THE FILIPPOV EQUILIBRIUM AND SLIDING MOTION IN AN
INTERNET CONGESTION CONTROL MODEL

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Abstract. We consider an Internet congestion control system which is pre-
sented as a group of differential equations with time delay, modeling the ran-
dom early detection (RED) algorithm. Although this model achieves success in
many aspects, some basic problems are not clear. We provide the result on the
existence of the equilibrium and the positivity and boundedness of the solution.
Also, we implement the model by route switch mechanism, based on the mini-
num delay principle, to model the dynamic routing. For the simple network
topology, we show that the Filippov solution exists under some restrictions on
parameters. For the case with a single user group and two alternative links,
we prove that the discontinuous boundary, or equivalently the sliding region,
always exists and is locally attractive. This result implies that for some cases
this type of routing may deviate from the purpose of the original design.

1. Introduction. Design of the congestion control of computer networks is receiv-
ing increasing attention as the demand for network service bursts in recent decades.
One central task of the network algorithm design is to find the optimal scheme to
regulate the traffic on the network. The sloppy design of the congestion control
algorithm may lead to disastrous consequences [13, 20]. Due to the complexity
of the network system, undoubtedly it is necessary to look into the problem from
a theoretic point of view to make simplifications and provide suggestions on the
network design.

To give a mathematical description of the process of delivering data packets, two
parallel schemes are developed: the method based on optimization and the contin-
uous flow approximation . The theory of optimization was employed by Kelly [14]
to build a framework in which many important issues related to congestion control
can be tackled uniformly. A nonlinear utility function is constructed and taken as
the objective function to be maximized, subject to a set of linear constraints. For a
well-designed problem, the existence of the globally optimal solution is guaranteed by Karush-Kuhn-Tucker theorem [2]. For a dynamic network system, such solution is tracked by a controller which is designed with the Lyapunov method [21]. As long as the steady state is globally asymptotically stable, the utility and fairness can be realized. Such kind of model has been extensively investigated from various points of view, such as the stability switch induced by parameter uncertainty, nonlinear dynamics and control [23, 24, 25].

However, to establish the model based on the theory of optimization, many details are ignored, some of which may be crucial in understanding the dynamical behavior of the system. Correspondingly, some authors focus on the original discrete TCP (transmission control protocol) and adopt the continuous flow assumption to approximate the process of data transferring. The merit of the treatment is that the model is sufficiently accurate and thus the prediction by such model may agree well with the experiments. In the present work, we consider a mathematical model that describes the RED (random early detection) algorithm [12, 19], by taking the window size of the source user and the queue size at the buffer of the router as its indicative variables. The window size is increased by one packet for each RTT (round trip time), and decreased by an quantity proportional to the current window size and the averaged number of lost packets. The RTT is the sum of the propagation delay and the queueing delay, which is in proportion with the queue size at the buffer. Besides, the function that marks or drops data packets is piece-wise smooth or even discontinuous. As a result, the model is given in terms of a group of differential equations with state-dependent time delay and piece-wise smooth or discontinuous right-hand side functions.

Due to the detailed information characterizing TCP/RED algorithm, such kind of model has been investigated by the authors from various points of view [4, 6]. However, some theoretical problems are still open and one of the most important concerns might be the existence and boundedness of the steady state solution, which greatly challenges the validity of the model. This motivates us to study the existence and boundedness of the equilibrium of the system.

During the recent several years, the dynamic routing has attracted much interest among the research field and industry since this scheme shows more flexibility. For example, for shortest-path routing algorithms with traffic-sensitive routing matrices, the routing table is calculated by measuring the RTT. If the RTT increases significantly during a short period, the primal link may become congested and the dynamic routing algorithm will direct the transmission to the other available links. Actually, for some current dynamic routing algorithms, the RTT is used to select the most preferable link. We attempt to implement such mechanism into the TCP/RED model with the assumption that the routing updates once the condition for the switch is triggered, resulting in a model of switch network system.

Filippov’s construction of differential inclusion is the most typical one to address the problem of the existence of the solution for non-smooth or discontinuous dynamical systems [9, 10, 18]. To analyze the dynamics and bifurcation around the non-smooth or discontinuous border, the vector field analysis together with the construction of Filippov vector field are employed for an intuitive understanding [11, 15, 16]. The singular perturbation [17, 18] and smooth approximation [4] are also used to investigate some problems involving switch network systems.

The main focus of this paper is the steady state and motion induced by the switch mechanism. We use the concept of differential inclusion and vector fields
to study the existence of the equilibrium and the motion near the discontinuous boundary. The rest of the paper is organized as follows. In Section 2, we introduce the underlying model and discuss several basic properties of the solution. In Section 3, the existence of the Filippov equilibrium solution is established for simple network topology. In Section 4, the sliding motion on the discontinuous boundary is investigated and numerical results are shown to provide illustrative examples. Conclusions are summarized in Section 5.

