ON THE ISOTYPIC DECOMPOSITION OF COHOMOLOGY MODULES OF SYMMETRIC SEMI-ALGEBRAIC SETS: POLYNOMIAL BOUNDS ON MULTIPlicITIES

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ABSTRACT. We consider symmetric (as well as multi-symmetric) real algebraic varieties and semi-algebraic sets, as well as symmetric complex varieties in affine and projective spaces, defined by polynomials of fixed degrees. We give polynomial (in the dimension of the ambient space) bounds on the number of irreducible representations of the symmetric group which acts on these sets, as well as their multiplicities, appearing in the isotypic decomposition of their cohomology modules with coefficients in a field of characteristic 0. We also give some applications of our methods in proving lower bounds on the degrees of defining polynomials of certain symmetric semi-algebraic sets, as well as better bounds on the Betti numbers of the images under projections of (not necessarily symmetric) bounded real algebraic sets.

We conjecture that the multiplicities of the irreducible representations of the symmetric group in the cohomology modules of symmetric semi-algebraic sets defined by polynomials having fixed degrees are computable with polynomial complexity, which would imply that the Betti numbers of such sets are also computable with polynomial complexity. This is in contrast with general semi-algebraic sets, for which this problem is provably hard (\#P-hard). We also formulate a question asking whether these multiplicities can be expressed as a polynomial in the number \(k\) of variables for all large enough \(k\).

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1. Introduction

Throughout this paper \( \mathbb{R} \) will denote a fixed real closed field and \( \mathbb{C} \) the algebraic closure of \( \mathbb{R} \). We also fix a field \( F \) of characteristic 0. For any closed semi-algebraic set \( S \) we will denote by \( b^i(S, F) \) the dimension of the \( i \)-th cohomology group, \( H^i(S, F) \), and by \( b(S, F) = \sum_{i \geq 0} b^i(S, F) \). (We refer the reader to [7, Chapter 6] for the definition of homology/cohomology groups of semi-algebraic sets defined over arbitrary real closed fields, noting that they are isomorphic to the singular homology/cohomology groups in the special case of \( \mathbb{R} = \mathbb{R} \).)

1.1. History and motivation. The problem of obtaining quantitative bounds on the topology measured by the the Betti numbers of real semi-algebraic as well as complex constructible sets in terms of the degrees and the number of defining polynomials is very well studied (see for example, [6] for a survey). For semi-algebraic (respectively, constructible) subsets of \( \mathbb{R}^k \) (respectively, \( \mathbb{C}^k \)) defined by \( s \) polynomials of degrees bounded by \( d \), these bounds are typically exponential in \( k \), and polynomial (for fixed \( k \)) in \( s \) and \( d \).

More precisely, suppose that \( S \) is a semi-algebraic (resp. constructible) subset of \( \mathbb{R}^k \) (resp. \( \mathbb{C}^k \)) defined by a quantifier-free formula involving \( s \) polynomials in \( \mathbb{R}[X_1, \ldots, X_k] \) (resp. \( \mathbb{C}[X_1, \ldots, X_k] \)) of degrees bounded by \( d \).

**Theorem 1** (Olešnik and Petrovskii [23], Thom [28], Milnor [21]),

\[
b(S) \leq (sd)^{O(k)}.
\]

The single exponential dependence on \( k \) of the bound in Theorem 1 is unavoidable. In the real case it suffices to consider the real variety

\[
V_k = \{-1, 1\}^k \subset \mathbb{R}^k
\]
defined by the polynomial

\[ F_k = \sum_{i=1}^{k} \prod_{i=1}^{d} (X_i - i)^2. \]

It is easy to see that \( \deg(F_k) = 2d \), and \( b_0(V_k) = d^k \).

In the complex case, it follows from a classical formula of algebraic geometry [20] that the sum of the Betti numbers of a non-singular hypersurface \( V_k \subset \mathbb{P}_C^k \) of degree \( d \) is asymptotically \( \Theta(d^k) \). Using a standard excision argument and induction on dimension, the same asymptotic estimate on the Betti numbers hold for the affine part of such a variety as well.

The problem of obtaining tighter estimates on the Betti numbers of semi-algebraic sets (motivated partly by applications in other areas of mathematics and theoretical computer science) has been considered by several authors [2, 19, 5]. The algorithmic problem of designing efficient algorithms for computing these invariants has attracted attention as well [8, 3]. Most of this work has concentrated on the real semi-algebraic case (since using the real structure a complex constructible subset \( S \subset \mathbb{C}^k \) can be considered as a real semi-algebraic subset of \( \mathbb{R}^{2k} \) defined by twice as many polynomials of the same degrees as those defining \( S \)), but the complex case has also being considered separately as well [27, 30]. From the point of view of algorithmic complexity, the problem of computing the Betti numbers is provably a hard problem – and so in its full generality a polynomial time algorithm for solving this problem is not to be expected, except in special situations (see [11, 4] for some of these exceptional cases). However, even an algorithm with a singly exponential complexity is not known for computing all the Betti numbers.

It is a (unproven) meta-theorem in algorithmic semi-algebraic geometry – that the worst-case topological complexity of a class of semi-algebraic sets (measured by the Betti numbers for example) serve as a rough lower bound for the complexity of algorithms for computing topological invariants or deciding topological properties of this class of sets. So the best complexity known for algorithms for determining whether a general semi-algebraic set is empty or connected is singly exponential, reflecting the singly exponential behavior of the topological complexity of such sets as exhibited by the example given in Example 1. This is true even if the degrees of the polynomials describing the given set is bounded by some constant \( > 2 \). On the other hand there are certain classes of semi-algebraic sets where the situation is better. For example, for semi-algebraic sets defined by few (i.e. any constant number of) quadratic inequalities, we have polynomial upper bounds on the Betti numbers [1], as well as algorithms with polynomial complexities for computing them [11].

The problem of proving bounds on Betti numbers of semi-algebraic sets defined by polynomials of degrees bounded by some constant and admitting an action of a product of symmetric groups is considered in [9]. (Note that the topological structure of varieties (also symmetric spaces) admitting actions of Lie groups is a very well-studied topic (see for example [22]). Here we concentrate on the action of finite reflection groups, which seems to be a less developed field of study.)

It is intuitively clear that the symmetry imposes strong restrictions on the topology of such sets. Nevertheless, as shown in Example 1 below, the Betti numbers of such sets can be exponentially large. However, when the degrees of the defining polynomials are fixed, a polynomial bound is proved on the equivariant Betti
numbers of such sets in [9]. Moreover, an algorithm with polynomially bounded complexity is given in [9] for computing the (generalized) Euler-Poincaré characteristics of such sets. Thus, from the point of view of the meta-theorem of the previous paragraph, symmetric semi-algebraic sets pose a dilemma. On one hand their Betti numbers can be exponentially large in the worst case, on the other hand there are reasons to believe that their topological invariants (when the degree is fixed) has some structure allowing for efficient computation. The polynomial bound on the equivariant Betti numbers proved in [9] is the first indication of such a structure.

In this paper, we generalize (as well as sharpen) the results in [9] in several directions. The action of the symmetric group $S_k$ on a symmetric semi-algebraic subset $S \subset \mathbb{R}^k$, induces an action on the cohomology groups $H^*(S, \mathbb{F})$, making $H^*(S, \mathbb{F})$ into a (finite dimensional) $S_k$-representation.

**Remark 1 (Homology versus cohomology).** Note that since $\mathbb{F}$ is a field, $H^*(S, \mathbb{F}) \cong \text{hom}(H_*(S, \mathbb{F}), \mathbb{F})$ as vector spaces. Moreover, from the basic property of $S_k$ that the conjugacy class of an element equals that of its inverse it follows that for any finite-dimensional representation $W$ of $S_k$, $\text{hom}(W, \mathbb{F}) \cong W$ as $S_k$-modules.

The equivariant cohomology group, $H^*_S(S, \mathbb{F})$, is isomorphic to the subspace of $H^*(S, \mathbb{F})^{S_k}$ that is fixed by $S_k$. Another description of $H^*(S, \mathbb{F})^{S_k}$ is that it is the isotypic component of $H^*(S, \mathbb{F})$ corresponding to the trivial one-dimensional representation of $S_k$, and the dimension of $H^*(S, \mathbb{F})^{S_k}$ is equal to the multiplicity of the trivial representation in $H^*(S, \mathbb{F})$. Thus, the main result of [9] can be expressed as saying that the multiplicity of the trivial representation in $H^*(S, \mathbb{F})$ is bounded polynomially. It is well known (see Section 3.6 below) that the irreducible representations of $S_k$ (the so called Specht-modules) are in correspondence with the finite set of partitions of $k$. The number of partitions of $k$ is exponentially large.

In this paper, we extend the results in [9] by proving a polynomial bound on both the number of irreducible representations appearing with positive multiplicities in $H^*(S, \mathbb{F})$ as well as on their multiplicities. Some of these representations can have dimensions which are exponentially large (as is unavoidable since the dimension of $H^*(S, \mathbb{F})$ can be exponentially large as in Example 1). Thus, we prove that while the Betti numbers of symmetric semi-algebraic sets can be exponentially large, they can be expressed as a sum of polynomially many numbers (the dimensions of the isotypic components), and each of these numbers is a product of a multiplicity (which is polynomially bounded) and the dimension of a Specht module (which can be exponentially large, but efficiently computable due to the hook formula (see Theorem 14).

We also conjecture that all these (polynomially bounded) multiplicities of all representations that appear in the isotypic decomposition of $H^*(S, \mathbb{F})$, and hence all the Betti numbers, $b_i(S, \mathbb{F})$, of any given symmetric semi-algebraic set $S$ defined by polynomials having degrees bounded by some fixed constant, are computable with polynomially bounded complexity. See Conjecture 1 for a more precise statement.

### 1.2. Basic notation and definition

In this section we introduce notation and definitions that we will use for the rest of the paper.

**Notation 1.** For $P \in \mathbb{R}[X_1, \ldots, X_k]$ (respectively $P \in \mathbb{C}[X_1, \ldots, X_k]$) we denote by $Z(P, \mathbb{R}^k)$ (respectively $Z(P, \mathbb{C}^k)$) the set of zeros of $P$ in $\mathbb{R}^k$ (respectively $\mathbb{C}^k$). More generally, for any finite set $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$ (respectively $\mathcal{P} \subset \mathbb{C}^k$).
C[X_1,\ldots, X_k]), we denote by Z(\mathcal{P}, R^k) (respectively Z(\mathcal{P}, C^k)) the set of common zeros of \mathcal{P} in R^k (respectively C^k). For a homogeneous polynomial \mathcal{P} \in R[X_0, \ldots, X_{k-1}] (respectively \mathcal{P} \in C[X_0, \ldots, X_{k-1}]) we denote by Z(\mathcal{P}, \mathbb{P}_C^{k-1}) (respectively Z(\mathcal{P}, \mathbb{P}_C^{k-1})) the set of zeros of \mathcal{P} in \mathbb{P}_C^{k-1} (respectively \mathbb{P}_C^{k-1}). And, more generally, for any finite set of homogeneous polynomials \mathcal{P} \subset R[X_0, \ldots, X_{k-1}] (respectively \mathcal{P} \subset C[X_0, \ldots, X_{k-1}]), we denote by Z(\mathcal{P}, \mathbb{P}_C^{k-1}) (respectively Z(\mathcal{P}, \mathbb{P}_C^{k-1})) the set of common zeros of \mathcal{P} in \mathbb{P}_C^{k-1} (respectively \mathbb{P}_C^{k-1}).

**Notation 2.** For any finite family of polynomials \mathcal{P} \subset R[X_1, \ldots, X_k], we call an element \sigma \in \{0, 1, -1\}^\mathcal{P}, a sign condition on \mathcal{P}. For any semi-algebraic set \mathcal{Z} \subset R^k, and a sign condition \sigma \in \{0, 1, -1\}^\mathcal{P}, we denote by \mathcal{R}(\sigma, \mathcal{Z}) the semi-algebraic set defined by

\[
\{ x \in \mathcal{Z} \mid \text{sign}(P(x)) = \sigma(P), P \in \mathcal{P} \},
\]

and call it the realization of \sigma on \mathcal{Z}. More generally, we call any Boolean formula \Phi with atoms, P\{=, >, \leq\}\}{0, P \in \mathcal{P}, to be a \mathcal{P}-formula. We call the realization of \Phi, namely the semi-algebraic set

\[
\mathcal{R}(\Phi, R^k) = \{ x \in R^k \mid \Phi(x) \}
\]
a \mathcal{P}-semi-algebraic set. Finally, we call a Boolean formula without negations, and with atoms P\{=, >, \leq\}\}{0, P \in \mathcal{P}, to be a \mathcal{P}-closed formula, and we call the realization, \mathcal{R}(\Phi, R^k), a \mathcal{P}-closed semi-algebraic set.

The notion of partitions of a given integer will play an important role in what follows, which necessitates the following notation that we fix for the remainder of the paper.

**Notation 3** (Partitions). We denote by Par(k) the set of partitions of k, where each partition \pi \in Par(k) (also denoted \lambda \vdash k) is a tuple (\pi_1, \pi_2, \ldots, \pi_\ell), with \pi_1 \geq \pi_2 \geq \cdots \geq \pi_\ell \geq 1, and \pi_1 + \pi_2 + \cdots + \pi_\ell = k. We call \ell the length of the partition \pi, and denote length(\pi) = \ell.

More generally, for any tuple k = (k_1, \ldots, k_\ell) \in N^\ell, we will denote by Par(k) = Par(k_1) \times \cdots \times Par(k_\ell), and for each \pi = (\pi^{(1)}, \ldots, \pi^{(\ell)}) \in Par(k), we denote by length(\pi) = \sum_{i=1}^{\ell} length(\pi^{(i)}). We also denote for each p = (p_1, \ldots, p_\ell) \in N^\ell,

\[
|p| = p_1 + \cdots + p_\ell,
\]

\[
F(k, p) = \text{card}\{\pi = (\pi^{(1)}, \ldots, \pi^{(\ell)}) \mid \text{length}(\pi^{(i)}) = p_i, 1 \leq i \leq \ell\}.
\]

**Notation 4** (Transpose of a partition and partitions of bounded lengths). For a partition \lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash k, we will denote by \lambda the transpose of \lambda. More precisely, \lambda = (\lambda_1, \ldots, \lambda_\ell), where \lambda_i = \text{card}\{i \mid \lambda_i \geq j\}. For k, d \geq 0, we denote

\[
\text{Par}(k, d) := \{ \lambda \in \text{Par}(k) \mid \text{length}(\lambda) \leq d \}
\]

More generally, for k = (k_1, \ldots, k_\ell), d = (d_1, \ldots, d_\ell) we denote

\[
\text{Par}(k, d) := \{ \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \mid \lambda^{(i)} \in \text{Par}(k_i), \text{length}(\lambda^{(i)}) \leq d_i, 1 \leq i \leq \ell\}
\]

When d = (d, \ldots, d), we will also use Par(k, d) to denote Par(k, d).

**Notation 5** (Products of symmetric groups). For each k \in N, we denote by \mathfrak{S}_k the symmetric group on k letters (or equivalently the Coxeter group A_{k-1}). For k = (k_1, \ldots, k_\ell) \in N^\ell we denote by \mathfrak{S}_k the product group \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_\ell}, and we will usually denote k = |k| = \sum_{i=1}^{\ell} k_i.
**Definition 1** ($\mathfrak{S}_k$-symmetric polynomials). Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Suppose that $k = (k_1, \ldots, k_\ell)$, $m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$, and let $P \in \mathbb{K}[X^{(1)}, \ldots, X^{(\ell)}]$ where for $1 \leq h \leq \ell$, $X^{(h)} = \left( X_{i,j}^{(h)} \right)_{1 \leq i \leq k_h, 1 \leq j \leq m_h}$.

