Quantum atom optics with Bose–Einstein condensates

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Abstract. The formal equivalence between bosonic atoms and photons, and the rich possibilities for manipulation of atomic states, has led to numerous proposals for generation of atomic states with properties similar to the ones of the non-classical states of light, e.g. squeezed states and Schrödinger cat states. We present a number of research directions for quantum optics in which atoms seize or have already seized the role of the photons.

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1. Introduction

Just like electromagnetic waves, atomic matter waves can, to a large extent, be manipulated in space. Interference phenomena have been observed, and a large number of high precision atom optics measurements have been carried out [1]. The spatial [2, 3] and temporal [4] coherence of condensates have been verified, and potential applications have been identified for Bose–Einstein condensates as a coherent matter wave source for interferometric time and frequency standards, detection of inertial effects and a host of related technological tasks. The analogy with optics has been emphasized with a number of experiments producing atom-laser outputs from condensates [5] and by implementation of atom-optical components such as beam-splitters and mirrors for the atoms [6]. Coherent amplification of matter waves [7] has further established the close analogy with laser and maser sources of light, and the production and guiding of atoms in the vicinity of chip surfaces and in optical and magnetic guides [8] mark the first attempts towards integrated atom optics which, in conjunction with efficient detection methods, may be a promising approach to various quantum information tasks [9].

Bosonic atoms and photons are described by the same commutator relations, and the analogies between atoms and light go further than to the display of wave interference. In a quantum optical perspective, what characterizes a Bose–Einstein condensate is its single-mode character and the resulting first-order coherence properties. Two- and three-body loss has been used to show that the normal state of a condensate displays higher-order intensity correlations similar to the ones of coherent state in quantum optics [10]. This paper focuses on the possibilities of manipulating the population statistics and higher-order correlations in condensates. By use of the atomic interactions as an adjustable non-linear element, strong quantum effects are foreseen [11]–[13] and, in combination with the long storage times for atoms, these effects are both potentially much larger for atoms than for light, and more useful or even indispensable, e.g. for the improvement of atomic clocks beyond the standard quantum limit [14].

With examples from the experimental and theoretical research of the past few years, we shall review how and why theory and experiments aimed at fundamental quantum optical studies of light are worthwhile revisiting with ultra-cold atoms playing the role of photons. We shall also describe some of the theoretical tools and approximations that have to be invoked in these studies.

In section 2, we recall the similarity of the photon field and the matter wave field in relation to single-particle descriptions in terms of photons and atoms. In section 3, we present the observation of interference between condensates as an effect of quantum mechanical state reduction, induced by measurements on the system. Section 4 presents atomic analogies of quantum optical effects in the ‘continuous variable regime’ of many photons and atoms, where interactions, for example, lead to squeezing of field variables. In section 5, we present analogies in the ‘discrete variable regime’ of few photons and atoms, where one may carry out controlled quantum evolution of the state vector of the system, conveniently represented in a number state basis. In section 6, we comment on the similarities and differences between the prospects for similar studies and between the meaning of certain concepts in condensed and non-condensed systems. In the final brief section 7 we try to argue that, despite the large number of examples, the proposed atomic states or processes are not mere analogies to similar states and processes in quantum optics: the research with atoms has its own goals and potentials, and quantum atom optics has its own individual profile.
2. Quanta and atoms

Light is classically described as a wave phenomenon, and, for example, canonical quantization establishes the operator character of the electric and magnetic fields and a Hamiltonian which is equivalent to that of a collection of material oscillators with equidistant energy eigenvalues $(n + \frac{1}{2})\hbar\omega$. This leads to identification of the quantum of energy, as carried by the massless photon, and the state of the field as a superposition state or a mixture with different numbers of photons in the different field modes. Maxwell’s equations can be written as the Dirac equation for a massless spin-1 particle [15], and it is formally possible to describe a state with a definite number of photons as a (symmetrized) product state of single-photon states, and to carry out calculations of the functioning of optical devices with such a product state approach [16]. Various quantum electrodynamical effects rely on a fully quantized theory of light but, ironically, the photoelectric effect, which is often cited as a clear signature of field quantization, is perfectly described by the interaction of classical radiation with the quantum mechanical atom. The quantum nature of light and the discreteness of the energy spectrum was not manifest from experiments, and theory did not point to clear cases of non-classical behaviour until around 1960 where it was realized that the fluctuations in recorded light signals probe the quantum nature of light fields. Experiments on anti-bunching and squeezing are part of a well-established experimental activity serving both to demonstrate the ‘non-classical’ character of light and to exploit the quantum properties for high precision purposes [17] and, for example, for various quantum communication purposes [18].

