ON SUMS OF BINOMIAL COEFFICIENTS
AND THEIR APPLICATIONS

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ABSTRACT. In this paper we study recurrences concerning the combinatorial sum
\[ \binom{n}{r}_m = \sum_{k \equiv r \pmod{m}} \binom{n}{k}, \]
and the alternate sum
\[ \left\{ \binom{n}{r}_m \right\} = \sum_{k \equiv r \pmod{m}} (-1)^{k-r/m} \binom{n}{k}. \]
where \( m > 0, n \geq 0 \) and \( r \) are integers. For example, we show that if \( n \geq m-1 \) then
\[ \left\lfloor \frac{(m-1)/2}{2} \right\rfloor \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} [n-2i] \equiv \frac{2^{n-m+1}}{m}. \]

We also apply such results to investigate Bernoulli and Euler polynomials. Our
approach depends heavily on an identity established by the author [Integers 2(2002)].

Keywords: Binomial coefficient; combinatorial sum; recurrence; Bernoulli polynomial; Euler polynomial.

1. Introduction and Main Results

As usual, we let
\[ \binom{x}{0} = 1 \text{ and } \binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!} \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots \}. \]

Following [Su2], for \( m \in \mathbb{Z}^+, n \in \mathbb{N} = \{0, 1, 2, \ldots \} \) and \( r \in \mathbb{Z} \) we set

\[ \binom{n}{r}_m = \sum_{k \equiv r \pmod{m}} \binom{n}{k} \quad \text{and} \quad \left\{ \binom{n}{r}_m \right\} = \sum_{k \equiv r \pmod{m}} (-1)^{k-r/m} \binom{n}{k}. \]
As \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \) for any \( k \in \mathbb{Z}^+ \), we have the following useful recursions:

\[
\binom{n+1}{r}_m = \binom{n}{r}_m + \binom{n}{r-1}_m \quad \text{and} \quad \{\binom{n+1}{r}\}_m = \{\binom{n}{r}\}_m + \{\binom{n}{r-1}\}_m.
\]

Let \( m, n \in \mathbb{Z}^+ \) and \( r \in \mathbb{Z} \). The study of the sum \( \binom{n}{r}_m \) dates back to 1876 when C. Hermite showed that if \( n \) is odd and \( p \) is an odd prime then \( \binom{n}{0}_{p-1} \equiv 1 \pmod{p} \) (cf. L. E. Dickson [D, p. 271]). In 1899 J. W. L. Glaisher obtained the following generalization of Hermite’s result:

\[
\binom{n+p-1}{r}_{p-1} \equiv \binom{n}{r}_{p-1} \pmod{p} \quad \text{for any prime } p.
\]

(See, e.g., [Gr, (1.11)].) If \( p \) is a prime with \( p \equiv 1 \pmod{m} \), then \( \binom{p}{r}_m \equiv \{\binom{p}{r}\}_m \pmod{p} \) since \( p \) divides any of \((p)_1, \ldots, (p)_{p-1}\), thus \( \binom{n+p-1}{r}_m \equiv \{\binom{n}{r}_m \pmod{p} \) by (1.2) and induction. This explains Glaisher’s result in a simple way. (Recently the author and R. Tauraso [ST] obtained a further extension of Glaisher’s congruence.) In the modern investigations made by Z. H. Sun and the author (cf. [SS], [S], [Su1] and [Su2]), \( \binom{n}{r}_m \) was expressed in terms of linear recurrences and then applied to produce congruences for primes. The sum \( \binom{n}{r}_m \) also appeared in C. Helou’s study of Terjanian’s conjecture concerning Hilbert’s residue symbol and cyclotomic units (cf. [H, Prop. 2 and Lemma 3]).

Now we state two theorems on the sums in (1.1) and give two corollaries. The proofs of them depend heavily on an identity established by the author in [Su3], and will be presented in Section 2.

**Theorem 1.1.** Let \( m \) be a positive integer. Then, for any integers \( k \) and \( n \geq 2\lfloor (m-1)/2 \rfloor \), we have

\[
\sum_{i=0}^{\left\lfloor (m-1)/2 \right\rfloor} (-1)^i \binom{m-1-i}{i}_m \binom{n-2i}{k-i}_m = 2^{n-m+1} + \delta_{m-2,n} \frac{(-1)^k}{2},
\]

where the Kronecker symbol \( \delta_{l,n} \) is 1 or 0 according to whether \( l = n \) or not.

**Corollary 1.1.** Let \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^+ \). For \( n \in \mathbb{N} \) set

\[
u_n = \binom{n}{\lfloor (k+n)/2 \rfloor}_m \quad \text{and} \quad v_n = mu_n - 2^n - \delta_{n,0} \delta_{(-1)m,1} (-1)^{\lfloor k/2 \rfloor},
\]

where \( \lfloor \alpha \rfloor \) denotes the integral part of a real number \( \alpha \). Then we have

\[
\sum_{i=0}^{\left\lfloor (m-1)/2 \right\rfloor} (-1)^i \binom{m-1-i}{i} u_{n-2i} = 2^{n-m+1} - \delta_{m-2,n} \frac{(-1)^{\lfloor (k+m)/2 \rfloor}}{2}
\]
for every integer \( n \geq 2 \left(\lfloor (m - 1)/2 \rfloor \right) \). Also, \((v_n)_{n \in \mathbb{N}}\) is a linear recurrence sequence, satisfying the recurrence:

\[
\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m - 1 - i}{i} v_{n-2i} = 0 \quad \text{for all } n \geq 2 \left(\lfloor m/2 \rfloor \right).
\]

Remark 1.1. (a) In fact, the author first proved (1.6) in the case \( 2 \nmid m \) on August 1, 1988, motivated by a conjecture of Z. H. Sun; after reading the author’s initial proof Z. H. Sun [S] noted that the equality in (1.6) also holds if \( 2 \mid m \) and \( n \geq m - 1 \).
(b) In light of the first equality in (1.2), on August 11, 1988 the author obtained the following result by induction: Let \( m, n \in \mathbb{Z}^+ \) and \( m > 2 \). If \( n \geq m - 1 \) then

\[
\left[ \frac{n}{n+1} \right]_m > \left[ \frac{n}{n+1} + 1 \right]_m > \cdots > \left[ \frac{n}{n+m} \right]_m,
\]

otherwise

\[
\left[ \frac{n}{n+1} \right]_m > \cdots > \left[ \frac{n}{n} \right]_m > \left[ \frac{n}{n+1} \right]_m = \cdots = \left[ \frac{n}{n+m} \right]_m = 0.
\]

