ON (ULTRA-) COMPLETENESS NUMBERS AND (PSEUDO-) PAVING NUMBERS

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Abstract. We study the completeness and ultracompleteness numbers of a convergence space. In the case of a completely regular topological space, the completeness number is countable if and only if the space is Čech-complete, and the ultracompleteness number is countable if and only if the space is ultracomplete. We show that the completeness number of a space is equal to the pseudopaving number of the upper Kuratowski convergence on the space of its closed subsets, at ∅. Similarly, the ultracompleteness number of a space is equal to the paving number of the upper Kuratowski convergence on the space of its closed subsets, at ∅.

1. Introduction

E. Čech generalized the traditional metric notion of completeness to the topological setting in [6], calling a topological space topologically complete if it is a $G_δ$-subset of a compact Hausdorff space. This notion is now usually called Čech-completeness.

Z. Frolik introduced in [15] the notion of a $G_δ$-space, that is, a topological space that is $G_δ$ in every Hausdorff space in which it is dense. He proved that the notion is equivalent to that of Čech for completely regular spaces, and that it can be characterized in terms of the existence of a complete sequence of covers.

Importantly, he generalized the notion to that of $G(m)$-space for an arbitrary cardinal $m$. For completely regular spaces, compactness is recovered when $m = 0$, local compactness (without compactness) when $m = 1$, and Čech-completeness when $m = \aleph_0$.

S. Dolecki was the first to realize [8] that such notions of completeness naturally extend from the setting of topological spaces to the wider context of convergence spaces, and that completeness is most naturally studied in that setting. He also introduced a completeness number, which, in the case of a completely regular topological space, is the least cardinal $m$ for which the space is a $G(m)$-space in the sense of Frolik.

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1In this paper, we assume that completely regular spaces, also called functionally regular in [12], are Hausdorff.

2Namely, a sequence of open cover of the space with the property that any filter with a filter-based formed by open sets that contains an element of each cover has non-empty adherence.

3A topological space that is the intersection of $m$ many open subsets, in every Hausdorff space in which it is dense.
One of the main reasons to study topological problems in the larger context of convergence spaces is the availability of a canonical function space structure (the so-called continuous convergence \cite{3,2}, also called natural convergence \cite{12}) satisfying the exponential law. Hence the question arises naturally whether this new cardinal invariant of a space (completeness number) can be related to a cardinal invariant of a dual space, where duality is considered with respect to a function space endowed with the natural convergence. Dolecki followed this line of investigation with \cite[Theorem 10.1]{8}, relating the completeness number of a convergence space \((X, \xi)\) with the local structure at the empty set of the set of closed subsets of \(\xi\) equipped with the upper Kuratowski convergence – a space that appears naturally in various contexts (see, e.g., \cite{10}) and can be identified with the space \([\xi, \$]\) of continuous functions from \((X, \xi)\) to the Sierpinski space \$, endowed with the natural convergence. While the approach was very insightful, it turns out that the result is not correct as stated.

More specifically, \cite[Theorem 10.1]{8} stated that the completeness number of a convergence space \((X, \xi)\) is the paving number of \([\xi, \$]\) at \(\emptyset\), that is, the minimal number of filters necessary to determine the convergence at \(\emptyset\) (See Section 4 for a proper definition). It turns out that the paving number of \([\xi, \$]\) at \(\emptyset\) is equal to the ultracompleteness number of \((X, \xi)\) (Theorem 26). Topological spaces with countable ultracompleteness number have been called cofinally Čech-complete \cite{26}, strongly complete \cite{25}, and ultracomplete \cite{4} and have been extensively studied.

On the other hand, the completeness number of \((X, \xi)\) is equal to the pseudopaving number of \([\xi, \$]\) at \(\emptyset\) (Theorem 28), another natural local invariant for convergence spaces.

The key results are in Section 6. To prepare for it, after introducing basic conventions and notions of convergence spaces (Section 2), we introduce and study the completeness and ultracompleteness numbers (Section 3).

It turns out that our convergence-theoretic approach allows us to refine \cite[Proposition 2]{16} of Garcia and Romaguera, which states that the open continuous image of an ultracomplete topological space is ultracomplete, in contrast to Čech-complete spaces. We show that it is enough for the map to be biquotient.

We then turn to the pseudopaving and paving numbers of a convergence \cite{4}, before particularizing these notions to the \(\$\)-dual \([\xi, \$]\) (Section 5).

We use notations and terminology consistent with the recent book \cite{12}, to which we refer the reader for a comprehensive treatment of convergence spaces. We only give the strictly necessary convergence-theoretic definitions in the next section.

\section{Notations and conventions}

\subsection{Set-theoretic conventions.}
Let \(2^X\) denote the powerset of \(X\). If \(\mathcal{P} \subset 2^X\), let

\[\mathcal{P}_c := \{X \setminus P : P \in \mathcal{P}\},\]

\[\mathcal{P}^\cup := \left\{ \bigcup_{P \in A} P : A \subset \mathcal{P}, \text{card} A < \infty \right\}\]

and \(\mathcal{P}^\cap := \left\{ \bigcap_{P \in A} P : A \subset \mathcal{P}, \text{card} A < \infty \right\}\),

\[\mathcal{P}^\uparrow := \{ A \subset X : \exists P \in \mathcal{P}, P \subset A \}\]

and \(\mathcal{P}^\downarrow := \{ A \subset X : \exists P \in \mathcal{P}, A \subset P \}\).\)

If \(\mathcal{D} \subset 2^{2^X}\), we write \(\mathcal{D}^a = \{D^a : D \in \mathcal{D}\}\), where \(a\) ranges over \(\cup, \cap, \uparrow\) and \(\downarrow\).
If $a : 2^X \to 2^X$ and $A \subseteq 2^X$, we write

$$a^\# A = \{a(A) : A \in A\}.$$  

We say that two families $A$ and $B$ of subsets of $X$ mesh, in symbols $A \# B$, if $A \cap B \neq \emptyset$ whenever $A \in A$ and $B \in B$. We also write

$$A^\# = \{B \subseteq X : \{B\} \# A\}.$$  

A family $D \subseteq 2^X$ of non-empty subsets of $X$ is a filter if $D = D \cap \uparrow$ and a family $P$ of proper subsets of $X$ is an ideal if $P = P \cup \downarrow$. Of course, $D$ is a filter if and only if $D^c$ is an ideal.

We denote by $\mathcal{F}X$ the set of filters on $X$, which we order by the inclusion order of families of subsets of $X$. Maximal filters are called ultrafilters. We denote by $UX$ the set of ultrafilters on $X$ and, given $F \in \mathcal{F}X$, by $\beta F$ the set of ultrafilters that are finer than $F$.

2.2. Convergences. A convergence $\xi$ on a set $X$ is a relation between filters on $X$ and points of $X$, where we write $x \in \lim_\xi F$ if $F$ and $x$ are $\xi$-related, subject to the conditions that $x \in \lim_\xi \{x\}^\uparrow$ for every $x \in X$ and that $\lim_\xi F \subseteq \lim_\xi G$ whenever $F \leq G$. We denote by $|\xi|$ the underlying set of a convergence $\xi$, that is, if $(X, \xi)$ is a convergence space, $|\xi| = X$. A function $f : |\xi| \to |\tau|$ is continuous if

$$f(\lim_\xi F) \subseteq \lim_\tau f[F]$$  

for every $F \in \mathcal{F}|\xi|$, where $f[F] = \{B \subseteq |\tau| : f^-(B) \in F\}$.

Given two convergences $\xi$ and $\theta$ with the same underlying set, we say that $\xi$ is finer than $\theta$ or that $\theta$ is coarser than $\xi$, in symbols $\xi \geq \theta$, if $\lim_\xi F \subseteq \lim_\theta F$ for every filter $F$.

