Log-amplitude statistics of non-Gaussian fluctuations

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Abstract. To quantify heavy-tailed distributions observed in financial time series, we propose a general method to characterize symmetric non-Gaussian distributions. In our approach, an observed time series is assumed to be described by the multiplication of Gaussian and amplitude random variables, where the amplitude variable describes fluctuations of the standard deviation. Based on this framework, it is shown that statistical properties of the log-amplitude fluctuations can be estimated using the logarithmic absolute moments of the observed time series.

1. Introduction
Recently, using statistical tools developed in nonequilibrium statistical physics, many studies have been conducted to establish universal properties in the financial market dynamics [1–3]. A remarkable property of the financial time series is inhomogeneity of variance, which results in non-Gaussian probability density functions (PDFs). To characterize such non-Gaussian fluctuations, it has been demonstrated that non-Gaussian PDFs are often described effectively by a superposition of Gaussian distributions with fluctuating variances [4; 5].

In the study of the velocity difference between two points in fully developed turbulent flows, Castaing et al. [4] introduced the following PDF:

\[ P(x) = \int_{0}^{\infty} \frac{1}{\sigma} P_L\left(\frac{x}{\sigma}\right) G(\ln \sigma) d(\ln \sigma), \]

(1)

where it is assumed that \( P_L \) is the Gaussian distribution and \( G \) is a distribution describing the fluctuation of the standard deviations. This PDF has been also applied to econophysics [1; 2], geophysics [6], and physiology [7]. However, in the previous studies, the estimation of the statistical properties of \( G \) depends on a priori knowledge of the functional form of \( G(\ln \sigma) \) such as log-normality, which would limit the applicability of this model. Thus, it is required to develop a method for determining statistical properties of \( G \) from the observed data.

As pointed out by Jung and Swinney [8], equation (1) can be linked with Beck and Cohen’s superstatistics [5] that has also been applied to a wide range of nonequilibrium systems (see, [5] and references therein). Hence, a general problem that is of great interest in experimental applications is how to objectively characterize the variance fluctuations described by \( G \) in Eq. (1). To solve this problem, we propose a novel method for estimating the variance and higher moments of \( G \) from the observed time series without assumptions on the shape of \( G \).
2. Log-amplitude variance and higher moments

To explain our approach, let us assume that an observed stationary time series \( \{x_i\} \) with zero mean is described by a multiplicative stochastic process,

\[
X_i = W_i e^{Y_i},
\]

where \( W \) is a Gaussian random variable with zero mean, and \( Y \) is the other random variable independent of \( W \). In this case, the PDF of \( X \) has the same functional form as Eq. (1), where \( G(y) \) is the PDF of \( Y \). Here, we refer to \( \{Y_i\} \) as the log-amplitude fluctuation.

To characterize the log-amplitude fluctuation, we consider the variance and higher central moments of \( Y \),

\[
\mu_n = \langle (Y - \langle Y \rangle)^n \rangle,
\]

(3)

where \( \langle \cdot \rangle \) denotes the statistical average. By the calculation of the logarithmic absolute moments of \( X \) and the assumption of Eq. (2), we can obtain the following relations for \( \mu_n \):

\[
\mu_2 = \langle (\ln |X| - \langle \ln |X| \rangle)^2 \rangle - \frac{\pi^2}{8},
\]

(4)

\[
\mu_3 = \langle (\ln |X| - \langle \ln |X| \rangle)^3 \rangle + \frac{7}{4} \zeta(3),
\]

(5)

\[
\mu_4 = \langle (\ln |X| - \langle \ln |X| \rangle)^4 \rangle - \frac{3}{4} \pi^2 \mu_2 - \frac{7}{64} \pi^4,
\]

(6)

where \( \zeta(n) \) is the Riemann zeta function (\( \zeta(3) = 1.2020569 \cdots \)). Note that the logarithmic absolute moments of \( X \) do not depend on the variance of \( X \), because

\[
\ln |\sigma_0 X| - \langle \ln |\sigma_0 X| \rangle = \ln |X| - \langle \ln |X| \rangle,
\]

(7)

where \( \sigma_0 \) is an arbitrary constant. Furthermore, we can obtain the estimator of the higher-order moment, if required.

