Super-Easy Quantum Groups with Complex Parameters

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Abstract. We discuss an extended easy quantum group formalism, with the Schur-Weyl theoretic Kronecker symbols being allowed to take values in $\mathbb{T} \cup \{0\}$. Our study includes an axiomatization of the theory, some structure and classification results, and the study of the basic examples of quantum unitary and reflection groups. We comment as well on some related questions, of algebraic and probabilistic nature.

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Introduction

It is well-known that any compact Lie group appears as a closed subgroup of a unitary group, $G \subset U_N$. Moreover, such subgroups are described by their Tannakian categories $C = (C(k, l))$, with $C(k, l) = Hom(u^\otimes k, u^\otimes l)$, where $u$ is the fundamental representation. One way of understanding the Tannakian correspondence $G \leftrightarrow C$, which is functional analytic and algebraic geometric in nature, is via the following formula:

$$C(G) = C(U_N) \left/ \langle T \in Hom(u^\otimes k, u^\otimes l), \forall k, l, \forall T \in C(k, l) \rangle \right.$$ 

Thanks to the results of Woronowicz in [43], [44], as modified by the contributions of Wang [37] and Malacarne [27], a quantum extension of this is available. To be more

2010 Mathematics Subject Classification. 46L65 (46L54).

Key words and phrases. Quantum Lie group, Super-easiness.
precise, the unitary group \( U_N \) has a free analogue \( U_N^+ \), the closed subgroups \( G \subset U_N^+ \) satisfy what one would expect from the “compact quantum Lie groups”, and a Tannakian correspondence \( G \leftrightarrow C \) constructed as above, with \( U_N \) replaced by \( U_N^+ \), holds indeed.

There are many interesting closed subgroups \( G \subset U_N^+ \), and generally speaking, these do not have an analogue of a Lie algebra. The methods for studying them are of algebraic geometric and probabilistic nature, based on the Tannakian duality \( G \leftrightarrow C \). Let us mention here for instance the Weingarten formula, which allows one to compute the integrals of type \( \int_G u_{i_1j_1} \cdots u_{i_pj_p} \), and so to effectively do probability theory over \( G \), once linear bases of each \( C(k, l) \) are known. For an introduction to this, see [5].

Getting back now to the classical case, \( G \subset U_N \), in the continuous case the Lie theory rules, and the traditional way of measuring the “complexity” of \( G \) is via the complexity of the associated Lie algebra. Indeed, when using the Lie theory for the study of \( G \), the groups of type \( A \) are of course “simpler” than those of type \( BCD \), and so on.

From the Tannakian viewpoint, however, all this is quite irrelevant, and some new criteria are needed, in order to judge the complexity of \( G \). In the classical case, the problem can be approached by having the Weingarten formula, and some related potential applications, in mind. To be more precise, the “simplest” compact groups \( G \subset U_N \) should be those having a simple Tannakian category \( C \), featuring explicit combinatorial bases for each \( C(k, l) \) space. This was the point of view of Collins and Śniady in [19].

In the quantum case now, \( G \subset U_N^+ \), the same philosophy can be applied. In addition, the problem is conceptually simpler here than in the classical case, because there is no temptation to use Lie algebras, and other differential geometry methods. This was the idea behind the notion of “easiness”, axiomatized in our paper with Speicher [11].

The definition of the easiness rests on the well-known fact that for the symmetric group \( S_N \), viewed as algebraic group, \( S_N \subset U_N \), via the standard permutation matrices, we have \( C = \text{span}(T_\pi | \pi \in P) \), where \( P = (P(k, l)) \) is the set of two-row partitions, and:

\[
T_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1 ... j_l} \delta_\pi \left( i_1 \ldots i_k \atop j_1 \ldots j_l \right) e_{j_1} \otimes \cdots \otimes e_{j_l}
\]

Here \( e_1, \ldots, e_N \) is the standard basis of \( \mathbb{C}^N \), and \( \delta_\pi \in \{0, 1\} \) is a Kronecker symbol, equal to 1 when each block of \( \pi \) contains equal indices, and equal to 0 otherwise.

Now since the Tannakian correspondence \( G \leftrightarrow C \) is contravariant, when assuming \( S_N \subset G \), the Tannakian category of \( G \) appears as a subcategory \( C \subset \text{span}(T_\pi | \pi \in P) \). In other words, the assumption \( S_N \subset G \) brings us into the realm of combinatorics.

Based on this observation, a quantum group \( S_N \subset G \subset U_N^+ \) can be called “easy” when its Tannakian category appears in the simplest possible way, namely \( C = \text{span}(T_\pi | \pi \in D) \), for a certain category of partitions \( D \subset P \). With this idea in mind, and by using technical ingredients from a long series of preliminary papers, mostly written by Bichon, Collins and us, the theory was axiomatized in [11], and studied in subsequent papers.
Going beyond easiness is quite a tricky task, and there have been numerous attempts here over the last years, notably by Freslon, Weber and al. [18], [22], [23], [31].

Philosophically speaking, there is a bit of a dilemma in dealing with this problem, in relation with the “direction” in which easiness should go. One would like of course to cover more examples, and these examples can only come from:

1. Classical groups. Bluntly put, as long as there are basic compact Lie groups $G \subset U_N$ which are not easy, in some generalized sense, the story is not over.
2. Group duals. The situation here is a priori a bit similar, with the big difference, however, that the world of discrete groups is extremely wide.
3. Quantum algebra. The world of easy quantum groups $G \subset U^+_N$ can be of course enlarged by using various abstract generalization methods.
4. Quantum physics. A main objective here would be to cover the examples $G \subset U^+_N$ which play an important role in physics. These are not known yet.

These methods are all quite powerful, and can be used in order to reach to concrete advances. Our personal belief is that all of them are useful.

The present paper is a continuation of our work on the subject, vaguely privileging (1) above. Our idea is that of allowing the Kronecker symbols to take complex values:

$$\delta_\pi \left( \begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_k \\ j_l \end{array} \right) \in \mathbb{T} \cup \{0\}$$

The origins of this approach go back to our work with Collins, Zinn-Justin [7], Skalski [10], to our paper [1], and our work with Nechita [9], the story being as follows:

1. The paper [7] was concerned with the fine probabilistic study of the standard coordinates $u_{ij} \in C(O^+_N)$. This technically requires the study of the deformations of $O^+_N$ as well, as constructed in [35], and in particular of $Sp^+_N$. Such deformations are easy in some generalized sense, further explained in [16].
2. The paper [10], and its recent continuation [3], were concerned with the general orthogonal case, where $u \sim \bar{u}$, and $u$ is irreducible. The equivalence $u \sim \bar{u}$ must come from a certain “super-structure” on $\mathbb{C}^N$, and this makes appear a generalized easy formalism, with signed Kronecker symbols, $\delta_\pi \in \{-1, 0, 1\}$.
3. The paper [1] was concerned with the $q = -1$ twisting of the basic quantum groups $O_N, O^+_N, U_N, U^+_N$ and of some other easy quantum groups. Once again, this made appear a generalized easy formalism, with signed Kronecker symbols, $\delta_\pi \in \{-1, 0, 1\}$, but which is different from the one in (2) above.
4. The paper [9] was concerned with the block-modification of the Wishart matrices, by using “easy maps”, of type $T_\pi$. In a recent follow-up, [3], the signed maps as in (3) above were tried, and the conclusion was that, in connection with certain Wishart matrix questions, an extension to $\delta_\pi \in \mathbb{T} \cup \{0\}$ is needed.
Summarizing, we have here several interesting generalizations of the easy formalism, partly exiting the quantum group world, and having as common feature the fact that the Kronecker symbols belong to $\mathbb{T} \cup \{0\}$. Thus, an axiomatization problem appears.

We will discuss here this axiomatization question, by using certain generalized Kronecker functions, valued in $\mathbb{T} \cup \{0\}$. At the concrete level, we will present two quite general constructions of such Kronecker functions, as follows:

(1) A first class of examples comes from “signature” functions $\varepsilon \in \mathbb{Z}_s$, which are by definition such that the map $\delta_\pi \varepsilon$ produces, via the usual formula for $T_\pi$, a correspondence $\pi \to T_\pi$ which is categorical. In the simplest non-trivial case, $s = 2$ and $\tau = \text{id}$, we recover in this way the signature function from [1].

(2) A second class of examples, once again depending on a number $s \in \mathbb{N}$, comes by using permutations $\tau \in S_N$ satisfying $\tau^s = \text{id}$. We can indeed construct a certain Kronecker function $\delta_\tau^s \in \{0, 1\}$, which at $s = 1$ is the usual one from [11], at $s = 2$ is the one from [3, 7, 10], and at general $s \in \mathbb{N}$ is related to [4, 9].

These two constructions generalize and unify all the known examples. However, our study is not exactly complete, because these two constructions still need to be unified. In addition, the unification with other known extensions of the easy quantum group formalism, which were partly mentioned above, remains as well an open problem.

The paper is organized as follows: 1-2 contain various preliminaries and generalities, in 3-4 we extend the easy formalism by using a non-trivial signature map $\varepsilon \in \mathbb{T}$, in 5-6 we review the super-easy constructions, and in 7-8 we perform the unification, and we discuss the general theory, the main examples, and the probabilistic aspects.

Acknowledgements. I would like to thank Pisunaka and her friends, for sharing with me some of their knowledge, and for general advice and support.

1. Quantum groups

We use Woronowicz’s quantum group formalism in [13, 14], under the extra assumption $S^2 = \text{id}$. To be more precise, the definition that we will need is:

**Definition 1.1.** Assume that $(A, u)$ is a pair consisting of a C*-algebra $A$, and a unitary matrix $u \in M_N(A)$ whose coefficients generate $A$, such that the formulae

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^*$$

define morphisms of C*-algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$, $S : A \to A^{\text{opp}}$. We write then $A = C(G)$, and call $G$ a compact matrix quantum group.
The basic examples are the compact Lie groups, \( G \subset U_N \). Indeed, given such a group we can set \( A = C(G) \), and let \( u_{ij} : G \to \mathbb{C} \) be the standard coordinates, \( u_{ij}(g) = g_{ij} \). The axioms are then satisfied, with \( \Delta, \varepsilon, S \) being the functional analytic transposes of the multiplication \( m : G \times G \to G \), unit map \( u : \{ . \} \to G \), and inverse map \( i : G \to G \).

The following key construction is due to Wang [37]:

**Proposition 1.2.** We have a compact quantum group \( U_N^+ \), defined via

\[
C(U_N^+) = C^* \left( (u_{ij})_{i,j=1,...,N} \mid u^* = u^{-1}, u^t = \bar{u}^{-1} \right)
\]

and the \( N \times N \) compact quantum groups are precisely the closed subgroups \( G \subset U_N^+ \).

**Proof.** It is routine to check that if \( u = (u_{ij}) \) is biunitary \( (u^* = u^{-1}, u^t = \bar{u}^{-1}) \), then so are the matrices \( u^\Delta = \left( \sum_k u_{ik} \otimes u_{kj} \right) \), \( u^\varepsilon = (\delta_{ij}) \), \( u^S = (u^*_{ji}) \). Thus we can construct \( \Delta, \varepsilon, S \) as in Definition 1.1, by using the universal property of \( C(U_N^+) \).

Regarding the last assertion, in the context of Definition 1.1 we have \( u^* = u^{-1} \), and by applying \( S \) we obtain \( u^t = \bar{u}^{-1} \). Thus \( u \) is biunitary, so we have a quotient map \( C(U_N^+) \to C(G) \), which corresponds to an inclusion of quantum groups \( G \subset U_N^+ \). \( \square \)

We will need Woronowicz’s Tannakian duality [44], as modified in [27]:

**Proposition 1.3.** The closed subgroups \( G \subset U_N^+ \) are in correspondence with their Tannakian categories, \( C = (C(k,l)) \) with \( C(k,l) = \text{Hom}(u^{\otimes k}, u^{\otimes l}) \), via the construction

\[
C(G) = C(U_N^+) / \left\langle T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall k,l, \forall T \in C(k,l) \right\rangle
\]

where we use colored integers as exponents, with the conventions \( u^o = u, u^* = \bar{u} \) and multiplicativity, and we agree to identify the full and reduced quantum group algebras.

**Proof.** Given a closed subgroup \( G \subset U_N^+ \), it follows from the results in [43] that the spaces \( C(k,l) = \text{Hom}(u^{\otimes k}, u^{\otimes l}) \) form indeed a Tannakian category, in the sense that:

1. \( C \) is stable by composition.
2. \( C \) is stable by tensor products.
3. \( C \) is stable by involution.
4. \( C(\circ, \circ) \) and \( C(\bullet, \bullet) \) contain the identity map \( e_i \to e_i \).
5. \( C(\ldots, \circ \bullet) \) and \( C(\ldots, \bullet \circ) \) contain the duality map \( 1 \to \sum_i e_i \otimes e_i \).

Conversely, given a collection of spaces \( C = (C(k,l)) \), the formula in the statement defines a certain closed subgroup \( G \subset U_N^+ \). The problem is that of proving that when \( C \) is the Tannakian category of \( G \), the quantum group that we obtain is \( G \) itself.

Thus, we are left with proving that the equality in the statement holds indeed, with \( C(k,l) = \text{Hom}(u^{\otimes k}, u^{\otimes l}) \). The fact that we have an arrow from right to left is clear. As for the reverse arrow, its existence follows either from [43], or from [27]. \( \square \)

For \( k, l \) being colored integers, let \( P(k,l) \) be the set of partitions between an upper row of points representing \( k \), and a lower row of points representing \( l \). We have:
Definition 1.4. A category of partitions is a collection of subsets $D(k, l) \subset P(k, l)$ which contains the identity and duality partitions $\mathbb{1}$, $\mathbb{1}$, $\mathbb{1}$, and is stable under:

1. The horizontal concatenation operation, $(\pi, \sigma) \rightarrow [\pi \sigma]$.
2. The vertical concatenation, with the middle components erased, $(\pi, \sigma) \rightarrow [\sigma \pi]$.
3. The upside-down turning, with switching of the colors, $\pi \rightarrow \pi^*$.

Also, given $\pi \in P(k, l)$ and multi-indices $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_l)$, we set $\delta_{\pi}(i, j) = 1$ when each block of $\pi$ contains equal indices, and $\delta_{\pi}(i, j) = 0$ otherwise.

We have now all the needed ingredients for formulating the founding result and usual definition of the easiness, from [11], in their modified unitary version from [33]:

Proposition 1.5. Given a category $D \subset P$, the formula $C = \text{span}(T_\pi | \pi \in D)$, where

$$T_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, \ldots, j_l} \delta_{\pi}(i_1, \ldots, i_k, j_1, \ldots, j_l) e_{j_1} \otimes \cdots \otimes e_{j_l}$$

defines a closed subgroup $G \subset U_N^+$. We call such quantum groups easy.

Proof. As explained in [11], the construction $\pi \rightarrow T_\pi$ has the following properties, where $c(\pi)$ denotes the number of components in the middle, when concatenating:

$$T_{[\pi \sigma]} = T_\pi \otimes T_\sigma, \quad T_{[\pi]} = N^{c(\pi)} T_\pi T_\sigma, \quad T_\pi^* = T_\pi^*$$

In addition, the partitions $\pi = |, \cap$ produce the identity and duality maps. Thus $C$ is a Tannakian category, and the result follows from Proposition 1.3.

In order to discuss now the main examples, let $O_N^+$ be the real version of $U_N^+$, constructed by assuming $u_{ij} = u_{ij}^*$. Consider also the hyperoctahedral group $H_N = S_N \wr \mathbb{Z}_2$ and its complex version $K_N = S_N \wr \mathbb{T}$, as well as their free analogues $H_N^+ = S_N \wr \mathbb{Z}_2$ and $K_N^+ = S_N \wr \mathbb{T}$. See [5], [6].

