Consistency of Lipschitz learning with infinite unlabeled data and finite labeled data*

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Abstract

We prove that Lipschitz learning on graphs is consistent with the absolutely minimal Lipschitz extension problem in the limit of infinite unlabeled data and finite labeled data. In particular, we show that the continuum limit is independent of the distribution of the unlabeled data, which suggests the algorithm is fully supervised (and not semi-supervised) in this setting. We also present some new ideas for modifying Lipschitz learning to incorporate the distribution of the unlabeled data.

1 Introduction

The problem of graph-based semi-supervised learning has drawn attention recently, due to the general availability of unlabeled data and the cost of labeling data [4]. In many problems, such as website classification or medical image analysis, an expert is required to label data, and this may be costly. On the other hand, the cost associated with acquiring unlabeled data is generally much smaller, and there is often no limit to the amount of unlabeled data available. This has led to a resurgence in algorithms that use both labeled and unlabeled data (i.e., semi-supervised algorithms) with the goal of improving the performance of fully supervised algorithms.

Let us describe in general the problem of graph-based semi-supervised learning. Let \( G = (X, W) \) be a weighted graph with vertices \( X \) and nonnegative edge weights \( W = \{w(x, y)\}_{x,y \in X} \). Assume we are given a label function \( g : O \to \mathbb{R} \) where \( O \subset X \) are the labeled vertices. The graph-based semi-supervised learning problem is to extend the labels from \( O \) to the remaining vertices of the graph \( X \setminus O \). The problem is not well-posed as stated, since there is no unique way to extend the labels. One generally makes the semi-supervised smoothness assumption, which says that the learned labels must very smoothly through dense regions of the graph.

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There are many ways to impose the semi-supervised smoothness assumption. We study the problem of Lipschitz learning \[8,9\], which solves the graph-based semi-supervised learning problem by seeking solutions of

\[
\begin{aligned}
\min_{u: X \rightarrow \mathbb{R}} \max_{x, y \in X} \{w(x, y)|u(x) - u(y)|\}
\text{ subject to } u(x) = g(x) \text{ for all } x \in \mathcal{O}.
\end{aligned}
\]

Thus, we are looking for a function \(u: \mathcal{X} \rightarrow \mathbb{R}\) that agrees with the known labels and whose gradient is minimal in some sense. We note that the minimizer in (1) is not unique, even when the graph is connected. Therefore, we interpret (1) as seeking the minimizer whose gradient \(\{w(x, y)|u(x) - u(y)|\}_{x, y \in \mathcal{X}}\) is smallest in the lexicographic ordering [8]. This is equivalent to seeking the absolutely minimal Lipschitz extension of the function \(g\), which in the continuous setting has a long history in analysis [2] (see [12] for the vector-valued version).

We consider here the case where the fraction of labeled points is vanishingly small, that is, we take the limit of infinite unlabeled data and finite labeled data. We show that Lipschitz learning is well-posed in this limit, and converges to an infinity harmonic function, which corresponds to the absolutely minimal Lipschitz extension in the ambient continuous space. Our result has very little dependence on the model chosen for the graph. We simply need that \(\mathcal{X} \subset \mathbb{R}^d\) and that the weights depend on the distance between points as in (4). In particular, our proof does not use probability, and the consistency result does not depend on the distribution of the unlabeled data. This work verifies a conjecture from [6], and shows that Lipschitz learning is in a sense a fully supervised learning algorithm.

It has recently been observed [6,11] that the commonly used Laplacian regularization [14] is ill-posed in the limit of infinite unlabeled and finite labeled data (the label function \(u\) degenerates into a constant label). Our work shows that Lipschitz learning does not suffer from the same ill-posedness, suggesting that Lipschitz learning may be superior to Laplacian regularization in some settings. However, this improvement is at the expense of losing all dependence on the distribution of the unlabeled data.

2 Main result

We work on the flat Torus \(\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d\), that is, we take periodic boundary conditions. For each \(n \in \mathbb{N}\) let \(X_n \subset \mathbb{T}^d\) be a collection of \(n\) points. Let \(\mathcal{O} \subset \mathbb{T}^d\) be a fixed finite collection of points and

\[
\mathcal{X}_n := X_n \cup \mathcal{O}.
\]

