Gray’s Decomposition on Doubly Warped Product Manifolds and Applications

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Abstract. A. Gray presented an interesting $O(n)$ invariant decomposition of the covariant derivative of the Ricci tensor. Manifolds whose Ricci tensor satisfies the defining property of each orthogonal class are called Einstein-like manifolds. In the present paper, we answered the following question: Under what condition(s), does a factor manifold $M_i, i = 1,2$ of a doubly warped product manifold $M = f_2 M_1 \times f_1 M_2$ lie in the same Einstein-like class of $M$? By imposing sufficient and necessary conditions on the warping functions, an inheritance property of each class is proved. As an application, Einstein-like doubly warped product space-times of type $A, B$ or $P$ are considered.

1. An introduction

Alfred Gray in [22] presented $O(n)$ invariant orthogonal irreducible decomposition of the space $W$ of all $(0,3)$ tensors satisfying only the identities of the gradient of the Ricci tensor $\nabla R_{ij}$. The space $W$ is decomposed into three orthogonal irreducible subspaces, that is, $W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{I}$. This decomposition produces seven classes of Einstein-like manifolds, that is, manifolds whose Ricci tensor satisfies the defining identity of each subspace. They are the trivial class $P$, the classes $\mathcal{A}, \mathcal{B}, \mathcal{I}$ and three composite classes $\mathcal{I} \oplus \mathcal{A}, \mathcal{I} \oplus \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$.

In class $P$, the Ricci tensor is parallel i.e. $\nabla R_{ij} = 0$ whereas class $\mathcal{A}$ contains manifolds whose Ricci tensor is Killing. The Ricci tensor of manifolds in class $\mathcal{B}$ is a Codazzi tensor i.e. $\nabla \nabla R_{ij} = \nabla R_{ij}$. The traceless part of the Ricci tensor vanishes in class $\mathcal{I}$ i.e. class $\mathcal{I}$ contains Sinyukov manifolds[26]. The tensor

$$\mathcal{L}_{ij} = R_{ij} - \frac{2R}{n+2} g_{ij}$$

is Killing in class $\mathcal{I} \oplus \mathcal{A}$ whereas the tensor

$$\mathcal{H}_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$$

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is a Codazzi tensor in class $I \oplus B$. The class $A \oplus B$ is identified by having constant scalar curvature. The same decomposition is discussed extensively in [3 Chapter 16] (see also [24, 26] and Section 3 for more details and equivalent conditions). Thereafter, Einstein-like manifolds have been studied by many authors such as G. Calvaruso in [7, 10] Mantica et al in [24, 26] and many others [2, 5, 6, 30, 36]. An interesting study in [26] shows Einstein-like generalized Robertson-Walker space-times are perfect fluid space-times except those in class $I$ which are not restricted. Sufficient conditions on generalized Robertson-Walker space-times in this class to be a perfect fluid are derived in [13].

Doubly warped products is a generalization of singly warped products introduced in [4]. The geometric properties of doubly warped product manifolds have been investigated by many authors such as pseudo-Riemannian manifolds (except those in class $I \oplus B$). The class $I \oplus B$ is discussed extensively in [3, Chapter 16] (see also [24, 26] and Section 3 for more details and equivalent conditions). Thereafter, Einstein-like manifolds have been studied by many authors such as G. Calvaruso in [7–10] Mantica et al in [24–26] and many others [2, 5, 6, 30, 36]. An interesting study in [26] shows Einstein-like generalized Robertson-Walker space-times are perfect fluid space-times such as $G. Calvaruso in [7–10] Mantica et al in [24–26] and many others [2, 5, 6, 30, 36]. An interesting study in [26] shows Einstein-like generalized Robertson-Walker space-times are perfect fluid space-times such as $G. Calvaruso in [7–10] Mantica et al in [24–26] and many others [2, 5, 6, 30, 36].

Inspired by the above studies of Einstein-like metrics and doubly warped product manifolds, we studied doubly warped products as exact solutions of Einstein’s field equations. Recently, the existence of compact Einstein doubly warped product manifolds is considered in [23]. Doubly warped products are widely used as exact solutions of Einstein’s field equations. Recently, the existence of compact Einstein doubly warped product manifolds is considered in [23].

