Relations for a class of terminating $4F_3(4)$ hypergeometric series

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ABSTRACT
We derive relations for a certain class of terminating $4F_3(4)$ hypergeometric series with three free parameters. The invariance group composed of these relations is shown to be isomorphic to the symmetric group $S_3$. We further study relations for terminating $3F_2(4)$ series that fall under two families. By using a series reversal, we examine the corresponding terminating $4F_3(1/4)$ and $3F_2(1/4)$ series relations. We additionally derive formulas for the sums of the first $n+1$ terms of several nonterminating $3F_2(4)$ and $3F_2(1/4)$ series. We also show how certain known summation formulas for terminating $3F_1(4)$ and $3F_2(4)$ series follow as limiting cases of some of our relations.

1. Introduction

The theory of the hypergeometric series of type $2F_1$ was systematically developed by Gauss [1]. Subsequently, generalized hypergeometric series of type $pF_q$, where $p$ and $q$ are nonnegative integers, and special summations and relations among such series, were investigated in the late nineteenth and early twentieth century by Thomae [2], Barnes [3,4], Ramanujan (see [5]), Whipple [6–8], Bailey [9,10], and others.

Over the last thirty-five years there has been a renewed interest in studying hypergeometric series. In particular, among other things, relations involving hypergeometric and basic hypergeometric series have been described in terms of group theory frameworks in papers by Beyer et al. [11], Srinivasa Rao et al. [12], Formichella et al. [13], Mishev [14], Green et al. [15,16], Van der Jeugt and Srinivasa Rao [17], Lievens and Van der Jeugt [18,19]. Other works include Groenevelt [20], van de Bult et al. [21], Krattenthaler and Rivoal [22].

There are numerous applications of hypergeometric series. There have been recent papers by Bump [23], Stade [24–27], and Stade and Taggart [28] with applications in the theory of automorphic functions. Some other recent works, with applications in physics, were written by Drake [29], Grozin [30], and Raynal [31].

In this paper, we study a certain class of terminating $4F_3(4)$ hypergeometric series with three free parameters (see Section 2 for the relevant definitions and terminology pertaining
Hypergeometric series with argument 4 have not been studied much in the past. Some of the few previous works are papers by Chu [32] studying certain terminating $2F_1(4)$ series summations perturbed by two integer parameters, and Chen and Chu [33,34] examining two classes of terminating $3F_2(4)$ series summations perturbed by two and three integer parameters, respectively.

The family of terminating $4F_3(4)$ series we consider is given by

$$4F_3\left(-n, \frac{a}{2}, \frac{a + 1}{2}, b \middle| 4\right),$$

(1.1)

where $n$ is a nonnegative integer and the complex numbers $a$, $b$, and $c$ are the three free parameters.

In Section 3, we employ a method of Bailey’s (see [10, Section 4.3]) along with the Chu–Vandermonde formula (see (2.2)) to obtain a transformation of the series in (1.1) into a terminating $4F_3(1/4)$ series. By reversing the order of summation of the latter series (see (2.4)), we obtain a transformation of (1.1) into another terminating $4F_3(4)$ series of the same family. By suitably normalizing that transformation, we obtain a group of six invariance relations. Furthermore, the invariance group composed of these six relations is shown to be isomorphic to the symmetric group $S_3$.

In Section 4, we study terminating $3F_2(4)$ series relations that follow from our terminating $4F_3(4)$ series relations in Section 3. In particular, we look at the two different families of terminating $3F_2(4)$ series given by

$$3F_2\left(-n, \frac{a}{2}, \frac{a + 1}{2} \middle| 4\right),$$

(1.2)

and

$$4F_3\left(-n, \frac{a}{2}, \frac{a + 1}{2} \middle| 4\right).$$

(1.3)

The first of these two families, under a suitable normalization, is shown to again have an invariance group isomorphic to the symmetric group $S_3$, while we do not have an $S_3$-symmetry in the second one of these families. We still have two nontrivial relations for the family (1.3), and one of those two relations is relevant to the works of Chu [32] and Chen and Chu [33]. In fact, by taking certain limits of that relation, we can obtain some of the formulas found by Chu in [32] and Chen and Chu in [33]. We further obtain formulas for the sums of the first $n + 1$ terms of certain divergent $3F_2(4)$ hypergeometric series.

By series reversal, we explore in Section 5 corresponding relations among terminating $4F_3(1/4)$ series, and the corresponding invariance group isomorphic to $S_3$.

Finally, in Section 6, we study, as consequence of our relations in Section 5, the different types of relations among terminating $3F_2(1/4)$ hypergeometric series which fall under two families. We also give a formula for the sum of the first $n + 1$ terms of a certain nonterminating $3F_2(1/4)$ series.
2. Preliminaries

The hypergeometric series of type \( pFq \) is the power series in \( z \) defined by

\[
pFq \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_p)_n}{n!(b_1)_n(b_2)_n \cdots (b_q)_n} z^n,
\]

(2.1)

where \( p \) and \( q \) are nonnegative integers, \( a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, z \in \mathbb{C} \), and the rising factorial \( (a)_n \) is given by

\[
(a)_n = \begin{cases} 
  a(a+1) \cdots (a+n-1), & n > 0, \\
  1, & n = 0.
\end{cases}
\]

In this paper we will consider the case where \( p = q + 1 \). The series of type \( q+1Fq \) converges absolutely if \(|z| < 1 \) or if \(|z| = 1 \) and \( \Re(\sum_{i=1}^{q+1} b_i - \sum_{i=1}^{q+1} a_i) > 0 \) (see [10, p. 8]). We assume that no denominator parameter \( b_1, b_2, \ldots, b_q \) is a negative integer or zero. When a numerator parameter \( a_1, a_2, \ldots, a_{q+1} \) is a negative integer or zero, the series has only finitely many nonzero terms and is said to terminate.

