Scalar–tensor gravitation and the Bakry–Émery–Ricci tensor

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Abstract
The Bakry–Émery generalized Ricci tensor arises in scalar–tensor gravitation theories in the conformal gauge known as the Jordan frame. Recent results from the mathematics literature show that standard singularity and splitting theorems that hold when an energy condition is applied in general relativity also hold when that energy condition is applied to the Bakry–Émery tensor. We show here that a direct consequence is that the Hawking–Penrose singularity theorem and the timelike splitting theorem hold for scalar–tensor theory in the Jordan frame. As examples, we consider dilaton gravity (including totally anti-symmetric torsion) and the Brans–Dicke family of scalar–tensor theories. For Brans–Dicke theory the theorems do not extend to cover the entire space of values of the Brans–Dicke family parameter $\omega$, and so may fail to hold for $\omega < -1$. Observations show that this range of values does not describe our Universe, but the result is in accord with examples in the literature of Brans–Dicke spacetimes that have no singularity in the Jordan frame and do not split as a Riemannian product.

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1. Introduction
Many theorems in general relativity hold under a positivity assumption on components of the nongravitational stress–energy tensor, usually called an energy condition, of which there are various inequivalent forms. Because of the Einstein equation, this assumption then becomes a condition on the Einstein curvature and so, ultimately, on the Ricci curvature. Once one has a condition on Ricci curvature, one has a tool to prove theorems. Examples of these theorems include the Hawking–Penrose singularity theorem ([6], p 266), and the timelike splitting theorem [4, 5], among others. The singularity theorem is sometimes interpreted as the statement that nonnegative Ricci curvature generically evolves to produce singularities,
while the splitting theorem shows that the nongeneric, nonsingular cases must have quite special geometry\(^1\).

Scalar–tensor gravitation theories may be expressed in various conformal gauges. The two most common such gauges are known as the **Einstein frame** and the **Jordan frame**. In the Einstein frame, it is usually a simple matter to show that standard theorems will follow from an energy condition imposed on stress–energy. Note, however, that this description of the theory is not ‘minimally coupled’. One may therefore prefer to impose the energy condition on the theory in the Jordan frame, but in this frame an energy condition imposed on nongravitational stress–energy does not directly lead to a condition on the Ricci curvature from which one could then derive these theorems (at least, in a direct manner by following the usual general relativity proofs).

As well, other assumptions, which limit the solution space of a scalar–tensor theory in the necessary manner to imply that these theorems hold\(^2\), are not invariant in form under the conformal transformation between ‘frames’. The net result is that, while singularity and splitting theorems may hold for Einstein-frame solutions under certain assumptions, similar theorems may not hold when assumptions of the same form (thus, not conformally transformed versions of Einstein-frame assumptions) are applied directly to Jordan-frame solutions of the theory.

In this short note, I point out that recent progress in the mathematics literature means that we now have available singularity and splitting theorems arising from energy conditions and other assumptions applied directly in the Jordan frame. The theorems concerned are not obtained as corollaries from easily-established Einstein frame theorems by simple methods. An interesting feature is that, for Brans–Dicke theory, the theorems do not cover the case of a Brans–Dicke parameter \(\omega < -1\) (and require an additional assumption in the special case of \(\omega = -1\)). This is consistent with results reported in [9], which found that for four-dimensional Brans–Dicke theory in the Jordan frame with a barotropic perfect fluid and with Brans–Dicke parameter \(\omega \leq -4/3\), there are nonsingular, nonsplit solutions.

To understand the mathematical progress that facilitates this, note that in applications that touch upon Riemannian geometry, such as optimal transportation [10], generalizations of the Ricci curvature arise. An important example is the Bakry–Émery (or Bakry–Émery–Ricci (BER)) tensor, which augments the Ricci tensor with another term given by the Hessian of a twice-differentiable weight function. Some familiar theorems in Riemannian geometry can be modified to hold under the assumption that the Bakry–Émery tensor, rather than the usual Ricci tensor, obeys a sign condition [7]. In Lorentzian geometry, Case [3] has shown that a similar sign condition on timelike components of the Bakry–Émery tensor—i.e. an energy condition—will, in an analogous fashion to the Riemannian case, imply that singularity theorems and the timelike splitting theorem hold.

