MULTIPLE-SHOT AND UNAMBIGUOUS
DISCRIMINATION OF VON NEUMANN MEASUREMENTS

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Abstract. We present an in-depth study of the problem of discrimination of von Neumann measurements in finite-dimensional Hilbert spaces. Specifically, we consider two scenarios: multiple-shot and unambiguous discrimination. In the case of multiple-shot discrimination, we focus on discrimination of measurements with the assistance of entanglement. Interestingly, we prove that in this setting all pairs of distinct von Neumann measurements can be distinguished perfectly (i.e. with the unit success probability) using only a finite number of queries. We also show that in this scenario queering the measurements in parallel gives the optimal strategy and hence any possible adaptive methods do not offer any advantage over the parallel scheme. In the second scenario we give the general expressions for the optimal discrimination probabilities with and without the assistance of entanglement. Finally, we show that typical pairs of Haar-random von Neumann measurements can be perfectly distinguished with only two queries.

\section{Introduction}

With the recent technological progress quantum information science is not anymore merely a collection of purely theoretical ideas. Indeed, quantum protocols of increasing degree of complexity are currently being implemented on more and more complicated quantum devices \cite{1,2} and are expected to soon yield practical solutions to some real-world problems \cite{3}. This situation motivates the need of certification and benchmarking of various building-blocks of quantum devices \cite{4,6}. Discrimination or quantum hypothesis testing constitute one of the paradigms for assessing the quality of parts of quantum protocols \cite{7,11}. In this work we present a comprehensive study of various scenarios of discrimination of von Neumann measurements on a finite-dimensional Hilbert space. Here, by von Neumann measurements we understand fine-grained projective measurements. The general problem of quantum channel discrimination has attracted a lot of attention in recent

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years. One of the first results was the study of discrimination of unitary operators \[12,13\]. Later, this has been extended to various settings such as multipartite unitary operations \[14\]. The follow-up paper \[15\] formulated necessary and sufficient the conditions under which quantum channels can be perfectly discriminated. Further works investigated the adaptive \[16–18\] and parallel \[19\] schemes for discrimination of channels. Finally, some asymptotic results on discrimination of typical quantum channels in large dimensions were obtained in \[20\] . Discrimination of quantum measurements, being a subset of quantum channels, is thus of particular interest. Some of the earliest results on this topic involve condition on perfect discrimination of two measurements \[21–25\].

Here we are interested in the following problem. Imagine we have an unknown device hidden in a black box. We know it performs one of the two possible von Neumann measurements \(\mathcal{P}_1\) or \(\mathcal{P}_2\). Generally, whenever a quantum state is sent through the box, the box produces, with probabilities predicted by quantum mechanics, classical labels corresponding to the measurement outcomes. Our goal is to find schemes that attain the optimal success probability for discrimination of measurements. The results contained in this work concern the following two scenarios:

**Multiple-shot discrimination**—In this setting we are allowed to use the black box containing von Neumann measurement many times. Furthermore, we can prepare any input state with an arbitrarily large quantum memory (i.e. we can use ancillas of arbitrary large dimensions) and we can perform any channel between usages of the black box. This allows to implement both parallel (see Fig. 1) as well as adaptive discrimination strategies (see Fig. 2). We focus on the case of entanglement-assisted discrimination. Interestingly, we prove that every pair of different von Neumann measurements can be distinguished perfectly (i.e. with zero error probability) using only a finite number of queries to the black box. Moreover, we show that in the multiple-shot scenario adaptive strategies do not offer advantage over the parallel queries to the black box measurement.

**Unambiguous discrimination**—This scenario is an analogue to the well-known scheme of unambiguous state discrimination \[26\]. Namely, for every query to the black box, the decision procedure outputs \(\mathcal{P}_1\), \(\mathcal{P}_2\), or the inconclusive answer. The latter means that the user cannot decide which measurement was contained in the black box. Importantly, we require that the procedure cannot wrongly identify the measurement (see Fig. 3).
The main contribution to this problem is the derivation of the general schemes which attain the optimal success probability both with and without the assistance of entanglement. We also present single-letter formulas for the optimal discrimination probability of von Neumann measurements for qubits.

The rest of the paper is organized as follows. First, in Section 2 we give a survey of the main concepts and notation used throughout this work (including the basic background on discrimination of quantum channels and measurements). In there, we show that the parallel discrimination scheme is optimal in this case. Then, in Section 3 we present our results for the scenario of multiple-shot measurement discrimination. The following Section 4 contains the results concerning the unambiguous discrimination of quantum measurements. We obtain an upper bound on the probability of success and show that the parallel scheme is again optimal. Lastly, in Section 5 we summarize our results and give some directions for future research.

2. Preliminaries and main concepts

By \( \mathcal{D}(\mathbb{C}^d) \) we will denote quantum states on \( \mathbb{C}^d \). A set of generalized measurements (POVMs) \( \mathbb{C}^d \) will be denoted by \( \text{POVM}(\mathbb{C}^d) \). A general quantum measurement \( \mathcal{M} \) on \( \mathbb{C}^d \) is a tuple of positive semidefinite operators \( \{M_i\} \) on \( \mathbb{C}^d \) that add up to identity on \( \mathbb{C}^d \) i.e. \( \mathcal{M} = (M_1, \ldots, M_n) \) with \( M_i \geq 0 \) and \( \sum_i M_i = 1 \). If a quantum state \( \sigma \) is measured by a measurement \( \mathcal{M} \), the outcome \( i \) is obtained with the probability \( p(i|\sigma, \mathcal{M}) = \text{tr}(\sigma M_i) \) (Born rule).

