PARAMETRIZATIONS OF DEGENERATE DENSITY MATRICES

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Abstract. It turns out that a parametrization of degenerate density matrices requires a parametrization of $\mathcal{F} = U(n)/(U(k_1) \times U(k_2) \times \cdots \times U(k_m))$, $n = k_1 + \cdots + k_m$ where $U(k)$ denotes the set of all unitary $k \times k$-matrices with complex entries. Unfortunately the parametrization of this quotient space is quite involved. Our solution does not rely on Lie algebra methods directly, but succeeds through the construction of suitable sections for natural projections, by using techniques from the theory of homogeneous spaces. We mention the relation to the Lie algebra back ground and conclude with two concrete examples.

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1. Introduction

In various parts of Physics density matrices, i.e., positive trace class operators of trace 1 on a complex separable Hilbert space play an important role, see [5]. Density matrices represent states of quantum systems. In many concrete applications the Hilbert space is typically finite dimensional and the Hilbert space then is $\mathbb{C}^n$, the space of $n$-tuples of complex numbers with its standard inner product. Thus the space $\mathcal{D}_n$ of all density matrices on $\mathbb{C}^n$ is the space of all $n \times n$ matrices $\rho$ with complex entries such that $\langle x, \rho x \rangle \geq 0$ for all $x \in \mathbb{C}^n$ and $\text{Tr}(\rho) = \sum_{j=1}^n \langle e_j, \rho e_j \rangle = 1$ for any orthonormal basis $\{e_j : j = 1, \ldots, n\}$ of $\mathbb{C}^n$. Through these two constraints the entries of a density matrix are not all independent and thus contain redundant parts. But for an effective description of quantum states one would like to get rid of these redundant parts of a density matrix, i.e., one would like to have a description of density matrices in terms of a set of independent parameters, that is a parametrization in the sense of Definition 1.1. The best known parametrization of density matrices seems to be the Bloch vector parametrization [4, 14]. While this parametrization is perfect for $n = 2$-level systems, it has a serious defect for $n \geq 3$-level systems in the sense that the parameter set cannot be determined explicitly (see for instance [9]). Thus various authors have been looking for alternative ways to parametrize density matrices, see for instance [2, 8, 10, 15]. Some time ago we started with a parametrization of density matrices based on their spectral representation [7, 6, 9].

The spectral representation of a density matrix $\rho \in \mathcal{D}_n$ reads

$$\rho = U D_n(\lambda_1, \ldots, \lambda_n) U^*$$

(1.1)
where $D_n(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of the eigenvalues $\lambda_1, \ldots, \lambda_n$ and $U$ is some unitary $n \times n$ matrix, i.e., $U \in U(n)$. These $n$ eigenvalues are not necessarily distinct; they occur in this list as many times as their multiplicity requires.

In this article we consider parametrizations in the strict sense as suggested in [9]. This definition reads:

**Definition 1.1.** A parametrization of density matrices is given by the following:

(a) Specification of a parameter set $Q_n \subset \mathbb{R}^m$ where $m$ depends on $n$, i.e., $m = m(n)$;

(b) Specification of a one-to-one and onto map $F_n : Q_n \longrightarrow D_n$.

When the spectral representation (1.1) is chosen as the starting point one obviously needs a suitable parametrization of unitary matrices.

The set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ can be ordered according to their size: We denote the set of eigenvalues ordered in this way by $\Lambda_n$, i.e.,

$$
\Lambda_n = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) : 0 \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1, \sum_{j=1}^{n} \lambda_j = 1 \right\}.
$$

We begin by addressing the question of uniqueness of the spectral representation (1.1). Accordingly suppose that for $\lambda, \lambda' \in \Lambda_n$ and $U, V \in U(n)$ we have

$$
U^* D_n(\lambda) U = V^* D_n(\lambda') V.
$$

Since the spectrum of a matrix is uniquely determined and since $\lambda, \lambda' \in \Lambda_n$ it follows $\lambda = \lambda'$ and therefore it follows $V U^* D_n(\lambda) = D_n(\lambda) V U^*$, i.e.,

$$
V U^* \in D_n(\lambda)'
$$

where $D_n(\lambda)'$ denotes the commutant of the diagonal matrix $D_n(\lambda)$ in $U(n)$. If a density matrix $\rho \in D_n$ has a non-degenerate spectrum, i.e., if

$$
\lambda \in \Lambda_n^\# = \{ \lambda \in \Lambda_n : 0 \leq \lambda_n < \lambda_{n-1} < \cdots < \lambda_2 < \lambda_1 \}
$$

then this commutant is easily determined and is given by

$$
D_n(\lambda)' = U(1) \times \cdots \times U(1), \quad n \text{ terms}
$$

Naturally there are many ways in which a density matrix can be degenerate. Suppose that the spectrum of $\rho_n \in D_n$ has $m$ different eigenvalues $\lambda_1, \ldots, \lambda_m$ with multiplicities $k_1, \ldots, k_m$ with

$$
\sum_{j=1}^{m} k_j = n.
$$

Thus $\lambda \in \Lambda_n$ is of the form

$$
(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_m, \ldots, \lambda_m)
$$

where each $\lambda_j$ is repeated $k_j$ times and $\sum_{j=1}^{m} k_j \lambda_j = 1$ and where we use the ordering $0 \leq \lambda_m < \lambda_{m-1} < \cdots < \lambda_1$ according to (1.2). Thus one has in this case

$$
D_n(\lambda) = \text{diag}_n(\lambda_1 I_{k_1}, \lambda_2 I_{k_2}, \ldots, \lambda_m I_{k_m})
$$

where $\text{diag}_n$ denotes the $n \times n$ diagonal matrix with entries as indicated and where $I_{k_j}$ denotes the $k_j \times k_j$ identity matrix. Therefore the commutant of the diagonal matrix $D_n(\lambda)$ is in this case

$$
D_n(\lambda)' = U(k_1) \times U(k_2) \times \cdots \times U(k_m).
$$

Thus in order to complete the parametrization problem for degenerate density matrices we need to find a suitable parametrization of

$$
\mathfrak{F} = U(n)/(U(k_1) \times U(k_2) \times \cdots \times U(k_m)) \quad n = k_1 + \cdots + k_m.
$$

We begin with a discussion of the simplest case, i.e., $k_j = 1$ for all $j$ and $m = n$. Note that in this case

$$
U(n)/(U(1) \times \cdots \times U(1)) = U(n)/\sim
$$
for the equivalence relation $\sim$ in $U(n)$ defined by
\[ U \sim U' \iff U^{-1}U' \in U(1) \times \ldots \times U(1). \]
Accordingly the elements of $U(n)/\sim$ are the equivalence classes
\[ [U] = \{UV; V \in U(1) \times \ldots \times U(1)\} \]
$U(n)/\sim = U(n)/(U(1) \times \ldots \times U(1))$ is called a (complex) full flag manifold (see [13]).

Introduce the natural projection
\[ \pi : U(n) \ni U \rightarrow [U] \in U(n)/(U(1) \times \ldots \times U(1)). \]
Then a map $\iota : U(n)/(U(1) \times \ldots \times U(1)) \rightarrow U(n)$ is called a section of $U(n)$ on $U(n)/(U(1) \times \ldots \times U(1))$ for $\pi$. The relation $U' = \iota([U])$ implies that $U'$ is a representative of the coset $[U]$. The mapping
\[ p : \Lambda_n^\neq \times U(n)/(U(1) \times \ldots \times U(1)) \ni ((\lambda_1, \ldots, \lambda_n), m) \rightarrow \iota(m)D_n(\lambda_1, \ldots, \lambda_n)\iota(m)^* \]
does not depend on the section $\iota$, and thus the mapping $p$ gives a parametrization of density matrices, if $U(n)/(U(1) \times \ldots \times U(1))$ is suitably parametrized. Since $U(n)/(U(1) \times \ldots \times U(1))$ is a manifold, it is parametrized locally. But unfortunately, this parametrization is not simple. Even though the mapping does not depend on $\iota$, the construction of a concrete section is necessary, but also not so simple.

Through the construction of a concrete section we will also achieve a parametrization of unitary matrices, an important problem in itself which has found considerable attention in the last 10–15 years (see the references mentioned above). The starting point of this construction is the so called canonical coset decomposition which gives in particular the well-known Jarlskog parametrization [11] [12].

Recall that the coset space $U(n)/(U(n - 1) \times U(1))$ is the projective space $CP^{n-1}$ (see [13]). Symbolically, the canonical coset decomposition is:
\[ U(n) = U(n)/(U(n - 1) \times U(1)) \cdot U(n - 1)/(U(n - 2) \times U(1)) \]
\[ \ldots \cdot U(2)/(U(1) \times U(1)) \cdot (U(1) \times \ldots \times U(1)). \]
In Section 3 we parametrize $U(n)$ by constructing sections $\iota_j : CP^{n-j} \rightarrow U(n - j + 1)$ for $j = 1, \ldots, n - 1$.

For the degenerate case we have only to use (complex) Grassmann manifolds $U(k)/(U(k_1) \times U(k_2))$ instead of the projective spaces $U(k)/(U(k - 1) \times U(1))$:
\[ U(n) = U(n)/(U(n - k_m) \times U(k_m)) \cdot U(n - k_m)/(U(n - k_{m-1} - k_m) \times U(k_{m-1})) \]
\[ \ldots \cdot U(k_1 + k_2)/(U(k_1) \times U(k_2)) \cdot (U(k_1) \times \ldots \times U(k_r)). \]
In Section 2 we study this case extensively because the parametrization of degenerate density matrices is the main new result of this paper (Propositions 2.10, 2.13). The result of Section 3 is the special case of $k_j = 1$ for $1 \leq j \leq n$. In Section 4 we study Jarlskog parametrization used in [12] for the non-degenerate case.

If $S(\mathbb{C}^n)$ denotes the unit sphere in $\mathbb{C}^n$, we can parametrize the subset $\Omega = \{[z]; z \in S(\mathbb{C}^n), z_n \neq 0\} \subseteq CP^{n-1}$ by $B(n - 1) = \{x \in \mathbb{C}^{n-1}; \|x\| < 1\}$. But for the boundary $\partial B(n - 1) = \{x \in \mathbb{C}^{n-1}; \|x\| = 1\}$, the mapping
\[ \partial B(n - 1) \ni x \rightarrow [x] \in CP^{n-1} \]
is not injective and consequently there are $x$ and $x'$ in $\partial B(n - 1)$ such that $W(x) \neq W(x')$ for $W(x)$ of [13,13] and there exist $V$ and $V'$ in $U(n - 1) \times U(1)$ such that
\[ W(x)V = W(x')V'. \]
Consequently, the parametrization of density matrices is not always unique. In Section 5 we present a way to construct a section on a Grassmann manifold by using sections on a suitable projective space, since, for concrete calculations, the construction of a section presented in Section 2 is fairly involved. In Section 7 we give simple concrete examples of degenerate
density matrices. In this paper, we mainly use the technique of homogeneous spaces. But there is the theory of Lie algebra behind it. In Section 4 a Lie algebraic back ground is presented.

