The Edwards–Wilkinson equation with drift and Neumann boundary conditions

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Abstract

The well-known scaling of the Edwards–Wilkinson equation is essentially determined by dimensional analysis. In a range of experimental setups, be it due to the presence of an electrical or a gravitational field, or the indirect effect of other terms or an expansion, an additional drift term has to be considered. Once the drift term is added, more sophisticated reasoning is required to determine the scaling, which initially suggests that the drift term dominates the diffusion. However, the diffusion term is dangerously irrelevant and the resulting scaling in fact non-trivial. In order to assess the universality of the Edwards–Wilkinson equation with drift and to describe a physically more relevant situation, we compare the scaling obtained with Neumann boundary conditions to the published case with Dirichlet boundary conditions.

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1. Introduction

The Edwards–Wilkinson equation [1] is very well understood. It is a stochastic equation of motion describing the most basic surface evolution of a growth model, consisting merely of diffusion and random particle deposition. Given its minimal parameterization in terms of a diffusion constant $D$, a noise amplitude $\Gamma$ as well as the system size $L$ and time $t$, equations (1) and (2), the scaling of its roughness in $L$ and $t$ is determined by dimensional analysis, because the only dimensionless quantity is $Dt/L^2$. This changes, as soon as another interaction term is added whose coupling allows an additional dimensionless quantity to be formed. In principle, the scaling is then not determined by simple dimensional arguments. In this situation, it is common to invoke physical arguments to show that one of the terms is (infrared) irrelevant and thus can be dropped [2–4]. There is no mathematically rigorous argument underpinning this procedure and it therefore can produce incorrect results. This has been analysed in the past.
for the addition of a drift (convection) term to the Edwards–Wilkinson equation, which first seems to render the diffusion irrelevant, when in fact it becomes dangerously irrelevant [8] and thus should not be dropped from the analysis.

The problem was originally introduced and subsequently solved by Derrida, Lebowitz, Speer and Spohn in the context of an interface in Toom’s cellular automaton [6], which was pinned on its two ends (Dirichlet boundary conditions): ‘the key idea is that the translational velocity of excitations along the interface converts a \( t^\alpha \) growth of fluctuations in time to a \( x^\alpha \) growth of fluctuations along the interface’. A drift term in a system with Dirichlet boundary conditions thus changes the scaling behaviour quite significantly, rendering the roughness exponent \( \alpha \) equal to the roughening exponent \( \beta \), which remains unchanged. On the other hand, it is trivial to show that in the presence of periodic boundary conditions the drift does not enter the roughness (in the definition below) at all. Thus, one might wonder how many fluctuations of the interface at the boundaries have to be suppressed to trigger the change in scaling as observed in the presence of Dirichlet boundary conditions. Secondly, although the mechanism quoted above seems to apply quite generally, it is a priori not clear whether boundary conditions other than Dirichlet produce the same scaling (regarding exponents and amplitudes). We therefore ask: how universal is the Edwards Wilkinson equation with drift?

These questions are addressed in the following by deriving the leading order behaviour of the Edwards–Wilkinson equation in the presence of drift and Neumann boundary conditions. The calculations turn out to be far more involved than in the Dirichlet case, due to the particular structure of Green’s function.

It is not surprising if boundary conditions change some universal quantities at the critical point [9], certainly not if there is some net transport across the system as in the present case. Given that the system is apparently sensitive to the choice of boundary condition, the finding below might come as a surprise: Neumann and Dirichlet boundary conditions produce, in fact, the same leading order behaviour.

Beyond its role of probing scaling and universality in the Edwards–Wilkinson equation, the drift term has a simple physical motivation. It could, for example, be caused directly by a gravitational [10] or an electric field, present during epitaxial growth, or effective, generated by the non-equilibrium conditions of the surface diffusion as found in many models of epitaxial growth [11]. A drift is thus a very common observation.

To make the connection to experiments complete, we would have liked to study a zero flux boundary conditions, i.e. impose that there is no net flux of particles across the boundary. This problem poses major technical difficulties and suggests to implement Neumann boundary conditions as a stepping stone first. These are a significant improvement over the Dirichlet case, which corresponds to particles falling off the edge of the substrate and being generated whenever a hole forms at the boundary. The issue of an imperfect implementation of zero-flux boundary conditions appears less serious if the Edwards–Wilkinson equation under consideration is to describe only the effective behaviour of a growth model, as it is derived from an expansion to lowest order. Some reservations remain, however, as zero flux boundary conditions render the deterministic part of the Langevin equation conservative, i.e. they add a symmetry, which we do not capture.

1.1. Preliminaries

The features of the original Edwards–Wilkinson (EW) equation [1]

\[
\partial_t \phi(x, t) = D \nabla^2 \phi(x, t) + \eta(x, t)
\]  

(1)
have been reviewed a number of times [2, 12–15]. The field $\phi(x, t)$ describes the height of an interface over a substrate of dimension $d$ and linear extension $L$ at position $x$ and time $t$. The noise $\eta(x, t)$ is Gaussian and white, characterized by the usual correlator
\begin{equation}
\langle \eta(x, t) \eta(x', t') \rangle = \Gamma^2 \delta(x-x') \delta(t-t'),
\end{equation}
with amplitude $\Gamma^2$ and $\langle \eta \rangle = 0$. The operation $\langle \cdot \rangle$ describes the averaging over the noise, such that, for example, $\langle \phi(x, t) \rangle = D \nabla^2 \langle \phi(x, t) \rangle$. In the context of growth phenomena, the observable of choice is often the roughness, which is the ensemble average of the spatial variance of the field $\phi(x, t)$,
\begin{equation}
\omega^2(L, t) = \langle \phi(x, t)^2 - \phi(x, t)^2 \rangle,
\end{equation}
where $\tau$ describes the spatial average
\begin{equation}
\tau = \frac{1}{L^d} \int_{L^d} dx
\end{equation}
over the entire substrate of volume $L^d$. The roughness is closely related to the correlation function $C(x, x', t) = \langle (\phi(x, t) - \phi(x', t))^2 \rangle$, as
\begin{equation}
\omega^2(L, t) = \frac{1}{2} \frac{1}{L^d} \int_{L^d} d^d x' \frac{1}{L^d} \int_{L^d} d^d x C(x, x', t).
\end{equation}

The scaling of the roughness in time is characterized by the universal dynamical exponent $z$ and the finite size scaling in $L$ by the universal roughness exponent $\alpha$, which is summarized in the Family–Vicsek scaling hypothesis [16]:
\begin{equation}
\omega^2(L, t) = a L^{2\alpha} G \left( \frac{t}{b L^z} \right),
\end{equation}
where $a$ and $b$ are two non-universal metric factors [9]. The (universal) dimensionless scaling function $G(x)$ mediates between two regimes, as $\lim_{x \to \infty} G(x) = G_{\infty}$ with $0 < G_{\infty} < \infty$ and $G(x) \sim x^{\beta z}$ for $x \to 0$ with the roughening exponent $\beta$ obeying $z \beta = \alpha$. Taking the limit $t \to \infty$ first, the roughness thus scales like $\omega^2(L, t) \propto L^{\alpha z}$ in increasing $L$; taking the thermodynamic limit $L \to \infty$ first, it scales like $\omega^2(L, t) \propto t^{2 \beta}$ with increasing $t$. Subleading terms in $L$ and $t$ respectively are expected to be suppressed with increasing $t$ and $L$, respectively.

