Thermal transport through non-ideal Andreev quantum dots

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Abstract
We consider the scenario of thermal transport through two types of Andreev quantum dots which are coupled to two leads, belonging to the Class D and Class C symmetry classes. Using the random matrix description we derive the joint probability density function (JPDF) in terms of hypergeometric function of matrix arguments when we consider one lead to be attached ideally and one lead non ideally. For the Class C ensemble we derive a more explicit representation of the JPDF which results in a new type of random matrix model.

Keywords: random matrix, quantum dots, symmetric functions

1. Introduction
Quantum dots form an important class of mesoscopic systems whose electric and thermal transport properties are being actively studied. Random matrix theory has had a tremendous success in describing these systems in the limit of low voltage and low temperature, when the classical motion of an electron is chaotic in the dot. In this regime transport through the dot boils down to characterizing how single (quasi-)particles are transmitted through a chaotic cavity. The Landauer–Büttiker approach does this trough the study of the scattering matrix. If the classical dynamics is chaotic in the cavity then the scattering matrix would be well described by a random uniformly distributed unitary matrix [1], an ensemble which had already been studied by Dyson. Taking the distribution to be uniform over the scattering matrix gave rise to three ensembles, namely the circular unitary, orthogonal and symplectic ensemble or CUE, COE and CSE. Each of these is determined by the presence or absence of time reversal symmetry and spin rotation symmetry of the electron in the chaotic cavity. In [2] Altland and Zirnbauer showed the existence of four more types of symmetry classes which appear when particle-hole symmetry is present. When a quantum dot is put in contact with a superconductor an electron moving inside the cavity can be reflected as a hole. This process is called Andreev reflection [3] and in [2] it was shown that these hybrid normal
metal-superconductor systems formed four new symmetry classes named Class D, C, DIII and CI. When considering a quantum dot within these symmetry classes and assuming that the scattering matrix is uniformly distributed these give rise to the circular real ensemble (CRE), circular quaternion ensemble (CQE), circular real time reversal symmetric ensemble (T-CRE) and the circular quaternion time reversal symmetric ensemble (T-CQE), [4, 5]. Since Andreev reflections change the charge of the particle moving in the cavity, electric transport is no longer the same as thermal transport, which in these systems is the same as particle transport. Put differently, a particle scattered through the cavity will transport a definite amount of energy but not a definite amount of charge since it can come out as an electron or a hole.

The uniform distribution over all of these circular ensembles is the most ‘simple’ scenario which, although it can be realized experimentally, need not be the case. It was shown in [6] that, if on average the scattering matrix was different from zero then the distribution over the scattering matrix is given by the Poisson kernel, $P(S)$. This was shown for the CUE, COE and CSE cases. A generalization of the Poisson kernel for the CRE, CQE, T-CRE and T-CQE was derived in [7]. When the distribution is given by the Poisson kernel, or its generalization, the system is said to be non ideal. When the leads are ideally coupled to the chaotic cavity the scattering matrix distribution is uniform.

Aside from being a more general description of the quantum dots, non ideal systems can be attractive for different reasons. For example, in [8] it was shown that semi-non-ideal quantum dots could be used to tune the amount of entanglement between two electrons scattering on the quantum dot.

Many transport observables, such as the conduction or the shot noise, can be written down in terms of the transmission or the reflection eigenvalues. The main obstacle when studying non ideal scenarios is that the joint probability density function (JPDF) for these eigenvalues is not available while it is available for the ideal case. Exceptions to this are the cases of the semi-non-ideal quantum dot with broken/preserved time reversal symmetry and spin rotation symmetry. These are non ideal versions of the CUE, solved in [9] and studied in [8] regarding entanglement and the COE and CSE analyzed in [10].

We will consider the problem of thermal transport through Andreev quantum dots where two leads are attached and we will consider only one lead to be non ideal (semi-non-ideal quantum dot). The symmetry classes we will analyze are the Class D and C. Class D systems correspond to those with broken time-reversal and spin-rotation symmetry while Class C has only broken time-reversal symmetry. This means we are looking at the non ideal version of the CRE and CQE, and refer to them as the Poisson real and quaternion ensemble (PRE and PQE). In these cases the scattering matrices are orthogonal $(O(N))$ and symplectic $(Sp(N))$ respectively. The orthogonal scattering matrices can be further split into two parts. Matrices with determinant 1 and matrices with determinant $-1$. The determinant is called the topological quantum number and when it is 1 (−1) we are in the topologically (non-)trivial phase. We will consider the case when the determinant is 1 and thus the scattering matrices form the group $SO(N)$.

Our strategy is analogous to the one used in previous work [10]. In section 2 we will review the Landauer–Büttiker approach and explain where the main hurdle lies to find the JPDF. In section 3 we will use the theory of symmetric function to derive how the JPDF can be expressed in terms of hypergeometric function of matrix arguments (HFMA) and in section 4 we compute the normalization constant. In section 5 we will derive a representation of the HFMA which will be useful to derive a more compact representation to the JPDF for the quaternion ensemble. In appendix B we review the main results from the theory of symmetric functions that we use in this derivation.
2. Landauer–Büttiker approach

The system we consider is an Andreev quantum dots with a left lead with \( n \) channels and a right lead with \( m \) channels. We take \( n \leq m \) and for the real ensemble we have included the spin and particle/hole quantum numbers. For the quaternion ensemble we do not include spin quantum number. The scattering matrix, \( S \), is then a \((n + m) \times (n + m)\) matrix for the real ensemble and for the quaternion ensemble it is a \((n + m) \times (n + m)\) matrix with quaternion elements. Which means the scattering matrix is either an orthogonal matrix in \( SO(N) \) or a symplectic matrix in \( Sp(N) \). The transmission matrix, \( t_{n \times m} \), is a sub-block of the scattering matrix.

\[
S = \begin{pmatrix} t_{n \times n} & t_{n \times m} \\ t_{m \times n} & t'_{m \times m} \end{pmatrix}.
\]  

The Landauer–Büttiker approach characterizes transport through a quantum dot by the eigenvalues of the product of the transmission matrix with its hermitian conjugate. That is to say, the eigenvalues \( T_j \) of the matrix \( tt^\dagger \) determine the thermal transport observables, such as the conductance \( G \), through the following formula:

\[
G = dG_0 \sum_{j=1}^n T_j,
\]  

with \( G_0 = \frac{\pi^2 k_B^2 T_0}{6 \hbar} \) and \( d \) denotes the degeneracy of the transmission eigenvalues.

Alternatively the reflection eigenvalues \( R_j \), the eigenvalues of the matrix \( rr^\dagger \), can be used. They are related to the transmission eigenvalues as \( R_j = 1 - T_j \) and we will use these instead of the transmission eigenvalues. The random matrix theory description of quantum dots starts with a given distribution, \( P(S) \), over the scattering matrix. Given this distribution the expectation value of an observable depending on the transmission eigenvalues, \( F(R_j) \), is given by

\[
\left\{ F \left( R_j \right) \right\} = \int d\mu(S) P(S) F \left( R_j \right),
\]  

where \( d\mu(S) \) is the uniform measure or Haar measure over \( S \). In order to characterize the statistics of observables depending on the reflection eigenvalues one needs to derive from \( P(S) \) the JPDF of the reflection eigenvalues, \( P(R_j) \).

By using the polar decomposition of the scattering matrix it is parametrized as follows

\[
S = U_1 \begin{pmatrix} 0 & r \\ 0 & -t' \end{pmatrix} U_2 \begin{pmatrix} V_1^+ & 0 \\ 0 & V_2^+ \end{pmatrix} = U \begin{pmatrix} r & t \\ -t' & r' \end{pmatrix} V^+, \tag{2.4}
\]  

where \( r, t, r' \) and \( r'' \) are diagonal matrices. \( r \) has as diagonal values \( n_1 \), \( \cdots \), \( n_n \), with \( r_j \in [0, 1] \), and the reflection eigenvalues \( R_j \) are given by \( r_j^2 \). \( r' \) the same values as \( r \) and an extra \( m - n \) of 1’s as diagonal values.

\[
r = \text{diag} \{ n_1, \cdots, n_n \} \\
r' = \text{diag} \{ n_1, \cdots, n_n, 1\cdots1 \}
\]

\( t \) on the other hand is rectangular \((n \times m)\) with \( n \leq m \) and has as diagonal elements \( t_1, \cdots, t_n \). While \( t' \) is the transpose of \( t \), \( t' = t^T \). In order for this parametrization to be unique we take...
$U_1, V_1 \in O(n), V_2 \in O(m) \text{ and } U_2 \in O(m)/O(m-n)$\text{ when } S \in SO(N).\text{ When } S \in Sp(N)\text{ we take } U_1 \in Sp(n)/Sp(1)\text{, } V_1 \in Sp(n), V_2 \in Sp(m)\text{ and } U_2 \in Sp(m)/Sp(m-n).\text{ However, as shown in section A, the integrals over the coset spaces can be extended to the full group once the Jacobian is computed. Additionally, for the orthogonal matrices the determinant is equal to 1 and we need to insure that this condition is fulfilled in the parametrization. The determinant is given by}

$$\det \left[ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} r & \tilde{r} \\ -\tilde{r} & r \end{pmatrix} \begin{pmatrix} V_1 \dagger & 0 \\ 0 & V_2 \dagger \end{pmatrix} \right] = \det[U_1]\det[U_2]\det[V_1]\det[V_2].$$