2. Model for TCP/RED with single user group and single link. Misra et al [19] employed the theory of stochastic differential equation to derive the mathematical model that describes the interaction of TCP and RED and provided some simulation results. Based on the model in [19], we ignore the filtering process for simplicity, and the model describing TCP/RED algorithm can be written as

\[
\frac{dW(t)}{dt} = \frac{1}{R(q(t))} - W(t) \frac{W(t - R(q(t)))P(q(t - R(q(t))))}{2R(q(t))},
\]

\[
\frac{dq(t)}{dt} = \frac{NW(t)}{R(q(t))} - C = \begin{cases} 
\frac{NW(t)}{R(q(t))} - C \quad \text{if } q(t) = 0, \\
\frac{NW(t)}{R(q(t))} - C \quad \text{if } 0 < q(t) < B, \\
\frac{NW(t)}{R(q(t))} - C \quad \text{if } q(t) = B,
\end{cases}
\]

where \(W(t)\) represents the window size for delivering data packets, \(q(t)\) the queue size at the buffer of the router, \(P\) the probability of packet marking or dropping, \(C\) the transmission capacity, \(N\) the number of users. In the user group, all the users share the same system resources and consequently can be considered as identical. Thus, we use a single \(W(t)\) to denote the averaged behavior of all users [12, 19]. \([\cdot]^+\) represents \(\max\{\cdot, 0\}\) and \([\cdot]^−\) represents \(\min\{\cdot, 0\}\). \(R(q(t))\) denotes the round trip time or time delay which consists of two parts, namely

\[R(q(t)) = \tau_0 + \frac{q(t)}{C},\]

with the propagation delay \(\tau_0\). Obviously, \(R(q(t))\) is a state-dependent delay. \(P\) is a nondecreasing function and takes value in \([0, 1]\). In [19], \(P\) is given as

\[
P(x) = \begin{cases} 
0 & \text{for } 0 \leq x < b_1 B, \\
\frac{x - b_1 B}{b_2 - b_1} P_{\max} & \text{for } b_1 B \leq x < b_2 B, \\
1 & \text{for } b_2 B \leq x \leq B,
\end{cases}
\]

where \(0 < P_{\max} \leq 1\), \(0 < b_i < 1\) (\(i = 1, 2\)) are RED parameters and \(B\) is the buffer size of the router. To show the configuration of \(P\), an illustrative example with \(b_1 = 0.2\), \(b_2 = 0.5\), \(P_{\max} = 0.3\) and \(B = 50\) is shown in Fig. 1.

The model can be briefly interpreted as follows. The equation of the window size \(W(t)\) describes the TCP dynamics. Namely, \(W(t)\) is increased by one (measured in packets) during each RTT (round trip time), and decreased by an quantity proportional to the current window size and the averaged number of lost packets during the same time period. The motion of \(q(t)\) indicates how the active queue management is implemented to TCP. Namely, the rate of change in the queue size is the difference between the number of newly arrived data packets per time unit during each RTT \((NW(t)/R(q(t)))\) and the speed \((C)\) of processing packets by the router.

The stability and dynamics of the above model have been extensively investigated. For example, in [20], the authors showed that there existed multi-attractors
in (1) by means of the numerical continuation and provided suggestions for the selection of TCP parameters. However, some basic aspects of (1) are still not clear and become topics of discussion in the following.

2.1. The positiveness and boundedness of the solution of (1). To analyze the dynamics in (1), we address the solution properties and the existence of positive equilibrium point of the system. Let $C([-\tau_M, 0], \mathbb{R}^2)$ be the Banach space consisting of all continuous functions from $[-\tau_M, 0]$ to $\mathbb{R}^2$ with norm $\|\phi\| = \sup_{-\tau_M \leq \theta \leq 0} \|\phi(\theta)\|$, $\forall \phi \in C([-\tau_M, 0], \mathbb{R}^2)$, where $\tau_M = \tau_0 + B/C$. Denote the initial conditions $D = \{(\varphi_1(t), \varphi_2(t)) | \varphi_1(t) \geq 0, \varphi_2(t) \geq 0 \text{ for } -\tau_M \leq t \leq 0\}$ and we assume that $D \subset C([-\tau_M, 0], \mathbb{R}^2)$. Following the solution definition given in [3],

**Definition 1.** Rewrite (1) as

$$\frac{d}{dt} X(t) = f(X(t), X(t-\tau(X(t))))$$

where $X(t) = (W(t), q(t))^T$ and $f$ is the right-hand side of (1). Then, with the initial condition in $D$, $X(t)$ is called a Filippov solution of (1) if

a) $X(t)$ is defined on a non-degenerate interval $I$ and absolutely continuous on any closed subinterval of $I$.

b) $\frac{d}{dt} X(t) \in \bigcap_{\delta > 0} \bigcap_{\mu(M_1) = 0} \co f(B(X(t), \delta) \setminus M_1, B(X(t-\tau(X(t))), \delta) \setminus M_2)$,

for almost all $t \in I$, where $\co$ represents the closure of the convex hull, $B(\cdot)$ is the ball centered at the first component with the second component as its radius, and $\mu$ represents the Lebesgue measure.

We can check that the equilibrium solution of (1) satisfies Definition 1. For the properties of the solutions, we have the following theorem.

**Theorem 1.** All the solutions in (1) with the initial condition in $D$ are nonnegative and ultimately bounded, i.e., $0 \leq W(t) \leq \Delta_1$, $0 \leq q(t) \leq \Delta_2$ for all $t > 0$, where $\Delta_1$ and $\Delta_2$ are constants.

**Proof.** For the positivity, $q(t) \geq 0$ is guaranteed by the model, thus we only need to show that $W(t) \geq 0$ for $t > 0$. If not, there exists a finite $t_0 > 0$ such that $W(t) \geq 0$ for $t \in [0, t_0]$ and $W(t_0) = 0$, $\dot{W}(t_0) \leq 0$. However, $\dot{W}(t_0) = \frac{1}{R(q(t_0))} > 0$ which gives a contradiction.
According to the model setup, it is clear that \( q(t) \leq B = \Delta_2 \). We only need to prove that \( W(t) \) is bounded above. Assume \( W(t) \) is unbounded, then for any positive number \( V \), there must exist a finite \( t_1 \) such that \( W(t) > V \) and \( W(t_1) \geq 0 \).

