A Sharp Decay Estimate
for Positive Nonlinear Waves

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Abstract. We consider a strictly hyperbolic, genuinely nonlinear system of conservation laws in one space dimension. A sharp decay estimate is proved for the positive waves in an entropy weak solution. The result is stated in terms of a partial ordering among positive measures, using symmetric rearrangements and a comparison with a solution of Burgers’ equation with impulsive sources.

1 - Introduction

Consider a strictly hyperbolic system of \( n \) conservation laws

\[
{u_t} + f(u)_x = 0
\]

and assume that all characteristic fields are genuinely nonlinear. Call \( \lambda_1(u) < \cdots < \lambda_n(u) \) the eigenvalues of the Jacobian matrix \( A(u) = Df(u) \). We shall use bases of left and right eigenvectors \( l_i(u), r_i(u) \) normalized so that

\[
\nabla \lambda_i(u) \cdot r_i(u) \equiv 1, \quad l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

Given a function \( u : \mathbb{R} \rightarrow \mathbb{R}^n \) with small total variation, following [BC], [B] one can define the measures \( \mu^i \) of \( i \)-waves in \( u \) as follows. Since \( u \in BV \), its distributional derivative \( D_x u \) is a Radon measure. We define \( \mu^i \) as the measure such that

\[
\mu^i \doteq l_i(u) \cdot D_x u
\]

restricted to the set where \( u \) is continuous, while, at each point \( x \) where \( u \) has a jump, we define

\[
\mu^i(\{x\}) \doteq \sigma_i,
\]
where \( \sigma_i \) is the strength of the \( i \)-wave in the solution of the Riemann problem with data \( u^- = u(x^-), u^+ = u(x^+) \). In accordance with (1.2), if the solution of the Riemann problem contains the intermediate states \( u^- = \omega_0, \omega_1, \ldots, \omega_n = u^+ \), the strength of the \( i \)-wave is defined as

\[
\sigma_i \doteq \lambda_i(\omega_i) - \lambda_i(\omega_{i-1}).
\]  

(1.5)

Observing that

\[
\sigma_i = l_i(u^+) \cdot (u^+ - u^-) + O(1) \cdot |u^+ - u^-|^2,
\]

we can find a vector \( l_i(x) \) such that

\[
\|l_i(x) - l_i(u(x^+))\| = O(1) \cdot |u(x^+) - u(x^-)|,
\]

(1.6)

\[
\sigma_i = l_i(x) \cdot (u(x^+) - u(x^-)).
\]

(1.7)

We can thus define the measure \( \mu^i \) equivalently as

\[
\mu^i \doteq l_i \cdot D_x u,
\]

(1.8)

where \( l_i(x) = l_i(u(x)) \) at points where \( u \) is continuous, while \( l_i(x) \) is some vector which satisfies (1.6)-(1.7) at points of jump. For all \( x \in \mathbb{R} \) there holds

\[
\|l_i(x) - l_i(u(x))\| = O(1) \cdot |u(x^+) - u(x^-)|.
\]

(1.9)

We call \( \mu^{i^+}, \mu^{i^-} \) respectively the positive and negative parts of \( \mu^i \), so that

\[
\mu^i = \mu^{i^+} - \mu^{i^-}, \quad |\mu^i| = \mu^{i^+} + \mu^{i^-}.
\]

(1.10)

It is our purpose to prove a sharp estimate on the decay of the density of the measures \( \mu^{i^+} \). This will be achieved by introducing a partial ordering within the family of positive Radon measures. In the following, \( \text{meas}(A) \) denotes the Lebesgue measure of a set \( A \).

**Definition 1.** Let \( \mu, \mu' \) be two positive Radon measures. We say that \( \mu \leq \mu' \) if and only if

\[
\sup_{\text{meas}(A) \leq s} \mu(A) \leq \sup_{\text{meas}(B) \leq s} \mu'(B) \quad \text{for every } s > 0.
\]

(1.11)

In some sense, the above relation means that \( \mu' \) is more singular than \( \mu \). Namely, it has a greater total mass, concentrated on regions with higher density. Notice that the usual order relation

\[
\mu \leq \mu' \quad \text{if and only if} \quad \mu(A) \leq \mu'(A) \quad \text{for every } A \subset \mathbb{R}
\]

(1.12)

is much stronger. Of course \( \mu \leq \mu' \) implies \( \mu \leq \mu' \), but the converse does not hold.
Following [BC], [B], together with the measures $\mu^i$ we define the Glimm functionals

$$V(u) = \sum_i |\mu^i| (\mathcal{R}),$$

(1.13)

$$Q(u) = \sum_{i<j} (|\mu^i| \otimes |\mu^j|) \{ (x, y); \ x < y \} + \sum_i (\mu^i- \otimes |\mu^i|) \{ (x, y); \ x \neq y \}.$$  

(1.14)

Let now $u = u(t, x)$ be an entropy weak solution of (1.1). If the total variation of $u$ is small and the constant $C_0$ is large enough, it is well known that the quantities

$$Q(t) = Q(u(t)),$$

$$\Upsilon(t) = V(u(t)) + C_0 Q(u(t))$$

(1.15)

are non-increasing in time. The decrease in $Q$ controls the amount of interaction, while the decrease in $\Upsilon$ controls both the interaction and the cancellation in the solution.

An accurate estimate on the measure $\mu^i_{t+}$ of positive $i$-waves in $u(t, \cdot)$ will be obtained by a comparison with a solution of Burgers’ equation with source terms.

**Theorem 1.** For some constant $\kappa$ and for every small BV solution $u = u(t, x)$ of the system (1.1) the following holds. Let $w = w(t, x)$ be the solution of the scalar Cauchy problem with impulsive source term

$$w_t + (w^2/2)_x = -\kappa \text{sgn}(x) \cdot \frac{d}{dt} Q(u(t)),$$

(1.16)

$$w(0, x) = \text{sgn}(x) \cdot \sup_{\text{meas}(A) < 2|x|} \frac{\mu^i_{t+}(A)}{2}. $$

(1.17)

Then, for every $t \geq 0$,

$$\mu^i_{t+} \leq D_x w(t).$$

(1.18)

As shown in the next section, the initial data in (1.17) represents the odd rearrangement of the function $v_i(x) \doteq \mu^i_{0+}([-\infty, x])$. The above theorem improves the earlier estimate derived in [BC]. For a scalar conservation law with strictly convex flux, a classical decay estimate was proved by Oleinik [O]. In the case of genuinely nonlinear systems, results related to the decay of nonlinear waves were also obtained in [GL], [L1], [L2], [BG]. An application of the present analysis will appear in [BY], where Theorem 1 is used to estimate the rate of convergence of vanishing viscosity approximations.

