ProtoDash: Fast Interpretable Prototype Selection

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Abstract

In this paper we propose an efficient algorithm ProtoDash for selecting prototypical examples from complex datasets. Our work builds on top of the learn to criticize (L2C) work by [12] and generalizes it to not only select prototypes for a given sparsity level $m$ but also to associate non-negative weights with each of them indicative of the importance of each prototype. Unlike in the case of L2C, this extension provides a single coherent framework under which both prototypes and criticisms (i.e. lowest weighted prototypes) can be found. Furthermore, our framework works for any symmetric positive definite kernel thus addressing one of the open questions laid out in [12]. Our additional requirement of learning non-negative weights introduces technical challenges as the objective is no longer submodular as in the previous work. However, we show that the problem is weakly submodular and derive approximation guarantees for our fast ProtoDash algorithm. Moreover, ProtoDash can not only find prototypical examples for a dataset $X$, but it can also find (weighted) prototypical examples from $X^{(2)}$ that best represent another dataset $X^{(1)}$, where $X^{(1)}$ and $X^{(2)}$ belong to the same feature space. We demonstrate the efficacy of our method on diverse domains namely; retail, digit recognition (MNIST) and on the latest publicly available 40 health questionnaires obtained from the Center for Disease Control (CDC) website maintained by the US Dept. of Health. We validate the results quantitatively as well as qualitatively based on expert feedback and recently published scientific studies on public health.

1 Introduction

Interpretable modeling has received a lot of attention in recent times [16, 12, 10, 19, 18]. The reason being that nearly every real application with a human making decisions at its helm needs to have confidence in the model before he/she can trust its judgment. Interestingly, interpretability has also become important in deep learning [9] given all the recent studies [3, 9] showing their susceptibilities to slightly perturbed adversarial examples.

In this paper we provide two algorithms for selecting prototypical examples from complex datasets. A more standard greedy algorithm which we call ProtoGreedy and a faster one called

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ProtoDash. Our work builds on top of the learn to criticize (L2C) work by [12] and generalizes it to not only select prototypes for a given sparsity level \( m \) but also to associate non-negative weights with each of them indicative of the importance of each prototype. This extension leads to multiple advantages over L2C: a) the weights allow for assessing the importance of the prototypes, b) the non-negativity aids in making this comparison more natural and hence more interpretable, c) it provides a single coherent framework under which both prototypes and criticisms – which are the lowest weighted prototypes – can be found and d) our framework works for any symmetric positive definite kernel which is not the case for L2C. Moreover, ProtoDash can not only find prototypical examples for a dataset \( X \), but it can also find (weighted) prototypical examples from \( X^{(2)} \) that best represent another dataset \( X^{(1)} \), where \( X^{(1)} \) and \( X^{(2)} \) belong to the same feature space. This aspect has applications in covariate shift [1] kind of settings, where the weights associated with the chosen samples has to be computed only for the \( m \) prototypes. We showcase the power of our method in the experiments, where on a large retail dataset the prototypes actually improve performance over using all the data. We also depict its efficacy on MNIST where we gradually skew the distribution of \( X^{(1)} \) from it being a representative sample to containing only a single digit, with \( X^{(2)} \) remaining unchanged. Our method adapts to this by picking more (and increased weight) representatives of the skewed digit in \( X^{(1)} \) from \( X^{(2)} \) leading to significantly better performance. In addition, our extensions induce an implicit metric that can be used to order \( k \) different datasets \( X_1, ..., X_k \) based on how well their prototypes represent \( X^{(1)} \). This aspect is used to create a directed graph based on the 40 health questionnaires available through Center for Disease Control (CDC). The graph depicts which questionnaire is best represented by which other questionnaires. Such a graph can be used to find surrogates or even further study causal relationships between the conditions/categories denoted by these questionnaires. For instance, our method finds that the income of an individual is primarily affected by his early childhood and occupation. Occupation is natural to think of, but it is interesting that early childhood was selected as the most important factor given 39 other possibilities. However, this interesting insight can be justified by a recent article in the Atlantic [17], which talks about the significant decrease in social mobility in the 2000s. We can thus obtain and recover socially impactful insights at low cost, which could be a starting point for deeper investigations in the future.

Our additional requirement of learning non-negative weights does not maintain the submodularity of the objective as in L2C. In fact, we see our work as addressing one of the open questions laid out in [12] and we quote, "For future work, we hope to further explore the properties of L2C such as the effect of the choice of kernel, and weaker conditions on the kernel matrix for submodularity." To this end, we show that having non-equal weights for the prototypes eliminates any additional conditions on the kernel matrix but at the expense of abandoning submodularity. However, we show that the resultant set function is still weakly submodular for which we provide a standard greedy and a fast ProtoDash algorithm. Our main algorithm ProtoDash, which although has slightly worse performance bounds than ProtoGreedy is much faster than it with its performance in practice being virtually indistinguishable, as is seen in the experiments. We provide approximation guarantees for both of these methods. For interested readers, analysis of time complexity is provided in the Appendix E.

We would like to emphasize that the additional non-negativity constraint on weights precludes us from directly leveraging the results of [7] or [12]. We need to explicitly prove that the set function is weakly submodular and reestablish all the guarantees established in [7] as they do not directly follow. This is a key point which is discussed in more detail in Section 3.
2 Problem Statement

Let $\mathcal{X}$ be the space of all covariates from which we obtain the samples $X^{(1)}$ and $X^{(2)}$. Consider a kernel function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and its associated reproducing kernel Hilbert space (RKHS) $\mathcal{K}$ endowed with the inner product $k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$ where $\phi_y(y) = k(x, y) \in \mathcal{K}$ is continuous linear functional satisfying $\phi_y : h \mapsto h(x) = \langle \phi_y, h \rangle$ for any function $h \in \mathcal{K} : \mathcal{X} \to \mathbb{R}$.

