The stochastic resonance (SR) is a counter-intuitive phenomenon, where background noise can be instrumental in enhancing coherence and resonance of a non-linear system to a small periodic signal [1,2]. It occurs when noise-induced hopping events synchronize with the signal, which itself is not sufficient to drive the system to cross over the barrier. Since the pioneering work of Benzi et al. on recurrence of ice ages [3], the idea of SR has been extended to a vast range of phenomena encompassing signal and information processing, medicine, and biological systems [4–6]. Recently a number of workers reported the new mechanism of entropic stochastic resonance for the Brownian particle, which is confined in a narrow space and thus subject to free energy rather than purely energetic potential [7]. For nano-scaled soft-matter/biological systems the ubiquitous entropic effects due to confinement can play an important role in noise-induced resonant effects as recently have been shown in the studies of SR in ion channels [8,9].

As interconnected and flexible systems they are, biopolymers manifest interesting cooperative dynamics in certain confined environments, external fields and noises. Its cooperative dynamics is important, not only in understanding how such a system self-organizes its flexible degrees of freedom, but also in a multitude of single-molecule biophysics applications such DNA separation, and biopolymer sequencing, etc. Nature can utilize the ambient noises (fluctuations) of various kinds in such biological soft-condensed matter to facilitate the barrier crossing seemingly difficult to surmount, typically assisted by conformational changes.

The study of dynamics of a polymer surmounting a potential barrier along with the associated SR provides a basic paradigm in which to understand the self-organization and cooperativity induced by the chain flexibility and fluctuations. The potential force on the chain can be traded with an entropic force caused by confining geometry as noted by [10]. This makes it possible to study the dynamics under an external potential by probing the equivalent dynamics within a channel with the cross-section modulating over mesoscopic or macroscopic length scale that can be fabricated [11,12]. For the characteristic dimension of a double-well potential much longer than the chain contour length, the crossing (Kramers) rate for a flexible chain was found to be always higher than expected for the rigid globule in the limit of infinitely high elastic constant [13,14]. Also the rate is found to depend on the conformation, coiled or stretched, which the flexible chain takes at the energy barrier. These behaviors suggest the emergence of a bona fide entropic SR originating from chain’s intrinsic flexibility, which we study in this letter.

Based on the crossing rates of such chain given in [14], whose validity is confirmed by simulations [15], we recast the many-body dynamics of segments to that of the center of mass (CM) of the chain under an effective potential

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Abstract – We study the stochastic resonance (SR) of a flexible polymer surmounting a bistable-potential barrier. Due to the flexibility that can enhance crossing rate and change chain conformations at the barrier, the SR behaviors manifest many features of the bona fide entropic SR, such as the power amplification peaks at optimal chain lengths and elastic constants as well as the optimal noise strengths. The pronounced peaks that emerge depending on the chain lengths and conformation states suggest novel means of manipulating biopolymers, such as efficient separation methods using undulating channels.

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or free energy. To identify the most salient features of this SR, we analyze the power amplification factor via a linear response theory. The power amplification factors manifest peaks as the typical signature of SR at optimal noise strengths, but also, remarkably, at optimal chain lengths that depend on the chain flexibility and the conformations at the barrier or a narrow constriction in a channel. Therefore the SR suggests novel possibilities for understanding the biopolymer living processes and manipulating single biopolymers utilizing their flexibility and sorting them depending on their lengths [16]. Earlier studies on SR of linearly coupled chains have mostly focused on the other extreme, i.e., the chains much longer than the width of the barrier. Here the barrier crossing is facilitated by a mode of excitation called “kink-anti-kink pair”, which yields the crossing rate [17–19] and SR [20,21] much different from discussed above. Because the kink is a localized object in the long-chain limit, the activation energy is independent of the chain lengths and conformations, unlike in many practical cases, with which we are concerned. On the other hand, a study on SR by Dikshtein et al. [22] dealt with a polymer in a symmetric bistable potential with the two ends pinned the barrier top. The situation is simply different from ours where all the segmentes move.

