Asymptotic Approximations of Apostol-Genocchi Numbers and Polynomials

Cristina B. Corcino

1 Research Institute for Computational Mathematics and Physics, Cebu Normal University, 6000 Cebu City, Philippines
2 Mathematics Department, Cebu Normal University, 6000 Cebu City, Philippines

Abstract. Asymptotic approximations of the Apostol-Genocchi numbers and polynomials are derived using Fourier series and ordering of poles of the generating function. Asymptotic formulas for the Apostol-Euler numbers and polynomials are obtained as consequence. Asymptotic formulas for special cases which include the Genocchi numbers and polynomials are also explicitly stated.

2020 Mathematics Subject Classifications: 11B68, 41A60

Key Words and Phrases: Asymptotic approximations, Genocchi polynomials, Bernoulli polynomials, Euler polynomials, Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Apostol-Genocchi polynomials

1. Introduction

The Apostol-Genocchi polynomials \( G_n(x; \lambda) \) are defined by the generating function

\[
\frac{2te^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!},
\]

where \(|t| < \pi\) when \( \lambda = 1 \) and \(|t + \log \lambda| < \pi\) when \( \lambda \neq 1 \). When \( \lambda = 1 \), the above equation gives the generating function of the Genocchi polynomials [3].

When \( x = 0 \), (1.1) reduces to the generating function of the Apostol-Genocchi numbers \( G_n(0; \lambda) \) given by

\[
\frac{2t}{\lambda e^t + 1} = \sum_{n=0}^{\infty} G_n(0; \lambda) \frac{t^n}{n!}.
\]

For \( \lambda \) not zero, the set of poles of the generating function (1.1) is

\[
T_\lambda := \{(2k + 1)\pi i - \log \lambda : k \in \mathbb{Z}\},
\]

DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.3976

Email address: corcinoc@cnu.edu.ph (C. Corcino)
which is also the set of poles of (1.2), where the logarithm is taken to be the principal branch.

Bayad [2] and Luo [13] derived Fourier series of Apostol-Genocchi polynomials expressed in terms of these poles. The Fourier series they obtained is given in the next section. Fourier expansion of higher-order Apostol-Genocchi polynomials was derived in [4] and was shown to be reducible to those obtained in [2] and [13] when the order is 1.

New identities involving the Apostol-Genocchi polynomials were established in [9]. Some generalizations and properties of these polynomials were presented in [14]. Multiplication and explicit recursive formulas of higher-order Apostol-Genocchi polynomials were obtained in [12]. A new generalization of Apostol type Hermite-Genocchi polynomials is studied in [1] while products of the Apostol-Genocchi polynomials were studied in [10]. Moreover, the higher-order convolutions of these polynomials using generating-function methods and summation-transform techniques were established in [11].

Inspired by the work of Kim and Kim [7], a new class of the Frobenius-Genocchi polynomials was considered in [6] by means of the polyexponential function and new relations and properties were obtained. New relations on q-Genocchi polynomials where the relations were stated by symmetric group of degree n were done in [5].

Navas, Ruiz and Varona [15] obtained asymptotic estimates of the Apostol-Bernoulli and Apostol-Euler numbers and polynomials and further analyzed the asymptotic behavior of the Apostol-Bernoulli polynomials in detail. The starting point of their analysis is the Fourier series of the polynomials on the closed interval [0, 1] followed by ordering the poles of the generating function.

In this paper, asymptotic approximations of the Apostol-Genocchi numbers and polynomials for \( \lambda \in \mathbb{C}\{0\} \) are obtained. The method used in [15] is applied to the Apostol-Genocchi numbers and polynomials to obtain asymptotic formulas of these numbers and polynomials. A more detailed proof of the results is provided so as to reach a bigger group of readers. Asymptotic formulas of Genocchi numbers and Euler numbers are obtained as special cases. Asymptotic formulas of the Apostol-Euler numbers and Apostol-Euler polynomials are also derived. The results in this paper will complete the results of [15] as the latter considered only the Apostol-Bernoulli and Apostol-Euler polynomials. Moreover, the results can be used as check formulas of those in [15].

2. Asymptotic Approximations

Fourier series of the Apostol-Genocchi polynomials in terms of the poles in \( T_\lambda \) is given in the following theorem.

**Theorem 2.1.** ([2], [13]) Let \( \lambda \in \mathbb{C}\{0\} \). For \( n \geq 1 \), \( 0 \leq x \leq 1 \),

\[
\frac{G_n(x; \lambda)}{n!} = \frac{2}{\lambda x} \sum_{k \in \mathbb{Z}} \frac{e^{(2k+1)\pi i x}}{[(2k+1)\pi i - \log \lambda]^n},
\]

where the logarithm is taken to be the principal branch.
Taking \(x = 0\) in (2.1) gives the Fourier series of the Apostol-Genocchi numbers given by
\[
G_n(0; \lambda) \equiv \frac{2}{n!} \sum_{k \in \mathbb{Z}} \frac{1}{(2k+1) \pi i - \log \lambda}|n|, \tag{2.2}
\]
where the logarithm is taken to be the principal branch.

Proceeding as in [15], ordering of the poles of the generating function (1.1) is done in the following lemma.

**Lemma 2.2.** Let \(u_k = (2k+1) \pi i - \log \lambda\) with \(k \in \mathbb{Z}\), \(\lambda \in \mathbb{C}\backslash\{0\}\) and \(\gamma = (\log \lambda)/2\pi i\), where the logarithm is taken to be the principal branch.