First, for any \( t \in I_1 = [t_1 - R(q(t_1)) - R(q(t_1 - R(q(t_1)))), t_1] \), integrating the first equation in (1) from \( t \) to \( t_1 \), we have

\[
W(t_1) - W(t) = \int_t^{t_1} \left( \frac{1}{R(q(s))} - \frac{W(s)W(t_1 - R(q(s)))}{2R(q(s - R(q(s))))} \right) P(q(s - R(q(s)))) ds
\]

\[
\leq \int_t^{t_1} \frac{ds}{\tau_0} = \frac{t_1 - t}{\tau_0} \leq \frac{R(q(t_1)) + R(q(t_1 - R(q(t_1))))}{\tau_0} \leq \frac{2(\tau_0 + B/C)}{\tau_0} = 2\alpha,
\]

where \( \alpha = 1 + \frac{B}{C} \). From (3), \( W(t_1) - W(t) \leq 2\alpha \Rightarrow W(t) \geq W(t_1) - 2\alpha \). When \( W(t_1) \geq 2\alpha + \frac{2\Delta_1 \alpha^2}{N} \), we have

\[
\frac{NW(t)}{R(q(t))} - C \geq \frac{NW(t)}{\tau_0 + B/C} - C \geq \frac{N(W(t_1) - 2\alpha)}{\tau_0 + B/C} - C
\]

\[
\geq \frac{2\Delta_1 \alpha^2}{\tau_0} - C = C(\alpha - 1) = \frac{B}{\tau_0} \geq 0,
\]

then \( q(t) \geq 0 \). If \( q(t) \not= B \), for any \( t \in I_2 = [t_1 - R(q(t_1)) - R(q(t_1 - R(q(t_1)))), t_1 - R(q(t_1))] \subset I_1 \), from the second equation of (1), we have

\[
q(t_1 - R(q(t_1))) - q(t_1 - R(q(t_1))) - R(q(t_1 - R(q(t_1)))) = \int_{t_1 - R(q(t_1)) - R(q(t_1) - R(q(t_1)))}^{t_1 - R(q(t_1))} \frac{dq(s)}{ds} ds
\]

\[
> \frac{B}{\tau_0} = B \Rightarrow q(t_1 - R(q(t_1))) > B,
\]

which yields a contradiction. This implies that there exists \( t_a \in I_2 \) such that \( q(t_a) = B \). Then \( q(t_a) = 0 \) and \( q(t) = B \) for \( t \geq t_a \) since we have shown that \( \dot{q}(t) \geq 0 \) for \( t \in I_1 \). Consequently, \( q(t_1) - R(q(t_1)) = B \) and \( P(q(t_1) - R(q(t_1))) = 1 \). Therefore,

\[
\dot{W}(t_1) = \frac{1}{\tau_0 + B/C} - W(t_1) \frac{W(t_1 - R(q(t_1)))}{2(\tau_0 + B/C)} \leq \frac{1}{2(\tau_0 + B/C)}(2 - W(t_1)(W(t_1) - 2\alpha)).
\]

If \( W(t_1) > \frac{2\alpha + \sqrt{4\alpha^2 + 8}}{2} \), then \( \dot{W}(t_1) < 0 \). Let \( V = \max\{2\alpha + \frac{2\Delta_1 \alpha^2}{N}, \alpha + \sqrt{\alpha^2 + 2}\} \), then \( W(t_1) > V \) and \( W(t_1) < 0 \) which contradict with the assumption \( W(t_1) \geq 0 \). Thus \( W(t) \) is bounded above. 

2.2. The existence of positive equilibrium of (1). Denote the equilibrium of (1) by \((W^*, q^*)^T\). We say \((W^*, q^*)^T\) is desirable if \( b_1B < q^* < b_2B \). Then we have the following theorem.

**Theorem 2.** If \( \left(\frac{C\alpha + b_2B}{N}\right)^2 > 2 \), then (1) has a unique positive desirable equilibrium.

**Proof.** Let the right-hand side of (1) be zero, then \((W^*, q^*)\) is determined by

\[
0 = \frac{1}{\tau_0 + q^*/C} - \frac{(W^*)^2 P(q^*)}{2(\tau_0 + q^*/C)}, \tag{4}
\]
\[ 0 = \frac{NW^*}{\tau_0 + q^*/C} - C. \tag{5} \]

From the second equation of \((5)\) we have \(q^* = NW^* - C\tau_0\). Let
\[ L_1(W) = \frac{1}{W^2} - \frac{1}{2}P(NW - C\tau_0). \]

Obviously, \(L_1(W)\) is a continuous decreasing function for \(W \in (\frac{C\tau_0 + b_1B}{N}, \frac{C\tau_0 + b_2B}{N})\). From \(L_1(\frac{C\tau_0 + b_1B}{N}) = N^2/(C\tau_0 + b_1B)^2 > 0\) and \(L_1(\frac{C\tau_0 + b_2B}{N}) < 0\) if the condition \((\frac{C\tau_0 + b_2B}{N})^2 > 2\) is provided, we know there is a unique \(W^*\) that satisfies \(L_1(W^*) = 0\) and
\[ L_1(W^*) = \frac{1}{(W^*)^2} - P(NW^* - C\tau_0), \]
with \(P(x) = \frac{x-b_1B}{(b_2-b_1)B}P_{max}, \frac{C\tau_0 + b_1B}{N} < W^* < \frac{C\tau_0 + b_2B}{N}\). Therefore, \(b_1B < q^* < b_2B\) which implies the equilibrium is desirable.

**Remark 1.** The positive equilibrium is possible to exist for \(b_2B < q(t) \leq B\). This means that at the equilibrium, the probability of packet loss or marking, i.e. \(P(q(t))\), is 1, implying even at the steady state, there are excessive data packets in the buffer of the router and consequently the network system is at high risk of congestion. In other words, the congestion control system is not well designed in this case. Therefore, in the above discussion, we don’t consider such case by assuming that \(q < b_2B\) at the equilibrium.

