Abstract. In this article, we review our recent works on fourth order Weyl gravity, and try to obtain same results for cosmological coefficients, using a different approach. Indeed we shall impose a de Sitter like condition on the original components of field equations. Afterwards using the reduced equations on a definite geometric background, we derive the same results as in [1], for same cosmological constraints.

1. Introduction

Einstein’s general theory of relativity seems to be a perfect theory, at least at the classical level where it is confirmed by classical tests. This theory has been obtained from the Hilbert-Einstein action

$$ I_{HE} = \frac{1}{16\pi G} \int dx^4 \sqrt{-g} R. $$

However, the theory faces with difficulties like cosmological constant, dark matter and dark energy problems. In a series of papers, Mannheim and Kazanas have suggested conformal Weyl gravity instead of Einstein’s gravity. With this theory, they could explain two outstanding astrophysical issues namely the cosmological constant and galactic rotation curve problems [2, 3]. Therefore a large amount of works can be found in higher order theories including the Weyl gravity. In the Weyl theory of gravity the Hilbert-Einstein action is replaced by the square of the conformal Weyl tensor

$$ I_W = -\alpha \int dx^4 \sqrt{-g} C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda}, $$

(1)

where $C_{\mu\nu\rho\lambda}$ is the Weyl tensor

$$ C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} - \frac{1}{2} (g_{\mu\lambda} R_{\nu\rho} - g_{\mu\rho} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\rho} + g_{\nu\rho} R_{\mu\lambda}) + \frac{R}{6} (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}). $$

Since there is no accepted system of sign convention in general relativity, different articles use different sign conventions. In this work we first review Weyl gravity with the usual sign convention of Ref. [4], and then we translate it to a de Sitter like background. Finally, taking the cosmological “constant” and the Hubble parameter as functions of $r$ and $t$ respectively, we derive some estimations for their values, which are in very good agreement with the known values of the parameters for the current state of the universe.
2. Weyl gravity

2.1. The solution

The action (1) can be written as follows

\[ I_W = -\alpha \int d^4x \sqrt{-g} \left( R^{\mu \nu \rho \lambda} R_{\mu \nu \rho \lambda} - 2 R^{\mu \nu} R_{\mu \nu} + \frac{1}{3} R^2 \right), \]

since \( \sqrt{-g} \left( R^{\mu \nu \rho \lambda} R_{\mu \nu \rho \lambda} - 4 R^{\mu \nu} R_{\mu \nu} + R^2 \right) \) is a total divergence, so it does not contribute to the equation of motion and one can simplify the action as follows [5]

\[ I_W = -2\alpha \int d^4x \sqrt{-g} \left( R_{\alpha \beta} R^{\alpha \beta} - \frac{1}{3} R^2 \right) = -2\alpha \int d^4x \left( S_1 - \frac{1}{3} S_2 \right). \]  

(2)

The variation of the action with respect to \( g_{\alpha \beta} \) results in

\[
\delta(S_1) = 2\sqrt{-g} R \delta R + R^2 \delta \sqrt{-g} = \sqrt{-g}\nabla_\gamma A^\gamma + \sqrt{-g} \left( 2\nabla^\alpha \nabla^\beta R - 2 g^{\alpha \beta} \nabla_\gamma \nabla^\gamma R - 2 R R^{\alpha \beta} + \frac{1}{3} g^{\alpha \beta} R^2 \right) \delta g_{\alpha \beta},
\]

(3)

where

\[ A^\gamma \equiv 2 g^{\gamma \lambda} g^{\rho \gamma} R (\nabla_\alpha \delta g_{\beta \lambda} - \nabla_\lambda \delta g_{\alpha \beta}) - 2 g^{\gamma \beta} g^{\alpha \beta} \nabla_\rho \nabla_\alpha R \delta g_{\beta \lambda} + 2 g^{\alpha \beta} g^{\gamma \lambda} \nabla_\lambda R \delta g_{\alpha \beta}. \]

Similarly, we can write

\[
\delta(S_2) = \sqrt{-g} \nabla_\gamma B^\gamma + \sqrt{-g} \left( \nabla_\rho \nabla^\alpha R^{\beta \rho} + \nabla_\rho \nabla^\beta R^{\rho \alpha} - \nabla_\lambda \nabla^\lambda R^{\alpha \beta} - g^{\alpha \beta} \nabla_\rho \nabla_\lambda R^{\rho \lambda} - 2 R^2 R^{\alpha \beta} + \frac{1}{3} g^{\alpha \beta} R^2 \right) \delta g_{\alpha \beta},
\]

