Admission Control for Double-ended Queues

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Abstract

We consider a controlled double-ended queue consisting of two classes of customers, labeled sellers and buyers. The sellers and buyers arrive in a trading market according to two independent renewal processes. Whenever there is a seller and buyer pair, they are matched and leave the system instantaneously. The matching follows first-come-first-match service discipline. Those customers who cannot be matched immediately need to wait in the designated queue, and they are assumed to be impatient with generally distributed patience times. The control problem is concerned with the trade-off between blocking and abandonment for each class and the interplay of statistical behaviors of the two classes, and its objective is to choose optimal queue-capacities (buffer lengths) for sellers and buyers to minimize an infinite horizon discounted linear cost functional which consists of holding costs and penalty costs for blocking and abandonment.

When the arrival intensities of both customer classes tend to infinity in concert, we use a heavy traffic approximation to formulate an approximate diffusion control problem (DCP), and develop an optimal threshold policy for the DCP. Finally, we employ the DCP solution to establish an easy-to-implement asymptotically optimal threshold policy for the original queueing control problem.

Keywords: Double-ended queues, matching queues, heavy-traffic regime, two-sided Skorokhod problems, local-time processes, diffusion processes and approximations, asymptotic optimality.

AMS Subject Classifications: 93E20, 60H30.

Abbreviated Title: Controlled double-ended queues.

1 Introduction

We consider a mathematical model of a matching platform that matches two classes of customers, labeled as buyers and sellers. The customers of each class arrive sequentially and wait in their respective queue to be matched with a customer of the other class. They are matched according to the order of arrival which is known as the first-come-first-match service discipline. Once matched, a trade occurs and the pair leaves the system immediately. Both classes of customers are assumed to be impatient and they leave the system without being matched, if their patience runs out. Due to

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instantaneous matching, there cannot be positive numbers of buyers and sellers simultaneously in the system. Such queueing models are known as double-ended queues (or matching queues) with impatient customers.

The arrivals of buyers and sellers are sequential and form two independent renewal processes and their patience-times are represented by two independent IID (independent and identically distributed) sequences. To avoid long queues, we introduce an admission control mechanism of controlling the queue-capacity by blocking the incoming customers. On one hand, when there is no blocking, long queues can occur, which leads to heavy customer abandonment. On the other hand, when too many arrivals are blocked, profits of the operation decrease. Furthermore, blocking one side of the system affects the other side. In this work the queue-capacities for the two customer classes are the system manager’s choice and at a given time, they are represented by a vector of two positive integer-valued, time-dependent random variables, which may depend on the past history as well as the current state of the system. The two components of the queue-capacity vector behave as barriers on the seller queue and the buyer queue. Incoming customers of each class are blocked (i.e., rejected), when the queue is full at capacity at the time of their arrival. Once in the queue, they abandon if their patience expires, which happens after random times, IID across each customer class. We introduce an infinite horizon discounted linear cost functional which consists of holding costs, abandonment costs, and blocking costs. We are interested in analyzing such a controlled system and to develop near optimal control policies which minimize the above described cost structure of the queueing control problem (QCP). However, this problem is too complex for direct analysis and therefore, we resort to a heavy traffic approximation. Approximating queueing systems in heavy traffic by Brownian models is an effective strategy in obtaining both qualitative and quantitative insights of the original queueing systems (cf. [2, 21, 34, 42]). Our techniques are closely related to the analysis of queueing systems in Halfin-Whitt heavy traffic regime where the number of servers tends to infinity (cf. [8, 12, 33, 41]).

To establish a heavy traffic approximation, we develop an asymptotic framework, under which the class-dependent arrival intensities are increasing to infinity in concert so that the system is critically loaded. To capture the behavior of both customer classes, we define the state of the system by an imbalance process whose value at time $t$ is given by the number of sellers at time $t$ minus the number of buyers at time $t$. We first establish the tightness of a sequence of diffusion-scaled state processes and identify the limit points of such a sequence as solutions to a stochastic differential equation (SDE). An associated diffusion control problem (DCP) is then formulated using such limiting SDEs. The DCP turns out to be a singular control problem, and its solution is obtained by finding a smooth solution to the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Using the solution to the HJB equation, we obtain an optimal threshold policy for the DCP. Finally, we employ it to establish a threshold regime which describes four different types of asymptotically optimal strategies for the original QCP. Therefore, our solution yields a simple, asymptotically optimal control strategy which is easy to implement in a double-ended queue and the involved threshold values are easy to compute from the given system parameters.

The threshold parameters are characterized as the ratios $T_b = (c_b + r_b \delta_b)/(\alpha + \delta_b)$ and $T_s = (c_s + r_s \delta_s)/(\alpha + \delta_s)$, where $c_b, r_b, \delta_b$ (resp. $c_s, r_s, \delta_s$) represent the holding cost rate per buyer (resp. seller), the abandonment cost per buyer (resp. seller), and the abandonment rate for buyers (resp. sellers), respectively, and the parameter $\alpha$ is
the discount factor in the cost functional (2.13). Now let $p_b$ and $p_s$ denote the penalty costs per each blocked buyer and seller, respectively. The strategy we develop says when $p_b \geq T_b$ (blocking is expensive), there should be no blocking on buyers, and when $p_b < T_b$ (blocking is cheap), an asymptotically optimal buffer size for buyers can be designed using the solution of the DCP. The policy for the seller side is the same with $p_b, T_b$ replaced by $p_s, T_s$. To gain an insight into our solution, let us consider a special Markovian setting where the double-ended queue has Poisson arrivals for both types of customers with exponential patience times. Considering the buyer side, the overall cost rate per buyer becomes $c_b + r_b \delta_b$. Our cost structure in (2.13) can be thought of as the expected total cost of the three types of costs over an “observation period $[0, \tau]$”, where $\tau$ is an exponential random variable which is independent of all the other processes in the system, and has a rate parameter given as the discount factor $\alpha$ in (2.13). Hence the waiting time for a buyer who is facing an “extremely long queue” upon arrival is the minimum of two independent exponential random variables, with parameters $\delta_b$ (for abandonment) and $\alpha$ (for the observation period), respectively. Thus $T_b = (c_b + r_b \delta_b) / (\alpha + \delta_b)$ represents the expected cost for this buyer. Now if the blocking cost $p_b$ is greater than $T_b$, it is reasonable to admit this buyer to the queue. On the other hand, if $p_b < T_b$, then it is better to block the buyer. Similar explanation can be given to the seller side. However, this intuition does not yield the values of optimal queue-capacities. In this work, relying on the DCP obtained by the heavy traffic approximation, we show that this intuition remains valid under more general assumptions for the arrival processes and patience time distributions, and develop an asymptotically optimal admission control policy. It should be noted that the proposed asymptotically optimal queue-capacity for buyers depend on the statistical behavior of sellers and vice versa.

This work is related to the authors’ previous works [41] and [25]. The work [25] develops the diffusion approximation of the uncontrolled double-ended queue with renewal arrivals and generally distributed patience times and establish a linear relationship between the diffusion-scaled queue length and the offered waiting time (the asymptotic Little’s law). In [41], a controlled $G/M/n/B + GI$ queue is studied in the Halfin-Whitt heavy traffic (also known as Quality and Efficiency-Driven) regime, where $B$ is the control process representing the queue-capacity. Both [41] and our work study the trade-off between abandonment and blocking and use the heavy traffic approximation to find near-optimal cost minimization strategies. However, a significant challenge faced in this work will be the dependence of the random quantities of the buyer queue (resp. seller queue) such as the queue-length, virtual waiting times etc, on the statistical behaviors of the quantities of the sellers (resp. buyers). This leads to develop original proofs such as Proposition 3.3. We summarize the novelty of the current work as follows. (i) We establish the tightness and the moment bounds for the diffusion-scaled state processes under any admissible control satisfying Assumption 2.5 (see Theorem 3.1), while [25] only studies the uncontrolled double-ended queue and [41] establishes the tightness result under constant control policies; (ii) To establish Theorem 3.1, we introduce the virtual waiting time processes for the unblocked buyers and sellers and establish the asymptotic Little’s law in Proposition 3.10; (iii) The proofs of both Theorem 3.1 and the asymptotic optimality result in Theorem 5.3 depend on the properties of the two-sided Skorokhod problem (SP) over a time varying interval. This is in contrast with [41] where only the one-sided SP is used. We establish new weak convergence and oscillation results for the two-sided SP with time-dependent barriers,
which will be of independent interest (see Appendix A); (iv) The study of the HJB equation of the DCP is much more involved and it leads to a free boundary problem. In the most interesting case, this free boundary includes two points \( a^* < 0 < b^* \), and the values \(|a^*|\) and \(b^*\) describe the optimal boundaries for the DCP. Obtaining these free boundary points is quite complex and one has to carefully analyze the solution profiles of the corresponding differential equations on \((-\infty, 0)\) and \((0, \infty)\). Then use “the principle of smooth fit” at the origin to find the smooth solution for the HJB equation. In comparison, the free boundary associated with the HJB equation in [41] consists of a single point and proving its existence is relatively easy.

Many articles in the literature of double-ended queues are driven by applications. In [18], a taxi queueing system with a limited waiting space is modeled as a double-ended queue with Markovian assumptions on the arrival processes of taxis and passengers, and the steady state behavior of the system is studied. In [6], the effect of impatience behavior of customers in double-ended queues with Markovian arrivals and exponential patience times was studied. Double-ended queues have also been used to study perishable production-inventory systems (cf. [19, 29]), where one side of the queue represents the inventory of products and the other side accepts the arrivals of orders. In a recent work [24], a production rate control problem is studied to minimize a finite horizon cost functional consisting of linear costs for inventory and waiting and a cost that penalizes rapid fluctuations of production rates. An asymptotic optimal production rate is developed under the fluid scaling given that the demand arrival rate is time and state dependent. Double-ended queues are the simplest matching systems and can be naturally generalized to have more than two classes of customers. The generalized multi-class matching queues have occurred in assembled products in manufacturing setting where each product is completed by combining multiple components upon their arrivals (cf. [13, 30, 11]). In his early work, Harrison [13] studied the behavior of vector waiting times in a model for an assembly line product made with several components in heavy traffic. Each input component arrives according to an independent renewal process. Once the server has one component from each category, it takes a random processing time to finish the product. In [11], such a matching system is studied with instantaneous processing, for which the authors consider the problem of minimizing finite horizon cumulative holding costs. A myopic discrete-review matching control is developed and shown to be asymptotically optimal in heavy traffic. Plumbeck and Ward [30] study a control problem of an assemble-to-order system to maximize an expected infinite-horizon discounted profit by choosing product prices, component production capacities, and a dynamic policy for sequencing customer orders for assembly. If both sides of the double-ended queue are generalized to have multiple classes, we end up with a bipartite matching system. Such systems are widely used to model the organ transplant systems, where one side represents patients with multiple classes and the other side represents organs of multiple types. We refer to [1, 20] for optimal allocation problems of the bipartite matching systems in fluid scaling. An application of the simple double-ended queue to organ transplant systems is studied in [4].

The rest of this article is organized as follows: In Section 2, we introduce the model, the blocking control structure, and the control problem for the double-ended queue. Section 3 is devoted to the moment bounds and C-tightness of the diffusion-scaled processes. In particular, in Theorem 3.1, we identify the limit points of any given sequence of state processes as a solution to a SDE. In Section 4, the DCP is formulated using such limiting SDE, and its explicit solution is established in Theorem
4.2. Section 5 establishes the asymptotic optimality. Theorem 5.1 shows that the DCP provides a lower bound for the diffusion-scaled QCP. Next in Theorem 5.3, we employ the solution to the DCP to obtain asymptotically optimal policy under four different parameter regimes. We establish new convergence and oscillation results for the two-sided SP in Appendix A, which are of independent interest. Finally, Appendix B collects a series of lemmas to find a solution to the HJB equation.

Notation. Let \( \mathbb{N} \) denote the set of positive integers and \( \mathbb{R} \) denote the one dimensional Euclidean space. For a function \( f : G \to \mathbb{R} \) where \( G \) is an open set, we write \( f \in C^k(G) \) if the \( k \)th derivative of \( f \) is continuous on \( G \). For \( 0 < T \leq \infty \), we denote the function space of \( \mathbb{R} \)-valued right-continuous functions with left limits (RCLL), defined on \([0, T]\), by \( D[0, T] \). This function space is endowed with the standard Skorokhod \( J_1 \) topology. For \( k \in \mathbb{N} \), let \( D^k[0, T] \) be the product of \( k \) of \( D[0, T] \) spaces. The uniform norm on \([0, T]\) for a stochastic process \( X \) in \( D^k[0, T] \) is defined by \( \|X\|_T = \sup_{0 \leq t \leq T} |X(t)| \). To describe the processes associated with the \( n \)th system, we typically use the superscript, such as in the case of \( X^n(\cdot) \), etc. Throughout, we use \( \Rightarrow \) to denote weak convergence of processes in \( D^k[0, T] \). For each \( f \in D^k[0, T] \), we let the oscillation of \( f \) in a sub-interval \([t_1, t_2]\) be defined by

\[
Osc(f, [t_1, t_2]) \equiv \sup\{|f(t) - f(s)|; s, t \in [t_1, t_2]\}.
\] (1.1)

We simply denote \( Osc(f, [0, T]) \) by \( Osc(f, T) \). For a given \( \delta > 0 \), its modulus of continuity \( \omega(f, \delta, T) \) is defined by

\[
\omega(f, \delta, T) = \sup\{|f(t) - f(s)| : |t - s| < \delta \text{ and } s, t \in [0, T]\}.
\] (1.2)

Using the modulus of continuity \( \omega(\cdot, \delta, T) \) is advantageous in our arguments, because of its sub-additive property: For \( f, g \in D^k[0, T] \) and \( \delta > 0 \),

\[
\omega(f + g, \delta, T) \leq \omega(f, \delta, T) + \omega(g, \delta, T).
\]

This also helps us to establish the C-tightness of several processes considered here. We also follow the convention that the infimum of an empty set is infinity. For any real number \( x \), \( x^+ = \max\{0, x\} \) and \( x^- = \max\{0, -x\} \). For any two real numbers \( a \) and \( b \), \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \).

2 Double-ended Queues

All our stochastic processes and random variables are defined on a complete probability space \((\Omega, \mathcal{F}, P)\). We have a sequence of queueing systems indexed by \( n \in \mathbb{N} \), where the scaling parameter \( n \) is used to model the scale and traffic intensity of the system (see the heavy traffic condition in Assumption 2.3). The \( n \)th system represents a simple trading market where buyers and sellers arrive according to two independent renewal processes \( A^n_0(\cdot) \) and \( A^n_{n}(\cdot) \). A trade occurs when a buyer meets a seller and thereafter the pair leaves the system instantaneously. The buyers and sellers are matched according to first-come-first-match policy and they wait in their respective queues if not getting matched immediately. Since the matching is instantaneous, it is not possible to have positive numbers of buyers and sellers waiting in their queues simultaneously. It is assumed that both buyers and sellers are impatient and if they have to wait in
the queue, they abandon the system when their patience expires. This abandonment mechanism works as follows: with each customer, there is an associated clock. This clock rings after a random time, and if the clock rings while the customer is waiting in the queue, then the customer abandons the system. These clocks are all IID and independent of the arrival processes, as well as the history of the system up to that time. The cumulative distribution function of the patience-time for buyers is represented by $F_b$ and for sellers, it is given by $F_s$. At any time instant, either the buyer queue or the seller queue is empty and therefore, the state description of the system can be given by the imbalance process $X^n$. At any time instant $t \geq 0$, if $X^n(t) \geq 0$, then there are no buyers at time $t$ and $X^n(t)$ represents the queue length of sellers. Similarly, if $X^n(t) < 0$ then there are no sellers at time $t$ and $-X^n(t)$ represents the queue length of buyers. Without loss of generality, we assume that the initial number of sellers is given by $X^n(0-) \geq 0$. We refer to [26, 25] and [11] for similar representations of the state process. Let the quantities $G^n_b(t)$ and $G^n_s(t)$ represent the numbers of sellers and buyers abandoning the system during $[0, t]$, respectively.

To minimize the costs associated with abandonment and waiting, the management is permitted to control the system by blocking new arrivals whenever the queue-length is sufficiently large. Therefore, we allow the system manager to choose queue-capacities or buffer lengths for buyers and sellers. Throughout, we use these two words intermittently. For each customer class, when the buffer is full, incoming customers will be rejected. Each rejected customer incurs a loss in profit. For the $n^{th}$ system, the vector valued stochastic process $m_n(\cdot) = (m^n_b(\cdot), m^n_s(\cdot))$ represents the processes of controlled queue-capacities (waiting-room sizes or buffer lengths), where $m^n_b(t) < 0 < m^n_s(t)$ for each $t \geq 0$. More precisely, $-m^n_b(t)$ represents the buffer length for buyers at time $t$, and $m^n_s(t)$ represents the buffer length for sellers at time $t$. This queue-capacity process $m_n(\cdot)$ is the only control at the disposal of the manager of the $n^{th}$ system. The choice of large queue-capacities reduces the blocking of customers and increases the profit margins. However, at the same time, such large capacities are likely to give rise to long queues, which will in turn, increase the number of abandonments from the system, leading to a loss of income. This trade-off, between blocking and abandonment, naturally leads to a cost minimization problem, which is the underlying theme of our paper.

Associated with a queue-capacity process $m_n(\cdot) = (m^n_b(\cdot), m^n_s(\cdot))$, we introduce a pair of non-decreasing processes $U^n_s$ and $U^n_b$. For any time $t \geq 0$, $U^n_s(t)$ and $U^n_b(t)$ represent the numbers of sellers and buyers rejected during $[0, t]$, respectively. To describe the dynamics of the controlled system, we assume the stochastic primitives $A^n_b$, $A^n_s$, $G^n_b$ and $G^n_s$ to be RCLL processes, and make the following assumptions about the model.

**Assumption 2.1 (Initial condition).** The number of initial customers $X^n(0)$ is assumed to be deterministic, non-abandoning, and for some real $x$,

$$\lim_{n \to \infty} \frac{X^n(0)}{\sqrt{n}} = x. \quad (2.1)$$

Throughout, we simply assume $X^n(0) \geq 0$ so that there are no buyers initially in the system. The non-abandonment of initial customers is not a restrictive assumption and it can be easily relaxed following the proof of Lemma 4.1 of [42].
**Assumption 2.2** (Arrival processes). We assume that the arrival processes \( A^n_b \) and \( A^n_s \) are independent renewal processes. More precisely, there exist two positive sequences of real numbers \( \{\lambda^n_b\}_{n \in \mathbb{N}} \) and \( \{\lambda^n_s\}_{n \in \mathbb{N}} \) and positive constants \( \varsigma_b \) and \( \varsigma_s \) so that for \( n \in \mathbb{N} \), the inter-arrival times of buyers and sellers in the \( n^{th} \) are independent IID sequences with mean-variance pairs \((1/\lambda^n_b, (\varsigma_b/\lambda^n_b)^2)\) and \((1/\lambda^n_s, (\varsigma_s/\lambda^n_s)^2)\), respectively.

**Assumption 2.3** (Heavy traffic conditions). There exists a constant \( \lambda_0 > 0 \) and \( \beta_b, \beta_s \in \mathbb{R} \) so that
\[
\lim_{n \to \infty} \frac{\lambda^n_b - \lambda_0 n}{\sqrt{n}} = \beta_b, \quad (2.2)
\]
\[
\lim_{n \to \infty} \frac{\lambda^n_s - \lambda_0 n}{\sqrt{n}} = \beta_s. \quad (2.3)
\]

From the well known functional central limit theorem for renewal processes, we have
\[
\left( \frac{A^n_b(t) - \lambda^n_b t}{\sqrt{n}}, \frac{A^n_s(t) - \lambda^n_s t}{\sqrt{n}} \right) \text{ converges weakly to } (\sigma_b B_1(t), \sigma_s B_2(t)) \quad (2.4)
\]
in the space \( D^2[0,T] \), where \( \sigma_b = \sqrt{2 \lambda_0} \) and \( \sigma_s = \sqrt{2 \lambda_0} \) and \((B_1, B_2)\) is a standard two dimensional Brownian motion. We also have the following moment condition: For \( T > 0 \),
\[
E \left[ \sup_{t \in [0,T]} \left( (A^n_b(t) - \lambda^n_b t)^2 + (A^n_s(t) - \lambda^n_s t)^2 \right) \right] \leq Cn(1 + T^m), \quad (2.5)
\]
where \( C > 0 \) and the integer \( m > 1 \) are constants independent of \( T \) (for details, we refer to Lemma 2 of [2] and [22]).

**Assumption 2.4** (Patience time distributions). The patience-times of buyers and sellers are independent of each other. They are IID with cumulative distribution functions \( F_b \) and \( F_s \), respectively. We further assume \( F_b(0) = F_s(0) = 0 \) and they are right-differentiable at the origin with positive derivatives. Thus for some positive constants \( \delta_b \) and \( \delta_s \),
\[
\lim_{h \to 0^+} \frac{F_b(h)}{h} = \delta_b, \quad (2.6)
\]
\[
\lim_{h \to 0^+} \frac{F_s(h)}{h} = \delta_s. \quad (2.7)
\]

Similar assumptions on patience-times of customers in many-server queues were imposed in the articles [8, 2, 41].

**Assumption 2.5** (Admission control). At a given time \( t \geq 0 \), the controlled queue-capacity of sellers is represented by \( m^n_s(t) \geq 1 \) and the queue-capacity of buyers is represented by \(-m^n_b(t) \geq 1 \). Thus, if a customer queue is empty, then an incoming arrival from the same customer class is always admitted. We assume that both \( m^n_s(t) \) and \( m^n_b(t) \) are integer valued, and the controlled queue-capacity process \( m^t_n(\cdot) = (m^n_b(\cdot), m^n_s(\cdot)) \) has paths which are RCLL with a finite number of jumps in each finite interval. Furthermore, the control \( m^t_n(t) \) is allowed to depend on the current state, as
well as the whole history of the system up to time \( t \). Therefore, the process \( m_n(t) \) is assumed to be adapted to \( \mathcal{F}^n_t \) for each \( t \geq 0 \), where

\[
\mathcal{F}^n_t = \sigma(X^n(u), A^n_b(u), A^n_s(u), G^n_b(u), G^n_s(u), U^n_b(u), U^n_s(u) -); 0 \leq u \leq t),
\]

(2.8) completed by all the null sets. This \( \sigma \)-algebra represents all the information available to the system manager at time \( t \).

The controller is allowed to remove the initial customers if necessary. Therefore, the initial customer population is represented by \( X^n(0) \) and after the initial removal, the customer population at time \( t = 0 \) is represented by \( X^n(0) \). We assume the process \( m_n \) also adheres to the following conditions:

(i) There is a constant \( M > 0 \) independent of \( n \), but which depends on the initial data \( x \) in (2.1) so that for each \( n \), \(-M \sqrt{n} < m^n_b(0) < 0 < m^n_s(0) < M \sqrt{n} \) and it is assumed that \( X^n(0) \in [m^n_b(0), m^n_s(0)] \). Thus the number of initially removed customers is given by \( \max\{X^n(0) - m^n_s(0), m^n_b(0) - X^n(0)\} \).

(ii) At any time \( t > 0 \), \( X^n(t) \in [m^n_b(t), m^n_s(t)] \). Thus, once allowed to enter the system, no customer will be removed from the queue in the future.

(iii) The process \( m_n \) satisfies

\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \frac{1}{\sqrt{n}} \omega(m_n, \delta, T) = 0 \text{ in probability},
\]

(2.9) where the modulus of continuity \( \omega \) is defined in (1.2). Loosely speaking, (2.9) imposes that a change of queue-capacities of sellers and buyers at any time \( t \) will be at most of order \( \sqrt{n} \). If the controlled queue-capacity is a deterministic time-dependent function \( m_n(\cdot) \), then (2.9) can be replaced by the following simple sufficient condition: For all \( s, t \) in \([0, T]\),

\[
|m_n(t) - m_n(s)| \leq \sqrt{n} p(T)[\rho(|t - s|) + f(n)].
\]

Here \( p: [0, \infty) \to \mathbb{R}_+ \) is a positive continuous function; \( \rho: [0, \infty) \to \mathbb{R}_+ \) is a positive bounded continuous function, which satisfies \( \lim_{r \to 0} \rho(r) = 0 \); the function \( f: \mathbb{N} \to \mathbb{R}_+ \) satisfies \( \lim_{n \to \infty} f(n) = 0 \).

Given such an \( m_n \), the solution to the SP with time dependent barriers described in [5] guarantees the existence of the state process \( X^n \) in \( D[0, \infty) \). Note that the above assumptions allow constant \( m_n \) policies. They also accommodate the situation where no buyers or sellers are ever rejected. This is typically associated with the infinite buffer capacity. However, this can also be achieved by simply choosing the finite buffer capacities of \( m^n_b(t) = \min\{\inf_{u \in [0,t]} X^n(u) - 2, -1\} \) and \( m^n_s(t) = \max\{\sup_{u \in [0,t]} X^n(u) + 2, 1\} \), where \( \{X^n(t) \geq 0\} \) is the queue length process for the uncontrolled \( n^{th} \) system.