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2. Preliminaries

A doubly warped product manifold is the (pseudo-)Riemannian product manifold $M = M_1 \times M_2$ of two (pseudo-)Riemannian manifolds $(M_i, g_i, D_i), i = 1, 2$, furnished with the metric tensor

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^*(g_2),$$

where the functions $f_i : M_i \to (0, \infty), i = 1, 2$ are the warping functions of $M$. $M$ is denoted by $f_2 M_1 \times f_1 M_2$. The maps $\pi_i : M_1 \times M_2 \to M_i$ are the natural projections $M$ onto $M_i$ whereas the pull-back operator on tensors. In particular, if for example $f_2 = 1$, then $M = M_1 \times f_1 M_2$ is called a singly warped product manifold (see [15, 33] for doubly warped products and [4, 14, 16, 27, 33, 34] for singly warped products).

**Notation 2.1.** Throughout this work, we use the following notations

1. All tensor fields on $M_1$ are identified with their lifts to $M$. For example, we use $f_i$ for a function on $M_i$ and for its lift $(f_i \circ \pi_1)$ on $M$.
2. The manifolds $M_i$ have dimensions $n_i$ where $n = n_1 + n_2$.
3. Ric is the Ricci curvature tensor on $M$ and $\text{Ric}^i$ is the Ricci tensor on $M^i$.
4. The gradient of $f_i$ on $M_i$ is denoted by $\nabla^i f_i$ and the Laplacian by $\Delta^i f_i$, whereas $f_i^* = f_i(\Delta^i f_i) + (n_i - 1)g_i(\nabla^i f_i, \nabla^i f_i), i \neq j$.
5. The indices $i$ and $j$ to denote the geometric objects of the factor manifolds $M_i$ and $M_j$.
6. The $(0,2)$ tensors $\mathcal{F}^i$ is defined as

$$\mathcal{F}^i (X_\nu, Y_\nu) = \frac{n_j}{f_i} H^i (X_\nu, Y_\nu),$$

for $X_\nu, Y_\nu \in \mathfrak{X}(M_i)$ and $i, j = 1, 2, i \neq j$.

The Levi-Civita connection $D$ on $M = f_2 M_1 \times f_1 M_2$ is given by

$$D_{X} X_i = X_i (\ln f_2) X_i + X_i (\ln f_1) X_i,$$

$$D_{Y} Y_i = Y_i f_i^2 g_i (X_\nu, Y_\nu) \nabla (\ln f_i),$$

where $\nabla$ is the Levi-Civita connection on $M$. The maps $\pi_1 : M_1 \times M_2 \to M_1$ are the natural projections $M$ onto $M_1$ whereas the pull-back operator on tensors. In particular, if for example $f_2 = 1$, then $M = M_1 \times f_1 M_2$ is called a singly warped product manifold (see [15, 33] for doubly warped products and [4, 14, 16, 27, 33, 34] for singly warped products).
where $i \neq j$ and $X_i, Y_i \in \mathfrak{X}(M_i)$. Then the Ricci curvature tensor $\text{Ric}$ on $M$ is given by

$$
\text{Ric}(X_i, Y_i) = \text{Ric}^i(X_i, Y_i) - \frac{n_i}{f_i}H^i(X_i, Y_i) - \frac{f_i}{f_i^2}g_i(X_i, Y_i),
$$

$$
\text{Ric}(X_i, Y_i) = (n - 2) X_i (\ln f_i) Y_j (\ln f_j),
$$

where $i \neq j$ and $X_i, Y_i \in \mathfrak{X}(M_i)$. The reader is referred to [11, 12, 19] for some studies of curvature conditions on warped product manifolds.

3. Einstein-like doubly warped product manifolds

The Einstein-like doubly warped product manifolds $M = f_i M_1 \times_f M_2$ are investigated in this section. Every subsection is devoted to the study of a class of Einstein-like doubly warped product manifolds. Sufficient and necessary conditions are derived on the warping functions $f_i$ for factor manifolds $M_i$ to acquire the same Einstein-like class type.

3.1. Class $\mathcal{A}$

A doubly warped product manifold $(M, g)$ whose Ricci tensor is Killing, that is,

$$(D_X \text{Ric})(Y, Z) + (D_Y \text{Ric})(Z, X) + (D_Z \text{Ric})(X, Y) = 0,$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ is called Einstein-like doubly warped product manifold of class $\mathcal{A}$. This condition equivalent to

$$(D_X \text{Ric})(X, X) = 0,$$

for any vector field $X \in \mathfrak{X}(M)$ and the Ricci tensor is also called cyclic parallel. The legacy of factor manifolds of $M$ in class $\mathcal{A}$ is as follows.

