POSITIVE RADIAL SOLUTIONS FOR COUPLED SCHRÖDINGER SYSTEM WITH CRITICAL EXPONENT IN $\mathbb{R}^N$ ($N \geq 5$)

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Abstract. We study the following coupled Schrödinger system

$$\begin{cases}
-\Delta u + u = u^{2^*-1} + \beta u\frac{2^*}{2^*-1}v^{2^*} + \lambda_1 u^{\alpha-1}, & x \in \mathbb{R}^N, \\
-\Delta v + v = v^{2^*-1} + \beta u\frac{2^*}{2^*-1}v^{2^*} + \lambda_2 v^{r-1}, & x \in \mathbb{R}^N,
\end{cases}$$

where $N \geq 5, \lambda_1, \lambda_2 > 0, \beta \neq 0, 2 < \alpha, r < 2^*, 2^* \triangleq \frac{2N}{N-2}$. Note that the nonlinearity and the coupling terms are both critical. Using the Mountain Pass Theorem, Ekeland’s variational principle and Nehari mainfold, we show that this critical system has a positive radial solution for positive $\beta$ and some negative $\beta$ respectively.

Keywords: Schrödinger system; critical exponent; positive solution
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1. Introduction

In this paper, we consider the following coupled nonlinear Schrödinger system

$$\begin{cases}
-\Delta u + u = u^{2^*-1} + \beta u\frac{2^*}{2^*-1}v^{2^*} + \lambda_1 u^{\alpha-1}, & x \in \mathbb{R}^N, \\
-\Delta v + v = v^{2^*-1} + \beta u\frac{2^*}{2^*-1}v^{2^*} + \lambda_2 v^{r-1}, & x \in \mathbb{R}^N,
\end{cases}$$

where $N \geq 5, \lambda_1, \lambda_2 > 0, \beta \neq 0, 2 < \alpha, r < 2^*, 2^* \triangleq \frac{2N}{N-2}$. We are interested in the existence of a nontrivial solution $(u, v)$ for (1.1), that is to say that $u \neq 0$ and $v \neq 0$. We call a solution $(u, v)$ semi-trivial if $(u, 0)$ or $(0, v)$.

In recent years, there have been a lot of studies on the following coupled system of the time-dependent nonlinear Schrödinger equations

$$\begin{cases}
-i \frac{\partial}{\partial t} \Phi_1 - \Delta \Phi_1 = \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, & x \in \Omega, \ t > 0, \\
-i \frac{\partial}{\partial t} \Phi_2 - \Delta \Phi_2 = \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, & x \in \Omega, \ t > 0, \\
\Phi_j = \Phi_j(x, t) \in \mathbb{C}, \ j = 1, 2, \\
\Phi_j(x, t) = 0, \ j = 1, 2, \ x \in \partial \Omega, \ t > 0,
\end{cases}$$

(1.2)
where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. $i$ is the imaginary unit, $\mu_1, \mu_2 > 0$ and a coupling constant $\beta \neq 0$. When $N \leq 3$, system (1.2) appears in many physical problems, especially in nonlinear optics. Physically, the solution $\Phi_j$ denotes the $j$th component of the beam in Kerr-like photorefractive media (see [2]). The positive constant $\mu_j$ is for self-focusing in the $j$th component of the beam. The coupling constant $\beta$ is the interaction between the two components of the beam. The interaction is attractive if $\beta > 0$ while it is repulsive if $\beta < 0$. System (1.2) also arises in the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in two different hyperfine states, see more details in [2, 14, 19].

To obtain solitary wave solutions of system (1.2), we set $\Phi_1(x, t) = e^{i\lambda u} u(x)$, $\Phi_2(x, t) = e^{i\lambda v} v(x)$, then (1.2) turns to be the following elliptic system

$$
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2, & x \in \Omega, \\
-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2v, & x \in \Omega, \\
u, v = 0, & x \in \partial \Omega,
\end{cases}
$$

(1.3)

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$. When $N \leq 3$, system (1.3) is a problem of subcritical growth. Problem (1.3) was first studied by Lin and Wei in [15] where they obtained a nontrivial solution when $\Omega = \mathbb{R}^3$ and $\beta > 0$ is sufficiently small. After that, the existence and multiplicity of positive and sign-changing solutions have been extensively studied, we can refer to [3, 4, 6, 8, 9, 10, 11, 13, 16, 17, 18, 20, 21, 25, 26]. In particular, in [10], Chen and Zou studied problem (1.3) for $N = 4$ and $\Omega$ is a bounded domain of $\mathbb{R}^4$. In such case, the nonlinearity and the coupling terms are both of critical growth. By Ekeland’s variational principle and the Mountain Pass Theorem, they showed that if $-\lambda_1(\Omega) < \lambda_1 \leq \lambda_2 < 0$, then there exist $\beta_1 \in (0, \min \{\mu_1, \mu_2\})$, $\beta_2 \geq \max \{\mu_1, \mu_2\}$ such that (1.3) has a positive least energy solution for $\beta \in (-\infty, 0) \cap (0, \beta_1) \cap (\beta_2, +\infty)$, where $\lambda(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Meanwhile, (1.3) does not have a nontrivial nonnegative solution if $\mu_2 \leq \beta \leq \mu_1$ and $\mu_2 < \mu_1$. In a similar way to [10], Chen and Zou in [11] considered the following critically coupled nonlinear Schrödinger equations:

$$
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^{2^*-1} + \beta u^{2^*/2-1}v^{2^*/2}, & x \in \Omega, \\
-\Delta v + \lambda_2 v = \mu_2 v^{2^*-1} + \beta u^{2^*/2}v^{2^*/2-1}, & x \in \Omega, \\
u, v > 0, & x \in \Omega, u = v = 0, x \in \partial \Omega,
\end{cases}
$$

(1.4)

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 5)$. However, since $N \geq 5$, different phenomenons may happen comparing to the case $N = 4$ and it is much more complicated to handle. In [11], they proved that if $-\lambda_1(\Omega) < \lambda_1 \leq \lambda_2 < 0$, (1.4) has a positive least energy solution for any $\beta \neq 0$. Furthermore, in [21], by using the Mountain Pass Theorem, Kim showed that problem (1.4) has a nontrivial solution in the following two cases: $\beta$ is sufficiently large or $|\beta|$ is small enough.

Problem (1.1) can be seen as a counterpart of the following problem:

$$
-\Delta u + u = |u|^{2^*-2}u + f(u), \quad x \in \mathbb{R}^N.
$$

(1.5)
In [12], under the following assumptions on \( f(t) \):
\[
(f_1) \ f \in C^2(\mathbb{R}^1), \quad \lim_{t \to 0^+} \frac{f(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{f(t)}{t^{2^*-1}} = 0 \text{ for } t \geq 0;
\]
\( (f_2) \) There exists an \( \varepsilon > 0 \) small enough such that
\[
 tf'(t) \geq (1 + \varepsilon) f(t) > 0
\]
for \( t > 0; \)
\( (f_3) \) \( f(t) \) is odd,
Deng proved that when \( N \geq 4 \), (1.5) has at least a positive least energy radial solution with its corresponding energy \( < \frac{1}{N} S_{\frac{N}{2}}^\frac{N}{2} \) by using the Mountain Pass Theorem, where \( S \) is the sharp constant of \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \), i.e.
\[
 S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} |u|^{2^*})^{\frac{2}{2^*}}},
\]
where \( D^{1,2}(\mathbb{R}^N) \triangleq \{ u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \} \).

In particular, in (1.5), letting \( f(u) = \lambda_1 |u|^\alpha u \) or \( f(u) = \lambda_2 |u|^r u \) with \( \lambda_1, \lambda_2 > 0, 2 < \alpha, r < 2^* \), then we conclude from [12] that the following two equations
\[
-\Delta u + u = |u|^{2^*-2} u + \lambda_1 |u|^{\alpha-2} u, \quad x \in \mathbb{R}^N
\]
and
\[
-\Delta u + u = |u|^{2^*-2} v + \lambda_2 |u|^{r-2} u, \quad x \in \mathbb{R}^N
\]
respectively have at least one positive radial solution, denoted by \( u_1, v_1 \). Moreover, their corresponding energy respectively satisfies that
\[
B_1 \triangleq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_1|^{2^*} + \left( \frac{1}{2} - \frac{1}{\alpha} \right) \lambda_1 \int_{\mathbb{R}^N} |u_1|^\alpha < \frac{1}{N} S_{\frac{N}{2}}^\frac{N}{2}
\]
and
\[
B_2 \triangleq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |v_1|^{2^*} + \left( \frac{1}{2} - \frac{1}{r} \right) \lambda_2 \int_{\mathbb{R}^N} |v_1|^r < \frac{1}{N} S_{\frac{N}{2}}^\frac{N}{2}.
\]

Based on the above papers, an interesting question is: whether we can extend the existence results of (1.5) to system (1.1). In this paper, we will mainly discuss the existence of positive solutions to (1.1) in \( \mathbb{R}^N \) with \( N \geq 5 \), and obtain an affirmative answer. As far as we know, there is no existence result for (1.1).