1. If \(\text{Im} \lambda > 0\) then \(0 < \text{Re} \gamma < \frac{1}{2}\) and for \(k \geq 1\),
\[
|u_0| < |u_1| < |u_2| < \cdots < |u_k| < \cdots \tag{2.3}
\]
2. If \(\text{Im} \lambda < 0\) then \(-\frac{1}{2} < \text{Re} \gamma < 0\) and for \(k \geq 1\),
\[
|u_1| < |u_2| < |u_3| < \cdots < |u_k| < \cdots. \tag{2.4}
\]
3. If \(\lambda > 0\) (positive real number), then \(\text{Re} \gamma = 0\), and for \(k \geq 1\),
\[
|u_0| = |u_1| = |u_2| = \cdots < |u_k| = |u_{k+1}| < \cdots. \tag{2.5}
\]
4. If \(\lambda < 0\) (negative real number), then \(\text{Re} \gamma = \frac{1}{2}\), and for \(k \geq 1\),
\[
|u_0| < |u_1| = |u_2| = \cdots < |u_k| = |u_{k+1}| < \cdots. \tag{2.6}
\]
Moreover, \(|u_k| \geq 2\pi(|k| - 1)\) if \(|k| \geq 1\).

**Proof.** With the logarithm taken to be the principal branch, \(\gamma\) (as a function of \(\lambda\)) maps \(\lambda \in \mathbb{C}\backslash\{0\}\) to the strip \(-\frac{1}{2} < \text{Re} \gamma \leq \frac{1}{2}\) (see [15]). To see this write
\[
\gamma = \frac{\theta}{2\pi} - i \frac{\ln |\lambda|}{2\pi},
\]
from which we have
\[
\text{Re} \gamma = \frac{\theta}{2\pi} \quad \text{and} \quad \text{Im} \gamma = -\frac{\ln |\lambda|}{2\pi}.
\]
With \(-\pi < \theta \leq \pi\),
\[
-\frac{\pi}{2\pi} \leq \text{Re} \gamma = \frac{\theta}{2\pi} \leq \frac{\pi}{2\pi} \Rightarrow -\frac{1}{2} < \text{Re} \gamma \leq \frac{1}{2}.
\]
where $\Re \gamma = 0$ when $\lambda > 0$ and $\Re \gamma = \frac{1}{2}$ when $\lambda < 0$.

If $\Im \lambda > 0$, then $0 < \theta < \pi$, hence $0 < \Re \gamma < \frac{1}{2}$. If $\Im \lambda < 0$, then $-\pi < \theta < 0$, hence $-\frac{1}{2} < \Re \gamma < 0$.

To verify the chains in (2.3), (2.4), (2.5), (2.6), let $x = \Re \gamma$ and $y = \Im \gamma$. Then for $k \in \mathbb{Z}$,

$$u_k = 2\pi \sqrt{(k + \frac{1}{2} - x)^2 + y^2}.$$

a) If $\Im \lambda > 0$, then $0 < x < \frac{1}{2}$ and

$$|u_0| = 2\pi \sqrt{\left(\frac{1}{2} - x\right)^2 + y^2}$$
$$|u_1| = 2\pi \sqrt{\left(\frac{3}{2} - x\right)^2 + y^2}$$
$$|u_2| = 2\pi \sqrt{\left(\frac{5}{2} - x\right)^2 + y^2}$$
$$|u_{-1}| = 2\pi \sqrt{\left(-\frac{1}{2} - x\right)^2 + y^2} = 2\pi \sqrt{\left(\frac{1}{2} + x\right)^2 + y^2}$$
$$|u_{-2}| = 2\pi \sqrt{\left(-\frac{3}{2} - x\right)^2 + y^2} = 2\pi \sqrt{\left(\frac{3}{2} + x\right)^2 + y^2}$$
$$|u_{-3}| = 2\pi \sqrt{\left(-\frac{5}{2} - x\right)^2 + y^2} = 2\pi \sqrt{\left(\frac{5}{2} + x\right)^2 + y^2}$$
$$|u_3| = 2\pi \sqrt{\left(\frac{7}{2} - x\right)^2 + y^2}$$

\[ \vdots \]

From which one can see that the order of magnitude of $u_k$, $k \in \mathbb{Z}$ given in (2.3) holds.

b) The second case can be derived similarly.

The last two cases are belonging to the case $\Im \lambda = 0$. This means that $\lambda$ is a real number which is either positive or negative but not zero. Hence the cases $c$ and $d$.

c) If $\lambda > 0$, then $\Re \gamma = 0$. For $k \geq 0$,

$$|u_k| = 2\pi \sqrt{(k + \frac{1}{2})^2 + y^2}.$$
In particular,

\[
|u_0| = 2\pi \sqrt{\left(\frac{1}{2}\right)^2 + y^2}
\]

\[
|u_1| = 2\pi \sqrt{\left(1 + \frac{1}{2}\right)^2 + y^2}
\]

\[
|u_{-1}| = 2\pi \sqrt{\left(-1 + \frac{1}{2}\right)^2 + y^2}
\]

\[
|u_2| = 2\pi \sqrt{\left(2 + \frac{1}{2}\right)^2 + y^2}
\]

\[
|u_{-2}| = 2\pi \sqrt{\left(-2 + \frac{1}{2}\right)^2 + y^2}
\]

\[
|u_3| = 2\pi \sqrt{\left(3 + \frac{1}{2}\right)^2 + y^2}
\]

From which we have the chain

\[
|u_0| = |u_{-1}| < |u_1| = |u_{-2}| < |u_2| < \cdots < |u_k| = |u_{-(k+1)}| < |u_{k+1}| < \cdots,
\]

which is exactly (2.5).

d) If \( \lambda < 0, \ \theta = \pi, \) hence \( x = \frac{1}{2}. \) For \( k \geq 0, \)

\[
|u_k| = 2\pi \sqrt{k^2 + y^2} = |u_{-k}|,
\]

d from which it can be observed easily that

\[
|u_0| < |u_1| = |u_{-1}| < |u_2| = |u_{-2}| < |u_3| = |u_{-3}| < \cdots < |u_k| = |u_{-k}| < \cdots,
\]

which is exactly the chain in (2.6).

