On The Relativistic Classical Motion of a Radiating Spinning Particle in a Magnetic Field

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Abstract

We propose classical equations of motion for a charged particle with magnetic moment, taking radiation reaction into account. This generalizes the Landau-Lifshitz equations for the spinless case. In the special case of spin-polarized motion in a constant magnetic field (synchrotron motion) we verify that the particle does lose energy. Previous proposals did not predict dissipation of energy and also suffered from runaway solutions analogous to those of the Lorentz-Dirac equations of motion.

1 Introduction

What is the classical motion of an electron in a constant magnetic field? Surprisingly, this question does not have a definitive theoretical answer even now. The complications arise when the radiation from the particle is considered. Without

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it, the answer is quite elementary: a helical path around the magnetic field\(^1\) obtained by solving the Lorentz equation of motion\(^1\)

\[
\frac{du_\mu}{d\tau} = \frac{q}{m} F_{\mu\nu} u^\nu
\]  
(1)

But we know that matters are not so simple in reality: an accelerated charge radiates, and the radiation emitted by it exerts a force back on it. If the energy is not replenished the path should be a spiral, the radius of the circle decreasing with the energy.

Determining the force exerted on a point particle by its own field is fraught with conceptual difficulties: the force is infinite. Dirac\(^2\) made a major step forward by showing that this divergence can be removed by a renormalization of the mass. The resulting equation (the Lorentz-Dirac equation of motion)\(^2\)

\[
\frac{du_\mu}{d\tau} = \frac{q}{m} F_{\mu\nu} u^\nu + \epsilon \left[ \eta_{\mu\nu} - u_\mu u_\nu \right] \frac{d^2 u^\nu}{d\tau^2}, \quad \epsilon = \frac{2}{3} \frac{q^2}{m}
\]  
(2)

is still not the correct answer. It predicts runaway solutions: even in the absence of any external fields, a particle can accelerate away to infinity with exponentially growing energy. Part of the problem is that the radiative terms persist even in the absence of external forces. Also, the equation involves the third derivative of position; so it needs a new initial condition. Dirac suggested that this be chosen such that the energy remains finite forever. However, such solutions suffer from another problem: a violation of causality; the particle would accelerate before the external field is turned on.

The textbook of Landau and Lifshitz\(^3\) proposed (without proof) a way out of this dilemma. Replace the derivative of acceleration by its leading approximation in an expansion in powers of \(\epsilon\):

\[
\frac{du_\mu}{d\tau} = \frac{q}{m} F_{\mu\nu} u^\nu + \epsilon \left[ \eta_{\mu\nu} - u_\mu u_\nu \right] \left[ \frac{q}{m} F_{\rho\sigma} u^\rho \right], \quad \epsilon = \frac{2}{3} \frac{q^2}{m}
\]  
(3)

The radiative term now vanishes when the external force is zero; and the equation now involves only the second order derivative of position. The free particle does not run away. Further analysis shows that the orbits in a constant magnetic field

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\(^1\) In units with \(c = 1\). \(q\) and \(m\) are the charge and mass respectively. Also, \(u^\mu = \frac{dx^\mu}{d\tau}\) is the four-velocity and \(\tau\) is proper time. Note the constraint \(\eta_{\mu\nu} u^\mu u^\nu = 1\) and its derivative \(\eta_{\mu\nu} u^\mu \frac{du^\nu}{d\tau} = 0\).

\(^2\) The last term includes a projection operator that ensures that the constraint \(\eta_{\mu\nu} u^\mu \frac{du^\nu}{d\tau} = 0\) is satisfied.
field [4] and in a Coulomb field [5] decay as expected. But why should we stop at first order in the iteration in powers of $\varepsilon$? Although it works, the ad hoc prescription of Landau and Lifshitz is unsatisfactory.

To get sensible solutions to the Lorentz-Dirac equation, its new degree of freedom must be continuously fine-tuned so that the trajectory stays on a submanifold of the phase space (the critical submanifold). Spohn used modern ideas in the theory of dynamical systems to determine the equation of motion projected to this critical submanifold: it is precisely the above Landau-Lifshitz equation! Thus we consider the century-old problem of determining the equation of motion of a radiating point charge to be settled finally.