The maximum mean discrepancy (MMD) is a measure of difference between two distributions $p$ and $q$ where if $\mu_p = \mathbb{E}_{x \sim p}[\phi_x]$ it is given by:

$$\text{MMD}(\mathcal{K}, p, q) = \sup_{h \in \mathcal{K}} (\mathbb{E}_{x \sim p}[h(x)] - \mathbb{E}_{y \sim q}[h(y)])$$

$$= \sup_{h \in \mathcal{K}} \langle h, \mu_p - \mu_q \rangle.$$

Our goal is to approximate $\mu_p$ by a weighted combination of $m$ sub-samples $Z \subseteq X^{(2)}$ drawn from the distribution $q$, i.e., $\mu_p(x) \approx \sum_{j:z_j \in Z} w_j k(z_j, x)$ where $w_j$ is the associated weight of the sample $z_j \in X^{(2)}$. We thus need to choose the prototype set $Z \subseteq X^{(2)}$ of cardinality $(|Z|)$ $m$ where $n^{(1)} = |X^{(1)}|$ and learn the weights $w_j$ that minimizes the finite sample MMD metric with the additional non-negativity constraint for interpretability, as given below:

$$\overline{\text{MMD}}(\mathcal{K}, X^{(1)}, Z, w) = \frac{1}{(n^{(1)})^2} \sum_{x_i, x_j \in X^{(1)}} k(x_i, x_j) - \frac{2}{n^{(1)}} \sum_{z_j \in Z} w_j \sum_{x_i \in X^{(1)}} k(x_i, z_j)$$

$$+ \sum_{z_i, z_j \in Z} w_i w_j k(z_i, z_j); \text{ subject to } w_j \geq 0, \forall z_j \in Z. \quad (2.1)$$

Index the elements in $X^{(2)}$ from 1 to $n^{(2)} = |X^{(2)}|$ and for any $Z \subseteq X^{(2)}$ let $L_Z \subseteq [n^{(2)}] = \{1, 2, \ldots, n^{(2)}\}$ be the set containing its indices. Discarding the constant terms in (2.1) that do not depend on $Z$ and $w$ we define the function

$$l(w) = w^T \mu_p - \frac{1}{2} w^T K w \quad (2.2)$$

where $K_{i,j} = k(y_i, y_j)$ and $\mu_{p,j} = \frac{1}{n^{(1)}} \sum_{x_i \in X^{(1)}} k(x_i, y_j); \forall y_j \in X^{(2)}$ is the point-wise empirical evaluation of the mean $\mu_p$. Our goal then is to find a index set $L_Z$ with $|L_Z| \leq m$ and a corresponding $w$ such that the set function $f : 2^{[n^{(2)}]} \to \mathbb{R}^+$ defined as

$$f(L_Z) = \max_{w : \text{supp}(w) \in L_Z, w \geq 0} l(w) \quad (2.3)$$

attains maximum. Here $\text{supp}(w) = \{j : w_j > 0\}$. We will denote the maximizer for the set $L_Z$ by $\zeta(L_Z)$.

3 Related Work

Recently, there have been a surge of papers proposing interpretable models motivated by diverse applications such as medical [4], information technology [10] and entertainment [16]. The strategies
involve building rule/decision lists \cite{19, 18}, to finding prototypes \cite{12} in an unsupervised manner like us or strictly in a supervised manner as \cite{2}, to taking inspiration from psychometrics \cite{10} and learning understandable models. Works such as \cite{16} differ from the above methods in that they focus on answering instance specific user queries by locally approximating a superior performing complex model with a simpler easy to understand one. The hope is that the insights conveyed by the simpler model will be consistent with the complex model.

In our work, as mentioned above, we generalize the setting in \cite{12} and propose algorithms that select prototypes with non-negative weights associated with them. On the technical side, one recent work that we leverage and extend with non-negativity constraints for our MMD objective is \cite{7}. We recover their bounds even with the non-negativity constraint. In fact, our bounds are tighter since the restricted concavity parameter \( c_\Omega \) and restricted smoothness parameter \( C_\Omega \) stated in Definition 4.2 are obtained by searching over only the non-negative orthant as opposed to the entire \( \mathbb{R}^b \) space, where \( b \) is the dimensionality. Moreover, given our specific functional form for the objective we show in Corollary 4.6 that choosing an element with the largest gradient in ProtoDash at each step is equivalent to maximizing a tight lower bound on \( l(\cdot) \), which is not necessarily true for the setting considered in \cite{7}. Additionally, the gradient in our case can be easily computed. The added technical difficulty when deriving the guarantees in our case comes from the fact that we cannot let the gradients go to zero as the non-negativity constraints would make our solution infeasible. As a consequence, we cannot directly leverage the results of \cite{7} or \cite{12}. The complexity lies in showing that our set function even with the additional non-negativity constraints imposed for interpretability is still weakly submodular as we prove in Theorem 4.3. Weak submodularity alone does not provide the bounds for ProtoDash. We prove these in Theorem 4.5. Lemma 4.4 proved in our work is essential for proving both Theorems 4.3 and 4.5, which is not the case in \cite{7}.

4 Prototype Selection Framework

In this section we first describe two greedy algorithms ProtoGreedy and the faster ProtoDash. ProtoGreedy is algorithmically similar to L2C described in \cite{12} as both the methods greedily select the next element that maximizes the increment of the set function. However, ProtoGreedy additionally learns (unequal) weights for each of the selected prototypes which is not the case for L2C. Our main contribution with respect to ProtoGreedy is in showing that the set function is weakly submodular with the non-negativity constraints on the weights based on revisiting concepts such as weak submodularity, restricted strong concavity (RSC) and restricted smoothness (RSM). We then prove that \( f(L) \) is monotonic followed by bounding its submodularity ratio \( \gamma \) away from zero, which implies that it is weakly submodular. Having established that, the approximation guarantee of \( (1 - e^{-\gamma}) \) for ProtoGreedy can be obtained using the results from \cite{7}. However, the guarantees for ProtoDash do not directly follow and we explicitly prove an approximation guarantee of \( (1 - e^{-\frac{3c}{c^2} \gamma}) \), where \( c \) and \( C \) are the RSC and RSM parameters respectively.

