ON THE NUMBER OF GRADINGS ON MATRIX ALGEBRAS

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Abstract. We determine the number of isomorphism classes of elementary gradings by a finite group on an algebra of upper block-triangular matrices. As a consequence we prove that, for a finite abelian group $G$, the sequence of the numbers $E(G, m)$ of isomorphism classes of elementary $G$-gradings on the algebra $M_m(\mathbb{F})$ of $m \times m$ matrices with entries in a field $\mathbb{F}$ characterizes $G$. A formula for the number of isomorphism classes of gradings by a finite abelian group on an algebra of upper block-triangular matrices over an algebraically closed field, with mild restrictions on its characteristic, is also provided. Finally, if $G$ is a finite abelian group, $\mathbb{F}$ is an algebraically closed field and $N(G, m)$ is the number of isomorphism classes of $G$-gradings on $M_m(\mathbb{F})$ we prove that $N(G, m) \sim \frac{m}{|G|} m^{G-1} \sim E(G, m)$.

1. Introduction

Let $\mathbb{F}$ be a field, all vector spaces, algebras and tensor products are considered over $\mathbb{F}$. Let $A$ be an algebra and $G$ a group. A $G$-grading on $A$ is a decomposition

$$A = \bigoplus_{g \in G} A_g$$

of $A$ as a direct sum of subspaces such that $A_g A_h \subset A_{gh}$ for all $g, h \in G$, we say that $A$ is a $G$-graded (or simply graded) algebra. If $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ are $G$-graded algebras, we recall that an isomorphism of $G$-graded algebras $\varphi : A \rightarrow B$ is an isomorphism of algebras such that $\varphi (A_g) \subset B_g$ for all $g \in G$. We also recall that the non-zero elements $a \in A_g$ are called homogeneous of degree $g$. The problem of classifying and counting all the possible $G$-gradings (up to isomorphisms) in an algebra $A$ has been investigated by several authors. The gradings on matrix algebras are described in [1], [2] and [3]; we refer to the monograph [12, Chapter 2] for an account of the classification of gradings on matrix algebras. The gradings on algebras of upper triangular matrices and on algebras of upper block-triangular matrices are described in [6] and [13], respectively, and the isomorphisms of gradings on algebras of upper block-triangular matrices are studied in [5] and [9]. The problem of counting gradings on such algebras has been investigated in [4], [7] and [8].

From now on $M_m(\mathbb{F})$ is the algebra of $m \times m$ matrices with entries in $\mathbb{F}$. Let $m = (m_1, \ldots, m_s)$ be an $s$-tuple of positive integers and let $m = m_1 + \cdots + m_s$. We denote by $UT(m)$ the algebra of upper block-triangular matrices in $M_m(\mathbb{F})$ of the form

$$
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1s} \\
0 & A_{22} & \cdots & A_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{ss}
\end{pmatrix}
$$

where $A_{ij}$ is a block of size $m_i \times m_j$.

The algebras of upper block-triangular matrices and their $\mathbb{Z}_2$-gradings appear in a fundamental way in the classification of minimal varieties of algebras presented in [10], [11]. We remark that the matrix algebra $M_m(F)$ and the algebra $UT_m(\mathbb{F})$ of $m \times m$ upper triangular matrices over $\mathbb{F}$ are examples of algebras of upper block-triangular matrices. A $G$-grading on $UT(m)$ is elementary if every elementary matrix $e_{ij}$ in $UT(m)$ is homogeneous, we recall that $e_{ij}$ is the matrix with 1 in the $(i, j)$-th entry and 0 elsewhere. In this case it is well known that there exists an $n$-tuple $g = (g_1, \ldots, g_n) \in G^n$ such that

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$e_{ij}$ is homogeneous of degree $g_jg_i^{-1}$. If $D$ is a $G$-graded algebra, the algebra $A = UT(m) \otimes D$ admits a $G$-grading $A = \bigoplus_{g \in G} A_g$ such that

$$A_g = \text{span}\{ e_{ij} \otimes d \mid d \in D_h, g_jh g_i^{-1} = g \}.$$  

We denote by $E(G, m)$ the number of isomorphism classes of elementary $G$-gradings on $UT(m)$ and by $N(G, m)$ the number of isomorphism classes of $G$-gradings on $UT(m)$. For the matrix algebra $M_m(F)$ we use the notations $E(G, m)$ and $N(G, m)$ for the numbers of isomorphism classes of elementary and arbitrary $G$-gradings, respectively. We provide in Theorem 3.3, for an arbitrary finite group $G$, a formula for $G$, $H$ and $G,m$.

We study the asymptotic growth of $E(G, m)$ and $N(G, m)$. We prove that $E(G, m) \sim \frac{1}{|G|} m^{|G|-1}$ for a finite group $G$, see Proposition 3.9. The main results of this paper provide the exact asymptotic growth of $N(G, m)$ for a finite abelian group $G$ and show that the sequence $(E(G, m))_{m=1}^{\infty}$ characterizes such groups:

**Theorem 1.1.** Let $G, H$ be finite abelian groups. If $E(G, m) = E(H, m)$ for every natural number $m$, then $G \cong H$. Moreover, in general, the result does not hold for non-abelian groups.