However, the electron has a magnetic moment as well as electric charge; radiation arises not only from the acceleration of the charge but also the precession of the magnetic moment. In the relativistic case, these are of comparable magnitude. Therefore, it is necessary to extend the Lorentz-Dirac and Landau-Lifshitz equations to the case of a spinning charged particle.

Bargman, Michel and Telegdi[6] proposed, more than a half century ago, an equation for a (non-radiating) classical particle with charge and magnetic moment. However, it does not seem to be the classical limit of the Dirac wave equation of the electron. The correct equation must have a canonical form whose quantization leads to the Dirac wave equation. The BMT equation does not appear to satisfy this condition.

A better approach was proposed more recently by Barut and Zanghi[7]. We will review their ideas, which show that a canonical quantization does lead to the Dirac equation. We will also find solutions of the Barut-Zanghi equation for a non-radiating charged particle in a constant magnetic field: itself a new result.

Barut and Unal[8] also derived the analogue of the Lorentz-Dirac equation, after a mass renormalization. However, it suffers from the same runaway solutions of the free particle, as the spinless case.

In this paper, we propose an equation of motion for a radiating spinning charged particle with gyromagnetic ratio 2 (e.g., the electron or the muon) by applying a prescription analogous to that of Landau and Lifshitz to the Barut-Zanghi equation. We will show that, at least for spin-polarized orbits in a constant magnetic field, the orbits decay as expected physically. It should be possible to test our equations using a small (few MeV) electron synchrotron or a muon storage ring, such as the one used for the determination of $g - 2$. Also, it should not be difficult to revise the predictions for an electron accelerated by a laser field[9]. Current predictions do not include the effect of the magnetic moment.

We do not attempt, in this paper, to derive our prescription from first principles.
as Spohn did.

2 Non-Radiating Spinning Charged Particle

Can we talk of the motion of a particle with spin in the classical limit? At first this would appear to be impossible. But we do it all the time. The motion of electrons in synchrotron of a radius of several meters (if not kilometers) is surely classical: even if the spin were polarized, the differences in the quantum energy levels of the electron are much too small to be of significance. If we can find a lagrangian whose canonical quantization is the Dirac wave equation, it can be the basis for such a classical treatment.

The lagrangian proposed by Barut and Zangh is,

$$L = i\frac{1}{2} [\bar{z} \gamma^\mu \dot{z} - \bar{z} \dot{\bar{z}}] + p_\mu \dot{x}^\mu - H, \quad H = \bar{z} \gamma^\nu [p_\nu - q A_\nu], \quad \bar{z} = \bar{z}^\dagger \gamma_0 \quad (4)$$

Here $z$ is a Dirac spinor with four complex (not Grassmann) numbers as components. The dimensions of $z$ are that of the square root of action (=mass*length) and that of $\tau$ is the inverse of mass, in units where $c = 1$ but $\hbar \neq 1$. Then $q F_{\mu \nu}$ has dimensions of mass over length.

The equations of motion are, with $\pi_\mu = p_\mu - q A_\mu$

$$\frac{dx^\mu}{d\tau} = \bar{z} \gamma^\mu z$$

$$-i \frac{dz}{d\tau} = \gamma^\mu \pi_\mu z$$

$$\frac{d\pi_\mu}{d\tau} = q F_{\mu \nu} \bar{z} \gamma^\nu z \quad (5)$$

Note that, $\tau$ is not proper time as $\dot{x}^\mu$ is not necessarily of constant length. But we have the orthogonality condition

$$\dot{x}^\mu \pi_\mu = 0. \quad (6)$$

These equations follow from the Hamiltonian

$$H = \bar{z} \gamma^\mu z \pi_\mu \quad (7)$$

3They use a slightly different system of units.
and Poisson Brackets (P.B.) are obtained using

$$\{f, g\} = i \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}} \right) + \epsilon_{\mu\nu} \left( \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial x^\nu} - \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p^\nu} \right)$$

(8)

$$\{\pi_\mu, \pi_\nu\} = -q F_{\mu\nu}, \quad \{\pi_\mu, x^\nu\} = \delta_\mu^\nu$$

$$\{z, \bar{z}\} = i, \quad \{\pi_\mu, z\} = 0 = \{z, x\}$$

(9)

All other pairs of P.B. are zero. We can introduce a constraint

$$\bar{z}z = a$$

(10)

consistent with the P.B. and the equations of motion. The quantity $a$, with the dimensions of action, sets the scale for the intrinsic angular momentum of the particle.

