Modeling tagged pedestrian motion: a mean-field type control approach*

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Abstract: This paper suggests a mean-field model for the movement of tagged pedestrians, distinguishable from a surrounding crowd, with a targeted final destination. The tagged pedestrians move through a dynamic crowd, interacting with it while optimizing their path. The model includes distribution-dependent effects like congestion and crowd aversion. The behavior in the presence of such effects is studied. For some special cases, closed-form expressions of the optimal path are found and numerical simulations illustrate the dynamics of the tagged pedestrians.

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1. Introduction

Over the years, a variety of mathematical approaches to the modeling of pedestrian motion have been proposed. Microscopic force-based models [18, 13], and in particular the social force model, represent pedestrian behavior as a reaction to forces and potentials, applied not only by surrounding environment but also by the pedestrian’s internal motivation and desire. A cellular automata approach has also been used to microscopic modeling of pedestrian crowds [10, 25, 24]. Macroscopic models view the crowd as a continuum described by averaged quantities such as density and pressure. The Hughes model [23, 20, 30] couples a conservation law, representing the physics of the crowd, with an eikonal equation modeling a common task of the pedestrians. Its variations are manifold. Kinetic and other multi-scale models [3, 14, 4] constitute an intermediate step between the micro- and the macroscales. Microscopic game and optimal control models for pedestrian crowd dynamics with their relevant continuum limits in the form of mean-field games [17, 26, 9, 29, 11, 15], and mean-field type optimal control [16, 15, 2] have been studied in the last decade. The mean-field approach rests on an exchangeability assumption; pedestrians are anonymous, they may have different paths but one individual can not be distinguished from another. While interesting when modeling circumstances where

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pedestrians can be considered indistinguishable, for instance a train station during rush hour or fast exits of an area in case of an alarm, there are situations where an anonymous crowd model is not satisfactory. Two ways to break the anonymity have been suggested, considering *multiclass agents* \([22, 21]\) and *major agents* \([19, \text{and references therein}]\).

This paper models pedestrians acting dissimilarly to a surrounding crowd using a mean-field type control approach. The central section of this paper is the solved examples, that illustrate pedestrian motion in the model. The pedestrian/group of pedestrians, hereafter referred to as *the tagged pedestrian* or simply *the tagged*, is distinguishable from the rest of the crowd due to its distinct behavior. The setup has the flavor of a major-minor agent model, where as minor the tagged does not necessarily influence the surrounding crowd. The tagged pedestrian has a pre-set final destination which she considers essential to reach; she is not satisfied if she ends up in the proximity of the target, the target has to be hit. Examples of such motion are deliveries and emergency personnel, such as doctors, fire-fighters or guards, that move through crowded areas to reach their service targets. The tagged pedestrian might or might not be influenced by the motion of the crowd and vice versa. In the example just mentioned, a single medic is likely to have a very minor influence on the surrounding crowd. The crowd, on the other hand, will influence the movement of the medic through effects like congestion that slows down and flow that drags. A security team however ideally has the capability to move through a crowd and effect it when necessary. Within a group acting abnormally, say a team of medics, interaction between individuals that influence the movement of the group. In the mean-field approach, interaction is captured by the probability distribution of a typical agent in the group. To catch typical crowd interactions, we consider dependencies on the full distribution of the tagged’s state. This leads to a variation, in a sense with greater generality, of the control problem studied in \([19]\).

The contribution of this paper is an optimal control model for the motion of a tagged pedestrian acting under general distribution dependent backward dynamics and cost, and the main focus is on solvable cases. We identify a few classes of problems with closed-form solutions, some of which we solve explicitly. Further directions of research are also outlined.

The tagged pedestrian model is presented in Section 2. A Pontryagin-type maximum principle that gives necessary and sufficient conditions for optimality is applied. Examples of tagged motion are studied in Section 3. All technical proofs and background theory is moved to the appendix.

### 2. The tagged pedestrian model

In this section, the model for the tagged pedestrian will be introduces in a few steps. We start out in a deterministic setting. The tagged adjusts her velocity \(c : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d\) with the control \(v(\cdot)\), taking values in the compact set \(U \subset \mathbb{R}^d\), and she moves in \(\mathbb{R}^d\) according to the dynamics

\[
\begin{align*}
\dot{Y}(t) &= c(t, Y(t), v(t)), \quad t \in [0, T], \\
Y(T) &= \chi.
\end{align*}
\]  

(2.1)
At each point in time, the tagged pays a cost $l : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ and at start, she pays an additional cost $h : \mathbb{R}^d \to \mathbb{R}$. The tagged picks her control $v(\cdot)$ so that her total cost $I$ is minimized,

$$I(v(\cdot)) = \int_0^T l(t, Y(t), v(t))dt + h(Y(0)).$$

(2.2)

The terminal condition and the function $c$ are model data together with the running cost $l$ and initial cost $h$. Examples of typical such data are given in Section 3. Letting the tagged pick her control grants her vision. She may take the surroundings into account, thereby she avoids collisions and obstacles, and is no longer only ruled by inertia. This is a shift in perspective, from a classical particle model to a decision-based model. Fundamental differences between classical and smart particle models are outlined in [27], where will to reach specific targets and repulsion from other pedestrians are listed as typical traits. Also, pedestrian motion can be considered deterministic if the crowd is sparse but appears to be partially random if the crowd is dense. Next, we extend the (2.1) to stochastic dynamics to capture this.

Let the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying the usual conditions, carry a $d$-dimensional $\mathbb{F}$-Wiener process $B$ and a square-integrable $\mathcal{F}_T$-measurable random variable $\chi$ independent of $B$. Given a control $v \in U[0, T] := \{ u : [0, T] \times \Omega \to U ; u \mathcal{F}_t\text{-adapted} \}$, the path of the tagged pedestrian satisfies the backward stochastic differential equation (BSDE)

$$\begin{cases}
    dY_t = c(t, Y_t, Z_t, v_t)dt + Z_t dB_t, \\
    Y_T = \chi.
\end{cases}
$$

(2.3)