If \( z = 1 \), we say that the series is of unit argument and of type \( q+1Fq(1) \). When \( \sum_{i=1}^{q} b_i - \sum_{i=1}^{q+1} a_i = 1 \), the series is called Saalschützian. If \( 1 + a_1 = b_1 + a_2 = \cdots = b_q + a_{q+1} \), the series is called well-poised. A well-poised series that satisfies \( a_2 = 1 + \frac{1}{2}a_1 \) is called very-well-poised.

We will use the classical Chu–Vandermonde formula (see [10, Section 1.3]), which gives us the sum of a terminating \( 2F1 \) series:

\[
2F1 \left( \begin{array}{c} -n, a \\ b \end{array} \right| 1 \right) = \frac{(b - a)_n}{(b)_n}.
\]

(2.2)

If a hypergeometric series terminates, we can reverse its order of summation. Using the readily verified identity

\[
(a)_{n-k} = \frac{(-1)^k(a)_n}{(1-a-n)_k},
\]

(2.3)

we can derive the following well-known ‘summation-reversal’ formula:

\[
p+1Fq \left( \begin{array}{c} -n, a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right| x \right) = \frac{(a_1)_n(a_2)_n \cdots (a_p)_n(-x)^n}{(b_1)_n(b_2)_n \cdots (b_q)_n} \times q+1Fp \left( \begin{array}{c} -n, 1 - b_1 - n, 1 - b_2 - n, \ldots, 1 - b_q - n \\ 1-a_1 - n, 1-a_2 - n, \ldots, 1-a_p - n \end{array} \right| \frac{(-1)^{p+q}}{x} \right).
\]

(2.4)

We will use Equation (2.4) in Sections 3 and 5 when expressing terminating \( 4F3(1/4) \) series in terms of terminating \( 4F3(4) \) series and vice versa.

3. Relations for the terminating \( 4F3(4) \) series

In this section, we derive our relations for the terminating \( 4F3(4) \) hypergeometric series and examine the structure of those relations. The relations developed in this section form the foundation for the rest of the paper.
We begin with a general proposition that expresses a certain terminating hypergeometric series as a sum of terminating hypergeometric series of lower order:

**Proposition 3.1:** If \( n \) is a nonnegative integer, the following general identity holds:

\[
\begin{align*}
\sum_{m=0}^{n} \left( \frac{(-n)_m (a_1)_m (a_2)_m \cdots (a_p)_m \left( \frac{x}{4} \right)^m}{m!(1+a-c)_m (b_1)_m (b_2)_m \cdots (b_q)_m} \right) \times \binom{p+1}{q+1} \left( \frac{-n + m + a_1 + m + a_2 + m + \cdots + a_p + m}{c, b_1 + m, b_2 + m, \ldots, b_q + m} \left( \frac{x}{4} \right) \right),
\end{align*}
\]  

(3.1)

**Proof:** By the Chu–Vandermonde formula, we have

\[
\begin{align*}
\binom{n}{1} = \frac{(a+n)n}{(1+a-c)n} = \frac{(a)_{2n}}{(a)_n(1+a-c)_n} = \frac{\left(\frac{a}{4}\right)_n \left(\frac{a+1}{4}\right)_n 4^n}{(a)_n(1+a-c)_n}.
\end{align*}
\]  

(3.2)

Using this in the summation expansion of the left-hand side of (3.1), we obtain

\[
\begin{align*}
\sum_{m=0}^{n} \left( \frac{(-n)_m (a_1)_m (a_2)_m \cdots (a_p)_m \left( \frac{x}{4} \right)^m}{m!(1+a-c)_m (b_1)_m (b_2)_m \cdots (b_q)_m} \right) \times \binom{p+1}{q+1} \left( \frac{-n + m + a_1 + m + a_2 + m + \cdots + a_p + m}{c, b_1 + m, b_2 + m, \ldots, b_q + m} \left( \frac{x}{4} \right) \right),
\end{align*}
\]  

(3.3)

where in our simplifications we have used

\[
\frac{(-k)_m}{k!m!} = \frac{(-1)_m}{m!(k-m)!}
\]
and, after the change in index $k = m + t$,

$$\frac{(1 - c - k)_m}{(c)_k} = \frac{(1 - c - m - t)_m}{(c)_{m+t}} = \frac{(-1)^m(c + t)_m}{(c)_{m+t}} = \frac{(-1)^m}{(c)_t}. $$

From here,

$$p_{+}^{3F_{q+3}} \left( \begin{array}{c}
-n, \frac{a}{2}, \frac{a + 1}{2}, a_1, a_2, \ldots, a_p \\
a, 1 + a - c, c, b_1, b_2, \ldots, b_q
\end{array} \left| \begin{array}{c}
x
\end{array} \right. \right)$$

$$= \sum_{m=0}^{n} \frac{(-n)_m(a_1)_m(a_2)_m \cdots (a_p)_m(\frac{x}{4})^m}{m!(1 + a - c)_m(b_1)_m(b_2)_m \cdots (b_q)_m} \times \sum_{t=0}^{n-m} \frac{(-n + m)_t(a_1 + m)_t(a_2 + m)_t \cdots (a_p + m)_t(\frac{x}{4})^t}{t!(c)_t(b_1 + m)_t(b_2 + m)_t \cdots (b_q + m)_t}. \quad (3.4)$$

The right-hand side above is equal to the right-hand side of (3.1) and the proof is complete.

