ON THE HOMOTOPY TYPE OF THE SPACE $\mathcal{R}^+(M)$

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Abstract. In this paper it is shown that the space of metrics of positive scalar curvature on a manifold is, when nonempty, homotopy equivalent to a space of metrics of positive scalar curvature that restrict to a fixed metric near a given submanifold of codimension greater or equal than 3. Our main tool is a parameterized version of the Gromov-Lawson construction, which was used to show that the existence of a metric of positive scalar curvature on a manifold was invariant under surgeries in codimension greater or equal than 3.

1. Introduction

The topology of the space of metrics of negative scalar curvature $\mathcal{R}^-(M^n)$ was described by Lohkamp \cite{Lohkamp92}, and it turns out that $\mathcal{R}^-(M^n)$ is nonempty and contractible, when $n \geq 3$ and $M^n$ is closed.

On the other hand, the questions of existence and classification for metrics of positive scalar curvature are much more complicated. There are manifolds that do not admit a metric of positive scalar curvature. Any spin manifold $M^n$ for which $n \geq 5$ and $\tilde{A}(M^n) \neq 0$ is such a manifold. On the classification side, R. Carr \cite{Carr88} shows that the space of positive scalar curvature metrics on a sphere $S^{4m-1}$, $m \geq 2$ has infinitely many connected components. Also, in general, it is not known when a given closed manifold admits a metric of positive scalar curvature.

In this paper, we concern ourselves with the homotopy type of the space $\mathcal{R}^+(M)$. Since Surgery Theory is a major tool in studying manifolds, it is important to understand how the homotopy type of $\mathcal{R}^+(M)$ changes under surgeries.

Let $M^n$ be a closed manifold and $\mathcal{R}^+(M^n)$ be the space of metrics of positive scalar curvature on $M^n$ (which is assumed to be nonempty throughout this paper). The topology on this space is defined by the collection of $C^k$-norms $\| \cdot \|_k$ on the space of all Riemannian metrics $\mathcal{R}(M^n)$ with respect to some reference metric $h$: $\|g\|_k = \max_{i \leq k} \sup_{M^n}|\nabla^i g|$. This topology does not depend on the choice of the metric $h$. Let $N^{n-k} \subset M^n$, $k \geq 3$, be a closed submanifold with a trivial normal bundle. We fix a tubular neighborhood $\tau: N \times D^k \to M$, an arbitrary metric $g_N$ on $N$, and a torpedo metric $g_0$ on $D^k$ (for the definition of a torpedo metric see below, page \ref{page:torpedo}), such that the metric $g_N + g_0$ has positive scalar curvature on $N \times D^k$.

Given these data, we define

(1) $\mathcal{R}_0^+(M^n) := \{ g \in \mathcal{R}^+(M^n) | \tau^*(g) = g_N + g_0 \}$.

Our main result is the following theorem.

Theorem 1.1. Suppose that $\mathcal{R}^+(M^n)$ is not empty. Then the inclusion map

$i: \mathcal{R}_0^+(M) \to \mathcal{R}^+(M^n)$

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is a homotopy equivalence.

As an easy corollary we have the Surgery Theorem.

**Theorem 1.2.** Let $S^k \to M_1^n$ be an embedding with the trivial normal bundle and $\tau_1: S^k \times D^{n-k} \to M_1^n$ be a tubular neighborhood for this embedding. Let $M_2$ be a manifold obtained by doing surgery on this tubular neighborhood. If $n - k \geq 3$ and $k \geq 2$ then $\mathcal{R}^+(M_1)$ and $\mathcal{R}^+(M_2)$ are homotopy equivalent.

The idea of the proof is as follows. From Palais [2], it follows that $\mathcal{R}^+(M)$ and $\mathcal{R}^+(M)$ are dominated by CW-complexes. Therefore, it suffices to show that the homotopy sets $\pi_k(\mathcal{R}^+(M), \mathcal{R}^+(M)) = 0$, all $k \geq 0$, and the inclusion map $i$ is a bijection on path components of both spaces.

The Gromov-Lawson construction gives us a deformation $\text{GL}$ of a compact family $g_\varepsilon \in \mathcal{R}^+(M)$ into $\mathcal{R}^+(M)$. The only problem is that $\mathcal{R}^+(M)$ is not invariant under this deformation. The saving grace of the Gromov-Lawson construction is its local $O(k)$-symmetry on normal fibres of $N$.

Thus, we overcome this problem by introducing the space $W(N, \tau)$ of metrics with a warped fibre. The space $W(N, \tau)$ is invariant under $\text{GL}$ and we show that the subspace $\mathcal{R}^0_\tau(M)$ is a weak deformation retract of $W(N, \tau)$.

The Greek letter $\kappa$ is used to denote the scalar curvature throughout the paper.

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2. The Gromov-Lawson construction

The Gromov-Lawson construction (see [GL80], [RS01]) allows one to conclude that if a closed manifold $M_2$ is obtained from a closed manifold $M_1$ by doing surgery in codimension $\geq 3$ and $M_1$ carries a metric of positive scalar curvature, then $M_2$ carries a metric of positive scalar curvature. It is our main tool in deforming a compact family $g_\varepsilon \in \mathcal{R}^+(M)$ into $\mathcal{R}^0_\tau(M)$, see Theorems 2.4, 2.6, and 2.7.

Throughout this paper, we fix a tubular neighborhood $\tau: N \times D^k_{T_0} \to M$, an arbitrary metric $g_\varepsilon$ on $N$, and a torpedo metric $g_0$ on $D^k_{T_0} \subset \mathbb{R}^k$ of radius $\epsilon_0$ and size $T_0 = \epsilon_0 T$ (see the definition below), such that $g_\varepsilon + g_0$ has positive scalar curvature on $N \times D^k$. Since we assume that $\mathcal{R}^+(M)$ is not empty, the original Gromov-Lawson construction gives us a metric $h_0$ on $M$ (of positive scalar curvature), which is isometric to $g_\varepsilon + g_0$ near $N$. We also fix a vector bundle isometry $\phi: (N \times \mathbb{R}^k, g_{\text{eucl}}) \to \left(\perp, h_0\right)$ (recall that one of our assumptions is that the normal bundle of $N$ is trivial). We take the restriction $\phi: N \times D^k_{T_0} \to N$ and define $\tau_0 := \exp^k_{h_0} \circ \phi: N \times D^k_{T_0} \to M$, where $\exp^k_{h_0}$ is the normal exponential map for the metric $h_0$. Then $\tau_0(h_0) = g_\varepsilon + g_0$. By the uniqueness theorem for tubular neighborhoods we may assume that $\tau = \tau_0$.

In general, by a torpedo metric in the disc $D^k_{T_0}$, we understand an $O(k)$-symmetric positive scalar curvature metric which is a Riemannian product with a standard euclidean $k - 1$ sphere near the boundary and is a standard euclidean $k$ sphere metric near the center of the disc. More specifically, we fix a curve $\gamma_1$ in the plane $(t, r)$ as in Figure 1, which follows a horizontal line near the point $(0, 1)$, and then joins smoothly an arc of a circle of radius $1$. The curve $\gamma_\epsilon$ is the image of $\gamma_1$ under the homothety of the plane with the coefficient $\epsilon$. If $T$ is the length of $\gamma_1$, then $\epsilon T$ is the length of $\gamma_\epsilon$. Let

$$T_\gamma := \{(x, t) \in \mathbb{R}^k \times \mathbb{R} \mid (t, |x|) \in \gamma_\epsilon\}.$$
Here the metrics on $\mathbb{R}^k$ and $\mathbb{R}$ are the standard euclidean metric. We take $T_{\gamma,\epsilon}$ together with the induced metric and call such a metric a torpedo metric of radius $\epsilon$ and size $\epsilon T$.

2.1. The Gromov-Lawson construction. We begin by briefly describing the main construction by following [RS01]. For details, see [GL80], [RS01].

We take a Riemannian product $M \times \mathbb{R}$ and consider the restriction of the product metric to the hypersurface

$$T_{\gamma}(g) = \{(y,t) \in M \times \mathbb{R} | (t, \|y\|) \in \gamma\},$$

where $\gamma$ is a $C^\infty$-curve of finite length which lies in the first (NE) quadrant of $\mathbb{R}^2$ plane with coordinates $(t,r)$, see Figure 2, and $\|y\|$ is the distance from $y$ to $N$ with respect to the metric $g$. We call this hypersurface a neck.

Definition 2.1. Let $g_s, s \in S$ be a family of pscm metrics continuously parameterized by a compact space $S$ and $\gamma: \mathbb{R} \to \mathbb{R}^2 = \{(t,r)\}$ be a $C^\infty$ isometric embedding. We call $\gamma$ an admissible curve for a family $g_s$ if the following holds:

1. $\gamma(0) = (0, r_0)$ with $r_0 > 0$ is such that $\gamma$ follows the $r$-axis linearly from $\gamma(-\infty) = (0, \infty)$ to $\gamma(0, r_0)$;
2. $\gamma$ intercepts the $t$-axis at a right angle and follows an arc of a circle (of possibly infinite radius) at this point;
3. the curve $\gamma$ crosses the line $r = r_0$ only once and is symmetric about the $t$-axis, i.e. $\gamma(L - s) = R_t \circ \gamma(L + s)$, where $L$ is a unique number such that $\gamma(L) \in t$-axis and $R_t$ is the reflection about $t$-axis;
4. the injectivity radius of the normal exponential map for all $g_s$ is strictly greater than $r_0$.

The space of all admissible curves is denoted by $\Gamma$ (note that the vertical segment is an admissible curve) and has a natural topology which is induced from the $C^\infty$-topology on $C^\infty(\mathbb{R}, \mathbb{R}^2)$.

Remark 2.1. (i) Any curve $\gamma \in \Gamma$ is uniquely determined by its part on $[0, L]$; (ii) For an admissible curve, the neck $T_{\gamma}$ can be defined over the tubular neighborhood of $N$ of radius $r_0$ by taking the part of the curve on $[0, L]$ and applying the formula 2 above.

