LECTURE NOTES
BERNOULLI POLYNOMIALS
AND APPLICATIONS

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Abstract. In this lecture notes we try to familiarize the audience with the theory of Bernoulli polynomials; we study their properties, and we give, with proofs and references, some of the most relevant results related to them. Several applications to these polynomials are presented, including a unified approach to the asymptotic expansion of the error term in many numerical quadrature formulæ, and many new and sharp inequalities, that bound some trigonometric sums.

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1. Introduction

There are many ways to introduce Bernoulli polynomials and numbers. We opted for the algebraic approach relying on the difference operator. But first, let us introduce some notation.

Let the real vector space of polynomials with real coefficients be denoted by $\mathbb{R}[X]$. For a nonnegative integer $n$, let $R_n[X]$ be the subspace of $\mathbb{R}[X]$ consisting of polynomials of degree smaller or equal to $n$.

If $P$ is a polynomial from $\mathbb{R}[X]$, we define $\Delta P \overset{def}{=} P(X + 1) - P(X)$, and we denote by $\Delta$ the linear operator, defined on $\mathbb{R}[X]$, by $P \mapsto \Delta P$.

**Lemma 1.1.** The linear operator $\Phi$ defined by

\[ \Phi : \mathbb{R}[X] \rightarrow \mathbb{R}[X] \times \mathbb{R}, P \mapsto (\Delta P, \int_0^1 P(t) \, dt) \]  

is bijective.

**Proof.** Consider $P \in \ker \Phi$, then $P \in \ker \Delta$ and $\int_0^1 P(t) \, dt = 0$. Now, if we consider $Q(X) = P(X) - P(0)$, then clearly we have

\[ Q(X + 1) = P(X + 1) - P(0) = P(X) - P(0) = Q(X) \]

This implies by induction that $Q(n) = 0$ for every nonnegative integer $n$, so $Q = 0$, since it has infinitely many zeros. Thus, $P(X) = P(0)$, but we have also $\int_0^1 P(t) \, dt = 0$, so $P(0) = 0$, and consequently $P = 0$. This proves that $\Phi$ is injective.

Clearly, for a nonnegative integer $n$ we have $\deg \Delta(X^{n+1}) = n$. Thus

\[ P \in \mathbb{R}_{n+1}[X] \implies \Delta P \in \mathbb{R}_n[X] \]

Therefore,

\[ \forall n \in \mathbb{N}, \quad \Phi(\mathbb{R}_{n+1}[X]) \subset \mathbb{R}_n[X] \times \mathbb{R}. \]

But the fact that $\Phi$ is injective implies that

\[ \dim \Phi(\mathbb{R}_{n+1}[X]) = \dim \mathbb{R}_{n+1}[X] = 1 + \dim \mathbb{R}_n[X] = \dim (\mathbb{R}_n[X] \times \mathbb{R}), \]

and consequently

\[ \forall n \in \mathbb{N}, \quad \Phi(\mathbb{R}_{n+1}[X]) = \mathbb{R}_n[X] \times \mathbb{R} \]

This, proves that $\Phi$ is surjective, and the lemma follows. \( \square \)

Let us consider the basis $E = (e_n)_{n \in \mathbb{N}}$ of $\mathbb{R}[X] \times \mathbb{R}$ defined by $e_0 = (0, 1)$ and $e_n = (nX^{n-1}, 0)$ for $n \in \mathbb{N}^*$. We can define the Bernoulli polynomials, in terms of this basis and of the isomorphism $\Phi$ of Lemma 1.1 as follows:

**Definition 1.2.** The sequence of Bernoulli polynomials $(B_n)_{n \in \mathbb{N}}$ is defined by

\[ B_n = \Phi^{-1}(e_n) \quad \text{for } n \geq 0. \]

According to Lemma 1.1, this definition takes a more practical form as follows:

**Corollary 1.3.** The sequence of Bernoulli polynomials $(B_n)_{n \in \mathbb{N}}$ is uniquely defined by the conditions:

1. $B_0(X) = 1.$
2. $\forall n \in \mathbb{N}^*, \quad B_n(X + 1) - B_n(X) = nX^{n-1}.$
3. $\forall n \in \mathbb{N}^*, \quad \int_0^1 B_n(t) \, dt = 0.$

For instance, it is straightforward to see that

\[ B_1(X) = X - \frac{1}{2}, \quad \text{and} \quad B_2(X) = X^2 - X + \frac{1}{6}. \]
2. Properties of Bernoulli Polynomials

In the next proposition, we summarize some simple properties of Bernoulli polynomials:

Proposition 2.1. The sequence of Bernoulli polynomials \((B_n)_{n \in \mathbb{N}}\) satisfies the following properties:

i. For every positive integer \(n\) we have \(B'_n(X) = nB_{n-1}(X)\).

ii. For every positive integer \(n\) we have \(B_n(1 - X) = (-1)^n B_n(X)\).

iii. For every nonnegative integer \(n\) and every positive integer \(p\) we have

\[
\frac{1}{p} \sum_{k=0}^{p-1} B_n \left( \frac{X + k}{p} \right) = \frac{1}{p^n} B_n(X).
\]

Proof. Consider the sequence of polynomials \((Q_n)_{n \in \mathbb{N}}\) defined by \(Q_n = \frac{1}{n+1} B'_{n+1}\). It is straightforward to see that \(Q_0(X) = 1\) and for \(n \geq 1\):

\[
\Delta Q_n = \frac{1}{n+1} (\Delta B_{n+1})' = \frac{1}{n+1}((n + 1)X^n)' = nX^{n-1}
\]

and

\[
\int_0^1 Q_n(t) \, dt = \frac{1}{n+1} \int_0^1 B'_{n+1}(t) \, dt = \frac{\Delta B_{n+1}(0)}{n+1} = 0.
\]

This proves that the sequence of \((Q_n)_{n \in \mathbb{N}}\) satisfies the conditions (1), (2) and (3) of Corollary [13] and (4) follows because of the unicity assertion.

(ii) Consider again the sequence \((Q_n)_{n \in \mathbb{N}}\) defined by \(Q_n(X) = (-1)^n B_n(1 - X)\). Clearly \(Q_0(X) = 1\) and for \(n \geq 1\):

\[
\Delta Q_n = (-1)^n(B_n(-X) - B_n(1 - X)) = (-1)^n \Delta B_n(-X) = (-1)^{n-1} n(-X)^{n-1} = nX^{n-1}.
\]

Moreover, for \(n \geq 1\), \(\int_0^1 Q_n(t) \, dt = (-1)^n \int_0^1 B_n(1 - t) \, dt = 0\). This proves that the sequence of \((Q_n)_{n \in \mathbb{N}}\) satisfies the conditions (1), (2) and (3) of Corollary [13] and (4) follows from the unicity assertion.

(iii) Similarly, consider the sequence of polynomials \((Q_n)_{n \in \mathbb{N}}\) defined by

\[
Q_n(X) = p^{n-1} \sum_{k=0}^{p-1} B_n \left( \frac{X + k}{p} \right)
\]

Clearly \(Q_0(X) = 1\) and for \(n \geq 1\):

\[
\Delta Q_n = p^{n-1} \sum_{k=0}^{p-1} B_n \left( \frac{X + k + 1}{p} \right) - p^{n-1} \sum_{k=0}^{p-1} B_n \left( \frac{X + k}{p} \right)
\]

\[
= p^{n-1} \left( \sum_{k=1}^{p} B_n \left( \frac{X + k}{p} \right) - \sum_{k=0}^{p-1} B_n \left( \frac{X + k}{p} \right) \right)
\]

\[
= p^{n-1} \left( B_n \left( \frac{X + p}{p} \right) - B_n \left( \frac{X}{p} \right) \right) = p^{n-1} \Delta B_n \left( \frac{1}{p} X \right) = nX^{n-1}.
\]

Moreover, for \(n \geq 1\),

\[
\int_0^1 Q_n(t) \, dt = p^{n-1} \sum_{k=0}^{p-1} \int_0^1 B_n \left( \frac{t + k}{p} \right) \, dt
\]

\[
= p^{n-1} \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} B_n(t) \, dt = p^{n-1} \int_0^1 B_n(t) \, dt = 0.
\]

This proves that the sequence of \((Q_n)_{n \in \mathbb{N}}\) satisfies the conditions (1), (2) and (3) of Corollary [13] and (4) follows by unicity. The proof of Proposition [2.1] is complete. □
Definition 2.2. The sequence of Bernoulli Numbers \((b_n)_{n \in \mathbb{N}}\) is defined by

\[ b_n = B_n(0) \quad \text{for } n \geq 0. \]

The following proposition summarizes some properties of Bernoulli numbers.

Proposition 2.3. The sequence of Bernoulli numbers \((b_n)_{n \in \mathbb{N}}\) satisfies the following properties:

i. For every positive integer \(n\) we have \(b_{2n+1} = 0\) and \(b_{2n} = B_{2n}(1)\).

ii. For every nonnegative integer \(n\) we have \(B_n \left( \frac{1}{2} \right) = (2^{1-n} - 1)b_n\).

iii. For every nonnegative integer \(n\) we have

\[ B_n(X) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} X^k. \]

iv. For every positive integer \(n\) we have

\[ b_n = -\frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} b_k. \]

Proof. (i) Using Proposition 2.1(ii) we have, for \(n \geq 2\) :

\[ B_n(1) - B_n(0) = \int_0^1 B_n'(t) \, dt = n \int_0^1 B_{n-1}(t) \, dt = 0 \]

and according to Proposition 2.1(ii) we have \(B_n(1) = (-1)^n B_n(0)\) for every \(n \geq 1\). This proves that \(b_n = 0\) if \(n\) is an odd integer greater than 2, and that \(B_{2n}(1) = b_{2n}\) for \(n \geq 1\). This is (i).

(ii) According to Proposition 2.1(ii) with \(p = 2\) we see that, for every nonnegative integer \(n\) we have

\[ B_n \left( \frac{X}{2} \right) + B_n \left( \frac{X + 1}{2} \right) = 2^{1-n} B_n(X) \]

Substituting \(X = 0\) we obtain (ii).

(iii) Consider \(n \in \mathbb{N}\), according Now, we use again Proposition 2.1(ii) to conclude that

\[ B_n^{(k)} = n(n-1) \ldots (n-k+1)B_{n-k}, \quad \text{for } 0 < k \leq n. \]

It follows that

\[ \frac{B_n^{(k)}(Y)}{k!} = \binom{n}{k} B_{n-k}(Y), \quad \text{for } 0 < k \leq n \]

and using Taylor’s formula for polynomials we conclude that

\[ B_n(X + Y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(Y) X^k \]

Finally, substituting \(Y = 0\), we obtain (iii).

(iv) Let \(n\) be an integer such that \(n \geq 2\). We have shown that \(B_n(1) = b_n\), and using the preceding point we conclude that \(B_n(1) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}\), that is \(b_n = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}\) or \(\sum_{k=0}^{n-1} \binom{n}{k} b_k = 0\). So

\[ \forall n \geq 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \]

which is equivalent to (iv). \(\square\)
Application 1. In Proposition 2.3 (iii), Bernoulli polynomials are expressed in terms of the canonical basis \((X^k)_{k \in \mathbb{N}}\) of \(\mathbb{R}[X]\). In fact, we have proved that

\[
B_n(X + Y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(X)Y^k
\]

and this can be used to, conversely, express the canonical basis \((X^k)_{k \in \mathbb{N}}\) of \(\mathbb{R}[X]\) in terms of Bernoulli polynomials.

Indeed, substituting \(Y = 1\) in (2.1) we obtain

\[
B_{n+1}(X + 1) = \sum_{k=0}^{n+1} \binom{n+1}{k} B_{n+1-k}(X),
\]

but according to Corollary 2.3 we have also \(B_{n+1}(X + 1) = B_{n+1}(X) + (n+1)X^n\). Thus

\[
(n + 1)X^n = \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k}(X) = \sum_{k=0}^{n} \binom{n+1}{k} B_k(X)
\]

Finally,

\[
X^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(X)
\]

Remark 2.4. Using the recurrence relation of Proposition 2.3 (iv) we can determine the sequence of Bernoulli Numbers. In particular, \(b_1 = -\frac{1}{2}\). We find in Table 1 the the values of the first six Bernoulli numbers with even indices. Also we find in Table 2 the list of the first six Bernoulli Polynomials.

| \(n\) | 0 | 1   | 2   | 3   | 4   | 5   |
|------|---|-----|-----|-----|-----|-----|
| \(b_{2n}\) | 1 | 1\(^{-}\frac{1}{2}\) | 1\(^{-}\frac{1}{3}\) | 1\(^{-}\frac{1}{4}\) | 1\(^{-}\frac{1}{5}\) | 1\(^{-}\frac{1}{6}\) |

**Table 1.** The first Bernoulli Numbers with even indices.

| \(B_0(X)\) | 1 |
|\(B_1(X)\) | \(X - \frac{1}{2}\) |
|\(B_2(X)\) | \(X^2 - X + \frac{1}{6}\) |
|\(B_3(X)\) | \(X^3 - \frac{1}{2}X^2 + \frac{1}{3}X\) = \(X (X - \frac{1}{2}) (X - 1)\) |
|\(B_4(X)\) | \(X^4 - 2X^3 + X^2 - \frac{1}{30}\) |
|\(B_5(X)\) | \(X^5 - \frac{5}{2}X^4 + \frac{5}{2}X^3 - \frac{5}{6}X\) = \(B_3(X) (X^2 - X - \frac{1}{3})\) |
|\(B_6(X)\) | \(X^6 - 3X^5 + \frac{5}{2}X^4 - \frac{1}{2}X^2 + \frac{1}{12}\) |

**Table 2.** The first Bernoulli Polynomials.
**Application 2.** For a nonnegative integer \( n \) and a positive integer \( m \), we define \( S_n(m) \) to be the sum

\[
S_n(m) = 1^n + 2^n + \cdots + m^n = \sum_{k=1}^{m} k^n.
\]

Noting that

\[
k^n = \frac{1}{n+1} (B_{n+1}(k + 1) - B_{n+1}(k))
\]

we see that

\[
S_n(m) = \frac{1}{n+1} (B_{n+1}(m + 1) - B_{n+1}(0)).
\]

In Table 3 we have listed the first values of these sums using the results from Table 2.

| Power | Sum |
|-------|-----|
| 1     | \( \frac{m(m+1)}{2} \) |
| 2     | \( \frac{m(m+1)(2m+1)}{6} \) |
| 3     | \( \frac{m^2(m+1)^2}{4} \) |
| 4     | \( \frac{m(m+1)(2m+1)(3m^2+m-1)}{30} \) |

**Table 3.** The sum of consecutive powers.

It was on studying these sums that Jacob Bernoulli introduced the numbers named after him.

It is clear that \( x \mapsto B_1(x) = x - \frac{1}{2} \) is negative on \((0, \frac{1}{2})\) and positive on \((\frac{1}{2}, 1)\). It follows that \( x \mapsto -B_2(x) = -x^2 + x - \frac{1}{6} \) is increasing on the interval \([0, \frac{1}{2}]\), and decreasing on the interval \([\frac{1}{2}, 1]\), and since it has opposite signs at 0 and \( \frac{1}{2} \) we conclude that \( B_2 \) vanishes exactly once on the interval \((0, \frac{1}{2})\) and exactly once on the interval \((\frac{1}{2}, 1)\).

These results can be generalized as follows:

**Figure 1.** The graphs of \( B_1 \) and \( -B_2 \) on \([0, 1]\).
**Proposition 2.5.** For every positive integer \( n \) we have

\( \mathcal{P}_n: \) The function \( x \mapsto (-1)^n B_{2n}(x) \) is increasing on \([0, \frac{1}{2}]\), and decreasing on \([\frac{1}{2}, 1]\), and consequently, it vanishes exactly once on each of the intervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, 1)\).

\( \mathcal{Q}_n: \) The function \( x \mapsto (-1)^n B_{2n+1}(x) \) is negative on \((0, \frac{1}{2})\), and positive on \((\frac{1}{2}, 1)\). Moreover, 0, \( \frac{1}{2} \) and 1 are simple zeros of \( B_{2n+1} \) in the interval \([0, 1]\).

**Proof.** We have already proved \( \mathcal{P}_1 \).

\( \mathcal{P}_n \Rightarrow \mathcal{Q}_n. \) Let \( f(x) = (-1)^n B_{2n+1}(x) \), then we have \( f'(x) = (2n+1)(-1)^n B_{2n}(x) \), and according to \( \mathcal{P}_n \), there exists an \( \alpha \) in \((0, \frac{1}{2})\) and a \( \beta \) in \((\frac{1}{2}, 1)\) such that \( f' \) is negative on each of the intervals \((0, \alpha)\) and \((\beta, 1)\), and positive on the interval \((\alpha, \beta)\). Therefore, \( f \) has the following table of variations:

| \( x \)   | 0   | \( \alpha \) | \( \frac{1}{2} \) | \( \beta \) | 1   |
|-----------|-----|--------------|----------------|----------|-----|
| \( f'(x) \) | -   | 0            | +              | 0        | -   |
| \( f(x) \)   | 0   | <            | \( \nearrow \) | \( \nearrow \) | \( \nearrow \) | 0   |

where we used Proposition 2.1 (m) and Proposition 2.3 (a) and (m) to conclude that \( B_{2n+1}(0) = B_{2n+1}\left(\frac{1}{2}\right) = B_{2n+1}(1) = 0 \).

Moreover, \( \mathcal{P}_n \) implies that \( f' \) does not vanish at any of the points 0, \( \frac{1}{2} \) and 1. This proves that 0, \( \frac{1}{2} \) and 1 are the only zeros of \( f \) in the interval \([0, 1]\) and that they are simple. \( \mathcal{Q}_n \) follows immediately.

