Wintgen ideal submanifolds with a low-dimensional integrable distribution (I)

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Abstract

A submanifold in space forms satisfies the well-known DDVV inequality due to De Smet, Dillen, Verstraelen and Vrancken. The submanifold attaining equality in the DDVV inequality at every point is called Wintgen ideal submanifold. As conformal invariant objects, Wintgen ideal submanifolds are studied in this paper using the framework of Möbius geometry. We classify Wintgen ideal submanifolds of dimension $m > 2$ and arbitrary codimension when a canonically defined 2-dimensional distribution $\mathcal{D}$ is integrable. Such examples come from cones, cylinders, or rotational submanifolds over super-minimal surfaces in spheres, Euclidean spaces, or hyperbolic spaces, respectively.

2000 Mathematics Subject Classification: 53A30, 53A55;

Key words: Wintgen ideal submanifolds, DDVV inequality, super-conformal surfaces, super-minimal surfaces.

1 Introduction

A basic idea in submanifold theory is to find certain universal inequalities (pointwise or global ones) between various invariants (intrinsic and extrinsic), then characterize and classify the optimal submanifolds attaining equality in such inequalities.

$^0$T. Li, X. Ma, C.P. Wang are partially supported by the grant No. 11171004 of NSFC; X. Ma is partially supported by the grant No. 10901006 of NSFC.
The simplest example of such pointwise inequalities is that for mean curvature $H$ and Gaussian curvature $K$ of a surface in $\mathbb{R}^3$, there is always $H^2 \geq K$, and the equality holds true exactly at those umbilic points. This was generalized by Chen \cite{chen} to other space forms and to arbitrary codimensional case. (Another universal inequality posed by Chen \cite{chen2} motivated a series of investigation on submanifolds attaining the equality in Chen’s inequality.)

De Smet, Dillen, Verstraelen and Vrancken proposed in 1999 a strengthened inequality \cite{DSDV} involving the scalar, mean curvature and the norm of the normal curvature tensor as below. Let $f : M^m \to Q_c^{m+p}$ be an isometric immersion of an $m$–dimensional Riemannian manifold into a space form of dimension $m + p$ and constant sectional curvature $c$. Let $R$ (resp. $R^\perp$) be the Riemannian curvature tensor (resp. the normal curvature tensor) of $f$. Their conjectured that at any point,

\begin{equation}
DDVV \text{ inequality : } s \leq c + ||H||^2 - s^\perp.
\end{equation}

Here $H$ denotes the mean curvature of $f$, and

$$s = \frac{2}{m(m - 1)} \sum_{1 \leq i < j \leq n} \langle R(e_i, e_j)e_j, e_i \rangle, \quad s^\perp = \frac{2}{m(m - 1)} ||R^\perp||.$$

This so-called DDVV conjecture was proved in 2008 by J. Ge, Z. Tang \cite{GeTang} and Z. Lu \cite{Lu} independently.

Moreover, in \cite{GeTang} the pointwise structure of the second fundamental form of $f$ that attains equality was determined. It was shown that equality holds at $x \in M^m$ if, and only if, there exists an orthonormal basis $\{e_1, \cdots, e_m\}$ in the tangent space $T_xM^m$ and an orthonormal basis $\{n_1, \cdots, n_p\}$ in the normal space $T_x^\perp M^m$, such that the shape operators $\{A_{n_i}, i = 1, \cdots, m\}$ have the form

\begin{equation}
A_{n_1} = \begin{pmatrix}
\lambda_1 & \mu_0 & 0 & \cdots & 0 \\
\mu_0 & \lambda_1 & 0 & \cdots & 0 \\
0 & 0 & \lambda_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_1
\end{pmatrix}, \quad A_{n_2} = \begin{pmatrix}
\lambda_2 + \mu_0 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 - \mu_0 & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_2
\end{pmatrix},
\end{equation}

and

$$A_{n_3} = \lambda_3 I_p, \quad A_{n_r} = 0, r \geq 4.$$
Wintgen first proved the inequality (1.1) for surfaces \( f : M^2 \to S^4 \). The equality holds at \( x \in M^2 \) if and only if the curvature ellipse of \( M^2 \) in \( S^4 \) at \( x \) is a circle \([18, 11]\). \( f : M^2 \to S^4 \) is called a super-conformal surface if this holds true at every point. It is well-known that such surfaces correspond to images of complex curves in \( \mathbb{CP}^3 \) via the Penrose twistor projection \( \pi : \mathbb{CP}^3 \to S^4 \). According to the suggestion of \([4, 16]\), we make the following definition.

**Definition 1.1.** A submanifold \( f : M^m \to \mathbb{Q}^{m+p}_c \) is called a Wintgen ideal submanifold if the equality is satisfied in (1.1) at every point of \( M^m \).

We briefly review known results on the classification of Wintgen ideal submanifolds. It was shown in \([11]\) that \( f : M^2 \to \mathbb{Q}^{2+p}_c \) is a Wintgen ideal surface if and only if the ellipse of curvature of \( f \) at \( x \) is a circle. When being also minimal surfaces in the specific space form, they were already known as super-minimal surfaces. Such examples are abundant.

For three dimensional Wintgen ideal subanifolds, when they are minimal, they belong to the class of (three dimensional) austere submanifolds. The later was classified locally by Bryant for Euclidean submanifolds \([1]\), by Dajczer and Florit in the unit sphere \([5]\), by Choi and Lu \([13, 15]\) in hyperbolic space.

Finally, in \([6]\), Dajczer and Tojeiro provided a parametric construction of Wintgen ideal submanifolds of codimension two and arbitrary dimension in terms of minimal surfaces in the Euclidean space.