In section 2 below, we state Case’s versions of the singularity and timelike splitting theorems and then explain some of the terminology. In particular, we define the generalized Ricci tensor and Bakry–Émery tensor and discuss the energy conditions. In section 3 we apply these theorems to obtain Jordan-frame singularity and timelike splitting theorems for Brans–Dicke theory and (1-loop) dilaton gravity, including dilaton gravity with totally skew torsion. A brief concluding section contains some final remarks.

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\(^1\) The term **generic** in this sentence will actually acquire a precise meaning below. Also, while splitting theorems endeavour to describe the nongeneric cases, they typically do not fully describe them, since some assumptions are needed; in particular, the assumption of a complete timelike line; see section 2.

\(^2\) An example of an assumption that is not conformally invariant is the assumption of the existence of a complete timelike line in the splitting theorem below.
2. Case’s theorems

In [3], Case has proved the following two theorems.

**Theorem 2.1 (Case’s singularity theorem).** Let \( M \) be a chronological spacetime with \( \dim M \geq 3 \). Let \( \varphi : M \to \mathbb{R} \) obey the \( \varphi \)-generic condition\(^3\). Assume that either

1. there is an integer \( q > 0 \) such that the generalized Ricci tensor \( GRic[g, \varphi, q] \) obeys the energy condition, or
2. the Bakry–Émery tensor \( BER[g, \varphi] \) obeys the energy condition and \( \varphi \) is bounded away from zero\(^4\) (\( \varphi \geq C > 0 \) for some \( C \in \mathbb{R} \)).

Assume also that either

(a) \( M \) has a point \( p \) such that, along each null geodesic \( \gamma \) through \( p \), the modified null expansion scalar \( \hat{\theta} := \theta + \nabla_\gamma \log \varphi \) of the null geodesic congruence through \( p \) is negative somewhere to the future or past of \( p \), or
(b) \( M \) has a closed \( \varphi \)-trapped surface, or\(^5\)
(c) \( M \) has a compact spacelike hypersurface.

Then \((M, g)\) is nonspacelike geodesically incomplete.

**Theorem 2.2 (Case’s timelike splitting theorem).** Let \((M, g)\) be a connected spacetime such that

1. \((M, g)\) is either timelike geodesically complete or globally hyperbolic,
2. \( M \) contains a complete timelike line\(^6\), and
3. either
   
   (a) there is an integer \( q > 0 \) such that the generalized Ricci tensor \( GRic[g, \varphi, q] \) obeys the energy condition, or
   
   (b) the Bakry–Émery tensor \( BER[g, \varphi] \) obeys the energy condition and \( \varphi \) is bounded away from zero.

Then \((M, g)\) is isometric to \((\mathbb{R} \times \Sigma, -dt^2 \oplus h)\), where \((\Sigma, h)\) is a complete Riemannian manifold and \( \varphi \) is constant along \( \mathbb{R} \).

We now undertake to explain some of the terminology used in these theorems. First, for \((M, g)\) a Lorentzian \( n \)-manifold and \( \varphi \) a twice-differentiable function, we can define a family, parametrized by \( q \), of generalizations of the Ricci tensor (see, e.g., [10]). Taking \( q \to \infty \) this yields the Bakry–Émery tensor.

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3 Our usage differs from that of Case, who uses ‘\( f \)-generic’ with \( f = -\log \varphi \). The \( \varphi \)-generic condition is that \( \gamma_{\mu\nu}[\varphi(\gamma)] \rho^\gamma \rho^\nu \) is nonzero somewhere along each inextendible timelike geodesic \( \gamma \) (parametrized so that \( \gamma(\gamma, \gamma') = -1 \), where \( \gamma_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} \left( \nabla_\gamma \varphi \nabla_\gamma \varphi - \frac{n}{\varphi} \right) \left( \delta_{\mu\nu} - \frac{\varphi}{\gamma} \xi \xi \right) \)). The \( \varphi \)-generic condition implies that \( S_{\mu\nu} \rho^\mu \rho^\nu \) will be nonzero somewhere along \( \gamma \); see definitions 2.7 and 3.1 of [3] (but in the present paper we use the curvature tensor and terminology conventions of [6]).