**2 - Lower semicontinuity**
Let $\mu$ be a positive Radon measure on $\mathbb{R}$, so that $\mu = D_x v$ is the distributional derivative of some bounded, non-decreasing function $v : \mathbb{R} \mapsto \mathbb{R}$. We can decompose

$$\mu = \mu^{\text{sing}} + \mu^{\text{ac}}$$

as the sum of a singular and an absolutely continuous part, w.r.t. Lebesgue measure. The absolutely continuous part corresponds to the usual derivative $z = v_x$, which is a non-negative $L^1$ function defined at a.e. point. We shall denote by $\hat{z}$ the symmetric rearrangement of $z$, i.e. the unique even function such that

$$\hat{z}(x) = \hat{z}(-x), \quad \hat{z}(x) \geq \hat{z}(x') \quad \text{if} \quad 0 < x < x', \quad (2.1)$$

$$\meas\left(\{x; \hat{z}(x) > c\}\right) = \meas\left(\{x; z(x) > c\}\right) \quad \text{for every} \quad c > 0. \quad (2.2)$$

Moreover, we define the odd rearrangement of $v$ as the unique function $\hat{v}$ such that (fig. 1)

$$\hat{v}(-x) = -\hat{v}(x), \quad \hat{v}(0+) = \frac{1}{2} \mu^{\text{sing}}(\mathbb{R}), \quad (2.3)$$

$$\hat{v}(x) = \hat{v}(0+) + \int_0^x z(y) \, dy \quad \text{for} \quad x > 0. \quad (2.4)$$

By construction, the function $\hat{v}$ is convex for $x < 0$ and concave for $x > 0$.

![figure 1](image-url)

The relation between the odd rearrangement $\hat{v}$ and the partial ordering (1.10) is clarified by the following result, which is an easy consequence of the definitions.

**Proposition 1.** Let $\mu = D_x v$ and $\mu' = D_x v'$ be positive Radon measures. Call $\hat{v}, \hat{v}'$ the odd rearrangements of $v, v'$, respectively. Then $\mu \preceq D_x \hat{v} \preceq \mu$ and moreover

$$\hat{v}(x) = \text{sgn}(x) \cdot \sup_{\meas(A) \leq 2|x|} \frac{\mu(A)}{2}, \quad (2.5)$$
\[ \mu \preceq \mu' \quad \text{if and only if} \quad \hat{v}(x) \leq \hat{v}'(x) \quad \text{for all} \ x > 0. \quad (2.6) \]

Two more results will be used in the sequel. By the restriction of a measure \( \mu \) to a set \( J \), we mean the measure

\[ (\mu[J])(A) = \mu(A \cap J). \]

**Proposition 2.** Let \( \mu, \mu' \) be positive measures. Consider any finite partition \( \mathcal{R} = J_1 \cup \cdots \cup J_N \). If the restrictions of \( \mu, \mu' \) to each set \( J_\ell \) satisfy \( \mu|_{J_\ell} \preceq \mu'|_{J_\ell} \), then \( \mu \preceq \mu' \).

**Proposition 3.** Assume that \( \mu \preceq D_s w \) for some nondecreasing odd function \( w \). If \( |\hat{v} - \hat{\mu}|(\mathcal{R}) \leq \varepsilon \), then

\[ \hat{v} \preceq D_s \left[ w + \text{sgn}(s) \cdot \frac{\varepsilon}{2} \right]. \]

The next result is concerned with the lower semicontinuity of the partial ordering \( \preceq \) w.r.t. weak convergence of measures.

**Proposition 4.** Consider a sequence of measures \( \mu_\nu \) converging weakly to a measure \( \mu \). Assume that the positive parts satisfy \( \nu^+ \preceq Dw_\nu \) for some odd, nondecreasing functions \( s \mapsto w_\nu(s) \), concave for \( s > 0 \). Let \( w \) be the odd function such that

\[ w(s) = \liminf_{\nu \to \infty} w_\nu(s) \quad \text{for} \quad s > 0. \]

Then the positive part of \( \mu \) satisfies

\[ \mu^+ \preceq D_sw. \quad (2.7) \]

**Proof.** By possibly taking a subsequence, we can assume that \( w_\nu(s) \to w(s) \) for all \( s \neq 0 \). Moreover, we can assume the weak convergence

\[ \nu^+_\nu \to \tilde{\mu}^+, \quad \nu^-_\nu \to \tilde{\mu}^-, \]

for some positive measures \( \tilde{\mu}^+ \), \( \tilde{\mu}^- \). We thus have

\[ \mu = \tilde{\mu}^+ - \tilde{\mu}^- \quad \mu^+ \leq \tilde{\mu}^+, \quad \mu^- \leq \tilde{\mu}^-. \quad (2.8) \]

By (2.8) it suffices to prove that \( \tilde{\mu}^+ \preceq D_sw \), i.e.

\[ \text{meas}(A) \leq 2s \quad \implies \quad \tilde{\mu}^+(A) \leq 2w(s), \quad (2.9) \]
for every $s > 0$ and every Borel measurable set $A \subset \mathcal{B}$. If (2.9) fails, there exists $s > 0$ and a set $A$ such that
\[ \text{meas} (A) = 2s, \quad \tilde{\mu}^+(A) > 2w(s) = 2 \lim_{\nu \to \infty} w_{\nu}(s). \]
Since $w$ is continuous for $s > 0$, we can choose an open set $A' \supset A$ such that, setting $s' = \text{meas} (A')/2$, one has $2w(s') < \tilde{\mu}^+(A)$. By the weak convergence $\mu^+_{\nu} \rightharpoonup \tilde{\mu}^+$ one obtains
\[ \tilde{\mu}^+(A') \leq \liminf_{\nu \to \infty} \mu^+_{\nu}(A') \leq 2w(s') < \tilde{\mu}^+(A), \]
reaching a contradiction. Hence (2.9) must hold.

Toward the proof of Theorem 1 we shall need a lower semicontinuity property for wave measures, similar to what proved in [BaB]. In the following, $C_0$ is the same constant as in (1.15).

