The remaining terms correspond to \( s' = s \) and \( s'' = 0 \). We conclude

\[ \mathbb{E}[h_{ij}^{s+1}m_N^{2q}(P^{r-1}\tilde{P}^r)] = \sum_{p=1}^{M} \frac{C_p+1}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[m_N^{2q}\partial_{ij}^{p-s}(P^{r-1}\tilde{P}^r)] + O_\prec(\mathbb{E}[\Phi^r]). \]

And by moving the term corresponding to \( p = 1 \) to the left, we can rewrite the above equation as

\[ \mathbb{E}\left[ h_{ij}^{s+1} - \frac{1(s=1)}{N} \right] m_N^{2q}(P^{r-1}\tilde{P}^r) = \sum_{p=2}^{M} \frac{C_p+1}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E}[m_N^{2q}\partial_{ij}^{p-s}(P^{r-1}\tilde{P}^r)] + O_\prec(\mathbb{E}[\Phi^r]). \]  

(2.29)

By repeatedly using the relation (2.29), we can rewrite terms as in (2.27), and have proved the following claim.

**Claim 2.12.** Under the assumptions of Proposition 2.6, we can rewrite (2.28) as:

\[ \mathbb{E}[m_N^{2q}L_\mathcal{F}(P^{r-1}\tilde{P}^r)] \]

\[ = \prod_{e \in E(\mathcal{F})} \mathbb{E} \left[ \sum_{x_1, \ldots, x_\vert V(\mathcal{F}) \vert} \left( \frac{1}{N^{g(\mathcal{F})}} \right) \prod_{e \in E(\mathcal{F})} \left( h_{\alpha, \beta}^{s_e+1} - \frac{1(s_e=1)}{N} \right) m_N^{2q}(P^{r-1}\tilde{P}^r) \right] + O_\prec(\mathbb{E}[\Phi^r]) \]  

(2.30)

where \( \mathcal{F} \) is a forest as in Definition 2.2, and \( x_1, x_2, \ldots, x_\vert V(\mathcal{F}) \vert \) enumerate the vertices of \( \mathcal{F} \). \( L_\mathcal{F} \) and \( w(\mathcal{F}) \) are as defined in (2.27) and (2.7). Moreover, all the weights \( s_e \) are odd positive integers with \( \sum_e (s_e + 1) = 2\ell \).

Thanks to Claims (2.11) and (2.12), up to an error \( O_\prec(\mathbb{E}[\Phi^r]) \), the first term on the right-hand side of (2.20) is in the form

\[ -\mathbb{E}[a_2m_N^4 + a_4m_N^4 + \cdots + a_{2L}m_N^{2q}(P^{r-1}\tilde{P}^r)] + O_\prec(\mathbb{E}[\Phi^r]), \]  

(2.31)
where \( a_{2\ell} \) is a sum of terms in the form \( w(\mathcal{F}) \) as in (2.7), where \( \mathcal{F} \) is a forest as in Definition 2.2. Moreover, all the weights \( s_r \) are odd positive integers with \( \sum (s_r + 1) = 2\ell \). We can use the expression (2.31) as the definition of the polynomial \( Q \). Thus the term (2.31) cancels with \( \mathbb{E}[Q^{P_{r-1}P_r}] \) in (2.17), and we conclude Proposition 2.6.

\[ \square \]

### 2.4 Proof of Proposition 2.8 and 2.9

Before proving Propositions 2.8 and 2.9, we collect some useful estimates of the derivatives of \( m_N(z) \) and \( a_{2\ell} \) in Propositions 2.13 and 2.14.

**Proposition 2.13.** Adopt the assumptions in Proposition 2.6, and take \( z = \tilde{L} + w \). Fix distinct indices \( i, j, m = \{m_1, m_2, \ldots, m_r\} \). We consider the differential operator \( \partial^\beta \) with \( \beta = \{\beta_{uv}\}_{u,v \in ijm} \)

\[
\partial^\beta = \prod_{u,v \in ijm} \partial_{uv}^{\beta_{uv}}, \quad D^\beta = \prod_{u,v \in ijm} D_{uv}^{\beta_{uv}}, \quad |\beta| = \sum_{u,v \in ijm} \beta_{uv} \geq 1.
\]

1. The derivative \( D^\beta m_N(z) \) of \( m_N(z) \) is a linear combination of terms in the following form (with bounded coefficients)

\[
\frac{\text{Im}[m_N(z)]}{N\eta} \sum_k G_{ki}^{a} X_{kmi} Y_{kmj}, \tag{2.32}
\]

where \( a \geq 0, k \) is an index set, and \( \sum_k |X_{kmi}|^2, \sum_k |Y_{kmj}|^2 = O_{\prec}(1) \).

2. The derivative \( \partial^\beta a_{2\ell} \) of \( a_{2\ell} \) (as defined in (2.7)) is a linear combination of terms in the following form (with bounded coefficients)

\[
\frac{1}{N} \sum_{k, m \not\in \{i,j,k\}} h_{ij}^{a} X_{kmi} Y_{kmj}, \tag{2.33}
\]

where \( a \geq 0, k \) is an index set, \( \sum_k |X_{kmi}|^2, \sum_k |Y_{kmj}|^2 = O_{\prec}(1) \).

**Proof.** The derivative \( D^\beta m_N \) is a linear combination of terms in the following form

\[
\frac{1}{N} \sum_{k=1}^{n} G_{kv_1} G_{v_2 v_3} \cdots G_{v_{2\ell-2} v_{2\ell-1}} G_{v_{2\ell} k}, \tag{2.34}
\]

where \( v_1, v_2, \ldots, v_{2\ell} \in ijm \). For the Green’s function entries in (2.34) we can regroup them depending if they contain indices \( i, j \)

\[
\frac{1}{N} \sum_{k=1}^{N} G_{ki}^{a} X_{kmi} Y_{kmj},
\]

where for each \( G_{xy} \) in (2.34), if the index set \( \{x, y\} \) only contains \( i \) we put it in \( \tilde{X}_{kmi} \); if it only contains \( j \) we put it in \( \tilde{Y}_{kmj} \); if it does not contain \( i, j \) we simply put it in \( X_{kmi} \).

There are two cases. In the first case, each of \( \tilde{X}_{kmi}, \tilde{Y}_{kmj} \) contains one of \( G_{kv_1}, G_{v_{2\ell} k} \); in the second case, both \( G_{kv_1}, G_{v_{2\ell} k} \) are in \( X_{kmi} \) or \( Y_{kmj} \).

In the first case, say \( G_{kv_1} \) is in \( \tilde{X}_{kmi} \), using \( |G_{v_{\ell}, v_{\ell+1}}| \prec 1 \) from Theorem 2.1 and Ward identity (2.3) we have

\[
\sum_{k=1}^{N} |\tilde{X}_{kmi}|^2 < \sum_{k=1}^{N} |G_{kv_1}|^2 < \frac{\text{Im}[m_N]}{\eta}, \quad \sum_{k=1}^{N} |\tilde{Y}_{kmj}|^2 < \sum_{k=1}^{N} |G_{v_{2\ell} k}|^2 < \frac{\text{Im}[m_N]}{\eta}.
\]
The claim (2.32) follows by taking \( k = \{k\} \) and \( \sqrt{\text{Im}[m_N]/\eta} X_{kmi} = \tilde{X}_{kmi} \), \( \sqrt{\text{Im}[m_N]/\eta} Y_{kmj} = \tilde{Y}_{kmj} \).

In the second case, say both \( G_{kvi}, G_{vjk} \) are in \( \tilde{X}_{kmi} \). Then \( \tilde{Y}_{kmj} = \tilde{Y}_{mij} \) does not depend on the index \( k \). Moreover, using \( |G_{v_{k+1}}| \times 1 \) from Theorem 2.1 and Ward identity (2.3)

\[
\sum_k \tilde{X}_{kmi} < \sum_{k=1}^N |G_{kvi} G_{vjk}| \leq \frac{1}{2} \sum_{k=1}^N |G_{kvi}|^2 + |G_{vjk}|^2 < \frac{\text{Im}[m_N]}{\eta}.
\]

The claim (2.32) follows by taking \( k = \emptyset \) and \( (\text{Im}[m_N]/\eta) X_{kmi} = \sum_k \tilde{X}_{kmi}, Y_{kmj} = \tilde{Y}_{mij} \). This finishes the proof of (2.32).

Next we prove (2.33). We recall the following estimates from [40, Appendix A], which follows from computing high moments

\[
\sum_{j=1}^N \left| h_{ij}^2 - \frac{1}{N} \right| < 1, \quad \sum_{j=1}^N h_{ij}^\alpha < \frac{1}{q^{\alpha-2}}, \quad \alpha \geq 2.
\]  

(2.35)

We recall from (2.7) that \( a_{2\ell} \) is a linear combination of terms in the form

\[
w(\mathcal{F}) = \sum_{x_1, x_2, \cdots, x_{|V(\mathcal{F})|}} \frac{1}{N^{\theta(\mathcal{F})}} \prod_{e \in E(\mathcal{F})} w(h_{\alpha, \beta}; s_e), \quad w(h; s) := h^{s+1} - \frac{1(s=1)}{N},
\]

(2.36)

where \( \mathcal{F} \) is a weighted forest as in Definition 2.2, \( x_1, x_2, \cdots, x_{|V(\mathcal{F})|} \) enumerate the vertices of \( \mathcal{F} \) and weights \( s_e \) satisfy \( \sum_e (s_e + 1) = 2\ell \).

When we compute \( \partial^\mathcal{F} w(\mathcal{F}) \), the derivative \( \partial_{uv} \) may hit \( w(h_{xy}; s) \), which is nonzero if and only if \( e = \{x, y\} = \{u, v\} \), and \( \partial_{uv} w(h_{uv}; s) = (s+1) h_{uv}^s \). Therefore up to certain constant, the derivative \( \partial^\mathcal{F} w(\mathcal{F}) \) is also in the form (2.7), i.e. a product over \( e \in E(\mathcal{F}) \). The difference is that we need to fix some edges to be \( \{u, v\} \) with \( u \neq v \in ij \) and no longer need to sum over these indices.

We denote the vertex set which are not fixed to be \( ij \) as \( \mathbf{v} := V(\mathcal{F}) \setminus ij \). After fixing some vertices to be \( ij \) in \( \mathcal{F} \), the edge set of \( E(\mathcal{F}) \) decomposes into two sets \( \{i, j\} \cup E_1(\mathcal{F}) = \{e = \{x, y\} : x, y \in ij \} \) and \( E_2(\mathcal{F}) = E(\mathcal{F}) \setminus \{(i, j)\} \). Then for each edge \( \{x, y\} \in E_2(\mathcal{F}) \), at least one of \( x, y \) is in \( \mathbf{v} \). Then \( \partial^\mathcal{F} w(\mathcal{F}) \) is a linear combination of terms

\[
\sum_{\{x, y\} \in E_2(\mathcal{F})} \prod_{\{x, y\} \in E_1(\mathcal{F})} h_{\alpha, x}^{a_{\alpha, x}} \left( h_{x}^2 - \frac{1}{N} \right)^{b_{xy}} \prod_{e \in E_2(\mathcal{F})} w(h_e, s_e).
\]

(2.37)

We can further rewrite the first product in (2.37) as

\[
\prod_{\{i, x\} \in E_1(\mathcal{F})} h_{\alpha, i}^{a_{\alpha, i}} \left( h_{i}^2 - \frac{1}{N} \right)^{b_{ix}} \prod_{\{j, x\} \in E_1(\mathcal{F})} h_{\alpha, j}^{a_{\alpha, j}} \left( h_{j}^2 - \frac{1}{N} \right)^{b_{jx}} \prod_{\{x, y\} : x \in E_2(\mathcal{F}), y \notin \{i, j\}} h_{\alpha, x}^{a_{\alpha, x}} \left( h_{x}^2 - \frac{1}{N} \right)^{b_{xy}}.
\]

(2.38)

For each edge \( e \in E_2(\mathcal{F}) \) there are several possibilities: i) \( e = \{i, x\} \) where \( x \in \mathbf{v} \); iii) \( e = \{j, x\} \) where \( x \in \mathbf{v} \); iv) \( e = \{x, y\} \) where \( x, y \notin \{i, j\} \). We can rewrite the last sum in (2.37) as

\[
\sum_{i, x \in E_2(\mathcal{F})} \prod_{i, x \in E_2(\mathcal{F})} w(h_{ix}, s_{\{i, x\}}) \prod_{\{j, x\} \in E_2(\mathcal{F})} \prod_{\{e, y\} \in E_2(\mathcal{F}), x \notin \{i, j\}} w(h_{xy}, s_{\{x, y\}}).
\]

(2.39)

Using (2.38), (2.39), we can rewrite (2.37) in the following form

\[
\frac{1}{N} h_{ij}^\alpha \sum_{i, j \in \mathbf{v}} \tilde{X}_{imn} \tilde{Y}_{imn},
\]

(2.40)
In (2.41), we have divided the vertex set \( v = v_0v_1v_2 \), where \( v_1 \) is the set of leaf vertices adjacent to \( i \): \( v_1 = \{ x : x \text{ is a leaf vertex}, (i, x) \in E(F) \} \); \( v_2 \) is the set of leaf vertices adjacent to \( j \): \( v_2 = \{ x : x \text{ is a leaf vertex}, (j, x) \in E(F) \} \); and \( v_0 \) is the set of remaining vertices. In this way, \( \tilde{X}_{imv_0v_1} \) does not depend on the vertex set \( v_2 \) and \( \tilde{Y}_{jmv_0v_2} \) does not depend on the vertex set \( v_1 \).

Next we show that

\[
\sum_{v_0} \left( \sum_{v_1} |\tilde{X}_{imv_0v_1}| \right)^2 < 1, \quad \sum_{v_0} \left( \sum_{v_2} |\tilde{Y}_{jmv_0v_2}| \right)^2 < 1. \tag{2.42}
\]

We prove (2.42) for the case that \( F \) is a tree. The case that \( F \) is a union of trees is the same. In this case, (2.41) simplifies

\[
|\tilde{X}_{imv_0v_1}| \prec \prod_{(i, x) \in E_2(F), \ x \in v_1} |w(h_{ix}, s_{ix})| \prod_{(i, x) \in E_2(F), \ x \notin v_1} |w(h_{ix}, s_{ix})| \prod_{(x, y) \in E_2(F), \ x \notin v_1} \sqrt{|w(h_{xy}, s_{xy})|}. \tag{2.43}
\]

The last two factors in (2.43) do not depend on the indices \( v_1 \).

