Traffic distributions and independence: permutation invariant random matrices and the three notions of independence

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ABSTRACT:

Voiculescu’s notion of asymptotic free independence is known for a large class of random matrices including independent unitary invariant matrices. This notion is extended for independent random matrices invariant in law by conjugation by permutation matrices. This fact leads naturally to an extension of free probability, formalized under the notions of traffic probability. We first establish this construction for random matrices. We define the traffic distribution of random matrices, which is richer than the $*$-distribution of free probability. The knowledge of the individual traffic distributions of independent permutation invariant families of matrices is sufficient to compute the limiting distribution of the join family. Under a factorization assumption, we call traffic independence the asymptotic rule that plays the role of independence with respect to traffic distributions. Wigner matrices, Haar unitary matrices and uniform permutation matrices converge in traffic distributions, a fact which yields new results on the limiting $*$-distributions of several matrices we can construct from them. Then we define the abstract traffic spaces as non commutative probability spaces with more structure. We prove that at an algebraic level, traffic independence in some sense unifies the three canonical notions of tensor, free and Boolean independence. A central limiting theorem is stated in this context, interpolating between the tensor, free and Boolean central limit theorems.

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Introduction

Presentation of the article

Motivated by the study of von Neumann algebras of free groups, Voiculescu introduces \[Voi85\] free probability theory as a non commutative probability theory equipped with the well known notion of free independence. Voiculescu shows \[Voi91\] that free independence describes the global asymptotic behavior of eigenvalues of independent unitarily invariant matrices and Wigner matrices. Asymptotic free independence holds for a large class of matrices, see for instance \[HP00, CC04, SSB05, Shl98, BG09\].

Recall that in free probability theory, a non commutative probability spaces is a unital (non commutative) algebra \(A\) equipped with a linear form \(\Phi\) playing the role of the expectation and called the sate, satisfying mild assumptions (Definition 4.1). This context is considered together with the notion of free independence which replaces the classical notion of tensor independence. It is a canonical rule that, given two probability spaces \(pA_1, \Phi_1 \) and \(qA_2, \Phi_2\), associate a state \(\Phi = \Phi_1 \cdot \Phi_2\) on the free product algebra \(A_1 \ast A_2\) whose restriction on \(A_i\) is \(\Phi_i\) for \(i = 1, 2\). Speicher proves in \[Spe97\] that tensor independence and free independence are the only universal notions of independence satisfying an associativity and universal calculation rules, and that in the non unital case the third notion of Boolean independence appears.

In the first part of this article we introduce a general method to study random matrices which are not invariant by conjugation by unitary matrices: we start by considering independent families \(A_N^{(1)}, \ldots, A_N^{(L)}\) of random matrices which are invariant by conjugation by permutation matrices (Definition 0.1). We consider the problem of characterizing the limiting joint \(*\)-distribution of the collection of all matrices \(A_N^{(1)} \cup \cdots \cup A_N^{(L)}\) (the \(*\)-distribution is the expectation of normalized trace of polynomials in the matrices, see Section 1.1) when the size of the matrices goes to infinity. If the families are not unitarily invariant, in general the knowledge of the limiting \(*\)-distribution of each \(A_N^{(i)}\) is not sufficient to characterize the limiting \(*\)-distribution of \(A_N^{(1)} \cup \cdots \cup A_N^{(L)}\).

This question is studied in \[Rya98\] when the matrices are independent and have independent and identically distributed entries. In particular if the matrices have size \(N \times N\) and their entries are Bernouilli random variables with parameter \(\frac{1}{2}\), \(p\) fixed, then the matrices are not asymptotically free. In another direction, an interpolation of the classical and free convolutions is introduced and studied in \[BGL11\]: given two distributions \(\mu\) and \(\nu\) on the real line, the distribution \(\mu^t \ast \nu, t \geq 0\) is the limit of the empirical spectral distribution of matrices \(A_N + U_N^t B_N U_N^t\) where \(U_N^t\) is a diffusion on the unitary group starting from a uniform permutation matrix, and \(A_N, B_N\) are diagonal matrices that converge in moments to \(\mu\) and \(\nu\) respectively.

The strategy therein is to consider a more general setting that free probability theory, where we need to enrich the notion of \(*\)-distribution, or equivalently to consider more observables on matrices. For a family \(A_N\) of matrices, we define the traffic distribution as a linear functional on finite connected directed graphs whose edges are labelled by matrices (Definition 1.2). It contains more information than the data of the normalized trace of monomials of \(A_N\). Hence the convergence in traffic distribution of a family of random matrices implies its convergence in \(*\)-distribution, but it is possible that two random matrices have the same \(*\)-distribution but not the same traffic distribution. Equivalently, the convergence in traffic distribution of \(A_N\) is the convergence in \(*\)-distribution of the matrices of the so-called traffic space generated by \(A_N\) (Lemma 1.6).

In the main result Theorem 1.8 we prove an analogue of Voiculescu’s theorem, by giving
a formula for the limiting traffic distribution of independent permutation invariant families of matrices $A_N^{(1)} \cup \cdots \cup A_N^{(L)}$, knowing the limiting traffic distribution of each one. Our method extends the moment method in a similar fashion as in the first chapter of [Gui09] and applies for a large class of random matrices. In Theorem 1.8, the limiting traffic distribution of $A_N^{(1)} \cup \cdots \cup A_N^{(L)}$ is called the product of the limiting traffic distributions of the $A_N^{(i)}$'s. We will also say that the random matrices $A_N^{(1)} \cup \cdots \cup A_N^{(L)}$ are asymptotically traffic independent.

With Theorem 1.8 and several propositions stated in the article, we illustrate new ways to test the asymptotically freeness of independent random matrices. We consider Wigner, unitary Haar and uniform permutation matrices, as well as matrices obtained from several operations on them (transpose, entry-wise product). We also point out new examples of matrices that are not asymptotically free independent. Other models are considered in [Mal, CM, MD].

In the second part of the article, the notion of non commutative probability space is enhanced to consider the limits in traffic distribution of random matrices. The notion of operads (see e.g. [May97]) formalizes the idea that a given set of operations with good compatibility conditions defines an algebraic structure. We define the operad of graph operations, which is a slight modification of Jones’ planar algebras [Jon, Example 2.6] and Spivak’s wiring diagram algebras [Spi13]. An algebraic traffic space is an algebra over the operad of graph operations endowed with a linear form satisfying mild assumptions, and elements of a traffic space are simply called traffics. In particular, it is a non commutative probability space. Roughly speaking, traffics have the property that we can compose them not only by linear operations, but thanks to schemes given by a finite connected graph with an input and an output.

The interest in defining traffic spaces is that we can consider traffic independence in a universal way and state the usual limit theorems of probability. Traffic independence is a notion more general that the notions of independence in non commutative probability. Since the $*$-distribution is just a part of the traffic distribution, traffic independence is proved to actually encode both the tensor and the free independence of $*$-distributions, and also a large class of relations. Moreover, the additional structure of observables implies the existence of another linear form than the usual trace. In a certain setting, with respect to this second linear map, traffic independence encodes the notion of Boolean independence.

This connection with Boolean independence is actually motivated by the limit theorems. Let $(a_n)_{n \geq 1}$ be a family of independent identically distributed self-adjoint traffics. We prove a law of large number for the sum $\frac{a_1 + \cdots + a_n}{\sqrt{n}}$, and assuming the variables are centered in a certain sense, we prove the convergence in traffic distribution of $\frac{a_1 + \cdots + a_n}{\sqrt{n}}$ and interpret the limit. We prove that $\frac{a_1 + \cdots + a_n}{\sqrt{n}}$ converges in $*$-distribution to the sum $x + z$ of a semicircular variable $x$ free independent from a Gaussian variable $z$. The degree of freedom between the classical and the free worlds is possible thanks to the wealth of information contained in the traffic distribution.

Moreover, the limiting traffic distribution of $\frac{x + y + z}{\sqrt{3}}$ is actually the distribution of the sum of three variables $x + y + z$. The variable $y$ has variance zero, which explains why it does not appear in the convergence in $*$-distribution. But it has a nonzero traffic distribution, and can actually be interpreted as the limit in the central limit theorem for Boolean independence. Remarkably, the variables $x, y$ and $z$ in the central limit theorem are not traffic independent in general. We give a matrix model $(X_N, Y_N, X_N)$ which converges in traffic distribution to $(x, y, z)$.

In parallel to the evolution of this paper, Gabriel gives in [Gab, Gabb, Gabc] another answer to Theorem 1.8 based on a different point of view. His approach involves the pertition algebras instead of the test graphs defined in this article. Gabriel’s theory is essentially equivalent to the one introduced in this article. A dictionary [GDCM] between the two approaches is in preparation.

The organization of the article is the following. In Chapter 1 we define the traffic distribution of large random matrices and present our main result (Theorem 1.8) and its applications. In Chapter 2 we define traffic independence and prove there a simple criterion to prove that random matrices are not asymptotically free independent. We then prove the main result, Theorem 1.8, in Section 2.3. Chapter 3 is dedicated to applications of this theorem for Wigner matrices, uniform permutation matrices and unitary Haar matrices.
In Chapter[4] we introduce the general traffic spaces. In Chapter[5] we relate traffic independence with the three universal notions of independence. The law of large numbers and the central limit theorem for traffic independence are stated in the last Chapter.

Notations and preliminaries
While considering a matrix $A_N$, we implicitly mean a sequence $(A_N)_{N \geq 1}$, the matrix $A_N$ being of size $N$. We study large matrices and the term "asymptotic" refers to the limit when $N$ goes to infinity. We consider random matrices $A_N$ whose entries admit moments of all orders, that is $\forall K \geq 1, \forall n, m = 1, \ldots, N, \mathbb{E}[|A_N(n, m)|^K] < \infty$. We denote by $I_N$ the identity matrix and for $A_N$ a complex matrix we denote by $A_N^*$ its conjugate transpose. We often consider families of matrices, denoted by bold characters e.g. $A_N = (A_j)_{j \in J}$. For a family $A_N$ of matrices, we denote by $A_N^+$ the family of their complex transpose.

**Definition 0.1 (Unitary random matrices and invariances).**

1. A unitary matrix $U_N$ is a matrix such that $U_N U_N^* = U_N^* U_N = I_N$. A **unitary Haar matrix** $U_N$ is a random unitary matrix distributed according to the Haar distribution on the unitary group, that is the unique probability measure on the unitary group invariant by right and left multiplication of elements of the group.

2. A permutation matrix $V_N$ matrix of size $N$ is a unitary matrix for which there is a permutation $\sigma$ of $\{1, \ldots, N\}$ such that the entry $(i, j)$ of $V_N$ is one if $i = \sigma(j)$ and zero otherwise. A **uniform permutation matrix** is a random permutation matrix uniformly chosen among all the $N!$ choices.

3. A family of random matrices $A_N = (A_j)_{j \in J}$ is said to be **unitarily invariant** whenever it is invariant in law by conjugation by unitary matrix, that is for any unitary matrix $U_N$

$$A_N \overset{\text{law}}{=} U_N A_N U_N^* := (U_N A_j U_N^*)_{j \in J}.$$  

Equivalently, $A_N$ has the same law as $U_N A_N U_N^*$, where $U_N$ is a unitary Haar matrix independent of $A_N$.

4. A family of random matrices is said to be **permutation invariant** whenever it is invariant in law by conjugation by any permutation matrix. Equivalently, $A_N$ has the same law as $V_N A_N V_N^*$ where $V_N$ is a uniform permutation matrix independent of $A_N$.

**Definition 0.2 (Wigner matrices).** A (centered) complex Wigner matrix is a Hermitian matrix $X_N = (\frac{x_{i,j}}{\sqrt{n}})$ whose sub-diagonal entries are independent and centered random variables such that:

1. the diagonal entries $(x_{i,i})_{i=1, \ldots, N}$, respectively the sub-diagonal entries $(x_{i,j})_{j<i}$, are identically distributed,

2. the distribution of $x_{i,j}$ does not depend on $N$, has finite moments of all orders $(\mathbb{E}[|x_{i,j}|^k] < \infty$ for any $k \geq 1$) and $\mathbb{E}[x_{i,j}] = 0$ for any $i, j = 1, \ldots, N$.

We call parameter of $X_N$ the common value $(\alpha, \beta)$ of $(\mathbb{E}[|x_{i,j}|^2], \mathbb{E}[x_{i,j}^2])$ for $i \neq j$. A real Wigner matrix is a complex matrix with real entries.

It is worth noting that a complex Wigner matrix is almost surely a real Wigner matrix if and only if $\alpha = \beta$.

**Lemma 0.3.** A unitary Haar, a uniform permutation matrix and a real Wigner matrix are permutation invariant. A complex Wigner matrix is permutation invariant if and only if the entries has the same distribution as their complex conjugate (i.e. $x_{i,j} = \overline{x_{i,j}}$).
Proof. Let $U_N$ be a unitary Haar matrix. For any permutation matrix $V_N$, the matrix $V_N U_N V_N^*$ has the same distribution as $U_N$ since $V_N$ is unitary and $U_N$ is Haar distributed.

Let $W_N = ((\mathbb{1}(i = \sigma_W(j)))_{i,j}$ be a uniform permutation matrix, associated to a uniform permutation $\sigma_W$ of $\{1, \ldots, N\}$. Then for any permutation matrix $V_N$ associated to a permutation $\sigma$, $V_N W_N V_N^*$ has the same distribution as $W_N$:

$$V_N W_N V_N^* = \left( \sum_{k,\ell} \mathbb{1}(i = \sigma(k), k = \sigma_W(\ell), j = \sigma(\ell)) \right)_{i,j}$$

$$= \left( \mathbb{1}(\sigma^{-1}(i) = \sigma_W(\sigma^{-1}(j))) \right)_{i,j}$$

$$= \left( \mathbb{1}(\sigma \circ \sigma_W \circ \sigma^{-1}(i) = j) \right)_{i,j} \overset{\text{Law}}{=} \left( \mathbb{1}(\sigma_W(i) = j) \right)_{i,j}. $$

Let $X_N$ be a Wigner matrix. Denote by $E_{i,j} = (\mathbb{1}(k = i, \ell = j))_{k,\ell=1,\ldots,N}$, the elementary matrix for each $i, j = 1, \ldots, N$. For any permutation matrix $V_N$ associated to a permutation $\sigma$, the Hermitian matrix $V_N X_N V_N^*$ has independent sub-diagonal entries:

$$V_N X_N V_N^* = \sum_{i,j} \left( X_N(\sigma^{-1}(i), \sigma^{-1}(j)) \right)_{i,j}$$

$$= \sum_i \sum_{i,j} \left( E_{\sigma(i),\sigma(j)}X_N(i,i) + \sum_{i,j} \left( E_{\sigma(i),\sigma(j)}X_N(i,j) + E_{\sigma(j),\sigma(i)}X_N(j,i) \right) \right)$$

The distribution of its diagonal entries is the distribution of those of $X_N$. The non-diagonal entry $(k,\ell)$ for $k < \ell$ is distributed as $X(\sigma(k),\sigma(\ell))$ if $\sigma(k) < \sigma(\ell)$ and as $X(\sigma(k),\sigma(\ell))$ if $\sigma(k) > \sigma(\ell)$. The invariance in distribution by complex conjugation of the non-diagonal entries of a Wigner matrix is then a necessary and sufficient condition for the permutation invariance of $X_N$. \hfill \Box
Part I

The asymptotic traffic distributions of random matrices
Chapter 1

Statement of the main theorem and applications

We first recall the classical theorem of asymptotic free independence for large random matrices. Then, in order to extend this result for permutation invariant matrices, we define the traffic distribution of large random matrices. In the last section we state our main result Theorem 1.8 and its applications.

1.1 Asymptotic free independence of large matrices

The random matrices under consideration are assumed to have entries with finite moments of all orders, that is
\[ \mathbb{E}[|A_{N}(i, j)|^{K}] < \infty \text{ for any } K \geq 1. \]

The (mean) empirical spectral distribution of a random matrix \( H_{N} \) is the probability measure
\[ \mathcal{L}_{H_{N}} : f \mapsto \mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^{N} f(\lambda_{i}) \right], \]
where \( \lambda_{1}, \ldots, \lambda_{N} \) are the eigenvalues of \( H_{N} \) and \( f : \mathbb{C} \to \mathbb{C} \) are integrable functions in \( \lambda \).

In random matrix and free probability, a very common notion is the \( \ast \)-distribution, which extends the notion of empirical spectral distribution for several matrices. Denote by \( \mathbb{C}(x, x^{*}) \) the space of \( \ast \)-polynomials, i.e. finite complex linear combinations of words in symbols \( x = (x_{j})_{j \in J} \) and \( x^{*} = (x_{j}^{*})_{j \in J} \). The \( \ast \)-distribution of a family of random matrices \( A_{N} \) is defined by
\[ \Phi_{A_{N}} : P \in \mathbb{C}(x, x^{*}) \mapsto \mathbb{E}\left[ \frac{1}{N} \text{Tr}(P(A_{N})) \right] \in \mathbb{C}, \]
where \( \text{Tr} \) is the usual trace of matrices. The family \( A_{N} \) converges in \( \ast \)-distribution whenever \( \Phi_{A_{N}} \) converges pointwise as \( N \) goes to infinity. The convergence in \( \ast \)-distribution of \( A_{N} \) is actually equivalent to the convergence in moments of the empirical spectral distribution of any Hermitian random matrix \( H_{N} = P(A_{N}) \).

Voiculescu’s asymptotic freeness theorem and its extensions [Vol91, Vol98, Dyk93, Col03, AGZ10] give a characterization of the limiting \( \ast \)-distribution of a collection of unitarily invariant independent families of matrices.

**Theorem 1.1 (Asymptotic free independence).**

Let \( A_{N}^{(1)}, \ldots, A_{N}^{(L)} \) be independent families of \( N \times N \) random matrices. Make the following hypotheses:

1. Each family, except possibly one, is unitarily invariant (Definition 0.1).
2. Each family converges in \( \ast \)-distribution, namely
\[ \Phi_{\ell}(P) := \lim_{N \to \infty} \mathbb{E}\left[ \frac{1}{N} \text{Tr}(P(A_{N}^{(\ell)})) \right] \]
exists for each \( \ell \in \{1, \ldots, L\} \) and any \( \ast \)-polynomial \( P \).
3. For each $\ell \in \{1, \ldots, L\}$, either the matrices of $A^{(\ell)}_N$ are uniformly bounded in operator norm and 
\[ \frac{1}{N} \text{Tr}[P(A^{(\ell)}_N)] \xrightarrow{N \to \infty} \Phi_\ell(P) \] almost surely for any $\ell$ and $P$, or $A^{(\ell)}_N$ is of the form $U_N A^{(\ell)}_N U_N^*$ where $U_N$ is a Haar unitary random matrix and $A^{(\ell)}_N$ is deterministic.

Then the families $A^{(1)}_N, \ldots, A^{(L)}_N$ are asymptotically free independent, namely:

1. There is a limiting joint $*$-distribution: denoting $A_N = A^{(1)}_N \cup \cdots \cup A^{(L)}_N$, for any $*$-polynomial $P$ the limit
\[ \Phi(P) := \lim_{N \to \infty} \frac{1}{N} \text{Tr}[P(A_N)] \] exists. \hfill (1.1)

2. There limiting joint $*$-distribution $\Phi$ is the free product of the limiting distributions $\Phi_\ell$: for any $n \geq 1$ and any indices $\ell_1, \ell_2, \ldots, \ell_n$ in $\{1, \ldots, L\}$ such that $\ell_j \neq \ell_{j+1}, \forall j = 1, \ldots, n - 1$, for any $*$-polynomials $P_1, P_2, \ldots$ where $P_j \in \mathbb{C}(x_{\ell_j}, x_{\ell_j}^*)$ satisfies $\Phi_\ell(P_j) = 0, \forall j \geq 1$, one has
\[ \Phi(P_1 \times \cdots \times P_n) = 0. \] \hfill (1.2)

Moreover, the conclusion of the theorem remains valid if a family consists in independent Wigner matrices.

Nica [Nic93] and Neagu [Nea05] proved that independent permutation matrices uniformly distributed are asymptotically free independent and asymptotically free from independent Wigner matrices. See [AGZ10, NS06] for more details.

1.2 Convergence in traffic distribution

Our motivation is to state an analogue of Theorem 1.1 where the independent families of matrices $A_1, \ldots, A_L$ are only assumed to be permutation invariant rather than unitarily invariant (Definition 0.1). For that task, we need more than the $*$-distribution of the families to compute their limiting joint $*$-distributions.

1.2.1 Definition

We introduce a generalization of $*$-polynomials in matrices. These operations are given by combinatorial graphs for which we now fix the notations. The considered graphs are directed and can have multiple edges in any directions and loops. Formally, a graph is a couple $(V, E)$, where $V$ is a non empty set, called the set of vertices, and $E$ is a multi-set (elements appear with a certain multiplicity) of ordered pairs of vertices $e = (v, w) \in V^2$. The set $E$ is possibly empty and is called the set of edges. We point out that graphs with no edges are allowed, but a graph must always have at least one vertex. A graph $(V, E)$ is finite if both $V$ and $E$ are finite. Graphs are considered up to isomorphisms. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are identified whenever there exists a bijection $\phi : V_1 \to V_2$ preserving the adjacency of vertices, the orientation of edges and their multiplicity.

Definition 1.2 (Graph polynomials and traffic distributions). Let $J$ be an index set and consider two families of formal variables $x = (x_j)_{j \in J}$ and $x^* = (x_j^*)_{j \in J}$, i.e. pairwise distinct symbols.

1. A $*$-test graph in the variables $x$ is a finite, connected, oriented graph whose edges are labelled by $x_j$ and $x_j^*$, $j \in J$. Formally, it consists in a quadruple $T = (V, E, \gamma, \varepsilon)$, where
   - (a) $(V, E)$ is a finite connected graph,
   - (b) $\gamma$ is a map $E \to J$,
   - (c) $\varepsilon$ is a map $E \to \{1, *\}$. 

1.2. CONVERGENCE IN TRAFFIC DISTRIBUTION

The maps $\gamma$ and $\varepsilon$ indicate that an edge $e \in E$ has the label $x^{(e)}_{\gamma(e)}$. For multiple edges, each edge has its own label. We simply call $T$ a test graph when $\varepsilon$ is constant to one and denote it $T = (V, E, \gamma)$.

2. A *-graph monomial is a *-test graph with two given vertices: an input and an output. It consists in a triplet $g = (T, \text{in}, \text{out})$ where $T$ is as above and $\text{in}, \text{out} \in V$ are possibly equal. We denote by $\mathcal{G}(x, x^*)$ the space of finite linear complex combinations of *-graph monomials in variable $x$ (graphs are considered up to isomorphisms of graphs preserving labeled and in/outputs).

3. Denote $[N] := \{1, \ldots, N\}$. For any *-graph monomial $g = (T, \text{in}, \text{out})$, $T = (V, E, \gamma, \varepsilon)$, in the variables $x = (x_j)_{j \in J}$, and for any family $A_N = (A_j)_{j \in J}$ of matrices, we define the matrix $g(A_N)$ whose $(i, j)$-entry is

$$g(A_N)(i, j) = \sum_{\phi:V \to [N]} \prod_{e = (v, w) \in E} A_{\gamma(e)}^{x_e}(\phi(v), \phi(w)). \quad (1.1)$$

This definition is extended for $g \in \mathcal{GC}(x, x^*)$ by linearity. Note that for the graph monomial consisting in a simple line $g = \left( \begin{smallmatrix} x_{\text{out}}^{(1)} \\ \vdots \\ x_{\text{out}}^{(K)} \\ x_{\text{in}}^{(1)} \\ \vdots \\ x_{\text{in}}^{(K)} \end{smallmatrix} \right)$, then $g(A_N)$ is the product of matrices $A_{\text{out}}^{(1)} \cdots A_{\text{out}}^{(K)}$.

4. The traffic space generated by $A_N$ is the sub-space of $M_N(\mathbb{C})$ consisting in all matrices $g(A_N)$ for $g \in \mathcal{GC}(x, x^*)$. The traffic distribution of $A_N$ is the map

$$\Phi_{A_N} : g \in \mathcal{GC}(x, x^*) \mapsto \mathbb{E} \left[ \frac{1}{N} \text{Tr} \left[ g(A_N) \right] \right] \in \mathbb{C}. $$

We say that a family $A_N$ of random matrices converges in traffic distribution (implicitly when $N \to \infty$) whenever $\lim_{N \to \infty} \Phi_{A_N}(g)$ exists for any *-graph polynomial $g$.

The term traffic refers to the matrices $A_N$ and is formalized abstractly in Section 4. Definition (1.1) is considered in [MS12]. The traffic distribution of family $A_N$ is denoted $\Phi_{A_N}$, to not be confused with the *-distribution $\Phi_{A_N}$.

**Example 1.3.** Consider $A_N = (A_j)_{j \in J}$ a family of matrices and let us describe $g(A_N)$ for the following *-graph monomials $g$ in variables $x = (x_j)_{j \in J}$.

1. **Identity:** Let $g$ be the graph monomial with two vertices $\text{in}$ and $\text{out}$ and one edge from $\text{in}$ to $\text{out}$ labelled $x_j$. Then $g(A_N) = A_j$. We denote in short $g = \left( \begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right)_{\text{in}}^{(j)}$. With the convention that the vertex $\text{out}$ is on the left and the vertex $\text{in}$ is on the right. This is the same convention as for a matrix entry $A_{k, \ell}$, where the indices $\ell$ and $k$ indicate the source and target respectively of the associated linear map $\mathbb{C}^N \to \mathbb{C}^N$.

2. **Constant:** For $\cdot$ the graph with a single vertex $\text{in} = \text{out}$ and no edge, one has that $(\cdot)(A_N) = 1_N$ the identity matrix.

3. **Product:** For $(\cdot, \cdot, \cdot)$ the graph with three vertices $\text{out}, v, \text{in}$, one edge from $\text{in}$ to $v$ labelled $x_j$ and another one from $v$ to $\text{out}$ labelled $x_j$, one has $(\cdot, \cdot, \cdot)(A_N) = A_j \times A_j$.

4. **Transpose:** For $(\cdot, \cdot, \cdot)$ the graph with two vertices $\text{in}$ and $\text{out}$ and one edge from $\text{out}$ to $\text{in}$ labelled $x_j$, $(\cdot, \cdot, \cdot)(A_N)$ is the transpose $A_j^T$ of $A_j$.

5. **Projection on the diagonal:** For $(\cdot^2)$ consisting in a single vertex $\text{in} = \text{out}$ and one loop labelled $x_j$, then $(\cdot^2)(A_N)$ is the diagonal matrix of diagonal elements of $A_N$. We denote it $\Delta(A_N)$. Note that $\Delta$ is a projection.

6. **Degree:** For $(\cdot^{(i)})$ consisting in two vertices $\text{in} = \text{out}$ and $v$ and an edge from $v$ to $\text{in} = \text{out}$ labelled $x_j$, then $(\cdot^{(i)})(A_N)$ is the diagonal matrix, that we denote $\text{deg}(A_j)$, whose $i$-th diagonal element is the sum of the entries of $A_j$ on its $i$-th row. It is also a projection.
7. **Entry-wise products**: For \((\cdot, x_j)\) with two vertices in and out, two edges from in to out, one labelled \(x_j\) and the other one labelled \(x_j'\), one has \((\cdot, x_j) (A_N) = A_j \circ A_j\), where \(\circ\) denotes the entry-wise product of matrices (also known as Hadamard or Schur product).

8. **Complex transpose**: By definition and the first item above, for \((\cdot, x_j)\) with two vertices in and out and one edge from in to out labelled \(x_j\) then \((\cdot, x_j) (A_N) = A_j^*\). Note also that for any \(*\)-graph monomial \(g\), we can write \((g(A_N))^* = g^*(A_N)\) where \(g^*\) is obtained by interchanging the input and the output of \(g\), reversing the orientation of its edges and reversing labels \(x_j\) and \(x_j^*\). We also denote \(g^\dagger\) obtained similarly without reversion of labels \(x_j\) and \(x_j^*\), which satisfies \((g(A_N))^* = g^\dagger(A_N^*)\).

| Identity | Transpose | Product |
|----------|-----------|---------|
| ![Identity Diagram](image) | ![Transpose Diagram](image) | ![Product Diagram](image) |

Figure 1.1: Example of graph monomials.

The following lemma tells that \(*\)-graph operations permute with the action by conjugation by permutation matrices, which is a crucial ingredient for our main Theorem \[1.8\].

**Lemma 1.4.** Let \(A_N = (A_j)_{j \in J}\) be a family of \(N\) \(N\) matrices. For any permutation matrix \(V_N\), denote \(V_N A_N V_N^* := (V_N A_j V_N^*)_{j \in J}\). Then, for any \(*\)-graph operation \(T\) and any permutation matrix \(V_N\), one has \(g(V_N A_N V_N^*) = V_N g(A_N) V_N^*\). In particular, \(A_N\) and \(V_N A_N V_N^*\) have the same traffic distribution.

**Proof.** If \(\sigma\) denotes the permutation associated to \(V_N\), then the entry \((n, m)\) of a matrix \(V_N A_j V_N^*\) is \(A_j(\sigma^{-1}(n), \sigma^{-1}(m))\). Hence, we obtain the results thanks to the change of variable \(\phi' = \sigma^{-1}\circ \phi\) below

\[
\left( g(V_N A_N V_N^*) \right)(i, j) = \sum_{\phi:V \to [N] \atop \phi(\text{in}) = j, \phi(\text{out}) = i} \prod_{e = (v, w) \in E} A^e(\gamma(e)(\sigma^{-1} \circ \phi(w), \sigma^{-1} \circ \phi(v)))
\]

\[
= \sum_{\phi':V \to [N] \atop \phi'(\text{in}) = \sigma^{-1}(j), \phi'(\text{out}) = \sigma^{-1}(i)} \prod_{e = (v, w) \in E} A^e(\gamma(e)(\phi'(w), \phi'(v)))
\]

\[
= \left( V_N g(A_N) V_N^* \right)(i, j).
\]

Note that for an arbitrary unitary matrix \(U_N\) then \(g(U_N A_N U_N^*) \neq U_N g(A_N) U_N^*\) in general. Hence the traffic distributions of two matrices of a same linear map in two different basis can be different. For instance, the matrices \(A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) are both the matrices of a reflexion in \(\mathbb{C}^2\), but \(\Delta(A) = A\) and \(\Delta(B) = 0\), \(deg(A) = A\) and \(deg(B) = I_2\), etc.
1.2.2 Link with other notions of convergence

The traffic distribution of $A_N$ encodes its $\ast$-distribution, as well as many other statistics on the matrices.

Example 1.5. Let $A_N$ be a family of random matrices.

1. The graph operations in matrices contains the $\ast$-polynomials, and so the traffic distribution $\Phi_{A_N}$ of $A_N$ encodes the $\ast$-distribution

$$\Phi_{A_N} : P \mapsto \mathbb{E} \left[ \frac{1}{N} \operatorname{Tr} P(A_N) \right]$$

by restriction of $\Phi$ on $\ast$-graph polynomials consisting in linear combinations of simple lines $(e_{out} \leftrightarrow e_{in})$.

2. The traffic distribution of $A_N$ induces the bilinear form

$$(P,Q) \mapsto \mathbb{E} \left[ \frac{1}{N} \operatorname{Tr} [P(A_N) \circ Q(A_N)] \right],$$

where $\circ$ denotes the entry-wise product of matrices.

3. The diagonal matrix $\operatorname{deg}(A_N) = \operatorname{diag}(\sum_{i=1}^N A_{i,i})$ is a linear function of the matrix $A_N = (A_{i,j})_{i,j}$, so the map

$$\Psi_{A_N} : P \mapsto \mathbb{E} \left[ \frac{1}{N} \operatorname{Tr} \operatorname{deg}(P(A_N)) \right]$$

is a linear map. It plays an important role later in the second part of the article, Section 5.4, in the context of asymptotic Boolean independence.

Let us compare the convergence in traffic distribution, the convergence in $\ast$-distribution and the convergence in spectral distribution.

Lemma 1.6. Let $A_N$ be a family of random matrices. The following are equivalent.

1. The family $A_N$ converges in traffic distribution, i.e. $\Phi_{A_N}$ converges pointwise on $CG\langle x, x^* \rangle$.

2. The family of matrices $(g(A_N))_g$ indexed by all $g \in CG\langle x, x^* \rangle$ converges in $\ast$-distribution.

3. For any $\ast$-graph polynomial $g \in CG\langle x, x^* \rangle$ such that the matrix $g(A_N)$ is Hermitian, the mean empirical spectral distribution of $g(A_N)$ converges in moments.

To prove the lemma we use the following fact, that will be important in all the paper. There is a composition law for graph operations consisting in replacing variables of a $\ast$-graph monomial $g$ by $\ast$-graph monomials.

Definition 1.7 (Substitution of edges of graph monomials). For a $\ast$-graph monomial $g$ in the variable $(g_1, \ldots, g_n)$ and for $\ast$-graph monomials $g_1, \ldots, g_n$ in the variables $x$, the $\ast$-graph monomial $g(g_1, \ldots, g_n)$ in the variables $x$ is the graph obtained from $g$ by replacing each edge $e$ of $g$ labeled $y_i$ by the graph $g_i$, the input of $g_i$ identified with the source of $e$ and the output of $g_i$ with the target of $e$.

This operation of substitution is compatible with the evaluation of matrices: with $g, g_1, \ldots, g_n$ as above, for any families of matrices $A_N$, we have

$$g(g_1(A_N), \ldots, g_n(A_N)) = (g(g_1, \ldots, g_n))(A_N).$$

(1.2)

This property implies a lot of relations between the graph polynomials, that can be see easily by drawing associated graphs, without writing a formula in terms of the entries of the matrices. For instance, the relation

$$\Delta(A_N \operatorname{deg}(B_N)) = \operatorname{deg}(\Delta(A_N)B_N),$$

valid for any $A_N$ and $B_N$, is obtained in Figure 1.2 below.
Proof of Lemma 1.6. Assume first (1), that is $A_N$ converges in traffic distribution, and let us prove (2). Let $g_1, \ldots, g_K$ be *-graph monomials and let $M = y_1^i \ldots y_L^i$ be a *-monomial, where $i_\ell \in \{1, \ldots, K\}$ and $i_\ell \in \{1, \ldots, L\}$. One can write

$$M(g_1(A_N), \ldots, g_K(A_N)) = g_{i_1}(A_N)^{x_1} \ldots g_{i_L}(A_N)^{x_L} = \tilde{g}(A_N)$$

where $\tilde{g} = (\cdot \ y_1^1 \ldots y_L^1 \cdot) (y_1^2 \ldots y_L^2 \cdot)$ is the graph monomial of Definition 1.7. Hence, by convergence in traffic distribution of $A_N$, $E[\frac{1}{N} \text{Tr} \tilde{g}(A_N)]$ converges. By the associativity in Formula 1.2 and the multi-linearity of the *-distribution, the family $(g(A_N))_{g \in \mathbb{C}[G]}$ converges in *-distribution.