Proposition 2.1. Let $\{g_s\}$ be a family of metrics parameterized by a compact space $S$ and $\Gamma$ be the space of admissible curves for this family. Then there exists
Thus we obtain an embedding of Figure 2, and a horizontal (along \( \mathbb{R} \))

We only add that the argument goes through for a compact space \( \mathcal{M} \). This is essentially proved in the original work [GL80] and improved in [RS01].

**Proof.** We fix a family of smooth increasing functions \( \phi: \phi(r) = r + r_0 - L \) if \( r \geq L \geq r_0 \), \( \phi(r) = r \) if \( r \leq \delta \), some \( 0 \leq \delta < r_0 \). Let \( N_\gamma \) be a submanifold of \( T_\gamma \) which is diffeomorphic to \( N \) under the projection map \( p: M \times \mathbb{R} \to N \) and \( g_\gamma \) is a metric on the neck induced from the product metric on \( M \times \mathbb{R} \).

Recall that \( L \) is the length of the curve \( \gamma \) between the point where it intersects the \( t \)-axis and the point \((0, r_0)\). The function \( \phi \) defines a diffeomorphism of the normal bundle \( \phi: \nu(N) \to \nu(N), (x, v) \mapsto (x, \frac{\phi(|v|)}{|v|} v) \). Then the composition of maps

\[
Tb_L(N_\gamma) \xrightarrow{(\exp_{g_N})^{-1}} \nu(N_\gamma) \xrightarrow{dp} \nu(N) \xrightarrow{\phi} T_{r_0}\nu(N)
\]

is a diffeomorphism of tubular neighborhoods which is the identity on all points whose distance from \( N \) is greater or equal than \( r_0 \) and less than \( r_0 + \epsilon'' \), for some small enough \( \epsilon'' > 0 \); all distances are taken with respect to the metric \( g \).

It is clear that we may add the points \( x \in M \) outside this neighborhood as \((x, 0)\). Thus we obtain an embedding of \( M \) into \( M \times \mathbb{R} \).

The induced metric on the neck can be considered as a metric on the manifold \( M \) via the pullback by the embedding that we obtain from the lemma above.

We call an admissible curve \( \gamma \) horizontal if (i) the coordinates \( r(s), t(s) \) are monotone functions of \( s \) when \( s \in (-\infty, 0] \); (ii) it follows a segment parallel to \( t \)-axis somewhere between \( r = r_0 \) and \( r = 0 \).

**Theorem 2.2.** Let \( N \) be a closed submanifold of \( M \) of codimension \( k \geq 3 \) and \( g_s \) be a continuous family of metrics of positive scalar curvature, parameterized by a compact space \( S \). Then there exist a number of points \( (t_i, r_i) \in \mathbb{R}^2 \), \( i = 0, 5 \), see Figure 2 and a horizontal (along \([t_4, t_5] \times r_4\)) admissible curve \( \gamma \) such that the scalar curvature of the neck \( T_\gamma(g_s) \) is positive for all \( s \in S \).

**Proof.** This is essentially proved in the original work [GL80] and improved in [RS01]. We only add that the argument goes through for a compact family of metrics.

The first bend is small and occurs between \( r_1 \) and \( r_2 \) and is followed by a straight line. The second bend occurs between \( r_3 \) and \( r_4 \) and ends in a horizontal segment. If \( k \) is the signed curvature of the curve \( \gamma \), then during the second bend \( k \leq \frac{3\sin \theta}{16r} \). Here, \( \theta \) is the angle between normal to \( \gamma \) and \( t \)-axis. In addition, we can arrange that

\[
k \leq \frac{9\sin \theta}{16r}
\]

will still assure the positivity of the scalar curvature during the second bend.

From the equations controlling the scalar curvature on the neck we see that the scalar curvature will always be positive on the part bending towards the \( t \)-axis.
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Figure 2.

(k ≤ 0) provided $r_4$ is small enough. We assume that the end part of $\gamma$ is exactly the torpedo curve $\gamma_c$ (for $\epsilon = r_4$).

2.2. Initial bend. Let $\gamma$ be a curve in the plane $(t, r)$, as in the statement of Theorem 2.2. In this section we prove that one can deform $\gamma$ through admissible curves into the $r$-axis, keeping the scalar curvature on the neck positive.

Any admissible curve is uniquely determined by its curvature function $k(s)$ on the interval $[0, L]$. So, we deform such a curve by deforming its curvature function.

Let $\delta_{s_0}$ be a $\delta$-cutoff function at a point $s_0 \in [s_3, s_4]$, i.e. a $C^\infty$-function equal to 1 for all $s \leq s_0$, and equal to 0 for all $s \geq s_0 + \delta$ (the interval $[s_3, s_4]$ corresponds to the part of the curve between $r_3$ and $r_4$). Let $\tilde{\gamma} := \gamma(\delta_{s_0})$ be the curve corresponding to the curvature function $\tilde{k}_{s_0}(s) := \delta_{s_0}(s)k(s)$.

Proposition 2.3. There exists a $\delta > 0$ such that for any $s_0 \in [s_3, s_4]$ the curvature $\tilde{k}$ (which is equal to $\tilde{k}_{s_0}$) of the curve $\tilde{\gamma}$ satisfies

$$\tilde{k}(s) \leq \frac{9}{16} \frac{\sin \tilde{\theta}(s)}{\tilde{r}(s)}$$

on this interval. Here, $\tilde{\theta}$ is the angle between the normal to the curve $\tilde{\gamma}$ and the $t$-axis.

Proof. Since $\gamma$ is the curve from Theorem 2.2, its curvature satisfies $k \leq \frac{\sin \theta}{2r}$ on the interval $[s_3, s_4]$. Consider the function

$$F: [s_3, s_4] \times [0, 1] \rightarrow \mathbb{R}$$

$$(s, t) \mapsto k(s + t) - \frac{9}{16} \frac{\sin \theta(s)}{r(s)}$$

We have that $F([s_3, s_4] \times 0) \leq A_0 < 0$, so there exists a $\delta > 0$ such that $F([s_3, s_4] \times [0, \delta]) \leq A_1 < 0$. We take a cutoff function corresponding to $\delta$ and observe that for
Figure 3.

all $0 \leq s \leq \delta$ and all $s_0 \in [s_3, s_4]$

\[
k(s_0 + s) = \delta s_0 (s_0 + s) k(s_0 + s) \\
\leq k(s_0 + s) < \frac{9}{16} \frac{\sin \theta(s_0)}{r(s_0)} \\
\leq \frac{9}{16} \frac{\sin \theta(s_0 + s)}{r(s_0 + s)} .
\]

Now we can prove the following theorem.

**Theorem 2.4.** Suppose $\gamma \in \Gamma$ is as in Theorem 2.2, where $\Gamma$ is the set of admissible curves for the family $g_s$. Then there exists a continuous map $\alpha_1 : [0, 1] \to \Gamma$ such that

1. $\alpha_1(1) = \{r\text{-axis}\}$, $\alpha_1(0) = \gamma$;
2. the scalar curvature on each $T_{\alpha_1(t)}$ is positive for all $g_s$.

**Proof.** From the proof of Theorem 2.2 we may assume that the signed curvature of $\gamma$ is given by a $C^\infty$ function $k : [0, L] \to \mathbb{R}$ with the graph as in Figure 3 and $\int_0^L k(s)ds = 0$. In this picture $s_4 = \sup \{s \geq 0 | f(s) > 0 \}$ and $s_5 = \inf \{s \geq 0 | f(s) < 0 \}$ are positive numbers that remain fixed throughout the argument.

We deform the curve by deforming its curvature function to 0. The delicate part is to keep the first and the second bends where they initially took place, and to keep the last bend confined within the region $0 \leq r \leq r_4$.

Now, we choose a small number $0 < \epsilon < \min(s_1, s_5 - s_4)$ and a linear path $p(t) = s_4 - s_4 * t$. We take $\epsilon$ to be equal to $\delta$ of Proposition 2.3 and take a cutoff function $\epsilon_{p(t)}$ at $p(t)$ with the parameter $\epsilon$. The deformation function $k_t$ is $\tilde{k}_{p(t)}$ on $[0, p(t) + \epsilon]$ and a carefully rescaled function on $[p(t) + \epsilon, L_t]$ where it is non-positive. The curve $\gamma(\tilde{k}_{p(t)})$ will satisfy the inequality 3 and therefore will give the positive scalar curvature on the neck.

We set $\alpha_1(t) := \gamma(k_t)$.

From Palais [Pal68] and Proposition 2.1, we conclude that the pullback map

\[
\text{Emb}^\infty(M, M \times \mathbb{R}) \times \mathcal{R}(M \times \mathbb{R}) \to \mathcal{R}(M)
\]

is continuous. Therefore, we may regard $\alpha_1$ as a continuous deformation of a family of metrics $g_s$ on $M$. 

2.3. Middle stage deformation. During the second step of our construction, we further deform a given metric to a metric which on some tubular neighborhood of \( N \) is a Riemannian product of \( g_N \) and a torpedo metric of some small radius \( r_4 \).

Given a smooth family of metrics \( \alpha: I \to \mathcal{R}^+(X) \) on a closed smooth manifold \( X \), we can put a metric on \( X \times \mathbb{R} \). However, in general, the scalar curvature of this obvious metric \( g(x, t) = \alpha(t)(x) + dt^2 \) will not be positive.

We fix a function \( F: \mathbb{R} \to [0, 1] \) such that \( 0 \leq F' < 2 \), \( F(t) = 0 \), \( t \in (-\infty, \epsilon] \), \( F(t) = 1 \), \( t \in [1 - \epsilon, \infty) \), for some \( 0 < \epsilon < 1/4 \), and for a positive number \( \tau \) we define a function \( F_\tau: \mathbb{R} \to \mathbb{R} \), by \( F_\tau(t) = F(\frac{t}{\tau}) \).

Let

\[
g_\tau^2(x, t) := \alpha(F_\tau(t))(x) + dt^2.
\]

By Lemma on page 184 in [Gaj87], there exists a number \( \tau > 0 \) such that this metric has positive scalar curvature and is a Riemannian product near \( X \times 0 \) and \( X \times t_0 \).

