On the vector conformal models in an arbitrary dimension

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Abstract The conventional model of the gauge vector field is invariant under the local conformal symmetry only in the four-dimensional space (4d). Conformal generalization to an arbitrary dimension \(d\) is impossible even for the free theory, differently from scalar and fermion fields. We discuss how to overcome this restriction and eventually construct four vector conformal actions. One of these models is the particular case of the previously known conformal theory of \(n\)-forms and others are new, up to our knowledge. In some of these models, the gauge invariance is preserved, two of the new models are described by local actions with auxiliary compensating scalar fields, and the extended version of one of these models is on shell equivalent to the last, non-analytic, purely metric version.

1 Introduction

It is common wisdom that conformal symmetry plays a very prominent role in both classical and semiclassical gravity theories. In the last case, the conformal anomaly [1–3] includes a classification of the possible terms satisfying the conformal Noether identities [4,5]. One can say that understanding conformal anomaly is the critically important issue in the semiclassical theory because it is in the heart of the most important applications in cosmology and black hole physics (see, e.g., [6,7] for reviews).

On the other hand, many interesting questions are still unanswered. This concerns, in the first place, the universality of signs in the anomaly in 4d (see, e.g., the recent discussion in [7] and [8]). It is known that the signs of the beta functions of the square of the Weyl tensor and the Gauss–Bonnet term are (++++) and (−−−−) for the conformal scalar, fermion, gauge vector, and (we can invoke quantum gravity too, at this point) conformal quantum gravity. However, for the “next generation” of conformal models, i.e., for the four-derivative scalar [9–11] and three-derivative fermion [12,13], the sign pattern suffers a flip, i.e., we

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meet (−−) and (++) in these two cases. Is all this just a series of accidents, or there is an unknown general rule behind this?

In the situation, when this question is without an answer, it looks interesting to gather more examples, and this makes it interesting to construct new conformal models. Indeed, this is the problem that has attracted attention for a long time [14,15]. We can also cite Refs. [16–20] and references therein. The generalization of the known conformal actions to other dimensions represents part of this general study. The subject of the present work is the discussion of this problem for the vector field with two derivatives in the action.

In what follows, we present the discussion of the generalization of the 4d conformal action of the gauge vector field to an arbitrary dimension \( d \). Such a generalization would enable treating the free Abelian vector field in the same way as scalar and fermion field, in the analysis of the one-loop renormalization [21]. For situations with preserved gauge symmetry, one can also construct conformally invariant gauge-fixing, similarly to what was done in [22,23].

Let us note that vector conformal operators with two derivatives may constitute a basis for conformal field theory (CFT) algebras of primary vector operators coupled to external geometry, including e.g., non-trivial background spacetime. Thus, they might give rise to new CFTs coupled to gravity. In addition, such operators may have long-ranging applications from studies of conformal anomalies through applications of vector fields in cosmology to model building for unified theories (GUT) in particle physics. The vector conformal models may be even useful in computer graphics, where the conformal methods and vector fields are heavily used.

Other possible applications include the description of the renormalization group flows near fixed points in the theories with vectors coupled to gravity, related to the asymptotic safety program in gravity, and in the condensed matter physics, e.g., concerning vector excitations of graphene curved sheets varying in time, in the 3d case [24,25].

Thus, there may be interesting applications of the results presented below, but we leave the corresponding discussion to possible further works and now concentrate on the formal aspect of the problem. The paper is organized as follows: In Sect. 2, one can find a brief introduction to the problem. Section 3 is a report on a direct search of the \( d \)-dimensional conformal vector operator. As a result, we find that such an operator exists, but the gauge symmetry should be sacrificed, such that the longitudinal mode of the vector becomes propagating. A surprising detail is that this propagation is related to the four-derivative Paneitz operator [11] in 4d. In Sect. 4, we present two models of \( d \)-dimensional conformal vector operators with the gauge symmetry preserved, but the auxiliary scalars are required for their construction. In Sect. 5, we discuss the universal prescription for constructing non-analytic \( d \)-dimensional conformal models, introduced originally in Appendix of Ref. [26]. It is shown that this model and also its purely gravitational analog are on shell equivalent to the metric-scalar model of Sect. 4. Finally, in Sect. 6, we draw our conclusions.