We now prove that (2) implies (3). Consider a Hermitian matrix of the form $B_N = g(A_N)$ where $g$ is a *-graph polynomial. We can write $g = \sum \alpha_i g_i$ for a finite complex linear combination, where the $g_i$’s are *-graph monomials. For any integer $L \geq 1$, we can write $B_N^L = (g(A_N))^L = \sum_{i_1, \ldots, i_L} \alpha_{i_1} \ldots \alpha_{i_L} g_{i_1} \ldots g_{i_L}(A_N)$, where $g_{i_1} \ldots g_{i_L} = (\cdot \ y_1^1 \ldots y_L^1 \cdot) (g_{i_1} \ldots g_{i_L})$. So the convergence $E[\frac{1}{N} \text{Tr} g(A_N)]$ for all *-graph polynomials $g$ implies the convergence of $E[\frac{1}{N} \text{Tr} B_N^L]$ for each $L$, and hence the convergence in moments of the empirical spectral distribution of $g(A_N)$.

Let conclude by proving that (3) implies (1). For any $g \in \mathbb{C}[G]$, one can write $g(A_N) = g_1(A_N) + ig_2(A_N)$ where $g_1(x) = \frac{g(x) + g(-x)}{2}$ and $g_2 = \frac{g(x) - g(-x)}{2i}$. The matrices $g_1(A_N)$ and $g_2(A_N)$ are self-adjoint, so $E[\frac{1}{N} \text{Tr} g_i(A_N)]$ converges for each $i = 1, 2$. Hence the convergence for $g(A_N)$.
1.3. STATEMENTS OF THE RESULTS OF PART 1

1. They have a limiting joint traffic distribution: for any *-graph polynomial $g \in \mathbb{C}G\langle x_1, x_2^*, \ldots, x_k^* \rangle$, $k \geq 1$,

$$\tilde{\Phi}(g) := \lim_{N \to \infty} \mathbb{E}\left[ \frac{1}{N} \text{Tr}\left( g(A_N^{(1)}, \ldots, A_N^{(L)}) \right) \right]$$

(1.3)

In particular, $A_N = A_N^{(1)} \cup \cdots \cup A_N^{(L)}$ has a limiting joint *-distribution.

2. The limiting traffic distribution $\tilde{\Phi}$ of $A_N$ depends only on the marginal limiting traffic distributions $\Phi_t$ of the $A_N^{(t)}$'s. It is called the product of the distributions $\Phi_t$ and is given explicitly in Definition 2.17.

Moreover, the family of matrices $A_N$ also satisfies the factorization property.

The proof of the theorem is given in Section 2.4.

Theorem 1.8 is applied for the matrix models of Definitions 0.1 and 0.2. In particular, it extends in a unified way known results of asymptotic *-freeness in the setting of Theorem 1.8.

Corollary 1.9. Let consider $D_N, V_N, U_N, X_N, A_N$, independent families of random matrices where

- $D_N$ are independent diagonal matrices with independent and identically distributed diagonal entries whose moments of all orders exist.
- $V_N$ are independent permutation matrices,
- $U_N$ are independent unitary Haar matrices,
- $X_N$ are independent Wigner matrices whose entries are invariant in law by complex conjugation,
- $A_N$ is a family of random matrices satisfying the assumptions (2) and (3) of Theorem 1.8.

Then the matrices of $D_N, V_N, U_N, X_N$ and the family $A_N$ are asymptotically traffic independent. Moreover, the matrices of $U_N, X_N$ and the family $D_N \cup V_N \cup A_N$ are asymptotically free independent.

Corollary 1.9 is the consequence of several results stated in the next three sections. We summarize its proof in Section 5.3. The convergence and asymptotic traffic independence of Wigner matrices is proved in Section 3.1 (independently of Theorem 1.8) to illustrate the combinatorial method of computation of Section 2.1. We prove that the matrices $U_N, V_N$ of Corollary 1.9 satisfy the assumptions of Theorem 1.8 in Section 3.2.

We prove two general criterions that are useful for a large class of matrices.

Proposition 1.10. Let $A_N$ and $B_N$ be two asymptotically traffic independent matrices.

1. Denote $\Phi_N(A_N) = \mathbb{E}\left[ \frac{1}{N} \text{Tr} A_N \right]$. If there exists two *-polynomials $P, Q$ such that

$$\mathcal{R}(P, Q, (A_N)_{N \geq 1}) := \lim_{N \to \infty} \left( \Phi_N\left[ P(A_N) \circ Q(A_N) \right] - \Phi_N\left[ P(A_N) \right] \Phi_N\left[ Q(A_N) \right] \right)$$

is nonzero, and the same holds for $B_N$, then $A_N$ and $B_N$ are not asymptotically free independent.

2. If $A_N$ has the same limiting traffic distribution as a unitarily invariant families of matrices, then $A_N$ and $B_N$ are asymptotically free independent.

Remark 1.11. A partial reciprocal is true for this criterion: for two asymptotically traffic independent families $A_N$ and $B_N$, if $\mathcal{R}(P, Q, (A_N)_{N \geq 1})$ and $\mathcal{R}(P, Q, (B_N)_{N \geq 1})$ are zero for any $P, Q$, then $A_N$ and $B_N$ are asymptotically free independent. This is proved in [CDM16] and relies on an equivalent formulation of traffic independence. If $\mathcal{R}(P, Q, (A_N)_{N \geq 1}) = 0$ and $\mathcal{R}(P, Q, (B_N)_{N \geq 1}) \neq 0$, then different scenarios are possible.
The first point is proved in Section 2.3 and the second point in Section 5.3.
Theorem 1.8 and these two propositions are used in [Ma], [CM] and [MD] for several matrix models. We apply Proposition 1.10 in the context of Corollary 1.9 as follow.

Corollary 1.12.

- **Asymptotic free independence with the transpose:** Let $A_N$ satisfying Assumptions (2) and (3) of Theorem 1.8. Independent complex Wigner matrices $X_N^{(j)}, j \in J$ with parameter of the form $(\alpha_j, 0)$, independent Haar matrices $U_N^{(j')}, j \in J'$ and the transposed matrices $X_N^{(j)}, U_N^{(j')}, j \in J, j' \in J$ are asymptotically free independent.

- **Entry wise product of matrices:** Let $W_N = (\omega_{i,j})_{i,j=1,...,N}$ be a random matrix with i.i.d. entries whose moments are finite and independent of $N$. For independent unitary Haar matrix $U_N$ and uniform permutation matrix $V_N$, set the entry-wise product $M_N = W_N \circ U_N$ and $\tilde{M}_N = W_N \circ V_N$. If the $\omega_{i,j}$ are centered then independent copies of $M_N$ are asymptotically free circular elements. If the modulus of $\omega_{i,j}$ is not constant then independent copies of $\tilde{M}_N$ are not asymptotically free.

Illustration: the convolutions. Let us point out some cases of applications on the limiting distribution of the sum of two matrices. Consider two independent Hermitian random matrices $A_N$ and $B_N$. Assume that $A_N$ is diagonal with independent entries identically distributed according to some distribution $\mu_a$. Note that $A_N$ is permutation invariant. Assume now that $B_N$ converges in traffic distribution. In particular its mean empirical spectral distributions converges to some measure $\mu_b$. It follows from Theorem 1.8, Corollary 1.9 and Remark 2.23, that $A_N$ and $B_N$ are asymptotically traffic independent.

In particular, the empirical spectral distribution of the matrix $H_N = A_N + B_N$ converges to some measure $\mu_h$, which depends on $\mu_a$, $\mu_b$, and on the limiting traffic distribution of $B_N$.

1. If $B_N$ is a diagonal matrix, then $\mu_h$ is the classical convolution $\mu_a \ast \mu_b$ of $\mu_a$ and $\mu_b$, that is the distribution of the sum of a real random variable distributed according to $\mu_a$ and an independent real random variable distributed according to $\mu_b$. Indeed it is straightforward to see that the empirical spectral distribution of $H_N$ is actually the convolution of the empirical spectral distribution of $A_N$ and $B_N$ for each $N \geq 1$.

2. If $B_N$ is unitary invariant, then $\mu_h$ is the so-called free convolution $\mu_a \boxplus \mu_b$ of Voiculescu by consequence of their asymptotically free independence. In the setting of free probability [NS06], it is the distribution of the sum of a non random variable distributed according to $\mu_a$ and a free independent random variable distributed according to $\mu_b$.

3. For many examples of matrix models, it happens that $A_N$ and $B_N$ are asymptotically traffic independent but the limiting distribution of $A_N + B_N$ is neither the classical or the free convolution (see for example [Ma]).

In conclusion, at the level of the limiting $\ast$-distribution of large matrices, traffic independence encodes both with $\ast$-freeness and statistical independence, but also a large class of operations.
Chapter 2

Definition of asymptotic traffic independence

We first introduce some combinatorial tools, transforms of the traffic distribution. Then we define asymptotic traffic independence, and prove natural properties we can expect from a notion of independence. We illustrate this notion with a computation, which leads to a criterion of lack of asymptotic free independence for matrices. We conclude this section with the proof of Theorem 1.8.

2.1 Combinatorial form of traffic distributions and injective version

Recall that a $\ast$-test graph in variables $x = (x_j)_{j \in J}$ is a collection $(V, E, \gamma, \varepsilon)$ where $(V, E)$ is a finite connected graph with at least one vertex (but possibly no edges), whose edges are labelled by symbols $x_j$ and $x^*_j$. An edge $e$ has label $x^{(e)}_{\varepsilon(e)}$, where $\gamma : E \to J$ and $\varepsilon : E \to \{1, *\}$.

Definition 2.1. The set of $\ast$-test graphs in variables $x$ is denoted $T(x, x^*)$ and we denote by $C T(x, x^*)$ the space of finite linear complex combinations of elements of $T(x, x^*)$.

Recall that a $\ast$-graph monomial $g$ is the data of a $\ast$-test graph $T_g$ and two vertices $in, out$ of $T_g$ and that the traffic distribution of $A_N = (A_j)_{j \in J}$ is the data of the map $\Phi_{A_N} : g \in CT(x, x^*) \mapsto \mathbb{E}\left[ \frac{1}{N} \text{Tr}(A_N^g) \right] \in \mathbb{C}$. Given such a $\ast$-graph monomial $g = (T_g, in, out)$, note that $\Phi_{A_N}(g) = \tilde{\Phi}_{A_N}(\Delta(g))$ where $\Delta(g)$ is obtained from $g$ by identifying its input and output. Moreover, this quantity does not depend on the position of the input of $\Delta(g)$, but only on the $\ast$-test graph $T$ such that $\Delta(g) = (T, in, in)$.

Definition 2.2 (Combinatorial form of traffic distributions of matrices).

1. For any family $A_N = (A_j)_{j \in J}$ of matrices and any $\ast$-graph monomial $g$ in variables $x = (x_j)_{j \in J}$, we denote $\tau_{A_N}[T] = \tilde{\Phi}_{A_N}(g)$ where $T$ is the $\ast$-test graph obtained by identifying the input and output of $g$ and forgetting their position in the new $\ast$-test graph.

2. Equivalently, given $T = (V, E, \gamma, \varepsilon)$ a $\ast$-test graph in variables $x = (x_j)_{j \in J}$ and $A_N = (A_j)_{j \in J}$ a family of random matrices, then $\tau_N[T(A_N) : = \mathbb{E}\left[ \frac{1}{N} \text{Tr}[T(A_N)] \right]$ where

\[
\text{Tr}[T(A_N)] := \sum_{\phi : V \to [N]} \prod_{e \in (v, w) \in E} A^{(e)}_{\gamma(e)}(\phi(w), \phi(v))
\]

is called trace of the $\ast$-test graph $T$ in the matrices $A_N$. For any matrix $A_N$, denoting by $\bigcup$ the $\ast$-test graph with a single loop, we then have

\[
\text{Tr} A_N = \text{Tr}[\bigcup(A_N)].
\]
CHAPTER 2. DEFINITION OF ASYMPTOTIC TRAFFIC INDEPENDENCE

3. The combinatorial distribution of $A_N$ is the linear map
$$\tau_{A_N} : T \in \mathbb{C}[T(x,x^*)] \mapsto \tau_N[T(A_N)] \in \mathbb{C}.$$

Remark 2.3. 1. Notation $\text{Tr}[T(A_N)]$ and terminology are abusive since we do not define the object $T(A_N)$ and we still call trace this map, which is a function of a $*$-test graph $T$ and of a family of matrices $A_N$ indexed as the variables of $T$. Nevertheless, there is no risk of confusion with the notation $\tau_N[T]$ which contrasts with the notation $\Phi(g)$.

2. As in Definition 1.7, for a $*$-test graph in variables $y_1, \ldots, y_n$ and $*$-graph monomials $g_1, \ldots, g_n$, we define the $*$-test graph $\hat{T}(g_1, \ldots, g_n)$ by replacing edges labeled $y_i$ of $T$ by the graph $g_i$. Then we have the compatibility property
$$\tau_N\left[T(g_1(A_N), \ldots, g_n(A_N))\right] = \tau_N\left[T(g_1, \ldots, g_n)(A_N)\right].$$

In the l.h.s. the $*$-test graph is $T$ and the matrices are the $g_i(A_N)$, whereas in the r.h.s. the $*$-test graph is $T(g_1, \ldots, g_n)$ and the matrices are those of $A_N$.

Example 2.4. Example 1.5 continued. Let $A_N$ be a family of random matrices.

1. Let $M$ be a $*$-monomial $M = x_1^{i_1} \cdots x_n^{i_n}$. Then one has $\text{Tr}\, M(A_N) = \text{Tr}[T(A_N)]$ where $T$ is the $*$-test graph consisting in a simple oriented cycle, with vertex set $\{1, \ldots, n\}$ and edges $(i + 1, i)$ labeled $x_i^{i_1}$, $i = 1, \ldots, n$ (with indices modulo $n$).

2. Let $M'$ be a second $*$-graph monomial and let $T'$ the $*$-test graph with vertex set $\{1, \ldots, n'\}$ defined $T$ is defined from $M$ in the previous item. Recalling that $\circ$ denotes the entry wise product of matrices, then $\text{Tr}\, M(A_N) \circ M'(A_N) = \text{Tr}[T(A_N)]$ where $T$ is the $*$-test graph consisting in a bunch of two simple oriented cycles, obtained by identifying the vertex 1 of the graphs $T$ and $T'$.

3. Recall that $\text{deg}(A_N)$ is the diagonal matrix whose $i$-th diagonal matrix is the sum of the elements of $A_N$ over the $i$-th column. Then $\text{Tr}\, [\text{deg}(A_N)] = \text{Tr}[T(A_N)]$ where $T$ is the $*$-test graph consisting in a simple edge.

We can now introduce the following transformation of traffic distributions.

Definition 2.5 (Injective trace of matrices). Let $T = (V,E,\gamma,\epsilon)$ be a $*$-test graph in variables $x = (x_j)_{j\in J}$ and let $A_N = (A_j)_{j\in J}$ be a family of matrices, possibly random. We call injective trace of the $*$-test graph $T$ in the matrices $A_N$ the quantity
$$\text{Tr}^0[T(A_N)] = \sum_{\phi : V \to [N]} \prod_{e = (v,w) \in E} A_{\phi(e)}^{\gamma(e)}(\phi(w),\phi(v)). \quad (2.1)$$
and we set $\tau_N^0[T(A_N)] := \mathbb{E}\left[\text{Tr}^0[T(A_N)]\right]$, where the expectation is relative to the matrices when they are random. The injective traffic distribution of $A_N$ is the linear map
$$\tau_{A_N}^0 : T \in \mathbb{C}[T(x,x^*)] \mapsto \tau_N^0[T(A_N)] \in \mathbb{C}.$$

Combinatorial distributions and their injective versions are related each other. Let $T$ be a $*$-test graph with vertex set $V$ and let $\mathcal{P}(V)$ denote the set of partitions of $V$. For any $\pi \in \mathcal{P}(V)$, we denote by $T^\pi$ the $*$-test graph obtained by identifying vertices in a same block of $\pi$ (the edges link the associated blocks). See an example Figure 2.1 Then for any $*$-test graph $T$, one has
$$\text{Tr}[T(A_N)] = \sum_{\pi \in \mathcal{P}(V)} \text{Tr}^0[T^\pi(A_N)]. \quad (2.2)$$
In this formula, the different cases of equality for the indices of the matrices in the definition of $\text{Tr}[T(A_N)]$ are classified by choosing the partition of indices whose values are equal. Reciprocally, it is possible to write $\text{Tr}^0$ in terms of $\text{Tr}$ using a generalized inclusion-exclusion principle (see NS06 Sta12).
Remark

Definition 2.7. Let

Figure 2.1: Left: a test graph with vertex set \{1, \ldots, 12\}. Right: the quotient graph \( T^\pi \) for the partition \( \pi = \{\{1, 3\}, \{2, 4, 8\}, \{5, 7\}, \{6\}, \{9, 11\}, \{10\}, \{12\}\} \).

Let \( \pi \) be a linear map. We call injective version of \( \pi \) the linear map \( \tau^0: CT(\mathbf{x}, \mathbf{x}^*) \to \mathbb{C} \) defined for any *-test graph \( T \) by

\[
\tau^0[T] = \sum_{\pi \in \mathcal{P}(V)} \mu_V(\pi)\tau[T^\pi],
\]

where \( \mu_V \) and \( T^\pi \) are as in (2.3), which implies that for any *-test graph \( T \)

\[
\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi],
\]

Remark 2.8

1. There is no meaning for an expression like \( \text{Tr}^0 A_N \) or \( \text{Tr}^0 g(A_N) \), the injective trace is always defined for *-test graphs.

2. A relation exists between the injective trace and the notions of free cumulants, but in the particular case of unitarily invariant matrices. It is the main motivation of [CDM16] to state this relation.

3. With notations as in the second item of Remark 2.3 in general we have

\[
\tau^0_N \left[ T(g_1(A_N), \ldots, g_n(A_N)) \right] = \tau^0_N \left[ T(g_1, \ldots, g_n)(A_N) \right],
\]

see Lemmas 2.15.

Lemma 2.9. Let \( A_N \) be a family of random matrices. There is equivalence between

1. The convergence in traffic distribution of \( A_N \), namely the pointwise convergence of \( \Phi_{A_N} : g \in \mathcal{C}G(\mathbf{x}, \mathbf{x}^*) \mapsto \mathbb{E} \left[ \frac{1}{N} \text{Tr} g(A_N) \right] \),

2. the pointwise convergence of \( \tau_{A_N} : T \in CT(\mathbf{x}, \mathbf{x}^*) \mapsto \mathbb{E} \left[ \frac{1}{N} \text{Tr} T(A_N) \right] \)
3. the pointwise convergence of $\tau^{0}_{\mathbf{A}_N} : T \in \mathcal{C}(\mathbf{x}, \mathbf{x}^*) \mapsto \mathbb{E}\left[\frac{1}{N} \text{Tr}^0 T(\mathbf{A}_N)\right]$

Assuming this convergence, then $\mathbf{A}_N$ satisfies the factorization property, assumption (3) of Theorem 1.5, if and only if for any $\ast$-test graphs $T_1, \ldots, T_K$, $K \geq 2$,

$$
\mathbb{E}\left[\prod_{k=1}^{K} \frac{1}{N} \text{Tr}[T_k(\mathbf{A}_N)]\right] \rightarrow \prod_{k=1}^{K} \tau[T_k].
$$

(2.6)

which is also equivalent to the same property for the injective trace, namely for any $\ast$-test graphs $T_1, \ldots, T_K$, $K \geq 2$,

$$
\mathbb{E}\left[\prod_{k=1}^{K} \frac{1}{N} \text{Tr}^0[T_k(\mathbf{A}_N)]\right] \rightarrow \prod_{k=1}^{K} \tau^0[T_k].
$$

(2.7)

Proof. The equivalence of the formulations in terms of $\Phi_{\mathbf{A}_N}$ and $\tau_{\mathbf{A}_N}$ is clear by definition. The equivalence between convergence in traffic distribution and pointwise convergence of $\tau^{0}_{\mathbf{A}_N}$ is a consequence of Formulas (2.2) and (2.1). Let $T_1, \ldots, T_K$ be $\ast$-test graphs whose vertex sets are denoted $V_1, \ldots, V_K$ respectively. The factorization property implies

$$
\mathbb{E}\left[\prod_{k=1}^{K} \frac{1}{N} \text{Tr}[T_k(\mathbf{A}_N)]\right] = \sum_{\pi_k \in \mathcal{P}(V_k)} \prod_{k=1}^{K} \mu_{V_k}(\pi_k) \mathbb{E}\left[\prod_{k=1}^{K} \frac{1}{N} \text{Tr}[\hat{T}_k^0(\mathbf{A}_N)]\right]
$$

$$
= \sum_{\pi_k \in \mathcal{P}(V_k)} \prod_{k=1}^{K} \mu_{V_k}(\pi_k) \left(\prod_{k=1}^{K} \tau_N[\hat{T}_k^0(\mathbf{A}_N)]\right)
$$

$$
+ \varepsilon(T^\pi_k)_{k=1, \ldots, K},
$$

where for each $k = 1, \ldots, K$, the term $\varepsilon(T^\pi_k)$ tends to zero. Since $\tau_N[\hat{T}_k^0(\mathbf{A}_N)]$ is bounded for each $\pi_1, \ldots, \pi_k$, we get

$$
\mathbb{E}\left[\prod_{k=1}^{K} \frac{1}{N} \text{Tr}^0[T_k(\mathbf{A}_N)]\right] = \left(\sum_{\pi_k \in \mathcal{P}(V_k)} \prod_{k=1}^{K} \mu_{V_k}(\pi_k) \tau_N[T^\pi_k(\mathbf{A}_N)]\right) + \tilde{\varepsilon}
$$

$$
= \prod_{k=1}^{K} \tau^0_N[T^\pi_k(\mathbf{A}_N)] + \tilde{\varepsilon}
$$

where $\tilde{\varepsilon}$ is bounded by a constant times the maximum of the $\varepsilon(T^\pi_k)$ for any $\pi_k \in \mathcal{P}(V_k), k = 1, \ldots, K$. The reciprocal is true by the same computation where, starting with the trace instead of the injective one, we omit the terms $\mu_{V}(\pi_k)$ in the above computation. \hfill \Box

### 2.2 Asymptotic traffic independence, properties

**Definition 2.10** (Graph of colored components).

1. Let $T$ be a $\ast$-test graph in variables $\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_p$, where the $\mathbf{x}_j$'s are families of pairwise disjoint variables (a variable appears at most in one family). A **colored component** of $T$ with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_p$ is a maximal connected subgraph of $T$, with at least one edge, whose edges are labeled by variables in only one family among $\mathbf{x}_1, \ldots, \mathbf{x}_p$. We denote by $\mathcal{GCC}(T)$ the set of colored components of $T$ with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_p$.

2. The graph of colored components of $T$ with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_p$, denoted $\hat{\mathcal{GCC}}(T)$, is the following bipartite undirected graph.

- The first kind of vertices are the colored components $T_1, \ldots, T_K$ of $T$. 

The second kind of vertices are the vertices \( v_1, \ldots, v_L \) of \( T \) that belong to at least two graphs among \( T_1, \ldots, T_K \).

There is an edge between \( T_i \) and \( v_j \) if \( v_j \) is a vertex of \( T_i, i = 1, \ldots, K, j = 1, \ldots, L \).

In the leftmost picture of Figure 2.2 we draw a \(*-test\) graph in three families, represented by three colors. In the intermediate figure, we have encircled the colored components of \( T \), and drawn in the rightmost one its graph of colored components (making apparent the origin of each vertex by a color code for the reader’s convenience).

![Figure 2.2](image)

**Figure 2.2**: An example of construction of \( T \). The graph \( \GCC(T) \) is not a tree since \( \bar{T} \) has a cycle. Removing the edge labelled \( z_4 \) or \( x_5 \) in \( T \), \( \GCC(T) \) becomes a tree.

**Definition 2.11** (Asymptotic traffic independence). For each \( \ell \) in \( \{1, \ldots, L\} \), let \( \tau_\ell : \mathbb{C}T(\langle x_1, x_\ell \rangle) \to \mathbb{C} \) be a linear map sending the graph with no edge to one. Up to a renaming of the symbols, we assume that the variables of the different families \( x_\ell \) for \( \ell = 1, \ldots, L \) are pairwise distinct.

The product of the maps \( \tau_1, \ldots, \tau_L \) is the linear map \( \tau : \mathbb{C}T(\langle x, x \rangle) \to \mathbb{C} \) defined by the following:

1. for each \( \ell = 1, \ldots, L \) and any \(*-test\) graph \( T \) in the variables \( x_\ell \), \( \tau(T) = \tau_\ell(T) \) with the abuse of notation that \( x \) are considered as the variables of \( T \), with no edges labeled \( x_{\ell'} \), \( \ell' \neq \ell \).

2. for any \(*-test\) graph \( T \) in the variables \( x_1, \ldots, x_p \),

\[
\tau^0(T) = \mathbb{1}(\GCC(T) \text{ is a tree}) \prod_{S \in \GCC(T)} \tau^0(S),
\]

where \( \GCC(T) \) (resp. \( \CC(C) \)) is the graph (resp. the set) of colored components of \( T \) with respect to \( x_1, \ldots, x_p \).

We say that \( \mathbf{A}_N^{(l)} \) are asymptotically traffic independent whenever \( \mathbf{A}_N := \bigcup_{\ell} \mathbf{A}_N^{(l)} \) converges in traffic distribution and its combinatorial distribution converges to the product of the limiting distributions of the \( \mathbf{A}_N^{(l)} \)’s.

Hence the asymptotic traffic independence of \( \mathbf{A}_N^{(l)} \) is equivalent to the convergence for any \(*-test\) graph \( T \) with vertex set \( V \),

\[
\tau_N \left[ T \langle \mathbf{A}_N^{(l)} \rangle \right] \xrightarrow{N \to \infty} \sum_{\pi \in \mathcal{P}(V)} \prod_{S \in \GCC(T)} \lim_{N \to \infty} \tau_N^0 \left[ \mathbf{S} \langle \mathbf{A}_N^{(\ell(S))} \rangle \right],
\]

where \( \ell(S) \) is the index of the labels of the colored component \( S \). Since the injective traces in the product can also be express in terms of (non-injective) traces, asymptotic traffic independence

---

This product should be called "free product" as it satisfies a universal property, see [CDM16]. Nevertheless we do not use this term to avoid confusion with free independence.
entirely characterize the limiting traffic distribution of $A_N^{(1)} \cup \cdots \cup A_N^{(L)}$ in terms of the traffic distributions of the $A_N^{(i)}$’s.

Specified for graphs $T$ consisting in simple oriented cycles (Example 2.4), Formula (2.2) gives an expression for the limiting *-distribution of $A_N^{(1)} \cup \cdots \cup A_N^{(L)}$.

**Proposition 2.12.** The product of Definition 2.11 is symmetric and associative, in the following sense. Let $\tau_1, \tau_2, \tau_3$ denote three linear maps $\tau_i : CT(x_i, x) \to \mathbb{C}$ sending the graph with no edge to one. Then the product of $\tau_1$ and $\tau_2$ is the product of $\tau_2$ and $\tau_1$ (with identification $x_1 \cup x_2 \sim x_2 \cup x_1$). Moreover, the product $\tau$ of $\tau_1, \tau_2, \tau_3$ is the product $\tau_{1,2}$ of $\tau_3$ with the product $\tau_{1,2}$ of $\tau_1$ and $\tau_2$. In particular, three large families of matrices $A_N, B_N, C_N$ are asymptotically traffic independent whenever $A_N$ and $B_N$ are asymptotically traffic independent and $A_N \cup B_N$ and $C_N$ are asymptotically traffic independent.

We shall need the following lemma, now and in the sequel of the article.

**Lemma 2.13** (Number of edges and vertices in a connected graph). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite connected graph. Then, one has $|\mathcal{V}| \leq |\mathcal{E}| + 1$, with equality if and only if $\mathcal{G}$ is a tree.

**Proof of Lemma 2.13** If $G$ is a tree, we count the difference between the number of edges and vertices by removing its branches successively until it remains a single vertex, this yields the formula. If $G$ is not a tree, then we can remove edges until it becomes a tree which yields the inequality. \(\square\)

**Proof of Proposition 2.12** The symmetry is a immediate consequence of the definition. Let $T$ be a *-test graph in the indeterminates $x = \cup_{i=1}^3 x_i$. Denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the graph of colored components ($GCC$) of $T$ with respect to $x_1, x_2, x_3$. Denote by $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ the GCC of $T$ with respect to $x_1 \cup x_2$ and $x_3$. For each colored component $S$ of $G'$ with label in $x_1 \cup x_2$, we denote by $\mathcal{G}(S) = (\mathcal{V}(S), \mathcal{E}(S))$ the GCC of $S$ with respect to $x_1$ and $x_2$. Denote by $CC(x_i), i = 1, \ldots, 3$, the set of colored components of $T$ labeled $x_i$.

We prove that $\mathcal{G}$ is a tree if and only if $\mathcal{G}'$ is a tree and the $\mathcal{G}(S)$’s are trees. Denote by

1. $a$ the number of vertices of $T$ that belong both to an element of $CC(x_1)$ and to a single other element of $CC(x_1) \cup CC(x_2)$,
2. $b$ the number of vertices of $T$ that belong to an element of $CC(x_3)$, of $CC(x_1)$ and of $CC(x_2)$,
3. $c$ the number of vertices of $T$ that belong both to an element of $CC(x_1)$ and of $CC(x_2)$, but not of $CC(x_3)$,
4. $|CC(x_1 \cup x_2)|$ the number of colored components of $\mathcal{G}'$.

One can enumerate easily the following quantities

$$|\mathcal{V}| = a + b + c + \sum_{i=1}^3 |CC(x_i)|, \quad |\mathcal{E}| = 2a + 3b + 2c,$$

$$\sum_S |\mathcal{V}(S)| = b + c + \sum_{i=1}^2 |CC(x_i)|, \quad \sum_S |\mathcal{E}(S)| = 2b + 2c,$$

$$|\mathcal{V}'| = a + b + |CC(x_3)| + |CC(x_1 \cup x_2)|, \quad |\mathcal{E}'| = 2a + 2b.$$

Since $|CC(x_1 \cup x_2)|$ is the number of colored components $S$ of $G'$ labeled in $x_1 \cup x_2$, this results in the equality

$$|\mathcal{V}| - 1 - |\mathcal{E}| = (|\mathcal{V}'| - 1 - |\mathcal{E}'|) + \sum_S (|\mathcal{V}(S)| - 1 - |\mathcal{E}(S)|),$$

telling by Lemma 2.13 that $\mathcal{G}$ is a tree if and only if $\mathcal{G}'$ is a tree and the $\mathcal{G}(S)$’s are trees.

If we assume that $\mathcal{G}, \mathcal{G}'$ and the $\mathcal{G}(S)$’s are trees, it is clear that

$$\prod_{i=1}^3 \prod_{S \in CC(x_i)} x^0[S] = \prod_{S \in CC(x_3)} x^0[S] \times \prod_{S \in CC(G')} \prod_{\mathcal{G}' \text{ of } S} x^0[S].$$

\(\square\)
2.2. ASYMMETRIC TRAFFIC INDEPENDENCE, PROPERTIES

Proposition 2.14. Asymptotic traffic independence of a family of matrices is a property of the traffic space it generates, in the following sense. Let \(A_N^{(1)}, \ldots, A_N^{(L)}\) be asymptotically traffic independent families of matrices. For each \(\ell = 1, \ldots, L\), let \(B_N^{(\ell)}\) be a family of matrices in the traffic space generated by \(A_N^{(\ell)}\), that is \(B_N^{(\ell)}\) is a collection of matrices of the form \(g(A_N^{(\ell)})\) for \(g\) - graph polynomials of \(\mathcal{G}(x^{(\ell)}, x^{(\ell)*})\). Then \(B_N^{(1)}, \ldots, B_N^{(L)}\) are asymptotically traffic independent.

In particular, if two matrices \(A_{N,1}\) and \(\tilde{A}_{N,1}\) are asymptotically traffic independent, then the family of matrices \((A_{N,1}, A_{N,1}, A_{N,1}, \text{Deg}(A_{N,1})), \text{Deg}(A_{N,1}), A_{N,1} \circ A_{N,1}, \ldots, A_{N,1}, \ldots)\), \(\ell = 1, \ldots, L\) asymptotically traffic independent.

In order to prove the proposition, let us first state a property of the injective version \(\tau^0\) of a map \(\tau\). It is an analogue for traffics of the formula for the free cumulants whose entries are products of random variables \([\tau:\mathcal{T}(x,x^*) \to \mathbb{C} \text{ a linear form. For a } ^*\text{-test graph } T \text{ in variables } y = (y_1, \ldots, y_n) \text{ and } ^*\text{-graph monomials } y_1, \ldots, y_n \text{ in variables } x, \text{ recall we define the } ^*\text{-test graph } T'(y_1, \ldots, y_n) \text{ in variables } x \text{ by replacing in } T \text{ the edges labeled } y_i \text{ by the graph } g_i. \text{ We define } \hat{\tau} : \mathcal{T}(y, y^*) \to \mathbb{C} \text{ by } \hat{\tau}[T] = \tau[T'(y_1, \ldots, y_n)].