For a given buffer length policy \( m_n(\cdot) \), the controlled state process \( X^n \) has interval-valued RCLL paths and it satisfies the following equation: For all \( t \geq 0 \),

\[
X^n(t) = X^n(0) + A^n_s(t) - G^n_s(t) + G^n_b(t) - U^n_s(t) + U^n_b(t),
\]

where the processes \( U^n_b \) and \( U^n_s \) are given by

\[
U^n_b(t) = \int_0^t 1_{\{X^n(u) = m^n_b(u)\}} dA^n_b(u) + [m^n_b(0) - X^n(0-)]^+, \quad (2.10)
\]

\[
U^n_s(t) = \int_0^t 1_{\{X^n(u) = m^n_s(u)\}} dA^n_s(u) + [X^n(0-) - m^n_s(0)]^+. \quad (2.11)
\]
The associated infinite-horizon discounted cost functional is defined as

\[
\hat{J}^n(X^n(0), U^n_s, U^n_b) = E \left( \int_0^\infty e^{-\alpha t} [\hat{C}(X^n(t))\, dt + r_s \, dG^n(t) + r_b \, d\hat{G}^n_b(t) + p_s \, dU^n_s(t) + p_b \, d\hat{U}^n_b(t)] \right),
\]

where \( \hat{C}(x) = c_s x^+ + c_b x^- \) for all \( x \in \mathbb{R} \), and \( \alpha, c_s, c_b, r_s, r_b, p_s, p_b \) are all positive constants. In particular, \( c_s \) and \( c_b \) are the linear holding cost rates per each waiting seller and buyer, \( r_s, r_b \) are linear penalty costs for each abandoning seller and buyer, \( p_s, p_b \) are linear penalty costs for each blocked seller and buyer, and finally, \( \alpha \) is the discount factor. Our objective is to find optimal strategies which minimize the above cost functional and are easy to implement from the design point of view.

We introduce the following fluid and diffusion scaled quantities, which are similar to those in Halfin-Whitt heavy traffic regime.

**Fluid-scaled processes:** For a process \( Y^n \in \{A^n_s, A^n_b, G^n_b, G^n_s, U^n_b, U^n_s, X^n\} \), its fluid-scaled version is defined to be \( \tilde{Y}^n(t) = Y^n(t)/n \) for each \( t \geq 0 \).

**Diffusion-scaled processes:** The diffusion-scaled renewal processes are given as \( \hat{A}^n_s(t) = (A^n_s(t) - \lambda_n t)/\sqrt{n} \), and \( \hat{A}^n_b(t) = (A^n_b(t) - \lambda_n t)/\sqrt{n} \), and for any other process \( Z^n \in \{G^n_s, G^n_b, U^n_b, U^n_s, X^n, m_n\} \), its diffusion-scaled version is given as \( \tilde{Z}^n(t) = Z^n(t)/\sqrt{n} \) for \( t \geq 0 \).

The diffusion-scaled state process can now be formulated as

\[
\hat{X}^n(t) = \hat{X}^n(0) + \hat{A}^n_s(t) - \hat{A}^n_b(t) - \hat{G}^n_s(t) + \hat{G}^n_b(t) - \hat{U}^n_s(t) + \hat{U}^n_b(t), \quad t \geq 0. \tag{2.12}
\]

The corresponding diffusion-scaled cost function is given by \( \hat{J}^n = \hat{J}^n/\sqrt{n} \). Since the holding cost function is piecewise linear, the diffusion-scaled cost functional can be formulated as a functional of the diffusion-scaled processes as follows:

\[
\hat{J}^n(\hat{X}^n(0), \hat{U}^n_s, \hat{U}^n_b) = E \left( \int_0^\infty e^{-\alpha t} [\hat{C}(\hat{X}^n(t))\, dt + r_s \, d\hat{G}^n_s(t) + r_b \, d\hat{G}^n_b(t) + p_s \, d\hat{U}^n_s(t) + p_b \, d\hat{U}^n_b(t)] \right). \tag{2.13}
\]

The corresponding value function is given by

\[
\hat{V}^n(x) = \inf_{\mathcal{A}^n_x} \hat{J}^n(x, \hat{U}^n_s, \hat{U}^n_b), \tag{2.14}
\]

where \( \mathcal{A}^n_x \) is the collection of all admissible processes \( (\hat{X}^n, \hat{U}^n_s, \hat{U}^n_b) \) with \( \hat{X}^n(0) = x \), and the process \( (\hat{X}^n, \hat{U}^n_s, \hat{U}^n_b) \) is said to be admissible if the corresponding admission control \( m_n \) satisfies Assumption 2.5.

In the next Section 3, we establish the tightness of the diffusion-scaled processes. In particular, the computations illustrate that \( \{\hat{X}_n\}_{n \geq 1} \) is stochastically bounded and hence \( \hat{J}^n(\hat{X}^n(0), \hat{U}^n_s, \hat{U}^n_b) \) is of order \( \sqrt{n} \) and \( \hat{J}^n(\hat{X}^n(0), \hat{U}^n_s, \hat{U}^n_b) \) is of order 1. Next in Section 4, we develop the DCP and derive an explicit optimal solution. Finally, in Section 5, the DCP is shown to be a good approximation for the \( n^{th} \) QCP. In particular, we show that \( \hat{V}^n \), the value function of the \( n^{th} \) QCP, converges to the value function of the DCP as \( n \) tends to infinity. We then propose a threshold type admission control policy described by the optimal solution of the DCP for the \( n^{th} \) QCP, and prove that
it is asymptotically optimal for the QCP as \( n \) tends to infinity. Here a sequence of admissible control policies \( \{ (\hat{U}^n_s, \hat{U}^n_b) \}_{n \in \mathbb{N}} \) is said to be \textit{asymptotically optimal} if for any sequence of admissible control policies \( \{ (\bar{U}^n_s, \bar{U}^n_b) \}_{n \in \mathbb{N}} \),
\[
\lim_{n \to \infty} \hat{J}^n(\bar{X}^n(0), \bar{U}^n_s, \bar{U}^n_b) \leq \liminf_{n \to \infty} \hat{J}^n(\hat{X}^n(0), \hat{U}^n_s, \hat{U}^n_b).
\]

\section{Weak convergence}

This section is devoted to establishing the tightness of the diffusion-scaled processes and to characterizing their weak limits. For a given queue-capacity process \( \mathbf{m}_n(\cdot) \), the controlled diffusion-scaled state process \( \hat{X}^n \) described in (2.12) can be written as follows: For all \( t \geq 0 \),
\[
\hat{X}^n(t) = \hat{\zeta}^n(t) - \hat{G}^n_b(t) + \hat{U}^n(t) + \hat{U}_b^n(t),
\]
where
\[
\hat{\zeta}^n(t) = \hat{X}^n(0) + \hat{A}^n_s(t) - \hat{A}_b^n(t).
\]

We present the main result of weak convergence in the following theorem, and the rest of this section will focus on its proof.

\textbf{Theorem 3.1.} Any sequence of the controlled diffusion-scaled processes \( \{ (\hat{X}^n, \hat{\zeta}^n, \hat{G}^n_b, \hat{G}^n_s, \hat{U}^n_s, \hat{U}^n_b) \}_{n \in \mathbb{N}} \) is C-tight in \( D^0[0, T] \) for each \( T \geq 0 \). In particular, there exist a constant \( C > 0 \) and an integer \( m > 1 \) independent of the queue-capacity \( \mathbf{m}_n \) such that for each \( n \in \mathbb{N} \) and \( T > 0 \),
\[
E[\| \hat{X}^n \|^2_T + \| \hat{\zeta}^n \|^2_T + \| \hat{G}^n_b \|^2_T + \| \hat{G}^n_s \|^2_T] \leq C(1 + T^m).
\]

Furthermore, let \( (X, \zeta, G_b, G_s, U_b, U_s) \) denote a limit point. Then the following hold.

(i) There exists a standard one-dimensional Brownian motion \( B \) such that for \( t \geq 0 \),
\[
\zeta(t) = x + \sigma B(t) + \beta t,
\]
where \( x \) is the limit initial value given in (2.1), \( \sigma^2 = \sigma_s^2 + \sigma_b^2 \), \( \beta = \beta_s - \beta_b \) and the constants \( \sigma_s, \sigma_b, \beta_s, \beta_b \) are given in (2.4), (2.2) and (2.3).

(ii) For \( t \geq 0 \),
\[
G_s(t) = \delta_s \int_0^t X^+(u)du, \quad G_b(t) = \delta_b \int_0^t X^-(u)du,
\]
where \( \delta_s \) and \( \delta_b \) are as in (2.7) and (2.6).

(iii) The process \( X \) satisfies the Itô equation
\[
X(t) = x + \sigma B(t) + \int_0^t [\beta - h(X(s))]ds - U(t),
\]
where \( h \) is a piecewise linear function given by \( h(x) = \delta_s x^+ - \delta_b x^- \) for \( x \in \mathbb{R} \) and \( U = U_s - U_b \) is a process of bounded variation, which is adapted to the filtration generated by \( (X, B) \) and satisfies
\[
U_s(t) = \int_0^t 1_{\{X(u) > 0\}} dU_s(u), \quad U_b(t) = \int_0^t 1_{\{X(u) < 0\}} dU_b(u).
\]
Remark 3.1. Theorem 3.1 (i) follows from Assumptions 2.2 and 2.3. More precisely, from (2.4), (2.2) and (2.3),

\[(\hat{A}_n^0(t), \hat{A}_n^s(t)) \text{ converges weakly to } (\sigma_0B_1(t) + \beta_0t, \sigma_sB_2(t) + \beta_st), \quad (3.5)\]

in the space \(D^2[0,T] \) for any \( T > 0 \). From the initial condition (2.1) and (3.5), it follows that the process \( \hat{\zeta}^n \) is convergent weakly in \( D[0,T] \) and its limiting process is given by \( x + \sigma B(t) + \beta t \) for \( t \in [0,T] \).

The remainder of the proof of Theorem 3.1 is divided into five subsections. Section 3.1 establishes the stochastic boundedness of \( \hat{X}^n \). In Section 3.2, we introduce the virtual waiting times for unblocked customers and show that the diffusion-scaled virtual waiting time processes are stochastically bounded. Section 3.3 is devoted to the C-tightness of \((\hat{G}^n_s, \hat{G}^n_b)\). The C-tightness of \((\hat{X}^n, \hat{U}^n_b, \hat{U}^n_s)\) and the asymptotic relationships between \( \hat{X}^n \) and \( \hat{G}^n_s \) and \( \hat{G}^n_b \) are obtained in Section 3.4. Finally, in Section 3.5, we complete the proof of Theorem 3.1 based on the results derived in Sections 3.1 – 3.4.

For notation convenience, we introduce

\[ \hat{\zeta}^n(t) = \hat{G}^n_s(t) - \hat{G}^n_b(t), \quad (3.6) \]
\[ \hat{U}^n(t) = \hat{U}^n_s(t) - \hat{U}^n_b(t). \quad (3.7) \]

### 3.1 Stochastic boundedness of \( \hat{X}^n \)

For a given queue-capacity process \( m_n(\cdot) \), consider the controlled diffusion-scaled process \((\hat{X}^n, \zeta^n, \hat{G}^n_s, \hat{G}^n_b, \hat{U}^n_s, \hat{U}^n_b)\) satisfying (3.1) and (3.2). When the state process \( \hat{X}^n \) deviates far away from the origin, the processes \( \hat{G}^n_s, \hat{G}^n_b, \hat{U}^n_s, \) and \( \hat{U}^n_b \) act as frictional forces. The proof of the following result is based on this fact.

Proposition 3.2. For any state process \( \hat{X}^n \) in \( D[0,T] \), we have

\[ E[\|\hat{X}^n\|^2_T] \leq C(1 + T^n), \quad (3.8) \]

where the constant \( C > 0 \) and the integer \( m > 1 \) are independent of \( n, T \) and of the queue-capacity \( m_n(\cdot) \). Consequently, the sequence \( \{\hat{X}^n\}_{n \geq 1} \) is stochastically bounded.

Proof. From (2.5), (2.2) and (2.3), it follows that

\[ E[\|\hat{A}^0_n\|^2_T + \|\hat{A}^s_n\|^2_T] \leq \hat{C}(1 + T^n), \quad (3.9) \]

where \( \hat{C} \) is a positive constant and \( m > 1 \) is an integer constant, and both constants are independent of \( n, T \), and the queue-capacity \( m_n(\cdot) \). Thus there exists a constant \( C > 0 \) so that \( E[\|\zeta^n\|^2] \leq C(1 + T^n) \). The constant \( C \) is also independent of \( n, T \) and the queue-capacity \( m_n(\cdot) \). We let \( Y_n(t) = \hat{X}^n(t) - \zeta^n(t) \) for \( t \geq 0 \) and then use a path-wise argument to obtain (3.8). We choose \( M \equiv M(\omega) = 1 + \|\zeta^n\|_T < \infty \) a.s. by (3.9). We claim that \( \|Y_n\|_T \leq 2M \). Suppose it doesn’t hold. Then there exists a \( t_0 \) in \([0,T]\) so that \( |Y_n(t_0)| > 2(1 + \epsilon)M \) for some \( \epsilon > 0 \).

First we consider the case \( Y_n(t_0) > 2(1 + \epsilon)M \) for some \( t_0 \). Then, for any such \( t_0 \), \( \hat{X}^n(t_0) = Y_n(t_0) + \zeta^n(t_0) > M \). Since \( Y_n(0) = 0 \), we let \( \hat{t} = \inf\{t \in (0,T) : Y_n(t) > 2(1 + \epsilon)M \} \). Then \( 0 < \hat{t} \leq T, Y_n(\hat{t}) > 2M \) and \( Y_n(\hat{t}) - Y_n(\hat{t}^-) > 0 \) since \( Y_n \) has piecewise constant RCLL paths. Since \( \hat{X}^n(\hat{t}) > M > 0 \), we have \( \hat{X}^n(\hat{t}^-) \geq \hat{X}^n(\hat{t}) - \frac{1}{\sqrt{n}} \geq 0 \).
and $\hat{G}_b^n(t) + \hat{U}_b^n(t) = \hat{G}_b^n(t-e) + \hat{U}_b^n(t-e)$. Then $Y_n(t) - Y_n(t-e) = -[\hat{G}_b^n(t) + \hat{U}_s^n(t)] - (\hat{G}_b^n(t-e) + \hat{U}_s^n(t-e))] \leq 0$. This is a contradiction.

A similar argument shows that $Y_n(t) < -2(1+\epsilon)M$ also not possible when $0 \leq t \leq T$. Therefore $\|Y_n\| \leq 2M$ holds a.s. Since $Y_n(t) = \hat{X}^n(t) - \hat{\zeta}^n(t)$, this yields that $\|\hat{X}^n\|_T \leq 3M = 3(1 + \|\zeta^n\|_T)$. Now the conclusion (3.8) can be obtained using the moment bound in (3.9). This completes the proof. $\Box$

The following corollary is an immediate consequence of the above result.

**Corollary 3.1.** For any state process $\hat{X}^n$ in $D[0,T]$, 
\[
E[\|\hat{X}^n\|^2_T] \leq \frac{C(1 + T^m)}{n},
\]
where $C$ and $m$ are as in Proposition 3.2. Consequently, 
\[
\lim_{n \to \infty} E[\|\hat{X}^n\|^2_T] = 0. \tag{3.10}
\]

### 3.2 Virtual waiting times

We need to introduce the virtual waiting time processes and obtain their stochastic boundedness to guarantee the tightness of the state process. However, the virtual waiting times can be undefined on the time intervals during which the corresponding buffer is full. To circumvent this difficulty, we introduce the auxiliary arrival processes of unblocked customers to introduce the virtual waiting time processes and obtain their stochastic boundedness to guarantee the tightness of the state process.

For each $t \geq 0$, 
\[
E^n_b(t) = A^n_b(t) - U^n_b(t) \quad \text{and} \quad E^n_s(t) = A^n_s(t) - U^n_s(t),
\]
where the constant $C > 0$ and the integer $m > 1$ are independent of $n$, $T$, and the queue-capacity $m_n$. Consequently, the processes $\hat{V}_b^n$ and $\hat{V}_s^n$ are stochastically bounded in $D[0,T]$. 

**Proposition 3.3.** For each $T > 0$, let $\hat{V}_b^n$ and $\hat{V}_s^n$ be the virtual waiting time processes in $D[0,T]$. Then 
\[
E[\|\hat{V}_b^n\|^2_T + \|\hat{V}_s^n\|^2_T] \leq C(1 + T^m), \tag{3.12}
\]
where $C > 0$ and $\epsilon > 0$ are independent of $n$, $T$, and the queue-capacity $m_n$. Consequently, the processes $\hat{V}_b^n$ and $\hat{V}_s^n$ are stochastically bounded in $D[0,T]$. 

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Proof. We establish a moment bound for $E[\|\hat{V}_b^n\|_2^2]$. The moment bound for $E[\|\hat{V}_b^n\|_T^2]$ is similar. Let $t^n_j$ and $d^n_j$ represent the arrival time and the patience time of the $j$th buyer (according to the order of arrival), respectively. Similarly, we let $\hat{t}^n_j$ and $\hat{d}^n_j$ represent the arrival time and the patience time of the $j$th seller (according to the order of arrival), respectively.

If there are $K_n$ sellers waiting in the queue initially, the first $K_n$ buyers will be matched upon their arrival. In this case, using Assumption 2.1, $0 \leq K_n \leq M \sqrt{n}$ and they are all non-abandoning. Let $T_n$ be the time it takes to match all these buyers. Then $T_n \leq \sum_{j=1}^{K_n} \tilde{t}^n_j$ and hence using Cauchy-Schwartz inequality, $E[T_n^2] \leq K_n^2 E[\hat{t}^2_n]$. By Assumption 2.2, $E[\hat{t}^2_n] \leq (1 + s^2_0)/(\lambda_0^2)^2$ and consequently, $E[(\sqrt{n} T_n)^2] \leq M^2(1 + s^2_0)n^2/(\lambda_0^2)^2$. Now using Assumption 2.3, $\lim_{n \to \infty} \lambda_0^n / n = \lambda_0 > 0$ and hence, $\sup_{n \geq 1} E[(\sqrt{n} T_n)^2] \leq C < \infty$, where $C > 0$ is a generic constant. Therefore, we can simply assume that there are no buyers initially.

Let $v^n_j$ be the amount of time the $j$th buyer spent as the head of the queue (i.e. the time spent in the first place of the queue). We simply take $v^n_j = 0$ if the $j$th buyer did not reach the head position due to blocking or abandonment. Then we can write

$$V^n_b(t) = \sum_{j=1}^{A^n_b(t)} v^n_j - \int_0^t 1_{\{V^n_b(s) > 0\}} ds.$$ 

We introduce $S^n_b(t)$ to be the virtual waiting time to reach the head of the queue for an infinitely patient unblocked buyer who arrived at time $t$. To simplify the notation, we write $s^n_j \equiv S^n_b(t^n_j -)$. With a simple algebraic manipulation, we observe that the diffusion scaled virtual waiting time $\hat{V}_b^n(t) = \sqrt{n} V^n_b(t)$ satisfies

$$\hat{V}_b^n(t) = Y_n(t) + \sqrt{n} \int_0^t 1_{\{\hat{V}_b^n(s) = 0\}} ds,$$

where $Y_n$ is given by

$$Y_n(t) = \sqrt{n} \left[ A^n_b(t) \sum_{j=1}^{A^n_b(t)} v^n_j - t \right]$$

for all $t \geq 0$. Therefore, the pair $\{(\hat{V}_b^n(t), \sqrt{n} \int_0^t 1_{\{\hat{V}_b^n(s) = 0\}} ds); t \geq 0\}$ is the unique solution to the SP with the input process $Y_n$ and the reflection barrier at the origin (see [23]). Let $\Gamma$ denote the one-sided Skorokhod map with reflection barrier at the origin. Then $\hat{V}_b^n(t) = \Gamma(Y_n)(t)$ for $t \in [0, T]$. From the Lipschitz continuity of $\Gamma$ (see [23] and [41]), we have $\|\hat{V}_b^n\|_T \leq 2\|Y_n\|_T$. In the following, we estimate the second moment of $\|Y_n\|_T$.

For each $j \geq 1$, introduce the set $\mathcal{V}_j^n = [s^n_j, s^n_j + v^n_j)$. Notice that the length (Lebesgue measure) of the set $|\mathcal{V}_j^n| = v^n_j$ and this holds even when $v^n_j = 0$. The sets $\{\mathcal{V}_j^n\}$ are disjoint and for each $j$, $\mathcal{V}_j^n \subset [\tilde{t}^n_k, \tilde{t}^n_{k+1})$ for some $k$. It is important to observe that the same set $[\tilde{t}^n_k, \tilde{t}^n_{k+1})$ may contain several $\mathcal{V}_j^n$ intervals. Next we consider the collection of the intervals

$$\mathcal{A}(t) = \{[\tilde{t}^n_k, \tilde{t}^n_{k+1}) : k \in \mathbb{N} \text{ and } [\tilde{t}^n_k, \tilde{t}^n_{k+1}) \cap \mathcal{V}_j^n \text{ is nonempty for some } j = 1, \ldots, A^n_b(t)\}.$$ 

This collection of intervals is finite and we let $N^n_b(t) \equiv |\mathcal{A}(t)|$ which represents the number of elements in this set. For a given $t \geq 0$, the intervals in $\mathcal{A}(t)$ are called
“good” intervals. For \( t \geq 0 \), let \( h(t,k) \) denote the number of \( \mathcal{V}^n_j \)'s in \( [\tilde{P}^n_k, \tilde{P}^n_{k+1}) \in \mathcal{A}(t) \) for each \( k \). Then \( h(t,k) \) is non-decreasing in \( t \) and \( A^b(t) - N^b(t) = \sum_{k=1}^{\infty} (h(t,k) - 1)^+ \). Thus, \( A^b(t) - N^b(t) \) is a non-negative, non-decreasing process and we will use this fact to obtain our final estimate (3.17). We further introduce the scaled quantities \( \tilde{N}^n(t) = N^b(t)/n \) and \( \tilde{N}^n(t) = (N^b(t) - \lambda^b t)/\sqrt{n} \). Next introduce the non-negative integer valued random variables

\[
\tau^*_1 = \min\{k \geq 0 : [\tilde{P}^n_k, \tilde{P}^n_{k+1}) \cap \mathcal{V}^n_j \text{ is non-empty for some } j = 1, \ldots, A^b_0(t)\},
\]

and \( \tau^*_1 \) is infinite if the above set is empty. Let

\[
\tau^*_l = \min\{k > \tau^*_j : [\tilde{P}^n_k, \tilde{P}^n_{k+1}) \cap \mathcal{V}^n_l \text{ is non-empty for some } l = 1, \ldots, A^b_0(t)\},
\]

and \( \tau^*_l \) is infinite if the above set is empty. Introduce the filtration \( \{\mathcal{G}^n_k\}_{k \in \mathbb{N}} \) by \( \mathcal{G}^n_k = \sigma\{(\tilde{P}^n_1, \tilde{d}^n_1), \ldots, (\tilde{P}^n_k, \tilde{d}^n_k) \text{ for all } l \geq 1\} \). Then \( \tau^*_n = k \) if and only if there are exactly \( j - 1 \) “good” intervals among \([0, \tilde{P}^n_1), \ldots, [\tilde{P}^n_{k-1}, \tilde{P}^n_k)\) and the interval \( [\tilde{P}^n_k, \tilde{P}^n_{k+1}) \) is also “good”. Consequently, \( \tau^*_n + 1 \) is a \( \{\mathcal{G}^n_k\} \) stopping time. Since the sets \( \{\mathcal{V}^n_j\} \) are disjoint and for each \( j \), \( \mathcal{V}^n_j \subseteq [\tilde{P}^n_k, \tilde{P}^n_{k+1}) \) for some \( k \), it follows that

\[
Y_n(t) = W_n(t) = \sqrt{n} \sum_{j=1}^{N^b_n(t)} \tilde{u}_{n+1}^n - t,
\]

where \( \{\tilde{u}_k^n\}_{k \in \mathbb{N}} \) are the inter-arrival times of the arrival process \( A^b_n(t) \) of the sellers. Focusing on \( W_n(t) \) now, we establish that \( E[\tilde{u}_{n+1}^n] = 1/\lambda^n_\sigma \) and the \( \text{Var}(\tilde{u}_{n+1}^n) = (\sigma/\lambda^n_\sigma)^2 \) for each \( j \). Indeed, keep \( k \) fixed and consider the filtration \( \{\mathcal{G}^n_k\} \) described above. The random variables \( \{\tilde{u}_1^n, \ldots, \tilde{u}_{n-1}^n\} \) are \( \mathcal{G}^n_k \) adapted and the sequence \( \{\tilde{u}_k^n, \tilde{u}_{k+2}^n, \ldots\} \) are independent of \( \mathcal{G}^n_k \). Hence, we can employ the Wald’s equation for random sums to conclude that \( E[\tilde{u}_{n+1}^n] = 1/\lambda^n_\sigma \). A similar proof yields \( E[(\tilde{u}_{n+1}^n)^2] = (1/\lambda^n_\sigma)^2 + (\sigma/\lambda^n_\sigma)^2 \) and consequently, \( \text{Var}(\tilde{u}_{n+1}^n) = (\sigma/\lambda^n_\sigma)^2 \). Next we define the stopped \( \sigma \)-algebra \( \mathcal{H}^n_j = \mathcal{G}^n_{\tau^*_j} \). Then \( \{\mathcal{H}^n_j\} \) is a filtration. We use this filtration to introduce a discrete-time martingale \( \{M^n(k), \mathcal{H}_j^n\}_{k \in \mathbb{N}} \) given by \( M^n(0) = 0 \) and \( M^n(k) = \sqrt{n} \sum_{j=1}^{k} [\tilde{u}_{n+1}^n - 1/\lambda^n_\sigma] \). It is straight forward to check that \( \{M^n(k), \mathcal{H}_j^n\}_{k \in \mathbb{N}} \) is a square integrable martingale. Next, we introduce \( \mathcal{H}_n^i = \mathcal{H}_n^{[nt]} \) and \( M^n(t) = M^n([nt]) \) for all \( t \geq 0 \), where \( [x] \) represents the integer part of \( x \). Then it can be easily checked that \( \{(M^n(t), \mathcal{H}_n^i)\}_{t \geq 0} \) is a square integrable, pure-jump martingale and its quadratic variation process is given by \( [M^n, M^n](t) = n \sum_{j=1}^{[nt]} [\tilde{u}_{n+1}^n - 1/\lambda^n_\sigma]^2 \) for all \( t \geq 0 \). With a simple algebraic manipulation, we can represent \( W_n(t) \) by

\[
W_n(t) = M^n(N^b_n(t)) + \frac{n}{\lambda^n_b} \left[ \tilde{N}^b_n(t) + \left( \frac{\lambda^n_b - \lambda^n_\sigma}{\sqrt{n}} \right) t \right], \quad \text{for } 0 \leq t \leq T. \tag{3.14}
\]