An important observation of Dajczer and Tojeiro in \([6]\) is that the inequality as well as the equality case (and the class of Wintgen ideal submanifolds) are conformally invariant property. This follows from the observation in \([8]\) that inequality (1.1) holds at a point \( x \in M^m \) if and only if

\[
\sum_{\alpha, \beta=1}^p ||[\bar{A}_\alpha, \bar{A}_\beta]||^2 \leq \left( \sum_{\alpha=1}^p ||\bar{A}_\alpha|| \right)^2
\]

is satisfied for the traceless shape operators \( \bar{A}_1, \cdots, \bar{A}_p \) at \( x \in M^m \), whereas \( \{\bar{A}_i\} \) are conformal invariant objects (up to a scalar factor). It follows that Wintgen ideal submanifolds in the sphere \( S^{m+p} \) or hyperbolic space \( \mathbb{H}^{m+p} \) are the pre-image of a stereographic projection of Wintgen ideal submanifolds in \( \mathbb{R}^{m+p} \).
Thus it is appropriate to put the study of Wintgen ideal submanifolds in the framework of Möbius geometry. For the same reason it is no restriction when we describe them in the Euclidean space. This is exactly our main goal in this paper.

As the main result, we give a classification of Wintgen ideal submanifolds with integrable canonical distribution \( D = \text{Span}\{e_1, e_2\} \). It is clear from (1.2) that \( D \) is well-defined when the submanifold is nowhere totally umbilic.

**Theorem 1.1.** Let \( f : M^m \to \mathbb{R}^{m+p} (m \geq 3) \) be a Wintgen ideal submanifold without umbilic points. If the canonical distribution \( D = \text{span}\{e_1, e_2\} \) is integrable, then locally \( f \) is Möbius equivalent to

(i) a cone over a super-minimal surface in \( S^{2+p} \);

(ii) or a cylinder over a super-minimal surface in \( \mathbb{R}^{2+p} \);

(iii) or a rotational submanifold over a super-minimal surface in \( \mathbb{H}^{2+p} \).

This paper is organized as follows. In section 2, we review the elementary facts about Möbius geometry for submanifolds in \( \mathbb{R}^{m+p} \). In section 3, we describe the construction of Wintgen ideal submanifolds as cylinders, cones, or rotational submanifolds. In section 4, we give the proof of our main theorem.

## 2 Submanifold theory in Möbius geometry

In this section we briefly review the theory of submanifolds in Möbius geometry. For details we refer to [17] and [12].

Let \( \mathbb{R}^{m+p+2} \) be the Lorentz space with inner product \( \langle \cdot , \cdot \rangle \) defined by

\[
\langle Y, Z \rangle = -Y_0Z_0 + Y_1Z_1 + \cdots + Y_{m+p+1}Z_{m+p+1},
\]

where \( Y = (Y_0, Y_1, \cdots, Y_{m+p+1}) \), \( Z = (Z_0, Z_1, \cdots, Z_{m+p+1}) \) \( \in \mathbb{R}^{m+p+2} \).

Let \( f : M^m \to \mathbb{R}^{m+p} \) be a submanifold without umbilics and assume that \( \{e_i\} \) is an orthonormal basis with respect to the induced metric \( I = df \cdot df \) with \( \{\theta_i\} \) the dual
basis. Let \( \{ \nu_r | 1 \leq r \leq p \} \) be a local orthonormal basis for the normal bundle. As usual we denote the second fundamental form and the mean curvature of \( f \) as
\[
II = \sum_{ij, \gamma} h_{ij}^{\gamma} \theta_i \otimes \theta_j \nu_r, \quad H = \frac{1}{m} \sum_{j,r} h_{jj}^{r} \nu_r = \sum_r H^r \nu_r.
\]

We define the Möbius position vector \( Y : M^m \to \mathbb{R}_1^{m+p+2} \) of \( f \) by
\[
Y = \rho \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right), \quad \rho^2 = \frac{m}{m-1} \left| II - \frac{1}{m} \text{tr}(II)I \right|^2.
\]

It is known that \( Y \) is a well-defined canonical lift of \( f \). Two submanifolds \( f, \bar{f} : M^m \to \mathbb{R}^{m+p} \) are Möbius equivalent if there exists \( T \) in the Lorentz group \( O(m+p+1,1) \) in \( \mathbb{R}_1^{m+p+2} \) such that \( \bar{Y} = YT \). It follows immediately that
\[
g = \langle dY, dY \rangle = \rho^2 dx \cdot dx
\]
is a Möbius invariant, called the Möbius metric of \( f \).

Let \( \Delta \) be the Laplacian with respect to \( g \). Define
\[
N = -\frac{1}{m} \Delta Y - \frac{1}{2m^2} \langle \Delta Y, \Delta Y \rangle Y,
\]
which satisfies
\[
\langle Y, Y \rangle = 0 = \langle N, N \rangle, \quad \langle N, Y \rangle = 1.
\]

Let \( \{ E_1, \cdots, E_m \} \) be a local orthonormal basis for \( (M^m, g) \) with dual basis \( \{ \omega_1, \cdots, \omega_m \} \). Write \( Y_j = E_j(Y) \). Then we have
\[
\langle Y_j, Y \rangle = \langle Y_j, N \rangle = 0, \quad \langle Y_j, Y_k \rangle = \delta_{jk}, \quad 1 \leq j, k \leq m.
\]

We define
\[
\xi_r = H^r \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right) + (f \cdot \nu_r, -f \cdot \nu_r, \nu_r).
\]
Then \( \{ \xi_1, \cdots, \xi_p \} \) be the orthonormal basis of the orthogonal complement of \( \text{Span}\{Y, N, Y_j | 1 \leq j \leq m \} \). And \( \{ Y, N, Y_j, \xi_r \} \) form a moving frame in \( \mathbb{R}_1^{m+p+2} \) along \( M^m \).