4.1 Methods

In this subsection we describe two greedy algorithms ProtoGreedy and a faster version, ProtoDash, which is our main contribution. For both algorithms the termination condition can be a given sparsity level \( m \) or a minimal increase in objective value \( \epsilon \) that is required for selecting more elements.
Algorithm 1 ProtoGreedy

**Input:** sparsity level \( m \) or lower bound \( \epsilon \) on increase in \( l(.) \), \( X^{(1)} \), \( X^{(2)} \)

\( L = \emptyset \)

while termination condition is false do

\{ i.e., if \( m \) is given then \( |L| \leq m \), else increase in objective value \( \geq \epsilon \). \}

\( \forall j \in [n^{(2)}] \setminus L, v_j = f(L \cup \{j\}) - f(L) \) \{ \( f(.) \) depends on \( X^{(1)} \) and \( X^{(2)} \). \}

\( j_0 = \arg\max \, v_j \)

\( L = L \cup \{j_0\} \)

\( \zeta^{(L)} = \arg\max_{w: \text{supp}(w) \subset L, w \geq 0} l(w) \) \{ \( l(.) \) depends on \( X^{(1)} \) and \( X^{(2)} \). \}

end while

return \( L, \zeta^{(L)} \)

Algorithm 2 ProtoDash

**Input:** sparsity level \( m \) or lower bound \( \epsilon \) on increase in \( l(.) \), \( X^{(1)} \), \( X^{(2)} \)

\( L = \emptyset, \zeta^{(L)} = 0 \)

\( g = \nabla l(0) = \mu_p \)

while termination condition is false do

\{ i.e., if \( m \) is given then \( |L| \leq m \), else increase in objective value \( \geq \epsilon \). \}

\( j_0 = \arg\max \, g_j \)

\( L = L \cup \{j_0\} \)

\( \zeta^{(L)} = \arg\max_{w: \text{supp}(w) \subset L, w \geq 0} l(w) \) \{ \( l(.) \) depends on \( X^{(1)} \) and \( X^{(2)} \). \}

\( g = \nabla l\left( \zeta^{(L)} \right) = \mu_p - K \zeta^{(L)} \)

end while

return \( L, \zeta^{(L)} \)

In algorithm 1 ProtoGreedy, we select an element \( j \) that produces the greatest increase in objective value \( f(.) \) given the current (selected) set \( L \). We then compute the weights for \( L \cup j \) that maximize the objective.

In algorithm 2 ProtoDash, the next best element is obtained by maximizing a tight lower bound on the objective as is shown in Corollary 4.6. In particular, we choose an element \( j \) whose gradient given by \( \mu_{p,j} - K \zeta^{(L)} \) is the highest over the current set of candidates. Then similar to ProtoGreedy the optimal weights are computed. While ProtoGreedy requires solving a quadratic program at each iteration to select the next element, ProtoDash requires only a search over \( n^{(2)} - |L| + 1 \) elements leading to a \( O(m^2) \) speedup.

4.2 Preliminaries

Given a positive integer \( b \), let \( [b] := \{1, \ldots, b\} \) denote the set of the first \( b \) natural numbers. Let \( \langle x, y \rangle \) denote dot product of vectors \( x \) and \( y \).

**Definition 1 (Submodularity Ratio):** Let \( L, S \subset [b] \) be two disjoint sets, and \( f : [b] \to R \). The
submodularity ratio \( \gamma \) of \( L \) with respect to (w.r.t.) \( S \) is given by:

\[
\gamma_{L,S} = \frac{\sum_{i \in S} (f(L \cup i) - f(L))}{f(L \cup S) - f(L)} \tag{4.1}
\]

The submodularity ratio of a set \( U \) w.r.t. a positive integer \( r \) is given by:

\[
\gamma_{U,r} = \min_{L,S:|L| \leq r} \gamma_{L,S} \tag{4.2}
\]

The function \( f(.) \) is submodular iff \( \forall L, S, \gamma_{L,S} \geq 1 \). However, if \( \gamma_{L,S} \) can be shown to be bounded away from 0, but not necessarily \( \geq 1 \), then \( f(.) \) is said to be weakly submodular.

**Definition 2 (RSC and RSM):** A function \( l: R^b \rightarrow R \) is said to be restricted strong concave (RSC) with parameter \( \Omega_l \) and restricted smooth (RSM) with parameter \( \Omega_m \) \([7]\) if \( \forall x, y \in \Omega \subset R^b; \)

\[
-\frac{\Omega_l}{2} \| y - x \|^2_2 \geq l(y) - l(x) - \langle \nabla l(x), y - x \rangle \geq -\frac{\Omega_m}{2} \| y - x \|^2_2. \tag{4.3}
\]

We denote the RSC and RSM parameters on the domain \( \Omega_m = \{ x : \| x \|_0 \leq m; x \geq 0 \} \) of all \( m \)-sparse non-negative vectors by \( c_m \) and \( C_m \) respectively. Also, let \( \Omega = \{ (x, y) : \| x - y \|_0 \leq 1 \} \) with the corresponding smoothness parameter \( \tilde{C}_1 \).

### 4.3 Theoretical Guarantees

Based on the above two definitions we now prove our results and propose two greedy algorithms with approximation guarantees.

**Lemma 4.1** (Monotonicity). The set function \( f \) defined in \([2.3]\) is monotonic, meaning that if \( L_1 \subseteq L_2 \) then \( f(L_1) \leq f(L_2) \).

**Proof.** Let \( |L_1| = n_1 \) and \( |L_2| = n_2 \) and \( n_1 \leq n_2 \). Index the elements in \( L_2 \) such that the first \( n_1 \) elements are those contained in \( L_1 \). Then,

\[
f(L_2) = \max_{w: \text{supp}(w) \subseteq L_2, w_j \geq 0} l(w) \geq \max_{w: \text{supp}(w) \subseteq L_1, w_j \geq 0} l(w) = f(L_1).
\]

\[\blacksquare\]

**Lemma 4.2** (Finite RSC and RSM). Given a symmetric positive definite kernel matrix \( K \), the function \( l(w) \) in \([2.3]\) has a positive RSC and finite RSM parameters.

**Proof.** For the concave function \( l(w) = -\frac{1}{2} w^T K w + w^T \mu_p \), we calculate \( l(w_1) - l(w_2) - \nabla l(w_2), w_1 - w_2 \) \(-0.5(w_1 - w_2)^T K (w_1 - w_2) \). If \( w_1 \) and \( w_2 \) are \( k_1 \) and \( k_2 \) sparse vectors respectively, then \( \Delta w = w_1 - w_2 \) has a maximum of \( k \leq k_1 + k_2 \) non-zero entries. For the constants \( c \) and \( C \) satisfying \(-c\| \Delta w \|^2 \geq -\Delta w^T K \Delta w \geq -C\| \Delta w \|^2 \) we obtain the bounds: \( c \geq k \)-sparse smallest eigen value of \( K \) and \( C \leq k \)-sparse largest eigen value of \( K \). In particular, when \( \text{supp}(w_2) \subset \text{supp}(w_1) \), \( \| \Delta w \|_0 \leq k_1 \) providing tighter bounds for \( c \) and \( C \).

\[\blacksquare\]
Detailed proofs for Theorems 4.3 and 4.5, Lemma 4.4 and Corollary 4.6 are in the Appendix.

