Abstract
Let $H$ be a complex Hilbert space. Every nonnegative operator $L \in L(H)$ admits a unique nonnegative square root $R \in L(H)$, i.e., an nonnegative operator $R \in L(H)$ such that $R^2 = L$. Let $GL^+_S(H)$ be the set of nonnegative isomorphisms in $L(H)$. First we will prove that $GL^+_S(H)$ is convex and is a (real) Banach manifold. Denote by $L^{1/2}$ the square root nonnegative of the operator $L$. The objective of this paper is to prove that application $R : GL^+_S(H) \rightarrow GL^+_S(H)$, defined by $R(L) = L^{1/2}$, is a homeomorphism.

Key-words: nonnegative operators, functions of operators, Hilbert spaces, spectral theory.

1. Introduction

Let $H$ be a complex Hilbert space. The Theorem 2.8 shows that every nonnegative bounded operator admits a unique nonnegative square root. The main goal of this paper is to prove that the application that for each nonnegative isomorphism associates its nonnegative square root is a homeomorphism.

P.M. Fitzpatrick, J. Pejsachowicz and L. Recht in [2], use this square root to show the existence of a parametrix cogredient to a path of self-adjoint Fredholm operators, which is used for to define the spectral flux for this path. The square root is also used to find the polar decomposition of a bounded operator (see, for example, T. Kato in [3], p. 334). This polar decomposition permits to determine the positive and negative spectral subspaces of any self-adjoint operator.
In the next chapter we will remember some notions as: spectrum of an
operator, self-adjoint operator (nonnegative, positive), nonnegative square
root of a nonnegative operator, among others. We will also see several known
results in functional analysis that will be used in the rest of the work.

We shall denote by $GL^+_S(H)$ the subset of $L(H)$ of positive isomorphisms.
In the third chapter we will prove that $GL^+_S(H)$ is convex (i.e., if $L$ and
$T \in GL^+_S(H)$, $tL + (1 - t)T \in GL^+_S(H)$ for every $t \in [0, 1]$) and is a (real)
Banach manifolds.

Let $L \in L(E)$, where $E$ is a complex Banach space. Based on the Cauchy’s
integral formula, in Chapter 4 we will see that if $f : \Delta \to \mathbb{C}$ is a holomorphic
application, where $\Delta$ is an open subset of $\mathbb{C}$ that contains the spectrum of
$L$, denoted by $\sigma(L)$, we can define the operator $f(L) \in L(E)$ as

$$f(L) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(L - \lambda I)^{-1} d\lambda,$$

where $\Gamma$ is a positively oriented closed path (or a finite number of closed
paths which do not intersect) simple, contained in $\Delta$ and that contains
$\sigma(L)$ therein.

Finally, in Chapter 5, we prove that if $L \in L(H)$ is positive, the nonne-
gative square root of $L$ can be expressed in the form $\gamma(L)$, where $\gamma : \Theta \to \mathbb{C}$
is a appropriate holomorphic application and $\Theta$ is an open subset of the
complex numbers, which contains the spectrum of $L$. Using this expression,
we shall show that the application square root $\mathcal{R} : GL^+_S(H) \to GL^+_S(H)$ is
continuous.

2. Preliminary

In this chapter we will remember some notions and results that will be
of much use at work. Throughout the work, $E$ and $F$ shall denote complex
Banach spaces and $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
Denote by $L(E, F)$ (or $L(E)$ if $F = E$) the Banach space of bounded linear
operators $L : E \to F$ with the norm

$$\|L\| = \sup_{x \in E} \frac{\|Lx\|}{\|x\|}.$$

We say that an operator $L \in L(E, F)$ is invertible (or is a isomorphism) if
its inverse, which we denote by $L^{-1}$, is limited. If $F = E$, $I \in L(E)$ denotes
the identity operator.

**Proposition 2.1.** Let $L \in L(E, F)$ be a invertible operator. If $A \in L(E, F)$
and $\|A - L\| < 1/\|L^{-1}\|$, then $A$ is invertible.

*Demonstração.* See, for example, [3], p. 31.

We shall denote by $\text{Im} f$ the image of a application $f$.

**Proposition 2.2.** Suppose that $L \in L(E, F)$ is injective. Then, $L^{-1} : 
\text{Im} L \to E$ is bounded if and only if there exists $c > 0$ such that $\|Lx\| \geq c\|x\|$
for all $x \in E$. Moreover, if $L^{-1} : \text{Im} L \to E$ is continuous, then $\text{Im} L$ is a Banach space.

Demonstração. Suppose first that there exists $c > 0$ such that $\|Lx\| \geq c\|x\|$ for all $x \in E$ and prove that $L^{-1} : \text{Im} L \to E$ is continuous. Indeed, it is clear that $L^{-1} : \text{Im} L \to E$ is bijective. If $y \in \text{Im} L$, we have

$$\|y\| = \|LL^{-1}y\| \geq c\|L^{-1}y\|.$$ 

That is, $\|L^{-1}y\| \leq (1/c)\|y\|$ for all $y \in \text{Im} L$, which proves that $L^{-1} : \text{Im} L \to E$ is continuous.

Conversely, if $L^{-1} : \text{Im} L \to E$ is continuous, there exists $C > 0$ such that $\|T^{-1}y\| \leq C\|y\|$ for all $y \in \text{Im} L$. Since $L$ is injective, then $LL^{-1}Lx = x$ for all $x \in E$. Hence,

$$\|x\| = \|L^{-1}Lx\| \leq C\|Lx\|$$ 

for all $x \in E$.

Consequently, there exists $1/C > 0$ such that $(1/C)\|x\| \leq \|Lx\|$ for all $x \in E$.

