Anisotropic Hardy-Sobolev inequality in mixed Lorentz spaces with applications to the axisymmetric Navier-Stokes equations

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Abstract

In this paper, we establish several new anisotropic Hardy-Sobolev inequalities in mixed Lebesgue spaces and mixed Lorentz spaces, which covers many known corresponding results. As an application, this type of inequalities allows us to generalize some regularity criteria of the 3D axisymmetric Navier-Stokes equations.

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1 Introduction

1.1 Hardy-Sobolev inequality

The classical Hardy-Sobolev inequality in [2, 14, 16, 31] reads

\begin{equation}
\left\| \frac{f(x)}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \nabla f \right\|_{L^p(\mathbb{R}^n)}, 1 < p < n, \\
\left\| \frac{f(x)}{|x|^s} \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Lambda^s f \right\|_{L^2(\mathbb{R}^n)}, 0 \leq s < \frac{n}{2}.
\end{equation}

(1.1)

A variant of Hardy-Sobolev inequality involving fractional derivatives in the $L^p$ framework is proved in [27, 38]

\begin{equation}
\left\| \frac{f(x)}{|x|^s} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \Lambda^s f \right\|_{L^p(\mathbb{R}^n)}, 0 \leq s < \frac{n}{p}.
\end{equation}

(1.2)

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The Hardy-Sobolev type inequality and its extension play an important role in the study of nonlinear elliptic equations \([3]\), heat equations with a singular potential \([15]\), Morawetz-type estimates of wave equations \([31]\) and Schrödinger equations \([38]\), and the Navier-Stokes equations \([10,11,36]\).

Two kinds of natural refinement of the Hardy-Sobolev inequality are to study it in more general spaces and to consider it in the anisotropic case. It is well known that the Lorentz spaces \(L^p(\mathbb{R}^n)\) and anisotropic Lebesgue spaces \(L^{p,q}(\mathbb{R}^n)\) are both generalizations of the Lebesgue spaces \(L^p(\mathbb{R}^n)\), where \(\vec{p} = (p_1, p_2, \cdots, p_n)\). Making use of the Hölder inequality in Lorentz spaces in \([29]\) and the famous fact that \(|x|^{-s} \in L^{p,\infty}(\mathbb{R}^n)\), Hajaiej-Yu-Zhai \([17]\) established the following Hardy-Sobolev inequality in Lorentz spaces

\[
\left\| \frac{f(x)}{|x|^s} \right\|_{L^{p,q}(\mathbb{R}^n)} \leq C \|\Lambda^s f\|_{L^{p,q}(\mathbb{R}^n)}, 1 < p < \infty, 0 < s < \frac{n}{p}, 1 \leq q \leq \infty. \tag{1.3}
\]

The first objective of this paper is to extend the Hardy-Sobolev inequality \((1.1)-(1.2)\) from usual Lebesgue spaces to anisotropic Lebesgue spaces. More precisely, we state our first result as follows.

**Theorem 1.1.** Assume that \(1 < p_i, q_i < \infty\) and \(0 < s < \frac{n}{p_i}, i = 1, 2, \cdots, n\). Then there exists a positive constant \(C\) such that

\[
\left\| \frac{f(x)}{|x|^s} \right\|_{L^{p_i, q_i}(\mathbb{R}^n)} \leq C \|\Lambda^s f\|_{L^{p_i, q_i}(\mathbb{R}^n)}. \tag{1.4}
\]

**Remark 1.1.** This theorem covers the known Hardy-Sobolev inequalities \((1.1)-(1.3)\). When \(p_i = q_i\), the inequality \((1.4)\) reduces to

\[
\left\| \frac{f(x)}{|x|^s} \right\|_{L^{p_i}(\mathbb{R}^n)} \leq C \|\Lambda^s f\|_{L^{p_i}(\mathbb{R}^n)}, \tag{1.5}
\]

which still extends the classical results \((1.1)\) and \((1.2)\).

To illustrate the proof of the Hardy-Sobolev inequality \((1.4)\), we give some comments on the role of Lorentz spaces in the study of Hardy-Littlewood-Sobolev inequality and Hardy-Sobolev inequality in usual Lebesgue spaces. Indeed, making full use of the fact \(|x|^{-s} \in L^{p,\infty}(\mathbb{R}^n)\) and the Young inequality in Lorentz spaces in \([29]\), one can conclude the classical Hardy-Littlewood-Sobolev inequality that

\[
\left\| f * \frac{1}{|x|^{n-s}} \right\|_{L^{\frac{pn}{n-pn}}(\mathbb{R}^n)} \leq C \left\| f * \frac{1}{|x|^{n-s}} \right\|_{L^{\frac{qn}{n-qn}}(\mathbb{R}^n)} \leq C \|f\|_{L^{q}(\mathbb{R}^n)}, 1 < q < \frac{n}{s}, 0 < s < n,
\]

which also means that

\[
\left\| f \right\|_{L^{\frac{qn}{n-qn}}(\mathbb{R}^n)} \leq C \|\Lambda^s f\|_{L^{q}(\mathbb{R}^n)}, 1 < q < \frac{n}{s}. \tag{1.6}
\]

On the other hand, replacing the Young inequality by the Hölder inequality in Lorentz spaces and using the Sobolev inequality \((1.6)\), one may derive the Hardy-Sobolev type inequality \((1.3)\). This strategy will also be applied to prove \((1.4)\). To this end, we observe that the Hardy-Sobolev inequality \((1.4)\) can be enhanced to the inequality

\[
\left\| \frac{f(x)}{\prod_{i=1}^{n} |x_i|^\gamma_i} \right\|_{L^{p_i, q_i}(\mathbb{R}^n)} \leq C \|\Lambda^s f\|_{L^{p_i, q_i}(\mathbb{R}^n)}.
\]
This enlightens us to consider the Hardy-Sobolev type inequality in anisotropic cases. It is worth remarking that this kind of anisotropic inequalities is very helpful to investigate the cylindrical (axial symmetry) solutions of partial differential equations. To consider the nonlinear elliptic equations arising in astrophysics, Badiale-Tarantello [1] first introduced the following anisotropic Hardy-Sobolev inequality

\[
\left\| r_k^{-\frac{\alpha - q}{2q(n - s)}} f \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \nabla f \right\|_{L^q(\mathbb{R}^n)}, s < k, 2 \leq k \leq n, \tag{1.7}
\]

where \( r_k = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2} \).

Chen-Fang-Zhang [10] proved the following generalization of Hardy-Sobolev inequality

\[
\left\| r_k^{-\frac{\alpha}{q}} f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \left\| \nabla f \right\|_{L^p(\mathbb{R}^n)}, s < k, 2 \leq k \leq n, \tag{1.8}
\]

and employed it to consider the regularity of the three-dimensional axial symmetry Navier-Stokes system. Subsequently, the anisotropic Hardy-Sobolev inequality below

\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{p_2^{1 - \alpha} |x_3|^\alpha} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \Lambda_{x_3}^{\alpha j} \Lambda_{x_1 x_2}^{\alpha j} f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \nabla f \right\|_{L^p(\mathbb{R}^n)}, 0 < \alpha < 1, \tag{1.9}
\]

was proved and applied to investigate the regularity of axial symmetry Navier-Stokes equations by Yu in [36]. Partially motivated by this and the proof of Theorem 1.1, the second target of this paper is to further extend the aforementioned Hardy-Sobolev inequality as follows.