The points \(\mathcal{X}_n\) will form the vertices of our graph. To select the edge weights, let \(\Phi: [0, \infty) \rightarrow [0, \infty)\) be a \(C^2\) function satisfying

\[
\begin{aligned}
\Phi(s) \geq 1, & \quad \text{if } s \in (0, 1) \\
\Phi(s) = 0, & \quad \text{if } s \geq 2.
\end{aligned}
\]
Select a length scale $h_n > 0$ and define

$$w_n(x, y) := \Phi \left( \frac{|x - y|}{h_n} \right).$$

Let $\mathcal{W}_n = \{w_n(x, y)\}_{x, y \in X_n}$ and let $\mathcal{G}_n = (X_n, \mathcal{W}_n)$ be the graph with vertices $X_n$ and edge weights $\mathcal{W}_n$. Let $g : \mathcal{O} \to \mathbb{R}$ and let $u_n : X_n \to \mathbb{R}$ be the solution of the Lipschitz learning problem (1) (we prove existence and uniqueness of the solution $u_n$ in Section 3). The function $u_n$ satisfies the optimality conditions (see [8])

$$\begin{cases}
    L_n u_n = 0 & \text{in } X_n \\
    u_n = g & \text{in } \mathcal{O},
\end{cases}$$

where $L_n$ is the graph $\infty$-Laplacian defined by

$$L_n u(x) := \max_{y \in X_n} w_n(x, y)(u(y) - u(x)) + \min_{y \in X_n} w_n(x, y)(u(y) - u(x)).$$

We also define

$$r_n = \sup \left\{ s > 0 | B(x, s) \cap X_n = \emptyset \text{ for some } x \in \mathbb{T}^d \right\}.$$

This is the radius of the largest ball whose interior contains no points from $X_n$. We note that the graph $\mathcal{G}_n$ is connected whenever $r_n < h_n/4\sqrt{d}$.

We also recall the $\infty$-Laplace operator is defined as

$$\Delta_{\infty} u(x) := \frac{1}{|\nabla u(x)|^2} \sum_{i,j=1}^d u_{x_i x_j}(x)u_{x_i x_j}(x)$$

provided $\nabla u(x) \neq 0$.

Our main result is the following theorem.

**Theorem 1.** Suppose that $h_n \to 0$ and $r_n \to 0$ as $n \to \infty$ so that

$$\lim_{n \to \infty} \frac{r_n^2}{h_n^3} = 0.$$ 

Then

$$u_n \to u \text{ uniformly on } \mathbb{T}^d \text{ as } n \to \infty,$$

where $u \in C^{0,1}(\mathbb{T}^d)$ is the unique viscosity solution of the $\infty$-Laplace equation

$$\begin{cases}
    \Delta_{\infty} u = 0 & \text{in } \mathbb{T}^d \setminus \mathcal{O} \\
    u = g & \text{on } \mathcal{O}.
\end{cases}$$
The theorem shows that Lipschitz learning is well-posed and consistent in the limit of infinite unlabeled data and finite labeled data. We note that the viscosity solution of (10) is in general only Lipschitz continuous, and is not a classical \( C^2 \) solution. The notion of viscosity solution is based on the maximum principle, and is the natural notion of weak solution for nonlinear elliptic equations. Viscosity solutions are only required to be continuous functions, and satisfy the partial differential equation in a weak sense. We define viscosity solution for (10) in Section 5. For more details on viscosity solutions, we refer the reader to the user’s guide [5].

**Remark 1.** Notice that Theorem 1 makes no assumptions on the distribution of the unlabeled data \( X_n \). The unlabeled data may be deterministic or random. In the case the data is random, the samples need not be independent. All that is required is that (8) holds. We can specialize Theorem 1 to the case where \( X_n = \{Y_1, Y_2, Y_3, \ldots, Y_n\} \) is a sequence of independent and identically distributed random variables with a strictly positive probability density function. In this case it is possible to show that there exists \( C > 0 \) such that

\[
\limsup_{n \to \infty} \frac{r_n n^{1/d}}{\log(n)^{1/d}} \leq C \quad \text{a.s.}
\]

It follows that (8) holds almost surely provided

\[
\lim_{n \to \infty} \frac{nh_n^{3d/2}}{\log(n)} = \infty.
\]

The reader should contrast this with the requirement that

\[
\lim_{n \to \infty} \frac{nh_n^d}{\log(n)} = \infty \quad \text{or} \quad \lim_{n \to \infty} \frac{nh_n^{d+2}}{\log(n)} = \infty
\]

for the consistency results in Laplacian based regularization [13, 7]. The reason for the difference is that for the graph Laplacian one needs to control the fluctuations in a sum of random variables, while here we are dealing with the maximum of a collection of random variables.

**Remark 2.** We note that the requirement \( \Phi \in C^2 \) is necessary; it is used in the proof of Lemma 1. If \( \Phi \in C^1 \) then the proof of Theorem 1 can be modified by using (21) in place of (20) from Lemma 1. The only difference is that the condition (8) must be replaced with

\[
\lim_{n \to \infty} \frac{r_n n^{1/d}}{h_n^2} = 0.
\]

Remark 1 remains true provided (11) is replaced by

\[
\lim_{n \to \infty} \frac{nh_n^{3d}}{\log(n)} = \infty.
\]
Remark 3. If instead of working on the Torus $\mathbb{T}^d$, we take our unlabeled points to be sampled from a domain $X_n \subset \Omega \subset \mathbb{R}^d$, then Theorem 1 holds under similar hypotheses. The only difference is that the infinity Laplace equation has an additional boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega.$$ 

Working on the Torus allows us to avoid this Neumann boundary condition, which would add additional details in the proof.

