Dark pressure in a non-compact and non-Ricci flat 5D Kaluza-Klein cosmology

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Abstract:
In the framework of noncompact Kaluza-Klein theory, we investigate a (4 + 1)-dimensional universe consisting of a (4 + 1) dimensional Robertson-Walker type metric coupled to a (4 + 1) dimensional energy-momentum tensor. The matter part consists of an energy density together with a pressure subject to 4D part of the (4 + 1) dimensional energy-momentum tensor. The dark part consists of just a dark pressure $\bar{p}$, corresponding to the extra-dimension endowed by a scalar field, with no element of dark energy. It is shown that for a flat universe, coupled with the non-vacuum states of the scalar field, the reduced field equations subject to suitable equations of state for matter and dark part may reveal inflationary behavior at early universe, deceleration for radiation dominant era and then acceleration in matter dominant era.
I. INTRODUCTION

The recent distance measurements from the light-curves of several hundred type Ia supernovae [1, 2] and independently from observations of the cosmic microwave background (CMB) by the WMAP satellite [3] and other CMB experiments [4, 5] suggest strongly that our universe is currently undergoing a period of acceleration. This accelerating expansion is generally believed to be driven by an energy source called dark energy. The question of dark energy and the accelerating universe has been therefore the focus of a large amount of activities in recent years. Dark energy and the accelerating universe have been discussed extensively from various point of views over the past few years [6–8]. In principle, a natural candidate for dark energy could be a small positive cosmological constant. This is usually studied in ΛCDM model. This model attempts to explain: cosmic microwave background observations, large scale structure observations, and supernovae observations of accelerating universe. It uses Friedmann-Robertson-Walker (FRW) metric, the Friedmann equations and the cosmological equations of state to describe the universe from right after the inflationary epoch to the present and future times, according to Einstein equation with a cosmological constant

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} + \Lambda g_{\mu\nu}. \]

Alternative approaches have also been pursued, a few example of which can be found in [9–11]. These schemes aim to improve the quintessence approach overcoming the problem of scalar field potential, generating a dynamical source for dark energy as an intrinsic feature. Quintessence is a scalar field with an equation of state which unlike cosmological constant, varies through the space and time. Many models of quintessence have a tracker behavior. In these models, the quintessence field has a density which closely tracks (but is less than) the radiation density until matter-radiation equality, which triggers the quintessence to start, having characteristics similar to dark energy and eventually dominating the universe and starting the acceleration of the universe. The goal would be to obtain a comprehensive model capable of linking the picture of the early universe to the one observed today, that is, a model derived from some effective theory of quantum gravity which, through an inflationary period would result in today accelerated Friedmann expansion driven by some $\Omega_\Lambda$-term.

Another approach in this direction is to employ what is known as modified gravity where an arbitrary function of the Ricci scalar, namely $f(R)$, is inserted into the Einstein-Hilbert
action instead of Ricci scalar $R$ as

$$S = -\frac{1}{2k} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \psi),$$

where $\psi$ is the matter field. This is a fully covariant theory based on the principle of least action. One of the main considerations with this modified gravity is to know how can we fit this theory with the local solar system tests as well as cosmological constraints [12, 13]. It has been shown that such a modification may account for the late time acceleration and the initial inflationary period in the evolution of the universe so that positive powers of $R$ lead to inflation while negative powers of $R$ result in current acceleration [14–16]. A scenario where the issue of cosmic acceleration in the framework of higher order theories of gravity in 4D is addressed can be found in [17]. One of the first proposals in this regard was suggested in second reference of [15] where a term of the form $R^{-1}$ was added to the usual Einstein-Hilbert action.

The idea that our world may have more than four dimensions is due to Kaluza [18], who unified Einstein’s theory of General Relativity with Maxwell’s theory of Electromagnetism in a 5D manifold. In 1926, Klein reconsidered Kaluza’s idea and treated the extra dimension as a topologically compact small circle [19]. Since then the Kaluza-Klein idea has been studied extensively from different angles. Notable amongst them is the so-called Space-Time-Matter (STM) theory, proposed by Wesson and his collaborators, which is designed to explain the origin of matter in terms of the geometry of the bulk space in which our 4D world is embedded, for reviews see [20]. More precisely, in STM theory, our world is a hypersurface embedded in a five-dimensional Ricci-flat ($R_{AB} = 0$) manifold where all the matter in our world can be thought of as being the manifestation of the geometrical properties of the higher dimensional space according to