A topology can be seen as a particular kind of convergence. Indeed, a topology $\tau$ defines a convergence $x \in \lim_\tau F \iff F \geq \mathcal{N}(x)$, where $\mathcal{N}(x)$ denotes the neighborhood filter of $x$ for the topology $\tau$. This convergence completely and uniquely determines the topology, which can thus be identified with the convergence. Such a convergence is called topological.

A subset $C$ of $|\xi|$ is closed if $\lim_\xi F \subseteq C$ whenever $C \in \mathcal{F}^\#$. The collection of closed sets satisfies the properties of the collection of closed sets for a topology, which we denote by $T \xi$. This is the finest topological convergence that is coarser than $\xi$. The map $T$ is a functor, indeed a concrete reflector.

A convergence is a pseudotopology if

$$\bigcap_{U \in \beta F} \lim_\xi U \subseteq \lim_\xi F.$$  

Given a convergence $\xi$ there is a finest pseudotopology $S\xi$ that is coarser than $\xi$, which is called pseudotopological modification of $\xi$ and is defined by

$$\lim_{S\xi} F = \bigcap_{U \in \beta F} \lim_\xi U.$$  

The map $S$ is a concrete reflector and $S \leq T$, for every topology is a pseudotopology.
The adherence of a filter \( \mathcal{F} \) for a convergence \( \xi \) is
\[
\text{adh}_\xi \mathcal{F} = \bigcup_{F \in \mathcal{F}} \lim_{\xi} H = \bigcup_{U \in \beta \mathcal{F}} \lim_{\xi} U.
\]
Evidently, \( \text{adh}_\xi \mathcal{F} = \text{adh}_\xi \mathcal{G} \) and thus \( \lim_{\xi} \mathcal{F} = \bigcap_{F \in \mathcal{F}} \text{adh}_\xi H \).

A subset \( A \) of \( [\xi] \) is called compactoid if \( \lim_{\xi} U \neq \emptyset \) whenever \( A \in U \) and \( U \in \bigcup [\xi] \), and compact if \( \lim_{\xi} U \cap A \neq \emptyset \) whenever \( A \in U \) and \( U \in \bigcup [\xi] \). The family \( \mathcal{K} \) of compactoid subsets (of a given convergence space) is clearly an ideal, hence the family \( \mathcal{K}_c \) of complements is a (possibly degenerate if the convergence is compact) filter called cocompactoid filter.

Given \( f : [\xi] \to Y \), there is a finest convergence on \( Y \) making \( f \) continuous, which is called final convergence for \( f \) and \( \xi \) and is denoted \( f\xi \). Dually, given \( f : X \to |\tau| \), there is a coarsest convergence on \( X \) making \( f \) continuous, which is called initial convergence for \( f \) and \( \tau \) and is denoted \( f^-\tau \). Note that a map \( f : [\xi] \to |\tau| \) is continuous if and only if \( \tau \leq f\xi \) if and only if \( f^-\tau \leq \xi \).

An onto map \( f : [\xi] \to |\tau| \) is convergence quotient, or almost open, if \( \tau \geq f\xi \) so that if \( f \) is continuous and almost open then \( \tau = f\xi \). We say that \( f \) is biquotient if \( \tau \geq S f\xi \) and quotient if \( \tau \geq T f\xi \). Of course, if \( \xi \) and \( \tau \) are topological, then \( f \) is almost open, biquotient or quotient in this sense if and only if it is in the classical sense of \([21]\).

3. Completeness and ultracompleteness

3.1. Basic notions. A family \( \mathcal{P} \) of subsets of a convergence space \((X, \xi)\) is a \( \xi \)-cover if \( \mathcal{F} \cap \mathcal{P} \neq \emptyset \) for every convergent filter \( \mathcal{F} \), and a \( \xi \)-pseudocover if this condition is restricted to convergent ultrafilters \( \mathcal{F} \).

Given a family \( \mathcal{P} \) of covers, a filter \( \mathcal{F} \) is \( \mathbb{P} \)-Cauchy if \( \mathcal{F} \cap \mathcal{P} \neq \emptyset \) for every \( \mathcal{P} \in \mathbb{P} \) and \( \mathbb{P} \)-preCauchy if \( \mathcal{F} \# \cap \mathcal{P} \neq \emptyset \) for every \( \mathcal{P} \in \mathbb{P} \). A family \( \mathcal{P} \) of \( \xi \)-covers is called complete if every \( \mathbb{P} \)-Cauchy filter has non-empty \( \xi \)-adherence, and ultracomplete if every \( \mathbb{P} \)-preCauchy filter has non-empty \( \xi \)-adherence.

Recall that a completely regular topological space is Čech-complete if and only if it has a countable complete collection of (open) covers. See for instance, \([15]\) \((\mathbb{4}, \mathbb{13}\) Theorem 3.9.2) \([8]\).

On the other hand, completely regular spaces with a countable ultracomplete collection of open covers have been called cofinally Čech-complete \([26]\), strongly complete \([25]\), and ultracomplete \([4]\). We refer the reader to \([19]\) for a good survey on ultracomplete spaces \([\mathbb{4}]\).

Note that the usual assumption that the space be completely regular is irrelevant, and the notions at hand can be considered for general convergence spaces, as they have been in \([8]\). We call a convergence space countably (ultra)complete if it admits a countable (ultra)complete collection of covers.

S. Dolecki showed in \([8]\) that completeness can be reformulated entirely in terms of filters. Ultracompleteness can be characterized similarly, proceeding along the lines of \([8]\). Namely,

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\(^4\)Frolík asks for every \( \mathbb{P} \)-Cauchy filter that additionally has a filter-base composed of open sets to have adherence points. Whether we omit the filter-base composed of open sets condition or not results in an equivalent notion in topological spaces.

\(^5\)In the uniform context, the notion goes back to \([13]\) and \([7]\).
Proposition 1. [8, Proposition 4.2] A family $P$ of covers is complete if and only if $\mathbb{P}^{\cup\downarrow}$ is.

Lemma 2. A filter is $\mathbb{P}$-preCauchy if and only if it is $\mathbb{P}^{\cup\downarrow}$-preCauchy.

Proof. If $F$ is $\mathbb{P}$-preCauchy, then $F^\# \cap P \neq \emptyset$ for every $P \in \mathbb{P}$, and a fortiori, $F^\# \cap P^{\cup\downarrow} \neq \emptyset$ because $P \subset P^{\cup\downarrow}$. Thus $F$ is $\mathbb{P}^{\cup\downarrow}$-preCauchy.

Conversely, if $F$ is $\mathbb{P}^{\cup\downarrow}$-preCauchy and $P \in \mathbb{P}$, then $P^{\cup\downarrow} \cap F^\# \neq \emptyset$ so that there is $P_1, \ldots, P_n \in \mathbb{P}$ and $\bigcup_{i=1}^n P_i \subset C \in F^\#$ and thus there is $i \in \{1, \ldots, n\}$ with $P_i \in F^\#$ because $F^\#$ is a filter-grill. Thus $F$ is $\mathbb{P}$-preCauchy. □

Moreover,

Proposition 3. [8, Theorem 3.1] A family $P$ is a $\xi$-cover if and only if

$$\text{adh}_\xi P_c = \bigcup_{F^\# \cap P_c} \text{lim}_\xi F$$

is empty.

Of course, if $P$ is an ideal cover, then $P_c$ is a filter, and $\text{adh}_\xi P_c$ is the adherence in the usual sense for filters. We will say that a filter is non-adherent if its adherence is empty.

In view of Proposition 1 and Lemma 2 we can reformulate the definitions of complete and ultracomplete collections in terms of collection of filters.

Accordingly, we call a family $D$ of non-adherent filters on $(X, \xi)$ cocomplete if for every $G \in \mathbb{P}X$

$$\text{adh}_\xi G = \emptyset \implies \exists D \in D, \ G \geq D,$$

and ultracocomplete if

$$\text{adh}_\xi G = \emptyset \implies \exists D \in D, \ G \geq D.$$

In view of the previous discussion,

Proposition 4. The following are equivalent for a collection $\mathbb{P}$ of $\xi$-covers:

1. $\mathbb{P}$ is (ultra)complete;
2. $\mathbb{P}^{\cup\downarrow}$ is (ultra)complete;
3. $\mathbb{P}_* = \{P_c : P \in \mathbb{P}^{\cup\downarrow}\}$ is an (ultra)cocomplete collection of non-adherent filters.