To study the stability of the equilibrium \((W^*, q^*)^T\), let \(W(t) = a_1e^{\lambda t} + W^*, q(t) = a_2e^{\lambda t} + q^*\) where \(a_1\) and \(a_2\) are constants and \(\lambda\) is the eigenvalue of the linearized system. From \([5]\), we know that for the terms with state-dependent delay, the local linearization method can be employed by treating the delay as a constant at the equilibrium. Namely, let \(W(t-R(q(t))) = W(t) + W^*, q(t-R(q(t))) = q(t) + q^*\). Then the linearized system near \((W^*, q^*)\) can be written as
\[
\begin{align*}
\dot{W}(t) &= \frac{W^* d_3}{2d_1}W(t) + \frac{W^* d_3}{2d_1}W(t-d_1) \\
&\quad - \frac{1}{C} \dot{q}(t) + (W^*)^2 \left( \frac{d_2}{2d_1} - \frac{2d_3}{C} \right) \dot{q}(t-d_1), \\
\dot{q}(t) &= \frac{N}{d_1} W(t) - \frac{NW^*}{C} \dot{q}(t),
\end{align*}
\]
where \(d_1 = \tau_0 + \frac{q^*}{C\tau_0}\), \(d_2 = \frac{P_{max}}{(b_2-b_1)B}\) and \(d_3 = \frac{q^* - b_1B}{(b_2-b_1)B}P_{max}\). Then the characteristic equation is
\[
\Delta(\lambda) = \text{Det} \begin{pmatrix}
\lambda - \frac{W^* d_3}{2d_1} & -\frac{W^* d_3}{2d_1} e^{-\lambda d_1} \\
-\frac{W^* d_3}{d_1} & \lambda + \frac{NW^*}{C} - \frac{(W^*)^2 d_2}{C} e^{-\lambda d_1}
\end{pmatrix} = 0,
\]
which is
\[
\Delta(\lambda) = \lambda^2 + r_1 \lambda + r_2 = 0, \tag{6}
\]
with
\[
r_1 = \frac{(2d_1 N - C d_3 (1 + e^{-\lambda d_1})) W^*}{2Cd_1},
\]
\[
r_2 = \frac{(3d_1 d_3 - C d_2) N (W^*)^2 e^{\lambda d_1} - (d_3 (W^*)^2 - 2) d_1}{2Cd_1^2}. \tag{7}
\]
In general, we know that, if all the roots of (6) have negative real parts, then \((W^*, q^*)\) is locally stable, and is unstable if at least one root has positive real part. Due to the complexity of the system, although we can not provide the explicit conditions to guarantee the stability, we will check this general condition numerically in examples.

3. Dynamic routing and its mathematical description. Now we extend (1) to a general model with single user group and \(n\) links, each of which has a transmission delay \(\tau_k\), capacity \(C_k\), buffer size \(B_k\), marking parameters \(b_{k,1}, b_{k,2}, P_{k,\text{max}}\) and queue size \(q_k(t)\), \(k = 1, 2, \cdots, n\). The topology of the network is then shown in Fig. 2.

For any particular moment, the user will choose only one of the \(n\) links to send data packets, according to the shortest path principle. More specifically, if the RTT of the route by passing through the \(i\)th link is the minimum among all the links, then the user chooses the \(i\)th link to form a route [22]. Namely, at the moment \(t = \tilde{t}\), the condition for selecting the \(i\)th route by the user as the actual route is

\[
R(q_i(\tilde{t})) = \min \{R(q_k(\tilde{t})) | k = 1, 2, \cdots, n\}.
\]

Since the data packets from the source will no longer be accumulated on the other links, we have

\[
\frac{dq_k(t)}{dt} = \begin{cases} 
\left[-C_k\right]^+ = 0 & \text{if } q_k(t) = 0, \\
-C_k & \text{if } 0 < q_k(t) < B_k, \\
\left[-C_k\right]^– = -C_k & \text{if } q_k(t) = B_k,
\end{cases}
\]

where \(k \neq i\). Then the dynamics of the window size of the user and the queue size are governed by

\[
\frac{dX(t)}{dt} = F^i(X(t)),
\]

where

\[
X(t) = (W(t), q_1(t), q_2(t), \cdots, q_n(t))^T,
\]

\[
F^i = (F^i, G^i_1, G^i_2, \cdots, G^i_n)^T
\]
with

\[ F^i = \frac{1}{R(q_i(t))} - W(t) \frac{W(t - R(q_i(t)))P_i(t - R(q_i(t))))}{2R(q_i(t) - R_i(q_i(t)))}, \]

\[ G_i^i = \begin{cases} [NW(t) \frac{R(q_i(t))}{NW(t)} - C_i]^+ & \text{if } q_i(t) = 0, \\ \frac{R(q_i(t))}{NW(t)} - C_i & \text{if } 0 < q_i(t) < B_i, \\ [NW(t) \frac{R(q_i(t))}{NW(t)} - C_i]^+ & \text{if } q_i(t) = B_i, \end{cases} \]

\[ G_j^i = \begin{cases} [-C_j]^+ = 0 & \text{if } q_j(t) = 0, \\ -C_j & \text{if } 0 < q_j(t) < B_j, \\ [-C_j]^+ & \text{if } q_j(t) = B_j, \end{cases} \]

\[ P_i(x) = \begin{cases} 0 & \text{for } 0 \leq x < b_{i,1} B_i, \\ \frac{x}{B_i - b_{i,1}} b_{i,2} - b_{i,1} P_{i,\text{max}} & \text{for } b_{i,1} B_i \leq x < b_{i,2} B_i, \\ 1 & \text{for } b_{i,2} B_i \leq x \leq B_i. \end{cases} \]