The number of isomorphism classes of elementary gradings by a finite group of prime exponent is given in Example 3.6. As a consequence $E(G, m) = E(H, m)$, for every natural number $m$, if $G$ and $H$ are finite groups of prime exponent and of the same order. This remark proves the final statement of Theorem 1.1, i.e., the result does not hold without the hypothesis that the groups are abelian.

**Theorem 1.2.** If the field is algebraically closed then for all finite abelian groups $G$, the sequences of $N(G, m)$ and $E(G, m)$ have the same asymptotic growth. More precisely, $N(G, m) \sim E(G, m) \sim \frac{1}{|G|} m^{|G|-1}$.

The paper is organized as follows. In Section 2 we prove some technical results; in Section 3 we obtain a closed formula for $E(G, m)$ for arbitrary finite groups (not necessarily abelian). In particular, we provide a very simple formula for $E(G, m)$ if $G$ is a finite group of prime exponent and we also obtain a formula for $E(\mathbb{Z}_n, m)$ in terms of Euler’s totient function. Our first main result (Theorem 1.1) is proved in the end of Section 3. In Section 4 we provide a formula for $N(G, m)$ for a finite abelian group $G$ and prove the second main result (Theorem 1.2).

2. Preliminaries

In this section we present the results on the classification of gradings on algebras of upper block-triangular matrices that will be used in the proof of the main results of the paper.

Let $G$ be a finite group, $m = (m_1, \ldots, m_s)$ be an $s$-tuple of positive integers and let $I_s = \{1, \ldots, s\}$. Denote by $\gamma(m, G)$ the set of the maps $a : I_s \times G \to \mathbb{Z}$ such that $a(i, g) \geq 0$ for every $(i, g) \in I_s \times G$ and $\sum_{g \in G} a(i, g) = m_i$ for $i = 1, \ldots, s$. Given $a \in \gamma(m, G)$ and $h \in G$ the map $a \cdot h$ such that $(a \cdot h)(i, g) := a(i, gh)$ lies in $\gamma(m, G)$. Note that $(a, h) \mapsto a \cdot h$ is a right action of $G$ on $\gamma(m, G)$.

Henceforth we consider this action of $G$ on $\gamma(m, G)$. Our main goal in this section is to prove that there exists a bijection from the set of elementary gradings on $UT(m)$ to the set of orbits of this action. Next we present the result in [5, Corollary] using the notation here instead of the notation in [5], which is in terms of rings of endomorphisms of graded flags (see [5, Definition 1, Proposition 1]).

Let $n = m_1 + \cdots + m_s$ and let $P_1, \ldots, P_s$ be the subsets of $I_n$ such that every element of $P_i$ is strictly smaller than every element of $P_j$ whenever $i < j$ and $|P_i| = m_i$ for $i = 1, \ldots, s$. Henceforth we denote by $S_{m_1} \times \cdots \times S_{m_s}$ the Young subgroup associated to the partition

$$I_n = \cup_i P_i,$$

of the set $I_n$, in this case $S_{m_i} = \{ \sigma \in S_n : \sigma(j) = j, \forall j \notin P_i \}$.

Let $D$ be an algebra with a grading by the group $G$ and let $g \in G$. We may endow the same algebra with a $G$-grading, denoted by $[g^{-1}]D^{[g]}$, such that for every $h \in G$ a non-zero element $d \in D_h$ is homogeneous of degree $g^{-1}hg$ in the $G$-grading $[g^{-1}]D^{[g]}$. This notation appears in the next result.


Corollary 2.1. [5, Corollary 1] Let \( m = (m_1, \ldots, m_s) \) and \( m' = (m'_1, \ldots, m'_s) \) be tuples of positive integers and let \( g = (g_1, \ldots, g_n) \) and \( g' = (g'_1, \ldots, g'_n) \) be tuples of elements of \( G \), where \( n = m_1 + \cdots + m_s \) and \( n' = m'_1 + \cdots + m'_s \). Let \( B \) and \( B' \) be the algebras \( UT(m) \) and \( UT(m') \) with the elementary gradings induced by \( g \) and \( g' \), respectively. Let \( D \) and \( D' \) be \( G \)-graded algebras with a division grading. The \( G \)-graded algebras \( B \otimes D \) and \( B' \otimes D' \) are isomorphic if and only if \( m = m' \), there exists \( a, g \in G \) such that \( [g^{-1}]D[g] \) is isomorphic to \( D' \) and there exist \( h_1, \ldots, h_n \in \text{supp} \, D \) and \( \sigma \in S_{m_1} \times \cdots \times S_{m_s} \) such that \( g_i = g_{\sigma(i)}h_{\sigma(i)}g \) for \( i = 1, \ldots, n \).