These quantum groups have some remarkable connections between them:

Proposition 1.6. The basic quantum unitary groups and quantum reflection groups, with the inclusions between them, are as follows:

Moreover, this is an intersection diagram, in the sense that for any of its subsquare diagrams $A \subset B, C \subset D$ we have $\bigcap A = B \cap C$. 
Proof. This follows by comparing the defining relations of the various quantum groups involved. For full details regarding this material, we refer to [5]. □

In order to discuss the easiness properties of these quantum groups, let $P_{\text{even}}$ be the category of partitions all whose blocks have even size, and consider its subcategories $P_2, P_{\text{even}}, NC_{\text{even}}$ consisting respectively of the pairings, of the partitions which are “matching”, in the sense that we have $\# \circ = \# \bullet$ in each block, when the upper legs are counted $+$, and the lower legs are counted $-$, and of the noncrossing partitions.

By pairwise intersecting these subcategories we obtain 3 more categories, denoted $P_2, NC_2, NC_{\text{even}}$, and by intersecting all of them we obtain one more category, $NC_2$.

We can now state the main result regarding the examples, as follows:

**Theorem 1.7.** The basic quantum unitary groups and quantum reflection groups are all easy, the corresponding categories of partitions being as follows:

\[ \begin{array}{c}
\downarrow & \downarrow & \downarrow \\
P_2 & \rightarrow & NC_2 \\
\downarrow & \downarrow & \downarrow \\
P_{\text{even}} & \rightarrow & NC_{\text{even}} \\
\end{array} \]

Both this diagram and the quantum group one are intersection/generation diagrams, in the sense that for any subsquare $A \subset B, C \subset D$ we have $A = B \cap C, D = < B, C >$.

Proof. According to some old results of Brauer [17], the groups $O_N, U_N$ correspond indeed, via Tannakian duality in our sense, to the categories $P_2, P_2$. Regarding now the various versions of $O_N, U_N$, obtained by liberation, or by taking the corresponding quantum reflection subgroups, or both, the proof is quite similar. To be more precise, the idea in each case is that of writing the defining relations in the form $T_\pi \in Hom(u^k, u^l)$ with $\pi \in P(k, l)$, and then by computing the categories generated by these partitions $\pi$.

As for the last assertion, this is clear for the diagram of categories, and for the diagram of quantum groups this follows by Tannakian duality, because the $\cap$ operation for categories corresponds to the $<, >$ operation for quantum groups, and vice versa.

For full details regarding this material, we refer to the lecture notes [5]. □

There are of course many other interesting examples of easy quantum groups, the first of which are the symmetric group $S_N$ and its free analogue $S^+_N$, from [38], which correspond respectively to $P$ itself, and to the category of noncrossing partitions $NC$.

However, for the purposes of this paper, where we will often need the assumption $D \subset P_{\text{even}}$, the above 8 examples are those that we will be mainly interested in.
We refer to the classification articles [30], [33] and to the lecture notes [5] for lists of further known examples, and for the status of the classification problem.

2. Generalized easiness

There are several possible ways of generalizing the easy quantum group formalism. The most straightforward one is by modifying the formula of $\pi \rightarrow T_\pi$, namely:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi (i_1 \ldots i_k) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Indeed, $\pi$ can be replaced by more general combinatorial objects, and the indices can be replaced as well by more general algebraic quantities. See [18], [22], [23], [31].

In what follows, our idea will be that of replacing the Kronecker symbols $\delta_\pi \in \{0,1\}$ by more general quantities $\tilde{\delta}_\pi \in \mathbb{T} \cup \{0\}$. Our motivation comes from the symplectic group $Sp_N$, which is covered by such a formalism, with $\tilde{\delta}_\pi \in \{-1,0,1\}$, and by a number of more technical examples, of “quantum” nature, which suggest using $\delta_\pi \in \mathbb{T} \cup \{0\}$.

Let us first work out the needed basic algebra. We first have:

**Definition 2.1.** A generalized Kronecker function on a category of partitions $D \subset P$ is a collection of numbers $\tilde{\delta}_{\pi}(i_j) \in \mathbb{T} \cup \{0\}$, with $\pi \in D$, such that the formula

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \tilde{\delta}_\pi (i_1 \ldots i_k) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

defines a correspondence $\pi \rightarrow T_\pi$ which is categorical, in the sense that we have

$$T_{[\pi \sigma]} = T_\pi \otimes T_\sigma, \quad T_{[\sigma \pi]} = N^{c(\pi)}T_\pi T_\sigma, \quad T_{\pi^*} = T_\pi^*$$

as well as $T_{\emptyset} = T_{\emptyset}^* = id$, and $T_{\bullet \bullet} = T_{\bullet \bullet}^* = (1 \rightarrow \sum_i e_i \otimes e_i)$.

To be more precise here, $N \in \mathbb{N}$ is given, and if we assume $\pi \in D(k,l)$, then the above numbers $\tilde{\delta}_{\pi}(i_j) \in \mathbb{T} \cup \{0\}$ must be defined for any multi-indices $i,j$ having respective lengths $k,l$, and taking their individual indices from the set $\{1, \ldots , N\}$.

In more concrete terms now, we have the following description:

**Proposition 2.2.** $\tilde{\delta} : D \rightarrow \mathbb{T} \cup \{0\}$ is a generalized Kronecker function when

$$\tilde{\delta}_\pi (i_1 \ldots i_p) \tilde{\delta}_\sigma (k_1 \ldots k_r) = \tilde{\delta}_{[\pi \sigma]} (i_1 \ldots i_p k_1 \ldots k_r)$$

$$\sum_{j_1 \ldots j_q} \tilde{\delta}_\sigma (i_1 \ldots i_p) \tilde{\delta}_\pi (j_1 \ldots j_q) = N^{c(\sigma)} \tilde{\delta}_{[\sigma \pi]} (i_1 \ldots i_p j_1 \ldots j_q)$$

$$\tilde{\delta}_\pi (i_1 \ldots i_p) = \tilde{\delta}_{\pi^*} (i_1 \ldots i_p)$$

and when $\tilde{\delta}_\pi = \delta_\pi$ for $\pi = \emptyset, \bullet, \bigcirc, \bigtriangleup$.  

Proof. The proof here is routine, as in [11]. We include it, in view of some further use, later on. The concatenation axiom follows from the following computation:

\[
(T_\pi \otimes T_\sigma)(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r})
= \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \bar{\delta}_\pi(i_1 \ldots i_p) \bar{\delta}_\sigma(k_1 \ldots k_r) e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s}
= \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \bar{\delta}_{[\pi\sigma]}(i_1 \ldots i_p \; k_1 \ldots k_r) e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s}
= \bar{T}_{[\pi\sigma]}(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r})
\]

The composition axiom follows from the following computation:

\[
\bar{T}_\pi \bar{T}_\sigma(e_{i_1} \otimes \ldots \otimes e_{i_p})
= \sum_{j_1 \ldots j_q} \bar{\delta}_\sigma(i_1 \ldots i_p) \sum_{k_1 \ldots k_r} \bar{\delta}_\pi(j_1 \ldots j_q \; k_1 \ldots k_r) e_{k_1} \otimes \ldots \otimes e_{k_r}
= \sum_{k_1 \ldots k_r} N^{c(z)} \bar{\delta}_{[\sigma]}(i_1 \ldots i_p \; k_1 \ldots k_r) e_{k_1} \otimes \ldots \otimes e_{k_r}
= N^{c(z)} \bar{T}^p_{[\sigma]}(e_{i_1} \otimes \ldots \otimes e_{i_p})
\]

The involution axiom follows from the following computation:

\[
\bar{T}_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q})
= \sum_{i_1 \ldots i_p} \langle \bar{T}_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q}), e_{i_1} \otimes \ldots \otimes e_{i_p} \rangle \geq 0, \quad e_{i_1} \otimes \ldots \otimes e_{i_p} > e_{i_1} \otimes \ldots \otimes e_{i_p}
= \sum_{i_1 \ldots i_p} \bar{\delta}_\pi(i_1 \ldots i_p \; j_1 \ldots j_q) e_{i_1} \otimes \ldots \otimes e_{i_p}
= \bar{T}_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q})
\]

Finally, the identity and duality axioms follow from the last conditions in the statement. As for the converse, this follows as well from the above computations. \(\square\)

At the quantum group level now, we are led to:

**Definition 2.3.** Given a category of partitions \(D \subset P\), and a generalized Kronecker function \(\tilde{\delta} : D \rightarrow T \cup \{0\}\), the formula \(C = \text{span}(\bar{T}_\pi | \pi \in D)\), where

\[
T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi(i_1 \ldots i_k \; j_1 \ldots j_l) e_{j_1} \otimes \ldots \otimes e_{j_l}
\]

defines a closed subgroup \(\bar{G} \subset U_N^+\). We call such quantum groups generalized easy.
Here we have used of course Woronowicz’ Tannakian duality results from [44], in their “soft” form from Proposition 1.3, as in the proof of Proposition 1.5 above.

As basic examples, we have of course the usual easy quantum groups $S_N \subset G \subset U_N^+$. We will see in what follows that there are many other interesting examples.

In order to compute such quantum groups, we will use the following result:

**Proposition 2.4.** Given a Kronecker function $\bar{\delta} : D \to \mathbb{T} \cup \{0\}$ as above, and assuming $D = \langle \pi_1, \ldots, \pi_r \rangle$ with $\pi_i \in D(k_i, l_i)$, the associated quantum group is given by

$$C(\bar{G}) = C(U_N^+)/\left\langle \bar{T}_{\pi_i} \in \text{Hom}(u^\otimes k_i, u^\otimes l_i) \left| i = 1, \ldots, r \right. \right\rangle$$

with the usual rules for the exponents, namely $u^o = u$, $u^* = \bar{u}$ and multiplicativity.

*Proof.* This follows indeed from Proposition 1.3 above, and from the fact that, according to Definition 2.1, the correspondence $\pi \mapsto \bar{T}_{\pi}$ is categorical.

More concretely now, we can try to compute the quantum groups associated to some basic subcategories $E \subset D$. We first have the following result:

**Proposition 2.5.** Consider a generalized Kronecker function $\bar{\delta} : D \to \mathbb{T} \cup \{0\}$.

1. We have $\bar{\delta} = \delta$ on the subcategory $NC_2 \subset D$.
2. If $D = NC_2$, the associated quantum group is $U_N^+$.

*Proof.* Observe first that we have indeed $NC_2 \subset D$, because $D$ must contain the identity and duality partitions \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), which generate $NC_2$.

1. This follows indeed from the fact, from Proposition 2.2, that we have $\bar{\delta}_\pi = \delta_\pi$ for the standard generators $\pi = \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array}$, \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \), \( \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \) of the category $NC_2$.
2. This follows from (1), and from Theorem 1.7 above.

In relation now with the full statement of Theorem 1.7, given a category of partitions $D \subset P$, we can associate to it the following categories of partitions:

Thus, we have 8 basic examples of generalized easy quantum groups, as follows:
Definition 2.6. Given a generalized Kronecker function $\tilde{\delta} : D \to \mathbb{T} \cup \{0\}$, we let

\[
\begin{array}{c}
U_N & U^+_N \\
\downarrow & \downarrow \\
\tilde{O}_N & \tilde{O}^+_N \\
\downarrow & \downarrow \\
\tilde{K}_N & \tilde{K}^+_N \\
\downarrow & \downarrow \\
H_N & H^+_N
\end{array}
\]

be the generalized easy quantum groups associated to the above 8 categories.

We already know from Proposition 2.5 above that we have $\tilde{U}_N^+ = U_N^+$. We can extend this observation, by analyzing the other quantum groups as well, and we obtain:

Theorem 2.7. The basic quantum unitary and reflection groups are as follows:

1. We have $\tilde{U}_N^+ = U_N^+$.
2. If $NC_2 \subset D$, then $\tilde{O}_N^+ \subset U_N^+$ appears from the following intertwiner:

\[
\tilde{T}_\pi(e_i) = \sum_j \tilde{\delta}_\pi \left( \begin{array}{c} i \\ j \end{array} \right) e_j : \pi = \oplus
\]

3. If $P_2 \subset D$, then $\tilde{U}_N \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi(e_i \otimes e_j) = \sum_{kl} \delta_i \left( \begin{array}{c} i \\ k \end{array} \right) e_k \otimes e_l : \pi = \ominus \oplus
\]

4. If $NC_{\text{even}} \subset D$, then $\tilde{K}_N^+ \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi(1) = \sum_i \delta_i(i i i) : \pi = \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus
\]

5. If $P_2 \subset D$ then $\tilde{O}_N \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi : \pi = \ominus \oplus \ominus \oplus
\]

6. If $NC_{\text{even}} \subset D$ then $\tilde{H}_N^+ \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi : \pi = \ominus \oplus \ominus \oplus
\]

7. If $P_{\text{even}} \subset D$ then $\tilde{K}_N \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi : \pi = \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \oplus \ominus \oplus
\]

8. If $P_{\text{even}} \subset D$ then $\tilde{H}_N \subset U_N^+$ appears from the following intertwiners:

\[
\tilde{T}_\pi : \pi = \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus \oplus \ominus
\]

In addition, the diagram formed by these quantum groups is a generation diagram.
Proof. All these assertions are elementary, and follow from some well-known presentation results for the categories in question. To be more precise:

(1) This is something that we already know, from Proposition 2.5.

(2,3) These results follow from the following well-known formulae:

\[
NC_2 = \langle \uparrow \rangle, \quad P_2 = \langle \wp, \wp \rangle
\]

(4) It is well-known that, in the uncolored setting, we have \(NC_{\text{even}} = \langle \square \rangle\). By restricting the attention to the category \(NC_{\text{even}} \subset NC_{\text{even}}\) of the partitions which are matching, it follows that this subcategory is generating by the 6 matching colorings of \(\square\). Thus, we have the following presentation formula, which gives the result:

\[
NC_{\text{even}} = \langle \square \square, \square \square \square, \square \square \square \square, \square \square \square \square \square, \square \square \square \square \square \square \rangle
\]

(5,6) These follow from the following presentations results, which are well-known:

\[
P_2 = \langle \wp, \wp \rangle, \quad NC_{\text{even}} = \langle \wp, \square \rangle
\]

(7,8) Indeed, by proceeding as in the proof of (4), we conclude, starting from the above presentation results, that we have the following presentation results as well:

\[
P_{\text{even}} = \langle \wp, \wp \rangle, \quad P_{\text{even}} = \langle \wp, \wp \rangle
\]

Finally, the last assertion is clear from Theorem 1.7, because the diagram of the categories producing our quantum groups is an intersection diagram. □

All this is of course quite theoretical. In what follows we will recall the basic examples, and enlarge the formalism, as to reach in the end to two main constructions, which are both of quite concrete nature, and cover all the known examples.

3. Abstract signatures

The basic example of a generalized Kronecker function is the usual Kronecker function \(\delta : P \rightarrow \{0, 1\}\). Besides being defined on the whole \(P\), and taking only positive values, this function \(\delta_\pi(i)\) has the remarkable property of depending only on \(\ker(i) \in P\).