Define \( H \triangleq H_1^1(\mathbb{R}^N) \times H_1^1(\mathbb{R}^N) \) with the norm
\[
\|(u, v)\| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) + \int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) \right)^{\frac{1}{2}},
\]
where \( H_1^1(\mathbb{R}^N) \triangleq \{ u \in H^1(\mathbb{R}^N) : u(x) = u(|x|) \} \). It is well known that weak solutions of (1.1) correspond to critical points of the energy functional \( I : H \to \mathbb{R} \) defined as follows
\[
I(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*} + 2\beta |u|^{\frac{2^*}{2}} |v|^{\frac{2^*}{2}}) - \frac{\lambda_1}{\alpha} \int_{\mathbb{R}^N} |u|^\alpha - \frac{\lambda_2}{r} \int_{\mathbb{R}^N} |v|^r,
\]
for any \((u,v) \in H\). We say \((u,v) \in H\) a positive solution of (1.1) if \((u,v)\) is a solution of (1.1) and \(u > 0, v > 0\).

To state our main results, we set

\[
M = \left\{ (u,v) \in H \mid u \neq 0, v \neq 0, \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = \int_{\mathbb{R}^N} (|u|^{2^*} + \beta |u|^{2^*} |v|^{2^*} + \lambda_1 |u|^\alpha), \int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) = \int_{\mathbb{R}^N} (|v|^{2^*} + \beta |u|^{2^*} |v|^{2^*} + \lambda_2 |v|^\alpha) \right\}.
\]

Then \(M \neq \emptyset\). In fact, take \(\varphi, \psi \in C_0^\infty(\mathbb{R}^N)\) with \(\varphi, \psi \neq 0\) and \(\text{supp}(\varphi) \cap \text{supp}(\psi) = \emptyset\), then there exist \(t_1, t_2 > 0\) such that \((t_1 \varphi, t_2 \psi) \in M\) since \(\alpha, r > 2\). So \(M \neq \emptyset\).

Denote

\[
B \triangleq \inf_{(u,v) \in M} I(u,v).
\]

It is easy to see that \(B > 0\) by the Sobolev embedding inequality.

Our main results are as follows:

**Theorem 1.1.** If \(N \geq 5, \lambda_1, \lambda_2 > 0, 0 < \alpha, r < 2^*,\) then problem (1.1) has a positive solution \((u,v) \in H\) with \(I(u,v) = B\) for any \(\beta > 0\).

**Theorem 1.2.** If \(N \geq 5, \lambda_1, \lambda_2 > 0, 0 < \alpha, r < 2^*,\) then problem (1.1) has a positive solution \((u,v) \in H\) with \(I(u,v) = B\) for any \(-\frac{1}{2} \leq \beta < 0\).

**Remark 1.3.** By the definition of \(M\), we see that the solutions obtained in Theorems 1 and 2 are least energy solutions in the radially symmetric Sobolev subspace \(H = H^1_0(\mathbb{R}^N) \times H^1_r(\mathbb{R}^N)\), however, we do not know that whether the solutions we obtained are the least energy solutions in the whole space \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

We try to use the Mountain Pass Theorem and Ekeland’s variational principle to prove Theorems 1.1 and 1.2. Our methods are inspired by the work of [10, 11], which deals with problems in bounded domains. However, since we deal with problems in \(\mathbb{R}^N\) and the appearance of two perturbation terms \(\lambda_1 u^{q-1}\) and \(\lambda_2 v^{r-1}\) in (1.1), we will encounter some new difficulties. First, as we know, a substantial difference between a bounded domain and the whole space \(\mathbb{R}^N\) is that the Sobolev embedding \(H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)\) with \(2 \leq q \leq 2^*\) is not compact. But Strauss’ radial Lemma tells us that the embedding \(H^1_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), 2 < q < 2^*\) is compact. Thus in this paper, we will discuss problem (1.1) in \(H = H^1_0(\mathbb{R}^N) \times H^1_r(\mathbb{R}^N)\) to recover the lack of compactness. Second, since the nonlinearity and coupling terms are of critical growth, we still face the difficulty that \(H^1_0(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)\) is not compact. For the case where \(\beta > 0\), it easily see that the functional \(I\) possesses a mountain pass structure around \(0 \in H\), then by the Mountain Pass Theorem, \(I\) has a critical point, denoted by \((u,v)\), in \(H\). But \((u,v)\) may be a trivial or semi-trivial critical point of \(I\). In order to prove that \((u,v)\) is nontrivial, inspired by [7, 10, 11], we try to pull the energy level down below some critical level to recover certain compactness condition. However, problem (1.1) is much more complicated comparing with (1.4) due to the effect of the perturbation terms, so some new ideas are needed in the process of pulling down the energy level; Meanwhile, In the case where \(\beta < 0\), we try to
follow the method used in [11] to prove the existence of positive solutions, i.e. we try to obtain a minimizing sequence of $B$ by Ekeland’s variational principle and then to show that the minimizing sequence weakly converges to a critical point of $I$ and finally to prove that the critical point is nontrivial by pulling down the energy level. However, the method used in [11] does not work here for all $\beta < 0$ because of the existence of the perturbation terms and need to be improved. We succeeded in doing so by proving the existence result for $\beta < 0$ restricted in some suitable interval and more careful analysis.

The paper is organized as follows. In Section 2, we will prove Theorem 1.1; In Section 3, we will give the proof of Theorem 1.2. Throughout this paper, for simplicity, we denote various positive constants as $C$ and omit $dx$ in integration. We use “$\to$, $\rightharpoonup$” to denote the strong and weak convergence in the related function space respectively and denote $B_r(x) \triangleq \{y \in \mathbb{R}^N | |x - y| < r\}$.

2. Proof of Theorem 1.1

Define

$$B = \inf_{h \in \Gamma} \max_{t > 0} I(h(t)),$$

where $\Gamma = \{h \in C([0, 1], H) | h(0) = 0, I(h(t)) < 0\}$. Since $2 < \alpha, r < 2^*$, it is easy to check that $B$ is well defined and that $B = \inf_{(u,v) \neq (0,0)} \max_{t > 0} I(tu, tv) = \inf_{(u,v) \in \mathcal{M}} I(u, v)$,

where $\mathcal{M}$ is the Nehari manifold associated to $I$, i.e.

$$\mathcal{M} = \left\{ (u, v) \in H \setminus \{(0, 0)\} | G(u, v) \triangleq \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2) - \int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*} + 2\beta|u|^{\frac{2^*}{2}}|v|^{\frac{2^*}{2}}) - \int_{\mathbb{R}^N} \left( \lambda_1 |u|^\alpha + \lambda_2 |v|^r \right) = 0 \right\}.$$

Note that $M \subset \mathcal{M}$, one has that $B \leq B$ and $B > 0$.

Since the nonlinearity and the coupling terms are both of critical growth in (1.1), the existence of nontrivial solutions to problem (1.1) depends heavily on that to the following limiting problem

$$\begin{cases}
-\Delta u = |u|^{2^*-2}u + \beta|u|^{\frac{2^*}{2}}-2u|v|^{\frac{2^*}{2}-1}, & x \in \mathbb{R}^N, \\
-\Delta v = |v|^{2^*-2}v + \beta|u|^{\frac{2^*}{2}}-1|v|^{\frac{2^*}{2}}-2v, & x \in \mathbb{R}^N, \\
u, v \in D^{1,2}(\mathbb{R}^N).
\end{cases} (2.1)$$

Define $D \triangleq D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and a $C^1$ functional $E : D \to \mathbb{R}$ given by

$$E(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*} + 2\beta|u|^{\frac{2^*}{2}}|v|^{\frac{2^*}{2}}).$$
Set
\[ \mathcal{N} = \left\{ (u, v) \in D \mid u \not\equiv 0, v \not\equiv 0, \int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} (|u|^{2^*} + \beta |u|^\frac{2^*}{2} |v|^\frac{2^*}{2}), \right. \]
\[ \left. \int_{\mathbb{R}^N} |\nabla v|^2 = \int_{\mathbb{R}^N} (|v|^{2^*} + \beta |u|^\frac{2^*}{2} |v|^\frac{2^*}{2}) \right\} \]
and
\[ \mathcal{N}' = \left\{ (u, v) \in D \setminus \{(0, 0)\} \mid \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) = \int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*} + 2\beta |u|^\frac{2^*}{2} |v|^\frac{2^*}{2}) \right\}, \]
i.e. \( \mathcal{N}' \) is the Nehari manifold associated to \( E \). Similarly to \( M \), we have that \( \mathcal{N} \neq \emptyset \) and \( \mathcal{N} \subset \mathcal{N}' \).