2. GRASSMANNIAN AND CANONICAL COSET DECOMPOSITION

The Grassmann manifold $G(k, \mathbb{C}^n)$ is the set of all complex $k$-dimensional subspaces of $\mathbb{C}^n$ (see [13]). Let $W$ be a $k$-dimensional subspace of $\mathbb{C}^n$. Then we choose a basis of column vectors $z_1, \ldots, z_k$ of $W$ and associate with it the matrix

$$M(W) = (z_1 \ z_2 \ \ldots \ z_k).$$

Since the matrix $M(W)$ depends on the choice of a basis of $W$, $M(W)$ is not determined uniquely by $W$. There is a freedom of multiplication by regular $k \times k$ matrices from the right. Thus we have

$$G(k, \mathbb{C}^n) = \{M; M = \text{complex } n \times k \text{ matrix of rank } k\}/GL(k, \mathbb{C}).$$

Let $S_{k,n}$ be the set of permutations $\sigma$ of $\{1, \ldots, n\}$ such that $1 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(n-k) \leq n$ and $1 \leq \sigma(n-k+1) < \sigma(n-k+2) < \cdots < \sigma(n) \leq n$. Let $M(W)_{\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)}$ be the $k \times k$-matrix which consists of the $\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)$-th rows of $M(W)$, and define

$$\Omega_\sigma = \{W \in G(k, \mathbb{C}^n); \det M(W)_{\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)} \neq 0\} \subset G(k, \mathbb{C}^n).$$

Since the rank of the matrix $M(W)$ is $k$, there is a $\sigma \in S_{k,n}$ such that $\det M(W)_{\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)} \neq 0$. Thus we have

$$G(k, \mathbb{C}^n) = \bigcup_{\sigma \in S} \Omega_\sigma.$$  

Let $M(n-k, k)$ be a set of all $(n-k) \times k$ complex matrices. Define the mapping

$$\phi_\sigma : \Omega_\sigma \ni W \mapsto M(W)_{\sigma(1), \ldots, \sigma(n-k)} M(W)^{-1}_{\sigma(n-k+1), \ldots, \sigma(n)} \in M(n-k, k).$$

Then $\phi_\sigma$ gives the homeomorphism

$$\Omega_\sigma \cong M(n-k, k) \cong \mathbb{C}^{(n-k)k}.$$

In fact, the element $W$ of $\Omega_\sigma$ corresponds in a 1-1 way to the matrix of the form

$$M(W)_{\sigma(n-k+1), \ldots, \sigma(n)} = \left( \begin{array}{cc} z_{11} & \cdots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{n-k,1} & \cdots & z_{n-k,k} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{array} \right),$$

where $M(W)_{\sigma}$ is the matrix whose $i$-th row is the $\sigma(i)$-th row of $M(W)$, $Z \in M(n-k, k)$ and $I$ is the $k \times k$ identity matrix. Then the set $\{(\Omega_\sigma, \phi_\sigma); \sigma \in S_{k,n}\}$ gives an atlas of $G(k, \mathbb{C}^n)$.

There is another parametrization of $\Omega_\sigma$ which is more convenient for our purpose. Let

$$B(n-k, k) = \{X \in M(n-k, k); X^*X < I_k\}.$$ 

Then the set $\Omega_\sigma$ can be parametrized by the set $B(n-k, k)$. We will show this in the following.

Let $S(k, \mathbb{C}^n)$ denote the set of all orthonormal frames $F = (x_1, x_2, \cdots, x_k)$ in $\mathbb{C}^n$ of length $k$, i.e., $(x_j, x_i) = \delta_{ij}$ for $i, j = 1, \ldots, k$. For $F \in S(k, \mathbb{C}^n)$ we associate a matrix $M(F)$ by

$$M(F) = (x_1, x_2, \ldots, x_k).$$

Let $M(F)_{\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)}$ ($F \in S(k, \mathbb{C}^n)$) be the $k \times k$-matrix which consists of $\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)$-th rows of $M(F)$, and let

$$\tilde{\Omega}_\sigma = \{F \in S(k, \mathbb{C}^n); \det M(F)_{\sigma(n-k+1), \sigma(n-k+2), \ldots, \sigma(n)} \neq 0\} \subset S(k, \mathbb{C}^n).$$
Since the rank of the matrix $M(F)$ is $k$, there is a $\sigma \in S_{k,n}$ such that $\det M(F)_{\sigma(n-k+1),\sigma(n-k+2),\ldots,\sigma(n)} \neq 0$. Thus we have

$$S(k, \mathbb{C}^n) = \bigcup_{\sigma \in S_{k,n}} \tilde{\Omega}_\sigma.$$  

**Definition 2.1.** For a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ define $U_\sigma \in U(n)$ by $U_\sigma e_j = e_{\sigma(j)}$.

Then $M(F)_{\sigma} = U_{\sigma}^{-1}M(F)$. Define $F_{\sigma}$ by $M(F_{\sigma}) = M(F)_{\sigma}$, and identify $F$ and $M(F)$. Let $\pi_2$ be the surjective mapping

$$\pi_2 : S(k, \mathbb{C}^n) \ni F \mapsto W = \text{span} F \in G(k, \mathbb{C}^n),$$

where $\text{span} F$ is the complex subspace of $\mathbb{C}^n$ spanned by the frame $F$. If $F, F' \in S(k, \mathbb{C}^n)$ define the same subspace, then $F' = FU$ for some $U \in U(k)$. Thus we have

$$G(k, \mathbb{C}^n) \cong S(n, \mathbb{C}^n)/U(k).$$

In order to parametrize $G(k, \mathbb{C}^n)$, we must choose a unique representative $F \in S(k, \mathbb{C}^n)$ from $\pi_2^{-1}(F)$. Note that

$$G(k, \mathbb{C}^n) \ni \Omega_\sigma = \{ W \in G(k, \mathbb{C}^n); \det F(W)_{\sigma(n-k+1),\sigma(n-k+2),\ldots,\sigma(n)} \neq 0 \} = \pi_2(\tilde{\Omega}_\sigma).$$

For $W \in \Omega_\sigma$, we can choose a unique representative from the coset $F(W)U(k)$. In fact, since the submatrix $Y_\sigma = F_{\sigma(n-k+1),\ldots,\sigma(n)}$ is nonsingular, from the uniqueness of the polar decomposition (see [3]) we have

$$Y_\sigma^* = U|Y_\sigma^*|$$

for a unique $U \in U(k)$. Consequently

$$Y_\sigma = |Y_\sigma^*|U^*$$

for a unique $U^* \in U(k)$. So, we select a unique representative $\begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix}$ which corresponds to $W \in \Omega_\sigma$ such that $Y'_\sigma = Y_\sigma U = |Y_\sigma^*|$ is a positive operator and $X'_\sigma = X_\sigma U = F_{\sigma(1),\ldots,\sigma(n-k+1)}U$. Since the column vectors of $\begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix}$ give an orthonormal frame, we have

$$X''_\sigma X'_\sigma + Y'^2_\sigma = (X''_\sigma, Y'_\sigma) \begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix} = I_k.$$

This shows that $X'_\sigma \in B(n-k,k)$ and $Y'_\sigma = (I_k - X''_\sigma X'_\sigma)^{1/2}$. Thus $\Omega_\sigma$ is parametrized by $B(n-k,k)$.

This can be also understood (by showing directly that there is a 1 to 1 correspondence between $B(n-k,k)$ and $M(n-k,k)$). For the matrix $Z$ of (2.3) introduce

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} Z \\ I \end{pmatrix} (Z^*Z + I)^{-1/2}$$

Then we have

$$I - X^*Z = Y^2 > 0,$$

and $0 \leq X^*X < I$ and $Y = (I - X^*X)^{1/2}$. Therefore the mappings

$$\begin{pmatrix} Z \\ I \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix}$$

and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} (I - X^*X)^{-1/2} = \begin{pmatrix} Z \\ I \end{pmatrix}$$

give a 1-1 onto correspondence, and the mappings

$$M(n-k,k) \ni Z \mapsto X = Z(Z^*Z + I)^{-1/2} \in B(n-k,k),$$

$$B(n-k,k) \ni X \mapsto Z = X(I - X^*X)^{-1/2} \in M(n-k,k)$$

give a 1-1 onto correspondence between $M(n-k,k)$ and $B(n-k,k)$.
Let \( \tilde{\psi}_\sigma \) be the mapping

\[
\tilde{\psi}_\sigma : \tilde{\Omega}_\sigma \ni F \to X_\sigma U \in B(n - k, k), \quad F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix}, \quad Y_\sigma U = |Y_\sigma^*|.
\]

Then \( \tilde{\psi}_\sigma \) induces the mapping

\[
\psi_\sigma : \Omega_\sigma \ni \pi_2(F) \to X_\sigma U \in B(n - k, k), \quad F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix}, \quad Y_\sigma U = |Y_\sigma^*|
\]

because \( \pi_2(F) = \pi_2(F') \) implies \( \psi_\sigma(F) = \psi_\sigma(F') \).

**Proposition 2.2.** The mapping \( \kappa_\sigma : B(n - k, k) \to \Omega_\sigma = \pi_2(\tilde{\Omega}_\sigma) \) defined by

\[
\kappa_\sigma : B(n - k, k) \ni X \to U_\sigma \pi_2 \left( \frac{X}{(I_k - X^*X)^{1/2}} \right) \in \Omega_\sigma
\]

satisfies \( \kappa_\sigma \circ \psi_\sigma = \text{id} \) and \( \psi_\sigma \circ \kappa_\sigma = \text{id} \).

**Proof.** This is obvious. \( \square \)

It is easily seen that for any \( F, F' \in S(k, \mathbb{C}^n) \) there exists \( U \in U(n) \) such that \( F = UF' \), that is \( U(n) \) acts transitively on \( S(k, \mathbb{C}^n) \). Let \( x = (e_{n-k+1}, e_{n-k+2}, \ldots, e_n) \in S(k, \mathbb{C}^n) \). Then the isotropy subgroup of \( U(n) \) at \( x \) is \( U(n - k) \times \{I_k\} \).

Let \( F, F' \in S(k, \mathbb{C}^n) \), and suppose \( \pi_2(F) = \pi_2(F') \). Then there exists \( Q \in U(k) \) such that \( F = FQ \). Let \( U \in U(n) \). Then \( UF = UF'Q \), i.e., \( \pi_2(UF) = \pi_2(UF') \). This shows that \( U(n) \) acts on \( G(k, \mathbb{C}^n) = S(k, \mathbb{C}^n)/U(k) \) by

\[
U \pi_2(F) = \pi_2(UF).
\]

Since \( U(n) \) acts on \( S(k, \mathbb{C}^n) \) transitively, \( U(n) \) acts on \( G(k, \mathbb{C}^n) = \pi_2(S(k, \mathbb{C}^n)) \) transitively. This shows that \( G(k, \mathbb{C}^n) \) is a homogeneous space of \( U(n) \) (see [13]). Let \( y \in G(k, \mathbb{C}^n) \) be a \( k \)-dimensional subspace of \( \mathbb{C}^n \) spanned by the vectors \( (e_{n-k+1}, e_{n-k+2}, \ldots, e_n) \). The isotropy subgroup of \( U(n) \) at \( y \) is \( U(n - k) \times U(k) \) and thus we have

\[
U(n)/(U(n - k) \times U(k)) \cong G(k, \mathbb{C}^n).
\]

For

\[
x = (e_{n-k+1}, e_{n-k+2}, \ldots, e_n) = \begin{pmatrix} O \\ I_k \end{pmatrix} \in S(k, \mathbb{C}^n)
\]

and

\[
g = \begin{pmatrix} W & X \\ V & Y \end{pmatrix} \in U(n),
\]

denote by \( F = \begin{pmatrix} X \\ Y \end{pmatrix} \) the last \( k \) columns of \( g \) with a \( k \times k \) matrix \( Y \). Now define the mapping \( \pi_1 : U(n) \to S(k, \mathbb{C}^n) \) by

\[
\pi_1(g) = gx = \begin{pmatrix} W & X \\ V & Y \end{pmatrix} \begin{pmatrix} O \\ I_k \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \in S(k, \mathbb{C}^n).
\]

Suppose that \( Y \) is a regular \( k \times k \) matrix, i.e., \( F \in \tilde{\Omega}_e \) (\( = \tilde{\Omega}_\sigma \) for \( \sigma = e \) the identity permutation). Then there is a unique \( Q \in U(k) \) such that \( YQ = |Y^*| \). Put

\[
\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} XQ \\ |Y^*| \end{pmatrix}.
\]

Then we have

\[
\pi_2 \begin{pmatrix} X \\ Y \end{pmatrix} = \pi_2 \begin{pmatrix} X' \\ Y' \end{pmatrix}.
\]

Let

\[
g' = W(X') = \begin{pmatrix} (I - X'^*X')^{1/2} & X' \\ -X'^* & Y' \end{pmatrix}.
\]
Then \( g' \in U(n) \) by Proposition 6.6 and we have
\[
\pi_1(g') = \begin{pmatrix}
(I - XX^*)^{1/2} & X \\
-X^* & Y^r
\end{pmatrix} x = \begin{pmatrix}
X' \\
Y'^r
\end{pmatrix}.
\]

For (2.11) we have
\[
\pi = \pi_2 \circ \pi_1.
\]

Definition 2.3. A mapping \( \iota : G(k, \mathbb{C}^n) \to U(n) \) is a section of \( U(n) \) on \( G(k, \mathbb{C}^n) \) for \( \pi : U(n) \to G(k, \mathbb{C}^n) \) if it satisfies
\[
\pi(\iota(x)) = x \quad \text{for all } x \in G(k, \mathbb{C}^n).
\]

If \( \iota \) is defined only on a subset \( \Omega \subset G(k, \mathbb{C}^n) \) and satisfies (2.12) there, \( \iota \) is called a local section.