The rest of the paper is organized as follows. In Section 2.1 we discuss some new ideas for modifying Lipschitz learning so that the algorithm utilizes the distribution of the unlabeled data. In Section 3 we discuss the maximum principle for the graph $\infty$-Laplacian and prove existence and uniqueness of solutions to (5). The proof of Theorem 1 is contained in Sections 4 and 5. In Section 4 we prove consistency of the graph $\infty$-Laplacian for smooth functions, and in Section 5 we give the proof of Theorem 1.

2.1 Discussion

Since the continuum limit of Lipschitz learning is independent of the distribution of the unlabeled data, Lipschitz learning is fully supervised, and not semi-supervised, in this limit. It is natural to ask whether we can modify Lipschitz learning to incorporate a dependence on the distribution of the unlabeled data. One natural idea is to replace the Lipschitz learning problem (1) by

$$\min_{u: \mathcal{X} \rightarrow \mathbb{R}} \max_{x,y \in \mathcal{X}} \{d_x d_y w(x,y) | u(x) - u(y) | \} \quad \text{subject to } u(x) = g(x) \text{ for all } x \in \mathcal{O},$$

where $d_x$ is the degree of vertex $x \in \mathcal{X}$ given by

$$d_x = \sum_{y \in \mathcal{X}} w(x,y).$$

For a random geometric graph, the degree asymptotics are given by

$$d_x \sim C n h_n^d f(x),$$

where $f(x)$ is the probability density function of the data. For a $k$-nearest neighbors graph, where all the degrees are identical, we could instead consider the inverse of the distance to the $k^{th}$-nearest neighbor. This model forces the gradient to be small in dense regions of the graph, while allowing the gradient to be large in sparse regions.

We leave experiments with this new model to future work. Let us briefly say how we expect the probability density to appear in the continuum limit. At the continuum level,
the problem amounts to minimizing \( f^2 |\nabla u| \), which is a weighted version of the absolutely minimal Lipschitz extension problem. Consider approximating this through the limit

\[
\lim_{p \to \infty} I_p(u) := \left( \int_{\Omega} f^{2p} |\nabla u|^p \, dx \right)^{1/p}.
\]

For a finite value of \( p \), the Euler-Lagrange equation for the problem of minimizing \( I_p \) is

\[
\text{div} \left( f^{2p} |\nabla u|^{p-2} \nabla u \right) = 0.
\]

Expanding the divergence term we have

\[
f^{2p-2} |\nabla u|^{p-2} (f^2 \Delta u + p f^2 \cdot \nabla u + (p - 2) f^2 \Delta_x u) = 0.
\]

If we divide by \( p \) and send \( p \to \infty \) then we find that

\[
(13) \quad \nabla f^2 \cdot \nabla u + f^2 \Delta_x u = 0.
\]

This is expected to be the continuum limit for (12). Notice the dependence on the density function \( f \). The term \( \Delta_x u \) is a nonlinear diffusion term, and \( \nabla f^2 \cdot \nabla u \) is an advection (or drift) term. We can think of the advection term as acting to propagate labels along the gradient of the distribution, from dense to sparse regions of the graph.

We expect that the proof of Theorem 1 can be adapted to prove that (13) is the continuum limit of (12). However, the proof will involve probability due to the appearance of the degree \( d_x \), so some parts will be substantially different. We leave this verification to future work.

### 3 The maximum principle

In this section we show that (5) is well-posed and establish a priori estimates on the solution \( u_n \). The proof relies on the maximum principle on a graph, which we review below.

We first introduce some notation. We say that \( y \) is adjacent to \( x \) whenever \( w_n(x, y) > 0 \). We say that the graph \( \mathcal{G}_n = (\mathcal{X}_n, \mathcal{W}_n) \) is connected to \( \mathcal{O} \subset \mathcal{X}_n \) if for every \( x \in \mathcal{X}_n \setminus \mathcal{O} \) there exists \( y \in \mathcal{O} \) and a path from \( x \) to \( y \) consisting of adjacent vertices.

We now present the maximum principle for the graph infinity Laplace equation.