The group $\mathfrak{S}_k$, acts on $\mathbb{K}[X^{(1)}, \ldots, X^{(\ell)}]$ by permuting for each $i, 1 \leq i \leq \ell$, the rows of $X^{(h)}$ by the group $\mathfrak{S}_{k_h}$. For $\pi \in \mathfrak{S}_k$, and $P \in \mathbb{K}[X^{(1)}, \ldots, X^{(\ell)}]$, we denote the by $\pi \cdot P$ the image of $P$ under $\pi$. We say that $P$ is $\mathfrak{S}_k$-symmetric if it is invariant under the action of $\mathfrak{S}_k$, i.e. if $\pi \cdot P = P$ for every $\pi \in \mathfrak{S}_k$. Similarly, we say that a subset $S \subset \mathbb{K}^k, K = \sum_{1 \leq i \leq \ell} k_i m_i$, is $\mathfrak{S}_k$-symmetric if it is stable under the above action of $\mathfrak{S}_k$.

When $\ell = 1, m_1 = 1$, and $K = k_1 m_1 = k$, the action defined above is the usual action of $\mathfrak{S}_k$ on $\mathbb{K}^k$ permuting coordinates.

**Remark 2.** Note in case $\mathbb{K} = \mathbb{C}$, the action of $\mathfrak{S}_k$ on $\mathbb{C}^K$ defined above in Definition 1 can also be seen as the action of $\mathfrak{S}_k$ on $\mathbb{R}^{2K}$ (considering $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$), replacing $m$ by $2m$.

1.3. Basic example. Before proceeding further, we discuss an example which is our guiding example for the rest of the paper. While explaining the example we will need to refer to certain classical notions from the representation theory of symmetric groups, which we recall later in the paper (and which the reader can refer to if needed).

**Example 1** (Real affine case). Let

$$F_k = \sum_{i=1}^{k} X_i^2 (X_i - 1)^2 - \varepsilon,$$

and

$$V_k = \mathbb{Z}(F_k, \mathbb{R}^k).$$

Then, for all $\varepsilon, 0 < \varepsilon \ll 1$, $V_k$ is a compact non-singular hypersurface in $\mathbb{R}^k$, (in fact also in $\mathbb{P}_k^{\ell}$), the semi-algebraic set $S_k$ defined by $F_k \leq 0$ is homotopy equivalent to the finite set of points $\{0, 1\}^k$, and is bounded by $V_k$.

Clearly, $b_0(V_k, \mathbb{F}) = 2^k$, and it follows from Poincaré duality applied to $V_k$ that $b_{k-1}(V_k, \mathbb{F}) = 2^k$ as well. It also follows from Alexander-Lefshetz duality that $H^i(V_k, \mathbb{F}) = 0$ for $0 < i < k - 1$.

The real algebraic variety $V_k$ is symmetric under the standard action of the symmetric group $\mathfrak{S}_k$ on $\mathbb{R}^k$ permuting the coordinates. This action induces an $\mathfrak{S}_k$-module structure on $H^i(V_k, \mathbb{F})$, and it is interesting to study the isotypic decomposition of this representation into its isotypic components corresponding to the various irreducible representations of $\mathfrak{S}_k$, namely the Specht modules $S^\lambda$ indexed by different partitions $\lambda \vdash k$ (see for example [24] for the definition of Specht modules).

We now describe this decomposition. Clearly,

$$H^0(V_k, \mathbb{F}) \cong \bigoplus_{0 \leq i \leq k} H^0(V_{k,i}, \mathbb{F}),$$
where for \(0 \leq i \leq k\), \(V_{k,i}\) is the \(S_k\)-orbit of the connected component of \(V_k\) infinitesimally close (as a function of \(\varepsilon\)) to the point \(x^i = (0, \ldots, 0, 1, \ldots, 1)\), and \(H^0(V_{k,i}, F)\) is a sub-representation of \(H^0(V_k, F)\).

It is also clear that the isotropy subgroup of the class in \(H^0(V_k, F)\) corresponding to \(V_{k,i}\) is isomorphic to \(S_i \times S_{k-i}\), and hence,

\[
H^0(V_{k,i}, F) \cong \text{Ind}_{S_i \times S_{k-i}}^{S_k} (S^i \boxtimes S^{(k-i)})
\]

\[
\cong M^{(i,k-i)} \text{ if } i \geq k - i,
\]

\[
\cong M^{(k-i,i)} \text{ otherwise.}
\]

where for any \(\lambda \vdash k\), we denote by \(M^\lambda\) the Young module corresponding to \(\lambda\) (see Definition 4).

Also, observe that \(H^0(V_{k,i}, F)\) and \(H^0(V_{k,k-i}, F)\) are isomorphic as \(S_k\)-modules. In the following, for partitions \(\mu, \lambda \vdash k\), we will denote by \(K(\mu, \lambda)\) the corresponding Kostka number (see Definition 6 below). For this example, it is sufficient to observe that if \(\mu \succeq \lambda\) (see Definition 5 for the definition of the dominance order \(\succeq\) on the set of partitions), and if \(\mu\) has at most 2 rows, then \(K(\mu, \lambda) = 1\). It now follows from Proposition 5 that for \(k\) odd,

\[
H^0(V_k, F) \cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq 2} (M^\lambda \oplus M^{\lambda^\vee})
\]

\[
\cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq 2} 2K(\mu, \lambda)S^\mu
\]

\[
\cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq 2} 2\ell^\mu
\]

\[
\cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq 2} m_\mu S^\mu,
\]

where for each \(\mu = (\mu_1, \mu_2) \vdash k\),

\[
m_\mu = 2(\mu_1 - \lfloor k/2 \rfloor)
\]

\[= 2\mu_1 - k + 1.
\]

(1.3)
For \( k \) even we have,

\[
\mathcal{H}^0(V_k, \mathbb{F}) \cong \left( \bigoplus_{\lambda \vdash k \atop \ell_{\lambda} \leq 2} (M^\lambda \oplus M^\lambda) \right) \bigoplus (M^{(k/2, k/2)\oplus M})
\]

\[
\cong \left( \bigoplus_{\lambda \vdash k \atop \ell_{\lambda} \leq 2} \mu \geq \lambda \leq 2 \mu \neq (k/2, k/2) \right) \bigoplus K(\mu, (k/2, k/2)S^\mu)
\]

\[
\cong \bigoplus_{\mu \vdash k \atop \ell_{\mu} \leq 2} m_\mu S^\mu,
\]

where for each \( \mu = (\mu_1, \mu_2) \vdash k \),

\[
m_{\mu} = 2(\mu_1 - k/2) + 1 = 2\mu_1 - k + 1.
\]

(1.4)

We deduce for all \( k \),

\[
m_{\mu} = 2\mu_1 - k + 1 \leq k + 1.
\]

(1.5)

For \( \mu = (\mu_1, \mu_2) \vdash k \), by the hook-length formula (Theorem 14) we have,

\[
\dim S^\mu = \frac{k! (\mu_1 - \mu_2 + 1)}{\mu_1! \mu_2!}.
\]

(1.6)

This completes the description of the isotypic decomposition of \( \mathcal{H}^0(V_k, \mathbb{F}) \).

In particular for \( k = 2, 3 \) we have:

\[
\mathcal{H}^0(V_2, \mathbb{F}) \cong 3S^{(2)} \oplus S^{(1,1)},
\]

\[
\mathcal{H}^0(V_3, \mathbb{F}) \cong 4S^{(3)} \oplus 2S^{(2,1)}.
\]

The isotypic decomposition of \( \mathcal{H}^{k-1}(V_k, \mathbb{F}) \) requires one further ingredient – namely, an \( S_k \)-equivariant version of the classical Poincaré duality theorem for oriented manifolds. We include a proof of this result (Theorem 15) in Section 3.7.

We note that \( V_k \) is a compact real orientable manifold, by Poincaré duality theorem there exists an isomorphism between \( \mathcal{H}^0(V, \mathbb{F}) \) and \( \mathcal{H}^{k-1}(V, \mathbb{F}) \). This isomorphism is not necessarily a \( S_k \)-module isomorphism. However, it follows from Theorem 15 (which is a stronger form of Poincaré duality for orientable symmetric manifolds) that the isotypic representation of \( \mathcal{H}^{k-1}(V_k, \mathbb{F}) \) is isomorphic (as an \( S_k \)-module) to \( \mathcal{H}^0(V_k, \mathbb{F}) \otimes \text{sign}_k \).

Thus, denoting for each \( \lambda \vdash k \), the transpose of the partition \( \lambda \) by \( \tilde{\lambda} \),

\[
\mathcal{H}^{k-1}(V_k, \mathbb{F}) \cong \bigoplus_{\mu \vdash k \atop \ell_{\mu} \leq 2} m_\mu S^\tilde{\mu},
\]
where for each \( \mu = (\mu_1, \mu_2) \vdash k \), \( m_\mu \) is defined above in (1.5). In particular for \( k = 2, 3 \) we have:

\[
\begin{align*}
H^1(V_2, \mathbb{F}) &\cong 3S^{(1,1)} \oplus S^{(2)}, \\
H^2(V_3, \mathbb{F}) &\cong 4S^{(1,1,1)} \oplus 2S^{(2,1)}.
\end{align*}
\]

Notice that the multiplicity \( m_{1^k} \) of the Specht module \( S^{1^k} = \text{sign} \) in \( H^0(V_k, \mathbb{F}) \) is equal to 0 for \( k > 2 \). This implies that the multiplicity of the trivial representation \( S^{(k)} \) is equal to 0 in \( H^{k-1}(V_k, \mathbb{F}) \), and thus \( H^{k-1}_{S_k}(V_k, \mathbb{F}) = 0 \) as well (for \( k > 2 \)).

Also, notice that the multiplicity of each Specht-module, \( S^\mu, \mu \vdash k, \) in the isotypic decomposition of \( H^*(V_k, \mathbb{F}) \) is bounded polynomially (in fact, linearly) in \( k \), but the dimension of \( H^*(V_k, \mathbb{F}) \) itself is exponentially large in \( k \).

As an aside note that since \( \dim H^0(V_k, \mathbb{F}) = 2^k \), we obtain as a consequence (from (1.5) and (1.6)) the somewhat interesting identity

\[
k! \left( \sum_{\mu_1 \geq \mu_2 \geq 0 \atop \mu_1 + \mu_2 = k} \frac{(\mu_1 - \mu_2 + 1)^2}{(\mu_1 + 1)!\mu_2!} \right) = 2^k.
\]

**Example 2** (Projective case). Let

\[
P = \sum_{0 \leq i < j \leq k-1} (X_i^2 - X_j^2)^2,
\]

and let \( W_k = Z(P, \mathbb{P}_{\mathbb{R}}^{k-1}) \). Then,

\[
W_k = \{(x_0 : \cdots : x_{k-1}) \mid x_i = \pm 1, 0 \leq i \leq k - 1 \},
\]

and is symmetric under the action of \( \mathfrak{S}_k \) on \( \mathbb{P}_{\mathbb{R}}^{k-1} \) permuting the homogeneous coordinates.

It is clear that

\[
H^0(W_k, \mathbb{F}) \cong H^0(V_k, \mathbb{F}),
\]

where \( V_k \) is the real affine variety defined in (1.2), and the stated isomorphism is an \( \mathfrak{S}_k \)-modules.

**1.4. Equivariant cohomology.** We recall also the definition of **equivariant cohomology groups** of a \( G \)-space for an arbitrary compact Lie group \( G \). For \( G \) any compact Lie group, there exists a **universal principal \( G \)-space**, denoted \( EG \), which is contractible, and on which the group \( G \) acts freely on the right. The **classifying space** \( BG \), is the orbit space of this action, i.e. \( BG = EG/G \).

**Definition 2.** (Borel construction) Let \( X \) be a space on which the group \( G \) acts on the left. Then, \( G \) acts diagonally on the space \( EG \times X \) by \( g(z, x) = (z \cdot g^{-1}, g \cdot x) \). For any field of coefficients \( \mathbb{F} \), the \( G \)-equivariant cohomology groups of \( X \) with coefficients in \( \mathbb{F} \), denoted by \( H^*_G(X, \mathbb{F}) \), is defined by \( H^*_G(X, \mathbb{F}) = H^*(EG \times X/G, \mathbb{F}) \).

It is well known (see for example [25]) that when a finite group \( G \) acts on a topological space \( X \), and \( \text{card}(G) \) is invertible in \( \mathbb{F} \) (and so in particular, if \( \mathbb{F} \) is a field of characteristic 0) that there is an isomorphism

\[
H^*(X/G, \mathbb{F}) \cong H^*_G(X, \mathbb{F}).
\]

The action of \( G \) on \( X \) induces an action of \( G \) on the cohomology ring \( H^*(X, \mathbb{F}) \), and we denote the subspace of \( H^*(X, \mathbb{F}) \) fixed by this action \( H^*(X, \mathbb{F})^G \). When \( G \)
is finite and \( \text{card}(G) \) is invertible in \( F \), the Serre spectral sequence associated to the map \( X \to X/G \), degenerates at its \( E_2 \)-term, where

\[
E_2^{p,q} = H^p(G, H^q(X, F))
\]

(see for example \([13, \text{Section VII.7}]\)). This is due to the fact that \( H^p(G, H^q(X, F)) = 0 \) for all \( p > 0 \), which implies that

\[
H^p(X/G, F) \cong H^0(G, H^p(X, F)),
\]

for each \( q \geq 0 \). Finally,

\[
H^0(G, H^q(X, F)) \cong H^q(X, F)^G.
\]

Thus, combining (3.3), (3.4) and (1.9) we have the isomorphisms

\[
H^*(X/G, F) \cong H_G^*(X, F) \cong H^*(X, F)^G.
\]

1.5. Prior work. The problem of bounding the equivariant Betti numbers of symmetric semi-algebraic subsets of \( R^k \) was investigated in \([9]\). We recall in this section a few results from \([9]\) that are generalized in the current paper.

We recall some definitions and notation from \([9]\).

**Notation 6.** For any \( \mathfrak{S}_k \) symmetric semi-algebraic subset \( S \subset R^k \) with \( k = (k_1, \ldots, k_\ell) \in N^\ell \), with \( k = \sum_{i=1}^\ell k_i \), and any field \( F \), we denote

\[
b^i_{\mathfrak{S}_k}(S, F) = b_i(S/\mathfrak{S}_k, F),
\]

\[
b_{\mathfrak{S}_k}(S, F) = \sum_{i \geq 0} b^i_{\mathfrak{S}_k}(S, F).
\]

The following theorem is proved in \([9]\).

**Theorem 2.** \([9]\) Let \( k = (k_1, \ldots, k_\ell) \in N^\ell \), with \( k = \sum_{i=1}^\ell k_i \). Suppose that \( P \in R[X^{(1)}, \ldots, X^{(\ell)}] \), where each \( X^{(i)} \) is a block of \( k_i \) variables, is a non-negative polynomial, such that \( V = Z(P, R^k) \) is invariant under the action of \( \mathfrak{S}_k \), permuting each block \( X^{(i)} \) of \( k_i \) coordinates. Let \( \deg_{X^{(i)}}(P) \leq d \) for \( 1 \leq i \leq \ell \). Then, for any field of coefficients \( F \),

\[
b(V/\mathfrak{S}_k, F) \leq \sum_{p=(p_1, \ldots, p_\ell), 1 \leq p_i \leq \min(2d, k_i)} F(k, p)d(2d - 1)^{\left| p \right| + 1}
\]

(where \( F(k, p) \) is defined in Notation 3). If for each \( i, 1 \leq i \leq \ell, 2d \leq k_i \), then

\[
b(V/\mathfrak{S}_k, F) \leq (k_1 \cdots k_\ell)^{2d}(O(d))^{2d+1}.
\]

More generally, the following bound holds for symmetric semi-algebraic sets.