Therefore we need to insure

$$\det[U_1]\det[U_2]\det[V_1]\det[V_2] = +1.$$ 

We set in our parametrization $\det[V_2]$ equal to $\det[V_1]$ and $\det[U_2]$ equal to $\det[U_1]$. Meaning the matrices $U_2$ and $V_2$ are constricted. Denoting by $J(r_j)$ the Jacobian of the transformation of equation (2.4) one gathers

$$\left\langle F(r_j) \right\rangle = \prod_{j=1}^{n} \int \! \! \int_{0}^{1} \! \! \! \! \! \! dr_j F(r_j)J(r_j) \int \! \! \! \! \! \! d\mu(U)d\mu(V)P \times \left( U \begin{pmatrix} r & \tilde{r} \\ -\tilde{r} & r \end{pmatrix} V^{\dagger} \right).$$

(2.5)

For the quaternion ensemble $d\mu(U)d\mu(V)$ is a product of Haar measures over independent matrices

$$d\mu(U)d\mu(V) = d\mu(U_1)d\mu(U_2)d\mu(V_1)d\mu(V_2).$$

(2.6)

For the orthogonal ensemble $d\mu(U)d\mu(V)$ is a product of Haar measures over matrices whose determinants are related

$$d\mu(U)d\mu(V) = d\mu(U_1)d\mu(U_2)d\mu(V_1)d\mu(V_2)\delta(\det[V_1] - \det[U_1]) \delta(\det[V_2] - \det[U_2]).$$

(2.7)

with $U_j, V_j \in O(n)$ for the real ensemble and $U_j, V_j \in Sp(n)$ for the quaternion ensemble. We note also that for the quaternion ensemble the matrix is made of quaternion elements and so the diagonal quaternion matrix $r$ of singular values has $n$ blocks $r_j \in \mathbb{H}^{2 \times 2}$. Thus the singular values $r_j$ are double degenerate. The JPDF, denoted by $P(R_j)$, can almost be read of equation (2.5). Given that $r_j^2 = R_j$ we only need to make a change of variables in equation (2.5).

$$\left\langle F(R_j) \right\rangle = \left\langle F(r_j^2) \right\rangle = \prod_{j=1}^{n} \int \! \! \! \! \! \! dR_j F(R_j)J(R_j) \int \! \! \! \! \! \! d\mu(U)d\mu(V)P \left( U \begin{pmatrix} r & \tilde{r} \\ -\tilde{r} & r \end{pmatrix} V^{\dagger} \right) \times \left( U \begin{pmatrix} \sqrt{R} & \sqrt{T} \\ -\sqrt{T} & \sqrt{R} \end{pmatrix} V^{\dagger} \right).$$

(2.8)
The JPDF is then given by

\[
P(R_j) = J \left( \sqrt{R_j} \right) \left( 2R_j \right)^{-\frac{1}{2}} \times \int d\mu(U) d\mu(V) P\left( U \left( \begin{array}{cc} \sqrt{R} & \sqrt{T} \\ -\sqrt{T} & \sqrt{R} \end{array} \right) V^\dagger \right).
\] (2.9)

When both leads are attached to the quantum dot ideally random matrix theory models the ensemble of scattering through circular ensembles, meaning \( P(S) = 1 \). The integrals over \( U \) and \( V \) in equation (2.9) are therefore constants. The JPDF is then determined by the Jacobian of the polar decomposition, equation (2.4). The ensemble generated by the orthogonal matrices is then called the CRE and the one generated by the symplectic matrices is called the CQE. The JPDF for this case was derived in [5, 11].

\[
P(R_j) \propto J \left( \sqrt{R_j} \right) \left( 2R_j \right)^{-\frac{1}{2}} \times \Delta(R_j)^{\beta} \left( 1 - R_j \right)^{\frac{1}{2}(m-n+1)-1} R_j^\sigma
\]

with the following values of \( \beta, \eta \) and \( d \) depending on the ensemble:

| Ensemble | \( \beta \) | \( \eta \) | \( d \) |
|----------|---------|---------|------|
| CRE      | 1       | -1      | 1    |
| CQE      | 4       | 2       | 4    |

A more general situation is described when one allows for a non-ideal coupling between the leads and the dot. In this situation the distribution over the scattering matrix is a Poisson type kernel [7]

\[
P(S) = \frac{1}{C(\hat{\gamma})} \left| \det \left[ 1 - \hat{\gamma} S \right] \right|^{-N_0},
\]

\[N_0 = N + \sigma,\] (2.10)

where \( N = n + m \) for the real ensemble and \( N = 2n + 2m \) for the quaternion ensemble. \( C(\hat{\gamma}) \) is the normalization constant to be computed later on. These ensembles are no longer circular and we will refer to them as the PRE and the PQE. The parameter \( \sigma \) depending on the ensemble is given below.

\[
\text{PRE} \sigma = -1 \\
\text{PQE} \sigma = 1
\] (2.11)

The matrix \( \hat{\gamma} \) encodes the coupling between the left/right lead and the dot.

\[
\hat{\gamma} = \begin{pmatrix} \hat{\gamma}_L & 0 \\ 0 & \hat{\gamma}_R \end{pmatrix}.
\]

The left lead is taken to be non ideally coupled, \( \hat{\gamma}_L \neq 0 \), while the right one is arbitrary \( \hat{\gamma}_R = 0 \). We call this the semi-non-ideal scenario. Since we are studying the case where the right lead has coupling \( \hat{\gamma}_R = 0 \) the Poisson like kernel simplifies to

\[
P(S) = \frac{1}{C(\hat{\gamma})} \left| \det \left[ 1 - \hat{\gamma}_L S \right] \right|^{-N_0}.
\]
For the semi-non-ideal system we gather then from equations \((2.6), (2.7)\) and \((2.9)\)

\[
P_R = J \left( \sqrt{R_j} \right) \left( 2R_j \right)^{-\frac{3}{2}} \times \int d\mu(U) d\mu(V) \left| \det \left[ 1 - \gamma_L U r V_1^T \right] \right|^{(N+\sigma)},
\]

where the integrals are either over \(O(n)\) or \(Sp(n)\). The problem of finding the JPDF thus boils down to performing the integration over the orthogonal/symplectic group. Since we rely heavily on the theory of symmetric functions we have included an appendix where the most important features of the theory, for our present calculations, are explained.

3. The JPDF

3.1. Poisson real ensemble

For both ensembles the strategy is the same but we will perform them separately for the sake of clearness. The idea is to expand the Poisson kernel in terms of symmetric functions in order to perform the integrations over the group using known results. The integration theorems used here are collected in appendix B and can be found in section 4 of chapter VII in [12]. Once this is done the result will turn out to be known as HFMA. In section 5 we will elaborate on different representations of these HFMA.

For the PRE we expand the inverse determinant using equation \((B.58)\) in terms of the Schur functions \(S_{\lambda}(X)\). The integral to be performed (equation \((2.12)\)) is denoted by \(I_{\text{PRE}}(\hat{\gamma}, R_j)\) and defined as follows:

\[
I_{\text{PRE}}(\hat{\gamma}, R_j) = \int_{O(n)} d\mu(U) d\mu(V) \left| \det \left[ 1 - V^\dagger \hat{\gamma} U r V_1^T \right] \right|^{(N+\sigma)}
\]

\[
= \sum_{\lambda} S_{\lambda}(1_{M}) \int_{O(n)} d\mu(U) d\mu(V) S_{\lambda}(V^\dagger \hat{\gamma} U r)
\]

with \(\sigma = -1\) for this ensemble and \(1_{M}\) denotes the identity matrix of dimension \(M\). The integral over \(U\) (or \(V\)) is zero unless the partition is even [12]. This means the integers \(\lambda = (\lambda_1, \lambda_2, \ldots\) defining the partition have to be even numbers. This is denoted by \(\lambda = 2\lambda' = (2\lambda_1', 2\lambda_2', \ldots)\). Thus the sum over partitions can be written as a sum over even partitions. For even partitions we have through equation \((B.63)\) \([12]\)

\[
\int_{O(n)} d\mu(U) S_{2\lambda}(AU) = \Omega_{\lambda}^{(2)}(A),
\]

where \(\Omega_{\lambda}^{(2)}(A)\) are called the spherical functions defined through their integral property equation \((B.60)\). Using equation \((B.60)\) \([12]\) yields

\[
\int_{O(n)} d\mu(U) d\mu(V) S_{2\lambda}(V^\dagger \hat{\gamma} U r V) = \int_{O(n)} d\mu(V) \Omega_{\lambda}^{(2)}(r V^\dagger \hat{\gamma})
\]

\[
= \Omega_{\lambda}^{(2)}(\hat{\gamma}) \Omega_{\lambda}^{(2)}(r).
\]

The spherical functions can be expressed in terms of Jack polynomials through equation \((B.62)\) \([12]\). We find then
\[
\int_{\Omega(n)} dU dV S_{2\lambda} \left(V^\dagger U r\right) = \frac{P^{(2)}_{\lambda}(\hat{\gamma}^2) P^{(2)}_{\lambda}(r^2)}{P^{(2)}_{\lambda}(1_n) P^{(2)}_{\lambda}(1_n)}.
\]