Thus, by (2.10), we have constructed a local section:

Proposition 2.4. Let
\[
W(X) = \begin{pmatrix}
(I_{n-k} - XX^*)^{1/2} & X \\
-X^* & (I_k - X^*X)^{1/2}
\end{pmatrix}
\]
for \( X \in B(n - k, k) \). Then the mapping
\[
\iota_e = W \circ \psi_e : \Omega_e \ni \pi(g) \to g' \in U(n).
\]
gives a local section of \( U(n) \) on \( \Omega_e \) for \( \pi : U(n) \to G(k, \mathbb{C}^n) \).

Proof. In Proposition 6.6 it is shown in detail that the matrix \( W(X) \) in (2.13) is unitary. The proof of the remaining part of the statement is straightforward.

From now on, we use the notation \( \hat{\Omega} \) for \( \hat{\Omega}_e \), and \( \Omega \) for \( \Omega_e \), where \( e \) is the identity permutation.

Proposition 2.5.
\[
\hat{\Omega}_e = U_\sigma \hat{\Omega}, \quad \text{and } \Omega_\sigma = U_\sigma \Omega.
\]

Proof. Let \( F \in \hat{\Omega}_e \). Then \( U_{\sigma^{-1}} F = F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix} \) belongs to \( \hat{\Omega} \) since \( Y_\sigma \) is regular. Let \( F \in \hat{\Omega} \) and \( F' = U_\sigma F \). Then \( F' \in \hat{\Omega}_e \). In fact, \( F_\sigma' = U_{\sigma^{-1}} F' = F \) and \( Y_\sigma' = Y \) is regular. Thus we have \( \hat{\Omega}_e = U_\sigma \hat{\Omega} \), and by (2.7)
\[
\Omega_\sigma = \pi_2(\hat{\Omega}_e) = \pi_2(U_\sigma \hat{\Omega}) = U_\sigma \pi_2(\hat{\Omega}) = U_\sigma \Omega.
\]
Corollary 2.7. The mapping
\[ \iota_\sigma = U_\sigma W \circ \psi_\sigma : \Omega_\sigma \ni x \rightarrow W(\psi_\sigma(x)) \in U(n) \]
gives a local section of \( U(n) \) on \( \Omega_\sigma \subset G(k, \mathbb{C}^n) \) for \( \pi : U(n) \rightarrow G(k, \mathbb{C}^n) \).

Proof. Proposition 2.2 shows that \( \psi_\sigma \) is bijective; by Proposition 2.6 we conclude. \( \square \)

Let \( S_n \) be the set of all permutations of \( \{1, 2, \ldots, n\} \), and define an order on \( S_n \) as follows (lexicographic ordering). Let \( \sigma, \sigma' \in S_n \) then \( \sigma < \sigma' \) if there exists \( s \in \{1, 2, \ldots, n\} \) such that \( \sigma(j) = \sigma'(j) \) \((j = 1, \ldots, s - 1)\) and \( \sigma(s) < \sigma'(s) \). Then \( S_{k,n} \subset S_n \) is a well-ordered set, and \( S_{k,n} = \{\sigma_1 < \ldots < \sigma_m\} \) for \( m = \binom{n}{k} \).

For
\begin{equation}
V_j = \Omega_{\sigma_j} \setminus \cup_{i=1}^{j-1} \Omega_{\sigma_i}
\end{equation}
on one finds that
\[ G(k, \mathbb{C}^n) = \bigcup_{j=1}^{m} V_j, \quad m = \binom{n}{k} \]
is a disjoint union. We can construct a section \( \iota : G(k, \mathbb{C}^n) \rightarrow U(n) \) for \( \pi : U(n) \rightarrow G(k, \mathbb{C}^n) \) as follows.

Definition 2.8. Let \( x \in G(k, \mathbb{C}^n) \). Define the section \( \iota(x) \) by
\[ \iota(x) = \iota_{\sigma_j}(x) \text{ if } x \in V_j. \]

Proposition 2.9. Let \( \iota \) be a section of \( U(n) \) on \( G(k, \mathbb{C}^n) \) for \( \pi \). Then for any \( g \in U(n) \), there is a unique \( h \) such that
\[ g = \iota(\pi(g))h, \quad h \in U(n - k) \times U(k). \]

Proof. Let \( g' = \iota(\pi(g)) \). Since \( \pi(g) = \pi(g') \) and
\[ \pi(g) = \{gh; h \in U(n - k) \times U(k)\}, \]
there exists \( h \in U(n - k) \times U(k) \) such that \( g = gh \). If \( g \) has two expression \( g = gh = g'h' \). Then we have \( h = g'^{-1}g = h' \). \( \square \)

Now we come to the parametrization of unitary matrices by the canonical coset decomposition. Symbolically the canonical coset decomposition is:
\[ U(n) = U(n)/(U(n - k_m) \times U(k_m)) \cdot U(n - k_m)/(U(n - k_{m-1} - k_m) \times U(k_{m-1})) \]
\[ \cdots \cdot U(k_1 + k_2)/(U(k_2) \times U(k_2)) \cdot (U(k_1) \times \cdots \times U(k_m)). \]
The following Proposition shows the precise meaning of the above formula.

Proposition 2.10. For any section \( \iota_j \) of \( U(n - k_{j+1} - \cdots - k_m) \) on \( G(k_j, \mathbb{C}^{n-k_{j+1} - \cdots - k_m}) \) for \( \pi_j : U(n - k_{j+1} - \cdots - k_m) \rightarrow G(k_j, \mathbb{C}^{n-k_{j+1} - \cdots - k_m}) \) there is a unique surjection \( f : U(n) \ni g \rightarrow (z_m, z_{m-1}, \ldots, z_2) \in G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \cdots \times G(k_2, \mathbb{C}^{n-k_{j+1} - \cdots - k_m}) \)
and a unique \( h \in U(k_1) \times \cdots \times U(k_m) \) such that
\[ g = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 & \cdots & 0 \\ I_{k_m} & 0 & \cdots & I_{k_3 + \ldots + k_m} \end{pmatrix} h. \]

Proof. According to Proposition 2.9 there exists a unique \( H_m = (g_m, h_m) \in U(n - k_m) \times U(k_m) \) such that
\[ g = \iota_m(z_m)H_m = \iota_m(z_m)(g_m, h_m) = \iota_m(z_m) \begin{pmatrix} g_m & 0 \\ 0 & h_m \end{pmatrix}, \]
where \( z_m = \pi_m(g) \).
In the same way, we have a unique element $H_{m-1} = (g_{m-1}, h_{m-1}) \in U(n-k_m-k_{m-1}) \times U(k_{m-1})$ such that

$$g_m = \iota_{m-1}(z_{m-1})H_{m-1} = \iota_{m-1}(z_{m-1})(g_{m-1}, h_{m-1}) = \iota_{m-1}(z_{m-1}) \begin{pmatrix} g_{m-1} & 0 \\ 0 & h_{m-1} \end{pmatrix},$$

for $z_{m-1} = \pi_{m-1}(g_m)$, and

$$g = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \begin{pmatrix} g_{m-1} & 0 \\ 0 & h_{m-1} \end{pmatrix} = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \begin{pmatrix} g_{m-1} & 0 \\ 0 & h_{m-1} \end{pmatrix}.$$
Proof. Since \((U(k_1) \times \cdots \times U(k_{j-1}) \times U(k_j)) \subset U(k_1 + \cdots + k_{j-1}) \times U(k_j)\), \([g_j] = [g_j']\) implies \(\pi_j(g_j) = \pi_j(g'_j)\) for \(g_j, g'_j \in U(k_1 + \cdots + k_{j-1} + k_j)\).

It follows from Proposition 2.9 that \(g_j = \iota(\pi(g_j))H\) holds for \(H = (g_{j-1}, h) \in U(k_1 + \cdots + k_{j-1}) \times U(k_j)\). Thus we have \([g_j] = [\iota(\pi(g_j))H] = [\iota(\pi(g_j))g_{j-1}]\).

Let \([g_j] = [g_j']\) then there is a \(g'_{j-1} \in U(k_1 + \cdots + k_{j-1})\) such that \([g'] = [\iota(\pi(g_j))g'_{j-1}]\). Therefore there exists \(v \in (U(k_1) \times \cdots \times U(k_{j-1}) \times U(k_j))\) such that \(\iota(\pi(g_j))g'_{j-1} = \iota(\pi(g_j))g_{j-1}v\), and \(g'_{j-1} = g_{j-1}v\). This shows \([g_{j-1}] = [g'_{j-1}]\). \(\square\)

In the same way as Proposition 2.10 one proves the following result.

**Proposition 2.12.** For any section \(\iota_j\) of \(U(n - k_{j+1} - \cdots - k_m)\) on \(G(k_j, \mathbb{C}^{n - k_{j+1} - \cdots - k_m})\) for \(\pi_j : U(n - k_{j+1} - \cdots - k_m) \to G(k_j, \mathbb{C}^{n - k_{j+1} - \cdots - k_m})\), there is a unique mapping
\[
\phi : U(n)/(U(k_1) \times \cdots \times U(k_m)) \ni [g] \mapsto (z_m, z_{m-1}, \ldots, z_2) \in G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n - k_m}) \times \cdots \times G(k_2, \mathbb{C}^{n - k_3 - \cdots - k_m})
\]

such that
\[
[g] = [g'] \quad \text{for} \quad g' = \psi(\phi([g])).
\]

where
\[
\psi(z_m, z_{m-1}, \ldots, z_2) = \tau_m(z_m)
\]
\[
\begin{pmatrix}
\tau_{m-1}(z_{m-1}) & 0 & \cdots & 0 \\
0 & I_{k_m} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{k_3 + \cdots + k_m}
\end{pmatrix}.
\]

Let \(\pi : U(n) \ni g \to [g] \in U(n)/(U(k_1) \times \cdots \times U(k_m))\). The above proposition shows that the mapping \(\pi \circ \psi\) is surjective. Since \(\iota_j\) is a section of \(U(n - k_{j+1} - \cdots - k_m)\) on \(U(n - k_{j+1} - \cdots - k_m)/(U(n - k_{j+1} - \cdots - k_m) \times U(k_j))\), and \(U(k_1) \times \cdots \times U(k_m) \subset U(n - k_{j+1} - \cdots - k_m) \times U(k_j)\), the mapping \(\pi \circ \psi\) is injective. Consequently the mappings \(\phi\) and \(\pi \circ \psi\) are inverses to each other, and thus we get

**Proposition 2.13.** There is a bijection \(\phi = (\pi \circ \psi)^{-1}\) between the flag manifold \(U(n)/(U(k_1) \times \cdots \times U(k_m))\) and the direct product \(G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n - k_m}) \times \cdots \times G(k_2, \mathbb{C}^{n - k_3 - \cdots - k_m})\) of Grassmann manifolds. The mapping \(\psi \circ \phi\) is a section of \(U(n)\) on \(U(n)/(U(k_1) \times \cdots \times U(k_m))\) with respect to \(\pi\).

3. Projective space

In this section, we summarize the results in Section 2 for \(k_j = 1\). The Grassmann manifold \(G(1, \mathbb{C}^n)\), the set of all complex 1-dimensional subspaces of \(\mathbb{C}^n\), is called the projective space and denoted by \(\mathbb{C}P^{n-1}\).

\[
\mathbb{C}P^{n-1} = G(1, \mathbb{C}^n) = \{z \in \mathbb{C}^n; z \neq 0\}/(\mathbb{C}\setminus\{0\}).
\]

\(S_1, n\) is a set of permutation \(\sigma\) such that \(1 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(n-1) \leq n \) and \(1 \leq \sigma(n) \leq n\).