2 The global conformal model with vector field

The action of a vector field in four spacetime dimension has the form

\[
S_4(A, g) = -\frac{1}{4} \int d^4x \sqrt{-g} \ F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int d^4 x \sqrt{-g} \ A_\mu \left( g^{\mu\nu} \Box - \nabla^\mu \nabla^\nu - R^{\mu\nu} \right) A_\nu,
\]

(1)
where we restricted our attention to the Abelian model for the sake of simplicity. When dealing with the generalization to arbitrary $d$, the case of a non-Abelian field should be considered separately.

Under the local conformal transformation,

$$A_\mu = \bar{A}_\mu, \quad g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}, \quad \sigma = \sigma(x).$$  

(2)

this action remains invariant, i.e., $S_4(A, g) = S_4(\bar{A}, \bar{g})$. Our purpose is to formulate the generalization of transformation (2), which would provide the invariance of the $d$-dimensional version of the action (1).

Direct generalization of (1) leads to the functional

$$S_d(A, g) = -\frac{1}{4} \int d^d x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^d x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}.$$  

(3)

Assuming that the metric still transforms as in (2)\(^1\), the problem reduces to whether there exists some real number $w$ and/or a modification in the transformation of $A_\mu$, leaving the action (3) invariant under

$$A_\mu = e^{w\sigma} \bar{A}_\mu, \quad g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}.$$  

(4)

The simplest version concerns global transformation, with $\sigma = \text{const}$. Direct replacement of (4) into (3) shows that the symmetry is achieved for

$$w = \frac{4 - d}{2}.$$  

(5)

The case of a local transformation is more complicated. It is easy to check that for $\sigma \neq \text{const}$ the transformation of $F_{\mu\nu}$ does not have the desired form

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \neq e^{w\sigma} \bar{F}_{\mu\nu},$$  

(6)

which is a necessary condition for the global conformal symmetry of (3) (being the prerequisite of the local symmetry). In this formula, $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$ and $\bar{A}_\mu$ is from (4). Thus, the invariance under local conformal symmetry requires changing the form of the action and of the transformation rule for $A_\mu$.

### 3 Construction of the vector conformal operator

As a first step, we consider the direct construction of a vector conformal operator. In this section, we shall derive the conformal model known from the work by Deser and Nepomechie [14]. Let us note that when the first version of the present paper was prepared, we were not, unfortunately, aware of this well-known paper. However, we decided to preserve this section in the subsequent version for the sake of generality and also owing to some new details described in what follows.

Our starting point is the two-derivative action quadratic in the vector field $V_\mu$ and without self-interactions, whose basis consists of a general combination of possible scalar invariants, which can be constructed using a covariant vector field $V_\mu$ on a general spacetime background. The corresponding action is defined as

$$S = -\frac{1}{2} \int d^d x \sqrt{-g} V_\mu O^{\mu\nu} V_\nu,$$  

(7)

\(^1\) Let us note that this does not reduce generality.
where the operator of the energy dimension two has the form

\[ O^{\mu\nu} = a_1 \nabla^\mu \nabla^\nu + a_2 g^{\mu\nu} \Box + a_3 g^{\mu\nu} R + a_4 R^{\mu\nu} \]  

(8)

and \( a_i \) are arbitrary real coefficients. We reserve the notation \( A_\mu \) for the gauge field and thus use \( V_\mu \) here since the gauge symmetry is not demanded, in general. We assume that the conformal weight of the covariant vector field \( V_\mu \) is \( w \), such that the conformal transformation is (cf. with (4))

\[ V_\mu \longrightarrow \tilde{V}_\mu = e^{-w\sigma} V_\mu, \quad g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = e^{-2\sigma} g_{\mu\nu}. \]  

(9)

Under the infinitesimal conformal transformation (first order in \( \sigma \)), the action (7) transforms as

\[ \tilde{S} = S(\tilde{g}_{\mu\nu}, \tilde{V}_\mu) = S + \delta_c S, \]  