This suggests formulating the following definition:

**Definition 3.1.** A generalized Kronecker function \(\tilde{\delta} : D \rightarrow \mathbb{T} \cup \{0\}\) is called pure when it is given by a formula of type \(\tilde{\delta}_\pi(i) = \varphi(\pi, [i])\), where

\[
\begin{bmatrix}
i_1 & \ldots & i_k \\
j_1 & \ldots & j_l
\end{bmatrix} = \ker\left(\begin{bmatrix}
i_1 & \ldots & i_k \\
j_1 & \ldots & j_l
\end{bmatrix}\right)
\]

and where \(\varphi : D \times P \rightarrow \mathbb{T} \cup \{0\}\) is a certain function.
As already mentioned, the usual Kronecker function is pure. In fact, we have the following formula, where $\leq$ is the order relation on $P$ obtained by merging blocks:

$$\delta_{\pi}(i_1 \ldots i_k j_1 \ldots j_l) = \begin{cases} 1 & \text{if } \begin{bmatrix} i_1 \ldots i_k \\ j_1 \ldots j_l \end{bmatrix} \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Another interesting example is the “twisted” version of $\delta$. Consider indeed the standard embeddings $S_\infty \subset P_2 \subset P_{\text{even}}$, with the convention that the permutations act vertically, from top to bottom. We have then the following result, from [1]:

**Proposition 3.2.** The signature of the permutations $S_\infty \rightarrow \{\pm 1\}$ extends into a signature map $\varepsilon_2 : P_{\text{even}} \rightarrow \{\pm 1\}$, given by $\varepsilon_2(\pi) = (-1)^{c(\pi)}$, where $c(\pi)$ is the number of switches needed for putting $\pi$ in noncrossing form, and the formula

$$\bar{\delta}_\pi(i_1 \ldots i_k j_1 \ldots j_l) = \delta_\pi(i_1 \ldots i_k j_1 \ldots j_l) \varepsilon_2(i_1 \ldots i_k)$$

defines a generalized Kronecker map on $P_{\text{even}}$, which is pure.

**Proof.** The idea indeed is that the number $c(\pi)$ in the statement is well-defined modulo 2, and so we have a signature map as above, extending the usual signature of the permutations. The proof of the categorical axioms is routine as well, see [1]. \hfill $\Box$

At the quantum group level now, the result, also from [1], is as follows:

**Proposition 3.3.** The basic quantum unitary and reflection groups associated to the generalized Kronecker map constructed above are as follows,

$$\begin{array}{ccc}
\bar{U}_N & \rightarrow & U_N^+ \\
\bar{O}_N & \rightarrow & O_N^+ \\
\bar{K}_N & \rightarrow & K_N^+ \\
H_N & \rightarrow & H_N^+ \\
\end{array}$$

with the bar symbols denoting $q = -1$ twists, obtained by stating that the standard coordinates and their adjoints anticommute on rows and columns, and commute otherwise.

**Proof.** The signature map $\varepsilon_2$ being trivial on $NC_{\text{even}}$, the free quantum groups, namely $O_N^+, U_N^+, H_N^+, K_N^+$, are not twistable. Regarding now $O_N, \bar{U}_N$, the point here is that the
linear map associated to the basic crossing is as follows:

\[
\tilde{T}_\pi(e_i \otimes e_j) = \begin{cases} 
-e_j \otimes e_i & \text{for } i \neq j \\
 e_i \otimes e_i & \text{for } i = j
\end{cases}
\]

Thus, the corresponding quantum groups $\tilde{O}_N, \tilde{U}_N$ are obtained from $O_N, U_N$ by replacing the commutation relations $ab = ba$ between coordinates by relations of type $ab = \pm ba$, and so appear as $q$-deformations of $O_N, U_N$, with deformation parameter $q = -1$.

As for the remaining quantum groups, $H_N, K_N$, these are not twistable. The proof here is quite tricky, using some combinatorics in order to express the twisted maps $T_\pi$ in terms of the untwisted ones, via a Möbius inversion type formula. See [1], [2].

We refer to [1], [2] for more details on the above construction, and for some further structure and classification results. To summarize the main results on the subject, it is known from [30] and from there that in the orthogonal case, $H_N \subset G \subset O_N^+$, the classification diagram for the easy quantum groups and their twists is as follows:

\[
\begin{array}{c}
O_N \\
\downarrow \\
H_N
\end{array} \quad \begin{array}{c}
H_N^x \quad H_N^+ \\
\uparrow \\
O_N
\end{array} \quad \begin{array}{c}
O_N^x \\
\downarrow \\
O_N^+
\end{array}
\]

To be more precise, here $O_N^x$ are the various versions of $O_N$, obtained via liberation and twisting, and $H_N^x$ stands for the various intermediate liberations of the hyperoctahedral group $H_N$, fully classified in [30], and which are equal to their own twists.

Now back to our considerations, regarding the pure Kronecker functions, the above construction suggests looking into functions of the following type:

\[
\tilde{\delta}_\pi \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right) = \delta_\pi \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right) \varepsilon \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right)
\]

In order to define such functions, we only need to define $\varepsilon$ on the partitions of type $\nu = \ker(i_j)$ which satisfy $\delta_\pi(i_j) = 0$, which means $\nu \leq \pi$. Thus, we are in need of:

**Definition 3.4.** The completion of a category of partitions $D$ is given by:

\[
\tilde{D} = \left\{ \nu \left| \exists \pi \in D, \nu \leq \pi \right. \right\}
\]

In other words, $\tilde{D}$ is obtained from $D$ by allowing the joining of blocks.

Observe that $\tilde{D}$ is a category of partitions as well. This follows indeed from definitions. As a basic example, the completion of $D = P_2$ is the category $\tilde{D} = P_{even}$. 

Let us go back now to Proposition 2.2 above, and its proof. If we carefully examine the proof, we are led to the following axioms for the generalized signatures:

**Definition 3.5.** A pre-signature on a complete category $\bar{D}$ is a map $\varepsilon : \bar{D} \to T$ which is trivial on $\bar{D} \cap NC$, and which satisfies the following conditions:

1. $\varepsilon(\rho) = \varepsilon(\pi)\varepsilon(\sigma)$, with $\rho \leq [\pi\sigma]$ being obtained by joining left and right blocks.
2. $\varepsilon(\rho) = \varepsilon(\pi)\varepsilon(\sigma)$, with $\rho \leq [\pi\sigma]$ being obtained by joining up and down blocks.

In the case where $\varepsilon(\pi^*) = \bar{\varepsilon}(\pi)$ for any $\pi$, we call $\varepsilon$ a signature.

As basic examples, we have the trivial signature $\varepsilon_1 : P \to \{1\}$, as well as the “standard” signature $\varepsilon_2 : P_{\text{even}} \to \{\pm 1\}$, from Proposition 3.2. We will see later on that these two maps are particular cases of a pre-signature construction which works at any $s \in \mathbb{N}$.

Regarding our various axioms for the signatures, these basically follow from the technical conditions in Proposition 2.2, and this will be explained below. The only point which can be probably improved is our assumption that $\varepsilon$ must be trivial on $\bar{D} \cap NC$. This is actually quite a strong assumption, that we will not fully need, but which is verified for all the examples that we have. We believe that this might be actually automatic.

The interest in the signatures comes from the following result:

**Proposition 3.6.** Given a category of partitions $D$, and a signature on its completion $\varepsilon : \bar{D} \to T$, the construction

$$\delta_{\pi}\left(\begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_k \\ j_k \end{array}\right) = \delta_{\pi}\left(\begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_k \\ j_k \end{array}\right) \varepsilon\left[\begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_k \\ j_k \end{array}\right]$$

produces a generalized Kronecker function, which is pure.

**Proof.** Our first claim is that a map $\varepsilon : \bar{D} \to T$ which is trivial on $\bar{D} \cap NC$ is a signature precisely when it satisfies the following conditions, for any choice of the multi-indices, such that the corresponding kernels belong to $\bar{D}$:

$$\varepsilon\left[\begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_p \\ j_q \end{array}\right] \varepsilon\left[\begin{array}{c} k_1 \\ l_1 \\ \vdots \\ k_r \\ l_s \end{array}\right] = \varepsilon\left[\begin{array}{c} i_1 \\ j_1 \\ \vdots \\ i_p \\ j_q \end{array}\right] \varepsilon\left[\begin{array}{c} k_1 \\ l_1 \\ \vdots \\ k_r \\ l_s \end{array}\right]$$

Indeed, these conditions are equivalent to the axioms in Definition 3.5:

1. With $\{i, j\} \cap \{k, l\} = \emptyset$, our first condition reads $\varepsilon([\pi\sigma]) = \varepsilon(\pi)\varepsilon(\sigma)$, where $\pi = [i]$ and $\sigma = [j]$. In the general case now, where $\{i, j\} \cap \{k, l\}$ is not necessarily empty, the partition $\rho = [i,k]$ satisfies $\rho \leq [\pi\sigma]$, and is in fact obtained from $[\pi\sigma]$ by joining certain left and right blocks. Moreover, by choosing suitable indices $i, j, k, l$, we see that each
such joining is allowed, and we conclude that our first axiom for $\varepsilon$ is equivalent to the condition $\varepsilon(\rho) = \varepsilon(\pi)\varepsilon(\sigma)$, for any such partition $\rho$, as stated.

(2) The proof here is similar to the proof of (1) above, with the horizontal concatenation operation replaced by the vertical concatenation operation, and with the remark that the middle indices $j$ won’t interfere, these indices being connected by definition.

(3) This is trivial, because we can set $\pi = \begin{bmatrix} j \end{bmatrix}$, and we obtain the result.

Now with this claim in hand, the result follows from Proposition 2.2 and its proof, because the conditions (1-3), together with our assumption that $\varepsilon$ is trivial on the non-crossing partitions, allow us to insert the signature terms, in the formulae there. □

Summarizing, the signatures produce natural examples of pure Kronecker functions. At the theoretical level, one interesting question is whether any pure Kronecker function appears from a signature, in the above sense. We do not know.

In the context of Proposition 3.6, the signed operators $\bar{T}_\pi$ can be computed in terms of the unsigned operators $T_\pi$, by using the following formula, coming from \cite{2}:

**Proposition 3.7.** Given a signature $\varepsilon : \bar{D} \rightarrow \mathbb{T}$, we have the formula

$$\bar{T}_\pi = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau)\mu(\sigma, \tau)T_\sigma$$

valid for any $\pi \in D$, where $\mu$ is the M"obius function of $D$.

**Proof.** The linear combinations $T = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$ act on tensors as follows:

$$T(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{\tau \leq \pi} \alpha_\tau T_\tau(e_{i_1} \otimes \ldots \otimes e_{i_k})$$

$$= \sum_{\tau \leq \pi} \alpha_\tau \sum_{\sigma \leq \tau} \sum_{[i]_j = \sigma} e_{j_1} \otimes \ldots \otimes e_{j_1}$$

$$= \sum_{\sigma \leq \pi} \left( \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau \right) \sum_{[i]_j = \sigma} e_{j_1} \otimes \ldots \otimes e_{j_1}$$

Thus, in order to have $\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$, we must have, for any $\sigma \leq \pi$:

$$\varepsilon(\sigma) = \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau$$

But this problem can be solved by using the M"obius inversion formula, and we obtain the coefficients $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau)\mu(\sigma, \tau)$ in the statement. □

In order to construct now some basic examples of signatures, the idea will be that of unifying the constructions of the trivial signature $\varepsilon_1 : P \rightarrow \mathbb{Z}_1$ and of the usual signature $\varepsilon_2 : P_{\text{even}} \rightarrow \mathbb{Z}_2$, with a general construction of type $\varepsilon_s : P^s \rightarrow \mathbb{Z}_s$.

We will need the following result, from \cite{3}:
Proposition 3.8. Given a number \( s \in \mathbb{N} \cup \{\infty\} \), the following hold:

1. The group \( H^s_N = S_N \wr \mathbb{Z}_s \) is easy, with the corresponding category of partitions \( P^s \) consisting of the partitions having the property that each block, when weighted according to the rules \( \circ \rightarrow +, \bullet \rightarrow - \), has as size a multiple of \( s \).

2. The quantum group \( H^s_+ = S^+_N \wr \ast \mathbb{Z}_s \), where \( S^+_N \) is the quantum permutation group, and \( \ast \) is a free wreath product, is easy as well, with the corresponding category of partitions being \( NC^s = P^s \cap NC \).

Proof. This is something standard, extending some well-known results at \( s = 1, 2, \infty \), where \( H^s_N \) is respectively the symmetric group \( S_N \), the hyperoctahedral group \( H_N \), and the complex reflection group \( K_N \). For full details here, we refer to [6]. \( \square \)

In order to construct our signature map, the idea will be similar to the one in Proposition 3.2, by counting the switches needed in order to put \( \pi \) in noncrossing form. However, since we have to make now a reasoning modulo \( s \in \mathbb{N} \) instead of modulo 2, we have to carefully fine-tune the counting of the switches, by counting each switch either as \( +1 \), or as \( -1 \). Of course, in the case \( s = 2 \) this subtlety dissapears, because \( 1 = -1(2) \).

So, let \( \pi \in P^s \). We label 1, 2, 3, \ldots the blocks of \( \pi \), with respect to the occurrence of the first leg, when going counterclockwise, starting from bottom left.

With this convention, the number of switches needed in order to put \( \pi \) in noncrossing form can be constructed, as an integer modulo \( s \), as follows:

Proposition 3.9. Given a partition \( \pi \in P^s \), the number of switches needed for making \( \pi \) noncrossing is well-defined modulo \( s \), when computing this number as follows:

1. A “natural” switch, tending to arrange the blocks as 1, 2, 3, \ldots counts as \( +1 \), and a “reverse” switch, tending to arrange the blocks as \( \ldots, 3, 2, 1 \) counts as \( -1 \).

2. A switch between two legs having the same labels, \( \circ \) and \( \circ \) or \( \bullet \) and \( \bullet \), counts as \( +1 \), and a switch between two legs having different labels, \( \circ \) and \( \bullet \), counts as \( -1 \).

3. The switches in the upper row count as \( +1 \), the switches in the lower row count as \( -1 \), and there are no switches between the upper and lower rows.

Proof. First of all, we can indeed put \( \pi \) in noncrossing form by performing a certain number of switches, between neighbors in the upper row, and neighbors in the lower row. We have to prove that the following number, where the sum is over all the needed switches \( \times \), and where \( a, b, c \in \{\pm 1\} \) are the various signs associated to each switch, constructed according to the rules in (1,2,3) above, is well-defined modulo \( s \):

\[
c = \sum_{\times} a(\times)b(\times)c(\times)
\]

But this is clear, because in order to put \( \pi \) in noncrossing form, each leg must travel over whole blocks, having as signed size a multiple of \( s \). \( \square \)
The above construction might seem of course a bit complicated. We refer to section 4 below for some concrete computations of such switching numbers.

We can now construct our signature map, as follows:

**Theorem 3.10.** We have a well-defined map \( \varepsilon_s : P^s \to \mathbb{Z}^s \), given by

\[
\varepsilon_s(\pi) = w^{c(\pi)} \quad , \quad w = e^{2\pi i/s}
\]

where \( c(\pi) \) is the number of switches needed for making \( \pi \) noncrossing. Moreover:

1. \( \varepsilon_s \) is a pre-signature.
2. We have \( \varepsilon_s(\pi^*) = \varepsilon_s(\pi) \), for any \( \pi \in P^s \).
3. In the real case, \( s = 1, 2 \), the map \( \varepsilon_s \) is a signature.
4. In fact, at \( s = 1, 2 \) we obtain the usual signatures \( \varepsilon_1, \varepsilon_2 \).