**Remark 3.1.** Let \(\sigma(n) = j\). Then we have \(\sigma(k) = k\) if \(1 \leq k \leq j - 1\) and \(\sigma(k) = k + 1\) if \(j \leq k \leq n - 1\). Thus \(\sigma \in S_1, n\) is characterized by \(j = \sigma(n) \in \{1, 2, \ldots, n\}\).

Let \(j = \sigma(n)\) and \(z_j = z_{\sigma(n)}\) is the \(j\)-th component of \(z \in \mathbb{C}^n\). Define
\[
\Omega_j = \{z \in \mathbb{C}^n; z_j \neq 0\}/(\mathbb{C}\setminus\{0\}) \subset \mathbb{C}P^{n-1}.
\]

Since \(z \neq 0\), there is a \(j \in \{1, 2, \ldots, n\}\) such that \(z_j \neq 0\). Thus we have
\[
\mathbb{C}P^{n-1} = \cup_{j \in \{1, 2, \ldots, n\}} \Omega_j.
\]

Define the mapping
\[
(3.1) \quad \phi_j = \phi_\sigma : \Omega_j = \Omega_j \ni w \to (z_{\sigma(1)}, \ldots, z_{\sigma(n-1)})/z_{\sigma(n)} = (z_1 \ldots z_{j-1}, z_{j+1}, \ldots, z_n)/z_j \in \mathbb{C}^{n-1},
\]
where \(\sigma(n) = j\). This map \(\phi_j\) gives the homeomorphism
\[
\Omega_j \cong \mathbb{C}^{n-1}.
\]
and the set \( \{ (\Omega_j, \phi_j); j \in \{1, 2, \ldots, n\} \} \) an atlas of \( CP^{n-1} \).

There is another parametrization of \( \Omega_j \) which is more convenient for our purpose. Introduce
\[
B(n - 1) = \{ x \in \mathbb{C}^{(n-1)}; x^* x < 1 \};
\]
the set \( \Omega_j \) can be parametrized by the set \( B(n - 1) \). We will show this in the following.

Note that \( S(1, \mathbb{C}^n) = S(\mathbb{C}^n) = \{ z \in \mathbb{C}^n; \|z\| = 1 \} \). With
\[
\Omega_j = \{ z \in S(\mathbb{C}^n); z_j \neq 0 \} \subset S(\mathbb{C}^n).
\]

one has
\[
S(\mathbb{C}^n) = \bigcup_{j \in \{1, 2, \ldots, n\}} \tilde{\Omega}_j.
\]

**Remark 3.2.** Let \( \sigma \in S_{1,n} \) such that \( \sigma(n) = j \) and denote \( U_\sigma \) by \( U_j \). Then we have \( U_j e_k = e_k \) if \( 1 \leq k \leq j - 1 \), \( U_j e_k = e_{k+1} \) if \( j \leq k \leq n - 1 \) and \( U_j e_n = e_j \).

Let \( \pi_2 \) be the surjective mapping
\[
\pi_2 : S(\mathbb{C}^n) \ni z \to w = \text{span} (z) \subset CP^{n-1},
\]
where \( \text{span} z \) is the complex line spanned by \( z \). If \( z, z' \in S(\mathbb{C}^n) \) define the same line, then \( z' = ze^{i\theta} \) for some \( e^{i\theta} \in U(1) \). Thus we have
\[
(3.2) \quad CP^{n-1} \cong S(\mathbb{C}^n)/U(1).
\]

In order to parametrize \( CP^{n-1} \), we must choose a unique representative \( z \in S(\mathbb{C}^n) \) from (3.2). Note that
\[
\Omega_j = \{ z \in S(\mathbb{C}^n); z_j \neq 0 \}/U(1) = \pi_2(\tilde{\Omega}_j).
\]

Let \( \tilde{\psi}_j \) be the mapping
\[
\tilde{\psi}_j : \tilde{\Omega}_j \ni z \to (z_1 \ldots z_{j-1}, z_{j+1} \ldots, z_n)^T/e^{i\theta} \in B(n - 1), \quad z_j = |z_j| e^{i\theta}.
\]

Then \( \tilde{\psi}_j \) induces the mapping
\[
\psi_j : \Omega_j = \pi_2(\tilde{\Omega}_j) \ni \pi_2(z) \to (z_1 \ldots z_{j-1}, z_{j+1} \ldots, z_n)^T/e^{i\theta} \in B(n - 1), \quad z_j = |z_j| e^{i\theta}
\]
because \( \pi_2(z) = \pi_2(z') \) implies \( \tilde{\psi}_j(z) = \tilde{\psi}_j(z') \).

**Proposition 3.3.** The mapping
\[
\kappa_j : B(n - 1) \ni x \to \pi_2((z_1 \ldots z_{j-1}, (1 - x^* x)^{1/2}, z_j \ldots, z_{n-1}))^T \in \Omega_j.
\]
satisfies \( \psi_j \circ \kappa_j = \text{id} \) and \( \kappa_j \circ \psi_j = \text{id} \).

Let \( x = e_n \). Since \( U(n) \) acts on \( S(\mathbb{C}^n) \) transitively, and the isotropy group of \( x \) is \( U(n - 1) \times \{1\} \), \( U(n) \) acts on \( CP^{n-1} = S(\mathbb{C}^n)/U(1) \) transitively by
\[
U \pi_2(z) = \pi_2(Uz).
\]

Let \( y \in CP^{n-1} \) be a complex line of \( \mathbb{C}^n \) spanned by the vector \( e_n \). The isotropy group of \( y \) is \( U(n - 1) \times U(1) \), and we have
\[
U(n)/(U(n - 1) \times U(1)) \cong CP^{n-1}.
\]

Let
\[
e = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} O \\ 1 \end{pmatrix} \in S(\mathbb{C}^n)
\]
and
\[
g = \begin{pmatrix} W \\ v \\ x \\ y \end{pmatrix} \in U(n),
\]
where \( z = \begin{pmatrix} x \\ y \end{pmatrix} \) is the last column of \( g \) and \( y \in \mathbb{C}^1 \).
Define the mapping $\pi_1 : U(n) \to S(C^n)$ by

$$\pi_1(g) = ge = \begin{pmatrix} W & x \\ v & y \end{pmatrix} \begin{pmatrix} O \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \in S(C^n).$$

Suppose that $y \neq 0$. Then there is a unique $e^{i\theta} \in U(1)$ such that $ye^{i\theta} = |y|$. For

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xe^{i\theta} \\ |y| \end{pmatrix},$$

we have

$$\pi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \pi_2 \begin{pmatrix} x' \\ y' \end{pmatrix}.$$ And with

$$g' = W(x') = \begin{pmatrix} (I_{n-1} - x'x^*)^{1/2} & x' \\ -x'^* & y' \end{pmatrix},$$

we get

$$\pi_1(g') = \begin{pmatrix} (I_{n-1} - x'x^*)^{1/2} & x' \\ -x'^* & y' \end{pmatrix} e = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$ Finally introduce $\pi = \pi_2 \circ \pi_1$ and $\Omega_\sigma = \pi_2(\tilde{\Omega}_\sigma)$. Then we have

$$\pi(g) = \pi(g').$$ Thus we have constructed a local section

$$\iota : \Omega_n \ni \pi(g) \to g' \in U(n)$$ by (3.3).

**Proposition 3.4.** For $x \in B(n-1)$ define

$$W(x) = \begin{pmatrix} (I_{n-1} - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix}.$$ Then the mapping

$$\iota_n = W \circ \psi_n : \Omega_n \to U(n)$$
gives a local section of $U(n)$ on $CP^{n-1}$ for $\pi : U(n) \to CP^{n-1}$.

**Proof.** The matrix $W(x)$ in (3.5) is unitary according to Proposition 6.6 the proof of the remaining part is obvious by the above preparations.

From now on, we use the notation $\tilde{\Omega}$ for $\tilde{\Omega}_n$, and $\Omega$ for $\Omega_n$. Propositions 3.4 are the special cases ($k = 1$) of Propositions 2.4 respectively, and we omit the proofs.

**Proposition 3.5.**

$$\tilde{\Omega}_j = U_j \tilde{\Omega}, \text{ and } \Omega_j = U_j \Omega.$$ Next we extend Proposition 3.4 to $\Omega_j$.

**Proposition 3.6.** The mapping

$$\iota_j = U_j W \circ \psi_j : \Omega_j \ni x \to W(\psi_j(x)) \in U(n)$$
gives a local section of $U(n)$ on $\Omega_j \subset CP^{n-1}$ for $\pi : U(n) \to CP^{n-1}$.
For $j = 1, \ldots, n - 1$ introduce

\[(3.7) \quad V_j = \Omega_j \setminus \cup_{i=j+1}^{n} \Omega_i.\]

Then

\[CP^{n-1} = \bigcup_{j=1}^{n} V_j\]

is a disjoint union. We can construct a section $\iota : CP^{n-1} \to U(n)$ for $\pi : U(n) \to CP^{n-1}$ as follows.

**Definition 3.7.** Let $x \in CP^{n-1}$. Define the section $\iota(x)$ by

\[(3.8) \quad \iota(x) = \iota_j(x) \text{ if } x \in V_j.\]

**Remark 3.8.** Let $\sigma, \sigma' \in S_{1,n}$ be such that $\sigma(n) < \sigma'(n)$. Then $\sigma > \sigma'$ according to the lexicographic ordering. This causes the difference between (3.7) and (2.16).

**Proposition 3.9.** Let $\iota$ be a section of $U(n)$ on $CP^{n-1}$ for $\pi$. Then for any $g \in U(n)$, there is a unique $h \in U(n-1) \times U(1)$ such that

\[g = \iota(\pi(g))h.\]

Now we are well prepared to present the parametrization of unitary matrices by the canonical coset decomposition. Symbolically, the canonical coset decomposition is:

\[U(n) = U(n)/(U(n-1) \times U(1)) \cdot U(n-1)/(U(n-2) \times U(1)) \cdot \ldots \cdot U(2)/(U(1) \times U(1)) \cdot (U(1) \times \ldots \times U(1)).\]

The following proposition shows the precise meaning of the above formula.

**Proposition 3.10.** For any section $\iota_j$ of $U(j)$ on $CP^{j-1}$ for $\pi_j : U(j) \to CP^{j-1}$, there is a unique surjection $f : U(n) \ni g \to (z_n, z_{n-1}, \ldots, z_2) \in CP^{n-1} \times CP^{n-2} \times \cdots \times CP^1$ and a unique $h \in U(1) \times \cdots \times U(1)$ such that

\[g = \iota_n(z_n) \begin{pmatrix} \iota_{n-1}(z_{n-1}) & 0 \\ 0 & I_1 \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{n-2} \end{pmatrix} h.\]

Nondegenerate density matrices are parametrized by

\[\Lambda_n^e \times CP^{n-1} \times CP^{n-2} \times \cdots \times CP^1.\]

Let $\pi$ be the natural map

\[\pi : U(n) \to U(n)/(U(1) \times \cdots \times U(1)).\]

**Corollary 3.11.** For any section $\iota_j$ of $U(j)$ on $CP^{j-1}$ for $\pi_j : U(j) \to CP^{j-1}$, there is a unique bijection

\[\phi : U(n)/(U(1) \times \cdots \times U(1)) \ni \pi(g) \mapsto (z_n, z_{n-1}, \ldots, z_2) \in CP^{n-1} \times CP^{n-2} \times \cdots \times CP^1\]

such that

\[\pi(g) = \pi(g') \text{ for } g' = \psi(\phi(\pi(g))),\]

where

\[\psi(z_n, z_{n-1}, \ldots, z_2) = \iota_n(z_n) \begin{pmatrix} \iota_{n-1}(z_{n-1}) & 0 \\ 0 & I_1 \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{n-2} \end{pmatrix}.\]

$\psi \circ \phi$ is a section of $U(n)$ on $U(n)/(U(1) \times \cdots \times U(1))$ for $\pi$. 
4. Local charts of Grassmannian

For $F \in S(k, \mathbb{C}^n)$ the submatrix $M(F)_{n-k+1,\ldots,n}$ is not necessarily nonsingular, unless $F \in \tilde{O}$. From the uniqueness of the polar decomposition (see $\S 3$) we get
\begin{equation}
M(F)_{n-k+1,\ldots,n} = V|M(F)_{n-k+1,\ldots,n}|
\end{equation}
for a unique partial isometry $V$ with null space $N(V) = N(M(F)^*_{n-k+1,\ldots,n})$ (cp. $(2.8)$). There exists $U \in U(k)$ such that the restriction $U|_{N(V)}$ of $U$ to the orthogonal complement $N(V)^\perp$ of $N(V)$ is $V$. Consequently we have
\[ M(F)_{n-k+1,\ldots,n} = |M(F)^*_{n-k+1,\ldots,n}|U^* \]
for some $U^* \in U(k)$. So, we can select some frame $F' = \left( \begin{array}{c} X' \\ Y' \end{array} \right) \in S(k, \mathbb{C}^n)$ such that $Y' = M(F)_{n-k+1,\ldots,n}U = |M(F)^*_{n-k+1,\ldots,n}|$ is a nonnegative operator and $X' = M(F)_{1,\ldots,n-k}U$. Since the column vectors of $\left( \begin{array}{c} X' \\ Y' \end{array} \right) = \left( \begin{array}{c} XU \\ YU \end{array} \right)$ give an orthonormal frame, we have
\[ X'^*X + Y'^2 = (X'^*, Y') \left( \begin{array}{c} X' \\ Y' \end{array} \right) = I_k. \]
This shows that $X' \in \tilde{B}(n-k, k)$ and $Y' = (I_k - X'^*X')^{1/2}$ where we use the notation
\[ \tilde{B}(n-k, k) = \{ X \in M(n-k, k); X^*X \leq I_k \}. \]

Remark 4.1. We selected a frame $F'$ among the coset $FU(k)$ under the condition that $Y'$ is nonnegative. But this condition cannot determine a unique frame $F'$, because there are many elements $U \in U(k)$ such that the restriction $U|_{N(V)}$ of $U$ to the orthogonal complement $N(V)^\perp$ of $N(V)$ is $V$.