The quantum mechanical description of individual bosons by Schrödinger’s equation parallels the Dirac equation description of the single-photon states. Second quantization proceeds formally the same way as for light, in that the macroscopic matter-wave field is introduced as an operator with bosonic commutator relations. It is, however, a conceptual difference that the matter-wave field is not the quantum version of a phenomenon which is described classically as a field: the field picture is mainly introduced as a convenient approach for book keeping which keeps track of symmetry properties. This leads, in addition, to a number of practical approximations and physical pictures which are helpful when dealing with the practical solution of many-body problems. The best example is the symmetry-breaking ansatz which approximates the many-body state by an eigenstate of the atom-field annihilation operator and which violates both common sense and strong statements from particle physics concerning the conservation of baryons (and hence atoms), but which is nevertheless an extremely efficient tool to obtain excellent approximations to the static and dynamic properties of Bose–Einstein condensates. In atomic physics one is of course less excited by the possibility of revealing the graininess of matter which would be the deviation of the behaviour of a collection of atoms from that of a ‘classical’ matter-wave field. Nonetheless, key experiments of quantum optics do have interesting atomic counterparts, as we shall see in the following sections.

3. Interference, macroscopic quantum state reduction

Interference is the main feature of coherent radiation and the experimental observation of interference between two condensate components [2] was met with great interest. If two condensates with definite mean field expectation values are overlapped in space, the atomic density shows interference fringes which are predicted straightforwardly from the expectation value of the atomic density operator. As shown by Javanainen and Yoo [19] the macroscopic interference observed in experiments with merging atomic samples in definite number states can
also be fully understood in terms of quantum mechanical state reduction occurring during the first few random detection events. Two independent condensates acquire a random relative phase while the interference signal is measured, as illustrated in [19] by a numerical simulation of two overlapping number state condensates arriving at a position-sensitive detector.

In a simple model analysed by Castin and Dalibard [20], atoms leave their respective separate condensates and after passage of an atom beam-splitter they are detected in two possible superposition states, but we are unable to distinguish from which condensate component the individual atoms originated. We write $|n_a, n_b\rangle$ for the initial separable product state of the atoms. The detection and annihilation of a single atom in the state $\sqrt{2} (\phi_a + \phi_b)$ leaves the remaining atoms in an entangled state 
\[
\left( \sqrt{n_a} |n_a-1, n_b\rangle + \sqrt{n_b} |n_a, n_b-1\rangle \right)/\sqrt{n_a + n_b}.
\] If we assume that $n_a \simeq n_b$ and both are large numbers, we observe that there is now a 75% probability that the subsequent atom is detected in the same state and only a 25% probability that it is detected in the state $\sqrt{2} (\phi_a - \phi_b)$ [20]. In this way an interference effect is obtained as a consequence of the random outcome of a measurement on the system, and subsequent measurements serve to definitely lock the interference signal. The same phenomenon occurs in optics, where the equivalence of symmetry-breaking approaches and more involved analyses with entangled states of the field modes and the light emitting atoms can be formally proven [21].

What we see here is an essential element of quantum mechanics which is almost exclusively observed in quantum optics experiments: in the continuous observation of a single quantum system, the state vector of the system changes, often abruptly, as a consequence of the measurements carried out on the system. The above paragraph is not only a theoretical account for the emergence of a macroscopic interference pattern, it presents a description of a measurement-induced dynamics of the state of a macroscopic system: the change of probability from 50 to 75% after detection in one of the detectors really implies that the ‘detector mode’ $\sqrt{2} (\phi_a + \phi_b)$ is now populated by 75% rather than 50% of the atoms. This author is still amazed by the conclusion that the detection of only a single atom raises the number of atoms populating one given single-particle wavefunction from a half to three quarters of the macroscopic number of atoms of the entire system. Subsequent detection of atoms raises this fraction rapidly towards unity.

There is a practical interest in merging condensates in order to form large condensates or to compensate for atom loss, e.g. by an atom laser output. In the presence of collisional interactions, evaporative cooling may provide such merging into a single quantum state as demonstrated in [22], and it has been proposed [23] to use alternatively the coupling to a lossy cavity to provide the necessary non-unitary element for the merging process

\[
\phi_a^{n_a} \otimes \phi_b^{n_b} \rightarrow \phi^{n_a+n_b}
\] (1)
to happen efficiently. Whereas a unitary single-particle evolution can certainly not convert two different states $\phi_a$ and $\phi_b$ to the same final state $\phi$, there is no argument that prevents the existence of a unitary process on the many-body Hilbert space, which leads to (1), but it seems prohibitively difficult to construct the appropriate many-body Hamiltonian. Under detection, however, such merging does take place into a state which is determined by the registration of just a few atoms [24].