\( \mathcal{Q}_n \Rightarrow \mathcal{P}_{n+1}. \) Let \( f(x) = (-1)^{n+1} B_{2n+2}(x) \), then we have \( f'(x) = -(2n+2)(-1)^n B_{2n+1}(x) \), and according to \( \mathcal{Q}_n \), the derivative \( f' \) is positive on \([0, \frac{1}{2}]\) and negative on \((\frac{1}{2}, 1)\). Therefore, \( f \) has the following table of variations:

| \( x \)   | 0  | \( \frac{1}{2} \) | 1   |
|-----------|----|----------------|-----|
| \( f'(x) \) | 0  | +              | 0   |
| \( f(x) \)   | A  | \( \nearrow \) | \(-1 - 2^{-1-2n})A \) | \( \nearrow \) | \( A \) |

with \( A = (-1)^{n+1} b_{2n+2} \). Clearly \( A \neq 0 \) because \( f \) is increasing on \((0, \frac{1}{2})\). Consequently, \( f(0) f\left(\frac{1}{2}\right) < 0 \) and \( f(1) f\left(\frac{1}{2}\right) < 0 \). Thus, \( f \) vanishes exactly once on each of intervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, 1)\), and that, it is increasing on \([0, \frac{1}{2}]\) and decreasing \([\frac{1}{2}, 1]\). This proves \( \mathcal{P}_{n+1} \), and achieves the proof of the proposition. \( \Box \)

![Figure 2. Illustration of Proposition 2.5](image)

**Remark 2.6.** It follows from the preceding proof that \((-1)^{n+1} b_{2n} > 0\) for every positive integer \( n \).
**Corollary 2.7.** For every positive integer $n$ we have
\[
\sup_{x \in [0,1]} |B_{2n}(x)| = |b_{2n}| \quad \text{and} \quad \sup_{x \in [0,1]} |B_{2n+1}(x)| \leq \frac{2n+1}{4} |b_{2n}|
\]

**Proof.** In fact, we conclude from Proposition 2.5 that
\[
\sup_{x \in [0,1]} |B_{2n}(x)| = \max \left( |B_{2n}(0)|, \left| B_{2n}\left(\frac{1}{2}\right) \right| \right)
\]

\[
= \max \left( |b_{2n}|, \left| 1 - \frac{1}{2^{2n-1}} \right| |b_{2n}| \right) = |b_{2n}|
\]

In order to show the second inequality we consider several cases:

- **If** $x \in \left[0, \frac{1}{4}\right)$, then we have
  \[
  B_{2n+1}(x) = B_{2n+1}(x) - B_{2n+1}(0) = \int_{0}^{x} B_{2n}(t) \, dt
  \]
  So
  \[
  |B_{2n+1}(x)| \leq (2n+1) \int_{0}^{x} |B_{2n}(t)| \, dt
  \]
  \[
  \leq (2n+1) |b_{2n}| x \leq \frac{2n+1}{4} |b_{2n}|
  \]

- **If** $x \in \left[\frac{1}{4}, \frac{1}{2}\right)$, then we have
  \[
  B_{2n+1}(x) = B_{2n+1}(1) - B_{2n+1}\left(\frac{1}{2}\right) = \int_{1/2}^{x} (2n+1)B_{2n}(t) \, dt
  \]
  Thus
  \[
  |B_{2n+1}(x)| \leq (2n+1) \int_{1/2}^{1} |B_{2n}(t)| \, dt
  \]
  \[
  \leq (2n+1) |b_{2n}| \left(\frac{1}{2} - x \right) \leq \frac{2n+1}{4} |b_{2n}|
  \]

- **Finally, when** $x \in \left[\frac{1}{2}, 1\right]$, we recall that $B_{2n+1}(x) = -B_{2n+1}(1-x)$ according to Proposition 2.1 (ii). Thus, in this case we have also
  \[
  |B_{2n+1}(x)| \leq \sup_{0 \leq t \leq \frac{1}{2}} |B_{2n+1}(t)| \leq \frac{2n+1}{4} |b_{2n}|
  \]

and the second part of the proposition follows. \(\square\)

**Remark 2.8.** We will show later in these notes that
\[
\sup_{x \in [0,1]} |B_{2n+1}(x)| \leq \frac{2n+1}{2\pi} |b_{2n}|
\]

which is, asymptotically, the best possible result. That is, we have also
\[
\lim_{n \to \infty} \frac{2\pi}{(2n+1) |b_{2n}|} \cdot \sup_{x \in [0,1]} |B_{2n+1}(x)| = 1
\]
3. Fourier series and Bernoulli polynomials

Extending periodically the restriction $B_n|_{[0,1)}$ of the Bernoulli polynomial $B_n$ to the interval $[0,1)$, we obtain a 1-periodic piecewise continuous function denoted by $\tilde{B}_n$. In fact, for every real $x$ we have $\tilde{B}_n(x) = B_n(\{x\})$ where $\{t\} = t - \lfloor t \rfloor$ is the fractional part of the real $t$. In Figure 3 the graphs of the functions $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$ are depicted.

In this section we consider the Fourier series expansion of these periodic functions.

**Proposition 3.1.**

i. For every $x \in (0,1)$, we have

$$B_1(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k}$$

ii. For every positive integer $n$, and every $x \in [0,1]$, we have

$$B_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^{2n}}$$

$$B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n + 1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k^{2n+1}}$$

**Proof.** First, let us consider the case of $\tilde{B}_1$. It is clear that

$$C_0(\tilde{B}_1) = \int_0^1 B_1(t)dt = 0.$$ 

and for $k \neq 0$ we have

$$C_k(\tilde{B}_1) = \int_0^1 B_1(t)e^{-2\pi i k t} dt = \int_0^1 \left(t - \frac{1}{2}\right)e^{-2\pi i k t} dt$$

$$= \left(-\frac{1}{2}\right) e^{-2\pi i k t} \bigg|_0^1 + \frac{1}{2i\pi k} \int_0^1 e^{-2\pi i k t} dt = \frac{i}{2\pi k}$$

Thus, according to Dirichlet’s theorem \[14\] Corollary 3.3.9 we conclude that for $x \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$\tilde{B}_1(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{i}{2\pi k} e^{2\pi i k x} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k}$$

and (ii) follows.
Let us now consider the case of $\tilde{B}_n$, for $n \geq 2$. According to Corollary 1.3 we have

$$C_0(\tilde{B}_n) = \int_0^1 B_n(t) dt = 0.$$ 

and for $k \neq 0$ we find that

$$C_k(\tilde{B}_n) = \int_0^1 B_n(t)e^{-2i\pi kt} dt = -\frac{B_n(0) - B_n(1)}{2i\pi k} + \frac{n}{2i\pi k} \int_0^1 B_{n-1}(t)e^{-2i\pi kt} dt$$

where we used Corollary 1.3 and Proposition 2.1 (ii). This allows us to prove by induction on $n$ that,

$$\forall n \in \mathbb{N}^*, \forall k \in \mathbb{Z} \setminus \{0\}, \quad C_k(\tilde{B}_n) = -\frac{n!}{(2i\pi k)^n}$$

Thus, because of the continuity of $\tilde{B}_n$ for $n \geq 2$, and of the uniform convergence of the Fourier series of $\tilde{B}_n$ in this case, we conclude that [20, Ch. I, Sec. 3], for $x \in \mathbb{R}$ we have

$$\tilde{B}_n(x) = -\sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{n!}{(2i\pi k)^n} e^{2i\pi kx} = -\frac{n!}{(2i\pi)^n} \sum_{k=1}^{\infty} e^{2i\pi kx} + (-1)^n \sum_{k=1}^{\infty} e^{-2i\pi kx}$$

and (ii) follows by considering separately the cases of $n$ even and $n$ odd. \hfill \Box

In particular, we have the following well-known result:

**Corollary 3.2.** For $n \geq 1$ we have

$$\zeta(2n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} b_{2n}$$

$$\eta(2n) \equiv \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} = (-1)^n \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \left( \frac{1}{2} \right) = (-1)^{n-1} \frac{(2^{2n} - 2)\pi^{2n}}{2 \cdot (2n)!} b_{2n}$$

(Here $\zeta(\cdot)$ is the well-know “Riemann Zeta function.”)

Using Bessel-Parseval’s identity [20, Ch. I, Sec. 5] we obtain the following corollary:

**Corollary 3.3.** If $n$ and $m$ are positive integers then

$$\int_0^1 B_n(x)B_m(x) dx = \frac{(-1)^{n-1} (2\pi)^{2n}}{(2n)!} b_{n+m}$$

In particular, for $n \in \mathbb{N}^*$ we have

$$\int_0^1 \left( \frac{B_n(x)}{n!} \right)^2 dx = (-1)^{n-1} \frac{b_{2n}}{(2n)!} = \left| \frac{b_{2n}}{(2n)!} \right|$$

**Proof.** Indeed, if $n + m \equiv 1 \mod 2$ then the change of variables $x \leftrightarrow 1 - x$ proves, using Proposition 2.1 (ii), that the considered integral $\int_0^1 B_n(x)B_m(x) dx$ equals 0, and the conclusion follows from Proposition 2.3 (ii).
So, let us suppose that \( n \equiv m \mod 2 \). In this case, by Bessel-Parseval’s identity, we have
\[
\int_0^1 B_n(x) B_m(x) \, dx = \sum_{k \in \mathbb{Z}} C_k(\tilde{B}_n) C_k(\tilde{B}_m) = (-1)^n \frac{n! \, m!}{(2\pi)^{n+m}} \sum_{k \in \mathbb{Z} \setminus \{0\}} 1 \frac{1}{n+m}
\]
and the desired conclusion follows by Corollary 3.2. \( \square \)

**Application 3.** One more formula for Bernoulli polynomials.

Let us consider the polynomial \( T_n \) defined by
\[
T_n(X) = \frac{1}{n+1} \sum_{k=0}^n B_k(X) B_{n-k}(X).
\]
Clearly, for \( n \geq 1 \), we have
\[
(n+1)T'_n(X) = \sum_{k=1}^n k B_{k-1}(X) B_{n-k}(X) + \sum_{k=0}^{n-1} (n-k) B_k(X) B_{n-k-1}(X)
\]
\[
= \sum_{k=0}^{n-1} (k+1) B_k(X) B_{n-k-1}(X) + \sum_{k=0}^{n-1} (n-k) B_k(X) B_{n-k-1}(X)
\]
\[
= (n+1) \sum_{k=0}^{n-1} B_k(X) B_{n-k-1}(X) = \frac{n+1}{n} T_{n-1}
\]
That is \( T'_n = nT_{n-1} \). Now, since \( T_n \) is a polynomial of degree \( n \) there are \((\lambda_k^{(n)})_{0 \leq k \leq n}\) such that \( T_n(X) = \sum_{k=0}^n \lambda_k^{(n)} B_k \), and from \( T'_n = nT_{n-1} \) we conclude that
\[
\sum_{k=0}^{n-1} (k+1) \lambda_{k+1}^{(n)} B_k = \sum_{k=0}^{n-1} n \lambda_k^{(n-1)} B_k.
\]
Thus \( (k+1) \lambda_k^{(n)} = n \lambda_k^{(n-1)} \) for \( 0 \leq k \leq n-1 \). It follows that \( \lambda_k^{(n)} = \binom{n}{k} \lambda_0^{(n-k)} \) for \( 0 \leq k \leq n \). So, we have proved that
\[
\frac{1}{n+1} \sum_{k=0}^n B_k(X) B_{n-k}(X) = \sum_{k=0}^n \binom{n}{k} \lambda_0^{(k)} B_{n-k}
\]
Integrating on \([0,1]\), and using Corollaries 1.3 and 3.3, we obtain \( \lambda_0^{(0)} = 1, \lambda_0^{(1)} = 0 \), and for \( n \geq 2 \):
\[
\lambda_0^{(n)} = \frac{1}{n+1} \sum_{k=0}^{n-1} \int_0^1 B_k(t) B_{n-k}(t) \, dt = \frac{b_n}{n+1} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{\binom{n}{k}}
\]
\[
= \frac{1}{n+1} \sum_{k=1}^{n-1} (-1)^{k-1} \frac{k!(n-k)!}{(n+1)!} = -b_n \int_0^1 \left( \sum_{k=1}^{n-1} t^{n-k}(t-1)^k \right) \, dt
\]
\[
= -b_n \int_0^1 (t^{n+1} - (t-1)^{n+1} - t^n - (t-1)^n) \, dt = \frac{1 + (-1)^n}{2(n+1)(n+2)} b_n.
\]
That is \( \lambda_0^{(2m+1)} = 0 \) and \( \lambda_0^{(2m)} = \frac{1}{(m+1)(2m+1)} b_{2m} \). Therefore, we have proved that
\[
\frac{1}{n+1} \sum_{k=0}^n B_k(X) B_{n-k}(X) = \sum_{k=0}^\lfloor n/2 \rfloor \binom{n}{2k} \frac{b_{2k}}{(k+1)(2k+1)} B_{n-2k}(X).
\]
In particular, taking \( n = 2m \) and \( x = 0 \) we find, for \( m \neq 1 \), that
\[
\sum_{k=0}^{m} b_{2k} b_{2m-2k} = \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{2m+2}{2k+2} \right) b_{2k} b_{2m-2k}.
\]
or equivalently, for \( m \neq 2 \):
\[
\sum_{k=1}^{m} b_{2(k-1)} b_{2(m-k)} = \frac{1}{m} \sum_{k=1}^{m} \left( \frac{2m}{2k} \right) b_{2(k-1)} b_{2(m-k)}.
\]
This is an unusual formula since it combines both convolution and binomial convolution.

4. Bernoulli polynomials on the unit interval \([0, 1]\)

Our first result concerns the sequence of Bernoulli numbers and it follows immediately from Corollary 3.2:

**Proposition 4.1.** The sequence of Bernoulli numbers \((b_{2n})_{n \geq 1}\) satisfies the following:

i. For every positive integer \( n \), we have \(|b_{2n}| < 2 \left( 1 + \frac{3}{2^{2n}} \right) \frac{(2n)!}{(2\pi)^{2n}} < \frac{4(2n)!}{(2\pi)^{2n}}\).

ii. Asymptotically, for \( n \) in the neighborhood of \(+\infty\), we have \(|b_{2n}| \sim +\infty 2 \left( \frac{2n}{2\pi} \right)^{2n}\).

**Proof.** Noting that, for \( t \in [k-1, k] \), we have \(k^{-2n} \leq t^{-2n} \leq k^{-2n} \) we conclude that

\[
\sum_{k=3}^{\infty} \frac{1}{k^{2n}} \leq \sum_{k=3}^{k} \int_{k-1}^{k} \frac{dt}{t^{2n}} = \int_{2}^{\infty} \frac{dt}{t^{2n}} = \frac{2}{2n-1} \cdot \frac{1}{2^{2n}}
\]

Thus, for every \( n \geq 1 \) we have

\[
1 < \zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \leq 1 + \frac{1}{4} + \frac{2}{2n-1} \cdot \frac{1}{2^{2n}}
\]
or, equivalently

\[
1 < \zeta(2n) < 1 + \frac{2n+1}{2n-1} \cdot \frac{1}{2^{2n}} \leq 1 + \frac{3}{2^{2n}} \leq 1 + \frac{3}{4} < 2
\]

Hence, we have proved that \(1 < \zeta(2n) < 2\) for every \( n \geq 1 \) and that \( \lim_{n \to \infty} \zeta(2n) = 1 \). This implies the desired conclusion using Corollary 3.2. \( \square \)

**Remark 4.2.** Using Stirling’s Formula [1, pp. 257] we see that for large \( n \) we have

\[
|b_{2n}| \sim +\infty 4 \sqrt{n} \left( \frac{n}{e\pi} \right)^{2n}
\]

It is clear, according to Proposition 2.3 that the function \( x \mapsto |B_{2n}(x)| \) attains its maximum on the interval \([0, 1]\) at \( x = 0 \). Thus, for \( n \geq 1 \) we have

\[
\sup_{x \in [0, 1]} |B_{2n}(x)| = |B_{2n}(0)| = |b_{2n}|
\]

Determining the maximum of \( x \mapsto |B_{2n+1}(x)| \) on the interval \([0, 1]\) is more difficult. In this regard, we have the following result.
Proposition 4.3. For every positive integer \( n \) we have

i. \[ \sup_{x \in [0,1]} |B_{2n+1}(x)| < \frac{2n+1}{2\pi} |b_{2n}|. \]

ii. \[ |B_{2n+1}\left(\frac{1}{4}\right)| \geq \left(1 - \frac{4}{2^{2n}}\right) \frac{2n+1}{2\pi} |b_{2n}|. \]

Proof. In fact, using Proposition 3.1 (i) we see that

\[ \sup_{x \in [0,1]} |B_{2n+1}(x)| \leq \frac{2(2n+1)!}{(2\pi)^{2n+1}} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} < \frac{2(2n+1)!}{(2\pi)^{2n+1}} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2n}}. \]

Thus, according to Corollary 3.2 we conclude that

\[ \sup_{x \in [0,1]} |B_{2n+1}(x)| < \frac{2(2n+1)!}{(2\pi)^{2n+1}} \cdot \frac{(2\pi)^{2n}}{2 \cdot (2n)!} |b_{2n}| = \frac{2n+1}{2\pi} |b_{2n}|. \]

which is (1).

On the other hand, for \( n \in \mathbb{N} \), using Proposition 3.1 (ii) once more, we obtain

\[ B_{2n+1}\left(\frac{1}{4}\right) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}, \]

but the series above is alternating, so

\[ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} > 1 - \frac{1}{2^{2n+1}}. \]

Thus

\[ (-1)^{n+1} B_{2n+1}\left(\frac{1}{4}\right) > \left(1 - \frac{1}{3^{2n+1}}\right) \frac{2(2n+1)!}{(2\pi)^{2n+1}}. \]

Now, using Proposition 4.1 (i) we see that

\[ (-1)^{n+1} B_{2n+1}\left(\frac{1}{4}\right) > \left(1 - \frac{3-2^{2n-1}}{1 + 3 \cdot 2^{-2n}}\right) \frac{2n+1}{2\pi} |b_{2n}|, \]

and since

\[ \forall n \in \mathbb{N}^*, \quad 1 - \frac{3-2^{2n-1}}{1 + 3 \cdot 2^{-2n}} \geq 1 - \frac{4}{2^{2n}} \]

we obtain (2). \( \square \)

Remark 4.4. Combining Corollary 2.7 and Propositions 4.1 and 4.3 we see that, for large \( n \) we have

\[ \sup_{x \in [0,1]} |B_n(x)| \sim \frac{2 \cdot n!}{(2\pi)^n}. \]

Next, we will study the behavior of the unique zero of \( B_{2n} \) in the interval \((0,1/2)\).

Proposition 4.5. For a positive integer \( n \), let \( \alpha_n \) be the unique zero of \( B_{2n} \) that belongs to the interval \((0,1/2)\). Then the sequence \( (\alpha_n)_{n \geq 1} \) satisfies the following inequality:

\[ \frac{1}{4} - \frac{1}{\pi} \cdot 2^{2n} < \alpha_n < \alpha_{n+1} < \frac{1}{4}. \]

Proof. First, note that \( \alpha_1 = \frac{1}{4} - \frac{1}{\pi} \cdot \frac{1}{\sqrt{3}} \) so \( \frac{1}{4} - \frac{1}{\pi} < \alpha_1 < \frac{1}{4} \). On the other hand, since \( \alpha_1 (1 - \alpha_1) = \frac{1}{60} \) and \( B_4(X) = X^2(1-X)^2 - \frac{1}{30} \) we conclude that

\[ B_4(\alpha_1) = -\frac{1}{180} < 0 \quad \text{and} \quad B_4\left(\frac{1}{4}\right) = \frac{269}{7680} > 0. \]

This proves the desired inequality for \( n = 1 \).
Now, let us suppose that \( n \geq 2 \). Using Proposition 2.1 \((\text{iii})\) with \( p = 4 \) and \( X = 0 \) we obtain
\[
B_{2n}(0) + B_{2n} \left( \frac{1}{4} \right) + B_{2n} \left( \frac{1}{2} \right) + B_{2n} \left( \frac{3}{4} \right) = \frac{1}{4^{2n-1}} B_{2n}(0).
\]
But, according to Proposition 2.1 \((\text{iii})\), \( B_{2n} \left( \frac{1}{4} \right) = B_{2n} \left( \frac{3}{4} \right) \), and using Proposition 2.3 \((\text{i})\), we have also
\[
B_{2n} \left( \frac{1}{2} \right) = (2^{1-2n} - 1) b_{2n}.
\]
Hence
\[
B_{2n} \left( \frac{1}{4} \right) = \frac{1}{2^{2n}} (2^{1-2n} - 1) b_{2n} = \frac{1}{2^{2n}} B_{2n} \left( \frac{1}{2} \right).
\]
This proves that \( B_{2n} \left( \frac{1}{4} \right) B_{2n} \left( \frac{1}{2} \right) > 0 \) and \( B_{2n} \left( \frac{1}{4} \right) B_{2n}(0) < 0 \). It follows that \( 0 < \alpha_n < \frac{1}{4} \).

Now, let us define the function \( h_n \) by
\[
h_n(x) = \sum_{k=1}^{\infty} \cos \left( \frac{2\pi k x}{k^{2n}} \right).
\]
Also, let \( x_n = \frac{1}{4} - \frac{1}{2^{2n}} \). Clearly we have
\[
h_n(x_n) = \cos \left( \frac{\pi}{2} - \frac{2}{2^{2n}} \right) + \sum_{k=2}^{\infty} \cos \left( \frac{2\pi k x_n}{k^{2n}} \right) 
\geq \sin \left( \frac{2}{2^{2n}} \right) - \sum_{k=2}^{\infty} \frac{1}{k^{2n}} > \sin \left( \frac{2}{2^{2n}} \right) - \frac{2n + 1}{2n - 1} \cdot \frac{1}{2^{2n}}
\]
where we used the inequality \( \zeta(2n) - 1 < \frac{2n + 1}{2n - 1} \cdot 2^{-2n} \) from Proposition 1.1.

