Notes on discrete torsion in orientifolds

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In this short note we discuss discrete torsion in orientifolds. In particular, we apply the physical understanding of discrete torsion worked out several years ago, as group actions on $B$ fields, to the case of orientifolds, and recover some old results of Braun and Stefanski concerning group cohomology and twisted equivariant K theory. We also derive new results including phase factors for nonorientable worldsheets and analogues for $C$ fields.

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1 Introduction

Orientifolds – orbifolds in which some of the group elements are combined with the worldsheet-orientation-reversing operation – have recently been of interest in the physics community, see e.g. [1, 2].

One can deform both orbifolds and orientifolds through twisted sector phase factors known as “discrete torsion.” In this note will we study discrete torsion in orientifolds. Specifically, we will extend our previous results on discrete torsion in orbifolds [3, 4, 5, 6, 7] to orientifolds, reproducing results of [2, 8] on the counting of degrees of freedom by elements of $H^2(G, U(1))$ with a nontrivial action on the coefficients, as well as finding new results, such as phase factors for nonorientable worldsheets, and formal analogues for M theory $C$ fields. (See also [9, 10, 11] for other examples of work on discrete torsion in orientifolds; however, the phase factors discussed in [9, 10] seem to be somewhat more restrictive than those discussed here.)

We assume implicitly throughout this paper that the $B$ field is characterized formally as a connection on a 1-gerbe, that its topological characteristic class lives in $H^3(\mathbb{Z})$. Although that statement is true for bosonic strings, it has very recently been argued [2, 12] that this statement is slightly incorrect in type II strings. Nevertheless, the overall counting of discrete-torsion-type degrees of freedom for orientifolds of type II strings announced in [2] matches our results. As our methods are in any event completely appropriate for bosonic strings, and appear to give correct results more generally, we hope that this paper will be of interest.

We begin in section 2 by briefly reviewing the existing derivation of discrete torsion and related phase factors from group actions on $B$ fields. Mathematically, these phase factors just reflect a mathematical ambiguity in defining group actions on $B$ fields (technically, a non-uniqueness in the choice of equivariant structure, when such exists). We review how the counting by $H^2(G, U(1))$ arises, derive phase factors associated to one- and two-loop twisted sector diagrams, and also derive how this leads to a projectivization of group actions on D-branes, as well as the analogues for other (“momentum-winding shift”) degrees of freedom which also arise from group actions on $B$ fields. In section 3 we extend these considerations to $B$ fields in orientifolds, by first discussing group actions and deriving from them the counting by $H^2(G, U(1))$ (but with a nontrivial action on the coefficients, distinguishing this group from that arising in orbifolds). In section 4 we derive projectivizations of group actions on D-branes in orientifolds, and also apply some tricks to give one derivation of the Klein bottle phase factor. In section 5 we give a first-principles derivation of phase factors for the Klein bottle and real projective plane, verifying the predictions of the previous section. In section 6 we formally extend these considerations to $C$ fields (modelled as objects classified topologically by $H^3(\mathbb{Z})$, or equivalently as connections on abelian 2-gerbes). After reviewing the $C$ field analogue of discrete torsion for orbifolds (i.e. a set of degrees of freedom counted
by $H^3(G, U(1))$, and phase factors for a cube), we derive a set of degrees of freedom counted by $H^3(G, U(1))$ (with a nontrivial action on the coefficients), as well as phase factors for a nonorientable analogue of a cube. Finally, in appendix A we briefly review some pertinent results on group cohomology.

2 Review

Let us briefly review previous results on discrete torsion in [3, 4, 5, 6, 7]. Briefly, it was argued in those works that discrete torsion could be understood at the level of supergravity, solely in terms of group actions on $B$ fields. In particular, at the time, there was much confusion on this point – because of works such as [13], and the fact that no one had found a purely mathematical description counted precisely by $H^2(G, U(1))$ in all cases, there was much speculation that discrete torsion was something inherent to conformal field theory, some inherently stringy phenomenon requiring new mathematics to understand.

The computations in [3, 4, 5, 6, 7] argued, by contrast, that discrete torsion is not specific to conformal field theory and does not require any new mathematics, but can be understood very simply as a consequence of defining group actions on $B$ fields, i.e. discrete torsion is a consequence of some straightforward standard mathematics applied to $B$ fields. This was done by showing how the degrees of freedom counted precisely by $H^2(G, U(1))$ arise in all cases, and by deriving Vafa’s phase factors [14] and Douglas’s projectivization on D-branes [15, 16, 17], as well as by extending to $C$ fields and other generalizations.

Specifically, it was argued that discrete torsion is the $B$ field analogue of “orbifold Wilson lines,” an ambiguity in defining group actions on gauge fields. Consider a principal $U(1)$ bundle $P$ with connection $A$ over some manifold $M$, on which a finite group $G$ acts effectively. It is a standard result that the action of $G$ on $M$ need not lift automatically to the bundle with connection. When it does, $P$ is said to be equivariantizable, and a particular choice of lift to the bundle $P$ with connection is known as an equivariant structure. Such equivariant structures are not unique: given any one equivariant structure, we can combine the group action with a set of gauge transformations to define a new equivariant structure. Specifically, for each group element $g \in G$, one needs a gauge transformation $U(g)$, obeying the group law, and to preserve a fixed choice of $U(1)$ gauge field, that gauge transformation must

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1 In the special case of trivial bundles, it will; this is why this difficulty is not seen in typical toroidal orbifold constructions, because the bundles there are all trivial. A necessary, but not sufficient, condition is that the Chern classes be invariant under the group action. For example, for group actions on compact Riemann surfaces, every $SU(n)$ bundle automatically has invariant Chern classes, for trivial reasons, but not every such bundle is equivariantizable. For example, consider line bundles of degree zero, as classified by the Picard group of the Riemann surface. The equivariant line bundles lift from the Picard group of the quotient, which has smaller genus. A related example is a non-equivariantizable $Z_2$ bundle in [13][section 5.7.2].
satisfy \(dU(g)\). If \(M\) is connected, that means that each \(U(g)\) is a constant element of \(U(1)\), so such a set of \(U(g)\)'s is determined by an element of

\[\text{Hom}(G, U(1))\]

These are the orbifold Wilson lines, for a \(U(1)\) bundle with connection. As the quotient of a bundle need not be a bundle on the quotient space, these often do not have a simple understanding\(^2\) on the quotient space \(M/G\).

For \(B\) fields there is a closely analogous story. Given a space \(M\) with a \(B\) field, if an equivariant structure exists\(^3\), it will not be unique, because of the possibility of combining the group action with gauge transformations. In order to preserve the \(B\) field, the gauge transformations must be defined by flat line bundles with connection. Denote the line bundles by \(T^g\), and the connection on \(T^g\) by \(\Lambda(g)\). These must preserve the group action, which in this case means there must exist connection-preserving isomorphisms

\[\omega(g, h) : T^h \otimes h^*T^g \sim T^{gh}\]

Furthermore, those isomorphisms must obey a consistency condition, which we can write as

\[
\begin{align*}
T^{g_1} \otimes g_3^*(T^{g_2} \otimes g_2^*T^{g_1}) & \overset{\omega(g_1,g_2)}{\longrightarrow} T^{g_1} \otimes g_3^*T^{g_1g_2} \\
T^{g_2g_3} \otimes (g_2g_3)^*T^{g_1} & \overset{\omega(g_1,g_2g_3)}{\longrightarrow} T^{g_1g_2g_3}
\end{align*}
\]

Ordinary discrete torsion is recovered as a special case of the data above, in which the bundles \(T^g\) are all trivializable, with connections gauge-equivalent to zero. In this case, if we choose to represent each bundle \(T^g\) by the trivial line bundle, and choose each connection \(\Lambda(g)\) to vanish. The connection-preserving isomorphisms \(\omega(g_1, g_2)\) reduce to constant elements of \(U(1)\), obeying the condition

\[\omega(g_1g_2, g_3)\omega(g_1, g_2) = \omega(g_1, g_2g_3)\omega(g_2, g_3)\]

which is precisely the 2-cocycle condition in group cohomology. There are residual gauge transformations; if we let \(\kappa_g\) denote a gauge-transformation on (trivial) line bundle \(T^g\), one which preserves the trivial connection (and so is a constant element of \(U(1)\)), then

\[\omega(g_1, g_2) \mapsto \kappa_{g_1g_2}\omega(g_1, g_2)\kappa_{g_2}\kappa_{g_1}\]

\(^2\) They do, however, descend to honest bundles on the quotient stack \([M/G]\), and have a trivial understanding there.

