PARABOLIC FREQUENCY MONOTONICITY ON RICCI FLOW
AND RICCI-HARMONIC FLOW WITH BOUNDED CURVATURES

CHUANHUA LI, YI LI, AND KAIRUI XU

ABSTRACT. In this paper, we study the monotonicity of parabolic frequency motivated by [16] under the Ricci flow and the Ricci-harmonic flow on manifolds. Here we consider two cases: one is the monotonicity of parabolic frequency for the solution of linear heat equation with bounded Bakry-Émery Ricci curvature, and another case is the monotonicity of parabolic frequency for the solution of heat equation with bounded Ricci curvature.

1. INTRODUCTION

In 1979, the (elliptic) frequency functional for a harmonic function $u(x)$ on $\mathbb{R}^n$ was introduced by Almgren in [1], which is defined by

$$N(r) = \frac{r \int_{B(r,p)} |\nabla u(x)|^2 dx}{\int_{\partial B(r,p)} u^2(x) dA},$$

where $dA$ is the induced $(n-1)$-dimensional Hausdorff measure on $\partial B(r,p)$ and $p$ is a fixed point in $\mathbb{R}^n$. Almgren obtained that $N(r)$ is monotone nondecreasing for $r$, and he used this property to investigate local regularity harmonic of functions and minimal surfaces. Next, Garofalo-Lin [8, 9] considered the monotonicity of frequency functional on Riemannian manifold to study the unique continuation for elliptic operators. The frequency functional was also used to estimate the size of nodal sets in [20, 21]. For more applications, see [6, 14, 15, 17, 26].

The parabolic frequency for the solution of heat equation on $\mathbb{R}^n$ was introduced by Poon in [24], and Ni [23] considered the case when $u(t)$ is a holomorphic function, both of them showed that the parabolic frequency is nondecreasing. Besides, on Riemannian manifolds, the monotonicity of the parabolic frequency was obtained by Colding-Minicozzi [7] through the drift Laplacian operator. Using the matrix Harnack’s inequality in [12], Li-Wang [19] investigated the parabolic frequency on compact Riemannian manifolds and the 2-dimensional Ricci flow. For higher dimension, the Ricci flow

$$\partial_t g(t) = -2\text{Ric}(g(t))$$

is introduced by Hamilton in [13] to study the compact three-manifolds with positive Ricci curvature, which is a special case of the Poincaré conjecture finally proved by Perelman in [10, 11]. Hamilton [13] obtained the short time existence and uniqueness of the Ricci flow on compact manifolds, and Shi [25] obtained a short time solution of the Ricci flow on a complete noncompact manifold whose uniqueness with bounded Riemann curvature was proved by Chen-Zhu in [5].
In [16], Julius-Dain defined the following parabolic frequency for a solution \( u(t) \) of the heat equation

\[
U(t) = \frac{\tau \| \nabla g(t)u(t) \|^2_{L^2(d\nu)}}{\| u(t) \|^2_{L^2(d\nu)}} e^{-\frac{1}{1-\alpha} t}
\]

where \( \tau(t) \) is the backwards time, \( \kappa(t) \) is the time-dependent function and \( d\nu \) is the weighted measure. And they proved that parabolic frequency \( U(t) \) for the solution of heat equation is monotone increasing along the Ricci flow with the bounded Bakry-Émery Ricci curvature.

In this paper, besides considering the Ricci flow, we also consider the Ricci-harmonic flow introduced in [18, 22], which was motivated by Einstein vacuum equations in general relativity, where \( g(t) \) is a family of Riemannian metrics, \( \phi(t) \) is a family of smooth functions, and \( \alpha(t) \) is a time-dependent positive function. Müller [22] got the short time existence and uniqueness of the Ricci-harmonic flow.

The purpose of this paper is to study the monotonicity of parabolic frequency on Ricci flow (1.1) and the Ricci-harmonic flow (1.2) on \([0, T)\) with \( T \in (0, +\infty)\). Here, we consider the following two cases.

**A. Parabolic frequency for linear heat equation.** We consider the following linear heat equation

\[
(\partial_t - \Delta_{g(t)})u(t) = a(t)u(t),
\]

where \( a(t) \) is a time-dependent smooth function, \( u(t) \) is a family of smooth functions on \( M \). Here, \( g(t) \) is evolved by the Ricci flow or the Ricci-harmonic flow.

The parabolic frequency \( U(t) \) for the solution of linear heat equation (1.3) on Ricci flow or Ricci-harmonic flow is denoted by

\[
U(t) = \exp \left\{ -\int_{t_0}^t h'(s) + \kappa(s) \frac{1}{h(s)} ds \right\} \frac{h(t) \int_M |\nabla g(t)u(t)|^2_{g(t)} dV_{g(t)}}{\int_M u^2(t) dV_{g(t)}}
\]

where \( h(t) \) and \( \kappa(t) \) are time-dependent functions, \([t_0, t] \subset (0, T), \) and \( dV_{g(t)} \) is defined in (2.4).

By calculating, on the Ricci flow, we obtain the monotonicity of the parabolic frequency for the solution of linear heat equation (1.3).

**Theorem 1.1.** Suppose \((M^n, g(t))_{t \in [0, T]}\) is the Ricci flow (1.1) with \( \text{Ric}_f(t) \leq \frac{\kappa(t)}{2M(t)} \) for \( g(t) \), where \( \text{Ric}_f(t) \) is defined in (2.7).

(i). If \( h(t) \) is a negative time-dependent function, then the parabolic frequency \( U(t) \) is monotone increasing along the Ricci flow.

(ii). If \( h(t) \) is a positive time-dependent function, then the parabolic frequency \( U(t) \) is monotone decreasing along the Ricci flow.

Similarly, for the Ricci-harmonic flow, we have the following:

**Theorem 1.2.** Suppose \((M^n, g(t), \phi(t))_{t \in [0, T]}\) is the Ricci-harmonic flow (1.2) with

\[
\text{Ric}_f(t) - \alpha(t) |\nabla g(t)\phi(t)|^2_{g(t)} \leq \frac{\kappa(t)}{2\alpha(t)} g(t)
\]
(i). If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Ricci-harmonic flow.

(ii). If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Ricci-harmonic flow.

**B. Parabolic frequency for heat equation.** As also in Li-Wang [19], we consider the following heat equation:

(1.5) \[ \partial_t u(t) = \Delta_{g(t)} u(t), \]

where $u(t)$ is a family of smooth functions on $M$.

We now investigate the monotonicity of the parabolic frequency for the solutions of heat equation (1.5) on the Ricci flow (1.1) or the Ricci-harmonic flow (1.2) with bounded Ric($g(t)$) instead of Ric$(t)$. Note that, here we assume $M$ is closed.