Finally, using the inequality \( \sin x \geq x - \frac{x^3}{6} \) which is valid for \( x \geq 0 \), and recalling that \( n \geq 2 \) we conclude that
\[
h_n(x_n) > \frac{1}{2^{2n}} \left( 2 - \frac{2n + 1}{2n - 1} - \frac{4}{3 \cdot 2^{2n}} \right)
\geq \frac{1}{3 \cdot 2^{2n}} \left( 1 - \frac{1}{2^n} \right) > 0
\]
This proves, according to Proposition 3.1 \((\text{i})\), that \( B_{2n}(x_n) B_{2n}(0) > 0 \) and consequently \( x_n < \alpha_n \).

Next, let us show that \( h_{n+1}(\alpha_n) > 0 \), because this implies, according to Proposition 3.1 \((\text{i})\), that \( B_{2n+2}(\alpha_n) B_{2n+2}(0) > 0 \) and consequently \( \alpha_n < \alpha_{n+1} \).

First, on one hand, we have
\[
0 = h_n(\alpha_n) = \sum_{k=1}^{\infty} \frac{\cos(2\pi k \alpha_n)}{k^{2n}} = \cos(2\pi \alpha_n) + \sum_{k=2}^{\infty} \frac{\cos(2\pi k \alpha_n)}{k^{2n}}
\]
and on the other
\[
h_{n+1}(\alpha_n) = \sum_{k=1}^{\infty} \frac{\cos(2\pi k \alpha_n)}{k^{2n+2}} = \cos(2\pi \alpha_n) + \sum_{k=2}^{\infty} \frac{\cos(2\pi k \alpha_n)}{k^{2n+2}}.
\]
Hence
\[
h_{n+1}(\alpha_n) = -\sum_{k=2}^{\infty} \left( 1 - \frac{1}{k^2} \right) \frac{\cos(2\pi k \alpha_n)}{k^{2n}}
= -\frac{3}{4} \cdot \cos(\pi \alpha_n) - \sum_{k=3}^{\infty} \left( 1 - \frac{1}{k^2} \right) \frac{\cos(2\pi k \alpha_n)}{k^{2n}}.
\]
But, from \( \frac{1}{4} - \frac{1}{2^{2n}} < \alpha_n < \frac{1}{4} \) we conclude that \( \pi - \frac{\alpha_n}{2} - 4\pi \alpha_n < \pi \), and consequently
\[
1 - \frac{8}{2^{2n}} \leq 1 - 2 \sin^2 \left( \frac{4}{2^{2n}} \right) = \cos \left( \frac{4}{2^{2n}} \right) < -\cos(4\pi \alpha_n),
\]
Thus, using the estimate \( \sum_{k=3}^{\infty} \frac{1}{k^2} \) obtained on the occasion of proving Proposition 4.1, we get

\[
h_{n+1}(\alpha_n) \geq \frac{3}{2^{2n+2}} - \frac{6}{2^{2n}} - \sum_{k=3}^{\infty} \left( 1 - \frac{1}{k^2} \right) \frac{1}{k^{2n}}
\]

\[
> \frac{3}{2^{2n+2}} - \frac{6}{2^{2n}} - \sum_{k=3}^{\infty} \frac{1}{k^{2n}} > \frac{3}{2^{2n+2}} - \frac{8}{2^{2n}} - \frac{2}{3 \cdot 2^{2n}}
\]

\[
\geq \frac{1}{2^{2n}} \left( \frac{1}{12} - \frac{8}{2^{4n}} \right) \geq \frac{1}{2^{2n}} \left( \frac{1}{12} - \frac{1}{32} \right) > 0
\]

This concludes the proof of the desired inequality. \( \square \)

**Remark 4.6.** The better inequality: \( \alpha_n > \frac{1}{4} - \frac{1}{\pi} \cdot 2^{-2n} \), is proved in [26], but the increasing behaviour of the sequence is not discussed there. Concerning the rational zeros of Bernoulli polynomials, it was proved in [18] that the only possible rational zeros for a Bernoulli polynomial are 0, \( \frac{1}{2} \) and 1. A detailed account of the complex zeros of Bernoulli polynomials can be found in [10].

In the next proposition, we will show how to estimate the “\( L^1 \)-norm” \( \int_0^1 |B_n| \) in terms of the “\( L^\infty \)-norm” \( \sup_{[0,1]} |B_{n+1}| \).

**Proposition 4.7.** The following two properties hold:

i. For every positive integer \( n \), we have

\[
\int_0^1 |B_n(x)| \, dx < 16 \frac{n!}{(2\pi)^{n+1}}.
\]

ii. Asymptotically, for large \( n \), we have

\[
\int_0^1 |B_n(x)| \, dx \sim_{+\infty} 8 \frac{n!}{(2\pi)^{n+1}}.
\]

**Proof.** According to Proposition 2.1 [12] we have \( |B_n(x)| = |B_n(1-x)| \). Thus

\[
\int_0^1 |B_n(x)| \, dx = 2 \int_0^{1/2} |B_n(x)| \, dx.
\]

So, we consider two cases:

(a) The case \( n = 2m \). We have seen (Proposition 2.5) that \( x \mapsto (-1)^m B_{2m}(x) \) is increasing on \([0, 1/2]\) and has a unique zero \( \alpha_m \) in this interval. Hence

\[
\int_0^{1/2} |B_n(x)| \, dx = (-1)^m \left( \int_0^{\alpha_m} B_n(x) \, dx + \int_{\alpha_m}^{1/2} B_n(x) \, dx \right)
\]

\[
= (-1)^m \left( \frac{B_{n+1}(x)}{n+1} \right)_{x=0}^{\alpha_m} + \frac{B_{n+1}(x)}{n+1} \right)_{x=\alpha_m}^{1/2}
\]

\[
= 2(-1)^{m+1} \frac{B_{2m+1}(\alpha_m)}{n+1} = 2 \left( \frac{n+1}{2m+1} \right) \sup_{x \in [0,1]} |B_{n+1}(x)|.
\]

Thus, if \( n \) is even, we have

\[
\int_0^1 |B_n(x)| \, dx = \frac{4}{n+1} \sup_{x \in [0,1]} |B_{n+1}(x)| \tag{1}
\]

(b) The case \( n = 2m+1 \). We have seen (Proposition 2.5) that \( x \mapsto (-1)^m B_{2m+1}(x) \) is increasing on \([0, 1/2]\). Hence

\[
\int_0^{1/2} |B_n(x)| \, dx = (-1)^m \left( \int_0^{\alpha_m} B_n(x) \, dx + \int_{\alpha_m}^{1/2} B_n(x) \, dx \right)
\]

\[
= (-1)^m \left( \frac{B_{n+1}(x)}{n+1} \right)_{x=0}^{\alpha_m} + \frac{B_{n+1}(x)}{n+1} \right)_{x=\alpha_m}^{1/2}
\]

\[
= 2(-1)^{m+1} \frac{B_{2m+2}(\alpha_m)}{n+1} = 2 \left( \frac{n+1}{2m+1} \right) \sup_{x \in [0,1]} |B_{n+1}(x)|.
\]

Thus, if \( n \) is odd, we have

\[
\int_0^1 |B_n(x)| \, dx = \frac{4}{n+1} \sup_{x \in [0,1]} |B_{n+1}(x)| \tag{1}
\]
(b) The case \( n = 2m + 1 \). Using again Proposition 2.5, we see that the function \( x \mapsto (-1)^{m+1}B_{2m+1}(x) \) is nonnegative on \([0, 1/2]\), thus

\[
\int_{0}^{1/2} |B_n(x)| \, dx = (-1)^{m+1} \int_{0}^{1/2} B_n(x) \, dx = (-1)^{m+1} \frac{B_{n+1}(x)}{n+1} \bigg|_{0}^{1/2} \\
= \frac{(-1)^{m+1}}{n+1} \left( B_{2m+2} \left( \frac{1}{2} \right) - B_{2m+2}(0) \right) \\
= \frac{(-1)^{m+1}}{n+1} \left( \frac{1}{2^{2m+1} - 1} \right) b_{2m+2} - b_{2m+2} \\
= \left( \frac{2 - \frac{1}{2^n}}{n+1} \right). \\
\]

So, according to Corollary 2.7, if \( n \) is odd, we have

\[
\int_{0}^{1} |B_n(x)| \, dx = \frac{4 - 2^{1-n}}{n+1} \sup_{x \in [0,1]} |B_{n+1}(x)|. \tag{\dagger}
\]

Thus, combining (\ref{eq:1}), (\ref{eq:2}) and Remark 4.4 we obtain

\[
\int_{0}^{1} |B_n(x)| \, dx \sim n+1 \sup_{x \in [0,1]} |B_{n+1}(x)| \sim n! \frac{8 \cdot n!}{(2\pi)^{n+1}}.
\]

Also, using Propositions 4.1 and 4.3 we obtain

\[
\int_{0}^{1} |B_n(x)| \, dx < 16 \cdot \frac{n!}{(2\pi)^{n+1}}
\]

which is the desired conclusion. \( \square \)

5. Asymptotic behavior of Bernoulli polynomials

Proposition 5.1 shows that, for \( x \in [0, 1] \), we have

\[
\left| (-1)^{n+1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}(x) - \cos(2\pi x) \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^{2n}} = \zeta(2n) - 1 < \frac{3}{2^{2n}}
\]

and

\[
\left| (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2 \cdot (2n + 1)!} B_{2n+1}(x) - \sin(2\pi x) \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^{2n+1}} = \zeta(2n + 1) - 1 < \frac{3}{2^{2n+1}}
\]

where we used the following simple inequality, valid for \( m \geq 2 \):

\[
\zeta(m) - 1 < \frac{1}{2^m} + \int_{1/2}^{\infty} \frac{dt}{t^m} = \frac{m + 1}{m - 1} \cdot \frac{1}{2^m} < \frac{3}{2^m}
\]

Thus, the sequence \( \left( (-1)^{n+1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \right)_{n \geq 1} \) converges uniformly on the interval \([0, 1]\) to \( \cos(2\pi \cdot) \), and similarly, the sequence \( \left( (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2 \cdot (2n + 1)!} B_{2n+1} \right)_{n \geq 1} \) converges uniformly on the interval \([0, 1]\) to \( \sin(2\pi \cdot) \). In fact, this conclusion is a particular case of a more general result proved by K. Dilcher [9].
Let us first introduce some notation. Let \((T_n)_{n \in \mathbb{N}}\) be the sequence of polynomials defined by the formula:

\[
T_n(z) = (-1)^{[n/2]} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2\pi z)^{n-2k}}{(n-2k)!}.
\]

So that

\[
T_{2n}(z) = \sum_{k=0}^{n} (-1)^k \frac{(2\pi z)^{2k}}{(2k)!},
\]

\[
T_{2n+1}(z) = \sum_{k=0}^{n} (-1)^k \frac{(2\pi z)^{2k+1}}{(2k+1)!}.
\]

With this notation we have:

**Proposition 5.1.** For every integer \(n\), with \(n \geq 2\), and every complex number \(z\) we have

\[
\left| (-1)^{[n/2]} \frac{(2\pi)^n}{2 \cdot n!} B_n \left( z + \frac{1}{2} \right) - T_n(z) \right| < \frac{e^{4|z|}}{2^n}
\]

**Proof.** Note that \(B_{2k+1} \left( \frac{1}{2} \right) = 0\) for every \(k \geq 0\), and according to Corollary \(3.2\) we have

\[
B_{2k} \left( \frac{1}{2} \right) = (-1)^k \frac{2 \cdot (2k)!}{(2\pi)^{2k}} \eta(2k), \quad \text{for } k \geq 1
\]

where \(\eta(2k) = \sum_{m=1}^{\infty} (-1)^{m-1}/m^{2k}\). Thus, using Taylor’s expansion we have

\[
B_n \left( z + \frac{1}{2} \right) = z^n + \sum_{k=1}^{n} \binom{n}{k} B_k \left( \frac{1}{2} \right) z^{n-k}
\]

\[
= z^n + \sum_{1 \leq k \leq n/2} \binom{n}{2k} B_{2k} \left( \frac{1}{2} \right) z^{n-2k}
\]

\[
= z^n + \frac{2 \cdot n!}{(2\pi)^n} \sum_{1 \leq k \leq n/2} (-1)^k \eta(2k) \frac{(2\pi z)^{n-2k}}{(n-2k)!}.
\]

Thus

\[
\frac{(2\pi)^n}{2 \cdot n!} B_n \left( z + \frac{1}{2} \right) - (-1)^{[n/2]} T_n(z) = -\frac{(2\pi z)^n}{2 \cdot n!} + \sum_{1 \leq k \leq n/2} (-1)^k \eta(2k) \frac{(2\pi z)^{n-2k}}{(n-2k)!}
\]

and consequently, since \(0 < 1 - \eta(2k) < 2^{-2k}\), we get

\[
\left| (-1)^{[n/2]} \frac{(2\pi)^n}{2 \cdot n!} B_n \left( z + \frac{1}{2} \right) - T_n(z) \right| \leq \frac{(2\pi |z|)^n}{2 \cdot n!} + \sum_{1 \leq k \leq n/2} (1 - \eta(2k)) \frac{(2\pi |z|)^{n-2k}}{(n-2k)!}
\]

\[
\leq \frac{1}{2^n} \left( \frac{(4\pi |z|)^n}{2 \cdot n!} + \sum_{1 \leq k \leq n/2} (4\pi |z|)^{n-2k} \right)
\]

\[
\leq \frac{1}{2^n} \left( \sum_{0 \leq k \leq n/2} (4\pi |z|)^{n-2k} \right)
\]

\[
\leq \frac{e^{4|z|}}{2^n}
\]

and the desired inequality follows. \(\Box\)

Clearly, the sequences of polynomial functions \((T_{2n})_{n \in \mathbb{N}}\) and \((T_{2n+1})_{n \in \mathbb{N}}\) converge uniformly on every compact subset of \(\mathbb{C}\) to \(z \mapsto \cos(2\pi z)\) and \(z \mapsto \sin(2\pi z)\) respectively. So, the next corollary is obtained on replacing \(z\) by \(z - 1/2\) in Proposition 5.1.
Proposition 6.1. For every plane $C$ of $\sum$, this implies the convergence of the series

Hence, for every $(z, w)$, the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$ is convergent and

Proof. Using Proposition 4.1 and the facts that $b_0 = 1, b_1 = -1/2$, we see that

So, using Proposition 2.3, we see that for every nonnegative integer $n$ and complex number $z$ we have:

Hence, for every $(z, w) \in C \times D(0, 2\pi)$ and every nonnegative integer $n$ we have

This implies the convergence of the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$. Therefore, we can define

Moreover, for $w \in D(0, 2\pi)$, the normal convergence of the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$ on every compact subset of $C$, implies, using Proposition 2.3, the normal convergence of the series $\sum_{n=0}^{\infty} \frac{B_n'(z)}{n!} w^n$ on every compact subset of $C$. So, the function $F(z, w)$ has a derivative on $C$ and

Corollary 5.2. The following two properties hold:

i. The sequence $\left( (-1)^n + \frac{(2\pi)^n}{2(2\pi)!} B_{2n} \right)_{n \geq 1}$ converges uniformly on every compact subset of $C$ to the function $\cos(2\pi z)$.

ii. The sequence $\left( (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2\pi)!} B_{2n+1} \right)_{n \geq 1}$ converges uniformly on every compact subset of $C$ to the function $\sin(2\pi z)$.

6. The generating function of Bernoulli polynomials

In what follows, we will write $D(a, r)$ to denote the open disk of center $a$ and radius $r$ in the complex plane $C$:

The next result gives the generating function of the sequence of Bernoulli polynomials.

Proposition 6.1. For every $(z, w) \in C \times D(0, 2\pi)$, the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$ is convergent and

$\forall (z, w) \in C \times D(0, 2\pi), \quad \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n = \frac{we^{zw}}{e^w - 1}$

Proof. Using Proposition 4.1 and 4.4, and the facts that $b_0 = 1, b_1 = -1/2$, we see that

So, using Proposition 2.3, we see that for every nonnegative integer $n$ and complex number $z$ we have:

Hence, for every $(z, w) \in C \times D(0, 2\pi)$ and every nonnegative integer $n$ we have

This implies the convergence of the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$. Therefore, we can define

Moreover, for $w \in D(0, 2\pi)$, the normal convergence of the series $\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n$ on every compact subset of $C$, implies, using Proposition 2.3, the normal convergence of the series $\sum_{n=0}^{\infty} \frac{B_n'(z)}{n!} w^n$ on every compact subset of $C$. So, the function $F(z, w)$ has a derivative on $C$ and

$\frac{\partial F}{\partial z}(z, w) = \sum_{n=0}^{\infty} \frac{B_n'(z)}{n!} w^n = \sum_{n=1}^{\infty} \frac{nB_{n-1}(z)}{n!} w^n = wF(z, w)$
Thus, there exists a function \( f : D(0, 2\pi) \rightarrow C \) such that \( F(z, w) = e^{zw}f(w) \) for every \((z, w)\) in \( C \times D(0, 2\pi) \). Now, since the series \( \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} w^n \) is normally convergent on the compact set \([0, 1]\), and using Corollary 1.3 we obtain
\[
\int_0^1 F(t, w) \, dt = \sum_{n=0}^{\infty} \frac{w^n}{n!} \int_0^1 B_n(t) \, dt = 1
\]
But, on the other hand, we have
\[
\int_0^1 F(t, w) \, dt = f(w) \int_0^1 e^{tw} \, dt = f(w) \frac{e^w - 1}{w}
\]
Hence, \( f(w) = \frac{w}{e^w - 1} \), and consequently \( F(z, w) = we^{zw}/(e^w - 1) \), which is the desired conclusion. \( \Box \)

The above result allows us to find the power series expansion of some well-known functions.

**Proposition 6.2.** The functions \( z \mapsto z \cot z \), \( z \mapsto \tan z \) and \( z \mapsto z/\sin z \) have the following power series expansions in the neighbourhood of zero:

i. \( \forall z \in D(0, \pi), \quad z \cot z = \sum_{n=0}^{\infty} 2^{2n}(-1)^n b_{2n} z^{2n} \).

ii. \( \forall z \in D \left( 0, \frac{\pi}{2} \right), \quad \tan z = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)(-1)^{n+1} b_{2n}}{(2n)!} z^{2n-1} \).

iii. \( \forall z \in D(0, \pi), \quad \frac{z}{\sin z} = \sum_{n=0}^{\infty} \frac{(2^{2n} - 2)(-1)^n b_{2n}}{(2n)!} z^{2n} \).

**Proof.** Indeed, choosing \( z = 0 \) in Proposition 6.1 and using Proposition 2.3 (i) we obtain
\[
\forall w \in D(0, 2\pi), \quad -\frac{1}{2} w + \sum_{n=0}^{\infty} \frac{b_{2n}}{(2n)!} w^{2n} = \frac{w}{e^w - 1}.
\]
Thus
\[
\forall w \in D(0, 2\pi), \quad \sum_{n=0}^{\infty} \frac{2b_{2n}}{(2n)!} w^{2n} = w \left( \frac{e^w + 1}{e^w - 1} \right).
\]
Substituting \( w = 2iz \) we obtain
\[
\forall z \in D(0, \pi), \quad \sum_{n=0}^{\infty} \frac{2^{2n}(-1)^n b_{2n}}{(2n)!} z^{2n} = iz \left( \frac{e^{2iz} + 1}{e^{2iz} - 1} \right) = z \cot z.
\]
This proves (i).

On the other hand. Noting that
\[
\tan z = \cot z - 2 \cot(2z)
\]
\[
\frac{1}{\sin z} = \cot \left( \frac{z}{2} \right) - \cot z
\]
we obtain (ii) and (iii).