Based on the formulation of \( W_n(t) \), we introduce the process

\[
Z_n(t) = M^n(N^b_n(t)) + \frac{n}{\lambda^n_b} \left[ A^b_n(t) + \left( \frac{\lambda^n_b - \lambda^n_\sigma}{\sqrt{n}} \right) t \right], \quad \text{for } 0 \leq t \leq T. \tag{3.15}
\]

Since \( A^b_n(t) - N^b_n(t) \) is a non-negative, non-decreasing process, it follows that

\[
Y_n(t) \leq W_n(t) \leq Z_n(t), \quad \text{for } 0 \leq t \leq T. \tag{3.16}
\]
Moreover, $W_n(t) - Y_n(t)$ and $Z_n(t) - Y_n(t)$ are non-negative, non-decreasing processes. Consequently, using the monotonicity property of $\Gamma$ (that is, for $f, g \in D[0, T]$ and if $g$ is a non-negative, nondecreasing function, then $\Gamma(f)(t) \leq \Gamma(f + g)(t)$ (see again [23] and section 2.3 of [41])), we obtain

$$
\hat{V}_b^n(t) = \Gamma(Y_n)(t) \leq \Gamma(Z_n)(t), \text{ for all } 0 \leq t \leq T.
$$

(3.17)

Thus it suffices to estimate the second moment of $\|Z_n\|_T$. Applying the heavy traffic condition in Assumption 2.3, and the fact that $\hat{N}_b^n(t) \leq \hat{A}_b^n(t) \leq \hat{A}_b^n(T)$ for all $0 \leq t \leq T$ to (3.15), we have

$$
0 \leq \|Z_n\|_T \leq \sup_{0 \leq s \leq \hat{A}_b^n(T)} |M^n(s)| + \frac{n}{\lambda^n_b} \|\hat{A}_b^n\|_T + KT,
$$

where $K > 0$ is a constant which depends on the heavy traffic limit $\beta_b - \beta_s$. Notice that $\{\hat{A}_b^n(T) \leq r\} = \{\hat{A}_b^n(T) \leq [nr]\} \in \mathcal{H}^n_r$ for each $r > 0$, and hence, $\hat{A}_b^n(T)$ is a stopping time with respect to the filtration $\{\mathcal{H}^n_r\}_{t \geq 0}$. In the rest of the proof, the generic constants $C_i > 0$ where $i = 1, ..., 5$ are independent of $T$ and $n$. We then obtain

$$
E \left[ \sup_{0 \leq t \leq \hat{A}_b^n(T)} |M^n(t)|^2 \right] \leq C_1 E([M^n, M^n](\hat{A}_b^n(T))) = C_1 E[E([M^n, M^n](\hat{A}_b^n(T)))|\hat{A}_b^n(T)]
$$

$$
= C_1 \frac{n(s_s/\lambda^n_s)^2}{\lambda^n_b} E[\hat{A}_b^n(T)] \leq C_1 \frac{n(s_{s_s}/\lambda^n_{s_s})^2}{\lambda^n_b} \left( \frac{\lambda^n_b T}{n} + C_2 \sqrt{\frac{(1 + T)^n}{n}} \right),
$$

where the first inequality follows from the Burkholder’s inequality (see [32]) and the second inequality follows from (2.5). Next, from (2.5), $E[\|\hat{A}_b^n\|_T^2] \leq C_3 (1 + T^n)$ and hence, it follows that $E[\|Z_n\|_T^2] \leq C_4 (1 + T^n)$ where the constant $C_4 > 0$ and the integer $m > 1$ are independent of $n$ and $T > 0$. Combining this estimate with (3.17) and using the properties of Skorokhod map $\Gamma$, we obtain

$$
E[\|\hat{V}_b^n\|_T^2] \leq E[\|\Gamma(Z_n)\|_T^2] \leq 4E[\|Z_n\|_T^2] \leq C_5 (1 + T^n).
$$

This yields the moment bound for $E[\|\hat{V}_b^n\|_T^2]$. The estimate for $E[\|\hat{V}_s^n\|_T^2]$ is similar. Hence (3.12) follows and consequently, $\{\hat{V}_b^n\}_{n \in \mathbb{N}}$ and $\{\hat{V}_s^n\}_{n \in \mathbb{N}}$ are stochastically bounded in $D[0, T]$. This completes the proof. \(\square\)

### 3.3 Tightness of $\{(G^n_b, G^n_s)\}_{n \geq 1}$

We first introduce a pair of “eventual” abandonment processes $\hat{L}_b^n$ and $\hat{L}_s^n$. For $t \geq 0$, let $L_b^n(t)$ (resp. $L_s^n(t)$) denote the number of buyers (resp. sellers) who enter the queue by time $t$ and eventually abandon the system. Then $L_b^n(0) = L_s^n(0) = 0$ and it is evident that $G_b^n(t) \leq L_b^n(t) \leq E_b^n(t)$ and $G_s^n(t) \leq L_s^n(t) \leq E_s^n(t)$ for all $t \geq 0$.

Notice that the processes $L_b^n$ and $L_s^n$ are not adapted to the natural filtration $\{\mathcal{F}^n_r\}$ defined in (2.8). Their diffusion-scaled versions are given by $\hat{L}_b^n(t) = L_b^n(t)/\sqrt{n}$ and $\hat{L}_s^n(t) = L_s^n(t)/\sqrt{n}$ for $t \geq 0$.

Our first step is to show that the processes $\{\hat{L}_b^n\}$ and $\{\hat{L}_s^n\}$ are tight. Here we appropriately modify a proof originally developed in [34] for a single-server queue in conventional heavy traffic.
Proposition 3.4. For the non-decreasing processes $\hat{L}_b^n$, $\hat{G}_b^n$, $\hat{L}_s^n$ and $\hat{G}_s^n$ in $D[0,T]$,

\begin{align}
E[\hat{L}_b^n(T)^2 + \hat{L}_s^n(T)^2] & \leq C(1 + T^m), \\
E[\hat{G}_b^n(T)^2 + \hat{G}_s^n(T)^2] & \leq C(1 + T^m),
\end{align}

where the constant $C > 0$ and the integer $m > 1$ are independent of $n,T$, and the queue-capacity $m_n$. Consequently, the processes $\{(L_b^n, G_b^n, L_s^n, G_s^n)\}_{n \in \mathbb{N}}$ are stochastically bounded in $D[0,T]$.

**Proof.** We will prove the results for buyers and the case of sellers is very similar. Let $t^n_j$ and $d^n_j$ represent the arrival time and the patience time of the $j$th unblocked buyer respectively, according to the order of arrival. Since initial customers are of infinite patience, we can write $\hat{L}_b^n((t^n_j - ))$ for all $t^n_j \geq 0$, where it will be adapted. Let $\tilde{t}^n_0$ and $\tilde{d}^n_0$ represent the arrival time and the patience time of the $n$th unblocked seller, respectively, according to the order of arrival. Consider $\mathcal{H}$ to be the $\sigma$–algebra generated by the sequence $\{(t^n_j, d^n_j)\}_{j \geq 1}$ and introduce $G_b^n = \sigma(\tilde{t}^n_0) \lor \mathcal{H}$ to be the $\sigma$–algebra generated by the random variable $\tilde{t}^n_0$ and the collection $\mathcal{H}$. Similarly, for each $k \geq 1$, we introduce the $\sigma$–algebra

\[ G_k^n = \sigma((t^n_{1}, d^n_{1}), \ldots, (t^n_{k}, d^n_{k}), t^n_{k+1}) \lor \mathcal{H}. \]

Notice $V_b^n((t^n_j - ))$ is adapted to $G^n_{j-1}$ and $d^n_j$ is independent of $G^n_{j-1}$. Let $M^n(0) = 0$ and

\[ M^n(k) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \left[ 1_{\{V_b^n((t^n_j - )) \geq d^n_j\}} - F_b(V_b^n((t^n_j - ))) \right], \]

where $F_b$ is in (2.6). Then $\{(M^n(k), G^n_k)\}_{k \geq 1}$ is a square integrable mean zero martingale. Next, let $G^n_t = G^n_{[nt]}$ and $M^n(t) = M^n([nt])$ for all $t \geq 0$. Consequently, $\{(M^n(t), G^n_t)\}_{t \geq 0}$ is a square integrable, mean zero, pure-jump martingale and

\[ [M^n, M^n](t) \leq 2t \]

for all $t \geq 0$ (for details, we refer to [34]). We can represent $\hat{L}_b^n$ by

\[ \hat{L}_b^n(t) = M^n(\hat{E}_b^n(t)) + \frac{1}{\sqrt{n}} \sum_{j=1}^{E^n_b(t)} F_b(V_b^n((t^n_j - ))) \]

for all $t \geq 0$, where $\sum_{j=1}^{0} = 0$ is interpreted as zero. Next, we use this representation to prove (3.18). Let $T > 0$ that is fixed. First we show that $\hat{E}_b^n(T)$ is a discrete-valued stopping time with respect to the filtration $\{G^n_t\}_{t \geq 0}$. When $nr$ is a positive integer, $\{\hat{E}_b^n(T) = r\} = \{E^n_b(T) = nr\} = \{t^n_{nr} \leq T < t^n_{nr+1}\} \in G^n_r$. Similar argument holds
when $r = 0$. Hence $\bar{E}^n_b(T)$ is a stopping time. In what follows, the positive constants $C_i, i = 1, \ldots, 5$ and integer $m > 1$ are independent of $T$ and $n$. By (3.9),

$$E \left( \sup_{t \in [0,T]} \left| \bar{A}^n_b(t) - \bar{F}^n_b(t) \right|^2 \right) \leq \frac{C_1(1 + T^m)}{n}. \quad (3.22)$$

Therefore, $E[(\bar{E}^n_b(T))]^2 \leq E[|\bar{A}^n_b(T)|^2] \leq (\lambda_0 T)^2 + C_1(1 + T^m)/n$. Using (3.20) together with the Burkholder’s inequality (see [32]), we obtain $E[\sup_{0 \leq t \leq \bar{E}^n_b(T)} |M^n(t)|^2] \leq C_2E[(M^n, M^n)(\bar{E}^n_b(T))] \leq 2C_2E[\bar{E}^n_b(T)] \leq C_3(1 + T^m)$. Consequently,

$$E \left[ \sup_{0 \leq t \leq T} |M^n(\bar{E}^n_b(t))|^2 \right] \leq E \left[ \sup_{0 \leq t \leq \bar{E}^n_b(T)} |M^n(t)|^2 \right] \leq C_3(1 + T^m). \quad (3.23)$$

We now consider the second term in the RHS of (3.21). By (2.6), we have $F_b(x) \leq Kx$ for all $x \geq 0$, where $K > 0$ is a constant independent of $n$. Since $0 \leq \bar{E}^n_b(T) \leq \bar{A}^n_b(T)$, we have the following estimates.

$$\sqrt{n}F_b(\|V^n_b\|_T)\bar{E}^n_b(T) \leq \sqrt{n}|F_b(\|V^n_b\|_T)|\bar{A}^n_b(T) - \lambda_0 T| + \lambda_0 T\sqrt{n}F_b(\|V^n_b\|_T)$$

$$\leq \sqrt{n}|\bar{A}^n_b(T) - \lambda_0 T| + K\lambda_0 T\|V^n_b\|_T,$$

and thus

$$E \left[ \left( \sum_{j=1}^{E^n_b(t)} \frac{1}{\sqrt{n}} F_b(V^n_b(t_j^n -)) \right)^2 \right] \leq E \left[ (\sqrt{n}F_b(\|V^n_b\|_T)\bar{E}^n_b(T))^2 \right]$$

$$\leq 2nE(|\bar{A}^n_b(T) - \lambda_0 T|^2) + 2(\lambda_0 KT)^2 E(\|\bar{V}_n\|_T^2).$$

Now using (3.22) and (3.12), we obtain

$$E \left[ \left( \sum_{j=1}^{E^n_b(t)} \frac{1}{\sqrt{n}} F_b(V^n_b(t_j^n -)) \right)^2 \right] \leq C_4(1 + T^m). \quad (3.24)$$

Finally, using the estimates in (3.23) and (3.24) in (3.21), we obtain $E[\tilde{L}^n_b(T)^2] \leq C_5(1 + T^m)$. Similar estimate can be obtained for $E[\tilde{L}^n_b(T)^2]$ and hence (3.18) follows.

Since $0 \leq \tilde{G}^n_b(t) \leq \tilde{L}^n_b(t)$ holds for all $t \geq 0$, (3.19) follows from (3.18), and the stochastic boundedness is an immediate consequence of (3.18) and (3.19). □

The next proposition is concerned with the fluid-scaled control processes and the fluid-scaled arrival processes of unblocked customers.

**Proposition 3.5.** The process sequence $\{(\tilde{U}^n_b, \tilde{U}^n_s)\}_{n \in \mathbb{N}}$ is stochastically bounded and $C$-tight in $D^2[0, T]$ and

$$\lim_{n \to \infty} E[\tilde{U}^n_b(T)^2 + \tilde{U}^n_s(T)^2] = 0. \quad (3.25)$$

Consequently,

$$\lim_{n \to \infty} E \left( \sup_{t \in [0, T]} \left[ |\tilde{E}^n_b(t) - \lambda_0 t|^2 + |\tilde{E}^n_s(t) - \lambda_0 t|^2 \right] \right) = 0. \quad (3.26)$$
Proof. We focus on $\bar{U}_b^n$ and the proof for $\bar{U}_s^n$ is similar. For a given controlled buffer length policy $m_n(\cdot)$, the fluid-scaled state equation is given by
\[
\bar{X}_n(t) = \bar{X}_n(0) + \bar{A}_b^n(t) - \bar{A}_b^n(t) - \bar{G}_s^n(t) + \bar{G}_b^n(t) + \bar{U}_b^n(t) - \bar{U}_s^n(t), \quad t \geq 0. \tag{3.27}
\]
From (2.10) and (2.11), the processes $\bar{U}_b^n$ and $\bar{U}_s^n$ satisfy
\[
\bar{U}_b^n(t) = \int_0^t 1_{\{\bar{X}_n(s) = \bar{m}_b^n(s)\}} d\bar{A}_b^n(s) + [\bar{m}_b^n(0) - \bar{X}_n(0)]^+, \tag{3.28}
\]
and
\[
\bar{U}_s^n(t) = \int_0^t 1_{\{\bar{X}_n(s) = \bar{m}_s^n(s)\}} d\bar{A}_s^n(s) + [\bar{X}_n(0) - \bar{m}_s^n(0)]^+. \tag{3.29}
\]
We can use $\bar{U}_b^n(T) \leq |\bar{X}_n(0)| + \bar{A}_b^n(T)$ together with (2.1) and (3.9) to obtain
\[
E[\bar{U}_b^n(T)^2] \leq C[1 + T^m], \tag{3.30}
\]
where the constants $C > 0$, and the integer $m > 1$ are independent of $n, T$ and the queue-capacity $m_n$. Since the process $\bar{U}_b^n$ is non-decreasing, (3.30) yields that $\{\bar{U}_b^n\}_{n \in \mathbb{N}}$ is stochastically bounded in $D[0, T]$.

To prove the C-tightness of $\{\bar{U}_b^n\}_{n \in \mathbb{N}}$ in $D[0, T]$, let $0 \leq t_1 \leq t_2 \leq T$, then from (3.28), we have $0 \leq \bar{U}_b^n(t_2) - \bar{U}_b^n(t_1) \leq \bar{A}_b^n(t_2) - \bar{A}_b^n(t_1)$. Consequently, for any $\delta > 0$,
\[
\sup_{|t_1 - t_2| < \delta} |\bar{U}_b^n(t_2) - \bar{U}_b^n(t_1)| \leq \lambda_0 \delta + 2 \sup_{t \in [0, T]} |\bar{A}_b^n(t) - \lambda_0 t|.
\]
Using (3.9), it follows that
\[
\limsup_{n \to \infty} E\left[\sup_{|t_1 - t_2| < \delta} |\bar{U}_b^n(t_2) - \bar{U}_b^n(t_1)| \right]^2 \leq \lambda_0^2 \delta^2 \tag{3.31}
\]
for any $\delta > 0$. The conditions (3.30) and (3.31) (see [3]) imply that the process $\{\bar{U}_b^n\}$ is C-tight in $D[0, T]$. Similarly, $\{\bar{U}_s^n\}$ can also been shown to be C-tight in $D[0, T]$. Let $(\bar{u}_b, \bar{u}_s)$ be a weak limit of the joint process through a subsequence. We relabel this subsequence as the original sequence $(\bar{U}_b^n, \bar{U}_s^n)$ for convenience. We next show that $\bar{u}_b$ and $\bar{u}_s$ are identically zero in $[0, T]$.

Now we let
\[
\tilde{\bar{X}}_n(t) = \bar{X}_n(0) + \bar{A}_b^n(t) - \bar{A}_b^n(t) - \bar{G}_s^n(t) + \bar{G}_b^n(t), \quad t \geq 0. \tag{3.32}
\]
Following Theorem 2.6 and Corollary 2.4 in [5], given the input process $\tilde{\bar{X}}_n$ in (3.32), the equations (3.27), (3.29) and (3.30) yield a unique solution $(\bar{X}_n, \bar{U}_b^n - \bar{U}_s^n)$ for the extended SP (ESP) associated with the reflection barriers $\{\bar{m}_b^n(t), \bar{m}_s^n(t)\}$ for $0 \leq t \leq T$. By (3.3) and (3.10), $\lim_{n \to \infty} (\tilde{\bar{X}}_n, \bar{X}_n) = (0, 0)$ uniformly on $[0, T]$. Moreover, by (2.9), $\lim_{n \to \infty} (\bar{m}_b^n, \bar{m}_s^n) = (0, 0)$ uniformly on $[0, T]$. Now by the closure property in Proposition 2.5 of [5], the pair of zero functions $(0, 0)$ yields a unique solution to the ESP in the degenerate interval $[0, 0]$. Hence, by the uniqueness of the solution in Theorem 2.6 of [5], the limiting processes $\bar{u}_b$ and $\bar{u}_s$ are identically zero in $[0, T]$. Consequently, $(\bar{U}_b^n, \bar{U}_s^n)$ converges to $(0, 0)$ in $D^2[0, T]$.

To obtain (3.25), we prove $\lim_{n \to \infty} E[\bar{U}_b^n(T)^2] = 0$. The proof of $\lim_{n \to \infty} E[\bar{U}_s^n(T)^2] = 0$ is similar. By (3.30), $\lim_{n \to \infty} \sup_{n \geq 1} P[\bar{U}_s^n(T)^2 > a] = 0$. Let $\epsilon > 0$ be arbitrary. We
can pick $a > 0$ so that $P[\bar{U}_b^n(T)^2 > a] < \epsilon$ for all $n$. Using $\bar{U}_b^n(T) \leq |\bar{X}^n(0)| + \bar{A}_b^n(T)$, we have

$$E \left[ \bar{U}_b^n(T)^2 1_{\{\bar{U}_b^n(T)^2 > a\}} \right] \leq 2E[|\bar{X}^n(0)|^2] + 2E \left[ \bar{A}_b^n(T)^2 1_{\{\bar{U}_b^n(T)^2 > a\}} \right].$$

From Assumption 2.1, $\lim_{n \to \infty} E[|\bar{X}^n(0)|^2] = 0$. Next, we observe that

$$E \left[ \bar{A}_b^n(T)^2 1_{\{\bar{U}_b^n(T)^2 > a\}} \right] \leq \left[ (\lambda_0 T)^2 1_{\{\bar{U}_b^n(T)^2 > a\}} + (\lambda_0 T)^2 \right] \text{sup}_{0 \leq t \leq T} \bar{U}_b^n(T)^2 \leq 2E[|\bar{A}_b^n(T)|^2] + 2(\lambda_0 T)^2 \epsilon.$$

Therefore, we have $\limsup_{n \to \infty} E[\bar{U}_b^n(T)^2 1_{\{\bar{U}_b^n(T)^2 > a\}}] \leq \limsup_{n \to \infty} E[\bar{U}_b^n(T)^2] \leq (\lambda_0 T)^2 \epsilon$. Next noting that $\bar{U}_b^n$ converges to zero in $D[0, T]$, by the bounded convergence theorem,

$$\limsup_{n \to \infty} E \left[ \bar{U}_b^n(T)^2 1_{\{\bar{U}_b^n(T)^2 \leq a\}} \right] = 0.$$

Hence, $\lim_{n \to \infty} E[\bar{U}_b^n(T)^2] = 0$. Similarly, $\lim_{n \to \infty} E[\bar{U}_s^n(T)^2] = 0$.

Since $\bar{U}_b^n(t)$ and $\bar{U}_s^n(t)$ are non-decreasing processes, (3.9) together with (3.25) yields the conclusion (3.26). This completes the proof. □

**Proposition 3.6.** For the process sequences $\{\bar{L}_b^n\}_{n \in \mathbb{N}}$ and $\{\bar{L}_s^n\}_{n \in \mathbb{N}}$ in $D[0, T]$, we have

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left| \bar{L}_b^n(t) - \sqrt{n} \int_0^t F_b(V_b^n(u-)) d\bar{A}_b^n(u) \right| = 0, \text{ in probability,} \quad (3.33)$$

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left| \bar{L}_s^n(t) - \sqrt{n} \int_0^t F_s(V_s^n(u-)) d\bar{A}_s^n(u) \right| = 0, \text{ in probability.} \quad (3.34)$$

**Proof.** To prove (3.33), we use the representation (3.21) for $\bar{L}_b^n(t)$ and first show that

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left| \bar{L}_b^n(t) - \sqrt{n} \int_0^t F_b(V_b^n(u-)) d\bar{E}_b^n(u) \right| = 0, \text{ in probability.} \quad (3.35)$$

Noting that $n^{-1/2} \sum_{j=1}^{E_b^n(t)} F_b(V_b^n(t_j^-)) = \sqrt{n} \int_0^t F_b(V_b^n(u-)) d\bar{E}_b^n(u)$, (3.35) will follow from (3.21) if we can show $\lim_{n \to \infty} \sup_{t \in [0, T]} |M^n(\bar{E}_b^n(t))| = 0$ in probability, where the martingale $\{M^n(t)\}_{t \geq 0}$ is described in (3.21). We intend to show that

$$[M^n, M^n](T) = \frac{1}{n} \sum_{j=1}^{[nT]} \left[ 1_{\{V_b^n(t_j^-) \geq \epsilon_j\}} - F_b(V_b^n(t_j^-)) \right]^2.$$

Moreover, $\bar{E}_b^n(T)$ is a stopping time with respect to the filtration $\{\mathcal{G}_t^n\}_{t \geq 0}$. Using the Burkholder’s inequality (see [32]), we have

$$E \left[ \sup_{t \in [0, T]} |M^n(\bar{E}_b^n(t))|^2 \right] \leq E \left[ \sup_{0 \leq t \leq \bar{E}_b^n(T)} |M^n(t)|^2 \right] \leq C_1 E(|M^n, M^n|)(\bar{E}_b^n(T)), \quad (3.36)$$

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where $C_1 > 0$ is a constant independent of $n, T$ and the queue-capacity $m_n$. Since $[M^n, M^n](t) \leq 2t$ and $E_b^n(t) \leq A_b^n(t)$ for all $0 \leq t \leq T$, we first pick a fixed constant $K > \lambda_0 + 2$ such that
\[
[M^n, M^n](E_b^n(T)) \leq [M^n, M^n](E_b^n(T))1_{\{E_b^n(T) < KT\}} + 2A_b^n(T)1_{\{E_b^n(T) \geq KT\}}. \tag{3.37}
\]
For the first term in the right hand of (3.37), we can write
\[
[M^n, M^n](E_b^n(T))1_{\{E_b^n(T) < KT\}} = \frac{1}{n} \sum_{j=1}^{KnT} H_j^n 1_{\{E_b^n(T) < KT\}},
\]
where $H_j^n = |1_{\{V_b^n(t_j^n) \geq d_j^n\}} - F_b(V_b^n(t_j^n)) |^21_{\{t_j^n \leq T\}}$. Since $d_j^n$ is independent of $V_b^n(t_j^n)$ and $t_j^n$, $E[H_j^n|V_b^n(t_j^n), t_j^n] = F_b(V_b^n(t_j^n))(1 - F_b(V_b^n(t_j^n)))1_{\{t_j^n \leq T\}}$. Therefore,
\[
E \left( [M^n, M^n](E_b^n(T))1_{\{E_b^n(T) < KT\}} \right) \leq \frac{1}{n} \sum_{j=1}^{KnT} E[H_j^n] \leq \frac{1}{n} \sum_{j=1}^{KnT} F_b(\|V_b^n\|T) = KTE[F_b(\|V_b^n\|T)].
\]
By the assumption (2.6), there exists a constant $C_0 > 0$ so that $F_b(x) \leq C_0 x$ for all $x \geq 0$, and thus $E[F_b(\|V_b^n\|T)] \leq C_0 E[\|V_b^n\|T] \leq \frac{C_0}{\sqrt{n}} E[\|V_b^n\|T] \to 0$, as $n \to \infty$, using the moment bound in (3.12). Consequently, $\lim_{n \to \infty} E \left( [M^n, M^n](E_b^n(T))1_{\{E_b^n(T) < KT\}} \right) = 0$. For the second term in the RHS of (3.37), recalling that $K > \lambda_0 + 2$, we observe that
\[
E \left( A_b^n(T)1_{\{E_b^n(T) \geq KT\}} \right) \leq E \left( A_b^n(T)1_{\{A_b^n(T) \geq \lambda_0 T + 2T\}} \right) \leq E \left( A_b^n(T) - \lambda_0 T \right) 1_{\{A_b^n(T) \geq \lambda_0 T + 2T\}} + \lambda_0 T P (A_b^n(T) \geq \lambda_0 T + 2T) \leq E[\|A_b^n(T) - \lambda_0 T\| + \lambda_0 T P(\|A_b^n(T) - \lambda_0 T\| > 2T) \to 0,
\]
where the last step follows by (3.22). Hence, in view of (3.37) and (3.36), (3.35) is established.