\(^3\) Just as for bundles, not every (nontrivial) gerbe admits an equivariant structure, \(i.e.\) group actions cannot always be lifted from base spaces to gerbes. For example, consider a \(U(1)\) gerbe on \(T^6\). A non-\(G\)-equivariantizable gerbe on \(T^6\) is defined by an element of \(H^3(T^6, \mathbb{Z})\) that is not invariant under the \(G\) action, and it should be clear that only a subset of degree three cohomology of \(T^6\) will be invariant under a group action on \(T^6\). Suffice it to say, lack of equivariantizability and non-uniqueness of equivariant structures is a very standard story.
which is precisely the action of coboundaries in group cohomology. Thus, we see that the remaining isomorphisms in this case are determined by group cohomology.

If $M$ is simply-connected, with no torsion in $H^2(M, \mathbb{Z})$, then all flat line bundles are trivializable, with connections that are gauge-equivariant to zero, and the case above is the most general case – discrete torsion characterizes all the degrees of freedom. On the other hand, if $M$ is not simply-connected, or if there is torsion in $H^2(M, \mathbb{Z})$, then there are additional degrees of freedom. In the case of toroidal orbifolds, it was remarked in [3, 7] that these extra degrees of freedom correspond to momentum-winding lattice shift phases. These are phase factors of the form

$$\exp (ip_La_R - i p_Ra_L)$$

where $p_{L,R}$ correspond to left-, right- momentum/winding lattice modes and $a$’s to lattice translations. These phase factors are commonly used in asymmetric orbifolds, but can also appear in symmetric orbifolds.

Returning to ordinary discrete torsion, it is straightforward to compute the twisted sector phase factors appearing in loop computations. For orbifold Wilson lines, this is the analogue of computing the holonomy along a line from $x$ to $gx$, and computing that it is

$$\varphi_g \exp \left(i \int_{x}^{g \cdot x} A \right)$$

where $\varphi_g \in \text{Hom}(G, U(1))$. For example, corresponding to the one-loop diagram

![One-loop diagram](image)

(where $gh = hg$ for this diagram to exist) we compute the holonomy of the $B$ field to be [3]

$$\omega_x (g, h) \omega_x (h, g)^{-1} \exp \left(i \int_{x}^{h \cdot x} \Lambda(g) - i \int_{x}^{g \cdot x} \Lambda(h) \right) \exp \left( \int B \right)$$

where the $B$ integral is over the interior of the polygon. (Briefly, the $\Lambda$ integrals arise from the boundaries in the obvious way, and the $\omega$ factors are determined from the corners and by gauge-invariance.) In the case of ordinary discrete torsion, this specializes to the factor

$$\frac{\omega(g, h)}{\omega(h, g)}$$
As this is $x$-independent, it weights all the one-loop contributions the same way, exactly right for discrete torsion. Similarly, we can also derive the momentum/winding lattice shift phases in the same way. For a toroidal orbifold without discrete torsion, the $\omega$ factors are gauge-trivial, and the only contribution to the holonomy arises from the $\Lambda$ factors. Describing the flat $U(1)$ connection on a torus in terms of a constant connection, $\Lambda(g) \equiv \Lambda(g)_i dx^i$, one computes \cite{section 3}  

\[ \int_x^{h-x} \Lambda(g) = \Lambda(g)_i \int_x^{h-x} \frac{dx^i}{d\sigma} d\sigma = \Lambda(g)_i \left( p^i_L - p^i_R \right) \]

\[ \int_x^{g-x} \Lambda(h) = \Lambda(h)_i \int_x^{g-x} \frac{dx^i}{d\tau} d\tau = \Lambda(h)_i \left( p^i_L + p^i_R \right) \]

from which we see that the holonomy reduces to

\[ \exp \left( i \int_x^{h-x} \Lambda(g) - i \int_x^{g-x} \Lambda(h) \right) = \exp \left( ip^i_L a_{Ri} - ip^i_R a_{Li} \right) \]

with

\[ a_{Ri} = \Lambda(g)_i + \Lambda(h)_i, \quad a_{Li} = \Lambda(g)_i - \Lambda(h)_i \]

The phases acting on the $g$-twisted sector of the Hilbert space are the phases of the $(1, g)$ one-loop diagram. On a $(1, g)$ twisted sector, $a_L = a_R$ and so we see that we have correctly recovered the symmetric orbifold phase factor.

For another example, consider the two-loop diagram

(\text{where we assume } h_1 g_1 h_1^{-1} g_1^{-1} = g_2 h_2 g_2^{-1} h_2^{-1} \text{ in order for the polygon to close}). In this case, the holonomy of the $B$ field is easily seen to be

\[ \left( \omega_{g_1^{-1}, x(h_1, g_1)} \right)^{-1} \left( \omega_{g_2^{-1}, x(h_1, g_1)} \right) \left( \omega_{g_1^{-1}, x(h_1 g_1 h_1^{-1}, h_1)} \right) \left( \omega_{g_2^{-1}, x(h_2, g_2)} \right)^{-1} \cdot \left( \omega_{g_1^{-1}, x(h_1, g_1)} \right) \left( \omega_{g_2^{-1}, x(g_2 h_2 g_2^{-1}, h_2)} \right)^{-1} \]
\[
\cdot \exp \left( -i \int_{g_1^{-1}x}^{h_1^{-1}g_1^{-1}x} \Lambda(g_1) + i \int_{g_1h_1^{-1}g_1^{-1}x}^{h_1^{-1}g_1^{-1}x} \Lambda(h_1) + i \int_{h_2^{-1}x}^{g_2^{-1}h_2^{-1}x} \Lambda(h_2) - i \int_{h_2g_2^{-1}h_2^{-1}x}^{g_2^{-1}h_2^{-1}x} \Lambda(g_2) \right)
\cdot \exp \left( \int B \right)
\]

(In \cite{3} we computed the genus two phase factor in the special case that the genus two diagram factorizes into a pair of genus one diagrams; here, we demonstrate the more general case.)

Let us compare to the result for the genus two phase factor computed in \cite{20}. There, it was argued that if \(a_1, b_1, a_2, b_2\) are four group elements such that

\[
a_1b_1a_1^{-1}b_1^{-1} = b_2a_2b_2^{-1}a_2^{-1}
\]

then the genus two discrete torsion phase factor is \cite{20}[equ’n (15)]

\[
\frac{\omega(a_1, b_1)}{\omega(\gamma_1b_1, a_1)\omega(\gamma_1b_1)} \cdot \frac{\omega(\gamma_1, a_2)\omega(\gamma_1a_2, b_2)}{\omega(b_2, a_2)}
\]

where \(\gamma_1 = a_1b_1a_1^{-1}b_1^{-1}\). If we identify

\[
a_1 = g_2, \quad b_1 = h_2, \quad a_2 = g_1, \quad b_2 = h_1
\]

then the phase factor in \cite{20}[equ’n (15)] can be written

\[
\frac{\omega(g_2, h_2)}{\omega(g_2h_2g_2^{-1}, g_2)\omega(g_2h_2g_2^{-1}h_2^{-1}, h_2)} \cdot \frac{\omega(h_1g_1h_1^{-1}g_1^{-1}, g_1)\omega(h_1g_1h_1^{-1}, h_1)}{\omega(h_1, g_1)}
\]

Using the cocycle identity

\[
\omega(g_2h_2g_2^{-1}, g_2)\omega(g_2h_2g_2^{-1}h_2^{-1}, h_2) = \omega(g_2h_2g_2^{-1}h_2^{-1}, h_2g_2)\omega(h_2, g_2)
\]

it is easy to check that the phase factor \cite{11} specializes to that in \cite{20}[equ’n (15)].