**Theorem 3.** \([9]\) Let \( k = (k_1, \ldots, k_\ell) \in N^\ell \), with \( k = \sum_{i=1}^\ell k_i \), and let \( \mathcal{P} \subset R[X^{(1)}, \ldots, X^{(\ell)}] \) be a finite set of polynomials, where each \( X^{(i)} \) is a block of \( k_i \) variables, and such that each \( P \in \mathcal{P} \) is symmetric in each block of variables \( X^{(i)} \). Let \( S \subset R^k \) be a \( \mathcal{P} \)-closed-semi-algebraic set. Suppose that \( \deg(P) \leq d \) for each \( P \in \mathcal{P} \), \( \text{card}(\mathcal{P}) = s \), and let \( D = D(k, d) = \sum_{i=1}^{\ell} \min(k_i, 5d) \). Then, for any field of coefficients \( F \),

\[
b(S/\mathfrak{S}_k, F) \leq \sum_{i=0}^{D-1} \sum_{j=1}^{D-i} \binom{2s}{j} G(k, 2d)
\]
where
\[ G(k, d) = \sum_{p=(p_1, \ldots, p_\ell), 1 \leq p_i \leq \min(2d, k_i)} F(k, p) d(2d - 1)^{|p|+1} \]
(and \( F(k, p) \) is defined in Notation 3).

Remark 3. In the particular case, when \( \ell = 1, d = O(1) \), the bound in Theorem 3 takes the following asymptotic (for \( k \gg 1 \)) form.

\[ b(S/\mathfrak{S}_k, F) \leq O(s^{5d_k^{4d-1}}). \]

The rest of the paper is organized as follows. In Section 2 we state the new results proved in this paper. In Section 3 we prove or recall certain preliminary facts that will be needed in the proofs of the main theorems. In Section 4 we prove the main theorems, and finally in Section 5 we end with some open problems.

### 2. Main Results

In view of the isomorphism (1.10), Theorem 2 (respectively, Theorem 3) gives a bound (which is polynomial for fixed \( d \)) on the multiplicity of the trivial representation in the \( \mathfrak{S}_k \)-module \( H^*(V, F) \) (respectively, \( H^*(S, F) \)). In the current paper we generalize both Theorems 2 and 3 by proving a polynomial bound on the multiplicities of every irreducible representation appearing in the isotypic decomposition of \( H^*(V, F) \) and \( H^*(S, F) \). Note that as Example 1 shows, the dimensions of \( H^*(V, F) \), where \( V \) is a symmetric real variety in \( \mathbb{R}^k \) defined by polynomials of degree bounded by \( d \) could be exponentially large in \( k \). We also extend these basic results in several directions – including more general actions of the symmetric group, and as a particular case symmetric varieties in \( \mathbb{C}^k \), as well as symmetric projective varieties.

#### 2.1. Affine algebraic case

We first state our results for symmetric real algebraic subvarieties of real affine space. The main structural result giving restrictions on the irreducible representations of \( \mathfrak{S}_k \), and their multiplicities, in the cohomology modules of such varieties is given in Theorem 5. The quantitative estimates that follow are stated in Theorem 6. The case of symmetric complex affine varieties is dealt with in Theorem 7.

**Notation 7.** Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell \), and \( K = \sum_{i=1}^{\ell} k_i m_i \). For any \( \mathfrak{S}_k \)-symmetric semi-algebraic subset \( S \subset \mathbb{R}^K \), any field \( F \), and \( \lambda \in \text{Par}(k) \), we denote

\[ m_{i, \lambda}(S, F) = \dim_F \text{hom}_{\mathfrak{S}_k}(S^\lambda, H^i(S, F)) \]
\[ = \text{mult}(S^\lambda, H^i(S, F)). \]
\[ m_{\lambda}(S, F) = \sum_i m_{i, \lambda}(S, F). \]

Note that in the particular case when \( \lambda = ((k_1), \ldots, (k_\ell)) \) (i.e. when \( S^\lambda \) is the trivial representation of \( \mathfrak{S}_k \)),

\[ m_{i, \lambda}(S, F) = b_i(S/\mathfrak{S}_k, F) \]
\[ m_{\lambda}(S, F) = b(S/\mathfrak{S}_k, F). \]

**Notation 8.** For \( d = (d_1, \ldots, d_\ell), m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell \), we denote by \( d^m = (d_1^{m_1}, \ldots, d_\ell^{m_\ell}) \).
Notation 9. For $k \in \mathbb{N}^\ell, \lambda \in \text{Par}(k)$, we denote (see also Notation 17)

\[(2.1) \quad \mathcal{I}(\lambda) = \bigcup_{\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k), m_{\lambda^{(i)}, \lambda^{(i)'}} > 0} (\lambda^{(i)})^{1 \leq i \leq \ell} \bigcup_{\mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \text{Par}(k), m_{\mu} > 0} (\mu^{(i)})^{1 \leq i \leq \ell}.
\]

For $k, d, m \in \mathbb{N}^\ell$, we denote

\[(2.2) \quad \mathcal{I}(k, d, m) := \bigcup_{\lambda \in \text{Par}(k, (2d)^m)} \mathcal{I}(\lambda).
\]

If $\ell = 1$, $k = (k)$, $d = (d)$, $m = (m)$, we will denote $\mathcal{I}(k, d, m)$ by $\mathcal{I}(k, d, m)$.

The following theorem gives a restriction on the partitions in $\mathcal{I}(k, d, m)$.

**Theorem 4.** Let $k, d, m > 0$, and $\mu = (\mu_1, \mu_2, \ldots) \in \mathcal{I}(k, d, m)$. Then,

\[\text{card}(\{i \mid \mu_i \geq (2d)^m\}) \leq (2d)^m, \text{card}(\{j \mid \mu_j \geq (2d)^m\}) \leq (2d)^m.\]

**Remark 4.** Note that Theorem 4 implies that the Young diagram for each $\mu \in \mathcal{I}(k, d, m)$ is contained in the union of $(2d)^m$ rows and $(2d)^m$ columns. This is shown in Figure 1 for fixed $d, m$ and large $k$. The shaded area inside the $k \times k$ sized box contains all possible Young diagrams of partitions of $k$. The darker part contains the partitions belonging to $\mathcal{I}(k, d, m)$.

**Theorem 5.** Let $k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell), d = (d, \ldots, d) \in \mathbb{N}^\ell$, and $K = \sum_{i=1}^\ell m_i k_i$. Let $P \in \mathbb{R}[X^{(1)}, \ldots, X^{(\ell)}]$ be a $\mathbb{G}_k$-symmetric polynomial, with $\deg(P) \leq d$. Let $V = Z(P, R^K)$. Then, for all $\mu \in \text{Par}(k)$, $m_{\mu}(V, \mathbb{F}) > 0$ implies that

\[(2.3) \quad \mu \in \mathcal{I}(k, d, m).
\]

Moreover, for each $\mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{I}(k, d, m)$,

\[m_{\mu}(V, \mathbb{F}) \leq \sum_{\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k, (2d)^m)} G(\mu, \lambda, d, m),
\]

where

\[G(\mu, \lambda, d, m) = \prod_{1 \leq i \leq \ell} \left( (2d)^{\text{length}(\lambda^{(i)})} \right) \max_{\lambda^{(i)} = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_{\ell})} m_{\lambda^{(i)}}.\]

(The maximum on the right hand side is taken over all decompositions $\lambda^{(i)} = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_{\ell})$.)

**Remark 5.** Note that the restriction on the Specht modules that are allowed to appear in the cohomology module $H^*(V, \mathbb{F})$ does not follow only from dimension considerations, and the Olek-Petrovskii-Thom-Milnor bound (Theorem 1) on $b(V, \mathbb{F})$.

For example, let $\ell = 1, m_1 = 1, k_1 = k = 2^p - 1$, and let $\lambda \vdash k$ be the partition $(2^{p-1}, \ldots, 1)$. In this case:

\[\text{dim}_{\mathbb{F}} S^\lambda \leq \text{dim}_{\mathbb{F}} \text{Ind}_{S^k_{\lambda}} \text{Ind}_{S^k_{\lambda}} (\text{since } K(\lambda, \lambda) = 1 \text{ (Definition 6 and Proposition 5)})
\]
\[= \binom{k}{2^{p-1}, \ldots, 1}
\]
\[\leq O(1)^k \text{ using Stirling’s approximation.}\]
Thus, if \( V \) is defined by a polynomial of degree bounded by \( d \), and \( k \) is large enough, \( S^\lambda \) is not ruled out of appearing with positive multiplicity in \( H^*(V, \mathbb{F}) \) just on the basis of the upper bound in Theorem 1. On the other hand, it follows from (2.2) that for all \( k \) large enough, and fixed \( d \),

\[
S^\lambda \notin \mathcal{I}(k, d, 1),
\]

and hence by Theorem 5 cannot appear with positive multiplicity in \( H^*(V, \mathbb{F}) \).

With the same hypothesis as in Theorem 5 we have:

**Theorem 6** (Symmetric real affine varieties). *For each \( \lambda \in \text{Par}(k, (2d)^m) \),

\[
\text{card}(\mathcal{I}(\lambda)) \leq \prod_{1 \leq i \leq \ell} k_i \Theta(d^{2m_i}),
\]

and for each \( \mu \in \mathcal{I}(k, d, m) \)

\[
m_\mu(V, \mathbb{F}) \leq \prod_{1 \leq i \leq \ell} k_i \Theta(d^{2m_i}) d^{m_i d}.
\]
In the particular case, when \( \ell = 1 \), and \( d_1 = d \) and \( m_1 = m \) are fixed, both bounds are polynomial in \( k_1 = k \).

Remark 6. We also remark here (without proof) that if we fix some \( i \geq 0 \), it is possible to obtain using techniques from [9] (proof of [9, Proposition 8]) slightly better bound on \( m_{0,\mu} \) compared to the bound in Theorem 5. For example taking \( i = 0 \) and with the same notation as in Theorem 5, it is possible to prove (using techniques from [9]) that for all \( \mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \text{Par}(k) \), \( m_{0,\mu}(V,F) > 0 \) implies that \( \text{length}(\mu^{(j)}) \leq 2d \) for \( 1 \leq j \leq \ell \).

Moreover, for each \( \mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \text{Par}(k) \) with \( m_{0,\mu}(V,F) > 0 \), since \( \text{length}(\mu^{(j)}) \leq 2d, 1 \leq j \leq \ell \), it follows with the same arguments presented in the proof of Theorem 6 that

\[
m_{0,\mu}(V,F) \leq \sum_{\lambda=(\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k,(2d)^m)} \prod_{1 \leq i \leq \ell} \left( (2d)^{m_{\mu}(\lambda^{(i)})} K(\mu^{(i)}, \lambda^{(i)}) \right).
\]

The quantitative estimates obtained on \( m_{0,\mu}(V,F) \) using the above result will be slightly better than that obtained directly from Theorem 5, but since it does not affect the polynomial dependence on the \( k \) we prefer not to expand on this further.

A particular case of Theorem 5 deserves mention, and we state it as a corollary.

Corollary 1. Suppose that \( k = (1, \ldots, 1, k) \), \( m = (1, \ldots, 1, m) \), and \( (2d)^m \leq k \). If \( \mu = ((1), \ldots, (1), (k)) \) (i.e. \( \mathbb{S}^\mu \) is the trivial representation), then

\[
m_\mu(V,F) = b(V/\mathbb{S}_k;F) \leq \sum_{\lambda=(\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k, (2d)^m)} \left( \prod_{1 \leq i \leq \ell} (2d)^{m_{\mu}(\lambda^{(i)})} \right) \leq k^2 (2d)^m O(d)^m (2d)^m + \ell.
\]

Notice that Corollary 1 generalizes to the case \( m > 1 \), Corollary 3 in [9].

We have the following theorem for symmetric complex affine varieties.

Theorem 7 (Symmetric complex affine varieties). Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell), d = (d, \ldots, d) \in \mathbb{N}_\ell \), and \( K = \sum_{i=1}^{\ell} k_i m_i \). Let \( \mathcal{P} \subset \mathbb{C}[X^{(1)}, \ldots, X^{(\ell)}] \) be a finite set of \( \mathbb{S}_k \)-symmetric polynomials, with \( \text{deg}(\mathcal{P}) \leq d \) for each \( P \in \mathcal{P} \). Let \( V = Z(\mathcal{P}, C^K) \).

Then, for all \( \mu \in \text{Par}(k) \), \( m_\mu(V,F) > 0 \) implies that \( \mu \in \mathcal{I}(k, 2d, 2m) \).

Moreover, for each \( \mu \in \mathcal{I}(k, 2d, 2m) \)

\[
m_\mu(V,F) \leq \prod_{1 \leq i \leq \ell} k_i^{O(d^{m_{\mu}(i)})} d^{2m_i d}.
\]

2.2. Affine semi-algebraic case. We now state our results in the semi-algebraic case. As before we state first a structural result (Theorem 8 below), and deduce quantitative estimates from it (Theorem 9).

Theorem 8. Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell), d = (d, \ldots, d) \in \mathbb{N}_\ell \), and \( K = \sum_{i=1}^{\ell} k_i m_i \). Let \( \mathcal{P} \subset \mathbb{R}[X^{(1)}, \ldots, X^{(\ell)}] \) be a finite set of \( \mathbb{S}_k \)-symmetric polynomials,
with \(\deg(P) \leq d\) for all \(P \in \mathcal{P}\), and let \(\text{card}(\mathcal{P}) = s\). Let \(S \subset \mathbb{R}^K\) be a \(\mathcal{P}\)-closed semi-algebraic set.

Then, for all \(\mu \in \text{Par}(k)\), \(m_\mu(S, \mathbb{F}) > 0\) implies that
\[
\mu \in \mathcal{I}(k, d, m).
\]

Moreover, let \(D = D(k, m, d) = \sum_{i=1}^\ell \min(m_i k_i, d^m_i)\). Then, for each
\[
\mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{I}(k, d, m),
\]
\[
m_\mu(S, \mathbb{F}) \leq \sum_{i=0}^{D-1} \sum_{j=1}^{D-i} \left(2s + 1\right)^j \left(\sum_{\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k, (2d)^m)} \lambda \cdot G(\mu, \lambda, d, m)\right),
\]
where
\[
G(\mu, \lambda, d, m) = \prod_{1 \leq i \leq \ell} \left(2d\right)^{m_i \text{length}(\lambda^{(i)})} \max_{\lambda^{(i)} = \lambda^{(i)'} \prod \lambda^{(i)'}, \lambda^{(i)''}} \mu^{m_{\lambda^{(i)'}}, \lambda^{(i)''}}
\]
(the maximum on the right hand side is taken over all decompositions \(\lambda^{(i)} = \lambda^{(i)'} \prod \lambda^{(i)'}, \lambda^{(i)''}\)).

Using the same notation as in Theorem 8:

**Theorem 9 (Symmetric affine semi-algebraic sets).** For each \(\mu \in \mathcal{I}(k, d, m)\)
\[
m_\mu(S, \mathbb{F}) \leq O(s)^D \cdot \prod_{1 \leq i \leq \ell} k_i^{O(d^m_i)} d^{m_i d^m_i}.
\]

In the particular case, when \(\ell = 1\), and \(d_1 = d\) and \(m_1 = m\) are fixed, both bounds are polynomial in \(s\) and \(k_1 = k\).

**2.3. Projective case.** We can apply our results obtained in the previous section to study the topology of symmetric projective varieties as well. We state one such result below.