Given that the Edwards–Wilkinson equation in the form (1) is fully parameterized by $L$, $t$ and $D$, $\Gamma^2$, the only possible scaling is
\begin{equation}
\omega^2(L, t; D, \Gamma^2) = \frac{\Gamma^2}{D} L^{2-d} G \left( \frac{Dt}{L^2} \right),
\end{equation}
so that $\alpha = (2-d)/2$ and $z = 2$, unless $\omega^2$ does not exist, diverges or vanishes. In fact, at and above the upper critical dimension $d_c = 2$, the roughness is controlled by a lower cutoff or lattice spacing $a$, and diverges for $d > d_c$, like $a^{d-d_c}$ as $a \to 0$ [12].

In the following, the Edwards–Wilkinson equation is studied in $d = 1$ dimensions, where
\begin{equation}
\alpha = 1/2 \quad \beta = 1/4 \quad z = 2 \quad \text{ (standard EW) .}
\end{equation}
Provided that no other scales are present, the result (7) from dimensional analysis determines the scaling. Boundary conditions generally affect the scaling function $G(x)$, and the metric factors $a$ and $b$, (6), but not the exponents. Some types of interactions can be added to the Edwards–Wilkinson equation without affecting its scaling behaviour, which hints at the universality alluded to earlier. This can be illustrated by adding a term $-v (\nabla^2)^2 \phi$ on the right-hand side of equation (1), which results in a scaling of the roughness like
\begin{equation}
\omega^2(L, t; D, \Gamma^2, v) = \frac{\Gamma^2}{D} L^{2-d} G' \left( \frac{Dt}{L^2}, \frac{tv}{L^4} \right).
\end{equation}
Irrespective of \( t \), the parameter \( tv/L^4 \) vanishes with increasing \( L \) much faster than the parameter \( tD/L^2 \), so that \( w^2(L, t; D, \Gamma^2, v) \) approaches \( w^2(L, t; D, \Gamma^2, 0) \), assuming \( \tilde{G}(Dt/L^2, tv/L^4) \approx \tilde{G}(Dt/L^2) \) for sufficiently large \( L \) and irrespective of \( t \), in particular irrespective of whether \( t \) is held fixed or the limit \( t \to \infty \) is taken. This amounts to the statement that \(-v(V^2)^2\phi\) is asymptotically irrelevant, so that the scaling of the Edwards–Wilkinson equation of the original form (1) is recovered.

2. The Edwards–Wilkinson equation with drift

Whether a term is deemed relevant is a matter of the canonical dimension of the coupling. Changing (1) to the one-dimensional Edwards–Wilkinson equation with drift (EWd),

\[
\partial_t \phi(x, t) = D \partial_x^2 \phi(x, t) + v \partial_x \phi(x, t) + \eta(x, t),
\]

at first seems to suggest that \( D \) is irrelevant at any finite \( v \), suggesting a scaling of the form

\[
w^2(L, t; v, \Gamma^2) = \frac{\Gamma^2}{v} L^{1-d} \tilde{G} \left( \frac{tv}{L} \right),
\]

so that

\[
\alpha = 0 \quad \beta = 0 \quad z = 1 \quad \text{ (suspected EWd)}.
\]

This result, however, is obviously flawed, if periodic boundary conditions (PBC) are applied. In that case the drift term can be transformed away by a Galilean transformation, \( \tilde{\phi}(x, t) = \phi(x - vt, t) \), so that \( \tilde{\phi}(x, t) \) follows the original EW equation (1) and thus the roughness of the interface displays the scaling derived in (7),

\[
\alpha = 1/2 \quad \beta = 1/4 \quad z = 2 \quad \text{ (EWd with PBC)}.
\]

Since the initial guess (12) is based on a purely physical argument (rather than a mathematical one), this result merely questions the validity of this type of reasoning. It is clear that if \( v \) does not actually enter into the observable \( w^2 \) as defined in (3) because it can be transformed away, then it cannot dominate its scaling. As done below, where \( v \) cannot be transformed away, one might equally argue that \( D \) never becomes truly irrelevant [8]. A priori nothing can thus be said about the relevance of the couplings \( D \) and \( v \). The scaling of the roughness has to be written as

\[
w^2(L, t; D, \Gamma^2, v) = \frac{\Gamma^2}{D} L \tilde{G} \left( \frac{Dt}{L^2}, \frac{vt}{L} \right),
\]

which no longer fixes the scaling exponents \( \alpha, \beta \) and \( z \), as defined through equation (6), based on a scaling function on a single argument. The problem is the appearance of the new dimensionless parameter \( vt/L \), which can alter the asymptotic behaviour of \( w^2 \) in a completely unknown way. In fact, with Dirichlet boundary conditions (BC), \( \phi(x = 0, t) = \phi(x = L, t) = 0 \), it was found [7] that \( \tilde{G}(Dt/L^2, vt/L) \) scales like \( (D/(vL))^{1/2} \) as \( t \to \infty \) at fixed \( L \) and like \( (Dt/L^2)^{1/2} \) for \( L \to \infty \) at fixed \( t \), in summary

\[
w^2(L, t; D, \Gamma^2, v) = \frac{\Gamma^2}{D} L \tilde{G} \left( \frac{Dt}{L^2}, \frac{vt}{L} \right),
\]

with \( \tilde{G}(x) \to \text{ const as } x \to 0 \) and \( \tilde{G}(x) \propto \sqrt{x} \) as \( x \to \infty \) (with corrections in powers of \( Dt/L^2 \)), so that

\[
\alpha = 1/4 \quad \beta = 1/4 \quad z = 1 \quad \text{ (EWd with Dirichlet BC)}.
\]

Depending on the boundary condition, the additional scale \( v \) thus either leaves the scaling of the Edwards–Wilkinson equation unchanged (as seen in the case of periodic BC), or
changes them to anomalous values, which cannot be recovered from a simple dimensional analysis (as in the case of Dirichlet BC). The remainder of the present paper is dedicated to the question which scaling behaviour is generated by Neumann boundary conditions, \( \partial_x \phi(x = 0, t) = \partial_x \phi(x = L, t) = 0 \), which could lead, in principle, to a third set of exponents.

