QUANTUM MECHANICS IN GRASSMANN SPACE,
SUPERSYMMETRY AND GRAVITY

NORMA MANKOČ BORŠTNİK
Department of Physics, University of Ljubljana, Jadranska 19,
J. Stefan Institute, Jamova 39, 61111 Ljubljana, Slovenia

ABSTRACT

A particle which lives in a d-dimensional ordinary and a d-dimensional Grassmann space manifests itself in an ordinary four-dimensional subspace as a spinor, a scalar or a vector with charges. Operators of the Lorentz transformations and translations in both spaces form the super-Poincaré algebra. It is the super-Pauli-Ljubanski vector which generates spinors. Vielbeins and spin connections with the Lorentz index larger than or equal to five may manifest in a four-dimensional subspace as an electromagnetic, a weak and a colour field.

1. INTRODUCTION

Not only we have shown\(^1\) that the supersymmetry can consistently be formulated in the context of a Grassmann space, but that this approach brings a new insight into the concept of supersymmetry and physics, either in the flat space or in the presence of gauge fields entering into the theory as vielbeins and spin connections.

We suppose that a particle lives in a d-dimensional ordinary and a d-dimensional Grassmann space of anticommuting coordinates. From the supersymmetric geodesics of the particle it follows\(^1-2\) that the action is invariant under the Lorentz transformations in an ordinary and a Grassmann space with the same transformation parameters in both spaces. The canonical quantization of the dynamics of a particle in the Grassmann space manifests in particle’s internal degrees of freedom; a particle behaves like a scalar, a spinor or a vector\(^1\), with a weak or a colour charge\(^3\), while the momentum in the fifth dimension appears as an electromagnetic charge. Generators of the Lorentz transformations in the Grassmann space are differential operators in the Grassmann space of an even Grassmann character and so are the Dirac $\gamma^\mu$ operators which require $d \geq 5$.

Vector space spanned over the Grassmann coordinate space has the dimension $2^d$. Half of vectors have an even, half of vectors have an odd Grassmann character. Canonical quantization of fields quantizes the former to bosons, the latter to fermions.

Generators of translations and the Lorentz transformations in the ordinary and
the Grassmann space form the super-Poincaré algebra\(^1\). The super-Pauli-Ljubanski vector can be defined as a generalization of the Pauli-Ljubanski vector with an odd Grassmann character. It defines spinor charges.

Supervielbeins, depending on ordinary and Grassmann coordinates, connect supervectors of a freely falling coordinate system to an external coordinate system\(^1\). A spin connection appears as a superpartner of a vielbein. While a vielbein has an even Grassmann character describing a spin 2 gravitational field, a spin connection has an odd Grassmann character and describes a fermionic part of a gravitational field. It is the dependence on the Grassmann coordinates which determines the internal spin of a gravitational field in the four dimensional subspace.

Vielbeins with a Lorentz index \(a \geq 5\) manifest in the four-dimensional subspace as gauge fields of Yang-Mills type, the corresponding charges being defined by either the generators of the Lorentz transformations in that part of the Grassmann space which has the index higher than five (this is the case for weak and colour charges) or by the momentum in the fifth dimension of the ordinary space (which is the case for the electromagnetic field).

A torsion and a curvature of the gravitational gauge field are found by the Poisson brackets between components of covariant momentum of a particle. Since vielbeins and spin connections depend on ordinary and Grassmann coordinates, the derivatives with respect to both types of coordinates appear in the Lagrange density and correspondingly in equations of motion, defining dynamics in both spaces.

2. A PARTICLE IN A FREELY FALLING COORDINATE SYSTEM

We assume that a particle lives in a d-dimensional ordinary \(x^a\) and in a d-dimensional Grassmann space \(\theta^a\) space of anticommuting coordinates: \(\{x^a, \theta^a\}\) and that the action is invariant under the Lorentz transformations with the same transformation parameters in both spaces. We found two types of generators of the Lorentz transformations corresponding to two different actions, representations of one defining spinors, of the other defining vectors.

For the action

\[
I = \int d\tau L(x^a, \theta^a, \dot{x}^a, \dot{\theta}^a), \quad \dot{x}^a = \frac{d}{d\tau}x^a, \quad \dot{\theta}^a = \frac{d}{d\tau}\theta^a, \quad (2.1)
\]

in which \(\tau\) is an ordinary time parameter, the generators of the Lorentz transformations are

\[
M^{ab} = L^{ab} + S^{ab}, \quad L^{ab} = x^a p^b - x^b p^a, \quad S^{ab} = \theta^a \dot{p}^b - \dot{\theta}^b p^a, \quad (2.2)
\]

where

\[
p_a = \frac{\partial L}{\partial \dot{x}^a}, \quad p_a^\theta = \frac{\partial L}{\partial \dot{\theta}^a}. \quad (2.2a)
\]
We shall use left derivatives defined as follows:

$$\frac{\overleftarrow{\partial}}{\partial \theta^a} \theta^b f = \delta^b_a f - \theta^b \frac{\overleftarrow{\partial}}{\partial \theta^a} f. \tag{2.3}$$