**Proof.** The first assertion follows from Proposition 3.9. Then, we have:

1. Our map is by definition trivial on \( NC \), so we are left with the verification of the first two conditions in Definition 3.5. The proof goes as follows:
   - Concatenation. Assume that \( \rho \leq [\pi\sigma] \), as in Definition 3.5 (1). By switching to the noncrossing form, \( \pi \to \pi' \) and \( \sigma \to \sigma' \), our partition transforms as \( \rho \to \rho' \leq [\pi'\sigma'] \). Now since \([\pi'\sigma']\) is noncrossing, its signature is trivial, and we obtain the result.
   - Composition. Assume now that \( \rho \leq [\sigma_\pi] \), as in Definition 3.5 (2). Our claim is that we can jointly switch \( \pi, \sigma \) to the noncrossing form. Indeed, we can switch the upper legs of \( \sigma \), and the lower legs of \( \pi \), as for both these partitions to become noncrossing. Now observe that when switching in this way to the noncrossing form, \( \pi \to \pi' \) and \( \sigma \to \sigma' \), our partition transforms as \( \rho \to \rho' \leq [\sigma'_\pi] \), which is noncrossing. Thus, we obtain the result.
2. This follows indeed from the construction of \( c(\pi) \).
3. This follows from (2). Observe also that, regarding the associated operators \( T_\pi \), in the context of Proposition 3.7 above, the conjugation formulae are as follows:

\[
\widehat{T}_{\pi^*} = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau) \mu(\sigma, \tau) T_{\sigma} \\
\widetilde{T}_{\pi^*} = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau) \mu(\sigma, \tau) T_{\sigma}
\]

Thus, in the real case, we obtain indeed \( \widehat{T}_{\pi^*} = \widetilde{T}_{\pi^*} \), as required by Definition 2.1.
4. This is clear indeed from the construction in Proposition 3.8. \( \square \)

Summarizing, we have so far a natural extension of the constructions of \( \varepsilon_1, \varepsilon_2 \).

4. Basic examples

In this section we compute the basic quantum groups associated to \( \varepsilon_s \). In view of the problems with the conjugation axiom at \( s \geq 3 \), we have to slightly exit our formalism, and use “generalized pre-Kronecker functions”, which are by definition functions satisfying the conditions in Proposition 2.2, except for the conjugation axiom. Let us begin with:
Proposition 4.1. Given a number \( s \in \mathbb{N} \cup \{\infty\} \) and a category \( D \subset P^s \), we have a generalized pre-Kronecker function \( \bar{\delta} : D \rightarrow \mathbb{Z}_s \cup \{0\} \), given by

\[
\bar{\delta}_s \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right) = \delta_s \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right) \varepsilon_s \left[ \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_l \end{array} \right]
\]

and an associated quantum group \( \bar{G} \subset U^+_N \), obtained via the corresponding maps \( \bar{T}_s \).

Proof. This follows by combining the various constructions in Definition 2.3, Proposition 3.6 and Theorem 3.10, with the remark that \( P^s \) is indeed complete. \( \square \)

In order to compute the basic examples of such quantum groups, let us first compute the associated categories of partitions. At \( s = 1, 2 \) we have \( P_{\text{even}} \subset P^s \), and so these categories are the usual ones, from Theorem 1.7 above, namely:

At \( s \geq 3 \), however, we don’t have \( P_{\text{even}} \subset P^s \), and when intersecting with \( P^s \), some of these categories get smaller. The precise result here is as follows:

Proposition 4.2. The categories of partitions associated to \( P^s \) with \( s \geq 3 \) are:

\[
P^2 \quad NC^2 \\
\vdots \\
P_{\text{even}} \quad NC_{\text{even}}
\]

where \( [2, s] \) denotes as usual the least common multiple of 2, \( s \).

Proof. We use the fact, coming from Proposition 3.8 above at \( s = 1, 2, \infty \), or simply from the definition of the categories involved, that we have:

\[
P^1 = P, \quad P^2 = P_{\text{even}}, \quad P^\infty = P_{\text{even}}
\]

We will use as well the following fact, which is clear from definitions:

\[
P^r \cap P^s = P^{[r,s]}
\]
It is enough to prove the results for the square on the left, because the square on the right will appear by intersecting with \( NC \). And here, the situation is as follows:

- We have \( P_{\text{even}} \cap P^s = P^2 \cap P^s = P^{[2,s]} \).
- We have \( P_{\text{even}} = P^\infty \subset P^s \), and so \( P_{\text{even}} \cap P^s = P_{\text{even}} \).
- Since we have \( P_2 \subset P_{\text{even}} \), we obtain as well \( P_2 \cap P^s = P_2 \).
- Finally, by using our assumption \( s \geq 3 \), we obtain \( P^2 \cap P^s = P^2 \). \( \square \)

Summarizing, we have to split our study, following the values of \( s \). At \( s = 1, 2 \) the quantum groups are those in Proposition 1.6 and Proposition 3.3 above, and there is of course nothing to add to this. In the case \( s \geq 3 \) now, we first have:

**Proposition 4.3.** We have the following formulae,

\[ \bar{U}_N^+ = \bar{O}_N^+ = U_N^+ \quad , \quad \bar{K}_N^+ = K_N^+ \quad , \quad \bar{H}_N^+ = H_N^{[2,s]+} \]

for the quantum groups coming from Proposition 4.1, at \( s \geq 3 \).

**Proof.** The quantum groups in the statement are those associated to the categories on the right in the diagram from Proposition 4.2. Now since all these categories consist of noncrossing partitions, and we have \( \bar{\delta}_\pi = \delta_\pi \) for such partitions, we therefore obtain the associated usual easy quantum groups. But these latter quantum groups are computed in Theorem 1.7 and Proposition 3.8, and we obtain the quantum groups in the statement. \( \square \)

Regarding now the remaining quantum groups, our first purpose will be that of computing \( \bar{U}_N = \bar{O}_N \), coming from the category \( P_2 \). So, let us compute the map associated to the basic crossing. The result here, similar to the one in [1], is:

**Proposition 4.4.** The linear map associated to the basic crossing is

\[ \bar{T}_\chi(e_i \otimes e_j) = \begin{cases} 
\varepsilon_i e_j e_i & \text{for } i \neq j \\
1 & \text{for } i = j 
\end{cases} \]

where the sign is 1 in the case \( \chi = \begin{cases} \varepsilon_i e_i & \text{for } i \neq j \\
1 & \text{for } i = j 
\end{cases} \) and is -1 in the case \( \chi = \begin{cases} \varepsilon_i e_i & \text{for } i \neq j \\
-1 & \text{for } i = j 
\end{cases} \).

**Proof.** This kind of formula can be deduced from the Möbius formula from Proposition 3.7. In our case, however, best is to proceed directly, the computations being elementary, and quite illustrating. According to the general formula of \( \bar{T}_\pi \), we have:

\[ \bar{T}_\chi(e_i \otimes e_j) = \sum_{kl} \delta_{\chi}^{ij} \left[ \begin{array}{c} i \\
\chi \\
l \end{array} \right] \varepsilon_s \left[ \begin{array}{c} i \\
k \\
l \end{array} \right] e_k \otimes e_l = \varepsilon_s \left[ \begin{array}{c} i \\
j \\
l \end{array} \right] e_j \otimes e_i 
\]

In the case \( i = j \) the partition on the right is \( \left[ \begin{array}{c} i \\
l \end{array} \right] \), which is noncrossing, and so having signature 1. We therefore obtain the vector \( e_i \otimes e_i \), as claimed.
In the case \( i \neq j \) the partition on the right \( [i,j] \) is the basic crossing:

\[
\begin{array}{c|c}
2 & 1 \\
\hline
1 & 2 \\
\end{array}
\]

Here we have labelled the blocks, or rather their legs, with numbers 1, 2, ..., as in Proposition 3.9 above. In order to compute now the signature, we have to put this partition in noncrossing form, and there are two ways of doing this:

1. By switching the upper legs. This switch is “natural” in the sense of Proposition 3.9, and also is a switch in the upper part, so its sign is \((+1)(+1) = +1\).
2. By switching the lower legs. This is a “reverse” switch in the sense of Proposition 3.9, and also is a switch in the lower part, so its sign is \((-1)(-1) = +1\).

Summarizing, in both cases we obtain +1. Thus, when the coloring is uniform, the signature is \(w\). In the case where the coloring is matching and non-uniform, there is an extra \(-1\) sign coming from the Proposition 3.9 (2), and the signature is \(w^{-1}\).

Now by plugging this signature in the formula of \(\bar{T}_\chi\), we obtain the result. \(\square\)

At the quantum group level now, we first have:

**Proposition 4.5.** The quantum group \(U_N^{(s)} = \bar{U}_N\) with \(s \geq 1\) appears inside \(U_N^+\) via the following relations, applied to the standard coordinates:

1. \([u_{ij}, u_{kl}] = [u_{ij}, u_{kl}^*] = 0\) for \(i \neq k, j \neq l\).
2. \(u_{ij}u_{ik} = w \cdot u_{ik}u_{ij}\) and \(u_{ij}^*u_{ik} = w \cdot u_{ik}^*u_{ij}\) for \(j \neq k\).
3. \(u_{ij}u_{kj} = w \cdot u_{kj}u_{ij}\) and \(u_{ij}^*u_{kj} = w \cdot u_{kj}^*u_{ij}\) for \(j \neq k\).

**Proof.** According to Theorem 2.7 (3) above, the quantum group \(U_N^{(s)} = \bar{U}_N\) appears via the following relations, applied to the standard coordinates of \(U_N^+\):

\[
T_\pi \in \text{Hom}(u \otimes u, u \otimes u) \quad : \quad \pi = \circlearrowleft
\]

\[
\bar{T}_\pi \in \text{Hom}(u \otimes \bar{u}, \bar{u} \otimes u) \quad : \quad \pi = \circlearrowright
\]

Let us first analyse the first relation. For simplifying the presentation, we omit the 4 circles in the writing of \(\pi\). Our relation states that we must have:

\[
(T_\chi \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes 1) = u^{\otimes 2}(T_\chi \otimes 1)(e_i \otimes e_j \otimes 1), \quad \forall i, j
\]

By using the formula of \(T_\chi\) in Proposition 4.4, we obtain:

\[
(T_\chi \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes 1) = (T_\chi \otimes 1) \sum_{kl} e_k \otimes e_l \otimes u_{kl}u_{lj}
\]

\[
= \sum_k e_k \otimes e_k \otimes u_{kl}u_{kj} + w \sum_{k \neq l} e_l \otimes e_k \otimes u_{kl}u_{lj}
\]
By using the same formula, from Proposition 4.4, we have as well:

\[
\begin{align*}
u \otimes_2 (T_\chi \otimes 1)(e_i \otimes e_j \otimes 1) &= \begin{cases} 
\sum_{kl} e_l \otimes e_k \otimes u_{li} u_{ki} & \text{if } i = j \\
w \sum_{kl} e_l \otimes e_k \otimes u_{l_j} u_{ki} & \text{if } i \neq j
\end{cases}
\end{align*}
\]

In order to compare these two quantities, we have two cases, as follows:

Case \(i = j\). Here the equality that must be satisfied is as follows:

\[
\sum_k e_k \otimes e_k \otimes u_{ki}^2 + w \sum_{k \neq l} e_l \otimes e_k \otimes u_{ki} u_{li} = \sum_{kl} e_l \otimes e_k \otimes u_{li} u_{ki}
\]

We are therefore led to the following condition:

\[
w \cdot u_{ki} u_{li} = u_{ti} u_{ki}, \quad \forall k \neq l
\]

Case \(i \neq j\). Here the equality that must be satisfied is:

\[
\sum_k e_k \otimes e_k \otimes u_{ki} u_{kj} + w \sum_{k \neq l} e_l \otimes e_k \otimes u_{ki} u_{lj} = w \sum_{kl} e_l \otimes e_k \otimes u_{lj} u_{ki}
\]

We are therefore led to the following two conditions:

\[
\begin{align*}
&u_{ki} u_{kj} = w \cdot u_{kj} u_{ki}, \quad \forall i \neq j \\
&u_{ki} u_{lj} = w \cdot u_{lj} u_{ki}, \quad \forall i \neq j, k \neq l
\end{align*}
\]

Summarizing, from our first formula we have obtained precisely the relations in the statement, concerning the commutation of the variables \(u_{ij}\).

Regarding now our second formula, involving the partition \(\pi = \ast\), the study is similar. Once again by omitting the colorings, our relation states that we must have:

\[
(T_\chi \otimes 1)(u \otimes \bar{u})(e_i \otimes e_j \otimes 1) = (\bar{u} \otimes u)(T_\chi \otimes 1)(e_i \otimes e_j \otimes 1), \quad \forall i, j
\]

Thus, we obtain similar results, with some conjugates appearing, and with \(w\) being replaced by \(w^{-1}\), according to the formula in Proposition 4.4. To be more precise:

Case \(i = j\). Here we obtain the following relation:

\[
w^{-1} \cdot u_{ki} u_{li}^* = u_{ti} u_{ki}, \quad \forall k \neq l
\]

By multiplying everything by \(w\), this relation is equivalent to:

\[
u_{ki} u_{li}^* = w \cdot u_{ti} u_{ki}, \quad \forall k \neq l
\]

Case \(i \neq j\). Here we obtain the following two relations:

\[
\begin{align*}
u_{ki} u_{kj}^* &= w^{-1} \cdot u_{kj}^* u_{ki}, \quad \forall i \neq j \\
u_{ki} u_{lj}^* &= u_{lj}^* u_{ki}, \quad \forall i \neq j, k \neq l
\end{align*}
\]

By applying the involution to the first relation, this takes the following form:

\[
u_{kj} u_{ki}^* = w \cdot u_{ki}^* u_{kj}, \quad \forall i \neq j
\]

Summarizing, we have obtained the missing relations, and we are done. \(\square\)
We already know, from Theorem 1.7 and from Proposition 3.3 above, that at $s = 1, 2$ we obtain in this way the usual unitary group $U_N$, and its $q = -1$ twist $\bar{U}_N$. We can in fact further process the above relations, and we obtain the following result:

**Proposition 4.6.** The quantum groups $U_N^{(s)}$ are as follows:

1. At $s = 1$ we obtain the group $U_N$.
2. At $s = 2$ we obtain the twist $\bar{U}_N$.
3. At $s \geq 3$ we obtain the group $K_N$.

**Proof.** We use the relations found in Proposition 4.5 above:

1. At $s = 1$ we have $w = 1$, the relations in the statement tell us that the variables $\{u_{ij}, u_{ij}^*\}$ must all commute, and so we obtain the usual unitary group $U_N$.
2. At $s = 2$ we have $w = -1$, the relations tell us that the variables $\{u_{ij}, u_{ij}^*\}$ anti-commute on rows and columns, and commute otherwise, so we obtain the twist $\bar{U}_N$.
3. At $s \geq 3$ we have $w^2 \neq 1$, and our claim is that, by using this fact, the relations (2,3) in Proposition 4.5 above, which are of the form $ab = wab$ and $ab^* = wb^*a$, collapse in fact to relations of type $ab = 0$, and $ab^* = 0$. Indeed:
   - Regarding the relations of type $ab = wba$, with $a, b \in \{u_{ij}\}$, by changing the indices $i, j$ we must have as well $ba = wab$, and so we obtain $ab = 0$, as claimed.
   - Regarding the relations $ab^* = wb^*a$, by applying $*$ we must have $ba^* = w^{-1}a^*b$, and then by changing the indices we obtain $ab^* = w^{-1}b^*a$. Thus, $ab^* = 0$.

Now with this claim in hand, it follows that the variables $\{u_{ij}, u_{ij}^*\}$ all commute. Thus the quantum group that we are looking for is the subgroup $U_N^{(s)} \subset U_N$ given by the fact that its coordinates satisfy $ab = 0$ on rows and columns, which is $K_N$. □

Regarding now $\bar{O}_N$, according to our conventions and to Proposition 4.2 above, this comes from the category $P_2$ as well, and so we have $\bar{O}_N = U_N^{(s)} = K_N$ at $s \geq 3$.

