Decay properties of beauty and charm mesons within Isgur–Wise function formalism

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Abstract We investigate the decay properties of some beauty and charm mesons with a phenomenological potential model. First, we consider the nonrelativistic Hamiltonian of the mesonic system with Coulomb plus exponential terms and study the wave function and the energy of the system using the variational approach. Thereby, we compute the masses, the decay constants, the leptonic branching fractions of heavy–light mesons and the mixing mass parameter \( \Delta m_{B_q} \). We study the radiative leptonic decay widths of \( D_s \rightarrow \gamma \ell \bar{\nu} \), \( D^- \rightarrow \gamma \ell \bar{\nu} \), and the semileptonic decay widths of \( \bar{D}_s \rightarrow D(\bar{s}) \ell \bar{\nu} \) and \( \bar{D}(\bar{s}) \rightarrow D^*(\bar{s}) \ell \bar{\nu} \). Using Isgur–Wise function, we calculate the branching ratios of \( B \rightarrow D(\bar{s}) \pi \) and two-body nonleptonic decay of \( D \rightarrow K \pi \). Our results are consistent with other theoretical models and the experimental results.

1 Introduction

There have been many valuable studies on the weak decays of \( B \) and \( D \) mesons using the factorization approach [1,2], the light-cone sum rules [3–6], QCD sum rules [7,8], Bethe–Salpeter equation approach [9,10] and the nonrelativistic constituent quark model [11]. The recent progress on computing \( B \)-meson decay form factors was described in Ref. [3]. Employing the light-cone sum rules technique, Wang et al. calculated the heavy-to-light \( B \)-meson decay form factors at large hadronic recoil [4]. Furthermore, using QCD light-cone sum rule, Wang and Shen studied the form factors of \( B \rightarrow \pi \) [5]. \( B \rightarrow \pi \) and \( B \rightarrow K \) decays at large recoil from the light-cone sum rules have been studied in Ref. [6]. Lü and Song investigated the branching fractions of \( D(\bar{s}) \rightarrow \gamma \ell \bar{\nu}(\ell = e, \mu) \) using the nonrelativistic constituent quark model and the effective Lagrangian for the heavy flavor decays [11]. Li et al. studied both the semileptonic decays of \( \bar{B}_s^0 \rightarrow D_s^+ \ell \bar{\nu}_\ell \) and nonleptonic decays \( B_s \rightarrow D_s^+ M \) where \( M \) is a light or charmed meson under the factorization approach [12]. Ivanov et al. analyzed the exclusive leptonic and semileptonic \( B \) decays \( B \rightarrow \ell \nu \ell \) and \( B \rightarrow D^{(*)} \ell \nu \ell \) with the covariant quark model [13]. Besides, using the covariant light-front quark model, the semileptonic \( B \) and \( B_s \) decays to the pseudoscalar, vector, scalar and axial-vector mesons were fully stud-
ied in Ref. [14]. The nonleptonic decays of $B$ mesons into two mesons were investigated by Kramer and Lü using two versions of the pole-dominance model in addition with a factorization assumption [15]. With a relativistic potential model, Sun and Yang obtained the wave functions, the decay constants and the branching fractions for the leptonic decays of bottom mesons [16]. The semileptonic decays of $D$ and $D_s$ were investigated in detail in Ref. [17] based on the quasipotential approach. Using hydrogenic and Gaussian wave functions, the spectroscopy and the decay properties of $B$ and $B_s$ mesons have been studied in Ref. [18]. Ebert et al. calculated the decay constants of the pseudoscalar and vector heavy–light mesons using the relativistic quark model [19]. The mass spectroscopy of $B$ and $D$ mesons was studied in the framework of the relativistic quark model [20]. Furthermore, Bakker investigated the heavy–light meson masses and their decay constants in the framework of the field correlator model [21].

In the study of semileptonic decays of $B$ mesons which contains the flavor-changing quark transitions $b \to c$, one needs to introduce a universal Isgur–Wise function (IWF) $\xi(v,v')$. The semileptonic $B$ decays provide a good opportunity to measure the Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{cb}|$ and the knowledge of the heavy–light meson dynamics. Furthermore, the study of two-body nonleptonic decays of $B$ and $D$ mesons are also caught much interest in particle physics because they contain valuable information on the electroweak interactions of the quarks, flavor mixing, $CP$ violation [1] and they are useful for testing some QCD-motivated models. The semileptonic and nonleptonic decay widths with $B \to D^{(*)}$ are expressed in terms of IWF in the heavy quark effective theory (HQET). There are different parameterizations for IWF within phenomenological approaches [2,22–24]. The decay constants of the heavy–light mesons play an important role in particle physics such as the $B^0 - \bar{B}^0$ mixing, the treatment of nonleptonic heavy flavor decays in the factorization approximation, the analysis of the CKM matrix element and also the connection of the leptonic constants to the wave function at the origin in the nonrelativistic quark models [25–28].

In the next section, we make a brief introduction of the Hamiltonian of a mesonic system using a variation method. In Sect. 3, we show the results of the masses of pseudoscalar and vector mesons in the beauty and charm sectors, and the leptonic decay widths of $B$ and $D$ mesons. Using the decay constants obtained, we also evaluate the radiative leptonic decay widths of charm mesons including $D$ and $D_s$. The mixing mass parameters of the $B$ and $B_s$ mesons are given in Sect. 4. In Sect. 5, we use a parameterization of IWF, which is dependent on the energy and the wave function, to analyze semileptonic decay widths of $B$ mesons. Using the differential semileptonic decay widths of $B$ mesons at maximum recoil, we study the nonleptonic decays of $B$ to $D$ and a pion in their final transitions in Sect. 6. The decays of $D$ to a pion and a kaon are also included. At the end, we present our conclusions.