Our key idea is to calculate logarithmic absolute moments. If logarithmic absolute moments of \( X \) are finite, it is possible to define the log-amplitude moments of \( X \). Even in the case where the PDF of \( X \) has power-law tails, \( P(x) \sim |x|^{-\alpha} \), with \( 1 < \alpha \), the logarithmic absolute moments are finite, although the variance is undefined or infinite. Therefore, our approach can characterize a wide range of symmetric unimodal distributions with heavy tails. In other words, the log-amplitude variance \( \mu_2 \) [Eq. (4)] is simply interpreted as the difference in the second logarithmic-absolute moment between the observed non-Gaussian and Gaussian PDFs, because the second logarithmic-absolute moment of a Gaussian distribution equals \( \pi^2/8 \). Thus, \( \mu_2 \) can be used as a measure of the deviation from a Gaussian distribution.

3. Numerical examples

To test our approach, we introduce illustrative examples of non-Gaussian stochastic processes and carry out numerical experiments. Here, we consider a stochastic process described by independent and identically distributed (IID) variables. The first example is a multiplicative log-normal process [9] based on experimental observations in the study of the turbulent velocity field [4], solar wind [6], foreign exchange rate [1], stock index [2], and human heartbeat [10; 11]. Neglecting the detailed structure of the intermittent dynamics, we mimic the PDFs observed in the log-normal processes. For comparison, we also introduce a multiplicative log-Poisson process. In the log-normal and log-Poisson processes, fluctuations of the standard deviations are assumed to obey log-normal and log-Poisson distributions, respectively.

In the log-normal process \( \{X_i^{(LN)}\} \), the standardized random variable \( X^{(LN)} \) with zero mean is described by

\[
X^{(LN)} = e^{-a^2} W e^{aY},
\]

(8)
Figure 1. (a) Standardized PDFs of log-normal (\(X^{(LN)}\)), log-Poisson (\(X^{(LP)}\)), and superstatistical (\(X^{(SS)}\)) IID processes with the same log-amplitude variance \(\mu_2\), where \(X^{(LN)}\), \(X^{(LP)}\), and \(X^{(SS)}\) are defined by Eqs. (8), (9), and (12), respectively. Solid lines: \(X^{(SS)}\) with \(k = 3\) (top) and \(k = 6\) (bottom); dashed lines: \(X^{(LP)}\) with \(\lambda = 20\) and \(r = \sqrt{\mu_2}/\lambda\); solid lines: \(X^{(LN)}\) with \(\alpha = \sqrt{\mu_2}\); dot-dashed lines: \(X^{(LP)}\) with \(\lambda = 20\) and \(r = -\sqrt{\mu_2}/\lambda\). The PDFs are shifted in vertical directions for convenience of presentation. (b) Double-logarithmic plot of (a).

where both \(W\) and \(Y\) are independent standard Gaussian random variables with zero mean and unit variance. In this process, the non-Gaussian nature is determined by the parameter \(\alpha\), which is the same as a non-Gaussian parameter \(\lambda\) defined in Ref. [9]. In this case, its log-amplitude moments are \(\mu_2 = \alpha^2\), \(\mu_3 = 0\) and \(\mu_4 = 3\alpha^4\).

In the log-Poisson process \(\{X_i^{(LP)}\}\), the standardized random variable \(X^{(LP)}\) is described by

\[
X^{(LP)} = \exp\left(-\lambda (\exp(2r) - 1)/2\right)W e^{rP},
\]

where \(W\) are independent standard Gaussian random variables, \(P\) are independent Poisson random variables with mean \(\lambda\) and variance \(\lambda\), \(r\) is a real valued parameter. In this case, its log-amplitude moments are \(\mu_2 = r^2\lambda\), \(\mu_3 = r^3\lambda\), and \(\mu_4 = r^4\lambda(1 + 3\lambda)\).

The next example is a stochastic process \(\{X_i^{(SS)}\}\) based on so-called superstatistics [5]. Superstatistics considers an inhomogeneous driven nonequilibrium system that consists of many subsystems with different values of some intensive parameter \(\beta\). Each subsystem is assumed to reach local equilibrium very quickly. In this case, if the local equilibrium distribution is Gaussian, we obtain

\[
P(x) = \int_0^\infty \sqrt{\beta} P_L\left(\sqrt{\beta}x\right) f(\beta) d\beta,
\]

where \(P_L(x)\) is the standard normal distribution and \(f(\beta)\) is the distribution of \(\beta\). If \(f(\beta)\) is a log-normal distribution, the PDF of superstatistics has the same form as the above log-normal process [8].