Our integral is then
\[
I_{\text{PRE}}(\hat{\gamma}, R) = \sum_{\lambda} S_{2\lambda}(I_{N_0}) \frac{P^{(2)}_{\lambda}(\hat{\gamma}^2) P^{(2)}_{\lambda}(R)}{P^{(2)}_{\lambda}(1_n) P^{(2)}_{\lambda}(I_n)}
= \sum_{\lambda} \epsilon^{(2)}_{\lambda}(2) \frac{P^{(2)}_{\lambda}(\hat{\gamma}^2) P^{(2)}_{\lambda}(R)}{P^{(2)}_{\lambda}(1_n) P^{(2)}_{\lambda}(I_n)}.
\]

where we have used equation (B.57) to obtain an expression for the Schur polynomial evaluated at even partitions. We can now express our result in terms of Pochhammer symbols using equations (B.56) and (B.55).

\[
I_{\text{PRE}}(\hat{\gamma}, R) = \sum_{\lambda} \frac{1}{d^{(2)}_{\lambda}(2)} \frac{1}{\epsilon^{(2)}_{\lambda}(2)} \frac{P^{(2)}_{\lambda}(\hat{\gamma}^2) P^{(2)}_{\lambda}(R)}{P^{(2)}_{\lambda}(1_n) P^{(2)}_{\lambda}(I_n)}.
\]

(3.13)

We recognize that equation (3.13) is the definition of a HFMA of two matrix arguments, HFMA\_2, equation (B.65). There are three types HFMA\_2, denoted by \( _2F^{(\alpha)}_{\alpha} (a, b; c|X, Y) \) (with index \( \alpha = 2, 1 \) or \( 1/2 \)) and defined in terms of the symmetric functions as follows:

\[
_{2F}^{(\alpha)}(a, b; c|X, Y) = \sum_{\lambda} \frac{d^{(\alpha)}_{\lambda}}{[d^{(\alpha)}_{\lambda}]} \frac{[a]^{(\alpha)}_{\lambda} [b]^{(\alpha)}_{\lambda} [c]^{(\alpha)}_{\lambda}}{p^{(\alpha)}_{\lambda}(X) P^{(\alpha)}_{\lambda}(Y)}.
\]

Thus we have setting \( \alpha = -1 \)

\[
I_{\text{PRE}}(\hat{\gamma}, R) = _2F^{(2)}_{\alpha} \left(\frac{N}{2}, \frac{N - 1}{2}; \frac{n}{2} \right) \hat{\gamma}, R \right).
\]

(3.14)

Very little is actually known about determinantal/Pfaffian representations of HFMA\_2. If the coupling of the left lead to the quantum dot is independent of the mode then we are in the case where \( \hat{\gamma} = \gamma A \). The result reduces then to a HFMA\_1, equation (B.66)

\[
I_{\text{PRE}}(\gamma A, R) = _2F^{(2)}_{\alpha} \left(\frac{N}{2}, \frac{N - 1}{2}; \frac{n}{2} \right) \gamma A, R \right).
\]

(3.15)

Before turning to the question of representations of the HFMA\_1 we analyze the PQE case in the same manner. The results will be HFMA\_1,2 with the index \( \alpha = 1/2 \).

### 3.2. Poisson quaternion ensemble

As mentioned in the previous section, the strategy for the quaternion ensemble is the same as for the orthogonal one. The integration theorems used in this section can be found
in section 6 of chapter VII in [12] and are collected in appendix B. For quaternion ensemble we need to perform the following integration (equation (2.12))

\[ I_{\text{PQE}}(\tilde{\gamma}, R_j) = \int_{\text{Sp}(n)} d\mu(U) d\mu(V) \left| \det \left[ 1 - \tilde{\gamma} U r V \right] \right|^{-N_\sigma}. \quad (3.16) \]

Using the fact that the unitary matrices are symplectic we have \( U^* = U^T = -Z U^T Z \) with \( Z = \mathbb{1}_n \otimes i \tau_z \) and \( \tau_z \) the Pauli matrix, we gather that the determinant is real even though the matrix is complex.

\[
\det \left[ 1 - \tilde{\gamma} U r V \right]^* = \det \left[ 1 - \tilde{\gamma} U^* r V^* \right] = \det \left[ 1 - \tilde{\gamma} Z U r Z V \right] = \det \left[ 1 - \tilde{\gamma} U r V \right].
\]

In the last step we have used the fact that \( r \) has a double degeneracy, \( r = \text{diag}\{\eta, \cdots \eta\} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \). Thus for the PQE we have

\[
I_{\text{PQE}}(\tilde{\gamma}, R_j) = \int_{\text{Sp}(n)} d\mu(U) d\mu(V) \det \left[ 1 - \tilde{\gamma} U r V \right]^{-N_\sigma} = \sum_{\lambda} \frac{[N_\sigma]_{1}^{(1)}}{h_{\lambda}(1)} \int_{\text{Sp}(n)} d\mu(U) d\mu(V) S_{\lambda}(\tilde{\gamma} U r V).
\]

Similarly to the PRE the integral will be zero for partitions which do not have a specific form, namely the form \( \lambda = \lambda' \cup \lambda' \) for any \( \lambda' \). The partition \( \lambda' \cup \lambda' \) is defined by having each integer twice. That is \( \lambda' \cup \lambda' = (\lambda_1', \lambda_2', \ldots) \). We have then

\[
I_{\text{PQE}}(\tilde{\gamma}, R_j) = \sum_{\lambda} \frac{[N_\sigma]_{\lambda \cup \lambda}^{(1)}}{h_{\lambda \cup \lambda}(1)} \int_{\text{Sp}(n)} d\mu(U) d\mu(V) S_{\lambda \cup \lambda}(\tilde{\gamma} U r V).
\]

For partitions which do have this form we can perform the integrations using equations (B.61), (B.62) and (B.64), [12]

\[
\int_{\text{Sp}(n)} d\mu(U) d\mu(V) S_{\lambda \cup \lambda}(\tilde{\gamma} U r V) = \int_{\text{Sp}(n)} d\mu(V) \Omega_{\lambda}(1/2) (r \tilde{\gamma} V) = \Omega_{\lambda}(1/2) (\tilde{\gamma} \sigma \lambda (1/2) (r) = \frac{P_{\lambda}^{(1/2)}(\tilde{\gamma} \sigma \lambda (1/2)}{P_{\lambda}^{(1/2)}(\mathbb{1}_n)} \frac{P_{\lambda}^{(1/2)}(\mathbb{1}_n)}{P_{\lambda}^{(1/2)}(\mathbb{1}_n)}.
\]

This leads to the following expression:

\[
I_{\text{PQE}}(\tilde{\gamma}, R_j) = \sum_{\lambda} \frac{\epsilon_{\lambda} (1/2, N_\sigma/2, 1/2, N_\sigma/2)}{d_{\lambda} (1/2, N_\sigma/2, 1/2, N_\sigma/2)} \frac{P_{\lambda}^{(1/2)}(\tilde{\gamma} \sigma \lambda (1/2)}{P_{\lambda}^{(1/2)}(\mathbb{1}_n)} \frac{P_{\lambda}^{(1/2)}(\mathbb{1}_n)}{P_{\lambda}^{(1/2)}(\mathbb{1}_n)}.
\]
we have used the identity equation (B.69). Using equations (B.55) and (B.56) we have

\[
I_{\text{PQE}}(\hat{\gamma}, R_j) = \sum_{j} \left( \frac{1}{2} \right)^{\frac{1}{2}} \left[ \frac{N_x}{2} - 1 \right]^{(1/2)} \left[ \frac{N_x}{2} \right]^{(1/2)} \frac{1}{d_j} \left( \frac{1}{2} \right) \left[ 2n \right]^{1/2} \frac{p_j(\hat{\gamma}) p_j^{(1)}(R)}{p_j^{(1)}(I_n)}
\]

we identify this solution with the HFMA2 the index with \(\alpha = 1/2\). Setting \(\sigma = 1\) we gather

\[
I_{\text{PQE}}(\hat{\gamma}, R_j) = \left( \begin{array}{c} 2 \end{array} \right) \left( \begin{array}{c} N, N + 1 \end{array} ; 2n \right| \hat{\gamma}^2, R \right) = \left( \begin{array}{c} 2 \end{array} \right) \left( 2(n + m), 2(n + m) + 1 \right| \hat{\gamma}^2, R \right),
\]

(3.17)

where we have set \(\sigma = 1\) and the case of \(\hat{\gamma}\) proportional to the identity yields

\[
I_{\text{PQE}}(\hat{\gamma}, R_j) = \left( \begin{array}{c} 2 \end{array} \right) \left( \begin{array}{c} N, N + 1 \end{array} ; 2n \right| R \right) = \left( \begin{array}{c} 2 \end{array} \right) \left( 2(n + m), 2(n + m) + 1 \right| \hat{\gamma}^2 R \right).
\]

(3.18)

For the two ensembles we have then the following JPDF when \(\hat{\gamma}\) is arbitrary

\[
P_\alpha(R_j) = \frac{1}{C(\hat{\gamma})} \prod_{j=1}^{n} \left( R_j - 1 \right) \left( \frac{1}{2} \right)^{(m-n+1)-1} R_j^{-1} \delta(R_j) \left[ \begin{array}{c} \hat{\gamma} \end{array} \right] \left[ \begin{array}{c} \frac{m + n}{\alpha} \end{array} \right] \left[ \begin{array}{c} \frac{m + n}{\alpha} + \frac{\eta}{2} \end{array} \right] \left[ \begin{array}{c} \frac{n}{2} \end{array} \right] \left| \hat{\gamma}^2, R \right).
\]

