Abstract

In 1945 Einstein concluded that [1]: “The present theory of relativity is based on a division of physical reality into a metric field (gravitation) on the one hand, and into an electromagnetic field and matter on the other hand. In reality space will probably be of a uniform character and the present theory be valid only as a limiting case. For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance.”. The dichotomy can be resolved by introducing a scalar field/potential algebraically related to the Ricci tensor for which the corresponding metric is free of additional singularities. Hence, although a fundamentally nonlinear theory, the scalar field/potential provides an analytic framework for interacting particles; described by linear superposition. The stress tensor for the scalar field includes both the sources of and the energy-momentum for the gravitational field, and has zero covariant and ordinary divergence. Hence, the energy-momentum for the gravitational field and sources are conserved. The theory’s predictions agree with the experimental results for General Relativity. By introducing the corresponding Lagrangian to analytic mechanics, what is experimentally known for GR can be accounted for.
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Abstract

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1 Introduction: Towards a Theory of General Relativity

1.1 Outline

After A. Einstein introduced Special Relativity (SR) 1905 [2], the general case, i.e., a General Theory of Relativity (GR) turned out to be elusive. Eventually, he made progress through a collaboration with M. Grossmann 1913 [3] who introduced him to the tensor calculus developed by: K. Gauss 1827 [4], B. Riemann 1854 [5], E. Christoffel 1865 [6], and G. Ricci and T. Levi-Civita 1901 [7]. According to Einstein, his student friend was somewhat reluctant (p. 152 [8, 9]):

"...with the restriction that he would not be responsible for any statements and won’t assume any interpretations of physical nature."

Similarly, in a letter to Levi-Civita 1917, regarding a controversy on the gravitational stress tensor [10], Einstein wrote [11]:

"I admire the elegance of your method and calculation. It must be nice to ride through these fields upon the horse of true mathematics, while the likes of us have to make our way laboriously on foot..."

At the same time D. Hilbert was developing a general mathematical framework for field theories [12]. So, shortly after publishing his 1916 GR paper [13], Einstein re-derived his field equations using Hilbert’s variational approach [14]. Given the time it took to establish a General Theory of Relativity, it is no surprise that the path was paved by a whole slew of intellectual explorers at a time it was ripe; e.g. space-time continuum, etc. For example, G. Nordström was one of the first to contribute with a self-consistent approach [15–17].

The outline of this essay is as follows:

1. The variational framework is used to generate:
   a) Einstein’s field equations,
   b) equations of motion for a perfect fluid,
   c) field equations for a massless scalar field.

2. An (differential) algebraic relation between the scalar field stress tensor and the Ricci tensor is obtained.

3. The effect on the metric is given by the Christoffel symbols. The metric is obtained by solving the system for an $\frac{1}{r}$ potential.

4. By evaluating and expanding the corresponding Lagrangian for analytic mechanics, it is transparent that the theory’s predictions agree with GR; to first order in the gravitational constant.

5. The theory is essentially a generalization of Nordström’s theory from 1913 [15]–[17]; i.e., a scalar field in curved space-time vs. Minkowski metric.

1.2 Hilbert’s Variational Framework

The action $S$ for a Lagrangian density $\mathcal{L}$ is [12]

$$ S = \int \mathcal{L} d^4x. \tag{1} $$

Variation of the metric $g_{ij} \to g_{ij} + \delta g_{ij}$ generates the canonical stress tensor density

$$ \mathcal{\mathcal{T}}^{ij} \equiv \sqrt{-g} T^{ij} = -\frac{\delta \mathcal{L}}{\delta g_{ij}} = -2\sqrt{-g} \frac{\delta (\frac{1}{\sqrt{-g}} \mathcal{L})}{\delta g_{ij}} + g^{ij} \mathcal{L} \tag{2} $$

which has zero covariant divergence

$$ \nabla_i \mathcal{T}^{ri} = \sqrt{-g} \nabla_i T^{ri} = 0 \tag{3} $$

where $\nabla_i$ is the covariant derivative, since (e.g. [18])

$$ \int \frac{\delta \mathcal{L}}{\delta g_{rs}} \delta g^{rs} d^4x = \int \sqrt{-g} (\nabla_i T^r_i) \delta x^r d^4x = 0. \tag{4} $$
However, the integral
\[ \int \mathcal{F}^k dS_k \] (5)
is conserved only if the ordinary divergence is zero
\[ \partial_r \mathcal{F}^r = 0. \] (6)

1.3 General Relativity
For General Relativity (GR) \[ 14 \]
\[ \mathcal{L} = \frac{1}{2\kappa} \sqrt{-g} R + \mathcal{L}_m, \quad \kappa \equiv \frac{8\pi G}{c^4} \] (7)
where \( \mathcal{L}_m \) is the Lagrangian density describing matter. A variation of the metric gives
\[ \delta S = \delta \int \mathcal{L} d^4x = \delta \int \left( \frac{1}{2\kappa} \sqrt{-g} g^{ij} R_{ij} + \mathcal{L}_m \right) d^4x = \frac{\sqrt{-g}}{2\kappa} \left( R_{rs} - \frac{1}{2} g_{rs} R - \kappa T_{rs} \right) g^{rs} d^4x = 0 \] (8)
which leads to Einstein's field equations \[ 13 \]
\[ R_{ij} - \frac{1}{2} g_{ij} R = \kappa T^{(m)}_{ij} \] (9)
with the contraction
\[ R = -\kappa T^{(m)}. \] (10).
For a static spherically symmetric body with mass \( M \) they have the solution (Schwarzschild metric) \[ 19 \]
\[ ds^2 = -\left( 1 + \frac{2\phi}{c^2} \right)^2 c^2 dt^2 + \frac{dr^2}{\left( 1 - \frac{2\phi}{c^2} \right)} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \] (11)
where
\[ \phi = -\frac{GM}{r}, \quad r > 0. \] (12)
It becomes singular for (Schwarzschild radius)
\[ r = \frac{2GM}{c^2}. \] (13)