Let us fix \(\hat{T} = T(g_1, \ldots, g_n)\) and denote by \(\hat{V}\) the vertex set of \(\hat{T}\). To each vertex \(v\) of \(T\) is associated a vertex \(\hat{v} = v \text{ of } \hat{T}\) which gives the position of \(v\) in the new graph. Note that this defines a map \(f : V \to \hat{V}\), not necessarily injective since some vertices of \(T\) can be identified in \(\hat{T}\). For a partition \(\hat{\pi} \in \mathcal{P}(\hat{V})\), we define a partition \(\pi_V \in \mathcal{P}(V)\) via \(\pi_{|V} := f^{-1}(\hat{\pi})\). We denote by \(0_V\) the partition whose blocks are singletons. We then have

\[
\tau^0[T] = \sum_{\hat{\pi} \in \mathcal{P}(\hat{V})} \tau^0[\hat{\pi}].
\]

Note that if there is an edge labeled \(y_i\) which is not a self loop and such that \(\Delta(g_i) = g_i\), then there is no \(\tilde{\pi} \in \mathcal{P}(\hat{V})\) such that \(\tilde{\pi}_{|\hat{V}} = 0\), so \(\tau^0[T] = 0\).

Note that this property is easy to obtain for the trace of \(^*\text{-test graphs in matrices by definition of the injective trace of } ^*\text{-test graph in matrices. We will need this general version only later in Section 3.}

Proof. We have clearly

\[
\hat{\tau}[T] := \tau[\hat{T}] = \sum_{\hat{\pi} \in \mathcal{P}(\hat{V})} \tau^0[\hat{\pi}]
= \sum_{\hat{\pi} \in \mathcal{P}(\hat{V})} \left[ \sum_{\pi \in \mathcal{P}(V)} \tau^0[\hat{\pi}] \right].
\]

Note that the term in the bracket depends only on \(T^\pi\), and not on \(T\); each \(\hat{\pi}\), where \(\hat{\pi}\) is as in the sum inside the bracket, is obtained from \(T^\pi\) by constructing \((T^\pi)(g_1, \ldots, g_n)\) (replacing each edge \(y_i\) by the graph \(g_i\)) and then identifying some vertices of the latter graph according to a partition \(\sigma\) of its vertex set, namely

\[
(T(g_1, \ldots, g_n))^\hat{\pi} = ((T^\pi)(g_1, \ldots, g_n))^\sigma.
\]

Hence the term in the bracket is well an linear form on elements of \(\mathcal{T}(x,x^*)\). This implies by uniqueness of the injective trace that this term is \(\hat{\tau}^0[T^\pi]\), and we get the result since \(\hat{\tau}^0[T] = \hat{\tau}^0[T^0]\).  

Proof of Proposition 2.14. It is sufficient to prove the result for \(L = 2\), the general case follow by associativity and by a simple induction on the number of families of matrices. We denote \(A_N = A_N^{(1)} \sqcup A_N^{(2)}\) (the union of the families remembering their origin \(\ell \in 1, 2\)). In the sequel
we prove the following. Let \( g \) be a *-graph polynomial and denote \( B = g(A_N^{(1)}) \). Then \((B, A_N^{(1)})\) and \( A_N^{(2)}\) are asymptotically traffic independent. By induction and same reasoning for matrices of the form \( B = g(A_N^{(2)}) \), we then obtain: for families \( B_N^{(1)} \) and \( B_N^{(2)} \) as in the statement of the lemma, the families \( A_N^{(1)} \cup B_N^{(1)} \) and \( A_N^{(2)} \cup B_N^{(2)} \) are traffic independent. Hence \( B_N^{(1)} \) and \( B_N^{(2)} \) are asymptotically traffic independent (heredity of traffic independence is immediate).

Hence in the following we introduce several times a matrix \( B \) in the traffic space generated by \( A_N^{(1)} \). For any *-test graph in variables \( b, a^{(1)}, a^{(2)} \), we denote \( \tau^0[T] = \lim_{N \to \infty} \tau_\gamma[T(B, A_N^{(1)}, A_N^{(2)}), \gamma] \). We prove the asymptotic independence of \((B, A_N^{(1)}) \) and \( A_N^{(2)} \), namely that \( \tau^0 \) satisfies Formula (2.3):

\[
\tau^0[T] = \mathbb{1}(\text{GCC}(T) \text{ is a tree}) \prod_{S \in \text{GCC}(T)} \tau^0[S], \tag{2.3}
\]

where the notion of colored components is with respect to \((b, a^{(1)}) \) and \( a^{(2)} \).

1. Consider a matrix \( A \) of \( A_N \) and a complex number \( \lambda \), and set the matrix \( B = \lambda A \). The map \( A_N \mapsto \tau^0_N[T(A_N)] \) is anti-multilinear with respect to its edges: for a *-test graph \( T \) in variables \( h, a^{(1)}, a^{(2)} \), denote \( \hat{T} \) the *-test graph in the variables \( a^{(1)}, a^{(2)} \) obtained from \( T \) by replacing the label of the edges labelled \( b \) (respectively \( b^* \)) by \( a \) (respectively \( a^* \)). Let us set \( p \) (respectively \( q \)) the number of edges of \( T \) with label \( b \) (respectively \( b^* \)). Then \( \tau^0[T] = \lambda^p\tau^0[\hat{T}] \). Moreover, by asymptotically traffic independent of \( A_N^{(1)} \) and \( A_N^{(2)} \), we have \( \tau^0[T] = \mathbb{1}(\text{GCC}(T) \text{ is a tree}) \prod_{S \in \text{GCC}(T)} \tau^0[S], \) where the notion of colored components of \( \hat{T} \) is with respect to \( a^{(1)} \) and \( a^{(2)} \). Furthermore, we have \( \lambda^p \prod_{\gamma \in \text{GCC}(T)} \tau^0[S] = \prod_{\gamma \in \text{GCC}(T)} \tau^0[S] \) since the total number of edges with first label remains unchanged in the union of colored components. Hence asymptotic traffic independence is stable by multiplication by scalars.

2. Let us consider now another matrix \( A' \) in the same family as \( A \), and denote \( B = A + A' \). Fix a *-test graph \( T \) in variables \( b, a^{(1)}, a^{(2)} \) and denote by \( E_1 \) the set of edges of \( T \) with label \( b \). For any map \( \gamma : E_1 \to \{a, a'\} \), we denote by \( T_\gamma \) the *-test graph in variables \( a^{(1)}, a^{(2)} \) obtained from \( T \) by replacing the label of \( b \) by \( \gamma(e) \). We then have \( \tau^0[T] = \sum_{\gamma : E_1 \to \{a, a'\}} \tau^0[T_\gamma] \). Since both matrices \( A \) and \( A' \) are in the same family, then the graph of colored components of \( T_\gamma \) is the same as \( T \) for any \( \gamma \). For a colored component \( S \) of \( T \) corresponds a colored component \( S_\gamma \) of \( T_\gamma \). With \( E_{1,S} \) denoting the set of edges of \( S \) with label \( b \), we have \( \sum_{\gamma : E_1 \to \{a, a'\}} \tau^0[S_\gamma] = \tau^0[S] \). Hence asymptotic traffic independence is stable by linear operations of vector space.

3. It remains to prove the stability under graph monomials. We first consider a matrix of the form \( B = \Delta(A), \) i.e. \( B \) consists in the diagonal elements of a matrix of \( A_N \). Let \( T \) be a *-test graph in variables \( b, a^{(1)} \) and \( a^{(2)} \) with at least one edge \((v, w)\) labelled \( b \) which is not a self loop. Then, by the last remark of Lemma 2.15, \( \tau^0[T] \) vanishes, and as well \( \tau^0[S] = 0 \) for the colored component of \( T \) containing \((v, w)\). On the other hand if all the edges labeled \( b \) in \( T \) are loops, then \( \hat{T} \) is obtained from \( T \) by replacing \( b \) by \( a \), one has \( \tau[T] = \tau[\hat{T}] \), and so the same equality is true for non-injective trace. Hence the formula.

4. At last we consider a matrix \( B = g(A_N^{(1)}) \) where \( g \) is a *-graph monomial such that the input and the output are distinct \((g \neq \Delta(g))\). By Lemma 2.15, with \( \hat{T} \) obtained from \( T \) by replacing edges labeled \( b \) by the graph monomial \( g \), we have \( \tau^0[T] = \sum_{\pi \in \mathcal{P}_V} \tau^0[\hat{T}_\pi] \). We know by asymptotic traffic independence of \( A_N^{(1)} \) and \( A_N^{(2)} \) that \( \tau^0[\hat{T}_\pi] = \mathbb{1}(\text{GCC}(\hat{T}_\pi) \text{ is a tree}) \prod_{S \in \text{GCC}(\hat{T}_\pi)} \tau^0[S], \) where the notion of colored components is with respect to \( a^{(1)} \) and \( a^{(2)} \). Let \( \pi \in \mathcal{P}(V) \) such that \( \hat{T}_\pi \) is a subset of \( \hat{V}_\pi \). In other words, the vertices of \( V \), which are not identified in \( \hat{V}_\pi \), are not identified in \( \hat{V}_\pi \) either. We see from now \( V \) as a subset of \( \hat{V}_\pi \).

We claim that \( \text{GCC}(\hat{T}_\pi) \) is a tree only if \( \text{GCC}(T) \) is a tree. To prove this claim, in the graph \( \hat{T} \) (where the edges labelled \( b \) are replaced by \( g \)) we chose an enumeration of the vertices \( \{v_1, \ldots, v_0\} \) of \( \hat{V} \setminus V \) (there are copies of the vertices of \( g \) that are not the input nor the output). For each \( q = 0, \ldots, Q \) we denote \( V_q = V \cup \{v_1, \ldots, v_Q\} \) \((V_0 = V)\). Moreover, denoting the partition \( \pi = \{\hat{B}_1, \ldots, \hat{B}_P\} \), we set the partition \( \pi(q) = \{\hat{B}_1 \cap V_q, \ldots, \hat{B}_P \cap V_q\} \). We have \( \pi(0) = \pi \) by definition. Moreover, we have \( \text{GCC}(\hat{T}_\pi^{(0)}) = \text{GCC}(T) \) since \( \pi(0) \) is made of singletons (we can replace \( g \) by every connected graph labeled in the same family without changing \( \text{GCC}(T) \)). Denote by \( |V_q| \) and
2.3. APPLICATION: A CRITERION OF LACK OF ASYMPTOTIC FREE INDEPENDENCE

Let $A_N = (A_1, A_2)$ and $B_N = (B_1, B_2)$ be two asymptotically traffic independent families of matrices. Let us first manipulate the definition of traffic independence to understand how to compute

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr} A_1 B_1 A_2 B_2\right].$$

We denote for all *-test graph $T$ and all *-polynomials $P, Q$,

$$\tau[T(a, b)] = \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr} T(A_N, B_N)\right],$$

$$\Phi[P(a, b)] = \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr} P(A_N, B_N)\right],$$

$$\Phi[P(a, b) \circ Q(a, b)] = \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr} [P(A_N, B_N) \circ Q(A_N, B_N)]\right].$$

Firstly, according to Definition 2.2, we introduce the *-test graph $T$ consisting in a simple cycle with edges labeled $b_2, a_2, b_1, a_1$ along it, so that we can write $\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr} A_1 B_1 A_2 B_2\right] = \tau[T(a, b)] = \sum_{x \in \mathcal{P}(V)} \tau^0[T(x, y)]$, where $V$ is the set of the four vertices of $T$. By Definition 2.11 the asymptotic traffic independence of $A_N$ and $B_N$ implies that there are three partitions $\pi \in \mathcal{P}(V)$ that contribute on the limit (the computation is illustrated in Figure 2.3). One has $\tau[T(a, b)] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T(\pi)]$ for the following *-test graph $T_1, T_2, T_3$.

- The *-test graph $T_1$ with one vertex and four loops, labelled $a_1, a_2, a_1, a_2$ respectively. By the factorization property, one has $\tau^0[T_1(a, b)] = \tau^0[T_1^0(a)] \times \tau^0[T_1^0(b)]$ where $T_1^0(x, y)$ is the *-test graph with one vertex and two loops, labeled $x$ and $y$ respectively. Since $T_1^0$ has a single vertex, one has $\tau_N[T_1^0(a)] = \tau_N[T_1^0_0(a)].$ By the second case of Example 2.4, we finally get $\tau[T_1(a)] = \Phi[a_1 \circ a_2]$ where $\circ$ denotes the entry-wise product of matrices, as well as the similar expression for $b.$
\[
\lim_{N \to \infty} E \left[ \frac{1}{N} \text{Tr}(A_1 B_1 A_2 B_2) \right] = \tau \left[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array} \right] = \tau^0 \left[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array} \right] + \tau^0 \left[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array} \right] + \tau^0 \left[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array} \right]
\]

Figure 2.3: Computation of (2.1)

- The *-test graph \( T_2 \) with two vertices, one loop on each vertex, and one edge between the two vertices in each direction. Labels are \( a_1 \) and \( a_2 \) on the loops, \( b_1 \) and \( b_2 \) on the simple edges, in such a way one can read the word \( b_2, a_2, b_1, a_1 \) by following a closed path. The factorization property yields \( \tau^0[T_2(a, b)] = \tau^0[T_0(a_1)] \times \tau^0[T_0(a_2)] \times \tau^0[T_2(b)] \) where \( T_0(a) \) is the *-test graph with one vertex and one loop labeled \( a \) and \( T_2(b) \) the *-test graph with two vertices and a pair twin edges labeled \( b_1, b_2 \) and opposite orientation. Since \( T_0 \) has a single vertex, one has \( \tau^0[T_0(a_1)] = \tau[T_0(a_1)] = \Phi(a_1) \). Moreover, the relation \( \tau[T_2(b)] = \tau^0[T_2(b)] + \tau^0[T_1(b)] \) yields \( \tau^0[T_2(b)] = \Phi(b_1 b_2) - \Phi(b_1 \circ b_2) \).

- The *-test graph \( T_3 \) obtained similarly with the roles of \( a \) and \( b \) interchanged.

This gives by summing the three contributions

\[
\lim_{N \to \infty} E \left[ \frac{1}{N} \text{Tr}[A_1 B_1 A_2 B_2] \right] = \Phi(a_1 \circ a_2) \times \Phi(b_1 \circ b_2)
+ \left[ \Phi(a_1^2) - \Phi(a_1 \circ a_2) \right] \times \Phi(b_1) \Phi(b_2)
+ \left[ \Phi(b_1^2) - \Phi(b_1 \circ b_2) \right] \times \Phi(a_1) \Phi(a_2). \tag{2.1}
\]

Using this formula for \( \hat{a} := a \circ \Phi(a) \) and \( \hat{b} := b \circ \Phi(b) \) yields \( \Phi(\hat{a}_1 \hat{b}_1 \hat{a}_2 \hat{b}_2) = \Phi(\hat{a}_1 \circ \hat{a}_2) \times \Phi(\hat{b}_1 \circ \hat{b}_2) \). If this quantity is not zero then \( A_N \) and \( B_N \) are not asymptotically free independent. Writing \( \Phi(\hat{a}_1 \circ \hat{a}_2) = \Phi(a_1 \circ a_2) \Phi(a_1) \Phi(a_2) \) yields the following criterion which is useful for applications.

**Proposition 2.16.** Let \( A_N \) and \( B_N \) be two asymptotically traffic independent families. Denote \( \Phi_N = E \left[ \frac{1}{N} \text{Tr} \cdot \right] \). Define for any self-adjoint *-polynomials \( P, Q \),

\[
\mathcal{R}(P, Q, (A_N)_{N \geq 1}) :=
\lim_{N \to \infty} \left( \Phi_N(P(A_N) \circ Q(A_N)) - \Phi_N(P(A_N)) \times \Phi_N(Q(A_N)) \right)
\]

where \( \circ \) denotes the entry-wise product of matrices, and similarly define \( \mathcal{R}(P, Q, (B_N)_{N \geq 1}) \). Assume that \( \mathcal{R}(P, Q, (A_N)_{N \geq 1}) \) and \( \mathcal{R}(\hat{P}, \hat{Q}, (B_N)_{N \geq 1}) \) are nonzero for some \( P, Q, \hat{P}, \hat{Q} \). Then \( A_N \) and \( B_N \) are not asymptotically free independent.

**Example 2.17.**

- **Diagonal matrices:** Denote \( D_N \) a family of diagonal matrices. Then \( \mathcal{R}(P, Q, (D_N)_{N \geq 1}) \) is the covariance

\[
\lim_{N \to \infty} \Phi_N(P(D_N) \times Q(D_N)) - \Phi_N(P(D_N)) \times \Phi_N(Q(D_N))
\]

of the limiting *-distribution. So, by the Cauchy Schwartz inequality, it is zero for all \( P \) and \( Q \) if and only if for all matrices \( D_N \) of \( D_N \) one has \( \lim_{N \to \infty} (\Phi_N(D_N^2) - \Phi_N(D_N)^2) = 0 \).
2.4. Proof of the main theorem

2.4.1 Presentation of the method

In order to compute the limiting traffic distribution of several random matrices (and to prove Theorem 1.8) we often use the injective trace \( \tau^0_N \) defined in Section 2.1. This interest is that it is easy to relate \( \tau^0_N \) with moments of the entries of the matrices.

Lemma 2.18. Let \( \mathbf{A}_N = (A_j)_{j \in J} \) be a family of random matrices. Let \( T = (V, E, \gamma, \epsilon) \) be an \( * \)-test graph and let \( \phi_N \) be a random injective map \( V \rightarrow [N] \), uniformly distributed, independent of \( \mathbf{A}_N \). Denote

\[
\delta^0_N[T(\mathbf{A}_N)] := \mathbb{E} \left[ \prod_{(v, w) \in E} A^\epsilon(\gamma)_{\phi_N(w), \phi_N(v)} \right].
\]

1. If \( \mathbf{A}_N \) is a permutation invariant family of random matrix, then for any \( T = (V, E, \gamma, \epsilon) \), one has

\[
\delta^0_N[T(\mathbf{A}_N)] = \mathbb{E} \left[ \prod_{(v, w) \in E} A^\epsilon(\gamma)_{\phi(w), \phi(v)} \right]
\]

for any injective map \( \phi : V \rightarrow [N] \). Hence the function \( \delta^0_N \) computes joint moments in the entries of the matrices of \( \mathbf{A}_N \).

2. For any \( * \)-test graph \( T \) with vertex set \( V \) and any family \( \mathbf{A}_N \) of matrices, one has

\[
\tau^0_N[T(\mathbf{A}_N)] = \frac{1}{N} \frac{N!}{(N - |V|)!} \delta^0_N[T(\mathbf{A}_N)]. \tag{2.1}
\]

• Matrices with independent entries: Let \( d > 0 \) and let \( A_N \) be a Hermitian matrix whose sub-diagonal entries are independent and distributed according to the Bernoulli distribution with parameter \( \frac{d}{N} \). Then, by independence of the entries of \( A_N \),

\[
\Phi_N(A_N^* \circ A_N) - \Phi_N(A_N^2) = \mathbb{E} \left[ \frac{1}{N} \sum_{i,j} (A_N(i,j)^2)^2 \right] - \mathbb{E} \left[ \frac{1}{N} \sum_{i,j} A_N(i,j)^2 \right]^2 = \mathbb{E} \left[ \frac{1}{N} \sum_{i,j} A_N(i,j)^4 \right] + o(1) = \frac{d}{N} \sum_{i,j} d = d > 0.
\]

This particular model is studied in a more general framework in [Ma] for which application of the convergence in traffic distribution is specified.

• Adjacency matrices of non regular graphs: Let \( G_N \) be a random graph on the set of vertices \([N]\). Assume that \( G_N \) is a directed simple graph (with no loops nor multiple edges). The adjacency matrix \( A_N \) of \( G_N \) is the random matrix with entries in \([0, 1]\), such that \( A_N(i,j) = 1 \) if and only if there is an edge from \( i \) to \( j \), and \( A_N(i,i) = 0 \) for all \( i \).

For a vertex \( v \) uniformly chosen at random, denote \( d_N = \sum_{j=1}^N A_N(v,j) \) (the number of neighbors of \( v \), in the sense of the direction of the edges). Then \( \Phi_N(A_N A^*_N) = \mathbb{E} \left[ \frac{1}{N} \sum_{i,j=1}^N A_N(i,j)^2 \right] = \mathbb{E}[d_N] \) since \( A_N(i,j) \in \{0, 1\} \) and

\[
\Phi_N(A_N A^*_N \circ A_N A_N^*) = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \left( \sum_{j=1}^N A_N(i,j)^4 \right) \right] = \mathbb{E}[d_N^2].
\]

Hence \( \delta(xx^*, xx^*, (A_N)_{N \geq 1}) = \text{Var}(d_N) \). So if the degree of a vertex of \( G_N \) uniformly chosen does not converge to a constant, then the adjacency matrix \( A_N \) of \( G_N \) satisfies Proposition 2.16.

2.4 Proof of the main theorem
CHAPTER 2. DEFINITION OF ASYMPTOTIC TRAFFIC INDEPENDENCE

Proof. 1. Let $\phi : V \to [N]$ be an injective map. Let $\sigma_N$ a random permutation of $[N]$ uniformly distributed and independent of $A_N$. By the proof of Lemma 1.3 since $A_N$ is permutation invariant one has

$$E\left[ \prod_{(v,w) \in E} A^c_{\gamma(v,w)}(\phi(w), \phi(v)) \right] = E\left[ \prod_{(v,w) \in E} A^c_{\gamma(v,w)}(\sigma_N \circ \phi(w), \sigma_N \circ \phi(v)) \right].$$

But $\sigma_N \circ \phi$ is distributed as a uniform injective map $\phi_N : V \to [N]$ independent of $A_N$, hence the result.

2. We get the result by the following interpretation of the sum as an expectation

$$\tau^0_N[T(A_N)] = \frac{1}{N} \times \text{Card} \left\{ \phi : V \to \{1, \ldots, N\} \mid \phi \text{ injective} \right\} \sum_{\phi V \to \{1, \ldots, N\}} \prod_{(v,w) \in E} A^c_{\gamma(v,w)}(\phi(w), \phi(v))$$

$$= \frac{1}{N} \frac{N!}{(N - |V|)!} E \left[ \prod_{(v,w) \in E} A^c_{\gamma(v,w)}(\phi_N(w), \phi_N(v)) \right].$$

Let see an application of this lemma, which is a sort of asymptotic independence theorem for the entry-wise product of matrices. We will use it later as illustrations of traffic independence.

**Corollary 2.19.** Let $A_N = (A_j)_{j \in J}$ and $B_N = (B_j)_{j \in J}$ be two independent families of random matrices. Assume the following hypotheses.

1. The family $A_N$ converges in traffic distribution, that is for any *-test graph $T$ the limit $\tau^0[T] := \lim_{N \to \infty} \tau^0_N[T(A_N)] = N^{|V|-1}(1 + O\left(\frac{1}{N}\right)) \delta^0_N[T(A_N)]$ exists.

2. For any *-test graph $T$, the limit $\delta^0[T] := \lim_{N \to \infty} \delta^0_N[T(B_N)]$ exists.

3. One of the two families $A_N$, $B_N$ is permutation invariant.

Denote $C_N = (A_j \circ B_j)_{j \in J}$, where $\circ$ stands for the entry-wise product of matrices. Then $C_N$ converges in traffic distribution. Moreover, for any *-test graph $T$, one has

$$\lim_{N \to \infty} \tau^0_N[T(C_N)] = \tau^0[T] \delta^0[T].$$

In the proof of the corollary we see an argument that appears also in the proof of Theorem 1.8

**Example 2.20.** Let $A_N$ be a random matrix converging in traffic distribution. Let $B_N = (\omega_{i,j})_{i,j=1,\ldots,N}$ be a matrix with i.i.d. entries, possibly up to the Hermitian symmetry. Assume that the distribution of the $\omega_{i,j}$’s is independent of $N$, invariant by complex conjugate, and that $E[|\omega_{i,j}|^K] < \infty$ for each $K \geq 1$. Then $A_N \circ B_N$ converges in traffic distribution. See Section 5.2 for applications.

**Proof.** Let $T = (V, E, \gamma, \varepsilon)$ be a *-test graph and $\phi_N$ a uniform injective map $V \to [N]$. Then $\tau^0_N[T(C_N)] = N^{|V|-1}(1 + o\left(\frac{1}{N}\right)) \times \delta^0_N[T(C_N)]$. Moreover, one has

$$\delta^0_N[T(C_N)] = E \left[ \prod_{e = (v,w) \in E} A^c_{\gamma(v,w)}(\phi_N(w), \phi_N(v)) \times B^c_{\gamma(v,w)}(\phi_N(w), \phi_N(v)) \right].$$

Assume that $B_N$ is permutation invariant. Let $V$ be a random permutation matrix, associated to a uniform permutation $\sigma_N$ independent of $A_N$. Then

$$\delta^0_N[T(C_N)] = E \left[ \prod_{e = (v,w) \in E} A^c_{\gamma(v,w)}(\phi_N(w), \phi_N(v)) \times B^c_{\gamma(v,w)}(\sigma_N \circ \phi_N(w), \sigma_N \circ \phi_N(v)) \right].$$
2.4. PROOF OF THE MAIN THEOREM

Since \( \sigma_N \) is uniform and independent of \((A_N, B_N, \phi_N)\), the quadruple \((A_N, B_N, \phi_N, \sigma_N \circ \phi_N)\) has the same law as \((A_N, B_N, \phi_N, \phi'_N)\) where \( \phi'_N \) is a uniform injection independent of \((A_N, B_N, \phi_N)\). Hence, by independence of \((A_N, \phi_N)\) and \((B_N, \phi'_N)\) we obtain

\[
\delta_N^0 \left[ T(C_N) \right] = \mathbb{E} \left[ \prod_{e=(v,w) \in E} A^{(e)}_{\gamma(e)}(\phi_N(v), \phi_N(w)) \right] \times \mathbb{E} \left[ B^{(e)}_{\gamma(e)}(\phi'_N(v), \phi'_N(w)) \right]
\]

\[
= \delta_N^0 \left[ T(A_N) \right] \times \delta_N^0 \left[ T(B_N) \right].
\]

Note that the role of \( A_N \) and \( B_N \) can be interchanged so there is no limitation in assuming \( B_N \) permutation invariant. We obtain

\[
\delta_N^0 \left[ T(C_N) \right] = N^{k-1} \left( 1 + o \left( \frac{1}{N} \right) \right) \delta_N^0 \left[ T(A_N) \right] \delta_N^0 \left[ T(B_N) \right] + o(1),
\]

as expected.

\[
\square
\]

An important technical aspect of Lemma \ref{lem:renorm} is that it tells how we have to renormalized the joint moments the entries of the matrices \( A_N \) in order to compute its limiting distribution, and relies directly this normalization to a topological constant of \( T \). Indeed, \( \tau_N^0 \left[ T(\cdot) \right] \) is multiple of \( \delta_N^0 \left[ T(\cdot) \right] \) by a constant equivalent to \( N^{V-1} \). In practice, since the map \( \delta_N^0 \left[ T(\cdot) \right] \) is multilinear with respect to the edges of \( T \), for several matrix models \( A_N \) it is possible to write \( \tau_N^0 \left[ T(A_N) \right] = N^{k-1+|E|} \times \eta_N \) where \( \mathcal{G} = (V, E) \) is a graph depending on \( T \) well chosen and \( \eta_N \) is bounded. Thank to Lemma \ref{lem:classification} we can classify the graphs \( T \) that contribute in the limit.

### 2.4.2 Proof

We prove the theorem for two families, denoted in short \( A_1 = A_N^{(1)} \) and \( A_2 = A_N^{(2)} \), assuming \( A_2 \) permutation invariant. The general case is obtained by recurrence on the number of families using that the unions of the families also satisfies the assumptions of the theorem.

**Two lemmas**

In this proof, we call \( * \)-graph the objects formally defined as the \( * \)-test graph with the connectedness condition on the graph omitted (a \( * \)-graph is a finite disjoint union of \( * \)-test graphs). The trace and invective trace of a \( * \)-graph in matrices are defined as for \( * \)-test graph. If the connected components of \( T \) are the \( * \)-test graphs \( T^{(1)}, \ldots, T^{(k)} \), one sees immediately that \( \text{Tr} \left[ T(A_N) \right] = \prod_{i=1}^k \text{Tr} \left[ T^{(i)}(A_N) \right] \).

**Lemma 2.21** (Splitting the contribution due to \( A_1 \) and \( A_2 \)). Let \( T \) be a \( * \)-graph in the variables \( x_1 \) and \( x_2 \). For \( i = 1, 2 \), we denote by \( T_i = (V_i, E_i, \gamma_i, \varepsilon_i) \) the \( * \)-graph obtained from \( T \) by considering only the edges with a label in \( x_i \) and by deleting the vertices that are not attached to any edge after this process. Hence \( T_i \) is the union of the colored component labeled \( x_i \) of \( T \). We have

\[
\tau_N^0 \left[ T(A_1, A_2) \right] = N \times \frac{(N-|V_1|)! (N-|V_2|)!}{(N-|V|)!} \left\{ \tau_N^0 \left[ T_1(A_1) \right] \times \tau_N^0 \left[ T_2(A_2) \right] \right\}.
\]

**Proof.** Denote \( A_N = A_1 \cup A_2 = (A_j)_{j \in J} \), as in Lemma \ref{lem:renorm} we denote for any \( * \)-graph \( T = (V, E, \gamma, \varepsilon) \),

\[
\delta_N^0 \left[ T(A_N) \right] = \mathbb{E} \left[ \prod_{e=(v,w) \in E} A^{(e)}_{\gamma(e)}(\phi_N(v), \phi_N(w)) \right],
\]

where \( \phi_N \) is a uniform injective map \( V \to [N] \) independent of \( A_N \), and we have

\[
\tau_N^0 \left[ T(A_N) \right] = \frac{1}{N} \times \frac{N!}{(N-|V|)!} \times \delta_N^0 \left[ T(A_N) \right],
\]

(2.3)
still valid when $T$ is not connected. Moreover, one has
\[
\delta_N^0 \left[ T(A_N) \right] = \mathbb{E} \left[ \prod_{e=(v,w) \in E_1} A^{(e)}_{\gamma_1(e)}(\phi_N(w), \phi_N(v)) \prod_{e=(v,w) \in E_2} A^{(e)}_{\gamma_2(e)}(\phi_N(w), \phi_N(v)) \right].
\]

Let $W$ be a random permutation matrix, associated to a uniform permutation $\sigma_N$ independent of $A_N$. The family $A_2$ has the same distribution as $WA_2W^*$, and denoting $\phi_N(e) = (\phi_N(w), \phi_N(v))$ for any edge $e = (v, w)$, then for $e \in E_2$ one has $(WA_2W^*\gamma(e))(\phi_N(e)) = A^{(e)}(\phi_N \circ \phi_N)$. Since $\sigma_N$ is uniform and independent of $(A_N, \phi_N)$, the triplet $(A_N, \phi_N, \sigma_N \circ \phi_N)$ has the same law as $(A_N, \phi_N, \phi_N')$ where $\phi_N'$ is a uniform injection independent of $(A_N, \phi_N)$. Hence we get
\[
\delta_N^0 \left[ T(1, A_2) \right] = \frac{1}{N} \times \left( \frac{N!}{(N-|V|)!} \right) \times \mathbb{E} \left[ \prod_{e \in E_1} A^{(e)}(\phi_N(e)) \right] \times \mathbb{E} \left[ \prod_{e \in E_2} A^{(e)}(\phi_N'(e)) \right] = \frac{1}{N} \left( \delta_N^0 [T_1(A_1)] \right) \delta_N^0 [T_2(A_2)].
\]

Using again (2.3) for the graphs $T_1$ and $T_2$ yields (2.2). \hfill \Box

**Lemma 2.22** (Decomposition of components).

For any $j = 1, 2$ and any finite $\ast$-graph $T = (V, E)$ in the variables $x_j$ whose connected components are denoted by $T_1, \ldots, T_n$, one has
\[
\mathbb{E} \left[ \frac{1}{N^n} T^0_n [T(A_j)] \right] - \mathbb{E} \left[ \prod_{i=1}^n \frac{1}{N} T^0_i [T[A_j]] \right] = O \left( \frac{1}{N} \right).
\]

**Proof.** By the relation between the injective and the standard one,
\[
T^0 [T(A_j)] = \sum_{\pi \in P(V)} \mu_V(\pi) \text{ Tr} [T^\pi (A_j)]. \tag{2.4}
\]

Denote by $K_\pi$ the number of components of $T^\pi$. By the factorization property, we have that $\mathbb{E} \left[ \frac{1}{N^n} \text{ Tr} [T^\pi (A_j)] \right]$ converges and so is bounded. Denote by $\Lambda$ the set of partitions $\pi$ of $P(V)$ such that two vertices of different components never belong to a same block of $\pi$. For $\pi \in \Lambda$ one can decompose $\pi$ into partitions $\pi_i$ of $V_i$, $i = 1, \ldots, n$ and we denote $\pi = \pi_1 \sqcup \cdots \sqcup \pi_n$. We then obtain, since $K_\pi = n$ if and only if $\pi \in \Lambda$,
\[
\mathbb{E} \left[ \frac{1}{N^n} T^0_n [T(A_j)] \right] = \sum_{\pi \in P(V)} \mu_V(\pi_1 \sqcup \cdots \sqcup \pi_n) \mathbb{E} \left[ \prod_{i=1}^n \frac{1}{N} \text{ Tr} [T^{\pi_i} (A_j)] \right] + O \left( \frac{1}{N} \right).
\]

By Lemma 2.6 and the formula for $\mu_V$, one has $\mu_V(\pi_1 \sqcup \cdots \sqcup \pi_n) = \mu_V(\pi_1) \times \cdots \times \mu_V(\pi_n)$. Hence the result. \hfill \Box

**Proof of the asymptotic traffic independence of $A_1$ and $A_2$.**

By Lemmas 2.21 and 2.22 for any $\ast$-test graph $T$ in the variables $x_1$ and $x_2$, one has
\[
\delta_N^0 \left[ T(A_1, A_2) \right] = \frac{N \times (N - |V_1|)! (N - |V_2|)!}{(N - |V|)! N!} \times N^{K_1-1} \times N^{K_2-1} \times \left( \prod_{i=1}^2 \mathbb{E} \left[ \prod_{k=1} K_i \frac{1}{N} T^0_{i,k} (A_i) \right] \right) + O \left( \frac{1}{N} \right).
\]
where the $T_{i,k}$`s are the connected components of $T$ that are labelled by $x_i$, for $i = 1, 2$ and $k = 1, \ldots, K_i$, $|V|$ is the number of vertices of $T$ attached to some edges labelled in $x_i$, for $i = 1, 2$, and $|V|$ is the number of vertices of $T$. Remark that $\Gamma_N = N^n(1 + O(1/N))$ where $\eta = K_1 + K_2 + |V| - |V_1| - |V_2| - 1$. Note that by the factorization property and Lemma 2.20 one has

$$E \left[ \prod_{k=1}^{K_i} \frac{1}{N} \text{Tr}^0[T_{i,k}(A_i)] \right] = \prod_{k=1}^{K_i} \lim_{N \to \infty} \tau_N^0[T_{i,k}(A_i)] + o(1).$$

So it remains to prove that $\Gamma_N \to 1$ if the graph $GCC(T)$ of colored components of $T$ with respect to $x_1, x_2$ is a tree, and $\Gamma_N \to 0$ otherwise.