(10)

where

\[
\delta_c S = \frac{1}{2} \int d^d x \sqrt{-g} \left\{ (d - 4 + 2w)\sigma V_\mu O^{\mu\nu} V_\nu + [(d - 2 + w)a_1 \\
- (d - 2)a_4] V_\mu (\nabla^\mu \nabla^\nu) V_\nu - [(1 - w)a_2 + 2(d - 1)a_3 + a_4] V_\mu g^{\mu\nu} (\Box \sigma) V_\nu \\
+ [(d - 2 + w)a_1 + 2a_2] V_\mu (\nabla^\nu \sigma) \nabla^\mu V_\nu + [(d - 4 + 2w)a_2] V_\mu g^{\mu\nu} (\nabla_\tau \sigma) \nabla^\tau V_\nu \\
- [(2 - w)a_1 + 2a_2] V_\mu (\nabla^\mu \sigma) \nabla^\nu V_\nu \right\}. 
\]  

(11)

In this and subsequent formulas, the parentheses restrict the action of covariant derivatives, e.g., \( \nabla A = A \nabla + (\nabla A) \).

The conformal invariance requires \( \delta_c S = 0 \), i.e., the integrand in (11) has to vanish. This condition gives the system of equations for the coefficients

\[
\begin{align*}
(\text{d} - 4 + 2w)\sigma = 0, \\
(\text{d} - 2 + w)a_1 - (\text{d} - 2)a_4 = 0, \\
(1 - w)a_2 + 2(d - 1)a_3 + a_4 = 0, \\
(\text{d} - 2 + w)a_1 + 2a_2 = 0, \\
(2 - w)a_1 + 2a_2 = 0.
\end{align*}
\]  

(12)

Note that some equations are degenerate for the dimensions \( \text{d} = 1 \) and \( \text{d} = 2 \). Let us first look at these special dimensions.

- In the case \( \text{d} = 1 \), we get \( a_1 = a_2 = a_4 = 0 \). Since, on top of that, \( R = 0 \), the operator is geometrically irrelevant in \( \text{d} = 1 \).
- For \( \text{d} = 2 \), we find \( w = 1 \), \( a_1 = a_2 = 0 \), and \( a_3 = -\frac{a_4}{2} \). Thus,

\[
O^{\mu\nu}\big|_{\text{d}=2} = a_4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)\big|_{\text{d}=2} = 0. 
\]  

(13)

Thus, in dimensions \( \text{d} = 1 \) and \( \text{d} = 2 \), the conformal vector operator of the given type does not exist. In other dimensions, it is easy to see that the system of equations in (12) can be solved with \( w \) from (5) and with the following coefficients:

\[
a_2 = -\frac{d}{4}a_1, \quad a_3 = \frac{(d - 4)d^2}{16(d - 2)(d - 1)}a_1 \quad \text{and} \quad a_4 = \frac{d}{2(d - 2)}a_1. 
\]  

(14)
The resulting conformal operator is uniquely defined, up to the overall arbitrary real coefficient $a_1$, i.e.,

$$ O^{\mu\nu} = a_1 \left[ \nabla^\mu \nabla^\nu - \frac{d}{4} g^{\mu\nu} \Box + \frac{(d - 4)d^2}{16(d - 2)(d - 1)} g^{\mu\nu} R + \frac{d}{2(d - 2)} R^{\mu\nu} \right]. \quad (15) $$

As it should be expected, this expression becomes singular at $d \to 1$ or $d \to 2$. Until the end of this section, we do not consider the special dimensions $d = 1, 2$. On the other hand, in $d = 4$, the coefficient $a_3 = 0$ and we arrive at the well-known operator for the Maxwell field on a general background,

$$ O^{\mu\nu} \big|_{d=4} = a_1 \left( \nabla^\mu \nabla^\nu - g^{\mu\nu} \Box + R^{\mu\nu} \right). \quad (16) $$

Choosing $a_1 = 1$, one obtains action (1). It is remarkable that it is possible to recover the action of the electromagnetic field theory in $d = 4$, using only the requirement of the local conformal invariance in curved space, i.e., without demanding the gauge symmetry. Let us stress that this feature is typical only for this special dimension.

Coming back to the case of the general dimension $d$, direct calculations (we skip the details in this part since they are cumbersome) demonstrate that action (7) is also invariant under the finite conformal transformations (9). One can note that the operator in (15) is unique and that the conformal invariance does not concern surface terms.