For any vertex \( x \in F \), and a subset of its neighborhoods \( y = \{ y_1, y_2, \cdots, y_n \} \). Using (2.35) and \( |w(h_{xy}, s_{xy})| \prec 1 \) for any \( (x, y) \in V(F) \), we have

\[
\sum_{x=1}^{N} n \prod_{i=1}^{n} |w(h_{xy}, s_{xy})| \leq \sum_{x=1}^{N} |w(h_{xy_1}, s_{xy_1})| \prec 1, \tag{2.44}
\]

provided \( n \geq 1 \). Using the bound (2.44), we can sum over the indices \( v_1 \) in (2.43)

\[
\left( \sum_{v_1} |\tilde{X}_{imv_0v_1}| \right)^2 \prec \prod_{(i, x) \in E_2(F), \ x \in v_1} |w(h_{ix}, s_{ix})|^2 \prod_{(x, y) \in E_2(F), \ x \notin v_1} \sqrt{|w(h_{xy}, s_{xy})|}.
\]

Then we can further sum over vertices \( v_0 \), by repeatedly using (2.44),

\[
\sum_{v_0} \left( \sum_{v_1} |\tilde{X}_{imv_0v_1}| \right)^2 < 1.
\]

This finishes the proof of the claim (2.41).
The summation in (2.37) is over indices \( v = v_0v_1v_2 \in [1,N] \) such that \( ijmv \) are all distinct. We can first sum over \( v_0 \) then \( v_1, v_2 \)

\[
\sum_{w : jmuv \text{ distinct}} = \sum_{v_0} \sum_{v_1, v_2} \sum_{w : jmuv \text{ distinct}}
\]

(2.45)

where in the second summation \( v_1, v_2 \) are distinct. By the inclusion-exclusion principle, we can rewrite the second summation in (2.45) as a linear combination of terms with \( v_1 = \{v'_0, v'_1\}, v_2 = \{v'_0, v'_2\} \) where \( v'_1, v'_2 \) may not be distinct:

\[
\sum_{v'_0, v'_1, v'_2} \sum_{v_0} \sum_{v_1, v_2} \sum_{w : jmuv \text{ distinct}}
\]

In this way, we can combine \( k = \{v_0, v'_0\} \), and conclude that the summation in (2.37) is a linear combination of terms in the form

\[
\frac{1}{N} h_{ij}^a \sum_{k : jmk \text{ distinct}} X_{imk}Y_{jmk}.
\]

\( X_{imk} := \sum_{v_0}' \sum_{v'_0, v'_1} X_{imvkv'_0v'_1}, Y_{imk} := \sum_{v_0}' \sum_{v'_1, v'_2} Y_{imvkv'_1v'_2}. \)

Thanks to (2.42)

\[
\sum_k |X_{imk}|^2 \leq \sum_k \left( \sum_{v'_0} |X_{imvkv'_0}| \right)^2 = \sum_{v_0} \sum_{v_0}' \left( \sum_{v'_0} |X_{imvkv'_0}| \right)^2 \leq \sum_{v_0} \sum_{v_0}' \left( \sum_{v'_0} |X_{imvkv'_0}| \right)^2 \leq 1.
\]

And we have the same estimate for \( \sum_k |Y_{jmk}|^2 < 1 \). This finishes the claim (2.33).

\[ \square \]

**Proposition 2.14.** Adopt the assumptions in Proposition 2.6, and take \( z = \tilde{L} + w \). Fix distinct indices \( m = \{m_1, m_2, \ldots, m_r\} \). We consider the differential operator \( \partial^\beta \) with \( \beta = \{\beta_{uv}\}_{u,v \in ijm} \)

\[
\partial^\beta = \prod_{u,v \in m} \partial^\beta_{uv}, \quad D^\beta = \prod_{u,v \in m} D^\beta_{uv}, \quad |\beta| = \sum_{u,v \in m} \beta_{uv} \geq 1.
\]

We have the following estimates

\[
\partial^\beta z = \partial^\beta \tilde{L} < \frac{1}{N}, \quad \partial^\beta G_{ab}(z) = D^\beta G_{ab}(z) + O_{\infty} \left( \frac{\text{Im}[m_N]}{N \eta} \right), \quad \partial^\beta m_N(z) < \frac{\text{Im}[m_N]}{N \eta}, \quad (2.46)
\]

and

\[
\partial^\beta m_N(z) = D^\beta m_N(z) + \partial_z m_N(z) \partial^\beta z + O_{\infty} \left( \frac{\text{Im}[m_N]}{(N \eta)^2} \right), \quad (2.47)
\]

\[
\partial^\beta P(z, m_N(z)) = (D^\beta m_N(z) + \partial_z m_N(z) \partial^\beta z) P' + ((\partial^\beta z)m_N(z) + \sum_{k=1}^L (\partial^\beta a_{2k}) m_N^2(z)) + O_{\infty} \left( \frac{\text{Im}[m_N]}{(N \eta)^2} \right), \quad (2.48)
\]

**Proof.** From (2.33), we have \( \partial^\beta a_{2k} \), is a sum of terms in the form

\[
\frac{1}{N} \sum_{k : jmk \text{ distinct}} h_{ij}^a X_{km1}Y_{kmj} \prec \frac{1}{N} \sum_k |X_{km1}||Y_{kmj}| \prec \frac{1}{N}
\]

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Thus $|\partial^\beta a_{2\ell}| \sim 1/N$. For $\partial^\beta z$, by the chain rule we have it is a linear combination of terms in the form

$$\left| \prod_{i=1}^m (\partial_{a_{2\ell_i}} z) \right| \prod_{i=1}^m (\partial^\beta a_{2\ell_i}) \leq \frac{1}{N}, \quad \sum_{i=1}^m \beta_i = \beta, \quad m \geq 1,$$

where we used Proposition 2.4 that $\left( \prod_{i=1}^m (\partial_{a_{2\ell_i}} z) \right)$ is bounded.

For the derivative $\partial^\beta G_{ab}$, each $\partial_{uv}$ either it hits $G_{ab}$ or it hits $z$. The derivative $\partial^\beta G_{ab}$ is a sum of terms in the form

$$(\partial^\beta \partial^m_z G_{ab}) \prod_{i=1}^m (\partial^\alpha z) \prec \frac{1}{N^{m+1}} |\partial^m \partial^\beta z G_{ab}|, \quad \beta + \sum_{i=1}^m \beta_i = \beta, \quad m \geq 1,$$

where we used $|\partial^\beta z| \sim 1/N$. We notice that for any Green’s function term $G_{ij}$, its derivative satisfies

$$|\partial^m_z G_{ij}| = m! |(G^{m+1})_{ij}| = m! \sum_{\alpha} \frac{u_{\alpha}(i)u_{\alpha}(j)}{(z - \lambda_\alpha)^{m+1}} \leq \frac{1}{N^{m+1}} \sum_{\alpha} \frac{1}{|z - \lambda_\alpha|^2} = \frac{\text{Im}[m_N(z)]}{\eta^m}, \quad m \geq 1,$$

(2.49)

where we used the delocalization of eigenvectors $\|u_\alpha\|_\infty \sim 1/\sqrt{N}$ from (2.2). Since $D^\beta G_{ab}$ is a polynomial of Green’s function entries, and $|G_{ij}|, |m_N| \sim 1$ from Theorem 2.1, (2.49) also implies for $m \geq 1$

$$(\partial^\beta \partial^m_z G_{ab}) \prod_{i=1}^m \partial^\beta z \prec \frac{1}{N^{m+1}} |\partial^m \partial^\beta z G_{ab}| \prec \frac{\text{Im}[m_N]}{(N\eta)^{m+1}}. \quad (2.50)$$

The second statement in (2.46) follows. For the bound of $\partial^\beta m_N$, we have

$$\partial^\beta m_N = \frac{1}{N} \sum_{i=1}^N \partial^\beta G_{ii} \prec \frac{\text{Im}[m_N]}{N\eta}.$$

Next for (2.47), the derivative $\partial^\beta m_N$ is a sum of terms in the form

$$(\partial^\beta \partial^m_z m_N) \prod_{i=1}^m (\partial^\beta z) \prec \frac{1}{N^m} |\partial^m \partial^\beta m_N|, \quad \beta + \sum_{i=1}^m \beta_i = \beta, \quad m \geq 1. \quad (2.51)$$

The first two terms in (2.47) correspond to the case when $m = 0$ and $m = 1, \beta' = 0$. When $m \geq 2$, using (2.50), we have that (2.51) is bounded by $O_\prec(\text{Im}[m_N]/(N\eta)^2)$. Next we estimate (2.51) for $m = 1$ and $|\beta'| \geq 1$,

$$\frac{1}{N} |D^\beta \partial_z m_N| = \frac{1}{N^2} |D^\beta \text{Tr} G^2| = \frac{1}{N^2} \sum_{\beta'_1 + \beta'_2 = \beta'} \left| \sum_{ij} D^\beta_1 G_{ij} D^\beta_2 G_{ij} \right|. \quad \left(2.52\right)$$

There are two cases, 1) $|\beta'_1|, |\beta'_2| \geq 1$, 2) one of $|\beta'_1|, |\beta'_2|$ is zero. In the first case, we have that both $D^\beta_1 G_{ij}, D^\beta_2 G_{ij}$ are sums of terms in the form $G_{ix_1}G_{x_2z_3} \cdots G_{x_2z_j}$ where $x_1, x_2, \cdots, x_{2\ell} \in m$. Then by the Ward identity (2.3), we have

$$\frac{1}{N^2} \sum_{ij} D^\beta_1 G_{ij} D^\beta_2 G_{ij} \leq \frac{1}{N^2} \sum_{ij} |D^\beta_1 G_{ij}|^2 \sum_{ij} |D^\beta_2 G_{ij}|^2 \prec \frac{\text{Im}[m_N]^2}{(N\eta)^2}.$$

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In the second case, say $|\beta'| = 0$. Since the $L_2$ norm of $G$ is bounded by $1/\eta$, we have

$$\frac{1}{N^2} \left| \sum_{ij} G_{ij} G_{ix_1} G_{x_2 x_3} \cdots G_{x_{2t}} \right| \leq \frac{1}{\eta N^2} \sqrt{\sum_i |G_{ix_1}|^2 \sum_j |G_{x_2 x_3} \cdots G_{x_{2t}}|^2} < \frac{\text{Im}[m_N]}{(N\eta)^2}.$$

This finishes the proof of (2.47). The claim (2.48) follows from (2.47).

**Proof of Proposition 2.8.** We first prove (2.15). The left hand side of (2.15) is a linear combination of terms (with bounded coefficients) in the form

$$\left( \frac{1}{N} + \frac{\text{Im}[m_N]}{N\eta} \right) I_1 I_2 \cdots I_t P^{r - 1 - t_1} \tilde{P}^{r - t_2}, \quad t_1 + t_2 = t,$$

where for $1 \leq s \leq t$, $I_s = \partial^{\beta_s} P$ or $I_s = \partial^{\beta_s} \tilde{P}$ with $|\beta_s| \geq 1$. Thanks to Proposition 2.14, $|\partial^{\beta} P| < \text{Im}[m_N]/N\eta$. Therefore we can conclude that

$$\left( \frac{1}{N} + \frac{\text{Im}[m_N]}{N\eta} \right) I_1 I_2 \cdots I_t P^{r - 1 - t_1} \tilde{P}^{r - t_2} \lesssim \sum_{s \geq 1} E \left[ \left( \frac{\text{Im}[m_N]}{N\eta} \right)^s |P|^{2r - s} \right] \ll \Phi_r.$$

This gives (2.15).

For (2.14), there are two cases: 1) $\sum_{e \in E(F)} (p_e - s_e) \geq 1$, 2) $\sum_{e \in E(F)} (p_e - s_e) = 0$. We first study the first case. As we will show later, the second case can be reduced to the first case. We recall the derivatives of $P$ from (2.48),

$$\partial^\beta P = (D^\beta m_N + (\partial_2 m_N) \partial^\beta z) P' + ((\partial^\beta z) m_N + \sum_{\ell=1}^{L} (\partial^\beta a_{2\ell}) m_{2\ell}^2) + O_{\infty} \left( \frac{\text{Im}[m_N]|P'| + \text{Im}[m_N]|}{(N\eta)^2} \right),$$

$$= (D^\beta m_N) P' + \sum_{\ell=1}^{L} (\partial^\beta a_{2\ell}) ((\partial_2 m_N P' + m_N) \partial_{a_{2\ell}} z + m_{2\ell}^2) + O_{\infty} \left( \frac{\text{Im}[m_N]|P'| + \text{Im}[m_N]|}{(N\eta)^2} \right).$$

(2.52)

We can rewrite (2.14) as a sum of terms in the form

$$\frac{1}{N^{r+2}} \sum_{ijm} \sum_{a}^{s} \text{I}_a \left[ G_{ij} I_1 I_2 \cdots I_t P^{r - 1 - t_1} \tilde{P}^{r - t_2} \right], \quad t_1 + t_2 = t,$$

(2.53)

where $I_s$ for $1 \leq s \leq t$ corresponds to terms in (2.52) defined in the following:

1. From Proposition 2.13, the first term in (2.52) $(D^\beta m_N) P'$ is a sum of terms in the form:

$$\frac{\text{Im}[m_N]|P'|}{N\eta} \sum_{a}^{s} G_{ij}^{a} X_{k, m_i} Y_{k, m_j}.$$

(2.54)

where $a_s \geq 1$ and $k_s$ is some index set, and $\sum_{k_s} |X_{k, m_i}|^2 \sum_{k_s} |Y_{k, m_j}|^2 = O(1)$. We take $I_s$ to be (2.54) or its complex conjugate.

2. From Proposition 2.13, the second term in (2.52) $\partial^\beta a_{2\ell} ((\partial_2 m_N P' + m_N) \partial_{a_{2\ell}} z + m_{2\ell}^2)$ is a sum of terms in the form:

$$\frac{(\partial_2 m_N P' + m_N) \partial_{a_{2\ell}} z + m_{2\ell}^2}{N} \sum_{k_s}^{d} h_{ij}^{a_{2\ell}} X_{k, m_i} Y_{k, m_j}.$$

(2.55)

$$= O \left( \frac{\text{Im}[m_N]|P'|}{N\eta} + \frac{1}{N} \right) \sum_{k_s}^{d} h_{ij}^{a_{2\ell}} X_{k, m_i} Y_{k, m_j}.$$
where \( a_s \geq 0, k_s \) is an index set, \( \sum_{k_s} |X^2_{k_s,m_i}|, \sum_{k_s} |Y^2_{k_s,m_j}| = O_\prec(1) \). We take \( I_s \) to be (2.55) or its complex conjugate.

3. \( I_s \) is bounded by \( O_\prec \left( \frac{\text{Im}[m_N]|(P'| + \text{Im}[m_N])}{(N\eta)^2} \right) \) corresponding to the last term in (2.48).

From the construction, we have \( |I_s| \prec \text{Im}[m_N]/N\eta \) for all \( 1 \leq s \leq t \). If there is one \( I_s \) corresponding to Item 1 with \( a_s \geq 1 \), then

\[
|I_s| \prec |G_{ij}| \frac{\text{Im}[m_N]|P'|}{N\eta}.
\]

And noticing \( |P'| \prec 1 \), we have

\[
\frac{1}{N^{1+2}} \sum_{ij} E[G_{ij} R_{ij,m} I_1 I_2 \cdots I_t P^{r-t_1-1} \bar{P}^{r-t_2}] \prec \frac{1}{N^2} \sum_{ij} E \left[ |G_{ij}|^2 \left( \frac{\text{Im}[m_N]}{N\eta} \right)^t |P^{2r-t_1-1}| \right] = E \left[ \left( \frac{\text{Im}[m_N]}{N\eta} \right)^{t+1} |P^{2r-t_1-1}| \right] \leq E[\Phi_r],
\]

where we used Ward identity (2.3).

