A GENERALIZATION OF SELBERG'S BETA INTEGRAL

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Abstract. We evaluate several infinite families of multidimensional integrals which are generalizations or analogs of Euler's classical beta integral. We first evaluate a $q$-analogue of Selberg's beta integral. This integral is then used to prove the Macdonald-Morris conjectures for the affine root systems of types $S(C_\ell)$ and $S(C_\ell)^\vee$ and to give a new proof of these conjectures for $S(BC_\ell)$, $S(B_\ell)$, $S(B_\ell)^\vee$ and $S(D_\ell)$.

1. Introduction

In 1944, A. Selberg [23] evaluated the following integral (see also Aomoto [1]):

\[
\int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2z} \prod_{j=1}^{n} t_j^{x-1} (1-t_j)^{y-1} \, dt_j \\
= \prod_{j=1}^{n} \frac{\Gamma(x + (j-1)z) \Gamma(y + (j-1)z) \Gamma(jz + 1)}{\Gamma(x + y + (n+j-2)z) \Gamma(z+1)},
\]

where $n$ is a positive integer, $x, y, z \in \mathbb{C}$ and $\text{Re}(x), \text{Re}(y) > 0$ and $\text{Re}(z) > -\max\{\frac{1}{n}, \text{Re}(x)/(n-1), \text{Re}(y)/(n-1)\}$. For $n = 1$, the integral (1) reduces to Euler's classical beta integral.

Now let $n \geq 1$ and $a_1, a_2, a_3, a_4, b, q \in \mathbb{C}$ with

\[
\max\{|a_1|, \ldots, |a_4|, |b|, |q|\} < 1.
\]

For $c \in \mathbb{C}$ define

\[
[c; q]_\infty = [c]_\infty = \prod_{k=0}^{\infty} (1 - cq^k).
\]

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If $T^n$ is the $n$-fold direct product of the unit circle $\{t \in \mathbb{C} | |t| = 1\}$ traversed in the positive direction, then we can evaluate the integral

\[
(2) \quad \frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} \frac{[t_j t_k^{-1}]_{\infty} [t_j^{-1} t_k]_{\infty} [t_j^-1 t_k^-1]_{\infty}}{[b t_j t_k^{-1}]_{\infty} [b t_j^-1 t_k]_{\infty} [b t_j^-1 t_k^-1]_{\infty}} \cdot \prod_{j=1}^{n} \frac{[t_j^2]_{\infty} [t_j^{-2}]_{\infty}}{\prod_{k=1}^{4} [a_k t_j]_{\infty} [a_k t_j^{-1}]_{\infty}} dt_j
\]

\[
= 2^n n! \prod_{j=1}^{n} \frac{[b]_{\infty} [b^{n+j-2}]_{\infty}}{\prod_{1 \leq k < l \leq 4} [a_k a_l b^{l-j-1}]_{\infty}}.
\]

Then $n = 1$ case of integral (2) is due to Askey and Wilson [4]. The integral (2) is a $q$-analog of (1) in the sense that after a change of variables and an appropriate specialization of (2) and limit as $q \to 1$, then (1) can be deduced from (2).

Selberg’s integral (1) has had diverse applications in fields ranging from number theory, physics, statistics, combinatorics, algebra and analysis. Two particular applications were a use by Bombieri to prove Mehta’s conjecture [18] and by Macdonald [17] to prove some of his conjectures ($q = 1$ case) for the affine root systems (for definition and properties see [15]) of types $S(BC_l)$, $S(B_l)$, $S(B_l)^\vee$, $S(C_l)$, $S(C_l)^\vee$ and $S(D_l)$ for all $l \geq 1$ (when defined). Just as Macdonald used integral (1) to prove some of his ($q = 1$) conjectures, we will use integral (2) to prove for the same set of affine root systems the corresponding Macdonald-Morris conjectures with arbitrary parameter $q$.

