Boson Realization of the SU(4) Model of High-Temperature Superconductivity

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The SU(4) algebraic model of high-temperature superconductivity is studied employing the boson mapping techniques. The bosonization of the model enables us to get exact numerical solution of the model. The order parameters are discussed. The situation close to the SO(5) dynamical symmetry limit is interpreted as a modelling case for the behaviour of D-wave superconducting and antiferromagnetic phases in cuprates.

The high-temperature superconductivity represents a challenging task for the theory of many-electron correlated systems. Data suggest that it is related to the singlet D-wave pairing (dSC) phase. One of the intriguing aspects is then a proximity of the dSC phase to the antiferromagnetic (AF) phase. Indeed, in cuprates, the AF order is observed at half filling whereas the dSC order develops as the material is doped by holes.

The interplay between the dSC and AF phases is treated by employing the symmetry principles in the SO(5) model proposed by S.-C. Zhang [1]. Two dSC order parameters and three AF order parameters are unified in a five dimensional vector. The SO(5) rotations induce then transitions between two phases. Microscopic models exhibiting the SO(5) symmetry have also been discussed [2,3].

Recently, Guidry and collaborators have embedded the SO(5) realization of the SU(4) model proposed by S.-C. Zhang [1]. Two dSC or- dination constant, charge, and quadratic spin terms is written as [4,5]

\[ g(k) = g_0(k) = g_{ij}(k) = g(k_x, k_y) \]

with properties \( g_{ij}(k+Q) = g_{ij}(k) \) and \( |g_{ij}(k)| = 1 \). [4,5]

There are three dynamical symmetry chains conserving total spin \( S \) and particle number \( n \) (or charge \( M = \frac{1}{2}(n - \Omega) \)) discussed in [3,6]:

i) the SO(4) limit comprising a subset of operators of staggered magnetization \( \sigma_\mu \), and spin \( S_\mu \),

ii) the SO(5) limit with the triplet \( \pi \)-wave creation and annihilation operators \( \pi_\mu^+ \) and \( \pi_\mu^- \), and \( S_\mu \),

iii) the SU(2) limit with the singlet \( D \)-wave creation and annihilation operators \( D_\mu^+ \) and \( D_\mu^- \) (and \( S_\mu \)).

One can move between the dynamical symmetry limits with a Hamiltonian

\[ H_1 = -\frac{G_1}{2} \left( (1 + x)D_\mu^+ D_\mu^- - (1 - x)\pi_\mu^+ \cdot \pi_\mu^- \right) \]

with \( x = 1 \) for the SU(2) limit, \( x = -1 \) for the SO(5) limit, and \( x = 0 \) for the SO(4) limit. The SU(4) quadratic Casimir operator

\[ C_{SU(4)} = D_\mu^+ D_\mu^- + \pi_\mu^+ \cdot \pi_\mu^- + Q \cdot Q + S \cdot S + M(M - 4) \]

gives for the most symmetric collective SU(4) subspace (which is only discussed in the present paper) the value \( \frac{1}{4} \Omega (\Omega + 8) \). Then, an alternative parameterization of the SU(4) Hamiltonian up to uninteresting for the present discussion constant, charge, and quadratic spin terms is written as [6,7]

\[ H_{II} = -G_{II}(1 - p)D_\mu^+ D_\mu^- + pQ \cdot Q \]

with \( p = 0 \) for the SU(2) limit, \( p = 0.5 \) for the SO(5) limit, and \( p = 1 \) for the SO(4) limit.
Even though the Hamiltonians (1) and (2) must generally be equivalent, the assumed range of the parameters $G_1, G_{II} \geq 0$, 1 $\geq x \geq -1$, and 1 $\geq p \geq 0$ leads to different physical picture. For the SU(2) limit, both parameterization agree. For the SO(4) and SO(5) limits and in-between, however, the positive values of the interaction strengths $G_1$ and $G_{II}$ imply different signs of the Hamiltonians (2) and (3). The spectra of two parameterization are reversed with the ground state of (2) being the highest-lying state of (3) and vice versa.