Denote
\[ A \triangleq \inf_{(u,v) \in \mathcal{N}} E(u,v), \quad A' \triangleq \inf_{(u,v) \in \mathcal{N}'} E(u,v). \]
Then \( 0 < A' \leq A \).

Lemma 2.1. ([11], Theorem 1.6, Proposition 2.1)
(i) If \( \beta < 0 \), then \( A \) is not attained and \( A = \frac{2}{N} S^N \).
(ii) If \( \beta > 0 \), then \( (2.1) \) has a positive least energy solution \((U, V) \in D \) with \( E(U, V) = A \),
which is radially symmetric decreasing. Moreover,
\[ U(x) + V(x) \leq C(1 + |x|)^{2-N}, \quad |\nabla U| + |\nabla V| \leq C(1 + |x|)^{1-N} \]
and
\[ A = A'. \]

Lemma 2.2. ([11], Lemma 3.3) Let \( u_n \to u, v_n \to v \) in \( H^1(\mathbb{R}^N) \) as \( n \to +\infty \), then passing to a subsequence, there holds
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} \left( |u_n|^\frac{2^*}{2} |v_n|^\frac{2^*}{2} - |u_n - u|^\frac{2^*}{2} |v_n - v|^\frac{2^*}{2} - |u|^\frac{2^*}{2} |v|^\frac{2^*}{2} \right) = 0. \]

Lemma 2.3. Let \( \beta > 0 \), then
\[ B < \min\{B_1, B_2, A\}, \]
where \( B_1, B_2 \) are given in (1.8), (1.9).

Proof. The arguments follow from that of Lemma 3.4 in [11], however, due to the effect of the perturbation terms, it is more complicated and some new ideas are needed. The proof consists of two steps.

Step 1: we prove that \( B < A \).

For \( \rho > 0 \), let \( \psi \in C_0^\infty(B_{2\rho}(0)) \) be a cut-off function with \( 0 \leq \psi \leq 1 \) and \( \psi \equiv 1 \) for \( |x| \leq \rho \). Recall that \((U, V) \) given in Lemma 2.1 (ii), for \( \varepsilon > 0 \), we define
\[ (U_\varepsilon(x), V_\varepsilon(x)) \triangleq (\varepsilon^{-\frac{N-2}{2}} U(\frac{x}{\varepsilon}), \varepsilon^{-\frac{N-2}{2}} V(\frac{x}{\varepsilon})). \]
By direct calculation, then
\[
\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 = \int_{\mathbb{R}^N} |\nabla U|^2, \quad \int_{\mathbb{R}^N} |U_\varepsilon|^2 = \int_{\mathbb{R}^N} |U|^2, \\
\int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 = \int_{\mathbb{R}^N} |\nabla V|^2, \quad \int_{\mathbb{R}^N} |V_\varepsilon|^2 = \int_{\mathbb{R}^N} |V|^2.
\]

Denote
\[(u_\varepsilon, v_\varepsilon) \triangleq (\psi U_\varepsilon, \psi V_\varepsilon).
\]

By the estimates given in Lemma 3.4 of [11], then we see that
\[
\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \leq \int_{\mathbb{R}^N} |\nabla U|^2 + O(\varepsilon^{N-2}), \quad (2.2)
\]
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^2 \geq \int_{\mathbb{R}^N} |U|^2 + O(\varepsilon^{N}), \quad (2.3)
\]
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^2 |v_\varepsilon|^2 \geq \int_{\mathbb{R}^N} |U|^2 |V|^2 + O(\varepsilon^{N}). \quad (2.4)
\]

Moreover, we still have the following inequality:
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^2 \leq C\varepsilon^2, \quad (2.5)
\]
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^\alpha \geq C\varepsilon^{N-\frac{N-2}{2}\alpha} + O(\varepsilon^{\frac{N-2}{2}\alpha}), \quad (2.6)
\]

where \(C\) denotes a positive constant. In fact, let \(0 < \varepsilon \ll \rho, \)
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^2 \leq \int_{|x| \leq 2\rho} \varepsilon^{2-N} U^2\left(\frac{x}{\varepsilon}\right) = \varepsilon^2 \int_{|x| \leq \frac{2\rho}{\varepsilon}} U^2(x) \leq \varepsilon^2 \int_{\mathbb{R}^N} U^2(x) \leq C\varepsilon^2.
\]

By Lemma 2.1(ii), we have that
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^\alpha \geq \int_{|x| \leq \rho} \varepsilon^{-\frac{N-2}{2}\alpha} |U\left(\frac{x}{\varepsilon}\right)|^\alpha = \int_{|x| \leq \frac{\rho}{\varepsilon}} \varepsilon^{N-\frac{N-2}{2}\alpha} |U(x)|^\alpha \geq C\varepsilon^{N-\frac{N-2}{2}\alpha} - C\varepsilon^{N-\frac{N-2}{2}\alpha} \int_{\frac{\rho}{\varepsilon}}^{\infty} r^{N-1-(N-2)\alpha} dr \geq C\varepsilon^{N-\frac{N-2}{2}\alpha} + O(\varepsilon^{\frac{N-2}{2}\alpha}).
\]

Similarly,
\[
\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \leq \int_{\mathbb{R}^N} |\nabla V|^2 + O(\varepsilon^{N-2}), \quad (2.7)
\]
\[
\int_{\mathbb{R}^N} |v_\varepsilon|^2 \geq \int_{\mathbb{R}^N} |V|^2 + O(\varepsilon^N),
\]
\[
\int_{\mathbb{R}^N} |v_\varepsilon|^2 \leq C \varepsilon^2,
\]
\[
\int_{\mathbb{R}^N} |v_\varepsilon|^r \geq C \varepsilon^{N-\frac{2}{N-r}} + O(\varepsilon^{\frac{2}{N-r}}),
\]
where \( C \) denotes a positive constant. Recall that \( E(U, V) = A \) and \( (U, V) \in N \), then
\[
NA = \int_{\mathbb{R}^N} |\nabla U|^2 + |\nabla V|^2 = \int_{\mathbb{R}^N} (|U|^2 + |V|^2 + 2\beta |U|^\frac{2}{r} |V|^\frac{2}{r}).
\]
For any \( t > 0 \), set
\[
g_\varepsilon(t) \equiv I(tu_\varepsilon, tv_\varepsilon) = \frac{t^2}{2} ||(u_\varepsilon, v_\varepsilon)||^2 - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + 2\beta |u_\varepsilon|^{\frac{2}{r}} |v_\varepsilon|^{\frac{2}{r}})
\]
\[
- \frac{\lambda_1 t^{\alpha}}{\alpha} \int_{\mathbb{R}^N} |u_\varepsilon|^\alpha - \frac{\lambda_2 t^r}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r.
\]
Since \( \alpha, r > 2 \), we easily see that \( g_\varepsilon(t) \) has a unique critical point \( t_\varepsilon \equiv t_{u_\varepsilon, v_\varepsilon} > 0 \), which corresponds to its maximum, i.e.
\[
I(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) = \max_{t > 0} I(tu_\varepsilon, tv_\varepsilon).
\]
By \( g'_\varepsilon(t_\varepsilon) = 0 \), then
\[
(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) \in M.
\]
We claim that \( \{t_\varepsilon\}_{\varepsilon > 0} \) is bounded from below by a positive constant. Otherwise, there exists a sequence \( \{\varepsilon_n\} \subset \mathbb{R}^+ \) satisfying \( \lim_{n \to \infty} t_{\varepsilon_n} = 0 \) and \( I(t_{\varepsilon_n} u_{\varepsilon_n}, t_{\varepsilon_n} v_{\varepsilon_n}) = \max_{t > 0} I(tu_{\varepsilon_n}, tv_{\varepsilon_n}) \), then
\[
0 < B \leq \lim_{n \to \infty} I(t_{\varepsilon_n} u_{\varepsilon_n}, t_{\varepsilon_n} v_{\varepsilon_n}) = 0,
\]
which is impossible. So there exists \( C > 0 \) independent of \( \varepsilon \) satisfying
\[
t_\varepsilon > C > 0 \text{ for all } \varepsilon > 0.
\]
Since \( N \geq 5 \) and \( 2 < \alpha, r < 2^* \),
\[
0 < N - \frac{N - 2}{2} \alpha < 2 < N - 2 < \frac{N - 2}{2} \alpha < N
\]
and
\[
0 < N - \frac{N - 2}{2} r < 2 < N - 2 < \frac{N - 2}{2} r < N.
\]
Then by (2.2)-(2.15), we see that
\[
I(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) = \frac{t_\varepsilon^2}{2} ||(u_\varepsilon, v_\varepsilon)||^2 - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + 2\beta |u_\varepsilon|^{\frac{2}{r}} |v_\varepsilon|^{\frac{2}{r}})
\]
\[
- \frac{\lambda_1 t_\varepsilon^{\alpha}}{\alpha} \int_{\mathbb{R}^N} |u_\varepsilon|^\alpha - \frac{\lambda_2 t_\varepsilon^r}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r
\]
\[
\leq \frac{t_\varepsilon^2}{2} \left[ \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2) + C \varepsilon^2 + O(\varepsilon^{N-2}) \right] - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} (|U|^{2^*} + |V|^{2^*})
\]
the Implicit Function Theorem, there exists a $u$ and $v$ such that