1.4 Gravitational Stress Tensor: The Einstein - Levi-Civita Controversy
Levi-Civita, inspired by Einstein's 1916 theory \[ 13 \], the same year introduced the notion parallel transport \[ 20 \] to simplify the geometric framework.
The Einstein - Levi-Civita controversy 1917 \[ 10 \] was regarding the interpretation of the Einstein equations
\[ R_{ij} - \frac{1}{2} g_{ij} R = \kappa T^{(m)}_{ij}. \] (14)
In particular, Einstein viewed them as field equations whereas Levi-Civita wrote them in the form
\[ T^{(g)}_{ij} + T^{(m)}_{ij} = 0 \] (15)
where
\[ T^{(g)}_{ij} = \frac{1}{\kappa} \left( R_{ij} - \frac{1}{2} g_{ij} R \right) \] (16).
Similarly, in 1947 Landau-Lifshitz introduced the (symmetric) pseudo-tensor \( T_{ij}^{(g)} \) and (mass conservation)

\[
T_{ij}^{(g)} = \frac{1}{2\kappa} \left[ G^* \delta^i_j - \left( \partial_{\delta_j} g^{rs} G^* \right) \partial_r g^{sj} \right] = \frac{1}{2\kappa} \delta^i_j g^{rs} \Gamma^s_{ru} \Gamma^u_{st} - g^{rs} \Gamma^i_{ru} \Gamma^u_{sj} \tag{17}
\]

where (affine scalar; only invariant under linear transformations)

\[
G^* = g^{rs} \left( \Gamma^i_{ru} \Gamma^u_{st} - \Gamma^t_{rs} \Gamma^u_{tu} \right) \tag{18}
\]

and it follows that

\[
\partial_r \left( \sqrt{-g} T_{i}^{(m)} + T_{i}^{(g)} \right) = 0. \tag{19}
\]

In particular

\[
\int \sqrt{-g} R d^4x = \int \sqrt{-g} G^* d^4x + \text{surface term} \tag{20}
\]

and \( G^* \) depends only on \( g_{ij} \) and their first derivatives; since \( R \) is linear in the second derivatives. Similarly, in 1947 Landau-Lifshitz introduced the (symmetric) pseudo-tensor \( T_{ij}^{(g)} \)

\[
t_{ij} = \frac{1}{2\kappa} \left[ \left( 2 \Gamma^t_{rs} \Gamma^u_{tu} + \Gamma^t_{ru} \Gamma^u_{st} - \Gamma^t_{rt} \Gamma^u_{su} \right) \left( g^{ir} g^{js} - g^{ij} g^{rs} \right) + g^{ir} g^{st} \left( \Gamma^j_{rs} \Gamma^u_{st} + \Gamma^j_{su} \Gamma^u_{rs} - \Gamma^j_{tu} \Gamma^u_{rs} - \Gamma^j_{rs} \Gamma^u_{tu} \right) + g^{ir} g^{st} \left( \Gamma^i_{ru} \Gamma^u_{st} - \Gamma^i_{tu} \Gamma^u_{rs} \right) + g^{rs} g^{tu} \left( \Gamma^i_{rt} \Gamma^u_{su} - \Gamma^i_{ru} \Gamma^u_{st} \right) \right]
\]

for which

\[
\partial_r \left( -g \left( T^{ir(m)} + T^{ir(g)} \right) \right) = 0. \tag{21}
\]

### 1.5 Perfect Relativistic Fluid

A Lagrangian approach for a perfect relativistic fluid requires variational calculus with constraints \[\text{\textsuperscript{22}}\]. For example, starting from \[\text{\textsuperscript{23}}\]

\[
\mathcal{L}_m (p^i) = -c_0 \sqrt{-g} \rho \sqrt{-u^s u^s} = -c_0 \sqrt{-p_r p^r}, \quad p^i \equiv \sqrt{-g} u^i \tag{22}
\]

and (mass conservation)

\[
\frac{dp^r}{ds} = u^r \partial_r \rho = u^r \nabla_r \rho = -\rho \nabla_r u^r. \]

variation of the momenta \( p^i \to p^i + \delta p^i \) and using the relations

\[
\delta \sqrt{-p_r p^r} = -\frac{1}{\sqrt{-p_r p^r}} p_r \delta p^r = \frac{1}{c_0} u_r \delta p^r, \quad \delta p^i = \partial_r \left( p^i \delta x^r - p^r \delta x^i \right), \quad \partial_i u_j - \partial_j u_i = \nabla_i u_j - \nabla_j u_i, \tag{23}
\]

\[
\nabla_i (u^r u_r) = 2u^r \nabla_i u_r = 0 \tag{24}
\]

leads to

\[
\delta S = \delta \int \mathcal{L}_m d^4x = \int \sqrt{-g} \rho u^r \left( \nabla_r u_s \right) \delta x^s d^4x \tag{25}
\]

\[\text{\textsuperscript{1}}\text{Similarly, Maxwell’s equations requires the constraint } \delta \nabla_r F^{rs} = 0; \text{ or the introduction of a vector potential.}\]
and the equations of motion (geodesic)

$$u^r \nabla_r u^i = \frac{du^i}{ds} + \Gamma^i_{rs} u^r u^s = 0. \tag{26}$$

The canonical stress tensor density (for $p^r$) is

$$\mathcal{T}^{i(m)}_{ij} = -2 \frac{\delta \mathcal{L}(p^r)}{\delta g_{ij}} = 2c_0 \frac{\delta \sqrt{-g} p^r}{\delta g_{ij}} = -\sqrt{-g} \rho u^u u^r \tag{27}$$

i.e., Eq (25.4) in ref. [23], with the trace

$$\mathcal{T}^r_{rr} = \sqrt{-g} \rho c_0^2 \tag{28}$$

and the covariant divergence is zero (equations of motion)

$$\nabla_r \mathcal{T}^r_{i(m)} = -\partial_r (\sqrt{-g} \rho u^u u^r) = -\nabla_r (p^r u_i) = u_i \nabla_r p^r + \sqrt{-g} \rho u^r \nabla_r u_i = 0 \tag{29}$$

i.e., for mass conservation $\nabla_r p^r = 0$ and along a geodesic $u^r \nabla_r u_i = 0$.