We shall now that $\text{Im} L$ is a Banach space. Let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\text{Im} L$. So, $y_n = Lx_n$ for a sequence $(x_n)_{n=1}^{\infty}$ in $E$. Now, by hypothesis,

$$\|x_n - x_m\| \leq (1/c)\|L(x_n - x_m)\| = (1/c)\|y_n - y_m\|.$$ 

This fact implies that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since $E$ is Banach, $(x_n)_{n=1}^{\infty}$ converges to someone $x \in E$. Therefore, by the continuity of $L$ we have $(y_n)_{n=1}^{\infty}$ converges to $L(x) \in \text{Im} L$. Hence $\text{Im} L$ is a Banach space. □

**Definition 2.3** (Spectrum of an operator). Let $L$ be an operator $L(E)$. A regular value of $L$ is a number $\lambda \in \mathbb{C}$ such that the operator $L - \lambda I$ is a isomorphism. The set of regular values $L$, denoted by $\rho(L)$, is called resolvent set of $L$. Its complement $\sigma(L) = \mathbb{C} - \rho(L)$ is called spectrum of $L$. For $\lambda \in \rho(L)$, the application $R(\lambda) = (L - \lambda I)^{-1}$ and called resolvent application of $L$.

It is well known that the spectrum of an operator $L \in L(E)$ is a subset not empty $\mathbb{C}$, compact and

$$(2.1) \quad \|\lambda\| \leq \|L\| \quad \text{for all } \lambda \in \sigma(L).$$

As we said in the introduction, the objective of this paper has to do with the square root of an nonnegative operator. Remember here in these notions.

**Definition 2.4** (Self-adjoint operator). Let $L \in L(H)$. We say that $L$ is self-adjoint if

$$\langle Lx, x \rangle = \langle x, Ly \rangle \quad \text{for all } x, y \in H.$$

A characteristic of the spectrum of self-adjoint operators is given in the following theorem, whose proof can be seen in [1], p. 324, Theorem 6.11-A.
Theorem 2.5. The spectrum of a self-adjoint operator \( L \in L(H) \) is contained in the interval \([m, M] \subseteq \mathbb{R}\), where
\[
m = \inf_{\|x\|=1} \langle Lx, x \rangle \quad \text{and} \quad m = \sup_{\|x\|=1} \langle Lx, x \rangle.
\]

Definition 2.6. Let \( L \in L(H) \) be a self-adjoint operator. If \( \langle Lx, x \rangle \geq 0 \) for all \( x \in H \), we say that \( L \) is nonnegative. If \( \langle Lx, x \rangle > 0 \) (\( \langle Lx, x \rangle < 0 \)) for all \( x \in H \) with \( \|x\| = 1 \), we say that \( L \) is positive.

It is not difficult to see that if \( L \) is positive, then \( L \) is injective.

Definition 2.7 (Square root). Let \( L \in L(H) \) be a nonnegative operator. A square root of \( L \) is an operator \( R \in L(H) \) such that \( R^2 = L \).

Of course, if \( R \) is a square root of a nonnegative operator \( L \), then \( R \) is also a square root of \( L \). In general, a nonnegative operator may have several square roots. The following theorem, whose proof can be found, for example, in [5], p. 476, Theorem 9.4-2, shows that each nonnegative operator has exactly one nonnegative square root.

Theorem 2.8 (Nonnegative square root). Each nonnegative operator \( L \in L(H) \) has a nonnegative square root \( R \), which is unique. The operator \( R \) commutes with each operator in \( L(H) \) that commute with \( L \).

The demonstration of the previous theorem, given by Kreyszig in [5], is to prove that the sequence \((R_n)_{n=1}^\infty\), where \( R_0 = 0 \) and
\[
R_{n+1} = R_n + \frac{1}{2}(L - R_n^2), \quad n = 0, 1, 2, ..., \tag{2.2}
\]
converges pointwise to nonnegative operator \( R \) such that \( R^2 = L \), this is, the nonnegative square root of \( L \). However, the theorem is not an explicit form of the operator \( R \).

In the case where \( H \) has finite dimension, it is not difficult to find the nonnegative square root of a nonnegative operator, as we will see in the following example.

Example 2.9. Suppose that \( \dim H = n < \infty \) and let \( L \in L(H) \) be a nonnegative operator. Since \( L \) is self-adjoint, there exists an orthonormal basis \( \{v_1, v_2, ..., v_n\} \) of \( H \) and complex numbers \( \lambda_1, \lambda_2, ..., \lambda_n \) (not necessarily different) such that
\[
L v_i = \lambda_i v_i \quad \text{for} \quad i = 1, 2, ..., n. \tag{2.3}
\]
Since \( L \) is nonnegative, then the \( \lambda_1, \lambda_2, ..., \lambda_n \) are nonnegative real numbers. Take \( R \in L(H) \), defined by
\[
R v_i = \sqrt{\lambda_i} v_i \quad \text{for} \quad i = 1, 2, ..., n.
\]
So, is \( R \) nonnegative and \( R^2 = L \), ie, \( R \) is the nonnegative square root of \( L \).
When $H$ has infinite dimension, we cannot always have the expressed in the equation (2.3). However, in Chapter 5 we will give an explicit formula for the square root for positive definite isomorphism $L$, that is not simple though, serve to reach our goal.

We shall end this chapter showing that the resolvent application is holomorphic.

**Definition 2.10** (Applications holomorphic). Let $\Delta$ be an open subset of $\mathbb{C}$. We say that an application $f : \Delta \rightarrow E$ is **holomorphic** on $\lambda_0 \in \Delta$ if there exists the limit

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f'(\lambda_0).$$

If $f$ is holomorphic at each point of $\Delta$, we say that $f$ is **holomorphic** on $\Delta$ or simply that $f$ is **holomorphic**.