4 Equivalently, \( f = -\log \varphi \) is bounded above.

5 Case uses the term ‘\( f \)-trapped’ with \( f = -\log \varphi \). We define a closed (compact, without boundary), co-dimension 2, \( C^2 \) surface \( \Sigma \) to be \( \varphi \)-trapped if the modified null expansion scalars \( \hat{\theta} \) of the two oppositely directed null congruences leaving it orthogonally obey \( \hat{\theta}|_p \leq 0 \) at each \( p \in \Sigma \). This amounts to saying that each congruence is initially converging, if convergence is measured with the rescaled metric \( \hat{g} = \varphi^{2/(\alpha-2)} g \).

6 Definition: a timelike line \( \gamma \) is an inextendible timelike geodesic such that, for each pair of points \( p, q \) along \( \gamma \) and every piecewise-smooth timelike curve joining these points, the proper time interval along such curves from \( p \) to \( q \) is maximized by \( \gamma \). For the theorem, the line must be complete, meaning that its affine parameter takes values throughout all of \( \mathbb{R} \).
Definition 2.3. Let \( \varphi : M \to (0, \infty) \) be twice differentiable and let \( q \in (0, \infty) \). The generalized Ricci tensor \( \text{GRic}[g, \varphi, q] \), also denoted \( G_{ij} \), is the tensor

\[
\text{GRic}[g, \varphi, q] := \text{Ric}[g] - \text{Hess}(\log \varphi) - \frac{1}{q} \nabla \log \varphi \otimes \nabla \log \varphi,
\]

\[
\equiv \text{Ric}[g] - \frac{\text{Hess}(\varphi)}{\varphi} + \left( 1 - \frac{1}{q} \right) \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi^2},
\]

(2.1)

where \( \text{Hess}(\varphi) := \nabla \nabla \varphi \) is the Hessian of \( \varphi \). The BER tensor \( \text{BER}[g, \varphi] \), also denoted \( B_{ij} \), is defined by

\[
\text{BER}[g, \varphi] := \text{Ric}[g] - \text{Hess}(\log \varphi),
\]

\[
\equiv \text{Ric}[g] - \frac{\text{Hess}(\varphi)}{\varphi} + \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi^2}.
\]

(2.2)

The theorems above assume that either BER or GRic obeys what is called the energy condition. We now define energy conditions. Our first such condition will apply to any \((0, 2)\)-tensor.

Definition 2.4. We say that a \((0, 2)\)-tensor \( S \) obeys the energy condition on \((M, g)\) if, for every \( p \in M \) and every timelike vector \( t \in T_p M \), then \( S|_p (t, t) \geq 0 \).

By continuity, if a \((0, 2)\)-tensor \( S \) obeys the energy condition, then \( S(l, l) \geq 0 \) for all null vectors \( l \) as well.

In general relativity, it is common to apply the energy condition directly to the stress–energy tensor \( T_{ij} \) of the theory. This is then called the weak energy condition, and in general relativity it implies that the Einstein tensor obeys the energy condition. If the energy condition is instead applied to \( T_{ij} - \frac{1}{n-2} g_{ij} T \), it is called the strong energy condition, and in general relativity this implies that the Ricci tensor obeys the energy condition. With that in mind, assume now that we are provided with a distinguished tensor \( T_{ij} \) called the stress–energy tensor, whether we are working in general relativity or within a more general Lorentzian framework, such as that provided another gravitation theory. We formulate an energy condition on \( T_{ij} \) that is natural in Brans–Dicke theory and which contains the weak and strong energy conditions as special cases.

Definition 2.5. We say the \( \omega \)-energy condition holds for a given \( \omega > -(n-1)/(n-2) \) if

\[
\left( T_{ij} - \frac{(1 + \omega)}{(n-1 + (n-2)\omega)} g_{ij} T \right) t^i t^j \geq 0
\]

(2.3)

for all timelike vectors \( t^i \in T_p M \) and all \( p \in M \). Taking \( \omega \to \infty \), we simply say that the strong energy condition holds if

\[
\left( T_{ij} - \frac{1}{n-2} g_{ij} T \right) t^i t^j \geq 0
\]

(2.4)

for all timelike vectors \( t^i \in T_p M \) and all \( p \in M \). On the other hand, we say the weak energy condition holds if the \( \omega \)-energy condition holds for \( \omega = -1 \); that is, if

\[
T_{ij} t^i t^j \geq 0
\]

(2.5)

for all timelike vectors \( t^i \in T_p M \) and all \( p \in M \).