Moreover,

\[
|u_k| = 2\pi \left| k + \frac{1}{2} - \gamma \right|
\]

\[
= 2\pi \sqrt{\left(k + \frac{1}{2} - x\right)^2 + y^2}
\]

\[
\geq 2\pi \sqrt{\left(k + \frac{1}{2} - x\right)^2}
\]
\[ = 2\pi \left| k + \frac{1}{2} - x \right|, \text{ with } -\frac{1}{2} \leq x \leq \frac{1}{2} \]
\[ = 2\pi \left| k - \left( x - \frac{1}{2} \right) \right| \]
\[ \geq 2\pi \left( |k| - \left| x - \frac{1}{2} \right| \right) \]
\[ \geq 2\pi \left( |k| - \left| \frac{1}{2} - x \right| \right) \]
\[ \geq 2\pi \left( |k| - 1 \right). \]

An asymptotic expansion of the Apostol-Genocchi numbers \( G_n(0; \lambda) \) is given in the next theorem.

**Theorem 2.3.** Given \( \lambda \in \mathbb{C} \setminus \{0\} \), let \( H \) be a finite subset of \( T_\lambda \) satisfying
\[
\max \{ |u| : u \in H \} < \min \{ |u| : u \in T_\lambda \setminus H \} := \nu.
\]
For all integers \( n \geq 2 \),
\[
\frac{G_n(0; \lambda)}{n!} = 2 \sum_{u \in H} \frac{1}{u^n} + O(\nu^{-n}).
\]

**Proof.** Write the series in (2.2) as \( \sum_k \frac{1}{(\mu_k)^n} \). By Lemma 2.2 we can relabel the set of poles in increasing order of magnitude as
\[
|\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_M| \leq \cdots.
\]
Since \( |\mu_k| \geq 2\pi(|k| - 1) \), for \( k \geq 2 \), the series \( \sum_k \frac{1}{(\mu_k)^n} \) is absolutely convergent for \( n \geq 2 \).

For any \( M > 2 \), the tail of the series is
\[
\sum_{k=M+1}^{\infty} \frac{1}{(\mu_k)^n} = \frac{1}{|\mu_{M+1}|^n} \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^n.
\]
Since for \( k > M + 1 \), \( \left| \frac{\mu_{M+1}}{\mu_k} \right| \leq 1 \), we have \( \left| \frac{\mu_{M+1}}{\mu_k} \right|^n \leq \frac{\mu_{M+1}}{\mu_k} \) for \( n \geq 2 \).

Hence,
\[
\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \leq \frac{1}{|\mu_{M+1}|^n} \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^2.
\]

Let
\[
C_{M,\lambda} = \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^2.
\]
Then
\[
\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \leq \frac{C_{M,\lambda}}{|\mu_{M+1}|^n}.\]
Consider $C_{M,\lambda}$:

$$
C_{M,\lambda} = \sum_{k=M+1}^{\infty} \frac{|\mu_{M+1}|^2}{|\mu_k|^2}
= |\mu_{M+1}|^2 \sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^2}
= (2\pi)^2 \left| M + 1 + \frac{1}{2} - \gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{(2\pi)^2} \left| k + \frac{1}{2} - \gamma \right|^2
\leq \left| M + \frac{3}{2} - \gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{(|k| - 1)^2}
\leq 2 \left| M + \frac{3}{2} - \gamma \right|^2 \sum_{l=0}^{\infty} \frac{1}{(M+l)^2}
\leq 2 \left| M + \frac{3}{2} - \gamma \right|^2 \left( \frac{1}{M^2} + \sum_{l=1}^{\infty} \frac{1}{(M+l)^2} \right).
$$

With

$$
\sum_{l=1}^{\infty} \frac{1}{(M+l)^2} \leq \int_{1}^{\infty} \frac{1}{(M+x)^2} \, dx = \frac{1}{M+1},
$$

$$
C_{M,\lambda} \leq 2 \left| M + \frac{3}{2} - \gamma \right|^2 \left( \frac{1}{M^2} + \frac{1}{M+1} \right)
= \frac{2 \left| M + \frac{3}{2} - \gamma \right|^2}{M^2} + 2 \left| M + \frac{3}{2} - \gamma \right|^2.
$$

Let

$$
\epsilon_1 = \frac{\left| M + \frac{3}{2} - \gamma \right|^2}{M^2} \leq \left| \frac{5}{2} - \gamma \right|^2,
$$

and

$$
\epsilon_2 = \frac{\left| M + \frac{3}{2} - \gamma \right|}{M + 1} \leq 1 + \frac{|1/2 - \gamma|}{|M + 1|} \leq 1 + \frac{1}{2} - \gamma.
$$

Consequently,

$$
\frac{C_{m,\lambda}}{|\mu_{M+1}|^n} \leq 2 \frac{\epsilon_1}{|\mu_{M+1}|^n} + 2 \frac{\epsilon_2}{|\mu_{M+1}|^n} \cdot \left| M + \frac{3}{2} - \gamma \right|
\leq \frac{2\epsilon_1}{|\mu_{M+1}|^n} + \frac{2\epsilon_2 \cdot |M + 3/2 - \gamma|}{|\mu_{M+1}|^n},
$$

$$
\frac{C_{m,\lambda}}{|\mu_{M+1}|^n} \leq 2 \frac{\epsilon_1}{|\mu_{M+1}|^n} + \frac{\epsilon_2}{|\mu_{M+1}|^n} \cdot \left| M + \frac{3}{2} - \gamma \right|.
$$
where

\[ | \mu_{M+1} | = \left| M + \frac{3}{2} - \gamma \right| = \sqrt{\left( M + \frac{3}{2} - \Re \gamma \right)^2 + (\Im \gamma)^2} \geq |M| - 2. \]

\[ C_{M,\lambda} \leq \frac{\epsilon_1}{2^{n-1} \pi^n |M + 3/2 - \gamma|^n} + \frac{\epsilon_2}{2^{n-1} \pi^n |M + 3/2 - \gamma|^{n-1}} \]
\[ \leq \frac{\epsilon_1}{2^{n-1} \pi^n (|M| - 2)^n} + \frac{\epsilon_2}{2^{n-1} \pi^n (|M| - 2)^{n-1}} \]
\[ \leq 2^{n-1} \pi^n (|M| - 2)^{n-1} + \frac{1}{2^{n-1} \pi^n (|M| - 2)^n} \]
\[ \leq \frac{\epsilon_1}{2^{n-1} \pi^n} + \frac{1}{2^{n-1} \pi^n}. \]

