CONSTRUCTION OF NON-TRIVIAL RELATIVISTIC QUANTUM FIELDS IN ARBITRARY SPACE-TIME DIMENSION VIA SUPERPOSITION OF FREE FIELDS

MARTIN GROTHAUS AND ANDREAS NONNENMACHER

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Abstract. We construct Schwinger functions as the superposition of Schwinger functions which correspond to those of free fields with sharp masses $m$. We prove that all axioms of Osterwalder and Schrader are satisfied. This construction works independently of the space-time dimension $d$. The Schwinger functions under consideration are the moments of a non-Gaussian measure on the space of tempered distributions $S'({\mathbb{R}}^d)$.

Key words: Axiomatic quantum field theory, Schwinger functions, non-Gaussian measure, Källen-Lehmann representation.

1 Introduction

The formulation of relativistic quantum field theory in terms of axioms by Gårding and Wightman in the sixties was a major achievement on the mathematical rigorous treatment of quantum phenomena. See e.g. [19] or [15][Section IX.7] for a complete list of these axioms. Since then different sets of (partially) equivalent axioms were formulated in terms of various different functional analytic, algebraic and probabilistic objects. Here we want to mention in particular the axioms of Osterwalder and Schrader (E0)-(E4) listed below which we work with in the course of this paper, see also [13, 14] for the original formulations. For an overview and the equivalence of these axioms we refer to [8] and [18] and the references therein. Since the mathematical objects of each of these axioms are so involved it is a challenging task to write down examples of such objects fulfilling these axioms. Fortunately, in any space-time dimension the so-called generalized free field models exists. This proves that the axioms are consistent. Unfortunately, these models don’t incorporate any interesting physics since they only describe non-interacting particles. A huge workload in mathematical physics was done to construct models which include an interaction of particles. Different strategies such as the so called Hamiltonian and Euclidean strategies were followed to obtain such models and providing proof of the fulfillment of all axioms under consideration. For example the models with polynomial self-interaction as the $P(\Phi)_2$ and the $(\Phi)^3_3$ model in space-time dimension $d = 2, 3$, respectively, were milestones in axiomatic quantum field theory. Interacting models in arbitrary space time dimension $d$ are for example given by the Albeverio-Høegh-Krohn model. All these models are given by their Schwinger functions which are moments of a corresponding probability measure $\mu$ on $(S'(\mathbb{R}^d), B)$, here $B$ denotes the Borel $\sigma$-field of the weak topology of $S'(\mathbb{R}^d)$, see e.g. [6] [7]. All these measures arise through a highly non-trivial renormalization procedure to remove so called cutoffs which make the objects under consideration well-defined, see e.g. [5] [12] [18] [2]. At this point we want to mention a further approach by the authors in [3, 11] in the context of Levy measures and white noise analysis, respectively, which works without the machinery of renormalization. It is closely related to our approach and might help to identify its underlying interaction. The Schwinger functions which we construct below are still given as moments
of a probability measure $\mu_\rho$, but not of (generalized) Levy measures as in $[11,3]$. The measure $\mu_\rho$ is simply given as the superposition of Euclidean free field measures of different mass $m$. The symbol $\rho$ denotes a measure on the positive real line describing which masses $m$ contribute to the superposition. In particular, the approach chosen here works for arbitrary space-time dimension $d \in \mathbb{N}$. The construction of these Schwinger functions is heavily inspired by the Källen-Lehmann representation of the two point function of a relativistic quantum field, see e.g. $[15]$[Theorem IX.34]. In particular, the measure $\rho$ is exactly the spectral measure from the Källen-Lehmann representation. To point out the difference of our approach to generalized free fields for the case that $\rho$ is a probability measure let us make the following observations. The superposition $\mu_\rho$ of Gaussian measures is in general non-Gaussian, see Example 2.11 below. On the other hand, Schwinger functions of generalized free fields are moments of a Gaussian measure. The corresponding covariance operator is the superposition of the covariance operators of Euclidean free field measures with fixed masses $m$ w.r.t. $\rho(dm)$.

To explain the idea for the construction below let us make the following observation. All axioms of Osterwalder and Schrader, except the cluster property are constraints which are linear in a family of distributions $(S_n)_{n \in \mathbb{N}_0}$, see (E0)-(E4) below. Indeed, if $(S^1_n)_{n \in \mathbb{N}_0}$ and $(S^2_n)_{n \in \mathbb{N}_0}$ are two families of distributions satisfying the conditions (E0)-(E3) given below, then we obtain immediately that their sum $(S_n)_{n \in \mathbb{N}_0} := (S^1_n + S^2_n)_{n \in \mathbb{N}_0}$ satisfies (E0)-(E3), too. Only the cluster property is a non-linear condition and therefore it is not directly clear whether $(S_n)_{n \in \mathbb{N}_0}$ satisfies (E4) if $(S^1_n)_{n \in \mathbb{N}_0}$ and $(S^2_n)_{n \in \mathbb{N}_0}$ do. Further, from the construction of generalized free fields, see e.g. $[16]$ and $[8]$, on sees that the truncated vacuum expectation values, see e.g. $[16]$[Section XI.16], are equal to zero. Equivalently, the truncated Schwinger functions of generalized free fields are equal to zero. Observe that the truncated Schwinger functions are given as the image of the Schwinger functions under a non-linear map. In particular, the truncated Schwinger functions of a superposition of Schwinger functions are in general not zero, even if this holds for the single summands. Hence, if one can show that the cluster property holds for a superposition of Schwinger functions one automatically obtains a non-generalized free field.

