DOUBLE-BOSONIZATION AND MAJID’S CONJECTURE, (II): CASES OF IRREGULAR R-MATRICES AND TYPE-CROSSINGS OF F_4, G_2

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Abstract. The purpose of the paper is to build up the related theory of weakly quasitriangular dual pairs suitably for non-standard R-matrices (irregular), and establish the generalized double-bosonization construction theorem for irregular R, which generalize Majid’s results for regular R in [M1]. As an application, the type-crossing construction for the exceptional quantum groups of types F_4, G_2 is obtained. This affirms the Majid’s expectation that the tree structure of nodes diagram associated with quantum groups can be grown out of the node corresponding to U_q(sl_2) by double-bosonization procedures. Notably from a representation perspective, we find an effective method to get the minimal polynomials for the non-standard R-matrices involved.

1. Introduction

1.1. The charming power of quantum group theory always attracts many mathematicians to find some better way in a suitable framework to understand the structure of quantum groups defined initially by generators and relations. Specially for the quasitriangular Hopf algebras (with universal R-matrices), there are several well-known general constructions, among which main representatives are: (1) The Drinfeld’s quantum double theory of any Hopf algebra introduced in [Dr1]. (2) The FRT-construction in [FRT] based on the (standard) R-matrices. (3) The Majid’s double-bosonization theory in [M1] (also see Sommerhäuser’s construction [So] independently working in a slightly different braided category, i.e., the Yetter-Drinfeld category), which improved the FRT-construction via extending Drinfeld’s quantum double to the generalized quantum double associated with dually-paired braided groups (named by Majid) coming from a braided category consisting of the (co-)modules of a (co-)quasitriangular Hopf algebra. Based on his series of earlier work [M2]–[M5], Majid [M1] in 1995 developed the double-bosonization theory to yield a direct construction for

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$U_q(\mathfrak{g})$ as an application of which one can view the Lusztig’s algebra $\mathfrak{f}(\mathfrak{L})$ as the braided group in a special braided category $\mathfrak{M}_H$ (or $\mathfrak{M}_B$), where $(H = kQ = k[K_i^\pm], A = kQ^\vee)$ is a weakly quasitriangular dual pair (see p. 169 of [M1]), where $Q$ (resp. $Q^\vee$) is a (dual) root lattice of $\mathfrak{g}$. (4) The Rosso’s quantum shuffle construction [Ro] (axiomatic approach) (for the entire construction see [HLR]) based on the braidings, which together with Nichols’ earlier work [N] has been closely related to the currently fruitful developments of classifications of the pointed Hopf algebras or Nichols algebras (or braided groups in terms of Majid), for instance, see [AS], [AHS], [HS], etc. Let us say more about the double-bosonization below.

1.2. Roughly speaking, Majid’s double-bosonization theory is as follows. Associated to any mutually dual braided groups $B^*, B$ covariant under a underlying quasitriangular Hopf algebra $H$, there is a new quantum group on the tensor space $B^* \otimes H \otimes B$ by double bosonization in [M1], consisting of $H$ extended by $B$ as additional positive roots and its dual $B^*$ as additional negative roots. Specially, Majid regarded $U_q(n^\pm)$ as the mutually dual braided groups in a braided category of right (left) $H$-modules with Hopf “Cartan subalgebra” $H$. On the other hand, based on a couple of examples of lower rank given in [M1, M6, M7], Majid claimed that many novel quantum groups, as well as the rank-inductive or type-crossing construction (we named) of $U_q(\mathfrak{g})$ can be obtained in principle by this theory, since he expected his double-bosonization construction would allow to generate a tree of quantum groups and at each node of the tree, there should be many choices to adjoin a certain dually-paired braided groups covariant under a certain quantum subgroup at that node. In fact, it is a challenge to elaborate the full tree structure because it involves some rather complicated technical points needed to be solved from representation theory. Moreover, it is difficult to explore the information encoded in the $R$-matrix corresponding to a certain representation of $U_q(\mathfrak{g})$, for example, the spectral decomposition of $R$-matrix, etc. By exploiting the well-known information for standard $R$-matrices (we mean those associated to the vector representations of classical types $ABCD$), the authors in [HH] obtained the rank-inductive construction of quantum groups $U_q(\mathfrak{g})$ for classical types in the quantum tree of Majid. Furthermore, can we elucidate the tree structure at the exceptional nodes? That is, How to construct the exceptional quantum groups via double-bosonization theory?

1.3. Recall that the starting point of Majid’s framework [M1] is to work with the weakly quasitriangular dual pairs of bialgebras $(\widehat{U}(R), A(R))$, where $\widehat{U}(R) := A(R) \bowtie A(R)$ is the double cross product defined by [M5], $A(R)$ is the FRT-bialgebra defined in [FRT]. He could seek the weakly quasitriangular dual pair $(U_q(\mathfrak{g}), O_q(G))$ as the quotient Hopf algebras of $(\widehat{U}(R), A(R))$ to establish his double-bosonization Theorem (see Corollary 5.5 [M1]) when $R$ is regular (for definition, see p. 174, [M1]). In [HH], when $R$ is one of standard $R$-matrices which is regular in the sense of Majid, we can work with a slight different
weakly quasitriangular dual pair \((U^\text{ext}_q(\mathfrak{g}), O_q(G))\) that fits into Majid’s original framework (Corollary 5.5 [M1]), where \(U^\text{ext}_q(\mathfrak{g})\) is the extended Hopf algebra of \(U_q(\mathfrak{g})\) of classical type, \(O_q(G)\) is the quotient Hopf algebra of \(A(R)\), the quantum coordinate algebra on the associated simple Lie group \(G\) of classical type (cf. [KS]), so that the braided vector algebra \(B = V(R', R) \in O_q(G)\mathfrak{m}\), the braided co-vector algebra \(B^\ast = V^\ast(R', R_2^{-1}) \in \mathfrak{m}O_q(G)^\ast\) can be transferred into the originally-required objects in the braided categories \(\mathfrak{m}U^\text{ext}_q(\mathfrak{g}), U^\text{ext}_q(\mathfrak{g})\mathfrak{m}\), respectively.