We can see that \( C_{M,\lambda} \to 0 \) as \( n \to \infty \) for \( |M| > 2 \). Thus, the tail of the series,

\[ \sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \to 0 \quad \text{as} \quad n \to \infty. \]

Moreover, for fixed \( M > 2 \) and \( n \gg 0 \), \( C_{M,\lambda} \) is bounded and independent of \( M \). Hence, we can replace \( C_{M,\lambda} \) by \( C_{\lambda} \). This completes the proof of the theorem.

When \( \lambda = 1 \), \( \log \lambda = 0 \) and \( u_k = (2k+1)\pi i \), \( k \in \mathbb{Z} \). Take \( H = \{ \pi i, -\pi i \} \). Then \( \nu = 3\pi \) and the ordinary Genocchi numbers \( G_n = G_n(0; 1) \) satisfy

\[ \frac{G_n}{2(n!)} = G_n(0; 1) = \frac{1}{(\pi i)^n} + \frac{1}{(-\pi i)^n} + O((3\pi)^{-n}). \quad (2.7) \]

An approximation of \( G_n(0; 1) \) is given by

\[ \frac{G_n}{2(n!)} \approx \frac{1}{(\pi i)^n} + \frac{1}{(-\pi i)^n}. \quad (2.8) \]

For odd \( n, n \geq 3 \), it is known that \( G_n = 0 \) which is also true when we use (2.8). For even indices,

\[ G_{2n} \approx \frac{(-1)^n 4((2n)!)}{\pi^{2n}}, \quad n \geq 2 \quad (2.9) \]

Taking \( n = 4 \),

\[ G_8 \approx \frac{4(8!)}{\pi^8} \approx 16.99. \]

This value is very close to the exact value of \( G_8 \) which is 17.

It is proved in the next theorem that an asymptotic approximation of the Apostol-Genocchi polynomials can be obtained from its Fourier series (2.1) by choosing an appropriate subset of \( T_{\lambda} \).
Theorem 2.4. Given \( \lambda \in \mathbb{C}\setminus\{0\} \), let \( H \) be a finite subset of \( T \) satisfying
\[
\max\{|u| : u \in H \} < \min\{|u| : u \in T \setminus H \} := \nu.
\]
For all integers \( n \geq 2 \), we have, uniformly for \( x \) in a compact subset \( K \) of \( \mathbb{C} \),
\[
\frac{G_n(x; \lambda)}{n!} = 2 \sum_{u \in H} \frac{e^{ux}}{u^n} + O\left(\frac{\nu^{|x|}}{\nu^n}\right),
\]
where the constant implicit in the order term depends on \( \lambda \), \( H \) and \( K \). Moreover, for \( n \gg 0 \), this constant can be made independent of \( K \), equal to the constant for the Apostol-Genocchi numbers, corresponding to the case \( x = 0 \).

Proof. From the generating function (1.1) we have
\[
\frac{2ze^{(x+y)z}}{\lambda e^z + 1} = \sum_{n=0}^{\infty} G_n(x+y; \lambda) \frac{z^n}{n!}.
\]
The LHS can be written
\[
\frac{2ze^{xz}}{\lambda e^z + 1} \cdot e^{yz} = \left( \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} (yz)^n \frac{n!}{n!} \right)
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} G_{n-k}(x; \lambda) \frac{z^{n-k}}{(n-k)!} \frac{(yz)^k}{k!}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(x; \lambda)y^k \right) \frac{z^n}{n!},
\]
from which
\[
G_n(x+y; \lambda) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(x; \lambda)y^k.
\]
For \( z \in \mathbb{C} \), writing \( z = 0 + z \) (here \( y = z, x = 0 \)),
\[
G_n(z; \lambda) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(0; \lambda)z^k,
\]
\[
\frac{G_n(z; \lambda)}{n!} = \sum_{k=0}^{n} \frac{G_{n-k}(0; \lambda)z^k}{(n-k)!} \\
= 2 \sum_{k=0}^{n} \left( \sum_{u \in H} \frac{1}{u^{n-k}} + O(\nu^{-(n-k)}) \right) \frac{z^k}{k!} \quad \text{by Theorem 2.3}
\]
\[
= 2 \sum_{k=0}^{n} \left( \sum_{u \in H} \frac{z^k}{u^{n-k} k!} \right) + \sum_{k=0}^{n} O(\nu^{-(n-k)}) \frac{z^k}{k!}.
\]
where the implicit constant $c$ in the order term is that corresponding to $z = 0$ and only depends on $H$ and $\lambda$. Note also that

$$
\left| \sum_{k=0}^{n} O(\nu^{-n+k}) \frac{z^k}{k!} \right| \leq \sum_{k=0}^{n} c \nu^{-n+k} \frac{|z^k|}{k!} \\
= c \nu^{-n} \sum_{k=0}^{n} \nu^k \frac{|z^k|}{k!} \\
\leq c \nu^{-n} e_n(\nu |z|),
$$

where $e_n = \sum_{k=0}^{n} \frac{w^k}{k!}$.

To prove the theorem, it remains to show that

$$
\frac{e_n^*(uz)}{u^n} = \frac{e^{uz} - c_n(uz)}{u^n}
$$

is bounded.