Finally, concerning the remaining quantum groups, we have:

**Proposition 4.7.** At $s \geq 3$, we have the formulae

$$\bar{K}_N = K_N, \quad \bar{H}_N = H_N^{[2,s]}$$

for the quantum groups coming from Proposition 4.1.

**Proof.** According to Proposition 4.2 we have inclusions $\bar{H}_N \subset \bar{K}_N \subset \bar{U}_N$, because these quantum groups come from the categories $P_2^{[2,s]} \supset P_{even} \supset P_2$. On the other hand, we know from Proposition 4.6 that we have $\bar{U}_N = K_N$. Thus, our inclusions become:

$$\bar{H}_N \subset K_N \subset K_N$$

Summarizing, we are left with two classical group computations. In order to compute the group $\bar{K}_N$, we can use the following well-known formula:

$$P_{even} = \langle P_2, NC_{even} \rangle$$
At the quantum group level we obtain, by using as well $K_N^+ = K_N^+$:

$$K_N = \bar{U}_N \cap K_N^+ = K_N \cap K_N^+ = K_N$$

Regarding now $\bar{H}_N$, we can use the following well-known formula:

$$P^{[2,s]} = \langle P_2, NC^{[2,s]} \rangle$$

Now since $NC^{[2,s]}$ consists of noncrossing partitions, the quantum group associated to it, via Proposition 4.1, is the usual easy quantum group associated to it, namely $H_N^{[2,s]+}$. Thus, we can perform an intersection as above, and we obtain:

$$\bar{H}_N = K_N \cap H_N^{[2,s]+} = H_N^{[2,s]}$$

Summarizing, we have obtained the formulae in the statement. □

We can now formulate our main result, in the spirit of [5], as follows:

**Theorem 4.8.** At $s \geq 3$ the basic quantum unitary and reflection groups are

![Diagram](image)

with the various collapsings essentially coming from $P_{even} \not\subset P^s$.

**Proof.** Indeed, the 4 quantum groups on the right come from Proposition 4.3, the remaining 2 quantum groups on top, on the left, come from Proposition 4.6, and the remaining quantum groups on the bottom, on the left, come from Proposition 4.7. □

As a conclusion, at the purely combinatorial level, we have a nice and natural generalization of the signature maps $\varepsilon_1, \varepsilon_2$, and of the corresponding linear maps $T_\pi, \bar{T}_\pi$. However, in what regards the quantum group applications, things rather collapse at $s \geq 3$, or at least we don’t know, so far, how to make them not to collapse.

All this seems to require some more work. Also, we intend to use the signatures $\varepsilon_s$ and linear maps $T_\pi$ constructed above in the context of the block-modified Wishart matrices, where our belief is that these maps can indeed help, in a continuation of [4].

Generally speaking, we believe that all this is related to the well-known fact that the $q$-deformations of the compact Lie groups, with parameter $q \in \mathbb{T} - \{\pm 1\}$, are not semisimple. The $*$-algebraic and operator algebraic aspects of these $q$-deformations were worked out in [25, 41], and it is an open question whether the above considerations can lead to something more concrete, in the spirit of [42]. We do not know.
5. Super-easiness

In this section and in the next two ones we generalize the easy quantum group formalism in another direction, coming from the work in [10], and in subsequent papers. The idea indeed is that an interesting class of examples of generalized easy quantum groups is provided by the “super-versions” of the easy quantum groups.

In order to explain this material, coming from [10], and from [3], we will need the following standard result, coming from the work in [14], [35], [39]:

**Proposition 5.1.** Given a closed subgroup $G \subset U^+_N$, with its fundamental corepresentation $u = (u_{ij})$ assumed to be irreducible, we have $u \sim \bar{u}$ precisely when $u = J\bar{u}J^{-1}$ for some linear map $J : \mathbb{C}^N \to \mathbb{C}^N$ satisfying $JJ^* = 1, J\bar{J} = \pm 1$.

*Proof.* The condition $u \sim \bar{u}$ tells us that we must have $u = J\bar{u}J^{-1}$, with $J \in GL_N(\mathbb{C})$. By conjugating we obtain $\bar{u} = J\bar{u}J^{-1}$, and so $u = (J\bar{J})u(J\bar{J})^{-1}$. Now since $u$ is assumed to be irreducible, we must have $J\bar{J} = c1$, with $c \in \mathbb{C}$. By conjugating we obtain $JJ = c1$, and hence $c \in \mathbb{R}$. Moreover, by rescaling we may assume $c = \pm 1$.

From $(id \otimes S)u = u^*$ we get $(id \otimes S)\bar{u} = u^t$. By applying $id \otimes S$ to $u = J\bar{u}J^{-1}$ we get $u^* = Ju^tJ^{-1}$, so $u = (J^*)^{-1}\bar{u}J^*$, so $\bar{u} = J^*u(J^*)^{-1}$. With $u = J\bar{u}J^{-1}$ this gives $JJ^* \in \text{End}(u)$, so $JJ^* = d1$ with $d \in \mathbb{C}$. We have $JJ^* > 0$, so $d > 0$. From $JJ = \pm 1$ and $JJ^* = d1$ we get $|\det J|^2 = (\pm 1)^N = d^N$, and so $d = 1$, and we are done. \(\square\)

We will need as well the following more elaborate version of the above result, coming also from [14], [35], and from the Tannakian duality results in [44]:

**Proposition 5.2.** Given $J : \mathbb{C}^N \to \mathbb{C}^N$ satisfying $JJ^* = 1, J\bar{J} = \pm 1$, we have a quantum group $O^+_N \subset U^+_N$ as follows, whose fundamental corepresentation is irreducible:

$$C(O^+_N) = C(U^+_N) / \langle u = J\bar{u}J^{-1} \rangle$$

The family $\{O^+_N\}$ is a free analogue of $\{O_N\}$, in the sense that any $G \subset U^+_N$ having an irreducible fundamental corepresentation must appear as subgroup of some $O^+_N$.

*Proof.* The fact that $O^+_N$ is indeed a quantum group follows from an elementary computation, from [35]. The fact that the fundamental corepresentation is irreducible is non-trivial, and follows from the fact, explained for instance in [7], that the family $\{O^+_N\}$, which contains of course $O^+_N$, consists of quantum groups having the same fusion rules. As for the last assertion, this follows from Proposition 5.1 above. \(\square\)

Summarizing, the notion of “generalized orthogonality” in the quantum group setting comes from the linear maps $J : \mathbb{C}^N \to \mathbb{C}^N$ satisfying $JJ^* = 1, J\bar{J} = \pm 1$. As explained in [14], in the case $J\bar{J} = 1$, up to an orthogonal base change, the map $J$ can be assumed to
be of the following form, for a certain decomposition $N = 2p + q$:

$$J = \begin{pmatrix}
0 & 1 & & & \\
1 & 0(1) & & & \\
& & \ddots & & \\
& & & 0 & 1 \\
& & & 1 & 0(p) \\
& & & 1(1) & \\
& & & \ddots & \\
& & & & 1(q)
\end{pmatrix}$$

As for the case $J\bar{J} = -1$, here $N$ must be even, and once again by [14], up to an orthogonal base change, $J$ must be as follows, with $N = 2p$:

$$J = \begin{pmatrix}
0 & 1 & & & \\
-1 & 0(1) & & & \\
& & \ddots & & \\
& & & 0 & 1 \\
& & & 1 & 0(p) \\
& & & & 1(1) \\
& & & & \ddots \\
& & & & & -1 \\
& & & & & 0(1)
\end{pmatrix}$$

Summarizing, we are led in this way to the following notion, from [3]:

**Definition 5.3.** A super-structure on $\mathbb{C}^N$ is a linear map $J : \mathbb{C}^N \to \mathbb{C}^N$ satisfying $JJ^* = 1, J\bar{J} = \pm 1$. Such a map is called normalized when it is of the form

$$J(e_i) = w_i e_{\tau(i)}$$

with $\tau \in S_N$ being an involution, $\tau^2 = id$, and with $w_i \in \{\pm 1\}$ being certain signs, which are either trivial, $w_i = 1$ for any $i$, or satisfy $w_i w_{\tau(i)} = -1$ for any $i$.

In relation now with our considerations, our claim is that such a super-structure naturally produces a generalized Kronecker function, and that the associated generalized easy quantum groups are $O^+_N$ and its unitary and discrete versions from [3], [7], [10].

This claim basically comes from the Tannakian computations in [3], [7], [10]. However, the computations there all exploit the self-adjointness condition $u \sim \bar{u}$, which allows one to restrict the attention to the “reduced” Tannakian category, formed by the linear spaces $C(k, l) = \text{Hom}(u^\otimes k, u^\otimes l)$, with the exponents $k, l \in \mathbb{N}$ being usual integers.

In what follows we will need the formula of $C(k, l)$ in the general case, where $k, l$ are colored integers, so we will basically have to do the computation again.

Let us first discuss the case $J\bar{J} = 1$. Here we have the following result:
Proposition 5.4. Associated to the map \( J(c_i) = e_{\tau(i)} \), with \( \tau \in S_N \) being an involution, is a generalized Kronecker function \( \delta^\tau_{\pi} \in \{0, 1\} \), defined on \( P_{\text{even}} \), given by

\[
\delta^\tau_{\pi} = \prod_{\beta \in \pi} \delta^\tau_{\beta}
\]

and whose values on the one-block partitions \( \beta = 1^k \) are given by \( \delta^\tau_{\beta}(j) = 1 \) precisely when the multi-index \( (j) \) is of the following form,

\[
\left( \tau^{x_1}(p) \quad \tau^{x_2}(p) \quad \tau^{x_3}(p) \quad \tau^{x_4}(p) \quad \ldots \ldots \right)
\]

\[
\left( \tau^{y_1}(p) \quad \tau^{y_2}(p) \quad \tau^{y_3}(p) \quad \tau^{y_4}(p) \quad \ldots \ldots \right)
\]

for a certain \( p \in \{1, \ldots, N\} \), where \( k = (x_1 \ldots x_k) \) and \( l = (y_1 \ldots y_l) \) are the writings of \( k, l \) using \( \circ, \bullet \) symbols, with the conventions \( \circ = \bullet, \bullet = \circ \), \( \tau^0 = \text{id}, \tau^* = \tau \).

Proof. We have to check the various categorical conditions from Proposition 2.2 above, for the function \( \delta^\tau_{\pi} \) from the statement. The proof goes as follows:

1. Regarding the tensor product, by using \( \delta^\tau_{\pi} = \prod_{\beta \in \pi} \delta^\tau_{\beta} \), along with the fact that the blocks of \([\pi\sigma]\) are the blocks of \( \pi \) or of \( \sigma \) we obtain, as desired:

\[
\delta^\tau_{\pi}(i_1 \ldots i_p) \delta^\sigma_{\gamma}(k_1 \ldots k_r) = \prod_{\beta \in \pi} \delta^\tau_{\beta}(i_1 \ldots i_p) \prod_{\gamma \in \sigma} \delta^\gamma_{\gamma}(k_1 \ldots k_r)
\]

\[
= \prod_{\rho \in [\pi\sigma]} \delta^\rho_{\rho}(i_1 \ldots i_p, k_1 \ldots k_r)
\]

2. Regarding now the composition, we have to verify here that:

\[
\sum_{j_1 \ldots j_q} \delta^\sigma_{\gamma}(i_1 \ldots i_p) \delta^\pi_{\rho}(j_1 \ldots j_q) = N^c(\pi, \sigma) \delta^\pi_{[\pi\sigma]}(i_1 \ldots i_p)
\]

In this formula, the sum on the left equals the cardinality of the following set:

\[
S = \left\{ j_1, \ldots, j_q \mid \delta^\tau_{\pi}(i_1 \ldots i_p) = 1, \delta^\tau_{\sigma}(j_1 \ldots j_q) = 1 \right\}
\]

In order to compute the cardinality of this set, we have two cases, as follows:

Case \( c(\pi) = 0 \). In this case there are no middle components when concatenating \( \pi, \sigma \), the above multi-indices \( j \in S \) are uniquely determined by \( \pi, \sigma, i, k \), and so we have \#\( S \in \{0, 1\} \). Moreover, since these \( j \) multi-indices are irrelevant with respect to the question of finding the precise cardinality \#\( S \in \{0, 1\} \), we have:

\[
\#S = \delta^\tau_{[\pi\sigma]}(i_1 \ldots i_p)
\]
Thus, we obtain the formula that we wanted to prove.

Case \( c(\sigma) > 0 \). In this case there are middle components when concatenating \( \pi, \sigma \), but we can isolate them when computing \#S, by using our condition \( \delta_\pi^\tau = \prod_{\beta \in \pi} \delta_\beta^\tau \). Thus, in order to prove the formula in this case, we just have to count the number of free multi-indices \( j \), and prove that their number equals \( N^{c(\sigma)} \). But the free indices for each component must come from a single index \( p \in \{1, \ldots, N\} \), via the formula for \( \delta_1^\tau \), from the statement, and so we obtain as multiplicity the number \( N^{c(\sigma)} \), as claimed.

(3) Regarding the involution axiom, here we must check that we have:

\[
\delta_\pi^\tau \begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_q \end{pmatrix} = \delta_\pi^\tau \begin{pmatrix} j_1 & \cdots & j_q \\ i_1 & \cdots & i_p \end{pmatrix}
\]

But this is clear, because we can use here the condition \( \delta_\pi^\tau = \prod_{\beta \in \pi} \delta_\beta^\tau \), together with the formula of \( \delta_1^\tau \), which is invariant under upside-down turning.

(4) The identity axiom follows from the following computation:

\[
\delta_3^\tau \begin{pmatrix} i \\ j \end{pmatrix} = 1 \iff \exists p, \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} \tau^o(p) \\ \tau^\circ(p) \end{pmatrix} \\
\iff \exists p, \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} p \\ p \end{pmatrix} \\
\iff i = j
\]

(5) The duality axiom follows from a similar computation, as follows:

\[
\delta_\tau^\bullet (ij) = 1 \iff \exists p, (i, j) = (\tau^o(p), \tau^\circ(p)) \\
\iff \exists p, (i, j) = (\tau^o(p), \tau^o(p)) \\
\iff \exists p, (i, j) = (p, p) \\
\iff i = j
\]

Thus, we have indeed a generalized Kronecker function, as claimed.

At the quantum group level now, we first have the following result, improving some previous findings from [10], and from [3]:
Proposition 5.5. The basic quantum unitary and reflection groups associated to the generalized Kronecker map constructed above are as follows,

\[
\begin{align*}
\text{where } O_{N}^{J+} & \text{ is the quantum group in Proposition 5.2, } \overline{O}_{N}^{J} = \{ U \in U_N | U = J \overline{U} J^{-1} \} \text{ is its classical version, and } \wr & \text{ is a free wreath product, and } \hat{\ast} \text{ is a dual free product.}
\end{align*}
\]

Proof. We denote as usual by \( \overline{G}_{N}^{X} \) the analogues of the basic easy quantum groups \( G_{N}^{X} \).

The computation of these quantum groups goes as follows:

1. We know from Proposition 2.5 above that we have \( \overline{U}_{N}^{+} = U_{N}^{+} \).

2. In order to identify the quantum group in the statement with \( O_{N}^{J+} \), we have to find inside our category the conditions which imply \( u = J \overline{u} J^{-1} \). There are two ways in doing so. First, we can use the identity partition, the computation being as follows:

\[
\delta_{\pi}(ij) = 1 \iff \exists p, (i, j) = (\tau(p), \tau^{\circ}(p)) \iff \exists p, (i, j) = (\tau(p), \tau^{\bullet}(p)) \iff i = \tau(j)
\]

We can equally use the duality partition, as follows:

\[
\delta_{\overline{\pi}}(ij) = 1 \iff \exists p, (i, j) = (\tau^{\circ}(p), \tau^{\circ}(p)) \iff \exists p, (i, j) = (\tau(p), \tau^{\bullet}(p)) \iff \exists p, (i, j) = (p, \tau(p)) \iff j = \tau(i)
\]

Summarizing, in both cases we have reached to the conclusion that we must have \( u = J \overline{u} J^{-1} \), and this shows that we have \( \overline{O}_{N}^{J} = O_{N}^{J+} \), as claimed.