(4)

where

\[ B^\gamma \equiv g^{\gamma \rho} g^{\alpha \beta} \left( \nabla_\alpha \delta g_{\beta \rho} + \nabla_\beta \delta g_{\alpha \rho} - \nabla_\rho \delta g_{\alpha \beta} \right) - g^{\alpha \beta} \left( \nabla_\alpha R^{\gamma \beta} \delta g_{\beta \rho} - \nabla_\beta R^{\gamma \rho} \delta g_{\alpha \beta} \right) - g^{\beta \rho} \left( R^{\alpha \gamma} \nabla_\alpha \delta g_{\beta \rho} + \nabla_\beta R^{\alpha \gamma} \delta g_{\alpha \rho} \right) + g^{\alpha \rho} \nabla_\rho R^{\alpha \beta} \delta g_{\alpha \beta}. \]

Note that when the equation of motion is considered, the first terms in the right hand side of Eq.s (3-4) have no contribution. Therefore we obtain

\[
S_{\alpha \beta} \equiv S_{\alpha \beta}^{(2)} - \frac{1}{3} S_{\alpha \beta}^{(1)} = \nabla^\rho \nabla_\alpha R_{\beta \rho} + \nabla^\rho \nabla_\beta R_{\alpha \rho} - \nabla_\lambda \nabla^\lambda R_{\alpha \beta} - g_{\alpha \beta} \nabla_\rho \nabla_\lambda R^{\rho \lambda} - 2 R_{\rho \beta} R^{\rho \alpha} + \frac{1}{3} g_{\alpha \beta} R^2.
\]

(5)

\[ S_{\alpha \beta} = 0 \] is the vacuum Weyl field equation. It is worth to mention that relation (5) has been written in the following sign convention

\[ R^{\alpha \gamma \beta \rho}_{\beta \gamma \rho} = \partial_\gamma \Gamma_{\beta \rho}^{\alpha} + \Gamma_{\gamma \delta}^{\rho} \Gamma_{\beta \rho}^{\delta} - \gamma \leftrightarrow \rho, \]

\[ R_{\beta \rho} = +R^{\alpha \beta}_{\beta \alpha \rho}, \text{ and } R = +R^{\alpha}_{\alpha}. \]

Note that in Mannheim’s works the opposite sign for the Ricci tensor and Ricci scalar has been chosen.

Now let us choose the line element which is static and spherically symmetric

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2. \]  

(6)
Having this metric, one can reach exactly the same relation for $S^{rr}$ as it was reported in [6].

$$S^{rr} = -\frac{1}{48B^2 A^4}(-8B^2 A^2 A'^4 + 4BB'' A^2 A'^4 - 7B^2 A^4 r^4$$

$$+4B^2 A^2 A'^4 r^4 + 12B^2 A A'' A'^4 r^4 + 8BB' A^2 A'^4 r^4 - 6BB' A^A'^4 r^4$$

$$-7B^2 A^2 A'^4 r^4 - 48B^2 A^2 A'^4 r^3 + 16B^2 A^A'^4 r^3 - 16BB' A^A'^4 r^3$$

$$+20B^2 A A'^4 r^3 + 16BB' A^2 A'^4 r^3 - 16BB'' A^A'^4 r^3 + 28B^2 A^3 A'^4 r^3$$

$$+32B^2 A A'' A'^4 r^2 + 4B^2 A^2 A'^4 r^2 - 8BB' A^A'^4 r^2 + 16BB'' A^A'^4 r^2$$

$$-28B^2 A^2 A'^4 r^2 - 32B^2 A A'^4 A'^4 r - 16B^4 A^4 + 16B^2 A^4), \quad (7)$$

in which the "Primes" denote differentiation with respect to $r$; as we expected the results are similar.