Proposition 3.1 above is of similar nature to [10, Equations (4.3.1) and (4.3.6)]. We employ this proposition to derive a relation between a terminating $4F_3(4)$ series and a terminating $4F_3(1/4)$ series:

**Proposition 3.2:** If $n$ is a nonnegative integer, the following relation holds:

$$4F_3 \left( \begin{array}{c}
-n, \frac{a}{2}, \frac{a + 1}{2}, b \\
a, 1 + a - c, c
\end{array} \left| \begin{array}{c}
4
\end{array} \right. \right)$$

$$= \frac{(c - b)_n}{(c)_n} 4F_3 \left( \begin{array}{c}
-n, 1 - c - n, b, 1 + b - c \\
1 + a - c, \frac{1 + b - c - n}{2}, \frac{2 + b - c - n}{2}
\end{array} \left| \begin{array}{c}
1
\end{array} \right. \right). \quad (3.5)$$

**Proof:** Letting $p = 1, q = 0, a_1 = b$, and $x = 4$ in Proposition 3.1, we obtain

$$4F_3 \left( \begin{array}{c}
-n, \frac{a}{2}, \frac{a + 1}{2}, b \\
a, 1 + a - c, c
\end{array} \left| \begin{array}{c}
4
\end{array} \right. \right)$$

$$= \sum_{m=0}^{n} \frac{(-n)_m(b)_m}{m!(1 + a - c)_m} 2F_1 \left( \begin{array}{c}
-n + m, b + m \\
c
\end{array} \left| \begin{array}{c}
1
\end{array} \right. \right). \quad (3.6)$$

Summing the $2F_1$ series in (3.6) by the Chu–Vandermonde formula and simplifying gives

$$2F_1 \left( \begin{array}{c}
-n + m, b + m \\
c
\end{array} \left| \begin{array}{c}
1
\end{array} \right. \right) = \frac{(c - b - m)_{n-m}}{(c)_{n-m}}.$$
I. D. MISHEV

Proof:

In Proposition 3.2, we reverse the order of summation in the right-hand side of (3.5) according to Equation (2.4). The right-hand side in (3.5) thus becomes

\[
\frac{(c - b)_{n}}{(c)_{n}} \frac{4F_{3} \left(-n, \frac{a}{2}, \frac{a + 1}{2}, b \right)}{(1 + a - c)_{n}} \left(1 + a - c, \frac{1 + b - c - n}{2}, \frac{2 + b - c - n}{2} \right)_{4}
\]

\[
= \frac{(c - b)_{n}(1 + b - c - n)_{n}(1 + b - c)_{n}\left(-\frac{1}{4}\right)_{n}}{(c)_{n}(1 + a - c)_{n} \left(\frac{1 + b - c - n}{2}\right)_{n} \left(\frac{2 + b - c - n}{2}\right)_{n}}
\]

\[
\times 4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}
\]

\[
= \frac{(c - b)_{n}(b)_{n}(1 + b - c)_{n}}{(1 + a - c)_{n}(1 + b - c - n)_{2n}}
\]

\[
\times 4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}
\]

\[
= \frac{(-1)^{n} (b)_{n}}{(1 + a - c)_{n}} \frac{4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}}{(1 + a - c)_{n}}
\]

(3.7)

Combining (3.6) and (3.7) yields the result.

Reversing the order of summation in the terminating \(4F_{3}(1/4)\) series on the right-hand side of (3.5) gives a relation between two terminating \(4F_{3}(4)\) series:

**Proposition 3.3:** If \(n\) is a nonnegative integer, the following relation between two terminating \(4F_{3}(4)\) series holds:

\[
4F_{3} \left(-n, \frac{a}{2}, \frac{a + 1}{2}, b \right)_{4} = \frac{(-1)^{n}(1 + b - c - n + 2m)_{n-m}(1 - c - n)_{m}}{(c)_{n}}
\]

\[
= \frac{(-1)^{n}(1 + b - c - n)_{n+m}(1 - c - n)_{m}}{(c)_{n}(1 + b - c - n)_{2m}}
\]

\[
= \frac{(c - b)_{n}(1 + b - c)_{m}(1 - c - n)_{m}}{(c)_{n} \left(\frac{1 + b - c - n}{2}\right)_{m} \left(\frac{2 + b - c - n}{2}\right)_{m} 4^{m}}
\]

(3.8)

**Proof:** In Proposition 3.2, we reverse the order of summation in the \(4F_{3}(1/4)\) series on the right-hand side according to Equation (2.4). The right-hand side in (3.5) thus becomes

\[
\frac{(c - b)_{n}}{(c)_{n}} \frac{4F_{3} \left(-n, \frac{a}{2}, \frac{a + 1}{2}, b \right)}{(1 + a - c)_{n}} \left(1 + a - c, \frac{1 + b - c - n}{2}, \frac{2 + b - c - n}{2} \right)_{4}
\]

\[
= \frac{(c - b)_{n}(1 + b - c - n)_{n}(1 + b - c)_{n}\left(-\frac{1}{4}\right)_{n}}{(c)_{n}(1 + a - c)_{n} \left(\frac{1 + b - c - n}{2}\right)_{n} \left(\frac{2 + b - c - n}{2}\right)_{n}}
\]

\[
\times 4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}
\]

\[
= \frac{(c - b)_{n}(b)_{n}(1 + b - c)_{n}}{(1 + a - c)_{n}(1 + b - c - n)_{2n}}
\]

\[
\times 4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}
\]

\[
= \frac{(-1)^{n}(b)_{n}}{(1 + a - c)_{n}} \frac{4F_{3} \left(-n, \frac{c - b - n}{2}, \frac{c - b - n + 1}{2}, c - a - n \right)_{4}}{(1 + a - c)_{n}}
\]