The Wiener process $B$ is the internal source of randomness in the tagged motion. Exogenous randomness will enter through the velocity $c : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times U$ and the terminal condition $\chi$. In contrast to an ODE with a terminal condition, that can be solved by reversing the time of an initial value ODE, one cannot reverse the time of an SDE to retrieve a meaningful solution to a BSDE. There is a natural “flow of information” in the model; the past behavior is known but future random events are not observed. All information in the model up to time $t$ is contained in the $\sigma$-algebra $\mathcal{F}_t$. A process that depends only on past and current information, i.e. that at time $t$ it is $\mathcal{F}_t$-measurable, is called an $\mathcal{F}_t$-adapted process. If the tagged has access to complete information about the past, it is very reasonable to require $\mathcal{F}_t$-adaptedness of a solution to the BSDE. Hence a time-shift technique can not be used to solve it. Instead, in addition to the velocity, (2.3) contains the martingale term $Z_t dB_t$. The role of the process $Z$ is to make the solution of (2.3) adapted. Consider the equation $d\tilde{Y}_t = c(t, \omega)dt$ with terminal condition $\tilde{Y}_T = \chi$. The solution $\tilde{Y}_t = \chi - \int_t^T c(s, \omega)ds$ is not necessarily $\mathcal{F}_t$-adapted since $\chi \in L^2(\mathcal{F}_T)$. On the other hand,

$$Y_t = \mathbb{E} \left[ \chi - \int_t^T c_s ds \mid \mathcal{F}_t \right]$$

(2.4)

is $\mathcal{F}_t$-adapted. Under the assumption that $Y_t$ is square-integrable for all $t \in [0, T]$ the martingale representation theorem gives existence of a unique square-integrable and $\mathcal{F}_t$-adapted process $Z$ such that

$$Y_t - \int_0^t c(s, \omega)ds = \int_0^t Z_s dB_s, \quad Y_T = \chi.$$

(2.5)
The conditional expectation (2.4) is the $L^2$-projection of the tagged pedestrian’s future path on the information currently available. Therefore $Z$, can be interpreted as a control used by the tagged to make her path towards $\chi$ the best prediction based on available information. This approach to BSDEs is standard and was first introduced by [5] for the linear case and by [28] in the general case. From a modeling point of view, the tagged pedestrian uses two controls:

- $v(\cdot)$ - to heed her preferences on congestion and energy use. This control is picked by an optimization procedure. More on $v(\cdot)$ below in Section 3.
- $Z$ - to predict the best path to $\chi$ given $v(\cdot)$. This is a square-integrable process given by the martingale representation theorem.

The control $v(\cdot)$ is picked as a minimizer to the tagged’s expected total cost $I$,

$$I(v(\cdot)) = \mathbb{E} \left[ \int_0^T l(t, Y_t, v_t) dt + h(Y_0) \right].$$

(2.6)

The tagged can also be viewed as a group of agents breaking away from the general behavior of the surrounding crowd, starting to move towards a new target. In the context of pedestrian crowds, the group can represent a medical team or a small security force. In an emergency situation, these groups take on a completely different role than the average pedestrian. Applications towards the dynamics of cancer cells and smart medicine in the human body and malware propagation in a network are also possible in this setup. The communication between individuals in the tagged group is modeled with a dependence on the law of a typical group member, $P_{Y_t} := P \circ (Y_t)^{-1} \in P_2(\mathbb{R}^d)$, and the group members follow the dynamics

$$\begin{cases}
    dY_t = c(t, Y_t, P_{Y_t}, Z_t, v_t) dt + Z_t dB_t, \\
    Y_T = \chi.
\end{cases}$$

(2.7)

The dependence on the law makes (2.7) a so-called mean-field BSDE, see Appendix A for a short background. Effects like congestion, the extra effort needed when moving in a high density area, and aversion, repulsion from other pedestrians, can be captured with law-dependence, so the law is also introduced in the running and initial cost. Minimizing $I$ under (2.3) is a mean-field type open loop optimal control problem,

$$\begin{cases}
    \min_{v(\cdot) \in U[0,T]} I(v(\cdot)) := \mathbb{E} \left[ \int_0^T l(t, Y_t, P_{Y_t}, v_t) dt + h(Y_0, P_{Y_0}) \right], \\
    \text{s.t. } dY_t = c(t, Y_t, P_{Y_t}, Z_t, v_t) dt + Z_t dB_t, \\
    Y_T = \chi.
\end{cases}$$

(2.8)

Under the assumption that the control problem (2.8) is well-defined, i.e. that the minimum is attained for some $v \in U[0,T]$, we have the following characterization of the optimal control.

**Theorem 2.1** (Maximum Principle - necessary conditions). Let $H$ be the Hamiltonian

$$H(t, y, \mu, z, v, p) := c(t, y, \mu, z, v)p - l(t, y, \mu, v),$$

(2.9)

let $\theta_t := (t, Y_t, P_{Y_t}, Z_t, v_t)$ and let $p_t$ solve the adjoint equation (cf. Appendix B)

$$\begin{cases}
    dp_t = - \left\{ \partial_y c(\theta_t)p_t + \mathbb{E} \left[ \partial_{\mu} c(\tilde{\theta}_t)p_t + \partial_{\mu} l(\tilde{\theta}_t) \right] \right\} dt - \partial_z c(\theta_t)p_t dB_t, \\
    p_0 = \partial_y h(\theta_0) + \mathbb{E} \left[ \partial_{\mu} (\tilde{\theta}_0) \right].
\end{cases}$$

(2.10)
Suppose that \((\hat{Y}, \hat{Z}, \hat{v})\) is optimal for the control problem (2.8) and let \(\hat{\theta}_t := (t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{v}_t)\). Then

\[
\hat{v}_t = \arg\max_{v \in U} H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, v, p_t|_{\hat{\theta}_t = \hat{\theta}_t}).
\]  

(2.11)

**Theorem 2.2** (Maximum Principle - sufficient condition). Suppose that \(\hat{v}_t\) satisfies (2.11) for all \(t \in [0, T]\), that \(H\) is concave in \((y, \mu, z, v)\) and that \(h\) is convex in \((y, \mu)\). Then \((\hat{Y}, \hat{Z}, \hat{v})\) is optimal for the control problem (2.8).

The proofs of the preceding theorems are found in Appendix B. As a corollary, we get sufficient conditions for the existence of an optimal control to (2.8).