Next we establish
\[
\lim_{n \to \infty} \sqrt{n} \int_0^T F_b(V_b^n(u-))d\bar{U}_b^n(u) = 0, \text{ in probability}. \tag{3.38}
\]
Then (3.35) and (3.38) implies (3.33). Let $\epsilon > 0$ be arbitrary. Since $\{\bar{V}_b^n\}_{n \in \mathbb{N}}$ is stochastically bounded, we pick a large constant $M > 1$ so that $P(\|\bar{V}_b^n\|T \geq M) < \epsilon$. To derive (3.38), it suffices to consider the set $\{\|\bar{V}_b^n\|T \leq M\}$. We observe that $\sqrt{n}F_b(\|\bar{V}_b^n\|T)1_{\{\|\bar{V}_b^n\|T \leq M\}} \leq \sqrt{n}F_b(M/\sqrt{n})$. By (2.6), $\sqrt{n}F_b(M/\sqrt{n})$ is a bounded sequence, say, bounded above by $K_M > 0$. Let $\delta > 0$ be arbitrary. Then
\[
P \left[ \sqrt{n} \int_0^T F_b(V_b^n(u-))d\bar{U}_b^n(u) > \delta, \|\bar{V}_b^n\|T \leq M \right] \leq P(\sqrt{n}F_b(M/\sqrt{n})) \bar{U}_b^n(T) > \delta \leq P(\bar{U}_b^n(T) > \delta/K_M) \to 0,
\]
where the convergence in the last step follows from (3.25). Then (3.38) holds. This completes the proof of (3.33). The proof of (3.34) will be similar and is omitted. \qed
Let $\omega M > C$-tightness. It suffices to prove $\lim_{n \to \infty} P[\omega(\hat{L}^n_b, \delta, T) > \epsilon] = 0$ for any $\epsilon > 0$ to obtain the desired C-tightness.

Let $\epsilon > 0$ be arbitrary. Since $\{\hat{V}^n_b\}$ is stochastically bounded, for a given constant $M > 0$, there is an integer $n_M$ so that $P[\|\hat{V}^n_b\|_T \geq M] < \epsilon$ whenever $n \geq n_M$. By (2.6), $\{\sqrt{n}F_b(M/\sqrt{n})\}_{n\in\mathbb{N}}$ is a bounded sequence, say, bounded above by $K_M > 0$. Since $\bar{E}^n_b(\cdot)$ is a non-negative increasing process, then on the set $[\|\hat{V}^n_b\|_T \leq M]$, we have $\sqrt{n} \int_s^t F_b(V^n_b(u))d\bar{E}^n_b(u) \leq K_M(\bar{E}^n_b(t) - \bar{E}^n_b(s))$ for every $0 \leq s \leq t \leq T$. Using this estimate together with (3.21) on the set $[\|\hat{V}^n_b\|_T \leq M]$ and when $s, t$ are in $[0, T]$, we obtain $|\hat{L}^n_b(t) - \hat{L}^n_b(s)| \leq \sup_{t \in [0, T]} |M^n(\bar{E}^n_b(t))| + K_M|\bar{E}^n_b(t) - \bar{E}^n_b(s)|$. Therefore, when $n \geq n_M$,

$$P[\omega(\hat{L}^n_b, \delta, T) > 2\epsilon] \leq P[\omega(\hat{L}^n_b, \delta, T) > 2\epsilon, \|\hat{V}^n_b\|_T \leq M] + P[\|\hat{V}^n_b\|_T > M]$$

$$\leq P \left[ \sup_{t \in [0, T]} |M^n(\bar{E}^n_b(t))| > \epsilon \right] + \epsilon + P[\omega(\bar{E}^n_b, \delta, T) > \epsilon/K_M] + \epsilon. \quad (3.39)$$

We know that $\lim_{n \to \infty} E[\sup_{t \in [0, T]} |M^n(\bar{E}^n_b(t))|^2] = 0$, which was established above (3.38), and $\lim_{n \to \infty} \sup_{t \in [0, T]} |\bar{E}^n_b(t) - \lambda t| = 0$ in probability from (3.26). Hence $P[\omega(\hat{L}^n_b, \delta, T) > 2\epsilon]$ in (3.39) converges to 0 as $n \to \infty$. This completes the proof. \qed

In the next proposition, we prove the C-tightness of $\{(\hat{G}^n_b, \hat{G}^n_s)\}_{n \in \mathbb{N}}$ in $D^2[0, T]$.\[Proposition 3.8.\] The process sequence $\{(\hat{G}^n_b, \hat{G}^n_s)\}_{n \in \mathbb{N}}$ is C-tight in $D^2[0, T]$, and furthermore, $\lim_{n \to \infty} \|\hat{L}^n_b - \hat{G}^n_b\|_T = 0$ and $\lim_{n \to \infty} \|\hat{L}^n_s - \hat{G}^n_s\|_T = 0$ in probability. Consequently,

$$\lim_{n \to \infty} \sup_{t \in [0, T]} |\hat{G}^n_s(t) - \sqrt{n} \int_0^t F_s(V^n_s(u))d\hat{A}^n_s(u)| = 0, \text{ in probability}, \quad (3.40)$$

$$\lim_{n \to \infty} \sup_{t \in [0, T]} |\hat{G}^n_b(t) - \sqrt{n} \int_0^t F_b(V^n_b(u))d\hat{A}^n_b(u)| = 0, \text{ in probability}. \quad (3.41)$$

\[Proof.\] The proof of this proposition is similar to the discussion in Section 4.4 of [8] and with appropriate changes, one can also easily follow the proof of Proposition 4.3 in [42]. Moreover, (3.40) and (3.41) follows directly by combining the first part of the proposition with (3.33) and (3.34) of Proposition 3.6. Hence, it will be omitted. \qed

### 3.4 Asymptotics for $\hat{X}^n$ and $(\hat{G}^n_b, \hat{G}^n_s)$

To obtain the tightness of the sequence of state processes $\{\hat{X}^n\}_{n \in \mathbb{N}}$, we are heavily dependent on the properties of the two-sided Skorokhod map in $D[0, T]$ as described in [5]. In particular, we make use of an oscillation inequality for the Skorokhod map in a time varying interval. The inequality is given in Proposition A.3 in Appendix A together with other results on the Skorokhod map.

\[Theorem 3.9.\] The processes $\{(\hat{X}^n, \hat{U}^n_b, \hat{U}^n_s)\}_{n \in \mathbb{N}}$ are C-tight in $D^3[0, T]$.\[21\]
Proof. To prove the C-tightness of \( \{\hat{X}^n\} \) in \( D[0,T] \), we will verify the two conditions of Theorem 13.2 in [3]. Proposition 3.2 implies the first condition on stochastic boundedness. To verify the second condition, by (3.1)–(3.7), notice that \( \hat{X}^n, U^n \) is the unique solution to the SP in \( D[0,T] \) for the input process \( \hat{\zeta}^n - \hat{G}^n \) on the time-dependent interval \( [\hat{l}^n, \hat{r}^n] \), where \( \hat{l}^n(t) = -m^n_b(t)/\sqrt{n} \) and \( \hat{r}^n(t) = m^n_s(t)/\sqrt{n} \). Since \( \hat{l}^n(t) \leq -\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \leq \hat{r}^n(t) \) for all \( t \geq 0 \), we can combine Corollary 2.4 and Theorem 2.6 of [5] to obtain the above unique solution. Hence by (A.13),

\[
\omega(\hat{X}^n, \delta, T) \leq 4[\omega(\hat{\zeta}^n - \hat{G}^n, \delta, T) + \omega(\hat{l}^n, \delta, T) + \omega(\hat{r}^n, \delta, T)].
\] (3.42)

Since \( \hat{\zeta}^n \) converges weakly to a process with continuous paths as in (3.3), and \( \{\hat{G}^n\} \) is C-tight from Proposition 3.8, using Corollary 3.33 of Chapter VI in [16], \( \{\hat{\zeta}^n - \hat{G}^n\} \) is C-tight, and it follows that \( \lim_{n \to \infty} P[\omega(\hat{\zeta}^n - \hat{G}^n, \delta, T) > \epsilon] = 0 \). Next, using (2.9), we have \( \lim_{\delta \to 0} \limsup_{n \to \infty} \omega(\hat{l}^n, \delta, T) = 0 \), and \( \lim_{\delta \to 0} \limsup_{n \to \infty} \omega(\hat{r}^n, \delta, T) = 0 \) in probability. Using these facts together with (3.42), we conclude \( \{\hat{X}^n\} \) is C-tight in \( D[0,T] \).

Now by (3.1), again from Corollary 3.33 of Chapter VI in [16], we conclude that \( \{\hat{U}^n\} \) is C-tight in \( D[0,T] \). Recall that \( \hat{U}^n(t) = \hat{U}^n_s(t) - \hat{U}^n_b(t) \) for all \( t \geq 0 \), and \( \hat{U}^n_s \) and \( \hat{U}^n_b \) are non-decreasing processes. We would like to show that the processes \( \hat{U}^n_s \) and \( \hat{U}^n_b \) are also tight in \( D[0,T] \). We prove it for \( \{\hat{U}^n_s\} \) and the proof for \( \{\hat{U}^n_b\} \) is similar. We know \( \hat{U}^n_b(0) = [\hat{m}^n_b(0) - \hat{X}^n(0-)]^+ \) is bounded (see Assumptions 2.1 and 2.5). By (A.22) of Corollary A.3, we have \( \omega(\hat{U}^n_b, \delta, T) \leq C[\omega(\hat{\xi}^n, \delta, T) + \omega(\hat{\zeta}^n, \delta, T)] \) and thus \( \lim_{\delta \to 0} \limsup_{n \to \infty} \omega(\hat{U}^n_b, \delta, T) = 0 \) in probability. Now we can follow Theorem 13.2 and its corollary in Chapter 3, page 140 of [3] (or see the discussion underneath Theorem 3.2 of [44]) to conclude \( \{\hat{U}^n_b\} \) is C-tight in \( D[0,T] \). □

In the following result, we build an asymptotic little’s law for the queue length process and the waiting time process, and the asymptotic linear relationship between the number of abandonments in \([0,t]\) and the integral of the queue length on \([0,t]\). The proof is related to similar results in [8] and [42].

**Proposition 3.10.** Let \( T > 0 \). We have the following convergence results.

(i) **Asymptotic little’s law.**

\[
\lim_{n \to \infty} E[\|\hat{X}^{n,+} - \lambda_0 \hat{V}^n_s\|_T] = 0, \quad \lim_{n \to \infty} E[\|\hat{X}^{n,-} - \lambda_0 \hat{V}^n_b\|_T] = 0,
\] (3.43)

(ii) **Asymptotic linear relationship between the abandonment and the integral of queue length.**

\[
\lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| \hat{G}^n_s(t) - \delta_s \int_0^t (\hat{X}^n(s))^+ ds \right| \right] = 0,
\] (3.44)

\[
\lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| \hat{G}^n_b(t) - \delta_b \int_0^t (\hat{X}^n(s))^+ ds \right| \right] = 0,
\] (3.45)

and consequently,

\[
\lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| \hat{G}^n(t) - \int_0^t [\delta_s (\hat{X}^n(s))^+ - \delta_b (\hat{X}^n(s))^+] ds \right| \right] = 0.
\] (3.46)
Proof. The idea of the proof of part (i) is similar to that of Theorem 4.5 (ii) in [42]. Any seller arrived by time \( t \) will be served by the time \( t + V^n_s(t) \) unless the seller has abandoned the system. Let \( K^n_s(t, t + r) \) be the number of sellers arrived after time \( t \), but abandoned the system by time \( t + r \) and let \( \hat{K}^n_s(t, t + r) = K^n_s(t, t + r)/\sqrt{n} \). Therefore,

\[
(X^n(t + V^n_s(t)))^+ = [A^n_s(t + V^n_s(t)) - A^n_s(t)] - [U^n_s(t + V^n_s(t)) - U^n_s(t)] - \hat{K}^n(t, t + V^n_s(t)),
\]

and

\[
(\hat{X}^n(t + V^n_s(t)))^+ = \lambda_0 \hat{V}^n_s(t) + [\hat{A}^n_s(t + V^n_s(t)) - \hat{A}^n_s(t)] - [\hat{U}^n_s(t + V^n_s(t)) - \hat{U}^n_s(t)] - \hat{K}^n(t, t + V^n_s(t)).
\]

Noting that \( 0 \leq \hat{K}^n(t, t + V^n_s(t)) \leq [\hat{L}^n_s(t + V^n_s(t)) - \hat{L}^n_s(t)] \), we have

\[
\| (\hat{X}^n(t + V^n_s(t)))^+ - \lambda_0 \hat{V}^n_s(t) \|_T \\
\leq \| \hat{A}^n_s(t + V^n_s(t)) - \hat{A}^n_s(t) \|_T + \| \hat{U}^n_s(t + V^n_s(t)) - \hat{U}^n_s(t) \|_T + \| \hat{L}^n_s(t + V^n_s(t)) - \hat{L}^n_s(t) \|_T.
\]

By (3.12), \( \lim_{n \to \infty} \| V^n_s \|_T = 0 \) in probability. Since \( \hat{A}^n_s, \hat{U}^n_s, \hat{L}^n_s \) and \( \hat{X}^n \) are all C-tight in \( D[0, T] \), the RHS of (3.47) tends to zero in probability. Hence \( \lim_{n \to \infty} \| \hat{X}^{n,+} - \lambda_0 \hat{V}^n_s \|_T = 0 \) in probability. Consequently, using (3.8) and (3.12), \( \lim_{n \to \infty} E[\| \hat{X}^{n,+} - \lambda_0 \hat{V}^n_s \|_T] = 0 \). The proof of the second result in part (i) is similar and is omitted.

We next establish (3.44) and the proof of (3.45) is similar. The proof consists of several steps. First we show that

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| G^n_s(t) - \sqrt{n} \delta_s \int_0^t V^n_s(u-)d\hat{A}^n_s(u) \| = 0 \quad \text{in probability.} \tag{3.48}
\]

With (3.40) in hand, it suffices to show

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \sqrt{n} \int_0^t \left| F_s(V^n_s(u-)) - \delta_s V^n_s(u-) \right| d\hat{A}^n_s(u) = 0 \quad \text{in probability.} \tag{3.49}
\]

Let \( \epsilon > 0 \) be arbitrary and \( I_{n,T} = \sup_{t \in [0, T]} | \sqrt{n} \int_0^t [F_s(V^n_s(u-)) - \delta_s V^n_s(u-)]d\hat{A}^n_s(u) | \). By (2.7), there is a \( \delta > 0 \) so that \( |F_s(x) - \delta_s x| < \epsilon x \) whenever \( 0 < x < \delta \). By (3.12), \( \{ V^n_s \} \) is stochastically bounded in \( D[0, T] \) and hence we can find \( M > 0 \) so that \( P[\| V^n_s \|_T \geq M] < \epsilon \) for all \( n \). We pick \( n_0 > 1 \) so that \( M/\sqrt{n_0} < \delta \). On the set \( \{ \| \hat{V}^n_s \|_T \leq M \} \), \( |F_s(V^n_s(u-)) - \delta_s V^n_s(u-)| \leq \epsilon V^n_s(u-) \) for all \( u \leq T \) when \( n > n_0 \). Hence, when \( n \geq n_0 \),

\[
E[I_{n,T}1_{\{\| \hat{V}^n_s \|_T \leq M\}}] \leq \epsilon E[\| \hat{V}^n_s \|_T \| \hat{A}^n_s \|_T].
\]

Since \( \epsilon \) is arbitrary, (3.22) and (3.12) together with the Holder’s inequality yield

\[
\limsup_{n \to \infty} E[I_{n,T}1_{\{\| \hat{V}^n_s \|_T \leq M\}}] = 0.
\]

Next, by (2.7) there is a \( K > \delta_s \) so that \( F_s(x) \leq Kx \) for all \( x \). Hence,

\[
I_{n,T}1_{\{\| \hat{V}^n_s \|_T > M\}} \leq K \| \hat{V}^n_s \|_T I_{n,T}1_{\{\| \hat{V}^n_s \|_T > M\}} \| \hat{A}^n_s \|_T.
\]

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By the Holder’s inequality, (3.22) and the uniform integrability of $\|\hat{V}_n\|_T$ from (3.12), we obtain $\limsup_{n \to \infty} E[I_n 1_{\{\|\hat{V}_n\|_T > M\}}] = 0$. Hence, (3.49) follows. Next, from part (i), we observe that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^t (\hat{X}_n(u))^+ - \lambda_0 \hat{V}_n(u^-) \right| d\hat{A}_n(u) = 0 \quad \text{in probability.}$$

(3.50)

Finally, we need to establish

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^t (\hat{X}_n(u))^+ (d\hat{A}_n(u) - \lambda_0 du) \right| = 0 \quad \text{in probability.}$$

(3.51)

We can use Lemma 4.6 of [42] with the use of Lipschitz function $g(x) = x^+$ and $T_n(t) = \hat{A}_n(t) - \lambda_0 t$ to obtain (3.51). Now combining (3.48), (3.50) and (3.51),

$$\sup_{t \in [0,T]} \left| \hat{G}_n(t) - \delta_n \int_0^t (\hat{X}_n(s))^+ ds \right| \to 0, \quad \text{in probability.}$$

(3.52)

Finally, to obtain (3.44), we establish the uniform integrability of the LHS of (3.52). Using (3.8) and (3.19),

$$E \left[ \sup_{t \in [0,T]} |\hat{G}_n(t) - \delta_n \int_0^t (\hat{X}_n(s))^+ ds|^2 \right] \leq 2E[(\hat{G}_n(T))^2 + (\delta_n T \|\hat{X}_n\|_T)^2] \leq C(1 + T^n),$$

where $C, m$ are positive constants independent of $n$ and $T$. Now (3.44) follows. The proof of (3.45) is similar, and (3.46) follows from (3.44) and (3.45). This completes the proof. \(\Box\)

### 3.5 Proof of Theorem 3.1

Part (i) follows from (3.3) in Remark 3.1. Introduce \(\hat{Z}^n = (\hat{X}_n, \hat{\zeta}_n, \hat{G}_n^b, \hat{G}_n, \hat{U}_n^b, \hat{U}_n)\) in $D^6[0,T]$. Then by (3.3), Proposition 3.8 and Theorem 3.9, each component of $\hat{Z}^n$ is C-tight. Using Corollary 3.33 of Chapter VI in [16], it follows that the sequence $\{\hat{Z}^n\}$ is C-tight in $D^6[0,T]$.

Let $Z = (X, \zeta, G_b, G_s, U_b, U_s)$ be any limit point of $\{\hat{Z}^n\}$ along a subsequence. Without loss of generality, we relabel the subsequence such that $\hat{Z}^n$ converges weakly to $Z$ as $n \to \infty$. The C-tightness guarantees the continuity of paths for $Z$. Then, using the Skorokhod representation theorem (see [9] Chapter 3, Theorem 1.8), we simply assume that $\lim_{n \to \infty} \hat{Z}_n(t) = Z(t)$ a.s. uniformly on $[0,T]$. Moreover, $U_b, U_s$ are non-decreasing. By (3.45), and the continuous mapping theorem, part (ii) follows. For part (iii), using (3.45), we can write

$$\hat{X}_n(t) = \hat{\zeta}_n(t) - \int_0^t h(\hat{X}_n(s)) ds - (\hat{U}_s^n(t) - \hat{U}_s^n(t)) + \epsilon_n(t),$$

(3.53)

where $h(x) = \delta_n x^+ - \delta_b x^-$ and $\lim_{n \to \infty} \|\epsilon_n\|_T = 0$ in probability. The processes $\hat{U}_s^n$ and $\hat{U}_s^n$ are monotone increasing, and with disjoint increment supports on the sets $\{X_n(s) > 0\}$ and $\{x_n(s) < 0\}$, respectively. Hence, $X$ satisfies $X(t) = x + \sigma B(t) + \int_0^t \beta - h(X_s(s)) ds - U(t)$ for each $0 \leq t \leq T$. The constants $\sigma$ and $\beta$ are as described in (3.3). The process $U$ is of bounded variation in $D^2[0,T]$ adapted to the filtration generated by $(X, B)$ and using Proposition 2.3 of [5], it can be expressed as $U(t) = U_s(t) - U_b(t)$, where $U_s(t) = \int_0^t 1_{\{X(r) > 0\}} dU_s(r)$ and $U_b(t) = \int_0^t 1_{\{X(s) < 0\}} dU_b(s)$. This completes the proof of part (iii).
4 Diffusion Control Problem (DCP)

4.1 Problem Formulation

We formulate a one-dimensional stochastic control problem for diffusion processes (i.e., the DCP) which can be considered as the limiting form of the cost minimization problem for the queueing systems (i.e., the QCP). An explicit solution for this DCP will be obtained here. In Section 5, we shall “translate” the optimal strategy of the DCP to obtain an asymptotically optimal strategy for the QCP.

We consider the limiting process of \((\hat{X}^n, \hat{U}^n)\) derived in Theorem 3.1. Rigorously, we define a controlled state-process

$$X_x(t) = x + \sigma B(t) + \int_0^t [\beta - h(X_x(s))] ds - U(t), \quad (4.1)$$

where \(X_x(0) = x\) is a real number as in (2.1), the parameters \(\sigma\) and \(\beta\) are constants as in Theorem 3.1 (i), \(B\) is a standard one-dimensional Brownian motion, adapted to a right-continuous filtration \(F = \{F_t : t \geq 0\}\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and the function

$$h(x) = \delta_x x^+ - \delta_b x^-, \quad (4.2)$$

with \(\delta_b\) and \(\delta_x\) being positive constants as in (2.7) and (2.6). Furthermore, the \(\sigma\)-algebra \(F_0\) contains all the null sets in \(\mathcal{F}\), the Brownian increments \(B(t + s) - B(t)\) are independent of \(F_t\) for all \(t \geq 0\) and \(s \geq 0\), and the control \(U\) is a right-continuous process with paths of bounded variation adapted to the filtration \(F\). It is assumed that \(U\) can be expressed as

$$U(t) = U_s(t) - U_b(t), \quad (4.3)$$

where \(U_s(t) = \int_0^t 1_{\{X(s) > 0\}} dU_s(s)\) and \(U_b(t) = \int_0^t 1_{\{X(s) < 0\}} dU_b(s)\), and the processes \(U_b\) and \(U_s\) are adapted, non-decreasing with RCLL paths. Thus no control can be enforced when the state process is at the origin where there is no holding cost in the cost structure. Throughout this section, the pair \((U_s, U_b)\) describes the control policy.

We introduce the following cost functional for the state process in (4.1):

$$J(X_x, U_s, U_b) = E \left( \int_0^{\infty} e^{-\alpha t} \left[ (\theta_s X_x^+(t) + \theta_b X_x^-(t)) dt + p_s dU_s(t) + p_b dU_b(t) \right] \right), \quad (4.4)$$

where \(\theta_s = c_s + r_s \delta_s, \theta_b = c_b + r_b \delta_b\) and \(\alpha, p_s,\) and \(p_b\) as well as \(c_s, r_s, \delta_s, c_b, r_b, \delta_b\) are positive constants as given in the QCP (2.13). Note that the holding cost

$$C(x) = \theta_s x^+ + \theta_b x^-, \quad x \in \mathbb{R}, \quad (4.5)$$

is a piecewise linear convex function.

