Entanglement of Purification for Multipartite States
and its Holographic Dual

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Abstract

We introduce a new information-theoretic measure of multipartite quantum/classical correlations $\Delta_P$, by generalizing the entanglement of purification to multipartite states. We provide proofs of its various properties, focusing on several entropic inequalities, in generic quantum systems. In particular, it turns out that the multipartite entanglement of purification gives an upper bound on multipartite mutual information, which is a generalization of quantum mutual information in the spirit of relative entropy. After that, motivated by a tensor network description of the AdS/CFT correspondence, we also define a holographic dual of multipartite entanglement of purification $\Delta_W$, as a sum of minimal areas of codimension-2 surfaces which divide the entanglement wedge into multi-pieces. We prove that this geometrical quantity satisfies all properties we proved for the multipartite entanglement of purification. These agreements strongly support the $\Delta_P = \Delta_W$ conjecture. We also show that the multipartite entanglement of purification gives an upper bound on multipartite squashed entanglement, which is a promising measure of multipartite quantum entanglement. We discuss potential saturation of multipartite squashed entanglement onto multipartite mutual information in holographic CFTs and its applications.
1 Introduction

Quantum entanglement has recently played significant roles in condensed matter physics [1, 2, 3], particle physics [4, 5, 6, 7] and string theory [8, 9]. To study the quantum entanglement in states of a quantum system, one often divide the total system into a subsystem $A$ and its complement $A^c$, and then compute the entanglement entropy $S_A := -\text{Tr}\rho_A \log \rho_A$, where $\rho_A$ is the reduced density matrix of a given total state $\rho_A := \text{Tr}_{A^c}\rho_{\text{tot}}$. In the gauge/gravity correspondence [10], the Ryu-Takayanagi formula [8, 9] allows one to compute the entanglement entropy in CFTs by a minimal area of codimension-2 surface in AdS. This discovery opens a new era of studying precise relations between spacetime geometry and quantum entanglement [11, 12, 13, 14, 15, 16, 17, 18, 19].

Recently the holographic counterpart of a new quantity independent of entanglement entropy, called the entanglement of purification $E_P$ [20], has been conjectured [21, 22]. The entanglement of purification quantifies an amount of total correlation, including quantum entanglement, for bipartite mixed states $\rho_{AB}$ acting on $H_{AB} \equiv H_A \otimes H_B$. On the other hand, the entanglement entropy truly measures quantum entanglement only for pure states $|\psi\rangle_{AB}$. Indeed, it is not even a correlation measure if a given total state is mixed. Thus, so called $E_P = E_W$ conjecture fills a gap between correlations in mixed states and spacetime geometry. This has been further studied in the literature [23, 24, 25, 26].

Another important direction is to explore multipartite correlations and its geometric dual. It is well known that there are richer correlation structures in quantum systems consisting of three or more subsystems, especially about quantum entanglement (see e.g. [27]). For example, there appear different separability criteria for multipartite states. To understand operational aspects of an amount of entanglement by means of (S)LOCC, one needs (infinitely) many kinds of standard states, such as W and GHZ states. Therefore multipartite entanglement is much more sophisticated than bipartite entanglement.

On the other hand, the holographic interpretation of multipartite correlations is less known, though it is crucial to understand the emergence of bulk geometry out of boundary renormalization group flows as well as the idea of ER=EPR [13, 28, 29, 30]. In the literature, one quantity $I_3$ called tripartite information, which is one particular generalization of mutual information $I(A : B) = S_A + S_B - S_{AB}$, attracts a lot of attention in holography [30, 31, 32, 33]. However, it can be either positive or negative, which makes a bit hard to use it to diagnose physical properties of the system at times.

In this paper, we first introduce a new measure of multipartite quantum and classical correlations $\Delta_P$ in generic quantum systems. It is given by generalizing the entanglement of purification to multipartite states. We study its information-theoretic properties, especially focusing on various entropic inequalities. It turns
out that $\Delta_P$ gives an upper bound on another generalization of quantum mutual information introduced in [34, 35], the so called multipartite mutual information.

We then propose a holographic dual of multipartite entanglement of purification $\Delta_W$, as a sum of minimal areas of codimension-2 surfaces in the entanglement wedge [36, 37, 38], motivated by the tensor network description of the AdS/CFT correspondence [11, 15, 16, 17]. One typical example of $\Delta_W$ was drawn in Fig.1.1. We demonstrate that it satisfies all the properties of the multipartite entanglement of purification by geometrical proofs. These agreements tempt us to provide a new conjecture $\Delta_P = \Delta_W$.

Finally, we clarify the relations between the defined new measure and a multipartite generalization [34, 35] of squashed entanglement, which is a promising measure of genuine quantum entanglement [39, 40, 41]. We also discuss a potential holographic counterpart of squashed entanglement, which coincides with holographic mutual information.

This paper is organized as follows: In Section 2, we give the definition of multipartite entanglement of purification $\Delta_P$ and prove its information-theoretic properties. In Section 3, we introduce a multipartite generalization of the entanglement wedge cross-section $\Delta_W$ in holography and study its properties geometrically, and find full agreement with those we proved in Section 2. Based on these facts, we propose the conjecture $\Delta_P = \Delta_W$. In Section 4 we study the universal relations between $\Delta_P$, multipartite squashed entanglement and multipartite mutual information, as well as the potential saturation condition of squashed entanglement in holography. We conclude in Section 5 and discuss future questions.

Note added: After all the results in this paper were obtained, a similar paper appeared in which they define essentially the same multipartite generalization of $E_P$, as well as propose its holographic dual, which is different from $\Delta_W$ [42].
2 Multipartite entanglement of purification

In this section, we will define a generalization of entanglement of purification for multipartite correlations and prove its various information-theoretic properties.

Let us start from recalling the definition and basic properties of the entanglement of purification [20, 43]. First we consider a quantum state \( \rho_{AB} \) on a bipartite quantum system \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). The entanglement of purification \( E_P \) for a bipartite state \( \rho_{AB} \) is defined by

\[
E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S_{AA'},
\]

where the minimization is taken over purifications \( \rho_{AB} = \text{Tr}_{A'B'}[|\psi\rangle \langle \psi|_{AA'BB'}] \). This is an information-theoretic measure of total correlations, namely, it captures both of quantum and classical correlations between \( A \) and \( B \). This point is the same as the quantum mutual information \( I(A : B) = S_A + S_B - S_{AB} \) does. Nevertheless, it is known that the regularized version of entanglement of purification \( E_P^\infty(\rho_{AB}) := \lim_{n \to \infty} E_P(\rho_{AB}^\otimes n)/n \) has an operational interpretation in terms of EPR pairs and local operation and asymptotically vanishing communication [20]. Note that we can also say that \( E_P \) measures a minimal value of quantum entanglement between \( AA' \) and \( BB' \) in an optimally purified system.

