Statefinder diagnostic for coupled quintessence

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Abstract

The problem of the cosmic coincidence is a longstanding puzzle. This conundrum may be solved by introducing a coupling between the two dark sectors. In this Letter, we study two cases of the coupled quintessence scenario. (a) Assume that the mass of dark matter particles depends exponentially on the scalar field associated to dark energy and meanwhile the scalar field evolves in an exponential potential; (b) Assume that the mass of dark matter particles depends on a power law function of the scalar field and meanwhile the scalar field evolves in a power law potential. Since the dynamics of this system is dominated by an attractor solution, the mass of dark matter particles is forced to change with time as to ensure that the ratio between the energy densities of dark matter and dark energy becomes a constant at late times, and one thus solve the cosmic coincidence problem naturally. We perform a statefinder diagnostic to both cases of this coupled quintessence scenario. It is shown that the evolving trajectory of this scenario in the $s - r$ diagram is quite different from those of other dark energy models.
There are more and more evidences [1–3] support that the present universe is dominated by dark sectors. Combined analysis of cosmological observations, esp. the Wilkinson Microwave Anisotropy Probe (WMAP) satellite experiment [2], shows that dark energy (DE) occupies about 73% of the energy of our universe, and dark matter (DM) about 23%. The usual baryon matter which can be described by our known particle theory occupies only about 4% of the total energy of the universe. The accelerated expansion of the present universe is attributed to that DE is an exotic component with negative pressure, such as the cosmological constant [4] or a scalar field with a proper potential (i.e. the so-called quintessence) [5]. The cosmological constant \( \Lambda \) (or vacuum energy) has the equation of state \( w = -1 \). The cosmological model that consists of a mixture of vacuum energy and cold dark matter (CDM) is called LCDM (or \( \Lambda \)CDM). While the so-called QCDM cosmology is based upon a mixture of CDM and quintessence field. The energy density and the negative pressure are provided by the quintessence scalar field \( \phi \) slowly evolving down its potential \( V(\phi) \). The equation of state of the quintessence \(-1 < w < -1/3\) is guaranteed by the slow evolution. However, as is well known, there are two difficulties arise from all of these scenarios, namely, the 'fine-tuning' problem and the 'cosmic coincidence' problem. The cosmic coincidence problem [6] states: Since the energy densities of DE and DM scale so differently during the expansion of the universe, why are they nearly equal today? To get this coincidence, it appears that their ratio must be set to a specific, infinitesimal value in the very early universe.

A possible solution to this cosmic coincidence problem may be provided by introducing a coupling between quintessence DE and CDM. This coupling is often described by the variable-mass particle (VAMP) scenario [7]. The VAMP scenario assumes that the CDM particles interact with the scalar DE field resulting in a time-dependent mass, i.e. the mass of the CDM particles evolves according to some function of the scalar field \( \phi \). In this Letter we study two cases of this coupled quintessence scenario: (a) The quintessence scalar field \( \phi \) evolves in an exponential potential and the DM particle mass also depends exponentially on \( \phi \); (b) The quintessence scalar field \( \phi \) evolves in a power law potential and the DM particle mass also depends on a power law function of \( \phi \). In both cases, the late time behavior of the cosmological equations will give accelerated expansion and, a constant ratio between DM energy density \( \rho_\chi \) and DE energy density \( \rho_\phi \) [8,9]. This behavior relies on the existence of an attractor solution, which makes the effective equation of state of DE mimic the effective equation of state of DM at late times so that the late time cosmology insensitive to the initial conditions for DE and DM. Therefore, the scenario containing coupled quintessence with VAMPs solves the cosmic coincidence problem in this sense.

In this Letter, we will first show the solution to the problem of cosmic coincidence given
by this scenario and, then we perform a statefinder diagnostic for both cases of this coupled quintessence model. The statefinder parameters introduced by Sahni et al. [10] are proven to be useful tools to characterize and differentiate between various DE models. We show in this Letter that the evolving trajectory of this scenario in the $s-r$ diagram is quite different from those of other DE models.