**Remark 6.3.** Recalling Corollary 5.2 we see that for \( z \in D(0, 1) \) we have
\[
\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n} \quad (6.1)
\]
Interchanging the signs of summation we find that
\[
\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{z}{k}\right)^{2n} = 1 - 2 \sum_{k=1}^{\infty} \frac{z^2}{k^2 - z^2}
\]
\[
= 1 + z \sum_{k=1}^{\infty} \left(\frac{1}{z - k} + \frac{1}{z + k}\right)
\]
This yields the following simple fraction expansion of the cotangent function:
\[
\pi \cot(\pi z) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{z - k} + \frac{1}{z + k}\right) = \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{1}{z - k}
\] (6.2)

Note that we have proved this for \(z \in D(0,1)\) but the result is valid for every \(z \in \mathbb{C} \setminus \mathbb{Z}\) using analytic continuation \[2, Chap. 8, §1\]. Similarly, for \(z \in D(0,1)\) we have
\[
\frac{\pi z}{\sin(\pi z)} = 1 - 2 \sum_{n=1}^{\infty} \eta(2n)z^{2n}
\] (6.3)

Interchanging the signs of summation we find that
\[
\frac{\pi z}{\sin(\pi z)} = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{z}{k}\right)^{2n} = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^2}{k^2 - z^2}
\]
\[
= 1 + z \sum_{k=1}^{\infty} (-1)^{k} \left(\frac{1}{z - k} + \frac{1}{z + k}\right)
\]
This yields the following simple fraction expansion of the cosecant function:
\[
\frac{\pi}{\sin(\pi z)} = 1 + \sum_{k=1}^{\infty} (-1)^{k} \left(\frac{1}{z - k} + \frac{1}{z + k}\right) = \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{(-1)^{k}}{z - k}
\] (6.4)

This is also valid for every \(z \in \mathbb{C} \setminus \mathbb{Z}\) using analytic continuation.

**Application 4.** Using the power series expansion of \(z \mapsto \tan z\) obtained in the previous result, we see that for every positive integer \(n\) we have
\[
\tan(2n+1)(0) = \frac{2^{2n}(2^{2n} - 1)}{2n}(-1)^{n+1}b_{2n}
\]
So, let us define \(a_n = \tan(n)(0)\). We note that \(a_{2n} = 0\) for every \(n \geq 0\) since “tan” is an odd function. If we use \(\tan' = 1 + \tan^2\) and the Leibniz formula, we obtain,
\[
\tan(2n+1)(z) = (1 + \tan^2)(2n) = \sum_{k=0}^{2n} \binom{2n}{k} \tan^{(k)}(z) \tan^{(2n-k)}(z)
\]
for every positive integer \(n\). Thus
\[
\forall n \geq 1, \quad a_{2n+1} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}a_{2k+1}a_{2(n-k)-1}
\]
But, \(a_1 = 1\) and the above formula shows inductively that \(a_{2n+1}\) is an integer for every \(n\). This proves that
\[
\forall n \geq 1, \quad \frac{2^{2n}(2^{2n}-1)}{2n}b_{2n} \in \mathbb{Z}
\]
and considering the separate case of \(b_1\) we see that
\[
\forall n \geq 1, \quad \frac{2^n(2^n-1)}{n}b_n \in \mathbb{Z}.
\]
This result is to be compared with Corollary \[7,5\].
Application 5. Let \( f(z) = z \cot z \). Since \( f(z) - zf'(z) = z^2 + f^2(z) \) we conclude from Proposition 6.2 that
\[
\sum_{n=0}^{\infty} \frac{2^{n}(1-2n)(-1)^n b_{2n}}{(2n)!} z^{2n} = z^2 + \sum_{n=0}^{\infty} \frac{2^{2n}(-1)^n}{(2n)!} \left( \sum_{k=0}^{n} \binom{n}{2k} b_{2k} b_{2n-2k} \right) z^{2n}
\]
Comparing the coefficients of \( z^{2n} \) we see that the sequence \((b_{2n})_{n \geq 1}\) can be defined recursively by the formula
\[
b_2 = \frac{1}{6}, \quad \forall n \geq 2, \quad b_{2n} = -\frac{1}{2n+1} \sum_{k=1}^{n-1} \binom{n-1}{2k} b_{2k} b_{2n-2k}.
\]

Application 6. A multiplication formula for Bernoulli polynomials. Consider an integer \( q \), with \( q \geq 2 \), clearly we have
\[
e^{qw} - 1 = \sum_{k=0}^{q-1} e^{kw} = 1 + \sum_{n=0}^{\infty} \left( \sum_{k=1}^{n-1} k^n \right) \frac{w^n}{n!},
\]
\[
= 1 + \sum_{n=0}^{\infty} S_n(q-1) \frac{w^n}{n!} = 1 + \sum_{n=0}^{\infty} \frac{B_{n+1}(q) - B_{n+1}(0)}{n+1} \frac{w^n}{n!},
\]
\[
= q + \sum_{n=1}^{\infty} \frac{B_{n+1}(q) - b_{n+1}}{n+1} \frac{w^n}{n!}.
\]
Where we used the notation of Application 2 Noting the identity
\[
q \cdot \frac{w e^{(qw)w}}{e^w - 1} = \frac{(qw) e^{z(qw)}}{e^{qw} - 1} \cdot \frac{e^{qw} - 1}{e^w - 1}
\]
we conclude that
\[
\sum_{n=0}^{\infty} q B_n(qz) \frac{w^n}{n!} = \left( \sum_{n=0}^{\infty} q^n B_n(z) \frac{w^n}{n!} \right) \left( q + \sum_{n=1}^{\infty} \frac{B_{n+1}(q) - b_{n+1}}{n+1} \frac{w^n}{n!} \right)
\]
\[
= \sum_{n=0}^{\infty} G_n(q, z) \frac{w^n}{n!}
\]
where
\[
G_n(q, z) = q^{n+1} B_n(z) + \frac{1}{n+1} \sum_{j=0}^{n-1} q^j \binom{n+1}{j} B_j(z)(B_{n+1-j}(q) - b_{n+1-j}).
\]
But, because a power series expansion is unique, we have \( q B_n(qz) = G_n(q, z) \) for every \( n \). Now, fix \( z \) in \( \mathbb{C} \) and consider the polynomial
\[
Q(X) = B_n(zX) - \left( X^n B_n(z) + \frac{1}{n+1} \sum_{j=0}^{n-1} \binom{n+1}{j} B_j(z)(B_{n+1-j}(X) - b_{n+1-j})X^{j-1} \right)
\]
(Note that \( X((B_{n+1-j}(X) - b_{n+1-j})) \) clearly deg \( Q \leq n \), and \( Q \) has infinitely many zeros, (namely, every integer \( q \) greater than 1.) Thus \( Q(X) = 0 \) and we have proved the following “Multiplication Formula”, valid for every complex numbers \( z \) and \( w \):
\[
B_n(zw) = w^n B_n(z) + \frac{1}{n+1} \sum_{j=0}^{n-1} \binom{n+1}{j} B_j(z)w^{j-1}(B_{n+1-j}(w) - b_{n+1-j}) \tag{6.5}
\]
For example, taking \( w = 2 \) and \( z = 0 \) we obtain, the following recurrence
\[
b_n = \frac{1}{2(1-2^n)} \sum_{j=0}^{n-1} 2^j \binom{n}{j} b_j
\]
since \( B_{n+1-j}(2) - B_{n+1-j}(0) = n + 1 - j \) for \( 0 \leq j < n \) according to Corollary 1.3. This recurrence was obtained in [27], and was generalized in [5]. All these generalizations follow from (6.5).

**Application 7. More formulæ for Bernoulli numbers.** The function \( w \mapsto e^{w-1} \) is entire, and has a power series expansion, that converges in the whole complex plane. Let \( \rho \) be defined by

\[
\frac{1}{\rho} = \sup_{w \in D(0,1)} \left| \frac{e^{w} - 1}{w} \right| > 1.
\]

Now, for the disk \( w \in D(0, \rho) \) we have \( |e^{w} - 1| < 1 \) and consequently

\[
w = \log(1 - (1 - e^{w})) = - \sum_{n=1}^{\infty} \frac{1}{n} (1 - e^{w})^{n}.
\]

Thus, for every \( m \geq 1 \) we have

\[
\frac{w}{e^{w} - 1} = \frac{m}{n+1} \sum_{n=0}^{m} \frac{(1 - e^{w})^{n}}{n+1} + g_{m}(w)
\]

where \( g_{m}(w) = \sum_{n=m+1}^{\infty} \frac{1}{n+1} (1 - e^{w})^{n} \). Clearly, \( w = 0 \) is a zero of \( g_{m} \) of order greater than \( m \). Thus \( g_{m}^{(m)}(0) = 0 \). But, using Proposition 6.1 the Bernoulli number \( b_{m} \) is the \( m \)th derivative of \( w \mapsto \frac{e^{w} - 1}{w} \) at 0 so

\[
b_{m} = \sum_{n=0}^{m} \frac{1}{n+1} \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \right)_{w=0}^{(m)}
\]

But \( (1 - e^{w})^{n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} e^{kw} \). Thus,

\[
b_{m} = \sum_{n=0}^{m} \frac{1}{n+1} \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \right), \quad \text{for } m \geq 1.
\]

This is quite an old formula for Bernoulli numbers (see [12] and the references therein.) Noting that

\[
\sum_{n=1}^{m} \frac{(1-x)^{n}}{n} = \int_{0}^{1-x} \left( \sum_{n=1}^{m} \frac{t^{n-1}}{n} \right) dt = \int_{0}^{1-x} \frac{t^{m-1}}{l-1} dt = \int_{0}^{1-x} \left( \sum_{n=1}^{m} \binom{m}{n} (-1)^{n} u^{n-1} \right) du = \sum_{n=1}^{m} \binom{m}{n} (-1)^{n} u^{n-1},
\]

we can rearrange our previous calculation, as follows

\[
\sum_{n=1}^{m} \frac{(1-e^{w})^{n-1}}{n} = \sum_{n=1}^{m} \binom{m}{n} (-1)^{n} e^{nw} - 1 = \sum_{n=1}^{m} \binom{m}{n} \frac{(-1)^{n}}{n} \left( \sum_{k=0}^{n-1} e^{kw} \right).
\]

Hence, taking as before the \( m \)th derivative at 0, another formula is obtained [5]:

\[
b_{m} = \sum_{n=1}^{m} \binom{m}{n} \frac{(-1)^{n}}{n} \left( \sum_{k=0}^{n-1} k^{m} \right), \quad \text{for } m \geq 1.
\]
7. The von Staudt-Clausen theorem

In this section we give the proof of a famous theorem that determines the fractional part of a Bernoulli number. First, let us introduce some notation, the reader is invited to take a look at [19, Chapter 15], and the references therein, for a deeper insight on the role played by Bernoulli numbers in Number Theory.

Let us denote by $\mathfrak{A}$ the set of functions $f$ that are analytic in the neighborhood of 0 and such that $f^{(n)}(0)$ is an integer for every nonnegative integer $n$. Let $f$ and $g$ be two members of $\mathfrak{A}$, and let $m$ be a positive integer. We will write $f \equiv g \pmod{m}$ if $f^{(n)}(0) \equiv g^{(n)}(0) \pmod{m}$ for every nonnegative integer $n$. Finally, for two functions $f$ and $g$ that are analytic functions in the neighborhood of 0, we write $f \equiv g \pmod{\mathfrak{A}}$ if $f - g \in \mathfrak{A}$.

**Lemma 7.1.** The following properties hold:

i. If $f$ belongs to $\mathfrak{A}$, then both $f'$ and $z \mapsto \int_0^z f(t) \, dt$ belong to $\mathfrak{A}$.

ii. If $f$ and $g$ belong to $\mathfrak{A}$, then $fg$ belongs also to $\mathfrak{A}$.

iii. If $f$ belongs to $\mathfrak{A}$ and $f(0) = 0$, then $\frac{1}{m} f^m$ belongs also to $\mathfrak{A}$ for every positive integer $m$.

**Proof.** Consider $f \in \mathfrak{A}$. There is a sequence of integers $(a_n)_{n \in \mathbb{N}}$ such that $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ in a neighbourhood of 0. But then

$$f'(z) = \sum_{n=0}^{\infty} \frac{a_n+1}{n!} z^n \quad \text{and} \quad \int_0^z f(t) \, dt = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n,$$

and consequently both $f'$ and $z \mapsto \int_0^z f(t) \, dt$ belong to $\mathfrak{A}$. This proves (i). Property (ii) follows from Leibniz's formula.

Property (iii) is proved by mathematical induction. It is true for $m = 1$, and if we suppose that $\frac{1}{(m-1)!} f^{m-1}$ belongs to $\mathfrak{A}$, then according to (ii) and (iii) the function $\frac{1}{(m-1)!} f^m$ belongs also to $\mathfrak{A}$, and using (ii) once more we conclude that

$$z \mapsto \frac{1}{m!} f^m(z) = \frac{1}{(m-1)!} \int_0^z f^{m-1}(t) f'(t) \, dt$$

also belongs $\mathfrak{A}$. This achieves the proof of the lemma. \hfill $\square$

**Lemma 7.2.** If $m$ is a composite positive number such that $m > 4$ then

$$(m-1)! \equiv 0 \pmod{m}.$$

**Proof.** Let $p$ be the smallest prime that divides $m$, and let $q = m/p$. Since $m$ is composite we conclude that $q \geq p$ so, there are two cases:

- $q > p$. In this case $1 < p < q < m$ and consequently $m = pq$ divides $(m-1)!$.
- $q = p$. That is $m = p^2$, but $m > 4$, implies that $p > 2$ and consequently $1 < p < 2p < m$. It follows that $2m = p \times (2p)$ divides $(m-1)!$.

and the lemma follows. \hfill $\square$

**Proposition 7.3.** The function $g$ defined by $g(z) = e^z - 1$ belongs to $\mathfrak{A}$ and it satisfies the following properties:

i. If $m$ is composite and greater than 4 then $g^{m-1} \equiv 0 \pmod{m}$.

ii. If $m = 4$ then $g^{m-1}(z) \equiv 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \pmod{m}$.

iii. If $m$ is prime then $g^{m-1}(z) \equiv -\sum_{k=1}^{\infty} \frac{z^{k(m-1)}}{(km-k)!} \pmod{m}$.
Proof. The fact that \( g \in \mathfrak{A} \) is immediate.

Suppose that \( m \) is a composite integer greater than 4. Using Lemma 7.1 [16] we see that \( \frac{m^n - 1}{m - 1} \in \mathfrak{A} \), and (4) follows from Lemma 7.2.

Consider the case \( m = 4 \). Noting that \( g^3(z) = e^{3z} - 3e^{2z} + 3e^z - 1 \) we conclude that \( g^3(z) = \sum_{n=0}^{\infty} \frac{b_n}{m^n} z^n \) with
\[
a_n = 3^n - 3 \cdot 2^n + 3.
\]

But, for \( n \geq 3 \) we have \( a_n \equiv (-1)^n - 1 \pmod{4} \) so \( a_{2k} \equiv 0 \pmod{4} \) and \( a_{2k+1} \equiv 2 \pmod{4} \). This proves (6).

Finally, suppose that \( m \) is a prime. Here
\[
g^{m-1}(z) = \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-k-1} e^{kz},
\]
and consequently \( g^{m-1}(z) = \sum_{n=m-1}^{\infty} \frac{b_n}{m^n} z^n \) with
\[
b_0 = b_1 = \ldots = b_{m-2} = 0, \quad b_{m-1} = (m-1)!,
\]
and for \( n \geq m - 1 \)
\[
b_n = \sum_{k=1}^{m-1} \binom{m-1}{k} (-1)^{m-k-1} k^n
\]
But according to Fermat’s Little Theorem [16, Theorem 71] we have \( k^{m-1} \equiv 1 \pmod{m} \) for \( 1 \leq k \leq m-1 \), and consequently \( b_{n+m-1} \equiv b_n \pmod{m} \) for every \( n \). Thus, \( b_n \equiv 0 \pmod{m} \) if \( n \) is not a multiple of \( m-1 \), and if \( n \) is a multiple of \( m-1 \) then
\[
b_n \equiv (m-1)! \equiv -1 \pmod{m}
\]
where the last congruence follows from Wilson’s Theorem [16, Theorem 80], and (12) follows. \( \Box \)

Theorem 7.4 (von Staudt-Clausen Theorem). For a given positive integer \( n \), let the set of primes \( p \) such that \( p-1 \) divides \( 2n \) be denoted by \( \mathfrak{p}_n \). Then
\[
b_{2n} + \sum_{p \in \mathfrak{p}_n} \frac{1}{p} \in \mathbb{Z}.
\]

Proof. Indeed, consider the function \( g \) of Proposition 7.3. Note that
\[
z = \log(1 + g(z)) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} g^m(z)
\]
Thus
\[
\frac{z}{e^z - 1} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} g^{m-1}(z) = 1 - \frac{g(z)}{2} - \frac{g^3(z)}{4} + \sum_{p > 2, \ p \text{ prime}} \frac{g^{p-1}(z)}{p} \pmod{\mathfrak{A}}
\]
\[
= 1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{z^{2k+1}}{(2k+1)!} - \sum_{p \geq 2, \ p \text{ prime}} \frac{1}{p} \sum_{k=1}^{\infty} \frac{z^{k(p-1)}}{(kp-k)!} \pmod{\mathfrak{A}}
\]
But, since \( z/(e^z - 1) = \sum_{n=0}^{\infty} \frac{b_n}{m^n} z^n \) according to Proposition 6.1 the desired conclusion follows from the above equality, on comparing the coefficients of \( z^{2n} \).

For instance, \( \mathfrak{p}_1 = \{2, 3\} \) and \( b_2 + \frac{1}{2} + \frac{1}{3} = 1 \). Also, \( \mathfrak{p}_2 = \{2, 3, 5\} \), and \( b_4 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 \). Generally, for a positive integer \( n \), we have \( \{2, 3\} \subset \mathfrak{p}_n \) and consequently the denominator of \( b_{2n} \) is always a multiple of 6.
Corollary 7.5. For every positive integer \( m \) and every nonnegative integer \( k \) the quantity \( m(m^k - 1)b_k \) is an integer.

Proof. We only need to consider the case \( k = 2n \) for some positive integer \( n \) because the other cases are trivial.

Consider a prime \( p \) such that \( p - 1 \) divides \( 2n \), (i.e. \( p \mid \tau) \). Consider also a positive integer \( m \).

- If \( p \mid m \) then clearly \( p \mid m(m^k - 1) \).
- If \( p \nmid m \) then, according to Fermat’s Little Theorem \[ \text{Theorem 71} \] we have \( m^{p-1} \equiv 1 \pmod{p} \), and since \( (p - 1) \mid k \) we conclude that \( m^k \equiv 1 \pmod{p} \). Thus \( p \mid m(m^k - 1) \) also in this case. It follows that \( m(m^k - 1) \) is a multiple of every prime \( p \in \mathfrak{p} \), and the result follows according to Proposition \[ \text{7.3} \].

Corollary 7.6. For every positive integer \( n \) there are infinitely many integers \( m \) such that \( B_{2m} - B_{2n} \) is an integer.

Proof. Consider \( \tau = \text{lcm}(d + 1 : d \mid (2n)) \); (the least common multiple of the numbers \( d + 1 \) where \( d \) is a divisor of \( 2n \), and let \( q \) be a prime number such that \( q \equiv 1 \pmod{\tau} \).

Now, if \( m = nq \) then \( \mathfrak{p}_m = \mathfrak{p}_n \). Indeed, if \( \lambda \) prime \( p' \in \mathfrak{p}_m \) then \( p' - 1 \) divides \( 2nq \), so, there are two cases:

- If \( p' - 1 \) divides \( 2n \) then clearly \( p' \in \mathfrak{p}_n \) since \( p' \) is prime.
- If \( p' - 1 = dq \) for some \( d \mid 2n \), then

\[
p' = 1 + dq = (d + 1)q + 1 - q = (d + 1)q - q\tau,
\]

so, \( p' \) is a multiple of \( d + 1 \), and since it is a prime, we conclude that \( p' = d + 1 \) which is absurd since \( q \neq 1 \).

