ON THE $L^\infty$ STABILITY OF PRANDTL EXPANSIONS IN GEVREY CLASS

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ABSTRACT. In this paper, we prove the $L^\infty \cap L^2$ stability of Prandtl expansions of shear flow type as $(U(y/\sqrt{\nu}), 0)$ for the initial perturbation in the Gevrey class, where $U(y)$ is a monotone and concave function and $\nu$ is the viscosity coefficient. To this end, we develop the direct resolvent estimate method for the linearized Orr-Sommerfeld operator instead of the Rayleigh-Airy iteration method introduced by Grenier, Guo and Nguyen.

1. INTRODUCTIONS

In this paper, we study the incompressible Navier-Stokes equations in $\Omega := \mathbb{T} \times \mathbb{R}_+$ when the viscosity coefficient $\nu$ tends to zero:

$$\begin{align*}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu - \nu \Delta u^\nu &= f^\nu \quad \text{in } [0, T] \times \Omega, \\
\nabla \cdot u^\nu &= 0 \quad \text{in } [0, T] \times \Omega, \\
\partial_t u^\nu|_{\partial \Omega} &= 0 \quad \text{on } [0, T] \times \partial \Omega, \\
u^\nu(0) &= u_0 \quad \text{in } \Omega.
\end{align*}$$

(1.1)

Here $u^\nu = (u_1^\nu, u_2^\nu)$ is the velocity field, $p^\nu$ is the pressure and $f^\nu$ is the external force.

In the absence of the boundary, the solution $u^\nu$ of the Navier-Stokes equations converges to the solution $u^e$ of the Euler equations as $\nu \to 0$:

$$\begin{align*}
\partial_t u^e + u^e \cdot \nabla u^e + \nabla p^e &= 0, \\
\nabla \cdot u^e &= 0.
\end{align*}$$

The inviscid limit has been justified in various functional settings \cite{19, 33, 3, 26, 5, 1, 25}.

In the presence of the boundary, the inviscid limit problem is difficult due to the appearance of boundary layer. For the Navier-slip boundary condition, the boundary layer is weak. In such case, the limit from the Navier-Stokes equations to the Euler equations was first justified in 2-D by Clopeau, Mikelic and Robert \cite{4}, and in 3-D by Itimie and Planas \cite{18}. See \cite{27, 31, 39, 38} for more relevant results. For the nonslip boundary condition, the boundary layer is strong. In such case, when $\nu \to 0$, the solution of (1.1) formally behaves as

$$\begin{align*}
\begin{cases}
\partial_t u^e_1 + u^e_1 \cdot \nabla u^e_1 + \nabla p^e_1 &= 0, \\
\nabla \cdot u^e_1 &= 0,
\end{cases}
\end{align*}$$

(1.2)

where $(u^p, v^p) = (u_1^e(t, x, 0) + u^{BL}(t, x, Y), \partial_y u_2^e(t, x, 0)Y + v^{BL}(t, x, Y))$ is the solution of the Prandtl equation

$$\begin{align*}
\begin{cases}
\partial_t u^p + u^p \partial_x u^p + v^p \partial_Y u^p + \partial_y p^e|_{y=0} &= \partial_Y^2 u^p, \\
\partial_x u^p + \partial_Y v^p &= 0, \\
u^p|_{Y=0} = v^p|_{Y=0} &= 0, \\
\lim_{Y \to +\infty} u^p(t, x, Y) &= u_1^e(t, x, 0).
\end{cases}
\end{align*}$$

(1.3)

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Let us mention some recent well-posedness results of the Prandtl equation \cite{32, 40, 1, 28, 10, 13, 6, 23, 8}, which are a key step toward the inviscid limit problem.

To our knowledge, the justification of the Prandtl expansion (1.2) is still a challenging problem except for some special cases: the analytic data \cite{33} (see \cite{37} for a new proof via direct energy method), and the initial vorticity away from the boundary \cite{24, 9}. In addition, the convergence was justified in \cite{22, 29} when the domain and the initial data have a circular symmetry. Initiated by Kato \cite{20}, there are many works devoted to the conditional convergence \cite{35, 36, 21}.

Recently, there is a lot of attention on the stability of some special boundary layer solutions. For example, Grenier studied the Prandtl expansion of shear flow type as

\begin{equation}
(1.4) \quad u^\nu = (U^e(t,y), 0) + \left( U^{BL}(t, \frac{y}{\sqrt{\nu}}, 0) \right).
\end{equation}

When the shear flow \( U^{BL}(0,Y) \) is linearly unstable for the Euler equations, he proved the instability of the expansion in the \( H^1 \) space by constructing the solution with the highly oscillating as \( e^{i\alpha x/\sqrt{\nu}} \) and the growth as \( e^{C|\nu|^{\frac{3}{4} \beta}} \) at the high frequency \( n = \frac{1}{\sqrt{\nu}} \gg 1 \). In a recent important work, Grenier and Nguyen \cite{15} proved the \( L^\infty \) instability of the Prandtl expansion (1.4). In another important work, Guo, Grenier and Nguyen proved that the shear flows which are linearly stable for the Euler equations could be linearly unstable for the Navier-Stokes equations when \( \nu \) is very small, where they constructed the solution with the growth as \( e^{C|\nu|^{\frac{3}{4} \beta}} \). Their result in particular implies that it is possible to prove the stability of monotone and concave shear flows for the Navier-Stokes equations in the Gevrey class \( \frac{3}{2} \). In a remarkable work \cite{12}, Gerard-Varet, Masmoudi and Maekawa proved the stability of the Prandtl expansion (1.4) for the perturbations in the Gevrey class when \( U^{BL}(t,Y) \) is a monotone and concave function. Roughly speaking, they showed that if the initial perturbation \( a(x,y) \) satisfies \( \|a\|_{G_{\gamma}} \leq \frac{1}{\nu^{\frac{3}{4} \beta}} \) for \( \beta = \frac{2(1-\gamma)}{\gamma} \), where \( G_{\gamma} \) is a norm of Gevrey class \( \frac{1}{\gamma} \) with \( \gamma \in (0,1] \) depending on the profile \( U^{BL}(t,Y) \), then

\[
\sup_{t \in [0,T]} (\|v^\nu(t)\|_{L^2} + (\nu t)^{\frac{\beta}{4}} \|\nabla v^\nu(t)\|_{L^2} + (\nu t)^{\frac{\beta}{4}} \|v^\nu(t)\|_{L^\infty}) \leq C\|a\|_{G_{\gamma}},
\]

where \( v^\nu(t,x,y) = u^\nu(t,x,y) - (U^e(t,y), 0) + (U^{BL}(t, \frac{y}{\sqrt{\nu}}, 0), 0) \). The \( L^\infty \) stability estimate, which is in fact an interpolation result between \( L^2 \) estimate and \( H^1 \) estimate, blows up when \( t \to 0 \) due to the prefactor \( t^{\frac{\gamma}{2}} \). Their proof relies on the resolvent estimates for the linearized Orr-Sommerfeld operator via the complicated Rayleigh-Airy iteration method introduced in \cite{14}.

For the steady Navier-Stokes equations, Gerard-Varet and Maekawa \cite{11} proved the stability of shear flows \( (U(\frac{y}{\sqrt{\nu}}), 0) \) for the external force \( f^\nu \) in the Sobolev space, and Guo and Iyer \cite{16} proved the stability of Blasius flows, which are self-similar solutions of the steady Prandtl equation. Guo and Nguyen \cite{17} also considered the Prandtl expansions of steady Navier–Stokes equations over a moving plate.

The goal of this paper is twofold. First of all, we would like to prove the \( L^\infty \) stability estimate of the Prandtl expansion (1.4) without the prefactor \( t^{\frac{\gamma}{2}} \), i.e. \( \nu^{\frac{\beta}{8}} \|v^\nu(t)\|_{L^\infty} \leq C\|a\|_{G_{\gamma}} \).

Secondly, we would like to develop a direct resolvent estimate method for the linearized Orr-Sommerfeld operator instead of the Rayleigh-Airy iteration method.
For the simplicity, we take $U^e(t, y) \equiv 1$ and $U^{BL}(t, Y) := U^P(Y) - 1$ in (1.4), where $U^P = U^P(Y)$ is a scalar function on $\mathbb{R}_+$ satisfying

$$\lim_{Y \to \infty} U^P(Y) = 1, \quad U^P(Y = 0) = 0.$$ 

After assuming that the external force

$$f = \left( \partial_y^2 U^P \left( \frac{y}{\sqrt{P}} \right), 0 \right),$$

we write the solution of (1.1) in the perturbation form

$$u''(t, x, y) = \left( 1 + U^{BL} \left( \frac{y}{\sqrt{P}} \right), 0 \right) + u(t, x, y)$$

with

$$\begin{cases}
    \partial_t u - \nu \Delta u + u \cdot \nabla u + U^P \left( \frac{y}{\sqrt{P}} \right) \partial_x u + \nu^{-1/2} u_2 \left( \partial_y U^P \left( \frac{y}{\sqrt{P}} \right), 0 \right) + \nabla p = 0, \\
    \nabla \cdot u = 0, \\
    u|_{\partial \Omega} = 0, \quad u(0) = a.
\end{cases}$$

The system (1.5) can be written as

$$\begin{cases}
    \partial_t u + A_u u = -P(u \cdot \nabla u), \\
    u|_{t=0} = a,
\end{cases}$$

where $P : L^2(\Omega)^2 \to L^2(\Omega)$ is the Helmholtz-Leray projection and

$$A_u u = -\nu \Delta u + P \left( U^P \left( \frac{y}{\sqrt{P}} \right) u + \nu^{-1/2} u_2 \left( \partial_y U^P \left( \frac{y}{\sqrt{P}} \right), 0 \right) \right)$$

with the domain

$$D(A_u) = W^{2,2}(\Omega)^2 \cap W^{1,2}_0(\Omega)^2 \cap L^2(\Omega).$$

Before stating main result, we introduce some notations and function spaces. Let

$$(P_n f)(y) = f_n(y) e^{inx}, \quad f_n(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x, y) e^{-inx} dx, \quad n \in \mathbb{Z},$$

be the projection on the Fourier mode $n$ in $x$. We then defined the following function spaces, for $\gamma \in (0, 1], d \geq 0$ and $K > 0$

$$X_{d, \gamma, K} := \left\{ f \in L^2(\Omega) \bigm| ||f||_{X_{d, \gamma, K}} = \sup_\Gamma (1 + |n|^d)e^{K|n|^\gamma} ||P_n f||_{L^2(\Omega)} < \infty \right\},$$

and

$$Y_{d, \gamma, K} := \left\{ f \in L^2(\Omega) \bigm| ||f||_{Y_{d, \gamma, K}} = \sup_\Gamma (1 + |n|^d)e^{K|n|^\gamma} ||P_n f||_{L^2_0 L^\infty(\Omega)} < \infty \right\}.$$ 

We also introduce the following strongly concave condition on shear flows:

**SC condition:** we consider $U(Y)$ with

$$||U|| := \sum_{k=0, 1, 2} \sup_{Y \geq 0} (1 + Y)^k |\partial^k_y U(Y)| < \infty.$$ 

Our assumption are

1. $U|_{Y=0} = 0$, $\lim_{Y \to \infty} U(Y) = 1$ and $U \in BC^2(\mathbb{R}_+)$. 
2. There exists $M > 0$ such that $-M \partial^2_y U \geq (\partial_y U)^2$ and $|\partial^2_y U/\partial^2_t U| + |\partial^2_y U/\partial_t^2 U| \leq M$ for any $Y \geq 0$.

Now we state our main result.
**Theorem 1.1.** Assume that $U^P$ satisfies (SC) condition. For any $\gamma \in [2/3,1)$, $d > \frac{11}{4} - \frac{3}{4}\gamma$ and $K > 0$, there exists $C, T', K' > 0$ such that the following statement holds for any sufficiently small $\nu > 0$. If $\|a\|_{X_{d,\gamma,K}} + \nu^\frac{1}{2}\|a\|_{Y_{d,\gamma,K}} \leq \nu^{\frac{1}{2} + \beta}$ with $\beta = \max\{\frac{2(1-\gamma)}{\gamma}, \frac{3}{10} + \frac{3(1-\gamma)}{2\gamma} \frac{1}{4} + \frac{1-\gamma}{2}\gamma\}$, then the system (1.3) admits a unique solution $u \in C([0,T']; L^2_\sigma(\Omega)) \cap L^2(0,T'; W^{1,2}_0(\Omega))$ satisfying

$$
\sup_{0 < t \leq T'} \left( \|u(t)\|_{X_{d,\gamma,K'}} + \nu^\frac{1}{2}\|u(t)\|_{L^\infty(\Omega)} + (\nu t)^\frac{3}{2}\|\nabla u(t)\|_{L^2(\Omega)} \right) \\
\leq C\|a\|_{X_{d,\gamma,K}} + C\nu^\frac{3}{4}\|a\|_{Y_{d,\gamma,K}}.
$$

Let us give two remarks on our result.

1. The main achievement of Theorem 1.1 is the $L^\infty$ stability estimate. To this end, the price we pay is that we require more restriction on $\beta$ compared with [12], where $\beta = \frac{2(1-\gamma)}{\gamma}$. However, we are actually interested in the case of $\gamma = \frac{2}{3}$. In this case, $\beta$ is still chosen as $\frac{2(1-\gamma)}{\gamma} = 1$ due to the definition of $\beta$.

2. Our proof could be easily modified to the case when $U(Y)$ satisfies weakly concave(WC) condition: $-M_\sigma \partial^2_\sigma U \geq (\partial Y U)^2$ for $Y \geq \sigma > 0$ or $U$ is time dependent and weakly concave. The same results hold but with the range of $\gamma$ replaced by $\gamma \in \left[\frac{2}{3}, 1\right]$ or $\gamma \in \left[\frac{2}{9}, 1\right]$ respectively.

The roadmap of the proof of Theorem 1.1 is similar to [12]. We focus on the linearized system of (1.5) and obtain the corresponding semigroup estimates via the resolvent estimates for the linearized Orr-Sommerfeld operator. Then using the Duhamel formula of the solution, we prove the nonlinear stability by combining the semigroup estimates with the estimates for nonlinear terms. However, there are two main differences with [12]:

1. Motivated by [7], the idea of achieving resolvent estimates is that we first consider the linearized system with Navier-slip boundary condition. Thanks to good boundary condition, we can obtain various resolvent estimates by choosing suitable multipliers and integration by parts arguments. Secondly, it is enough to build up a boundary corrector via the Airy function to match nonslip boundary condition instead of slow mode and fast mode used in [12]. The advantage of this idea is that we not only avoid to use the Rayleigh-Airy iteration, but also we still can obtain the resolvent estimates as sharp as in [12]. The resolvent estimate method we develop is of independent interest, and could be used to the relevant problems in hydrodynamic stability.

2. To obtain $L^\infty$ semigroup estimate, we establish a weighted $H^1$ resolvent estimate, which is new and easy to obtain by our resolvent estimate method.

Finally, let us introduce the formulation in Fourier series of (1.6). Since the shear flow is assumed to be $x$-independent, it is natural to study $A_\nu$ on each Fourier mode with respect to the $x$ variable. We define $A_{\nu,n}$ by the restriction of $A_\nu$ on the subspace $P_n L^2_\sigma(\Omega)$. By the Duhamel formula, the solution $u(t)$ of (1.6) could be written into the following integral form with respect to each Fourier mode $n$:

$$
(1.9) \quad P_n u(t) = e^{-t A_{\nu,n}} P_n a - \int_0^t e^{-(t-s) A_{\nu,n}} P_n [P(u \cdot \nabla) u](s) ds.
$$

The rest of this paper is organized as follow. Section 2 is devoted to the resolvent estimates, which is the main part of this paper. In section 3, we show the semigroup estimate. In section 4 we finish the proof of Theorem 1.1 with proving the estimate of nonlinear term. At last, we show Hardy’s type inequality and several properties of Airy functions in the appendix.
2. Resolvent Estimates in Middle Frequency

In the middle range of frequency $O(1) \leq |n| \leq O(\nu^{-3/4})$, we cannot obtain a useful estimate of semigroup $e^{-\tau A_{\nu,n}}$ via simple energy method. To obtain a better estimate of semigroup $e^{-\tau A_{\nu,n}}$, we consider the corresponding resolvent problem in this section.

2.1. Reformulation and main results. As in [12], it is more convenient to introduce the rescaled velocity

$$u(t, x, y) = v(\tau, X, Y), \quad (\tau, X, Y) = \left(\frac{t}{\sqrt{\nu}}, \frac{x}{\sqrt{\nu}}, \frac{y}{\sqrt{\nu}}\right).$$

If $u(t) = e^{-tA_{\nu,n}}$, then $v$ is the solution to

$$\begin{align*}
\partial_{\tau}v - \sqrt{\nu}\Delta_{X,Y}v + (v_2 \partial_Y V, 0) + V \partial_X v + \nabla_{X,Y} q &= 0 \quad \text{in } \Omega_{\nu}, \\
\text{div}_{X,Y} v &= 0, \quad v|_{Y=0} = 0, \quad v|_{\tau=0} = v^{(\nu)},
\end{align*}$$

where $\Omega_{\nu} := (\nu^{-1/2} \mathbb{T}) \times \mathbb{R}_+$ and $v^{(\nu)} := a(\nu^{1/2} X, \nu^{1/2} Y)$. We also introduce the notation of Fourier series after scaling:

$$(P_{\nu,n}f)(Y) = f_n(Y)e^{in\pi X}, \quad f_n(Y) = \frac{\sqrt{\nu}}{2\pi} \int_0^{2\pi} f(X, Y)e^{-in\sqrt{\nu}X} dX.$$

To obtain the estimates of solutions to (2.2), we study the corresponding resolvent problem for the operator:

$$\begin{align*}
\mathbb{L}_{\nu} v &= -\sqrt{\nu}D_{\nu,\nu} \Delta v + D_{\nu}(V \partial_X v + v_2(\partial_Y V, 0)) \\
\text{with } D(\mathbb{L}_{\nu}) &= W^{2,2}(\Omega_{\nu}) \cap W^{1,2}_0(\Omega_{\nu}) \cap L^2(\Omega_{\nu}).
\end{align*}$$

In details, we consider the following resolvent problem of $\mathbb{L}_{\nu}$ by assuming that $V(Y)$ satisfies (SC) condition,

$$\begin{align*}
\mu v - \sqrt{\nu}\Delta_{X,Y}v + (v_2 \partial_Y V, 0) + V \partial_X v + \nabla_{X,Y} q &= f, \quad Y \geq 0, \\
\text{div}_{X,Y} v &= 0, \quad v|_{Y=0} = 0.
\end{align*}$$

Here $\mu \in \mathbb{C}$ is a resolvent parameter, and $v, \nabla q$ and $f$ are assumed to be $\frac{2\pi}{\sqrt{\nu}}$-periodic in $X$. Let $w = \partial_X v_2 - \partial_Y v_1$ be the vorticity field of $v$. Direct computation gives

$$\mu w - \sqrt{\nu} \Delta w - v_2 \partial_Y^2 V + V \partial_x w = \partial_X f_2 - \partial_Y f_1.$$

The corresponding stream function denoted by $\psi$ which solves

$$\Delta \phi = w, \quad \phi|_{Y=0} = 0.$$

Then $v = (-\partial_Y \phi, \partial_X \phi)$. Let $\phi_n(Y)$ be the $n\sqrt{\nu}$ fourier modes of $\phi(X, Y)$, namely

$$\phi_n(Y) = \int_{T/\sqrt{\nu}} \phi(X, Y)e^{-in\sqrt{\nu}X} dX, \quad \phi(X, Y) \sim \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \phi_n(Y)e^{in\sqrt{\nu}X}.$$

Then we know that for each $n\sqrt{\nu}$ mode with respect to variable $x$, the stream function $\phi_n$ solves the following system

$$\begin{align*}
\mu(\partial_Y^2 - n^2 \nu) \phi_n - \sqrt{\nu}(\partial_Y^2 - n^2 \nu)^2 \phi_n - in\sqrt{\nu}\phi_n(\partial_Y^2 V) + in\sqrt{\nu}V(\partial_Y^2 - n^2 \nu) \phi_n \\
= in\sqrt{\nu}f_{2,n} - \partial_Y f_{1,n}, \\
\phi_n|_{Y=0} = \partial_Y \phi_n|_{Y=0} = 0,
\end{align*}$$

(2.5)
where
\[ f_{i,n}(Y) := \int_{T/\sqrt{\nu}} f_i(X,Y)e^{-in\sqrt{\nu}X}dX, \quad i = \{1, 2\}. \]

As you will see later, we only need to consider the case
\[ \delta_0^{-1} \leq |n| \leq \delta_0^{-1}\nu^{-3/4}, \quad \text{where} \quad \delta_0 = \frac{1}{2(1 + \|V\|)}. \]

Let \( \lambda := \frac{i\mu}{|n|\sqrt{\nu}} \) and \( \alpha := \nu^{1/2}|n| \). Then we obtain the Orr-Sommerfeld equations
\[
\begin{cases}
-\sqrt{\nu}(\partial_\nu^2 - \alpha^2)\phi + i\alpha((V - \lambda)(\partial_\nu^2 - \alpha^2)\phi - (\partial_\nu^2 V)\phi) = i\alpha f_2 - \partial_\nu f_1, \\
\phi|_{Y=0} = \partial_\nu\phi|_{Y=0} = 0,
\end{cases}
\]
and the regime of parameter \( \alpha \) we need to study is \( \delta_0^{-1}\nu^{1/2} \leq \alpha \leq \delta_0^{-1}\nu^{-1/4} \). Moreover, we consider the case
\[
\text{Re}\mu = \alpha\text{Im}\lambda \geq \frac{\nu^{\frac{3}{4}}|n|^\gamma}{\delta}
\]
for some \( \gamma \in [0, 1] \) and for sufficiently small but fixed positive number \( \delta \). We denote \( \lambda_r = \text{Re}\lambda, \lambda_i = \text{Im}\lambda \). We also introduce the weight function
\[
\rho(Y) = \begin{cases}
|n|^{\frac{3}{2} + \frac{2}{\gamma} + \nu^\frac{3}{2}}Y, & \text{if } 0 \leq Y \leq \frac{\delta^\frac{3}{2}}{|n|^{\frac{3}{2} + \frac{2}{\gamma} + \nu^\frac{3}{2}}}, \\
1, & \text{if } Y \geq \frac{\delta^\frac{3}{2}}{|n|^{\frac{3}{2} + \frac{2}{\gamma} + \nu^\frac{3}{2}}},
\end{cases}
\]
Our key resolvent estimates are stated as follows.