For \(x \in \mathbb{R}\), we call the sextuple \(((\Omega, \mathcal{F}, \mathbb{P}), F, B, X_x, U_s, U_b)\) an admissible control system if

(i) \((X_x, U_s, U_b)\) is a weak solution to (4.1), and

(ii) the cost functional \(J(X_x, U_s, U_b)\) is finite.

When there is no ambiguity, we simply use \((X_x, U_s, U_b)\) to represent an admissible control system. To define the value function, we introduce the set

$$\mathcal{A}(x) = \{(X_x, U_s, U_b) : (X_x, U_s, U_b) \text{ is admissible}\}. \quad (4.6)$$
This set is nonempty for each \( x \) in \( \mathbb{R} \) since the zero control policy \( U_s = U_b = 0 \) leads to a diffusion process \( X_x \) for which \( J(X_x, 0, 0) \) is finite. The value function of the DCP is thus well defined and given by

\[
V(x) = \inf_{A(x)} J(X_x, U_s, U_b), \ x \in \mathbb{R}. \tag{4.7}
\]

At last we describe the formal HJB equation associated with our DCP. We introduce the differential operator \( \mathcal{G} \) by

\[
\mathcal{G} = \frac{\sigma^2}{2} \frac{d^2}{dx^2} + (\beta - h(x)) \frac{d}{dx} - \alpha, \tag{4.8}
\]

where the constants \( \alpha, \beta, \) and \( \sigma \) and the function \( h(x) \) are as described earlier in (4.1). The formal HJB-equation for the above control problem can now be written as

\[
\min \left\{ \mathcal{G} F(x) + C(x), \ F'(x) + p_b, \ p_s - F'(x) \right\} = 0, \tag{4.9}
\]

where \( C(x) \) is the holding cost function (4.5). Our proofs show that the value function \( V \) is the unique smooth solution to the HJB equation (4.9).

**A verification lemma.** The following verification lemma guarantees that any smooth function which satisfies the HJB equation (4.9) is a lower bound for the value function and helps us to identify an optimal strategy.

**Lemma 4.1.** Assume that \( F \in C^2(\mathbb{R}) \) and satisfies (4.9) on \( \mathbb{R} \). Then \( F(x) \leq V(x) \) for all \( x \in \mathbb{R} \), where \( V \) is the value function defined in (4.7).

**Proof.** Let \( (X_x, U_s, U_b) \) be an admissible control system, and \( F \in C^2(\mathbb{R}) \) satisfy (4.9) on \( \mathbb{R} \). Using the generalized Itô’s lemma (see [27], p. 285),

\[
F(X_x(T))e^{-\alpha T} = F(x) + \sigma \int_0^T e^{-\alpha s} F'(X_x(s-))dB(s) + \int_0^T e^{-\alpha s} \mathcal{G} F(X_x(s-))ds - \int_0^T e^{-\alpha s} F'(X_x(s-))dU(s) + \sum_{0 \leq s \leq T} e^{-\alpha s} \Delta F(X_x(s)) + F'(X_x(s-))\Delta U(s),
\]

where \( \Delta F(X_x(s)) = F(X_x(s)) - F(X_x(s-)), \) and \( \Delta U(s) = U(s) - U(s-) \). Let \( U^c_b, U^c_s, \) and \( U^c = U^c_s - U^c_b \) be the continuous parts of the processes \( U_b, U_s \) and \( U \), respectively. By (4.9), \( F'(x) \in [-p_b, p_s] \) and hence \( E[\int_0^T e^{-\alpha s} F'(X_x(s-))dB(s)] = 0 \) for each \( T \). Then by (4.10), we obtain

\[
E \left[ F(X_x(T))e^{-\alpha T} \right] = F(x) + E \left[ \int_0^T e^{-\alpha s} \mathcal{G} F(X_x(s-))ds \right] - E \left[ \int_0^T e^{-\alpha s} F'(X_x(s-))dU^c(s) \right] + E \left[ \sum_{0 \leq s \leq T} e^{-\alpha s} \Delta F(X_x(s)) \right]. \tag{4.11}
\]

When \( X_x(t) > 0 \) for some \( t > 0 \), \( \Delta X_x(t) = -\Delta U_b(t) \) and \( \Delta F(X_x(t)) = F'(\xi)\Delta X_x(t) \), where \( \xi \) is between \( X_x(t) \) and \( X_x(t-) \). Since \( F'(x) \leq p_s \), we obtain \( \Delta F(X_x(t)) \geq -p_s \Delta U_b(t) \).
$-p_s \Delta U_s(t) = -[p_s \Delta U_s(t) + p_b \Delta U_b(t)]$, where $\Delta U_b(t) = 0$ when $X_s(t) > 0$. By a similar argument when $X_s(t) < 0$, $\Delta F(X_s(t)) \geq -[p_s \Delta U_s(t) + p_b \Delta U_b(t)]$. Moreover, $\dot{F}'(X_s(t))dU_b(t) = F'(X_s(t-))(dU_b^+(t) - dU_b^-(t)) \leq p_s dU_b^+(t) + p_b dU_b^-(t)$. By (4.9), $\mathcal{G}F(x) \geq -C(x)$ holds for each $x \in \mathbb{R}$, and hence from (4.11),

$$E \left[ F(X_x(T))e^{-\alpha T} \right] \geq F(x) - E \left[ \int_0^T e^{-\alpha s} C(x(s)) ds \right] - E \left[ \int_0^T e^{-\alpha s}(p_s dU_s(s) + p_b dU_b(s)) \right].$$

Consequently, $E[F(X_x(T))e^{-\alpha T}] + J(X_x, U_s, U_b) \geq F(x)$ for each $x$ and $T > 0$. Since $\dot{F}'(x)$ is bounded, there exist some $c_1, c_2 > 0$, $E[|F(X_x(T))|e^{-\alpha T}] \leq (c_1 + c_2 E[X_x(T)])e^{-\alpha T}$. Noting that $xh(x) \geq 0$ and $X(t)dU(t) \geq 0$, we can apply Itô’s lemma for $X_x(t)^2$ to obtain $\lim_{T \to \infty} e^{-\alpha T} E[|X_x(T)|^2] = 0$. Hence $\lim_{T \to \infty} e^{-\alpha T} E[F(X_x(T))] = 0$. Therefore, $J(X_x, U_s, U_b) \geq F(x)$ for each $x$. Consequently, $V(x) \geq F(x)$ holds for each $x$. This completes the proof. \(\Box\)

### 4.2 Optimal solutions of the DCP

When the blocking costs $p_s$ and $p_b$ are high relative to the holding and abandonment costs, blocking customers is not cost effective, whereas if the costs $p_s$ and $p_b$ are sufficiently low, then it is optimal to use finite buffer sizes. Here we clarify the threshold values for $p_s$ and $p_b$ and establish a threshold optimal solution to the DCP under four different regimes.

**Theorem 4.2.** Let $T_s = \theta_s/(\alpha + \delta_s)$ and $T_b = \theta_b/(\alpha + \delta_b)$. Then the DCP admits the following optimal solution.

(i) When $p_s \geq T_s$ and $p_b \geq T_b$, an optimal strategy is given by the zero control policy, which is described by $U^*_s(t) = U^*_b(t) = 0$ for all $t$, with the state equation

$$X^*_x(t) = x + \sigma B(t) + \int_0^t [\beta - h(X^*_x(s))] ds, \quad t \geq 0. \quad (4.12)$$

(ii) When $0 < p_s < T_s$ and $0 < p_b < T_b$, there exist two points $a^* < b^* < 0$ such that the reflected diffusion process $X^*_x$ on $[a^*, b^*]$ described by

$$X^*_x(t) = x + \sigma B(t) + \int_0^t [\beta - h(X^*_x(s))] ds - L_{b^*}(t) + L_{a^*}(t) \quad (4.13)$$

is an optimal state process, and the optimal control pair $(U^*_s, U^*_b)$ is given by the local-time processes $(L_{b^*}, L_{a^*})$.

(iii) When $p_s \geq T_s$ and $0 < p_b < T_b$, there exists a point $\tilde{a}^* < 0$ such that the reflected diffusion process $X^*_x$ on $[\tilde{a}^*, \infty)$ described by

$$X^*_x(t) = x + \sigma B(t) + \int_0^t [\beta - h(X^*_x(s))] ds + L_{\tilde{a}^*}(t), \quad (4.14)$$

is an optimal state process, and the optimal control pair $(U^*_s, U^*_b)$ is given by the local-time processes $(0, L_{\tilde{a}^*})$. 

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(iv) When $0 < p_s < T_s$ and $p_b \geq T_b$, there exists a point $\hat{b}^* > 0$ such that the reflected diffusion process $X^*_x$ on $(-\infty, \hat{b}^*)$ described by

$$X^*_x(t) = x + \sigma B(t) + \int_0^t [\beta - h(X^*_x(s))]ds - L_{\hat{b}^*}(t)$$

is an optimal state process, and the optimal control pair $(U^*_s, U^*_b)$ is given by the local-time processes $(L_{\hat{b}^*}, 0)$.

**Remark 4.1.** In Theorem 4.2 (ii), (iii) and (iv), if the initial value $x$ is not in the desired interval, there will be an initial jump to the nearest point in the interval. For example, in (ii), if $x$ is outside the interval $[a^*, b^*]$, then there is an initial jump to $a^*$ when $x < a^*$ or $b^*$ when $x > b^*$.

The proof of Theorem 4.2 relies on the construction of solutions to the HJB equation (4.9) with different boundary conditions. In the following Sections 4.2.1 and 4.2.2, we construct the solution of the HJB (4.9) and provide the proof of Theorem 4.2. At last, in Section 4.2.3, we summarize how the optimal buffer sizes $a^*, b^*, \tilde{a}^*$ and $\tilde{b}^*$ are computed and present a numerical example.

### 4.2.1 Optimality of Zero Control

In this section we verify that the zero control (no blocking of customers) strategy is optimal when $p_b$ and $p_s$ are above the threshold values $T_b$ and $T_s$. To this end, we need to construct a solution $Q$ to (4.9) of the the form:

$$GQ(x) + C(x) = 0 \text{ and } -p_b < Q'(x) < p_s, \quad x \in \mathbb{R}. \quad (4.16)$$

Next let $\gamma(x) = h'(x)$ and it can be written in the form

$$\gamma(x) = \delta_b 1_{\{x < 0\}} + \delta_s 1_{\{x > 0\}}, \quad x \in \mathbb{R}\setminus\{0\}. \quad (4.17)$$

We let $W(x) = Q'(x)$ and differentiating $Q$, we obtain

$$G W(x) - \gamma(x) W(x) + C'(x) = 0, \quad x \in \mathbb{R}\setminus\{0\}. \quad (4.18)$$

First we observe that $W(x) = -T_{\hat{b}}$ is a constant solution on $(-\infty, 0)$, and $W(x) = T_s$ is a constant solution on $(0, \infty)$. They play an important role in our analysis. Since $\alpha Q(0) = \frac{\sigma^2}{2} W'(0) + \beta W(0), Q$ can be obtained by knowing $W$. Therefore, we construct a sufficiently smooth bounded solution $W$ to (4.18), with the boundary data

$$W(-\infty) = -T_{\hat{b}} \text{ and } W(\infty) = T_s. \quad (4.19)$$

The following proposition presents the existence and uniqueness of such a function $W$. We defer its proof in Appendix B.

**Proposition 4.3.** There exists a unique function $W \in C^1(\mathbb{R})$ satisfying (4.18) and (4.19) with the following properties.

(i) The function $W$ is bounded and strictly increasing.

(ii) $W$ is twice continuously differentiable everywhere except at the origin.
(iii) The one sided limits $W''(0-)\) and $W''(0+)$ exist and they satisfy

$$\frac{\sigma^2}{2} W''(0-) + \beta W'(0) = (\alpha + \delta_b) W(0) + \theta_b, \quad (4.20)$$

$$\frac{\sigma^2}{2} W''(0+) + \beta W'(0) = (\alpha + \delta_s) W(0) - \theta_s. \quad (4.21)$$

**Proof of Theorem 4.2 (i).** Using the function $W$ derived in Proposition 4.3, we introduce the function $Q$ by

$$\alpha Q(x) = \frac{\sigma^2}{2} W'(0) + \beta W(0) + \alpha \int_0^x W(u) du, \quad x \in \mathbb{R}.$$ 

Moreover, $Q'(x) \equiv W(x)$ is strictly increasing. Then it is straightforward to check that $Q$ satisfies $\mathcal{G}Q(x) + C(x) = 0$, and $-T_b < Q'(x) < T_s$.

By the assumptions in Theorem 4.2 (i), $-p_b < Q'(x) < p_s$ for all $x$ and thus $Q \in C^2(\mathbb{R})$ and is a convex function which satisfies (4.9). Therefore, by using the verification Lemma 4.1, we can conclude that $Q(x) \leq V(x)$ for all $x$.

Next we consider the state process $X_x$ described in (4.12) which corresponds to zero control strategy. We apply the Itô’s lemma to $Q(X_x(t))e^{-\alpha t}$ and use the facts that $Q'$ is bounded and $Q$ satisfies the HJB equation (4.9) to obtain

$$Q(X_x(T))e^{-\alpha T} = Q(x) - E \left( \int_0^T e^{-\alpha s} C(X_x(s)) ds \right). \quad (4.22)$$

From the proof of Lemma 4.1, we have $\lim_{T \to \infty} e^{-\alpha T} E[|X_x(T)|] = 0$. Since $Q'(\cdot)$ is bounded, it follows that $\lim_{T \to \infty} e^{-\alpha T} E[Q(X_x(T))] = 0$. Therefore, by letting $T$ tends to infinity in (4.22), we have $Q(x) = E \left[ \int_0^\infty e^{-\alpha s} C(X_x(s)) ds \right]$, which says $Q(x)$ represents the pay-off function from the zero control strategy. Hence $Q(x) \geq V(x)$. This completes the proof. $\square$

### 4.2.2 Optimality of Reflected Diffusion Processes

We first focus on the case when both costs $p_b$ and $p_s$ are below the threshold values $T_b$ and $T_s$. The following proposition lays the groundwork to derive the optimal policy in Theorem 4.2 (ii), and its proof is presented in Appendix B.

**Proposition 4.4.** Let the cost parameters $p_b$ and $p_s$ satisfy $0 < p_b < T_b$ and $0 < p_s < T_s$ Then there exist two points $a^* < 0 < b^*$ and a function $W^*: [a^*, b^*] \to \mathbb{R}$ satisfying the following free boundary problem:

$$\mathcal{G} W^*(x) - \gamma(x) W^*(x) + C'(x) = 0, \quad a^* \leq x \leq b^*, \quad (4.23)$$

$$W^*(a^*) = -p_b, \quad W^*(b^*) = p_s, \quad \text{and} \quad W^{**}(a^*) = W^{**}(b^*) = 0. \quad (4.24)$$

Moreover, $-p_b < W^*(x) < p_s$ when $a^* < x < b^*$.

**Proof of Theorem 4.2 (ii).** We first extend the function $W^*$ in Proposition 4.4 to $\mathbb{R}$ by defining $W^*(x) = -p_b$ if $x < a^*$ and $W^*(x) = p_s$ if $x > b^*$. Hence, $W^* \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{a^*, b^*\})$. Then introduce $Q \in C^2(\mathbb{R})$ by

$$\alpha Q(x) = \left[ \frac{\sigma^2}{2} W''(0+) + \beta W'(0) \right] + \alpha \int_0^x W^*(u) du, \quad x \in \mathbb{R}.$$
To verify $Q$ satisfies (4.9), we notice $W^*$ satisfies (4.23) in $[a^*, b^*]$. Moreover, $\mathcal{G}Q(b^*) + C(b^*) = 0$ and for $x > b^*$,

$$[\mathcal{G}Q(x) + C(x)] - [\mathcal{G}Q(b^*) + C(b^*)] = (\alpha + \delta_s) \left[ \frac{\theta_s}{\alpha + \delta_s} - p_b \right] (x - b^*) > 0.$$  

Thus $\mathcal{G}Q(x) + C(x) > 0$ and $Q'(x) = p_s$ on $(b^*, \infty)$. Consequently, $Q$ satisfies (4.9) on $[b^*, \infty)$. Similarly, $Q$ satisfies (4.9) on $(-\infty, a^*)$. Therefore, $Q \in C^2(\mathbb{R})$ satisfies the HJB equation (4.9) and by the verification Lemma 4.1, we obtain $Q(x) \leq V(x)$ for $x \in \mathbb{R}$.

Next, we show that $X^*_x$ in (4.13) yields the pay-off $Q(x)$. For this, first we consider $x \in [a^*, b^*]$ and apply Itô’s lemma to $Q(X^*_x(t))e^{-\alpha t}$ to obtain

$$Q(x) = E \left[ Q(X^*_x(T))e^{-\alpha T} \right] + E \left( \int_0^T e^{-\alpha t} [C(X^*_x(t))dt + p_b dL_a^*(t) + p_a dL_b^*(t)] \right).$$

Since $Q$ is bounded on $[a^*, b^*]$, $\lim_{T \to \infty} E[Q(X^*_x(T))e^{-\alpha T}] = 0$. Hence, when $x \in [a^*, b^*]$,

$$Q(x) = E \left( \int_0^\infty e^{-\alpha t} [C(X^*_x(t))dt + p_b dL_a^*(t) + p_a dL_b^*(t)] \right).$$

It is evident that the process $(X^*_x, L^*_a, L^*_b)$ forms an admissible policy. When $x > b^*$, there is an initial jump of $X^*$ to $b^*$ so that $L^*_a(0+) = (x - b^*)$. Then it follows that $Q(x) = Q(b^*) + p_a(x - b^*)$. Similar analysis follows for the case $x < a^*$. Consequently, $Q(x) \geq V(x)$ holds and the process $X^*_x$ together with the controls $L^*_a$ and $L^*_b$ describes an optimal policy. This completes the proof. \(\square\)

We now discuss the situations where one-sided reflected diffusion processes are optimal.

**Proof of Theorem 4.2 (iii) and (iv).** To prove part (iii), we find a point $\tilde{a}^* < 0$ and construct a bounded strictly increasing function $W_L : [\tilde{a}^*, \infty) \to \mathbb{R}$ satisfying

$$\begin{align*}
\mathcal{G}W_L(x) - \gamma(x)W_L(x) + C^r(x) &= 0 \text{ for } x > \tilde{a}^*, \\
W_L(\tilde{a}^*) &= -p_b, \quad W'_L(\tilde{a}^*) = 0, \quad W_L(\infty) = T^r.
\end{align*}$$

(4.25)

This function $W_L(\cdot)$ will be obtained in the Appendix B (see Remark B.2). Thereafter, we can essentially follow the proof of Theorem 4.2 (ii), since this is essentially the case $b^* = \infty$. Hence, we omit the details. Proof of part (iv) is quite similar and is omitted. This completes the proof. \(\square\)

### 4.2.3 Computing the optimal buffer sizes

We first assume that both blocking cost parameters are below the thresholds, i.e., $0 < p_s < T_s$ and $0 < p_b < T_b$. We summarize the solution construction process in Appendix B and find $a^*$ and $b^*$ in Theorem 4.2 (ii).

- For $a < 0$, we consider a function $W_a$ that satisfies the linear equation (4.18) on $(-\infty, 0]$, and $W_a(a) = -p_b$ and $W_a'(a) = 0$. For each $a < 0$, we can extend $W_a$ to $(0, \infty)$ so that it satisfies (4.18) on $(0, \infty)$ with the available initial data $W_a(0)$ and $W_a'(0)$. More precisely, $W_a$ satisfies

$$\begin{align*}
\frac{\sigma^2}{2} W''_a(x) + (\beta + \delta_b x)W'_a(x) - (\alpha + \delta_b)W_a(x) - \theta_b &= 0, \quad x \leq 0, \\
\frac{\sigma^2}{2} W''_a(x) + (\beta - \delta_s x)W'_a(x) - (\alpha + \delta_s)W_a(x) + \theta_s &= 0, \quad x > 0,
\end{align*}$$

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with the given $W_a(a) = -p_b$ and $W'_a(a) = 0$.

- We summarize some properties of $W_a(\cdot)$ for $a < 0$.
  - When $|a|$ is sufficiently large, from Lemma B.4, $\lim_{x \to \infty} W_a(x) = \infty$ and $W_a(\cdot)$ is strictly increasing on $(a, \infty)$.
  - Let
    
    $$c = \sup \{ a < 0 : \lim_{x \to \infty} W_a(x) = \infty \}$$

    From Lemma B.5, $c < 0$ and $\lim_{x \to \infty} W_c(x) = T_s$.
  - From Lemma B.7, for $a \in (c, 0)$, $\lim_{x \to \infty} W_a(x) = -\infty$ and $W_a(\cdot)$ has a unique maximum on $(0, \infty)$.

- For $a \in (c, 0)$, introduce
  
  $$M(a) = \max_{x \geq a} W_a(x), \quad r_a = \arg \max_{x \geq a} W_a(x).$$

  Then $a^*$ is chosen from the interval $(c, 0)$ such that $M(a^*) = p_s$ (this can be achieved because $p_s < T_s$) and $b^* = r_{a^*}$.

- Let $W^*(t) = W_{a^*}(t), t \in \mathbb{R}$. Then $W^*$ satisfies all conditions in Proposition 4.4.

Consider now the case when one cost parameter is below the threshold and the other one is above its threshold. In the aforementioned construction process, when $p_s$ increases and approaches the threshold $T_s$, the corresponding $a^*$ will approach $c$, and the $b^*$ will grow to infinity. More precisely, when $0 < p_b < T_b$ and $p_s \geq T_s$, we have $\tilde{a}^* = c$, where $c$ is as in (4.26). When $0 < p_s < T_s$ and $p_b \geq T_b$, one can switch the two sides, and calculating the corresponding $c$.

![Plots of $W_a(x)$ for $x \geq a$](image)

Figure 1: The functions $W_a(x), x \geq a$, for different values of $a$.

**Example 4.1.** We set $\sigma^2 = 1, \beta = 2, \alpha = 1, \delta_b = 2, \delta_s = 4, \theta_b = 4, \theta_s = 5$, and $p_b = 0.4, p_s = 0.1$. It is easily seen that $p_b < T_b = \theta_b/(\alpha + \delta_b) = 4/3$ and $p_s < T_s = \theta_s/(\alpha + \delta_s) = 5/3$. However, $p_s < T_s$ is not necessarily true in all cases.
\( \theta_s/(\alpha + \delta_s) = 1. \) We present two figures. In Figure 1, the plots of \( W_a(x), x \geq a \) for \( a = -1.2, -1.1, \ldots, -0.7, -0.6 \) are derived along with the value \( c = -0.944. \) One can observe that when \( a < c, W_a(x) \to \infty \) as \( x \to \infty, \) while when \( a \in (c, 0) \), \( W_a(x) \to -\infty \) as \( x \to \infty. \) In Figure 2, we find the values \( a^* = -0.5248 \) and \( b^* = 0.1104. \)

![Plot of \( W_a(x) \), where \( a^* = -0.5248 \)](image)

**Figure 2:** Finding the values \( a^* \) and \( b^*. \)

**Example 4.2.** We set \( \sigma^2 = 1, \beta = 2, \alpha = 1, \delta_b = 2, \delta_s = 4, \theta_b = 4 \) and \( \theta_s = 5 \) as in Example 4.1. We consider different values of \( p_b \) and \( p_s \) and observe the corresponding changes of the optimal buffer sizes. For convenience, we fix the value of \( p_b \), and let the value of \( p_s \) vary. In this setting, the value of \( c \) does not change. We summarize the numerical results in the following table. We observe that as \( p_s \) approaches \( T_s, \) both \(-a^*\) and \( b^*\) increase and in particular, \( a^* \) goes to \( c.\)

| \( p_s \) | \( p_b \) | \( T_b \) | \( T_b \) | \( c \) | \( a^* \) (buyer side) | \( b^* \) (seller side) |
|---------|---------|---------|---------|-------|---------------------|---------------------|
| 0.1     | 0.4     | 1       | 4/3     | -0.9440 | -0.5248             | 0.1104              |
| 0.3     | 0.4     | 1       | 4/3     | -0.9440 | -0.6568             | 0.1333              |
| 0.5     | 0.4     | 1       | 4/3     | -0.9440 | -0.7707             | 0.1935              |
| 0.7     | 0.4     | 1       | 4/3     | -0.9440 | -0.8671             | 0.2876              |
| 0.9     | 0.4     | 1       | 4/3     | -0.9440 | -0.9345             | 0.5501              |

Table 1: The value of \( p_b \) is fixed to be 0.4 and the value of \( p_s \) varies from 0.1 to 0.9 approaching \( T_s = 1.\)

5 **Asymptotic Optimality**

In this section, we establish the following two main results: In the first theorem (see Theorem 5.1), we prove the value function \( V(x) \) of the DCP given in (4.7) is an asymptotic lower bound for the sequence of value functions \( \{ \hat{V}^n(x) \} \) of the QCPs given in
We first show (5.1). In (2.13), using the Fubini’s theorem, we have

\[ E \left( \int_0^\infty e^{-\alpha t} G_s^n(t) \right) = E \left( \int_0^\infty \int_t^\infty e^{-\alpha u} dG_s^n(t) \right) = \alpha E \left( \int_0^\infty G_s^n(u)e^{-\alpha u} du \right). \] (5.3)

Using (3.8) and (3.19), we have

\[ E[\sup_{t \in [0,T]} |\hat{G}_s^n(t) - \delta_s \int_0^t (\hat{X}^n(s))^+ ds|^2] \leq C(1 + T^m) \]

where the constant \( C > 0 \) and the integer \( m > 1 \) are independent of \( n \) and \( T \). Hence using (3.44) and Fubini’s theorem, we conclude that

\[ \lim_{n \to \infty} E \left( \int_0^\infty e^{-\alpha t} \left| \hat{G}_s^n(t) - \delta_s \int_0^t (\hat{X}^n(s))^+ ds \right| dt \right) = 0. \] (5.4)

By reversing the above procedure, we have

\[ \alpha E \left( \int_0^\infty e^{-\alpha t} \int_0^t (\hat{X}^n(s))^+ dsdt \right) = E \left( \int_0^\infty e^{-\alpha t} (\hat{X}^n(t))^+ dt \right). \] (5.5)

From (5.3) – (5.5), \( \lim_{n \to \infty} |E \int_0^\infty e^{-\alpha t} \hat{G}_s^n(t) - \delta_s E \int_0^\infty e^{-\alpha t} (\hat{X}^n(t))^+ dt| = 0 \), and similarly, \( \lim_{n \to \infty} |E \int_0^\infty e^{-\alpha t} \hat{G}_b^n(t) - \delta_b E \int_0^\infty e^{-\alpha t} (\hat{X}^n(t))^+ dt| = 0 \). Recall the following parameters from the DCP (4.4), \( \theta_s = c_s + r_s \delta_s > 0 \) and \( \theta_b = c_b + r_s \delta_b > 0 \), and the effective running cost function \( C(x) = \theta_s x^+ + \theta_b x^-, x \in \mathbb{R} \), defined in (4.5). Hence, the cost functional can be written in the form of (5.1).