The first observation is that any convergence space $(X, \xi)$ admits an ultracomplete family of filters, namely,

$$D_\emptyset := \{F \in \mathbb{P}X : \text{adh}_\xi F = \emptyset\},$$

which is non-empty unless $\xi$ is compact, in which case any family of filters is ultracomplete, including the empty family.

Thus we can define the completeness number $\text{compl}(\xi)$ and ultracompleteness number $\text{ncompl}(\xi)$ of $\xi$, which are the smallest cardinality of a cocomplete collection of filters, and of an ultracomplete collection of filters respectively.

Remark 5. One may consider variants of the notions at hand where adherence is replaced by limit. For instance, Fletcher and Lindgren [14] call strongly Čech-complete a $T_1$ topological space admitting a countable collection $\mathbb{P}$ of open covers so that every $\mathbb{P}$-Cauchy filter converges. This property, however, is not equivalent to its counterpart in terms of filters, for using covers or ideal covers does make
a difference here. In fact, no $T_1$ convergence with more than one point would admit any collection of filters that satisfy the corresponding strong cocompleteness condition, namely that

$$\lim_\xi G = \emptyset \implies \exists D \in \mathcal{D}, G#D.$$  

3.2. Interpretation in the space of ultrafilters. Given a convergence space $(X, \xi)$, let $UX$ denote the set of ultrafilters on $X$, endowed with the usual Stone topology $\beta$ for which the sets

$$\beta A = \{ U \in UX : A \in U \}$$  

where $A$ ranges over the subsets of $X$ form a basis for the topology. Recall that closed subsets of $UX$ are of the form

$$\beta F = \{ U \in UX : U \geq F \}$$  

where $F \in FX$.

Let $U_\xi X$ denote the set of ultrafilters on $X$ that converge for $\xi$. The following was already observed implicitly in [15], and explicitly in [8] and [12], but we include a proof for the sake of comparison with Theorem 8. We call a subset of a topological space a $G_\kappa$-subset (resp. $F_\kappa$-subset) if it is an intersection of $\kappa$ many open sets (resp. a union of $\kappa$ many closed sets).

**Theorem 6.** [8] The following are equivalent:

1. $\text{compl}(\xi) \leq \kappa$;
2. $U_\xi X$ is a $G_\kappa$-subset of $(UX, \beta)$;
3. $UX \setminus U_\xi X$ is an $F_\kappa$-subset of $(UX, \beta)$.

**Proof.** That $(2) \iff (3)$ is obvious. $(1) \implies (3)$: Let $\mathcal{D}$ be a complete family of filters of cardinality at most $\kappa$. Then each $D \in \mathcal{D}$ has empty adherence so that $\beta D \subset UX \setminus U_\xi X$. Moreover, if $U \in UX \setminus U_\xi X$, then $\lim_\xi U = \text{adh}_\xi U = \emptyset$ so that there is $D \in \mathcal{D}$ with $\beta D \not\subset U\xi U$, equivalently, $U \in \beta D$. Thus $UX \setminus U_\xi X = \bigcup_{D \in \mathcal{D}} \beta D$ is an $F_\kappa$-subset of $(UX, \beta)$.

$(3) \implies (1)$: If $UX \setminus U_\xi X$ is an $F_\kappa$-subset of $(UX, \beta)$, then $UX \setminus U_\xi X = \bigcup_{D \in \mathcal{D}} \beta D$

for a family $\mathcal{D}$ of filters of cardinality $\kappa$. Because $\beta D \subset UX \setminus U_\xi X$, each $D \in \mathcal{D}$ has empty adherence. If now $G \in FX$ with $\text{adh}_\xi G = \emptyset$ then $\beta G \subset UX \setminus U_\xi X = \bigcup_{D \in \mathcal{D}} \beta D$ so that there is $D \in \mathcal{D}$ with $\beta G \cap \beta D \neq \emptyset$ so that $G \# D$. Thus $\mathcal{D}$ is complete.

Since closed and compact subsets of $(UX, \beta)$ coincide, $F_\sigma$-subsets and $\sigma$-compact subsets coincide. Hence,

**Corollary 7.** A convergence $\xi$ is countably complete if and only if $UX \setminus U_\xi X$ is $\sigma$-compact.

Recall that a $k$-cover of a topological space $X$ is a family $C \subset \mathcal{P}(X)$ such that for every compact subset $K$ of $X$, there is $C \in C$ with $K \subset C$. Recall (e.g., [20]) that the $k$-Arens number $ka(X)$ of $X$ is the smallest cardinality of a family of compact sets that is a $k$-cover of $X$. A topological space is hemicompact if $ka(X) \leq \omega$. 
Clearly, these definitions make sense for arbitrary convergence spaces as well. As compactness is absolute, compact subsets of $S \subseteq UX$ are of the form $\beta H$ for some $H \in \mathcal{F}X$.

If $A$ is a subset of a topological space $X$, we denote by $\chi(A)$ the character of $A$ in $X$, that is, the smallest cardinality of a filter base of the filter

$$O_X(A) := \{ U \in O_X : A \subseteq U \}.$$ 

**Theorem 8.**

$$ucompl(\xi) = \chi(\xi X) = k\alpha(UX \setminus U_{\xi}X).$$

**Proof.** That $\chi(\xi X) = k\alpha(UX \setminus U_{\xi}X)$ is easily seen.

Let $D$ be an ultracomplete collection of non-adherent filters. Because each $D \in \mathcal{D}$ is non-adherent, $\bigcup_{D \in \mathcal{D}} \beta D \subseteq UX \setminus U_{\xi}X$. On the other hand, if $K$ is a compact subset of $UX \setminus U_{\xi}X$, there is $G \in \mathcal{F}X$ with $\beta G = K \subseteq UX \setminus U_{\xi}X$. Hence, $\text{ad} G = \emptyset$, so that, by (1.2), there is $D \in \mathcal{D}$ with $G \subseteq D$, equivalently, $\beta G \subseteq \beta D$. Thus $\{ \beta D : D \in \mathcal{D} \}$ is a $\beta$-cover of $UX \setminus U_{\xi}X$ composed of compact subsets of $UX \setminus U_{\xi}X$. Thus $k\alpha(UX \setminus U_{\xi}X) \leq ucompl(\xi)$.

Conversely, a $\beta$-cover of $UX \setminus U_{\xi}X$ composed of compact subsets of $UX \setminus U_{\xi}X$ is of the form $\{ \beta D : D \in \mathcal{D} \}$ for some family $\mathcal{D}$ of filters. Because $\beta D \subseteq UX \setminus U_{\xi}X$, each $D \in \mathcal{D}$ is non-adherent. If now $G$ is another non-adherent filter on $(X, \xi)$ then $\beta G$ is a compact subset of $UX \setminus U_{\xi}X$, so that there is $D \in \mathcal{D}$ with $\beta G \subseteq \beta D$, equivalently, $D \subseteq G$. Thus $D$ is ultracomplete. Hence $ucompl(\xi) = k\alpha(UX \setminus U_{\xi}X)$. □

**Corollary 9.** A convergence $\xi$ is countably ultracomplete if and only if $UX \setminus U_{\xi}X$ is hemicompact.

**Remark 10.** Corollaries [7] and [9] are reminiscent of the classical facts that a completely regular space is Čech-complete if and only if its remainder in one (equivalently all) of its compactifications is $\sigma$-compact (e.g., [13] Theorem 3.9.1), and ultracomplete if and only if its remainder in one (equivalently all) of its compactifications is hemicompact [14]. Of course, corollaries [7] and [9] make sense whether compactifications of $\xi$ in the usual sense exist or not, so we can see them as generalization of the classical statements. Let us clarify further the relationship:

As $(UX, \beta)$ is the Čech-Stone compactification of $X$ endowed with the discrete topology, it has the universal property that any map $f : X \to K$ where $K$ is a compact Hausdorff topological space has a unique continuous extension $\bar{f} : UX \to K$ (defined by $\bar{f}(U) = \lim_k f(U)$), e.g., [12] Theorem IX.5.11. Hence if $X$ is a completely regular space and $Y$ is a (Hausdorff) compactification of $X$, then the inclusion map $i : X \to Y$ has a unique continuous extension $\bar{i} : UX \to Y$ defined by $\bar{i}(U) = \lim_Y i(U)$, where $i(U) = U^Y$ is the filter generated on $Y$ by $U$.

