Real-world ballistics: A dropped bucket

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Abstract: I discuss an apparently simple ballistics problem: the time it takes an object to fall a small vertical distance near the surface of the Earth. It turns out to be not so simple; I spend a great deal of time on the quantitative assessment of the assumptions involved, especially with regards to the influence of the air. The point is not to solve the problem; indeed I don’t even end up solving the problem exactly. I introduce dimensional analysis to perform all of the calculations approximately. The principal theme of the lecture is that real physics can be very different from “textbook” physics, since in the real world you aren’t ever told what equations are appropriate, or why.

I was walking to work one morning and above me on the third story of a scaffolding there was a man washing windows. As I was walking under him, he accidentally knocked his bucket and the bucket started to fall from the scaffolding towards me. How much time did I have to jump out of the way?

1. Real physics, not textbook physics

I expect that almost anyone reading this has had some physics instruction of some kind in secondary school or from equivalent books or classes. I want to emphasize that the “physics” one learns in these contexts might not be all that useful or relevant to understanding the real physics we are going to discuss here and in the lectures that follow. This is true even for—maybe especially for—students especially good at that kind of “textbook” physics. Real physics is very different from textbook physics.

Doing real physics involves intuition, approximation, and quantitative reasoning to understand what matters in a physical situation. We use formal techniques (such as calculus and proofs) only once we have understood the fundamental properties of the physics problem at hand. There are scalings we can find, which tell us how the problem solution depends, functionally, on each input parameter. We can analyze limiting behavior of the system.

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2Story may be apocryphal.
when it is simplified in various ways, many of which are physically illuminating. We can determine, quantitatively, the dominant processes, the physical effects or forces that make the biggest contributions to the system’s behavior. We are rarely concerned with the particulars of an exact solution to a well-defined mathematically solvable problem, although sometimes one will emerge beautifully after our—much more important—rough work on the problems.

In contrast, in textbook physics, a lot of what you do involves equations: how to use and derive them. The equations are related to physics, of course, but the focus is on the equations themselves. You probably remember many of them: “\(v_0 t + \frac{1}{2} a t^2\),” “\(\frac{1}{2} m v^2\),” “\(F = ma\).” You also solve many problems in a textbook class, but those problems are tuned carefully to make use of the very equations you had been given. They are not physics questions, really, they are cleverly disguised mathematical “word problems.” Textbook physics classes teach techniques for solving those problems—problems well-matched to the store of equations. In these lectures we not going to concentrate on those textbook skills, though we will use some of them.

Don’t get me wrong: We are going to solve problems. We are going to use equations and some beautiful mathematics. But we are not going to make our goal the mathematical solutions of those particular problems. The problems we do—and we will do several per lecture—will be done with the over-riding goal of making ourselves better physicists; better at real physics.

Let us return to the problem: I am under the window-washer’s platform and the bucket has begun to fall. What should I do? There is a clear textbook “strategy” for the solution of this problem: Find the relevant equation (from the book), plug in numbers, and get an answer. There are two problems with this strategy: (1) In the real world, no-one tells you what equations to use, and (2) scaffolding and buckets aren’t labeled with their heights and masses and contents. In other words, the problem is ill-posed. I haven’t told you what to assume, physically, and I haven’t told you what numbers to plug in.

In the real world, all questions of importance are ill-posed. An ill-posed problem is “explain the motions of the planets.” A well-posed problem is “compute the relative semi-major axes of Mars and the Earth, using the tabulated sidereal periods of the planets and Kepler’s laws.” An ill-posed problem is “can we generate our electricity with wind power?” A well-posed

\(^3\)If we say “dominant” we usually mean not just the biggest, but so much bigger that we can ignore the others.
problem is “what is the torque on a propeller with blades of a certain length, width, pitch angle, and cross-sectional shape, given a steady wind of a certain speed?” Well-posed questions can only be asked once significant progress has been made on the ill-posed questions.