Recall that the set of vertices of $GCC(T)$ is the disjoint union of the set of colored components $T_{i,k}$, $i = 1, 2$, $k = 1, \ldots, K_i$, and the set $\delta V$ of vertices of $T$ that belong to simultaneously to $T_1$ and $T_2$. For each $v$ in $\delta V$ there is an edge toward each connected component of $T_1$ and $T_2$ the vertex $v$ belongs to. Hence the graph has $|V| := |GCC(T)| + |\delta V|$ vertices, it has $|E| := 2|\delta V|$ edges and, noting that $K_1 + K_2 = |GCC(T)|$ we have $\eta = |V| - |E| - 1$. By the relation between the number of vertices and the number of edges in a graph applied to $GCC(T)$ (Lemma 2.13), we get that $\eta < 0$ with equality if and only if $GCC(T)$ is a tree. We then get the expected result: for any $*-$test graph $T$,

$$\tau_N^0[T(A_1, A_2)] = \left( 1 (GCC(T) \text{ is a tree }) + O(\frac{1}{N}) \right) \times \left( \prod_{i=1}^{K_i} \prod_{k=1}^{K_i} \tau_N^0[T_{i,k}(A_i)] + o(1) \right).$$

Remark 2.23. 1. Suppose that $A_1$ and $A_2$ do not satisfy the factorization property, namely assumption (3) of Theorem 1.8. Suppose instead that $E \left[ \prod_{i=1}^{K_i} \frac{1}{N} \text{Tr}^0[T_{i,k}(A_i)] \right]$ converges for any test graphs $T_1, \ldots, T_{K_j}$, $j = 1, 2$. Then it remains true that $(A_1, A_2)$ converges in traffic distribution. The families of matrices are not asymptotically traffic independent. Indeed, we see from Lemma 2.9 that $E \left[ \prod_{i=1}^{K_i} \frac{1}{N} \text{Tr}^0[T_{i,k}(A_i)] \right]$ converges also. It will be clear in the next step that we also get that $E \left[ \prod_{i=1}^{K_i} \frac{1}{N} \text{Tr}[T_{i,k}(A_1, A_2)] \right]$ converges.

2. Assume that $A_1$ consists in a family of diagonal matrices. Then it is clear that $\tau_N^0[T(A_1, A_2)] = \tau_N^0[\tilde{T}(A_1, A_2)]$ where $\tilde{T}$ is obtained by identifying source and target of each edge of $T$. Hence, there is a single colored component of $\tilde{T}$ whose labels correspond to $A_2$. Hence we do not need to assume the factorization property for $A_2$.

Proof of the factorization property for $(A_1, A_2)$.

Lemma 2.24. Let $A_N$ be a family of matrices and $S$ be a $*-$graph whose connected components are $T_1, \ldots, T_K$. Then,

$$\text{Tr}^0[T_1(A_N)] \cdots \text{Tr}^0[T_K(A_N)] = \sum_{\pi \in \mathcal{P}^*(V)} \text{Tr}^0[S^\pi(A_N)],$$

where $\mathcal{P}^*(V)$ is the set of partitions of $V$ that contain at most one vertex of each $S_k$, $k = 1, \ldots, K$.

Proof of Lemma 2.24. We write $S = (V, E, \gamma, \varepsilon)$ and denote by $V_k$ the set of vertices of $S_k$, $k = 1, \ldots, K$. Then, denoting $\phi(e) = (\phi(w), \phi(v))$ for any edge $e = (v, w)$,

$$\text{Tr}^0[T_1(A_N)] \cdots \text{Tr}^0[T_K(A_N)] = \sum_\phi \prod_{e \in E} A_{e}^\varepsilon(\phi(e)),$$

where the sum is over all maps $\phi : V \to \{1, \ldots, N\}$ such that $\phi|_{V_1}, \ldots, \phi|_{V_K}$ are injective. The sum over $\pi$ in the lemma represents all the possible situations of overlapping of the images of $\phi|_{V_1}, \ldots, \phi|_{V_K}$. \qed
Let $S, T_1, \ldots, T_K$ be as in the lemma:

$$E\left[\prod_{i=1}^{K} \frac{1}{N} \text{Tr}^0[\mathbf{T}_i(A_1, A_2)]\right] = \sum_{\pi \in \mathcal{P}(V)} \frac{1}{N^{K-1}} \tau^0_N \left[S^\pi(A_1, A_2)\right]. \tag{2.5}$$

Let $\pi \in \mathcal{P}(V)$ and denote by $K_\pi \leq K$ the number of components of $S^\pi$. We apply Lemmas 2.21 and 2.22 as in the previous step with $T = S^\pi$, with the same notations:

$$\frac{1}{N^{K-1}} \tau^0_N [T(A_1, A_2)] = \tilde{\Gamma}_N \times \left(\prod_{i=1}^{2} \left(\prod_{k=1}^{K_i} \frac{1}{N} \text{Tr}^0[T_{i,k}(A_i)]\right) + O\left(\frac{1}{N}\right)\right)$$

where the $T_{i,k}$ are the colored components of $T$ and now $\tilde{\Gamma}_N = N^{|V|-|E|-K_\pi} \times (1 + O\left(\frac{1}{N}\right))$. Note that $K_\pi$ is also the number of connected components of $\text{GCC}(T)$, so by Lemma 2.13, we get that $\tilde{\Gamma}_N \xrightarrow{N \to \infty} 1$ if $\text{GCC}(T)$ is a forest and it tends to zero otherwise. Hence the only partition $\pi$ which contributes in (2.5) is the trivial partition and we get

$$E\left[\prod_{i=1}^{K} \frac{1}{N} \text{Tr}^0[\mathbf{T}_i(A_1, A_2)]\right] \xrightarrow{N \to \infty} \prod_{i=1}^{K} \lim_{N \to \infty} \tau^0_N \left[T_i(A_1, A_2)\right].$$

This yields, together with Lemma 2.9, that $(A_1, A_2)$ satisfies the factorization property.
Chapter 3

Examples and applications for classical large matrices

We prove the convergence in traffic distribution of a family of independent Wigner matrices and give several applications. We also prove the convergence in traffic distribution of a uniform permutation matrix and of a Haar unitary matrix, and compare these two models. The last section is devoted to the case of Diagonal matrices and to a remark on the factorization property.

Denote $\tau_N^0 = \mathbb{E}[\frac{1}{N} \text{Tr}^0 \cdot]$ the mean normalized injective trace (Definition 2.5).

3.1 Wigner matrices

3.1.1 Limiting distribution of independent matrices

**Proposition 3.1** (The limit of Wigner matrices).

Let $X_N = (X_j)_{j \in J}$ be a family of independent Wigner matrices (Definition 0.2), whose entries have the same law as their complex conjugates. Then $X_N$ has a limiting traffic distribution. Denote the parameter of $X_j$ by $p_{\alpha_j, \beta_j}$ for any $j \in J$. Let $T = (V, E, \gamma)$ be a test graph in variables $x = (x_j)_{j \in J}$, with no edge labeled $x^\dagger$ (we deduce the general distribution as the matrices are Hermitian).

Say that $T$ is a fat tree whenever it becomes a tree if the multiplicity of the edges and the orientation are forgotten. It is called a double tree if moreover the multiplicity of the edges is always two. Let call twin edges of a double tree two edges that share the same vertices. We say that $T$ is colored if twin edges $e, e'$ of $T$ have the same label $\gamma(e) = \gamma(e') \in J$ so they correspond to the same matrix $X_{\gamma(e)}$. For a double tree $T$ we denote by $\alpha_j^T, \beta_j^T$ the number of twin edges labeled $x_j$ with opposing (respectively similar) orientation.

Then for any test graph $T$ one has

$$\tau_N^0 \left[ T(X_N) \right] \xrightarrow{N \to \infty} \#(T \text{ is a colored double tree}) \prod_{j \in J} \alpha_j^T \beta_j^T.$$ (3.1)

In particular, if the matrices are real Wigner matrices with parameter $(1, 1)$, $\tau_N^0 \left[ T(X_N) \right]$ is asymptotically the indicator that $T$ is a colored double tree, and if they are complex matrices with parameter $(1, 0)$ the assumptions that twin edges of $T$ have different orientation is required.

**Proof of Lemma 3.1** Without lose of generality we assume $\alpha_j = 1$ for any $j \in J$. Denote $M_N = (M_j)_{j \in J}$. Consider a test graph $T = (V, E, \gamma)$ in variables $x$ but not the adjoint. By multi-linearity of $\tau_N^0 \left[ T(\cdot) \right]$ with respect to the edges of $T$ and since the family is permutation invariant, one has

$$\tau_N^0 \left[ T(X_N) \right] = N^{-1} \tau_N^0 \left[ T(M_N) \right] = N^{|V|-1} \tau_N^0 \left[ T(M_N) \right] + O(N^{-1}),$$

where we set $\delta_N^0 \left[ T(M_N) \right] = \mathbb{E}[\prod_{(v, w) \in E} M_{\gamma(v)}(\phi(w), \phi(v))]$, as in Formula (2.1), for any injection $\phi : V \to [N]$. The quantity $\delta_N^0 \left[ T(M_N) \right]$ is bounded and can be computed since the entries of $M_j$
are independent and independent of $N$. By centering of the entries it is zero whenever $T$ has an edge of multiplicity one. Let $T$ such that each edge has at least of multiplicity two.

We apply Lemma 2.13 to the graph $\mathcal{G} = (V, E)$ obtained from $G$ by forgetting the multiplicity if its edges (and their orientation and labels). Since $N^{|V|-1-|E|/2} = N^{|V|-1-|E|} \times N^{|E|-|E|/2}$, we then get that $\tau_N^0[T(X_N)]$ converges to zero if it $T$ is not a double tree. Moreover, by independence of the entries, when $T$ is a double tree then $\tau_N^0[T(X_N)] = \delta_N^0[T(M_N)]$ is the product of terms of the form $\mathbb{E}[M_N(k, \ell)^2]$ or $\mathbb{E}[M_N(\ell, k)^2]$ for $k \neq \ell$ along each twin edges. Hence if the double tree is not colored then $\delta_N^0[T(X_N)]$ converges to zero. Otherwise, the independence of the matrices and their entries give the expected formula. □

3.1.2 Applications

Practical computations.

The limiting traffic distribution of independent Wigner matrices yields the following computations.

1. Limit of $\Phi_N(P(X_N))$. Let $n \geq 1$ be an integer and let us prove Wigner’s law, namely for $\alpha = 1$ the convergence of $m_{n}^{(N)} := \mathbb{E}\left[\frac{1}{N} \text{Tr} X_N^{\alpha}\right] \underset{N \rightarrow \infty}{\longrightarrow} a_{\alpha}$ where $a_{\ell}$ is zero if $\ell$ is not an integer and is the $\ell$-Catalan number $\frac{1}{\ell + 1} \binom{2\ell}{\ell}$ otherwise. This proof is similar to the one in [Gui09].

We know from Example 1.5 that we can write $m_{n}^{(N)} = \tau_N^{0}[T_n(X_N)]$, where $T_n$ is the simple circle of length $n$ with edges oriented along the cycle (and all edges have the same label, associated to the matrix $X_N$). Now, denoting by $V_n$ the vertex set of $T_n$, by the relation (2.2) relating the trace and the injective trace and by Lemma 3.1 one gets

$$m_{n}^{(N)} = \sum_{\pi \in V_n} \tau_N^{0}[T_n^{\pi}(X_N)] = \sum_{\pi \in V_n} \mathbb{I}(T_n^{\pi} \text{ is a double tree}) + o(1).$$

We then see that the $n$-th moment $m_{n}^{(N)}$ converges to the number of double trees $T_n^{\pi}$ we can obtain from the simple cycle $T_n$. This number is zero if $n$ is odd. Moreover, choosing a double tree is equivalent to choose the pairs that form double edges. These pair partitions are the non crossing partitions [Gui09], for which it is known that the cardinals are the Catalan numbers [NS06].

2. Limit of $\Phi_N(P(X_N) \circ Q(X_N))$. We can see that for any *-polynomials $P, Q$, one has $\Phi_N(P(X_N) \circ Q(X_N)) = \Phi_N(P(X_N)) \Phi_N(Q(X_N)) + o(1)$. Indeed, according to the second application of Example 1.5 for any monomials $P, Q$, one has $\Phi_N(P(X_N) \circ Q(X_N)) = \tau_N[T(X_N)]$ where $T$ is the *-test graph consisting in a simple cycle $T_1$ of length the degree of $P$ and a simple cycle $T_2$ of length the one of $Q$, both oriented and attached together by identifying a single vertex of each cycle. Edges are labeled accordingly to the monomials $P$ and $Q$ following the orientation of the respective cycle.

The partitions $\pi$ of $T$ such that $T^{\pi}$ is a double tree are those for which the induced subgraphs of $T_1$ and $T_2$ are double trees that have in common a single vertex. They are given by two
partitions $\tau_1$ of $T_1$ and $\tau_2$ of $T_2$ such that $T_1^{\tau_1}$ and $T_2^{\tau_2}$ are double trees and then form the smallest partition $\bar{\tau}$ of $V$ whose blocks contain both those of $\tau_1$ and $\tau_2$. Hence we have

$$\tau_N[T(X_N)] = \sum_{\tau \in P(V)} \tau^0_N[T_1(X_N)] \tau^0_N[T_2(X_N)] + o(1)$$

We have to prove that $X$ the conjugates are asymptotically traffic independent.

**Proof.** Let $X_N = (X_j)_{j \in J}$ be such a family. We have seen under the assumptions of Lemma 3.1 that $X_N$ converges in traffic distribution. Let $T = (V, E, \gamma, \epsilon)$ be a $*$-test graph in variables $x$. Without using Theorem 1.8, we get from Proposition 3.1 (with respect to $x_j, j \in J$) denote by $V_S$ its set of vertices. Denote also $\delta V_S$ the set of vertices of $S$ that are also contained in another colored component, i.e $\delta V_S = V_S \cap (\bigcup_{S \neq S'} V_{S'})$. Denote $\hat{V}_S = V_S \setminus \delta V_S$ and $\delta V = \cup_S \delta V_S$. Note that we have the equality

$$\sum_S (|V_S| - 1) = \sum_S (|\delta V_S| + |\hat{V}_S| - 1) = |V| - |\delta V| + \sum_S |\delta V_S| - |\bar{\gamma}(T)|.$$ 

**First example of asymptotic traffic independent matrices.**

Without using Theorem 1.8 we get from Proposition 3.1

**Lemma 3.2.** Independent Wigner matrices whose entries have the same law as their complex conjugates are asymptotically traffic independent.

**Proof.** Let $X_N = (X_j)_{j \in J}$ be such a family. We have seen under the assumptions of Lemma 3.1 that $X_N$ converges in traffic distribution. Let $T = (V, E, \gamma, \epsilon)$ be a $*$-test graph in variables $x$. We have to prove that $T$ is a colored double tree if and only if its graph of colored components is a tree and its colored components are double trees. This fact can be clearly seen in pictures. We prove it with the same method as in Lemma 3.1.

![Figure 3.2: The decomposition in colored components of a colored double tree.](image)

If $T$ has an edge of multiplicity one then $T$ is not a double tree and it has a colored component that is not a double tree. So both the limiting distribution of Wigner matrices and the product of the marginal limits vanish on $*$-test graph that have an edge of multiplicity one. Assume from now $T$ has no edge of multiplicity one. For each $S \in \bar{\gamma}(T)$ colored component of $T$, denote by $E_S$ the set of edges of $T$ and by $|E_S|$ its number of edges without multiplicity. By Lemma 2.13 and the proof of Lemma 3.1, for a graph $T$ with no simple edge, the quantity

$$\eta(T) = \left(|\hat{V} - \sum_{S \in \bar{\gamma}(T)} |E_S|\right) + \left(|\hat{E}| - \frac{|E|}{2}\right) + \left(|V| - 1 - |\hat{E}|\right)$$

is zero if and only if $T$ is a colored double tree. We can rewrite $\eta(T)$ in terms of quantities for the graph of colored components of $T$ as follow. First for each colored component $S$ of $T$ (with respect to the $x_j, j \in J$) denote by $V_S$ its set of vertices. Denote also $\delta V_S$ the set of vertices of $S$ that are also contained in another colored component, i.e $\delta V_S = V_S \cap (\bigcup_{S \neq S'} V_{S'})$. Denote $\hat{V}_S = V_S \setminus \delta V_S$ and $\delta V = \cup_S \delta V_S$. Note that we have the equality

$$\sum_S (|V_S| - 1) = \sum_S (|\delta V_S| + |\hat{V}_S| - 1) = |V| - |\delta V| + \sum_S |\delta V_S| - |\bar{\gamma}(T)|.$$ 

Moreover, denoting $|V|$ and $|E|$ the number of vertices and edges of the graph of colored components of $T$, one has

$$|V| - 1 - |E| = |CC(T)| + |\delta V| - 1 - \sum_S |\delta V_S|.$$
Hence summing the two above identities yields an expression for $|V| - 1$ that gives

$$\eta(T) = \left( |E| - \frac{|E|}{2} \right) + (|V| - 1 - |E|) + \sum_{S} (|V_S| - 1 - |E_S|).$$

Since $T$ has no simple edges, the three terms in the r.h.s are nonpositive the above equation is zero if and only the multiplicity of the edges of $T$ is two, the graph of colored components of $T$ is a tree and the colored components are double trees. In particular, we get that for real Wigner matrices the limiting traffic distribution of $X_N$ is well the product of the limiting distributions of the $X_i$’s.

Let now consider the case of complex Wigner matrices. It is clear that the twin edges of $X_N$ are asymptotically traffic independent. The graph of colored component of the graph $(\cdot \frac{X_N}{X_N} \cdot)$ with respect to the variables $x$ and $x'$ is not a tree, so we should have $\Phi_N(X_N^2) \xrightarrow{N \to \infty} 0$. But $\Phi_N(X_N^2)$ converges to the variance of $\sqrt{N}X_N(i, j)$ for $i \neq j$ which is nonzero since $X_N$ is a nonzero Wigner matrix.

**Asymptotic free independence with the transpose**

Recall from Theorem 1.1 that independent complex Wigner matrices are asymptotically free independent (it will also be a consequence of Proposition 5.7).

**Lemma 3.3.** Let $X_N = (X_j)_{j \in J}$ be a family of random matrices. Assume that it converges in traffic distribution and has the same limiting distribution as a family of independent complex Wigner matrices with parameter $(\alpha_j, 0)_{j \in J}$. Then the matrices $X_j, X_j^t, j \in J$, are asymptotically free independent.

Asymptotic free independence with the transposed ensemble is studied for a larger class of examples in [JM, CDM16].

**Proof.** We show that $(X_j, X_j^t, j \in J)$ has the same limiting *-distribution as $(Y_j, Y_j^t, j \in J)$, where the $Y_j, Y_j^t, j \in J$ are independent and $Y_j, Y_j^t$ are distributed as $X_j$ for each $j \in J$. Since it is well known that the $Y_j, Y_j^t, j \in J$, are asymptotically *-free this will yield the expected result.

By Lemma 1.6 the limiting *-distribution of the $X_N \cup X_N^t, j \in J$ is entirely determined by the traffic distribution of $X_N$. More precisely, let $T$ be a *-test graph labelled in the variables $(y, y')$ and not their adjoint (no edges are labeled $y^*$ or $(y')^*$, the matrices are self-adjoint). When there is a cycle visiting each edge once in the sense of their orientation, we say that $T$ is cyclic. By Example 1.5 and since the quotient of cyclic graph is cyclic, the limits of $\tau_N^0[T(X_N, X_N^t)]$ for any cyclic *-test graph characterize the limiting joint *-distribution of $(X_N, X_N^t)$.

Moreover, we have $\tau_N^0[T(X_N, X_N^t)] = \tau_N^0[T(X_N)]$, where the orientation of the edges labelled $y'$ has been reversed and labels $y'$ where replaced by $y$. Hence, by the expression of the limiting distribution of $X_N$, $\tau_N^0[T(X_N, X_N^t)]$ converges to one whenever $T$ is a double tree whose twin edges have same label and opposite orientation or different labels and same orientation, and it converges to zero otherwise. But since $T$ is cyclic, it is not possible in a double tree that twin edges have same orientation. Hence for these graphs $T$ the quantities $\tau_N^0[T(X_N, X_N^t)]$ and $\tau_N^0[T(X_N, X_N^t)]$ have the same limit, which concludes the proof.

**Non asymptotic traffic independent matrices**

**Lemma 3.4.** A nonzero complex Wigner matrix $X_N$ is not asymptotically traffic independent from its transpose $X_N^t$.

Together with Lemma 3.3 this gives an example of asymptotically free independent matrices that are not asymptotically traffic independent.

**Proof.** We have $\Phi_N(X_N^2) = \Phi_N(X_N^t X_N^t) = \tau[\cdot \frac{X_N}{X_N} \cdot] = \tau^0[\cdot \frac{X_N}{X_N} \cdot] + o(1)$. Assume that $X_N$ and $X_N^t$ are asymptotically traffic independent. The graph of colored component of the graph $(\cdot \frac{X_N}{X_N} \cdot)$ with respect to the variables $x$ and $x'$ is not a tree, so we should have $\Phi_N(X_N^2) \xrightarrow{N \to \infty} 0$. But $\Phi_N(X_N^2)$ converges to the variance of $\sqrt{N}X_N(i, j)$ for $i \neq j$ which is nonzero since $X_N$ is a nonzero Wigner matrix.
3.2. APPLICATION TO UNITARY HAAR AND UNIFORM PERMUTATION MATRICES

3.1.3 Factorization property

Lemma 3.5. A Wigner matrix $X_N$ satisfies the factorization property (1.3).

Proof. Let $T_1, \ldots, T_n$ be test graphs in one variable, and denote by $T$ the graph obtained as the disjoint union of $T_1, \ldots, T_n$. By Lemma 2.24,

$$E \left[ \prod_{i=1}^{n} \frac{1}{N} \text{Tr}^0 \left[ T_i(X_N) \right] \right] = \sum_{\pi} \frac{1}{N^n} E \left[ \text{Tr}^0 \left[ T^\pi(X_N) \right] \right] = \sum_{\pi} N^{V_\pi - \frac{n}{2} - n} \left(1 + O \left( \frac{1}{N} \right) \right) \delta_N^0 \left[ T^\pi(\sqrt{N}X_N) \right],$$

where the sum is over all partitions whose blocks contain at most one vertex of each $T_i$ and $V_\pi$ denotes the vertex set of $T^\pi$. We have by Lemma 2.13 that $V_\pi - \frac{n}{2} - n \leq 0$ with equality if and only if $\pi$ is the trivial partition with only singleton blocks and the graphs $T_1, \ldots, T_n$ are double trees. By the independence of the entries of $X_N$, we obtain the factorization property

$$E \left[ \prod_{i=1}^{n} \frac{1}{N} \text{Tr}^0 \left[ T_i(X_N) \right] \right] = \prod_{i=1}^{n} \text{1 if } T_i \text{ is a double tree} \delta_N^0 \left[ T_i(\sqrt{N}X_N) \right] + o(1) \xrightarrow{N \to \infty} \prod_{i=1}^{n} \lim_{N \to \infty} \tau_N^0 \left[ T_i(X_N) \right].$$

3.2 Application to unitary Haar and uniform permutation matrices

3.2.1 Limits in traffic distribution

We have already studied the limiting traffic distributions of Wigner matrices in Proposition 3.1. Let now compare the limiting traffic distributions of uniform permutation matrices and unitary Haar matrices.

Proposition 3.6 (The limiting distribution of a uniform permutation matrix). A uniform permutation matrix $V_N$ has a limiting traffic distribution and satisfies the factorization property. Say that a *-test graph is a directed line whenever there is an integer $K \geq 1$ such that the vertices of $T$ are $1, \ldots, K$ and its directed edges are $(1,2), \ldots, (K-1,K)$ labelled $x$ and $(2,1), \ldots, (K,K-1)$ labelled $x^*$, with arbitrary multiplicity. Then, for any *-test graph $T$ in one variable,

$$\tau_N^0 \left[ T(V_N) \right] \xrightarrow{N \to \infty} \begin{cases} 1 & \text{if } T \text{ is a directed line} \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

Proposition 3.7 (The limiting distribution of a unitary Haar matrix). A unitary matrix $U_N$ distributed according to the Haar measure on the unitary group converges in traffic distribution and satisfies the factorization property. Let call simple oriented cycle of a *-test
3.2.3 Practical computations

Before proving this statement (see Section 3.2.4), we show some applications.

3.2.2 Applications

Before proving this statement (see Section 3.2.4), we show some applications.

1. Limiting *-distribution: Let \( V_N \) be a uniform permutation matrix and \( U_N \) be a unitary Haar matrix. For any unitary matrix \( W_N \), the *-distribution of \( W_N \) depends only the limits of \( \mathbb{E} \left[ \frac{1}{N} \text{Tr} W_N^k(W_N^*)^\ell \right] \) for any \( k, \ell \). By [Nic93] and [Voi91], for both \( V_N \) and \( U_N \) this limit is \( \mathbb{I}(k = \ell) \). Let prove this result from the above propositions.

Let \( W_N \in \{ V_N, U_N \} \). Let \( T = T_{k, \ell} \) be the *-test graph in variables \( w, w^* \) consisting in a simple cycle with \( k \) edges labelled \( w \) followed by \( \ell \) edges labelled \( w^* \) in such a way \( \mathbb{E} \left[ \frac{1}{N} \text{Tr} W_N^k(W_N^*)^\ell \right] = \tau_N \left[ T(W_N) \right] \): denote by \( 1, \ldots, k + \ell \) its vertices, by \( e_i = (i, i + 1), i = 1, \ldots, k - 1 \), its edges labelled \( w \) and by \( e_{k+i} = (k+i, k+i+1), i = 1, \ldots, \ell \) its edges labelled \( w^* \) (with the convention \( i_{k+\ell+1} = 1 \)).

There is a partition \( \pi \) of the vertex set of \( T \) such that \( T^{\pi} \) is a line directed if and only if \( k = \ell \) and in that case this partition is necessarily \( \{1\}, \{k\}, \{i, 2k-i\}, i = 2, \ldots, k-1 \}. Hence \( \tau_N \left[ T(W_N) \right] \xrightarrow{N \to \infty} \mathbb{I}(k = \ell) \). Moreover, there is a partition \( \pi \) such that \( T^{\pi} \) is a *-test graph that contributes to the limiting traffic distribution of a unitary Haar matrix if and only if \( k = \ell \), and then \( \pi \) must be the same partition. Since all simple cycles of \( T^{\pi} \) are of length two, when \( k = \ell \) one has \( \tau_N \left[ T(U_N) \right] \xrightarrow{N \to \infty} \mathbb{I}(k = \ell) \). We then obtained \( \tau_N \left[ T_{k, \ell}(W_N) \right] \xrightarrow{N \to \infty} \mathbb{I}(k = \ell) \) for \( W_N = V_N, U_N \).

2. Limits of entry-wise products: We prove that for \( W_N \) a uniform permutation or unitary Haar matrix, one has \( \Phi_N \left( P(W_N) \circ Q(W_N) \right) = \Phi_N \left( P(W_N) \right) \times \Phi_N \left( Q(W_N) \right) = o(1) \).
3.2. APPLICATION TO UNITARY HAAR AND UNIFORM PERMUTATION MATRICES

As before we can assume \( P(w) = w^{k_1}(w^*)^{k_2} \) and \( Q = w^{k_3}(w^*)^{k_4} \). Denote by \( T \) the *-test graph in variables \( w, w^* \) consisting in two simple cycles, one with \( k_1 \) edges labelled \( w \) followed by \( k_2 \) edges labeled \( w^* \) and the second defined similarly with \((k_3, k_4)\) instead of \((k_1, k_2)\), in such a way \( \mathbb{E} \left[ \frac{1}{N} \text{Tr}(W_N^k W_N^{k*}) \right] = \mathbb{E} \left[ \frac{1}{N} \text{Tr}(W_N^k W_N^{k*}) \right] \). There is a partition \( \pi \) of the vertex set of \( T \) such that \( T^\pi \) is a directed line if and only if \( k_1 = \ell_1 \) and \( k_2 = \ell_2 \). In that case the partition must be the one which makes the identification of each circle into a directed line, and then identifying the vertices of the shortest line into the one of the longest to create a directed line by starting for the point of identification of the vertices. Hence \( \tau_N^0 \left[ T(V_N) \right] = \tau_N^0 \left[ T_1(V_N) \right] \times \tau_N^0 \left[ T_2(V_N) \right] + o(1) \). Let \( \mathbb{I} \) be the identity matrix. Then \( \mathbb{I} (k_1 = \ell_1) \mathbb{I} (k_2 = \ell_2) \).

Unitary Haar matrices and their transpose

Let \( U_N = \{ U_j \}_{j \in J} \) be a family of independent unitary Haar matrices. By Voiculescu’s theorem (Theorem 1.1), the matrices \( U_j, \ j \in J \) are asymptotically free independent (it will also be a consequence of Proposition 5.7). We also have a similar statement as for complex Wigner matrices, Lemma 3.3, the matrices \( U_j, U^*, \ j \in J \) are asymptotically free independent.

The proof is the same as for Lemma 3.3; we show that \( \{ U_N, U_N^* \} \) has the same limiting *-distribution as \( \{ U_N, \hat{U}_N \} \), where \( \hat{U}_N \) is an independent copy of \( U_N \). The proof is the same but the twin edges are now simple cycles of arbitrary length.

Multiplication of entries by independent weights

We now give a concrete example of Corollary 2.19.

**Proposition 3.8.** Let \( U_N \) be a unitary Haar matrix and let \( V_N \) be a uniform permutation matrix. Let \( W = (\omega_{i,j})_{i,j \geq 1} \) be an infinite array of independent identically distributed complex entries, independent of the parameter \( N \), independent of \( (U_N, V_N) \), such that \( \mathbb{E}[|\omega_{i,j}|^2] \) is finite for any \( K \geq 0 \). Denote by \( W_N \) the \( N \) by \( N \) matrix \( (\omega_{i,j})_{i,j=1,...,N} \). Recall that the symbol \( \circ \) denotes the entry wise product of matrices.

1. Denote \( M_N = U_N \circ W_N \) and \( M_N \) a family of independent copies of \( M_N \). If \( \omega_{i,j} \) is centered, then the matrices of \( M_N \) are asymptotically free independent. Moreover, the matrix \( \frac{1}{\sqrt{N}} \tilde{W}_N = \frac{1}{\sqrt{N}} (\tilde{\omega}_{i,j})_{i,j=1,...,N} \) with centered independent complex Gaussian entries with covariance \( \mathbb{E}[|\tilde{\omega}_{i,j}|^2] = \mathbb{E}[|\omega_{i,j}|^2] \) and \( \mathbb{E}[\tilde{\omega}_{i,j}^2] = 0 \) converges in traffic distribution to the same limit as \( M_N \).

2. Denote \( M_N = V_N \circ W_N \) and \( M_N \) a family of independent copies of \( M_N \). If the modulus of \( \omega_{i,j} \) is not deterministic, then the matrices of \( M_N \) are not asymptotically free independent but are asymptotically traffic independent.

**Proof.** The convergence in traffic distribution of \( M_N \) is consequence of Corollary 2.19 and Theorem 1.3.

1. By Corollary 2.19, for any *-test graph \( T \), one has \( \tau_N^0 \left[ T(M_N) \right] = \tau_N^0 \left[ T(U_N) \right] \times \delta_N^0 \left[ T(W_N) \right] \). Since the entries of \( W_N \) are independent and centered, then \( \delta_N^0 \left[ T(W_N) \right] = 0 \) if \( T \) has an edge of multiplicity one. Moreover, if \( \tau_N^0 \left[ T(U_N) \right] \) does not converges to zero and has no edge of multiplicity one, then \( T \) is a double tree whose twin edges have different label and orientation. For such \( T \) with \( 2k \) edges, the quantity \( \delta_N^0 \left[ T(W_N) \right] \) is \( \mathbb{E}[|\omega_{i,j}|^2]^k \). On the other hand let \( W_N \) the matrix with independent Gaussian entries of the lemma. By Lemma 2.18, one has \( \tau_N^0 \left[ T(W_N) \right] = N^{4V-1}(1+o(N^{-1})) N^{-2E} \delta_N^0 \left[ T(W_N) \right] \) where \( V \) and \( E \) are the vertex and edge set of \( T \). This quantity is zero if \( T \) has an edge of multiplicity one, and we assume from now it is not so. Denote by \( |E| \) its number of vertices without multiplicity and write \( |V| - 1 - \frac{|E|}{2} = (|V| - 1 - |E|) + (|E| - \frac{|E|}{2}) \).
Applying Lemma 2.13 to the graph obtained from $T$ by forgetting the multiplicity of its edges, we get that the only *-test graphs $T$ for which $\tau_N^0[T(W_N)]$ possibly does not vanish at infinity are the double trees. By independence of the entries, $\delta_N^0[T(W_N)]$ is multiplicative with respect to the twin edges of $T$. Since for the considered complex Gaussian random variable $\omega$ one has $E[\omega^2] = 0$, then the only graphs that contribute are those for which the twin edges have opposite orientation and different labels. Hence, $M_N$ and $\sqrt{N} \tilde{W}_N$ have the same limiting traffic distribution.

In particular $M_N$ has the same limiting *-distribution as independent copies of $\frac{1}{\sqrt{N}} \tilde{W}_N$. This is known that such matrices are asymptotically free independent (this is also a consequence of Theorem 5.7). Hence the result.

2. Let now denote $M_N = V_N \circ W_N$ and let prove that $\kappa_N := \Phi(M_N M_N^\ast \circ M_N M_N^\ast) - \Phi(M_N M_N^\ast) \Phi(M_N M_N^\ast)$ has a non zero limit. By permutation invariance of $M_N$, for each $i$ in $[N]$ one has $\kappa_N = \var\left( \sum_{i=1}^N |M_N(i,j)|^2 \right)$. But $|M_N(i,j)|^2 = 0$ if $j$ is not the image of $i$ by the permutation associated to $\sigma$. Hence $\kappa_N = \var\left( |\omega_{i,j}|^2 \right)$ for any $i, j$. We get the result thanks to Proposition 2.16.