## 2 General framework

For the heavy–light bound states of $B$ and $D$ mesons, we consider the nonrelativistic Hamiltonian to obtain the wave function of the mesonic system,

$$H = \frac{p^2}{2\mu} + V(r),$$

where $\mu$ is the reduced mass of the mesons, and $V(r)$ the interaction potential. We should consider a potential that satisfies the behavior of the quark constituents of the meson and describes the quark–antiquark bound states. Our formalism is based on the QCD-motivated
quark potential models. Therefore the interaction of the quarks should take into account both the confinement and asymptotic freedom, where the interactions become stronger for large $r$ and weaker at the short distance. Our considered potential includes the Coulomb term $-\frac{4\alpha_s}{r}$, that arises from one gluon exchange contribution and describes the short distance behavior, and the exponential term $\frac{b(1-e^{-\lambda r})}{\lambda}$, that can be expanded in different powers of $r$ to confine the quarks at long distances. Note that the exponential term, which is called the screening term, behaves like a linear confinement potential at short distances and changes to $\frac{b}{\lambda}$ at long distances. In 1985, motivated by QCD, Godfrey and Isgur constructed a quark model to study mesons in Ref. [29], where they proposed a universal one gluon exchange plus linear confinement potential. Besides, for the Cornell potential, which includes the summation of usual Coulomb and linear confinement, Lucha et al. discussed various QCD-inspired potential models, such as power-law $-\frac{\alpha_s}{r^p} + \kappa' r^n$ with $(\alpha_s' > 0, \kappa' > 0, 2 > p > 0, n > 0)$ and $A' \ln(\frac{r}{\rho_0})$ [30]. A Cornell-like potential, improved by an exponential term $-b'e^{-\frac{r}{\lambda'}}$ with $b' = 1.378$ GeV and $c' = 1.20$ GeV$^{-1}$, had also been proposed for a quarkonium system [31] and later extrapolated to baryon spectrum with potential parameters as $b' = 0.956$ GeV and $c' = 2.05$ GeV$^{-1}$ [32]. Song et al. calculated the mass spectra and strong decays of the $D_s$ meson by taking a screening effect in the Godfrey–Isgur model [33]. Li and Chao also used the screened potential model $(-\frac{4\alpha_s}{r} + \frac{\lambda'(1-e^{-\mu r})}{\mu'})$ with $\mu' = 0.0979$ GeV and $\lambda' = 0.21$ GeV$^2$ for charmonium spectrum [34]. Thus, considering the one gluon exchange for the short-distance contributions and the exponential term for the confinement effects at a long distance, the potential $V(r)$ can be taken as [29,30,35,36],

$$V(r) = \left[ -\frac{3}{4} \left( \frac{b(1-e^{-\lambda r})}{\lambda} + V_0 \right) + \frac{\alpha_s}{r} \right] \vec{F}_1 \cdot \vec{r},$$

(2)

where $V_0$ is a free parameter and will be determined by fitting to the experimental masses of the heavy–light mesons. We take the potential parameters as $b = 0.18$ GeV$^2$ and $\lambda = 0.11$ GeV from Ref. [37], where the $B$ and $D$ mesons are also studied with a nonrelativistic potential model. Furthermore, the force $\vec{F}_i$ is defined as $\vec{F}_i = \vec{F}_2 - \vec{F}_1$ with $i = 1, 2$ for quarks and antiquarks, respectively, where $\vec{F}_i$ is the Gell-Mann matrix. For mesons we have $\left( \vec{F}_1 \cdot \vec{F}_2 \right) = -\frac{4}{3}$ [29]. Besides, $\alpha_s$ is the QCD coupling constant, which depends on the quark masses $\mu$ and is given by [38],

$$\alpha_s(\mu^2) = \frac{4\pi}{\left( 11 - 2\frac{n_f}{3} \right) \ln \left( \frac{4\mu^2 + M_B^2}{\Lambda^2} \right)},$$

(3)

where the background mass is taken as $M_B = 0.95$ GeV, the QCD scale $\Lambda_{QCD} = 0.2$ GeV and the number of flavors $n_f = 3$ [39]. We show the considered potentials of Eq. (2) in Fig. 1 for $B$ and $D$ mesons, where the differences for $B$ and $D$ mesons are very small even though the parameter of $\alpha_s$ obtained with Eq. (3) has different values.

Note that different methods have been applied in solving the problem of the bound state systems. One of the satisfaction techniques for solving the Schrödinger equation without perturbation, is the variational method [30,41,42]. In this method, one can choose a trial wave function, which depends on one or more adjustable parameters known as the variation parameters. There is a definite lower bound to the energy eigenvalue spectrum, and thus, the bound state energy can be determined by the variation of the expectation value of the Hamiltonian under the variational parameters. This can be achieved by minimizing $\langle H \rangle = \langle H \rangle_\text{var}$.
\( \langle \psi | H | \psi \rangle = E(g) \) with respect to the variation parameter, \( g \), where one can obtain the physical quantities such as the energy of the system and the wave function at the origin. Due to the fact that the hydrogen-like wave function is reasonably valid for the lower states in the low \( r \) region where the Coulomb term is dominant, one can see that the chosen wave function is a virtually exact solution for Eq. (1). Thus, using a variational method, we take a trial hydrogen-like wave function for different mesons \([18,40]\),

\[
\psi_{n,l}(g,r) = N g^{\frac{3}{2}} (gr)^l e^{-gr} L^{(2l+1)}_{n-l-1}(gr),
\]

where \( N \) is the normalization constant, \( g \) the variational parameter and \( L^{(2l+1)}_{n-l-1}(gr) \) are the associated generalized Laguerre polynomials. We take the quantum numbers \( n=1, l=0 \) in the present work. Taking into account the normalized condition

\[
\int_0^\infty 4\pi r^2 \psi_{1,0}^2(g,r)dr = 1,
\]

\( N \) will be \( \frac{1}{2\sqrt{2\pi}} \). Note that, once the value of the variational parameter, \( g \), is determined, one can obtain the wave functions for each mesonic system. Therefore, one can also minimize the trial energy by taking the expectation value of the Hamiltonian and the derivative with respect to \( g \), and then setting it equal to zero, having

\[
\frac{\partial}{\partial g} \left( \langle \psi_{n,l}(g,r) | H | \psi_{n,l}(g,r) \rangle \right) = 0,
\]

where we can solve it for \( g \) and obtain the energy \( E_{n,l} \) of the mesonic system by means of

\[
E_{1,0} = \langle \psi_{1,0}(g,r) | H | \psi_{1,0}(g,r) \rangle,
\]

which is relied on the variation parameter \( g \) in the wave function of Eq. (4). We have considered the ground state of mesons and obtained the variation parameter \( g \) for different \( D \) and \( B \) mesons using Eqs. (4) and (5), see our results of the second column of Table 1 in Sect. 3, where one can see that \( g \) depends on the quantum numbers \( n, l \) and the masses of quarks.