Here, we choose the \(\chi^2\) distribution with degree \(k\),

\[
f(\beta) = \frac{1}{\Gamma(k/2)} \left(\frac{k}{2\beta_0}\right)^{k/2} \beta_0^{k/2-1} e^{-\frac{k\beta}{2\beta_0}},
\]

which is one of the universality classes proposed in superstatistics [5]. In this case, equation (10) results in the Student’s \(t\)-distribution, which exhibits power-law tails, \(P(x) \sim |x|^{-(k+1)}\) for
Figure 2. Estimation of $\mu_n$ for log-normal ($X^{(LN)}$, circles), log-Poisson ($X^{(LP)}$, triangles), and superstatistical ($X^{(SS)}$, squares) IID processes, where $\lambda = 20$ for $X^{(LP)}$. The sample means of $\mu_n$ were estimated from 100 samples of length $N = 10^6$. The error bars indicate the sample standard deviation. The solid lines indicate the theoretical predictions.

large $|x|$. In this superstatistical process, the random variable $X^{(SS)}$ is described as

$$X^{(SS)} = W \sqrt{\frac{k}{\beta_0 Q}}$$

(12)

where $W$ are independent standard Gaussian random variables, $Q$ are independent $\chi^2$ random variables with $k$ degrees of freedom. When $k > 2$, $X^{(SS)}$ can be standardized by $\beta_0 = k/(k - 2)$. In Castaing’s description [Eq. (1)], the corresponding $G(y)$ is defined by

$$G(y) = \frac{2}{\Gamma(k/2)} \exp \left\{ -k \left( y - \frac{1}{2} \ln \frac{k}{2\beta_0} \right) - e^{-2 \left( y - \frac{1}{2} \ln \frac{k}{2\beta_0} \right)} \right\}.$$  

(13)

which is the Gumbel distribution when $k = 2$. In this case, its log-amplitude moments are $\mu_2 = \psi^{(1)}(k/2)/4$, $\mu_3 = -\psi^{(2)}(k/2)/8$, and $\mu_4 = (3 \psi^{(1)}(k/2)^2 + \psi^{(3)}(k/2))/16$, where $\psi^{(n)}(x)$ is $n$th derivative of the Euler’s psi function $\psi(x)$.

It is important to note that the log-amplitude moments $\mu_n$ can be defined for $k > 0$. When $k = 1$, the PDF of $X^{(SS)}$ reduces to a Cauchy distribution,

$$P(x) = \frac{\gamma}{\pi (x^2 + \gamma^2)},$$  

(14)
where the scale parameter $\gamma$ is chosen as $\gamma = 1/\sqrt{\beta_0}$. Because this distribution has power-law tails, $P(x) \sim |x|^{-2}$, for large $|x|$, its second and higher moments are infinite. On the other hand, all of the log-amplitude moments $\mu_n$ are finite, which demonstrates that the log-amplitude statistics is applicable to a variety of heavy tailed distributions.

As shown in Fig. 1, the center parts of the PDFs have similar shapes, if the values of $\mu_2$ are equal. To test our approach, we numerically generate data sets using the above models, and then estimate the value of $\mu_n$. To estimate the logarithmic absolute moments, $\langle (\ln |X| - \langle \ln |X| \rangle)^n \rangle$, in Eqs. (4)-(6), here we use the following estimators,

$$M_n = \frac{1}{N} \sum_{i=1}^{N} (\ln |X_i| - M_1)^n \quad (n = 2, 3, 4),$$

where

$$M_1 = \frac{1}{N} \sum_{i=1}^{N} \ln |X_i|.$$  

As shown in Fig. 2, the theoretical values are estimated well from the observed time series. In particular, in the plot of $\mu_3$ vs $\mu_2$ [Fig. 2 (d)], we find significant differences between the models, although in the plot of $\mu_4$ vs $\mu_2$ [Fig. 2 (e)] we find no differences between the log-normal and log-Poisson processes.

4. Conclusion

We proposed log-amplitude statistics to characterize non-Gaussian time series. Both turbulence statistics by Castaing et al. [4] and superstatistics by Beck and Cohen [5] have been very successful in describing non-Gaussian fluctuations. Including such examples, our method can be used to characterize non-Gaussian fluctuations.

As discussed in the previous section, a crucial advantage in our approach is that a priori knowledge of the variance fluctuations is not assumed. Hence, its increasingly widespread application in the analysis of financial time series is foreseen.

References

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