Therefore

\[
\left[ \frac{n}{m} \right]_m > \left[ \frac{2n}{m} \right]_m > \left[ \frac{2n}{m+n} \right]_m.
\]

Theorem 1.2. Let \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^+ \). Then

\[
\sum_{i=0}^{\lfloor (m+1)/2 \rfloor} (-1)^i c_m(i) \left[ \binom{n - 2i}{k - i} \right]_m = 2(-1)^k \delta_{m,n}
\]

for each integer \( n \geq 2 \left(\lfloor (m + 1)/2 \rfloor \right) \), and

\[
\sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i d_m(i) \left\{ \binom{n - 2i}{k - i} \right\}_m = (-1)^k \delta_{m-1,n}
\]

for any integer \( n \geq 2 \left(\lfloor m/2 \rfloor \right) \), where \( c_1(1) = 4 \), and

\[
c_m(i) = \frac{m^2 + m - 2i}{(m-i)(m+1-i)} \binom{m + 1 - i}{i} \in \mathbb{Z} \text{ and } d_m(i) = \frac{m}{m-i} \binom{m-i}{i} \in \mathbb{Z}
\]

for every \( i = 0, \ldots, m - 1 \).

Remark 1.2. Let \( p \) be an odd prime. It is easy to check that

\[
(-1)^{i-1} c_{p-1}(i) \equiv (-1)^i d_{p-1}(i) \equiv C_i \pmod{p} \quad \text{for } i = 1, 2, \ldots, \frac{p-1}{2},
\]

where \( C_i = \binom{2i}{i}/(i+1) = \binom{2i}{i} - \binom{2i}{i+1} \) is the \( i \)-th Catalan number.
Corollary 1.2 (A. Fleck, 1913). Let $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. If $p$ is a prime, then

$$\sum_{0 \leq k \leq n \atop p \mid k-r} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^\lceil (n-1)/(p-1) \rceil}.$$  

By [Su2, Remark 2.1], for $k \in \mathbb{Z}$, $l \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $\varepsilon \in \{1,-1\}$, we have

$$\sum_{\gamma_m = \varepsilon} \gamma^k (2 - \gamma - \gamma^{-1})^l = (-1)^k m \times \left\{ \begin{array}{ll}
\left\lceil \frac{2l}{k+l} \right\rceil_m & \text{if } \varepsilon = (-1)^m, \\
\{ \frac{2l}{k+l} \}_m & \text{otherwise}.
\end{array} \right.$$  

So Theorem 1.2 is closely related to the following materials on Bernoulli and Euler polynomials.

Let $m, n \in \mathbb{Z}^+$, $q \in \mathbb{Z}$ and $(q, m) = 1$, where $(q, m)$ is the greatest common divisor of $q$ and $m$. If $\gamma_m = 1$ and $\gamma(q,m) = \gamma \neq 1$, then $\gamma_q \neq 1$ and hence $2 - \gamma^q - \gamma^{-q} \neq 0$.

We define a linear recurrence $(U_l(q)(m,n))_{l \in \mathbb{N}}$ of order $\lfloor m/2 \rfloor$ by

$$U_l(q)(m,n) = \frac{1}{2m} \sum_{\gamma_m = -1 \atop \gamma \neq 1} \frac{2 - \gamma^qn - \gamma^{-qn}}{2 - \gamma^q - \gamma^{-q}} (2 - \gamma - \gamma^{-1})^l.$$  

Note that $U_l(-q)(m,n) = U_l(q)(m,n)$ and

$$mU_l(q)(m,n) = (1 - (-1)^{(m-1)n}) 2^{2l-2} + \sum_{d \mid m \atop d > 2} u_l(q)(d,n),$$  

where

$$u_l(q)(d,n) = \sum_{0 < c < d/2 \atop (c,d) = 1} \frac{2 - e^{2\pi i \frac{qn}{d}} - e^{-2\pi i \frac{qn}{d}}}{2 - e^{2\pi i \frac{q}{d}} - e^{-2\pi i \frac{q}{d}}} (2 - e^{2\pi i \frac{2c}{d}} - e^{-2\pi i \frac{2c}{d}})^l = \sum_{0 < c < d/2 \atop (c,d) = 1} \frac{(\sin(\pi nc/d))^2}{\sin(\pi qc/d)} \frac{(4 \sin^2 \pi c)^l}{d}.$$  

Obviously $u_l(q)(m,n) = mU_l(q)(m,n)$ if $m$ is an odd prime. Later we will see that $U_0(q)(m,n) = n(m-n)/(2m)$ if $1 \leq n \leq m$, and $U_l(q)(m,n) \in \mathbb{Z}$ if $l > 0$. When $(q,2m) = 1$, for $l \in \mathbb{N}$ we also define

$$V_l(q)(m,n) = \frac{1}{2m} \sum_{\gamma_m = -1} \frac{2 - \gamma^qn - \gamma^{-qn}}{2 - \gamma^q - \gamma^{-q}} (2 - \gamma - \gamma^{-1})^l;$$
clearly $V_l^{(\pm q)}(m, n) = 2U_l^{(q)}(2m, n) - U_l^{(q)}(m, n)$ since $\gamma^m = -1$ if and only if $\gamma^{2m} = 1$ but $\gamma^m \neq 1$.

Let $p$ be an odd prime, and let $m, n > 0$ be integers with $p \nmid m$ and $m \nmid n$. A. Granville and the author [GS, pp. 126–129] proved the following surprising result for Bernoulli polynomials: If $p \equiv \pm q \pmod{m}$ where $q \in \mathbb{Z}$, then

\begin{equation}
B_{p-1} \left( \left\{ \frac{pn}{m} \right\} \right) - B_{p-1} \equiv \frac{m}{2p} \left( U_p^{(q)}(m, n) - 1 \right) \quad \pmod{p}
\end{equation}

where we use $\{ \alpha \}$ to denote the fractional part of a real number $\alpha$. (The reader may consult [Su4] for other congruences concerning Bernoulli polynomials.) With the help of Theorem 1.2, we can write the recurrent coefficients of the sequence $(U_l^{(q)}(m, n))_{l \in \mathbb{N}}$ in a simple closed form.