Consider, now, the interacting DE model in which we postulate that the DM component $\chi$ interacts with the DE field $\phi$ through the interaction term $Q$ according to

$$\dot{\rho}_\chi + 3H\rho_\chi = -Q, \quad (1)$$

$$\dot{\rho}_\phi + 3H\rho_\phi(1 + w_\phi) = Q, \quad (2)$$

where $\rho_\chi$ and $\rho_\phi$ are energy densities of DM and DE, respectively, dot denotes a derivative with respect to time $t$, $H = \dot{a}/a$ represents the Hubble parameter, in which $a(t)$ is the scale factor of the universe, and

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \quad (3)$$

is the usual parameter of equation of state for the homogeneous scalar field $\phi$ associated to DE, where $V(\phi)$ is some potential of the quintessence field. For convenience we can define the effective equations of state for DM and DE through the parameters

$$w_{\chi}^{(e)} = \frac{Q}{3H\rho_\chi}, \quad w_\phi^{(e)} = w_\phi - \frac{Q}{3H\rho_\phi}. \quad (4)$$

If the effective equation of state parameters $w_{\chi}^{(e)}$ and $w_\phi^{(e)}$ evolve to be an equal constant at late times, the field system is then proven to have a stable attractor solution. In what follows we will discuss two cases of this coupled quintessence scenario — the exponential case and the power law case.

**Exponential case**

Assume that the DM particle $\chi$ with mass $M$ depending exponentially on the DE field $\phi$,

$$M_\chi(\phi) = M_\ast e^{-\lambda\phi}, \quad (5)$$

where $\phi$ is expressed in units of the reduced Planck mass $M_p$ ($M_p \equiv 1/\sqrt{8\pi G} = 2.436 \times 10^{18}$ GeV), and $\lambda$ is a positive constant. The scalar field has an exponential potential

$$V(\phi) = V_\ast e^{\eta\phi}, \quad (6)$$
where \( \eta \) is a positive constant. As a consequence, the interaction term \( Q \) in this case can be given

\[
Q = \lambda \dot{\phi} \rho_\chi .
\]  
(7)

The equation of motion is then

\[
\frac{1}{3} \left( \frac{\rho_\chi + \rho_b + \rho_{\text{rad}} + V}{1 - \dot{\phi}^2/6} \right) \ddot{\phi} + \frac{1}{2} \left( \rho_\chi + \rho_b + \frac{2}{3} \rho_{\text{rad}} + 2V \right) \dot{\phi}' = \lambda \rho_\chi - \eta V .
\]  
(8)

where \( \rho_b \) and \( \rho_{\text{rad}} \) are energy densities of baryons and radiation, respectively, and we have assumed a spatially flat universe. Primes denote derivatives with respect to \( u = \ln(a/a_0) = -\ln(1 + z) \), in which \( z \) is the red-shift, and \( a_0 \) represents the current scale factor. Since we are interested in the late-time behavior, we can assume \( \rho_b, \rho_{\text{rad}} \ll \rho_\chi, \rho_\phi \). In this limit it is easy to see that there is a solution

\[
\phi = \phi_0 - \frac{3}{\lambda + \eta} u ,
\]  
(9)

such that

\[
\Omega_\phi \simeq 1 - \Omega_\chi = \frac{3 + \lambda (\lambda + \eta)}{(\lambda + \eta)^2} ,
\]  
(10)

and

\[
w^{(e)}_\phi = w^{(e)}_\chi = W = -\frac{\lambda}{\lambda + \eta} .
\]  
(11)

This is an attractor in field space for \( \eta > (-\lambda + \sqrt{\lambda^2 + 12})/2 \). When the attractor is reached, the energy densities of DM and DE will evolve at a constant ratio depending only on \( \lambda \) and \( \eta \), thus solving the cosmic coincidence problem.

It is shown by (11) that \( W \) is negative and may lead, if \( W < -1/3 \), to an accelerated expansion of the universe. To understand how it is possible to get both acceleration and constant ratio between DM and DE one may look at the scaling behavior of the energy densities on the attractor (9),

\[
\rho_\chi \sim e^{-\lambda \phi - 3u} \sim \rho_\phi \sim e^{\eta \phi} \sim e^{-3(1+W)u} .
\]  
(12)

The scaling behavior of DM deviates from the usual scaling way \( e^{-3u} \) due to the \( \phi \)-dependence of the DM mass. The interaction between DM and DE forces their effective equation of state parameters to become an equal negative constant \( W \), and thus solving the coincidence problem and at the same time resulting in an accelerated expansion.
FIG. 1. A typical solution to the exponential case. The evolution of the density parameters for different components and the effective equation of state parameters for DE and DM. The corresponding model parameters are: $\eta = 2$, $\lambda = 3$, $M_* = 0.4\rho_0/n_{\lambda 0}$ and $V_* = 0.1\rho_0$.