Definition 4.2. Define the mapping $\tilde{\kappa}_e : \tilde{B}(n-k, k) \rightarrow G(k, \mathbb{C}^n)$ by
\[ \tilde{\kappa}_e : \tilde{B}(n-k, k) \ni X \rightarrow \pi_2 \left( \begin{array}{c} X \\ (I_k - X^*X)^{1/2} \end{array} \right) \in G(k, \mathbb{C}^n). \]

Remark 4.3. The mapping
\[ \kappa_e : B(n-k, k) \ni X \rightarrow \pi_2 \left( \begin{array}{c} X \\ (I_k - X^*X)^{1/2} \end{array} \right) \in \Omega. \]
of Proposition $(2.2)$ is bijective. But the mapping $\tilde{\kappa}_e$ of the above definition is not bijective but only surjective.

For $X \in \tilde{B}(n-k, k)$ denote
\[ W(X) = \left( \begin{array}{c} (I - XX^*)^{1/2} \\ -X^* \\ X \\ Y \end{array} \right), \quad Y = (I - X^*X)^{1/2}. \]

Proposition 4.4. For any $g \in U(n)$ we can find $X \in \tilde{B}(n-k, k)$, $V \in U(k)$ and $h \in U(n-k) \times I_k$ such that
\[ g = \left( \begin{array}{c} (I - XX^*)^{1/2} \\ -X^* \\ X \\ Y \end{array} \right) \left( \begin{array}{cc} I_{n-k} & O \\ O & V \end{array} \right) h \]
\[ = \left( \begin{array}{c} (I - XX^*)^{1/2} \\ -X^* \\ X \\ Y \end{array} \right) \left( \begin{array}{cc} U & O \\ O & V \end{array} \right), \]
with $U \in U(n-k)$ and $\tilde{B}(m, k) = \{ X \in M(m, k); X^*X \leq I_k \}$. 


Remark 4.5. The above proposition is the counterpart of Proposition 2.10. Note that $V \in U(k)$ is not unique, and consequently, $X$ and $U$ are also not unique. If we use $B(n - k, k)$ instead of $\bar{B}(n - k, k)$, then we have the uniqueness. The above proposition only holds if $g$ satisfies $\pi(g) \in \Omega$. The question of uniqueness is addressed in the following proposition.

Proposition 4.6. Let $g, g' \in U(n - k_1 - \cdots - k_j)$ and $W(X_j), W(X'_j)$ for $X_j, X'_j \in \bar{B}(n - k_1 - \cdots - k_j, k_j)$ such that $W(X_j)g = W(X'_j)g'$. Then $X_j = X'_j$ and $g = g'$.

Proof. Let

$$\pi_1 : U(n - k_1 - \cdots - k_j) \to \mathbb{C}^{n-k_1-\cdots-k_j}$$

be the projection defined by (2.9). Then $(X_j, Y_j) = \pi(W(X_j)g) = \pi(W(X'_j)g') = (X'_j, Y'_j)$, and $X_j = X'_j$, $g = W(X_j)^{-1}W(X'_j)g' = g'$.

It follows from Proposition 4.14 that for any $U_1 \in U(n - k_1)$, there is $W(X_2)$ for $X_2 \in \bar{B}(n - k_1 - k_2, k_2)$ such that

$$U_1 = W(X_2) \begin{pmatrix} U_2 & O \\ O & V_2 \end{pmatrix} = \begin{pmatrix} (I - X_2X_2^*)^{1/2}U_2 & X_2V_2 \\ -X_2^*U_2 & Y_2V_2 \end{pmatrix},$$

where $U_2 \in U(n - k_1 - k_2)$ and $V_2 \in U(k_2)$. Then (5) and (6) imply, for $g \in U(n)$,

$$g = W(X_1) \begin{pmatrix} (I - X_2X_2^*)^{1/2}U_2 & X_2V_2 \\ -X_2^*U_2 & Y_2 \end{pmatrix} \begin{pmatrix} O \\ \circ \end{pmatrix} = W(X_1) \begin{pmatrix} (I - X_2X_2^*)^{1/2}U_2 & X_2 \\ -X_2^* & Y_2 \end{pmatrix} \begin{pmatrix} U_2 \\ \circ \end{pmatrix} \begin{pmatrix} O \\ V_1 \end{pmatrix}.$$ 

Iteration of this procedure gives

Proposition 4.7. (1) For any $g \in U(n)$ there exists $(X_m, \ldots, X_2) \in \bar{B}(n - k_m, k_m) \times \cdots \times \bar{B}(n - k_3 - \cdots - k_m, k_2)$ and $(V_1, \ldots, V_m) \in U(k_1) \times \cdots \times U(k_m)$ such that

$$g = W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3+\cdots+k_m} \end{pmatrix} \begin{pmatrix} V_1 \\ \circ \end{pmatrix}.$$ 

(2) The mapping

$$\bar{B}(n - k_m, k_m) \times \cdots \times \bar{B}(n - k_3 - \cdots - k_m, k_2) \ni (X_m, \ldots, X_2) \to W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3+\cdots+k_m} \end{pmatrix} \in U(n)$$

is injective.

Proof. The existence of $X_j$ and $V_j$ follows from Proposition 4.14 and the injectivity of the mapping follows from Proposition 4.6. \qed

Remark 4.8. The above proposition is the counterpart of Proposition 2.10 and the following proposition is the counterpart of Proposition 3.11.

Proposition 4.9. Let $x_j \in \bar{B}(j - 1) = \{ x \in \mathbb{C}^{j-1}; \| x \| \leq 1 \}$ and

$$W(x_j) = \begin{pmatrix} (I_j - x_jx_j^*)^{1/2}x_j \\ -x_j^* \\ y_j \end{pmatrix}, \quad y_j = (1 - x_j^*x_j)^{1/2}.$$
(1) For any \( g \in U(n) \) there exists \((x_n, x_{n-1}, \ldots, x_2) \in \tilde{B}(n-1) \times \tilde{B}(n-2) \times \cdots \times \tilde{B}(1) \) and 
\((V_1, \ldots, V_n) \in U(1) \times \cdots \times U(1)\) such that

\[
g = W(x_n) \begin{pmatrix} W(x_{n-1}) & O \\ O & I_1 \end{pmatrix} \cdots \begin{pmatrix} W(x_2) & O \\ O & I_{n-2} \end{pmatrix} \begin{pmatrix} V_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & V_n \end{pmatrix}.
\]

(2) The mapping

\[
\tilde{B}(n-1) \times \cdots \times \tilde{B}(1) \ni (x_n, \ldots, x_2) \rightarrow W(x_n) \begin{pmatrix} W(x_{n-1}) & O \\ O & I_1 \end{pmatrix} \cdots \begin{pmatrix} W(x_2) & O \\ O & I_{n-2} \end{pmatrix} \in U(n)
\]
is injective.

Instead of \( \tilde{B}(j-1) \) in \([7, 9]\) the following parameter space \( \bar{Q}_j \)

\[
\bar{Q}_j = \{(\theta_j, \zeta_j); 0 \leq \theta \leq \pi/2, \zeta_j \in S(\mathbb{C}^{j-1})\}, \quad S(\mathbb{C}^{j-1}) = \{\zeta \in \mathbb{C}^{j-1}; \|\zeta\| = 1\}
\]
is introduced for the Jarlskog parametrization \([11], [12]\). The mapping

\[
\bar{Q}_j \ni (\theta_j, \zeta_j) \rightarrow \sin \theta_j \zeta_j = x \in \tilde{B}(j-1) = \{x \in \mathbb{C}^{j-1}; \|x\| \leq 1\}
\]
shows that both parameter spaces are the same.

The formula in \([11]\) which correspond to the formula (4.3) is

\[
U_n = A_{n,n}A_{n,n-1} \cdots A_{n,2}D(e^{ia_1}, \ldots, e^{ia_n}),
\]

where, using bra and ket notation of Physics,

\[
A_{n,j} = \begin{pmatrix} V_j(\theta_j, \zeta_j) & 0 \\ 0 & I_{n-j} \end{pmatrix}, \quad V_j(\theta_j, \zeta_j) = \begin{pmatrix} I_{j-1} - (1 - \cos \theta_j)|\zeta_j\rangle \langle \zeta_j| & \sin \theta_j|\zeta_j\rangle \\ -\sin \theta_j\langle \zeta_j| & \cos \theta_j \end{pmatrix}.
\]

For \( x = \sin \theta_j \zeta_j \), \( V_j(\theta_j, \zeta_j) \) and \( W(x) \) of (4.2) are precisely the same.

In \([11]\) there is a statement that a typical coset representative in the coset space \( U(2)/(U(1) \times U(1)) \) is

\[
V = \begin{pmatrix} \cos \alpha & e^{i\phi} \sin \alpha \\ -e^{-i\phi} \sin \alpha & \cos \alpha \end{pmatrix}.
\]

\( V \) is \( V_1(\theta_1, \zeta_1) \) for \((\theta_1, \zeta_1) = (\alpha, \phi) \in \bar{Q}_1\). But if \( \cos \alpha = 0 \) and \( e^{i\phi} \neq 1 \),

\[
\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}.
\]

This is an example showing that the expression of Proposition 4.14 is not unique. We find

\[
\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix},
\]

which shows that the density matrix \( \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \) has two parametrization, i.e., \((\lambda, \mu), 1)\) and \((\lambda, \mu), e^{i\alpha}\).

However, if we use the parameter space

\[
Q_j = \{(\theta_j, \zeta_j); 0 \leq \theta < \pi/2, \zeta_j \in S(\mathbb{C}^{j-1})\}, \quad S(\mathbb{C}^{j-1}) = \{\zeta \in \mathbb{C}^{j-1}; \|\zeta\| = 1\}
\]
instead of $Q_j$ of (1.4), then the uniqueness is recovered. But not every $g$ can not be expressed by such parameters (see Remark 4.6). Let $g = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}$. Then $\pi(g) = \pi_2 \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix}$ does not belong to $\Omega = \Omega_2$ but belongs to $\Omega_1 = \pi_2(\tilde{\Omega}_1)$, where $\tilde{\Omega}_j = \{z = (z_1, z_2) \in S(\mathbb{C}^2); z_j \neq 0\}$. The unitary matrix $U_1$ in Remark 3.2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the mapping $\kappa_1$ which gives bijection between $B(1)$ and $\Omega_1$ is $\kappa_1 = U_1 \kappa$:

$$\kappa_1 : B(1) \ni 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pi_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pi_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_2 \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix},$$

and

$$\lambda_1 = U_1 \lambda : B(1) \ni 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \iota_1(\pi(g)).$$

Thus $g$ has the unique form

$$g = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} = \iota_1(\pi(g))h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

It follows from Remarks 4.3 and 4.6 that Proposition 4.14 for the parameter space $B(n-k, k)$ is valid only if $\pi(g) \in \Omega$. As shown in the following, $G(k, \mathbb{C}^n)\setminus \Omega = \partial \Omega$. So, in most cases $\partial \Omega_\sigma$ is negligible.