1.4. While for those non-standard \(R\)-matrices we encountered when we deal with the type-crossing construction of the exceptional quantum groups, the weakly quasitriangular dual pair of bialgebras \((\hat{U}(R), A(R))\), previously served as the starting point of Majid’s work, has to be replaced. The new weakly quasitriangular dual pair needs to be created since the quantum coordinate algebra \(O_q(G)\) doesn’t meet our requirement and its substitute is no longer the quotient of \(A(R)\). Such \(R\)-matrices are not regular in the sense of Majid, we name them irregular. This means that it is desirable to establish the related theory of the weakly quasitriangular dual pairs suitably for irregular \(R\)-matrices. Roughly speaking, thanks to Theorem 8 [FRT], we can exploit their defining bialgebra \(H_R\) as a suitable candidate, which will be a Hopf algebra when the \(R\) involved satisfies the FRT-condition. Also, we can still use the double cross product \(\hat{U}(R)\) to yield a suitable quotient Hopf algebra \(U(R)\) we define. Fortunately, we prove that such a pair \((\hat{U}(R), H_R)\) forms a weakly quasitriangular dual pair in the sense of Majid as we expected. When \(R\) is standard, \(\hat{U}(R)\) coincides with \(U^\text{ext}_q(\mathfrak{g})\) by [KS], and \(H_R\) is isomorphic to \(O_q(G)\) due to the Remark of Theorem 8 [FRT]. For convenience, we still write \(U^\text{ext}_q(\mathfrak{g})\) instead of \(\hat{U}(R)\). On the other hand, we further argue that \(H_R\) is coquasitriangular when \(R\) satisfies our more assumption (than the FRT-condition), so that the categories of left (right) comodules over \(H_R\) are braided in the sense of Majid and \(V(R', R) \in H_R\mathfrak{m}, V^\ast(R', R_2^{-1}) \in \mathfrak{m}H_R^\ast\).

1.5. After clarifying the rough destination above, in order to deduce the generalized version of double-bosonization construction theorem suitably for general \(R\)-matrices, especially for the double-bosonization inductive construction of the exceptional quantum groups, we have to refine the above framework. Firstly, we need to replace \(U^\text{ext}_q(\mathfrak{g})\) by its central extension object \(\widetilde{U}^\text{ext}_q(\mathfrak{g}) := U^\text{ext}_q(\mathfrak{g}) \otimes \mathbb{C}\langle c, c^{-1}\rangle\), and \(H_{R_{VV}}\) by \(\widetilde{H}_{R_{VV}} := H_{R_{VV}} \otimes \mathbb{C}[g, g^{-1}]\), where \(R_{VV}\) is the \(R\)-matrix corresponding to a certain chosen irreducible representation of \(U_q(\mathfrak{g})\). Secondly, to normalize \(R_{VV}\) at its certain eigenvalue to gain a quantum group normalization constant \(\lambda\) such that \(\langle c, g \rangle = \lambda\) and \((\widetilde{U}^\text{ext}_q(\mathfrak{g}), \widetilde{H}_{R_{VV}})\) forms a new weakly quasitriangular dual pair of Hopf algebras as we desired. Thirdly, to get the minimal polynomial of the braiding \(PR_{VV}\), from which we achieve the pair \((R, R')\), where \(R\) is determined by the above normalization of \(R_{VV}\), subsequently, \(R'\) can be figured out from the minimal polynomial. Notice that
\[ A(R) = A(R_{VV}) \subset H_{VV} \subset \tilde{H}_{VV} \] as bialgebras, we can regard the braided objects \( B = V(R', R) \in \mathbb{H}(\mathfrak{g}) \) and \( B^* = V^*(R', R^{-1}) \in \mathfrak{g}^\widehat{\text{sl}_n} \) as the braided objects in \( \tilde{H}_{VV} \mathfrak{g} \) and \( \mathfrak{g} \tilde{H}_{VV} \), respectively. Via the dual pairing between \( U_q^{\text{ext}}(\mathfrak{g}) \) and \( \tilde{H}_{VV} \), we get \( B = V(R', R) \in \mathfrak{g} \tilde{H}_{VV} \) and \( B^* = V^*(R', R^{-1}) \in \tilde{H}_{VV} \mathfrak{g} \). This affords us a prerequisite for building the generalized double-bosonization construction Theorem for the irregular \( R \)'s.

1.6. The paper is organized as follows. In Section 2, we recall some basic facts about the FRT-construction [FRT] and the Majid’s double-bosonization [M1]. In Section 3, for the irregular \( R \)-matrices, we start with the weakly quasitriangular dual pair of bialgebras \((\tilde{U}(\mathfrak{g}), H_R)\) to get the required weakly quasitriangular dual pair of Hopf algebras \((\tilde{U}(\mathfrak{g}), H_R)\) when the \( R \) involved meets certain strict conditions. Finally, we establish the generalized double-bosonization construction theorem for irregular \( R \)'s. Section 4 is devoted to applying this theorem to give the type-crossing double-bosonization constructions of \( U_q(F_4) \) and \( U_q(G_2) \). First of all, in order to work out the construction of \( U_q(F_4) \), we start from the nodes diagram associated to \( U_q(B_3) \) and its 8-dimensional spin representation \( T_\mathbb{C} \), and prove that the braiding \( \tilde{R}_{VV} \) is of diagonal type and find an ingenious method to figure out its minimal polynomial, which captures/denies on some features of the defining representation rather than the whole information of \( R \)-matrix itself. Consequently, making the normalization for \( R_{VV} \) at a certain eigenvalue of its braiding to get the pair \((R, R')\) from which determine our required dually-paired braided groups, we prove the quantum group we constructed is just \( U_q(F_4) \). Next, for constructing \( U_q(G_2) \), we begin with \( U_q(A_1) \) and its spin \( \frac{3}{2} \) representation, which is built on the 4-dimensional homogeneous submodule of degree 3 in the \( U_q(sl_2) \)-module algebra (cf. [H]) \( \mathcal{A}_q = \mathbb{C}_q(x, y) \) with \( xy = qyx \) (namely, the Manin’s quantum plane). Similar information necessary for \( U_q(G_2) \) is provided. We believe that the results and methods of this paper will be useful for seeking new quantum groups or Hopf algebras arising from various Nichols algebras currently focused on by many Hopf algebraists (for instance, see [AHS], [AS], [CL], [HS], etc. and references therein).