This bucket problem is ill-posed, but is it important? It has the great virtue that—despite being ill-posed—it has a well-posed form and an exact solution under certain conditions (to be explored below), so we can use it to test our methods. It has this exact solution, and it shares with important ill-posed problems the requirement that we need to figure out the relevant physics on our own; this makes it a good problem with which to start. In general—in these lectures and in the world—we have to figure out what matters; we aren’t going to be told. If we are lucky, the dominant physical effect will be something we can represent, approximately, with an equation. If we can, we will derive that equation. Then we are going to use things we know about the world (scaffolding, buildings, buckets, and the Earth) to figure out what numerical quantities to insert into the variables of the equation we derived.

I am spending a lot of time on this polemic, but for a good reason: Most of what you have learned about physics up to this point is highly misleading. I have just mentioned that “if we are lucky” we will be able to represent the dominant physical effect with an equation. Most questions you can ask about the physical world do not have an answer of this kind; most things you observe have no good description in terms of an equation you can write down. For example, if a window-washer drops not a bucket but the front page of a broadsheet newspaper, there is no equation we can write down that even gives a good approximation of the probability that the paper hits me (depending, as it does, on the distribution of eddies and breezes in the air between us), let alone how long takes. If the window-washer drops the bucket not from the third story but from the tenth, there are approximate equations we can use, but if we want an answer accurate at the few-percent level, we require a computer.

Most textbook presentations of physics you have seen previously have “led you down the garden path” in which have been planted simple, solvable problems, described by simple, solvable equations. In the universe of all physics problems, the problems in this garden comprise the exception, not

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4A similar point is well made in Mazur, E., “The problem with problems,” Optics & Photonics News, 6 59–60 (June 1996).
the rule. Worse, some presentations of physics have inappropriately applied simple, solveable equations to situations where they are physically inapplicable, even approximately. What we are going to do differently here is that we are going to spend our time not solving or using the equations, but we are going to spend our time discovering and interrogating them. To use an equation is one thing, to show that it is the appropriate equation is quite another.

2. The ingredients of a solution

A real solution to a physics problem involves interrogating your physical intuition to make some kind of guess or prediction, figuring out all of the possibly relevant physical effects, figuring out which are most important, making the necessary approximations to get an equation or other mathematical representation, doing the math, comparing your answer to your intuitive guess, and then finally checking quantitatively your answer and your approximations using non-trivial diagnostics. The making of approximations is what turns the ill-posed problem into a well-posed problem. Exact formulae are only used at the point that there is a well-posed problem, and they comprise only the tiniest part of this kind of complete solution—a real solution—of a physics problem.

So once again: The bucket is starting to fall. Imagine it. How long do you think it will take to fall? Will it happen in the blink of an eye? Will I watch it fall, and have time to think about it, perhaps time to jump out of the way? Or will it take many seconds, such that I could have a substantial conversation with a friend as it falls? In general, when a physics problem describes a situation that you have actually seen, it is a good idea to use that experience. So take a moment to visualize this, and predict an answer, just from your memory of similar events.

An important theme of these lectures is that fundamental principles—fundamental properties of the Universe—appear in all sorts of situations around you.

What can matter, physically, in the problem? One quantity that clearly matters is the vertical distance $h$ that the bucket is going to fall. I think we would all agree that the larger the distance, the longer the fall, or that time increases with distance. What is the height of a third-story window? Well, stories of buildings are about 3 m, so the third story has a height $h \approx 10$ m. You might imagine that it matters whether the bucket falls from the floor
level or window level; in principle that matters, but at the level of precision
of this calculation it won’t. Don’t take it from me, however: We will check
this at the end. This is another theme that will recur in this lectures and
the lectures to follow: There is no need to determine any numbers to better
precision than the calculation supports. With a question as ill-posed as this
one, we certainly don’t need to know the height $h$ with great accuracy.

What else can matter, physically, in the problem? Clearly the size, mass,
contents, and composition of the bucket all matter.