We simply write \( E_P(\rho_{AB}) \) as \( E_P(A : B) \) unless otherwise a given state specified. We summarize a part of known properties of \( E_P \) for the reader’s convenience:

(I) It reduces to the entanglement entropy for pure states \( \rho_{AB} = |\psi\rangle \langle \psi|_{AB} \).

\[
E_P(A : B) = S_A = S_B, \text{ for pure states.} \tag{2.2}
\]

(II) It vanishes if and only if a given state \( \rho_{AB} \) is product,

\[
E_P(A : B) = 0 \iff \rho_{AB} = \rho_A \otimes \rho_B. \tag{2.3}
\]

(III) It monotonically decreases upon discarding ancilla,

\[
E_P(A : BC) \geq E_P(A : B). \tag{2.4}
\]

(IV) It is bounded from above by the entanglement entropy,

\[
E_P(A : B) \leq \min\{S_A, S_B\}. \tag{2.5}
\]

(Va) It is bounded from below by a half of mutual information,

\[
E_P(A : B) \geq \frac{I(A : B)}{2}. \tag{2.6}
\]
(Vb) For tripartite states $\rho_{ABC}$, it has a lower bound,

$$E_P(A : BC) \geq \frac{I(A : B) + I(A : C)}{2}. \quad (2.7)$$

(VI) For any tripartite pure states $|\psi\rangle_{ABC}$, it is polygamous,

$$E_P(A : BC) \leq E_P(A : B) + E_P(A : C). \quad (2.8)$$

(VIIa) For a class of states that saturate the subadditivity, i.e. $S_{AB} = S_B - S_A$, it reduces to the entanglement entropy,

$$E_P(A : B) = S_A \text{ when } S_{AB} = S_B - S_A. \quad (2.9)$$

(VIIb) For a class of states that saturate the strong subadditivity, i.e. $S_{AB} + S_{AC} = S_B + S_C$, it reduces to the entanglement entropy,

$$E_P(A : B) = S_A \text{ when } S_{AB} + S_{AC} = S_B + S_C. \quad (2.10)$$

These properties are not independent each other and one can prove (VI) from (I) and (Va), and also does (VIIa) (or (VIIb)) from (III) and (Va) (or (Vb)), respectively. All these properties are proven in generic quantum systems. (I) allows us to regard $E_P$ as a generalization of entanglement entropy for quantifying an amount of correlations for mixed states. Refer to [20, 43] for detailed proofs and discussion.

### 2.1 Definition

We define a new quantity that captures total multipartite correlations by generalizing the entanglement of purification. We first note that we can rewrite the definition of $E_P$ (2.1) as

$$E_P(\rho_{AB}) = \frac{1}{2} \min_{|\psi\rangle_{AA’BB’}} [S_{AA’} + S_{BB’}], \quad (2.11)$$

because $S_{AA’} = S_{BB’}$ holds for any purifications $|\psi\rangle_{AA’BB’}$. This form of $E_P$ motivate us to define a generalization of entanglement of purification for a $n$-partite state $\rho_{A_1 \cdots A_n}$ as follows.

**Definition 1.** For $n$-partite quantum states $\rho_{A_1 \cdots A_n}$, we define the multipartite entanglement of purification $\Delta_P$ by

$$\Delta_P(\rho_{A_1 \cdots A_n}) := \min_{|\psi\rangle_{A_1A_1’ \cdots A_nA_n’}} \sum_{i=1}^n S_{A_iA_i’}, \quad (2.12)$$

where the minimization is taken over all possible purifications of $\rho_{A_1 \cdots A_n}$. 


We call it multipartite entanglement of purification and write \( \Delta_P(\rho_{A_1: \cdots : A_n}) = \Delta_P(A_1: \cdots : A_n) \) unless we need to specify a given state. This \( \Delta_P \) can be regarded as the minimal value of sum of quantum entanglement in an optimal purification \( |\psi\rangle_{A_1A'_1 \cdots A_nA'_n} \) between one of \( n \)-parties and the other \( n-1 \) parts. For example, for tripartite state \( \rho_{ABC} \), it can be represented as

\[
\Delta_P(A : B : C) := \min_{|\psi\rangle_{AA'B'B'C'C'}} [S_{AA'} + S_{BB'} + S_{CC'}],
\]

and the entanglement entropies \( S_{AA'}, S_{BB'} \) and \( S_{CC'} \) in the brackets represent quantum entanglement between \( AA' : BB'C'C' \), \( BB' : AA'C'C' \) and \( CC' : AA'BB' \), respectively. We also note that a purification that gives the minimum of \( \Delta_P \) may not be unique in general, as is so for the entanglement of purification \( E_P \).

### 2.2 Other measures

Besides the entanglement of purification, there have been a lot of measures of quantum and/or classical correlations which quantify a bipartite correlations for mixed states - including mutual information or squashed entanglement [39, 40]. Their generalization for multipartite correlations also have been proposed in the literature [34, 35]. To see that multipartite correlations are not just a sum of bipartite ones, let us consider for example the famous GHZ state in a tripartite qubit system:

\[
|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).
\]

This state exhibits a strong correlation between 3-parties as we will see the same result of spin measurements. The point is that, after one of the subsystems is traced out, the remaining bipartite state is just a separable state and there is no more quantum entanglement. This shows that the structure of multipartite quantum entanglement is much richer than bipartite ones.

To quantify an amount of quantum/classical correlations for multipartite states, there have been several ways proposed by generalizing the mutual information. One well known quantity is based on the Ven diagram, and for example so-called tripartite information is defined by

\[
\bar{I}_3(A : B : C) := S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC} = I(A : C) + I(A : B) - I(A : BC).
\]

This quantity plays an important role in the context of holography and has been studied e.g. in [30, 31, 32, 33]. However, the tripartite information \( \bar{I}_3 \) takes all of negative, zero, or positive values in generic quantum systems, which might make it rather hard to regard as an ordinary measure of correlations.
The other approach is introduced in \cite{34, 35} and it is called multipartite mutual information (or multiparty information), defined as a distance measured by the relative entropy between a given original state and its local product state:

$$I(A : B : C) := S(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C) = S_A + S_B + S_C - S_{ABC}, \quad (2.16)$$

where $S(\rho || \sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma$. This definition is motivated by an expression of the bipartite mutual information,

$$I(A : B) = S(\rho_{AB} || \rho_A \otimes \rho_B). \quad (2.17)$$

In general, the multipartite mutual information for $n$-partite states $\rho_{A_1 \cdots A_n}$ is defined in the same manner:

$$I(A_1 : \cdots : A_n) := S(\rho_{A_1 \cdots A_n} || \rho_{A_1} \otimes \cdots \otimes \rho_{A_n}) = \sum_{i=1}^{n} S_{A_i} - S_{A_1 \cdots A_n}. \quad (2.18)$$

This is clearly a positive semi-definite and is monotonic under local operations \cite{34}. Thus one may consider it as a more promising measure of total multipartite correlations than $\tilde{I}$. Note that we can rewrite $I(A_1 : \cdots : A_n)$ to a suggestive form that is a summation of bipartite mutual information:

$$I(A_1 : \cdots : A_n) = I(A_1 : A_2) + I(A_1A_2 : A_3) + \cdots + I(A_1 \cdots A_{n-1} : A_n). \quad (2.19)$$

We will carefully distinguish these two types of generalizations of mutual information, $\tilde{I}$ and $I$, and in the following we mainly discuss the latter one.

### 2.3 Properties of multipartite entanglement of purification

Here we explore properties of $\Delta_P$ in generic quantum systems. Firstly, we can immediately see the following property from the definition.

**Lemma 2.** If one of the subsystems is decoupled $\rho_{A_1 \cdots A_n} = \rho_{A_1 \cdots A_{n-1}} \otimes \rho_{A_n}$, then $\Delta_P(A_1 : \cdots : A_n) = \Delta_P(A_1 : \cdots : A_{n-1})$.