Indeed, the solution of Eq. (5) is different for each mesonic system because the reduced mass of mesons, \( \alpha_s \), and also \( V_0 \) are different for the bottom and charmed sectors. For instance, we find \( g=0.882 \) for \( B \) meson, see also the results in Table 1. Therefore, the result of Eq. (4) is different for each meson due to \( g \). With the variation parameter \( g \) obtained for \( B, B_s, D_s \)
Table 1 Masses and decay constants for the heavy–light mesons

| Meson   | g  | Our mass | Exp. mass [45] | Our $f_{P/V}$ | Others’ $f_{P/V}$ |
|---------|----|----------|----------------|----------------|-------------------|
| $D^\pm$ | 0.829 | 1.901 GeV | 1.869 GeV | 294 MeV | 205.8 MeV [45] |
|         |      |          | 243 MeV [19] |               |                   |
| $B^\pm$ | 0.882 | 5.298 GeV | 5.279 GeV | 195 MeV | 198 $\pm$ 14 MeV [27] |
|         |      |          | 187.2$^{+4.0}_{-4.3}$ MeV [28] |       | 178 MeV [19] |
| $D_s^{\pm}$ | 0.938 | 1.958 GeV | 1.968 GeV | 354 MeV | 233.1$^{+5.0}_{-5.4}$ MeV [28] |
|         |      |          | 341$^{+7}_{-5}$ MeV [48] |       | 384 MeV [19] |
| $B_s^0$ | 1.017 | 5.347 GeV | 5.366 GeV | 244 MeV | 237 $\pm$ 17 MeV [27] |
|         |      |          | 278 MeV [19] |               |                   |
| $D^*$  | 0.829 | 2.022 GeV | 2.006 GeV | 285 MeV | 226.6$^{+5.9}_{-10.2}$ MeV [28] |
|         |      |          | 315 MeV [19] |               |                   |
| $B^*$  | 0.882 | 5.343 GeV | 5.324 GeV | 194 MeV | 193.1$^{+4.3}_{-4.6}$ MeV [28] |
|         |      |          | 195 MeV [19] |               |                   |
| $D_s^*$ | 0.938 | 2.082 GeV | 2.112 GeV | 343 MeV | 254.7$^{+6.3}_{-6.7}$ MeV [28] |
|         |      |          | 326$^{+21}_{-17}$ MeV [48] |       | 335 MeV [19] |
| $B_s^*$ | 1.017 | 5.395 GeV | 5.415 GeV | 243 MeV | 272 $\pm$ 20 MeV [26] |
|         |      |          | 214 MeV [19] |               |                   |

and $D$ mesons, we show our results for the wave functions of $B$, $B_s$, $D_s$ and $D$ mesons in Fig. 2, which have been normalized to one. Furthermore, to see the behavior of the wave functions in the momentum space, one can have the variation of the wave function in the momentum space by means of

$$
\psi_{n,l}(g, p) = \int_0^\infty r \psi_{n,l}(g, r) \sin(pr)dr.
$$

(6)

We show the results of Eq. (6) for $B_s$ and $D$ mesons in Fig. 3, where one can see that vanish in the zero-momentum limit and then reduces fast by increasing the momentum. Note that, our results of Fig. 3 for the wave functions in momentum space are compatible with the ones obtained with the relativistic potential models in Refs. [16,43] and QCD sum rule calculation [44].

3 Masses and leptonic branching fractions

Using the wave function and the energy of the system obtained in the previous section, we can get the masses of mesons. Thus, the mass for the pseudoscalar ($P$) or vector ($V$) meson is given by

$$
M_{P/V} = m_1 + m_2 + E_{1,0} + \langle H_{sd} \rangle,
$$

(7)
where $m_1$ and $m_2$ are the quark masses. Besides, the expectation value $\langle H_{sd} \rangle$ is also dependent on the wave function at the origin and the spin dependent, which is given by

$$\langle H_{sd} \rangle = \begin{cases} 
\frac{8\pi\alpha_s}{9m_1m_2} \left| \psi_{1,0}^{(P/V)}(g, 0) \right|^2 & \text{for } S = 1, \\
\frac{8\pi\alpha_s}{3m_1m_2} \left| \psi_{1,0}^{(P/V)}(g, 0) \right|^2 & \text{for } S = 0,
\end{cases}$$

(8)
where the wave function at the origin satisfies the relation

$$|\psi_{1,0}^{(P/V)}(g, 0)|^2 = \frac{\mu}{2\pi\hbar^2} \left( \int dV(r) \right),$$

(9)

with the potential $V(r)$ given by Eq. (2). The free parameter $V_0$ in Eq. (2) is used to fit the masses. Note that, to derive the wave function at the origin of Eq. (9), we have followed the method of Ref. [30]. Starting with Schrödinger equation for $S$-wave states, $-V^2\psi(r) = 2\mu(E - V(r))\psi(r)$, using $V\psi = \frac{1}{\sqrt{4\pi}} \frac{Vy}{r}$, along with multiplying both sides with $\frac{y}{\sqrt{4\pi}r}$, and then integrating over $d^3r$, we have,

$$-\int d^3r \frac{yy'}{4\pi r^2} = -\frac{1}{2} \int_0^\infty \! dr (y^2)' = -\frac{1}{2} (y^2)|_0^\infty = -2\pi (\psi + r\psi')^2|_0^\infty = 2\pi |\psi(0)|^2,$$

(10)

and

$$2\mu \int d^3r [E - V] \frac{yy'}{4\pi r^2} = \mu \int_0^\infty \! dr [E - V](y^2)' = \mu \int_0^\infty \! dr V'(r)y^2 = \mu \langle V' \rangle,$$

(11)

which will lead to the relation of Eq. (9) for the wave function at the origin of the mesonic system.