As $X$ is dense, $i$ is onto, and thus, as a continuous onto map between compact Hausdorff spaces, it is also perfect. It is clear that $\bar{i}(UX \setminus U_{\xi}X) = Y \setminus X$. Moreover, as $i$ is continuous and perfect, $Y \setminus X$ is $\sigma$-compact (respectively hemicompact), if and only if $UX \setminus U_{\xi}X$ is. In other words, the classical statements for completely regular spaces are simple corollaries of our corollaries [7] and [9]. More generally,

**Theorem 11.** Let $X$ be a completely regular topological space.

1. The following are equivalent:
   (a) $\text{compl}(X) \leq \kappa$;
   (b) $U_{\xi}X$ is a $G_\kappa$-subset of $(UX, \beta)$;
   (c) $X$ is a $G_\kappa$-subset of some compactification of $X$;
(d) $X$ is a $G_\kappa$-subset of all compactifications of $X$.

(2) The following are equivalent:
(a) $\text{ucompl}(X) \leq \kappa$;
(b) $\chi(\mathcal{U}_\xi X) \leq \kappa$ in $(\mathcal{U}X, \beta)$;
(c) $\chi(X) \leq \kappa$ in some compactification of $X$;
(d) $\chi(X) \leq \kappa$ in all compactifications of $X$.

3.3. Finite ultracompleteness numbers. Since $\text{compl}(\xi) = 0$ or $\text{ucompl}(\xi) = 0$ if the empty family of covers is (ultra)complete, and since every filter is $\emptyset$-Cauchy,

$$\text{compl}(\xi) = 0 \iff \text{ucompl}(\xi) = 0 \iff \xi \text{ is compact.}$$

On the other hand,

**Proposition 12.** The following are equivalent:

(1) $\text{ucompl}(\xi) < \infty$
(2) $\text{ucompl}(\xi) \leq 1$;
(3) $\text{compl}(\xi) < \infty$
(4) $\text{compl}(\xi) \leq 1$;
(5) The cocompactoid filter is non-adherent (and possibly degenerate if $\xi$ compact);
(6) $\xi$ is locally compactoid.

Note that the case of the completeness number, that is, equivalences (3) to (6), is [8, Proposition 7.3].

**Proof.** The equivalence between (3), (4), (5) and (6) is established in [8, Proposition 7.3]. Moreover, (2) $\implies$ (1) $\implies$ (3) is clear, for $\text{compl}(\xi) \leq \text{ucompl}(\xi)$. Hence it is enough to show that (5) $\implies$ (2). We claim that if the cocompactoid filter $\mathcal{K}_c$ is non-adherent, then the family $\{\mathcal{K}_c\}$ is ultracomplete. Indeed, if $\mathcal{F}$ is a non-adherent filter, then it does not mesh any compactoid set, that is, $K^c \in \mathcal{F}$ for every $K \in \mathcal{K}$ and thus $\mathcal{F} \geq \mathcal{K}_c$. \qed

3.4. Basic facts on ultracompleteness numbers. In view of Proposition 12, a locally compact space is countably ultracomplete, and a countably ultracomplete space is countably complete. None of these implications can be reversed. For instance, the set of irrationals with the topology induced from $\mathbb{R}$ is a non-ultracomplete completely metrizable, hence countably complete, space [4, Example 3.2]. On the other hand, the space $[0, 1] \setminus \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ with its natural topology is an ultracomplete metrizable space that is not locally compact [4, Example 3.1]. At any rate, there are countably complete (metrizable topological) spaces that are not countably ultracomplete, so that, in view of corollaries 7 and 9

**Proposition 13.** There is a set $X$ and a subspace $S$ of $(\mathcal{U}X \setminus X, \beta)$ which is $\sigma$-compact but not hemicompact.

This observation will turn out to be useful when considering the paving number of a convergence, and its variants.

Countable (ultra)completeness is obviously hereditary with respect to closed subsets, but, unlike countable completeness which is hereditary with respect to $G_\omega$-sets among regular convergences, countable ultracompleteness is not hereditary (among completely regular spaces) even with respect to open sets [5, Example 2.3].
On the other hand, continuous perfect onto maps preserve countable completeness (e.g., [13, Theorem 3.9.10]) and countable ultracompleteness [16, Theorem 1] in both directions. The latter extends to the ultracompleteness number. This can be shown, for instance, with the obvious variant of the proof of [12, Theorem XI.6.4(1)] stating that if \( f : |\xi| \to |\tau| \) is a continuous surjective perfect map then \( \operatorname{compl}(\tau) = \operatorname{compl}(\xi) \), to the effect that:

**Proposition 14.** If \( f : |\xi| \to |\tau| \) is a continuous surjective perfect map then \( \operatorname{ucompl}(\tau) = \operatorname{ucompl}(\xi) \).

On the other hand, while countable completeness is not always preserved under open continuous image, countable ultracompleteness is [16, Proposition 2]. This, too, extends to ultracompleteness numbers, and more importantly can be refined to continuous biquotient maps, that is, continuous onto maps \( f : |\xi| \to |\tau| \) with \( \tau \geq S(f\xi) \). Every onto open map is almost open (that is, \( \tau \geq f\xi \)) and thus biquotient. See [12, Section XV.1] for details.

**Theorem 15.** Let \( f : |\xi| \to |\tau| \) be a continuous biquotient map. Then \( \operatorname{ucompl}(\tau) \leq \operatorname{ucompl}(\xi) \).

**Lemma 16.** If

1. \( f : |\xi| \to |\tau| \) is almost open and onto, and \( \mathcal{P} \) is a \( \xi \)-cover then \( f[\mathcal{P}] = \{f(P) : P \in \mathcal{P}\} \) is a \( \tau \)-cover;
2. \( f : |\xi| \to |\tau| \) is a biquotient map and \( \mathcal{P} \) is an ideal \( \xi \)-cover then \( f[\mathcal{P}] \) is an (ideal) \( \tau \)-cover.

**Proof.** (1) Under these assumptions \( \tau \geq f\xi \). Let \( \mathcal{F} \) be a \( \tau \)-convergent filter. Then there is a \( \xi \)-convergent filter \( \mathcal{G} \) with \( \mathcal{F} \geq f[\mathcal{G}] \). As \( \mathcal{P} \) is a \( \xi \)-cover, there is \( P \in \mathcal{P} \cap \mathcal{G} \), hence \( f(P) \in f[\mathcal{P}] \cap \mathcal{F} \).