As a consequence of the corollary above we prove the main result of this section.

Proposition 2.2. Let \( m = (m_1, \ldots, m_s) \) be an \( s \)-tuple of positive integers and let \( G \) be a group. There exists a bijection between the set of isomorphism classes of elementary \( G \)-gradings on \( UT(m) \) and the set of orbits of the right \( G \)-action on \( \gamma(m, G) \).

Proof. Let \( n = m_1 + \cdots + m_s \) and let \( P_1, \ldots, P_n \) be the sets in the partition (1) of \( I_n \). We associate to the tuple \( h = (h_1, \ldots, h_n) \in G^n \) the map \( a(h) \) such that

\[
a(h)(i, g) := \{ j \in P_i \mid h_j = g \}.
\]

Note that \( a(h) \in \gamma(m, G) \) and \( h \mapsto a(h) \) is a surjective map. If \( g = (g_1, \ldots, g_n) \) and \( g' = (g'_1, \ldots, g'_n) \) are tuples in \( G^n \) then \( a(g) = a(g') \) if and only if there exists \( \sigma \in S_{m_1} \times \cdots \times S_{m_s} \) such that \( g'_i = g_{\sigma(i)} \) for \( i = 1, \ldots, n \). Also, \( a(g) \cdot g^{-1} \) is the map associated to the tuple \( (g_1g, \ldots, g_ng) \). Hence \( a(g) \) and \( a(g') \) lie in the same orbit if and only if there exist \( \sigma \in S_{m_1} \times \cdots \times S_{m_s} \) and \( g \in G \) such that \( g'_i = g_{\sigma(i)}g \) for \( i = 1, \ldots, n \). Corollary 2.1 implies that this holds if and only if the elementary gradings induced by \( g \) and \( g' \) are isomorphic. Therefore the map that associates to the isomorphism class of the elementary grading on \( UT(m) \) induced by \( g \in G^n \) the orbit of \( a(g) \) is a bijection.

We end this section with the results on the classification of gradings on algebras of upper block-triangular matrices that will be used in the proof of Theorem 4.3 that provides a formula for \( N(G, m) \).

Theorem 2.3. [13] Let \( G \) be any group, let \( m = (m_1, m_2, \ldots, m_s) \) be an \( s \)-tuple of positive integers and consider any \( G \)-grading the algebra \( A = UT(m) \) of upper block-triangular matrices over a field \( F \). Suppose that either \( \text{char} \, F = 0 \) or \( \text{char} \, F > \text{dim} \, A \). Then there exists a division \( G \)-grading \( D \) on \( M_n(F) \) and an algebra \( B = UT(n) \), where \( n = (n_1, \ldots, n_s) \), of upper block-triangular matrices endowed with an elementary grading, such that \( A \cong B \otimes D \).

We remark that for a matrix algebra, i.e., for \( s = 1 \) the above result holds for an arbitrary field \( F \).

Remark 2.4. It follows from the known classification results of gradings on matrix algebras that the above result holds for matrix algebras without the hypothesis on the characteristic of the field \( F \), see [12, Corollary 2.12].

As commented in [13], it is an interesting question if Theorem 2.3 holds for an arbitrary field.

3. Counting elementary gradings and the proof of Theorem 1.1

We begin this section determining the number of isomorphism classes of elementary gradings by a finite group \( G \) on algebras of upper block-triangular matrices. We use following well known result.

Lemma 3.1. [Burnside–Cauchy–Frobenius] Let \( G \) be a finite group acting on a finite set \( X \). Then the number of orbits equals the average number of fixed points:

\[
|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|,
\]

where

\[
\text{Fix}(g) = \{|x \in X : x \cdot g = x\}.
\]

The number of elements of order \( m \) in \( \mathbb{Z}_m \) is given by Euler’s totient function \( \phi : \mathbb{N} \to \mathbb{N} \); we recall that \( \phi(n) = n \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) \). In Proposition 3.7 we determine the map, in terms of \( \phi \), that associates to a positive integer \( t \) the number of elements of order \( t \) in \( \mathbb{Z}_m \). This is a particular case of the map in the next definition.
Definition 3.2. Let $G$ be a finite group. We denote by $\varphi_G$ the map $\mathbb{N} \to \mathbb{N}$ that associates to $t \in \mathbb{N}$ the number of elements of $G$ of order $t$.

We are now ready to provide a formula for the number of isomorphism classes of elementary gradings by a finite group in an algebra of upper block triangular matrices. We recall that for a finite group $G$ its exponent, denoted by $\operatorname{Exp}(G)$, is the least common multiple of the orders of its elements.

Theorem 3.3. Let $G$ be a finite group and let $m = (m_1, \ldots, m_s)$ be a tuple of positive integers. The number $E(G, m)$ of isomorphism classes of elementary $G$-gradings on the algebra of upper block-triangular matrices $UT(m)$ is

$$E(G, m) = \frac{1}{|G|} \left( \sum_{t \mid d} \prod_{i=1}^s \left( \frac{m_i}{t} + \frac{|G|}{t} - 1 \right) \varphi_G(t) \right),$$

where $d = \gcd(m_1, \ldots, m_s, \operatorname{Exp}(G))$.