**Remark 2.1.** Geometrically, \( \xi_r \) corresponds to the unique sphere tangent to \( M^m \) at one point \( x \) with normal vector \( n_r \) and the same mean curvature \( H^r(x) \). We call \( \{ \xi_r \} \) the mean curvature spheres of \( M^m \).
We will use the following range of indices in this section: $1 \leq i, j, k \leq m; 1 \leq r, s \leq p$. We can write the structure equations as below:

$$
\begin{align*}
  dY &= \sum_i \omega_i Y_i, \\
  dN &= \sum_{ij} A_{ij} \omega_i Y_j + \sum_{i,\gamma} C^r_i \omega_i \xi_r, \\
  dY_i &= -\sum_j A_{ij} \omega_j Y - \omega_i N + \sum_{j,\gamma} B^r_{ij} \omega_j Y_r, \\
  d\xi_r &= -\sum_i C^r_i \omega_i Y - \sum_{ijr} \omega_i B^r_{ij} Y_j + \sum_s \theta_{rs} \xi_s,
\end{align*}
$$

where $\omega_{ij}$ are the connection 1-forms of the Möbius metric $g$ and $\theta_{rs}$ the normal connection 1-forms. The tensors

$$
A = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad B = \sum_{ijr} B^r_{ij} \omega_i \otimes \omega_j \xi_r, \quad \Phi = \sum_{jr} C^r_j \omega_j \xi_r
$$

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of $x$, respectively. The covariant derivatives of $C^r_i, A_{ij}, B^r_{ij}$ are defined by

$$
\begin{align*}
  \sum_j C^r_{i,j} \omega_j &= dC^r_i + \sum_{ij} C^r_{j} \omega_j + \sum_s C^r_s \theta_{sr}, \\
  \sum_k A_{ij,k} \omega_k &= dA_{ij} + \sum_{ik} A_{ik} \omega_k + \sum_k A_{kj} \omega_{ki}, \\
  \sum_k B^r_{ij,k} \omega_k &= dB^r_{ij} + \sum_{ik} B^r_{ik} \omega_{kj} + \sum_k B^r_{kj} \omega_{ki} + \sum_s B^r_{ij} \theta_{sr}.
\end{align*}
$$

The integrability conditions for the structure equations are given by

$$
\begin{align*}
  (2.3) & \quad A_{ij,k} - A_{ik,j} = \sum_r B^r_{ik} C^r_j - B^r_{ij} C^r_k, \\
  (2.4) & \quad C^r_{i,j} - C^r_{j,i} = \sum_k (B^r_{ik} A_{kj} - B^r_{jk} A_{ki}), \\
  (2.5) & \quad B^r_{ij,k} - B^r_{ik,j} = \delta_{ij} C^r_k - \delta_{ik} C^r_j, \\
  (2.6) & \quad R_{ijkl} = \sum_r B^r_{ik} B^r_{jl} - B^r_{il} B^r_{jk} + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \\
  (2.7) & \quad R^l_{rskl} = \sum_k B^s_{ik} B^r_{kj} - B^s_{ik} B^r_{kj}.
\end{align*}
$$

Here $R_{ijkl}$ denote the curvature tensor of $g$, $\kappa = \frac{1}{n(n-1)} \sum_i R_{ijij}$ is its normalized Möbius scalar curvature. It follows from (2.6) that the Ricci tensor of $g$ satisfies

$$
R_{ij} := \sum_k R_{ikjk} = -\sum_{kr} B^r_{ik} B^r_{kj} + (\text{tr} A) \delta_{ij} + (m-2) A_{ij}.
$$
Other restrictions on tensors $A, B$ are

\begin{align}
\sum_j B_{jj} &= 0, \quad \sum_{ijr} (B_{ij}^r)^2 = \frac{m-1}{m}, \\
tr A &= \sum_j A_{jj} = \frac{1}{2m}(1 + m^2 \kappa). \tag{2.9} \tag{2.10}
\end{align}

We know that all coefficients in the structure equations are determined by $\{g, B\}$ and the normal connection $\{\theta_{\alpha\beta}\}$.

3 Examples of Wintgen ideal submanifolds

In this section we will use minimal Wintgen ideal submanifolds in space forms $\mathbb{R}^n, S^n$ or $\mathbb{H}^n$ to construct other general examples. Note that being a minimal submanifold is not a conformal invariant property.

We remark that a minimal Wintgen ideal submanifold in any space form is an austere submanifold of rank two, and the converse is also true. Super-minimal surfaces in space forms are special examples and there are plenty of them, including all minimal 2-spheres in $S^n$ and all complex curves in $\mathbb{C}^n = \mathbb{R}^{2n}$.

**Definition 3.1.** Let $u : M^r \rightarrow \mathbb{R}^{r+p}$ be an immersed submanifold. We define the cylinder over $u$ in $\mathbb{R}^{m+p}$ as

$$f = (u, id) : M^r \times \mathbb{R}^{m-r} \rightarrow \mathbb{R}^{r+p} \times \mathbb{R}^{m-r} = \mathbb{R}^{m+p},$$

where $id : \mathbb{R}^{m-r} \rightarrow \mathbb{R}^{m-r}$ is the identity map.

**Proposition 3.1.** Let $u : M^r \rightarrow \mathbb{R}^{r+p}$ be an immersed submanifold. Then the cylinder $f = (u, id) : M^r \times \mathbb{R}^{m-r} \rightarrow \mathbb{R}^{m+p}$ is a Wintgen ideal submanifold if and only if $u : M^r \rightarrow \mathbb{R}^{r+p}$ is a minimal Wintgen ideal submanifold.

**Proof.** Let $\eta_1, \cdots, \eta_p$ be an orthonormal frame in the normal bundle of $u$ in $\mathbb{R}^{r+p}$. Then $e_r = (\eta_r, \bar{0}) \in \mathbb{R}^{m+p}$ is such a frame of $f$. The first and second fundamental forms $I, II$ of $f$ are related with corresponding forms $I_u, II_u$ of $u$ by

$$I = I_u + I_{\mathbb{R}^{m-r}}, \quad II = II_u.$$

\[7\]
where \( I_{\mathbb{R}^m-r} \) denotes the standard metric of \( \mathbb{R}^{m-r} \). Clearly \( f \) is a Wintgen ideal submanifold if and only if \( u \) is a minimal Wintgen ideal submanifold. This completes the proof to Proposition 3.1.

**Remark 3.1.** The Möbius position vector \( Y : M^r \times \mathbb{R}^{m-r} \to \mathbb{R}_1^{m+p+2} \) of the cylinder \( f \) is

\[
Y = \rho_0 \left( \frac{1 + |u|^2 + |y|^2}{2}, \frac{1 - |u|^2 - |y|^2}{2}, u, y \right),
\]

(3.12)

where \( \rho_0 = \frac{m}{m-1}(|II_u|^2 - mH_u^2) : M^r \to \mathbb{R} \), and \( y : \mathbb{R}^{m-r} \to \mathbb{R}^{m-r} \) is the identity map.