(3.9)

and the result follows.
We define now
\[ T_n(a, b, c) = (1 + a - c)_n(c)_n \, _4F_3 \left( \begin{array}{c} -n, \frac{a}{2}, \frac{a + 1}{2}, b \\ a, 1 + a - c, c \end{array} \right). \] (3.10)

The function \( T_n \) defined above has the trivial invariance
\[ T_n(a, b, c) = T_n(a, b, 1 + a - c). \] (3.11)

Proposition 3.3 gives us a nontrivial invariance for \( T_n \):
\[ T_n(a, b, c) = T_n(c - b - n, c - a - n, c). \] (3.12)

The invariances (3.11) and (3.12) generate an invariance group \( G \) for the function \( T_n \). The next theorem lists all six resulting relations in the invariance group \( G \) and describes \( G \) as isomorphic to the symmetric group \( S_3 \):

**Theorem 3.4:** Let \( G \) be the invariance group for the function \( T_n(a, b, c) \) generated by (3.11) and (3.12) above. Then \( G \) is isomorphic to the symmetric group \( S_3 \) of order 6. Furthermore, the resulting six invariances for \( T_n(a, b, c) \) are given by:
\[ T_n(a, b, c) = T_n(a, b, c), \] (3.13)
\[ T_n(a, b, c) = T_n(a, b, 1 + a - c), \] (3.14)
\[ T_n(a, b, c) = T_n(c - b - n, c - a - n, c), \] (3.15)
\[ T_n(a, b, c) = T_n(c - b - n, c - a - n, 1 - b - n), \] (3.16)
\[ T_n(a, b, c) = T_n(1 + a - b - c - n, 1 - c - n, 1 + a - c), \] (3.17)
\[ T_n(a, b, c) = T_n(1 + a - b - c - n, 1 - c - n, 1 - b - n). \] (3.18)

**Proof:** By combining (3.11) and (3.12) in all possible ways, it is straight-forward to check that we obtain the relations (3.13)–(3.18).

It is well-known that there are two groups of order 6: the cyclic group \( \mathbb{Z}_6 \) and the symmetric group \( S_3 \). We directly compute the orders of the relations (3.13)–(3.18) in \( G \) to be 1, 2, 2, 3, 3, and 2, respectively. Based on the orders of these elements, in particular the lack of an element of order 6, the invariance group \( G \) must be isomorphic to the symmetric group \( S_3 \).

We next reparameterize the function \( T_n \) to emphasize the \( S_3 \)-symmetry further:

**Theorem 3.5:** Define the function \( U_n \) by
\[ U_n(x, y, z) = T_n \left( x - y - z, \frac{1 + 3x - y - z - 2n}{2}, \frac{1 + x + y - 3z}{2} \right). \] (3.19)

Then \( U_n \) is a symmetric function in its parameters \( x, y, z \), i.e. \( U_n \) is invariant under any of the six possible permutations of \( x, y, z \). Furthermore, the six invariances of \( U_n \) given by
\[ U_n(x, y, z) = U_n(x, y, z), \] (3.20)
\[ U_n(x, y, z) = U_n(x, z, y), \] (3.21)

\[ U_n(x, y, z) = U_n(y, x, z), \] (3.22)

\[ U_n(x, y, z) = U_n(y, z, x), \] (3.23)

\[ U_n(x, y, z) = U_n(z, x, y), \] (3.24)

\[ U_n(x, y, z) = U_n(z, y, x) \] (3.25)

correspond to the invariances (3.13)–(3.18), respectively, of \( T_n(a, b, c) \) given in Theorem 3.4.

**Proof:** The result in this theorem follows from a direct check.

\[ \square \]

### 4. Relations for the terminating \( 3F_2(4) \) series

In this section, we study consequences of the relations from the previous section. We derive as special cases relations between terminating \( 3F_2(4) \) hypergeometric series.

If we let \( b = a \) in Proposition 3.3, we obtain a relation between two terminating \( 3F_2(4) \) series:

\[
\begin{align*}
3F_2\left(-n, a, \frac{a+1}{2}, \frac{1}{1+a-c}, c \left| \frac{4}{4}\right.\right) \\
= \frac{(-1)^n(a)_n}{(1+a-c)_n} 3F_2\left(-n, c-a-n, \frac{1-a-n}{2}, c \left| \frac{4}{4}\right.\right).
\end{align*}
\] (4.1)

Let us define the function

\[
\widetilde{T}_n(a, c) = (1+a-c)n(c)_n 3F_2\left(-n, \frac{a+1}{2}, \frac{1}{1+a-c}, c \left| \frac{4}{4}\right.\right).
\] (4.2)

A trivial relation for \( \widetilde{T}_n \) is

\[
\widetilde{T}_n(a, c) = \widetilde{T}_n(a, 1+a-c).
\] (4.3)

Furthermore, Equation (4.1) gives the nontrivial relation

\[
\widetilde{R}_n(a, c) = \widetilde{R}_n(c-a-n, c).
\] (4.4)

Just like the invariance group for the function \( T_n(a, b, c) \) for the terminating \( 4F_3(4) \) series, the invariance group for \( \widetilde{T}_n(a, c) \) is isomorphic to the symmetric group \( S_3 \) as well. In fact, the invariance relations for \( \widetilde{T}_n \) follow from the invariance relations (3.13)–(3.18) for \( T \) upon setting \( b = a \) in each of the latter relations. Below is a list of the six invariances for \( \widetilde{T}_n(a, c) \):

\[
\begin{align*}
\widetilde{T}_n(a, c) &= \widetilde{T}_n(a, c), \\
\widetilde{T}_n(a, c) &= \widetilde{T}_n(a, 1+a-c), \\
\widetilde{T}_n(a, c) &= \widetilde{T}_n(c-a-n, c), \\
\widetilde{T}_n(a, c) &= \widetilde{T}_n(c-a-n, 1-a-n).
\end{align*}
\] (4.5)
\[
\tilde{T}_n(a, c) = \tilde{T}_n(1 - c - n, 1 + a - c), \quad (4.9)
\]
\[
\tilde{T}_n(a, c) = \tilde{T}_n(1 - c - n, 1 - a - n). \quad (4.10)
\]

Invariances (4.5)–(4.10) correspond to invariances (3.13)–(3.18), respectively.