Since we derived action (7) with (15) without demanding gauge invariance, the next step is to check out how it behaves under the Abelian gauge transformation, i.e.,

$$ V^\mu \rightarrow V'^\mu = V^\mu + \nabla^\mu f, \quad (17) $$

where $f = f(x)$ is an arbitrary scalar field.

Performing transformation (17), after some integrations by parts, and using the contracted Bianchi identity, we find that the conformal action transforms as

\begin{align*}
S' &= S - \frac{1}{2} a_1 \int d^dx \sqrt{-g} \left\{ \frac{(d - 4)}{2} \nabla_\nu V^\mu \left[ g^{\mu\nu} \Box - \frac{d^2}{4(d - 2)(d - 1)} g^{\mu\nu} R \right. \\
&+ \frac{d}{(d - 2)} R^{\mu\nu} \big] f + \frac{d(d - 4)}{8(d - 1)} V^\mu (\nabla_{\mu} R)f + \frac{d - 4}{4} f \left[ \Box^2 + \frac{d}{d - 2} R^{\mu\nu} \nabla_\mu \nabla_\nu \right. \\
&\left. \left. - \frac{d^2}{4(d - 2)(d - 1)} R \Box + \frac{d}{4(d - 1)} \left( \nabla^\mu R \right) \nabla_\mu \right] f \right\}. \quad (18)
\end{align*}

Thus, for $d \neq 4$ the action is not gauge invariant. Since the gauge invariance of action (1) means the absence of a longitudinal mode, it is worth exploring this mode in model (15). Let us perform a York decomposition of the vector field, $V^\mu = V^\mu_\perp + V^\mu_\parallel$. Here the field $V^\mu_\parallel$ is regarded as a divergence of a scalar field $V^\mu_\parallel = \nabla_\mu \varphi$, and the transversality conditions read $\nabla^\mu V^\perp_\mu = 0$. \footnote{This operator was originally found in [14]. Furthermore, it is the particular $n = 2$ case of the conformal operator for the $n$-forms of gauge field strength as described in Refs. [16,20].}

In the new variables, the action reads

\begin{align*}
S &= - \frac{a_1}{2} \int d^dx \sqrt{-g} \left\{ V^\perp_\mu \left[ - \frac{d}{4} g^{\mu\nu} \Box + \frac{d^2(d - 4)}{16(d - 1)(d - 2)} g^{\mu\nu} R + \frac{d}{2(d - 2)} R^{\mu\nu} \right] V^\perp_\nu \\
&+ \frac{d - 4}{4} \varphi \left[ \Box^2 + \frac{d}{d - 2} R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{d^2}{4(d - 1)(d - 2)} R \Box + \frac{d}{4(d - 1)} \left( \nabla^\mu R \right) \nabla_\mu \right] \varphi \right\}.
\end{align*}

\footnote{We do not intend to discuss the practical implementation of these requirements on an arbitrary metric background, but simply suppose that it can be done.}
\begin{equation}
\frac{d(d-4)}{2(d-2)} \left\{ \left( \nabla_\nu V_\mu^\perp \right) R^{\mu\nu} \varphi + \frac{d-2}{4(d-1)} V_\mu^\perp (\nabla^\mu R) \varphi \right\}. \tag{19}
\end{equation}

It is easy to see that the longitudinal mode decouples from the transverse one only in \(d = 4\), where it disappears. Thus, we can conclude that the generalization of the \(d = 4\) action (1) to an arbitrary dimension \(d\) makes the longitudinal part of the vector field propagating. This means we gain the new degree of freedom compared to the original gauge-invariant action. This is the price one has to pay for the conformal symmetry in this model. One can say that this is a kind of a conformal Stückelberg procedure, being, however, quite different from the well-known one of Deser [27].