**Theorem 3.1.** In a doubly warped product manifold $M = f_i M_1 \times_f M_2$ where $M$ is of class type $\mathcal{A}$, a factor manifold $(M_i, g_i)$ is an Einstein-like manifold of class $\mathcal{A}$ if and only if

$$(D^i_X f^i)(X_i, X_i) = \frac{2}{f_i}X_i (f_i) g_i (X_i, X_i) \left[ f_i^2 + (n - 2) (\nabla f_i)(f_i) \right],$$

where $i, j = 1, 2, i \neq j$ and $X_i \in \mathfrak{X}(M_i)$.

**Proof.** In a doubly warped product manifold $M = f_i M_1 \times_f M_2$ of class $\mathcal{A}$, it is

$$
0 = (D_X \text{Ric})(X, X) = X (\text{Ric}(X, X)) - 2 \text{Ric}(D_X X, X).
$$

Thus, for a the special case where $X = X_i$ lands on one factor, one may get

$$
0 = (D_X \text{Ric})(X_i, X_i)
$$

$$
= X_i \left( \text{Ric}^i(X_i, X_i) - f_i^2 g_i (X_i, X_i) \right) - 2 \text{Ric} \left( D^i_{X_i} f^i, X_i \right) + 2 f_i^2 g_i \left( D^i_{X_i} f^i, X_i \right)
$$

$$
+ 2(n - 2) \frac{1}{f_i^2} (\nabla f_i)(f_i) X_i (f_i) g_i (X_i, X_i).
$$
Thus, after lengthy computations, it is
\[
0 = \left( D_{X_i} \text{Ric} \right)(X_i, X_i) - \left( D_{Y_i} F^i \right)(X_i, X_i) \\
+ \frac{2}{f_i} X_i (f_i) g_{i}(X_i, X_i) \left[ f_i^2 + (n - 2) \left( \nabla^i f_j \right) f_i \right].
\]

These equations complete the proof. □

It is now easy to recover a similar result on singly warped product manifolds.

**Corollary 3.2.** In a singly warped product manifold \( M = M_1 \times_f M_2 \) where \( M \) is of class type \( \mathcal{A} \), \((M_1, g_1)\) is an Einstein-like manifold of class \( \mathcal{A} \) if and only if \( F^i \) is Killing. In addition, \((M_2, g_2)\) is of class type \( \mathcal{A} \).

### 3.2. Class \( \mathcal{B} \)

Let \( M \) be as Einstein-like doubly warped product manifold of class \( \mathcal{B} \). Then, the Ricci tensor is a Codazzi tensor, that is,
\[
\left( D_{X_i} \text{Ric} \right)(Y_i, Z_i) = \left( D_{Y_i} \text{Ric} \right)(X_i, Z_i).
\]

The above condition is equivalent to:

1. \( M \) has a harmonic Riemann tensor, that is, \( \nabla \varepsilon R^\varepsilon_{\alpha\beta\gamma} = 0 \), or
2. \( M \) admits a harmonic Weyl conformal tensor and the scalar curvature is constant, that is, \( \nabla \varepsilon C^\varepsilon_{\alpha\beta\gamma} = 0 \) and \( \nabla \varepsilon R = 0 \).

The base manifold and the fiber manifold gain the Einstein-like class type \( \mathcal{B} \) according to.

**Theorem 3.3.** In a doubly warped product manifold \( M = f_2 M_1 \times_f M_2 \) where \( M \) is of class type \( \mathcal{B} \), the factor manifold \((M_i, g_i)\) is an Einstein-like manifold of class \( \mathcal{B} \) if and only if
\[
\left( D_{X_i} F^i \right)(Y_i, Z_i) = \left( D_{Y_i} F^i \right)(X_i, Z_i) \\
+ \frac{1}{f_i^2} Y_i (f_i) g_{i}(X_i, Z_i) \left( 2 f_i^2 - (n - 2) \left( \nabla^i f_j \right) f_i \right) \\
- \frac{1}{f_j^2} X_j (f_j) g_{i}(X_i, Z_i) \left( 2 f_i^2 - (n - 2) \left( \nabla^i f_j \right) f_i \right),
\]
where \( i, j = 1, 2, i \neq j \) and \( X_i, Y_i, Z_i \in \mathfrak{X}(M_i) \).