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

Applications of the idea of induced matter or induced geometry can also be found in other situations [22]. The STM theory allows for the metric components to be dependent on the extra dimension and does not require the extra dimension to be compact. The sort of cosmologies stemming from STM theory is studied in [23–27]. The evolution of the universe has also been studied extensively based on this noncompact vacuum Kaluza-Klein theory [28] where a 5D mechanism is developed to explain, by a single scalar field, the evolution
of the universe including inflationary expansion and the present day observed accelerated expansion.

Since in a variety of inflationary models the scalar fields have been used in describing the transition from the quasi-exponential expansion of the early universe to a power law expansion, it is natural to try to understand the present acceleration of the universe by constructing models where the matter responsible for such behavior is also represented by a scalar field. Such models are worked out, for example, in Ref [29]. In this chapter, based on the above mentioned ideas on higher dimension and scalar field, a 5D cosmological model is introduced which is not Ricci flat, but is extended to be coupled to a higher dimensional energy momentum tensor. It is shown that the higher dimensional sector of this model may induce a dark pressure, through a scalar field, in four dimensional universe so that for a flat universe under specific conditions one may have early inflation and current acceleration even for the non-vacuum states of the scalar field.

II. SPACE-TIME-MATTER VERSUS KALUZA-KLEIN THEORY

The Kaluza-Klein theory is essentially general relativity in 5D subject to two conditions: 1) the so called “cylinder” condition, introduced by Kaluza, to set all partial derivations with respect to the 5th coordinate to zero 2) the “compactification” condition, introduced by Klein, to set a small size and a closed topology for the 5th coordinate. Physically, both conditions have the motivation of explaining why we perceive the 4 dimensions of space-time and do not observe the fifth dimension. In perfect analogy with general relativity, one may define a $5 \times 5$ metric tensor $g_{AB}$ ($A, B = 0, 1, 2, 3, 4$) where 4 denotes the extra coordinate which is referred to “internal” coordinate. Moreover, one can form a 5D Ricci tensor $R_{AB}$, a 5D Ricci scalar $R$ and a 5D Einstein tensor $G_{AB} = R_{AB} - \frac{1}{2}Rg_{AB}$. In Principle, the field equations are expected to be

$$G_{AB} = kT_{AB},$$

where $k$ is an appropriate coupling constant and $T_{AB}$ is a 5D momentum tensor. In Kaluza-Klein theory, however, it is usually assumed that the universe in 5D is empty, so we have
the vacuum field equations which may equivalently be defined as

\[ R_{AB} = 0, \]  

(2)

where \( R_{AB} \) is the 5D Ricci tensor. These 15 relations serve to determine the 15 components of the metric \( g_{AB} \). To this end, some assumptions are to be made on \( g_{AB} = g_{AB}(x^C) \) as the choice of coordinates or gauges. Interested in electromagnetism and its unification with gravity, Kaluza realized that \( g_{AB} \) may be expressed in the following form that involves the electromagnetic 4-potentials \( A_\alpha \) as well as an scalar field \( \Phi \)

\[
\begin{pmatrix}
(g_{\alpha\beta} - \kappa^2\Phi^2 A_\alpha A_\beta) & -\kappa\Phi^2 A_\alpha \\
-\kappa\Phi^2 A_\beta & -\Phi^2 \\
\end{pmatrix}
\]  

(3)

The five dimensional Ricci tensor and Christoffel symbols exactly as in four dimension are defined in terms of the metric as

\[
R_{AB} = \partial_C \Gamma^C_{AB} - \partial_B \Gamma^C_{AC} + \Gamma^C_{AB} \Gamma^D_{CD} - \Gamma^C_{AD} \Gamma^D_{BC},
\]

(4)

\[
\Gamma^C_{AB} = \frac{1}{2} g^{CD} (\partial_A g_{DB} + \partial_B g_{DA} - \partial_D g_{AB}).
\]

(5)

Then the field equations \( R_{AB} = 0 \) reduces to

\[
G_{\alpha\beta} = \frac{\kappa^2 \Phi^2}{2} T_{\alpha\beta} - \frac{1}{\Phi} (\nabla_\alpha \nabla_\beta \Phi - g_{\alpha\beta} \Box \Phi),
\]