2.1. Neumann boundary conditions

In the following the Neumann condition \( \partial_x \phi(x, t) = 0 \) for \( x = 0, L \) will be analysed. To ease notation, it is helpful to absorb the various couplings and other dimensionful quantities in (10) into a redefinition of time, space, field and noise. With \( y = x/L \in [0, 1] \), \( \tau = Dt/L^2 \) and coupling \( q = vL/D \), equation (10) becomes

\[
\partial_\tau \phi(y, \tau) = \partial_y^2 \phi(y, \tau) + q \phi(y, \tau) + \xi(y, \tau),
\]

where

\[
\phi(y, \tau) = \sqrt{\frac{D}{L\Gamma^2}} \phi(x, t)
\]

and

\[
\xi(y, \tau) = \sqrt{\frac{L^3}{D\Gamma^2}} \eta(x, t),
\]

so that

\[
\langle \xi(y, \tau) \xi(y', \tau') \rangle = \delta(y - y') \delta(\tau - \tau').
\]

The Neumann boundary conditions correspond to

\[
\partial_y \phi(0, \tau) = \partial_y \phi(1, \tau) = 0.
\]

As suggested by naïve dimensional analysis, the coupling \( q \) seems to play an ever increasingly important role, as it diverges in the thermodynamic limit \( L \to \infty \). An alternative reparameterization of the drift that does not suffer from this problem is the dimensionless quantity \( u = tv^2/D \), which will be of great use below.

The formal solution of equation (17) for a given realization of the noise \( \xi(y, \tau) \)

\[
\phi(y, \tau) = \int_0^\tau d\tau' \int_0^1 dy' G(y, \tau - \tau'; y', q) \xi(y', \tau'),
\]

is based on Green’s function \( G(y, \tau; \tau', q) \) which describes the propagation of a Dirac delta peak at \( y' \) at time \( \tau' \) to \( y \) at time \( \tau \). It is determined by considering the deterministic, homogeneous partial differential equation (PDE)

\[
\partial_\tau G = \left( \partial_y^2 + q \partial_y \right) G,
\]

with initial condition \( \lim_{\tau \to 0} G(y, \tau; y_0, q) = \delta(y - y_0) \) and Neumann boundary conditions, which in turn can be constructed from a complete set of eigenfunctions of the Sturm–Liouville problem

\[
\lambda_n g_n(y) = \left( \partial_y^2 + q \partial_y \right) g_n(y).
\]

The operator can be made self-adjoint with the help of a suitable weight function [17]. \( \langle f | g \rangle = \int_0^1 dy e^{\eta y} f(y) g(y) \). The set of normalized eigenfunctions orthogonal with respect to this scalar product is then found as

\[
g_n(y) = e^{-\frac{1}{4} \eta y} \sqrt{\frac{2}{k_n^2 + (1/4)q^2}} \left[ k_n \cos(k_n y) + \frac{1}{2} q \sin(k_n y) \right],
\]

and \( \lambda_n = -(1/4)q^2 - k_n^2 \) for \( n \geq 1 \), where \( k_n = \pi n \). The eigenfunction without a node, \( n = 0 \), deviates from that pattern,

\[
g_0(y) = \sqrt{\frac{q}{e^\eta - 1}}.
\]
and $\lambda_0 = 0$. Having introduced a scalar product above that renders the differential operator self-adjoint, the temporal evolution of any initial distribution under the PDE (22) can be determined. On this basis, or equivalently, on the basis of completeness, Green’s function is found to be [18, p 63]

$$G(y, \tau; y_0, q) = \frac{q e^{qy}}{e^q - 1} + e^{-\frac{i}{2}q(y-y_0) - \frac{1}{2}q^2\tau} \sum_{n=1}^{\infty} \frac{2e^{-k_n^2\tau}}{k_n^2 + (1/4)q^2} \left(k_n \cos(k_n y) + \frac{1}{2} q \sin(k_n y_0)\right) \left(k_n \cos(k_n y) + \frac{1}{2} q \sin(k_n y)\right).$$

(26)

2.1.1. Poisson summation. In principle, Green’s function (26) could now be used in (21) and the asymptotic properties of the roughness (3) be determined. However, it very quickly turns out that the real space limit of large $L$ is very difficult to handle in Fourier space, $k_n$, and the sums appearing in (26) are practically intractable. After performing a Poisson summation [19, p 373], $G(y, \tau; y_0, q)$ can be written as

$$G(y, \tau; y_0, q) = \frac{q e^{qy}}{e^q - 1} - e^{-\frac{i}{2}q(y-y_0) - \frac{1}{2}q^2\tau} + e^{-\frac{i}{2}q(y-y_0) - \frac{1}{2}q^2\tau} \sum_{n=1}^{\infty} \frac{qe^{-k_n^2\tau}}{k_n^2 + (1/4)q^2} \left(k_n \sin(k_n (y + y_0)) - \frac{1}{2} q \cos(k_n (y + y_0))\right) + e^{-\frac{i}{2}q(y-y_0) - \frac{1}{2}q^2\tau} \frac{1}{\sqrt{4\pi \tau}}$$

$$\times \sum_{n=-\infty}^{\infty} \left(e^{-\frac{q(y-y_0+n^2\tau)^2}{2\tau}} + e^{-\frac{q(y+y_0+n^2\tau)^2}{2\tau}}\right).$$

(27)

In the following, the aim is to express equation (27) in terms of the free propagator on $\mathbb{R}$,

$$\Phi_0(y, \tau; q) = \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{(y-y_0)^2}{2\tau}},$$

(28)

which solves (22), so that none of the terms is expressed any longer in Fourier space, which facilitates integration and the determination of the asymptote using a saddle point approximation. In fact, including all pre-factors, the last sum in (27) can be written as

$$\sum_{n=-\infty}^{\infty} e^{qy} (\Phi_0(y - y_0 + 2n, \tau; q) + e^{qy_0}\Phi_0(y + y_0 + 2n, \tau; q)).$$