Proof. We prove that for an element $h \in G$ of order $t$

$$\text{Fix}(h) = \begin{cases} 0 & \text{if } t \nmid m_i \text{ for some } i \\ \prod_{i=1}^s \left( \frac{q_i + k - 1}{q_i} \right) & \text{if } m_i = q_i t, \end{cases}$$

where $k = \frac{|G|}{t}$. The result then follows from Lemma 3.1. Let $a : I_s \times G \to \mathbb{Z}$ be a map such that $a : h = a$. Note that $a(i, gh) = a(i, g)$ for every $(i, g) \in I_s \times G$, and it follows by induction that

$$a(i, gh^n) = a(i, g),$$

for every non-negative integer $n$. Let $H = \{e, h_1, \ldots, h^t \}$ and let $g_1 H, \ldots, g_a H$ be the lateral classes of $H$ in $G$. We have

$$m_i = \sum_{g \in G} a(i, g) = \sum_{j=1}^u \sum_{h' \in H} a(i, g_j h') = t \left( \sum_{j=1}^u a(i, g_j) \right),$$

therefore $t \mid m_i$ for $i = 1, \ldots, s$. Hence $\text{Fix}(h) = 0$ if $t \nmid m_i$ for some $i \in I_s$. Now assume that there exist integers $q_1, \ldots, q_s$ such that $m_i = q_i t$ for $i \in I_s$. In this case the number of elements of $\gamma(m, G)$ fixed by $h$ is equal to the number of functions $\tilde{a} : I_s \times G/H \to \mathbb{Z}$ such that $\tilde{a}(i, gH) \geq 0$ for every $(i, gH) \in I_s \times G/H$ and $\sum_{gH \in G/H} \tilde{a}(i, gH) = q_i$ for every $i \in I_s$. The number of such maps is $\prod_{i=1}^s \left( \frac{q_i + t - 1}{q_i} \right)$.

A particular case of the previous theorem, for matrix algebras, is the following corollary.

Corollary 3.4. Let $G$ be a finite group. For be a positive integer $m$, the number $E(G, m)$ of isomorphism classes of elementary $G$-gradings on $M_m(\mathbb{F})$ is

$$E(G, m) = \frac{1}{|G|} \left( \sum_{t \mid d} \left( \frac{m}{t} + \frac{|G|}{t} - 1 \right) \varphi_G(t) \right),$$

where $d = \gcd(m, \operatorname{Exp}(G))$.

Next we apply of Theorem 3.3 for the algebra of upper triangular matrices.

Corollary 3.5. The number of isomorphism classes of gradings by a finite group $G$ on the algebra $UT_n(\mathbb{F})$ of upper triangular matrices with entries in the field $\mathbb{F}$ is $|G|^{n-1}$.

Proof. As a consequence of [6, Theorem 7] every $G$-grading on $UT_n(\mathbb{F})$ is isomorphic to an elementary grading. Hence the number of isomorphism classes of $G$-gradings on $UT_n(\mathbb{F})$ coincides with the number of isomorphism classes of elementary gradings on this algebra. The result now follows directly from Theorem 3.3. □
The formula above is given in terms of the map $\varphi_G$ in Definition 3.2. If $\text{Exp}(G)$ is a prime number $p$ it is plain that

$$\varphi_G(t) = \begin{cases} 1 & \text{if } t = 1 \\ |G| - 1 & \text{if } t = p \\ 0 & \text{if } t \nmid p \end{cases}$$

As a consequence we obtain the following formula for $E(G, m)$ if $G$ is a finite group of prime exponent.

**Example 3.6.** Let $G$ be a group of prime exponent $p$ and order $p^n$. Then

$$E(G, m) = \begin{cases} \frac{1}{p^n} (\frac{m+p^n-1}{m}) & \text{if } \gcd(p, m) = 1 \\ \frac{1}{p^n} \left( \frac{m+p^n-1}{m} + \left( \frac{m}{p^n} + \frac{m-1}{m} \right) (p^n-1) \right) & \text{if } \gcd(p, m) = p \end{cases}$$

A recurrence formula for $E(\mathbb{Z}_m^n, m)$ was obtained in [4], explicit formulas are given for $n = 1, 2$. We remark that as a consequence of the previous example if $G$ and $H$ are groups of prime exponent $p$ and order $p^n$ then $E(G, \ldots) = E(H, \ldots)$ even if $G$ and $H$ are not isomorphic. In Theorem 1.1 we prove that if $G$ and $H$ are abelian groups then the previous equality implies that $G$ and $H$ are isomorphic.