**Proof.** One can separately purify $\rho_{A_n}$ and remaining parts, and then it directly follows by definition. \qed

Especially, for bipartite states $\rho_{AB}$, it reduces to the twice of entanglement of purification: $\Delta_P(A : B) = 2E_P(A : B)$. \footnote{The normalization factor is not essential in our discussion, so we take it so that the entropic inequalities below become simple.} It guarantees that $\Delta_P$ is a generalization of $E_P$ to multipartite states.

We expect that $\Delta_P$ is a natural generalization of $E_P$ for multipartite correlations and has similar properties. Indeed, one can prove the following properties which are the counterparts of those of $E_P$ mentioned above.
Proposition 3. If a given \( n \)-partite state is pure \( |\phi\rangle_{A_1 \cdots A_n} \), then the multipartite entanglement of purification is given by the summation of entanglement entropy of each single subsystem,

\[
\Delta_P(A_1 : \cdots : A_n) = \sum_{i=1}^{n} S_{A_i} \quad \text{for pure states.} \tag{2.20}
\]

Proof. Notice that \( |\phi\rangle_{A_1 \cdots A_n} \) itself is a purification, and that all the other purifications should have a form \( |\psi\rangle_{A_1 A'_1 \cdots A_n A'_n} = |\phi\rangle_{A_1 \cdots A_n} \otimes |\phi'\rangle_{A'_1 \cdots A'_n} \). Thus adding ancillary systems always increases the sum of entanglement entropy of purified systems, and the minimum is achieved by the original state \( |\psi\rangle_{A_1 A'_1 \cdots A_n A'_n} = |\phi\rangle_{A_1 \cdots A_n} \).

This is a generalization of property (I) and makes it easy to calculate \( \Delta_P \) for pure states. Note that in the case of pure states, the multipartite mutual information also reduces to the sum of entanglement entropy:

\[
I(A_1 : \cdots : A_n) = \sum_{i=1}^{n} S_{A_i} - S_{A_1 \cdots A_n} = \sum_{i=1}^{n} S_{A_i}. \tag{2.21}
\]

Proposition 4. \( \Delta_P \) vanishes if and only if a given \( n \)-partite state is fully product,

\[
\Delta_P(A_1 : \cdots : A_n) = 0 \iff \rho_{A_1 \cdots A_n} = \rho_{A_1} \otimes \cdots \otimes \rho_{A_n}. \tag{2.22}
\]

Even though it can be directly proven, we postpone its proof after the proposition 8. This property indicates that \( \Delta_P \) is not a measure of multipartite merely quantum entanglement but of both quantum and classical correlations. It is expected as a generalization of \( E_P \). Note also that \( \Delta_P \) is not a measure of genuine \( n \)-partite correlations, but also captures all of \( 2, \cdots, n-1 \)-partite correlations included in \( \rho_{A_1 \cdots A_n} \).

As a measure of correlations, it is natural to decrease when we trace out a part of one subsystem. Actually, this is true for \( \Delta_P \).

Proposition 5. \( \Delta_P \) monotonically decrease upon discarding ancilla,

\[
\Delta_P(X A_1 : \cdots : A_n) \geq \Delta_P(A_1 : \cdots : A_n). \tag{2.23}
\]

Proof. It follows from the fact that all purifications of \( \rho_{XA_1 \cdots A_n} \) are included in these of \( \rho_{A_1 \cdots A_n} \). Namely, if \( |\psi\rangle_{XX'A_1'\cdots A_n'A_n'} \) is a optimal purification for \( \rho_{XA_1 \cdots A_n} \), then it is also one of the (not optimal in general) purification of \( \rho_{A_1 \cdots A_n} \); thus

\[
\Delta_P(X A_1 : \cdots : A_n) = S_{A_1(A_1'X)} + S_{A_2A_2'} + \cdots + S_{A_nA_n'} \geq \Delta_P(A_1 : \cdots : A_n). \tag{2.24}
\]
We give an upper bound on $\Delta_P$ in terms of a certain sum of entanglement entropy, as a generalization of property (IV).

**Proposition 6.** $\Delta_P$ is bounded from above by

$$\Delta_P(A_1 : \cdots : A_n) \leq \min_i[S_{A_1} + \cdots + S_{A_{i-1}A_{i+1} \cdots A_n} + \cdots + S_{A_n}].$$

(2.24)

**Proof.** For simplicity, we firstly prove this bound for tripartite states $\rho_{ABC}$. Let us consider a standard purification of a given tripartite state $\rho_{ABC} = \sum p_k |\phi^k\rangle \langle \phi^k|_{ABC}$ such that

$$|\psi\rangle_{AA'BB'CC'} = \sum_{k=1}^{\text{rank}[\rho_{ABC}]} \sqrt{p_k} |\phi^k\rangle_{ABC} \otimes |0\rangle_{A'} \otimes |0\rangle_{B'} \otimes |k\rangle_{C'}.$$  (2.25)

Then we have $\Delta_P(A : B : C) \leq S_{AA'} + S_{BB'} + S_{CC'}$ for this purification. However, by noting that $\rho_{AA'} = \rho_A \otimes |0\rangle \langle 0|_A$ and the same for $B$, and $S_{CC'} = S_{AA'BB'}$, it can be easily shown that

$$S_{AA'} = S_A, \ S_{BB'} = S_B, \ S_{CC'} = S_{AB},$$

for this state. Thus we get $\Delta_P(A : B : C) \leq S_A + S_B + S_{AB}$. Commuting $A, B, C$, we get three upper bounds on $\Delta_P$,

$$\Delta_P(A : B : C) \leq \min\{ S_A + S_B + S_{AB}, S_B + S_C + S_{BC}, S_C + S_A + S_{CA} \}.$$  (2.27)

To generalize this proof for $n$-partite cases is straightforward. \(\square\)

These upper bounds tell us that if one has a bipartite state $\rho_{AB}$, and considers extensions $\rho_{ABC}$, the upper bound of $\Delta_P(A : B : C)$ is totally determined by the information included in $\rho_{AB}$. In other words, we can not arbitrarily increase the multipartite correlations by adding ancillary systems (and the upper bound can be reached by any purification $|\psi\rangle_{ABC}$ of $\rho_{AB}$, while it is not the only way to saturate these bounds as we will see in corollary 10 and 11).

Next we state universal lower bounds on $\Delta_P$. First, we observe the following inequality satisfied for any tripartite state $\rho_{ABC}$.

**Proposition 7.** Tripartite entanglement of purification $\Delta_P(A : B : C)$ is bounded from below by

$$\Delta_P(A : B : C) \geq \max\{ S_A + S_B + S_C - S_{ABC}, 2(S_A + S_B + S_C) - S_{AB} - S_{BC} - S_{CA} \}.$$  (2.28)
Proof. Let us take an optimal purification $|\psi\rangle_{AA'BB'CC''}$. For this state we have

$$\Delta_P(A : B : C) = S_{AA'} + S_{BB'} + S_{CC''} = I(AA' : BB') + I(AA'BB' : CC'') \geq I(A : B) + I(AB : C) = S_A + S_B + S_C - S_{ABC}, \quad (2.29)$$

where in the third line we used the monotonicity of mutual information

$$I(AX : B) \geq I(A : B). \quad (2.30)$$

Moreover, for tripartite pure states $|\psi\rangle_{AA'BB'CC''}$, we have $I(AA'BB' : CC'') = I(AA' : CC'') + I(BB' : CC'')$. Thus we get

$$\Delta_P(A : B : C) = I(AA' : BB') + I(BB' : CC'') + I(CC' : AA') \geq I(A : B) + I(B : C) + I(C : A) = 2(S_A + S_B + S_C) - S_{AB} - S_{BC} - S_{CA}. \quad (2.31)$$

These bounds have a suggestive form in terms of tripartite mutual information\(^2\):

$$\Delta_P(A : B : C) \geq \max\{I(A : B : C), I(A : B : C) + \tilde{I}_3(A : B : C)\}. \quad (2.32)$$

Especially, $\Delta_P$ is always greater or equal than the multipartite mutual information $I(A : B : C)$. Indeed, this is also true for $n$-partite states, as we see in the following proposition.