### 3.2.4 Proof of the convergence and factorization property

We now go back to the proofs of Propositions 3.6 and 3.7. We first prove the convergence in traffic distribution for uniform permutation and unitary Haar matrices. Then prove the factorization property for these two models.

**Proof of Proposition 3.4.** Let $V_N$ be a uniform permutation matrix. First, remark that since the entries of $V_N$ are in $\{0,1\}$, then for any *-test graph $T$ in one variable, $\tau_N^0[T(V_N)] = \tau_N^0[\tilde{T}(V_N)]$ where $\tilde{T}$ is obtained by

- reversing the orientation of edges labelled $x^*$ and replacing this label by $x$,
- forgetting the multiplicity of each edges (assuming the multiplicity is one).

Hence, we can assume $T = \tilde{T}$ without loss of generality.

Moreover, each row and column of $V_N$ has a single nonzero entry. Hence, $\frac{1}{N} \text{Tr}_N^0[T(V_N)]$ is zero as soon as two distinct edges leave (or start from) a same vertex. Hence, there are only two kinds of test graphs that possibly contribute: for any $K \geq 1$,

- the test graph $T_K^c$ with vertices $1, \ldots, K$ and edges $(1, 2), \ldots, (K - 1, K), (K, 1)$ ($c$ stands for closed path).
- the test graph $T_K^o$ with vertices $1, \ldots, K$ and edges $(1, 2), \ldots, (K - 1, K)$ ($o$ stands for open path).

Let $\sigma_N$ be the random permutation associated to $V_N$. Then, $\tau_N^0[T_K^c(V_N)]$ is the probability that a given integer $i$ in $\{1, \ldots, N\}$ belongs to a cycle of $\sigma_N$ of length $K$. This probability is $\frac{1}{N^K}$. Indeed, there are $\frac{(N-1)!}{(N-K)!} \times (N-K)!$ permutations such that $i$ is contained in a cycle of length $K$ (the first term counts the number of ways to chose the cycle of length $K$ containing $i$, while the second term accounts for the remaining freedom in the permutation). Then we get

$$\tau_N^0[T_K^c(V_N)] \underset{N \to \infty}{\longrightarrow} 0.$$  

At the contrary, $\tau_N^0[T_K^o(V_N)]$ is the probability that a given integer $i$ in $\{1, \ldots, N\}$ belongs to a cycle of $\sigma_N$ of length bigger than $K$. By the above, one has

$$\tau_N^0[T_K^o(V_N)] \underset{N \to \infty}{\longrightarrow} 1.$$  

Let now prove the factorization property. Let $\sigma_N$ be the permutation of $\{1, \ldots, N\}$ associated to $V_N$. For any $K_1, \ldots, K_n, L_1, \ldots, L_m \geq 1$, the number

$$E\left[ \prod_{i=1}^n \frac{1}{N} \text{Tr}_N^0[T_{K_i}(V_N)] \times \prod_{i=1}^m \frac{1}{N} \text{Tr}_N^0[T_{L_i}(V_N)] \right]$$
is the probability that, choosing $i_1, \ldots, i_n, j_1, \ldots, j_m$ uniformly and independently on $\{1, \ldots, N\}$ one has

- $i_k$ belongs to a cycle of length $K_k$ of $\sigma_N$ for any $k = 1, \ldots, n$,
- $j_k$ belongs to a cycle of length bigger than $L_k$ of $\sigma_N$ for any $k = 1, \ldots, m$.

By a straightforward computation, this probability tends to zero or one, depending if $n$ is positive or not respectively.

**Proof of Proposition** Let $U_N$ be a unitary Haar matrix. Let $T = (V, E, \varepsilon)$ be a *-graph in the variable $x$. By Formula (2.3), one has $\tau^0_N[T(U_N)] = N^{|V|-1}(1 + O(N^{-1}))\delta^0_{W^g}(T(U_N))$. To compute the asymptotic of this quantity, we use Weingarten calculus (see [NS06 Lecture 23] and [CS06]). Denote by $(j_1, i_1), \ldots, (j_k, i_k)$ the edges of $T$ with label $x$ and $(j'_1, i'_1), \ldots, (j'_\ell, i'_\ell)$ the edges with label $x^*$, where the $i_n, j_n, i'_n, j'_n$ are integers. The invariance of $U_N$ by conjugation by permutation matrices and Weingarten formula [NS06 Lemma 23.5] tell us that

$$\delta^0_N[T(U_N)] = E[U_N(i_1, j_1) \ldots U_N(i_k, j_k) U^*_N(j'_1, i'_1) \ldots U^*_N(j'_\ell, i'_\ell)]$$

is zero if $k \neq k'$, and otherwise is equal to

$$\sum_{\sigma, \tau \in S_N} \delta_{\sigma(1), i'_1} \ldots \delta_{\sigma(k), i'_k} \delta_{\tau(1), j'_1} \ldots \delta_{\tau(\ell), j'_\ell} W_N^g(\sigma \circ \tau^{-1}, N),$$

(3.2)

where $S_N$ is the set of permutation of $\{1, \ldots, k\}$ and $W_N^g$ has asymptotic $W_N^g(\sigma \circ \tau^{-1}, N) = \phi(\sigma \circ \tau^{-1})N^{-|E|+|E_\varepsilon|} \times (1 + O(N^{-2}))$. The number $\phi(\sigma \circ \tau^{-1})$ is $\prod_{i=1}^L \lambda_{i_{\sigma^{-1}}}(1-1)^{L_i-1}$ where $\ell_1, \ldots, \ell_L$ are the sizes of the cycles of $\sigma \circ \tau^{-1}$ by [CS06 Equation (14)], and $\lambda(\sigma \circ \tau^{-1})$ is the number of cycles of $\sigma \circ \tau^{-1}$ (counting cycles of size one).

Choosing $\sigma$ and $\tau$ as in the sum above for which all the indicator functions are nonzero amount to cover the edges of $T$ by disjoint cycles alternating between edges labeled $x$ and $x^*$ as follow. Firstly, we think $\sigma$ (resp. $\tau$) as the map sending the edge $(i'_n, j'_n)$ to $(j_{\tau(n)}, i_{\tau(n)})$ (resp. $(j_{\sigma(n)}, i_{\sigma(n)})$). Then with this convention we denote by $\pi$ the permutation of $E$ defined by $\pi(e) = \sigma(e)$ if e has label $x^*$ and $\pi(e) = \tau^{-1}(e)$ if e has label $x$. Note that $\sigma \circ \tau^{-1}$ is $\pi^2$ restricted to the edges labeled $x^*$. In particular, the number of cycles of $\pi$ is the one of $\sigma^{-1} \circ \tau$, the length of the cycle of $\pi$ is two times those of $\sigma^{-1} \circ \tau$.

We introduce the undirected graph (with possibly multiple edges) $G = (V, E)$, where $V$ is the union of $V$ and of the set of simple cycles of $\pi$, and $E$ is the multi-set where for each edge $e$ of $E$ we add in $E$ edges between the goal of $e$ and the cycles of $\pi$ that contain $e$. Note that assuming that their exist $\sigma$ and $\tau$ such that in the associated term of (3.2) the indicator function is nonzero and constructing the partition $\pi$, we get that each edge belong at least to a directed simple cycle of $T$.

The existence of $\pi$ actually forces $G$ to be connected. Let $v$ and $v'$ be two vertices of $G$ that correspond to vertices of $V$. In $T$ there is a path from $v$ to $v'$. While looking at consecutive steps of the path that stay in a same simple cycle $c_i$, the vertices that are visited are all linked tp $c_i$ in $G$. Now when from a step to another we move to a different cycle $c_{i+1}$, the vertex between these two steps belong both to the cycles $c_i$ and $c_{i+1}$. Since each vertex of $G$ that corresponds to a simple cycle of $T$ are linked to some vertices of $T$ in $G$, we get that $G$ is connected.

One has $|V| = |V| + |\pi|$ and $|E| = |\pi|$. We then get from the formulas for $\tau^0_N, \delta^0_N, W_N^g(\cdot, N)$ above and Lemma 2.13

$$\tau^0_N[T(U_N)] = \sum_{\pi} N^{|V|-1-|E|} \times \phi(\pi) = \sum_{\pi} (\mathbb{I}_G \text{ is a tree } + O(N^{-1})) \phi(\pi),$$

(3.3)

where $\phi(\pi) = \prod_{i=1}^L c_{i_{\sigma^{-1}}}(1-1)^{L_i-1}$ for $k_1, \ldots, k_L$ are the sizes of the cycles of $\pi$.

When $G$ is a tree we claim that $T$ is a directed cactus, that is each edge of $T$ belongs exactly to one simple cycle which is directed. Indeed, assume that a vertex $(v, w)$ belongs to two distinct simple cycles $c_1$ and $c_2$. Then there is a link between $w$ and $c_1$ and a link between $w$ and $c_2$. But $v$
has the same property since it is both the target of an edge in \( c_1 \) and an edge in \( c_2 \), in contradiction with the fact that \( G \) is a tree.

Given that \( T \) is a cactus, there is no choice in the partition \( \pi \) in the sum of Equation (3.3), its cycles must consist in the simple cycles of \( T \). Indeed, otherwise there is a cycle of \( \pi \) with two distinct edges with same target, and then \( G \) has an edge of multiplicity two and \( G \) is not a tree. We get the expected formula.

Let now prove the factorization property and let \( T_1 = (V_1, E_1), \ldots, T_n = (V_n, E_n) \) be \(*\)-test graphs in one variable \( x \). We first use Lemma 2.24

\[
E\left[ \frac{1}{N} \text{Tr}^0[T_1(U_N)] \cdots \frac{1}{N} \text{Tr}^0[T_n(U_N)] \right] = \sum_\sigma \frac{1}{N^n} E\left[ \text{Tr}^0[T^\sigma(U_N)] \right],
\]

where the sum is over the partition \( \sigma \) that possibly identify vertices of different graphs \( T_i \). Fix \( \sigma \) as in the sum. Computing \( E\left[ \frac{1}{N} \text{Tr}^0[T^\sigma(U_N)] \right] \), we reason in the proof of Proposition 3.7. The difference is that now \( T^\sigma \) is not connected in general. Hence, we now deduce from (3.3) that

\[
\frac{1}{N^n} \tau^0_N[T^\sigma(U_N)] = \sum_\pi (1_G \text{ is a forest} + O(N^{-1})) \phi(\pi) \times N^{n_\pi - n},
\]

where \( n_\pi \) is the number of connected components of \( T^\pi \). Hence, this quantity goes to zero unless the partition \( \pi \) is trivial. Then, the sum over the choice of covering of \( T^\pi \) by cycles splits into \( n \) sums of coverings of the \( T_j \)'s by cycles. As the map \( \phi(\pi) \) is multiplicative with respect to its cycles, we get as expected

\[
E\left[ \prod_{\ell=1}^n \frac{1}{N} \text{Tr}^0[T_\ell(U_N)] \right] = \prod_{\ell=1}^n E\left[ \frac{1}{N} \text{Tr}^0[T_\ell(U_N)] \right] + O(N^{-1}).
\]

\( \square \)

### 3.3 Diagonal matrices and remark on the factorization property

**Diagonal matrices**

In the section we first consider diagonal matrices in the context of Theorem 1.8. Denote by \( \Phi_N : P \mapsto E\left[ \frac{1}{N} \text{Tr} P \right] \) the expectation of the normalized trace of \(*\)-polynomials, \( T_N : T \mapsto E\left[ \frac{1}{N} \text{Tr} T \right] \) the expectation of the normalized trace of \(*\)-test graphs and \( \tau^0_N \) its injective version.

**Lemma 3.9.** Let \( D_N = (D_j)_{j \in J} \) be a family of random diagonal matrices. Then \( D_N \) converges in traffic distribution whenever it converges in \(*\)-distribution: for any \(*\)-test graph \( T = (V, E, \gamma, \varepsilon) \), one has

\[
\tau_N[T(D_N)] = \Phi_N\left[ \prod_{e \in E} D^{(e)}(\gamma(e)) \right], \quad \tau^0_N[T(D_N)] = \mathbb{1}[|V| = 1] \times \Phi_N\left[ \prod_{e \in E} D^{(e)}(\gamma(e)) \right].
\]

It satisfies the factorization property (Assumption B3 of Theorem 1.8) if and only if for any \( K \geq 2 \) and any \(*\)-polynomials \( P_1, \ldots, P_K \) one has

\[
\lim_{K \to \infty} \Phi_N\left[ P_K(D_N) \right] - E\left[ \prod_{k=1}^K \frac{1}{N} \text{Tr} P_k(D_N) \right] \to 0.
\]

The proof follows immediately from the fact that non diagonal entries of \( D_N \) are zero.

**Example 3.10.** Let \( D_N \) be a diagonal matrix with independent and identically distributed diagonal entries, distributed according to a probability distribution \( \mu \) on \( \mathbb{C} \), whose moments of all orders exist. Then, the traffic distribution of \( D_N \) does not depend on \( N \). For any \(*\)-test graph \( T = (V, E, \gamma, \varepsilon) \) in one variable \( x \), one has \( \tau_N[T(D_N)] = \int z^n \bar{z}^m \, dz \) where \( n \) and \( m \) are respectively the number of edges labeled \( x \) and \( x^* \). Hence \( D_N \) converges in traffic distribution. It is immediate to see that it also satisfies the factorization property.
We have now all the ingredient for the proof of Corollary 1.9.

**Proof of Corollary 1.9.** Let $X_N, U_N, V_N, D_N$ be independent families of matrices as in the Corollary, namely:

1. $X_N$ is a family of independent Wigner matrices whose entries are invariant in law by complex conjugation, for which the convergence in traffic distribution is proved in Proposition 3.1. and the factorization property is proved at the end of Section 3.2.

2. $U_N$ and $V_N$ are respectively a family of independent unitary Haar and uniform permutation matrices. Their convergence in traffic distribution and the factorization property are proved in Section 3.2.

3. $D_N = (D_j)_{j \in J}$ is a family of independent diagonal matrices with independent and identically distributed diagonal entries whose moments of all orders exist. By Lemma 3.9 above and its example, the matrices of $D_N$ also satisfy these assumptions.

The matrices of $X_N, U_N, V_N, D_N$ are permutation invariant. Hence by Theorem 1.8, for any family of matrices $A_N$ converging in traffic distribution and satisfying the factorization property, the matrices of $X_N, U_N, V_N, D_N$ and the family $A_N$ are asymptotically traffic independent.

The limiting *-distribution of $X_N, U_N, V_N, D_N, A_N$ depends only of the limit for each family of normalized trace of test graphs $T$ such that there exists a cycle visiting each edge of $T$ is the sense of their orientation. Hence this limiting *-distribution is unchanged if the Wigner matrices are replaced by unitarily invariant Gaussian Wigner matrices with parameters of the form $(a_0, 0)$.

By Theorem 1.1, the matrices of $X_N \cup U_N$ are asymptotic free independent and $X_N \cup U_N$ is asymptotically free independent from $V_N \cup D_N \cup A_N$. Instead of Theorem 1.1 one can use Proposition 5.7 stated later in the article.

**On the factorization property**

Recall that the factorization property (1.2) is Assumption B3 of Theorem 1.8.

The convergence of the traffic distribution of $A_N = A_N^{(1)} \cup \cdots \cup A_N^{(L)}$ remains true if we do not assume the factorization property but only the convergence of $(g_1, \ldots, g_K) \mapsto \mathbb{E} \left[ \prod_k \frac{1}{T_k} \text{Tr} [g_k(A_N^{(l)})] \right]$. See Remark 2.23 in the proof of Theorem 1.8. When the factorization property is not satisfied, the limiting traffic distribution of $A_N$ is not the product of the limiting distribution distribution of the $A_N^{(l)}$ in the sense of Definition 2.11.

Let $A_N$ and $B_N$ be two independent families of matrices, and assume that the matrices of $A_N$ are diagonal. Assume that $A_N$ and $B_N$ converges in traffic distribution, that one of the family is permutation invariant, and that only the family of diagonal matrices $A_N$ satisfies the factorization property. Then the conclusion of Theorem 1.8 is valid for $A_N, B_N$, even if $B_N$ does not necessarily satisfies the factorization property. See Remark 2.23 in the proof of Theorem 1.8.

Let $A_N$ be a random matrix which is a uniform permutation matrix with probability one half and a unitary Haar matrix otherwise. Since independent uniform permutation and unitary Haar matrices have same limiting *-distribution and are asymptotically free independent, independent copies of $A_N$ are asymptotically free independent. Nevertheless we will see in Propositions 3.6 and 3.7 that a uniform permutation matrix and a unitary Haar matrix do not have the same limiting traffic distribution. It follows that $A_N$ does not satisfies the factorization property (1.2) (this is a simple exercise left to the reader). So according to the first point of the remark, independent copies of $A_N$ are not asymptotically traffic independent. Let now consider a random matrix $B_N$ of the form of a block matrix

$$B_N = \tilde{V}_N \begin{pmatrix} U_{p_N} & 0 \\ 0 & V_{N-p_N} \end{pmatrix} \tilde{V}_N^*,$$

where $V_{N-p_N}, \tilde{V}_N$ are independent uniform permutation matrices, independent from a unitary Haar matrix $U_{p_N}$, and $p_N$ a sequence of positive integers such that $\frac{p_N}{N} \to 0$. Then it is easy to see that $B_N$ has the same limiting traffic distribution as $A_N$ and it satisfies the factorization property (1.2), so independent copies of $B_N$ are asymptotically traffic independent.
Part II

Traffic and their independence
Chapter 4

Algebraic traffic spaces

In the previous sections we have considered the traffic distributions of matrices and their point-wise convergence. After a recall on free probability (see [AGZ10, NS06] for detailed presentations), we define in this section the abstract traffics which model the limits of large matrices for this mode of convergence.

4.1 Non commutative probability spaces

Definition 4.1 (Spaces of non commutative random variables).

A non commutative probability space is a pair \((\mathcal{A}, \Phi)\), where

- \(\mathcal{A}\) is a unital algebra over \(\mathbb{C}\),
- \(\Phi : \mathcal{A} \to \mathbb{C}\) is a unital linear functional. It is called a trace when it satisfies \(\Phi(ab) = \Phi(ba)\) for any \(a, b \in \mathcal{A}\).

A \(\ast\)-probability space is a non commutative probability space \((\mathcal{A}, \Phi)\) such that

- \(\mathcal{A}\) is a \(\ast\)-algebra, i.e., it is endowed with an anti linear involution \(\ast\) such that \((ab)\ast = b\ast a\ast\) for any \(a, b \in \mathcal{A}\),
- \(\Phi\) is a state, that is it satisfies the positivity condition \(\Phi(a\ast a) \geq 0\) for any \(a \in \mathcal{A}\). It is called faithful if moreover \(\Phi(a\ast a) = 0\) implies \(a = 0\) for any \(a \in \mathcal{A}\).

Elements of a non commutative probability space are called non commutative random variables. The non commutative distribution of a family \(a = (a_j)_{j \in J}\) of non commutative random variables is the linear map \(\Phi_a\), defined on polynomials in indeterminates \(x\), by

\[
\Phi_a : P \mapsto \Phi(P(a))
\]

To emphasize that the notion of distribution is relative to the linear form we also that that \(\Phi_a\) is the distribution of \(a\) w.r.t. \(\Phi\). For \(a\) in a \(\ast\)-probability space, the \(\ast\)-distribution of \(a\) is the non commutative distribution of \((a, a\ast)\), or equivalently the map \(\Phi_a\) defined as above but for \(\ast\)-polynomials. The convergence of a family \(a_X\) of non commutative random variables is the point-wise convergence of their non commutative distribution.

Remark 4.2. In a \(\ast\)-probability space \((\mathcal{A}, \Phi)\), the positivity of \(\Phi\) yields the Cauchy-Schwarz inequality, that is \(\Phi(ab)^2 \leq \Phi(a\ast a)\Phi(b\ast b)\) for any \(a, b \in \mathcal{A}\) and faithfulness of \(\Phi\) implies that \((a, b) \mapsto \Phi(a\ast b)\) is actually a scalar product.

Example 4.3. 1. Classical random variables: Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space in the usual sense. The algebra \(L^{-\infty}(\Omega, \mathbb{C})\) of measurable maps \(\Omega \to \mathbb{C}\) with finite moments of all orders is a \(\ast\)-probability space, equipped with the complex conjugate and the expectation \(\mathbb{E}\). Its quotient \(L^{-\infty}(\Omega, \mathbb{C})\) by measurable maps null almost everywhere is a \(\ast\)-probability space with faithful state.
2. Random matrices: The space $M_N(C)$ of deterministic $N$ by $N$ matrices is a $\ast$-probability space with trace $\frac{1}{N} \text{Tr}$. Let consider an algebra $\mathcal{L}^{\infty}(\Omega, M_N(C))$ of random matrices whose entries, defined in a same probability space, have finite moments of all orders. It is a $\ast$-probability space with faithful trace $\tau_N = E[\frac{1}{N} \text{Tr}]$.

In the next section, we define the algebraic traffic spaces, that can be seen as non commutative probability spaces with more structure. We do not define the associated notion of positivity in this article, which do not play an important role in the questions considered here (see [CDM16]).

### 4.2 Algebraic traffic spaces

#### 4.2.1 $\mathcal{G}$-algebras

The main point in defining algebraic traffic spaces is to formalize the good structure that replaces the notion of algebra. For that task, we use the notion of symmetric operads. An operad is a set endowed with operations of composition

$$\mathcal{G} = \bigcup_{K \geq 0} \mathcal{G}_K$$

and a fixed element $\text{id}_\mathcal{G}$ of $\mathcal{G}_1$, called the identity of the operad, satisfying the following properties.

- The element $\text{id}_\mathcal{G}$ is a unit for composition, namely for any $g \in \mathcal{G}$ one has $g = \text{id}_\mathcal{G} g = g \text{id}_\mathcal{G}$.
- The composition is associative: for any $g \in \mathcal{G}_K$, any $g_k \in \mathcal{G}_{L_k}$ and any $g_{k,\ell} \in \mathcal{G}$, $k = 1, \ldots, K, \ell = 1, \ldots, L_k$ one has

$$g(g_1, \ldots, g_{1,L_1}, \ldots, g_K, \ldots) = (g(g_1, \ldots, g_K))(g_{1,1}, \ldots, g_{1,L_1}, \ldots, g_{K,1}, \ldots, g_{K,L_K}).$$

A symmetric operad is an operad $\mathcal{G} = \bigcup_{K \geq 0} \mathcal{G}_K$ endowed with an action $(\sigma, g) \mapsto g_\sigma$ of the symmetric group $S_K$ on $\mathcal{G}_K$ such that

$$g_\sigma(g_1, \ldots, g_K) = g(g_{\sigma(1)}, \ldots, g_{\sigma(K)}),
\quad g((g_1)_{\sigma_1}, \ldots, (g_K)_{\sigma_K}) = g(g_1, \ldots, g_K)_{\sigma_1 \times \cdots \times \sigma_K}.$$

For the purpose of defining traffic spaces, we use the following operad.

**Definition 4.5.**

- A graph operation $g$ of $K$ elements is a finite connected oriented graph $(V, E)$ with $K$ edges, with the data of an ordering of the edges and of two vertices $\text{in}, \text{out}$, (possibly equal) called the input and the output. An ordering of the edges can be thought as a bijection $\gamma : E \rightarrow [K]$. We denote $g = (T, \text{in}, \text{out})$ and $T = (V, E, \gamma)$. We set $\mathcal{G}_K$ the set of graph operations of $K$ elements and $\mathcal{G} = \bigcup_{K \geq 0} \mathcal{G}_K$.

- The graph operation of $\mathcal{G}_1$ with exactly two vertices $\text{in} \neq \text{out}$ and one single edge from $\text{in}$ to $\text{out}$ is denoted by its graph $(\cdot \leftrightarrow \cdot)$, where implicitly in this picture the input is on the right and the output on the left.

- As we have seen before for graph monomials, we define the operation consisting edge replacement of a graph operation $g$ of $\mathcal{G}_K$ by $K$ graphs $g_1, \ldots, g_K$ of $\mathcal{G}$. The input (respectively the output) vertex of $g_k$ replaces the source (respectively the target) vertex of the $k$-th edge of $g$. The induced order for this new graph is the lexical order, namely the edges of $g_k$ come before edges of $g_{k+1}$ for $i = 1, \ldots, n - 1$. This produces an new graph operation $g(g_1, \ldots, g_K)$ in $\mathcal{G}$.
4.2. ALGEBRAIC TRAFFIC SPACES

- For any permutation $\sigma$ of $\{1, \ldots, K\}$ and any $g = (T, in, out) \in \mathcal{G}_K$, $T = (V, E, \gamma)$, denote $g_\sigma$ the graph operation obtained from $g$ by considering the $i$-th edge of $g$ as the $\sigma(i)$-th one for $g_\sigma$, namely $g_\sigma = (T, in, out), T = (V, E, \sigma^{-1} \circ \gamma)$. We let the reader verify that

**Definition 4.6.**  1. An algebra over the operad $\mathcal{G}$ of graph operations (in short a $\mathcal{G}$-algebra) is a vector space $A$ over $\mathbb{C}$ endowed with an action of $\mathcal{G}$ as follow.

- **Linearity:** For any $K \geq 0$, $Z_g : A^{\otimes K} \to A$ is a linear map.

- **Unitality:** By convention, for the single graph operation $(\cdot) \in \mathcal{G}_0$ with one vertex and no edge, $Z(\cdot)$ is a fixed element $1$ of $A$.

- **Identity:** The identity of the operad $id_\mathcal{G} = (\cdot \mapsto \cdot)$ is associated to the identity map, that is $Z_{id_\mathcal{G}} = id_A$ or equivalently $Z(\cdot)(a) = a$ for any $a \in A$.

- **Equivariance:** An action of a graph operation consists in replacing edges by elements of $A$, so it depends on the locations of the edges not on their ordering: for any $a_1, \ldots, a_K \in A$, one has

$$Z_{g\sigma}(a_1 \otimes \cdots \otimes a_K) = Z_g(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(K)}).$$

- **Substitution:** The action of graph operations on $A$ are compatible with the substitution of graph operations: for any $g \in \mathcal{G}_K, g_1, \ldots, g_K \in \mathcal{G}$,

$$Z_g(Z_{g_1} \otimes \cdots \otimes Z_{g_K}) = Z_{g(g_1, \ldots, g_K)}.$$

2. A $*$-algebra over the operad $\mathcal{G}$ (in short, a $\mathcal{G}^*$-algebra) is a $\mathcal{G}$-algebra endowed with an antilinear involution $a \mapsto a^*$ with the following property. Given a graph operation $g$, we call transpose of $g$ and denote $g^t$ the graph operation obtained by reversing the orientation of the edges and interchanging input and output. Then for any graph operation $g$ and any $a_1, \ldots, a_n$ in $A$, one has $(Z_g(a_1 \otimes \cdots \otimes a_n))^* = Z_{g^t}(a_1^* \otimes \cdots \otimes a_n^*)$.

3. The $\mathcal{G}$-algebra spanned by a subset $A \subset A$ of a $\mathcal{G}$-algebra is the linear space generated by \{Z_g(a_1 \otimes \cdots \otimes a_K), g \in \mathcal{G}_K, a_1, \ldots, a_K \in A\}. The $\mathcal{G}^*$-algebra spanned a subset $A \subset A$ of a $\mathcal{G}^*$-algebra is defined similarly with $a_1, \ldots, a_K \in A \cup A^*$.

**Example 4.7.**  1. **Abelian algebras:** Let $A$ be an abelian unital algebra with product $(a, b) \mapsto a \times b$ and unit $1$. There is a trivial structure of $\mathcal{G}$-algebra on $A$: for any family $(a_1, \ldots, a_K)$ of elements of $A$ and any graph operation $g \in \mathcal{G}_K$, we set $Z_g(a_1 \otimes \cdots \otimes a_K) = a_1 \times \cdots \times a_K$. This well defines a structure of $\mathcal{G}$-algebra on $A$. If moreover $A$ is a $*$-algebra with involution $\cdot^*$ then $A$ is a $\mathcal{G}^*$-algebra. Note that if $A$ is not abelian, then $Z_g(a_1 \otimes \cdots \otimes a_K)$ can be defined similarly but depends on the ordering of its edges of $g$. Hence the equivariance axiom is not verified.

2. **Matrix algebras:** The space of $N \times N$ matrices with complex entries is a $\mathcal{G}^*$-algebra for the operations given with similar formula as Definition 1.2. For any matrices $A_1, \ldots, A_K$ and any graph operation $g = (T, in, out)$ with $T = (V, E, \gamma)$ and $|E| = K$, the entry $(i, j)$ of $Z_g(A_1 \otimes \cdots \otimes A_K)$ is

$$Z_g(A_1 \otimes \cdots \otimes A_K)(i, j) = \sum_{\phi: V \rightarrow [N]} \prod_{k=1}^K A_k(\phi(w_k), \phi(v_k)),$$

where the $k$-th edge is denoted $(v_k, w_k)$.

4.2.2 Polynomials and graph polynomials

In a $\mathcal{G}$-algebra, we define a product $\times$ as the bilinear map $(a, b) \mapsto Z_{(\cdot \mapsto \cdot)}(a \otimes b)$, where the graph operation consists in two consecutive edges $e_1 = (v, out)$ and $e_2 = (in, v)$ with $out, v, in$ pairwise distinct. The product is associative since by the axiom of substitution one has $(a \times b) \times c =
between the operations. For instance, the relation monomials as for matrices in Example 1.3. In any A
set of test graphs labeled in p
where p
Definition 4.8.

Traffic spaces come together with the following space of observables.

\[ Z_{\ell} \cdot Z_{\ell} \cdot Z_{\ell} \cdot (a, b, c) = a \times (b \times c). \]
This defines a structure of algebra on \( \mathcal{G} \)-algebras for which \( I = Z_{\ell} \) is the unit and we simply denote \( ab = a \times b \). Moreover, in a \( \mathcal{G}^* \)-algebras, we have by definition
\[ (ab)^* = Z_{\ell} \cdot Z_{\ell} \cdot (a^* \otimes b^*) = Z_{\ell} \cdot Z_{\ell} \cdot (a^* \otimes b^*) = b^* a^*. \]
Hence a \( \mathcal{G}^* \)-algebra is in particular a *-algebra.

Recall that a graph monomial is a finite connected directed graph whose edges are labeled by formal variables with an input and an output, and that \( \mathbb{C}G(x) \) denotes their linear space. Note that contrary to graph operations \( g = (V, E, \gamma) \), in a graph monomial, we do not consider an ordering of the edges as for graph operations but a labeling \( \gamma : E \rightarrow J \), and a same label can appear on several edges. The space of graph polynomials in some variables is a \( \mathcal{G} \)-algebra: for \( g \)
a graph operation with \( K \)-edges and \( g_1, \ldots, g_K \) graph monomials, the graph \( Z_g(g_1 \otimes \cdots \otimes g_K) \) is given by edge substitution as usual.

Let \( a = (a_j)_{j \in J} \) be a family of elements of a \( \mathcal{G} \)-algebra \( \mathcal{A} \) and let \( g \) be a graph monomial in the variables \( x = (x_j)_{j \in J} \) with \( K \) edges. Choosing an arbitrary ordering of its edges gives a graph operation that we denote \( \tilde{g} \) and allow us to consider the element \( Z_g(a_{j_1} \otimes \cdots \otimes a_{j_K}) \) of \( \mathcal{A} \), where \( j_k \in J \) is the label of the \( k \)-th edge of \( \tilde{g} \). By the equivariance for the action of operads, this element depends only on \( a \) and \( g \), not on the ordering of the edges. It is denoted \( g(a) \) and this definition is extended by linearity for \( g \in \mathbb{C}G(x) \).

By definition of the product and thanks to the axiom of substitution on \( \mathcal{G} \)-algebras, the subspace of \( \mathbb{C}G(x) \) generated by simple directed lines \( (\begin{array}{ccc} z_{10} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{K0} & \cdots & z_{Kn} \end{array}), j_k \in J \), can be identified with the space of non commutative polynomials \( \mathbb{C}(x) \).

Similarly, recall that a \( * \)-graph monomial in variables \( x \) is a graph monomial with edges labeled by variables \( x \) and \( x^* \). With same notations we define \( g(a) = Z_g(a_{j_1}^* \otimes \cdots \otimes a_{j_K}^*) \) where \( \epsilon_k = 1 \) or \( * \) depending if the \( k \)-edge is labeled by a variable in \( x \) or \( x^* \) respectively.

The axioms for the action of graph operations implies similar properties for graph polynomials.

1. The substitution axiom implies that for any \( g, h_j, j \in J \), elements of \( \mathcal{G}(x) \) and any \( a \in \mathcal{A}^J \), the element \( g(h_j(a), j \in J) \) is equal to \( (g(h_j))_{j \in J}(a) \) where \( (g(h_j))_{j \in J} \) is obtained from \( g \) by replacing each edge labeled \( x_j \) by the graph \( h_j \).

2. The linearity of the \( Z_g \)'s implies that a graph monomial is multi-linear with respect to its edges, in the following sense. Let \( g \) be a graph monomial in variables \( x \) and \( y \), let \( a \in \mathcal{A}^J \), \( b_1, b_2 \in \mathcal{A} \) and \( \lambda \in \mathbb{C} \). Denote by \( E_y \) the set of edges of \( g \) labeled \( y \). Then one has \( g(a, \lambda b) = \lambda |E_y| g(a, b) \) and \( g(a, b_1 + b_2) = \sum_{|E_y|=\{1,2\}} g_y(a, b_1, b_2) \) where \( g_y \) is obtained from \( g \) by declaring that an edge \( e \in E_y \) has label associated to \( b_{y_0(1)}(1) \).

For elements \( a, b \) of a \( \mathcal{G} \)-algebra, we define \( a^\dagger \), \( \Delta(a) \), \( \deg(a) \) and \( a \circ b \) with the same graph monomials as for matrices in Example 1.5. In any \( \mathcal{G} \)-algebras, there are infinitely many relations between the operations. For instance, the relation
\[ \Delta(a \times \deg(b)) = \deg(\Delta(a) \times b), \]
valid for any \( a, b \in \mathcal{A} \), was obtained in Figure 1.2.

