A note on the qualitative analysis of Volterra integro-differential equations

Cemil Tunç and Osman Tunç
Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, Van, Turkey

ABSTRACT
The aim of paper is to analyse some qualitative properties of solutions of nonlinear Volterra integro-differential equations (VIDDEs) with constant delay and Volterra integro-differential equations (VIDEs) without delay by means of the Razumikhin method. A suitable Lyapunov function is constructed and applied to these IDEs such that some former results can be obtained under weaker conditions and additionally some new results are given on the qualitative properties of that IDEs such as uniformly stability and integrability of solutions. For the VIDDEs with constant delay, the improved less conservative conditions of the stability, asymptotically stability and the boundedness of solutions and the new conditions of uniformly stability, integrability of solutions are all derived subject to the functions appeared in the considered IDEs. Moreover, the established stability, asymptotically stability and boundedness conditions of VIDDEs with constant delay simplify, extend and improve some previous works can be found in the literature and remove some unnecessary conditions. Finally, the validity of the presented results is indicated by some numerical examples using MATLAB-Simulink.

1. Introduction
Two of the main properties in the qualitative theory of integral equations, differential and integro-differential equations are stability and boundedness of solutions. Like ordinary and functional differential equations, integral and integro-differential equations have wide applications in physics, mechanics, heat transfer, viscoelasticity, electrical circuit, electrochemistry, dynamics, economics, control and some other many scientific areas (see [1–4]). There are several approaches in the literature to investigate the qualitative features of integro-differential equations, one of which is the second method of Lyapunov. During the last three decades, many researchers have done studies on the stability and boundedness of solutions of certain mathematical models of integral and integro-differential equations. And some basic results on the stability, boundedness and other related problems are obtained by the second method of Lyapunov (see [5–25]). For more details about the stability and boundedness results and the methods available to analyse the stability and boundedness of integro-differential equations, the reader can refer to the recent papers or books (see [1–25]) and the references therein.

Recently, El-Hajji [7] considered a VIDE given by

$$x'(t) = A(t)x(t) + B(t) + \int_0^t C(t,s)f(x(s))\,ds + g(x(t)),$$

where \( t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty), x \in \mathbb{R} \), \( A(t) \) and \( B(t) \) are continuous scalar functions on \( \mathbb{R}^+, f(x) \) and \( g(x) \) are continuous scalar functions on \( \mathbb{R} \), and \( C(t,s) \) is a scalar continuous function for \( 0 \leq s \leq t < \infty \).

El-Hajji [7] proved the following results on the stability, asymptotically stability of the zero solution and boundedness of solutions of VIDE (1), respectively, when \( B(t) = 0 \) and \( B(t) \neq 0 \). The results of El-Hajji [7] are stated by the following theorems, Theorems 1.1–1.3.

Consider VIDE (1) and let

\[ B(t) \neq 0. \]

Theorem 1.1 (\([7]\)):
We assume the following assumptions hold:

(A1) There exist positive constants \( \lambda_1, \lambda_2 \) and \( M \) such that the functions \( f, g \) and \( B \) satisfy

$$|g(x)| \leq \lambda_1 |x|, \quad \forall x \in \mathbb{R},$$

$$|f(x)| \leq \lambda_2 |x|, \quad \forall x \in \mathbb{R},$$

$$|B(t)| \leq M, \quad \forall x \in \mathbb{R}^+, \; \mathbb{R}^+ = [0, \infty).$$

(A2) There exists a positive and differentiable scalar function \( \psi : [0, \infty) \to [0, \infty) \) satisfying \( \psi'(t) \leq 0, \forall t \geq 0 \) and \( \psi \in L^1[0, \infty) \).

