EIGENVALUE ESTIMATES FOR DIRAC OPERATOR WITH THE GENERALIZED APS BOUNDARY CONDITION

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Abstract. Under two boundary conditions: the generalized Atiyah-Patodi-Singer boundary condition and the modified generalized Atiyah-Patodi-Singer boundary condition, we get the lower bounds for the eigenvalues of the fundamental Dirac operator on compact spin manifolds with nonempty boundary.

1. INTRODUCTION

Let $M$ be a compact Riemannian spin manifold without boundary. In 1963, Lichnerowicz [21] proved that any eigenvalue of the Dirac operator satisfies

$$\lambda^2 > \frac{1}{4} \inf_M R,$$

where $R$ is the positive scalar curvature of $M$. The first sharp estimate for the smallest eigenvalue $\lambda$ of the Dirac operator $D$ was obtained by Friedrich [8] in 1980. The idea of the proof is based on using a suitably modified Riemannian spin connection. He proved the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R,$$

on closed manifolds $(M^n, g)$ with the positive scalar curvature $R$. Equality gives an Einstein metric. In 1986, combining the technique of the modified spin connection with a conformal change of the metric, Hijazi [11] showed, for $n \geq 3$

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu_1,$$

where $\mu_1$ is the first eigenvalue of the conformal Laplacian given by $L = \frac{4(n-1)}{n-2} \Delta + R$. When the equality holds, the manifold becomes an Einstein manifold.

Let $M$ be a compact Riemannian spin manifold with nonempty boundary. In this case, the boundary conditions for spinors become crucial to make the Dirac operator elliptic, and, in general, two types of the boundary conditions are considered, i.e. the local boundary condition and the Atiyah-Patodi-Singer (APS) boundary condition (see, e.g., [6]). In 2001, Hijazi, Montiel and Zhang [16] proved the generalized version of Friedrich-type inequalities under both the local boundary condition and the Atiyah-Patodi-Singer boundary condition. In 2002, Hijazi, Montiel and Roldán [15] investigated the Friedrich-type inequality for eigenvalues of the fundamental Dirac operator under four elliptic boundary conditions and under some curvature assumptions (the non-negative
mean curvature). In particular, they introduced a new global boundary condition, namely
the modified Atiyah-Patodi-Singer (mAPS) boundary condition.

In the present paper, we study the spectrum of the fundamental Dirac operator on compact
Riemannian spin manifolds with nonempty boundary, under the two new boundary
conditions: the generalized Atiyah-Patodi-Singer (gAPS) boundary condition and the
modified generalized Atiyah-Patodi-Singer (mgAPS) boundary condition. Following the
terminology of [6] those are obtained composing a so-called spectral projection with the
identity in the first case, and with the zero order differential operator \( \text{Id} + \gamma(e_0) \) in the
second case (where \( \gamma(e_0) \) is the Clifford product with the unit normal \( e_0 \) on the boundary
\( \partial M \)).

2. Preliminaries

Let \((M, g)\) be an \((n+1)\)-dimensional Riemannian spin manifold with nonempty-boundary \( \partial M \). We denote by \( \nabla \) the Levi-Civit\`a connection on the tangent bundle \( TM \). For the
given structure \( \text{Spin}M \) (and so a corresponding orientation) on manifold \( M \), we denote
by \( S \) the associated spinor bundle, which is a complex vector bundle of rank \( 2^{\lfloor \frac{n+1}{2} \rfloor} \).

Then let
\[
\gamma : \mathbb{C}l(M) \to \text{End}_\mathbb{C}(S)
\]
be the Clifford multiplication, which is a fibre preserving algebra morphism. It is well
known [22] that there exists a natural Hermitian metric \( \langle , \rangle \) on the spinor bundle \( S \) which
satisfies the relation
\[
(\gamma(X)\varphi, \gamma(X)\psi) = |X|^2(\varphi, \psi),
\]
for any vector field \( X \in \Gamma(TM) \), and for any spinor fields \( \varphi, \psi \in \Gamma(S) \). We denote also by
\( \nabla \) the spinorial Levi-Civit\`a connection acting on the spinor bundle \( S \). Then the connection
\( \nabla \) is compatible with the Hermitian metric \( \langle , \rangle \) and Clifford multiplication "\( \gamma \)". Recalling
the fundamental Dirac operator \( D \) is the first order elliptic differential operator acting on
the spinor bundle \( S \), which is locally given by
\[
D = \sum_{i=0}^{n} \gamma(e_i) \nabla_i,
\]
where \( \{e_0, e_1, \ldots, e_n\} \) is a local orthonormal frame of \( TM \).