**Theorem 2** (Maximum principle). Assume the graph \( \mathcal{G}_n = (\mathcal{X}_n, \mathcal{W}_n) \) is connected to \( \mathcal{O} \subset \mathcal{X}_n \). Let \( u, v : \mathcal{X}_n \to \mathbb{R} \) satisfy

\[
L_n u(x) \geq 0 \geq L_n v(x).
\]

Then

\[
(14) \quad \max_{\mathcal{X}_n} (u - v) = \max_{\mathcal{O}} (u - v).
\]
The proof of Theorem 2 in a similar setting was proved in [10]. We include a simple proof here for completeness.

**Proof.** Define
\[ M = \{ x \in \mathcal{X}_n : u(x) - v(x) = \max_{\mathcal{X}_n}(u - v) \}. \]

If \( M \cap \mathcal{O} \neq \emptyset \) then we are done, so we may assume that \( M \cap \mathcal{O} = \emptyset \). Let \( x \in M \). Then we have
\[ u(x) - u(y) \geq v(x) - v(y) \quad \text{for all} \quad y \in \mathcal{X}_n. \]

It follows that \( L_n u(x) \leq L_n v(x) \). The opposite inequality is true by hypothesis, and hence \( L_n u(x) = 0 = L_n v(x) \) whenever \( x \in M \). This implies that
\[
M = \{ x \in \mathcal{X}_n : u(x) - v(x) = \max_{\mathcal{X}_n}(u - v) \}.
\]

Define
\[
A := \max_{y \in \mathcal{X}_n} w_n(x,y)(u(x) - u(y)) = \max_{y \in \mathcal{X}_n} w_n(x,y)(v(x) - v(y)),
\]
\[
B := \min_{y \in \mathcal{X}_n} w_n(x,y)(u(x) - u(y)) = \min_{y \in \mathcal{X}_n} w_n(x,y)(v(x) - v(y)),
\]
and \( A + B = -L_n u(x) = -L_n v(x) = 0 \). We now have two cases.

1. If \( A > 0 \) and \( B < 0 \) then there exists \( z \in \mathcal{X}_n \) such that
\[
w_n(x,z)(u(x) - u(z)) = \min_{y \in \mathcal{X}_n} w_n(x,y)(u(x) - u(y)) < 0.
\]

Therefore \( u(z) > u(x) \) and
\[
u(z) - v(z) = -(u(x) - u(z)) + v(x) - v(z) + u(x) - v(x) \geq u(x) - v(x).
\]

It follows that
\[
u(z) - u(z) = u(x) - v(x), \quad u(z) > u(x), \quad \text{and} \quad v(z) > v(x).
\]

2. If \( A = B = 0 \) then \( u(x) = u(y) \) and \( v(x) = v(y) \) for all \( y \) adjacent to \( x \), and so
\[
u(y) - v(y) = u(x) - v(x)
\]

for all \( y \) adjacent to \( x \).

Let \( Q_1 \subset M \) be the collection of points for which case 1 holds, and let \( Q_2 \subset M \) be the points for which case 2 holds. We construct a path in \( M \) inductively as follows. Let \( x_0 \in M \) and suppose we have chosen \( x_0, \ldots, x_k \). If \( x_k \in Q_1 \), we choose \( x_{k+1} = z \) as in case 1 above. If \( x_k \in Q_2 \), then we find a path \( x_k = y_1, \ldots, y_\ell \) from \( x_k \) to \( y_\ell \in \mathcal{O} \). Let
\[
j = \max \{ i : y_q \in Q_2 \text{ for all } 1 \leq q \leq i \}.
\]

Since \( y_j \in Q_2 \), case 2 holds and so we have \( y_{j+1} \in M \). Therefore \( j + 1 \leq \ell - 1 \), \( y_{j+1} \not\in \mathcal{O} \), and \( y_j+1 \in Q_1 \). Choose \( x_{k+1} = z \) as in case 1. We terminate the construction when \( x_{k+1} \in \mathcal{O} \).
This constructs a path $x_0, x_1, \ldots, x_k, \ldots$ belonging to $M$ such that $u - v$ is constant along the path, and $u$ is strictly increasing, i.e.,

$$u(x_0) < u(x_1) < \cdots < u(x_k) < \cdots.$$ 

Therefore, the path cannot revisit any point, and must eventually terminate at some $x_T \in \mathcal{O}$. Since $u - v$ is constant along the path, we have

$$\max_{\mathcal{X}_n} (u - v) = u(x_0) - v(x_0) = u(x_T) - v(x_T) \leq \max_{\mathcal{O}} (u - v),$$

which completes the proof.

\begin{proof}
\end{proof}

**Corollary 1.** Assume the graph $\mathcal{G}_n = (\mathcal{X}_n, \mathcal{W}_n)$ is connected to $\mathcal{O}$. Let $u, v : \mathcal{X}_n \to \mathbb{R}$ satisfy

$$L_n v(x) = 0 = L_n u(x) \quad \text{for all} \quad x \in \mathcal{X}_n.$$ 

Then

$$\max_{\mathcal{X}_n} |u - v| = \max_{\mathcal{O}} |u - v|.$$ 

**Remark 4.** Corollary 1 shows that (5) has at most one solution, and the solution is stable under perturbations in the boundary conditions.