$$H(t) = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) + 2\beta |U|^{2^*} |V|^{2^*} + O(\varepsilon^N)$$

By the definition of $u$, it is easy to check that

$$F(t) = (tu_1, tv_1)$$

and

$$F(t(s), s) = 0, \quad \forall \ s \in (-\delta, \delta),$$

which implies that

$$t(s)u_1, t(s)sv_1) \in M, \quad \forall \ s \in (-\delta, \delta).$$

By direct calculation, we obtain that

$$F_s(t, s) = 2st^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + |v|^2) - 2^* t^{2^*-1} \int_{\mathbb{R}^N} |v|^2 + 2^* \beta t^{2^*-1} \int_{\mathbb{R}^N} |u|^2 + 2^* \int_{\mathbb{R}^N} |u|^2 |v|^2$$

and

$$F_t(t, s) = 2t\|u_1, sv_1\|^2 - 2^* t^{2^*-1} \int_{\mathbb{R}^N} |u|^2 + s^{2^*} \int_{\mathbb{R}^N} |v|^2 + 2\beta s^{2^*} \int_{\mathbb{R}^N} |u|^2 |v|^2$$

$$- \alpha \lambda_1 t^{\alpha-1} \int_{\mathbb{R}^N} |u_1|^2 - r \lambda_2 t^{\alpha-1} s^{\alpha} \int_{\mathbb{R}^N} |v|^2.$$
Since $N \geq 5$ and $r \in (2, 2^*)$, $\frac{2^*}{2} < r < 2^*$, then we have that
\[
\lim_{s \to 0} \frac{t'(s)}{|s|^{\frac{2^*}{2} - 2}s} = \lim_{s \to 0} \frac{-F_s/F_t}{|s|^{\frac{2^*}{2} - 2}s} = \frac{-2^* \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha},
\]
i.e.
\[
t'(s) = \frac{-2^* \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2} - 2}s(1 + o(1)) \text{ as } s \to 0.
\]
So
\[
t(s) = 1 - \frac{2\beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \text{ as } s \to 0,
\]
which implies that
\[
t^{2^*}(s) = 1 - \frac{22^* \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \text{ as } s \to 0, \tag{2.17}
\]
\[
t^\alpha(s) = 1 - \frac{2\alpha \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \text{ as } s \to 0 \tag{2.18}
\]
and
\[
t^r(s) = 1 - \frac{2r \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \text{ as } s \to 0. \tag{2.19}
\]
Thus by (1.8),(2.16)-(2.19) and $\frac{2^*}{2} < 2 < r$, we see that for $\forall s \in (-\delta, \delta)$,
\[
\mathcal{B} \leq \int (t(s)u_1, t(s)v_1)
\]
\[
= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2^*} \right) t(s) \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) + s^2 (|\nabla v_1|^2 + |v_1|^2) - \frac{t(s)}{2^*} \int_{\mathbb{R}^N} (|u_1|^{2^*} + |s|^{2^*} |v_1|^{2^*})
+ 2\beta |s|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}} |v_1|^{\frac{2^*}{2}} - \frac{\lambda_1 t(s)^\alpha}{\alpha} \int_{\mathbb{R}^N} |u_1|^\alpha - \frac{\lambda_2 t(s)^r}{r} \int_{\mathbb{R}^N} |s|^r |v_1|^r
\]
\[
= \left( \frac{1}{2} - \frac{1}{2^*} \right) t(s) \int_{\mathbb{R}^N} (|u_1|^{2^*} + |s|^{2^*} |v_1|^{2^*} + 2\beta |s|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}} |v_1|^{\frac{2^*}{2}})
+ \left( \frac{1}{2} - \frac{1}{\alpha} \right) \lambda_1 t(s)^\alpha \int_{\mathbb{R}^N} |u_1|^\alpha + \left( \frac{1}{2} - \frac{1}{r} \right) \lambda_2 t(s)^r \int_{\mathbb{R}^N} |s|^r |v_1|^r
\]
\[
= \left( \frac{1}{2} - \frac{1}{2^*} \right) \left[ 1 - \frac{22^* \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \right]
\]
\[
\times \int_{\mathbb{R}^N} (|u_1|^{2^*} + |s|^{2^*} |v_1|^{2^*} + 2\beta |s|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}} |v_1|^{\frac{2^*}{2}})
+ \left[ 1 - \frac{2\alpha \beta \int_{\mathbb{R}^N} |u_1|^\frac{2^*}{2} |v_1|^\frac{2^*}{2}}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1(\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha} |s|^{\frac{2^*}{2}}(1 + o(1)) \right].
\]
Hence by Brezis-Lieb lemma and Lemma 2.2, we have that
\[
\times \left( \frac{1}{2} - \frac{1}{\alpha} \right) \lambda_1 \int_{\mathbb{R}^N} |u_1|^\alpha + \left[ \frac{2r\beta}{(2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} |v_1|^{2^*}} (2^* - 2) \int_{\mathbb{R}^N} |u_1|^{2^*} + \lambda_1 (\alpha - 2) \int_{\mathbb{R}^N} |u_1|^\alpha |s|^{2^*} \right] (1 + o(1)) \right]
\times \left( \frac{1}{2} - \frac{1}{r} \right) \lambda_2 \int_{\mathbb{R}^N} |s|^r |v_1|^r
\leq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_1|^{2^*} + \left( \frac{1}{2} - \frac{1}{\alpha} \right) \lambda_1 \int_{\mathbb{R}^N} |u_1|^\alpha - C |s|^{2^*} + o(|s|^{2^*}) \quad (\exists \ C > 0)
\leq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_1|^{2^*} + \left( \frac{1}{2} - \frac{1}{\alpha} \right) \lambda_1 \int_{\mathbb{R}^N} |u_1|^\alpha = B_1, \quad \text{as } |s| > 0 \text{ small enough,}
\]
where \( C > 0 \) is a constant independent of \( s \). Hence \( \mathcal{B} < B_1 \).

Similarly, we have that \( \mathcal{B} < B_2 \). Hence the proof of the Lemma is completed. \( \square \)

**Proof of Theorem 1.1**

Since \( \beta > 0 \), it is easy to check that \( I(u, v) \) possesses a mountain pass structure, then by the Mountain Pass Theorem in [5, 24], there exists a sequence \( \{(u_n, v_n)\} \subset \mathcal{H} \) such that
\[
\lim_{n \to +\infty} I(u_n, v_n) = \mathcal{B}, \quad \lim_{n \to +\infty} I'(u_n, v_n) = 0.
\]

It is standard to see that \( \{(u_n, v_n)\} \) is bounded in \( \mathcal{H} \), so we may assume that \( (u_n, v_n) \rightharpoonup (u, v) \) in \( \mathcal{H} \) for some \( (u, v) \in \mathcal{H} \). Set \( \omega_n = u_n - u, \sigma_n = v_n - v \). Then
\[
\omega_n \to 0, \quad \sigma_n \to 0 \quad \text{in } H^1_r(\mathbb{R}^N),
\]
\[
\omega_n \to 0, \quad \sigma_n \to 0 \quad \text{in } L^{2^*}(\mathbb{R}^N)
\]
and
\[
\omega_n \to 0 \quad \text{in } L^\alpha(\mathbb{R}^N), \quad \sigma_n \to 0 \quad \text{in } L^r(\mathbb{R}^N).
\]