2. Preliminaries

Let us fix some general notation which will be kept throughout this paper. The letters \( \mathbb{C}, \mathbb{Z} \) always stand for the complex field, the set of integer numbers, respectively. \( 0 \neq q \in \mathbb{C} \) and \( n \in \mathbb{Z}_+ \), then

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad \binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}.
\]

Let \( \mathbb{R} \) be the real field, \( E \) the Euclidean space \( \mathbb{R}^n \) or a suitable subspace. Denote by \( e_i \)'s the usual orthogonal unit vectors in \( \mathbb{R}^n \). Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie
algebra with simple roots \( \alpha_i \)'s, \( \lambda_i \) the fundamental weight corresponding to simple root \( \alpha_i \). Cartan matrix of \( \mathfrak{g} \) is \( (a_{ij}) \), where \( a_{ij} = \frac{2(a_i, a_j)}{(a_i, a_i)} \), and \( d_i = \frac{(a_i, a_i)}{2} \). Let \((H, \mathcal{H})\) be a quasitriangular Hopf algebra, where \( \mathcal{H} \) is the universal \( R \)-matrix, \( \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \), \( \mathcal{T}_1 = \mathcal{H}^{(2)} \otimes \mathcal{H}^{(1)} \). Denote by \( \Delta, \eta, \epsilon, S \) its coproduct, counit, unit, and antipode of \( H \), respectively. We shall use Sweedler’s notation: \( \Delta(h) = h_1 \otimes h_2 \), for \( h \in H \). \( H^{\text{op}} \) (\( H^{\text{cop}} \)) denotes the opposite (co)algebra structure of \( H \). \( \mathcal{H}_H (\mathcal{M}) \) denotes the braided category consisting of right (left) \( H \)-modules. If \( (A, \tau) \) is a quasitriangular Hopf algebra, then \( \mathcal{H}_A (\mathcal{M}^A) \) denotes the braided category consisting of left (right) \( A \)-comodules. For a detailed description of these theories, we left the readers to refer to Drinfeld’s and Majid’s papers [Dr2], [M2], [M3], and so on. By a braided group we mean a braided bialgebra or Hopf algebra in some braided category. In order to distinguish from the ordinary Hopf algebras, we denote by \( \Delta, S \) its coproduct and antipode, respectively.

2.1. **FRT-construction.** An invertible solution of the quantum Yang-Baxter equation (in brief, QYBE) \( R_{12}R_{13}R_{23} = R_{23}R_{12}R_{13} \) is called \( R \)-matrix. There is a bialgebra \( A(R) \) [FRT] corresponding to any invertible \( R \)-matrix, called the FRT-bialgebra.

**Definition 2.1.** \( A(R) \) is generated by 1 and \( t = [t^i_j] \), having the following structure:

\[
RT_1T_2 = T_2T_1R, \quad \Delta(T) = T \otimes T, \quad \epsilon(T) = I \text{ (unit matrix)}.
\]

\( A(R) \) is a coquasitriangular bialgebra with \( \mathcal{D} : A(R) \otimes A(R) \rightarrow k \) such that \( \mathcal{D}(t^i_j \otimes t^l_m) = R^{ik}_{jm} \). Here \( T_1 = T \otimes I, T_2 = I \otimes T, R^{ik}_{jm} \) denotes the entry at row \((ik)\) and column \((jl)\) in \( R \).

There is a general procedure to construct the Hopf algebra \( H_R \) associated with the FRT-bialgebra \( A(R) \) in [FRT]. The algebra \( H_R \) is generated by 1 and \( t^i_j, \tilde{t}^i_j, i, j = 1, \cdots, n \), with the relations

1. \( RT_1T_2 = T_2T_1R, \quad R^{i'}_j \tilde{t}^{i'}_j = \tilde{t}^{i'}_j R^{i'}_j, \quad (R^{i'})^{-1}T_1 \tilde{t}^{i'}_j = \tilde{t}^{i'}_j T_1(R^{i'})^{-1}, \)

where \( (R^i)_kl = R^i_{jk}, (R^i)_kj = R^i_{jk}, \quad (R^i)_lij = R^i_{lj} \). Specially, assume that \( R^{i'} \) is nonsingular and

2. \( (R^{i'})^l_j P(R^{i'})^{-1}PK_0 = \text{const} \cdot K_0, \)

where \( P \) is the permutation matrix with the entry \( P^l_j = \delta_{lj} \delta_{jk} \), “const” means a diagonal matrix consisting of constants, and \( (K_0)_ij := \delta_{ij} \delta_{kl} \), then \( H_R \) is a Hopf algebra with the structure maps

\[
\Delta(T) = T \otimes T, \quad \Delta(\tilde{T}) = \tilde{T} \otimes \tilde{T}, \quad \epsilon(T) = \epsilon(\tilde{T}) = I; \quad S(T) = (\tilde{T})^t, \quad S(\tilde{T}) = DT^tD^{-1}.
\]

Here \( D = \text{tr}_2(P((R^{i'})^{-1})) \in M_n(\mathbb{C}) \), and \( \text{tr}_2 \) denotes matrix trace in the second factor in the tensor product \( \mathbb{C}^n \otimes \mathbb{C}^n \). In particular, for the standard \( R \)-matrices, \( H_R \)'s are isomorphic to quantum coordinate functions algebras \( O_q(G)'s \) on the corresponding Lie groups \( G \) (see
the Remark of Theorem 8 [FRT]). We will consider those Hopf algebras \( H_R \) for the general \( R \)-matrices in the next section.

In the braided category \( A(R) \mathcal{M} \) (\( \mathcal{M}^{A(R)} \)), there are two classical braided groups \( V(R', R) \) and \( \overline{V}(R', R^{-1}) \) in [M4], called the braided (co-)vector algebras, where \( R' \) is another matrix satisfying

\[
R_{12}R_{13}R_{23}' = R_{23}'R_{13}R_{12}, \quad R_{23}R_{13}R_{12}' = R_{12}'R_{13}R_{23}, \quad (PR + 1)(PR' - 1) = 0, \quad R_{21}'R_{12}' = R_{21}'R_{12}.
\]

(3) \( (PR + 1)(PR' - 1) = 0, \) \( R_{21}'R_{12}' = R_{21}'R_{12}. \)

\( R \) is a \( \delta \)-matrix in the next section.