Existence of a solution to (5) was proved in [8, 12] as the absolutely minimal Lipschitz extension on the graph. We record this in the following theorem.

**Theorem 3.** Assume the graph $\mathcal{G}_n = (\mathcal{X}_n, \mathcal{W}_n)$ is connected to $\mathcal{O}$. Then there exists a unique solution $u_n : \mathcal{X}_n \to \mathbb{R}$ of (5). Furthermore, there exists a constant $C > 0$ depending only on $\mathcal{O}$ and $g$ such that

$$\min_{\mathcal{O}} g \leq u \leq \max_{\mathcal{O}} g,$$

and

$$\max_{x, y \in \mathcal{X}_n} w_n(x, y) |u_n(x) - u_n(y)| \leq C h_n.$$ 

**Proof.** Existence of a solution follows from [8, 12] and uniqueness follows from Corollary 1. The \textit{a priori} estimate (17) follows from Theorem 2. All that is left to prove is (18). Let $\varphi \in C^1(\mathbb{T}^d)$ such that $\varphi(x) = g(x)$ for all $x \in \mathcal{O}$. Since $u_n$ is the absolutely minimal Lipschitz extension of $g$ to the graph $\mathcal{G}_n$, we have that

$$\max_{x, y \in \mathcal{X}_n} w_n(x, y) |u_n(x) - u_n(y)| \leq \max_{x, y \in \mathcal{X}_n} w_n(x, y) |\varphi(x) - \varphi(y)| \leq C \|
abla \varphi\|_{L^\infty(\mathbb{T}^d)} h_n.$$
4 Consistency for smooth functions

In this section we prove consistency for the graph infinity Laplacian for smooth functions. Even though the viscosity solution of the infinity Laplace equation (10) is not smooth, the viscosity solution framework allows for checking consistency only with smooth functions.

We define the operator

\[ H_n u(x) := \max_{y \in \mathbb{T}^d} w_n(x, y)(u(y) - u(x)) + \min_{y \in \mathbb{T}^d} w_n(x, y)(u(y) - u(x)). \]

The proof of consistency is split into two steps. First, in Lemma 1 we show that \( L_n \) can be approximated by \( H_n \). Then, in Lemma 2 we prove that \( H_n \) is consistent with the infinity Laplace operator \( \Delta_\infty \) in the limit as \( n \to \infty \).

Lemma 1. Let \( \varphi \in C^2(\mathbb{R}^d) \). Then

\[ |L_n \varphi(x) - H_n \varphi(x)| \leq C (\|\nabla \varphi\|_\infty + h_n \|\nabla^2 \varphi\|_\infty) r_n^2 h_n^{-1}, \]

and

\[ |L_n \varphi(x) - H_n \varphi(x)| \leq C \|\nabla \varphi\|_\infty r_n h_n^{-1}. \]

Proof. Let \( h = h_n \), set

\[ \psi(y) = \Phi \left( \frac{|x - y|}{h} \right) (\varphi(y) - \varphi(x)), \]

and note that

\[ \|\nabla^2 \psi\|_\infty \leq C \left( \frac{1}{h} \|\nabla \varphi\|_\infty + \|\nabla^2 \varphi\|_\infty \right). \]

Let \( y_0 \neq x \) be a point in \( \mathbb{T}^d \) at which \( \psi \) attains its maximum value. Since \( \psi \) is \( C^2 \) and \( \nabla \psi(y_0) = 0 \) we have

\[ \psi(y) \geq \psi(y_0) - \frac{1}{2} \|\nabla^2 \psi\|_\infty |y - y_0|^2 \]

for all \( y \). Set \( t = \psi(y_0) - \max_{\mathcal{X}_n} \psi \). Then for any \( y \in \mathcal{X}_n \) we have

\[ \frac{1}{2} \|\nabla^2 \psi\|_\infty |y - y_0|^2 \geq \psi(y_0) - \psi(y) \geq t. \]

It follows that

\[ r_n^2 \geq \frac{2t}{\|\nabla^2 \psi\|_\infty}, \]

and hence

\[ \max_{\mathbb{T}^d} \psi - \max_{\mathcal{X}_n} \psi \leq \frac{1}{2} \|\nabla^2 \psi\|_\infty r_n^2. \]
A similar argument shows that
\[ \min_{x_n} \psi - \min_{T^d} \psi \leq \frac{1}{2} \| \nabla^2 \psi \|_\infty r^2. \]

The proof of (20) is completed by noting that
\[ L_n \varphi(x) - H_n \varphi(x) = \min_{x_n} \psi - \min_{T^d} \psi - (\max_{x_n} \psi - \max_{T^d} \psi). \]

The proof of (21) is similar.