**Proposition 2.5.** Let \( \nu = (E, B, p) \) be a finite dimensional vector bundle, and \( g_1, g_2 \) be two bundle metrics on \( \nu \). Then there exists a canonical vector bundle isometry:

\[
\begin{array}{ccc}
(E, g_1) & \xrightarrow{\sqrt{(g_1, g_2)}} & (E, g_2) \\
P \downarrow & & \downarrow P \\
B & & B
\end{array}
\]

**Proof.** For each fiber \( V \) there exists a unique positive symmetric operator \( A \) on \( V \), such that \( g_1(u,v) = g_2(Au,v) \). From the spectral theorem, there exists a unique positive symmetric operator \( B \) with the property \( B^2 = A \). We have: \( g_1(u,v) = g_2(BBu,v) = g_2(Bu,Bv) \). The operator \( \sqrt{(g_1, g_2)} := B: V \to V \) provides the required isometry. \( \square \)

Let \( \gamma \) be the curve from Theorem 2.2. By \( \tilde{g}_s \) we denote the family of metrics obtained by pulling back the induced metrics on the necks \( T_\gamma(g_s) \subset M \times \mathbb{R} \) via our canonical diffeomorphisms.

Our goal is to deform the family \( \tilde{g}_s \) further, so that the resulting metrics near \( N \) are isometric to \( g_N + g_0 \), where \( g_0 \) is a torpedo metric of radius 1.

The key observation here is the following. If we start with an arbitrary metric on \( M \) (not necessarily of positive scalar curvature) then the metric induced on the \( \epsilon \)-sphere bundle of \( N \) will have positive scalar curvature if \( \epsilon \) is sufficiently small. Here, the assumption that codimension is \( \geq 3 \) is crucial. In the same fashion, if \( T_\gamma \) is a “cap” (i.e. the neck for some torpedo metric curve \( \gamma_\epsilon \)) of some small radius \( \epsilon \) over a neighborhood of \( N \), then the scalar curvature of \( T_\gamma \) is positive.

**Theorem 2.6.** There exists a continuous map

\[
\alpha_2: S \times I \to \mathcal{R}^+(M)
\]

such that: (i) \( \alpha_2(s, 0) = \tilde{g}_s \); (ii) \( \tau_\epsilon^*(\alpha_2(s, 1)) = g_N + g_0 \), where \( g_N \) is a fixed metric on \( N \), \( g_0 \) is the fixed torpedo metric of radius \( \epsilon_0 \), and \( \tau_\epsilon := \exp_{\alpha_2(s, 1)}^{-1} \circ \sqrt{[\tilde{h}_0, \alpha_2(s, 1)]} \circ \phi: N \times D^2_{\tilde{h}_0} \to M \) is a tubular map for the metric \( \alpha_2(s, 1) \).
Proof. Given a compact family of metrics \( \bar{g}_s \) on \( M \), we define a family of metrics on the tubular neighborhood \( \tau(N) \):

\[
\bar{\alpha} : S \times I \rightarrow \mathcal{M}(\tau(N)),
(s, t) \mapsto (1-t)g_s + t(g_N + g_{eucl}).
\]

Here, \( \bar{g}_s \) denotes the restriction of \( g_s \) to \( \tau(N) \), the metric \( g_{eucl} \) is the standard flat metric on \( D^k \) and the product is taken with respect to our fixed tubular map \( \tau : N \times D^k \rightarrow M \). We take a curve \( \gamma \) corresponding to a torpedo metric of radius \( \epsilon \) and a cap

\[
T_\gamma, (\bar{\alpha}(s, t)) := \{(y, t) \mid (y, \|y\|) \in \gamma\}
\]

in \( \tau(N) \times R \). The distance \( \|y\| \) from \( y \) to \( N \) is taken with respect to \( \bar{\alpha}(s, t) \). We choose \( \epsilon \) such that for all \( (s, t) \in S \times I \) the metric \( \bar{\alpha}(s, t) \) is a diffeomorphism from the normal disc bundle of radius \( \epsilon \) into \( \tau(N) \), and the scalar curvature of the induced metric on the cap \( T_\gamma, (\bar{\alpha}(s, t)) \) is positive. For such a choice of \( \epsilon \) all caps are canonically diffeomorphic via normal exponential maps and the canonical isometry

\[
\sqrt{(g_1, g_2)} : (\perp N, g_1) \rightarrow (\perp N, g_2)
\]

for any two such metrics \( g_1 \) and \( g_2 \).

Let \( \gamma \in \) be the curve in the plane \( (t, r) \), constructed in Theorem 2.1.2. Recall that this curve comes with a choice of a number of parameters \( r_1, t_1 \). In particular, when \( r = r_4 \), it follows the horizontal line between \( t_4 \) and \( t_5 \). Without loss of generality we may assume that \( \epsilon = r_4 \).

Let \( s \in S \) and \( T_\gamma \subset M \times R \) be the neck corresponding to \( \gamma \) and the metric \( g_s \). Let \( N' \) be the part of the neck for \( t \geq t_5 \), \( N'' \) the part for \( t_4 \leq t \leq t_5 \), and \( N''' \) the part for \( t \leq t_4 \). Recall that the part of the curve that defines \( N' \) is exactly the curve \( \gamma_t \). The boundary of \( N' \) is the spherical bundle \( S(N) \) of \( N \) of radius \( \epsilon \), which we consider as a submanifold of \( M \).

We pull the induced metric on the cap for \( g_t := \bar{\alpha}(s, t) \) to \( N' \) via our canonical diffeomorphism. This gives us a smooth family of metrics on \( N' \), \( \beta_1 : I \rightarrow R^+(N') \). The metric \( \beta_1(1) \) is isometric to \( g_N + g_0 \), where \( g_0 \) is the torpedo metric of radius \( \epsilon \). There is a canonical path \( \beta_2 \) from \( g_0 \) to our fixed torpedo metric \( g_0 \). It is obtained by taking a linear path from \( \epsilon \) to \( \epsilon_0 \). Applying this path to fibers of the normal bundle we obtain a path \( \beta : I \rightarrow R^+(N') \) as follows

\[
\beta(\tau) := \begin{cases} 
\beta_1(2\tau), & \tau \in [0, 1/2] \\
\beta_2(2(\tau - 1/2)), & \tau \in [1/2, 1] 
\end{cases}
\]

We have, \( \beta(1) = g_N + g_0 \). The path constructed in such a way is not necessarily smooth in \( \tau \) at the point \( \tau = 1/2 \). However, we can always apply the standard smoothing procedure for piecewise smooth paths. So, we assume \( \beta \) is smooth in \( \tau \) everywhere.

By the formula \[11\] we obtain a metric \( g(x, t) := \tilde{\beta}(F(t_0(t-t_4))(x) + dt^2 \) of positive scalar curvature on \( S(N) \times [t_4, t_5] \), where \( t_0 = \frac{1}{t_5-t_4} \). In this formula, \( \tilde{\beta} \) is the restriction of \( \beta \) to \( S(N) \). For a fixed \( t_4 \) we can always choose \( t_5 \) large enough, so that \( t_0 \) is small enough to assure positivity of the scalar curvature of \( g(x, t) \). Near the boundary \( S(N) \times t_4 \) this metric is a Riemannian product of the restriction of \( g_s \) to \( S(N) \) and the standard metric on \( R \), near the boundary \( S(N) \times t_5 \) it is a Riemannian product of the metric \( \beta(1)|_{S(N)} \) and the standard metric on \( R \).
We now define a map

\[
\alpha_2 : S \times I \to \mathcal{R}^+(M)
\]

\[
\alpha_2(s, \tau) = \begin{cases} 
  g_s|_{T_0}, & \text{on } N'' \\
  \beta(\tau F_{t_0} (t - t_4)) + dt^2, & \text{on } N'' \\
  \beta(\tau), & \text{on } N'
\end{cases}
\]

This map is continuous and it gives us the required deformation.

2.4. Final deformation. After completing the second deformation, all metrics in the new family \( \alpha_2(s, 1) \), \( s \in S \) have the required form \( g_N + g_0 \) near the submanifold \( N \) with respect to their individual tubular maps. However, these tubular maps are not necessarily the same as our fixed tubular map \( \tau \). Therefore, we need an explicit version of the unique tubular neighborhood theorem for a family of metrics.

Let \( \{g_s\} \) a family of metrics on \( M \) parameterized by a smooth compact manifold \( S \).

For our fixed metric \( h_0 \) we define a new family of metrics \( \{g_{s,t} := (1 - t) h_0 + t g_s\}_{s,t \in [0,1]} \) on \( M \).

For any two numbers \( \epsilon^*, \epsilon^{**} > 0 \) we fix a radial diffeomorphism \( \psi : \mathbb{R}^k \to \mathbb{R}^k \) with the properties:

1. \( \psi(\mathbb{R}^k) \subset D^{*} + \epsilon^{**} \mathbb{R}^k \)
2. \( \psi|_{D^{*}} = \text{id} \)

Radial means \( \psi(x) = \lambda(|x|) x \) with \( \lambda > 0 \).

Since \( S \) and \( N \) are compact, we may find two positive numbers \( \epsilon^* \) and \( \epsilon^{**} \) such that for each pair \( (s, t) \) the map \( \tau_{s,t} := \exp^{\epsilon^*} \circ (\psi) \circ \sqrt{(h_0, g_{s,t})} \circ \phi : N \times \mathbb{R}^k \to M^n \) is an embedding. Here, the map \( \sqrt{(h_0, g_{s,t})} \) is the canonical isometry between \((\perp N, h_0)\) and \((\perp N, g_{s,t})\). Taking the restriction \( h_{s,t} := \tau_{s,t}|_{N \times D^{*}} \), for a fixed \( s \in S \), we obtain an isotopy of tubular neighborhoods of \( N \), where each neighborhood has radius \( \epsilon^* \) with respect to the metric \( g_{s,t} \). From the parameterized version of the Thom's embedding theorem it follows that this family of isotopies is embeddable, i.e. there exist a compact neighborhood \( V_0 \) of \( N \) and a \( C^0 \)-family \( \{H_s\} \) of diffeotopies of \( M^n \) such that \( h_{s,t} = H_{s,t} \circ h_{s,0} \) and all \( H_{s,t} \) leave points outside \( V_0 \) fixed.

We deform the family \( g_s \) further, so that the resulting family is equal to \( h_0 \) on the fixed tubular neighborhood \( \tau(N) \subset M \).