**Theorem 1.3.** Let $m, n \in \mathbb{Z}^+$, $q \in \mathbb{Z}$ and $(q, m) = 1$. Then we have the recursions:

\begin{equation}
U_l^{(q)}(m, n) = \sum_{0 < i \leq \lfloor m/2 \rfloor} (-1)^{i-1} a_m(i) U_{i-l}^{(q)}(m, n) \quad \text{for } l \geq \left\lfloor \frac{m}{2} \right\rfloor,
\end{equation}

and

\begin{equation}
V_l^{(q)}(m, n) = \sum_{j=1}^{\lfloor (m+1)/2 \rfloor} (-1)^{j-1} b_m(j) V_{l-j}^{(q)}(m, n) \quad \text{for } l \geq \left\lfloor \frac{m+1}{2} \right\rfloor
\end{equation}

provided $(q, 2m) = 1$, where the integers $a_m(i)$ and $b_m(j)$ are given by

\begin{equation}
a_m(i) = \begin{cases} c_m(i) & \text{if } 2 \mid m, \\ d_m(i) & \text{if } 2 \nmid m, \end{cases} \quad \text{and} \quad b_m(j) = \begin{cases} d_m(j) & \text{if } 2 \mid m, \\ c_m(j) & \text{if } 2 \nmid m. \end{cases}
\end{equation}

If $m$ does not divide $n$, and $p$ is an odd prime with $p \equiv \pm q \pmod{2m}$, then

\begin{equation}
(-1)^{\lfloor pn/m \rfloor} E_{p-2} \left( \left\{ \frac{pm}{m} \right\} \right) + \frac{2p-2}{p} \equiv \frac{m}{p} \left( V_p^{(q)}(m, n) - 1 \right) \quad \pmod{p},
\end{equation}

where the Euler polynomials $E_k(x)$ ($k = 0, 1, \ldots$) are given by

\[ \frac{2e^{xz}}{e^z + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{z^k}{k!}. \]

In Section 3 we will first deduce Theorem 1.3 from Theorem 1.2, and then give another proof of (1.13) via Chebyshev polynomials. Section 4 is an appendix containing the explicit values of $a_m(i)$ and $b_m(j)$ for $m = 2, 3, \ldots, 12$. 
Lemma 2.1. Let \( l \) be any nonnegative integer. Then

\[
\sum_{j=0}^{l} (-1)^{l-j} \binom{x+y+j}{l-j} \binom{y+2j}{j} = \sum_{j=0}^{l} \binom{l-x}{j}.
\]

Proof. Since both sides of (2.1) are polynomials in \( x \) and \( y \), it suffices to show (2.1) for all \( x \in \{l, l+1, \ldots \} \) and \( y \in \{0, 2, 4, \ldots \} \).

Let \( x = l + n \) and \( y = 2k \) where \( n, k \in \mathbb{N} \). Set \( m = k + l \). Then

\[
\sum_{j=0}^{l} (-1)^{l-j} \binom{x+y+j}{l-j} \binom{y+2j}{j} = \sum_{i=k}^{m} (-1)^{i-(i-k)} \binom{x+2k+i-k}{l-(i-k)} \binom{2k+2(i-k)}{i-k}
\]

\[
= (-1)^m \sum_{i=k}^{m} (-1)^i \binom{m+n+i}{m-i} \binom{2i}{k+i}
\]

\[
= \sum_{j=0}^{l} (-1)^j \binom{n+j-1}{j} \quad \text{(by [Su3, (3.2)])}
\]

\[
= \sum_{j=0}^{l} \binom{-n}{j} = \sum_{j=0}^{l} \binom{l-x}{j}.
\]

This concludes the proof. □

Remark 2.1. Lemma 2.1, an equivalent version of [Su3, (3.2)], played a key role when the author established the following curious identity in [Su3]:

\[
(x+m+1) \sum_{i=0}^{m} (-1)^i \binom{x+y+i}{m-i} \binom{y+2i}{i} = \sum_{i=0}^{m} \binom{x+i}{m-i} (-4)^i + (x-m) \binom{x}{m},
\]

where \( m \) is any nonnegative integer. The reader is referred to [C], [CC], [EM], [MS] and [PP] for other proofs of (2.2), and to [SW] for an extension of (2.2). In the case \( x \in \{0, \ldots , l\} \) the right-hand side of (2.1) turns out to be \( 2^{l-x} \), so (2.1) implies identity (3) in [C], which has a nice combinatorial interpretation.

Proof of Theorem 1.1. Let \( n \in \mathbb{Z} \) and \( n \geq 2h \), where \( h = \lfloor (m-1)/2 \rfloor \). Then \( n+1 \geq m-1 > m-2 \). Suppose that (1.3) holds for all \( k \in \mathbb{Z} \). Then, for any given
For any $k \in \mathbb{Z}$, we have

$$
\sum_{i=0}^{h}(-1)^i \binom{m-1-i}{i} \left[ n+1-2i \right]_{k-i} \\
= \sum_{i=0}^{h}(-1)^i \binom{m-1-i}{i} \left( \binom{n-2i}{k-i} + \binom{n-2i}{k-1-i} \right) \\
= \sum_{i=0}^{h}(-1)^i \binom{m-1-i}{i} \binom{n-2i}{k-i} + \sum_{i=0}^{h}(-1)^i \binom{m-1-i}{i} \binom{n-2i}{k-1-i} \\
= 2^{n-m+1} + \delta_{m-2,n} \left( \frac{(-1)^k}{2} \right) + \left( 2^{n-m+1} + \delta_{m-2,n} \frac{(-1)^{k-1}}{2} \right) = 2^{(n+1)-m+1}.
$$

In view of the above, it suffices to show (1.3) for $n = 2h$ and $k \in \{0, 1, \ldots, m-1\}$. For any $i \in \mathbb{N}$ with $i \leq h$, we have $k-i+m > n-2i$ since $n-m < 0 \leq k+i$, thus

$$
\binom{n-2i}{k-i} = \begin{cases} 
\binom{n-2i}{k-i} & \text{if } i \leq k, \\
0 & \text{if } i > k.
\end{cases}
$$

Let $x = m-1-n+k$, $y = n-2k$, and $\Sigma$ denote the left hand side of (1.3). Then

$$
\Sigma = \sum_{j=0}^{k}(-1)^k \left( \frac{m-1-i}{i} \right) \left( \frac{n-2i}{k-i} \right) \\
= \sum_{j=0}^{k}(-1)^{k+j} \binom{x+y+j}{k-j} \binom{y+2j}{j} \\
= \sum_{j=0}^{k} \binom{k-x}{j} \sum_{j=0}^{k} \binom{n-(m-1)}{j}
$$

with the help of Lemma 2.1. If $m$ is odd, then $n = m-1$ and hence $\Sigma = \sum_{j=0}^{k} \binom{0}{j} = 1 = 2^{n-m+1}$. If $m$ is even, then $n = m-2$ and

$$
\Sigma = \sum_{j=0}^{k} \binom{-1}{j} = \sum_{j=0}^{k}(-1)^j = \frac{1+(-1)^k}{2} = 2^{n-m+1} + \frac{(-1)^k}{2}.
$$

So we do have $\Sigma = 2^{n-m+1} + \delta_{m-2,n} (-1)^k/2$ as required.  