Upon quantization, the wave function can be thought of as a function of $x$, $p_\mu = \hbar \frac{\partial}{\partial x^\mu}$; it is also a polynomial in $\bar{z}$, with $z = -\hbar \frac{\partial}{\partial \bar{z}}$ satisfying the constraint

$$-\hbar \bar{z} \frac{\partial}{\partial \bar{z}} \psi = a \psi.$$

(11)

Thus the parameter $a$ is quantized in multiples of $\hbar$. The smallest value is $a = \hbar$ for which $\psi(x, \bar{z}) = \bar{z} \psi(x)$ is a linear function.

The Dirac wave equation is the eigenvalue equation for $H$, with eigenvalue $am$:

$$i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \psi = m \psi$$

(12)

Other quantizations, for which $\psi(\bar{z})$ is a higher degree polynomial, correspond to the wave equations of Bhabha and Harish-Chandra. They could be interesting as wave equations for composite particles like hadrons[10].

It is convenient to introduce the velocity and spin variables

$$v_\mu = \bar{z} \gamma^\mu z, \quad S^{\mu\nu} = \frac{i}{4} \bar{z} [\gamma^\mu, \gamma^\nu] z$$

(13)

so that
\[
\begin{align*}
\frac{d\mathbf{x}^\mu}{d\tau} &= v^\mu \\
\frac{dv^\mu}{d\tau} &= 4S^{\mu\nu} \pi^\nu \\
\frac{d\pi^\mu}{d\tau} &= qF_{\mu\nu} v^\nu \\
\frac{dS^{\mu\nu}}{d\tau} &= v^\nu \pi^\mu - v^\mu \pi^\nu
\end{align*}
\]

(14)

These equations describe a relativistic analogue of the isotropic precessing top of Poinsot [11], as we explain in the Appendix. The hamiltonian

\[
H = v^\mu \pi_\mu
\]

(15)

describes a coupling to the external field which causes a precession of the velocity and spin. For a constant electromagnetic field, \( K = \pi \cdot \pi - qS^{\mu\nu} F_{\mu\nu} \) is conserved as well:

\[
\frac{d}{d\tau} [\pi \cdot \pi - qS^{\mu\nu} F_{\mu\nu}] = 2qF_{\mu\nu} \pi^\mu v^\nu - 2qF_{\mu\nu} v^\nu \pi^\mu = 0.
\]

(16)

This is the classical analogue of the square of the Dirac operator, corresponding to the gyromagnetic ratio 2. We will see later that it is proportional to the angular momentum in the plane of the electromagnetic field.

3 A Non-Radiating Spinning Particle in a Constant Magnetic Field

Now let us turn to solving the equations of motion in the case of a magnetic field pointed along the third direction: \( F_{12} = -F_{21} > 0 \); and all other components are zero. Before we solve the problem including radiation damping, we must recover the solution without radiation. Even this appears to be new: Barut and collaborators only solved the case of a free particle.

We consider only orbits that lie in the 12 plane. That is, we restrict to the case where all the tensor indices \( \mu, \nu, \rho \cdots \) are in the 012 subspace. It is useful to introduce the ‘dual’ variables
\[ q F_{\mu \nu} = \varepsilon_{\mu \nu \rho} F^\rho \]
\[ S^{\mu \nu} = \varepsilon^{\mu \nu \rho} S_\rho \]
\[ S_\rho = \frac{1}{2} \varepsilon^{\mu \nu \rho} S_{\mu \nu} \]  
\[ (17) \]

For a constant magnetic field, only the time component \( F^0 \) is non-zero. We can assume it to be positive without loss of generality. Then

\[ \frac{d \pi_\mu}{d \tau} = \varepsilon_{\mu \nu \rho} v^\nu F^\rho \]
\[ \frac{d v^\mu}{d \tau} = -4 \varepsilon_{\mu \nu \rho} S_\nu \pi_\rho \]
\[ \frac{d S_\mu}{d \tau} = -\varepsilon_{\mu \nu \rho} v^\nu \pi^\rho \]  
\[ (18) \]