**Proposition 3.7.** Let $n$ be a positive integer. Then

$$\varphi_{\mathbb{Z}_n}(t) = \begin{cases} \phi(t) & \text{if } t \mid n \\ 0 & \text{if } t \nmid n. \end{cases}$$

**Proof.** Let $t$ be a natural number. If $t \nmid n$ then it is clear that $\varphi_{\mathbb{Z}_n}(t) = 0$. Now assume that $t \mid n$. The elements of $\mathbb{Z}_n$ of order $t$ lie in the subgroup $H$ of $\mathbb{Z}_n$ of order $t$, hence $\varphi_G(t) = \varphi_H(t)$. Since $H \cong \mathbb{Z}_t$ it follows that $\varphi_H(t) = \phi(t)$. 

The proposition above combined with Corollary 3.4 allow us to provide a formula for $E(\mathbb{Z}_m^n, m)$ in terms of Euler’s totient function. For a cyclic group of order $n$, a prime power of a prime, an explicit formula is given in the next example.

**Example 3.8.** Let $p$ be a prime number and let $n$ be a natural number. If $m = p^k m'$, where $p \nmid m'$ then

$$E(\mathbb{Z}_{p^k m'}, m) = \frac{1}{p^k} \left( \frac{m+p^k m'-1}{m} \right) + \sum_{i=1}^{k} \left( \frac{p^{n-i} - 1}{p^{n-i} m'} \right) \left( p^i - p^{i-1} \right).$$

Note that the maps $E(\mathbb{Z}_m^n, \cdot)$ and $E(\mathbb{Z}_{p^k}, \cdot)$ are different, however their asymptotic growth is the same. In the next proposition we prove that the asymptotic growth of $E(G, \cdot)$ is determined by the order of $G$.

**Proposition 3.9.** Let $G$ be a finite group. If $d \mid \text{Exp}(G)$ then there exists a polynomial $p^G_d(x)$ of degree $|G| - 1$, leading coefficient $\frac{1}{|G|!}$ and $p^G_d(m') \geq 0$ for every $m' \geq 0$, such that $E(G, m) = p^G_d(m)$ whenever $\gcd(\text{Exp}(G), m) = d$. Moreover $p^G_d(m) \leq E(G, m) \leq p_{\text{Exp}(G)}^G(m)$ for every natural number $m$, in particular $E(G, m) \sim \frac{1}{|G|!} m^{|G| - 1}$.

**Proof.** Let $f^G_d(x) = \frac{1}{(\frac{x}{d} - 1)!} \left( \frac{|G|}{x} - 1 \right) \left( \frac{|G|}{x} - 2 \right) \cdots \left( \frac{|G|}{x} - d + 1 \right)$ and let $p^G_d(x) = \frac{1}{|G|!} \left( \sum_{i=0}^{d} f^G_d(x) \varphi_G(t) \right)$. Note that $p^G_d(x)$ is a polynomial of degree $|G| - 1$ and leading coefficient $\frac{1}{|G|!}$. Corollary 3.4 implies that $E(G, m) = p^G_d(m)$ whenever $\gcd(\text{Exp}(G), m) = d$. Moreover since $f^G_d(m) > 0$ for every $m$ it follows that $p^G_d(m) \leq E(G, m) \leq p_{\text{Exp}(G)}^G(m)$ for every natural number $m$. Since $\lim_{m \to \infty} \frac{p^G_d(m)}{\frac{m^{|G| - 1}}{|G|!}} = 1$ for every divisor $d$ of $\text{Exp}(G)$ it follows from the previous inequalities that $E(G, m) \sim \frac{1}{|G|!} m^{|G| - 1}$. 

An immediate consequence of the proposition above is the following result.

**Corollary 3.10.** If $G$ and $H$ are finite groups then $E(G, m) \sim E(H, m)$ if and only if $|G| = |H|$. 

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Now we prove that the map \( E(G, \cdot) \) determines \( \varphi_G \).

**Proposition 3.11.** Let \( G, H \) be finite groups. The equality \( E(G, m) = E(H, m) \) holds for every \( m \) if and only if \( \varphi_G = \varphi_H \).

**Proof.** If \( \varphi_G = \varphi_H \) then \( |G| = |H| \) and \( \text{Exp}(G) = \text{Exp}(H) \); thus Corollary 3.4 implies that \( E(G, m) = E(H, m) \) for every \( m \). Now assume that the equality \( E(G, m) = E(H, m) \) holds for every \( m \). Corollary 3.10 implies that \( |G| = |H| \). We have \( \varphi_G(1) = 1 = \varphi_H(1) \). Let \( t' \) be a natural number and assume that \( \varphi_G(t') = \varphi_H(t') \) for every \( t < t' \). We prove that \( \varphi_G(t') = \varphi_H(t') \) by induction that \( \varphi_G(t) = \varphi_H(t) \) for every \( t \in \mathbb{N} \). If \( t' \mid k \) then \( \varphi_G(t') = 0 = \varphi_H(t') \). Now assume that \( t' \not| k \). Since \( E(G, t') = E(H, t') \), \( |G| = |H| \) and \( \varphi_G(t) = \varphi_H(t) \) for every \( t < t' \) it follows from Corollary 3.4 that \( \varphi_G(t') = \varphi_H(t') \).