3. In order to compute \( \overline{U}_{N} \), we must impose to \( U_{N}^{+} = U_{N}^{+} \) the intertwining conditions coming from the basic crossing \( \chi \), colored with its possible 4 matching colorings. By using the condition \( \delta_{\pi}^{\ast} = \prod_{\beta \in \pi} \delta_{\beta}^{\ast} \), along with the identity axiom for \( \delta^{\ast} \), and its conjugate, we conclude that for all the 4 matching versions of \( \pi = \chi \), we have \( \overline{T}_{\pi} = T_{\pi} \). Now since we
have \( T_\pi(e_i \otimes e_j) = e_j \otimes e_i \), we obtain in this way the usual commutation relations between the variables \( \{u_{ij}, u_{ij}^*\} \). Thus \( \bar{U}_N \) is classical, and so we have \( \bar{U}_N = U_N \).

(4) Regarding the analogue of \( O_N \), we can compute it by intersecting, as follows:

\[
\bar{O}_N = \bar{U}_N \cap \bar{O}_N^+ = U_N \cap O_N^{J+} = O_N^J
\]

(5) Let us compute now \( \bar{K}_N \). We know that this appears as a subgroup of \( \bar{U}_N = U_N \), via the relations coming from the intertwiner \( \bar{T}_\pi \), where \( \pi = \begin{array}{ccc}
\circ & \bullet & \circ \\
\bullet & \circ & \bullet
\end{array} \), with any of its 6 matching colorings. Best here is to choose an alternating coloring, as follows:

\[
\pi = \begin{array}{ccc}
\circ & \bullet & \circ \\
\bullet & \circ & \bullet
\end{array}
\]

By Frobenius duality, we can say as well that \( \bar{K}_N \subset U_N \) appears via the relations coming from the intertwiner \( \bar{T}_\sigma \), where \( \sigma \) is the rotated version of \( \pi \), given by:

\[
\sigma = \begin{array}{ccc}
\circ & \bullet & \circ \\
\bullet & \circ & \bullet
\end{array}
\]

According to our conventions, from Proposition 5.4 above, the linear map associated to this partition is independent of \( \tau \), given by the following formula:

\[
\bar{T}_\sigma(e_i \otimes e_j) = \delta_{ij}e_i \otimes e_i
\]

By using this formula, we obtain the following relations:

\[
\bar{T}_\sigma(u \otimes \bar{u})(e_i \otimes e_j) = \sum_{ab} \delta_{ab}u_{ai}u_{bj}^* \otimes e_a \otimes e_b
\]

\[
(u \otimes \bar{u})\bar{T}_\sigma(e_i \otimes e_j) = \sum_{ab} \delta_{ij}u_{ai}u_{bj}^* \otimes e_a \otimes e_b
\]

Thus, the condition \( \bar{T}_\sigma \in \text{End}(u \otimes \bar{u}) \), which defines \( \bar{K}_N \subset U_N \), is equivalent to:

\[
\delta_{ab}u_{ai}u_{bj}^* = \delta_{ij}u_{ai}u_{bj}^* \quad \forall a, b, i, j
\]

But this latter condition holds trivially when \( a = b, i = j \) or when \( a \neq b, i \neq j \). As for the remaining two cases, namely \( a = b, i \neq j \) and \( a \neq b, i = j \), here this condition tells us that the distinct matrix entries of any group element \( U \in \bar{K}_N \) must have products 0, on each row and each column. We conclude that we have \( \bar{K}_N = K_N \), as claimed.

(6) Let us compute now \( \bar{H}_N \). This group appears as an intersection, as follows:

\[
\bar{H}_N = \bar{K}_N \cap \bar{O}_N
\]

\[
= K_N \cap O_N^{J+}
\]

\[
= \{ U \in K_N | U = J\bar{U}J^{-1} \}
\]
Now observe that, since the matrices $UJ, JU$ are obtained from $U, \bar{U}$ by interchanging the rows and columns $i, i + 1$, with $i = 1, 3, \ldots, 2p - 1$, the relation $UJ = JU$ reads:

$$U = \begin{pmatrix} a & b & \ldots & v \\ \bar{b} & \bar{a} & \ldots & \bar{v} \\ \ldots & \ldots & \ldots & \ldots \\ w & \bar{w} & \ldots & X \end{pmatrix}$$

To be more precise, here $a, b, \ldots \in \mathbb{C}, v, \ldots \in \mathbb{C}^q$ are row vectors, $w, \ldots \in \mathbb{C}^q$ are column vectors, and $X \in M_q(\mathbb{C})$. In the case $U \in K_N$, as only one entry in each row/column may be non-zero, the vectors $v, \ldots$ and $w, \ldots$ must vanish, we must have $X \in K_q$, and all $(a, b)$ blocks must be either $(0, 0)$ or the following form, with $z \in \mathbb{T}$:

$$\left( \begin{array}{cc} z & 0 \\ 0 & \bar{z} \end{array} \right), \quad \left( \begin{array}{cc} 0 & z \\ \bar{z} & 0 \end{array} \right)$$

Thus, we obtain in this way an identification $\tilde{H}_N = (K_p \wr \mathbb{Z}_2) \times K_q$, as claimed.

(7) In order to compute now $\tilde{K}_N^+$, we use the fact that this appears inside $\tilde{U}_N^+ = \tilde{U}_N^+$ by imposing the relations associated to the partition $\otimes \otimes$, with its six possible matching decorations for the legs, namely $\circ \circ \bullet \bullet, \circ \bullet \circ \bullet, \circ \bullet \bullet \circ, \bullet \circ \circ \bullet, \bullet \circ \bullet \circ, \bullet \bullet \circ \circ$.

By using now Frobenius duality, as in the proof of (5) above, we are led to the relations coming from the following 6 partitions, obtained by rotating:

\[
\begin{array}{cccc}
\circ & \circ & \circ & \bullet \\
\circ & \circ & \bullet & \bullet \\
\circ & \bullet & \circ & \bullet \\
\bullet & \circ & \circ & \bullet \\
\bullet & \bullet & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

We already know, from the proof of (5) above, that the conditions coming from the 2\textsuperscript{nd} partition simply state that we must have $ab^* = 0$, for any two distinct entries $a, b \in \{u_{ij}\}$ on the same row or the same column. Regarding the 1\textsuperscript{st}, 5\textsuperscript{th}, 6\textsuperscript{th} partitions, the situation here is similar, with the relations being respectively $a = 0, a^*b = 0, a^*b = 0$, with $a, b \in \{u_{ij}\}$ being as above. As for the remaining relations, coming from the 3\textsuperscript{rd}, 4\textsuperscript{th} partitions, these tell us that the entries of $u_{ij}$ must be normal. Summing up, we have reached to the definition of $K_N^+$, and so we have $\tilde{K}_N^+ = K_N^+$, as claimed.

(8) Finally, the quantum group $\tilde{H}_N^+$ appears as an intersection, as follows:

$$\tilde{H}_N^+ = \tilde{K}_N^+ \cap \tilde{O}_N^+ = K_N^+ \cap O_N^{J+}$$

In order to compute this intersection, we use the interpretation of the commutation relation $uJ = J\bar{u}$ given in the proof of (6) above, with all the complex conjugates there replaced of course by adjoints. By reasoning as there, we first conclude that the vectors $v, \ldots$ and $w, \ldots$ must vanish, and so that $X$ must be the fundamental corepresentation of $K_q^+$, and that we are in a dual free product situation. The study of the $2 \times 2$ blocks
is similar, and this gives a free wreath product decomposition $K^+_p \bowtie \mathbb{Z}_2$ for the first component. Thus, we obtain $H^+_N = (K^+_p \bowtie \mathbb{Z}_2) \ast K^+_q$, as claimed. \hfill \Box

We can improve the above result by using the following observation, from [10]:

**Proposition 5.6.** With the fundamental corepresentation $V = CUC^*$, where

\[
C = \begin{pmatrix}
\Gamma_{(1)} & & \\
& \ddots & \\
& & \Gamma_{(p)} \\
1_{(1)} & & \\
& \ddots & \\
& & 1_{(q)}
\end{pmatrix} : \quad \Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix}
\rho & \rho^7 \\
\rho^3 & \rho^5
\end{pmatrix}, \quad \rho = e^{\pi i/4}
\]

the relations defining $O^+_N$ become $V = \bar{V} = \text{unitary}$, and so we have $O^+_N \sim O^+_N$.

**Proof.** Observe indeed that the above matrix $\Gamma$ is unitary, and that we have $\Gamma(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})\Gamma^t = 1$. Thus the matrix $C$ is unitary as well, and satisfies $CJC^t = 1$. It follows that in terms of $V = CUC^*$ the relations $U = JUJ^{-1} = \text{unitary}$ defining $O^+_N$ simply read $V = \bar{V} = \text{unitary}$, so we obtain an isomorphism $O^+_N \simeq O^+_N$, as in the statement. See [10]. \hfill \Box

We can now formulate an improved version of Proposition 5.5, as follows:

**Theorem 5.7.** The basic quantum unitary and reflection groups associated to the generalized Kronecker map constructed above are as follows:

\[
\begin{array}{c}
U_N \rightarrow O^+_N \\
O_N \rightarrow O^+_N \rightarrow K^+_N \\
(K_p \bowtie \mathbb{Z}_2) \times K_q \rightarrow (K^+_p \bowtie \mathbb{Z}_2) \ast K^+_q
\end{array}
\]

where the quantum groups in the upper part of the diagram are taken with respect to the fundamental corepresentation $V = CUC^*$ constructed above.

**Proof.** Let us denote by $\tilde{G}_N^\times$ the quantum groups $G_N^\times$, with fundamental corepresentation $V = CUC^*$. The computation of these quantum groups goes as follows:

1. We know from Proposition 5.5 above that we have $\bar{U}_N^+ = U_N^+$. Since the matrix $C$ is unitary, it follows that we have $\bar{U}_N^+ = U_N^+$ as well.

2. According to Proposition 5.5 we have $\bar{O}_N^+ = O^+_N$, and by Proposition 5.6, when changing the fundamental corepresentation, we obtain $\bar{O}_N^+ = O^+_N$. 

(3) We know from Proposition 5.5 that the subgroups $\bar{U}_N \subset \bar{U}_N^+$ and $\bar{O}_N \subset \bar{O}_N^+$ simply appear by taking the classical version. Thus, the subgroups $\hat{U}_N \subset \hat{U}_N^+$ and $\hat{O}_N \subset \hat{O}_N^+$ appear as well by taking the classical version, and so $\hat{U}_N = U_N$, $\hat{O}_N = O_N$.

(4) Finally, the quantum groups on the bottom are those from Proposition 5.5, with the remark of course that the 4 vertical embeddings are not the standard ones. □

It is probably possible to further improve the above result, by recomputing the quantum groups on the bottom, by using the fundamental corepresentation $V = CUC^*$. However, the isomorphism classes of these quantum groups will remain of course the same, and so this would result just in a better understanding of the 4 vertical inclusions.

6. SYMPLECTIC GROUPS

We discuss now the case $J\bar{J} = -1$. Here, as already mentioned, $N$ must be even, and up to an orthogonal base change we must have, with $N = 2p$:

$$J = \begin{pmatrix}
0 & 1 \\
-1 & 0_{(1)} \\
& & \ddots \\
& & & 0 & 1 \\
& & & -1 & 0_{(p)}
\end{pmatrix}$$

As a main motivation for the study of this case, our easiness theory here will cover the symplectic group $Sp_N$ and its free version $Sp_N^+$, because we have:

**Proposition 6.1.** For a super-structure on $\mathbb{C}^N$ satisfying $J\bar{J} = -1$, the quantum group $Sp_N^+ = O_N^{+J}$ appears as a liberation of the symplectic group in $N$ dimensions, $Sp_N = \left\{ U \in U_N \big| U = J\bar{U}J^{-1} \right\}$ in the sense that we have $(Sp_N^+)^{\text{class}} = Sp_N$. Moreover, $Sp_N \subset Sp_N^+$ is a “true liberation”, in the sense that the laws of the truncated characters

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

with $t \in [0,1]$ are related, in the $N \to \infty$ limit, by the Bercovici-Pata bijection.
Proof. Observe first that, with $J$ as above, for an arbitrary $N \times N$ scalar matrix $U$, the relation $JU = UJ$ tells us that $U$ must be of the following special form:

$$U = \begin{pmatrix} a & b & \ldots & u & v \\ -\bar{b} & \bar{a} & \ldots & -\bar{v} & \bar{u} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & s & x & y \\ -\bar{s} & \bar{r} & -\bar{y} & \bar{x} \end{pmatrix}$$

When $U$ is unitary, we obtain in this way the standard condition for the symplectic matrices. Thus $Sp_N$ is the group in the statement, and $Sp_N^+ = O_J^+$ is a liberation of it.

Regarding now the last assertion, this is a standard consequence of the Weingarten integration formula, by using categorical input from [7], [19], or from the various results below. To be more precise, the laws in question are Gaussian distributions for $Sp_N$ with $N \to \infty$, and semicircular variables for $Sp_N^+$ with $N \to \infty$, in agreement with [12].

We refer to [7], [19] for more on these facts, as well as to the more recent paper [16], and to the paper [5] as well, for more on the “true” half-liberations. \qed

As a first comment, it would be of course desirable as well to have a purely algebraic explanation for the fact that $Sp_N \subset Sp_N^+$ is indeed a “correct” liberation. We have two statements here, which are both conjectural, as follows:

1. $Sp_N \subset Sp_N^+$ is a maximal liberation, in the sense that there is no proper intermediate quantum group $Sp_N^+ \subset G \subset U_N^+$ satisfying $G_{\text{class}} = Sp_N$.
2. $Sp_N \subset Sp_N^+$ is the universal liberation of $Sp_N$, in the sense that given $G \subset U_N^+$ satisfying $G_{\text{class}} = Sp_N$, we must have $G \subset Sp_N^+$.

Observe that (2) implies (1). However, both questions are non-trivial. In fact, most of such questions are open even for the basic examples of easy quantum groups.

Finally, in view of some future use, let us record as well the following basic fact:

**Proposition 6.2.** At $N = 2$ we have $Sp_2 = Sp_2^+ = SU_2$.

Proof. It is well-known that we have $Sp_2 = SU_2$, the reason for this being that:

$$Sp_2 = SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| |a|^2 + |b|^2 = 1 \right\}$$

Our claim is that we have $Sp_2^+ = SU_2$ as well. Indeed, the relation $Ju = J\bar{u}$ tells us that the fundamental corepresentation $u$ must be of the following type:

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

Now since we must have $uu^* = u^*u = 1$, we obtain:

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
We conclude that \(\{a, a^*, b, b^*\}\) pairwise commute, and this gives the result. \(\square\)

In relation now with our present considerations, let us go back to Definition 5.3 above. We agree to call “sign” of the super-structure the sign \(\pm 1 = JJ\) appearing there. We have seen in the previous section how to associate a generalized Kronecker function to a positive super-structure. In the negative case, the result is as follows:

**Proposition 6.3.** Associated to a negative super-structure, \(J(e_i) = w_i e_{\tau(i)}\), is the generalized Kronecker function \(\delta^J_\pi \in \{-1, 0, 1\}\), defined on \(P_{\text{even}}\), given by

\[
\delta^J_\pi(i_1 \ldots i_k) = \delta^\tau_\pi(j_1 \ldots j_l) \delta^J_\pi(i_1 \ldots i_k)
\]

where \(\delta^\tau_\pi \in \{0, 1\}\) is the generalized Kronecker function constructed in section 5 above, in the positive case, and where the sign on the right is given by

\[
w^J_\pi(i_1 \ldots i_k) = \prod_{r=1}^{k} w^x_{i_r - r} \prod_{s=1}^{l} w^y_{j_s - s}
\]

where \(k = (x_1 \ldots x_k)\) and \(l = (y_1 \ldots y_l)\) are the writings of the colored integers \(k, l\) by using 0, 1 symbols, with the conventions \(\circ = 0, \bullet = 1\).