**Proof of Corollary 1.1.** Let $n \in \mathbb{N}$ and $n \geq 2(m-1)/2$. By Theorem 1.1,

$$
\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} \left[ \frac{n-2i}{k+n} - i \right]_m = 2^{n-m+1} + \delta_{m-2,n} \frac{(-1)^{\lfloor (k+n)/2 \rfloor}}{2}.
$$
If \( m - 2 = n \), then \( 2 \mid m \) and \( (k + n)/2 = (k + m)/2 - 1 \). So (1.5) holds.

For \( 0 \leq i \leq 2[(m - 1)/2] \), if \( n - 2i = 0 \) and \( 2 \mid m \), then we must have \( n/2 = i = [(m - 1)/2] = m/2 - 1 \). Note also that

\[
\sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-1-i}{i} 2^{m-1-2i} = m
\]

by (1.60) of [G] or (4) of [C]. Therefore

\[
\sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-1-i}{i} v_{n-2i} =
\]

\[
\sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-1-i}{i} u_{n-2i} - \sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-1-i}{i} 2^{n-2i}
\]

\[
= -m \delta_{m-2,n} (\frac{(-1)\lfloor(k+m)/2\rfloor}{2}) - \delta_{m-2,n} (\frac{(-1)\lfloor(m-1)/2\rfloor m}{2} (-1)\lfloor k/2 \rfloor) = 0.
\]

This concludes the proof. \( \square \)

Proof of Theorem 1.2. i) Clearly \( c_m(0) = 1 \). As \( \lfloor m/2 \rfloor + \lfloor (m+1)/2 \rfloor = m \), whether \( m = 1 \) or not, we have

\[
c_m \left( \left\lfloor \frac{m+1}{2} \right\rfloor \right) = 4 \left( \left\lfloor \frac{m}{2} \right\rfloor \right) = 4 \left( m - \left\lfloor \frac{m+1}{2} \right\rfloor \right).
\]

If \( 0 < i < m/2 \) then

\[
c_m(i) = \frac{(m-i)!}{i!(m-2i)!} \cdot \frac{m^2 + m - 2i}{(m-i)(m+1-2i)} = \frac{(m-i)!}{i!(m-2i)!} \left( \frac{m-2i}{m-i} + \frac{4i}{m+1-2i} \right)
\]

\[
= \frac{(m-1-i)!}{i!(m-1-2i)!} + 4 \frac{(m-i)!}{(i-1)!(m+1-2i)!} = \binom{m-1-i}{i} + 4 \binom{m-i}{i-1}.
\]

Let \( n \in \mathbb{N} \) and \( n \geq 2 \lfloor (m+1)/2 \rfloor \). Set \( h = \lfloor (m-1)/2 \rfloor \). As \( n > n - 2 \geq 2h \), by Theorem 1.1 we have

\[
\sum_{i=0}^{h} (-1)^i \binom{m-1-i}{i} \left\lfloor \frac{n-2i}{k-i} \right\rfloor = 2^{n-m+1}
\]

and

\[
\sum_{i=0}^{h} (-1)^i \binom{m-1-i}{i} \left\lfloor \frac{n-2-2i}{k-1-i} \right\rfloor = 2^{n-2-m+1} + \delta_{m,n} \frac{(-1)^{k-1}}{2}.
\]
Therefore
\[
0 = 2^{n-m+1} - 4 \cdot 2^{n-2-m+1} = \sum_{i=0}^{h} (-1)^i \binom{m-1-i}{i} \left[ n-2i \right]_m \\
- 4 \left( \sum_{i=0}^{h} (-1)^i \binom{m-1-i}{i} \left[ n-2-2i \right]_{k-1-i} + \delta_{m,n} \frac{(-1)^k}{2} \right)
\]
and hence
\[
2(-1)^k \delta_{m,n} = \binom{n}{k} + \sum_{0<i<m/2} (-1)^i \binom{m-1-i}{i} \left[ n-2i \right]_m \\
+ 4 \sum_{j=1}^{h+1} (-1)^j \binom{m-j}{j-1} \left[ n-2j \right]_m \\
= \binom{n}{k} + \sum_{0<i<m/2} (-1)^i \left( \binom{m-1-i}{i} + 4 \binom{m-i}{i-1} \right) \left[ n-2i \right]_m \\
+ (-1)^{\left\lfloor \frac{m+1}{2} \right\rfloor} 4 \left( \binom{m-\left\lfloor \frac{m+1}{2} \right\rfloor}{\left\lfloor \frac{m+1}{2} \right\rfloor} - 1 \right) \left[ n-2\left\lfloor \frac{m+1}{2} \right\rfloor \right]_m \\
= \sum_{i=0}^{h+1} (-1)^i c_m(i) \left[ n-2i \right]_{k-i}.
\]

This proves the first part of Theorem 1.2.

ii) Observe that
\[
\frac{m}{m-i} \binom{m-i}{i} = 2 \binom{m-i}{i} - \binom{m-1-i}{i} \in \mathbb{Z} \quad \text{for } i = 0, \ldots, m-1.
\]