By defining the generalized coordinates

$$\tilde{a}^a := i (p^a - i \theta^a), \quad \tilde{\tilde{a}}^a := -(p^a + i \theta^a). \tag{2.4}$$

we may write

$$S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \quad \tilde{S}^{ab} = -\frac{i}{4} (\tilde{a}^a \tilde{a}^b - \tilde{a}^b \tilde{a}^a), \quad \tilde{\tilde{S}}^{ab} = -\frac{i}{4} (\tilde{\tilde{a}}^a \tilde{\tilde{a}}^b - \tilde{\tilde{a}}^b \tilde{\tilde{a}}^a). \tag{2.5}$$

Either $L^{ab}$ or $S^{ab}$ or $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$ form the Lie algebra of the Lorentz group. It appears$^1$ that $S^{ab}$ define vector and scalar representations while $\tilde{S}^{ab}$ and $\tilde{\tilde{S}}^{ab}$ define spinor representations in the Grassmann space. The Hamilton function $H = \dot{x}^\mu p_\mu + \dot{\theta}^\mu \theta_\mu - L$ defines the Hamilton equations:

$$\frac{\partial H}{\partial x^a} = -\frac{d}{dt} p_a, \quad \frac{\partial H}{\partial p_a} = \dot{x}^a, \quad \frac{\partial H}{\partial \theta^a} = -\frac{d}{dt} \theta^a = -\dot{\theta}^a \quad \text{and the Poisson brackets in the ordinary and the Grassmann space:}$$

$$\{B, A\}_p = -\frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_a} + \frac{\partial A}{\partial p_a} \frac{\partial B}{\partial x^a} - \left( \frac{\overleftarrow{\partial}}{\partial \theta^a} \frac{\overleftarrow{\partial}}{\partial p_a} + \frac{\overleftarrow{\partial}}{\partial \theta^a} \frac{\overleftarrow{\partial}}{\partial p_a} \right) (1)^n_A, \quad \tag{2.6}$$

where $n_A$ is either one or two depending on whether $A$ has an odd or an even Grassmann character, respectively. It may be checked that the Poisson brackets have the following properties:

$$\{A, B\}_p = (-1)^{n_A n_B} \{B, A\}, \quad \{A, BC\}_p = \{A, B\}_p C + (-1)^{n_A n_B} B \{A, C\}_p, \quad \{AB, C\}_p = A \{B, C\}_p + (-1)^{n_B n_C} \{A, C\}_p B, \quad \tag{2.6a}$$

and fulfill the Jacobi’s identity:

$$(-1)^{n_A n_C} \{A, \{B, C\}_p\}_p + (-1)^{n_C n_B} \{C, \{A, B\}_p\}_p + (-1)^{n_B n_A} \{B, \{C, A\}_p\}_p = 0. \tag{2.6b}$$

In the quantization procedure $-i \{A, B\}_p$ goes to either commutators or to anti-commutators, according to the Poisson brackets (2.6).

2.1 SPINORS
Defining a particle supergeodesics by supercoordinates\textsuperscript{1−2} \(X^a = x^a + \varepsilon \xi \theta^a\), which depend on two parameters: on an ordinary time parameter \(\tau\) and on a Grassmann odd parameter \(\xi\), \(\varepsilon\) is here an ordinary complex coordinate, the action follows

\[
I = \frac{1}{2} \int d\tau d\xi ED_A X^a D_B X^b \eta_{ab} \eta^{AB},
\]

where \(D_A = E^i_A \partial_i, \partial_i := (\partial_\tau, \partial_\xi), \tau^i = (\tau, \xi)\) may be written which is invariant under general coordinate transformations in a two dimensional superspace \(\tau^i\) and under the Lorentz transformations in a 2d-dimensional superspace. By choosing \(\eta_{AA} = 0, \eta_{12} = \eta_{21} = 1\), and

\[
E^i_A = \left( 1, \frac{-\varepsilon M}{\xi}, N - \xi M \right), \quad E = \frac{1}{N},
\]

while \(\eta_{ab} = (1, -1, -1, -1, -1, \ldots, -1)\) and integrating the action (2.7) over the Grassmann odd parameter \(\xi\), the action for a superparticle follows

\[
I = \int d\tau \left( \frac{1}{N} \dot{x}^a \dot{x}_a + \varepsilon^2 \dot{\theta}^a \theta_a - \frac{2\varepsilon^2 M}{N} \dot{x}^a \theta_a \right),
\]

which requires that the coordinate in the Grassmann space is proportional to its conjugate momentum

\[
p_a^\theta := \frac{\partial L}{\partial \dot{\theta}} = \varepsilon^2 \theta^a,
\]

bringing into the theory the spinorial degrees of freedom. For \(\varepsilon^2 = -i\), it follows that \(\tilde{a}^a = 0\) and so are \(\tilde{S}^{ab} = 0\), while \(S^{ab} = \tilde{S}^{ab}\).