If we define

\[ \lambda_\mu = \frac{1}{2} (S_\mu + \frac{1}{2} v_\mu), \quad \rho_\mu = \frac{1}{2} (S_\mu - \frac{1}{2} v_\mu) \]

they reduce to

\[ \frac{d \pi_\mu}{d \tau} = 2 \varepsilon_{\mu \nu \rho} [\lambda^\nu - \rho^\nu] F^\rho \]
\[ \frac{d \lambda_\mu}{d \tau} = -2 \varepsilon_{\mu \nu \rho} \lambda_\nu \pi_\rho \]
\[ \frac{d \rho_\mu}{d \tau} = 2 \varepsilon_{\mu \nu \rho} \rho^\nu \pi^\rho \]  
\[ (19) \]

### 3.1 Spin-Polarized Orbits

There is a special class of ‘left-handed’ orbits for which \( \rho_\mu = 0 \). (Also, the opposite case with \( \lambda_\mu = 0 \).) For these, the spin and velocity are synchronized in the following way:
\[ S_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho} v^\rho \]  

(20)

In the non-relativistic limit, this means the spin is pointed along the magnetic field (times \( q \)).

Since these solutions are periodic (not chaotic), orbits of this kind in a synchrotron should be of special interest. If the gyromagnetic ratio is slightly different from two, the spin would depart from this orientation, which is the principle behind a famous experiment for the measurement of \( g - 2 \) for the muon.

The equations reduce to

\[ \frac{d\pi_\mu}{d\tau} = \epsilon_{\mu\nu\rho} v^\nu F^\rho \]
\[ \frac{dv_\mu}{d\tau} = -2\epsilon_{\mu\nu\rho} v_\nu \pi_\rho \]  

(21)

The tensor \( \epsilon_\sigma^{\mu\nu} = \epsilon_{\mu\nu\rho} \eta^{\rho\sigma} \) form the structure constants of the three dimensional Lorentz Lie algebra \( SO(1,2) \). This is analogous to the way that the Levi-Civita tensor gives the structure constants of the Lie algebra of rotations in Euclidean three dimensional space. Denote the Lie product of \( SO(1,2) \) as a relativistic cross product

\[ (u \times w)_\mu = \epsilon_\mu^{\nu\rho} u_\nu w_\rho \]  

(22)

This differs from the familiar cross product in Euclidean geometry by some signs:

\[ (u \times w)_0 = u_1 w_2 - u_2 w_1, \quad (u \times w)_1 = -u_2 w_0 + u_0 w_2, \quad (u \times w)_2 = -u_0 w_1 + u_1 w_0 \]  

(23)

Then our equations are a relativistic version of the equation for a precessing top:

\[ \dot{\pi} = v \times F, \quad \dot{v} = -2v \times \pi \]  

(24)

It follows that

\[ v \cdot \dot{v} = 0 \]  

(25)
so that $v \cdot v = a^2$ is conserved; $a$ is some number (like Planck’s constant) with the dimensions of action; the spin is of magnitude $\frac{a}{2}$. Also, the parameter $\tau$ is proportional to proper time in this special case.

The energy $\pi_0 = E$ and

$$K = \pi_\mu \pi^\mu - v^\mu F_\mu$$

are conserved \(^4\). Now,

$$\dot{v}_0 = 2(v_2 \pi_1 - v_1 \pi_2) \quad (27)$$

and

$$v_0^2 = 4 \left( [v_1^2 + v_2^2] \left[ \pi_1^2 + \pi_2^2 \right] - [v_1 \pi_1 + v_2 \pi_2]^2 \right) \quad (28)$$

Re-expressing the r.h.s. in terms of conserved quantities,

$$\dot{v}_0 = 2 \sqrt{\left[ v_0^2 - a^2 \right]} \left[ E^2 - K + v_0 F_0 \right] - \left[ H - Ev_0 \right]^2 \quad (29)$$

Weierstrass elliptic function satisfy this differential equation. The remaining components of $\pi, v$ can be obtained by solving the other constraint equations. It is also possible to understand the solution as an integrable hamiltonian system, a relativistic analogue of the precessing top. (See the Appendix)

4 The Equation of Motion for Radiating Spinning Particles

Barut and Unal followed the method of Dirac to derive the analogue of the Lorentz-Dirac equation of motion including the self-force. Only the equation for $\pi_\mu$ is affected:

$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}v^\nu + q^2 \left[ \eta_{\mu\nu} - \frac{v_\mu v_\nu}{v \cdot v} \right] \left\{ \frac{2}{3} \frac{v^\nu}{v \cdot v} - \frac{9}{4} \frac{v \cdot \dot{v} v^\nu}{(v \cdot v)^2} \right\}$$

The last term is the contribution from self-interactions, after a renormalization. It is a singular perturbation i.e. $q^2$ is small and it changes the order of the equations. It has runaway solutions, just like the Lorentz-Dirac equation. Let us

\[^4K\] is proportional to angular momentum.
apply to it a prescription like that of Landau and Lifshitz, to get a second order equation of motion.

Naively, the Landau-Lifshitz prescription amounts to using

\[ \ddot{v}^\mu = 4 S^{\mu\nu} \pi_\nu + 4 S^{\mu\nu} \dot{\pi}_\nu \] (31)

and replacing \( \dot{\pi} \) by its zeroth order contribution:

\[ \ddot{v}^\mu \rightarrow 4 S^{\mu\nu} \pi_\nu + 4 S^{\mu\nu} q F_{\nu \rho} v^\rho = 4 [v^\nu \pi^\mu - v^\mu \pi^\nu] \pi_\nu + 4 S^{\mu\nu} q F_{\nu \rho} v^\rho \] (32)

But this can’t be right: for a free particle (\( F_{\mu\nu} = 0 \)), the radiation reaction force should be zero. Dropping the terms not proportional to \( F_{\mu\nu} \),

\[ \frac{d\pi_\mu}{d\tau} = q F_{\mu\nu} v^\nu + \frac{8 q^3}{3 v \cdot v} \left[ \eta_{\mu\nu} - \frac{v_\mu v_\nu}{v \cdot v} \right] S^{\nu \rho} F_{\rho \sigma} v^\sigma \] (33)

Along with the previous equations for \( v \) and \( S \), this system of ordinary differential equations is our proposal for the equation of motion of a spinning radiating charged particle. We must verify that the solutions are physically sensible: there should not be runaway solutions. We will study the case of polarized orbits in a constant magnetic field and show that the energy decreases with time. The ultimate test of our proposal must be experimental, especially since we do not yet have a derivation from first principles.

## 5 Radiating Spin-Polarized Particles in a Magnetic Field

Now let us turn to our proposed equations for a radiating spinning charged particle. Again, we will look for solutions in the special case where the spin is polarized

\[ S_\mu = \frac{1}{2} v_\mu \] (34)

Since the radiative terms do not change the equations for \( v^\mu \), \( S^{\mu\nu} \) this condition is still preserved by time evolution i.e. \( \dot{S}_\mu = \frac{1}{2} \dot{v}_\mu \). Unpolarized orbits are chaotic and the radiation damping is likely to be even larger. (See Appendix)

The equations then reduce to
\[
\begin{align*}
\frac{d\pi_\mu}{d\tau} &= qF_{\mu\nu}v^\nu + \frac{4q^3}{3v \cdot v} \left[ \eta_{\mu\nu} - \frac{v_\mu v_\nu}{v \cdot v} \right] \epsilon^{\nu\rho\alpha\beta} v_\rho v_\sigma F_{\beta\alpha}
\frac{dv_\mu}{d\tau} &= -2\epsilon^{\nu\rho\sigma} v_\nu \pi_\rho
\end{align*}
\]

(35)

We know that, \(v \cdot \dot{v} = 0\) and \(v \cdot v = v_0^2 - (v_1^2 + v_2^2) = a^2\).

\[
\begin{align*}
\frac{d\pi_\mu}{d\tau} &= \epsilon_{\mu\nu\rho} v^\nu F^\rho + \left[ a^2 \eta_{\mu\nu} - v_\mu v_\nu \right] \gamma F^\nu \\
\end{align*}
\]

where

\[
\gamma = \frac{4q^2}{3a^2}
\]

(36)

In terms of relativistic cross products

\[
\dot{\pi} = v \times [F + \gamma F \times v], \quad \dot{v} = -2v \times \pi
\]

(38)

Thus, even the radiation damping terms have a simple meaning in terms of the relativistic cross product of the Lie algebra \(SO(1,2)\).