Or do they? A little knowledge can be a very dangerous thing. From
a textbook-physics setting, you may recall the observation, attributed\footnote{I say “attributed” here because I am not a historian; I don’t want to assert anything
historical. However, from here on, I \textit{will} attribute this to Galileo.} to
Galileo, that all things fall at the same rate. That is, the size, mass, contents,
and composition of the bucket \textit{can’t matter}.

Or can they? Galileo’s observation is true only in a very certain \textit{regime},
or set of physical conditions. It is true in the regime in which \textit{the air does not
matter}. Here are two examples of situations in which the air \textit{does matter}:
In one, the window washer drops not a bucket but a page from a broadsheet
newspaper. That sheet will flip and flop and fold and drift down, because
its interaction with the air is just as important as the gravitational force.
In another, the bucket is not a tin bucket filled with water, but a rubber
balloon filled with helium gas. In this case, when the window washer “drops”
(releases?) it, the bucket falls not downwards, but \textit{upwards}; for a helium
balloon, the buoyant force exceeds the direct gravitational force.

How can we tell whether we are in the air-doesn’t-matter regime, and how
does that depend on the mass, size, contents, and composition of the bucket?
Those who cannot answer that question, quantitatively, have \textit{never really
understood} Galileo’s famous observation. Use of Galileo’s result without
the ability to quantitatively test its applicability constitutes only textbook
physics. The purpose of these lectures will be to develop the real, \textit{physical}
understanding.

Our problem is ill-posed, so we \textit{don’t know} the size, mass, contents, or
composition of the bucket for sure. But, in fact, we all \textit{do} have very good
ideas of these, from our own experiences of buckets. A typical bucket holds
about a cubic foot (which, since a foot is about $1/3$ m, is about 0.03 m$^3$), and
if it is filled with water, its mass is dominated by the water. Water has a
density of 1000 kg m$^{-3}$ (yes, that is 1 g cm$^{-3}$ because there are 100 cm in 1 m
and you get to cube that to make 1 m\(^3\), so an absolutely full bucket of water
has a mass of nearly 30 kg (that’s really, really heavy). An empty bucket is
much less massive, maybe even less than 1 kg, but in detail it depends on
whether it is a metal bucket or a plastic one, and how thick are its walls.

What else can matter, physically, in the problem? Clearly the properties
of the air must matter, because we have to assess whether we are in the
air-doesn’t-matter regime! Let’s start with the density. Unfortunately, it is
hard to estimate the density of the air without some memory.\(^4\) You may
recall from a chemistry class that 1 mole of gas at standard temperature and
pressure (both close to the air temperatures and pressures you are used to)
fills a volume of about 22 ℓ = 0.022 m\(^3\) and that the air is mainly N\(_2\) gas with
a molecular weight of about 28 g mol\(^{-1}\). That makes air about 1.3 kg m\(^{-3}\) or
about 1000 times less dense than water.

In general, other properties of the air can matter to this, including the
viscosity. We are going to cheat a bit by just assuming that it is only the
density of the air that matters here and not the viscous properties; I will
leave the demonstration of that as a problem for the ambitious reader. It
also matters in detail whether there is a strong wind. For now we will assume
not.

What else can matter, physically, in the problem? We know that the
bucket will fall. What makes it fall? Gravity, of course! So we need to know
about the gravitational force. As I have noted, we all know—not really from
our general experiences (which are complex) but rather from the textbook-
physics classes we have taken (which are over-simplified)—that “all things
fall at the same rate.” What is meant by this? First of all, as I have said,
there ought to be a qualifier “in the absence of air resistance” and another
“in the absence of buoyancy”; that is, the statement is only true when the air
doesn’t matter, and gravity is the only important force. But what is meant
be the word “rate”? Really the gravitational force sets not the “rates” of
falling objects (whatever they would be) but the accelerations of them.