(6)

\[
\nabla^\alpha F_{\alpha\beta} = -3 \frac{\nabla^\alpha \Phi}{\Phi} F_{\alpha\beta},
\]

(7)

\[
\Box \Phi = -\frac{\kappa^2 \Phi^3}{4} F_{\alpha\beta} F^{\alpha\beta},
\]

(8)

where \( G_{\alpha\beta}, F_{\alpha\beta} \) and \( T_{\alpha\beta} \) are the usual 4D Einstein tensor, electromagnetic field strength tensor and energy-momentum tensor for the electromagnetic field, respectively. The Kaluza’s case \( \Phi^2 = 1 \) together with the identification \( \kappa = (16\pi G)^{\frac{1}{2}} \) leads to the Einstein-Maxwell equations

\[
G_{\alpha\beta} = 8\pi G T_{\alpha\beta},
\]

(9)

\[
\nabla^\alpha F_{\alpha\beta} = 0.
\]

(10)

The STM or “induced matter theory” is also based on the vacuum field equation in 5D as \( R_{AB} = 0 \) or \( G_{AB} = 0 \). But, it differs mainly from the Kaluza-Klein theory about the
cylinder condition so that one may now keep all terms containing partial derivatives with respect to the fifth coordinate. This results in the 4D Einstein equations

\[ G_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \] (11)

provided an appropriate definition is made for the energy-momentum tensor of matter in terms of the extra part of the geometry. Physically, the picture behind this interpretation is that curvature in \((4 + 1)\) space induces effective properties of matter in \((3 + 1)\) space-time. The fact that such an embedding can be done is supported by Campbell’s theorem [21] which states that any analytical solution of the Einstein field equations in \(N\) dimensions can be locally embedded in a Ricci-flat manifold in \((N + 1)\) dimensions. Since the matter is induced from the extra dimension, this theory is also called the induced matter theory. The main point in the induced matter theory is that these equations are a subset of \(G_{AB} = 0\) with an effective or induced 4D energy-momentum tensor \(T_{\alpha\beta}\), constructed by the geometry of higher dimension, which contains the classical properties of matter.

Taking the metric (3) and choosing coordinates such that the four components of the gauge fields \(A_\alpha\) vanish, then the 5-dimensional metric becomes

\[ g_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \epsilon \Phi^2 \end{pmatrix}, \] (12)

where the factor \(\epsilon\) with the requirement \(\epsilon^2 = 1\) is introduced in order to allow a timelike, as well as spacelike signature for the fifth dimension. Using the definitions (4), (5) and keeping derivatives with respect to the fifth dimension, the resultant expressions for the \(\alpha\beta\), \(\alpha 4\) and \(44\) components of the five dimensional Ricci tensor \(R_{AB}\) are obtained

\[ \hat{R}_{\alpha\beta} = R_{\alpha\beta} - \nabla_\beta (\partial_\alpha \Phi) \frac{\Phi}{\Phi} \]

\[ + \frac{\epsilon}{2\Phi^2} \left( \frac{\partial_\alpha \Phi \partial_\beta g_{\alpha\beta}}{\Phi} - \partial_\beta g_{\alpha\beta} + g^{\gamma\delta} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\alpha\beta} - \frac{g^{\gamma\delta} \partial_\gamma g_{\alpha\beta} \partial_\delta g_{\alpha\beta}}{2} \right). \] (13)

\[ \hat{R}_{\alpha 4} = \frac{g^{44} g^{\beta\gamma}}{4} (\partial_\alpha g_{\beta\gamma} \partial_\alpha g_{44} - \partial_\gamma g_{44} \partial_\alpha g_{\alpha\beta}) \]

\[ + \frac{\partial_\beta g^{\beta\gamma} \partial_\alpha g_{\gamma\alpha}}{2} + \frac{g^{\beta\gamma} \partial_\alpha g_{\gamma\alpha}}{2} - \frac{\partial_\alpha g^{\beta\gamma} \partial_\delta g_{\beta\gamma}}{2} \]