Finally, we are able to prove Theorem 1.1 stated in the Introduction.

**3.1. Proof of Theorem 1.1.** Proposition 3.11 implies that \( \varphi_G = \varphi_H \). We shall prove that this implies that \( G \cong H \). Let \( p \) be a prime number and let \( G(p) \) denote the set of elements of \( G \) whose order is a power of \( p \). We define \( H(p) \) analogously. Corollary 3.10 implies that \( |G| = |H| = n \). For every prime number \( p \) that divides \( n \), \( G(p) \) and \( H(p) \) are non-trivial subgroups of \( G \) and \( H \), respectively and

\[
G = \bigoplus_{p \mid n} G(p) \quad \text{and} \quad H = \bigoplus_{p \mid n} H(p).
\]

Note that for every prime number \( p \) and every natural number \( s \) we have \( \varphi_G(p^s) = \varphi_H(p^s) \) and \( \varphi_H(p^s) = \varphi_H(p^s) \). Since \( \varphi_G = \varphi_H \) we conclude that \( \varphi_G(p) = \varphi_H(p) \). The result follows if we prove that \( G(p) \cong H(p) \) for every prime number \( p \) that divides \( n \). There exist natural numbers \( t, t', u_1, \ldots, u_t, v_1, \ldots, v_{t'} \) satisfying \( u_1 \geq u_2 \geq \cdots \geq u_t > 0 \) and \( v_1 \geq v_2 \geq \cdots \geq v_{t'} > 0 \) such that

\[
G(p) \cong \mathbb{Z}_{p^{u_1}} \times \cdots \times \mathbb{Z}_{p^{u_t}} \quad \text{and} \quad H(p) \cong \mathbb{Z}_{p^{v_1}} \times \cdots \times \mathbb{Z}_{p^{v_{t'}}}.
\]

Let \( \psi_G(p)(t) \) be the number of elements of \( G(p) \) of order at most \( t \). We define \( \psi_H(p) \) analogously. Since \( \varphi_G(p) = \varphi_H(p) \) we conclude that \( \psi_G(p^s) = \psi_H(p^s) \). In particular \( G(p) \) and \( H(p) \) have the same order. Let \( p^a = |G(p)| \). Let \( s \) be a natural number with \( 1 \leq s \leq u_1 \) and let \( l \) be the greatest integer such that \( u_i \geq s \), we have \( \psi_G(p^s) = p^{a-(u_1+\cdots+u_i)+ls} \). Note that \( u_i \) is the greatest integer that satisfies \( \psi_G(p^s) > \psi_G(p^{s-1}) \). Since \( \psi_G(p^s) = \psi_H(p^s) \) we conclude that \( u_i = v_i \). Let \( r \) and \( r' \) be the greatest integers such that \( u_i = v_i = v_i \); for \( s = u_i-1 \) we have

\[
\psi_G(p^s) = p^{a-rv_i+vrs} \quad \text{and} \quad \psi_H(p^s) = p^{a-r'v_i+v's}.
\]

Since \( \psi_G(p^s) = \psi_H(p^s) \) and \( v_i = u_i \) we conclude that \( r = r' \). We proceed in this way and conclude that \( t = t' \) and \( u_i = v_i \) for \( i = 1, \ldots, t \). This implies that \( G(p) \cong H(p) \). \( \square \)

Let \( p \) be a prime number and let \( \lambda = (u_1, \ldots, u_t) \vdash \alpha \) be a partition of \( \alpha \), here \( u_1 \geq \cdots \geq u_t > 0 \), and let \( n = p^\alpha \). We denote by \( G_{p,\lambda} \) the group \( \mathbb{Z}_{p^{u_1}} \times \cdots \times \mathbb{Z}_{p^{u_t}} \). Let \( \alpha \) be the function given by \( \alpha_x(s) = \alpha \) for \( s > u_1 \) and \( \alpha_x(s) = \alpha - (u_1 + \cdots + u_i) + ls \) for \( s \leq u_i \), where \( l \) is the greatest integer such that \( u_i \geq s \). It follows from the proof of the previous result that \( \varphi_{G_{p,\lambda}}(p^s) = p^\alpha \). Therefore \( \varphi_{G_{p,\lambda}}(p^s) = p^\alpha \lambda(s) - p^\alpha \lambda(s-1) \), for \( s \geq 1 \) and \( \psi_G_{p,\lambda}(1) = 1 \). Now let \( G \) be an abelian group of order \( n \) and let \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) be the decomposition of \( n \) as a product of prime numbers. There exist partitions \( \lambda_i \vdash \alpha_i \), for \( i = 1, \ldots, k \) such that \( G \cong G_{p_1,\lambda_1} \times \cdots \times G_{p_k,\lambda_k} \). It will be convenient to assume that \( \alpha_{\lambda(-1)} = -\infty \) and that \( p^\alpha \lambda(-1) = 0 \). Given \( t \equiv p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) be a divisor of \( n \) have

\[
\varphi_G(t) = \left( p_1^{-\lambda_1(s)} - p_1^{-\lambda_1(s-1)} \right) \cdots \left( p_k^{-\lambda_k(s)} - p_k^{-\lambda_k(s-1)} \right).
\]

Together with Corollary 3.4 this allows us to obtain a formula for \( E(G, m) \) if \( G \) is a finite abelian group.