**Lemma 1.** Consider a sequence of functions $u_{\nu}$ with uniformly small total variation and call $\mu^+_i$ the corresponding measures of positive $i$-waves. Let $s \mapsto w_{\nu}(s), \nu \geq 1$, be a sequence of odd, nondecreasing functions, concave for $s > 0$, such that
\[ \mu^+_i \preceq D_s \left[ w_{\nu} + C_0 \text{sgn}(s)(Q_0 - Q(u_{\nu})) \right] \quad \text{(2.10)} \]
for some $Q_0$. Assume that $u_{\nu} \to u$ and $w_{\nu} \to w$ in $L^1_{\text{loc}}$. Then the measure of positive $i$-waves in $u$ satisfies
\[ \mu^+_i \preceq D_s \left[ w + C_0 \text{sgn}(s)(Q_0 - Q(u)) \right]. \quad \text{(2.11)} \]

**Proof.** The main steps follow the proof of Theorem 10.1 in [B].

1. By possibly taking a subsequence we can assume that $u_{\nu}(x) \to u(x)$ for every $x$ and that the measures of total variation converge weakly, say
\[ |\mu_{\nu}| \equiv |D_x u_{\nu}| \rightharpoonup \mu^z \quad \text{(2.12)} \]
for some positive Radon measure $\mu^z$. In this case one has $\mu^z \geq |\mu|$, in the sense of (1.12).

2. Let any $\varepsilon > 0$ be given. Since the total mass of $\mu^z$ is finite, one can select finitely many points $y_1, \ldots, y_N$ such that
\[ \mu^z(\{x\}) < \varepsilon, \quad \text{for all} \quad x \notin \{y_1, \ldots, y_N\}. \quad \text{(2.13)} \]
We now choose disjoint open intervals $I_k \equiv [y_k - \rho, y_k + \rho[$ such that
\[ \mu^z(I_k \setminus \{y_k\}) < \frac{\varepsilon}{N} \quad \text{for all} \quad k = 1, \ldots, N. \quad \text{(2.14)} \]
Moreover, we choose $R > 0$ such that
\[ \bigcup_{k=1}^{N} I_k \subset [-R, R], \quad \mu^\sharp([-\infty, -R] \cup [R, \infty]) < \varepsilon. \] (2.15)

Because of (2.13), we can now choose points $p_0 < -R < p_1 < \cdots < R < p_r$ which are continuity points for $u$ and for every $u_\nu$, such that
\[ \mu^\sharp(\{p_h\}) = 0, \quad u_\nu(p_h) \to u(p_h) \quad \text{for all } h = 0, \ldots, r, \] (2.16)
and such that either
\[ p_h - p_{h-1} < \frac{\varepsilon}{N}, \quad p_{h-1} < y_k < p_h, \quad [p_{h-1}, p_h] \subset I_k, \] (2.17)
for some $k \in \{1, \ldots, N\}$, or else
\[ |\mu([p_{h-1}, p_h])| \leq \mu^\sharp([p_{h-1}, p_h]) < \varepsilon. \] (2.18)

Call $J_h = [p_{h-1}, p_h]$. If (2.18) holds, by weak convergence for some $\nu_0$ sufficiently large one has
\[ |\mu_\nu|(J_h) < \varepsilon \quad \text{for all } \nu \geq \nu_0. \] (2.19)

On the other hand, if (2.17) holds, from (2.14) it follows
\[ |\mu|(J_h \setminus \{y_k\}) \leq \mu^\sharp(J_h \setminus \{y_k\}) < \frac{\varepsilon}{N}. \] (2.20)

In the remainder of the proof, the main strategy is as follows.

- On the intervals $J_{h(k)}$ containing a point $y_k$ of large oscillation, we first replace each $u_\nu$ by a piecewise constant function $\tilde{u}_\nu$ having a single jump at $y_k$. The relations between the corresponding measures $\mu^i_\nu$ and $\tilde{\mu}^i_\nu$ are given by Lemma 10.2 in [B]. Then we take the limit as $\nu \to \infty$.

- On the remaining intervals $J_h$ with small oscillation, we replace the left eigenvectors $l_i(u_\nu)$ by a constant vector $l_i(u_\nu^*)$. Then we use Proposition 4 to estimate the limit as $\nu \to \infty$.

3. We first take care of the intervals $J_h$ containing a point $y_k$ of large oscillation, so that (2.17) holds. For each $k = 1, \ldots, N$, let $h = h(k) \in \{1, \ldots, r\}$ be the index such that $y_k \in J_h = [p_{h-1}, p_h]$. For every $\nu \geq 1$ consider the function
\[ \tilde{u}_\nu(x) = \begin{cases} u_\nu(x) & \text{if } x \notin \bigcup_k J_{h(k)}, \\ u_\nu(p_{h(k)-1}) & \text{if } x \in [p_{h(k)-1}, y_k], \\ u_\nu(p_h) & \text{if } x \in [y_k, p_{h(k)}]. \end{cases} \]
Observe that all functions $u, \tilde{u}_\nu$ are continuous at every point $p_0, \ldots, p_r$ and have jumps at $y_1, \ldots, y_N$. Call $\tilde{\mu}_\nu, i = 1, \ldots, n$, the corresponding measures, defined as in (1.8) with $u$ replaced by $\tilde{u}_\nu$. Clearly $\tilde{\mu}_\nu = \mu^+_\nu$ outside the intervals $J_{h(k)}$ of large oscillation. By Lemma 10.2 at p.203 in [B], there holds

$$Q(\tilde{u}_\nu) \leq Q(u_\nu), \quad V(\tilde{u}_\nu) + C_0 Q(\tilde{u}_\nu) \leq V(u_\nu) + C_0 \cdot Q(u_\nu),$$

$$\tilde{\mu}^+_{i\nu}(\mathbb{R}) - \mu^+_{i\nu}(\mathbb{R}) \leq C_0 \left[ Q(u_\nu) - Q(\tilde{u}_\nu) \right].$$

As a consequence, from (2.10) we deduce

$$\tilde{\mu}^+_{i\nu} \leq D_s \left[ T^\nu \omega_{i\nu} + C_0 \text{sgn}(s) (Q_0 - Q(\tilde{u}_\nu)) \right],$$

where

$$T^\nu \omega(s) = \begin{cases} w(s + \varepsilon/2) & \text{if } s > 0, \\ w(s - \varepsilon/2) & \text{if } s < 0. \end{cases}$$

Indeed, all the mass which in $\mu^+_{i\nu}$ lies on the set

$$\Omega \doteq \bigcup_{k=1}^N J_{h(k)}, \quad J_h \doteq [p_{h-1}, p_h]$$

is replaced in $\tilde{\mu}^+_{i\nu}$ by point masses at $y_1, \ldots, y_N$. We obtain (2.21) by observing that, by (2.17), $\text{meas}(\Omega) < \varepsilon$. Moreover, the increase in the total mass is $\leq C_0 [Q(u_\nu) - Q(\tilde{u}_\nu)]$.