2.2. The effective potential

Weyl theory of gravity is a theory of fourth order and this makes it problematic. In order to
find the vacuum solution for this theory, we must attain a solution for $S^{rr} = 0$. The differential
equation has been solved ([6]) and the metric parameter $A(r)$ and $B(r)$ have been derived as:

$$A(r) = -\frac{\beta(2 - 3\beta\gamma)}{r} + (1 - 3\beta\gamma) + \gamma r - kr^2 \quad (8)$$

where it has been adopted that

$$B(r) = A^{-1}(r) \quad \gamma, \beta, k = \text{constant.}$$

Since this theory is of fourth order, its solution possesses three constants of integration (the familiar Schwarzschild solution contains only one constant, namely $\beta$ in (8)). These constants become very important in cosmology.

As a first step let us consider the geodesics in Weyl gravity. The geodesic equations
$(dx^\mu/d\tau + \Gamma^\mu_{\nu\rho} dx^\nu/d\tau = 0$ and $\tau$ is the affine parameter) are:

$$\frac{d^2 t}{d\tau^2} + \frac{(2\beta - 3\beta^2\gamma + \gamma r^2 - 2kr^3)(\frac{dt}{d\tau})(\frac{dx}{d\tau})}{r(r - 2\beta + 3\beta^2\gamma - 3\beta\gamma r + \gamma r^2 - k r^3)} = 0;$$

$$\frac{d^2 r}{d\tau^2} + \frac{1}{2} \frac{(r - 2\beta + 3\beta^2\gamma - 3\beta\gamma r + \gamma r^2 - k r^3)(2\beta - 3\beta^2\gamma + \gamma r^2 - 2kr^3)(\frac{dt}{d\tau})^2}{r^3}$$

$$- \frac{1}{2} \frac{(2\beta - 3\beta^2\gamma + \gamma r^2 - 2kr^3)(\frac{dx}{d\tau})^2}{r(r - 2\beta + 3\beta^2\gamma - 3\beta\gamma r + \gamma r^2 - k r^3)} + (2\beta - r - 3\beta^2\gamma + 3\beta\gamma r - \gamma r^2 + k r^3)(\frac{d\theta}{d\tau})^2$$

$$- (r - 2\beta + 3\beta^2\gamma - 3\beta\gamma r + \gamma r^2 - k r^3) \sin^2(\theta)(\frac{d\phi}{d\tau})^2 = 0;$$

$$\frac{d^2 \theta}{d\tau^2} + \frac{2(\frac{d\theta}{d\tau})(\frac{d\phi}{d\tau})}{r} - \sin(\theta) \cos(\theta)(\frac{d\phi}{d\tau})^2 = 0;$$

$$\frac{d^2 \phi}{d\tau^2} + \frac{2(\frac{d\theta}{d\tau})(\frac{d\phi}{d\tau})}{r} + 2 \cot(\theta)(\frac{d\theta}{d\tau})(\frac{d\phi}{d\tau}) = 0. \quad (9)$$
Let us take
\[ \epsilon \equiv -g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{d\tau}. \] (10)

For \( \theta = \pi/2 \) along the geodesics, the two conserved quantities, namely the total energy and the total angular momentum, are:
\[ A(r) \frac{dt}{d\tau} \equiv E, \quad \text{and} \quad r^2 \frac{d\phi}{d\tau} \equiv L. \]

Expanding (10) for \( \epsilon \) and maintaining the expression for \( A(r) \), we get
\[-(1 - \frac{\beta(2 - 3\beta \gamma)}{r}) - 3\beta\gamma + \gamma r - kr^2) \left( \frac{dt}{d\tau} \right)^2 + \left( 1 - \frac{\beta(2 - 3\beta \gamma)}{r} \right)
-3\beta\gamma + \gamma r - kr^2) \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2 = -\epsilon. \] (11)

Multiplying (11) by \( A(r) \) yields
\[-E^2 + \left( \frac{dr}{d\tau} \right)^2 + \left( 1 - \frac{\beta(2 - 3\beta \gamma)}{r} - 3\beta\gamma + \gamma r - kr^2) \right) \left( \frac{L^2}{r^2} + \epsilon \right) = 0. \] (12)

Rewrite (12) as
\[ \frac{1}{2} E^2 = \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V(r), \] (13)
where
\[ V(r) = \frac{1}{2} \left( 1 - \frac{\beta(2 - 3\beta \gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 \right) \left( \frac{L^2}{r^2} + \epsilon \right). \] (14)