**Proposition 2.1.** Assume that (SC) condition holds and \( \delta_0^{-1} \leq |n| \leq \delta_0^{-1}\nu^{-\frac{3}{4}} \). Then there exist \( \delta_1, \delta_2, \delta_* \in (0, 1) \) satisfying \( \delta_1, \delta_2 \leq \delta_0 \) and \( \delta_* \leq \min\{\delta_1, \delta_2\} \) such that the following statements hold true. Assume that (2.7) holds for some \( \delta \in (0, \delta_*] \).

1. Let \( n \in \mathbb{Z} \) and \( \gamma \in [0, 1] \). Then there exists \( \theta \in (\frac{\pi}{2}, \pi) \) such that the set
\[ S_{\nu,n}(\theta) = \{ \mu \in C|\text{Im}\mu| \geq (\tan \theta)\text{Re}\mu + \delta_1^{-1}(\nu^\frac{1}{2}|n| + |\tan \theta||n|^{\frac{1}{2}}), |\mu| \geq \delta^{-1}\nu^\frac{1}{2}|n| \} \]
is in the resolvent set of \( -L_{\nu,n} \) and
\[ \| (\mu + L_{\nu,n})^{-1}f \|_{L^2(\Omega_\nu)} \leq \frac{C}{|\mu|^{\frac{3}{2}}}\|f\|_{L^2(\Omega_\nu)}, \]
\[ \| \nabla(\mu + L_{\nu,n})^{-1}f \|_{L^2(\Omega_\nu)} + \| \rho^\frac{3}{2}(\text{curl}(\mu + L_{\nu,n})^{-1}f) \|_{L^2(\Omega_\nu)} \leq \frac{C}{\nu^\frac{3}{2}|\mu|^{\frac{3}{2}}}\|f\|_{L^2(\Omega_\nu)}, \]
\]
for all \( \mu \in S_{\nu,n}(\theta) \) and \( f \in \mathcal{P}_{\nu,n}L^2_{\sigma}(\Omega_\nu). \)

2. If \( |n| \geq \delta^{-1}_0 \) and \( \text{Re}\mu + n^2\nu^\frac{3}{2} \geq \delta^{-1}_2 \), then \( \mu \) belongs to the resolvent set of \( -L_{\nu,n} \) and the following estimates hold: for all \( f \in \mathcal{P}_{\nu,n}L^2_{\sigma}(\Omega_\nu) \)
\[ \| (\mu + L_{\nu,n})^{-1}f \|_{L^2(\Omega_\nu)} \leq \frac{C}{\text{Re}\mu}\|f\|_{L^2(\Omega_\nu)}, \]
\[ \| \nabla(\mu + L_{\nu,n})^{-1}f \|_{L^2(\Omega_\nu)} + \| \rho^\frac{3}{2}(\text{curl}(\mu + L_{\nu,n})^{-1}f) \|_{L^2(\Omega_\nu)} \leq \frac{C}{\nu^\frac{3}{2}(\text{Re}\mu)^{\frac{3}{2}}}\|f\|_{L^2(\Omega_\nu)}. \]
(3) Let $\delta_0^{-1} \leq |n| \leq \delta_0^{-1} \nu^{-\frac{2}{3}}$ and $\gamma \in \left[ \frac{2}{3}, 1 \right]$. Then the set

\begin{equation}
O_{\nu,n} := \{ \mu \in \mathbb{C} \mid |\mu| \leq \delta_1^{-1}|n| \nu^{\frac{1}{2}}, \quad \text{Re} \mu \geq \frac{|n| \nu^{\frac{1}{2}}}{\delta} \}
\end{equation}

is included in the resolvent set of $-\mathbb{L}_{\nu,n}$. Moreover, if $\mu \in O_{\nu,n}$ satisfies $\text{Re} \mu = \frac{|n| \nu^{\frac{1}{2}}}{\delta}$ and $\text{Re} \mu + n^2 \nu^2 \leq \delta_2^{-1}$, then

\begin{equation}
\|(\mu + \mathbb{L}_{\nu,n})^{-1}f\|_{L^2(\Omega_\nu)} \leq \frac{C n^{1-\gamma}}{\text{Re} \mu} \|f\|_{L^2(\Omega_\nu)},
\end{equation}

\begin{equation}
\|\nabla (\mu + \mathbb{L}_{\nu,n})^{-1}f\|_{L^2(\Omega_\nu)} \leq \frac{C n^{1-\gamma}}{\text{Re} \mu} (|n|^{\frac{1}{2}} + |n|^\frac{1}{2} n^{-\frac{4}{3}(1-\gamma)}) \|f\|_{L^2(\Omega_\nu)},
\end{equation}

and

\begin{equation}
\|\rho^{\frac{1}{2}}(\text{curl}(\mu + \mathbb{L}_{\nu,n})^{-1}f)\|_{L^2(\Omega_\nu)} \leq \frac{C}{\nu^{\frac{1}{2}}(\text{Re} \mu)^{\frac{1}{2}}} \|f\|_{L^2(\Omega_\nu)}.
\end{equation}

Proof. We point out that we can deduce (2.12)-(2.17) directly from Proposition 2.3 and 2.13 respectively. Hence, we only prove the first statement of Proposition 2.1. Let $\theta$ such that $\theta \geq \frac{n}{2}$ and for any $\eta \in P_{\nu,n}$
\begin{equation}
\|(\mu + \mathbb{L}_{\nu,n})^{-1}f\|_{L^2(\Omega_\nu)} \leq \frac{C n^{1-\gamma}}{|\mu|} \|f\|_{L^2(\Omega_\nu)},
\end{equation}

which implies that the ball $B_r(\mu) := \{ \eta \in \mathbb{C} \mid |\eta - \mu| \leq r \}$ with $r = \frac{|\mu|}{C}$ belongs to the resolvent set of $-\mathbb{L}_{\nu,n}$ and for any $\eta \in B_r(\mu)$,

\begin{equation}
\|(\eta + \mathbb{L}_{\nu,n})^{-1}f\|_{L^2(\Omega_\nu)} \leq \frac{4C}{|\mu|} \|f\|_{L^2(\Omega_\nu)} \leq \frac{8C}{|\mu|} \|f\|_{L^2(\Omega_\nu)}.
\end{equation}

Hence, by taking $\theta = \frac{n}{2} + \theta_0$ with $\theta_0 = \frac{1}{2C}$, we obtain

\begin{equation}
S_{\nu,n} \subset \bigcup_{\mu \in E_{\nu,n}} B_{r}(\mu) \subset \rho(-\mathbb{L}_{\nu,n}),
\end{equation}

where

\begin{equation}
E_{\nu,n} := \{ \mu \in \mathbb{C} \mid \text{Re} \mu \geq \delta_1^{-1}|n| \nu^{\frac{1}{2}}, |\mu| \geq \delta_1^{-1} \alpha \}.
\end{equation}

Hence, we complete the proof of (2.12) and (2.13). Similarly, we can obtain (2.17).

In the sequel, we always assume $\delta_0^{-1} \leq |n| \leq \delta_0^{-1} \nu^{-\frac{2}{3}}$, that is $\delta_0^{-1} \nu^{\frac{1}{2}} \leq |\alpha| \leq \delta_0^{-1} \nu^{-\frac{2}{3}}$, and we assume that $n > 0$ for convenience.

2.2. Resolvent estimates when $\lambda$ is far away from origin. In this part, we deal with the case $|\lambda|$ large and $\text{Im} \lambda$ large. We first notice that (2.6) can be written as

\begin{equation}
\begin{cases}
-\nu(\partial^2_\theta - \alpha^2)\partial^2_\phi + i\alpha((V - \lambda_\nu)(\partial^2_\theta - \alpha^2)\phi - (\partial^2_\phi V)\phi) = i\alpha f_2 - \partial_\theta f_1,
\phi|_{\gamma_0} = \partial_\theta \phi|_{\gamma_0} = 0,
\end{cases}
\end{equation}

where $\lambda_\nu = \lambda + i\sqrt{\nu} \alpha$. Note that $\text{Im} \lambda_\nu = \lambda + n \nu$. The following Proposition shows the resolvent estimates for large $|\lambda|$.
Proposition 2.2. There exists \( \delta_1 \in (0, \delta_0] \) such that the following statements hold. Let \( |\lambda| \geq \delta_1^{-1} \) and \( n \in \mathbb{N} \). Suppose that (2.7) holds for some \( \gamma \in [0, 1] \) and \( \delta \in (0, \delta_1] \). Then for any \( f = (f_1, f_2) \in L^2(\mathbb{R}_+)^2 \), the corresponding weak solution \( \phi \in H^1_0(\mathbb{R}_+) \) to (2.6) satisfies

\[
\| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq \frac{C}{|\alpha \lambda|} \| f \|_{L^2},
\]

(2.19)

\[
\| (\partial_Y^2 - \alpha^2) \phi \|_{L^2} \leq \frac{C}{v^{1/4}|\alpha \lambda|^{1/2}} \| f \|_{L^2},
\]

(2.20)

where \( C \) is a constant only depending on \( \| V \| \).

Proof. Let \( f = (f_1, f_2) \in L^2(\mathbb{R}_+)^2 \) and \( \phi \) be the corresponding unique weak solution to (2.6). Then by the previous argument, \( \phi \) also satisfies (2.18). Assume that \( |\lambda| \geq \delta_1^{-1} \) with \( \delta_1 := (32(1 + \| U \|))^{-1} < \delta_0 \). Since \( |\mu| = |\alpha | |\lambda| \) and \( \text{Im} \lambda_\nu = \text{Im} \lambda + \nu n \geq \text{Im} \lambda > 0 \), we have \( |\lambda_\nu| \geq |\lambda| \leq \delta_1^{-1} \), which implies

\[
\frac{|\lambda|}{2} \leq |V - \lambda_\nu| \leq 2|\lambda|.
\]

Multiplying both sides of the first equation of (2.18) by \((V - \lambda_\nu)\tilde{\phi})\), we have

\[
\| (\partial_Y \phi, \alpha \phi) \|_{L^2}^2 \leq \text{Re} \int_0^\infty \frac{i \alpha f_2 + \partial_Y f_1}{\alpha(V - \lambda_\nu)} \tilde{\phi} dY + \int_0^\infty \frac{|\partial_Y V|}{|V - \lambda_\nu|} |\phi|^2 dY
\]

(2.21)

\[
+ \frac{6}{n \text{Im} \lambda_\nu} \int_0^\infty \frac{|\partial_Y V|^2}{|V - \lambda_\nu|^2} |\phi|^2 dY + \frac{6}{n \text{Im} \lambda_\nu} \int_0^\infty \left( \frac{|\partial_Y V|^4}{|V - \lambda_\nu|^4} + \alpha^2 \frac{|\partial_Y V|^2}{|V - \lambda_\nu|^2} \right) |\phi|^2 dY.
\]

We first notice that

\[
\text{Re} \int_0^\infty \frac{i \alpha f_2 + \partial_Y f_1}{\alpha(V - \lambda_\nu)} \tilde{\phi} dY \leq \frac{1}{4} \| (\partial_Y \phi, \alpha \phi) \|_{L^2}^2 + \frac{C}{\alpha (V - \lambda_\nu)} \| f \|_{L^2}^2
\]

(2.22)

\[+ C \| V \| \| f \|_{L^2}^2 \]

The estimates of the other terms on the right hand side of (2.21) are similar with each other. For example,

\[
\frac{1}{n \text{Im} \lambda_\nu} \int_0^\infty \frac{|\partial_Y V|^4}{|V - \lambda_\nu|^4} |\phi|^2 dY \leq C \frac{25 \delta_1}{n \text{Im} \lambda_\nu} \int_0^\infty \| V \|^2 (1 + Y^{-1}) |\phi|^2 dY
\]

\[+ \frac{1}{128} \| (\partial_Y \phi, \alpha \phi) \|_{L^2}^2,
\]

which combined with (2.21), (2.22) and the fact \( |\lambda|/2 \leq |V - \lambda_\nu| \leq 2|\lambda| \) implies

\[
\| (\partial_Y \phi, \alpha \phi) \|_{L^2}^2 \leq \frac{C}{\alpha^2 |\lambda|^2} \| f \|_{L^2}^2.
\]

(2.23)

This shows (2.19).

Now we turn to prove (2.20). We multiply both sides of (2.6) by \( \tilde{\phi} \) and integrate over \( (0, \infty) \). Then we have

\[-\sqrt{v} \| (\partial_Y^2 - \alpha^2) \phi \|_{L^2}^2 + (i \alpha (V - \lambda) (\partial_Y - \alpha^2) \phi, \phi)_{L^2} - (i \alpha (\partial_Y^2 V) \phi, \phi)_{L^2} = \langle (i \alpha f_2 - \partial_Y f_1), \phi \rangle_{L^2},
\]

which implies

\[
\nu^{\frac{1}{2}} \| (\partial_Y^2 - \alpha^2) \phi \|_{L^2}^2 + \alpha \lambda \| (\partial_Y \phi, \alpha \phi) \|_{L^2}^2
\]

(2.24)
We also notice that
\[(2.30)\]
and
\[(2.31)\]
Then by taking the real part of the above equality, we get
\[(2.32)\]

**Proof.** The proposition and the definition of \(\lambda_\nu\), we have \(\alpha \Im \lambda_\nu \geq \delta_2^{-1}\). By taking \(L^2\)-inner product on both sides of (2.18) with \(\phi\), we obtain
\[(2.29)\]
Then by taking the real part of the above equality, we get
\[(2.30)\]
We also notice that
\[(2.31)\]
and
\[(2.32)\]
Hence, after \(\delta_2 \leq \frac{1}{\sqrt{1 + |\nu|}}\) and collecting (2.30) and (2.31) and (2.32), we obtain
\[(2.33)\]
which gives (2.27) and (2.28). \(\square\)
2.3. **Resolvent estimates when** $|\lambda| \leq \delta_1^{-1}$. The purpose of this part is to give the resolvent estimates when $|\lambda| \leq \delta_1^{-1}$. However, the boundary condition in (2.6) brings a lot of troubles to obtain an appropriate bound. Our main idea to overcome the difficulty generated by the boundary is the following:

1. We first obtain the resolvent estimates under the Navier-slip boundary condition, which allows us to use some special structures of the first equation of (2.6) by using integration by parts argument.
2. We show the bounds of the boundary corrector. Such corrector is built around the Airy function and perfectly matches the boundary layer.
3. By combining the controls of the above two, we obtain the resolvent estimates for non-slip boundary condition.

2.3.1. **Resolvent estimates for Navier-slip boundary condition.** In this part, we replace the non-slip boundary condition of (2.6) by Navier-slip boundary condition, on which we obtain a more delicate estimate. In detail, we consider the following system

\[
\begin{cases}
-\sqrt{\nu}(\partial_Y^2 - \alpha^2)w + i\alpha((V - \lambda)w - (\partial_Y^2 V)\phi) = F, \\
F = -\partial_Y F_1 + i\alpha F_2, \\
(\partial_Y^2 - \alpha^2)\phi = w, \; w|_{Y=0} = \phi|_{Y=0} = 0.
\end{cases}
\]

(2.34)

Since the source term $F$ actually belongs to $H^{-1}(\mathbb{R}_+)$ by $F_1, F_2 \in L^2(\mathbb{R}_+)$, we decompose $w = w_1 + w_2$ with $w_1$ and $w_2$ satisfying

\[
\begin{cases}
-\sqrt{\nu}(\partial_Y^2 - \alpha^2)w_1 + i\alpha((V - \lambda)w_2 - (\partial_Y^2 V)\phi_1) - (V - \lambda)\alpha^2 \phi_1 - V''\phi_1 = F, \\
F = -\partial_Y F_1 + i\alpha F_2, \\
(\partial_Y^2 - \alpha^2)\phi_1 = w_1, \\
w_1|_{Y=0} = w_1|_{Y=+\infty} = \phi_1|_{Y=0} = \phi_1|_{Y=+\infty} = 0,
\end{cases}
\]

(2.35)

and

\[
\begin{cases}
-\sqrt{\nu}(\partial_Y^2 - \alpha^2)w_2 + i\alpha((V - \lambda)w_2 - V''\phi_2) = V'h, \\
(\partial_Y^2 - \alpha^2)\phi_2 = w_2, \\
h = i\alpha \partial_Y \phi_1 \\
w_2|_{Y=0} = w_2|_{Y=+\infty} = \phi_2|_{Y=0} = \phi_2|_{Y=+\infty} = 0.
\end{cases}
\]

(2.36)

**Proposition 2.4.** Let $\nu \leq 1$ and $|\lambda| \leq \delta_1^{-1}$. Suppose that (SC) condition holds. Then there exists $\delta_4 \in (0, \delta_1]$ such that if $\lambda$ satisfies (2.7) for some $\delta \in (0, \delta_4]$, then the unique solution to (2.34) satisfies

\[
\nu^{\frac{1}{2}}(\lambda Y^2 - \frac{1}{2})\|w\|_{L^2} + C\lambda^{-1}\|\|\partial_Y \phi, \alpha \phi\|_{L^2} \leq C\lambda^{-1}(F_1, F_2)\|_{L^2},
\]

where the constant $C$ only depends on $\|V\|$.

**Proof.** Let $w$ be the solution to (2.34) and $\phi$ be the corresponding stream function. As we mention above, we decompose $w$ as $w = w_1 + w_2$, where

\[
\begin{cases}
-\sqrt{\nu}(\partial_Y^2 - \alpha^2)w_1 + i\alpha((V - \lambda)w_1' - (V - \lambda)\alpha^2 \phi_1 - V''\phi_1) = F, \\
-\sqrt{\nu}(\partial_Y^2 - \alpha^2)w_2 + i\alpha((V - \lambda)w_2 - V''\phi_2) = i\alpha V'\partial_Y \phi_1, \\
F = -\partial_Y F_1 + i\alpha F_2, \\
(\partial_Y^2 - \alpha^2)\phi_i = w_i, \; i \in \{1, 2\}, \\
w_i|_{Y=0} = w_i|_{Y=+\infty} = \phi_i|_{Y=0} = \phi_i|_{Y=+\infty} = 0, \; i \in \{1, 2\}.
\end{cases}
\]
Then considering the real part, we get
\[ \delta \]
Then there exists \( \| \) for (2.36), then by Lemma 2.5, we get
\[ \lambda \] according to our assumption that (2.36), satisfies (2.7) for some \( \delta \), \( \lambda \) such that if \( \bar{\phi} \), \( \phi \) \( \in \) \( \mathbb{R}_+ \) then \( \delta \leq \delta \), \( \lambda \) \( \in \) \( (0, \delta) \) for some \( \delta \leq \delta /100 \), we obtain
\[ \lambda^{-2} - \nu^{-\frac{3}{2}}(\alpha \lambda) - \frac{1}{2} = \nu^{-\frac{3}{2}}(\alpha \lambda) i - \frac{1}{2} (\alpha^{-\frac{1}{2}} \lambda^{-\frac{3}{2}} \nu^{-\frac{3}{2}} + 1) \leq \nu^{-\frac{3}{2}}(\alpha \lambda) - \frac{1}{2} (\lambda^{-\frac{1}{2}} \delta^{-\frac{1}{2}} + 1) \leq \nu^{-\frac{3}{2}}(\alpha \lambda) - \frac{1}{2}.
\]
Finally, we have
\[ \| w \|_{L^2} \leq C \nu^{-\frac{3}{2}}(\alpha \lambda) - \frac{1}{2} \| (F_1, F_2) \|_{L^2}.
\]
This finishes the proof.

The following lemma is about the control of the solution to (2.36).

**Lemma 2.5.** Let \( 1 < \nu \leq 1 \), \( |\lambda| \leq \delta_1 \) and \( h \in L^2(\mathbb{R}_+) \). Suppose that (SC) condition holds. Then there exists \( \delta_1 \in (0, \delta_1) \) such that if \( \lambda \) satisfies (2.7) for some \( \delta \leq \delta /100 \), then the unique solution to (2.36) satisfies
\[ \alpha \lambda \| (\partial \phi, \alpha \phi) \|_{L^2} \leq C \| h \|_{L^2},
\]
\[ \nu^{\frac{3}{2}}(\alpha \lambda) \| (\partial \phi, \alpha \phi) \|_{L^2} \leq C \| h \|_{L^2},
\]
\[ \nu^{\frac{3}{2}}(\alpha \lambda) \| (\partial \phi, \alpha \phi) \|_{L^2} \leq C \| h \|_{L^2}.
\]
Proof. Let \( \zeta \) be an arbitrary fixed small positive number.