**Definition 4.10.** Let $(\Omega_\sigma, \phi_\sigma)_{\sigma \in S}$ be the system of (at most) countable coordinate neighborhoods of a $m$-dimensional manifold $M$. A subset $A \subset M$ is said to have the measure zero if for every coordinate neighborhood $(\Omega_\sigma, \phi_\sigma)$ the set $\phi_\sigma(A \cap \Omega_\sigma)$ has Lebesque measure zero in $\mathbb{R}^m$.

**Proposition 4.11.** Let $(\Omega_\sigma, \phi_\sigma)_{\sigma \in S}$ be the atlas of the Grassmann manifold $G(k, \mathbb{C}^n)$. Then $\Omega_\sigma \cap \Omega_\sigma'$ is an open and dense subset of $\Omega_\sigma$.

**Proof.** Assume $\sigma \neq \sigma'$. Let

$$M_\sigma M_{\sigma(n-k+1), \ldots, \sigma(n)}^{-1} = \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{n-k+1} & \cdots & z_{n-k,k} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = N.$$

Define a function $f(Z)$ of $Z \in M(n-k, k)$ by

$$f(Z) = \det N_{\sigma^{-1}\iota_\sigma(n-k+1), \sigma^{-1}\iota_\sigma(n-k+2), \ldots, \sigma^{-1}\iota_\sigma(n)}.$$

Then $f(Z)$ is a non-constant polynomial (provided $\sigma \neq \sigma'$), and

$$\Omega_\sigma \supset \Omega_\sigma \cap \Omega_\sigma' = \{Z \in M(n-k, k); f(Z) \neq 0\},$$

$$\Omega_\sigma \setminus \Omega_\sigma' = \{Z \in M(n-k, k); f(Z) = 0\}.$$

Since $f(Z)$ is a polynomial of $Z$, $f(Z)$ is continuous and $\Omega_\sigma \cap \Omega_\sigma'$ is open subset of $\Omega_\sigma$. For any neighborhood $V$ of $Z_0 \in \Omega_\sigma \setminus \Omega_\sigma'$ there exists $Z_1 \in V$ such that $Z_1 \in \Omega_\sigma \cap \Omega_\sigma'$. Otherwise, there exists a neighborhood $V$ of $Z_0$ such that $f(Z) = 0$ on $V$. Since $f(Z)$ is an analytic function, we have $f(Z) \equiv 0$. This is a contradiction. □

**Corollary 4.12.** $\Omega_\sigma$ is an open and dense subset of $G(k, \mathbb{C}^n)$, and therefore $G(k, \mathbb{C}^n)\setminus \Omega_\sigma = \partial \Omega_\sigma$ and $\partial \Omega_\sigma$ is a set of measure zero.

**Proof.** This follows from Relation (2.1). □
Note that \((U(n), G(k, \mathbb{C}^n), \pi, U(n-k) \times U(k))\) has the structure of a fiber bundle, where \(U(n)\), \(G(k, \mathbb{C}^n)\), and \(U(n-k) \times U(k)\) are the total space, the base space, and the fiber respectively, and \(\pi : U(n) \to G(k, \mathbb{C}^n)\) is a continuous surjection satisfying a local triviality condition: For every \(z \in G(k, \mathbb{C}^n)\), there is an open neighborhood \(\Omega_\sigma\) of \(z\) (which will be called a trivializing neighborhood) such that there is a homeomorphism 
\[
\phi : \Omega_\sigma \times (U(n-k) \times U(k)) \ni (z, h) \mapsto \phi(z, h) = \iota_\sigma(z)h \in \pi^{-1}(\Omega_\sigma).
\]

**Proposition 4.13.** \(\pi^{-1}(\partial\Omega_e)\) is a set of measure zero.

**Proof.** Since \(\partial\Omega_e \cap \Omega_e\) is the boundary of \(\Omega_e \cap \Omega_\sigma\) in \(\Omega_e\), \(\partial\Omega_e \cap \Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\partial\Omega_e \cap \Omega_\sigma)\) is the boundary of \(\Omega_e \cap \Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\Omega_e \cap \Omega_\sigma)\) in \(\Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\Omega_\sigma)\). Thus \(\pi^{-1}(\partial\Omega_e)\) is the boundary of \(\pi^{-1}(\Omega_e)\) in \(U(n) = \pi^{-1}(G(k, \mathbb{C}^n))\) and a set of measure zero. \(\Box\)

**Proposition 4.14.** For almost all \(g \in U(n)\), there is a unique \(X \in B(n-k, k)\) and \(h \in U(n-k) \times U(k)\) such that 
\[
g = W(X)h.
\]

**Proof.** The proposition follows from the fact that \(\pi^{-1}(\partial\Omega_e) \cup \pi^{-1}(\Omega_e) = U(n)\) and the previous proposition. \(\Box\)

**Proposition 4.15.** For almost all \(g \in U(n)\) we have a mapping \(U(n) \ni g \mapsto (X_3, X_2) \in B(n-k_3, k_3) \times B(n-k_3 - k_2, k_2)\) and a unique \(h \in U(k_1) \times U(k_1) \times U(k_3)\) such that
\[
g = W(X_3) \begin{pmatrix} W(X_2) & O \\ O & I_{k_3} \end{pmatrix} h.
\]

**Proof.** Let \(m = n - k_3\) and consider the two fiber bundles \(F = (U(n), G(k_3, \mathbb{C}^n), \pi, U(n-k_3) \times U(k_3))\) and \(F' = (U(m), G(k_2, \mathbb{C}^m), \pi', U(m-k_2) \times U(k_2))\). Let \(\Omega_{e}'\) be a trivializing neighborhood of \(F'\). Then \(\partial\Omega_{e}' \times (U(m-k_2) \times U(k_2)) \cong \pi^{-1}(\partial\Omega_{e}')\) is the boundary of \(\Omega_{e}' \times (U(m-k_2) \times U(k_2)) \cong \pi^{-1}(\Omega_{e}')\) in \(U(m)\) and \(\Omega_{e} \times \pi^{-1}(\partial\Omega_{e}') \times U(k_3) \cong \phi(\Omega_{e}, \pi^{-1}(\partial\Omega_{e}') \times U(k_3))\) is the boundary of \(\Omega_{e} \times \pi^{-1}(\Omega_{e}') \times U(k_3) \cong \phi(\Omega_{e}, \pi^{-1}(\Omega_{e}') \times U(k_3))\) in \(\Omega_{e} \times U(m) \times U(k_3) \cong \pi^{-1}(\Omega_{e})\). Thus \(\phi(\Omega_{e}, \pi^{-1}(\Omega_{e}') \times U(k_3))\) is a set of measure zero. Since 
\[
\phi(\Omega_{e}, \pi^{-1}(\Omega_{e}') \times U(k_3)) \cup \phi(\Omega_{e}, \pi^{-1}(\partial\Omega_{e}') \times U(k_3)) = \pi^{-1}(\Omega_{e}),
\]
almost all \(g \in U(n)\) are expressed as \(g = \phi(\Omega_{e}, \pi^{-1}(\Omega_{e}') \times U(k_3))\), which is just (4.5). \(\Box\)

Continuation of the above chain of arguments implies the following theorem.

**Theorem 4.16.** For almost all \(g \in U(n)\) we have a mapping \(U(n) \ni g \mapsto (X_m, \ldots, X_2) \in B(n - k_m, k_m) \times \ldots \times B(n - k_3 - \ldots - k_2, k_2)\) and a unique \(h \in U(k_1) \times \ldots \times U(k_m)\) such that
\[
g = W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \ldots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3 + \ldots + k_m} \end{pmatrix} h.
\]

5. **Section on Grassmannian**

The mapping 
\[
W(X) = \begin{pmatrix} (I_{n-k} - XX^*)^{1/2} & X \\ -X^* & (I_k - X^*X)^{1/2} \end{pmatrix}
\]
of (2.13) for \(X \in B(n-k, k)\) which gives a local section \(\iota_e = W \circ \psi_e : \Omega_e \to U(n)\) is not really suitable for concrete calculations. So here we construct a local section using simpler ones 
\[
W(x) = \begin{pmatrix} (I_{n-k} - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix}
\]
of (5.3) for \(x \in B(n-1)\) which gives a local section \(\iota_n = W \circ \psi_n : \Omega_n \to U(n)\).

We begin with the embedding of \(C P^{m-1}\) in \(C P^{m-1} (m < n)\). \(S(\mathbb{C}^n)\) can be embedded in \(S(\mathbb{C}^n)\) by 
\[
\iota_{n,m} : S(\mathbb{C}^n) \ni (z_1, \ldots, z_m)^T \mapsto (z_1, \ldots, z_m, 0, \ldots, 0)^T \in S(\mathbb{C}^n).
\]
This mapping $\iota_{n,m}$ induces the embedding $\iota_{n,m} : CP^{m-1} \to CP^{m-1}$.

Let $\sigma_{j,n} \in S_{1,n}$ such that $\sigma_{j,n}(n) = j$. Then $\tilde{\Omega}_m = \tilde{\Omega}_{m,m} = \tilde{\Omega}_{\sigma_{m,m}}$ (resp. $\Omega_m = \Omega_{m,m} = \Omega_{\sigma_{m,m}}$) is identified with $\tilde{\Omega}_{m,n} = \tilde{\Omega}_{\sigma_{m,n}}$ (resp. $\Omega_{m,n} = \Omega_{\sigma_{m,n}}$). Let $\tilde{\psi}_{m,n}$ be the mapping

$$\tilde{\psi}_{m,n} : \tilde{\Omega}_{m,n} \ni z = (z_1, \ldots, z_{m-1}, 0, \ldots, 0, z_m)^T \to (z_1, \ldots, z_{m-1}, 0, \ldots, 0)^T/e^{i\theta} \in B(n-1), \ z_m = |z_m|e^{i\theta}.$$ 

Then $\tilde{\psi}_{m,n}$ induces the mapping

$$\psi_{m,n} : \Omega_{m,n} \ni \pi_2(z) \to (z_1, \ldots, z_{m-1}, 0, \ldots, 0)^T/e^{i\theta} \in B(n-1), \ z_m = |z_m|e^{i\theta},$$

where $\pi_2 : S(C^n) \to CP^{m-1}$ is the canonical projection.