Hence by Brezis-Lieb lemma and Lemma 2.2, we have that \( I'(u, v) = 0 \) and
\[
\int_{\mathbb{R}^N} |\nabla \omega_n|^2 - \int_{\mathbb{R}^N} \left( |\omega_n|^{2^*} + \beta |\omega_n|^{\frac{2^*}{r}} |\sigma_n|^{\frac{2^*}{r}} \right) = o(1), \tag{2.20}
\]
\[
\int_{\mathbb{R}^N} |\nabla \sigma_n|^2 - \int_{\mathbb{R}^N} \left( |\sigma_n|^{2^*} + \beta |\omega_n|^{\frac{2^*}{r}} |\sigma_n|^{\frac{2^*}{r}} \right) = o(1) \tag{2.21}
\]
and
\[
I(u_n, v_n) = I(u, v) + E(\omega_n, \sigma_n) + o(1). \tag{2.22}
\]

Passing to a subsequence, we may assume that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla \omega_n|^2 = b_1, \quad \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla \sigma_n|^2 = b_2.
\]

Then by (2.20),(2.21), we have that
\[
E(\omega_n, \sigma_n) = \frac{1}{N} (b_1 + b_2) + o(1). \tag{2.23}
\]
Letting \( n \to +\infty \) in (2.22), we have that
\[
0 \leq I(u, v) \leq I(u, v) + \frac{1}{N}(b_1 + b_2) = \lim_{n \to +\infty} I(u_n, v_n) = \mathcal{B}.
\] (2.24)

**Case 1:** \( u \equiv 0, v \equiv 0 \).

By (2.24), we have that \( 0 < N\mathcal{B} = b_1 + b_2 < +\infty \), then we may assume that \( (\omega_n, \sigma_n) \neq (0, 0) \) for \( n \) large. By the definition of \( \mathcal{N} \) and (2.20),(2.21), similar to the proof of (2.12), we see that there exists a sequence \( \{t_n\} \subset \mathbb{R}_+ \) such that \( (t_n\omega_n, t_n\sigma_n) \in \mathcal{N} \) and \( t_n \to 1 \) as \( n \to +\infty \). Then by (2.20),(2.23) and Lemma 2.1 (ii), we have that
\[
\mathcal{B} = \frac{1}{N}(b_1 + b_2) = \lim_{n \to +\infty} E(\omega_n, \sigma_n) = \lim_{n \to +\infty} E(t_n\omega_n, t_n\sigma_n) \geq \mathcal{A} = A,
\]
which contradicts to Lemma 2.3. So Case 1 is impossible.

**Case 2:** \( u \neq 0, v \equiv 0 \) or \( u \equiv 0, v \neq 0 \).

Without loss of generality, we may assume that \( u \neq 0, v \equiv 0 \). Then \( u \in H^1_1(\mathbb{R}^N) \) is a nontrivial solution of \( -\Delta u = |u|^{2^* - 2}u + \lambda_1|u|^{\alpha - 2}u \), and so \( \mathcal{B} \geq I(u, 0) \geq B_1 \), which contradicts to Lemma 2.3. So Case 2 is impossible.

Since Case 1 and Case 2 are both impossible, we see that \( u \neq 0, v \neq 0 \). By \( I'(u, v) = 0 \), then we have that \( (u, v) \in \mathcal{M} \). By \( \mathcal{B} \leq \mathcal{B} \) and (2.24), we have that
\[
I(u, v) = \mathcal{B} = B.
\]
Moreover, it is easy to see that \( (|u|, |v|) \in \mathcal{M} \subset \mathcal{M} \) and \( I(|u|, |v|) = \mathcal{B} = B \) since the functional \( I \) and the manifolds \( \mathcal{M} \) and \( \mathcal{M} \) are symmetric, hence we may assume that such a minimizer of \( B \) does not change sign, i.e. \( u \geq 0, v \geq 0 \). This means that \( (u, v) \in H \) is a nontrivial nonnegative solution of (1.1) with \( I(u, v) = B \). By the maximum principle, we see that \( u(x), v(x) > 0 \) for all \( x \in \mathbb{R}^N \). Thus, \( (u, v) \) is a positive solution of (1.1) in \( H \) with \( I(u, v) = B \). This completes the proof of Theorem 1.1.

### 3. Proof of Theorem 1.2

For \( \varepsilon > 0 \) and \( y \in \mathbb{R}^N \), the following Aubin-Talenti instanton \( U_{\varepsilon, y} \in D^{1,2}(\mathbb{R}^N) \) (see [1, 23])
\[
U_{\varepsilon, y}(x) = [N(N - 2)]^{\frac{N-2}{4}} \left( \frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2}{2}}.
\]
Then \( U_{\varepsilon, y} \) solves \( -\Delta u = |u|^{2^* - 2}u \) in \( \mathbb{R}^N \) and
\[
\int_{\mathbb{R}^N} |\nabla U_{\varepsilon, y}|^2 = \int_{\mathbb{R}^N} |U_{\varepsilon, y}|^{2^*} = S^F_N.
\] (3.1)
Furthermore, \( \{U_{\varepsilon, y} : \varepsilon > 0, y \in \mathbb{R}^N\} \) contains all positive solutions of the equation \( -\Delta u = |u|^{2^* - 2}u \) in \( \mathbb{R}^N \).

As has been mentioned in Section 1, \( u_1, v_1 \in H^1_1(\mathbb{R}^N) \) are positive least energy radial solutions of (1.6) and (1.7) respectively, moreover, by the standard regularity arguments, \( u_1, v_1 \in C(\mathbb{R}^N) \).
Lemma 3.1. Suppose that $-\frac{1}{2} \leq \beta < 0$, then we have that

$$B < \min \{ B_1 + \frac{1}{N} S_\frac{N}{2}, B_2 + \frac{1}{N} S_\frac{N}{2}, A \}.$$ 

Proof. The proof is similar to that of Lemma 3.1 in [11], however, due to the effect of the perturbation terms, the arguments need to be slightly improved. We give its detailed proof.

Let $-\frac{1}{2} \leq \beta < 0$, consider the following function

$$F(t) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) - \frac{t^{2^*}}{22^*} \int_{\mathbb{R}^N} |u_1|^{2^*} - \frac{\lambda t^\alpha}{\alpha} \int_{\mathbb{R}^N} |u_1|^{\alpha} + \frac{2^{N-2}}{N} S_\frac{N}{2}, \quad t > 0.$$ 

Then by $\alpha > 2$, there exists a $t_0 > 0$ such that

$$F(t) < 0, \quad \forall \ t > t_0. \quad (3.2)$$

For $y_0 \in \mathbb{R}^N, R > 0$, let $\psi \in C_0^\infty (B_{2R}(y_0))$ be a cut-off function with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ for $|x - y_0| \leq R$. Define $v_\varepsilon = \psi U_{\varepsilon,y_0}$, where $U_{\varepsilon,y_0}$ is defined in (3.1). Then by [7], we have the following estimates:

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 = S_\frac{N}{2} + O(\varepsilon^{N-2}), \quad \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} = S_\frac{N}{2} + O(\varepsilon^N), \quad (3.3)$$

and

$$\int_{\mathbb{R}^N} |v_\varepsilon|^2 \leq C \varepsilon^2, \quad \int_{\mathbb{R}^N} |v_\varepsilon|^r \geq C \varepsilon^{N-N-r} + O(\varepsilon^{N-r}) \quad (3.4)$$

and

$$\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} \leq \int_{B(y_0,2R)} |U_{\varepsilon,y_0}|^{2^*} \leq C \int_{B(0,2R)} \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N}{2}} \leq C \varepsilon^{\frac{N}{2}} (\ln \frac{2R}{\varepsilon} + 1) = o(\varepsilon^2), \quad (3.5)$$

where $C > 0$ is a constant.