Step 1. Estimate of the vorticity \( w \) in a weighted norm

By taking \( L^2 \)-inner product with \(-w/(V'' - \zeta)\) on both sides of the first equation of (2.36), we obtain

\[
\text{Re}\langle -\sqrt{\nu}(\partial_Y^2 - \alpha^2)w + i\alpha((V - \lambda)w - V''\phi), -\frac{w}{V'' - \zeta} \rangle \leq |\langle V' h, \frac{w}{V'' - \zeta} \rangle|.
\]

For each terms on the left-hand side of (2.37), we first have

\[
\langle (\partial_Y^2 - \alpha^2)w, w/(V'' - \zeta) \rangle = \left\| \frac{(\partial_{Y}w, \alpha w)}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 + \left\langle \partial_{Y}w, \frac{wV''}{(V'' - \zeta)^2} \right\rangle,
\]

which gives

\[
\text{Re}\langle \sqrt{\nu}(\partial_Y^2 - \alpha^2)w, w/(V'' - \zeta) \rangle \]
\[
= \sqrt{\nu} \left\| \frac{(\partial_{Y}w, \alpha w)}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 + \sqrt{\nu} \text{Re}\left\langle \partial_{Y}w, \frac{wV''}{(V'' - \zeta)^2} \right\rangle
\]
\[
\geq \sqrt{\nu} \left\| \frac{(\partial_{Y}w, \alpha w)}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - \sqrt{\nu} \left\| \frac{\partial_{Y}w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2} \left\| \frac{V''}{V'' - \zeta} \right\|_{L^\infty} \left\| \frac{w}{V'' - \zeta^{\frac{1}{2}}} \right\|_{L^2}
\]
\[
\geq \frac{\sqrt{\nu}}{2} \left\| \frac{(\partial_{Y}w, \alpha w)}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - C\sqrt{\nu} \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2.
\]

In the last inequality of (2.38) we used \(|V''/V| + |V''/V'| \leq C\). We also notice that

\[
\text{Im}\langle (V - \lambda)w - V''\phi, w/(V'' - \zeta) \rangle
\]
\[
= \text{Im}\left( \langle V - \lambda, |w|^2/(V'' - \zeta) \rangle + \|(\partial_{Y}\phi, \alpha\phi)\|_{L^2}^2 - \zeta\langle \phi, w/(V'' - \zeta) \rangle \right)
\]
\[
= \lambda_i \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - \zeta\text{Im}\langle \phi, w/(V'' - \zeta) \rangle,
\]

from which, we deduce that

\[
\text{Re}\left( i\alpha(V - \lambda)w - V''\phi, -\frac{w}{V'' - \zeta} \right) = \alpha\text{Im}\langle (V - \lambda)w - V''\phi, w/(V'' - \zeta) \rangle
\]
\[
= \alpha\lambda_i \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - \zeta\alpha\text{Im}\langle \phi, w/(V'' - \zeta) \rangle
\]
\[
\geq \alpha\lambda_i \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - \zeta^{\frac{1}{2}} \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2} \alpha\phi \right\|_{L^2}
\]
\[
\geq \frac{\alpha\lambda_i}{2} \left\| \frac{w}{|V'' - \zeta|^{\frac{1}{2}}} \right\|_{L^2}^2 - C\zeta^{\frac{3}{2}} \alpha\phi \|_{L^2}^2.
\]
According to the above inequality, (2.37) and (2.38), we obtain
\begin{equation}
\sqrt{\nu} \left\| \frac{(\partial Y w) \phi}{|V'' - \zeta|^{1/2}} \right\|_{L^2}^2 + \alpha \lambda_i \left\| \frac{w}{|V'' - \zeta|^{3/2}} \right\|_{L^2}^2 
\leq 2 \|h\|_{L^2} \left\| \frac{V'}{|V'' - \zeta|^{1/2}} \right\|_{L^\infty} \left\| \frac{w}{|V'' - \zeta|^{3/2}} \right\|_{L^2}^2 + 2 C \sqrt{\nu} \left\| \frac{w}{|V'' - \zeta|^{3/2}} \right\|_{L^2}^2 + 2 C \zeta^{\frac{3}{2}} |\phi|^2_{L^2},
\end{equation}
which gives
\begin{equation}
\sqrt{\nu} \left\| \frac{(\partial Y w, \alpha w)}{|V'' - \zeta|^{1/2}} \right\|_{L^2}^2 + \alpha \lambda_i \left\| \frac{w}{|V'' - \zeta|^{3/2}} \right\|_{L^2}^2 \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2}^2 + C \zeta (\alpha \lambda_i)^{-1} |\phi|^2_{L^2},
\end{equation}
along with the (SC) condition and the fact that
\[ \alpha \lambda_i \geq \frac{\delta_0}{\delta} \sqrt{\nu}, \] with \( \delta \) small enough.

**Step 2. Estimates via the Rayleigh structure**

Denote \( R\phi := (V - \lambda)(\partial Y - \alpha^2)\phi - V'' \phi \), then we can write the equation as
\[ i\alpha (R\phi) = \sqrt{\nu} (\partial Y - \alpha^2) w + V'h. \]
Applying Lemma 2.7 by taking \( h_1 = h/(i\alpha) \), \( h_2 = \sqrt{\nu} \partial Y w/(i\alpha) \), \( h_3 = \sqrt{\nu} w \), we get
\[ \| (\partial Y \phi, \alpha \phi) \|_{L^2} \leq C \left( \lambda_i^{-1} \|h_1\|_{L^2} + \lambda_i^{-2} \|h_2, h_3\|_{L^2} \right) \]
\[ \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2} + C \nu^{\frac{3}{2}} \lambda_i^{-2} \alpha^{-1} \| (\partial Y w, \alpha w) \|_{L^2} \]
\[ \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2} + C \nu^{\frac{3}{2}} \lambda_i^{-2} \alpha^{-1} \|V'' - \zeta\|_{L^\infty} \left\| \frac{(\partial Y w, \alpha w)}{|V'' - \zeta|^{1/2}} \right\|_{L^2} \]
\[ \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2} + C \nu^{\frac{3}{2}} \lambda_i^{-2} \alpha^{-1} \left\| \frac{(\partial Y w, \alpha w)}{|V'' - \zeta|^{1/2}} \right\|_{L^2}. \]
Substituting (2.40) into the above inequality, we deduce that
\[ \| (\partial Y \phi, \alpha \phi) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2} + C \nu^{\frac{3}{2}} (\alpha \lambda_i)^{-\frac{3}{2}} \lambda_i^{-1} \|h\|_{L^2} + C \zeta \nu^{\frac{1}{2}} (\alpha \lambda_i)^{-\frac{3}{2}} \lambda_i^{-1} |\phi|^2_{L^2}. \]
Letting \( \zeta \to 0^+ \), we infer that
\[ (2.41) \quad \| (\partial Y \phi, \alpha \phi) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2} + C \nu^{\frac{3}{2}} (\alpha \lambda_i)^{-\frac{3}{2}} \lambda_i^{-1} \|h\|_{L^2}. \]
On the other hand, we notice that by (2.7)
\[ \nu^{\frac{1}{2}} (\alpha \lambda_i)^{-\frac{3}{2}} \leq \alpha^{-1} (\nu^{3(\gamma-2/3)} \alpha^{-2(\gamma-2/3)/2} \delta^{3/2}) \leq \alpha^{-1} (\delta_0^{3(\gamma-2/3)/2} \delta^{3/2}) \leq \alpha^{-1}, \]
provided that \( \gamma \in [2/3, 1] \) and \( \delta \leq \delta_0 < 1 \). Then (2.41) becomes
\[ (2.42) \quad \| (\partial Y \phi, \alpha \phi) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} \|h\|_{L^2}. \]
Putting this into (2.40), we conclude that
\[ \nu^{\frac{1}{2}} \left\| \frac{(\partial Y w, \alpha w)}{|V'' - \zeta|^{1/2}} \right\|_{L^2} + (\alpha \lambda_i)^{\frac{3}{2}} \left\| \frac{w}{|V'' - \zeta|^{3/2}} \right\|_{L^2} \leq C (\alpha \lambda_i)^{-\frac{1}{2}} (1 + \zeta^{\frac{3}{2}} (\alpha \lambda_i)^{-1}) \|h\|_{L^2}. \]
Applying \( 1/|V'' - \zeta|^{1/2} \geq CM^{\frac{3}{2}} /\|V'\| + (M\zeta)^{\frac{1}{2}} \), and letting \( \zeta \to 0^+ \), we deduce that
\[ \nu^{\frac{1}{2}} (\alpha \lambda_i)^{\frac{3}{2}} \left\| \frac{(\partial Y w, \alpha w)}{|V''|} \right\|_{L^2} + \alpha \lambda_i \left\| \frac{w}{|V'|} \right\|_{L^2} \leq C \|h\|_{L^2}, \]
which gives the second inequality.

The following lemma is about Rayleigh’s trick for the strong concave shear flow.

**Lemma 2.6.** Let \( \phi \in H^1_0(\mathbb{R}_+) \cap H^2(\mathbb{R}_+) \) and we denote the Rayleigh operator as \( R\phi := (V - \lambda)(\partial_Y^2 - \alpha^2)\phi - V\phi \). If \((SC)\) condition holds, then we have

\[
\text{Re}\left( \frac{1 - \lambda}{i\lambda_i} \int_0^{+\infty} (R\phi)\frac{\bar{\phi}}{V - \lambda} dY \right) \geq \| (\partial_Y\phi, \alpha\phi) \|_{L^2}^2 + M^{-1} \left\| \frac{(1 - V)^{\frac{1}{2}} V\phi}{V - \lambda} \right\|_{L^2}^2.
\]

Moreover, if \( \lambda_r \geq 1 \), then we have

\[
\text{- Re}\left( \int_0^{+\infty} (R\phi)\frac{\bar{\phi}}{V - \lambda} dY \right) \geq \| (\partial_Y\phi, \alpha\phi) \|_{L^2}^2 + M^{-1} \left\| \frac{(1 - V)^{\frac{1}{2}} V\phi}{V - \lambda} \right\|_{L^2}^2.
\]

Here \( M \) is the constant in second property of the \((SC)\) condition.

**Proof.** Taking inner product with \( \frac{-\bar{\phi}}{V - \lambda} \), we get

\[
\int_0^{+\infty} (R\phi)\frac{\bar{\phi}}{V - \lambda} dY = \| (\partial_Y\phi, \alpha\phi) \|_{L^2}^2 + \int_0^{+\infty} \frac{(\partial_Y^2 V)|\phi|^2}{V - \lambda} dY.
\]

Considering the real and imaginary part respectively, we obtain

\[
\text{Re}\left( \int_0^{+\infty} (R\phi)\frac{\bar{\phi}}{V - \lambda} dY \right) = \| (\partial_Y\phi, \alpha\phi) \|_{L^2}^2 + \int_0^{+\infty} \frac{(V - \lambda_r)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY, \tag{2.45}
\]

\[
\text{Im}\left( \int_0^{+\infty} (R\phi)\frac{\bar{\phi}}{V - \lambda} dY \right) = \lambda_i \int_0^{+\infty} \frac{(-\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY. \tag{2.46}
\]

By \((SC)\) condition: \(-\partial_Y^2 V \geq M^{-1}(\partial_Y V)^2\), and applying \((2.46)\), we have

\[
-\int_0^{+\infty} \frac{(V - \lambda_r)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY
= -\int_0^{+\infty} \frac{(1 - \lambda_r)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY - \int_0^{+\infty} \frac{(V - 1)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY
\]

\[
= \frac{1 - \lambda_r}{\lambda_i} \left( \int_0^{+\infty} \frac{\lambda_i(-\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY \right) - \int_0^{+\infty} \frac{(V - 1)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY
\]

\[
\leq \frac{1 - \lambda_r}{\lambda_i} \text{Im} \left( \int_0^{+\infty} \frac{(R\phi)\bar{\phi}}{V - \lambda} dY \right) - \int_0^{+\infty} \frac{(1 - V)(\partial_Y V)^2|\phi|^2}{M|V - \lambda|^2} dY
\]

\[
\leq \frac{1 - \lambda_r}{\lambda_i} \text{Im} \left( \int_0^{+\infty} \frac{(R\phi)\bar{\phi}}{V - \lambda} dY \right) - \int_0^{+\infty} \frac{(1 - V)^{\frac{1}{2}} (\partial_Y V)\phi}{V - \lambda} dY.
\]

Putting this into \((2.45)\), we obtain

\[
\| (\partial_Y\phi, \alpha\phi) \|_{L^2}^2 = -\int_0^{+\infty} \frac{(V - \lambda_r)(\partial_Y^2 V)|\phi|^2}{|V - \lambda|^2} dY - \text{Re} \left( \int_0^{+\infty} \frac{(R\phi)\bar{\phi}}{V - \lambda} dY \right)
\]

\[
\leq \frac{1 - \lambda_r}{\lambda_i} \text{Im} \left( \int_0^{+\infty} \frac{(R\phi)\bar{\phi}}{V - \lambda} dY \right) - \int_0^{+\infty} \frac{(1 - V)^{\frac{1}{2}} (\partial_Y V)\phi}{V - \lambda} dY - \text{Re} \left( \int_0^{+\infty} \frac{(R\phi)\bar{\phi}}{V - \lambda} dY \right)
\]
\[\text{Proof.} \quad \text{Notice that in the proof of Lemma 2.5.} \]

\[\left| \frac{1 - \lambda r}{i \lambda_i} \int_0^{+\infty} \left( \frac{(R\bar{\phi})\bar{\phi}}{V - \lambda} \right) \right| - \left| \int_0^{+\infty} \left( \frac{(R\bar{\phi})\bar{\phi}}{V - \lambda} \right) \right| - M^{-1} \left\| \frac{(1 - V)\frac{1}{2}(\partial_Y \phi)}{V - \lambda} \right\|_{L^2}^2,\]

This gives the first inequality.

If \(\lambda_r \geq 1\), again by (2.45) and (SC) condition: \(-\partial_Y^2 V \geq M^{-1}(\partial_Y V)^2\), we have

\[-\text{Re} \left( \int_0^{+\infty} \left( \frac{(R\bar{\phi})\bar{\phi}}{V - \lambda} \right) \right) = \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2}^2 + \int_0^{+\infty} \frac{(V - \lambda_r)(\partial_Y^2 V)\phi^2}{|V - \lambda|^2} \, dY \]

\[\geq \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2}^2 + \int_0^{+\infty} \frac{(1 - V)(-\partial_Y^2 V)\phi^2}{|V - \lambda|^2} \, dY \]

\[\geq \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2}^2 + M^{-1} \left\| \frac{(1 - V)\frac{1}{2}(\partial_Y \phi)}{V - \lambda} \right\|_{L^2}^2.\]

This gives the second inequality. \(\square\)

From the above lemma about Rayleigh’s structure, we have the following lemma is used in the proof of Lemma 2.5.

**Lemma 2.7.** Let \(\phi \in H_0^1(\mathbb{R}_+)\) solve \(R\phi = \tilde{h} := V'h_1 + \partial_Y h_2 + i\alpha h_3\) with \(h_i \in L^2(\mathbb{R}_+)\) for \(i = 1, 2, 3\). If \(|\lambda| \leq \delta^{-1}_3\). Then it holds that

\[\left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2} \leq C \left( \lambda_i^{-1} \left\| h_1 \right\|_{L^2} + \lambda_i^{-2} \left\| (h_2, h_3) \right\|_{L^2} \right).\]

**Proof.** Notice that

\[\left| \int_0^{+\infty} \frac{\tilde{h}\bar{\phi}}{V - \lambda} \, dY \right| = \left| \int_0^{+\infty} \left( \frac{V'h_1 + \partial_Y h_2 + i\alpha h_3}{V - \lambda} \bar{\phi} \right) \, dY \right| \]

\[\leq \left| \int_0^{+\infty} \left( \frac{V'h_1}{V - \lambda} \right) \bar{\phi} \, dY \right| + \left| \int_0^{+\infty} \frac{h_2\partial_Y \bar{\phi}}{V - \lambda} \, dY \right| + \left| \int_0^{+\infty} \frac{\alpha h_3 \bar{\phi}}{V - \lambda} \, dY \right| \]

\[\leq \left\| \sqrt{-V''\phi} \right\|_{L^2} \left( \left\| \frac{V'h_1}{V - \lambda} \right\|_{L^2} + \left\| \frac{V'h_2}{\sqrt{-V''}} \right\|_{L^2} \right) + \left\| \partial_Y \phi \right\|_{L^2} \left\| \frac{h_2}{V - \lambda} \right\|_{L^2} \]

\[+ \|\alpha\|_{L^2} \left\| \frac{h_3}{V - \lambda} \right\|_{L^2} \]

\[\leq M^{-\frac{1}{2}} \left\| \sqrt{-V''\phi} \right\|_{L^2} \left( \left\| h_1 \right\|_{L^2} + \left\| \frac{h_2}{(V - \lambda)} \right\|_{L^2} \right) + C\lambda_i^{-1} \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2} \left\| (h_2, h_3) \right\|_{L^2}.\]

Here we used the strong concave condition. And (2.46) gives

\[\left| \int_0^{+\infty} \frac{\tilde{h}\bar{\phi}}{V - \lambda} \, dY \right| \geq \text{Im} \left( \int_0^{+\infty} \frac{\tilde{h}\bar{\phi}}{V - \lambda} \, dY \right) = \lambda_i \left\| \sqrt{-V''\phi} \right\|_{L^2}^2.\]

Then we obtain

\[\left| \int_0^{+\infty} \frac{\tilde{h}\bar{\phi}}{V - \lambda} \, dY \right| \leq C\lambda_i^{-1} M \left( \left\| h_1 \right\|_{L^2} + \left\| \frac{h_2}{(V - \lambda)} \right\|_{L^2} \right)^2 + C\lambda_i^{-1} \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2} \left\| (h_2, h_3) \right\|_{L^2} \]

\[\leq C\lambda_i^{-1} \left\| h_1 \right\|_{L^2}^2 + C\lambda_i^{-3} \left\| h_2 \right\|_{L^2}^2 + C\lambda_i^{-1} \left\| (\partial_Y \phi, \alpha \phi) \right\|_{L^2} \left\| (h_2, h_3) \right\|_{L^2}.\]
By Lemma 2.6 and $|\lambda| \leq \delta_i^{-1}$, we get
\[
\| (\partial_Y \phi, \alpha \phi) \|^2_{L^2} \leq C \lambda_i^{-1} \left| \int_0^{+\infty} \frac{\tilde{h} \bar{\phi}}{V - \lambda} dY \right|
\leq C \lambda_i^{-1} (\lambda_i^{-1} \|h_1\|^2_{L^2} + \lambda_i^{-4} |\lambda| \|h_2\|^2_{L^2} + \lambda_i^{-1} \| (\partial_Y \phi, \alpha \phi) \|_{L^2} \| (h_2, h_3) \|_{L^2}),
\]
which gives
\[
\| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq C (\lambda_i^{-1} \|h_1\|_{L^2} + \lambda_i^{-2} \| (h_2, h_3) \|_{L^2}).
\]

2.3.2. Boundary corrector. Since we have got the resolvent estimate under the Navier-slip boundary condition, we need to re-correct the boundary condition from Navier-slip boundary condition to nonslip one. The boundary corrector we need is the solution to the following system
\[
\begin{cases}
- \sqrt{\nu} (\partial_Y^2 - \alpha^2) W_b + i \alpha ((V - \lambda) W_b - V'' \Phi_b) = 0, \\
(\partial_Y^2 - \alpha^2) \Phi_b = W_b, \\
\Phi_b|_{Y=0} = 0, \quad \partial_Y \Phi_b|_{Y=0} = 1.
\end{cases}
\tag{2.47}
\]
Hence, our goal in this part is to show the control of $(W_b, \Phi_b)$. However, instead of considering the above system directly, we first pay our attention to study the system as follow
\[
\begin{cases}
- \sqrt{\nu} (\partial_Y^2 - \alpha^2) W + i \alpha ((V - \lambda) W - V'' \Phi) = 0, \\
(\partial_Y^2 - \alpha^2) \Phi = W, \\
\Phi|_{Y=0} = 0, \quad \partial_Y \Phi|_{Y=0} = 1.
\end{cases}
\tag{2.48}
\]
The reason we consider this system is that
\begin{itemize}
\item The solution $W$ to (2.48) actually is a small perturbation around the Airy function, which is known very well by us. Moreover, the corresponding perturbation satisfies the Navier-slip condition, on which we can apply the result in the previous part.
\item We observe that $\partial_Y \Phi$ on the boundary $Y = 0$ is also positive. Hence, to drive the estimates from (2.48) to (2.47), we just need to normalized the value of $\partial_Y \Phi$ on the boundary.
\end{itemize}

For convenience, we introduce the following notation
\[
A = |n|^{\frac{1}{2}} (1 + |n|^{-\frac{1}{2}} |\lambda_i|)^{\frac{1}{2}}.
\tag{2.49}
\]
We first present the following lemma, which will be used frequently.

**Lemma 2.8.** Let $\delta_0^{-1} \nu^{\frac{1}{2}} \leq \alpha \leq \delta_0^{-1} \nu^{-\frac{1}{2}}$ with $\alpha = \sqrt{\nu} n$, $\lambda_i \geq \frac{n^{\gamma - 1}}{\delta}$ for some $\gamma \in [\frac{2}{3}, 1]$. Then it holds that
\[
\max \left( \delta_0 \alpha, \delta_0^{-\frac{1}{2}} \lambda_i \right) \leq |n|^{\frac{1}{2}} \quad \text{and} \quad |n|^{-\frac{1}{2}} \leq C (1 + \alpha)^{-1}.
\]
Moreover, we have
\[
|n|^{\frac{1}{2}} \lambda_i \geq \delta_0^{-(\gamma - 2/3)} \delta^{-1} \geq \delta^{-1}.
\]
Lemma 2.9. We point out that $\alpha = |n|^{\frac{1}{4}}|\nabla_\theta a|^2 \leq |n|^{\frac{1}{4}}\delta_0^{-\frac{1}{4}}$. Due to $\delta_0^{-1}\nu^{-\frac{3}{4}} \leq n, |n|^{\frac{1}{4}} \geq \delta_0^{-\frac{1}{4}}$. This deduces the first inequality. We also have

$$|n|^{\frac{1}{4}}\lambda_i \geq \frac{\alpha^{\frac{1}{2}} \nu^{(1-\gamma)/2} \alpha^{\gamma-1}}{\nu^6} \delta^{-\frac{1}{3}} = \nu^{(2/3-\gamma)/2} \alpha^{\gamma-2/3} \delta^{-1} = (\alpha/\sqrt{\nu})^{\gamma-2/3} \delta^{-1} \geq \delta_0^{-(\gamma-2/3)} \delta^{-1},$$

which gives the third inequality.