The variation of the action (2.7a) with respect to M and N, the former having an odd the later an even Grassmann character, gives the two constraints:

\[
p^a \tilde{a}_a = 0 = \tilde{p}^a p_a.
\]

which, according to the Poisson brackets (2.6), in the quantization procedure define the Dirac and the Klein-Gordon equation, respectively. The operators \(\theta^a, p^a\) (in the coordinate representation \(\theta^a \rightarrow \theta^a, p^a \rightarrow -i \frac{\partial}{\partial \theta^a}\)) fulfill the Grassmann algebra, while the operators \(\tilde{a}^a\) and \(\tilde{a}^a\) fulfill the Clifford algebra:

\[
\{ \tilde{a}^a, \tilde{a}^b \}_+ = 2\eta^{ab} = \{ \tilde{a}^a, \tilde{a}^b \}_+,
\]

with \(\{ \tilde{a}^a, \tilde{a}^b \} = 0 = \{ \tilde{S}^{ab}, \tilde{S}^{cd} \}_-\) and \(\tilde{S}^{ab} = -\frac{i}{4} [\tilde{a}^a, \tilde{a}^b]_-, \tilde{S}^{ab} = -\frac{i}{4} [\tilde{a}^a, \tilde{a}^b]_-\). The constraints (2.10) lead to the Dirac and the Klein-Gordon equation

\[
p^a \tilde{a}_a |\tilde{\Psi} > = 0, \quad p^a p_a |\tilde{\Psi} > = 0.
\]
In the case that \( < \tilde{\psi} | p^5 | \tilde{\psi} > = m \) and \( d = 5 \), it follows
\[
(\tilde{a}^b p_b - \tilde{a}^5 p^5 | \tilde{\psi} > = 0 = (\tilde{\gamma}^b p_b - m) | \tilde{\psi} > , b = 0, 1, 2, 3. \tag{2.13}
\]

Since the Dirac \( \gamma^b \) operators have an even Grassmann character, we assume that
\[
\tilde{\gamma}^b = -\tilde{a}^5 \tilde{a}^b = -2i \tilde{S}^{5b} , b = 0, 1, 2, 3. \tag{2.14}
\]
are the Dirac operators. It can be checked that in the four-dimensional subspace \( \tilde{\gamma}^b \) fulfil the Clifford algebra \( \{ \tilde{\gamma}^b, \tilde{\gamma}^c \} = \eta^{ab} \), while \( \tilde{S}^{cd} = -\frac{1}{4} [\tilde{\gamma}^c, \tilde{\gamma}^d] \), \( c, d = 0, 1, 2, 3 \).

The existence of \( \tilde{\gamma}^c \) as an even Grassmann operator requires that \( d \geq 5 \). The operator \( \tilde{\Gamma} = i \tilde{a}^0 \tilde{a}^1 \tilde{a}^2 \tilde{a}^3 = i \tilde{\gamma}^0 \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \) is one of the two Casimir operators of the Lorentz group.

There are \( 2^d \) products of operators \( \tilde{a}^\mu \) which form the Clifford algebra, half of them with an even Grassmann character form by themselves the Dirac algebra. Generators of the infinitesimal translations in the ordinary space \( p^\mu \) and in the Grassmann space \( \tilde{a}^\mu \) and the generators of the Lorentz transformations in both spaces \( M^{\mu \nu} \) form the super-Poincaré algebra\(^1\). The super-Pauli-Ljubanski vector \( \tilde{G}^\mu = \frac{1}{3!} \varepsilon_{\alpha \beta \gamma \rho} \tilde{M}^{\alpha \beta} \tilde{a}^\gamma p^\rho \) can be defined so that \( \tilde{G}^\mu \tilde{G}^\nu, p^\mu p^\nu \) and \( \tilde{a}^\mu \tilde{a}^\nu \) commute with all generators of the super-Poincaré group as well as with \( \tilde{G}^\rho \) itself. The super-Pauli-Ljubanski vector define spinor charges\(^1\), which being applied on a vacuum scalar state define spinors.

### 2.2 SCALARS AND VECTORS

Dynamics of spinors occurs their momenta in the Grassmann space is proportional to their coordinates. Dynamics of scalars and vectors occur when their momenta in the Grassmann space are proportional to the derivative of Grassmann coordinates with respect to the time parameter \( \dot{\theta}^\mu \). The constraints then lead to the Klein-Gordon equation. In this case the generators of the Lorentz transformations and the translations in the Grassmann space are \( S^{\mu \nu} = \dot{\theta}^\mu p^\nu - \dot{\theta}^\nu p^\mu \) and \( p^\theta \), respectively.