(3.19)

and when \(\hat{\gamma} = \gamma\) it simplifies to

\[
P_\alpha(R_j) = \frac{1}{C(\hat{\gamma})} \prod_{j=1}^{n} \left( R_j - 1 \right) \left( \frac{1}{2} \right)^{(m-n+1)-1} R_j^{-1} \delta(R_j) \left[ \begin{array}{c} \hat{\gamma} \end{array} \right] \left[ \begin{array}{c} \frac{m + n}{\alpha} \end{array} \right] \left[ \begin{array}{c} \frac{m + n}{\alpha} \end{array} \right] \left[ \begin{array}{c} \frac{n}{2} \end{array} \right] \left| \hat{\gamma}^2 R \right).
\]

(3.20)

where we have added the index \(\alpha\) to the JPDF of equation (2.12) that specifies the ensemble. \(\alpha = 2\) for the real ensemble and \(\frac{1}{2}\) for the quaternion one. \(C\) denotes the normalization constant which we compute now.
4. Normalization

From equation (3.19) we gather the normalization constant is given by the following integral

\[
C(\hat{j}) = \prod_{j=1}^{n} \int_{0}^{1} dR_j \left( R_j - 1 \right)^{\frac{1}{2}(m-n+1)-1} R_j^2 \\
\times |\Delta(R_j)|^2 P_{\hat{j}}^{(n)} \left( \frac{m+n}{\alpha}, \frac{m+n}{\alpha} + \frac{n}{\alpha} \right)^{2^{\hat{j}}; R} \\
= \sum_{j} a^{\hat{j}+1} \left[ \frac{m+n}{\alpha} \right]^{(a)} \left[ \frac{m+n+\eta}{\alpha} \right]^{(a)} P_{\hat{j}}^{(a)} (p^2) \\
\times \prod_{j=1}^{n} \int_{0}^{1} dR_j \left( R_j - 1 \right)^{\frac{1}{2}(m-n+1)-1} R_j^2 |\Delta(R_j)|^{2} P_{\hat{j}}^{(n)} (R).
\]

Given the Selberg integral over Jack polynomials [13]

\[
\prod_{j=1}^{n} \int_{0}^{1} dR_j \left( 1 - R_j \right)^{\frac{1}{2}} R_j^2 |\Delta(R_j)|^{2} P_{\hat{j}}^{(n)} (R) \\
= P_{\hat{j}}^{(n)} (1_{\eta}) \left[ \frac{x+1+n-1}{\alpha} \right]^{(a)} S_{\eta}(x, y, \alpha) \\
S_{\eta}(x, y, \alpha) = \prod_{j=1}^{n} \int_{0}^{1} dR_j \left( 1 - R_j \right)^{\frac{1}{2}} R_j^2 |\Delta(R_j)|^{2} \\
= \prod_{j=0}^{n-1} \frac{\Gamma \left( x+1+j+\frac{1}{\alpha} \right) \Gamma \left( y+1+j+\frac{1}{\alpha} \right) \Gamma \left( 1+j+\frac{1}{\alpha} \right)}{\Gamma \left( x+y+2 + \frac{n+j-1}{\alpha} \right) \Gamma \left( 1+\frac{1}{\alpha} \right)} \tag{4.21}
\]

we have

\[
\prod_{j=1}^{n} \int_{0}^{1} dR_j \left( R_j - 1 \right)^{\frac{1}{2}(m-n+1)-1} R_j^2 |\Delta(R_j)|^{2} P_{\hat{j}}^{(n)} (R) \\
= C_n P_{\hat{j}}^{(n)} (1_{\eta}) \left[ \frac{n}{2} + 1 - \frac{1}{\alpha} + \frac{n}{\alpha} \right]^{(a)} \left[ \frac{n}{2} + 1 - \frac{1}{\alpha} + \frac{1}{\alpha} \left( m+n \right) \right]^{(a)} \\
C_n = S_{\eta} \left( \frac{\eta}{2}, 1, \alpha \right) (m-n+1)-1, \alpha \right). \tag{4.22}
\]

\(C_n\) is the normalization constant for the circular ensemble (\(\hat{j} = 0\)). Given that \(\frac{\eta}{2} + 1 - \frac{1}{\alpha} = 0\) for both ensembles we have
\[ \int_0^1 dR_j (R_j - 1)^{\frac{1}{2}(m-n+1)-1} R_j^\gamma |\Delta(R_j)| \hat{P}_j^{(\alpha)}(R) \]
\[ = C_n \hat{P}_j^{(\alpha)}(I_n) \left[ \frac{n - \eta}{\alpha} \right]_2^\eta \left[ \frac{1}{\alpha (m + n)} \right]_2 \]  
(4.23)

and so
\[ C(\hat{\gamma}) = C_n \sum_p \frac{d^{(p)}}{d j^{(\alpha)}} \left[ \frac{m + n}{\alpha} + \frac{\eta}{2} \right]_2^{(\alpha)} \hat{P}_j^{(\alpha)}(\hat{\gamma}^2) \]
\[ = C_n \prod_{j=1}^n \left[ 1 - \gamma_j^2 \right]^{\frac{(m+n+\eta)}{2}} \]
\[ = C_n \det \left[ 1 - \hat{\gamma}^2 \right]^{\frac{(m+n+\eta)}{2}}. \]  
(4.24)

For the orthogonal ensembles this yields \( \det \left[ 1 - \hat{\gamma}^2 \right]^{\frac{N-1}{2}} \) and for the quaternion ensemble we have \( \det \left[ 1 - \hat{\gamma}^2 \otimes I_2 \right]^{\frac{N+1}{2}} \).

### 5. Representations of HFMA1

In this section we will derive another integral representation for the HFMA1. We will show that the following matrix integral
\[ F_{\alpha,\beta}^{(\alpha)}(X) = \frac{1}{Z_p(a, b)} \prod_j \int_0^\infty dy_j |\Delta(y_j)| \hat{y}_j^{a} \left( \frac{y_j}{1 + y_j} \right)^b \prod_{k=1}^{n} \left( 1 + x_k y_j \right), \]  
(5.25)

with \( \alpha = 1/2, 1, 2 \) and \( Z_p(a, b) \) the normalization constant
\[ Z_p(a, b) = \prod_{j=1}^p \int_0^\infty dy_j |\Delta(y_j)| \hat{y}_j^{a} \left( \frac{y_j}{1 + y_j} \right)^b \]
\[ = 2^{p-1}(1+\frac{1}{p-1}) \prod_{j=0}^{p-1} \left( a + 1 + \frac{j}{a} \right)^{\frac{1}{a}} \left( b - a - 1 + \frac{j}{a} \right)^{\frac{1}{a}} \prod_{j=0}^{p-1} \left( a + 1 + \frac{j}{a} \right)^{\frac{1}{a}} \left( b - 1 + \frac{j}{a} \right)^{\frac{1}{a}} \]  
(5.26)

is a HFMA1. We first use the dual Cauchy identity [12]
\[ \prod_{j,k} \left( 1 + x_j y_k \right) = \sum_{\alpha} P^{(\frac{1}{2})}_{\alpha}(X) P^{(\alpha)}_{\alpha}(Y) \]
leading to

\[
F_{a,b}^{p,q}(X) = \frac{1}{Z_p(a, b)} \sum_{\lambda} P^{(\frac{1}{2})}_{\lambda}(X) \prod_{j=1}^{\infty} \int_0^\infty dy_j \left| \Delta(y_j) \right|^\frac{q}{2} \frac{y_j^p}{(1 + y_j)^b} P^{(\alpha)}_{\lambda}(y_j). \tag{5.27}
\]

The sum over $\lambda$ is over partitions such that $l(\lambda) \leq p$ and $l(\lambda') \leq n$. The dual generalized Selberg integrals states the following identity holds \cite{13}

\[
\prod_{j=1}^{\infty} \int_0^\infty dy_j \left| \Delta(y_j) \right|^\frac{q}{2} \frac{y_j^a}{(1 + y_j)^b} P^{(\alpha)}_{\lambda}(y_j) = Z_p(a, b) P^{(\alpha)}_{\lambda}(1_p) \left[ a + 1 + \frac{p - 1}{\alpha} \right]^{(a)}_{\lambda} \tag{5.28}
\]

provided $l(\lambda') < b - a - 1 - \frac{2(p - 1)}{\alpha}$. Since the sum is over partitions such that $l(\lambda') \leq n$, the condition is fulfilled for all partitions if $n < b - a - 1 - \frac{2(p - 1)}{\alpha}$. Let us assume this last inequality holds. Using the dual generalized Selberg integral in equation (5.27) we have

\[
F_{a,b}^{p,q}(X) = \sum_{\lambda} (-1)^{\lambda_1} \left[ \frac{P^{(\alpha)}_{\lambda}}{h_\lambda(\alpha)} \right]^{(a)}_{\lambda} P^{(\frac{1}{2})}_{\lambda'}(X) P^{(\alpha)}_{\lambda}(1_p). \tag{5.28}
\]