\[ - \frac{g^{\beta\gamma} \partial_\delta g_{\beta\gamma} \partial_\alpha g_{\gamma\alpha}}{4} + \frac{g^{\beta\gamma} \partial_\delta g_{\gamma\alpha} \partial_\beta g_{\delta\epsilon}}{4} + \frac{\partial_\delta g^{\beta\gamma} \partial_\alpha g_{\beta\gamma}}{4}. \] (14)
\[ \hat{R}_{44} = -\epsilon \Phi \Box \Phi - \frac{\partial_4 g^{\alpha \beta} \partial_4 g_{\alpha \beta}}{2} - \frac{g^{\alpha \beta} \partial_4 (\partial_4 g_{\alpha \beta})}{2} \]

\[ + \frac{\partial_4 \Phi g^{\alpha \beta} \partial_4 g_{\alpha \beta}}{2\Phi} - \frac{g^{\alpha \beta} g_{\gamma \delta} \partial_4 g_{\gamma \delta} \partial_4 g_{\alpha \beta}}{4}. \]  

(15)

Assuming that there is no higher dimensional matter, the Einstein equations take the form

\[ R_{AB} = 0 \]

which produces the following expressions for the 4-dimensional Ricci tensor

\[ R_{\alpha \beta} = \frac{\nabla_{\beta}(\partial_{\alpha} \Phi)}{\Phi} - \frac{\epsilon}{2\Phi^2} \left( \frac{\partial_4 \Phi \partial_4 g_{\alpha \beta}}{\Phi} - \partial_4 g_{\alpha \beta} + g^{\gamma \delta} \partial_4 g_{\alpha \gamma} \partial_4 g_{\beta \delta} - \frac{g^{\gamma \delta} \partial_4 g_{\gamma \delta} \partial_4 g_{\alpha \beta}}{2} \right), \]

(16)

\[ \nabla_{\beta} \left[ \frac{1}{2\sqrt{g_{44}}} (g^{\beta \gamma} \partial_4 g_{\gamma \delta} - \delta_\alpha^\beta g^{\gamma \epsilon} \partial_4 g_{\gamma \epsilon}) \right] = 0, \]

(17)

\[ \epsilon \Phi \Box \Phi = - \frac{\partial_4 g^{\alpha \beta} \partial_4 g_{\alpha \beta}}{4} - \frac{g^{\alpha \beta} \partial_4 (\partial_4 g_{\alpha \beta})}{2} + \frac{\partial_4 \Phi g^{\alpha \beta} \partial_4 g_{\alpha \beta}}{2\Phi}. \]

(18)

The first equation introduces an induced energy-momentum tensor as

\[ 8\pi GT_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta}. \]

(19)

The four dimensional Ricci scalar is obtained

\[ R = g^{\alpha \beta} R_{\alpha \beta} = \frac{\epsilon}{4\Phi^2} \left[ \partial_4 g^{\alpha \beta} \partial_4 g_{\alpha \beta} + (g^{\alpha \beta} \partial_4 g_{\alpha \beta})^2 \right]. \]

(20)

Inserting \( R \) and \( R_{\alpha \beta} \) into eq. (19) one finds that

\[ 8\pi GT_{\alpha \beta} = \frac{\nabla_{\beta}(\partial_{\alpha} \Phi)}{\Phi} \]

\[ - \frac{\epsilon}{2\Phi^2} \left[ \frac{\partial_4 \Phi \partial_4 g_{\alpha \beta}}{\Phi} - \partial_4 g_{\alpha \beta} + g^{\gamma \delta} \partial_4 g_{\alpha \gamma} \partial_4 g_{\beta \delta} - \frac{g^{\gamma \delta} \partial_4 g_{\gamma \delta} \partial_4 g_{\alpha \beta}}{2} + \frac{g_{\alpha \beta}}{4} \left( \partial_4 g^{\gamma \delta} \partial_4 g_{\gamma \delta} + (g^{\gamma \delta} \partial_4 g_{\gamma \delta})^2 \right) \right]. \]

(21)

Therefore, the 4-dimensional Einstein equations \( G_{\alpha \beta} = 8\pi GT_{\alpha \beta} \) are automatically contained in the 5-dimensional vacuum equations \( G_{AB} = 0 \), so that the matter \( T_{\alpha \beta} \) is a manifestation of pure geometry in the higher dimensional world and satisfies the appropriate requirements: it is symmetric and reduces to the expected limit when the cylinder condition is re-applied.