One can also derive the effect of projectivization action of discrete torsion in D-branes. The reason for the link is the fact that gauge transformations \(B \mapsto B + d\Lambda\) induce the action \(A \mapsto A + \Lambda I\) on the Chan-Paton factors of open strings. Thus, the choice of equivariant structure on the \(B\) field directly affects the equivariant structure on the Chan-Paton gauge field. As described in \cite{3 6}, in a suitable basis of open sets, the modified equivariant structure can be written

\[
g^* A^\alpha = (\gamma^g_\alpha) A^\alpha (\gamma^g_\alpha)^{-1} + (\gamma^g_\alpha) d (\gamma^g_\alpha)^{-1} + I\Lambda(g)^\alpha
\]

\[
g^* g_{\alpha\beta} = \left( \gamma^g_{\alpha\beta} \right) \left[ (\gamma^g_\alpha) (g_{\alpha\beta}) (\gamma^g_\beta)^{-1} \right]
\]

\[
(h^g_{\alpha\beta}) (\gamma^g_{\alpha\beta}) = (g^2_{\alpha\beta} (\gamma^g_\alpha)) (\gamma^g_\alpha)
\]

where \(A^\alpha\) is the Chan-Paton gauge field on patch \(U_\alpha\), \(g_{\alpha\beta}\) are transition functions for the Chan-Paton bundle, \(\gamma^g_\alpha\) define the equivariant structure on the Chan-Paton bundle, and
\( \Lambda(g)^\alpha, \nu_{\alpha\beta}^g, \) and \( h_{\alpha_1,\alpha_2}^{g_1,g_2} \) are data defining the equivariant structure on the \( B \) field. If we start with a topologically trivial \( B \) field, and a topologically-trivial Chan-Paton bundle, and only consider the effect of discrete torsion, then we can take \( \Lambda(g) \equiv 0, \nu^g \equiv 1, \) \( h_{g_1, g_2}^{g_1, g_2} \equiv \omega(g_1, g_2), \) and then the equivariant structure above reduces to

\[
g^* A = (\gamma^g) A (\gamma^g)^{-1} + (\gamma^g_\alpha) d (\gamma^g_\alpha)^{-1}
g^* g_{\alpha\beta} = (\gamma^g_\alpha) (g_{\alpha\beta}) (\gamma^g_\beta)^{-1}
(h_{g_1, g_2}^{g_1, g_2}) (\gamma^g_{1, g_2}^{g_1, g_2}) = (\gamma^g_1) (\gamma^g_2)
\]

which is precisely the projectivized orbifold group action described in [15, 16].

For completeness, let us also outline the same result for momentum/winding lattice shift phases of toroidal orbifolds. In such cases, taking the line bundles \( P^g \) to be trivial with flat connections \( \Lambda(g) \), the equivariant structure above reduces to

\[
g^* A = (\gamma^g) A (\gamma^g)^{-1} + (\gamma^g_\alpha) d (\gamma^g_\alpha)^{-1} + I \Lambda(g)
g^* g_{\alpha\beta} = (\gamma^g_\alpha) (g_{\alpha\beta}) (\gamma^g_\beta)^{-1}
\gamma^{g_1, g_2} = (\gamma^g_1) (\gamma^g_2)
\]

### 3 Orientifolds and \( B \) fields

For ordinary group actions, the work in [3, 4, 5, 6, 7] assumed that the group action preserved the \( B \) field up to a gauge transformation:

\[
g^* B = B + \text{(gauge transformation)}
\]

In more detail, including the gauge transformations on each coordinate patch, their coordinate transformations, and so forth, the full set of data was summarized in [3] as

\[
g^* B^\alpha = B^\alpha + d\Lambda(g)^\alpha
\]

\[
g^* A^{\alpha\beta} = A^{\alpha\beta} + d\ln\nu_{\alpha\beta}^g + \Lambda(g)^\alpha - \Lambda(g)^\beta
\]

\[
g^* h_{\alpha\beta\gamma} = h_{\alpha\beta\gamma} + \nu_{\alpha\beta}^g \nu_{\beta\gamma}^g \nu_{\gamma\alpha}^g
\]

\[
\Lambda(g_1 g_2)^\alpha = \Lambda(g_2)^\alpha + g_2^* \Lambda(g_1)^\alpha - d\ln h_{\alpha_1, \alpha_2}^{g_1, g_2}
\]

\[
\nu_{\alpha_1, \alpha_2}^{g_1, g_2} = \left( \nu_{\alpha_3}^{g_2} \right) \left( g_2^* \nu_{\alpha_3}^{g_1} \right) \left( h_{\alpha_1, \alpha_2}^{g_1, g_2} \right) \left( h_{\alpha_1, \alpha_2}^{g_1, g_2} \right)^{-1}
\]

\[
(h_{\alpha_1, \alpha_2}^{g_1, g_2}) (h_{\alpha_1, \alpha_2}^{g_1, g_2}) = (g_2^* h_{\alpha_1, \alpha_2}^{g_1, g_2}) (h_{\alpha_1, \alpha_2}^{g_1, g_2})
\]

where \( A^{\alpha\beta}, h_{\alpha\beta\gamma} \) define the \( B \) field globally:

\[
B^\alpha = B^\beta = dA^{\alpha\beta}
A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} = d\ln h_{\alpha\beta\gamma}
\delta(h_{\alpha\beta\gamma}) = 1
\]
and where $\Lambda(g)^\alpha, \nu^\beta_{\alpha\beta}$, and $h^{g_1,g_2}_{\alpha}$ are structures introduced to define the action of the orbifold group on the $B$ field. (As noted previously, equivariant structures need not exist on all gerbes; we assume implicitly that the gerbe with connection described here admits an equivariant structure.)

In the case of an orientifold, instead of equation (2), we have instead

$$g^*B = -B + \text{(gauge transformation)}$$

for some elements $g$ of the orientifold group. Physically, $B$ is mapped to $-B$ (modulo gauge transformations) because the orientifold action reverses worldsheet orientation. Ultimately this modifies the conditions satisfied by the data $\Lambda(g)^\alpha, \nu^\beta_{\alpha\beta}$, and $h^{g_1,g_2}_{\alpha}$, and will give rise to a modified form of discrete torsion.

To see this, first let us be a little more careful in our description of the orientifold action. If the orientifold group is $G$, then in general some elements of $G$ will act by orientation-reversal on the target, and others will not. Following [8], let $\epsilon: G \to \mathbb{Z}_2$ be a homomorphism that expresses whether a given element of the orientifold group acts as an orientation-reversal on the target space. Then, schematically, we can write

$$g^*B = \epsilon(g)B + \text{(gauge transformation)}$$

where we identify $\mathbb{Z}_2$ with $\{\pm 1\}$. From the global definition of the $B$ field, we see immediately that under such a group action,

$$g^*B^\alpha = \epsilon(g)B^\alpha + \frac{d\Lambda(g)^\alpha}{g_1}$$

$$g^*A^{\alpha\beta} = \epsilon(g)A^{\alpha\beta} + \frac{d\ln \nu^\beta_{\alpha\beta}}{g_1} + \Lambda(g)^\alpha$$

$$g^*h^{\alpha\beta\gamma}_{\alpha\beta} = h^{\epsilon(g)}_{\alpha\beta\gamma}$$

for some $\Lambda(g)^\alpha, \nu^\beta_{\alpha\beta}$, and $h^{g_1,g_2}_{\alpha}$. Furthermore, following the same procedure as in for example [3], that overlap data must satisfy the coherence conditions:

$$\Lambda(g_1 g_2)^\alpha = \epsilon(g_1)\Lambda(g_2)^\alpha + g_2^*\Lambda(g_1)^\alpha - d\ln h^{g_1,g_2}_{\alpha}$$

$$\nu^{g_1,g_2}_{\alpha\beta} = (\nu^{g_2}_{\alpha\beta})^{\epsilon(g_1)} (g_2^*\nu^{g_1}_{\alpha\beta}) (h^{g_1,g_2}_{\alpha})^{-1}$$

$$h^{g_1,g_2,g_3}_{\alpha} = (g_3 h^{g_1,g_2}_{\alpha})^{\epsilon(g_1)}$$

The first two can be derived by demanding that $g_2^*g_1^* = (g_1 g_2)^*$ on the data defining the $B$ field globally; the third can be derived by demanding that

$$\nu^{g_1,g_2,g_3}_{\alpha\beta} = \nu^{(g_1 g_2)g_3}_{\alpha\beta}$$

and using a coherence condition just derived.