The braided vector algebra \( V(R', R) \) defined as a quadratic algebra with generators \( 1, \{ e^i \mid i = 1, \ldots, n \} \) and relations \( e^i e^j = \sum_{a,b} R_{ab}^i e^a \otimes e^b, \) forms a braided group with \( \Delta(e^i) = e^i \otimes 1 + 1 \otimes e^i, \) \( S(e^i) = 0, S(e^i) = -e^i, \) \( \Psi(e^i \otimes e^j) = \sum_{a,b} R_{ab}^i e^a \otimes e^b \) in braided category \( A(R) \mathcal{M}. \) Under the duality \( \langle f_j, e^i \rangle = \delta_{ij}, \) the braided co-vector algebra \( \overline{V}(R', R^{-1}) \) defined by \( 1, \{ f_j \mid j = 1, \ldots, n \}, \) and relations \( f_i f_j = \sum_{a,b} f_a R_{ij}^{ab}, \) forms another braided group with \( \Delta(f_j) = f_j \otimes 1 + 1 \otimes f_j, \) \( S(f_i) = -f_i, \) \( \Psi(f_i \otimes f_j) = \sum_{a,b} f_b \otimes f_a R_{ij}^{ab} \) in braided category \( \mathcal{M}^{A(R)}. \)

\textbf{Remark 2.1.} In fact, when we know the minimal polynomial equation \( \prod_i (PR - x_i) = 0 \) of the braiding \( PR \), we can normalize \( R \) at a certain eigenvalue \( x_i \) of \( PR \) so that \( x_i = -1, \) then set \( R' = P + P \prod_{j \neq i} (PR - x_j) \). This gives us a way to get the pair \( (R, R') \) that satisfy conditions (3) and (4).

\textbf{2.2. Majid’s double-bosonization.} Let \( C, B \) be a pair of braided groups in \( \mathcal{M}_{H}, \) which are called dually paired if there is an intertwiner \( ev : C \otimes B \rightarrow k \) with \( ev(c, d, b) = ev(d, b, c)ev(c, b) \), \( ev(c, ab) = ev(c, a)ev(c, b) \), \( \forall a, b \in C, d \in C. \) Then \( C^{op/cop} \) (with opposite product and coproduct) is a Hopf algebra in \( \mathcal{M}_{H} \), which is dual to \( B \) in the sense of an ordinary duality pairing \( \langle , \rangle \) being \( H \)-bicovariant. Let \( \overline{C} = (C^{op/cop})^{cop} \), then \( \overline{C} \) is a braided group in \( \overline{\mathcal{M}}_{\overline{H}}, \) where \( \overline{H} \) is \( (H, \overline{\mathcal{R}}^{-1}) \). With these, Majid gave the following double-bosonization theorem and some results in [M1]:

\textbf{Theorem 2.1.} (Majid) On the tensor space \( C \otimes H \otimes B \), there is a unique Hopf algebra structure \( U = UC(H, B) \) such that \( H \triangleleft B \) (bosonization) and \( C \vartriangleright H \) (bosonization) are sub-Hopf algebras by the canonical inclusions and

\[
bc = (\overline{\mathcal{R}}^{(2)} b_1 (\overline{\mathcal{R}}_2^{(2)} b_2) \triangleleft \overline{\mathcal{R}}_1^{(1)} b_3) \triangleleft \mathcal{R}_2^{(1)} (\overline{\mathcal{R}}^{(2)} b_1) \triangleleft \mathcal{R}_1^{(1)} b_3 \triangleleft \mathcal{R}_2^{(1)}).
\]

\[
\langle \mathcal{R}_1^{(2)} b_1 \rangle \triangleleft \mathcal{R}_2^{(2)} b_2 \mathcal{R}_3^{(1)} \triangleleft \mathcal{R}_2^{(1)} b_3 \mathcal{R}_1^{(1)}.
\]
for all $b \in B$, $c \in \mathbb{C}$ viewed in $U$. The product, coproduct of $U$ are given by

\[(c \otimes h \otimes b) \cdot (d \otimes g \otimes a) = c(h(1), \mathcal{R}(2) \triangleright d(2)) \otimes h(2), \mathcal{R}(2) \triangleright \mathcal{R}^{-1}(1) g(1) \otimes (b(2), \mathcal{R}(1) \triangleright \mathcal{R}^{-1}(2) g(2)) a\]

\[
\Delta(c \otimes h \otimes b) = c(1) \otimes \mathcal{R}^{-1}(1) h(1) \otimes h(1) \triangleright \mathcal{R}(1) \otimes \mathcal{R}^{-2}(1) \triangleright b(2), \mathcal{R}(1) \triangleright \mathcal{R}^{-2}(2) \triangleright c(2) \otimes h(2), \mathcal{R}(2) \triangleright b(2).
\]

Moreover, the antipodes of $H \ltimes B$ and $C \rtimes H$ can be extended to an antipode $S : U \rightarrow U$ by the two extensions $S(\mathbb{C}) = U \otimes U$ and $S(\mathbb{C}) = (S(a)) \cdot (S(c))$.

Here, $\triangleright$, $\triangleleft$ refer to left, right actions respectively, $\mathcal{R}_1, \mathcal{R}_2$ are distinct copies of the quasitriangular structure $\mathcal{R}$ of $H$.

Moreover, Majid [M1] proposed the concept of a weakly quasitriangular dual pair via his insight on more examples on matched pairs of bialgebras or Hopf algebras in [M5].

**Definition 2.2.** Let $(H, A)$ be a pair of Hopf algebras equipped with a dual pairing $(\cdot, \cdot)$ and convolution-invertible algebra/anti-coalgebra maps $\mathcal{R}, \mathcal{R}^*: A \rightarrow H$ obeying

\[\langle \mathcal{R}(a), b \rangle = \langle \mathcal{R}^{-1}(b), a \rangle, \quad \partial^R h = \mathcal{R}^* (\partial^L h) * \mathcal{R}^{-1}, \quad \partial^L h = \mathcal{R}^* (\partial^L h) * \mathcal{R}^{-1}\]

for $a, b \in A, h \in H$. Here $*$ is the convolution product in $\text{hom}(A, H)$ and $(\partial^L h)(a) = \langle h(1), a \rangle h(2)$, $(\partial^R h)(a) = h(1) \langle a, h(2) \rangle$, are left, right “differentiation operators” regarded as maps $A \rightarrow H$ for any fixed $h$.