(2) Let \( \mathcal{F} \) be a \( \tau \)-convergent filter. Then every \( U \in \beta \mathcal{F} \) is \( f\xi \)-convergent, and by (1) contains \( f(P_U) \) for some \( P_U \in \mathcal{P} \). Thus, there is a finite subset \( S \) of \( \beta \mathcal{F} \) (e.g., by [12, Proposition II.6.5]) such that

\[
\bigcup_{U \in S} f(P_U) = f\left( \bigcup_{U \in S} P_U \right) \in \mathcal{F}.
\]

Since \( \mathcal{P} \) is an ideal cover, \( \bigcup_{U \in S} P_U \in \mathcal{P} \) and thus \( f[\mathcal{P}] \) is a \( \tau \)-cover. \( \square \)

**Proof of Theorem 15.** Suppose \( \mathcal{P} \) is an ultracomplete collection of \( \xi \)-covers, which we can assume to be ideal covers by Proposition 3. Then

\[
f[\mathcal{P}] = \{f[\mathcal{P}] : \mathcal{P} \in \mathcal{P}\}
\]

is a collection of \( \tau \)-covers by Lemma 16. Moreover, if \( \mathcal{F} \) is \( f[\mathcal{P}] \)-preCauchy, then for every \( \mathcal{P} \in \mathcal{P} \) there is \( P_{\mathcal{P}} \in \mathcal{P} \) with \( f(P_{\mathcal{P}}) \in \mathcal{F}^\# \), equivalently, \( P_{\mathcal{P}} \in (f^-[\mathcal{F}])^\# \). Since \( \mathcal{P} \) is ultracomplete, there is an \( \operatorname{adh}_\xi f^-[\mathcal{F}] \neq \emptyset \) and thus, by continuity of \( f \), \( \operatorname{adh}_\tau \mathcal{F} \neq \emptyset \).

In contrast, countable ultracompleteness is not preserved under closed maps [4, Example 3.3], hence not under hereditarily quotient maps.

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6See [9] for further generalization of this result.
4. (Pseudo)paving Number

A family $\mathcal{D}$ of filters on a convergence space $(X, \xi)$ is a pavement at $x$ if every $\mathcal{D} \in \mathcal{D}$ converges to $x$ and, for every filter $\mathcal{F}$ converging to $x$, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \leq \mathcal{F}$. The family $\mathcal{D}$ is a pseudopavement at $x$ if every $\mathcal{D} \in \mathcal{D}$ converges to $x$ and, for every ultrafilter $\mathcal{U}$ converging to $x$, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{U} \in \beta \mathcal{D}$.

Let $p(\xi, x)$ denote the paving number of $\xi$ at $x$, that is, the smallest cardinality of a pavement at $x$ for $\xi$, and let $pp(\xi, x)$ denote the pseudopaving number of $\xi$ at $x$, that is, the smallest cardinality of a pseudopavement at $x$ for $\xi$.

Let

$$U_\xi(x) := \{U \in UX : x \in \lim_\xi U\}.$$  

**Theorem 17.** Let $(X, \xi)$ be a pseudotopological space, and let $\mathcal{D}$ be a family of filters on $X$. The following are equivalent:

1. $\mathcal{D}$ is a pseudopavement at $x$;
2. $x \in \lim_\xi \mathcal{D}$ for every $\mathcal{D} \in \mathcal{D}$, and, for every $\mathcal{F} \in FX$,
   
   $$x \in \lim_\xi \mathcal{F} \Rightarrow \exists \mathcal{D} \in \mathcal{D}, \mathcal{D}\#\mathcal{F};$$

3. $$U_\xi(x) = \bigcup_{\mathcal{D} \in \mathcal{D}} \beta \mathcal{D}.$$  

**Proof.** (1) $\Rightarrow$ (2): If $x \in \lim_\xi \mathcal{F}$ and $\mathcal{U} \in \beta \mathcal{F}$, then $x \in \lim_\xi \mathcal{U}$, so that there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \leq \mathcal{U}$, because $\mathcal{D}$ is a pseudopavement. Hence $\mathcal{D}\#\mathcal{F}$.

(2) $\Rightarrow$ (3): Because $x \in \lim_\xi \mathcal{D}$ for every $\mathcal{D} \in \mathcal{D}$,

$$\bigcup_{\mathcal{D} \in \mathcal{D}} \beta \mathcal{D} \subset U_\xi(x).$$

If now $\mathcal{U} \in U_\xi(x)$, then by (2), there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D}\#\mathcal{U}$, equivalently, $\mathcal{U} \in \beta \mathcal{D}$. Hence, we have (3).

(3) $\Rightarrow$ (1): If $\beta \mathcal{D} \subset U_\xi(x)$ and $\xi$ is pseudotopological, $x \in \lim_\xi \mathcal{D}$. If $x \in \lim_\xi \mathcal{U}$ for $\mathcal{U} \in UX$, then $\mathcal{U} \in U_\xi(x)$, so that by (3), there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{U} \in \beta \mathcal{D}$. □

**Corollary 18.**

$$pp(\xi, x) = \inf \{\kappa \in \text{Ord} : U_\xi(x) \text{ is a } F_\kappa \text{-subset of } UX\}.$$  

In particular, $pp(\xi, x) \leq \omega$ if and only if $U_\xi(x)$ is $\sigma$-compact.

Let us return to characterizing the usual paving number.

**Theorem 19.** Let $(X, \xi)$ be a pseudotopological space. A family $\mathcal{D}$ of filters on $X$ is a pavement at $x$ for $\xi$ if and only if

$$\{\beta \mathcal{D} : \mathcal{D} \in \mathcal{D}\}$$

is a $k$-cover of $U_\xi(x)$ by compact subsets of $U_\xi(x)$.

**Proof.** Let $\mathcal{D}$ be a family of compact subsets of $U_\xi(x)$ that is a $k$-cover of $U_\xi(x)$. Since closed subsets of $UX$ are of the form $\beta \mathcal{F}$ for some filter $\mathcal{F} \in FX$, there is $\mathcal{B} \subset FX$ with $\mathcal{D} = \{\beta \mathcal{B} : \mathcal{B} \in \mathcal{B}\}$. We claim that $\mathcal{B}$ is a $\xi$-pavement at $x$. Since $\beta \mathcal{B} \subset U_\xi(x)$, each $\mathcal{B} \in \mathcal{B}$ converges to $x$, because $\xi$ is a pseudotopology. If $x \in \lim_\xi \mathcal{G}$, then $\beta \mathcal{G}$ is a compact subset of $U_\xi(x)$, so that there is $\mathcal{B} \in \mathcal{B}$ with $\beta \mathcal{G} \subset \beta \mathcal{B}$, that is, $\mathcal{B} \subset \mathcal{G}$.

Conversely, if $\mathcal{B}$ is a $\xi$-pavement at $x$, then $\mathcal{D} = \{\beta \mathcal{B} : \mathcal{B} \in \mathcal{B}\}$ is a family of compact subsets of $U_\xi(x)$, and is a $k$-cover of $U_\xi(x)$. Indeed, compact sets are
of the form $\beta G$ for some filter $G$ on $X$, and $\beta G \subset U_\xi(x)$ means that $x \in \lim_\xi G$ because $\xi$ is a pseudotopology. Because $B$ is a pavement, there is $B \in B$ with $B \leq G$, equivalently, $\beta G \subset \beta B$.

\textbf{Corollary 20.}

$$p(\xi, x) = ka(U_\xi(x)).$$

In particular, $p(\xi, x) \leq \omega$ if and only if $U_\xi(x)$ is hemicompact.

\textbf{Example 21} (A pseudotopology of countable pseudopaving number and uncountable paving number). In view of Proposition \[13\], there is a set $X$ and $S \subset UX \setminus X$ where $S$ is $\sigma$-compact but not hemicompact. Let $x_0 \in X$ and let $\xi$ be the pseudotopology on $X$ defined by $U_\xi(x_0) = \{x_0\} \cup S$ and $U_\xi(x) = \{x\}$ for all $x \in X$, $x \neq x_0$. In view of corollaries \[18\] and \[20\] the pseudotopology $\xi$ satisfies

$$pp(\xi, x_0) = \omega < p(\xi, x_0).$$

Comparing Theorem \[6\] with Corollary \[18\] and Theorem \[8\] with Corollary \[20\] suggests a duality between the pseudopaving number on the local side and the completeness number on the global side, and similarly between the paving number on the local side and the strong completeness number on the global side. As we will see, this duality is made explicit via the $\$-dual.