Measurement-induced processes represent a promising way to engineer useful states of a quantum system. Note that also for non-degenerate gases, measurements present a quite robust means to correlate and entangle atomic states [25] and we foresee that spectacular ‘non-classical’ states of Bose-condensed atoms will also be prepared this way. In [25], the collective atomic
states are being probed by optical radiation; for condensates, interactions with an atomic beam may be a more sensitive probe.

4. From non-linear atom optics to quantum atom optics—continuous variables

For high precision purposes it was realized a long time ago that so-called non-classical states of light may have advantages compared to classical field states [17]. It is therefore natural to consider the production of such quantum correlated states of atoms as well. We find the term ‘quantum correlated’ more suitable than ‘non-classical’ for the states of atoms, since what is very non-classical for light, e.g. a number state, seems perfectly classical for atoms. It is natural to divide the discussion into two sections. This section deals with large numbers of atoms and mechanisms which, for light, are applied typically to propagating beams and which the theorist usually attempts to describe by mean values and variances of operators with quasi-continuous spectra. In the next section we shall deal with typically smaller numbers of atoms—states that a theorist would describe by wavefunctions and density matrices in a discrete number state representation and which the experimentalist might attempt to control at that level, similar to the very advanced work being done on single-field modes trapped in high-$Q$ cavities.

Bouyer and Kasevich proposed [26] that atomic interferometry with Bose–Einstein condensates should be able to reach a phase sensitivity $\delta \theta \sim 1/N$ rather than the normal dependence $\sim 1/\sqrt{N}$ on the number of quanta (atoms). This proposal requires a Fock state input to the interferometer and it points to the interest in dealing constructively with the fluctuations in atomic distributions, and here there is a lot to be learned from quantum optics.

Collision processes in which the internal state of atoms change, so that the atoms are liberated in pairs from the condensate state, represents a non-linearity similar to parametric generation of photons, and hence a means to produce entangled beams or trapped ensembles of atoms [13, 27].

Due to the collisional interaction between atoms in a Bose condensate, the system contains a non-linearity, equivalent to the Kerr effect for light, and, apart from its significance for the shape and the energy of the condensate, this collisional interaction has been demonstrated in a matter-wave analogue of the non-linear optical effect of four-wave mixing [28] and in the tunnelling dynamics of condensates [29]. Non-linearities are the source of non-classical states of light, and the Kerr-effect Hamiltonian may indeed both be applied to produce spin squeezed states of atoms [12] and to produce Schrödinger cat-like states [11]. The collisional effects on the population statistics have been demonstrated experimentally in the large atom number limit [30, 31].

It is not easy to control precisely the quantum features of interest; due to the spatial degrees of freedom, the problem is a genuine multi-mode one and, since squeezing effects are small (reduction of fluctuations below the level of Poisson statistics, which is often already too small to be experimentally discernible), one has to pay attention to the role of non-condensed atoms as well. Let us review the basic idea behind spin squeezing due to collisional interaction and let us illustrate with this example the attempts to present a quantitative treatment of such processes.

4.1. Spin squeezing—two-mode description

A collection of $N$ identical two-level atoms with interparticle interactions and interactions with the environment which are invariant under permutation of the particles are conveniently described in a collective spin picture. Every two-level system is described as a spin-$1/2$ particle with, for
example, the population difference between states $|1\rangle$ and $|2\rangle$ represented by the $z$ component of the spin $s_z = \frac{1}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|)$. The collective spin vector

$$\vec{S} = \sum_i \vec{s}_i$$

has a $z$ component, which represents the total population difference in the entire system between state-1 and state-2 atoms. This representation was introduced by Dicke [32] to describe superradiant decay of a collection of atoms, and it has been pointed out that the precision of atomic clocks and spectroscopic studies can be addressed by the noise properties of the collective spin components. In particular it has been suggested to improve this precision by application of spin squeezed states with reduced noise in a single spin component [33]–[35].

Let us consider spin squeezing as resulting from the collisional interaction between condensate atoms, proposed in [12]. In a single condensate, this interaction leads to an effective quadratic term in the Hamiltonian proportional to $N(N - 1)$, where $N$ is the total number of atoms. That term affects only the global phase of the condensate, but if the condensate is divided into two components, $N = n_1 + n_2$, interaction terms $n_1(n_1 - 1), n_2(n_2 - 1)$ and $n_1 n_2$ representing the interactions between atoms belonging to the same and to different condensate components appear, and they will change the noise properties of the collective spin.