Thus, we have proved that \( \mathfrak{p}_m \subset \mathfrak{p}_n \). But, the inverse inclusion is trivially true, and \( \mathfrak{p}_m = \mathfrak{p}_n \), or equivalently \( B_{2m} - B_{2n} \) is an integer.

Finally, using Dirichlet’s Theorem \[ \text{19, Chapter 16} \], we know that there are infinitely many primes \( q \) such that \( q \equiv 1 \pmod{\tau} \), and the corollary follows.

8. The Euler-Maclaurin’s Formula

For a function \( g \) defined on the interval \([0, 1]\) we introduce the notation \( \delta g \) to denote the difference \( g(1) - g(0) \). Also we recall the notation \( \widetilde{B}_n \) for the 1-periodic function that coincides with \( x \mapsto B_n(x) \) on the interval \([0, 1]\), or equivalently,

\[
\forall x \in \mathbb{R}, \quad \widetilde{B}_n(x) = B_n(\{x\})
\]

where \( \{t\} = t - [t] \) is the fractional part of \( t \).

Proposition 8.1. Consider a positive integer \( m \) and a function \( f : [0, 1] \to \mathbb{C} \) having a continuous \( m^{th} \) derivative. For every \( x \) in \([0, 1]\) we have

\[
\int_0^1 f(t) \, dt - f(x) + \sum_{k=0}^{m-1} \frac{B_{k+1}(x)}{(k+1)!} \delta f^{(k)} = \frac{1}{m!} \int_0^1 \widetilde{B}_m(x-t)f^{(m)}(t) \, dt
\]

Proof. For an integer \( k \) with \( 0 \leq k \leq m \) we define \( F_k(x) \) by the formula

\[
F_k(x) = \frac{1}{k!} \int_0^1 \widetilde{B}_k(x-t)f^{(k)}(t) \, dt.
\]

Clearly we have

\[
F_0(x) = \int_0^1 \widetilde{B}_0(x-t)f(t) \, dt = \int_0^1 f(t) \, dt.
\]
Also, for $0 \leq k < m$ and $x \in [0, 1]$, we have

\[
F_{k+1}(x) = \int_0^1 \frac{B_{k+1}(x-t)}{(k+1)!} f^{(k+1)}(t) \, dt
\]

\[
= \int_0^x \frac{B_{k+1}(x-t)}{(k+1)!} f^{(k+1)}(t) \, dt + \int_x^1 \frac{B_{k+1}(1+x-t)}{(k+1)!} f^{(k+1)}(t) \, dt
\]

\[
= \frac{B_{k+1}(x-t)}{(k+1)!} f^{(k)}(t) \bigg|_{t=0}^{t=x} + \int_0^x \frac{B_k(x-t)}{k!} f^{(k)}(t) \, dt
\]

\[
= \frac{B_{k+1}(1+x-t)}{(k+1)!} f^{(k)}(t) \bigg|_{t=x}^{t=1} + \int_x^1 \frac{B_k(1+x-t)}{k!} f^{(k)}(t) \, dt
\]

\[
= \frac{B_{k+1}(0) - B_{k+1}(1)}{(k+1)!} f^{(k)}(x) + \frac{B_{k+1}(x)}{(k+1)!} \delta f^{(k)} + \int_0^x \frac{B_k(x-t)}{k!} f^{(k)}(t) \, dt
\]

\[
= \frac{B_{k+1}(0) - B_{k+1}(1)}{(k+1)!} f^{(k)}(x) + \frac{B_{k+1}(x)}{(k+1)!} \delta f^{(k)} + F_k(x)
\]

Hence, we have proved that

\[
F_0(x) = \int_0^1 f(t) \, dt
\]

\[
F_1(x) = -f(x) + B_1(x) \delta f + F_0(x)
\]

\[
F_{k+1}(x) = \frac{B_{k+1}(x)}{(k+1)!} \delta f^{(k)} + F_k(x) \quad \text{for } 1 \leq k < m
\]

Adding these equalities as $k$ varies from 0 to $m-1$ we obtain the desired formula. \(\square\)

The next corollary corresponds to the particular case $x = 1$.

**Corollary 8.2.** Consider a positive integer $m$, and a function $f$ that has a continuous $(2m-1)^{\text{st}}$ derivative on $[0, 1]$. If $f^{(2m-1)}$ is decreasing, then

\[
\int_0^1 f(t) \, dt = \frac{f(1) + f(0)}{2} - \frac{m-1}{(2m-1)!} \delta f^{(2k-1)} + (-1)^m R_m
\]

with

\[
R_m = \int_0^{1/2} |B_{2m-1}(t)| \left( f^{(2m-1)}(t) - f^{(2m-1)}(1-t) \right) \, dt
\]

and

\[
0 \leq R_m \leq \frac{6}{(2\pi)^{2m}} \left( f^{(2m-1)}(0) - f^{(2m-1)}(1) \right).
\]

**Proof.** Indeed, choosing $x = 1$ in Proposition 8.1 with $2m-1$ for $m$, we obtain

\[
\int_0^1 f(t) \, dt - f(1) + \sum_{k=0}^{2m-2} \frac{B_{k+1}(1)}{(k+1)!} \delta f^{(k)} = \frac{1}{(2m-1)!} \int_0^1 B_{2m-1}(t) f^{(2m-1)}(t) \, dt
\]

Now, using Proposition 2.1 (iii), Proposition 2.3 (iii) and the fact that $B_1(1) = 1/2$, we see that

\[
\int_0^1 f(t) \, dt - \frac{f(1) + f(0)}{2} + \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} \delta f^{(2k-1)} = -\frac{r_m}{(2m-1)!}
\]

with

\[
r_m = \int_0^1 B_{2m-1}(t) f^{(2m-1)}(t) \, dt
\]
But,
\[
\begin{align*}
r_m &= \int_0^{1/2} B_{2m-1}(t) f^{(2m-1)}(t) \, dt + \int_{1/2}^1 B_{2m-1}(t) f^{(2m-1)}(t) \, dt \\
&= \int_0^{1/2} B_{2m-1}(t) f^{(2m-1)}(t) \, dt + \int_0^{1/2} B_{2m-1}(1-t) f^{(2m-1)}(1-t) \, dt \\
&= \int_0^{1/2} B_{2m-1}(t) \left( f^{(2m-1)}(t) - f^{(2m-1)}(1-t) \right) \, dt 
\end{align*}
\]

Now, according to Proposition 2.5, we know that \((-1)^m B_{2m-1}\) is positive on \((0, 1/2)\). Thus
\[
r_m = (-1)^m \int_0^{1/2} |B_{2m-1}(t)| \left| f^{(2m-1)}(t) - f^{(2m-1)}(1-t) \right| \, dt,
\]
and the expression of \(R_m\) follows.

In particular, when \(f^{(2m-1)}\) is decreasing, the maximum on the interval \([0, 1/2]\) of the quantity \(f^{(2m-1)}(t) - f^{(2m-1)}(1-t)\) is \(-\delta f^{(2m-1)}\), attained at \(t = 0\), and its minimum on the same interval is 0 and it is attained at \(t = 1/2\). Consequently, using Proposition 4.4 we have
\[
0 \leq R_m \leq (-1)^{m+1} \left( \int_0^{1/2} \frac{B_{2m}(t) - B_{2m}(0)}{(2m)!} \right) \delta f^{(2m-1)} \\
\leq (-1)^{m+1} \frac{B_{2m}(1/2) - B_{2m}(0)}{(2m)!} \delta f^{(2m-1)} \\
\leq (2 - 2^{1-2m}) \frac{|B_{2m}|}{(2m)!} \delta f^{(2m-1)} \\
\leq 4 \left( 1 - \frac{1}{2^{2m}} \right) \left( 1 + \frac{3}{2^{2m}} \right) \frac{1}{(2\pi)^{2m}} \delta f^{(2m-1)} \\
\leq 4 \frac{1 + 2^{1-2m}}{(2\pi)^{2m}} \delta f^{(2m-1)} \leq \frac{6}{(2\pi)^{2m}} \delta f^{(2m-1)}
\]
and the desired conclusion follows. \(\square\)

Before proceeding to the next result, we will prove the following property that generalises the well-known “Riemann Lebesgue’s lemma”.

**Lemma 8.3.** Consider an integrable function \(h : [0, 1] \rightarrow \mathbb{C}\), and a piecewise continuous 1-periodic function \(g : \mathbb{R} \rightarrow \mathbb{C}\). Then
\[
\lim_{p \to \infty} \int_0^1 g(pt) h(t) \, dt = \left( \int_0^1 g(t) \, dt \right) \cdot \left( \int_0^1 h(t) \, dt \right)
\]

**Proof.** First, suppose that \(\int_0^1 g = 0\). This implies that \(x \mapsto G(x) = \int_0^x g(t) \, dt\) is a continuous 1-periodic function. Particularly \(G\) is bounded, and we can define \(M = \sup_{x} |G|\).

Now, assume that \(h = \chi_{[a, \beta]}\); the characteristic function of an interval \([a, \beta]\). In this case
\[
\int_0^1 g(pt) h(t) \, dt = \int_a^\beta g(pt) \, dt = \frac{G(p\beta) - G(p\alpha)}{p}
\]
So
\[
\left| \int_0^1 g(pt) h(t) \, dt \right| \leq \frac{2M}{p}
\]
and consequently \(\lim_{p \to \infty} \int_0^1 g(pt) h(t) \, dt = 0\).
Using linearity, we see that the same conclusion holds if \( h \) is a step function. Finally, the density of the space of step functions in \( L^1([0, 1]) \) implies that \( \lim_{p \to \infty} \int_0^1 g(pt)h(t) \, dt = 0 \) for every integrable function \( h : [0, 1] \to \mathbb{C} \).

Applying the preceding case to the function \( \tilde{g} = g - \int_0^1 g, \) that satisfies \( \int_0^1 \tilde{g}(t) \, dt = 0, \) we conclude that
\[
\lim_{p \to \infty} \int_0^1 \tilde{g}(pt)h(t) \, dt = 0
\]
for every \( h \in L^1([0, 1]) \). Which is the desired conclusion. \( \square \)

The next theorem is the main result of this section.

**Theorem 8.4.** For a positive integer \( p \), and a function \( f \) having at least a continuous \( m \)-th derivative on the interval \([0, 1]\), we define the quantities \( \mathcal{H}_p(f; x) \) and \( \mathcal{E}(p, m, f; x) \) for \( x \in [0, 1] \) by
\[
\mathcal{H}_p(f; x) = \frac{1}{p} \sum_{k=0}^{p-1} f \left( \frac{k + x}{p} \right)
\]
and
\[
\mathcal{E}(p, m, f; x) = \int_0^1 f(t) \, dt - \mathcal{H}_p(f; x) + \sum_{k=1}^{m} \frac{B_k(x)}{k!} \cdot \frac{\delta f^{(k-1)}}{p^k}.
\]
Then,

i. The quantity \( \mathcal{E}(p, m, f; x) \) has the following expression in terms of \( \tilde{B}_m \):
\[
\mathcal{E}(p, m, f; x) = \frac{1}{p^m} \int_0^1 \tilde{B}_m(x - pt) \, f^{(m)}(t) \, dt.
\]

ii. It satisfies also the following inequality:
\[
|\mathcal{E}(p, m, f; x)| \leq \frac{8}{\pi} \frac{1}{(2\pi)^m} \sup_{[0,1]} \left| f^{(m)} \right|.
\]

iii. Moreover,
\[
\lim_{p \to \infty} p^m \cdot \mathcal{E}(p, m, f; x) = 0.
\]

**Proof.** Applying Proposition 8.11 to the function \( x \mapsto \mathcal{H}_p(f; x) \) we obtain
\[
\int_0^1 \mathcal{H}_p(f; t) \, dt - \mathcal{H}_p(f; x) + \sum_{k=0}^{m-1} \frac{B_{k+1}(x)}{(k+1)!} \delta \mathcal{H}_p^{(k)}(f; \cdot) = \frac{1}{m!} \int_0^1 \tilde{B}_m(x - t) \mathcal{H}_p^{(m)}(f; t) \, dt. \tag{*}
\]
But
\[
\int_0^1 \mathcal{H}_p(f; t) \, dt = \frac{1}{p} \sum_{k=0}^{p-1} \int_0^1 f \left( \frac{k + t}{p} \right) \, dt = \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} f(u) \, du = \int_0^1 f(u) \, du.
\]
Also,
\[
\mathcal{H}_p^{(k)}(f; x) = \frac{1}{p^{k+1}} \sum_{k=0}^{p-1} f^{(k)} \left( \frac{k + x}{p} \right) = \frac{1}{p^k} \mathcal{H}_p(f^{(k)}; x).
\]
Thus
\[
\delta \mathcal{H}_p^{(k)}(f; \cdot) = \frac{1}{p^{k+1}} \left( \sum_{k=0}^{p-1} f^{(k)} \left( \frac{k + 1}{p} \right) - \sum_{k=0}^{p-1} f^{(k)} \left( \frac{k}{p} \right) \right),
\]
\[
= \frac{f^{(k)}(1) - f^{(k)}(0)}{p^{k+1}} = \frac{1}{p^{k+1}} \delta f^{(k)}.
\]
Replacing the above results in (9.1) we conclude that
\[ E(p,m,f;x) = \frac{1}{m! \cdot p^m} \int_0^1 \tilde{B}_m(x-t)H_p(f^{(m)};t) \, dt, \]
and (9.2) follows, since
\[
\int_0^1 \tilde{B}_m(x-t)H_p(f^{(m)};t) \, dt = \frac{1}{p} \sum_{k=0}^{p-1} \int_0^1 \tilde{B}_m(x-t) f^{(m)} \left( \frac{k+t}{p} \right) \, dt \\
= \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} \tilde{B}_m(x+k-pt) f^{(m)}(t) \, dt \\
= \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} \tilde{B}_m(x-pt) f^{(m)}(t) \, dt \\
= \int_0^1 \tilde{B}_m(x-pt) f^{(m)}(t) \, dt
\]

Using (9.2), and recalling that \( \tilde{B}_m \) is 1-periodic, we see that
\[
|E(p,m,f;x)| \leq \frac{1}{m! \cdot p^m} \int_0^1 \left| \tilde{B}_m(x-pt) \right| \left| f^{(m)}(t) \right| \, dt \\
\leq \frac{1}{m! \cdot p^m} \sup_{t \in [0,1]} \left| f^{(m)}(t) \right| \int_0^1 \left| \tilde{B}_m(x-pt) \right| \, dt \\
\leq \frac{1}{m! \cdot p^m} \sup_{t \in [0,1]} \left| f^{(m)}(t) \right| \frac{1}{p} \int_0^p \left| \tilde{B}_m(x-u) \right| \, du \\
= \frac{1}{m! \cdot p^m} \sup_{t \in [0,1]} \left| f^{(m)}(t) \right| \int_0^1 \left| \tilde{B}_m(t) \right| \, dt,
\]
and (9.3) follows using Proposition 4.17 (4).

Finally, applying Lemma 8.3 to the 1-periodic function \( u \mapsto g(t) = \tilde{B}_m(x-t) \) and the integrable function \( t \mapsto h(t) = f^{(m)}(t) \) we obtain (9.3) because \( \int_0^1 g = 0 \) in this case. \( \square \)

9. Asymptotic expansions for numerical quadrature formulæ

In this section we only consider functions defined on the interval \([0,1]\). The more general case of a functions defined on \([a,b]\) can be obtained by applying the results after using the change of variable \( t \mapsto a + t(b-a) \).

We consider a function \( f : [0,1] \rightarrow \mathbb{C} \), having a continuous \( m \)th derivative, with \( m \geq 2 \), and we will use freely the notation of the previous section.

9.1. Riemann sums. The Riemann sum of \( f \) obtained by taking the values of the function \( f \) at the lower bound of each subdivision interval, is given by
\[ R_p^L(f) = \frac{1}{p} \sum_{k=0}^{p-1} \left( \frac{k}{p} \right) f \left( \frac{k}{p} \right). \tag{9.1} \]

According to Theorem 8.4 we have \( R_p^L(f) = H_p(f,0) \), so
\[
\int_0^1 f(t) \, dt = R_p^L(f) + \frac{\delta f}{2p} - \sum_{1 \leq k \leq \frac{p}{2}} b_{2k} \frac{1}{(2k)!} \cdot p^{2k} \cdot \delta f^{(2k-1)} + E(p,m,f;0) \tag{9.2}
\]
Similarly, the Riemann sum of \( f \) obtained by taking the values of the function \( f \) at the upper bound of each subdivision interval, is given by

\[
R^R_p(f) = \frac{1}{p} \sum_{k=1}^{p} f\left(\frac{k}{p}\right)
\]  

(9.3)

And again using Theorem 8.4 we have \( R^R_p(f) = H_p(f, 1) \), so

\[
\int_0^1 f(t)dt = R^R_p(f) - \frac{\delta f}{2p} - \sum_{1 \leq k \leq \frac{p}{2}} \frac{b_{2k}}{(2k)!} \cdot \frac{\cdot \delta f^{(2k-1)}}{p^{2k}} + \mathcal{E}(p, m, f; 0)
\]  

(9.4)

where we noted that \( \mathcal{E}(p, m, f; 1) = \mathcal{E}(p, m, f; 0) \), since \( \tilde{B}_m \) is 1-periodic.

Also, the Riemann sum of \( f \) obtained by taking the values of the function \( f \) at the midpoint of each subdivision interval, is given by

\[
R^M_p(f) = \frac{1}{p} \sum_{k=0}^{p-1} f\left(\frac{2k+1}{2p}\right)
\]  

(9.5)

This is the “Midpoint Quadrature Rule”. By Theorem 8.4 we have \( R^M_p(f) = H_p(f, \frac{1}{2}) \), so

\[
\int_0^1 f(t)dt = R^M_p(f) - \sum_{1 \leq k \leq \frac{p}{2}} \frac{(2^{1-2k}) - 1}{(2k)!} \frac{b_{2k}}{p^{2k}} \cdot \delta f^{(2k-1)} + \mathcal{E}\left(p, m, f; \frac{1}{2}\right)
\]  

(9.6)

For example, taking \( m = 2 \), we obtain from Theorem 8.4 (iii):

\[
\lim_{p \to \infty} p^2 \left( \int_0^1 f(t)dt - R^M_p(f) \right) = \frac{f'(1) - f'(0)}{24}.
\]  

(9.7)

Thus, the midpoint quadrature rule is a second order rule.

9.2. The trapezoidal rule. Taking the half sum of \( R^L_p(f) \) and \( R^R_p(f) \) we obtain the trapezoidal rule that corresponds to approximating \( f \) linearly on each interval of the subdivision.

\[
T_p(f) = \frac{1}{2} \left( R^L_p(f) + R^R_p(f) \right) = \frac{f(0) + f(1)}{2} + \frac{1}{p} \sum_{k=1}^{p-1} f\left(\frac{k}{p}\right)
\]  

(9.8)

Using (9.2) and (9.4) we see that

\[
\int_0^1 f(t)dt = T_p(f) - \sum_{1 \leq k \leq \frac{p}{2}} \frac{b_{2k}}{(2k)!} \cdot \frac{\cdot \delta f^{(2k-1)}}{p^{2k}} + \mathcal{E}(p, m, f; 0)
\]  

(9.9)

In particular, choosing \( m = 2 \), we obtain from Theorem 8.4 (iii):

\[
\lim_{p \to \infty} p^2 \left( \int_0^1 f(t)dt - T_p(f) \right) = -\frac{f'(1) - f'(0)}{12}.
\]  

(9.10)

Thus, the trapezoidal quadrature rule is a second order rule.