**Theorem 2.7.** Let \( g_s = \alpha_2(s, 1) \), \( s \in S \) be a continuous family of metrics parameterized by a compact manifold \( S \), such that \( \tau^*_s(g_s) = g_N + g_0 \). Then there exists a continuous map

\[
\alpha_3 : S \times I \to \mathcal{R}^+(M),
\]

such that \( \alpha_3(s, 0) = g_s \); \( \tau^*(\alpha_3(s, 1)) = g_N + g_0 \) on \( N \times D^{k}_{T_0} \), where \( \tau : N \times D^{k}_{T_0} \to M \) is the fixed tubular map.

**Proof.** Without loss of generality, we can assume that for each \( g_s \) the normal exponential map is a diffeomorphism on the normal disc bundle of radius \( T_1 > T_0 \). For a number \( 0 < c \leq 1 \) and a metric \( g_s \) we define a diffeotopy of \( M \) as follows. Fix a family of increasing functions \( \phi_t \) on \([0, T_1]\) parameterized by \( t \in [0, 1] \), such that \( \phi_0(\tau) = \tau \) and \( \phi_1(\tau) = c \tau \) on \([0, T_0]\), \( \phi_1(\tau) = \tau \) near \( T_1 \). This family defines a family \( \Phi_{s,t} \) of diffeotopies of \( M \) by a diffeomorphism of a normal bundle \( (x, v) \mapsto (x, \frac{\phi_t(v)}{|v|} v) \) and then applying a normal exponential map of \( g_s \). Now, let \( H_{s,t} \) be a family of diffeotopies of \( M \) corresponding to the family \( g_{s,t} \) defined above,
and \( \epsilon^* \) be the radius of the tubular neighborhood which is embedded by this family of diffeotopies. We may always choose \( \epsilon^* \) so that \( c = \frac{\epsilon^*}{T_0} \leq 1 \). We define

\[
\bar{\Phi}_{s,t} := \begin{cases} 
\Phi_{s,3t}, & t \in [0, 1/3] \\
H_{s,3t}^{-1} \circ \Phi_{s,1}, & t \in [1/3, 2/3], \\
\Phi_{0,3t-2} \circ H_{s,1}^{-1} \circ \Phi_{s,1}, & t \in [2/3, 1]
\end{cases}
\]

where \( \Phi_{0,t} \) is the family of diffeotopies for the fixed metric \( h_0 \).

The map \( \alpha_3 \) is defined as

\[
\alpha_3(s, t) := (\bar{\Phi}_{s,t})^* (g_s).
\]

The differential of \( \bar{\Phi}_{s,1} \), when restricted to the fibers of the normal bundle is the canonical vector bundle isometry \( \sqrt{(g_s, h_0)} \) between \( (\perp N, g_s) \) and \( (\perp N, h_0) \).

We have \( \tau^*(\alpha_3(s, 1)) = (\Phi_{s,1}^{-1} \circ H_{s,1} \circ \Phi_{0,1} \circ \tau)^* (g_s) \). An easy computation shows that \( \Phi_{s,1}^{-1} \circ H_{s,1} \circ \Phi_{0,1} \circ \tau = \tau_s \).

\[\square\]

3. Locally warped metrics

We fix a nonnegative number \( B \) and define a space

\[ W := \{ h \in \mathcal{R}^+(D^n_{2B}) \mid h = g(t)^2 dt^2 + f(t)^2 d\xi^2 \} \]

of warped metrics in the disc with the additional requirement that the scalar curvature of \( h \) is greater than \( B \) everywhere in the disc. The metric \( d\xi^2 \) is the standard metric on the euclidean sphere of radius 1.

The subspace \( W^\text{loc} \subset W \) consists of metrics that are isometric to the torpedo metric \( g_0 \) near the origin \( 0 \in D^n_{2B} \), see Definition 3.3.

The main result of this section is Theorem 3.10 which states that there exists a deformation of \( W \) into \( W^\text{loc} \) such that it does not change the metric near the boundary of the disc.

3.1. Warped products. Here we briefly recall some facts about warped Riemannian products.

**Definition 3.1.** Let \((B, \bar{g})\) and \((F, \hat{g})\) be Riemannian manifolds, and \( f \) a positive function on \( B \). Then the Riemannian manifold \((B \times F, \bar{g} + f \hat{g})\) is called a warped product.

In particular, any warped product is a Riemannian submersion over the base \((B, \bar{g})\) with respect to the projection \( \pi \) on the first factor. To any Riemannian submersion \( \pi: (M, g) \to (B, \bar{g}) \) we can correspond two fundamental invariants \( A \) and \( T \), which are \((2, 1)\) tensors on \( M \).

**Definition 3.2.** For any two vector fields \( E_1 \) and \( E_2 \) on \( M \)

\[ T_{E_1}E_2 = \mathcal{H}D_{VE_1}VE_2 + \mathcal{V}D_{VE_1}\mathcal{H}E_2, \]

and

\[ A_{E_1}E_2 = \mathcal{H}D_{HE_1}VE_2 + \mathcal{V}D_{HE_1}\mathcal{H}E_2. \]

In the above definition, \( \mathcal{H} \) is the projection on the horizontal subspace of \( TM \), and \( \mathcal{V} \) is the projection on the vertical subspace of \( TM \).

The scalar curvature of the Riemannian submersion can be expressed in terms of the invariants \( A \) and \( T \) and their covariant derivatives. Let \( (X_i) \) be a local...
orthonormal basis of $H_x$ and $(U_i)$ be a local orthonormal basis of $V_x$. The horizontal vector field on $M$

$$N = \sum_j T_{U_j} U_j$$

is called the mean curvature vector. Denote

$$\delta N = - \sum_i \sum_j \langle (D_X T_{U_j}) U_{U_i}, X_i \rangle,$$

$$|A|^2 = \sum_i \sum_j \langle A_X U_j, A_X U_j \rangle,$$

$$|T|^2 = \sum_i \sum_j \langle T_{U_j} X_i, T_{U_j} X_i \rangle.$$

The following is Corollary 9.37 from [Bes87].

**Proposition 3.1.** Let $(M, g)$ be a Riemannian submersion over $(B, \tilde{g})$ and $(F_b, \hat{g}_b)$ is the fiber over $b \in B$ with the restriction metric $\hat{g}_b = g|_{F_b}$. Let $\kappa, \tilde{\kappa}, \hat{\kappa}$ be the scalar curvatures of the corresponding metrics. Then

$$\kappa = \tilde{\kappa} \circ \pi + \hat{\kappa} - |A|^2 - |T|^2 - |N|^2 - 2\delta N.$$  \hfill (6)

3.2. **Warped products over one-dimensional base with spherical fiber.** We now restrict to the case where the base $B$ is one-dimensional and the fiber is an $(n-1)$-dimensional sphere. Let $B = (0, T)$ and $S^{n-1}$ is the standard Riemannian sphere with the metric $d\xi^2$. The object of our investigation is the space of warped metrics over $B$ with the fiber $S^{n-1}$. Any such metric can be written as

$$h = dt^2 + f^2 d\xi^2.$$  \hfill (7)

The conditions under which such a metric can be extended to a smooth metric on an $n$-dimensional disc are given in the following proposition, which is Lemma 9.114 in [Bes87].

**Proposition 3.2.** If we identify $\{x \in \mathbb{R}^n | 0 < |x| < T\}$ with $(0, T) \times S^{n-1}$ in polar coordinates, the smooth Riemannian metric $dt^2 + f(t)^2 d\xi^2$ extends to a smooth Riemannian metric on $\{x \in \mathbb{R}^n | |x| < T\}$ if and only if $f$ is the restriction to $(0, T)$ of a smooth odd function on $(-T, T)$ with $f'(0) = 1$.

**Proposition 3.3.** The scalar curvature $\kappa$ for the metric $h$ is

$$\kappa = (n - 1) \left( (n - 2) \frac{1 - f'^2}{f^2} - 2 \frac{f''}{f} \right).$$  \hfill (7)

**Proof.** It is clear that in this case the tensorial invariant $A$ is 0. From Proposition 9.104 [Bes87] it follows that $N$ is basic and $\pi$-related to the vector field $-(n - 1) \frac{f'}{f}$ on $B$. It follows that

$$|N|^2 = (n - 1)^2 \frac{f'^2}{f^2}.$$  \hfill (8)

From the formula 9.105e in [Bes87] we obtain

$$\delta N = (n - 1) \left( \frac{f''}{f} + \frac{f'^2}{f^2} \right) \circ \pi.$$
Also, from 9.103 and 9.104 we get \( T U_j X_i = -\frac{(N X_j)}{(n-1)} U_j \) and
\[
|T|^2 = \sum_{i,j} (T U_j X_i, T U_j X_i) = \sum_{i,j} \left( \frac{(N X_i)}{(n-1)} U_j \right) \left( \frac{(N X_j)}{(n-1)} U_j \right)
= \sum_i \frac{1}{(n-1)} (N X_i)^2 = \frac{|N|^2}{(n-1)} = (n-1) f'^2 \frac{f'}{f^2}.
\]

Since our base is 1-dimensional, the scalar curvature \( \kappa = 0 \). The scalar curvature of \((n-1)\)-dimensional standard Riemannian sphere of radius \( f \) is equal to \((n-1)(n-2)\). Substituting into \( \kappa \) we obtain
\[
\kappa = \frac{(n-1)(n-2)}{f^2} - 2(n-1) \left( \frac{f'}{f} + \frac{f'^2}{f^2} \right) - (n-1)^2 \frac{f'^2}{f^2} - (n-1) \frac{f'^2}{f^2}
= (n-1) \left( (n-2) \frac{1-f'^2}{f^2} - 2 \frac{f''}{f} \right).
\]

Any smooth odd function \( G \) on \((-T_0, T_0)\) gives rise to a radial diffeomorphism of euclidean discs
\[
\Upsilon(G) : D_{T_0} \to D_{G(T_0)},
\]
defined by the formula
\[
x \to \frac{G(|x|)}{|x|} x.
\]

For an even function \( g \) on \([-T_0, T_0]\), we set \( G(t) = \int_0^t g(\tau) d\tau \) to obtain a diffeomorphism from \( D_{T_0} \) to \( D_T \), where \( T = G(T_0) \).