**Theorem 1.2.** Let \( \sum_{j=1}^{k_j} \leq n \) and \( k_j \in \mathbb{Z}^+, 1 \leq j \leq i \). Take any \( k_1 \) elements from the set \( A_0 = \{ x_1, x_2, \cdots, x_n \} \) and denote \( A_1 = \{ x_{i_1}, x_{i_2}, \cdots, x_{i_{k_1}} \} \), \( r_1 = \sqrt{x_{i_1}^2 + x_{i_2}^2 + \cdots + x_{i_{k_1}}^2} \). Then choose any \( k_2 \) elements from the complementary set \( A_0 \setminus A_1 \) and write \( A_2 = \{ x_{i_{k_1}+1}, x_{i_{k_1}+2}, \cdots, x_{i_{k_1}+k_2} \} \), \( r_2 = \sqrt{x_{i_{k_1}+1}^2 + x_{i_{k_1}+2}^2 + \cdots + x_{i_{k_1}+k_2}^2} \). Continue until we take any \( k_i \) elements from the complementary set \( A_0 \setminus \bigcup_{j=1}^{i-1} A_j \) and denote \( A_i = \{ x_{i_{\sum_{j=1}^{i-1} k_j}+1}, x_{i_{\sum_{j=1}^{i-1} k_j}+2}, \cdots, x_{i_{\sum_{j=1}^{i-1} k_j}+k_i} \} \), \( r_i = \sqrt{x_{i_{\sum_{j=1}^{i-1} k_j}+1}^2 + x_{i_{\sum_{j=1}^{i-1} k_j}+2}^2 + \cdots + x_{i_{\sum_{j=1}^{i-1} k_j}+k_i}^2} \).

Suppose that \( 1 < p < \infty \) and \( 0 < \alpha_j < \frac{k_j}{p} \), \( 1 \leq j \leq i \). Then for all \( f \in C_0^\infty(\mathbb{R}^n) \), there holds

\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{i} r_j^{\alpha_j}} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \Lambda_{x_{i_1} \cdots x_{i_{\sum_{j=1}^{i-1} k_j}}}^{\alpha_j} f \right\|_{L^p(\mathbb{R}^n)}. \tag{1.10}
\]

**Remark 1.2.** The Mihlin-Hörmander multipliers theorem on \( L^p(\mathbb{R}^n) \) guarantees that

\[
\left\| \Lambda_{x_{i_1} \cdots x_{i_{\sum_{j=1}^{i-1} k_j}}}^{\alpha_j} f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \sum_{j=1}^{i} \alpha_j \right\|_{L^p(\mathbb{R}^n)}/ \tag{1.11}
\]

Therefore, one can replace \( \left\| \Lambda_{x_{i_1} \cdots x_{i_{\sum_{j=1}^{i-1} k_j}}}^{\alpha_j} f \right\|_{L^p(\mathbb{R}^n)} \) by \( \left\| \Lambda^{\alpha_1 + \cdots + \alpha_i} f \right\|_{L^p(\mathbb{R}^n)} \) in inequality \( (1.10) \).

**Remark 1.3.** Setting \( i = j = 1 \) and choosing \( k_1 = k \) and \( A_1 = \{ x_1, x_2, \cdots, x_k \} \) in this theorem, one may derive from (1.10) and (1.11) that

\[
\left\| r_k^{-\frac{\alpha}{q}} f \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| \Lambda^{\alpha} f \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \left\| \nabla f \right\|_{L^p(\mathbb{R}^n)} \left\| \frac{n-k}{p} - \frac{n-\alpha}{q} + 1 \right\|_{L^p(\mathbb{R}^n)}, s < k, 1 \leq k \leq n,
\]

3
where the Gagliardo-Nirenberg inequality was used. It is worth remarking that this inequality implies Badiale-Tarantello’s inequality [17] and Chen-Fang-Zhang’s inequality [18].

**Remark 1.4.** Taking \( A_1 = \{ x_1, x_2 \}, A_2 = \{ x_3 \}, \alpha_1 = 1 - \alpha, \alpha_2 = \alpha \) in this theorem, one has

\[
\left\| \frac{f}{|r|^{1-\alpha} |x_3|^\alpha} \right\|_{L^p(\mathbb{R}^3)} \leq C \| \nabla f \|_{L^p(\mathbb{R}^3)}, \quad \max\{1 - \frac{2}{p}, 0\} < \alpha < \frac{1}{p}, \quad p > 1,
\]

which is a variant of Yu’s inequality [19].

By a slightly modified proof of Theorem [1.2] we can further obtain the following result.

**Corollary 1.3.** Suppose that \( \mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \cdots \times \mathbb{R}^{k_i} \times \mathbb{R}^{n-\sum_{j=1}^{i} k_j} \) and \( n \geq \sum_{j=1}^{i} k_j \). Let \( r_1 = \sqrt{x_1^2 + x_2^2 + \cdots + x_{k_j}^2} \), \( r_2 = \sqrt{x_{k_1+1}^2 + x_{k_1+2}^2 + \cdots + x_{k_1+k_2}^2} \), \( r_i = \sqrt{x_{\sum_{j=1}^{i-1} k_j+1}^2 + x_{\sum_{j=1}^{i-1} k_j+2}^2 + \cdots + x_{\sum_{j=1}^{i} k_j}^2} \), \( 0 < p, q \leq \infty \), and \( 1 < (\frac{p}{p_j})_i \leq \infty \), \( 1 \leq (\frac{q}{q_j})_i \leq \infty \). Then for all \( f \in C_0^\infty (\mathbb{R}^n) \), there holds

\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{i} |r_j|^\alpha_j} \right\|_{L^{p_j} \cap \mathbb{R}^1 (\mathbb{R}^{k_1}) \cdots L^{p_j} \cap \mathbb{R}^i (\mathbb{R}^{k_i}) L^{p} (\mathbb{R}^{n-\sum_{j=1}^{i} k_j})} \leq C \left\| \sum_{j=1}^{i} \alpha_j \right\|_{L^{p_j} \cap \mathbb{R}^1 (\mathbb{R}^{k_1}) \cdots L^{p_j} \cap \mathbb{R}^i (\mathbb{R}^{k_i}) L^{p} (\mathbb{R}^{n-\sum_{j=1}^{i} k_j})}. \tag{1.12}
\]

Next, we present an application of Hardy-Sobolev inequalities obtained above to the axisymmetric Navier-Stokes equations. In what follows, for the convenience of presentation, we set \( r = r_2 = \sqrt{x_1^2 + x_2^2} \) in \( \mathbb{R}^3 \).