Using MVT for Banach spaces (see also [15])

$$
e_n^*(w) = \frac{w^{n+1}}{(n+1)!} + \frac{w^{n+2}}{(n+2)!} + \cdots \\
= \frac{w^{n+1}}{(n+1)!} \left\{ 1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots \right\},
$$

from which

$$
|e_n^*(w)| \leq \frac{|w|^{n+1}}{(n+1)!} \left| 1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots \right| \\
\leq |w|^{n+1} e^{\Re^+(w)},
$$

where $\Re^+(w) = \max\{\Re(w), 0\}$.

Since $|u| \leq \nu$, for all $u \in H$, we have

$$
\frac{|e_n^*(uz)|}{|u^n|} \leq \frac{e^{|uz|}|u z|^{n+1}}{|u^n|(n+1)!} \\
= |u| e^{|uz|} \frac{|z|^{n+1}}{(n+1)!} \\
< \nu e^{\nu |z|} \frac{|z|^{n+1}}{(n+1)!},
$$

so that

$$
\sum_{u \in H} \frac{e_n^*(uz)}{u^n} \leq \sum_{u \in H} \frac{|e_n^*(uz)|}{|u^n|}
$$
where \( \#H = \text{no. of elements in } H \).

We give the argument that
\[
\#H \nu e^{\nu|z|} \frac{|z|^{n+1}}{(n+1)!} < c e^{\nu|z| \nu - n}
\]
if
\[
\#H \frac{(\nu|z|)^{n+1}}{(n+1)!} < c,
\]
which certainly holds for \( n \gg 0 \), uniformly for \( z \) in a compact subset \( K \subset \mathbb{C} \).

**Corollary 2.5.** Let \( K \) be an arbitrary compact subset of \( \mathbb{C} \). The Genocchi polynomials satisfy uniformly on \( K \) the estimates
\[
G_{2n}(x) = \frac{(-1)^n 4 \cos \pi x}{\pi^{2n}} + O \left( \frac{e^{3\pi|x|}}{(3\pi)^n} \right), \quad n \geq 2,
\]
\[
G_{2n+1}(x) = \frac{(-1)^n 4 \sin \pi x}{\pi^{2n+1}} + O \left( \frac{e^{3\pi|x|}}{(3\pi)^n} \right), \quad n \geq 3,
\]
where the implicit constant in the order term depends on the set \( K \). Moreover, for \( n \gg 0 \), this constant can be made independent of \( K \), equal to the constant for the Genocchi numbers, corresponding to the case \( x = 0 \).

**Proof.** The Genocchi polynomials correspond to the case \( \lambda = 1 \) so that \( u_k = (2k+1)\pi i \), for \( k \in \mathbb{Z} \). Thus, \( T_1 = \{(2k+1)\pi i : k \in \mathbb{Z}\} \). Taking \( H = \{(2k+1)\pi i \mid k = -1, 0\} = \{-\pi i, \pi i\} \), then \( \nu = |3\pi i| = 3\pi \). From Theorem 2.4,
\[
\frac{G_n(x; 1)}{n!} = 2 \sum_{u \in H} \frac{e^{\nu x}}{u^n} + O \left( \frac{e^{\nu|z|}}{\nu^n} \right)
\]
\[
= 2 \left( \frac{e^{-\pi ix}}{(-\pi i)^n} + \frac{e^{\pi ix}}{(\pi i)^n} \right) + O \left( \frac{e^{3\pi|x|}}{(3\pi)^n} \right).
\]

For even indices,
\[
\frac{G_{2n}(x)}{(2n)!} = \frac{G_{2n}(x; 1)}{(2n)!}
\]
\[
= 2 \left( \frac{e^{-\pi ix}}{(\pi i)^{2n}} + \frac{e^{\pi ix}}{(\pi i)^{2n}} \right) + O \left( \frac{e^{3\pi|x|}}{(3\pi)^{2n}} \right)
\]
\[
= 4 \cos \pi x \frac{x}{(\pi i)^{2n}} + O \left( \frac{e^{3\pi|x|}}{(3\pi)^{2n}} \right).
For odd indices, \( G_{2n+1}(x) = \frac{G_{2n+1}(x; 1)}{(2n+1)!} \)
\[
= 2 \left( \frac{e^{-\pi i x}}{(-\pi i)^{2n+1}} + \frac{e^{\pi i x}}{(\pi i)^{2n+1}} \right) + O \left( \frac{e^{3\pi |x|}}{(3\pi)^{2n+1}} \right)
\]
\[
= 2 \left( \frac{(-1)^n 2 \sin \pi x}{(\pi)^{2n+1}} \right) + O \left( \frac{e^{3\pi |x|}}{(3\pi)^{2n+1}} \right)
\]
\[
= \frac{(-1)^n (4 \sin \pi x)}{\pi^{2n+1}} + O \left( \frac{e^{3\pi |x|}}{(3\pi)^n} \right).
\]

Notice the resemblance of the results in Corollary 2.5 and of (33) in [3].

Since, for \( k = 2n \),
\[
\cos \left( \pi x - \frac{k\pi}{2} \right) = \pm \cos \pi x = (-1)^n \cos \pi x,
\]
(33) in [3] can be written as

\[
G_{2n}(x) = \frac{4((2n)!)\pi}{2n} \left[ (-1)^n \cos \pi x + O(3^{-n}) \right]
\]
\[
G_{2n}(x) = \frac{(-1)^n 4 \cos \pi x}{\pi^{2n}} + O \left( \frac{3^{-n}}{(\pi)^{2n}} \right)
\]
\[
= \frac{(-1)^n 4 \cos \pi x}{\pi^{2n}} + O \left( \frac{1}{(3\pi)^n} \right)
\]
\[
= \frac{(-1)^n 4 \cos \pi x}{\pi^{2n}} + O \left( \frac{e^{3\pi |x|}}{(3\pi)^n} \right), \text{ for } x \in K.
\]