In the proof of Theorem 5.11 we will need to deform such a diffeomorphism to a diffeomorphism which, near the origin, is multiplication by a constant.

**Proposition 3.4.** Let \( \Upsilon(G) : D_{T_0} \to D_T \) be the diffeomorphism of euclidean discs, corresponding to an even function \( g \) on \([0, T_0]\), \( T := \int_0^{T_0} g(\tau) d\tau \). Given any number \( \nu < T \), let \( \nu^* \) be such that \( \int_0^{\nu^*} g(\tau) d\tau = \nu \). Then there exists a canonical continuous deformation \( \Upsilon_\lambda(G) : D_{T_0} \to D_T \), \( \lambda \in [0, 1] \) such that (i) \( \Upsilon_0(G) = \Upsilon(G) \); (ii) \( \Upsilon_1(G)(x) = \frac{\nu}{\nu^*} x \) on \( D_{\nu^*} \); (iii) \( \Upsilon_\lambda(G) = \Upsilon(G) \) on an annulus \( \frac{\nu^*+\nu}{2} \leq |x| \leq T_0 \).

**Proof.** For the proof it is enough to canonically construct a function \( g_1 \) such that \( g_1(t) = \frac{\nu}{\nu^*} \) on \([0, \nu^*]\), \( g_1 = g \) on \([\nu^*+T_0, T_0]\), and \( \int_0^{T_0} g_1(\tau) d\tau = \int_0^{T_0} g(\tau) d\tau \). Then the required deformation is given by
\[
\Upsilon_\lambda(G) = \Upsilon((1-\lambda)G + \lambda G_1).
\]
Let
\[
A = \int_{\nu^*}^{(\nu^*+T_0)/2} g(\tau) d\tau,
A' = \min \left( A, \frac{\nu}{T_0} \left( \frac{T_0}{\nu^*} - 1 \right) \right).
\]
From the Lemma 5.14 below we can canonically construct two cutoff functions \( \phi_1 \) and \( \phi_2 \) with the following properties: (i) \( \phi_1 \) is a decreasing function, equal to \( \frac{\nu}{\nu^*} \)
The function $g \leq h$ of $g$ is an isometry between functions $\tilde{\psi}$ which are radial, i.e. $\Upsilon_T$ in Lemma 3.8.

Large amount of curvature, we can bend out a collar. This deformation is realized arbitrarily close to the center of the disc. Lemma 3.7 tells us that once we have this deformation process. Lemma 3.5 together with the curvature equation 7 tells us that for any warped metric we can create an arbitrarily large amount of curvature $D$ deformation of warped metrics in the disc.

3.3. Deformation of warped metrics in the disc. We deform a warped metric by taking a composition $f(\tilde{\psi}(t))$, where $\tilde{\psi}$ is a family of nondecreasing smooth functions. For example, if the family $\tilde{\psi}$ is such that for some point $c$, $\tilde{\psi}^{(n)}(c) = 0$ for all $n \geq 1$, then it means we have created a collar for the metric $(1, \tilde{\psi}(t))$.

Creating a collar with certain properties is the most delicate and hard step in the deformation process. Lemma 3.5 together with the curvature equation 7 tells us that for any warped metric we can create an arbitrarily large amount of curvature arbitrarily close to the center of the disc. Lemma 3.7 tells us that once we have this large amount of curvature, we can bend out a collar. This deformation is realized in Lemma 3.8.

In Lemmas 3.5 and 3.7 below, we construct a family of functions defined on varying size intervals. From the formula 10 it makes sense to talk about continuity of such deformations with respect to the Fréchet topology on $C^\infty[0, T_0]$. 

**Lemma 3.5.** Let $C_1 \leq 1$, $t^* \leq T_0/2$, and $\alpha \in (0, t^*/2)$ be positive numbers. Then there exist positive continuous functions $C(C_1, t^*)$, $T(\lambda)$, $\lambda \in [0, 1]$, and a family of functions $\tilde{\psi}_{\lambda, \alpha}$ on $[0, T(\lambda)]$, see Figure 4 below, continuously depending on $C_1$, $\alpha$, $t^*$, and $\lambda$ with the following properties:

1. $\tilde{\psi}_{\lambda, \alpha}(0) = 0$, $\tilde{\psi}_{\lambda, \alpha}(T(\lambda)) = t^*$;
2. $\tilde{\psi}_{0, \alpha}(t) = t$;
3. $\tilde{\psi}_{\lambda, \alpha}'' \leq C_1$;
4. $\tilde{\psi}_{\lambda, \alpha}'' \leq 0$ on $\left[ \frac{t}{2}, \frac{t}{2} \right]$;
5. $\tilde{\psi}_{\lambda, \alpha}'' \geq 0$, when $\tilde{\psi}_{\lambda, \alpha}(t) \in \left[ \frac{\lambda}{10} t^*, \frac{\alpha t^*}{11} \right]$;
\( \tilde{\psi}'_{\lambda,\alpha}(t) = 1 \), for \( \tilde{\psi}_{\lambda,\alpha}(t) \in \left[ 0, \frac{\alpha t^*}{10} \right] \cup \left[ \frac{\alpha t^*}{20}, t^* \right] \); 

(7) \( 0 < 1 - C(C_1, t^*) \leq \tilde{\psi}'_{\lambda,\alpha}(t) \), all \( t \in [0, T(\lambda)] \); 

(8) on an interval \( \left[ \frac{\alpha t^*}{10}, \frac{8t^*}{10} \right] \), \( \tilde{\psi}_{\lambda,\alpha} \) follows a straight line with the slope \( 1 - \lambda C(C_1, t^*) \).

**Proof.** We fix a smooth function \( \theta_0 \) on \((-\infty, \infty)\), not identically 0, such that \( \theta_0 \) is 0 on \((-\infty, 0] \cup \left[ \frac{1}{20}, \infty \right)\), \( 0 \leq \theta_0 \leq 1 \). The function \( C \) can now be defined as

\[
C(C_1, t^*) = C_1 \int_0^{t^*/20} \theta_0 \left( \frac{\sigma}{t^*} \right) d\sigma = C_1 t^* \cdot \text{Const}.
\]

The family \( \tilde{\psi}_{\lambda,\alpha} \) will be defined as a solution to the differential equation

\[
\tilde{\psi}''_{\lambda,\alpha}(t) = \theta_{\lambda,\alpha}(t)
\]

with the initial conditions \( \tilde{\psi}_{\lambda,\alpha}(0) = 0 \), \( \tilde{\psi}'_{\lambda,\alpha}(0) = 1 \), and a careful choice of the function \( \theta_{\lambda,\alpha}(t) \) on \([0, T(C_1, t^*)] \). To begin constructing function \( \theta_{\lambda,\alpha}(t) \), we consider the equation \( \tilde{\psi}''_{\lambda,\alpha}(t) = \theta_{\lambda,\alpha}(t) \) with the same initial conditions and a function \( \theta_0, \lambda, \alpha(t) = -\frac{2}{\alpha} \lambda C_1 \theta_0 \left( \frac{2t}{\alpha t^*} - 1/20 \right) \) on \( \left[ \frac{\alpha t^*}{40}, \frac{\alpha t^*}{20} \right] \) and 0 otherwise. It is clear that the solution will be a smooth increasing function whose graph in the \((t, s)\)-plane is a curve that follows a straight line with the slope \( 1 - \lambda C(C_1, t^*) > 0 \) on \( t > \frac{\alpha t^*}{20} \). It will intercept the horizontal line \( s = \frac{8t^*}{10} \) at some point \( t_\lambda \) which continuously depends on all possible parameters. We can now define the function \( \theta_{\lambda,\alpha} \) as follows:

\[
\theta_{\lambda,\alpha}(t) = \begin{cases} 
-\frac{2}{\alpha} \lambda C_1 \theta_0 \left( \frac{2t}{\alpha t^*} - 1/20 \right) & \text{if } t \in \left[ \frac{\alpha t^*}{40}, \frac{\alpha t^*}{20} \right] \\
\lambda C_1 \theta_0 \left( \frac{t-t_\lambda}{t^*} \right) & \text{if } t \in \left[ t_\lambda, t_\lambda + \frac{t^*}{20} \right] \\
0 & \text{otherwise}
\end{cases}
\]

Clearly \( \theta_{\lambda,\alpha} \) is a smooth function. One can easily check that the solution

\[
\tilde{\psi}_{\lambda,\alpha}(t) = t + \int_0^t \int_0^\tau \theta_{\lambda,\alpha}(\sigma) d\sigma d\tau
\]

to the differential equation \( \tilde{\psi}''_{\lambda,\alpha}(t) = \theta_{\lambda,\alpha}(t) \) with this function has all the required properties. This finishes the proof. \( \square \)
Lemma 3.6. Let $\phi_0$ be a smooth nonincreasing cutoff function on $[0, 1]$, i.e. $\phi_0(t) = 1$ for $t \leq 0$, and $\phi_0(t) = 0$ for $t \geq 1$. Let $0 < c < 1$. Then, for any continuous nonnegative $f$ on $[a, b]$, $a \neq b$, which is not identically 0, there exists a unique number $\mu > 0$ such that

$$
\int_a^b \phi_0 \left( \frac{t-a}{b-a} \right)^\mu f(t) \, dt = c \int_a^b f(t) \, dt.
$$

The number $\mu$ is a continuous function of $f$ and $c$.

Proof. This is a corollary of the Dominated Convergence theorem. \qed

Lemma 3.7. Let $C_1 \leq 1$ and $t^* \leq T_0/2$ be two positive numbers. Then there exist positive continuous functions $\alpha = \alpha(C_1, t^*)$, $T(\lambda)$, $\alpha \in (0, t^*/2)$ and $\lambda \in [0, 1]$ and a family of functions $\psi_\lambda$ on $[0, T(\lambda)]$, as in Figure 5 below, continuously depending on $\lambda$ with the following properties:

1. $\psi_\lambda(0) = 0$, $\psi_\lambda(T(\lambda)) = t^*$;
2. $\psi_0(t) = t$;
3. $\psi_\lambda''(t) \leq \frac{C_1}{t}$ for all $t > 0$;
4. $\psi_\lambda'(t) = 1$, for $\psi_\lambda(t) \in \left[0, \frac{\alpha}{10}\right] \cup \left[\frac{\alpha}{10}, t^*\right]$;
5. $0 \leq \psi_\lambda' \leq 1$;
6. $\psi_\lambda^{(n)}(\alpha) = 0$, all $n \geq 1$.