**Theorem 10 (Symmetric complex projective varieties).** Let \(V \subset \mathbb{P}^k_{\mathbb{C}}\) be defined by symmetric homogeneous polynomials in \(\mathbb{C}[X_0, \ldots, X_k]\) of degrees bounded by \(d\). Then, the irreducible representations, \(\mathcal{S}^\mu, \mu + k + 1\), that occur in \(H^*(V, \mathbb{F})\) with positive multiplicities belong to the set \(\mathcal{I}(k, 2d, 2)\), and the multiplicity of each such representation is bounded by
\[
k^{O(d^2)} d^{2d}.
\]

**Remark 7.** Suppose \(V \subset \mathbb{P}^k_{\mathbb{C}}\) be defined by symmetric homogeneous polynomials in \(\mathbb{C}[X_0, \ldots, X_k]\) of degrees bounded by \(d\). Unlike in the affine case, it is not true that dimensions of equivariant cohomology, \(\dim \mathcal{H}^*_\mathcal{G}_{k+1}(V, \mathbb{F})\), are bounded by a function of \(d\) independent of \(k\). For example,
\[
H^*(\mathbb{P}^k_{\mathbb{C}}/\mathcal{G}_{k+1}, \mathbb{F}) \cong H^*(\mathbb{P}^k_{\mathbb{C}}/\mathcal{G}_{k+1}, \mathbb{F}),
\]
and thus
\[
\dim \mathcal{H}^*(\mathbb{P}^k_{\mathbb{C}}/\mathcal{G}_{k+1}, \mathbb{F}) = k + 1,
\]
which clearly grows linearly with \(k\).
2.4. Application to bounding topological complexity of images of polynomial maps. In this section we discuss an application of Corollary 1 to bounding the Betti numbers of images of real algebraic varieties under linear projections. In [9], similar results were proved in the very special case of projections of the form \( \pi : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^k \). In this paper, since we consider more general actions of the symmetric group, we are able to handle projections along more than one variables, and so are able to strengthen as well as generalize the results in [9].

In order to state our results more precisely we first introduce some notation. Let \( P \in \mathbb{R}[Y_1, \ldots, Y_k, X_1, \ldots, X_m] \) be a non-negative polynomial with \( \deg(P) \leq d \). Let \( \pi : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k \) be the projection map to the first \( k \) co-ordinates, and let \( V = Z(P, \mathbb{R}^{m+k}) \). We consider the problem of bounding the Betti numbers of the image \( \pi(V) \). Bounding the complexity of the image under projection of semi-algebraic sets is a very important and well-studied problem related to quantifier elimination in the first order theory of the reals, and has many ramifications – including in computational complexity theory (see for example [10]).

There are two different approaches. One can first obtain a semi-algebraic description of the image \( \pi(V) \) with bounds on the degrees and the number of polynomials appearing in this description (via results in effective quantifier elimination in the first order theory of the reals), and then apply known bounds on the Betti numbers of semi-algebraic sets in terms of these parameters. Another approach (due to Gabrielov, Vorobjov and Zell [18]) is to use the “descent spectral sequence” of the map \( \pi|_V \) which abuts to the cohomology of \( \pi(V) \) and bound the Betti numbers of \( \pi(V) \) by bounding the dimensions of the \( E_1 \)-terms of this spectral sequence. For this approach it is essential that the map \( \pi \) is proper (which is ensured by requiring that \( V \) is bounded) since in the general case the spectral sequence might not converge to \( H^*(S, \mathbb{F}) \). The second approach produces a slightly better bound. The following theorem (in the special case of algebraic sets) whose proof uses the second approach appears in [18].

**Theorem 11.** [18] With the same notation as above,

\[
\Betti(\pi(V), \mathbb{F}) = (O(d))^{(m+1)k}.
\]

Notice that in the exponent of the bound in (2.4), there is a factor of \((m + 1)\) which is linear in the dimension of the fibers of the projection \( \pi \). This factor is also present if one uses effective quantifier elimination method to bound the Betti numbers of \( \pi(V) \). Using Theorem 9 we are able to remove this multiplicative factor of \((m + 1)\) in the exponent of the bound in (2.4) at the expense of an extra additive term that depends just on \( d \) and \( m \).

We now state the result more precisely. In [9], the following bound on the Betti numbers of the image under projection to a subspace of dimension one less than that of the ambient space of real algebraic varieties (i.e. with \( m = 1 \)), as well as of semi-algebraic sets (not necessarily symmetric).

**Theorem 12.** [9] Let \( P \in \mathbb{R}[Y_1, \ldots, Y_k, X] \) be a non-negative polynomial and with \( \deg(P) \leq d \). Let \( V = Z(P, \mathbb{R}^{k+1}) \) be bounded, and \( \pi : \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}^k \) be the projection map to the first \( k \) coordinates. Then,

\[
\Betti(\pi(V), \mathbb{F}) \leq \left( \frac{k}{d} \right)^{2d} (O(d))^{k+2d+1}.
\]
In this paper we generalize the above results to the case \( m > 1 \). We prove the following theorem.

**Theorem 13.** Let \( P \in \mathbb{R}[Y_1, \ldots, Y_k, X_1, \ldots, X_m] \) be a non-negative polynomial and with \( \deg(P) \leq d \). Let \( V = Z(P, \mathbb{R}^{k+m}) \) be bounded, and \( \pi : \mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^k \) be the projection map to the first \( k \) coordinates. Then,

\[
b(\pi(V), F) \leq k (2d)^m (O(d))^{k+m(2d)^m+1}.
\]

2.5. **Application to proving lower bounds on degrees.** The upper bounds in the theorems stated above can be potentially applied to prove lower bounds on the degrees of polynomials needed to define symmetric varieties having certain prescribed geometry. We describe one such example.

**Example 3.** Let \( k = 2^p - 1 \), and let \( \tilde{V}_k \) be any non-empty compact semi-algebraic set contained in the subset of \( \mathbb{R}^k \) defined by

\[
X_1 = \cdots = X_{2^p - 1} \\
X_{2^p - 1 + 1} = \cdots = X_{2^p - 1 + 2^p - 2} \\
X_{2^p - 1 + 2^p - 2 + 1} = \cdots = X_{2^p - 1 + \cdots + 2^2 + 1 + 2^{p-1} + 2^1} \\
X_{2^p - 1 + \cdots + 2^{p-1} + 2^1}.
\]

Then, the stabilizer of \( \tilde{V}_k \) under the action of \( \mathcal{S}_k \) on \( \mathbb{R}^k \), is the Young subgroup \( \mathcal{S}_{\lambda(k)}(k) \), where \( \lambda(k) = (2^p - 1, 2^p - 2, \ldots, 1) \). Let \( V_k \) be the orbit of \( \tilde{V}_k \) under the action of \( \mathcal{S}_k \). In other words,

\[ V_k = \mathcal{S}_k \cdot \tilde{V}_k. \]

Then,

\[
b_0(V_k, F) = b_0(\tilde{V}_k, F) \cdot \binom{k}{2^p-1, 2^p-2, \ldots, 2^1} = b_0(\tilde{V}_k, F) \cdot (\Theta(1))^k \] using Stirling’s approximation.

We claim that that for any constant \( d_0 \), for all \( k \) large enough, \( V_k \) cannot be described as the set of real zeros of a polynomial \( P \in \mathbb{R}[X_1, \ldots, X_k] \) with \( \deg(P) \leq d_0 \). To see this observe that

\[
H^0(V_k, F) \cong \mathcal{S}_k b_0(\tilde{V}_k, F) \cdot M^{\lambda(k)},
\]

with \( \lambda(k) = (2^p - 1, 2^p - 2, \ldots, 1) \), and it follows that \( m_{0, \lambda(k)}(V_k, F) > 0 \). However, it follows from the definition of \( T(k, d, m) \) (see (2.2)) that for any fixed \( d_0 \),

\[
\lambda(k) \notin T(k, d_0, 1),
\]

for all \( k \) large enough, and hence by Theorem 5 for all large enough \( k \), \( V_k \) cannot be defined by polynomials with degrees bounded by \( d_0 \).

Note that in the case, when \( \tilde{V}_k \) is a finite set of points, the same result can also be deduced from Proposition 2.

3. **Preliminaries**

Before proving the new theorems stated in the previous section we need some preliminary results. These are described in the following subsections.
3.1. Real closed extensions and Puiseux series. In this section we recall some basic facts about real closed fields and real closed extensions.

We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [7] for further details.

Notation 10. For R a real closed field we denote by R⟨ε⟩ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation R⟨ε1,...,εm⟩ to denote the real closed field R⟨ε1⟩...⟨εm⟩. Note that in the unique ordering of the field R⟨ε1,...,εm⟩, 0 < εm ≪ εm−1 ≪ ... ≪ ε1 ≪ 1.

Notation 11. If R′ is a real closed extension of a real closed field R, and S ⊂ Rk is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by Ext(S,R′) ⊂ R′k the semi-algebraic subset of R′k defined by the same formula. It is well-known that Ext(S,R′) does not depend on the choice of the formula defining S [7].

Notation 12. For x ∈ Rk and r ∈ R, r > 0, we will denote by Bk(x,r) the open Euclidean ball centered at x of radius r. If R′ is a real closed extension of the real closed field R and when the context is clear, we will continue to denote by Bk(x,r) the extension Ext(Bk(x,r),R′). This should not cause any confusion.

3.2. Tarski-Seidenberg transfer principle. In some proofs involving Morse theory (see for example the proof of Lemma 2), where integration of gradient flows is used in an essential way, we first restrict to the case R = R. After having proved the result over R, we use the Tarski-Seidenberg transfer theorem to extend the result to all real closed fields. We refer the reader to [7, Chapter 2] for an exposition of the Tarski-Seidenberg transfer principle.

3.3. Equivariant Morse theory. In this section we develop some basic results in equivariant Morse theory that we will need for the proof of Theorem 5. Since the results of this section applies to more general (finite) groups acting on a manifold, we state and prove our results in a more general setting than what we need in this paper.

Let G be a finite group acting on a compact semi-algebraic S ⊂ Rk, defined by Q ≤ 0, and W = Z(Q,Rk) = ∂S a bounded non-singular real algebraic hypersurface. Let e : W → R be a G-equivariant regular function with isolated non-degenerate critical points on W. For each such critical point x, we will denote by ind−(x) the dimension of the negative eigenspace of the Hessian of e at x. More precisely, the Hessian Hess(e)(x) is a symmetric, non-degenerate quadratic form on the tangent space T_xW, and ind−(x) is the number of negative eigenvalues of Hess(e)(x).

Consider the set of critical points, C, of the function e restricted to V. For any subset I ⊂ R, we will denote by S_I = S ∩ e^−1(I). If I = [x,c] we will denote S_I = S_{≤c}.

In the next two lemmas we will write R = R since we will use properties of gradient flows.

Lemma 1. Let v_1 < ⋅⋅⋅ < v_N be the critical values of e restricted to W. Then, for 1 ≤ i < N, and for each v ∈ [v_i,v_{i+1}), S_{≤v} is a deformation retract of S_{≤v}, and the retraction can be chosen to be G-equivariant.

Proof. See for example the proof of Theorem 7.5 (Morse Lemma A) in [7] with W = Z(Q,Rk), a = v_i and b = v, noting since W is symmetric and e is symmetric,
the retraction of $W_{\leq v}$ to $W_{\leq v_i}$ that is constructed in the proof of Theorem 7.5 in [7] is symmetric as well. □

We also need the following equivariant version of Morse Lemma B.

**Lemma 2.** Let $v \in e(C)$ be a critical value of $e$. Let $C_v^+, C_v^-, C_v \subset C$ be defined by

- $C_v^+ = \{ x \in C \mid e(x) = v, \langle \text{grad}(e), \text{grad}(Q) \rangle (x) > 0 \}$,
- $C_v^- = \{ x \in C \mid e(x) = v, \langle \text{grad}(e), \text{grad}(Q) \rangle (x) < 0 \}$,
- $C_v = C_v^+ \cup C_v^-$

($\cup$ denotes disjoint union). Then for all $0 < \varepsilon \ll 1$, $S_{\leq v + \varepsilon}$ retracts $G$-equivariantly to a space

$$S_{\leq v - \varepsilon} \cup_B A,$$

where

$$(A, B) = \prod_{y \in C_v} (A_y, B_y),$$

and for each $y \in C_v$, $(A_y, B_y)$ is $G$-equivariantly homotopy equivalent to the pair

$$(\text{Dom}^{-\varepsilon}(y) \times [0, 1], \partial \text{Dom}^{-\varepsilon}(y) \times [0, 1] \cup \text{Dom}^{-\varepsilon}(y) \times \{1\})$$

if $y \in C_v^+$, or to the pair

$$(\text{Dom}^{-\varepsilon}(y), \partial \text{Dom}^{-\varepsilon}(y)),$$

if $y \in C_v^-$. □

**Proof.** See proof of Proposition 7.19 in [7], noting again that the retraction constructed in that proof is symmetric in case $Q$ is a symmetric polynomial and the Morse function $e$ is symmetric as well.

Now, let $\mathcal{C}$ be a set containing a unique representative from each $G$-orbit of $C$.

Let $\overline{\mathcal{C}}_i \subset \mathcal{C}$ be the set of representatives of the different orbits corresponding to the critical value $v_i$ – in other words, $e(\overline{\mathcal{C}}_i) = \{ v_i \}$. Note that the cardinality of $\overline{\mathcal{C}}_i$ can be greater than one. For each $x \in \overline{\mathcal{C}}_i$, let $G_x \subset G$ denote the stabilizer subgroup of $x$.

Let also for each $i, 0 \leq i \leq N$, and $0 < \varepsilon \ll 1$, $j_{i, \varepsilon}$ denote inclusion $S_{\leq v_{i+1} + \varepsilon} \hookrightarrow S_{\leq v_i}$, and $j_{i, \varepsilon}^*: H^*(S_{\leq v_{i+1} + \varepsilon}, \mathbb{F}) \rightarrow H^*(S_{\leq v_i}, \mathbb{F})$ the induced homomorphism (which is in fact a homomorphism of the corresponding $G$-modules).

Let $\overline{\mathcal{C}}_i = \{ x^{i,1}, \ldots, x^{i,N_i} \}$ (choosing an arbitrary order).

**Proposition 1.** The homomorphism $j_{i, \varepsilon}^*$ factors through $N_i$ homomorphisms as follows:

$$H^*(S_{\leq v_{i+1} + \varepsilon}, \mathbb{F}) = M_0 \xrightarrow{j_{i, \varepsilon, 1}} M_1 \xrightarrow{j_{i, \varepsilon, 2}} \cdots \xrightarrow{j_{i, \varepsilon, N_i}} M_{N_i+1} = H^*(S_{\leq v_i}, \mathbb{F}),$$

where each $M_h$ is a finite dimensional $G$-module, and for each $1 \leq h \leq N_i$, either

(a) $j_{i, \varepsilon, h}$ is injective, and $M_h \cong M_h \oplus \text{Ind}^G_{G_{x^{i,h}}} (W_{x^{i,h}})$,

for some one-dimensional representation $W_{x^{i,h}}$ of $G_{x^{i,h}}$, or

(b) the homomorphism $j_{i, \varepsilon, h}$ is surjective, and $M_h \cong M_{h+1} \oplus \text{Ind}^G_{G_{x^{i,h}}} (W_{x^{i,h}})$,

for some one-dimensional representation $W_{x^{i,h}}$ of $G_{x^{i,h}}$. □
Proof: We first assume that $R = \mathbb{R}$. Using Lemma 2 (equivariant Morse Lemma B), we have that $S_{\leq v_i - \varepsilon}$ can be retracted $G$-equivariantly to a semi-algebraic set
\[
\tilde{S}_i = S_{\leq v_i - \varepsilon} \bigcup_{1 \leq j \leq N_i} \prod_{y \in G \setminus \mathbf{x}^{i,j}} D^{\text{ind}^-}(y) / \sim,
\]
where the identification $\sim$ identifies the boundaries of the disks $D^{\text{ind}^-}(y)$ with spheres of the same dimension in $S_{\leq v_i - \varepsilon}$.