We start below by introducing the complete list of Osterwalder-Schrader axioms and some related notations. Afterwards we show for a certain class of probability measures $\rho(dm)$ that the superposition of Schwinger corresponding to the Euclidean free field of mass $m$ satisfies (E0)-(E4). The major part consist of course of proving the cluster property as explained above. We conclude this chapter by showing that the corresponding truncated Schwinger functions don’t vanish identically.

## 2 Construction of Schwinger functions as the moments of a superposition of Gaussian measures

In the following we fix an arbitrary space-time dimension $d \in \mathbb{N}$. Further, all function spaces in this chapter consist of complex-valued functions. For sake of simplicity we don’t distinguish between a continuous $n$-linear mapping on $S(\mathbb{R}^d)^n$ and a tempered distribution in $S'(\mathbb{R}^{dn})$, $n \in \mathbb{N}$, which is legitimated by the kernel theorem, see $[11]$. We denote by $\{\|\cdot\|_p\}_{p \in \mathbb{N}}$ a family of semi-norms on $S(\mathbb{R}^d)$ which induces the topology of $S(\mathbb{R}^d)$. In the following let $n \in \mathbb{N}$. A function $f \in S(\mathbb{R}^{dn})$ we always consider as a function in $n$ variables $x_1, ..., x_n \in \mathbb{R}^d$. The first component of a vector $x \in \mathbb{R}^d$ is called the time component of $x$ and we usually write $x = (x^0, \bar{x})$ with $x^0 \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^{d-1}$. By $S_+(\mathbb{R}^{dn})$ we denote the subspace of $S(\mathbb{R}^{dn})$ consisting of the functions which vanish together with their partial derivatives of any order at $(x_1, ..., x_n) \in \mathbb{R}^{dn}$ unless $0 < x^0_1 < x^0_2 < ... < x^0_n$. In the following let $a \in \mathbb{R}^d$, $\Lambda \in \text{SO}(d)$ and $\pi \in \Sigma_n$ a permutation of $n$ elements. On $S(\mathbb{R}^{dn})$ we introduce the following linear
operators \( \cdot^*, \Theta, \cdot_{(a,\Lambda)} \) and \( \cdot^n \). To this end let \( f \in S(\mathbb{R}^{d_n}) \) and \( (x_1, ..., x_n) \in \mathbb{R}^d \) be arbitrary and define
\[
f^\ast(x_1, ..., x_n) := \tilde{f}(x_n, ..., x_1),
\]
\[
\Theta f(x_1, ..., x_n) := f((-x_0^0, \tilde{x}_1), (-x_0^0, \tilde{x}_2), ..., (-x_0^0, \tilde{x}_n))
\]
\[
f_{(a,\Lambda)}(x_1, ..., x_n) := f(\Lambda x_1 + a, \Lambda x_2 + a, ..., \Lambda x_n + a),
\]
\[
f^n(x_1, ..., x_n) := f(x_{n(1)}, ..., x_{n(n)}).
\]

In particular, for \( t \geq 0 \) we denote by \( T_t \) the operator \( \cdot_{(t,1)} \), where \( \tilde{f} \in \mathbb{R}^d \) is given by \( \tilde{t}_i = t \delta_{1,i} \), \( i = 1, ..., d \). \( (T_t)_{t \geq 0} \) is called the time translation semigroup. The space of finite sequences \( \tilde{f} = (f_0, f_1, f_2, ...) \) with \( f_i \in S(\mathbb{R}^{d_i}) \) \( (S_+(\mathbb{R}^{d_i})) \), \( i = 0, 1, 2, ... \), we denote by \( S_+ \) \( (S_+) \). The operators \( \cdot^*, \Theta \) and \( \cdot_{(a,\Lambda)} \) lift to operators on \( S_+ \) via componentwise application and their extensions are denoted by the same symbols. We say that a sequence of real numbers \( (\sigma_k)_{k \in \mathbb{N}} \) is of factorial growth if there are positive constants \( \alpha \) and \( \beta \) s.t. \( \sigma_k \leq \alpha(k!)^\beta \) for all \( k \in \mathbb{N} \). Now we are ready to define the Osterwalder-Schrader axioms.

**Definition 2.1.** Let \( S_0 = 1 \) and \( (S_n)_{n \in \mathbb{N}} \) be a sequence of distributions s.t. \( S_n \in S'(\mathbb{R}^{dn}) \).