If there is one \( I_s \) corresponding to Item 2 with \( a_s \geq 1 \), then

\[
|I_s| \prec \left( \frac{\text{Im}[m_N]|P'|}{N\eta} + \frac{1}{N} \right) |h_{ij}|^{a_s}.
\]

Then by the Cauchy-Schwartz inequality, we have

\[
\frac{1}{N^{1+2}} \sum_{ij} E[G_{ij} R_{ij,m} I_1 I_2 \cdots I_t P^{r-t_1-1} \bar{P}^{r-t_2}] \prec \frac{1}{N^2} \left( \frac{\text{Im}[m_N]|P'|}{N\eta} + \frac{1}{N} \right) \sum_{ij} E \left[ |G_{ij}| |h_{ij}|^{a_s} \left( \frac{\text{Im}[m_N]}{N\eta} \right)^{t-1} |P^{2r-t_1-1}| \right] \leq \frac{1}{N^{1+2}} \left( \frac{\text{Im}[m_N]|P'|}{N\eta} + \frac{1}{N} \right) \sum_{ij} E \left[ \frac{1}{N} \left( \frac{\text{Im}[m_N]}{N\eta} \right)^{t-1} |P^{2r-t_1-1}| \right] \leq E[\Phi_r],
\]

where in the last to second line we used

\[
\sum_{ij} |G_{ij}|^2 = \frac{N \text{Im}[m_N]}{\eta}, \quad \sum_{ij} |h_{ij}|^{2a_s} \prec N.
\]

If there is one \( I_s = O_\prec \left( \frac{\text{Im}[m_N]/|P'| + \text{Im}[m_N])}{(N\eta)^2} \right) \) as in Item 3, we have

\[
\frac{1}{N^{1+2}} \sum_{ij} E[G_{ij} R_{ij,m} I_1 I_2 \cdots I_t P^{r-t_1-1} \bar{P}^{r-t_2}] \prec E \left[ \frac{\text{Im}[m_N]|P'| + \text{Im}[m_N])}{(N\eta)^2} \left( \frac{\text{Im}[m_N]}{N\eta} \right)^{t-1} |P^{2r-t_1-1}| \right] \leq E[\Phi_r].
\]
In the rest, we can assume that each $I_s$ either corresponds to Item 1 with $a_s = 0$, or corresponds to Item 2 with $a_s = 0$. Say $I_1, I_2, \cdots, I_t$ correspond to Item 1 with $a_s = 0$, and $I_{t+1}, I_{t+2}, \cdots, I_t$ correspond to Item 2 with $a_s = 0$, where $t_3 + t_4 = t$. Then we have

$$
\frac{1}{N^{r+2}} \sum_{i,j,m} E[G_{ij} R_{ijm} I_1 I_2 \cdots I_t P^{r-t_1 - 1} \bar{P}^{r-t_2}] = \frac{1}{N^{r+2}} \sum_{k'} \sum_{k''} \sum_{i,j,m,k''} E \left[ G_{ij} X_{kmi} Y_{kmj} O \left( \frac{\text{Im}[mN]}{N\eta} + \frac{1}{N} \right)^t \right] P^{r-t_1 - 1} \bar{P}^{r-t_2} ,
$$

(2.56)

where $k = k' \cup k''$ with $k' = \cup_{1 \leq s \leq t_3} k_s$, $k'' = \cup_{t_3+1 \leq s \leq t} k_s$, and

$$X_{kmi} = \prod_s X_{k,mi}, \quad Y_{kmj} = \prod_s Y_{k,mj} .$$

They are bounded

$$
\sum_k |X_{kmi}|^2 = \sum_k \prod_s |X_{k,mi}| = \prod_s \sum_k |X_{k,mi}| < 1, \quad \sum_k |Y_{kmj}|^2 < 1 .
$$

(2.57)

Then, we use the norm of $G$ is bounded by $1/\eta$, and $|G_{ii}| < 1$,

$$
\left| \frac{1}{N^2} \sum_{i,j,m,k''} G_{ij} X_{kmi} Y_{kmj} \right| \leq \frac{1}{N^2} \sum_{i,m,k''} \sum_{j} G_{ij} X_{kmi} Y_{kmj} \
+ \frac{1}{N^2} \sum_{i,m,k''} G_{ii} X_{kmi} Y_{kmj} \leq \frac{1}{N^2 \eta} \sqrt{\sum_i |X_{kmi}|^2 \sum_j |Y_{kmj}|^2} .
$$

(2.58)

Further, using the Cauchy-Schwartz inequality, and (2.57) we have

$$
\frac{1}{N^r} \sum_{k'} \sum_{k''} \sum_{m,k''} \sum_{i,j} \frac{1}{N^2} \sum_{i,j} G_{ij} X_{kmi} Y_{kmj} \leq \frac{1}{N^r} \sum_{m} \frac{1}{N^2 \eta} \sum_{k''} \sqrt{\sum_i |X_{kmi}|^2 \sum_j |Y_{kmj}|^2} \
\leq \frac{1}{N^r} \sum_{m} \frac{1}{N^2 \eta} \sqrt{\sum_i |X_{kmi}|^2 \sum_j |Y_{kmj}|^2} \leq \frac{1}{N^r} - \frac{1}{N^2 \eta} \sum_{m} \sum_{k''} \sum_{j} |X_{kmi}|^2 |Y_{kmj}|^2 = \frac{1}{N^r \eta} ,
$$

(2.59)

where we also used that $|m| = r$. By plugging (2.59) into (2.56), we get

$$
\frac{1}{N^{r+2}} \sum_{i,j,m} E[G_{ij} R_{ijm} I_1 I_2 \cdots I_t P^{r-t_1 - 1} \bar{P}^{r-t_2}] \leq \frac{1}{N^r \eta} \left( \frac{\text{Im}[mN]}{N\eta} + \frac{1}{N} \right)^t |P|^{2r-t-1} \leq E[ \Phi_r] .
$$

This finishes the proof of the first case that $\sum_{e \in E(F)} (p_e - s_e) \geq 1$.

For the second case, when $\sum_{e \in E(F)} (p_e - s_e) = 0$. We use the identities $G_{ij} = \sum_{k \neq t} G_{ij} h_{ik} G_{kji}^{(i)}$, and $G_{kji}^{(i)} = G_{kj} - G_{ki} G_{ji} / G_{ii}$. We denote

$$U = R_{ijm} P^{r-t_1} \bar{P}^{r-t_2} .$$

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Then by the cumulant expansion, we have

\[
\frac{1}{N^{r+2}} \sum_{ijm} E[G_{ij}U] = \frac{1}{N^{r+2}} \sum_{ijm}^* E \left[ \sum_{k \neq i} h_{ik} G_{kj}^{(i)} G_{ii} U \right]
\]

\[
= \sum_{p=1}^M \frac{C_{p+1}}{N^{r+3q^{p-1}}} \sum_{ijm}^* E \left[ \sum_{k \neq i} \partial_{ik}^p (G_{kj}^{(i)} G_{ii} R_{ijm} P^{r-1} \bar{P}^r) \right] + O_\prec(\mathbb{E}[\Phi_r]) \tag{2.60}
\]

\[
= \sum_{p=1}^M \sum_s \frac{C_{p+1}}{N^{r+3q^{p-1}}} \left( \frac{p}{s} \right) \sum_{ijm}^* E \left[ \sum_{k \neq i} \partial_{ik}^p (G_{kj}^{(i)} G_{ii} R_{ijm}) \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) \right] + O_\prec(\mathbb{E}[\Phi_r])
\]

\[
= \sum_{p=1}^M \sum_s \frac{C_{p+1}}{N^{r+3q^{p-1}}} \left( \frac{p}{s} \right) \sum_{ijm} \sum_{k \neq i}^* D_{ik}^s (G_{kj}^{(i)} G_{ii} R_{ijm}) \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) + O_\prec(\mathbb{E}[\Phi_r]).
\]

where in the last line we replaced \( \partial_{ik}^p \) by \( D_{ik}^p \) using (2.46), and the error term can be bounded by \( O_\prec(\mathbb{E}[\Phi_r]) \) using (2.15). The sum of terms with \( k \in jm \) is also bounded

\[
\lesssim \frac{1}{N^{r+2}} \sum_{ijm} E \left[ \sum_{k \neq i} \frac{1}{N} \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) \right] \lesssim \frac{1}{N^{r+2}} \sum_{ijm}^* \mathbb{E}[\Phi_r] \lesssim \mathbb{E}[\Phi_r].
\]

Therefore, we can further restrict the summation in (2.60) to \( \sum_{ijkm}^* \). Since \( G_{kj}^{(i)} \) is independent of \( h_{ik} \), we can further rewrite (2.60) as

\[
= \sum_{p=1}^M \sum_s \frac{C_{p+1}}{N^{r+3q^{p-1}}} \left( \frac{p}{s} \right) \sum_{ijkm} E \left[ G_{kj}^{(i)} D_{ik}^s (G_{ii} R_{ijm}) \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) \right] + O_\prec(\mathbb{E}[\Phi_r])
\]

\[
= \sum_{p=1}^M \sum_{s} \frac{C_{p+1}}{N^{r+3q^{p-1}}} \left( \frac{p}{s} \right) \sum_{ijkm} E \left[ (G_{kj} - \frac{G_{ki} G_{ji}}{G_{ii}}) D_{ik}^s (G_{ii} R_{ijm}) \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) \right] + O_\prec(\mathbb{E}[\Phi_r])
\]

\[
= \sum_{p=1}^M \sum_s \frac{C_{p+1}}{N^{r+3q^{p-1}}} \left( \frac{p}{s} \right) \sum_{ijkm} E \left[ G_{kj} D_{ik}^s (G_{ii} R_{ijm}) \partial_{ik}^{p-s} (P^{r-1} \bar{P}^r) \right] + O_\prec(\mathbb{E}[\Phi_r]).
\]

Here for the last line we used that if there are two off-diagonal Green’s function entries, the sum is bounded by \( O_\prec(\mathbb{E}[\Phi_r]) \) using Ward identity (2.3) and (2.15). The derivative \( D_{ik}^s (G_{ii} R_{ijm}) \) is again a sum of monomials in Green’s function entries. If \( p > s \), the last line of (2.60) is in the form of (2.14) with \( \sum \rho_e (p_e - s_e) \geq 1 \). Thus, from the discussion of the first case, they are bounded by \( O_\prec(\mathbb{E}[\Phi_r]) \). The terms in (2.61) with \( p = s \) are given by

\[
\sum_{p=1}^M \sum_{s} \frac{C_{p+1}}{N^{r+3q^{p-1}}} \sum_{ijkm}^* E \left[ G_{kj} D_{ik}^p (G_{ii} R_{ijm}) (P^{r-1} \bar{P}^r) \right].
\]

The derivative \( D_{ik}^p (G_{ii} R_{ijm}) \) is a sum of monomials in Green’s function entries. For these monomials containing at least one off-diagonal Green’s function entry, the total number of off-diagonal Green’s function is at least two. The sum is bounded by \( O_\prec(\mathbb{E}[\Phi_r]) \). If there are monomials containing only diagonal Green’s function entries, it is necessary that \( p \geq 2 \) (\( D_{ik} G_{xx} = -2G_{xi} G_{xx} \) with \( x \in jm \) contains at least one off-diagonal Green’s function entry \( G_{kx} \)). These terms are again in the form of (2.14) with some extra \( 1/q \) factors. They are of higher order (recall from (2.16)). Then we can repeat the above
procedure, until all the terms have order bigger than $M$. Then we can trivially bound them by $1/q^M \prec \Phi_r$ as in (2.16).

\[\]

**Proof of Proposition 2.9.** By the definition of the Green’s function, we have

1. $1 = -zG_{jj} + (HG)_{jj}$, \hspace{1cm} (2.62)
2. $1 = -zG_{ii} + (HG)_{ii}$. \hspace{1cm} (2.63)

Multiplying (2.62) and (2.63) by $G_{ii}$ and $G_{jj}$ respectively, averaging over the indices, and then taking the difference, we get

\[G_{ii} = m_N + \frac{1}{N} \sum_{j=1}^{N} (G_{ii}(HG)_{jj} - m_N(HG)_{ii}).\] \hspace{1cm} (2.64)

We will use the above relation (2.64) to replace a copy of $G_{ii}$ on the lefthand side of (2.22) to $m_N$.

Denote

$U := R_m V$, $V := \prod_{e \in E(x)} \partial_{\alpha e}^{\alpha e - \beta e}(P_r - 1)\bar{P}_r$.

Then using (2.64), we can rewrite each term on the righthand side of (2.20) (up to some constant) as

\[
\frac{1}{N^{r+1}} \sum_{im}^* \sum_j^* E[G_{ii}^a U] = \frac{1}{N^{r+1}} \sum_{im}^* \sum_j^* E[m_N G_{ii}^{a-1} U] \\
+ \frac{1}{N^{r+2}} \sum_{im}^* \sum_j^* \sum_p^* E[(G_{ii}(HG)_{jj} - m_N(HG)_{ii})G_{ii}^{a-1} U].
\] \hspace{1cm} (2.65)

It turns out the second term on the righthand side of (2.65) is of order at least 1 (recall from (2.16)).