The two components may be two different internal states of the atoms with different collision strengths [12] or they may be different spatial wavefunctions populated after passage of an atomic beam-splitter [36]. In either case, a quick coherent transfer from the initial state $|1\rangle$ to the superposition state $(|1\rangle + |2\rangle)/\sqrt{2}$ of every individual atom implies that the entire condensate of $N$ atoms now populates a two-mode entangled state:

$$|\Psi\rangle = \sum_{n_1=0}^{N} c_{n_1,n_2=N-n_1} |n_1; \phi_1\rangle |n_2; \phi_2\rangle$$

where $c_{n_1,n_2} = \sqrt{2^{-N} \binom{N}{n_1}}$, and where $|n_i; \phi_i\rangle$ denotes a state with $n_i$ atoms populating the internal state $\phi_i$ or the spatial wavefunction $\phi_i(\vec{r})$. An approximate treatment of the dynamics of the system consists in solving self-consistent equations for the single-particle state $\phi_1$ and $\phi_2$ (a pair of coupled Gross–Pitaevskii equations), identifying the interaction strengths from these states, and solving finally the resulting two-mode dynamics treating the amplitudes $c_{n_1,n_2}$ in (3) as dynamical variables. Since the collisions are assumed not to change the state of the atoms, only the phase of the amplitudes changes due to the interaction. As pointed out in [12] the collisional interaction terms $n_1(n_1 - 1), n_2(n_2 - 1)$ and $n_1 n_2$ can be expressed in terms of $S_z$ and $S_y^2$. The Heisenberg equations of motion for $\vec{S}$ can be solved analytically with this interaction and it was shown by Ueda and Kitagawa [35] that a particular linear combination of the collective spin components $S_z$ and $S_y$ becomes squeezed.

The process is illustrated in figure 1 on a collective spin Bloch sphere. In the initial state $|1\rangle$ of all atoms, the individual spins and the collective spin point vertically down. A rapid pulse prepares the superposition state with a macroscopic collective spin pointing along the $x$ axis. In the state (3), the atomic population of states 1 and 2 is binomial with a variance $\text{var}(S_z) = N/4$ and this variance is not changed by the interactions, but the interactions cause a rotation of the different components of (3), illustrated by the upper part of the $(S_x, S_z)$ uncertainty ellipse shifting towards the east, and the lower part shifting towards the west on the Bloch sphere, as
Figure 1. Diagrammatic representation of the collective spin on a Bloch sphere where the south pole (north pole) represents all atoms occupying the state $|1\rangle$ ($|2\rangle$). The vertical component of the Bloch vector accounts for the occupancy of the atomic states and its horizontal components account for the real and imaginary parts of the coherence in the two-state system. The uncertainty ellipse or circle attached to the end of the Bloch vector describes the uncertainty of the collective spin components orthogonal to the mean spin, e.g. a sample populating an even superposition of $|1\rangle$ and $|2\rangle$ has a binomial distribution of the total population of both states. The collisional interaction does not change the populations of states $|1\rangle$ and $|2\rangle$, but it smears out the phase as indicated by the tilted uncertainty ellipses. If this ellipse is subsequently rotated corresponding to a coherently driven transition between $|1\rangle$ and $|2\rangle$, we can obtain a number or phase squeezed states illustrated by the dark ellipse in figure 1. The uncertainty of $S_y$ is increased but, as indicated in the figure, it is possible to identify a spin component $S_\theta = \cos \theta S_z - \sin \theta S_y$ which has a reduced variance. Experimentally one must coherently couple the states $\phi_1$ and $\phi_2$ again in a new rapid process, which can rotate the ellipse either by $-\theta$ to obtain a state with even mean populations but strongly reduced population fluctuations or by $(90^\circ - \theta)$ to obtain a state with a very well defined phase.

A word of caution is necessary: however easy it seems to rotate the ellipse by a simple coupling of states 1 and 2 in the spin representation in figure 1, this may be completely impossible, since the states may describe atoms at different locations in space or states which, for other reasons, are not coupled by an available interaction mechanism. Experimentally one must thus seek to ‘bring the system together again’ at the final stage where the squeezing ellipse is rotated to provide the experimental signature of squeezing. Until that point the two separated components are strongly entangled, but counting noise will be Poissonian or worse. One must, of course, also seek to make the recombination as careful as possible, similarly to the importance of mode matching in work with squeezed light where the introduction of vacuum mode components leads to higher noise. A careful examination of this effect in the case of a spatial interferometer is presented in [36].
4.2. Descriptions beyond the two-mode ansatz

The Gross–Pitaevskii equation identifies single-particle wavefunctions populated by the atoms in the interacting sample. It is an ansatz that such privileged wavefunctions exist and that they are really macroscopically populated. In the above analysis we made the ansatz that only two atomic states are populated and the dynamics is restricted to the time evolution of the amplitudes $c_{n_1,n_2}$ of the state vector (3) or of the collective spin operator in the associated simple two-mode Fock space. Such an ansatz has to be justified by treatments going beyond the two-mode description. Bogoliubov theory is perturbation theory in the interaction beyond the Gross–Pitaevskii ansatz, and among its main results are the calculation of the excitation spectrum of the condensate and an estimate of the quantum depletion, i.e. the total atomic population which cannot be accounted for by a single one-particle wavefunction.