**Proposition 5.1.** Let

$$S_k(C^n) \ni F = \begin{pmatrix} X \\ Y \end{pmatrix}, \ X \in M(n-k, k), \ Y \in M(k, k)$$

be given with $\det Y \neq 0$. Then there exists a unique $U \in U(k)$ such that $Y' = YU = T$ where $T$ is a lower triangular matrix:

$$T = \begin{pmatrix} t_{11} & 0 & \ldots & 0 \\ t_{21} & t_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \ldots & * & t_{kk} \end{pmatrix}, \ t_{jj} > 0, \ 1 \leq j \leq k.$$

**Proof.** Let $C^{\ast k}$ be the set of all complex row $k$ vectors $z = (z_1, \ldots, z_k)$ with an inner product $(z, z') = \sum_{j=1}^{k} z_j \bar{z}_j'$, and $z^* = (\bar{z}_1, \ldots, \bar{z}_k)^T$. Let $y_j$ be the $j$-th row vector of $Y$ and $\{u_1, \ldots, u_k\}$ be the Schmidt's orthogonalization of $\{y_1, \ldots, y_k\}$, i.e., $u_1 = y_1/\|y_1\|, \ u_j = x_j/\|x_j\|, \ x_j = y_j - \sum_{i=1}^{j-1} (y_j, u_i) u_i$. Define $U \in U(k)$ by

$$U = (u_1^*, \ldots, u_k^*).$$

Since span $\{y_1, \ldots, y_j\} = \text{span} \{u_1, \ldots, u_j\}, \ (y_j, u_i) = 0$ if $i < j$, and $(y_j, u_j) > 0$. Thus we have $YU = T$. For the uniqueness of $U$, suppose $Y = TU^* = T'U'^*$ for

$$U^* = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}, \ U'^* = \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_k' \end{pmatrix}, \ T' = \begin{pmatrix} t'_{11} & 0 & \ldots & 0 \\ t'_{21} & t'_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \ldots & * & t'_{kk} \end{pmatrix}.$$ 

Then we have $y_1 = t_{11} u_1 = t'_{11} u'_1$ and therefore $t_{11} = t'_{11}$ and $u_1 = u'_1$. From $y_2 = t_{21} u_1 + t_{22} u_2 = t'_{21} u_1 + t'_{22} u_2'$, we have $t_{21} = (y_2, u_1) = t'_{21}$ and $y_2 - t_{21} u_1 = t_{22} u_2 = t'_{22} u_2'$. Thus we have $t_{22} = t'_{22}$ and $u_2 = u_2'$. Continuing these procedures, we get $U^* = U'^*$, i.e., the uniqueness of $U$. \hfill $\square$

**Proposition 5.2.** Let $g_n \in U(n)$ be given with

$$g_n = \begin{pmatrix} W & X \\ V & T \end{pmatrix},$$

where $W \in M(n-k, n-k), \ X \in M(n-k, k), \ V \in M(k, n-k)$ and $T \in M(k, k)$ is a lower triangular matrix with positive diagonal elements as $[5.1]$. Then there exist $x \in C^{n-1}$ and $g_{n-1} \in U(n-1)$ such that

$$g_n = W(x) \cdot (g_{n-1} \times I_1),$$

where

$$W(x) = \begin{pmatrix} (I_{n-1} - xx^*)^{1/2} & x \\ -x^* & (1 - x^* x)^{1/2} \end{pmatrix},$$

$$g_{n-1} \times I_1 = \begin{pmatrix} \lambda_1 \cdots \lambda_{n-1} \\ 0 \cdots 0 \end{pmatrix}.$$
and \( g_{n-1} \) has the form
\[
(5.4) \quad g_{n-1} = \begin{pmatrix} W' & X' \\ V' & T' \end{pmatrix},
\]
where \( W' \in M(n-k, n-k), X' \in M(n-k, k-1), V' \in M(k-1, n-k) \) and \( T' \in M(k-1, k-1) \) is a lower triangular matrix with positive diagonal elements of the form
\[
(5.5) \quad T = \begin{pmatrix} T' & 0 \\ * & I_{kk} \end{pmatrix}.
\]

Proof. We only have to show that \( T' \in M(k-1, k-1) \) is of the form of (5.5). Let \((x, t)^T(x \in \mathbb{C}^{n-1}, t \in \mathbb{C})\) be the last column of the matrix \( g_n \).

Since \( T \) is lower triangular, \( x = (x_1, \ldots, x_{n-k}, 0, \ldots, 0)^T \), \((I_{n-1} - xx^*)^{1/2} = (I_{n-k} - x'x'^*)^{1/2} \times I_{k-1} \), where \( x' = (x_1, \ldots, x_{n-k})^T \). Since \( W(x)^{-1}g_n = g_{n-1} \times I_1 \),
\[
T' \times I_1 = \begin{pmatrix} O_1 & I_{k-1} & O_2 \\ x^* & (1 - x^*x)^{1/2} & X \end{pmatrix} T,
\]
where \( O_1 \) (resp. \( O_2, O_3 \)) is the \((k-1) \times (n-k)\) (resp. \((k-1) \times 1, 1 \times (k-1)\)) zero matrix. Therefore
\[
(T' \ O_4) = (I_{k-1} \ O_2) T,
\]
where \( O_4 \) is the \((k-1) \times 1\) zero matrix and \( T' \) is the \((k-1) \times (k-1)\)-submatrix of \( T \). \( \Box \)

Proposition 5.3. Let \( \pi : U(n) \to G(k, \mathbb{C}^n) \) be the canonical projection. Then there is a unique surjection
\[
f : U(n) \supset \pi^{-1}(\Omega_e) \ni g \to (z_k, z_{k-1}, \ldots, z_1)
\]
\[
\in \Omega_{m,n} \times \Omega_{m,n-1} \times \ldots \times \Omega_{m,n-k+1} \subset CP^{n-1} \times CP^{n-2} \times \ldots \times CP^{n-k}
\]
for \( m = n - k + 1 \) and \( \Omega_e \subset G(k, \mathbb{C}^n) \), and a unique \( h \in U(n-k) \times U(k) \) such that
\[
(5.6) \quad g = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \ldots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix} h.
\]

Proof. Since \( g \in \pi^{-1}(\Omega_e) \), \( g \) has the form of (5.2) with \( \det Y \neq 0 \) and there exists \( U \in U(k) \) such that \( T = YU \) has the form of (5.1). Let
\[
g_n = g \begin{pmatrix} I_{n-k} & 0 \\ 0 & U \end{pmatrix}.
\]

Then it follows from Proposition 5.2 that there exists \( g_{n-1} \in U(n-1) \) which satisfies (5.3) and has the form of (5.4) where \( T' \) again satisfies the hypothesis of Proposition 5.2. Iterating this argument, we get
\[
g_n = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \ldots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix} h,
\]
where \( h = g_{n-k} \times I_k \), \( g_{n-k} \in U(n-k) \). This shows (5.6) with \( h = g_{n-k} \times U \). The relation \( f(g') = (z_k, z_{k-1}, \ldots, z_1) \) for \((z_k, z_{k-1}, \ldots, z_1) \in \Omega_{m,n} \times \Omega_{m,n-1} \times \ldots \times \Omega_{m,n-k+1} \) and
\[
g' = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \ldots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix}
\]
show the surjectivity of \( f \). \( \Box \)

Corollary 5.4. Let \( \pi : U(n) \to G(k, \mathbb{C}^n) \) be the canonical projection, and \( i_j \) the section of \( U(j) \) on \( CP^{j-1} \) defined by (3.3). Then there is a unique bijection
\[
\phi_e : G(k, \mathbb{C}^n) \ni \Omega_e \ni \pi(\phi_e) \to (z_k, z_{k-1}, \ldots, z_1) \in \Omega_{m,n} \times \Omega_{m,n-1} \times \ldots \times \Omega_{m,n-k+1} \subset CP^{n-1} \times CP^{n-2} \times \ldots \times CP^{n-k}
\]
such that
\[
\pi(g) = \pi(g') \text{ for } g' = \psi(\phi_e(\pi(g)));
\]
where
\[
\psi(z_k, z_{k-1}, \ldots, z_1) = \iota_n(z_k) \begin{pmatrix}
t_{n-1}(z_{k-1}) & 0 & 0 \\
0 & I_1 & 0 \\
0 & 0 & I_{k-1}
\end{pmatrix}.
\]
\[
\psi \circ \phi_e \text{ is a section of } U(n) \text{ on } \Omega_e \subset G(n, \mathbb{C}^n) \text{ for } \pi.
\]

6. Lie Algebraic Background

In the articles which we have mentioned many statements are based on the use of the Lie algebra \(u(n)\) of Lie group \(U(n)\). We comment here on the connection with the approach presented above.

The Lie algebra \(u(n)\) of the Lie group \(U(n)\) is defined by
\[
u(n) = \{X \in M(n, n); \forall t \in \mathbb{R}, \exp tX \in U(n)\}.
\]
From the relation
\[
\exp tX^* = (\exp tX)^* = (\exp tX)^{-1} = \exp -tX
\]
it follows
\[
u(n) = \{X \in M(n, n); X^* = -X\}.
\]
Let \(n = k_1 + k_2\). Then the Lie algebra of the Lie group \(U(k_1) \times U(k_2)\) is \(u(k_1) \oplus u(k_2)\), namely, the set of the elements of the form
\[
\begin{pmatrix}X_1 & 0 \\ 0 & X_2\end{pmatrix}, \quad X_j \in u(k_j).
\]
Let \(p\) be a subset of \(u(n)\) such that
\[
u(n) = u(k_1) \oplus u(k_2) \oplus p.
\]
Then \(p\) consists of the elements of the form
\[
K = K(B) = \begin{pmatrix} O_1 & B \\ -B^* & O_2 \end{pmatrix},
\]
where \(O_j (j = 1, 2)\) is the \(k_j \times k_j\) matrix whose entries are all zero and \(B\) is an \(k_1 \times k_2\) complex matrix. Since the space
\[
u(n)/(u(k_1) \oplus u(k_2)) \cong p = \{K(B); B \in M(k_1, k_2)\}
\]
is considered to be the tangent space of the homogeneous space \(U(n)/(U(k_1) \times U(k_2))\) at \(o = \pi(e)\), where \(e\) is the identity of \(U(n)\) and \(\pi : U(n) \to U(n)/(U(k_1) \times U(k_2))\) of (2.11), we study \(\exp K(B)\). First, we have
\[
K^2 = \begin{pmatrix} -BB^* & O \\ O^* & -B^*B \end{pmatrix},
\]
where \(BB^*\) is an \(k_1 \times k_1\)-matrix, \(B^*B\) an \(k_2 \times k_2\)-matrix and \(O\) is the \(k_1 \times k_2\)-matrix whose entries are all zero. Observe now
\[
K^{2n+2} = \begin{pmatrix} -BB^* & O \\ O^* & -B^*B \end{pmatrix}^{n+1} = \begin{pmatrix} -\sqrt{BB^*}^{2n+2} & O \\ O^* & -\sqrt{B^*B}^{2n+2} \end{pmatrix}.
\]
\[
\text{(2.11)} \quad I_n + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} K^{2n+2} = \begin{pmatrix} \cos \sqrt{BB^*} & O \\ O^* & \cos \sqrt{B^*B} \end{pmatrix}
\]
and similarly
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n} K = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( (-BB^*)^n O \begin{pmatrix} O_1 & B \\ O^* & (-B^*B)^n \end{pmatrix} \left( O_1 \begin{pmatrix} O_1 & B \\ O^* & (-B^*B^2)^n \end{pmatrix} \right) \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( (-B^*B)^n (-B^*) \begin{pmatrix} O_1 & (-BB^*)^n B \\ O^* & O_2 \end{pmatrix} \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( (-B^*B)^n (-B^*) \begin{pmatrix} O_1 & B(-B^*B^2)^n \\ O^* & O_2 \end{pmatrix} \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( (-B^*B)^n (-B^*) \begin{pmatrix} O_1 & \sqrt{B^*B}^{-1} (-1)^n \sqrt{B^*B}^{-1} \end{pmatrix} \right)
\]
\[
= \left( \sqrt{B^*B}^{-1} \sin \sqrt{B^*B} (-B^*) \begin{pmatrix} O_1 & \sqrt{B^*B} \sin \sqrt{B^*B} \\ O^* & \cos \sqrt{B^*B} \end{pmatrix} \right).
\]

Thus we conclude
\[
e^K = I_n + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} K^{2n+2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n+1} = \begin{pmatrix} \cos \sqrt{B^*B} & B \sin \sqrt{B^*B} \\ -\sin \sqrt{B^*B} B^* & \cos \sqrt{B^*B} \end{pmatrix}.
\]

**Remark 6.1.** Since
\[
(-BB^*)^{n+1} = (-1)^{n+1} B(B^*B)^n B^*;
\]
\[
\cos \sqrt{BB^*} = I_{k_1} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-BB^*)^{n+1}
\]
\[
= I_{k_1} + B \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (B^*B)^n B^*
\]
\[
= I_{k_1} + B(\sqrt{B^*B})^{-2} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (\sqrt{B^*B})^{2n+2} B^*
\]
\[
= I_{k_1} + B(\sqrt{B^*B})^{-2} (\cos \sqrt{B^*B} - I_{k_2}) B^*.
\]

**Remark 6.2.** Since
\[
(-B^*B)^{n+1} = (-1)^{n+1} B^*(BB^*)^n B;
\]
\[
\cos \sqrt{B^*B} = I_{k_2} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-B^*B)^{n+1}
\]
\[
= I_{k_2} + B^* \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (BB^*)^n B
\]
\[
= I_{k_2} + B^*(\sqrt{BB^*})^{-2} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (\sqrt{BB^*})^{2n+2} B
\]
\[
= I_{k_2} + B^*(\sqrt{BB^*})^{-2} (\cos \sqrt{BB^*} - I_{k_2}) B.
\]