### 1.2 An application of Hardy-Sobolev inequality to the Navier-Stokes equations

The three-dimensional axisymmetric Navier-Stokes equations can be written in the cylindrical coordinates

\[
\begin{aligned}
\partial_t u_r + (u_r \partial_r + u_x \partial_x) u_r - (\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r) u_r + \frac{1}{r^2} u_r - \frac{1}{r} (u_\theta)^2 + \partial_r p &= 0, \\
\partial_t u_\theta + (u_r \partial_r + u_x \partial_x) u_\theta - (\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r) u_\theta + \frac{1}{r^2} u_\theta + \frac{1}{r} u_\theta u_r &= 0, \\
\partial_t u_x + (u_r \partial_r + u_x \partial_x) u_x - (\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r) u_x + \partial_x p &= 0, \\
\partial_r(r u_r) + \partial_\theta (r u_\theta) &= 0, \\
u_r |_{r=0} = 0, \quad u_\theta |_{r=0} = 0, \quad u_x |_{r=0} = u_0,
\end{aligned}
\tag{1.13}
\]

where, \( u_r, u_\theta \) and \( u_x \) denote the radial component, swirl component, vertical component of the velocity \( u \), respectively. It is valid that \( u = u_r e_r + u_\theta e_\theta + u_x e_x \), where \( e_r = (\frac{2}{r^2}, \frac{2}{r}, 0) \), \( e_\theta = (-\frac{2x_2}{r^2}, \frac{2x_1}{r}, 0) \) and \( e_x = (0, 0, 1) \). Let \( \omega_r = -\partial_x u_\theta, \omega_\theta = \partial_x u_r - \partial_r u_x, \omega_x = \partial_\theta (r u_\theta), \omega = \text{curl} u = \omega_r e_r + \omega_\theta e_\theta + \omega_x e_x \). The equations of \( (\omega_r, \omega_\theta, \omega_x) \) is determined by

\[
\partial_t \omega_r + (u_r \partial_r + u_x \partial_x) \omega_r - (\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r) \omega_r + \frac{1}{r^2} \omega_r - (\omega_r \partial_r + \omega_x \partial_x) u_r = 0,
\]
For the 3D axisymmetric Navier-Stokes equations (1.13) without swirl, it is well-known that the global well-posedness was established independently by Ukovskii and Yudovich [32] and Ladyzhenskaya [22]. See also [25] for a refined proof. However, in the presence of swirl, the global well-posedness problem is still open. By Caffarelli, Kohn, Nirenberg [4], the problem is reduced to how to remove the possible singularities on the symmetry axis. Recently, using DeGeogi-Nash-Moser iterations and a blow-up approach respectively, Chen, Strain-Tsai-Yau [9] and Koch-Nadirashvili-Seregin-Sverák [19] obtained an interesting and important development on this problem. They proved that if the solution satisfies (1) the gradient norm of $u$ is sufficiently small norm of $u$, then the solution satisfies (1) and (2) $|u(x,t)| \leq C$ for $0 < t < T^*$, where $C > 0$ is an arbitrary and absolute constant and $(0, T^*)$ is the maximal existence interval of the solution. They showed that these conditions are scaling invariant and imply the possible blow-up rate of the solution. See also various extensions by Pan [30] and Lei and Zhang [23].

Very recently, Chen, Fang and Zhang [11] proved that the lifespan condition (1.14) can be replaced by

$$r^\alpha u^\theta \in L^q(0, T; L^{p_1}(\mathbb{R})) L^{p_2}(\mathbb{R}^2), \frac{2}{q} + \frac{1}{p_3} = 1 - \alpha, 0 \leq \alpha < \frac{1}{2},$$

(1.16)

Let us also mention that Yu [36] recently showed that $|x_3|u^\theta \in L^\infty(0, T; L^\infty(\mathbb{R}^3))$ and the sufficiently small norm of $||x_3|u^\theta||_{L^\infty} = L^\infty$ ensure the regularity of solutions of axisymmetric Navier-Stokes equations with the help of Hardy-Sobolev inequality (1.9). The other regularity criteria and recent studies can be seen in [1, 12, 24, 34] and references therein.

We now state our main theorems. We firstly invoke the Hardy-Sobolev inequality (1.10) to generalize Yu’s regularity class in [36]. Our results can be formulated as
**Theorem 1.4.** Suppose that \( u \) be an axisymmetric weak solution of the Navier-Stokes system (1.13) associated with the axisymmetric divergence-free initial data \( u_0 \in H^2(\mathbb{R}^3) \). If
\[
|x_3|^\alpha u_\theta \in L^p(0, T; L^p(\mathbb{R}^3)),
\]
\[
\frac{2}{q} + \frac{3}{p} = 1 - \alpha, 0 \leq \alpha < \frac{1}{4}, \quad \frac{3}{1 - \alpha} < p \leq \infty, \quad \frac{2}{1 - \alpha} \leq q < \infty.
\]
(1.17)
or \( u_\theta |x_3|^\alpha \in L^\infty(0, T; L^{\frac{3}{1 - \alpha}}(\mathbb{R}^3)) \) and the norm of \( \|u_\theta |x_3|^\alpha\|_{L^\infty(0, T; L^{\frac{3}{1 - \alpha}}(\mathbb{R}^3))} \) is sufficiently small, then \( u \) is smooth in \((0, T] \times \mathbb{R}^3\).

Besides, the Hardy-Sobolev inequality (1.12) allows us to slightly improve the result by Chen, Fang and Zhang [11]. The corresponding results can be stated as follows.

**Theorem 1.5.** Let \( u \) be an axisymmetric weak solution of the Navier-Stokes system (1.13) associated with the axisymmetric initial data \( u_0 \in H^2(\mathbb{R}^3) \) satisfying \( \text{div} u_0 = 0 \). If
\[
r^\alpha u_\theta \in L^p(0, T; L^{p_1, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_3, \infty}(\mathbb{R})), \quad \frac{2}{q} + \frac{1}{p_3} + \frac{1}{p_2} + \frac{1}{p_1} = 1 - \alpha,
\]
(1.18)
\[
0 \leq \alpha < \frac{1}{2}, \quad \frac{3}{1 - \alpha} < p_i \leq \infty, \quad \frac{2}{1 - \alpha} \leq q < \infty.
\]
or \( r^\alpha u_\theta \in L^\infty(0, T; L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R})) \) with \( \frac{1}{p_3} + \frac{1}{p_2} + \frac{1}{p_1} = 1 - \alpha \) and the sufficiently small norm of \( \|r^\alpha u_\theta\|_{L^\infty(0, T; L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R}))} \), then \( u \) is smooth in \((0, T] \times \mathbb{R}^3\).

**Remark 1.5.** In view of inclusion relationship \( L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R}) \subset L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R}) \) and \( L^{p_1, \infty}(\mathbb{R}) \subset L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R}) \), we know that the condition
\[
r^\alpha u_\theta \in L^q(0, T; L^{\hat{q}}(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{1}{p_3} + \frac{1}{p_2} + \frac{1}{p_1} = 1 - \alpha, 0 \leq \alpha < \frac{1}{2}, \quad \frac{3}{1 - \alpha} < p_i \leq \infty, \quad \frac{2}{1 - \alpha} \leq q \leq \infty,
\]
or
\[
r^d u_\theta \in L^q(0, T; L^{p_3, \infty}(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p_1} = 1 - \alpha, 0 \leq \alpha < \frac{1}{2}, \quad \frac{3}{1 - d} < p_i \leq \infty, \quad \frac{2}{1 - \alpha} \leq q \leq \infty,
\]
ensures the non-breakdown of solutions of the Navier-Stokes equations, which is also a generalization of (1.14).

**Remark 1.6.** As Theorem 1.5, one can generalize (1.16) to the anisotropic Lorentz spaces. We leave this to the interesting readers.

In the spirit of Theorem 1.5, we can show similar result involving radial component \( r^\alpha u_r \).