We consider a linear, harmonic chain of N beads (monomers) undergoing a Brownian motion in three dimension. Assuming each bead has a friction coefficient \(\gamma\), the dynamics of the \(N\) beads \((n=1,2,3 \ldots N)\) under a one-dimensional external potential \(U(x)\) and thermal activation is governed by the Langevin equation,

\[
\gamma \frac{dx_n}{dt} = -k(2x_n - x_{n-1} - x_{n+1}) - \frac{\partial U(x_n)}{\partial x_n} + \xi_n(t),
\]

where the \(k\) is the spring (elastic) constant of the chain, \(\xi_n(t)\) is assumed to be Gaussian and white noise satisfying

\[
\langle \xi_n(t) \rangle = 0, \quad \langle \xi_n(t) \xi_n(t + \tau) \rangle = 2D\gamma \delta(\tau)
\]

with \(D = k_B T\) is the strength of thermal noise. The external potential energy each bead experiences is given by

\[
U(x) = -\frac{1}{2} \omega_B^2 x^2(1 - \frac{1}{2} x^2/m^2)
\]

as shown in fig. 1. The positions \(x = \pm x_m\) represent the two minima of the potential separated by a barrier of height \(U_B = x_m^2 \omega_B^2/4\) which is centered at \(x = 0\). The parameter \(\omega_B^2 = 1/\omega_0^2\) and \(\omega_0^2\) denote the potential curvatures of the barrier top and the well minima, respectively.

We ask a fundamental question: how does the linearly coupled flexible system like a polymer chain in the double well cooperatively respond to a small time-periodic force in the presence of the ambient thermal noise? The determinant parameters are the noise strength (temperature), the chain length, and the elastic constant, given the double-well potential parameters as fixed. By such coupling the SR is enhanced above that of a single bead as already has been studied in earlier work [20,21]. In the globular limit where the coupling (spring) constant is infinity, the polymer moves as a single Brownian particle (a rigid globule) in the bistable potential, whose SR behavior is well known. Now turning on the internal degrees of freedom that give rise to the chain conformational flexibility and variability, how will the SR be affected?

We base our theoretical development of SR on previous theoretical results [13,14] for the rate of the chain crossing over the potential barrier by thermal activation in the absence of driving force, which is found to be consistent with numerical simulations [15]. We consider the case of the potential where \(NU_B \gg D\), and \(L\) (chain contour length) \(\gg x_m\) so that the potential is designed to vary on a mesoscopic or macroscopic scale. In this case the multidimensional Kramers theory [13,14] yields

\[
R = R_0 e^{-\Delta H/2k_B T},
\]

Fig. 1: Schematic diagram for a flexible polymer chain in a symmetric double-well potential \(U(x) = -\frac{1}{2} \omega_B^2 x^2(1 - \frac{1}{2} x^2/m^2)\). The potential wells and the barrier top are located at \(x = \pm x_m\) and at \(x = 0\).
where

\[ R_0 = \frac{\omega_B^2 \omega_0}{2 \pi} e^{-\frac{N \omega_B^2}{2}}. \]  

(5)

Here \( R_0 \) is the rate in the globular limit, \( k \to \infty \). As the chain attains the flexibility by decreasing \( k \), the rate is enhanced by the factor \( e^{-\frac{N \omega_B^2}{2}} \) where \( \Delta F' \) is the free-energy change of the chain during the barrier crossing, which depends on temperature.

It is because the chain, a flexible extended object as it is, experiences an entropy increase in surpassing the potential, i.e., from the confining well to the unstable barrier top, thereby reducing the activation energy barrier [14]. \( \Delta F' \) depends on \( D \) as shown below. Depending upon the chain length \( N \) and the scaled spring constant \( k = k/\omega_B^2 \), the flexible chain takes either the coiled or stretched conformation at the barrier top (at the transition state). For the regime of parameters satisfying \( q > 0 \), where

\[ q = \frac{k \pi}{N \omega_B^2} - 1 \]  

(6)

the chain retains the coiled conformation, the free-energy change is expressed by

\[ \Delta F' = -D \ln \left[ \left( \frac{\omega_B^2}{\omega_0^2} \right)^{1/4} \left( \frac{\sinh(N \sqrt{\frac{\omega_0^2}{\omega_B^2}})}{\sin(N \sqrt{\frac{\omega_0^2}{\omega_B^2}})} \right)^{1/2} f(\alpha) \right] = -T \Delta S_c, \]  