That is, the weak energy condition holds if the energy condition holds for \( S_{ij} = T_{ij} \), and the strong energy condition holds when the energy conditions hold for \( S_{ij} = T_{ij} - \frac{1}{n-2} g_{ij} T \).
The terms weak, strong, and $\omega$-energy condition will always refer to $T_{ij}$, whereas we have used energy condition to refer to any tensor.

The $\omega$-energy condition holds for all $\omega \geq -1$ if and only if matter obeys both the weak and the strong energy condition.

The $\omega$-energy condition for any $\omega$ reduces to the weak energy condition if matter consists only of massless radiation, since then $T = 0$.

3. Applications to scalar–tensor gravitation

3.1. Dilaton gravity

It is believed that low energy string theory is described by a nonlinear sigma model at a fixed point of its renormalization group flow. The precise model depends on whether torsion in the guise of the so-called $B$-field is present\(^7\), and whether the string theory in question is bosonic, supersymmetric, or heterotic. The fixed point condition means that the so-called beta-functions of the theory vanish. Of particular interest will be the condition for the vanishing of the graviton beta-function, $\beta_g$. This is a condition on the metric of the target manifold and can be expressed as

$$0 = \beta_g \equiv \alpha' \left( R_{ij} + 2 \nabla_i \nabla_j \Phi - \frac{1}{4} H_{ijkl} H^{ijkl} \right) + O(\alpha'^2). \quad (3.1)$$

Here $\alpha'$ is a constant, $\Phi$ is a scalar field called the dilaton field, and the 3-form $H$ is the field strength tensor $H = dB$ of a 2-form field $B$. The $O(\alpha'^2)$ terms are in fact a power series in $\alpha'$, and for the bosonic nonlinear sigma model the terms consist of terms of quadratic and higher order in the Riemann tensor, $\nabla \Phi$, $H$, and their derivatives (for other models, additional fields can eventually appear). These terms are exactly determined, but are known only to rather low order in $\alpha'$ (the precise order depends on the sigma model). Therefore, we will write (3.1) in the form

$$R_{ij} + 2 \nabla_i \nabla_j \Phi = \frac{1}{4} H_{ijkl} H^{ijkl} + 8 \pi \alpha' \tau_{ij}, \quad (3.2)$$

where $\tau_{ij}$ is a power series in $\alpha'$ that would be completely determined if we had the ability to compute $\beta_g$ to all orders in $\alpha'$. Instead of $\tau_{ij}$, to make contact with general relativity we will use the corresponding 'stress–energy tensor' $T_{ij}$ defined by $T_{ij} := \tau_{ij} - \frac{1}{4} g_{ij} \tau$, with $\tau := g^{ij} T_{ij}$. We also prefer to replace the dilaton by $\varphi = e^{-2\Phi}$. \(\quad (3.3)\)

The left-hand side of (3.2) becomes

$$R_{ij} + 2 \nabla_i \nabla_j \Phi = R_{ij} - \frac{1}{\varphi} \nabla_i \varphi \nabla_j \varphi + \frac{1}{\varphi^2} \nabla_i \varphi \nabla_j \varphi \equiv B_{ij}[g, \varphi], \quad (3.4)$$

and so the condition for the vanishing of $\beta_g$ becomes

$$B_{ij}[g, \varphi] = \frac{1}{4} H_{ijkl} H^{ijkl} + 8 \pi \tau_{ij} \equiv \frac{1}{4} H_{ijkl} H^{ijkl} + 8 \pi \alpha' \left( T_{ij} - \frac{1}{n-2} g_{ij} T \right). \quad (3.5)$$

Proposition 3.1. Let $(M, g)$ be a chronological spacetime with dim $M \geq 3$. Say that $T_{ij}$ obeys the strong energy condition, that $H_{ijkl}$ is zero or totally skew, that (3.5) holds, that there is a $C \in \mathbb{R}$, $C > 0$, such that $\varphi(p) \geq C$ for all $p \in M$, and that the $\varphi$-generic condition holds. Assume further that at least one of the conditions (a), (b), or (c) from theorem 2.1 hold. Then $(M, g)$ is nonspacelike geodesically incomplete.