(E0) (Distribution property) There exists a number \( p \in \mathbb{N} \) and a sequence of real numbers \( (\sigma_k)_{k \in \mathbb{N}} \) of factorial growth s.t. for every \( n \in \mathbb{N} \) and \( f_1, ..., f_n \in S(\mathbb{R}^d) \) it holds
\[
|S_n(f_1 \otimes f_2 \otimes ... \otimes f_n)| \leq \sigma_n n \prod_{i=1}^{n} \|f_i\|_p.
\]

(E1) (Euclidean invariance) For every \( a \in \mathbb{R}^d, \Lambda \in SO(d) \) and \( f \in S(\mathbb{R}^{dn}) \) it holds
\[
S_n(f) = S_n(f_{(a,\Lambda)}).
\]

(E2) (Reflection positivity) For every \( \tilde{f} \in S_+ \) it holds
\[
\sum_{k=0}^{\infty} S_{n+k}(\Theta f^\ast_n \otimes f_k) \geq 0.
\]

(E3) (Symmetry) For \( n \in \mathbb{N}, f \in S(\mathbb{R}^{dn}) \) and every permutation \( \pi \in \Sigma_n \) it holds
\[
S_n(f) = S_n(f_{\pi}).
\]

(E4) (Cluster property) For every \( \tilde{f}, \tilde{g} \in S_+ \) it holds
\[
\lim_{t \to \infty} \sum_{n,k=0}^{\infty} S_{n+k}(\Theta f^\ast_n \otimes T_t g_m) = \sum_{n=0}^{\infty} S_n(\Theta f^\ast_n) \sum_{k=0}^{\infty} S_k(g_k).
\]

**Remark 2.2.**
(i) Observe that the sums in (E2) and (E4) extend only over finitely many indices.
(ii) The formulation of the axioms (E0) – (E5) is not the most general one. In particular, the distribution property can be weakened. For more details we refer to the original papers [13, 14], see also [18].
(iii) The cluster property (E4) is not the original condition from [13]. Under the assumption of (E1) the formulation used here and the one in [13] are obviously equivalent.

**Assumption 2.3.** Let \( \rho \) be a probability measure on \( ((0, \infty), \mathcal{B}(0, \infty)) \) s.t. for some \( m_0 \in (0, \infty) \) it holds \( \text{supp}(\rho) \subseteq [m_0, \infty) \).

**Remark 2.4.** In the following we tacitly work with the completion \( \mathcal{B}^0((0, \infty)) \) of \( \mathcal{B}((0, \infty)) \) w.r.t. \( \rho \) and denote its extension to \( \mathcal{B}^0([0, \infty)) \) by \( \rho \), too.
Let $m \in (0, \infty)$ and denote by $S_{n,m}$, $n \in \mathbb{N}_0$, the $n$–th Schwinger function of the free field of mass $m$. I.e., the second Schwinger function $S_{2,m}$ is given by

$$S_{2,m}(f_1, f_2) = \int_{\mathbb{R}^d} \frac{1}{(|p|^2 + m^2)^2} \mathcal{F}f_1(p) \mathcal{F}f_2(p) \, dp, \quad f_1, f_2 \in S(\mathbb{R}^d).$$

(2.1)

And further for $n \in \mathbb{N}_0$ and $f_1, ..., f_{2n}, f_{2n+1} \in S(\mathbb{R}^d)$ it holds

$$S_{2n,m}(f_1, ..., f_{2n}) = \sum_{\text{pairings}} S_{2,m}(f_{i_1}, f_{j_1})...S_{2,m}(f_{i_n}, f_{j_n}),$$

(2.2)

$$S_{2n+1,m}(f_1, ..., f_{2n+1}) = 0,$$

(2.3)

where the sum $\sum_{\text{pairings}}$ in (2.2) extends over all $(2n - 1)!! = \frac{(2n)!}{2^n n!}$ ways of writing $1, ..., 2n$ as $n$ distinct (unordered) pairs $(i_1, j_1), ..., (i_n, j_n)$, see also [18][Proposition I.2]. In particular, the Schwinger functions $(S_{n,m})_{n \in \mathbb{N}_0}$ are the moments of the Gaussian measure $\mu_m$ which is given by the Bochner-Minlos theorem via its characteristic function

$$\hat{\mu}_m(\varphi) = \exp \left( -\frac{1}{2} [C_m \varphi, \varphi]_{L^2(\mathbb{R}^d)} \right), \quad \varphi \in S(\mathbb{R}^d),$$

(2.4)

where the linear operator $C_m = (-\Delta + m^2)^{-1}$ on $L^2(\mathbb{R}^d)$ is defined through the Fourier transform

$$\mathcal{F}(C_m \varphi)(p) = \frac{1}{|p|^2 + m^2} \mathcal{F}\varphi(p), \quad p \in \mathbb{R}^d, \varphi \in S(\mathbb{R}^d).$$