Since the different balls $D^{\text{ind}^-}(y)$ are disjoint, we can decompose the glueing process by glueing disks belonging to each orbit successively, choosing the order arbitrarily, and thus obtain a filtration,
\[
S_{| \leq v_i - \varepsilon} = S_{i,0} \subset S_{i,1} \subset \cdots \subset S_{i,N_i'} = \tilde{S}_i,
\]
where $S_{i,0} = S_{i,j} = S_{i,j-1} \bigcup \left( \prod_{y \in \text{orbit}(y)} D^{\text{ind}^-}(y) / \sim \right)$ for some $j'$, $1 \leq j' \leq N_i$.

Let $D_{i,j'}$ denote the disjoint union of the balls $D^{\text{ind}^-}(y)$, and $C_{i,j'} \subset D_{i,j'}$ the disjoint union of their boundaries.

We have the Mayer-Vietoris exact sequence
\[
\cdots \rightarrow H^p(C_{i,j'},\mathbb{F}) \rightarrow H^p(D_{i,j'},\mathbb{F}) \oplus H^p(S_{i,j-1},\mathbb{F}) \rightarrow H^p(S_{i,j},\mathbb{F}) \rightarrow H^{p-1}(C_{i,j'},\mathbb{F}) \rightarrow \cdots
\]
which is also equivariant. Let $n = \text{ind}^-(x^{i,j'})$, and assume $n \neq 0$. In this case, $H^p(C_{i,j'},\mathbb{F}) = 0$ unless $p = 0$, $n-1$. Now, $H^{n-1}(C_{i,j'},\mathbb{F})$ is a direct sum of card(orbit($x^{i,j'}$)), each summand is stable under the action of a subgroup of $G$ each isomorphic to $G_{x^{i,j'}}$, and is thus a one-dimensional representation of $G_{x^{i,j'}}$, which we denote by by $W_{x^{i,j'}}$. It follows that the representation $H^{n-1}(C_{i,j'},\mathbb{F})$ is the induced representation $\text{Ind}_{G_{x^{i,j'}}}^G(W_{x^{i,j'}})$. From the Mayer-Vietoris sequence it is evident that either
\[
\text{(i)} \quad H^n(S_{i,j},\mathbb{F}) = H^n(S_{i,j-1},\mathbb{F}) \oplus H^{n-1}(C_{i,j'},\mathbb{F}),
\]
or
\[
\text{(ii)} \quad H^{n-1}(S_{i,j},\mathbb{F}) \oplus H^{n-1}(C_{i,j'},\mathbb{F}) \cong H^n(S_{i,j-1},\mathbb{F}).
\]
These two cases corresponds to (a) and (b) respectively. Finally, we extend the proof to general $R$ using the Tarski-Seidenberg transfer principle in the usual way (see [7, Chapter 7] for example).

3.4. Degree principle for the action of symmetric group on $R^k$. In this section we prove an important proposition that forms the basis of all our quantitative results. It generalizes to the multi-symmetric case (i.e. for $m$ not necessarily equal to $(1, \ldots, 1)$) a similar result proved earlier (see [26, 29, 9]).

Notation 13. Let $k = (k_1, \ldots, k_\ell)$, $m = (m_1, \ldots, m_\ell), p = (p_1, \ldots, p_\ell) \in \mathbb{N}^\ell$, and let
\[
k = \sum_{1 \leq i \leq \ell} k_i m_i.
We denote by \( A^p_{k,m} \) the subset of \( \mathbb{R}^k \) defined by

\[
A^p_{k,m} = \left\{ x = (x^{(1)}, \ldots, x^{(\ell)}) \mid \text{card} \left( \bigcup_{j=1}^k \{x^{(i)}_j\} \right) = p \right\}.
\]

In the special case when \( \ell = 1, m = 1 \), we define for \( p \leq k \),

\[
A^p_k = \left\{ x = (x_1, \ldots, x_k) \mid \text{card} \left( \bigcup_{j=1}^k \{x_j\} \right) = p \right\}.
\]

Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell), d = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell \), let \( K = \sum_{i=1}^\ell k_i \), and \( P \in R[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}] \), where each for \( 1 \leq h \leq \ell \), \( X^{(h)} \) is a block of \( m_h \times k_h \) variables, \( (X^{(h)}_{i,j})_{1 \leq i \leq m_h, 1 \leq j \leq k_h} \), such that that \( P \) is non-negative and \( \mathcal{S}_{k,m} \)-symmetric. Also, suppose that \( \deg_{X^{(h)}(P)} \leq d_h, 1 \leq h \leq \ell \).

The following proposition generalizes (to the case \( m \neq (1, \ldots, 1) \)) Proposition 6 in [9].

**Proposition 2** (Degree Principle). Let

\[
e = \sum_{1 \leq h \leq \ell, 1 \leq i \leq m_h, 1 \leq j \leq k_h} X^{(h)}_{i,j}.
\]

Let \( C \) denote the set of critical points of \( e \) restricted to \( W = \mathbb{Z}(P, \mathbb{R}^K) \), and suppose that \( C \) is a finite set. Then,

\[
C \subseteq \bigcup_{p \leq dm} A^p_{k,m}.
\]

**Proof.** For \( m = 1 := (1, \ldots, 1) \), the proposition follows immediately from [9, Proposition 6]. Suppose that \( x = (x^{(1)}, \ldots, x^{(\ell)}) \in C \). For \( x = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{R}^k \), and \( i = (i_1, \ldots, i_\ell) \in [1, m_1] \times \cdots \times [1, m_\ell] \) denote by \( x_i = (x^{(1)}_{i_1}, \ldots, x^{(\ell)}_{i_\ell}) \in \mathbb{R}^k \). It follows from the case \( 1 = (1, \ldots, 1) \) ([9, Proposition 6]), that for each \( i = (i_1, \ldots, i_\ell) \in [1, m_1] \times \cdots \times [1, m_\ell] \),

\[
x_i \in \bigcup_{p \leq d} A^p_{k,1}.
\]

This proves that for each \( x = (x^{(1)}, \ldots, x^{(\ell)}) \in C \), each row of each \( m_h \times k_h \)-matrix \( x^{(h)} \) has at most \( d \) distinct entries, and this implies that the matrix \( x^{(h)} \) has at most \( d^{m_h} \) distinct columns. This implies that

\[
x \in \bigcup_{p \leq dm} A^p_{k,m},
\]

which proves the proposition.

\[\square\]

3.5. **Deformation.** Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell, K = \sum_{i=1}^\ell k_i m_i \), and \( d \geq 0 \). Following the notation introduced previously,

**Notation 14.** For any \( P \in R[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}] \) we denote

\[
\text{Def}(P, \zeta, d) = P - \zeta \left( 1 + \sum_{1 \leq h \leq \ell} \sum_{1 \leq i \leq m_h} \sum_{1 \leq j \leq k_h} (X^{(h)}_{i,j})^d \right),
\]

where \( \zeta \) is a new variable.
Notice that if $P$ is $\mathfrak{S}_k$-symmetric, then so is $\text{Def}(P, \zeta, d)$.

**Proposition 3.** [9, Proposition 4] Let $d$ be even, and suppose that $V = Z(P, R^K)$ is bounded. The variety $\text{Ext}(V, R(\zeta)^K)$ is a semi-algebraic deformation retract of the (symmetric) semi-algebraic subset $S$ of $R(\zeta)^K$ defined by the inequality

$$\text{Def}(P, \zeta, d) \leq 0,$$

and hence is semi-algebraically homotopy equivalent to $S$.

**Proposition 4.** [9, Proposition 5]. Let $P \in R[X^{(1)}, \ldots, X^{(\ell)}]$, and $d$ be an even number with $\deg(P) \leq d = p + 1$, with $p$ a prime. Let

$$e = \sum_{1 \leq h \leq \ell} \sum_{1 \leq \ell \leq m_h} \sum_{1 \leq j \leq k_h} (X_{i,j}^{(h)})^d,$$

and

$$V_{\zeta} = Z(\text{Def}(P, \zeta, d), R(\zeta)^K).$$

Suppose also that $\gcd(p, K) = 1$. Then, the critical points of $e$ restricted to $V_{\zeta}$ are finite in number, and each critical point is non-degenerate.

### 3.6. Representation theory of products of symmetric groups.

We begin with some notation.

**Notation 15** (Product of symmetric groups and certain subgroups). For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \text{Par}(k)$, we will denote by $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_d}$ the subgroup of $\mathfrak{S}_k$ which is the direct product of the subgroups $G_i \cong \mathfrak{S}_{\lambda_i}$, where $G_i$ is the subgroup of permutations of $[1, k]$ fixing $[1, k] \setminus [\lambda_1 + \cdots + \lambda_i - 1 + 1, \lambda_1 + \cdots + \lambda_i]$.

More generally, for $k = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$, $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k)$, we denote by $\mathfrak{S}_\lambda$ the subgroup $\mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(\ell)}}$ of $\mathfrak{S}_k$, where for $1 \leq j \leq \ell$, $\mathfrak{S}_{\lambda^{(j)}}$ is the subgroup of $\mathfrak{S}_{k_j}$ defined above.

**Notation 16** (Irreducible representations of symmetric groups). For $\lambda \in \text{Par}(k)$, we will denote by $S^\lambda$ the irreducible representation of $\mathfrak{S}_k$ corresponding to $\lambda$ (see [24] for definition). Note that $S^k$ is the trivial representation (corresponding to the partition $(k) \in \text{Par}(k)$ (which we also denote by $1_{\mathfrak{S}_k}$), and $S^{(1^\ell)}$ is the sign representation, which we will also denote by $\text{sign}_k$. It is well known fact that for any $\lambda \in \text{Par}(k)$,

$$S^{(\lambda)} \cong S^\lambda \otimes \text{sign}_k.$$

For $k = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$, $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k)$, we denote by $S^\lambda$ the irreducible representation $S^{\lambda^{(1)}} \boxtimes \cdots \boxtimes S^{\lambda^{(\ell)}}$ of $\mathfrak{S}_k$.

The following classical formula (due to Frobenius) gives the dimensions of the representations $S^\lambda$ in terms of the *hook lengths* of the partition $\lambda$ defined below.

**Definition 3** (Hook lengths). Let $B(\lambda)$ denote the set of boxes in the Young diagram corresponding to a partition $\lambda \vdash k$. For a box $b \in B(\lambda)$, the length of the hook of $b$, denoted $h_b$ is the number of boxes strictly to the right and below $b$ plus 1.

**Theorem 14** (Hook length formula). Let $\lambda \vdash k$. Then,

$$\dim S^\lambda = \frac{k!}{\prod_{b \in B(\lambda)} h_b}.$$
Definition 4 (Young module). For $\lambda \vdash k$, we will denote
$$M^\lambda = \text{Ind}^{S^\lambda}_{S^\ast}(1_{S^\ast})$$
(where $1_{S^\ast}$ denotes the trivial one-dimensional representation of $S^\ast$).

Notation 17 (Induced representations and multiplicities). For each $\lambda \vdash k$, we denote by $\text{Par}(\lambda)$ the set of partitions $\mu \vdash k$ such that, there exists a decomposition $\lambda = \lambda' \coprod \lambda''$, $\lambda' = (\lambda'_1, \ldots, \lambda'_s), \lambda'' = (\lambda''_1, \ldots, \lambda''_t)$, $\ell' + \ell'' = \ell = \text{length}(\lambda)$, such that $S^\mu$ occurs with positive multiplicity in the representation
$$S^\lambda,\lambda'' := \text{Ind}^{S^k}_{S^\lambda \times S^{(1)}(\lambda'')} \left( \bigotimes_{i=1}^{s} S^{(\lambda'_i)} \bigotimes \left( \bigotimes_{j=1}^{t} S^{(1,\lambda''_j)} \right) \right),$$
and we denote the multiplicity of $S^\mu$ in $S^\lambda,\lambda''$ by $m^\mu_{\lambda,\lambda''}$.

More generally, for $k \in \mathbb{N}$, and $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(t)}) \in \text{Par}(k)$, we denote by $\text{Par}(\lambda) = \text{Par}(\lambda^{(1)}) \times \cdots \times \text{Par}(\lambda^{(t)})$.

Definition 5 (Dominance order). For any two partitions $\mu = (\mu_1, \mu_2, \ldots) \in \text{Par}(k)$, we say that $\mu \succeq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order on $\text{Par}(k)$. More generally, for $k = (k_1, \ldots, k_\ell) \in \mathbb{N}$, and $\mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}), \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \text{Par}(k)$, we denote $\mu \succeq \lambda$ if and only if $\mu^{(i)} \geq \lambda^{(i)}$ for each $i, 1 \leq i \leq \ell$.

We also need the definitions of Kostka numbers and the Littlewood-Richardson coefficients.

Definition 6 (Kostka numbers). For $\lambda, \mu \vdash k$, $K(\mu, \lambda)$ denotes the number of semi-standard Young tableaux of shape $\mu$ and weight $\lambda$ (see [24] for definitions of semi-standard Young tableaux, and also their shape and weight).

The following fact is very basic (see for example [14, Theorem 3.6.11] or [24, page 541, Section 7.3]).

Proposition 5 (Young’s rule). Let $k \in \mathbb{N}$, and $\lambda \in \text{Par}(k)$. Then,
$$\text{Ind}^{S^k}_{S^\lambda} (S^{(\lambda_1)} \otimes \cdots \otimes S^{(\text{length}(\lambda))}) \cong \bigoplus_{\mu \succeq \lambda} K(\mu, \lambda) S^\mu.$$

Definition 7 (Littlewood-Richardson coefficients). For $\lambda \vdash m, \mu \vdash n, \nu \vdash m + n$, $c^\nu_{\lambda, \mu}$ is the multiplicity of the irreducible representation $S^{\nu}$ in $\text{Ind}^{S^{m+n}}_{S^\lambda \times S^n} (S^\lambda \otimes S^n)$.

Proposition 6. Let $k, d > 0, \lambda \in \text{Par}(k, d)$ such that $\lambda = \lambda' \coprod \lambda''$, and $\mu \in \text{Par}(\lambda)$.

Then,
(A) \quad \text{card}(\{i | \mu_i \geq d\}) \leq d, \text{card}(\{j | \mu_j \geq d\}) \leq d,
(B) \quad m^\mu_{\lambda',\lambda''} = \sum_{\nu' \vdash |\lambda'|, \nu' \succeq \lambda', \nu'' \vdash |\lambda''|, \nu'' \succeq \lambda''} K(\nu', \lambda') \cdot K(\nu'', \lambda'') \cdot c^\nu_{\nu',\nu''},
(C) \quad \sum_{\mu \vdash k} m^\mu_{\lambda',\lambda''} \leq k^O(d^2).
Remark 8. It is well known that $K(\mu, \mu) = 1$ for all $\mu \in \text{Par}(k)$, $K(\mu, \lambda) = 0$ unless $\mu \geq \lambda$. Finally, if $\mu$ is the maximal element in the dominance ordering $\succeq$ on $\text{Par}(k)$, that is $\mu = (k)$, then $K(\mu, \lambda) = 1$ for all $\lambda \in \text{Par}(k)$. In particular, in conjunction with Schur’s lemma the above fact implies, that the trivial representation, $S^k$ occurs with multiplicity equal to $1 (= K(\lambda, \lambda))$ in $\text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k} \bigoplus_{j=1}^{\lambda} S^{(\lambda)}$.