### III. THE EXTENDED MODEL

As explained above, both Kaluza-Klein and STM theories use the vacuum field equations in 5 dimensions. In Kaluza-Klein case, the energy-momentum tensor is limited to Electromagnetic and scalar fields. In STM case, \( T_{\alpha \beta} \) includes more types of matter but is limited
to those obtained for an specific form of the $5D$ metric. We are now interested in a $5D$ model with a general $5D$ energy-momentum tensor which is, in principle, independent of the geometry of higher dimension and can be set by physical considerations on $5$-dimensional matter. We start with the $5D$ line element

$$dS^2 = g_{AB} dx^A dx^B.$$  \hfill (22)

The space-time part of the metric $g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha)$ is assumed to define the Robertson-Walker line element

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),$$  \hfill (23)

where $K$ takes the values $+1, 0, -1$ according to a close, flat or open universe, respectively. We also take the followings

$$g_{4\alpha} = 0, \quad g_{44} = \epsilon \Phi^2 (x^\alpha),$$

where $\epsilon = 1$ and the signature of the higher dimensional part of the metric is left general. This choice has been made because any fully covariant $5D$ theory has five coordinate degrees of freedom which can lead to considerable algebraic simplification, without loss of generality. Unlike the noncompact vacuum Kaluza-Klein theory, we will assume the fully covariant $5D$ non-vacuum Einstein equation

$$G_{AB} = 8\pi G T_{AB},$$  \hfill (24)

where $G_{AB}$ and $T_{AB}$ are the $5D$ Einstein tensor and energy-momentum tensor, respectively. Note that the $5D$ gravitational constant has been fixed to be the same value as the $4D$ one.

In the following we use the geometric reduction from $5D$ to $4D$ as appeared in [27]. The $5D$ Ricci tensor is given in terms of the $5D$ Christoffel symbols by

$$\hat{R}_{\alpha\beta} = \partial_C \Gamma^C_{\alpha\beta} - \partial_B \Gamma^C_{\alpha\beta} + \Gamma^C_{\alpha\beta} \Gamma^D_{CD} - \Gamma^C_{AD} \Gamma^D_{BC}.$$  \hfill (25)

The $4D$ part of the $5D$ quantity is obtained by putting $A \to \alpha, B \to \beta$ in (25) and expanding the summed terms on the r.h.s by letting $C \to \lambda, 4$ etc. Therefore, we have

$$\hat{R}_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} + \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\lambda} \Gamma^\mu_{\beta\mu}.$$  \hfill (26)

where $\hat{\ast}$ denotes the $4D$ part of the $5D$ quantities. One finds the $4D$ Ricci tensor as a part of this equation which may be cast in the following form

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\lambda} \Gamma^\mu_{\beta\mu} - \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu}.$$  \hfill (27)
Evaluating the Christoffel symbols for the metric \( g_{AB} \) gives
\[
\hat{\nabla}_\alpha \nabla_\beta \Phi - \frac{\nabla_\alpha \nabla_\beta \Phi}{\Phi}.
\] (28)

Putting \( A = 4, B = 4 \) and expanding with \( C \to \lambda, 4 \) in Eq.(25) we obtain
\[
R_{44} = \partial_\lambda \Gamma^\lambda_{44} - \partial_4 \Gamma^\lambda_{4\lambda} + \Gamma^\lambda_{44} \Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{4\mu} \Gamma^\mu_{4\lambda} - \Gamma^4_{4\mu} - \Gamma^\mu_{44}.
\] (29)

Evaluating the corresponding Christoffel symbols in Eq.(29) leads to
\[
R_{44} = -\epsilon \Phi \Box \Phi.
\] (30)

We now construct the space-time components of the Einstein tensor
\[
\hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_{(5)}.
\]