Equation (14) gives us the opportunity to define the effective potential as
\[ V_{eff} = \frac{1}{2} \{ \epsilon - 3\beta\gamma\epsilon - kL^2 \} - \frac{1}{2} \{ \epsilon\beta(2 - 3\beta \gamma) - \gamma L^2 \} \frac{1}{r} - \frac{1}{2} \{ L^2(3\beta\gamma - 1) \} \frac{1}{r^2}
+ \frac{1}{2} \{ L^2\beta(3\beta \gamma - 2) \} \frac{1}{r^3} + \frac{1}{2} \gamma \epsilon r - \frac{1}{2} k \epsilon r^2, \] (15)

which is the Weyl effective potential for a particle, supposed to move in Weyl geometry. This potential has some extra terms which may have significant physical meaning, for example in large scale the last two terms play important roles. Now taking \( \beta = GM \), and \( \epsilon = 1, \gamma = k = 0 \), one obtains the famous effective potential of general relativity, namely:
\[ V_{eff} = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2 r^2} - \frac{GML^2}{r^3}. \] (16)

Other values of these constants give us various kinds of potentials. For example, taking \( \epsilon, \gamma, k = 0 \) and \( \beta = GM \) we obtain:
\[ V_{eff} = \frac{1}{2} \frac{L^2}{2 r^2} - \frac{GM}{r^3}. \] (17)

This potential has two terms. The term \( \frac{1}{2} \frac{L^2}{2 r^2} \) denotes the Newtonian effective potential for which \( L = m r^2 \dot{\phi}^2 \) and \( m = 1 \). The potential covers the central force problem. The second, namely \( \frac{GM}{r^3} \), denotes the Schwarzschild effective potential. Thus equation (17) with the above conditions
indicates the effective potential due to general relativity. For $\epsilon = 0$ but $\gamma, k \neq 0$ and $\beta = GM$, equation (15) yields

$$V_{\text{eff}} = -\frac{1}{2}kL^2 + \frac{\gamma L^2}{2r} - \left(\frac{3}{2}\gamma GM L^2 - \frac{1}{2}L^2\right)\frac{1}{r^2} + \left(\frac{3}{2}\gamma G^2 M^2 L^2 - L^2\right)\frac{1}{r^3}. \quad (18)$$

This effective potential has two important additional terms. The term $\frac{\gamma L^2}{2r}$ in (18) denotes the potential due to the term $\gamma r$ in the metric. This is a repulsive potential which is proportional to the inverse of distance. The term $\frac{1}{2}kL^2$ denotes the potential due to the term $kr^2$ in the metric. This is a constant potential with respect to $r$. In analogy to general relativity, the terms contain $\frac{1}{r^2}$ and $\frac{1}{r^3}$ have also two additional terms say $-\frac{3\gamma GM L^2}{2r^2}$ and $\frac{3\gamma G^2 M^2 L^2}{2r^3}$ respectively.

Note that the coefficient $\gamma$, is one of the most important constants, introduced in [6], in order to explain the strange flat galactic rotation curves without maintaining dark matter. Its value has been calculated to be $2 \times 10^{-28}$ cm$^{-1}$. Also it has been semi-analytically obtained in [7], resulting in same value.

3. Geometric background and the de Sitter like condition
We start with the line element (6) by imposing the following de Sitter like condition on it:

$$R_{\mu\nu} = \frac{1}{4}g_{\mu\nu} R, \quad (19)$$

in which $R$ may have a $r$ or $t$ dependence. Note that this essentially requires a constant total curvature, i.e. $R = 4\Lambda$, which is the de Sitter condition (this is also mentioned in the appendix). However, in this article the question we want to examine is: if we assume this condition is retained for variable total curvature in a definite geometric background, would it be possible to regain our previous results for cosmological parameters or not? Therefore from now on, we shall consider a de Sitter like background geometry, and equation (19) will help us to reduce the original field equations.