Taking the input values of the quark masses as $m_d = m_u = 0.336$ GeV, $m_s = 0.465$ GeV, $m_c = 1.55$ GeV and $m_b = 4.97$ GeV [38], we obtain the masses of the $B$ and $D$ mesons as shown in the third column of Table 1, where we determined $V_0 = -0.534$ GeV for the bottom mesons and $V_0 = -0.490$ GeV for the charm ones. As one can see from Table 1, in most cases our results are in agreement with the experimental data from Particle Data Group (PDG) [45]. For instance, the differences between our obtained masses and the experimental ones [45] are 19 MeV, 32 MeV, 10 MeV and 19 MeV for the pseudoscalar mesons $B, D, D_s$ and $B_s$, respectively. Moreover, one can compute the mean square deviations for the obtained masses with experimental data of PDG [45], using the formula of

$$\sigma^2 = \frac{1}{N} \sum_{n=1}^{N} \left( \frac{M_{n, \text{Ours}} - M_{n, \text{Exp}}}{M_{n, \text{Exp}}} \right)^2,$$

(12)

where we obtained $\sigma = 0.009$ for our results. With the relativistic quark model, Ebert et al. [20] calculated the mass spectrum of $D, D_s, B, B_s$, obtained as 1.875 GeV, 1.981 GeV, 5.285 GeV and 5.375 GeV, respectively. Also for the excited states, they got $M_{D^*} = 2.009$ GeV, $M_{D_s^*} = 2.111$ GeV, $M_{B^*} = 5.324$ GeV and $M_{B_s^*} = 5.412$ GeV. Our masses are also consistent with theirs. Furthermore, using a relativistic quantum field model, the work of [46] obtained $M_D = 1892$ MeV, $M_{D_s} = 1998$ MeV, $M_B = 5117$ MeV and $M_{B_s} = 5232$ MeV, which are similar to what we have.

Next, we make a further investigation on the leptonic decays of the heavy–light mesons, which contain the flavor-changing transitions. In these decay processes, a quark and antiquark annihilate via a virtual $W$ boson. Therefore, these decay processes caught much attentions both in the theoretical and experimental aspects since these decay procedures can be provided many data points about the weak interactions, specified by the CKM matrix elements, and a hint for the new physics beyond the Standard Model (SM). For instance, the decay of $B \rightarrow \tau \nu_\tau$ can be related to $|V_{ub}|$. Thus, aiming at looking for the weak interaction information, at
first we study the leptonic decays of the $B$ and $D$ mesons. Let $P$ denote the heavy pseudoscalar mesons as above, which includes $B$, $D$ and $D_s$. In the SM, the purely leptonic decay widths of the heavy–light mesons can be obtained by

$$
\Gamma(P \rightarrow l\nu) = \frac{G_F^2 |V_{q_1q_2}|^2 f_P^2}{8\pi m_l^2} \left(1 - \frac{m_l^2}{M_P^2}\right)^2 M_P,
$$

where $l$ denotes the lepton and $\nu$ the corresponding neutrino, $G_F$ is the Fermi coupling constant, and $V_{q_1q_2}$ is the CKM matrix element between the quarks $q_1q_2$ inside $P$, which is $V_{cs}$, $V_{cd}$ and $V_{ub}$ for the case of $D_s, D$ and $B$, respectively. Besides, the leptonic widths are also dependent on the mass of lepton $m_l$, the decayed meson mass $M_P$ and the decay constant $f_P$.

In the nonrelativistic limit, the decay constants of the pseudoscalar ($f_P$) and vector mesons ($f_V$) can be expressed through the meson wave function at the origin by Van-Royen–Weisskopf formula [47],

$$
f_{P/V} = \sqrt{\frac{12}{M_{P/V}}} |\psi_{1,0}^{(P/V)}(g, 0)|
$$

where $|\psi_{1,0}^{(P/V)}(g, 0)|$ can be obtained from Eq. (9). Furthermore, by multiplying the lifetime with the decay width, we calculate the leptonic decay rates by $Br = \tau_P \Gamma(P \rightarrow l\nu)$, see our results later.

In the present work, we take the input parameters as follows: $G_F = 1.16637 \times 10^{-5}$ GeV$^{-2}$, $|V_{ub}| = 0.00351$, $|V_{cs}| = 0.97344$, $|V_{cd}| = 0.22520$, $m_\mu = 0.510 \times 10^{-3}$ GeV, $m_\mu = 0.105$ GeV and $m_\tau = 1.776$ GeV [45]. Our results for the decay constants $f_{P/V}$ of the beauty and charm mesons can be found in the fifth column of Table 1, which are consistent with the other works, see the ones in the sixth column. From the results of Table 1, we obtain the ratio of the charm decay constants $f_B^*/f_D^*$ = 1.203 and the one for the bottom decay constants $f_B^*/f_B^*$ = 1.251, which are consistent with the experimental ones found in PDG [45], $f_B^*/f_D^* = 1.175$ and $f_B^*/f_B^* = 1.209$, respectively, where there are the differences of about 2.37 % and 3.54 % compared with theirs, whereas the results of Ref. [49] are 1.15 for these fractions using the constituent quark model and the complete relativistic potential. Based on light-front holographic QCD, Qin Chang et al. got the ratios of $f_B^*/f_D^* = 1.129$, $f_B^*/f_B^* = 1.028$ [28], whereas we have found $f_B^*/f_D^* = 1.203$, $f_B^*/f_B^* = 0.970$, $f_{D_s}^*/f_{D_s}^* = 0.970$, $f_{D_s}^*/f_{D_s}^* = 0.970$, $f_{B_s}^*/f_{B_s}^* = 1.251$, $f_{B_s}^*/f_{B_s}^* = 0.996$ and $f_{B_s}^*/f_{B_s}^* = 0.996$, which are generally in compatible with their results within the uncertainties. The ratios of the decay constants for the bottom mesons had also been given by the lattice QCD calculations [50], $f_{B_s}^*/f_{B_s}^* = 0.958 \pm 0.022$ and $f_{B_s}^*/f_{B_s}^* = 0.974 \pm 0.010$, which are similar to what we get. It is also reasonable to apply the mentioned method for $B_c$ meson. PDG reported 6.274 GeV [45] for the mass of this meson and we have obtained $M_{B_c} = 6.225$ GeV. We have also calculated $f_{B_c} = 0.476$ GeV, which agrees with the 0.470 GeV quoted in Ref. [51] and is close to the results of relativistic quark models such as 438 MeV in Ref. [21] and 433 MeV in Ref. [19]. With mass and decay constant of $B_c$, the total decay width of $B_c$ can be calculated in the spectator approximation [47] as $20.24 \times 10^{-4}$ eV, and the predicted $B_c$ lifetime would be 0.325 ps where $19.17 \times 10^{-4}$ eV and 0.344 ps were reported for the mentioned quantities in a quark model [52].
With the leptonic decay widths that we obtain from Eq. (13), the leptonic decay branching ratios for the beauty and charm mesons are given in the second column of Table 2, where we compare with the experimental ones [45] and find that they are consistent. Recently, Fleischer et al. have given their results for the branching ratios of leptonic $D_s$ decays with the low-energy effective Hamiltonian approach as $Br(D_s^+ \rightarrow e^+\nu_e) = (1.24 \pm 0.02) \times 10^{-7}$, $Br(D_s^+ \rightarrow \mu^+\nu_\mu) = (5.28 \pm 0.08) \times 10^{-3}$ and $Br(D_s^+ \rightarrow \tau^+\nu_\tau) = (5.15 \pm 0.08) \times 10^{-7}$ [53], where our results are almost two times bigger than what they had. In the case of leptonic $B$-decay, the work of Ivanov et al. [13] obtained the branching fractions $Br(B^- \rightarrow e^-\nu_e) = 1.16 \times 10^{-11}$, $Br(B^- \rightarrow \mu^-\nu_\mu) = 0.49 \times 10^{-6}$ and $Br(B^- \rightarrow \tau^-\nu_\tau) = 1.10 \times 10^{-4}$, where our results in Table 2 are about 30% larger than theirs.