The parabolic frequency $U(t)$ on $[t_0, t_1] \subset (0, T)$ is denoted by

$$U(t) = \exp \left\{ - \int_{t_0}^{t} \left( \frac{h'(s)}{h(s)} + 2Kn + \frac{C(s)n}{2} + \frac{n}{s} \right) ds \right\} \frac{h(t) \int_M |\nabla_{g(t)} u(t)|^2_{g(t)} dV_{g(t)}}{\int_M u^2(t) dV_{g(t)}}$$

where $h(t)$ is a time-dependent function, $K$ is a positive constant, $n$ is the dimension of $M$, $C(t) = N/t$, $N = \log \frac{4}{\pi}$, $\eta = \min_M u(0)$ and $A = \max_M u(0)$. Observe that, $A$ and $\eta$ are both positive constants.

Then using Li-Yau Harnack estimate and Hamilton’s estimate on Ricci flow (1.1), we have the following:

**Theorem 1.3.** If $M^n$ is a closed Riemannian manifold, $(M, g(t))_{t \in [0, T]}$ is the solution of the Ricci flow (1.1) with bounded Ricci curvature, $0 \leq \text{Ric}(g(t)) \leq Kg(t)$, where $K$ is a positive constant, and $u(t)$ is a positive solution of heat equation (1.5) with $u(\cdot, 0) \leq A$, then the following holds:

(i). If $h(t)$ is a negative time-dependent function, the parabolic frequency $U(t)$ is monotone increasing along the Ricci flow.

(ii). If $h(t)$ is a positive time-dependent function, the parabolic frequency $U(t)$ is monotone decreasing along the Ricci flow.

With the same discussion of the Ricci-harmonic flow (1.2), we have

**Theorem 1.4.** Suppose $M^n$ is a closed Riemannian manifold, $(M, g(t), \phi(t))_{t \in [0, T]}$ is the solution of the Ricci-harmonic flow (1.2) with bounded Ricci curvature, $0 \leq \text{Ric}(g(t)) \leq Kg(t)$, $K$ is a positive constant, and $u(t)$ is a positive solution of heat equation (1.5) with $u(\cdot, 0) \leq A$. Moreover, if we assume that $\alpha(t)$ is a non-increasing function, bounded from below by $\tilde{\alpha}$, and $0 \leq d\phi(t) \otimes d\phi(t) \leq C g(t)$, where $C$ is a constant depending on $n$ and $\tilde{\alpha}$, then the following holds:

(i). If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Ricci-harmonic flow.

(ii). If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Ricci-harmonic flow.

2. Notations and definitions

In this section, we introduce some notations and definitions which will be used in the sequel. We use the notations in Hamilton’s paper [13], $\nabla_g$ is the Levi-Civita connection induced by $g$, Ric, R, $d\mu_g$ are Ricci curvature, scalar curvature and
volume form, respectively. The Laplacian of smooth time-dependent function $f(t)$ with respect to a family of Riemannian metrics $g(t)$ is

$$\Delta_{g(t)} f(t) = g^{ij}(t)[\partial_i \partial_j f(t) - \Gamma^k_{ij}(t) \partial_k f(t)]$$

where $\Gamma^k_{ij}(t)$ is Christoffel symbols of $g(t)$ and $\partial_i = \frac{\partial}{\partial x^i}$.

Under the Ricci flow (1.1), let $\tau(t) = T - t$ be the backwards time. For any time-dependent function $f(t) \in C^\infty(M)$, we denote

$$K(t) = \frac{1}{4\pi \tau(t)} e^{-f(t)}$$

to be the positive solution of the conjugate heat equation

$$(2.1) \quad \partial_t K(t) = -\Delta_{g(t)} K(t) + R(g(t))K(t).$$

From the definition of $K(t)$, we can prove the smooth function $f(t)$ satisfies the following equation

$$(2.2) \quad \partial_t f(t) = -\Delta_{g(t)} f(t) - R(g(t)) + |\nabla_{g(t)} f(t)|^2_{g(t)} + \frac{n}{2\tau(t)}.$$

With the same discussion, the backward heat-type equation with the conjugate heat equation under the Ricci-harmonic flow (1.2) is given by

$$(2.3) \quad \partial_t f(t) = -\Delta_{g(t)} f(t) - R(g(t)) + |\nabla_{g(t)} f(t)|^2_{g(t)} + \frac{n}{2\tau(t)} + \alpha(t) |\nabla_{g(t)} \phi(t)|^2_{g(t)}.$$

For (2.3), we can also define $K(t)$ as

$$K(t) = \frac{1}{4\pi \tau(t)} e^{-f(t)}$$

which satisfies the following conjugate heat equation

$$(2.4) \quad \partial_t K(t) = -\Delta_{g(t)} K(t) + R(g(t))K(t) - \alpha(t) |\nabla_{g(t)} \phi(t)|^2_{g(t)} K(t).$$

Then, under Ricci flow (1.1) or Ricci-harmonic flow (1.2), we can define the weighted measure

$$dV_{g(t)} := K(t) d\mu_{g(t)} = \frac{1}{4\pi \tau(t)} e^{-f(t)} d\mu_{g(t)}, \quad \int_M dV_{g(t)} = 1.$$

On the weighted Riemannian manifold $(M^n, g(t), dV_{g(t)})$, the weighted Bochner formula for $u$ is as follow

$$\Delta_{g(t), f(t)}(|\nabla_{g(t)} u|^2_{g(t)}) = 2|\nabla_{g(t)} u|^2_{g(t)} + 2\langle \nabla_{g(t)} u, \nabla_{g(t)} \Delta_{g(t), f(t)} u \rangle_{g(t)} + 2\text{Ric}_{f(t)}(\nabla_{g(t)} u, \nabla_{g(t)} u)$$

where

$$(2.7) \quad \text{Ric}_{f(t)} := \text{Ric}(g(t)) + \nabla_{g(t)}^2 f(t)$$

is the Bakry-Émery Ricci tensor introduced in [3], and

$$\Delta_{g(t), f(t)} u := e^{f(t)} \text{div}_{g(t)}(e^{-f(t)} \nabla_{g(t)} u) = \Delta_{g(t)} u - \langle \nabla_{g(t)} f(t), \nabla_{g(t)} u \rangle_{g(t)}$$

is the drift Laplacian operator for any smooth function $u$.