We begin with (19). Using the 00 component of metric (6) taking $B = A^{-1}$. We have:

$$\left( r^2 A'' + 4 A' r - 2 + 2 A \right) = 2r \left( r A'' + 2 A' \right), \quad (20)$$

The solution can be written as follows

$$A(r) = 1 + \frac{b_1}{r} + b_2 r^2. \quad (21)$$

Substituting $b_1 = -2GM$ and $b_2 = -\frac{1}{3}\Lambda$, we obtain the Schwarzschild-de Sitter metric. Now let us work with the following component of metric, forming a de Sitter like geometric background, in which $\Lambda$ has been chosen to be a function of $r$:

$$g_{00} = -c^2 \left( 1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda(r) r^2}{c^2} \right),$$

$$g_{11} = \left( 1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda(r) r^2}{c^2} \right)^{-1},$$

$$g_{22} = r^2,$n

$$g_{33} = r^2 \sin^2(\theta). \quad (22)$$
The components of the Ricci tensor, due to metric (22) become as follows:

\[
R_{00} = \frac{1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda r^2}{c^2}}{2rc^2} \left[ 2rc^2 \left( -4 \frac{GM}{r^3c^2} - \frac{1}{3} \frac{\Lambda'' r^2}{c^2} - \frac{4}{3} \frac{\Lambda'}{c^2} - \frac{2}{3} \frac{\Lambda}{c^2} \right) \right] \]
\[
+ 4c^4 \left( 2 \frac{GM}{r^2c^2} - \frac{1}{3} \frac{\Lambda' r^2}{c^2} - \frac{2}{3} \frac{\Lambda r}{c^2} \right),
\]

\[
R_{11} = \frac{1}{4c^4} \left( 1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda r^2}{c^2} \right) \left[ -2rc^2 \left( -4 \frac{GM}{r^3c^2} - \frac{1}{3} \frac{\Lambda'' r^2}{c^2} - \frac{4}{3} \frac{\Lambda'}{c^2} - \frac{2}{3} \frac{\Lambda}{c^2} \right) \right] \]
\[
- 4c^4 \left( 2 \frac{GM}{r^2c^2} - \frac{1}{3} \frac{\Lambda' r^2}{c^2} - \frac{2}{3} \frac{\Lambda r}{c^2} \right),
\]

\[
R_{22} = \frac{1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda r^2}{c^2}}{2c^2} \left[ -2 \left( 2 \frac{GM}{r^2c^2} - \frac{1}{3} \frac{\Lambda' r^2}{c^2} - \frac{2}{3} \frac{\Lambda r}{c^2} \right) rc^2 \left( 1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda r^2}{c^2} \right)^{-1} \right] \]
\[
+ 2c^2 \left( 1 - 2 \frac{GM}{rc^2} - \frac{1}{3} \frac{\Lambda r^2}{c^2} \right)^{-1} - 2c^2 \right] = \frac{1}{\cos^2(\theta)} R_{33}. \tag{23}
\]

And the Ricci scalar \( R = R^\mu_\mu \) would be:

\[
R = \frac{1}{3} \left( \frac{\Lambda'' r^2}{c^2} + 8 \frac{\Lambda'}{c^2} r + 12 \frac{\Lambda}{c^2} \right). \tag{24}
\]

Note that, a constant \( \Lambda \) results in \( R = \frac{1}{3c^2} \Lambda \), which is the de Sitter condition.

3.1. Estimating the cosmological constant

The 00 component of the Weyl field equations (5), using metric (22) and the values in (23) and (24), gives

\[
S_{00} = \frac{1}{324r^3c^4} \left[ 396 r^4c^4 \Lambda''' + 36 r^5c^4 \Lambda'''' + 720 GMA r^3 \Lambda' + 1440 GMA r^4 \Lambda'' + 48 \Lambda' r GMA''\right]

- 1548 GMr^3 \Lambda''' + 540 GMA r^5 \Lambda'''' + 12 \Lambda' r^6 GMA'''' - 144 r^4c^2 \Lambda'''' GMA + 48 GMA''' r^6 \Lambda

- 3116 GMr^2c^2 \Lambda'' - 1584 GMr^2c^2 \Lambda' + 864 G^2 M^2 \Lambda' - 24 (\Lambda')^2 r^5 c^2 + 8 (\Lambda')^2 r^7 \Lambda

+ 96 \Lambda^{2r^6 \Lambda'} + 576 r^2c^4 \Lambda' + 3 (\Lambda'')^2 r^7 c^2 - (\Lambda'')^2 r^9 \Lambda + 144 \Lambda^2 r^7 \Lambda'' + 48 \Lambda^2 r^8 \Lambda'''

+ 1044 r^5c^4 \Lambda'' + 4 \Lambda^2 r^9 \Lambda''' + 48 (\Lambda')^2 r^4 GMA - 480 \Lambda r^4c^2 \Lambda' + 3456 G^2 M^2 \Lambda'' r