The time evolution of the density parameters for different components (including also $\rho_b$ and $\rho_{\text{rad}}$) and the effective equation of state parameters for DE and DM for a typical solution is plotted in Fig.1. Notice that the attractor solution is going to be reached currently in this example.

**Power law case**

In this case we assume that the DM particle $\chi$ with mass $M$ depending on a power law function of the DE field $\phi$,

$$M_\chi(\phi) = M_* \phi^{-\alpha},$$

and the scalar field has a power law potential

$$V(\phi) = V_* \phi^\beta,$$

where $\alpha, \beta > 0$. The interaction term $Q$ in this case is then expressed as

$$Q = \alpha \frac{\dot{\phi}}{\phi} \rho_\chi.$$

The equation of motion can be given

$$\frac{1}{3}\left(\rho_\chi + \rho_b + \rho_{\text{rad}} + V\right)\phi'' + \frac{1}{2}\left(\rho_\chi + \rho_b + \frac{2}{3}\rho_{\text{rad}} + 2V\right)\phi' = \frac{\alpha}{\phi} \rho_\chi - \frac{\beta}{\phi} V,$$
and it can be proven that there is a stable attractor solution in the field space \([9,11]\)
\[
\phi = \phi_0 e^{-\frac{3}{\alpha + \beta} u},
\]
(17)
such that
\[
\Omega_\phi \simeq 1 - \Omega_\chi = \frac{\alpha}{\alpha + \beta},
\]
(18)
and
\[
w_\phi^{(e)} = w_\chi^{(e)} = W = -\frac{\alpha}{\alpha + \beta}.
\]
(19)
When the attractor is reached, the energy densities of DM and DE will evolve at a constant ratio depending only on \(\alpha\) and \(\beta\), thus solving the cosmic coincidence problem. We see that in this case \(W\) is also a negative constant and can thus lead to an accelerated expansion of the universe. The scaling behavior of the energy densities on the attractor (17) is exhibited as
\[
\rho_\chi \sim \phi^{-\alpha} e^{-3u} \sim \rho_\phi \sim \phi^\beta \sim e^{-3(1+W)u}.
\]
(20)

FIG. 2. A typical solution to the power law case. The evolution of the density parameters for different components and the effective equation of state parameters for DE and DM. The corresponding model parameters are: \(\alpha = 11, \beta = 4, M_\ast = 230\rho_0/n_\chi_0\) and \(V_\ast = 0.1\rho_0\).
In Fig. 2 we plot a typical solution, including also $\rho_b$ and $\rho_{\text{rad}}$. Notice that the attractor solution is going to be reached today in this example.

In what follows we will perform a statefinder diagnostic to this coupled quintessence scenario. Since more and more DE models are constructed for interpreting or describing the cosmic acceleration, there exists the problem of discriminating between the various contenders. In order to be able to differentiate between those competing cosmological scenarios involving DE, a sensitive and robust diagnostic for DE models is a must. For this purpose a diagnostic proposal that makes use of parameter pair \( \{ r, s \} \), the so-called ”statefinder”, was introduced by Sahni et al. [10]. The statefinder probes the expansion dynamics of the universe through higher derivatives of the expansion factor $\ddot{a}$ and is a natural companion to the deceleration parameter which depends upon $\dddot{a}$. The statefinder pair \( \{ r, s \} \) is defined as follows

\[
    r \equiv \frac{\ddot{a}}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)}. \tag{21}
\]

The statefinder is a ‘geometrical’ diagnostic in the sense that it depends upon the expansion factor and hence upon the metric describing space-time.

Trajectories in the \( s - r \) plane corresponding to different cosmological models exhibit qualitatively different behaviors. The spatially flat LCDM scenario corresponds to a fixed point in the diagram

\[
    \{ s, r \}_{\text{LCDM}} = \{ 0, 1 \}. \tag{22}
\]

Departure of a given DE model from this fixed point provides a good way of establishing the ‘distance’ of this model from LCDM [12]. As demonstrated in [10,12–14] the statefinder can successfully differentiate between a wide variety of DE models including the cosmological constant, quintessence, the Chaplygin gas, braneworld models and interacting DE models. The interacting DE model analyzed in Ref. [14] cannot solve but only alleviate the cosmic coincidence problem. We in this Letter will perform a diagnostic for the coupled quintessence scenario which can provide a natural solution to the coincidence problem and show explicitly the difference between this scenario and other DE models.