An interesting detail is a similarity between the longitudinal \(\varphi\varphi\) mode in (19) and the Paneitz conformal operator [11] (see also [28]).

\begin{equation}
\Delta_4 = \Box^2 + \frac{4}{d-2} R_{\mu\nu} \nabla^\mu \nabla^\nu - \frac{d-6}{2(d-1)} (\nabla_\mu R) \nabla^\mu - \frac{d^2 - 4d + 8}{2(d-1)(d-2)} R \Box
+ (d-4) \left[ \frac{d^3 - 4d^2 + 16d - 16}{16(d-1)^2(d-2)^2} R^2 - \frac{1}{(d-2)^2} R_{\mu\nu} R^{\mu\nu} - \frac{1}{4(d-1)} (\Box R) \right]. \tag{20}
\end{equation}

One can note that the \(\varphi\varphi\) terms in the brackets of (19) coincide with (20) in the \(d = 4\) limit. This coincidence does not hold in other dimensions; however, this can be seen as a shortcut to the Paneitz operator in \(d = 4\) (another link between the two operators was discussed in [23]). The origin of this relation looks unclear and perhaps can be added to the list of open problems concerning conformal operators. Another interesting aspect is that the \(V_\mu^\perp \varphi\) sector of Eq. (19) also includes the coefficient \(\frac{d-2}{d(d-1)}\), that is a typical value for the two-derivative scalar operator [28].

All considerations presented above address only the free field model (7). One can consider various possible extensions including interaction terms. The addition of these terms may be seen as a kind of analogy with the non-Abelian gauge symmetry, but for just one vector field. The first option is the term \((V^\mu V_\mu)^m\). This interaction term in \(d = 4\) and, respectively, for \(m = 2\), is introduced in Eq. (8.122) of [29] for the axial vector field related to torsion (see also related discussion of the conformal transformations with torsion in [30] and [31]). The \(d\)-dimensional action with such a term has the form

\begin{equation}
S_2 = -\frac{1}{2} \int d^d x \sqrt{-g} \left\{ V_\mu O^{\mu\nu} V_\nu + \lambda (V^\mu V_\mu)^m \right\}, \tag{21}
\end{equation}

where \(\lambda\) is a coupling constant. Taking the infinitesimal conformal variation of the action (21) and requiring that \(\delta_c S_2 = 0\), we arrive at the value of \(m\),

\begin{equation}
m = \frac{d}{d-2}, \tag{22}
\end{equation}

independent of the coupling \(\lambda\). The conditions for the quadratic part of the action are the same as in (14), with \(w = 2 - d/2\).

### 4 Conformal models with auxiliary scalars

In this section, we consider a few simpler ways to generalize action (1) into an arbitrary dimension \(d\), preserving the invariance under the local conformal symmetry.

\footnote{In the dimensions \(d \geq 6\) one can construct more conformal interactions, e.g., using the products of \(V^\mu V_\mu\) and \((\nabla_\alpha V_\beta - \nabla_\beta V_\alpha)^2\).}
4.1 Extended connection

One possible solution of the problem is based on the modification of the definition of $F_{\mu\nu}$ in (6). This solution is rooted in the similar approach used to explore the conformal transformations in the models of gravity with torsion (see, e.g., [31] and references therein, pioneer work [27], and also subsequent papers with a similar procedure, e.g., [32–34] and [35,36]). We assume that the covariant derivative of the vector field is constructed with a modified affine connection,

$$D_\mu A_\nu = \nabla_\mu A_\nu - K^\lambda_{\nu\mu} A_\lambda,$$

where $\varphi$ is an additional (can be called auxiliary) scalar field. We assume that the transformation rule for this field has form

$$\varphi = \bar{\varphi} + \gamma \sigma,$$

where $\gamma$ is the parameter to be found from the modified version of the l.h.s. of (6), with a new version of the field tensor $F_{\mu\nu}$, based on a new covariant derivative $D_\mu$. Equation (24) is called to supplement the transformations of the metric and gauge vector field (4).

Substituting (23) into the new definition of the field tensor

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\nu \partial_\mu \varphi - A_\mu \partial_\nu \varphi,$$

after a small algebra, we find that the condition

$$F_{\mu\nu} = e^{w\sigma} \bar{F}_{\mu\nu},$$

is satisfied for

$$\gamma = -w = \frac{d - 4}{2}.$$

Finally, the conformally symmetric action has the form

$$\tilde{S}_d(A, g) = -\frac{1}{4} \int d^d x \sqrt{-g} F_{\mu\nu} F^{\mu\nu},$$

that coincides with (1) in the limit $d = 4$ and $\varphi = 0$. One of the main features of (25) and (28) is that these two objects do not have the gauge symmetry, at least not under the usual gauge transformation. One can, of course, try to look for the modified form of this transformation. However, this issue was explored in the models with torsion [37,38]; hence, we skip this part.