\(^7\) Most authors use $B$ for this field, but we use $\mathcal{B}$ to distinguish it from the Bakry–Émery tensor $B_{ij}$.
Proof. Choose any timelike vector $t'$ and construct an orthonormal frame $\{e_0, e_\alpha : \alpha = 1, \ldots , n - 1\}$ aligned with $t'$, so $t' = |t|e_0$ where $|t| := \sqrt{-g_{ij}t^it^j}$. For totally skew torsion, we then have $\mathcal{H}_{ij}^k\mathcal{H}_{jk}^it^i = + |t|^2(\mathcal{H}_{ijk})^2$ since total skewness implies that neither of the $\mathcal{H}$ factors can have more than one 0-index. Using this and the strong energy condition, we see that the right-hand side of (3.5) is nonnegative whenever it is contracted against $t't'$ for any timelike vector $t'$. Then the result is an immediate consequence of theorem 2.1. □

Proposition 3.2. Let $\dim M \geq 3$. Say that $T_{ij}$ obeys the strong energy condition, that $\mathcal{H}_{ijk}$ is zero or totally skew, that BER is given by (3.5), and that there is a $c \in \mathbb{R}$, $c > 0$, such that $\psi(p) \geq c$ for all $p \in M$. If $(M, g)$ is either globally hyperbolic or timelike geodesically complete, and admits a complete timelike line, then $(M, g) \simeq (\mathbb{R} \times M, -dt^2 + \tilde{g})$, so $(M, g)$ splits off the timelike line and $\psi$ is constant.

Proof. This follows from the same calculation as in the previous proof, but now we apply theorem 2.2 instead of theorem 2.1. □

3.2. Brans–Dicke theory in the Jordan frame

The Brans–Dicke theory [2] on a manifold $M$ is actually a family of theories, parametrized by the Brans–Dicke parameter $\omega > - (n - 1)$. There are two gravitational fields, a Lorentzian metric tensor $g_{ij}$ and a scalar field $\psi$ which is taken to be everywhere positive. These fields obey the system of equations ([11], p 123)

$$R_{ij} = \frac{1}{2}g_{ij}R = \frac{8\pi}{\psi}T_{ij} + \frac{\omega}{\psi^2}\nabla_i\nabla_j\psi - \frac{1}{2}g_{ij}\omega\nabla_k\nabla_l\psi + \frac{1}{\psi^2}(\nabla_i\nabla_j\psi - g_{ij}\nabla^2\psi),$$

(3.6)

$$\Box\psi = \frac{1}{2\psi^2}\nabla_i\nabla_j\psi + \frac{1}{2\omega}R\psi = 0,$$

(3.7)

where $\Box\psi := \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\psi) = g^{ij}\nabla_i\nabla_j\psi$ is the d’Alembertian of $\psi$.

Conformal transformations will change the form of these equations. The conformal choice that leads to the above form is usually called the Jordan frame in the literature. Note that notions like closed trapped surface and timelike line are not conformally invariant.

Equation (3.6) can be rewritten as

$$R_{ij} = \frac{1}{\psi}\nabla_i\nabla_j\psi = \frac{8\pi}{\psi}
\left( T_{ij} \frac{1}{(n - 2)}g_{ij}T \right) + \frac{\omega}{\psi^2}\nabla_i\nabla_j\psi + \frac{1}{(n - 2)}\Box\psi \frac{1}{\psi}g_{ij}.$$  

(3.8)

We can also rewrite (3.7) as

$$\Box\psi = \frac{8\pi T}{[n - 1 + (n - 2)\omega]}.$$  

(3.9)

Inserting this in (3.8) yields

$$R_{ij} = \frac{1}{\psi}\nabla_i\nabla_j\psi - \frac{\omega}{\psi^2}\nabla_i\nabla_j\psi = \frac{8\pi}{\psi}
\left( T_{ij} - \frac{(1 + \alpha)}{[n - 1 + (n - 2)\omega]}g_{ij}T \right).$$  