Proposition 8. The multipartite entanglement of purification $\Delta_P(A_1 : \cdots : A_n)$ is bounded from below by the multipartite mutual information,

$$\Delta_P(A_1 : \cdots : A_n) \geq I(A_1 : \cdots : A_n). \quad (2.33)$$

Proof. The proof is same as in tripartite case. Let us consider an optimal purification $|\psi\rangle_{A_1A'_1\cdots A_nA'_n}$ for $\rho_{A_1\cdots A_n}$. Then we have

$$\Delta_P(A_1 : \cdots : A_n) = \sum_{i=1}^n S_{A_iA'_i} = \sum_{i=1}^n S_{A_iA'_i} - S_{A_1A'_1\cdots A_nA'_n} = I(A_1A'_1 : \cdots : A_nA'_n) \geq I(A_1 : \cdots : A_n), \quad (2.34)$$

\(^2\)Note that it leads a general relationship $\Delta_P(A : B : C) \geq \tilde{I}_3(A : B : C)$ for any quantum states.
where we used the property of multipartite mutual information [34]
\[
I(\mathcal{A}_1 X : \cdots : \mathcal{A}_n) \geq I(\mathcal{A}_1 : \cdots : \mathcal{A}_n). 
\] (2.35)

This is a generalization of property (Va), (Vb) and gives us a general relationship between two types of multipartite total correlation measures $I$ and $\Delta_P$. These two quantities behave very similarly. Indeed, the propositions 2, 3, 4, 5, and 6 are also true for multipartite mutual information. One exception is, for example, the lower bound of tripartite case, $\Delta_P(\mathcal{A} : \mathcal{B} : \mathcal{C}) \geq I(\mathcal{A} : \mathcal{B} : \mathcal{C}) + \tilde{I}_3(\mathcal{A} : \mathcal{B} : \mathcal{C})$, which is obviously violated for $I(\mathcal{A} : \mathcal{B} : \mathcal{C})$ when $\tilde{I}_3(\mathcal{A} : \mathcal{B} : \mathcal{C})$ is positive.

The proposition 8 also allows us to give a simple proof of the proposition 4.

\textbf{Proof.} If a $n$-partite state is totally product $\rho_{\mathcal{A}_1 \cdots \mathcal{A}_n} = \rho_{\mathcal{A}_1} \otimes \cdots \otimes \rho_{\mathcal{A}_n}$, then one can get $\Delta_P = 0$ by purifying each subsystems independently. On the other hand, if $\Delta_P(\mathcal{A}_1 : \cdots : \mathcal{A}_n) = 0$, then $I(\mathcal{A}_1 : \cdots : \mathcal{A}_n) = S(\rho_{\mathcal{A}_1 \cdots \mathcal{A}_n} || \rho_{\mathcal{A}_1} \otimes \cdots \otimes \rho_{\mathcal{A}_n}) = 0$ from the proposition 8. Thus, the non-degeneracy of relative entropy leads to $\rho_{\mathcal{A}_1 \cdots \mathcal{A}_n} = \rho_{\mathcal{A}_1} \otimes \cdots \otimes \rho_{\mathcal{A}_n}$. \hfill \Box

Using the above arguments, some properties of $\Delta_P$ follow as corollaries.

\textbf{Corollary 9.} For any pure $n$-partite state, $\Delta_P$ is polygamous:
\[
\Delta_P(\mathcal{A}_1 : \cdots : \mathcal{A}_{n-1} : BC) \leq \Delta_P(\mathcal{A}_1 : \cdots : \mathcal{A}_{n-1} : B) + \Delta_P(\mathcal{A}_1 : \cdots : \mathcal{A}_{n-1} : C). 
\] (2.36)

\textbf{Proof.} For any pure state $|\phi\rangle_{\mathcal{A}_1 \cdots \mathcal{A}_n BC}$ it follows from the proposition 3 that
\[
\Delta_P(\mathcal{A}_1 : \cdots : \mathcal{A}_{n-1} : BC) = \sum_{i=1}^{n-1} S_{A_i} + S_{BC} = \sum_{i=1}^{n-1} S_{A_i} + S_{\mathcal{A}_1 \cdots \mathcal{A}_{n-1}} 
\leq 2 \sum_{i=1}^{n-1} S_{A_i} 
= \sum_{i=1}^{n-1} S_{A_i} + S_B - S_{\mathcal{A}_1 \cdots \mathcal{A}_{n-1} B} + \sum_{i=1}^{n-1} S_{A_i} + S_C - S_{\mathcal{A}_1 \cdots \mathcal{A}_{n-1} C} 
\leq \Delta(\mathcal{A}_1 : \cdots : \mathcal{A}_{n-1} : B) + \Delta(\mathcal{A}_1 : \cdots : \mathcal{A}_n : C), 
\] (2.37)
where in the first inequality we used the subadditivity of von Neumann entropy recursively, in the third line $S_B = S_{\mathcal{A}_1 \cdots \mathcal{A}_{n-1} C}$, $S_C = S_{\mathcal{A}_1 \cdots \mathcal{A}_{n-1} B}$ for pure states, and in the last inequality the proposition 8. \hfill \Box
As $E_P$ is so, the multipartite entanglement of purification is difficult to calculate in general, because of the minimization over infinitely many purifications. Nevertheless, there is a class of quantum states for which one can rigorously calculate $\Delta_P$ in terms of entanglement entropy.

**Corollary 10.** For a class of tripartite states $\rho_{ABC}$ that saturate the subadditivity $S_{ABC} = S_C - S_{AB}$, we have $\Delta_P(A : B : C) = S_A + S_B + S_{AB}$.

**Proof.** From proposition 6 and 7, we have

$$S_A + S_B + S_C - S_{ABC} \leq \Delta_P(A : B : C) \leq S_A + S_B + S_{AB}. \tag{2.38}$$

Therefore $S_C - S_{ABC} = S_{AB}$ leads $\Delta_P(A : B : C) = S_A + S_B + S_{AB}$. $\square$

**Corollary 11.** For a class of tripartite states $\rho_{ABC}$ that saturate both of two forms of the strong subadditivity, $S_A + S_C = S_{AB} + S_{BC}$ and $S_B + S_C = S_{AB} + S_{AC}$, then we have $\Delta_P(A : B : C) = S_A + S_B + S_{AB}$.

**Proof.** From proposition 6 and 7, we have

$$2(S_A + S_B + S_C) - S_{AB} - S_{BC} - S_{CA} \leq \Delta_P(A : B : C) \leq S_A + S_B + S_{AB}, \tag{2.39}$$

where the lower bound can be expressed as

$$2(S_A + S_B + S_C) - S_{AB} - S_{BC} - S_{CA} = S_A + S_B + S_{AB} + (S_A + S_C - S_{AB} - S_{BC}) + (S_B + S_C - S_{AB} - S_{AC}). \tag{2.40}$$

Thus if the two strong subadditivity are simultaneously saturated, we get $\Delta_P(A : B : C) = S_A + S_B + S_{AB}$. $\square$

We also mention another lower bound on $\Delta_P$ in terms of bipartite entanglement of purification.