**Lemma 2.** Let \( \varphi \in C^3(\mathbb{R}^d) \) and \( x_0 \in T^d \). If \( \nabla \varphi(x_0) \neq 0 \) and \( \Delta \varphi(x_0) < 0 \) then there exists \( C > 0 \) depending only on \( \Phi \) such that
\[
\limsup_{n \to \infty} \frac{1}{h_n^2} H_n \varphi(x) \leq C \Delta \varphi(x_0).
\]

**Proof.** Let \( h = h_n \) and \( x \in T^d \). Write
\[ B(y) = \max_{y \in T^d} \Phi(h^{-1}|x-y|)(\varphi(x) - \varphi(y)). \]

Let \( p = \nabla \varphi(x) \) and \( A = \nabla^2 \varphi(x) \). Then we have
\[
B(y) = \max_{y \in T^d} \left\{ \Phi(h^{-1}|x-y|) \left( p \cdot (x-y) - \frac{1}{2} (x-y) \cdot A (x-y) + O(|x-y|^3) \right) \right\}
\]
\[ = \max_{0 \leq r \leq 2h} \left\{ \Phi(h^{-1}r) \max_{|v|=r} \left\{ p \cdot v - \frac{1}{2} v \cdot Av \right\} + O(r^3) \right\}. \]

Define
\[ C_r(p, A) = \max_{|v|=r} \left\{ p \cdot v - \frac{1}{2} v \cdot Av \right\}. \]

Since \( \nabla \varphi(x_0) \neq 0 \) we have that \( p \neq 0 \) for \( x \) near \( x_0 \). If \( A = 0 \) then \( C_r(p, A) = r|p| \).

Now suppose \( A \neq 0 \) and \( p \neq 0 \). Setting \( v = rp/|p| \) we have
\[ C_r(p, A) \geq r|p| - \frac{r^2}{2|p|^2} p \cdot Ap = r|p| + O(r^2). \]

Therefore \( C_r(p, A) > 0 \) for \( r > 0 \) sufficiently small. The optimality conditions for the optimization problem defining \( C_r(p, A) \) are
\[ p - Av = \frac{\lambda}{r} v \quad \text{and} \quad |v| = r, \]
where \( \lambda \) is a Lagrange multiplier. From now on, \( v \) will denote a minimizer of \( C_r \), which satisfies the optimality conditions above. It follows that

\[
|\lambda| = |p - Av| = |p| + O(r).
\]

By the optimality conditions we have

\[
C_r(p, A) = p \cdot v - \frac{1}{2} v \cdot Av = \lambda r + \frac{1}{2} v \cdot Av \leq \lambda r + Cr^2.
\]

Since \( C_r(p, A) > 0 \) for sufficiently small \( r > 0 \), it follows that \( \lambda > 0 \) for small \( r \), thus \( |\lambda| = |p| + O(r) \) and \( \lambda^{-1} = |p|^{-1} + O(r) \).

By a Taylor expansion we have

\[
v = \lambda^{-1} r (I + \lambda^{-1} r A)^{-1} p
\]

\[
= \lambda^{-1} r (p - \lambda^{-1} r A p + O(r^2))
\]

\[
= \lambda^{-1} r p - \lambda^{-2} r^2 A p + O(r^3).
\]

Therefore

\[
\frac{\lambda}{r} v = p - \lambda^{-1} r A p + O(r^2).
\]

Since \( |v| = r \) and \( \lambda^{-1} = |p|^{-1} + O(r) \) we have

\[
\lambda = |p - |p|^{-1} r A p + O(r^2)| = |p| - |p|^{-2} p \cdot A p r + O(r^2).
\]

It follows that

\[
\lambda^{-1} = |p|^{-1} + |p|^{-4} (p \cdot A p) r + O(r^2) \quad \text{and} \quad \lambda^{-2} = |p|^{-2} + O(r).
\]

Substituting this into the Taylor expansion for \( v \) we have

\[
v = |p|^{-1} p r + |p|^{-4} (p \cdot A p) r^2 - |p|^{-2} (A p) r^2 + O(r^3).
\]

Therefore

\[
p \cdot v = r |p| + O(r^3),
\]

and

\[
v \cdot A v = |p|^{-2} (p \cdot A p) r^2 + O(r^3).
\]

Finally we have

\[
C_r(p, A) = p \cdot v - \frac{1}{2} v \cdot Av
\]

\[
= r |p| - \frac{1}{2} |p|^{-2} (p \cdot A p) r^2 + O(r^3)
\]

\[
= |\nabla \varphi(x)| r - \frac{1}{2} \Delta_{\infty} \varphi(x) r^2 + O(r^3),
\]
since \( p = \nabla \varphi(x) \) and \( A = \nabla^2 \varphi(x) \).