**Proposition 2.5.** For every $m > 0$ the Schwinger functions $(S_{n,m})_{n \in \mathbb{N}_0}$ satisfy (E0) – (E4).

**Proof:** See e.g. [18].

Observe that for $f_1, f_2 \in S(\mathbb{R}^d)$ the map

$$R_{f_1, f_2} : (0, \infty) \to C, \quad m \mapsto S_{2,m}(f_1, f_2)$$

(2.5)

is analytic and satisfies the estimate $|R_{f_1, f_2}(m)| \leq \frac{1}{m^2} \|f_1\|_{L^2(\mathbb{R}^d)} \|f_2\|_{L^2(\mathbb{R}^d)}$. In particular, it holds for $f_1, ..., f_{2n} \in S(\mathbb{R}^d)$

$$|S_{2n,m}(f_1, ..., f_{2n})| \leq \frac{(2n - 1)!!}{m^{2n}} \prod_{i=1}^{2n} \|f_i\|_{L^2(\mathbb{R}^d)}.$$  

(2.6)

Hence, if $\rho$ satisfies the Assumption 2.3 we can define for $n \in \mathbb{N}_0$ and $f_1, ..., f_{2n}, f_{2n+1} \in S(\mathbb{R}^d)$

$$S_{2n,\rho}(f_1, ..., f_{2n}) := \int_{[m_0, \infty)} S_{2n,m}(f_1, ..., f_{2n}) \rho(dm),$$

(2.7)

$$S_{2n+1,\rho}(f_1, ..., f_{2n+1}) := 0.$$

(2.8)

**Remark 2.6.** Observe that for a general relativistic quantum field the corresponding second Schwinger function is determined by a polynomial bounded measure on the positive real axis. This is basically the content of the Källen-Lehmann representation, see [18]/Theorem IX.34.

**Theorem 2.7.** Let $\rho$ satisfies the Assumption 2.3. Then it holds that the family $(S_{n,\rho})_{n \in \mathbb{N}_0}$ fulfills the axioms (E0) – (E4). Furthermore, the family $(S_{n,\rho})_{n \in \mathbb{N}_0}$ are the moments of a unique probability measure $\mu_\rho$ on $(S'(\mathbb{R}^d), \mathcal{B})$.

**Remark 2.8.**

(i) For $\rho = \delta_m$, $m \in (0, \infty)$, we simply obtain $S_{n,\rho} = S_{n,m}$ for all $n \in \mathbb{N}_0$. 


(ii) A sufficient criteria for a measure \( \nu \) on \( (S'(\mathbb{R}^d), \mathcal{B}) \) to be non-Gaussian is that the even truncated moments don't vanish. Indeed, let \( \nu \) be a probability measure on \( (S'(\mathbb{R}^d), \mathcal{B}) \) s.t. all moments \( M_n, \ n \in \mathbb{N}_0 \), exist. Now let \( f_1, ..., f_n \in S(\mathbb{R}^d) \) and \( n \in \mathbb{N} \). Then the truncated moments \( M_n^T \) are recursively defined via

\[
M_n^T = 1, \\
M_n(f_1, ..., f_n) = \sum_{I \in P^{(n)}} \prod_{i \in I} M_k^T(f_{i_1}, ..., f_{i_k}),
\]

where \( P^{(n)} \) denotes the set of all partitions of the set \( \{1, ..., n\} \), see also [3][Definition 4.4]. Then one concludes from Wicks Theorem, see e.g. [13][Proposition 1.2], that if \( \nu \) is Gaussian with mean \( 0 \), it holds \( M_{2n}^T = 0 \). Observe that the condition \( M_{2n}^T = 0 \) for all \( n \in \mathbb{N}_0 \) is a highly non-linear condition on the family \( (M_n)_{n \in \mathbb{N}} \). Hence, a superposition of measures with zero truncated moments does in general not inherit this property.

In the following we denote for an linear operator \( (A, D(A)) \) on a Hilbert space \( \mathcal{H} \) and a complex number \( \lambda \in \mathbb{C} \) the eigenspace of \( A \) with corresponding eigenvalue \( \lambda \) by \( \text{Eig}(A, \lambda) \). We need a some results from the theory of symmetric semigroups.

**Proposition 2.9.** Let \( (S_t)_{t \geq 0} \) be a strongly continuous contraction semigroup of symmetric operators with generator \( (L, D(L)) \) on a Hilbert space \( \mathcal{H} \).

(i) The orthogonal projection \( P_0 \) onto \( \text{Eig}(L, 0) \) is given by \( P_0 = \lim_{t \to \infty} S_t \), where the limit is taken in the strong operator topology.

(ii) It holds \( \cap_{t \geq 0} \text{Eig}(S_t, 1) = \text{Eig}(L, 0) \).

**Proof:** We first proof (ii): Let \( x \in \text{Eig}(L, 0) \). The orthogonal projection \( P_0 \) onto \( \text{Eig}(L, 0) \) is given via the spectral theorem for self-adjoint operators by \( \chi_{\{0\}}(L) \), where \( \chi_{\{0\}} \) is the indicator function of the set \( \{0\} \), see e.g. [17][Section VIII.3]. Hence, for \( t \geq 0 \) it holds by [17][Theorem VIII.5(a)]

\[
S_t x = \exp(-tL)\chi_{\{0\}}(L)x = (\exp(-t\cdot)\chi_{\{0\}})(L)x = \chi_{\{0\}}(L)x = x.
\]

The second inclusion is trivial.