### 3. A PARTICLE IN A GRAVITATIONAL FIELD

We suppose\(^1,\,\,4\) that supervielbeins transform vectors of a freely falling coordinate system into vectors of an external coordinate system. Due to two types of derivatives \( \partial_i (i = 1, 2) \) (eq.2.7) we assume two types of supervielbeins: \( e_{\mu}^a, i = 1, 2 \), the index \( a \) refers to a freely falling coordinate system(a Lorentz index), the index \( \mu \) refers to external coordinate system(an Einstein index). Supervielbeins depend on ordinary and Grassmann coordinates. We write
\[
\partial_i X^a = e^{ia}_{\mu} \partial_i X^\mu , \partial_i X^\mu = f^{i\mu}_{a} \partial_i X^a , i = 1, 2. \tag{3.1}
\]
It follows (eq. (3.1)) that
\[ e^{ia}_\mu f^{\mu b}_i = \delta^a_b, \quad f^{\mu a}_i e^{ia}_\nu = \delta^\mu_\nu. \]  

(3.2)

Supervielbeins are vectors with respect to the Lorentz and the Einstein index. Making a Taylor expansion of supervielbeins in terms of \( \xi (\xi^2 = 0) \) we find
\[ e^{ia}_\mu = e^{ia}_\mu + \varepsilon \xi \theta^b e^{ib}_{\mu b}, \quad f^{\mu a}_i = f^{\mu a}_i - \varepsilon \xi \theta^b f^{ib}_{ab}, \quad i = 1, 2. \]  

(3.3)

Both expansion coefficients are fields, which depend on ordinary and Grassmann coordinates. While \( e^{ia}_\mu \) have an even Grassmann character and describe the spin 2 part of a gravitational field, \( \varepsilon \theta^b e^{ib}_{\mu b} \) have an odd Grassmann character (\( \varepsilon \) is a complex constant) and are as superpartners of \( e^{ia}_\mu \) candidates for spinorial part of a gravitational field.

From eqs (3.2) and (3.3) it follows that
\[ e^{ia}_\mu f^{\mu b}_i = \delta^a_b, \quad f^{\mu a}_i e^{ia}_\nu = \delta^\mu_\nu, \quad e^{ia}_\mu f^{\mu b}_c = e^{ia}_\mu f^{ib}_{cb}, \quad i = 1, 2. \]  

(3.2a)

We find super tensors \( g^{i}_{\mu \nu} = e^{ia}_\mu e^{ib}_{\nu b}, \quad g^i_{\mu \nu} = f^{ia}_\mu f^{ib}_{\nu b}, \quad i = 1, 2 \) with an even Grassmann character and the properties \( g^{i}_{\mu \nu} g^i_{\nu \sigma} = \delta^\mu_\sigma = g^{i}_{\mu \sigma} g^i_{\sigma \nu} \), with \( g^{i}_{\mu \sigma} = e^{ia}_\mu e^{ia}_{\sigma}. \)

We see from eq. (3.1) that vectors in an ordinary and a Grassmann space are connected as follows
\[ \dot{x}^a = e^{1a}_a \dot{\mu}, \quad \dot{x}^a = f^{1a}_a \dot{\mu}, \quad \theta^a = e^{2a}_a \theta^\mu, \quad \dot{\theta}^a = f^{2a}_a \theta. \]  

(3.4)

\[ \dot{x}^a = e^{1a}_a \dot{\mu}, \quad \dot{x}^a = f^{1a}_a \dot{\mu}, \quad \theta^a = e^{2a}_a \theta^\mu, \quad \dot{\theta}^a = f^{2a}_a \theta. \]  

(3.4a)

We use the notation \( e^{2a}_{\nu,\mu x} \) to denote the following relations among fields
\[ e^{2a}_{\nu,\mu x} = 0, \quad e^{2a}_{\nu,\mu b} \theta^\nu = e^{1a}_a \theta^\mu e^{2a}_{\nu,\mu b} \]  

which means that a point particle with a spin sees a spin connection \( \theta^b e^{ib}_{\mu b} \) related to a vielbein \( e^{2a}_a \).  

Rewriting the action (2.7) in terms of an external coordinate system according to eqs (3.1), using the Taylor expansion of supercoordinates \( X^\mu \) and superfields \( e^{ia}_\mu \) and integrating the action over the Grassmann odd parameter \( \xi \) the action
\[ I = \int d\tau \{ \frac{1}{N} g^{1}_{\mu \nu} \dot{x}^\nu \dot{x}^\mu - \varepsilon^2 \frac{2M}{N} \theta_a e^{ia}_\mu \dot{x}^\mu + \frac{\varepsilon}{2} (\dot{\theta}^\mu \theta_a - \theta_a \dot{\theta}^\mu) e^{ia}_\mu + \frac{\varepsilon}{2} (\dot{\theta}^b \theta_a - \theta_a \dot{\theta}^b) e^{ia}_{\mu b} \dot{x}^\mu \}. \]  