Since the normal bundle to the boundary hypersurface is trivial, the Riemannian mani-
fold \( \partial M \) is also spin and so we have a corresponding spinor bundle \( S\partial M \), a spinorial
Levi-Civit\`a connection \( \nabla^{\partial M} \) and an intrinsic Dirac operator \( D^{\partial M} = \gamma^{\partial M}(e_i) \nabla_i^{\partial M} \).

Then a simple calculation yields the spinorial Gauss formula
\[
\nabla_i \varphi = \nabla_i^{\partial M} \varphi + \frac{1}{2} h_{ij} \gamma(e_j) \gamma(e_0) \varphi,
\]
for \( 1 \leq i, j \leq n, \varphi \in \Gamma(S|_{\partial M}) \), \( e_0 \) is a unit normal vector field compatible with the induced
orientation, \( h_{ij} \) is the second fundamental form of the boundary hypersurface.

It is easy to check (see [4] [11] [15] [17]) that the restriction of the spinor bundle \( S \) of \( M \)
to its boundary is related to the intrinsic Hermitian spinor bundle \( S\partial M \) by
\[
S := S|_{\partial M} \cong \begin{cases} 
S\partial M & \text{if } n \text{ is even} \\
S\partial M \oplus S\partial M & \text{if } n \text{ is odd}.
\end{cases}
\]
For any spinor field $\psi \in \Gamma(S)$ on the boundary $\partial M$, define on the restricted spinor bundle $S$, the Clifford multiplication $\gamma^S$ and the spinorial connection $\nabla^S$ by
\[
\gamma^S(e_i)\psi := \gamma(e_i)\gamma(e_0)\psi,
\]
\[
\nabla^S_i\psi := \nabla_i\psi - \frac{1}{2}h_{ij}\gamma^S(e_j)\psi = \nabla_i\psi - \frac{1}{2}h_{ij}\gamma(e_j)\gamma(e_0)\psi.
\]
One can easily find that $\nabla^S$ is compatible with the Clifford multiplication $\gamma^S$, the induced Hermitian inner product $(, )$ from $M$ and together with the following additional identity
\[
\nabla^S_i(\gamma(e_0)\psi) = \gamma(e_0)\nabla^S_i\psi.
\]
As a consequence, the boundary Dirac operator $D$ associated with the connection $\nabla^S$ and the Clifford multiplication $\gamma^S$, is locally given by
\[
D\psi = \gamma^S(e_i)\nabla^S_i\psi = \gamma(e_i)\gamma(e_0)\nabla_i\psi + \frac{H}{2}\psi,
\]
where $H = \sum h_{ii}$ is the mean curvature of $\partial M$. In fact, $\gamma(e_0)(D - \frac{1}{2}H) = \gamma(e_i)\nabla_i$ is the hypersurface Dirac operator defined by Witten [24] to prove the positive energy conjecture in general relativity.

Now by (2.5), we have the supersymmetry property
\[
D\gamma(e_0) = -\gamma(e_0)D.
\]
Hence, when $\partial M$ is compact, the spectrum of $D$ is symmetric with respect to zero and coincides with the spectrum of $D^{\partial M}$ for $n$ even and with $(\text{Spec} D^{\partial M}) \cup (-\text{Spec} D^{\partial M})$ for $n$ odd.

From [14] or [15], one gets the integral form of the Schrödinger-Lichnerowicz formula for a compact Riemannian spin manifold with compact boundary
\[
\int_{\partial M} \langle \varphi, D\varphi \rangle - \frac{1}{2} \int_{\partial M} H|\varphi|^2 = \int_M |\nabla\varphi|^2 + \frac{R}{4}|\varphi|^2 - |D\varphi|^2.
\]
By (2.7), $D$ is self-adjoint with respect to the induced Hermitian metric $(, )$ on $S$. Therefore, $D$ has a discrete spectrum contained in $\mathbb{R}$ numbered like
\[
\cdots \leq \lambda_{-j} \leq \cdots \leq \lambda_{-1} < 0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots
\]
and one can find an orthonormal basis $\{\varphi_j\}_{j \in \mathbb{Z}}$ of $L^2(\partial M; S)$ consisting of eigenfunctions of $D$ (i.e. $D\varphi_j = \lambda_j\varphi_j, j \in \mathbb{Z}$) (see e.g. Lemma 1.6.3 in [9]). Such a system $\{\lambda_j; \varphi_j\}_{j \in \mathbb{Z}}$ is called a spectral decomposition of $L^2(\partial M; S)$ generated by $D$, or, in short, a spectral resolution of $D$. According to the definition 14.1 in [9], we have