The statefinder parameters can be expressed in terms of the total energy density $\rho$ and the total pressure $p$ in the universe:

\[
    r = 1 + \frac{9(\rho + p)p}{2\rho \dot{\rho}}, \quad s = \frac{(\rho + p)\dot{\rho}}{p \dot{\rho}}. \tag{23}
\]

Since the total energy of the universe is conserved, we have $\dot{\rho} = -3H(\rho + p)$. Then making use of $\dot{\rho}_\phi = -3H(1 + w^{(c)}_\phi)\rho_\phi$ and $\dot{\rho}_{\text{rad}} = -4H\rho_{\text{rad}}$, we can get
\[
\frac{\dot{p}}{H} = p' = [w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\rho_\phi - \frac{4}{3}\rho_{\text{rad}}.
\]  

(24)

Hence, the statefinder parameters for the coupled quintessence scenario can be obtained

\[
\begin{align*}
    r &= 1 - \frac{3}{2}[w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\Omega_\phi + 2\Omega_{\text{rad}}, \\
    s &= -\frac{3[w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\Omega_\phi + 4\Omega_{\text{rad}}}{9w_\phi\Omega_\phi + 3\Omega_{\text{rad}}}.
\end{align*}
\]

(25)

(26)

The deceleration parameter is also given

\[
q = \frac{1}{2}(1 + 3w_\phi\Omega_\phi + \Omega_{\text{rad}}).
\]

(27)

We first apply a statefinder analysis on the exponential case. In Fig. 3, we show the time evolution of the statefinder pair \{r, s\}. The plot is for variable interval \(u \in [-2, 2]\), and the selected evolution trajectories of \(r(s)\) correspond to \(\eta = 2\) and \(\lambda = 3, 2, 1\), respectively, and the other model parameters are taken to be the same values as those used in Fig. 1. We see clearly that the distant from this model to LCDM scenario is somewhat far. It is of interest to find that the trajectory of \(r(s)\) will form swirl before reaches the attractor, which is quite different from other DE models (see [10,12–14]). It is demonstrated again that the statefinder can successfully characterize and differentiate between various DE models. As
complementarity for the diagnostic, we also plot the evolution trajectories of statefinder pair \( \{r, q\} \) in Fig.4. We see that the cosmic acceleration is ensured by sufficient strong coupling \( \lambda \).

**FIG. 4.** The \( q-r \) diagram of the exponential case: evolution trajectories of \( r(q) \) in the variable interval \( u \in [-2, 2] \). Selected curves \( r(q) \) for \( \lambda = 3, 2, \) and 1, respectively. Dots locate the current values of the statefinder pair \( \{q, r\} \).

**FIG. 5.** The \( s-r \) diagram of the power law case: evolution trajectories of \( r(s) \) for the variable interval \( u \in [-2, 2] \). Selected curves \( r(s) \) for \( \alpha = 11, 13, \) and 15, respectively. Dots locate the current values of the statefinder pair \( \{s, r\} \).
Next we apply a statefinder diagnostic to the power law case. In Fig.5, we show the time evolution of the statefinder pair \( \{r, s\} \). The plot is also for variable interval \( u \in [-2, 2] \), and the selected evolution trajectories of \( r(s) \) correspond to \( \beta = 4 \) and \( \alpha = 11, 13 \) and 15, respectively, and the other model parameters are as the same as those used in Fig.2. It can be seen that the trajectories of this case will pass through LCDM fixed point. And the swirls in this case are more evident than those of exponential case. We also plot the evolution trajectories of statefinder pair \( \{r, q\} \) in Fig.6.

![Diagram](image_url)

**FIG. 6.** The \( q - r \) diagram of the power law case: evolution trajectories of \( r(q) \) for the variable interval \( u \in [-2, 2] \). Selected curves \( r(q) \) for \( \alpha = 11, 13, \) and 15, respectively. Dots locate the current values of the statefinder pair \( \{q, r\} \).

In summary, we study in this Letter the statefinder of the coupled quintessence scenario. We analyze two cases of this scenario — the exponential case and the power law case. It is shown that both cases of this scenario have attractor behaviors and can thus provide a natural solution to the cosmic coincidence problem. Then we perform a statefinder diagnostic to both cases of this coupled quintessence scenario. It is shown that the evolving trajectory of this scenario in the \( s - r \) plane is quite different from those of other DE models. We hope that the future high precision observations (e.g. SNAP) will be capable of determining these statefinder parameters and consequently shed light on the nature of DE.

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