**Theorem 4.3** (Weak submodularity). The set function $f$ in (2.3) is weakly submodular with the submodularity ratio $\gamma > 0$.

**Proof Sketch.** Given a set $U$, let $l(.)$ be $(c|U|+m, C|U|+m)$ strongly concave and smooth respectively on $|U| + m$-sparse non-negative vectors and $\tilde{C}_1$ smooth on $(x, y) \in \tilde{\Omega}$ where $x, y \in R^b$. We then show that $\gamma_{U,m} \geq c|U|+m \tilde{C}_1 \geq c|U|+m C|U|+m$.

The key challenge in showing the above result is the non-negativity constraint, since to respect it we cannot let the gradients go to 0 as in [7]. We thus have to analyze the resultant KKT conditions with this added complexity. To this end, the following Lemma is useful.

**Remark** Lemma 4.4 and Theorem 4.3 imply that algorithm 1, ProtoGreedy, has an approximation of $(1 - e^{-\gamma})$ [14].

**Lemma 4.4.** For $j / \in L$, if $\nabla l_j \left( \zeta^{(L)} \right) \leq 0$ then $\zeta^{(L \cup \{j\})} = \zeta^{(L)}$. In particular $\zeta_j^{(L \cup \{j\})} = \zeta_j^{(L)} = 0$. Hence, if $\zeta_j^{(L \cup \{j\})} > 0$ then $\nabla l_j \left( \zeta^{(L)} \right) > 0$.

**Proof Sketch.** The proof is based on the observation that the corresponding Langrange multiplier $\lambda_j$ satisfies dual feasibility and KKT conditions which are necessary and sufficient for optimality.

**Theorem 4.5** (ProtoDash Guarantees). If $L_D$ is the $m$ sparse set selected by ProtoDash and $L^*$ is the optimal $m$ sparse set then,

$$f \left( L_D \right) \geq \left( 1 - e^{-\frac{3c}{4C}} \right) f \left( L^* \right)$$  \hspace{1cm} (4.4)

where $\gamma$, $c$ and $C$ are the submodularity ratio, RSC and RSM parameters respectively.

**Proof Sketch.** Theorem 4.3 doesn’t directly imply that we have a bound for ProtoDash. Based on Lemma 4.4 and analyzing KKT conditions with the additional non-negativity constraint. With this added constraint we upper bound the loss between ProtoGreedy and ProtoDash at each iteration showing that it is no worse than a multiplicative factor $\frac{3c}{4C}$ in the exponent.

Algorithm 2 ProtoDash, with the above guarantees, will select a prototype $j$ at a particular iteration if $\mu_{p,j} - K_j, \zeta_j^{(L)}$ is the highest over all candidates.

**Corollary 4.6.** In ProtoDash, at each iteration, selecting the next prototype with the maximum gradient is equivalent to choosing a prototype that maximizes a tight lower bound on the function maximized by ProtoGreedy for its selection of the next prototype.

**Proof Sketch.** Let $l_j(.)$ attain its optimum at $w^j$. Then, $l_j(w^j) \geq l_j(w^*), \forall j$. The choice $j^D$ by our ProtoDash method has the property that $l_{j^D}(w^*) \geq l_{j}(w^*)$. Ergo, ProtoDash selects the prototype $j^D$ that maximizes a lower bound on $l_j(w^j)$. Using KKT conditions we are able to show that this lower bound is tight.
5 Experiments

In this section we quantitatively as well as qualitatively validate our algorithms on three diverse domains. The first is a dataset from a large retailer. The second is MNIST which is a handwritten digit dataset. The third are 40 health questionnaires obtained from the CDC website.

We compare ProtoDash (or PrDash) with five other methods. The first is our slower but potentially better performing greedy method ProtoGreedy (or PrGrdy). The second is L2C. The third is P-Lasso (or P-Las), i.e., lasso with the non-negativity constraint [8]. The fourth is K-Medoids (or K-Med) [2]. The fifth is RandomW (or RndW), where prototypes are selected randomly, but the weights are computed based on our strategy. ProtoGreedy’s and ProtoDash’s superior performance to this baseline as well as to L2C implies that selecting high quality prototypes in conjunction with determining their weights are important for obtaining state-of-the-art results and that neither of these strategies suffices in isolation. We use Gaussian kernel in all the experiments. The kernel width is chosen by cross-validation.

More experiments with an adapted version of L2C for going across datasets, and results with our methods on MNIST where we choose the top \( m \) prototypes from the top \( 2m \) or \( 3m \) prototypes based on the magnitude of the learned weights are given in Appendix F. This is also another benefit of learning the weights where we can first oversample and then choose the desired number of prototypes that have the largest weights, which leads to even better results.

5.1 Retail

The first dataset we consider is from a large retailer. We have 2 years of online customer data from the beginning of 2015 to the end of 2016. This is information of roughly 80 million customers. Around 2 million of which are loyalty customers and we know of 9878 customers who were regular customers in 2015 but became loyalty in 2016.

The goal is to accurately predict the total expenditure of a customer and to evaluate if being a loyalty or a regular customer has any effect on his behavior independent of factors such as the number of online visits, his geo or zip, average time per visit, average number of pages viewed per visit, brand affinities, color and finish affinities, which are the attributes in the dataset.

To answer this counter-factual we build a SVM-RBF [5] model using 10-fold cross-validation on the 2016 data and evaluate its performance on the 2015 data for the 9878 customers that were not loyalty then. In essence we test our model by evaluating how accurately we predict the expenditure of these 9878 customers in 2015, with a model that is built using the 2016 data as described next.

The 2016 data that we use to train the SVM-RBF depends on the prototype selection methods. The entire loyalty group is always part of the training. The question is what subset of the regular customers we should also add to training. For our methods we choose prototypes from the regular customer base that best represent the loyalty group. We select around 1.5 million customers because the improvement in objective is incremental beyond this point. We select the same number of prototypes for the competing methods. For this experiment we have an additional baseline which is training using all the data which we aver to as PrAll. Moreover, we also pass instance specific weights for training, for the methods that learn weights. We use a Spark cluster for this experiment.