Alternatively, one may start from the Lagrangian density

$$\mathcal{L}_m (u^i) = -\frac{1}{2} p^r u_r, \tag{30}$$

since

$$\mathcal{L}_p \equiv -\int p^r du_r = -\int \sqrt{-g} \rho u^r du_r = -\frac{1}{2} p^r u_r, \tag{31}$$

to generate the canonical stress tensor

$$\mathcal{T}^{(m)}_{ij} = -2 \sqrt{-g} \frac{\delta (\frac{1}{2} \mathcal{L}_p)}{\delta g_{ij}} + g^{ij} \mathcal{L} = \sqrt{-g} \rho \left( u^i u_j + \frac{c_0^2}{2} g^{ij} \right) \tag{32}$$

and proceed directly to the equations of motion (see last paragraph of section 4.c in ref. [22])

$$\nabla_r \mathcal{T}^r_{i(m)} = \nabla_r \left( \sqrt{-g} \rho u^u u_i + \frac{c_0^2}{2} \sqrt{-g} \delta_i^u \right) = u_i \nabla_r p^r + \sqrt{-g} \rho u^r \nabla_r u_i + \frac{c_0^2}{2} \sqrt{-g} \nabla_i \rho = 0. \tag{33}$$

For a fluid for which (mass conservation)

$$\nabla_r p^r = 0 \tag{34}$$

they simplify to

$$u^r \nabla_r u^i = \frac{du^i}{ds} + \Gamma^i_{rs} u^r u^s + \frac{c_0^2}{2} g^{rs} \partial_r (\rho \ln (\rho)) = 0 \tag{35}$$

which differ from Eq. (26) by the last term.

For an incompressible fluid $\rho = \text{const}$.

$$u^r \nabla_r u^i = \frac{du^i}{ds} + \Gamma^i_{rs} u^r u^s = 0 \tag{36}$$

2 Massless Scalar Field

2.1 Lagrangian Density and Field Equations

For a free, massless scalar field

$$\mathcal{L} (\phi, \phi_i) = \frac{1}{8\pi G} \sqrt{-g} \phi^r \phi_r = \frac{1}{8\pi G} \sqrt{-g} g^{rs} \phi_s \phi_r, \quad \phi_i \equiv \partial_i \phi. \tag{37}$$

A (covariant) variation of the field $\phi \to \phi + \delta \phi$ gives (Euler-Lagrange equations) [24]

$$\partial_\phi \mathcal{L} - \partial_r (\partial_r \mathcal{L}) = 0 \tag{38}$$
which leads to the field equations

\[ \partial_r (\sqrt{-g} \phi^r) = \sqrt{-g} \nabla_r \phi^r = \frac{1}{2} \partial_\phi (\sqrt{-g} \phi^r) = \frac{1}{2} (\partial_\phi \sqrt{-g}) \phi^r \phi_r \]  

(39)

since \( \partial_\phi (\phi^r \phi_r) = 0 \). The stress tensor density is

\[ T_{ij} (\phi) = -\frac{2}{\sqrt{-g}} \frac{\delta L}{\delta g_{ij}} = \frac{1}{4\pi G} \left( \phi^i \phi^j - \frac{1}{2} \phi^r \phi_r g_{ij} \right) = -\phi_j \partial_\phi \mathcal{L} + g^{ij} \mathcal{L}. \]

(40)

with the trace

\[ T_{rr} (\phi) = \frac{\sqrt{-g}}{4\pi G} \phi^r \phi_r \]

(41)

and zero covariant as well as ordinary divergence

\[ \partial_r T_{rr} (\phi) = \partial_r (-\phi_j \partial_\phi \mathcal{L} + \delta_j^i \mathcal{L}) = 0 \]

(42)

where we have used the Euler-Lagrange equations Eq. (38) and

\[ \delta L = \frac{\delta L}{\delta \phi} \delta \phi + \frac{\delta L}{\delta g} \delta g \Rightarrow \]

\[ \partial_i \mathcal{L} = (\partial_\phi \mathcal{L}) \phi_i + (\partial_\phi \mathcal{L}) \partial_r \phi_r = (\partial_\phi \mathcal{L}) \partial_r \phi_r. \]

(43)

(44)

By using the expression for the trace, the field equations can be written

\[ \nabla_r (\sqrt{-g} \phi^r) = \frac{1}{2} (\partial_\phi \sqrt{-g}) \phi^r \phi_r = \frac{2\pi G}{\sqrt{-g}} (\partial_\phi \sqrt{-g}) T_{rr} (\phi) = 2\pi G (\partial_\phi \ln (\sqrt{-g})) T_{rr} (\phi) \]

(45)

i.e., the RHS is the trace of the stress tensor density.