**Proposition 2.11.** Let $L \in L(E)$. The application $R : \rho(L) \rightarrow L(E)$, defined by $R(\lambda) = (L - \lambda I)^{-1}$ for $\lambda \in \rho(L)$, is holomorphic.

**Demonstração.** It is easy to see that $R$ is continuous, because is composition of continuous applications. Let us fix $\lambda_0$ in $\rho(L)$. Note that if $L$ and $T \in L(E, F)$ are invertible, then

$$L^{-1} - T^{-1} = -L^{-1}(L - T)T^{-1}. \quad (2.4)$$

In fact,

$$L^{-1} - T^{-1} = L^{-1}L(L^{-1} - T^{-1})TT^{-1}$$

$$= L^{-1}(I - LT^{-1})TT^{-1}$$

$$= -L^{-1}(L - T)T^{-1}.$$ 

Thus, if $\lambda \in \rho(L)$, by (2.4) we have

$$(L - \lambda I)^{-1} - (L - \lambda_0 I)^{-1} = -(L - \lambda I)^{-1}(\lambda_0 I - \lambda I)(L - \lambda_0 I)^{-1},$$

that is,

$$\frac{(L - \lambda I)^{-1} - (L - \lambda_0 I)^{-1}}{(\lambda - \lambda_0)} = (L - \lambda I)^{-1}(L - \lambda_0 I)^{-1}.$$ 

Consequently,

$$\lim_{\lambda \to \lambda_0} \frac{(L - \lambda I)^{-1} - (L - \lambda_0 I)^{-1}}{(\lambda - \lambda_0)} = (L - \lambda_0 I)^{-2},$$

which proves the proposition. \qed
3. Some topological properties of $GL_S^+(H)$

Recall that we denote by $GL_S^+(H)$ the set of positive isomorphisms in $L(H)$. In this section we shall show that $GL_S^+(H)$ is convex and, moreover, is a (real) Banach manifold.

First, assume that $H$ is finite-dimension. Take $L, T \in GL_S^+(H)$ and $t \in [0,1]$. Note that $tL + (1-t)T$ is positive, because if $x \neq 0 \in H$,

$$\langle tL + (1-t)T x, x \rangle = \langle tLx, x \rangle + \langle (1-t)Tx, x \rangle = t\langle Lx, x \rangle + (1-t)\langle Tx, x \rangle > 0.$$  

Thus, $tL + (1-t)T$ is injective. Since $H$ is finite-dimensional, $tL + (1-t)T$ is an isomorphism. Therefore $tL + (1-t)T \in GL_S^+(H)$. Hence $GL_S^+(H)$ is a convex subset of $L(H)$.

To show that $GL_S^+(H)$ is convex in the case where $H$ is infinite-dimensional, first let's look at the following known result in the theory of operators in spaces of Hilbert, which we give a proof for reasons of precision.

**Lemma 3.1.** If $L : H \rightarrow H$ is a positive isomorphism, then there exists $c > 0$ such that

$$\inf_{\|x\|=1} \langle Lx, x \rangle \geq c.$$

**Demonstração.** Let $x, y \in H$ be such that $\langle Lx, y \rangle \neq 0$. For $a \in \mathbb{R}$, we take $w_a = x + a \langle Lx, y \rangle y$. Since $L$ is positive, we have

$$0 \leq \langle Lw_a, w_a \rangle = \langle L(x + a \langle Lx, y \rangle y), x + a \langle Lx, y \rangle y \rangle = \langle Lx, x \rangle + \langle Lx, a \langle Lx, y \rangle y \rangle + \langle L(a \langle Lx, y \rangle y), x \rangle + \langle L(a \langle Lx, y \rangle y), a \langle Lx, y \rangle y \rangle = \langle Lx, x \rangle + a \langle Lx, y \rangle \langle Lx, y \rangle + a^2 \langle Lx, y \rangle \langle Ly, y \rangle = \langle Lx, x \rangle + \frac{a^2}{\langle Lx, y \rangle} \langle Lx, y \rangle + \frac{a^2}{\langle Ly, y \rangle} \langle Ly, y \rangle.$$

Thus, for all $a \in \mathbb{R}$,

$$\langle Lx, x \rangle + 2a \langle Lx, y \rangle \langle Lx, y \rangle + a^2 \langle Lx, y \rangle \langle Ly, y \rangle \geq 0.$$  

Consequently, taking the left half of the earlier inequality as a polynomial in $a$, we have that the discriminant of this polynomial is less or equal than 0, that is,

$$\begin{align*}
(2\langle Lx, y \rangle \langle Lx, y \rangle)^2 - 4\langle Lx, y \rangle \langle Ly, y \rangle \langle Lx, x \rangle &\leq 0. \\
\end{align*}$$

Since $\langle Lx, y \rangle \neq 0$, $\langle Lx, y \rangle \langle Lx, y \rangle > 0$. Therefore, by the inequality (3.1), we have $\langle Lx, y \rangle \langle Lx, y \rangle - \langle Ly, y \rangle \langle Lx, x \rangle \leq 0$, that is,

$$\langle Lx, y \rangle \langle Lx, y \rangle \leq \langle Ly, y \rangle \langle Lx, x \rangle.$$
Is clear that the previous inequality also holds when \( \langle Lx, y \rangle = 0 \) because \( L \) is positive, hence holds for all \( x, y \in H \). Thus, taking \( x \in H \), with \( \|x\| = 1 \), and \( y = Lx \), by (3.2) we have

\[
\|Lx\|^4 \leq \langle L^2 x, Lx \rangle \langle Lx, x \rangle \leq \|L^2\| \|Lx\|^2 \langle Lx, x \rangle
\]

that is,

(3.3) \quad \|Lx\|^2 \leq \|L\| \langle Lx, x \rangle \quad \text{for all } x \in H \text{ with } \|x\| = 1.