There also ought to be another qualifier: “when the two things are near
one another.” Acceleration is a vector; it has a magnitude and a direction.
Two objects close in space obtain the same acceleration due to gravity, in
both magnitude in direction. But two bodies far from one another will not

\(^4\)Actually, you can figure it out if you know that atmospheric pressure is about 15 lb in\(^{-2}\)
or 10\(^5\) N m\(^{-2}\) and that the scale-height of the atmosphere is about 10 km, but most students
are more likely to remember secondary-school chemistry.
necessarily have the same gravitational acceleration. For example, bricks dropped in New York and Paris fall with very similar acceleration *magnitudes* but in rather different acceleration *directions*, since New York and Paris are in different directions relative to the center of the Earth. For another example, bricks dropped from sea level and from a (theoretically possible) platform thousands of km above sea level—or from a location inside a (theoretically possible) hole dug thousands of km into the ground—will fall with different acceleration magnitudes. These bricks were dropped from locations that are *not* close in space.

The statement of Galilean gravity—that all things fall at the same rate—ought to be—more precisely—“under conditions in which gravity is the dominant force, all bodies that are close in space move with the same acceleration.”

Whenever a word like “close” or “far” or “big” or “small” appears in a physics problem (or physics solution), it is necessary to ask “with respect to what?”. When we say “close in space” in our statement of the Galilean observation (which we will later come to call the “equivalence principle” as Einstein and others promoted it to a fundamental symmetry of the Universe), we mean “close in space” relative to what? There is only one scale in the problem. We mean that the bodies are much closer to one another than they are to the source of the gravitational acceleration which, in this case, is the Earth. If you think a bit longer about the New York and Paris example, you will see that the scale is the distance to the *center* of the Earth and the condition is “much closer together than their distance to the center of the gravitating body.”

Now, when the bucket falls from its third-story platform, it traverses a distance $h$. That distance (a matter of meters) is much, much smaller than the distance to the center of the Earth (thousands of kilometers). Therefore, throughout its trajectory, the acceleration of the bucket due to gravity is constant and the same. It has magnitude $|g| \equiv g \approx 10 \text{ m s}^{-2}$ and direction *straight down*.

Do we have all of the possibly relevant physical quantities for this prob-

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7It was not trivial for Newton to demonstrate, mathematically, that it is only this distance that matters, when the bodies are spherical, and not some more complicated distance computed from the shape of the body. But even if he had found a more complicated relationship it would nonetheless be the case that the distance to the center of the Earth sets the approximate *scale* of the problem, and the condition would be true at the order-of-magnitude level.
lem? I am not sure we do, but let’s proceed. One unfortunate thing about the study of physics is that you never really know if you have all of the relevant physical effects under control, and the history of the field is riddled with interesting examples of wrong conclusions when the most relevant effect was ignored or unknown (think “perihelion precession of Mercury,” which led to the discovery of general relativity, or “the energy source for the Sun,” which was at one time thought to be gravitational, and caused Darwin to doubt his theory of evolution[8]). I will give you a rule of thumb for everyday mechanics however: If you can account for gravity, and if you can account for everything that mechanically touches the body of interest, then you have probably accounted for all of the forces. We did gravity, and the only thing that mechanically touches the bucket as it falls is the air, the density of which we have estimated, so we satisfy this rule of thumb in this case. Of course all bets are off if you add electric and magnetic fields, but these rarely matter in macroscopic mechanics problems like this one.

Now, how do we solve the problem?

3. Can we ignore air resistance?

The first question is: Does the air matter, or can we ignore it? It turns out—as you may know—it is straightforward to solve this problem if the air doesn’t matter. Purely gravitational trajectories are the stuff of textbooks. We will perform this calculation shortly. But what to do if the air does matter? Since the bucket is more dense than the air (even if empty), you might think that it is plausible to ignore air resistance and buoyancy. But of course a sheet of newspaper is much more dense than the air, and yet it’s trajectory can not reasonably be approximated by ignoring air resistance.