(7)

where \( f(\alpha) = \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dz e^{-z^2/2} \) is the Gaussian integral with \( \alpha = 2q(\frac{\omega_B^2}{\omega_0^2})^{1/2} \). It should be noted that \( \Delta F' \) is given solely by the entropy change \( \Delta S_c \) mentioned above. On the other hand, when \( q < 0 \), the chain takes a stretched conformation at the barrier top, thereby reducing the energy barrier by the amount

\[ \Delta F' = -N \omega_B^2 x_m^2 q^2 / 6 - D \ln \left[ \sqrt{2} \left( \frac{\omega_B^2}{\omega_0^2} \right)^{1/4} \left( \frac{\sinh(N \sqrt{\frac{\omega_0^2}{\omega_B^2}})}{\sin(N \sqrt{\frac{\omega_0^2}{\omega_B^2}})} \right)^{1/2} g(\alpha) \right] \]  

(8)

for small values of \( |q| \). In eq. (8) \( g(\alpha) = \sqrt{\frac{\pi}{2}} e^{-\alpha^2/2} \int_{-\infty}^{\infty} dz e^{-(\alpha^2/2)z^2 - 3/8} \) with \( (\omega_B^2)^2 = (1 - 2|q|) \omega_0^2 \). Equation (8) includes the decrease in the internal energy due to stretching (the first term) as well as the entropy increase (the second term) in crossing the barrier in stretched state. The transition from coiled to stretched conformation at the barrier is incurred as the chain length increases above \( N_c \) or the spring constant decreases below a critical value \( k_c \) with the potential parameter fixed.

Driven by a force \( f(t) = f_0 \cos(\Omega t) \) on each bead, where \( f_0 \) and \( \Omega \) are the amplitude and angular frequency, the center-of-mass (CM) position \( X \) of the chain, the reaction coordinate for the barrier crossing, evolves following the Langevin equation under an effective potential or free-energy function \( F(X) \)

\[ N \gamma \frac{dX}{dt} = -\frac{\partial F(X)}{\partial X} + N f_0 \cos(\Omega t) + \xi(t), \]  

(9)

where \( \xi(t) \) represents the random force characterized by \( \langle \xi(t) \rangle = 0, \langle \xi(t) \xi(t + \tau) \rangle = 2D N \gamma \delta(\tau) \).

The \( F(X) \) is the free energy associated with the CM position at \( X \). Being given as the total potential averaged over all the chain configurations with this given CM position, it is different from \( NU(X) \) but retains the bistability along with same minimum and maximum positions due to the potential symmetry around them. It is reasonable to assume the free energy \( F(X) \) associated with the CM position \( X \) takes a form similar to eq. (3),

\[ F(X) = -\frac{\Omega_B^2}{2} X^2 \left( 1 + \frac{X^2}{2x_m^2} \right), \]  

(10)

with the barrier curvature \( \Omega_B^2 \) and height \( \Delta F = \frac{\Omega_B^2}{2} x_m^2 \). The Kramers rate for the CM regarded effectively as a single Brownian particle is

\[ R = \frac{\Omega_B^2 \Omega_0}{2 \pi N \gamma} e^{-\frac{\alpha^2}{2}}, \]  

(11)

which is equated with eq. (4) to yield

\[ \Delta F \approx NU_B + \Delta F', \]  

(12)

and thus

\[ \Omega_B = \sqrt{N \omega_B} \left( 1 + \frac{4(\Delta F')^2}{NU_B x_m^2} \right)^{1/2}, \]  

(13)

both of which depend on \( N \) as well as on the \( D \) via \( \Delta F' \). The free-energy function, rather than purely energetic potential, under which the particle representative of the chain is moving, signals an occurrence of an entropic stochastic resonance [7].

It is well known that the double-well potential \( U \), to which each segment is subject, can be realized experimentally using optical tweezers [23]. As another experimental means of emulating the situation, the one-dimensional free energy \( F(x) \) in particular, we consider the dynamics of the chain flowing within the channel of undulating cross-section \( A(X) \). Since we envision the CM as a single-particle—moving subject to \( F(X) \) as shown in eq. (9), such consideration is possible, if \( F(X) = -D \ln A(X)/A_0 \), or \( A(X) = A_0 \exp(-F(X)/D) \) where \( A_0 \) is the cross-section at the point where \( F(X) = 0 \), and if the hydrodynamic interactions between segments and from the channel wall are neglected. Then the crossing dynamics and the associated SR can be achieved for a chain within such a fluidic channel subject to an external periodic forcing, for example a single-stranded DNA within a channel subject to an AC field.