**Definition 3.2.** Let \( u : M^r \to S^{r+p} \subset \mathbb{R}_1^{r+p+1} \) be an immersed submanifold. We define the cone over \( u \) in \( \mathbb{R}^{m+p} \) as

\[
f : R^+ \times \mathbb{R}^{m-r-1} \times M^r \to \mathbb{R}^{m+p},
\]

\[
f(t, y, u) = (y, tu),
\]

**Proposition 3.2.** Let \( u : M^r \to S^{r+p} \) be an immersed submanifold. Then the cone \( f = (y, tu) : R^+ \times \mathbb{R}^{m-r-1} \times M^r \to \mathbb{R}^{m+p} \) is a Wintgen ideal submanifold if and only if \( u \) is a minimal Wintgen ideal submanifold in \( S^{r+p} \).

**Proof.** The first and second fundamental forms of \( f \) are, respectively,

\[
I = t^2 I_u + I_{\mathbb{R}^m-r}, \quad II = t II_u,
\]

where \( I_u, II_u, I_{\mathbb{R}^m-r} \) are understood as before. The conclusion follows easily.

**Remark 3.2.** The Möbius position vector \( Y : R^+ \times \mathbb{R}^{m-r-1} \times M^r \to \mathbb{R}_1^{m+p+2} \) of the cone \( f \) is

\[
Y = \rho_0 \left( \frac{1 + t^2 + |y|^2}{2t}, \frac{1 - t^2 - |y|^2}{2t}, u, y \right),
\]

where \( \rho_0 = \frac{m}{m-1}(|II_u|^2 - mH_u^2) : M^r \to \mathbb{R} \), and \( y : \mathbb{R}^{m-r-1} \to \mathbb{R}^{m-r-1} \) is the identity map. Let

\[
\mathbb{H}^{m-r} = \{(y_0, y) \in \mathbb{R}^{m-r+1} | -y_0^2 + |y|^2 = -1, y_0 \geq 1\} \cong R^+ \times \mathbb{R}^{m-r-1},
\]

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then \( \left( \frac{1+r^2+|y|^2}{2t}, \frac{1-t^2-|y|^2}{2t} \right) : \mathbb{R}^+ \times \mathbb{R}^{m-r-1} = \mathbb{H}^{m-r} \to \mathbb{H}^{m-r} \) is nothing else but the identity map. And the Möbius position vector of the cone \( f \) is

\[
Y = \rho_0(id, u) : \mathbb{H}^{m-r} \times M^r \to \mathbb{H}^{m-r} \times \mathbb{S}^{r+p} \subset \mathbb{R}^{1+m+p+2},
\]

where \( \rho_0 \in C^\infty(M^r) \) and \( id : \mathbb{H}^{m-r} \to \mathbb{H}^{m-r} \) is a identity map.

**Definition 3.3.** Let \( \mathbb{R}^{r+p}_+ = \{(x_1, \ldots, x_{r+p}) \in \mathbb{R}^{r+p} | x_{r+p} > 0 \} \) be the upper half-space endowed with the standard hyperbolic metric

\[
ds^2 = \frac{1}{x_{r+p}^2} \sum_{i=1}^{r+p} dx_i^2.
\]

Let \( u = (x_1, \ldots, x_{r+p}) : M^r \to \mathbb{R}^{r+p}_+ \) be an immersed submanifold. We define rotational submanifold over \( u \) in \( \mathbb{R}^{m+p} \) as

\[
f : M^r \times \mathbb{S}^{m-r} \to \mathbb{R}^{m+p},
\]

\[
f(u, \phi) = (x_1, \ldots, x_{r+p-1}, x_{r+p}\phi).
\]

where \( \phi : \mathbb{S}^{m-r} \to \mathbb{R}^{m-r+1} \) is the standard sphere.

**Proposition 3.3.** Let \( u = (x_1, \ldots, x_{r+p}) : M^r \to \mathbb{R}^{r+p}_+ \) be an immersed submanifold. Then the rotational submanifold \( f : M^r \times \mathbb{S}^{m-r} \to \mathbb{R}^{m+p} \) is a Wintgen ideal submanifold if and only if \( u \) is a minimal Wintgen ideal submanifold.

**Proof.** Let \( \mathbb{R}_+^{r+p+1} \) be the Lorentz space with inner product

\[
\langle y, y \rangle = -y_1^2 + y_2^2 + \cdots + y_{r+p+1}^2, \quad y = (y_1, \ldots, y_{r+p+1}).
\]

Let \( \mathbb{H}^{r+p} = \{y \in R^{r+p+1}_+ | \langle y, y \rangle = -1, y_1 > 0 \} \) be the hyperbolic space. Introduce isometry \( \tau : \mathbb{R}^{r+p}_+ \to \mathbb{H}^{r+p} \) as below:

\[
\tau(x_1, \ldots, x_{r+p}) = \left( \frac{1+x_1^2+\cdots+x_{r+p}^2}{2x_{r+p}}, \frac{1-x_1^2-\cdots-x_{r+p}^2}{2x_{r+p}}, \frac{x_1}{x_{r+p}}, \cdots, \frac{x_{r+p-1}}{x_{r+p}} \right).
\]

Let \( \eta^1, \ldots, \eta^p \) be the unit normal vectors of \( u \) in \( \mathbb{R}^{r+p}_+ \). Write \( \eta^i = (\eta^i_1, \cdots, \eta^i_{r+p}) \).

Since \( \eta^i \) is the unit normal vector, then

\[
\frac{(\eta^i_1)^2 + \cdots + (\eta^i_{r+p})^2}{x_{r+p}^2} = 1, \quad 1 \leq i \leq p.
\]
Thus the unit normal vector of \( f \) in \( \mathbb{R}^{m+p} \) is
\[
\xi_i = \frac{1}{x_{r+p}} (\eta^1_i, \cdots, \eta^{r+p}_i \phi).
\]
The first fundamental form of \( u \) is
\[
I_u = \frac{1}{x_{r+p}} (dx_1 \cdot dx_1 + \cdots + dx_{r+p} \cdot dx_{r+p}).
\]
The second fundamental form of \( u \) is
\[
II^i_u = -\langle \tau_u(du), \tau_u(dn^1) \rangle = \frac{1}{x_{r+p}^2} (dx_1 \cdot dn^1_1 + \cdots + dx_{r+p} \cdot dn^i_{r+p}) - \frac{\eta^{r+p}_i}{x_{r+p}} I_u.
\]
Now we can write out the first and the second fundamental forms of \( f \):
\[
I = dx \cdot dx = x^2_{r+p} (I_u + I_{S^{m-r}}), \quad II^i = x^2_{r+p} II^i_u - \eta^{r+p}_i (I_u + I_{S^{m-r}}),
\]
where \( I_{S^{m-r}} \) is the standard metric of \( S^{m-r} \). This completes the proof.