We would like to rewrite this expression solely in terms of $\lambda'$ and $\frac{1}{\alpha}$ so as to compare it with the definition of $\text{HFMA}_1$. For the Jack polynomial evaluated at identity we have

\[
P^{(\alpha)}_{\lambda}(1_p) = \frac{\alpha^{|\lambda|} \left[ \frac{P^{(\alpha)}_{\lambda}}{h_\lambda(\alpha)} \right]}{h^{\frac{1}{\alpha}}(\frac{1}{\alpha})}. \tag{5.29}
\]

and using the following relationship between Pochhammer symbols of different index $\alpha$, \cite{13}

\[
[s]^{(\alpha)}_{\lambda} = \frac{(-1)^{|\lambda|}}{\alpha^{|\lambda|}} [-\alpha s]^{(\frac{1}{\alpha})}_{\lambda}, \tag{5.30}
\]

we can express the Jack polynomial evaluated at the identity $P^{(\alpha)}_{\lambda}(1_p)$ as

\[
P^{(\alpha)}_{\lambda}(1_p) = \frac{(-1)^{|\lambda|} [\frac{1}{\alpha}]^{\frac{1}{2}}}{h^{1/\alpha}(\alpha)}. \tag{5.31}
\]

Using equation (5.30) we can also rewrite the ratios of Pochhammer symbols in the sum equation (5.28) as
In addition we have the following relations [13]

\[ h_{\lambda}(a) = a! d_{\lambda'}^{(1)} \left( \frac{1}{a} \right) \] (5.33)

Combining equations (5.31), (5.32) and (5.33) in equation (5.28) we gather

\[
\begin{align*}
F_{a,b}^{p,a}(X) &= \sum_{\lambda; l(\lambda) \leq p, l(\lambda') \leq a} \frac{[-p]_k^{(a)}}{a! d_{\lambda'}^{(1)} \left( \frac{1}{a} \right)} \\
& \times \frac{[-\alpha(a + 1) + 1 - p]_{\lambda}^{(a)}}{[-\alpha(a + 2) + 2(1 - p) + ab]_{\lambda}^{(a)}} P_{\lambda'}^{(a)}(X).
\end{align*}
\] (5.34)

Since there is a one to one correspondence between partitions and their conjugates, summing over all partitions is the same as summing over all conjugate partitions. We make the change in notation \( \lambda' \rightarrow \lambda \) and denote \( \alpha' = \frac{1}{\alpha} \) leading to

\[
\begin{align*}
F_{a,b}^{p,a}(X) &= \sum_{\lambda; l(\lambda) \leq p, l(\lambda') \leq a} \frac{(\alpha')! [-p]_k^{(a')}}{d_{\lambda'}^{(a')} \left[ -\frac{1}{\alpha'}(a + 1) + 1 - p \right]_{\lambda}^{(a')}} \\
& \times P_{\lambda'}^{(a')} (X).
\end{align*}
\] (5.35)

The Pochhammer symbol \([-p]_k^{(a')}\) is zero if \( \lambda_1 > p \). Since \( \lambda_1 = l(\lambda') \) the restriction \( l(\lambda') \leq p \) is automatically satisfied in the sum. Thus we have

\[
\begin{align*}
F_{a,b}^{p,a}(X) &= \sum_{\lambda; l(\lambda) \leq a} \frac{(\alpha')! [-p]_k^{(a')} \alpha'}{d_{\lambda'}^{(a')} [c]_{\lambda'}^{(a')}} P_{\lambda'}^{(a')} (X)
\end{align*}
\] (5.36)

with

\[
\begin{align*}
-q &= -\frac{1}{\alpha'}(a + 1) + 1 - p \quad (5.37) \\
c &= -\frac{1}{\alpha'}(a + 2) + 2(1 - p) + \frac{b}{\alpha'}. \quad (5.38)
\end{align*}
\]

The sum in equation (5.36) is known to be a HFMA_1, equation (B.66).

\[
F_{a,b}^{p,a}(X) = 2 F_{1}^{(a')}(-p, -q, c |X). \quad (5.39)
\]

This identity holds subjected to the condition which came from the use of the dual Selberg integral.
Thus for the HFMA1 \(2F_1(a(-p, q; c|X))\) the condition translates into (with \(c = \frac{1}{\alpha'(b-a-2) - 2(p-1)}\))

\[
\frac{n-1}{\alpha'} < c.
\] (5.40)

If the condition is met the HFMA1 has the following integral representation

\[
2F_1(a(-p, -q; c|X)) = \frac{1}{Z_p} \prod_{j=1}^{p} \int_0^\infty \text{dy}_j \left| \Delta\left(y_j\right) \right|^\frac{n}{2} \frac{y^{a(q-p+1)-1}}{(1+y_j)^{a'(c+p+q-1)+1}} \prod_{k,j}^{n,p} (1 + x_k y_j)
\]

\[
Z_p = \prod_{j=1}^{p} \int_0^\infty \text{dy}_j \left| \Delta\left(y_j\right) \right|^\frac{n}{2} \frac{y^{a(q-p+1)-1}}{(1+y_j)^{a'(c+p+q-1)+1}}
\]

\[
= 2^{2(p-1)(1+a'(p-1))} \prod_{j=0}^{p-1} \times \frac{\Gamma(a'(q-j))\Gamma(a'(c+j)+1)\Gamma(1+j\alpha')}{\Gamma(a'(c+q+j)+1)\Gamma(1+a')}. \] (5.41)

Performing the change of variables \(y_j = \frac{1 + \lambda_j}{1 - \lambda_j}\) we have

\[
2F_1(a(-p, -q; c|X)) = \frac{1}{Z_p} \prod_{k=1}^{n} (1 - x_k) \prod_{j=1}^{p} \int_{-1}^1 \text{d}\lambda_j \left| \Delta\left(\lambda_j\right) \right|^\frac{n}{2} (1 + \lambda_j)^{a'(q-p+1)-1}(1 - \lambda_j)^{a'-n}
\]

\[
\times \prod_{k,j}^{n,p} \left(\frac{1 + x_k}{1 - x_k} - \lambda_j\right)
\]

\[
\tilde{Z}_p = \frac{Z_p}{2^{2(p-1)(1+p\alpha')-a'(c+p+q-1)}} \] (5.42)

and if we set \(z_k = \frac{1 + x_k}{1 - x_k}\) then the integral is the average of a product of characteristic polynomials of a Jacobi Ensemble, when \(\frac{2}{\alpha} = 1, 2\) and the average of a product of square roots of characteristic polynomials when \(\frac{2}{\alpha} = 4\).

6. Representation of the JPDF

The representation of HFMA1 derived above applies only for negative integer values of the first two arguments \(-p, -q\) while in equation (3.20) the first two arguments of the HFMA1 has clearly positive values. To obtain a representation of the JPDF we use the well known Kummer’s relations for the HFMA1,

\[
2F_1(a, b; c|X) = \frac{2F_1(a-c, a-b; c|X)}{\text{det}[1-X]^{p+b-c}}. \] (6.43)
We first look at the PQE, \( \alpha = \frac{1}{2} \). For the PQE we have from equation (3.18)
\[
I_{\text{PQE}}(\gamma, R) = \frac{\zeta F_1^{(2)}(-2m, -2m - 1; 2n \vert y^2 R)}{\det[1 - y^2 R]^{2m + 2n + 1}}.
\]

In equation (5.42) we set \( p = \frac{m}{2} \) and \( q = \frac{m}{2} + \frac{n}{2} \), and have then \( p = 2m, q = 2m + 1 \). For these values of \( p, q, \) and \( c \) the weight in (5.42) simplifies to \((1 + \lambda_j)^{q-p+1}(1 - \lambda_j)^{n-q} = 1\). The HFMA appearing can be expressed as a Pfaffian over a Vandermonde determinant through equations (C.73), (C.76), [14], depending on whether \( n \) is even or odd. We assume \( n \) is even and using the result of equation (C.76) we gather
\[
\frac{\zeta F_1^{(2)}(-2m, -2m - 1; 2n \vert y^2 R)}{\Delta(\gamma)} \prod_{k,j} \left( \frac{1 + y^2 R_k - \lambda_j}{1 - y^2 R_k} \right)^{\frac{1}{2}} \prod_{k,j} \left( \frac{1 + y^2 R_k}{1 - y^2 R_k} \right)\]
\[
= \frac{1}{\mathcal{Z}_{2m}(2y^2)^{\frac{n}{2}+1}} \prod_{k=1}^{n} \left( 1 - y^2 R_k \right)^{2m+n-1} \frac{1}{\Delta(R_k)} \prod_{j,k \in n} f_{jk}.
\]

with
\[
\mathcal{Z}_{2m} = \frac{1}{\gamma^{2m+2n}} \prod_{j=0}^{2m-1} \frac{\Gamma\left(\frac{1}{2}(2m + 1 - j)\right) \Gamma\left(\frac{1}{2}(2n + j) + 1\right) \Gamma\left(\frac{1}{2} + \frac{j-1}{2}\right)}{\Gamma\left(\frac{1}{2}(2n + 2m + 1 + j) + 1\right) \Gamma\left(1 + \frac{1}{2}\right)}
\]

and the entries \( f_{jk} \) are given in terms of the Jacobi skew orthogonal polynomials by equations (C.77) with the weight \( w(u) = 1 \). The arguments \( v_k \) in equations (C.77) are \( 1 + y^2 R_k \). The JPDF is then for the PQE
\[
P_{\gamma=\frac{1}{2}}(R_j) = \frac{1}{\mathcal{C}(\gamma) \Delta(R_j)^3} \prod_{j=k}^{n} \frac{R_k^2 \left(1 - R_k\right)^{2m+n+1}}{\left(1 - y^2 R_k\right)^{2m+n+2}} \prod_{j,k \in n} f_{jk}.
\]
\[
\mathcal{C}(\gamma) = C(\gamma) \mathcal{Z}_{2m}(2y^2)^{\frac{n}{2}+1}.
\]