For odd \( k (k = 2n + 1) \),
\[
\cos \pi x - \frac{k\pi}{2} = (-1)^n \sin \pi x.
\]

Then (33) in [3] can be written as

\[
G_{2n+1}(x) = \frac{4((2n+1)!)\pi}{2n+1} \left[ (-1)^n \sin \pi x + O \left( 3^{-(2n+1)} \right) \right]
\]
\[
G_{2n+1}(x) = \frac{(-1)^n 4 \sin \pi x}{\pi^{2n+1}} + O \left( \frac{3^{-(2n+1)}}{(\pi)^{2n+1}} \right)
\]
\[
= \frac{(-1)^n 4 \sin \pi x}{\pi^{2n+1}} + O \left( \frac{1}{(3\pi)^{2n+1}} \right)
\]
\[
\begin{align*}
&= (-1)^n \frac{4 \sin \pi x}{\pi^{2n+1}} + O \left( \frac{e^{3\pi|x|}}{(3\pi)^{2n+1}} \right) \\
&= (-1)^n \frac{4 \sin \pi x}{\pi^{2n+1}} + O \left( \frac{e^{3\pi|x|}}{(3\pi)^n} \right).
\end{align*}
\]

Thus, the asymptotic formulas in Corollary 2.5 are equivalent to (33) in [3].

3. \( \lambda \) is a negative real number

When \( \lambda \) is a negative real number, writing \( \lambda = -|\lambda| \), the generating function is given by
\[
\frac{2t \mathrm{e}^{xt}}{-|\lambda| \mathrm{e}^{t} + 1} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!}.
\]
(3.1)

The poles of the generating function (3.1) is
\[
T_{-|\lambda|} = \{2k\pi i - \log |\lambda| : k \in \mathbb{Z}\}.
\]

The next theorem follows from Theorem 2.4.

**Theorem 3.1.** Given that \( \lambda \) is a negative real number, let \( F \) be a finite subset of \( T_{-|\lambda|} \) satisfying
\[
\max \{|a| : a \in F\} < \min \{|a| : a \in T_{-|\lambda|} \setminus F\} := \mu.
\]
For all integers \( n \geq 2 \), we have, uniformly for \( x \) in a compact subset \( K \) of \( \mathbb{C} \),
\[
\frac{G_n(x; \lambda)}{n!} = 2 \sum_{a \in F} \frac{e^{ax}}{a^n} + O \left( \frac{e^{\mu|x|}}{\mu^n} \right),
\]
(3.2)
where the constant implicit in the order term depends on \( \lambda \), \( F \) and \( K \).

The Apostol-Genocchi numbers \( G_n(0; -1) \) corresponding to the case \( \lambda = -1 \) has generating function
\[
\frac{2t}{-\mathrm{e}^{t} + 1} = \sum_{n=0}^{\infty} G_n(0; -1) \frac{t^n}{n!}.
\]
(3.3)

The set of poles is \( T_{-1} = \{2k\pi i : k \in \mathbb{Z} \setminus \{0\}\} \). An asymptotic formula for \( G_n(0; -1) \) is given in the following theorem.

**Theorem 3.2.** For \( n \geq 3 \), the Apostol-Genocchi numbers \( G_n(0; -1) \) satisfy
\[
\frac{G_n(0; -1)}{n!} = 2 \left( \frac{1}{(-2\pi i)^n} + \frac{1}{(2\pi i)^n} \right) + O \left( (4\pi)^{-n} \right).
\]
(3.4)

In particular,
\[
\frac{G_{2n}(0; -1)}{(2n)!} = \frac{(-1)^n 4}{(2\pi)^{2n}} + O \left( (4\pi)^{-2n} \right), \quad n \geq 2.
\]
(3.5)
Proof. Taking $x = 0$, $F = \{-2\pi i, 2\pi i\}$ in Theorem 3.1, then $\mu = 4\pi$. Hence,
\[ \frac{-\frac{1}{2} G_n(0; -1)}{n!} = -\left( \frac{1}{(-2\pi i)^n} + \frac{1}{(2\pi i)^n} \right) + O\left((4\pi)^{-n}\right), \] (3.6)
from which (3.4) follows.

For $(n \geq 3)$, (3.6) gives $G_{2n+1}(0; -1) \approx 0$. Indeed $G_{2n+1}(0; -1) = 0, \forall n \geq 1$.

For $n \geq 2$,
\[ \frac{G_{2n}(0; -1)}{(2n)!} = 4\left( \frac{(-1)^n}{(2\pi)^{2n}} \right) + O\left((4\pi)^{-2n}\right). \] (3.7)

From (3.7) we have the approximation
\[ G_{2n}(0; -1) \approx \frac{(-1)^n 4(2n)!}{(2\pi)^{2n}}. \] (3.8)

Taking $n = 4$,
\[ G_8(0; -1) = \frac{4(8)!}{(2\pi)^8} \approx 0.06638. \]

The actual value of $G_8(0; -1) = -2B_8 = \frac{1}{15} \approx 0.06667$.

The Apostol-Genocchi polynomials, $G_n(x; -1)$ correspond to the case $\lambda = -1$. These polynomials have generating function
\[ \frac{2te^{xt}}{-e^t + 1} = \sum_{n=0}^{\infty} G_n(x; -1) \frac{t^n}{n!}. \] (3.9)

We will prove the following theorem.