Now fix \( x \in \mathbb{R} \). Let \((X_x, U_s, U_b)\) be an arbitrary admissible strategy in (4.6). We first establish \( E[X_x(t)^2] \leq x^2 + Kt \) for all \( t \geq 0 \), where \( K > 0 \) is a constant independent
of \( t \) as well as the strategy \((X_t, U_s, U_b)\). Using the Itô’s formula to \(X_x(t)^2\) in (4.1) and the fact \(X_x(t)du(t) \geq 0\), we obtain

\[
E[X_x(t)^2] \leq x^2 - 2E\left(\int_0^t [\delta_0 X_x(s)^2 - \beta X_x(s)]ds\right),
\]

(5.6)

where \(\delta_0 = \min\{\delta_b, \delta_s\} > 0\). Notice that \(\delta_0 y^2 - \beta y = \delta_0 (y - c_0)^2 - c_0^2 \geq -K\), where \(c_0 = \beta/(2\delta_0)\) and \(K = \delta_0 c_0^2 = \beta/(2\delta_0)\). Using this in the RHS of (5.6), we obtain

\[
E[X_x(t)^2] \leq x^2 + Kt\quad\text{for all } t \geq 0.
\]

Let \(\epsilon > 0\) be arbitrary and pick \(T_0 > 0\) so that \(\int_{T_0}^{\infty} e^{-\alpha t} (x^2 + Kt)dt < \epsilon\). We now consider \(T > T_0\). Let \((\hat{X}_n, \hat{U}_s^n, \hat{U}_b^n)\) be an admissible process which satisfies (3.53). By Theorem 3.1, we consider any limit point \((X_n, U_s, U_b)\) in \(D^3[0,T]\) which satisfies (3.4). Using (5.1), integration by parts, and the Fatou’s lemma, we obtain

\[
\liminf_{n \to \infty} \hat{J}^n(\hat{X}_n(0), \hat{U}_s^n, \hat{U}_b^n) = \liminf_{n \to \infty} E\left(\int_0^\infty e^{-\alpha t}[C(\hat{X}_n(t))dt + p_s d\hat{U}_s^n(t) + p_b d\hat{U}_b^n(t)]\right)
\]

\[
= \liminf_{n \to \infty} \left[ E\left(\int_0^\infty e^{-\alpha t} C(\hat{X}_n(t))dt\right) + \alpha E\int_0^\infty e^{-\alpha t}[p_s \hat{U}_s^n(t) + p_b \hat{U}_b^n(t)]dt\right]
\]

\[
\geq E\left(\int_0^T e^{-\alpha t} C(X_x(t))dt\right) + \alpha E\int_0^T e^{-\alpha t}[p_s U_s(t) + p_b U_b(t)]dt
\]

\[
= E\left(\int_0^T e^{-\alpha t}[C(X_x(t))dt + p_s dU_s(t) + p_b dU_b(t)]\right)
\]

Next, we extend \((X_x, U_s, U_b)\) on \([0, T]\) to the process \((\hat{X}_x, \hat{U}_s, \hat{U}_b)\) on \([0, \infty)\) in \(D^3[0, \infty)\) by letting \(\hat{U}_s(t) = U_s(T)\) and \(\hat{U}_b(t) = U_b(T)\) for all \(t \geq T\) and \((X_x, U_s, U_b)\) satisfies (3.4). Then \((\hat{X}_x, \hat{U}_s, \hat{U}_b)\) is an admissible strategy for the DCP in (4.6) and \(J(\hat{X}_x, \hat{U}_s, \hat{U}_b)\) satisfies \(J(\hat{X}_x, \hat{U}_s, \hat{U}_b) \leq E\left(\int_0^T e^{-\alpha t}[C(X_x(t))dt + p_s dU_s(t) + p_b dU_b(t)]\right) + M\epsilon\), where \(M = \max\{\theta_s, \theta_b\}\). Hence \(\liminf_{n \to \infty} \hat{J}^n(\hat{X}_n(0), \hat{U}_s^n, \hat{U}_b^n) + M\epsilon > J(\hat{X}_x, \hat{U}_s, \hat{U}_b) \geq V(x)\), for every \(x\) and \(\epsilon > 0\). Letting \(\epsilon \to 0\), then (5.2) follows and the proof is complete.

Our next aim is to construct an asymptotically optimal sequence of processes \((\hat{X}_n, \hat{U}_s^n, \hat{U}_b^n)\) to achieve the lower bound in (5.2). There will be four different types of asymptotically optimal state process sequences corresponding to the four types of optimal controls for the DCP constructed in Theorem 4.2.

For each \(n \geq 1\), we consider a diffusion-scaled state process \((\hat{X}_n, \hat{U}_s^n, \hat{U}_b^n)\) which satisfies (2.12) with time-independent reflection barriers at \(a^*_n < 0 < b^*_n\) and initial data \(\hat{X}_n(0) = x_n\). Here \(\hat{X}_n(t)\) takes values in a lattice of the form \(\{j/\sqrt{n} : j \text{ is an integer}\}\) and hence, the initial position \(x_n\), as well as the reflecting barriers \(a^*_n\) and \(b^*_n\) are assumed to take values in the same lattice as well. We choose the sequences \(\{a^*_n\}\) and \(\{b^*_n\}\) in the lattice so that \(\lim_{n \to \infty} a^*_n = a^*\) and \(\lim_{n \to \infty} b^*_n = b^*\), where \(a^* < 0 < b^*\). In the following lemma, for the simplicity of the presentation, we assume \(a^*_n \leq x_n \leq b^*_n\), but if \(x_n\) is outside the interval \([a^*_n, b^*_n]\), then there is an initial jump at time \(0^-\) to the nearest point in \(\{a^*_n, b^*_n\}\) from \(x_n\). Since the jump size of this possible jump is \(\min\{|x_n - a^*_n|, |x_n - b^*_n|\}\) and it is bounded, the conclusion of the lemma clearly holds for this general case.

**Lemma 5.2.** Let \(n \geq 1\) and \((\hat{X}_n, \hat{U}_s^n, \hat{U}_b^n)\) satisfy (2.12) with time-independent reflection barriers at \(a^*_n < 0 < b^*_n\) and initial data \(\hat{X}_n(0) = x_n \in [a^*_n, b^*_n]\). We assume there is a \(\delta > 0\) so that \(a^*_n < -\delta < 0 < \delta < b^*_n\) for all \(n\), and for each \(n\), the process \(\hat{X}_n(t)\) and
constraint processes, we consider for $\zeta < b_0$ (c) of [39] we obtain for some $f$

**Proof.** We notice (6) of Theorem 5.3.

When (ii) $p \geq T_0$, the input process $\tilde{G}(t) - \tilde{G}^n(t)$ as given in (3.2) and (3.6). To use the comparison theorem for the control policy $\tilde{A}$.

From (5.8) and (5.9), we have

\[ \hat{U}^n_b \leq \tilde{U}^n_b \leq \hat{U}^n_s + G^n_s, \quad \hat{U}^n_s \leq \hat{U}^n_t + G^n_s. \]  

Next using the comparison theorem for $\tilde{X}^n$ and $\hat{Z}^n$, we have

\[ \hat{U}^n_b \leq \hat{U}^n_b \leq \hat{U}^n_s + \hat{G}^n_s, \quad \hat{U}^n_s \leq \hat{U}^n_t + \hat{G}^n_s. \]  

From (5.8) and (5.9), we have

\[ \hat{U}^n_b \leq \hat{U}^n_b \leq \hat{U}^n_s + \hat{G}^n_s. \]  

We note that the processes $A^n_b$ and $A^n_s$ are independent. Hence $\tilde{A}^n_b(t) - \hat{A}^n_b(t)$ is a pure jump process with jump size $\pm 1/\sqrt{n}$. Next, we rely on the oscillation inequalities (see Proposition 4.1 part (ii) (e) in [39] or [7]) to obtain an upper bound for $\hat{U}^n_b(T)$. For any $f$ in $D[0, T]$, let $Osc(f, T)$ be defined by (1.1), then using Proposition 4.1 part (e) of [39] we obtain for some $\kappa > 0$ independent of $n$ and $T$ so that

\[ \hat{U}^n_s(T) \leq \kappa \left( Osc(x_n + \hat{A}^n_s - \hat{A}^n_b, T) + \frac{1}{\sqrt{n}} \right) \leq \kappa (2\|\hat{A}^n_b\|T + 2\|\hat{A}^n_b\|T + 1). \]  

A similar estimate can be obtained for $\hat{U}^n_b(T)$. Next, using this estimate, (5.10), (3.9) and Proposition 3.8 (i), we can easily obtain (5.7). This completes the proof. □

**Theorem 5.3 (Asymptotic optimality).** Under Assumptions 2.1 – 2.5, the following results hold. Let $a^*_n, b^*_n$ be as in Theorem 4.2.

(i) When $p_s \geq T_s$ and $p_b \geq T_b$, the sequence of processes $(\hat{X}^n, \hat{U}^n_s, \hat{U}^n_b)$ with the zero control policy $\hat{U}^n_s(t) \equiv 0$, and $\hat{U}^n_b(t) \equiv 0$ for all $t \geq 0$ is asymptotically optimal.

(ii) When $0 < p_s < T_s$ and $0 < p_b < T_b$, for $a^*_n < a^* < 0 < b^* < b^*_n$ satisfying $\lim_{n \to \infty} a^*_n = a^*$ and $\lim_{n \to \infty} b^*_n = b^*$, the associated sequence of the two-sided reflected processes $(\tilde{X}^n, \tilde{U}^n_s, \tilde{U}^n_b)$ with reflecting boundaries at $a^*_n$ and $b^*_n$ provides an asymptotically optimal sequence.
(iii) When \( p_s \geq T_s \) and \( 0 < p_b < T_b \), for \( \bar{a}_n^* < \bar{a}^* < 0 \) satisfying \( \lim_{n \to \infty} \bar{a}_n^* = \bar{a}^* \),
the corresponding sequence of the one-sided reflected processes \( \hat{X}^n, \hat{U}_s^n, \hat{U}_b^n \) with
reflecting boundary at \( \bar{a}_n^* \) yields an asymptotically optimal sequence.

(iv) When \( 0 < p_s < T_s \) and \( p_b \geq T_b \), for \( \hat{b}_n^* > \hat{b}^* > 0 \) satisfying \( \lim_{n \to \infty} \hat{b}_n^* = \hat{b}^* \),
the corresponding sequence of the one-sided reflected processes \( \hat{X}^n, \hat{U}_s^n, \hat{U}_b^n \) with
reflecting boundary at \( \hat{b}_n^* \) yields an asymptotically optimal sequence.

**Proof.** Let \( x \) be fixed. By Theorem 4.2 (i), zero control strategy is optimal for the
DCP when \( p_s \geq T_s \) and \( p_b \geq T_b \). The corresponding state process \( X_x \) is the unique strong solution to (4.12). Moreover, the value function of the DCP is given by \( V_{\text{DCP}} \) when

\[
E[\cdot] \quad \text{and} \quad \hat{\epsilon} \quad \text{where} \lim_{\epsilon \to 0} \hat{\epsilon} > 0 \quad \text{obtain} \lim \inf_{n \to \infty} \alpha_{nT} \neq 0 \quad \text{is arbitrary,} \lim \inf_{n \to \infty} \\|\alpha_{nT}\|T = 0 \quad \text{in probability for any} \quad T > 0 \quad \text{Using Theorem 11.4.5 of [43], (3.3), the Lipschitz continuity of} h(\cdot) \quad \text{and following Theorem 2.11 of [38], we conclude that} \quad \hat{X}^n(\cdot) \quad \text{converges weakly to} \quad X_x(\cdot) \quad \text{in} \quad D[0,T]. \quad \text{Let} \epsilon > 0 \quad \text{be arbitrary. Using (3.8) and the Lipschitz continuity of} C(\cdot), \quad \text{we can find a large} \quad T > 0 \quad \text{so that}
\]

\[
\sup_{n \geq 1} E \int_T^\infty e^{-\alpha t} C(\hat{X}^n(t))dt < \epsilon.
\]

Since \( \hat{X}^n \) converges weakly to \( X_x \) in \( D[0,T] \), using (3.8), we obtain

\[
\lim_{n \to \infty} E \int_0^T e^{-\alpha t} C(\hat{X}^n(t))dt = E \int_0^T e^{-\alpha t} C(X_x(t))dt,
\]

and therefore,

\[
\lim_{n \to \infty} \inf E \int_0^T e^{-\alpha t} C(\hat{X}^n(t))dt \leq E \int_0^T e^{-\alpha t} C(X_x(t))dt + \epsilon \leq V(x) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, \( \lim \inf_{n \to \infty} E \int_0^\infty e^{-\alpha t} C(\hat{X}^n(t))dt \leq V(x) \). Using (5.2), we obtain \( \lim \inf_{n \to \infty} E \int_0^\infty e^{-\alpha t} C(\hat{X}^n(t))dt = V(x) \). Hence part (i) follows.

To prove part (ii), consider \((X_x^*, L_{a^*}, L_{b^*})\) satisfying (4.13) with reflection barriers at \( a^* \) and \( b^* \). It is an optimal control for the DCP in this parameter regime as proved in Theorem 4.2 (ii). Our aim is to construct an asymptotically optimal sequence \((\hat{X}^n, \hat{U}_s^n, \hat{U}_b^n)\) which converges weakly to \((X_x^*, L_{a^*}, L_{b^*})\). For each \( n \geq 1 \), we consider a state process \((\hat{X}^n, \hat{U}_s^n, \hat{U}_b^n)\) which satisfies (2.12) with time-independent reflection barriers at \( a_n^* < 0 < b_n^* \) and initial data \( \hat{X}^n(0) = x_n \) as described in Lemma 5.2. Here we choose \( \{a_n^*\} \) and \( \{b_n^*\} \) so that \( a_n^* < a^* < 0 < b^* < b_n^* \), \( \lim_{n \to \infty} a_n^* = a^* \) and \( \lim_{n \to \infty} b_n^* = b^* \).

We assume (2.1) and consider the case \( x \in [a^*, b^*] \). Then the state equation can be written in the form (3.53) and \( \hat{\zeta}^n \) converges weakly as in (3.3). Using the Skorokhod representation theorem, we assume that all these processes are defined in a same probability space and \( \lim_{n \to \infty} \sup_{t \in [0,T]} |\hat{\zeta}^n(t) - (x + \sigma B(t) + \beta t)| = 0 \) a.s. The reflected diffusion process \((X_{a^*}^*, L_{a^*}, L_{b^*})\) is a strong solution to (4.13) in this probability space with respect to the same Brownian motion \( B \). Next, we use Proposition A.2 in Appendix A to conclude \( \lim_{n \to \infty} \|\hat{X}^n - X_{a^*}^*\|_T = 0 \), \( \lim_{n \to \infty} \|\hat{U}_s^n - L_{a^*}\|_T = 0 \) and
\[ \lim_{n \to \infty} \| \hat{U}_b^n - L_a \|_T = 0 \] for any \( T > 0 \). Next let \( \epsilon > 0 \) be arbitrary. Using the moment bounds in (3.8) and (5.7), we can find a \( T > 0 \) so that

\[
\sup_{n \geq 1} E \int_T^{\infty} e^{-\alpha t} \left[ C(\hat{X}_n(t)) dt + p_s d\hat{U}_s(t) + p_b d\hat{U}_b(t) \right] < \epsilon
\]

and similarly,

\[
E \int_T^{\infty} e^{-\alpha t} \left[ C(X_\alpha^*(t)) dt + p_s dL_a(t) + p_b dL_b(t) \right] < \epsilon.
\]

Using the convergence of \((\hat{X}_n, \hat{U}_a, \hat{U}_b)\) to \((X_\alpha^*, L_a^*, L_b^*)\) almost surely on \([0, T]\), and by an argument similar to part (i), we can conclude that \( |\hat{J}^n(\hat{X}_n(0), \hat{U}_a^n, \hat{U}_b^n) - J(X_\alpha^*, L_a^*, L_b^*)| < 2\epsilon \) and consequently, \( \lim_{n \to \infty} \hat{J}^n(\hat{X}_n(0), \hat{U}_a^n, \hat{U}_b^n) = J(X_\alpha^*, L_a^*, L_b^*) \). This completes the proof of part (ii).

Proofs of parts (iii) and (iv) uses only the one-sided Skorokhod map and hence the proofs are much simpler than part (ii). In each case, the proof is very similar to that of Theorem 4.2 in [41]. Therefore, we omit it here. \( \Box \)

### A Two-sided Skorokhod maps

In this sub-section, we establish several results to supplement the work of [23, 34] and Chapter 14 of [43]. These results enable us to obtain asymptotically optimal strategies in Theorem 5.3. We first revisit the definition of the two-sided Skorokhod map (see Definition 1.2 in [23]). We follow the notation in [23], and let \( \Gamma_{a,b} : D[0, \infty) \to D[0, \infty) \) represent the two-sided Skorokhod map on the (time-independent) interval \([a, b]\).

**Definition A.1.** Let the constants \( a < b \). Given \( \psi \in D(0, \infty) \), there exists a unique pair of functions \( \phi \in D(0, \infty) \) and \( \eta \) of bounded variation such that

(i) for each \( t \geq 0 \), \( \phi(t) = \psi(t) + \eta(t) \in [a, b] \);

(ii) \( \eta(0-) = 0 \), \( \eta(0) \geq 0 \), and \( \eta \) has the decomposition \( \eta = \eta^l - \eta^r \) satisfying that \( \eta^l \) and \( \eta^r \) are non-decreasing, and

\[
\int_0^{\infty} 1_{\{ \phi(s) > a \}} d\eta^l(s) = 0 \quad \text{and} \quad \int_0^{\infty} 1_{\{ \phi(s) < b \}} d\eta^r(s) = 0.
\]

The map \( \Gamma_{a,b} : D[0, \infty) \to D[0, \infty) \) that takes \( \psi \) to the corresponding \( \phi \) is referred to as the two-sided Skorokhod map on \([a, b]\), and the triple \((\phi, \eta^l, \eta^r)\) is referred to as the Skorokhod decomposition of \( \psi \) on \([a, b]\). From the comparison properties of the Skorokhod map on \([a, b]\) (Theorem 1.7 of [23]), the Skorokhod decomposition is unique.

**Lemma A.1.** Let \( c < a < b < d \). Then for any \( f \) in \( D(0, \infty) \),

\[
\| \Gamma_{c,d}(f) - \Gamma_{a,b}(f) \|_T \leq 3|a - c| + |b - d|.
\]

**Proof.** Using equation (1.14) in [23] and the notation therein, for any \( f \) in \( D[0, \infty) \),

\[
\Gamma_{a,b}(f) = \Lambda_{a,b} \circ \Gamma_a(f).
\]
where for \( f \in D[0, \infty) \) and \( t \geq 0 \), \( \Gamma_a(f)(t) = f(t) + \sup_{s \in [0,t]} [a - f(s)]^+ \), and for \( g \in D[a, \infty) \), \( \Lambda_{a,b}(g)(t) = g(t) - \sup_{s \in [0,t]} ((g(s) - b)^+ \wedge \inf_{u \in [a,t]} (g(u) - a)) \). We notice that

\[
\|\Gamma_a(f) - \Gamma_c(f)\|_T = \sup_{t \in [0,T]} \left| \sup_{s \in [0,t]} [a - f(s)]^+ - \sup_{\tilde{s} \in [0,t]} [c - f(\tilde{s})]^+ \right| \leq |a - c|,
\]

and

\[
\|\Lambda_{a,b}(f) - \Lambda_{c,d}(f)\|_T = \sup_{t \in [0,T]} \left| \Lambda_{a,b}(f)(t) - \Lambda_{c,d}(f)(t) \right|
\]

\[
= \sup_{t \in [0,T]} \left| \sup_{s \in [0,t]} ((g(s) - b)^+ \wedge \inf_{u \in [a,t]} (g(u) - a)) - \sup_{\tilde{s} \in [0,t]} ((g(\tilde{s}) - d)^+ \wedge \inf_{u \in [a,t]} (g(u) - c)) \right|
\]

\[
\leq |a - c| + |b - d|.
\]

From (1.16) of [23], we have \( \|\Lambda_{c,d}(f) - \|\Lambda_{c,d}(g)\|_T \leq 2\|f - g\|_T \) for any \( f \) and \( g \) in \( D[0, \infty) \). Hence

\[
\|\Gamma_{c,d}(f) - \Gamma_{a,b}(f)\|_T \leq \|\Lambda_{c,d}(\Gamma_c(f)) - \Lambda_{c,d}(\Gamma_a(f))\|_T
\]

\[
+ \|\Lambda_{c,d}(\Gamma_a(f)) - \Lambda_{a,b}(\Gamma_a(f))\|_T
\]

\[
\leq 2\|\Gamma_c(f) - \Gamma_a(f)\|_T + |a - c| + |b - d| \leq 3(|a - c| + |b - d|).
\]

(\text{A.3})

\[\square\]

In the next result, we consider two convergent sequences \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) so that \( a_n < a < b < b_n \), \( a_n \) is increasing to \( a \), and \( b_n \) is decreasing to \( b \) as \( n \to \infty \). Let \( \{Y_n\}_{n \in \mathbb{N}} \) be a convergent sequence in \( D[0, \infty) \) so that \( \lim_{n \to \infty} \|Y_n - Y_\infty\|_T = 0 \) for each \( T > 0 \), where \( Y_\infty \) is a continuous function. We introduce

\[
W_n(t) = Y_n(t) + \int_0^t h(\Gamma_{a_n,b_n}(W_n)(s))ds,
\]

(A.4)

and

\[
W_\infty(t) = Y_\infty(t) + \int_0^t h(\Gamma_{a,b}(W_\infty)(s))ds,
\]

(A.5)

where \( h \) is a Lipschitz continuous function.

Consider the Skorokhod decomposition \( (Z_n, \eta^n, \eta^n_a) \) of the function \( W_n \) on the interval \( [a_n, b_n] \), and the Skorokhod decomposition \( (Z_\infty, \eta^\infty, \eta^\infty_a) \) of the function \( W_\infty \) on \( [a, b] \). Next, we obtain the convergence results of the Skorokhod decompositions of \( W_n \) and \( W_\infty \). This proof is closely related to that of Proposition 4.2 of [34].