To gain understanding of the major salient features of the distinctive nature of the polymer SR, it suffices to
consider the linear response of the chain CM to the small driving forces. In response to the weak driving force, the average CM position is given by

\[ \bar{X}(t) = N f_0 \bar{\chi}(\Omega) \cos(\Omega t - \phi), \]

where \( \bar{\chi}(\Omega) \) is the Fourier transform of the response function \cite{24} given by \( \bar{\chi}(\Omega) = \bar{\chi}'(\Omega) + i \bar{\chi}''(\Omega) = \frac{\langle X^2 \rangle}{D} \frac{2R}{2\pi \Omega + i \Omega^2} \) and \( \phi \) is the phase delay given by \( \tan^{-1} \frac{\bar{\chi}''(\Omega)}{\bar{\chi}'(\Omega)} \). While the coherence of the system is measured by the real part of \( \bar{\chi}(\Omega) \), the SR intensity is quantified by the power amplification factor (the ratio of the power stored in the response of the system to the power of the driving force with frequency):

\[ \eta = |\bar{\chi}(\Omega)|^2 = \left( \frac{\langle X^2 \rangle}{D} \right)^2 \frac{4R^2(D)}{4R^2(D) + \Omega^2}, \]

where \( \langle X^2 \rangle = \int X^2 e^{-\frac{E(X)}{D}} dX/ \int e^{-\frac{E(X)}{D}} dX \).

The \( \eta \) indeed shows non-monotonic noise strength dependence manifesting a peak at an optimal noise strength, which is the typical signature of SR. Via dependence of the rate \( R \) and \( \langle X^2 \rangle \) on the conformational variability, the SR for the flexible polymer manifests distinctive dependence on the noise strength, and additional dependence on the chain length characteristic of a polymer entropic SR. Below we quantitatively discuss the power amplification factor \( \eta \) for the polymer conformation both below and above coil-to-stretch transition. We introduce dimensionless parameters: \( \tilde{D} = D/\omega_B^2 \langle X \rangle_m^2 \), \( \tilde{\Omega} = \gamma \Omega/\omega_B^2 \) and \( \tilde{k} = k/\omega_B^2 \). From now on all the quantities are rescaled (dimensionless) so the bars will be dropped.

Let us first plot \( \eta \) as a function of rescaled noise strength \( D \) for the rescaled spring constant \( k = 200 \) which can be reasonable for a real flexible polymer if \( \omega_B^2 \) is chosen properly. Figures 2(a) and (b) depict the rescaled \( \eta \) as a function of noise strength \( D \) for \( \Omega = 10^{-5} \) and \( \Omega = 1 \), respectively, for various transition states of polymer conformation; the coiled state with \( N = 30 \) and the stretched state with \( N = 50 \), both of which are compared to globular state (the case of infinite \( k \)) with \( N = 30 \). Hereafter in the figures, the coiled, stretched and globular states are depicted as dashed, solid and dotted lines. As shown in figs. 2(a) and (b), for all cases, the \( \eta \) reach their maximum value \( \eta_R \) at certain optimal values of \( D \), denoted by \( D_R \). When the chain is in coiled state, \( \eta_R \) is much larger than that of globular state. This signifies that its intrinsic flexibility facilitating faster crossing allows the polymer chain to respond in a more cooperative and coherent manner to the external signal.

For very small \( \Omega \), as depicted in fig. 2(a) the optimal noise strength corresponding to the peaks satisfy \( D_{RC} < D_{RG} < D_{RS} \), where the subscripts \( C, G \) and \( S \) denote the three conformational states. To understand this, we note that the time scale matching condition between the mean crossing time and half of the period of oscillation, \( \frac{1}{2\langle D \rangle} = \frac{\pi}{\Omega} \), which provides a reasonable approximation for the resonant condition. With \( R(D_R) \approx 3 \times 10^{-6} \) given thus, and from the \( R \) obtained from eqs. (7) and (8) (see fig. 3(a)), the above inequality indeed holds. Figure 2(a) shows that \( \eta_{RC} > \eta_{RG} > \eta_{RS} \) which can be understood as follows. At very low frequency, eq. (15) is given by \( \eta \approx |\bar{\chi}(0)|^2 = \left( \frac{\langle X^2 \rangle}{D} \right)^2 \) which implies that the resonance mechanism is governed by chain static susceptibility \( |\bar{\chi}(0)| = \frac{\langle X^2 \rangle}{D} \) of the CM in response to a constant force. Finding the static susceptibility for the \( D_R \) of each transition state (\( D_{RC} < D_{RG} < D_{RS} \)), one indeed affirms the inequality for the \( \eta_R \).