Since $|\beta| \leq \frac{1}{2}$, we have for any $t, s > 0$ that

$$2|\beta| t^2 s^{\frac{N}{2}} \int_{\mathbb{R}^N} |u_1|^{2^*} |v_\varepsilon|^{2^*} \leq \frac{t^{2^*}}{2} \int_{\mathbb{R}^N} |u_1|^{2^*} + \frac{s^{2^*}}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*}$$
and then

\[ \begin{align*}
I(tu_1, sv_\varepsilon) &= \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) + \frac{s^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) \\
&\quad - \frac{\lambda_1^{\frac{1}{2^*}}}{\alpha} \int_{\mathbb{R}^N} |u_1|^\alpha - \frac{\lambda_2 s^r}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r \\
&\leq \left[ \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) - \frac{\lambda_1^{\alpha}}{\alpha} \int_{\mathbb{R}^N} |u_1|^\alpha \right] \\
&\quad + \left[ \frac{s^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) - \frac{\lambda_2 s^{2r}}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r \right] \triangleq f(t) + g(s). \tag{3.6}
\end{align*} \]

Similar to the proof of (2.13), since \( r > 2 \),

\[ g(s) = \frac{s^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) - \frac{s^2}{22^*} \int_{\mathbb{R}^N} v_\varepsilon^{2^*} - \frac{\lambda_2 s^r}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r, \ s > 0, \]

has a unique critical point \( s_\varepsilon > 0 \) corresponding to its maximum and there exists \( C > 0 \) independent of \( \varepsilon \) such that

\[ s_\varepsilon > C > 0 \quad \text{for all} \ \varepsilon > 0. \tag{3.7} \]

Then by (3.3),(3.4),(3.7) and \( r > 2 \), by direct calculation, it is easy to check that

\[ \max_{s > 0} g(s) = g(s_\varepsilon) < \frac{1}{N} 2^{\frac{N-2}{2}} S_\varepsilon^N \quad \text{for} \ \varepsilon \ \text{small enough}. \tag{3.8} \]

Then by (3.2),

\[ f(t) + g(s) < 0, \ \forall \ t > t_0, \ s > 0. \]

Thus by (3.6),

\[ \max_{t,s > 0} I(tu_1, sv_\varepsilon) = \max_{0 < t < t_0, s > 0} I(tu_1, sv_\varepsilon). \]

Set

\[ g_\varepsilon(s) = \frac{s^2}{2} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) - \frac{s^2}{22^*} \int_{\mathbb{R}^N} v_\varepsilon^{2^*} - \frac{\lambda_2 s^r}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r, \ s > 0. \]

Similar to the proof of (3.7), \( g_\varepsilon(s) \) has a unique critical point \( s(\varepsilon) > 0 \) such that

\[ g_\varepsilon(s(\varepsilon)) = \max_{s > 0} g_\varepsilon(s) \quad \text{and} \quad s(\varepsilon) > C_0 > 0, \]

where \( C_0 > 0 \) is a constant independent of \( \varepsilon \). Since \( g_\varepsilon(s) \) is strictly increasing for \( 0 < s \leq s(\varepsilon) \), for any \( 0 < s < C_0 \), we have that \( g_\varepsilon(s) < g_\varepsilon(C_0) \) and then

\[ I(tu_1, sv_\varepsilon) < I(tu_1, C_0 v_\varepsilon), \ \forall \ t > 0, \ 0 < s < C_0. \]

Hence

\[ \max_{t,s > 0} I(tu_1, sv_\varepsilon) = \max_{0 < t < t_0, s \geq C_0} I(tu_1, sv_\varepsilon). \tag{3.9} \]
For $0 < t < t_0, s \geq C_0$, we see from (3.5), $-\frac{1}{2} \leq \beta < 0$ and $u_1 \in C(\mathbb{R}^N)$ that there exists a $C \triangleq \max_{B_{R_0}(R_0)} u_1 > 0$ such that

$$|\beta| t^{\frac{2}{T}} s^{\frac{2}{T}} \int_{\mathbb{R}^N} |u_1|^{\frac{2}{T}} |v_\varepsilon|^{\frac{2}{T}} \leq CC^0_{\varepsilon} - \frac{2t_0}{2} s^2 \int_{\mathbb{R}^N} |v_\varepsilon|^{\frac{2}{T}} \leq s^2 o(\varepsilon^2),$$

so

$$I(tu_1, sv_\varepsilon) = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) + s^2 \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2)$$

$$- \frac{2t}{2} \int_{\mathbb{R}^N} (t^2 |u_1|^2 + s^2 |v_\varepsilon|^2 + 2\beta t^{\frac{2}{T}} s^{\frac{2}{T}} |u_1|^{\frac{2}{T}} |v_\varepsilon|^{\frac{2}{T}})$$

$$- \frac{\lambda_1 t^\alpha}{\alpha} \int_{\mathbb{R}^N} |u_1|^\alpha - \frac{\lambda_2}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r$$

$$\leq \left[ \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2) - \frac{t^2}{2} \int_{\mathbb{R}^N} |u_1|^2 - \frac{\lambda_1 t^\alpha}{\alpha} \int_{\mathbb{R}^N} |u_1|^\alpha \right]$$

$$+ \left[ \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) + o(\varepsilon^2) \right] - \frac{s^2}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^2 - \frac{\lambda_2}{r} \int_{\mathbb{R}^N} |v_\varepsilon|^r$$

$$\triangleq f_1(t) + g_1(s).$$

Note that max$_{t>0} f_1(t) = f_1(1) = B_1$. Similar to the proof of (3.8), we have that

$$\max_{s>0} g_1(s) < \frac{1}{N} S^\frac{
abla}{2} \varepsilon \quad \text{for } \varepsilon \text{ small enough.}$$

By (3.9)(3.10), we see that

$$\max_{t,s>0} I(tu_1, sv_\varepsilon) = \max_{0 < t \leq t_0, s \geq 1} I(tu_1, sv_\varepsilon)$$

$$\leq \max_{t>0} f_1(t) + \max_{s>0} g_1(s)$$

$$< B_1 + \frac{1}{N} S^\frac{
abla}{2} \varepsilon \quad \text{for } \varepsilon \text{ small enough.}$$

Denote

$$D_1 = \int_{\mathbb{R}^N} |u_1|^2, \quad D_2 = \beta \int_{\mathbb{R}^N} |u_1|^{\frac{2}{T}} |v_\varepsilon|^{\frac{2}{T}} < 0, \quad D_3 = \int_{\mathbb{R}^N} |v_\varepsilon|^2,$$

$$D_4 = \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2), \quad E_1 = \lambda_1 \int_{\mathbb{R}^N} |u_1|^\alpha, \quad E_2 = \lambda_2 \int_{\mathbb{R}^N} |v_\varepsilon|^r.$$

Then

$$D_1 + E_1 = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |u_1|^2)$$

and

$$D_2^2 = |\beta|^2 \left( \int_{\mathbb{R}^N} |u_1|^{\frac{2}{T}} |v_\varepsilon|^{\frac{2}{T}} \right)^2 \leq \frac{1}{4} D_1 D_3 < D_1 D_3.$$

(3.12)
We claim that there exist \( t_\varepsilon, s_\varepsilon > 0 \) such that \((t_\varepsilon u_1, s_\varepsilon v_\varepsilon) \in M\), i.e. \((t_\varepsilon, s_\varepsilon)\) solves the following system

\[
\begin{cases}
    t^2(D_1 + E_1) = t^{2^*} D_1 + t^{\frac{2^*}{2}} s^{\frac{2^*}{2}} D_2 + t^\alpha E_1, \\
    s^2 D_4 = t^{2^*} D_3 + t^{\frac{2^*}{2}} s^{\frac{2^*}{2}} D_2 + t^r E_2, \\
    t, s > 0.
\end{cases}
\]  

(3.13)

System (3.13) is equivalent to

\[
\begin{cases}
(t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 = s^{\frac{2}{2}} D_2 + (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1, \\
s^{\frac{2}{2}} D_4 = s^{\frac{2}{2}} D_3 + t^{\frac{2}{2}} s^{\frac{2}{2}} D_2 + t^r E_2, \\
\end{cases}
\]  

(3.14)

then

\( (t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 < (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1, \)

hence \( t > 1 \) since \( 2 < \alpha < 2^* \) and \( 1 < \frac{2^*}{2} < 2 \). Moreover, by (3.14), we have that

\[
s^{\frac{2}{2}} = \frac{(t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 - (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1}{D_2}.
\]

Then (3.14) is equivalent to

\[
G(t) \triangleq \frac{[(t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 - (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1]}{D_2} \cdot \frac{1 - \frac{2^r}{2}}{D_4}
\]

\[
- \frac{[(t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 - (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1]}{D_2} \cdot D_3 - t^{\frac{2^r}{2}} D_2
\]

\[
- \frac{[(t^{\frac{2}{2}} - t^{\frac{2}{2}})D_1 - (t^{\alpha - \frac{2}{2}} - t^{2 - \frac{2}{2}})E_1]}{D_2} \cdot \frac{2^r - 2}{2} E_2 = 0, \quad t > 1.
\]

Since \( N > 5 \) and \( 2 < r < 2^*, \ 2^* < 4 < 2r \). Then \( G(1) = -D_2 > 0 \) and \( \lim_{t \to +\infty} \frac{G(t)}{t^{\frac{2^r}{2}}} = \frac{D_1D_3 - D_2^2}{D_2} < 0 \), hence \( G(t) = 0 \) has a solution \( t > 1 \). So (3.14) has a solution \( t_\varepsilon, s_\varepsilon > 0 \), i.e. (3.13) has a solution \( t_\varepsilon, s_\varepsilon > 0 \). Therefore \((t_\varepsilon u_1, s_\varepsilon v_\varepsilon) \in M\). By (3.11), we have that

\[
B \leq I(t_\varepsilon u_1, s_\varepsilon v_\varepsilon) \leq \max_{t,s > 0} I(tu_1, sv_\varepsilon) < B_1 + \frac{1}{N} S_{\frac{N}{2}}.
\]

Similarly, we can also prove that \( B < B_2 + \frac{1}{N} S_{\frac{N}{2}} \). Moreover, by Lemma 2.1(i) and (1.8),(1.9), we see that

\[
A > \max \left\{ B_1 + \frac{1}{N} S_{\frac{N}{2}}, B_2 + \frac{1}{N} S_{\frac{N}{2}} \right\}.
\]

Then we complete the proof of Lemma 3.1. \( \square \)
Remark 3.2. In [10, 11], Chen and Zou proved the same result for any $\beta < 0$, however, for equation (1.1), because of the effect of the perturbation terms, we only obtain the estimate for $\beta \in [-\frac{1}{2}, 0)$.