**Definition 2.1.** For self-adjoint Dirac operator $D$ and for any real $b$ we shall denote by
\[
P_{\geq b} : L^2(\partial M; S) \rightarrow L^2(\partial M; S)
\]
the spectral projection, that is, the orthogonal projection of $L^2(\partial M; S)$ onto the subspace spanned by $\{\varphi_j | \lambda_j \geq b\}$, where $\{\lambda_j; \varphi_j\}_{j \in \mathbb{Z}}$ is a spectral resolution of $D$.

**Definition 2.2.** For $\varphi \in \Gamma(S)$ and $b \leq 0$, the projective boundary conditions
\[
P_{\geq b}\varphi = 0,
\]
\[
P^m_{\geq b}\varphi = P_{\geq b}(Id + \gamma(e_0))\varphi = 0
\]
are called the generalized Atiyah-Patodi-Singer (gAPS) boundary condition and the modified generalized Atiyah-Patodi-Singer (mgAPS) boundary condition respectively.

**Remark 2.1.** We shall adopt the notation $P_{<b} = Id - P_{\geq b} ; P_{[b,-b]} = P_{\geq b} - P_{>-b}$ for $b < 0$ or $P_{[b,]-b]} = P_{\geq b} - P_{>-b}$ for $b > 0$. It is well known that the spectral projection $P_{\geq b}$ is a pseudo-differential operator of order zero (see e.g. the proposition 14.2 in [4]).

**Remark 2.2.** If $b = 0$, the gAPS boundary condition is exactly the APS boundary condition (see Atiyah, Patodi & Singer[1],[2],[3]).

In case of a closed manifold, the fundamental Dirac operator is a formally self-adjoint operator and so its spectrum is discrete and real. While in case of a manifold with nonempty boundary, we have the following formula

$$\int_M (D\varphi, \psi) - \int_M (\varphi, D\psi) = - \int_{\partial M} (\gamma(e_0)\varphi, \psi).$$

(2.9)

for $\varphi, \psi \in \Gamma(\partial M)$, and where $e_0$ is the inner unit field along the boundary. In the following, we’ll show ellipticity and self-adjointness for the Dirac operator under both the gAPS boundary condition and the mgAPS boundary condition.

On one hand, if $P_{\geq b}\varphi = 0$, $P_{\geq b}\psi = 0$, for $\varphi, \psi \in \Gamma(S)$ and $b \leq 0$, then $P_{\geq b}(\gamma(e_0)\varphi) = \gamma(e_0)P_{\geq b}$. Therefore

$$\int_{\partial M} (\gamma(e_0)\varphi, \psi) = 0.$$

(2.10)

On the other hand, for $\varphi, \psi \in \Gamma(S)$ and $b \leq 0$, under the mgAPS boundary conditions, i.e.

$$P_{\geq b}(Id + \gamma(e_0))\varphi = 0 \quad \text{and} \quad P_{\geq b}(Id + \gamma(e_0))\psi = 0,$$

a simple calculation implies that (2.10) also holds.

These facts imply that the fundamental Dirac operator $D$ is self-adjoint under the -gAPS boundary condition and the gmAPS boundary condition and hence it has real and discrete eigenvalues.

The ellipticity of both gAPS and mgAPS boundary conditions the fundamental Dirac operator $D$ follows straightforward from [6, Prop. 14.2], [16, Prop. 1] and [16, Thm. 5].

**Lemma 2.1.** Under the gAPS boundary condition $P_{\geq b}\varphi = 0$, the inequality

$$\int_{\partial M} (\varphi, D\varphi) \leq b \int_{\partial M} |\varphi|^2$$

holds for $b \leq 0$.