The purpose of designing the dynamic routing algorithm is to provide the users with more available resources to improve the QoS (quality of service). Thus it is important to show the performance of the system when it has accesses to all the available links. For this, we combine all the possibilities of the route selection together and consider the following system of differential equations

\[ \frac{dX(t)}{dt} = \sum_{k=1}^{n} \eta_k F^k(X(t)), \]

with

\[ \eta_1 = S_1, \eta_2 = S_2(1 - S_1), \ldots, \eta_{n-1} = S_{n-1} \prod_{j=1}^{n-2} (1 - S_j), \eta_n = \prod_{j=1}^{n-1} (1 - S_j), \]

satisfying \( \sum_{k=1}^{n} \eta_k = 1, \)

\[ S_k = \begin{cases} 1 & \text{if } \tau_k + \frac{q_k(t)}{C_k} = \min \{ \tau + \frac{q_i(t)}{C_i} : l = 1, 2, \ldots, n \}, \\ 0 & \text{else.} \end{cases} \]

When \( n = 2, \) the system becomes

\[ \frac{dX(t)}{dt} = \sum_{k=1}^{2} \eta_k F^k(X(t)), \]

where

\[ \eta_1 = S, \eta_2 = 1 - S, \]

\[ S = \begin{cases} 1 & \text{if } \tau_1 + \frac{q_1(t)}{C_1} < \tau_2 + \frac{q_2(t)}{C_2} \\ 0 & \text{else.} \end{cases} \]
4. The sliding motion of (13). Following the result given in Theorem 2, we know under certain conditions, there exist two desirable equilibrium points in (13), namely, $X^{*1} = (W^{1*}, q^{1*}_1, q^{1*}_2)^T = (W^1, q^1_1, 0)^T$ and $X^{*2} = (W^{2*}, q^{2*}_1, q^{2*}_2)^T = (W^2, 0, q^2_2)^T$. This makes sense because each single link is supposed to work well separately to be considered as a potential resource. In addition, there is a possibility that the two links are utilized simultaneously, as shown in Fig. 3, where the motion of (13) is governed by the joint contribution of $F^1(X(t))$ and $F^2(X(t))$ by noticing that both $q^1(t)$ and $q^2(t)$ are not zero in the long time evolution. This indicates that both of the links are utilized which implies $R(q^1(t)) = R(q^2(t))$ is satisfied. This observation suggests that for this case, the motion of $X(t)$ is actually restricted to the hyperplane defined by $\Sigma = \{(W, q^1, q^2)^T | H(X) = 0, 0 \leq q^1 \leq B_1, 0 \leq q^2 \leq B_2\}$ where

\[
H(X) = \tau_1 + \frac{q^1}{C_1} - \tau_2 - \frac{q^2}{C_2}.
\]
On $\Sigma$, the motion of the system is determined by neither $F^1(X(t))$ nor $F^2(X(t))$ but some combination of them. Rewrite the system as follows

$$\frac{dX}{dt} = \begin{cases} F^1(X), & X \in \Sigma_1, \\ K(X), & X \in \Sigma, \\ F^2(X), & X \in \Sigma_2, \end{cases} \quad (14)$$

where

$$\Sigma_1 = \{(W,q_1,q_2)^T | H(X) < 0, 0 \leq q_1 \leq B_1, 0 \leq q_2 \leq B_2\},$$

$$\Sigma_2 = \{(W,q_1,q_2)^T | H(X) > 0, 0 \leq q_1 \leq B_1, 0 \leq q_2 \leq B_2\}.$$

$K(X)$ is the vector field that governs the evolution of the system in $\Sigma$ and is to be determined. In the following, we will focus our attention on the study of the dynamics around $\Sigma$.

For $X \in \Sigma$, let

$$\sigma_1(X) = \langle H_X(X), F^1(X) \rangle,$$

$$\sigma_2(X) = \langle H_X(X), F^2(X) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product, $H_X(X)$ represents the normal vector of $\Sigma$ which is

$$H_X(X) = \left( \frac{\partial H}{\partial W}, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2} \right)^T = \left( 0, \frac{1}{C_1}, -\frac{1}{C_2} \right)^T.$$

According to the Filippov convex method [9, 16], the crossing region $\Sigma_c \subset \Sigma$ and sliding region $\Sigma_s \subset \Sigma$ are defined as

$$\Sigma_c = \{X \in \Sigma | \sigma_1(X)\sigma_2(X) > 0\},$$

$$\Sigma_s = \{X \in \Sigma | \sigma_1(X)\sigma_2(X) < 0\}.$$

Then the dynamics of the system restricted to $\Sigma_s$ is called sliding motion. It is clear that on both sides of $\Sigma_s$, the orbits are attracted to or repelled from $\Sigma_s$. Indeed, if

$$\sigma_1(X) > 0, \sigma_2(X) < 0,$$

the sliding region is stable in the normal direction of $\Sigma$, while

$$\sigma_1(X) < 0, \sigma_2(X) > 0,$$

the sliding region is unstable. At the points in $\Sigma_c$, the orbits of the system cross $\Sigma$ because the vector fields on both sides are pointing in the same direction.