In so doing, we first obtain the 5D Ricci scalar \( R_{(5)} \) as
\[
R_{(5)} = g^{AB} R_{AB} = \hat{g}^{\alpha\beta} \hat{R}_{\alpha\beta} + g_{44} R_{44} = \hat{g}^{\alpha\beta} (\hat{R}_{\alpha\beta} - \frac{\nabla_\alpha \nabla_\beta \Phi}{\Phi}) + \frac{\epsilon}{\Phi^2} (-\epsilon \Phi \Box \Phi)
\]
\[= R - \frac{2}{\Phi} \Box \Phi,
\] (31)

where the \( \alpha 4 \) terms vanish and \( R \) is the 4D Ricci scalar. The space-time components of the Einstein tensor is written \( \hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} \hat{g}_{\alpha\beta} R_{(5)} \). Substituting \( \hat{R}_{\alpha\beta} \) and \( R_{(5)} \) into the space-time components of the Einstein tensor gives
\[
\hat{G}_{\alpha\beta} = G_{\alpha\beta} + \frac{1}{\Phi} (g_{\alpha\beta} \Box \Phi - \nabla_\alpha \nabla_\beta \Phi).
\] (32)

In the same way, the 4-4 component is written \( G_{44} = R_{44} - \frac{1}{2} g_{44} R_{(5)} \), and substituting \( R_{44}, R_{(5)} \) into this component of the Einstein tensor gives
\[
G_{44} = -\frac{1}{2} \epsilon R \Phi^2.
\] (33)

We now consider the 5D energy-momentum tensor. The form of energy-momentum tensor is dictated by Einstein’s equations and by the symmetries of the metric \( g_{AB} \). Therefore, we may assume a perfect fluid with nonvanishing elements
\[
T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta - pg_{\alpha\beta},
\] (34)
\[
T_{44} = -\bar{p} g_{44} = -\epsilon \bar{p} \Phi^2,
\] (35)
where $\rho$ and $p$ are the conventional density and pressure of perfect fluid in the 4D standard cosmology and $\bar{p}$ acts as a pressure living along the higher dimensional sector. Hence, the field equations (24) are to be viewed as constraints on the simultaneous geometric and physical choices of $G_{AB}$ and $T_{AB}$ components, respectively.

Substituting the energy-momentum components (34), (35) in front of the 4D and extra dimensional part of Einstein tensors (32) and (33), respectively, we obtain the field equations

$$G_{\alpha\beta} = 8\pi G[(\rho + p)u_\alpha u_\beta - \bar{p}g_{\alpha\beta}] + \frac{1}{\Phi} [\nabla_\alpha \nabla_\beta \Phi - \Box \Phi g_{\alpha\beta}],$$

and

$$R = 16\pi G\bar{p}.$$  

By evaluating the $g^{\alpha\beta}$ trace of Eq.(36) and combining with Eq.(37) we obtain

$$\Box \Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p})\Phi.$$

This equation infers the following scalar field potential

$$V(\Phi) = -\frac{1}{6}(8\pi G(\rho - 3p) + 16\pi G\bar{p})\Phi^2,$$

whose minimum occurs at $\Phi = 0$, for which the equations (36) reduce to describe a usual 4D FRW universe filled with ordinary matter $\rho$ and $p$. In other words, our conventional 4D universe corresponds to the vacuum state of the scalar field $\Phi$. From Eq.(38), one may infer the following replacements for a nonvanishing $\Phi$

$$\frac{1}{\Phi} \Box \Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p}),$$

$$\frac{1}{\Phi} \nabla_\alpha \nabla_\beta \Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p})u_\alpha u_\beta.$$  

Putting the above replacements into Eq.(36) leads to

$$G_{\alpha\beta} = 8\pi G[(\rho + \bar{p})u_\alpha u_\beta - \bar{p}g_{\alpha\beta}],$$

where

$$\bar{p} = \frac{1}{3}(\rho + 2\bar{p}).$$

This energy-momentum tensor effectively describes a perfect fluid with density $\rho$ and pressure $\bar{p}$. The four dimensional field equations lead to two independent equations

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho,$$
\[ \frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} = -8\pi G\tilde{p}. \]  

(45)

Differentiating (44) and combining with (45) we obtain the conservation equation

\[ \frac{d}{dt}(\rho a^3) + \tilde{p} \frac{d}{dt}(a^3) = 0. \]  

(46)

The equations (44) and (45) can be used to derive the acceleration equation

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\tilde{p}) = -\frac{8\pi G}{3} (\rho + \tilde{p}). \]  

(47)

The acceleration or deceleration of the universe depends on the negative or positive values of the quantity \((\rho + \tilde{p})\).