\footnote{For simplicity we restrict our attention to measurements with a finite number of outcomes.}
Therefore, a quantum measurement $M$ can be uniquely identified with a quantum channel

$$
\Psi_M(\sigma) = \sum_{i=1}^{n} \text{tr}(M_i \sigma) |i\rangle \langle i|
$$

where states $|i\rangle \langle i|$ are perfectly distinguishable (orthogonal) pure states that can be regarded as states describing the state of a classical register. In what follows we will abuse the notation and simply treat quantum measurements (denoted by symbols $M, N, P, ...$) as quantum channels having the classical outputs. Using this interpretation one can readily use the results concerning the discrimination of quantum channels for generalized measurements. In particular for entanglement-assisted discrimination of quantum channels we have a classic result due to Helstrom [27]. It states that the probability of correct discrimination $p_{\text{disc}}(\Phi, \Psi)$ between two quantum channels $\Phi$ and $\Psi$ is given by

$$
p_{\text{disc}}(\Phi, \Psi) = \frac{1}{2} + \frac{1}{4} \| \Phi - \Psi \|_\diamond,
$$

where $\| S \|_\diamond = \max_{\| X \|_1 = 1} \| (S \otimes \mathbb{I})(X) \|_1$ denotes the diamond norm of the superoperator $S$, any optimal $X$ is called a discriminator. Thus, if the value of the diamond norm of the difference of two channels is strictly smaller than two, then the two channels cannot be distinguished perfectly in a single-shot scenario.

In this work we will be concerned with von Neumann measurements i.e. projective and fine-grained measurements on a Hilbert space of a given dimension $d$. Von Neumann measurements $P$ in $\mathbb{C}^d$ are tuples of $d$ orthogonal projectors on vectors forming an orthonormal basis $\{|\psi_i\rangle\}_{i=1}^d$ in $\mathbb{C}^d$ i.e.

$$
P = (|\psi_1\rangle \langle \psi_1|, |\psi_2\rangle \langle \psi_2|, \ldots, |\psi_d\rangle \langle \psi_d|).
$$

In what follows we will use $P_1$ to denote the measurement in the standard computational basis. We will also use $P_U$ to denote the von Neumann measurement in the basis $|\psi_i\rangle = U|i\rangle$ for a unitary $d \times d$ matrix $U \in \mathcal{U}_d$. In other words, vectors $|\psi_i\rangle$ from Eq. (3) are columns of the matrix $U$. Consider now a general task of discriminating between two projective measurements $P_{U_1}$, $P_{U_2}$, and let $p_{\text{succ}}(P_{U_1}, P_{U_2})$ be the optimal probability for discriminating between these measurements (we do not specify what kind of discrimination task we have in mind). Then, due to the unitary invariance of the discrimination problem and from the (easily verifiable) identity $VP_UV^\dagger = P_{UV}$ we obtain that $p_{\text{succ}}(P_{U_1}, P_{U_2}) = p_{\text{succ}}(P_1, P_{U_1^\dagger U_2})$. Therefore, for any reasonable discrimination tasks, without loss of generality, we can limit ourselves to considering the problem of distinguishing between the measurement in the standard basis $P_1$ and another projective measurement $P_U$.

**Remark.** Our definitions distinguish projective measurements that differ only by ordering of elements of the basis. On the other hand, the notation $P_U$ is ambiguous because of the relation $P_U = P_{UE}$ valid for all diagonal $d \times d$ unitaries $E \in \mathcal{U}_d$. Note however that a set $\{UE| E \in \mathcal{U}_d\}$ uniquely specifies a projective measurement.
Distinguishability of quantum measurements is strictly related to the distinguishability of unitary channels. The prominent result \[28, 29\] gives an expression which makes calculating the diamond norm of the difference of unitary channels \(\Phi_U, \Phi_1\) substantially easier. It says that for a unitary matrix \(U\) we have

\[
\|\Phi_U - \Phi_1\|_\diamond = 2\sqrt{1 - \nu^2},
\]

where \(\nu = \min_{x \in W(U)} |x|\) and \(W(X)\) denotes the numerical range of the operator \(X\). Building on this result, in \[24\] the following characterization of the diamond between measurements was obtained

\[
\|P_U - P_1\|_\diamond = \min_{E \in DU} \|\Phi_{UE} - \Phi_1\|_\diamond.
\]

Therefore the distance between two von Neumann measurements is the minimal value of the diamond norm on the difference between optimally coherified channels \[30\].

Equation (5) will be of significant importance throughout this work. We will also make use of the dephasing channel denoted \(\Delta(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|\).

Finally, when talking about the eigenvalues of a unitary matrix \(U \in U_d\), we will follow convention that \(\lambda_1, \ldots, \lambda_d\) are ordered by their phases. The angle between \(\lambda_1\) and \(\lambda_d\) will be denoted by \(\Theta(U)\).

3. Multiple-shot discrimination

Multiple-shot discrimination is a natural generalization of the single-shot scheme studied in \[24\]. In there, we show that the problem at hand is closely related to the task of discriminating of unitary channels. Our goal here is to reduce parallel distinction of von Neumann measurements to a similar task for unitary channels. In this section, the probability of discrimination of von Neumann measurements will be studied on the grounds of the diamond norm.

Before we proceed to presenting our results on the discrimination of quantum measurements, we will show the optimality of the parallel scheme (see Fig. 4) in the case of distinguishability of unitary channels. This is stated in the following proposition, which is based on the results presented in \[13\].

**Proposition 1.** Let \(V_1 = UX_1U \cdots X_{N-1}U\) and \(V_2 = X_1 \cdots X_{N-1}\). Then

\[
\max_{X_1, \ldots, X_{N-1}} \|\Phi_{V_1} - \Phi_{V_2}\|_\diamond = \|\Phi_{U^{\otimes N}} - \Phi_{1^{\otimes N}}\|_\diamond.
\]