**Proof.** Let us define the deviation tensor \( B(X, Y, Z) \) as follows
\[
B(X, Y, Z) = (D_X \text{Ric})(Y, Z) - (D_Y \text{Ric})(X, Z).
\]

There are three different cases. Let us consider the first case, that is,
\[
B(X_i, Y_i, Z_i) = (D_{X_i} \text{Ric})(Y_i, Z_i) - (D_{Y_i} \text{Ric})(X_i, Z_i).
\]

(1)
It is enough to find \((D_X \text{Ric})(Y_i, Z_i)\) as
\[
(D_X \text{Ric})(Y_i, Z_i) = X_i \left( \text{Ric}^c(Y_i, Z_i) \right) - X_i \left( F^i(Y_i, Z_i) \right) - f_i^2 X_i \left( \frac{1}{f_i^2} g_i(Y_i, Z_i) \right)
\]
\[= -\text{Ric}^c \left( D_{X_i} Y_i, Z_i \right) + F^i \left( D_{X_i} Y_i, Z_i \right) + \frac{f_i}{f_i^2} g_i \left( D_{X_i} Y_i, Z_i \right)
\]
\[= -\text{Ric}^c \left( Y_i, D_{X_i} Z_i \right) + F^i \left( Y_i, D_{X_i} Z_i \right) + \frac{f_i}{f_i^2} g_i \left( Y_i, D_{X_i} Z_i \right)
\]
\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i, Y_i) \nabla^i f_i \left( f_i \right) Z_i \left( f_i \right)
\]
\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i, Z_i) \nabla^i f_i \left( f_i \right) Y_i \left( f_i \right).
\]
Simplifying this expression, it is
\[
(D_X \text{Ric})(Y_i, Z_i) = \left( D_X^i \text{Ric}^c \right)(Y_i, Z_i) - \left( D_X^i F^i \right)(Y_i, Z_i) + 2 \frac{f_i}{f_i^2} X_i(f_i) g_i(Y_i, Z_i)
\]
\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i, Y_i) \nabla^i f_i \left( f_i \right) Z_i \left( f_i \right)
\]
\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i, Z_i) \nabla^i f_i \left( f_i \right) Y_i \left( f_i \right).
\](2)

By exchanging \(X_i\) and \(Y_i\) in the last equation and substitution in Equation \((\text{1})\), one gets the deviation tensor. For Einstein-like manifolds of class \(\mathcal{B}\), the deviation tensor vanishes from which the result hold. \(\square\)

It is easy to retrieve a similar result on a singly warped product manifold.

**Corollary 3.4.** In a singly warped product manifold \(M = M_1 \times_{f_1} M_2\) where \(M\) is of class type \(\mathcal{B}\), \((M_1, g_1)\) is an Einstein-like manifold of class \(\mathcal{B}\) if and only if
\[
\left( D^i_X F^i \right)(Y_1, Z_1) = \left( D^i_X F^i \right)(X_1, Z_1),
\]
where \(X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)\). In addition, \((M_2, g_2)\) is Einstein-like of class type \(\mathcal{B}\).

3.3. Class \(\mathcal{P}\)

Let \(M\) be an Einstein-like doubly warped product manifold of class \(\mathcal{P}\). Thus, \(M\) has a parallel Ricci tensor, that is,
\[
(D_X \text{Ric})(Y, Z) = 0.
\]
Manifolds in this class are usually called Ricci symmetric.

**Theorem 3.5.** In a doubly warped product manifold \(M = f \times_{f_1} M_1 \times_{f_2} M_2\) where \(M\) is of class type \(\mathcal{P}\), \((M_1, g_1)\) is an Einstein-like manifold of class \(\mathcal{P}\) if and only if
\[
\left( D^i_X F^i \right)(Y_i, Z_i) = \frac{n - 2}{f_i^3} \left[ g_i(X_i, Y_i) Z_i(f_i) + g_i(X_i, Z_i) Y_i(f_i) \right] \left( \nabla^i f_i \right) f_i
\]
\[+ 2 \frac{f_i}{f_i^3} X_i(f_i) g_i(Y_i, Z_i),
\]
where \(i, j = 1, 2, i \neq j\) and \(X_i, Y_i, Z_i \in \mathfrak{X}(M_i)\).
Proof. Let $M = f_1 M_1 \times f_2 M_2$ be a Ricci symmetric doubly warped product manifold, that is,

\[0 = (D_X \text{Ric})(Y, Z)\]