In view of (1.2), it suffices to verify (1.8) in the case \( n = 2 \lfloor m/2 \rfloor \) and \( 0 \leq k < m \).
For any \( i \in \mathbb{N} \) with \( i \leq n/2 = \lfloor m/2 \rfloor \), we have \( k-i+m > n-2i \) if and only if \( i = k = 0 \) and \( m = n \), and thus
\[
\begin{align*}
\left\{ \begin{array}{ll}
0 & \text{if } i > k, \text{ or } i = k = 0 \& m = n, \\
\binom{n-2i}{k-i} & \text{otherwise}.
\end{array} \right.
\end{align*}
\]
Therefore

\[
\sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i d_m(i) \binom{n-2i}{k-i}_m
\]

\[
= \sum_{i=0}^{k} (-1)^i \frac{m}{m-i} \frac{m-i}{i} \binom{n-2i}{k-i} - \delta_{k,0} \delta_{m,n}
\]

\[
= 2 \sum_{i=0}^{k} (-1)^i \binom{m-i}{i} \binom{n-2i}{k-i} - \sum_{i=0}^{k} (-1)^i \binom{m-i}{i} \binom{n-2i}{k-i} - \delta_{k,0} \delta_{m,n}
\]

\[
= 2 \sum_{j=0}^{k} \binom{n-m}{j} - \sum_{j=0}^{k} \binom{n-(m-1)}{j} - \delta_{k,0} \delta_{m,n} = \delta_{m-1, n} (-1)^k
\]

with the help of Lemma 2.1.

The proof of Theorem 1.2 is now complete. □

Proof of Corollary 1.2. The case \( p = 2 \) can be verified directly, so let \( p > 2 \). Clearly, (1.9) holds if and only if \( p^{\lfloor (n-1)/(p-1) \rfloor} \mid \{n\}_{r}^{p} \). If \( n \geq p \), then \( \{n\}_{r}^{p} = \sum_{i=1}^{[p/2]} (-1)^{i-1} d_p(i) \{n-2i\}_{r}^{p} \) by Theorem 1.2. Since \( p \mid d_p(i) \) for \( i = 1, \ldots, [p/2] \), we have the desired result by induction on \( n \). □

3. Proof of Theorem 1.3

Let \( m \in \mathbb{Z}^+ \), \( n \in \mathbb{N} \) and \( r \in \mathbb{Z} \). Set

\[
\begin{align*}
\binom{n}{r}_m &= \begin{cases} \{n\}_{r}^{m} & \text{if } 2 \mid m, \\ \{n\}_{r}^{m} & \text{if } 2 \nmid m; \end{cases} \\
\binom{n}{r}^* &= \begin{cases} \{n\}_{r}^{m} & \text{if } 2 \mid m, \\ \{n\}_{r}^{m} & \text{if } 2 \nmid m. \end{cases}
\end{align*}
\]

Clearly

\[
(-1)^r \binom{n}{r}_m = \sum_{k=0}^{n} \binom{n}{k} (-1)^k, \quad (-1)^r \binom{n}{r}^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+(k-r)/m}
\]

and

\[
\binom{n}{r}_m + \binom{n}{r}^*_m = \left[ \binom{n}{r}_m \right] + \binom{n}{r}^*_m = 2 \left[ \binom{n}{r}^* \right]_{2m} = 2 \binom{n}{r}_{2m}.
\]

Since \( \left[ \binom{n-1}{r} \right]_m = \binom{n}{r}_m \) and \( \{\binom{n}{r}\}_m = \{\binom{n}{r}\}_m \), we have \( \binom{n}{r}_m = \binom{n}{r}_m \) and also \( \binom{n}{r}^*_m = \binom{n}{r}^*_m \).
Lemma 3.1. Let \( l \in \mathbb{N}, m, n \in \mathbb{Z}^+, q \in \mathbb{Z} \) and \((q, m) = 1\). Then

\[
U_l^{(q)}(m, n) = \sum_{r=0}^{n} \frac{n-r}{1+\delta_{r,0}} \left((-1)^{qr} \left(\frac{2l}{l+qr}\right)_m - \delta_{l,0}\right),
\]

and \(U_l^{(q)}(m, n) \in \mathbb{Z}\) if \(1 \leq l \leq \lfloor (m+1)/2 \rfloor\). When \((q, 2m) = 1\), we have

\[
V_l^{(q)}(m, n) = \sum_{r=0}^{n} \frac{n-r}{1+\delta_{r,0}} (-1)^{qr} \left(\frac{2l}{l+qr}\right)_m^*,
\]

and also \(V_l^{(q)}(m, n) \in \mathbb{Z}\) provided \(1 \leq l \leq \lfloor (m+1)/2 \rfloor\).

**Proof.** Let \( \rho = 1 \), or \( \rho = -1 \) and \((q, 2m) = 1\). With the help of the identity

\[
\frac{2 - x^n - x^{-n}}{2 - x - x^{-1}} = n + \sum_{r=1}^{n} (n-r)(x^r + x^{-r}) = \sum_{r=-n}^{n} (n-|r|)x^r
\]

(cf. [GS, (2.2)]), we have

\[
\sum_{\gamma^m = \rho} \left(\sum_{\gamma \neq 1} \frac{2 - \gamma^q - \gamma^{-q}}{2 - \gamma^q - \gamma^{-q}} (2 - \gamma - \gamma^{-1})^l\right)
\]

\[
= \sum_{\gamma^m = \rho} \sum_{r=-n}^{n} (n-|r|)\gamma^{qr} (2 - \gamma - \gamma^{-1})^l
\]

\[
= \sum_{r=-n}^{n} (n-|r|) \left(\sum_{\gamma^m = \rho} \gamma^{qr} (2 - \gamma - \gamma^{-1})^l - \delta_{\rho,1} \delta_{l,0}\right)
\]

\[
= \sum_{r=-n}^{n} (n-|r|) \times \begin{cases} (-1)^{qr} m (2l)_{(l+qr)} - \delta_{l,0} & \text{if } \rho = 1, \\ (-1)^{qr} m (2l)_{(l+qr)}^* & \text{if } \rho = -1, \end{cases}
\]

where in the last step we apply Remark 2.1 of [Su2]. Note that \((2l)_{(l+qr)} = (2l)_{(l-qr)}^*\) and \((2l)_{(l+qr)}^* = (2l)_{(l-qr)}^*\). So we have (3.2), also (3.3) holds if \((q, 2m) = 1\).

Suppose that \(1 \leq l \leq \lfloor (m+1)/2 \rfloor\). In the case \(m = 1\), both \((2l)_{m} = 0\) and \((2l)_{m}^* = 4\) are even. If \(m > 1\), then \(l + m > 2l\) and hence \((2l)_{m} = (2l)_{m}^* = (2l)_{m} = 2^{(2l-1)}\). Therefore \(U_l^{(q)}(m, n) \in \mathbb{Z}\), and also \(V_l^{(q)}(m, n) \in \mathbb{Z}\) when \((q, 2m) = 1\).