Remark 9. Note also that the representation $\text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k} \bigoplus_{j=1}^{\lambda} S^{(\lambda)}$ is isomorphic to the permutation representation of $\mathfrak{S}_k$ on the set of cosets $\mathfrak{S}_k/\mathfrak{S}_\lambda$, and in particular

$$\dim \text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k} \bigoplus_{j=1}^{\lambda} S^{(\lambda)} = \frac{k!}{\prod_{1 \leq j \leq \lambda} \lambda_j !}.$$ 

Definition 8 (Skew partitions, horizontal and vertical strips). For any two partitions, $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$, $\mu = (\mu_1, \mu_2, \ldots) \vdash n$, $m \leq n$, we say that $\lambda \subset \mu$, if $\lambda_i \leq \mu_i$ for all $i$.

Identifying $\lambda, \mu$ with their respective Young diagrams, we say that the skew partition $\mu/\lambda$ is a horizontal strip if no two cells of $\mu/\lambda$ belong to the same column. We say that $\mu/\lambda$ is a vertical strip if no two cells of $\mu/\lambda$ belong to the same row.

Proposition 7 (Pieri’s rule). For $\lambda \vdash m$, and $n \geq 0$, we have the two following relations.

$$\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (S^\lambda \boxtimes S^{(n)}) \cong \bigoplus_{\mu/\lambda \text{ is a horizontal strip}} S^\mu,$$

$$\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (S^\lambda \boxtimes S^{1^n}) \cong \bigoplus_{\mu/\lambda \text{ is a vertical strip}} S^\mu.$$

We also have the following associativity relationship that allows us to apply Pieri’s rule (Proposition 7) iteratively.

Proposition 8. Let $n = m_1 + \cdots + m_\ell$, where for each $i, 1 \leq i \leq \ell$. Then,

$$\text{Ind}_{\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_\ell}}^{\mathfrak{S}_n} (V_1 \boxtimes \cdots \boxtimes V_\ell)$$

is isomorphic to

$$\text{Ind}_{\mathfrak{S}_{m_1 + \cdots + m_{\ell-1}} \times \mathfrak{S}_{m_\ell}}^{\mathfrak{S}_{m_1 + \cdots + m_\ell}} \left( \text{Ind}_{\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_{\ell-1}}}^{\mathfrak{S}_{m_1 + \cdots + m_{\ell-1}}} (V_1 \boxtimes \cdots \boxtimes V_{\ell-1}) \boxtimes V_\ell \right),$$

where for each $i, 1 \leq i \leq \ell$, $V_i$ is an $\mathfrak{S}_{m_i}$-module.

Proof of Proposition 6. We first prove (B). Let $k' = |\lambda'|$ and $k'' = |\lambda''|$. Then, using Young’s rule (Proposition 5)

$$\text{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_{k'}} \left( \bigotimes_{i=1}^{\lambda'} S^{(\lambda_i)} \right) \cong \bigoplus_{\nu' \vdash k', \nu' \geq \lambda'} K(\nu', \lambda') S^{\nu'},$$

$$\text{Ind}_{\mathfrak{S}_{\lambda''}}^{\mathfrak{S}_{k''}} \left( \bigotimes_{i=1}^{\lambda''} S^{(1^{\lambda''})} \right) \cong \bigoplus_{\nu'' \vdash k'', \nu'' \geq \lambda''} K(\nu'', \lambda'') S^{\nu''}.$$
is isomorphic to
\[ \bigoplus_{\nu' \vdash \lambda', \nu'' \vdash \lambda''} K(\nu', \lambda') K(\nu'', \lambda'') \mathbb{S}^{\nu'} \boxtimes \mathbb{S}^{\nu''}. \]

Eqn. (3.1) then follows from the isomorphism
\[ S_{\lambda', \lambda''} \cong \text{Ind}_{\mathbb{S}_{\nu'} \times \mathbb{S}_{\nu''}} \left( \left( \boxplus_{i=1}^{\mu} \mathbb{S}^{(\lambda_i')} \right) \boxtimes \left( \boxplus_{j=1}^{\nu} \mathbb{S}^{(1+\nu_j''')} \right) \right) \]
and the definition of the Littlewood-Richardson's coefficients, \( \ell_{\mu, \nu', \nu''} \) (Definition 7).

An alternative way of obtaining the multiplicities \( m_{\lambda', \lambda''}^{\mu} \) is by applying Pieri's rule iteratively at most \( d \) times using Propositions 8 and 7. Let \( \text{length}(\lambda') = \ell', \text{length}(\lambda'') = \ell'' \), so that \( \ell' + \ell'' = \text{length}(\lambda) \leq d \).

Let for \( 1 \leq i \leq \ell' \),
\[ M_i = \text{Ind}_{\mathbb{S}_{\lambda_i'} \times \cdots \times \mathbb{S}_{\lambda_i'} \times \mathbb{S}_{\lambda_i'} \times \mathbb{S}_{\lambda_i'} \times \mathbb{S}_{\lambda_i'} \times \mathbb{S}_{\lambda_i'}} \left( M_{i-1} \boxtimes \mathbb{S}^{(\lambda_i')} \right), \]
with the convention that \( M_0 = 1 \). For \( \nu \vdash \lambda_1' + \cdots + \lambda_i' \), let \( m_i^{\nu} \) denote the multiplicity of \( \mathbb{S}^{\nu} \) in \( M_i \), and \( m_i = \sum_{\nu \vdash \lambda_i'} m_i^{\nu} \). We prove by induction on \( i \) the following two statements.

(a) \[ m_i \leq m_{i-1} \cdot \binom{\lambda_i' + i - 1}{i - 1}. \]

(b) For each \( \nu \vdash \lambda_1' + \cdots + \lambda_i' \), such that \( m_i^{\nu} > 0 \), \( \text{length}(\nu) \leq i \).

Assuming statements (a) and (b) hold for \( i - 1 \) we prove them for \( i \). By induction for each \( \nu' \vdash \lambda_1' + \cdots + \lambda_{i-1}' \) with \( m_{i-1}^{\nu'} > 0 \), \( \text{length}(\nu') \leq i - 1 \). Applying Pieri’s rule (Proposition 7) we obtain
\[ \text{Ind}_{\mathbb{S}_{\lambda_i'} \times \cdots \times \mathbb{S}_{\lambda_i'} \times \mathbb{S}_{\lambda_i'}} \left( S^{\nu'} \boxtimes S^{(\lambda_i')} \right) \cong \bigoplus_{\nu' \vdash \lambda_i' + \cdots + \lambda_i'} \mathbb{S}^{\nu'}. \]

Observe that each choice of \( \nu \vdash \lambda_1' + \cdots + \lambda_i' \) such that \( \nu/\nu' \) is a horizontal strip, corresponds uniquely to a composition of \( \lambda_i' \) into at most \( \text{length}(\nu') \) parts, and the number of such compositions is clearly bounded by \( \binom{\lambda_i' + i - 1}{i - 1} \). This proves part (a).

Part (b) also follows from (3.2) noting that the length of each \( \mu \) that occurs on the right is at most \( \text{length}(\nu') + 1 \) which is \( \leq i \) using the induction hypothesis. This complete the proof of parts (a) and (b).

Now let for \( 1 \leq j \leq \ell'' \),
\[ N_j = \text{Ind}_{\mathbb{S}_{\nu'' \vdash \lambda''_1 + \cdots + \lambda''_j}} \left( N_{j-1} \boxtimes S^{(\lambda''_j)} \right), \]
with the convention that \( N_0 = M_{\ell''} \). For \( \nu \vdash \lambda''_1 + \cdots + \lambda''_j \), let \( n_j^{\nu} \) denote the multiplicity of \( S^{\nu} \) in \( N_j \), and \( n_j = \sum_{\nu \vdash \lambda''_1 + \cdots + \lambda''_j} n_j^{\nu} \).

The following two statements from an induction on \( j \). The proofs are very similar to the proofs of (a) and (b) above and are omitted.

(c) \[ n_j \leq n_{j-1} \cdot \binom{\lambda''_j + \ell' + j - 1}{\ell' + j - 1}. \]
(d) For each $\nu \vdash \lambda''_1 + \cdots + \lambda''_j$, such that $n''_j > 0$, $\text{length}(\bar{\nu}) \leq \ell' + j$.

It follows from (a), (b), (c), and (d), that

$$\sum_{\mu \vdash k} m^{\mu}_{\lambda', \lambda''} \leq kO(d^2),$$

which proves (C). Finally, it is easy to check that for each $\mu$ with $m^{\mu}_{\lambda', \lambda''} > 0$ that arises in the above process satisfies

$$\text{card}(\{i \mid \mu_i \geq d\}) \leq d, \text{card}(\{j \mid \tilde{\mu}_j \geq d\}) \leq d,$$

which proves (A). \qed

**Remark 10.** The following particular cases of Proposition 6 will be of interest.

(A) If $\lambda' = \lambda$ (and hence, $\lambda''$ is the empty partition),

$$m^{\mu}_{\lambda', \lambda''} = K(\mu, \lambda).$$

(B) If $\mu = (k)$,

$$m^{\mu}_{\lambda', \lambda''} = 1.$$

**Proof of Theorem 4.** The theorem follows from Part (A) of Proposition 6. \qed

### 3.7. Equivariant Poincaré duality.

**Theorem 15.** Let $V \subset \mathbb{R}^k$ be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of $\mathfrak{S}_k$ on $\mathbb{R}^k$. Then, for each $p, 0 \leq p \leq k$, there is an $\mathfrak{S}_k$-module isomorphism

$$H^p(V; \mathbb{F}) \cong H^{k-p-1}(V; \mathbb{F}) \otimes \text{sign}_k.$$

**Proof.** If $M$ is a $C^0$-manifold of dimension $\ell$, then the following sheaf-theoretic statement of Poincaré duality is well known (see for example [12, Corollary 5.5.6]).

\begin{equation}
\hom_{\mathbb{F}}(H^*_c(M; \mathbb{F}_M), \mathbb{F}) \cong H^*(M; \mathbb{F}_M)[\ell].
\end{equation}

In our case, with $M = V$. The $\mathfrak{S}_k$-action on the ambient space $\mathbb{R}^k$, induces an $\mathfrak{S}_k$-module structure on $H^*(V; \mathbb{F}_V)$ by the induced isomorphisms $\pi^*: H^*(V; \mathbb{F}_V) \cong H^*(V; \mathbb{F}_V)$, $\pi \in \mathfrak{S}_k$.

Now for $\pi \in \mathfrak{S}_k$ (and also denoting by $\pi$ the induced map $\pi: V \to V$), we have that $\pi$ induces the sign representation on the one dimensional vector space, $\Gamma(V; \mathfrak{or}_V)$, of global sections of the orientation sheaf on $V$. This implies the following $\mathfrak{S}_k$-isomorphism for each $p \geq 0$,

\begin{equation}
H^p(V; \mathfrak{or}_V) \cong H^p(V; \mathbb{F}_V) \otimes \text{sign}_k.
\end{equation}

The theorem follows from (3.3) and (3.4), after noting that since $V$ is assumed to be compact

$$\hom_{\mathbb{F}}(H^*(V, \mathbb{C}), \mathbb{F}) \cong H^*_c(V, \mathbb{F}) \cong H^*(V, \mathbb{F})$$

where all isomorphisms are $\mathfrak{S}_k$-module isomorphisms. \qed
3.8. Equivariant Mayer-Vietoris inequalities. Suppose that $S_1, S_2 \subset R^K$ are $G_k$-symmetric closed semi-algebraic sets. Then $S_1 \cup S_2$, and $S_1 \cap S_2$ are also $G_k$-symmetric closed semi-algebraic sets, and there is the classical Mayer-Vietoris exact sequence,

$$\cdots \to H^i(S_1 \cup S_2, F) \to H^i(S_1, F) \oplus H^i(S_2, F) \to H^i(S_1 \cap S_2, F) \to H^{i+1}(S_1 \cup S_2, F) \to \cdots$$

where all the homomorphisms are $G_k$-equivariant. Denoting by $H^*(S, F)_\mu$ the isotypic component of $H^*(S, F)$ corresponding to $\mu \in \text{Par}(k)$ for any $G_k$-symmetric closed semi-algebraic set $S \subset R^K$, we obtain using Schur’s lemma for each $\mu \in \text{Par}(k)$, an exact sequence,

$$\cdots \to H^i(S_1 \cup S_2, F)_\mu \to H^i(S_1, F)_\mu \oplus H^i(S_2, F)_\mu \to H^i(S_1 \cap S_2, F)_\mu \to H^{i+1}(S_1 \cup S_2, F)_\mu \to \cdots$$

The following inequalities follow from the above exact sequence (the proofs are similar to the non-equivariant case and can be found in [7]).

Let $S_1, \ldots, S_s \subset R^K$, $s \geq 1$, be $G_k$-symmetric closed semi-algebraic sets of $R^K$, contained in a $G_k$-symmetric closed semi-algebraic set $T$.

For $1 \leq t \leq s$, let $S_{\leq t} = \bigcap_{1 \leq j \leq t} S_j$, and $S_{\leq t} = \bigcup_{1 \leq j \leq t} S_j$. Also, for $J \subset \{1, \ldots, s\}$, $J \neq \emptyset$, let $S_J = \bigcap_{i \in J} S_i$, and $S_J = \bigcup_{i \in J} S_i$. Finally, let $S^0 = T$.

**Proposition 9.**

(a) For $\mu \in \text{Par}(k)$ and $i \geq 0$,

$$m_{i, \mu}(S_{\leq s}, F) \leq \sum_{j=1}^{i+1} \sum_{J \subset \{1, \ldots, s\}, \text{card}(J) = j} m_{i-j+1, \mu}(S_J, F).$$

(b) For $\mu \in \text{Par}(k)$ and $0 \leq i \leq K$,

$$m_{i, \mu}(S_{\leq s}, F) \leq \sum_{j=1}^{K-i} \sum_{J \subset \{1, \ldots, s\}, \text{card}(J) = j} m_{i+j-1, \mu}(S_J, F) + \left(\sum_{j=1}^{s} m_{K, \mu}(S^0, F)\right).$$

**Proof.** Follows from the proof of [7, Proposition 7.33] and Schur’s lemma. \(\square\)

**Proposition 10.** If $S_1, S_2$ are $G_k$-symmetric closed semi-algebraic sets, then for $\mu \in \text{Par}(k)$, any field $F$ and every $i \geq 0$

$$m_{i, \mu}(S_1, F) + m_{i, \mu}(S_2, F) \leq m_{i, \mu}(S_1 \cup S_2, F) + m_{i, \mu}(S_1 \cap S_2, F).$$

**Proof.** It follows from the proof of [7, Proposition 6.44] and Schur’s lemma. \(\square\)

3.9. Descent spectral sequence. We recall here a result proved in [9] that will be needed in the proof of Theorem 13.

We first introduce a notation.

**Notation 18** (Symmetric product). We denote for each $p \geq 0$, $\text{Sym}^{(p)}(X)$ the $(p+1)$-fold symmetric product of $X$ i.e.

$$\text{Sym}^{(p)}(X) = \frac{X \times \cdots \times X}{\Sigma_{p+1}}.$$
More generally, given a semi-algebraic map \( f : X \to Y \), we denote for each \( p \geq 0 \), by \( \text{Sym}_f^p(X) \) the quotient \( X \times_f \cdots \times_f X / \mathfrak{S}_{p+1}m \) where \( X \times_f \cdots \times_f X \) denote the \((p + 1)\)-fold fiber product of \( f \).