**Remark 3.3.** The Möbius position vector \( Y : M^r \times S^{m-r} \rightarrow \mathbb{R}^{m+p+2} \) of the rotational submanifold \( f \) is
\[
Y = \rho_0 \left( \frac{1+|u|^2}{2x_{r+p}}, \frac{1-|u|^2}{2x_{r+p}}, \frac{x_1}{x_{r+p}}, \cdots, \frac{x_{r+p-1}}{x_{r+p}} \right),
\]
where \( \rho_0 = \frac{m}{m-1} (|II_u|^2 - mH_u^2) : M^r \rightarrow \mathbb{R} \), and \( \phi : S^{m-r} \rightarrow S^{m-r} \) is the identity map. Since \( \frac{1+|u|^2}{2x_{r+p}}, \frac{1-|u|^2}{2x_{r+p}}, \frac{x_1}{x_{r+p}}, \cdots, \frac{x_{r+p-1}}{x_{r+p}} = \tau(u) : M^r \rightarrow \mathbb{H}^{r+p}, \) then the Möbius position vector of the rotational submanifold \( f \) is
\[
Y = \rho_0 (\tau(u), \phi) : M^r \times S^{m-r} \rightarrow \mathbb{H}^{r+p} \times S^{m-r} \subset \mathbb{R}_1^{m+p+2},
\]
where \( \rho_0 \in \mathcal{C}^\infty (M^r) \) and \( \phi : \mathbb{H}^{m-r} \rightarrow \mathbb{H}^{m-r} \) is the identity map.

From (3.12), (3.13) and (3.13), we have

**Proposition 3.4.** Let \( f : M^m \rightarrow \mathbb{R}^{m+p} \) be an immersed submanifold without umbilical points.
(1) If there exists a submanifold \( u : M^r \rightarrow \mathbb{R}^{r+p} \) such that the Möbius position vector of \( f \) is
\[
Y = \rho_0 \left( \frac{1+|u|^2 + |y|^2}{2}, \frac{1-|u|^2 - |y|^2}{2}, u, y \right)
\]
\[
Y : M^r \times \mathbb{R}^{m-r} \rightarrow \mathbb{R}^{r+p+2} \times \mathbb{R}^{m-r} \subset \mathbb{R}_1^{m+p+2},
\]
where $\rho_0 \in C^\infty(M^r)$, and $y : \mathbb{R}^{m-r} \to \mathbb{R}^{m-r}$ is the identity map. Then $f$ is a cylinder over $u$.

(2) If there exists a submanifold $u : M^r \to S^{r+p}$ such that the Möbius position vector of $f$ is

$$Y = \rho_0(id, u) : \mathbb{H}^{m-r} \times M^r \to \mathbb{H}^{m-r} \times S^{r+p} \subset \mathbb{R}_1^{m+p+2},$$

where $\rho_0 \in C^\infty(M^r)$ and id : $\mathbb{H}^{m-r} \to \mathbb{H}^{m-r}$ is the identity map. Then $f$ is a cone over $u$.

(3) If there exists a submanifold $u : M^r \to \mathbb{R}^{r+p}$ such that the Möbius position vector of $f$ is

$$Y = \rho_0(\tau(u), \phi) : M^r \times S^{m-r} \to \mathbb{H}^{r+p} \times S^{m-r} \subset \mathbb{R}_1^{m+p+2},$$

where $\rho_0 \in C^\infty(M^r)$, $\phi : S^{m-r} \to S^{m-r}$ is the identity map, and $\tau(u)$ is defined as in (3.14). Then $f$ is the rotational submanifold over $u$.

### 4 Proof of the Main theorem

A submanifold $f : M^m \to \mathbb{R}^{m+p}$ is a Wintgen ideal submanifold if and only if, at each point of $M^m$, there is a suitable frame such that the second fundamental form has the form (1.2). If $\mu_0 = 0$ in (1.2), then the Wintgen ideal submanifold is totally umbilical submanifold. Next we consider non-umbilical Wintgen ideal submanifolds, that is $\mu_0 \neq 0$ on $M^m$ and $m \geq 3$.

Since $\mu_0 \neq 0$, we can choose an orthonormal basis $\{E_1, \cdots, E_m\}$ of $T_xM^m$ with respect to the Möbius metric $g$ and an orthonormal basis $\{\xi_1, \cdots, \xi_p\}$ of $\mathbb{T}_x^\perp M^m$ such that the coefficients of the Möbius second fundamental form $B$ has the form

$$B^1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad B^2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

and $B^\alpha = 0$ for all $\alpha \geq 3$. By (2.9), the norm of $B$ is constant and $\mu = \sqrt{\frac{m-1}{4m}}$. Clearly the distribution $\mathbb{D} = \text{span}\{E_1, E_2\}$ is well-defined.
For convenience we adopt the convention below on the range of indices:

\[ 1 \leq i, j, k, l \leq m, \quad 3 \leq a, b, c \leq m, \quad 3 \leq \alpha, \beta, \gamma \leq p. \]

Then our assumption means that except

\[ B_{12} = B_{21} = B_{21}^2 = B_{22}^2 = \mu = \sqrt{\frac{m-1}{4m}}, \]

any other coefficient of the Möbius second fundamental form vanishes. In particular,

(4.17) \[ B_{11} = B_{22} = B_{ij}^1 = 0, \quad B_{12}^2 = B_{21}^2 = B_{ij}^2 = 0, \quad B_{ij}^\alpha = 0. \]