Since \( L \) is a isomorphism, it follows from Proposition 2.2 that there exists \( c_1 > 0 \) such that

\[
c_1 \leq \inf_{\|x\|=1} \|Lx\|^2.
\]

Taking \( c = c_1/\|L\| \), by (3.3) we have

\[
c \leq \inf_{\|x\|=1} \langle Lx, x \rangle,
\]

which proves the lemma.

**Theorem 3.2.** The set \( GL^+_S(H) \) is convex.

**Demonstração.** Let \( L \) and \( T \) be positive isomorphisms and \( t \in [0, 1] \). Since \( tL + (1 - t)T \) is positive, we have

\[
\text{Ker}(tL + (1 - t)T) = \{0\}.
\]

Now, we shall show that \( tL + (1 - t)T \) is surjective. To this end, let us first see that \( \text{Im}(tL + (1 - t)T) \) is closed. By the previous lemma, we have that there exists positive real numbers \( c_1 \) and \( c_2 \) such that

\[
c_1 \leq \inf_{\|x\|=1} \langle Lx, x \rangle \quad \text{and} \quad c_2 \leq \inf_{\|x\|=1} \langle Tx, x \rangle.
\]

Therefore, if \( x \in H \) with \( \|x\| = 1 \), by the Cauchy-Schwarz inequality we obtain that

\[
\|(tL + (1 - t)T)x\| \geq \langle (tL + (1 - t)T)x, x \rangle = t\langle Lx, x \rangle + (1 - t)\langle Tx, x \rangle \geq (tc_1 + (1 - t)c_2)\|x\|.
\]

Hence, since \( tc_1 + (1 - t)c_2 > 0 \), it follows from Proposition 2.2 that \( \text{Im}(tL + (1 - t)T) \) is closed. Thus, since \( tL + (1 - t)T \) is self-adjoint and knowing the fact that for any operator \( S \in L(H) \), \( [\text{Ker}S]^\perp = \overline{\text{Im}(S^*)} \), we obtain that

\[
\text{Im}(tL + (1 - t)T) = \overline{\text{Im}(tL + (1 - t)T^*)} = \overline{\text{Im}(tL + (1 - t)T)^*} = [\text{Ker}(tL + (1 - t)T)]^\perp = H,
\]

that is, \( tL + (1 - t)T \) is surjective. Consequently, \( tL + (1 - t)T \) is a positive isomorphism. \( \Box \)
We denote by $L_S(H)$ the space of self-adjoint operators in $L(H)$. Note that $L_S(H)$ is not vector subspace $L(H)$, because if $L \neq 0 \in L_S(H)$, $iL \notin L_S(H)$. However, it is not difficult to prove that $L_S(H)$ is a vector space over the field of real numbers. Whereas $L_S(H)$ with the topological subspace structure of $L(H)$, is easy to see that it is a real Banach space.

We shall now show that $GL^+_S(H)$ is an open subset of $L_S(H)$.

**Proposition 3.3.** The set $GL^+_S(H)$ is open in $L_S(H)$.

**Demonstração.** Let $L \in GL^+_S(H)$. If follows from Lemma 3.1 that there exists $c > 0$ such that

$$c \leq \inf_{\|x\|=1} \langle Lx, x \rangle.$$

Let $T \in L_S(H)$ be such that $\|L - T\| < \min\{1/\|L\|^{-1}, c\}$. Thus, $T$ is an isomorphism (see Lemma 2.1) and, furthermore, for $x \neq 0 \in H$, we have

$$\langle Lx, x \rangle \geq c\langle x, x \rangle > \|L - T\|\langle x, x \rangle = \langle \|L - T\|x, x \rangle \geq \langle (L - T)x, x \rangle,$$

that is,

$$0 < \langle Lx, x \rangle - \langle (L - T)x, x \rangle = \langle Tx, x \rangle \quad \text{for all } x \in H \text{ with } x \neq 0.$$

So, $T$ is a positive isomorphism. □

Since $L_S(H)$ is a real Banach space, it follows from Proposition 3.3 that $GL^+_S(H)$ is a differentiable Banach manifold, with

$$T_LGL^+_S(H) \cong L_S(H) \quad \text{for } L \in GL^+_S(H),$$

where $T_LGL^+_S(H)$ denotes the tangent space of $L \in GL^+_S(H)$.

4. Functions of operators

Let $L \in L(E)$ be fixed. Based on the Cauchy’s integral formula, in this chapter we will see that if $f : \Delta \to \mathbb{C}$ is a holomorphic application, where $\Delta$ is an open subset of $\mathbb{C}$ containing $\sigma(L)$, we can define the operator $f(L) \in L(E)$ as

$$f(L) = -\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(L - \lambda I)^{-1}d\lambda,$$

where $\Gamma$ is a path (or a finite number of paths that do not intersect) closed, simple, positively oriented, contained in $\Delta$ and containing $\sigma(L)$ in its interior.

In Chapter 5 we shall prove that if $L \in GL^+_S(H)$, the nonnegative square root of $L$ can be expressed in the form $\gamma(L)$, where $\gamma : \Theta \to \mathbb{C}$ is a appropriated holomorphic application and $\Theta$ is an open subset of the complex numbers containing $\sigma(L)$.

We shall first remember some concepts such as: rectifiable paths (closed, single or positively oriented), Riemann’s integral of holomorphic applications, among others.
Definition 4.1. Let $\Gamma : [a, b] \to \mathbb{C}$ be a path, that is, a continuous application. For any partition of $[a, b]$, given by $P = \{t_0, t_1, ..., t_m\}$, we define

$$\Lambda_\Gamma(P) = \sum_{k=1}^{m} \|\Gamma(t_k) - \Gamma(t_{k-1})\|.$$ 

If

$$\Lambda_\Gamma = \sup\{\Lambda_\Gamma(P) : P \text{ is a partition of } [a, b]\}$$

is finite, thus we say that $\Gamma$ is rectifiable and its length is $\Lambda_\Gamma$.