**4. Counting gradings and the proof of Theorem 1.2.**

Our goal now is to determine the number of gradings by a finite abelian group \( G \) on an algebra of upper block-triangular matrices. We first state the following theorem that gives the classification of division gradings on matrix algebras, over an algebraically closed field, with support a finite abelian group.
Theorem 4.1. [12, Theorem 2.15] Let $T$ be a finite abelian group and let $F$ be an algebraically closed field. There exists a grading on the matrix algebra $M_n(F)$ with support $T$ making $M_n(F)$ a graded division algebra if and only if $\text{char } F$ does not divide $n$ and $T \cong \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_r}, l_1 \cdots l_r = n$. The isomorphism classes of such gradings are in one-to-one correspondence with the non-degenerate alternating bicharacters $\beta : T \times T \to F^\times$.

Definition 4.2. Let $T(G, k) = \{ T \leq G \mid T \cong \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_r}, l_1 \cdots l_r = k \}$. Let $D(T, k)$ denote the number of isomorphism classes of division gradings with support $T$ on $M_k(F)$.

As a consequence of Corollary 2.1 and the previous theorem we have the following result.

Theorem 4.3. Let $G$ be a finite abelian group. Let $\mathbf{m} = (m_1, \ldots, m_s)$ be an $s$-tuple of positive integers and let $F$ be an algebraically closed field. If $s > 1$ we assume that $\text{char } F = 0$ or $\text{char } F > \dim UT(\mathbf{m})$. Given $k \neq 0$ we denote by $\frac{\mathbf{m}}{k}$ the $s$-tuple whose $i$-th entry is $\frac{m_i}{k}$. The number of isomorphism classes of $G$-gradings on $UT(\mathbf{m})$ is

$$\sum_{k \mid m_i, 1 \leq i \leq s} \left( \sum_{T \in T(G, k)} D(T, k)E \left( \frac{G/T}{m/k} \right) \right).$$

Proof. Let $(k, T, \beta, \sigma)$ be a quadruple such that $k \mid m_i$ for $i = 1, \ldots, s$, $T \in T(G, k)$, $\beta : T \times T \to F^\times$ is a non-degenerate alternating bicharacter and $\sigma$ is an orbit of the $G/T$-action on $\gamma(\frac{\mathbf{m}}{k}, G/T)$. Denote by $Q$ the set of such quadruples. Let $q = (k, T, \beta, \sigma) \in Q$. Let $D$ be a division grading on $M_k(F)$ that is a representative of the isomorphism class that corresponds to $\beta : T \times T \to F^\times$ under the one-to-one correspondence in Theorem 4.1. Let $\tilde{B}$ be an elementary grading in the class that corresponds to $\sigma$ under the one-to-one correspondence in Proposition 2.2. Let $q = (g_1T, \ldots, g_sT)$, where $n = m_1 + \cdots + m_s$, be a tuple of elements of $G/T$ that induces the elementary grading on $\tilde{B}$ and let $B$ be the elementary $G$-grading on $UT(\frac{\mathbf{m}}{k})$ induced by $(g_1, \ldots, g_s)$. The algebra $B \otimes D$ is isomorphic to $UT(\mathbf{m})$ via the Kronecker product. Let $A$ be the corresponding grading on $UT(\mathbf{m})$ and denote by $[q]$ the isomorphism class of $A$. Corollary 2.1 implies that $[q]$ is independent of the choices of $D$, $\tilde{B}$ and $g_1, \ldots, g_s$. Hence we obtain a map $q \mapsto [q]$ from $Q$ to the set of isomorphism classes of $G$-gradings on $UT(\mathbf{m})$. Theorem 2.3 implies that this map is surjective. Let $q = (k, T, \beta, \sigma)$ and $q' = (k', T', \beta', \sigma')$ be quadruples in $Q$ such that $q = [q']$. Let $B \cong D$ and $B' \cong D'$ be the algebras obtained from the construction above for $q$ and $q'$, respectively. Then $D$ and $D'$ are the division gradings on $M_k(F)$, $M_k(F)$ associated to $\beta$ and $\beta'$, respectively, and $\tilde{B}, \tilde{B}'$ are the elementary gradings associated to $\sigma, \sigma'$ respectively. The equality $[q] = [q']$ implies that $B \cong D \cong B' \cong D'$. Since $G$ is abelian we have $[\gamma^{-1}D] = D$ and Corollary 2.1 implies that $B \cong D \cong B' \cong D'$ if and only if $D \cong D'$ and $\tilde{B} \cong \tilde{B}'$. Since $D \cong D'$ it follows that $k = k'$, moreover Theorem 4.1 implies that $T = T'$ and $\beta = \beta'$. The isomorphism $\tilde{B} \cong \tilde{B}'$ together with Proposition 2.2 imply that $\sigma = \sigma'$. Hence $q = q'$ and the map is also injective.