Proof. We construct $\psi_\lambda$ as a solution to a second order ordinary differential equation

$$
\psi''_\lambda(t) = \theta_\lambda(t)
$$

with the initial conditions $\psi_\lambda(0) = 0$ and $\psi_\lambda'(0) = 1$. The function $\theta_\lambda$ will be defined as $\theta_\lambda = \lambda \theta_1$, where $\theta_1$ is constructed in a few steps. First, we start with the differential equation

$$
f''(t) = \frac{C_1}{t}
$$

with the initial conditions $f(\alpha) = \epsilon$, $f'(\alpha) = 0$ with $\epsilon \leq \alpha$. The solution to this equation is $f(t) = C_1(t \ln t - t) - (C_1 \ln \alpha) + \alpha C_1 + \epsilon$. Let $t_0 = \alpha e^{\frac{\epsilon}{C_1}}$. Then $f'(t_0) = 1.1$ and $f(t_0) = \alpha \left(1.1 e^{\frac{\epsilon}{C_1}} + C_1 - C_1 e^{\frac{\epsilon}{C_1}}\right) + \epsilon$. The function $h(t) := e^{\frac{\epsilon}{t}} - te^{\frac{\epsilon}{t^*}} + t$ is a decreasing positive function on $(0, \infty)$ that goes to $\infty$ as $t$ goes to 0, and goes to 0 as $t$ goes to $\infty$. Also, if $t > 0$, then $h(t) < e^{\frac{\epsilon}{t}}$. If $\alpha$ is small enough then
\( f(t_0) \) will be less than any prescribed positive number. We set \( \alpha = \frac{9}{20} t^* e^{-\frac{1}{t^*}} \). For this choice of \( \alpha \) we have that \( \alpha < \frac{t^*}{2}, \) and \( f(t_0) < \frac{9}{10} t^* \). Let \( \alpha_1 \) and \( t_1 \) be unique points such that \( \int_0^{\alpha_1} C_1 \, d\tau = \int_{t_1}^{t_0} C_1 \, d\tau = 0.1 \). From the Lemma 3.6 we can canonically construct two cutoff functions \( \phi_1 \) and \( \phi_2 \) such that for the smooth function

\[
\tilde{\theta}_1(t) = \begin{cases} 
\phi_1(t) \frac{C_1}{t} & t \in [\alpha, \alpha_1] \\
\phi_2(t) \frac{C_1}{t} & t \in [\alpha_1, t_1] \\
0 & \text{otherwise}
\end{cases}
\]

we have \( \int_0^{\alpha_1} \tilde{\theta}_1(\tau) \, d\tau = 1 \). We fix a smooth nonnegative bump function \( \tilde{\theta}_2 \) which is 0 outside \([0,1]\) and \( \int_0^1 \tilde{\theta}_2(\tau) \, d\tau = 1 \) and define \( \theta_1 \) as

\[
\theta_1(t) = \begin{cases} 
- \frac{10}{9\alpha} \tilde{\theta}_2 \left( \frac{10t - \alpha}{9\alpha} \right) & t \in \left[ \frac{\alpha}{10}, \alpha \right] \\
\tilde{\theta}_1(t) & t \in [\alpha, t_0] \\
0 & \text{otherwise}
\end{cases}
\]

The solution to the equation (16) is

\[
\psi_\lambda(t) = t + \int_0^t \int_0^{\tau} \lambda \theta_1(\sigma) d\sigma d\tau.
\]

On the half interval \([t_0, \infty)\) the function \( \psi_\lambda \) is strictly increasing for all \( \lambda \). Therefore, its graph in the \((t,s)\)-plane has a unique intersection \( T(\lambda) \) with the horizontal line \( s = t^* \).

The family \( \psi_\lambda \) has all the required properties. Properties 1, 2, 3, and 5 are obvious from the construction. The property 6 follows from the properties of cutoff functions. The first part of the property 4 is clear. For the second part, it is enough to show that \( \psi_\lambda(t_0) \leq \frac{9t^*}{10} \). From the construction we have

\[
\int_0^{t_0} \int_0^{\tau} \theta_1(\sigma) d\sigma d\tau < \frac{9t^*}{10} - \alpha.
\]

Then,

\[
\psi_\lambda(t_0) \leq \alpha + \int_0^{t_0} (1 - \lambda) + \lambda \int_0^{t_0} \int_0^{\tau} \theta_1(\sigma) d\sigma d\tau < \alpha + (1 - \lambda)(t_0 - \alpha) + \lambda \left( \frac{9t^*}{10} - \alpha \right) < \alpha + (1 - \lambda) \left( \frac{9t^*}{10} - \alpha \right) + \lambda \left( \frac{9t^*}{10} - \alpha \right) = \frac{9t^*}{10}.
\]

\( \square \)

3.4. A deformation of \( W \) into \( W^{\text{loc}} \). Recall that we defined \( W \) to be the space of warped metrics in the disc \( D_{r_0} \) whose scalar curvature is strictly bounded from below by \( B \geq 0 \)

\[
\kappa = (n - 1) \left( (n - 2) \frac{1 - f'^2}{f^2} - 2 f'' \right) > B.
\]
Definition 3.3. Let \( h \in \mathbb{W} \). We call \( h \) a locally torpedo metric if, for some number \( c \) with \( T_0 \geq c > 0 \), the metric \( h \) is a warped metric \((\tilde{h}_0, f_0(\tilde{h}_0 t))\) in the disc \( D_c \). Here, \( f_0 \) is the warping function of our fixed torpedo metric \( g_0 \). The space of all such metrics together with the induced topology is denoted by \( W^{\text{loc}} \).

Remark 3.1. The number \( c \) in the above definition is a continuous function on \( W^{\text{loc}} \), \( c : W^{\text{loc}} \to (0, \bar{T}_0] \). To see this, fix a vector \( u_0 \in T_0D_{T_0} \) which has unit length with respect to the standard flat metric on \( D_{T_0} \). Then, \( c(h) = \frac{T_0}{\sqrt{h(u_0, u_0)}} \). It is easy to see that \( c(h) \) does not depend on the choice of \( u_0 \).

The following two lemmas are cornerstones in the construction of a deformation of \( W \) into \( W^{\text{loc}} \).

Lemma 3.8. There exists a continuous function \( \sigma : W \to (0, T_0/2] \) and a continuous map
\[
\Psi_1 : W \times I \to W
\]
such that \( \Psi_1(\cdot, 0) = \text{Id}_W \) and for the metric \( \tilde{h} = \Psi_1(h, 1) = (1, \tilde{f}) \) we have for all \( n \geq 1 \), the \( n \)-th derivative \( \tilde{f}^{(n)}(\sigma(h)) = 0 \).

Proof. We write a metric \( h \in W \) as a pair \((1, f)\) on \([0, T(h)]\) and define two continuous functions
\[
\rho_0(t) = \min_{0 \leq \tau \leq t} \frac{f''(\tau)}{2f'(0)} T(h)
\]
and
\[
\rho_1(t) = \min_{0 \leq \tau \leq t} \frac{f'(\tau)}{2f'(0)} T(h)
\]
be two nonincreasing functions on \([0, T(h)/2] \), which are equal to \( T(h)/2 \) at 0. It follows that there exist unique numbers \( t_0, t_1 \in [0, T(h)/2] \) such that \( t_0 = \rho_0(t_0) \), \( t_1 = \rho_1(t_1) \). We define \( t^* := \min(t_0, t_1, T_0/2) \). It follows that on \([0, t^*] \) the function \( f \) satisfies \( 0 < f'(t) < 1 \) and \( f''(t) < 0 \), when \( t > 0 \) and \( f'(0) = 1 \) and \( f''(0) = 0 \).

Let \( B' = \max(\frac{1}{2}, B') \), where
\[
\bar{B}' = \max_{[0, \frac{1}{2} + B')} \frac{(n-1)(n-2)(1-f'^2) - Bf^2}{2(n-1)f f''},
\]
and
\[
B'' = \min_{[\frac{1}{2} + B', 1]} \left( \frac{(n-2)}{8} \frac{1-f'^2}{f f''} - \frac{f''}{4f'} - \frac{Bf}{8(n-1)f'} \right).
\]
If \( C_1 = \min(\tilde{C}_1, B'') \), where \( \tilde{C}_1 \) is such that \( 1 - C(\tilde{C}_1, t^*) = \sqrt{1 + \frac{B'}{2}} \), then \( 1 - C(C_1, t^*) \geq \sqrt{1 + \frac{B'}{2}} \). For a choice
\[
t^{**} = \min \left( \frac{8}{10} t^*, \sqrt{\frac{(n-1)}{B} \frac{1-(1-C(C_1, t^*))^2}{1-(1-C(C_1, t^*))^2}} \right),
\]
and
\[
C_{1,2} = \min \left( \frac{(n-2)}{8} (1-(1-C(C_1, t^*))^2), 1 \right)
\]
we set \( \alpha = \alpha(C_{1,2}, t^{**}) \).
Let \( \Psi_{\lambda, \alpha} \) be a family of curves from lemma 8.1 above, corresponding to \( C_1, t^* \), and \( \psi_{\alpha} \) a family of curves from lemma 8.2 above, corresponding to \( C_{1,2} \) and \( t^{**} \). We define

\[
\Psi_1(h, \lambda) = \begin{cases} 
(1, f(\psi_{2\lambda, \alpha}(t))), & \lambda \in [0, 1/2] \\
(1, f(\psi_{1, \alpha}(\psi_{2(\lambda-1/2)}(t)))), & \lambda \in [1/2, 1]. 
\end{cases}
\]  