**Proof.** We will show first that the inequality \(\|\Phi_{V_1} - \Phi_{V_2}\|_\diamond \leq \|\Phi_{U^{\otimes N}} - \Phi_{1^{\otimes N}}\|_\diamond\) holds for every choice of \(X_1 \cdots X_{N-1}\). We will use the fact that the diamond norm of unitary channels is a function of \(\Theta\) and use the induction. The first step is trivial so assume that for \(k < N - 1\) we have

\[
\Theta \left( X_k^\dagger \cdots X_1^\dagger UX_1 U \cdots UX_k U \right) \leq k\Theta(U).
\]
Then, by the use of the inequality from Lemma 1 from \[13\] we obtain
\[
\Theta \left( X_{k+1}^{\dagger}X_{k}^{\dagger} \ldots X_{1}^{\dagger}UX_{1} \ldots UX_{k}X_{k+1}^{\dagger} \right) \\
= \Theta \left( X_{k}^{\dagger} \ldots X_{1}^{\dagger}UX_{1} \ldots UX_{k}X_{k+1}X_{k+1}^{\dagger} \right) \\
\leq \Theta \left( X_{k}^{\dagger} \ldots X_{1}^{\dagger}UX_{1} \ldots UX_{k}X_{k+1} \right) + \Theta \left( X_{k+1}X_{k+1}^{\dagger} \right) \\
\leq k \Theta(U) + \Theta(U) = (k + 1) \Theta(U).
\]

The equality is achieved for \( X = 1 \), thus the equality in \((6)\) is proven. \( \square \)

3.1. Parallel discrimination. The first step in our study is to extend Eq. \((5)\) to the parallel setting. Here we study the optimal form of the optimal matrix \( E \) in the parallel scheme. The following theorem states that it has a tensor product form. The proof of the Theorem requires a technical lemma which was used in the proof of Eq. \((5)\) in \[24\]. It is stated in Appendix \(A\).

**Theorem 1.** Let \( U \in \mathcal{U}_{d} \), \( \mathcal{P}_{U} \) be a von Neumann measurement and \( \mathcal{D} \mathcal{U}_{d} \) be the set of unitary diagonal matrices. Then
\[
\| \mathcal{P}_{U}^{\otimes N} - \mathcal{P}_{1} \|_{\diamond} = \min_{E \in \mathcal{D} \mathcal{U}_{d}^{N}} \| \Phi_{U}^{\otimes N} E^{\otimes N} - \Phi_{1} \|_{\diamond}.
\]

**Proof.** Consider first the case when \( \mathcal{P}_{U} \) and \( \mathcal{P}_{1} \) are perfectly distinguishable. Then \( \| \mathcal{P}_{U} - \mathcal{P}_{1} \|_{\diamond} = \min_{E \in \mathcal{D} \mathcal{U}_{d}^{N}} \| \Phi_{U} - \Phi_{1} \|_{\diamond} = 2 \). This happens if and only if there exists \( \rho \in \Omega_{d} \) such that \( \text{diag}(\rho U) = 0 \). Then for \( U^{\otimes N} \) we have
\[
\text{diag}(\rho^{\otimes N} U^{\otimes N}) = \text{diag}( (\rho U)^{\otimes N} ) = 0.
\]

This means that \( P_{U}^{\otimes N} \) and \( P_{1} \) are perfectly distinguishable and
\[
\| \mathcal{P}_{U}^{\otimes N} - \mathcal{P}_{1} \|_{\diamond} = \min_{F \in \mathcal{D} \mathcal{U}_{d}^{N}} \| \Phi_{U}^{\otimes N} F - \Phi_{1} \|_{\diamond} = 2.
\]

As the minimum taken over a subset cannot be smaller that the minimum of the set, then
\[
\| \mathcal{P}_{U}^{\otimes N} - \mathcal{P}_{1} \|_{\diamond} = \min_{E \in \mathcal{D} \mathcal{U}_{d}^{N}} \| \Phi_{U}^{\otimes N} E^{\otimes N} - \Phi_{1} \|_{\diamond}.
\]

Now, consider the second case when \( \mathcal{P}_{U} \) and \( \mathcal{P}_{1} \) are not perfectly distinguishable. Then, there exists an optimal matrix \( E_{0} \in \mathcal{D} \mathcal{U}_{d} \) such that \( \| \mathcal{P}_{U} - \mathcal{P}_{1} \|_{\diamond} = \| \Phi_{U} - \Phi_{1} \|_{\diamond} \). Let \( 1 = \lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{d} \), where \( \lambda_{k} = e^{i\alpha_{k}} \leq \lambda_{l} = e^{i\alpha_{l}} \iff \alpha_{k} \leq \alpha_{l} \) for \( \alpha \in [0, 2\pi) \).

Let us take states \( \rho_{1}, \rho_{d} \) defined as in Lemma \[1\].

We will study two cases. As for the first one, we assume that \( 0 \not\in \text{W}(U^{\otimes N}E^{\otimes N}) \).
Hence, as \( \text{diag}(\rho_{1}) = \text{diag}(\rho_{d}) \), then
\[
\text{diag}(\rho_{1}^{\otimes N}) = \text{diag}(\rho_{d}^{\otimes N})
\]

and \( \rho_{k}^{\otimes N} \) for \( k \in \{1, d\} \) lie on the subspaces spanned by the eigenvectors of the eigenvalues \( \lambda_{1} \) and \( \lambda_{d}^{N} \) respectively. As all the assumptions of Lemma \[1\] are fulfilled, we obtain
\[
\min_{\rho} \text{tr} \left( \rho U^{\otimes N} E^{\otimes N} \right) = \max_{F \in \mathcal{D} \mathcal{U}_{d}} \min_{\rho} \text{tr} \left( \rho U^{\otimes N} F \right).
\]
Hence
\begin{equation}
\|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond = \min_{F \in DU_d} \|\Phi_{U\otimes NF} - \Phi_1\|_\diamond = \|P_{U\otimes N} - P_1\|_\diamond
\end{equation}
where the last equality follows from Theorem 1 from [24]. Finally, as
\begin{equation}
\min_{E \in DU_d} \|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond \leq \|P_{U\otimes N_E\otimes N} - P_1\|_\diamond
\end{equation}
and
\begin{equation}
\min_{E \in DU_d} \|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond \geq \min_{F \in DU_d} \|\Phi_{U\otimes NF} - \Phi_1\|_\diamond,
\end{equation}
then eventually
\begin{equation}
\min_{E \in DU_d} \|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond = \|P_{U\otimes N} - P_1\|_\diamond.
\end{equation}