Quantitative Evaluation: In Figure [1], we observe the root mean squared errors (RMSE) of the different methods. We see that our methods are significantly better than that the competitors. Using all the data is not a good idea probably due to the high size imbalance between the two
Figure 1: We observe the quantitative and qualitative results of the different methods on retail dataset.

We also observe that ProtoDash is almost as good as ProtoGreedy.

In Figure 1b, we observe the running time of the different prototype selection methods. Here we see that ProtoDash is close in running time to L2C and over 3 times faster than ProtoGreedy.

Hence, from Figures 1a and 1b we can conclude that ProtoDash would be the method of choice for this application.

**Qualitative Evaluation:** We did further investigation of our prototypes. We found that our prototype group had high number of visits i.e. the (weighted) average number of visits for this group was in the top 1% of the visits by regular customers and they also had relatively high expenditure i.e. in the top 2% in this group.

The more reassuring fact that our prototypes and weights had reasonable predictive power was when we shared the top 100 prototypes based on our weights from the 1.5 million with the domain experts and they told us that around 83 of those actually became loyalty customers in 2017. This is depicted in Figure 1c.

5.2 MNIST

We employ the (global) nearest neighbor prototype classifier as described in [12]. Since we want to obtain more robust generalization results especially as we skew the test distribution towards a single digit, we flip the MNIST training and test sets. In particular, we use the MNIST training set of 60000 images to form multiple test sets of size 5420, which is the cardinality of least frequent digit in this set. We then randomly select 1500 images from the original MNIST test set to form our training set. The first test set we form is representative of the population and contains an equal number (i.e. ∼ 10%) of all the digits. We now create skewed test sets for percentages of \( s = 30 \), 50, 70, 90 and 100. For each value of \( s \) we create 10 test sets where a particular digit is \( s \) fraction of the test set and the remaining portion of test set contains representative population of the other digits. For example, when \( s = 70 \) one of the test sets will have 70% 0s and the remaining 30% is shared equally by the other 9 digits. By averaging our results for each \( s \) we can observe the performance of the different methods for varying levels of skew. The reported results are over 100 such resamplings of the training set.

**Quantitative Evaluation:** If we average over all percentages \( s \) and plot our objective for different levels of sparsity as shown in Figure 2, we find that around \( m = 200 \) the gain in objective is incremental. We thus choose 200 prototypes for all the methods.

In Figure 3a, we see that the performance of the closest competitors is almost unaffected by skew.
Figure 2: We see convergence behavior of the different methods for varying sparsity on MNIST.

![MNIST performance graph]

Figure 3: We observe the quantitative and qualitative results of the different methods on MNIST.

Our methods are a little worse than K-Medoids initially but their performance drastically improves as the skew increases. This scenario shows the true power of our methods in being able to adapt to non-representative test distributions that are significantly different than the train. Additionally, the performance of ProtoDash again is indistinguishable from ProtoGreedy.

In Figure 3b, we see that ProtoDash is orders of magnitude faster than K-Medoids and ProtoGreedy which are its closest competitors.

**Qualitative Evaluation:** We further wanted to understand the reason why our methods outperform the other methods at high skews. Thus at 100% skew, i.e. where the test set is just samples of a single digit, we tried to see what fraction of the prototypes picked by the different methods were that digit. In this case we use the top 100 prototypes from the 200 based on weight for our methods and RandomW, while just based on order for the other methods, since in a training set of size 1500 there only a little over 100 copies of a digit.

In Figure 3c, we report the percentages of the target digit picked by the different methods from the training averaged over the 10 digits. We see that our methods adapt swiftly picking almost exclusively images of the target digit with all the weight concentrated on them. This provides a strong qualitative justification for the superior performance of our methods.

### 5.3 CDC Questionnaires Data

The US dept. of health conducts surveys consisting of 10s of questionnaires sent to over thousands of people every couple of years. This is a rich repository of anonymized human health facts that are publicly available. We in this study use the latest available health questionnaires collected over the
There are 43 questionnaires but we had some data issues with 3 of them so we used the remaining 40. The entire list is in the supplement but to name a few we have *Alcohol Use, Occupation, Income, Early Childhood, Depression, Diet.*

An expert in public health we collaborated with wanted to see 1) if we could rank order the questionnaires based on some measure of importance so that henceforth they could either send fewer questionnaires for people to fill or at least tell them in which order to fill them so as to reduce time and effort on both sides. 2) If for a given questionnaire we could find others that are most representative or indicative of it. Such an insight could potentially lead to early interventions that could help improve health of the populous.

We attacked both problems with our prototype selection framework. In fact, accomplishing the 2nd task is a big step in resolving the 1st. For each questionnaire $Q_i \in Q$ where $Q = \{Q_1, ..., Q_{40}\}$ we found prototypes ($m = 10$) after which the improvement in objective was incremental. We then evaluated the quality of these prototypes on the other 39 questionnaires based on our objective. Thus, for a particular $Q_i$ we now rank ordered the other $Q/Q_i$ based on our objective value. Ergo, the rank $r_{ij}$ signifies how well the prototypes of a questionnaire $Q_j$ represents $Q_i$. This resolves the ask 2) above. Note that the rank is not commutative and hence graphically it can be viewed as a directed graph. Now to satisfy ask 1), for each $Q_j$ we found its average rank i.e. $r_j = \frac{1}{39} \sum_{i \in \{1,...,40\}, i \neq j} r_{ij}$ across other questionnaires and sorted the $r_j$s in ascending order. Thus, lower the $r_j$ more important the questionnaire.

**Quantitative Evaluation:** We obtained from the expert a list of 10 questionnaires he thought would be most important of the 40. We intersected this list with the top 10 by the different methods and report the overlap percentage in Figure 4a. P-Lasso didn’t produce results possibly due to bad condition number on most datasets so we omitted it. We see that our methods have the largest overlap and thus have the most agreement with the expert. We also see again in Figure 4b that ProtoDash is highly efficient in addition to having superior performance.