Now let us show (i): Denote by \( E \) the spectral measure of \( (L, D(L)) \). Let \( x \in \mathcal{H} \) be arbitrary. We need to show \( \|P_0 x - S_t x\|_{\mathcal{H}} \to 0 \) as \( t \to \infty \). By the spectral theorem it holds

\[
\|P_0 x - S_t x\|_{\mathcal{H}}^2 = \int_{[0, \infty)} |\chi_{\{0\}}(\lambda) - \exp(-t\lambda)|^2 dE(\lambda, x)
\]

Hence, the claim follows from the dominated convergence theorem. \( \square \)

**Proof:** [Proof of Theorem 2.7] The distribution property \( [E0] \) follows immediately from \( [2.6] \) and the definition of \( S_{n,0}, \ n \in \mathbb{N} \). In particular, it holds

\[
|S_{2n,0}(f_1, ..., f_{2n})| \leq \frac{(2n - 1)!!}{m_0^{2n}} \prod_{i=1}^{2n} \|f_i\|^2_{L^2(\mathbb{R}^d)}, \quad f_1, ..., f_{2n} \in S(\mathbb{R}^d).
\]

(2.11)

The properties \( [E1] \) – \( [E3] \) above are linear in the family \( (S_n)_{n \in \mathbb{N}_0} \). Hence, by Proposition 2.5 the properties \( [E1] \) – \( [E3] \) are satisfied by \( (S_{n,0})_{n \in \mathbb{N}_0} \). The cluster property \( [E4] \) is non-linear in \( (S_n)_{n \in \mathbb{N}_0} \). To show the cluster property we use ideas from the proof of the Osterwalder-Schrader reconstruction theorem and translate the cluster property into the corresponding property for the Gårding-Wightman theory, i.e., into the uniqueness of the vacuum vector. Hence, we introduce some objects from the original proof of Osterwalder and Schrader from [13], [14] and we refer the reader to the last mentioned
references for more details. Let \( S_+ \) be as above. We equip the space \( S_+ \) with several semi-definite inner products. For \( \tilde{f}, \tilde{g} \in S_+ \) and \( m \in (m_0, \infty) \) we define

\[
(\tilde{f}, \tilde{g})_m := \sum_{l,k=0}^{\infty} S_{l+k,m}(\Theta f_l^\ast \otimes g_k)
\]

\[
(\tilde{f}, \tilde{g})_\rho := \sum_{l,k=0}^{\infty} S_{l+k,\rho}(\Theta f_l^\ast \otimes g_k)
\]

\[
= \int_{[m_0,\infty)} (\tilde{f}, \tilde{g})_m \rho(\mathrm{d}m).
\]

Observe that the sums in (2.12) and (2.13) are finite. Next we define the subspaces

\[
N_m := \left\{ \tilde{f} \in S_+ \mid (\tilde{f}, \tilde{f})_m = 0 \right\},
\]

\[
N_\rho := \left\{ \tilde{f} \in S_+ \mid (\tilde{f}, \tilde{f})_\rho = 0 \right\}.
\]

Now we form the quotient spaces \( \tilde{H}_m := S_+/N_m \) and \( \tilde{H}_\rho := S_+/N_\rho \) and define the Hilbert spaces \( H_m \) and \( H_\rho \) as the abstract completions of \( \tilde{H}_m \) and \( \tilde{H}_\rho \), respectively. The extensions of \( (\cdot, \cdot)_m \) and \( (\cdot, \cdot)_\rho \) to scalar products on \( \tilde{H}_m (H_m) \) and \( \tilde{H}_\rho (H_\rho) \) we denote by the same symbol and the induced norms by \( \|\cdot\|_m \) and \( \|\cdot\|_\rho \), respectively. For \( \tilde{f} \) we denote by \( [\tilde{f}]_m \) and \( [\tilde{f}]_\rho \) the respective equivalence class in \( \tilde{H}_m (H_m) \) and \( \tilde{H}_\rho (H_\rho) \), respectively. We also define \( \Omega := (1,0,0,...) \in S_+ \). The time translation operators \( T_t \), \( t \geq 0 \), defined above lift to continuous and symmetric linear operators \( T^m_t \) and \( T^\rho_t \) on the Hilbert spaces \( H_m \) and \( H_\rho \), respectively, see [13]. Furthermore, \( (T^m_t)_{t \geq 0} \) and \( (T^\rho_t)_{t \geq 0} \) form strongly continuous semigroups of contractions. Their respective self-adjoint generator we denote by \( (H^m, D(H^m)) \) and \( (H^\rho, D(H^\rho)) \). Observe that \( H^m \) and \( H^\rho \) are positive. Now we reformulate the cluster property in the following way. Denote by \( P^\rho_0 \) the orthogonal projection onto the eigenspace \( \text{Eig}(H^\rho,0) \). Then the assertions in (2.15)-(2.17) are equivalent:

\[
(E4) \text{ holds for } (S_{n,\rho})_{n \in \mathbb{N}_0},
\]

\[
P^\rho_0 = ([\Omega]_\rho, \cdot)_\rho[\Omega]_\rho, \text{ i.e. } \text{Eig}(H^\rho,0) = \text{span}_C([[\Omega]_\rho]),
\]

\[
\text{If } \Psi \in H_\rho \text{ satisfies } T^\rho_t \Psi = \Psi \text{ for all } t \geq 0 \text{ then it holds } \Psi \in \text{span}_C([\Omega]_\rho).
\]