5. $\$-DUAL OF A CONVERGENCE SPACE

5.1. $\$-dual convergence, reduced and erected filters. The \textit{natural convergence} \[\mathbb{F}\]$[\xi, \sigma]$ on the set $C(\xi, \sigma)$ of continuous functions from $\xi$ to $\sigma$ is the coarsest convergence on $C(\xi, \sigma)$ making the \textit{evaluation map} $ev = \langle \cdot, \cdot \rangle : |\xi| \times C(\xi, \sigma) \to |\sigma|$ (defined by $ev(x, f) = (x, f) = f(x)$) jointly continuous. See \[3\], \[2\], \[12\] for a systematic study of the natural convergence.

We focus here on the case where $\sigma$ is the Sierpiński space $\$\$, where $|\$\$| = \{0, 1\}$ and the open sets are $\emptyset$, $\{1\}$ and $\{0, 1\}$. We call \textit{indicator function} of $A \subset X$ the function $\chi_A : X \to |\$\$|$ defined by $\chi_A(x) = 0$ if and only if $x \in A$. Then, a function $f : |\xi| \to |\$\$|$ is continuous if and only if it is the indicator function of a closed subset of $\xi$, so that we identify here the underlying set of $[\xi, \$\$]$ with $\xi$-closed subsets of $|\xi|$. We call $[\xi, \$\$]$ the \textit{$\$-dual of $\xi$}.

If $A \subset [\xi, \$\$]$, then $rdc(A) := \bigcup_{A \in A} A$ and if $G \in F([\xi, \$\$])$ then

$$rdc^5 G := \{rdc G : G \in G\}$$

is a (possibly degenerate) filter on $X$, called \textit{reduced filter of $G$}. Then

\textbf{Proposition 22.} \[10\] Let $\xi$ be a convergence. If $G \in F([\xi, \$\$])$ and $A \in [\xi, \$\$]$ then

$$A \in \lim_{[\xi, \$\$]} G \iff \text{adh}_\xi \text{rdc}^5 G \subset A.$$  

In particular, if $\xi$ is a topology then

$$A \in \lim_{[\xi, \$\$]} G \iff \bigcap_{G \in G} \text{cl}_\xi \left( \bigcup_{C \in G} C \right) \subset A,$$

and the convergence $[\xi, \$\$]$ is then the \textit{upper Kuratowski convergence} on the $\xi$-closed subsets of $[\xi]$, which has been extensively studied, e.g., \[10\], \[24\], \[1\], \[23\], \[11\]. This

\[7\]also often called \textit{continuous convergence}
convergence can also be seen as the Scott convergence on the complete lattice \((C_\xi, \supseteq)\) of closed subsets of \(\xi\) ordered by reversed inclusion (e.g., [17, p. 132]).

Let \((X, \xi)\) be a convergence space. If \(F \subset X\), let
\[
e(F) = \{C = \text{cl}_\xi C \subset F\}
\]
denote the erected image of \(F\). If \(\mathcal{F} \in \mathbb{F}X\), let
\[
e^\xi \mathcal{F} = \{e(F) : F \in \mathcal{F}\}^\uparrow
\]
denote its erected filter on \([|\xi|, \$]|\).

It is readily seen (See e.g., [11, Section 8]) that if \(\mathcal{F} \in \mathbb{F}[|\xi|, \$]\) and \(\mathcal{G} \in \mathbb{F}[|\xi|, \$]\) then
\[
\text{rdc}^\xi (e^\xi \mathcal{F}) \geq \mathcal{F}
\]
and
\[
\mathcal{G} \geq e^\xi (\text{rdc}^\xi \mathcal{G}).
\]

Thus
\[
\text{rdc}^\xi \mathcal{G} = rdc^\xi \left(e^\xi (\text{rdc}^\xi \mathcal{G})\right)
\]
and \([|\xi, \$]|\) admits at every point a pavement consisting of saturated filters, that is, of filters of the form \(e^\xi (\text{rdc}^\xi \mathcal{G})\). In fact, if \(\mathbb{P}\) is a pavement of \([|\xi, \$]|\) at \(A\), then
\[
\mathbb{P}' := \left\{e^\xi (\text{rdc}^\xi \mathcal{G}) : \mathcal{G} \in \mathbb{P}\right\}
\]
is another pavement at \(A\), of the same cardinality and composed of saturated filters.

Note also [22] that
\[
\text{rdc}^\xi (e^\xi \mathcal{F}) = \text{int}^\xi_* \mathcal{F},
\]
where \(\xi^*\) is an Alexandroff topology on \(|\xi|\) defined by
\[
\text{cl}_{\xi^*} B = \{x \in |\xi| : \text{cl}_{\xi} \{x\} \cap B \neq \emptyset\} = \bigcup_{b \in B} \text{cl}_{\xi} \{b\}.
\]
The dual Alexandroff topology \(\xi^*\) is defined by
\[
\text{cl}_{\xi^*} B = \bigcup_{b \in B} \text{cl}_{\xi} \{b\}.
\]
A subset of \(|\xi|\) is \(\xi^*\)-open if and only if it is \(\xi^*\)-closed. Moreover,
\[
\text{Adh}_{\xi^*} B \iff \text{cl}_{\xi^*} B = \text{adho}_{\xi} B.
\]
We call a convergence \(\ast\)-regular (resp. \(\bullet\)-regular) if \(\lim_\xi \text{cl}_{\xi}^\xi \mathcal{F} = \lim_\xi \mathcal{F}\) (resp. \(\lim_\xi \text{cl}_{\xi^*}^\xi \mathcal{F} = \lim_\xi \mathcal{F}\)) for every filter \(\mathcal{F}\). In view of (5.4), we have:

**Lemma 23.** Let \(\mathcal{F}\) be a filter on \(|\xi|\). If \(\xi\) is \(\ast\)-regular then
\[
\text{adho}_{\xi} \mathcal{F} = \text{adho}_{\xi} \text{cl}_{\xi}^\xi \mathcal{F}
\]
and if \(\xi\) is \(\bullet\)-regular, then
\[
\text{adho}_{\xi} \mathcal{F} = \text{adho}_{\xi} \text{cl}_{\xi^*}^\xi \mathcal{F}.
\]

**Proof.** Since \(\text{cl}_{\xi}^\xi \mathcal{F} \leq \mathcal{F}\), \(\text{adho}_{\xi} \mathcal{F} \subset \text{adho}_{\xi} \text{cl}_{\xi}^\xi \mathcal{F}\). On the other hand, \(x \in \text{adho}_{\xi} \text{cl}_{\xi}^\xi \mathcal{F}\) if there is \(\mathcal{G} \# \text{cl}_{\xi}^\xi \mathcal{F}\), equivalently, \(\text{cl}_{\xi}^\xi \mathcal{G} \# \mathcal{F}\), with \(x \in \lim_\xi \mathcal{G}\). Since \(\xi\) is \(\ast\)-regular, \(x \in \lim_\xi \text{cl}_{\xi^*} \mathcal{G}\) and thus \(x \in \text{adho}_{\xi} \mathcal{F}\). The other equality is proved similarly. \(\square\)
Moreover,
\[ \text{cl}_\xi^* F \leq F \leq \text{rdc}_\xi^*(E F) = \text{int}_\xi^* F \]
and \( F \) is a reduced filter if and only if both inequalities are equalities. Note that in particular, \( \text{cl}_\xi^* F \) is a reduced filter and thus
\[ \text{cl}_\xi^* F = \text{rdc}_\xi^*(E \text{cl}_\xi^* F). \]

As a consequence,
\begin{equation}
\tag{5.5}
F \geq \text{rdc}_\xi^*(E \text{cl}_\xi^* F).
\end{equation}

In particular, if \( \xi \) is \( T_1 \), then \( \text{rdc}_\xi^*(E F) = F \).