**Proof.** In order to check the various categorical conditions from Proposition 2.2, we just have to insert \(w^J\) signs in the proof of Proposition 5.4, a bit in the same way as we did in section 3 above, for the signatures. This can be done as follows:

1. Regarding the tensor product, here we can use the following formula:

\[
w^J_\pi(i_1 \ldots i_p, j_1 \ldots j_q) w^J_\sigma(k_1 \ldots k_r, l_1 \ldots l_s) = w^J_{[\pi\sigma]}(i_1 \ldots i_p, k_1 \ldots k_r, j_1 \ldots j_q, l_1 \ldots l_s)
\]

2. For the composition, we can use the following formula:

\[
w^J_\pi(i_1 \ldots i_p, j_1 \ldots j_q) w^J_\sigma(j_1 \ldots j_q, k_1 \ldots k_r) = w^J_{[\pi\sigma]}(i_1 \ldots i_p, k_1 \ldots k_r)
\]

3. For the involution axiom, we can use the following formula:

\[
w^J_\pi(i_1 \ldots i_p, j_1 \ldots j_q) = w^J_{\pi^*}(i_1 \ldots i_p, j_1 \ldots j_q)
\]

4. The identity axiom follows from the following computation:

\[
w^J_\pi(p, p) = w_p^{0-1} w_p^{0-1} = w_p^2 = 1
\]

5. As for the duality axiom, this follows from a similar computation, namely:

\[
w^J_{\pi^*}(p, p) = w_p^{0-1} w_p^{1-2} = w_p^2 = 1
\]

Thus, we have indeed a generalized Kronecker function, as claimed. \(\square\)
At the quantum group level now, we first have:

**Proposition 6.4.** The quantum group associated to the category of pairings $P_2$, via the above generalized Kronecker function, is the quantum group $O_{N}^{J+} = Sp_{N}^+$. 

**Proof.** In order to prove the result, we have to find inside our category the conditions which imply $u = J\bar{u}J^{-1}$. For the identity partition, we recall that we have:

$$\delta_{i}^{\tau} \left( \begin{array}{c} i \\ j \end{array} \right) = 1 \iff \exists p, \left( \begin{array}{c} i \\ j \end{array} \right) = \left( \begin{array}{c} \tau^{\bullet}(p) \\ \tau^{\circ}(p) \end{array} \right)$$

$$\iff \exists p, \left( \begin{array}{c} i \\ j \end{array} \right) = \left( \begin{array}{c} \tau(p) \\ p \end{array} \right)$$

$$\iff i = \tau(j)$$

In the case of negative structures, we have to add the following sign:

$$w^{J}_{\tau} \left( \begin{array}{c} \tau(p) \\ p \end{array} \right) = w^{1-1}_{\tau(p)}w^{0-1}_{p} = 1 \cdot w_{p} = w_{p}$$

We can equally use the duality partition, where we have:

$$\delta_{i}^{\tau} \left( ij \right) = 1 \iff \exists p, (i, j) = (\tau^{\circ}(p), \tau^{\circ}(p))$$

$$\iff \exists p, (i, j) = (\tau^{\circ}(p), \tau^{\bullet}(p))$$

$$\iff j = \tau(i)$$

In the case of negative structures, we have to add the following sign:

$$w^{J}_{\tau} (p, \tau(p)) = w^{0-1}_{p}w^{0-2}_{\tau(p)} = w_{p} \cdot 1 = w_{p}$$

Summarizing, in both cases we have $u = J\bar{u}J^{-1}$, and this gives the result. \hfill \Box

More generally, we have the following result, improving previous findings from [3]:

**Theorem 6.5.** The basic quantum unitary and reflection groups associated to the generalized Kronecker map constructed above are as follows,

\[
\begin{array}{cccc}
U_{N} & \longrightarrow & U_{N}^{+} \\
\downarrow & & \downarrow \\
Sp_{N} & \longrightarrow & Sp_{N}^{+} \\
\downarrow & & \downarrow \\
K_{N} & \longrightarrow & K_{N}^{+} \\
\downarrow & & \downarrow \\
K_{p} \wr \mathbb{Z}_{2} & \longrightarrow & K_{p}^{+} \wr \mathbb{Z}_{2} \\
\end{array}
\]

with respect to the usual fundamental corepresentation.
Proof. We follow the proof of Proposition 5.5 above, with the usual convention that $G^\times_K$ denote the various quantum groups to be computed.

(1) We know from Proposition 2.5 that we have $\bar{U}_N^+ = U_N^+$.
(2) We know from Proposition 6.4 that we have $\bar{O}_N^+ = S_{p_N}^+$.
(3) Regarding now $\bar{U}_N$, here we must impose to the standard coordinates of $\bar{U}_N^+ = U_N^+$ the relations coming from the basic crossing $\chi$, colored with its possible 4 matching colorings. But for any such matching coloring, the sign to be added is:

$$w^J_\chi \begin{pmatrix} i & j \\ j & i \end{pmatrix} = w^2_i \cdot w^2_j = 1 \cdot 1 = 1$$

Thus we have $\bar{T}_\pi(e_i \otimes e_j) = e_j \otimes e_i$, and we obtain in this way usual commutation relations between the variables \{u_{ij}, u^*_{ij}\}. Thus $\bar{U}_N$ is classical, and so $\bar{U}_N = U_N$.

(4) The quantum group $\bar{O}_N$ appears as an intersection, as follows:

$$\bar{O}_N = \bar{U}_N \cap \bar{O}_N^+ = U_N \cap S_{p_N}^+ = S_{p_N}$$

(5) Regarding now $\bar{K}_N$, we can follow again the method from the proof of Proposition 5.5. Indeed, in the main computation there, we have to add the following sign:

$$w^J_\chi \begin{pmatrix} i & i \\ i & i \end{pmatrix} = w^2_i = 1$$

(6) The quantum group $\bar{H}_N$ appears as an intersection, as follows:

$$\bar{H}_N = \bar{K}_N \cap \bar{O}_N^+ = K_N \cap S_{p_N}$$

In order to compute this intersection, we recall from the proof of Proposition 6.1 above that for a $N \times N$ scalar matrix $U$, the relation $JU = UJ$ reads:

$$U = \begin{pmatrix} a & b & \ldots \\ -b & a & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

Here $a, b, \ldots \in \mathbb{C}$. In the case $U \in K_N$, the $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ blocks can be either $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or of the following form, with $z \in \mathbb{T}$:

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$$

By using once again the orthogonality condition, we must have exactly one nonzero block on each row and column, so we obtain $\bar{H}_N = K_p \wr \mathbb{Z}_2$, as claimed.

(7) The proof of $\bar{K}_N^+ = K_N^+$ follows by using the same argument as in the proof of Proposition 5.5 above, because in each case, the sign to be added is 1.

(8) Finally, the quantum group $\bar{H}_N^+$ appears as an intersection, as follows:

$$\bar{H}_N^+ = \bar{K}_N^+ \cap \bar{O}_N^+ = K_N^+ \cap S_{p_N}^+$$

In order to compute this intersection, we use the interpretation of the commutation relation $uJ = J\bar{u}$ given in the proof of (6) above, with all the complex conjugates there
replaced of course by adjoints. By reasoning as there, we conclude that we have a free
wreath product decomposition $H_N^+ = K_p^+ \ast \mathbb{Z}_2$, as claimed. □

Summarizing, we have now an improved version of the twisting material in [1], [2], and
of the super-easiness considerations in [3], [10].

7. Cyclic extension

We discuss here a joint generalization of the usual Kronecker function, from section
1, and of the constructions of generalized Kronecker functions from sections 5-6. As a
starting observation, these constructions have the following common features:

(1) The functions have the multiplicativity property $\bar{\delta}_\pi^{(i_j)} = \prod_{\beta \in \pi} \bar{\delta}_\beta^{(i_{j_\beta})}$, where
the product is over all blocks of $\pi$, and $i_\beta, j_\beta$ are restricted multi-indices.

(2) The values on the blocks are nonzero when the indices come from a single index
$p \in \{1, \ldots, N\}$, via a certain algebraic procedure, involving permutations.

In order to have a common framework for these constructions, we can use the following
straightforward extension of Definition 5.3 above:

**Definition 7.1.** A complex super-structure on $\mathbb{C}^N$, having level $s \in \mathbb{N}$, is a linear map $J : \mathbb{C}^N \to \mathbb{C}^N$, which with respect to the standard basis of $\mathbb{C}^N$ is of the form

$$J(e_i) = w_i e_{\tau(i)}$$

where $\tau \in S_N$ is a permutation satisfying $\tau^s = \text{id}$, and where $w_i \in \mathbb{Z}_s$ satisfy

$$w_i w_{\tau(i)} \ldots w_{\tau^{s-1}(i)} = w, \quad \forall i$$

where $w \in \mathbb{Z}_s$ is a fixed number, called sign of the super-structure.

Observe that the condition on the numbers $w_i$ tells us that we must have $J^s = w$. In
fact, this condition is satisfied precisely when $J^s$ is a scalar multiple of the identity.

Observe also that at $s = 2$ we obtain precisely the notion in Definition 5.3.

In the simplest case, where $w_i = 1$ for any $i$, we have the following result:

**Proposition 7.2.** Let $s \in \mathbb{N}$, and let $\tau \in S_N$ satisfying $\tau^s = \text{id}$. We can define then a
Kronecker function $\delta^r_\tau \in \{0, 1\}$, for any $\pi \in P^s$, by setting

$$\delta^r_\tau = \prod_{\beta \in \pi} \delta^r_\beta$$

with the values on the blocks being $\delta^r_{1^l} = 1$ when the index is of the form

$$\left( \begin{array}{cccc}
\tau^{x_1-1}(p) & \tau^{x_2-2}(p) & \ldots & \tau^{x_k-k}(p) \\
\tau^{y_1-1}(p) & \tau^{y_2-2}(p) & \ldots & \tau^{y_l-l}(p)
\end{array} \right)$$

for a certain $p \in \{1, \ldots, N\}$, where $k = (x_1 \ldots x_k)$ and $l = (y_1 \ldots y_l)$ are the writings of
$k, l$ using 0, 1 symbols, with the conventions $\circ = 0, \bullet = 1$. In the cases $s = 1, 2$ we obtain
respectively the usual Kronecker function $\delta_\pi$, and the old Kronecker function $\delta^r_\pi$. 
Proof. Let us first check the last assertion. At \( s = 1 \) we have \( \tau = id \), and the fact that we obtain indeed \( \delta_\pi \) comes from the fact that the fitting indices must be as follows:

\[
\binom{i}{j} = \left( \begin{array}{cccc} p & p & \ldots & p \\ p & p & \ldots & p \end{array} \right)
\]

At \( s = 2 \) now, we have \( \tau^2 = id \), all the exponents can be reduced modulo 2, and the fitting indices must be as follows, with the conventions \( \bar{0} = 1, \bar{1} = 0 \):

\[
\binom{i}{j} = \left( \begin{array}{cccc} \tau^{\bar{x}_1}(p) & \tau^{\bar{x}_2}(p) & \tau^{\bar{x}_3}(p) & \ldots \\ \tau^{\bar{y}_1}(p) & \tau^{\bar{y}_2}(p) & \tau^{\bar{y}_3}(p) & \ldots \end{array} \right)
\]

In terms of \( q = \tau(p) \), this formula takes the following form:

\[
\binom{i}{j} = \left( \begin{array}{cccc} \tau^{x_1}(q) & \tau^{x_2}(q) & \tau^{x_3}(q) & \ldots \\ \tau^{y_1}(q) & \tau^{y_2}(q) & \tau^{y_3}(q) & \ldots \end{array} \right)
\]

But this is exactly the formula in Proposition 5.4 above, with \( p \) replaced by \( q \), because our exponentiation conventions from there coincide with those used here.

Summarizing, the construction in the statement extends the previous ones, which appear at \( s = 1, 2 \). Regarding now the proof of the categorical axioms, this is similar to the proof of Proposition 5.4, with just a few adjustments needed, as follows:

(1) Regarding the tensor product, by using \( \delta_\pi^\gamma = \prod_{\beta \in \pi} \delta_\beta^\gamma \), along with the fact that the blocks of \( [\pi\sigma] \) are the blocks of \( \pi \) or of \( \sigma \) we obtain, as desired:

\[
\begin{align*}
\delta_\pi^\tau \binom{i_1 \ldots i_p}{j_1 \ldots j_q} \delta_\sigma^\gamma \binom{k_1 \ldots k_r}{l_1 \ldots l_s} &= \prod_{\beta \in \pi} \delta_\beta^\tau \binom{i_1 \ldots i_p |_{\beta}}{j_1 \ldots j_q |_{\beta}} \prod_{\gamma \in \sigma} \delta_\gamma^\tau \binom{k_1 \ldots k_r |_{\gamma}}{l_1 \ldots l_s |_{\gamma}} \\
&= \prod_{\rho \in [\pi\sigma]} \delta_\rho^\tau \binom{i_1 \ldots i_p k_1 \ldots k_r |_{\rho}}{j_1 \ldots j_q l_1 \ldots l_s |_{\rho}} \\
&= \delta_{[\pi\sigma]}^\tau \binom{i_1 \ldots i_p k_1 \ldots k_r}{j_1 \ldots j_q l_1 \ldots l_s}
\end{align*}
\]

(2) Regarding now the composition, we have to verify here that:

\[
\sum_{j_1 \ldots j_q} \delta^\tau_{\sigma} \binom{i_1 \ldots i_p}{j_1 \ldots j_q} \delta^\tau_{\pi} \binom{j_1 \ldots j_q}{k_1 \ldots k_r} = N_{c(\tau)}^{e(\tau)} \delta^\tau_{[\pi\sigma]} \binom{i_1 \ldots i_p}{k_1 \ldots k_r}
\]

In this formula, the sum on the left equals the cardinality of the following set:

\[
S = \left\{ j_1, \ldots, j_q \, | \, \delta^\tau_{\sigma} \binom{i_1 \ldots i_p}{j_1 \ldots j_q} = 1, \delta^\tau_{\pi} \binom{j_1 \ldots j_q}{k_1 \ldots k_r} = 1 \right\}
\]

In order to compute the cardinality of this set, we can proceed as in the proof of Proposition 5.4. Indeed, the middle components obtained when concatenating \( \pi, \sigma \) can be isolated by using our condition \( \delta^\tau_{\pi} = \prod_{\beta \in \pi} \delta^\tau_{\beta} \). Thus, we just have to count the number of free multi-indices \( j \), and prove that their number equals \( N_{c(\tau)}^{e(\tau)} \). But the free indices for
each component must come from a single index \( p \in \{1, \ldots, N\} \), via the formula for \( \delta^\tau_{1k} \) from the statement, and so we obtain as multiplicity the number \( Nc(\pi) \), as claimed.