By introducing the conformal factor

\[ \sqrt{-g} = e^{-2\phi/\phi_0} \]

(46)

they simplify to (covariant d’Alambert equation)

\[ \nabla_r (\sqrt{-g} \phi^r) = \nabla_r (\phi^r) = \Box (\phi) = -\frac{4\pi G}{\phi_0} T_{rr} (\phi) = -\frac{4\pi G}{\phi_0} \sqrt{-g} T_{rr} (\phi) \]

(47)

where we have introduced the scalar field density \( \phi^i \); the dual field to \( \phi_i \). Hence, the metric \( g_{ij} \) is a constitutive relation describing the properties of the space-time continuum

\[ \phi^i = \sqrt{-g} g^{ir} \phi_r \]

(48)

between the abstract mathematical field \( \phi_i \) and measurable physical field \( \phi^i \).

By inserting the stress tensor Eq. (40) into the R.H.S. of Einstein’s field Eqs. one obtains the algebraic relation

\[ R^i_j - \frac{1}{2} \delta^i_j R = \kappa T^i_j = -\frac{2}{\phi_0} \left( \phi^i \phi_j - \frac{1}{2} \delta^i_j \phi^r \phi_r \right) \]

(49)

one obtains the algebraic relation

\[ R^i_j = -\frac{2}{\phi_0} \phi^i \phi_j. \]

(50)
2.2 Static Spherically Symmetric Solution

Generalizing, one may take the trace of the stress tensor of matter as the source of the scalar field. For a static point source, the covariant d’Alambert equation Eq. (47) with the stress tensor trace for a perfect fluid Eq. (28) has the solution

\[ \phi = -\frac{GM}{r}, \quad r > 0 \]  

(51)

which were simplified by introducing the conformal factor Eq. (46)

\[ \sqrt{-g} = e^{-2\phi/c^2_0}. \]  

(52)

The scalar field stress tensor is Eq. (40)

\[ T^i_j(\phi) = -\frac{1}{4\pi G} \left( \phi^i \phi_j - \frac{1}{2} \delta^i_j \phi^r \phi_r \right) = \frac{GM^2}{8\pi} \begin{pmatrix} \frac{2\phi/c^2_0}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{2\phi/c^2_0}{r^4} & 0 & 0 \\ 0 & 0 & \frac{2\phi/c^2_0}{r^4} & 0 \\ 0 & 0 & 0 & \frac{2\phi/c^2_0}{r^4} \end{pmatrix}, \quad T(\phi) = \frac{GM^2 e^{2\phi/c^2_0}}{4\pi r^4}. \]  

(53)

From the Christoffel symbols \( \Gamma^i_{jk} \) from tensor calculus

\[ R_{ij} = \partial_r \Gamma^r_{ij} - \partial_j \Gamma^r_{ir} + \Gamma^r_{ij} \Gamma^s_{rs} - \Gamma^s_{ir} \Gamma^r_{js}, \]
\[ \Gamma^i_{jk} = \frac{1}{2} g^{ir} (\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}) \]  

(54)

and Einstein’s field equations

\[ T^i_j(\phi) = \frac{1}{\kappa} \left( R^i_j - \frac{1}{2} \delta^i_j R \right) \]  

(55)

one can solve for the metric, which leads to

\[ ds^2 = -e^{2\phi/c^2_0} c^2 dt^2 + e^{-2\phi/c^2_0} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right). \]  

(56)

It is noteworthy that it is free of additional singularities. Also, for a classical field theory (e.g. gravitation, elasticity, electromagnetism, fluid dynamics, and thermodynamics), invariance of the potential \( \phi \) under a group (of transformations) is described by

\[ \phi \rightarrow \phi + \alpha \]  

(57)

where \( \alpha \) is a constant. Generalizing, for a function \( f(\phi) \) to be invariant at two different locations \( i \) and \( j \) requires that

\[ \frac{f(\phi_i)}{f(\phi_j)} = \frac{f(\phi_i + \alpha)}{f(\phi_j + \alpha)} \]  

(58)

which leads to

\[ d \ln (f(\phi_i)) = d \ln (f(\phi_j)) = \text{cst} \]  

(59)

with the solution

\[ f(\phi_i) = f(\phi_j) e^{a(\phi_i - \phi_j)} \]  

(60)

where \( a \) is a constant. Clearly, e.g. the Schwarzschild metric for GR Eq. (11) does not have this property. Hence the introduction of “test particles” (in an “external field”) in GR text books; vs. interacting particles. Alternatively, like Einstein concluded in 1945 [1], one may view it as a leading order theory.

The stress tensor density in mixed form has a particularly simple expression

\[ \mathcal{T}^i_j(\phi) = \frac{\sqrt{g}}{\kappa} \left( R^i_j - \frac{1}{2} \delta^i_j R \right) = \frac{GM^2}{8\pi} \begin{pmatrix} \frac{1}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{r^4} & 0 & 0 \\ 0 & 0 & \frac{1}{r^4} & 0 \\ 0 & 0 & 0 & \frac{1}{r^4} \end{pmatrix}, \quad \mathcal{T}(\phi) = \frac{GM^2}{4\pi r^4}. \]  

(61)
i.e., like a static point charge in electromagnetism. Comparing with the Schwarzschild metric \[19\]

\[
T^{(m)}_{ij} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad T = 0, \quad r > 0
\] (62)

because the GR stress tensor only contains the sources for the gravitational field; not the energy-momentum for the field itself.

Not surprisingly, much has been written on this topic since the inception of the theory; e.g. the Levi-Civita - Einstein controversy 1917 \[10\]. For a perspective, one may compare with electromagnetism, for which the Maxwell stress and electro-magnetic field tensors are given by

\[
\mathcal{T}^{(em)}_{ij} = \frac{1}{\mu_0} \left( F^{ir} F_{jr} - \frac{1}{4} \delta^{ir} F^{rs} F_{rs} \right), \quad F_{ij} = \partial_i A_j - \partial_j A_i
\] (63)

and

\[
\partial_r \mathcal{T}^{(em)} = \mu_0 \mathcal{F}.
\] (64)

For e.g. a point charge \(Q\) the stress tensor is

\[
T^{(em)}_{ij} = \frac{Q^2}{32\pi^2\varepsilon_0} \begin{bmatrix}
-\frac{1}{r^2} & 0 & 0 & 0 \\
0 & -\frac{1}{r^2} & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r^2}
\end{bmatrix}, \quad T^{(em)} = 0
\] (65)

which may be compared with the previous result Eq. (61).