**Proof.** Let $\{\lambda_j; \varphi_j\}_{j \in \mathbb{Z}}$ be a spectral resolution of the hypersurface Dirac operator $D$. Then any $\varphi \in \Gamma(S)$ can be expressed as follows,

$$\varphi = \sum_j c_j \varphi_j,$$

where $c_j = \int_{\partial M} (\varphi, \varphi_j)$. Then we have

$$\int_{\partial M} (\varphi, D\varphi) = \sum_{\lambda_j < b} \lambda_j |c_j|^2 \leq b \sum_{\lambda_j < b} |c_j|^2 = b \int_{\partial M} |\varphi|^2.$$

\[\square\]
3. LOWER BOUNDS FOR THE EIGENVALUES

In this section, we adapt the arguments used in [18] and [16] to the case of compact Riemannian spin manifolds with nonempty boundary. We use the integral identity (2.8) together with an appropriate modification of the Levi-Civita connection to obtain the eigenvalue estimates.

**Theorem 3.1.** Let $M^n$ be a compact Riemannian spin manifold of dimension $n \geq 2$, with the nonempty boundary $\partial M$, and let $\lambda$ be any eigenvalue of $D$ under the gAPS boundary condition $P_{\geq b} \varphi = 0$, for $\varphi \in \Gamma(S)$ and $b \leq 0$. If there exist real functions $a, u$ on $M$ such that

$$b + a \, du(e_0) \leq \frac{1}{2} H$$
onumber

on $\partial M$, then

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup a,u \inf_M R_{a,u},$$

where

$$R_{a,u} := R - 4a \Delta u + 4\nabla a \cdot \nabla u - 4(1 - \frac{1}{n})a^2 |du|^2$$

$\Delta$ is the positive scalar Laplacian, $R$ is scalar curvature of $M$.

**Proof.** For any real function $a$ and $u$, define the modified spinorial connection (see [16] [18]) on $\Gamma(S)$ by

$$\nabla_{a,u}^i = \nabla_i + a \nabla_i u + \frac{a}{n} \nabla_j u \gamma(e_i) \gamma(e_j) + \frac{\lambda}{n} \gamma(e_i).$$

A simple calculation yields

$$\int_M |\nabla_{a,u}^i \varphi|^2 = \int_M \left[ (1 - \frac{1}{n})\lambda^2 - \frac{R_{a,u}}{4} \right] |\varphi|^2 + \int_{\partial M} (\varphi, D\varphi) + \int_{\partial M} \left[ a \, du(e_0) - \frac{H}{2} \right] |\varphi|^2.\tag{3.4}$$

Considering Lemma 2.1 and $b + a \, du(e_0) \leq \frac{1}{2} H$, we infer

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup a,u \inf_M R_{a,u}.\tag{3.5}$$

If the equality in (3.5) holds, then by the Lemma 3 in [16], we have $a = 0$ or $u = constant$ and $b \leq \frac{1}{2} H$. Since $\varphi$ is a non-degenerated killing spinor, $|\varphi|^2$ is a nonzero constant. Let $\{\lambda_j; \varphi_j\}_{j \in \mathbb{Z}}$ be a spectral resolution of the hypersurface Dirac operator $D$. Under the gAPS boundary condition $P_{\geq b} \varphi = 0$, one gets $\varphi = \sum c_j \varphi_j$, where $c_j = \int_{\partial M} (\varphi, \varphi_j)$. Then we have

$$0 = \int_{\partial M} (\varphi, D\varphi) - \int_{\partial M} \frac{1}{2} H |\varphi|^2$$

$$= \sum_{\lambda_j < b} \lambda_j |c_j|^2 - \frac{1}{2} \sum_{\lambda_j, \lambda_k < b} c_j c_k \int_{\partial M} H(\varphi_j, \varphi_k)$$

$$\leq \sum_{\lambda_j < b} (\lambda_j - b) |c_j|^2 < 0.$$

This is a contradiction. Thus the inequality (3.1) holds. \qed
Remark 3.1. If \( b = 0 \), the gAPS boundary condition becomes the APS boundary condition and then the theorem is exactly the Theorem 3.1 in [16].

Now we make use of the energy-momentum tensor, introduced in [12] and used in [17][18][19], to get lower bounds for the eigenvalues of \( D \). For any spinor field \( \varphi \in \Gamma(S) \), we define the associated energy-momentum 2-tensor \( Q_\varphi \) on the complement of its zero set by

\[
Q_{\varphi,ij} = \frac{1}{2} \Re(\gamma(e_i) \nabla_j \varphi + \gamma(e_j) \nabla_i \varphi, \varphi/|\varphi|^2).
\]

Obviously, \( Q_{\varphi,ij} \) is a symmetric tensor. If \( \varphi \) is the eigenspinor of the Dirac operator \( D \), the tensor \( Q_\varphi \) is well-defined in the sense of distribution.