The equivalence of (2.15) and (2.16) is now a direct consequence of Proposition 2.9(i) and the continuity of \( P^\rho_0 \) and \( ([\Omega]_\rho, \cdot)_\rho[\Omega]_\rho \). The equivalence of (2.16) and (2.17) follows directly from Proposition 2.9(ii).

For the choice \( \rho = \delta_m \), \( m \in (0, \infty) \), we obtain by Proposition 2.9 that the equivalent statements (2.15)-(2.17) for the Schwinger functions \( (S_{n,m})_{n \in \mathbb{N}_0} \) hold true. Our goal is to show that (2.17) holds true for \( (T^\rho_t)_{t \geq 0} \). The idea is to use that the operator \( T^\rho_t \) \( t \geq 0 \), 'factorizes' along the operators \( (T^m_t)_{m \in [m_0,\infty)} \) and use the corresponding result for \( T^m_t \). We make the idea precise below. The formula (2.14) for the scalar product \( (\cdot, \cdot)_\rho \) indicates that \( H_\rho \) is isometric isomorphic to a subspace of the direct integral of Hilbert spaces \( \int_{[m_0,\infty)} H_m \rho(\mathrm{d}m) \). Indeed, the spaces \( (H_m)_{m \in [m_0,\infty)} \) form a measurable field of Hilbert spaces in the sense of [1][Definition 1]. Hence we can form the direct integral of Hilbert spaces \( \mathcal{H} := \int_{[m_0,\infty)} H_m \rho(\mathrm{d}m) \), see also [1][Definition 5]. We define the following map

\[
\tilde{U} : \tilde{H}_\rho \rightarrow \mathcal{H}, \quad [\tilde{f}]_\rho \rightarrow \left( [\tilde{f}]_m \right)_{m \in [m_0,\infty)}.
\]

One obtains just by the definitions of the involved spaces that \( \tilde{U} \) is well-defined, linear and an isometry. Hence, \( \tilde{U} \) extends to an isometry \( U \) from \( H_\rho \) to \( \tilde{K} := \text{Im}(\tilde{U}) \), where the closure is taken in \( \mathcal{H} \).
Next we claim: For every $\Psi \in H_\rho$ and $t \geq 0$ there exists a $\rho$-negligible set $N_{t,\Psi}$ s.t. it holds
\[ T^m_t U \Psi (m) = (UT^0 m \Psi)(m) \text{ for all } m \in N_{t,\Psi}. \]

We prove that claim in two steps. First let $\Psi = [t]_\rho \in H_\rho$. Then the statement follows from the definition of the operators $T^m_t$, $T^0_t$ and the definition of $U$ as an extension of $U$. For an arbitrary $\Psi$ choose $\Psi_n \in H_\rho$ s.t. $\Psi_n \xrightarrow{n \to \infty} \Psi$ in $H_\rho$. Now we define $N_{1,\Psi} := \cup_{n \in N} N_{t,\Psi_n}$. Via [I][Proposition 5.(ii)], we can switch to a subsequence which we also denote by $(\Psi_n)_{n \in N}$ which converges $\rho$-a.e. outside a $\rho$-negligible set $N^2_{t,\Psi}$. The set $N_{1,\Psi} := N_{1,\Psi} \cup N^2_{t,\Psi}$ has the desired properties.

Now let us show the property (2.17). Let $\Psi \in H_\rho$ s.t. $T^m_t \Psi = \Psi$ for all $t \geq 0$. We define the $\rho$-negligible set $N_\Psi := \cup_{t \in [0,\infty) \cap \mathbb{Q}} N_{t,\Psi}$. For arbitrary $t \geq 0$ we choose a sequence $(t_n)_{n \in \mathbb{N}}$ s.t. $t_n \xrightarrow{n \to \infty} t$. Then for $m \in N_\Psi$ it holds by the strong continuity
\[ T^m_t (U \Psi)(m) = \lim_{n \to \infty} T^m_{t_n} (U \Psi)(m) = \lim_{n \to \infty} (UT^0 m \Psi)(m) = (U \Psi)(m). \quad (2.19) \]