### 4.2.3 Algebraic traffic spaces

Defining the traffic distribution of a family of matrices \( \mathbb{A}_N \), we have firstly considered the map
\[ \Phi_{\mathbb{A}_N} : \mathbb{C}G(x, x^*) \rightarrow \mathbb{C} \] which generalizes the notion of \( * \)-distribution. Then we have introduced the equivalent data given by the combinatorial distribution \( \tau_{\mathbb{A}_N} : \mathbb{C}\mathcal{T}(x, x^*) \rightarrow \mathbb{C} \) in Definition 2.2 which allowed us to defined in a simple way its injective version \( \tau_{\mathbb{A}_N}^0 \). In the definition of abstract traffics below (Definition 4.9), we consider the data of the combinatorial distribution to be the intrinsic one. For this reason, the \( \mathcal{G} \)-algebras considered to introduce the traffic spaces come together with the following space of observables.

**Definition 4.8.** Let \( \mathcal{A} \) be an arbitrary set. A test graph labeled in \( \mathcal{A} \) is a quadruple \( T = (V, E, \gamma) \) where \( (V, E) \) is a finite connected graph, endowed with a map \( \gamma : E \rightarrow \mathcal{A} \). We define by \( \mathcal{T}(\mathcal{A}) \) the set of test graphs labeled in \( \mathcal{A} \) and by \( \mathbb{C}\mathcal{T}(\mathcal{A}) \) its linear space.
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A test graph $T$ in variables $\mathbf{x} = (x_j)_{j \in J}$ and a family $\mathbf{a} = (a_j)_{j \in J}$ of elements of $\mathcal{A}$ defines an element $T(\mathbf{a}) \in T(\mathcal{A})$ by obvious evaluation.

**Definition 4.9.** An algebraic traffic space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a $\mathcal{G}$-algebra and $\tau : \mathcal{CT}(\mathcal{A}) \to \mathbb{C}$ is a linear form satisfying the following properties.

1. **Unity:** $\tau$ sends the graph with no edges to one.

2. **Substitution:** For any $T \in \mathcal{T}(\mathcal{A})$ having an edge $e_0$ labeled $h(\mathbf{a})$ for a graph monomial $h$, then $\tau[T] = \tau[T_b]$ where $T_b$ is obtained from $T$ by replacing the edge $e_0$ by the graph $h$, with labels as in the evaluation $h(\mathbf{a})$.

3. **Multi-linearity w.r.t. the edges:** For any $T \in \mathcal{T}(\mathcal{A})$ having an edge $e_0$ labeled $a + \lambda b$, we have $\tau[T] = \tau[T_a] + \lambda \tau[T_b]$ where $T_a$ and $T_b$ are obtained from $T$ by declaring that the label of $e_0$ is now $a$ and $b$ respectively.

Elements of $\mathcal{A}$ are called **traffics** and $\tau$ is called the combinatorial trace on $\mathcal{A}$. The **traffic distribution** of a family $\mathbf{a} = (a_j)_{j \in J}$ of elements of $\mathcal{A}$ is the data of the map 

$$\tau_\mathbf{a} : T \in \mathcal{CT}(\mathbf{x}) \mapsto \tau[T(\mathbf{a})] \in \mathbb{C}.$$ 

As we will see equivalent formulation of the traffic distribution, we name specifically $\tau_\mathbf{a}$ the **combinatorial distribution** of the family $\mathbf{a}$. Let $\mathbf{a}_N = (a_{N,j})_{j \in J}$, for each $N \geq 1$, and $\mathbf{a} = (a_j)_{j \in J}$ be families of traffics, possibly in different spaces. We say that $\mathbf{a}_N$ converges to $\mathbf{a}$ in traffic distribution as $N$ goes to infinity whenever $\tau_{\mathbf{a}_N}$ converges point-wise to $\tau_\mathbf{a}$.

A motivation for using the term **traffics** is because of the algebraic structure of $\mathcal{G}$-algebra which allows to compose them not only by multiplication, but thanks to a scheme given by graph monomials. The term **traffic space** means space of traffics as for the term vector space.

**Example 4.10.** Example \[L3\] continued.

- Let $(\mathcal{A}, \Phi)$ be an abelian non commutative probability space. We endow $\mathcal{A}$ with its trivial structure of $\mathcal{G}$-algebra structure. Moreover we define $\tau : \mathcal{CT}(\mathcal{A}) \to \mathbb{C}$ by $\tau[T] = \Phi[\prod_e a_e]$ where the product is over the edges $e$ of $T$ and $a_e$ denotes their associated label. Hence $\tau$ is simply the expectation of the product of edge labels of $T$. Then $(\mathcal{A}, \tau)$ is an algebraic traffic space.

- The $\mathcal{G}$-algebra of $N$ by $N$ matrices is an algebraic traffic space when endowed with the combinatorial trace of test graphs in matrices defined in the first part of the article. Note yet that the traffic distribution of a family of matrices $\mathbf{A}_N$ as defined in the first part (defined on $^*$-test graphs) corresponds to the traffic distribution of $(\mathbf{A}_N, \mathbf{A}_N^*)$ in the sense of algebraic traffic spaces.

**Lemma 4.11.** For each $N \geq 1$, let $\mathbf{a}_N = (a_{N,j})_{j \in J}$ be a family of traffics in some space $(\mathcal{A}_N, \tau_N)$. Assume that $\tau_{\mathbf{a}_N}$ converges point-wise. Then there exists a family $\mathbf{a} = (a_j)_{j \in J}$ in some space $(\mathcal{A}, \tau)$ such that $\tau_{\mathbf{a}_N}$ converges to $\mathbf{a}$ in traffic distribution.

In particular, for a family of matrices $(\mathbf{A}_{N,j})_{j \in J}$ converging in traffic distribution in the sense of the first part of the article, the family $(\mathbf{A}_{N,j}, \mathbf{A}_{N,j}^*)_{j \in J}$ converges to a family $(a_j, a_j^*)_{j \in J}$ in an algebraic traffic space.

**Proof.** Let $\tau : \mathcal{CG}(\mathbf{x}) \to \mathbb{C}$ be the limiting distribution of $\mathbf{a}_N$, where $\mathbf{x} = (x_j)_{j \in J}$. The $\mathcal{G}$-algebra $\mathcal{A} := \mathcal{CG}(\mathbf{x})$ of graph polynomials in variables $\mathbf{x}$ endowed with $\tau$ is a traffic space and $\mathbf{a} := \left((\tau_{\mathbf{a}}(x_j)), \mathbf{a}_N^*\right)_{j \in J}$ is limit in traffic distribution of $\mathbf{a}_N$.  

Similarly if $\mathbf{a}_N$ is a family of traffics in a $\mathcal{G}^*$-algebra and $\tau_{\mathbf{a}_N, \mathbf{a}_N^*}$ converges point-wise, then $(\mathbf{a}_N, \mathbf{a}_N^*)$ has a limit in traffic distribution $(\mathbf{a}, \mathbf{a}^*)$ in some algebraic traffic space which is a $\mathcal{G}^*$-algebra.
4.2.4 Trace, anti-trace and injective trace

A traffic can be seen as an non commutative random variable, but there are different points of views to do so, as shows the following proposition.

**Proposition 4.12.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space. We define two functions $\Phi$ and $\Psi$ from $\mathcal{A}$ to $\mathbb{C}$ by
\[
\Phi(a) = \tau[\Delta(a)] = \tau[\Delta(a)], \quad \forall a \in \mathcal{A},
\]
amely the combinatorial trace of a self-loop and a simple edge respectively. Then $\Phi$ and $\Psi$ are unital linear forms on $\mathcal{A}$, defining two structures of non commutative probability space. The form $\Phi$ is tracial and is called the trace associated to $\tau$. The linear form $\Psi$ is called the anti-trace associated to $\tau$ (it is not a trace in general).

**Proof.** We have $\Phi(1) = \tau[\Delta] = \tau[\Delta]$ and by convention, $1 = 1(\cdot)$ is associated to the graph monomial with no edges, so that the substitution axioms implies $\tau[\Delta(a)] = \tau[\cdot]$. By the unity axiom for $\tau$, $\Phi$ is unital. Moreover, $\Phi$ is linear since $\Phi(a + \lambda b) = \tau[\Delta + \lambda \Delta(b)] = \tau[\Delta(a)] + \lambda \tau[\Delta(b)]$ by the linearity of $\tau$ with respect to the edges of graphs. The same reasoning implies that $\Psi$ is linear. It remains to prove that $\Phi$ is tracial. But we have $\Phi(ab) = \tau[\Delta(a)] = \tau[\Delta(a)]$ by the substitution axiom. The expression is symmetric in $a$ and $b$, which yields $\Phi(ab) = \Phi(ba)$ as expected. Concrete examples where $\Psi$ is not a trace appear for instance in Section 3.4.2 below.

Let us first consider the trace $\Phi$ and find a characterization of this map in $(\mathcal{A}, \tau)$.

**Lemma 4.13.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space. The trace $\Phi$ associated to $\tau$ satisfies the following properties.

1. **Diagonality:** Recalling that $\Delta$ is the graph monomial with a single vertex and a single self-loop, one has $\Phi = \Phi(\Delta(\cdot))$.

2. **Input-independence:** For any graph monomial $g = (T, in, in)$ with same input and output and for any family $a$ of elements of $\mathcal{A}$, $\Phi(g(a))$ does not depend on the place of input in $g(a)$ but only on the test graph $T(a)$ labeled in $\mathcal{A}$.

Reciprocally, in a $\mathcal{G}$-algebra $\mathcal{A}$, assume that $\Phi$ is a unital linear map satisfying the two above properties. We define the linear form $\tau : \mathcal{G}T(\mathcal{A}) \to \mathbb{C}$ by $\tau[T(a)] = \Phi(g(a))$ for any $g = (T, in, in)$. Then $(\mathcal{A}, \tau)$ is an algebraic traffic space for which $\Phi$ is the associated trace.

**Proof.** For any $a \in \mathcal{A}$, we have $\Phi(\Delta(a)) = \tau[\Delta(a)]$. Using the substitution axiom for $\tau$ means replacing a self-loop by a self-loop, which yields $\Phi(\Delta(a)) = \tau[\Delta(a)] = \Phi(a)$. Hence $\Phi$ is diagonal. Let $g = (T, in, in)$ be a graph monomial with same input and output. Then $\Phi(g(a)) = \tau[\Delta(a)] = \tau[T(a)]$ by substitution axiom, which then depends only on the labeled graph $T(a)$.

Let now $\Phi$ be a unital linear form on a $\mathcal{G}$-algebra satisfying the two properties of the lemma, namely the diagonality and input-independence, and let $\tau$ as defined therein. Since $\Phi$ is unital and by definition of the action of the graph $g_0$ with no edges, we have $1 = \Phi(1) = \Phi(g_0) = \tau[\cdot]$. Hence $\tau$ satisfies the unity axiom. Let now $T$ be a test graph labeled in $\mathcal{A}$ and write $\tau[T] = \Phi(g(a))$ for a graph polynomial obtained from $T$ by choosing an arbitrary vertex as common value of input and output. Assume that $T$ has an edge labeled $h(\bar{a})$ for a graph monomial $h$. By the substitution property for graph monomials stated in the previous section, $g(a) = g_h(\bar{a}, a)$, where $g_h$ is obtained from $g$ by replacing the edge evaluated in $h(\bar{a})$ by the graph $h$. And then we get $\tau[T] = \tau[T_h]$ as expected in the Substitution axiom. The property of multi-linearity w.r.t. the edges of $\tau$ follows similarly from the same property for graph monomials stated in the previous section.

**Lemma 4.14.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space with trace $\Phi$. Assume moreover that $\mathcal{A}$ is a $\mathcal{G}^*$-algebra and that $\Phi$ is a state, that is $\Phi(a^*a) \geq 0$ for any $a \in \mathcal{A}$. Then $\Psi$ is a state. Moreover, for a test graph $T \in \mathcal{T}(\mathcal{A})$, denote $T^*$ the test graph in same variables obtained by exchanging the orientations of each edge and replacing each label $a$ by its adjoint $a^*$. Then for any $T \in \mathcal{T}(\mathcal{A})$, one has $\tau[T^*] = \tau[T]$. 


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Proof. If \( \Phi \) is a state, then \( \Psi \) is a state since
\[
\Psi(a^*a) = \tau[\tilde{a}^* \tilde{a}] = \tau[\tilde{a}^* \tilde{a}] = \Phi[\deg(a)^* \deg(a)] \geq 0.
\]
Moreover, if \( \Phi \) is a state recall that \( \Phi(a^*) = \Phi(a) \) \cite{NS06} Remarks 1.2. Hence, given \( \tau[T(a)] = \Phi(g(a)) \), one verifies that \( \tau[T^*(a)] = \Phi(g(a)^*) \) which is equal to \( \Phi(g(a)) = \tau[T(a)] \).

We now consider the anti-trace. We omit the proof of the lemma below that follows with same arguments as in proof of the analogue statement for \( \Phi \).

Lemma 4.15. Let \((A, \tau)\) be an algebraic traffic space. The anti-trace \( \Psi \) associated to \( \tau \) satisfies the following input/output-independence property. For any graph monomial \( g = (T, \text{in}, \text{out}) \) (with arbitrary input and output) and for any family \( a \) of elements of \( A \), \( \Psi(g(a)) \) does not depend on the place of the input and the output but only on the test graph \( T(a) \) labeled in \( A \).

Reciprocally, in a \( G \)-algebra \( A \), assume that \( \Psi \) is a unital linear map satisfying the above input/output-independence property. Then the linear form \( \tau : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C} \) defined by \( \tau[T(a)] = \Phi(g(a)) \) for any \( g = (T, \text{in}, \text{out}) \) defines on \( A \) a structure of algebraic traffic space, for which \( \Psi(a) = \tau[\tilde{a}] \).

We can also relate directly \( \Phi \) and \( \Psi \) as follow. Recall that \( \deg \) denotes the graph monomial with two distinct vertices \( \text{in} = \text{out} \) and \( v \) and an edge from \( v \) to \( \text{in} = \text{out} \).

Lemma 4.16. Let \((A, \tau)\) be an algebraic traffic space with \( \Phi \) and \( \Psi \) defined as above. Then we have \( \Psi(a) = \Phi(\deg(a)) \) and \( \Phi(a) = \Psi(\Delta(a)) \).

Proof. We have \( \Phi(\deg(a)) = \tau[\deg(a)^*] \). Using the substitution axiom for \( \tau \) means replacing the self-loop labeled \( \deg(a) \) by a simple edge, identifying the vertex of the loop with the vertex \( \text{in} = \text{out} \) of the graph of \( \deg \). Hence \( \Phi(\deg(a)) = \tau[\tilde{a}] = \Psi(a) \).

We then obtain the three following formulation for distribution of a family \( a \) of traffics: the combinatorial distribution \( \tau_a : T \in \mathcal{C}(\mathcal{A}) \mapsto \tau[T(a)] \), the map \( \Phi_a : g \in \mathcal{G}(\mathcal{X}) \mapsto \Phi(g(a)) \), and the map \( \Psi_a : g \in \mathcal{G}(\mathcal{X}) \mapsto \Psi(g(a)) \). The traffic distribution is generic term for the three maps. The restriction \( \Phi_a \) (resp. \( \Psi_a \)) of \( \Phi_a \) (resp. \( \Psi_a \)) on \( \mathcal{C}(\mathcal{X}) \) is the non commutative distribution of \( a \) with respect to \( \Phi \) (resp. \( \Psi \)).

Moreover, from Chapter 3 we can also characterize the traffic distribution of \( a \) thanks to the injective version \( \tau_0 \) of \( \tau_a \). The same definition is valid to define from \( \tau : T(\mathcal{A}) \rightarrow \mathcal{C} \) the linear map \( \tau^0 : T(\mathcal{A}) \rightarrow \mathcal{C} \), namely for any test graph \( T \in T(\mathcal{A}) \) with vertex set \( V \), one has \( \tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi] \), where \( T^\pi \) is the quotient graph identifying vertices of \( T \) in a same block of \( \pi \).

Lemma 4.17. Let \((A, \tau)\) be an algebraic traffic space. The injective version \( \tau^0 \) of \( \tau \) satisfies the following property.

1. It sends the graph with no edges to one.

2. It is multi-linear with respect to the edges of the graphs in the sense of Definition 3.2.

3. It satisfies the following property. For any \( T \in T(\mathcal{A}) \) having an edge \( e_0 \) labeled \( h(a) \) for a graph monomial \( h \), denote \( T_h \) is obtained from \( T \) by replacing the edge \( e_0 \) by the graph \( h(a) \) and by \( V_h \) its vertex set. Then one has
\[
\tau^0[T] = \sum_{\pi \in \mathcal{P}(V_h)} \tau^0[T^\pi],
\]
where the notation \( \pi[V] = 0_V \) means that the sum is over the partitions \( \pi \) of \( V_h \) such that any couple of vertices \( v \) and \( w \) in \( T \), when seen in \( T_h \) after insertion of \( h \), belong to two distinct blocks of \( \pi \).
Reciprocally, in a $\mathcal{G}$-algebra $\mathcal{A}$, assume that $\tau^0 : \mathcal{C}T(\mathcal{A}) \to \mathbb{C}$ is a linear map satisfying the three above properties. Then $\tau^0$ is the injective version of a map $\tau$ which defines a structure of algebraic traffic space on $\mathcal{A}$.

Proof. We have $\tau[\cdot] = \tau^0[\cdot]$ so the unity axioms coincides for both maps. Likewise, since $\tau$ and $\tau^0$ are related each other by linear combinations, the multi-linearity of the maps are also equivalent.

Let $\tau^0$ be the injective version of a linear form $\tau$ on $\mathcal{C}T(\mathcal{A})$. Let $T, \epsilon_0, h(a)$ and $T_h$ be as in the statement. Assuming that $\tau$ satisfies the substitution axiom, we already proved that $\tau^0$ satisfies the expected formula in Proposition 2.14.

So we assume now that $\tau^0$ satisfies the formula of the lemma and prove that $\tau[T] = \tau[T_h]$. On the one hand, we have $\tau[T] = \sum_{\pi \in \mathbb{P}(V)} \tau^0[T^\pi]$. Let us denote by $(T^\pi)_h$ the graph obtained from $T^\pi$ by replacing the edge $\epsilon_0$ by $h$ and by $V_{\pi,h}$ its vertex set. Then the formula for $\tau^0$ tells that

$$\tau[T] = \sum_{\pi \in \mathbb{P}(V)} \sum_{\sigma \in \mathbb{P}(V_{\pi,h})} \sum_{s.t. \: \sigma \sigma_s = 0 \forall s} \tau^0((T^\pi)_h)^{\sigma} = \sum_{\sigma \in \mathbb{P}(V_h)} \tau^0[T_h^\sigma].$$

Using again the definition of $\tau^0$ yields $\tau[T] = \tau[T_h]$.

Lemma 4.18. Let $(\mathcal{A}, \tau)$ be an algebraic traffic space. For any test graph $T$, one has $\tau[T(\cdot, \cdot)] = \tau[\tilde{T}(\cdot, \cdot)]$ where $\tilde{T}$ is obtained from $T$ by identifying source and target of each edge labelled $\mathbb{I}$, and removing these edges. Moreover, with same notations, $\tau^0[T(\cdot, \cdot)] = \tau^0[\tilde{T}(\cdot, \cdot)]$ if all the edges of $T$ labelled $\mathbb{I}$ are self loops and it vanishes otherwise.

Proof. We have $\Delta(\mathbb{I}) = \mathbb{I}$, where $\Delta$ always denote the graph monomial consisting in a single loop. And so, by definition of $\mathbb{I}$ and by the substitution axiom, $\tau[T(\mathbb{I}, \cdot)]$ is not modified if we identify source and target of edges labelled $\mathbb{I}$ and removing them. Similarly, let $T$ be a test graph with an edge $\epsilon_0 = (v, w)$ labelled $\mathbb{I}$ which is not a self loop. Substituting the edge with its label as in Lemma 4.17, the resulting graph $T_h$ is obtained by identifying $v$ and $w$. Hence $\tau^0[T] = 0$, since in the sum of the formula for $\tau^0$ in the lemma, there is no partition that separate $v$ and $w$. Let now $T$ be a test graph such that all edges labeled $\mathbb{I}$ are self loops. By the direct definition of $\tau^0$, one has $\tau^0[T(\mathbb{I}, \cdot)] = \sum_{\pi \in \mathbb{P}(V)} \mu_V(\pi)\tau[T^\pi(\cdot, \cdot)]$ where $V$ is the vertex set of $T$. Denoting by $\tilde{T}^\pi$ the graph obtained from $T^\pi$ by erasing self loops labeled $\mathbb{I}$, by the above we have $\tau[\tilde{T}^\pi(\cdot, \cdot)] = \tau[\tilde{T}(\cdot, \cdot)]$, and then $\tau^0[T(\mathbb{I}, \cdot)] = \sum_{\pi \in \mathbb{P}(V)} \mu_V(\pi)\tau[\tilde{T}(\cdot, \cdot)] = \tau^0[\tilde{T}(\cdot, \cdot)]$.

The reasoning shows actually that $\tau^0[T(\Delta(a), \cdot)] = 0$ for any traffic $a$ and any $T$ such that there is at least one edge labeled $\Delta(a)$. 

Chapter 5

Traffic independence and the three classical notions

The product of traffic distribution, discovered in the first part for large matrices, is considered in the context of algebraic traffic spaces, for which it defines the notion of traffic independence. It unifies the three classical notions of non commutative independence.

5.1 Definition and statement

For convenience we recall in a same place the classical notions of non commutative independence and the definition of the product of traffic distribution.

Definition 5.1. 1. The unital subalgebras $A_1, \ldots, A_L$ of a non commutative probability space $(A, \Phi)$ are said to be freely independent if and only if for any $n \geq 1$, any $a_j \in A_\ell_j$, $j = 1, \ldots, n$, such that $\Phi(a_j) = 0$ and $\ell_j \neq \ell_{j+1}$, $\ell_j \in \{1, \ldots, L\}$ one has $\Phi(a_1 \ldots a_n) = 0$.

2. The sub-algebras $A_1, \ldots, A_L$ (non necessarily unital) of an algebra $A$ are said to be Boolean independent with respect to a linear form $\Psi$ on $A$ if and only if for any $n \geq 1$, any $a_j \in A_\ell_j$, $j = 1, \ldots, n$, such that $\ell_j \neq \ell_{j+1}$, one has $\Psi(a_1 \ldots a_n) = \Psi(a_1) \times \cdots \times \Psi(a_n)$.

3. The unital subalgebras $A_1, \ldots, A_L$ of a non commutative probability space $(A, \Phi)$ are said to be tensor independent if and only if they commute (i.e. $ab = ba$, $\forall a \in A_\ell, b \in A_m, \ell \neq m$) and for any $a_\ell \in A_\ell$, $\ell = 1, \ldots, L$, one has $\Phi(a_1 \ldots a_L) = \Phi(a_1) \times \cdots \times \Phi(a_L)$.

Definition 5.2. Let $(A, \tau)$ be an algebraic traffic space and let $A_1, \ldots, A_L$ be unital $G$-subalgebras of $A$. We say that $A_1, \ldots, A_L$ are independent (or traffic independent if necessary to avoid confusion) whenever the restriction of $\tau$ on the $G$-algebra spanned by $A_1, \ldots, A_L$ is the product of $\tau_1, \ldots, \tau_L$ in the sense of Definition 2.11: for every family $a_\ell$ of $A_\ell$, $\ell = 1, \ldots, L$, for any test graph $T$ in variables $x_1, \ldots, x_L$,

$$\tau^0[T(a_1, \ldots, a_L)] = \mathbb{1}(GCC(T) \text{ is a tree}) \prod_{S \in CC(T)} \tau^0[S],$$

(5.1)

where $GCC(T)$ (respectively $CC(T)$) is the graph (respectively the set) of colored components of $T$ with respect to $x_1, \ldots, x_L$ (see Definition 2.10).

Subsets of $A$ or families of elements of $A$ are said to be (traffic) independent whenever the $G$-subalgebra they spanned are independent. Let $(A_N, \tau_N)_{N \geq 1}$ be a sequence of algebraic traffic spaces. A sequence of families $a_1^{(N)}, \ldots, a_L^{(N)}$ of element of $A_N$ is said to be asymptotically independent whenever it converges toward independent variables.

The map $\tau$ on the $G$-subalgebra spanned by independent $G$-subalgebras is completely determined by the restriction of $\tau$ on each $G$-subalgebras: for any test graph $T$ with vertex set $V_T$ and
Lemma 5.3. Independence of traffics subspaces is symmetric and associative, in the sense stated in Proposition 2.12. Moreover, if the traffic distribution of \( a_1, \ldots, a_L \) is the product of the traffic distributions of the \( a_i \)'s (Formula (5.1) is true for \( a_1, \ldots, a_L \) given and for all \( T \)), then \( a_1, \ldots, a_L \) are independent (Formula (5.1) is true for any graph polynomials \( g(a_i) \)).

The first part of the Lemma is a direct consequence of Proposition 2.12. The second part follows thanks to Proposition 2.14. The classical notions of independence in each case. Hence families \( A^{(1)}, \ldots, A^{(L)} \) are asymptotically traffic independent in the sense of the first part of the article if and only if the families \( A_N^{(1)} \cup A_N^{(1)*}, \ldots, A_N^{(L)} \cup A_N^{(L)*} \) are asymptotically traffic independent in the sense of the above definition.

Lemma 5.4. If \( a \) and \( b \) are independent traffics then \( \Phi(ab) = \Phi(a)\Phi(b) \) and \( \Psi(ab) = \Psi(a)\Psi(b) \).

We can distinguish three particular kinds of traffics, for which traffic independence can be interpreted in terms of one of the classical notions of independence in each case.

Theorem 5.5. Let \( (A, \tau) \) be an algebraic traffic space, with trace \( \Phi : a \mapsto \tau[C_a] \) and anti-trace \( \Psi : a \mapsto \tau[-C_a] \). Let \( a_1, \ldots, a_L \) be independent families of elements of \( A \).

1. Say that \( a = (a_j)_{j \in J} \) is "unitarily invariant" in \( (A, \tau) \) if and only if it has the same traffic distribution as \( uu^* = (uu^*)_{j \in J} \) where \( (u, u^*) \) is independent from \( a \) and is the limit in traffic distribution of \((U_N, U_N^*)\) for \( U_N \) a large unitary Haar matrix and \( uu^* = uu = \mathbf{1} \). If each \( a_\ell \) is unitarily invariant for each \( \ell = 1, \ldots, L \) (except possibly one), then \( a_1, \ldots, a_L \) are free independent in the non commutative probability space \((A, \Phi)\).

2. Say that \( a = (a_j)_{j \in J} \) is "of Boolean type" in \( (A, \tau) \) if and only if its combinatorial distribution is supported on trees. If \( a_\ell \) is of Boolean type for each \( \ell = 1, \ldots, L \), then \( a_1, \ldots, a_L \) are Boolean independent with respect to \( \Psi \).

3. Say that an element of a \( G \)-algebra \( A \) is "diagonal" whenever \( a = \Delta(a) \) for \( \Delta \) the graph operation with a single vertex and a single self loop. If the traffics of \( a_\ell \) are diagonal for each \( \ell = 1, \ldots, L \), then \( a_1, \ldots, a_L \) are tensor independent, both in the non commutative probability spaces \((A, \Phi)\) and \((A, \Psi)\). Reciprocally the tensor independence of diagonal families of traffics in the non commutative probability space \((A, \Phi)\) (resp. \((A, \Psi)\)) characterizes their traffic-independence.

The theorem is proved in the three following sections. In the case of Boolean independence, we exhibit examples of random matrices whose limit in traffic distribution are as in the theorem.

Given traffic spaces \( A_1, \ldots, A_L \), it is natural to ask if there exists a traffic space \( A \) containing the \( A_\ell \) as independent spaces. This fact is true and is proved in [CDM16]. This implies the existence, for any traffic \( a \), of a space containing a sequence \((a_n)_{n \geq 1}\) of independent traffics distributed as \( a \).

In this article, we always assume for granted the existence of such sequences.

5.2 Link with free independence

In order to prove the first point of Theorem 5.5 it suffices to prove the following.

Lemma 5.6. Let \( a, b \) be two arbitrary families of traffics.

1. Assume that \( (a, b) \) is traffic independent from \( (u, u^*) \) limit in traffic distribution of \((U_N, U_N^*)\) for a Haar unitary matrix \( U_N \) and \( uu^* = uu^* = \mathbf{1} \). Then \( a \) and \( b \) \( uu^* \) are free independent.

2. Assume that \( a, b \) are traffic independent and unitary invariant. Then the joint family \( a \cup b \) is unitary invariant.
5.2. LINK WITH FREE INDEPENDENCE

Indeed, let \( a^{(1)}, \ldots, a^{(L)} \) be as in Proposition 5.7. Assume that \( a^{(2)}, \ldots, a^{(L)} \) are unitarily invariant and denote \( b = a^{(2)} \cup \cdots \cup a^{(L)} \). Let \((u, u^*)\) be traffic independent of \((a^{(1)}, \ldots, a^{(L)})\) and limit of \((U_N, U_N^*)\) as in the lemma. Then it implies then that \( a^{(1)} \) and \( ubu^* \) are free independent.

On the other hand, we know that \( a^{(1)} \) and \( b \) are traffic independent, and the associativity of traffic independence implies that \( a^{(1)} \) and \( ubu^* \) are also traffic independent. Their joint traffic distribution depends only on the marginal distributions. But thanks to the second part of the lemma, \( ubu^* \) has the same traffic distribution as \( b \) which is unitarily invariant. Hence \((a^{(1)}, ubu^*)\) has the same traffic distribution as \((a^{(1)}, b)\). Hence they have the same \(*\)-distribution and so \( a^{(1)} \) and \( b \) are free independent. We get the proposition by induction on \( L \) and thanks to the associativity of free independence [NS06].

This implies a useful criterion of asymptotic free independence in the context of the asymptotic traffic independence theorem.

**Corollary 5.7.**

Let \( A_N^{(1)}, \ldots, A_N^{(L)} \) be as in Theorem 1.8. Assume moreover that each family \( A_N^{(l)} \) has the same limiting traffic distribution as \( U_N A_N^{(l)} U_N^* \) for any unitary matrix \( U_N \), except possibly one index \( l \in \{1, \ldots, L\} \). Then \( A_N^{(1)}, \ldots, A_N^{(L)} \) are asymptotically free independent.

Note that the additional assumption in Proposition 5.7 is much weaker than assuming the unitary invariance of the \( A_N^{(l)} \) (that is \( A_N^{(l)} \approx U_N A_N^{(l)} U_N^* \) for each \( N \) and any unitary matrix \( U_N \)). For instance, independent Wigner matrices satisfy this proposition. This result is applied for a class of large random graphs with large degree in [CM]. A much detailed analysis of the relation between traffic independence, free independence and unitarily invariance is made in [CDM16].

**Proof of Lemma 5.6.**

We first prove the first part of the lemma. Let \( n \geq 1 \) be an integer and \( P_1, \ldots, P_n, Q_1, \ldots, Q_n \) be non commutative polynomials and denote \( y_i = P_i(a), z_i = Q_i(b), y = (y_1)_1, \ldots, n \) and \( z = (z_1)_1, \ldots, n \). Note that \( uy_iu^* = P_i(uaa^*) \) for any \( i = 1, \ldots, n \) since \( uu^* = u^*u = I \). We assume \( \Phi(y_1, \ldots, y_n) = 0 \), \( \Phi(z_1) = 0 \). Proving for any \( i = 1, \ldots, n \) that \( \Phi(y_1, \ldots, z_i, \ldots, z_n) \) converges to zero we will get the free independence of \( uaa^* \) and \( b \).

One has \( \Phi(y_1, \ldots, y_n) = \Phi(uy_1u^*z_1 \cdots uy_nu^*) \). Let \( T = (V, E, \gamma, \epsilon) \) be the test graph in variables \( u, u^*, y_1, \ldots, y_n, z_1, \ldots, z_n \) such that \( \Phi(uy_1u^*z_1 \cdots uy_nu^*) = \tau[T] \), namely

- the set of vertices is \( V = \{1, 2, \ldots, 4n\} \),
- the edges are \((1, 4n), (4n, 4n - 1), (4n - 1, 4n - 2), \ldots, (3, 2), (2, 1)\),
- with notation of indices modulo \( 2n \), the edges \((4i + 2, 4i + 1)\) are labelled \( u \), the edges \((4i + 3, 4i + 2)\) are labelled \( y_i \), the edges \((4i + 4, 4i + 3)\) are labelled \( u^* \), and the edge \((4i + 5, 4i + 4)\) are labelled \( z_i \).

By the relation between the trace and its injective version (Formula (2.3)), one has \( \tau[T] = \sum_{\pi \in P(V)} \tau^\pi[T] \). Moreover, since \((u, u^*)\) and \((a, b)\) are traffic independent, then \((u, u^*)\) and \((y_1, \ldots, z_i)\) are traffic independent. Hence by Definition of traffic independence, we get

\[
\tau^\pi[T^\pi] = \mathbb{I}(GCC(T^\pi) \text{ is a tree}) \prod_{S \in \mathcal{CC}(T^\pi)} \tau^\pi[S],
\]

where \( GCC(T^\pi) \) (respectively \( \mathcal{CC}(T^\pi) \)) is the graph (the set) of colored components of \( T^\pi \) with respect to the variables \((u, u^*)\) and \((y, z)\).

Recall that the injective distribution of \( u \) is supported on cacti (Proposition 3.7), namely graphs such that each edge belongs exactly to one simple cycle. More precisely, the edges of a same cycle must be oriented in the same direction and labels must alternate between \( u \) and \( u^* \).

Given \( \pi \) as in the sum, denote by \( S(T^\pi) \) the \(*\)-test graph obtained from \( T^\pi \) by identifying the vertices attached to a same connected component labelled in \( y \) or \( z \), and forgetting the edges labelled in \( y, z \). To ensure that \( \pi \) contributes, each connected component of \( T^\pi \) labeled in \( \{x, x^*\} \) must be a cactus. When \( GCC(T^\pi) \) is a tree, the graph \( S(T^\pi) \) itself must be a cactus.