(29)

The first three terms in (27), the two exponentials and the sum, are much more difficult to handle. Taking the limit $\tau \to 0$ for these terms, they can be regarded as propagating by (28), an initial source $s(y, y_0)$, which can be written as

$$s(y, y_0) = \frac{q e^{qy}}{e^q - 1} + \frac{1}{2} q e^{-\frac{i}{2}q(y-y_0)} \sum_{n=1}^{\infty} \frac{1}{k_n^2 + (1/4)q^2} \left(k_n \sin(k_n (y + y_0)) - \frac{1}{2} q \cos(k_n (y + y_0))\right).$$

(30)

so that

$$G(y, \tau; y_0, q) = \int_{-\infty}^{\infty} d\tilde{y} s(\tilde{y}, y_0) \Phi_0(y - \tilde{y}, \tau; q)$$

$$+ \sum_{n=-\infty}^{\infty} e^{q(\Phi_0(y - y_0 + 2n, \tau; q) + e^{qy_0}\Phi_0(y + y_0 + 2n, \tau; q)).}$$

(31)
Taking the limit $\tau \to 0$ for the entire $G(y, \tau; y_0, q)$, \(27\) reveals

$$
\lim_{\tau \to 0} G(y, \tau; y_0, q) = s(y, y_0) + e^{-\frac{i}{4}(y-y_0)^2} \sum_{n=-\infty}^{\infty} \left( \delta(y - y_0 + 2n) + \delta(y + y_0 + 2n) \right),
$$
\(32\)

which means that $s(y, y_0) = 0$ for $y, y_0 \in [0, 1]$, since $\lim_{\tau \to 0} G(y, \tau; y_0, q) = \delta(y - y_0)$ on that interval, which implies that

$$
\sum_{n=-\infty}^{\infty} \frac{1}{k_n^2 + (1/4)q^2} \left(k_n \sin(k_n z) - \frac{1}{2} q \cos(k_n z) \right) = -\frac{2}{e^q - 1} e^{izq},
$$
\(33\)

for any $z \in (0, 2)$. Using the periodicity of the left-hand side, the domain is easily extended to $z \in \mathbb{R}$:

$$
\sum_{n=-\infty}^{\infty} \frac{1}{k_n^2 + (1/4)q^2} \left(k_n \sin(k_n z) - \frac{1}{2} q \cos(k_n z) \right) = -\sum_{n=-\infty}^{\infty} \frac{2}{e^q - 1} e^{i\tau z} I_\Omega(z),
$$
\(34\)

with the indicator function

$$
I_\Omega(z) = \begin{cases} 
1 & \text{for } z \in \Omega \\
0 & \text{otherwise}.
\end{cases}
$$
\(35\)

On this basis, Green’s function can be written as

$$
G(y, \tau; y_0, q) = \int_0^\tau d\tau' q e^{iq\tau} \sum_{n=-\infty}^{\infty} \left(1 - e^{i\tau q}(\Phi_0(y + y_0 + 2n - \bar{y}, \tau; q) + \Phi_0(y + y_0 + 2n, \tau; q)\right)
$$
\(36\)

Problems of convergence have not been addressed here in any detail, which could affect some of the manipulations done above, in particular when the order of integration and summation is changed. Yet, because of the free propagator $\Phi_0$ effectively posing an exponential cutoff on the sums as well as the integrals, all such convergence issues turn out to be harmless.

To verify that the propagator \(36\) indeed solves the PDE \(22\) is a matter of a straightforward calculation. Similarly, the initial condition $\lim_{\tau \to 0} G(y, \tau; y_0, q) = \delta(y - y_0)$ for $y, y_0 \in [0, 1]$ can be identified by mere inspection. The only slight difficulty are the boundary conditions $\partial_1 G(y, \tau; y_0, q) = 0$, which are most easily verified by writing the gradient as

$$
\partial_1 G(y, \tau; y_0, q) = e^{-\frac{i}{4}(y-y_0)^2} q \tau \frac{1}{\sqrt{4\pi \tau}} \sum_{n=-\infty}^{\infty} \frac{y_0 - 2n - q \tau}{2\tau} \left( e^{-\frac{(y_0 - 2n)^2}{4\tau}} - e^{-\frac{(y_0 + 2n)^2}{4\tau}} \right)
$$
\(37\)

\[2.2. Calculation of the roughness\]

The propagator \(36\) can be used in the formal solution \(21\), which provides the basis for calculating the roughness via \(3\) and \(18\),

$$
w^2(L, t) = \frac{L \Gamma^2}{D} \int_0^1 dy_1 \int_0^1 dy_2 \delta(y_1 - y_2) - 1 |\varphi(y_1, \tau)\varphi(y_2, \tau)|,
$$
\(38\)

where the ensemble average enters only through \(19\).
The integral on the right-hand side is the dimensionless roughness,
\[ \int_0^1 dy_1 \int_0^1 dy_2 [\delta(y_1 - y_2) - 1] \varphi(y_1, \tau) \varphi(y_2, \tau), \]
(39)
which depends on only two dimensionless parameters, \( \tau = Dt/L^2 \) and \( q = vL/D \). However, to prevent the relevant limits \( L \to \infty \) and \( t \to \infty \) from affecting both parameters at the same time,
\[ \tilde{w}^2(q, u) = q \int_0^1 dy_1 \int_0^1 dy_2 [\delta(y_1 - y_2) - 1] \varphi(y_1, \tau) \varphi(y_2, \tau), \]
(40)
with \( u = tv^2/D = q^2 \tau \) is advantageous, so that \( w^2(L, t) = (\Gamma_1^2/v) \tilde{w}^2(q, u) \), which also means that all scaling of the roughness can be read off the scaling of \( \tilde{w}^2 \) without any further prefactor such as \( L \) in (38).

Using the parameter \( u \) in favour of \( \tau \) needs to be implemented in the propagator as well, (36), using \( \tau = u/q^2 \). It is also useful to replace the integral over \( \tau \) in the formal solution (21) correspondingly by an integral over \( u \), so that
\[ \frac{v}{\Gamma^2} w^2(L, t) = \tilde{w}^2(q, u) = q^{-1} \int_0^1 dy_1 \int_0^1 dy_2 [\delta(y_1 - y_2) - 1] \]
\[ \times \int_0^1 dy' \int_0^u du' \tilde{G}(y_1, u'; y', q) \tilde{G}(y_2, u'; y', q), \]
(41)
where \( \tilde{G} \) is the re-parameterized propagator with all parameters explicitly appearing as arguments.