If we reparameterize \( \tilde{T}_n(a, c) \) by
\[
\tilde{U}_n(x, y, z) = \tilde{T}_n \left( \frac{1 + 2x - y - z - 2n}{3}, \frac{2 + x + y - 2z - n}{3} \right), \quad (4.11)
\]
then \( \tilde{U}_n \) is invariant under all six permutations of \( x, y, z \). In fact, the invariances
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(x, y, z), \quad (4.12)
\]
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(x, z, y), \quad (4.13)
\]
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(y, x, z), \quad (4.14)
\]
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(y, z, x), \quad (4.15)
\]
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(z, x, y), \quad (4.16)
\]
\[
\tilde{U}_n(x, y, z) = \tilde{U}_n(z, y, x) \quad (4.17)
\]
correspond to the invariances (4.5)–(4.10), respectively, of \( \tilde{T}_n(a, c) \).

We next let \( b = 1 + a - c \) in Proposition 3.3. We obtain the following relation between two terminating \( 3F_2(4) \) series:
\[
\begin{align*}
3F_2 \left( \begin{array}{c}
-n, a, a + 1 \\
\frac{a}{2}, \frac{a + 1}{2}
\end{array} \right | 4 \\
\end{align*}
\]
\[
= (-1)^n 3F_2 \left( \begin{array}{c}
-n, 2c - a - n - 1, 2c - a - n \\
2, 2c - a - n - 1, c
\end{array} \right | 4 \). \quad (4.18)
\]
We note that relation (4.18) is different from relation (4.1) as the two \( 3F_2(4) \) series in (4.18) are different in type from the two \( 3F_2(4) \) series in (4.1).

If we define
\[
Q_n(a, c) = (1 + a - c)_n(c) \ n \ 3F_2 \left( \begin{array}{c}
-n, a, a + 1 \\
\frac{a}{2}, \frac{a + 1}{2}
\end{array} \right | 4 \), \quad (4.19)
\]
then (4.18) leads to
\[
Q_n(a, c) = Q_n(2c - a - n - 1, c), \quad (4.20)
\]
which is a relation of order 2. The function \( Q_n(a, c) \) does not have any trivial relations besides the identity, and thus the invariance group for \( Q_n(a, c) \) is isomorphic to the
symmetric group $S_2$ of order 2. In fact, if we define

$$W_n(x, y) = Q_n \left( x, \frac{1 + x + y + n}{2} \right),$$  \hspace{1cm} (4.21)

the nontrivial relation (4.20) for $Q_n(a, c)$ can be written as

$$W_n(x, y) = W_n(y, x).$$  \hspace{1cm} (4.22)

There is one more nontrivial relation for the series $3F_2 \left( \begin{array}{c} -n, \frac{a}{2}, \frac{a + 1}{2} \\ \frac{a}{2}, a, c \end{array} \right | 4 \right)$ that can be obtained from the relations (3.13)–(3.18) in Theorem 3.4: letting $b = 1 + a - c$ in (3.17) (or in (3.18)), we have

$$3F_2 \left( \begin{array}{c} -n, \frac{a}{2}, \frac{a + 1}{2} \\ \frac{a}{2}, a, c \end{array} \right | 4 \right) = (-1)^n (1 + a - c)_n 3F_2 \left( \begin{array}{c} -n, \frac{1 - n}{2}, 1 - c - n \\ \frac{1}{2}, c - a - n, 1 + a - c \end{array} \right | 4 \right).$$  \hspace{1cm} (4.23)

**Remark 4.1:** Let $a = 1$ in Equation (4.23) above. We obtain the relation

$$2F_1 \left( \begin{array}{c} -n, \frac{1}{2} \\ c \end{array} \right | 4 \right) = (-1)^n (2 - c)_n 3F_2 \left( \begin{array}{c} -n, \frac{1 - n}{2}, 1 - c - n \\ \frac{1}{2}, c - n - 1, 2 - c \end{array} \right | 4 \right).$$  \hspace{1cm} (4.24)

The special case $c = 1 + \gamma + \lceil \frac{n}{2} \rceil$, where $\gamma$ is an integer and $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number $x$, in the $2F_1(4)$ series on the left-hand side of (4.24) coincides with the special case $\alpha = 0$ of the function $\Omega_n(\alpha, \gamma)$ defined in [32, Equation (1)]. If we take the limit as $c \rightarrow 1 + \gamma + \lceil \frac{n}{2} \rceil$ in (4.24), we can find the corresponding values of $\Omega_n(0, \gamma)$. For example, the specific values of $\Omega_n(0, \gamma)$ given in [32] for $\gamma = -2, -1, 0, 1, 2$ can all be obtained as limits of our relation (4.24) as just described.