Trajectories of bodies in mechanics are set by considering accelerations, and accelerations are set by forces (via the famous equation $\mathbf{F} = m \mathbf{a}$). (If that sentence is mysterious to you, don’t worry, we will come back to forces and accelerations again and again in future lectures.) So it makes sense for us to compare any air resistance or buoyancy forces to the force of gravity. By the famous law “$\mathbf{F} = m \mathbf{a}$,” the force on an object accelerating with the acceleration $\mathbf{g}$ due to gravity—what you might call the “gravitational force”

[8] Although I am not a historian, it is clear from text in The Origin of the Species that the contemporaneous physical understanding of the Sun made Darwin doubt his theory, because he could see that evolution must take place over very long times, much longer than the few million years the Sun would last if it were powered only by its own gravitational energy.
is just \( F_g = mg \). We want to compare this force to the forces of air resistance and buoyancy. What are the magnitudes of these forces?

Buoyancy is the simplest. The buoyant force on the bucket has the magnitude of the gravitational force on the air displaced by the object, but the opposite direction. One argument for this is that parcels of still air neither rise nor fall because they have exactly balanced gravitational and buoyant forces. The buoyant force on an object is a small correction to the gravitational force whenever the object is much more dense than the air. No problem there, for either the bucket or the water it may contain.

Air resistance is more difficult, though not difficult. We all have the (correct) intuition that the air resistance force increases with the object’s speed and size. We can attempt to estimate the magnitude of the air resistance force by dimensional analysis; that is, we can find quantities that have the same dimensions or units of force, depend in an increasing way on the object’s speed and size, and which relate somehow to the properties of the air (of which we have only estimated one, its density). We can then hope that the quantities so found are close to what we are trying to determine.

There is a miracle of physics that when you find a dimensionally correct answer, and when you have properly identified the most important physical processes, you are almost always very close to the exact answer, as I will not argue, but as I will show by example in what follows. The “dimensions” of a physical quantity is hard to define, but it is the part of it described by the units; for example in this problem the height \( h \) is measured in m or \( \text{ft} \) and therefore has dimensions of length. The bucket has a property measured in kg; it has dimensions of mass. The acceleration due to gravity is measured in m s\(^{-1}\); it has dimensions of length over time squared. In mechanics, all quantities have dimensions which can be reduced to (possibly complex) combinations of mass, Every correct expression in physics has correct dimensions; that is, the dimensions on the left side of the equality must be the same as those on the right side; this is an incredibly restrictive property of all physical expressions!

\[^9\text{It sometimes comes as a surprise that all objects, including buckets and people and automobiles, are subject to a buoyant force at all times. Because these bodies are not in a vacuum but rather surrounded by air, there are energies and forces involved in the displacement of that air. Your kitchen scale measures not the mass of your tomato, but the difference between the mass of the tomato and the mass of the air it displaces. Actually, it might be more complex than that, depending on how the scale has been calibrated, but that is a subject for a later lecture.}\]
The technique of dimensional analysis—finding the only dimensionally correct expression and then assuming or hoping that the exact expression is close—is an incredibly powerful technique in physics. We will use it in almost every lecture in this series. It allows us to rapidly explore hypotheses without doing a great deal of math or analysis, and it provides very robust results. The results aren’t precise but they are robust because although in general the exact answer is unknown, every correct physical expression is required to have the correct dimensions; this “symmetry” is not just restrictive but fundamental.\footnote{I am not sure it is correct to say that dimensional analysis relies on a symmetry, but the requirement of correct dimensions is very like a symmetry, in the way that symmetries are used in physics. For example, it is similar to the restrictive requirement that if the left-hand side of an equation is a vector expression, then the right-hand side must also be a vector expression.}

Forces are described by the law $F = ma$ and therefore must have dimensions of mass times acceleration. An acceleration is a length over a time squared. So the dimensions of force are

\begin{equation}
[\text{force}] = [\text{mass}] [\text{acceleration}] = [\text{mass}] [\text{length}] [\text{time}]^{-2} .
\end{equation}