Given the anti symmetry of the Vandermonde determinant and the Pfaffian under exchange of two variables \( R_j \) and \( R_k \) we gather
\[
\mathcal{P}_{\alpha=1}(R_j) = \frac{1}{C(\gamma)} \Delta(R_j)^3 \\
\times \prod_{k=1}^{n} R_k^{\frac{1}{2}} \left( 1 - R_k \right)^{2(m-n)+1} \prod_{j=1}^{\frac{n}{2}} F \left( \frac{1 + \gamma^2 R_{2j-1}}{1 - \gamma^2 R_{2j-1}}, \frac{1 + \gamma^2 R_{2j}}{1 - \gamma^2 R_{2j-1}} \right),
\]

where the function \( F(u, v) \) is given by equation (C.78). For \( \alpha = 2 (\beta = 1, \text{the PRE case}) \) we have from equation (3.15)

\[
I_{\text{PRE}}(\gamma^2, R) = \frac{2F_1^{(2)}\left( -\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} \mid \gamma^2 R \right)}{\det \left[ 1 - \gamma^2 R \right]^{\frac{n-1}{2} - 1}}.
\]

There are two possibilities, \( m \) even or odd. For \( m \) even we take \( p = \frac{m}{2}, q = \frac{m-1}{2} \). For these values of \( p, q \) and \( c \) the weight in (5.42) simplifies to \( (1 + \lambda_j)^{\frac{1}{2}(q-p+1)-1}(1 - \lambda_j)^{\frac{1}{2}n-1} = 1. \)

\[
2F_1^{(2)}\left( -\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} \mid \gamma^2 R \right) = \frac{1}{Z_{\frac{n-1}{2}}} \prod_{k=1}^{n} \left( 1 - \gamma^2 R_k \right)^{\frac{1}{2}} \times \prod_{j=1}^{\frac{n}{2}} \int_{-1}^{1} d\lambda_j \left| \Delta(\lambda_j) \right|^{\frac{1}{2}} \prod_{k,j}^{n} \left( 1 + \gamma^2 R_k \right) \left( 1 - \gamma^2 R_k \right) \lambda_j.
\]

For \( m \) odd we take \( p = \frac{m-1}{2}, q = \frac{m}{2} \). For these values of \( p, q \) and \( c \) the weight in (5.42) simplifies to \( (1 + \lambda_j)^{\frac{1}{2}(q-p+1)-1}(1 - \lambda_j)^{\frac{1}{2}n-1} = (1 + \lambda_j)^{\frac{1}{2}}. \)

\[
7F_1^{(2)}\left( -\frac{m}{2}, -\frac{m-1}{2}; \frac{n}{2} \mid \gamma^2 R \right) = \frac{1}{Z_{\frac{n-1}{2}}} \prod_{k=1}^{n} \left( 1 - \gamma^2 R_k \right)^{\frac{1}{2}} \prod_{j=1}^{\frac{n-1}{2}} \int_{-1}^{1} d\lambda_j \left| \Delta(\lambda_j) \right|^{\frac{1}{2}} (1 + \lambda_j)^{\frac{1}{2}} \times \prod_{k,j}^{n} \left( 1 + \gamma^2 R_k \right) \left( 1 - \gamma^2 R_k \right) \lambda_j.
\]

The products appearing in the average here are not characteristic polynomials but rather square roots of characteristic polynomials and this is why we can not follow the same type of calculation as for the PQE.

7. Conclusion

We have shown that when considering thermal transport through a semi-non-ideal Andreev quantum dot the JPDF is related to HFMA quite analogously to the case of electric transport studied in [9] and [10]. In addition we have derived for the quaternion ensemble a different representation of JPDF and found a type of random matrix model where the JPDF is of the form of a Vandermonde determinant to the third power times a Pfaffian. This is similar in structure to the JPDF found in [10]. These results can be used as a starting point for further analyzing thermal transport through such quantum dots. In particular it could be used to compute the moments/cumulants of observables such as the conductance but it would also be
interesting to use them to compute the particle entanglement as in [8, 10] or to compute systematically corrections to weakly non ideal quantum dots [15]. We note that even though we do not have a Pfaffian representation for the PRE case, it is likely we could still compute the entanglement for low number of channels as in [10]. Lastly our results could be used to study dephasing, i.e. the loss of phase coherence of particles in the quantum dot [16]. When studying dephasing the parameter \( \gamma \), representing the coupling between the dot and the leads, cannot be treated perturbatively and so an exact expression in \( \gamma \) for the JPDF would be extremely helpful.

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Appendix A. Polar decomposition

In this section we will discuss some details about the polar decomposition and the uniqueness of the decomposition. The scattering matrix can belong to \( SO(N) \) or \( Sp(N) \). The polar decomposition for the scattering matrix states that it can be decomposed as follows.

\[
S = \begin{pmatrix}
    r_{nxn} & t_{n\times m} \\
    t_{m\times n} & r'_{m\times m}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    U_1 & 0 \\
    0 & U_2
\end{pmatrix}
\begin{pmatrix}
    r & t \\
    -t' & r'
\end{pmatrix}
\begin{pmatrix}
    V_1^\dagger & 0 \\
    0 & V_2^\dagger
\end{pmatrix}
\]

\[
= U \left( \begin{pmatrix}
    r & t \\
    -t' & r'
\end{pmatrix} \right) V^\dagger,
\]

where \( r, t, t', r' \) are diagonal matrices and we have taken \( n \leq m \). The matrices \( U_1, V_1 \in O(n) \) and \( U_2, V_2 \in O(m) \) when \( S \in SO(N) \) while the matrices \( U_1, V_1 \in Sp(n), U_2, V_2 \in Sp(m) \) when \( S \in Sp(N) \). Given the unitarity condition on \( S \) we have the following relations among the diagonal elements of \( r, t, t', r' \)

\[
\begin{align*}
    r_j^2 + t_j^2 &= 1 \\
    t_j' &= t_j \\
    \text{for } j &\leq n \quad r_j' = r_j \\
    \text{for } j &> n \quad r_j' = 1.
\end{align*}
\]  

(A.48)

However this decomposition is not unique. The matrix \( r' \) has all diagonal elements equal to 1 when \( j > n \) which means it is invariant under a unitary transformation in this sector. The rectangular matrices \( t \) and \( t' \) is filled with 0’s in this sector. Thus we can restrict \( U_2 \) to the coset space \( O(m)/O(m - n) \) when the scattering matrix is in \( SO(N) \) and to the coset space \( Sp(m)/Sp(m - n) \) when the scattering matrix is in \( Sp(N) \). The number of degrees of freedom of \( SO(N) \) is given by \( \frac{N(N - 1)}{2} \). In our parametrization we have \( \frac{n(n - 1)}{2} \) degrees of freedom for \( U_1, V_1 \), \( \frac{m(m - 1)}{2} \) degrees of freedom for \( V_2 \) and \( \frac{m(m - 1)}{2} - \frac{(m - n)(m - n - 1)}{2} \) degrees of freedom for \( U_2 \). Adding to these the \( n \) degrees of
freedom from the $r_j$ variables we have in total $\frac{n^2 + m^2 - n - m}{2} + mn$ which accounts for all the number of degrees of freedom of $SO(N)$, $\frac{N(2N-1)}{2}$.

A similar situation presents itself for the decomposition of $Sp(N)$. The number of degrees of freedom for $Sp(N)$ is $N(2N+1)$. To make the parametrization unique we take $U_1 \in Sp(n)$, $V_1 \in Sp(n)/Sp(1)^n$, $U_2 \in Sp(m)/Sp(m-n)$ and $V_2 \in Sp(m)$ Summing up the degrees of freedom we have $2n^2 + 2m^2 + 4mn + m + n$ which correspond to the $N(2N+1)$ degrees of freedom of $Sp(N)$. A unique parametrization is necessary to compute the Jacobian. However the scattering matrix is invariant under the subgroup $O(m-n)$ for the case of $SO(N)$ and invariant under $Sp(m-n)$ and $Sp(1)^n$ in the case of $Sp(N)$. This means that we can extend the integration over the coset space to the group, the difference being a proportionality constant. More precisely we have for every matrix $U$ of the form

\[
U = \begin{pmatrix} I_m & 0 \\ 0 & U' \end{pmatrix}
\]

we have

\[
F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 r' V_1 & U_2 r' V_2 \end{pmatrix} \right) = F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 r' U r V_1 & U_2 r' U r V_2 \end{pmatrix} \right)
\]

Therefore the identity holds when integrating over $U'$

\[
F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 r' V_1 & U_2 r' V_2 \end{pmatrix} \right) = \int_{O(m-n)} d\mu(U') F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 r' U r V_1 & U_2 r' U r V_2 \end{pmatrix} \right).
\]

We have then

\[
\int_{SO(N)} d\mu(S) F(S) = \int d\tau j(\tau) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) \times \int_{O(n)/O(m-n)} d\mu(U_2) F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 r' V_1 & U_2 r' V_2 \end{pmatrix} \right)
\]

\[
= \int d\tau j(\tau) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) \int_{O(n)/O(m-n)} d\mu(U_2) \times \int_{O(m-n)} d\mu(U) F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 U r' V_1 & U_2 U r' V_2 \end{pmatrix} \right)
\]

\[
= \int d\tau j(\tau) \int_{O(n)} d\mu(U_1) d\mu(V_1) d\mu(V_2) d\mu(U_2) F \left( \begin{pmatrix} U_1 rV_1 & U_1 rV_2 \\ -U_2 U r' V_1 & U_2 U r' V_2 \end{pmatrix} \right). \tag{A.49}
\]

**Appendix B. Theory of symmetric functions**

In this appendix we review briefly some properties of partitions, Jack polynomials and zonal spherical functions. For more details see [12, 17] and in particular for the integral theorems
see sections 4, 5 and 6 from chapter VII in [12]. We use the theory of symmetric functions to expand a given symmetric function of multiple variables $f(x_1, \cdots, x_n)$ in terms of Jack polynomials and subsequently integrate using known integration properties of these polynomials.