(3.5)
defines the two momenta of the system

\[ p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = p_{0\mu} + \varepsilon^2 \theta^a \theta^b \epsilon^1_{a\mu}, \quad p_{0\mu} = \frac{2}{N} (\dot{x}_\mu - \varepsilon^2 Mp^\theta_\mu), \]

\[ p^\theta_\mu = \varepsilon^2 \theta_a \epsilon^{1\alpha}_\mu = \varepsilon^2 (\theta_\mu + \varepsilon^2 \epsilon^{2\alpha}_\nu \theta^c_{\alpha\beta} \theta^{\nu\beta}). \tag{3.6} \]

For \( p^\theta_\mu = p^\theta_\mu f^{1\mu}_a \) it follows that \( p^\theta_\mu \) is proportional to \( \theta_\mu \). For a choice \( \varepsilon^2 = -i \), \( \tilde{a}_a = i(p^\theta_a - i\theta_a), \tilde{a}^\mu = f^{1\mu}_a \), while \( \tilde{a}_a = 0 \). In this case we may write

\[ p_\mu = p_{0\mu} + \frac{1}{2} \tilde{S}^{ab}_1 \epsilon^1_{ab\mu} = p_{0\mu} + \frac{1}{2} \tilde{S}^{ab}_1 \omega_{ab\mu}, \omega_{ab\mu} = \frac{1}{2} (\epsilon^{1, ab}_a \epsilon^{1, ab}_b), \tag{3.6a} \]

and obtain the Hamilton function

\[ H = \frac{N}{4} g^{1\mu\nu} p_{0\mu} p_{0\nu} + \frac{i}{2} Mp_\mu f^{1\mu}_a \tilde{a}_a. \tag{3.7} \]

and the two constraints

\[ p^0_{0\mu} p_{0\mu} = 0 = p_{0\mu} f^{1\mu}_a \tilde{a}_a. \tag{3.8} \]

In the quantization procedure the two constraints in eqs. (3.8) \( p_{0\mu} f^{1\mu}_a \tilde{a}_a \) have to be symmetrized properly, due to the fact that fields depend on ordinary and Grassmann coordinates, in order that the Klein-Gordon and the Dirac equations in the presence of gravitational fields follow correspondingly.

A torsion and a curvature follow from the Poisson brackets \( \{p_{0a}, p_{0b}\} \), with \( p_{0a} = f^{1\mu}_a (p_\mu + \frac{1}{2} \tilde{S}^{cd}_1 \omega_{cd\mu}). \)

We find

\[ \{p_{0a}, p_{0b}\} = -\frac{1}{2} \tilde{S}^{cd}_1 R_{cdab} + p_{0c} T^c_{ab}, \tag{3.9} \]

\[ R_{cdab} = f^{1\mu}_a f^{1\nu}_b \omega_{cd\mu\nu} + \omega^{e\mu}_c \omega_{cd\nu} \overrightarrow{\omega}_{cd\mu,\nu} \theta^{e}_{\mu,\nu}, \]

\[ T^c_{ab} = \epsilon^1_{epsilon} (f^{1\nu}_b \delta^{1\mu}_a, \omega^{f}_{\nu} \omega^{1\mu}_b \theta^{f}_{\nu} \omega^{1\mu}_a), \]

with \( A_{[a} B_{b]} = A_a B_b - A_b B_a. \)

If the action for a free gravitational field is

\[ I = \int d^4 x d^4 \theta \omega L, \tag{3.10} \]

where \( \omega \) is a scalar density in the Grassmann space, the Lagrange density \( L \) includes \( \text{det}(e^{1\mu}_a) R, R = R^{ab}_{cd}, \) or (and) \( \text{det}(e^{1\mu}_a) T^a_{cd} T^{cd}_a. \)

Generators of the Lorentz transformations in the Grassmann space form a group \( SO(1, N - 1) \) which can be decomposed (\( SO(1, 17) \) for example) into \( SO(1, 3) \times \)
SU(2) × SU(3) × .... This means that the generators of the Lorentz transformations in the Grassmann space in higher then four dimensions may define weak and colour charges, while vielbeins and spin connections with the Lorentz index higher then four manifest in the four-dimensional ordinary subspace as weak and colour fields. Since the momentum in the fifth dimension of the ordinary space manifests as an electromagnetic charge of a particle in the four-dimensional ordinary subspace and accordingly the spin connection with the Lorentz index equal to five manifests as an electromagnetic field, can Lorentz indices only greater then five be connected with fields, which manifest in the four-dimensional subspace as non-Abelian gauge fields.

4. ELECTRODYNAMICS AS A GRAVITATION IN THE FIFTH DIMENSION

We shall present in this section how a vielbein and a spin connection, depending on the ordinary and Grassmann coordinates, with the Lorentz index equal to five manifest in the four-dimensional subspace as an electromagnetic field. The dependence of the field on the Grassmann coordinates determines the internal spin one of the field.