**Remark 4.2:** The special case $c = \gamma + a + \lceil \frac{n}{2} \rceil$, where $\gamma$ is an integer, in the $3F_2(4)$ series on the left-hand side of Equation (4.23) coincides with the special case $\alpha = 0$ of the function $\Omega^{n}_{\alpha, \gamma}(a)$ defined in [33, Equation (1.3)]. In fact, by taking the limit as $c \rightarrow \gamma + a + \lceil \frac{n}{2} \rceil$ in (4.23), we can find the corresponding values of $\Omega^{n}_{0, \gamma}(a)$. For example, the specific values of $\Omega^{n}_{0, \gamma}(a)$ given in [33] for $\gamma = -2, -1, 0, 1, 2$ can all be obtained as limits of our relation (4.23) as just described.
For the hypergeometric series \( pF_q \left( \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \mid z \right) \), we let the expression 
\[ pF_q \left( \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \mid z \right) \]
\( n \) denote the sum of the first \( n + 1 \) terms of the series, i.e.
\[
\left[ pF_q \left( \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \mid z \right) \right]_n = \sum_{k=0}^{n} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{k!(b_1)_k(b_2)_k \cdots (b_q)_k} z^k.
\]
(4.25)

Now let \( c = 1 + a + n \) in Proposition 3.3. We obtain the following curious result regarding
the sum of the first \( n + 1 \) terms of the divergent series \( 3F_2 \left( \frac{a + 1, a + n}{a + 1 + a + n} \mid 4 \right) \):
\[
\left[ 3F_2 \left( \frac{a + 1, a + n}{a + 1 + a + n} \mid 4 \right) \right]_n
= \frac{(b)_n}{n!} F_3 \left( \begin{array}{l}
-n, 1 + a - b, 2 + a - b, 1 + a + n - 1 + a - b, 1 - b - n, 1 + a + n
\end{array} \mid 4 \right).
\]
(4.26)

Letting \( c = 1 + a + n \) in (3.17) (or in (3.18)) and then combining the result with
(4.26), we obtain a formula for the sum of the first \( n + 1 \) terms of the divergent series
\( 3F_2 \left( \frac{-b - n, -b - a - 2n}{-b - 2n, 1 - b - n} \mid 4 \right) \):
\[
\left[ 3F_2 \left( \frac{-b - n, -b - a - 2n}{-b - 2n, 1 - b - n} \mid 4 \right) \right]_n
= \frac{(-1)^n(1 + a + n)^2n}{n!(1 + a)_n} F_3 \left( \begin{array}{l}
-n, 1 + a - b, 2 + a - b, 1 + a - b, 1 - b - n, 1 + a + n - 1 + a - b, 1 - b - n, 1 + a + n
\end{array} \mid 4 \right).
\]
(4.27)

5. Relations for the terminating \( 4F_3(1/4) \) series

We explore relations for terminating \( 4F_3(1/4) \) hypergeometric series in this section and
describe the structure of those relations. The relations for the terminating \( 4F_3(1/4) \) series correspond to the series reversals of the relations for the terminating \( 4F_3(4) \) series in
Section 3.

**Proposition 5.1:** If \( n \) is a nonnegative integer, the following relation between two terminating
\( 4F_3(1/4) \) series holds:
\[
4F_3 \left( \begin{array}{l}
-n, a, a - c - n, c
\end{array} \begin{array}{c}
\frac{1}{2}
\end{array} \mid 4 \right)
= \frac{(1 + c - a)_n(b - c)_n}{(1 - a)_n(b)_n}
\]
\[
\times 4F_3 \left( \begin{array}{l}
-n, 1 + c - b, 1 - b - n, c
\end{array} \begin{array}{c}
\frac{1}{2}
\end{array} \mid 4 \right).
\]
(5.1)

**Proof:** Reverse the order of summation in the two \( 4F_3(4) \) series in Proposition 3.3
according to (2.4), and then re-label \( 1 - a - n, 1 - b - n, \) and \( 1 - c - n \) with \( a, b, \) and \( c, \)
respectively.
\]
Let
\[ R_n(a, b, c) = (1 - a)_n(b)_n \frac{\Gamma(-n, a, a - c - n, c)}{\Gamma(a - n, 1 + a - n, b) \frac{1}{4}}. \]  
(5.2)

The function \( R_n \) has the trivial invariance
\[ R_n(a, b, c) = R_n(a, b, a - c - n). \]  
(5.3)

Proposition 5.1 gives us a nontrivial invariance for \( R_n \):
\[ R_n(a, b, c) = R_n(1 + c - b, 1 + c - a, c). \]  
(5.4)

The invariance group for the function \( R_n(a, b, c) \) generated by (5.3) and (5.4) is, just like the invariance group for \( T_n(a, b, c) \), isomorphic to the symmetric group \( S_3 \) of order 6. The invariance relations for \( R_n \) are given by
\[ R_n(a, b, c) = R_n(a, b, c), \]  
(5.5)
\[ R_n(a, b, c) = R_n(a, b, a - c - n), \]  
(5.6)
\[ R_n(a, b, c) = R_n(1 + c - b, 1 + c - a, c), \]  
(5.7)
\[ R_n(a, b, c) = R_n(1 + c - b, 1 + c - a, 1 - b - n), \]  
(5.8)
\[ R_n(a, b, c) = R_n(1 + a - b - c - n, 1 - c - n, a - c - n), \]  
(5.9)
\[ R_n(a, b, c) = R_n(1 + a - b - c - n, 1 - c - n, 1 - b - n), \]  
(5.10)

and these invariance relations correspond to the summation reversals of (3.13)–(3.18), respectively.