Now we construct the solution $W$ to (2.48) via the Airy function. Denote that $d = -\lambda \nu/V'(0)$, where $\lambda \nu = \lambda + i\nu$. We introduce

$$W_a(Y) = A_i(e^{i\pi} |nV'(0)|^\frac{3}{4}(Y + d))/A_i(e^{i\pi} |nV'(0)|^\frac{3}{4}d).$$

here $A_i(y)$ is the Airy function defined in the appendix, which satisfies $A''/(y) - yA(y) = 0$. Then we have

$$W_a(Y) = \frac{\partial^2 W_a}{\partial Y^2} - \frac{\partial^2 A_i}{\partial Y^2} W_a + i \alpha (V'(0)Y - \lambda) W_a = 0,$n
$$W_a(0) = 1.$n

We denote the perturbation $W_e = W - W_a$, which satisfies

$$\begin{cases}
- \sqrt{\nu} (\partial^2_y - \alpha^2) W_e + i \alpha (V - \lambda) W_e - V'' \Phi_e = -i \alpha (V - V'(0)Y) W_a + i\alpha V'' \Phi_a, \\
(\partial^2_y - \alpha^2) \Phi_e = W_e, \\
\Phi_a(0) = \Phi_e(0) = 0, \quad W_e(0) = 0.
\end{cases}$$

We point out that $W_e$ satisfies the Navier-slip boundary condition. As a consequence, we have the following lemma.

**Lemma 2.9.** Let $W_e$ solve (2.51) and $|\lambda| \leq \delta^{-1}_1$. Then it holds that

$$\nu^\frac{1}{4} \alpha^\frac{1}{2} \lambda_i^{-\frac{1}{2}} ||W_e||_{L^2} + \alpha \lambda_i ||(\partial_y \Phi_e, \Phi_e)||_{L^2} \leq C \alpha \lambda_i^{-1} A^{-\frac{3}{2}}.$$

Moreover, we have

$$||\partial_y \Phi_e||_{L^\infty} \leq C |n|^{-\frac{3}{4}} (A \lambda_i)^{-\frac{3}{4}}.$$

**Proof.** We first notice that $W_a$ satisfies

$$(V - V'(0)Y) W_a = \partial_y [(V - V'(0)Y) \partial_y \Phi_a] - \partial_y [(V' - V'(0)) \Phi_a] + V'' \Phi_a - \alpha^2 (V - V'(0)Y) \Phi_a.$$

Then we have

$$-i \alpha (V - V'(0)Y) W_a + i\alpha V'' \Phi_a = -i \alpha \partial_y [(V - V'(0)Y) \partial_y \Phi_a - (V' - V'(0)) \Phi_a] + i\alpha^3 (V - V'(0)Y) \Phi_a$$

$$= -\partial_y F_{1,1} + i\alpha F_{1,2},$$

where

$$F_{1,1} = i \alpha [(V - V'(0)Y) \partial_y \Phi_a - (V' - V'(0)) \Phi_a],$$
\[ F_{1,2} = \alpha^2(V - V'(0)Y)\Phi_a. \]

Since we have
\[
|V(Y) - V'(0)Y| = \left| \int_0^Y \int_0^Z V''(Z_1)dZ_1dZ \right| \leq Y^2\|V''\|_{L^\infty}/2 \leq CY^2,
\]
\[
|V'(Y) - V'(0)| = \left| \int_0^Y V''(Z)dZ \right| \leq Y\|V''\|_{L^\infty} \leq CY,
\]
we infer that
\[
|F_{1,1}(Y)| \leq C\alpha(\|Y^2\partial_Y\Phi_a\| + |Y\Phi_a|), \quad |F_{1,2}(Y)| \leq C\alpha^2|Y^2\Phi_a|.
\]

By taking \( \kappa = |nV'(0)|^{\frac{1}{2}}, \eta = -\lambda_0/V'(0) \), then \( \kappa(1 + |\kappa\eta|)^{\frac{1}{2}} \sim A \), then applying Lemma [3.2] and Lemma [2.5] we obtain
\[
\|(F_{1,1}, F_{1,2})\|_{L^2} \leq C\alpha \left( \|Y^2 \partial_Y \Phi_a\|_{L^2} + \|Y\Phi_a\|_{L^2} + \alpha \|Y^2 \Phi_a\|_{L^2} \right)
\leq C\alpha \left( A^{-\frac{\nu}{2}} + \alpha A^{-\frac{\nu}{2}} \right) = C\alpha A^{-\frac{\nu}{2}}(1 + \alpha A^{-1}) \leq C\alpha A^{-\frac{\nu}{2}}(1 + \alpha|n|^{-\frac{1}{2}})
\leq C\alpha A^{-\frac{\nu}{2}}(1 + \delta_0^{-\frac{1}{2}}) \leq C\alpha A^{-\frac{\nu}{2}}.
\]

Now we come to estimate \( W_e \). We know that \( (W_e, \Phi_e, F_{1,1}, F_{1,2}) \) fits the structure of \( 2.34 \). Then applying Proposition [2.4] we get
\[
\nu^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\lambda_i^{-\frac{3}{2}}\|W_e\|_{L^2} + \alpha \lambda_i \|(\partial_Y \Phi_e, \alpha \Phi_e)\|_{L^2} \leq CA^{-\frac{\nu}{2}} \|(F_{1,1}, F_{1,2})\|_{L^2} \leq CA^{-\frac{\nu}{2}} \lambda_i^{-1}.
\]
This gives the first inequality. And we have
\[
A\|\partial_Y \Phi_e\|_{L^\infty} \leq CA\|W_e\|_{L^2}^{\frac{1}{2}} \|(\partial_Y \Phi_e, \alpha \Phi_e)\|_{L^2}^{\frac{1}{2}}
\leq C\alpha A^{-\frac{\nu}{2}} \lambda_i^{-1} - \nu^{-\frac{1}{2}}\alpha^{-\frac{3}{2}}\lambda_i^{-\frac{3}{2}} = C(|n|^{\frac{1}{2}}/A)^{\frac{1}{2}} \lambda_i^{\frac{3}{2}}(A\lambda_i)^{-\frac{\nu}{2}} \leq C(A\lambda_i)^{-\frac{\nu}{2}}.
\]
Here we used \( |n|^{\frac{1}{2}}/A \leq 1, \lambda_i \leq \delta_i^{-1}. \)

Combining with Lemma [2.9] and Lemma [B.2] we get

**Lemma 2.10.** Let \( \lambda \) satisfy \( |\lambda| \leq \delta_i^{-1} \), and let \( W \) solve \( 2.48 \). Then it holds that
\[
\|(\partial_Y \Phi, \alpha \Phi)\|_{L^2} \leq CA^{-\frac{\nu}{2}}, \quad \|W\|_{L^2} \leq A^{-\frac{\nu}{2}}.
\]

**Proof.** Applying Lemma [B.2] and Lemma [2.9] we deduce that
\[
\|(\partial_Y \Phi, \alpha \Phi)\|_{L^2} \leq \|(\partial_Y \Phi_a, \alpha \Phi_a)\|_{L^2} + \|(\partial_Y \Phi_e, \alpha \Phi_e)\|_{L^2}
\leq CA^{-\frac{\nu}{2}} + C\lambda_i^{-2}A^{-\frac{\nu}{2}} \leq CA^{-\frac{\nu}{2}}(1 + (A\lambda_i)^{-2})
\leq CA^{-\frac{\nu}{2}}(1 + (|n|^{\frac{1}{2}}\lambda_i)^{-2}) \leq CA^{-\frac{\nu}{2}},
\]
here we used \( (A\lambda_i)^{-2} \leq (|n|^{\frac{1}{2}}\lambda_i)^{-2} \leq \delta^2 \leq 1 \). This gives the first inequality. Still applying Lemma [B.2] and Lemma [2.9] we infer that
\[
\|W\|_{L^2} \leq \|W_a\|_{L^2} + \|W_e\|_{L^2} \leq CA^{-\frac{\nu}{2}} + C\lambda_i^{-\frac{1}{2}}A^{-\frac{\nu}{2}}\lambda_i^{-\frac{1}{2}}
\leq CA^{-\frac{\nu}{2}}(1 + (L/A)^{\frac{1}{2}}(A\lambda_i)^{-\frac{3}{2}}\lambda_i^{-1}) \leq CA^{-\frac{\nu}{2}}.
\]
provide that \( (|n|^{\frac{1}{2}}/A) \leq 1, (A\lambda_i)^{-\frac{1}{2}} \leq (|n|^{\frac{1}{2}}\lambda_i)^{-\frac{1}{2}} \leq \delta^2 \leq 1, \lambda_i \leq \delta_i^{-1}. \)
Now we denote
\[ J := - \int_{0}^{+\infty} W(Y)e^{-\alpha Y} dY. \]
Then clearly by Lemma 2.9, we have that \( J = \partial_Y \Phi(0) \). Hence, the task of the following lemma is to show that the lower bound of \( J \) is positive strictly.

**Lemma 2.11.** Let \( \lambda \) satisfy \( |\lambda| \leq \delta^{-1}_1 \). Recall that \( J = \int_{0}^{+\infty} W(Y)e^{-\alpha Y} dY \), and let \( W \) be the solution of (2.48). Then it holds that
\[ |J| \geq C^{-1}A^{-1}. \]

**Proof.** Thanks to Lemma B.3, we have
\[ \left| \int_{0}^{+\infty} W_a(Y)e^{-\alpha Y} dY \right| = \partial_Y \Phi_a(0) \geq C^{-1}(1 + |n|^{\frac{1}{2}}|\lambda|)^{-\frac{1}{4}}(|n|^{\frac{1}{4}} + \alpha)^{-1}, \]
which along with Lemma 2.8 gives
\[ (2.52) \quad \left| \int_{0}^{+\infty} W_a(Y)e^{-\alpha Y} dY \right| \geq C^{-1}(1 + |n|^{\frac{1}{2}}|\lambda|)^{-\frac{1}{4}}n^{-\frac{1}{4}} = C^{-1}A^{-1}. \]
And by Lemma 2.9 and Lemma 2.8, we get
\[ \left| \int_{0}^{+\infty} W_e(Y)e^{-\alpha Y} dY \right| = |\partial_Y \Phi_e(0)| \leq \| \partial_Y \Phi_e \|_{L^\infty} \leq CA^{-1}(A\lambda)\frac{\pi}{4} \leq CA^{-1}(n)^{-\frac{1}{4}} \leq CA^{-1} \delta^{-\frac{1}{4}}. \]
Then we deduce that
\[ |J| \geq \left| \int_{0}^{+\infty} W_a(Y)e^{-\alpha Y} dY \right| - \left| \int_{0}^{+\infty} W_e(Y)e^{-\alpha Y} dY \right| \geq C^{-1}A^{-1} (1 - C\delta^{-\frac{1}{4}}). \]
Taking \( \delta \) sufficiently small so that \( C\delta^{-\frac{1}{4}} \leq 1/2 \), we arrive at
\[ |J| \geq C^{-1}A^{-1}. \]
\[ \square \]

From the above lemma, we can define \( W_b(Y) = W(Y)/J \). By combining with Lemma 2.11 and Lemma 2.10, we get

**Proposition 2.12.** Let \( W_b \) solve (2.47) and \( |\lambda| \leq \delta^{-1}_1 \). Then it holds that
\[ \| (\partial_Y \Phi_b, \alpha \Phi_b) \|_{L^2} \leq CA^{-\frac{1}{2}} = C, \quad \| W_b \|_{L^2} \leq A^\frac{1}{2}. \]

2.3.3. **Resolvent estimates for nonslip boundary condition.** Now we are back to consider the estimate for the solution to
\[ (2.53) \quad \begin{cases} - \sqrt{\nu} (\partial_{Y}^2 - \alpha^2)w + i\alpha ((V - \lambda)w - (\partial_{Y}^2 V)\phi) = F, \\ F = -\partial_Y F_1 + i\alpha F_2, \\ (\partial_{Y}^2 - \alpha^2)\phi = w, \quad \partial_Y \phi|_{Y=0} = \phi|_{Y=0} = 0. \end{cases} \]
Let \( w_{Na} \) solve
\[ \begin{cases} - \sqrt{\nu} (\partial_{Y}^2 - \alpha^2)w_{Na} + i\alpha ((V - \lambda)w_{Na} - (\partial_{Y}^2 V)\phi_{Na}) = F, \\ F = -\partial_Y F_1 + i\alpha F_2, \\ (\partial_{Y}^2 - \alpha^2)\phi_{Na} = w_{Na}, \quad w_{Na}|_{Y=0} = \phi_{Na}|_{Y=0} = 0. \end{cases} \]
By matching the boundary condition, we have
\[ w(Y) = w_{Na}(Y) + \partial_Y \phi_{Na}(0)W_b(Y). \]

Then we have the following proposition.

**Proposition 2.13.** There exists \( \delta_* \in (0, \delta_1) \) and \( \nu_0 \) such that the following statements hold. Let \( |\lambda| \leq \delta^{-1}_1, n \in \mathbb{N} \) and \( \nu \leq \nu_0 \). Suppose that (2.7) holds for some \( \gamma \in [0, 1] \) and \( \delta \in (0, \delta_*). \) Then for any \( f = (f_1, f_2) \in L^2(\mathbb{R}_+)^2, \) the corresponding weak solution \( \phi \in H^0_0(\mathbb{R}_+) \) to (2.53) satisfies

\[
\begin{align*}
\| (\partial_Y \phi, \alpha \phi) \|_{L^2} & \leq C(\alpha \lambda_i)^{-1} \lambda_i^{-1} \| (F_1, F_2) \|_{L^2}, \\
\| w \|_{L^2} & \leq C \nu^{-\frac{1}{4}} \alpha^{-\frac{1}{2}} \lambda_i^{-\frac{5}{4}} \| (F_1, F_2) \|_{L^2}, \\
\| \rho^{\frac{1}{2}} w \|_{L^2} & \leq C \nu^{-\frac{1}{4}} \alpha^{-\frac{1}{2}} \lambda_i^{-\frac{5}{4}} \| (F_1, F_2) \|_{L^2}.
\end{align*}
\]

**Proof.** By Proposition 2.4, we obtain

\[
\nu^{\frac{1}{4}} \alpha^{\frac{1}{2}} \lambda_i^{-\frac{5}{4}} \| w_{Na} \|_{L^2} + \alpha \lambda_i \| (\partial_Y \phi_{Na}, \alpha \phi_{Na}) \|_{L^2} \leq C \alpha^{-1} \lambda_i^{-\frac{5}{4}} \| (F_1, F_2) \|_{L^2}.
\]

Then by the interpolation, we get

\[
\| \partial_Y \phi_{Na}(0) \|_{L^\infty} \leq C \| w_{Na} \|_{L^2}^{\frac{1}{2}} \| (\partial_Y \phi_{Na}, \alpha \phi_{Na}) \|_{L^2}^{\frac{3}{2}} \leq C \nu^{-\frac{1}{4}} \alpha^{-\frac{1}{2}} \lambda_i^{-\frac{5}{4}} \| (F_1, F_2) \|_{L^2} = C |n|^{\frac{1}{4}} \alpha^{-1} \lambda_i^{-\frac{5}{4}} \| (F_1, F_2) \|_{L^2}.
\]

This along with Proposition 2.12 gives

\[
\| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq \| (\partial_Y \phi_{Na}, \alpha \phi_{Na}) \|_{L^2} + \| \partial_Y \phi_{Na}(0) \|_{L^2} \| (\partial_Y \phi_{Na}, \alpha \phi_{Na}) \|_{L^2} \leq C (|n|^{\frac{1}{4}} \alpha^{-1} \lambda_i^{-\frac{5}{4}} A^{-\frac{1}{2}}) \| (F_1, F_2) \|_{L^2} = C (|n|^{\frac{1}{4}} \alpha^{-1} \lambda_i^{-\frac{5}{4}} A^{-\frac{1}{2}}) \| (F_1, F_2) \|_{L^2} \leq C (|n|^{\frac{1}{4}} \alpha^{-1} \lambda_i^{-\frac{5}{4}} A^{-\frac{1}{2}}) \| (F_1, F_2) \|_{L^2}.
\]

provided that \( |n|^{\frac{1}{4}} \lambda_i^{-\frac{1}{2}} / A \leq 1, \lambda_i \leq \delta^{-1}_1. \) This gives the first inequality. We also notice that

\[
\| w \|_{L^2} \leq \| w_{Na} \|_{L^2} + \| \partial_Y \phi_{Na}(0) \|_{L^2} \leq C (\nu^{-\frac{1}{4}} \alpha^{-\frac{1}{2}} \lambda_i^{-\frac{5}{4}} + |n|^{\frac{1}{4}} \alpha^{-1} \lambda_i^{-\frac{5}{4}} A^{-\frac{1}{2}}) \| (F_1, F_2) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} \lambda_i^{-\frac{1}{2}} \| (F_1, F_2) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} \lambda_i^{-\frac{1}{2}} \| (F_1, F_2) \|_{L^2} \leq C (\alpha \lambda_i)^{-1} |n|^{\frac{1}{4}} \lambda_i^{-\frac{1}{2}} \| (F_1, F_2) \|_{L^2}.
\]
where in the last line, we used $|n|^{-\frac{1}{4}} + \lambda_i + |\nu| \leq C$. This gives the second inequality. We also have
\[
\|\rho^{\frac{1}{2}}w\|_{L^2} \leq \|w_{Na}\|_{L^2} + |\partial_Y \phi_{Na}(0)| \|\rho^{\frac{1}{2}}W_b\|_{L^2} \\
\leq C(\nu^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}\lambda_i^{-\frac{1}{2}} + |n|^{-\frac{1}{4}}\alpha^{-1}\lambda_i^{-\frac{2}{3}}(\|n\|^{-\frac{1}{4}}\lambda_i^{-\frac{1}{2}}))^3 \|(F_1, F_2)\|_{L^2} \\
= C\nu^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}\lambda_i^{-\frac{1}{2}} \|(F_1, F_2)\|_{L^2}.
\]
This finishes the proof of this proposition. \( \square \)

3. Semigroup estimates of $e^{-t\nu}$

This section is devoted to the semigroup estimates of $e^{-t\nu}$. We first introduce the $L^2$ estimates, which is a combination of Proposition 4.1, 4.2 and 5.35 in [12] in the case when the shear flow satisfies the (SC) condition.

**Proposition 3.1** (Semigroup estimate). Assume that (SC) condition holds. Then there exist $\delta_1, \delta_2, \delta_s \in (0, 1)$ satisfying $\delta_1 \leq \delta_0$ and $\delta_s \leq \min\{\delta_1, \delta_2\}$ such that the following statements hold true. Assume that (2.4) holds for some $\delta \in (0, \delta_s]$ and $\gamma \in [\frac{2}{3}, 1]$. Then the following estimates hold for all $f \in P_n L^2(\Omega)$ and $t > 0$.

1. If $|n| \leq \delta_0^{-1}$, then
\[
\|e^{-t\nu, n} f\|_{L^2} \leq Ce^{ct} \|f\|_{L^2}, \\
\|\nabla e^{-t\nu, n} f\|_{L^2} \leq \frac{C}{\sqrt{vt}} (1 + te^{ct}) \|f\|_{L^2}.
\]

2. If $\delta_0^{-1} \leq |n| \leq \delta_0^{-1} \nu^{-3/4}$ and $|n|^{-\frac{1}{2}} \nu^n < 1$, then
\[
\|e^{-t\nu, n} f\|_{L^2} \leq C|n|^{2(1-\gamma)} e^{\frac{|n|\gamma}{\nu^2} t} \|f\|_{L^2}, \\
\|\nabla e^{-t\nu, n} f\|_{L^2} \leq \frac{C}{\nu^1/2} \left(t^{-1/2} + |n|^{2(1-\gamma)} (|n|^{1/4} + |n|^{-\frac{1}{4}} (1-\gamma) \nu^n t)\right) \|f\|_{L^2}.
\]

3. If $|n|^{-\frac{1}{2}} \nu^n \geq 1$ and $|n| \leq \delta_0^{-1} \nu^{-\frac{1}{4}}$, then
\[
\|e^{-t\nu, n} f\|_{L^2} \leq C|n|^{-1} e^{-\frac{|n|\gamma}{\nu^2} t} \|f\|_{L^2}, \\
\|\nabla e^{-t\nu, n} f\|_{L^2} \leq \frac{C}{\nu^{1/2}} \left(t^{-1/2} + |n|^{(1-\gamma)/2} e^{\frac{|n|\gamma}{\nu^2} t}\right) \|f\|_{L^2}.
\]

4. If $|n| \geq \delta_0^{-1} \nu^{-3/4}$, then
\[
\|e^{-t\nu, n} f\|_{L^2} \leq e^{-\frac{1}{4} |n|\nu^2 t} \|f\|_{L^2}, \\
\|\nabla e^{-t\nu, n} f\|_{L^2} \leq \frac{Ce^{-\frac{1}{4} |n|\nu^2 t}}{\sqrt{vt}} (1 + |n| t) \|f\|_{L^2}.
\]

Here $C$ and $c$ are universal constants only depending on $U^P$.