**Theorem 3.3.** Let $K$ be a compact subset of $\mathbb{C}$. The Apostol-Genocchi polynomials $G_n(x; -1)$ satisfy uniformly on $K$ the estimates
\[ \frac{G_{2n}(x; -1)}{(2n)!} = \frac{(-1)^n 4 \cos 2\pi x}{(2\pi)^{2n}} + O\left(\frac{e^{4|x|}}{(4\pi)^n}\right), \] (3.10)
\[ \frac{G_{2n+1}(x; -1)}{(2n+1)!} = \frac{(-1)^n 4 \sin 2\pi x}{(2\pi)^{2n+1}} + O\left(\frac{e^{4|x|}}{(4\pi)^n}\right), \] (3.11)

where the implicit constant in the order term depends on the set $K$. Moreover, for $n \gg 0$, this constant can be made independent of $K$, equal to the constant for the Apostol-Genocchi numbers $G_n(0; -1)$ corresponding to the case $x = 0$. 

Proof. Taking $F = \{-2\pi i, 2\pi i\}$, then $\mu = 4\pi$. Hence, it follows from Theorem 3.1 that

$$\frac{-1}{2}G_n(x; -1) = -\frac{e^{2\pi ix}}{(2\pi i)^n} - \frac{e^{-2\pi ix}}{(-2\pi i)^n} + O\left(\frac{e^{4\pi |x|}}{(4\pi)^n}\right).$$  \hfill (3.12)

For odd indices,

$$\frac{-1}{2}G_{2n+1}(x; -1) = \left(\frac{e^{2\pi ix}}{(2\pi i)^{2n+1}} + \frac{e^{-2\pi ix}}{(-2\pi i)^{2n+1}}\right) + O\left(\frac{e^{4\pi |x|}}{(4\pi)^{2n+1}}\right)$$  \hfill (3.13)

$$G_{2n+1}(x; -1) = (1)^{n}4\sin 2\pi x \frac{e^{4\pi |x|}}{(4\pi)^n}. \hfill (3.14)$$

For even indices,

$$G_{2n}(x; -1) = 2\left(\frac{e^{2\pi ix}}{(2\pi i)^{2n}} + \frac{e^{-2\pi ix}}{(-2\pi i)^{2n}}\right) + O\left(\frac{e^{4\pi |x|}}{(4\pi)^{2n}}\right)$$  \hfill (3.15)

$$= (1)^{n}4\cos 2\pi x \frac{e^{4\pi |x|}}{(4\pi)^n}. \hfill (3.16)$$

4. Apostol-Euler Numbers and Polynomials

The Apostol-Euler numbers are defined by the generating function

$$\frac{2\lambda e^t + 1}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(0; \lambda) \frac{t^n}{n!}. \hfill (4.1)$$

Multiplying both sides of (4.1) by $t$ gives

$$\sum_{n=0}^{\infty} G_n(0; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (n + 1)E_n(0; \lambda) \frac{t^{n+1}}{(n + 1)!},$$

from which we have, for $n \geq 1$

$$E_{n-1}(0; \lambda) = \frac{G_n(0; \lambda)}{n} = (n - 1)! \frac{G_n(0; \lambda)}{n!}.$$  \hfill (4.2)

Thus, from Theorem 2.3,

$$E_{n-1}(0; \lambda) = 2(n - 1)! \left[\sum_{\nu=0}^{\infty} \frac{1}{\nu^n} + O\left(\nu^{-n}\right)\right], \hfill (4.3)$$

where $F \subseteq T_\lambda = \{(2k + 1)\pi i - \log \lambda \mid k \in \mathbb{Z}\}$ and $F$ satisfies
\[ \max\{|u| : u \in F\} < \min\{|u| : u \in T_{\lambda} \setminus F\} = \nu. \]

For odd \( n \), say \( n = 2k + 1 \), from (4.2), we have
\[ E_{2k} (0; \lambda) = \frac{G_{2k+1} (0; \lambda)}{2k + 1}, \quad (4.4) \]
while for even \( n \), say \( n = 2k \),
\[ E_{2k-1} (0; \lambda) = \frac{G_{2k} (0; \lambda)}{2k}. \quad (4.5) \]
The case \( \lambda = 1 \), corresponds to the Euler numbers \( E_n \). From (4.2),
\[ E_{n-1} = \frac{G_n}{n}. \quad (4.6) \]
Since \( G_n = 0 \) for all odd \( n \geq 3 \), \( E_{2k} = 0 \) for \( k \geq 1 \).

For odd indices, using (2.9) we have
\[ E_{2n-1} = (2n - 1)! \frac{G_{2n}}{2n} = (2n - 1)! \left( \frac{(-1)^n (4)}{\pi^{2n}} + O \left( (3\pi)^{-n} \right) \right), \quad n \geq 2. \quad (4.7) \]
Taking \( n = 2 \),
\[ E_3 \approx 3! \left( \frac{4}{\pi^4} \right) = \frac{24}{\pi^4} = 0.24638. \]
The Actual value of \( E_3 = 0.25 \).

The Apostol-Euler Polynomials \( E_n (x; \lambda) \) are defined by the generating function
\[ \frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n (x; \lambda) \frac{t^n}{n!}, \quad (4.8) \]
which can be written
\[ \sum_{n=0}^{\infty} \frac{G_n (x; \lambda) t^n}{n!} = \sum_{n=0}^{\infty} (n + 1) E_n (x; \lambda) \frac{t^{n+1}}{(n + 1)!}. \quad (4.9) \]
Thus,
\[ E_{n-1} (x; \lambda) = \frac{G_n (x; \lambda)}{n}. \quad (4.10) \]

From Theorem 2.4,
\[ E_{n-1} (x; \lambda) = \frac{G_n (x; \lambda)}{n} \cdot \frac{(n - 1)!}{(n - 1)!} \]
\[ = (n - 1)! \frac{G_n (x; \lambda)}{n!} \]
\[ = (n - 1)! \left( 2 \sum_{u \in F} u^n e^{\nu |x|} + O \left( \frac{e^{\nu |x|}}{\nu^n} \right) \right). \]

Hence, we have the following corollary.
Corollary 4.1. Given \( \lambda \in \mathbb{C}\backslash\{0\} \), let \( F \) be a finite subset of \( T_\lambda \) satisfying
\[
\max\{|u| : u \in F\} < \min\{|u| : u \in T_\lambda \backslash F\} = \nu.
\]
Let \( K \) be an arbitrary compact subset of \( \mathbb{C} \). The Apostol-Euler polynomials satisfy uniformly on \( K \) the estimates,
\[
\frac{E_{n-1}(x; \lambda)}{(n-1)!} = 2 \sum_{u \in F} \frac{e^{u x}}{u^n} + O\left(\frac{e^{\nu|x|}}{\nu^n}\right),
\]
where the constant implicit in the order term depends on \( \lambda, F \) and \( K \). Moreover, for \( n \gg 0 \), this constant can be made independent of \( K \), equal to the constant for the Apostol-Euler numbers, corresponding to the case \( x = 0 \).