**B.1. Preliminaries and notation: partitions**

A set of non increasing integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is called a partition of $\kappa$ if

\[
\sum_{j=1}^{l} \lambda_j = \kappa,
\]

$\kappa$ is called the weight of the partition and the length of the partition, $l(\lambda) = l$, is the number of integers $\lambda_j$. Often one also writes $(\lambda_1, \lambda_2, \cdots, \lambda_l, 0, \cdots, 0) = (\lambda_1, \lambda_2, \cdots, \lambda_l)$. To each partition there is an associated diagram made of rows of consecutive boxes. For a partition $\lambda$ there are $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row and so on. The length, $l(\lambda)$, then denotes the amount of rows. Each box is then denoted by two coordinates $s_{ij}$, where $i$ denotes the row and $j$ the column where the box is located. The arm of a box $a_{ij}(s)$ is equal to the number of boxes directly to the right of the box $s_{ij}$. The leg of a box $l_{ij}(s)$ is equal to the number of boxes directly below of the box $s_{ij}$. Similarly the co-arm and co-leg are defined as the boxes directly to the left and above the box $s_{ij}$.

From a given partition $\lambda$ other partition can be constructed. The conjugate of a partition $\lambda$ is a partition denoted by $\lambda^\top$ and defined as

\[
\lambda^\top_k = \#\{\lambda_j \in \lambda : \lambda_j \geq k\}.
\]

Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, the sum of the partition $\lambda$ with itself is given by

\[
\lambda + \lambda = (\lambda_1 + \lambda_1, \lambda_2 + \lambda_2, \cdots) = (2\lambda_1, 2\lambda_2, \cdots) = 2\lambda
\]

and the union of the partition $\lambda$ with itself is defined as

\[
\lambda \cup \lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots).
\]

The partition $2\lambda$ is called an even partition because every entry is an even number and the transposed of the union of the partition with itself, $(\lambda \cup \lambda)^\top$, is an even partition. The generalized Pochhammer symbol is given by

\[
[u]_\lambda^{(\alpha)} = \prod_{j=1}^{\lambda} \frac{\Gamma\left(u - \frac{j - 1}{\alpha} + \lambda_j\right)}{\Gamma\left(u - \frac{j - 1}{\alpha}\right)}.
\]
We define the following coefficients

\[
\begin{align*}
  d_j(\alpha) &= \prod_{s \in \lambda} (aa(s) + \alpha + l(s) + 1) \\
  d_j'(\alpha) &= \prod_{s \in \lambda} (aa(s) + \alpha + l(s)) \\
  e_j(\alpha, n) &= \prod_{s \in \lambda} (aa'(s) + \alpha + n - l'(s)) \\
  e_j'(\alpha, n) &= \prod_{s \in \lambda} (aa'(s) + \alpha + n - l'(s) - 1) \\
  h_j(\alpha) &= \prod_{s \in \lambda} (aa(s) + l(s) + 1) \\
  b_j(\alpha, n) &= \prod_{s \in \lambda} (aa'(s) + n - l'(s)),
\end{align*}
\]

(B.52)

We have then in terms of the Pochhammer symbol

\[
\begin{align*}
  b_j(\alpha, n) &= a^{|\lambda|} \left[ \frac{n}{\alpha} \right]_j^{(\alpha)}, \quad (B.53) \\
  e_j(\alpha, n) &= a^{|\lambda|} \left[ 1 + \frac{n}{\alpha} \right]_j^{(\alpha)}, \quad (B.54) \\
  e_j'(\alpha, n) &= a^{|\lambda|} \left[ 1 + \frac{n-1}{\alpha} \right]_j^{(\alpha)}. \quad (B.55)
\end{align*}
\]

B.2. Jack polynomials

We review here some properties of Jack polynomials and the zonal spherical functions. In particular for integral properties of the spherical functions and representations in terms of Jack polynomials (equations (B.60), (B.61), (B.62), (B.63) and (B.64)) see sections 4, 5 and 6 from chapter VII in [12]. The Jack polynomials, denoted by \( P_\lambda^{(\alpha)}(x_1, \ldots, x_n) \), are multi-variable polynomials which are symmetric under the permutation of the variables and they form a basis for expanding other symmetric functions. \( \alpha \) is a real index and in our case will be related to some kind of symmetry but we can also view them as a different set of symmetric polynomials that is orthogonal with respect to a different scalar product. When \( \alpha = 1 \) the Jack polynomial is equal to the Schur polynomial, \( P_\lambda^{(1)}(x_1, \ldots, x_n) = S_\lambda(x_1, \ldots, x_n) \). The variables of the Jack polynomials can also be seen as the eigenvalues of a matrix, which is our case. One has then the notation

\[
P_\lambda^{(\alpha)}(X) = P_\lambda^{(\alpha)}(x_1, \ldots, x_n)
\]

with \( x_j \) the eigenvalues of the matrix \( X \). For our purposes we are only interested in the Jack polynomials with \( \alpha = 2, 1, \frac{1}{2} \) which corresponds in the random matrix perspective to \( \beta = 1, 2, 4 \) respectively \( \left( \alpha = \frac{2}{\beta} \right) \). The Jack polynomials evaluated at the identity matrix is known and given in terms of the Pochhammer symbol as
In addition there exist relations between the different Jack polynomials evaluated at the identity and the Schur polynomials. Then the following identities hold \[13\]

\[
\frac{e_j^\alpha(n)}{d_j^\alpha(n)} P_j^{(\alpha)}(1_n) = \begin{cases} 
S_j^\alpha(1_n) & \alpha = 1 \\
S_{2j}(1_n) & \alpha = 2 \\
S_{j,\cup j}(1_{2n}) & \alpha = \frac{1}{2}
\end{cases}. \tag{B.57}
\]

These can be proved by expressing the Jack polynomials and Schur functions in terms of Pochhammer symbol, equation (B.56), and using equations (B.52). We are mainly interested in the expansion of the determinant raised to some power in terms of symmetric functions. In terms of Schur polynomials it is given as follows (generalized binomial summation formula [12, 13, 17])

\[
\det[1 - X]^{-\beta} = \sum_j S_j(1_n)S_j(X)
\tag{B.58}
\]

\[
= \sum_j \frac{[\alpha]_j^{(1)}}{d_j^{(1)}} S_j(X). \tag{B.59}
\]

In this expansion the coefficients in front of the Jack polynomials are given themselves in terms of Jack polynomials evaluated at the identity. The Zonal spherical functions defined in Macdonald, \(\Omega_k^{(\beta)}(x)\), are defined by the following integration property ([12] pp 425, 453)

\[
\int_{O(n)} dU \Omega_k^{(\beta)}(AUB) = \Omega_k^{(\beta)}(A) \Omega_k^{(\beta)}(B), \tag{B.60}
\]

\[
\int_{Sp(n)} dU \Omega_k^{(1/2)}(AUB) = \Omega_k^{(1/2)}(A) \Omega_k^{(1/2)}(B) \tag{B.61}
\]

and are given in terms of Jack polynomials as ([12] pp 422, 444, 453)

\[
\Omega_k^{(\beta)}(x_n) = \frac{P_k^{(\beta)}(x_n)}{P_k^{(\beta)}(1_n)} \tag{B.62}
\]

and can be related to the Schur functions via an integration over a group that depends on the symmetry index \(\beta\).

\[
\Omega_k^{(2)}(x_n) = \int_{O(n)} dk S_{2j}(kx), \tag{B.63}
\]

\[
\Omega_k^{(1/2)}(x_n) = \int_{Sp(n)} dk S_{j,\cup j}(kx). \tag{B.64}
\]

A careful use of the expansion and of the integration theorems (B.60), (B.61), (B.63) and (B.64), is what ultimately will allow us to compute the JPDF. The HFMA_2, is given as follows
The HFMA of one matrix argument, HFMA$_1$, is given as follows

\[ 2F^{(a)}_1(a; b; c|X) = \sum_{\lambda} \frac{a(a)_\lambda}{d_j(a)} \frac{b(b)_\lambda}{(c(c)_\lambda} \frac{P^{(a)}_\lambda(X) \overline{P}^{(a)}_\lambda(Y)}{P^{(a)}_\lambda(1)}. \] (B.65)

and \( 2F^{(a)}(a; b; c|X) = 2F^{(a)}(a; b; c|X, 1). \)