For the dynamical symmetry limits, the SU(4) Hamiltonian is analytically solvable. Outside the limits, the numerical diagonalization must be performed. In both these tasks, an application of the boson mapping techniques proves to be useful. The Dyson boson realization of the SU(4) algebra is constructed by mapping the bifermion operators onto bosonic operators formed from the scalar boson operators $d$ and vector boson operators $p$.

\begin{equation}
\begin{aligned}
D^\dagger &\to (\Omega + 2 - n) d^\dagger + (d^\dagger d) p^\dagger p d \\
D &\to d \\
\pi_{\mu}^\dagger &\to (\Omega + 2 - n) p_\mu^\dagger + (d^\dagger d - p^\dagger p) \tilde{p}_\mu \\
Q_{\mu} &\to p_\mu d - d^\dagger \tilde{p}_\mu \\
S_{\mu} &\to \sqrt{2}[p_\mu^\dagger \tilde{p}_\mu^\dagger] \\
n &\to 2(d^\dagger d - p^\dagger p) 
\end{aligned}
\end{equation}

(4)

By employing (4), one can easily construct the boson image of the SU(4) Hamiltonian. The bosonized task is relatively simple to solve numerically in the boson space SU(1)$_d$$\otimes$SU(3)$_p$. To obtain the spectrum of $S = 0$ states including the ground state, it is sufficient to diagonalize a matrix of dimension $n/2$. That represents an enormous truncation of the original fermion space.

In the boson treatment, we obtain the exact and full solution of the fermion problem. The eigenstates states have good charge and spin quantum numbers which reflect the symmetry of the original task. For them, we relate the dSC order parameter to the matrix element of the $D$-pair transfer between the ground states

$$
\alpha_0 = \langle n + 2 | D^\dagger | n \rangle,
$$

whereas the AF order parameter is connected to the reduced matrix element of the $Q$ operator between the $S=0$ ground state and the lowest lying $S=1$ state

$$
\beta_0 = \langle nS = 1 | Q | nS = 0 \rangle.
$$

Even if the $D^\dagger$ and $Q$ order parameters do not always belong to the generators of the particular dynamical symmetry limit chains, we have succeeded to obtain the analytical formulas by inspecting the numerical results. We have got

$$
\begin{align*}
\text{SU(2) parameterizations (2) and (3)} \\
\alpha_0 &= \frac{1}{4}(n + 2)(2\Omega - n) \\
\beta_0 &= \frac{3}{4}\Omega - n \\
\text{SO(4) parameterization (3)} \\
\alpha_0 &= \frac{1}{4}\Omega - n(n + 4) \\
\beta_0 &= \frac{1}{4}(n + 4).
\end{align*}
$$

(5)

The results (4) and (6) for the SO(5) and SO(4) limits of the parameterization (3) are remarkable. In the SO(4) limit, the AF order parameter $\beta_0$ increases linearly with $n$ up to the value $\Omega/2$ for $n = \Omega$. The dSC order parameter $\alpha_0$ goes to zero when $n \to 0$ and $n \to \Omega$, and reaches the maximum value $\Omega/2$ at the quarter filling $n = \Omega/2$.

In the SO(5) limit, the order parameters for $n \ll \Omega$ agree with the values of the SC case (5) since then the SO(5) ground state of the parameterization (3) has the form of the D-pair condensate $|g.s.\rangle \propto D^\dagger \frac{1}{2} |0\rangle$. The situation changes for $n \to \Omega$ when a steep increase appears in the AF order parameter whereas the SC order parameter goes to zero. One may get more insight into this phase transition by introducing the fractional doping of holes $\delta = 1 - n/\Omega$. Then for $\Omega \to \infty$, Eqs.(6) are rewritten as