In addition, we take $\Lambda(1)^\alpha \equiv 0, \nu^{1}_{\alpha\beta} \equiv 1$, and $h^{1,1}_{\alpha} = 1 = h^{1,1}_{\alpha}$. Then, in the case $G = \mathbb{Z}_2$, with $\epsilon: G \to \mathbb{Z}_2$ the identity, the data above precisely specializes to the Jandl structures discussed in [19][section 1].
Discrete torsion for ordinary orbifolds arises as the difference between any two group actions on a given B field. Specifically, for any two group actions defined by \( \left( \Lambda(g)^\alpha, \nu^g_{\alpha\beta}, h^g_{\alpha} \right) \), \( \left( \tilde{\Lambda}(g)^\alpha, \tilde{\nu}^g_{\alpha\beta}, \tilde{h}^g_{\alpha} \right) \), we get a bundle \( T^g \) defined by transition functions

\[
\frac{\nu^g_{\alpha\beta}}{\tilde{\nu}^g_{\alpha\beta}}
\]

with a connection defined by \( \tilde{\Lambda}(g)^\alpha - \Lambda(g)^\alpha \), and with connection-preserving bundle isomorphisms \( \omega^h : T^h \otimes h^* T^g \longrightarrow T^{gh} \) defined in local trivializations by

\[
\frac{h^g_{\alpha}}{\tilde{h}^h_{\alpha}}
\]

obeying the condition that the diagram

\[
\begin{array}{ccc}
T^{g_3} \otimes g_3^* (T^{g_2} \otimes g_2^* T^{g_1}) & \longrightarrow & T^{g_1} \otimes g_3^* T^{g_1 g_2} \\
\omega^{g_2, g_3} & & \omega^{g_1, g_2, g_3} \\
T^{g_2 g_3} \otimes (g_2 g_3)^* T^{g_1} & \longrightarrow & T^{g_1 g_2} g_3
\end{array}
\]

commute. (Verification that these ratios have the interpretations listed is straightforward from the Cech identities, and is discussed in detail in [3].) Discrete torsion specifically arises as the special case of group actions differing by data in which the bundles \( T \) are all trivial with zero connection, so that the \( \omega^{g_1, g_2} \) are constant gauge transformations. In other words, the \( \omega^{g_1, g_2} \) define maps \( G \times G \rightarrow U(1) \), which we shall denote \( \omega(g_1, g_2) \). Commutivity of the diagram above implies that

\[
\omega(g_1 g_2, g_3) \omega(g_1, g_2) = \omega(g_1, g_2 g_3) \omega(g_2, g_3)
\]

which is the condition for a group 2-cocycle. (Note that the condition \( h^1_{\alpha} = h^2_{\alpha} \) implies that \( \omega(1, g) = \omega(g, 1) = 1 \) for all \( g \), so this is a normalized cocycle.) Furthermore, a constant gauge transformation \( \lambda^g \) on each \( T^g \) will rotate the \( \omega^{g_1, g_2} \)'s, and hence modify the group cochains by factors \( \lambda(g) \) (determined by \( \lambda^g \)) as

\[
\omega(g_1, g_2) \mapsto \omega(g_1, g_2) \lambda(g_1 g_2) (\lambda(g_1))^{-1} (\lambda(g_2))^{-1}
\]

which is exactly how group 2-cocycles are shifted by group coboundaries. (Furthermore, \( \lambda^1 = 1 \), so this is a normalized group coboundary.) More general group actions on \( B \) fields are certainly possible, and as discussed in [7], are interpreted as momentum/winding lattice shifts.

Now, let us repeat the analysis above for the case of orientifold group actions, rather than orbifold group actions. We can define bundles \( T^g \), connections, and bundle morphisms...
\( \omega^{g_1,g_2} \) from the Cech data as previously, but the interpretation now changes. For example, from the coherence condition

\[
\nu_{\alpha \beta}^{g_1,g_2} = \left( \nu_{\alpha \beta}^{g_2} \right)^{\epsilon(g_1)} (h_{\alpha}^{g_1,g_2}) (h_{\beta}^{g_1,g_2})^{-1}
\]

we see that the \( \omega^{g_1,g_2} \) should be interpreted as bundle maps

\[
\omega^{g,h} : \left( T^h \right)^{\epsilon(g)} \otimes h^* T^g \to T^g
\]

which, because of the coherence condition

\[
(h_{\alpha}^{g_1,g_2,g_3}) (h_{\alpha}^{g_2,g_3})^{\epsilon(g_1)} = (g_3^* h_{\alpha}^{g_1,g_2}) (h_{\alpha}^{g_1,g_2,g_3})
\]

make the diagram

\[
\begin{array}{c}
(T^{g_1})^{\epsilon(g_1)} \otimes g_3^* \left( (T^{g_2})^{\epsilon(g_1)} \otimes g_2^* T^{g_1} \right) \xrightarrow{\omega^{g_1,g_2}} (T^{g_3})^{\epsilon(g_1)} \otimes g_3^* T^{g_1,g_2} \\
\downarrow \quad \downarrow \omega^{g_1,g_2,g_3} \\
(T^{g_2,g_3})^{\epsilon(g_1)} \otimes (g_2 g_3)^* T^{g_1} \xrightarrow{\omega^{g_1,g_2,g_3}} T^{g_1,g_2,g_3}
\end{array}
\]

commute. Proceeding as before, we extract the orientifold analogue of discrete torsion by restricting to the special case that the \( T^g \) are all trivial with vanishing connection, so that the \( \omega^{g,h} \) become constant gauge transformations. Thus, the \( \omega^{g,h} \) define (normalized) group 2-cochains, which we shall denote \( \omega(g,h) \), subject to the condition

\[
\omega(g_1 g_2, g_3) \omega(g_1, g_2) = \omega(g_1, g_2 g_3) (g_1 \cdot \omega(g_2, g_3))
\]

Furthermore, the residual constant gauge transformations on the bundles \( T^g \) means we must mod out the identifications

\[
\omega(g,h) \sim \omega(g,h) \lambda(gh) (\lambda(g))^{-1} (g \cdot \lambda(h))^{-1}
\]

The result is \( H^2(G,U(1)) \) with nontrivial action on the coefficients, as discussed in appendix A. This is precisely the same group cohomology discussed by [2] \& [8].

### 4 D-branes and projectivized group actions

As discussed earlier in section 2, for D-branes discrete torsion has the effect of projectivizing the orbifold group action.
Let us now quickly sketch out the corresponding results for orientifolds. Repeating the same analysis as in [3], and reviewed earlier, one quickly finds that an equivariant structure on the bundle with connection defined by \((g_{\alpha\beta}, A^\alpha)\) satisfying
\[
A^\alpha - g_{\alpha\beta} A^\beta g_{\beta\gamma} g_{\gamma\alpha} = h_{\alpha\beta} I
\]
is defined by (using \(\epsilon\) exponents to describe complex conjugation)
\[
g^* A^\alpha = \epsilon(g) \gamma_{\alpha}^g A^\alpha (\gamma_{\alpha}^g)^{-1} + (\gamma_{\alpha}^g) d (\gamma_{\alpha}^g)^{-1} + I \Lambda(g)^\alpha
\]
\[
g^* g_{\alpha\beta} = \nu_{\alpha\beta}^g (\gamma_{\alpha}^g) g_{\alpha\beta} (\gamma_{\beta}^g)^{-1}
\]
\[
h_{\alpha\beta}^{g_1g_2\gamma_1g_2} = (g_2^{\gamma_{\alpha}^g_1} (\gamma_{\alpha}^g)^{g_1})
\]

so long as the transition functions and bundle maps are invariant under the action of the orientifold:
\[
g_{\alpha\beta} = g_{\alpha\beta}^{\epsilon(h)}, \quad \gamma_{\alpha}^g = (\gamma_{\alpha}^g)^{\epsilon(h)}
\]

for all \(g\) and \(h\). Our notation above is such that if \(\epsilon(g) = -1\), then \(g_{\alpha\beta}^{\epsilon(g)} = g_{\beta\alpha}\), complex conjugation, hence the invariance constraint implies that the vector bundle is real, as expected for a bundle on the fixed-point set of an antilinear involution.

We have not referenced K theory or type II strings specifically so far; however, if one were working in a type II string theory and the \(B\) field were described by a 1-gerbe (which recent analysis [12] slightly contradicts), then the structure above would be the key part of twisted equivariant K theory, recovering another part of [8].

With an eye to the next section, let us use the structure above to outline a derivation of the phase factor for a Klein bottle, following the analysis of [20]. To that end, consider a Klein bottle with boundary, as shown below: where \(g, h,\) are group elements such that
\[
gh = hg^{-1}, \quad \epsilon(g) = +1, \quad \epsilon(h) = -1.
\]

Following [20], let \(s\) denote a projective representation of the orientifold group, obeying
\[
s(a) s(b)]^{\epsilon(a)} = \omega(a, b) s(ab)
\]

Figure 1: A Klein bottle sector.
for any two group elements $a, b$, where $\omega(a, b)$ is a normalized group 2-cocycle, and such that $s(1) = 1$. Then, the Klein bottle phase factor should be given by

$$s(g)s(h) [h \cdot s(g)] s(h)^{-1}$$

$$= [\omega(g, h)s(gh)] [\omega(g, g^{-1})s(g^{-1})^{-1}]^{-1} s(h)^{-1}$$

$$= [\omega(g, h)s(gh)] \omega(g, g^{-1})^{-1} [s(h)s(g^{-1})^{-1}]^{-1}$$

$$= [\omega(g, h)s(gh)] \omega(g, g^{-1})^{-1} \left[ \omega(h, g^{-1})s(hg^{-1}) \right]^{-1}$$

$$= \omega(g, h)\omega(g, g^{-1})^{-1} \omega(h, g^{-1})^{-1}$$

giving us the Klein bottle phase factor

$$\frac{\omega(g, h) [h \cdot \omega(g, g^{-1})]}{\omega(h, g^{-1})}$$

(in terms of normalized cocycles) which is easily checked to descend to group cohomology, and furthermore generalizes the corresponding result for orbifolds [21][equ’n (16)], namely

$$\frac{\omega(g, h)\omega(g, g^{-1})}{\omega(h, g^{-1})}$$

We will independently derive the same phase factor for Klein bottles in the next section, as the $B$ field holonomy.