**Proposition 12.** $\Delta_P$ is bounded from below by

$$\Delta_P(A_1 : \cdots : A_n) \geq \sum_{i=1}^n E_P(A_i : A_1 \cdots A_{i-1} A_{i+1} \cdots A_n). \tag{2.41}$$

**Proof.** Let us prove it for tripartite cases for simplicity. For a state $\rho_{ABC}$, we have

$$\Delta_P(A : B : C) = \min_{|\psi\rangle_{AA'BB'CC'}} [S_{AA'} + S_{BB'} + S_{CC'}] \geq \min_{|\psi\rangle_{AA'BB'CC'}} S_{AA'} + \min_{|\psi\rangle_{AA'BB'CC'}} S_{BB'} + \min_{|\psi\rangle_{AA'BB'CC'}} S_{CC'} = E_P(A : BC) + E_P(B : CA) + E_P(C : AB), \tag{2.42}$$

then the bound follows. $\square$
3 Holography: multipartite entanglement wedge cross section

In this section, we define a multipartite generalization of entanglement wedge cross-section introduced in [21, 22], motivated by tensor network descriptions of AdS geometry [11, 15, 16, 17].

3.1 Definition

We start by setting our conventions in holography. In the case of computation of entanglement entropy in quantum field theories, we often choose a (maybe disconnected) subsystem $A$ on a time slice, and the Hilbert space of the field theory is factorized into $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$. Then the entanglement entropy for subsystem $A$ in a state $\rho_{\text{tot}}$, is defined as the von Neumann entropy of the reduced density matrix $\rho_A$,

$$S_A := -\text{Tr} \rho_A \log \rho_A \, , \quad \rho_A = \text{Tr}_{A^c} \rho_{\text{tot}} \, .$$

(3.1)

In the AdS/CFT correspondence, the holographic entanglement entropy formula [8, 9] tells us how to calculate entanglement entropy in dual gravity side. Consider a state in $d$-dimensional holographic CFT which has a classical $d+1$ dimensional gravity dual. In the present paper, we will restrict ourselves to work in only static cases. The dual gravity solution of the given state $\rho_{\text{tot}}$ will be a canonical time slice $M$ in gravity side. To compute the entanglement entropy for a chosen subsystem $A \subset \partial M$, we are looking for a $d-1$-dimensional surface $\Gamma_A$ in $M$ with minimal area, under the conditions that $\partial \Gamma_A = A$ and $\Gamma_A$ is homologous to $A$. Then the holographic entanglement entropy is determined by the minimal area\(^4\),

$$S_A = \frac{\text{Area}(\Gamma_A^{\text{min}})}{4G_N} \, .$$

(3.2)

Let us start to define our main interest, i.e. a multipartite generalization of entanglement wedge cross-section $\Delta_W$. We mostly focus on tripartite case for simplicity, though the generalization to more partite cases is rather straightforward. We take subsystems $A$, $B$ and $C$ on the boundary $\partial M$. In general $\rho_{ABC}$ is a mixed state. Then one can compute the holographic entanglement entropy $S_A, S_B, S_C$ and also $S_{ABC}$ following (3.2). The corresponding minimal surfaces are

\(^4\)Even though we can define a covariant version of multipartite entanglement wedge cross-section in a similar way for $E_W$ [21], the holographic proofs of its entropic inequalities for more than 4-partite cases is not straightforward [45]. We leave it as a future problem.

\(^4\)Including this definition, as well as discussion on multipartite entanglement wedge cross-section, we work at the leading order of large $N$ limit through the whole of present paper.
denoted by $\Gamma^\text{min}_A$, $\Gamma^\text{min}_B$, $\Gamma^\text{min}_C$, and $\Gamma^\text{min}_{ABC}$, respectively. The entanglement wedge $M_{ABC}$ [36, 37, 38] is defined as a region of $M$ with boundary $A, B, C$ and $\Gamma^\text{min}_{ABC}$:

$$\partial M_{ABC} = A \cup B \cup C \cup \Gamma^\text{min}_{ABC}. \quad (3.3)$$

Notice that $M_{ABC}$ gets disconnected when some of $A, B, C$ or all of them are decoupled. Also note that $\partial M_{ABC}$ may include bifurcation surfaces in the bulk such as in AdS black hole geometry.

Next, we divide arbitrarily the boundary $\partial M_{ABC}$ (not $M_{ABC}$ itself) into three parts $\tilde{A}, \tilde{B}, \tilde{C}$ so that they satisfy

$$\tilde{A} \cup \tilde{B} \cup \tilde{C} = \partial M_{ABC}, \quad (3.4)$$

and

$$A \subset \tilde{A}, B \subset \tilde{B}, C \subset \tilde{C}. \quad (3.5)$$

The boundary of $\tilde{A}, \tilde{B}, \tilde{C}$ is denoted by $\mathcal{D}_{ABC}$. Regarding each $\tilde{A}, \tilde{B}, \tilde{C}$ as a subsystem of a geometric pure state, we can calculate

$$S_{\tilde{A}} + S_{\tilde{B}} + S_{\tilde{C}}, \quad (3.6)$$

by using holographic entanglement entropy formula (3.2). This is performed by finding a minimal surface $\Sigma^\text{min}_{ABC}$ that consists of three parts $\Sigma_A, \Sigma_B, \Sigma_C$, which share the boundary $\mathcal{D}_{ABC}$, such that

$$\Sigma^\text{min}_{ABC} = \Sigma_A \cup \Sigma_B \cup \Sigma_C, \quad \partial \Sigma^\text{min}_{ABC} = \mathcal{D}_{ABC}, \quad (3.7)$$

and

$$\Sigma_{A,B,C} \text{ is homologous to } \tilde{A}, \tilde{B}, \tilde{C} \text{ inside } M_{ABC}. \quad (3.8)$$

Since $\partial M_{ABC}$ is codimension-2, the surfaces $\mathcal{D}_{ABC}$ which plays the role of the division of $\partial M_{ABC} = \tilde{A} \cup \tilde{B} \cup \tilde{C}$, is codimension-3. In the case of AdS$_3$/CFT$_2$, $\mathcal{D}_{ABC}$ is in general three separated points on $\Gamma^\text{min}_{ABC}$, see Fig 3.1, 3.2.

\footnote{More precisely, we consider a constant time slice of entanglement wedges and call it also entanglement wedge, while the former is codimension-0 and the latter is codimension-1.}
Figure 3.1: An example of tripartite entanglement wedge cross-section. The black bold dashed lines represents the minimal surface $\Gamma_{ABC}^{\text{min}}$, giving a part of the boundary of $M_{ABC}$. The yellow thin dashed lines represents $\Sigma_{ABC}^{\text{min}}$ whose area (divided by $4G_N$) is $\Delta_W$.

Figure 3.2: An example of tripartite entanglement wedge cross-section in a black hole geometry. Each surface of $\Sigma_{ABC}^{\text{min}}$ is doubled.