5.2. Graph induced by a convergence and case of the \$dual\$. A convergence \( \xi \) determines a directed graph whose set of vertices is its underlying set \( X \), with the relation
\[ y \rightarrow x \iff x \in \lim_\xi \{ y \}^\top. \]

If the convergence is centered, then \( x \rightarrow x \), so that there is a loop at each vertex. If the convergence is topological then \( \rightarrow \) is a transitive relation, but in general, it is not. Consider the forward and backward neighborhoods in this graph:
\[ N^+(y) := \{ x : y \rightarrow x \} = \lim_\xi \{ y \}^\top \text{ and } N^-(y) := \{ x \in [\xi] : x \rightarrow y \}. \]

Let us call a point \( r \) of the graph a root if \( r \rightarrow x \) for all \( x \in X \), and an end if \( x \rightarrow r \) for all \( x \in X \). Let \( R(\xi) \) and \( E(\xi) \) denote the set of roots, and the set of ends, of the graph induced by \( \xi \), respectively. Of course, in many cases, \( R(\xi) = \emptyset \) and/or \( E(\xi) = \emptyset \), but not always. For instance \( R([\xi, 8]) = \{ \emptyset \} \) and \( E([\xi, 8]) = \{ [\xi] \} \).

Let
\[ y^+ := \{ x \in [\xi] : N^-(y) \cap N^-(x) \subset R(\xi) \}, \quad y^- := \{ x \in [\xi] : N^-(y) \cap N^-(x) \subset E(\xi) \}. \]

Given \( A \subset X \), consider
\[ \text{cl}_\xi^* A = \bigcap_{c \in \bigcap_{A \in A} a^+} c^+. \]

A closure operator \( c : 2^X \rightarrow 2^X \) satisfies \( A \subset c(A) \) for all \( A \subset X \), and \( c(A) \subset c(B) \) whenever \( A \subset B \). It is grounded if additionally \( c(\emptyset) = \emptyset \), and additive if \( c(A \cup B) = c(A) \cup c(B) \).

**Proposition 24.** Given a convergence \( \xi \), the induced operator \( \text{cl}_\xi^* \) is an idempotent (not necessarily additive) closure operator. It is grounded if and only if \( \xi \) has no root.

**Proof.** For every \( A \subset [\xi] \), we have \( A \subset \text{cl}_\xi^* A \) because if \( a_0 \in A \) and \( c \in \bigcap_{a \in A} a^+ \), then in particular, \( c \in a_0^+ \), equivalently, \( a_0 \in c^+ \).

If \( A \subset B \), then \( \bigcap_{b \in B} b^+ \subset \bigcap_{a \in A} a^+ \) and thus \( \text{cl}_\xi^* A \subset \text{cl}_\xi^* B \).

Note that
\begin{equation}
\tag{5.6}
\bigcap_{a \in \text{cl}_\xi^* A} a^+ = \bigcap_{a \in A} a^+.
\end{equation}

\[ \text{We do not introduce the dual notion using } y^-_\xi \text{ instead, only because we do not have any particular use for it here.} \]
so that $\text{cl}_{\xi^*} A$ is idempotent. To see (5.6), note that the inclusion $\subseteq$ is clear because $A \subseteq \text{cl}_{\xi^*} A$. On the other hand, if $c \in \bigcap_{a \in A} a^{x^*}$ and $x \in \text{cl}_{\xi^*} A$, then $x \in c^{x^*}$, equivalently, $c \notin x^{x^*}$, which shows the reverse inclusion.

Finally, $\text{cl}_{\xi^*}$ may not be grounded if $\xi$ has roots, for $\text{cl}_{\xi^*} \emptyset = R(\xi)$. Indeed, if $x \in \text{cl}_{\xi^*} \emptyset$ then $x \notin x^{x^*}$ for every $c \in X$, so that in particular $x \notin x^{x^*}$ and thus $x$ is a root. Clearly, every root is in $c^{x^*}$ for every $c \in X$.

We call a convergence $\xi^{\downarrow}$-regular if $\lim_{\xi} F = \lim_{\xi^*} \text{cl}_{\xi} F$ for every $F \in F[\xi]$.

Of course, if $\xi$ is $T_1$ and has more than one point, then $N^{-}(y) = N^{<}(y) = \{ y \}$ and $E(\xi) = R(\xi) = \emptyset$ for every $y$ so that $y^{x^*} = y^{x^*} = X \setminus \{ y \}$. Thus

$$\bigcap_{a \in A} a^{x^*} = \bigcap_{a \in A} \{ a \}^{c} = \left( \bigcup_{a \in A} \{ a \} \right)^{c} = A^{c},$$

and therefore $\text{cl}_{\xi} A = \bigcap_{c \subseteq X} c^{x^*} = A$.

In the upper Kuratowski convergence, note that

$$\lim_{\xi} C^{\uparrow} = \{ D \in \mathbb{C}_{\xi} : C \subseteq D \},$$

which is, $C \rightarrow D \iff C \subseteq D$.

Hence, $N^{-}(C) = \{ D \in \mathbb{C}_{\xi} : C \subseteq D \}$, $N^{<}(C) = \{ D \in \mathbb{C}_{\xi} : D \subseteq C \}$, so that $C_{x^*} = \{ D \in \mathbb{C}_{\xi} : C \cap D = \emptyset \}$

and $C_{x^*} = \{ D \in \mathbb{C}_{\xi} : C \cup D = X \} = \{ D \in \mathbb{C}_{\xi} : X \setminus C \subseteq D \}$.

Moreover, given $G \subseteq \mathbb{C}_{\xi}$,

$$\text{cl}_{\xi} G = \bigcap_{C \subseteq \bigcap_{D \in G} D^{x^*}} C^{x^*} = \{ F \in \mathbb{C}_{\xi} : (C \cap \text{rdc } G = \emptyset) \Rightarrow C \cap F = \emptyset \}.$$ 

In other words, $F \in \text{cl}_{\xi} G$ if and only if $O(\text{rdc } G) \subseteq O(F)$, if and only if $\ker O(\text{rdc } G) = \{ x : \text{cl}_{\xi} \{ x \} \cap \text{rdc } G \neq \emptyset \} = \text{cl}_{\xi^*} (\text{rdc } G)$,

that is,

$$\text{(5.7)} \quad \text{cl}_{\xi} G = \text{e} (\text{cl}_{\xi^*} (\text{rdc } G)).$$

A convergence is reciprocal if its induced graph is symmetric, that is, if

$$x \rightarrow y \Rightarrow y \rightarrow x.$$

In a reciprocal convergence $\xi$, the two topologies $\xi^*$ and $\xi^*$ coincide so that $\text{cl}_{\xi^*} (\text{rdc } G) = \text{rdc } G$ because $\text{rdc } G$ is always $\xi^*$-open. Thus

**Proposition 25.** If $\xi$ is reciprocal, then for every $G \subseteq C_{\xi}$,

$$\text{cl}_{\xi} G = \text{e} (\text{rdc } G).$$

6. **Duality theorems**

Recall (e.g., [11], [12]) that the epitopological reflection $\text{Epi } \xi$ of a convergence $\xi$ is given by the initial convergence for the point evaluation map $i : [\xi] \rightarrow [[\xi], \xi]$, defined by $i(x) = ev(x, \cdot)$, and for the convergence $[[\xi], \xi]$. Since $[\xi, \xi] = [\text{Epi } \xi, \xi]$ and $\text{Epi } \xi$ is a always a $\star$-regular pseudotopology with closed limits (e.g., [12] Proposition XVII.4.2]), the assumption that $\xi$ be $\star$-regular in the two duality theorems below is natural and not much to ask.
Theorem 26. Let $\xi$ be a $*$-regular convergence. Then
\[ \text{ucompl}(\xi) = p((\xi, \emptyset), \emptyset). \]

Proof. Let $\mathbb{D}$ be a strongly cocomplete collection of non-adherent filter on $|\xi|$. We claim that the collection
\[ \{ e^2D : D \in \mathbb{D} \} \]
is a pavement of $[\xi, \emptyset]$ at $\emptyset$. That $\emptyset \in \text{lim}_{[\xi, \emptyset]} e^2D$ for every $D \in \mathbb{D}$ follows from (5.1) and $\text{adh}_{\xi}D = \emptyset$. Hence, any choice of one ultrafilter $U$ forms a pavement of $[\xi, \emptyset]$ with $\text{rdc}_{\xi}D = \emptyset$, so that there is $D \in \mathbb{D}$ with $\text{rdc}_{\xi}D \geq D$, because $\mathbb{D}$ is strongly cocomplete. Hence,
\[ G \geq e^2(\text{rdc}_{\xi}D) \geq e^2D \]
by (5.2).