Since the Schwinger functions $(S_{n,m})_{n \in \mathbb{N}}$ satisfy (E4) for every $m > 0$, we conclude that for $m \in N_\Psi$ it holds $(U \Psi)(m) = [\Omega]_m$. Eventually, we obtain $\Psi = [\Omega]_\rho$ since $U$ is injective which finishes the proof. It remains to show that the family $(S_{n,m})_{n \in \mathbb{N}}$ are the moments of a unique probability measure $\mu_\rho$ on $(S' (\mathbb{R}^3), B)$. We simply define the characteristic function of the measure $\mu_\rho$ via the Schwinger functions, i.e., define
\[ \hat{\mu}_\rho (\varphi) := \sum_{n=0}^{\infty} \frac{i^n}{n!} S_{n,m} (\varphi^{\otimes n}), \quad \varphi \in S(\mathbb{R}^d). \quad (2.20) \]

Due to (2.11) the series in (2.20) is absolutely convergent. Furthermore, since for every $\in [m_0, \infty)$ the family $(S_{n,m})_{n \in \mathbb{N}}$ are the moments of the measure $\mu_\rho$, we obtain that $\hat{\mu}_\rho$ satisfies all assumptions of the Bochner-Minlos theorem from which we conclude the proof.

\begin{remark}
(i) The assumption $\text{supp}(\rho) \subseteq [m_0, \infty)$ with $m_0 > 0$ in Theorem 2.7 is used to make the integral in (2.14) convergent, see also the estimate (2.6). It is clear that the result in Theorem 2.7 can be generalized in various ways in terms of more general class of measures $\rho$. In particular, one could replace the assumption $\text{supp}(\rho) \subseteq [m_0, \infty)$ by some growth condition of $\rho$ near zero. Furthermore, one could also allow $\rho$ to be non-finite, since the integrand in (2.14) has polynomial decrease, see again 2.6.

(ii) Recall the Källen-Lehmann representation which in general holds for the two point function or equivalently the second Schwinger function. The measure $\rho$ of our approach is in terms of the two point function by definition the spectral measure of the Källen-Lehmann representation of the field corresponding to $(S_{n,\rho})_{n \in \mathbb{N}}$. Hence, this approach allows to construct a large class of fields with Källen-Lehmann measure given through $\rho$.
\end{remark}

\begin{example}
Let $\rho = \frac{1}{2}(\delta_{m_1} + \delta_{m_2})$, where $m_1 \neq m_2$ and $m_1, m_2 > 0$, and $f_1, ..., f_4 \in S(\mathbb{R}^d)$. We calculate the fourth truncated Schwinger function $S_{4,\rho}^m$. It holds
\begin{align*}
4S_{4,\rho}^m (f_1, ..., f_4)
= & 4 (S_{4,\rho} (f_1, ..., f_4) - S_{2,\rho} (f_1, f_2) S_{2,\rho} (f_3, f_4) - S_{2,\rho} (f_1, f_3) S_{2,\rho} (f_2, f_4) - S_{2,\rho} (f_1, f_4) S_{2,\rho} (f_2, f_3)) \\
= & S_{4,m_1} (f_1, ..., f_4) - S_{2,m_1} (f_1, f_2) S_{2,m_1} (f_3, f_4) - S_{2,m_1} (f_1, f_3) S_{2,m_1} (f_2, f_4) - S_{2,m_1} (f_1, f_4) S_{2,m_1} (f_2, f_3) \\
& + S_{4,m_2} (f_1, ..., f_4) - S_{2,m_2} (f_1, f_2) S_{2,m_2} (f_3, f_4) - S_{2,m_2} (f_1, f_3) S_{2,m_2} (f_2, f_4) - S_{2,m_2} (f_1, f_4) S_{2,m_2} (f_2, f_3) \\
& + S_{4,m_1} (f_1, ..., f_4) - S_{2,m_2} (f_1, f_2) S_{2,m_2} (f_3, f_4) - S_{2,m_2} (f_1, f_3) S_{2,m_2} (f_2, f_4) - S_{2,m_2} (f_1, f_4) S_{2,m_2} (f_2, f_3) \\
& + S_{4,m_2} (f_1, ..., f_4) - S_{2,m_1} (f_1, f_2) S_{2,m_1} (f_3, f_4) - S_{2,m_1} (f_1, f_3) S_{2,m_1} (f_2, f_4) - S_{2,m_1} (f_1, f_4) S_{2,m_1} (f_2, f_3) \\
& - S_{2,m_1} (f_1, f_3) S_{2,m_2} (f_2, f_4) - S_{2,m_2} (f_1, f_3) S_{2,m_1} (f_2, f_4) - S_{2,m_1} (f_1, f_4) S_{2,m_2} (f_2, f_3) - S_{2,m_2} (f_1, f_4) S_{2,m_1} (f_2, f_3) \\
& - S_{2,m_2} (f_1, f_4) S_{2,m_1} (f_2, f_3) 
\end{align*}