Note that

\[
H = v \cdot \pi
\]

(39)

is still conserved and \(m = \frac{H}{a}\) continues to have the physical meaning of mass. But energy \(E = \pi_0\) and \(K\) are no longer conserved. The evolution of \(v_0\) continues to be determined by the equation we derived earlier. Thus we get the system

\[
\begin{align*}
\frac{dE}{d\tau} &= -\gamma F_0 (v_0^2 - a^2) \\
\frac{dK}{d\tau} &= 2\gamma F_0 \left[ a^2 E - Hv_0 \right] \\
v_0 &= 2\sqrt{[v_0^2 - a^2] \left[ E^2 - K + v_0 F_0 \right] - \left[ H - E v_0 \right]^2}
\end{align*}
\]

(40)

Since \(E\) and \(K\) are no longer constants, the solution is no longer an elliptic function. Also, noting that \(F_0 > 0\) in our convention and \(v_0^2 - a^2 = (v_1^2 + v_2^2) > 0\), we see that energy is monotonically decreasing.
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Appendix: Hamiltonian Formalism

The Barut-Zanghi equations for a spinning charges particle follow from the P. B. forming the Lie algebra $SO(2,3)$:

\[
\begin{align*}
\{v^\mu, v^\nu\} &= 4S^{\mu\nu} \\
\{S^{\mu\nu}, v^\rho\} &= \eta^{\mu\rho} v^\nu - \eta^{\nu\rho} v^\mu \\
\{S^{\mu\nu}, S^{\rho\sigma}\} &= \eta^{\mu\rho} S^{\nu\sigma} + \eta^{\nu\sigma} S^{\mu\rho} - \eta^{\nu\rho} S^{\mu\sigma} - \eta^{\mu\sigma} S^{\nu\rho}
\end{align*}
\]  

(41)

Using previous relations we can also find the following P.B. :

\[
\begin{align*}
\{v_\mu, v_\nu\} &= 4S_{\mu\nu} \\
\{S_\mu, S_\nu\} &= S_{\mu\nu} \\
\{S_\mu, v_\nu\} &= \epsilon_{\mu\nu\sigma} v^\sigma
\end{align*}
\]  

(42)

which generalizes the rotation algebra. When the third axis is ignored, this reduces to $SO(2,2) \approx SO(1,2) \oplus SO(1,2)$. This splitting can be seen by in terms of the linear combinations $\lambda_\mu, \rho_\mu$ :

\[
\begin{align*}
\{\lambda_\mu, \lambda_\nu\} &= \epsilon_{\mu\nu\sigma} \lambda^\sigma \\
\{\rho_\mu, \rho_\nu\} &= \epsilon_{\mu\nu\sigma} \rho^\sigma \\
\{\lambda_\mu, \rho_\nu\} &= 0
\end{align*}
\]  

(43)

We have also the P.B.,
\begin{align*}
\{ \lambda_\mu, \pi_\nu \} &= 0 \\
\{ \rho_\mu, \pi_\nu \} &= 0 \\
\{ \pi_\mu, \pi_\nu \} &= -\epsilon_{\mu\nu\rho} F^\rho
\end{align*}

(44)

The hamiltonian is then

\[ H = 2[\lambda - \rho] \cdot \pi \]  

(45)

**Non-Radiating Spin-Polarized Orbits in Constant Magnetic Field**

Since

\[ v^0 = \sqrt{a^2 + v_b v_b} \]  

(46)

and \( \pi_0 = E \) is conserved we only have four co-ordinates in the phase space: \( \pi_a, v_a \). The P.B. are

\[ \{ \pi_1, \pi_2 \} = -F_0 \]  

(47)

and

\[ \{ v_1, v_2 \} = 2\sqrt{a^2 + v_1^2 + v_2^2} \]  

(48)

The latter corresponds to the standard symplectic form on hyperbolic space. It is not hard to find canonical co-ordinates

\[ \left\{ \frac{\pi_1}{\sqrt{F_0}}, \frac{\pi_2}{\sqrt{F_0}} \right\} \rightarrow \{ \sqrt{2p} \cos \theta, \sqrt{2p} \sin \theta \}, \{ v_1, v_2 \} \rightarrow \{ a \sinh \theta \cos \phi, a \sinh \theta \sin \phi \} \]

so that-

\[ \{ p, \theta \} = 1, \quad p = \frac{1}{2F_0}(\pi_1^2 + \pi_2^2), \quad \theta = \arctan \frac{\pi_2}{\pi_1} \]  