Since $u_\nu(p_h) \rightarrow u(p_h)$ for every $h$, there holds

$$\left| \mu^i(\{y_k\}) - \tilde{\mu}^i_{\nu}(\{y_k\}) \right| = O(1) \cdot \left\{ |u(y_k^-) - u(p_{h(k)-1})| + |u(y_k^+) - u(p_{h(k)})| + \left| u(p_{h(k)-1}) - u_\nu(p_{h(k)-1}) \right| + \left| u(p_{h(k)}) - u_\nu(p_{h(k)}) \right| \right\}$$

$$= O(1) \cdot \frac{\varepsilon}{N}$$

for each $k = 1, \ldots, N$ and all $\nu$ sufficiently large. By construction we also have

$$|\tilde{\mu}^i_{\nu}(J_{h(k)} \setminus \{y_k\})| = 0, \quad |\mu^i(\{y_k\}) \setminus \{y_k\})| = O(1) \cdot \frac{\varepsilon}{N}.$$ (2.23)

4. Next, call $\mathcal{S} \doteq \{ h : \mu^i(J_h) < \varepsilon \}$ the family of intervals where the oscillation of every $u_\nu$ is small, so that (2.18) holds. If $h \in \mathcal{S}$, for every $x, y \in J_h$ and $\nu$ sufficiently large, one has

$$|u_\nu(x) - u_\nu(y)| \leq \left| \mu_\nu(J_h) \right| < \varepsilon,$$

$$|u(x) - u(y)| \leq |\mu(J_h)| \leq \mu^i(J_h) < \varepsilon.$$
Set \( u_h^* \equiv u(p_h) \). By the pointwise convergence \( u_\nu(p_h) \to u(p_h) \) and the two above estimates it follows

\[
|u_\nu(x) - u_h^*| < \varepsilon, \quad |u(x) - u_h^*| < \varepsilon, \quad \text{for all } x \in J_h.
\]

\[ (2.24) \]

5. We now introduce the measures \( \hat{\mu}_\nu^i \) such that

\[
\hat{\mu}_\nu^i = l_i(u_h^*) \cdot D_x u_\nu
\]

restricted to each interval \( J_h, h \in S \) where the oscillation is small, while

\[
\hat{\mu}_\nu^i = \hat{\mu}_\nu^i
\]

on each interval \( J_h = J_{h(k)} \) where the oscillation is large. Observe that the restriction of \( \hat{\mu}_\nu^i \) to \( J_{h(k)} \) consists of a single mass at the point \( y_k \). Namely, \( \hat{\mu}_\nu^i(\{y_k\}) \) is precisely the size of the \( i \)-th wave in the solution of the Riemann problem with data \( u^- = u_\nu(p_{h(k)-1}), u^+ = u_\nu(p_{h(k)}) \).

We define \( \hat{w}_\nu \) as the non-decreasing odd function such that

\[
\hat{w}_\nu(s) \equiv \sup_{\meas(A) \leq 2s} \frac{\hat{\mu}_\nu^{i+}(A)}{2}, \quad s > 0.
\]

\[ (2.25) \]

By possibly taking a further subsequence we can assume the convergence

\[
Q(\bar{u}_\nu) \to \overline{Q}, \quad \hat{\mu}_\nu^i \to \hat{\mu}_\nu^i, \quad \hat{w}_\nu(s) \to \hat{w}(s).
\]

Using (2.16), we can apply Proposition 4 on each interval \( J_h \) and obtain

\[
\hat{\mu}_\nu^{i+} \preceq D_s \hat{w}.
\]

\[ (2.26) \]

6. Observe that, by (2.24) and (2.19),

\[
|\hat{\mu}_\nu^i - \mu_\nu^i|(J_h) = \mathcal{O}(1) \cdot \varepsilon \mu_\nu^2(J_h) \quad h \in S,
\]

\[ (2.27) \]

From (2.21) and the definition of \( \hat{w}_\nu \) at (2.25) it thus follows

\[
\hat{w}_\nu(s) \leq T^\varepsilon w_\nu(s) + C_0 [Q_0 - Q(\bar{u}_\nu)] + \mathcal{O}(1) \cdot \varepsilon \quad s > 0.
\]

\[ (2.28) \]

Letting \( \nu \to \infty \) we obtain

\[
\hat{w}(s) \leq T^\varepsilon w(s) + C_0 [Q_0 - \overline{Q}] + \mathcal{O}(1) \cdot \varepsilon \quad s > 0,
\]

\[ (2.29) \]

\[
\overline{Q} = \lim_{\nu \to \infty} Q(\bar{u}_\nu) \geq \lim_{\nu \to \infty} Q(u_\nu) - \mathcal{O}(1) \cdot \varepsilon \geq Q(\bar{u}) - \mathcal{O}(1) \cdot \varepsilon,
\]

\[ (2.30) \]

because of the lower semicontinuity of the functional \( u \mapsto Q(u) \). From (2.26), (2.29) and (2.30) we deduce

\[
\hat{\mu}_\nu^{i+} \preceq D_s \left[ T^\varepsilon w + \sgn(s) \left( C_0 [Q_0 - Q(\bar{u})] + \mathcal{O}(1) \cdot \varepsilon \right) \right].
\]

By (2.22)-(2.24), our construction of the measure \( \hat{\mu}_\nu^i \) achieves the property

\[
|\mu_\nu^{i+} - \hat{\mu}_\nu^{i+}|(\mathbb{R}) = \mathcal{O}(1) \cdot \varepsilon.
\]

Hence, by Proposition 3,

\[
\mu_\nu^{i+} \preceq D_s \left[ T^\varepsilon w + \sgn(s) \left( C_0 [Q_0 - Q(\bar{u})] + \mathcal{O}(1) \cdot \varepsilon \right) \right].
\]

Since \( \varepsilon > 0 \) was arbitrary, this proves (2.11). \( \square \)
3 - A decay estimate

The second basic ingredient in the proof is the following lemma, which refines the estimate in [BC].