**Qualitative Evaluation:** We tried to validate some of the rankings (entire list in supplement) we got from task 2) based on prior studies. The insights are depicted in Figure 4c. We found that for the Income questionnaire its best representative prototypes, other than itself of course, came from the Early Childhood questionnaire which has information about the environment in which the child was born. The second best questionnaire was Occupation. Occupation is intuitive to understand as affecting income. However, Early childhood is interesting and the expert mentioned that there is validation of this based on a recent study which talks about significant decrease in social mobility in recent years [17]. The ranking of the other methods differed with no such justification.
We also analyzed the Demographic data questionnaire from the same year in terms of how it fared with representing the 40 questionnaires. It turned out that it was the top ranked for multiple questionnaires such as; Alcohol Use, Blood Pressure & Cholesterol, Smoking - Household Smokers, Smoking - Secondhand Smoke Exposure, Volatile Toxicant, Weight History - Youth, Disability, Preventive Aspirin Use, Kidney Conditions - Urology, Hepatitis, Housing Characteristics, Immunization and Dermatology. Amongst the prototypes some of the most common and highest weighted were white Americans with education levels that were at best AA. This is highly consistent with the recent study [13] which shows that the death toll among middle aged uneducated white Americans is on the rise due to financial and health related stresses.

6 Discussion

In this paper we provided a fast interpretable prototype selection method ProtoDash. We derived approximation guarantees for it and showed in the experiments that its performance is as good as our other standard greedy version ProtoGreedy with it being much faster. Learning non-negative weights and being able to find prototypes across datasets leads to its superior performance over L2C and other competitors, while still outputting interpretable results.

In the future, it would be interesting to further close the theoretical gap between these two greedy algorithms maybe based on the ideas in [11], although it is not clear if they would generalize to our setting. The other extension may be to obtain a convex combination of weights rather than them being just non-negative. In terms of practical applications, we are in the process of further studying how demographics and other behavioral traits relate to statistics on increased mortality rates [13], which has been a major concern in the recent decades. Our prototype selection methods could also have applications in transfer learning and lifelong learning applications, where one can use the prototypes to efficiently and accurately learn models for new tasks. We plan to explore such avenues in the future, where we would also want to investigate extensions of our current methods to online settings making them even more scalable.

Appendix

A Proof of Lemma 4.4

Recall that the optimization problem for computing the set function $f(L)$ requires that for $j \notin L$, $x_j = 0$. Let $\lambda_j$ denote the corresponding Lagrange multiplier. The stationarity condition of the unconstrained problem implies that at the optimum $\zeta^{(L)}$, $\lambda_j = -\nabla l_j (\zeta^{(L)}) \geq 0$. In the optimization problem for computing $f(L \cup \{j\})$, $\lambda_j$ is the KKT multiplier for the constraint $x_j \geq 0$. As $\lambda_j$ satisfies the dual feasibility condition which together with other KKT conditions are both necessary and sufficient for the optimality of maximizing concave functions $l$, we get $\zeta^{(L \cup \{j\})} = \zeta^{(L)}$.

B Proof of weak sub-modularity (Theorem 4.3)

We lower bound the numerator and upper bound the denominator. Let $\bar{m} = |L| + |S|$. Recall that $\zeta^{(L)}, \zeta^{(L \cup S)} \in \mathbb{R}^{b^+}$ are the maximizer $l \left( \zeta^{(L)} \right) = f(L)$ and $l \left( \zeta^{(L \cup S)} \right) = f(L \cup S)$ respectively. By
the definition of \( RSC \) and \( RSM \) constants we find

\[
\frac{c_m}{2} \left\| \zeta^{(L\cup S)} - \zeta^{(L)} \right\|_2^2 \leq l \left( \zeta^{(L)} \right) - l \left( \zeta^{(L\cup S)} \right) + \left\langle \nabla l \left( \zeta^{(L)} \right), \zeta^{(L\cup S)} - \zeta^{(L)} \right\rangle.
\]

Noting that \( f \) is monotone for increasing supports we get

\[
0 \leq l \left( \zeta^{(L\cup S)} \right) - l \left( \zeta^{(L)} \right) \leq \left\langle \nabla l \left( \zeta^{(L)} \right), \zeta^{(L\cup S)} - \zeta^{(L)} \right\rangle - \frac{c_m}{2} \left\| \zeta^{(L\cup S)} - \zeta^{(L)} \right\|_2^2
\]

\[
\leq \max_{v : v^{(L\cup S)} \neq 0, v > 0} \left\langle \nabla l \left( \zeta^{(L)} \right), v - \zeta^{(L)} \right\rangle - \frac{c_m}{2} \left\| v - \zeta^{(L)} \right\|_2^2.
\]

(B.1)

The vector \( v \) with the support restricted to the coordinates specified by \( L \cup S \) attains maximum at

\[
v_{L\cup S} = \max \left\{ \frac{1}{c_m} \nabla l_{L\cup S} \left( \zeta^{(L)} \right) + \zeta^{(L)}_{L\cup S}, 0 \right\}.
\]

It then follows

\[
\left( v - \zeta^{(L)} \right)_{L\cup S} = \max \left\{ \frac{1}{c_m} \nabla l_{L\cup S} \left( \zeta^{(L)} \right), -\zeta^{(L)}_{L\cup S} \right\}.
\]

The KKT conditions at the optimum \( \zeta^{(L)} \) for the function \( f(L) \) necessitates that \( \forall j \in L, \)

\[
\zeta_j^{(L)} > 0 \implies \nabla l_j \left( \zeta^{(L)} \right) = 0,
\]

\[
\zeta_j^{(L)} = 0 \implies \nabla l_j \left( \zeta^{(L)} \right) \leq 0
\]

and hence we have \( \left( v - \zeta^{(L)} \right)_j = 0 \). Further, for \( j \in S, \) \( \zeta_j^{(L)} = 0 \) implying that \( \left( v - \zeta^{(L)} \right)_j = \max \left\{ \frac{1}{c_m} \nabla l_j \left( \zeta^{(L)} \right), 0 \right\} \). Defining \( \nabla l^+_S \left( \zeta^{(L)} \right) = \max \left\{ \nabla l_S \left( \zeta^{(L)} \right), 0 \right\} \) and plugging the quantities computed at the maximum value \( v \) in (B.1) we get the bound

\[
0 \leq l \left( \zeta^{(L\cup S)} \right) - l \left( \zeta^{(L)} \right) \leq \frac{1}{2c_m} \left\| \nabla l^+_S \left( \zeta^{(L)} \right) \right\|_2^2.
\]

(B.2)