As for the second case, assume that $0 \in W(U\otimes N_E\otimes N)$. Then there exists $k < N$ such that $0 \in \text{conv}(\lambda_1, \lambda_d^{N-k}, \lambda_d^N)$ and there exists a probability vector $p = (p_1, p_2, p_3)$ such that
\begin{equation}
p_1 \lambda_1 + p_2 \lambda_d^{N-k} + p_3 \lambda_d^N = 0.
\end{equation}

Let us again take $\rho_1$ and $\rho_d$ as in Lemma [1]. Define a state
\begin{equation}
\rho = p_1 \rho_1^{\otimes N} + p_2 \left(\rho_1^{\otimes k} \otimes \rho_d^{\otimes N-k}\right) + p_3 \rho_d^{\otimes N}.
\end{equation}

We will show that $\text{diag}(\rho U_{\otimes N}) = 0$. Indeed
\begin{equation}
\begin{aligned}
\text{diag}(\rho U_{\otimes N}) &= \text{diag}\left(p_1 \lambda_1 \rho_1^{\otimes N} + p_2 \lambda_d^{N-k} \left(\rho_1^{\otimes k} \otimes \rho_d^{\otimes N-k}\right) + p_3 \lambda_d^N \rho_d^{\otimes N}\right) \\
&= p_1 \lambda_1 \text{diag}(\rho_1^{\otimes N}) + p_2 \lambda_d^{N-k} \text{diag}\left(\rho_1^{\otimes k} \otimes \rho_d^{\otimes N-k}\right) + p_3 \lambda_d^N \text{diag}(\rho_d^{\otimes N}) \\
&= \left(p_1 \lambda_1 + p_2 \lambda_d^{N-k} + p_3 \lambda_d^N\right) \text{diag}(\rho_1^{\otimes N}) = 0.
\end{aligned}
\end{equation}

Thus $\rho$ is an optimal state and
\begin{equation}
2 = \min_{F \in DU_d} \|\Phi_{U\otimes NF} - \Phi_1\|_\diamond \leq \min_{E \in DU_d} \|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond = 2.
\end{equation}

Hence
\begin{equation}
\min_{E \in DU_d} \|\Phi_{U\otimes N_E\otimes N} - \Phi_1\|_\diamond = \|P_{U\otimes N} - P_1\|_\diamond.
\end{equation}

\[\square\]

Now we are interested in calculating the number of usages of the black-box required for perfect discrimination. Let us recall here that in the case of distinguishing unitary operations this can always be achieved in a finite number of steps $N = \lceil \frac{\pi}{\Theta(U)} \rceil_{\mathbb{N}}$.

The following corollary states that in the parallel discrimination scheme any two (not equal) von Neumann measurements can always be perfectly distinguished by a finite number of uses $N = \lceil \frac{\pi}{\Theta(U)} \rceil$, where $\Theta(U)$ is an optimized version of $\Theta(U)$ i.e.
\begin{equation}
\Theta(U) = \min_{E \in DU} \Theta(UE).
\end{equation}
Using this notation we may rewrite Theorem 1 as

**Corollary 1.** Let $N \in \mathbb{N}$, $U \in \mathcal{U}_d$. The following holds

(i) if $NY(U) \geq \pi$, then $\|P_{U^\otimes N} - P_1\|_o = 2$;

(ii) if $NY(U) < \pi$, then $\|P_{U^\otimes N} - P_1\|_o = 2 \sin(\frac{N}{2} \Upsilon(U))$.

The structural characterization of multiple-shot discrimination of von Neumann measurements given above allows us to draw strong conclusions about distinguishability of generic pairs von Neumann measurements. In this work we restrict our attention to pairs of measurements distributed independently according to the natural distribution coming from the Haar measure $\mu(\mathcal{U}_d)$ [31].

**Theorem 2.** Consider two independently distributed Haar-random von Neumann measurements on $\mathbb{C}^d$, i.e. $P_U, P_V$, where $U \sim \mu(\mathcal{U}_d)$, $V \sim \mu(\mathcal{U}_d)$. Let $p_{\text{opt}}(P_{U^\otimes 2}, P_{V^\otimes 2})$ be the optimal probability of discrimination measurements $P_U$ and $P_V$ using two queries and assistance of entanglement. Then, we have the following bound

$$Pr_{U,V \sim \mu(\mathcal{U}_d)}(p_{\text{opt}}(P_{U^\otimes 2}, P_{V^\otimes 2}) < 1) \leq \frac{1}{2^d - 1}.$$ 

In other words, in the limit of large dimensions $d$, typical Haar-random von Neumann measurements are perfectly distinguishable with the usage of two queries and assistance of entanglement (the probability that they cannot be perfectly distinguished is exponentially suppressed as a function of $d$).

**Proof.** From the unitary invariance of the Haar measure and the symmetry of the problem of measurement discrimination it follows that the distribution of the random variable $p_{\text{opt}}(P_{U^\otimes 2}, P_{V^\otimes 2})$ is identical to that of $p_{\text{opt}}(P_{V^\otimes 2}, P_{U^\otimes 2})$. Consequently, we have

$$Pr_{U,V \sim \mu(\mathcal{U}_d)}(p_{\text{opt}}(P_{U^\otimes 2}, P_{V^\otimes 2}) < 1) = Pr_{U \sim \mu(\mathcal{U}_d)}(p_{\text{opt}}(P_{U^\otimes 2}, P_{V^\otimes 2}) < 1).$$