Lemma 2. For some constant \( \kappa > 0 \) the following holds. Let \( u = u(t,x) \) be any entropy weak solution of (1.1), with initial data \( u(0,x) = \bar{u}(x) \) having small total variation. Then the measure \( \mu^{i+}_t \) of positive \( i \)-waves in \( u(t,\cdot) \) can be estimated as follows.

Let \( w : [0,\tau] \times \mathbb{R} \rightarrow \mathbb{R} \) be the solution of Burgers’ equation

\[
    w_t + \left( \frac{w^2}{2} \right)_x = 0
\]

with initial data

\[
    w(0,x) = \text{sgn}(x) \cdot \sup_{\text{meas}(A) \leq 2|x|} \frac{\mu^{i+}_0(A)}{2}.
\]

Set

\[
    w(\tau,x) = w(\tau-,x) + \kappa \text{sgn}(x) \cdot \left[ Q(\bar{u}) - Q(u(\tau)) \right].
\]

Then

\[
    \mu^{i+}_\tau \leq D_x w(\tau).
\]

Proof. The main steps follow the proof of Theorem 10.3 in [B]. We first prove the estimate (3.3) under the additional hypothesis:

(H) There exist points \( y_1 < \cdots < y_m \) such that the initial data \( \bar{u} \) is smooth outside such points, constant for \( x < y_1 \) and \( x > y_m \), and the derivative component \( l_i(u)u_x \) is constant on each interval \( ]y_\ell, y_{\ell+1}[ \). Moreover, the Glimm functional \( t \mapsto Q(u(t)) \) is continuous at \( t = \tau \).

1. The solution \( u = u(t,x) \) can be obtained as limit of front tracking approximations. In particular, we consider a particular converging sequence \( (u_\nu)_\nu \geq 1 \) of \( \varepsilon_\nu \)-approximate solutions with the following additional properties:

   (i) Each \( i \)-rarefaction front \( x_\alpha \) travels with the characteristic speed of the state on the right:

   \[
   \dot{x}_\alpha = \lambda_i(u(x_\alpha^+)).
   \]

   (ii) Each \( i \)-shock front \( x_\alpha \) travels with a speed strictly contained between the right and the left characteristic speeds:

   \[
   \lambda_i(u(x_\alpha^+)) < \dot{x}_\alpha < \lambda_i(u(x_\alpha^-)).
   \]
(iii) As $\nu \to \infty$, the interaction potentials satisfy
\[ Q(u_\nu(0, \cdot)) \to Q(\bar{u}). \] (3.6)

2. Let $u_\nu$ be an approximate solution constructed by the front tracking algorithm. By a (generalized) $i$-characteristic we mean an absolutely continuous curve $x = x(t)$ such that
\[ \dot{x}(t) \in [\lambda_i(u_\nu(t, x^-)), \lambda_i(u_\nu(t, x^+))] \]
for a.e. $t$. If $u_\nu$ satisfies the above properties (i)-(ii), then the $i$-characteristics are precisely the polygonal lines $x : [0, \tau] \mapsto \mathbb{R}$ for which the following holds. For a suitable partition $0 = t_0 < t_1 < \cdots < t_m = \tau$, on each subinterval $[t_{j-1}, t_j]$ either $\dot{x}(t) = \lambda_i(u_\nu(t, x))$, or else $x$ coincides with a wave-front of the $i$-th family. For a given terminal point $\bar{x}$ we shall consider the minimal backward $i$-characteristic through $\bar{x}$, defined as
\[ y(t) = \min \{ x(t) ; \ x \text{ is an } i \text{-characteristic, } x(\tau) = \bar{x} \}. \]
Observe that $y(\cdot)$ is itself an $i$-characteristic. By (3.5), it cannot coincide with an $i$-shock front of $u$ on any nontrivial time interval.

In connection with the exact solution $u$, we define an $i$-characteristic as a curve
\[ t \mapsto x(t) = \lim_{\nu \to \infty} x_\nu(t) \]
which is the limit of $i$-characteristics in a sequence of front tracking solutions $u_\nu \to u$.

3. Let $\varepsilon > 0$ be given. If the assumption (H) holds, the measure $\mu^i_\tau$ of $i$-waves in $u(\tau)$ is supported on a bounded interval and is absolutely continuous w.r.t. Lebesgue measure. We can thus find a piecewise constant function $\psi^\tau$ with jumps at points $x_1(\tau) < \bar{x}_2(\tau) < \cdots < \bar{x}_N(\tau)$ such that
\[ \int |d\mu^i_\tau/dx - \psi^\tau| \, dx \leq \varepsilon, \quad \int_{x_j(\tau)}^{x_{j+1}(\tau)} \left( \frac{d\mu^i_\tau}{dx} - \psi^\tau \right) \, dx = 0 \quad j = 1, \ldots, N - 1. \] (3.7)
To prove the lemma in this special case, relying on Proposition 2, it thus suffices to find $i$-characteristics $t \mapsto x_j(t)$ such that the following holds (fig. 2)

(i) For each $j = 1, \ldots, N$, the function $\psi^\tau$ is constant on the interval $[x_j(\tau), x_{j+1}(\tau)]$ and (3.7) holds. Moreover, either $x_j(0) = x_{j+1}(0)$, or else the derivative component $\psi^0 \equiv t^i(u)x(t, \cdot)$ is constant on the interval $[x_j(0), x_{j+1}(0)]$.

(ii) An estimate corresponding to (3.3)-(3.4) holds restricted to each subinterval $[x_j(\tau), x_{j+1}(\tau)]$.

We need to explain in more detail this last statement. Define

$$I_j(t) = [x_j(t), x_{j+1}(t)], \qquad \Delta_j = \{(t, x); \ t \in [0, \tau], \ x \in I_j(t)\}.$$ 

For each $j$, we denote by $\Gamma_j$ the total amount of wave interaction within the domain $\Delta_j$. This is defined as in [B], first for a sequence of front tracking approximations $u_\nu$, then taking a limit as $\nu \to \infty$. Furthermore, we define the constant values

$$\psi^\tau_j = \psi^\tau(x) \quad x \in I_j(\tau),$$

$$\psi^0_j = \psi^0(x) \quad x \in I_j(0),$$

Call

$$\sigma^0_j = \lim_{t \to 0^+} \mu^{i^+}(I_j(t))$$

the initial amount of positive $i$-waves inside the interval $I_j$.