From extra dimensional equation (47) (or 4-dimensional Eqs. (43), (44) and (45)) we obtain

\[ -\frac{6(k + \dot{a}^2 + \ddot{a}a)}{a^2} = 16\pi G \tilde{p}. \]  

(48)

Using power law behaviors for the scale factor and dark pressure as \(a(t) = a_0 t^\alpha\) and \(\tilde{p}(t) = \tilde{p}_0 t^\beta\) in the above equation, provided \(k = 0\) in agreement with observational constraints, we obtain \(\beta = -2\).

Based on homogeneity and isotropy of the 4D universe we may assume the scalar field to be just a function of time, then the scalar field equation (38) reads as the following form

\[ \ddot{\Phi} + 3\frac{\dot{a}}{a} \dot{\Phi} - \frac{8\pi G}{3} ((\rho - 3\tilde{p}) + 2\tilde{p}) \Phi. \]  

(49)

Assuming \(\Phi(t) = \Phi_0 t^\gamma\) and \(\rho(t) = \rho_0 t^\delta\) \((\rho_0 > 0)\) together with the equations of state for matter pressure \(p = \omega \rho\) and dark pressure \(\tilde{p} = \Omega \rho\) we continue to calculate the required parameters for inflation, deceleration and then acceleration of the universe. In doing so, we rewrite the acceleration equation (47), scalar field equation (49) and conservation equation (46), respectively, in which the above assumptions are included as

\[ \alpha(\alpha - 1) + \frac{8\pi G}{3} \rho_0 (1 + \Omega) = 0, \]  

(50)

\[ \gamma(\gamma - 1) + 3\alpha\gamma - \frac{8\pi G}{3} \rho_0 ((1 - 3\omega) + 2\Omega) = 0, \]  

(51)

\[ 2\rho_0 [(2 + \Omega)\alpha - 1] = 0, \]  

(52)

where \(\delta = -2\) has been used due to the consistency with the power law behavior \(t^{3\alpha-3}\) in the conservation equation. The demand for acceleration \(\ddot{a} > 0\) through Eq. (47) with the
assumptions $\rho(t) = \rho_0 t^\delta$ and $\bar{p} = \Omega \rho$, requires $\rho_0 (1 + \Omega) < 0$ or $\Omega < -1$ which accounts for a negative dark pressure. This negative domain of $\Omega$ leads through the conservation equation (46) to $\alpha > 1$ which indicates an accelerating universe as expected.

In both radiation dominant and matter dominant eras $\omega = \frac{1}{3}$, $\omega = 0$, respectively, the scalar field equation (51) leads to the following inequality

$$\gamma (\gamma - 1 + 3\alpha) < 0, \quad (53)$$

which through $\alpha > 1$ means

$$1 - 3\alpha < \gamma < 0. \quad (54)$$

It is easily seen in the acceleration equation (50) that as $|\Omega|$ becomes larger and larger, the values of $\alpha$ required for acceleration ($\alpha > 1$) become larger and larger, as well. This result is not surprising, since the more negative dark pressure we have, the more acceleration is expected.

On the other hand, using Friedmann equation we obtain $\alpha = \frac{1}{2 + \Omega}$ which together with the condition $\alpha > 1$ requires that $-2 < \Omega < -1$. Now, one may recognize two options as follows.

The first option is to attribute an intrinsic evolution to the parameter $\Omega$ along the higher dimension so that it can produce the 4D expansion evolution in agreement with standard model including early inflation and subsequent deceleration, and also current acceleration of the universe. Ignoring the phenomenology of the evolution of the parameter $\Omega$, we may require

$$\begin{cases} 
\Omega \gtrsim -2 \text{ for inflation} \\
\Omega > -1 \text{ for deceleration} \\
\Omega \lesssim -1 \text{ for acceleration.}
\end{cases} \quad (55)$$

The first case corresponds to highly accelerated universe due to a large $\alpha >>> 1$. This can be relevant for the inflationary era if one equate the power law with exponential behavior. The second case corresponds to a deceleration $\alpha < 1$, and the third case represents a small acceleration $\alpha \gtrsim 1$. In this option, there is no specific relation between the physical phase along extra dimension, namely $\Omega$, and the ones defined in 4D universe by $\omega$. Therefore, an unexpected acceleration in the "middle" of matter dominated phase $\omega = 0$ is justified due to the beginning of a new phase of $\Omega$. 