5 Phase factors for nonorientable worldsheets

5.1 The Klein bottle

Let us now compute a Klein bottle twisted sector phase factor, following the same pattern as in [3]. (Interested readers should also consult [19][sections 3.2, 3.3], where essentially the same formal holonomy expressions are outlined, though the specific Klein bottle holonomy below is not computed.) Since discrete torsion arises from the difference between two group actions, for simplicity let us assume the $B$ field is defined by a trivial gerbe, and take one group action to be the canonical trivial action on a trivial gerbe. In principle, and referring to figure 1 (though with $x$ shifted to $g^{-1} \cdot x$ to clean up the result), the phase factor can be computed by starting with

$$\exp \left( i \int B \right) \exp \left( i \int_{g^{-1} \cdot x}^{x} \Lambda(h) + i \int_{g^{-1} \cdot x}^{h \cdot x} \Lambda(g) \right)$$
and adding factors of $\omega$ needed to ensure gauge-invariance of the result. The factors of $\exp(i \int \Lambda)$ arise in order to take into account the group action across boundaries. The data at the edges of the integrals amount to four lines:

\[
\left( T^h_x \otimes (g^{-1} T^h_x)^{-1} \right) \otimes \left( h^* T^g_x \otimes (g^{-1} T^g_x)^{-1} \right)
\]

and can be fixed with the following factor:

\[
\omega^{g,h} \cdot (\omega^{g,g^{-1}})^{-1} \cdot \omega^{h,g^{-1}} : T^h \otimes h^* T^g \otimes (g^{-1} T^g)^{-1} \otimes (g^{-1} T^h)^{-1} \rightarrow O
\]

Thus, the complete gauge-invariant phase factor is

\[
\exp \left( i \int B \right) \exp \left( i \int_{g^{-1} x} \Lambda(h) + i \int_{g^{-1} x} \Lambda(g) \right) \omega^{g,h} \cdot (\omega^{g,g^{-1}})^{-1} \cdot \omega^{h,g^{-1}}
\]

from which we read off that the Klein bottle orientifold discrete torsion phase factor is given by

\[
\frac{\omega(g,h)^{h} \cdot \omega(g,g^{-1})}{\omega(h,g^{-1})}
\]

(as obtained by restricting to trivial bundles with trivial connections), matching that obtained in the last section by other means.

Vafa’s original discrete torsion phase factor was partially defined by the property of being modular invariant. Therefore, it is natural to ask whether the phase factor we have derived obeys an analogous constraint. The modular transformations $SL(2, \mathbb{Z})$ of the two-torus are naturally understood as its mapping class group, and the Klein bottle has a nontrivial mapping class group $[22, 23, 24, 25]$, albeit merely $\mathbb{Z}_2 \times \mathbb{Z}_2$. This mapping class group is generated by a combination of a Dehn twist and the “Y-homeomorphism,” but unfortunately do not seem $[26]$ to have a natural action on $g, h$ above.

### 5.2 The real projective plane

Another nonorientable twisted sector one should also consider is the real projective plane. This also can be described by a polygon with sides identified, as in the figure below:

\[
\begin{align*}
& \overline{x} \quad g \\
& \overline{g \cdot x}
\end{align*}
\]

\[4\] In $[3]$, the phase factor involved the difference, rather than the sum, of the same two integrals. Here, because of nonorientability, there is an ambiguity, which can be resolved by demanding gauge invariance – the difference can not be made gauge-invariant through $\omega$ factors, whereas the sum can be.
where \( g^2 = 1 \), i.e. \( g = g^{-1} \).

We can compute the phase factor proceeding exactly as before. Taking into account the edges, the phase factor should be

\[
\exp \left( i \int B \right) \exp \left( i \int_{x}^{g \cdot x} \Lambda(g) \right)
\]

The lines at the corners are of the form

\[
(g^* T^g_x) \otimes (T^g_x)^{-1}
\]

Since

\[
\omega^{g,g} : (T^g)^{(g)} \otimes g^* T^g \longrightarrow T^{g^2} = T^1
\]

we find by demanding gauge-invariance that the complete phase factor must be

\[
\exp \left( i \int B \right) \exp \left( i \int_{x}^{g \cdot x} \Lambda(g) \right) \omega^{g,g}
\]

In particular, the analogue of the discrete torsion phase factor for the orientifold real projective plane is the phase

\[
\omega(g, g)
\]

where \( \omega \) is a normalized group 2-cocycle. It is easy to check that this descends to group cohomology.

One can also trivially derive the same result from open string theories along the lines of [20], just as we did for the Klein bottle in the last section, from the phase factor \( s(g)g \cdot s(g) \).

As a consistency check, we can ‘square’ the polygon giving the real projective plane, to get that for a sphere:

![Diagram of real projective plane]

This diagram should be associated with the square of the phase associated to a single real projective plane, i.e.

\[
\omega(g, g) \omega(g, g)
\]

On the other hand, there is no twisted sector phase, indeed no twisted sector, on \( S^2 \). Hence, this phase factor ought to be unity:

\[
\omega(g, g) \omega(g, g) = 1
\]

It is straightforward to check that this statement is true, a consequence of the cocycle condition corresponding to the three group elements \( g_1 = g_2 = g_3 = g \).
6  

$C$ fields  

As discrete torsion is not specific to conformal field theory, but rather is a mathematical property of defining group actions on theories with tensor field potentials, one should correctly expect that there is an analogue of discrete torsion for other tensor field potentials than just the $B$ field, even though conformal-field-theoretic descriptions of more general cases are problematic. In [4], we worked out the formal analogue of discrete torsion for $C$ fields. In this section, we shall first review that analysis, then extend it to orientifolds.

As in [4], we shall assume that the $C$ fields in question are well-described by 2-gerbes. Now, as remarked in [4], that assumption is not quite accurate: in type IIA strings, for example, $C$ fields are better defined using K theory. The K theoretic description takes into account interactions, and so gives a more nearly complete accounting of the degrees of freedom in the entire theory.

6.1 Review: $C$ field analogue of discrete torsion

In this section we shall review the results of [4] concerning $C$ fields an orbifolds. We argue that $C$ fields have a degree of freedom analogous to discrete torsion, counted by $H^3(G, U(1))$ instead of $H^2(G, U(1))$, and work out the corresponding phase factors. We also discuss analogues of momentum/winding lattice shift phase factors in this case.

Briefly, in [4] we argued that any two equivariant structures on the same $C$ field differed by a set of flat gauge transformations, defined by the following data:

1. A set of flat 1-gerbes $\Upsilon^g$ with connection, $\mathcal{B}(g)$ (such that $d\mathcal{B} = 0$ in every coordinate patch).

2. Connection-preserving isomorphisms $(\Omega^{g,h}, \theta(g, h))$ between the 1-gerbes with connection

   $$\Upsilon^h \otimes h^* \Upsilon^g \xrightarrow{\sim} \Upsilon^{gh}$$

   preserving the group law.