For convenience, we give in brief the definition of coquasitriangular Hopf algebra.

**Definition 2.3.** A coquasitriangular Hopf algebra is a Hopf algebra $A$ equipped with a linear form $\tau : A \otimes A \rightarrow \mathbb{C}$ such that the following conditions hold:

(i) $\tau$ is invertible with respect to the convolution, that is, there exists another linear form $\bar{\tau} : A \otimes A \rightarrow \mathbb{C}$ such that $\tau \bar{\tau} = \bar{\tau} \tau = \epsilon \otimes \epsilon$ on $A \otimes A$,

(ii) $\tau \circ (\cdot \otimes \text{id}) = \tau_{13} \ast \tau_{23}$, $\tau \circ (\text{id} \otimes \cdot) = \tau_{11} \ast \tau_{12}$, $\cdot \circ \tau = \tau \ast \tau^{-1}$.

**Remark 2.2.** If there exists a coquasitriangular Hopf algebra $A$ such that $(H, A)$ is a weakly quasitriangular dual pair, and $b, c$ are primitive elements. Then many cross relations in the above theorem can be simplified, for example,

\[[b, c] = \mathcal{R}(b^{(1)}) \langle c, b^{(2)} \rangle - \langle c^{(1)}, b \rangle \mathcal{R}(c^{(2)})\]

\[\Delta b = b^{(1)} \otimes \mathcal{R}(b^{(2)}), \quad \Delta c = c \otimes 1 + \mathcal{R}(c^{(2)}) \otimes c^{(1)}\]

3. **Generalized Double-bosonization Construction Theorem**

Throughout the following sections, let $T_V$ be an irreducible representation of $U_q(\mathfrak{g})$ or the $h$-adic Drinfeld-Jimbo algebra $U_h(\mathfrak{g})$ with $V$ a finite-dimensional vector space.
3.1. The universal $R$-matrix and $L$-functionals of $\mathcal{U}_h(\mathfrak{g})$. In order to provide an explicit expression for the universal $R$-matrix $\mathcal{R}$ of $\mathcal{U}_h(\mathfrak{g})$, the key step is to determine dual bases of the $h$-adic vector spaces $\mathcal{U}_h(\mathfrak{b}_+)$ and $\mathcal{U}_h(\mathfrak{b}_-)$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$:

\begin{align}
\langle H_i', \tilde{H}_j \rangle' &= \delta_i^j \frac{h}{\cosh h - e^{-dh}}, \quad \forall i, j \in \Delta, \quad h \ll 1, \\
\langle H_i, L_j \rangle &= \delta_i^j a_{ij}, \quad \langle H_i, \tilde{H}_j \rangle' = \delta_i^j \frac{h}{q_i - q_i},
\end{align}

where $\delta_i^j$ is the Kronecker delta. To do this, there is the following Lemma 3.1.

**Lemma 3.1.** If a positive root $\beta$ and a simple root $\alpha_i$ of $\mathfrak{g}$ belong to the same Weyl group $W$-orbit, and they satisfy

\begin{align}
\langle E_{\beta}, \tilde{F}_{\beta} \rangle' &= \langle E_i, F_i \rangle' = \frac{h}{q_i - q_i},
\end{align}

then using the above relations (5), the dual bases of $\mathcal{U}_h(\mathfrak{b}_+)$ and $\mathcal{U}_h(\mathfrak{b}_-)$ are obtained.

These dual bases are denoted by $e_i$ resp. $f_i$, and then $\mathcal{R}_D := \sum_i (1 \otimes e_i) \otimes (f_i \otimes 1)$ is a universal $R$-matrix of quantum double $\mathcal{D}(\mathcal{U}_h(\mathfrak{b}_+), \mathcal{U}_h(\mathfrak{b}_-))$. Taking advantage of the canonical homomorphism $\pi$ of $\mathcal{D}(\mathcal{U}_h(\mathfrak{b}_+), \mathcal{U}_h(\mathfrak{b}_-))$ to $\mathcal{U}_h(\mathfrak{g})$, then $\mathcal{R} := (\pi \otimes \pi)\mathcal{R}_D$ is a universal $R$-matrix of $\mathcal{U}_h(\mathfrak{g})$. Especially, the root elements $E_\beta$ and $F_\beta$, obtained by Lusztig’s automorphism $T_\beta$’s [L], satisfy the condition (6), and then an corresponding explicit form of the universal $R$-matrix is [KS]:

\begin{align}
\mathcal{R} &= \exp(h \sum_{i,j} B_{ij}(H_i \otimes H_j)) \prod_{\beta \in \Delta} \exp_{q_\beta}\left((1 - q_\beta^2)(\mathcal{E}_\beta \otimes \mathcal{F}_\beta)\right).
\end{align}

Here the matrix $(B_{ij})$ is the inverse of the matrix $(C_{ij}) = (d_j^{-1}a_{ij})$, the $q$-exponential function $\exp_q x$ is defined by $\exp_q x = \sum_{r=0}^{\infty} \frac{q^{r^2}}{(q^r; q^r)_r} x^r$.