**Corollary 2.1.** Problem (2.8) admits an optimal control if \(H\) and \(h\) satisfy the conditions of Theorem 2.2.

3. Closed-form solutions in terms of conditional expectations

Consider a tagged pedestrian moving through a crowded environment. Suppose the crowd is described by a mean-field model, for example as in [2], and is influenced by the tagged. If all pedestrians are rational, this leads to a set of inequalities describing their best response to each others actions,

\[
\begin{align*}
J(\hat{u}(); \hat{v}()) &\leq J(u(); \hat{v}()) , \quad \forall u() \in \mathcal{U}[0, T], \\
I(\hat{v}(); \hat{u}()) &\leq I(v(); \hat{u}()) , \quad \forall v() \in \mathcal{U}[0, T],
\end{align*}
\]

(3.1)

where \(J\) is the expected total cost of the crowd and \(u()\) is the control of the crowd. The system (3.1) constitutes a non-zero sum game and allows for numerous types of interactions. Since the tagged influences the crowd, it is a so-called major agent. The tagged could then represent an evacuation team with the ability to guide the crowd towards a target exit. If the crowd is independent of the behavior of the tagged pedestrian, i.e. \(J(u(); \hat{v}()) = J(u(); \cdot)\), the tagged is a minor agent, representing for example a single medic, security guard or fire fighter passing through the indifferent crowd towards a patient, disturbance or fire. Note that if the dependence on \(\hat{v}()\) is suppressed in \(J\), (3.1) becomes two separate optimization problems where the solution of the first is an input into the second. Here are some possible interactions:

(i) In dense parts of the crowd there is a higher risk of accidents and panic, therefore pedestrians exhibit an aversion to these areas. Moreover, it takes more effort to move through a dense crowd than a sparse crowd. This effect, congestion, results is lower walking speed and discomfort. A tagged pedestrian with crowd- and/or congestion-averse preferences observes the distribution of the crowd reacts to it.

(ii) While walking through the crowd, the tagged will adapt her velocity to match the velocity of the crowd as a mean of reducing energy use; she can not without effort stand still in an advancing crowd nor walk against its movement. This effect can be introduced as a penalty in the expected total cost and is reminiscent of the desired velocity in Helbing’s social force model [18].

(iii) If the crowd acts under a penalty on deviation from the tagged’s path, the tagged has the role of a leader or guide whom the crowd wants to follow. Oppositely, if the penalty is on the proximity to the tagged’s path the she pushes the crowd away.
The rest of this section will not consider the game (3.1) but is devoted to examples of the one-way interactions described in (i) and (ii). Two special case is solved numerically, otherwise the optimal path of the tagged pedestrian given as a conditional expectation.

### 3.1. Aversion towards a surrounding crowd

Here, a case similar to the non-local model considered in [2] will be solved. Movement in densely crowded areas is penalized, this gives rise to a crowd averse behavior. Consider a crowd with known distribution $P_X_t$ at time $t$, indifferent to the actions of the tagged pedestrian. The tagged pedestrian takes the crowd into account through her cost,

$$
c := v(t) + \lambda_0 Z_t, \quad h(Y_0, P_{Y_0}) := h(Y_0),
$$

$$
l := \frac{\lambda_1}{2} |v(t)|^2 + \lambda_2 \int_{\mathbb{R}} \phi(Y_t - z) P_X_t(dz),
$$

where $\phi$ is a smooth, symmetric and positive compactly supported density function. The function $\phi$ models the area within which other pedestrians influence the tagged pedestrian and the levels of this influence therein. In view of Theorem 2.2, the optimal control is $\hat{v}(t) = p_t/\lambda_1$ where $p$ solves the adjoint equation (cf. (2.10))

$$
\begin{cases}
    dp_t = \lambda_2 \varphi'(\hat{Y}_t) dt - \lambda_0 p_t dB_t, \\
p_0 = \partial_y h(\hat{Y}_0),
\end{cases}
$$

where $\varphi(y) := \int_{\mathbb{R}} \phi(y - z) P_X_t(dz)$. Equation (3.3) has the explicit solution

$$
p_t = U_t \left( \partial_y h(\hat{Y}_0) + \int_0^t \lambda_2 \varphi'(\hat{Y}_s) U_s^{-1} ds \right),
$$

$$
U_t := \exp \left( -\lambda_0 B_t - \frac{1}{2} \lambda_0^2 t \right).
$$

Let $d\tilde{B} := \lambda_0 dt + dB_t$, the dynamics of the tagged becomes

$$
\begin{cases}
dY_t = v(t) dt + Z_t d\tilde{B}_t, \\
Y_T = \chi.
\end{cases}
$$

The optimally controlled backward dynamics can be summarized in the following formula

$$
\hat{Y}_t = E \left[ \chi - \int_t^T \frac{U_s}{\lambda_1} \left( \partial_y h(\hat{Y}_0) + \int_0^s \lambda_2 \varphi'(\hat{Y}_\tau) U_\tau^{-1} d\tau \right) ds \mid F_t \right],
$$

where the expectation is taken with respect to the measure $\tilde{P}$ under which $\tilde{B}$ is a standard Brownian motion. Note that the optimal control is not Markovian since $\hat{v}_t$ depends on $(\hat{Y}_s)_{0 \leq s \leq t}$, the path up to time $t$. 
A special case: aversion to the mean

Assume that the tagged can track the mean of the crowd and let its crowd aversion preference be \( \varphi(Y_t) := \mathbb{E}[(Y_t - \bar{X}_t)^2] \). Depending on the sign of \( \lambda_2 \), this will yield mean-seeking or mean-averse behavior. The adjoint equation becomes

\[
\begin{cases}
    dp_t = \lambda_2(\hat{Y}_t - \mathbb{E}[X_t])dt - pt dB_t, \\
p_0 = \partial_y h(\hat{Y}_0).
\end{cases}
\]  

(3.7)

Furthermore, assume that \( h \) is a quadratic penalization on the deviation from a preferred initial position \( y_0 \), so that

\[ p_0 = \lambda_3(\hat{Y}_0 - y_0). \]  

(3.8)

We will solve \( d\hat{Y}_t = \dot{v}(t) + \lambda_0 \hat{Z}_t dt + \hat{Z}_t dB_t, \hat{Y}_T = \chi \) with the ansatz

\[ \hat{Y}_t = \gamma(t)p_t + \theta(t), \quad \gamma(T) = 0, \quad \theta(T) = \chi. \]  

(3.9)

Differentiating (3.9) gives

\[
\begin{cases}
    d\hat{Y}_t = \left(\dot{\gamma}(t)p_t + \dot{\theta}(t)\right) dt + \gamma(t)\lambda_2(\hat{Y}_t - \mathbb{E}[X_t]) dt - \lambda_0 \gamma(t)p_t dB_t, \\
    \hat{Y}_T = \chi.
\end{cases}
\]  

(3.10)