(3.10)

For completeness, the conformal transformation $g \mapsto \tilde{g} = \phi^{1+\omega}g$ takes us from the Jordan frame to the Einstein frame, in which (3.6) becomes, with obvious notation, $\tilde{R}_{ij} = \frac{1}{\phi^{\omega}}\tilde{g}_{il}\tilde{g}^{jk}T_{jk} + \frac{\omega}{\phi}T_{ij}^\alpha + \frac{\omega}{\phi}T_{ij}^\alpha$, where $T_{ij}^\alpha$ is the stress–energy tensor for a free, massless scalar field $\psi = \frac{1}{\sqrt{-g}}\log \phi$ and $\epsilon = (1 + \frac{\omega}{\psi^2} + \omega)$. It is important to note that the matter $T_{ij}$ is minimally coupled in the Jordan frame but not minimally coupled in the Einstein frame, since factors of the metric (or sometimes the connection; e.g., for spinor fields) that appear inside $T_{ij}$ give rise to additional factors of $\psi$ due to the conformal transformation.

\footnote{For completeness, the conformal transformation $g \mapsto \tilde{g} = \phi^{1+\omega}g$ takes us from the Jordan frame to the Einstein frame, in which (3.6) becomes, with obvious notation, $\tilde{R}_{ij} = \frac{1}{\phi^{\omega}}\tilde{g}_{il}\tilde{g}^{jk}T_{jk} + \frac{\omega}{\phi}T_{ij}^\alpha + \frac{\omega}{\phi}T_{ij}^\alpha$, where $T_{ij}^\alpha$ is the stress–energy tensor for a free, massless scalar field $\psi = \frac{1}{\sqrt{-g}}\log \phi$ and $\epsilon = (1 + \frac{\omega}{\psi^2} + \omega)$. It is important to note that the matter $T_{ij}$ is minimally coupled in the Jordan frame but not minimally coupled in the Einstein frame, since factors of the metric (or sometimes the connection; e.g., for spinor fields) that appear inside $T_{ij}$ give rise to additional factors of $\psi$ due to the conformal transformation.}
Proposition 3.3. Let \((M, g)\) be a chronological spacetime with \(\dim M \geq 3\). Say that either (i) for some fixed \(\omega > -1\), the pair \((g, \varphi)\) obeys the system (3.6), (3.7), where \(Tij\) obeys the \(\omega\)-energy condition, or (ii) for \(\omega = -1\) the pair \((g, \varphi)\) obeys the system (3.6), (3.7) where \(Tij\) obeys the weak energy condition and \(\varphi \geq C > 0\) for some constant \(C\). Assume further that the \(\varphi\)-generic condition holds and that at least one of the conditions (a), (b), or (c) from theorem 2.1 hold. Then \((M, g)\) is nonspacelike geodesically incomplete.

Proof. Equations (3.6), (3.7) imply (3.10), which in turn can be written as

\[
G_{ij}
\left[
\frac{g}{\varphi}, \frac{1}{q}
\right] = \frac{8\pi}{\varphi}
\left[
(T_{ij} - \frac{(1 + \omega)}{[n - 1 + (n - 2)\omega]}g_{ij}T)
\right] + \left[1 + \omega - \frac{1}{q}\right]
\frac{\nabla_{i} \nabla_{j} \varphi}{\varphi^{2}}.
\] (3.11)

If \(\omega > -1\), choose a positive integer \(q \geq \frac{1}{1 + \omega}\). For such a \(q\), and using the \(\omega\)-energy condition, the right-hand side of (3.11) is \(\geq 0\) when contracted with \(t^{i}t^{j}\) for any timelike vector \(t^{i}\). Thus, \(G_{ij}\left[\frac{g}{\varphi}, \frac{1}{q}\right]t^{i}t^{j} \geq 0\). The result then follows from theorem 2.1.