Under the Ricci flow (1.1), the volume form $d\mu_{g(t)}$ satisfies

$$\partial_t (d\mu_{g(t)}) = -R(g(t)) d\mu_{g(t)}$$

while under the Ricci-harmonic flow (1.2), the volume form $d\mu_{g}$ satisfies

$$\partial_t (d\mu_{g(t)}) = \left( -R(g(t)) + \alpha(t) |\nabla_{g(t)} \phi(t)|^2_{g(t)} \right) d\mu_{g(t)}.$$
Thus, the conjugate heat kernel measure \(dV_{g(t)}\) are both evolved by

\[
\partial_t (dV_{g(t)}) = -(\triangle_{g(t)} K(t)) d\mu_{g(t)} = -\frac{\triangle_{g(t)} K(t)}{K(t)} dV_{g(t)}.
\]

For convenient, we use \(\triangle_f\), \(\text{Ric}, \nabla, | \cdot |, dV\) to replace \(\triangle_{g(t),f(t)}\), \(\text{Ric}(g(t))\), \(\nabla_{g(t)}, | \cdot |_{g(t)}, dV_{g(t)}\). We always omit the time variable \(t\).

### 3. PARABOLIC FREQUENCY FOR THE LINEAR HEAT EQUATION

In this section, we consider parabolic frequency \(U(t)\) for the solution of the linear heat equation (\ref{heat equation}) under the Ricci flow (\ref{Ricci flow}) and the Ricci-harmonic flow (\ref{Ricci-harmonic flow}), respectively.

For a time-dependent function \(u = u(t) : M \times [t_0, t_1] \to \mathbb{R}\) with \(u(t), \partial_t u(t) \in W^{2,2}_0(dV_{g(t)})\) for all \(t \in [t_0, t_1] \subset (0, T)\), we denote by

\[
I(t) = \int_M u^2(t)dV_{g(t)},
\]

\[
D(t) = h(t) \int_M |\nabla_{g(t)} u(t)|^2_{g(t)} dV_{g(t)} = -h(t) \int_M (u(t), \triangle_{g(t),f(t)} u(t))_{g(t)} dV_{g(t)},
\]

\[
U(t) = \exp \left\{ -\int_{t_0}^t \frac{h'(s) + \kappa(s)}{h(s)} ds \right\} D(t),
\]

where \(h(t)\) and \(\kappa(t)\) are both time-dependent smooth functions.

#### 3.1. Parabolic frequency for the linear heat equation under Ricci flow.

At first, we study the parabolic frequency \(U(t)\) under the Ricci flow. Before doing it, we give some lemmas.

**Lemma 3.1.** For all \(u, v \in W^{1,2}_0(dV_{g(t)})\), the drift Laplacian operator \(\triangle_{g(t),f(t)}\) satisfies:

\[
\int_M (\triangle_{g(t),f(t)} u) v dV_{g(t)} = \int_M u(\triangle_{g(t),f(t)} v) dV_{g(t)}
\]

**Proof.** By a straight computation, we obtain

\[
\int_M (\triangle_{g(t),f(t)} u) v dV_{g(t)} = - (4\pi t)^{-\frac{d}{2}} \int_M e^{-f(t)} \langle \nabla_{g(t)} u, \nabla_{g(t)} v \rangle_{g(t)} d\mu_{g(t)}
\]

\[
= \int_M u(\triangle_{g(t),f(t)} v) dV_{g(t)}
\]

Thus, we get the desired result. \(\square\)

**Lemma 3.2.** For any \(u \in W^{2,2}_0(dV_{g(t)})\), we have

\[
\int_M |\nabla_{g(t)} u|^2_{g(t)} dV_{g(t)} = \int_M \left( (\triangle_{g(t),f(t)} u)^2_{g(t)} - \text{Ric}(f(t)) \langle \nabla_{g(t)} u, \nabla_{g(t)} u \rangle_{g(t)} \right) dV_{g(t)}
\]

**Proof.** This result has been proved in Lemma 1.13 of \cite{10}. \(\square\)

**Lemma 3.3.** Suppose \(u(t)\) is a family of smooth functions. Under the Ricci flow (\ref{Ricci flow}), the norm of \(|\nabla_{g(t)} u(t)|^2_{g(t)}\) satisfies the evolution equation:

\[
(\partial_t - \triangle_{g(t)}) |\nabla_{g(t)} u(t)|^2_{g(t)} = -2|\nabla_{g(t)} u(t)|^2_{g(t)} + 2\langle \nabla_{g(t)} u(t), \nabla_{g(t)} (\partial_t - \triangle_{g(t)}) u(t) \rangle_{g(t)}
\]
Proof. By calculating straightly, we have
\[ \partial_t |\nabla u|^2 = 2 \text{Ric}(\nabla u, \nabla u) + 2 \langle \nabla u, \nabla \partial_t u \rangle. \]
(3.5)
Together with the Bochner formula, yields
\[ (\partial_t - \nabla) |\nabla u|^2 = 2 \langle \nabla u, \nabla (\partial_t - \nabla) u \rangle - 2 |\nabla^2 u|^2. \]
(3.6)
Then we get the desired result. \( \square \)

**Theorem 3.4.** Suppose \((M^n, g(t))_{t \in [0,T]}\) is the Ricci flow (1.1) with \(\text{Ric}_f(t) \leq \frac{\kappa(t)}{2h(t)} g(t)\).

(i) If \(h(t)\) is a negative time-dependent function, then the parabolic frequency \(U(t)\) is monotone increasing along the Ricci flow.

(ii) If \(h(t)\) is a positive time-dependent function, then the parabolic frequency \(U(t)\) is monotone decreasing along the Ricci flow.

Proof. We only give the proof of (i). Our main purpose is to compute the \(I'(t)\) and \(D'(t)\). Under the Ricci flow (1.1), combining with the linear heat equation (1.3) and Lemma 3.3, we can obtain
\[ (\partial_t - \nabla) |\nabla u|^2 = 2 \langle \nabla u, \nabla (\partial_t - \nabla) u \rangle - 2 |\nabla^2 u|^2 \]
(3.7)
According to (2.9) and integration by parts, we get the derivative of \(I(t)\) as follow
\[ I'(t) = \int_M \left( 2 u \partial_t u - u^2 \frac{\triangle K}{K} \right) dV \]
(3.8)
\[ = \int_M (2 u \partial_t u - \triangle(u^2)) dV \]
\[ = \int_M (2 u \partial_t u - 2 u \triangle u - 2 |\nabla u|^2) dV \]
\[ = \frac{2}{h} D(t) + 2a(t)I(t). \]