We shall treat the case with \( d = 5 \) and with no gravitational field in the ordinary four-dimensional subspace:

\[ e^m_{\mu} = \delta^m_{\mu}, \quad e^{i5}_{\alpha} = e^{i5}_{\alpha}(x^\beta, \theta^\mu), \quad e^{i5}_5 = e^{i5}_5(x^\beta, \theta^\mu), \quad (4.1a) \]

\[ m = 0, 1, 2, 3; \quad \alpha, \beta, \gamma = 0, 1, 2, 3; \quad \mu = \alpha, 5; \quad i = 1, 2. \]

It follows then from eq.(3.2) that

\[ f^{i\alpha} = \delta^{i\alpha} \beta, \quad f^{i5}_m = -(e^5_5)^{-1} e^{i5}_m, \quad f^{i5}_5 = (e^5_5)^{-1}. \quad (4.1b) \]

If we take into account the choice of the fields of eqs.(4.1) in the action(3.5) and in eq.(3.6a) we find a covariant momenta

\[ p_{0\alpha} = p_\alpha - e^{15}_5 e^{15}_5^{-1} p_5 - \frac{i}{4} e^2_{5[a,a]} e^{i5}_5, \quad \alpha = 0, 1, 2, 3, \quad (4.2) \]

with \( p_{0\mu} = \frac{2}{N} e^{15}_5(e_{5[a,a]}^1 \dot{x}^r + \frac{i}{2} Me^{1a}_5 \tilde{a}_a) \) taken as a constant. Indices \( a \) as well as \( \alpha \) are, due to the supposition (4.1a), raised or lowered by the Minkowski matrix \( \eta_{\alpha,\beta} \).

The Hamilton function

\[ H = \frac{N}{4} p_{0\alpha} \tilde{p}^{0\alpha} + \frac{i}{2} M p^a_0 \tilde{a}_\alpha, \quad \alpha = 0, 1, 2, 3 \quad (4.3) \]

and the two constraints can be found

\[ p^a_0 \tilde{a}_\alpha - m \tilde{a}_a = 0 = p^a_0 p_{0\alpha} - m^2 = 0, \quad (4.4a, b) \]
where \( m = -p_5 e^{15}_5 -1 \).

The two fields \( e^{5}_i \), \( i = 1, 2 \), depending on ordinary and Grassmann coordinates, have an even Grassmann character and are Lorentz vectors. They are related to each other and to the field of an odd Grassmann character \( \theta^b e^{1a}_{ab} \) (3.4a)

\[
e^{15}_a = e^{25}_a + e^{25}_\nu \theta^\nu, \quad e^{15}_{\alpha \beta} = e^{25}_\beta \alpha + e^{25}_a \alpha \gamma_{5}^{\beta}, e^{15}_5 = 0. \quad (4.5)
\]

In the gauge \(^1\) in which

\[
e^{-\frac{i}{2} \omega_{\alpha \beta} S^{\alpha \beta} e^{25}_a e^{25}_\beta} = \Lambda_{\alpha}^\beta e^{25}_\beta, \quad (4.5a)
\]

with \( S^{\mu \nu} \) being the differential operator in the Grassmann space defined in eq.(2.2) (after quantization) and \( \Lambda_{\alpha}^\beta \) a tensor of the Lorentz transformations, it follows

\[
e^{15}_a e^{15}_5 -1 p_5 = e A_\alpha. \quad (4.6)
\]

The first constraint of eqs.(4.4a) can then be written in the form

\[
\frac{1}{2} \left[ \gamma^\alpha (p_\alpha + e A_\alpha) - \frac{e}{4m} \tilde{\gamma}^{\alpha \beta} F_{\alpha \beta} + (p_\alpha + e A_\alpha) \tilde{\gamma}^\alpha - \frac{e}{4m} F_{\alpha \beta} \tilde{\gamma}^{\alpha \beta} \right] - m = 0. \quad (4.7)
\]

In the quantization procedure (according to the Poisson brackets (2.6)), eq.(4.7) has to be symmetrized since the field depends on ordinary and Grassmann coordinates.

In a nonhomogenous field the term with \( F_{\alpha \beta} \) causes an anomalous magnetic moment of a charged particle, unless the particle is considered as a (almost massless) cluster of very heavy constituents. In an homogenous field the contribution of that term vanishes since \( [p_\alpha, e A_\alpha + \frac{i}{8} \gamma^\beta F_{\alpha \beta}] = e A_{\beta, \alpha} \).

The Poisson brackets of eq.(3.9) bring in the gauge (4.5a) a torsion

\[
T^{m}_{\alpha \beta} = 0, \quad T^{5}_{mn} = F_{mn} - \omega^{5}_{mn} e^{5}_5 -1 \quad (4.8)
\]

and a Riemann tensor which depends on \( \omega_{mn} \) only.