Furthermore, with the obtained masses and decay constants, we also investigate the radiative leptonic decays of the charm mesons, since the helicity suppression can be overcome by a photon radiated from the charged particles compared to the one of the pure-leptonic decay process as commented in Ref. [11]. Wang investigated the radiative leptonic decays of $B \rightarrow \gamma \ell \nu$ with the QCD factorization approach in Ref. [54], where the sub-leading power contribution to the $B \rightarrow \gamma$ form factor is taken into account with the dispersion relation and these perturbative QCD corrections to the soft two-particle contribution were found to have nontrivial contributions to the leading power ones. One step further, considering the sub-leading power correction with the light-cone sum rules, in the work of [55] Wang and Shen revisited the decays of $B \rightarrow \gamma \ell \nu$ and found that the twist-two hadronic photon contribution leaded to about 30% correction to the leading power one from the perturbative QCD factorization approach. Note that, the results of the branching fractions for $B \rightarrow \gamma \ell \nu$ in Refs. [54,55] from the QCD factorization approach were the inverse-moment $\lambda_B(\mu_0)$ and the cut of the photon energy $E^\gamma_{\text{cut}}$ dependent. Thus, the experimental upper bound for the branching fractions for $B \rightarrow \gamma \ell \nu$ cannot be precise enough to the determination of $\lambda_B(\mu_0)$ for their predictions. Therefore, in the present work, we studied the radiative leptonic decays of the $D$ and $D_s$ mesons with the nonrelativistic constituent quark model following Ref. [11], which are sensitive to the decay constants of $f_P$ for the $D$ and $D_s$ mesons. But, the masses and decay constants have been obtained in our potential model above. Thus, the decay width for the radiative leptonic decay of the $D_s$ mesons can be given by [11],

$$
\Gamma(D_s \rightarrow \gamma \ell \bar{\nu}) = \frac{\alpha}{2592 \pi^2} \frac{G_F^2 |V_{cs}|^2}{M_{D_s}^3} f_{D_s}^2 [x_s + x_c] ,
$$

(15)
Table 3 Branching ratios for the radiative leptonic decays

| Decay       | Our branching ratio | Result of Ref. [11] |
|-------------|---------------------|---------------------|
| $D \rightarrow \gamma \ell \nu$ | $13.62 \times 10^{-6}$ | $4.6 \times 10^{-6}$ |
| $D_s \rightarrow \gamma \ell \nu$ | $4.48 \times 10^{-5}$ | $1.8 \times 10^{-5}$ |

with

$$x_s = \left(3 - \frac{M_{D_s}}{m_s}\right)^2, \quad x_c = \left(3 - \frac{2M_{D_s}}{m_c}\right)^2.$$ (16)

For the case of $D$ mesons, we have [11]

$$\Gamma(D^- \rightarrow \gamma \ell \bar{\nu}) = \frac{\alpha G_F^2 |V_{cd}|^2}{2592 \pi^2} f_D^2 M_D^3 \left[ x_d + x_c \right].$$ (17)

where

$$x_d = \left(3 - \frac{M_D}{m_d}\right)^2, \quad x_c = \left(3 - \frac{2M_D}{m_c}\right)^2.$$ (18)

Using $\alpha = \frac{1}{137}$, $\tau_{D_s} = 5.04 \times 10^{-13}$ s, $\tau_D = 1.04 \times 10^{-12}$ s [45], we obtained the decay widths $\Gamma(D \rightarrow \gamma \ell \nu) = 8.613 \times 10^{-18}$ GeV and $\Gamma(D_s \rightarrow \gamma \ell \nu) = 5.842 \times 10^{-17}$ GeV, which are in the same magnitudes with the results obtained in Ref. [11], $\Gamma(D \rightarrow \gamma \ell \nu) = 2.9 \times 10^{-18}$ GeV and $\Gamma(D_s \rightarrow \gamma \ell \nu) = 2.3 \times 10^{-17}$ GeV. Thus, we have the branching ratios of the radiative leptonic decays for the charm mesons as given in Table 3, where our results are about 2.5 times bigger than the ones obtained Ref. [11] as a consequence of our larger decay widths, whereas they are in the same magnitudes. As one can see from the results in Table 3 that the branching ratio of $D \rightarrow \gamma e^+ \nu_e$ is $13.62 \times 10^{-6}$, which is close to the upper bound of the experimental one in PDG [45] $< 3.0 \times 10^{-5}$. From Table 3, we also obtained $4.48 \times 10^{-5}$ for the ratio of $D_s$ radiative leptonic decay, which is in agreement with the one shown in PDG [45] $Br(D_s \rightarrow \gamma e^+ \nu_e) < 1.3 \times 10^{-4}$.