**Theorem 1.6.** Let \( u \) be an axisymmetric weak solution of the Navier-Stokes system (1.13) associated with the axisymmetric initial data \( u_0 \in H^2(\mathbb{R}^3) \) satisfying \( \text{div} u_0 = 0 \). If
\[
r^\alpha u_r \in L^p(0, T; L^{\hat{p}}(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{1}{p_3} + \frac{1}{p_2} + \frac{1}{p_1} = 1 - \alpha,
\]
(1.19)
\[
0 \leq \alpha < \frac{1}{2}, \quad \frac{3}{1 - \alpha} < p_i \leq \infty, \quad \frac{2}{1 - \alpha} \leq q \leq \infty,
\]
or \( r^\alpha u_\theta \in L^\infty(0, T; L^{p_3, \infty}(\mathbb{R})L^{p_2, \infty}(\mathbb{R})L^{p_1, \infty}(\mathbb{R})) \) with \( \frac{1}{p_3} + \frac{1}{p_2} + \frac{1}{p_1} = 1 - \alpha \) and its norm is sufficiently small, then \( u \) is smooth in \((0, T] \times \mathbb{R}^3\).
Remark 1.7. This theorem is an improvement of corresponding results in [21].

This paper is organized as follows. In section 2, we list some basic fact of the various functions spaces used in this paper and the auxiliary lemma for the discussion of the axisymmetric Navier-Stokes equations. The section 3 is devoted to the proof of Hardy-Sobolev type inequalities. In Section 4, we are concerned with the sufficient regularity conditions for weak solutions of the 3D axisymmetric Navier-Stokes equations by applications of the Hardy-Sobolev type inequalities established in last section.

2 Function spaces and key auxiliary lemmas

For \( p \in [1, \infty] \), the notation \( L^p(0, T; X) \) stands for the set of measurable function \( f \) on the interval \((0, T)\) with values in \( X \) and \( \|f\|_X \) belonging to \( L^p(0, T) \). The Fourier transform \( \hat{f} \) of a Schwartz function \( f \) on \( \mathbb{R}^n \) is defined as \( \hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx \), and the inverse Fourier transform \( f^\vee \) is given by \( f^\vee(\xi) = \hat{f}(-\xi) \) for all \( \xi \in \mathbb{R}^n \). Furthermore, for \( s \geq 0 \), we define \( \Lambda^s f \) by \( \Lambda^s \hat{f}(\xi) = (\sum_{i=1}^n |\xi_i|^2)^{s/2} \hat{f}(\xi) \), where the notation \( \Lambda \) stands for the square root of negative Laplacian \( (-\Delta)^{1/2} \). Similarly, we denote \( \Lambda^s_{x_1, \ldots, x_k} f(\xi) = (\sum_{i=1}^k |\xi_i|^2)^{s/2} \hat{f}(\xi) \). We will use \( C \) to denote an absolute constant which may be different from line to line unless otherwise stated.

Next, we recall the definition of Lorentz spaces. Denote the distribution function of a measurable function \( f \) on \( \mathbb{R}^n \) by \( f_\ast \) defined on \([0, \infty)\) by

\[
f_\ast(\alpha) = |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|.\]

The decreasing rearrangement of \( f \) is the function \( f^\ast \) defined on \([0, \infty)\) by

\[
f^\ast(t) = \inf\{\alpha > 0 : f_\ast(\alpha) \leq t\}.
\]

For \( p, q \in (0, \infty) \), we write

\[
\|f\|_{L^{p,q} (\mathbb{R}^n)} = \left\| \left( \int_0^\infty \left( \int_0^t f^\ast(s) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L^p (\mathbb{R}^n)} = \left\{ \begin{array}{ll}
\left( \int_0^\infty \left( \int_0^t f^\ast(s) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} , & \text{if } q < \infty, \\
\sup_{t>0} t^{\frac{1}{q}} f^\ast(t) , & \text{if } q = \infty.
\end{array} \right.
\]

Furthermore,

\[
L^{p,q} (\mathbb{R}^n) = \{ f : f \text{ is a measurable function on } \mathbb{R}^n \text{ and } \|f\|_{L^{p,q} (\mathbb{R}^n)} < \infty \}.
\]

It is well-known that \( L^{\infty, \infty} = L^\infty \), \( L^{q,q} = L^q \) and \( L^{\infty, q} = \{0\} \) for \( 0 < q < \infty \).

The study of anisotropic Lebesgue spaces \( L^q (\mathbb{R}^n) \) originated from Benedek-Panzone’s work [3]. A function \( f \) belongs to the anisotropic Lebesgue space \( L^q (\mathbb{R}^n) \) if

\[
\|f\|_{L^q (\mathbb{R}^n)} = \left\| f \right\|_{L^{q_1}_{\mathbb{R}} L^{q_2}_{\mathbb{R}} \cdots L^{q_n}_{\mathbb{R}} (\mathbb{R}^n)} = \left\| \cdots \left\| f \right\|_{L^{q_1}_{\mathbb{R}} (\mathbb{R})} \left\| f \right\|_{L^{q_2}_{\mathbb{R}} (\mathbb{R})} \cdots \left\| f \right\|_{L^{q_n}_{\mathbb{R}} (\mathbb{R})} \right\|_{L^\infty (\mathbb{R})} < \infty.
\]
Mixed Lorentz space $L^{\vec{p}, \vec{q}}(\mathbb{R}^n)$ was introduced in [4, 13, 18] and its norm is determined by

$$
\|f\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)} = \|f\|_{L^{p_1,q_1}_1 L^{p_2,q_2}_2 \cdots L^{p_n,q_n}_n(\mathbb{R}^n)} \quad \\quad = \| \cdots \| f \|_{L^{p_1,q_1}(\mathbb{R})} \| L^{p_2,q_2}(\mathbb{R}) \cdots \| L^{p_n,q_n}(\mathbb{R}) < \infty.
$$

For the convenience of readers, we present some properties of mixed Lorentz spaces as follows.

- Hölder’s inequality in mixed Lorentz spaces [4, 18]

$$
\|fg\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)} \leq C \|f\|_{L^{r_1,\vec{s}}(\mathbb{R}^n)} \|g\|_{L^{r_2,\vec{s}}(\mathbb{R}^n)},
$$
  
  with \( \frac{1}{\vec{p}} = \frac{1}{r_1} + \frac{1}{r_2}, \frac{1}{\vec{q}} = \frac{1}{s_1} + \frac{1}{s_2}, 0 < r_1, r_2, s_1, s_2 \leq \infty \).

- The mixed Lorentz spaces increase as the exponent $\vec{q}$ increases [4, 18]

  For $0 < \vec{p} \leq \infty$ and $0 < \vec{q}^* < \vec{q} \leq \infty$,

$$
\|f\|_{L^{\vec{p}, \vec{q}^*}(\mathbb{R}^n)} \leq C \|f\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)}. \quad \quad (2.1)
$$

- Sobolev inequality in mixed Lorentz spaces [4, 18], for $1 \leq \vec{\ell} \leq \infty$

$$
\|f\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)} \leq C \|(-\Delta)^{\frac{s}{2}} f\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)}, \quad \text{with} \quad \sum_{i=1}^{n} \left( \frac{1}{r_i} - \frac{1}{p_i} \right) = s, 1 < r_i < p_i < \infty. \quad \quad (2.2)
$$

- Young inequality in mixed Lorentz spaces [4, 18]

  Let $1 < \vec{p}^*, \vec{q}, \vec{r}^* \leq \infty$, $0 < \vec{s}_1, \vec{s}_2 \leq \infty$, $\frac{1}{\vec{p}^*} + \frac{1}{\vec{q}} = \frac{1}{\vec{r}^*} + 1$, and $\frac{1}{\vec{s}} = \frac{1}{\vec{s}_1} + \frac{1}{\vec{s}_2}$. Then there holds

$$
\|fg\|_{L^{\vec{p}, \vec{q}}(\mathbb{R}^n)} \leq C \|f\|_{L^{\vec{p}, \vec{r}^*}(\mathbb{R}^n)} \|g\|_{L^{\vec{s}, \vec{r}^*}(\mathbb{R}^n)}. \quad \quad (2.3)
$$

Finally, we recall two known results involving the regularity of the 3D axisymmetric Navier-Stokes equations.