For each interval $I_j$, we consider on one hand the function $w^\tau_j$ corresponding to (3.2)-(3.3), namely

$$w^\tau_j(s) = \min \left\{ \sigma^0_j, \frac{s}{\tau + (\psi^0_j)^{-1}} \right\} + \kappa \Gamma_j \cdot \text{sgn}(s).$$

Here $(\psi^0_j)^{-1} \equiv 0$ in the case where $x_j(0) = x_{j+1}(0)$. This may happen when the initial data has a jump at $x_j(0)$, and the corresponding measure $\mu^{i^+}$ has a Dirac mass (with infinite density) at that point.

On the other hand, we look at the nondecreasing, odd function $\eta_j$ such that

$$\eta_j(s) = \min \left\{ \psi^\tau_j s, \ \psi^\tau_j \left[ x_{j+1}(\tau) - x_j(\tau) \right] \right\} \quad s > 0.$$ 

Our basic goal is to prove that (fig. 3)

$$\eta_j(s) \leq w^\tau_j(s) \quad \text{for all } s > 0.$$ 

(3.8)

Indeed, by (3.7), for $s > 0$ one has

$$\sup_{\text{meas}(A) \leq 2s} \frac{\mu^{i^+}(A \cap I_j(\tau))}{2} \leq \eta_j(s) + \varepsilon_j.$$
with
\[ \sum_j \varepsilon_j < \varepsilon. \]

Proving (3.8) for each \( j \) will thus imply
\[ \mu_{\tau}^{i+} \leq w(\tau, x) = w(\tau-, x) + \kappa \text{sgn}(x) \cdot \left[ Q(\bar{u}) - Q(u(\tau)) + \mathcal{O}(1) \cdot \varepsilon \right]. \]

Since \( \varepsilon > 0 \) was arbitrary, this establishes the lemma under the additional assumptions (H).

4. We now work toward a proof of (3.8), in three cases.

Case 1: \( \sigma_j^0 = 0. \)

Case 2: \( x_j(0) = x_{j+1}(0) \) and \( \sigma_j^0 > 0. \)

Case 3: \( x_j(0) < x_{j+1}(0) \) and \( \sigma_j^0 = (x_{j+1}(0) - x_j(0)) \psi_j^0 > 0. \)

In Case 1 the proof is easy. Indeed, the total amount of positive \( i \)-waves in \( I_j(\tau) \) is here bounded by a constant times the total amount of interaction taking place inside the domain \( \Delta_j \), i.e.
\[ \mu_{\tau}^{i+}(I_j(\tau)) \leq C_0 \cdot \Gamma_j \]
for some constant \( C_0 \). On the other hand
\[ w_j^\tau(s) = \kappa \Gamma_j \cdot \text{sgn}(s). \]

Choosing \( \kappa > C_0 \) we achieve (3.8).
5. Since Case 2 can be obtained from Case 3 in the limit as $x_{j+1} - x_j \to 0$, we shall only give a proof for Case 3.

We can again distinguish two cases. If the amount of interaction $\Gamma_j$ is large compared with the initial amount of $i$-waves, say
\[
\Gamma_j \geq \frac{1}{6C_0} \sigma_j^0,
\]
then the bound (3.8) is readily achieved choosing $\kappa > 8C_0$. Indeed, for $s > 0$ we have
\[
\eta_j(s) \leq \frac{1}{2} \mu_{\tau}^+ (I_j(\tau)) \leq C_0 \Gamma_j + \sigma_j^0 \leq 7C_0 \Gamma_j.
\]

The more difficult case to analyse is when $\Gamma_j$ is small, say
\[
\Gamma_j < \sigma_j^0 / 6C_0.
\]
Looking at figure 3, it clearly suffices to prove (3.8) for the single value
\[
s = s_j^* = \frac{x_{j+1}(\tau) - x_j(\tau)}{2}.
\]
Equivalently, calling
\[
z_j(t) = x_{j+1}(t) - x_j(t)
\]
the length of the interval $I_j(t)$ and
\[
\sigma_j^\tau = \mu_{\tau}^+ (I_j(\tau)) = z_j(\tau) \psi_j^\tau
\]
the total amount of positive $i$-waves inside $I_j(\tau)$, we need to show that
\[
\sigma_j^\tau \leq 2\kappa \Gamma_j + \min \left\{ \sigma_j^0, \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}} \right\}.
\]
(3.10)

By the approximate conservation of $i$-waves over the region $\Delta_j$, we can write
\[
\sigma_j^\tau \leq \sigma_j^0 + C_0 \Gamma_j.
\]
(3.11)
Using (3.11) in (3.10), our task is reduced to showing that
\[
\sigma_j^\tau \leq 2\kappa \Gamma_j + \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}}
\]
(3.12)
for a suitably large constant $\kappa$. Because of (3.11), it suffices to show that
\[
z_j(\tau) \geq (\sigma_j^0 - C' \Gamma_j)(\tau + (\psi_j^0)^{-1})
\]
\[= [z_j(0) + \tau \sigma_j^0] - C'(\tau + (\psi_j^0)^{-1}) \Gamma_j.
\]
(3.13)
for a suitable constant $C'$.

6. We now prove (3.13). Notice that, by genuine nonlinearity and the normalization (1.2), if no other waves were present in the region $\Delta_j$ we would have $\Gamma_j = 0$ and

$$\frac{d}{dt} z_j(t) \equiv \sigma_0^j.$$  

In this case, the equality would hold in (3.13).

To handle the general case, we represent the solution $u$ as a limit of front tracking approximations $u_\nu$, where for each $\nu \geq 1$ the function $u_\nu(0, \cdot)$ contains exactly $\nu$ rarefaction fronts equally spaced along the interval $I_j(0)$. Each of these fronts has initial strength $\sigma_\alpha(0) = \sigma_j^0 / \nu$. For $\alpha = 1, \ldots, \nu$, let $y_\alpha(t) \in I_j(t)$ be the location of one of these fronts at time $t \in [0, \tau]$, and let $\sigma_\alpha(t) > 0$ be its strength. Moreover, call

$$J_\alpha(t) \doteq [y_\alpha(t), y_{\alpha+1}(t)] , \quad \Delta_\alpha \doteq \{(t, x) ; \ t \in [0, \tau], \ x \in J_\alpha(t)\},$$

and let $\Gamma_\alpha$ be the total amount of interaction in $u_\nu$ taking place inside the domain $\Delta_\alpha$.