Equation 4 infers

\[
(D_X \text{Ric})(Y_i, Z_i) = \left(D_X \text{Ric}^i\right)(Y_i, Z_i) - \left(D_X \text{F}^i\right)(Y_i, Z_i) + 2 \frac{f_i}{f_i^2} X_i(f_i) g_i(Y_i, Z_i)
\]

\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i Y_i) \nabla f_i(f_i) Z_i(f_i)
\]

\[+ (n - 2) \frac{1}{f_i^2} g_i(X_i Z_i) \nabla f_i(f_i) Y_i(f_i).
\]

Thus, having a parallel Ricci tensor implies

\[
\left(D_X \text{Ric}^i\right)(Y_i, Z_i) = \left(D_X \text{F}^i\right)(Y_i, Z_i) - 2 \frac{f_i}{f_i^2} X_i(f_i) g_i(Y_i, Z_i)
\]

\[= \frac{n - 2}{f_i^2} g_i(X_i Y_i) Z_i(f_i) + g_i(X_i Z_i) Y_i(f_i) \nabla f_i(f_i).
\]

This equation completes the proof. □

The corresponding result on singly warped product manifolds is as follows.

**Corollary 3.6.** In a singly warped product manifold $M = M_1 \times f_2 M_2$ where $M$ is of class type $\mathcal{P}$. Then $(M_1, g_1)$ is an Einstein-like manifold of class $\mathcal{P}$ if and only if

\[
\left(D_X \text{F}^i\right)(Y_i, Z_i) = 0,
\]

where $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$. Also, $(M_2, g_2)$ is Einstein-like of class type $\mathcal{P}$.

### 3.4. Class $I \oplus B$

A doubly warped product manifold $M$ is of class type $I \oplus B$ if its Ricci tensor satisfies

\[
\nabla_i \left[ R_{ab} - \frac{R}{2(n-1)} g_{ab} \right] = \nabla_i \left[ R_{ab} - \frac{R}{2(n-1)} g_{ab} \right],
\]

that is, the tensor $\mathcal{H}_{ab} = R_{ab} - \frac{R}{2(n-1)} g_{ab}$ is a Codazzi tensor. This condition is equivalent to

\[
\nabla_i \mathcal{C}_{apb} = 0,
\]

where $\mathcal{C}$ is the Weyl conformal curvature tensor and $n \geq 3$, i.e., $M$ has a harmonic Weyl tensor. Let $g_{\phi} = \phi^2 g_{\phi}$ be a conformal change of on a manifold $M$. It is well known that the Weyl tensor $\mathcal{C}_{apb}^{\phi}$ remains invariant, that is, $\mathcal{C}_{apb}^{\phi} = \mathcal{C}_{apb}^{\phi}$ however $C_{apb}^{\phi} = \phi^2 \mathcal{C}_{apb}^{\phi}$. The divergence of the Weyl tensor is given by

\[
\nabla_i \mathcal{C}_{apb}^{\phi} = \nabla_i \mathcal{C}_{apb}^{\phi} - \frac{n - 3}{\phi} (\nabla_i \phi) \mathcal{C}_{apb}^{\phi}.
\]

The doubly warped product metric may be rewritten as follows

\[
g = f_1^2 f_2^2 \left(f_1^2 g_1 + f_2^2 g_2\right)
\]

\[= f_1^2 f_2^2 (g_1 + g_2)
\]

\[= f_1^2 f_2^2 g
\]
where \( g_i = f_i^2 \hat{g}_i \) and \( \hat{g} = \hat{g}_1 + \hat{g}_2 \). The doubly warped product manifold \((M, \hat{g})\) has harmonic Weyl tensor if and only
\[
\nabla_\epsilon C_{\alpha\beta\gamma}^\epsilon = \frac{n-3}{\varphi} \left( \nabla_\epsilon \varphi \right) C_{\alpha\beta\gamma}^\epsilon
\]
where \( \varphi = f_1 f_2 \). Assume that \( \nabla_\epsilon (f_1 f_2) C_{\alpha\beta\gamma}^\epsilon = 0 \), then
\[
\nabla_\epsilon C_{\alpha\beta\gamma}^\epsilon = 0.
\]