First we compute the covariant derivatives of \( B_{\alpha}^r \). Denote

(4.18) \[ \theta = 2\omega_{12} + \theta_{12}. \]

Since the Möbius second fundamental form \( B \) has the form (4.16), we have

(4.19) \[ B_{\delta}^{\alpha,\beta} = 0, 1 \leq \delta \leq p, 1 \leq \kappa \leq m, \quad B_{i\alpha,\beta}^1 = 0, B_{2a,\beta}^1 = 0. \]

(4.20) \[ \omega_{2a} = \sum_i \frac{B_{1a,i}^1}{\mu} \omega_i = -\sum_i \frac{B_{2a,i}^2}{\mu} \omega_i, \quad \omega_{1\alpha} = \sum_i \frac{B_{2a,i}^1}{\mu} \omega_i = \sum_i \frac{B_{1a,i}^2}{\mu} \omega_i. \]

(4.21) \[ \theta = \sum_i \frac{-B_{11,i}^1}{\mu} \omega_i = \sum_i \frac{B_{22,i}^2}{\mu} \omega_i = \sum_i \frac{B_{12,i}^2}{\mu} \omega_i, \]

\[ B_{12}^{i} = 0, \quad B_{11}^{1} = B_{22}^{2} = 0. \]

It follows from (3) that

\[ C_1^\alpha = B_{aa,1}^\alpha - B_{a1,a}^\alpha = 0; C_2^\alpha = B_{aa,2}^\alpha - B_{a2,a}^\alpha = 0. \]

\[ C_a^\alpha = B_{11,a}^\alpha - B_{a1,1}^\alpha = B_{22,a}^\alpha - B_{2a,2}^\alpha = B_{22,a}^\alpha, \]

Since \( \sum_i B_{ij,k}^\delta = 0, 1 \leq \delta \leq p, 1 \leq k \leq m, \) we have

\[ C^\alpha = 0. \]
From (4.20) and (4.21), we obtain
\[ B_{2a,2}^1 = B_{22,a}^1 = B_{1a,2}^2, \quad B_{1a,1}^2 = 0, B_{2a,2}^2 = 0. \]
This implies that \( C_a^1 = 0, \quad C_a^2 = B_{11,a}^2 = B_{22,a}^2, \) thus \( C_a^2 = 0. \)

The other coefficients of \( \{ C_j^r \} \) are obtained similarly as below:

\[
C_1^1 = -B_{1a,a}^1 = -\mu \omega_{2a}(e_a), \quad C_2^2 = -B_{2a,a}^2 = \mu \omega_{2a}(e_a), \\
C_2^1 = -B_{2a,a}^1 = -\mu \omega_{1a}(e_a), \quad C_1^2 = -B_{1a,a}^2 = -\mu \omega_{1a}(e_a).
\]

In particular we have

\[
C_1^1 = -C_2^2, \quad C_2^1 = C_1^2.
\]

The covariant derivative of coefficients of \( C \) are

\[
C_{1,i}^1 = -C_{2,i}^2, \quad C_{1,i}^1 = C_{2,i}^2, \quad C_{a,i}^\alpha = 0.
\]

From (4.20) and (4.21), we write out the connection forms

\[
\theta = \sum_i -\frac{B_{1a,1}^{1,i}}{\mu} \omega_i = \sum_i \frac{B_{22,i}^1}{\mu} \omega_i = \sum_i \frac{B_{12,i}^2}{\mu} \omega_i,
\]

\[
\omega_{1a} = \frac{B_{22,a}^1}{\mu} \omega_2 + \frac{B_{2a,a}^1}{\mu} \omega_a = \frac{B_{1a,2}^2}{\mu} \omega_2 + \frac{B_{1a,a}^2}{\mu} \omega_a,
\]

\[
\omega_{2a} = \frac{B_{1a,1}^1}{\mu} \omega_1 + \frac{B_{1a,a}^1}{\mu} \omega_a = -\frac{B_{2a,1}^2}{\mu} \omega_1 - \frac{B_{2a,a}^2}{\mu} \omega_a.
\]

Now we use the assumption that the distribution \( \mathbb{D} = \text{span}\{E_1, E_2\} \) is integrable, which says

\[ d\omega_a \equiv 0, \mod \{\omega_a\}. \]

From (4.25), we obtain

\[ B_{11,a}^1 = -B_{22,a}^1 = -B_{12,a}^2 = 0. \]

\[
\theta = \frac{C_1^1}{\mu} \omega_1 - \frac{C_1^1}{\mu} \omega_2, \omega_{1a} = -\frac{C_1^2}{\mu} \omega_a, \quad \omega_{2a} = -\frac{C_1^1}{\mu} \omega_a.
\]
\[-\frac{1}{2} \sum_{ij} R_{1aij} \omega_i \wedge \omega_j = - \sum_i \frac{C^{2,i}_{1}}{\mu} \omega_i \wedge \omega_a + \frac{(C^{1}_{1})^2 + (C^{1}_{2})^2}{\mu^2} \omega_1 \wedge \omega_a, \]

\[-\frac{1}{2} \sum_{ij} R_{2aij} \omega_i \wedge \omega_j = - \sum_i \frac{C^{1,i}_{1}}{\mu} \omega_i \wedge \omega_a + \frac{(C^{1}_{1})^2 + (C^{1}_{2})^2}{\mu^2} \omega_2 \wedge \omega_a, \]

\[\text{(4.27)}\]

From (4.27), we obtain

\[R_{1a1a} = A_{11} + A_{aa} = \frac{C^{1,1}_{1}}{\mu} - \frac{(C^{1}_{1})^2 + (C^{1}_{2})^2}{\mu^2}, \]

\[R_{2a2a} = A_{22} + A_{aa} = \frac{C^{1,1}_{2}}{\mu} - \frac{(C^{1}_{1})^2 + (C^{1}_{2})^2}{\mu^2} \]

\[\text{(4.28)}\]

\[R_{1a2a} = \frac{C^{1,2}_{1}}{\mu}, \quad R_{2a1a} = \frac{C^{1,1}_{1}}{\mu}, \quad R_{1a12} = A_{2a} = 0, \quad R_{2a12} = -A_{1a} = 0, \]

\[R_{121a} = A_{2a} = \frac{C^{1,a}_{1}}{\mu}, \quad R_{122a} = -A_{1a} = \frac{C^{1,a}_{2}}{\mu}. \]