If instead \(\omega = -1\), then taking \(q \to \infty\) in (3.11) and again invoking the \(\omega\)-energy condition, the right-hand side of (3.11) is again \(\geq 0\) when contracted with \(t^{i}t^{j}\) for any timelike vector \(t^{i}\), implying now that \(B_{ij}\left[\frac{g}{\varphi}\right]t^{i}t^{j} \geq 0\). Using the boundedness of \(\varphi\), the result again follows from theorem 2.1.

Proposition 3.4. Let \(\dim M \geq 3\). Say that either (i) for some fixed \(\omega > -1\), the pair \((g, \varphi)\) obeys the system (3.6), (3.7), where \(Tij\) obeys the \(\omega\)-energy condition, or (ii) for \(\omega = -1\) the pair \((g, \varphi)\) obeys the system (3.6), (3.7) where \(Tij\) obeys the weak energy condition and \(\varphi \geq C > 0\). If \((M, g)\) is either globally hyperbolic or timelike geodesically complete, and admits a complete timelike line, then \((M, g) \simeq (\mathbb{R} \times \tilde{M}, -dr^{2} + \tilde{g})\) and \(\varphi\) is constant.

Thus \((M, g)\) is a static solution of general relativity.

Proof. Follow the same argument as in the proof of the previous proposition, but instead of theorem 2.1, invoke theorem 2.2.

In the case of globally hyperbolic spacetimes, a nice example of the necessity of the assumption that the timelike line in the splitting theorem must be complete is provided by the O’Hanlon–Tupper family [8], each member of which has incomplete timelike lines and otherwise satisfies all assumptions of proposition 3.4, and which only splits as a warped product with nonconstant \(\varphi\) (which is monotonic along the timelike line and not bounded away from zero).

4. Concluding remark

The theorems do not cover the case of \(\omega < -1\). Consider \(n = 4\) dimensions. Of course, this is the physical case, and observations currently imply that Brans–Dicke theory cannot describe gravity in the solar system unless \(\omega > 4 \times 10^{2}\) [1]. Nevertheless, we can consider solutions with \(3/2 < \omega < -1\). As mentioned in the introduction, there are known \(n = 4\) nonsingular solutions for \(-3/2 < \omega \leq -4/3\) which obey the assumptions of the splitting theorem, including the \(\omega\)-energy condition\(^{9}\) and the existence of a complete timelike line, but do not split [9] (they do split as a warped product, but not as a product). Thus, the splitting

\(^{9}\) Solutions are found in [9] for values of a parameter \(\gamma\), the barotropic index of the perfect fluid, in the range \(0 < \gamma < 2\). For \(\omega < -1\) and \(n = 4\) dimensions, the \(\omega\)-energy condition holds when \(\gamma \leq 1\). For \(\omega \leq -4/3\), as in the [9] solutions, the \(\omega\)-energy condition holds for \(\gamma \leq 5/4\).
theorem cannot be extended to all allowed $\omega$. Whether there can be $n = 4$ nonsplit, nonsingular solutions of Brans–Dicke theory with $-4/3 < \omega < -1$ is an open question.

Finally, we return to the issue of whether the Brans–Dicke results are genuinely unrelated to the Einstein frame. With the splitting theorem, there is no apparent reason to question that it is an entirely independent result. For the singularity theorem, this may be less clear. On the one hand, by working entirely in the Jordan frame, we see that $\omega = -1$ is singled out as a boundary case, previous results [9] having already indicated that there should be a boundary case. As well, the form of the theorem that employs assumption (c) of theorem 2.1 seems satisfactory. However, the forms of the theorem that employ assumption (a) or (b) seem rather less satisfactory, since they both are phrased as a condition on $\hat{\theta}$, which is related to the null expansion $\theta$ by the same conformal transformation that relates the Einstein and Jordan frames. Hence, these assumptions still echo an Einstein frame formulation. Perhaps this is the best that can be done, since it may be necessary to modify the mean curvature of a trapped surface to overcome a defocusing effect due to the interaction of curvature with the Brans–Dicke scalar field\textsuperscript{10}. For now, whether these assumptions can be modified to refer only to $\theta$ remains an open question.

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\textsuperscript{10} This can work both ways: one can have $\theta > 0$ and yet $\hat{\theta} \leq 0$, trapping surfaces that would be, in general relativity, not trapped; then the scalar field would contribute an additional focusing, rather than defocusing, effect.