Matching the coefficients of the optimally controlled dynamics, where \( \dot{v}_t = p_t/\lambda_1 \) by Theorem 2.2, with the previous equation yields

\[
\begin{cases}
    \dot{\gamma}(t) = -\lambda_2 \gamma(t)^2 + \lambda_0 \gamma(t) + 1/\lambda_1, \quad \gamma(T) = 0, \\
    \dot{\theta}(t) = -\lambda_2 \gamma(t)(\theta(t) - \mathbb{E}[X_t]), \quad \theta(T) = \chi,
\end{cases}
\]  

(3.11)

and \( p_t \) is then given by (3.7)-(3.8) with \( \hat{Y}_0 \) replaced by the ansatz \( \gamma(0)p_0 + \theta(0) \). Realizations of \( \hat{Y} \) are presented in Figure 3.1 for three parameter sets and mean process

\[ \mathbb{E}[X_t] := -1 + \sin(2\pi t). \]  

(3.12)

The trade-off between the three penalties can be noticed in the plots: the tagged pedestrian wants to keep distance/proximity to the mean process, while not moving too fast and not starting too far from \( y_0 = -2 \). Also, the difficulty to accurately predict the optimal path for small \( t \) is visible; as the filtration grows, the tagged has more information to base the choice of \( Z_t \) on.

A variation: desired velocity

Let the setup be as above, except the running cost which is changed to

\[ l := \frac{\lambda_{1,1}}{2}|v_t|^2 + \frac{\lambda_{1,2}}{2}|v_t - u(t, Y_t)|^2 + \frac{\lambda_2}{2}\varphi(Y_t). \]  

(3.13)
Fig 1. Top: Mean-averse behavior. Middle: Mean-seeking behavior. Bottom: Mean-averse behavior with high energy penalization. Realizations of the optimally controlled path in three cases, illustrating the trade off between initial position, walking speed and aversion of the dashed mean process. Common for the three cases are that at start, the path is rugged but the prediction of path to $\chi = -1$ improves as time increases.
where \( u \) is a desired velocity of the tagged. It may be used to model either internal preferences on walking speed, or the extra effort required to move in a different direction than a surrounding crowd. By Theorem 2.2, the optimal control is

\[
\dot{v}_t = \frac{p_t + \lambda_{1,2}u(t,\hat{Y}_t)}{\lambda_{1,1} + \lambda_{1,2}},
\]

and \( p \) solves the adjoint equation

\[
\begin{cases}
    dp_t = \frac{\lambda_{1,2}}{\lambda_{1,1} + \lambda_{1,2}} \partial_y u(t,\hat{Y}_t) \left( p_t - \lambda_{1,1}u(t,\hat{Y}_t) \right) + \lambda_2 \varphi'({\hat{Y}_t}) dt - \lambda_0 p_t dB_t, \\
p_0 = \partial_y h(\hat{Y}_0).
\end{cases}
\]

With an ansatz and a matching argument, the optimal path can be found in terms of a system of ODEs. In Figure 3.1, the simulation under two desired-velocity profiles is presented. In the upper plot, the tagged wants to first walk downwards, then after \( t = 0.5 \) upwards. In the second case, the tagged observes the velocity of the mean of the crowd, \( \frac{d}{dt} \mathbb{E}[X_t] = 2\pi \cos(2\pi t) \) and desires to follow it.

### 3.2. Congestion caused by a surrounding crowd

The key feature of congestion models is that the cost of moving depends on the density in an increasing manner. In this example, it will be more tiresome for the tagged pedestrian to walk in a crowded area than in free space. Let the setup be as in (3.2), except \( l \) which is changed to

\[
l := \frac{v_t^2}{2} \left( \lambda_{1,1} + \lambda_{1,2} \left( \int_{\mathbb{R}} \phi_r(Y_t - z) \mathbb{P}_X(dz) \right)^a \right) + \lambda_2 \int_{\mathbb{R}} \phi_r(Y_t - z) \mathbb{P}_X(dz),
\]

where \( a \geq 0 \) is a congestion parameter. By Theorem 2.2, the optimal control is given by

\[
\dot{v}_t := \frac{p_t}{\lambda_{1,1} + \lambda_{1,2} \varphi(\hat{Y}_t)^a},
\]

and \( p \) solves the adjoint equation

\[
\begin{cases}
    dp_t = \frac{v_t^2}{2} \left( \lambda_{1,2} a \varphi(\hat{Y}_t)^{a-1} \varphi'({\hat{Y}_t}) \right) + \lambda_2 \varphi'({\hat{Y}_t}) dt - \lambda_0 p_t dB_t, \\
p_0 = \partial_y h(\hat{Y}_0).
\end{cases}
\]

Equation (3.18) can be solved by separation of variables. Let \( p_t = \alpha_t \beta_t \), where \( \alpha_t \) is a stochastic integral and \( \beta_t \) a deterministic integral. Then

\[
\begin{cases}
    d\alpha_t = -\lambda_0 \alpha_t dB_t, \quad \alpha_0 = p_0, \\
    d\beta_t = A_t \alpha_t \beta_t^2 + \lambda_2 \varphi'({\hat{Y}_t}) \alpha_t^{-1} dt, \quad \beta_0 = 1,
\end{cases}
\]

where

\[
A_t := -\frac{\lambda_{1,2} a \varphi(\hat{Y}_t) \varphi(\hat{Y}_t)^{a-1}}{2(\lambda_{1,1} + \lambda_{1,2} \varphi(\hat{Y}_t)^a)^2}.
\]
Fig 2. Realizations of the optimally controlled path under two desired-velocity profiles. The velocity of the tagged plotted is estimated with a simple finite difference scheme applied to the averaged process $1/N \sum_{i=1}^{N} Y_{t}^{i}$. 

$(\lambda_{0}, \lambda_{11}, \lambda_{12}, \lambda_{3}, \chi, \gamma_{6}) = (0.2, 0.5, 0.5, -2.1, -1, 0)$

- Tagged path
- $E[X]$
Hence $\alpha_t$ is an exponential martingale while $\beta_t$ is the solution of a Riccati equation. The Riccati equation has a solution if and only if there is a solution $y_t = (y_{1t}, y_{2t})$ of the system

$$
\partial_t y_t + \begin{bmatrix}
0 & A_t \alpha_t \\
\lambda_2 \varphi'(\hat{Y}_t) \alpha_t^{-1} & 0 \\
\end{bmatrix} y_t = 0
$$

such that $y_{1t} \neq 0$. In that case, $\beta_t = y_{2t}/y_{1t}$. Again, the control is not Markovian. Also note that if $a \to 0$, the example on crowd aversion is retrieved with $\lambda_1 = \lambda_{1,1} + \lambda_{1,2}$. The same change of measure that was done in the example on crowd aversion yields the optimally controlled trajectory of the tagged as a conditional expectation,

$$
\hat{Y}_t = \tilde{\mathbb{E}} \left[ \chi - \int_t^T \frac{p_s}{\lambda_{1,1} + \lambda_{1,2} \varphi(\hat{Y}_s)^a} ds \mid \mathcal{F}_t \right].
$$

### 4. Concluding remarks and outlook

In this paper a mean-field model for the movement of pedestrians is presented which enables non-anonynmity and allows for non-local interactions. To the best of our knowledge, it is the first optimal control of mean-field BSDE pedestrian crowd model with a general law-dependence on its law.