The bulk of the work is performing the integration (41). This can be done more conveniently by splitting the propagator up into four terms,
\[ \tilde{G}(y_i, u'; y', q) = A_i - B_i + C_i + D_i, \]
(42)
where
\[ A_i \equiv \frac{q e^{y_i'} e^{y_i}}{e^{q} - 1} \]
(43a)
\[ B_i \equiv \frac{q e^{y_i'} e^{y_i}}{e^{q} - 1} \sum_{n_i = -\infty}^{\infty} e^{q n_i} \int_0^2 d\tilde{y}_i \Phi_0(y_i + y' + 2n_i - \tilde{y}_i, u; q) \]
(43b)
\[ C_i \equiv \sum_{n_i = -\infty}^{\infty} e^{q n_i} \Phi_0(y_i - y' + 2n_i, u; q) \]
(43c)
\[ D_i \equiv \sum_{n_i = -\infty}^{\infty} e^{q (n_i + y')} \Phi_0(y_i + y' + 2n_i, u; q), \]
(43d)
with
\[ \Phi_0(y, u; q) = \frac{q}{\sqrt{4\pi u}} e^{-\frac{(y - \Gamma_1^2 q u)^2}{4u}}. \]
(44)
Because of the prefactor $[\delta(y_1 - y_2) - 1]$, for each of the ten distinct terms generated by using (2.2) in (41), effectively two different integrations have to be performed, namely one over $y_1$ and $y_2$ distinct, and one over $y_1 = y_2$. In addition, an integral over $y'$ and $u'$ needs to be performed, as well as over $\tilde{y}$ for term $B_i$. In all cases, a saddle point approximation (SPA) is used, specifically

$$
\frac{q}{\sqrt{4\pi u}} \int_\mathbb{A} dy \ e^{-\frac{(y - y_0)^2}{4q^2u}} = I_\mathbb{A}(y_0 - \frac{\tilde{u}}{q}) \quad \text{for} \quad \frac{\tilde{u}}{q^2} \ll 1.
$$

where $I_\mathbb{A}(y)$ is again an indicator function, (35). It is worth stressing that there are no further algebraic terms, i.e. all other contributions from the integral vanish exponentially in large $q^2/\tilde{u}$.

In most cases, further integrals over variables contained in the argument of the indicator function are to be performed. If the intersection of the integration range of the other variables and the set $\mathbb{A}$ above has finite measure, the result is straightforward to calculate. If, however, the intersection contains a single point, then the SPA has to be considered to be of higher order. In general, for each ‘marginal variable’ the resulting integral acquires an additional factor $\sqrt{u/q}$. For example, to leading order $\tilde{u}/q^2$,

$$
\int_0^1 dy_1 \int_0^1 dy_2 e^{-\frac{(y_1 - y_2 - 1/3)^2}{4q^2u}} \approx \frac{\sqrt{4\pi u}}{q} \int_0^1 dy_2 I(0,1)(y_1 + 1/3) = \frac{4\sqrt{\pi u}}{3q},
$$

using (45) in a non-marginal case (maximum of the exponential at $y_1 - y_2 = 1/3 \in (-1, 1)$), whereas replacing $1/3$ by 1 produces

$$
\int_0^1 dy_1 \int_0^1 dy_2 e^{-\frac{(y_1 - y_2)^2}{4q^2u}} \approx \frac{2u}{q^2},
$$

rather than $\frac{\sqrt{4\pi u}}{q} \int_0^1 dy_2 I(0,1)(y_1 + 1) = 0$. Integrals like that result in subleading terms, whose amplitude is not normally calculated in the following. Instead, only the power of $q$ is noted and by comparison to other terms it is verified that it is safe to ignore it. The same result is recovered by power counting in the four terms (2.2), where each integral gives rise to a leading order $q^{-1}$, and thus each of the four terms has the same algebraic dependence on $q$.

Further details of the calculation are exemplified in the appendix. Combining all contributions gives

$$
u^2_{\text{physical}} = \frac{\Gamma^2}{v} q^{-1} \int_0^1 dy_1 \int_0^1 dy_2 \delta(y_1 - y_2) - 1 \ (A_1 - B_1 + C_1 + D_1) \ (A_2 - B_2 + C_2 + D_2).
$$

For $u$ small, looking at the limit $q \to \infty$, to leading order:

$$
\lim_{q \to \infty} \nu^2_{\text{physical}} = \frac{\Gamma^2}{v} \sqrt{\frac{u}{2\pi}}.
$$

For $q$ large and considering the limit $u \to \infty$, to leading order:

$$
\lim_{u \to \infty} \nu^2_{\text{physical}} = \frac{\Gamma^2}{v} \frac{2}{3\sqrt{2\pi}} \sqrt{q}.
$$
3. Discussion and conclusion

Upon replacing $u$ and $q$ by their definitions, $u = tv^2/D$ and $q = vL/D$, the two asymptotes for the roughness derived in equations (49) and (48) are

$$w^2(t, L) = \Gamma^2 \times \begin{cases} \sqrt{\frac{t}{2\pi D}} & \text{for } L \to \infty \\ \frac{2}{3} \sqrt{\frac{L}{2\pi Dv}} & \text{for } t \to \infty, \end{cases}$$

which is, to leading order, identical to the result in [7] for Dirichlet boundary conditions. The exponents as defined in (6),

$$\alpha = \frac{1}{4}, \quad \beta = \frac{1}{4}, \quad z = 1 \quad \text{(EWd with Neumann BC)}$$

therefore reproduce (16). On the one hand, this is a surprising result, because after realizing that the usual dimensional arguments do no longer apply, any exponents are mathematically possible. The fact that Neumann boundary conditions reproduce the anomalous results of the Dirichlet case (contrasting those for periodic boundary conditions) even down to the amplitudes therefore point to some universal mechanism at work in both equations. Physically, on the other hand, arguments very similar to those discussed in [5–7] apply: the drift with velocity $v$ effectively constantly re-initializes the interface with vanishing slope from one side to the other, while constantly under the influence of the external noise. This mechanism erases practically all features that develop over times exceeding $L/v$ and thus in saturation, $t \to \infty$, reduces the roughness to about the value after time $L/v$. In fact, the stationary roughness is $2/3$ of what is extrapolated from initial roughening:

$$\lim_{t \to \infty} w^2(t, L) = \frac{2}{3} \lim_{L \to \infty} w^2(L/v, L').$$