The situation is different for large driving frequency where the resonance conditions are met for larger \( D \) (see fig. 2(b)). While the inequality for \( D_R \) is unchanged, that of \( \eta_R \) is not; \( \eta_{RG} \) for stretched state is highest. This is understandable if we note for large \( \Omega \) the \( \eta \) is largely governed by the rate, rather than the static susceptibility. As the chain length becomes longer than the critical value for this large \( D \), this flexible chain can extend to lower the free-energy barrier and enhance the crossing rate, as shown in fig. 3(b). This results in the stretched conformation enhancing SR (solid line) than the coiled or globular
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Fig. 3: (a) Polymer crossing rate $R \times 10^6$ vs. $D$ for small values of $D$. The dotted, dashed and solid lines stand for globular ($N = 30$ and $N = 50$), coiled ($N = 30$) and stretched ($N = 50$) states at the top of the barrier. (b) Polymer crossing rate $R$ vs. $D$ for large values of $D$.

In both figures we fix $\Omega = 0.01$. States at these noise strengths as eq. (15) indicates. This means that the chain flexibility allows the conformational transition to enhance the cooperativity and coherence to external influences.

Another aspect of this entropic SR is shown in figs. 4(a) and (b) which depict $\eta$ a function of chain length with $k = 200$ and $\Omega = 10^{-2}$ for three different noise strengths, $D = 1, 2,$ and $4$. For $D = 1$, the chain manifest the response of coiled conformation for all values of $N$ which are mostly smaller than $N_c = 44.4$. For $D = 2$, the chain responses in coiled conformation for $N < N_c$ but responses in stretched conformation for $N > N_c$. For $D = 4$ the chain manifest smaller $\eta$ for all chain of length $N > N_c$ which are all in stretched conformation at the barrier top. Each of these lines have a peak at a certain optimal $N$ which increases with $D$. Even with the temperature fixed, as is usual in biological systems, the polymer by stretching can escape the barrier or the narrowest constriction in a most coherent and resonant way to an external signal.

Let us now fix the polymer length $N = 50$ and vary the rescaled spring constant $k$ where $k_c = 253.3$ is the critical spring constant that demarcates the polymer conformations at the barrier. Figure 5(a) exhibits $\eta$ vs. $D$, where the curves depict $k = 0.8k_c$, $k = 1.5k_c$, and $k = \infty$, respectively representative of stretched, coiled, and...
globular conformation at the barrier. As the spring constant (stiffness) increases, the chain shows less resonant peaks at larger optimal noise strength. This means that larger thermal energy is required to drive the stiffer chain to move coherently with an external signal. On the other hand, fig. 5(b) exhibits the plot of $\eta$ as a function of $k$ for $N = 50$. As $k$ becomes very small, $\eta$ tends to be very small as the whole monomers becomes non-interacting (non-cooperative). When $k \to \infty$, $\eta$ becomes small again as globular polymer lacks the sufficient flexibility to conform to external driving. At a certain optimal value, in between $\eta$ attains an optimum value which is an increasing function of $D$.

In summary, we have studied the stochastic resonance of a single flexible polymer moving in a bistable potential, or flowing within an equivalent fluidic channel, both of which modulate over a mesoscopic or even a macroscopic scale. At an optimal noise strength, the dynamics of the chain, otherwise random, shows SR, i.e., it moves in coherence and resonance with a periodic driving force. Even with the noise strength (temperature) fixed, the polymer displays an entropic SR by responding cooperatively to external driving in a most efficient way at optimal chain lengths and elastic constants, owing to chain flexibility and concomitant conformational transition. Utilizing their self-organizing behaviors, we may learn biomolecular machineries of living and clever ways of manipulating them such as efficient separation methods.

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