Using the cumulant expansion, we get

\[
\frac{1}{N^{r+2}} \sum_{im}^* \sum_j^* \sum_{jk}^* \sum_{p=1}^{M} \frac{C_p}{N^{q_p-1}} \sum_{s=1}^{p} E[\partial_{jk}^p (G_{ii}^a G_{jk} R_m) \partial_{jk}^{p-s} V] - \partial_{ik}^p (m_N G_{ik} G_{ii}^{a-1} R_m) \partial_{ik}^{p-s} V)] + O \prec \left(\frac{1}{q^M}\right)
\] \hspace{1cm} (2.66)

For the first term on the righthand side of (2.66), using Proposition 2.8, we can replace $\partial_{jk}^p$ by $D_{jk}^s$, $\partial_{jk}^p (G_{ii}^a G_{jk} R_m) = D_{jk}^s (G_{ii}^a G_{jk} R_m) + O(\text{Im}[m_N]/N^\eta)$. Thanks to (2.15), the error term is bounded by $O_\prec(\mathbb{E}[\tilde{\Phi}_r])$.

\[
\frac{1}{N^{r+2}} \sum_{im}^* \sum_{jk}^* \sum_{p=1}^{M} \frac{C_p}{N^{q_p-1}} \left(\frac{\text{Im}[m_N]}{N^\eta}\right) |\partial_{jk}^{p-s} V| \prec \Phi_r.
\]
Using (2.15) again, for the sum when \( j = k, j \in im, \) or \( k \in im, \) the sum is bounded by

\[
\frac{1}{{\cal{N}}^r+2} \sum_{ijkm} \sum_{p=1}^{+1} \sum_{s=1}^{Nq^p-1} \left( \begin{array}{c} p \\ s \end{array} \right) E[D_{jk}^s (G_{ii}^a G_{jk} R_m)] \leq \Phi_r. 
\]

Thus we can restrict the summation to the case that \( i j k m \) are distinct. Finally, using Proposition 2.8, terms in \( D_{jk}^s (G_{ii}^a G_{jk} R_m) \) with at least one off-diagonal Green’s function entries can be bounded by \( O_\prec (E[\varphi_r]), \)

\[
D_{jk}^s (G_{ii}^a G_{jk} R_m) = -1 (s \text{ is odd}) s! G_{ii}^a G_{jj}^{\frac{r+1}{2}} G_{kk}^{\frac{r+1}{2}} R_m + \{\text{terms with off-diagonal entries}\}.
\]

Therefore, the leading terms in first term on the righthand side of (2.66) are those which do not contain any off-diagonal Green’s function terms,

\[
\frac{1}{{\cal{N}}^r+2} \sum_{ijkm} \sum_{p=1}^{+1} \sum_{s=1}^{Nq^p-1} \left( \begin{array}{c} p \\ s \end{array} \right) E[D_{jk}^s (G_{ii}^a G_{jk} R_m)] 
\]

\[
= -\frac{1}{{\cal{N}}^r+2} \sum_{ijkm} \sum_{p=1}^{+1} \sum_{s=1}^{Nq^p-1} \left( \begin{array}{c} p \\ s \end{array} \right) E[G_{ii}^a G_{jj}^{\frac{r+1}{2}} G_{kk}^{\frac{r+1}{2}} R_m] + O_\prec (E[\varphi_r]).
\]

Similarly for the second term on the righthand side of (2.66), we have

\[
\frac{1}{{\cal{N}}^r+1} \sum_{ikm} \sum_{p=1}^{+1} \sum_{s=1}^{Nq^p-1} \left( \begin{array}{c} p \\ s \end{array} \right) E[D_{ik}^s (m_N G_{ik} G_{ii}^a R_m)] 
\]

\[
= \frac{1}{{\cal{N}}^r+1} \sum_{ikm} \sum_{p=1}^{+1} \sum_{s=1}^{Nq^p-1} \left( \begin{array}{c} p \\ s \end{array} \right) s^\alpha (\frac{s+1}{2} - \frac{s-1}{2}) E[m_N G_{ii}^a G_{jk}^{\frac{r+1}{2}} G_{kk}^{\frac{r+1}{2}} R_m] + O_\prec (E[\varphi_r]),
\]

where we used that

\[
D_{ik}^s (G_{ik} G_{ii}^a R_m) = -1 (s \text{ is odd}) s! (\frac{s+1}{2} - \frac{s-1}{2}) G_{ii}^a G_{jk}^{\frac{r+1}{2}} G_{kk}^{\frac{r+1}{2}} + \{\text{terms with diagonal entries}\}.
\]

By comparing (2.67) and (2.68), the terms corresponding to \( p = 1, s = 1 \) cancel out:

\[
-\frac{1}{{\cal{N}}^r+2} \sum_{ijkm} C_2 E[G_{ii}^a G_{jk} R_m V] = -\frac{1}{{\cal{N}}^r+2} \sum_{ikm} C_2 E[m_N G_{ik} G_{ii}^a R_m V] + O_\prec (E[\varphi_r])
\]

\[
= -\frac{1}{{\cal{N}}^r+2} \sum_{ikm} C_2 E[m_N G_{ik} G_{jk} R_m V] + O_\prec (E[\varphi_r]).
\]

Then the claim (2.23) follows from combining (2.67) and (2.68).

\[
\square
\]

2.5 Proof of Theorem 1.6

In this section we prove Theorem 1.6 by analyzing the high order moment estimates of \( P(z, m_N(z)) \) from Proposition 2.6. We recall the shifted spectral domain \( D \) from (2.9), and the following Proposition from [31, Proposition 2.11].

Proposition 2.15. There exists a constant \( \varepsilon > 0 \) such that the following holds. Suppose that \( \delta : D \to \mathbb{R} \) \( (D \text{ is as defined in (2.9)}) \) is a function so that

\[
|P(\tilde{L} + w, m_N(\tilde{L} + w))| \leq \delta(w), \quad w \in D.
\]
Suppose that $N^{-2} \leq \delta(w) \ll 1$ for $w \in D$, that $\delta$ is Lipschitz continuous with Lipschitz constant $N$ and moreover that for each fixed $\kappa$ the function $\eta \mapsto \delta(\kappa + i\eta)$ is nonincreasing for $\Re \eta > 0$. Then,

$$|m_N(\bar{\kappa} + w) - \bar{\kappa}m(\bar{\kappa} + w)| = O \left( \frac{\delta(w)}{\sqrt{|\kappa| + \eta + \delta(w)}} \right),$$

where $\bar{\kappa}$ is from Proposition 2.4. The implicit constant is independent of $N$.

Before proving Theorem 1.6, we first prove a weaker estimate.

**Proposition 2.16.** Let $H$ be as in Definition 1.1 with $N^\varepsilon \leq q \lesssim N^{1/2}$. Let $m_N(z)$ be the Stieltjes transform of its eigenvalue density, and $\bar{m}(z)$ as defined in Proposition 2.4. Uniformly for any $z = \bar{\kappa} + w, w = \kappa + i\eta \in D$ as defined in (2.9), we have

$$|m_N(z) - \bar{m}(z)| \ll \sqrt{|\kappa| + \eta}.$$

**Proof.** By Proposition 2.4, we have

$$\Im[\bar{m}(z)] \approx \Phi(w) := \left\{ \begin{array}{ll} \sqrt{|\kappa| + \eta}, & \kappa \leq 0, \\ \eta/\sqrt{|\kappa| + \eta}, & \kappa \geq 0. \end{array} \right.$$

and

$$|\partial_z P(z, \bar{m}(z))| \ll \sqrt{|\kappa| + \eta}.$$

We denote

$$\Lambda_N(w) := |m_N(\bar{\kappa} + w) - \bar{\kappa}m(\bar{\kappa} + w)|.$$

Then we have

$$\Im[m_N(z)] \lesssim \Phi(w) + \Lambda_N(w),$$

and by Proposition 2.4

$$\partial_z P(z, m(z)) = \partial_z P(z, \bar{m}(z)) + O(|m_N(z) - \bar{m}(z)|) = O(\sqrt{|\kappa| + \eta} + \Lambda_N(w)).$$

By Hölder’s inequality we obtain from Proposition 2.6,

$$\mathbb{E}[|P(z, m_N(z))|^{2r}] \leq \frac{1}{(N\eta)^{2r}} \mathbb{E} \left[ \Lambda_N(w)^{2r} + (|\kappa| + \eta)^{r/2}(\Phi(w)^r + \Lambda_N(w)^r) \right].$$

With overwhelming probability we have the following Taylor expansion,

$$P(z, m_N(z)) = P(z, \bar{m}(z)) + \partial_z P(z, \bar{m}(z))(m_N(z) - \bar{m}(z)) + \frac{\partial^2_z P(z, \bar{m}(z))}{2}(m_N(z) - \bar{m}(z))^2 + O(1) + (m_N(z) - \bar{m}(z))^2,$$

where we used that $\partial^2_z P(z, \bar{m}(z)) = 2 + O(1/q)$ and $\Lambda_N(w) \ll 1$ with overwhelming probability. Rearranging the last equation and using the definition of $\Lambda_N(w)$, we have arrived at

$$\Lambda_N(w)^2 \ll \Lambda_N(w)(\sqrt{|\kappa| + \eta}) + |P(z, m_N(z))|,$$

and thus

$$\mathbb{E}[\Lambda_N(w)^{4r}] \lesssim (|\kappa| + \eta)^r \mathbb{E}[\Lambda_N(w)^{2r}] + \mathbb{E}[|P(z, m_N(z))|^{2r}] + \mathbb{E}[|P(z, m_N(z))|^{2r}] + \mathbb{E}[\Lambda_N(w)^{4r}] \ll (|\kappa| + \eta)^{2r},$$

On the domain $D$, we have $1/N\eta \leq \sqrt{|\kappa| + \eta}$. We replace $\mathbb{E}[|P(z, m_N(z))|^{2r}]$ in (2.72) by (2.69), and get

$$\mathbb{E}[\Lambda_N(w)^{4r}] \ll (|\kappa| + \eta)^{2r},$$

It follows from Markov’s inequality that $\Lambda_N(w) \ll \sqrt{|\kappa| + \eta}$. \qed
**Proof of Theorem 1.6.** We assume that there exists some deterministic control parameter \( \Lambda(w) \) such that the prior estimate holds

\[
|m_N(L + w) - \tilde{m}(\tilde{L} + w)| < \Lambda(w) \lesssim \sqrt{|\kappa| + \eta}.
\]

Since \( \Phi(w) \gtrsim \sqrt{|\kappa| + \eta} \) and \( \Lambda(w) \lesssim \sqrt{|\kappa| + \eta} \) from Proposition 2.16, (2.69) combining with Markov’s inequality leads to

\[
|P(z, m_N(z))| < \frac{1}{N\eta} \left( (\Lambda(w) + \Phi(w))\sqrt{|\kappa| + \eta} \right)^{1/2}.
\]  
(2.73)

If \( \kappa \geq 0 \), then \( \Phi(w) = \eta/\sqrt{|\kappa| + \eta} \), and (2.73) simplifies to

\[
|P(z, m_N(z))| < \frac{1}{N\eta^{1/2}} + \frac{(|\kappa| + \eta)^{1/4}\Lambda(w)^{1/2}}{N\eta}.
\]  
(2.74)

Thanks to Proposition 2.15, by taking \( \delta(w) \) the righthand side of (2.74) times \( N^\varepsilon \) with arbitrarily small \( \varepsilon \), we have

\[
|m_N(z) - \tilde{m}(z)| < \frac{1}{\sqrt{|\kappa| + \eta}} \left( \frac{1}{N\eta^{1/2}} + \frac{(|\kappa| + \eta)^{1/4}\Lambda(w)^{1/2}}{N\eta} \right).
\]  
(2.75)

By iterating (2.75), we get

\[
|m_N(z) - \tilde{m}(z)| < \frac{1}{\sqrt{|\kappa| + \eta}} \left( \frac{1}{N\eta^{1/2}} + \frac{1}{(N\eta)^{2}} \right).
\]  
(2.76)

If \( \kappa \leq 0 \), then \( \Phi(w) = \sqrt{|\kappa| + \eta} \) and \( \Lambda(w) \lesssim \sqrt{|\kappa| + \eta} \), (2.73) simplifies to

\[
|P(z, m_N(z))| < \frac{(|\kappa| + \eta)^{1/2}}{N\eta}.
\]  
(2.77)

It follows from Proposition 2.15, by taking \( \delta(z) \) the righthand side of (2.77) times \( N^\varepsilon \) with arbitrarily small \( \varepsilon \), we have

\[
|m_N(z) - \tilde{m}(z)| < \frac{1}{N\eta}.
\]  
(2.78)

The claim (1.4) follows from the estimates of the Stieltjes transform (2.76) and (2.78), see [22, Section 11].

\[\square\]

3 **Edge statistics of \( H(t) \)**

Let \( H \) be as in Definition 1.1. In this section we consider the Gaussian divisible ensemble

\[
H(t) := e^{-t/2}H + (1 - e^{-t})^{1/2}W,
\]  
(3.1)

where \( H(0) = H \) and \( W \) is an independent GOE matrix. We denote the eigenvalues of \( H(t) \) as \( \lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t) \), and the Stieltjes transform of its empirical eigenvalue distribution as

\[
m_t(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(t) - z}, \quad m_0(z) = m_N(z).
\]
Conditioning on $H$, the matrix ensemble $H(t)$ has the same law as the matrix Brownian motion starting from $H$ with each entry given by an Ornstein–Uhlenbeck process. The dynamic of the eigenvalues of the matrix Brownian motion is given by Dyson’s Brownian motion

$$d\lambda_i(t) = \frac{d B_i(t)}{\sqrt{N}} + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)} - \frac{\lambda_i(t)}{2} dt,$$

where for one time slice, $(\tilde{\lambda}_1(t), \tilde{\lambda}_2(t), \cdots, \tilde{\lambda}_N(t))$ has the same law as $(\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t))$.

For sufficiently regular initial data, it has been proven in [38], after short time the eigenvalue statistics at the spectral edge of (3.2) agree with GOE. A modified version of this theorem was proven in [1], which assumes that the initial data is sufficiently close to a nice profile. To use these results, we need to restrict $H$ to a subset, on which the optimal rigidity holds. We denote $A$ to be the set of sparse random matrices $H$, such that (2.76) and (2.78) hold at edges $\pm \tilde{L}$:

$$A := \{H: (2.76) \text{ and } (2.78) \text{ hold}\}.$$

By Theorem 1.6, we know that the event $A$ holds with probability $\mathbb{P}(A) \gg 1 - N^{-D}$ for any $D \geq 0$.

We denote $\rho_{sc}(x)$ the semicircle law which is the limit eigenvalue density of a Gaussian orthogonal ensemble $W$. The limit eigenvalue density of $(1 - e^{-t})^{1/2} W$, is given by $(1 - e^{-t})^{-1/2} \rho_{sc}((1 - e^{-t})^{-1/2} x)$, and the empirical eigenvalue distribution of $e^{-t/2} H$ concentrates around $e^{-t/2} \tilde{\rho}(e^{t/2} x)$ (as defined in Proposition 2.4). We denote the free convolution of $(1 - e^{-t})^{-1/2} \rho_{sc}((1 - e^{-t})^{-1/2} x)$ and $e^{-t/2} \tilde{\rho}(e^{-t/2} x)$ by $\hat{\rho}_t$, and its Stieltjes transform by $\tilde{m}_t$. Then $\tilde{m}_t$ satisfies the functional equations

$$e^{-t/2} \tilde{m}_t(z) = \int \frac{\tilde{\rho}(x) dx}{x - \xi_t(z)} = \tilde{m}(\xi_t(z)), \quad \xi_t(z) := e^{t/2} z + e^{t/2}(1 - e^{-t}) \tilde{m}_t(z).$$

By the definition we have $\tilde{m} = \tilde{m}_0$. Recall from Proposition 2.4, $\tilde{m}_0(z)$ satisfies the functional equation

$$1 + z \tilde{m}_0(z) + \tilde{m}_0(z)^2 + Q(\tilde{m}_0(z)) = 0,$$

The next proposition states that $\tilde{m}_t$ satisfies a similar equation

**Proposition 3.1.** Adapt the assumptions in Theorem 1.6, and recall $\tilde{m}_t(z)$ from (3.4). It is the Stieltjes transform of a measure $\hat{\rho}_t$, which is the free convolution of $(1 - e^{-t})^{-1/2} \rho_{sc}((1 - e^{-t})^{-1/2} x)$ and $e^{-t/2} \tilde{\rho}(e^{-t/2} x)$ by $\hat{\rho}_t$. The measure $\hat{\rho}_t$ is symmetric and supported on $[-\tilde{L}_t, \tilde{L}_t]$. Moreover, $\tilde{m}_t(z)$ satisfies the following equation

$$1 + z \tilde{m}_t(z) + \tilde{m}_t(z)^2 + Q(e^{-t/2} \tilde{m}_t(z)) = 0.$$

**Proof.** By taking $z$ to be $\xi_t(z)$ in (3.5), and using the relation (3.4)

$$1 + \xi_t(z) e^{-t/2} \tilde{m}_t(z) + e^{-t} \tilde{m}_t^2(z) + Q(e^{-t/2} \tilde{m}_t(z)) = 0.$$

From the definition of $\xi_t(z)$, we have

$$\xi_t(z) e^{-t/2} + (e^{-t} - 1) \tilde{m}_t(z) = z,$$

and (3.6) simplifies to

$$1 + z \tilde{m}_t(z) + \tilde{m}_t^2(z) + Q(e^{-t/2} \tilde{m}_t(z)) = 0.$$

For any $t \geq 0$, The same argument as for Proposition 2.4, $\tilde{m}_t(z)$ is the Stieltjes transform of a measure $\hat{\rho}_t$, which is symmetric and supported on $[-\tilde{L}_t, \tilde{L}_t]$. Moreover, it has square root behavior.
We remark that $Q$ is a random polynomial which depends on certain averaged quantities of $h_{ij}$, so $\tilde{L}_t$ is also random. But once we condition on $H$, both of them are deterministic. Next we prove the following theorem. It states that for time $t \gg N^{-1/3}$ the fluctuations of extreme eigenvalues of $H(t)$ conditioning on $H(0) \in \mathcal{A}$ as in (3.3) are given by the Tracy-Widom distribution.