Macdonald’s root system conjectures in [17] were motivated partly by a conjecture of Dyson [7] related to the root system $A_n$, a $q$-analog of Dyson’s conjecture made by Andrews [2] and some conjectures of Morris [19] for the root system of type $G_2$. Dyson’s conjecture was proved by Gunson [10] and Wilson [25]. The Andrews-Dyson conjecture was proved by Zeilberger and Bressoud [28].

Morris’ Conjecture A in [19] for arbitrary parameter $q$ and any reduced irreducible affine root system $S$ extends Macdonald’s Conjectures 2.3 and 3.1 in [17]. In the simplest case of these Macdonald-Morris conjectures, let $R$ be a reduced finite (not affine) root system of rank $l$ with basis $\{\alpha_1, \ldots, \alpha_l\}$. For each $\alpha \in R$, let $e^\alpha$ be the formal exponential, which is an element of the group ring of the lattice generated by $R$. Let $d_1, \ldots, d_l$
be the degrees of the fundamental invariants of the Weyl group \( W(R) \).

**Conjecture (Macdonald [17, Conjecture 3.1]).** With the above notation, the constant term (i.e. involving \( q \) but no exponential \( e^\alpha \)) in

\[
\prod_{\alpha > 0} \prod_{i=1}^{k} (1 - q^{l-1} e^{-\alpha})(1 - q^l e^\alpha)
\]

where \( k \) is a positive integer or \(+\infty\) is

\[
\prod_{i=1}^{l} \left[ \frac{k d_i}{k} \right]
\]

where

\[
\begin{pmatrix} n \\ r \end{pmatrix}
\]

is the "\( q \)-binomial coefficient"

\[
\frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}
\]

We will actually prove the more general Morris’ Conjecture A [19] for the affine root systems \( S \) of types \( S(BC_I), S(B_I), S(B_I)^\vee, S(C_I), S(C_I)^\vee \) and \( S(D_I) \) for all \( l \geq 1 \) (when defined) and for arbitrary parameter \( q \). Macdonald’s Conjecture 3.1 stated above, where \( R \) is a finite root system of type \( B_I, C_I \) or \( D_I \), then follows as a special case of Morris’ Conjecture A for \( S(B_I), S(C_I) \) and \( S(D_I) \). Kadell [14] has previously proved these conjectures for all affine root systems of type \( S(BC_I) \) and hence \( S(B_I), S(B_I)^\vee \) and \( S(D_I) \). The Macdonald-Morris conjectures for \( R = G_2 \) have been proved by Habsieger [13] and Zeilberger [26]. See Garvan [8] for \( F_4 \), Garvan and Gonnet [9] for \( S(F_4)^\vee \), Zeilberger [27] for \( S(G_2)^\vee \) and Opdam [20] for the \( q = 1 \) conjectures. There is also the conjecture of Rahman [21] which seems related to the special case of integral (2) where \( a_2 = q^{1/2}a_1 \) and \( a_4 = q^{1/2}a_3 \).

2. Proof of integral (2)

Since the \( n = 1 \) case of (2) is proved in [4], we may assume that \( n \geq 2 \). Denote the integral on the left-hand side of (2) by \( I_n(a_1, a_2, a_3, a_4; b; q) \). Let \( c_j \in \mathbb{C}, |c_j| < 1 \), for \( 1 \leq j \leq 2n+2 \).
with \( q \) and \( T \) as above. In [11] we have evaluated the integral

\[
\frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} \left\{ [t_j t_k^{-1}]_\infty [t_j^{-1} t_k]_\infty [t_j t_k]_\infty [t_j^{-1} t_k^{-1}]_\infty \right\} 
\times \prod_{j=1}^{n+2} \prod_{k=1}^n [c_j t_k]_\infty [c_j t_k^{-1}]_\infty 
\times \prod_{j=1}^n \frac{[t_j^2]_\infty [t_j^{-2}]_\infty \, dt_j}{t_j} 
\times 2^n n! \left[ \prod_{j=1}^{n+2} c_j \right]_\infty 
\frac{[q]_\infty^n \prod_{1 \leq j < k \leq 2n+2} [c_j c_k]_\infty \cdot}{[q]_\infty^n \prod_{1 \leq j < k \leq 2n+2} [c_j c_k]_\infty}.
\]