The proof of Lemma 3.1 is now complete. \(\square\)

**Remark 3.1.** Let \( q \in \mathbb{Z}, m \in \mathbb{Z}^+ \) and \((q, m) = 1\). In view of (3.2), we have

\[
U_0^{(q)}(m, n) = \frac{n}{2} \left(1 - \frac{1}{m}\right) = \frac{n(m-n)}{2m} \text{ for } n = 1, \ldots, m.
\]
If $m > 1$, then
\begin{equation}
U_l^{(q)}(m, 1) = \frac{1}{2} \binom{2l}{l} = \binom{2l - 1}{l} \quad \text{for } l = 1, \ldots, \left\lfloor \frac{m + 1}{2} \right\rfloor.
\end{equation}

When $(q, 2m) = 1$, $V^{(q)}_0(m, n) = 2U^{(q)}_0(2m, n) - U^{(q)}_0(m, n) = n/2$ for $n = 1, \ldots, m$, and also $V^{(q)}_l(m, 1) = 2U^{(q)}_l(2m, 1) - U^{(q)}_l(m, 1) = \binom{2l - 1}{l}$ if $m > 1$ and $1 \leq l \leq \left\lfloor (m + 1)/2 \right\rfloor$.

For positive integers $m$ and $n$, it is known that
\begin{equation}
\sum_{r=0}^{m-1} B_n \left( \frac{x + r}{m} \right) = m^{1-n} B_n(x)
\end{equation}
(due to Raabe), and
$$E_{n-1}(x) = \frac{2}{n} \left( B_n(x) - 2^n B_n \left( \frac{x}{2} \right) \right).$$

**Lemma 3.2.** Let $n$ be a positive integer, and let $x$ be a real number. Then
$$nE_{n-1} \{x\} = 2(-1)^{\lfloor |x| \rfloor} \left( B_n \{x\} - 2^n B_n \left( \frac{x}{2} \right) \right).$$

**Proof.** Clearly $2\{x/2\} - \{x\} = \lfloor x \rfloor - 2\lfloor x/2 \rfloor \in \{0, 1\}$. If $2 \mid \lfloor x \rfloor$, then
$$B_n \{x\} - 2^n B_n \left( \frac{x}{2} \right) = B_n \{x\} - 2^n B_n \left( \frac{x}{2} \right) = \frac{n}{2} E_{n-1} \{x\}.$$ 

By Raabe’s formula (3.6),
$$B_n \left( \frac{x}{2} \right) + B_n \left( \frac{x + 1}{2} \right) = 2^{1-n} B_n \{x\}.$$ 

So, if $2 \nmid \lfloor x \rfloor$ then
\begin{align*}
B_n \{x\} - 2^n B_n \left( \frac{x}{2} \right) &= B_n \{x\} - 2^n B_n \left( \frac{x + 1}{2} \right) \\
&= B_n \{x\} - 2^n \left( 2^{1-n} B_n \{x\} - B_n \left( \frac{x}{2} \right) \right) \\
&= 2^n B_n \left( \frac{x}{2} \right) - B_n \{x\} = -\frac{n}{2} E_{n-1} \{x\}.
\end{align*}

This concludes the proof. \(\square\)

From Lemma 3.2 we have
Lemma 3.3. Let $p$ be an odd prime, and let $m, n \in \mathbb{Z}^+$ and $p \nmid m$. Then

$$
\frac{(-1)^{|pn/m|}}{2} E_{p-2} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) + \frac{2^{p-1} - 1}{p} 
\equiv B_{p-1} \left( \begin{pmatrix} pm \\ 2m \end{pmatrix} \right) - B_{p-1} - \left( B_{p-1} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) - B_{p-1} \right) \pmod{p}.
$$

(3.7)

Proof. By Lemma 3.2,

$$
\frac{(-1)^{|pn/m|} p - 1}{2} E_{p-2} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) + (2^{p-1} - 1)B_{p-1} 
= B_{p-1} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) - B_{p-1} - 2^{p-1} \left( B_{p-1} \left( \begin{pmatrix} pm \\ 2m \end{pmatrix} \right) - B_{p-1} \right)
$$

As $2^{p-1} \equiv 1 \pmod{p}$ by Fermat’s little theorem, and $pB_{p-1} \equiv -1 \pmod{p}$ by [IR, p. 233], the desired (3.7) follows at once. □

Remark 3.2. Let $p$ be an odd prime not dividing $m \in \mathbb{Z}^+$. By [GS, pp. 125–126] or [Su5, Corollary 2.1],

$$
B_{p-1} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) - B_{p-1} \equiv - \sum_{k=1}^{[pm/m]} \frac{1}{k} \pmod{p} \quad \text{for } n = 0, \ldots, m - 1.
$$

Combining this with (3.7) we get that

$$
\frac{(-1)^{|pn/m|}}{2} E_{p-2} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) + \frac{2^{p-1} - 1}{p} 
\equiv \sum_{k=1}^{[pm/m]} \frac{1}{k} - \sum_{k=1}^{[pm/(2m)]} \frac{1}{k} = \sum_{k=1}^{[pm/m]} \frac{(-1)^{k-1}}{k} \pmod{p}
$$

for every $n = 0, \ldots, m - 1$. In light of Lemma 3.3, we can also deduce from (3) and (4) of [GS] the following congruences with $n \in \mathbb{Z}^+$ and $(m, n) = 1$.

$$
(-1)^{|pn/m|} E_{p-2} \left( \begin{pmatrix} pm \\ m \end{pmatrix} \right) \equiv \begin{cases} 
\left( \frac{2}{n} \right) \frac{1}{p} P_{p-(2^n)} \pmod{p} & \text{if } m = 4, \\
\left( \frac{3}{n} \right) \frac{5}{p} F_{p-(3^n)} + \frac{2^{p-2} - 2}{p} \pmod{p} & \text{if } m = 5, \\
\left( \frac{3}{n} \right) \frac{5}{p} S_{p-(3^n)} \pmod{p} & \text{if } m = 6,
\end{cases}
$$

(3.8)

where $(-)$ denotes the Jacobi symbol, and the sequences $(F_k)_{k \in \mathbb{N}}$, $(P_k)_{k \in \mathbb{N}}$ and $(S_k)_{k \in \mathbb{N}}$ are defined as follows:

- $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \in \mathbb{N}$;
- $P_0 = 0$, $P_1 = 1$, and $P_{k+2} = 2P_{k+1} + P_k$ for $k \in \mathbb{N}$;
- $S_0 = 0$, $S_1 = 1$, and $S_{k+2} = 4S_{k+1} - S_k$ for $k \in \mathbb{N}$.
Proof of Theorem 1.3. Let \( k \in \mathbb{Z} \). By Theorem 1.2, for any integer \( l \geq \lfloor m/2 \rfloor \) we have

\[
\sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i a_m(i) \left( (-1)^k \left( \begin{array}{c} 2l - 2i \\ k + l - i \end{array} \right)_m - \frac{\delta_{l-i,0}}{m} \right) = (1 + \delta_{(-1)^m,1})(-1)^l \delta_{l,\lfloor m/2 \rfloor} - \frac{\delta_{l,\lfloor m/2 \rfloor}}{m} \left(-1\right)^{l-1} a_m \left( \frac{m}{2} \right) = 0;
\]

also

\[
\sum_{j=0}^{\lfloor (m+1)/2 \rfloor} (-1)^j b_m(j) \left( \begin{array}{c} 2l - 2j \\ k + l - j \end{array} \right)_m = 0
\]

for all integers \( l \geq \lfloor (m+1)/2 \rfloor \). This, together with Lemma 3.1, yields (1.13), and also (1.14) in the case \((q,2m) = 1\).

By Lemma 3.1, \( U_l^{(q)}(m,n) \in \mathbb{Z} \) for every \( l = 1, \ldots, \lfloor (m+1)/2 \rfloor \); by Theorem 1.2, \( a_m(i) \in \mathbb{Z} \) if \( 0 < i \leq \lfloor m/2 \rfloor \). Thus, in view of (1.13), we have \( U_l^{(q)}(m,n) \in \mathbb{Z} \) for each \( l = 1, 2, 3, \ldots \). If \((q,2m) = 1\), then \( V_l^{(q)}(m,n) = 2U_l^{(q)}(2m,n) - U_l^{(q)}(m,n) \in \mathbb{Z} \) for all \( l \in \mathbb{Z}^+ \).

Now assume that \( m \nmid n \), and let \( p \) be an odd prime with \( p \equiv \pm q \pmod{2m} \). By Lemma 3.3 and (1.12),

\[
(-1)^{\lfloor pm/m \rfloor} E_{p-2} \left( \left\{ \frac{pm}{m} \right\} \right) + \frac{2p - 2}{p} = 2 \left( B_{p-1} \left( \left\{ \frac{pm}{2m} \right\} \right) - B_{p-1} \right) - 2 \left( B_{p-1} \left( \left\{ \frac{pm}{m} \right\} \right) - B_{p-1} \right)
\]

\[
= \frac{2m}{p} \left( U_p^{(q)}(2m,n) - 1 \right) - \frac{m}{p} \left( U_p^{(q)}(m,n) - 1 \right) = \frac{m}{p} \left( V_p^{(q)}(m,n) - 1 \right) \quad (\text{mod } p).
\]

This proves (1.16). We are done. \( \square \)

We can also prove (1.13) by determining the characteristic polynomial

\[
f_m(x) := \prod_{0 < k \leq \lfloor m/2 \rfloor} \left( x - \left( 2 - e^{2\pi i k/m} - e^{-2\pi i k/m} \right) \right)
\]

of the recurrence \((U_l^{(q)}(m,n))_{l \in \mathbb{N}}\) of order \( \lfloor m/2 \rfloor \). If \( m \) is even, then

\[
f_m(x) = \prod_{0 < k \leq m/2} \left( x - 2 - e^{2\pi i (m/2 - k)/m} - e^{-2\pi i (m/2 - k)/m} \right)
\]

\[
= \prod_{j=0}^{m/2-1} \left( x - 2 - 2 \cos \frac{2j\pi}{m} \right) = (x-4) \prod_{0 < k < m/2} \left( x - 4 \cos^2 \frac{k\pi}{2m} \right).
\]
If $m$ is odd, then

$$f_m(x) = \prod_{0 < j \leq (m-1)/2} \left( x - 2 - e^{2\pi i (m-2j)/(2m)} - e^{-2\pi i (m-2j)/(2m)} \right)$$

$$= \prod_{0 < k < m \atop 2|k} \left( x - 2 - 2 \cos \frac{2k\pi}{2m} \right) = \prod_{0 < k < m \atop 2|k} \left( x - 4 \cos^2 \frac{k\pi}{2m} \right).$$

So $f_m(x)$ can be determined with the help of the following lemma.

**Lemma 3.4.** Let $n$ be any positive integer. Then

$$\prod_{0 < k < n \atop 2|k - \delta} \left( x - 4 \cos^2 \frac{k\pi}{2n} \right) = \begin{cases} C_n(x) & \text{if } \delta = 0, \\ D_n(x) & \text{if } \delta = 1, \end{cases}$$

where

$$C_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n-1-i}{i} x^{\lfloor (n-1)/2 \rfloor - i}$$

and

$$D_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} x^{\lfloor n/2 \rfloor - i}.$$

**Proof.** It is well known that $\cos(n\theta) = T_n(\cos \theta)$ and $\sin(n\theta) = \sin \theta \cdot U_{n-1}(\cos \theta)$, where the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$ are given by

$$T_n(x) = \frac{n}{2} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{(n-1-i)!}{i!(n-2i)!} (2x)^{n-2i}$$

and

$$U_{n-1}(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \frac{(n-1-i)!}{i!(n-2i)!} (2x)^{n-1-2i}.$$

If $n$ is even, then $T_n(x) = D_n(4x^2)/2$ and $U_{n-1}(x) = 2xC_n(4x^2)$; if $n$ is odd, then $T_n(x) = xD_n(4x^2)$ and $U_{n-1}(x) = C_n(4x^2)$.