Next, we reparameterize \( R_n \) by defining \( V_n \) according to
\[ V_n(x, y, z) = R_n \left( x - y - z, \frac{2 + 3x - y - z - n}{2}, \frac{x + y - 3z - n}{2} \right). \]  
(5.11)

Then \( V_n(x, y, z) \) is invariant under all six permutations of \( x, y, z \). Furthermore, the invariances
\[ V_n(x, y, z) = V_n(x, y, z), \]  
(5.12)
\[ V_n(x, y, z) = V_n(x, z, y), \]  
(5.13)
\[ V_n(x, y, z) = V_n(y, x, z), \]  
(5.14)
\[ V_n(x, y, z) = V_n(y, z, x), \]  
(5.15)
\[ V_n(x, y, z) = V_n(z, y, x), \]  
(5.16)
\[ V_n(x, y, z) = V_n(z, y, x) \]  
(5.17)
correspond to the invariances (5.5)–(5.10), respectively, of \( R_n(a, b, c) \).
6. Relations for the terminating $3F_2(1/4)$ series

In this final section, we derive some consequences of the relations for the $4F_3(1/4)$ hypergeometric series from the previous section. We obtain relations for terminating $3F_2(1/4)$ series. We note that the relations for the terminating $3F_2(1/4)$ series in this section correspond to the series reversals of the relations for the terminating $3F_2(4)$ series in Section 4.

Let $b = a$ in Proposition 5.1. We obtain the following relation between two terminating $3F_2(1/4)$ series:

$$
3F_2\left(\frac{-n, a-c-n, c}{a-n, 1+a-n} \bigg| \frac{1}{4} \right) = \frac{(1+c-a)_n(a-c)_n}{(1-a)_n(a)_n}
\times 3F_2\left(\frac{-n, 1-a-n, c}{1+c-a-n, 2+c-a-n} \bigg| \frac{1}{4} \right).
$$

(6.1)

Define the function

$$
\tilde{R}_n(a, c) = (1-a)_n(a)_n 3F_2\left(\frac{-n, a-c-n, c}{a-n, 1+a-n} \bigg| \frac{1}{4} \right).
$$

(6.2)

A trivial relation for $\tilde{R}_n$ is

$$
\tilde{R}_n(a, c) = \tilde{R}_n(a, a-c-n).
$$

(6.3)

Furthermore, Equation (6.1) gives the nontrivial relation

$$
\tilde{R}_n(a, c) = \tilde{R}_n(1+c-a, c).
$$

(6.4)

The invariance group for the function $\tilde{R}_n$ is isomorphic to the symmetric group $S_3$. The six invariances of $\tilde{R}_n$ are given by

$$
\tilde{R}_n(a, c) = \tilde{R}_n(a, a),
$$

(6.5)

$$
\tilde{R}_n(a, c) = \tilde{R}_n(a, a-c-n),
$$

(6.6)

$$
\tilde{R}_n(a, c) = \tilde{R}_n(1+c-a, c),
$$

(6.7)

$$
\tilde{R}_n(a, c) = \tilde{R}_n(1+c-a, 1-a-n),
$$

(6.8)

$$
\tilde{R}_n(a, c) = \tilde{R}_n(1-c-n, a-c-n),
$$

(6.9)

$$
\tilde{R}_n(a, c) = \tilde{R}_n(1-c-n, 1-a-n),
$$

(6.10)

and they correspond to setting $b = a$ in relations (5.5)–(5.10), respectively, for the function $R_n(a, b, c)$.

If we reparameterize $\tilde{R}_n(a, c)$ by

$$
\tilde{V}_n(x, y, z) = \tilde{R}_n\left(\frac{2+2x-y-z-n}{3}, \frac{1+x+y-2z-2n}{3} \bigg| \frac{1}{4} \right),
$$

(6.11)

then $\tilde{V}_n$ is invariant under all six permutations of $x, y, z$. The invariances

$$
\tilde{V}_n(x, y, z) = \tilde{V}_n(x, y, z),
$$

(6.12)
\[ \tilde{V}_n(x, y, z) = \tilde{V}_n(x, z, y), \quad (6.13) \]
\[ \tilde{V}_n(x, y, z) = \tilde{V}_n(y, x, z), \quad (6.14) \]
\[ \tilde{V}_n(x, y, z) = \tilde{V}_n(y, z, x), \quad (6.15) \]
\[ \tilde{V}_n(x, y, z) = \tilde{V}_n(y, x, z), \quad (6.16) \]
\[ \tilde{V}_n(x, y, z) = \tilde{V}_n(z, y, x) \quad (6.17) \]
correspond to the invariances (6.5)–(6.10), respectively, of \( \tilde{R}_n(a, c) \).

Next, let us set \( b = a - c - n \) in Proposition 5.1. We obtain the following relation between two terminating \( 3F_2(1/4) \) series:

\[
3F_2\left(\frac{-n, a, c}{2}, \frac{a - n, 1 + a - n}{2} \bigg| \frac{1}{4}\right) = \frac{(1 + 2c - a)_n}{(1 - a)_n} 3F_2\left(\frac{-n, 1 + 2c - a + n, c}{2}, \frac{1 + 2c - a, 2 + 2c - a}{2} \bigg| \frac{1}{4}\right). \quad (6.18)
\]

We define the function

\[ M_n(a, c) = (1 - a)_n(1 + c - a)_n 3F_2\left(\frac{-n, a, c}{2}, \frac{a - n, 1 + a - n}{2} \bigg| \frac{1}{4}\right). \quad (6.19) \]

We note that Equation (6.18) gives us

\[ M_n(a, c) = M_n(1 + 2c - a + n, c), \quad (6.20) \]

which is a relation of order 2. Since the function \( M_n(a, c) \) does not have any trivial relations besides the identity, the invariance group for \( M_n(a, c) \) is isomorphic to the symmetric group \( S_2 \) of order 2. Furthermore, if we define

\[ L_n(x, y) = M_n\left(\frac{x + y - n - 1}{2}\right), \quad (6.21) \]

the nontrivial relation (6.20) for \( M_n(a, c) \) can be written as

\[ L_n(x, y) = L_n(y, x). \quad (6.22) \]