**Proposition A.2.** There exists a constant \( C_T > 0 \) which depends only on \( T \) and the Lipschitz constant of the function \( h \) such that

\[
\|Z_n - Z_\infty\|_T \leq C_T \left( |a_n - a| + |b_n - b| + \|Y_n - Y_\infty\|_T \right).
\]

(A.6)

Consequently, \( \lim_{n \to \infty} \|Z_n - Z_\infty\|_T = 0 \) and

\[
\lim_{n \to \infty} \|\eta^n - \eta^\infty\|_T + \|\eta^n_a - \eta^\infty_a\|_T = 0.
\]

(A.7)
Proof. The existence and uniqueness of solutions to (A.4) and (A.5) depend only on the Lipschitz continuity of the function $h$ as shown in Lemma 4.2 of [34]. Using the Lipschitz continuity of $h$ in (A.4) and (A.5), we obtain $|W_n(t) - W_\infty(t)| \leq |Y_n(t) - Y_\infty(t)| + C \int_0^t |\Gamma_{a,b}(W_n)(s) - \Gamma_{a,b}(W_\infty)(s)| ds$, where $C > 0$ is a constant depending on the Lipschitz constant of the function $h$. Then,

$$
|\Gamma_{a,b}(W_n)(s) - \Gamma_{a,b}(W_\infty)(s)| \leq |\Gamma_{a,b}(W_n)(s) - \Gamma_{a,b}(W_n)(s)| + |\Gamma_{a,b}(W_n)(s) - \Gamma_{a,b}(W_\infty)(s)|.
$$

(A.8)

Now using (A.3) and the Lipschitz property of $\Gamma_{a,b}$, for each $0 \leq t \leq T$ we obtain,

$$
\|W_n - W_\infty\|_t \leq \|Y_n - Y_\infty\|_t + 3T(\|a_n - a\| + |b_n - b|) + 2C \int_0^t \|W_n - W_\infty\|_s ds. \quad (A.9)
$$

Employing the Gronwall's inequality leads to

$$
\|W_n - W_\infty\|_T \leq (\|Y_n - Y_\infty\|_T + 3T(\|a_n - a\| + |b_n - b|)e^{2CT}, \quad (A.10)
$$

and hence $\lim_{n \to \infty} \|W_n - W_\infty\|_T = 0$. Next we note that $\|Z_n - Z_\infty\|_T \leq \|\Gamma_{a,b}(W_n) - \Gamma_{a,b}(W_\infty)\|_T + \|\Gamma_{a,b}(W_n) - \Gamma_{a,b}(W_\infty)\|_T$. Using (A.1) and the Lipschitz property of $\Gamma_{a,b}$ we obtain $\|Z_n - Z_\infty\|_T \leq C(\|a_n - a\| + |b_n - b| + \|W_n - W_\infty\|_T)$. Consequently, using (A.10), (A.6) follows and thus the proof of part (i) is complete.

To prove part (ii), we let $K_n(t) = \eta_n^r(t) - \eta_n^l(t)$ and $K_\infty(t) = \eta_\infty^r(t) - \eta_\infty^l(t)$ for all $t \geq 0$. By part (i), $\lim_{n \to \infty} \|K_n - K_\infty\|_T = 0$. Next we follow the proof of Theorem 14.8.1 in [43]. Let $\varepsilon > 0$ be arbitrarily small satisfying $0 < \varepsilon < (b - a)/100$. Let $t_1 = \min\{T, \tau_1\}$ where $\tau_1 = \inf\{t \geq 0 : Z_\infty(t) < a + \varepsilon\}$. Let $t_2 = \min\{T, \tau_2\}$ where $\tau_2 = \inf\{t \geq t_1 : Z_\infty(t) > b - \varepsilon\}$. Inductively, let $t_{2j-1} = \min\{T, \tau_{2j-1}\}$ where $\tau_{2j-1} = \inf\{t \geq t_{2j-2} : Z_\infty(t) < a + \varepsilon\}$ and $t_{2j} = \min\{T, \tau_{2j}\}$ where $\tau_{2j} = \inf\{t \geq t_{2j-1} : Z_\infty(t) > b - \varepsilon\}$. Since $Y_\infty$ is a continuous function, $Z_\infty$ also continuous on $[0, \infty)$. Consequently, there are only finitely many points $0 \leq t_1 < t_2 < \cdots < t_m \leq T$. Let $n_0 > 1$ so that $\|Z_n - Z_\infty\|_T < \varepsilon/10$ for any $n \geq n_0$. For $n \geq n_0$, $l_n$ as well as $b_n$ increases only on a finite number of intervals of the form $[t_{2j-1}, t_{2j}]$ say $j = 1, 2, \ldots, r$. On those intervals, $u_n$ and $w_n$ remain constant. Moreover, $\eta_n^l$ and $\eta_n^r$ remain constant on the intervals $(t_{2j}, t_{2j+1})$. On each interval $[t_{2j-1}, t_{2j}]$, $j = 1, 2, \ldots, r$, only the one sided Skorokhod map (see equation (1.4) in [23]) will be applied. Hence, it is evident that $\lim_{n \to \infty} \|\eta_n^l - \eta_\infty^l\|_{[t_{2j-1}, t_{2j}]} = 0$ on each interval. Consequently, $\lim_{n \to \infty} \|\eta_n^l - \eta_\infty^l\|_T = 0$ follows. This completes the proof. □

The following results are immediate from the above proposition.

**Corollary A.1.** The Skorokhod decomposition $(Z_n, \eta_n^l, \eta_n^r)$ converges to $(Z_\infty, \eta_\infty^l, \eta_\infty^r)$ in $D^3[0, \infty)$ in Skorokhod $J_1$-topology.

**Corollary A.2.** Let $Y$ be in $D[0, \infty)$ be fixed. Consider the Skorokhod decompositions $(Z_n, \eta_n^l, \eta_n^r)$ and $(Z_\infty, \eta_\infty^l, \eta_\infty^r)$ of $Y$ on $[a_n, b_n]$ and $[a, b]$, respectively. Assume that $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$. Then

$$
\lim_{n \to \infty} \|Z_n - Z_\infty\|_T = 0, \quad (A.11)
$$

and

$$
\lim_{n \to \infty} \|\eta_n^l - \eta_\infty^l\|_T + \|\eta_n^r - \eta_\infty^r\|_T = 0. \quad (A.12)
$$
At last we consider the two-sided SP on time dependent intervals. The following definition is from [5].

**Definition A.2.** Let \( l, r \in D[0, \infty) \), where \( l \) could take \(-\infty\) and \( r \) could take \( \infty \). Given \( \psi \in D[0, \infty) \) so that \( l(t) < r(t) \) for \( t \geq 0 \), a pair of functions \( (\phi, \eta) \in D^2[0, \infty) \) is said to solve the SP for \( \psi \) on the time-dependent interval \([l(\cdot), r(\cdot)]\), if and only if it satisfies the following properties.

(i) for each \( t \geq 0 \), \( \phi(t) = \psi(t) + \eta(t) \in [l(t), r(t)] \);

(ii) \( \eta \) has the decomposition \( \eta = \eta^l - \eta^r \), where \( \eta^l \) and \( \eta^r \) are non-decreasing functions, and

\[
\int_0^\infty 1_{\{\phi(s) > t(s)\}} d\eta^l(s) = 0 \quad \text{and} \quad \int_0^\infty 1_{\{\phi(s) < r(s)\}} d\eta^r(s) = 0.
\]

If there is a unique solution \( (\phi, \eta) \) to the SP for \( \psi \) on \([l, r]\), then we write \( \phi = \Gamma_{l,r}(\psi) \), where \( \Gamma_{l,r} \) will be referred to as the two-sided Skorokhod map on \([l, r]\), and the triple \((\phi, \eta^l, \eta^r)\) is referred to as the Skorokhod decomposition of \( \psi \) on \([l, r]\).

From Theorem 2.5 and Corollary 2.4 of [5], if \( \inf_{t \geq 0}(r(t) - l(t)) > 0 \), then there is a unique pair \((\phi, \eta)\) that solves the SP for \( \psi \), and from the composition properties (Section 3 in [5]), the corresponding Skorokhod decomposition \((\phi, \eta^l, \eta^r)\) is unique.

Next we develop the oscillation inequalities for the constrained processes of Skorokhod decomposition. For a function \( f \) in \( D[0, T] \), we recall its oscillation \( \text{Osc}(f, [t_1, t_2]) \) in an interval \([t_1, t_2] \subset [0, T]\) is defined by (1.1), and the modulus of continuity \( \omega(f, \delta, T) \) is given in (1.2).

**Proposition A.3.** Let the functions \( l \) and \( r \) be in \( D[0, \infty) \) and \( \inf_{t \geq 0}(r(t) - l(t)) > 0 \) for all \( t \geq 0 \). Given a function \( \psi \in D[0, \infty) \), let \((\phi, \eta)\) be the unique solution to the SP in \( D^2[0, \infty) \) satisfying \( l(t) \leq \phi(t) \leq r(t) \) for all \( t \geq 0 \) and \( \eta \) be a function of bounded variation on \([0, \infty)\) as described in Theorem 2.6 of [5]. Then for all \( \delta > 0 \) and \( T > 0 \),

\[
\omega(\phi, \delta, T) \leq 4[\omega(\psi, \delta, T) + \omega(l, \delta, T) + \omega(r, \delta, T)]. \tag{A.13}
\]

**Proof.** Using Theorem 2.6 and Corollary 2.4 of [5], we can write \( \phi(t) = \psi(t) - \Theta(\psi)(t) \) for \( t \geq 0 \), where

\[
\Theta(\psi)(t) = \max\{b(t), h(t)\}, \tag{A.14}
\]

where

\[
b(t) = \min\{(\psi(0) - r(0))^+, \inf_{u \in [0, t]} (\psi(u) - l(u))\}, \tag{A.15}
\]

and

\[
h(t) = \sup_{s \in [0, t]} \min\{(\psi(s) - r(s)), \inf_{u \in [s, t]} (\psi(u) - r(u))\}, \tag{A.16}
\]

for all \( t \geq 0 \).

Since \( \phi = \psi - \Theta(\psi) \), it is evident that

\[
\omega(\phi, \delta, T) \leq 4[\omega(\psi, \delta, T) + \omega(\Theta(\psi), \delta, T)]. \tag{A.17}
\]

From (A.14), it follows that

\[
\omega(\Theta(\psi), \delta, T) \leq \omega(b, \delta, T) + \omega(h, \delta, T). \tag{A.18}
\]

Next, we estimate \( \omega(b, \delta, T) \) and \( \omega(h, \delta, T) \) carefully. Using (A.15), we obtain \( |b(t) - b(s)| \leq \sup_{u \in [s, t]} |\psi(u) - \psi(s)| + \sup_{u \in [s, t]} |l(u) - l(s)| \) whenever \( 0 \leq s \leq t \). Therefore, \( \omega(b, \delta, T) \leq 40 \).
\(\omega(\psi, \delta, T) + \omega(l, \delta, T)\). Similarly, for \(\omega(h, \delta, T)\), we can use (A.16) together with the simple inequality \(|\min\{a, b\} - \min\{c, d\}| \leq |a - c| + |c - d|\) to obtain

\[
\omega(h, \delta, T) \leq [2\omega(\psi, \delta, T) + \omega(r, \delta, T) + \omega(l, \delta, T)].
\] (A.19)

Now combining (A.17) through (A.19), we obtain (A.13). This completes the proof. \(\Box\)

**Remark A.1.** Following the above proof, one can obtain the inequality

\[
\sup_{u \in [s, t]} |\phi(u) - \phi(v)| \leq 4 \left[ \sup_{u \in [s, t]} |\psi(u) - \psi(v)| + \sup_{u \in [s, t]} |l(u) - l(v)| + \sup_{u \in [s, t]} |r(u) - r(v)| \right]
\]
on any given interval \([s, t]\).

**Proposition A.4.** Let \((\phi, \eta^l, \eta^r)\) in \(D^3[0, T]\) be the solution to the SP with the input function \(\psi\) in \(D[0, T]\) and the time-dependent barriers \(l\) and \(r\) which are also in \(D[0, T]\). Assume \(\inf_{t \in [0, T]} |r(t) - l(t)| > 0\), and \(l(0) \leq \psi(0) \leq r(0)\). Then the following oscillation inequalities hold for any \(0 \leq t_1 \leq t_2 \leq T\):

\[
\text{Osc}(\eta^l, [t_1, t_2]) \leq C[\text{Osc}(\psi, [t_1, t_2]) + \text{Osc}(l, [t_1, t_2])],
\] (A.20)

and

\[
\text{Osc}(\eta^r, [t_1, t_2]) \leq C[\text{Osc}(\psi, [t_1, t_2]) + \text{Osc}(r, [t_1, t_2])],
\] (A.21)

where \(C > 0\) is a generic constant independent of \(\psi, l\) and \(r\) and \([t_1, t_2]\).

**Proof.** We focus on proving (A.20) the oscillation property for \(\eta^l\) below and the proof of (A.21) is similar. The proof is divided into the following three steps.

**Step 1.** Oscillation inequality in \([0, b]\), where \(b > 0\) is a constant:

Using Theorem 4.2 of [7], \(\text{Osc}(\eta^l, [t_1, t_2]) \leq \kappa(\text{Osc}(\psi, [t_1, t_2]) + \|\Delta \eta^l\|_{[t_1, t_2]})\). The constant \(\kappa\) is independent of the functions \(\psi, l, r\) and \(\phi, t_1, t_2\) as well as the domain \([0, b]\). Since set of piecewise constant functions are dense in \(D[0, T]\) with respect to uniform norm (see [43], Theorem 12.2.2.), we assume \(\psi\) is piecewise constant. Then \((\phi, \eta^l, \eta^r)\) also piecewise constant with the same points of discontinuity. We can use the construction of \((\phi, \eta^l, \eta^r)\) in the equation (8.2) of Chapter 14 of [43]. Let \(0 \leq s_1 < s_2 < ... < s_m \leq T\) be the points of discontinuity of \(\psi\). Then \(\eta^l(s_i) - \psi(s_i) \leq |\psi(s_i) - \psi(s_i)|\) holds at each jump point \(s_i\). Hence we obtain \(\text{Osc}(\eta^l, [t_1, t_2]) \leq 2\kappa(\text{Osc}(\psi, [t_1, t_2]))\) when \(\psi\) is a piecewise constant function. This inequality can be generalized for any \(\psi\) in \(D[0, T]\) using Proposition A.2 and (A.6) by taking the function \(h\) to be identically zero. Hence \(\text{Osc}(\eta^l, [t_1, t_2]) \leq 2\kappa(\text{Osc}(\psi, [t_1, t_2]))\) follows.

**Step 2.** Oscillation inequality in \([0, \beta(t)]\), where \(\beta(\cdot) > 0\) is in \(D[0, T]\):

We assume \(\inf_{t \in [0, T]} \beta(t) > 0\). Consider the triple \((\phi, \eta^l, \eta^r)\) corresponding to the input function \(\psi\). We pick \(t_1 < t_2\) in \([0, T]\). We pick \(b > 0\) so that \(0 < b \leq \inf_{t \in [0, T]} \beta(t)\). For convenience assume \(t_1\) is not a point of discontinuity of \(\beta\). Now consider the interval \([t_1, T]\). If \(\beta(t_1) > b\) choose \(c_0 = \phi(t_1) - b\) and \(\tilde{\psi} = \psi - c_0\). If \(\beta(t_1) \leq b\) choose \(\tilde{\psi} = \psi\). Now \((\tilde{\phi}, \eta^l(t) - \eta^l(t_1), \eta^r(t) - \eta^r(t_1))\) is the solution to the SP for \(\tilde{\psi}\) on \([t_1, T]\) and let \((\tilde{\phi}, \tilde{\eta}^l, \tilde{\eta}^r)\) be the solution to the SP for \(\tilde{\psi}\) on \([t_1, T]\). Then by the comparison theorem in Proposition 3.5 of [6], (with \(\nu\) identically zero), we have \(\eta^l(t_2) - \eta^l(t_1) \leq \tilde{\eta}^l(t_2) - \tilde{\eta}^l(t_1)\). Next we compare the solution \((\tilde{\phi}, \tilde{\eta}^l, \tilde{\eta}^r)\) of \(\tilde{\psi}\) on the time dependent interval \([0, \beta]\) with the solution \((\phi, \eta^l, \eta^r)\) of \(\psi\) on \([0, b]\). We use the comparison theorem in Proposition 3.3 of [5] to conclude \(\tilde{\eta}^l(t_2) - \tilde{\eta}^l(t_1) \leq \eta^l(t_2) - \eta^l(t_1)\). Using Step 1, we obtain
\( \dot{\eta} (t_2) - \dot{\eta} (t_1) \leq 2\kappa (Osc (\psi, [t_1, t_2])) \). Consequently, \( \eta (t_2) - \eta (t_1) \leq 2\kappa (Osc (\psi, [t_1, t_2])) \). Hence \( Osc (\eta, [t_1, t_2]) \leq 2\kappa (Osc (\psi, [t_1, t_2])) \) holds in \([0, \beta (\cdot)]\), when \( \beta (\cdot) \) is in \( D [0, T] \) and \( \inf_{t \in [0, T]} \beta (t) > 0 \).

**Step 3.** Oscillation inequality in \([l (t), r (t)]\) with \( \inf_{t \in [0, T]} [r (t) - l (t)] > 0 \):

We can use the proof of Lemma 2.2 and equation (2.26) in [36] to observe that the corresponding constraining processes are unchanged under translation by the function \( \alpha \) there. Let \( \beta (t) = r (t) - l (t) \) for all \( t \) in \([0, T] \). Hence using Step 2, we obtain \( Osc (\eta, [t_1, t_2]) \leq 2\kappa (Osc (\psi, [t_1, t_2])) \). But \( Osc (\psi - l, [t_1, t_2]) \leq Osc (\psi, [t_1, t_2]) + Osc (l, [t_1, t_2]) \). Hence we have \( Osc (\eta', [t_1, t_2]) \leq 2\kappa (Osc (\psi, [t_1, t_2]) + Osc (l, [t_1, t_2])) \) and this yields (A.20) and the proof is complete. \( \Box \)

For a given \( \delta > 0 \), by choosing intervals \([t_1, t_2]\) of length less than \( \delta \), and using (A.20) and (A.21), we can obtain the following corollary.

**Corollary A.3.** Let \((\phi, \eta', \eta^*)\) in \( D^3 [0, T] \) be the solution to SP for \( \psi \) in \( D [0, T] \) and the time-dependent barriers \( l \) and \( r \) in \( D [0, T] \) so that \( \inf_{t \in [0, T]} [r (t) - l (t)] > 0 \), and \( l (0) \leq \psi (0) \leq r (0) \). Let \( \omega (f, \delta, T) \) be as in (1.2). Then

\[
\omega (\eta', \delta, T) \leq C [\omega (\psi, \delta, T) + \omega (l, \delta, T)],
\]

and

\[
\omega (\eta^*, \delta, T) \leq C [\omega (\psi, \delta, T) + \omega (r, \delta, T)],
\]

where \( C > 0 \) is a generic constant independent of \( \psi, l, r \) and \([0, T] \).

## B Constructing the solutions of the HJB

### B.1 Proof of Proposition 4.3

Our approach here is to find two bounded solutions in the domains \((-\infty, 0)\) and \((0, \infty)\) so that we can paste them smoothly at the origin. On \((0, \infty)\), (4.18) can be written as

\[
\frac{\sigma^2}{2} W'' (x) + (\beta - \delta \alpha) W' (x) - (\alpha + \delta \alpha) W (x) + \theta = 0.
\]

Since \( W (x) \equiv T_x \) is a particular solution, \( W \) has the representation \( W (x) = u (x) + \theta / (\alpha + \delta \alpha) \), where \( u \) satisfies the homogeneous equation

\[
\frac{\sigma^2}{2} u'' (x) + (\beta - \delta \alpha) u' (x) - (\alpha + \delta \alpha) u (x) = 0, \quad x > 0.
\]

Since we expect \( W \) to be bounded, we seek for a bounded non-trivial solution for \( u \). A fundamental set for this homogeneous equation consists of one bounded function and another unbounded function as \( x \) tends to infinity. We need only to seek for this bounded solution. Such a solution exists and has a stochastic representation (see Section 50 of Chapter V in [35]): Consider a solution to the Ornstein-Uhlenbeck type equation \( Z (t) = x + \sigma B (t) + \int_0^t [\beta - \delta \alpha] Z (u)] du \), where \( x \geq 0 \), \( B \) is a standard Brownian motion and consider the stopping time \( \tau_0 = \inf\{t \geq 0 : Z (t) = 0\} \). Then \( \Psi_0 (x) = E_x [e^{-\alpha + \delta \alpha \tau_0}] \) for \( x \geq 0 \) is such a bounded solution to the homogeneous equation (B.1) and any other bounded solution in \((0, \infty)\) is a constant multiple of \( \Psi_0 \). Moreover, \( \Psi_0 (0) = 1 \) and it is a strictly decreasing function on \((0, \infty)\). We expect \( W \) to be of the form \( W (x) = k_s \Psi_0 (x) + \theta / (\alpha + \delta \alpha) \) in the interval \((0, \infty)\), where \( k_s \) is a constant which needs to be determined. Since \( \lim_{x \rightarrow \infty} \Psi_0 (x) = 0 \), it follows that \( \lim_{x \rightarrow \infty} W (x) = \theta / (\alpha + \delta \alpha) \).
We can perform a similar analysis on the interval \((-\infty, 0)\). Notice that \(W(x) = -T_b\) is a particular solution to (4.18) on \((-\infty, 0)\). Let \(Z(t) = x + \sigma B(t) + \int_0^t [\beta - \delta_b Z(s)]ds\), where \(x \leq 0\), and \(B\) is a standard Brownian motion. Introduce the stopping time 
\[\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}\] and let \(\Phi_0(x) = E_x[e^{-(\alpha + \delta_b)\tau_0}]\) for \(x \leq 0\). Then \(\Phi_0\) is a bounded solution to the homogeneous equation

\[\frac{\sigma^2}{2} u''(x) + (\beta - \delta_b)u'(x) - (\alpha + \delta_b)u(x) = 0, \quad x < 0.\]

Moreover, \(\lim_{x \to -\infty} \Phi_0(x) = 0\), \(\Phi_0(0) = 1\), and it is strictly increasing in \((-\infty, 0)\) (for details, see Section 50 of Chapter V in [35]). Then similar to above analysis, we expect \(W\) to be of the form \(W(x) = k_b\Phi_0(x) - \delta_b/(\alpha + \delta_b)\) when \(x < 0\). The constant \(k_b\) needs to be determined. Then \(\lim_{x \to -\infty} W(x) = -\delta_b/(\alpha + \delta_b)\) also follows.

Finally, to determine the constants \(k_s\) and \(k_b\), we impose the “smooth fit conditions” for \(W\) across the origin as they are described by \(W(0-) = W(0+)\) and \(W'(0-) = W'(0+)\). We use \(\Psi_0(0) = \Phi_0(0) = 1\) and \(\Psi_0'(0+) < 0 < \Phi_0'(0-)\). Then, \(W(0-) = W(0+)\) implies \(k_b = k_s + T_s + T_b\). The condition \(W'(0-) = W'(0+)\) implies \(k_b = k_s\Psi_0(0+)/\Phi_0(0-)\). By solving these equations and using the condition \(\Psi_0'(0+) < 0 < \Phi_0'(0-)\), we obtain

\[k_s = -\frac{T_s + T_b}{1 - \frac{\psi_0'(0+)}{\phi_0'(0-)}} < 0, \quad k_b = k_s\frac{\psi_0'(0+)}{\phi_0'(0-)} > 0.\]

Consequently, for this pair of \(k_s\) and \(k_b\), the above described \(W \in C^1(\mathbb{R})\), strictly increasing function on \(\mathbb{R}\) which satisfies (4.18). Moreover, \(W''\) is continuous everywhere except at \(x = 0\). It also satisfies (4.20) and (4.21) at the origin. Hence the proofs of parts (i), (ii) and (iii) are complete and it remains to show the uniqueness of \(W\).

To prove the uniqueness, we consider \(Z(t) = x + \sigma B(t) + \int_0^t [\beta - h(Z(s))]ds\), where \(B\) is a standard Brownian motion. The condition \(xh(x) \geq 0\) implies the explosion time of \(Z\) is infinite. Since \(W \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})\), \(W''(0-)\) and \(W''(0+)\) are finite, we can apply Itô’s lemma (see [17]) to \(W(Z(T))e^{-\int_0^T (\alpha + \gamma(Z(t)))dt}\) to obtain the representation \(W(x) = \int_0^\infty e^{-\int_0^t (\alpha + \gamma(Z(s)))ds} C'(Z(s))ds\). Hence the uniqueness of \(W\) follows. \(\square\)

### B.2 Proof of Proposition 4.4

Let the cost parameters \(p_b\) and \(p_s\) satisfy \(0 < p_b < T_b\) and \(0 < p_s < T_s\). Throughout, we use continuity properties of the solutions with respect to initial data and other parameters and we refer to Chapter V of [15]. First we gather a few useful facts about the differential equation (4.18). Recall that \(W(x) = -T_b\) is a constant solution on the interval \((-\infty, 0)\) and on \((0, \infty)\), \(W(0) = T_s\) is a constant solution for (4.18).

**Lemma B.1.** Let \(W\) satisfy (4.18) in the neighborhood of a point \(x = c\) and assume \(W'(c) = 0\). Then the following hold.

(i) If \(c < 0\) and \(W(c) > -\delta_b/(\alpha + \delta_b)\), then \(W''(c) > 0\) and \(x = c\) is a strict local minimum.

(ii) If \(c > 0\) and \(W(c) < \delta_b/(\alpha + \delta_b)\), then \(W''(c) < 0\) and \(x = c\) is a strict local maximum.
Proof. If \( c < 0 \) and \( W \) satisfies (4.18) together with \( W'(c) = 0 \), then we observe that
\[
\frac{a^2}{2}W''(c) = (\alpha + \delta c)\left(W(c) + \frac{\theta_c}{\alpha + \delta c}\right).
\]
The conclusion of part (i) is straightforward. The proof of part (ii) is similar and is omitted. \( \Box \)

Remark B.1. Notice that if \( c < 0 \), \( W'(c) = 0 \) and \( W(c) = -\theta b/(\alpha + \delta b) \), then by the uniqueness of solutions to the differential equation (4.18), it follows that \( W(x) = -T_b \) for all \( x < 0 \). Similar conclusion holds when \( c > 0 \), \( W'(c) = 0 \) and \( W(c) = T_a \).