**Theorem 3.2.** Let $H$ be as in Definition 1.1 with $N^\varepsilon \leq q \lesssim N^{1/2}$. Conditioning on $H \in \mathcal{A}$ as in (3.3), let $H(t)$ be as in (3.1), with eigenvalues denoted by $\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)$, and $t = N^{-1/3+\varepsilon}$. Let $k \geq 1$ and $F : \mathbb{R}^k \to \mathbb{R}$ be a bounded test function with bounded derivatives. There is a universal constant $\epsilon > 0$ depending on $\mathcal{A}$, it holds

$$\mathbb{E}_{\tilde{H}}[F(N^{2/3}(\lambda_1(t) - \tilde{L}_t), \ldots, N^{2/3}(\lambda_k(t) - \tilde{L}_t)) | H]$$

$$= \mathbb{E}_{\text{GOE}}[F(N^{2/3}(\mu_1 - 2), \ldots, N^{2/3}(\mu_k - 2))] + O(N^{-\varepsilon}),$$

where the expectation on the righthand side is with respect to a GOE matrix with eigenvalues denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$.

**Proof.** Take $\eta_\kappa = N^{-2/3+\beta/2}$, and $z = \tilde{L} + w$, where $\tilde{L}$ is from Proposition 2.4 and $w = \kappa + i \eta \in \mathcal{D}$ from (2.9). For any $H \in \mathcal{A}$, from the defining relations of $\mathcal{A}$, i.e. (2.76) and (2.78), we have

$$|m_N(z) - \tilde{m}(z)| \prec \frac{1}{N^\eta},$$

for $0 \leq \kappa \leq 1$ and $\eta_\kappa \leq \eta \leq 1$, and

$$|m_N(z) - \tilde{m}(z)| \prec \frac{1}{N^{\eta_1/2} \sqrt{\kappa} + \eta} + \frac{1}{(N\eta)^2},$$

for $-1 \leq \kappa \leq 0$ and $\eta_\kappa \leq \eta \leq 1$. Moreover, (2.76) also implies (1.4) such that $\lambda_1(0) - \tilde{L} \leq N^{-2/3+\beta/2}$. Hence, $H$ is $\eta_\kappa$-regular in the sense of [1, Assumption 4.1], and the result of [1, Theorem 6.1] applies for $t = N^{-1/3+\varepsilon}$ as above. This result gives the limiting distribution of the extreme eigenvalues of $H(t)$, and Theorem 3.2 follows.

It was also proven in [1, Proposition 4.6] that the Stieltjes transform $m_t(z)$ concentrates around $\tilde{m}_t(z)$. We collect the result in the following Proposition, which will be used in the next Section.

**Proposition 3.3.** Adapt the assumptions in Theorem 3.2. Conditioning on $H \in \mathcal{A}$ as in (3.3), let $H(t)$ be as in (3.1), with Stieltjes transform $m_t(z)$. Then for any $0 \leq \kappa \leq (\log N)^{-3}$, the following holds uniformly for $z = \tilde{L} + w$, with $w \in \mathcal{D}$ from (2.9)

$$|m_t(z) - \tilde{m}_t(z)| \prec \frac{1}{N \text{Im}|w|}.$$

4 Comparison

We recall $H(t)$ from (3.1). We denote the Stieltjes transform of its empirical eigenvalue density as $m_t(z)$, and the Stieltjes transform of the eigenvalue density of $H$ as $m_0(z) = m_N(z)$. In this section we prove the following theorem, which states that for $t \ll N^{-1/3}q$, the rescaled extreme eigenvalues of $H$ and $H(t)$ have the same distribution. Then Theorem 1.7 follows from combining Theorem 3.2 and Theorem 4.1.

**Theorem 4.1.** Let $H$ be as in Definition 1.1 with $N^\varepsilon \leq q \lesssim N^{1/2}$, and $H(t)$ be as in (3.1), with eigenvalues denoted by $\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)$, and $t = N^{-1/3+\varepsilon}$ with $\varepsilon \leq 2/20$. Fix $k \geq 1$ and numbers $s_1, s_2, \ldots, s_k$, there is a universal constant $\epsilon > 0$ so that,

$$\mathbb{P}_H \left( N^{2/3}(\lambda_i(0) - \tilde{L}) \geq s_i, 1 \leq i \leq k \right)$$

$$= \mathbb{P}_{H(t)} \left( N^{2/3}(\lambda_i(t) - \tilde{L}_t) \geq s_i, 1 \leq i \leq k \right) + O(N^{-\varepsilon}),$$

(4.1)
where \( \tilde{L}_t \) is as defined in Proposition 3.1. The analogous statement holds for the smallest eigenvalues.

Theorem 4.1 is a consequence of the following Green’s function comparison result.

**Proposition 4.2.** Adapt the assumptions in Theorem 4.1. We fix \( c > 0, E_1, E_2, \cdots, E_k = O(N^{-2/3}) \), \( \eta_0 = N^{-2/3-c} \) and \( F : \mathbb{R}^k \rightarrow \mathbb{R} \) a bounded test function with bounded derivatives. For \( t \ll 1 \) we have

\[
\mathbb{E}_H \left[ F \left( \left\{ \text{Im} \left[ N \int_{E_i}^{N^{-2/3+c}} m_N(\tilde{L} + y + i\eta_0) \right] \right\}_i \right) \right] = \mathbb{E}_{H(t)} \left[ F \left( \left\{ \text{Im} \left[ N \int_{E_i}^{N^{-2/3+c}} m_t(\tilde{L}_t + y + i\eta_0) \right] \right\}_i \right) \right] + O(N^{10c} \left( N^{1/3} \frac{1}{q} \right)).
\]

(4.2)

Next we prove Theorem 4.1 using Proposition 4.2 as an input. The proof of Proposition 4.2 will occupy the remaining of this section.

**Proof of Theorem 4.1.** We need to first introduce some notations. For any \( E \in \mathbb{R} \), we define

\[
\mathcal{N}_t(E) := | \{ i : \lambda_i(t) \geq \tilde{L}_t + E \} |
\]

and we write \( \mathcal{N}_0(E) \) as \( \mathcal{N}(E) \). We fix \( c > 0 \), and take \( \ell = N^{-2/3-c/3} \) and \( \eta_0 = N^{-2/3-c} \). Both are smaller than \( N^{-2/3} \). Then with overwhelming probability, from (1.4), we have that \( \lambda_i(t) \leq \tilde{L}_t + N^{-2/3+c} \). We define:

\[
\chi_E(x) = 1_{E,N^{-2/3+c}}(x - \tilde{L}_t), \quad \theta_\eta(x) := \frac{\eta}{\pi(x^2 + \eta^2)} = \frac{1}{\pi} \frac{\text{Im} \frac{1}{x+i\eta}}.
\]

From the same argument as in [35, Lemma 2.7], we get that

\[
\text{Tr}(\chi_{E+\ell} * \theta_\eta)(H(t)) - N^{-\epsilon/9} \leq \mathcal{N}_t(E) \leq \text{Tr}(\chi_{E-\ell} * \theta_\eta)(H(t)) + N^{-\epsilon/9},
\]

(4.3)

hold with overwhelming probability. Let \( K_i : \mathbb{R} \rightarrow [0,1] \) be a monotonic smooth function satisfying,

\[
K_i(x) = \begin{cases} 0 & x \leq i - 2/3, \\ 1 & x \geq i - 1/3. \end{cases}
\]

We have that \( 1_{\mathcal{N}_t(E) \geq i} = K_i(\mathcal{N}_t(E)) \), and since \( K_i \) is monotonically increasing, and so

\[
K_i\left( \text{Tr}(\chi_{E+\ell} * \theta_\eta)(H(t)) \right) + O(N^{-\epsilon/9}) \leq 1_{\mathcal{N}_t(E) \geq i} \leq K_i\left( \text{Tr}(\chi_{E-\ell} * \theta_\eta)(H(t)) \right) + O(N^{-\epsilon/9}),
\]

(4.4)

In this way we can express the locations of eigenvalues in terms of the integrals of the Stieltjes transform of the empirical eigenvalue densities. We have,

\[
\mathbb{E}_{H(t)} \left[ \prod_{i=1}^k K_i \left( \text{Im} \left[ \frac{N}{\pi} \int_{s_i, N^{-2/3+\ell}} m_t(\tilde{L}_t + y + i\eta) \right] d\eta \right) \right] + O\left( N^{-\epsilon/9} \right)
\]

\[
\leq \mathbb{P}_{H(t)} \left( N^{2/3}(\lambda_i(t) - \tilde{L}_t) \geq s_i, 1 \leq i \leq k \right) = \mathbb{E} \left[ \prod_{i=1}^k 1_{\mathcal{N}_t(s_i, N^{-2/3}) \geq i} \right]
\]

\[
\leq \mathbb{E}_{H(t)} \left[ \prod_{i=1}^k K_i \left( \text{Im} \left[ \frac{N}{\pi} \int_{s_i, N^{-2/3-\ell}} m_t(\tilde{L}_t + y + i\eta) \right] d\eta \right) \right] + O\left( N^{-\epsilon/9} \right).
\]
Since \( q \geq N^{\varepsilon} \) and \( t = N^{-1/3+\delta} \), we can take \( \varepsilon \) and \( \delta \) smaller than \( \varepsilon/20 \), and then the error terms in (4.2) are of order \( O(N^{-\varepsilon}) \). By combining (4.4) and (4.2), we get

\[
\begin{align*}
&\leq \mathbb{P}_H(t) \left( N^{2/3} (\lambda_i(t) - \bar{L}_i) \geq s_i + 2N^{2/3} \ell, 1 \leq i \leq k \right) + O(N^{-\varepsilon/9}) \\
&\leq \mathbb{E}_H(t) \left[ \prod_{i=1}^{k} K_i \left( \operatorname{Im} \left[ \frac{N}{\pi} \int_{L_i + y + i\eta}^{N^{-2/3+\varepsilon}} m_i(y) \, dy \right] \right) \right] + O(N^{-\varepsilon/9}) \\
&\leq \mathbb{P}_H \left( N^{2/3} (\lambda_i(0) - \bar{L}_i) \geq s_i, 1 \leq i \leq k \right) \\
&\leq \mathbb{E}_H(t) \left[ \prod_{i=1}^{k} K_i \left( \operatorname{Im} \left[ \frac{N}{\pi} \int_{L_i + y + i\eta}^{N^{-2/3+\varepsilon}} m_i(y) \, dy \right] \right) \right] + O(N^{-\varepsilon/9}) \\
&\leq \mathbb{P}_H(t) \left( N^{2/3} (\lambda_i(t) - \bar{L}_i) \geq s_i - 2N^{2/3} \ell, 1 \leq i \leq k \right) + O(N^{-\varepsilon/9}).
\end{align*}
\]

Since \( N^{2/3} \ell = N^{-\varepsilon/3} \ll 1 \), (4.1) follows. \hfill \Box

For simplicity of notation we only prove Proposition 4.2 in the case \( k = 1 \). The general case can be proved in the same way. Let,

\[
X_t := X_t(H(t), \bar{L}_t) = \operatorname{Im} \left[ N \int_{E}^{N^{-2/3+\varepsilon}} m_i(\bar{L}_i + y + i\eta_0) \, dy \right].
\]

We prove the \( k = 1 \) case of (4.2)

\[
|\mathbb{E}[F(X_t)] - \mathbb{E}[F(X_0)]| \leq N^{10\varepsilon} \left( \frac{N^{1/3}t}{q} \right). \tag{4.5}
\]

In the rest of this section, we recall \( H(t) \) from (3.1)

\[
H(t) := e^{-t/2} H + \left(1 - e^{-t}\right)^{1/2} W. \tag{4.6}
\]

We denote the Green’s function of \( H(t) \) by \( G(z; t) = (H(t) - z)^{-1} \). If the context is clear, we will simply write \( G(z; t) \) as \( G \) or \( G(z) \). We write the derivatives \( \partial_{ij} = \partial_{h_{ij}} \). For the remaining part of this section, we will take \( z = \bar{L}_i + w \), with \( w = y + i\eta \), \( |y| \leq N^{-2/3+\varepsilon} \) and \( \eta \geq N^{-2/3-\varepsilon} \). Then \( z \) depends on \( h_{ij} \) through \( \bar{L}_i \). From Theorem 2.1, we have that \( |G_{ij}(z, t)| \ll 1 \). Both \( z \) and \( \bar{L}_i \) are independent of \( W \). The derivative \( \partial_{ij} \) in \( \partial_{ij} G(z; t) \) may hit \( G \) or \( z \). We introduce the notation \( D_{ij} G(z; t) := \partial_{h_{ij}(t)} G(z; t) = -G(z; t) (E_{ij} + E_{ji}) G(z; t) \), where the derivative does not hit \( z \), and \( E_{ij} \) is the \( N \times N \) matrix whose \((i, j)\)-th entry is one and other entries are zero. With this notation, we have

\[
\partial_{ij} G(z; t) = \partial_{h_{ij}} G(z; t) = e^{-t/2} D_{ij} G(z; t) + (\partial_{h_{ij}} \bar{L}_i) \partial_z G(z; t).
\]

In the rest of this section, we prove the following proposition on the time derivative of \( \mathbb{E}[F(X_t)] \). The claim (4.5) follows from plugging (4.8) into (4.7) and integrating from time 0 to \( t \).