On the other hand, corresponding to an irreducible representation $T_V$ of $\mathcal{U}_h(\mathfrak{g})$, there are uniquely determined elements $l_{ij}^\pm \in \mathcal{U}_h(\mathfrak{g})$ such that

\begin{align}
(T_V \otimes \text{id})\mathcal{R}(1 \otimes v_j) &= \sum_i l_{ij}^+ \otimes v_i, \\
(T_V \otimes \text{id})(\mathcal{R}^{-1})(v_j \otimes 1) &= \sum_i v_i \otimes l_{ij}^-.
\end{align}

These $l_{ij}^\pm$ are called $L$-functionals associated with the representation $T_V$ [KS]. Moreover, these matrices $L^\pm = (l_{ij}^\pm)$ generate a bialgebra, denoted by $\mathcal{U}(L^\pm)$, satisfying the following relations

\begin{align}
L_1^+ L_2^+ R_{VV} &= R_{VV} L_2^+ L_1^+, \\
L_1^- L_2^- R_{VV} &= R_{VV} L_2^- L_1^-, \\
\Delta(L^\pm) &= L^\pm \otimes L^\pm, \\
e(L^\pm) &= I.
\end{align}
3.2. R-matrices for representations of \( U_q(\mathfrak{g}) \). By a certain algebra automorphism of the completion \( \hat{U}_q^+(\mathfrak{g}) \otimes \hat{U}_q^+(\mathfrak{g}) \) (see P.264 in [KS]), the above universal R-matrix (7) can yield a universal R-matrix of \( U_q(\mathfrak{g}) \), that is,

\[
\mathcal{R} = \sum_{r_1, \ldots, r_n=0}^{\infty} \prod_{j=1}^{n} \left( 1 - q_{\beta_j}^{-2} \right) r_j E_{\beta_j}^{r_j} \otimes F_{\beta_j}^{r_j}.
\]

Then the R-matrix datum \( R_{VV} \) is obtained by \( R_{VV} = B_{VV} \circ (T_V \otimes T_V)(\mathcal{R}) \), where \( B_{VV} \) denotes the linear operator on \( V \otimes V \) given by \( B_{VV}(v \otimes w) := q^{(\mu^\prime \mu)}v \otimes w \), for \( v \in V_\mu, w \in V_{\mu^\prime} \). We can take a basis \( \{ v_i \} \) of \( V, i = 1, \cdots, \dim V \), then

\[
B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = R_{VV_{ij}}^{mn}(v_n \otimes v_m).
\]

It need to pay attention to that the R-matrix in Majid’s paper [M1] is the \( P \circ \cdot \circ P \) of the ordinary R-matrix. In general, \( B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = R_{VV_{ij}}^{mn}(v_n \otimes v_m) \) in some other references. However, \( P \circ R \circ P = R' \) when \( PR \) is symmetric, so then we write occasionally \( B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = R_{VV_{ij}}^{mn}(v_n \otimes v_m) \) for convenience. Notice that we will use Majid’s form of R-matrix in the remaining sections of this paper.

3.3. The extended Hopf algebra \( U^{\text{ext}}_q(\mathfrak{g}) \). Majid gave the concept of “weak antipodes” for the dually paired bialgebras in [M5]. Dually paired bialgebras \( H_1, H_2 \) are said to possess weak antipodes if the canonical maps \( H_1 \rightarrow H_2 ' \) and \( H_2 \rightarrow H_1 ' \) defined by the pairing are invertible in the convolution algebras Hom\( (H_1, H_2 ') \) and Hom\( (H_2, H_1 ') \) respectively. These weak antipodes are denoted by \( S_{H_1}, S_{H_2} \). On the other hand, Majid defined \( \tilde{U}(\mathbb{K}) = A(R) \cong A(R) \) the double cross product bialgebra constructed in [M5]. It consists of two copies of \( A(R) \) generated by 1 and \( m^\pm \) with the cross relations and coalgebra structure:

\[
R m^+_1 m^-_2 = m^-_2 m^+_1 R, \quad R m^+_1 m^-_2 = m^-_2 m^+_1 R, \quad \Delta((m^+)_{ij}^l) = (m^+)_{il}^j \otimes (m^+)_{lj}^i, \quad \epsilon((m^+)_{ij}^l) = \delta_{ij}.
\]

For any invertible R-matrix \( R \), the bialgebras \( \tilde{U}(\mathbb{K}) \) and FRT-bialgebra \( A(R) \) are dually-paired by \( (m^+)_{ij}^l, (m^-)_{ij}^l = (R^{-1})_{lj}^{ki} \) in [M1], moreover, we have the following (revised version of [M5])

**Proposition 3.1.** Let \( R \) be an invertible matrix solution of the QYBE. If matrices \( R^\pm \) and \( (R^{-1})^\pm \) are also invertible, and set \( \tilde{R} = ((R^\pm)^{-1})^\pm, \quad \bar{R}^{-1} = [((R^{-1})^\pm)^{-1}]^\pm \), then the dual pairing \( (\tilde{U}(\mathbb{K}), A(R)) \) possesses weak antipodes, which are defined by

\[
(S_{A(R)}(\mathcal{R}))_{ij}^{kl}(m^+)_{ij}^l = (S_{\tilde{U}(\mathbb{K})}(m^+)_{ij}^l)(\mathcal{R})_{ij}^{kl} = \bar{R}^{ij}_{kl} = ((R^{-1})^\pm)^{ij}_{kl},
\]

\[
(S_{A(R)}(\mathcal{R}))_{ij}^{kl}(m^-)_{ij}^l = (S_{\tilde{U}(\mathbb{K})}(m^-)_{ij}^l)(\mathcal{R})_{ij}^{kl} = \bar{R}^{-1}_{ij} = [((R^{-1})^\pm)^{-1}]_{ij}^l.
\]
Their equivalent expressions in matrices are

\[
(\mathcal{S}_{A(R)}(t_2))(m_1^\dagger) = (\mathcal{S}_{\overline{U(R)}}(m_1^\dagger))(t_2) = \tilde{R},
\]

\[
(\mathcal{S}_{A(R)}(t_2))(m_1^\dagger) = (\mathcal{S}_{\overline{U(R)}}(m_1^\dagger))(t_2) = P \circ \overline{R^{-1}} \circ P.
\]

**Proof.** The corresponding canonical maps for the dual pair \((\overline{U(R)}, A(R))\) are

\[
j_{A(R)}(t'_j) = \langle \cdot, t'_j \rangle \in (\overline{U(R)})^*, \quad j_{\overline{U(R)}}((m^+)_{j}) = \langle (m^+)_{j}, \cdot \rangle \in (A(R))^*.
\]