Thus, we arrive at the new form of the vector field action, remaining conformally invariant in an arbitrary dimension $d$. The price to pay is the $O(d - 4)$-modification of the affine connection for a scalar field and the $O(d - 4)$-violation of gauge invariance. Indeed, both issues do not contradict the scheme of the proof of conformal one-loop renormalizability, given in [8] (and more complete one in [21]).

Furthermore, it remains unclear how to make a generalization for a non-Abelian vector field, due to the nonlinearity in the field tensor $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g^{abc} A^b_\mu A^c_\nu$ and the respective inhomogeneity of $G^a_{\mu\nu}$ with respect to $A^a_\mu$.

4.2 Simpler way of using scalar field

Consider an alternative way of providing conformal symmetry in the modified version of the Abelian vector field action (1). In this case, we do not change the transformation rule for the
vector field in (2), but instead insert the auxiliary scalar in the action. In this case, there is no problem with the gauge invariance and with the non-Abelian version of the model. Thus, we directly consider the $d$-dimensional action

$$S^*_d(A, g) = -\frac{1}{4} \int d^d x \sqrt{-g} \, \Phi^\lambda \, G^{a}_\mu \gamma_g^{\rho} \, g^{\mu \rho} \, g^{\nu \sigma}, \quad (29)$$

where $\lambda = \frac{d-2}{2}$. Indeed, for the sake of quantum theory, it is useful to have a free part of the vector action, which enables one to construct propagator and separate vertices. To get this, let us set $\Phi = \chi_0 + \chi$, where $\chi_0$ is a constant of the mass dimension $(d - 2)$, and regard $\chi$ to be a new scalar field. To achieve the conformal invariance of (29), the transformation law for the new scalars should be

$$\Phi = \tilde{\Phi} e^{(2-d)\sigma} \quad \text{or} \quad \chi_0 + \chi = (\chi_0 + \tilde{\chi}) e^{(2-d)\sigma}. \quad (30)$$

With these definitions, the combination $\sqrt{-g} \, \Phi^\lambda \, g^{\mu \rho} \, g^{\nu \sigma}$ is conformally invariant, therefore providing the invariance of the action in (29).

It is worth mentioning that the Faddeev–Popov procedure needs only a minor change, i.e., the gauge fixing action should be

$$S^*_{gf, d}(A, g) = -\frac{1}{2\omega} \int d^d x \sqrt{-g} \, t^a \, \Phi^\lambda \, t^a = \frac{1}{2} \int d^d x \sqrt{-g} \, t^a \, Y_{ab} \, t^b, \quad t^a = \nabla_\mu A^a \mu. \quad (31)$$

Thus, the change concerns only the weight operator $Y_{ab}$ and not the gauge conditions. The new weight operator is local and therefore does not contribute to the divergences. The action of gauge ghosts constructed in this model has a standard form. It is not conformally invariant, but this is true even in $d = 4$. This fact does not imply the non-invariance of the one-loop divergences [21]. In general, since $\lambda = O(d - 4)$, all the diagrams with the field $\chi$ come with the factors $(d - 4)$; hence, the one-loop divergences are not expected to change.

5 Non-analytic approach to conformal models

For the sake of completeness, let us mention the existing, albeit not very much discussed (except the appendix of [26]) method of constructing conformal actions in an arbitrary dimension $d$. Let us illustrate how this non-analytic approach to construct $d$-dimensional conformal model works for the massless vector field. In this case, a single formula can replace many words, so the desired result is

$$S^*_d(A, g) = -\frac{1}{4} \int d^d x \sqrt{-g} \left( G^{a}_\mu \gamma_g^{a} \, G^{a} \mu \nu \right)^d, \quad (32)$$

that is certainly conformal for any $d$. In Ref. [26], the same idea was used with the square of the Weyl tensor $C^{\mu \nu \rho \delta}_{\gamma} C^{\mu \nu \rho \delta}_{\gamma}$ instead of the square $G^{a}_\mu \gamma_g^{a} \, G^{a} \mu \nu$ in Eq. (32).