So we have

\[
B(y) = \max_{0 \leq r \leq 2h} \left\{ \Phi(h^{-1}r)\|\nabla \varphi(x)\|_r - \frac{1}{2} \Phi(h^{-1}r)\Delta_\infty \varphi(x) r^2 + O(r^3) \right\}
\]

\[
= h \max_{0 \leq s \leq 2} \left\{ s\Phi(s)\|\nabla \varphi(x)\|_r - \frac{1}{2} s^2\Phi(s)\Delta_\infty \varphi(x) h \right\} + O(h^3).
\]

By a similar argument we have

\[
B'(y) := \min_{y \in \mathbb{R}^d} \Phi(h^{-1}|x - y|)(\varphi(x) - \varphi(y))
\]

\[
= -h \max_{0 \leq s \leq 2} \left\{ s\Phi(s)\|\nabla \varphi(x)\| + \frac{1}{2} s^2\Phi(s)\Delta_\infty \varphi(x) h \right\} + O(h^3).
\]

Let \( s_n \in [0, 2] \) such that

\[
B'(y) = -hs_n\Phi(s_n)\|\nabla \varphi(x)\| - \frac{1}{2} s_n^2\Phi(s_n)\Delta_\infty \varphi(x) h^2 + O(h^3).
\]

Then

\[
H_n \varphi(x) = -B(y) - B'(y) \leq s_n^2\Phi(s_n)\Delta_\infty \varphi(x) h^2 + O(h^3).
\]

Notice that

\[
s_n\Phi(s_n) \to \max_{0 \leq s \leq 2} \{s\Phi(s)\} \quad \text{as } n \to \infty.
\]

Therefore, there exists \( c > 0 \) depending only on \( \Phi \) such that \( s_n \geq c \) for all \( n \). Since \( h = h_n \to 0 \) as \( n \to \infty \)

\[
\limsup_{n \to \infty} \frac{1}{h_n} H_n \varphi(x) \leq c \max_{0 \leq s \leq 2} \{s\Phi(s)\} \Delta_\infty \varphi(x_0).
\]

5 Proof of main result

In this section we prove our main result, Theorem 1. The first step is to prove a Lipschitz estimate on the sequence \( u_n \) (Lemma 3), which gives us compactness. Then we introduce the notion of viscosity solution for the infinity Laplace equation, and complete the proof of Theorem 1.

**Lemma 3.** There exist \( C, c > 0 \) such that whenever \( r_n \leq ch_n \) we have

\[
|u_n(x) - u_n(y)| \leq C(|x - y| + h_n) \quad \text{for all } x, y \in \mathcal{X}_n.
\]
Proof. By Theorem 3 there exists $C > 0$ such that

$$w_n(x, y)|u_n(x) - u_n(y)| \leq Ch_n \quad \text{for all } x, y \in \mathcal{X}_n.$$  

Since $w_n(x, y) \geq 1$ whenever $|x - y| \leq h_n$ we have

$$|u_n(x) - u_n(y)| \leq Ch_n \quad \text{for all } x, y \in \mathcal{X}_n \text{ such that } |x - y| \leq h_n.  \quad (24)$$

Partition $\mathbb{R}^d$ into cubes of side lengths $h_n/2\sqrt{d}$. Assume $r_n \leq h_n/8\sqrt{d} < h_n/4\sqrt{d}$. Then every cube must have at least one point from $\mathcal{X}_n$. Therefore, for any $x, y \in \mathcal{X}_n$ there exists a path $x = x_1, x_2, x_3, \ldots, x_\ell = y$ with $x_i \in \mathcal{X}_n$ and $|x_i - x_{i+1}| \leq h_n$ for all $i$ and

$$\ell \leq d \left( \frac{2\sqrt{d}|x - y|}{h_n} + 1 \right).$$

Therefore

$$|u_n(x) - u_n(y)| \leq \sum_{i=1}^{\ell-1} |u_n(x_i) - u_n(x_{i+1})| \leq C\ell h_n. \quad \Box$$

We recall the definition of viscosity solution for the partial differential equation

$$\Delta_\infty u = 0 \quad \text{in } \Omega, \quad (25)$$

where $\Omega \subset \mathbb{R}^d$ is open.