4 Mixing mass parameter

Particle–antiparticle mixing phenomena have fundamental importance in testing the SM. With the calculated meson masses and pseudoscalar decay constants, we can also study the oscillation frequency $\Delta m_{B_q}$. $q = d, s$. This parameter is a measure of the frequency of the change from $B$ into $\bar{B}$ and can be parameterized as [56–58],

$$\Delta m_{B_q} = \frac{G_F^2 m_t^2 M_{B_q} f_{B_q}^2}{8\pi} \left[ \frac{1}{4} + \frac{9}{4(1-x_t)} - \frac{3}{2(1-x_t)^2} - \frac{3x_t^2}{2(1-x_t)^3} \right] \eta_t |V_{tq}^* V_{tb}|^2 B; \quad q = d, s$$ (19)

which depends on parameters of the SM, such as the Fermi coupling $G_F$, the CKM matrix elements ($V_{td} = 7.4 \times 10^{-3}$, $V_{tb} = 1$, $V_{ts} = 40.6 \times 10^{-3}$), the top quark ($m_t = 174$ GeV) and $W$ boson ($m_W = 80.403$ GeV) masses [58]. Moreover, we also take $x_t = \frac{m_t^2}{m_W^2}$, $B = 1.34$, representing the correction to the vacuum insertion and the numerical factor.
Table 4  Mixing parameters for the B mesons

| Meson  | Our $\Delta m_B (ps^{-1})$ | Other results               |
|--------|----------------------------|-----------------------------|
| $B_d$  | 0.38                       | 0.55 [58]                   |
|        |                             | 0.507 [45]                  |
| $B_s$  | 18.01                      | 23.88 [58]                  |
|        |                             | $17.3 \pm 2.6$ [57]        |

$\eta_t = 0.55$ standing for QCD corrections [58]. We show our calculated values for the mixing parameters for the B mesons in Table 4, where our results are comparable with the experimental data in PDG [45] and the other theoretical values in Refs. [57, 58].

5 Semileptonic decays of B mesons

It is worthwhile to study the semileptonic $B \to D^{(s)}$ decays theoretically, since these decay processes are related to the heavy quark transition of $b \to c$, which also catch the experimental interest to determine the CKM element of $V_{cb}$. In the heavy quark limit, the decay rate of $B_{(s)} \to D_{(s)} \ell \nu_\ell$ is dependent on the hadronic matrix element that coincides with the IWF. Thus, with the obtained wave functions for mesons, which are important for the determination of IWF, we continue to investigate the semileptonic B decays. The differential semileptonic decay width can be written as [59, 60],

$$
\frac{d\Gamma (\bar{B}_{(s)} \to D_{(s)} \ell \bar{\nu})}{d\omega} = \frac{G_F^2 |V_{cb}|^2}{48\pi^3} (M_{B_{(s)}} + M_{D_{(s)}})^2 M_{D_{(s)}}^3 (\omega^2 - 1)^{\frac{3}{2}}
\times |h^- (\omega) - \frac{M_{B_{(s)}} - M_{D_{(s)}}}{M_{B_{(s)}} + M_{D_{(s)}}} h^+ (\omega)|^2
$$

(20)

in terms of $\omega$, the dot product of four velocities of initial and final hadrons which is placed in the range $1 \leq \omega \leq \frac{M_{B_{(s)}}^2 + M_{D_{(s)}}^2}{2M_{B_{(s)}} M_{D_{(s)}}}$, and the form factors $h^-(\omega)$ and $h^+(\omega)$, which are given by

$$
h^-(\omega) = 0,
$$
$$
h^+(\omega) = \xi (\omega),
$$

(21)

in the heavy quark symmetry, associated with the IWF $\xi (\omega)$. In the range of semileptonic decay width, we have shown the behavior of $\frac{d\Gamma}{d\omega} (B_{(s)} \to D_{(s)} \ell \nu)$ as a function of $\omega$ in Fig. 4, where one can see that the differential decay widths grow up when $\omega$ increases.

From the lattice results based on the heavy quark symmetry, it is found that at the zero recoil point the IWF can be determined by specifying $\rho^2$, the slope parameter [23]. In Ref. [23], several phenomenological parameterizations were proposed, such as the exponential type and the pole-type, where they assumed the parameter $\rho^2$ as the important parameter for IWF. In the phenomenological approach, Hioki also took the linear form for IWF as $1 + \rho^2 (1 - \omega)$ [2]. Due to the fact that the kinematic region for the semileptonic decays lies in a small range of $\omega = 1 \to 1.43$, we took only the first two terms in the expansion of IWF near zero recoil point ($\omega = 1$). Thus, the IWF can be parameterized within a phenomenological model [23],

$$
\xi = 1 - \rho^2 (\omega - 1),
$$

(22)
Differential semileptonic decay widths of $B^s \rightarrow D^s \ell \nu$ versus $\omega$

where the IWF is related to the energy Eq. (5) and the wave function of the system through the following relation [23],

$$\rho^2 = \frac{1}{2} + \frac{E^2}{3} \int_0^\infty r^4 \psi_{n,l}^2(g,r) dr. \quad (23)$$

Using Eqs. (4) and (5), we can obtain the parameter of IWF for a certain meson as $\rho_{B_s}^2 = 0.57$, $\rho_{B_s}^2 = 0.63$, $\rho_{D_s}^2 = 0.60$ and $\rho_{D_s}^2 = 0.66$. The slope of IWF was reported as $\rho^2 = 0.81 \pm 0.22$ in QCD light-cone sum rules [61]. Also in a potential model, the IWF parameters were given as 0.6699, 0.8792, 0.6821 and 0.7968 for the $B$, $B_s$, $D$ and $D_s$ mesons, respectively [62]. In the limit of zero lepton mass, we can write for the differential semileptonic decay width of $\bar{B}^s \rightarrow D^*_s \ell \bar{\nu}$ in terms of the IWF as [47,63],

$$\frac{d\Gamma(B^s \rightarrow D^*_s \ell \bar{\nu})}{d\omega} = \frac{G_F^2}{48\pi^3} M_{D^*_s}^3 (M_{B^s} - M_{D^*_s})^2 [1 + \beta^{s1}(1)]^2 \times \sqrt{\omega^2 - 1} (\omega + 1)^2 |V_{cb}|^2$$

$$\times \xi^2(\omega) \left[ 1 + \frac{4\omega}{\omega + 1} \left( \frac{M_{B^s}^2 - 2\omega M_{B^s} M_{D^*_s}^* + M_{D^*_s}^2}{M_{B^s}^2 - M_{D^*_s}^2} \right) \right] K(\omega), \quad (24)$$

where $\beta^{s1}(1) = -0.01$ and $K(\omega) = 1$. By integrating the differential semileptonic decay widths of Eqs. (20) and (24) over the parameter $\omega$, using the life times of the $B$ mesons, $\tau_B = 1.638 \, ps$, $\tau_{B_s} = 1.512 \, ps$, and $|V_{cb}| = 0.04$ [45], we can calculate the semileptonic branching fractions of the $B$ mesons. We show the differential semileptonic decay widths of $B^s \rightarrow D^*_s \ell \bar{\nu}$ in Fig. 5, which are not much different from decays to the $D_s$ mesons. As we can see from Fig. 6, the IWFs have been normalized to unity at $\omega = 1$. We have shown our results for the semileptonic decay widths in Table 5 using IWF of Eq. (22) and the comparison of the other results.

