A Lower Density Operator for the Borel Algebra

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Abstract. We prove that the existence of a Borel lower density operator (a Borel lifting) with respect to the $\sigma$-ideal of countable sets, for an uncountable Polish space, is equivalent to $\text{CH}$. One of the implications is known (due to K. Musial) and the remaining implication is derived from a general abstract result dealing with the negation of $\text{GCH}$. We observe that there is no lower density Borel operator with respect to the $\sigma$-ideal of countable sets, whose range is of bounded level in the Borel hierarchy.

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1. A General Negative Result on Lower Densities

Let $S$ be a $\sigma$-algebra of subsets of a nonempty set $X$ and let $J \subseteq S$ be a $\sigma$-ideal. We write $A \sim B$ whenever the symmetric difference $A \triangle B$ is in $J$. This is an equivalence relation on $S$ and its quotient space is denoted by $S/J$. A mapping $\Phi: S \to S$ is called a lower density operator (respectively, a lifting) with respect to $J$ if it satisfies the following conditions (1)–(4) (respectively, (1)–(5)):

1. $\Phi(X) = X$ and $\Phi(\emptyset) = \emptyset$,
2. $A \sim B \implies \Phi(A) = \Phi(B)$ for every $A, B \in S$,
3. $A \sim \Phi(A)$ for every $A \in S$,
4. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for every $A, B \in S$,
5. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for every $A, B \in S$. 

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The problem of the existence of liftings together with their various applications were widely discussed in the monograph [5] and in the later survey [14]. Lower density operators play an important role in constructions of density like topologies; see [4, 11, 15]. Several solutions of interesting problems on liftings require some set-theoretic tools. Our paper also use such methods.

We will start from a general negative result dealing with the negation of generalized continuum hypothesis (GCH). Then we will apply it to obtain a theorem stating the existence of a lower density operator on a Borel algebra in an uncountable Polish space, with respect to the \(\sigma\)-ideal of countable sets.

Let \(\kappa\) and \(\lambda\) be uncountable cardinals with \(\kappa \leq \lambda\). Let \(|X| = \lambda\). A family \(F\) of subsets of \(X\) is called \(\kappa\)-additive if \(\bigcup G \in F\) whenever \(G \subseteq F\) and \(|G| < \kappa\). Note that an \(\omega_1\)-additive family is simply a \(\sigma\)-ideal. An algebra of subsets of \(S \times X\) is called a cross-type algebra if it contains all sets of the form \(\{x\} \times X\) and \(X \times \{x\}\) for \(x \in X\). Of course, all singletons \(\{(x, y)\}\), with \(x, y \in X\), belong to a cross-type algebra. Let \(J_\kappa\) denote the ideal of all subsets of \(S \times X\) of cardinality \(< \kappa\). Note that \(J_\kappa\) is contained in every \(\kappa\)-additive cross-type algebra of subsets of \(S \times X\).

**Theorem 1.** Assume that \(\kappa\) and \(\lambda\) are infinite cardinals such that \(\kappa^+ < \lambda\).

**Proof.** Fix a set \(X\) with \(|X| = \lambda\). Then for every \(\kappa^+\)-additive cross-type algebra \(S\) of subsets of \(X \times X\), there is no lower density operator \(\Phi : S \rightarrow S\) with respect to the \(\sigma\)-ideal \(J_{\kappa^+}\). In particular, this is true if \(\lambda := 2^\kappa\) and we assume that \(\kappa^+ < 2^\kappa\) (a part of \(\neg\)GCH).