(A3) There exists a constant \( \lambda_3 \) such that

$$\lambda_2 |C(t,s)| + \lambda_3 \psi'(t - s) \leq 0, \quad \forall 0 \leq s \leq t < \infty.$$
(A4) There exists a negative scalar function \( A(t) \) defined on \([0, \infty)\) and a positive constant \( \alpha \) such that
\[
A(t) + \lambda_1 + \lambda_3 \psi(0) \leq -\alpha, \quad \forall 0 \leq s \leq t < \infty.
\]

(A5) There exists a positive uniformly continuous scalar function \( \beta(t) \) and a positive continuous scalar function \( H(t) \) such that
\[
H(t) = \beta(t) + \lambda_3 \int_0^t \psi(t-s) \beta(s) ds, \quad \lambda_3 > 0,
\]
\[
H'(t) = -\alpha \beta(t), \quad \alpha > 0, \quad \beta(0) = 1,
\]
which implies that
\[
\beta(t) + \int_0^t (\lambda_3 \psi(t-s) + \alpha) \beta(s) ds = 1, \quad |\beta(t)| \in L^1[0, \infty)
\]
and
\[
\lim_{t \to +\infty} \beta(t) = 0.
\]

Then all solutions of VIDE (1) are bounded.

It should be noted that to prove this theorem, the Lyapunov functional
\[
V = |x| + \lambda_3 \int_0^t \psi(t-s) |x(s)| ds, \quad \forall t \geq 0,
\]
is used as a basic tool by El-Hajji [7].

Let
\[
B(t) = 0.
\]

**Theorem 1.2 ([7]):** We assume the following assumptions hold:

(C1) There exist positive constants \( \lambda_1 \) and \( \lambda_2 \) such that the functions \( f \) and \( g \) satisfy
\[
|g(x)| \leq \lambda_1 |x|, \quad \forall x \in \mathbb{R},
\]
\[
|f(x)| \leq \lambda_2 |x|, \quad \forall x \in \mathbb{R}.
\]

(C2) The scalar function \( A(t) \) is negative definite, that is, \( A(t) \leq 0 \) such that there exist constants \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) satisfying
\[
|A(s)| \geq \lambda_1 + \lambda_2 \int_s^t |C(u,s)| du, \quad \forall 0 \leq s \leq t < \infty.
\]

Then the zero solution of VIDE (1) is stable.

**Theorem 1.3 ([7]):** In addition to assumptions (C1) and (C2), suppose that there exist two positive constants \( t_2 \geq 0 \) and \( \alpha > 0 \) such that the following assumptions hold.

(C3) \( |A(s)| \geq \lambda_1 + \lambda_2 \int_0^s |C(u,s)| du + \alpha, \quad \forall 0 \leq t_2 \leq s \leq t < \infty. \)

(C4) Both \( |A(s)| \) and \( \int_0^s |C(u,s)| ds \) are bounded.

Then the zero solution of VIDE (1) is asymptotically stable.

It is also worth mentioning that to prove both theorems, Theorems 1.2 and 1.3, the Lyapunov functional
\[
H(t, x(s)) = |x| + \int_0^t \left( |A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \right) x(s) ds, \quad \forall 0 \leq s \leq t < \infty.
\]
is taken as a basic tool by El-Hajji [7].

Motivated form VIDE (1), we first consider the VIDDEs with the constant delay of the form
\[
x'(t) = A(t) x(t) + B(t, x(t)) + \int_{t-\tau}^t C(t, s) f(x(s)) ds + g(x(t)), \tag{2}
\]
with the initial condition
\[
x(t_0 + \theta) = \phi_0(\theta) \quad \text{for} \quad \theta \in [-\tau, 0], x(t_0) = \phi_0(0), \phi_0 \in C([-\tau, 0], \mathbb{R}),
\]
and \( t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty), x \in \mathbb{R}, \mathbb{R} = (-\infty, \infty), \tau > 0, \tau \in \mathbb{R}, \tau \) is the fixed constant delay, \( A(t) \) and \( B(t, x) \) are continuous scalar functions on \( \mathbb{R}^+ \) and \( \mathbb{R}^+ \times \mathbb{R} \), respectively, \( f(x) \) and \( g(x) \) are continuous scalar functions on \( \mathbb{R} \) and \( C(t, s) \) is a continuous scalar function on \( \mathbb{R}^+ \times [-\tau, \infty), -\tau \leq s \leq t < \infty. \) When need it is supposed that \( x \) represents \( x(t) \).

**2. Qualitative analysis of solutions**

It should be noted that in the qualitative analysis of solutions of VIDDE (2) and VIDE (5), which will be given below, we will use Razumikhin method combined with the derivative of a suitable Lyapunov function. Therefore, before giving the main results of this paper, we present the following theorem, which can be easily derived and adopted from the book of Samoilenko and Perestyuk [3].