### 2.3 Static Spherically Symmetric Body with Charge: Reissner-Nordström Metric

Because the Maxwell stress tensor has zero trace, the electro-magnetic field is not a source of the scalar field. Hence, it does not affect the metric. However, it can contribute to the rest mass of the body; so called Maxwell stresses.

For GR, Einstein’s field equations Eq. (9) for a charged spherically symmetric body with mass \(M\) and charge \(Q\) are

\[
R_{ij} - \frac{1}{2} g_{ij} R = \kappa T^{(em)}_{ij} = \kappa \frac{Q^2}{32\pi^2\varepsilon_0} \begin{bmatrix}
-\frac{1}{r^2} & 0 & 0 & 0 \\
0 & -\frac{1}{r^2} & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r^2}
\end{bmatrix}
\] (66)

where the R.H.S is the Maxwell stress tensor for a point charge Eq. (65). The solution is \[27, 28\]

\[
ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)c_0^2 dt^2 + \frac{1}{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2
\] (67)

where

\[
r_s = \frac{2GM}{c_0^2}, \quad r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c_0^4}
\] (68)

which has additional singularities for

\[
r = \frac{1}{2} \left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right).
\] (69)

### 2.4 Lagrangian Formulation and Post-Newtonian Approximation

A Lagrangian formulation is obtained from the Lagrangian

\[
S = \int L dt = -\int \frac{m_0 c_0^2}{\gamma} dt = -\int m_0 c_0^2 \frac{ds}{c_0 dt} dt
\] (70)
where one obtains the Post-Newtonian approximation for the Schwarzschild metric in isotropic form \[30\]

In 1913 G. Nordström developed a self-consistent scalar field theory for gravity with Minkowski metric to analytic mechanics, what is experimentally known for GR can be accounted for. Hence, by introducing this Lagrangian which agrees with Eq. (74) to leading order in the gravitational constant.

A variation of the field gives

\[\partial_\phi \mathcal{L} - \partial_r (\partial_\phi, \mathcal{L}) = 0\]

which leads to the field equation

\[\Box \phi = -4\pi G m_0\]

and the scalar field stress tensor

\[T^{ij}(\phi) = -\frac{1}{4\pi G} \left( \phi^i \phi^j - \frac{1}{2} \eta^{ij} \phi^r \phi_r \right) = -\phi^i \partial_\phi \mathcal{L} + \eta^{ij} \mathcal{L}\]

2.5 Generalized Nordström Theory

In 1913 G. Nordström developed a self-consistent scalar field theory for gravity with Minkowski metric $\eta_{ij}$ \[15\]. To quote A. Pais \[16\]:

"Though it was not to survive, it deserves to be remembered as the first logically consistent relativistic field theory of gravitation ever formulated."

His field equations can be generated by the Lagrangian density \[15\] \[17\]

\[\mathcal{L} = \frac{1}{8\pi G} \phi^r \phi_r - m_0 c_0 \left( 1 + \phi^2 \right) \sqrt{-u_r u^r} = \frac{1}{8\pi G} \phi^r \phi_r - c_0 \sqrt{-p_r p^r}, \quad p^i \equiv m_0 \left( 1 + \phi^2 \right) u^i.\]

A variation of the field gives

\[\partial_\phi \mathcal{L} - \partial_r (\partial_\phi, \mathcal{L}) = 0\]

which leads to the field equation

\[\Box \phi = -4\pi G m_0\]

and the scalar field stress tensor

\[T^{ij}(\phi) = -\frac{1}{4\pi G} \left( \phi^i \phi^j - \frac{1}{2} \eta^{ij} \phi^r \phi_r \right) = -\phi^i \partial_\phi \mathcal{L} + \eta^{ij} \mathcal{L}\]
with the trace

\[ T^{(\phi)} = -\frac{1}{4\pi G} \phi^r \phi_r. \]  \hfill (82)

Similarly, the matter stress tensor is

\[ T_{ij}^{(m)} = -m_0 \left( 1 + \frac{\phi}{c_0^2} \right) u^i u^j \]  \hfill (83)

with the trace

\[ T^{(m)} = m_0 c_0^2 \left( 1 + \frac{\phi}{c_0^2} \right). \]  \hfill (84)

Hence, the field equations can be written

\[ \square \phi = -\frac{4\pi G}{c_0^2} \frac{T^{(m)}}{1 + \frac{\phi}{c_0^2}} \]  \hfill (85)

The divergence of the total stress tensor is zero (equations of motion)

\[ \partial_r T^{rj}_{\text{tot}} = \partial_r \left( T^{rj}_{\phi} + T^{rj}_{m} \right) = 0. \]  \hfill (86)

The field equations may be compared with Eq. (47) for the generalized theory (in curved space-time)

\[ \square \phi = -\frac{4\pi G}{c_0^2} \frac{T^{(m)}}{1 + \frac{2\phi}{c_0^2} + \ldots}. \]  \hfill (87)

generated by the Lagrangian density

\[ L = \sqrt{-g} \left( \frac{1}{8\pi G} \phi^r \phi_r - m_0 c_0 \sqrt{-u^i u^i} \right) = e^{-\phi/c_0^2} \left( \frac{1}{8\pi G} \phi^r \phi_r - m_0 c_0 \sqrt{-u^i u^i} \right) \]  \hfill (88)

and the metric Eq. (56)

\[ ds^2 = -e^{2\phi \phi/c_0^2} dt^2 + e^{-\phi/c_0^2} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right). \]  \hfill (89)

### 3 Metric: Constitutive Relations for the Space-Time Continuum

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."
A. Einstein (1921) [31].