The optimal control is characterized under quite restrictive conditions, however not too strict for the application in mind. Similar control problems, but with forward dynamics, have been studied under less restrictive conditions and since our derivation of a maximum principle follows a standard path, the conditions can certainly be relaxed.

A full game between a crowd and a tagged pedestrian is discussed, but not analyzed. This game would extend current mean-field major-minor models and we would like to work on this system of inequalities in the near future.

The solved examples show that closed-form solutions can be found in interesting situations, but the backward-forward nature of the system consisting of the dynamics and the adjoint equation results in non-Markovian optimal controls. Simulation of the optimally controlled path can therefore not be done by the standard backward time-stepping. Since in general the model yields solutions expressed as a sequence of conditional expectations, using sequential Monte Carlo methods to approximate this sequence could be fruitful. We would like to look into this and devise a fitting particle scheme in the near future.

### Appendix A: Mean-field BSDE

In the non-mean-field case, the following setup is standard. Let

$$
S^2_\mathbb{F}(0,T) = \left\{ Y \text{ F-prog. meas.} \mid (t, \omega) \mapsto Y_t(\omega) \text{ F-adapted, } \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^2 \right] < \infty \right\},
$$

$$
H^2_\mathbb{F}(0,T) = \left\{ Z \text{ F-prog. meas.} \mid \mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] < \infty \right\}.
$$

(A.1)
Definition A.1. A solution to the BSDE given by \((\chi, c)\),
\[
\begin{aligned}
- dY_t &= c(t,Y_t,Z_t)dt - Z_t dW_t, \quad 0 \leq t \leq T, \\
Y_T &= \chi,
\end{aligned}
\] (A.2)
is a pair of processes \((Y_t,Z_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T) \times \mathbb{H}^2(0,T)\).

Theorem A.1. Let \(\chi\) be \(\mathcal{F}_T\)-measurable and \(\mathbb{E} [\|\chi\|^2] < \infty\). Let \(c : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) satisfy
\begin{itemize}
  \item[a)] \(c(t,y,z) \in \mathbb{H}^2(0,T)\) for all \(y \in \mathbb{R}^d\), \(z \in \mathbb{R}\),
  \item[b)] \(|c(t,y,z)-c(t,y',z')| \leq L(|y-y'|+|z-z'|)\) for all \(y,y' \in \mathbb{R}^d\), \(z,z' \in \mathbb{R}\) and \(a.e. \,(t,\omega)\).
\end{itemize}
Then (A.2) has a unique solution.

Proof. For a proof, see [28]. \(\square\)

The conditions of Theorem A.1 can be relaxed, see the references in [8]. What now follows is the definition, the existence and the uniqueness of a solution of the mean-field BSDE (2.7). For any \(\beta > 0\), let \(\mathcal{M}_\beta(0,T)\) be the Banach space
\[
\mathcal{M}_\beta(0,T) := \mathcal{S}^2(0,T) \times \mathbb{H}^2(0,T)
\] (A.3)
equipped with the norm
\[
\|(Y,Z)\|_\beta := \mathbb{E} \left[ \sup_{t \in [0,T]} e^{2\beta t} |Y_t|^2 + \int_0^T e^{2\beta t} |Z_t|^2 dt \right]^{1/2}. \tag{A.4}
\]
Note that if \(Y \in \mathcal{S}^2(0,T)\), then \(\mathbb{P}_{Y_t} \in \mathcal{P}_2(\mathbb{R}^d)\) for all \(t \in [0,T]\) since
\[
\int_{\mathbb{R}^d} |y|^2 \mathbb{P}_{Y_t}(dy) = \mathbb{E}[|Y_t|^2] \leq \mathbb{E}\left[ \sup_{t \in [0,T]} |Y_t|^2 \right] < \infty. \tag{A.5}
\]

Let \(U\) be a compact subset of \(\mathbb{R}^d\). Given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), let \(\mathcal{U}[0,T]\) be the set of \(U\)-valued progressively measurable processes \((a_t)_{t \in [0,T]}\) such that \(\mathbb{E}\left[ \int_0^T |a_s|^2 ds \right] < \infty\). An element of \(\mathcal{U}[0,T]\) will be called an admissible control.

Definition A.2. Given an admissible control \(v(\cdot) \in \mathcal{U}[0,T]\), a pair of processes \((Y,Z) \in \mathcal{M}_\beta(0,T)\) is called an adapted solution of the mean-field BSDE
\[
\begin{aligned}
dY_t &= c(t,Y_t,\mathbb{P}_{Y_t},Z_t,v_t)dt + Z_t dB_t, \quad t \in [0,T], \\
Y_T &= \chi,
\end{aligned}
\] (A.6)
if
\[
Y_t = \chi - \int_t^T c(s,Y_s,\mathbb{P}_{Y_s},Z_s,v_s)ds - \int_t^T Z_s dB_s, \quad \forall t \in [0,T], \mathbb{P} - a.s. \tag{A.7}
\]
Moreover, (A.6) is said to have a unique adapted solution if for any two adapted solutions \((Y,Z)\) and \((Y',Z')\), it must hold that
\[
\mathbb{P}\{Y_t = Y'_t, \forall t \in [0,T] \text{ and } Z_t = Z'_t, \mathbb{P} - a.e. \, t \in [0,T]\} = 1. \tag{A.8}
\]
To prove existence and uniqueness of an adapted solution to (A.6), the following assumption on $c$ is needed.

A.1 For any $(y, \mu, z, a)$, $c(t, y, \mu, z, a)$ is $(\mathcal{F}_t)$-adapted and $c(\cdot, 0, \delta_0, 0, a)$ is square integrable for any $a \in U$. Moreover, there exists an $L > 0$ such that for all $t \in [0, T]$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d \times d}$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,
\begin{equation}
|c(t, y, \mu, z, a) - c(t, y', \mu', z', a)| \leq L \left( |y - y'| + |z - z'| + d_2(\mu, \mu') \right),
\end{equation}
where $d_2$ is the square-Wasserstein metric.