If we write
\[ \hat{I}(t) = \exp \left\{ - \int_{t_0}^t 2a(s) ds \right\} I(t), \]
then we can easily find
\[ \hat{I}'(t) = -\frac{2}{h} \exp \left\{ - \int_{t_0}^t 2a(s) ds \right\} D(t). \]
(3.9)
Next it’s turn to compute the derivative of \(D(t)\). Using (2.9), (3.7) and the boundedness of Bakry-Émery Ricci tensor, we obtain
\[ D'(t) = h' \int_M |\nabla u|^2 dV + h \int_M \left( \partial_t |\nabla u|^2 - |\nabla u|^2 \frac{\triangle K}{K} \right) dV \]
(3.10)
\[ = h' \int_M |\nabla u|^2 dV + h \int_M (\partial_t - \nabla) |\nabla u|^2 dV \]
\[ = (2ah + h') \int_M |\nabla u|^2 dV - 2h \int_M |\nabla^2 u|^2 dV \]
\[ = (2ah + h') \int_M |\nabla u|^2 dV - 2h \int_M [|\nabla u|^2 - \text{Ric}_f(\nabla u, \nabla u)] dV \]
\[
\geq (\kappa + 2ah + h') \int_M |\nabla u|^2 dV - 2h \int_M |\triangle_f u|^2 dV
\]
\[
= \left( 2a + \frac{h' + \kappa}{h} \right) D(t) - 2h \int_M |\triangle_f u|^2 dV.
\]
Similarly, if we write
\[
\hat{D}(t) = \exp \left\{ - \int_{t_0}^t \left[ 2a(s) + \frac{h'(s) + \kappa(s)}{h(s)} \right] ds \right\} D(t),
\]
then we can find
\[
(3.11) \quad \hat{D}'(t) \geq -2h \exp \left\{ - \int_{t_0}^t \left[ 2a(s) + \frac{h'(s) + \kappa(s)}{h(s)} \right] ds \right\} \int_M |\triangle_f u|^2 dV.
\]
Finally, the parabolic frequency \( U(t) \) can be written as \( U(t) = \frac{\hat{D}(t)}{\hat{I}(t)} \). By (3.9) and (3.11), we can compute the derivative of \( U(t) \)
\[
(3.12) \quad \hat{I}^2(t) U''(t) = \hat{D}'(t) \hat{I}(t) - \hat{I}'(t) \hat{D}(t)
\]
\[
\geq -2h \exp \left\{ \int_{t_0}^t \left[ 4a(s) + \frac{h'(s) + \kappa(s)}{h(s)} \right] ds \right\} \cdot \left[ \left( \int_M |\triangle_f u|^2 dV \right) \cdot \left( \int_M |u|^2 dV \right) - \left( \int_M |\nabla u|^2 dV \right)^2 \right]
\]
\[
\geq -2h \exp \left\{ \int_{t_0}^t \left[ 4a(s) + \frac{h'(s) + \kappa(s)}{h(s)} \right] ds \right\} \cdot \left[ \left( \int_M \langle u(t), \triangle_f u \rangle dV \right)^2 - \left( \int_M |\nabla u|^2 dV \right)^2 \right]
\]
\[
= 0
\]
the last inequality is directly obtained by the definition of \( D(t) \) the Cauchy-Schwarz inequality.

Then we have the following corollary.

**Corollary 3.5.** Let \((M^n, g(t))_{t \in [0,T]}\) be Ricci flow \( \text{(1.1)} \) with \( \text{Ric}_f(t) \leq \frac{\kappa(t)}{2h(t)} g(t) \), where \( h(t) \) is negative. If \( u(\cdot, t_1) = 0 \), then \( u(\cdot, t) \equiv 0 \) for any \( t \in [t_0, t_1] \subset (0, T) \).

**Proof.** Recalling the definition of \( U(t) \), we get
\[
(3.13) \quad \frac{d}{dt} \log(U(t)) = \frac{I'(t)}{I(t)} = - \frac{2D(t)}{h(t) \hat{I}(t)} + 2a(t)
\]
\[
= - \frac{2}{h(t)} \exp \left\{ \int_{t_0}^t \frac{h'(s) + \kappa(s)}{h(s)} ds \right\} \hat{I}(t) = 2 \int_{t_0}^t a(t) dt.
\]
According to Theorem 3.4 and integrating (3.13) from \( t' \) to \( t_1 \) for any \( t' \in [t_0, t_1] \), yields
\[
\log I(t_1) - \log I(t') \geq -2 \int_{t'}^{t_1} \exp \left\{ \int_{t_0}^t \frac{h'(s) + \kappa(s)}{h(s)} ds \right\} \frac{U(t)}{h(t)} dt + 2 \int_{t'}^{t_1} a(t) dt
\]
\[
\geq -2U(t_0) \int_{t'}^{t_1} \exp \left\{ \int_{t_0}^t \frac{h'(s) + \kappa(s)}{h(s)} ds \right\} \frac{dt}{h(t)} + 2 \int_{t'}^{t_1} a(t) dt
\]
Since \(a(t), h(t)\) are finite, it follows from the last inequality that

\[
(3.14) \quad \frac{I(t_1)}{I(t')} \geq \exp \left( -2U(t_0) \int_{t'}^{t_1} \exp \left\{ \int_{t_0}^{t} \frac{h'(s) + \kappa(s)}{h(s)} \, ds \right\} \, dt \right) + 2 \int_{t'}^{t_1} a(t) \, dt
\]

which implies Corollary 3.5. \(\square\)

### 3.2. Parabolic frequency for the linear heat equation under the Ricci-harmonic flow

Next, we consider parabolic frequency under the Ricci-harmonic flow \((1.2)\). Similarly, we will give the following Lemma.

**Lemma 3.6.** Under the Ricci-harmonic flow \((1.2)\), for a family of smooth functions \(u(t)\) on \(M\), the norm of \(\nabla_{g(t)}(t)^2\) satisfies the evolution equation:

\[
(\partial_t - \triangle_{g(t)})|\nabla_{g(t)}(t)|^2 = 2\langle \nabla_{g(t)}(t), \nabla_{g(t)}(\partial_t - \triangle_{g(t)})u(t) \rangle_{g(t)} - 2|\nabla_{g(t)}(t)|^2_{g(t)} - 2\alpha(t)\langle \nabla_{g(t)}(t), \nabla_{g(t)}(t) \rangle_{g(t)}^2
\]

**Proof.** By calculating straightly, we have

\[
(3.15) \quad \partial_t |\nabla u|^2 = 2\text{Ric} \langle \nabla u, \nabla u \rangle - 2\alpha(t)\langle \nabla \phi, \nabla u \rangle^2 + 2\langle \nabla u, \nabla \partial_t u \rangle.
\]

Together with the Bochner formula, yields

\[
(3.16) \quad (\partial_t - \triangle)|\nabla u|^2 = 2\text{Ric} \langle \nabla u, \nabla u \rangle - 2\alpha(t)\langle \nabla \phi, \nabla u \rangle^2 + 2\langle \nabla u, \nabla \partial_t u \rangle
\]

Then we get the desired result. \(\square\)

**Theorem 3.7.** Suppose \((M^n, g(t), \phi(t))_{t \in [0, T]}\) is the Ricci-harmonic flow \((1.2)\) with \(\text{Ric}_{f(t)} - \alpha(t)\langle \nabla g(t)\phi(t), \cdot \rangle_{g(t)}^2 \leq \frac{\kappa(t)}{2h(t)} g(t)\).