As one can see in Table 5, our results for the $B$ and $B_s$ decays are consistent with those obtained in Ref. [14] within the uncertainties, which are obtained with the covariant light-front quark model. From the results in Table 5, we have found that the ratios of $B$ semileptonic decays as $\frac{\Gamma(B \rightarrow D^*_s \ell \bar{\nu})}{\Gamma(B \rightarrow D \ell \bar{\nu})} = 2.81$ and $\frac{\Gamma(B_s \rightarrow D^*_s \ell \bar{\nu})}{\Gamma(B_s \rightarrow D \ell \bar{\nu})} = 2.72$, which are in accordance with the
Fig. 5  Differential semileptonic decay widths of $B_{(s)} \rightarrow D^{*}_{(s)} \ell \nu$ versus $\omega$

Fig. 6  Variation of IWFs

ones obtained in Ref. [64], $\frac{\Gamma(\bar{B}_{s} \rightarrow D^{*+} \ell \bar{\nu})}{\Gamma(\bar{B} \rightarrow D^{+} \ell \bar{\nu})} = 3.2^{+3}_{-2} \pm 1.0$ and $\frac{\Gamma(B_{s} \rightarrow D^{+*} \ell \nu)}{\Gamma(B_{s} \rightarrow D_{s} \ell \bar{\nu})} = 3.3^{+2}_{-1} \pm 1.0$, with the differences of about 12.12 % and 17.62 %, respectively. Moreover, our branching ratios of semileptonic $B$ decays are close to the reported values of Hiller et al., which are $\text{Br}(B^{0} \rightarrow D^{+} \ell \bar{\nu}) = (2.23 \pm 0.24) \times 10^{-2}$ and $\text{Br}(B^{0} \rightarrow D^{*+} \ell \nu) = (5.34 \pm 0.40) \times 10^{-2}$ for $\ell = e, \mu$ [65]. In the heavy quark limit, Ivanov et al. obtained the semileptonic decay branching fractions of the $B$ mesons $\text{Br}(B^{0} \rightarrow D^{+} \ell \bar{\nu}) = 2.65$ and $\text{Br}(B^{0} \rightarrow D^{*+} \ell \bar{\nu}) = 7.21$ [13], where our results in Table 5 are in a good agreement with theirs, as well as with the ones of Ref. [8] where they found $\text{Br}(B^{0} \rightarrow D^{*+} \ell \bar{\nu}) = 4.57 \sim 9.12$. In the case of branching ratios $B_{s}$, Azizi et al. got the branching ratios of $\text{Br}(B_{s} \rightarrow D_{s} \ell \nu) = 2.8 \sim 3.5$ [7] and $\text{Br}(B_{s} \rightarrow D^{*}_{s} \ell \nu) = 1.89 \sim 6.61$ [8] using the framework of three point QCD sum rules, where our results are within their predicted range and compatible with theirs well.
Applying HQET, two general formulae for the form factors of $B \to D$ transitions can be expressed in terms of IWF [67],

$$f_+(q^2) = \xi(\omega) \frac{M_B + M_D}{2\sqrt{M_B M_D}},$$  

$$f_0(q^2) = \frac{\sqrt{M_B M_D}}{M_B + M_D} \xi(\omega) \Delta(\omega)(1 + \omega) \frac{1 + \kappa}{1 - \kappa},$$

where the scalar density is approximated about $\Delta(\omega) = 0.46 \pm 0.02$ [67] and $\kappa = \frac{M_D}{M_B}$. Note that, the range of $q^2$ places in $0 \leq q^2 \leq (M_B - M_D)^2 = q_{\text{max}}^2$. In the work of [67], with the leading order and next leading order perturbative QCD predictions, Xiao et al. reported $f_0(0) = 0.52^{+0.12}_{-0.10}$ and $f_+(0) = 0.52^{+0.12}_{-0.10}$ for the form factors of the transition $B \to D$. Taking into account the perturbative QCD corrections to the $B \to D$ form factor from the light-cone sum rules, Wang et al. obtained its value as $f_+(0) = 0.673$ [4]. With the IWF in our potential model, we have the semileptonic $B \to D$ form factors at $q^2 = 0$ as $f_+(0) = 0.726$ and $f_0(0) = 0.726$, which are a bit larger than what they obtained [4,67]. Moreover, the differential decay width of $B \to D\ell\nu$ in the rest frame of $B$ meson can be given by [4],

$$\frac{d\Gamma(B \to D\ell\nu)}{dq^2} = \frac{\eta_{EW}^2 G_F^2 |V_{cb}|^2}{24 \pi^3 M_B^2} \left( 1 - \frac{m_\ell^2}{q^2} \right)^2 |\vec{p}_D|^2 \left[ \left( 1 + \frac{m_\ell^2}{2q^2} \right) M_B^2 |\vec{p}_D|^2 |f_+(q^2)|^2 + \frac{3m_\ell^2}{8q^2} (M_B^2 - M_D^2)^2 |f_0(q^2)|^2 \right].$$

where the three momentum of $D$ meson is $|\vec{p}_D| = \sqrt{\lambda(M_B^2, M_D^2, q^2)}$ with the factor $\lambda(M_B^2, M_D^2, q^2)$ is the usual Källen triangle function and given by