**Theorem A.2.** Under A.1 the mean-field BSDE (A.6) admits a unique adapted solution $(Y, Z) \in \mathcal{M}_\beta(0, T)$ for any square-integrable, $\mathcal{F}_T$-adapted $\chi$ and any $v \in \mathcal{U}[0, T]$.

**Proof.** A proof can be found in [8]. \hfill $\square$

**Appendix B: Optimal control of the tagged pedestrian**

This appendix contains the proofs of Theorem 2.1 and Theorem 2.2. The Pontryagin maximum principle approach to mean-field type control problems was introduced by [1], who discovered the duality relationship between the adjoint process and the first variation process (cf. Lemma B.1), and extended in [6, 7].

To do the proofs, we need to consider derivatives of measure-valued functions and we adopt the approach presented in [12] to deal with them. We assume that every probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ we use is rich enough so that for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there is an $\mathcal{F}$-measurable random variable $\vartheta \in L^2(\mathbb{R}^d)$ such that $\mathbb{P} \circ \vartheta^{-1} = \mu$. The main idea is as follows, for any function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ a function $f^\vartheta : L^2(\mathbb{R}^d) \to \mathbb{R}$ is induced so that $f^\vartheta(\vartheta) := f(\mathbb{P} \circ \vartheta^{-1})$. One can then compute $Df^\vartheta$, the Fréchet derivative of $f^\vartheta$, and define the derivative $f$ at $\mathbb{P} \circ \vartheta^{-1}$ as
\begin{equation}
\partial_\mu f(\mathbb{P} \circ \vartheta^{-1}; \vartheta) := Df^\vartheta(\vartheta).
\end{equation}
For a rigorous definition of the derivative of a $\mathcal{P}_2(\mathbb{R}^d)$-valued function $f$, see [7, Definition 2.1].

To ease notation, let $\theta_t := (t, Y_t, \mathbb{P}_{Y_t}, Z_t, v_t)$ and $\theta^\varphi_t := (t, Y^\varphi_t, \mathbb{P}_{Y^\varphi_t}, Z^\varphi_t, v^\varphi_t)$ denote arguments for $c$ and $l$, even though $l$ does not have a $Z_t$-dependence. The symbol $\partial_x$ will denote derivatives with respect to $x$ and $\delta$ denotes variation in the control variable, i.e. $\delta c(t) := c(t, Y_t, \mathbb{P}_{Y_t}, Z_t, v_t) - c(t, Y_t, \mathbb{P}_{Y_t}, Z_t, v_t)$. Let $(Y^1, Z^1)$ solve the first order variation equation
\begin{equation}
\begin{cases}
 dY^1_t = \left\{ \partial_y c(\theta_t)Y^1_t + \partial_z c(\theta_t)Z^1_t \right\} + \mathbb{E}\left[ \partial_\mu c(\tilde{\theta}_t \varphi)\hat{Y}^1_t \right] + \delta c(t) 1_{E_\varphi}(t) \right\} dt + Z^1_t dB_t, \\
 Y^1_0 = 0,
\end{cases}
\end{equation}
and let $p$ solve the adjoint equation,
\begin{equation}
\begin{cases}
 dp_t = - \left\{ \partial_y c(\theta_t)p_t + \mathbb{E}\left[ \partial_\mu c(\tilde{\theta}_t \varphi)\tilde{p}_t \right] \right\} - \partial_y l(\theta_t) - \mathbb{E}\left[ \partial_\mu l(\tilde{\theta}_t) \right] dt - \partial_z c(\theta_t)p_t dB_t, \\
p_0 = \partial_y h(\theta_0) + \mathbb{E}\left[ \partial_\mu (\tilde{\theta}_0) \right].
\end{cases}
\end{equation}
The notation used in connection with the measure derivatives distinguishes whether it is the original or the induced random variable (cf. \( \vartheta \) in (B.1)) that is being integrated,

\[
\partial_\mu c(\bar{\theta}_t) := \partial_\mu c(t, Y_t, \bar{P}_{Y_t}, Z_t, v_t; \bar{Y}_t), \\
\partial_\mu c(\bar{\theta}_t^+) := \partial_\mu c(t, \bar{Y}_t, \bar{P}_{Y_t}, \bar{Z}_t, \bar{v}_t; Y_t).
\]

(B.4)

Note that since the lifted random variable is an independent copy of the original, \( \bar{P}_{Y_t} = \bar{P}_{\bar{Y}_t} \).

**Lemma B.1.** The following duality relation holds,

\[
\mathbb{E} [p_0 Y_0^1] = -\mathbb{E} \left[ \int_0^T p_t \partial c(t) \mathbb{I}_{E_\epsilon}(t) + Y_t^1 \left( \partial_y l(\theta_t) + \mathbb{E} \left[ \partial_\mu (\bar{\theta}_t^+) \right] \right) \right] dt.
\]

(B.5)

**Proof.** The relation (B.5) follows by Itô’s formula, (B.2) and (B.3).

Let the Hamiltonian of the control problem be defined as

\[
H(t, y, \mu, z, v, p) := c(t, y, \mu, z, v)p - l(t, y, \mu, v)
\]

and let \( v^\epsilon \) be a spike variation of \( v \),

\[
v_t^\epsilon := \begin{cases} v_t, & t \in [0, T] \setminus E_\epsilon, \\ a, & t \in E_\epsilon, \end{cases}
\]

(B.7)

where \( a \) is an arbitrary element of \( U \) and \( E_\epsilon \) is a measurable subset of \( [0, T] \) of size \( \epsilon \). Towards a maximum principle, we study the difference in cost,

\[
I(v^\epsilon) - I(v) = \mathbb{E} \left[ \int_0^T l(\theta_t^\epsilon) - l(\theta_t) dt \right] + \mathbb{E} \left[ h(\theta_0^\epsilon) - h(\theta_0) \right].
\]

(B.8)

A Taylor expansion of the second term yields

\[
h(\theta_0^\epsilon) - h(\theta_0) = \partial_y h(\theta_0)(Y_0^\epsilon - Y_0) + \mathbb{E} \left[ \partial_\mu h(\bar{\theta}_0)(\bar{Y}_0^\epsilon - \bar{Y}_0) \right] \\
+ o(Y_0^\epsilon - Y_0)
\]

(B.9)

\[
= \partial_y h(\theta_0)\bar{Y}_0 + \mathbb{E} \left[ \partial_\mu h(\bar{\theta}_0)\bar{Y}_0 \right] + o(|\bar{Y}|),
\]

where \( \bar{Y} := Y^\epsilon - Y \). Changing the order of integration, we get

\[
\mathbb{E} \left[ h(\theta_0^\epsilon) - h(\theta_0) \right] = \mathbb{E} \left[ p_0 \bar{Y}_0 + o(|\bar{Y}_0|) \right].
\]

(B.10)

The following lemma permits an exchange of \( \bar{Y}_0 \) for \( Y_0^1 \) in (B.10) and bounds the remainder.