It follows from Corollary 2.5 that the Euler polynomials which correspond to \( \lambda = 1 \), satisfy, uniformly on a compact subset \( K \) of \( \mathbb{C} \) the estimates
\[
\frac{E_{2n-1}(x)}{(2n-1)!} = G_{2n}(x) = (-1)^n 4 \sin \frac{\pi x}{2} + O\left(\frac{e^{4|\pi x|}}{(4\pi)^n}\right),
\]
\[
\frac{E_{2n}(x)}{(2n)!} = G_{2n+1}(x) = (-1)^n 4 \cos \frac{\pi x}{2} + O\left(\frac{e^{4|\pi x|}}{(4\pi)^n}\right),
\]
as \( n \to \infty \), for \( n \geq 1 \).

The Apostol-Euler polynomials \( E_{n-1}(x; -1) \) correspond to the special case \( \lambda = -1 \). From (4.10),
\[
E_{n-1}(x; -1) = \frac{G_n(x; -1)}{n}.
\]
It follows from (3.10) and (3.11), respectively that
\[
\frac{E_{2n}(x; -1)}{(2n)!} = (-1)^n 4 \sin 2\pi x + O\left(\frac{e^{4|\pi x|}}{(4\pi)^n}\right),
\]
\[
\frac{E_{2n-1}(x; -1)}{(2n-1)!} = (-1)^n 4 \cos 2\pi x + O\left(\frac{e^{4|\pi x|}}{(4\pi)^n}\right),
\]
on a compact subset \( K \) of \( \mathbb{C} \).

5. Conclusion

Asymptotic approximations of the Apostol-Genocchi numbers and polynomials were obtained for values of the parameter \( \lambda \) in \( \mathbb{C}\backslash\{0\} \). Unlike in [15] we have considered explicitly the case when \( \lambda \) is negative and obtained corresponding asymptotic formulas.
Moreover, the asymptotic formulas for $\lambda = 1$ are explicitly obtained for each of the Apostol-Genocchi and Apostol-Euler numbers and polynomials. The tangent polynomials [8] have generating function very similar to that of the Apostol-Genocchi polynomials. The author recommends finding Fourier expansion and asymptotic approximations of these polynomials.

Acknowledgements

This research project is funded by Cebu Normal University through its Center for Research and Development.

References

[1] S. Araci, W.A Khan, M. Acikgoz, C. Ozel and P. Kumam, A new generalization of Apostol type Hermite-Genocchi polynomials and its applications, Springerplus, 5(2016), Art. ID 860.

[2] A. Bayad, Fourier expansions for Apostol-Bernoulli, Apostol- Euler and Apostol-Genocchi polynomials, Math Comp., 80 (2011), 2219–2221.

[3] C. B. Corcino, R.B. Corcino, Asymptotics of Genocchi polynomials and higher order Genocchi polynomials using residues. Afr. Math. (2020), https://doi.org/10.1007/s13370-019-00759-z

[4] C. B. Corcino, R.B. Corcino, Fourier expansions for higher-order Apostol-Genocchi, Apostol-Bernoulli and Apostol-Euler polynomials, Advances in Difference Equations, 2020(1), 1-13.

[5] A. Duran, M. Acikgoz, S. Araci, A note on the symmetric relations of q-Genocchi polynomials under symmetric group of degree n, TAMAP Journal of Mathematics and Statistics, 2017 (Article ID 7), 8 pages.

[6] U. Duran, M. Acikgoz, S. Araci, Construction of the type 2 poly-Frobenius-Genocchi polynomials with their certain applications, Advances in Difference Equations, 2020, 432(2020).

[7] D.-S. Kim, T. Kim, A note on polyexponential and unipoly functions, Russ. J. Math. Phys. 2019 26(1), 40-49.

[8] B. Kurt, Identities and Relations on Hermite-based Tangent polynomials, arXiv:1811.03587v1[math.NT]8 Nov 2018.

[9] Y. He, S. Araci, H.M. Srivastava and M. Acikgoz, Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials, Appl. Math. Comput., 262 (2015), 31-41.
[10] Y. He, Some new results on products of the Apostol-Genocchi polynomials, *J. Comput. Anal. Appl.*, **22**(4) (2017), 591-600.

[11] Y. He, S. Araci, H.M. Srivastava and M. Abdel-Aty, Higher-order convolutions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Mathematics **6**, Article ID 329 (2019). https://doi.org/10.3390/math6120329

[12] H. Jolany and H. Sharifi, Some results for the Apostol-Genocchi polynomials of higher order, *arXiv preprint arXiv:1104.1501*, (2011)

[13] Q.M. Luo, Extensions of the Genocchi polynomials and their Fourier expansions and integral representations. *Osaka J. Math.* **48**, 291309 (2011)

[14] Q. M. Luo, H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, *Appl. Math. Comput.*, **217**(12), (2011), 5702-5728.

[15] L. Navas, F. Ruiz, and J.Varona, Asymptotic estimates for Apostol-Bernoulli and Apostol-Euler polynomials, *Mathematics of Computation*, **Volume 81**, (Number 279), (July 2012), 1707-1722.