**Theorem 2.1 ([3]):** In addition to the basic hypotheses given above, we assume that there exists a Lyapunov function \( V = V(t, x) \) such that the following hypotheses hold:

(D1) \( V(t, 0) \equiv 0 \) for \( t \geq 0, \quad \alpha(||x||) \leq V(t, x) \) for \( t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \quad \alpha \in K, \)
\[
K = \{ \omega \in C(\mathbb{R}^+, \mathbb{R}^+)_+ : \omega(s) \text{ is strictly increasing and } \omega(0) = 0 \},
\]

(D2) for any initial data \( (t_0, \phi_0) \in \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}) \) and any point \( s > t_0 \) such that \( V(s + \theta, x(s + \theta)) < V(s, x(s)) \) for \( \theta \in [-\tau, 0] \) we have
\[
\frac{d}{dt} V(t, x) \leq 0, \quad t \in (t_0, s),
\]
where \( x(t) = x(t, t_0, \phi_0) \) is the corresponding solution of IVP for VIDDE (2). Then, then the zero solution of VIDDE (2) is stable.
We now state the main results of this paper. Consider VIDDE (2) and suppose that
\[ B(t, x) = 0. \]

**Theorem 2.2:** We assume that the following hypotheses hold:

1. (H1) There exist positive constants \( \lambda_1 \) and \( \lambda_2 \) such that the functions \( f \) and \( g \) satisfy
   \[ |g(x)| \leq \lambda_1|x|, \quad \forall x \in \mathbb{R}, \]
   \[ |f(x)| \leq \lambda_2|x|, \quad \forall x \in \mathbb{R}. \]
2. (H2) There exists a negative scalar function \( A(t) \) defined on \([0, \infty)\) such that
   \[ -A(t) \geq \lambda_2 \int_{t-\tau}^{t} |C(t, s)| ds + \lambda_1. \]

Then, the zero solution of VIDDE (2) is stable.

**Proof:** We define a quadratic Lyapunov function \( V = V(t, x) \) by
\[ V(t, x) = \frac{1}{2} x^2. \]

Hence, we have
\[ V(t, 0) = 0 \]
and
\[ V(t, x) \geq \frac{1}{2} p_0 x^2, \tag{3} \]
where \( 0 < p_0 < 1, p_0 \in \mathbb{R} \).

Differentiating the Lyapunov function \( V \) along the solutions of nonlinear VIDDE (2), we have
\[
\dot{V} = xx' = x[A(t)x + \int_{t-\tau}^{t} C(t, s)f(x(s))ds + g(x)]
= A(t)x^2 + x \int_{t-\tau}^{t} C(t, s)f(x(s))ds + xg(x)
\leq A(t)x^2 + x \int_{t-\tau}^{t} |C(t, s)||f(x(s))|ds + xg(x).
\]

Applying the hypothesis (H1) and the inequality \( a^2 + b^2 \geq 2ab \), we obtain
\[
\dot{V} \leq A(t)x^2 + \lambda_2 |x| \int_{t-\tau}^{t} |C(t, s)||x(s)|ds + xg(x)
\leq A(t)x^2 + 2^{-1} \lambda_2 \int_{t-\tau}^{t} |C(t, s)||x^2(t) + x^2(s)|ds + xg(x)
= A(t)x^2 + 2^{-1} \lambda_2 x^2 \int_{t-\tau}^{t} |C(t, s)|ds
+ 2^{-1} \lambda_2 \int_{t-\tau}^{t} |C(t, s)|x^2(s)ds + \lambda_1 x^2.
\]

At next step, we apply the Razumikhin method or condition (see, Samoilenko and Perestyuk [3]). Hence, let \( t > t_0 \) such that
\[ V(t + s, x(t + s)) < V(t, x(t)), \quad \forall s \in [-r, 0), \]
that is,
\[ (x(t + s))^2 < (x(t))^2, \quad \forall s \in [-r, 0). \]

Then, applying the substitution \( s = t - \xi \) to the third right term of \( V \) and the Razumikhin condition, it follows that
\[
\dot{V} \leq A(t)x^2 + 2^{-1} \lambda_2 x^2 \int_{t-\tau}^{t} |C(t, s)|ds
+ 2^{-1} \lambda_2 x^2 \int_{t-\tau}^{t} |C(t, s)|ds + \lambda_1 x^2
= [A(t) + \lambda_2] \int_{t-\tau}^{t} |C(t, s)|ds + \lambda_1 x^2 \leq 0.
\]

This inequality implies that the zero solution of nonlinear VIDDE (2) is stable.