Lemma 3.3. Suppose that $-\frac{1}{2} \leq \beta < 0$, then there exist $C_2 > C_1 > 0$, such that for any $(u, v) \in M$ with $I(u, v) \leq C$, then we have that

$$C_1 \leq \int_{\mathbb{R}^N} |u|^{2^*}, \int_{\mathbb{R}^N} |v|^{2^*} \leq C_2.$$ 

Proof. Since $(u, v) \in M$ and $2 < \alpha, r < 2^*$ and $-\frac{1}{2} \leq \beta < 0,$

$$I(u, v) = \frac{1}{N} \int_{\mathbb{R}^N} (|u|^{2^*} + 2\beta |u|^{\frac{2^*}{2}} |v|^{\frac{2^*}{2}} + |v|^{2^*}) + \left(\frac{1}{2} - \frac{1}{r}\right) \lambda_1 \int_{\mathbb{R}^N} |u|^{\alpha}$$

$$+ \left(\frac{1}{2} - \frac{1}{r}\right) \lambda_2 \int_{\mathbb{R}^N} |v|^{r} \geq \frac{1}{N} (1 - |\beta|) \int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*}),$$

then there exists $C_2 > 0$ such that $\int_{\mathbb{R}^N} |u|^{2^*}, \int_{\mathbb{R}^N} |v|^{2^*} \leq C_2.$

Now, we prove that $\int_{\mathbb{R}^N} |u|^{2^*}, \int_{\mathbb{R}^N} |v|^{2^*} \geq C_1.$

By $(u, v) \in M$, $\beta < 0$ and the Sobolev embedding inequality, we have that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = \int_{\mathbb{R}^N} |u|^{2^*} + \beta |u|^{\frac{2^*}{2}} |v|^{\frac{2^*}{2}} + \lambda_1 \int_{\mathbb{R}^N} |u|^\alpha$$

$$\leq \int_{\mathbb{R}^N} |u|^{2^*} + \lambda_1 \int_{\mathbb{R}^N} |u|^\alpha$$

$$\leq S^{-\frac{2^*}{2^*}} \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \right)^\frac{2^*}{2} + C \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \right)^\frac{2^*}{4}. \tag{3.15}$$

We conclude that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \geq C \quad \text{for some } C > 0$$

since $0 < \frac{\alpha}{2} + \frac{2^*}{2}$. Moreover, $I(u, v) \leq C$ implies that $(u, v)$ is bounded in $H$, i.e. $||(u, v)|| \leq C$ for some $C > 0$, then by (3.15) and the interpolation inequality, we have that

$$C \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \leq \int_{\mathbb{R}^N} |u|^{2^*} + \lambda_1 \int_{\mathbb{R}^N} |u|^\alpha$$

$$\leq \int_{\mathbb{R}^N} |u|^{2^*} + \lambda_1 \left( \int_{\mathbb{R}^N} |u|^{2^*} \right)^\theta (\int_{\mathbb{R}^N} |u|^2)^{\frac{(1-\theta)\alpha}{2}}$$

$$\leq \int_{\mathbb{R}^N} |u|^{2^*} + C \left( \int_{\mathbb{R}^N} |u|^{2^*} \right)^\theta.$$
where \( \frac{1}{\alpha} = \frac{\theta}{2^*} + \frac{1-\theta}{2} \) and \( 0 < \theta < 1 \). So there exists a \( C > 0 \) such that

\[
\int_{\mathbb{R}^N} |u|^{2^*} \geq C.
\]

Similarly, we have that \( \int_{\mathbb{R}^N} |v|^{2^*} \geq C \) and we complete the proof of the Lemma. \( \Box \)

**Proof of Theorem 1.2**

The main idea of the proof comes from [10, 11], but more careful analysis is needed. Note that \( I \) is coercive and bounded from below on \( M \), then by the Ekeland’s variational principle (see [22]), there exists a minimizing sequence \( \{ (u_n, v_n) \} \subset M \) satisfying

\[
I(u_n, v_n) \leq \min \{ B + \frac{1}{n}, A \},
\]

\[
I(u, v) \geq I(u_n, v_n) - \frac{1}{n} \| (u_n, v_n) - (u, v) \|, \quad \forall (u, v) \in M.
\]

Then it is easy to see that \( \{ (u_n, v_n) \} \) is bounded in \( H \). For any \((\varphi, \psi) \in H \) with \( \| \varphi \|_{H^1(\mathbb{R}^N)}, \| \psi \|_{H^1(\mathbb{R}^N)} \leq 1 \) and each \( n \in N \), define \( h_n, g_n : \mathbb{R}^3 \to \mathbb{R} \) by

\[
h_n(t, s, l) = \int_{\mathbb{R}^N} |\nabla (u_n + t\varphi + su_n)|^2 + \int_{\mathbb{R}^N} |u_n + t\varphi + su_n|^2 - \int_{\mathbb{R}^N} |u_n + t\varphi + su_n|^{2^*} - \beta \int_{\mathbb{R}^N} |u_n + t\varphi + su_n|^{\frac{2^*}{2}} |v_n + t\phi + lv_n|^{\frac{2^*}{2}} - \lambda_1 \int_{\mathbb{R}^N} |u_n + t\varphi + su_n|^{\alpha}
\]

and

\[
g_n(t, s, l) = \int_{\mathbb{R}^N} |\nabla (v_n + t\phi + lv_n)|^2 + \int_{\mathbb{R}^N} |v_n + t\phi + lv_n|^2 - \int_{\mathbb{R}^N} |v_n + t\phi + lv_n|^{2^*} - \beta \int_{\mathbb{R}^N} |u_n + t\varphi + su_n|^{\frac{2^*}{2}} |v_n + t\phi + lv_n|^{\frac{2^*}{2}} - \lambda_2 \int_{\mathbb{R}^N} |v_n + t\phi + lv_n|^r
\]

Let \( 0 = (0, 0, 0) \). Then \( h_n, g_n \in C^1(\mathbb{R}^3, \mathbb{R}) \) and \( h_n(0) = g_n(0) = 0 \).

\[
\frac{\partial h_n}{\partial s}(0) = -(2^* - 2) \int_{\mathbb{R}^N} |u_n|^{2^*} - \frac{2^*}{2} \beta \int_{\mathbb{R}^N} |u_n|^{\frac{2^*}{2}} |v_n|^{\frac{2^*}{2}} - (\alpha - 2) \lambda_1 \int_{\mathbb{R}^N} |u_n|^{\alpha},
\]

\[
\frac{\partial h_n}{\partial l}(0) = \frac{\partial g_n}{\partial s}(0) = -\frac{2^*}{2} \beta \int_{\mathbb{R}^N} |u_n|^{\frac{2^*}{2}} |v_n|^{\frac{2^*}{2}},
\]

\[
\frac{\partial g_n}{\partial l}(0) = -(2^* - 2) \int_{\mathbb{R}^N} |v_n|^{2^*} - \frac{2^*}{2} \beta \int_{\mathbb{R}^N} |u_n|^{\frac{2^*}{2}} |v_n|^{\frac{2^*}{2}} - (r - 2) \lambda_2 \int_{\mathbb{R}^N} |v_n|^r.
\]