Conversely, let $\mathbb{P}$ be a pavement of $[\xi, \emptyset]$ at $\emptyset$. Let
\[ \mathbb{D} = \{ \text{rdc}_{\xi}P : P \in \mathbb{P} \}. \]
Because $\emptyset \in \text{lim}_{[\xi, \emptyset]} P$, $\text{adh}_{\xi} \text{rdc}_{\xi}P = \emptyset$ for every $P \in \mathbb{P}$. If moreover $\mathcal{F}$ is another filter on $|\xi|$ with $\text{adh}_{\xi}\mathcal{F} = \emptyset$, then $\text{adh}_{\xi}\text{cl}_{\xi}^2\mathcal{F} = \emptyset$ by Lemma (2.3) and thus $\emptyset \in \text{lim}_{[\xi, \emptyset]} e^2(\text{cl}_{\xi}^2\mathcal{F})$ by (5.1). Therefore, there is $P \in \mathbb{P}$ with $e^2(\text{cl}_{\xi}^2\mathcal{F}) \geq P$, and thus $\text{rdc}_{\xi}(e^2(\text{cl}_{\xi}^2\mathcal{F})) \geq \text{rdc}_{\xi}P$. Since $\mathcal{F} \geq \text{rdc}_{\xi}(e^2(\text{cl}_{\xi}^2\mathcal{F}))$ by (5.2), we conclude that $\mathbb{D}$ is strongly cocomplete. □

We call a family $\mathbb{P}$ of filters converging to $x$ a $\dagger$-pseudopavement at $x$ if for every $\mathcal{F} \in \mathbb{P}X$ with $x \in \text{lim}_{\xi}\mathcal{F}$ there is $P \in \mathbb{P}$ with $P \# \text{cl}_{\xi}^2\mathcal{F}$. Let $pp^\dagger(\xi, x)$ denote the smallest cardinality of a $\dagger$-pseudopavement at $x$. Of course, every pseudopavement is a $\dagger$-pseudopavement because $\text{cl}_{\xi}^2\mathcal{F} \leq \mathcal{F}$ for every $\mathcal{F}$, so that $pp^\dagger(\xi, x) \leq pp(\xi, x)$. If $\xi$ is $T_1$ then $\text{cl}_{\xi}^2A = A$ for all $A$ and thus $pp^\dagger(\xi, x) = pp(\xi, x)$. On the other hand, without any separation, the inequality may be strict:

Example 27 (We may have $pp^\dagger(\xi) = 1$ and $pp(\xi, x)$ arbitrary large). Consider the ultrafilter convergence $\xi$ of the antidiscrete topology of an infinite set $X$. Then every point of $X$ is a root (and an end) and thus $\text{cl}_{\xi}^2A = X$ for every $A \subset X$. As a result, any choice of one ultrafilter $\mathcal{U}_0$ forms a $\dagger$-pseudopavement (at any point), for $\text{lim}_{\xi}\mathcal{U}_0 = X$ and for any other ultrafilter $\mathcal{U}$, we have $\text{cl}_{\xi}^2\mathcal{U} = \{ X \}$ so that $\mathcal{U}_0 \# (\text{cl}_{\xi}^2\mathcal{U})$. Hence $pp^\dagger(\xi, x) = 1$ at every $x \in X$. On the other hand, $pp(\xi, x) \geq |X|$. Indeed a pseudopavement (at any point $x_0$) is composed of ultrafilters on $X$, and for every $y \in X$, $x_0 \in \text{lim}\{ y \}^\dagger$ but the only ultrafilter meshing with $\{ y \}^\dagger$ is $\{ y \}^\dagger$. Thus, a pseudopavement needs to contain at least all principal ultrafilters and is thus of cardinality at least $|X|$.

Theorem 28. Let $\xi$ be a $*$-regular convergence. Then
\[ \text{compl}(\xi) = pp^\dagger([\xi, \emptyset], \emptyset). \]

Proof. Let $\mathbb{D}$ be a cocomplete family of non-adherent filters on $|\xi|$, and let $\mathbb{P} = \{ e^2D : D \in \mathbb{D} \}$. Each element of $\mathbb{P}$ converges to $\emptyset$ for $[\xi, \emptyset]$. Let $\mathcal{G}$ be another filter with $\emptyset \in \text{lim}_{[\xi, \emptyset]} \mathcal{G}$, that is, $\text{adh}_{\xi} \text{rdc}_{\xi}^2 \mathcal{G} = \emptyset$. Since $\mathbb{D}$ is cocomplete, there is $D \in \mathbb{D}$ with $D \# \text{rdc}_{\xi}^2 \mathcal{G}$, so that $(e^2D) \# (e^2(\text{rdc}_{\xi}^2 \mathcal{G}))$. In view of (5.7), $e^2(\text{rdc}_{\xi}^2 \mathcal{G}) \geq e^2 \text{cl}_{\xi}^2 \mathcal{G}$, and we conclude that $\mathbb{P}$ is a $\dagger$-pseudopavement of $[\xi, \emptyset]$ at $\emptyset$. 
Conversely, let \( \mathcal{P} \) be a \( \uparrow \)-pseudopavement of \( [\xi, \mathcal{S}] \) at \( \emptyset \). Then

\[
\mathcal{D} = \left\{ \text{cl}^\uparrow_{\xi} (\text{rdc}^\downarrow \mathcal{P}) : \mathcal{P} \in \mathcal{P} \right\}
\]

is a cocomplete collection of non-adherent filters on \( [\xi] \). Indeed, every \( \text{rdc}^\downarrow \mathcal{P} \in \mathcal{D} \) is non-adherent because \( \emptyset \in \lim_{[\xi, \mathcal{S}]} \mathcal{P} \) and thus every \( \text{cl}^\uparrow_{\xi} (\text{rdc}^\downarrow \mathcal{P}) \) is non-adherent by *-regularity, using Lemma \[23\]. Moreover, if \( \mathcal{F} \) is a non-adherent filter on \( [\xi] \) then \( \emptyset \in \lim_{[\xi, \mathcal{S}]} e^2 \mathcal{F} \), so that there is \( \mathcal{P} \in \mathcal{P} \) with \( \mathcal{P} \# \text{cl}^\uparrow_{\xi} (e^2 \mathcal{F}) \). Moreover, \( \text{cl}^\uparrow_{\xi} (e^3 \mathcal{F}) = e^3 \left( \text{cl}^\uparrow_{\xi} \left( \text{rdc}^\downarrow (e^3 \mathcal{F}) \right) \right) \) by \([5.7]\). Then \( (\text{rdc}^\downarrow \mathcal{P}) \# \text{cl}^\uparrow_{\xi} (\text{rdc}^\downarrow (e^3 \mathcal{F})) \)

equivalently, \( \text{cl}^\uparrow_{\xi} (\text{rdc}^\downarrow \mathcal{P}) \# \text{rdc}^\downarrow (e^3 \mathcal{F}) \).

Since \( \text{rdc}^\downarrow (e^3 \mathcal{F}) \geq \mathcal{F} \), we conclude that \( \text{cl}^\uparrow_{\xi} (\text{rdc}^\downarrow \mathcal{P}) \# \mathcal{F} \).

As a result, we see that \[8\] Theorem 10.1 stating

\[ \text{compl}(\xi) = p([\xi, \mathcal{S}], \emptyset) \]

is erroneous because, as we have seen, there are countably complete topological spaces that are not countably ultracomplete, so that

\[ \text{compl}(\xi) = \omega < \text{ucompl}(\xi) = p([\xi, \mathcal{S}], \emptyset). \]

Yet the statement is corrected by the pair of theorems \[26\] and \[28\].

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