3. Isomorphisms

   $$\omega(g_1, g_2, g_3) : \Omega^{g_1 g_2 g_3} \circ g_3^* \Omega^{g_1 g_2} \xrightarrow{\sim} \Omega^{g_1 g_2 g_3} \circ \Omega^{g_2 g_3}$$

\[5\] As explained in [4], there are two potential physical problems. The first is that in type II strings, $C$ fields are understood in terms of differential K theory, not 2-gerbes; for this reason, we only speak of M theory $C$ fields, ignoring gravitational corrections (hence our results are of a very formal nature). The second is that once we move to M theory, one could reasonably object that the form of string orbifolds is specific to theories with a perturbative description as string theories – we do not truly know whether M theory makes sense on stacks as well as spaces. As discussed in [4], our analysis for $C$ fields is meant to be a formal guide, not a definitive final answer to all such issues.
enforcing the higher coherence relation
\[
\gamma^{g_1} \otimes \gamma^{g_2} \otimes \gamma^{g_3} \rightarrow \gamma^{g_1, g_2, g_3}
\]
and themselves obeying an even higher-order coherence relation
\[
\omega(g_1, g_2, g_3, g_4) = \omega(g_2, g_3) \circ \omega(g_1, g_2) \circ \omega(g_3, g_4)
\]
where both sides are functions
\[
\Omega^{g_1, g_2, g_3, g_4} \circ g_4 \circ (g_3 g_4)^* \circ \Omega^{g_1, g_2, g_3} \rightarrow \Omega^{g_1, g_2, g_3, g_4} \circ g_3 \circ g_2 \circ g_3^* \circ \Omega^{g_1, g_2, g_3}
\]

In the special case that all flat 1-gerbes with connection are topologically-trivial with gauge-trivial connection, the data above reduces to a set of flat line bundles \(\Omega^{g, h}\), with connection-preserving isomorphisms \(\omega(g_1, g_2, g_3)\). If in addition, all flat line bundles are topologically trivial with gauge-trivial connection, then after a suitable gauge transformation the data above reduces to a set of constant \(U(1)\) elements
\[
\omega(g_1, g_2, g_3)
\]
obeying (by virtue of the coherence relation) the 3-cocycle condition
\[
\omega(g_1, g_2, g_3, g_4) \omega(g_1, g_2, g_3) = \omega(g_1, g_2, g_3) \omega(g_2, g_3, g_4)
\]
modulo the residual gauge transformations defined by constant \(U(1)\) elements \(\kappa(g_1, g_2)\):
\[
\omega(g_1, g_2, g_3) \mapsto \omega(g_1, g_2, g_3) \frac{\kappa(g_2, g_3) \kappa(g_1, g_2 g_3)}{\kappa(g_1, g_2) \kappa(g_1, g_3) \kappa(g_1, g_2)}
\]
which is the action of a coboundary, as reviewed in appendix A. Thus, in this case, all of the degrees of freedom are encapsulated by elements of \(H^3(G, U(1))\). In more general cases, there are \(C\)-field-analogues of the momentum/winding lattice shift phase factors.

It is also straightforward to compute the phase factors that would be seen by membranes. Below we have illustrated an example of a membrane twisted sector:
where, in order for the cube to close, we assume that $g_1$, $g_2$, and $g_3$ commute with one another. Using the obvious boundaries and gauge-invariance, it is straightforward to show that the holonomy is \[4\][section 3.1]

\[
(\omega_x(g_1, g_2, g_3)) (\omega_x(g_2, g_1, g_3))^{-1} (\omega_x(g_3, g_2, g_1))^{-1} (\omega_x(g_3, g_1, g_2)) (\omega_x(g_2, g_3, g_1)) (\omega_x(g_1, g_3, g_2))^{-1}
\cdot \exp \left( -i \int_{x}^{g_3} \left[ \theta(g_1, g_2) - \theta(g_2, g_1) \right] - i \int_{x}^{g_1} \left[ \theta(g_2, g_3) - \theta(g_3, g_2) \right] \right)
\cdot \exp \left( -i \int_{g_2}^{x} \left[ \theta(g_1, g_3) - \theta(g_3, g_1) \right] \right) \exp \left( i \int_{1} B(g_1) + i \int_{2} B(g_2) + i \int_{3} B(g_3) \right)
\cdot \exp \left( i \int C \right)
\]

where $\theta(g_1, g_2)$ is part of the data together with $\Omega^{g_1,g_2}$ defining a map between 1-gerbes (and which in simple cases, in which $\Omega^{g_1,g_2}$ becomes a bundle, reduces to a connection on that bundle).

In the special case of degrees of freedom counted by $H^3(G, U(1))$, the phase factor above reduces to

\[
\frac{\omega(g_1, g_2, g_3) \omega(g_3, g_1, g_2) \omega(g_2, g_3, g_1)}{\omega(g_2, g_1, g_3) \omega(g_3, g_2, g_1) \omega(g_1, g_3, g_2)} \quad (5)
\]
in terms of group cocycles. This expression is invariant under group coboundaries, and hence is well-defined on group cohomology. Furthermore, it was shown in \[4\][section 3.2] that the expression above is invariant under $SL(3, \mathbb{Z})$ transformations. Now, unlike two-dimensional string theories, there is no analogue of a modular invariance constraint, but the $SL(3, \mathbb{Z})$ invariance here (and the $SL(2, \mathbb{Z})$ invariance of one-loop discrete torsion phases) arises because of the condition that the phase factor be well-defined on a torus. We do not impose $SL(3, \mathbb{Z})$ at the beginning, we do not impose it as a constraint that must be satisfied, but we instead discover after a derivation that does not mention $SL(3, \mathbb{Z})$ that the result does happen to possess $SL(3, \mathbb{Z})$ invariance.

For a recent application of the ideas in this subsection, see e.g. [1].

### 6.2 Orientifolds and $C$ fields

In this section we shall perform the analogous analysis for $C$ fields in orientifolds.

First, for ordinary group actions which preserve the $C$ field, in the sense

\[
g^*C = C + \text{(gauge transformation)} \quad (6)
\]

following \[4\] the full set of data was given by

\[
g^*C^\alpha = C^\alpha + dA^{(2)}(g)^\alpha
\]
\[ g^* B^{\alpha\beta} = B^{\alpha\beta} + d\Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(2)}(g)^\alpha - \Lambda^{(2)}(g)^\beta \]
\[ g^* A^{\alpha\beta\gamma} = A^{\alpha\beta\gamma} + d\ln \nu^{(2)}_{\alpha\beta\gamma} + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(1)}(g)^{\beta\gamma} + \Lambda^{(1)}(g)^\gamma \]
\[ g^* h_{\alpha\beta\gamma\delta} = \left( h^{(g)}_{\alpha\beta\gamma\delta} \right) \left( \nu^{g}_{\beta\gamma\delta} \right)^{-1} \left( \nu^{g}_{\alpha\beta\gamma} \right)^{-1} \nu^{g}_{\alpha\beta\gamma\delta} \]
\[ \Lambda^{(2)}(g_1 g_2)^\alpha = \Lambda^{(2)}(g_2)^\alpha + g_2^* \Lambda^{(2)}(g_1)^\alpha + d\Lambda^{(3)}(g_1, g_2)^\alpha \]
\[ \Lambda^{(1)}(g_1 g_2)^{\alpha\beta} = \Lambda^{(1)}(g_2)^{\alpha\beta} + g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} - \Lambda^{(3)}(g_1, g_2)^{\alpha\beta} + \Lambda^{(3)}(g_1, g_2)^{\beta\alpha} \]
\[ \Lambda^{(3)}(g_2, g_3)^\alpha + \Lambda^{(3)}(g_1, g_2)^{\alpha\beta} = g_3^* \Lambda^{(3)}(g_1, g_2)^\alpha + \Lambda^{(3)}(g_1, g_2, g_3)^\alpha + d\ln \gamma_{\alpha^{g_1, g_2, g_3}} \]

where \( B^{\alpha\beta}, A^{\alpha\beta\gamma}, \) and \( h_{\alpha\beta\gamma\delta} \) define the \( C \) field globally:

\[ C^\alpha - C^\beta = dB^{\alpha\beta} \]
\[ B^{\alpha\beta} + B^{\beta\gamma} + B^{\gamma\alpha} = dA^{\alpha\beta\gamma} \]
\[ A^{\beta\gamma\delta} - A^{\alpha\gamma\delta} + A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma\delta} = d\ln h_{\alpha\beta\gamma\delta} \]
\[ \delta h_{\alpha\beta\gamma\delta} = 1 \]

and where \( \nu^{g}_{\alpha\beta\gamma}, \lambda^{g_1, g_2}_{\alpha\beta}, \gamma_{\alpha^{g_1, g_2, g_3}}, \Lambda^{(1)}(g)^{\alpha\beta}, \Lambda^{(2)}(g)^\alpha, \) and \( \Lambda^{(3)}(g_1, g_2)^{\alpha\beta} \) are structures introduced to define the orbifold group action.

In the case of an orientifold, equation (8) is replaced by

\[ g^* C = -C + (\text{gauge transformation}) \]  
(7)

for some elements \( g \) of the orientifold group, just as in our discussion of \( B \) fields. As previously, this modifies the conditions satisfied by the gauge-transformation data.