**Proposition 4.3.** Adapt the assumptions in Proposition 4.2. Let \( w = y + i\eta_0 \) with \( \eta_0 = N^{-2/3-\varepsilon} \) and \( z = \bar{L}_i + w \), we have the following estimates

\[
\frac{d}{dt} \mathbb{E}[F(X_t)] = \sum_{p=2}^{M} \frac{e^{-t/2} C_p}{2N^{q-1}} \sum_{ij} \mathbb{E}[\partial_{ij}^p G_{ij}(z; t) F'(X_t)] + \frac{N}{2} \mathbb{E}[Q(e^{-t/2} m_t(z)) F'(X_t)] \bigg|_{y = E} \tag{4.7}
\]

\[
+ O\left( \frac{N^{10\varepsilon+1/3}}{q} \right),
\]

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and uniformly for any $|y| \leq N^{-2/3+\epsilon}$,  
\[
\frac{1}{N} \sum_{p=2}^{M} e^{-t/2}C_{p} N^{q-1} \sum_{ij} \mathbb{E}[\partial_{ij}^{p}(G_{ij}(z; t)F'(X_{t}))] + \mathbb{E}[Q(e^{-t/2}m_{t}(z))F'(X_{t})] \propto \frac{N^{8\epsilon}}{N^{2/3}q}. \tag{4.8}
\]

### 4.1 Proof of (4.7)

We compute the time derivative of $\mathbb{E}[F(X_{t})]$  
\[
\frac{d}{dt} \mathbb{E}[F(X_{t})] = \mathbb{E} \left[ F'(X_{t}) \frac{dX_{t}}{dt} \right] = \mathbb{E} \left[ F'(X_{t}) \operatorname{Im} \int_{E} \left( \sum_{ijk} \hat{h}_{ij}(t)D_{ij}G_{kk}(z, t) + N\partial_{t}\tilde{L}_{t}\partial_{z}m_{t}(z) \right) \bigg|_{z=\tilde{L}_{t}+y+im_{0}} \right], \tag{4.9}
\]
where from the definition (4.6) of $H(t)$,  
\[
\hat{h}_{ij}(t) = \frac{1}{2} e^{-t/2} h_{ij} + \frac{e^{-t}}{2\sqrt{1-e^{-t}}} w_{ij}. 
\]

The key to understand the righthand side of (4.9) is to compute the time derivative of $\tilde{L}_{t}$, which is given by the following Proposition.

**Proposition 4.4.** Adapt the assumptions in Proposition 4.2. We have the following estimate with high probability, uniformly for any $w = y + \eta_{0}$ with $|y| \leq N^{-2/3+\epsilon}$, $\eta_{0} = N^{-2/3-\epsilon}$  
\[
\left| \partial_{t}\tilde{L}_{t} - \frac{1}{2} \partial_{m}Q(e^{-t/2}m_{t}(\tilde{L}_{t} + w)) \right| \propto \frac{N^{4\epsilon}}{N^{1/3}q}. \tag{4.10}
\]

where the polynomial $Q(m)$ is from (2.6), and $\tilde{L}_{t}$ is constructed in Proposition 3.1.

Before proving Proposition 4.4, we first state some useful estimates, which will be used repeatedly in the rest of this section. Their proofs are postponed to the next section.

**Proposition 4.5.** Adapt the assumptions in Proposition 4.2. Uniformly for any $w = y + \eta$ with $|y| \leq N^{-2/3+\epsilon}$, $\eta \geq N^{-2/3-\epsilon}$, we have the following estimates  
\[
\left| \operatorname{Im}[m_{t}(\tilde{L}_{t} + w)] \right| \propto N^{-1/3+3\epsilon}, \quad \left| \partial_{u}m_{t}(\tilde{L}_{t} + w) \right| \propto N^{1/3+4\epsilon}, \quad \left| \partial_{u}G_{ab}(\tilde{L}_{t} + w) \right| \propto N^{1/3+4\epsilon}. \tag{4.11}
\]

Fix distinct indices $i, j, m = \{m_{1}, m_{2}, \cdots, m_{r}\}$, we consider the differential operators  
\[
\partial^{\beta} = \prod_{u,v \in i m} \partial_{uv}^{\beta_{uv}}, \quad D^{\beta} = \prod_{u,v \in i m} D_{uv}^{\beta_{uv}}, \quad |\beta| = \sum_{u,v \in i m} \beta_{uv} \geq 1. \tag{4.12}
\]

The following holds

1. The derivatives of $\tilde{L}_{t}$ satisfy: $\partial^{\beta}_{\beta} \tilde{L}_{t} \propto 1/N$. In the special case $\beta_{ij} + \beta_{ji} = 1$, we have slightly stronger estimate $\partial^{\beta}_{\beta} \tilde{L}_{t} \propto |\beta_{ij}|/N$.

2. The derivatives of the Green’s function $G$ and Stieltjes transform $m_{N}$ satisfy  
\[
\partial^{\beta}G_{ab}(\tilde{L}_{t} + w) = e^{-|\beta|/2} D^{\beta}G_{ab}(\tilde{L}_{t} + w) + O_{\sim} \left( N^{-2/3+4\epsilon} \right) \propto 1, \tag{4.13}
\]
and  
\[
\left| \partial^{\beta}m_{N}(\tilde{L}_{t} + w) \right| \propto N^{-2/3+4\epsilon}, \quad \left| \partial^{\beta}X_{t} \right| \propto N^{-1/3+5\epsilon}, \quad \left| \partial^{\beta}F'(X_{t}) \right| \propto N^{-1/3+5\epsilon}. \tag{4.14}
\]
3. For any monomial $R_{ijm}$ of the Green’s function entries $G(\tilde{L}_t + w, t)$ as in Definition 2.7, if $\beta_{ij} = \beta_{ji} = 0$, we have the estimates

$$\frac{1}{N} \sum_{ijm \text{ distinct}} \mathbb{E} \left[ \partial_{wij} (R_{ijm} \partial^\beta (F'(X_t))) \right] = e^{t/2} \sqrt{1 - e^{-t}} \sum_{ijm \text{ distinct}} \mathbb{E} \left[ \partial_{ij} (R_{ijm} \partial^\beta (F'(X_t))) \right] + O_N \left( \frac{N^{1/3+5\varepsilon} \sqrt{1 - e^{-t}}}{q} \right). \quad (4.15)$$

As an easy consequence of Proposition 4.5, for any $w = y + i\eta$ with $|y| \leq N^{-2/3+\varepsilon}$, $\eta \geq N^{-2/3-\varepsilon}$ by Ward identity (2.3)

$$\frac{1}{N^2} \sum_{ij} |G_{ij}(\tilde{L}_t + w)|^2 = \frac{\text{Im}[m_t(\tilde{L}_t + w)]}{N\eta} < N^{-2/3+4\varepsilon},$$

$$\frac{1}{N^2} \sum_{ij} |G_{ij}(\tilde{L}_t + w)| \leq \sqrt{\frac{\text{Im}[m_t(\tilde{L}_t + w)]}{N\eta}} < N^{-1/3+2\varepsilon}. \quad (4.16)$$

**Proof of Proposition 4.4.** The spectral edge $\tilde{L}_t$ is characterized by

$$\tilde{L}_t = -\frac{1}{\zeta_t} - \zeta_t - \frac{Q_t(\zeta_t)}{\zeta_t}, \quad \partial_m \left( -\frac{1}{m} - m - \frac{Q_t(m)}{m} \right) \bigg|_{m = \zeta_t} = 0, \quad (4.17)$$

where $\zeta_t = \tilde{m}_t(\tilde{L}_t)$, and $\tilde{m}_t$ is the solution of $1 + z\tilde{m}_t + \tilde{m}_t^2 + Q_t(\tilde{m}_t) = 0$. By taking time derivative on both sides of (4.17), we get

$$\partial_t \tilde{L}_t = \partial_m \left( -\frac{1}{m} - m - \frac{Q_t(m)}{m} \right) \bigg|_{m = \zeta_t} \partial_t \zeta_t - \frac{(\partial_t Q_t(\zeta_t)}{\zeta_t}$$

$$= -\partial_t Q_t(\zeta_t) = \sum_{\ell=1}^L \ell a_{2\ell} e^{-\ell t} \zeta_t^{2\ell - 1} = \frac{1}{2} \partial_m Q_t(m) \bigg|_{m = \zeta_t}. \quad (4.18)$$

Let $w' = y + N^{-2/3+\varepsilon}$, then it is easy to see that $w' \in \mathcal{D}$ (recall from 2.9). By (4.11) and the optimal rigidity estimates (3.8)

$$|m_t(\tilde{L}_t + w') - \zeta_t| \leq |m_t(\tilde{L}_t + w') - m_t(\tilde{L}_t + w)| + |m_t(\tilde{L}_t + w') - \tilde{m}_t(\tilde{L}_t)|$$

$$\leq |w' - w| N^{1/3+4\varepsilon} + |\tilde{m}_t(\tilde{L}_t + w') - \tilde{m}_t(\tilde{L}_t)| + |m_t(\tilde{L}_t + w') - \tilde{m}_t(\tilde{L}_t + w')|$$

$$\leq N^{-1/3+5\varepsilon} + |\tilde{m}_t(\tilde{L}_t + w') - \tilde{m}_t(\tilde{L}_t)| + O_N \left( \frac{1}{N \text{Im}[w']} \right),$$

where $\text{Im}[w] = \eta_0 = N^{-1/3-\varepsilon}$. Thanks to the square root behavior of $\tilde{m}_t$, close to the spectral edge we have $|\tilde{m}_t(\tilde{L}_t + w') - \tilde{m}_t(\tilde{L}_t)| \lesssim |w'|^{1/2} \lesssim N^{-1/3+\varepsilon/2}$. Therefore it follows that

$$m_t(\tilde{L}_t + w) - \zeta_t \sim \frac{N^{5\varepsilon}}{N^{1/3}}. \quad (4.19)$$

By plugging (4.19) into (4.18), and using that $\partial_m Q_t(m)$ is a finite polynomial in $m$ with coefficients bounded by $O_N(1/q)$, we conclude

$$\partial_t \tilde{L}_t = \frac{1}{2} \partial_m Q_t(m) \bigg|_{m = \zeta_t} = \frac{1}{2} \partial_m Q_t(m_t(\tilde{L}_t + w)) + O_N \left( \frac{N^{5\varepsilon}}{N^{1/3} q} \right).$$

This finishes the proof of Proposition 4.4. \(\square\)
Proof of (4.7). Let \( w = y + i\eta \), for the first term on the righthand side of (4.9), \( \sum_k D_{ij}G_{kk} = -\sum_k G_{ik}G_{jk} = \partial_w G_{ij} \), we can rewrite it as

\[
\sum_{ij} \mathbb{E} \left[ h_{ij}(t)F'(X_t)\partial_w G_{ij}(\tilde{L}_t + w) \right].
\] (4.20)

By using Proposition 4.4, we can rewrite the second term on the righthand side of (4.9) as

\[
NE \left[ F'(X_t)\partial_h \tilde{L}_t \partial_z m_t(z) \right] = \frac{N}{2} \mathbb{E} \left[ F'(X_t) (\partial_h Q_t)(m_t) \partial_z m_t(z) \right] + O_{\prec} \left( \frac{N^{2/3 + 5\varepsilon}}{q} \right) \mathbb{E}[|\partial_z m_t(z)|],
\]

\[
= \frac{N}{2} \mathbb{E} \left[ F'(X_t) \partial_z (Q_t(m_t(z))) \right] + O_{\prec} \left( \frac{N^{1+9\varepsilon}}{q} \right),
\] (4.21)

where we used Proposition 4.4 in the first equality, and (4.11) in the second line.

By plugging (4.20) and (4.21) into (4.9), we get

\[
\frac{d}{dt} \mathbb{E}[F(X_t)] = O_{\prec} \left( \frac{N^{10\varepsilon + 1/3}}{q} \right)
\]

\[+ \text{Im} \int_E \mathbb{E} \left[ F'(X_t) \left( \sum_{ij} h_{ij}(t) \partial_w G_{ij}(\tilde{L}_t + w) + \frac{N}{2} \partial_w (Q(e^{-t/2}m_t(\tilde{L}_t + w))) \right) \right] dy
\]

\[= \text{Im} \mathbb{E} \left[ F'(X_t) \left( \sum_{ij} h_{ij}(t) G_{ij}(\tilde{L}_t + w) + \frac{N}{2} Q(e^{-t/2}m_t(\tilde{L}_t + w)) \right) \right]_{y = E} + O_{\prec} \left( \frac{N^{10\varepsilon + 1/3}}{q} \right).
\] (4.22)

For the first term on the righthand side of (4.22), by the cumulant expansion formula, we have

\[
-\sum_{ij} \mathbb{E} \left[ h_{ij}(t) F'(X_t) G_{ij} \right] = \frac{1}{2} \sum_{ij} \mathbb{E} \left[ e^{-t/2} h_{ij} F'(X_t) G_{ij} \right] - \sum_{ij} \mathbb{E} \left[ \frac{e^{-t}}{2\sqrt{1 - e^{-t}}} w_{ij} F'(X_t) G_{ij} \right]
\]

\[= \sum_{p=1}^M \frac{e^{-t/2}C_p}{2N q^{p-1}} \sum_{ij} \mathbb{E}[\partial^p_{ij}(F'(X_t) G_{ij})] - \frac{e^{-t}}{2N \sqrt{1 - e^{-t}}} \sum_{ij} \mathbb{E}[\partial_{w_{ij}}(F'(X_t) G_{ij})] + O_{\prec} \left( \frac{1}{q^M} \right),
\]

\[= \sum_{p=1}^M \frac{e^{-t/2}C_p}{2N q^{p-1}} \sum_{ij} \mathbb{E}[\partial^p_{ij}(F'(X_t) G_{ij})] - \frac{e^{-t}}{2N \sqrt{1 - e^{-t}}} \sum_{ij} \mathbb{E}[\partial_{w_{ij}}(F'(X_t) G_{ij})] + O_{\prec} (1)
\] (4.23)

\[= \sum_{p=2}^M \frac{e^{-t/2}C_p}{2N q^{p-1}} \sum_{ij} \mathbb{E}[\partial^p_{ij}(F'(X_t) G_{ij})] - \frac{e^{-t/2}}{2N} \sum_{ij} \mathbb{E}[\partial_{ij}(F'(X_t) G_{ij})] + O_{\prec} \left( \frac{N^{1/3 + 5\varepsilon}}{q} \right)
\]

\[= \sum_{p=2}^M \frac{e^{-t/2}C_p}{2N q^{p-1}} \sum_{ij} \mathbb{E}[\partial^p_{ij}(F'(X_t) G_{ij})] + O_{\prec} \left( \frac{N^{1/3 + 5\varepsilon}}{q} \right),
\]

where in the third line, we used that the contribution from terms corresponding to \( i = j \) is of order \( O_{\prec}(1) \); in the fourth line, we used the relation (4.15) between \( \partial_{ij} \) and \( \partial_{w_{ij}} \). The claim (4.7) follows from plugging (4.23) into (4.22).

\[\square\]

4.2 Proof of (4.8)

If we replace \( F'(X_t) \) by \( P^{r-1} \tilde{P}^r \), the expression on the righthand side of (4.8) is essentially the same as (2.17), up to some \( e^{-t/2} \) factors (In (2.17), the term corresponds to \( p = 1 \) cancels with \( \mathbb{E}[m_N P^{r-1} \tilde{P}^r] \)).
We have these $e^{-t/2}$ factors in (4.8), because the cumulant expansion formula with respect to $h_{ij}(t)$ is slightly different from the cumulant expansion formula with respect to $h_{ij}$. We record the cumulant expansion formula with respect to $h_{ij}(t)$. Take $U = R_{ijm} \partial^\beta(F'(X_t))$, for any monomial $R_{ijm}$ of the Green’s function entries $G(L_t + w, t)$ as in Definition 2.7 and $\beta = \{\beta\}_{u,v \in ijm}$ with $\beta_{ij} = \beta_{ji} = 0$, then

$$
\frac{1}{N^{r+2}} \sum_{ijm} E[h_{ij}(t)U] = \frac{1}{N^{r+2}} \sum_{ijm} E \left[ e^{-t/2}h_{ij}U \right] + \frac{1}{N^{r+2}} \sum_{ijm} E \left[ \sqrt{1 - e^{-t}} w_{ij}U \right]
$$

$$
= \frac{1}{N^{r+2}} \sum_{ijm} \sum_{p=1}^M e^{-t/2} C_p \left( \frac{N^{q-1}}{p^{q-1}} \right) E[\partial_p U] + \sqrt{1 - e^{-t}} \sum_{ijm} E[\partial_{w_{ij}} U] + O_s \left( \frac{1}{q^M} \right)
$$

$$
= \frac{1}{N^{r+2}} \sum_{ijm} \sum_{p=2}^M \sum_{s=0}^p e^{-t/2} C_p \left( \frac{N^{q-1}}{p^{q-1}} \right) E[\partial_p U] + \sqrt{1 - e^{-t}} \sum_{ijm} E[\partial_{w_{ij}} U] + O_s \left( \frac{N^{5\epsilon}}{qN^{2/3}} \right),
$$

(4.24)

where we used (4.15) to replace $\partial_{w_{ij}}$ in the second line.