The equations (4.28) implies that

\[L := A_{aa} = A_{bb}, \quad A_{1a} = A_{2a} = A_{ab} = 0, \quad a \neq b. \]

Define new frame vectors

\[\hat{Y} = \frac{- (C^{1}_{1})^2 - (C^{1}_{2})^2}{2 \mu^2} Y + N - \frac{C^{1}_{1}}{\mu} Y_2 - \frac{C^{1}_{2}}{\mu} Y_1, \]

\[\eta_1 = Y_1 + \frac{1}{\mu} C^{1}_{1} Y, \quad \eta_2 = Y_2 + \frac{1}{\mu} C^{1}_{2} Y, \quad K = 2L + \frac{(C^{1}_{1})^2 + (C^{1}_{2})^2}{\mu^2}. \]

Then we have the moving frame \(\{ Y, \hat{Y}, \eta_1, \eta_2, Y_3, \ldots, Y_m, \xi_1, \xi_2, \xi_3, \ldots, \xi_p \} \), such that \(\mathbb{R}^{m+p+2} = \text{span}\{ Y, \hat{Y} \} \bigoplus \text{span}\{ \eta_1, \eta_2, Y_3, \ldots, Y_m, \xi_1, \xi_2, \xi_3, \ldots, \xi_p \} \), \(\langle Y, \hat{Y} \rangle = 1\) and \(\{ \eta_1, \eta_2, Y_3, \ldots, Y_m, \xi_1, \xi_2, \xi_3, \ldots, \xi_p \} \) are orthonormal vector fields.

Using (4.28) and (4.27), we have

\[d \xi_1 = -\mu \omega_1 \eta_2 - \mu \omega_2 \eta_1 + \sum_{s=1}^{p} \theta_{1s} \xi_s, \]

\[\text{(4.29)}\]

\[d \xi_2 = -\mu \omega_1 \eta_1 + \mu \omega_2 \eta_2 + \sum_{s=1}^{p} \theta_{2s} \xi_s, \]

\[d \xi_\alpha = -\theta_{1\alpha} \xi_1 - \theta_{2\alpha} \xi_2 + \sum_{\beta} \theta_{\alpha \beta} \xi_\beta. \]
\( d\eta_1 = \left[ \omega_{12} - \frac{C_1^1}{\mu} \omega_1 + \frac{C_1^2}{\mu} \omega_2 \right] \eta_2 + \omega_1 \left( \frac{K}{2} Y - \hat{Y} \right) + \mu \omega_2 \xi_1 + \mu \omega_1 \xi_2, \)

\( d\eta_2 = - \left[ \omega_{12} - \frac{C_1^1}{\mu} \omega_1 + \frac{C_1^2}{\mu} \omega_2 \right] \eta_1 + \omega_2 \left( \frac{K}{2} Y - \hat{Y} \right) + \mu \omega_1 \xi_1 - \mu \omega_2 \xi_2, \)

\( d \left( \frac{K}{2} Y - \hat{Y} \right) = K \left[ \omega_1 \eta_1 + \omega_2 \eta_2 \right] + \left[ \frac{C_1^2}{\mu} \omega_1 + \frac{C_1^1}{\mu} \omega_2 \right] \left( \frac{K}{2} Y - \hat{Y} \right). \)

\( E_1(K) = 2 \frac{C_1^2}{\mu} K, \quad E_2(K) = 2 \frac{C_1^1}{\mu} K, \quad E_a(K) = 0. \)

From (4.29) and (4.30), we know that the subspace
\[ V = \text{span} \{ \left( \frac{K}{2} Y - \hat{Y} \right), \eta_1, \eta_2, \xi_1, \xi_2, \cdots, \xi_p \} \]
is parallel along \( M^m \). The orthogonal complement \( V^\perp \) also is parallel along \( M^m \). In fact,
\[ V^\perp = \text{span} \{ \left( \frac{K}{2} Y + \hat{Y} \right), Y_3, \cdots, Y_m \}. \]

Using (4.28) and (4.27), we can obtain
\( d \left( \frac{K}{2} Y + \hat{Y} \right) = \left( \frac{C_1^2}{\mu} \omega_1 + \frac{C_1^1}{\mu} \omega_2 \right) \left( \frac{K}{2} Y + \hat{Y} \right) + K \sum \omega_a Y_a. \)

Clearly, the distribution \( D^\perp = \text{span} \{ E_3, \cdots, E_m \} \) also is integrable. From (4.29) and (4.30), we know that the mean curvature spheres \( \xi_1, \xi_2 \) induce 2-dimensional submanifolds in the de sitter space \( S_1^{m+p+1} \)
\[ \xi_1, \xi_2 : M^2 = M^m / F \rightarrow S_1^{m+p+1}, \]
where fibers \( F \) are integral submanifolds of distribution \( D^\perp \). In other words, \( \xi_1, \xi_2 \) form 2-parameter family of \((m+p-1)\)-spheres enveloped by \( f : M^m \rightarrow \mathbb{R}^{m+p} \).

Since \( \langle \frac{K}{2} Y - \hat{Y}, \frac{K}{2} Y - \hat{Y} \rangle = - \langle \frac{K}{2} Y + \hat{Y}, \frac{K}{2} Y + \hat{Y} \rangle = -K \) satisfies a linear first-order PDE (4.31), we see that \( K \equiv 0 \) or \( K \neq 0 \) on the connected open set of \( M^m \). Thus there are three possibilities for the induced metric on the fixed subspace \( V, V^\perp \subset \mathbb{R}_1^{m+p+2} \).