As in our discussion of \( B \) fields, let \( \epsilon : G \to \mathbb{Z}_2 \) be a homomorphism that expresses whether a given element of the orientifold group acts as an orientation-reversal on the target space. Then, schematically,

\[ g^* C = \epsilon(g) C + (\text{gauge transformation}) \]  
(8)

From the global definition of the \( C \) field, we see immediately that

\[ g^* C^\alpha = \epsilon(g) C^\alpha + d\Lambda^{(2)}(g)^\alpha \]
\[ g^* B^{\alpha\beta} = \epsilon(g) B^{\alpha\beta} + d\ln \nu^{(2)}_{\alpha\beta} + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(2)}(g)^\alpha - \Lambda^{(2)}(g)^\beta \]
\[ g^* A^{\alpha\beta\gamma} = \epsilon(g) A^{\alpha\beta\gamma} + d\ln \nu^{g}_{\alpha\beta\gamma} + \Lambda^{(1)}(g)^{\alpha\beta}\gamma + \Lambda^{(1)}(g)^{\beta\gamma} + \Lambda^{(1)}(g)^\gamma \]
\[ g^* h_{\alpha\beta\gamma\delta} = \left( h^{\epsilon(g)}_{\alpha\beta\gamma\delta} \right) \left( \nu^{g}_{\beta\gamma\delta} \right)^{-1} \left( \nu^{g}_{\alpha\beta\gamma} \right)^{-1} \nu^{g}_{\alpha\beta\gamma\delta} \]
Following the same procedure as in [4], it can be shown this overlap data must satisfy the coherence conditions

\[
\Lambda^{(2)}(g_1g_2)^\alpha = \epsilon(g_1)\Lambda^{(2)}(g_2)^\alpha + g_2^*\Lambda^{(2)}(g_1)^\alpha + d\Lambda^{(3)}(g_1, g_2)^\alpha
\]

\[
\Lambda^{(1)}(g_1g_2)^{\alpha\beta} = \epsilon(g_1)\Lambda^{(1)}(g_2)^{\alpha\beta} + g_2^*\Lambda^{(1)}(g_1)^{\alpha\beta} - \Lambda^{(3)}(g_1, g_2)^\alpha + \Lambda^{(3)}(g_1, g_2)^\beta - d\ln\Lambda^{g_1, g_2}_{\alpha\beta}
\]

\[
\epsilon(g_1)\Lambda^{(3)}(g_2, g_3)^\alpha + \Lambda^{(3)}(g_1, g_2g_3)^\alpha = g_3^*\Lambda^{(3)}(g_1, g_2)^\alpha + \Lambda^{(3)}(g_1g_2, g_3)^\alpha + d\ln\gamma^{g_1, g_2, g_3}_{\alpha}
\]

(For example, the expression for \(\Lambda^{(3)}\)'s can be checked by expanding out \(\Lambda^{(2)}\) in two different ways.)

As in our discussion of orientifolds and \(B\) fields, we take \(\Lambda^{(1)}(1)^{\alpha\beta} = 0\), \(\Lambda^{(2)}(1)^\alpha = 0\), \(\Lambda^{(3)}(1, g)^\alpha = \Lambda^{(3)}(g, 1)^\alpha = 0\), \(\nu^{1}_{\alpha\beta\gamma} = 1\), \(\lambda^{g}_{\alpha\beta} = \lambda^{g, 1}_{\alpha\beta} = 1\), and \(\gamma^{1, g, h}_{\alpha} = \gamma^{g, 1, h}_{\alpha} = \gamma^{g, h, 1}_{\alpha} = 1\). This will lead to normalized 3-cocycles for the orientifold \(C\) field analogue of discrete torsion, in very close analogy with the \(B\) field case.

Proceeding as in [4], we consider the differences between group actions. Using tildes to denote different group actions, it is straightforward to check that

\[
\Upsilon^{g}_{\alpha\beta\gamma} = \frac{\nu^{g}_{\alpha\beta\gamma}}{\nu^{g}_{\alpha\beta\gamma}}
\]

define Čech cocycles defining a 1-gerbe, with connection defined by

\[
\mathcal{B}(g)^\alpha = \Lambda^{(2)}(g)^\alpha - \tilde{\Lambda}^{(2)}(g)^\alpha
\]

\[
\mathcal{A}(g)^{\alpha\beta} = \tilde{\Lambda}^{(1)}(g)^{\alpha\beta} - \Lambda^{(1)}(g)^{\alpha\beta}
\]

Furthermore, this connection is constrained to be flat: \(d\mathcal{B}(g)^\alpha = 0\). In addition, there are connection-preserving maps

\[
\Omega^{g_1, g_2} : (\Upsilon^{g_2})^\epsilon(g_1) \otimes g_2^*\Upsilon^{g_1} \rightarrow \Upsilon^{g_1, g_2}
\]

defined locally by

\[
\Omega^{g_1, g_2}_{\alpha\beta} = \frac{\lambda^{g_1, g_2}_{\alpha\beta}}{\lambda^{g_1, g_2}_{\alpha\beta}}
\]
Associated to the \( \Omega(g_1, g_2) \) are

\[
\theta(g_1, g_2)^\alpha \equiv \tilde{\Lambda}^{(3)}(g_1, g_2)^\alpha - \Lambda^{(3)}(g_1, g_2)^\alpha
\]

(In special cases when the \( \Omega(g_1, g_2) \) reduce to bundles, the \( \theta(g_1, g_2) \) reduce to connections on those bundles.) The coherence condition

\[
\left( \lambda_{\alpha\beta}^{g_1 g_2 g_3} \right) \left( g_3^* \lambda_{\alpha 3}^{g_1 2} \right) = \left( \lambda_{\alpha\beta}^{g_1 g_2 g_3} \right) \left( g_2^* \lambda_{\alpha\beta}^{g_2 3} \right) \epsilon(g_1) \left( \gamma_{\alpha}^{g_1 g_2 g_3} \right) \left( \gamma_{\beta}^{g_1 g_2 g_3} \right)^{-1}
\]

implies that the following diagram commutes:

\[
\begin{array}{ccc}
(\Upsilon^{g_3})^{(g_1, g_2)} \otimes g_3^* (\Upsilon^{g_2})^{(g_1)} \otimes g_2^* \Upsilon^{g_1} & \overset{g_3^* \Omega^{g_1, g_2}}{\longrightarrow} & (\Upsilon^{g_3})^{(g_1, g_2)} \otimes g_3^* \Upsilon^{g_1, g_2}
\end{array}
\]

up to isomorphisms

\[
\omega(g_1, g_2, g_3) : \Omega^{g_1 g_2 g_3} \circ g_3^* \Omega^{g_1, g_2} \sim \Omega^{g_1, g_2} \circ (\Omega^{g_2, g_3})^{(g_1)}
\]

These isomorphisms are defined locally by

\[
\omega^{g_1, g_2, g_3} = \frac{\gamma^{g_1 g_2 g_3}}{\gamma^{g_1 g_2 g_3}}
\]

and because of the identity

\[
(\gamma^{g_1 g_2 g_3, g_4}) (\gamma^{g_1 g_2, g_3, g_4}) = (\gamma^{g_1 g_2, g_3, g_4}) (\gamma^{g_2, g_3, g_4})\epsilon(g_1) (g_4^* \gamma^{g_1 g_2, g_3, g_4})
\]

themselves obey the higher coherence condition

\[
\omega(g_1, g_2, g_3, g_4) \circ \omega(g_1 g_2, g_3, g_4) = (\omega(g_2, g_3, g_4))^{(g_1)} \circ \omega(g_1, g_2 g_3, g_4) \circ g_4^* \omega(g_1, g_2, g_3)
\]

where both sides map

\[
\begin{array}{ccc}
\Omega^{g_1 g_2 g_3, g_4} \circ g_4^* \Omega^{g_1 g_2} \circ (g_3 g_4)^* \Omega^{g_1, g_2} & \sim \Omega^{g_1 g_2} \circ (g_2 g_3 g_4)^* \epsilon(g_1) \circ (\Omega^{g_1, g_4})^{(g_1)} & \rightarrow (g_2 g_3 g_4)^* \Upsilon^{g_1} \rightarrow \Upsilon^{g_1 g_2 g_3, g_4}
\end{array}
\]