- 780 \Lambda r^5c^2 \Lambda'' - 6 (\Lambda')^2 r^6 GMA - 276 \Lambda r^6c^2 \Lambda''' + 144 G^2 M^2 \Lambda'' r^3 + 8 \Lambda^2 r^8 \Lambda''

+ 1512 G^2 M^2 \Lambda'' r^2 - 6 r^7c^2 \Lambda'' + 2 \Lambda' r^9 \Lambda'' + 24 r^7c^2 \Lambda'' + 24 \Lambda' r^6 \Lambda'' \right]. \tag{25}
\]
In order to find an analytical solution for $\Lambda(r)$, we should set $S_{00} = 0$. This evidently generates a rather complicated equation. Hence, we shall reduce the vacuum Weyl equations of (5), in a de Sitter like background, by imposing the condition (19). This results in the following field equation:
\[
S^{(dS)}_{\alpha\beta} = -\frac{1}{12}g_{\alpha\beta}\nabla_\lambda \nabla^\lambda R - \frac{1}{6}g_{\alpha\beta}\nabla R.
\]  
(26)

Using (22) and (24) in (26) yields:
\[
S^{(dS)}_{00} = \frac{1}{27r^2 c^4}(-3r^2 c^2 + 12GM + \Lambda r^3)(2r^2 \Lambda' + 6r^2 \Lambda' - 3r^2 \Lambda'' + 6GM \Lambda'' + r^3 \Lambda'' - 6c^2 \Lambda'),
\]
\[
S^{(dS)}_{11} = -\frac{1}{3(-3r^2 c^2 + 6GM - r^3 \Lambda)}(2r^2 \Lambda^2 + 6r^2 \Lambda' - 9r^2 \Lambda'' + 18GM \Lambda'' + 3r^3 \Lambda'' - 6c^2 \Lambda'),
\]
\[
S^{(dS)}_{22} = \frac{1}{9c^3}(18GM \Lambda' + r^4 \Lambda^2 + 6r^3 \Lambda' - 3r^2 c^2 \Lambda'' + 6GM r \Lambda'' + r^4 \Lambda'' - 12r c^2 \Lambda') = \frac{1}{\sin^2(\theta) S^{(dS)}_{33}}.
\]  
(27)

And the trace due to these components will be:
\[
S^{(dS)}_{\alpha} = \frac{1}{9r^2 c^4}(4r^4 \Lambda^2 + 18r^3 \Lambda - 9r^2 c^2 \Lambda'' + 18GM r \Lambda'' + 3r^4 \Lambda'' - 30r c^2 \Lambda' + 2r^4 c^2 \Lambda^2 + 6r^3 c^2 \Lambda' - 9r^2 c^2 \Lambda'' + 18GM c^2 \Lambda'' + 3r^4 c^2 \Lambda'' - 6r c^4 \Lambda' + 36GM \Lambda').
\]  
(28)

Since the conformal Weyl gravity is a traceless theory, therefore we can obtain a differential equation with respect to $\Lambda(r)$:
\[
64r^4 \Lambda^2 + 18r^3 \Lambda - 9r^2 c^2 \Lambda'' + 18GM r \Lambda'' + 3r^4 \Lambda'' - 30r c^2 \Lambda' + 2r^4 c^2 \Lambda^2 + 6r^3 c^2 \Lambda' - 9r^2 c^2 \Lambda'' + 18GM c^2 \Lambda'' + 3r^4 c^2 \Lambda'' - 6r c^4 \Lambda' + 36GM \Lambda' = 0,
\]  
(29)

for which a proper solution is
\[
\Lambda(r) = \frac{3c^2}{r^2} - \frac{6GM}{r^3}.
\]  
(30)

This can be applied to obtain an estimation for the current value of the cosmological constant $\Lambda_0$. To do this, we consider
\[
\Lambda_0 = \Lambda(r_{ob}),
\]
in which $r_{ob} = 4.39937 \times 10^{28} cm$ is the distance to the particle horizon [8]. Other important measurable value in (30) is the mass $M$, which for the current state of the universe, is the mass of the observable universe, calculated by Sir Fred Hoyle to be $M_{ob} = 8 \times 10^{52} kg$ [9]. Also the speed of light is $c = 299792458 \frac{cm}{s}$ and the gravitational constant $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$. These values, all together in (30) result in
\[
\Lambda_0 = 1.020631735 \times 10^{-36} \text{ s}^{-2}.
\]  
(31)