The second option is to assume a typical relation between the parameters \( \Omega \) and \( \omega \) as \( \Omega = f(\omega) \) so that

\[
\begin{align*}
\Omega & \gtrsim -2 \quad \text{for } \omega = -1 \\
\Omega & > -1 \quad \text{for } \omega = \frac{1}{3} \\
\Omega & \lesssim -1 \quad \text{for } \omega = 0.
\end{align*}
\]

(56)

The physics of \( \omega \) is well known in the standard cosmology (see below) but that of the parameter \( \Omega \) clearly needs more careful investigation based on effective representation of higher dimensional theories, for instance string or Brane theory. In fact, at early universe when it is of Plank size, it is plausible that all coordinates including 3-space and higher dimension would be symmetric with the same size. However, during the GUT era when some spontaneous symmetry breakings could have happened to trigger the inflation, one may assume that such symmetry breakings could lead to asymmetric compact 3-space and non-compact higher dimension. This could also result in demarcation between three spatial pressures on 3-space and one higher dimensional pressure. Therefore, it is possible to consider a relation \( \Omega = f(\omega) \) which is based on the physics of early universe (phase transitions) along with initial conditions to justify this demarcation.

The case \( \omega = -1 \) corresponds to the early universe and shows a very high acceleration due to \( \alpha \gg> 1 \). The case \( \omega = \frac{1}{3} \) corresponds to the radiation dominant era and shows a deceleration \( \alpha < 1 \). Finally, the case \( \omega = 0 \) corresponds to the matter dominant era and shows an small acceleration \( \alpha \gtrsim 1 \) at the “beginning” of this era.
CONCLUSION

A \((4 + 1)\)-dimensional universe consisting of a \((4 + 1)\) dimensional metric of Robertson-Walker type subject to a \((4 + 1)\) dimensional energy-momentum tensor in the framework of noncompact Kaluza-Klein theory is studied. In the matter part, there is energy density \(\rho\) together with pressure \(p\) subject to \(4D\) part of the \((4 + 1)\) dimensional energy-momentum tensor, and a dark pressure \(\bar{p}\) corresponding to the extra-dimensional part endowed by a scalar field. The reduced \(4D\) and extra-dimensional components of \(5D\) Einstein equations together with different equations of state for pressure \(p\) and dark pressure \(\bar{p}\) may lead to a \(4D\) universe which represents inflation for early universe, deceleration for radiation dominant and acceleration for matter dominant eras. This is done by assuming a typical relation \((??)\) between the two parameters in equations of state. This relation is not unique and one may propose other suitable relations, satisfying the requirements for early inflation, then deceleration and finally recent acceleration of the universe in a more realistic way. The important point of the present model is that there is no longer “coincidence problem”. This is because, in the present model there is no element of “dark energy” at all and we have just one energy density \(\rho\) associated with ordinary matter. So, there is no notion of coincidental domination of dark energy over matter densities to trigger the acceleration at the present status of the universe. In fact, a dark pressure with negative values have existed along the \(5^{th}\) dimension for the whole history of the \(4D\) universe including inflationary, radiation, and matter dominant eras. These stages of the \(4D\) universe have occurred because of negative, positive and zero values of the four dimensional pressure, respectively, which leads through relation \((??)\) to a competition between energy density \(\rho\) and dark pressure \(\bar{p}\) in the acceleration equation \((17)\). For the same reason that there is no element of dark energy in this model, the apparent phantom like equation of state for dark pressure \(\Omega < -1\) is free of serious problems like unbounded from below dark energy or vacuum instability \([30]\).

The above results are independent of the signature \(\epsilon\) by which the higher dimension takes part in the \(5D\) metric. Moreover, the role played by the scalar field along the \(5^{th}\) coordinate in the \(5D\) metric is very impressed by the role of scale factor over the \(4D\) universe. At early universe during the inflationary era the scalar field is highly suppressed and the \(5^{th}\) coordinate is basically ignored in \(5D\) line element. At radiation dominant era the scalar field is much less suppressed and the \(5^{th}\) coordinate becomes considerable in \(5D\) line element.
Finally, at matter dominant era the scalar field and its possible fluctuations starts to be more suppressed and the observable effect of 5th coordinate becomes vanishing in 5D line element at $t \simeq 10^{17} \text{Sec}$, leaving a 4D universe in agreement with observations.
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