We obtain in this way a bijection from $Q$ to the set of isomorphism classes of $G$-gradings on $UT(\mathbf{m})$. Let $k$ be a positive integer such that $k \mid m_i$ for $i = 1, \ldots, s$ and let $T$ be a group in $T(G, k)$. Theorem 4.1 implies that the number of non-degenerate alternating bicharacters on $T$ is $D(T, k)$ and Proposition 2.2 implies that the number of orbits of the $G/T$-action on $\gamma(\frac{\mathbf{m}}{k}, G/T)$ is $E(\frac{\mathbf{m}}{k}, G/T)$. Hence the number of quadruples $(k', T', \beta', \sigma') \in Q$ such that $k' = k$ and $T' = T$ is $D(T, k)E \left( \frac{G/T}{\frac{\mathbf{m}}{k}} \right)$. As a consequence the number of quadruples in $Q$ and hence the number of isomorphism classes of $G$-gradings on $UT(\mathbf{m})$ is $\sum_{k \mid m_i, 1 \leq i \leq s} \sum_{T \in T(G, k)} D(T, k)E \left( \frac{G/T}{m/k} \right)$.

Now we finish the paper by proving Theorem 1.2.

4.1. Proof of Theorem 1.2. Corollary 4.3 implies that

$$N(G, \mathbf{m}) = \sum_{k \mid \mathbf{m}} \sum_{T \in T(G, k)} D(T, k)E \left( \frac{G/T}{m/k} \right).$$

Proposition 3.9 implies that

$$E(G, \mathbf{m}) \leq N(G, \mathbf{m}) \leq E(G, \mathbf{m}) + \sum_{1 \leq k \mid \mathbf{m}} D(T, k)p_{\text{Exp}(G/T)}(m/k).$$
The sum on the right side of the inequalities above is a polynomial of degree strictly smaller than $|G| - 1$. Hence the above inequalities and Proposition 3.9 imply that $N(G, m) \sim \frac{1}{|G|} m^{|G| - 1}$.

□

References

[1] Y. A. Bahturin, S. K. Sehgal, and M. V. Zaicev, Group gradings on associative algebras, J. Algebra 241 (2001), no. 2, 677–698.
[2] Y. A. Bahturin and M. V. Zaicev, Group gradings on matrix algebras, Canad. Math. Bull. 45 (2002), no. 4, 499–508, Dedicated to Robert V. Moody.
[3] Y. A. Bahturin and M. V. Zaicev, Graded algebras and graded identities, Polynomial identities and combinatorial methods (Pantelleria, 2001), Lecture Notes in Pure and Appl. Math., vol. 235, Dekker, New York, 2003, pp. 101–139.
[4] C. Boboc, S. Dascalescu, Good gradings of matrix algebras by finite abelian groups of prime index, Bull. Math. Soc. Sc. Math. Roumanie Tome 49 (97), No 1, 2006, 5–11.
[5] A. R. Borges, C. Fidelis and D. Diniz, Graded isomorphisms on upper block triangular matrix algebras, Linear Algebra and its Applications 543 (2018) 92–105.
[6] A. Valenti, M. Zaicev, Group gradings on upper triangular matrices, Arch. Math. 89 (2007) 33–40.
[7] Di Vincenzo, O. M., Koshlukov, P., Valenti, A., Gradings on the algebra of upper triangular matrices and their graded identities, J. Algebra 275 (2004) 550–566.
[8] M. Kochetov, N. Parsons, S. Sadov, Counting fine grading on matrix algebras and on classical simple Lie algebras, Internat. J. Algebra Comput. 23 (2013), no. 7, 1755–1781.
[9] M. Kochetov, F. Yasumura, Group gradings on the Lie and Jordan algebras of block-triangular matrices, J. Algebra 537 (2019), 147–172.
[10] A. Giambruno, M. Zaicev, Codimension growth and minimal superalgebras, Trans. Amer. Math. Soc. 355 (2003), no. 12, 5091–5117.
[11] A. Giambruno, M. Zaicev, Minimal varieties of algebras of exponential growth, Adv. Math. 174 (2003), no. 2, 310–323.
[12] A. Elduque, M. Kochetov, Gradings on simple Lie algebras, Mathematical Surveys and Monographs, 189. American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013.
[13] F-Y. Yasumura, Group gradings on upper block triangular matrices, Arch. Math. 4 (2018) 327–332.

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