The interaction term in the many-body Hamiltonian can be approximately diagonalized by a Bogoliubov transformation. A similar Bogoliubov transformation also diagonalizes the quantum optical squeezing Hamiltonian, where the canonical mixing of creation and annihilation operators yields a scaling of suitable field quadrature components and hence of their noise properties. The lowest energy state of the atomic system is the Bogoliubov vacuum, i.e. the zero eigenstate for such a set of transformed annihilation operators, and this naturally associates this state with a displaced squeezed state [37] rather than the simple coherent (displaced vacuum) state discussed above, see also [38]. A promising way to address the squeezing effects inherent in the Bogoliubov analysis is to consider the spatial correlation functions [37] and to look for quantum correlations between excitation modes of the system [39]. Correlations between excitation modes can actually be introduced in a number of ways, e.g. when Bragg scattering in two different condensates is stimulated by a common probe [40], but we shall now return to the simpler ‘few-mode’ case where it is the mode(s) occupied by almost all the atoms that are manipulated.

One must pay attention if one wants to analyse the two-mode squeezing by the Bogoliubov theory, since the collisional effect is then really only accounted for to lowest order, and the validity of assigning a privileged role to the squeezed mode is not clear. Sørensen [41] has shown that it is possible to retain the privileged role of two dominant modes and to reserve the perturbative treatment to the weak depletion beyond those modes only. This enables calculations into a regime with substantial spin squeezing where corrections compared to the achievements of the simple $S^2$ Hamiltonian are sizable and non-trivial.

Prior to this perturbative treatment, another elegant approach had been applied to study the validity of a single- or two-mode ansatz for the atoms. The interaction depends on the number of atoms, and hence the self-consistent wavefunctions $\phi_1$ and $\phi_2$ could carry a, supposedly weak, dependence on $n_1$ and $n_2$. This implies that the ansatz (3) should be replaced by one where the wavefunctions are allowed to differ for different values of the populations, and hence a whole lot of these functions have to be determined. Sinatra and Castin [42] have developed the equations to be solved for these functions, and in [12] this approach was applied to the problem of spin squeezing in a trap, where it yielded results in complete agreement with the Bogoliubov treatment [41]. The equations of motion for the $n$-dependent wavefunctions are identified as a set of Gross–Pitaevskii-like equations, which have to be solved with different populations. This makes it an appealing approach with a clear physical picture although it lacks, to some extent, the systematic character of the Bogoliubov approach and a natural measure or criterion of validity of the results obtained by the method.

Finally, there is an exact approach, which is numerically far too demanding to allow a
general approach to all the interesting many-body problems of the field, but which seems well suited for finite-time studies, such as the transient dynamics during atomic spin squeezing. The approach, which is called the positive $P$ method, was developed in quantum optics to describe non-linear effects [43] and it works by first expanding the system density matrix in a coherent state basis (eigenstates of the field annihilation operators: $\hat{\psi}(\vec{r})|\psi\rangle = \psi(\vec{r})|\psi\rangle$):

$$\rho(t) = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 |\psi_1\rangle \langle \psi_2| P(\psi_1, \psi_2, t),$$

(4)

where the integration is over all values of the spatially dependent functions $\psi_1(\vec{r}), \psi_2(\vec{r})$. The Schrödinger equation for the density matrix is then transformed into an equation of motion for the phase space function $P$, making use of the operator correspondence (obtained by explicit insertion in equation (4)):

$$\hat{\psi}\rho \rightarrow \psi_1 P(\psi_1, \psi_2, t),$$

$$\rho\hat{\psi}^\dagger \rightarrow \psi_2 P(\psi_1, \psi_2, t),$$

$$\hat{\psi}^\dagger \rho \rightarrow \left(\psi_2 - \frac{\partial}{\partial \psi_1}\right) P(\psi_1, \psi_2, t),$$

$$\rho\hat{\psi} \rightarrow \left(\psi_1 - \frac{\partial}{\partial \psi_2}\right) P(\psi_1, \psi_2, t),$$