**Lemma 2.1.** ([10, 26]) Assume $u$ is the smooth axisymmetric solution of equations (1.13), $\omega = \text{curl } u$, then there holds

$$
\frac{u_r}{r} = \partial_{x_3} \Delta^{-1} \Gamma - 2 \frac{\partial}{r} \Delta^{-1} \partial_{x_3} \Delta^{-1} \Gamma. \quad \quad (2.5)
$$

In addition, for $1 < p < \infty$, it is valid that

$$
\left\| \nabla \frac{u_r}{r} \right\|_{L^p} \leq C \| \Gamma \|_{L^p}, \quad \quad (2.6)
$$

and

$$
\left\| \nabla^2 \frac{u_r}{r} \right\|_{L^p} \leq C \| \partial_{x_3} \Gamma \|_{L^p}. \quad \quad (2.7)
$$

**Lemma 2.2.** ([10]) Let $u \in C([0, T); H^2(\mathbb{R}^3)) \cap L^2([0, T); H^2(\mathbb{R}^3))$ be the axisymmetric solution of the Navier-Stokes equations with the axisymmetric divergence-free initial data $u_0$. If $T < \infty$ and $\Gamma \in L^\infty([0, T); L^2(\mathbb{R}^3))$, then $u$ can be continued beyond $T$. 


3 Proof of anisotropic Hardy-Sobolev inequality

This section is devoted to the proof of various Hardy-Sobolev type inequalities, in which the main tool is the mixed Lorentz spaces. Our critical observation is that the function

$$\left[ \prod_{j=1}^{i} \left( \prod_{\ell=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{\alpha_j} \right| \right) \right]^{-1} \in \text{anisotropic Lorentz space } L^{p_i, \infty}_{\alpha_i} (\mathbb{R}^{k_1}) \cdots L^{p_i, \infty}_{\alpha_i} (\mathbb{R}^{k_i}).$$

To illustrate the above argument, we begin with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Notice that $|x|^{-s} \in L^{\alpha_i, \infty}(\mathbb{R}^n)$. Combing this with the Hölder inequality (2.1) in anisotropic Lorentz spaces, we infer that

$$\left\| \frac{f(x)}{|x|^s} \right\|_{L^{\alpha, \infty}(\mathbb{R}^n)} \leq C \left\| |x|^{-s} \right\|_{L^{\alpha, \infty}(\mathbb{R}^n)} \left\| f \right\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C \| f \|_{L^{p, \infty}(\mathbb{R}^n)},$$

where $p_i^* = \frac{np_i}{n - np_i}$ for $1 \leq i \leq n$.

The Sobolev inequality (2.3) in mixed Lorentz spaces further guarantees that

$$\| f \|_{L^{p, \infty}(\mathbb{R}^n)} \leq C \| \Lambda^s f \|_{L^{\alpha, \infty}(\mathbb{R}^n)}.$$

The proof of this theorem is finished. \(\square\)

Now we turn our attention to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Thanks to the celebrated Fubini’s theorem for the $L^p(\mathbb{R}^n)$ norm, it suffices to prove (1.10) for the case that $r_m = m$ with $1 \leq m \leq \Sigma_{j=1}^{i} k_j$. According to the definition of $r_j$, we see that, for $1 \leq j \leq i$ and $1 \leq \ell \leq k_j$, $r_j \geq |x_{\sum_{m=1}^{j-1} k_m + \ell}|$.

It turns out that

$$r_{ij} = (r_j)^{k_j} \geq \prod_{\ell=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{\alpha_j} \right|. $$

Hence, we see that

$$\left\| f(x_1, x_2, \cdots, x_n) \right\|_{L^p(\Pi_{j=1}^{i} \mathbb{R}^{k_j})} \leq \left\| \prod_{j=1}^{i} \left( \prod_{\ell=1}^{k_j} \frac{x_{\sum_{m=1}^{j-1} k_m + \ell}^{\alpha_j}}{r_{ij}} \right) \right\|_{L^p(\Pi_{j=1}^{i} \mathbb{R}^{k_j})},$$

It is clear that

$$\left\| \prod_{j=1}^{i} \left( \prod_{\ell=1}^{k_j} \frac{x_{\sum_{m=1}^{j-1} k_m + \ell}^{\alpha_j}}{r_{ij}} \right) \right\|_{L^{p_i, \infty}(\mathbb{R}^{k_1}) \cdots L^{p_i, \infty}(\mathbb{R}^{k_i})} < \infty.$$
By the Hölder inequality (2.1) and Sobolev inequality (2.3) in mixed Lorentz spaces, we observe that
\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{a_j / k_j} \right|} \right\|_{L^p(\Pi_j^l R^{k_j})} \leq C \left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{a_j / k_j} \right|} \right\|_{L^p(\Pi_j^l R^{k_j})} \leq C \left\| \Lambda_{\alpha_1, \cdots, \alpha_i} f \right\|_{L^p(\Pi_j^l R^{k_j})}. \]

It follows that
\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{a_j / k_j} \right|} \right\|_{L^p(\Pi_j^l R^n)} \leq C \left\| \Lambda_{\alpha_1, \cdots, \alpha_i} f \right\|_{L^p(\Pi_j^l R^n)}. \]

Consequently, we complete the proof of this theorem. \(\square\)

**Proof of Corollary 1.3.** It suffices to notice that
\[
\left\| \frac{f(x_1, x_2, \cdots, x_n)}{\prod_{j=1}^{k_j} \left| x_{\sum_{m=1}^{j-1} k_m + \ell}^{a_j / k_j} \right|} \right\|_{L^p(\Pi_j^l R^{k_j})} \leq C \left\| \Lambda_{\alpha_1, \cdots, \alpha_i} f \right\|_{L^p(\Pi_j^l R^{k_j})}, \]

where \(\frac{1}{p_j} = \frac{1}{p_j} - \frac{\alpha_j}{k_j}\) for \(1 \leq j \leq i\). Arguing in the same manner as above, we can achieve the proof of this corollary. We leave this to the interested readers. \(\square\)

### 4 An application of Hardy-Sobolev inequality to the axisymmetric Navier-Stokes equations

In this section, we will show that the anisotropic Hardy-Sobolev inequality derived in the last section is useful for studying the solutions of 3D axisymmetric Navier-Stokes equations.

**Proof of Theorem 1.4.** For an axisymmetric function \(f\), there holds \(|\partial_r f|^2 + |\partial_{x_3} f|^2 = |\nabla f|^2\). Hence, taking the inner product of \(\Phi\) equation in (1.15) with \(\Phi\) and integrating by parts yields
\[
\frac{1}{2} \frac{d}{dt} \|\Phi\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Phi\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \Phi(\omega_r \partial_r + \omega_3 \partial_{x_3}) \frac{u_r}{r} dx = \int_{\mathbb{R}^3} u_0(\partial_r \Phi \partial_r + \partial_3 \partial_{x_3} \Phi - \partial_{x_3} \frac{u_r}{r} \partial_r \Phi) dx. \tag{4.1} \]

Similarly,
\[
\frac{1}{2} \frac{d}{dt} \|\Gamma\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Gamma\|_{L^2(\mathbb{R}^3)}^2 = -2 \int_{\mathbb{R}^3} u_0 \frac{\Gamma}{r} \Phi dx. \tag{4.2} \]
We estimate the terms on the right hand side of (4.1) and (4.2) in the following two cases respectively.