To find the exact expression of $K(X)$, we employ the Filippov construction [9] which associates $K(X)$ with the combination of $F^1(X)$ and $F^2(X)$. Namely, for any $X \in \Sigma$, let

$$K(X) = \lambda F^1(X) + (1 - \lambda) F^2(X). \quad (15)$$

When the vector field $K(X)$ is perpendicular to the normal of $\Sigma$, that is,

$$\langle H_X(X), K(X) \rangle = 0,$$

we can obtain a specific coefficient $\lambda$ so that the evolution of the system can be restricted to $\Sigma$,

$$\dot{\lambda} = \frac{\langle H_X(X), F^2(X) \rangle}{\langle H_X(X), F^2(X) - F^1(X) \rangle} = \frac{\sigma_2(X)}{\sigma_2(X) - \sigma_1(X)}. \quad (16)$$
Noticing that \( \hat{\lambda} = \frac{1}{\sigma_1(X)/\sigma_2(X)} \), when \( \sigma_1(X)\sigma_2(X) < 0 \Rightarrow 0 < \hat{\lambda} < 1 \) and \( \sigma_1(X)\sigma_2(X) > 0 \Rightarrow 1 < \hat{\lambda} \) or \( \hat{\lambda} < 0 \). Thus we have

**Lemma 1.** If \( \lambda = \hat{\lambda} \) and \( 0 < \hat{\lambda} = \frac{\sigma_2(X)}{\sigma_2(X) - \sigma_1(X)} < 1 \), \( X \in \Sigma_* \), while if \( \hat{\lambda} > 1 \) or \( \hat{\lambda} < 0 \), \( X \in \Sigma_* \).

4.1. **The existence of the pseudo-equilibrium of (14).** Particularly, we are curious about whether there will be steady state in (14) for which both of the links are utilized. This corresponds a special type of sliding motion, namely, the equilibrium on the sliding region.

**Definition 2.** (16) If there exist \( 0 < \lambda < 1 \) and \( \hat{X} \in \Sigma \) such that \( K(\hat{X}) = 0 \) where \( K(.) \) is given by (15), then \( \hat{X} \) is a pseudo-equilibrium of (14).

It is reasonable to assume that at the pseudo-equilibrium, the queue size of the two links is neither too big nor too small. According to the physical interpretation of the model, if the queue size of the \( i \text{th} \) link is so small that \( P_i(.) = 0 \), then it is clear that the network resources are not fully utilized and consequently the utility of the user is not to be maximized. In turn, if the queue size is big enough to yield \( P_i(.) = 1 \), then the system load is too heavy and consequently the network is at high risk of congestion. In either case, the steady state is not well designed and even if the stability of the pseudo-equilibrium is guaranteed, the network system can not provide the best service.

As we already know that the pseudo-equilibrium \( \hat{X} = (\hat{W}, \hat{q}_1, \hat{q}_2)^T \) is desirable if \( b_{1,1}B_1 < \hat{q}_1 < b_{1,2}B_1 \) and \( b_{2,1}B_2 < \hat{q}_2 < b_{2,2}B_2 \). Parallel to the result in Theorem 2, we have

**Theorem 3.**
\[
L_2(W) = \frac{C_1}{C_1 + C_2} \left( \frac{1}{W} - \frac{1}{2} P_1(C_1(NW/C_1 + C_2 - \tau_1)) \right) + \frac{C_2}{C_1 + C_2} \left( \frac{1}{W} - \frac{1}{2} P_2(C_2(NW/C_1 + C_2 - \tau_2)) \right),
\]
and \( m = \min\{b_{i,1}B_i + C_i\tau_i\} \) for \( i = 1, 2 \). If
\[
L_2(M) > 0, \quad L_2(m) < 0,
\]
then (14) with (15) has a desirable pseudo-equilibrium.

**Proof.** It suffices to show the existence of positive \( (\hat{\lambda}, \hat{W}, \hat{q}_1, \hat{q}_2)^T \) such that
\[
\begin{align*}
\hat{\lambda} (N\hat{W})^2 (\tau_1 + \hat{q}_1/C_1) &- \frac{P_1(\hat{q}_1)}{\hat{q}_1 + \hat{q}_2/C_1} + \frac{1}{\hat{W}} (N\hat{W})^2 (\tau_2 + \hat{q}_2/C_2) - \frac{P_2(\hat{q}_2)}{\hat{q}_1 + \hat{q}_2/C_2} = 0, \quad (a) \\
\hat{\lambda} \tau_1 + \hat{q}_1/C_1 &- C_1 = 0, \quad (b) \\
(1 - \hat{\lambda}) \tau_2 + \hat{q}_2/C_2 &- C_2 = 0, \quad (c) \\
\tau_1 + \hat{q}_1/C_1 &- \tau_2 - \hat{q}_2/C_2 = 0. \quad (d)
\end{align*}
\]
(17)

Substituting (17)(d) to (17)(b) and (17)(c) we obtain
\[
\frac{\hat{\lambda}}{1 - \hat{\lambda}} = \frac{C_1}{C_2} \Rightarrow \hat{\lambda} = \frac{C_1}{C_1 + C_2},
\]
and
\[
\hat{q}_1 = C_1(N\hat{W}/C_1 + C_2 - \tau_1), \quad \hat{q}_2 = C_2(N\hat{W}/C_1 + C_2 - \tau_2).
\]
(18)
Substituting (15) to (17)(a), then solving (17) is reduced to finding the root for $L_2(W) = 0$. Obviously, $L_2(W)$ is a continuous and decreasing function for $W \in (M, m)$. From $L_2(M) > 0$ and $L_2(m) < 0$, we claim that $L_2(W) = 0$ must have a unique root in $(M, m)$. Therefore, $b_{1,1}B_1 < q_i^* < b_{i,2}B_i$ for $i = 1, 2$, which implies that the pseudo-equilibrium is desirable. \hfill \Box