Considering, however, the whole range of possible boundary conditions, one can ask how many fluctuations at the boundary are needed to restore the scaling of the Edwards–Wilkinson equation without drift term. Drift in the presence of periodic boundaries, $\phi(0, t) = \phi(L, t)$, does not affect the scaling of the roughness, because the drift is gauged away in the co-moving frame. That suggests that time-dependent Dirichlet boundary conditions might have a similar effect, $\phi(0, t) = \psi_0(t)$, $\phi(L, t) = \psi_L(t)$, provided that $\psi_0, L(t)$ displays a suitable spectrum of fluctuations. In fact, only one boundary, namely the one the drift points away from (the right boundary for $v > 0$) can affect the scaling. One might conclude that in the presence of drift, any boundary condition that suppresses fluctuations leads to the exponents confirmed in this paper for the Neumann case: $\beta$ remains unchanged compared to the system without drift, whereas $\alpha$ changes to $\beta$; so that $z = 1$, unless $z < 1$ initially, in which case the drift is not expected to have any impact at all, because the interface saturates before being re-initialized [7].

One might wonder whether the mechanism discussed above can be translated to interfacial phenomena in the presence of quenched noise, i.e. whether the scaling of a system with quenched noise and drift contains information about the system without drift. Unfortunately, quenched noise makes the situation much more complicated. In one dimension with periodic boundary conditions, the drift seems to render the quenched noise thermal, as can be seen using a Galilean transform [20]. However, the same scaling is observed in the presence of fixed boundary conditions [21, 22], i.e. boundary conditions do not seem to matter in one dimension. In higher dimensions, on the other hand, drift in the presence of periodic boundary conditions does not seem to change the asymptotes at all [23–25]. Not much seems to be known for other boundary conditions in higher dimensions.
In summary, we have shown that the Edwards–Wilkinson equation with drift and Neumann boundary conditions produces anomalous scaling different from what is expected from naive dimensional analysis and easily derived for periodic boundary conditions. The scaling and the amplitudes of the Dirichlet case are reproduced, consistent with a simple physical scenario of an interface that is constantly re-initialized. Neumann boundary conditions are physically more relevant than Dirichlet ones, yet far more difficult to handle analytically.

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Appendix A. Integral (41) term by term

In this appendix some details of the integral (41) are shown, considering the propagator term by term as defined in (42) and (2.2).

A.1. \( A_1 X_2 \) term

Since there is no \( y_1 \) or \( y_2 \) dependence in \( A_1 \), (43a):

\[
\int_0^1 dy_1 \int_0^1 dy_2 [\delta(y_1 - y_2) - 1] \int_0^1 dy' \int_0^u du' A_1 X_2 = 0
\]

for \( X \) being any of the \( A, B, C \) or \( D \) terms.

A.2. \( B_1 X_i \) term

Since all \( B_1 X_i \) integrals follow similar arguments, we exemplify the procedure on \( B_1 C_2 \). The integrals to be calculated (to leading order) are

\[
w_{BC}^2 = \frac{\Gamma^2}{v} q^{-1} \int_0^1 dy_1 \int_0^1 dy_2 [\delta(y_1 - y_2) - 1] \int_0^1 dy' \int_0^u du' q e^{q y'} e^q - 1 \\
\times \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{\theta n_1} \int_0^2 d\tilde{y}_1 q \sqrt{4\pi u} \frac{e^{(q y_1 + y_1 + 2 n_1 - \tilde{y}) \tilde{y}}}{4} \sum_{n_2=-\infty}^{\infty} e^{\theta n_2} q \sqrt{4\pi u} \frac{e^{(q y_2 + y_2 + 2 n_2 + u') \tilde{y}}}{4}.
\]

(A.1)

We consider the total exponential of \( B_1 C_2 \) in the form

\[
\frac{1}{e^q - 1} \exp (q (n_1 + n_2) - f(y', \tilde{y})),
\]

(A.2)

where

\[
f(y', \tilde{y}) = -q y' + \frac{q^2}{4u'} \left( y_1 + y' + 2 n_1 - \tilde{y} + \frac{u'}{q} \right)^2 + \frac{q^2}{4u'} \left( y_2 - y' + 2 n_2 + \frac{u'}{q} \right)^2.
\]

(A.3)

A contribution in the form (A.2) vanishes for large \( q \) unless

\[
n_1 + n_2 = \frac{f(y', \tilde{y})}{q} \geq 1.
\]

(A.4)
Applying the SPA to the integral over $y'$ means to evaluate the integral at $y'$ that minimizes $f$:

$$y'_0 = \frac{1}{2}(y_2 - y_1) + (n_2 - n_1) + \frac{1}{2} \tilde{y} + \frac{u'}{q}$$  \hspace{1cm} (A.5)

Applying the SPA again to the integral over $\tilde{y}$ gives, with $f(y'_0, \tilde{y})$,

$$\tilde{y}_0 = y_1 + y_2 + 2(n_1 + n_2) + \frac{4u'}{q},$$  \hspace{1cm} (A.6)

where now

$$f(y'_0, \tilde{y}_0) = -q(y_2 + 2n_2) - 2u'.$$  \hspace{1cm} (A.7)

Determining the asymptotic behaviour is greatly facilitated by writing $\frac{u'}{q} = \sigma + m \geq 0$, with $\sigma \in [0, 1)$ and $m \in \mathbb{N}^0$, so that $\lfloor u'/q \rfloor = m$. Using the final result of $f$, (A.7), the inequality in (A.4) becomes

$$n_1 + n_2 - \frac{f}{q} = y_2 + 2\sigma + n_1 + 3n_2 + 2m \geq 1.$$  \hspace{1cm} (A.8)

As $y_2 \in [0, 1]$ and $\sigma \in [0, 1)$, so $y_2 + 2\sigma \in [0, 3)$, the contribution (A.2) vanishes for large $q$ unless

$$n_1 + 3n_2 + 2m > -2.$$  \hspace{1cm} (A.9)

Since $n_1, n_2$ and $m$ are integers, we have the inequality

$$n_1 + 3n_2 + 2m \geq -1.$$  \hspace{1cm} (A.10)

Both SPAs produce an indicator function as well. Writing $\frac{u'}{q}$ in terms of $\sigma$ and $m$ gives

$$I_{(0,1)}(y'_0) = I_{(0,1)}[(y_2 - y_1)/2 + (n_2 - n_1) + \tilde{y}/2 + \sigma + m]$$  \hspace{1cm} (A.11)

$$I_{(0,2)}(\tilde{y}_0) = I_{(0,2)}[y_1 + y_2 + 2(n_1 + n_2) + 4\sigma + 4m].$$  \hspace{1cm} (A.12)