Finally we minimize the area of $\Sigma_{ABC}^{\text{min}}$, over all possible divisions $\tilde{A}, \tilde{B}, \tilde{C}$ that satisfy the conditions (3.4) and (3.5). This gives now a quantity we call the multipartite entanglement wedge cross section

$$\Delta_W(\rho_{ABC}) := \min_{\tilde{A}, \tilde{B}, \tilde{C}} \left[ \frac{\text{Area}(\Sigma_{ABC}^{\text{min}})}{4G_N} \right].$$  \hspace{1cm} (3.9)
For $n$-partite boundary subsystems, in general, the multipartite entanglement wedge cross-section is defined in a similar manner. Notice that when it is reduced to the bipartite case, this definition is actually twice of the bipartite entanglement wedge cross-section defined in [21, 22]. We sometimes write \( \Delta_W(\rho_{ABC}) = \Delta_W(A : B : C) \) to clarify a way of partition of subsystems.

In summary, \( \Delta_W \) computes the multipartite cross-sections of the entanglement wedge \( M_{ABC} \) and it is a natural generalization of the bipartite entanglement wedge cross-section. This can be used as a total measure of how strongly multiple parties are holographically connected with each other. Below we study the properties of \( \Delta_W \).

### 3.2 Properties of multipartite entanglement wedge cross-section and the conjecture \( \Delta_W = \Delta_P \)

In the following, we investigate holographic properties of \( \Delta_W \), inspired by those of \( \Delta_P \). To avoid unnecessary complexity, we mostly consider tripartite case only, and it should be understood that the properties are easily generalized for \( n \)-partite cases in somewhat trivial ways unless otherwise emphasized.

First, if \( \rho_{ABC} \) is pure, from the definition (3.7), \( \Sigma_{ABC}^{\text{min}} \) coincides with \( \Gamma_A^{\text{min}} \cup \Gamma_B^{\text{min}} \cup \Gamma_C^{\text{min}} \). Therefore \( \Delta_W \) is equal to the sum of the entanglement entropy of \( A, B \) and \( C \):

\[
\Delta_W(A : B : C) = S_A + S_B + S_C.
\]

(3.10)

As we mentioned above, for a partly decoupled entanglement wedges i.e. if \( M_{ABC} = M_{AB} \bigcup M_C \), where \( \bigcup \) denote that the geometries \( M_{AB}, M_C \) are totally separated, then \( \Delta_W \) is reduced to (twice of) entanglement wedge cross-section. Similarly, it clearly leads that \( \Delta_W = 0 \) if and only if the total entanglement wedge is totally decoupled \( M_{A_1 \cdots A_n} = \bigcup_{i=1}^n M_{A_i} \) for multipartite setups.

One can easily show that \( \Delta_W \) decreases when we reduce one of the subregions in \( A, B, C \equiv C_1 \cup C_2 \):

\[
\Delta_W(A : B : C_1 \cup C_2) \geq \Delta_W(A : B : C_1),
\]

(3.11)

by using so-called entanglement wedge nesting,

\[
M_X \subset M_{XY},
\]

(3.12)

which holds for any boundary subregions \( X, Y \) [36, 37, 38].
One can also easily show an upper bound of $\Delta_W$ by a graph proof (Fig. 3.3):

$$\Delta_W(A : B : C) \leq S_A + S_B + S_{AB}, \quad (3.13)$$

By commuting $A, B, C$, one can further get

$$\Delta_W(A : B : C) \leq \min[S_A + S_B + S_{AB}, S_B + S_C + S_{BC}, S_A + S_C + S_{AC}]. \quad (3.14)$$

![Figure 3.3: The proof of an upper bound of $\Delta_W$. The sum of blue real lines is $S_A + S_B + S_{AB}$ and the sum of dashed yellow lines are $\Delta_W(\rho_{ABC})$. Clearly $\Delta_W(\rho_{ABC}) \leq S_A + S_B + S_{AB}$, somewhat trivially since $S_A + S_B + S_{AB}$ have UV divergences while $\Delta_W$ are not. When two of subsystems share the boundary, $\Delta_W$ also diverges, but it is always weaker than $S_A + S_B + S_{AB}$ shown by a graph.](image)

Furthermore, for tripartite setups, one can show two lower bounds of $\Delta_W$ also by graph proofs (Fig. 3.4, Fig. 3.5):

$$\Delta_W(A : B : C) \geq I(A : B : C) = S_A + S_B + S_C - S_{ABC}. \quad (3.15)$$

$$\Delta_W(A : B : C) \geq I(A : B : C) + \tilde{I}_3(A : B : C) = 2(S_A + S_B + S_C) - S_{AB} - S_{BC} - S_{AC}. \quad (3.16)$$

Note that, however, in holography we always have $I(A : B : C) \geq I(A : B : C) + \tilde{I}_3(A : B : C)$ because of negative tripartite information $\tilde{I}_3 \leq 0$. Therefore the former is always tighter.

Similarly, for $n$-partite setup, one can easily show that

$$\Delta_W(A_1 : \cdots : A_n) \geq I(A_1 : \cdots : A_n), \quad (3.17)$$

by drawing graphs.
Figure 3.4: The proof of a lower bound of $\Delta W$. The sum of dashed yellow lines is $\Delta_{ABC} + S_{ABC}$ and the sum of real blue lines are $S_A + S_B + S_C$. Clearly $\Delta W(\rho_{ABC}) + S_{ABC} \geq S_A + S_B + S_C$ follows since the entanglement entropy are defined as minimal surfaces.

Figure 3.5: The proof of a lower bound of $\Delta W$. The sum of dashed yellow lines is $\Delta_{ABC} + S_{AB} + S_{BC} + S_{CA}$, and the sum of real doubled blue lines are $2(S_A + S_B + S_C)$. Clearly $\Delta W + S_{AB} + S_{BC} + S_{CA} \geq 2(S_A + S_B + S_C)$ follows, since the entanglement entropy are defined as minimal surfaces.

Note that the corollaries in the previous section automatically follows for $\Delta W$ from the above discussion, while we can also show them by drawing graphs. We also note that the proposition 12 for $\Delta W$ can be easily shown by graphs.
One can view the above properties are the multipartite generalization of the holographic properties of bipartite entanglement wedge cross-section. Motivated by the same properties of $\Delta_W$ and $\Delta_P$, we now make a conjecture that, the multipartite entanglement wedge cross-section $\Delta_W$ we defined in this section is nothing but the holographic counterpart of the multipartite entanglement of purification $\Delta_P$ we defined in the last section (at the leading order $O(N^2)$):

$$\Delta_W = \Delta_P .$$  \hspace{1cm} (3.18)

### 3.3 Computation of $\Delta_W$ in pure AdS$_3$

Now we compute $\Delta_W$ in the simple examples of AdS$_3$/CFT$_2$. We work in Poincaré patch, and a static ground state of a CFT$_2$ on a infinite line is described by a bulk solution with the metric

$$ds^2 = \frac{dx^2 + dz^2}{z^2} , \quad x \in (-\infty, +\infty), z \in [0, +\infty) .$$  \hspace{1cm} (3.19)

The three subsystems we choose are the intervals $A = [-b, -a - r], B = [-a + r, a - r], C = [a + r, b]$, where $b > a > 0$ and $r$ is relatively small compared to both $a$ and $b$. We require that the entanglement wedge of $ABC$ is connected, as shown in Fig.3.6. Follow the definition of $\Delta_W$ given in (3.9), in this set up the problem becomes to find a triangle type configuration with the minimal length of three connected geodesics, where the ending points of the geodesics are located on 3 semi-circles separately, as shown in Fig.3.6. Since we focus on the case of 3 intervals $A, B, C$ which have a reflection symmetry $x \rightarrow -x$, the reasonable minimal configuration should also keep the reflection symmetry. This consideration reduces the problem to find a special angle $\theta$ such that the length of 3 geodesics is minimal. Then, the tripartite entanglement wedge cross-section $\Delta_W$ is given by

$$\Delta_W(A : B : C) = \min_{\theta} \left[ \frac{L(\theta)}{4G_N} \right] .$$  \hspace{1cm} (3.20)

We obtained a compact formula of the length $L$ as a function of $a, b, r$ and $\theta$, however this formula is rather complicated. We instead show numerical $\theta$ dependence of $L$ as plotted in Fig.3.7 and evaluate the special values of both $\theta$ and $L$ satisfying the minimal length condition for a given $a, b, r$.