**Theorem 2.3:** If hypotheses (H1) and (H2) hold, then the zero solution of nonlinear VIDDE (2) is asymptotically stable and all solutions of VIDDE (2) are bounded as \( t \to \infty \).

**Proof:** From Theorem 2.1, it is known that, in the light of the hypotheses (H1) and (H2), we have
\[ V(t, x) \geq \frac{1}{2} p_0 x^2 \]
and
\[ \dot{V} \leq [A(t) + \lambda_2] \int_{t-\tau}^{t} |C(t, s)|ds + \lambda_1 x^2 \leq 0. \]

We now consider the set defined by
\[ I_5 = \{ x : \dot{V}(t, x) = 0 \}. \]

If we apply the LaSalle’s invariance principle, we observe both \( V(t, x) = 0 \) and \( (t, x) \in I_5 \) imply that \( x = 0 \). From this point, both \( x = 0 \) and nonlinear VIDDE (2), together, necessarily imply that \( x = 0 \), when \( B(t, x) = 0 \). In fact, this result shows that the largest invariant set contained in \( I_5 \) is \((t, 0) \in I_5\). Hence, we can conclude that the zero solution of nonlinear VIDDE (2) is asymptotically stable. This fact complete the proof of the asymptotic stability of the zero solution of the considered equation.
Now, as far the boundedness of solutions, integrating the inequality \( V \leq 0 \) from \( t_0 \) to \( t \), we have
\[
V(t, x(t)) \leq V(t_0, x(t_0)), \quad \forall t \geq t_0. \quad (4)
\]

It is obvious that
\[
V(t, 0) = 0 \quad \text{and} \quad V(t_0, x(t_0)) = \frac{1}{2}x^2(t_0) = K_0 > 0, \quad K_0 \in \mathbb{R},
\]
when
\[
x(t_0) \neq 0.
\]

Then
\[
\frac{1}{2}p_0x^2 \leq V(t, x) \leq K_0
\]
so that
\[
|x(t)| \leq \sqrt{2p_0^{-1}K_0}
\]
for all \( t \geq t_0 \). From this relation, we can conclude that all solutions of nonlinear VIDDE (2) are bounded as \( t \to \infty \).

**Corollary 2.1:** If hypotheses (H1) and (H2) hold, then the zero solution of nonlinear VIDDE (2) is uniformly stable. We ignore the proof of this corollary.

We now consider a general form of VIDE (1) and a particular case of VIDDE (2) as
\[
x'(t) = A(t)x(t) + B(t, x(t)) + \int_0^t C(t, s)f(x(s))ds + g(x(t)). \quad (5)
\]

We suppose all basic assumptions on the given functions in VIDDE (2) are satisfied for that in VIDE (5) for \( \forall t \in \mathbb{R^+} \) and \( \forall x \in \mathbb{R} \). Here, we will prove two theorems on the integrability and boundedness of solutions of VIDE (5).

Consider VIDE (5) and suppose that
\[
B(t, x) = 0.
\]

**Theorem 2.4:** Let hypothesis (H1) be hold. Further, we assume that the following hypothesis holds:
\[
\text{(H3)} \quad \text{There exists a negative scalar function } A(t) \text{ defined on } [0, \infty) \text{ and a positive constant } \mu \text{ such that}
\]
\[
-A(t) - \lambda_2 \int_0^t |C(t, s)|ds - \lambda_1 \geq \mu.
\]

Then, all solutions of VIDE (5) are integrable.