The units in the convolution algebras \(\text{Hom}(A(R), (\overline{U(R)})^*)\) and \(\text{Hom}(\overline{U(R)}, (A(R))^*)\) are \(\eta_{(\overline{U(R)})^*} \circ \varepsilon_{A(R)}\) and \(\eta_{(A(R))^*} \circ \varepsilon_{\overline{U(R)}}\), respectively, and

\[
[(\eta_{(\overline{U(R)})^*} \circ \varepsilon_{A(R)})(t'_j)][(m^+)_{j}] = \varepsilon_{A(R)}(t'_j)(1, (m^+)_{j}) = \varepsilon_{A(R)}(t'_j)\eta_{\overline{U(R)}}((m^+)_{j}) = \delta_{ij}\delta_{kl} = I^j_{ik}.
\]

\(\eta_{(A(R))^*} \circ \varepsilon_{\overline{U(R)}}((m^+)_{j})(t^j) = I^j_{ik}\) can also be obtained. For any \(t'_j\) and \((m^+)_{j}\), we have

\[
[(\mathcal{S}_{A(R)} \circ j_{A(R)})(t'_j)][(m^+)_{j}] = [(\mathcal{S}_{A(R)}(t'_j))j_{A(R)}(t'_j)][(m^+)_{j}]
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\langle (m^+)_{j}, t'_j \rangle)(m^+)_{j}
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\varepsilon_{A(R)}(t'_j))(m^+)_{j}
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\varepsilon_{A(R)}(t'_j))(m^+)_{j} = (R^2)^{-1}_{i} R^2_{b_{a}} = (R^2)^{-1}_{i} R^2_{b_{a}}
\]

\[
= (R^2)^{-1}_{i} R^2_{b_{a}} R^2_{b_{a}} = 1 = I_{i}^j = \delta_{ij}\delta_{kl}
\]

\[
= [(\eta_{(\overline{U(R)})^*} \circ \varepsilon_{A(R)})(t'_j)][(m^+)_{j}].
\]

(11)

\[
[(\mathcal{S}_{A(R)} \circ j_{A(R)})(t'_j)][(m^+)_{j}] = [(\mathcal{S}_{A(R)}(t'_j))j_{A(R)}(t'_j)][(m^+)_{j}]
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\langle (m^+)_{j}, t'_j \rangle)(m^+)_{j}
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\varepsilon_{A(R)}(t'_j))(m^+)_{j}
\]

\[
= (\mathcal{S}_{A(R)}(t'_j))(\varepsilon_{A(R)}(t'_j))(m^+)_{j} = (R^2)^{-1}_{i} R^2_{b_{a}} = (R^2)^{-1}_{i} R^2_{b_{a}}
\]

\[
= (R^2)^{-1}_{i} R^2_{b_{a}} R^2_{b_{a}} = 1 = I_{i}^j = \delta_{ij}\delta_{kl}
\]

\[
= [(\eta_{(\overline{U(R)})^*} \circ \varepsilon_{A(R)})(t'_j)][(m^+)_{j}].
\]

(12)

\[
[(j_{A(R)} \circ \mathcal{S}_{A(R)})(t'_j)][(m^+)_{j}] = [(j_{A(R)}(t'_j))\mathcal{S}_{A(R)}(t'_j)][(m^+)_{j}]
\]

\[
= (j_{A(R)}(t'_j))(\langle (m^+)_{j}, \mathcal{S}_{A(R)}(t'_j) \rangle)(m^+)_{j}
\]

\[
= (m^+)_{j} j_{A(R)}(t'_j) \mathcal{S}_{A(R)}(t'_j)(m^+)_{j}
\]

\[
= (m^+)_{j} j_{A(R)}(t'_j) \mathcal{S}_{A(R)}(t'_j)(m^+)_{j} = R^2_{i} R^2_{b_{a}} = R^2_{i} R^2_{b_{a}}
\]

\[
= (R^2)^{-1}_{i} R^2_{b_{a}} R^2_{b_{a}} = 1 = I_{i}^j = \delta_{ij}\delta_{kl}
\]

\[
= [(\eta_{(\overline{U(R)})^*} \circ \varepsilon_{A(R)})(t'_j)][(m^+)_{j}].
\]

(13)
According to relations (11), we obtain the following equalities in a similar way.

\[
[(j_{A(R)} \ast S_{A(R)})(t'_j)]((m^-)_j^i) = [(j_{A(R)}(t'_j))S_{A(R)}(t'_j)]((m^-)_j^i)
\]
\[
= (j_{A(R)}(t'_j))((m^-)_j^i)S_{A(R)}((m^-)_j^i)
\]
\[
= \langle (m^-)_j^i, t'_j \rangle S_{A(R)}((m^-)_j^i)
\]
\[
= (R^{-1})_{j\ell}^i ((R^{-1})_{\ell\alpha}^b (R^{-1})_{\alpha\beta}^c)_{j\beta}^{1ab}
\]
\[
= ((R^{-1})_{\ell\alpha}^b (R^{-1})_{\alpha\beta}^c)_{j\beta}^{1ab} = I_{j\ell} = \delta_{ij}\delta_{kl}
\]
\[
= [\eta_{\tilde{U}(R)} \circ \epsilon_{A(R)}((m^-)_j^i)](t_j).
\]

In view of equalities (11)–(14), we obtain

\[
S_{A(R)} \ast j_{A(R)} = j_{A(R)} \ast S_{A(R)} = \eta_{\tilde{U}(R)} \circ \epsilon_{A(R)}.
\]

We obtain the following equalities in a similar way.

\[
[(S_{\tilde{U}(R)} \ast j_{\tilde{U}(R)})(m^+)_j^i)](t'_j) = (S_{\tilde{U}(R)}((m^+)_j^i)S_{\tilde{U}(R)}((m^+)_j^i))(t'_j)
\]
\[
= (R^{i\ell}R_{j\ell}^{-1}ba)_{\ell\alpha}^c = I_{j\ell} = \delta_{ij}\delta_{kl}
\]
\[
= [\eta_{\tilde{U}(R)} \circ \epsilon_{\tilde{U}(R)}((m^+)_j^i)](t_j).
\]

According to relations (15) and (16), we prove that \(S_{\tilde{U}(R)}, S_{A(R)}\) defined by (9) in the Proposition are actually the weak antipodes of the dual pairing \((\tilde{U}(R), A(R))\). \(\square\)

Now, we are only concerned with the bialgebra \(\tilde{U}(R)\) generated by the upper (lower) triangular matrices \(m^+\) (\(m^-\)) for any invertible upper triangular matrix \(n^2 \times n^2\) R-matrix \(R\). Starting from \(\tilde{U}(R)\), we give the following

**Definition 3.1.** Define \(\tilde{U}(R)\) as the quotient algebra of \(\tilde{U}(R)\) modulo the biideal \(D\) generated by

\[
(m^+)_j^i(m^-)_j^i - 1, \quad (m^-)_j^i(m^+)_j^i - 1, \quad i = 1, 2, \ldots, n.
\]

The generators of \(\tilde{U}(R)\) are still denoted by \(m^\pm\) without confusion.