In fact, for the case of the trapezoidal rule we have a more refined result in some cases. This is the object of the following proposition.
Proposition 9.1. Consider a positive integer \( m \), and a function \( f \) that has a continuous \((2m - 1)\)st derivative on \([0, 1]\). If \( f^{(2m-1)} \) is decreasing, then, for every positive integer \( p \) we have

\[
\int_0^1 f(t) \, dt = T_p(f) - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)! \cdot p^{2k}} \cdot \delta f^{(2k-1)} + (-1)^{m+1} R_{m,p}
\]

with

\[
0 \leq R_{m,p} \leq \frac{6}{(2\pi p)^{2m}} \left( f^{(2m-1)}(0) - f^{(2m-1)}(1) \right).
\]

Proof. Our starting point will be Corollary 8.2 applied to the function \( x \mapsto f \left( \frac{j}{p} \right) \), with \( 0 \leq j < p \). It follows that

\[
p \int_{j/p}^{(j+1)/p} f(t) \, dt = \frac{1}{2} \left( f \left( \frac{j+1}{p} \right) + f \left( \frac{j}{p} \right) \right) - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)! p^{2k}} \left( f^{(2k-1)} \left( \frac{j+1}{p} \right) - f^{(2k-1)} \left( \frac{j}{p} \right) \right) + (-1)^{m+1} R_{m,p,j}
\]

with

\[
0 \leq R_{m,p,j} \leq \frac{6}{(2\pi p)^{2m-1}} \left( f^{(2m-1)} \left( \frac{j}{p} \right) - f^{(2m-1)} \left( \frac{j+1}{p} \right) \right).
\]

Adding these inequalities, for \( 0 \leq j < p \), and recalling (9.8) we see that

\[
\int_0^1 f(t) \, dt = T_p(f) - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)! p^{2k}} \delta f^{(2k-1)} + (-1)^{m+1} R_{m,p}
\]

with \( R_{m,p} = \frac{1}{p} \sum_{j=0}^{p-1} R_{m,p,j} \). Now, using (4) we get

\[
0 \leq R_{m,p} \leq \frac{6}{(2\pi p)^{2m}} \left( f^{(2m-1)}(0) - f^{(2m-1)}(1) \right)
\]

which is the desired conclusion. \( \square \)

Application 8. An asymptotic expansion for a trigonometric sum. For a positive integer \( p \), we consider the trigonometric sum

\[
J_p = \sum_{j=1}^{p-1} j \cot \left( \frac{j\pi}{p} \right).
\]

This sum will be studied in detail later, but we want here to illustrate the use of the Proposition 9.1.

Proposition 9.2. For every positive integers \( p \) and \( m \), there is a real number \( \theta_{p,m} \) such that

\[
J_p = -\frac{1}{\pi} p^2 H_p + \frac{\ln(2\pi) \cdot p^2}{2\pi} - p - \sum_{k=1}^{m-1} \frac{b_{2k}(1 + 2\zeta(2k))}{2\pi k \cdot p^{2k}} + \frac{(-1)^m}{p^{2m-2}} \theta_{p,m}
\]

and

\[
0 < \theta_{p,m} < \frac{|b_{2m}| \cdot (1 + 2\zeta(2m))}{2\pi m},
\]

where \( H_p = \sum_{j=1}^{p} 1/j \) is the \( p \)th harmonic number.

Proof. Indeed, let \( \varphi \) be the function defined by

\[
\varphi(x) = \pi x \cot(\pi x) + \frac{1}{1-x}.
\]
According to formula (6.2) we know that
\[ \varphi(x) = 2 + \frac{x}{x+1} + \sum_{n=2}^{\infty} \left( \frac{x}{x-n} + \frac{x}{x+n} \right). \]

Thus, \( \varphi \) is defined and analytic on the interval \((-1, 2)\). Let us show that, for every positive integer \( k \), the derivative \( \varphi^{(2k)} \) is negative on the interval \([0, 1]\). To this end, we note, using (6.1), that
\[ \varphi(x) = \pi x \cot(\pi x) + \frac{2}{1-x^2} - \frac{1}{1+x} \]
\[ = 3 - \frac{1}{1+x} - 2 \sum_{n=1}^{\infty} (\zeta(2n) - 1)x^{2n}. \]
Hence,
\[ \frac{\varphi^{(2k)}(x)}{(2k)!} = -\frac{1}{(1+x)^{2k+1}} - 2 \sum_{n=k}^{\infty} \left( \frac{2n}{2k} \right) (\zeta(2n) - 1)x^{2n-2k}, \]
which is clearly negative on \([0, 1]\).

Now, we can apply Proposition 9.1 to \( \varphi \). We only need to calculate \( \delta \varphi^{(2k-1)} \) for every \( k \). Note that
\[ \varphi(x) + \varphi(1-x) = \frac{1 - \pi \cot(\pi x)}{x} + 2 \pi x \cot(\pi x) + \frac{1}{1-x} \]
Thus, using (6.2) again we obtain, for \( |x| < 1 \),
\[ \varphi(x) + \varphi(1-x) = 3 + \sum_{n=1}^{\infty} (2\zeta(2n) + 1)x^{2n-1} + \sum_{n=1}^{\infty} (1 - 4\zeta(2n))x^{2n} \]
Taking the \((2k-1)^{st}\) derivative at \( x = 0 \), we get
\[ \frac{\delta \varphi^{(2k-1)}}{(2k-1)!} = -1 - 2\zeta(2k). \]
So, applying Proposition 9.1, we obtain
\[ \int_{0}^{1} \varphi(t) \, dt = \mathcal{T}_p(\varphi) + \sum_{k=1}^{m-1} b_{2k}(1 + 2\zeta(2k)) - (-1)^{m+1} R_{m,p} \]
with
\[ 0 \leq R_{m,p} \leq \frac{6(2m-1)!}{(2\pi)^{2m}} \cdot \frac{1 + 2\zeta(2m)}{(2\pi)^{2m}} \]
But
\[ \mathcal{T}_p(\varphi) = \frac{\varphi(0) + \varphi(1)}{2p} + \frac{1}{p} \sum_{j=1}^{p-1} \varphi \left( \frac{j}{p} \right) \]
\[ = \frac{3}{2p} + \frac{\pi}{p^2} \sum_{j=1}^{p-1} j \cot \left( \frac{\pi j}{p} \right) \]
\[ + \sum_{j=1}^{p-1} \frac{1}{p - j} = H_p + \frac{1}{2p} + \frac{\pi}{p^2} J_p \]
Also, for \( x \in [0, 1) \), we have
\[ \int_{0}^{x} \varphi(t) \, dt = -\ln(1-x) + x \ln \sin(\pi x) - \int_{0}^{x} \ln \sin(\pi t) \, dt \]
and, letting \( x \) tend to 1 we obtain
\[ \int_{0}^{1} \varphi(t) \, dt = \ln(\pi) - \int_{0}^{1} \ln \sin(\pi t) \, dt = \ln(2\pi) \]
where we used the fact \( \int_0^1 \ln(\sin(\pi t)) \, dt = -\ln 2 \), (see [13, 4.224 Formula 3]). Thus \((\ref{9.1})\) is equivalent to

\[
\ln(2\pi) = H_p + \frac{1}{2p} + \frac{\pi}{p^2} J_p + \sum_{k=1}^{m-1} \frac{b_{2k}(1 + 2\zeta(2k))}{2k \cdot p^{2k}} + (-1)^{m+1} R_{m,p}
\]

or

\[
\frac{\pi}{p^2} J_p = \ln(2\pi) - H_p - \frac{1}{2p} - \sum_{k=1}^{m-1} \frac{b_{2k}(1 + 2\zeta(2k))}{2k \cdot p^{2k}} + (-1)^{m} R_{m,p}
\]

Thus, we have shown that for every nonnegative integer \( m \) we have

\[
\frac{\pi}{p^2} J_p < \ln(2\pi) - H_p - \frac{1}{2p} - \sum_{k=1}^{2m} \frac{b_{2k}(1 + 2\zeta(2k))}{2k \cdot p^{2k}}
\]

and

\[
\frac{\pi}{p^2} J_p > \ln(2\pi) - H_p - \frac{1}{2p} - \sum_{k=1}^{2m+1} \frac{b_{2k}(1 + 2\zeta(2k))}{2k \cdot p^{2k}}
\]

So, for every positive integer \( m \) we have,

\[
0 < (-1)^m \left( J_p - \frac{p^2}{\pi} (\ln(2\pi) - H_p) + \frac{p}{2\pi} + \sum_{k=1}^{m-1} \frac{b_{2k}(1 + 2\zeta(2k))}{2\pi k \cdot p^{2k-2}} \right) < \frac{|b_{2m}| (1 + 2\zeta(2m))}{2\pi m \cdot p^{2m-2}}.
\]

Which is the desired conclusion. \( \Box \)

The result of this proposition is not completely satisfactory, because of the sum \( H_p \). That is why it is just the beginning of the story! It will be pursued in a later section.

9.3. Simpson’s rule. Comparing \((\ref{9.10})\) and \((\ref{9.7})\) we see that

\[
\lim_{p \to \infty} p^2 \left( \int_0^1 \frac{f - T_p(f) + 2R_p^M(f)}{3} \right) = 0,
\]

so the quantity \( \frac{1}{3} \left( T_p(f) + 2R_p^M(f) \right) \) is a better quadrature rule than the second order ones. Hence, let us define the “Simpson quadrature rule” by

\[
S_p(f) = \frac{1}{3} \left( T_p(f) + 2R_p^M(f) \right)
\]

Using \((\ref{9.9})\) and \((\ref{9.6})\) we see that

\[
\int_0^1 f(t) \, dt = S_p(f) + \sum_{2 \leq k \leq \infty} (1 - 4^{1-k}) \frac{b_{2k}}{3(2k)! \cdot p^{2k}} \cdot \delta f^{(2k-1)} + \mathcal{E}^S(p, m, f)
\]

with

\[
\mathcal{E}^S(p, m, f) = \frac{1}{3} \left( \mathcal{E}(p, m, f; 0) + 2\mathcal{E} \left( p, m, f; \frac{1}{2} \right) \right)
\]

Using Theorem \([8, 4]\) we see that

\[
\mathcal{E}^S(p, m, f) \leq \frac{8}{\pi} \cdot \frac{1}{(2p)^{2m}} \cdot \sup_{[0,1]} |f^{(2m)}| \quad \text{and} \quad \lim_{p \to \infty} p^{2m} \cdot \mathcal{E}^S(p, m, f) = 0
\]

In particular, choosing \( m = 4 \), we obtain:

\[
\lim_{p \to \infty} p^2 \left( \int_0^1 f(t) \, dt - S_p(f) \right) = -\frac{f^{(3)}(1) - f^{(3)}(0)}{2880}.
\]

Thus, the Simpson quadrature rule is a forth order rule.
9.4. The two point Gauss rule. Applying Theorem [3.4] at $x$ and $1 - x$ and using Proposition 2.1 (ii), we obtain after taking the half sum:

$$
\int_0^1 f(t) dt = \frac{1}{2p} \sum_{k=0}^{p-1} \left( f\left( \frac{k+x}{p} \right) + f\left( \frac{k+1-x}{p} \right) \right) - \sum_{1 \leq k \leq 2p} \frac{B_{2k}(x)}{(2k)!} \frac{1}{p^{2k}} \delta f^{(2k-1)} + \tilde{E}(p, m, f; x)
$$

with

$$
\tilde{E}(p, m, f; x) = \frac{1}{2} \left( E(p, m, f; x) + E(p, m, f; 1 - x) \right).
$$

Here $\tilde{E}$ satisfies the same properties as $E$ in Theorem 3.4. The case $x = 0$ corresponds to the trapezoidal rule, and the case $x = \frac{1}{2}$ corresponds the midpoint point rule. But the best choice for $x$ is when $x = \alpha = \frac{1}{2} - \frac{1}{\sqrt{12}}$ which is a zero of $B_2$. Then, we obtain the two point Gauss quadrature rule:

$$
G_p(f) = \frac{1}{2p} \sum_{k=0}^{p-1} \left( f\left( \frac{k + \frac{1}{2} - \frac{1}{\sqrt{12}}}{p} \right) + f\left( \frac{k + \frac{1}{2} + \frac{1}{\sqrt{12}}}{p} \right) \right)
$$

(9.14)

with

$$
\int_0^1 f(t) dt = G_p(f) - \sum_{2 \leq k \leq 2p} \frac{B_{2k}(\alpha)}{(2k)!} \frac{1}{p^{2k}} \delta f^{(2k-1)} + \tilde{E}(p, m, f; \alpha)
$$

(9.15)

For example, with $m = 4$ we find that

$$
\lim_{p \to \infty} p^4 \left( \int_0^1 f(t) dt - G_p(f) \right) = \frac{f^{(3)}(1) - f^{(3)}(0)}{4320}
$$

(9.16)

Thus, the two point Gauss quadrature rule is a fourth order rule.

9.5. Romberg's rule. Let us consider again the case of the trapezoidal rule (9.8), and the error asymptotic expansion:

$$
\int_0^1 f(t) dt = T_p(f) - \sum_{1 \leq k \leq \infty} \frac{b_{2k}}{(2k)!} \cdot \frac{1}{p^{2k}} \delta f^{(2k-1)} + E(p, m, f; 0)
$$

(9.17)

We define $T_p^{(0)}(f) = T_p(f)$ and for simplicity we write $E_m^{(0)}(p)$ for $E(p, m, f; 0)$. Next, we define inductively

$$
T_p^{(\ell)}(f) = \frac{4^\ell}{2p} T_{2p}^{(\ell-1)}(f) - \frac{T_p^{(\ell-1)}(f)}{4^\ell - 1}
$$

$$
E_m^{(\ell)}(p) = \frac{4^\ell}{2p} E_m^{(\ell-1)}(2p) - \frac{E_m^{(\ell-1)}(p)}{4^\ell - 1}
$$

for $\ell = 1, 2, \ldots$.

It is easy to prove by induction, starting from (9.17) that

$$
\int_0^1 f(t) dt = T_p^{(1)}(f) - \sum_{1 \leq k \leq \infty} \frac{4^{1-k} - 1}{4 - 1} \frac{b_{2k}}{(2k)!} \cdot \frac{1}{p^{2k}} \delta f^{(2k-1)} + E_m^{(1)}(p)
$$

$$
\int_0^1 f(t) dt = T_p^{(2)}(f) - \sum_{1 \leq k \leq \infty} \frac{4^{1-k} - 1}{4 - 1} \cdot \frac{4^{2-k} - 1}{4^2 - 1} \frac{b_{2k}}{(2k)!} \cdot \frac{1}{p^{2k}} \delta f^{(2k-1)} + E_m^{(2)}(p)
$$

$$
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
$$

$$
\int_0^1 f(t) dt = T_p^{(\ell)}(f) - \sum_{1 \leq k \leq \infty} \prod_{j=1}^{\ell} \frac{4^{j-k} - 1}{4^j - 1} \frac{b_{2k}}{(2k)!} \cdot \frac{1}{p^{2k}} \delta f^{(2k-1)} + E_m^{(\ell)}(p)
$$
In particular, when 
\[
\prod_{j=1}^{\ell} \left( \frac{4j-k-1}{4j-1} \right) = 0 \quad \text{for } k = 1, 2, \ldots, \ell,
\]
so, in fact, we have
\[
\int_0^1 f(t) dt = T_p^{(\ell)}(f) - \sum_{\ell < k \leq \frac{\ell}{2}} \prod_{j=1}^{\ell} \left( \frac{4j-k-1}{4j-1} \right) \frac{b_{2k}}{(2k)!} \cdot p^{2k} \delta f^{(2k-1)} + \mathcal{E}_m^{(\ell)}(p) \tag{9.18}
\]
In order to simplify a little bit the notation we recall that the finite \(q\)-Pochhammer \((z;a)_n\) symbol is defined as the product
\[
(z;a)_n = \prod_{k=1}^n (1 - za^{-1}).
\]
The limit as \(n\) tend to \(+\infty\) defines the \(q\)-Pochhammer symbol \((z;a)_\infty\) when \(|a| < 1\). Also, we define the \(q\)-binomial coefficient \(\binom{n}{m}_q\) by the formula
\[
\binom{n}{m}_q = \frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}, \quad \text{for } 0 \leq m \leq n. \tag{9.19}
\]
With this notation we see that for \(k > \ell\) and \(a = 1/4\), we have
\[
\prod_{j=1}^{\ell} \left( \frac{4j-k-1}{4j-1} \right) = \frac{(-1)^{\ell}}{2^{(1+\ell)}} \prod_{j=1}^{\ell} \left( \frac{1-a^{-j}}{1-a^{-j}} \right)
\]
\[
= \frac{(-1)^{\ell}}{2^{(1+\ell)}} \frac{(q;q)_{k-1}}{(q;q)_{k-\ell}(a;q)_{\ell}} = \frac{(-1)^{\ell}}{2^{(1+\ell)}} \binom{k-1}{\ell}_q.
\]
Thus, we can write (9.18) as follows
\[
\int_0^1 f(t) dt = T_p^{(\ell)}(f) - \frac{(-1)^{\ell}}{2^{(1+\ell)}} \sum_{\ell < k \leq \frac{\ell}{2}} \binom{k-1}{\ell}_q \frac{b_{2k}}{(2k)!} \cdot p^{2k} \delta f^{(2k-1)} + \mathcal{E}_m^{(\ell)}(p) \tag{9.20}
\]
Also, since according to Theorem S.3 (iii), we have \(\lim_{p \to \infty} p^m \mathcal{E}_m^{(0)}(p) = 0\), we conclude by induction on \(\ell\) that we have
\[
\lim_{p \to \infty} p^m \mathcal{E}_m^{(\ell)}(p) = 0, \quad \text{for } \ell = 0, 1, 2, \ldots.
\]
In particular, when \(m \geq 2\ell + 2\) we have
\[
\lim_{p \to \infty} p^{2\ell+2} \left( \int_0^1 f(t) dt - T_p^{(\ell)}(f) \right) = -\frac{|b_{2\ell+2}|}{2^{(1+\ell)} (2\ell + 2)!} \delta f^{(2\ell+1)}.
\]
On the other hand, using Theorem S.4 (ii) we have
\[
\left| \mathcal{E}_m^{(0)}(p) \right| \leq \frac{8}{(2\pi p)^m} \sup_{[0,1]} |f^{(m)}|
\]
and, for \(\ell = 1, 2, \ldots\) we have
\[
\left| \mathcal{E}_m^{(\ell)}(p) \right| \leq \frac{4^\ell}{4^\ell - 1} \left| \mathcal{E}_m^{(\ell-1)}(2p) \right| + \left| \mathcal{E}_m^{(\ell-1)}(p) \right|
\]
So

\[
|\mathcal{E}^{(1)}_m(p)| \leq \frac{8}{(2\pi p)^m} \left( \frac{1 + 4/2^m}{4 - 1} \right) \sup_{[0,1]} |f^{(m)}|
\]

\[
|\mathcal{E}^{(2)}_m(p)| \leq \frac{8}{(2\pi p)^m} \left( \frac{1 + 4/2^m}{4 - 1} \right) \left( \frac{1 + 4^2/2^m}{4^2 - 1} \right) \sup_{[0,1]} |f^{(m)}|
\]

\[\vdots\]

\[
|\mathcal{E}^{(t)}_m(p)| \leq \frac{8}{(2\pi p)^m} \prod_{j=1}^t \left( \frac{1 + 4^j/2^m}{4^j - 1} \right) \sup_{[0,1]} |f^{(m)}|
\]

(9.21)

But, for \(m \geq 2\ell + 2\) we have

\[
\prod_{j=1}^\ell \left( 1 + 4^j/2^m \right) \leq \prod_{j=1}^\ell \left( 1 + 4^{-(\ell-j+1)} \right) = \prod_{j=1}^\ell \left( 1 + 4^{-j} \right)
\]

So

\[
\prod_{j=1}^\ell \left( \frac{1 + 4^j/2^m}{4^j - 1} \right) \leq \frac{1}{2^\ell(1+\ell)} \prod_{j=1}^\ell \frac{1 + 4^{-j}}{1 - 4^{-j}}
\]