**Proof:** In the light of the hypotheses of Theorem 2.4, we can proceed
\[
V(t, x(t)) \leq -\mu|x|.
\]

Integrating this inequality from \( t_0 \) to \( t \), it follows that
\[
V(t, x(t)) - V(t, x(t_0)) \leq -\mu \int_{t_0}^t |x(s)|ds.
\]

Then, this inequality is arranged as
\[
\mu \int_{t_0}^t |x(s)|ds \leq V(t, x(t)) - V(t, x(t_0)) \leq V(t_0, x(t_0)).
\]

It is clear that \( V(t_0, x(t_0)) \) is a positive constant, that is, \( V(t_0, x(t_0)) = K_0 \). Then
\[
\int_{t_0}^t |x(s)|ds \leq \mu^{-1}K_0.
\]

Hence, we can conclude that
\[
\int_{t_0}^\infty |x(s)|ds < \infty.
\]

This inequality completes the proof of Theorem 2.4. Consider VIDE (5) and let
\[
B(t, x) \neq 0.
\]

**Theorem 2.5:** Assume hypotheses (H1) and the both below hypotheses hold.

\( (H4) \) There exists a negative scalar function \( A(t) \) defined on \( [0, \infty) \) such that
\[
-A(t) \geq \lambda_2 \int_0^t |C(t, s)|ds + \lambda_1.
\]
\( (H5) \) \( |B(t, x)| \leq \frac{1}{2}|Q(t)||x|, \forall t, x \in \mathbb{R^+}, Q \in L^1[0, \infty) \).

Then all solutions of VIDE (5) are bounded.

**Proof:** In the light of hypotheses (H1), (H4), (H5) and the inequality \( |B(t, x)| \leq \frac{1}{2}|Q(t)||x| \), it can be followed that
\[
V(t, x) \leq |B(t, x)||x| \leq \frac{1}{2}|Q(t)||x^2| = |Q(t)||V(t, x)|.
\]

By integration of this inequality, we can proceed
\[
V(t, x(t)) \leq V(t_0, x(t_0)) \exp\int_{t_0}^\infty |Q(s)|ds.
\]

Then
\[
\frac{1}{2}x^2 \leq V(t, x) \leq V(t_0, x(t_0)) \exp\int_{t_0}^\infty |Q(s)|ds.
\]

This inequality is the end of the proof.

**Example 2.1:** We consider the following scalar VIDDE with constant delay, \( \tau = 1 \),
\[
x' = -(20 + \frac{1}{1+\tau})x + \frac{1}{36}x \cos x + \int_{t-1}^t \exp(-t + s - 1) \sin(x(s))ds. \quad (6)
\]
By comparison of VIDDE (6) and VIDE (2) we can see that

\[ A(t) = -20 - \frac{1}{1 + t^4}, \]

\[ g(x) = \frac{1}{36} x \cos x, \quad g(0) = 0, \]

\[ |g(x)| = \frac{1}{36} |x| |\cos x| \leq \frac{1}{36} |x|, \quad \forall x \in \mathbb{R}, \]

\[ \lambda_1 = \frac{1}{36}, \]

\[ f(x) = \sin x, \quad f(0) = 0, \]

\[ |f(x)| = |\sin x| \leq |x|, \quad \lambda_2 = 1, \quad \forall x \in \mathbb{R}, \]

\[ C(t,s) = \exp(-t + s - 1), \]

\[ \int_{t-1}^{t} |C(t,s)|ds = \int_{t-1}^{t} \exp(-t + s - 1)ds = \frac{e^{-1}}{e^2}, \]

\[ -A(t) - \lambda \int_{t-1}^{t} |C(t,s)|ds - \lambda_1 = 20 + \frac{1}{1 + t^4} - \frac{e^{-1}}{e^2} - \frac{1}{36} > 19 = \mu. \]

We now reconsider the Lyapunov function by

\[ V(t,x) = \frac{1}{2} x^2. \]

The time derivative of this function along VIDDE (6) implies

\[ \dot{V} = xx' = - (20 + \frac{1}{1 + t^4}) x^2 + \frac{1}{36} x^2 \cos x \]

\[ + x(t) \int_{t-1}^{t} \exp(-t + s - 1) \sin(x(s))ds \]

\[ \leq - \left( 20 + \frac{1}{1 + t^4} \right) x^2 + \frac{1}{36} x^2 \]

\[ + |x(t)| \int_{t-1}^{t} \exp(-t + s - 1) |\sin(x(s))|ds \]

\[ \leq - \left( 20 + \frac{1}{1 + t^4} \right) x^2 + \frac{1}{36} x^2 \]