#### 3.1 Metric for the Space-Time Continuum

In 1907 A. Einstein concluded that the finite transformation for time dilatations must be translation invariant (section 18) [32]

\[ \Delta t = \Delta t_0 e^{a \Delta x/c_0^2} \]  \hfill (90)

where \( a = \text{const.} \) is the acceleration. But oddly, when generalizing to GR, limited the considerations to the infinitesimal part, his Eq. (30)

\[ \Delta t = \Delta t_0 \left( 1 + \frac{a \Delta x}{c_0^2} + \ldots \right), \]  \hfill (91)

by using his Eq. (30a); although GR is a fundamentally nonlinear theory. Similarly, in 1911 he obtained the infinitesimal transformation for the redshift due to a gravitational field [33]

\[ f = f_0 \left( 1 + \frac{\phi}{c_0^2} + \ldots \right). \]  \hfill (92)

The finite transformation is [25]

\[ f = f_0 e^{\phi/c_0^2} = f_0 \left( 1 + \frac{\phi}{c_0^2} + \ldots \right). \]  \hfill (93)
Similarly, the rest-mass is not conserved in a gravitational field; only mass ratios. Starting from the SR relation
\[ E = m_0 c_0^2, \tag{94} \]
Nordström obtained for an accelerated frame \[ m c_0^2 = m_0 c_0^2 e^{\phi/c_0^2}. \tag{95} \]
Alternatively, the same result is obtained for the redshift Eq. (93) of a de Broglî (“matter”) wave \[ E = h \nu, \tag{96} \]
where \( h \) is Planck’s constant, which leads to
\[ E = m c_0^2 = h \nu = h \nu_0 e^{\phi/c_0^2} = m_0 c_0^2 e^{\phi/c_0^2}. \tag{97} \]
Also, in the 1911 paper he concluded that
\[ c(\phi) = c_0 \left( 1 + \frac{\phi}{c_0^2} \right). \tag{98} \]
Actually, since for light \( ds = 0 \), the Schwarzschild metric in isotrophic form Eq. (73) gives for the local speed of light (section 83 (c)) \[ c(\phi) = \frac{dr}{dt} = c_0 \left( 1 + \frac{2\phi}{c_0^2} + \ldots \right) \tag{99} \]
a factor 2 difference; because there is a length contraction as well as a time dilatation (observed by “radar echo delay” experiments \[ 34 \]). Similarly, for the exponential metric one obtains
\[ c(\phi) = \frac{dr}{dt} = e^{2\phi/c_0^2} c_0 = c_0 \left( 1 + \frac{2\phi}{c_0^2} + \ldots \right) \tag{100} \]
i.e., they agree to leading order. The exponential metric can be written in the form
\[ ds^2 = -e^{2\phi/c_0^2} c_0^2 dt^2 + e^{-2\phi/c_0^2} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right) = -e^{-2\phi/c_0^2} \left( c^2(\phi) dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2 \right) \tag{101} \]
for which its conformal structure becomes transparent.

### 3.2 Light Propagation: Fermat’s Principle

The Lagrangian for light propagation is (Fermat’s principle)
\[ S = \int c_0 dt = \int \frac{c_0}{c} ds = \int n ds \tag{102} \]
where \( n \) is the refractive index
\[ n \equiv \frac{c_0}{c} \tag{103} \]
To evaluate the bending of light, Einstein in his 1916 paper \[ 13 \] used the local speed of light obtained for \( ds = 0 \)
\[ c(\phi) = \frac{dr}{dt} = c_0 \left( 1 - \frac{2GM}{c_0^2 r} + \ldots \right) \tag{104} \]
where we have used the Schwarzschild metric in isotrophic form Eq. (73). If the light moves along the \( x \)-axis in the \( z = 0 \) plane the transverse differential deflection is
\[ \frac{d\theta}{dx} = \frac{1}{c(\phi)} \frac{dc(\phi)}{dy} \tag{105} \]
which gives
\[
\frac{d\theta}{dx} \approx \frac{1}{c_0} \frac{dc}{dy} = \frac{2GM}{c_0^2} \frac{R + y}{[x^2 + (R + y)^2]^{3/2}} \approx \frac{2GM}{c_0^2} \frac{R}{[x^2 + R]^2} \tag{106}
\]

where \( R \) is the distance of closest approach. An integration over \( x \) gives
\[
\theta = \frac{\int_{-\infty}^{\infty} d\theta}{dx} dx = \frac{2GM}{c_0^2} \frac{x}{R\sqrt{x^2 + R^2}} \biggr|_{-\infty}^{\infty} = \frac{4GM}{c_0^2 R} \tag{107}
\]
as expected.