Now suppose that \( X, Y \) are compact semi-algebraic sets and \( f : X \to Y \) a continuous surjection, and let \( S \subset R^m \times R^k \) be a closed and bounded semi-algebraic set, and \( \pi : R^m \times R^k \to R^k \) the projection to the second factor.

**Theorem 16.** [9] With the above notation and for any field of coefficients \( \mathbb{F} \)

\[
 b(\pi(S), \mathbb{F}) \leq \sum_{0 \leq p < k} b(\text{Sym}_f^p(S), \mathbb{F}).
\]

4. PROOFS OF THE MAIN THEOREMS

4.1. Proofs of Theorems 5, 6 and 7.

**Proof of Theorem 5.** First replace \( V \) by the set \( S \) defined by \( \text{Def}(P, d', \zeta) \leq 0 \), where \( d' \) is the least even number such that \( d' > d \) and where \( d' - 1 \) is prime. It follows from Bertrand’s postulate that \( d' \leq 2d \). The Theorem follows from Proposition 4, Lemma 1, Proposition 1, Proposition 2, and Proposition 5. \( \square \)

**Proof of Theorem 6.** Theorem 6 follows from Theorem 5 and Proposition 6. \( \square \)

**Proof of Theorem 7.** Substituting \( X^{(j)} = Y^{(j)} + iZ^{(j)}, 1 \leq j \leq \ell \) in \( P \) and separating the real and imaginary parts, obtain another family of polynomials, \( Q \subset R[Y^{(1)}, Z^{(1)}, \ldots, Y^{(\ell)}, Z^{(\ell)}] \) with \( \deg_{Y^{(j)}(Q)}, \deg_{Z^{(j)}(Q)} \leq d, 1 \leq j \leq \ell \), such that the polynomials in \( Q \) are \( \mathfrak{S}_k \)-symmetric.

Now apply Theorem 6 with \( k = (k_1, \ldots, k_\ell), m = (2m_1, \ldots, 2m_\ell) \), and \( d = (d, \ldots, d) \). \( \square \)

4.2. Proofs of Theorems 8 and 9. We first need a few preliminary definitions and results.

**Definition 9.** For any finite family \( P \subset R[X_1, \ldots, X_k] \) and \( \ell \geq 0 \), we say that \( P \) is in \( \ell \)-general position with respect to a semi-algebraic set \( V \subset R^k \) if for any subset \( P' \subset P \), with \( \text{card}(P') > \ell \), \( Z(P', V) = \emptyset \).

Let \( k = (k_1, \ldots, k_\ell), m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell \), and \( K = \sum_{i=1}^\ell k_im_i \). Let \( \mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X^{(1)}, \ldots, X^{(\ell)}] \) be a finite set of \( \mathfrak{S}_k \)-symmetric polynomials, with \( \deg_{X^{(i)}(P_1)} \leq d \) for \( 1 \leq i \leq \ell, 1 \leq j \leq s \). Let \( S \subset R^k \) be a \( P \)-closed semi-algebraic set. Let \( \zeta = (\varepsilon_1, \ldots, \varepsilon_s) \) be a tuple of new variables, and let \( \mathcal{P}_\zeta = \bigcup_{1 \leq i \leq s} \{P_i \pm \varepsilon_i\} \).

We have the following two lemmas.

**Lemma 3.** Let

\[
 D(k, m, d) = \sum_{i=1}^\ell \min(k_im_i, d^{m_i}).
\]

The family \( \mathcal{P}_\zeta \subset R'[X^{(1)}, \ldots, X^{(\ell)}] \) is in \( D \)-general position with respect to any semi-algebraic subset \( Z' \subset R'^k \), where \( R' = R(\zeta) \) (cf. Notation 10), and where \( Z' = \text{Ext}(Z, R^k) \) (cf. Notation 11), and \( Z \subset R^k \) is a semi-algebraic set stable under the action of \( \mathfrak{S}_k \).
Proof. The lemma follows from the fact that the ring of multi-symmetric polynomials is generated by the multi-symmetric power sum polynomials [16, Theorem 1.2], and the cardinality of the set of multi-symmetric power sum polynomials in the variables $X^{(i)}$ of degree bounded by $d$ is bounded by $d^m_i$. \hfill $\square$

Let $\Phi$ be a $P$-closed formula, and let $S = \mathcal{R}(\Phi, V)$ be bounded over $R$.

**Notation 19.** For $\mu \in \text{Par}(k)$ and $i \geq 0$, we will denote
\[
\begin{align*}
m_{i,\mu}(\Phi, F) &= m_{i,\mu}(S, F), \\
m_{\mu}(\Phi, F) &= m_{\mu}(S, F).
\end{align*}
\]

Let $\Phi_\tau$ be the $P_\tau$-closed formula obtained from $\Phi$ be replacing for each $i, 1 \leq i \leq s$,

i. each occurrence of $P_i \leq 0$ by $P_i - \varepsilon_i \leq 0$, and
ii. each occurrence of $P_i \geq 0$ by $P_i + \varepsilon_i \geq 0$.

Let $R' = R\langle \varepsilon_1, \ldots, \varepsilon_s \rangle$, and $S'_\tau = \mathcal{R}(\Phi_\tau, R^{K})$.

**Lemma 4.** For any $r > 0$, $r \in R$, the semi-algebraic set set $\text{Ext}(S \cap B_K(0, r), R')$ is contained in $S'_\tau \cap B_K(0, r)$, and the inclusion $\text{Ext}(S \cap B_K(0, r), R') \hookrightarrow S'_\tau \cap B_K(0, r)$ is a semi-algebraic homotopy equivalence. The induced isomorphism,
\[
\mathcal{H}(S'_\tau \cap B_K(0, r), F) \cong \text{Ext}(S \cap B_K(0, r), R') \mathcal{H}(F)
\]
is an isomorphism of $\mathfrak{S}_k$-modules.

**Proof.** The proof is similar to the one of Lemma 16.17 in [7]. \hfill $\square$

**Remark 11.** In view of Lemmas 3 and 4 we can assume (at the cost of doubling the number of polynomials) after possibly replacing $P$ by $P_\tau$, and $R$ by $R\langle \varepsilon_1, \ldots, \varepsilon_s \rangle$, that the family $P$ is in $D(k, m, d)$-general position.

Now, let $\delta_1, \ldots, \delta_s$ be new infinitesimals, and let $R' = R\langle \delta_1, \ldots, \delta_s \rangle$.

**Notation 20.** We define $P_{>i} = \{P_{i+1}, \ldots, P_s\}$ and

\[
\begin{align*}
\Sigma_i &= \{P_i = 0, P_i = \delta_i, P_i = -\delta_i, P_i \geq 2\delta_i, P_i \leq -2\delta_i\}, \\
\Sigma_{\leq i} &= \{\psi \mid \psi = \bigwedge_{j=1,\ldots,i} \psi_j, \psi_j \in \Sigma_i\}.
\end{align*}
\]

Note that for each $\psi \in \Sigma_i$, $\mathcal{R}(\psi, R\langle \delta_1, \ldots, \delta_i \rangle^K)$ is symmetric with respect to the action of $\mathfrak{S}_k$, and for $\psi \neq \psi', \psi, \psi' \in \Sigma_{\leq i}$,
\[
\mathcal{R}(\psi, R\langle \delta_1, \ldots, \delta_i \rangle^K) \cap \mathcal{R}(\psi', R\langle \delta_1, \ldots, \delta_i \rangle^K) = \emptyset.
\]

If $\Phi$ is a $P$-closed formula, we denote
\[
\mathcal{R}_i(\Phi) = \mathcal{R}(\Phi, R\langle \delta_1, \ldots, \delta_i \rangle^K),
\]
and
\[
\mathcal{R}_i(\Phi \wedge \psi) = \mathcal{R}(\psi, R\langle \delta_1, \ldots, \delta_i \rangle^K) \cap \mathcal{R}_i(\Phi).
\]

The proof of the following proposition is very similar to Proposition 7.39 in [7] where it is proved in the non-symmetric case.
Proposition 11. For every \( \mathcal{P} \)-closed formula \( \Phi \), and \( \mu \in \text{Par}(k) \),
\[
m_{\mu}(\Phi, \mathcal{F}) \leq \sum_{\Psi \in \Sigma_{\leq s}} m_{\mu}(\Psi, \mathcal{F}).
\]

Proof. The symmetric spaces \( \mathcal{R}(\Psi, \text{Ext}(V, R')) \), \( \Psi \in \Sigma_{\leq s} \), are disjoint by (4.1). The proposition now follows from Schur’s lemma, and the proof of Proposition 7.39 in [7]. \( \square \)

Proposition 12. Suppose for \( \mu \in \text{Par}(k) \) and \( i \geq 0 \), \( m_{i,\mu}(S, \mathcal{F}) > 0 \). Then,
\[
(4.2)
\]
where \( d = (d, \ldots, d) \). For \( i \geq 0 \), and \( \mu \in \mathcal{I}(k, d, m) \),
\[
\sum_{\Psi \in \Sigma_{\leq s}} m_{i,\mu}(\Psi, \mathcal{F}) \leq \sum_{j=0}^{D(k,m,d)} \binom{s}{j} 6^j F(\mu, k, m, d),
\]
where
\[
F(\mu, k, m, d) = \sum_{\lambda = (\lambda^{(1)}, \ldots, \lambda^{(t)}) \in \text{Par}(k, (2d)^m)} G(\mu, \lambda, d, m),
\]
and
\[
G(\mu, \lambda, d, m) = \prod_{1 \leq i \leq t} \left( (2d)^{m_{\lambda^{(i)}}} \max_{\lambda^{(i)} = \lambda^{(i)}, \lambda^{(i)}' \in \lambda^{(i)}} m_{\lambda^{(i)}, \lambda^{(i)}'} \right)
\]

(the maximum on the right hand side is taken over all decompositions \( \lambda^{(i)} = \lambda^{(i)}, \lambda^{(i)}', \lambda^{(i)}'' \)).

In order to prove Proposition 12 we first need the following lemmas.

Let for \( 1 \leq i \leq s \), \( Q_i = P_i^2(P_i^2 - \delta_i^2)^2(P_i^2 - 4\delta_i^2) \).

For \( j \geq 1 \) let,
\[
V_j' = \mathcal{R}\left( \bigvee_{1 \leq i \leq j} Q_i = 0, R(\delta_1, \ldots, \delta_j)^K \right),
\]
\[
W_j' = \mathcal{R}\left( \bigvee_{1 \leq i \leq j} Q_i \geq 0, R(\delta_1, \ldots, \delta_j)^K \right).
\]

Lemma 5. Let \( I \subset [1, s] \), \( \sigma = (\sigma_1, \ldots, \sigma_s) \in \{0, \pm 1, \pm 2\}^s \) and let \( \mathcal{P}_{I, \sigma} = \bigcup_{i \in I} \{ P_i + \sigma_i \delta_i \} \). Then, \( Z(P_{I, \sigma}, R^K) = \emptyset \), whenever \( \text{card}(I) > D \).

Proof. This follows from the fact that \( \mathcal{P} \) is in \( D \)-general position by Remark 11. \( \square \)

Lemma 6. For each \( \mu \in \mathcal{I}(k, d, m) \), and \( i \geq 0 \),
\[
m_{i,\mu}(V_j', \mathcal{F}) \leq (6^j - 1) F(k, m, 2d).
\]

Proof. The set \( \mathcal{R}((P_j^2(P_j^2 - \delta_j^2)^2(P_j^2 - 4\delta_j^2) = 0), R(\delta_1, \ldots, \delta_j)^K) \) is the disjoint union of
\[
\mathcal{R}(P_i = 0, R(\delta_1, \ldots, \delta_j)^K),
\]
\[
\mathcal{R}(P_i = \delta_i, R(\delta_1, \ldots, \delta_j)^K),
\]
\[
\mathcal{R}(P_i = -\delta_i, R(\delta_1, \ldots, \delta_j)^K),
\]
\[
\mathcal{R}(P_i = 2\delta_i, R(\delta_1, \ldots, \delta_j)^K),
\]
\[
\mathcal{R}(P_i = -2\delta_i, R(\delta_1, \ldots, \delta_j)^K).
\]
It follows from part (a) of Proposition 9 that \( m_{i,\mu}(V_j', \mathbb{F}) \) is bounded by the sum for \( 1 \leq p \leq i + 1 \), of the multiplicities of \( S^\mu \) in the \((i - p + 1)\)-th cohomology module of all possible \( p \)-ary intersections amongst the sets listed in (4.3). It is clear that the total number of such non-empty \( p \)-ary intersections is at most \( \binom{i}{j} 5^p \). It now follows from Theorem 2 applied to the non-negative symmetric polynomials \( P_i^2, (P_i + \delta_i)^2, (P_i + 2\delta_i)^2 \), and noting that the degrees of these polynomials are bounded by \( 2d \), that

\[
m_{i,\mu}(V_j', \mathbb{F}) \leq \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\mu, k, m, 2d).
\]

\( \square \)

**Lemma 7.** For each \( \mu \in I(k, d, m) \), and \( i \geq 0 \),

\[
m_{i,\mu}(W_j', \mathbb{F}) \leq \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p (F(\mu, k, m, 2d)) + m_{i,\mu}(R(\delta_1, \ldots, \delta_j)^K, \mathbb{F}).
\]

**Proof.** Let

\[
F = \mathcal{R}\left( \bigwedge_{1 \leq i \leq j} Q_i \leq 0 \lor \bigvee_{1 \leq i \leq j} Q_i = 0, \operatorname{Ext}(Z, R(\delta_1, \ldots, \delta_i)) \right).
\]

Now, from the fact that \( W_j' \cup F = R(\delta_1, \ldots, \delta_j)^K, W_j' \cap F = V_j' \),

it follows immediately that

\[
(W_j' \cup F) = W_j' \cup F = R(\delta_1, \ldots, \delta_j)^K,
\]

and

\[
W_j' \cap F = (W_j' \cap F) = V_j'.
\]

Using Proposition 10 we get that

\[
m_{i,\mu}(W_j', \mathbb{F}) \leq m_{i,\mu}((W_j' \cap F), \mathbb{F}) + m_{i,\mu}((W_j' \cup F), \mathbb{F}) = m_{i,\mu}(V_j', \mathbb{F}) + m_{i,\mu}(R(\delta_1, \ldots, \delta_j)^K, \mathbb{F}).
\]

We conclude using Lemma 6. \( \square \)

Now, let

\[
T_i = \mathcal{R}\left( P_i^2 (P_i^2 - \delta_i^2) (P_i^2 - 4\delta_i^2) \geq 0, \operatorname{Ext}(Z, R(\delta_1, \ldots, \delta_s)) \right), 1 \leq i \leq s,
\]

and let \( T \) be the intersection of the \( T_i \) with the closed ball in \( R(\delta_1, \ldots, \delta_s, \delta)^K \) defined by \( \delta^2 \left( \sum_{1 \leq i \leq k} X_i^2 \right) \leq 1 \). Then, it is clear from Lemma 4 that

\[
\sum_{\Psi \in \Sigma \leq s} m_{i,\mu}(\Psi, \mathbb{F}) = m_{i,\mu}(T, \mathbb{F}).
\]

**Proof of Proposition 12.** Using part (b) of Proposition 9 we get that

\[
\sum_{\Psi \in \Sigma \leq s} m_{i,\mu}(\Psi, \mathbb{F}) \leq \sum_{j=1}^{\min(D, K-i)} \sum_{\text{card}(J) = j, J \subset \{1, \ldots, s\}} m_{i+j-1,\mu}(S^J, \mathbb{F}) + \left( \begin{array}{c} 5^p \\ K-i \end{array} \right) m_{K,\mu}(S^0, \mathbb{F}).
\]
It follows from Lemma 7 that,

\[ m_{i+j-1,\mu}(S^d) \leq \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\mu, k, m, 2d) + m_{K,\mu}(R^K, \mathbb{F}). \]

Hence,

\[
\sum_{\Psi \in \Sigma_{\leq s}} m_{s,\mu}(\Psi, \mathbb{F}) \leq \sum_{j=1}^{D} \sum_{J \subseteq \{1, \ldots, s\}} m_{i+j-1,\mu}(S^d, \mathbb{F}) + \left( \binom{s}{K - i} \right) m_{K,\mu}(S^0, \mathbb{F})
\]

\[
\leq \sum_{j=1}^{D} \binom{s}{j} \left( \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\mu, k, m, 2d) \right)
\]

\[
\leq \sum_{j=1}^{D} \binom{s}{j} 6^j F(\mu, k, m, 2d).
\]

\[ \square \]

**Proof of Theorem 8.** Follows from Propositions 11 and 12. \[ \square \]

**Proof of Theorem 9.** Follows immediately from Theorem 8 and Proposition 6. \[ \square \]

### 4.3. Proof of Theorem 10.

**Proof of Theorem 10.** Let \( S^{2k+1} \subset C^{k+1} \) denote the unite sphere defined by \( |Z_0|^2 + \cdots + |Z_k|^2 = 1 \). Consider the Hopf fibration \( \phi : S^{2k+1} \to \mathbb{P}_C^k \), defined by \((z_0, \ldots, z_k) \mapsto (z_0 : \cdots : z_k)\). We denote by \( \tilde{V} = \phi^{-1}(V) \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{i} & S^{2k+1} \\
\downarrow{\phi|_{\tilde{V}}} & & \downarrow{\phi} \\
V & \xrightarrow{i} & \mathbb{P}_C^k \\
\end{array}
\]

Note that \( \tilde{V} \) is a \( S^1 \)-bundle over \( V \), and using the fact that \( \mathbb{P}_C^k \) is simply connected, there is a \( S_{k+1} \)-equivariant spectral sequence degenerating at its \( E_3 \) term converging to the cohomology of \( \tilde{V} \).