With notation as above, consider the integral

\[
\frac{1}{(2\pi i)^{2n-1}} \int_{T^n} \int_{T^{n-1}} \prod_{1 \leq j < k \leq n} \left\{ [t_j t_k^{-1}]_\infty [t_j^{-1} t_k]_\infty [t_j t_k]_\infty [t_j^{-1} t_k^{-1}]_\infty \right\} 
\times \prod_{j=1}^{n+1} \prod_{k=1}^n [a_j t_j]_\infty [a_k t_j^{-1}]_\infty 
\times \prod_{j=1}^n \frac{[t_j^2]_\infty [t_j^{-2}]_\infty \, dt_j}{t_j} 
\times \prod_{j=1}^{n-1} \prod_{k=1}^n \left\{ [b_1/2 s_j t_j]_\infty [b_1/2 s_k t_j]_\infty [b_1/2 s_j t_j^{-1}]_\infty [b_1/2 s_k t_j^{-1}]_\infty \right\} 
\times \prod_{k=1}^{n-1} \frac{[s_k^2]_\infty [s_k^{-2}]_\infty \, ds_k}{s_k} \prod_{j=1}^n \frac{dt_j}{t_j},
\]

where \( b^{1/2} \) is any fixed square root of \( b \). In the integral (4) we may use identity (3) to evaluate the interior integral either with respect to the set of variables \( \{s_1, \ldots, s_{n-1}\} \) or, by changing the order of integration, with respect to the set of variables \( \{t_1, \ldots, t_n\} \). Equating the resulting integrals we obtain

\[
\frac{2^{n-1} (n-1) \prod_{j=1}^4 [a_j]_\infty}{[a_1 b]_\infty^n} I_n(a_1, a_2, a_3, a_4; b; q) 
\times 2^n n! \prod_{j=1}^{n-1} [a_j]_\infty 
\frac{[b]_\infty^{n-1} \prod_{1 \leq j < k \leq 4} [a_j a_k]_\infty}{[a_1 b^{1/2}, \ldots, a_4 b^{1/2}; b; q].}
\]
We finish the proof of identity (2) by doing induction on $n$, using identity (5) and the Askey-Wilson integral for the case $n = 1$.

3. Morris' Conjecture A

We sketch a proof of Morris' Conjecture A [19] for the affine root systems $S$ of types $S(BC_l)$, $S(B_l)$, $S(C_l)$, $S(C_l)^{\vee}$ and $S(D_l)$ where $l \geq 1$ (when defined) and for arbitrary parameter $q$. The proof consists of specializing the parameters in identity (2) and making use of the identity found in Theorem 2.8 of [16]. As an illustration of this method of proof of Morris' Conjecture A, consider the case $S = S(C_l)$ where $l \geq 2$. Consider the integral $I_t(a^{1/2}, -a^{1/2}, q^{1/2}a^{1/2}, -q^{1/2}a^{1/2}; b; q)$ where $|a|, |b| < 1$. Multiply the integrand in this integral by

$$\prod_{1 \leq j < k \leq l} \frac{(1 - bw(t_j^{-1}t_k))(1 - bw(t_j^{-1}t_k^{-1}))}{(1 - w(t_j^{-1}t_k))(1 - w(t_j^{-1}t_k^{-1}))} \prod_{j=1}^l (1 - aw(t_j^{-2})),$$