Similarly to (2.16), all the terms we will get in the expansion are in the form

$$
\frac{1}{q^q} \times \frac{1}{N^r} \sum_m E \left[ R_m \left( \prod_{e \in E(F)} \partial_{\alpha_e \beta_e}^{p_e - s_e} \right) (F'(X_t)) \right],
$$

(4.25)

where $F$ is a weighted forest with vertex set $V(F) = m = \{m_1, m_2, \cdots, m_r\}$, $R$ is a monomial as in Definition 2.7, $p = \{p_e\}_{e \in E(F)}$ are nonnegative integers, and $\alpha \geq 0$ is the order parameter. Since the second factor in (2.16) can be trivially bounded by $O_s(1)$, the whole expression can be bounded by $O_s(1/q^q)$. For terms with order at least $M$, we will trivially bound them by $O_s(1/q^q)$.

**Proof of (4.8).** We follow the three step strategy as in the proof of Proposition 2.6.

**Step 1 (eliminate off-diagonal Green’s function terms)** The first term on the righthand side of (4.8) is in the following form

$$
\frac{1}{N} \sum_{ij} \sum_{p=2}^M \sum_{s=0}^p e^{-t/2} C_{p+1} \left( \frac{N^{q-1}}{p^{q-1}} \right) E[\partial_p G_{ij}(\tilde{L}_t + w; t)] \partial_{ij}^{p-s} (F'(X_t))].
$$

(4.26)

Thanks to Proposition 4.5, we can replace $\partial_p G_{ij}(\tilde{L}_t + w; t) = e^{-st/2} D_{ij}^s G_{ij}(\tilde{L}_t + w; t) + O_s(N^{-2q + 4\epsilon})$. The error term is bounded by

$$
\frac{1}{N} \sum_{ij} \sum_{p=2}^M \sum_{s=0}^p e^{-t/2} C_{p+1} \left( \frac{N^{q-1}}{p^{q-1}} \right) E[N^{-2q + 4\epsilon} | \partial_{ij}^{p-s} (F'(X_t))] | \times \frac{1}{N^2} \sum_{ij} \sum_{p=2}^M \sum_{s=0}^p N^{-2q + 4\epsilon} \frac{q^{p-1}}{p^{q-1}} \lesssim \frac{N^{4\epsilon}}{N^{2/3} q^q},
$$

where we used that $|\partial_{ij}^{p-s} (F'(X_t))| \lesssim 1$, from (4.14). For the term in (4.26) with $i = j$, we can similarly bound them as

$$
\frac{1}{N} \sum_{i} \sum_{p=2}^M \sum_{s=0}^p e^{-t/2} C_{p+1} \left( \frac{N^{q-1}}{p^{q-1}} \right) E[(D_{ii}^s G_{ii}) | \partial_{ii}^{p-s} (F'(X_t))] | \times \frac{1}{N^2} \sum_{i} \sum_{p=2}^M \sum_{s=0}^p \frac{1}{q^{p-1}} \lesssim \frac{1}{N^q}.
$$

Therefore we can further restrict the summation in (4.26) to $i \neq j$.

$$
\frac{1}{N} \sum_{ij} \sum_{p=2}^M \sum_{s=0}^p e^{-t/2} C_{p+1} \left( \frac{N^{q-1}}{p^{q-1}} \right) E[(D_{ij}^s G_{ij}) | \partial_{ij}^{p-s} (F'(X_t))]].
$$

(4.27)
The derivative $D_{ij}^s G_{ij}$ is a sum of monomials in the form $G_{ii}^a G_{jj}^b G_{ij}^c$,

$$D_{ij}^s G_{ij} = -1(s \text{ is odd})! G_{ii}^{s-1} \overrightarrow{G_{jj}}^s + \{\text{terms with off-diagonal entries}\}.$$ 

If the monomial contains at least two off-diagonal terms, i.e. $c \geq 2$, then it is bounded by $|G_{ij}|^2$ and (4.16) gives

$$\frac{1}{N^{2q-1}} \sum_{ij} E[|G_{ij}|^2 | \partial_{ij}^{p-s} (F'(X_t))|] \leq \frac{1}{N^{2q-1}} \sum_{ij} E[|G_{ij}|^2] \leq \frac{N^{4c}}{N^{2/3} q}.$$ 

Analogous to Proposition 2.8, terms with exactly one off-diagonal term are negligible. We have the following

**Proposition 4.6.** Adopt the assumptions of Proposition 4.2. Given a weighted forest $F$ with vertex set $V(F) = ijm$, where $m = \{m_1, m_2, \cdots, m_r\}$. Then for any monomial $R_{ijm}$ of Green’s function entries $G(L + w, t)$ as in Definition 2.7 and nonnegative integers $p = \{p_c\}_{c \in E(F)}$ such that $p_c \geq s_c$, we have

$$\frac{1}{N^{r+2}} \sum_{ijm}^* E \left[ G_{ij} R_{ijm} \left( \prod_{e \in E(F)} \partial_{\alpha_e}^{p_e-s_e} \right)(F'(X_t)) \right] \leq \frac{N^{7c}}{N^{2/3} q}. \quad (4.28)$$

As a consequence of Proposition 4.6, we have that the terms in $D_{ij}^s G_{ij}$ with exactly one off-diagonal term, i.e. $c = 1$, are bounded by

$$\frac{1}{N^{2q-1}} \sum_{ij} E[|G_{ij}| | \partial_{ij}^{p-s} (F'(X_t))|] \leq \frac{N^{7c}}{N^{2/3} q},$$

where $p \geq 2$.

Combining the discussion above, the leading term in (4.27) comes from the monomials with only diagonal Green’s functions entries,

$$-\frac{1}{N} \sum_{p=2}^M \sum_{s} \sum_{ij} e^{-(s+1)t/2} c_{p+1}^{s}! \frac{q}{N^{p-1}} \sum_{ijm}^* E[|G_{ii}^{s+1} G_{jj}^{s+1} \partial_{ij}^{p-s} (F'(X_t))|] + O \left( \frac{N^{7c}}{N^{2/3} q} \right). \quad (4.29)$$

**Step 2 (replace diagonal Green’s function entries by $m_t$)** Analogous to Proposition 2.9, we can use the following proposition to replace diagonal Green’s function entries by $m_t$.

**Proposition 4.7.** Adopt the assumptions of Proposition 4.2. Given a weighted forest $F$ with vertex set $V(F) = ijm$, where $m = \{m_1, m_2, \cdots, m_r\}$. Then for any monomial $R_m$ as in Definition 2.7 with no off-diagonal Green’s function entries, i.e. $\chi(R_m) = 0$, and integers $p = \{p_c\}_{c \in E(F)}$ with $p_c - s_c \geq 0$, we have

$$\frac{1}{N^{r+1}} \sum_{ijm}^* E[|G_{ii}^a R_m V|] = \frac{1}{N^{r+1}} \sum_{ijm}^* E \left[ G_{ii}^{a-1} m_N R_m V \right] + \Omega_1 + \Omega_2 + O(N^{6c-2/3}),$$

where $\Omega_1, \Omega_2$ are given by

$$\Omega_1 = -\frac{1}{N^{r+2}} \sum_{p=2}^M \sum_{s} \sum_{ijm} e^{-(s+1)t/2} c_{p+1}^{s}! \frac{q}{N^{p-1}} \sum_{ijm}^* E[|G_{ii}^{s+1} G_{jj}^{s+1} R_m \partial_{ij}^{p-s} V |],$$

$$\Omega_2 = \frac{1}{N^{r+1}} \sum_{p=2}^M \sum_{s} e^{-(s+1)t/2} c_{p+1}^{s}! \frac{q}{N^{p-1}} \sum_{ijm}^* \left( \frac{s+1}{\alpha} - 1 \right) \sum_{i}^* E[m_N G_{ii}^{a+1} G_{kk}^{s+1} R_m \partial_{ij}^{p-s} V]. \quad (4.31)$$

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Comparing these terms $\Omega_1, \Omega_2$ with (4.30), since $p \geq 2$ they are of order at least 1 (recall from (4.25)).

**Remark 4.8.** These terms $\Omega_1, \Omega_2$ in (4.31) are in the same form as in (4.30). With given $p$, the term in $\Omega_1$ is associated with a weighted forest $F_1$, which is from $F$ by adding vertices $j, k$ and an edge $\{j, k\}$ with weight $s$. In total $\Omega_1$ has $r + 2$ vertices. $\Omega_2$ is associated with a weighted forest $F_2$, which is from $F$ by adding one vertex $k$ and an edge $\{i, k\}$ with weight $s$. In total $\Omega_2$ has $r + 1$ vertices. For given $s$, both $\Omega_1, \Omega_2$ have an extra factor $e^{-(s+1)/t}$ and derivative $\partial^p - s$. Moreover, the total number of diagonal Green’s function entries increases by $s + 1$.

By repeatedly using Proposition 4.7, we can replace the product $G_i^{(s+1)/2}G_j^{(s+1)/2}$ in (4.29) by $m_t^{s+1}$, to get the leading terms

$$- \frac{1}{N} \sum_{p=2}^M \sum_{s \text{ odd}} \sum_{ij} \frac{e^{-(s+1)/2}C_{p+1}^{s+1}}{Nq^{p-1}} \left( \frac{p}{s} \right)^s \mathbb{E} \left[ m_t^{s+1} \partial^{p - s} F'(X_t) \right],$$

(4.32)

with higher order terms (which have at least one more copies of $1/q$) as linear combination of terms (with bounded coefficients) in the form

$$\frac{1}{N^r} \sum_{2 \leq \sum_{p=1}^{r} C_{p+1}^{s+1}} \sum_{x \in E(F)} \mathbb{E} \left[ \prod_{e \in E(F)} \frac{e^{-(s_e+1)/2}C_{p_e+1}^{s_e+1}}{q^{p_e-1}} \left( \frac{p_e}{s_e} \right)^{s_e} \partial^{p_e - s_e} \right] \left( F'(X_t) \right),$$

(4.33)

where $F$ is a weighted forest as in Definition 2.2 with vertex set $\{m_1, m_2, \ldots, m_r\}$; the monomial $m_t$ has no off-diagonal entries, i.e. $\chi(R_m) = 0$ and $\deg(R_m) = \sum_{e \in E(F)} (s_e + 1)$.

We can repeat Step 2 for these higher order terms (4.33). If a term has order bigger than $M$, we can trivially bound it by $O_<(1/q^M) < N^{8c}/qN^{2/3}$ as in (4.25). The final expression is a linear combination (with bounded coefficients) of terms in the form:

$$\mathbb{E} \left[ m_t^{2f} L_F^{t} \left( P^{r-1} P' \right) \right],$$

(4.34)

$$L_F^{t} = \sum_{2 \leq \sum_{p=1}^{r} C_{p+1}^{s+1}} \sum_{x \in V(F)} \mathbb{E} \left[ \prod_{e \in E(F)} \frac{e^{-(s_e+1)/2}C_{p_e+1}^{s_e+1}}{q^{p_e-1}} \left( \frac{p_e}{s_e} \right)^{s_e} \partial^{p_e - s_e} \right] \left( F'(X_t) \right),$$

where $F$ is a forest as in Definition 2.2, $x_1, x_2, \ldots, x_{V(F)}$ enumerate the vertices of $F$. Moreover, all the weights $s_e$ are odd positive integers, and the total weights satisfies $\sum_e (s_e + 1) = 2f$. The above discussion leads to the following claim.

**Claim 4.9.** Under the assumptions of Proposition 4.7, (4.27) is a finite sum of terms in the form, with an error $O_<(N^{8c}/qN^{2/3})$:

$$\mathbb{E} \left[ m_t^{2f} L_F^{t} F'(X_t) \right],$$

(4.35)

where $L_F^{t}$ is as defined in (4.34).