(9.22)

Now, since \(x \mapsto \ln (\frac{1 + x}{1 - x})\) is convex on the interval \([0, 1/4]\) we conclude that for \(x \in [0, 1/4]\) we have

\[
\ln \left( \frac{1 + x}{1 - x} \right) \leq 4 \ln \left( \frac{1 + x}{1 - x} \right) x = 4 \ln \left( \frac{1 + x}{1 - x} \right) \approx x
\]

Thus

\[
\sum_{j=1}^\ell \ln \left( \frac{1 + 4^{-j}}{1 - 4^{-j}} \right) \leq 4 \ln \left( \frac{5}{3} \right) \sum_{j=1}^\infty \frac{1}{4^j} = 4 \ln \left( \frac{5}{3} \right)
\]

Finally, since \((5/3)^{4/3} < 2\) we obtain from (9.22) that

\[
\forall m \geq 2\ell + 2, \quad \prod_{j=1}^\ell \left( \frac{1 + 4^j/2^m}{4^j - 1} \right) \leq \frac{2}{2^\ell(1+\ell)}
\]

Thus, (9.21) implies the following more appealing form

\[
|\mathcal{E}^{(t)}_m(p)| \leq \frac{16}{\pi} \cdot \frac{1}{2^\ell(1+\ell)(2\pi p)^m} \sup_{[0,1]} |f^{(m)}|
\]

(9.23)

We have proved the following result:

**Proposition 9.3.** Let \(m\) be a positive integer, and let \(f\) be a function having a continuous \(m^{\text{th}}\) derivative on \([0, 1]\). For a positive \(p\), let \(T_p^{(0)}(f)\) be the trapezoidal quadrature rule applied to \(f\) defined by (9.3). Next, for \(\ell \geq 1\), define inductively the Romberg’s rule of order \(\ell\) by

\[
T_p^{(\ell)}(f) = \frac{4^\ell T_p^{(\ell-1)}(f) - T_p^{(\ell-1)}(f)}{4^\ell - 1}
\]

Then

\[
\int_0^1 f(t)dt = T_p^{(\ell)}(f) - \frac{(-1)^\ell}{2^\ell(1+\ell)} \sum_{\ell < k \leq 2^\ell} \left( \begin{array}{c} k - 1 \\ \ell \end{array} \right) \frac{b_{2k}}{(2k)! \cdot p^{2k}} \delta f^{(2k-1)} + \mathcal{E}^{(t)}_m(p)
\]

with \(\lim_{p \to \infty} p^m \mathcal{E}^{(t)}_m(p) = 0\), (where the \(q\)-binomial is defined by (9.19).) Moreover, for \(m \geq 2\ell + 2\) we have

\[
|\mathcal{E}^{(t)}_m(p)| \leq \frac{16}{\pi} \cdot \frac{1}{2^\ell(1+\ell)(2\pi p)^m} \sup_{[0,1]} |f^{(m)}|
\]
Consider the particular case where $m = 2\ell + 2$. In this case, Proposition 9.3 implies
\[
\int_0^1 f(t) dt - T_p(t)(f) \leq \frac{1}{2^{(\ell+1)}(2\pi\ell)^2} \left( \frac{(2\pi)^{2\ell+2}|b_{2\ell+2}|}{(2\ell+2)!} \left| \delta f(2\ell+1) \right| + \frac{16}{\pi} \sup_{[0,1]} |f(2\ell+2)| \right)
\]
\[
\leq \frac{1}{2^{(\ell+1)}(2\pi\ell)^2} \left( 4 \left| \delta f(2\ell+1) \right| + \frac{16}{\pi} \sup_{[0,1]} |f(2\ell+2)| \right)
\]
\[
\leq \frac{4 + 16/\pi}{2^{(\ell+1)}(2\pi\ell)^2} \sup_{[0,1]} |f(2\ell+2)|
\]
\[
\leq \frac{10}{2^{(\ell+2)(\ell+1)}(\pi p)^2} \sup_{[0,1]} |f(2\ell+2)|
\]
where we used Proposition 4.1 (4), and the fact that \( |\delta f(2\ell+1)| \leq \sup_{[0,1]} |f(2\ell+2)| \).

Note that the Romberg’s rule \( T_p(t)(f) \) subdivides the interval \([0, 1]\) into \( 2^p \ell \) equal subintervals. In the particular case \( p = 1 \), we obtain
\[
\int_0^1 f(t) dt - T_1(t)(f) \leq \frac{10}{2^{(\ell+2)(\ell+1)}\pi 2^{2\ell+2}} \sup_{[0,1]} |f(2\ell+2)|.
\]

10. Asymptotic expansions for the sum of certain series related to harmonic numbers

Recall that the sequence of harmonic numbers \((H_n)_{n\in\mathbb{N}}\) is defined by \( H_n = \sum_{k=1}^{n} 1/k \) (with the convention \( H_0 = 0 \)). It is well-known that \( \lim_{n \to \infty} (H_n - \ln n) = \gamma \), where \( \gamma \approx 0.57721 56649 \) is the so-called Euler-Mascheroni Constant.

In the next proposition, the asymptotic expansion of \((H_n)_{n\in\mathbb{N}}\) is presented.

**Proposition 10.1.** For every positive integer \( n \) and nonnegative integer \( m \), we have
\[
H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{m-1} \frac{b_{2k}}{2k} \cdot \frac{1}{n^{2k}} + (-1)^m R_{n,m},
\]
with
\[
R_{n,m} = \int_0^{1/2} |B_{2m-1}(t)| \sum_{j=n}^{\infty} \left( \frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}} \right) dt
\]
Moreover, \( 0 < R_{n,m} < \frac{|b_{2m}|}{2m \cdot n^{2m}} \).

**Proof.** Note that for \( j \geq 1 \) we have
\[
\frac{1}{j} - \ln \left( 1 + \frac{1}{j} \right) = \int_0^1 \left( \frac{1}{j-t} - \frac{1}{j+t} \right) dt = \int_0^1 \frac{t}{j(j+t)} dt
\]
Adding these equalities as \( j \) varies from 1 to \( n-1 \) we conclude that
\[
H_n - \ln n - \frac{1}{n} = \int_0^1 \left( \sum_{j=1}^{n-1} \frac{t}{j(j+t)} \right) dt.
\]
Thus, letting \( n \) tend to \( \infty \), and using the monotone convergence theorem [4, Corollary 2.3.5], we conclude
\[
\gamma = \int_0^{1} \left( \sum_{j=1}^{\infty} \frac{t}{j(j+t)} \right) dt.
\]
It follows that
\[ \gamma + \ln n - H_n + \frac{1}{n} = \int_0^1 \left( \sum_{j=n}^{\infty} \frac{t}{j(j+t)} \right) dt. \]

So, let us consider the function \( f_n : [0,1] \rightarrow \mathbb{R} \) defined by
\[ f_n(t) = \sum_{j=n}^{\infty} \frac{t}{j(j+t)} \]

Note that \( f_n(0) = 0 \), \( f_n(1) = 1/n \), and that \( f_n \) is infinitely continuously derivable with
\[ \frac{f_n^{(k)}(t)}{k!} = (-1)^{k+1} \sum_{j=n}^{\infty} \frac{1}{(j+t)^{k+1}} \quad \text{for } k \geq 1. \]

In particular,
\[ \frac{f_n^{(2k-1)}(t)}{(2k-1)!} = \sum_{j=n}^{\infty} \frac{1}{(j+1)^{2k}} - \sum_{j=n}^{\infty} \frac{1}{j^{2k}} = -\frac{1}{n^{2k}} \]

So, \( f_n^{(2m-1)} \) is decreasing on the interval \([0,1]\), and
\[ \frac{\delta f_n^{(2k-1)}}{(2k-1)!} = \sum_{j=n}^{\infty} \frac{1}{(j+1)^{2k}} - \sum_{j=n}^{\infty} \frac{1}{j^{2k}} = -\frac{1}{n^{2k}} \]

Applying Corollary 8.2 to \( f_n \), and using the above data, we get
\[ \gamma + \ln n - H_n + \frac{1}{2n} = \sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot n^{2k}} + (-1)^{m+1} R_{n,m} \]

with
\[ R_{n,m} = \int_0^{1/2} |B_{2m-1}(t)| \left( \sum_{j=n}^{\infty} \frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}} \right) dt \]

and
\[ 0 < R_{n,m} < 6 \cdot \frac{(2m-1)!}{(2\pi)^{2m} n^{2m}}. \]

What is important in this estimate is the lower bound, i.e. \( R_{n,m} > 0 \). In fact, considering separately the cases \( m \) odd and \( m \) even, we obtain, for every nonnegative integer \( m' \):
\[ H_n < \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{2m'} \frac{b_{2k}}{2k \cdot n^{2k}}, \]

and
\[ H_n > \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{2m'+1} \frac{b_{2k}}{2k \cdot n^{2k}}. \]

This yields the following more precise estimate for the error term:
\[ 0 < (-1)^m \left( H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot n^{2k}} \right) < \frac{|b_{2m}|}{2m \cdot n^{2m}} \quad (10.1) \]

which is valid for every positive integer \( m \). (see [15], Chapter 9.)
In Table 4 we find the values of these bounds for Euler’s γ. It is worth noting that formula (10.1) with \( m = 251 \) and \( n = 10^4 \) was used by Knuth in 1962 to obtain 1271 decimal digits of Euler’s constant [21].

\[
\begin{array}{|c|c|c|}
\hline
n & \gamma^-_n & \gamma^+_n \\
\hline
1 & 0.5750000000 & 0.5833333333 \\
2 & 0.5771653194 & 0.5776861528 \\
4 & 0.5772147535 & 0.5772473055 \\
8 & 0.5772156500 & 0.5772176845 \\
16 & 0.5772156647 & 0.5772157918 \\
32 & 0.5772156649 & 0.5772156728 \\
64 & 0.5772156649 & 0.5772156654 \\
128 & 0.5772156649 & 0.5772156649 \\
\hline
\end{array}
\]

Table 4. Euler’s γ belongs to the interval \( (\gamma^-_n, \gamma^+_n) \) for each \( n \).

Now, consider the two sequences \( (c_n)_{n \geq 1} \) and \( (d_n)_{n \geq 1} \) defined by

\[
c_n = H_n - \ln n - \gamma - \frac{1}{2n} \quad \text{and} \quad d_n = H_n - \ln n - \gamma
\]

For a positive integer \( p \), we know according to Proposition 10.1 that \( c_{pn} = \mathcal{O} \left( \frac{1}{n^m} \right) \), it follows that the series \( \sum_{n=1}^{\infty} c_{pn} \) is convergent. Similarly, since \( d_{pn} = c_{pn} + \frac{1}{2pn} \) and the series \( \sum_{n=1}^{\infty} (-1)^{n-1}/n \) is convergent, we conclude that \( \sum_{n=1}^{\infty} (-1)^{n-1} d_{pn} \) is also convergent. In what follows we aim to find asymptotic expansions, (for large \( p \),) of the following sums:

\[
C_p = \sum_{n=1}^{\infty} c_{pn} = \sum_{n=1}^{\infty} \left( H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} \right) \quad (10.2)
\]

\[
D_p = \sum_{n=1}^{\infty} (-1)^{n-1} d_{pn} = \sum_{n=1}^{\infty} (-1)^{n-1} (H_{pn} - \ln(pn) - \gamma) \quad (10.3)
\]

**Proposition 10.2.** If \( p \) and \( m \) are positive integers and \( C_p \) is defined by (10.2) then

\[
C_p = -\sum_{k=1}^{m-1} \frac{b_{2k} \zeta(2k)}{2^k \cdot p^{2k}} + (-1)^m \frac{\zeta(2m)}{2^m \cdot p^{2m}} \varepsilon_{p,m}, \quad \text{with} \quad 0 < \varepsilon_{p,m} < |b_{2m}|
\]

where \( \zeta \) is the well-known Riemann zeta function.

**Proof.** Indeed, we conclude from Proposition 10.1 that

\[
H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} = -\sum_{k=1}^{m-1} \frac{b_{2k} \cdot 1}{2^k \cdot p^{2k}} + \frac{(-1)^{n-1}}{2^m \cdot p^{2m}} \frac{r_{pn,m}}{n^{2m}}
\]

with \( 0 < r_{pn,m} \leq |b_{2m}| \). It follows that

\[
C_p = -\sum_{k=1}^{m-1} \frac{b_{2k} \cdot 1}{2^k \cdot p^{2k}} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) + (-1)^m \frac{\zeta(2m)}{2^m \cdot p^{2m}} \varepsilon_{p,m}.
\]
where \( \tilde{r}_{p,m} = \sum_{n=1}^{\infty} \frac{r_{p,m,n}}{n^{2m}} \).

Hence,

\[
0 < \tilde{r}_{p,m} = \sum_{n=1}^{\infty} \frac{r_{p,m,n}}{n^{2m}} < |b_{2m}| \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = |b_{2m}| \zeta(2m)
\]

and the desired conclusion follows with \( \varepsilon_{p,m} = \tilde{r}_{p,m}/\zeta(2m) \).

\( \square \)

For example, when \( m = 3 \), we obtain

\[
\sum_{n=1}^{\infty} \left( H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} \right) = -\frac{\pi^2}{72p^2} + \frac{\pi^4}{10800p^4} + O \left( \frac{1}{p^6} \right).
\]

Similarly, in the next proposition we have the analogous result corresponding to \( D_p \).

**Proposition 10.3.** If \( p \) and \( m \) are positive integers and \( D_p \) is defined by (10.3), then

\[
D_p = \frac{\ln 2}{2p} - \sum_{k=1}^{m-1} \frac{b_{2k} \eta(2k)}{2k \cdot p^{2k}} + (-1)^m \frac{\eta(2m)}{2m \cdot p^{2m}} \varepsilon_{p,m}, \quad \text{with } 0 < \varepsilon_{p,m} < |b_{2m}|,
\]

where \( \eta(2k) \) is defined in Corollary 3.3.

**Proof.** Indeed, let us define \( a_{n,m} \) by the formula

\[
a_{n,m} = H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot n^{2k}}
\]

with empty sum equal to 0. We have shown in the proof of Proposition 10.1 that

\[
(-1)^m a_{n,m} = \int_0^{1/2} |B_{2m-1}(t)| g_{n,m}(t) dt
\]

where \( g_{n,m} \) is the positive decreasing function on \([0,1/2]\) defined by

\[
g_{n,m}(t) = \sum_{j=n}^{\infty} \left( \frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}} \right).
\]

Now, for every \( t \in [0,1/2] \) the sequence \( (g_{n,p,m}(t))_{n \geq 1} \) is positive and decreasing to 0. So, Using the alternating series criterion [3] Theorem 7.8, and Corollary 7.9] we see that, for every \( N \geq 1 \) and \( t \in [0,1/2] \),

\[
\sum_{n=1}^{\infty} (-1)^{n-1} g_{n,p,m}(t) \leq g_{N,p,m}(t) \leq g_{N,p,m}(0) = \frac{1}{(Np)^{2m}}.
\]

This proves the uniform convergence on \([0,1/2]\) of the series

\[
G_{p,m}(t) = \sum_{n=1}^{\infty} (-1)^{n-1} g_{n,p,m}(t).
\]

Consequently

\[
(-1)^m \sum_{n=1}^{\infty} (-1)^{n-1} a_{p,m,n} = \int_0^{1/2} |B_{2m-1}(t)| G_{p,m}(t) dt.
\]

Now using the properties of alternating series, we see that for \( t \in (0,1/2) \) we have

\[
0 < G_{p,m}(t) < g_{p,m}(t) < g_{p,m}(0) = \sum_{j=p}^{\infty} \frac{1}{j^{2m}} - \frac{1}{(j+1)^{2m}} = \frac{1}{p^{2m}}
\]

Thus,

\[
\sum_{n=1}^{\infty} (-1)^{n-1} a_{p,m,n} = \frac{(-1)^m}{p^{2m}} \rho_{p,m}
\]
with $0 < \rho_{p,m} < \sqrt{\int_0^{1/2} |B_{2m-1}(t)| \, dt}$.

On the other hand we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_{pn,m} = D_p - \frac{1}{2p} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} + \sum_{k=1}^{m-1} \frac{b_{2k}}{2k p^{2k}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2k}}$$

Thus

$$D_p = \ln \frac{2}{2p} - \sum_{k=1}^{m-1} \frac{b_{2k} \eta(2k)}{2k \cdot p^{2k}}.$$  

Now, the important estimate for $\rho_{p,m}$ is the lower bound, i.e. $\rho_{p,m} > 0$. In fact, considering separately the cases $m$ odd and $m$ even, we obtain, for every nonnegative integer $m'$:

$$D_p < \ln \frac{2}{2p} - \sum_{k=1}^{m'-1} \frac{b_{2k} \eta(2k)}{2k \cdot p^{2k}},$$

and

$$D_p > \ln \frac{2}{2p} - \sum_{k=1}^{m'+1} \frac{b_{2k} \eta(2k)}{2k \cdot p^{2k}}.$$

This yields the following more precise estimate for the error term:

$$0 < (-1)^m \left( D_p - \ln \frac{2}{2p} + \sum_{k=1}^{m-1} \frac{b_{2k} \eta(2k)}{2k p^{2k}} \right) < |b_{2m} \eta(2m)| \frac{p^{2m}}{2m \cdot p^{2m}},$$

and the desired conclusion follows.  \[\square\]

In the next lemma, we will show that there are other alternating series that can be expressed in terms of $D_p$. This lemma will be helpful in §11.

**Lemma 10.4.** For a positive integer $p$, we have

$$E_p \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n (H_{p(n+1)} - H_{pn}) = \ln p + \gamma - \ln \left( \frac{\pi}{2} \right) + 2D_p,$$

where $D_p$ is the sum defined by (10.3).
Proof. Indeed

\[2D_p = d_p + \sum_{n=2}^{\infty} (-1)^{n-1}d_{pn} + \sum_{n=1}^{\infty} (-1)^{n-1}d_{pn}\]

\[= d_p + \sum_{n=1}^{\infty} (-1)^n d_{p(n+1)} + \sum_{n=1}^{\infty} (-1)^{n-1}d_{pn}\]

\[= d_p + \sum_{n=1}^{\infty} (-1)^{n-1}(d_{pn} - d_{p(n+1)})\]

\[= d_p + \sum_{n=1}^{\infty} (-1)^{n-1}(d_{pn} - d_{p(n+1)}) + \sum_{n=1}^{\infty} (-1)^{n-1} \ln \left(\frac{n+1}{n}\right)\]

Using Wallis formula for \(\pi\) [13, Formula 0.262], we have

\[\sum_{n=1}^{\infty} (-1)^{n-1} \ln \left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right)\]

\[= -\ln \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \ln \left(\frac{\pi}{2}\right)\]

and the desired formula follows. \(\square\)

11. Asymptotic expansions for certain trigonometric sums

In this section we aim to exploit the results of the previous sections to study the following trigonometric sums defined for a positive integer \(p\) by the formulæ:

\[I_p = \sum_{j=1}^{p-1} \frac{1}{\sin(k\pi/p)} = \sum_{j=1}^{p-1} \csc \left(\frac{k\pi}{p}\right)\] (11.1)

\[J_p = \sum_{j=1}^{p-1} k \cot \left(\frac{k\pi}{p}\right)\] (11.2)

with empty sums interpreted as 0.

While there is a of favourable result [24] concerning the sum \(\sum_{k=1}^{p} \sec \left(\frac{2k\pi}{2p+1}\right)\), and many favourable results [6] concerning the power sums \(\sum_{k=1}^{p-1} \csc^{2n}(k\pi/p)\), it seems that there is no known closed form for \(I_p\), and the same can be said about the sum \(J_p\). Therefore, we will look for asymptotic expansions for these sums and will give some tight inequalities that bound \(I_p\) and \(J_p\). This investigation complements the work of H. Chen in [7, Chapter 7.], and answers an open question raised there.