**Remark 2.** The result obtained in Theorem 3 is easy to be extended to a system with $n$ links. Consider the system given by (8)-(12), namely,

\[
\text{Substituting (21)(c) to (21)(b) we obtain}
\]

\[
\frac{dX(t)}{dt} = \sum_{k=1}^{n} \lambda_k F^k(X(t)),
\]

where

\[
0 < s_k < 1 \text{ for } k = 1, 2, \cdots, n. \text{ Obviously } \sum_{k=1}^{n} \lambda_k = 1. \text{ Based on the above notations, we list the equations that the desirable pseudo-equilibrium } \hat{X}^* = (W^*, \hat{q}_1^*, \hat{q}_2^*, \cdots, \hat{q}_n^*)^T \text{ needs to satisfy:}
\]

a) There exist $s_1, s_2, \cdots, s_n$ such that $\sum_{k=1}^{n} \lambda_k F^k(\hat{X}^*) = 0$ where $\lambda_k$ is given by (20), $1 \leq k \leq n$;

b) $b_{k,1}B_k < \hat{q}_k^* < b_{k,2}B_k$, $1 \leq k \leq n$.

Now we can state the following result parallel to Theorem 3.

**Theorem 4.** Let

\[
L_3(W) = \sum_{k=1}^{n} \frac{C_i}{C_i} \left(1 - \frac{1}{2} P_i(C_i(\sum_{i=1}^{n} C_i - \tau_k))) \right)
\]

and $\bar{m} = \min \{\frac{(b_{i,2}B_i+C_i)\tau_i}{\sum_{i=1}^{n} C_i}\}$, $\bar{M} = \max \{\frac{(b_{i,1}B_i+C_i)\tau_i}{\sum_{i=1}^{n} C_i}\}$ for $i = 1, 2, \cdots, n$. If $L_3(\bar{M}) > 0$ and $L_3(\bar{m}) < 0$,

then (19) has a desirable pseudo-equilibrium.

**Proof.** It suffices to show the existence of positive $(\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n, \hat{W}^*, \hat{q}_1^*, \hat{q}_2^*, \cdots, \hat{q}_n^*)^T$ such that

\[
\begin{aligned}
&\sum_{k=1}^{n} \lambda_k \frac{1}{(W^*)^2(\tau_i + \hat{q}_i^*/C_i)} - \frac{P_i(\hat{q}_i^*)}{2(\tau_i + \hat{q}_i^*/C_i)} = 0, \quad (a) \\
&\hat{\lambda}_i \frac{NW^*}{\tau_i + \hat{q}_i^*/C_i} - C_i = 0, \quad i = 1, 2, \cdots, n, \quad (b) \\
&\tau_1 + \hat{q}_1^*/C_1 = \tau_2 + \hat{q}_2^*/C_2 = \cdots = \tau_n + \hat{q}_n^*/C_n. \quad (c)
\end{aligned}
\]

Substituting (21)(c) to (21)(b) we obtain

\[
\frac{\hat{\lambda}_1}{C_1} = \frac{\hat{\lambda}_2}{C_2} = \cdots = \frac{\hat{\lambda}_n}{C_n}.
\]
It follows from (20) that \( \sum_{i=1}^{n} \hat{\lambda}_i = 1 \), thus

\[
\hat{\lambda}_i = \frac{C_i}{\sum_{k=1}^{n} C_k}, \quad i = 1, 2, \ldots, n.
\] (22)

Substituting (22) to (21)(b) we have

\[
\hat{q}^*_i = C_i (NW^*/\sum_{k=1}^{n} C_k - \tau_i), \quad i = 1, 2, \ldots, n.
\] (23)

Substituting (23) to (21)(a), then solving (21) is reduced to locating the root of \( L_3(W) = 0 \). Obviously, \( L_3(W) \) is continuous and decreasing for \( W \in (\hat{M}, \hat{m}) \). From \( L_3(\hat{M}) > 0 \) and \( L_3(\hat{m}) < 0 \), we conclude that \( L_2(W) = 0 \) has a unique root in \((\hat{M}, \hat{m})\). Consequently, \( b_{i1}B_i \leq q^*_i < b_{i2}B_i \) for \( i = 1, 2, \ldots, n \), which implies the existence of the desirable pseudo-equilibrium.

\[ \square \]

4.2. The attractivity of the sliding region of (14). After proving the existence of the pseudo-equilibrium, we now study the attractivity of the sliding region of (14). On the discontinuous boundary \( \Sigma \), from \( \tau_1 + \frac{q_1}{C_1} = \tau_2 + \frac{q_2}{C_2} \) and \( 0 \leq q_i \leq B_i \) for \( i = 1, 2 \), it is easy to see that \( \tau_1 \leq \tau_1 + \frac{q_1}{C_1} = \tau_2 + \frac{q_2}{C_2} \leq \tau_2 + \frac{B_2}{C_2} \) and \( \tau_2 \leq \tau_2 + \frac{q_2}{C_2} = \tau_1 + \frac{B_1}{C_1} \leq \tau_1 + \frac{B_1}{C_1} \). In fact, we have

**Lemma 2.** If

\[
\tau_1 + \frac{B_1}{C_1} \geq \tau_2 \quad \text{and} \quad \tau_2 + \frac{B_2}{C_2} \geq \tau_1.
\] (24)

Then there exists the sliding region on \( \Sigma \) in (14).