(5)

which, for the interaction terms quadratic in density, leads to second-order derivatives and thus to an equation of Fokker–Planck type for $P(\psi_1, \psi_2, t)$. This equation is, of course, prohibitively complicated to solve, but for the case of particles with single-particle Hamiltonian

$$\hat{h} = -\frac{\hbar^2}{2m}\Delta + V(\vec{r})$$

and a single-component interaction term $\int d\vec{r} \frac{g^2}{2} (\hat{\psi}^\dagger)^2\hat{\psi}^2$, it may be replaced by the Langevin equations

$$i\hbar d\psi_1 = (\hat{h} + g\psi_2\psi_1)\psi_1 dt + \sqrt{-ig}\psi_1 dW_1,$$

$$i\hbar d\psi_2 = (\hat{h} + g\psi_1\psi_2)\psi_2 dt + \sqrt{-ig}\psi_1 dW_2,$$

$$\frac{dW_i}{dW_j} = 0,$$

$$\frac{dW_i(\vec{r}, t) dW_j(\vec{r}', t')} = dt \delta(t - t')\delta(\vec{r} - \vec{r}')\delta_{ij}.$$

(6)

These are ‘noisy’ Gross–Pitaevskii equations, i.e. if we drop the noise terms we recover the equations for the Gross–Pitaevskii wavefunction and its complex conjugate. Drummond suggested to solve these equations for a large statistical ensemble of realizations to describe evaporative cooling towards condensation [44], but divergences in the method, occurring also in the quantum optical applications limit its use to a few atoms or short-time behaviour (the density term $\psi_1\psi_2$ in (6) is only real on average and its fluctuating imaginary parts spoil the statistical properties of the method). Considerable progress has been made to modify the approach and deal with the divergences [45], but already, without these modifications, the positive $P$ approach offers the possibility to study short-time behaviour (until the fluctuations become numerically intractable). Two- or multiple-component systems are readily accounted for by adding extra field variables $\psi_1, \psi_2$ for each new component (basically because the product of two coherent states is parametrized by a pair of amplitudes). In our first application of the method, we studied the equivalent of down conversion in the optical parametric oscillator: dissociation of a molecular condensate, in which one prepares a ‘squeezed atomic vacuum’, where an initially empty atomic mode develops into a non-classical state with a vanishing mean field amplitude [27]. Also spin
squeezing and the generation of squeezed light from a spin squeezed two-component sample have been studied by this method [46]. The treatment makes no assumption about the atoms belonging to definite modes during the process, and it is hence a good test of the expectations based on the few-mode approximations.

5. Atoms in lattices—discrete variable quantum optics

When a Bose–Einstein condensate is fragmented by a multi-well potential, the interaction between atoms occupying the same well represents a non-linearity with various consequences. In the limit of many atoms per well, the interaction causes a dephasing like the one described in the previous section, and in the case of a reasonably large number of wells, it is possible to investigate this effect experimentally by observation of the diffraction pattern when the atoms are released from the potential [31]. Unlike the case of interference between two condensates, many condensates will only show interference if there is a prefixed phase relationship between the different components. If, for example, each component is populated by a number state, taken at random from within a Poisson distribution, the situation is equivalent to one of a collection of coherent states with truly random phases, which cannot be absorbed in a single random phase difference as in the case of two condensates.

5.1. Thousands of high-\(Q\) cavities within a millimetre

In the case of a few atoms per potential minimum in a multi-well situation, it has been possible to observe the superfluid/Mott-insulator phase transition in the atomic system [47, 48]. This phase transition is a true many-body phenomenon, but the states obtained, and the possibilities for experiments with these states, have many connections to quantum optics. With only a few atoms per site, we may effectively describe the system by wavefunctions expanded in the number state basis:

\[
|\psi\rangle = \sum_n c_n |n\rangle
\]  

(7)

referring to the occupancy of the single-particle state located on the relevant site. The Mott-insulator case corresponds to only one of the \(c_n\) amplitudes being different from zero (commensurate filling) and it results if the potential is raised slowly enough that the collisional interactions can keep the system in its lowest energy state. If the potential is raised more rapidly, but still slowly enough that only the lowest motional state in each potential well is populated, a state with fluctuating numbers in each well is obtained. Each atom is in a superposition of being localized in several wells, and like the binomial distribution above (3), the population of a single well coherently attains several values. For identical particles it does not make sense to specify which atoms go where, but the states of the second quantized fields characterizing the population in the wells are entangled with each other. The multi-well problem is too complicated to be described as a product of state vectors for each well, but equation (7) can be taken as an ansatz (Gutzwiller), lending support from the fact that, if interactions do not enforce the Mott-insulator dynamics, a macroscopic coherent state is exactly split into a product of ‘microscopic’ coherent states occupying each well. Hence also the entangled state resulting from the splitting of a macroscopic number state is correctly described by a product state of coherent states with number fluctuations which describe how the entanglement manifests itself. Technically, one can apply a projection to the product state of the many coherent states, putting it into a state...
with a definite total number of atoms, but this projection will not affect any observables at the microscopic level.