We have to show that the scalar curvature of \( \Psi_1(h, \lambda) \) is greater than \( B \) for all \( \lambda \in [0, 1] \). If the warping function is \( f(\psi(t)) \) then the scalar curvature is

\[
(n - 1) \left( (n - 2) \frac{1 - f'(\psi)^2 \psi'}{f'(\psi)} - 2 \frac{f''(\psi) \psi'}{f'(\psi)} - 2 \frac{f'(\psi)\psi''}{f(\psi)} \right).
\]

During the first part of the deformation \( \psi''(t) \leq 0 \) until \( t \) reaches \( \frac{8}{10t^*} \). For the scalar curvature to be greater than \( B \) for these values of \( t \), it suffices to have

\[
\frac{(n - 1)(n - 2)(1 - f'^2(\psi)\psi') - B f(\psi)^2}{2(n - 1)f(\psi)f''(\psi)} < \psi'^2.
\]

This is a consequence of our choice of \( C_1 \). The only place where \( \psi'' \) might be nonnegative is the interval \([\frac{8}{10t^*}, \frac{4}{10t^*}]\). On this interval it suffices for \( \psi \) to satisfy

\[
\frac{(n - 2)(1 - f'(\psi)^2\psi')}{2} \frac{f''(\psi)}{f'(\psi)} - \frac{f''(\psi)}{f'(\psi)} \psi'^2 - \frac{B f(\psi)}{2(n - 1)f'(\psi)} > \psi''
\]

for the scalar curvature to be greater than \( B \). However, on this interval \( \frac{4}{10t^*} \geq \psi(t) \geq \frac{2\sqrt{2}}{5} t^* \) and \( \psi'^2 \geq \left( \frac{1}{2} + \frac{B}{4} \right) \). Therefore,

\[
\frac{(n - 2)(1 - f'^2(\psi))}{2} \frac{f''(\psi)}{f'(\psi)} - \frac{f''(\psi)}{f'(\psi)} \left( \frac{1}{2} + \frac{B}{2} \right) - \frac{B f(\psi)}{2(n - 1)f'(\psi)} > \frac{(n - 2)(1 - f'^2(\psi))}{2} \frac{f''(\psi)}{2f'(\psi)} - \frac{f''(\psi)}{2f'(\psi)} \left( \frac{1}{2} \right) - \frac{B f(\psi)}{4(n - 1)f'(\psi)}
\]

This shows that the scalar curvature remains strictly bounded from below by \( B \) during the first part of the deformation.

To estimate the scalar curvature during the second deformation it is enough to look at the inequality 20. This is, because after the first deformation on the interval \([\frac{8}{10t^*}, t^{**}]\) the inequalities \( 0 < f' \leq \tilde{C} = (1 - C(C_1, t^*)) < 1 \) will be satisfied. We
have
\[
\frac{(n-2)\left(1 - f^2(\psi)\psi'^2\right)}{2} - \frac{B}{f(\psi)} - \frac{f'(\psi)}{2(n-1)\psi'} \geq \frac{(n-2)(1 - \tilde{C}^2)}{2} - \frac{B_0}{2(n-1)}f^2(\psi)
\]
\[
\geq \frac{(n-2)(1 - \tilde{C}^2)}{2} - \frac{B}{t} - \frac{t^2}{2}
\]
\[
\geq 2\frac{C_1\lambda}{t} > \psi''(t).
\]

We define \(\sigma(h) := \alpha\). Let \((1, f) := \Psi(h, 1)\). Then the property \(\hat{f}^{(n)}(\sigma(h)) = 0\) follows from the construction and the properties of cutoff functions.

**Definition 3.4.** Let \(D^n_{[T_0, T_1]} := \{x \in \mathbb{R}^k \mid 0 < T_0 \leq |x| \leq T_1\}\) be a euclidean annulus and \(h\) is a metric on it. We call \(h\) an annular metric if (i) \(h\) is warped with the fibre metric \(d\xi^2\) the standard euclidean sphere of radius 1; (ii) near the boundary components \(S^n_{T_i}, i = 0, 1\), we have that \(h = dt^2 + r_i^2d\xi^2\) for some constants \(r_i > 0\).

We fix a family of annular metrics \(\hat{h}\) which is continuous in \(T_i, r_i\) and the scalar curvature of \(\hat{h}\) is strictly bounded from below by \(B \geq 0\). This definition makes sense since one can think of an annular metric as a warped metric on \(\mathbb{R} \times S^{n-1}\).

This family \(\hat{h}\) allows us to consider a deformation defined on a subdisc (sub-annulus) as a deformation in the whole disc (annulus) that is constant on some neighborhood of the boundary.

**Lemma 3.9.** Let \(h = (1, f)\) be a warped metric on the disc \(D_{T(h)}\) and \(\sigma \in (0, T(h))\), \(\sigma < T_0/2\), be such that \(f^{(n)}(\sigma) = 0\) for all \(n \geq 1\); \(0 \leq f' \leq 1, f'' \leq 0\) on \([0, \sigma]\). Then there exists a positive number \(\delta_0\) and a family of metrics \(\Psi_2(h, \lambda, \sigma) = (g_\lambda, f_\lambda), \lambda \in [0, 1]\) on the disc \(D_T\) such that \(\Psi_2(h, 0, \sigma) = h, g_1 = 1, f_1 = f_0\) on \([0, T_0]\), and the restriction of \(h\) to an annulus \(D_{[T(h) - \delta_0, T(h)]}\) is isometric to the restriction of \(\Psi_2(h, \lambda, \sigma)\) to an annulus \(D_{[T_1 - \delta_0, T_1]}\) for all \(\lambda\). The family \(\Psi_2(h, \lambda, \sigma)\) is continuous in \(h, \sigma, \lambda\).

**Proof.** We take a linear deformation \(f_\lambda = (1 - \lambda)f + \lambda f_0\) on the interval \([0, \sigma]\). In general, the condition in the formula above is not convex. However, it is convex when \(B = 0\) and \(f\) is such that \(0 \geq f'(t) \leq 1, f''(t) < 0\) for all \(f \in [0, \sigma]\) \(f_0\) can always be chosen to satisfy these conditions. That is, the scalar curvature for \((1, f_\lambda)\) is positive for all \(\lambda \in [0, 1]\), if the scalar curvature for \(f\) were positive. Since the family \(h_\lambda = (1, f_\lambda)\) is compact, it is possible to find a number \(1 \geq \nu > 0\) continuously depending on \(f\) such that the scalar curvature of the family \(\nu h_\lambda\) in the disc \(D_0\) is bounded from below by \(B\). We then plug this family in the disc \(D_{T_0}\) using the fixed family of annular metrics \(\hat{h}\). The existence of \(\delta_0\) follows from the compactness of \([0, 1]\).

**Theorem 3.10.** There exist a number \(\delta > 0\) and a continuous map
\[
\Psi: W \times I \to W,
\]
such that: (i) \(\Psi(h, 0) = h\); (ii) \(\Psi(h, 1) \in W^{1,\infty}\); (iii) \(\Psi(h, t) = h\) on the annulus \(D_{[T_0 - \delta, T_0]}\) for all \(t \in [0, 1]\).
Proof. Let \( W' = \Psi_1(W \times 1) \) be the space from the deformation of Lemma 3.8. A metric \( h = (\tilde{g}, \tilde{f}) \in W' \) has the following property. After reparametrization, we can write \( h = (1, f) \). Then the warping function \( f \) on \([0, T(\tilde{h})]\) is an odd smooth function, \( 0 \leq f' \leq 1, f'' \leq 0 \) on the interval \([0, \mu(\tilde{h})]\), where \( \mu(\tilde{h}) \) is a continuous function of metric (\( \mu \) is the composition of \( \sigma \) and the diffeomorphism function of \( \tilde{g} \)) and \( f^{(n)}(\mu(h)) = 0, \) all \( n \geq 1 \). We also notice from the proof of Lemma 3.8 that \( \Psi_1 \) does not change the metric on the collar of the boundary of \( D^n_T \), of some fixed size \( \delta_1 \).

The required deformation is defined as

\[
\Psi(h, t) := \begin{cases} 
\Psi_1(h, 2t), & t \in [0, 1/2] \\
\Psi_2(\Psi_1(h, 1), 2t - 1, \mu(\Psi_1(h, 1))), & t \in [1/2, 1]
\end{cases}
\]

where \( \Psi_2 \) is the deformation from Lemma 3.8. We take \( \delta := \min(\delta_1, \delta_0) \), where \( \delta_0 \) is from Lemma 3.9.

After Proposition 3.8 we may assume that \( \Psi(h, 1) \in W^{\text{loc}} \). This completes the proof.

We finish this section with a lemma that we will need in the proof of Theorem 4.1.

Lemma 3.11. Let \( D_{[t_1, t_2]} \) be a euclidean annulus, and \( h \) be a warped metric on this annulus, i.e., in polar coordinates, \( h = g(t)^2 dt^2 + f(t)^2 d\xi^2 \), where \( d\xi^2 \) is the standard metric on \( S^{k-1} \). Let \( B \geq 0 \) be a lower bound for the scalar curvature of \( h \) in \( D_{[t_1, t_2]} \). Suppose that for some \( 0 < \epsilon < (t_2 - t_1)/2 \) and some \( 0 < r_0 \) we have \( g \equiv 1 \) and \( f \equiv r_0 \) on \([t_1, t_1 + \epsilon] \cup [t_2 - \epsilon, t_2]\). Then there exists a continuous family of warped metrics \( h_\lambda, \lambda \in [0, 1] \) such that:

(i) \( h_0 = h \);
(ii) \( h_1 = dt^2 + r_0^2 d\xi^2 \);
(iii) \( \exists \delta > 0 \) such that \( h_\lambda \equiv h \) whenever \( |x| \in [t_1, t_1 + \delta] \cup [t_2 - \delta, t_2] \);
(iv) \( \kappa_{h_\lambda}(x) > B, \) all \( x \in D_{[t_1, t_2]}, \lambda \in [0, 1] \).

Proof. (i) The case \( B = 0 \). We may assume \( g \equiv 1 \). Set \( f_\lambda := (1 - \lambda)f + \lambda r_0, \lambda \in [0, 1] \). From equation (4) there exists \( A \geq 1 \) such that the scalar curvature of the family \( A^2 dt^2 + f_\lambda^2(t)d\xi^2 \) is positive. One can easily write down the deformation \( h_\lambda \).