**Remark 6.3.** $\cos \sqrt{B^*B}$, $\sqrt{B^*B}^{-1} \sin \sqrt{B^*B}$ and $(\sqrt{B^*B})^{-2} (\cos \sqrt{B^*B} - I_{k_2})$ are entire functions of $B^*B$. 
Remark 6.4.
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n} K = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-BB^*)^n B \\ (-B^*)^n & O_2 \end{pmatrix}
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-BB^*)^n B \\ (-B^*)^n & O_2 \end{pmatrix}
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-1)^n \sqrt{BB^*}^2 B \\ (-B^*)^n & O_2 \end{pmatrix}
\]
\[
= \begin{pmatrix} O_1 & \sqrt{B^* B}^{-1} \sin \sqrt{B^* B} \\ (-B^*) & \sqrt{B^* B}^{-1} \sin \sqrt{B^* B} \end{pmatrix}.
\]

Let \( X = B \sin \sqrt{B^* B} \) and \( Y = \cos \sqrt{B^* B} \). Then
\[
X^* X = \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} B^* B \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} = \sin^2 \sqrt{B^* B},
\]
and
\[
X^* X + Y^2 = \sin^2 \sqrt{B^* B} + \cos^2 \sqrt{B^* B} = I, \ Y = (I - X^* X)^{1/2}.
\]

In the same way, we have
\[
XX^* = B \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} B^*
\]
\[
= B \frac{\sin^2 \sqrt{B^* B}}{B^* B} B^* = \frac{\sin^2 \sqrt{B^* B}}{BB^*} BB^* = \sin^2 \sqrt{BB^*},
\]
where we used the fact that for an entire function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \),
\[
B f(B^* B)B^* = \sum_{n=0}^{\infty} a_n B(B^* B)^n B^* = \sum_{n=0}^{\infty} a_n (BB^*)^n BB^* = f(BB^*) BB^*,
\]
and \( \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} \) is an entire function of \( B^* B \). This shows
\[
\cos \sqrt{BB^*} = (I - XX^*)^{1/2}.
\]

Since \( K(B) \in \mathfrak{u}(n) \), \( \exp K(B) \in \mathfrak{u}(n) \) and
\[(6.1) \quad \exp K(B) = \begin{pmatrix} (I_{k_1} - XX^*)^{1/2} & X \\ -X^* & (I_{k_2} - X^* X)^{1/2} \end{pmatrix} = W(X) \in \mathfrak{u}(n).
\]

Without knowing such background, we can show directly the unitarity of the matrix \( (6.1) \).

**Proposition 6.5.** For \( X \in M(k_1, k_2) \), \( X^* X \leq I_{k_2} \iff XX^* \leq I_{k_1} \).

**Proof.** Here is the elementary proof.
\[
X^* X \leq I_{k_2} \iff \forall e \in C^k(\|e\| = 1) \Rightarrow (e, X^* X e) = 1 \Rightarrow (e, I_{k_2} e) = 1 \leq 1
\]
\[
\iff \forall e \in C^k(\|e\| = 1) \Rightarrow (X e, X e) \leq 1 \iff \forall e \in C^k(\|e\| = 1) \Rightarrow \|X e\|_1 \leq 1
\]
\[
\iff \forall e \in C^k(\|e\| = 1) \Rightarrow \|d\| = 1 \Rightarrow (d, X e) \leq 1
\]
\[
\iff \forall e \in C^k(\|e\| = 1) \Rightarrow \|d\| = 1 \Rightarrow (X^* d, e) \leq 1
\]
\[
\iff \forall d \in C^k(\|d\| = 1) \Rightarrow \|X^* d\|_2 \leq 1 \iff \forall d \in C^k(\|d\| = 1) \Rightarrow (X^* d, X^* d) \leq 1
\]
\[
\iff \forall d \in C^k(\|d\| = 1) \Rightarrow (d, X X^* d) \leq 1 \Rightarrow XX^* \leq I_{k_1}.
\]
\[\square\]
Proposition 6.6. Let \( X \in \tilde{B}(k_1, k_2) = \{ X \in M(k_1, k_2); X^*X \leq I_{k_2} \} \). Then \((I_{k_2} - X^*X)^{1/2}\) and \((I_{k_1} - XX^*)^{1/2}\) are well defined, and \(W(X)\) of (6.1) which appeared in Proposition 2.4 is unitary.

Proof.

\[
\begin{pmatrix}
(I_{k_1} - XX^*)^{1/2} & -X \\
X^* & (I_{k_2} - X^*X)^{1/2}
\end{pmatrix}
\begin{pmatrix}
(I_{k_1} - XX^*)^{1/2} & X \\
X^* & (I_{k_2} - X^*X)^{1/2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
I_{k_1} - XX^* + XX^* \\
X^*(I_{k_1} - XX^*)^{1/2} - (I_{k_2} - X^*X)^{1/2}X^* + I_{k_2} - X^*X
\end{pmatrix}
\begin{pmatrix}
(I_{k_1} - XX^*)^{1/2}X - X(I_{k_2} - X^*X)^{1/2}X \\
X^*X + I_{k_2} - X^*X
\end{pmatrix}.
\]

Since \((1 + x)\alpha = 1/2\) is expanded as \(1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n\) for \(-1 < x < 1\),

\[
(I_{k_1} - XX^*)^{1/2}X = (I_{k_1} + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-XX^*)^n)X
\]

\[
= X + \sum_{n=1}^{\infty} \binom{\alpha}{n} X(-X^*X)^n = X(I_{k_1} - X^*X)^{1/2}
\]

and

\[
X^*(I_{k_1} - XX^*)^{1/2} = X^*(I_{k_2} + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-XX^*)^n)
\]

\[
= X^* + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^nX^* = (I_{k_2} - X^*X)^{1/2}X^*
\]

hold for \(X \in B(k_1, k_2) = \{ X \in M(k_1, k_2); X^*X < I_{k_2} \}\). In order to show the above two formulae for \(X \in B(k_1, k_2)\), we employ a limiting process \(B(k_1, k_2) \ni X_n \to X \in B(k_1, k_2)\) as \(n \to \infty\) with respect to a norm \(\|X\|^2 = \text{Tr}X^*X = \sum_{i=1}^{k_1} \sum_{j=1}^{k_1} |x_{ij}|^2\). Thus we have \(W(X)^*W(X) = I_n\) for all \(X \in B(k_1, k_2)\).

\(\Box\)

Remark 6.7. The following relation is useful. Here \(\alpha = 1/2\).

\[
(I_{k_1} - XX^*)^{1/2} = I_{k_1} - X \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^{n-1}X^* = I_{k_1} + X(X^*X)^{-1/2} \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^n(X^*X)^{-1/2}X^* = I_{k_1} + X(X^*X)^{-1/2}[(1 - X^*X)^{1/2} - 1](X^*X)^{-1/2}X^*.
\]

Remark 6.8. Since a Lie algebra describes only the local properties of its Lie group, the mapping \(B \to \pi(\exp K(B))\) gives a local homeomorphism, that is, there is a neighborhood \(V\) of 0 in \(M(k_1, k_2)\) and \(U\) of \(e\) in \(U(n)\) such that \(V \ni B \to \pi(\exp K(B)) \in \pi(U)\) is homeomorphic (see [13]). But for \(W(X)\), Proposition 2.2 says that the mapping \(\kappa_\sigma : B(k_1, k_2) \ni X \to \pi(W(X) \in \Omega_\sigma \subset \pi(U(n)))\) is bijective and Proposition 4.4 says the mapping \(\bar{\kappa} : B(k_1, k_2) \ni B \to \pi(W(X) \in \pi(U(n)))\) is surjective.
7. Examples

Here we give two examples of the parametrization of degenerate density matrices with diagonal matrices of eigenvalues of the forms:

1) $D_4(\lambda) = \text{diag}_4 (\lambda_1 I_3, \lambda_2 I_1)$.
2) $D_4(\lambda) = \text{diag}_4 (\lambda_1 I_2, \lambda_2 I_2)$.

For the first case, the density matrices are parametrized by $\Lambda_2^x \times G(1, \mathbb{C}^4) = \Lambda_2^x \times CP^3$ (see [217]).

Since $\Omega_4 \subset CP^3$ is an open dense subset of $CP^3$ and $\Omega_4$ is parametrized by $B(3) = \{ z \in \mathbb{C}^3 ; |z| < 1 \}$, almost all density matrices are parametrized by $\Lambda \times B(3)$.

Concretely, we have the following parametrization:

$$\Lambda_2^x \times B(3) \ni ((\lambda_1, \lambda_2), x) \rightarrow \rho = \begin{pmatrix} (I_3 - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1 I_3 \\ 0 \end{pmatrix} \begin{pmatrix} (I_3 - xx^*)^{1/2} \\ -x^* \end{pmatrix} (1 - x^*x)^{1/2}.$$ 

$(I_3 - xx^*)^{1/2}$ can be calculated as:

$$(I_3 - xx^*)^{1/2} = I_3 + (x^*x)^{-1}[(1 - x^*x)^{1/2} - 1]xx^*$$

(see Remark [6.7]).

For the second case, the density matrices are parametrized by $\Lambda_2^x \times G(2, \mathbb{C}^4)$.

Since $\Omega_e \subset G(2, \mathbb{C}^4)$ is an open dense subset of $G(2, \mathbb{C}^4)$ and $\Omega_e$ is parametrized by $B(2, 2) = \{ X \in M(2, 2); X^*X < I_2 \}$, almost all density matrices are parametrized by $\Lambda \times B(2, 2)$.

Concretely, we have the following parametrization:

$$\Lambda_2^x \times B(2, 2) \ni ((\lambda_1, \lambda_2), x) \rightarrow \rho = \begin{pmatrix} (I_2 - xx^*)^{1/2} & x \\ -x^* & (I_2 - x^*x)^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1 I_2 \\ 0 \end{pmatrix} \begin{pmatrix} (I_2 - xx^*)^{1/2} \\ -x^* \end{pmatrix} (I_2 - x^*x)^{1/2}.$$ 

But unfortunately, $(I_2 - xx^*)^{1/2}$ and $(I_2 - x^*x)^{1/2}$ are not easy to calculate. So, we employ Corollary [5.4] which states that there is a bijection $\phi_e : \Omega_e \rightarrow \Omega_{2,4} \times \Omega_{2,3} \subset CP^3 \times CP^2$, and $\psi \circ \phi_e$ is a local section of $U(4)$ on $\Omega_e$, where

$$\psi(z_2, z_1) = \iota_4(z_2) \begin{pmatrix} \iota_3(z_1) \\ 0 \\ I_1 \end{pmatrix},$$

and $\iota_j$ is the section defined by [5.8]. Concretely, we have the following parametrization:

$$\Lambda_2^x \times B(2)^2 \ni ((\lambda_1, \lambda_2, x_2, x_1) \rightarrow \rho = U(x_2, x_1) \text{diag}_4(\lambda_1 I_2, \lambda_2 I_2) U(x_2, x_1)^*,$$

$$U(x_2, x_1) = \begin{pmatrix} (I_2 - x_2 x_1^*)^{1/2} & 0 & x_2 \\ 0 & 1 & 0 \\ -x_2^* & 0 & (1 - x_2 x_1^*)^{1/2} \end{pmatrix} \begin{pmatrix} (I_2 - x_1 x_1^*)^{1/2} & x_1 & 0 \\ -x_1^* & 0 & (1 - x_1 x_1^*)^{1/2} \\ 0 & 0 & 1 \end{pmatrix}.$$ 

8. Conclusion

The problem of parametrizing degenerate density matrices required to develop a new approach using techniques from the theory of homogeneous spaces as outlined in sections 1-5. This approach is not based on the use of Lie algebra methods. Actually our approach helps to detect some short comings of the Lie algebra approach as used the the given references, i.e., the exponential map from Lie algebra to Lie group is not one to one and onto, and to correct these, also in the case of non-degenerate density matrices. These short comings are due to the non-injectivity of the given map at the boundary of the respective parameter domain.
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