Theorem 3.2. Let \( M^n \) be a compact Riemannian spin manifold of dimension \( n \geq 2 \), whose boundary \( \partial M \) is nonempty, and let \( \lambda \) be any eigenvalue of \( D \) under the gAPS boundary condition \( P_{\geq b} \varphi = 0 \), for \( b \leq 0, \varphi \in \Gamma(S) \). If there exist real functions \( a, u \) on \( M \) such that

\[
b + adu(e_0) \leq H/2
\]

on \( \partial M \), where \( H \) is the mean curvature of \( \partial M \), then

\[
\lambda^2 \geq \sup_{a, u} \inf_M \left( \frac{1}{4} R_{a, u} + |Q_\varphi|^2 \right),
\]

where \( R_{a, u} \) is given in (3.2).

Proof. For any real function \( a \) and \( u \), define the modified spinorial connection (see [16][18]) on \( \Gamma(S) \) by

\[
\nabla^{Q,a,u}_i = \nabla_i + a \nabla_i u + \frac{a}{n} \nabla_j u \gamma(e_i) \gamma(e_j) + Q_{\varphi,ij} \gamma(e_j).
\]

One can easily compute

\[
\int_M |\nabla^{Q,a,u}_i \varphi|^2 = \int_M \left[ \lambda^2 - \left( \frac{R_{a, u}}{4} + |Q_\varphi|^2 \right) \right] |\varphi|^2 + \int_{\partial M} (\varphi, D \varphi) + \int_{\partial M} \left[ adu(e_0) - H/2 \right] |\varphi|^2.
\]

Considering the boundary conditions, we immediately arrive at the asserted formula. \( \square \)

Remark 3.2. By Cauchy-Schwarz inequality and \( tr Q_\varphi = \lambda \), we have

\[
|Q_\varphi|^2 = \sum_{i,j} |Q_{\varphi,ij}|^2 \geq \sum_i |Q_{\varphi,ii}|^2 \geq \frac{1}{n} (tr Q_\varphi)^2 = \frac{1}{n} \lambda^2.
\]

Therefore, one gets the inequality (3.5).

Remark 3.3. If \( b = 0 \), the above theorem becomes Theorem 5 in [16]. Under the gAPS boundary condition, taking \( a = 0 \) or \( u = constant \) in (3.6) and assuming \( H \geq 0 \), then one gets the following inequality [12]

\[
\lambda^2 \geq \inf_M \left( \frac{R}{4} + |Q_\varphi|^2 \right).
\]

In the following, we state the eigenvalue estimates for the Dirac operator under the mgAPS boundary condition.
Theorem 3.3. Let $M^n$ be a compact Riemannian spin manifold of dimension $n \geq 2$, whose boundary $\partial M$ is nonempty, and let $\lambda$ be any eigenvalue of $D$ under the mgAPS boundary condition $P^{m}_{b} = P_{b}(Id + \gamma(e_0)) = 0$, for $b \leq 0$. If there exist real functions $a, u$ on $M$ such that

$$a \ du(e_0) \leq \frac{1}{2}H$$

on $\partial M$, then

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a, u} \inf_{M} R_{a,u}$$

where $R_{a,u}$ is given in (3.2). Moreover, the equality holds if and only if $M$ carries a nontrivial real killing spinor field with a killing constant $-\frac{\lambda}{n} < \frac{b}{n-1}$ and the boundary $\partial M$ is minimal.

Proof. Assuming

$$P^{m}_{b} \varphi = P_{b}(\varphi + \gamma(e_0) \varphi) = 0 \quad \text{and} \quad P^{m}_{b} \psi = P_{b}(\psi + \gamma(e_0) \psi) = 0,$$

for $\varphi, \psi \in \Gamma(S)$ and $b \leq 0$, then we have

$$P_{>b} \psi + \gamma(e_0) P_{<b} \psi = 0 \quad \text{and} \quad P_{[b, -b]} \psi + \gamma(e_0) P_{[b, -b]} \psi = 0,$$

i.e.

$$P^{m}_{b} \psi = P_{>b} \psi + \gamma(e_0) P_{<b} \psi = 0 \quad \text{and} \quad P_{[b, -b]} \psi = 0$$

(3.10)

These imply

$$P^{m}_{b} \varphi = P_{>b} (\varphi + \gamma(e_0) \varphi) = 0,$$

$$P^{m}_{b} \psi = P_{>b} (\psi + \gamma(e_0) \psi) = 0.$$

Then one gets

$$\gamma(e_0) \psi - \psi = P_{>b} (\gamma(e_0) \psi - \psi).$$

(3.11)