having a harmonic Weyl tensor is equivalent to the condition
\[
0 = \bar{\mathcal{I}}_{\alpha\beta\gamma} = \bar{\mathcal{I}}_{\alpha\beta\gamma} - \frac{1}{2(n-1)} \left( \nabla_\gamma \bar{R} g_{\alpha\beta} - \left( \nabla_\gamma \bar{R} \right) g_{\alpha\beta} \right),
\]
where \( \bar{\mathcal{I}} \) is the Cotton tensor. The metric \( \bar{g} \) splits as \( \bar{g} = \bar{g}_1 + \bar{g}_2 \) and consequently the divergence of the Cotton tensor \( \bar{\mathcal{I}} \) splits on the factor manifolds \((M_i, \bar{g}_i)\) as
\[
0 = \bar{\mathcal{I}}_{\alpha\beta\gamma}^i + \frac{n_i}{2(n-1)(n_i-1)} \left( \nabla_\gamma R^i (\bar{g}_i) g_{\alpha\beta} - \left( \nabla_\gamma R^i \right) (\bar{g}_i) g_{\alpha\beta} \right) \]
where \( \bar{\mathcal{I}}^i_{\alpha\beta\gamma} \) is constant if and only if the cotton tensor \( \bar{\mathcal{I}}^i_{\alpha\beta\gamma} \) on the doubly warped factor manifolds \((M^i, \bar{g}_i)\) vanishes i.e.
\[
\nabla_\epsilon C_{\alpha\beta\gamma}^\epsilon = 0.
\]
The Weyl tensors \( C^\epsilon \) on doubly warped product factor manifolds \((M_i, g_i)\) satisfy
\[
0 = \nabla_\epsilon C_{\alpha\beta\gamma}^\epsilon = \nabla_\epsilon C_{\alpha\beta\gamma}^\epsilon + \frac{n_i-3}{f_i} \left( \nabla_\epsilon f_i \right) C_{\alpha\beta\gamma}^\epsilon.
\]
It is time now to write the following result.

**Theorem 3.7.** In a doubly warped product manifold \( M = f_i M_1 \times f_i M_2 \) where \( M \) is of class type \( I \oplus B \). Assume that \( \nabla_\epsilon (f_1 f_2) C_{\alpha\beta\gamma}^\epsilon = 0 \) and the conformal change \( (M_i, f_i^{-2} g_i) \) has a constant scalar curvature. Then \((M_i, g_i)\) is an Einstein-like manifold of class \( I \oplus B \) if and only if \( \left( \nabla_\epsilon f_i \right) C_{\alpha\beta\gamma}^\epsilon = 0 \) for each \( i = 1, 2 \).

A. Gebarowski proved an inheritance property of this class in [18, Theorem 2].

3.5. Class \( I \oplus A \)

Doubly warped product manifolds where the tensor
\[
\mathcal{L} = \text{Ric} - \frac{2R}{n+2} g
\]
is Killing lies the class \( I \oplus A \). The above condition is equivalent to
\[
0 = \left( D_X \mathcal{L} \right) (X, X).
\]
The following theorem draw the inheritance property of this class.
\textbf{Theorem 3.8.} In a doubly warped product manifold \( M = f_j M_1 \times f_i M_2 \) where \( M \) is of class type \( I \oplus A \), the factor manifold \((M, g_i)\) is of class type \( I \oplus A \) if and only if

\[
(D_Xe^F)(X_i, X_i) = \frac{2}{f_i^2} X_i (f_i) g_i (X_i, X_i) \left[ f_i^2 + (n-2) \left( \nabla^i f_j \right) (f_j) \right] - \frac{2}{n+2} \left( D_X R - \frac{n+2}{n_i+2} D_X R^j \right) g_i (X_i, X_i).
\]

\textit{Proof.} Assume that \( M = f_j M_1 \times f_i M_2 \) be a doubly warped product manifold of class type \( I \oplus A \). Then

\[
0 = (D_X) \left( \text{Ric} - \frac{2R}{n+2} g \right)(X, X)
= (D_X \text{Ric})(X, X) - \frac{2}{n+2} g(X, X) D_X R.
\]