**Lemma B.2.** Under A.2, there is a constant \( C > 0 \) such that

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ (Y_t^1)^2 \right] \leq C \epsilon^2,
\]

(B.11)

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ |Y_t^\epsilon - Y_t - Y_t^1|^2 \right] \leq C \epsilon^2.
\]

(B.12)
Proof. The proof is found in Appendix C.

By Lemma B.2,
\[ E[h(\theta_0^s) - h(\theta_0)] = E[p_0 Y_0^1] + o(\epsilon). \] (B.13)

Furthermore, a Taylor expansion of the running cost yields
\[ l(\theta_t^s) - l(\theta_t) = \partial_y l(\bar{\theta}_t) \bar{Y}_t + \tilde{E} \left[ \partial_{\mu} l(\bar{\theta}_t) \bar{Y}_t \right] + \partial_{\nu} l(\bar{\theta}_t)(v^s_t - v_t) + o(|\bar{Y}_t|) + o(|v^s_t - v_t|). \] (B.14)

Note that \( v^s_t - v_t = a I_{E_\epsilon}(t) \) and therefore \( \int_0^T o(|v^s_t - v_t|) = o(\epsilon) \) by compactness of \( U \). Integrating (B.14) yields
\[
E \left[ \int_0^T l(\theta_s^s) - l(\theta_s) \, ds \right] + E[p_0 Y_0^1] = E \left[ \int_0^T \partial_y l(\bar{\theta}_s) Y_s^1 + \tilde{E} \left[ \partial_{\mu} l(\bar{\theta}_s) Y_s^1 \right] + \partial_{\nu} l(\theta_s) a I_{E_\epsilon}(s) \, ds \right] - E \left[ \int_0^T p_1 \delta c(s) I_{E_\epsilon}(s) + Y_s^1 \left( \partial_y l(\theta_s) + \tilde{E} \left[ \partial_{\mu} l(\bar{\theta}_s) \right] \right) \, ds \right] + o(\epsilon)
\] (B.15)
\[
= E \left[ \int_0^T ( - p_1 \delta c(s) + \partial_{\nu} l(\theta_s) a) I_{E_\epsilon}(s) \, ds \right] + o(\epsilon).
\]

Note that \( \partial_{\nu} l(\theta_s) a I_{E_\epsilon}(s) = \delta l(s) I_{E_\epsilon}(s) + o(\epsilon) \), hence
\[
I(v^s) - I(v) = -E \left[ \int_0^T \delta H(s) I_{E_\epsilon}(s) \, ds \right] + o(\epsilon).
\] (B.16)

Proof of Theorem 2.1. Choose \( E_\epsilon := [s, s + \epsilon] \), for an arbitrary \( s \in [0, T - \epsilon] \), and let \( A \) be an arbitrary set of \( \mathcal{F}_s \). Consider the control
\[
v_t^s := \begin{cases} 
 a I_A + \hat{v}_t I_{AC}, & t \in E_\epsilon, \\
 \hat{v}_t, & t \in [0, T] \setminus E_\epsilon,
\end{cases}
\]
for an arbitrary \( a \in U \). Denote by \( \hat{\theta}_t := (t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{v}_t) \) and \( \hat{p}_t := p_t |_{\theta = \hat{\theta}} \). Then
\[
H(\hat{\theta}_t, \hat{p}_t) - H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, v^s_t, \hat{p}_t) = \left( H(\hat{\theta}_t, \hat{p}_t) - H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, a, \hat{p}_t) \right) I_A I_{E_\epsilon}(t)
\] (B.18)

By applying (B.16), one obtains
\[
\frac{1}{\epsilon} E \left[ \int_s^{s+\epsilon} \left( H(\hat{\theta}_t, \hat{p}_t) - H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, a, \hat{p}_t) \right) I_A \, dt \right] \geq \frac{1}{\epsilon} o(\epsilon).
\] (B.19)

Sending \( \epsilon \) to zero,
\[
E \left[ \left( H(\hat{\theta}_s, \hat{p}_s) - H(s, \hat{Y}_s, \mathbb{P}_{\hat{Y}_s}, \hat{Z}_s, a, \hat{p}_s) \right) I_A \right] \geq 0, \quad \text{a.e. } s \in [0, T].
\] (B.20)
Since the last inequality holds for all $A \in \mathcal{F}_s$, it follows that
\[
\mathbb{E} \left[ (H(\hat{\theta}_s, \hat{\rho}_s) - H(\hat{\theta}_s, \hat{\rho}_s, \hat{\xi}_s, \hat{\xi}_s, a, \hat{\rho}_s) \left| \mathcal{F}_s \right. \right] \geq 0, \quad \text{a.e. } s \in [0, T], \mathbb{P}\text{-a.s.} \tag{B.21}
\]
Hence, by measurability of what is inside the conditional expectation,
\[
\hat{v}_t = \arg\max_{a \in U} H(t, \hat{Y}_t, \hat{\xi}_t, \hat{\xi}_t, a, \hat{\rho}_t), \quad \text{a.e. } t \in [0, T], \mathbb{P}\text{-a.s.} \tag{B.22}
\]

**Proof of Theorem 2.2.** By assumption, $\delta H(t) \leq 0$ for all $t \in [0, T]$ and for all $v \in U$. Applying the convexity and concavity assumptions to the first equality of (B.15) gives
\[
0 \leq -\mathbb{E} \left[ \int_0^T \delta H(s) \mathbb{1}_{E_v}(s) ds \right] \leq I(v) - I(\hat{v}), \tag{B.23}
\]
and the result follows. \qed