**Case 1.** \( K < 0 \) on \( M^m \); \( V \) is a fixed space-like subspace, \( V^\perp \) is a fixed Lorentz subspace in \( \mathbb{R}_1^{m+p+2} \). We can assume that \( V = \mathbb{R}^{3+p}, \quad V^\perp = \mathbb{R}_1^{m-1} \). From (4.29), (4.30) and (4.31), we know
\[ u = \frac{1}{\sqrt{-K}} \left( \frac{K}{2} Y - \hat{Y} \right) : M^2 \rightarrow S_2^{2+p}. \]
On the other hand, the equation (4.32) implies that
\[ \phi = \frac{1}{\sqrt{-K}} \left( \frac{K}{2} Y + \hat{Y} \right) : \mathbb{H}^{m-2} \to \mathbb{R}_1^{m-1} \]
is the embedding of the hyperbolic space \( \mathbb{H}^{m-2} \) in \( \mathbb{R}_1^{m-1} \). Then
\[ Y = 2\sqrt{-K}(u, \phi) : M^2 \times \mathbb{H}^{m-2} \to \mathbb{S}^{2+p} \times \mathbb{H}^{m-2} \subset \mathbb{R}_1^{m+p+2}, \]
where \( 2\sqrt{-K} \in C^\infty(M^2) \) and \( \phi : \mathbb{H}^{m-2} \to \mathbb{H}^{m-2} \) is a identity map. From Proposition (3.4), we know that \( f \) is a cone over \( u : M^2 \to \mathbb{S}^{2+p} \).

Case 2. \( K > 0 \) on \( M^m \); \( V \) is a fixed Lorentz subspace, \( V^\perp \) is a fixed space-like subspace in \( \mathbb{R}_1^{m+p+2} \). We can assume that \( V = \mathbb{R}_1^{3+p} \), \( V^\perp = \mathbb{R}^{m-1} \). From (4.29), (4.30) and (4.31), we know
\[ u = \frac{1}{\sqrt{K}} \left( \frac{K}{2} Y - \hat{Y} \right) : M^2 \to \mathbb{H}^{2+p}. \]
On the other hand, the equation (4.32) implies that
\[ \phi = \frac{1}{\sqrt{K}} \left( \frac{K}{2} Y + \hat{Y} \right) : \mathbb{S}^{m-2} \to \mathbb{R}_1^{m-1} \]
is the embedding of the sphere \( \mathbb{S}^{m-2} \) in \( \mathbb{R}^{m-1} \). Then
\[ Y = 2\sqrt{K}(u, \phi) : M^2 \times \mathbb{S}^{m-2} \to \mathbb{H}^{2+p} \times \mathbb{S}^{m-2} \subset \mathbb{R}_1^{m+p+2}, \]
where \( 2\sqrt{K} \in C^\infty(M^2) \) and \( \phi : \mathbb{S}^{m-2} \to \mathbb{S}^{m-2} \) is the identity map. From Proposition (3.4), we know that \( f \) is the rotational submanifold over \( u : M^2 \to \mathbb{H}^{2+p} \).

Case 3. \( K = 0 \) on \( M^m \). From (4.32), we can assume that \( \hat{Y} = e^g(-1, 1, 0, \ldots, 0) \), where \( g \in C^\infty(M^2) \). On the other hand, \( V, \ V^\perp \) are two fixed spaces endowed with a degenerate inner product. we can assume that \( V = \text{span}\{ (\frac{K}{2} Y - \hat{Y}), \eta_1, \eta_2, \xi_1, \xi_2, \ldots, \xi_p \} = \mathbb{R}_0^{3+p}, \ V^\perp = \mathbb{R}_0^{m-1} \). We write vector \( v \in \mathbb{R}_0^{3+p} \) and \( w \in \mathbb{R}_0^{m-1} \) by
\[ u = (u_0, -u_0, u_1, \ldots, u_{p+2}, 0, \ldots, 0), \ w = (w_0, -w_0, 0, \ldots, 0, w_1, \ldots, w_{m-2}); \]
and we write
\[ e^\sigma Y = \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right), \ f = (u_1, \ldots, u_{p+2}, w_1, \ldots, w_{m-2}) \in \mathbb{R}^{m+p}. \]
From (4.29) and (4.30), we know that
\[ u = (u_1, \ldots, u_{p+2}) : M^2 \to \mathbb{R}^{2+p} \]
is an immersed surface, and

\[ w = (w_1, \cdots, w_{m-2}) : \mathbb{R}^{m-2} \to \mathbb{R}^{m-2} \]

is the identity map. From Proposition (3.4), we know that \( f \) is the cylinder over \( u : M^2 \to \mathbb{R}^{2+p} \).

Combining Proposition 3.1, 3.2 and 3.3, we complete the proof to Theorem 1.1.

Remark 4.1. From (4.29) and (4.30), we obtain

\[
\begin{align*}
  dY &= -\left( \frac{C_1^2}{\mu} \omega_1 + \frac{C_1^1}{\mu} \omega_2 \right) Y + \omega_1 \eta_1 + \omega_2 \eta_2 + \sum_a \omega_a Y_a, \\
  d\hat{Y} &= \left( \frac{C_2^1}{\mu} \omega_1 + \frac{C_2^1}{\mu} \omega_2 \right) \hat{Y} + \frac{K}{2} \sum_a \omega_a Y_a - \frac{K}{2} (\omega_1 \eta_1 + \omega_2 \eta_2).
\end{align*}
\]

Thus we have

\[
\langle d\hat{Y}, d\hat{Y} \rangle = \frac{K^2}{4} \langle dY, dY \rangle = \frac{K^2}{4} g.
\]

Let \( \hat{f} : M^m \to R^{m+p} \) be an immersed submanifold such that the Möbius position vector is \( \hat{Y} \). From \( \langle \hat{Y}, \xi_1 \rangle = \cdots = \langle \hat{Y}, \xi_p \rangle = 0 \) and (4.33), we know that the submanifold \( \hat{f} : M^m \to R^{m+p} \) envelops the mean curvature spheres \( \{ \xi_1, \cdots, \xi_p \} \). And if \( \hat{f} \) is an immersed submanifold, then \( \hat{f} \) also is a Wintgen ideal submanifold and is conformal to \( f \). This is analogous to the duality phenomenon for Willmore surfaces in \( \mathbb{S}^3 \), but simpler than that. Here \( \hat{f} \) either differ from \( f \) by an antipodal map of the sphere in Case 1, or by an inversion/reflection with respect to the boundary at infinity of the hyperbolic space in Case 2, or degenerate to the single point at infinity of the Euclidean space in Case 3.

Acknowledgements: The authors thank Dr. Yuquan Xie for helpful discussions.

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