(i) If \(h(t)\) is a negative time-dependent function, then the parabolic frequency \(U(t)\) is monotonically increasing along the Ricci-harmonic flow.

(ii) If \(h(t)\) is a positive time-dependent function, then the parabolic frequency \(U(t)\) is monotonically decreasing along the Ricci-harmonic flow.

**Proof.** With the same discussion in \((3.8)\), using integration by parts, we have

\[
(3.17) \quad I'(t) = -\frac{2}{h(t)} D(t) + 2a(t)I(t)
\]

According to the Bochner formula, together with Lemma 3.2 and Lemma 3.6, we have the derivative of \(D(t)\)

\[
(3.18) \quad D'(t) = h'(t) \int_M |\nabla u|^2 \, dV + h(t) \int_M \partial_t |\nabla u|^2 \, dV + \int_M |\nabla u|^2 \partial_t (dV)
\]

\[
= h'(t) \int_M |\nabla u|^2 \, dV + 2h(t) \int_M (\partial_t - \triangle)|\nabla u|^2 \, dV
\]

\[
= h'(t) \int_M |\nabla u|^2 \, dV + 2h(t) \int_M \left[ \langle \nabla u, \nabla (\partial_t - \triangle) u \rangle - |\nabla^2 u|^2 - \alpha(t)\langle \nabla \phi, \nabla u \rangle^2 \right] \, dV
\]

\[
= \left( \frac{h'(t)}{h(t)} + 2a(t) \right) D(t) - 2h(t) \int_M |\triangle_f u|^2 \, dV
\]
Combining with (3.17) and (3.21), we obtain

\[
\text{(3.21)}
\]

Thus, combining (3.17) and (3.19), applying Hölder’s inequality, we have the following corollary by Theorem 3.7:

\[
D'(t) \geq \left( \frac{h'(t) + \kappa(t)}{h(t)} + 2\alpha(t) \right) D(t) - 2h(t) \int_M |\triangle_f u|^2 dV
\]

Similarly, if \( h(t) > 0 \) and recalling (3.18), we have

\[
D'(t) \leq \left( \frac{h'(t) + \kappa(t)}{h(t)} + 2\alpha(t) \right) D(t) - 2h(t) \int_M |\triangle_f u|^2 dV.
\]

Combining with (3.17) and (3.21), we obtain \( t^2(t)U'(t) \leq 0 \). Thus, we prove this theorem.

We define the first nonzero eigenvalue of the Ricci-harmonic flow \( (M^n, g(t), \phi(t)) \) with the weighted measure \( dV_{g(t)} \) by

\[
\lambda(t) = \inf \left\{ \frac{\int_M |\nabla_g(t) u|^2 dV_{g(t)}}{\int_M u^2 dV_{g(t)}} \mid u \in C^\infty(M) \setminus \{0\} \right\}.
\]

Then we have the following corollary by Theorem 3.7.

**Corollary 3.8.** If \( (M^n, g(t), \phi(t)) \in [0, T) \) is the Ricci-harmonic flow with \( \text{Ric}_f(t) - \alpha(t) \langle \nabla_{g(t)} \phi(t), \cdot \rangle_{g(t)}^2 \leq \frac{\phi(t)}{2d(t)} g(t) \), then for any \( t \in [t_0, t_1] \subset (0, T) \), the following holds.

(i) If \( h(t) < 0 \), then \( h(t) \lambda(t) \) is a monotone increasing function.

(ii) If \( h(t) > 0 \), then \( h(t) \lambda(t) \) is a monotone decreasing function.

If we take \( \phi(t) \) as a time-dependent smooth function, we can get similar conclusion with Corollary 3.8 for Ricci flow.
Corollary 3.9. Suppose \((M^n, g(t), \phi(t))_{t \in [0,T)}\) is the Ricci-harmonic flow (1.2) with Ric\(_f(t) - \alpha(t)(\nabla g(t)\phi(t), \cdot)^2 \leq \frac{\kappa(t)}{2} g(t)\), where \(h(t)\) is negative. If \(h(t) < 0\) and \(u(\cdot, t_1) = 0\), then \(u(\cdot, t) \equiv 0\) for any \(t \in [t_0, t_1] \subset (0, T)\).

Proof. With the same discussion of Corollary 3.5, we obtain the desired result. □

4. PARABOLIC FREQUENCY FOR HEAT EQUATION

In this section, we study the parabolic frequency for the solution of heat equation (1.5) under the closed Ricci flow (1.1) and the closed Ricci-harmonic flow (1.2), here we use Li-Yau Harnack inequality in [2] and Hamilton's estimate in [12] to replace bounded Bakry-Émery Ricci curvature by bounded Ricci curvature.

For a function \(u = u(t) : M \times [t_0, t_1] \to \mathbb{R}^+\) with \(u(t), \partial_t u(t) \in W^2_0(dV_{g(t)})\) and for all \(t \in [t_0, t_1] \subset (0, T)\), we define the parabolic frequency \(U(t)\)

\[
I(t) = \int_M u^2(t) dV_{g(t)},
\]

\[
D(t) = h(t) \int_M |\nabla_{g(t)} u(t)|^2 dV_{g(t)},
\]

\[
U(t) = \exp \left\{ - \int_{t_0}^t \left( \frac{h'(s)}{h(s)} + 2Kn + \frac{C(s)}{2} n + \frac{n}{s} \right) ds \right\} \frac{D(t)}{I(t)}.
\]

where \(h(t)\) is a time-dependent function, \(K\) is a positive constant, \(n\) is the dimension of \(M\), \(C(t) = N/t\), \(N = \log \frac{A}{\eta}\), \(\eta = \min_M u(0)\) and \(A = \max_M u(0)\). Observe that, \(A\) and \(\eta\) are both positive constants.

4.1. Parabolic frequency for heat equation under closed Ricci flow. Before computing the monotonicity of \(U(t)\), we give some lemmas from [2] and [12].

Lemma 4.1. If \(M^n\) is a closed Riemannian manifold, \((M^n, g(t))\) is the solution of the Ricci flow (1.1) and \(u(t)\) is a positive solution of heat equation (1.5) with \(u(\cdot, 0) \leq A\), then we have the following estimate:

\[
t \frac{|\nabla_{g(t)} u(t)|^2_{g(t)}}{u(t)} \leq u^2(t) \log \left( \frac{A}{u(t)} \right).
\]

Proof. This Lemma has been proved in Theorem 2.4 of [2] and Theorem 3.1(b) of [27].