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,$$

and QED corrections $\eta_{EW} = 1.0066$. With our obtained masses of the mesons and considered form factors in terms of IWF, we can plot the differential decay widths for two decays of

| Decay | $\Gamma$ (in GeV) | Our branching ratio | Other branching ratios |
|-------|------------------|---------------------|-----------------------|
| $\bar{B} \to \bar{D}\ell\bar{\nu}$ | $1.063 \times 10^{-14}$ | 2.65 | $1.8 \pm 0.5$ [2] |
| | | | $2.31 \pm 0.09$ [9] |
| | | | $2.61 \pm 0.29$ [14] |
| | | | $2.35 \pm 0.09$ [45] |
| $\bar{B} \to \bar{D}^*\ell\bar{\nu}$ | $2.989 \times 10^{-14}$ | 7.44 | $4.9 \pm 0.8$ [2] |
| | | | $6.71 \pm 0.74$ [14] |
| | | | $5.66 \pm 0.22$ [45] |
| $\bar{B}_s \to \bar{D}_s\ell\bar{\nu}$ | $1.158 \times 10^{-14}$ | 2.66 | $2.45 \pm 0.27$ [14] |
| | | | $2.1 \pm 0.2$ [66] |
| $\bar{B}_s \to \bar{D}_s^*\ell\bar{\nu}$ | $3.148 \times 10^{-14}$ | 7.23 | $7.49 - 7.66$ [24] |
| | | | $6.05 \pm 0.67$ [14] |
| | | | $5.3 \pm 0.5$ [66] |
Fig. 7 Differential decay widths for two semileptonic $B \to D$ decays as a function of the momentum transfer squared $q^2$

$B \to D \ell\bar{\nu}$ for $\ell = \mu, \tau$ as a function of the momentum transfer squared $q^2$, see the results of Fig. 7, which is in agreement with the ones obtained in Ref. [4].

6 Nonleptonic decays of $B$ and $D$ mesons

Analogously to the semileptonic decays discussed in the last section, the nonleptonic decays of $B$ and $D$ mesons can also be used as a test of the SM, since these decay procedures are also associated with the flavor changing, where some of them can also have the CP violation effects. Thus, in the present work, we study the two-body nonleptonic decays of the $B$ and $D$ mesons with a pion and the other one in their final states, based on the results of the mesonic masses and the semileptonic decay widths discussed above. For these decay channels, the nonleptonic decay width for the $B \to D(\ast)\pi$ process is given by the semileptonic differential decay width at maximal recoil [63],

$$\frac{d\Gamma(B \to D(\ast)\pi\ell\bar{\nu})}{dq^2} \times \frac{10^{12}}{|V_{cs}|^2} \text{GeV}^{-1}$$

$$\begin{array}{c}
\frac{d\Gamma(B \to D(\ast)\pi)}{dq^2} \times \frac{10^{12}}{|V_{cs}|^2} \text{GeV}^{-1} \\
\| \omega = \omega_{\text{max}} \end{array}$$

(29)

where $C|V_{ud}| \approx 1$, the pion decay constant $f_\pi = 0.130$ GeV [28].

On the other hand, the decay width of $D$ decay is given by [68],

$$\Gamma(D^0 \to K^-\pi^+) = C_f \frac{G_F^2 |V_{cs}|^2 |V_{ud}|^2 f_\pi^2}{2\pi M_D^3} (\lambda(M_D^2, M_{K^-}^2, M_{\pi}^2)^2 f_\pi^2(q^2))$$

(30)

for the case of $c \to s + u + \bar{d}$ and

$$\Gamma(D^0 \to K^+\pi^-) = C_f \frac{G_F^2 |V_{cd}|^2 |V_{us}|^2 f_\pi^2}{2\pi M_D^3} (\lambda(M_D^2, M_{K^+}^2, M_{\pi}^2)^2 f_\pi^2(q^2))$$

(31)

for the case of $c \to d + u + \bar{d}$, and the color factor is $C_f = C_A^2 + C_B^2$, with

$$C_A = \frac{1}{2}(C_+ + C_-), \quad C_B = \frac{1}{2}(C_+ - C_-),$$

(32)
Table 6 Nonleptonic decay rates of $B$ and $D$ mesons

| Decay             | Our branching ratio | Exp. results [45] | Other results |
|-------------------|---------------------|-------------------|---------------|
| $B \to D^* \pi$   | $4.884 \times 10^{-3}$ | $(4.90 \pm 0.17) \times 10^{-3}$ | $4.74 \times 10^{-3}$ [15] |
| $B \to D \pi$     | $4.720 \times 10^{-3}$ | $(4.68 \pm 0.13) \times 10^{-3}$ | $5.91 \times 10^{-3}$ [15] |
| $D^0 \to K^- \pi^+$ | $2.185 \times 10^{-2}$ | $(3.950 \pm 0.031) \times 10^{-2}$ | $4.03 \times 10^{-2}$ [1] |
| $D^0 \to K^+ \pi^-$ | $0.626 \times 10^{-4}$ | - | $(1.12 \pm 0.05) \times 10^{-4}$ [1] |

Besides, we take the form of $f_+(q^2)$, weak transition form factor as shown in Eq. (25) [13,69]. In Fig. 8, we show the decay width of $D \to K \pi$ as a function of $\omega$. Considering the IWF of Eq. (22), and taking $\tau_{D^0} = 0.410 \, \text{ps}^{-1}$ [45], $M_\pi = 0.139 \, \text{GeV}$ [45], $M_K = 0.493 \, \text{GeV}$ [45], $|V_{us}| = 0.225$ [45] and $|V_{ud}| = 0.974$ [45], we obtain the nonleptonic decay rates with Eqs. (30), (31) as well as Eq. (29) and show them in Table 6, where one can see that our results are consistent with the other results in Refs. [1,15,45]. Besides, our results of $Br(B^0 \to D^+ \pi^-) = 0.345\%$ and $Br(B^0 \to D^+ \pi^-) = 0.331\%$ are in agreement with the ones obtained in Ref. [10], $Br(B^0 \to D^+ \pi^-) = 0.345\%$ and $Br(B^0 \to D^+ \pi^-) = 0.331\%$.

7 Conclusions

In the present work, we have presented a phenomenological model based on the screened potential, and obtained the energy, the wave functions and the masses for the beauty and charm mesons using a variational method. Consequently, we study the decay properties of the heavy–light mesons, such as the leptonic decays, the radiative leptonic decays, the semileptonic decays and two-body nonleptonic decays. Through this work, we have got the results of Tables 1, 2, 3, 4, 5 and 6, where the results we provided give a satisfactory
description of the properties of the beauty and charm mesons and are compatible with the other theoretical and experimental results. Thus, our results can be useful for the further studying of the properties of the $B$ and $D$ mesons and their branching fractions for the leptonic decays, the radiative decays, the semileptonic decays and the nonleptonic decays.

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