We expect the air resistance force to relate somehow to the speed of the object, which has dimensions

\begin{equation}
[\text{speed}] = [\text{length}] [\text{time}]^{-1} ,
\end{equation}

and to the density of air, which has dimensions

\begin{equation}
[\text{density}] = [\text{mass}] [\text{volume}]^{-1} = [\text{mass}] [\text{length}]^{-3} .
\end{equation}

To get [mass] on top, and [time]$^2$ on the bottom, the expression must involve density times speed squared. This leaves the dimensions wrong by a factor of [length]$^2$, or area. So there is a dimensionally correct expression for the air resistance force of the form

\begin{equation}
[\text{force}] = [\text{density}] [\text{area}] [\text{speed}]^2 .
\end{equation}

It turns out that if we make the “area” the cross-sectional area of the object—that is, a measure of the object’s size—and the density the density of air, this is a reasonable expression for the air resistance force! It is wrong in detail, because there is a dimensionless prefactor of order unity that depends on
the specific shape and state of rotation of the object. But right now we are just trying to estimate if there is any chance that air resistance matters in this problem.

Before we continue, allow me to remark\footnote{This remark is more for the teachers of this material than for the students} that any book or class that claims to “solve” a problem like this—a problem relating to an object falling near the surface of the earth—in which a quantitative analysis of the effect of the air has not been considered, has failed, utterly, to solve the problem. This is generic in the textbook approach to physics. The use of the no-air equations is an exercise in mathematics, not physics; it is the identification of the relevant physical effects that is the most important work of physicists. It certainly isn’t physics if there is no consideration or analysis or justification of the approximations.

Unfortunately, we don’t yet know the speed of the falling bucket. We have a chicken-and-egg problem here. We can’t calculate the speed of the bucket properly if we can’t ignore air resistance, but we can’t figure out whether we can ignore air resistance if we can’t calculate the speed of the bucket! What to do? We break the impasse by ignoring air resistance and then evaluating, after the calculation, whether we were justified in doing so\footnote{Since air resistance will only slow the fall, and the air resistance force depends in an increasing way on speed, this one-sided approach (assume air doesn’t matter and then evaluate after) is safe. There are pathological situations where this will not work.}. If we ignore air resistance, what is the speed of the bucket? Well, it increases as it falls, because gravity sets the acceleration, not the speed. It falls a distance \( h \), a measure of length, and gravity sets the acceleration \( g \), a length per time squared. What is the speed of the bucket at its fastest point? The only quantity with dimensions of speed in this problem is obtained by taking the square root of the product, or

\[
\sqrt{\text{[length]} \, \text{[acceleration]}} = \sqrt{\text{[length]^2 [time]^{-2}}} = \text{[speed]}. \tag{5}
\]

This, up to a factor of order unity (a factor of \( \sqrt{2} \) but it just doesn’t matter at this level of precision), is the speed of the bucket when it hits the ground, if we ignore air resistance.

Putting it all together, we can safely ignore air resistance when the magnitude \( F_a \) of the force from the air is much smaller than the magnitude \( F_g \) of the force of gravity, or, in dimensionless language, when the ratio is much
smaller than unity. Symbolically, we are okay when

$$\frac{F_a}{F_g} \approx \frac{\rho A v^2}{mg} \approx \frac{\rho A h}{m} \ll 1,$$

where $\rho$ is the density of the air, $A$ is the cross-sectional area of the bucket as it falls, $h$ is the height, $m$ is the total mass of the bucket and its contents, and we have used our previous result (from dimensional analysis) that $v^2 \approx g h$.

The astute reader will notice that the gravitational acceleration $g$ has factored out, and the limit is that in which the bucket is much more massive than the column of air through which it falls.

Can we ignore the air? I have that $\rho \approx 1 \text{ kg m}^{-3}$, $A \approx 0.1 \text{ m}^2$, $h \approx 10 \text{ m}$, and $1 < m < 30 \text{ kg}$ (depending on how full the bucket is). I estimated the area and maximum mass for the bucket by assuming that it is about a cubic foot, and I assumed 3 m per story in the building.