The relation (2.7) implies that

$$(D \psi, \psi) = \frac{1}{2} (D(\psi + \gamma(e_0)\psi), \psi - \gamma(e_0)\psi).$$

The combination $P^{m}_{>b} \psi = P^{m}_{>b} \psi = P_{>b} (\psi + \gamma(e_0) \psi) = 0$ with (3.11) yields

$$\int_{\partial M} (D(\psi + \gamma(e_0)\psi), \psi - \gamma(e_0)\psi) = 0.$$

This implies

$$\int_{\partial M} (D \psi, \psi) = 0.$$  

(3.12)

From (3.3), (3.4) and (3.12), one can infer

$$\int_{M} \left| \nabla^{a,u} \varphi \right|^2 = \int_{M} \left[ (1 - \frac{1}{n}) \lambda^2 - \frac{R_{a,u}}{4} \right] |\varphi|^2 + \int_{\partial M} \left[ a \ du(e_0) - \frac{H}{2} \right] |\varphi|^2.$$  

(3.13)

Then the inequality (3.9) holds.

If the equality occurs, we deduce

$$\nabla^{a,u} \varphi = 0 \quad \text{and} \quad \frac{1}{2} H = adu(e_0).$$
By the Lemma 3 in [16], we get $a = 0$ or $u = constant$. Therefore

$$H = 0 \quad \text{and} \quad \nabla_i \varphi = -\frac{\lambda}{n} \gamma(e_i) \varphi.$$  

From the supersymmetry property (2.7), we get

$$\mathcal{D}(\varphi + \gamma(e_0) \varphi) = -\frac{n-1}{n} \lambda(\varphi + \gamma(e_0) \varphi).$$

Since $P_m^b \varphi = 0$, we deduce that $-\frac{n-1}{n} \lambda < b$.

Conversely, if $M$ is a compact Riemannian spin manifold with minimal boundary $\partial M$ and a nontrivial killing spinor $\varphi$ with a real killing constant $-\lambda/n < b_{n-1}$, then we have $D \varphi = \lambda \varphi$. Moreover, from the fact that $\nabla_{e_0} \varphi = -\frac{2}{n} \gamma(e_0) \varphi$, we infer that the restriction of $\varphi$ to the boundary satisfies

$$\mathcal{D} \varphi = -\frac{n-1}{n} \gamma(e_0) \varphi.$$ 

Finally, we have

$$\mathcal{D}(\varphi + \gamma(e_0) \varphi) = -\frac{n-1}{n} \lambda(\varphi + \gamma(e_0) \varphi).$$

Since the spinor field $\varphi + \gamma(e_0) \varphi$ is an eigenspinor of $\mathcal{D}$ with an eigenvalue $-\frac{(n-1)}{n} \lambda < b$, one can infer $P_m^b \varphi = 0$. □

**Remark 3.4.** Under the mgAPS boundary condition, taking $a = 0$ or $u = constant$ in (3.8) and (3.13), one gets Friedrich’s inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R.$$ 

In particular, if we take $b = 0$, the above theorem is exactly the Theorem 5 in [15].

Using the energy-momentum tensor, (3.7) and (3.12), we get

$$\int_M |\nabla^{Q,a,u} \varphi|^2 = \int_M \left[ \lambda^2 - \left( \frac{R_{a,u}}{4} + |Q \varphi|^2 \right) \right] |\varphi|^2 + \int_{\partial M} \left[ adu(e_0) - H/2 \right] |\varphi|^2.$$ 

Thus we obtain the following theorem:

**Theorem 3.4.** Let $M^n$ be a compact Riemannian spin manifold of dimension $n \geq 2$, with nonempty boundary $\partial M$, and let $\lambda$ be any eigenvalue of $\mathcal{D}$ under the mgAPS boundary condition $P_m^b \varphi = P_m^b(\varphi + \gamma(e_0) \varphi) = 0$, for $b \leq 0$, $\varphi \in \Gamma(S)$. If there exist real functions $a, u$ on $M$ such that

$$a \ du(e_0) \leq \frac{1}{2} H$$

on $\partial M$, then

$$\lambda^2 \geq \sup_{a,u} \ inf_M \left( \frac{1}{4} R_{a,u} + |Q \varphi|^2 \right). \quad (3.14)$$

where $R_{a,u}$ is given in (3.2).

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