It is also straightforward to check the properties of $\Delta_W$ studied before in this particular setup.
3.4 Computation of $\Delta W$ in BTZ black hole

Now we turn to the BTZ black holes. A planar BTZ black hole describes a 2d CFT on an infinite line at finite temperature. The metric of a fixed time slice of BTZ is given by

$$ds^2 = \frac{1}{z^2} \left( \frac{dz^2}{f(z)} + dx^2 \right), \quad f(z) = 1 - \frac{z^2}{\sqrt{\bar{H}}},$$ (3.21)
where the temperature is related to the horizon by $\beta = 2\pi z_H$. For simplicity we choose 3 subsystems $A$, $B$, $C$ as intervals $[-\ell, 0]$, $[0, \ell]$ and the remaining part of the infinite line, respectively.

![Diagram of subsystems A, B, C and z-axis](image)

**Figure 3.8:** The computation of $\Delta W$ in BTZ.

As studied in [21], the geodesic length between the boundary and the horizon is

$$L_1 = \log \frac{\beta}{\pi \epsilon},$$  \hspace{1cm} (3.22)

where $\epsilon$ is the UV cutoff. The geodesic length between $(-\ell, 0)$ and $(0, 0)$ is \(^6\)

$$L_2 = 2 \log \frac{\beta \sinh(\pi \ell / \beta)}{\pi \epsilon},$$  \hspace{1cm} (3.23)

and the geodesic length between $(-\ell, 0)$ and $(\ell, 0)$ is

$$L_3 = 2 \log \frac{\beta \sinh(2\pi \ell / \beta)}{\pi \epsilon}.$$  \hspace{1cm} (3.24)

At high temperature, $\Sigma_{ABC}$ consists of six short lines of length $L_1$, so the tripartite cross-section is given by

$$\Delta W = 6L_1 = 6 \log \frac{\beta}{\pi \epsilon} =: A^{(1)} .$$  \hspace{1cm} (3.25)

At low temperature, $\Sigma_{ABC}$ consists of 3 geodesic lines connecting $(-\ell, 0)$, $(0, 0)$ and $(\ell, 0)$, so the $\Delta W$ is given by

$$\Delta W = 2L_2 + L_3 = 4 \log \frac{\beta \sinh(\pi \ell / \beta)}{\pi \epsilon} + 2 \log \frac{\beta \sinh(2\pi \ell / \beta)}{\pi \epsilon} =: A^{(2)} .$$  \hspace{1cm} (3.26)

\(^6\)It is the same as the geodesic length between $(0, 0)$ and $(\ell, 0)$. 

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The tripartite entanglement wedge cross-section $\Delta_W$ is thus given by
\[
\Delta_W(A : B : C) = \frac{1}{4G_N} \min[A^{(1)}, A^{(2)}].
\] (3.27)

We compare $A^{(1)}$ and $A^{(2)}$ and find the critical temperature
\[
\beta_* = \frac{\log \sqrt{y_*}}{\pi \ell},
\] (3.28)
where $y_*$ is the positive root of
\[
(y + y^{-1} - 2)(y - y^{-1}) = 8.
\] (3.29)

Furthermore, one can confirm that there are only two phases $A^{(1)}, A^{(2)}$ separated by the above critical temperature. When $\beta < \beta_*$, $A^{(1)}$ is favored and when $\beta > \beta_*$, $A^{(2)}$ is favored.

### 4 The multipartite squashed entanglement and holography

There have been a lot of measures of genuine quantum entanglement proposed. In particular, the squashed entanglement [39, 40] is the most promising measure of quantum entanglement for mixed states, as it satisfies all known desirable properties e.g. additivity. The squashed entanglement is defined by
\[
E_{sq}(\rho_{AB}) := \frac{1}{2} \inf_{\rho_{ABE}} I(A : B|E),
\] (4.1)
where $I(A : B|E) = I(A : BE) - I(A : E) = S_{AE} + S_{BE} + S_{ABE} - S_E$ is the conditional mutual information, and the minimization is taken over all possible extensions $\rho_{AB} = \text{Tr}_E[\rho_{ABE}]$.

In [34, 35], multipartite generalizations of $E_{sq}$ were also introduced. To define the one we are interested in, we first introduce a conditional multipartite mutual information,
\[
I(A_1 : \cdots : A_n|E) = I(A_1 : A_2|E) + I(A_1A_2 : A_3|E) + \cdots + I(A_1 \cdots A_{n-1} : A_n|E).
\] (4.2)

Using this, the ($q$-)multipartite squashed entanglement for $n$-partite states $\rho_{A_1 \cdots A_n}$ is defined by
\[
E_{sq}^q(A_1 : \cdots : A_n) := \inf_{\rho_{A_1 \cdots A_nE}} I(A_1 : \cdots : A_n|E),
\] (4.3)
where the minimization is taken over all possible extensions of $\rho_{A_1 \cdots A_n}$. 21
Noting that a trivial extension \( \rho_{A_1 \cdots A_n E} = \rho_{A_1 \cdots A_n} \otimes |0\rangle \langle 0|_E \) gives \( I(A_1 : \cdots : A_n | E) = I(A_1 : \cdots : A_n) \), we get a generic bound,

\[
E_{sq}^n(A_1 : \cdots : A_n) \leq I(A_1 : \cdots : A_n). \tag{4.4}
\]

Combining it with the proposition 8, we get a generic order of three types of measures of multipartite correlations.

**Corollary 13.** It holds for any \( n \)-partite quantum states that

\[
E_{sq}^n(A_1 : \cdots : A_n) \leq I(A_1 : \cdots : A_n) \leq \Delta_P(A_1 : \cdots : A_n). \tag{4.5}
\]

Note that for any pure \( n \)-partite states these bounds are saturated and we have

\[
E_{sq}^n = I = \Delta_P.
\]

Generally, \( E_{sq}^n \leq \Delta_P \) is a desirable property, since \( \Delta_P \) is expected to be a measure of both quantum and classical correlations while \( E_{sq}^n \) is only of quantum ones.

### 4.1 Holographic counterpart of \( E_{sq} \)

The definition of squashed entanglement (4.1) is similar to that of the entanglement of purification (2.1). Both of them use a certain type of extension, indeed, purification is a special set of extension. This observation motivates us to seek for a holographic counterpart of \( E_{sq} \) in the same spirit of \( E_P \) or \( \Delta_P \).

Let us regard a time slice of AdS as a tensor network which describes a quantum state of CFT. In a gravity background with a tensor network description, one can define a pure or mixed state for any codimension two convex surface, called the surface/state correspondence [15, 16, 17]. Let us now take use of this picture to study what the holographic counterpart of squashed entanglement could be if it indeed exists\(^7\).