\[ + |x(t)| \int_{t-1}^{t} \exp(-t + s - 1) |x(s)|ds \]

\[ \leq - \left( 20 + \frac{1}{1 + t^4} \right) x^2 + \frac{1}{36} x^2 \]

\[ + \frac{1}{2} \int_{t-1}^{t} \exp(-t + s - 1) |x^2(s) + x^2(s)ds \]

\[ = - \left( 20 + \frac{1}{1 + t^4} \right) x^2 + \frac{1}{36} x^2 \]

\[ + \frac{1}{2} \int_{t-1}^{t} \exp(-t + s - 1)ds \]

\[ + \frac{1}{2} \int_{t-1}^{t} \exp(-t + s - 1) x^2(s)ds. \]

We now apply the Razumikhin method (see, Samoilenko and Perestyuk [3]) to the fourth term in the right of the last equality. Hence, let \( t > t_0 \) such that

\[ V(t+s,x(t+s)) < V(t,x(t)), \quad \forall s \in [-1,0), \]

that is,

\[ (x(t+s))^2 < (x(t))^2, \quad \forall s \in [-1,0). \]

Then, applying the substitution \( s - t = \xi \) to the fourth right term of \( \dot{V} \) and using Razumikhin condition, it follows that

\[ \dot{V} \leq - \left[ (20 + \frac{1}{1 + t^4}) - \frac{1}{36} - \int_{t-1}^{t} \exp(-t + s - 1)ds \right] x^2 \]

\[ = - \left[ (20 + \frac{1}{1 + t^4}) - \frac{1}{36} - \frac{e^{-1}}{e^2} \right] x^2 \]

\[ \leq -19x^2. \]

**Figure 1.** Trajectories of \( x(t) \) for Example 2.1.
Figure 2. Trajectories of \( x(t) \) for Example 2.2.

Thus, all hypotheses of Theorems 2.2 and 2.3 hold. Hence, if we exclude the term \( B(t, x) \) of VIDDE (2), then the zero solution of VIDDE (6) is stable, uniformly stable, and asymptotically stable and all solutions of VIDDE (6) are bounded as \( t \to \infty \) (Figure 1).

**Example 2.2:** We consider the following scalar VIDE for \( \forall t \in [0, \infty) \),

\[
x' = -\left(20 + \frac{1}{1 + t^4}\right)x + \frac{1}{36}x \cos x + \int_0^t \exp(-t + s - 2) \sin(x(s))ds + \frac{x}{1 + t^2}.
\]

(7)

Comparing both VIDE (7) and VIDE (5), we follow that

\[
A(t) = -20 - \frac{1}{1 + t^4},
\]

\[
C(t, s) = \exp(-t + s - 2),
\]

\[
\int_0^t |C(t, s)|ds = \int_0^t \exp(-t + s - 2)ds
\]

\[
= \frac{1}{e^t} - \frac{1}{e^{t+2}} > 0, \quad t > 0,
\]

\[
B(t, x) = \frac{x}{1 + t^2},
\]

\[
|B(t, x)| \leq \frac{1}{1 + t^2}|x|,
\]

\[
\int_0^\infty |Q(s)|ds = \int_0^\infty \frac{1}{1 + s^2}ds = \frac{\pi}{2} < \infty.
\]

\[
-A(t) - \lambda_2 \int_0^t |C(t, s)|ds - \lambda_1 = 20 + \frac{1}{1 + t^4} - \frac{1}{e^{t+2}} - \frac{1}{36} > 17.
\]

The other details for the satisfaction of the hypotheses of Theorems 2.4 and 2.5 can be shown easily. We omit the details. Hence, we can conclude that all solutions of VIDE (7) are integrable and bounded (Figure 2).

3. Conclusion

In this paper, we use a classical quadratic Lyapunov to derive new sufficient conditions for the qualitative analysis of solutions of a class of nonlinear IDEs. The derived sufficient conditions are expressed in terms of that nonlinear IDEs. By this study, we not only weaken and delete some reasonless conditions of the related theorems in [7], but also improve the results of [7]. At the same time, additionally, we study uniformly stability and integrability of solutions. Further, two examples are given to illustrate the validity and feasibility of main results.

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ORCID

Cemil Tunç © http://orcid.org/0000-0003-2909-8753

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