Like the motion of a harmonically trapped ion, our physical system is equivalent to light in the high-$Q$ cavities of micro-masers and micro-lasers. It is a well controlled system with long lifetimes, discrete energy states, and hence various transient quantum processes can be observed. In addition, the existence of many identical replicas of the same ‘cavity system’ in the neighbouring wells in the lattice makes it possible to observe few-atom quantum dynamics without the need for single-atom detection capabilities. The collapse and revival of the atomic phase due to interactions in a quasi-coherent state with only a few atoms has thus been revealed experimentally by the absence and presence of interference between wells in diffraction patterns when the atoms are released [49], and a host of experimental possibilities appear for this system. As shown in [50], for a finite number of particles the collective interaction $S^z$ which is responsible for squeezing in large samples may lead to production of a Schrödinger-cat state, i.e. a coherent superposition of all atoms in one or another internal state [51]. It should also be mentioned that proposals exist for quantum computation with lattice-trapped atoms [52], relying on a means to move the atoms and couple them to each other in a state selective way. Precisely this state selective coupling may be applied to all nearest neighbour pairs at once in the lattice and serve to implement Ising and Heisenberg Hamiltonians, and thus to generate and study ferromagnetic and anti-ferromagnetic orders in the system [53]. The first successful experiments with state selective displacement of atoms trapped in lattices have been reported in [54].

5.2. Quantum molecular optics

Another appealing possibility is to prepare molecules in the lattice by coherent photoassociation. Molecule formation in free space or in a wide trap is a free–free transition [55], whereas in a lattice well the atoms and molecules are bound, and both the higher density of atomic pairs and the fact that the process can be driven resonantly between discrete states makes the lattice an ideal environment for photoassociation. One strategy is to have the atoms undergo the Mott-insulator transition to a state with precisely two atoms per well and then form a molecular Mott insulator [56], a process that can also be performed with two different species with the purpose of forming a condensate of heteronuclear molecules [57]. After formation of the molecules, the lattice potential should be removed slowly to give the insulator time to ‘melt’.

Alternatively, one may work with a small coherent state atomic content in each well $\langle n \rangle < 1$ which, by application of photoassociation lasers, is transformed into a coherent blend of atoms and molecules:

$$c_0|\text{vacuum}\rangle + c_1|1 \text{ atom}\rangle + c_2|2 \text{ atoms}\rangle \rightarrow c_0|\text{vacuum}\rangle + c_1|1 \text{ atom}\rangle + c_2|1 \text{ molecule}\rangle,$$

in what is approximately a product of coherent states (the first component in (8) serves as the vacuum state (zero-quantum component) for both species and the population of states with more than two atoms has been neglected for simplicity) [58].

The introduction of multiple species and of the photoassociation process is like opening Pandora’s box. A number of quantum optical effects may be studied in a system with completely different interaction, decoherence and storage properties than what has been addressed with quantum states of light. We have, for example, proposed [59] to implement the quantum optical Jaynes–Cummings Hamiltonian [60] through photo-association of a system with a quantum state like (7) of one atomic population and precisely one atom of another species per well. Then, the
formation of a molecule annihilates an atom in the same way:

\[ |n \text{ A-atoms, 1 B-atom} \rangle \leftrightarrow |n - 1 \text{ A-atoms, 1 AB-molecule} \rangle, \]  

(9)
as the excitation of an atom in the Jaynes–Cummings model annihilates a photon. This system thus offers a pure matter-wave version of the Jaynes–Cummings dynamics. The Jaynes–Cummings model offers a wide range of possibilities for state preparation and for dynamical studies, including, for example, quantum state tomography on the population statistics of the matter field in the lattice potential.

6. Bosons, ‘bolzons’, and fermions

Some, but not all, of the above phenomena and mechanisms rely on the bosonic properties of atoms.

Matter-wave interference is a single atom phenomenon, and it does not rely on any single-mode character of the system, which is also why Young could observe optical interference long before the laser was invented. This, however, only holds true for the interference of waves emerging from a single source. Atoms, or photons, that never met will not have a predictable interference pattern, and the interference between two independent condensates linked with the macroscopic state reduction works because the atoms are indistinguishable and they emerge from the same pair of macroscopically populated modes.