Lemma 4.2. If \(M^n\) is a closed Riemannian manifold, \((M, g(t))\) is the solution of the Ricci flow (1.1) with bounded Ricci curvature, \(0 \leq \text{Ric}(g(t)) \leq Kg(t)\), \(K\) is a positive constant, and \(u(t)\) is a positive solution of heat equation (1.5), then we have the following estimate:

\[
\frac{|\nabla_{g(t)} u(t)|^2_{g(t)}}{u(t)} - \partial_t u(t) \leq \frac{n}{2t} u(t) + Kn u(t)
\]

Proof. This lemma is from Theorem 2.9 in [2], here we give another method from [12] to prove it. Recalling Lemma 4.1, we have

\[
\frac{\partial}{\partial t} \frac{|\nabla u|^2}{u} = \Delta \frac{|\nabla u|^2}{u} - 2 \frac{\nabla_i \nabla_j u}{u} - 2 \frac{\nabla_i u \nabla_j u}{u} \frac{|\nabla u|^2}{u},
\]
taking trace over the second term, yields

\[ (4.5) \quad \left| \nabla_i \nabla_j u - \frac{\nabla_i u \nabla_j u}{u} \right|^2 \geq \frac{1}{n} \left( \Delta u - \frac{|\nabla u|^2}{u} \right)^2 = \frac{1}{n} \left( \partial_t u - \frac{|\nabla u|^2}{u} \right)^2. \]

Next, we need calculate the following

\[ (4.6) \quad \frac{\partial}{\partial t} (\partial_t u) = 2 R_{ij} \nabla_i \nabla_j u + \Delta (\partial_t u) \]

If we let

\[ \psi = \partial_t u - \frac{|\nabla u|^2}{u} + \frac{n}{2t} u + Kn u \]

together with (4.4)-(4.6), then

\[ (4.7) \quad \frac{\partial \psi}{\partial t} \geq \Delta \psi + \frac{2}{nu} \left( \partial_t u - \frac{|\nabla u|^2}{u} \right)^2 + 2 R_{ij} \nabla_i \nabla_j u - \frac{n}{2t^2} u. \]

Now, when \( \psi \leq 0 \), we have

\[ 0 \leq Kn u + \frac{n}{2t} u \leq \frac{|\nabla u|^2}{u} - \partial_t u. \]

which implies

\[ \frac{\partial \psi}{\partial t} \geq \Delta \psi \quad \text{when} \quad \psi \leq 0. \]

Note that \( \psi \to +\infty \) as \( t \to 0 \), according to the maximum principle, we obtain \( \psi \geq 0 \) for all time. Then we proved this Lemma.

**Theorem 4.3.** If \( M^n \) is a closed Riemannian manifold, \( (M, g(t))_{t \in [0, T)} \) is the solution of the Ricci flow (1.1) with bounded Ricci curvature, \( 0 \leq \text{Ric}(g(t)) \leq Kg(t) \), where \( K \) is a positive constant, and \( u(t) \) is a positive solution of heat equation (1.5) with \( u(\cdot, 0) \leq A \), then the following holds.

(i). If \( h(t) \) is a negative time-dependent function, the parabolic frequency \( U(t) \) is monotone increasing along the Ricci flow.

(ii). If \( h(t) \) is a positive time-dependent function, the parabolic frequency \( U(t) \) is monotone decreasing along the Ricci flow.

**Proof.** Before discussing the monotonicity of \( U(t) \), we need calculate the derivative of \( I(t) \) and \( D(t) \). Using Young’s identity and Lemma 4.1, we get the derivative of \( I(t) \).

\[ (4.8) \quad I'(t) = \frac{d}{dt} \left( \int_M u^2 dV \right) \]
\[ = 2 \int_M (u \cdot \partial_t u - |\nabla u|^2) dV - 2 \int_M u \Delta u dV \]
\[ \geq - \left( \frac{n}{t} + 2Kn \right) I(t) - 2 \int_M u \Delta u dV \]
\[ \geq - \left( \frac{n}{t} + 2Kn + \frac{C(t)}{2n} \right) I(t) - \frac{2}{C(t)n} \int_M |\nabla u|^2 dV. \]

For the derivative of \( D(t) \), from Lemma 3.3, we have

\[ (4.9) \quad D'(t) = h'(t) \int_M |\nabla u|^2 dV + h(t) \frac{d}{dt} \left( \int_M |\nabla u|^2 dV \right) \]
\[ h'(t) \int_M |\nabla u|^2 dV + h(t) \int_M (\partial_t - \triangle) |\nabla u|^2 dV \]
\[ = h'(t) \int_M |\nabla u|^2 dV - 2h(t) \int_M |\nabla^2 u|^2 dV. \]

If \( h(t) < 0 \), then combing (4.8) and (4.9), together with Lemma 4.1, yields
\[ I^2(t)U'(t) \geq \exp \left\{ - \int_{t_0}^t \left( \frac{h'(s)}{h(s)} + \frac{2Kn + C(s) + \frac{n}{s}}{n} \right) ds \right\} \]
\[ \cdot \left[ -2hI(t) \left( \int_M |\nabla^2 u|^2 dV \right) + \frac{2h}{C(t)n} \left( \int_M |\Delta u|^2 dV \right) \left( \int_M |\nabla u|^2 dV \right) \right] \]
\[ \geq \exp \left\{ - \int_{t_0}^t \left( \frac{h'(s)}{h(s)} + \frac{2Kn + C(s) + \frac{n}{s}}{n} \right) ds \right\} \]
\[ \cdot \left[ -2hI(t) \left( \int_M |\Delta u|^2 dV \right) + \frac{2h}{C(t)n} \cdot \frac{M}{t} I(t) \left( \int_M |\Delta u|^2 dV \right) \right] \]
\[ = 0 \]

where we take trace over \( |\nabla^2 u|^2 \) and let \( C(t) = \frac{n}{s} \).

On the other hand, if \( h(t) > 0 \), similarly, we have
\[ I^2(t)U'(t) \leq \exp \left\{ - \int_{t_0}^t \left( \frac{h'(s)}{h(s)} + \frac{2Kn + C(s) + \frac{n}{s}}{n} \right) ds \right\} \]
\[ \cdot \left[ -2hI(t) \left( \int_M |\nabla^2 u|^2 dV \right) + \frac{2h}{C(t)n} \left( \int_M |\Delta u|^2 dV \right) \left( \int_M |\nabla u|^2 dV \right) \right] \]
\[ \leq \exp \left\{ - \int_{t_0}^t \left( \frac{h'(s)}{h(s)} + \frac{2Kn + C(s) + \frac{n}{s}}{n} \right) ds \right\} \]
\[ \cdot \left[ -2hI(t) \left( \int_M |\Delta u|^2 dV \right) + \frac{2h}{C(t)n} \cdot \frac{M}{t} I(t) \left( \int_M |\Delta u|^2 dV \right) \right] \]
\[ = 0. \]

Thus we get our result. \( \square \)

We define the first nonzero eigenvalue of the Ricci flow \( (M^n, g(t)) \) with the weighted measure \( dV_{g(t)} \) by
\[ \lambda(t) = \inf \left\{ \frac{\int_M |\nabla u|^2 dV_{g(t)}}{\int_M u^2 dV_{g(t)}} \mid 0 < u \in C^\infty(M) \setminus \{0\} \right\} \]

Then we have the following corollary by Theorem 4.3.