**Proof of Claim.** There is \(x \in X\) such that \(|\{\beta < \lambda : x \in \pi_2[Q_\beta]\}| \geq \kappa^+\).

**Proof of Claim.** Suppose that \(|\{\beta < \lambda : x \in \pi_2[Q_\beta]\}| < \kappa^+\) for each \(x \in X\). Let

\[
L_\alpha := \{\beta < \lambda : x_\alpha \in \pi_2[Q_\beta]\} \quad \text{for} \ \alpha < \lambda.
\]

Then \(|\bigcup_{\alpha < \kappa^+} L_\alpha| \leq \kappa^+\) by our supposition. Since \(\kappa^+ < \lambda\), the set \(\lambda \setminus \bigcup_{\alpha < \kappa^+} L_\alpha\) is nonempty (of cardinality \(\lambda\)). Take \(\xi \in \lambda \setminus \bigcup_{\alpha < \kappa^+} L_\alpha\). Then

\[
\{x_\alpha : \alpha < \kappa^+\} \subseteq X \setminus \pi_2[Q_\xi] = \pi_2[P_\xi \setminus Q_\xi] \subseteq \pi_2[P_\xi \setminus Q_\xi]
\]

which gives a contradiction since \(|\pi_2[P_\xi \setminus Q_\xi]| \leq |P_\xi \setminus \Phi(P_\xi)| < \kappa^+\) by (3).

Take \(x \in X\) as in the Claim. Consider \(P := X \times \{x\}\). Then \(|P \cap P_\alpha| = 1\) and \(\Phi(P) \cap Q_\alpha = \Phi(P) \cap \Phi(P_\alpha) = \Phi(P \cap P_\alpha) = \emptyset\) for each \(\alpha < \lambda\), by (4), (2) and (1). Therefore \(\Phi(P) \cap \bigcup_{\alpha < \lambda} Q_\alpha = \emptyset\). On the other hand, \(|P \cap \bigcup_{\alpha < \lambda} Q_\alpha| \geq \ldots\)
\[ \kappa^+ \] by the choice of \( x \). Thus \( P \setminus \Phi(P) \notin J_{\kappa^+} \) which yields a contradiction with (3).

\[ \square \]

2. A Theorem on the Existence of Borel Liftings

If \( S \) is the \( \sigma \)-algebra of Borel sets in a given Hausdorff space, then the respective operator \( \Phi \) satisfying conditions (1)–(5) is called a Borel lifting. Note that von Neumann and Stone [9] proved the existence of a lifting for a Borel measure space on \([0,1]\) under the assumption of the continuum hypothesis (CH). A simple proof of the same result was then given by Musial [8]. This was later generalized by Mokobodzki [6] and Fremlin [3] who showed that, subject to CH, any \( \sigma \)-finite measure space with the measure algebra of cardinality \( \leq \omega_2 \) has a lifting. On the other hand, Shelah [12] proved that \( 2^{\omega} = \omega_2 \) is consistent with the nonexistence of Borel lifting for the Lebesgue measure algebra. Later in [2], it was shown that \( 2^{\omega} = \omega_2 \) is consistent with the existence of Borel lifting for the Lebesgue measure algebra.

We focus on Borel liftings in the following case. We consider the \( \sigma \)-algebra \( \mathcal{B}(X) \) of Borel subsets of an uncountable Polish space \( X \) and the \( \sigma \)-ideal \( \{X\} \subseteq \omega \) of all countable subsets of \( X \). Since any two uncountable Borel subsets of Polish spaces are Borel isomorphic [13, Theorem 3.3.13], if we seek a lifting from \( \mathcal{B}(X) \) into \( \mathcal{B}(X) \) with respect to \( \{X\} \subseteq \omega \), it does not matter which Polish space is considered.

**Theorem 2.** For an uncountable Polish space \( X \), the following conditions are equivalent:

(i) CH;

(ii) there exists a lifting \( \Phi: \mathcal{B}(X) \to \mathcal{B}(X) \) with respect to \( \{X\} \subseteq \omega \);

(iii) there exists a lower density operator \( \Phi: \mathcal{B}(X) \to \mathcal{B}(X) \) with respect to \( \{X\} \subseteq \omega \).

**Proof.** Implication (i) \( \implies \) (ii) follows from [8, Theorem 1] where it is shown that, for any \( \sigma \)-algebra \( S \) and any \( \sigma \)-ideal \( J \subseteq S \), if \( |S/J| \leq \omega_1 \), there exists a lifting from \( S \) to \( S \), with respect to \( J \).