It should be noticed that when \( \tau_1 + \frac{B_1}{C_1} = \tau_2 \) or \( \tau_2 + \frac{B_2}{C_2} = \tau_1 \), \( \Sigma \) contains only the point \((q_1, q_2)^T = (B_1, 0)^T\) or \((q_1, q_2)^T = (0, B_2)^T\). This implies that the switch between the two links is inactivated and \( \tau_1 + \frac{B_1}{C_1} = \tau_2 \) or \( \tau_2 + \frac{B_2}{C_2} = \tau_1 \) can not guarantee the existence of the sliding region. Thus, in the following, we assume \( \tau_1 + \frac{B_1}{C_1} > \tau_2 \) and \( \tau_2 + \frac{B_2}{C_2} > \tau_1 \) in \( \Sigma \). In most of the real-world situations, there is a link with highest priority among all the accessible links. In other words, the propagation delay of this link is the smallest among all the links, so it is reasonable to let \( \tau_1 \neq \tau_2 \). Let

\[
\Sigma_{s,1} = \{(W, q_1, q_2)^T | (W, q_1, q_2)^T \in \Sigma; W > 0, 0 \leq q_1 \leq B_1, 0 \leq q_2 \leq B_2\},
\]

\[
\Sigma_{s,2} = \{(W, q_1, q_2)^T | (W, q_1, q_2)^T \in \Sigma; q_1 = 0, q_2 = 0\}.
\]

Then based on Lemma 2, we have the following result.

**Theorem 5.** Assume the conditions given by (24) are satisfied. Then,

(i) the sliding region is \( \Sigma_s = \Sigma_{s,1} \setminus \Sigma_{s,2} \).

(ii) \( \Sigma_s \) is locally attractive when \( \tau_1 \neq \tau_2 \).

**Proof.** First, notice that

\[
H_x(X) = (0, \frac{1}{C_1}, -\frac{1}{C_2})^T,
\]

\[
F^1(X) = (F^1, [\frac{NW}{R(q_1)} - C_1]^+, -C_2)^T,
\]
The pseudo-equilibrium is stable in $\Sigma_s$.

**b)** The sliding region is attractive;

**c)** There exists such pseudo-equilibrium.

From the results given in Theorems 3 and 5, we know that the sliding region of (14) is locally attractive and under certain conditions, the system possesses a pseudo-equilibrium. Although the theoretical analysis for the stability of the pseudo-equilibrium in $\Sigma_s$ is out of the scope of this article, we can carry out the numerical investigation on the original switch system (13) when we choose $\tau_1 = 0.11, \tau_2 = 0.13, N = 10, C_1 = 200, C_2 = 150, B_1 = 50, b_{1,1} = b_{2,1} = 0.2, b_{1,2} = b_{2,2} = 0.95, P_{1,\max} = P_{2,\max} = 0.4$ and $B_2 = 30$ in Fig.4 and $B_2 = 15$ in Fig.5. For both groups of parameters, the conditions in Theorems 3 and 5 are satisfied, implying the pseudo-equilibrium does exist and the sliding region is locally attractive. As shown in Fig.4(a), for $B_2 = 30$, all the eigenvalues of the linearized system around the pseudo-equilibrium of the Filippov vector field have negative real parts, indicating the pseudo-equilibrium is stable in $\Sigma_s$. When we reduce the value of $B_2$ to $B_2 = 15$ (Fig.5(a)), one pair of eigenvalues crosses the imaginary axis and therefore the pseudo-equilibrium becomes unstable, as a consequence of a bifurcation similar to the Hopf bifurcation. Even though the sliding region is locally attractive, the system is oscillatory in $\Sigma$, as shown in Figs.5(b)-(d), which is different from the dynamical system without switch.

5. **Conclusion and discussion.** In this paper, we consider some basic problems for a TCP/RED congestion control model, which is described by a state-dependent delayed system with discontinuous right-hand side function. We prove that all the solutions of the system are bounded, and under some conditions on the parameters, there exists a unique positive equilibrium in the system. For a TCP/RED network system with a single user group and two optional links, we have shown that the sliding motion always exists and is locally stable in a relatively simple framework. This result deserves particular attention. The sliding motion can be realized only under the Filippov vector field constructed by convexly combining the individual vector fields with coefficients selected from $(0,1)$. However, in real-world applications, the switch of the system is represented by Boolean variables which jump between 0
Figure 4. The numerical continuation (a) by DDE-BIFTOOL [7] and simulation (b)-(e) by XPP-AUT [8] for (13) as $B_2 = 30$. Initial conditions: $W(0) = 6$, $q_1(0) = 11$, $q_2(0) = 7$. (a) shows the distribution of the real and imaginary parts of the eigenvalues of $\dot{X}(t) = K(X(t))$, $\lambda = \hat{\lambda}$ given by (16). The real parts of all the eigenvalues are negative and consequently the pseudo-equilibrium is stable in $\Sigma_s$ which is confirmed by the time history plots (b), (c) and (d). (e) shows that the dynamics of the system is restricted to $\Sigma$, in other words, $\Sigma_s$ is locally attractive.
Figure 5. The numerical continuation (a) and simulation (b)-(e) for \((13)\) when \(B_2 = 15\). \(W(0) = 6, q_1(0) = 11, q_2(0) = 7\). Red dots in (a) represents the eigenvalue with positive real part. (a) shows the distribution of the eigenvalues of \(\dot{X}(t) = K(X(t))\). The maximum of the real parts of the eigenvalues is positive and consequently the pseudo-equilibrium is unstable in \(\Sigma_s\) which is confirmed by the time history plots (b), (c) and (d). (e) shows that the dynamics of the system is restricted to \(\Sigma\), implying the local attractivity of \(\Sigma_s\).
and 1. Therefore, even an equilibrium in the sliding region is stable, the system could be oscillatory. In other words, the resources have been consumed but the stability is not improved remarkably. Some may argue that the continuous switch between individual systems, which is unlikely to be realized in the real-world cases, is a reason for the phenomenon observed in the mathematical model. However, the attractivity of the sliding region implies that the trajectory of the system has a tendency to move towards the sliding region and oscillates, no matter at what period the routing table is updated. This may suggest that it is necessary to modify the current routing switch algorithm.

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