As $U_{n-1}(\cos \frac{k\pi}{2n}) = 0$ for those even $0 < k < n$, the $2\lfloor (n-1)/2 \rfloor$ distinct numbers $\pm \cos \frac{k\pi}{2n}$ ($0 < k < n$, $2 \mid k$) are zeroes of the polynomial $C_n(4x^2)$ of degree $2\lfloor (n-1)/2 \rfloor$. Similarly, since $T_n(\cos \frac{k\pi}{2n}) = 0$ for those odd $0 < k < n$, the $2\lfloor n/2 \rfloor$
distinct numbers $\pm \cos \frac{k\pi}{2n}$ ($0 < k < n$, $2 \nmid k$) are zeroes of the polynomial $D_n(4x^2)$ of degree $2[n/2]$. So

$$C_n(4x^2) = \prod_{0<k<n \atop 2 \nmid k} \left(4x^2 - 4\cos^2 \frac{k\pi}{2n}\right) \text{ and } D_n(4x^2) = \prod_{0<k<n \atop 2 \nmid k} \left(4x^2 - 4\cos^2 \frac{k\pi}{2n}\right).$$

Therefore (3.10) holds. □

**Remark 3.3.** For each $n \in \mathbb{Z}^+$, by Lemma 3.4 we have

$$C_n(x) = \prod_{0<k<n/2} \left(x - 4\cos^2 \frac{k\pi}{n}\right) = \prod_{d|n} A_d(x),$$

where

$$A_d(x) = \prod_{0<c<d/2 \atop (c,d)=1} \left(x - 4\cos^2 \frac{c\pi}{d}\right).$$

Applying the Möbius inversion formula we obtain

$$A_n(x) = \prod_{d|n} C_d(x)^{\mu(n/d)},$$

which makes the polynomial $A_n(x)$ (introduced in [Su2]) computable.

4. **Appendix: Explicit Values of** $a_m(i)$ **and** $b_m(j)$ **for** $2 \leq m \leq 12$

| $m$ | $i$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | 4   |     |     |     |     |     |     |
| 3   |     | 3   |     |     |     |     |     |
| 4   |     | 6   | 8   |     |     |     |     |
| 5   |     | 5   | 5   |     |     |     |     |
| 6   |     | 8   | 19  | 12  |     |     |     |
| 7   |     | 7   | 14  | 7   |     |     |     |
| 8   |     | 10  | 34  | 44  | 16  |     |     |
| 9   |     | 9   | 27  | 30  | 9   |     |     |
| 10  |     | 12  | 53  | 104 | 85  | 20  |     |
| 11  |     | 11  | 44  | 77  | 55  | 11  |     |
| 12  |     | 14  | 76  | 200 | 259 | 146 | 24  |
Table 2: Values of $b_m(j)$ with $2 \leq m \leq 12$

| $j$ | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|
| 2   | 2  |    |    |    |    |    |
| 3   | 5  | 4  |    |    |    |    |
| 4   | 4  | 2  |    |    |    |    |
| 5   | 7  | 13 | 4  |    |    |    |
| 6   | 6  | 9  | 2  |    |    |    |
| 7   | 9  | 26 | 25 | 4  |    |    |
| 8   | 8  | 20 | 16 | 2  |    |    |
| 9   | 11 | 43 | 70 | 41 | 4  |    |
| 10  | 10 | 35 | 50 | 25 | 2  |    |
| 11  | 13 | 64 | 147| 155| 61 | 4  |
| 12  | 12 | 54 | 112| 105| 36 | 2  |

Acknowledgment. The author thanks the referee for his/her helpful comments.

References

[C] D. Callan, *A combinatorial proof of Sun’s “curious” identity*, Integers: Electron. J. Combin. Number Theory 4 (2004), #A05, 6pp.

[CC] W. Chu and L.V.D. Claudio, *Jensen proof of a curious binomial identity*, Integers: Electron. J. Combin. Number Theory 3 (2003), #A20, 3pp.

[D] L. E. Dickson, *History of the Theory of Numbers*, Vol. I, AMS Chelsea Publ., 1999.

[EM] S. B. Ekhad and M. Mohammed, *A WZ proof of a “curious” identity*, Integers: Electron. J. Combin. Number Theory 3 (2003), #A6, 2pp.

[G] H. W. Gould, *Combinatorial Identities*, Morgantown, W. Va., 1972.

[Gr] A. Granville, *Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers*, in: Organic mathematics (Burnaby, BC, 1995), 253–276, CMS Conf. Proc., 20, Amer. Math. Soc., Providence, RI, 1997.

[GS] A. Granville and Z. W. Sun, *Values of Bernoulli polynomials*, Pacific J. Math. 172 (1996), 117–138.

[H] C. Helou, *Norm residue symbol and cyclotomic units*, Acta Arith. 73 (1995), 147–188.

[IR] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Graduate Texts in Math., Vol. 84, Springer, New York, 1990.

[MS] D. Merlini and R. Sprugnoli, *A Riordan array proof of a curious identity*, Integers: Electron. J. Combin. Number Theory 2 (2002), #A8, 3pp.

[PP] A. Panholzer and H. Prodinger, *A generating functions proof of a curious identity*, Integers: Electron. J. Combin. Number Theory 2 (2002), #A6, 3pp.

[S] Z. H. Sun, *Combinatorial sum $\sum_{k \equiv r \pmod{m}} \binom{r}{k}$ and its applications in number theory. I*, Nanjing Univ. J. Math. Biquarterly 9 (1992), no. 2, 227–240.

[SS] Z. H. Sun and Z. W. Sun, *Fibonacci numbers and Fermat’s last theorem*, Acta Arith. 60 (1992), 371–388.

[Su1] Z. W. Sun, *A congruence for primes*, Proc. Amer. Math. Soc. 123 (1995), 1341–1346.
[Su2] Z. W. Sun, *On the sum $\sum_{k \equiv r \mod m} \binom{n}{k}$ and related congruences*, Israel J. Math. **128** (2002), 135–156.

[Su3] Z. W. Sun, *A curious identity involving binomial coefficients*, Integers: Electronic J. Combin. Number Theory **2** (2002), #A4, 8pp.

[Su4] Z. W. Sun, *General congruences for Bernoulli polynomials*, Discrete Math. **262** (2003), 253–276.

[Su5] Z. W. Sun, *Binomial coefficients and quadratic fields*, Proc. Amer. Math. Soc. **134** (2006), 2213–2222.

[ST] Z. W. Sun and R. Tauraso, *Congruences for sums of binomial coefficients*, J. Number Theory **126** (2007), 287–296.

[SW] Z. W. Sun and K. J. Wu, *An extension of a curious binomial identity*, Int. J. Mod. Math. **2** (2007), 247–251.