Definition 4.2. Let $\Gamma : [a, b] \to \mathbb{C}$ be a path.

We say that $\Gamma$ is closed if $\Gamma(a) = \Gamma(b)$. In this case the interior of $\Gamma$, denoted by $\hat{\Gamma}$, is a region of $\mathbb{C}$ bounded by $\Gamma$.

We say that $\Gamma$ is simple if $\Gamma(t_1) \neq \Gamma(t_2)$ for $t_1, t_2 \in [a, b]$, with $t_1 \neq t_2$ and at least one of them is an interior point $[a, b]$. If $t_1 < t_2$, we use the notation $\Gamma(t_1) < \Gamma(t_2)$, whenever $\Gamma(t_1) \neq \Gamma(t_2)$. For simplicity, the image $\Gamma([t_1, t_2])$ is denoted by $[\Gamma(t_1), \Gamma(t_2)]$. Furthermore, we shall write $\lambda \in \Gamma$ to denote that $\lambda$ belongs to $\Gamma([a, b])$.

A closed path $\Gamma$ is positively oriented if it traversed in a counterclockwise direction, i.e., its interior is on the left, go to the $\Gamma$.

For simplicity, a closed, simple, positively oriented and rectifiable path $\Gamma : [a, b] \to \mathbb{C}$ is called closed path, and moreover, it will be denoted by $\Gamma$.

Now recall the definition of the integral of an application along a path contained in the complex plane. Consider a rectifiable path $\Gamma : [a, b] \to \Delta$. A partition of $\text{Im } \Gamma$ is a subset $P = \{\lambda_0, \lambda_1, \lambda_2, ..., \lambda_n\} \subseteq \text{Im } \Gamma$, where $\lambda_0 = \Gamma(a)$ and $\lambda_n = \Gamma(b)$, such that

$$\lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n.$$ 

The norm of the partition $P$ of $\text{Im } \Gamma$ is defined by

$$\|P\| = \max\{\lambda_i - \lambda_{i-1} : i = 1, ..., n\}.$$ 

Now take one application $f : \Delta \to E$ and a rectifiable path $\Gamma : [a, b] \to \Delta$. For a partition $P = \{\lambda_0, \lambda_1, \lambda_2, ..., \lambda_n\}$ of the image of $\Gamma$, let $Q = \{\zeta_1, \zeta_2, ..., \zeta_n\}$, where $\zeta_i \in [\lambda_{i-1}, \lambda_i]$, for $i = 1, 2, ..., n$. Consider the sum

$$S(P, Q, f) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})f(\zeta_i).$$

(4.1)

Definition 4.3 (Integrable applications). We say that $f : \Delta \to E$ is integrable in the path $\Gamma$ if there exists a number $A$ with the following property: For given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|S(P, Q, f) - A| < \varepsilon$$

whenever $\|P\| < \delta$.

The number $A$ is called integral of $f$ on $\Gamma$ and we shall denote by

$$\int_{\Gamma} f(\lambda)d\lambda.$$
In the case of real continuous functions of a real variable, the analog notion of integral above is equivalent to the classical definition of the Riemann integral, given with the upper and lower sums.

**Theorem 4.4.** If \( f : \Delta \to E \) is continuous thus is integrable in any rectifiable path contained in \( \Delta \).

We can to find a prove of the above Theorem when \( E = \mathbb{C} \), for example, in [4], Chapter 3, §9. However, it is not difficult to see that the same proof also applies to the case where \( E \) is any complex Banach space.

It follows from Theorem 4.4 and Definition 4.3 that, if \( f \) is continuous, for any sequence \((P_n, Q_n)_{n=1}^{\infty}\) of partitions of \( \Gamma \) such that \( \lim_{n \to \infty} \|P_n\| = 0 \), thus

\[
\int_{\Gamma} f(\lambda) d\lambda = \lim_{n \to \infty} S(P_n, Q_n, f).
\]

A property of the above integral is given in the following lemma (see, for example, in [4], p. 45, Theorem 5).

**Lem 4.5.** If \( f : \Delta \to E \) is holomorphic, thus, for any rectifiable path \( \Gamma \subseteq \Delta \),

\[
\left\| \int_{\Gamma} f(\lambda) d\lambda \right\| \leq Ml,
\]

where \( M = \sup_{\lambda \in \Gamma}|f(\lambda)| \) and \( l \) is the length of \( \Gamma \).

We shall below present some classical results of complex functions theory that will be used in this section. The following definition is analogous to the Cauchy integral formula for holomorphic complex applications (see, for example, [4], p. 61).

**Definition 4.6.** Suppose that \( L \in L(E) \) and let \( f : \Delta \to \mathbb{C} \) be a holomorphic application such that \( \sigma(L) \subseteq \Delta \). Let \( \omega \subseteq \mathbb{C} \) be a open set such that its boundary consists of a finite number of closed paths \( \Gamma_1, \ldots, \Gamma_n \) and

\[
\sigma(L) \subseteq \omega = \bigcup_{i=1}^{n} \Gamma_i \subseteq \bigcup_{i=1}^{n} \hat{\Gamma}_i \subseteq \Delta.
\]

The operator \( f(L) \) is defined by

\[
f(L) = -\frac{1}{2\pi i} \int_{\partial \omega} f(\lambda)(L - \lambda I)^{-1} d\lambda.
\]

The existence of above integral follows from Theorem 4.4 because the application \( \lambda \mapsto f(\lambda)(L - \lambda I)^{-1} \) is continuous (Proposition 2.11). Therefore, \( f(L) \in L(E) \).