From Corollary 1 it follows that the condition $\|P_U - P_1\|_o \geq \sqrt{2}$ implies $\|P_{U^\otimes 2} - P_{1^\otimes 2}\|_o = 2$ and consequently it also perfect discrimination of two copies of measurements: $p_{\text{opt}}(P_{U^\otimes 2}, P_{1^\otimes 2}) = 1$. Therefore we have

$$Pr_{U \sim \mu(\mathcal{U}_d)}(P_{U^\otimes 2} < 1) \leq Pr_{U \sim \mu(\mathcal{U}_d)}(\|P_U - P_1\|_o \leq \sqrt{2}).$$
Using now the characterization given in Eq. (5) in conjunction to the formula Eq. (4) we obtain \[ \| \mathcal{P}_U - \mathcal{P}_1 \| \leq 2\sqrt{1 - |U_{11}|^2} \] (note that \( U_{11} = \text{tr}(|1\rangle \langle 1| U) \in W(U) \)). Using this and simple algebra we get

\[ (28) \quad \text{Pr}_{U \sim \mu(U)}(\mathcal{P}_{U_1 \otimes 2}, \mathcal{P}_{\varphi_2}) < 1) \leq \text{Pr}_{U \sim \mu(U)}\left(|U_{11}|^2 \geq \frac{1}{2}\right). \]

The right-hand side of the above inequality can be computed exactly using the property that for Haar-distributed \( U \) the random variable \( X = |U_{11}|^2 \) is distributed according to the Beta distribution \( p(X) = (d-1)(1-X)^{d-2} \) (see for instance Eq. (9) in [32]). The simple integration gives \( (1/2)^{d-1} \), which together with Eq. (26) gives the claimed result. \( \square \)

3.2. Optimality of the parallel scheme. In this subsection we will prove the following theorem

**Theorem 3.** Let \( U \in \mathcal{U}_d \). Consider the distinguishability of general quantum network with \( N \) uses of the black box in which there is one of two measurements - either \( \mathcal{P}_U \) or \( \mathcal{P}_1 \). Then the probability of correct distinction cannot be better then in the parallel scenario.

This theorem is in the spirit of the results obtained in [33] for discrimination of unitary channels. The rest of this section is devoted to the proof.

Without loss of generality we may assume that the adaptation is performed using only unitary operations. Indeed, using Steinspring dilation theorem, any channel might be represented via a unitary channel on a larger system followed by the partial trace operation. What is left to observe is that \( \| \text{tr}_B(X_{AB})\|_1 \leq \|X_{AB}\|_1 \) for arbitrary bipartite matrix \( X_{AB} \).

The sequential scheme is shown in Fig. 5 and can be expressed as a channel

\[ (29) \quad \Psi_U = (\Delta_{1,\ldots,N} \otimes \mathbb{1}) \Phi_{A_U}, \]

associated with a matrix \( A_U \). Here \( \Delta_{1,\ldots,N} \) is the dephasing channel on subsystems \( 1,\ldots,N \). The channel \( \Phi_{A_U} \) is shown in Fig. 6 and the exact form of this transformation can be found in Appendix B.
MULTIPLE-SHOT AND UNAMBIGUOUS DISCRIMINATION

\[ V(k) = \sum_{i_1, \ldots, i_k} |i_1, \ldots, i_k\rangle \langle i_1, \ldots, i_k| \otimes V^{(k)}_{i_1, \ldots, i_k}. \]

Assuming that matrix \( U \) is chosen in the optimal form as in (24) i.e. \( \mathcal{X}(U) = \Theta(U) \) we may calculate the distance between \( \Psi_U \) and \( \Psi_1 \) as

\[
\max_{\rho} \| (\Psi_U - \Psi_1)(\rho) \|_1 = \max_{\rho} \| [\Delta_{1, \ldots, N} \otimes \mathbb{1}](\Phi_A - \Phi_A') \|_1 \\
\leq \max_{\rho} \| (\Phi_A - \Phi_A')(\rho) \|_1 \leq \max_{\rho} \| (\Phi_U \otimes \mathbb{1} - \Phi_1')(\rho) \|_1 \\
= \| \Phi_{U \otimes N} - \Phi_1' \|_\diamond = \| P_{U \otimes N} - P_1' \|_\diamond,
\]

where we maximize over states \( \rho \) of appropriate dimensions. The first inequality follows from the properties of the induced trace norm and the second one follows from the optimality of the parallel scheme. Therefore the adaptive scenario does not give any advantage over the parallel scheme.

4. UNAMBIGUOUS DISCRIMINATION

The unambiguous discrimination of measurements \( P_1 \) and \( P_U \) can be understood as unambiguous discrimination [26] of states generated by the corresponding channel. Specifically, for a fixed input state \( \sigma \) the output states \( P_1(\sigma), P_U(\sigma) \) can be unambiguously discriminated using the measurement strategy \( \mathcal{M} = (M_1, M_U, M_I) \), where the first two effects represent conclusive answers and the last one corresponds to the inconclusive output of the procedure. For equal a priori probabilities of occurrence of \( P_1 \) and \( P_U \), as well as fixed \( \sigma \) and \( \mathcal{M} \), the success probability is given by

\[
p_u(P_1, P_U; \sigma, \mathcal{M}) = \frac{1}{2} \left( \text{tr}(M_1 P_1(\sigma)) + \text{tr}(M_U P_U(\sigma)) \right),
\]

where additionally the unambiguity condition has to be satisfied:

\[
\text{tr}(M_U P_1(\sigma)) = \text{tr}(M_1 P_U(\sigma)) = 0.
\]

The optimal success probability of unambiguous discrimination of measurements \( P_1, P_U \) without the assistance of entanglement can be now defined as as the maximum of (31) over all strategies that do not use entanglement.
Formally, we have
\begin{equation}
\tilde{p}_u(P_1, P_U) := \max_{\sigma \in D(C^d)} \max_{\mathcal{M} \in \text{POVM}(C^d)} p_u(P_1, P_U; \sigma, \mathcal{M}),
\end{equation}
where $\mathcal{M} \in \text{POVM}(C^d)$ is a three-outcome measurement on $C^d$ that additionally satisfies constrains (32). Likewise, the optimal entanglement-assisted unambiguous discrimination probability is given by
\begin{equation}
p_u(P_1, P_U) := \max_{\sigma \in D(C^d \otimes C^{d'})} \max_{M \in \text{POVM}(C^d \otimes C^{d'})} p_u(P_1, P_U; \sigma, M),
\end{equation}
where this time $\sigma$ is a (possibly entangled) state on the extended Hilbert space $C^d \otimes C^{d'}$. In what follows we will show that without loss of generality it is enough to consider the dimension $d'$ of ancilla equal to the dimension of the system $d$.