(1) If
\[
|x_3|^\alpha u_0 \in L^q(0, T; L^p(\mathbb{R}^3)),
\]
\[
\frac{2}{q} + \frac{3}{p} = 1 - \alpha, 0 \leq \alpha < \frac{1}{4}, \ \frac{3}{1 - \alpha} < p \leq \infty, \ \frac{2}{1 - \alpha} \leq q < \infty.
\]

For the first term on the right hand of (4.1), it follows from the Hölder inequality that
\[
\left| \int_{\mathbb{R}^3} u_0 \partial_r \frac{u_r}{r} \partial_{x_3} \Phi \, dx \right| = \left| \int_{\mathbb{R}^3} u_0 |x_3|^\alpha \frac{\partial_r u_r}{|x_3|^\alpha} \partial_{x_3} \Phi \, dx \right|
\leq \| u_0 |x_3|^\alpha \|_{L^p(\mathbb{R}^3)} \left\| \frac{\partial_r u_r}{|x_3|^\alpha} \right\|_{L^{\frac{2p}{3-\alpha}}(\mathbb{R}^3)} \left\| \partial_{x_3} \Phi \right\|_{L^2(\mathbb{R}^3)}.
\]

Since \( p > \frac{3}{1-\alpha} \) and \( \alpha < \frac{1}{4} \) ensure \( \alpha < \frac{p-2}{2p} \), the Hardy inequality in Theorem 1.2 can be applied
\[
\left\| \frac{\partial_r u_r}{|x_3|^\alpha} \right\|_{L^{\frac{2p}{3-\alpha}}(\mathbb{R}^3)} \leq C \left\| \Lambda^\alpha \partial_r u_r \right\|_{L^{\frac{2p}{p-3-\alpha}}(\mathbb{R}^3)}.
\]

Inserting (4.5) into (4.4) and using the Young inequality yield
\[
\left| \int_{\mathbb{R}^3} u_0 \partial_r \frac{u_r}{r} \partial_{x_3} \Phi \, dx \right| \leq C \| u_0 |x_3|^\alpha \|_{L^p(\mathbb{R}^3)} \left\| \partial_\Gamma \right\|_{L^2(\mathbb{R}^3)^2} + \frac{1}{4} \left( \left\| \nabla \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2 \right).
\]

Likewise,
\[
\left| \int_{\mathbb{R}^3} u_0 \partial_\Gamma \frac{u_r}{r} \partial_{x_3} \Phi \, dx \right| \leq C \| u_0 |x_3|^\alpha \|_{L^p(\mathbb{R}^3)} \left\| \nabla \right\|_{L^2(\mathbb{R}^3)^2} + \frac{1}{4} \left( \left\| \nabla \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2 \right).
\]

It remains to bound the term on the right hand side of (4.2). To this end, using the Hölder inequality, we conclude that
\[
\left| \int_{\mathbb{R}^3} u_0 \frac{\Gamma}{r} \Phi \, dx \right| = \left| \int_{\mathbb{R}^3} u_0 |x_3|^\alpha \frac{\Gamma}{|x_3|^\alpha} \frac{\Phi}{|x_3|^\alpha} \, dx \right|
\leq \| u_0 |x_3|^\alpha \|_{L^p(\mathbb{R}^3)} \left\| \frac{\Gamma}{|x_3|^\alpha} \right\|_{L^{\frac{2p}{3-\alpha}}(\mathbb{R}^3)} \left\| \frac{\Phi}{|x_3|^\alpha} \right\|_{L^{\frac{2p}{3-\alpha}}(\mathbb{R}^3)}.
\]

For \( p > \frac{3}{1-\alpha} \) and \( \alpha < \frac{1}{4} \), using Hardy-Sobolev inequality (1.10) and the Gagliardo-Nirenberg inequality again yield
\[
\left\| \frac{\Gamma}{|x_3|^\alpha} \right\|_{L^{\frac{2p}{3-\alpha}}(\mathbb{R}^3)} \leq C \left\| \Lambda^\alpha \Gamma \right\|_{L^{\frac{2p}{p-3-\alpha}}(\mathbb{R}^3)} \leq C \left\| \nabla \right\|_{L^2(\mathbb{R}^3)}^2 \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2.
\]
Similarly, 
\[ \left\| \frac{\Phi}{|x_3|^{\frac{2p}{2p-\alpha}}} \right\|_{L^p}\leq C \left\| \Lambda^{\frac{1+\alpha}{2}} \alpha \right\|_{L^p} \left\| \Phi \right\|_{L^p} \leq C \left\| \Phi \right\|_{L^p} \left\| \nabla \Phi \right\|_{L^p}. \]

Hence, by using the Young inequality, we arrive at
\[ \left| \int_{\mathbb{R}^3} u_\theta \frac{\Gamma}{r} \Phi dx \right| \leq C \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \left( \left\| \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} \right) + \frac{1}{4} \left( \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right). \] (4.8)

Combining (4.1) and (4.2) with (4.6), (4.7) and (4.8), we find
\[ \frac{d}{dt} \left( \left\| \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right) + \left( \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right) \]
\[ \leq C \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \left( \left\| \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right). \]

The Gronwall lemma and (1.7) enables us to obtain
\[ \left\| \Gamma \right\|_{L^\infty(0,T;L^2)} \leq C < \infty. \]

(2) If \( u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \) and the norm of \( \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \) is sufficiently small.

Similarly to (4.4), we know that
\[ \left\| \frac{\partial_x u_r}{|x_3|^\alpha} \right\|_{L^{\frac{6}{2\alpha}}(\mathbb{R}^3)} \leq C \left\| \Lambda^\alpha \frac{\partial_x u_r}{|x_3|^\alpha} \right\|_{L^{\frac{6}{2\alpha}}(\mathbb{R}^3)} \leq C \left\| \nabla \partial_x u_r \right\|_{L^{2}(\mathbb{R}^3)} \leq C \left\| \nabla \Gamma \right\|_{L^{2}(\mathbb{R}^3)}, \]

which turns out that
\[ \left| \int_{\mathbb{R}^3} u_\theta \partial_x \frac{u_r}{r} \partial_x \Phi \right| \leq C \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \left( \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right). \] (4.9)

In the same manner as above, there holds
\[ \left| \int_{\mathbb{R}^3} u_\theta \partial_x \frac{u_r}{r} \partial_x \Phi \right| \leq C \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \left( \left\| \nabla \Phi \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \Gamma \right\|^2_{L^2(\mathbb{R}^3)} \right). \] (4.10)

Using the Hölder inequality, we see that
\[ \left| \int_{\mathbb{R}^3} u_\theta \frac{\Gamma}{r} \Phi \right| \leq C \left\| u_\theta \right\|_{L^{1+\alpha}(\mathbb{R}^3)} \left( \left\| \Gamma \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \right) \left( \left\| \Phi \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \right). \]