**Appendix C: Proof of Lemma B.2**

**Proof.** Let $\beta \in (0, 1)$. Hölder’s inequality together with a change of order of integration yields
\[
\mathbb{E} \left[ (Y^1_t)^2 + \beta \int_t^T (Z^1_s)^2 ds \right]
\leq \mathbb{E} \left[ 2 \left( \int_t^T \partial_y c(\theta_s) Y^1_s + \partial_z c(\theta_s) Z^1_s + \mathbb{E} \left[ \partial_y c(\theta_s) Y^1_s \right] ds \right)^2 \right]
+ \mathbb{E} \left[ 2 \left( \int_t^T \delta c(s) \mathbb{1}_{E_v}(s) ds \right)^2 \right] + \mathbb{E} \left[ \int_t^T (Z^1_s)^2 ds \right]
\leq 6C^2(T - t) \int_t^T (Y^1_s)^2 ds + (6C^2 + 1)(T - t) \int_t^T (Z^1_s)^2 ds
+ 2\mathbb{E} \left[ \left( \int_{[t,T] \cap \mathcal{F}_s} \mathbb{1}_{L_{ds}} \right)^2 \right]
\leq 6C^2 T \int_t^T \mathbb{E} \left[ (Y^1_s)^2 \right] ds + (6C^2 + 1)(T - t) \mathbb{E} \left[ \int_t^T (Z^1_s)^2 ds \right] + C^2 \epsilon^2.
\tag{C.1}
\]
For all $t \in [T - \tau, T]$, where $\tau := \beta/(12C^2 + 2),$
\[
\mathbb{E} \left[ (Y^1_t)^2 \right] + \frac{\beta}{2} \mathbb{E} \left[ \int_t^T (Z^1_s)^2 ds \right] \leq 6C^2 T \int_t^T \mathbb{E} \left[ (Y^1_s)^2 \right] ds + C^2 \epsilon^2 \tag{C.2}
\]
and by Grönwall’s inequality, $\mathbb{E} \left[ (Y^1_t)^2 \right]$ and $\mathbb{E} \left[ \int_t^T (Z^1_s)^2 ds \right]$ are bounded by $C^2 \epsilon^2$ for all $t \in [T - \tau, T]$. Repeating the calculations, the same result can be proven for all $t \in [T - 2\tau, T - \tau]$ and after a finite number of iterations,
\[
\mathbb{E} \left[ (Y^1_t)^2 \right] \leq C^2 \epsilon^2, \quad \mathbb{E} \left[ \int_t^T (Z^1_s)^2 ds \right] \leq C^2 \epsilon^2, \quad \forall t \in [0, T]. \tag{C.3}
\]
On the other hand, by assumption A.2 all derivatives are uniformly bounded in \( t \) and hence
\[
\sup_{t \in [0,T]} (Y_1^1)^2 \leq 6(C_1^2 + 1) \int_0^T (Y_1^1)^2 + (Z_1^1)^2 ds + \sup_{t \in [0,T]} \left( \int_t^T \delta c(s) I_{E_1}(s) ds \right)^2 \tag{C.4}
\]
which, using the calculations above, yields
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} (Y_1^1)^2 \right] \leq C_4 \epsilon^2. \tag{C.5}
\]
This proves the estimate (B.11). The constant \( C_4 \) depends on \( T, L \) the uniform bound for the derivatives and the size of the control space.

Moving on to the next estimate, a Taylor expansion of the dynamics gives
\[
Y_t^\epsilon - Y_t - Y_1^1 = \int_t^T c(\theta_s^\epsilon) - c(\theta_s) - \partial_y c(\theta_s) Y_s^1 - \partial_z c(\theta_s) Z_s^1
\]
\[
= \int_t^T \partial_y c(\theta_s^\epsilon) (\tilde{Y}_s - Y_s^1) + \partial_z c(\theta_s^\epsilon) (\tilde{Z}_s - Z_s^1) + \mathbb{E} \left[ \partial_y c(\theta_s^\epsilon) (\tilde{Y}_s - Y_s^1) \right]
\]
\[
+ \partial_z c(\theta_s^\epsilon) I_{E_1}(s) a - \delta c(s) I_{E_1}(s) ds + \int_t^T \tilde{Z}_s - Z_s^1 dB_s
\]
\[
+ \int_t^T o(|\tilde{Y}_s|) + o(|\tilde{Z}_s|) + o(|a I_{E_1}(s)|) ds. \tag{C.6}
\]
The inequality is due to the concavity of \( c \). Taking expectations, we use Hölder’s and Young’s inequalities, Itô’s isometry and (C.3) to get
\[
\mathbb{E} \left[ \left( \int_t^T o(|\tilde{Y}_s|) + o(|\tilde{Z}_s|) + o(|a I_{E_1}(s)|) ds \right)^2 \right] \leq \mathbb{E} \left[ C_0 \int_t^T (\tilde{Y}_s - Y_s^1)^2 + (\tilde{Z} - Z_s^1)^2 ds \right] + C_0 \epsilon^2 \tag{C.7}
\]
and
\[ E \left( |\bar{Y}_t - Y^1_t|^2 \right) \leq E \left[ 6 \left( \int_t^T \partial_{\theta}(\theta_s)(\bar{Y}_s - Y^1_s) + \partial_{\zeta}(\theta_s)(\bar{Z}_s - Z^1_s) + \bar{E} \left[ \partial_{\mu}(\tilde{\theta}_s) \right] (\bar{Y}_s - Y^1_s) ds \right)^2 \right] \]
\[ + E \left[ 6 \left( \int_t^T (\partial_{\nu}(\theta_s) - \delta_{\nu}(s)) I_{E_\nu}(s) ds \right)^2 + 6 \int_t^T (\bar{Z}_s - Z^1_s)^2 ds \right] \]
\[ + E \left[ C_0 \int_t^T (\bar{Y}_s - Y^1_s)^2 + (\bar{Z}_s - Z^1_s)^2 ds \right] + C_0 \epsilon^2 \]
\[ = E \left[ 36(C_1^2 + C_2 + 1) \int_t^T (\bar{Y}_s - Y^1_s)^2 + (\bar{Z}_s - Z^1_s)^2 ds \right] \]
\[ + E \left[ 6 \left( \int_t^T (C_1 - La) I_{E_\nu}(s) ds \right)^2 \right] + C_0 \epsilon^2 \]
\[ \leq E \left[ C_5 \int_t^T (\bar{Y}_s - Y^1_s)^2 + (\bar{Z}_s - Z^1_s)^2 ds \right] + C_6 \epsilon^2. \]
\[(C.8)\]

The same arguments used to prove (B.11) yields
\[ \sup_{t \in [0,T]} E \left( |\bar{Y}_s - Y^1_s|^2 \right) \leq C_7 \epsilon^2, \]
\[(C.9)\]
where \( C_7 \) is a constant that depends on \( T, L \), the bound on the derivatives of \( c \) and the size of the control space. \( \square \)

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