In a similar fashion one obtains coherence conditions on \( \mathcal{B}(g)^\alpha, \theta(g_1, g_2)^\alpha \):

\[
\mathcal{B}(g_1 g_2)^\alpha = \epsilon(g_1) \mathcal{B}(g_2)^\alpha + g_2^* \mathcal{B}(g_1)^\alpha - d\theta(g_1, g_2)^\alpha
\]

\[
\epsilon(g_1) \theta(g_2, g_3)^\alpha + \theta(g_1, g_2 g_3)^\alpha
\]

\[
= g_3^* \theta(g_1, g_2)^\alpha + \theta(g_1 g_2, g_3)^\alpha - d \ln \omega^{g_1 g_2, g_3}
\]
As before, to recover the precise analogue of ordinary discrete torsion, we restrict to topologically-trivial 1-gerbes with gauge-trivial connection, so that the data above reduces to a set of flat line bundles $\Omega^{g_1,g_2}$ with connection-preserving isomorphisms $\omega_{g_1,g_2,g_3}$, and then further restrict to the case that those flat line bundles are all topologically-trivial with gauge-trivial connection. In general, there will be more degrees of freedom, generalizations of momentum/winding shift phases, but in this very special case, after suitable equivalences the data above reduces to a set of constant $U(1)$ elements $\omega(g_1,g_2,g_3)$ obeying the condition

$$\omega(g_1,g_2,g_3)\omega(g_1g_2,g_3,g_4) = (g_1 \cdot \omega(g_2,g_3,g_4))\omega(g_1,g_2g_3,g_4)\omega(g_1,g_2,g_3)$$

which is precisely the condition for a 3-cocyle in group cohomology, as reviewed in appendix A. Furthermore, there are residual constant gauge transformations $\kappa(g_1,g_2)$, arising from the fact that $\omega(g_1,g_2,g_3)$ maps

$$\Omega^{g_1g_2,g_3} \circ g_3^*\Omega^{g_1,g_2} \xrightarrow{\sim} \Omega^{g_1,g_2g_3} \circ (\Omega^{g_2,g_3})^{\kappa(g_1)}$$

which act as

$$\omega(g_1,g_2,g_3) \mapsto \omega(g_1,g_2,g_3)\kappa(g_1g_2,g_3)\kappa(g_1,g_2)\kappa(g_1,g_2g_3)^{-1}(g_1 \cdot \kappa(g_2,g_3))^{-1}$$

The reader will recognize this from appendix A as the action of coboundaries.

Thus, we see these degrees of freedom are counted by $H^3(G,U(1))$ with a nontrivial action on the coefficients, (realized physically via normalized 3-cocycles,) exactly as one would naively expect from our conclusions for $B$ fields.

Next, let us compute the phase factor for a nonorientable 3-manifold, built by identifying edges of a box as shown below:

where, in order for the cube to close, we assume

$$g_2g_3 = g_3g_2, \quad g_1g_3 = g_3g_1, \quad g_1 = g_2g_1g_2$$

The actions of $g_2, g_3$ preserve orientation, but $g_1$ flips orientation horizontally in the figure shown.
Let us now compute the holonomy, following the same procedure as in [4]. A first approximation is given by

$$\exp \left( i \int C \right) \exp \left( i \int_\varepsilon B(g_1) + i \int_\zeta B(g_2) + i \int_\eta B(g_3) \right)$$

As in [4], we must take into account the one-dimensional edges of the cube. For the most part, this analysis is identical to that in [4], except for the vertical edges in the figure above. Their contribution is determined by the relation

$$B(g_1) - g_2^{-1}B(g_1) + B(g_2) - g_2^{-1}B(g_2) = d \left[ \theta(g_2, g_1) - \theta(g_1, g_2^{-1}) - \theta(g_2, g_2^{-1}) \right]$$

(closely mirroring the two-dimensional Klein bottle computation earlier in this paper). With the modification above, the edges contribute a phase factor

$$\exp \left( \int_{g_2^{-1}}^g \left[ \theta(g_3, g_1) - \theta(g_1, g_3) \right] + i \int_{g_2^{-1}}^{g_1} \left[ \theta(g_3, g_2) - \theta(g_2, g_3) \right] \right)$$

Taking into account the corners, to make the phase factor gauge-invariant, it is straightforward to compute (following [4]) that one gets the final contribution

$$\frac{\omega(g_1, g_2^{-1}, g_3)\omega(g_2, g_3, g_1)\omega(g_3, g_1, g_2^{-1})}{\omega(g_2, g_1, g_3)\omega(g_1, g_3, g_2^{-1})\omega(g_3, g_2, g_1)}$$

When one restricts to the degrees of freedom counted by $H^3(G, U(1))$ (with a nontrivial action on the coefficients), the phase factor above reduces to its final factor

$$\frac{\omega(g_1, g_2^{-1}, g_3)\omega(g_2, g_3, g_1)\omega(g_3, g_1, g_2^{-1})}{\omega(g_2, g_1, g_3)\omega(g_1, g_3, g_2^{-1})\omega(g_3, g_2, g_1)}$$

It is straightforward to check that this is invariant under coboundaries, and so descends to group cohomology. Formally, the expression above can be obtained as the antisymmetrization of $\omega(g_1, g_2^{-1}, g_3)$, just as the expression for the oriented cube [3], with the difference that whenever a pair $g_1, g_2$ are exchanged, the $g_2$ becomes $g_2^{-1}$ and one picks up an additional cocycle factor in which the $g_1$ is replaced by $g_2$.

### 7 Conclusions

In this paper, we have reviewed how discrete torsion can be concretely understood in terms of group actions on $B$ fields, and generalized both discrete torsion (as well as momentum/winding phase factors and analogues for $C$ fields) to orientifolds. We have recovered
older results of [8] as well as derived some new results, including phase factors and $C$ field analogues.

There are a number of directions for further generalizations. One example involves the physical role of more general group cohomologies, with more general operations on coefficients. The original discrete torsion of [14] was classified by group cohomology with trivial action on the coefficients, and in orientifolds we have seen in this paper that one has group cohomology with a nontrivial (though still very special) action on the coefficients. It has sometimes been speculated that heterotic string orbifolds may contain more general examples of group cohomology. Discrete torsion in heterotic strings was briefly outlined in [5], where it was argued that if one holds fixed the group action on the gauge bundle and varies only the action on the $B$ field, one recovers ordinary discrete torsion. One might find generalizations arising from more general mixings.

Another general question involves the role of non-equivariantizable fluxes in orbifolds of flux vacua. There have been a number of papers in the literature over the last few years attempting to estimate numbers of distinct string vacua often obtained by various orbifolds of supergravity backgrounds with nontrivial fluxes. As we remarked earlier, invariance of the curvature under a group action does not guarantee the existence of an equivariant structure on the corresponding tensor potential, nor are such equivariant structures typically unique.

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A Group cohomology review

Group cohomology groups $H^*(G, U(1))$ are defined as follows (see e.g. [27][section III.1] for an exhaustive discussion). In degree $n$, one has cochains which are maps

$$\omega : G^n \longrightarrow U(1)$$

and coboundary operations

$$(\delta \omega)(g_1, \ldots, g_{n+1}) \equiv (g_1 \omega(g_2, \ldots, g_{n+1})) (\omega(g_1, g_2, g_3, \ldots, g_{n+1}))^{-1} \cdots (\omega(g_1, \cdots, g_n))^{(n+1)}$$

In this paper, we usually work with “normalized” cochains, in which $\omega(g_1, \cdots, g_n) = 1$ if any of the $g_i = 1$. These yield the same group cohomology [27][section III.1], and are more
convenient for orientifold discussions. The group cohomology group $H^n(G, U(1))$ is then the group of degree $n$ cocycles (cochains annihilated by $\delta$) modulo the degree $n$ coboundaries (cochains in the image of $\delta$).

Note that there is an action of the group on the coefficients $U(1)$ implicit in the definition above, in the first term in the action of $\delta$. In the group cohomology used in ordinary discrete torsion in [14], this action is trivial, and so the coboundary operator acts as

$$(\delta \omega)(g_1, \cdots, g_{n+1}) \equiv (\omega(g_2, \cdots, g_{n+1})) (\omega(g_1g_2, g_3, \cdots, g_{n+1}))^{-1} \cdots (\omega(g_1, \cdots, g_n))^{(-)^{n+1}}$$

In more general cases, however, the group action is nontrivial.

For example, for a nontrivial group action, degree-2 group cohomology is defined by functions $\omega : G \times G \to U(1)$ such that

$$(g_1 \cdot \omega(g_2, g_3)) \omega(g_1, g_2g_3) = \omega(g_1g_2, g_3) \omega(g_1, g_2)$$

modulo multiplication of functions of the form

$$\frac{(g_1 \cdot f(g_2)) f(g_1)}{f(g_1g_2)}$$

Similarly, degree-3 group cohomology is defined by functions $\omega : G \times G \times G \to U(1)$ such that

$$(g_1 \cdot \omega(g_2, g_3, g_4)) \omega(g_1, g_2g_3g_4) \omega(g_1, g_2, g_3) = \omega(g_1g_2, g_3, g_4) \omega(g_1, g_2, g_3g_4)$$

modulo multiplication of functions of the form

$$\frac{(g_1 \cdot f(g_2, g_3)) f(g_1, g_2g_3)}{f(g_1g_2, g_3) f(g_1, g_2)}$$

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