We define a subset of indices $\mathcal{I} \subseteq \{1, \ldots, \nu\}$ by setting $\alpha \in \mathcal{I}$ if

$$5C_0 \Gamma_\alpha > \sigma_\alpha(0) = \sigma_j^0 / \nu.$$  

(3.14)

Observe that, if $\alpha \notin \mathcal{I}$, then

$$\left| \frac{\sigma_\alpha(t)}{\sigma_\alpha(0)} - 1 \right| < \frac{1}{2} \quad \text{for all } t \in [0, \tau].$$
In particular, if \( \alpha, \alpha + 1 \notin \mathcal{I} \), then the interval \( J_\alpha(t) \) is well defined for all \( t \in [0, \tau] \). Its length

\[
z_\alpha(t) \doteq y_{\alpha+1}(t) - y_\alpha(t)
\]
satisfies the differential inequality

\[
\frac{d}{dt}z_\alpha(t) \geq W_\alpha(t) - C_1 \cdot \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta|
\]
for some constant \( C_1 \). Here

\[
W_\alpha(t) \doteq \left[ \text{amount of } i \text{-waves inside the interval } J_\alpha(t) \right]
\]
\[
\geq \sigma_\alpha(0) - C_0 \Gamma_\alpha,
\]
while \( \mathcal{C}_\alpha(t) \) refers to the set of all wave fronts of different families which are crossing the interval \( J_\alpha \) at time \( t \). Calling \( W'_\alpha \) the total amount of waves of families \( \neq i \) which lie inside \( J_\alpha(0) \), we can now write

\[
\int_0^\tau \left( \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \right) dt \leq \left( \max_{t \in [0, \tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha - \mathcal{O}(1) \cdot \tau \Gamma_\alpha + \mathcal{O}(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W'_\alpha.
\]

Indeed, by strict hyperbolicity, every front \( \sigma_\beta \) of a different family can spend at most a time \( \mathcal{O}(1) \cdot z_\alpha \) inside \( J_\alpha \). Either it is located inside \( J_\alpha \) already at time \( t = 0 \), or else, when it enters, it crosses \( y_\alpha \) or \( y_{\alpha+1} \). In this case, since \( \alpha, \alpha + 1 \notin \mathcal{I} \), by (3.14) it will produce an interaction of magnitude \( |\sigma_\beta \sigma_\alpha| \geq |\sigma_\beta \sigma_j^0|/2\nu \). The second term on the right hand side of (3.17) takes care of the new wave fronts which are generated through interactions inside \( J_\alpha \). The last term takes into account wave front of different families that initially lie already inside \( J_\alpha \) at time \( t = 0 \). Integrating (3.15) over the time interval \([0, \tau]\) and using (3.16)-(3.17) one obtains

\[
z_\alpha(\tau) \geq z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} - \mathcal{O}(1) \cdot \tau \Gamma_\alpha - \mathcal{O}(1) \cdot \left( \max_{t \in [0, \tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha - \mathcal{O}(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W'_\alpha.
\]

7. To proceed in our analysis, we now show that

\[
\max_{t \in [0, \tau]} z_\alpha(t) \leq 2z_\alpha(\tau).
\]

Indeed, let \( \tau' \in [0, \tau] \) be the time where the maximum is attained. If our claim (3.19) does not hold, there would exists a first time \( \tau'' \in [\tau', \tau] \) such that \( z_\alpha(\tau'') = z_\alpha(\tau')/2 \). Then from (3.15) and the assumption \( W_\alpha(t) \geq 0 \) it follows

\[
\int_{\tau'}^{\tau''} C_1 \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| dt \geq \frac{z_\alpha(\tau')}{2}.
\]
Using the smallness of the total variation, a contradiction is now obtained as follows. Call
\[ \Phi(t) \doteq C_0 Q(t) + \sum_{k, \beta \neq i} \phi_{k, \beta}(t, x_{\beta}(t)) |\sigma_\beta|, \]
where the sum ranges over all fronts of strength \( \sigma_\beta \) located at \( x_{\beta} \), of a family \( k, \beta \neq i \). The weight functions \( \phi_j \) are defined as
\[
\phi_j(t, x) \doteq \begin{cases} 
0 & \text{if } x > y_{\alpha + 1}(t), \\
\frac{y_{\alpha + 1}(t) - x}{y_{\alpha + 1}(t) - y_{\alpha}(t)} & \text{if } x \in [y_{\alpha}(t), y_{\alpha + 1}(t)], \\
1 & \text{if } x < y_{\alpha}(t), 
\end{cases}
\]
in the case \( j > i \), while
\[
\phi_j(t, x) \doteq \begin{cases} 
1 & \text{if } x > y_{\alpha + 1}(t), \\
\frac{x - y_{\alpha}(t)}{y_{\alpha + 1}(t) - y_{\alpha}(t)} & \text{if } x \in [y_{\alpha}(t), y_{\alpha + 1}(t)], \\
0 & \text{if } x < y_{\alpha}(t), 
\end{cases}
\]
in the case \( j < i \). Because of the term \( C_0 Q(t) \), the functional \( \Phi \) is non-increasing at times of interactions. Moreover, outside interaction times a computation entirely similar to the one at p.213 of [B] now yields
\[ -\frac{d}{dt} \Phi(t) \geq \sum_{\beta \in C_\alpha(t)} |\sigma_\beta| \cdot \frac{c_0}{\gamma(t)}, \]
for some small constant \( c_0 > 0 \) related to the gap between different characteristic speeds. From (3.20) and (3.21) respectively we now deduce
\[ \int_{\tau'}^{\tau''} \sum_{\beta \in C_\alpha(t)} |\sigma_\beta| dt \geq \frac{z_{\alpha}(\tau')}{2C_1}, \]
\[ \int_{\tau'}^{\tau''} \sum_{\beta \in C_\alpha(t)} |\sigma_\beta| dt \leq \int_{\tau'}^{\tau''} \frac{|d\Phi(t)|}{c_0} dt \leq \frac{\Phi(\tau')}{c_0} z_{\alpha}(\tau'). \]
Since \( \Phi(t) = O(1) \cdot \text{Tot.Var.}\{u(t)\} \), by the smallness of the total variation we can assume \( \Phi(\tau') < 2C_1/c_0 \). In this case, the two above inequalities yield a contradiction.