\end{example}
\[
S_{4,m_1}(f_1, \ldots, f_4) + S_{4,m_2}(f_1, \ldots, f_4) \\
-S_{2,m_1}(f_1, f_2)S_{2,m_2}(f_3, f_4) - S_{2,m_2}(f_1, f_2)S_{2,m_1}(f_3, f_4) - S_{2,m_1}(f_1, f_3)S_{2,m_2}(f_2, f_4) \\
-S_{2,m_2}(f_1, f_3)S_{2,m_1}(f_2, f_4) - S_{2,m_1}(f_1, f_4)S_{2,m_2}(f_2, f_3) - S_{2,m_2}(f_1, f_4)S_{2,m_1}(f_2, f_3).
\]

(2.21)

Now choose \(m_1 \) and \(f_1, \ldots, f_4 \in S(\mathbb{R}^d) \) s.t. \(S_{4,m_1}(f_1, \ldots, f_4) \neq 0 \) and observe that all but the first term in (2.21) depend on \(m_2 \). Hence, from the estimate (2.10) we conclude that for fixed \(m_1 \) and sufficiently large \(m_2 \) the distribution \(S_{4,m_2} \) is not equal to zero. In particular, the measure \(\mu_\rho \) defined in Theorem 2.7 is non-Gaussian for suitable choices of \(m_1 \) and \(m_2 \) and the Schwinger function don’t factorize. This shows that the Wightman-theory corresponding to \((S_{n,\rho})_{n \in \mathbb{N}} \) is non-trivial in the sense that it is not a generalized free field.

**Remark 2.12.** (i) The idea of constructing non-Gaussian measures from Gaussian ones via superposition is also used in [10, 9] in the context of white noise and Mittag-Leffler analysis.

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**References**

[1] Chapter 1: Fields of hilbert spaces. In Jacques Dixmier, editor, *Von Neumann Algebras*, volume 27 of *North-Holland Mathematical Library*, pages 161–178. Elsevier, 1981.

[2] S. Albeverio and R. Hoegh Krohn. Uniqueness of the physical vacuum and the Wightman functions in the infinite volume limit for some non polynomial interactions. *Communications in Mathematical Physics*, 30(3):171–200, 1973.

[3] S. Albeverio, H. Gottschalk, and J. Wu. Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions. *Reviews in Mathematical Physics*, 8(6):763–817, 1996.

[4] Y. M. Berezansky and Y. G. Kondratiev. *Spectral Methods in Infinite Dimensional Analysis*. Mathematical Physics and Applied Mathematics; 12/1-2. Kluwer Academic Publishers, 1995.

[5] J. S. Feldman and K. Osterwalder. The wightman axioms and the mass gap for weakly coupled \((\varphi_1^4) \) quantum field theories. *Annals of Physics*, 97(1):80–135, 1976.

[6] I. Gelfand and G. Shilov. *Generalized Functions 1. Properties and operations*. AMS Chelsea Publishing, Rhode Island, 1964.

[7] I. Gelfand and G. Shilov. *Generalized Functions 2. Spaces of fundamental and generalized functions*. AMS Chelsea Publishing, Rhode Island, 1968.

[8] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. Methods of modern mathematical physics; vol. 3. Springer, 1981.

[9] M. Grothaus and F. Jahnert. Mittag-Leffler analysis II: Application to the fractional heat equation. *Journal of Functional Analysis*, 270(7):2732–2768, 2016.

[10] M. Grothaus, F. Jahnert, F. Riemann, and J. L. da Silva. Mittag-Leffler analysis I: Construction and characterization. *Journal of Functional Analysis*, 268(7):1876–1903, 2015.

[11] M. Grothaus and L. Streit. Construction of relativistic quantum fields in the framework of white noise analysis. *Journal of Mathematical Physics*, 40:5387, 1999.

[12] J. Magnen and R. Sénéor. The infinite volume limit of the \(\varphi_1^4 \) model. *Annales de l’Institut Henri Poincaré*, 24(2):95–159, 1976.

[13] K. Osterwalder and R. Schrader. Axioms for Euclidean Green’s functions. *Communications in Mathematical Physics*, 31(2):83–112, 1973.

[14] K. Osterwalder and R. Schrader. Axioms for Euclidean Green’s functions. II. *Communications in Mathematical Physics*, 42(3):281–305, 1975.

[15] M. Reed and B. Simon. *Fourier analysis, Self-Adjointness*. Methods of modern mathematical physics; vol. 2. Academic Press, New York, 1975.
[16] M. Reed and B. Simon. *Scattering theory*. Methods of modern mathematical physics; vol. 3. Academic Press, 1979.

[17] M. Reed and B. Simon. *Functional Analysis*. Methods of modern mathematical physics; vol. 1. Academic Press, New York, 1980.

[18] B. Simon. *The \( \mathcal{P}(\Phi)_2 \) Euclidean (quantum) field theory*. Princeton Univ. Pr., Princeton, NJ, 1974.

[19] R. Streater and A. Wightman. *PCT, spin and statistics, and all that*. Princeton University Press, New Jersey, 1980.

Mathematics Department, TU Kaiserslautern,
PO Box 3049, 67653 Kaiserslautern, Germany

Email address: grothaus@mathematik.uni-kl.de

Email address: nonnenmacher@mathematik.uni-kl.de