The \( E_3 \)-term of the spectral sequence is given by

\[
E_2^{p,q} = H_p(V, F), \text{ if } q = 0, 1,
\]

\[
E_2^{p,q} = 0, \text{ else ,}
\]

and the differentials \( d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1} \) shown below.
Fix $\lambda \vdash k+1$, and recall that we denote for each $i \geq 0$, $m_{i,\lambda}(V,F)$ (resp. $m_{i,\lambda}(\tilde{V},\tilde{F})$) the multiplicity of $S^\lambda$ in $H^i(V,F)$ (resp. $H^i(\tilde{V},\tilde{F})$).

We observe that since $H^0(V,F) \cong_{\mathfrak{S}_{k+1}} H^0(\tilde{V},\tilde{F})$, we have for all $\lambda \vdash k+1$,
\begin{equation}
(4.5) \quad m_{0,\lambda}(V,F) = m_{\lambda,0}(\tilde{V},\tilde{F}).
\end{equation}

Also, note that it follows from the fact that the spectral sequence $E^p,q$ degenerates at its $E_\infty$ term that,
\[ H^1(V,F) \oplus \ker(d_2) \cong_{\mathfrak{S}_{k+1}} H^1(\tilde{V},\tilde{F}), \]
and we obtain from the fact that the spectral sequence $E^r_{p,q}$ is $\mathfrak{S}_{k+1}$-equivariant that
\begin{equation}
(4.6) \quad m_{1,\lambda}(V,F) \leq m_{1,\lambda}(\tilde{V},\tilde{F}).
\end{equation}

More generally, we have from the $E^2$-term of the spectral sequence that
\[ H^i(\tilde{V},\tilde{F}) \cong_{\mathfrak{S}_{k+1}} \operatorname{coker}(d_2^{-i,2}) \oplus \ker(d_2^{-i-1,1}). \]

For $\lambda \vdash k+1$, $i \geq 0$, and any finite dimensional $F$-representation $W$ of $\mathfrak{S}_{k+1}$, we denote by $\operatorname{mult}_\lambda(W,F)$ the multiplicity of $S^\lambda$ in $W$.

Since,
\[ H^i(V,F) \cong_{\mathfrak{S}_{k+1}} \operatorname{Im}(d_2^{-i,2}) \oplus \operatorname{coker}(d_2^{-i-1,1}), \]
we have for all $\lambda \vdash k+1$, $i \geq 0$,
\[ \operatorname{mult}_\lambda(\operatorname{coker}(d_2^{-i-1,1}),F) = m_{i,\lambda}(V,F) - \operatorname{mult}_\lambda(\operatorname{Im}(d_2^{-i-1,1}),F), \]
and we also have for $i \geq 2$,
\begin{equation}
(4.7) \quad \operatorname{mult}_\lambda(\operatorname{Im}(d_2^{-i-2,1}),F) \leq m_{i-2,\lambda}(V,F). \]

This implies that for all $\lambda \vdash k+1$, $i \geq 0$
\[ m_{i,\lambda}(\tilde{V},\tilde{F}) = (m_{i,\lambda}(V,F) - \operatorname{mult}_\lambda(\operatorname{Im}(d_2^{-i-2,1}),F)) + \operatorname{mult}_\lambda(\operatorname{coker}(d_2^{-i-1,1}),F). \]

It follows that
\[ m_{i,\lambda}(V,F) = m_{i,\lambda}(\tilde{V},\tilde{F}) + \operatorname{mult}_\lambda(\operatorname{Im}(d_2^{-i-2,1}),F) - \operatorname{mult}_\lambda(\operatorname{coker}(d_2^{-i-1,1}),F) \leq m_{i,\lambda}(\tilde{V},\tilde{F}) + \operatorname{mult}_\lambda(\operatorname{Im}(d_2^{-i-2,1}),F) \leq m_{i,\lambda}(\tilde{V},\tilde{F}) + m_{i-2,\lambda}(V,F) \text{ using (4.7).} \]

Finally we have shown that for each $\lambda \vdash k+1$ and $i \geq 2$,
\[ m_{i,\lambda}(V, F) \leq m_{i,\lambda}(\tilde{V}, F) + m_{i-2,\lambda}(V, F) \]
\[ \leq \sum_{0 \leq j \leq \lfloor \frac{i}{2} \rfloor} m_{i-2j,\lambda}(\tilde{V}, F) \] using induction.

(4.8)

The Theorem follows from applying Theorem 6 to the set \( \tilde{V} \), and inequalities (4.5), (4.6), and (4.8).

\[ \square \]

Remark 12. Note that in the proof of Theorem 10, since \( \tilde{V} / S_{k+1} \sim V / S_{k+1} \), and hence \( H^*_{S_{k+1}}(\tilde{V}, F) \sim H^*_{S_{k+1}}(V, F) \), we can avoid the argument involving the spectral sequence if we are only interested in the equivariant cohomology of \( V \). Also note that it is possible to replace the spectral sequence argument altogether by an argument using the equivariant version of the Gysin exact sequence.

4.4. Proof of Theorem 13.

Proof of Theorem 13. First notice that
\[ \text{Sym}^{(p)}(V) = V^{(p)} / S_{k(p)}, \]
where
\[ V^{(p)} = Z(P^{(p)}, R^{k+(p+1)m}), \]
\( P^{(p)} \in R[X, Y_0, \ldots, Y_p] \) is defined by
\[ P^{(p)} = P(X, Y_0) + \cdots + P(X, Y_p), \]
and \( k(p) = (1, \ldots, 1, p+1) \).

Notice that since \( V \) is bounded, so is \( V^{(p)} = Z(P^{(p)}, R^{k+(p+1)m}) \). Moreover, \( \deg(P^{(p)}) = \deg(P) \), and \( P^{(p)} \) is symmetric in \((Y_0, \ldots, Y_p)\), and is thus \( S_{k(p)} \)-symmetric.

By Theorem 16,
\[ b(\pi(V), F) \leq \sum_{0 \leq p < k} b(\text{Sym}^{(p)}(V), F). \]

Now using Corollary 1,
\[ b(\text{Sym}^{(p)}(V), F) \leq (p + 1)^{(2d)m} (O(d))^{k + m(2d)^m + 1}, \]
and hence,
\[ b(\pi(V), F) \leq \sum_{0 \leq p < k} b(\text{Sym}^{(p)}(V), F) \]
\[ \leq \sum_{0 \leq p < k} (p + 1)^{(2d)m} (O(d))^{k + m(2d)^m + 1} \]
\[ \leq k^{(2d)^m} (O(d))^{k + m(2d)^m + 1}. \]

This completes the proof of the theorem. \[ \square \]
5. Conclusion and open problems

In this paper we have proved polynomial bounds on the number and the multiplicities of the irreducible representations of the symmetric group (or more generally product of symmetric groups) that appear in the cohomology modules of symmetric real algebraic and more generally real semi-algebraic sets. We have given several applications of the main results, including to improve existing bounds on the topological complexity of sets defined as images of semi-algebraic maps, and proving lower bounds on the degrees etc. We end with some open problems and future research directions.

5.1. Representational Stability Question. The bounds on the multiplicities that we prove in this paper are all polynomial in the number of variables (for fixed degrees). Motivated by the recently developed theory of FI-modules [15] it makes sense to ask whether it is possible to prove some stability result as \( k \to \infty \). We formulate one such question below.

Let \( F \in \mathbb{R}[X_1, \ldots, X_d]_{\leq d} \) be a symmetric polynomial of degree at most \( d \), and let for \( k \geq d \) \( F_k = \phi_{d,k}(F) \in \mathbb{R}[X_1, \ldots, X_k]_d \) where \( \phi_{d,k} : \mathbb{R}[X_1, \ldots, X_d]_d \hookrightarrow \mathbb{R}[X_1, \ldots, X_k]_d \) is the canonical injection, defined by

\[
F_k = G(p_1^{(k)}(X_1, \ldots, X_k), \ldots, p_d^{(k)}(X_1, \ldots, X_k)),
\]

where \( G = G(p_1^{(d)}, \ldots, p_d^{(d)}) \) denoting for each \( n > 0 \), \( p_i^{(n)} = \sum_{j=1}^n X_j^{i} \) (the \( i \)-th power sum polynomial). Let \( V_k = \mathbb{R}(F_k, \mathbb{R}^k) \). Also, let \( \mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0 \) be any fixed partition, and for all \( k \geq k_0 + \mu_1 \), let

\[
\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k.
\]

It is a consequence of the hook-length formula (Theorem 14) that

\[
\dim_F(\mathbb{S}^\ell)^{\mu}) = \frac{\dim_F(\mathbb{S}^{\mu})}{|\mu|!} P_\mu(k),
\]

where \( P_\mu(T) \) is a monic polynomial having distinct integer roots, and \( \deg(P_\mu) = |\mu| \) (see [17, 7.2.2]).

Finally, for a fixed number \( p \geq 0 \) we pose the following question.

**Question 1.** Does there exist a polynomial \( P_{F,p,\mu}(k) \) such that for all sufficiently large \( k \), \( m_{p,\{\mu\}_k}(V_k, \mathbb{F}) = P_{F,p,\mu}(k) \)? In conjunction with (5.2), a positive answer would imply that

\[
\dim_F(H^\ell(V_k, \mathbb{F}))_{\{\mu\}_k} = \frac{\dim_F(\mathbb{S}^{\mu})}{|\mu|!} P_{F,p,\mu}(k) P_\mu(k)
\]

is also given by a polynomial for all large enough \( k \).

In particular, taking \( \mu = () \) to be the empty partition, is it true that \( m_{p,\{\mu\}_k}(V_k, \mathbb{F}) = b_{\mathbb{S}^k}(V_k, \mathbb{F}) \) (that is the \( p \)-th equivariant Betti number of \( V_k \) cf. Notation 6) is given by a polynomial in \( k \)?

A stronger question is to ask for a bound on the degree of \( P_{F,p,\mu}(k) \) as a function of \( d, \mu \) and \( p \).

**Remark 13.** Note that it follows from the results of this paper (Theorem 6) that there exists a polynomial \( P_{F,p,\mu}(k) \) of degree \( O(d^2) \), with the property that

\[
m_{p,\{\mu\}_k}(V_k, \mathbb{F}) \leq P_{F,p,\mu}(k)
\]
for all $k \geq 0$. In the particular case when $\mu$ is the empty partition, it follows from Corollary 1 that the degree of $P_{F,\mu}(k)$ can be bounded by $2d$.

**Remark 14.** Question 1 has a positive answer for the sequence of polynomials $F_k$ introduced in Example 1. In that example we can take $F$ to be the following polynomial:

$$F = \sum_{i=1}^{4} X_i^2(X_i - 1)^2 - \varepsilon \in \mathbb{R}[\varepsilon][X_1, X_2, X_3, X_4]_{\leq 4}.$$

From the discussion in Example 1 we deduce that for each $p > 0, \mu \vdash k_0$, and for all large enough $k$,

$$m_{p,\mu}(V_k, \mathbb{F}) = 0.$$  

For $p = 0$, and partitions $\mu$ with $\text{length}(\mu) > 1$, we again have for all large enough $k$,

$$m_{p,\mu}(V_k, \mathbb{F}) = 0.$$  

Finally, for $p = 0$, and any partition $(k_0)$ of length $\leq 1$, and for all $k \geq 2k_0$,

$$m_{0,\mu}(V_k, \mathbb{F}) = 2(k - k_0) - k + 1 \text{ (using (5.1) and (1.5))} = k - 2k_0 + 1.$$

Thus, $m_{p,\mu}(V_k, \mathbb{F})$ is given by a polynomial for all large $k$, for any fixed $p$ and $\mu$. Notice also that the degree of this polynomial is bounded by 1.

### 5.2. Algorithmic Conjecture.

As mentioned in the Introduction, a polynomial bound on any topological invariant of a class of semi-algebraic sets usually implies also that there exists an algorithm with polynomially bounded complexity for computing it. Since we have we proved that the multiplicities of the irreducible representations of $S_k$ appearing in the cohomology group of a symmetric $\mathcal{P}$-semi-algebraic set $S \subset \mathbb{R}^k$, where $\deg(P), P \in \mathcal{P}$ is bounded by a constant, is bounded by a polynomial function of $\text{card}(\mathcal{P})$ and $k$, the mentioned principle implies that these multiplicities should be computable by an algorithm with polynomially bounded complexity (for fixed $d$). If this holds, then since the number of irreducibles that are allowed to appear with positive multiplicity is also polynomially bounded, and their respective dimensions are polynomially computable using the hook length formula (Theorem 14), we deduce that once these multiplicities are computed, the dimensions of the cohomology groups of $S$ (with coefficients in $\mathbb{Q}$) can be computed with polynomially bounded complexity.

This leads us to make the following algorithmic conjecture.

**Conjecture 1.** For any fixed $d > 0$, there is an algorithm that takes as input the description of a symmetric semi-algebraic set $S \subset \mathbb{R}^k$, defined by a $\mathcal{P}$-closed formula, where $\mathcal{P}$ is a set symmetric polynomials of degrees bounded by $d$, and computes $m_{i,\lambda}(S, \mathbb{Q})$, for each $\lambda \vdash k$, and $m_{i,\lambda}(S, \mathbb{Q}) > 0$, as well as all the Betti numbers $b_i(S, \mathbb{Q})$, with complexity which is polynomial in $\text{card}(\mathcal{P})$ and $k$.

**Remark 15.** We note that Conjecture 1 is not completely unreasonable, since an analogous result for computing the generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets has been proved in [9]. However, computing the Betti numbers of a semi-algebraic set is usually a much harder task than computing the Euler-Poincaré characteristic.
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