8. Using (3.19), from (3.18) we obtain
\[
z_j(\tau) = \sum_{1 \leq \alpha \leq \nu} z_{\alpha}(\tau) \geq \sum_{\alpha \notin I} z_{\alpha}(\tau)
\geq \sum_{\alpha \notin I} \left\{ z_{\alpha}(0) + \frac{\tau \sigma_j^0}{\nu} \frac{1}{1 + C_2 (\nu/\sigma_j^0 \Gamma_\alpha)} - O(1) \cdot \tau \Gamma_j - O(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W_\alpha' \right\}
\geq \sum_{\alpha \notin I} \left( z_{\alpha}(0) + \frac{\sigma_j^0}{\nu} \right) \left( 1 - C_2 \frac{\nu}{\sigma_j^0 \Gamma_\alpha} \right) - O(1) \cdot \tau \Gamma_j - O(1) \cdot \frac{z_j(0) + 1}{\nu}
\geq \sum_{\alpha \notin I} \left( z_{\alpha}(0) + \frac{\sigma_j^0}{\nu} \right) - C_2 \frac{z_j(0) + 1}{\sigma_j^0 \Gamma_j} - O(1) \cdot \tau \Gamma_j - O(1) \cdot \frac{z_j(0) + 1}{\nu}. \]
By (3.14) the cardinality of the set $\mathcal{I}$ satisfies

$$\#\mathcal{I} \cdot \frac{\sigma_0}{5C_0\nu} \leq \sum_{\alpha \in \mathcal{I}} \Gamma_\alpha \leq \Gamma_j,$$

hence

$$\frac{\#\mathcal{I}}{\nu} \leq \frac{5C_0}{\sigma_0^j} \Gamma_j.$$  

In turn, this implies

$$\sum_{\alpha \in \mathcal{I}} \left( z_\alpha(0) + \tau \frac{\sigma_0^j}{\nu} \right) \geq (z_j(0) + \tau \sigma_0^j) \left( 1 - \frac{\#\mathcal{I}}{\nu} \right) \geq (z_j(0) + \tau \sigma_0^j) - 5C_0 \Gamma_j \frac{z_j(0)}{\sigma_0^j} \Gamma_j - 5C_0 \tau \Gamma_j. \quad (3.23)$$

Using (3.23) in (3.22), observing that

$$\frac{z_j(0)}{\sigma_0^j} = \frac{x_{j+1}(0) - x_j(0)}{\sigma_0^j} = (\psi_j^0)^{-1}.$$  

and letting $\nu \to \infty$ we conclude

$$z_j(\tau) \geq (z_j(0) + \tau \sigma_0^j) - \mathcal{O}(1) \cdot (\psi_j^0)^{-1} \Gamma_j - \mathcal{O}(1) \cdot \tau \Gamma_j.$$  

This establishes (3.13), for a suitable constant $C'$.  

9. In the general case, without the assumptions (H), the lemma is proved by an approximation argument. We construct a convergent sequence of initial data $\bar{u}_\nu \to \bar{u}$ which satisfy (H) and such that

$$\bar{u}_\nu \to \bar{u}, \quad Q(\bar{u}_\nu) \to Q(\bar{u}), \quad |\mu_{i^+,0} - \mu_{i^+,\nu}| \to 0.$$  

Calling $w_\nu$ the solution of (3.1) with initial data

$$w_\nu(0,x) = \text{sgn}(x) \cdot \sup_{\text{meas}(A) \leq 2|x|} \frac{\mu_{i^+,0}(A)}{2},$$

by the previous analysis we have

$$\mu_{i^+,\nu,\tau_\nu} \leq D_x \left[ w_\nu(\tau_\nu-) + \text{sgn}(x) \cdot \left( Q(\bar{u}_\nu) - Q(u_\nu(\tau_\nu)) \right) \right].$$

Observe that $w_\nu(\tau_\nu-) \to w(\tau-) \text{ in } L^1_{\text{loc}}$. Choosing $\kappa \geq C_0$, by the lower semicontinuity result stated in Lemma 1 we now conclude

$$\mu_{i^+,\tau} \leq D_x \left[ w(\tau-) + \kappa \text{sgn}(x) \cdot \left( Q(\bar{u}) - Q(u(\tau)) \right) \right].$$
4 - Proof of the main theorem

Using the previous lemmas, we now give a proof of Theorem 1. For a given interval $[0, \tau]$, the solution of the impulsive Cauchy problem (1.17)-(1.18) can be obtained as follows. Consider a partition $0 = t_0 < t_1 < \cdots < t_N = \tau$. Construct an approximate solution by requiring that $w(0, x) = \hat{v}_i(x)$,

$$w_t + (w^2/2)_x = 0$$

(4.1)
on each subinterval $[t_{k-1}, t_k[$, while

$$w(t_k, x) = w(t_{k-1}, x) + \kappa \operatorname{sgn}(x) \cdot [Q(t_{k-1}) - Q(t_k)].$$

(4.2)

We then consider a sequence of partitions $0 = t^\nu_0 < t^\nu_1 < \cdots < t^\nu_N = \tau$, and the corresponding solutions $w^\nu$. If the mesh of the partitions approaches zero, i.e.

$$\lim_{\nu \to \infty} \sup_k |t^\nu_k - t^\nu_{k-1}| = 0,$$

then the approximate solutions $w^\nu$ converge to a unique limit, which yields the solution of (1.17)-(1.18).

Call $\mathcal{F}$ the set of nondecreasing odd functions, concave for $x > 0$. This set is positively invariant for the flow of Burgers’ equation (4.1). Moreover, this flow is order preserving. Namely, if $w, w' \in \mathcal{F}$ are solutions of (4.1) with initial data such that $w(0, x) \leq w'(0, x)$ for all $x > 0$, then also

$$w(t, x) \leq w'(t, x) \quad \text{for all } t, x > 0.$$

Equivalently,

$$D_xw(0) \leq D_xw'(0) \quad \implies \quad D_xw(t) \leq D_xw'(t)$$

for every $t > 0$. For each fixed $\nu$, we can apply Lemma 2 on each subinterval $[t^\nu_{k-1}, t^\nu_k]$ and obtain

$$\mu^{i+}_{k} \leq D_xw^\nu(t^\nu_k) \quad \implies \quad \mu^{i+}_{k+1} \leq D_xw^\nu(t^\nu_{k+1}).$$

By induction on $k$, this yields

$$\mu^{i+}_{\tau} \leq D_xw^\nu(\tau),$$

(4.3)

where $w^\nu$ is the approximate solution constructed according to (4.1)-(4.2). Letting $\nu \to \infty$ and using Lemma 1, we achieve a proof of Theorem 1.

\[\Box\]
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