In the remaining of this section we will give our results for unambiguous discrimination of von Neumann measurements both with and without the assistance of entanglement.

4.1. Unambiguous discrimination without assistance of entanglement. At the first sight the problem of finding the optimal unambiguous discrimination of $P_1$ and $P_U$ looks very complicated due to high dimensional optimization from Eq. (33). However, the problem can be greatly simplified because states form the output of a quantum measurements are purely classical (diagonal in the fixed basis).

Let us fix the input state $\sigma \in D(C^d)$. Then, for every Hermitian operator $X$ on $C^d$ we have
\begin{equation}
\text{tr}(P_1(\sigma)X) = \text{tr}(P_1(\sigma)\tilde{X}), \quad \text{tr}(P_U(\sigma)X) = \text{tr}(P_U(\sigma)\tilde{X}),
\end{equation}
where $\tilde{X} = \text{diag}(X)$ is a dephased version of the operator $X$. Let now $\mathcal{M} = (M_1, M_U, M_?)$ be a measurement on $C^d$ satisfying Eq. (32). Then, by the virtue of (35) we have
\begin{equation}
\tilde{p}_u(P_1, P_U; \sigma, M) = \tilde{p}_u(P_1, P_U; \sigma, \tilde{M}),
\end{equation}
where $\tilde{M} := (\text{diag}(M_1), \text{diag}(M_U), \text{diag}(M_?))$ (it can be easily checked that $\tilde{M}$ is a POVM on $C^d$). Consequently, without loss of generality, for any fixed input state $\sigma$, we can restrict our attention to measurements $\mathcal{M}$ with diagonal effects.

From the unambiguity condition Eq. (32) we obtain that $M_1 \perp \text{supp}(P_U(\sigma))$ and $M_U \perp \text{supp}(P_1(\sigma))$. Therefore, the optimal measurements can be always chosen as projectors onto disjoint subsets $\Gamma, \Delta$ of $\{1, \ldots, d\}$. For any subset $A \subset \{1, 2, \ldots, N\}$ we define projectors
\begin{equation}
\Pi_A := \sum_{i \in A} |i\rangle \langle i|, \quad \Theta_A := U \Pi_A U^\dagger.
\end{equation}
Now, if the result of the final measurement is from the set $\Gamma$, we know with certainty that in the unknown measurement was $P_1$. Analogously, if the result was from the set $\Delta$, then we can be sure that there was the second
measurement in the unknown device. The formula for the success probability reads
\[
\tilde{p}_u(P_1, P_U; \sigma, \Gamma, \Delta) = \frac{1}{2} \text{tr}(\Pi_{\Gamma} \sigma) + \frac{1}{2} \text{tr}(\Theta_{\Delta} \sigma).
\]
Importantly, the input state \(\sigma\) satisfies \(\sigma \perp \Pi_{\Gamma^c}\) and \(\sigma \perp \Theta_{\Delta^c}\), where \(A^c\) denotes the complement of \(A\). For fixed subsets \(\Delta, \Gamma\), due to linearity, the maximum over \(\sigma\) equals \(\|P_{\Gamma^c}(\Pi_{\Gamma} + \Theta_{\Delta})P_{\Gamma^c}\|\), where \(\| \cdot \|\) denotes the operator norm and \(P_{\Gamma,\Delta}\) is the orthogonal projector onto \(\text{Span (}\{|i\rangle\}_{i \in \Gamma^c}) \cap \text{Span (}\{|j\rangle\}_{j \in \Delta^c})\). By optimizing over disjoint subsets \(\Delta, \Gamma \subset \{1, \ldots, d\}\) we obtain the following result.

**Theorem 4.** The optimal success probability of unambiguous discrimination, without the use of entanglement, between von Neumann measurements \(P_1\) and \(P_U\) is given by
\[
\tilde{p}_u(P_1, P_U) = \frac{1}{2} \max_{\Gamma, \Delta \subset \{1,2,\ldots,N\} : \Gamma \cap \Delta = \emptyset} \|P_{\Gamma^c}(\Pi_{\Gamma} + \Theta_{\Delta})P_{\Gamma^c}\|
\]
with \(P_{\Gamma,\Delta}\) defined as above.

**Remark.** The projector \(P_{\Gamma,\Delta}\) projects onto the intersection of supports of \(\Pi_{\Delta^c}\) and \(\Theta_{\Gamma^c}\). By the use of Theorem 4 from [34], we can express the optimal probability of unambiguous discrimination as
\[
\tilde{p}_u(P_1, P_U) = 2 \max_{\Gamma, \Delta \subset \{1,2,\ldots,N\} : \Gamma \cap \Delta = \emptyset} \|\Pi_{\Delta^c}(\Pi_{\Delta^c} + \Theta_{\Gamma^c})^{-1} \Theta_{\Delta}(\Pi_{\Delta^c} + \Theta_{\Gamma^c})^{-1} \Pi_{\Delta^c}

+ \Theta_{\Gamma^c}(\Pi_{\Delta^c} + \Theta_{\Gamma^c})^{-1} \Pi_{\Gamma^c}(\Pi_{\Delta^c} + \Theta_{\Gamma^c})^{-1} \Theta_{\Gamma^c}\|
\]
where \((\cdot)^{-1}\) denotes Moore-Penrose pseudo inverse [33]. Moreover, the optimal input state is the one which gives the above norm.

The calculation of \(\tilde{p}_u(P_1, P_U)\) is especially simple since the optimization over disjoint subsets can be carried out explicitly.