Uh oh! We can’t ignore air resistance if the bucket is empty! The empty bucket gets a ratio $(F_a/F_g)$ of unity. The full bucket is fine (or fine enough for our precision today). I think this jives with our intuition or memory of similar events. Imagine an empty bucket falling from the third story. You can imagine it getting slowed a bit by the air, approaching a constant speed, maybe having its trajectory wander a bit left and right as it falls. But a full, heavy bucket slams straight downwards, impervious to the air or breeze.

4. The solution and some discussion

How to proceed? For now let’s simply note that the bucket must be filled with water if we are going to safely ignore air resistance (we have learned something!) but then assume that the bucket is filled with water and continue. In this case, we can finally answer the question posed at the beginning of this lecture: How long does the bucket take to fall?

We seek a time; continuing with our dimensional analysis, in the context of gravity, we have an acceleration $g$, a mass $m$ and a height (length) $h$. The only one of these that includes time at all is $g$, and the only combination with dimensions of purely of time is

$$\sqrt{[\text{length}] [\text{acceleration}]^{-1}} = [\text{time}].$$

\[13\] See also Hogg, “Air resistance,” (arXiv: physics/0609156), and Mahajan & Hogg, “Introductory physics: The new scholasticism,” (arXiv: physics/0412107).
With \( h \approx 10 \text{ m} \) and \( g \approx 10 \text{ m s}^{-2} \), this characteristic time is 1 s. And indeed, this probably agrees with your initial guess based on your memory of similar incidents you have experienced.

This answer, \( \sqrt{h/g} \), obtained by dimensional analysis, is in fact off of an exact calculation by a factor of \( \sqrt{2} \). Do we care? Sure we do! We do have to get right answers sometimes. This characteristic time answer is not quite correct, but it contains almost all of the physics in the problem. Everything after this is just math. We just have to write down the correct equations and solve them. But notice that before we start writing and manipulating equations, we already know that our answer has to be some unknown dimensionless constant times \( \sqrt{h/g} \). That’s pretty good.

The dimensional part of our approximate answer \( \sqrt{h/g} \) is correct, but more importantly, it is close—in an order-of-magnitude sense—to the right answer. After all, we could have manipulated numbers or symbols and obtained a characteristic time of 100 ns, or we could have obtained 35 yr. But in fact when we wrote down the only combination of things that we could possibly have written down that could satisfy the dimensions requirements, the quantity we wrote down is in fact close to correct! That’s significant. We have got almost all of the way to the complete, exact answer using nothing but common sense and some simple facts about gravity near the surface of the Earth.

But we learned more than this, we also learned something about the scaling of the answer with changes to the problem. For example, our answer (as we might have predicted) shows that the time gets longer as the height increases (and how it gets longer) or as the acceleration due to gravity decreases (if, say, I saw this happen on the Moon). We also learned, without putting it in explicitly, that the mass \( m \) of the bucket does not matter. It matters to the estimation of the importance of air resistance, but it does not matter to the free-fall time in the absence of air resistance. This, one of the main and most counter-intuitive of Galileo’s discoveries, falls naturally out of the dimensional argument (which starts, of course, with Galileo’s observation that gravity sets the accelerations of falling objects to \( g \)).

More precisely, what we have obtained from our dimensional argument is the following: When the bucket is full and air can be ignored, the free-fall time \( t \) can be written in the form

\[ t = Q \sqrt[2]{\frac{h}{g}} , \]  

(8)
where $Q$ is a dimensionless prefactor. All of the *math* you did when you solved this problem in textbook-physics classes was all about just determining this one—relatively uninteresting—dimensionless factor $Q$, which turns out to be $\sqrt{2}$.