$$
\begin{align*}
\alpha_0 / \Omega &= \frac{1}{2} \left[ \frac{\Omega\delta}{\Omega\delta + 3} (1 - \delta^2) \right]^{\frac{1}{2}} \\
\beta_0 / \Omega &= \frac{1}{2} \left[ \frac{3}{\Omega\delta + 5} (1 - \delta^2) \right]^{\frac{1}{2}},
\end{align*}
$$

from which form the difference between the half filled case ($\delta = 0$) and the hole doped situation ($\delta > 0$) is easily seen.
FIG. 1. AF order parameter (bottom panel) and dSC order parameter (top panel) as a function of the fractional doping of holes $\delta$. Degeneracy of the lattice $\Omega=2000$. Curves are shown for the values of the parameter $p$ in (3) $p=1$ (SO(4) limit), $p=0.5$ (SO(5) limit) and two intermediate cases $p=0.52$ and $p=0.75$. Outside the dynamical symmetry limits, the order parameters are close to the SO(5) values for $\delta \to 1$ and to the SU(4) values for $\delta \to 0$. The results for the AF order parameter in the case $p=0.52$ are very much similar to those obtained in Ref. [5] with the coherent state approach.

In Fig. 1, the order parameters are shown as obtained with Hamiltonian (3) for $\Omega = 2000$. Besides the dynamical symmetry limits SO(5) ($p=0.5$) and SO(4) ($p=1$), two calculations are given for the values of the parameter $p=0.52$ and 0.75. Outside the dynamical symmetry limits, the order parameters are close to the SO(5) values for $\delta \to 1$ and to the SU(4) values for $\delta \to 0$. The results for the AF order parameter in the case $p=0.52$ are very much similar to those obtained in Ref. [5] with the coherent state approach.

We thus confirm the findings of Refs. [3] and [4] that the SU(4) algebraic model in a situation close to its SO(5) dynamical symmetry limit is a convenient tool to model behaviour of high-temperature-superconducting cuprates. The transition from the AF phase to the dSC phase naturally appears as the system is doped by holes starting from the half-filled case.

Attractive feature of the SU(4) algebraic model is its exact solvability. Then despite the fact that some portion of the real physics might not be present in the model (there is no kinetic-energy term in the SU(4) Hamiltonian), it can still be useful in testing approximate many-body procedures. For the dynamical symmetry limits we have obtained closed expressions from which behaviour of the order parameters in the phase transition region is seen explicitly.

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[5] L.-A. Wu, M. Guidry, Y. Sun, and C.-L. Wu, preprint cond-mat/9905436 in electronic xxx.lanl.gov basis.
[6] Our $\pi^\dagger$ operator differs by a multiplicative factor i from $\pi^\dagger$ of Ref. [4].
[7] Note, that the SU(4) algebra can be embedded into the SO(8) algebra containing the Cooper $S$-pair ($S^\dagger = \sum_k c^\dagger_k c^\dagger_{-k}$) and quasi-spin $\eta$-pair operators ($\eta^\dagger = \sum_k c^\dagger_{k+Q} c^\dagger_{-k-Q}$), besides the $D$-pair and $\pi$-pair operators.
[8] The spurious (nonphysical) states arise in the boson space for $n > \Omega$ which is outside the focus of our analysis. The Dyson mapping (4) is nonunitary leading to nonhermitian boson Hamiltonians. Standard procedures exist how to treat the original hermitian fermion problem in the nonunitary boson method (see for example A. Klein and E.R. Marshalek, Rev. Mod. Phys. 63, 375 (1991)).
[9] Of course, an alternative methods for the numerical solution may employ the algebraic properties of the fermionic SU(4) algebra and any of its dynamical symmetry chains.
[10] In the dynamical symmetry limits both in the parameterizations (3) and (4), the first excited $S=1$ state exhausts fully the $Q$-strength from the ground state. Similarly, the $D$-pair transfer from the ground state goes fully or almost fully to the ground state of the adjacent system. An exception is the SO(5) limit in the parameterization (4) for $n = 4k + 2$ case.