First, we assume that the optimal extension from \( \rho_{AB} \) to \( \rho_{ABE} \) has a classical geometric description. Then we found all the geometric extensions with a nontrivial \( E \) cause \( I(A : B | E) \geq I(A : B) \). We illustrate our graph proof of \( I(A : B | E) \geq I(A : B) \) in Fig.4.1 , Fig.4.2. This eventually makes that the optimal extension is given by the trivial one if it is indeed geometrised, which leads to

\[
E_{sq}(A : B) = \frac{1}{2} I(A : B). \tag{4.6}
\]

This tells that the upper bound of \( E_{sq}(A : B) \) is saturated in terms of mutual information in holography.

\(^7\)We are grateful to Tadashi Takayanagi for lots of fruitful comments about the following discussion.
Figure 4.1: Proof of $I(A : B|E) > I(A : B)$ for the extension $\rho_{AB}$ to $\rho_{ABE}$ on the purified geometry of the entanglement wedge. One can easily check that $I(A : B|E) - I(A : B)$ in this case is given by the orange lines minus the blue lines, which is certainly positive. Notice that here we consider a small $E$, but the result holds for large $E$ (near the purification).

Figure 4.2: Proof of $I(A : B|E) > I(A : B)$ for the extension $\rho_{AB}$ to $\rho_{ABE}$ on the original boundary geometry. One can check that in this case $I(A : B|E) - I(A : B)$ is given by dashed orange lines minus solid blue lines, which is positive. Here we also consider small $E$, but the result holds for large $E$.  

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The saturation in holography is also implicated in [30], there it was shown that the mutual information in holographic CFTs always satisfies the so-called monogamy relation

\[ I(A : BC) \geq I(A : B) + I(A : C), \tag{4.7} \]

for any tripartite states. The proof is based on holographic properties of CFTs, as it is known that it can be violated in generic quantum systems.

On the other hand, it is well known that the squashed entanglement does satisfy this monogamy property for any quantum system [41]. We consider this observation as another evidence for \( E_{sq} = I/2 \).

Now, we expect that this kind of saturation also happens for multipartite cases due to the same reason in bipartite cases. Namely, we are tempted to test the saturation of (4.4) in holography,

\[ E_{sq}^n(A_1 : \cdots : A_n) \geq I(A_1 : \cdots : A_n). \tag{4.8} \]

One such test is the strong superadditivity of multipartite squashed entanglement [34]:

\[ E_{sq}^n(A_1B_1 : \cdots : A_nB_n) \geq E_{sq}^n(A_1 : \cdots : A_n) + E_{sq}^n(B_1 : \cdots : B_n). \tag{4.9} \]

This is true in any \( 2n \)-partite state \( \rho_{A_1B_1A_2B_2A_3B_3} \) for the multipartite squashed entanglement. Note that the multipartite mutual information does not satisfy this property in general. For example, let us consider a quantum state

\[ \rho_{A_1B_1A_2B_2A_3B_3} = \frac{1}{\sqrt{2}}(|000000\rangle \langle 000000|_{A_1B_1A_2B_2A_3B_3} + |111111\rangle \langle 111111|_{A_1B_1A_2B_2A_3B_3}), \tag{4.10} \]

This will lead to \( I(A_1B_1 : A_2B_2 : A_3B_3) = I(A_1 : A_2 : A_3) = I(B_1 : B_2 : B_3) \), which clearly violates the bound.

However, in holographic CFTs, one can show that the multipartite mutual information does satisfy the strong superadditivity. This comes as follows:

\[
I(A_1B_1 : \cdots : A_nB_n) = I(A_1B_1 : A_2B_2) + I(A_1B_1A_2B_2 : A_3B_3) + \cdots + I(A_1B_1A_2B_2 \cdots A_{n-1}B_{n-1} : A_nB_n) \\
\geq I(A_1 : A_2) + I(A_1A_2 : A_3) + \cdots + I(A_1A_2 \cdots A_{n-1} : A_n) \\
+ I(B_1 : B_2) + I(B_1B_2 : B_3) + \cdots + I(B_1B_2 \cdots B_{n-1} : B_n) \\
= I(A_1 : \cdots : A_n) + I(B_1 : \cdots : B_n), \tag{4.11}
\]

where we used the monogamy of holographic mutual information recursively. This observation therefore gives further support for our conjectured saturation in holography (4.8), or generally speaking, for the pictures of purified geometry.
To our best knowledge, there is no counter example for such saturation conjecture. If there is a property which $E_{sq}$ always satisfies but $I$ does not always, then one can try to test whether it holds for holographic mutual information. If it does, this can be considered as additional positive evidence for our conjecture.

5 Conclusion

In this paper, we defined a generalization of entanglement of purification to multipartite states denoted by $\Delta_P$, and proved its various properties focusing on bounds described by other entropic quantities. We demonstrate that $\Delta_P$ satisfies desired properties as a generalization of $E_P$ and provides an upper bound on the multipartite mutual information introduced in [34, 35]. In particular, we show that the tripartite entanglement of purification $\Delta_P(A : B : C)$ is bounded from below by a sum of two different generalizations of mutual information. We also show that for a class of tripartite quantum states which saturate the subadditivity or the strong subadditivity, there is a closed expression of $\Delta_P$ in terms of entanglement entropy.

Based on the holographic conjecture of entanglement of purification [21, 22], we defined a generalization of entanglement wedge cross-section as the holographic counterpart of the multipartite entanglement of purification $\Delta_W$, and show that all the properties $\Delta_P$ has are indeed satisfied by $\Delta_W$. It leads us to propose a new conjecture $\Delta_P = \Delta_W$ at the leading order $O(N^2)$ of large $N$ limit. Alternatively speaking, this implies the naive picture of purified geometry based on tensor network description [15, 16, 17] still works for multipartite cases. As explicit examples, we calculated $\Delta_W$ for several simple setups in AdS$_3$/CFT$_2$ including pure AdS$_3$ and black hole geometry.

Last but not least, we clarified the relation between our new measure of total multipartite correlation and purely quantum one. The latter is quantified by the so called multipartite ($q$-)squashed entanglement, first introduced in [34, 35]. We show that the multipartite squashed entanglement is bounded from above by the multipartite mutual information, thus by the multipartite entanglement of purification too. This is actually a desired property of our $\Delta_P$ because total correlation including both classical and quantum ones should be greater than the purely quantum one. We also studied the validity of the purified geometry picture in holography by proposing a test, examining the relation between mutual information and squashed entanglement in holographic CFTs. Our multipartite investigations between different measures eventually sharpen this proposal.

Several future questions are in order: First, proof of $\Delta_W$ properties in time-dependent background geometry. The properties in this case are expected to be

\footnote{Note that related properties about holographic entanglement entropy are intensively studied in [44] for multipartite setups.}
very useful in understanding the dynamical process in quantum gravity systems, since many interesting physical set up go beyond the bi-partite pattern. The entropic inequalities for multipartite cases were actually discussed in [45] for covariant cases, there they argued that new techniques apart from the maximin surfaces [37] are needed to prove them for 5 or more partite cases. Second, we expect that there is an operational interpretation for $\Delta_P$ just like $E_P$ has, such as the one based on SLOCC for multipartite qubits. Third, looking for the new properties satisfied by $\Delta_W$ but not by $\Delta_P$ is certainly interesting, because these are essentially the new constraints on holographic states, such as the conjectured strong superadditivity of entanglement of purification in holography. We shall report the progress in future publications.

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