(3) Regarding the involution axiom, here we must check that we have:

\[
\delta^\tau_\pi(i_1 \ldots i_p) = \delta^\tau_\pi(j_1 \ldots j_q) = \delta^\tau_\pi(i_1 \ldots i_p)
\]

But this is clear, because we can use here the condition \( \delta^\tau_\pi = \prod_{\beta \in \pi} \delta^\tau_\beta \), together with the formula of \( \delta^\tau_{1k} \), which is invariant under upside-down turning.

(4) The identity axiom follows from the following computation:

\[
\delta^\tau_\pi(i) = 1 \iff \exists p, (i) = (\tau^{r-1}(p)) \\
\iff \exists p, (i) = (\tau^{-1}(p)) \\
\iff i = j
\]

(5) The duality axiom follows from a similar computation, as follows:

\[
\delta^\tau_\pi(ij) = 1 \iff \exists p, (i, j) = (\tau^{r-1}(p), \tau^{-1}(p)) \\
\iff \exists p, (i, j) = (\tau^{-1}(p), \tau^{-1}(p)) \\
\iff i = j
\]

Thus, we have indeed a generalized Kronecker function, as claimed. \( \square \)

In the general case now, where the super-structure is as in Definition 7.1, with non-trivial parameters \( w_i \in \mathbb{Z}_s \), the result is as follows:

**Proposition 7.3.** Associated to general complex super-structure, \( J(e_i) = w_i e_{\tau(i)} \), is the generalized Kronecker function \( \delta^J_\pi \in \mathbb{Z}_s \cup \{0\} \), defined on \( P^s \), given by

\[
\delta^J_\pi(i_1 \ldots i_k) = \delta^\tau_\pi(i_1 \ldots i_k) w^J_\pi(i_1 \ldots i_k)
\]

where \( \delta^\tau_\pi \in \{0,1\} \) is the generalized Kronecker function constructed in Proposition 7.2 above, in the positive case, and where the scalar on the right is given by

\[
w^J_\pi(i_1 \ldots i_k) = \prod_{r=1}^k w^{(-1)^r(x_r-r)}_{j_r} \prod_{s=1}^l w^{(-1)^s(y_s-y_s)}_{j_s}
\]

where \( k = (x_1 \ldots x_k) \) and \( l = (y_1 \ldots y_l) \) as usual. In the cases \( s = 1,2 \) we obtain respectively the usual Kronecker function \( \delta_\pi \), and the old Kronecker function \( \delta^\tau_\pi \).

**Proof.** The extension can be done as in the proof of Proposition 6.3 above, by inserting \( w^J \) terms in the proof of Proposition 7.2. The situation is as follows:
(1) Regarding the tensor product, here we can use the following formula:

\[ w_{\pi}^{\sigma} \left( i_1 \ldots i_p \right) \left( j_1 \ldots j_q \right) = w_{\pi \sigma}^{\sigma} \left( i_1 \ldots i_p, k_1 \ldots k_r \right) \]

(2) For the composition, we can use the following formula:

\[ w_{\pi}^{\sigma} \left( i_1 \ldots i_p \right) w_{\sigma}^{\rho} \left( j_1 \ldots j_q \right) = w_{\pi \sigma \rho}^{\sigma} \left( i_1 \ldots i_p \right) \]

(3) For the involution axiom, we can use the following formula:

\[ w_{\pi}^{\sigma} \left( i_1 \ldots i_p \right) = w_{\pi}^{\sigma} \left( j_1 \ldots j_q \right) \]

(4) The identity axiom follows from the following computation:

\[ w_{\pi}^{\rho} \left( p \right) = w_{\pi}^{1-0} w_{p}^{0-1} = w_{p} \cdot w_{p}^{-1} = 1 \]

(5) As for the duality axiom, this follows from a similar computation, namely:

\[ w_{\pi}^{\rho} \left( p, p \right) = w_{p}^{0-1} w_{p}^{2-1} = w_{p}^{1} \cdot w_{p} = 1 \]

Thus, we have indeed a generalized Kronecker function, as claimed.

Regarding now the last assertion, observe first that our construction extends indeed the construction in Proposition 7.2, which can be recovered with trivial scalars, \( w_i = 1 \) for any \( i \). It is also clear that at \( s = 1 \) we obtain the usual Kronecker function \( \delta_\pi \).

As for the case \( s = 2 \), the first remark here is that the complex super-structures of level 2, in the sense of Definition 7.2 above, coincide with the old super-structures, as axiomatized in Definition 5.3. With this identification made, the generalized Kronecker functions that we construct here coincide with those from sections 5-6 above.

At the quantum group level now, we recall from Proposition 4.2 above that at \( s \geq 3 \) the basic diagram of categories of partitions, obtained from the diagram in Theorem 1.7 above by intersecting with the category \( P^s \), collapses to the following diagram:

\[ \begin{array}{cccc}
P_2 & \rightarrow & NC_2 \\
\downarrow & \nearrow & \downarrow \\
NC_2 & \rightarrow & NC_{even}
\end{array} \]

Thus, in order to have results regarding the basic examples, we must compute the quantum groups associated to these categories. We have here:
Theorem 7.4. For a complex super-structure $J$ having level $s \geq 3$, the associated basic quantum unitary and reflection groups are as follows, independently of $J$:

Also, in the case of the super-structures of level $s = 1, 2$, we recover in this way the quantum groups in Theorem 1.7, Theorem 5.7, Theorem 6.5.

Proof. First of all, the last assertion is clear. Regarding now the case $s \geq 3$, here we must intersect the categories with $P^s$, and we are led to the computation of the quantum groups associated to the categories of partitions given above.

For this purpose, let us first discuss the case where $w_i = 1$ for any $i$, corresponding to the formalism of Proposition 7.2 above. The idea here is to follow the proof of Proposition 5.5 above, and to perform some changes, where needed:

(1) We know from Proposition 2.5 that we have $\bar{U}_N^+ = U_N^+$.

(2) Since the quantum group $\bar{O}_N^+$ comes at $s \geq 3$, according to our conventions, from the same category of partitions as $\bar{U}_N^+$, we have as well $\bar{O}_N^+ = U_N^+$.

(3) Regarding now $\bar{U}_N$, this appears inside $U_N^+$ from the relations coming from the basic pairing, with matching colorings. Since we have $\delta_\tau^\pi = \prod_{\beta \in \pi} \delta_\beta^\tau$, the values of our generalized Kronecker function on this pairing coincide with the values of the usual Kronecker function. Thus, exactly as in the proof of Proposition 5.5, we obtain $\bar{U}_N = U_N$.

(4) Since the quantum group $\bar{O}_N$ comes by definition at $s \geq 3$ from the same category of partitions as $\bar{U}_N$, we have as well $\bar{O}_N = U_N$.

(5) Regarding now $\bar{K}_N$, as explained in the proof of Proposition 5.5, we have to examine the linear map associated to the following partition:

According to our conventions for the Kronecker function, from Proposition 7.2 above, the indices which fit into this partition are those of the following form:

$$
\begin{pmatrix}
\tau^{0-1}(p) & \tau^{1-2}(p) \\
\tau^{0-1}(p) & \tau^{1-2}(p)
\end{pmatrix} =
\begin{pmatrix}
\tau^{-1}(p) & \tau^{-1}(p) \\
\tau^{-1}(p) & \tau^{-1}(p)
\end{pmatrix}
$$
Thus, the situation here is identical to the one in the proof of Proposition 5.5, and by reasoning as there, we obtain that we have \( \bar{K}_N = K_N \).

(6) Regarding now \( \bar{K}^+_N \), the study is once again similar to the one in the proof of Proposition 5.5, and we obtain \( \bar{K}^+_N = K^+_N \).

(7) Let us compute now \( \bar{H}_N \). This appears as a subgroup, \( \bar{H}_N \subset \bar{K}_N = K_N \), coming from the category of partitions \( P^{[2,s]} \). With the notation \( [2,s] = 2r \), the standard generators for this category are the 1-block partitions having \( r \) upper legs, \( r \) lower legs, and with the labels being such that this partition belongs indeed to \( P^{2r} \). By performing a computation similar to the one in [6], we obtain \( \bar{H}_N = H^r_N \), as claimed.

(8) Finally, regarding \( \bar{H}^+_N \), here we can use once again the partitions from the proof of (7) above, which are the standard generators of \( NC^m \), and we obtain \( \bar{H}^+_N = H^r_N \).

Summarizing, we are done with the case where all the parameters are trivial, \( w_i = 1 \) for any \( i \). In the general case now, where we have parameters \( w_i \in \mathbb{Z}_s \), the situation is similar, because for all the partitions considered above, the \( w^J \) factors won’t affect the computations. Thus, we obtain the same quantum groups, and this finishes the proof. □

Summarizing, we are in a collapsing situation a bit similar to the one that we met in section 4 above, in connection with the notion of complex signatures.

8. Open questions

We discuss in this section a number of open questions, in relation with the general theory that we developed above. These questions all look quite tractable, but we have not investigated them, in order to keep the present work at a reasonable length.

Problem 8.1. Further examples.

There are several interesting questions, regarding the computation of some further basic examples of generalized easy quantum groups. As explained for instance in [5], [15], we can construct half-liberations \( G_N \subset G^*_N \subset G_N^+ \), by imposing the half-commutation relations \( abc = cba \) to the standard coordinates of \( G_N^+ \), and their adjoints. The basic 8-diagram of easy quantum groups extends in this way into a 12-diagram, as follows:

![Diagram](image-url)

We have as well some natural intermediate objects for the complexification inclusions, going from front to back, obtained by generalizing the construction of the intermediate
subgroup $O_N \subset TO_N \subset U_N$. We obtain in this way an 18-diagram, as follows:

As explained in [5], this is an intersection and generation diagram, which can be thought of as being the diagram of “extended basic” easy quantum groups. The computation of the corresponding generalized easy quantum groups looks like an interesting question. In practice, in view of [5], we have to analyze the following two diagrams:

The computations here are quite technical, especially for the first diagram, in the context of the complex signatures, and we have no results here.

In general, the classification problem for the easy quantum groups and for the corresponding categories of partitions, which could bring more input for the present considerations, is an open problem. For the status of this question, and for a list of known examples, we refer here to the recent papers [24], [30], [33], [34].

Problem 8.2. Unification questions.

The formalism of generalized easy quantum groups, from section 2 above, is probably a bit too general, because it requires the verification of the somewhat obscure conditions in Proposition 2.2. It would be very interesting to work out a “softer” formalism, unifying the signature constructions from sections 3-4 above with the super-easiness constructions from sections 5-7, with generalized Kronecker functions of the following type:

$$\bar{\delta}_\pi \left( \begin{array}{ccc} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{array} \right) = \delta_\pi \left( \begin{array}{ccc} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{array} \right) w^t \left( \begin{array}{ccc} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{array} \right) \varepsilon [i_1, \ldots, i_k]$$

However, the subject is quite technical, and we have no results so far. As a concrete question here, to be solved first, we have the twisting problem for the symplectic group
Indeed, at $N = 2$ we have $Sp_2 = Sp_2^+ = SU_2$, and according to the general philosophy from [1, 2] that “the free versions are not twistable”, the symplectic group $Sp_2$ should therefore not be twistable. It is not very clear if this is indeed the case, nor whether this phenomenon should extend to $Sp_N$, with $N \in 2\mathbb{N}$ arbitrary.

There are many other unification questions left as well. Indeed, there are several interesting combinatorial generalizations of the easy quantum group framework, from [18] on one hand, and from [31], [40], on the other. As a main question regarding this subject, however, there are many similarities with Freslon’s work in [22], waiting to be understood. Generally speaking, we believe that a unification with [22] could emerge on something quite conceptual, that we can call “easiness 2.0” afterwards.

**Problem 8.3. Weingarten integration.**

We recall that, assuming that $G \subset U_N^+$ has Tannakian category $C = (C(k, l))$, the Haar integration over $G$ is given by the following Weingarten formula, where $D_k$ is a linear basis of $C(0, k)$, $\delta_\pi(i) = \langle \pi, e_i \otimes \ldots \otimes e_i \rangle$, and $W_{kN} = G_{kN}^{-1}$, with $G_{kN}(\pi, \sigma) = \langle \pi, \sigma \rangle$:

$$
\int_G u_{i_1j_1} \ldots u_{i_kj_k} = \sum_{\pi, \sigma \in D_k} \delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi, \sigma)
$$

Indeed, according to [42], the integrals in the statement form altogether the orthogonal projection $P^K$ onto $Fix(u^{\otimes k}) = \text{span}(D_k)$. By a standard linear algebra computation, it follows that we have $P = WE$, where $E(x) = \sum_{\pi \in D_k} \langle x, \xi_\pi \rangle \xi_\pi$, and where $W$ is the inverse on $\text{span}(T_\pi|\pi \in D_k)$ of the restriction of $E$, and this gives the result. See [5].

In the complex signature setting, the Gram matrix is given by:

$$
\langle \xi_\pi, \xi_\sigma \rangle = \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k)\delta_\sigma(i_1, \ldots, i_k)\varepsilon[i_1, \ldots, i_k]\varepsilon[i_1, \ldots, i_k]
$$

Thus, the Gram matrix, and so the Weingarten matrix as well, are invariant under twisting, and this generalizes some previous $s = 2$ observations from [1]. In the case of the super-easy quantum groups, however, nothing much is known about the Gram and Weingarten matrices, and these questions are waiting to be investigated.

**Problem 8.4. Poisson laws.**

At a more elementary level, it follows from the Peter-Weyl theory from [42] that the asymptotic spectral measure of the main character $\chi = \sum_i u_{ii}$ with respect to the Haar measure of $G$ is the $*$-distribution having the following $*$-moments:

$$
\int_G \chi^k = \#D(0, k)
$$
As a consequence, in the quantum reflection group case, we have many new examples of quantum groups having as spectral measure a compound free Poisson law, in the sense of free probability theory [36]. This suggests the following general questions:

(1) When is the law of $G \subset U_N$ compound Poisson?
(2) When is the law of $G \subset U_N^+$ compound free Poisson?
(3) What is the generalized easy formalism, covering all these examples?

In relation with (1), there are several solutions of type $G = F \wr S_N$, with $F$ finite. Regarding now (2), there are some examples of type $G = F \wr_* S_N^+$, and some further examples of the same type, coming from [13], [21], [26], [32]. Further examples come from the present work. In general, the classification problem for such quantum groups is probably related to the notion of monoidal equivalence, from [14], [28], [29].

**Problem 8.5. Gram determinants.**

Getting back now to the general Weingarten integration, a key problem is that of computing the determinants of the associated Gram matrices. In view of the work in [20], and from [8], of particular interest is the quantum reflection group case.

In the classical case the determinant formula is very simple, coming from the semilattice property of the corresponding category of partitions, as follows:

$$\det(G_{kN}) = \prod_{\pi \in D(N)} \frac{N!}{(N - |\pi|)!}$$

In the free case the computation is substantially more complicated, corresponding to the meander determinants computed in [20], and there are some open problems, even in the usual easy case. According to [8], the goal here is that of finding a factorization formula for the determinant similar to the one from the classical case, as follows:

$$\det(G_{kN}) = \prod_{\pi \in D(N)} \varphi(\pi)$$

We refer to [8] for more details here, and for some open questions, in the usual easy case. One hope here would be that our present point of view, which provides a lot more flexibility, could be used in order to solve some of these questions. We do not know.

**Problem 8.6. Wishart matrices.**

There are some interesting questions as well in relation with the block-modified Wishart matrices [9]. To be more precise, the problem is whether the computations in [9], which are based on diagrammatic structures reminding the orthogonal easy quantum groups, have or not a unitary extension, perhaps by using the formalism from the present paper, as to cover as well, and in $*$-moments, the free Bessel laws constructed in [6].
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