Implication (ii) \( \implies \) (iii) is obvious.

To prove (iii) \( \implies \) (i) assume \( \neg \text{CH} \). We work with \( \mathbb{R} \times \mathbb{R} \) as a Polish space. It suffices to apply Theorem 1 where \( \kappa := \omega \) and \( \lambda := 2^{\omega} = |\mathbb{R}| \). Then \( J_{\omega_1} \) consists of all countable subsets of the plane and the role of a cross-type \( \sigma \)-algebra is played by \( \mathcal{B}(\mathbb{R} \times \mathbb{R}) \).

Note that implication (iii) \( \implies \) (ii) follows from the final part of [11] or from [4, Theorem 2.8]. In fact, the existence of a lower density operator implies the existence of a lifting in a general case.
Theorem 2 answers a question posed by Jacek Hejduk during his invited talk given on the Conference on Real Function Theory in Stará Lesná in September 2016. He asked about the existence of a lower density operator on $\mathcal{B}(\mathbb{R})$ with respect to $[\mathbb{R}]^{\leq \omega}$.

3. Nonexistence of a Range of Bounded Borel Level

It can happen that the values of a lower density operator $\Phi: S \to S$ with respect to $J \subseteq S$ are located in a proper subfamily of $S$. Denote by $\mathcal{L}$ the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Recall a canonical lower density operator $\Phi: \mathcal{L} \to \mathcal{L}$ with respect to the $\sigma$-ideal of null sets. Namely, $\Phi$ is given by

$$
\Phi(A) := \left\{ x \in \mathbb{R} : \lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1 \right\}
$$

for $A \in \mathcal{L}$ where $\lambda$ stands for Lebesgue measure on $\mathbb{R}$ (see [10, 15]). It is shown in [15] that the values of $\Phi$ hit into the Borel class $\Pi_3^0$ (that is, $F_{\omega, 3}$ in the classic notation; cf. [13, 3.6]). In [1], an exact Borel complexity of sets $\Phi(A)$ was studied where, instead of $\mathbb{R}$, the Cantor space with the respective measure is considered.

In the above context, we return to lower density operators $\Phi: \mathcal{B}(X) \to \mathcal{B}(X)$ which exist under CH by Theorem 2. We may ask whether the range of $\Phi$ can be contained in $\Sigma_\alpha^0$ for some $\alpha < \omega_1$ (we then say that this range is of bounded Borel level). We will show that the answer is negative. Pick $\mathbb{R}$ as a Polish space and let $\mathcal{B} := \mathcal{B}(\mathbb{R})$.

**Proposition 3.** There is no lower density operator $\Phi: \mathcal{B} \to \mathcal{B}$ with respect to $[\mathbb{R}]^{\leq \omega}$, whose range is of bounded Borel level.

**Proof.** Suppose that there exists $\Phi$ with the range $\Phi(\mathcal{B}) \subseteq \Sigma_\alpha^0$ for some $\alpha < \omega_1$. We may assume that $\alpha \geq 3$. Fix $A \subseteq \mathbb{R}$ with $A \in \Pi_\alpha^0 \setminus \Sigma_\alpha^0$ (cf. [13, Corollary 3.6.8]). We will show that $\Phi(A) \notin \Sigma_\alpha^0$ which yields a contradiction. Suppose that $\Phi(A) \in \Sigma_\alpha^0$. Since $A \bigtriangleup \Phi(A)$ is countable, we have $A = (\Phi(A) \cap B^c) \cup C$ for some countable sets $B, C \subseteq \mathbb{R}$. Then $B^c \in \Pi_3^0 \subseteq \Sigma_\alpha^0$ and $C \in \Sigma_2^0 \subseteq \Sigma_\alpha^0$; see [13, Proposition 3.6.1]. Consequently, $A \in \Sigma_\alpha^0$ which is impossible. \qed

Finally, let us mention another negative result obtained in the recent paper [7]. Namely, it is proved that there is no reasonably definable selector that chooses representatives for the equivalence relation on the Borel sets of having countable symmetric difference.

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