**Step 3 (rewrite differential operators as an expectation)** Finally use the cumulant expansion, the same as in (2.29), we have

$$\mathbb{E} \left[ e^{-(s+1)/t} \left( \frac{1}{m_t} - \frac{1}{N} \right) \right] m_t^{2f} F'(X_t)$$

$$= \frac{e^{-(s+1)/2}C_{p+1}^{s+1}}{Nq^{p-1}} \left( \frac{p}{s} \right) \mathbb{E} \left[ m_t^{2f} \partial^{p - s} \left( F'(X_t) \right) \right] + O_<(1/q^M + N^{4c}/N^{5/3}q).$$

(4.36)
By repeatedly using the relation (4.36), and in the same way as in Claim 2.12, we can rewrite (4.35) in the following form

\[
E[m_t^{2L}F'(X_t)] = \prod_{e \in E(F)} \mathbb{E} \left[ \sum_{x_1, \ldots, x_{|V(F)|}} \left( \frac{1}{N^{t|F|}} \prod_{e \in E(F)} \left( b_{\alpha e}^{s_e+1} - \frac{1}{N} \right) \right) e^{-t \cdot m_t^{2L} F'(X_t)} + O_e \left( \frac{N^{4t}}{q N^{3/3}} \right) \right]
\]

(4.37)

\[
= \prod_{e \in E(F)} \mathbb{E} \left[ (w(F)) \cdot e^{-\ell t/2} m_t^{2L} F'(X_t) \right] + O \left( \frac{N^{4t}}{q N^{3/3}} \right).
\]

So far every estimate is parallel to those from Proposition 2.8, except for the extra factor \( e^{-\ell t/2} \) in (4.37). Thanks to Claim 4.9 and (4.37), the first term on the righthand side of (4.8) is in the form

\[
-\mathbb{E}[e^{-t(a_2 m_t^2 + e^{-2t} a_4 m_t^4 + \cdots + e^{-t} a_{2L} m_t^{2L}) F'(X_t)] + O_\prec \left( \frac{N^{8t}}{q N^{3/3}} \right),
\]

(4.38)

where \( a_{2L} \) is a sum of terms in the form \( w(F) \) as in (2.7), where \( F \) is a forest as in Definition 2.2. Moreover, all the weights \( s_e \) are odd positive integers with \( \sum_e s_e = 2 \ell \). The expression (4.38) is precisely the definition of the polynomial \( Q(e^{-t} m_t) F'(X_t) \). Thus the term (4.38) cancels with \( \mathbb{E}[Q(e^{-t} m_t) F'(X_t)] \) in (4.8), and we conclude Proposition 4.3.

4.3 Proof of Propositions from Sections 4.1 and 4.2

Proof of Proposition 4.5. Let \( w' = y + N^{-2/3+i} \), then it is easy to see that \( w' \in D \) (recall from 2.9), and Proposition 3.3 gives

\[
\text{Im}[m_\ell (\tilde{L}_t + w)] \leq \text{Im}[	ilde{m}_\ell (\tilde{L}_t + w')] + O_\prec \left( \frac{1}{N \text{Im}[w']} \right) \leq \sqrt{|w'|} + O_\prec \left( \frac{1}{N \text{Im}[w']} \right) \leq \frac{N^{\epsilon/2}}{N^{1/3}}
\]

where we used that \( \tilde{m}_\ell \) has square root behavior. The derivative of \( m_\ell \) satisfies

\[
|\partial_z \text{Im}[m_\ell(z)]| \leq |\partial_z m_\ell(z)| \leq \frac{\text{Im}[m_\ell(z)]}{\text{Im}[z]},
\]

(4.39)

which gives that \( \text{Im}[m_\ell(E + i \eta / M)] \leq M \text{Im}[m_\ell(E + i \eta)] \) for any \( M \geq 1 \). In particular, we have \( \text{Im}[m_\ell(\tilde{L}_t + w)] \leq (\text{Im}[w']/\text{Im}[w]) \cdot \text{Im}[m_\ell(\tilde{L}_t + w')] \leq N^{2/3} N^{-1/3+t/2} \leq N^{-1/3+3\epsilon} \). Using (4.39) again, we have \( |\partial_z m_\ell(\tilde{L}_t + w)| \leq N^{1/3+4\epsilon} \). For the derivative of the Green’s function, Ward identity (2.3) implies

\[
|\partial_a G_{ab}(\tilde{L}_t + w)| \leq \sum_{i=1}^{N} |G_{ai}(\tilde{L}_t + w) G_{hi}(\tilde{L}_t + w)| \leq \frac{1}{2} \sum_{i=1}^{N} \left( |G_{ai}(\tilde{L}_t + w)|^2 + |G_{hi}(\tilde{L}_t + w)|^2 \right) \leq \frac{\text{Im}[m_N(\tilde{L}_t + w)]}{\eta} \leq N^{1/3+4\epsilon}.
\]

(4.40)

Since \( \tilde{L}_t \) is the spectral edge of \( \tilde{\rho}_t \), which is characterized by \( 1 + z\tilde{m}_\ell(z) + \tilde{m}_\ell(z) + Q(e^{-t} \tilde{m}_\ell(z)) = 0 \). The same as in Proposition 2.4, \( \tilde{L}_t \) depends smoothly on the coefficients of \( Q \). In particular, its derivatives with respect to \( a_2, a_4, \ldots, a_{2L} \) are bounded. Thus the bounds on \( |\partial^\beta \tilde{L}_t| \) follow from Proposition 2.13. The estimates in (4.13) can be proven the same way as (2.46), and using that \( \text{Im}[m_\ell(\tilde{L}_t + w)]/\eta \approx N^{-1/3+3\epsilon} \).

For (4.14), we can rewrite the derivative \( \partial^\beta m_\ell \) as

\[
\partial^\beta m_\ell = \frac{1}{N} \sum_{i=1}^{N} e^{\beta |t/2} D^\beta G_{ii} + O_\prec \left( N^{-2/3+4\epsilon} \right).
\]
$D^\beta G_{ii}$ is a monomial of Green’s function entries, and each contains at least two off-diagonal entries. We can bound the sum using the Ward identity (2.3) as in (4.40),

$$\frac{1}{N} \sum_{i=1}^{N} e^{\beta t/2} D^\beta G_{ii} < \frac{N^{1/3+4\epsilon}}{N} = \frac{N^{4\epsilon}}{N^{2/3}}.$$ 

The second and third relation in (4.14) follows from

$$|\partial^\beta X_t| = \left| \text{Im} \left[ N \int_E N^{-2/3+\epsilon} \partial^\beta m_t(\tilde{L}_t + w)dy \right] \right| < N^{-1/3+5\epsilon},$$

and the fact that $F$ has bounded derivatives.

For either $U = R_{ijm}$ or $U = F'(X_t)$, $U = U(H(t), \tilde{L}_t)$ is a function of both $H(t)$ and $\tilde{L}_t$. Since $\tilde{L}_t$ depends only on $H$ but not on $W$, the derivatives of $\partial_{ij}$ and $\partial_{w_{ij}}$ are related by the following relation

$$\partial_{w_{ij}} U = e^{t/2} \sqrt{1 - e^{-t}}(\partial_{ij} U - (\partial_{ij} \tilde{L}_t) \partial_{L_t} U). \tag{4.41}$$

Using the relation (4.41), we can rewrite the lefthand side of (4.15)

$$\frac{1}{N} \sum_{ij,ijm \text{ distinct}} E \left[ \partial_{w_{ij}} (R_{ijm} \partial^\beta (F'(X_t))) \right] = \frac{e^{t/2} \sqrt{1 - e^{-t}}}{N} \sum_{ij,ijm \text{ distinct}} E \left[ \partial_{ij} (R_{ijm} \partial^\beta (F'(X_t))) \right]$$

$$+ \frac{e^{t/2} \sqrt{1 - e^{-t}}}{N} \sum_{ij,ijm \text{ distinct}} E \left[ (\partial_{ij} \tilde{L}_t)(\partial_{L_t} R_{ijm}) \partial^\beta (F'(X_t)) + R_{ijm} \partial^\beta ((\partial_{ij} \tilde{L}_t) \partial_{L_t} F'(X_t)) \right]. \tag{4.42}$$

Using (4.11), (4.13), (4.14) for any $\beta'$ with $\beta'_{ij} = \beta'_{ji} = 0$, we have

$$|\partial_{ij} \tilde{L}_t| \prec \frac{|h_{ij}|}{N}, \quad |\partial^\beta \partial_{ij} \tilde{L}_t| \prec \frac{|h_{ij}|}{N}, \quad |R_{ijm}| \prec 1, \quad |\partial_{L_t} R_{ijm}| \prec N^{1/3+4\epsilon}, \quad \partial^\beta X_t \prec N^{1/3+5\epsilon}.$$ 

Moreover we also have

$$\partial^\beta \partial_{L_t} X_t = \left| \text{Im} \left[ N \int_E N^{-2/3+\epsilon} \partial^\beta \partial_{L_t} m_t(\tilde{L}_t + w)dy \right] \right| < N^{1/3+5\epsilon}.$$ 

Then we can bound the second term on the righthand side of (4.42) as

$$\frac{1}{N} \sum_{ij,ijm \text{ distinct}} E \left[ |h_{ij}| \right] \left[ N^{1/3+5\epsilon} \right] \prec \frac{N^{1/3+5\epsilon}}{N^{1/3+5\epsilon}} \frac{q}{N}.$$ 

The claim (4.15) follows.

\textbf{Proof of Proposition 4.6.} If $\sum_e (p_e - s_e) \geq 1$, then (4.14) implies that

$$\left| \left( \prod_{e \in E(F)} \partial_{\alpha_e, \beta_e}^{p_e - s_e} \right) (F'(X_t)) \right| \prec N^{-1/3+5\epsilon},$$

and

$$\frac{1}{N^{r+2}} \sum_{ijm} E \left[ G_{ij} R_{ijm} \left( \prod_{e \in E(F)} \partial_{\alpha_e, \beta_e}^{p_e - s_e} \right) (F'(X_t)) \right] \prec \frac{1}{N^{r+2}} \sum_{ijm} E \left[ |G_{ij}| N^{-1/3+3\epsilon} \right] \prec \frac{N^{7\epsilon}}{N^{2/3}}. \tag{4.43}$$
where in the last inequality we used (4.16). If \( \sum_{c}(p_{c} - s_{c}) = 0 \), then we use the identities \( G_{ij} = \sum_{k \neq i} G_{ii}h_{ik}G_{kj}^{(i)} \) and \( G_{kj}^{(i)} = G_{kj} - G_{ki}G_{ji}/G_{ii} \). We denote

\[
U = R_{ijm}F'(X_{t}).
\]

Then by the cumulant expansion (4.24), we have

\[
\frac{1}{N^{r+2}} \sum_{ijm}^{*} \mathbb{E}[G_{ij}U] = \frac{1}{N^{r+2}} \sum_{ijm}^{*} \mathbb{E} \left[ \sum_{k \neq i} h_{ik}(t)G_{kj}^{(i)}G_{ii}U \right]
\]

\[
= \frac{1}{N^{r+2}} \sum_{ijm}^{*} \sum_{p=1}^{M} e^{-t/2}C_{p}N^{q_{p}+1-1} \mathbb{E}[\partial_{ik}^{p}(G_{kj}^{(i)}G_{ii}U)] + \frac{1}{N^{r+3}} \sum_{ijm}^{*} \mathbb{E}[\partial_{ik}(G_{kj}^{(i)}G_{ii}U)] + O(\frac{1}{qM})
\]

\[
= \frac{1}{N^{r+2}} \sum_{ijm}^{*} \sum_{p=1}^{M} e^{-t/2}C_{p}N^{q_{p}+1-1} \mathbb{E}[\partial_{ik}^{p}(G_{kj}^{(i)}G_{ii}U)] + \frac{1}{N^{r+3}} \sum_{ijm}^{*} \mathbb{E}[\partial_{ik}(G_{kj}^{(i)}G_{ii}U)] + O(\frac{1}{N^{2/3} + \frac{1}{qM}})
\]

(4.44)

where to get the third line, we used that the summation for terms with \( k \in jm \) is bounded by \( 1/N \). Then (4.44) can be analyzed in the same way as for (2.60), by using \( |F'(X_{t})| \leq 1 \) and \( |\partial^{3}F'(X_{t})| \ll N^{-1/3} \) from (4.14). This leads to the claim (4.28).

\[\square\]

**Proof of Proposition 4.7.** The proof is similar to that of Proposition 2.9. We will use (2.22) to replace a copy of \( G_{ii} \) to \( m_{i} \). Denote

\[
U := R_{im}V, \quad V := \left( \prod_{e \in E(F)} \partial_{\alpha_{e}, \beta_{e}}^{p_{e}} \right) (F'(X_{t})).
\]

Then we have exactly the same expression as in (2.65). For the second term on the right-hand side of (2.65), we can rewrite it as

\[
\frac{1}{N^{r+2}} \sum_{ijm}^{*} \mathbb{E}[(G_{ii}(HG)_{jj} - m_{i}(HG)_{ii})G_{ii}^{\alpha-1}U]
\]

\[
= \frac{1}{N^{r+2}} \sum_{ijm}^{*} \sum_{p=1}^{M} e^{-t/2}C_{p}N^{q_{p}+1-1} \mathbb{E}[\partial_{jk}^{p}(G_{ii}^{\alpha}G_{jj}R_{im}V) - \partial_{ik}^{p}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im}V)]
\]

\[
+ \frac{1}{N^{r+3}} \sum_{ijm}^{*} \mathbb{E}[\partial_{w_{jk}}(G_{ii}^{\alpha}G_{jj}R_{im}V) - \partial_{w_{ik}}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im}V)] + O\left(\frac{1}{qM}\right)
\]

(4.45)

\[
= \frac{1}{N^{r+2}} \sum_{ijm}^{*} \sum_{p=1}^{M} e^{-t/2}C_{p}N^{q_{p}+1-1} \mathbb{E}[\partial_{jk}^{p}(G_{ii}^{\alpha}G_{jj}R_{im}V) - \partial_{ik}^{p}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im}V)]
\]

\[
+ \frac{1}{N^{r+3}} \sum_{ijm}^{*} \mathbb{E}[\partial_{w_{jk}}(G_{ii}^{\alpha}G_{jj}R_{im}V) - \partial_{w_{ik}}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im}V)] + O\left(\frac{1}{N} + \frac{1}{qM}\right)
\]

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where we used that the summation for terms with \( k \in ijm \) is bounded by \( 1/N \). Then we can replace the derivatives \( \partial_{w_{jk}} \) and \( \partial_{w_{ik}} \) by \( \partial_{jk} \) and \( \partial_{ik} \) using (4.15), and get

\[
\frac{1}{N^{r+2}} \sum_{ijm}^{*} \mathbb{E}[(G_{ii}(HG)_{jj} - m_{i}(HG)_{ii})G_{ii}^{\alpha-1}U]
\]

\[
= \frac{1}{N^{r+2}} \sum_{ijkm}^{*} \frac{e^{t/2}}{N} \mathbb{E}[\partial_{jk}(G_{ii}^{\alpha}G_{jj}R_{im}V) - \partial_{ik}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im}V)] + O_{<} \left( \frac{1}{q^{M}} \right)
\]

\[
+ \frac{1}{N^{r+2}} \sum_{ijkm}^{*} \sum_{p=2}^{M} \sum_{s} e^{-(s+1)/2 \frac{C_{p+1}}{Nq^{p-1}}} \frac{1}{C_{p+1}} \mathbb{E}[\partial_{jk}^{p}(G_{ii}^{\alpha}G_{jj}R_{im})\partial_{ik}^{p-1}V - \partial_{ik}^{p}(G_{jj}G_{ik}G_{ii}^{\alpha-1}R_{im})\partial_{jk}^{p-1}V].
\]

(4.46)

Then (4.46) can be analyzed in the same way as for (2.66), by using \(|F'(X_{t})| \lesssim 1 \) and \(|\partial^{\beta}F'(X_{t})| \lesssim N^{-1/3+\varepsilon} \) from (4.14). This leads to the claim (4.31). □

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