Thinking about the applications to quantum gravity, up to some extent, this solution looks less interesting and less useful compared to the three others discussed above. The reason is that, in this case, it is unclear how to use the Faddeev–Popov procedure. As it was mentioned in [26], the similar gravitational term also has only restricted interest since, for example, it cannot give rise to the free propagation of gravitons around flat spacetime for $d > 4$. 
On the other hand, the situation may be different if we trade the action (32) to the extended version of the model. Introducing one more scalar-dependent term in (29), we get

\[ S_{d, \text{ext}}^*(A, g) = \int d^d x \sqrt{-g} \left\{ -\frac{1}{4} \Phi^\lambda G^{\lambda} + \frac{\tau}{4} \Phi^\alpha \right\}, \tag{33} \]

where \( \tau \) is a new coupling constant and we used condensed notation

\( G^{\mu\nu} G^{\rho\sigma} g_{\mu\rho} g_{\nu\sigma}. \)

It is easy to check that the local conformal invariance of this expression requires fixing \( \alpha = \frac{d}{d-2} \). The point is that, using the equation of motion for \( \Phi^{\lambda} \), model (33) is equivalent to (32). In detail, the on-shell condition gives

\[ \Phi = \left( \frac{\lambda}{\alpha \tau} G^2 \right)^{1/(\alpha - \lambda)}. \tag{34} \]

After some algebra, the on-shell action is found in the form

\[ S_{d, \text{ext}}^*(A, g) \bigg| = \int d^d x \sqrt{-g} \left\{ -\frac{1}{4} + \frac{d - 4}{4d} \right\} \left( \frac{d - 4}{\tau d} \right)^{\frac{d}{d-4}} (G^2)^{\frac{d}{2}}. \tag{35} \]

After an obvious constant reparametrization of the gauge field \( A^{\mu}_a \) and the gauge coupling \( g \), this expression coincides with (32). Notice that in the limit \( \tau \to 0 \) and for \( d > 4 \), the on-shell action (35) is singular, as it should, because when the last term in action (33) vanishes there is no equivalence between the two models.

The procedure described above can be used also in the purely gravitational action [26], with the Weyl tensor used instead of the Yang–Mills field tensor,

\[ S_{d}^*(g) = -\frac{1}{2\lambda} \int d^d x \sqrt{-g} \left( C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)^{\frac{d}{4}}. \tag{36} \]

The procedure described above (including the values of \( \alpha, \lambda \) and other coefficients) remains the same, and as a result, we arrive at the on-shell equivalent representation of the conformal action.

The Lagrangian in (32) and similar gravitational version (36) are non-analytic for odd dimensions. However, we have shown that they give rise to classical dynamics which can be described by the equivalent models which are local and analytic.

The last observation concerns the new conformal model of the duality-invariant conformal extension of a modified Maxwell’s theory with self-interactions [39]. The approach analogous to (32) can be certainly applied to generalizing it to dimensions \( d \neq 4 \), but this extension may not be unique. It would be interesting to find a scalar mapping for both conformal models (32), (36), and for that of [39], using the procedure described in [40]. One can also see [41–43] and further references therein, for different utilizations of the Power–Maxwell models, such as (32).

### 6 Conclusions

We considered four different ways to provide the invariance under local conformal transformations for the action of massless vector field in an arbitrary spacetime dimension \( d \). These constructions teach us a few small, but potentially useful lessons. In the first model, we learned that the requirement of conformal symmetry in the presence of the metric leads to the standard action of Abelian vector model in \( d = 4 \), but not in any other dimension, and without demanding the Abelian gauge symmetry. For \( d = 4 + \epsilon \), we have found that...
the violation of gauge symmetry in the lowest-order $O(\epsilon)$-terms is partially related to the 4d scalar Paneitz operator [9–11].

The next two models show that the desired generalization of conformal model to $d \neq 4$ is always achieved by means of an extra scalar field. All these models can be seen as different versions of the conformal Stückelberg procedure. Out of these models, the (29) is certainly the most appropriate for the Lagrangian quantization. Furthermore, it can be shown that its extended version possess the on-shell equivalence with the non-analytic conformal model described in Sect. 5.

The aforementioned non-analytic model is interesting mostly by its universality. It is clear that one can use the procedure qualitatively similar to (32) for constructing conformal models in an arbitrary $d$ starting from a special value $d_0$ where the initial model is conformal. This procedure is supposed to work for different field contents and also for different numbers of derivatives.

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