where $w$ is an element of the Weyl group $W$ of $C_l$, i.e. a permutation of the variables $t_1, \ldots, t_l$ together with inversions $t_j \rightarrow t_j^{-1}$ and the corresponding action on $t_1^{-1}, \ldots, t_l^{-1}$. The resulting integral is independent of $w \in W$. Now summing over $w \in W$ and using the identity [16, Theorem 2.8] for $C_l$ we obtain

$$\frac{1}{(2\pi i)^l} \int_{T} \prod_{1 \leq j < k \leq l} \frac{[t_j^2 t_k^{-2}]_{\infty}[q t_j t_k^{-1} t_k^{-1}]_{\infty}}{[b t_j^2 t_k^{-2}]_{\infty}[q b t_j t_k^{-1} t_k^{-1}]_{\infty}} \prod_{j=1}^l \frac{[t_j^2]_{\infty}[q t_j^{-2}]_{\infty}}{[a t_j^2]_{\infty}[q a t_j^{-2}]_{\infty}} dt_j$$

$$= \prod_{j=1}^l \frac{[qb]_{\infty}[q a^2 b^{l+j-2}]_{\infty}[q a b^{l+1-j}]_{\infty}}{[q]_{\infty}[q b^j]_{\infty}[q a^2 b^{2(j-1)}]_{\infty}},$$

which is equivalent to Morris' Conjecture A for $S(C_l)$ [19, p. 131]. Setting $a = b$ in (6), this also proves Macdonald's Conjecture 3.1 for $R = C_l$ as stated above.

4. Some integral evaluations

We state some integral identities whose proofs are similar to that of (2), making use of integral identities from [11 and 12].
Details of the proofs of these and related integral identities should be given elsewhere.

Let \( n \geq 1 \) and \( z_1, \ldots, z_n, \alpha_1, \ldots, \alpha_4, a_1, \ldots, a_4, \beta_1, \beta_2, b, \delta \in \mathbb{C} \) and \( m_1, \ldots, m_n \in \mathbb{Z} \). Choose \( z_1, \ldots, z_n \) so that the integrands in the integrals (9) and (10) below have no poles. Then

\[
\begin{align*}
(7) \quad & \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{1 \leq j < k \leq n} \left\{ \frac{\Gamma(\delta + t_j - t_k)\Gamma(\delta + t_k - t_j)}{\Gamma(t_j - t_k)\Gamma(t_k - t_j)} \right\} \\
& \cdot \frac{\Gamma(\delta + t_j + t_k)\Gamma(\delta - t_j - t_k)}{\Gamma(t_j + t_k)\Gamma(-t_j - t_k)} \prod_{j=1}^{n} \prod_{k=1}^{4} \frac{\Gamma(\alpha_k + t_j)\Gamma(\alpha_k - t_j)}{\Gamma(2t_j)\Gamma(-2t_j)} \\
& = 2^n n! \prod_{j=1}^{n} \prod_{1 \leq k < l \leq 4} \frac{\Gamma(\alpha_k + \alpha_l + (j - 1)\delta)}{\Gamma(\delta)\Gamma((n + j - 2)\delta + \sum_{k=1}^{4} \alpha_k)},
\end{align*}
\]

where the contours of integration are the imaginary axis and

\[
\min\{\text{Re}(\delta), \text{Re}(\alpha_1), \ldots, \text{Re}(\alpha_4)\} > 0;
\]

\[
\begin{align*}
(8) \quad & \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{1 \leq j < k \leq n} \left\{ \frac{\Gamma(\delta + t_j - t_k)}{\Gamma(t_j - t_k)} \right\} \\
& \cdot \prod_{j=1}^{n} \left\{ \prod_{k=1}^{2} \left[ \frac{\Gamma(\alpha_k + t_j)\Gamma(\beta_k - t_j)}{\Gamma(\beta_k - t_j)} \right] \right\} \\
& = n! \prod_{j=1}^{n} \frac{\Gamma(j\delta) \prod_{k, l=1}^{2} \Gamma(\alpha_k + \beta_l + (j - 1)\delta)}{\Gamma(\delta)\Gamma((n + j - 2)\delta + \sum_{k=1}^{4} (\alpha_k + \beta_k))},
\end{align*}
\]