Since \( 0 < \alpha < \frac{1}{4} \), then \( \frac{2}{3} < \frac{2+\alpha}{2} \) and \( \frac{2}{3} < \frac{2+\alpha}{6} \). As a result, we can use Hardy-Sobolev inequality (1.10) and Sobolev embedding theorem again to get
\[ \left\| \frac{\Gamma}{|x_3|^\alpha} \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \leq C \left\| \Lambda^{\frac{2+\alpha}{2}} \Gamma \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \leq C \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)} \]
\[ \left\| \frac{\Phi}{|x_3|^\alpha} \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \leq C \left\| \Lambda^{\frac{2+\alpha}{2}} \Phi \right\|_{L^{\frac{6}{2+\alpha}}(\mathbb{R}^3)} \leq C \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}. \]
Therefore,
\[ \left| \int_{\mathbb{R}^3} u_\theta \Gamma \Phi \right| \leq \|u_\theta|_{x_3}^\alpha\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|\nabla \Phi\|_{L^2(\mathbb{R}^3)} \|\nabla \Gamma\|_{L^2(\mathbb{R}^3)}. \] (4.11)

Plugging (4.10), (4.11) into (4.2) and (4.1), we find out that
\[
\frac{d}{dt}(\|\Phi\|_{L^2(\mathbb{R}^3)}^2 + \|\Gamma\|_{L^2(\mathbb{R}^3)}^2) + C(\|\nabla \Phi\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Gamma\|_{L^2(\mathbb{R}^3)}^2)
\leq C\|u_\theta|_{x_3}^\alpha\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}(\|\nabla \Phi\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Gamma\|_{L^2(\mathbb{R}^3)}^2).
\]

Because the norm of \(\|u_\theta|_{x_3}^\alpha\|_{L^\infty(0;T; L^{\frac{3}{2}}(\mathbb{R}^3))}\) is sufficiently small, we can obtain \(\Gamma \in L^\infty(0; T; L^2(\mathbb{R}^3))\).

At this position, Lemma 2.2 helps us to complete the proof of this theorem. \(\Box\)

Proof of Theorem 7.3. Taking advantage of the Hölder inequality (2.1) in mixed-Lorentz space, we have
\[
\left| \int_{\mathbb{R}^3} u_\theta \partial_r \frac{u_r}{r} \partial_{x_3} \Phi \right| \leq C\|r^\alpha u_\theta\|_{L^{p_1,\infty}_x L^{p_2,\infty}_r L^{p_3, \infty}_t} \|\partial_r \frac{u_r}{r}\|_{L^{p_1, \infty}_x L^{p_2, \infty}_r L^{p_3, \infty}_t} \|\partial_{x_3} \Phi\|_{L^2(\mathbb{R}^3)}.
\] (4.12)

In view of \(p_1, p_2 > \frac{2}{1-\alpha}\), we can apply the Hardy-Sobolev inequality (1.12) in Corollary 1.3 to conclude that
\[
\left\| \partial_r \frac{u_r}{r} \right\|_{L^{p_1, \infty}_x L^{p_2, \infty}_r L^{p_3, \infty}_t} \leq C \left\| \Delta^{\frac{3}{2}} \partial_{x_1} \partial_{x_2} \frac{u_r}{r} \right\|_{L^2(\mathbb{R}^3)}.
\]

The Sobolev embedding (2.3) and the Gagliardo-Nirenberg inequality further help us to derive that
\[
\left\| \Delta^{\frac{3}{2}} \partial_{x_1} \partial_{x_2} \frac{u_r}{r} \right\|_{L^{p_1, \infty}_x L^{p_2, \infty}_r L^{p_3, \infty}_t} \leq C \left\| \nabla \phi \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)} \alpha + \sum_{i=1}^3 \frac{1}{p_i}.
\]

Therefore, we arrive at
\[
\left\| \partial_r \frac{u_r}{r} \right\|_{L^{p_1, \infty}_x L^{p_2, \infty}_r L^{p_3, \infty}_t} \leq C \left\| \nabla \phi \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)} \alpha + \sum_{i=1}^3 \frac{1}{p_i}.
\]

Substituting this into (4.12), we infer that
\[
\left| \int_{\mathbb{R}^3} u_\theta \partial_r \frac{u_r}{r} \partial_{x_3} \Phi \right| \leq C\|r^\alpha u_\theta\|_{L^{p_1,\infty}_x L^{p_2,\infty}_r L^{p_3, \infty}_t} \|\nabla \phi \|_{L^2(\mathbb{R}^3)} \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)} \alpha + \sum_{i=1}^3 \frac{1}{p_i}.
\]

Exactly as in the above derivation, we know that
\[
\left| \int_{\mathbb{R}^3} u_\theta \partial_r \frac{u_r}{r} \partial_{x_3} \Phi \right| \leq C\|r^\alpha u_\theta\|_{L^{p_1,\infty}_x L^{p_2,\infty}_r L^{p_3, \infty}_t} \|\nabla \phi \|_{L^2(\mathbb{R}^3)} \alpha + \sum_{i=1}^3 \frac{1}{p_i} \frac{1}{p_i} \left\| \nabla \phi \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \phi \right\|_{L^2(\mathbb{R}^3)} \alpha + \sum_{i=1}^3 \frac{1}{p_i}.
\]
Using the Hölder inequality (2.1) once again, we get

\[
\left| \int_{\mathbb{R}^3} u_\alpha \frac{\Gamma}{r} \Phi \right| = \int_{\mathbb{R}^3} u_\alpha \left| r^{\frac{\alpha}{2}} \frac{\Gamma}{r^{\frac{1}{2}}} \right| \leq \| r^{\alpha} u_\alpha \|_{L^{p_3,\infty}_x L^{p_2,\infty}_x L^{p_1,\infty}_x(\mathbb{R}^3)} \left\| \frac{\Gamma}{r^{\frac{1}{2}}} \right\|_{L^{\frac{p_3}{2},\infty}_x L^{\frac{p_2}{2},\infty}_x L^{\frac{p_1}{2},\infty}_x(\mathbb{R}^3)} \left\| \frac{\Phi}{r^{\frac{1}{2}}} \right\|_{L^{\frac{p_3}{2},\infty}_x L^{\frac{p_2}{2},\infty}_x L^{\frac{p_1}{2},\infty}_x(\mathbb{R}^3)}.
\]

(4.13)

According to \( \frac{1+\alpha}{2} < \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \), we derive from the Hardy-Sobolev inequality (1.12) in Corollary 1.3 and the inclusion relation (2.2) that

\[
\left\| \frac{\Gamma}{r^{\frac{1}{2}}} \right\|_{L^{\frac{p_3}{2},\infty}_x L^{\frac{p_2}{2},\infty}_x L^{\frac{p_1}{2},\infty}_x(\mathbb{R}^3)} \leq C \left\| \Lambda_{x_1,x_2} \frac{\Gamma}{r^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^3)}.
\]

(4.13)

From the Sobolev embedding (2.3) and the Gagliardo-Nirenberg inequality, we get

\[
\left\| \Lambda_{x_1,x_2} \frac{\Gamma}{r^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \Lambda \sum_{i=1}^3 \frac{1}{2p_i} + \frac{1+\alpha}{2} \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^{\frac{1-\alpha-\sum_{i=1}^3 \frac{1}{p_i}}{4}} \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)}^{\frac{1+\alpha+\sum_{i=1}^3 \frac{1}{p_i}}{4}}.
\]