**B.3. A combinatorial identity**

We will prove here the identity (B.67) holds for \( n \) real which we will need in section 3.2 and have not found in the literature. We have the following identity for \( a = 1/2 \), equation (B.57)

\[ S_{j \cup \lambda(2n)} = \frac{e_j(1/2, n)b_j(1/2, n)}{d_j(1/2)h_j(1/2)}. \]

The right hand side can be written down in terms of Pochhammer symbols and because we have on the left hand side a Schur function of the identity we can express this side also in terms of the Pochhammer symbol.

\[ \frac{(2n)_{{\lambda}_\cup \lambda \lambda}}}{h_j(1/2)} = \frac{(1/2)^{2l_j}[2n - 1]_{\lambda_j}^{(1/2)}[2n]_{\lambda_j}^{(1/2)}}{d_j(1/2)h_j(1/2)}. \] (B.67)

This identity holds for \( n \) integer and we will show now it holds for \( n \) real. We denote by \( \mu_j \) the \( j \)th integer of the partition \( \lambda \cup \lambda \). Meaning \( \lambda \cup \lambda = \{\mu_1, \mu_2, \ldots \mu_{l(\lambda \cup \lambda)}\} \). Thus \( \mu_{j-1} = \lambda_j \) and \( \mu_j = \lambda_j \). The Pochhammer symbol on the right can be decomposed as follows

\[
2\lambda_\cup \lambda = \prod_j \frac{\Gamma\left(2x - j + 1 + \mu_j\right)}{\Gamma(2x - j + 1)} = \prod_j \frac{\Gamma\left(2x - (2j - 1) + 1 + \lambda_j\right)}{\Gamma(2x - (2j - 1) + 1)} \prod_j \frac{\Gamma\left(2x - 2j + 1 + \lambda_j\right)}{\Gamma(2x - 2j + 1)}
\]

\[
\prod_j \frac{\Gamma\left(2x - 2(j - 1) + 1 + \lambda_j\right)}{\Gamma(2x - 2(j - 1))} \prod_j \frac{\Gamma\left(2x - 1 - 2(j - 1) + \lambda_j\right)}{\Gamma(2x - 1 - 2(j - 1))} = [2\lambda_\cup \lambda]_{\lambda _\lambda} = [2\lambda_\cup \lambda]_{\lambda _\lambda} (2x - 1)^{1/2}. \] (B.68)

We have not assumed \( x \) to be an integer here so this identity is valid for \( x \) real. If we use this in the identity, equation (B.67) above, we gather

\[ \frac{(1/2)^{2l_j}}{d_j(1/2)h_j(1/2)} = \frac{1}{h_{\lambda \cup \lambda}(1)}. \]

This no longer depends on \( n \) and so it is a combinatorial relation. Thus we have by multiplying the right by \([2\lambda_\cup \lambda]_{\lambda _\lambda}\) and the left by \([2\lambda_\cup \lambda]_{\lambda _\lambda} (2x - 1)^{1/2}\)
\[
\frac{[2x]^{(1)}_{\lambda(1)}}{h_{\lambda(1)}} = \frac{(1/2)^{2|k|}[2x]^{(1/2)}[2x-1]^{(1/2)}}{d_{\lambda}^{(1/2)}h_{\lambda}^{(1/2)}}
\]
\[
= \frac{e_{1/2}(1/2, x)b_{1/2}(1/2, x)}{d_{\lambda}^{(1/2)}h_{\lambda}^{(1/2)}}.
\] 

(B.69)

Appendix C. Average of characteristic polynomials

We wish to compute the average of characteristic polynomials of Jacobi ensembles.

\[
\prod_{j=1}^{p} \int_{-1}^{1} d_{\lambda} \Delta(\lambda_{j})^{\beta} (1 - \lambda_{j})^{a}(1 + \lambda_{j})^{b} \prod_{k,d}^{n_{p}} (v_{k} - \lambda_{j}).
\] 

(C.70)

We briefly state here the results of [14] where it was found that these averages of characteristic polynomials can be written down as a Pfaffian. We note that in [18] another Pfaffian representation was derived. We introduce the skew-orthogonal polynomials \( q_{\ell}(x) \) which satisfy the following orthogonality relations

\[
\langle q_{2l}, q_{2\ell} \rangle = 0
\]
\[
\langle q_{2l+1}, q_{2\ell+1} \rangle = 0
\]
\[
\langle q_{2l}, q_{2\ell+1} \rangle = \eta \delta_{lp}
\]

with the scalar product

\[
\langle f, g \rangle = \int_{-1}^{1} dv \int dvf(v)g(v)w(v)sign(x - y),
\] 

(C.71)

\[
w(u) = (1 + u)^{a}(1 - u)^{b}.
\] 

(C.72)

There are four possible cases depending if \( p \) and \( n \) are even or odd. In our case \( p \) will be even and \( n \) arbitrary. When \( p \) is even and \( n \) even we have from [14]

\[
\left\langle \prod_{k=1}^{n} \det[v_{k} - Y] \right\rangle = e_{n,p} Pf_{j,k\leq\ell+1}^{f_{jk}} \Delta(v_{k})
\] 

(C.73)

with \( f_{jk} \) a \( n \times n \) anti symmetric matrix with the following entries

\[
f_{j,k} = F(v_{j}, v_{k})
\] 

(C.74)

with

\[
F(v, u) = \sum_{l=1}^{\infty} \frac{1}{\gamma_{l-1}}(q_{2l-2}(u)q_{2l-1}(v) - q_{2l-2}(v)q_{2l-1}(u)).
\] 

(C.75)

When \( p \) is even and \( n \) is odd we have

\[
\left\langle \prod_{k=1}^{n} \det[v_{k} - Y] \right\rangle = e_{n,p} Pf_{j,k\leq\ell+1}^{f_{jk}} \Delta(v_{k})
\] 

(C.76)
with \( f_{jk} \) a \( n+1 \times n+1 \) anti symmetric matrix with the following entries

\[
\begin{align*}
    f_{1,1} &= 0 \\
    f_{j,k} &= F(v_{j-1}, v_{k-1}) \quad \text{for } j, k = 2, \cdots n+1 \\
    f_{j,j+1} &= q_{n+p-1}(v_{j-1}) \quad \text{for } j = 1, \cdots n
\end{align*}
\] (C.77)

with

\[
F(v, u) = \sum_{j=1}^{p+1} \frac{1}{2^{j-1}} \left( q_{2j-2}(u)q_{2j-1}(v) - q_{2j-2}(v)q_{2j-1}(u) \right). \quad (C.78)
\]

References

[1] Blümel R and Smilansky U 1990 Random-matrix description of chaotic scattering: semiclassical approach Phys. Rev. Lett. 64 241
[2] Altland A and Zirnbauer M 1997 Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures Phys. Rev. B 55 1142
[3] Andreev A F 1964 Thermal conductivity of the intermediate state of superconductors Sov. Phys. JETP 19 1228
[4] Serban I, Béri B, Akhmerov A R and Beenakker C W J 2010 Domain wall in a chiral p-wave superconductor: a pathway for electrical current Phys. Rev. Lett. 104 147001
[5] Dahlhaus J P, Béri B and Beenakker C W J 2010 Random-matrix theory of thermal conduction in superconducting quantum dots Phys. Rev. B 82 014536
[6] Mello P A, Pereya P and Seligman T H 1985 Information theory and statistical nuclear reactions: I. General theory and applications to few-channel problems Ann. Phys. 161 254
[7] Béri B 2009 Random scattering matrices for Andreev quantum dots with nonideal leads Phys. Rev. B 79 214506
[8] Villamaina D and Vivo P 2013 Entanglement production in non-ideal cavities and optimal opacity Phys. Rev. B 88 041301
[9] Vidal P and Kanzieper E 2012 Statistics of reflection eigenvalues in chaotic cavities with non-ideal leads Phys. Rev. Lett. 108 206806
[10] Jarosz A, Vidal P and Kanzieper E 2015 Random matrix theory of quantum transport in chaotic cavities with non-ideal leads Phys. Rev. B 91 180203
[11] Forrester P J 2006 Random scattering matrices and the jacobi ensemble J. Phys. A: Math. Gen. 39 6861
[12] Macdonald I G 1995 Symmetric Functions and Hall Polynomials (Oxford: Clarendon)
[13] Feng Z M and Song J P 2009 Integrals over the circular ensembles relating to classical domains J. Phys. A: Math. Theor. 42 325204
[14] Nagao T and Forrester P J 1998 The smallest eigenvalue distribution at the spectrum edge of random matrices Nucl. Phys. B 509 561
[15] Rodríguez-Pérez S, Marino R, Novaes M and Vivo P 2013 Statistics of quantum transport in weakly nonideal chaotic cavities Phys. Rev. E 88 052912
[16] Brouwer P and Beenakker C W J 1997 Voltage-probe and imaginary-potential models for dephasing in a chaotic quantum dot Phys. Rev. B 55 4695
[17] Forrester P J 2010 Log-Gases and Random Matrices (Princeton, NJ: Princeton University Press)
[18] Borodin A and Strahov E 2006 Averages of characteristic polynomials in random matrix theory Commun. Pure Appl. Math. 59 161–253