I have attempted to separate the physical reasoning part of the problem from the mathematical details, and I have written here only about the physical reasoning. And although we did not get that prefactor $Q$—we need to delve into the mathematical details to get that—we did get a lot, like the scaling, and the order of magnitude. We learned that I *do* have time jump out of the way of the falling bucket, but we also learned that I don’t have time to make and install a sign to warn others! We also got lots of time to think about physics and not have mathematical techniques interfere with that.

We have some loose ends: What would we have done if the bucket were empty? Unfortunately, with a force—the air resistance “ram pressure” force $\rho A v^2$—that goes like speed squared, there is no analytic trajectory. You have to *integrate* the trajectory with a computer. Interestingly, this is very easy to do with an accounting spreadsheet or any simple programming language or data analysis package or anything equivalent, as we will see in later lectures. So we would not be *doomed* if we were forced to consider the empty bucket. But our textbook physics would have failed us, because in textbook physics you only consider problems that have analytic solutions, which, as I have said, is a vanishingly small minority of problems you might care about.

Another loose end we left was on the question of the accuracy with which we estimate the height of the third story. Our guess of 10 m is not good to better than, say, 30 percent. However, since the height appears in our answer inside a square-root, a 30 percent change to the height leads to only about a 15 percent change in the free-fall time.\footnote{We will return to this kind of scaling argument in future lectures.} That uncertainty is smaller than our missing factor of $\sqrt{2}$ and our uncertainties about the effects of air resistance, and therefore not important at the level of accuracy we have adopted.

That brings up a final note. In this entire lecture, I have been careless with the numbers and performed all estimation and mathematics with little rigour. I have found no precise or accurate answer to anything. This is not strange or unusual; this is absolutely *generic* for the study of physics. One of the pervasive myths about physicists is that we spend our time making very precise predictions and measurements; after all, we know that the
magnetic moment of the electron is \(1.001159652187 \mu_B\). But of course that can happen only in a very specialized part of physics, it can be done only in those rare circumstances in which we have an exact physical description with which we can exactly understand a physical experiment and we have a precise experiment. Only then do we perform precise calculations and make precise measurements. In most of the physics we are going to be talking about—things like falling buckets—there are a very large number of different physical effects that are important if we want high precision. Furthermore, many of the effects are so complex that they are not computable at all! So it would be pointless in this problem to attempt to get a highly precise answer for the time. Generically, we will have to make approximations and get an approximate answer. This is another very important distinction between the physics we do here and the textbook physics you might find elsewhere. We will not compute precise answers when problems do not warrant them.

5. Problems

**Problem 1:** From what story will the bucket take twice as long to fall as it does from the third story? Is air resistance important for that fall, assuming the bucket is full?

**Problem 2:** At what height \(h\) does air resistance become important for the free fall of a full bucket of water?

**Problem 3:** A ball of mass \(m\) is thrown upwards with speed \(v\). Use dimensional analysis to estimate the height \(h\) to which it will ascend. Ignore air resistance. How wrong is your answer obtained by dimensional analysis alone?

**Problem 4:** A can throw with twice the velocity of B. How much farther can A throw than B, if you ignore air resistance? Qualitatively, how will your answer change if air resistance becomes important?

**Problem 5:** Rank these objects in terms of the effect of air resistance on their fall; that is, by how far they can fall before air resistance becomes important: a penny, a brick, and a tennis ball.

**Problem 6:** A great baseball outfielder can throw a baseball 120 m. Is that throw strongly affected by air resistance? Make the calculation by assuming that it isn’t strongly affected, compute the (approximate) speed of the throw, and then assess.
Problem 7: A professional golfer can hit a golf ball 300 m. Is that shot strongly affected by air resistance? Make the calculation by assuming that it isn’t strongly affected, compute the (approximate) speed of the shot, and then assess.

Problem 8: The air resistance force we computed in this lecture is a kind of “ram pressure.” There is also a “viscous drag” term, involving the viscosity of air. Use a textbook value for the viscosity of air and dimensional analysis to evaluate whether I was justified in ignoring the viscous term relative to ram pressure for the falling bucket. What do you need to assume?