For each \( a < 0 \), we let \( W_a \) be the solution to (4.18) on \((-\infty, 0]\) with the initial data \( W_a(a) = -p_a \) and \( W'_a(a) = 0 \). Since \(-T_b\) is a particular solution to (4.18) on the interval \((-\infty, 0]\), we can express \( W_a \) as follows:
\[
W_a(x) = U_a(x) - T_b, \quad x \in (-\infty, 0)
\]
where \( U_a \) is a solution to the homogeneous equation given below:
\[
\frac{a^2}{2}U''_a(x) + (\beta - \delta b)xU'_a(x) - (\alpha + \delta b)U_a(x) = 0, \quad \text{for } x < 0,
\]
with the boundary conditions
\[
U_a(a) \equiv d_b = T_b - p_b > 0, \quad \text{and } U'_a(a) = 0.
\]

For the homogeneous differential equation (B.3), there exists a fundamental set of solutions \( \{\Phi_0, \Phi_1\} \) on \((-\infty, 0]\) satisfying \( \Phi_0(-\infty) = 0 \), \( \Phi_1(-\infty) = \infty \), \( \Phi_0(0) = 1 \), \( \Phi_1(0) = 1 \). These solutions \( \Phi_0 \) and \( \Phi_1 \) have stochastic representations as described in Chapter V of [35] (see Section 50 and in particular Proposition 50.3.). Moreover, \( \Phi_0 \) is strictly increasing and \( \Phi_1 \) is strictly decreasing in the domain \((-\infty, 0]\).

Lemma B.2. Let \( a < 0 \) then the following hold for each solution \( W_a \).

(i) Each \( W_a \) is strictly decreasing on \((-\infty, a)\), and strictly increasing on \((a, 0)\). Moreover, \( W_a(0) \) and \( W'_a(0) \) are finite.

(ii) \( W_a(-\infty) = \infty \), and \( \lim_{a \to -\infty} W_a(0) = \infty \).

Proof. By (B.2), it suffices to prove the lemma for \( U_a \). To show part (i), we observe that for any \( a < 0 \), if \( U'_a(c) = 0 \) and \( U_a(c) > 0 \) for some \( c < 0 \), then by (B.3), \( U''_a(c) > 0 \) and \( x = c \) is necessarily a strict local minimum. In particular, for \( a < 0 \), \( U_a(a) = d_b > 0 \), and by (B.4), \( x = a \) is a strict local minimum. With the absence of any local maxima when \( U_a(x) > 0 \), we observe that \( U_a \) is strictly decreasing in \((-\infty, a)\) and strictly increasing in \((a, 0)\). Moreover, the coefficients of the differential equation (B.3) can be linearly and continuously extended to \([0, \infty)\). Hence \( U_a(0) \) and \( U'_a(0) \) are finite. This proves part (i).

For part (ii) let \( a < 0 \). Using the fundamental set we can write \( U_a(x) = p_a \Phi_0(x) + q_a \Phi_1(x) \) for all \( x \leq 0 \), where \( p_a \) and \( q_a \) are constants. Since \( U_a \) has a positive, strict local minimum at \( x = a \), \( U_a \) is strictly decreasing in \((a - \delta, a)\) and strictly increasing in \((a, a + \delta)\) for some \( \delta > 0 \). Since \( \Phi_0 \) is strictly increasing and \( \Phi_1 \) is strictly decreasing, it follows that \( p_a > 0 \) and \( q_a > 0 \). But \( \Phi_0 \) is a bounded function on \((-\infty, 0)\) and \( \Phi_1(-\infty) = \infty \). Hence, \( U_a(-\infty) = \infty \) and consequently, \( W_a(-\infty) = \infty \) holds.

To show \( \lim_{a \to -\infty} U_a(0) = \infty \), let \( a_1 < a_2 < 0 \). By (B.4), \( U_{a_1} \) and \( U_{a_2} \) meet at a point in the interval \((a_1, a_2)\) and \( U_{a_1}(a_2) > U_{a_2}(a_2) = d_b \). But by the uniqueness
of solutions, two solutions to a homogeneous equation cannot meet more than once. Therefore, \( U_a(0) > U_{a_2}(0) > 0 \). Consequently, \( \lim_{a \to -\infty} U_a(0) \) is finite or else it is \(+\infty\).

Suppose \( \lim_{a \to -\infty} U_a(0) = L > 0 \) is finite. Then \( U_a(0) = p_a + q_a \leq L \) for all \( a < 0 \). Moreover, \( p_a > 0, q_a > 0 \), and \( \Phi_1'(0) < 0 \), \( U_a'(0) \leq p_a \Phi_1'(0) \leq L \Phi_0'(0) \) for each \( a < 0 \). By integrating (B.3) and using \( U_a(a) = d_b > 0 \), and \( U_a \) is a non negative increasing function in \((a, 0)\), we obtain the bound \( \frac{s^2}{2} U_a'(0) \geq (\beta - \delta a) d_b - \beta U_a(0) \). Since \( U_a(0) \) is bounded, RHS of this inequality tends to \(+\infty\), when \( a \) tends to \(-\infty\). But \( U_a'(0) \) is bounded above by the constant \( L \Phi_0'(0) \) which is finite. This yields a contradiction. Hence, \( L = \infty \) and this completes the proof. \( \Box \)

From the above discussion, \( W_a \) satisfies the linear equation (4.18) on \((-\infty, 0)\), and \( W_a(0) \) and \( W_a'(0) \) are finite. For each \( a < 0 \), we can extend \( W_a \) to \((0, \infty)\) so that it satisfies (4.18) on \((0, \infty)\) with the available initial data for \( W_a(0) \) and \( W_a'(0) \). More precisely, on the interval \((0, \infty)\), \( W_a \) satisfies

\[
\frac{\sigma^2}{2} W_a''(x) + (\beta - \delta x) W_a'(x) - (\alpha + \delta) W_a(x) + \theta_s = 0. \tag{B.5}
\]

We let

\[
U_a(x) = W_a(x) - T_s, \quad x \in (0, \infty). \tag{B.6}
\]

Then \( U_a \) satisfies the homogeneous equation

\[
\frac{\sigma^2}{2} U_a''(x) + (\beta - \delta x) U_a'(x) - (\alpha + \delta) U_a(x) = 0, \quad x \in (0, \infty), \tag{B.7}
\]

together with the boundary data \( U_a(0) = W_a(0) - T_s \) and \( U_a'(0) = W_a'(0) \). Similar to \( U_a \) on \((-\infty, 0)\), there is a fundamental set of solutions \( \{\Psi_0, \Psi_1\} \) for the above homogeneous equation on \((0, \infty)\) so that \( \hat{\Psi}_0(0) = \hat{\Psi}_1(0) = 1 \), \( \hat{\Psi}_0(\infty) = 0 \), \( \hat{\Psi}_1(\infty) = \infty \), \( \hat{\Psi}_0 \) is strictly decreasing and \( \hat{\Psi}_1 \) is strictly increasing on \((0, \infty)\). (See Chapter V of [35] for a stochastic representation of \( \Psi_0 \) and \( \Psi_1 \).) Hence each \( U_a \) can be written as

\[
U_a(x) = r_a \Psi_0(x) + t_a \Psi_1(x), \quad x \in (0, \infty), \tag{B.8}
\]

where \( r_a \) and \( t_a \) are constants.

In the next two results, we show that when \( a < 0 \), there are two types of solution profiles of \( W_a \) which exhibits very different behavior on \([0, \infty)\). When \(|a|\) is very large, \( W_a(x) \) tends to \(+\infty\) as \( x \to \infty \). When \(|a|\) is small, \( W_a(x) \) tends to \(-\infty\) as \( x \to \infty \). These two profiles are separated by the solution curve of \( W_c \), where \( a = c < 0 \) is a special point. The solution \( W_c \) is bounded on \([0, \infty)\) and it approaches \( T_b \) as \( x \to \infty \).

**Lemma B.3.** There exists a \( \delta > 0 \) so that if \( -\delta < a < 0 \), then

(i) \( W_a \) has a local maximum in \((0, \infty)\);

(ii) \( \lim_{x \to \infty} W_a(x) = -\infty \).

**Proof.** We first consider \( W_0 \) which corresponds to \( a = 0 \). Then \( W_0 \) satisfies (4.18) everywhere except at \( x = 0 \) and \( W_0(0) = -p_b \), and \( W_0'(0) = 0 \). Moreover, \( W_0 \) and \( W'_0 \) are continuous everywhere, \( W''_0 \) has a jump discontinuity at \( x = 0 \), \( \frac{\delta}{2} W''_0(0-) = (\alpha + \delta_b)(T_b - p_b) > 0 \), and \( \frac{\delta^2}{2} W''_0(0+) = - (\alpha + \delta_s)(T_s + p) < 0 \). Following the proof of Lemma B.1 (i), \( W_0 \) is strictly decreasing on \((-\infty, 0)\). On \((0, \infty)\), since \( W'_0(0) = 0 \) and \( W''_0(0) < 0 \), by a similar argument, \( W_0 \) is strictly decreasing.
Lemma B.4. Let $K > x$ strictly concave and strictly decreasing for large $W$. The corresponding extended solution equation (B.3) can be continuously and linearly extended to $(0, \infty)$. 

Proof. From part (i), $W_a(0) = -p_b < 0$ so that $W_a(0) < -p_b - 2\epsilon$. Note that both $W_0$ and $W_a$ satisfy (4.18) on $(-\infty, 0)$. Hence, using the continuity of the solutions in initial data, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ so that $|W_a(0) - W_0(0)| < |W'_a(0) - W'_0(0)| < \epsilon$ whenever $-\delta_1 < a < 0$. The functions $W_0$ and $W_a$ satisfy (B.5) in $(0, \infty)$. Again using the same continuity property, we can find a $\delta_2 > 0$ so that $|W_a(x_0) - W_0(x_0)| < \epsilon_0$ whenever $|W_a(0) - W_0(0)| + |W'_a(0) - W'_0(0)| < \delta_2$. Now, by letting $\epsilon = \delta_2$, we can find a $\delta > 0$ so that $W_a(x_0) < -p_b - \epsilon_0$ whenever $-\delta < a < 0$. Hence, each $W_a$ is strictly increasing in $(a, 0)$, $W_a(a) = -p_b$ and $W_a(x_0) < -p_b - \epsilon_0$. Thus, $W_a$ has a local maximum in $(0, x_0)$. This completes part (i).

From part (i), $W_a$ has a local maximum at a point $x = c$ in $(0, x_0)$. Thus, $W''_a(c) = 0$, $W'_a(c) = 2(\alpha + \delta_0) (W_a(c) - T_s) \leq 0$, which yields that $W_a(c) \leq T_s$. But $W_a(c)$ cannot be equal to $T_s$; otherwise $W_a(x) = T_s$ becomes the unique solution as explained in Remark B.1, which contradicts with the fact that $W_a$ is strictly decreasing on $(0, \infty)$. Hence $W_a(c) < T_s$ and $W''_a(c) < 0$ for $x > c$. Using (B.5), then it follows that $W_a$ is strictly concave and strictly decreasing for large $x > 0$. Hence, part (ii) follows. □

Next we consider the properties of the solution $W_a$ when $a < 0$ and $|a|$ is very large. From Lemma B.2 (ii), we can find a constant $K > 0$ so that

$$\text{when } a < -K, \quad W_a(0) > T_s. \quad \text{(B.9)}$$

Lemma B.4. Let $a < -K$ where $K > 0$ is given in (B.9). Then

(i) $W_a$ is strictly increasing on the interval $(a, \infty)$;

(ii) $\lim_{x \to \infty} W_a(x) = \infty$.

Proof. By Lemma B.2 (i), $U'_a(0) \geq 0$ (equivalently, $W'_a(0) \geq 0$). The differential equation (B.3) can be continuously and linearly extended to $(0, \infty)$. Hence we consider the corresponding extended solution $U_a$ as given in (B.2) and (B.6). Next, we show that $U'_a(0) > 0$. Suppose $U'_a(0) = 0$. Then by (B.9) and (B.7), we have $U''_a(0) = \frac{2(\alpha + \delta_0)}{a^2} U_a(0) > 0$. Thus $U_a$ has a strict local minimum at the origin. This is a contradiction since $U_a$ is strictly increasing in $(a, 0)$. Hence $U'_a(0) > 0$ and consequently, $W'_a(0) > 0$. Thus, we can find a $\delta > 0$ so that $W'_a(x) > 0$ when $0 < x < \delta$. Let $c = \sup\{x > 0 : W'_a(u) < 0 \text{ for } 0 < u < x\}$. Suppose $c$ is finite. Then $W'_a(c) = 0$, and $W_a(c) > T_s/(\alpha + \delta)$. Hence, $W''_a(c) > 0$ and $x = c$ is a strict local minimum. This is a contradiction since $W''_a(x) > 0$ in $(0, c)$. Hence $c = \infty$ and part (i) follows.

By part (i) and (B.6), $U_a$ is strictly increasing on $(a, \infty)$. Then $\lim_{x \to \infty} U_a(x)$ exists and suppose it is finite. Then, $t_a = 0$ and $r_a < 0$ in (B.8). Thus, $\lim_{x \to \infty} U_a(x) = 0$ and by (B.6), $\lim_{x \to \infty} W_a(x) = T_s$. It contradicts with (B.9) since $W_a$ is increasing on $(a, \infty)$. Hence, $\lim_{x \to \infty} U_a(x) = \infty$ and by (B.6), part (ii) follows. □

Lemma B.5. There exists a point $c < 0$ so that

(i) the solution $W_c$ is strictly increasing in $(c, \infty)$;

(ii) $W_c$ is bounded in $(c, \infty)$ and $\lim_{x \to \infty} W_c(x) = T_s$.

Proof. Let $c = \sup\{a < 0 : \lim_{x \to \infty} W_a(x) = \infty\}$. By Lemmas B.3 and B.4, such a point $c$ exists and $-\infty < c < 0$. Moreover, $W_c$ is strictly increasing on $(c, 0)$ and thus $W_c'(0) \geq 0$. Suppose $W_c'(0) = 0$, then $W''(0-) > 0$ from (B.5) and by the fact that $W_c(0) > -p_b > -T_b$. This is a contradiction since $W_c$ is strictly increasing in
(c, 0). Hence \( W'_c(0) > 0 \) and there is a \( \delta > 0 \) so that \( W'_c > 0 \) over \((c, \delta)\). We let 
\[ x_0 = \sup\{x > 0 : W'_c(u) > 0 \text{ for all } 0 < u < x\}. \]
Clearly, \( x_0 > \delta \). Suppose that \( x_0 \) is finite. Then \( W'_c(x_0) = 0 \). If \( W_c(x_0) > \theta_s/(\alpha + \delta_s) \), using (B.5), \( W'_c(x_0) > 0 \). This is a contradiction since \( W_c \) is strictly increasing in \((0, x_0)\). Thus \( W_c(x_0) = T_s \). Now if \( W_c(x_0) = T_s \), by the uniqueness of solutions, \( W_c \) is a constant on \((0, \infty)\) and this contradicts with \( W'_c(0) > 0 \). Hence the only possibility is \( W_c(x_0) < T_s \), in which case \( W''_c(x_0) < 0 \) and \( x = x_0 \) is a strict local maximum and \( W_c \) is strictly decreasing on \((x_0, \infty)\). Similar to the proof of Lemma B.3 (ii), we see that \( \lim_{x \to \infty} W_c(x) = -\infty \).

Now using the continuity of the solutions \( W_a \) with respect to the parameter \( a \), we can find a \( \delta_2 > 0 \) so that for each \( c - \delta_2 < a < c \), \( \lim_{x \to \infty} W_a(x) = -\infty \). This contradicts with the definition of \( c \). Hence \( x_0 \) is infinite and part (i) follows.

From part (i), \( \lim_{x \to \infty} W_c(x) = L \leq \infty \) exists. Suppose \( L = \infty \). We can pick \( M > 0 \) so that \( W_c(M) > 1 + T_s \). Using the continuity of \( W_a \) and \( W'_a \) with respect to \( a \), we can find a \( \delta_3 > 0 \) so that for each \( c < a < c + \delta_3 \), \( W_a(M) > T_s \), and \( W'_a(x) > 0 \) on \((0, M)\).

By the proof in Lemma B.4, each such \( W_a \) increases to \( \infty \) and this contradicts with the definition of \( c \). Hence \( L < \infty \) and \( W_c \) is bounded in \([c, \infty)\) and equivalently, \( U_c \) defined in (B.6) is also bounded. By (B.8), we can write \( U_c(x) = r_c \Psi_0(x) + t_c \Psi_1(x) \)
\[ \text{for } x > 0. \]
Since \( U_c \) is bounded, we have \( t_c = 0 \) and thus, \( U_c(x) = r_c \Psi_0(x) \) for \( x > 0 \). But \( \lim_{x \to \infty} \Psi(x) = 0 \) and consequently \( \lim_{x \to \infty} U_c(x) = 0 \). Using this together with (B.6), we obtain \( \lim_{x \to \infty} W_c(x) = T_s \). This completes the proof. \( \square \)

In the next lemma, we establish the uniqueness of the point \( c \).

**Lemma B.6.** For \( a < 0 \), let \( W_a \) be the corresponding solution of (4.18) so that \( \lim_{x \to \infty} W_a(x) \) is finite. Then \( a = c \) and \( W_a(x) \equiv W_c(x) \) for all \( x \).

_Proof._ Suppose \( a \neq c \) and \( W_a \) satisfies the assertion. Without loss of generality, let \( c < a \). We note that \( W_a(a) = W_c(c) = -p_b \) and from Lemma B.2, \( W_a \) and \( W_c \) are strictly decreasing on \((-\infty, a)\) and \((-\infty, c)\) respectively, and are strictly increasing on \((a, 0)\) and \((c, 0)\) respectively. Hence \( W_a \) and \( W_c \) meet at a point \( x_0 \), where \( c < x_0 < a \), and moreover, \( W_a(a) = -p_a \). Next, following the argument of Lemma B.5 (ii), we conclude that \( \lim_{x \to \infty} W_a(x) = \lim_{x \to \infty} W_c(x) = T_s \).

We now consider \( H(x) = W_c(x) - W_a(x) \) for \( x_0 \leq x < \infty \). Then \( H(x_0) = 0 \), \( H'(\infty) = 0 \) and \( H'(a) > 0 \). Moreover, \( H \) is a bounded solution to the homogeneous equation \( \mathcal{G}H(x) - \gamma(x)H(x) = 0 \). We let \( m > x_0 \) so that \( H(m) = \max_{x \in [x_0, \infty)} H(x) \). Then \( H(m) > 0 \) and \( H'(m) = 0 \). When \( m \neq 0 \), using the differential equation for \( H \) at \( x = m \), we obtain \( H''(m) > 0 \) and thus, \( x = m \) is a local minimum, which contradicts with the definition of \( H(m) \). Hence \( m = 0 \). Again, by the differential equation, we obtain \( H''(0-) > 0 \) and thus \( H \) is strictly decreasing in \((-\delta, 0)\) for some \( \delta > 0 \), which is again a contradiction because \( H(m) \) is the maximum. Hence \( c = a \) and the conclusion of the lemma follows. \( \square \)

Let \( c = \sup\{a < 0 : \lim_{x \to \infty} W_a(x) = \infty\} \) be as in Lemma B.5.

**Lemma B.7.** For \( a \in (c, 0) \), \( W_a(x) < W_c(x) \) for all \( x > a \), \( W_a \) has a unique local maximum on \((0, \infty)\) and thereafter it is decreasing to \(-\infty\), i.e., \( \lim_{x \to \infty} W_a(x) = -\infty \).

Moreover, if \( c < a_1 < a_2 < 0 \), then \( W_{a_2}(x) < W_{a_1}(x) < W_c(x) \) for all \( x > a_2 \).

_Proof._ Similar to the argument in the proof of Lemma B.6, \( W_c \) and \( W_a \) meet at a point \( x_0 \in (c, a) \), and \( W_c(a) > W_a(a) \). Suppose that \( W_c \) and \( W_a \) meet at a point \( z > a \). We consider \( H(x) = W_c(x) - W_a(x) \) for \( x_0 \leq x \leq z \), and define \( m = \arg \max_{x \in [x_0, z]} H(x) \). Following an argument similar to the proof of Lemma B.6, we can obtain a contradiction.
on $H(m)$, and thus $W_a(x) < W_c(x)$ for all $x > a$. Consequently, $W_a(x) < T_s$ for all $x > a$. By Lemma B.1, $W_a$ can have at most one local maximum and no local minima. If $W_a$ has no local maxima, then it is increasing and bounded above by $\theta_s/(\alpha + \delta_s)$, and $\lim_{x \to \infty} W_a(x)$ is finite. Using Lemma B.6, we have $a = c$ and this is a contradiction. Therefore, each $W_a$ has exactly one local maximum in $(0, \infty)$ and thereafter $W_a$ is decreasing. Again using Lemma B.6, we must have $\lim_{x \to \infty} W_a(x) = -\infty$. The proof of the last assertion is very similar to the above proof of $W_a(x) < W_c(x)$ for all $x > a$. And it is omitted. This completes the proof. □

For $a \in (c, 0)$, assume that each $W_a$ achieves the unique local maximum at $x = r_a$. We introduce the function $M : (c, 0) \to (-p_b, T_s)$ by

$$
M(a) \equiv \max_{x \in [a, \infty)} W_a(x) = W_a(r_a).
$$

(B.10)

**Proposition B.8.** The following results hold for the function $M$ defined in (B.10).

(i) The function $M$ is a continuous strictly decreasing function.

(ii) $\lim_{a \to c+} M(a) = T_s$ and $\lim_{a \to 0-} M(a) = -p_b$ and $M$ is an on-to function.

(iii) Given $0 < p_s < T_s$, there is a unique $a^*$ in $(c, 0)$ so that $M(a^*) = p_s$. Let $b^* \equiv r_{a^*} > 0$. Then the corresponding $W_{a^*}$ is strictly increasing in $(a^*, b^*)$ and $W_{a^*}(b^*) = M(a^*) = p_s > 0$ and the triple $(W_{a^*}, a^*, b^*)$ satisfies all the conditions in Proposition 4.4.

**Proof.** The strictly decreasing property of the function $M$ follows from Lemma B.7. To show the continuity of $M$, we fix $a_0$ so that $c < a_0 < 0$. Since the solution to (4.18) is continuous with respect to the initial data, for a given $\epsilon_1 > 0$, there exists a $\delta > 0$ so that $a \in (a_0 - \delta, a_0 + \delta) \subset (c, 0)$ implies $|W_a(0) - W_{a_0}(0)| + |W'_a(0) - W'_{a_0}(0)| < \epsilon_1$. Let $b \in (c, a_0 - \delta)$, and choose $z_0 > 0$ so that $W_b(z_0) < 0$. Next, $W_a$ satisfies (B.5) on $(0, \infty)$ and the coefficients of this differential equation can be smoothly extended to $(c, \infty)$. Using the continuity property of initial data for this extended linear equation, for any $\epsilon > 0$, we can find $\delta_2 > 0$ so that $|W_a(0) - W_{a_0}(0)| + |W'_a(0) - W'_{a_0}(0)| < \delta_2$ guarantees $\sup_{x \in [0, z_0]} |W_a(x) - W_{a_0}(x)| < \epsilon$. Now choose $\epsilon_1 = \delta_2$. Then $|a_0 - a| < \delta$ implies $\sup_{x \in [0, z_0]} |W_a(x) - W_{a_0}(x)| < \epsilon$. By Lemma B.7, $W_b(z_0) < 0$ implies $W_a(z) < 0$ for all $a \in (b, 0)$ and $z \geq z_0$. Consequently, when $a - \delta < a_0 < a$, from Lemma B.7, $0 < W_a(r_{a_0}) \leq W_a(r_a) < W_{a_0}(r_a) \leq W_a(r_{a_0})$. Thus, $|M(a) - M(a_0)| = W_{a_0}(r_{a_0}) - W_a(r_{a_0}) \leq \sup_{x \in [0, z_0]} |W_a(x) - W_{a_0}(x)| < \epsilon$. Similarly, when $a_0 - \delta < a < a_0$, it follows that $|M(a) - M(a_0)| \leq \sup_{x \in [0, z_0]} |W_a(x) - W_{a_0}(x)| < \epsilon$. This proves the continuity of $M$ at $a_0$. The proof of part (i) is complete.

Since $M$ is strictly decreasing and $T_s$ is an upper bound, $\lim_{a \to c+} M(a)$ exists and it is less than or equal to $T_s$. Again by the continuity property of the initial data, $W_c$ is close to $W_a$ when $a$ is very close to $c$. Using this together with Lemma B.5 (ii), it follows that $\lim_{a \to c+} M(a) = T_s$. The proof of $\lim_{a \to 0-} M(a) = -p_b$ is similar. These limits together with part (i) implies that $M$ is an onto function.

Finally, part (iii) is a consequence of parts (i) and (ii). This completes the proof. □

**Remark B.2.** For the proof of Theorem 4.2 (iii), let $(c, W_c)$ be as in the Lemma B.5. Take $\tilde{a}^* = c$, and let $W_L$ identically equal to $W_c$. Then the pair $(\tilde{a}^*, W_L)$ satisfies (4.25).

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