We mention that the results of (1) and (4) are obtained via standard energy method. However, for the result (2) and (4) which are estimates in Gevrey class for the mid-range frequency, we need to use the corresponding resolvent estimates in this case. The idea of the proof is similar to [12]. For completeness, we still present the details of this proof.
Proof. Let $n \in \mathbb{Z}$ and $f \in \mathcal{P}_n L^2_\nu(\Omega)$. We denote $u^{(n)} := e^{-t\nu\Delta} f$. Then by the definition of semigroup $e^{-t\nu\Delta}$, we know that $u^{(n)}$ satisfies
\[
\partial_t u^{(n)} - \nu \Delta u^{(n)} + U^p \left( \frac{y}{\sqrt{\nu}} \right) \partial_x u^{(n)} + \frac{1}{\nu} \left( u^{(n)} \partial_y U^p \left( \frac{y}{\sqrt{\nu}} \right), 0 \right) + \mathcal{P}_n \nabla p = 0, \quad u^{(n)}|_{t=0} = f,
\]
which by introducing $Y = \frac{y}{\sqrt{\nu}}$ can be written as
\[
(3.1) \quad \partial_t u^{(n)} - \nu \Delta u^{(n)} + U^p(Y) \partial_x u^{(n)} + (y^{-1} u^{(n)}_2 Y \partial_y U^p(Y), 0) + \mathcal{P}_n \nabla p = 0.
\]
Recall that
\[
(3.2) \quad \delta_0 = \frac{1}{2(1 + \|U^p\|)},
\]
where $\| \cdot \|$ is defined in (SC) condition.

Now we turn to prove the first statement of Proposition 3.1. We first notice that by the boundary condition and divergence free condition
\[
(3.3) \quad \|y^{-1} u^{(n)}_2\|_{L^2(\Omega)} \leq 2 \|\partial_y y_2\|_{L^2(\Omega)} = 2 \|\partial_x u^{(n)}_1\|_{L^2(\Omega)} \leq 2 |n| \|u^{(n)}\|_{L^2(\Omega)}.
\]
By taking $L^2$-inner product with $u^{(n)}$ on both sides of (3.1), we have
\[
\frac{d}{dt} \|u^{(n)}\|^2_{L^2(\Omega)} = -2\nu \|\nabla u^{(n)}\|^2_{L^2(\Omega)} - 2 \text{Re} \left( y^{-1} u^{(n)}_2(y) Y \partial_y U^p(Y), u^{(n)}_1 \right)_{L^2}
\leq -2\nu \|\nabla u^{(n)}\|^2_{L^2(\Omega)} + 2 \|Y \partial_y U^p\|_{L^\infty} |n| \|u^{(n)}\|^2_{L^2(\Omega)}.
\]
Hence by Gronwall’s inequality, we obtain that for any $n \in \mathbb{Z}$,
\[
(3.4) \quad \|u^{(n)}(t)\|_{L^2(\Omega)} \leq e^{\frac{|n| t}{\delta_0}} \|f\|_{L^2(\Omega)}.
\]
To prove the derivative estimates, by the Duhamel formula, we have
\[
(3.5) \quad u^{(n)}(t) = e^{t\nu\Delta} f - \int_0^t e^{(t-s)\nu\Delta} \mathcal{P}(U^p(Y) \partial_x u^{(n)}(s) + (y^{-1} u^{(n)}_2(s) Y \partial_y U^p(Y), 0)) ds,
\]
which along with the well-known derivative estimate of heat kernel gives
\[
\|\nabla u^{(n)}(t)\|_{L^2(\Omega)} \leq C(\nu t)^{-\frac{1}{2}} \|f\|_{L^2(\Omega)} + C |n| \|U^p\| \sup_{0 \leq s \leq t} \|u^{(n)}(s)\|_{L^2(\Omega)} \int_0^t \frac{1}{\nu s(t-s)^{\frac{1}{2}}} ds.
\]
Combining the above inequality and (3.4), we obtain
\[
(3.6) \quad \|\nabla u^{(n)}(t)\|_{L^2} \leq \frac{C}{\sqrt{\nu t} \left( 1 + te^{\frac{|n| t}{\delta_0}} \right)} \|f\|_{L^2}.
\]
This shows the first statement in the proposition by taking $|n| \leq \delta_0^{-1}$.

Now we turn to prove the second and third statement in Proposition 3.1. By rescaling, we have
\[
u^{\frac{1}{2}} u^{(n)}(t, x, y) = e^{-\tau \nu \Delta} f(x, y) = e^{-\tau L_{\nu, n} \Delta} f_{\nu}(X, Y),
\]
where $(\tau, X, Y) = (t/\sqrt{\nu}, x/\sqrt{\nu}, y/\sqrt{\nu})$, $f_{\nu}(X, Y) = f(\nu^{\frac{1}{2}} X, \nu^{\frac{1}{2}} Y)$ and the operator $L_{\nu, n}$ is defined as in (2.3). According to (3.3), after rescaling, we have already known that $-L_{\nu, n}$ generates a $C_0$-semigroup acting on $\mathcal{P}_{\nu, n} L^2_{\nu}(\Omega_{\nu})$. Here $\Omega_{\nu} := (\nu^{-1/2} \mathbb{T}) \times \mathbb{R}_+$ and
\[
(\mathcal{P}_{\nu, n} f)(Y) = f_{\nu}(Y) e^{in\sqrt{\nu} X}, \quad f_{\nu}(Y) = \frac{\sqrt{\nu}}{2\pi} \int_0^{2\pi} f(X, Y) e^{-in\sqrt{\nu} X} dX.
\]
In details, from (3.4) we obtain that for any \( g \in \mathcal{P}_{\nu,n}L^2(\Omega_\nu) \)

\[
\|e^{-\tau\mathbb{L}_{\nu,n}}g\|_{L^2(\Omega_\nu)} \leq e^{\frac{s}{\delta}n^\frac{\gamma}{2}} \|g\|_{L^2(\Omega_\nu)},
\]

which deduces the results of the second and third statement for short time \( 0 \leq \tau \leq \nu^{-\frac{1}{2}}|n|^{-1}. \)

Hence, we now only need to consider the case \( \tau \geq \nu^{-\frac{1}{2}}|n|^{-1}. \) According to Proposition 2.1 we know that the set

\[
\Sigma_{\nu,\gamma} := S_{\nu,n}(\theta) \cup \{ \mu \in \mathbb{C} | \text{Re}\mu \geq \frac{n\gamma\nu_1^\frac{1}{2}}{\delta} \}
\]

is included in the resolvent set of \(-\mathbb{L}_{\nu,n}\), where \( S_{\nu,n}(\theta) \) is the set defined in Proposition 2.1. The first statement in Proposition 2.1 ensures that the semigroup \( e^{-\tau\mathbb{L}_{\nu,n}} \) can be represented as

\[
e^{-\tau\mathbb{L}_{\nu,n}} = \frac{1}{2\pi i} \int_{\Gamma} e^{\tau\mu}(\mu + \mathbb{L}_{\nu,n})^{-1}d\mu,
\]

where the curve \( \Gamma \) is taken as \( \Gamma = \Gamma_+ + \Gamma_- + l_+ + l_- + l_0 \) with

\[
\Gamma_\pm := \{ \mu \in \mathbb{C} | \pm \text{Im}\mu = (\tan \theta)\text{Re}\mu + \delta_1^{-1}(\sqrt{\nu}|n| + |\tan \theta||n\gamma\nu_1^\frac{1}{2}), \text{Re}\mu \leq 0 \}
\]

\[
l_\pm := \{ \mu \in \mathbb{C} | \pm \text{Im}\mu = \delta_1^{-1}(\sqrt{\nu}|n| + |\tan \theta||n\gamma\nu_1^\frac{1}{2}), 0 \leq \text{Re}\mu \leq \frac{|n\gamma\nu_1^\frac{1}{2}}{\delta} \}
\]

\[
l_0 := \{ \mu \in \mathbb{C} | 0 \leq |\text{Im}\mu| \leq \delta_1^{-1}(\sqrt{\nu}|n| + |\tan \theta||n\gamma\nu_1^\frac{1}{2}), \text{Re}\mu = \frac{|n\gamma\nu_1^\frac{1}{2}}{\delta} \}.
\]

By (2.10), we have that for any \( g \in \mathcal{P}_{\nu,n}L^2(\Omega_\nu) \)

\[
\| \frac{1}{2\pi i} \int_{\Gamma_\pm} e^{\tau\mu}(\mu + \mathbb{L}_{\nu,n})^{-1}gd\mu \|_{L^2(\Omega_\nu)} \leq C\|g\|_{L^2(\Omega_\nu)} |\int_{\Gamma_\pm} e^{\tau\text{Re}\mu}|\mu|^{-1}d\mu|
\]

\[
\leq C\|g\|_{L^2(\Omega_\nu)} \int_0^\infty e^{-\tau s} \frac{s}{s + |\tan \theta|s + \delta_1^{-1}(\sqrt{\nu}|n| + |\tan \theta||n\gamma\nu_1^\frac{1}{2})} \text{ds}
\]

\[
\leq C(\sqrt{\nu}|n|^\frac{1}{2})^{-1}\|g\|_{L^2(\Omega_\nu)},
\]

which implies that for any \( \tau \geq (\nu^{1/2}/n)^{-1} \)

\[
\| \frac{1}{2\pi i} \int_{\Gamma_\pm} e^{\tau\mu}(\mu + \mathbb{L}_{\nu,n})^{-1}gd\mu \|_{L^2(\Omega_\nu)} \leq C\|g\|_{L^2(\Omega_\nu)}, \quad g \in \mathcal{P}_{\nu,n}L^2(\Omega_\nu).
\]

Again by (2.10), we have

\[
\| \frac{1}{2\pi i} \int_{l_\pm} e^{\tau\mu}(\mu + \mathbb{L}_{\nu,n})^{-1}gd\mu \|_{L^2(\Omega_\nu)}
\]

\[
\leq C\|g\|_{L^2(\Omega_\nu)} \int_0^{|n|\gamma\nu_1^\frac{1}{2}} \frac{s}{s + \delta_1^{-1}(\sqrt{\nu}|n| + |\tan \theta||n\gamma\nu_1^\frac{1}{2})} \text{ds}
\]

\[
\leq C|n|^{-1}e^{\frac{|n|\gamma\nu_1^\frac{1}{2}}{s}} \|g\|_{L^2(\Omega_\nu)}.
\]

On \( l_0 \), we apply (2.12) and (2.15) for the case \( |n|\gamma\nu_1^\frac{1}{2} \geq 1 \) and \( |n|\gamma\nu_1^\frac{1}{2} \leq 1 \) respectively.
We obtain the estimates on (3.16) by (3.11), (3.12), (3.13) and (3.14). In a similar way, on (3.15) we have

\[ \| \nabla \gamma \nu, n \|_{L^2(\Omega_v)} \leq C|\nu|^{1-\gamma} e^{|\nabla \gamma \nu, n|_{L^1}} \| g \|_{L^2(\Omega_v)}. \]

Therefore, we prove the estimates in the second and third statement of Proposition 3.1 by (3.11), (3.12), (3.13) and (3.14).

Next we consider the derivative estimates. We use the representation (3.9) again. For the integral on \( \Gamma_{\pm} \), we have that by (2.11),

\[
\| \nabla \gamma \nu, n \|_{L^2(\Omega_v)} \leq C|\nu|^{1-\gamma} e^{|\nabla \gamma \nu, n|_{L^1}} \| g \|_{L^2(\Omega_v)}.
\]

In a similar way, on \( l_{\pm} \), we have

\[
\| \nabla \gamma \nu, n \|_{L^2(\Omega_v)} \leq C|\nu|^{1-\gamma} e^{|\nabla \gamma \nu, n|_{L^1}} \| g \|_{L^2(\Omega_v)}.
\]

We obtain the estimates on \( l_0 \) in the same way as in (3.13) and (3.14). In details, for \( |\nabla \gamma \nu, n|_{L^1} \geq 1 \), we have

\[
\| \nabla \gamma \nu, n \|_{L^2(\Omega_v)} \leq C|\nu|^{1-\gamma} e^{|\nabla \gamma \nu, n|_{L^1}} \| g \|_{L^2(\Omega_v)}.
\]
and for $|n|^\gamma \nu ^{\frac{1}{2}} \leq 1$, we have

\begin{equation}
\| \nabla_{x,y} \frac{1}{2\pi i} \int_{I_0} e^{\tau\mu} (\mu + L_{\nu,n})^{-1} g d\mu \|_{L^2(\Omega)} \leq C n^{2(1-\gamma)} (|\nu|^{\frac{1}{4}} + |n|^\frac{\gamma}{2}(1-\gamma)) e^{\frac{1}{4}\nu t^{\frac{1}{2}}} \|g\|_{L^2(\Omega)}.
\end{equation}

Combining (3.15)–(3.18) and scaling back to the original variables, we finish the proof of the second and third statement of Proposition 3.1.

Finally, we deal with the last statement of the proposition. In this situation, we are back to the system (3.1). Recall that we consider the case $|n| \geq \delta_0^{-1} \nu ^{-\frac{3}{2}}$. By the standard energy method, we obtain

$$
\frac{d}{dt} \| u^{(n)} \|_{L^2(\Omega)}^2 = -2\nu \| \nabla u^{(n)} \|_{L^2(\Omega)}^2 - 2\nu^{-\frac{3}{2}} \Re(e^{u^{(n)}_2(y)} \partial Y U^P(Y), u^{(n)\dagger})_{L^2} \\
\leq -\nu \| \nabla u^{(n)} \|_{L^2(\Omega)}^2 - \nu n^2 \| u^{(n)} \|_{L^2}^2 + 2\nu^{-\frac{3}{2}} \| \partial Y U^P \|_{L^\infty} |n| \| u \|_{L^2}^2.
$$

Then the above inequality and $|n| \geq \delta_0^{-1} \nu ^{-\frac{3}{2}}$ lead to

$$
\frac{d}{dt} \| u^{(n)} \|_{L^2(\Omega)}^2 \leq -\frac{\nu n^2}{2} \| u^{(n)} \|_{L^2(\Omega)}^2.
$$

Hence by the Gronwall’s inequality, we have

$$
\| e^{-\nu t \nabla_{\nu,n}} f \|_{L^2} \leq e^{-\frac{\nu n^2 t}{2}} \| f \|_{L^2}.
$$

As in the proof of the first statement, by the Duhamel formula and the above $L^2$-estimates, we can obtain the derivative estimates in the last statement.

Next we state the $L^\infty$-estimates of semigroup.

**Proposition 3.2.** Assume that $(SC)$ condition holds. Then there exist $\delta_1, \delta_2, \delta_3 \in (0, 1)$ satisfying $\delta_1, \delta_2 \leq \delta_0$ and $\delta_3 \leq \min(\delta_1, \delta_2)$ such that the following statements hold true. Assume that (2.7) holds for some $\delta \in (0, \delta_3]$ and $\gamma \in [\frac{2}{3}, 1]$.

1. If $|n|^\gamma \nu ^{\frac{1}{2}} \leq 1$, then

$$
\| e^{-\nu t \nabla_{\nu,n}} f \|_{L^2_{\nu,n} L^\infty} \leq C \| f \|_{L^2_{\nu,n} L^\infty} + C \nu^{-\frac{5}{4}} t^{\gamma} (|n| + |n|^{3-2\gamma}) e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2(\Omega)},
$$

and

$$
\| e^{-\nu t \nabla_{\nu,n}} f \|_{L^2_{\nu,n} L^\infty} \leq C (1-\gamma) |n|^\frac{\gamma}{2}(1-\gamma) + C \nu^{-\frac{3}{4}} t^{\gamma} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2_{\nu,n}} + C |n|^{1-\gamma} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2(\Omega)},
$$

$$
+ C \nu^{-\frac{3}{4}} (1-\gamma) \log t (|n| + 1) \| n \|^{\frac{\gamma}{2}-\frac{3}{4}} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2_{\nu,n}}.
$$

2. If $|n|^\gamma \nu ^{\frac{1}{2}} \geq 1$ and $|n| \leq \delta_0^{-1} \nu ^{-\frac{3}{2}}$, then

$$
\| e^{-\nu t \nabla_{\nu,n}} f \|_{L^2_{\nu,n} L^\infty} \leq C \| f \|_{L^2_{\nu,n} L^\infty} + C \nu^{-\frac{5}{4}} t^{\gamma} |n|^{2-\gamma} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2(\Omega)},
$$

$$
\| e^{-\nu t \nabla_{\nu,n}} f \|_{L^2_{\nu,n} L^\infty} \leq C \nu^{-\frac{5}{4}} t^{\gamma} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2_{\nu,n}} + C \nu^{-\frac{5}{4}} |n|^{1-\gamma} e^{\frac{|n|^\gamma}{2}} \| f \|_{L^2_{\nu,n}}.
$$
\[ (3) \text{ If } |n| \geq \delta_0^{-1} \nu^{-\frac{3}{4}}, \text{ then} \]
\[ \|e^{-t\Delta_{Y,n}}f\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}|n|e^{-\frac{1}{4}\nu\eta^2t}\|f\|_{L^2(\Omega)}, \]
\[ \|e^{-t\Delta_{Y,n}}f\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\frac{1}{\nu^{3/4}}(1 + |n|^{3/4})e^{-\frac{1}{4}\nu|n|^2t}\|f\|_{L^2(\Omega)}. \]

**Proof.** Let \( n \in \mathbb{Z} \) and \( f \in \mathcal{P}_nL^2_{\mathcal{L}_Y^\infty}(\Omega) \cap \mathcal{P}_nL^2_\nu(\Omega) \). We still denote \( u^{(n)} = e^{-t\Delta_{Y,n}}f \). By Duhamel’s formula, \( u^{(n)} \) can be written as
\[ u^{(n)}(t) = e^{t\Delta}f - \int_0^t e^{(t-s)\Delta}\mathbb{P}(U_\gamma(Y)\partial_xu^{(n)}(s) + (y^{-1}u_2^{(n)}(s)Y\partial_YU_\gamma(Y), 0))ds. \]

Using the standard estimate of heat kernel, we have
\[ \|u^{(n)}(t)\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\sup_{0 \leq s \leq t}\|\mathbb{P}(U_\gamma(Y)\partial_xu^{(n)}(s) + (y^{-1}u_2^{(n)}(s)Y\partial_YU_\gamma(Y), 0))\|_{L^2} \int_0^t \frac{1}{(\nu(t-s))^{3/4}}ds \]
\[ \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}\|\partial_xu^{(n)}(s)\|_{L^2} + \|y^{-1}u_2^{(n)}(s)\|_{L^2}, \]
which along with Hardy inequality and divergence free condition implies
\[ \|u^{(n)}(t)\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}|n| \sup_{0 \leq s \leq t} \|u^{(n)}(s)\|_{L^2}. \]

Now we consider the case \( |n|^{\gamma}\nu^{\frac{1}{2}} \leq 1 \). In this case, by the results (1) and (2) in Proposition 3.1, we have known that
\[ \|u^{(n)}(t)\|_{L^2(\Omega)} \leq C(1 + |n|^{2(1-\gamma)})e^{\frac{|n|^{\gamma}t}{\delta}}\|f\|_{L^2(\Omega)}, \]
which combined with (3.20) deduces that
\[ \|u^{(n)}(t)\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}(|n| + |n|^{3-2\gamma})e^{\frac{|n|^{\gamma}t}{\delta}}\|f\|_{L^2(\Omega)} \]

In a similar way, we have that for \( 1 \leq |n|^{\gamma}\nu^{\frac{1}{2}} \leq \delta_0^{-1} \nu^{-\frac{3}{4}} \),
\[ \|u^{(n)}(t)\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}|n|^{2-\gamma}e^{\frac{|n|^{\gamma}t}{\delta}}\|f\|_{L^2(\Omega)}, \]
and for \( |n| \geq \delta_0^{-1} \nu^{-\frac{3}{4}} \),
\[ \|u^{(n)}(t)\|_{L^2_{\mathcal{L}_Y^\infty}} \leq C\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}t^{\frac{3}{4}}|n|e^{-\frac{1}{4}\nu\eta^2t}\|f\|_{L^2(\Omega)} \]

By the standard interpolation and Proposition 3.1 we obtain that for any \( |n|^{\gamma}\nu^{\frac{1}{2}} \geq 1 \) and \( |n| \leq \delta_0^{-1} \nu^{-\frac{3}{4}} \)
\[ \|u^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}} \leq \|u^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}}^{\frac{1}{2}}\|\partial_yu^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}}^{\frac{1}{2}} \]
\[ \leq C\frac{|n|^{(1-\gamma)/2}}{\nu^{3/4}}e^{\frac{|n|^{\gamma}t}{\delta}}\|f\|_{L^2_{\mathcal{L}_Y^\infty}} + C\nu^{-\frac{1}{4}}|n|^{\frac{3}{4}-\frac{\gamma}{2}}e^{\frac{|n|^{\gamma}t}{\delta}}\|f\|_{L^2_{\mathcal{L}_Y^\infty}}. \]

We also obtain that for any \( |n| \geq \delta_0^{-1} \nu^{-\frac{3}{4}} \)
\[ \|u^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}} \leq \|u^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}}^{\frac{1}{2}}\|\partial_yu^{(n)}\|_{L^2_{\mathcal{L}_Y^\infty}}^{\frac{1}{2}} \]
\[ \leq C\frac{1}{\nu^{3/4}}(1 + |n|^{3/4})e^{-\frac{1}{4}\nu\eta^2t}\|f\|_{L^2_{\mathcal{L}_Y^\infty}}. \]
Now we have proven this Proposition except: \(|n|^\gamma \nu^{\frac{1}{2}} \leq 1\).