Note that the cosmological constant has been measured to be of order $10^{-35} \text{ s}^{-2}$ or $10^{-29} \frac{\text{cm}^3}{\text{gr}^4}$ [10]. This latter one, has been also mentioned in [1], which means that we have achieved the same results, using a different method.
3.2. Estimating the Hubble constant

Another important cosmological parameter is the Hubble parameter, and here we pursue the previous method, to evaluate it. Now Let us assume a flat and time dependent metric as:

\[ ds^2 = -c^2 dt^2 + f(t) dx_i^2, \]

where \( i = 1, 2, 3 \). We try to find a definite expression for \( f(t) \) under the condition (19). The Ricci tensor components due to the metric (32) are:

\[ R_{00} = -\frac{3}{4} \frac{\ddot{f}(t) f(t) - \dot{f}(t)^2}{f(t)^2}, \]

\[ R_{11} = R_{22} = R_{33} = \frac{1}{4} \frac{2 \ddot{f}(t) f(t) + \dot{f}(t)^2}{c^2 f(t)}, \]

where dot stands for the time derivative, and the Ricci scalar will be

\[ R = \frac{3 \ddot{f}(t)}{c^2 f(t)} \]

By imposing (19) we obtain

\[ \ddot{f}(t) f(t) - \dot{f}(t)^2 = 0. \]  

A proper solution to (35) may be found as

\[ f(t) = f_1 e^{f_2 t}, \]

in which \( f_1 \) and \( f_2 \) are integration constants. Considering \( f_1 = 1 \) and \( f_2 = 2H \), turns the metric in (32) to

\[ ds^2 = -c^2 dt^2 + e^{2Ht} dx_i^2, \]

which is exactly the de Sitter’s flat slicing metric. To investigate the metric in (37) we firstly suppose that \( H \) is itself a function of \( t \). Therefore the Ricci tensor components and the Ricci scalar change to

\[ R_{00} = -3 \left( \dot{H} t + 2 \dot{H} + (\dot{H} t + H)^2 \right), \]

\[ R_{ij} = \frac{c^2 H t}{c^2} \left( \dot{H} t + 2 \ddot{H} + 3(\dot{H} t + H)^2 \right), \]

\[ R = \frac{6}{c^2} \left( \dot{H} t + 2 \ddot{H} + 2 \dot{H} H t + 4 \dot{H} H t + 2H^2 \right). \]

Note that if \( H \) be a constant then we will also get the de Sitter condition for the Ricci scalar. Now using the de Sitter like condition in (19) for the Ricci tensor components in (38) we have

\[ \frac{3}{2} \dot{H} t + 3 \dot{H} = 0, \]

which has a solution of the form:

\[ H(t) = c_1 + \frac{c_2}{t}. \]

Substituting the Ricci tensor components of (38) in (5) vanishes all of the components, i.e.

\[ S_{\alpha\beta} = 0. \]

This means that (37) even when \( H = H(t) \), is a vacuum solution to Weyl gravity.
Now let us compute these relations in de Sitter like background. The components of $S^{(dS)}_{\mu\nu}$:

$$S^{(dS)}_{00} = -\frac{3}{2} (H^{(4)} t + 4H^{(3)} + 4\dot{H}^2 t^2 + 31 \dot{H} \ddot{H} t + 5\dot{H} H^{(3)} t^2 + 16\dddot{H} + 5H^{(3)} H t$$

$$+ 15\dddot{H} + 4\dddot{H} \dddot{H} t^3 + 8\dddot{H}^2 t^2 + 8\dddot{H} \dot{H} H t^2 + 16H \dot{H}^2 t + 4H \dot{H}^2 t + 8H^2 \dot{H}), \quad (42)$$

$$S^{(dS)}_{11} = S^{(dS)}_{22} = S^{(dS)}_{33} = \frac{e^{2Ht}}{2c^4} (H^{(4)} t + 4H^{(3)} + 4\dot{H}^2 t^2 + 43 \dot{H} \dddot{H} t + 9H H^{(3)} t^2$$

$$+ 16\dot{H}^2 + 9H^{(3)} H t + 27\dddot{H} H + 20\dddot{H} \dddot{H} t^3 + 40\dddot{H}^3 t^2 + 40\dddot{H} \dot{H} H t^2 + 80H \dot{H}^2 t + 20\dot{H} H^2 t + 40H^2 \dot{H}). \quad (43)$$