Considering the ranges of $y_1, y_2, \tilde{y}$ and $\sigma$, after some algebra, it turns out that we require $n_2 = m = 0$ to prevent either indicator function from vanishing. Therefore, the two indicator functions become

$$I_{(0,1)}(y'_0) = I_{(0,1)}[y_2 + 3\sigma]$$  \hspace{1cm} (A.13)

$$I_{(0,2)}(\tilde{y}_0) = I_{(0,2)}[y_1 + y_2 + 4n_1 + 4\sigma].$$  \hspace{1cm} (A.14)

This also constrains the range of $n_1$ to $-1 \leq n_1 \leq 0$, whereas (A.12) indicates

$$1 > y_2 + 3\sigma > 0.$$  \hspace{1cm} (A.15)

If $n_1 = -1$, then (A.12) implies $y_2 + 3\sigma > 3 - \sigma$. Since $\sigma \in [0, 1)$, this contradicts (A.14). Therefore, we conclude that $n_1 = 0$. However, using $n_1 = n_2 = m = 0$ in (A.8) gives $y_2 + 2\sigma > 1$, which, after comparing to (A.14), implies $\sigma < 0$ which is not within its range. As the conditions cannot all be fulfilled simultaneously, the integrals vanish exponentially in large $q$.

Considering in addition any marginal cases will produce terms of lower algebraic order in $q$. For the present integral, the marginal cases are $n_1 = n_2 = m = 0$ with $y_2 = 0$ and $\sigma = 0$ (double marginal), as well as $n_1 = 1, n_2 = m = 0$ with $y_1 = y_2 = 0$ and $\sigma = 0$ (triple marginal). Power counting in the initial integral (A.1) thus gives overall contributions to $w_{BC}^2$ of order $O(q^{-2})$ and $O(q^{-3})$, respectively. Extra care must be taken when considering the integral with upper bound $u$, as $u \to \infty$ might be taken before $q \to \infty$, in which case the integral over $u'$ might give rise to a term of order $q$ itself (see, for example, (A.16) versus (A.17)). In the present case this does not apply, because $u' = q(m + \sigma)$ and both $m$ as well as $\sigma$ are fixed by the SPA.

By similar arguments, the terms $B_1B_2$ and $B_1D_2$ vanish.
$A.3. \mathcal{C}_1\mathcal{X}_2$ and $D_1D_2$ term

The calculations for the terms $\mathcal{C}_1\mathcal{X}_2$ and $D_1D_2$ are very similar and we therefore exemplify the procedure for $D_1D_2$ only. We write contribution by $D_1D_2$ as

$$w^{2}_{\mathcal{D}_D} = \frac{\Gamma^2}{v} q^{-1} \int_0^1 \text{d}y \int_0^1 \text{d}y_2 \left[ \delta(y_1 - y_2) - 1 \right] \int_0^1 \text{d}y' \int_0^u \text{d}u' \times \sum_{n_1=-\infty}^{\infty} e^{\eta_{n_1}} \frac{q}{\sqrt{4\pi u'}} e^{\frac{-q^2}{2u'} \left[ \frac{1}{2} (y_1 - y_2) + (n_1 - n_2) \right]^2} e^{\sigma y'} \sum_{n_2=-\infty}^{\infty} e^{\eta_{n_2}} \frac{q}{\sqrt{4\pi u''}} e^{\frac{-q^2}{2u''} \left[ \frac{1}{2} (y_1 - y_2) + \sigma \right]} e^{\eta y'} e^{\frac{-q^2}{2u'} (y_1 - y_2)} e^{\frac{-q^2}{2u''} (n_1 - n_2)}. $$

Similar to above, we apply an SPA to the integral over $y'$ and find the constraints $n_1 + n_2 = m$ and $n_1 + n_2 = m + 1$. Incorporating them by means of Kronecker $\delta$-functions, the total contribution becomes

$$w^{2}_{\mathcal{D}_D} = \frac{\Gamma^2}{v} q^{-1} \int_0^1 \text{d}y \int_0^1 \text{d}y_2 \left[ \delta(y_1 - y_2) - 1 \right] \times \int_0^u \text{d}u' \sum_{n_1,n_2} \frac{q}{\sqrt{8\pi u'}} \exp \left\{ -\frac{q^2}{2u'} \left( \frac{1}{2} (y_1 - y_2) + (n_1 - n_2) \right)^2 \right\} \times 2 q \left( \frac{1}{2} (y_1 + y_2) + (n_1 + n_2) \right) \exp \left\{ -\frac{q^2}{2u'} (n_1 - n_2)^2 - 2q (y + (n_1 + n_2)) \right\} \times \left[ e^{-q^2m} I_{(0,1)}(\sigma - y) \delta_{m,n_1+n_2} + e^{-q^2(1-m)} I_{(-1,0)}(\sigma - y) \delta_{m-1,n_1+n_2} \right].$$

At this stage, it is sensible to consider the $\delta$-term and 1-term in $\left[ \delta(y_1 - y_2) - 1 \right]$ separately. For $y_1 = y_2 = y$, i.e. the contribution of the $\delta$-term, the integral to consider is

$$(D_1D_2)_1 = \int_0^1 \text{d}y \int_0^u \text{d}u' \sum_{n_1,n_2} \frac{q}{\sqrt{8\pi u'}} \exp \left\{ -\frac{q^2}{2u'} (n_1 - n_2)^2 - 2q (y + (n_1 + n_2)) \right\} \times \left[ e^{-q^2m} I_{(0,1)}(\sigma - y) \delta_{m,n_1+n_2} + e^{-q^2(1-m)} I_{(-1,0)}(\sigma - y) \delta_{m-1,n_1+n_2} \right].$$

using $I_{(0,1)}(y+1) = I_{(-1,0)}(y)$. This term can in turn be split into three parts, each accounting for one of the Kronecker $\delta$-functions and the possible values of $m$. To prevent exponential suppression, $m = 0$ is required for the prefactor $e^{-q^2m}$ and $m = 0, 1$ for the prefactor $e^{-q^2(1-m)}$, which gives rise to three terms, $(D_1D_2)_1 = (D_1D_2)_0 + (D_1D_2)_1 + (D_1D_2)_2$.