(49)

\[ \{ v_0, \phi \} = 1, \quad \phi = \arctan \frac{v_2}{v_1} \]  

(50)

Note that \( J = p + \frac{1}{2} v_0 = \frac{K}{2F_0} \) is the generator of rotations; it is a conserved quantity corresponding to angular momentum. The hamiltonian is then
\begin{equation}
H = Ev_0 - \sqrt{2F_0 \left[v_0^2 - a^2\right]} \left(J - \frac{1}{2}v_0\right) \cos (\theta - \phi) \tag{51}
\end{equation}

Since
\begin{equation}
pd\theta + \frac{1}{2}v_0d\phi = Jd\theta + v_0d\alpha, \quad \alpha = \frac{1}{2}(\phi - \theta) \tag{52}
\end{equation}
we can use the pairs \(J, \theta\) and \(v_0, \alpha\) as canonical variables:
\begin{equation}
H = Ev_0 - \sqrt{2F_0 \left[v_0^2 - a^2\right]} \left(J - \frac{1}{2}v_0\right) \cos 2\alpha \tag{53}
\end{equation}

The equation of motion is
\begin{equation}
\frac{dv_0}{d\tau} = -\frac{\partial H}{\partial \alpha} = -2\sqrt{F_0 \left[v_0^2 - a^2\right]} \left(2J - v_0\right) \sin 2\alpha \tag{54}
\end{equation}

Thus
\begin{equation}
\left[\frac{dv_0}{d\tau}\right]^2 = 4F_0 \left[v_0^2 - a^2\right] \left(2J - v_0\right) - 4(H - Ev_0)^2 \tag{55}
\end{equation}

which is the Weierstrass equation. Because of its polarization, this also determines the spin.

Also
\begin{equation}
\frac{d\theta}{d\tau} = \frac{\partial H}{\partial J} = -\sqrt{\frac{F_0 \left[v_0^2 - a^2\right]}{2J - v_0}} \cos 2\alpha \tag{56}
\end{equation}

Thus
\begin{equation}
\frac{d\theta}{d\tau} = \frac{H - Ev_0}{2J - v_0} \tag{57}
\end{equation}

which determines \(\theta\) also in terms of elliptic integrals.

To determine the position variable, we note the that
\begin{equation}
\frac{d}{d\tau} \left[\pi_\mu - \epsilon_{\mu\nu\rho}x^\nu F^\rho\right] = 0 \tag{58}
\end{equation}

This leads to the conserved quantity (corresponding to translation invariance)
\begin{equation}
X_\mu = \pi_\mu - \epsilon_{\mu\nu\rho}x^\nu F^\rho \tag{59}
\end{equation}
Thus (in a co-ordinate system in which \( F_0 > 0, F_a = 0 \) and \( X_c = 0 \)) we have

\[
x_b = -\frac{1}{F_0} e_{bc} \pi_c
\]  

(60)

In other words, knowing \( \pi_1, \pi_2 \) is the same knowing the spatial orbit, except for a rotation by 90° and a scaling by \( F_0 \). In polar co-ordinates this amounts to knowing \( p = J - \frac{1}{2} v_0 \) and \( \theta \) as functions of \( \tau \), a kind of elliptic curve:

\[
\left[ \frac{dp}{d\tau} \right]^2 = 2F_0 \left[ 4(J-p)^2 - a^2 \right] p - (H - 2EJ + 2Ep)^2, \quad \frac{d\theta}{d\tau} = \frac{H - EJ}{2p} + \frac{E}{2}
\]  

(61)

It is not correct to interpret the \( x \) as the position of the electron: due to the phenomenon of zitterbewegung. Even for a free particle, \( x^\mu(\tau) \) is a spiral and not a straight line. Barut et. al. have shown how to recover a center of mass variable for the electron from this complicated solution.