The following theorem, whose proof can be found, for example, in [6], p. 136, Theorem 6.12, we see a first property of the operator defined above.
Theorem 4.7. Let $L$ be an operator on $L(E)$ and $\Gamma$ be a closed path such that $\sigma(L) \subseteq \Gamma$. Then, for each positive integer $k$, we have

$$L^k = -\frac{1}{2\pi i} \int_{\Gamma} \lambda^k (L - \lambda I)^{-1} d\lambda.$$ 

It is well known that if $L \in L(E)$ and $f : \mathbb{C} \to \mathbb{C}$ is a polynomial given by $f(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, where $a_0, a_1, \ldots, a_n \in \mathbb{C}$, the operator $f(L)$ is defined as

$$f(L) = \sum_{k=0}^{n} a_k L^k,$$ 

where $L^0 = I$.

As a consequence of the previous theorem we have

$$f(L) = \sum_{k=0}^{n} a_k L^k = -\frac{1}{2\pi i} \int_{\Gamma} \lambda^k (L - \lambda I)^{-1} d\lambda = -\frac{1}{2\pi i} \int_{\Gamma} \sum_{k=0}^{n} a_k \lambda^k (L - \lambda I)^{-1} d\lambda = -\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(L - \lambda I)^{-1} d\lambda.$$ 

Thus, $\sum_{k=0}^{n} a_k L^k$ coincides with the definition given in the formula (4.3).

We shall end this chapter by presenting the following properties of operator $f(L)$, whose proofs can be found, for example, in [6], p. 138, Lemma 6.15 and p. 139, Theorem 6.17, respectively.

Lem 4.8. Let $L$ be an operator in $L(E)$. Suppose that $f : \Delta \to \mathbb{C}$ and $g : \Delta \to \mathbb{C}$ are holomorphic applications and $\sigma(L) \subseteq \Delta$. If $h : \Delta \to \mathbb{C}$ is defined by $h(\lambda) = f(\lambda)g(\lambda)$, then

$$h(L) = f(L)g(L).$$

Lem 4.9. Let $L$ be an operator in $L(E)$. If $f : \Delta \to \mathbb{C}$ is holomorphic in an open neighborhood of $\sigma(L)$, then

$$\sigma(f(L)) = f(\sigma(L)),$$

that is, $\lambda \in \sigma(f(L))$ if and only if $\lambda = f(\zeta)$ for someone $\zeta \in \sigma(L)$.

5. Continuity of nonnegative square root application

As stated in the introduction, we will prove that the application that for each nonnegative isomorphism $L \in L(H)$ associates its nonnegative square root $R \in L(H)$ is a homeomorphism. For this purpose, we shall see that there exists a holomorphic application $\gamma : \Theta \to \mathbb{C}$, with $\sigma(L) \subseteq \Theta \subseteq \mathbb{C}$, such that $R = \gamma(L)$. 
In fact, let $\Theta = \{ a + ib \in \mathbb{C} : a > 0 \}$ and let

$$\gamma : \Theta \to \mathbb{C}, \quad \text{give by } \lambda \mapsto |\lambda|^{1/2} e^{i \text{Arg } z},$$

where $\text{Arg } z$ denotes principal argument of $z \in \mathbb{C}$. We can to see, for example, in [1], p. 64, Example 15, that $\gamma$ is holomorphic in $\Theta$. It is easy to see that

$$\gamma(\lambda) = \gamma(\lambda) \quad \text{and} \quad \gamma(\lambda)^2 = \gamma(\lambda)\gamma(\lambda) = \lambda \quad \text{for all } \lambda \in \Theta.$$

Let $L$ be an operator in $GL_S^+(H)$. It follows from Theorem 2.5 that $\sigma(L)$ is a subset of positive real numbers. Hence, $\sigma(L) \subseteq \Theta$. Consequently, as we saw (Definition 4.6), for an appropriate path $\Gamma$, we can to define

$$\gamma(L) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda) (L - \lambda I)^{-1} d\lambda.$$

We shall now that $\gamma(L)$ is the nonnegative square root of $L$. For this purpose, we shall need the following basic results.

**Lem 5.1.** Let $L \in L(H)$ be an isomorphism and $R \in L(H)$ be an operator such that $R^2 = L$, then $R$ is an isomorphism.

Demonstração. It is clear.

**Lem 5.2.** If $L \in L(H)$ is a nonnegative isomorphism, then $L$ is positive.

Demonstração. We shall show that $\langle Lx, x \rangle > 0$ for each $x \in H$, with $\|x\| = 1$. Let $R$ be the nonnegative square root of $L$ (Theorem 2.8). Since $L$ is a isomorphism, $R$ is an isomorphism (Lemma 5.1). Let $x \in H$, with $\|x\| = 1$. Since $R^2 = L$ and $R$ is self-adjoint, we find that

$$\langle Lx, x \rangle = \langle R^2 x, x \rangle = \langle Rx, Rx \rangle = \|Rx\|^2 > 0.$$

**Lem 5.3.** Let $L \in L(H)$ be self-adjoint, then

$$\inf_{\|x\|=1} \langle Lx, x \rangle = \min_{\lambda \in \sigma(L)} \lambda.$$

Demonstração. Let

$$\lambda_0 = \min_{\lambda \in \sigma(L)} \lambda \quad \text{and} \quad m = \inf_{\|x\|=1} \langle Lx, x \rangle.$$