**Corollary 4.4.** If \( M^n \) is a closed Riemannian manifold, \( (M, g(t))_{t \in [0, T)} \) is the solution of the Ricci flow (1.1) with bounded Ricci curvature, \( 0 \leq \text{Ric}(g(t)) \leq Kg(t) \), where \( K \) is a positive constant, and \( u(t) \) is a positive solution of heat equation (1.3) with \( u(0) \leq A \), then for any \( t \in [t_0, t_i] \subset (0, T) \), the following holds.

(i) If \( h(t) \) is a negative time-dependent function, then \( h(t)\lambda(t) \) is a monotone increasing function.
(ii). If \( h(t) \) is a positive time-dependent function., then \( h(t)\lambda(t) \) is a monotone decreasing function.

4.2. Parabolic frequency for heat equation under closed Ricci-harmonic flow. For Ricci-harmonic flow, we can prove that the conclusions of Lemma 4.1 and Lemma 4.2 still hold.

Lemma 4.5. Suppose \( M^n \) is a closed Riemannian manifold, \((M^n, g(t), \phi(t))\) is the solution of the Ricci-harmonic flow \((1.2)\), and \( u(t) \) is a positive solution of heat equation \((1.5)\) with \( u(\cdot,0) \leq A \). Moreover, if we assume \( \alpha(t) \geq \bar{\alpha} > 0 \) and \( d\phi \otimes d\phi \geq 0 \), then we have the following estimate:

\[
 t |\nabla u(t)|^2_{g(t)} \leq u^2(t) \log \left( \frac{A}{u(t)} \right)
\]

Proof. The proof of this lemma is similar with Theorem 3.1(b) of [27]. According to the maximum principle and the heat equation \((1.5)\), combining \( u(\cdot,0) \leq A \), we have

\[
 \sup_{M \times [0,T]} u(x,t) \leq A.
\]

Using the Bochner formula \((2.6)\), we compute

\[
 \frac{\partial}{\partial t} \frac{|\nabla u|^2}{u} = \Delta \frac{|\nabla u|^2}{u} - 2 \frac{u}{u} \left| \nabla_i \nabla_j u - \frac{\nabla_{ij} u}{u} \right|^2 - \frac{2}{u} \alpha(t) (\nabla \phi, \nabla u)^2
\]

and

\[
 \frac{\partial}{\partial t} \left( u \log \frac{A}{u} \right) = \Delta \left( u \log \frac{A}{u} \right) + \frac{|\nabla u|^2}{u}.
\]

Then we put

\[
 \psi = t |\nabla u|^2 - u \log \frac{A}{u}.
\]

Obviously, \( \psi \leq 0 \) at \( t = 0 \), together with \((4.10)\) and \((4.11)\), throwing away \( \alpha(t) (\nabla \phi, \nabla u)^2 \) with \( d\phi \otimes d\phi \geq 0 \) and \( \alpha(t) \geq \bar{\alpha} > 0 \), yields

\[
 \frac{\partial}{\partial t} \psi \leq \Delta \psi.
\]

Recalling the maximum principle, we have \( \psi \leq 0 \) for all time, which means

\[
 t |\nabla u|^2 \leq u^2 \log \left( \frac{A}{u} \right).
\]

Thus we obtain this lemma.

Lemma 4.6. Suppose \( M^n \) is a compact Riemannian manifold, \((M,g(t),\phi(t))\) is the solution of the Ricci-harmonic flow \((1.2)\) with bounded Ricci curvature, \( 0 \leq \text{Ric}(g(t)) \leq Kg(t) \), \( K \) is a positive constant, and \( u(t) \) is a positive solution of heat equation \((1.5)\). Moreover, if we assume \( \alpha(t) \) is a non-increasing function, bounded from below by \( \bar{\alpha} \), and \( 0 \leq d\phi \otimes d\phi \leq \frac{C}{T} g(t) \), where \( C \) is a constant depending on \( n \) and \( \bar{\alpha} \), then we have the following estimate:

\[
 \frac{|\nabla g(t) u(t)|^2_{g(t)}}{u(t)} - \partial_t u(t) \leq \frac{C_n}{2t} u(t) + Ku(t)
\]

where \( C_n = \frac{n}{2} + 4nC\alpha(0) \).
Proof. For more details, see \[3\].

**Theorem 4.7.** Suppose $M^n$ is a closed Riemannian manifold, $(M, g(t), \phi(t))_{t \in [0, T)}$ is the solution of the Ricci-harmonic flow (1.2) with bounded Ricci curvature, $0 \leq \text{Ric}(g(t)) \leq Kg(t)$, $K$ is a positive constant, and $u(t)$ is a positive solution of heat equation (1.5) with $u(t, 0) \leq A$. Moreover, if we assume that $\alpha(t)$ is a non-increasing function, bounded from below by $\hat{\alpha}$, and $0 \leq d\phi(t) \otimes d\phi(t) \leq \frac{C}{T}g(t)$, where $C$ is a constant depending on $n$ and $\hat{\alpha}$, then the following holds.

(i). If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Ricci-harmonic flow.

(ii). If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Ricci-harmonic flow.

**Proof.** The estimate of $I'(t)$ is same as the proof of Theorem 4.3. That is

$$I'(t) \geq - \left( \frac{C_n}{t} + 2Kn + \frac{C(t)}{2n} \right) I(t) - \frac{2}{C(t)n} \int_M |\Delta u|^2 dV.$$  

If we write

$$\hat{I}(t) = \exp \left\{ \int_t^t \left( \frac{C_n}{s} + 2Kn + \frac{C(s)}{2n} \right) ds \right\} I(t)$$

then we can easily find

$$\hat{I}'(t) \geq - \frac{2}{C(t)n} \exp \left\{ \int_t^t \left( \frac{C_n}{s} + 2Kn + \frac{C(s)}{2n} \right) ds \right\} \int_M |\Delta u|^2 dV.$$  

For the computation of $D'(t)$, according to Lemma 3.6, we have

$$D'(t) = h'(t) \int_M |\nabla u|^2 dV + h(t) \int_M (\partial_t - \Delta) |\nabla u|^2 dV$$

$$= h'(t) \int_M |\nabla u|^2 dV - 2h(t) \int_M (\alpha(t)d\phi \otimes d\phi(\nabla u, \nabla u) + |\nabla^2 u|^2) dV$$

$$= \frac{h'(t)}{h(t)} D(t) - 2h(t) \int_M (\alpha(t)d\phi \otimes d\phi(\nabla u, \nabla u) + |\nabla^2 u|^2) dV.$$  