Define the matrix

\[
F_n \triangleq \begin{pmatrix}
\frac{\partial h_n}{\partial s}(0) & \frac{\partial h_n}{\partial l}(0) \\
\frac{\partial g_n}{\partial s}(0) & \frac{\partial g_n}{\partial l}(0)
\end{pmatrix}.
\]
Then by \(-\frac{1}{2} \leq \beta < 0\), the Hölder inequality and Lemma 3.3, we have that

\[
\text{det}(F_n) = \left(2^* - 2\right)\int_{\mathbb{R}^N} |u_n|^2 + \frac{\beta (2^*_n - 2)}{2} \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} + 2\beta (\frac{2^*_n}{2} - 2) \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} - (\frac{\beta}{2})^2 \frac{2^*_n}{2} \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} \int_{\mathbb{R}^N} |v_n|^\frac{2^*_n}{2} + 2\beta (\frac{2^*_n}{2} - 2) \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} |v_n|^\frac{2^*_n}{2} - \frac{2^*_n}{2} \frac{2^*_n}{2} \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} |v_n|^\frac{2^*_n}{2}\]

\[
> (2^* - 2)\left(\int_{\mathbb{R}^N} |u_n|^2 + \int_{\mathbb{R}^N} |v_n|^2 + \frac{\beta (2^*_n - 2)}{2} \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} \int_{\mathbb{R}^N} |v_n|^\frac{2^*_n}{2} - (\frac{\beta}{2})^2 \frac{2^*_n}{2} \int_{\mathbb{R}^N} |u_n|^\frac{2^*_n}{2} \int_{\mathbb{R}^N} |v_n|^\frac{2^*_n}{2}\right)
\]

where \(C\) is independent of \(n\) and we have used the fact that

\[
f(t) \triangleq -2t^2 + (2^* - 4)t + 2^* - 2 \geq \min\{f(-\frac{1}{2}), f(0)\} > 0, \quad \forall \, t \in [-\frac{1}{2}, 0].
\]

By the Implicit Function Theorem, there exist \(\delta_n > 0\) and functions \(s_n(t), l_n(t) \in C^1(-\delta_n, \delta_n)\). Moreover, \(s_n(0) = l_n(0) = 0\),

\[
h_n(t, s_n(t), l_n(t)) = 0, \quad g_n(t, s_n(t), l_n(t)) = 0, \quad \forall \, t \in (-\delta_n, \delta_n)
\]

and

\[
\begin{align*}
\begin{cases}
    s_n'(0) = \frac{1}{\text{det}F_n} \left( \frac{\partial g_n}{\partial t}(0) \frac{\partial h_n}{\partial t}(0) - \frac{\partial g_n}{\partial l}(0) \frac{\partial h_n}{\partial t}(0) \right) \\
    l_n'(0) = \frac{1}{\text{det}F_n} \left( \frac{\partial g_n}{\partial s}(0) \frac{\partial h_n}{\partial t}(0) - \frac{\partial g_n}{\partial t}(0) \frac{\partial h_n}{\partial s}(0) \right).
\end{cases}
\end{align*}
\]

Since \(\{(u_n, v_n)\}\) is bounded in \(H\), it is easy to see that

\[
|\frac{\partial h_n}{\partial s}(0)|, |\frac{\partial h_n}{\partial t}(0)|, |\frac{\partial g_n}{\partial s}(0)|, |\frac{\partial g_n}{\partial t}(0)| \leq C,
\]

where \(C\) is independence of \(n\). Then by Lemma 3.3, we also have that

\[
|s_n'(0)|, |l_n'(0)| \leq C,
\]

Hence, we can conclude that

\[
|s_n(0)|, |l_n(0)| \leq C, \quad (3.20)
\]

where \(C\) is independence of \(n\).
Denote
\[ \varphi_{n,t} \triangleq u_n + t \varphi + s_n(t)u_n, \quad \phi_{n,t} \triangleq v_n + t \phi + l_n(t)v_n, \]
then \((\varphi_{n,t}, \phi_{n,t}) \in M\) for \(\forall t \in (-\delta_n, \delta_n)\). It follows from (3.17) that
\[ I(\varphi_{n,t}, \phi_{n,t}) - I(u_n, v_n) \geq -\frac{1}{n} \| (t \varphi + s_n(t)u_n, t \phi + l_n(t)v_n) \|. \quad (3.21) \]
By \((u_n, v_n) \in M\) and the Taylor Expansion we have that
\[ I(\varphi_{n,t}, \phi_{n,t}) - I(u_n, v_n) = \langle I'(u_n, v_n), (t \varphi + s_n(t)u_n, t \phi + l_n(t)v_n) \rangle + r(n, t) \]
where \(r(n, t) = o(\| (t \varphi + s_n(t)u_n, t \phi + l_n(t)v_n) \|)\) as \(t \to 0\). By (3.20), we see that
\[ \limsup_{t \to 0} \| (\varphi + \frac{s_n(t)}{t}u_n, \phi + \frac{l_n(t)}{t}v_n) \| \leq C, \]
where \(C\) is independence of \(n\). Hence \(r(n, t) = o(t)\). By (3.21)-(3.23) and letting \(t \to 0\), we have that
\[ | \langle I'(u_n, v_n)(\varphi, \phi) \rangle | \leq \frac{C}{n}, \]
where \(C\) is independence of \(n\). Hence
\[ \lim_{n \to +\infty} I'(u_n, v_n) = 0. \quad (3.24) \]
Since \(\{(u_n, v_n)\}\) is bounded in \(H\), passing to a subsequence, we may assume that \((u_n, v_n) \rightharpoonup (u, v)\) in \(H\) for some \((u, v) \in H\). Set \(\omega_n = u_n - u, \sigma_n = v_n - v\) and use the same notations as in the proof of Theorem 1.1, we also see that \(I'(u, v) = 0\) and (2.20)-(2.23) hold. Furthermore,
\[ 0 \leq I(u, v) \leq I(u, v) + \frac{1}{N}(b_1 + b_2) = \lim_{n \to +\infty} I(u_n, v_n) = B. \quad (3.25) \]

**Case 1:** \(u \equiv 0, \ v \equiv 0\).

By (3.25), we have \(0 < NB = b_1 + b_2 < +\infty\). Then we conclude from Lemma 3.3 that \(0 < b_1 < +\infty, 0 < b_2 < +\infty\). Hence we may assume that \(\omega_n \neq 0, \sigma_n \neq 0\) for \(n\) large. Similar to the argument used in the proof of Theorem 1.3 (P. 32-33) in [11], for \(n\) large, there exist \(t_n, s_n > 0\) such that \((t_n \omega_n, s_n \sigma_n) \in N\) and \(\lim_{n \to +\infty} (|t_n - 1| + |s_n - 1|) = 0\). Then we have that
\[ \frac{1}{N}(b_1 + b_2) = \lim_{n \to +\infty} E(\omega_n, \sigma_n) = \lim_{n \to +\infty} E(t_n \omega_n, s_n \sigma_n) \geq A. \]
By (3.25), we see that \(B \geq A\), which is a contradiction with Lemma 3.1. Therefore, Case 1 is impossible.

**Case 2:** \(u \neq 0, \ v \equiv 0\) or \(u \equiv 0, \ v \neq 0\).

Without loss of generality, we may assume that \(u \neq 0, v \equiv 0\). Then \(u \in H_1^r(\mathbb{R}^N)\) is a nontrivial solution of \(-\Delta u + u = |u|^{2r-2}u + \lambda_1 |u|^{\alpha-2}u\), so \(I(u, 0) \geq B_1\). By Lemma 3.3, we have that \(b_2 > 0\). By Case 1, we may assume that \(b_1 = 0\). Then
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\omega_n|^{\frac{2}{r}} |\sigma_n|^{\frac{2}{r}} = 0, \]
hence
\[ \int_{\mathbb{R}^N} |\nabla \sigma_n|^2 = \int_{\mathbb{R}^N} |\sigma_n|^2 + o(1) \leq S^{-\frac{2}{N}} \left( \int_{\mathbb{R}^N} |\nabla \sigma_n|^2 \right)^{\frac{2}{N}} + o(1), \]
which implies that \( b_2 \geq S^{\frac{N}{2}} \). By (3.25) we have that
\[ B \geq B_1 + \frac{1}{N} b_2 \geq B_1 + \frac{1}{N} S^{\frac{N}{2}}, \]
which is a contradiction with Lemma 3.1. Therefore Case 2 is impossible.

Since Case 1 and Case 2 are both impossible, we see that \( u \not\equiv 0, v \not\equiv 0 \). By \( I'(u, v) = 0 \), then we have that \( (u, v) \in M \), hence by (3.25), \( I(u, v) = B \). By a similar argument as in the proof of Theorem 1.1, we see that \( (u, v) \in H \) is a positive solution of (1.1) and \( I(u, v) = B \). This completes the proof of Theorem 1.2.

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