The other nontrivial relation for the series \( 3F_2\left(\frac{-n, a, c}{2}, \frac{1 + a - n}{2} \bigg| \frac{1}{4}\right) \) is given in the next proposition and it has two cases depending on whether \( n \) is even or odd:

**Proposition 6.1:** If \( n \) is a nonnegative integer, then the following two relations hold:

\[
3F_2\left(\frac{a}{2} - n, \frac{a + 1}{2} - n \bigg| \frac{1}{4}\right) = \frac{(-1)^n(2n)!c_n}{n!(1 - a)_{2n}} \times 3F_2\left(\frac{-n, 1 + c - a + n, a - c - n}{2}, \frac{1}{2}, 1 - c - n \bigg| \frac{1}{4}\right) \quad (6.23)
\]
and

\[
\begin{align*}
\quad 3F_2 \left( \begin{array}{c} -2n - 1, a, c \\ a - \frac{1}{2} - n, \frac{a}{2} - n \\
\end{array} \right| \frac{1}{4} \right) &= \frac{(-1)^n(1 + c - a + n)(2n + 1)!}{n!(1 - a)_{2n+1}} \\
\times 3F_2 \left( \begin{array}{c} -n, 2 + c - a + n, a - c - n \\ \frac{3}{2}, 1 - c - n \\
\end{array} \right| \frac{1}{4} \right) .
\end{align*}
\]

(6.24)

**Proof:** To prove (6.23), we start by replacing \( n \) with \( 2n \) in relation (5.9) (or (5.10)) for the terminating \( 4F_3(1/4) \) series. The resulting relation can be written as

\[
\begin{align*}
\quad 4F_3 \left( \begin{array}{c} -2n, a, a - c - 2n, c \\ \frac{a}{2} - n, \frac{a + 1}{2} - n, b \\
\end{array} \right| \frac{1}{4} \right) &= \frac{(c)_{2n}(1 + a - b - c - 4n)_{2n}}{(1 - a)_{2n}(b)_{2n}} \\
\times 4F_3 \left( \begin{array}{c} -2n, 1 + a - b - c - 2n, 1 - b - 2n, a - c - 2n \\ \frac{1 + a - b - c - 4n}{2}, \frac{2 + a - b - c - 4n}{2}, 1 - c - 2n \\
\end{array} \right| \frac{1}{4} \right) .
\end{align*}
\]

(6.25)

We now let \( b \to a - c - 2n \) in (6.25). The left-hand side of (6.25) turns into the left-hand side of (6.23). For the right-hand side of (6.25), we have

\[
\begin{align*}
\quad \lim_{b \to a - c - 2n} \left( \frac{(c)_{2n}(1 + a - b - c - 4n)_{2n}}{(1 - a)_{2n}(b)_{2n}} \right) \\
\times 4F_3 \left( \begin{array}{c} -2n, 1 + a - b - c - 2n, 1 - b - 2n, a - c - 2n \\ \frac{1 + a - b - c - 4n}{2}, \frac{2 + a - b - c - 4n}{2}, 1 - c - 2n \\
\end{array} \right| \frac{1}{4} \right) \\
= \frac{(c)_{2n}}{(1 - a)_{2n}(1 + c - a)_{2n}} \lim_{b \to a - c - 2n} \left( \frac{(1 + a - b - c - 4n)_{2n}}{b - a - c - 2n} \right) \\
\times \sum_{k=0}^{2n} \frac{(-2n)_k(1 + a - b - c - 2n)_k(1 - b - 2n)_k(a - c - 2n)_k}{k!(1 + a - b - c - 4n)_{2k}(1 - c - 2n)_k} \\
= \frac{(c)_{2n}}{(1 - a)_{2n}(1 + c - a)_{2n}} \\
\times \lim_{b \to a - c - 2n} \sum_{k=0}^{2n} \frac{(-2n)_k(1 + a - b - c - 2n)_k(1 - b - 2n)_k(a - c - 2n)_k}{k!(1 + a - b - c - 2n)_{2k-2n}(1 - c - 2n)_k} \\
= \frac{(c)_{2n}}{(1 - a)_{2n}(1 + c - a)_{2n}} \\
\times \sum_{k=n}^{2n} \frac{(-2n)_k(1 + c - a)_k(a - c - 2n)_k}{(1)_{2k-2n}(1 - c - 2n)_k} .
\end{align*}
\]

(6.26)

After the change of index \( k = n + m \), the right-hand side above becomes

\[
\frac{(c)_{2n}}{(1 - a)_{2n}(1 + c - a)_{2n}}
\]
which simplifies to the right-hand side of Equation (6.23), thus proving (6.23).

The proof of (6.24) follows similar lines: we start by replacing \( n \) with \( 2n + 1 \) in relation (5.9) (or (5.10)) for the terminating \( _4F_3(1/4) \) series and then let \( b \to a - c - 2n - 1 \) in the resulting relation in the same manner as above.

As a final result, we obtain an equation for the sum of the first \( n + 1 \) terms of a certain nonterminating \( _3F_2(1/4) \) hypergeometric series. We do this by letting \( b = -n \) in Proposition 5.1. We consequently obtain an equation for the sum of the first \( n + 1 \) terms of the nonterminating series \( _3F_2 \left( \frac{a,a-c-n}{a-n}, \frac{a-n+1+a-n}{2} \left| \frac{1}{4} \right. \right) \):

\[
\left[ _3F_2 \left( \frac{a,a-c-n}{a-n}, \frac{a-n+1+a-n}{2} \left| \frac{1}{4} \right. \right) \right]_n = \frac{(1 + c - a)_n(1 + c)_n}{n!(1 - a)_n} _4F_3 \left( \frac{-n,1+c+n,c,1}{2,2+c,1+c-a} \left| \frac{1}{4} \right. \right).
\]  

(6.28)

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