### A Symplectic Reduction of the General Planar Orbit

The general case of planar but not spin-polarized orbits does not appear to be integrable. It corresponds to chaotic motion of an electron in a synchrotron. We can use the conserved quantities

\[
\pi_0 = E, \quad \lambda \cdot \lambda \equiv a^2, \quad \rho \cdot \rho \equiv b^2
\]  

(62)

to reduce it to a conservative system with three degrees of freedom: the phase space is a plane \( (\pi_a) \) times a pair of hyperboloids parametrized by \( (\lambda_\mu, \rho_\mu) \). It is easy to check that

\[
\{ p, \theta \} = 1
\]  

(63)

with

\[
p = \frac{1}{2F_0} \pi_a \pi_a, \quad \theta = \arctan \frac{\pi_2}{\pi_1}
\]  

(64)

Also,
\{\lambda_0, \phi \} = 1, \quad \phi = \arctan \frac{\lambda_2}{\lambda_1}
\{\rho_0, \chi \} = 1, \quad \chi = \arctan \frac{\rho_2}{\rho_1}

(65)

In terms of these

\[
H = 2E(\lambda_0 - \rho_0) - 2(\lambda_a - \rho_a)\pi_a
= E(\lambda_0 - \rho_0) - \sqrt{2F_0 [\lambda_0^2 - a^2]} p \cos[\phi - \theta] + \sqrt{2F_0 [\rho_0^2 - b^2]} p \cos[\chi - \theta]
\]

(66)

Another conserved quantity is the generator of rotations\(^5\)

\[J = p + \lambda_0 + \rho_0\]

(67)

The reduced system has three degrees of freedom and two conserved quantities \(H, J\) : just one conserved quantity short of an integrable system. When \(\lambda_0 = a\) (or \(\rho_0 = b\) ) one degree of freedom decouples and we get an integrable system. Circular orbits correspond to the case \(\phi = \chi, \lambda_0 + \rho_0 = \text{constant}\).

It may be more convenient to use \(\alpha = \phi - \theta, \beta = \chi - \theta, \theta\) as the independent position variables. Then

\[
p_\alpha d\alpha + p_\beta d\beta + p_\theta d\theta = pd\theta + \lambda_0 d\phi + \rho_0 d\chi = pd\theta + \lambda_0 d[\theta + \alpha] + \rho_0 d[\theta + \beta]
\]

(68)

\[
p_\alpha = \lambda_0
p_\beta = \rho_0
p_\theta = p + \lambda_0 + \rho_0 = J
\]

(69)

\(^5\)This is linearly related to the quantity \(K\) we introduced earlier:

\[K = 2F_0J\]
Then we get a system with two degrees of freedom with hamiltonian

\[ H = E(\lambda_0 - \rho_0) - \sqrt{2F_0 \left[ \lambda_0^2 - a^2 \right]} \left[ J - \lambda_0 - \rho_0 \right] \cos \alpha + \sqrt{2F_0 \left[ \rho_0^2 - b^2 \right]} \left[ J - \lambda_0 - \rho_0 \right] \cos \beta \]

This appears to be a chaotic system. In general, the information about the polarization of spin in the initial condition will be lost rapidly and the particle beam will look unpolarized. For the special case \( \rho_0 = 0 = b \) the spin remains polarized and solutions can be obtained in terms of elliptic functions as we saw above.

References

[1] J. D. Jackson, *Classical Electrodynamics*, Wiley (1998); F. Rohrlich, *Classical Charged Particles*, Westview Press (1994);

[2] P. A. M. Dirac, Proc. Roy. Soc. Lond A167 (1938) 148

[3] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* Butterworth-Heinemann (1982); especially Section 76

[4] H. Spohn, Europhys. Lett. 50 287 (2000); H. Spohn, *Dynamics of Charged Particles And Their Radiation Field*, Cambridge University Press (2004)

[5] S. G. Rajeev Ann. Phys. 323, 2654 (2008)

[6] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. 2, 435 (1959)

[7] A. O. Barut and N. Zanghi, Phys. Rev. Lett. 52, 2009–2012 (1984)

[8] A. O. Barut and N. Unal, Phys. Rev. A 40, 5404–5406 (1989)

[9] Y. Hadad, L. Labun, J. Rafelski, N. Elkina, C. Klier and H. Ruhl, arxiv/1005.3980

[10] S. G. Rajeev, Ann. Phys. 323, Pages 2873 (2008)

[11] L. D. Landau and E. M. Lifshitz, *Mechanics*, Butterworth-Heinemann (1982)