Denote by \( S(n) \) the set of cacti with \( 2n \) edges. Then one has

\[
\tau[T] = \sum_{S \in S(n)} \sum_{\pi \in P(V)} \mathbb{I}(GCC(T^\pi) \text{ is a tree}) \prod_{S \in \mathcal{CC}(T^\pi)} \tau^\pi[S].
\]
For any \( S_0 \in S(n) \) and any \( \pi \) in \( \mathcal{P}(V) \) such that \( S(T^\pi) = S_0 \), there is a colored component \( S \in \mathcal{C}(T^\pi) \) labeled in \( y \) or \( z \) consisting in a loop. Indeed, there is a vertex of \( S(T^\pi) \) which belong to a single cycle (each cycle has length at least two). So since the variables \( y_i, y_j, u^*, z_i \) in \( T \) alternates there is such a loop in \( T^\pi \) corresponding to this vertex. This implies that \( \tau^0[T^\pi] \) is zero since the combinatorial trace of a loop is the usual trace of the centered variables \( y_i, z_i \).

It remains to prove the second part of the lemma. Let \( a, b \) be traffic independent and unitary invariant. Let \( u, v, w \) be traffics, limits of Haar unitary matrices (as long with their adjoint), such that \( a, b, u, v, w \) are independent. We prove that \((aua^*, ubu^*)\) has the same traffic distribution as \((a, b)\). By unitary invariance and associativity of independence, \((aua^*, ubu^*)\) has the same distribution as \((uv^*u^*, uwbw^*u^*)\). But \((uv, uw)\) and \((a, b)\) are independent and the matrix approximation (Theorem 4.8 for independent Haar unitary matrices) shows that \((uv, uw)\) has the same traffic distribution as \((v, w)\). Hence \((aua^*, ubu^*)\) has the same distribution as \((v^*, w^*w)\), and so the same distribution as \((a, b)\).

\[\square\]

### 5.3 Link with tensor independence

**Lemma 5.8.** Let \((A, \tau)\) be an algebraic traffic space with trace \(\Phi\) and anti-trace \(\Psi\). The space \(\Delta(A) = \{\Delta(a), a \in A\}\) of diagonal elements is a commutative \(G\)-subalgebra of \(A\). Moreover, for any family \(a\) of diagonal traffics and for any test graph \(T = (V, E, \gamma)\), one has

\[
\tau[T(a)] = \Phi\left[ \prod_{e \in E} a_{\gamma(e)} \right] = \Psi\left[ \prod_{e \in E} a_{\gamma(e)} \right], \quad \tau^0[T(a)] = 1(|V| = 1) \times \tau[T(a)].
\]

**Proof.** Let \( a \) be a family of elements of \(\Delta(A)\) and \(g = (T, in, out), T = (V, E, \gamma)\), a graph monomial. Since \(\Delta(A) = a\) and by the associativity of the composition of graph monomials, \(g(a) = \hat{g}(a)\) where \(\hat{g}\) is obtained from \(g\) by identifying source and goal of each edge, which results in a bunch of self-loops independently of the geometry of the initial graph \(g\). Hence, we have \(\hat{g}(a) = \prod_{e \in E} a_{\gamma(e)}\) and \(g(a)\) is diagonal. Since \(\Delta\) is linear, we get that \(\Delta(A)\) is a \(G\)-subalgebra. Moreover \(\Delta(a) \times \Delta(b) = \Delta(a, b) = \Delta(b) \times \Delta(a)\), where \(\Delta\) is the graph operation with two self-loops attached to a single edge and the chain of equality comes from the associativity of composition and equivariance.

The formula for \(\tau[T(a)]\) follows from the previous paragraph since we can write \(\tau[T(a)] = \Phi(g(a)) = \Psi(h(a))\) for some graph monomials \(g, h\). It remains to prove the formula for \(\tau^0\). Let \(T = (V, E, \gamma)\) be a test graph. Denote by \(1_{P_{\gamma}} = \{V\}\) the partition of \(V\) with a single block. By the previous point we have \(\tau[T] = \tau[T^{1_{P_{\gamma}}}]\). On the other hand,

\[
\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi], \quad \tau[T^{1_{P_{\gamma}}}] = \tau^0[T^{1_{P_{\gamma}}}].
\]

Hence we get that \(\tau[T] = \tau^0[T^{1_{P_{\gamma}}}], \quad \tau[T^{1_{P_{\gamma}}}] = 0, \quad \tau[T^{1_{P_{\gamma}}}] = 0, \quad \tau[T^{1_{P_{\gamma}}}] = 0, \quad \tau^0[T] = 2\) by an induction on \(|V|\). \[\square\]

**Proof of the third item of Theorem 5.3.** Assume that \(a^{(1)}, \ldots, a^{(L)}\) are traffic independent diagonal variables. For each \(\ell = 1, \ldots, L\), let \(P_\ell\) be a commutative monomial. We can write \(\Phi([^L_{\ell=1} P_\ell(a^{(\ell)})]} = \tau[T(a)]\) where \(T\) is the graph with a single vertex and on self-loop for each variable appearing in the monomials.

Since \(T\) has a single vertex, \(\tau[T(a)] = \tau^0[T(a)]\), and by traffic independence, \(\tau^0[T(a)] = \prod_{\ell=1}^L \tau[T_\ell(a)]\), where \(T_\ell\) is the subgraph of \(T\) whose edges labels correspond to \(a^{(\ell)}\). But for each \(\ell\), by the same reasoning, one has

\[
\tau^0[T_\ell(a)] = \tau[T_\ell(a)] = \Phi[P_\ell(a^{(\ell)})].
\]

Hence \(\Phi([^L_{\ell=1} P_\ell(a^{(\ell)})]} = \prod_{\ell=1}^L \Phi[P_\ell(a^{(\ell)})]\) so the families of matrices are tensor independent with respect to \(\Phi\). Moreover, \(\Phi\) and \(\Psi\) are equal for diagonal traffics so the families are also tensor independent with respect to \(\Psi\).

Reciprocally, assume now the tensor independence of diagonal elements \(a^{(1)}, \ldots, a^{(L)}\). Let \(T\) be a test graph. If \(T\) has more than a single vertex, then there is a colored component \(T^0\) of \(T\) which has
the same property and so \(\tau^0(T(a)) = 0\) and \(\tau^0(T'(a)) = 0\). Hence the rule of traffic independence is satisfied for these graphs. If now \(T\) has a single vertex, we have \(\tau^0(T(a)) = \prod_{t=1}^n \tau^0(T_t(a))\) by the above reasoning. Hence we get the traffic independence of the families.

\[\square\]

5.4 Link with Boolean independence

Note that from the illustration of Theorem \(\text{[1.8]}\) given at the very end of Section \(\text{[1.3]}\), the link between traffic independence and tensor and free independence is not a surprise. Nevertheless, these two notions are not sufficient if we want to explain a central limit theorem as we intend to in the last section. This was the original motivation to find the second item of Theorem \(\text{[5.5]}\).

5.4.1 Generalities and proof

**Lemma 5.9.** Let \(a\) be a family of elements in an algebraic traffic space \((A, \tau)\) with associated trace \(\Phi\).

1. If \(a\) is of Boolean type, then the \(^*\)-distribution of \(a\) with respect to \(\Phi\) is the distribution of the null element.

2. \(a\) is of Boolean type if and only if the injective combinatorial distribution of \(a\) is supported on trees, in which case for any \(^*\)-test graph \(T\) one has \(\tau(T(a)) = \tau^0(T(a))\).

3. \(a_1, \ldots, a_L\) are traffic independent and of Boolean type, then \(a_1 \cup \cdots \cup a_L\) is of Boolean type.

**Proof.** (1) Let \(a\) be of Boolean type. For any \(^*\)-monomials \(M, \Phi(M(a))\) is equal to \(\tau\left(\bigcirc (M(a))\right) = 0\).

(2) Recall that for any test graph \(T\), we have \(\tau(T) = \sum_{\pi \in P(V)} \tau^0(T^\pi)\) and \(\tau^0(T) = \sum_{\pi \in P(V)} \mu_V(\pi) \tau(T^\pi)\). But \(T^\pi\) is a tree if and only if \(T\) is a tree and \(\tau\) is the partition \(0_V\) consisting in singletons. Hence if \(a\) is of Boolean type then necessarily \(\tau^0(T) = \tau(T^0\pi) = \tau(T)\) for any \(T\). If \(\tau^0\) is supported on trees, similarly \(\tau(T) = \mu_V(0_V) \tau^0(T) = \tau^0(T)\). Hence the first point of the lemma.

(3) Let now \(a_1, \ldots, a_L\) be of Boolean type and traffic independent. For any test graph \(T\), one has \(\tau^0(T) = 1 (\mathcal{GCC}(T)\) is a tree\) \(\prod_{\pi \in S_{\mathcal{GCC}(T)}} \tau^0(S)\). If \(T\) is not a tree then either \(\mathcal{GCC}(T)\) is not a tree or a colored component of \(T\) is not a tree. Hence \(\tau^0(T) = 0\).

\[\square\]

**Proof of the second item of Theorem \(\text{[5.5]}\)** Let \(a_1, \ldots, a_L\) be traffic independent and such that the combinatorial distribution of each \(a_i\) is supported on trees. Let \(M_1, \ldots, M_n\) be non constant monomials. Let us prove that \(\Psi(M_1(a_{i_1}) \cdots M_n(a_{i_n})) = \prod_{j=1}^n \Psi(M_j(a_{i_j}))\) for any \(i_1 \neq i_2 \neq \cdots \neq i_n\).

Denoting \(M = M_1 \cdots M_n\), by the substitution axiom, we have \(\Psi(M(a_1, \ldots, a_L)) = \tau(T_M)\) where \(T_M\) consists in a simple line, namely \(T_M = (\overset{\cdots}{x_i} \cdots \overset{\cdots}{x_L})\) whenever \(M = x_1 \cdots x_Q\). By the above lemma, we have \(\tau(T_M) = \tau^0(T_M) = \prod_{\pi \in S_{\mathcal{GCC}(T)}} \tau^0(S)\). But the colored components of \(T\) are the graphs \(T_{M_1}, \ldots, T_{M_n}\) constructed as \(T_M\) for \(M\) replaced by \(M_1, \ldots, M_n\). Hence \(\tau(T_M) = \prod_{j=1}^n \tau^0(T_M_{j_i}) = \prod_{j=1}^n \tau(T_{M_j}) = \prod_{j=1}^n \Psi(M_j(a_{i_j}))\).

\[\square\]

5.4.2 Application to random matrices

Recall that \(\deg(A_N)\) denotes the diagonal matrix \(\text{diag}(\sum_{j=1}^N A_N(i, j))\).

**Corollary 5.10.** The anti-trace \(\Psi_N\) of the algebraic traffic space of matrices is given by

\[
\Psi_N[A_N] = E\left[\frac{1}{N} \text{Tr} \deg A_N\right] = E\left[\frac{1}{N} \sum_{i,j=1}^N A_N(i, j)\right] = \langle e_N, A_N e_N \rangle.
\]
where \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product in \( \mathbb{C}^N \) and \( e_N \), the column vector whose all entries are \( \frac{1}{\sqrt{N}} \). If \( A_N \) converges in traffic distribution, then it has a limiting distribution with respect to \( \Psi_N \), that is \( \Psi_N[P(A_N)] \) converges non commutative polynomial \( P \). If \( A_N^{(1)}, \ldots, A_N^{(\ell)} \) are asymptotically traffic independent families of matrices whose limiting combinatorial distributions are supported on simple trees, then the families \( A_N^{(1)}, \ldots, A_N^{(\ell)} \) are asymptotically Boolean independent with respect to \( \Psi_N \).

Let us now give examples of such matrices. The matrix \( J_N \), whose all entries are \( \frac{1}{\sqrt{N}} \), converges to a traffic of Boolean type: for any \(*\)-test graph \( T \), by Lemmas 2.18 and 2.13,

\[
\tau_N[T(J_N)] = N^{V-1-|E|} \left( 1 + O \left( \frac{1}{N} \right) \right) = \mathbb{1}(T \text{ is a tree}) + o(1).
\]

Note that one has \( \Psi(N(A_N)) = \mathbb{E}[\text{Tr}(A_N J_N)] \) and \( J_N = e_N e_N^* \) where \( e_N \) is as in Proposition 5.10. Furthermore, \( J_N \) is a deterministic permutation invariant matrix, and any permutation invariant deterministic matrix \( A_N \) is of the form \( \Phi_N(A_N) I_N + (\Psi_N(A_N) - \Phi_N(A_N)) J_N \), where \( \Phi_N(A_N) = \mathbb{E} \left( \frac{1}{N} \text{Tr} A_N \right) \) and \( I_N \) is the identity matrix. It is also a projection matrix, namely \( J_N^2 = J_N \).

The limiting distribution of \( J_N \) with respect to \( \Psi_N \) is the distribution of a variable constant to one. Indeed, one has \( \text{deg}(J_N) = 1 \) so for any monomial \( k \geq 1 \), one has \( \Psi(N[J_N^k]) = \mathbb{E} \left( \frac{1}{N} \text{Tr} I_N \right) = 1 \) and so for any polynomial \( P \) one has \( \Psi(N[P(J_N)]) = P(1) \).

Let see now a non constant example.

**Lemma 5.11.** Let \( X_1, \ldots, X_N \) be independent and identically distributed complex random variables. Assume all the moments of the \( X_i \)'s are finite and do not depend on \( N \). We denote the permutation invariant matrix \( M_N = \left( \frac{X_i + X_j}{\sqrt{N}} \right)_{i,j=1,\ldots,N} \). Then \( M_N \) has a limiting traffic distribution supported on trees and it satisfies the factorization property (Assumption B3 of Theorem 1.8): denoting \( \alpha_{\ell,v} = \mathbb{E}[X_i X_j^{*\ell}], \) for any test graph \( T = (V,E) \), one has

\[
\tau_N^0[T(M_N)] \rightarrow_N \mathbb{1}(T \text{ is a tree}) \sum_{\pi \in \mathcal{P}(E[V])} \prod_{B_i \in \pi} \alpha_{\ell_{B_i},k_{B_i}},
\]

where \( \mathcal{P}(E[V]) \) is the set of partitions \( \pi = \{ B_v, v \in V \} \) of edges of \( T \) into blocks \( B_v \) having in common a same vertex \( v \in V \), and \( \ell_{B_v} \) (respectively \( k_{B_v} \)) is the number of edges of \( B_v \) for which \( v \) is the source (respectively the target). Moreover, if the \( X_j \) are centered the limiting distribution of \( M_N \) with respect to \( \Psi_N \) is the Rademacher distribution \( \frac{\delta_{-1,1} + \delta_{1,1}}{2}, (\alpha_{1,1} = \mathbb{E}[[X_j]^2]) \), namely for any \( k \geq 1 \), \( \Psi(N[M_N^k]) \rightarrow_N \alpha_{1,1}^k(K \text{ even}) \).

In particular, if the variables \( X_j \) are real Gaussian random variables centered with variance one, then \( \alpha(T) \) is the number of partitions of edges of \( T \) whose blocks consist in two edges having a vertex in common. If the \( X_j \) are complex Gaussian random variables such that \( \mathbb{E}[X_j] = \mathbb{E}[X_j^2] = 0 \) and \( \mathbb{E}[[X_j]^2] = 1 \), then \( \alpha(T) \) is the number of partitions of edges of \( T \) whose blocks consist in two edges having a vertex in common, which is the source for one edge and the target of the other one.

The former lemma, Theorem 1.8 and Proposition 5.10 yield the first example of asymptotic Boolean independent matrices.

**Corollary 5.12.** Let \( M_N \) be a family of independent matrices as in Lemma 5.11 and let \( J_N \) be the matrix whose all entries are \( \frac{1}{\sqrt{N}} \). Then the matrices of \( M_N \) and \( J_N \) are asymptotically traffic independent and are asymptotically Boolean independent with respect to \( \Psi_N \). Moreover, if the variables \( X_i \) defining \( M_N \) are complex Gaussian variables such that \( \mathbb{E}[X_i] = \mathbb{E}[X_i^2] = 0 \) and \( \mathbb{E}[[X_i]^2] = 1 \), then the matrices of \( M_N \) and their transpose are asymptotically Boolean independent.

**Proof of Lemma 5.11.** For any \(*\)-test graph \( T = (V,E,\gamma) \) in one variable, one has (Lemma 2.18)

\[
\tau_N^0[T(M_N)] = N^{V-1-|E|} \delta_N^0[T(M_N)] \times \left( 1 + O \left( \frac{1}{N} \right) \right) = N^{V-1-|E|} \delta_N^0[T(N M_N)] \times \left( 1 + O \left( \frac{1}{N} \right) \right),
\]
where $\delta_N^{\tau}[T(NM_N)] = \mathbb{E}\left[\prod_{(v,w) \in E} (X_{\phi(v)} + \overline{X_{\phi(w)}})\right]$ for any injection $\phi: V \to [N]$. It does not depend on $\phi$ and $N$. Hence by Lemma 2.13, $\tau_N^{\tau}[T(M_N)]$ converges and its limit is zero if $T$ is not a tree. Let $T$ be a tree and denote $\alpha_N(T) := \delta_N^{\tau}[T(NM_N)]$. To compute the limit of $\alpha_N(T)$ we expand the product over $E$ and the sums in its definition, which amounts for each edge $e = (v, w)$ to keep either the variable attached to its source $X_{\phi(e)}$ or to its target $X_{\phi(w)}$. Since the variables $X_1, \ldots, X_N$ are independent, this yields Formula (5.1).

The proof of the factorization property is the same as for Wigner matrices. Let $T_1, \ldots, T_n$ be test graphs in one variable, and denote by $T$ the graph obtained as the disjoint union of $T_1, \ldots, T_n$. By Lemma 2.24

$$\mathbb{E}\left[\prod_{i=1}^{n} \frac{1}{N} \text{Tr}^0[T_i(M_N)]\right] = \sum_{\pi} \frac{1}{N^n} \mathbb{E}\left[\text{Tr}^0[T^\pi(M_N)]\right],$$

where the sum is over partitions that do not identify two vertices of a same $T_i$ and $V_\pi, E_\pi$ denote the vertex and edge set of $T^\pi$. We have by Lemma 2.13 that $V_\pi - E_\pi - n \leq 0$ with equality if and only if $\pi$ is the trivial partition with only singleton blocks and the graphs $T_1, \ldots, T_n$ are trees. Moreover, the matrix entries in $\delta_N^{\tau}[T^\pi(NM_N)]$ associated to edges of different components of $T^\pi$ are independent. Hence we obtain the factorization property

$$\mathbb{E}\left[\prod_{i=1}^{n} \frac{1}{N} \text{Tr}^0[T_i(M_N)]\right] = \prod_{i=1}^{n} \mathbb{E}(T_i \text{ is a tree}) \delta_N^{\tau}[T_i(NM_N)] + o(1).$$

Let compute now the limiting distribution of $M_N$ with respect to $\Psi_N$, assuming the $X_i$ defining $M_N$ centered. If $T$ is a directed line of odd length $2n+1$, then it is not possible to find an arm which is not zero (the partitions $\pi$ of $\mathcal{P}(E(V))$ possess a block of size one), and so $\Psi_N[M_N^{2n+1}] \overset{N \to \infty}{\to} 0$. If $T$ has an even length $2n$, then there is a unique way to get a non zero term in the expansion of $\alpha_N(T)$, which gives $\alpha_N(T) = \mathbb{E}[|X_i|^2]^n$. This proves the convergence of $M_N$ with respect to $\Psi_N$ to the expected limit.

The formulas for $\alpha(T)$ when the $X_i$ are Gaussian follows from Wick formula: for a Gaussian random variable $X$, in the real case $\mathbb{E}[X^k]$ is equal to the number of pair partitions of $\{1, \ldots, k\}$ and in the Gaussian case (with $\mathbb{E}[X^2] = 0$) $\mathbb{E}[X^kX^l]$ is the number of bijections $\{1, \ldots, k\} \to \{1, \ldots, l\}$. This is a direct consequence of the stability of the Gaussian distribution.

\[ \square \]

**Proof of Corollary 5.12.** The first part of Corollary 5.12 is consequence of Theorem 1.8 and Corollary 5.10. Let prove that when the variables defining $M_N$ are complex Gaussian random variables $\omega_{i,j}$ such that $\mathbb{E}[\omega^2] = 0$, the matrices of $M_N$ are asymptotically Boolean independent along with the transposed matrices. The proof is the same as for the analogue statement (Lemma 5.3) for Wigner matrices. Let $P$ be a monomial in $M_N$ and $M_N^\dagger$. We can write $\Psi_N[P(M_N, M_N^\dagger)] = \tau_N[T(M_N)]$ where the test graph $T$ is a simple line, with edges corresponding to $M_N$ in one direction and those corresponding to $M_N^\dagger$ in the other direction. But in (5.1), the partitions must pair edges with same orientation. Hence the couple $(M_N, M_N^\dagger)$ has the same limiting distribution with respect to $\Psi_N$ as $(M_N, M_N^\dagger)$ where $M_N$ is an independent copy of $M_N$. Hence their asymptotic Boolean independence.

\[ \square \]

**Corollary 5.13.** Let $M_N = \frac{X_i+X_j}{\sqrt{N}}$ as in Lemma 5.11 where $\mathbb{E}[X_j] = \alpha \neq 0$ and $\mathbb{E}[|X_j-\alpha|^2] = 1$. Then the limiting distribution of $M_N$ with respect to $\Psi_N$ is

$$\frac{1}{2} \left( 1 + \frac{\mathbb{E}[\omega]}{\sqrt{\mathbb{E}[\omega]^2} + 1} \right) \delta_\alpha + \left( 1 - \frac{\mathbb{E}[\omega]}{\sqrt{\mathbb{E}[\omega]^2} + 1} \right) \delta_0.$$
Proof. Note that for any matrices $A_N, B_N$, one has $\Psi_N(A_N J_N B_N) = \Psi_N(A_N) \times \Psi_N(B_N)$. Hence with the above notation we have

$$
\Psi_N(M_N^{K+2}) = \Psi_N(\tilde{M}_N^2 M_N^K) + 2\Re(\alpha)\Psi_N(\tilde{M}_N)\Psi_N(M_N^K) + 2\Re(\alpha)\Psi_N(M_N^{K+1}).
$$

By the expression of the limiting distribution of $\tilde{M}_N$, we have $\Psi_N(\tilde{M}_N^2 M_N^K) = \Psi_N(M_N^K) + o(1)$ and $\Psi_N(M_N) = o(1)$. Hence the sequence of moments $(m_K)_{K \geq 0}$ of the limiting distribution of $M_N$ with respect to $\Psi_N$ satisfies the recurrence relation $m_{K+2} = \beta m_{K+1} + m_K$, $m_0 = 1$ and $m_1 = \beta$, where $\beta = 2\Re(\alpha)$. We get that for any $K \geq 0$,

$$
m_K = \frac{1}{2} \left( 1 + \frac{\beta}{\gamma} \right) \left( \frac{\beta + \gamma}{2} \right)^K + \frac{1}{2} \left( 1 - \frac{\beta}{\gamma} \right) \left( \frac{\beta - \gamma}{2} \right)^K, \quad \beta = 2\Re(\alpha), \gamma = \sqrt{\beta^2 + 4}.
$$

Hence the distribution has two Dirac mass, it is characterized by its moments, and we get the result after simplification. \qed
Chapter 6

Limit theorems for independent traffics

We state the law of large number and the central limit theorem in the context of traffic independence. We see in both cases that the situation is much richer than for the classical notions of independence.

6.1 Constant traffics and law of large numbers

In a non commutative probability space, a constant non commutative random variable is an element distributed as a multiple of the identity, or equivalently an element freely independent with itself.

Recall that \( J_N \) denotes the matrix whose entries are \( \frac{1}{N} \). We have seen in Section 5.3 that \( J_N \) converges in traffic distribution. Denote by \( J \) a traffic distributed as the limit of \( J_N \). Recall that \( \text{deg}(\cdot) \) is the graph monomial with two vertices \( in = out \) and \( v \) and one edge from \( in \) to \( v \).

**Proposition 6.1.** Let \((\mathcal{A}, \tau)\) be an algebraic traffic space, let \( \Phi \) denote the trace associated to \( \tau \) and set \( \Psi = \Phi \circ \text{deg} \).

1. An element \( a \) of \( \mathcal{A} \) is traffic independent from itself if and only if it has the same distribution as \( \Phi(a)I + (\Psi(a) - \Phi(a))J \).

2. Law of large numbers: Let \((a_n)_{n \geq 1}\) be a sequence of identically distributed independent traffics in \( \mathcal{A} \) and let \( a \) be distributed as the \( a_n \)'s. For each \( n \geq 1 \), denote \( m_n = \frac{\Phi(a) + \cdots + \Phi(a)}{n} \). Then, as \( n \) goes to infinity, \( m_n \) converges to \( \Phi(a)I + (\Psi(a) - \Phi(a))J \) in traffic distribution.

**Proof of Proposition 6.1.**

Assume that \( a \) is traffic independent from itself. Let \( T \) be a test graph in one variable \( x \) and let \( e_1, \ldots, e_K \) be an enumeration of its edges. Let \( \tilde{T} \) be the test graph in \( K \) variables \( x_1, \ldots, x_K \) obtained from \( T \) by replacing label \( x \) of the \( k \)-th edge of \( T \) by \( x_k \). Let \( a_1, \ldots, a_K \) be independent copies of \( a \). By associativity of traffic independence, \((a_1, \ldots, a_K)\) has the same distribution as \((a, \ldots, a)\) and so \( \tau^0[T(a)] = \tau^0[\tilde{T}(a_1, \ldots, a_K)] \). By definition of traffic independence, this quantity is nonzero only if the graph of colored components of \( \tilde{T} \) with respect to \( x_1, \ldots, x_K \) is a tree. Since the labels of the edges are pairwise distinct, this means that one obtains a tree when removing the self loops of \( T \). For such a graph \( T \), denote by \( \ell \) its number of self loops labeled \( x \) and by \( m \) its number of simple edges (edges that are not loops) labeled \( x \). Then we get by definition of traffic independence \( \tau^0[T(a)] = \tau^0[\bigotimes^a]^\ell \tau^0[igotimes^a]_m \).

But \( \tau^0[\bigotimes^a] = \tau[\bigotimes^a] = \Phi(a) \) and \( \tau^0[\bigotimes^a]_m = \tau[\bigotimes^a]_m - \tau[\bigotimes^a] = \Psi(a) - \Phi(a) = \Psi(a-\Delta(a)) \), where we used (4.1) in the last equality. Hence

\[ \tau^0[T(a)] = \Phi(a)^\ell \Psi(a-\Delta(a))^m. \]

Let now prove that \( a \) has the same distribution as \( b = \Phi(a)I + \Psi(a-\Delta(a))J \). Let \( T \) be a test graph in one variable with \( K \) edges \( e_1, \ldots, e_K \). Denoting \([K] = \{1, \ldots, K\}\), for any map \( \gamma : [K] \to \{1,2\} \), let \( T_\gamma \) be the test graph in two variables \( i \) and \( j \) (for \( \ell \) and \( J \) respectively)
CHAPTER 6. LIMIT THEOREMS FOR INDEPENDENT TRAFFICS

obtained from $T$ by putting label $i$ for edges $e_k$ with $\gamma(k) = 1$ and label $j$ otherwise. Denote by $\ell_\gamma$ (resp. $m_\gamma$) the number of edges labeled $i$ (resp. $j$) in $T_\gamma$. Then, the multi-linearity of $\tau^0$ w.r.t. the edges of the graphs implies that

$$\tau^0[T(b)] = \sum_{\gamma \in [K] \rightarrow \{1, 2\}} \Phi(a)^{\ell_\gamma} \Psi(a - \Delta(a))^m \tau^0[T_\gamma(I, J)].$$

Denoting by $\tilde{T}_\gamma$ the graph obtained from $T_\gamma$ by erasing self-loops labeled $I$, we get $\tau^0[T_\gamma(I, J)] = \tau^0[\tilde{T}_\gamma(J)]$ if all edges labeled $i$ are simple loops and zero otherwise. Finally, recall from Section 5.4.2 that the injective distribution of $\mathcal{J}$ is the indicator of simple trees. Hence $\tau^0[T(b)]$ vanishes if $T$ is not a graph consisting in a tree decorated with simple loops. Otherwise, $\tau^0[T_\gamma(I, J)]$ is non zero only for the map $\gamma_0$ sending self-loops to 1 and simple edge to 2, for which $\tau^0[T_\gamma(I, J)]$ is one. Hence $\tau^0[T(b)] = \Phi(a)^0 \Psi(a - \Delta(a))^m$, with $\ell = \ell_{\gamma_0}$ and $m = m_{\gamma_0}$, so we get as expected that $a$ is distributed as $b$.

It remains to mention that $b = \Phi(a)I + \Psi(a - \Delta(a))\mathcal{J}$ is traffic independent from itself. Let $T$ be a test graph in two variables $x$ and $y$. One the one hand we know that $\tau^0[T(b, b)] = 0$ if $T$ is not a simple tree decorated with loops. This is equivalent to say that the graph of colored component of $T$ with respect to $x$ and $y$ is a tree and the colored components are trees decorated with loops. Therefore, we get $\tau^0[T(b, b)] = \Phi(a)^0 \Psi(a - \Delta(a))^m$ with the same notation as above, and for each colored components $S$ of $T$ we have $\tau^0[S(b)] = \Phi(a)^0 \Psi(a - \Delta(a))^m$ with similar notations. Since $\ell = \sum S \ell_S$ and $m = \sum S m_S$, we get the result.

2. We now prove the law of large numbers. For any test graph $T$ in a single variable $x$ with edge set $E$ and any map $\gamma : E \rightarrow [n] := \{1, \ldots, n\}$, denote by $T_\gamma$ the test graph in $n$ variables $x_1, \ldots, x_n$ obtained from $T$ by putting the label $x_{\gamma(e)}$ on edge $e$. Then by multi-linearity w.r.t. the edges for $\tau^0$, we get

$$\tau^0[T\left(\frac{a_1 + \cdots + a_n}{n}\right)] = \sum_{\gamma : E \rightarrow [n]} n^{-|E|} \tau^0[T_\gamma(a_1, \ldots, a_n)].$$

For any $\gamma$ let $\pi_\gamma$ be the partition of $E$ such that two edges belong to the same block whenever they have same label. Since the $a_i$ are identically distributed and independent, $\tau^0[T_\gamma(a_1, \ldots, a_n)]$ depends only on $\pi_\gamma$ and we denote by $\eta(\pi_\gamma)$ this quantity. Denote by $\mathcal{P}(E)$ the set of partitions of $E$ and by $|\pi|$ the number of blocks of an element $\pi$ of $\mathcal{P}(E)$. We then get

$$\tau^0[T\left(\frac{a_1 + \cdots + a_n}{n}\right)] = \sum_{\pi \in \mathcal{P}(E)} n(n - 1) \ldots (n - |\pi| + 1) \times n^{-|E|} \eta(\pi) = \sum_{\pi \in \mathcal{P}(E)} n^{|\pi| - |E|} (1 + o(1)) \eta(\pi).$$

The only partition that contributes is the partitions consisting in singletons. Hence $\tau^0[T\left(\frac{a_1 + \cdots + a_n}{n}\right)] = \tau^0[\tilde{T}(a_1, \ldots, a_{|E|})] + o(1)$ where $\tilde{T}$ is obtained by putting different labels for its edges. The $a_i$ are independent and identically distributed so we have seen in the proof of the previous point that $\tau^0[T(a_1, \ldots, a_{|E|})] = \tau^0[T(\Phi(a)I + \Psi(a - \Delta(a))\mathcal{J})].$

\[\square\]

6.2 Central limit theorem

We recall the classical CLTs and then state the "traffic" version.

**Theorem 6.2.** Let consider the three situations (1,2,3), of a non-commutative *-probability space $(\mathcal{A}, \Phi)$ for (1) and (3) and of a *-algebra $\mathcal{A}$ endowed with a state $\Phi$ for (2). Consider a sequence $\{a_n\}_{n \geq 1}$ of identically distributed, self adjoint elements of $\mathcal{A}$, either free (1), Boolean (2) or tensor (3) independent. Assume that $\Phi(a_n) = 0$ and $\Phi(a_n^2) = 1$. Then $m_n = \frac{a_1 + \cdots + a_n}{\sqrt{n}}$ converges in distribution to
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(1) a semicircular variable \(x\), i.e. \(\Phi(x^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k \sqrt{4 - t^2} dt\),

(2) a Rademacher distribution \(y\), i.e. \(\Phi(y^k) = 1\) (\(k\) is even),

(3) a Gaussian variable \(z\), i.e. \(\Phi(z^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k e^{-t^2/2} dt\),

The limit in CLT for traffics will be written as a sum of three terms that represent each of these variables.

**Definition 6.3.** Let \((\mathcal{A}, \tau)\) be an algebraic traffic space with associated trace \(\Phi\) and anti-trace \(\Psi\) such that \(\mathcal{A}\) is a \(\mathbb{G}^*\)-algebra. We say that an element \(a \in \mathcal{A}\) is self-adjoint if \(a^* = a\) and off-diagonal whenever \(\Delta(a) = 0\).

1. A centered semicircular traffic \(x\) with parameter
   \[
   (\alpha, \beta) = (\Phi(x^2), \Phi(x^4)), \quad |\beta| \leq \alpha,
   \]
is a self-adjoint off-diagonal traffic with distribution given as follow: for any test graph \(T\)
   \[
   \tau^0[T(x)] = \mathbb{1}(T\text{ is a double tree})\alpha^{\ell(T)}\beta^{k(T)},
   \]
   where \(\ell(T)\) (respectively \(k(T)\)) is the number of double edges with opposing (respectively similar) orientation (Proposition 3.1).

2. A centered simple Boolean traffic \(y\) with parameter given by a symmetric non-negative matrix
   \[
   (\alpha_{\ell,k})_{\ell,k \geq 0} = \left(\Phi(\text{deg}(y)^\ell \text{deg}(y)^k)\right)_{\ell,k \geq 0},
   \]
is a self-adjoint off-diagonal traffic with distribution given as follow: for any test graph \(T\)
   \[
   \tau^0[T(y)] = \mathbb{1}(T\text{ is a tree}) \sum_{\pi \in \mathcal{P}(E|V)} \prod_{v \in \pi} \alpha_{\ell(v),k(v)},
   \]
   where \(\mathcal{P}(E|V)\) is the set of partitions of edges of \(T\) into blocks \(B_v\) of edges having in common a same vertex \(v \in V\), and \(\ell(v)\) (respectively \(k(v)\)) is the number of edges of \(B_v\) for which \(v\) is the source (respectively the target).