Hence, there holds

\[
\left\| \frac{\Gamma}{r^{\frac{1}{2}}} \right\|_{L^{\frac{p_3}{2},\infty}_x L^{\frac{p_2}{2},\infty}_x L^{\frac{p_1}{2},\infty}_x(\mathbb{R}^3)} \leq C \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^{\frac{1-\alpha-\sum_{i=1}^3 \frac{1}{p_i}}{4}} \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)}^{\frac{1+\alpha+\sum_{i=1}^3 \frac{1}{p_i}}{4}}. \tag{4.14}
\]

Similarly, we have

\[
\left\| \frac{\Phi}{r^{\frac{1}{2}}} \right\|_{L^{\frac{p_3}{2},\infty}_x L^{\frac{p_2}{2},\infty}_x L^{\frac{p_1}{2},\infty}_x(\mathbb{R}^3)} \leq C \left\| \Phi \right\|_{L^2(\mathbb{R}^3)}^{\frac{1-\alpha-\sum_{i=1}^3 \frac{1}{p_i}}{4}} \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^{\frac{1+\alpha+\sum_{i=1}^3 \frac{1}{p_i}}{4}}. \tag{4.15}
\]

Putting (4.13), (4.14), (4.15) together, we obtain

\[
\left| \int_{\mathbb{R}^3} u_\alpha \frac{\Gamma}{r} \Phi \right| \leq C \left\| r^{\alpha} u_\alpha \right\|_{L^{p_3,\infty}_x L^{p_2,\infty}_x L^{p_1,\infty}_x(\mathbb{R}^3)} \left( \left\| \Gamma \right\|_{L^2} + \left\| \Phi \right\|_{L^2(\mathbb{R}^3)} \right)^{1-\alpha-\sum_{i=1}^3 \frac{1}{p_i}} \times \left( \left\| \nabla \Phi \right\|_{L^2} + \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)} \right)^{1+\alpha+\sum_{i=1}^3 \frac{1}{p_i}}.
\]

Putting the above estimates together, we arrive at

\[
\frac{d}{dt} \left( \left\| \Phi \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 \right) + C \left( \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 \right) \leq C \left\| r^{\alpha} u_\alpha \right\|_{L^{p_3,\infty}_x L^{p_2,\infty}_x L^{p_1,\infty}_x(\mathbb{R}^3)} \left( \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \Phi \right\|_{L^2(\mathbb{R}^3)}^2 \right)^{1-\alpha-\sum_{i=1}^3 \frac{1}{p_i}} \times \left( \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)} \right)^{1+\alpha+\sum_{i=1}^3 \frac{1}{p_i}}.
\]

Case 1: If \( \alpha + \sum_{i=1}^3 \frac{1}{p_i} < 1 \), we conclude by the Young inequality that

\[
\frac{d}{dt} \left( \left\| \Phi \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 \right) + C \left( \left\| \nabla \Phi \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 \right) \leq C \left\| r^{\alpha} u_\alpha \right\|_{L^{p_3,\infty}_x L^{p_2,\infty}_x L^{p_1,\infty}_x(\mathbb{R}^3)} \left( \left\| \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \Phi \right\|_{L^2(\mathbb{R}^3)}^2 \right) + \frac{1}{8} \left( \left\| \nabla \Gamma \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \partial_{x_4} \Phi \right\|_{L^2(\mathbb{R}^3)}^2 \right).
\]
The Gronwall inequality and Lemma 2.2 entail us to achieve the proof.

Case 2: If $\alpha + \sum_{i=1}^{3} \frac{1}{p_i} = 1$

$$\frac{d}{dt} \left( \| \Phi \|_{L^2(\mathbb{R}^3)} + \| \Gamma \|_{L^2(\mathbb{R}^3)}^2 \right) + C \left( \| \nabla \Phi \|_{L^2(\mathbb{R}^3)}^2 + \| \Gamma \|_{L^2(\mathbb{R}^3)}^2 \right)$$

$$\leq C \| r^\alpha u_\theta \|_{L_{p_1}^{3,\infty} L_{p_2}^{3,\infty} L_{p_3}^{3,\infty}} (\| \nabla \Phi \|_{L^2} + \| \nabla \Phi \|_{L^2(\mathbb{R}^3)})^2$$

$$\leq \frac{1}{2} \left( \| \nabla \Phi \|_{L^2} + \| \nabla \Phi \|_{L^2(\mathbb{R}^3)} \right)^2,$$

which leads to $\Phi, \Gamma \in L^\infty (L^2(0, T; \mathbb{R}^3)).$ At this stage, one can use the Lemma 2.2 once again to finish the proof of this theorem. \qed

Proof of Theorem 1.6. According to the result in [20, 28, 37], it suffices to show that $\frac{u_\theta}{r} \in L^4(0, T; L^4(\mathbb{R}^3))$. Indeed, from (1.13) and (1.13), we conclude by the standard energy estimate that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |u_\theta|^\ell dx + \ell - 1 \int_{\mathbb{R}^3} |\nabla u_\theta|^2 dx + \int_{\mathbb{R}^3} \frac{|u_\theta|^\ell}{r} dx = \int_{\mathbb{R}^3} \frac{1}{r} u_r |u_\theta|^\ell dx. \quad (4.16)$$

The Young inequality ensures that

$$\left| \int_{\mathbb{R}^3} \frac{1}{r} u_r |u_\theta|^\ell dx \right| \leq \int_{\mathbb{R}^3} u_r r^\alpha |u_\theta|^\ell dx \leq \int_{\mathbb{R}^3} |u_r r^\alpha| \frac{\ell - \ell(1+\alpha)}{r^{1+\alpha}} |u_\theta|^{\ell(1+\alpha)} dx$$

$$\leq \int_{\mathbb{R}^3} |u_r r^\alpha| \frac{2}{r^{1+\alpha}} |u_\theta|^\ell + \frac{1}{8} \int_{\mathbb{R}^3} |u_\theta|^\ell dx.$$ 

Taking advantage of the H"older inequality, we see that

$$\left| \int_{\mathbb{R}^3} (u_r r^\alpha)^{\frac{2}{1+\alpha}} |u_\theta|^\ell dx \right| \leq C \| u_r r^\alpha \|_{L_{1+\alpha}^{\infty}} \| u_\theta \|_{L_{2}^{2 \frac{2}{1+\alpha}} \times \infty}(\mathbb{R}^3)^2 \| u_\theta \|_{L_{2}^{2(1+\alpha)}} \| u_\theta \|_{L_{2}^{2 \frac{2}{1+\alpha}} \times \infty}(\mathbb{R}^3)^2.$$ 

It follows from the Sobolev embedding theorem (2.3) and the interpolation inequality that

$$\| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}} \leq C \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}}.$$ 

Based on this, there holds that

$$\int_{\mathbb{R}^3} (u_r r^\alpha)^{\frac{2}{1+\alpha}} |u_\theta|^\ell \leq C \| u_r r^\alpha \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}}.$$ 

Substituting this into (4.16), we know that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |u_\theta|^\ell dx + \ell - 1 \int_{\mathbb{R}^3} |\nabla u_\theta|^2 dx + \int_{\mathbb{R}^3} \frac{|u_\theta|^\ell}{r} dx \leq C \| u_r r^\alpha \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}} \| u_\theta \|_{L_{2}^{\frac{2}{1+\alpha}}}.$$ 

With this in hand, by arguing as was done to prove previous theorems, we complete the proof of this theorem. \qed
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