**Corollary 2.** In the case of qubit measurements optimal probability of unambiguous discrimination of \(P_1\) and \(P_U\) is is given by the following (discontinuous) function
\[
\tilde{p}_u(P_1, P_U) = \begin{cases} 
1 & \text{if } |U_{12}|^2 = 1 \\
\frac{1}{2}|U_{1,2}|^2 & \text{if } |U_{12}|^2 < 1 
\end{cases}
\]
In both cases the optimal input state can be chosen to be \(|1\rangle\langle 1|\).

**Remark.** The above considerations can be extended to unambiguous discrimination of multiple copies of von Neumann measurements applied in parallel. To this end, if we have access to \(N\) pararell queries to a black box measurements, it suffices to replace unitaries \(1\) by \(1^{\otimes N}\) and \(U\) by \(U^{\otimes N}\) in the above computations. Interestingly, in the contrast to unambiguous discrimination of quantum states [26], having access to two copies of black box measurements, sometimes allows to attain perfect discrimination. Specifically, consider the problem of discriminating between \(P_1\) and \(P_H\), where
MULTIPLE-SHOT AND UNAMBIGUOUS DISCRIMINATION

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]. Explicit computation shows that by taking the input state as \( |\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \) allows to perfectly distinguish between \( P_1 \otimes I_H \) and \( P_2 \otimes I_H \).

4.2. Unambiguous discrimination with assistance of entanglement.

Knowing that for perfect discrimination of quantum measurements the use of entanglement significantly improves the discrimination \[23\], we ask whether the same is true for unambiguous discrimination. As explained earlier, we treat the quantum measurement \( M \) with effects \( \{M_i\}_{i=1}^n \) as a measure and prepare channel with the classical output (see Eq.(1)). Consider now the situation in which a black-box measurement acts on a system extended by the ancilla space \( H_B \) (of some dimension \( d_1 \)). Without loss of generality we assume that the input state is pure i.e. \( \sigma = |\psi_{AB}\rangle\langle\psi_{AB}| \). Let \( X \) be a matrix such that \( |\psi_{AB}\rangle = \sum_{i,j=1}^{d,d_1} X_{ij} |i\rangle|j\rangle \). The action of the channel \( M \) on \( |\psi_{AB}\rangle\langle\psi_{AB}| \) can be expressed as

\[ (42) \quad M \otimes I_B (|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_{i=1}^n |i\rangle\langle i| \otimes X^TM_i^T X. \]

Now, if we apply one of two von Neumann measurements \( P_1 \) or \( P_U \), we get the following states

\[ (43) \quad P_1 \otimes I_B (|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_{i=1}^d |i\rangle\langle i| \otimes X^T|i\rangle\langle i| X, \]

\[ (44) \quad P_U \otimes I_B (|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_{i=1}^d |i\rangle\langle i| \otimes X^T\bar{U}|i\rangle\langle i| U^TX. \]

By using analogous reasoning to the one given in the preceding section it can be shown that for a given input state \( |\psi_{AB}\rangle \), the optimal measurement unambiguously discriminating the above two states has the following structure

\[ (45) \quad N = \sum_{i=1}^d |i\rangle\langle i| \otimes T_i, \]

where \( T_i \) is a POVM on \( H_B \). In other words, the optimal discrimination strategy amounts to first reading off a classical label \( i \) and then performing a measurement \( T_i \) on the auxiliary register. From Eq.(43) we see that upon obtaining the label \( i \), the state of the auxiliary subsystem is either

\[ (46) \quad |x_i\rangle\langle x_i| = p_i^{-1} X^T|i\rangle\langle i| X, \]

when measurement \( P_1 \) was performed, or it is given by

\[ (47) \quad |y_i\rangle\langle y_i| = q_i^{-1} X^T\bar{U}|i\rangle\langle i| U^TX, \]

if \( P_U \) was implemented. In the above formulas \( p_i, q_i \) are responsible for normalization. We assume that \( p_i > 0 \) and \( q_i > 0 \) (otherwise the specific outcome \( i \) does not occur). We see that states \( |x_i\rangle\langle x_i|, |y_i\rangle\langle y_i| \) are pure and
Theorem 5. The optimal success probability of unambiguous discrimination between von Neumann measurements $\mathcal{P}_1$ and $\mathcal{P}_U$ is given by

\[
\rho_u(\mathcal{P}_1, \mathcal{P}_U) = 1 - \min_{\rho} \sum_i |\langle i|\rho U|i\rangle|.
\]
These results give a nice geometric interpretation for the relationship between the diamond norm and the probability of unambiguous discrimination. This is depicted in Fig. 7. We start with a von Neumann measurement in a basis given by some unitary matrix $U$ and try to distinguish it from the measurement in the computational basis. We denote $U$’s eigenvalues as $\lambda_1, \ldots, \lambda_d$ ordered according to their phases and put $\angle(\lambda_1, \lambda_d) = \Upsilon(U)$. The dependence of the diamond norm and probability of unambiguous discrimination is clearly shown.

Remark 1. The above calculations can be easily extended to the case of parallel discrimination scheme. It suffices to substitute $U$ with $U^{\otimes N}$ and then we obtain that

$$p_u(P_{U^{\otimes N}}, P_1) = 1 - \min_\rho \sum_i |\langle i | \rho U^{\otimes N} | i \rangle|.$$ (55)

The following corollary states that in the qubit case the unambiguous discrimination with the assistance of entanglement always outperforms the unambiguous discrimination without the use of entanglement. On top of that, the special cases for which the use of entanglement does not give any advantage are described.

Corollary 3. Let $P_1$ and $P_U$ be two von Neumann measurements on a qubit. If $|U_{1,1}| \not\in \{0, 1\}$, then the probability in the case of entanglement-assisted unambiguous discrimination is given by

$$p_{\text{opt}} = 1 - |U_{1,1}|$$ (56)

and it is always greater than the probability without assistance of entanglement

$$p_u = \begin{cases} 1 & \text{if } |U_{1,2}|^2 = 1 \\ \frac{1}{2} |U_{1,2}|^2 & \text{if } |U_{1,2}|^2 < 1. \end{cases}$$ (57)

Moreover, if $|U_{1,1}| \in \{0, 1\}$, then $p_{\text{opt}} = p_{\text{opt}}$.