It follows from Theorem 2.5 that $\lambda_0 \geq m$. Suppose that $\lambda_0 > m$. Thus $L - mI$ is a isomorphism and, for all $x \in H$, with $\|x\| = 1$,

$$\langle (L - mI)x, x \rangle = \langle Lx, x \rangle - \langle mx, x \rangle = \langle Lx, x \rangle - m \geq 0,$$

that is, $L - mI$ is a nonnegative operator. Since $L - mI$ is a nonnegative isomorphism, $L - mI$ is positive by Lemma 5.2. It follows from Lemma 3.1 that there exists $c > 0$ such that

$$\inf_{\|x\|=1} \langle (L - mI)x, x \rangle \geq c.$$
However,
\[
\inf_{\|x\|=1} \langle (L - mI)x, x \rangle = \inf_{\|x\|=1} \left[ \langle Lx, x \rangle - \langle mx, x \rangle \right]
\]
\[
= \inf_{\|x\|=1} \langle Lx, x \rangle - m
\]
\[
= m - m = 0.
\]
It is contradicting that
\[
\inf_{\|x\|=1} \langle (L - mI)x, x \rangle > 0.
\]
Consequently, \( \lambda_0 = m \). □

**Theorem 5.4.** Let \( L \in GL^+_S(H) \). There exists a closed path \( \Gamma \subseteq \Theta \), with \( \sigma(L) \subseteq \Gamma \), such that

\[
L^{1/2} = \gamma(L) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(L - \lambda I)^{-1}d\lambda,
\]
where \( L^{1/2} \) denotes the nonnegative square root of \( L \).

**Demonstração.** It is clear that there are infinitely many paths \( \Gamma \) such that the integral in (5.3) is well defined (it is sufficient to find a closed path contained in \( \Theta \) such that contains \( \sigma(L) \) in its interior). The definition of this integral doesn’t depend of these paths, however, we will choose a path \( \Gamma \) that facilitate the remaining the proof. By Lemma 3.1 there exists \( c > 0 \) such that
\[
\inf_{\|x\|=1} \langle Lx, x \rangle \geq c.
\]
It follows from equation (2.1), Theorem 2.5 and Lemma 3.1 that \( \sigma(L) \subseteq [c, \|L\|] \). Then,
\[
\sigma(L) \subseteq (c/2, \|L\| + c/2).
\]
Take \( \Gamma \) as the circumference (positively oriented) centered in \( c + \|L\|/2 \) (the middle point of \((c/2, \|L\| + c/2)\)) and passing through the points \( c/2 \) and \( \|L\| + c/2 \). Hence,
\[
\sigma(L) \subseteq \Gamma \subseteq \Theta.
\]
Consequently, the integral in (5.3) is well defined. We shall now prove that \( \gamma(L) \) is the nonnegative square root of \( L \). By the uniqueness in the Theorem 2.8 it is sufficient to prove that \( \gamma(L)^2 = L \) and that \( \gamma(L) \) is nonnegative. In fact, since \( \gamma(\lambda)\gamma(\lambda) = \lambda \) for all \( \lambda \in \Theta \), by Lemma 4.8 we obtain that
\[
\gamma(L)^2 = \gamma(L)\gamma(L) = L.
\]
See now that \( \gamma(L) \) is self-adjoint. For this purpose, we will use the definition of above integral as the limit of a sequence of sums as in (4.1) (see (4.2)). In fact, para cada \( n \in \mathbb{N} \), take a partition \( P_n = \{\lambda_0, \lambda_1, ..., \lambda_n\} \) of \( \Gamma \) such that, for \( k = 0, 1, ..., n \),
\[
|\lambda_k - \lambda_{k-1}| \to 0 \quad \text{when } n \to \infty.
\]
Moreover, for \( k = 0, 1, \ldots, n \), let \( \xi_k = \frac{1}{n-1} \). Therefore, \( \xi_k \in \Gamma \) (by definition of \( \Gamma \)) and, since \( \lambda_0 < \lambda_1 < \ldots < \lambda_n = \lambda_0 \), then \( \xi_0 < \xi_1 < \ldots < \xi_n = \xi_0 \). Take \( x, y \in H \). Since \( [(L - \lambda I)^{-1}]^* = (L - \lambda I)^{-1} \) and \( \gamma(\lambda) = \gamma(\lambda) \), we have

\[
\sum_{k=1}^{n} \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} x, y
\]

\[
= \langle x, \sum_{k=1}^{n} \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} y \rangle
\]

\[
= \langle x, \sum_{k=1}^{n} \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} y \rangle
\]

\[
= \langle x, \sum_{k=1}^{n} \gamma(\xi_{n-k})(\xi_{n-k} - \xi_{n-k+1})(L - \xi_{n-k} I)^{-1} y \rangle
\]

\[
= \langle x, \sum_{k=1}^{n} \gamma(\xi_{n-k})(\xi_{n-k} - \xi_{n-k+1})(L - \xi_{n-k} I)^{-1} y \rangle
\]

\[
= \langle x, - \sum_{k=1}^{n} \gamma(\xi_{n-k})(\xi_{n-k} - \xi_{n-k})(L - \xi_{n-k} I)^{-1} y \rangle
\]

\[
= \langle x, - \sum_{j=1}^{n} \gamma(\xi_{j-1})(\xi_{j} - \xi_{j-1})(L - \xi_{j-1} I)^{-1} y \rangle.
\]

Therefore,

\[
\langle \frac{1}{2\pi i} \sum_{k=1}^{n} \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} x, y \rangle
\]

\[
= \langle x, \frac{1}{2\pi i} \sum_{j=1}^{n} \gamma(\xi_{j-1})(\xi_{j} - \xi_{j-1})(L - \xi_{j-1} I)^{-1} y \rangle.
\]

By (4.2),

\[
\lim_{n \to \infty} - \frac{1}{2\pi i} \sum_{k=1}^{n} \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} = \gamma(L)
\]

\[
= \lim_{n \to \infty} - \frac{1}{2\pi i} \sum_{j=1}^{n} \gamma(\xi_{j-1})(\xi_{j} - \xi_{j-1})(L - \xi_{j-1} I)^{-1}.
\]

Thus \( \langle \gamma(L)x, y \rangle = \langle x, \gamma(L)y \rangle \) for \( x, y \in H \). This fact proves that \( \gamma(L) \) is self-adjoint.