To lower bound the numerator, consider a single coordinate \( j \in S \). It suffices to restrict to those coordinates \( j \) where \( \nabla l_j \left( \zeta^{(L)} \right) > 0 \). Otherwise, by Lemma 4.4 \( f \left( L \cup \{ j \} \right) = f(L) \). Let \( 1^{(j)} \) be a vector with a value one only at the \( j \)th coordinates and zero elsewhere. For a \( \alpha \geq 0 \), define \( y^{(j)} = \zeta^{(L)} + \alpha 1^{(j)} \) such that \( \left( \zeta^{(L)}, y^{(j)} \right) \in \bar{\Omega} \). As \( \zeta^{(L\cup \{ j \})} \) is the optimal point for \( f \left( L \cup \{ j \} \right) \) we have

\[
l \left( \zeta^{(L\cup \{ j \})} \right) - l \left( \zeta^{(L)} \right) \geq l \left( y^{(j)} \right) - l \left( \zeta^{(L)} \right)
\]

\[
\geq \left\langle \nabla l \left( \zeta^{(L)} \right), \alpha 1^{(j)} \right\rangle - \frac{\tilde{C}_1}{2} \alpha^2.
\]

Maximizing w.r.t \( \alpha \) we get \( \alpha = \frac{\nabla l_j \left( \zeta^{(L)} \right)}{\tilde{C}_1} \geq 0 \). Substituting this maximum value we get

\[
l \left( \zeta^{(L\cup \{ j \})} \right) - l \left( \zeta^{(L)} \right) \geq \frac{1}{2\tilde{C}_1} \left( \nabla l_j \left( \zeta^{(L)} \right) \right)^2
\]

\[
\implies \sum_{j \in S} \left[ l \left( \zeta^{(L\cup \{ j \})} \right) - l \left( \zeta^{(L)} \right) \right] \geq \frac{1}{2\tilde{C}_1} \left\| \nabla l^+_S \left( \zeta^{(L)} \right) \right\|_2^2.
\]

(B.3)

From the equations (B.2) and (B.3) we get \( \gamma_{L,S} \geq \frac{c_m}{\tilde{C}_1} \). The minimum over all sets \( L, S \) proves the theorem.
C Guarantees for ProtoDash (Theorem 4.5)

Let $L = L_i^D$ be the set chosen by the ProtoDash up to the iteration $i$ such that $L_m^D = L^D$ and $L^*$ be the optimal set. Define the residual set $L_R = L^* \setminus L$. Given $L$, let $v$ and $u$ be the indexes that would be selected by running next step of ProtoDash and the ProtoGreedy respectively. Let $D(i + 1) = f(L \cup \{v\}) - f(L)$ and $G(i) = f(L \cup \{u\}) - f(L)$. Defining $y^{(v)} = \zeta^{(L)} + \alpha 1^{(v)}$ for some $\alpha \geq 0$ and recalling that $\zeta^{(L \cup \{v\})}$ is the maximizing point for $f(L \cup \{v\})$ we get

$$D(i + 1) \geq l \left( y^{(v)} \right) - l \left( \zeta^{(L)} \right) \geq \left\langle \nabla l \left( \zeta^{(L)} \right), \alpha 1^{(v)} \right\rangle - \frac{C_1}{2} \alpha^2$$

where the last inequality follows from recalling that ProtoDash chooses the coordinate $v$ that maximizes the gradient value $\nabla l \left( \zeta^{(L)} \right)$. As $u \notin L$, $\zeta^{(L)}_u = 0$ and hence $\zeta^{(L \cup \{u\})}_u - \zeta^{(L)}_u \geq 0$. We let $\alpha = \eta \left( \zeta^{(L \cup \{u\})}_u - \zeta^{(L)}_u \right)$ for some $\eta \geq 0$ and obtain $D(i + 1) \geq \eta \nabla l \left( \zeta^{(L)} \right) \left[ \zeta^{(L \cup \{u\})}_u - \zeta^{(L)}_u \right] - \frac{C_1}{2} \alpha^2$.

Consider a coordinate $j \in L$. The stationarity and complementary slackness KKT conditions enforce that if $\zeta^{(L)}_j > 0$ then $\nabla l_j \left( \zeta^{(L)} \right) = 0$ and if $\zeta^{(L)}_j = 0$ then $\nabla l_j \left( \zeta^{(L)} \right) \leq 0$. As $\zeta^{(L \cup \{u\})}_j \geq 0$ we derive the inequality

$$\sum_{j \in L} \nabla l_j \left( \zeta^{(L)} \right) \left[ \zeta^{(L \cup \{u\})}_j - \zeta^{(L)}_j \right] \leq 0. \quad (C.1)$$

We then conclude that

$$D(i + 1) \geq \eta \left\langle \nabla l \left( \zeta^{(L)} \right), \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\rangle - \eta^2 \frac{C_1}{2} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \quad (C.2)$$

where we have removed the restriction of the vectors to the coordinate $u$. Combining (C.2) with the definition of RSC constant $c_i + 1$ for $i + 1$ sparse vectors in the non-negative orthant, we can infer that

$$D(i + 1) \geq \eta \left[ G(i) + \frac{c_i + 1}{2} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \right] - \eta^2 \frac{C_1}{2} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2$$

which when maximized w.r.t $\eta$ leads to

$$D(i + 1) \geq \frac{c_i + 1}{2C_1} G(i) + \frac{G^2(i)}{2C_1 \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2} + \frac{c_i^2 + 1}{8C_1} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \quad (C.3)$$

Alluding again to the definition of $c_i + 1$ we have

$$-\frac{c_i + 1}{2} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \geq l \left( \zeta^{(L)} \right) - l \left( \zeta^{(L \cup \{u\})} \right) - \left\langle \nabla l \left( \zeta^{(L \cup \{u\})} \right), \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\rangle \quad (C.4)$$

Following the same line of argument that lead to the inequality in (C.1), we see that

$$\left\langle \nabla l \left( \zeta^{(L \cup \{u\})} \right), \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\rangle \leq 0.$$
Using it in (C.4) we find \(-\frac{c_{i+1}}{2} \left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \geq -G(i)\) and hence
\[
\left\| \zeta^{(L \cup \{u\})} - \zeta^{(L)} \right\|^2 \leq \frac{2G(i)}{c_{i+1}}.
\]

Using this inequality in (C.3) and dropping the last non-negative term of (C.3) gives us the bound
\[
D(i + 1) \geq \frac{3c_{i+1}}{4C_1} G(i). \quad (C.5)
\]

As the set function \(f\) is non-decreasing increasing supports and \(L_R \subseteq L^*\), we get \(f(L_R) \leq f(L^*)\) and \(|L_R| = |L^*| = m\). Further, \(\forall j \in L_R, G(i) \geq f(L \cup \{j\}) - f(L)\) as ProtoGreedy choose that next coordinate \(u\) that maximally increases the set function \(f\). Let \(B(i) = f(L^*) - f(L)\).