### 3.3 Phenomenological Approach: Polarizable Vacuum Interpretation

The Lagrangian density for the electromagnetic field is
\[
\mathcal{L}_{em} = -\frac{1}{4\mu_0} \mathcal{F}^{rs} F_{rs} \tag{108}
\]
where (constitutive relation for vacuum)
\[
\mathcal{F}^{ij} = \frac{\sqrt{-g}}{\mu_0} g^{ir} g^{js} F_{rs} \quad c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \tag{109}
\]

From the relations
\[
D = \varepsilon_0 E, \quad H = \frac{1}{\mu_0} B \tag{110}
\]
for the conformal metric Eq. (101) one may introduce (vacuum polarization)
\[
\varepsilon_0 \rightarrow \varepsilon_0 e^{-\phi/c_0^2}, \quad \mu_0 \rightarrow \mu_0 e^{-\phi/c_0^2} \tag{111}
\]
and it follows that
\[
c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \rightarrow c(\phi) = c_0 e^{\phi/c_0^2}. \tag{112}
\]

Landau-Lifshitz made a similar analysis (section 90) [18]. However, because the latter is based on the approximate relations
\[
F^{ij} = \frac{1}{\mu_0} g^{ir} g^{js} F_{rs}, \quad D^\alpha = \sqrt{-g_0} g^{\alpha\beta} F^{\alpha\beta}, \quad H^{\alpha\beta} = -\sqrt{-g_0} \varepsilon^{\alpha\beta\gamma}, \quad H_\alpha = -\frac{1}{2} \sqrt{\frac{g}{g_0}} \varepsilon_{\alpha\mu
u} H^{\mu\nu} \tag{113}
\]
vs. the constitutive relation Eq. (109), it leads to
\[
\varepsilon_0 \rightarrow \frac{\varepsilon_0}{\sqrt{-g_0}} = \varepsilon_0 e^{-\phi/c_0^2}, \quad \mu_0 \rightarrow \frac{\mu_0}{\sqrt{-g_0}} = \mu_0 e^{-\phi/c_0^2}, \quad c_0 \rightarrow c(\phi) = c_0 e^{\phi/c_0^2} = c_0 \left(1 + \frac{\phi}{c_0^2} + \ldots\right) \tag{114}
\]
vs. Eq. (99) and (100).

From the redshift Eq. (97) it follows that
\[
m_0 c_0^2 \rightarrow m(\phi) c^2(\phi) = \frac{m_0 c_0^2}{\sqrt{-g}} = m_0 c_0^2 e^{\phi/c_0^2} \tag{115}
\]
but mass ratios are preserved. Similarly, the Compton wavelength is reduced by
\[
\lambda_c = \frac{\hbar}{m_0 c_0} \rightarrow \frac{\lambda_c}{\sqrt{-g}} = \lambda_c e^{\phi/c_0^2}. \tag{116}
\]

However, the “vacuum impedance” remains invariant
\[
Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} \rightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{117}
\]
so the fine structure constant does not change

\[ \alpha \equiv \frac{e_0^2}{4\pi\varepsilon_0 c_0 \hbar} = \frac{e_0^2 Z_0}{4\pi \hbar}. \]  

Collecting the results, from the SR Lagrangian Eq. (70)

\[ S = \int L dt = -\int m_0 c_0^2 \frac{d\gamma}{dt} dt = -\int m_0 c_0^2 \frac{d\gamma}{dt} dt. \]  

the generalized Lagrangian is simply

\[ L = -\frac{m_0 c_0^2}{\gamma} \rightarrow m(\phi) c^2(\phi) \sqrt{1 - \frac{c^2(\phi)}{c_0^2}} \sqrt{1 - \left( \frac{\nu}{c_0 e^{2\phi/ c_0}} \right)^2} \]

\[ = -\frac{m_0 c_0^2}{\gamma} \left( e^{2\phi/ c_0} - e^{-2\phi/ c_0} \frac{\nu}{c_0} \right)^2 \]

in agreement with the earlier obtained Eq. (77).

The equations of motion are

\[ \frac{d}{dt} (\partial_\mu L) - \partial_\mu L = \frac{d\bar{\rho}}{dt} + m_0 \left[ 1 + \left( \frac{\nu}{c_0 e^{2\phi/ c_0}} \right)^2 \right] e^{2\phi/ c_0} \gamma_\phi \nabla \phi = 0 \]  

where we have introduced

\[ \bar{\rho} \equiv m_0 \gamma_\phi \bar{e} e^{-2\phi/ c_0}, \quad \gamma_\phi \equiv \frac{1}{e^{\phi/ c_0} \sqrt{1 - \left( \frac{\nu}{c_0 e^{2\phi/ c_0}} \right)^2}}. \]

To summarize:

- The gravitational effects on Maxwell’s equations can be interpreted as a “vacuum polarization” by the gravitational field governed by Eqs. (111) and (112).

- The theory is conformally invariant \[36\] but not scale invariant (a dilatation symmetry); because the relativistic fluid stress tensor has non-zero trace Eq. (28). Hence, e.g. the “vacuum impedance” and fine structure constant are unaffected whereas the Compton wavelength is changed; by the local gravitational field.

- The generalized Lagrangian can be obtained by: either by introducing the conformal metric Eq. (101) or by retaining the Minkowski (flat) space-time but with a varying the local speed of light and rest masses governed by Eqs. (112) and (115); leading to the same Lagrangian Eqs. (77) and (120). A matter of perspective.

- GR can be accounted for in analytic mechanics by introducing the generalized Lagrangian.

The latter approach dates back to H. Wilson 1921 and R. Dicke 1957 \[37,38\]. One may compare with e.g. fluid dynamics: Euler vs. Lagrange (moving frame) formulation. Or the theory of elasticity: the stress tensor in the original coordinates before vs. after deformation; due to the introduction of strain, see e.g. the Cosserat brothers 1909 \[39\].