Using equation (2), it is

\[
0 = \left( D_X^i \text{Ric}^c \right)(X_i, X_i) - \left( D_X^i e^F \right)(X_i, X_i)
+ \frac{2}{f_i^2} X_i (f_i) g_i (X_i, X_i) \left[ f_i^2 + (n-2) \left( \nabla^i f_j \right) (f_j) \right]
- \frac{2}{n+2} (D_X R) g_i (X_i, X_i)
\]

and consequently, one has

\[
0 = \left( D_X^i \text{Ric}^c \right)(X_i, X_i) - \frac{2}{n_i+2} g_i (X_i, X_i) D_X^i
- \left( D_X^i e^F \right)(X_i, X_i)
+ \frac{2}{f_i^2} X_i (f_i) g_i (X_i, X_i) \left[ f_i^2 + (n-2) \left( \nabla^i f_j \right) (f_j) \right]
- \frac{2}{n+2} \left( D_X R - \frac{n+2}{n_i+2} D_X R^j \right) g_i (X_i, X_i)
\]

which completes the proof. \( \square \)

\[3.6. \text{Class } A \oplus B\]

This class is identified by having a constant scalar curvature. Let \( M = f_i M_1 \times f_i M_2 \) be a doubly warped product manifold of class type \( A \oplus B \), that is, the scalar curvature \( R \) of \( M \) is constant, say \( c \). The use of Equation 7 in [18] implies that \( M_i \) is of class \( A \oplus B \) if there are two constants \( c_i \) and \( c_j \) such that

\[
\frac{c_i}{f_i^2} + \frac{c_j}{f_j^2} - \frac{n_i (n_i - 1)}{f_i^2} \Delta_i f_i - \frac{n_j (n_j - 1)}{f_j^2} \Delta_j f_j - \frac{2n_i}{f_i} F_i - \frac{2n_j}{f_j} F_j = c,
\]

where \( F_i = g_i^a \nabla^i a^j f_i \).
4. Einstein-like doubly warped Relativistic space-times

Let \((M, g)\) be a Riemannian manifold, \(f : M \to (0, \infty)\) and \(\sigma : I \to (0, \infty)\) are smooth functions. The manifold \(\bar{M} = f I \times \sigma M\) furnished with the metric tensor \(\bar{g} = -f^2 dt^2 \oplus \sigma^2 g\) is called a doubly warped space-time. For \(U, V \in \mathfrak{X}(M)\), the covariant derivative \(\bar{D}\) on \(\bar{M}\) is given by

\[
\bar{D}_a \partial_t = \frac{f}{\sigma^2} \nabla f,
\]

\[
\bar{D}_a U = D_U \partial_t - \frac{\sigma}{f} U (f) \partial_t,
\]

\[
\bar{D}_a V = D_v \partial_t - \frac{\sigma f}{f^2} g(U, V) \partial_t,
\]

whereas the Ricci tensor \(\bar{Ric}\) on \(\bar{M}\) is given by

\[
\bar{Ric}(\partial_t, \partial_t) = \frac{n}{\sigma} \frac{f^2}{\sigma^2},
\]

\[
\bar{Ric}(U, V) = \bar{Ric}(U, V) - \frac{1}{f^3} H f' (U, V) - \frac{\sigma}{f^2} g(U, V),
\]

\[
\bar{Ric}(\partial_t, U) = (n - 1) \frac{\sigma}{f} U (\ln f).
\]

For the definition and relativistic significance of doubly warped space-times, the reader is referred to [15, 32] and references therein.

**Theorem 4.1.** In a doubly warped space-time \(\bar{M} = f I \times \sigma M\) of class type \(A\), \(M\) is an Einstein-like manifold of class type \(A\) if and only if

\[
(\bar{D}_V \mathcal{F})(V, V) = \left((n - 1) \frac{\sigma}{f} + \sigma^2\right) \frac{2}{f^3} V(f) g(V, V).
\]

**Theorem 4.2.** In a doubly warped space-time \(\bar{M} = f I \times \sigma M\) of class type \(B\), \(M\) is an Einstein-like manifold of class type \(B\) if and only if

\[
(\bar{D}_V \mathcal{F})(U, V) = (\bar{D}_V \mathcal{F})(W, V) + \left(2\sigma^2 - (n - 1) \frac{\sigma}{f} \right) \frac{1}{f^3} W(f) g(U, V) - \left(2\sigma^2 + (n - 1) \frac{\sigma}{f} \right) \frac{1}{f^3} U(f) g(W, V).
\]

**Theorem 4.3.** In a doubly warped space-time \(\bar{M} = f I \times \sigma M\) of class type \(P\), \(M\) is an Einstein-like manifold of class type \(P\) if and only if

\[
(\bar{D}_V \mathcal{F})(U, V) = \frac{2\sigma^2}{f^3} W(f) g(U, V) + \frac{\sigma^2}{f^3} (n - 1) (g(W, V) U(f) + g(W, U) V(f)).
\]

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