Similarly, if we write

$$\hat{D}(t) = \exp \left\{ - \int_t^t \frac{h'(s)}{h(s)} ds \right\} D(t)$$

then we will find

$$\hat{D}'(t) = -2h(t) \exp \left\{ - \int_t^t \frac{h'(s)}{h(s)} ds \right\} \int_M (\alpha(t)d\phi \otimes d\phi(\nabla u, \nabla u) + |\nabla^2 u|^2) dV.$$  

Thus, the parabolic frequency can be defined by $U(t) = \frac{D(t)}{\hat{I}(t)}$. Combining the boundedness of $\alpha$ and $\phi$, if $h(t)$ is a negative time-dependent function, we see that

(4.12)

$$\hat{I}^2(t)U'(t) = \hat{D}'(t)\hat{I}(t) - \hat{D}(t)\hat{I}'(t)$$

$$\geq \exp \left\{ \int_t^t \left( - \frac{h'(s)}{h(s)} + \frac{C_n}{s} + 2Kn + \frac{C(s)}{2n} \right) ds \right\}$$

$$\cdot \left[ -2h \left( \int_M u^2 dV \right) \left( \int_M (\alpha(t)d\phi \otimes d\phi(\nabla u, \nabla u) + |\nabla^2 u|^2) dV \right) \right]$$
\[ \geq \exp \left\{ \int_{t_0}^t \left( -\frac{h'(s)}{h(s)} + \frac{C_n}{s} + 2Kn + \frac{C(s)}{2} n \right) ds \right\} \times \left[ -\frac{2h}{n} \left( \int_M u^2 dV \right) \left( \int_M |\triangle u|^2 dV \right) + \frac{2h}{C(t)n} \frac{N}{t} \left( \int_M |\triangle u|^2 dV \right) \left( \int_M u^2 dV \right) \right] \]

= 0

where we take \( C(t) = \frac{N}{t} \).

With the same discussion, if \( h(t) \) is a positive time-dependent function, the parabolic frequency \( U'(t) \leq 0 \). Thus, we get the desired results.

\[ \square \]

REFERENCES
1. Almgren, Frederick J., Jr. Dirichlet’s problem for multiple valued functions and the regularity of mass minimizing integral currents. Minimal submanifolds and geodesics. pp. 1–6, North-Holland, Amsterdam-New York, 1979.
2. Băileşteanu, Mihai; Cao, Xiaodong; Pulemotov, Artem. Gradient estimates for the heat equation under the Ricci flow. J. Funct. Anal. 258 (2010), no. 10, 3517-3542.
3. Băileşteanu, Mihai. Gradient estimates for the heat equation under the Ricci-harmonic map flow. Adv. Geom. 15 (2015), no. 4, 445–454.
4. Bakry, D.; Émery, Michel. Diffusions hypercontractives. Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math., 1123, Springer, Berlin, 1985.
5. Chen, Bing-Long; Zhu, Xi-Ping. Uniqueness of the Ricci flow on complete noncompact manifolds. J. Differential Geom. 74 (2006), no. 1, 119–154.
6. Colding, Tobias H.; Minicozzi, William P., II. Harmonic functions with polynomial growth. J. Differential Geom. 46 (1997), no. 1, 1–77.
7. Colding, Tobias H.; Minicozzi, William P., II. Parabolic frequency on manifolds. International Mathematics Research Notices. (2021).
8. Garofalo, Nicola; Lin, Fang-Hua. Monotonicity properties of variational integrals, Ap weights and unique continuation. Indiana Univ. Math. J. 35 (1986), no. 2, 245–268.
9. Garofalo, Nicola; Lin, Fang-Hua. Unique continuation for elliptic operators: a geometric-variational approach. Comm. Pure Appl. Math. 40 (1987), no. 3, 347–366.
10. Grisha Perelman. Ricci flow with surgery on three-manifolds. arXiv:math/0303109
11. Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159
12. Hamilton, Richard S. A matrix Harnack estimate for the heat equation. Comm. Anal. Geom. 1 (1993), no. 1, 113-126.
13. Hamilton, Richard S. Three-manifolds with positive Ricci curvature. J. Differential Geometry 17 (1982), no. 2, 255–306.
14. Han, Qing; Hardt, Robert; Lin, Fanghua. Geometric measure of singular sets of elliptic equations. Comm. Pure Appl. Math. 51 (1998), no. 11-12, 1425–1443.
15. Han, Qing; Lin, Fang-Hua. Nodal sets of solutions of parabolic equations. II. Comm. Pure Appl. Math. 47 (1994), no. 9, 1219–1238.
16. Julius, Baldauf; Dain, Kim. Parabolic frequency on Ricci flows. arXiv:2201.05505
17. Lin, Fang-Hua. Nodal sets of solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 44 (1991), no. 3, 287–308.
18. List, Bernhard. Evolution of an extended Ricci flow system. Comm. Anal. Geom. 16 (2008), no. 5, 1007–1048.
19. Li, Xiaolong; Wang, Kui. Parabolic frequency monotonicity on compact manifolds. Calc. Var. Partial Differential Equations 58 (2019), no. 6, Paper No. 189, 18 pp.
20. Logunov, Alexander. Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. Ann. of Math. (2) 187 (2018), no. 1, 221–239.
21. Logunov, Alexander. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. Ann. of Math. (2) 187 (2018), no. 1, 241–262.
22. Müller, Reto. *Ricci flow coupled with harmonic map flow*. Ann. Sci. Éc. Norm. Supér. (4) 45 (2012), no. 1, 101–142.

23. Ni, Lei. *Parabolic frequency monotonicity and a theorem of Hardy-Pólya-Szegő*. Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong, 203–210, Contemp. Math., 644, Amer. Math. Soc., Providence, RI, 2015.

24. Poon, Chi-Cheung. *Unique continuation for parabolic equations*. Comm. Partial Differential Equations 21 (1996), no. 3-4, 521–539.

25. Shi, Wan-Xiong. *Deforming the metric on complete Riemannian manifolds*. J. Differential Geom. 30 (1989), no. 1, 223–301.

26. Zelditch, Steve. *Local and global analysis of eigenfunctions on Riemannian manifolds*. In: Handbook of geometric analysis. No. 1, 545–658, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

27. Zhang, Qi S. *Some gradient estimates for the heat equation on domains and for an equation by Perelman*. Int. Math. Res. Not. 2006, Art. ID 92314, 39 pp.

School of Mathematics, Southeast University, Nanjing 211189, China
Email address: chli@seu.edu.cn

School of Mathematics and Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing 211189, China
Email address: yilicms@gmail.com, yilicms@seu.edu.cn

School of Mathematics, Southeast University, Nanjing 211189, China
Email address: karry_xu@seu.edu.cn