Then find
\[
m D(i + 1) \geq m \frac{3c_{i+1}}{4C_1} G(i) \geq \frac{3c_{i+1}}{4C_1} \sum_{j \in L_R} [f(L \cup \{j\}) - f(L)] = \frac{3c_{i+1}}{4C_1} \gamma_{L,L_R} [f(L \cup L_R) - f(L)] \geq \frac{3c_{i+1}}{4C_1} \gamma_{L^*,m} B(i).
\]

Letting \(\kappa = \frac{3c_{i+1}}{4C_1} \gamma_{L^*,m} m\) and noting that \(D(i + 1) = B(i) - B(i + 1)\) we get the recurrence relation
\[
B(i + 1) \leq (1 - \kappa) B(i)
\]
which when iterated \(i\) times starting from step 0 gives \(B(i) \leq (1 - \kappa)^i B(0)\). Plugging in \(B(k) = f(L^*) - f(L^D)\) and \(B(0) = f(L^*)\) gives us the required inequality
\[
f(L^D) \geq f(L^*) [1 - (1 - \kappa)^m] \geq f(L^*) \left[1 - e^{-\frac{3c_{i+1}}{4C_1} \gamma_{L^*,m}}\right].
\]

## D Proof of Corollary 4.6

Let \(L\) be the set chosen by the ProtoDash up to the current iteration. For every \(j \not\in L\), define the vector \(s_j\) of length \(|L|\) whose \(i^{th}\) element \(s_{j,i} = K_{j,i}\) for \(i \in L\). Let \(w^* = \zeta^{(L)}_L, w^j = \zeta^{(L \cup \{j\})}_L\), \(\mu_{p,L}\) and \(K_L\) be the restriction of the corresponding entities on the coordinates specified by \(L\) and similarly let \(w^j = \zeta^{(L \cup \{j\})}_j\). Recall that in the next iteration, ProtoDash chooses the prototype \(j^D\) such that \(j^D = \text{argmax}_{j} \nabla \zeta^{(L)}_j = \text{argmax} \mu_{p,j} - s^T_j w^*\). Pursuant to Lemma 4.4 we have, if \(\mu_{p,j} - s^T_j w^* \leq 0\), then \(\zeta^{(L \cup \{j\})}_j = \zeta^{(L)}_j\) and specifically, \(w^j = w^*\). Otherwise, the stationarity and complementary slackness KKT conditions entails that \(w^j = \frac{\mu_{p,j} - s^T_j w^*}{K_j}\). Using this value of \(w^j\), we see that the optimization problem that maximizes
\[
l_j(w) = -\frac{1}{2} w^T K_L w + \mu^T_{p,L} w + \frac{1}{2K_j} (\mu_{p,j} - s^T_j w^*)^2
\]
subject to \(w \geq 0\), and \(s^T_j w \leq \mu_{p,j}\)

attains its optimum at \(w = w^j\). Particularly, \(l_j(w^j) \geq l_j(w^*), \forall j\). The choice \(j^D\) by our ProtoDash method has the property that \(l_{j^D}(w^*) \geq l_j(w^*)\) assuming that the prototypes are normalized so that their self-norm \(K_j = 1, \forall j\), where as ProtoGreedy choose that index \(j^G\) where \(l_{j^G}(w^{j^G}) \geq l_{j^D}(w^{j^D})\).


E Time Complexity

For both ProtoGreedy and ProtoDash we need to compute the mean inner product of $X^{(2)}$ for instances in $X^{(1)}$, which takes $O(n^{(1)}n^{(2)})$ time. The time complexity to compute inner products between points in data set $X^{(1)}$ to build the kernel matrix $K$ is $O(mn^{(1)})$.

For ProtoGreedy the selection of the next best element requires running a quadratic program. Hence the time complexity for choosing the next best element is $O(m^4n^{(1)})$. The total time complexity of ProtoGreedy is $O(n^{(1)}(n^{(2)} + m^4))$.

For ProtoDash each iteration $i$ requires a search over $(n^{(1)} - i + 1)$ elements to determine the next best element. For each element searched, we need to compute a inner product between vectors of length $i - 1$ to compute the gradient value. Hence the complexity of choosing the next best element is $O(m^2n^{(1)})$. For each iteration $i$, we need to run a quadratic program to compute weights. This is $O(i^3)$ for each $i$. Hence, overall its $O(m^4)$. Consequently, the the total time complexity for ProtoDash is $O(n^{(1)}(n^{(2)} + m^2) + m^4)$.

F Experimental results with adapted L2C

Though the L2C algorithm described in [12] was originally designed to select prototypes that characterizes a given population where the prototypes are chosen from the same underlying dataset, it can be adapted to pick prototypes from one dataset, say $X^{(2)}$, that best represents a different population $X^{(1)} (X^{(2)} \neq X^{(1)})$, similar in spirit to our ProtoDash algorithm. In Figure 5 we provide experimental results with this adapted version of L2C, labeled L2C-A, and evaluate its performance against other algorithms. Results on retail and MNIST are presented since on CDC the performance of L2C and L2C-A are exactly the same. L2C and L2C-A also have the same time complexity so we do not create additional plots for it.

The results indicate that the performance of L2C-A is better than L2C on both retail and MNIST, which is not surprising. However, ProtoDash and ProtoGreedy are still the best performing algorithms. The RMSE of ProtoDash and ProtoGreedy is less than half that of L2C-A on the retail dataset as seen from Figure 5a. Moreover, their classification accuracy for different skewness of the target distribution and for different sparsity levels as shown in Figures 5b and 5c respectively are superior compared to L2C-A. When the target distribution is completely skewed where all the samples are from the same class (one of the digits $0, \ldots, 9$) and we measure the % of chosen prototypes belonging to the target class, we observe from Figure 5d that both ProtoDash and ProtoGreedy are 10% more likely to choose prototypes from the target class compared to L2C-A when we first select $r*m$ prototypes for $r = 2$ or $3$ while running ProtoDash and then pick the top $m$ with maximum weights. This also stresses the importance of having weights as we can oversample and then choose the desired sparsity level which leads to better results.
Figure 5: Experimental results with adapted L2C (L2C-A)

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