The variation of the action (3.10) with \( \mathcal{L} = T^{5}_{mn} T^{mn} \) with respect to \( A_\alpha \) and \( \omega^{5}_{mn} \) brings the ordinary Maxwell equations for a free field and the constraints (4.5) in the gauge(4.5a). This is the solution also for the case that \( \mathcal{L} = R^{mn}_{mn} \).

5.CONCLUSIONS

The theory in which the space has d ordinary and d Grassmann coordinates possesses the supersymmetry and enables not only the canonical quantization of
coordinates and fields but offers also the possibility of unifying gauge fields: generators of the super-Poincaré algebra (in the ordinary and the Grassmann space) define internal degrees of freedom: spins of particles and fields and charges. Spins of particles and fields appear as the dynamics of particles and fields in the Grassmann five-dimensional subspace. Momentum in the fifth ordinary dimension manifests as a charge (and a mass) of particles and fields in a four-dimensional ordinary subspace. Generators of the Lorentz transformations in higher than five dimensions manifest in the four-dimensional subspace charges of non-Abelian gauge fields, while gravitational fields in higher dimensions manifest in four-dimensional subspace as corresponding fields. Spin connections of an odd Grassmann character appear as superpartners of vielbeins with an even Grassmann character. Both depend on ordinary and Grassmann coordinates, the later defining the spin of fields. Generators of translations and the Lorentz transformations in the ordinary and the Grassmann space define super-Poincaré algebra, while super-Pauli-Ljubanski vector defines spinor charges. For d=5 there are four four-spinors defined by super-Pauli-Ljubanski vector and by generators of the Lorentz transformations in the Grassmann space. Half of them have a negative and half of them a positive internal parity, defined by discrete Lorentz transformations in the Grassmann space. If these degrees of freedom manifest on the level of quarks and leptons, the two generations of quarks and leptons would have a positive and the two a negative internal parity. There are also two scalars, two three vectors and two four vectors, which define internal spins of gauge fields.

8. APPENDIX - SUPER POINCARÉ ALGEBRA

The operators $\tilde{M}^{\mu\nu} = L^{\mu\nu} + \tilde{S}^{\mu\nu} , p^{\mu}, \tilde{a}^{\mu}$ from the super-Poincaré algebra $^1$.

$$\{p^{\mu}, p^{\nu}\} = 0 = \{p^{\mu}, \tilde{a}^{\nu}\}$$  \hspace{1cm} (A.1a)

$$\{\tilde{M}^{\mu\nu}, p^{\alpha}\} = i(p^{\nu} \eta^{\mu\alpha} - p^{\mu} \eta^{\nu\alpha})$$  \hspace{1cm} (A.1b)

$$\{\tilde{a}^{\mu}, \tilde{a}^{\nu}\} = 2\eta^{\mu\nu}$$  \hspace{1cm} (A.1c)

$$\{\tilde{M}^{\mu\nu}, \tilde{a}^{\alpha}\} = i(\tilde{a}^{\nu} \eta^{\mu\alpha} - \tilde{a}^{\mu} \eta^{\nu\alpha})$$ \hspace{1cm} (A.1d)

$$\{\tilde{M}^{\mu\nu}, \tilde{M}^{\rho\beta}\} = -i(\tilde{M}^{\mu\beta} \eta^{\nu\alpha} - \tilde{M}^{\nu\beta} \eta^{\mu\alpha} - \tilde{M}^{\mu\alpha} \eta^{\nu\beta} + \tilde{M}^{\nu\alpha} \eta^{\mu\beta}).$$ \hspace{1cm} (A.1f)

The two Lorentz scalars can be defined: $I_e = \frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \tilde{M}^{\mu\nu} \tilde{M}^{\rho\sigma} p^{\tau}$ and $I_o = -\frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \tilde{M}^{\mu\nu} \tilde{M}^{\rho\sigma} \tilde{a}^{\tau}$, the first with an even, the second with an odd Grassmann
character. The commutator of $I_e$ with $\tilde{a}^\mu$ or of $I_o$ with $p_\mu$ defines the super-Pauli-Ljubanski vector:

$$\tilde{G}^\mu = \{ I_e, \tilde{a}^\mu \} = \{ I_o, p^\mu \} = \frac{1}{3!} \varepsilon^{\mu \alpha \beta \rho} \tilde{M}^{\alpha \beta} \tilde{a}^\gamma p^\rho = \frac{1}{3!} \varepsilon^{\mu \alpha \beta \rho} \tilde{S}^{\alpha \beta} \tilde{a}^\gamma p^\rho.$$  \hfill (A.2)