Basing on Remark 1, we note that the angle $\Upsilon(U)$ increases in the multiple-shot case with the number of queries. This is depicted in Fig. 8 for two- and three-shot scenarios.
4.3. Optimality of the parallel scheme. In the general scheme we are allowed to use conditional unitary transformations \( \{ X_i \} \) after each measurement, so our setting for discrimination is the same as in Fig. 6. Henceforth, we will assume that matrix \( U \) is chosen in the optimal form as in (24) i.e. \( \Upsilon(U) = \Theta(U) \). Let us denote
\[
|\tilde{x}_i\rangle = (|i\rangle \otimes \mathbb{1}_{N+1}) A_1 |\psi_{A,B}\rangle \\
|\tilde{y}_i\rangle = (|i\rangle \otimes \mathbb{1}_{N+1}) A_U |\psi_{A,B}\rangle.
\]
Using the same notation as was used in Section 3.2 and repeating the calculation from the single-shot scenario we can upper-bound the probability of success discrimination.
\[
p_u(\Psi_U, \Psi_1) \leq 1 - \sum_i |\langle \tilde{x}_i | \tilde{y}_i \rangle| \leq 1 - \sum_i |\langle \tilde{x}_i | \tilde{y}_i \rangle| = 1 - |\langle \psi_{A,B} | A_1^\dagger A_U |\psi_{A,B}\rangle|.
\]
According to Proposition 1 we know there exists a state \( |\phi\rangle \) such that
\[
|\langle \psi_{A,B} | A_1^\dagger A_U |\psi_{A,B}\rangle| \geq |\langle \phi | U^{\otimes N} |\phi\rangle|.
\]
This implies that our probability is upper-bounded by \( 1 - \min_\rho \sum_i |\langle i | \rho U^{\otimes N} |i\rangle| \), which is achievable in parallel scheme as in Remark 1.

5. Conclusions and open problems

We have presented a comprehensive treatment of the problem of discrimination of von Neumann measurements. First of all, we showed that for two measurements \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), \( \mathcal{P}_1 \neq \mathcal{P}_2 \), there exists a finite number \( N \) of uses of the black box which allows us to achieve perfect discrimination. This is formally stated in Corollary 1. We also proved that the parallel discrimination scheme is optimal in the scenario of multiple-shot discrimination of von Neumann measurements (see Theorem 3).

Moreover, we studied unambiguous discrimination of von Neumann measurements. Our main contribution to this problem was the derivation of the general schemes that attain the optimal success probability both with (see Theorem 5) and without (see Theorem 4) the assistance of entanglement. Interestingly, for entanglement-assisted unambiguous discrimination the optimal success probability is functionally related to the corresponding
success probability for minimal error discrimination. Finally, we show that the parallel scheme is also optimal for unambiguous discrimination.

There are many interesting directions for further study that still remain to be explored. Below we list the most important (in our opinion) open research problems:

- Generalization of our results from projective measurements to other classes of measurements such as projective-simulable measurements \[37\], measurements with limited number of outcomes \[38\], or general quantum measurements (POVMs).
- Systematical study of the problem of unambiguous discrimination of projective measurements in the multiple-shot regime.
- Can typical pairs of Haar-random projective measurements on \(\mathbb{C}^d\) be discriminated perfectly using only one query and the assistance of entanglement as \(d \to \infty\)?
- How much entanglement is needed to attain the optimal success probability of multiple-shot discrimination of generic projective measurements on \(\mathbb{C}^d\)? In the same scenario, is it necessary to adopt the final measurement to the pair of measurements to be discriminated?

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APPENDIX A. LEMMA 5 FROM \[24\]

In this section we recall our previous results. This lemma is the basis of some of the proofs presented in the main part of this work.

**Lemma 1.** Let

- \(E_0 \in \mathcal{D}\mathcal{U}_d\) and \(U \in \mathcal{U}_d\), \(D(E) = \min_{\rho \in \mathcal{D}(\mathbb{C}^d)} |\text{Tr} \rho U E|\),
- \(D(E_0) > 0\),
- \(\lambda_1, \lambda_d\) denote the eigenvalues of \(UE_0\) such that the arc between them is the largest,
- \(P_1, P_d\) denote the projectors on the subspaces spanned by the eigenvectors corresponding to \(\lambda_1, \lambda_d\).

Then, the function \(|\text{Tr}(\rho U E)|\) has saddle point in \((\rho_0, E_0)\) if and only if there exist states \(\rho_1, \rho_d\) such that

- \(\rho_1 = P_1 \rho_1 P_1\),
- \(\rho_d = P_d \rho_d P_d\),
- \(\text{diag}(\rho_1) = \text{diag}(\rho_d)\).
APPENDIX B. EXPLICIT FORM OF THE MATRIX $A_U$

The matrix $A_U$ is a general operation which allows for adaptive information processing in the sequential discrimination scenario.

\[
A_U = (\mathbb{1}_{1, \ldots, N-1} \otimes U \otimes \mathbb{1}_{N+1}) \left( \sum_{i_{N-1}} |i_{N-1}\rangle \langle i_{N-1}| \otimes V_{i_{N-1}, N+1}^{i_{N-1}} \right) \\
(\mathbb{1}_{1, \ldots, N-2} \otimes U \otimes \mathbb{1}_{N,N+1}) \left( \sum_{i_{N-2}} |i_{N-2}\rangle \langle i_{N-2}| \otimes V_{N-1, N-1, N+1}^{i_{N-2}} \right) \\
\ldots \\
(\mathbb{1} \otimes U \otimes \mathbb{1}_{2, \ldots, N+1}) \left( \sum_{i_1} |i_1\rangle \langle i_1| \otimes V_{2, N+1}^{i_1} \right) (U \otimes \mathbb{1}_{2, \ldots, N+1}).
\]

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