Requiring the traceless condition results in the following equation:

$$\frac{c^4}{3} S^{\mu(dS)}_{\mu} = H^{(4)} t + 4H^{(3)} + 4\dot{H}^2 t^2 + 37 \dot{H} \dddot{H} t + 7H H^{(3)} t^2 + 16\dddot{H} + 7H^{(3)} H t$$

$$+ 21\dddot{H} H + 12\dddot{H} \dddot{H} t^3 + 42\dddot{H}^3 t^2 + 24\dddot{H} \dot{H} H t^2 + 48H \dot{H}^2 t + 12\dot{H} H^2 t + 24H^2 \dot{H} = 0. \quad (44)$$

This fourth order differential equation has a special solution as follows

$$H(t) = \frac{c_1}{t} + c_2. \quad (45)$$

A glance on equation (45) leads us to conclude that the Hubble parameter has a hyperbolic behavior with respect to time. As an example we try to find the recently derived Hubble parameter by taking the simplest guess namely $c_1 = 1$, $c_2 = 0$ and using the age of the universe. The age of the universe is now calculated to be $13.75 \times 10^9 \pm 0.17 \ yrs$ [11]. Substituting this value in (45) yields the Hubble constant for this time.

$$H_0 = 2.31 \times 10^{-18} \ s^{-1}. \quad (46)$$

The value for the Hubble constant, derived in [1], was the same. The measured Hubble constant in 2006 was $2.5 \times 10^{-18} \pm 0.15s^{-1}$ [12].

4. Conclusion

There have been intense research activities in higher order theories of gravity since a couple of decades ago. In this context, much attention has been given to Weyl gravity, where the usual Einstein-Hilbert action is replaced by the square of the conformal Weyl tensor. This leads to a gravitational theory of fourth order. Within a fourth order theory of gravity, we first reviewed this theory in the sign convention of Ref. [4]. Moreover, the added terms in the effective potential have been reviewed. We studied the time and $r$ dependence of the Hubble and cosmological parameters in de Sitter background and finally we presented an estimation for them. In this article we used a different method, consisting of imposing a de Sitter like condition on the original Weyl field equations, to reach the same values which have been derived formerly in our previous articles. It turned out that they are in very good agreement with other approaches and observational results.
Appendix A. de Sitter spacetime
The de Sitter space-time is a vacuum solution for both Einstein and Weyl gravity. Recent Astrophysical data indicate that our universe has a positive and non-vanishing acceleration, so it might be currently in de Sitter phase. de Sitter space-time is a maximally symmetric space so we can write
\[
R_{abcd} = H^2 (g_{ae} g_{bd} - g_{ad} g_{be}), \quad R_{ab} = 3 H^2 g_{ab}, \quad R = 12 H^2 = 4 \Lambda.
\]
In de Sitter background the following variations become important
\[
\delta g_{ab} = - \delta g^{ab},
\]
\[
\delta \Gamma^a_{bd} = \frac{1}{2} g^{ae} \left( \nabla_b \delta g_{ed} + \nabla_d \delta g_{be} - \nabla_e \delta g_{bd} \right),
\]
\[
\delta R^a_{bcd} = \frac{1}{2} g^{ae} \left( \nabla_c \nabla_b \delta g_{de} + \nabla_c \nabla_d \delta g_{be} - \nabla_e \nabla_c \delta g_{bd} - \nabla_d \nabla_b \delta g_{ce} - \nabla_e \nabla_d \delta g_{be} + \nabla_d \nabla_e \delta g_{be} \right),
\]
\[
\delta R_{ab} = \frac{1}{2} g^{cd} \left( \nabla_c \nabla_a \delta g_{bd} + \nabla_c \nabla_d \delta g_{ab} - \nabla_e \nabla_a \delta g_{cd} - \nabla_b \nabla_c \delta g_{ad} \right),
\]
\[
\delta R = \left( \nabla^a \nabla^b \delta g_{ab} - g^{ab} \nabla^\lambda \delta g_{ab} \right) - 3 H^2 g^{ab} \delta g_{ab}.
\]

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