### 4 Lagrangian Density, Field Equations, and Equations of Motion

The Lagrangian density \( \mathcal{L} \) for the scalar, matter, and electromagnetic fields is

\[ \mathcal{L} = \sqrt{-g} \left( \frac{1}{8\pi G} g^{rs} \phi_r \phi_s - \frac{1}{2} \rho g^{rs} u_r u_s - \frac{1}{4\mu_0} g^{rs} g^{tu} F_{rt} F_{su} \right) \]

\[ = \frac{1}{8\pi G} e^{-2\phi/ c_0} \phi^r \phi_r - \frac{1}{2} \rho e^{-2\phi/ c_0} u^r u_r - \frac{e^{-2\phi/ c_0}}{4\mu_0} F^{rs} F_{rs} \]
and
\[ p^i \equiv \sqrt{-g} u^i = \rho e^{-2\phi/c^2} u^i \] (123)

with the conformal metric Eq. (101) (the constitutive relations for space-time continuum)
\[
\begin{align*}
ds^2 &= -e^{2\gamma/\tilde{c}^2} c_0^2 dt^2 + e^{-2\gamma/\tilde{c}^2} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right) \\
&= -e^{2\gamma/\tilde{c}^2} \left( c^2(\phi) dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2 \right) \\
&= - \left( e^{2\gamma/\tilde{c}^2} - \frac{\tilde{c}^2}{c_0^2} e^{-2\gamma/\tilde{c}^2} \right) c_0^2 dt^2 = -e^{2\gamma/\tilde{c}^2} \left( 1 - \frac{\tilde{c}^2}{c_0^2} e^{-2\gamma/\tilde{c}^2} \right) c_0^2 dt^2 = -\frac{1}{\gamma^2} \frac{c_0^2}{\tilde{c}^2} dt^2 \\
\end{align*}
\] (124)

where
\[ \gamma^2(\phi) = \frac{1}{e^{2\gamma/\tilde{c}^2} \left( 1 - \frac{\tilde{c}^2}{c_0^2} e^{-2\gamma/\tilde{c}^2} \right)} \] (125)

The equations of motion Eq. (129) can be written
\[
\begin{align*}
p^i \nabla_r (\sqrt{-g} u_i) &= \frac{d}{ds} (\sqrt{-g} u_i) - \Gamma_{irs} \left( \frac{\sqrt{-g}}{2} g^{ir} u^s + \frac{\gamma^2}{c_0^2} \frac{\sqrt{-g}}{2} g^{ir} \partial_r \ln(\rho) \right) \\
&= \frac{d}{ds} (\sqrt{-g} u_i) - \frac{\sqrt{-g}}{2} (\partial_i g_{rs}) u^r u^s + \frac{\gamma^2}{c_0^2} \frac{\sqrt{-g}}{2} \partial_i \ln(\rho) = 0. \\
\end{align*}
\] (126)

For a particle with mass \( m_0 \)
\[
\begin{align*}
\frac{d}{ds} (\sqrt{-g} u_i) - \frac{\sqrt{-g}}{2} (\partial_i g_{rs}) u^r u^s &= \frac{1}{m_0} \frac{dp_i}{dt} + \frac{1}{2} (\partial_i g_{rs}) u^r u^s = \gamma^2 \frac{\gamma^2}{m_0 c_0^2} \frac{dp_i}{dt} - \frac{\gamma^2}{2 c_0^2} (\partial_i g_{rs}) \frac{dx^r}{dt} \frac{dx^s}{dt} \\
&= \gamma^2 \frac{\gamma^2}{m_0 c_0^2} \frac{dp_i}{dt} + \gamma^2 \frac{e^{2\gamma/\tilde{c}^2}}{c_0^2} \left[ 1 + \left( \frac{v}{c_0 e^{2\gamma/\tilde{c}^2}} \right) \right] \partial_i \phi = 0 \\
\end{align*}
\] (127)

it follows that
\[
\begin{align*}
\frac{dp_i}{dt} + m_0 \left[ 1 + \left( \frac{v}{c_0 e^{2\gamma/\tilde{c}^2}} \right) \right] e^{2\gamma/\tilde{c}^2} \gamma^2 \partial_i \phi = 0 \\
\end{align*}
\] (127)

i.e., the same result as obtained from the corresponding Lagrangian Eq. (127) or (129) and the equations of motion Eq. (129).

The field equations for the scalar field are
\[
\Box \phi = -\frac{4\pi G}{c_0^2} \phi^r \phi_r + 4\pi G \rho e^{-2\phi/c^2} + \frac{4\pi G}{c_0^2} \varepsilon_0 e^{-2\phi/c^2} \frac{F^r s F_{rs}}{2}. \\
\] (128)

Since the trace for the stress tensors for the scalar and electromagnetic fields are zero (massless), by a suitable choice of gauge
\[ \phi \rightarrow \phi + f(x) \] (129)

and
\[ F_{ij} \equiv \partial_i A_j - \partial_j A_i, \quad A_i \rightarrow A_i + \partial_i g(x) \] (130)

they simplify to
\[ \Box \phi = 4\pi G \rho e^{-2\phi/c^2}. \] (131)

Maxwell’s equations are equivalent to a medium (space-time continuum) with local speed of light \( c(\phi) \).
5 Conclusions

By introducing a scalar field/potential, and relating it algebraically to the Ricci tensor, using Einstein’s field equations, a metric with an exponential dependance on the potential is obtained; which is free of additional singularities. Hence, although the theory is fundamentally nonlinear, the scalar field/potential provides an analytic framework for interacting particles described by linear superposition. The stress tensor for the scalar field includes both the sources of, i.e., matter, and the energy-momentum for the gravitational field. The stress tensor has zero covariant and ordinary divergence. Hence, the energy-momentum for the gravitational field and sources are conserved. The theory’s predictions agree with the experimental results for General Relativity. By introducing the corresponding Lagrangian in analytic mechanics, what is experimentally known for GR can be accounted for. The theory is essentially a generalization of Nordström’s theory \[15–17\]; replacing the Minkowski (flat) metric with the exponential (curved) metric Eq. (56). Alternatively, the Minkowski metric can be retained by introducing a “vacuum polarization” due to the gravitational field that leads to varying local speed of light and rest masses. The theory is conformally invariant but not scale invariant (a dilatation symmetry); e.g. the “vacuum impedance” and fine structure constant are unaffected but the Compton wavelength is changed by the local field.

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