For the middle-range frequency, we prove a sharper estimates by introducing the following weight function

\[
\rho(Y) = \begin{cases} 
\frac{|n|^\frac{\gamma}{2} + \frac{1}{2} \nu^{\frac{1}{2}} Y}{\delta^{\frac{1}{2}}} & \text{if } 0 \leq Y \leq \frac{\delta^{\frac{1}{2}}}{|n|^\frac{\gamma}{2} + \frac{1}{2} \nu^{\frac{1}{2}}}, \\
1, & \text{if } Y \geq \frac{\delta^{\frac{1}{2}}}{|n|^\frac{\gamma}{2} + \frac{1}{2} \nu^{\frac{1}{2}}},
\end{cases}
\]

In detail, we first recall that after rescaling, we have

\[
v^{(n)}(t, X, Y) := (e^{-\tau^n_{\omega,n}} f_\nu)(X, Y) = (e^{-t h_\nu,n}) f(x, y) = u^{(n)}(t, x, y),
\]

where \((\tau, X, Y) = (t/\sqrt{\nu}, x/\sqrt{\nu}, y/\sqrt{\nu})\), \(f_\nu(X, Y) = f(\nu^{\frac{1}{2}} X, \nu^{\frac{1}{2}} Y)\) and the operator \(L_{\nu,n}\) is defined as in (2.3). Hence, we directly have

\[
\|v^{(n)}(t)\|_{L^2_\nu L^\infty(\Omega)} \leq \nu^{\frac{1}{2}} \|v^{(n)}(\tau)\|_{L^2_\nu L^\infty(\Omega, t)} + \nu^{\frac{1}{2}} \nu\|f_\nu\|_{L^2_\nu L^\infty(\Omega, t)},
\]

On the other hand, we observe that by the interpolation

\[
\|v^{(n)}(\tau)\|_{L^2_\nu L^\infty} \leq C\|\rho^{\frac{1}{2}} \omega^{(n)}(\tau)\|_{L^2_{X,Y}} + C\|1 - \rho^{\frac{1}{2}}\omega^{(n)}(\tau)\|_{L^2_{X,Y}}
\]

where \(\omega^{(n)} := \text{curl}_{X,Y} v^{(n)}\). For the second term on the right hand side, we notice that by taking \(h_0 = (|n|^{\gamma/2}/\delta)^{-\frac{1}{2}}\) and \(h_1 := h_0|n|^{(\gamma - 1)/2}\)

\[
\|(1 - \rho^{\frac{1}{2}})\omega^{(n)}\|_{L^2_\nu L^1} \leq \|\omega^{(n)}\|_{L^2_\nu L^1 (h_1, h_2)} + \|\omega^{(n)}\|_{L^2_\nu L^1 (0, h_1)}
\]

Hence, by the above two estimates, we get

\[
\|v^{(n)}(\tau)\|_{L^2_\nu L^\infty} \leq C\|\rho^{\frac{1}{2}} \omega^{(n)}(\tau)\|_{L^2_{X,Y}} + C\|1 - \rho^{\frac{1}{2}}\omega^{(n)}(\tau)\|_{L^2_{X,Y}}
\]

Applying Proposition 3.31 and after scaling, we deduce that for \(\delta_{0}^{-1} \leq |n| \leq \delta_{0}^{-1} \nu^{-3/4}\) and \(|n|^\gamma \nu^{\frac{1}{2}} < 1\),

\[
\|v^{(n)}(\tau)\|_{L^2_\nu L^\infty} \leq C|n|^{2(1 - \gamma)} e^{\frac{|n|^\gamma}{\delta} \nu^{\frac{1}{2}}} \|f_\nu\|_{L^2_{X,Y}},
\]

\[
\|\omega^{(n)}(\tau)\|_{L^2_\nu L^\infty} \leq C \left( \frac{1}{\nu^{1/2}} + n^{2(1 - \gamma)} (|n|^{1/4} + |n|^{\frac{1}{2} - \frac{3}{4}(1 - \gamma)} e^{\frac{|n|^\gamma}{\delta} \nu^{\frac{1}{2}}} ) \right) \|f_\nu\|_{L^2_{X,Y}}.
\]

For \(1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_{0}^{-1} \nu^{-\frac{1}{4}}\),

\[
\|v^{(n)}(\tau)\|_{L^2_\nu L^\infty} \leq C|n|^{-\gamma} e^{\frac{|n|^\gamma}{\delta} \nu^{\frac{1}{2}}} \|f_\nu\|_{L^2_{X,Y}},
\]
\[
\|\omega^{(n)}(\tau)\|_{L^2} \leq C \left( \frac{1}{\nu^{1/4} \tau^{2}} + n^{(1-\gamma/2)} e^{n^{\gamma/4} \frac{1}{\nu^{2}}} \right) \|f_\nu\|_{L^2_{X,Y}}.
\]

Hence, in order to obtain the estimate of \(\|v^{(n)}(\tau)\|_{L^2_{X,Y}}\), we are left with the control of \(\|\rho^{1/2} \omega^{(n)}(\tau)\|_{L^2_{X,Y}}\). For this purpose, we are back to the formula

\[
v^{(n)}(\tau) = e^{-\tau L_{\nu,n}} f_\nu = \frac{1}{2\pi i} \int_{\Gamma} e^{\tau \mu} (\mu + \mathbb{I}_{\nu,n})^{-1} f_\nu d\mu,
\]

where \(\Gamma\) is given in (3.10). According to this formula, (2.11), (2.13) and (2.17), we have that for any \(\delta_0^{-1} \leq |n| \leq \delta_0^{-1} \nu^{-\frac{3}{4}},\)

\[
(3.27) \quad \|\rho^{1/2} \omega^{(n)}(\tau)\|_{L^2} \leq C \left( \frac{1}{\nu^{1/4} \tau^{2}} + n^{(1-\gamma/2)} e^{n^{\gamma/4} \frac{1}{\nu^{2}}} \right) \|f_\nu\|_{L^2_{X,Y}},
\]

by using a similar argument in the proof of derivative estimates in Proposition 3.1.

Collecting the above estimates, we obtain that for \(\delta_0^{-1} \leq |n| \leq \delta_0^{-1} \nu^{-3/4}\) and \(|n|^{\gamma} \nu^{\frac{1}{2}} < 1,\)

\[
\|v^{(n)}(\tau)\|_{L^2_{X,Y}} \leq C|n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \left( \frac{1}{\nu^{1/4} \tau^{2}} + |n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \right) \|f_\nu\|_{L^2_{X,Y}}
\]

\[
+ C(1-\gamma)^{\frac{3}{2}} n^{1-\gamma} (\log(n))^{\frac{1}{2}} \left( \frac{1}{\nu^{1/4} \tau^{2}} + n^{(1-\gamma/2)} e^{n^{\gamma/4} \frac{1}{\nu^{2}}} \right) \|f_\nu\|_{L^2_{X,Y}}
\]

\[
+ Cn^{-\frac{1}{2}} \left( \frac{1}{\nu^{1/4} \tau^{2}} + n^{2(1-\gamma)} (|n|^{1/4} + |n|^{1-\gamma}) e^{n^{\gamma/4} \frac{1}{\nu^{2}}} \right) \|f_\nu\|_{L^2_{X,Y}}
\]

\[
+ C\nu^{\frac{1}{2}} |n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \|f_\nu\|_{L^2_{X,Y}}
\]

\[
\leq C((1-\gamma)|n|^{3(1-\gamma)} + 1) \nu^{\frac{1}{2}} |n|^{1-\gamma} \|f_\nu\|_{L^2_{X,Y}} + C|n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \|f_\nu\|_{L^2_{X,Y}}
\]

\[
+ C((1-\gamma) \log(n) + 1)|n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \|f_\nu\|_{L^2_{X,Y}}.
\]

Hence, by scaling back to the original variables and applying (3.25), we obtain that for any \(|n|^{\gamma} \nu^{\frac{1}{2}} \leq 1\)

\[
\|u^{(n)}\|_{L^2_{X,Y}} \leq C((1-\gamma)|n|^{3(1-\gamma)} + 1) \nu^{\frac{1}{2}} |n|^{1-\gamma} \|f\|_{L^2_{X,Y}} + C|n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \|f\|_{L^2_{X,Y}}
\]

\[
+ C\nu^{-\frac{1}{2}} ((1-\gamma) \log(n) + 1)|n|^{1-\gamma} e^{n^{\gamma} \frac{1}{28}} \|f\|_{L^2_{X,Y}}.
\]

This completes the proof. \(\square\)

4. Nonlinear stability in Gevrey class

In this section, we prove the nonlinear stability. By the Duhamel formula, the solution \(u(t)\) of the system (1.6) with initial data \(a\) is given by

\[
(4.1) \quad u(t) = e^{-t \nu a} - \int_0^t e^{-t \nu (t-s)} \mathbb{P}(u \cdot \nabla u) ds, \quad t > 0.
\]
Proof of Theorem 1.1. Let \( Y \) and the space \((4.2)\) denote nonlinear estimates using the results in Proposition (3.1) and (3.2).

Before we start our proof, we recall some function spaces which we shall work on. For \( \gamma \in [0, 1] \), \( d \geq 0 \) and \( K > 0 \), the Banach space \( X_{d, \gamma, K} \) is given by
\[ X_{d, \gamma, K} = \{ f \in L^2_\sigma(\Omega) \| f \|_{X_{d, \gamma, K}} = \sup_{n \in \mathbb{Z}} \| 1 + |n|^d e^{K|n|\gamma} \| P_n f \|_{L^2(\Omega)} < \infty \} , \]
and the space \( Y_{d, \gamma, K} \) is defined as
\[ Y_{d, \gamma, K} = \{ f \in L^2_\sigma(\Omega) \| f \|_{Y_{d, \gamma, K}} = \sup_{n \in \mathbb{Z}} \| 1 + |n|^d e^{K|n|\gamma} \| P_n f \|_{L^2 L^\infty(\Omega)} < \infty \} . \]

**Proof of Theorem 1.1.** Let \( \gamma \in \left[ \frac{2}{3}, 1 \right) \) and \( u(t) \) be the solution to the system (1.6) with initial data \( a \). Set
\[ q := d - \frac{7}{4} (1 - \gamma) \in (1, d), \quad K(t) := K - 2\delta^{-1} t. \]

We would like to establish an a priori estimate of \( u(t) \) in the space
\[ Z_{\gamma, K, T'} := \{ f \in C([0, T']; L^2_\sigma(\Omega)) \| f \|_{Z_{\gamma, K, T'}} := \sup_{0 < t < T'} \| f(t) \|_{X_{q, \gamma, K(t)}} + \nu^{\frac{1}{2}} \| f(t) \|_{Y_{q, \gamma, K(t)}} + (\nu t)^{\frac{1}{2}} \| \nabla f(t) \|_{X_{q, \gamma, K(t)}} \} < +\infty \].

For each \( n \in \mathbb{Z} \), we have already known that
\[ (4.5) \quad P_n u(t) = e^{-t\delta u \cdot n} \| P_n a - \int_0^t e^{-t\delta u \cdot n} \| P_n (u \cdot \nabla u)(s) ds. \]

We first show a basic estimate for nonlinear term \( P_n (u \cdot \nabla u) \), which will be used frequently later. We notice that
\[ \| P_n (u \cdot \nabla u) \|_{L^2(\Omega)} \leq \| P_n (u_1 \partial_x u) \|_{L^2(\Omega)} + \| P_n (u_2 \partial_y u) \|_{L^2(\Omega)} \]
\[ \leq \| \sum_{j \in \mathbb{Z}} (e^{-ij \cdot x} P_j u_1) \cdot (e^{-i(n-j)x} \partial_x P_{n-j} u) \|_{L^2_\sigma(\mathbb{R}^+)} \]
\[ + \| \sum_{j \in \mathbb{Z}} (e^{-ij \cdot x} P_j u_2) \cdot (e^{-i(n-j)x} \partial_x P_{n-j} u) \|_{L^2_\sigma(\mathbb{R}^+)} \]
From Gagliardo-Nirenberg inequality and divergence free condition, we obtain
\[ \| \sum_{j \in \mathbb{Z}} (e^{-ij \cdot x} P_j u_1) \cdot (e^{-i(n-j)x} \partial_x P_{n-j} u) \|_{L^2_\sigma(\mathbb{R}^+)} \]
\[ \leq C \sum_{j \in \mathbb{Z}} \| P_j u \|_{L^2(\Omega)} \| \nabla P_j u \|_{L^2(\Omega)} \| n - j \cdot \frac{2}{3} \| P_{n-j} u \|_{L^2(\Omega)} \| \nabla P_{n-j} u \|_{L^2(\Omega)} \]
and
\[ \| \sum_{j \in \mathbb{Z}} (e^{-ij \cdot x} P_j u_2) \cdot (e^{-i(n-j)x} \partial_y P_{n-j} u) \|_{L^2_\sigma(\mathbb{R}^+)} \leq C \sum_{j \in \mathbb{Z}} \| P_j u \|_{L^2(\Omega)} \| \nabla P_{n-j} u \|_{L^2(\Omega)} \]
Therefore, for \( u \in Z_{\gamma,K,T'} \), we have that for any \( 0 < t \leq T' \),

\[
\| \mathcal{P}_n u(t) \|_{L^2(\Omega)} \leq C(1 + |n|^{2(1-\gamma)}) e^{\frac{|n|^\gamma}{3}} \| \mathcal{P}_n a \|_{L^2(\Omega)}
\]

\[+ C(1 + |n|^{2(1-\gamma)}) \int_0^t e^{\frac{|n|^\gamma}{3}(s)} \| \mathcal{P}_n u(s) \|_{L^2(\Omega)} ds \]

\[\leq C(1 + |n|^{2(1-\gamma)}) e^{\frac{|n|^\gamma}{3}} \| \mathcal{P}_n u \|_{L^2(\Omega)}
\]

\[+ C(1 + |n|^{2(1-\gamma)}) \int_0^t e^{\frac{|n|^\gamma}{3}(s)} \| u(s) \|_{L^2(\Omega)} ds \]

On the other hand, we notice that

\[
\int_0^t e^{\frac{|n|^\gamma}{3}(s)} e^{-K(s)n|\gamma|} s^{-\frac{1}{2}} ds = e^{-(K-\delta-1)t|n|\gamma} \int_0^t e^{\frac{|n|^\gamma}{3}s} s^{-\frac{1}{2}} ds
\]

\[\leq C e^{-(K(t)|n|\gamma)} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, t^\frac{1}{2}\}.\]

Therefore, we obtain

\[
\| \mathcal{P}_n u(t) \|_{L^2(\Omega)} \leq C e^{-2K(t)|n|\gamma} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, t^\frac{1}{2}\} \| u \|_{Z_{\gamma,K,T'}}^2.
\]

Hence if \( T' < \frac{\delta K}{2} \), then for \( \beta_0 = \frac{3(1-\gamma)}{2\gamma} \),

\[
\sup_{0 < t \leq T'} \sup_{|n|^{\frac{1}{2}} < 1} (1 + |n|^q) e^{K(t)|n|^\gamma} \| \mathcal{P}_n u(t) \|_{L^2(\Omega)}
\]

\[\leq C \left( \| a \|_{X_{d,\gamma,K}} + \nu^{-\frac{1}{2}} \| u \|_{Z_{\gamma,K,T'}}^2 \right) \sup_{|n|^{\frac{1}{2}} < 1} (1 + |n|^{\frac{1}{2}+2(1-\gamma)} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, T^\frac{1}{2}\})
\]

\[\leq C \left( \| a \|_{X_{d,\gamma,K}} + \nu^{-\frac{1}{2}} - \beta_0 \| u \|_{Z_{\gamma,K,T'}}^2 \right) \sup_{|n|^{\frac{1}{2}} < 1} (1 + |n|^{\frac{1}{2}+2(1-\gamma)-2\beta_0} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, T^\frac{1}{2}\})
\]

\[\leq C \left( \| a \|_{X_{d,\gamma,K}} + \nu^{-\frac{1}{2}} - \beta_0 \| u \|_{Z_{\gamma,K,T'}}^2 \right) \sup_{|n|^{\frac{1}{2}} < 1} (1 + |n|^{\frac{1}{2}-(1-\gamma)} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, T^\frac{1}{2}\}).
\]

We also notice that for the case \( T^\frac{1}{2} \leq \delta^\frac{1}{2} |n|^{-\gamma} \),

\[
\sup_{|n|^{\frac{1}{2}} < 1} (1 + |n|^{\frac{1}{2}-(1-\gamma)} \min\{\delta^\frac{1}{2} |n|^{-\gamma}, T^\frac{1}{2}\}) \leq (1 + |n|^{\frac{1}{2}}) T^\frac{1}{2} \leq C(1 + T^\frac{1}{2} - 1) T^\frac{1}{2} \leq CT^\frac{1}{2} - \frac{1}{4}.
\]
and for $T'^{1/4} \geq \delta^{1/2}|n|^{-\gamma/2}$,
\[
\sup_{|n|^{\nu/2} \leq 1} (1 + |n|)^{1/2-(1-\gamma)} \min \{ \delta^{1/2}|n|^{-\gamma/2}, T'^{1/2} \} \leq C \delta^{1/2} (|n|^{-\gamma/2(1-\gamma)} + |n|^{-\gamma/2}) \leq C (T'^{1/2} - \frac{1}{2} + T'^{1/2}) \leq CT'^{1/4} - \frac{1}{2}.
\]

Finally, we obtain that for $T' < \frac{5K}{2}$,
\[
\sup_{0 \leq t \leq T'} \sup_{|n|^{\nu/2} \leq 1} (1 + |n|^\eta) e^{K(t)|n|^\gamma} \| P_n u(t) \|_{L^2(\Omega)} \leq C \left( \| a \|_{X_{d,\nu, K}} + \nu^{-\frac{1}{2}} \| u \|_{Z_{\nu, K, T'}} \right)^{1/2} (1 - \gamma) \log_T (\| n \| + 1) |n|^{\frac{3}{2} - 2\gamma} e^{\frac{|n|^\gamma}{2}} \int_0^t e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds,
\]
which along with (4.6) implies
\[
\| P_n u(t) \|_{L^2 L^\infty_y(\Omega)} \leq C \| P_n a \|_{L^2 L^\infty_y(\Omega)} + C \nu^{-\frac{1}{2}} t^{\frac{3}{4}} |n|^{3 - 2\gamma} e^{\frac{|n|^\gamma}{2}} \| P_n u(t) \|_{L^2(\Omega)} + C (1 - \gamma) |n|^{3/4} e^{\frac{|n|^\gamma}{2}} \int_0^t (t - s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds + C \nu^{-\frac{1}{2}} (1 + |n|^\eta) e^{\frac{|n|^\gamma}{2}} \int_0^t e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds.
\]

On the other hand, we notice that
\[
\int_0^t (t - s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq C e^{-K(t)|n|^\gamma} \min \left\{ \frac{1}{|n|^{\frac{1}{2} \gamma} t^{\frac{1}{2}}}, t^{\frac{3}{4}} \right\},
\]
and
\[
\int_0^t (t - s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq C e^{-K(t)|n|^\gamma} \min \left\{ \frac{1}{|n|^{\frac{1}{2} \gamma} t^{\frac{1}{2}}}, t^{\frac{3}{4}} \right\}.
\]

Collecting (4.7) and the above two inequalities, we obtain
\[
\| P_n u(t) \|_{L^2 L^\infty_y(\Omega)} \leq C \| P_n a \|_{L^2 L^\infty_y(\Omega)} + C \nu^{-\frac{1}{2}} t^{\frac{3}{4}} |n|^{3 - 2\gamma} e^{\frac{|n|^\gamma}{2}} \| P_n a \|_{L^2(\Omega)} + C (1 - \gamma) |n|^{3/4} e^{\frac{|n|^\gamma}{2}} \int_0^t (t - s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq C \left( \| a \|_{X_{d,\nu, K}} + \nu^{-\frac{1}{2}} \| u \|_{Z_{\nu, K, T'}} \right)^{1/2} (1 - \gamma) \log_T (\| n \| + 1) |n|^{\frac{3}{2} - 2\gamma} e^{\frac{|n|^\gamma}{2}} \int_0^t e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds.
\]
Hence if $T' < \min\{\frac{\delta K}{3}, 1\}$ and $t \leq 1$, then

\[
\begin{align*}
&\sup_{0 < t \leq T'} \sup_{|n| \frac{1}{2} < 1} \nu^{\frac{1}{2}} (1 + |n|^\frac{1}{2}) e^{-K(t)|n|^\gamma} \|P_n u(t)\|_{L^2_x L^\infty_y (\Omega)} \\
&\leq C \nu^{\frac{1}{2}} \|a\|_{Y_{\gamma,K}} C \|a\|_{X_{\gamma,K}} + C \nu^{\frac{1}{2}} (1 + |n|^\frac{1}{2}) e^{-K(t)|n|^\gamma} \|u\|_{Z_{\gamma,K,T'}}^2.
\end{align*}
\]

For $I$, we notice that for $\beta_1 = \frac{(1-\gamma)}{\gamma} + \frac{1}{8\gamma}$ and $T' \leq 1$

\[
I \leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 \sup_{0 < t \leq T'} \sup_{|n| \frac{1}{2} < 1} (1 + |n|)^{\frac{1}{2} + \frac{3}{4}(1-\gamma) - 2\gamma \beta_1} \min\left\{\frac{1}{|n|^\gamma t^\frac{1}{2}}, t^\frac{1}{2}\right\}
\]

\[
\leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 \sup_{0 < t \leq T'} \sup_{|n| \frac{1}{2} < 1} \min\left\{\frac{1}{|n|^\gamma t^\frac{1}{2}}, t^\frac{1}{2}\right\}.
\]

On the other hand, we notice that for any fixed $0 \leq t \leq T'$, if $t \leq |n|^\frac{-2}{2\gamma}$, then for any $n \neq 0$ \(min\{(|n|^\frac{1}{2}\gamma t^\frac{1}{2})^{-1}, t^\frac{1}{2}\} \leq t^\frac{1}{4}\), and if $t \geq |n|^\frac{-2}{2\gamma}$, then for any $n \neq 0$, \(min\{(|n|^\frac{1}{2}\gamma t^\frac{1}{2})^{-1}, t^\frac{1}{2}\} \leq |n|^\frac{-2}{2\gamma} t^{-\frac{1}{2}} \leq t^\frac{1}{4}\), which implies \(\sup_{|n| \frac{1}{2} < 1} \min\{(|n|^\frac{1}{2}\gamma t^\frac{1}{2})^{-1}, t^\frac{1}{2}\} \leq T'\frac{2}{4}\). Therefore, we obtain

\[
I \leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 T'\frac{2}{4}.
\]