It is worthwhile mentioning that the collective spin representation was introduced and successfully applied to superradiance from atoms which were not Bose-condensed [32]. The Bose-exchange symmetry in that case dealt only with the internal population of the atoms in ground and excited states, while the atoms were individually distinguishable (though not by the super-fluorescence from the sample). Atomic systems, monitored by a laser beam which couples with equal strength to all atoms (and for which spontaneous scattering does not yield too much information about the individual scatterers [61]), are thus well described by the Bose symmetry even if they are classical ‘bolzons’ or fermions. In [50] an \( S_z^2 \) interaction is presented for distinguishable ions through the coupling of the internal state of all ions to their joint centre-of-mass motion, and the internal state of the ions, indeed, stays in the symmetric part of the full Hilbert space, and a Schrödinger-cat state has been observed [62]. The ‘Bose-enhanced’ processes occur with the stimulated rates because of the combinatorial factors associated with the change from states with one and another distribution on internal states [32]. Notably, it has been shown that the matter-wave amplification, associated with four-wave mixing, works also for fermions [63]. In section 4, however, the incorporation of the two-particle interactions in a single collective spin operator needs, of course, all pairs of atoms to collide with identical collision strengths, depending on the internal state, and this is ensured by all atoms populating the same state of motion. To ensure similar synchronous collisions of distinguishable particles, we have to steer their motion, as in the quantum computing proposals with atoms trapped in optical lattices [52]. For that system, spin squeezing should therefore also be observable in the macroscopic properties of the atoms [53].

The collective states of fermions comprise a large research field with a wealth of exotic phenomena, of which BCS superfluidity attracts much experimental attention for the moment. We recall that the intensity (shot) noise was used as the decisive probe of the fractional quantum Hall effect, and that Hanbury-Brown–Twiss correlation measurements are used to investigate the reactions in heavy-nucleon collisions. It is thus very likely that the characteristic focus in
quantum optics on fluctuations rather than on mean values will inspire a whole new ‘Fermi quantum optics’ without direct analogies in the quantum optical systems studied so far.

Let us finally turn to the question about entanglement of the ‘quantum correlated’ atomic states: a question which turns out to be linked intimately with the quantum statistical and distinguishability properties of the atoms. It has been shown explicitly that a spin squeezed sample of two-level systems are in an entangled state [12, 64]. These proofs, however, treat the atoms as distinguishable particles and analyse the possibility of writing the joint state as a convex sum of product states of all particles. For atoms in a Bose–Einstein condensate, we cannot say that one particular atom is in an entangled state or not with the other atoms of the system. A meaningful definition of entanglement would be that the matter-wave field defined by one detector and by another detector are in entangled states, for example the atomic components in separate spatial regions. For a more formal discussion, see, e.g., [65]. This, of course, establishes the normal definition of entanglement of distinguishable particles and of separate samples of particles, and it is the way that we would also define entanglement between optical fields. This definition has the effect that, for example, the state of \( N \) atoms which are all in the same superposition of two state vectors is in an entangled state with respect to the detection of the Fock space over these two states, cf equation (3). And even the state of a single atom will always be in an entangled state with respect to detection in two separate regions in which the atomic wavefunction does not vanish. The coherent states and the vacuum are the only pure states of bosons which factorize for any subdivision in detector modes of the one-particle Hilbert space, and a pure state with a definite atom number will therefore always be an entangled state under some subdivision. Useful entanglement, however, may not come for free but may precisely require the kind of correlations discussed in this paper.

7. Discussion

We have pointed to analogies between atomic and optical states and processes, but the research should not restrict itself to make atomic analogies to optical effects. Atoms or photons, but not both, (1) can travel far, (2) can travel fast, (3) require a good vacuum, (4) can be stored, (5) can be detected effectively non-destructively, (6) can be effectively manipulated with light, (7) have a multi-level structure, (8) are fundamental frequency standards and (9) can be deposited to form spatial structures, . . . . It is clear that atoms and photons are unique systems with different potential roles in physics studies and technological applications. Cold atoms may have a role as quantum memory storage devices and quantum registers for quantum information processing, but most likely not as carriers of quantum information between distant locations. Both the continuous variable and the discrete variable quantum states are potentially useful for these tasks.

Concerning the importance of matching precisely the single-mode character and the bosonic properties of the photon field to see useful correlation and squeezing effects, we have examples where this is certainly important, but we have also seen that it may be circumvented, and distinguishable particles and even fermions may show strong collective behaviour. It may well turn out that the main advantage, albeit not an insignificant one, of using Bose–Einstein condensed atoms is their first and most elementary property which is that they are really cold: the condensation process itself is remarkably effective at cooling and hence preparing many particles in a pure, well-controlled quantum state—a necessary prerequisite for a number of quantum information tasks.
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