In the next lemma we give some equivalent forms for the trigonometric sums under consideration.
Lemma 11.1. For a positive integer \( p \) let

\[
K_p = \sum_{k=1}^{p-1} \tan \left( \frac{k\pi}{2p} \right), \quad \tilde{K}_p = \sum_{j=1}^{p-1} \cot \left( \frac{k\pi}{2p} \right),
\]

\[
L_p = \sum_{k=1}^{p-1} \frac{k}{\sin(k\pi/p)}, \quad M_p = \sum_{k=0}^{p-1} (2k+1) \cot \left( \frac{(2k+1)\pi}{2p} \right)
\]

Then,

i. \( K_p = \tilde{K}_p = I_p \).

ii. \( L_p = \left( \frac{p}{2} \right) I_p \).

iii. \( M_p = \left( \frac{p}{2} \right) J_{2p} - 2J_p = -p I_p \).

Proof. First, note that the change of summation variable \( j \leftarrow p-j \) proves that \( K_p = \tilde{K}_p \). So, using the trigonometric identity \( \tan \theta + \cot \theta = 2 \csc(2\theta) \) we obtain

\[
2K_p = K_p + \tilde{K}_p = \sum_{k=1}^{p-1} \left( \tan \left( \frac{k\pi}{2p} \right) + \cot \left( \frac{k\pi}{2p} \right) \right) = 2 \sum_{k=1}^{p-1} \csc \left( \frac{k\pi}{p} \right) = 2I_p
\]

This proves (i).

Similarly, (ii) follows from the change of summation variable \( j \leftarrow p-j \) in \( L_p \):

\[
L_p = \sum_{j=1}^{p-1} \frac{p-k}{\sin(k\pi/p)} = pI_p - L_p
\]

Also,

\[
M_p = \sum_{1 \leq k < 2p \atop k \text{ odd}} k \cot \left( \frac{k\pi}{2p} \right) = \sum_{k=1}^{2p-1} k \cot \left( \frac{k\pi}{2p} \right) - \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{2p} \right) = \sum_{k=1}^{2p-1} k \cot \left( \frac{k\pi}{2p} \right) - 2 \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{p} \right) = J_{2p} - 2J_p.
\]

But

\[
J_{2p} = \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{2p} \right) + \sum_{k=p+1}^{2p-1} k \cot \left( \frac{k\pi}{2p} \right)
\]

\[
= \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{2p} \right) - \sum_{k=1}^{p-1} (2p-k) \cot \left( \frac{k\pi}{2p} \right)
\]

\[
= 2 \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{2p} \right) - 2p \tilde{K}_p
\]

Thus, using (i) and the trigonometric identity \( \cot(\theta/2) - \cot \theta = \csc \theta \) we obtain

\[
M_p = J_{2p} - 2J_p = 2 \sum_{k=1}^{p-1} k \left( \cot \left( \frac{k\pi}{2p} \right) - \cot \left( \frac{k\pi}{p} \right) \right) - 2pI_p
\]

\[
= 2 \sum_{k=1}^{p-1} k \csc \left( \frac{k\pi}{p} \right) - 2pI_p = 2L_p - 2pI_p = -p I_p
\]

This concludes the proof of (iii).
Proposition 11.2. For \( p \geq 2 \), let \( I_p \) be the sum of cosecants defined by the \((11.1)\). Then
\[
I_p = -\frac{2 \ln 2}{\pi} + \frac{2 \ln 2}{\pi} E_p,
\]
where \( D_p \) and \( E_p \) are defined by formulæ \((10.3)\) and \((10.4)\) respectively.

Proof. Indeed, our starting point will be the “simple fractions” expansion \((6.4)\) of the cosecant function:
\[
\frac{\pi}{\sin(\pi \alpha)} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{\alpha - n} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{\alpha - n} + \frac{1}{\alpha + n} \right)
\]
which is valid for \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \). Using this formula with \( \alpha = k/p \) for \( k = 1, 2, \ldots, p - 1 \) and adding, we conclude that
\[
\frac{\pi}{p} I_p = \frac{p-1}{k} + \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{p-1} \left( \frac{1}{k - np} + \frac{1}{k + np} \right)
\]
and this result can be expressed in terms of the Harmonic numbers as follows
\[
\frac{\pi}{p} I_p = H_{p-1} + \sum_{n=1}^{\infty} (-1)^n \left( -H_{pn-1} + H_{p(n-1)} + H_{p(n+1)-1} - H_{pn} \right)
\]
\[
= H_{p-1} + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n+1)} - 2H_{pn} + H_{p(n-1)} \right) + \frac{1}{p} \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]
\[
= H_{p-1} + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n+1)} - 2H_{pn} + H_{p(n-1)} \right) + \frac{1}{p} \left( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} + \sum_{n=2}^{\infty} (-1)^n \frac{1}{n} \right)
\]
\[
= H_p + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n+1)} - 2H_{pn} + H_{p(n-1)} \right) - \frac{2}{p} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}
\]
\[
= H_p - \frac{2 \ln 2}{p} + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n+1)} - 2H_{pn} + H_{p(n-1)} \right).
\]
Thus
\[
\frac{\pi}{p} I_p + \frac{2 \ln 2}{p} = H_p + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n+1)} - H_{pn} \right) + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n-1)} - H_{pn} \right)
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \left( H_{p(n+1)} - H_{pn} \right) + \sum_{n=1}^{\infty} (-1)^n \left( H_{p(n-1)} - H_{pn} \right)
\]
\[
= E_p + E_p = 2E_p,
\]
and the desired formula follows according to Lemma \((10.4)\). \(\square\)

Combining Proposition \((11.2)\) and Proposition \((10.3)\) we obtain:

Proposition 11.3. For \( p \geq 2 \) and \( m \geq 1 \), we have
\[
\pi I_p = 2p \ln p + 2(\gamma - \ln(\pi/2))p - \sum_{k=1}^{m-1} \frac{2b_{2k}\eta(2k)}{k \cdot p^{2k-1}} + (-1)^m \frac{2\eta(2m)}{m \cdot p^{2m-1}} \varepsilon_{p,m},
\]
with \( 0 < \varepsilon_{p,m} < |b_{2m}| \).
Using the values of the $\eta(2k)$’s from Corollary 3.2 and considering separately the cases $m$ even and $m$ odd we obtain the following corollary.

**Corollary 11.4.** For every positive integer $p$ and every nonnegative integer $n$, the sum of cosecants $I_p$ defined by (11.1) satisfies the following inequalities:

$$I_p < \frac{2p}{\pi} (\ln p + \gamma - \ln(\pi/2)) + \sum_{k=1}^{2n} (-1)^k \frac{(2^{2k} - 2)b_{2k}^2}{k \cdot (2k)!} \left( \frac{\pi}{p} \right)^{2k-1},$$

and

$$I_p > \frac{2p}{\pi} (\ln p + \gamma - \ln(\pi/2)) + \sum_{k=1}^{2n+1} (-1)^k \frac{(2^{2k} - 2)b_{2k}^2}{k \cdot (2k)!} \left( \frac{\pi}{p} \right)^{2k-1}. $$

As an example, for $n = 0$ we obtain the following inequality, valid for every $p \geq 1$:

$$\frac{2p}{\pi} (\ln p + \gamma - \ln(\pi/2)) - \frac{\pi}{36p} < I_p < \frac{2p}{\pi} (\ln p + \gamma - \ln(\pi/2)).$$

This answers positively the open problem proposed in [7, Section 7.4].

**Remark 11.5.** The asymptotic expansion of $I_p$ was proposed as an exercise in [17, Exercise 13, p. 460], and it was attributed to P. Waldvogel, but the result there is less precise than Corollary 11.4.

Now we turn our attention to the other trigonometric sum $J_p$. Next we find an analogous result to Proposition 11.2.

**Proposition 11.6.** For a positive integer $p$, let $J_p$ be the sum of cotangents defined by (11.2). Then

$$\pi J_p = -p^2 \ln p + (\ln(2\pi) - \gamma)\pi^2 - p + 2p^2C_p$$

where $C_p$ is defined by the formula (10.2).

**Proof.** Recall that $c_n = H_n - \ln n - \gamma - \frac{1}{2n}$ satisfies $c_n = O(1/n^2)$. Thus, both series

$$C_p = \sum_{n=1}^{\infty} c_{pn} \quad \text{and} \quad \tilde{C}_p = \sum_{n=1}^{\infty} (-1)^{n-1}c_{pn}$$

are convergent. Further, we note that $\tilde{C}_p = D_p - \frac{\ln^2}{2p}$ where $D_p$ is defined by (10.3). According to Remark 11.10 we have

$$\tilde{C}_p = \frac{\ln(\pi/2) - \gamma - \ln p}{2} + \frac{\pi}{4p}I_p. \quad (11.3)$$

Now, noting that

$$C_p = \sum_{n \geq 1} \sum_{\substack{n \geq 1 \quad n \text{ even}}} c_{pn} + \sum_{n \geq 1} \sum_{\substack{n \geq 1 \quad n \text{ odd}}} c_{pn} = \sum_{n \geq 1} c_{pn} + \sum_{n=1}^{\infty} c_{2pn}$$

$$\tilde{C}_p = \sum_{n \geq 1} \sum_{\substack{n \geq 1 \quad n \text{ even}}} c_{pn} - \sum_{n \geq 1} \sum_{\substack{n \geq 1 \quad n \text{ odd}}} c_{pn} = \sum_{n \geq 1} c_{pn} - \sum_{n=1}^{\infty} c_{2pn}$$

we conclude that $C_p - \tilde{C}_p = 2C_{2p}$, or equivalently

$$C_p - 2C_{2p} = \tilde{C}_p \quad (11.4)$$

On the other hand, for a positive integer $p$ let us define $F_p$ by

$$F_p = \frac{\ln p + \gamma - \ln(2\pi)}{2} + \frac{1}{2p} + \frac{\pi}{2p^2}J_p. \quad (11.5)$$
It is easy to check, using Lemma 11.1 (iii), that
\[
F_p - 2F_{2p} = \frac{\ln(\pi/2) - \ln p - \gamma}{2} - \frac{\pi}{4p^2}(J_{2p} - 2J_p) = \frac{\ln(\pi/2) - \ln p - \gamma}{2} + \frac{\pi}{4p^2}F_p \tag{11.6}
\]
We conclude from (11.4) and (11.6) that \(C_p - 2C_{2p} = F_p - 2F_{2p}\), or equivalently
\[
C_p - F_p = 2(C_{2p} - F_{2p}).
\]
Hence,
\[
\forall m \geq 1, \quad C_p - F_p = 2^m(C_{2^m p} - F_{2^m p}) \tag{11.7}
\]
Now, using Proposition 11.1 to replace \(H_p\) in Proposition 11.2, we obtain
\[
\frac{\pi}{p^2}F_p = \ln(2\pi) - H_p - \frac{1}{2p} + O\left(\frac{1}{p^2}\right) = \ln(2\pi) - \ln p - \gamma - \frac{1}{p} + O\left(\frac{1}{p^2}\right)
\]
Thus \(F_p = O\left(\frac{1}{p}\right)\). Similarly, from the fact that \(c_n = O\left(\frac{1}{p}\right)\) we conclude also that \(C_p = O\left(\frac{1}{p}\right)\). Consequently, there exists a constant \(\kappa\) such that, for large values of \(p\) we have \(|C_p - F_p| \leq \kappa/p^2\). So, from (11.7), we see that for large values of \(m\) we have
\[
|F_p - p| \leq \frac{\kappa}{2m^2p^2}
\]
and letting \(m\) tend to \(+\infty\) we obtain \(C_p = F_p\), which is equivalent to the announced result. \(\square\)

Combining Proposition 11.6 and Proposition 11.7, we obtain:

**Proposition 11.7.** For \(p \geq 2\) and \(m \geq 1\), we have
\[
\pi J_p = -p^2 \ln p + (\ln(2\pi) - \gamma)p^2 - p - \sum_{k=1}^{m-1} \frac{b_{2k}\zeta(2k)}{k \cdot p^{2k-2}} + (-1)^m \frac{\zeta(2m)}{m \cdot p^{2m-2}} + \varepsilon_{p,m},
\]
with \(0 < \varepsilon_{p,m} < |b_{2m}|\), where \(\zeta\) is the well-known Riemann zeta function.

Using the values of the \(\zeta(2k)\)'s from Corollary 5.2 and considering separately the cases \(m\) even and \(m\) odd we obtain the next corollary.

**Corollary 11.8.** For every positive integer \(p\) and every nonnegative integer \(n\), the sum of cotangents \(J_p\) defined by (11.2) satisfies the following inequalities:
\[
J_p < \frac{1}{\pi} (-p^2 \ln p + (\ln(2\pi) - \gamma)p^2 - p) + 2\pi \sum_{k=1}^{2n} (-1)^k \frac{b_{2k}}{k \cdot (2k)!} \left(\frac{2\pi}{p}\right)^{2k-2},
\]
and
\[
J_p > \frac{1}{\pi} (-p^2 \ln p + (\ln(2\pi) - \gamma)p^2 - p) + 2\pi \sum_{k=1}^{2n+1} (-1)^k \frac{b_{2k}}{k \cdot (2k)!} \left(\frac{2\pi}{p}\right)^{2k-2}.
\]
As an example, for \(n = 0\) we obtain the following double inequality, which is valid for \(p \geq 1\):
\[
0 < \frac{1}{\pi} (-p^2 \ln p + (\ln(2\pi) - \gamma)p^2 - p) - J_p < \frac{\pi}{36}
\]
Remark 11.9. It is not clear that one can prove Proposition 11.7 by combining Propositions 9.2 and 10.1 directly.

Remark 11.10. Note that we have proved the following results. For a positive integer $p$:

\begin{align*}
\sum_{n=1}^{\infty} (-1)^{n-1} (H_{pn} - \ln(pn) - \gamma) &= \frac{\ln(\pi/2) - \gamma - \ln p}{2} + \frac{\ln 2}{2p} + \frac{\pi}{4p} \sum_{k=1}^{p-1} \csc \left( \frac{k\pi}{p} \right), \\
\sum_{n=0}^{\infty} (-1)^n (H_{p(n+1)} - H_{pn}) &= \frac{\ln 2}{p} + \frac{\pi}{2p} \sum_{k=1}^{p-1} \csc \left( \frac{k\pi}{p} \right), \\
\sum_{n=1}^{\infty} \left( H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} \right) &= \frac{\ln p + \gamma - \ln(2\pi)}{2} + \frac{1}{2p} + \frac{\pi}{2p^2} \sum_{k=1}^{p-1} k \cot \left( \frac{k\pi}{p} \right).
\end{align*}

These results are to be compared with those in [22], see also [23].

12. Endnotes

Bernoulli numbers appeared in the work Jacob Bernoulli (1654-1705, Basel, Switzerland) in connection with evaluating the sums of the form $\sum_{i=1}^{k}$. They appear in Bernoulli’s most original work Ars Conjectandi, (“The Art of Conjecturing”) published by his nephew in Basel in 1713, eight years after his death. They also appear independently in the work of the Japanese Mathematician Seki Takakazu (Seki Kōwa) (1642-1708). Bernoulli numbers are related to Fermat’s Last Theorem by Kummer’s theorem [25], this opened a very large and fruitful field of investigation (see [19, Chapter 15].)

The Euler-Maclaurin’s Summation Formula was developed independently by Leonhard Euler (1736) and Colin MacLaurin (1742). The aim was to provide a powerful tool for the computation of several sums like the harmonic numbers.

Finally, a thorough bibliography on Bernoulli numbers and their applications that contains more than 3000 entries can be found on the web [11].

References

[1] Abramowitz, M. and Stegan, I. A., *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover Books on Mathematics, Dover Publication, Inc., New York, (1972).

[2] Ahlfors, L. V., *Complex Analysis*, McGraw-Hill, Inc. (1979).

[3] Amann, H. and Escher, J., *Analysis I*, Birkhäuser Verlag, Basel -Boston-Berlin. (2005).

[4] Athreya, K. B. and Lahiri, S. N., *Measure Theory and Probability Theory*, Springer Science+Business Media, LLC. (2006).

[5] Bergmann, H., *Eine explizite Darstellung der Bernoullischen Zahlen*, Math. Nachr., 34, (1967), pp.377-378.

[6] Chen, H., *On some trigonometric power sums*, Internat. J. Math. Math. Sci, 30, (2002), 185–191.

[7] **Excursions in Classical Analysis**, Mathematical Association of America, Inc. (2010).

[8] Deeba, E. Y. and Rodrigues, D. M., *Stirling’s Series and Bernoulli Numbers*, The American Mathematical Monthly, 98, 5 (1991), pp.423–426.

[9] Dilcher, K., *Asymptotic Behaviour of Bernoulli, Euler, and Generalized Bernoulli Polynomials*, Journal of Approximation Theory, 49, (1987), 321–330.

[10] **Zeros of Bernoulli, generalized Bernoulli and Euler polynomials**, Mem. Amer. Math. Soc., Number 386, (1988).

[11] Dilcher, K. and Slavutskii, I. Sh., *A Bibliography of Bernoulli Numbers (1713-2007).* [ONLINE : http://http://www.mathstat.dal.ca/~dilcher/bernoulli.html]

[12] Gould, H. W., *Explicit formulas for Bernoulli numbers*, The American Mathematical Monthly, 79, 1 (1972), pp.44–51.

[13] Gradshteyn, I. and Ryzhik, I., *Tables of Integrals, Series and Products*, 7th ed., Academic Press, (2007).

[14] Grafakos, L., *Classical Fourier Analysis, Second edition.*, Graduate Texts in Mathematics, Springer Science+Business Media, LLC. (2008).
[15] Graham, R. L., Knuth, D. E., and Patashnik, O. Concrete Mathematics : a foundation for computer science, 2nd ed. Addison-Wesley Publishing Company, Inc, (1994).
[16] Hardy, G. H. and Wright, E. M., An introduction to the theory of numbers, 6th ed., Oxford University Press, (2007).
[17] Henrici, P., Applied and Computational Complex Analysis, Vol. 2, John Wiley & Sons, New York, (1977).
[18] Inkeri, K., The real roots of Bernoulli polynomials, Ann. Univ. Turku. Ser. A 137 (1959), 20pp.
[19] Ireland, K. and Rosen, M., A Classical Introduction to Modern Number Theory, 2nd ed., Springer-Verlag, New York, Inc, (1990).
[20] Katznelson, Y., Introduction To Harmonic Analysis, 3rd ed., Cambridge University Press, (2004).
[21] Knuth, D. E., Euler's constant to 1271 places, Math. Comput., 16, (1962), pp. 275–281.
[22] Kouba, O., The sum of certain series related to harmonic numbers, Octogon Mathematical Magazine, 19, No. 1 (2011), pp.3–18. [ONLINE : http://www.uni-miskolc.hu/~matsefi/Octogon/].
[23] , Proposed Problem 11499, The American Mathematical Monthly, 117, 7 (2010), p.371.
[24] Kouba, O., and Andreescu, T., Solution to Problem U207, Mathematical Reflections, Issue 5, (2011), p.16. [ONLINE : http://www.awesomemath/home].
[25] Kummer E. E., Allgemeiner Beweis des Fermat'schen Satzes, dass die Gleichung $x^\lambda + y^\lambda = z^\lambda$ durch ganze Zahlen unlösbar ist, für alle diejenigen Potenz-Exponenten $\lambda$, welche ungerade Primzahlen sind und in den Zählern der ersten $(\lambda - 3)/2$ Bernoulli'schen Zahlen als Factoren nicht vorkommen, J. Reine Angew. Math. 40 (1850), pp.131–138.
[26] Lehmer, D. H., On the maxima and minima of Bernoulli Polynomials, The American Mathematical Monthly, 47, 8 (1940). pp. 533–358.
[27] Namias, V., A simple Derivation of Stirling’s Asymptotic Series, The American Mathematical Monthly, 93, 1 (1986). pp. 25–29.

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