For $II$, in a similar way, we can obtain

\[
II \leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 \sup_{0 < t \leq T'} \sup_{|n| \frac{1}{2} < 1} (1 + |n|)^{\frac{1}{2} + (1-\gamma) - 2\gamma \beta_1} \min\left\{\frac{1}{|n|^\gamma t^\frac{3}{2}}, t^\frac{3}{2}\right\}
\]

\[
\leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 T'\frac{3}{2} \left(1 + (1-\gamma) \log (1 + T')\right).
\]

Again by a similar argument, we obtain the following bound of $III$

\[
III \leq \nu^{-\frac{1}{2} - \beta_1} \|u\|_{Z_{\gamma,K,T'}}^2 T'\frac{3}{2} \left(1 + (1-\gamma) \log (1 + T')\right).
\]
By (4.10), (4.11) and (4.12), we finally obtain
\[
\sup_{0 < t \leq T'}\sup_{\|u\|_{\gamma,K} < 1} \nu^{\frac{1}{2}}(1 + |n|^q)e^{K(t)|n|^\gamma}\|P_n u(t)\|_{L^2_x L^\infty_T(\Omega)}
\leq C\nu^{\frac{1}{2}}\|a\|_{Y_{\gamma,K}} + C\|a\|_{X_{\gamma,K}} \\
+ C\nu^{-\frac{1}{2} - \beta_0}\|u\|_{Z_{\gamma,K,T}}^2(T^{\frac{22}{45}} - \frac{19}{34} + T^{\frac{1}{47}} - \frac{1}{4}(1 + (1 - \gamma)\log(1 + T'))) 
\] (4.13)
Now we prove the derivative estimates in this case. From Proposition 3.1, we have
\[
\|\nabla P_n u(t)\|_{L^2(\Omega)} \leq \frac{C}{\nu^{\frac{1}{2}}} \left( t^{-1/2} + |n|^{1/2 + \frac{q}{2}(1-\gamma)}e^{\frac{|n|^\gamma \nu t}{s}} \right)\|P_n a\|_{L^2} \\
+ \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \left( (t-s)^{-1/2} + |n|^{1/2 + \frac{q}{2}(1-\gamma)}e^{\frac{|n|^\gamma \nu (t-s)}{s}} \right)\|P_n (u \cdot \nabla u)(s)\|_{L^2(\Omega)} ds,
\]
which combined with (4.6) and the fact \(e^{\frac{|n|^\gamma \nu t}{s}} \leq C \left( \frac{|n|^\gamma}{s} \right)^{\frac{q}{2}} e^{\frac{|n|^\gamma \nu t}{s}} \) implies
\[
\|\nabla P_n u(t)\|_{L^2(\Omega)} \leq \frac{C}{\nu(t)^{\frac{1}{2}}} \left( 1 + |n|^{\frac{q}{2}(1-\gamma)} \right)\|P_n a\|_{L^2(\Omega)} \\
+ \frac{C}{\nu(1 + |n|^q)^{\frac{1}{2}}} \|u\|_{Z_{\gamma,K,T}}^2 \int_0^t \left( (t-s)^{-1/2} + |n|^{1/2 + \frac{q}{2}(1-\gamma)}e^{\frac{|n|^\gamma \nu (t-s)}{s}} \right) e^{-K(s)|n|^\gamma} \frac{ds}{s^{\frac{1}{2}}},
\] (4.14)
On the other hand, we notice that
\[
\int_0^t (t-s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} \frac{ds}{s^{\frac{1}{2}}} \leq C e^{-K(t)|n|^\gamma} \min \{ (|n|^\gamma t)^{-\frac{1}{2}}, t^{\frac{1}{2}} \},
\]
and
\[
\int_0^t e^{\frac{|n|^\gamma}{2}(t-s)} e^{-K(s)|n|^\gamma} \frac{ds}{s^{\frac{1}{2}}} \leq C e^{-K(t)|n|^\gamma} \min \{ |n|^\gamma t^{-\frac{1}{2}}, t^{\frac{1}{2}} \}.
\]
Hence, (4.14) becomes for \(T' < \frac{\Delta K}{2}\)
\[
(\nu t)^{\frac{1}{2}}\|\nabla P_n u(t)\|_{L^2(\Omega)} \leq \frac{C e^{-K(t)|n|^\gamma}}{(1 + |n|)^d - \frac{7}{4}(1-\gamma)} \|a\|_{X_{\gamma,K}} + C(1 + |n|)^{\frac{1}{2} - \frac{q}{2}(1-\gamma)} e^{-K(t)|n|^\gamma} \min \{ (|n|^\gamma t)^{-\frac{1}{2}}, t^{\frac{1}{2}} \} \|u\|_{Z_{\gamma,K,T}}^2 \\
+ \frac{C(1 + |n|)^{1 + \frac{q}{2}(1-\gamma)} e^{-K(t)|n|^\gamma}}{\nu^{\frac{1}{2}}(1 + |n|^q)} \min \{ |n|^{-\gamma}, t \} \|u\|_{Z_{\gamma,K,T}}^2.
\]
Recall that \(q = d - \frac{7}{4}(1-\gamma)\) and \(\beta_0 = \frac{3(1-\gamma)}{27}\). Then we obtain
\[
\sup_{0 < t \leq T'} \sup_{|n|^\gamma \nu^{\frac{1}{2}} < 1} (1 + |n|^q)e^{K(t)|n|^\gamma}(\nu t)^{\frac{1}{2}}\|\nabla P_n u(t)\|_{L^2(\Omega)} \\
\leq C\|a\|_{X_{\gamma,K}} + C\nu^{-\frac{1}{2} - \beta_0}\|u\|_{Z_{\gamma,K,T}}^2 \sup_{|n|^\gamma \nu^{\frac{1}{2}} < 1} (1 + |n|)^{\frac{1}{2} - 3(1-\gamma)} \min \{ |n|^{-\frac{1}{2}}, T' \} \\
+ C\nu^{-\frac{1}{2} - \beta_0}\|u\|_{Z_{\gamma,K,T}}^2 \sup_{|n|^\gamma \nu^{\frac{1}{2}} < 1} (1 + |n|)^{1 - \frac{7}{4}(1-\gamma)} \min \{ |n|^{-\gamma}, T' \} \\
\leq C\|a\|_{X_{\gamma,K}} + C\nu^{-\frac{1}{2} - \beta_0}\|u\|_{Z_{\gamma,K,T}}^2 \left( T'^{\frac{15}{87}(1-\gamma)} + T^{\frac{3}{47}(1-\gamma)} \right).
\] (4.15)
Case 2. $1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}$. The argument is similar to Case 1. According to Propositions 3.1 and 4.6, we have that by a similar argument as in Case 1 and taking $\beta = \frac{3(1-\gamma)}{2\gamma}$,

$$\sup_{0 < t \leq T'} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} (1 + |n|^q)e^{K(t)|n|^\gamma} \|\mathcal{P}_nu(t)\|_{L^2(\Omega)}$$

$$\leq C\|a\|_{x_{d,\gamma,K}} + C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{2} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} (1 + |n|)^{\frac{3}{2}-\gamma}\min\{ |n|^{-\frac{3}{2}}, T^{\frac{3}{2}} \}$$

$$\leq C\|a\|_{x_{d,\gamma,K}} + C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{2} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} (1 + |n|)^{\frac{3}{2}(1-\gamma)}$$

$$\leq C\|a\|_{x_{d,\gamma,K}} + C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{2} = C\|a\|_{x_{d,\gamma,K}} + C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{2}.$$

For the $L^\infty$-estimate, we have that by Proposition 3.2,

$$\|\mathcal{P}_nu(t)\|_{L^2 L^\infty(\Omega)} \leq C\|\mathcal{P}_a\|_{L^2 L^\infty(\Omega)} + C\nu^{-\frac{1}{4}+\frac{1}{2}}|n|^{2-\gamma} e^{\frac{|n|^\gamma}{\nu}\min\{ |n|^{-\frac{3}{2}}, T^{\frac{3}{2}} \}}$$

$$\sup_{0 < t \leq T'} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} (1 + |n|^q)e^{K(t)|n|^\gamma} \|\mathcal{P}_nu(t)\|_{L^2 L^\infty(\Omega)}$$

Therefore, for $T' \leq \frac{\delta K}{2} < 1$ and $\beta_2 = \frac{3}{16} + \frac{3(1-\gamma)}{2\gamma}$, we have

$$\sup_{0 < t \leq T'} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} \nu^{\frac{3}{4}}(1 + |n|^q)e^{K(t)|n|^\gamma} \|\mathcal{P}_nu(t)\|_{L^2 L^\infty(\Omega)}$$

$$\leq C\nu^{\frac{3}{4}}\|a\|_{Y_{d,\gamma,K}} + C\|a\|_{x_{d,\gamma,K}}$$

$$+ C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{\frac{3}{4}+\frac{1}{2}} \sup_{0 < t \leq T'} \sup_{1 \leq |n|^\gamma \nu^{\frac{1}{2}} \leq \delta_0^{-}\gamma \nu^{-\frac{3}{4}+\frac{1}{2}}} (1 + |n|)^{\frac{3}{4}+\frac{1}{2}(1-\gamma)} \min\{ |n|^{-\frac{3}{2}}, T^{\frac{3}{2}} \}$$

$$\leq C\nu^{\frac{3}{4}}\|a\|_{Y_{d,\gamma,K}} + C\|a\|_{x_{d,\gamma,K}} + C\|a\|_{x_{d,\gamma,K}} + C\|a\|_{x_{d,\gamma,K}} + C\nu^{-\frac{1}{2}}\|u\|_{Z_{\gamma,K,T'}}^{\frac{3}{4}+\frac{1}{2}(1-\gamma)} \min\{ |n|^{-\frac{3}{2}}, T^{\frac{3}{2}} \}$$

Here we used $\frac{3}{4}(\frac{7}{28} + \frac{22}{28}(1-\gamma)) \leq \frac{3}{16} + \frac{15}{16}(1-\gamma)$.
For the derivative estimate, again by Proposition 3.1 and (4.6), we obtain that for $\beta_3 = \frac{2(1-\gamma)}{\gamma}$

$$\sup_{0 < t \leq T'} \sup_{1 \leq |n| |\nu|^2 \leq \delta_0^{-\gamma} \nu^{-\frac{3}{2} + \frac{1}{2}} \frac{1}{2}} (ut)^{\frac{1}{2}} (1 + |n|^q) e^{K(t)|n|^\gamma} \|P_n u(t)\|_{L^2(\Omega)}$$

$$\leq C \|a\|_{X_{\delta_0, \gamma, K}} + C \nu^{-\frac{1}{2}} \|u\|_{Z_{\gamma, K, T'}} \sup_{1 \leq |n| |\nu|^2 \leq \delta_0^{-\gamma} \nu^{-\frac{3}{2} + \frac{1}{2}} \frac{1}{2}} (1 + |n|)^{\frac{1}{2}} \min \{ |n|^{-\frac{1}{2}}, T' \frac{1}{2} \}$$

$$+ C \nu^{-\frac{1}{2}} \|u\|_{Z_{\gamma, K, T'}} \sup_{1 \leq |n| |\nu|^2 \leq \delta_0^{-\gamma} \nu^{-\frac{3}{2} + \frac{1}{2}} \frac{1}{2}} (1 + |n|)^{1 + \frac{1}{2}} (1 - \gamma) \min \{ |n|^{-\gamma}, T' \}$$

$$\leq C \|a\|_{X_{\delta_0, \gamma, K}} + C \nu^{-\frac{1}{2}} \|u\|_{Z_{\gamma, K, T'}} \sup_{1 \leq |n| |\nu|^2 \leq \delta_0^{-\gamma} \nu^{-\frac{3}{2} + \frac{1}{2}} \frac{1}{2}} (1 + |n|)^{\frac{1}{2}} \min \{ |n|^{-\frac{1}{2}}, T' \}$$

(4.18)

$$\leq C \|a\|_{X_{\delta_0, \gamma, K}} + C \nu^{-\frac{1}{2}} \|u\|_{Z_{\gamma, K, T'}}$$

Case 3. $|n| > \delta_0^{-\gamma} \nu^{-\frac{3}{2}}$. By Proposition 3.1 and (4.6), we first have

$$\|P_n u(t)\|_{L^2(\Omega)} \leq C \|P_n a\|_{L^2(\Omega)} + \frac{C (1 + |n|)^{\frac{1}{2}}}{\nu^{\frac{1}{2}} (1 + |n|^q)} \int_0^t e^{-\frac{1}{2} |n|^2 (t-s)} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \|u\|_{Z_{\gamma, K, T'}}.$$  

We claim that for all $|n| > \delta_0^{-\gamma} \nu^{-\frac{3}{2}}$

$$I_\nu := (1 + |n|)^{\frac{1}{2}} \int_0^t e^{-\frac{1}{2} |n|^2 (t-s)} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq \frac{Ce^{-K(t)|n|^\gamma}}{\nu^{\frac{1}{2}} (1 - \gamma)}.$$  

In fact, when $\delta_0^{-\gamma} \nu^{-\frac{3}{2}} \leq |n| \leq \nu^{-1}$, we have

$$I_\nu \leq C |n|^{\frac{1}{2}} \int_0^t e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq C |n|^{\frac{1}{2}} e^{-K(t)|n|^\gamma} |n|^{-\frac{1}{2}} \leq C \nu^{-\frac{1}{2}} (1 - \gamma) e^{-K(t)|n|^\gamma}$$

and when $|n| \geq \nu^{-1}$, we have

$$I_\nu \leq C \int_0^t \nu^{-\frac{3}{2}} (t-s)^{-\frac{1}{2}} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \leq C \nu^{-\frac{1}{2}} |n|^{-\frac{1}{2}} e^{-K(t)|n|^\gamma} \leq C \nu^{-\frac{1}{2}} (1 - \gamma) e^{-K(t)|n|^\gamma}$$

which implies (4.19). Then we obtain that for $\beta_4 = \frac{1}{\gamma}$

$$\sup_{0 < t \leq T'} \sup_{|n| |\nu|^2 > \delta_0^{-1} \nu^{-\frac{3}{2}}} (1 + |n|^q) e^{K(t)|n|^\gamma} \|P_n u(t)\|_{L^2(\Omega)}$$

$$\leq C \|a\|_{X_{\delta_0, \gamma, K}} + C \nu^{-\frac{1}{2}} (1 - \gamma) \|u\|_{Z_{\gamma, K, T'}}$$

$$\leq C \|a\|_{X_{\delta_0, \gamma, K}} + C \nu^{-\frac{1}{2}} \beta_4 \nu^{\frac{1}{2}} (1 - \gamma) \|u\|_{Z_{\gamma, K, T'}}.$$  

For the $L^\infty_{t} L^\infty_{x}$-estimate, we notice that from Proposition 3.2 and (4.6),

$$\|P_n u(t)\|_{L^\infty_{t} L^\infty_{x}(\Omega)} \leq C \|P_n a\|_{L^\infty_{t} L^\infty_{x}(\Omega)} + C |n| \|P_n a\|_{L^2(\Omega)}$$

(4.21)

$$+ \frac{C (1 + |n|)^{\frac{1}{2}}}{\nu^{\frac{1}{2}} (1 + |n|^q)} \int_0^t \frac{1 + |n|^{\frac{1}{2}} (t-s)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} e^{-\frac{1}{2} |n|^2 (t-s)} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \|u\|_{Z_{\gamma, K, T'}}.$$  

On the other hand, we notice that for all $|n| > \delta_0^{-1} \nu^{-\frac{3}{2}}$

$$I_{II} := (1 + |n|)^{\frac{1}{2}} \int_0^t \frac{1 + |n|^{\frac{1}{2}} (t-s)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} e^{-\frac{1}{2} |n|^2 (t-s)} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds$$
\[
\leq C \int_0^t \left( \frac{1}{\nu^2(t-s)^{\frac{3}{2}}} + \frac{1}{\nu^2(t-s)^{\frac{3}{2}}} \right) e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds
\]
\[
\leq C \nu^{-\frac{3}{4}} \min \{ |n|^{-\frac{3}{2}} t^{-\frac{3}{2}}, t^2 \} + \nu^{-\frac{3}{4}} \min \{ |n|^{-\frac{3}{2}} t^{-\frac{3}{2}}, t^2 \}
\]
\[
\leq C \nu^{-\frac{3}{4}} n^{-\frac{3}{2}} + C \nu^{-\frac{3}{4}} |n|^{\frac{9}{2}} \leq C \nu^{-\frac{3}{4} + \frac{9}{2}},
\]
which implies that for \( T' \leq \frac{6K}{\nu^2} < 1 \) and \( \beta_5 = \frac{1}{4} + \frac{1}{2\gamma} \)
\[
\sup_{0 < t \leq T'} \sup_{|n| > \delta_0^{-1} \nu^{-\frac{3}{4}}} \nu^\frac{1}{4} (1 + |n|^q) e^{K(t)|n|^\gamma} \| \mathcal{P}_n u(t) \|_{L_2^2 L_\infty^\gamma(\Omega)}
\]
\[
\leq C \nu^\frac{1}{4} \| a \|_{Y_{d,\gamma,K}} + C \| a \|_{X_{d,\gamma,K}} + C \nu^{-\frac{1}{4} - \frac{1}{2} - \frac{3}{4}} \| u \|_{Z_{\gamma,K,T'}}^2
\]
\[
\leq C \nu^\frac{1}{4} \| a \|_{Y_{d,\gamma,K}} + C \| a \|_{X_{d,\gamma,K}} + C \nu^{-\frac{1}{4} - \beta_5 \nu^\frac{1}{4} (1 - \gamma)} \| u \|_{Z_{\gamma,K,T'}}^2.
\]
By a similar argument as in (4.19), we obtain
\[
(1 + |n|)^3 \int_0^t \left( \frac{1}{(t-s)^{\frac{3}{2}}} + |n|(t-s)^{\frac{3}{2}} \right) e^{-\nu n^2(t-s)} e^{-K(s)|n|^\gamma} s^{-\frac{1}{2}} ds \| u \|_{Z_{\gamma,K,T'}}^2
\]
\[
\leq \frac{C e^{-K(t)|n|^\gamma}}{\nu^\frac{1}{4} (1 - \gamma)} t^\frac{3}{2},
\]
which implies that for \( \beta_0 = \frac{3(1-\gamma)}{2\gamma} \)
\[
\nu^\frac{1}{4} (1 + |n|^q) e^{K(t)|n|^\gamma} \| \mathcal{P}_n u(t) \|_{L_2^2 L_\infty^\gamma(\Omega)}
\]
\[
\leq C \frac{1 + |n| t^{\frac{3}{4} q}}{1 + |n|^q} e^{2|n|^q q} \nu^\frac{1}{4} n^2 t \| a \|_{X_{d,\gamma,K}} + C \nu^{-\frac{3}{4} - \frac{3}{2} (1 - \gamma)} \| u \|_{Z_{\gamma,K,T'}}^2
\]
\[
\leq C \frac{1 + |n|^{1-\gamma}}{1 + |n|^q} \| a \|_{X_{d,\gamma,K}} + C \nu^{-\frac{3}{4} - \beta_0 \nu^\frac{1}{4} (1 - \gamma)} \| u \|_{Z_{\gamma,K,T'}}^2.
\]
Since \( d - q - 1 + \gamma > 0 \) due to the choice of \( q \), we obtain
\[
\sup_{0 < t \leq T'} \sup_{|n| > \delta_0^{-1} \nu^{-\frac{3}{4}}} \nu^\frac{1}{4} (1 + |n|^q) e^{K(t)|n|^\gamma} \| \mathcal{P}_n u(t) \|_{L_2^2(\Omega)}
\]
\[
\leq C \frac{1 + |n|^{1-\gamma}}{1 + |n|^q} \| a \|_{X_{d,\gamma,K}} + C \nu^{-\frac{3}{4} - \beta_0 \nu^\frac{1}{4} (1 - \gamma)} \| u \|_{Z_{\gamma,K,T'}}^2.
\]
Now we denote
\[
\beta := \max \{ \beta_0, \ldots, \beta_4 \} = \max \left\{ \frac{2(1-\gamma)}{\gamma}, \frac{3}{16} + \frac{3(1-\gamma)}{2\gamma}, \frac{1}{4} + \frac{1-\gamma}{2\gamma} \right\}
\]
and
\[
\kappa(T', \nu) := \max \left\{ T'^{\frac{1}{4} (1-\gamma)} (1 + (1 - \gamma) \log(1 + T'))^{\nu^\frac{1}{4}}, \nu^\frac{1}{4} \right\}.
\]
Now we take (4.26)

\[ T \]

Therefore, we obtain that for small enough \( \gamma \in \left[ \frac{1}{2}, 1 \right] \) and \( T' < \frac{\delta K}{3} < 1, \]

\[ (4.26) \quad \| u \|_{Z_{\gamma, K, T'}} \leq C \left( \| a \|_{X_{d, \gamma, K}} + \nu^\frac{1}{2} \| a \|_{Y_{d, \gamma, K}} + \kappa(T', \nu) \nu^{-\frac{1}{2} - \beta} \| u \|_{Z_{\gamma, K, T'}}^2 \right). \]

Now we take \( T' \) and \( \nu \) small enough, then we can close the above estimates as long as

\[ \| u \|_{Z_{\gamma, K, T'}} \leq C \nu^{\frac{3}{2} + \beta}, \]

which is consistent with the condition \( \| a \|_{X_{d, \gamma, K}} + \nu^\frac{1}{2} \| a \|_{Y_{d, \gamma, K}} \leq \nu^{\frac{3}{2} + \beta}. \)