It has an odd Grassmann character. We can find it as one of constants of motion for the action (2.7a). The following properties can be prooved:

$$\{ \tilde{G}^\mu, p^\nu \} = 0, \quad \{ \tilde{a}^\mu, \tilde{G}^\nu \} = -\varepsilon^{\mu \nu \alpha \beta \rho} \tilde{S}^{\alpha \beta} p^\rho, \quad \{ M^{\mu \nu}, \tilde{G}^\rho \} = i(\eta^{\mu \nu} \tilde{G}^\rho - \eta^{\rho \nu} \tilde{G}^\mu),$$

$$\{ \tilde{G}^\mu, \tilde{G}^\nu \} = \frac{1}{3!} \varepsilon^{\mu \alpha \beta \gamma \rho} \tilde{M}^{\alpha \beta} \tilde{S}^{\gamma \beta} \tilde{a}^\rho, \quad \tilde{G}^\mu p_\mu = 0, \quad \{ \tilde{G}^\mu, \tilde{a}^\nu p_\nu \} = 0. \quad \hfill (A.4)$$

There are $\tilde{G}^\mu \tilde{G}_\mu, p^\mu p_\mu$ and $\tilde{a}^\mu \tilde{a}_\mu$ which commute with all generators of the super-Poincaré group as well as with $\tilde{G}^\rho$ itself.

We may write $\tilde{G}_\mu$ for $d = 5$ in an explicit form as follows:

$$\tilde{G}^0 = S_3 \tilde{a}^3 p^5 - \tilde{\rightarrow} \cdot \tilde{p} \tilde{a}^5, \quad \tilde{G} = \tilde{\rightarrow} \tilde{a}^0 p^5 - \tilde{a}^5 p^0 + (\tilde{\rightarrow} \times \tilde{p}) \tilde{a}^5,$$

$$\tilde{G}^5 = S_3 \tilde{a}^3 p^0 - \tilde{\rightarrow} \cdot \tilde{p} \tilde{a}^0, \quad \text{with} \quad \tilde{S}_i = \frac{1}{2} \varepsilon_{ijk} \tilde{S}^{jk}, \quad \tilde{K}_i = \tilde{S}_i^{0i}. \quad \hfill (A.5)$$

Being applied on a vacuum state which is a scalar in the Grassmann space, $\tilde{G}_\mu$ produces spinors. For the case that a scalar fulfils the Klein-Gordon equation and in the representation in which $p^\mu = (p^0, 0, 0, p^3, 0)$ the two operators

$$Q_{1,2}^- = \frac{\mp 1}{\sqrt{|p_0|}} (\tilde{G}_1 \pm i \tilde{G}_2), \quad \hfill (A.6)$$

define four spinors according to the choice of either $p^0 = p^3$ or $p^0 = -p^3$. We define accordingly four operators $\tilde{Q}^\pm_{1,2}$. The four spinors, which are chosen to be eigensatates of $\tilde{S}_3$ and fulfill the Dirac equation $p^\mu \tilde{a}_\mu \tilde{Q}^\pm_{1,2} = 0$ are presented on Table I.

All vectors of an even Grassmann character can be obtained from spinors by the application of an operator of an odd Grassmann character. On Table II the eigenvectors of the two Casimir opertors $\frac{1}{2} \tilde{S}^{\alpha \beta} \tilde{S}_{\alpha \beta}$ and $\Gamma = \frac{1}{3!} \varepsilon^{\alpha \beta \gamma \delta} \tilde{S}_{\alpha \beta} \tilde{S}_{\gamma \delta}, \alpha, \beta \cdots = 0, 1, 2, 3$ and $S_3$ are presented which are for $d = 5$ two scalars, two three-vectors and two
four-vectors.

Table I: The four spinors generated by the operators $\hat{Q}_{\pm 1,2}$ on a scalar vacuum state are presented (printed in bold face) together with the rest of spinors, which form the Weyl four spinors.

Table II: The two scalars, two three-vectors and two four-vectors for $d=5$ are presented. They are defined by the Casimir operators for the generators of the Lorentz transformations of the vectorial character in the Grassmann space.

7. ACKNOWLEDGEMENTS

The work was supported by Ministry of Science and Technology of Slovenia and the National Science Foundation through funds available to the US-Slovenia Joint Board for Scientific and Technological Cooperation (no.NSF 899).

8. REFERENCES
1 N. Mankoč-Borštnik, Phys.Lett. B 292 (1992) 25, Il nuovo Cimento A 105 (1992) 1461, Journ. of Math. Phys. 34 (1993) 8, Int. Journal of Modern Phys. A 9 (1994) 1731, IC/91/369, IC/91/370, IJS.TP.92/5, IJS.TP.92/22, IJS.TP.93/5, IJS.TP.93/14, Proceedings to the Edirne conference Frontier in Theoretical Physics,Edirne, Dec.1993, IJS.TP.94/7

2 H. Ikemori, Phys.Lett. B 199 (1987) 239

3 Work in progress

4 J.Wess, B.Zumino, Nucl.Phys. B70 (1974) 139, J.Wess, J.Bagger, Supersymmetry and supergravity, Princeton Series in Physics (Princeton U.P., Princeton, N.J., 1983)

5 B.S.De Witt, Rev. Mod.Phys.29 (1957) 377

6 J. Maalampi, M. Ross, Phys. Rep. 2 (1990) 53