We shall now prove that \( \gamma(L) \) is nonnegative. The Lemma 4.9 implies that

\[ \sigma(\gamma(L)) = \gamma(\sigma(L)). \]

Thus, since \( \sigma(L) \subseteq \mathbb{R}^+ \), then \( \sigma(\gamma(L)) \subseteq \mathbb{R}^+ \). It is a consequence of Lemma 4.3 that

\[ 0 \leq \min_{\lambda \in \sigma(\gamma(L))} \lambda = \inf_{\|x\|=1} \langle \gamma(L)x, x \rangle, \]

that is, \( \gamma(L) \) is nonnegative. \( \square \)
It follows from Lemma 5.1 that, if \( L \in GL_S^+(H) \), then \( L^{1/2} \in GL_S^+(H) \). Then, by Theorem 2.8 we obtain that the application

\[ R : GL_S^+(H) \to GL_S^+(H) \]

\[ L \mapsto L^{1/2} \]

is well defined. Using the above Theorem, we shall now prove that this application is a homeomorphism.

**Theorem 5.5.** The application

\[ R : GL_S^+(H) \to GL_S^+(H) \]

\[ L \mapsto L^{1/2} \]

is a homeomorphism.

**Demonstração.** Let \( L \in GL_S^+(H) \) and \( c \) be as in the previous theorem. Consider \( r = \min\{1/\|L^{-1}\|, c/2\} \) and

\[ B(L, r) = \{T \in L_S(H) : \|L - T\| < r\} \]

Thus \( B(L, r) \subseteq GL_S^+(H) \) (see Proposition 3.3). Let \( T \in B(L, r) \). We shall now prove that

(5.4) \[ \sigma(T) \subseteq (c/3, \|L\| + c/2) \]

Since \( T \) is positive, \( \sigma(T) \) is a subset of the positive real numbers (Theorem 2.5). If \( \lambda \in \sigma(T) \), then

\[ \lambda \leq \|T\| \leq \|L\| + \|T - L\| < \|L\| + c/2. \]

Now, suppose that \( \lambda \in \mathbb{R} \) with \( 0 < \lambda \leq c/3 \). Let \( x \in H \) be of norm 1. Since

\[ \|Lx\| \geq \langle Lx, x \rangle \geq c\|x\|, \]

\[ \|Tx\| \geq \|Lx\| - \|Lx - Tx\| \geq c - c/2, \]

that is, \( \|Tx\| \geq c/2 \). Consequently,

\[ \|(T - \lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq \frac{c}{2} - |\lambda| > 0. \]

This fact proves that \( \text{Ker}(T - \lambda I) = \{0\} \). It is a consequence of Proposition 2.2 that \( T - \lambda I \) is closed. Hence, since \( T - \lambda I \) is self-adjoint,

\[ \text{Im}(T - \lambda I) = \overline{\text{Im}(T - \lambda I)^*} = [\text{Ker}(T - \lambda I)]^\perp = H. \]

Therefore, \( \lambda \in \rho(T) \).

Let \( \Gamma \) be the circumference (positively oriented) with center in the middle point of the interval \((c/3, \|L\| + c/2)\) and passing through the points \( c/3 \) and \( \|L\| + c/2 \). It follows from (5.4) that

\[ \sigma(T) \subseteq \Gamma \subseteq \Theta \quad \text{for all} \quad T \in B(L, r). \]

Hence, we can to define

\[ \gamma(T) = \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1} d\lambda \quad \text{for} \quad T \in B(L, r). \]
The above theorem implies that, for $T \in B(L, r)$, the operator $T^{1/2}$ is $\gamma(T)$, that is,
\[ R(T) = \gamma(T) \quad \text{for } T \in B(L, r). \]
We shall prove that $R$ is continuous using the above equality. Let $T \in B(L, r)$. By (2.4) we obtain that
\[ (T - \lambda I)^{-1} - (L - \lambda I)^{-1} = -(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1} \quad \text{for } \lambda \in \Gamma. \]
Thus,
\[
\gamma(T) - \gamma(L) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1}d\lambda + \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(L - \lambda I)^{-1}d\lambda
\]
\[ = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)[(T - \lambda I)^{-1} - (L - \lambda I)^{-1}]d\lambda
\]
\[ = \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1}d\lambda. \]
Consequently,
\[
\gamma(T) - \gamma(L) = \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1}d\lambda.
\]
Let $M > 0$ be such that
\[ \| (T - \lambda I)^{-1} \| \leq M \quad \text{for } \lambda \in \Gamma \text{ and } T \in B(L, r). \]
Thus
\[ \| \gamma(\lambda)(L - \lambda I)^{-1}(L - T)(T - \lambda I)^{-1} \| \leq mM^2\| L - T \|, \]
where $m = \max_{\lambda \in \Gamma} |\gamma(\lambda)|$.
Let $\varepsilon > 0$ given. If $\| L - T \| < \varepsilon$, then
\[ \| R(L) - R(T) \| = \| \gamma(L) - \gamma(T) \| < \frac{1}{2\pi} mM^2l\varepsilon, \]
where $l$ is the length of $\Gamma$ (Lemma 4.5). Consequently, $R$ is continuous.
On the other hand, it is not difficult to show that the application
\[ C : GL_{\mathbb{R}}^+(H) \to GL_{\mathbb{R}}^+(H) \]
\[ L \mapsto L^2 \]
is the inverse of $R$. Therefore, $R$ is a homeomorphism. \qed

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