**Definition 1.** We say that $u \in C(\Omega)$ is a viscosity subsolution of $(25)$ if for every $x_0 \in \Omega$ and $\varphi \in C^\infty(\Omega)$ such that $u - \varphi$ has a strict local maximum at $x_0$ and $\nabla \varphi(x_0) \neq 0$ we have

$$\Delta_\infty \varphi(x_0) \geq 0.$$

We say that $u \in C(\Omega)$ is a viscosity supersolution of $(25)$ if $-u$ is a viscosity subsolution of $(25)$. We say that $u \in C(\Omega)$ is a viscosity solution of $(25)$ if $u$ is both a viscosity subsolution and supersolution of $(25)$.

Let $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$ be the projection operator.

**Definition 2.** A function $u \in C(\mathbb{T}^d)$ is a viscosity solution of $(10)$ if $u = g$ on $\mathcal{O}$ and $v(x) := u(\pi(x))$ is a viscosity solution of $(25)$ with

$$\Omega := \pi^{-1}(\mathbb{T}^d \setminus \mathcal{O}).$$

A slight modification of the argument from [1] can be used to show that there is at most one viscosity solution of $(10)$.

We now give the proof of our main result.
Proof of Theorem 1. Let \( p_n : \mathbb{T}^d \to \mathcal{X}_n \) be the closest point projection. That is
\[
|x - p_n(x)| = \min_{y \in \mathcal{X}_n} |x - y|.
\]
Define \( v_n : \mathbb{T}^d \to \mathbb{R} \) by \( v_n(x) = u_n(p_n(x)) \). Since \( |x - p_n(x)| \leq r_n \), it follows from Lemma 3 that for any \( x, y \in \mathbb{T}^d \)
\[
|v_n(x) - v_n(y)| \leq C(|p_n(x) - p_n(y)| + h_n)
\]
\[
= C(|p_n(x) - x + y - p_n(y) + x - y| + h_n)
\]
\[
\leq C(|x - y| + 2r_n + h_n).
\]
Since \( r_n, h_n \to 0 \) as \( n \to \infty \), we can use a variant of the Arzelà-Ascoli Theorem (see the appendix in [3]) to show that there exists a subsequence, which we again denote by \( v_n \), and a Lipschitz continuous function \( u \in C^{0,1}(\mathbb{T}^d) \) such that \( v_n \to u \) uniformly on \( \mathbb{T}^d \) as \( n \to \infty \).

Since \( u_n(x) = v_n(x) \) for all \( x \in \mathcal{X}_n \) we have
\[
\lim_{n \to \infty} \max_{x \in \mathcal{X}_n} |u_n(x) - u(x)| = 0.
\]
We claim that \( u \) is the unique viscosity solution of (10). Once this is verified, we can apply the same argument to any subsequence of \( u_n \) to show that the entire sequence converges uniformly to \( u \).

We first show that \( u \) is a viscosity subsolution of (10). Let \( x_0 \in \mathbb{T}^d \) and \( \varphi \in C^\infty(\mathbb{R}^d) \)
\[
\text{such that } u - \varphi \text{ has a strict global maximum at the point } x_0 \text{ and } \nabla \varphi(x_0) \neq 0.
\]
We need to show that
\[
\Delta_\infty \varphi(x_0) \geq 0.
\]
Assume, by way of contradiction, that \( \Delta_\infty \varphi(x_0) < 0 \). By (26) there exists a sequence of points \( x_n \in \mathcal{X}_n \) such that \( u_n - \varphi \) attains its global maximum at \( x_n \) and \( x_n \to x_0 \) as \( n \to \infty \).
Therefore
\[
|u_n(x_n) - u_n(x)| \geq \varphi(x_n) - \varphi(x) \quad \text{for all } x \in \mathcal{X}_n.
\]
Since \( x_0 \notin \mathcal{O} \), we have that \( x_n \notin \mathcal{O} \) for \( n \) sufficiently large. By Lemma 1
\[
0 = L_nu_n(x_n) \leq L_n\varphi(x_n) \leq H_n\varphi(x_n) + C r_n^2 h_n^{-1}
\]
for all \( n \geq 1 \).
By Lemma 2 and the assumption \( r_n^2 h_n^{-3} \to 0 \) as \( n \to \infty \)
\[
0 \leq \limsup_{n \to \infty} \frac{1}{h_n^3} H_n \varphi(x_n) + C r_n^2 h_n^{-3} \leq C \Delta_\infty \varphi(x_0).
\]
This is a contradiction. Thus \( u \) is a viscosity subsolution of (10).

To verify the supersolution property, we simply set \( v_n = -u_n \) and note that \( L_n v_n = -L_n u_n = 0 \) and \( v_n \to -u \) uniformly as \( n \to \infty \). The argument given above for the subsolution property shows that \( -u \) is a viscosity subsolution, and hence \( u \) is a viscosity supersolution. This completes the proof.\[\square\]
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