Therefore, we obtain that for small enough \( T' \) and \( \nu \)

\[ (4.27) \quad \| u \|_{Z_{\gamma, K, T'}} \leq C \left( \| a \|_{X_{d, \gamma, K}} + \nu^\frac{1}{2} \| a \|_{Y_{d, \gamma, K}} \right). \]

Here \( T' \) is taken uniformly with respect to \( \nu \). The proof is completed. \( \Box \)

**Appendix A. Hardy’s Type Inequality**

**Lemma A.1.** If \( \alpha \geq M_2, V'' \leq 0, V' \geq 0, f(Y_{\lambda}) = 0 \) for \( Y_{\lambda} \in [0, +\infty], \) and \( |V'(Y)| \leq C|V(Y)| \) for some constant \( C, \) then it holds that

\[
\left\| \frac{(\partial_Y V)f}{V - V(Y_{\lambda})} \right\|_{L^2} \leq C \left\| (\partial_Y f, f) \right\|_{L^2}.
\]

Specially, taking \( Y_{\lambda} = 0, \) if \( f(0) = 0, \) then

\[
\left\| \frac{(\partial_Y V)f}{V} \right\|_{L^2} \leq C \left\| (\partial_Y f, f) \right\|_{L^2}.
\]

Taking \( Y_{\lambda} = +\infty, \) if \( f(\infty) = 0, \) then

\[
\left\| \frac{(\partial_Y V)f}{V - 1} \right\|_{L^2} \leq C \left\| (\partial_Y f, f) \right\|_{L^2}.
\]

Here the constant \( C \) may depend on \( M_2. \)

**Proof.** Let \( I_1 = [Y_{\lambda}, +\infty), I_2 = [0, +\infty) \setminus I_1. \) If \( Y \in I_1, \) i.e, \( Y \geq Y_{\lambda}, \) then

\[
|V - V(Y_{\lambda})| = \int_{Y_{\lambda}}^{Y} (\partial_Y V)(Z) dZ \geq \int_{Y_{\lambda}}^{Y} (\partial_Y V)(Y) dZ = |Y_{\lambda} - Y| \partial_Y V(Y),
\]

\[
\Rightarrow \left| \frac{\partial_Y V}{V - V(Y_{\lambda})} \right| \leq (Y - Y_{\lambda})^{-1}, \quad Y \in I_1.
\]

Then by Hardy’s inequality, we get

\[
(A.1) \quad \left\| \frac{(\partial_Y V)f}{V - V(Y_{\lambda})} \right\|_{L^2(I_1)} \leq \left\| \frac{f}{Y - Y_{\lambda}} \right\|_{L^2(I_1)} \leq 2 \left\| \partial_Y f \right\|_{L^2(I_1)}.
\]

And for \( Y \in I_2, \) i.e \( Y \leq Y_{\lambda}, \) we have

\[
|f(Y)V'(Y)|^2 = \left| \int_{Y}^{Y_{\lambda}} (f'(Z)V'(Z) + f(Z)V''(Z)) dZ \right|^2
\]

\[
\leq C \left( \int_{Y}^{Y_{\lambda}} |(f'(Z), f(Z))||V''(Z)| dZ \right)^2
\]

\[
\leq C \int_{Y}^{Y_{\lambda}} V'(Z)(V(Y_{\lambda}) - V(Z))^{-\frac{1}{2}} dZ \times \int_{Y}^{Y_{\lambda}} V'(Z)|f'(Z), f(Z)|^2 (V(Y_{\lambda}) - V(Z))^\frac{1}{2} dZ
\]
\[ = C \int_Y V'(Z)(V(Y_\lambda) - V(Y))^\frac{1}{2}(V(Y_\lambda) - V(Z))^\frac{1}{2} \left| (f'(Z), f(Z)) \right|^2 dZ, \]

and then
\[
\left\| \frac{\partial Y V}{V - V(Y_\lambda)} \right\|_{L^2(I_2)}^2 \leq C \int_Y \frac{V'(Y)\left| f'(Y) \right|^2}{(V(Y) - V(Y_\lambda))^\frac{1}{2}} dY
\]

\[
\leq C \int_Y \int_Y \frac{V'(Z)(V(Y_\lambda) - V(Y))^\frac{1}{2}(V(Y_\lambda) - V(Z))^\frac{1}{2} \left| (f'(Z), f(Z)) \right|^2 dZ dY}{(V(Y) - V(Y_\lambda))^\frac{1}{2}}
\]

\[
= C \int_Y \int_Y \frac{V'(Z)(V(Y_\lambda) - V(Y))^\frac{1}{2}(V(Y_\lambda) - V(Z))^\frac{1}{2} \left| (f'(Z), f(Z)) \right|^2 dZ dY}{(V(Y_\lambda) - V(Y))^\frac{1}{2}}
\]

\[
= C \int_Y \int_Y \frac{V'(Z)(V(Y_\lambda) - V(Y))^\frac{1}{2}(V(Y_\lambda) - V(Z))^\frac{1}{2} \left| (f'(Z), f(Z)) \right|^2 dZ dY}{(V(Y) - V(Y_\lambda))^\frac{1}{2}}
\]

here we used the Fubini's identity. Since
\[
\int_Y \frac{V'(Y)}{(V(Y_\lambda) - V(Y))^\frac{1}{2}} dY \leq 2(V(Y_\lambda) - V(Z))^\frac{1}{2},
\]

we conclude
\[
\left\| \frac{\partial Y V}{V - V(Y_\lambda)} \right\|_{L^2(I_2)}^2 \leq C \int_Y \left| (f'(Z), f(Z)) \right|^2 dZ = C\| (\partial Y f, f) \|_{L^2(I_2)}^2,
\]

hence,
\[
\left\| \frac{\partial Y V}{V - V(Y_\lambda)} \right\|_{L^2}^2 = \left\| \frac{\partial Y V}{V - V(Y_\lambda)} \right\|_{L^2(I_1)}^2 + \left\| \frac{\partial Y V}{V - V(Y_\lambda)} \right\|_{L^2(I_2)}^2 \leq C\| (\partial Y f, f) \|_{L^2}^2.
\]

\[ \Box \]

**Lemma A.2.** If \( w \in L^2(\mathbb{R}_+) \), and \( \phi \in H^2 \) satisfies
\[
(\partial_Y^2 - \alpha^2)\phi = w,
\]

then it holds that
\[
- \partial_Y \phi(Y) = \int_Y^{+\infty} w(Z)e^{-\alpha(Z-Y)} dZ.
\]

Specially, we have
\[
- \partial_Y \phi(0) = \int_0^{+\infty} w(Y)e^{-\alpha Y} dY.
\]

**Proof.** Direct calculation gives
\[
\int_Y^{+\infty} w(Z)e^{-\alpha Y} dZ = \int_Y^{+\infty} ((\partial_Z^2 - \alpha^2)\phi(Z))e^{-\alpha Y} dZ
\]

\[
= \int_Y \phi((\partial_Z^2 - \alpha^2)e^{-\alpha Y})dZ + ((\partial_Z\phi)e^{-\alpha Y})|_{Y}^{+\infty} - \partial_Y \phi(Y)e^{-\alpha Y}.
\]
Lemma A.3. There exists a positive constant $C > 0$, such that $\forall z \in \mathbb{C}$, $t > 0$, it holds

$$
\int_0^t |z - s|^{1/2} ds \geq C^{-1}|z|^{1/2}t.
$$

Proof. Let $z_r = \text{Re}(z)$, $z_i = \text{Im}(z)$. Let us first claim that

(A.2) $$
\int_0^t |z_r - s|^{1/2} ds \geq C^{-1}|z_r|^{1/2}t.
$$

Once (A.2) holds, we have

$$
\int_0^t |z - s|^{1/2} ds \geq C^{-1} (\int_0^t |z_r - s|^{1/2} ds + \int_0^t |z_i|^{1/2} ds) \geq C^{-1} (|z_r|^{1/2}t + |z_i|^{1/2}t) \geq C^{-1}|z|^{1/2}t.
$$

It remains to prove (A.2).

Case 1. $z_r \leq 0$. In this case, we have

$$
\int_0^t |z_r - s|^{1/2} ds \geq \int_0^t |z_r|^{1/2} ds = |z_r|^{1/2}t.
$$

Case 2. $0 \leq z_r \leq t/2$. In this case, we have

$$
\int_0^t |z_r - s|^{1/2} ds \geq \int_{t/2}^t (s - z_r)^{1/2} ds = \frac{2(t - t/2)^{3/2}}{3} = \frac{2(t/2)^{3/2}}{3} \geq \frac{z_r^{3/2} t}{3} = \frac{|z_r|^{1/2} t}{3}.
$$

Case 3. $z_r \geq t/2$. In this case, we have

$$
\int_0^t |z_r - s|^{1/2} ds \geq \int_0^{t/4} |z_r - s|^{1/2} ds = \int_0^{t/4} |z_r - s|^{1/2} ds \geq \int_0^{t/4} (z_r - t/4)^{1/2} ds
$$

$$
= \frac{(z_r - t/4)^{1/2} t}{4} \geq \frac{|z_r/2|^{3/2} t}{4},
$$

here we used $z_r - t/4 \geq z_r/2 = |z_r|/2$.

Combining three cases, we conclude our result. \hfill \Box

APPENDIX B. SOME ESTIMATES OF AIRY FUNCTION

Let $Ai(y)$ be the Airy function, which is a nontrivial solution of $f'' - yf = 0$. We denote

$$
A_0(z) = \int_{e^{i\pi/6}z}^{\infty} Ai(t) dt = e^{i\pi/6} \int_z^{\infty} Ai(e^{i\pi/6}t) dt.
$$

The following lemma comes from [7].

Lemma B.1. There exists $c > 0$ and $\delta_0 > 0$ so that for $\text{Im}(z) \leq \delta_0$,

(B.1) $$
|\frac{A'_0(z)}{A_0(z)}| \leq 1 + |z|^{1/2}, \quad \text{Re}\frac{A'_0(z)}{A_0(z)} \leq \min(-1/3, -c(1 + |z|^{1/2})).
$$

Moreover, for $\text{Im}z \leq \delta_0$, we have

$$
|\frac{A'_0(z)}{A_0(z)}| \leq C(1 + |z|).
$$
We denote \( \tilde{A}(Y) := Ai(e^{i\frac{\pi}{4}}k(Y + \eta))/Ai(e^{i\frac{\pi}{4}}k\eta) \) with \( \kappa > 0 \) and \( \text{Im}\eta < 0 \), and define \( \tilde{\Phi}(Y) \) as the solution of \( (\partial^2_t - \alpha^2)\tilde{\Phi} = \tilde{A} \) and \( \tilde{\Phi}(0) = 0 \).

Lemma B.2. Let \( \kappa > 0 \) and \( \text{Im}\eta < 0 \). Then we have

\[
|\tilde{A}(Y)| \leq Ce^{-c\kappa(1+|\kappa\eta|^{\frac{1}{2}})} , \quad ||\tilde{A}||_{L^2} \leq C\kappa^{-\frac{1}{2}}(1 + |\kappa\eta|)^{-\frac{1}{4}}, \\
||Y \tilde{A}||_{L^2} \leq C\kappa^{-\frac{1}{2}}(1 + |\kappa\eta|)^{-\frac{1}{4}}, \quad ||Y^2 \tilde{A}||_{L^2} \leq C\kappa^{-\frac{5}{2}}(1 + |\kappa\eta|)^{-\frac{5}{4}}, \\
||(|\partial Y\tilde{\Phi}, \alpha\tilde{\Phi})||_{L^2} \leq C\kappa^{-\frac{\alpha}{2}}(1 + |\kappa\eta|)^{-\frac{\alpha}{4}}.
\]

Moreover, there holds

\[
|\partial Y\tilde{\Phi}(Y)| \leq C\kappa^{-1}(1 + |\kappa\eta|)^{-\frac{1}{4}}e^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})}, \\
||\tilde{\Phi}||_{L^2} + ||Y\partial Y\tilde{\Phi}||_{L^2} \leq C\kappa^{-\frac{5}{2}}(1 + |\kappa\eta|)^{-\frac{5}{4}}, \\
||Y\tilde{\Phi}||_{L^2} + ||Y^2\partial Y\tilde{\Phi}||_{L^2} \leq C\kappa^{-\frac{7}{2}}(1 + |\kappa\eta|)^{-\frac{7}{4}}, \\
||Y^2\tilde{\Phi}||_{L^2} + ||Y^3\partial Y\tilde{\Phi}||_{L^2} \leq C\kappa^{-\frac{9}{2}}(1 + |\kappa\eta|)^{-\frac{9}{4}}.
\]

Proof. By Lemma B.1 and Lemma A.3 we have

\[
\left| \frac{A_0(t + B)}{A_0(B)} \right| = \left| \exp \left( \ln(A_0(t + B)) - \ln(A_0(B)) \right) \right| = \exp \left( \int_0^t \frac{A_0'(s + B) - A_0'(s + B)}{A_0(s + B)} \right) \leq \exp \left( - \int_0^t \max(1/3, c(1 + |s + B|^{\frac{1}{2}})) ds \right),
\]

which along with Lemma A.3 implies

(B.2) \[
\left| \frac{A_0(t + B)}{A_0(B)} \right| \leq \exp \left( - \max \left( t/3, c(1 + |B|^{\frac{1}{2}}) \right) \right).
\]

Thanks to \( \text{Re} \frac{A_0'(z)}{A_0(z)} \leq \min(-1/3, -c(1 + |z|^{\frac{1}{2}})) < 0 \), \( \text{Re} \frac{A_0'(z)}{A_0(z)} \geq c(1 + |z|^{\frac{1}{2}}) \) and

(B.3) \[
\left| \frac{A_0(z)}{A_0'(z)} \right| = \left| \frac{A_0'(z)}{A_0(z)} \right|^{-1} \leq \left| \text{Re} \frac{A_0'(z)}{A_0(z)} \right|^{-1} \leq c^{-1}(1 + |z|^{\frac{1}{2}})^{-1}.
\]

Now we are ready to show the estimates about \( \tilde{A}(Y) \). Lemma B.1 gives

\[
|\tilde{A}(Y)| = \left| \frac{A_0'(\kappa(Y + \eta))}{A_0'(\kappa\eta)} \right| \leq \left| \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} \right| \left| \frac{A_0'(\kappa(Y + \eta))}{A_0'(\kappa\eta)} \right| \left| \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} \right| \\
\leq C(1 + |\kappa\eta|)^{-\frac{1}{4}}(1 + |\kappa\eta| + kY)^{\frac{1}{2}}e^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})} \\
\leq Ce^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})}.
\]

Then we find that

\[
||\tilde{A}||_{L^2} \leq C||e^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})}||_{L^2} \leq C\kappa^{-\frac{1}{2}}(1 + |\kappa\eta|)^{-\frac{1}{4}}, \\
||Y \tilde{A}||_{L^2} \leq C||Ye^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})}||_{L^2} \leq C\kappa^{-\frac{3}{2}}(1 + |\kappa\eta|)^{-\frac{3}{4}}, \\
||Y^2 \tilde{A}||_{L^2} \leq C||Y^2e^{-c\kappa Y(1+|\kappa\eta|^{\frac{1}{2}})}||_{L^2} \leq C\kappa^{-\frac{5}{2}}(1 + |\kappa\eta|)^{-\frac{5}{4}}.
\]
Now we turn to deal with $\tilde{\Phi}(Y)$. By duality argument and Hardy’s inequality, we obtain
\[
\|(\partial_Y \tilde{\Phi}, \alpha \tilde{\Phi})\|_{L^2} = |\langle \tilde{A}, \tilde{\Phi}\rangle| \leq \|Y \tilde{A}\|_{L^2} \|\tilde{\Phi}/Y\|_{L^2} \leq 2\|Y \tilde{A}\|_{L^2} \|\partial_Y \tilde{\Phi}\|_{L^2},
\]
which gives
\[
\|(\partial_Y \tilde{\Phi}, \alpha \tilde{\Phi})\|_{L^2} \leq 2\|Y \tilde{A}\|_{L^2} \leq C\kappa^{-\frac{3}{2}}(1 + |\kappa\eta|)^{-\frac{1}{4}}.
\]
By Lemma A.2 and $|\tilde{A}(Y)| \leq C e^{-c\kappa(1 + |\kappa\eta|^{\frac{1}{2}})Y}$, we have
\[
|\alpha Y \partial_Y \tilde{\Phi}(Y)| = \left|\int_{Y}^{+\infty} \tilde{A}(Z)e^{-\alpha Z} dZ\right| \leq \int_{Y}^{+\infty} C e^{-c\kappa(1 + |\kappa\eta|^{\frac{1}{2}})Y} e^{-\alpha Z} dZ
\]
\[
\leq C (c\kappa(1 + |\kappa\eta|^{\frac{1}{2}}) + \alpha)^{-1} e^{-c\kappa(1 + |\kappa\eta|^{\frac{1}{2}})Y - \alpha Y}
\]
\[
\leq C\kappa^{-1}(1 + |\kappa\eta|^{\frac{1}{2}})^{-1} e^{-c\kappa(1 + |\kappa\eta|^{\frac{1}{2}})Y - \alpha Y}.
\]
Therefore, we obtain
\[
|\partial_Y \tilde{\Phi}(Y)| \leq C\kappa^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} e^{-c(1 + |\kappa\eta|^{\frac{1}{2}})Y},
\]
which implies
\[
\|Y \partial_Y \tilde{\Phi}\|_{L^2} \leq C\kappa^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} \|Y e^{-c(1 + |\kappa\eta|^{\frac{1}{2}})Y}\|_{L^2} \leq C\kappa^{-\frac{3}{2}}(1 + |\kappa\eta|)^{-\frac{1}{2}},
\]
\[
\|Y^2 \partial_Y \tilde{\Phi}\|_{L^2} \leq C\kappa^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} \|Y^2 e^{-c(1 + |\kappa\eta|^{\frac{1}{2}})Y}\|_{L^2} \leq C\kappa^{-\frac{7}{2}}(1 + |\kappa\eta|)^{-\frac{1}{2}},
\]
\[
\|Y^3 \partial_Y \tilde{\Phi}\|_{L^2} \leq C\kappa^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} \|Y^3 e^{-c(1 + |\kappa\eta|^{\frac{1}{2}})Y}\|_{L^2} \leq C\kappa^{-\frac{9}{2}}(1 + |\kappa\eta|)^{-\frac{2}{4}}.
\]
For $\beta = \{0, 1, 2\}$, we have
\[
(2\beta + 1)\|Y^\beta \tilde{\Phi}\|_{L^2}^2 = \langle |\tilde{\Phi}|^2, Y Y^{2\beta+1}\rangle = -\langle \partial_Y (|\tilde{\Phi}|^2), Y Y^{2\beta+1}\rangle
\]
\[
= -2\text{Re} \langle Y^\beta \tilde{\Phi}, Y^{2\beta+1} \partial_Y \tilde{\Phi}\rangle \leq 2\|Y^\beta \tilde{\Phi}\|_{L^2} \|Y^{2\beta+1} \partial_Y \tilde{\Phi}\|_{L^2}.
\]
Then we obtain
\[
\|Y^\beta \tilde{\Phi}\|_{L^2} \leq \frac{2}{2\beta + 1} \|Y^{2\beta+1} \partial_Y \tilde{\Phi}\|_{L^2} \leq C\kappa^{-\frac{5+2\beta}{2}}(1 + |\kappa\eta|)^{-\frac{5+2\beta}{4}}.
\]

\textbf{Lemma B.3.} \textit{Let $\kappa > 0$ and $\text{Im}\eta < 0$. Then it holds that}
\[
|\partial_Y \Phi(0)| \geq C^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}}(\kappa + 3\alpha)^{-1}.
\]

\textit{Proof.} By Lemma A.2 we have
\[
-\partial_Y \tilde{\Phi}(0) = \int_{0}^{+\infty} \tilde{A}(Y)e^{-\alpha Y} dY = \int_{0}^{+\infty} \frac{A_i(e^{\frac{i\pi}{2}}\kappa(Y + \eta))}{A_i(e^{\frac{i\pi}{2}}\kappa\eta)} e^{-\alpha Y} dY
\]
\[
= \int_{0}^{+\infty} \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} e^{-\alpha Y} dY
\]
\[
= -\frac{A_0(\kappa\eta)}{\kappa A_0(\kappa\eta)} - \int_{0}^{+\infty} \frac{A_0(\kappa(Y + \eta))}{\kappa A_0(\kappa\eta)} \partial_Y (e^{-\alpha Y}) dY,
\]
which along with Lemma B.2 gives
\[
\kappa |A_0(\kappa\eta)||\partial_Y \tilde{\Phi}(0)| \geq |A_0(\kappa\eta)| - \int_{0}^{+\infty} |A_0(\kappa(Y + \eta))||\partial_Y (e^{-\alpha Y})| dY
\]
\[ \geq |A_0(\kappa \eta)| - \int_0^{+\infty} e^{-\kappa Y/3} |A_0(\kappa \eta)| |\partial_Y (e^{-\alpha Y})| dY \\
= |A_0(\kappa \eta)| + |A_0(\kappa \eta)| \int_0^{+\infty} e^{-\kappa Y/3} \partial_Y (e^{-\alpha Y}) dY \\
= \frac{\kappa}{3} |A_0(\kappa \eta)| \int_0^{+\infty} e^{-\kappa Y/3 - \alpha Y} dY \\
= |A_0(\kappa \eta)| \frac{\kappa}{\kappa + 3 \alpha}, \]

which along with Lemma B.1 gives

\[ |\partial_Y \Phi(0)| \geq \frac{|A_0(\kappa \eta)|}{(\kappa + 3 \alpha)|A'_0(\kappa \eta)|} \geq C^{-1}(1 + |\kappa \eta|)^{-\frac{1}{2}}(\kappa + 3 \alpha)^{-1}. \]

□

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