(ii) The case \( B > 0 \). We introduce a function

\[
S(h) := \min_{x \in B} \kappa_h(x),
\]

where \( D \) is the annulus on which \( h \) is defined. For the family \( h_\lambda \) defined above the condition \( S(h_\lambda) > B \) does not have to hold. However, the condition \( S(h_\lambda) > 0 \) (which was satisfied initially, i.e. when \( \lambda = 0 \)) will still hold. This allows to rescale the whole family \( h_\lambda \) with a single scaling factor \( \nu \), so that the scalar curvature of each metric in the rescaled family is greater than \( B \). We plug this family \( \nu h_\lambda \) into the annulus using the fixed family of annular metrics \( \tilde{h} \).

4. PROOF OF THE MAIN THEOREM

Let

\[
B = \max \left(-\min_{x \in N} \kappa_{g_N}(x), 0\right)
\]

and

\[
W(N, \tau) = \{ g \in \mathcal{R}^+(M^n) | \tau^*(g) = g_N + g_w \},
\]
where $g_w$ is a warped metric in the disc $D^2_{T_0}$ such that the scalar curvature of $g_w$ is greater than $B$. Analogously, we define the subspace $W^{\text{loc}}(N, \tau) \subset W(N, \tau)$, cf. Definition 3.3.

The following theorem holds.

**Theorem 4.1.** The subspace $R^+_0(M)$ is a weak deformation retract of $W(N, \tau)$.

*Proof.* Without loss of generality we may assume that the tubular map $\tau$ is an embedding on $N \times D^2_{T_1}$, where $T_1 > T_0$.

We fix a continuous family $\Upsilon_{\lambda, T}$, with $(\lambda, T) \in [0, 1] \times (0, T_0]$, of radial diffeomorphisms of $D_{T_1}$ with the following properties:

(i) $\Upsilon_{0, T} = \text{Id}$, all $T \in (0, T_0]$;
(ii) $\Upsilon_{1, T}$ acts by multiplication by $\frac{T}{T_0}$ on $D_{T_0}$;
(iii) if $T^* \geq T_0$ is such that $\Upsilon_{1, T}(S_{T^*}) = S_{T_0}$, then $\Upsilon_{1, T}$ is a radial isometry in some small radial neighborhood of $S_{T^*}$.

From Remark 3.1 we have a continuous map

$$ c: W^{\text{loc}}(N, \tau) \rightarrow (0, T_0]. $$

The retraction map $r: W(N, \tau) \rightarrow R^+_0(M)$ can now be defined as a composition

$$ (22) \quad W(N, \tau) \xrightarrow{\Psi_1} W^{\text{loc}}(N, \tau) \xrightarrow{\Upsilon} R^+_0(M), $$

where $\Psi_1 = \Psi(\cdot, 1)$ is from the deformation $\Psi: W \times I \rightarrow W$ that was constructed in Theorem 3.10 and $\Upsilon(h) = \Upsilon_{1, c(h)}(h)$.

The deformation $D: W(N, \tau) \times I \rightarrow W(N, \tau)$ from $\text{Id}$ to $r$ is given as

$$ D(h, \lambda) = \begin{cases} \Psi(h, 2\lambda) & \lambda \in [0, 1/2] \\ \Upsilon_{(2\lambda-1), c(h')}(h') & \lambda \in [1/2, 1] \end{cases}, $$

where $h' = \Psi(h, 1)$. This proves that $i \circ r \simeq \text{Id}$, i.e. that the map $r$ is the right homotopy inverse for the natural inclusion map $i$.

To show that $r \circ i \simeq \text{Id}$ on $R^+_0(M)$ we begin with the following observation. For all $h \in R^+_0(M)$ the function $c(\Psi(h, 1)) = T = \text{const} \leq T_0/2$. This follows from the construction of $\Psi$ (see the proofs of Lemma 3.3 and Theorem 3.10 for details). This implies that for all such metrics there exists a constant $T^* > T_0$ such that $r(h)$ is equal to the torus metric $g_0$ on $D_{T_0}$, is equal to a fixed warped metric $h_w = g(t)^2 dt^2 + f(t)^2 d\xi^2$ on the annulus $D_{[T_0, T^*]}$ with $g(t) \equiv 1$ in some small left neighborhood of $T^*$, and is equal to a pullback metric on the annulus $D_{(T^*, T_1]}$. The deformation that we are seeking consists of (a) applying Proposition 3.11 to deform $h_w$ to $dt^2 + r_0^2 d\xi^2$; (b) contracting the annulus $T_0 \leq |x| \leq T^*$ to the sphere $S_{T_0}$ and at the same time stretching the annulus $D_{[T^*, T_1]}$ back to the identity on $D_{[T_0, T_1]}$ by some family of diffeomorphisms which are radial isometries near the boundary. \(\square\)

Let $g_s \in R^+(M), s \in S$, be a continuous family of metrics and $S$ is a compact manifold. Then from Theorems 2.4, 2.6 and 2.8 we conclude that there exists a continuous map

$$ \text{GL}: S \times I \rightarrow R^+(M) $$

such that (i) $\text{GL}(s, 0) = g_s$; (ii) $\text{GL}(s, 1) \in R^+_0(M)$; (iii) $\text{GL}(s, 0) \in W(N, \tau)$ implies that $\text{GL}(s, t) \in W(N, \tau)$ for all $t \in [0, 1]$. The last property follows from the construction of the map $\text{GL}$, the definition of the space $W(N, \tau)$, and the fact that we can always choose $\text{GL}$ to take place inside the image of the tubular map.
\[ \tau. \] This deformation is by no means unique and we call it a GL deformation for the family \( g_t. \)

**Proof of Theorem 1.1** It is enough to show that all the relative homotopy sets
\[ (23) \quad \pi_r(\mathcal{R}^+(M^n), W(N, \tau)) = 0, \quad r \geq 1, \]
and the inclusion map \( i: W(N, \tau) \to \mathcal{R}^+(M) \) is a bijection between path connected components of both spaces.

We first show that \( i \) induces a bijection between the path components. Let \( h_1, h_2 \in W(N, \tau) \) and \( \alpha: I \to \mathcal{R}^+(M^n) \) be a path between \( h_1 \) and \( h_2 \). Take some GL: \( I \times I \to \mathcal{R}^+(M) \) for the family \( \alpha \). Then \( GL(t, 1) \in \mathcal{R}_0^+(M) \subset W(N, \tau) \) and \( \text{GL}(h_1, t), \text{GL}(h_2, t) \in W(N, \tau) \) for all \( t \in [0, 1] \). The path
\[
\alpha'(t) := \begin{cases} 
\text{GL}(h_1, 3t) & t \in [0, 1/3] \\
\text{GL}(\alpha(3(t - 1/3)), 1) & t \in [1/3, 2/3] \\
\text{GL}(h_2, 3(1 - t)) & t \in [2/3, 1]
\end{cases}
\]
is the required path.

Let \( [\alpha] \in \pi_r(\mathcal{R}^+(M^n), W(N, \tau)) \), where \( \alpha: (D^r, S^{r-1}) \to (\mathcal{R}^+(M^n), W(N, \tau)) \) is a continuous map. Take a deformation
\[
\text{GL}: D^r \times I \to \mathcal{R}^+(M^n)
\]
for the family \( \alpha \) to conclude that \( [\alpha] = 0. \)

**Proof of Theorem 1.2** We fix the standard euclidean metrics of radius 1 on \( S^k \) and on \( S^{n-k-1} \) and torpedo metrics on \( D^{n-k} \) and on \( D^{k+1} \) of size \( T_0 \) and radius 1. From the definition of a torpedo metric there exist a number \( \epsilon > 0 \) such that, near the boundary of the disc, both torpedo metrics are products of the standard sphere with an interval of length \( \epsilon \).

Let \( M_0 := M_1 - (S^k \times (D_{T_0-\epsilon/2})^{n-k}) \). Then
\[
M_2 = M_0 \bigcup_{S^k \times S^{n-k-1}} D_{T_0-\epsilon/2}^{k+1} \times S^{n-k-1}.
\]
This gives us an embedding of \( S^{n-k-1} \to M_2^n \) and a tubular neighborhood
\[
\tau_2: S^{n-k-1} \times D_{T_0-\epsilon/2}^{k+1} \to M_2^n.
\]
From Theorem 1.1, we have \( \mathcal{R}^+(M_1) \simeq \mathcal{R}_0^+(M_1) \) and \( \mathcal{R}^+(M_2) \simeq \mathcal{R}_0^+(M_2) \). But the space \( \mathcal{R}_0^+(M_1) \) is homeomorphic to \( \mathcal{R}_0^+(M_2) \) as can be seen by removing \( S^k \times D_{T_0-\epsilon/2}^{n-k} \) from \( M_1 \) and removing \( S^{n-k-1} \times D_{T_0-\epsilon/2}^{k+1} \) from \( M_2 \). In both cases we obtain the manifold \( M_0 \) with the fixed collar \( S^k \times S^{n-k-1} \times [0, \epsilon/2] \to M_0 \) and the space of positive scalar curvature metrics on \( M_0 \) that restrict to the fixed product metric on this collar. \[ \square \]

**REFERENCES**

[Bes87] Arthur L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

[Car88] Rodney Carr, *Construction of manifolds of positive scalar curvature*, Trans. Amer. Math. Soc. 307 (1988), no. 1, 63–74.

[Gaj87] Paweł Gajer, *Riemannian metrics of positive scalar curvature on compact manifolds with boundary*, Ann. Global Anal. Geom. 5 (1987), no. 3, 179–191.

[GL80] Mikhail Gromov and H. Blaine Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) 111 (1980), no. 3, 423–434.
[Loh92] Joachim Lohkamp, *The space of negative scalar curvature metrics*, Invent. Math. 110 (1992), no. 2, 403–407.

[Pal66] Richard S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), 1–16.

[Pal68] ———, *Foundations of global non-linear analysis*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[RS01] Jonathan Rosenberg and Stephan Stolz, *Metrics of positive scalar curvature and connections with surgery*, Surveys on surgery theory, Vol. 2, Princeton Univ. Press, Princeton, NJ, 2001, pp. 353–386.

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