In the first term $(D_1D_2)_0$, we have $m = 0$ and $n_1 = -n_2$ from the Kronecker $\delta$. Ensuring the indicator function is non-zero, we change the integration limits of $y$ and using $\delta_{0,n_1+n_2}$ we write $n_1 = -n_2 = n$.

$$(D_1D_2)_0 = \int_0^1 \text{d}y \int_0^u \text{d}u' \sum_{n} \frac{q}{\sqrt{8\pi u'}} \exp \left\{ -\frac{q^2}{u'} (2n^2) - 2q y \right\} = \int_0^u \text{d}u' \sum_{n} \frac{1}{2q} \exp \left( -\frac{q^2}{u'} (2n^2) \right) \frac{q}{\sqrt{8\pi u'}} [1 - e^{-2q\sigma}].$$

In the limit of large $q$, the first exponential will asymptotically vanish unless $n = 0$. Since $[u'/q] = m = 0$, we change the upper integration limit of $u'$ to $\min(q, u)$:

$$(D_1D_2)_0 = \frac{1}{4} \sqrt[4]{2\pi} \left[ 2 \min \left( 1, \frac{u}{q} \right) - \frac{1}{\sqrt{2\pi}} \Gamma \left( \frac{1}{2}, 2q \min \left( 1, \frac{u}{q} \right) \right) \right]$$

\[13\]
using \( u' = \sigma q \) and
\[
\int_0^y dx \ x^\mu e^{-x} = \Gamma(\mu + 1, y),
\]
which converges to \( \sqrt{\pi} \) for \( \mu = 1/2 \) in the limit of large \( y \).

Similar arguments can be used for \((D_1D_2)_b\) and \((D_1D_2)_s\) terms. We finally arrive at
\[
(D_1D_2)_b = (D_1D_2)_b + (D_1D_2)_s + (D_1D_2)_s = \frac{1}{2\sqrt{\pi}} \left[ \min \left( 1, \frac{u}{q} \right) \right].
\] (A.15)

For \( u \) small and by taking the limit \( q \to \infty \), it is obvious that
\[
\lim_{q \to \infty} (D_1D_2)_b \to 0.
\] (A.16)

Yet, taking \( u \to \infty \) first gives \min \left( 1, \frac{u}{q} \right) = 1 and we find to leading order in \( q \):
\[
\lim_{u \to \infty} (D_1D_2)_b = \frac{1}{v} q^{-1} \frac{1}{2\sqrt{\pi}} \ = \ \frac{1}{v} \frac{1}{2\sqrt{\pi q}}.
\] (A.17)

Now we consider the roughness contribution of the 1-term, i.e. \( y_1 \neq y_2 \):
\[
(D_1D_2)_1 = \int_0^1 dy_1 \int_0^1 dy_2 \int_0^u du' \sum_{n_1,n_2} \frac{q}{\sqrt{8\pi u'}} e^{-u' \left[ \frac{1}{2} (y_1 - y_2) + \frac{1}{2} (y_1 + y_2 + \sigma) \right] \delta_{m,n_1+n_2} - 2q \left( \frac{1}{2} (y_1 + y_2) + (n_1 + n_2) \right)} \delta_{m,n_1+n_2} + e^{-q|m-1|} I_{(1-0)} \left[ \frac{1}{2} (y_1 + y_2) + \sigma \right] \delta_{m-1,n_1+n_2}.
\]

This term can be split into three parts as above, \((D_1D_2)_1 = (D_1D_2)_1 + (D_1D_2)_1 + (D_1D_2)_1\), again, each accounting for one of the Kronecker \( \delta \)-functions and the possible values of \( m \).

Following a similar procedure as for the \( \delta \)-term, the \((D_1D_2)_1\) contribution vanishes for large \( q \) unless \( m = 0 \), which implies \( n_1 = -n_2 = n \). An SPA applied to the integral over \( y_1 \) gives a minimum at
\[
y_{1b} = y_2 = \frac{4u'}{q} - 4n.
\]

The first term therefore gives
\[
(D_1D_2)_1 = \int_0^1 dy_2 \sum_{n} \int_0^u du' \frac{q}{\sqrt{8\pi u'}} e^{-u' \left( -q \left( 2y_2 - \frac{2u'}{q} - 4n \right) \right)} \times I_{(0,1)} \left[ -\frac{1}{2} \left( y_2 - \frac{4u'}{q} - 4n + y_2 \right) + \sigma \right] I_{(0,1)} \left[ y_2 - \frac{4u'}{q} - 4n \right]
\]
\[= \int_0^1 dy_2 \int_0^u du' \exp \left( -q(2y_2 - 2\sigma - 4n) \right) \times I_{(0,1)} \left[ -y_2 + 3\sigma + 2n I_{(0,1)} [y_2 - 4\sigma - 4n].
\]

Both indicator functions suggest \( n = 0 \) or \( n = -1 \); otherwise the \((D_1D_2)_1\) term will not contribute. In the limit of large \( q \), the term will be exponentially suppressed for \( n = -1 \) so that the only case to be considered is \( n = 0 \). However, using \( n = 0 \) in both indicator functions implies \( \sigma < 0 \) for this term to contribute which contradicts \( \sigma \geq 0 \). We conclude that \((D_1D_2)_1\) does not contribute.
Similar arguments apply to \((D_1 D_2)_{12}\) and \((D_1 D_2)_{13}\). In both cases the indicator functions require negative \(n\), so that the term is exponentially suppressed.

Hence in total, considering both limits we have to leading order:

\[
\lim_{q \to \infty} w_2^{2 DD} = 0 \quad \text{and} \quad \lim_{u \to \infty} w_2^{2 DD} = \frac{\Gamma^2}{v} \frac{1}{2 \sqrt{2 \pi q}}. \tag{A.18}
\]

Any marginal cases are necessarily subleading compared to (A.15) and thus can be safely ignored.

By similar procedures, we obtain the leading order behaviour of \(w_2^{2 CC}\) and \(w_2^{2 CD}\) as follows:

\[
\lim_{q \to \infty} w_2^{2 CC} = \frac{\Gamma^2}{v} \sqrt{\frac{u}{2 \pi}} \quad \text{and} \quad \lim_{u \to \infty} w_2^{2 CC} = \frac{\Gamma^2}{v} \frac{2}{3 \sqrt{2 \pi}} \sqrt{q}, \tag{A.19}
\]

and

\[
\lim_{q \to \infty} w_2^{2 CD} = 0 \quad \text{and} \quad \lim_{u \to \infty} w_2^{2 CD} = 0.
\]

In summary, the only non-zero contributions are (A.18) and (A.19).

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