A centered Gaussian Boolean traffic \(y\) with parameter
   \[
   (\alpha, \beta) = (\Psi(y^2), \Psi(yy^t)), \quad |\beta| \leq \alpha,
   \]
is a centered simple Boolean traffic for which the parameter is \((\mathbb{E}[Y^tY^k])_{k,\ell \geq 0}\) for a complex Gaussian random variable \(Y\).

3. A real centered Gaussian diagonal traffic \(z\) with parameter \(\alpha = \Phi(z^2)\) is a self-adjoint diagonal traffic whose distribution with respect to \(\Phi\) is the centered real Gaussian law with variance \(\alpha\).

A semicircular traffic \(x\) is limit of a Wigner matrix by Lemma 5.1. A centered Gaussian Boolean traffic \(y\) is limit of a matrix of the form \(\frac{X_i + X_j}{\sqrt{2}}\) as in Lemma 5.11. A real centered Gaussian diagonal traffic \(z\) is the limit of a diagonal matrix with real Gaussian entries by Lemma 6.9. The non commutative distributions of \(x\) and \(z\) with respect to \(\Phi\) are respectively the semicircular and the Gaussian distributions, the non commutative distribution of \(y\) with respect to the \(\Psi\) is the Rademacher distribution.

**Theorem 6.4.** Let \((\mathcal{A}, \tau)\) be an algebraic traffic space with associated trace \(\Phi\) and anti-trace \(\Psi\) which is a \(\mathbb{G}^*\)-algebra. Let \((a_n)_{n \geq 1}\) be a sequence of self-adjoint, independent and identically distributed traffics in \(\mathcal{A}\) and let \(a\) be distributed as the \(a_n\)'s. Assume that \(\Phi(a) = \Psi(a) = 0\). Then \(m_n = \frac{a_1 + \ldots + a_n}{\sqrt{n}}\) converges to a traffic \(m\). If \(\Phi\) is a state, then \(m = x + y + z\) is the sum of a semicircular traffic \(x\), a Gaussian Boolean traffic \(y\) and a Gaussian diagonal traffic \(z\) which in general are not traffic independent.
Nevertheless, $x$ and $z$ are traffic independent. Hence, seen as a non commutative random variable in $(\mathcal{A}, \Phi)$, $x$ and $z$ are free independent ($y$ has the null distribution w.r.t. $\Phi$). Denoting $\rho = \Phi(\Delta(a)\Delta(a)) = \Phi(a \circ a)$ and assuming $\Phi(a^2) = 1$, then the distribution of $m = x + z$ w.r.t. $\Phi$ is the free convolution of the Gaussian law with mean zero and variance $\rho$ and the semicircular law with mean zero and variance $1 - \rho$.

Remark 6.5. Since we assume that $\Phi$ is a state, $\rho \geq 0$ and so the free convolution is well defined. It is the limit in distribution w.r.t. $\Phi$ of $\sqrt{\rho} D_N + \sqrt{1 - \rho} X_N$, where $D_N$ is a diagonal matrix with i.i.d. Gaussian entries independent from a Wigner matrix $X_N$ (by Corollary 1.9 and Proposition 5.7). We give below in Proposition 6.9 a matrix model for the sum $m = x + y + z$.

- One must be careful with the statement that $x, y$ and $z$ are not independent, since there is no unicity in the decomposition of $m$ as a sum of semicircular, Gaussian Boolean and Gaussian diagonal traffics, as we will see later. The are situations where the variables given by the theorem are not independent but $m$ can be written as a sum of independent variables, see Example 6.11.

Example 6.6. Let us state an application which is due to Au in [Au], from a preliminary version of Theorem 6.4 which contains only the second part of the theorem. We present his result and reasoning which is an interesting use of the traffic CLT in order to find a new proof of a result of Theorem 6.4 which contains only the second part of the theorem. We present his result and positioning $pX$. Hence $\Phi$ is a free convolution of a semicircular distribution and a Gaussian distribution.

\[ \Phi(J) = \begin{cases} \Phi(a)I + (\Phi(a) - \Phi(a))J + b & \text{if } b = 0, \\ \Phi(b)J + \Psi(b) & \text{if } b \neq 0. \end{cases} \]

Remark 6.7. In an algebraic traffic space $(\mathcal{A}, \tau)$ with trace $\Phi$ and anti-trace $\Psi$, when it exists we denote by $J \in \mathcal{A}$ a distinguished element independent from all $A$ and such that $\Phi(J) = 0$, $\Psi(J) = 1$.

Lemma 6.8. In the setting of the previous definition, for any traffics $b_1, b_2$, one has $\Phi(b_1J) = 0$ and $\Psi(b_1Jb_2) = \Psi(b_1)\Psi(b_2)$.

We can now gives an explicit description of the limit in traffic distribution of the central limit theorem.

Proposition 6.9. Let $a, a_n = \frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{n}}$ and $m$ be as in Theorem 6.4 and denote $\bar{a} = a - \Delta(a)$. Assume that in the traffic space generated by $a$, the trace $\Phi$ is a state. We now consider three independent centered variables in some space containing $J$ as in Definition 6.7.

1. a semicircular traffic $x'$ with parameter $(\alpha_1, \beta_1) = (\Phi(\bar{a}^2), \Phi(\bar{a}^2))$

2. a Gaussian Boolean traffic $y'$ with parameter $(\alpha_2, \beta_2) = (\Psi(\bar{a}), \Psi(\bar{a}))$

3. a Gaussian diagonal traffic $z'$ with parameter $\alpha_3 = \Phi[\Delta(a)^2] - \gamma \Psi[\Delta(a)^2]$, where $\gamma = \frac{\Psi[I + \Psi(z')]}{\alpha_2 + \Psi(\beta_2)}$ (if $\alpha_2 + \Psi(\beta_2) = 0$).
We denote \( q(x') = x' \mathbb{J} + Jx' \) and \( r(y') = \frac{\deg(y') + \deg'(y')}{\sqrt{2}} \). Then \( m \) has the same distribution as 
\( m' = (x' - q(x')) + y' + (\gamma r(y') + z') \). The centered traffics \( x, y = -q(x') + y' \) and \( z = \gamma r(y') + z' \) are respectively semicircular, Gaussian Boolean and Gaussian diagonal traffics.

Remark 6.10. Since \( \Phi \) is a state, the variables are well defined. Indeed, the Cauchy-Schwarz inequality \( |\Phi(ab)|^2 \leq \Phi(aa^*)\Phi(bb^*) \) implies that \( \alpha_1 = |\beta_1| \) and \( x' \) is well defined. As well, since \( \Psi(ab) = \text{Tr}(\Phi\Phi'(ab)) = \Phi([\deg^2(a)\deg(b)] \) for any \( a, b \), one has \( \alpha_2 \geq |\beta_2| \) and \( y' \) is well defined. To see that \( \alpha_3 \) is non-negative, note that \( \Psi(\Delta(a)b) = \Phi([a\deg(b)]) \) for any \( a, b \) and so

\[
\gamma \Psi[\Delta(a) \bar{a} + \bar{a}^t] = \frac{1}{\alpha_2 + \Re(\beta_2)}\Phi[\Delta(a)r(\bar{a})]^2 \leq \frac{1}{\alpha_2 + \Re(\beta_2)}\Phi[\Delta(a)^2]\Phi[r(\bar{a})^2].
\]

But since \( \Psi(ab) = \Phi(\deg(a)\deg'(b)) \) then

\[
\Phi[r(\bar{a})^2] = \Psi[\bar{a}\bar{a}^*] + \Phi[\deg(\bar{a})^2 + \deg'(\bar{a})^2].
\]

Moreover, by Lemma 4.11 one has \( \Phi[(\deg'(\bar{a})^2)] = \Phi[(\deg(\bar{a})^2) \] and since \( \bar{a}^* = \bar{a} \) we get \( \Phi[r(\bar{a})^2] = \alpha_2 + \Re(\beta_2) \) and then \( \alpha_3 \geq 0 \).

- Let \( X_N \) be a Wigner matrix with parameter \((\alpha_1, \beta_1)\), \( Y'_N \), \( Y'_N \) be a matrix as in Lemma 5.11 with parameter \((\alpha_2, \beta_2)\), \( Z_N \) be a diagonal matrix with i.i.d. real centered Gaussian entries with variance \( \alpha_3 \), and \( J_N \) be the matrix whose all entries are \( \frac{1}{\sqrt{N}} \), the random matrices being independent. By Theorem 1.8, the matrices are asymptotically traffic independent and \( m' \) is distributed as the limit of \( M_N = X'_N - \eta q(X'_N) + Y'_N + \gamma r(Y'_N) + Z_N \), where \( q_N(X'_N) = X'_NJ_N + J_NX'_N \) and \( r(Y'_N) = \frac{\deg(Y'_N) + \deg'(Y'_N)}{\sqrt{2}} \). In particular a traffic space as in the proposition well exists.

Example 6.11. Let \( m_n \) be the normalized sum \( \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \) of independent semicircular traffics with parameter \((1, \eta)\). Since \( (x_i)_{i \geq 1} \) is the limit of a family of independent Gaussian Wigner matrices, \( m_n \) and its limit \( m \) as \( n \) goes to infinity are also semicircular traffics with same parameter. Yet, the answer given by the above proposition looks more complicated. Definition 5.3, we compute the parameters for Proposition 6.9 is as follow. Let \( x \) be distributed as the \( x_i \).

- \( \Phi(x) = 0 \) by extra-diagonality of \( x \) (that is \( \Delta(x) = 0 \)) by noting that \( \Phi(x) = \Phi(\Delta(x)) \).
- \( \Psi(x) = \tau[. \xrightarrow{\mathbb{J}}.] = \tau^0[. \xrightarrow{\mathbb{J}}.] = 0. \)
- \( \Phi(x^2) = 1 \) and \( \Phi(xx^t) = \eta \) by definition of the parameters.
- \( \Psi(x^2) = \tau[. \xrightarrow{\mathbb{J}} . . \xrightarrow{\mathbb{J}} . \xrightarrow{\eta} .] \). The only quotient of the latter graph that is a double tree (a graph which becomes a tree when the multiplicity of edges is forgotten) is \( (. \xrightarrow{x} .) \), and so \( \Psi(x^2) = \tau^0[. \xrightarrow{x}.] = 1. \)
- \( \Psi(xx^t) = \tau[. \xrightarrow{\mathbb{J}} . . \xrightarrow{\mathbb{J}} . . \xrightarrow{x} . \xrightarrow{\eta} . \]. Similarly we have \( \Psi(xx^t) = \tau^0[. \xrightarrow{x} .] = \eta. \)
- \( \Phi(\Delta(x)^2) = \Phi(\Delta(x)r(x)) = 0 \) by extra-diagonality of \( x \).

Hence \( m \) has the distribution of \( (x' - q(x')) + y' \), where \( x' \) and \( y' \) are independent, semicircular and Gaussian Boolean respectively, with same parameter as the \( x_i \).

Corollary 6.12. Let \( x', y' \) be independent traffics, semicircular and Gaussian Boolean respectively, with same parameter. Then \( x = x' - q(x') + y' \) is a semicircular traffic with same parameter.
Although \( m \) as the same distribution as \( x' \), the proposition invites us to first remove a part of \( x' \) with the term \(-q(x')\), and then to sample it again by adding \( y' \). The reason of this approach will be clear with the third example below. Note that when the \( x_i \) are nonzero, then the semicircular part \( x' \) and the Gaussian Boolean part \(-q(x') + y'\) are not independent. Indeed, by Lemmas 5.4 and 6.8 one can see that \( \Psi(x' - q(x') + y') = -\Psi(x'q(x')) = -\Phi((x')^2) > 0 \), which should vanishes if they were independent. Of course this is irrelevant since \( m \) is actually semicircular.

2. Let \( m_n \) be the limit as \( N \to \infty \) of the normalized sum \( \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} (U_N^{(i)} + U_N^{(i)*}) \) of independent Haar unitary matrices and their adjoint. By Theorem 1.8 and Proposition 3.7 \( m_n \) is the normalized sum of i.i.d. self-adjoint traffics. Then the central limit theorem implies that the limit \( m \) of \( m_n \) is a semicircular traffic with parameter \((1,0)\), thanks to the same detour as in the previous example.

Indeed, let us compute the parameters we need in the central limit theorem, using the limiting distribution of \( (U_N, U_N^*) \) given in Proposition 3.7. We denote the traffics \((u,u^*) = \lim_{N \to \infty} (U_N^{(i)}, U_N^{(i)*}) \) and \( a = u + u^* \). Note that \( a \) has the distribution of an extra-diagonal traffic since for a test graph \( T \) labeled in \((u,u^*)\), \( \tau^0[T] = 0 \) if it has a single loop. In particular \( a \) has the same distribution as \( \tilde{a} \).

- \( \Phi(a) = 0 \) by extra-diagonality.
- \( \Psi(a) = 2\Re \Psi(u) = 2\Re \tau[\tau_{u}] = 2\Re (\tau^0[\tau_{u} + \tau^0[\tau_{u}]] = 0 \text{ by Lemma 4.14} \)
- \( \Phi(a^2) = 2\Re \Phi(u^2) + 2 \). But
  \[
  \Phi(u^2) = \tau\left[\begin{array}{c} u \\ u^* \end{array}\right] = 0
  \]
  since \( \tau^0[T] \neq 0 \) only if \( T \) has the same number of edges labeled \( u \) and \( u^* \) and that the quotient of a graph has the same number of edges labeled \( u \) and \( u^* \).

- \( \Phi(uu^t) = \Phi(uu^t + uu^*t + u^*u^t + uu^*uu^t) = 0 \) since
  \[
  \Phi(uu^t) = \tau\left[\begin{array}{c} u \\ u^* \end{array}\right] = \tau\left[\begin{array}{c} u \\ u^* \end{array}\right] = \tau^0\left[\begin{array}{c} u \\ u^* \end{array}\right] + \tau^0\left[u \cdot u\right] = 0
  \]
  and \( \Phi(u^*u^t) = 0 \) with the same computation, and the two other terms vanishes as the number of edges labeled \( u \) and \( u^* \) are not equal.

- \( \Psi(a^2) = 2\Re \Psi(u^2) + 2 = 2 \) for the same reason as before.
- \( \Psi(aa^t) = \Psi(uu^t + uu^*t + u^*u^t + uu^*uu^t) = 0 \) since \( \Psi(uu^t) = \Psi(u^*u^t) = 0 \) as before and
  \[
  \Psi(uu^t) = \tau\left[\begin{array}{c} u \\ u^* \end{array}\right] = \tau\left[\begin{array}{c} u \\ u^* \end{array}\right] = \tau^0\left[\begin{array}{c} u \\ u^* \end{array}\right] + \tau^0\left[u \cdot u\right] = 0
  \]
  and the same computation yields \( \Psi(u^*u^t) = 0 \)
- \( \Phi(\Delta(a)^2) = \Phi(\Delta(a)r(a)) = 0 \) since \( a \) has the distribution of an extra-diagonal traffic. Hence by Proposition 6.9 and Corollary 6.12 \( m \) is a semicircular traffic with parameter \((1,0)\).

3. In the previous examples, we can wonder why we do not chose the Gaussian Boolean part \( y' \) in such a way \( \Psi(y') = (\Psi - \Phi)(\tilde{a}^2) \), in order to not remove a part that is sampled again. The reason is that the quantity in the r.h.s. term is possible negative.

Let \( m_n \) be the limit as \( N \to \infty \) of the standardized sum \( \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} (V_N^{(i)} + V_N^{(i)*} - 2J_N) \) of independent uniform permutation matrices \( V_N^{(i)} \) and their transpose, where \( J_N \) denotes the matrix whose all entries are \( \frac{1}{2} \). By Theorem 1.8 Proposition 3.6 and the convergence of \( J_N \), \( m_n \) is the normalized sum of i.i.d. traffics. Then the central limit theorem implies that the limit \( m \) of \( m_n \) has the distribution of \( x' - q(x') \) where \( x' \) is a semicircular traffic with parameter \((1,1)\).

Indeed, let us denote \( v = \lim_{N \to \infty} V_N^{(i)} \) and \( a = v + v^t - 2J \). As in the previous case, \( a \) has the distribution of an extra-diagonal traffic.
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- \( \Phi(a) = 0 \) by extra-diagonality.
- \( \Psi(v) = \tau\left(\begin{array}{c} v \\ v \end{array}\right) = \tau^0\left(\begin{array}{c} v \\ v \end{array}\right) + \tau^0\left(\begin{array}{c} v \otimes v \end{array}\right) = 1 + 0 \) and so \( \Psi(a) = 0. \)
- By Lemma 6.8, \( \Psi(v^c) = \Psi(Jv^c) = 0 \) for \( e \in \{1, t\} \), so one has \( \Phi(a^2) = \Phi((v + v')^2) = 2 + 2\Phi(v^2) = 2 \) since

\[
\Phi(v^2) = \tau\left(\begin{array}{c} v \\ v \end{array}\right) = \tau^0\left(\begin{array}{c} v \\ v \end{array}\right) + \tau^0\left(\begin{array}{c} v \otimes v \end{array}\right) = 0
\]

- Since \( a' = a \) we have \( \Phi(aa') = \Phi(a^2) = 2. \)
- We can anticipate that \( \Psi(a^2) = 0. \) Indeed, the matrix \( V_N^{(i)} + V_N^{(i)t} \) is the adjacency matrix of a random graph (the graph of cycles of the associated permutation). The degree of each vertex of the graph (the number of neighbors) is two, and \( \Psi((V_N^{(i)} + V_N^{(i)t} - 2J_N)) \) is nothing else than the variance of the degree of a vertices uniformly chosen at random. More generally, we have the following fact.

**Lemma 6.13.** Let \( (\mathcal{A}, \tau) \) be an algebraic traffic space with anti-trace \( \Psi \), such that \( J \in \mathcal{A} \) as in Definition 6.7 exists. Let \( a \in \mathcal{A} \) having the same distribution as \( b - (b\| J) \) for some \( b \in \mathcal{A} \) such that \( \Psi(b) = 0. \) Then one has \( \Psi(b^2) = 0. \)

**Proof.** Recall that by Lemma 6.8 one has \( \Psi(b_1b_2) = \Psi(b_1)\Psi(b_2) \) for any \( b_1, b_2 \in \mathcal{A}. \) Hence we get

\[
\Psi(a^2) = \Psi(b^2) - \Psi((b\| J) + (b\| J)b) + \Psi((b\| J)(b\| J)b) = 0
\]

For the matrix \( A_N = V_N^{(i)} + V_N^{(i)t} - 2J_N \), one has \( \Psi(A_N) = 0 \) and \( (A_NJ_N + J_NA_N) = 0. \) Hence the limit \( a \) of \( A_N \) satisfies the assumption of the lemma.

- \( \Psi(aa') = \Psi(a^2) = 0. \)
- \( \Phi(\Delta(a)^2) = \Phi(\Delta(a)\tau(a)) = 0 \) since \( a \) has the distribution of an extra-diagonal traffic.

Hence the expected result: the limit \( m \) in the central limit theorem has the distribution of \( x' - q(x') \) for a semicircular traffic with parameter \( (1, 1) \). The traffics \( x' \) and \( -q(x') \) are not independent since \( \Psi(x'q(x')) = \Psi(x'^2) = 1 \), and there is no known way to decompose \( x' - q(x') \) as a sum of independent semicircular and Gaussian Boolean variables.

We first prove in Theorem 6.4 the convergence in traffic distribution of \( (m_n)_{n \geq 1} \), giving an expression for the distribution of its limit \( m \) in formula (6.3) below. Then we compute the distribution w.r.t. \( \Phi \) to prove that it is a convolution of a Gaussian and of a semicircular distribution. The rest of the proof will be dedicated to Proposition 6.9.

**Proof of the convergence in Theorem 6.4.** Let \( T = (V, E) \) be a test graph in one variable. With the same computation as for the law of large numbers, the multilinearity of \( \tau^0 \) with respect to the edges gives

\[
\tau^0\left(\frac{A_1 + \cdots + A_n}{\sqrt{n}}\right) = \sum_{\pi \in \Pi(E)} n_{|\pi|}^{-\frac{|\pi|}{2}} (1 + o(1)) \eta(\pi).
\]

Here \( \eta(\pi) \) equals \( \tau^0[T_{e_1}(a_1, \ldots, a_{|\pi|})] \), where \( \gamma : E \rightarrow \{1, \ldots, |\pi|\} \) is such that \( e \sim f \) if and only if \( \gamma(e) = \gamma(f) \), and \( T_{e} \) is obtained from \( T \) by putting for each edge \( e \) the label corresponding to \( a_{\gamma(e)}. \) Assume that \( \pi \) has a block of size one. Then, either \( GGC(T, \gamma) \) is not a tree and \( \eta(\pi) = 0 \), or by the factorization property of traffic independence, one can factorize in \( \eta(\pi) \) the term \( \tau^0[T(a)] \) where \( T \) has a single edge. If \( T = \gamma \) is a simple loop, then \( \tau^0[\gamma(a)] = \tau[\gamma(a)] = \Phi(a) = 0. \) If \( T \) is a simple edge, since \( \tau^0[T(a)] + \tau^0[\gamma(a)] = \tau[T(a)] = \Phi(\deg(a)) = 0 \) we get as well \( \tau^0[T(a)] = 0. \) Hence the only partitions that contribute are those for which \( \pi \) is a pair partition. This proves the convergence of \( m_N \) in traffic distribution.
Moreover, by definition of traffic independence, the partitions \( \pi \) that contribute are those for which the graph of colored components of \( T_\pi \) is a tree. Since \( \Phi(a) = \Phi(\deg(a)) = 0 \) and by the factorization property, the partitions must pair adjacent edges (that share at least one vertex).

Hence, the only test graphs \( T \) for which \( \tau^0[T(m_n)] \) does not vanish at infinity are graphs that become trees if we forget the multiplicity of the edges and delete the loops, such that the multiplicity of simple edges (that are not self loops) is one or two. Denote by \( T_0 \) the set of such graphs. Denote by \( P_0(T) \) the set of pair partitions \( \pi \) of adjacent edges of \( T \) such that \( \{e_1, e_2\} \in \pi \) for any twin edges \( e_1, e_2 \) of \( T \). For \( \pi \in P_0(T) \), let us identify \( S \in \pi \) with the subgraph of \( T \) consisting in the edges of \( S \). We then get

\[
\tau^0[T(m_n)] \xrightarrow{n \to \infty} \mathbb{1}(T \in T_0) \sum_{\pi \in P_0(T)} \prod_{S \in \pi} \tau^0[S].
\]  

(6.3)

**Description of the limiting distribution w.r.t. \( \Phi \) in Theorem 6.4** Let now compute the distribution w.r.t. \( \Phi \) of \( m \). Let \( T \) consisting in a cycle of length \( k \). Denoting by \( V \) its vertex set, we have \( \Phi(m^k) = \tau[T(m)] = \sum_{\sigma \in \mathcal{P}(V)} \tau^0[T^\sigma(m)] \). For any partition \( \sigma \) of \( V \), the graph \( T^\sigma \) has no cutting edge (edge whose removal disconnect the graph). Hence, a graph \( T^\sigma \) that is in \( T_0 \) for some \( \sigma \) in \( \mathcal{P}(V) \) consists in a double tree \( T_0 \) for which at each vertex \( v \in V \) is attached an ensemble of self-loops \( F_v \). Computing \( \tau^0[T^\sigma(m)] \) with the above formula, the pair partitions \( \pi \in P_0(T^\sigma) \) in the sum must gather twin edges of \( T_0 \) and pair of loops attached at a same vertex. Hence \( |F_v| \) must be even for this term to not vanish. In this case, denote by \( 2\ell \), the number of loops attached to a vertex \( v \in V \). By Lemma 2.13 that gives the relation between the number of vertices and edges in a tree, the number of edges of \( T_0 \) is \( 2|V| - 1 \).

We get, computing for the graph \( S \) consisting in a double edge \( \tau^0[S(a)] = \Phi(a^2) - \Phi(\Delta(a)^2) = 1 - \rho \),

\[
\tau^0[T^\sigma(m)] = (1 - \rho)^{|V| - 1} \prod_{v \in V} \rho^\ell \text{Card } \mathcal{P}_2(2m_v),
\]

where \( \mathcal{P}_2(2m) \) denotes the set of pair partitions of \( 2m \) elements. But

\[
\text{Card } \mathcal{P}_2(2m) = (2m - 1) \times (2m - 3) \ldots 5 \times 3 \times 1 = E[X^{2m}]
\]

where \( X \) is a random variable distributed according to the gaussian measure centered of unit variance (by a basic enumeration and by integration by part respectively).

Now, let \( x \) be a centered semicircular traffic with parameter \((1, 0)\), traffic independent from \( z \) a centered Gaussian diagonal traffic with parameter \(1\). Let prove that \( m \) has the same distribution w.r.t. \( \Phi \) as \( m = \sqrt{\rho z} + \sqrt{1 - \rho x} \). Let \( T \) consisting in a cycle of length \( k \) and for any partition \( \sigma \) of its vertex set,

\[
\tau^0[T^\sigma(\bar{m})] = \sum_{\gamma \in \bar{E} \to \{1, 2\}} \tau^0[T^\gamma(\sqrt{\rho z}, \sqrt{1 - \rho x})],
\]

where in \( T^\gamma \) an edge \( e \) has label \( \gamma(e) \). By the definition of traffic independence, the support of the injective distribution of \( (x, z) \) consists in double trees \( T_0 \) with a bunch of self loops \( F_v \) attached at each vertex \( v \). If \( T^\sigma \) is such a test graph, the only map \( \gamma \) which makes \( \tau^0[T(\sqrt{\rho z}, \sqrt{1 - \rho x})] \) possibly non zero consists in labeling the edge of \( T_0 \) with labels 1 (for \( x \) and the self-loops by label 2 (for \( z \). By multi-linearity for \( \tau \) we get

\[
\tau^0[T(\sqrt{\rho z} + \sqrt{1 - \rho x})] = (1 - \rho)^{|V| - 1} \prod_{v \in V} \rho^{\ell_v} E[X^{2m_v}]
\]

as expected. By the first case in Theorem 5.5, since \( x \) is unitarily invariant and \( x \) and \( z \) are traffic independent, they are free independent.

\[ \square \]
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Proof of Proposition 6.9. In order to prove that $m$ and $m'$ are equal in distribution, we can compute directly the distribution of $m' = (x' - q(x')) + y + (\gamma r(y') + z')$ using the distributions of $x', y', z'$ and their traffic independence. But it is much simpler to use An's argument [An]. The variable $m'$ is the limit of the matrix model $M_N = X_N' - q(X_N') + Y_N + \gamma r(Y_N') + Z_N'$ as in the above remark where the matrices $X_N', Y_N', Z_N'$ have Gaussian entries. Since $M_N$ is linear in these matrices, it is also Gaussian and so it is stable: it can be written $M_N = M_{N,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_{(i)}$ where the $M_{(i)}$ are independent copies of $M_N$. By Theorem 1.8 of asymptotic traffic independence, the limit $m'$ of $M_N$ is distributed as a limit of the central limit theorem (Theorem 6.1).

To prove that the variables $m = \lim_{n \to \infty} \frac{m'+m''}{\sqrt{n}}$ and $m' = (x' - q(x')) + y + (\gamma r(y') + z')$ of Proposition 6.9 have the same distribution, by Formula (6.3) it remains to show that for any test graph $T$ with two edges, $\tau^0[T(m')] = \tau^0[T(m)]$. Note that from (6.3) we have $\tau^0[T(m)] = \tau^0[T(a)]$ for test graphs with two edges, where we recall that $a$ is distributed as $a_1, \ldots, a_n$. Moreover, it is sufficient to prove the equality $\tau^0[T(m')] = \tau^0[T(a)]$ for these graphs (the number of edges of a test graph is unchanged when identifying vertices). There is a total of height connected directed graphs with two edges, but since the variables are self-adjoint two pairs of quantities are related each other: by Lemma 4.14 we have

$$\tau[\cdot \leftrightarrow \cdot] = \tau[(\cdot \leftrightarrow \cdot)^*]$$

and

$$\tau[\cdot \leftrightarrow \cdot \circ] = \tau[(\cdot \leftrightarrow \cdot \circ)^*]$$

Equivalently, it is then sufficient to prove that the six quantities

$$\Phi(\tilde{m'}(\tilde{m}')^\varepsilon), \Psi(\tilde{m'}(\tilde{m}')^\varepsilon), \Psi(\Delta(m') \frac{\tilde{m'} + (\tilde{m}')^t}{\sqrt{2}}), \Phi(\Delta(m')^2)$$

where $\tilde{m'} = m' - \Delta(m')$ and $\varepsilon \in [1, t]$, are equal to the same quantities where $m'$ is replaced by $a$.

Note that by Lemma 6.8 for any traffics $b_1, b_2$ one has $\Phi(b_1 q(b_2)) = 0$ and $\Psi(b_1 q(b_2)) = \Psi(b_1) \Phi(b_2) + \Psi(b_1) \Psi(b_2)$. Recall form Lemma 5.4 that if $b_1$ and $b_2$ are independent then $\Phi(b_1 b_2) = \Phi(b_1) \Phi(b_2)$ and $\Phi(b_1) = \Phi(b_1) \Psi(b_2) + \Psi(b_1) \Psi(b_2)$. Hence, using the fact that $\Phi(y(y')^\varepsilon) = 0$ since it is of Boolean type and that the variables are centered, we get for $\varepsilon \in [1, t]$

$$\Phi(\tilde{m'}(\tilde{m}')^\varepsilon) = \Phi((x' - q(x'))(x' - q(x'))^\varepsilon) + \Phi(y(y')^\varepsilon) = \Phi(x'(\tilde{m}')^\varepsilon),$$

$$\Psi(\tilde{m'}(\tilde{m}')^\varepsilon) = \Psi((x' - q(x'))(x' - q(x'))^\varepsilon) + \Psi(y(y')^\varepsilon) = \Psi(y(y')^\varepsilon)$$

Moreover, we have

$$\Psi(\Delta(m') \frac{\tilde{m'} + (\tilde{m}')^t}{\sqrt{2}}) = \Psi(\gamma r(y') \frac{y' + (y')^t}{\sqrt{2}}) = \Psi(r(a) \frac{\tilde{a} + \tilde{a}^t}{\sqrt{2}}) \cdot \eta_{\alpha_2 + \beta_2}.$$

where, using the definition of $\text{deg}$ in terms of graph operations and Lemma 4.14, we have

$$\eta = \frac{1}{2} \Psi(\text{deg}(y') y' + \text{deg}(y') (y')^t \Delta m \frac{(y')^t + (y')}{\sqrt{2}}) = \frac{1}{2} \Psi((y')^t y' + (y') y' + y' (y')^t) = \alpha_2 + \beta_2.$$

At last, we have

$$\Phi(\Delta(m')^2) = \gamma^2 \Phi(r(y')^2) + \Phi((z')^2)$$

$$= \gamma \frac{\Psi(\Delta(a) \text{deg}(a)^t)}{(\Psi(\tilde{a}^2) + \Psi(a^2) + \Psi(\tilde{a}))} \cdot \eta_{\alpha_2 + \beta_2}$$

$$+ \Phi(\Delta(a)^2) = \Phi(\Delta(a)^2).$$
This finishes the proof that $m$ is distributed as $m'$.

It remains to prove the last statement of the proposition, telling that $-q(x') + y'$ is a Gaussian Boolean traffic and that $\gamma r(y') + z'$ is a Gaussian diagonal traffic. By the model of large random matrices, the sum of two independent Gaussian Boolean (respectively Gaussian diagonal) traffics are Gaussian Boolean (respectively Gaussian diagonal). So the following lemma concludes the proof.

Lemma 6.14. 1. If $x$ is a centered semicircular traffic with parameter $(\alpha, \beta)$ then $q(x)$ is Gaussian Boolean with parameter $(\alpha, \beta)$.

2. If $y$ is a centered Gaussian Boolean with parameter $(\alpha, \beta)$, then

$$r(y) = \frac{\text{deg}(y) + \text{deg}^t(y)}{\sqrt{2}}$$

is centered Gaussian diagonal random variable with parameter $(\alpha + \Re(\beta))$.

Proof. 1. Let $X_N = \left(\frac{x_{i,j}}{\sqrt{N}}\right)_{i,j}$ be a Wigner matrix with centered Gaussian entries such that $\mathbb{E}[|x_{i,j}|^2] = \alpha$ and $\mathbb{E}[x_{i,j}^2] = \beta$. Then $q_N(X_N) := X_N \mathbb{I}_N + \mathbb{I}_N X_N = \left(\frac{Y_i + Y_j}{\sqrt{N}}\right)_{i,j}$ where $Y_i = \sum_j x_{i,j}$ is a Gaussian variable such that $\mathbb{E}[|Y_i|^2] = \alpha$ and $\mathbb{E}[Y_i^2] = \beta$. Then $X_N$ converges to the semicircular variable $x$ and $q_N(X_N)$ converges to a centered Gaussian Boolean traffic $y$. Hence $q(x) = y$ in distribution.

2. Firstly, $r(y)$ is diagonal since $\text{deg}(y)$ and $\text{deg}^t(y)$ are diagonal. Moreover $\text{deg}^*(y) = \text{deg}^t(y)$ since $y$ is self-adjoint and so $r(y)$ is self-adjoint. It remains to prove that $\Phi(r(y)^K) = \mathbb{E}[Z^K]$ for a real centered Gaussian variable $Z$ such that $\mathbb{E}[Z^2] = (\alpha + \Re(\beta))$, for any $K \geq 1$. But, with $Y$ denoting a centered Gaussian random variable such that $\mathbb{E}[|Y|^2] = \alpha$, $\mathbb{E}[Y^2] = \beta$, one has

$$\Phi(r(y)^K) = \Phi\left(\frac{\text{deg}(y) + \text{deg}^t(y)}{\sqrt{2}}\right) = 2^{-\frac{K}{2}} \sum_{k=0}^{K} \binom{K}{k} \Phi\left(\text{deg}(y)^k \text{deg}^t(y)^{K-k}\right)$$

$$= \Phi\left(\frac{\text{deg}(y) + \text{deg}^t(y)}{\sqrt{2}}\right) = 2^{-\frac{K}{2}} \sum_{k=0}^{K} \binom{K}{k} \mathbb{E}[Y^k Y^{K-k}]$$

$$= \mathbb{E}\left[\left(\frac{Y + Y}{\sqrt{2}}\right)^K\right] = \mathbb{E}[Z^K].$$

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