where the contours of integration are the imaginary axis and

\[
\min\{\text{Re}(\delta), \text{Re}(\alpha_1), \text{Re}(\alpha_2), \text{Re}(\beta_1), \text{Re}(\beta_2)\} > 0;
\]
(9) \[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq j, k \leq n \atop j \neq k} \frac{\Gamma(1 + z_j + t_j - z_k - t_k)}{\Gamma(1 + \delta + z_j + t_j - z_k - t_k)} \cdot \prod_{j=1}^{n} \frac{e^{2\pi im_j t_j}}{\prod_{k=1}^{2} \Gamma(1 + \alpha_k + z_j + t_j)\Gamma(1 + \beta_k - z_j - t_j)} \] 

\[ = \begin{cases} \prod_{j=1}^{n} \frac{\Gamma(1 + \delta)\Gamma(1 + (n + j - 2)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k))}{\prod_{k, l=1}^{2} \Gamma(1 + \alpha_k + \beta_l + (j - 1)\delta)} , & \text{if } m_1 = \cdots = m_n = 0 \\
0, & \text{otherwise} \end{cases} \]

where

\[ \min \left\{ \text{Re} \left( (n - 1)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k) \right) , \text{Re} \left( 2(n - 1)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k) \right) \right\} > -1 ; \]

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq n} \left\{ \frac{[bq^{1+z_j+t_j-z_k}]_{\infty}[bq^{1-z_j-t_j+z_k+t_k}]_{\infty}}{[q^{1+z_j+t_j-z_k}]_{\infty}[q^{1-z_j-t_j+z_k+t_k}]_{\infty}} \cdot \frac{[bq^{1+z_j+t_j+z_k+t_k}]_{\infty}[bq^{1-z_j-t_j-z_k-t_k}]_{\infty}}{[q^{1+z_j+t_j+z_k+t_k}]_{\infty}[q^{1-z_j-t_j-z_k-t_k}]_{\infty}} \right\} \]

\[ \cdot \prod_{j=1}^{n} \{a_k q^{1+z_j+t_j}\}_{\infty}\{a_k q^{1-z_j-t_j}\}_{\infty} \cdot e^{2\pi i m_j t_j} \] 

(10) \[ = \begin{cases} \prod_{k=1}^{n} \frac{[q]_{\infty}[qb^j]_{\infty} \prod_{1 \leq k < l \leq 4} [qa_k a_l b^{l-1}]_{\infty}}{[qb]_{\infty} [q b^{n+j-2} a_k]_{\infty} , & \text{if } m_1 = \cdots = m_n = 0 \\
0, & \text{otherwise} \end{cases} \]
where
\[
\max \left\{ \left| q b^{n-1} \prod_{k=1}^{4} a_k \right|, \left| q b^{2(n-1)} \prod_{k=1}^{4} a_k \right| \right\} < 1
\]
and for simplicity we assume that \( q \in \mathbb{R}, 0 < q < 1 \). The \( n = 1 \) case of (7) is due to de Branges [6] and Wilson [24], of (8) to Barnes [5], of (9) to Ramanujan [22] and (10) essentially to Askey [3].

Remarks. The integrals (9) and (10) are equivalent to multiple series summation theorems which generalize classical bilateral hypergeometric series summation theorems: Dougall’s \(_2H_2\) sum and Bailey’s \(_6\psi_6\) sum. A similar connection between some related integral evaluations and the corresponding multiple series identities is explained in [12]. As we plan to describe elsewhere, we are led to conjecture a family of multiple series summation identities which are equivalent to the Macdonald-Morris conjectures and contain the Macdonald identities [15] as special cases.

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