Specially, for the quotient algebra \(\tilde{U}(R)\), we have...
Theorem 3.1. $\overline{U(R)}$ obviously satisfies the following relations:

\begin{equation}
Rm^+_2 m^-_2 = m^+_2 m^+_R, \quad Rm^+_1 m^-_2 = m^-_1 m^+_R,
\end{equation}

\begin{equation}
(m^+_j)(m^-_j) = (m^-_j)(m^+_j) = 1, \quad i = 1, 2, \cdots, n.
\end{equation}

Moreover, $\overline{U(R)}$ is a Hopf algebra with the comultiplication $\Delta$, counit $\epsilon$, and the antipode $S$ determined by

\begin{equation}
\Delta((m^+_j)^i) = (m^+_j)^i \otimes (m^+_j)^i, \quad \epsilon((m^+_j)^i) = \delta_{ij}, \quad S((m^+_j)^i) = ((m^+_j)^i)^{-1}.
\end{equation}

In order to prove the above theorem, we give the following Lemma, which can be checked easily.

Lemma 3.2. Suppose $R$ is any invertible $R$-matrix, $A, B$ are matrices with non-commutative entries. Then we have the following equivalent relations

\begin{equation}
RA_1B_2 = B_2A_1R \iff R'B_2'A_1 = A_1'B_2'R',
\end{equation}

\begin{equation}
RA_2B_1 = B_1A_2R \iff R'B_1'A_2 = A_2'B_1'R'.
\end{equation}

Proof of Theorem 3.1. The other relations in Theorem 3.1 can be proved easily, so we only focus on the antipode. The lower (upper) triangular matrices $(m^+_j)^i', (m^-_j)^i)$ are invertible in virtue of $(m^+_j)(m^-_j)^i = (m^-_j)(m^+_j)^i = 1$ in $\overline{U(R)}$, then we denote the corresponding matrices $[((m^+_j)^i)^{-1}]^i_j$ by $m^\pm_1$, and the entry located at row $i$ and column $j$ by $(m^+_j)^i$. We build the following $\mathbb{C}[q, q^{-1}]$-linear map

\begin{equation}
S : \overline{U(R)} \rightarrow \overline{U(R)}
\end{equation}

\begin{equation}
(m^+_j)^i \rightarrow (m^\pm_1)^i_j : \quad S(m^\pm) = m^\mp.
\end{equation}

According to Lemma 3.2, we have

\begin{equation}
Rm^+_2 m^-_2 = m^+_2 m^+_R
\end{equation}

\begin{equation}
\iff R'(m^+_j)^i (m^-_j)^i_1 = (m^+_j)^i_1 (m^-_j)^i R'
\end{equation}

\begin{equation}
\iff ((m^+_j)^{-1}_2) (m^-_j)^i_1 = R'((m^+_j)^{-1}_1) (m^-_j)^i_2
\end{equation}

\begin{equation}
\iff R'[((m^+_j)^{-1}_2) [(m^-_j)^i_1)]_1 = [(m^+_j)^{-1}_1]_1 [(m^-_j)^i_2]_2 R
\end{equation}

\begin{equation}
\iff Rm^+_2 m^-_1 = m^+_1 m^+_2 R.
\end{equation}

\begin{equation}
Rm^-_2 m^-_2 = m^-_2 m^+_R
\end{equation}

\begin{equation}
\iff R'((m^-_j)^i_1 (m^-_j)^i_2 R'
\end{equation}

\begin{equation}
\iff ((m^-_j)^{-1}_2) (m^-_j)^i_1 = (m^+_j)^i_1 (m^-_j)^i_2
\end{equation}

\begin{equation}
\iff R'[(m^-_j)^{-1}_2 [(m^-_j)^i_1)]_1 = [(m^+_j)^{-1}_1]_1 [(m^-_j)^i_2]_2 R
\end{equation}

\begin{equation}
\iff Rm^-_2 m^-_1 = m^-_1 m^-_2 R.
\end{equation}
We obtain the diagonal entries \((\hat{m}^+)_{ij} = (m^+)_{ij}\) in view of the definition of \(\hat{m}^\pm\), then
\[
(25) \quad (\hat{m}^+)_{ij}(\hat{m}^-)_{ij} = (\hat{m}^-)_{ij}(\hat{m}^+)_j = 1.
\]

So the map \(S\) is the anti-algebra automorphism of \(\overline{U(R)}\) in virtue of \((23)-(25)\). By the equalities \((m^+)_{ij}((m^+)_{ij})^{-1} = ((m^+)_{ij})^{-1}(m^+)_{ij} = I\), we obtain the following equivalent relations
\[
\begin{align*}
((m^+)_{ij}((m^+)_{ij})^{-1})_{ij}^j &= [(m^+)_{ij}((m^+)_{ij})^{-1}]_{ij}^j = \delta_{ij} \\
\iff (m^+)_{ij}(m^+)_{ij} = (\hat{m}^-)_{ij} = \delta_{ij} \\
\iff (m^+)_{ij}(S m^+)_{ij} = (S m^+)_{ij} = \delta_{ij} \\
\iff [m \circ (\text{id} \otimes S) \circ \Delta]((m^+)_{ij}) = [m \circ (\text{id} \otimes \Delta) \circ \Delta]((m^+)_{ij}) = \delta_{ij} = (\eta \circ \epsilon)((m^+)_{ij}).
\end{align*}
\]

Obviously, we obtain \(m \circ (\text{id} \otimes S) \circ \Delta = m \circ (\text{id} \otimes \Delta) \circ \Delta = \eta \circ \epsilon\). So we prove that the map \(S\) is the antipode of \(\overline{U(R)}\). \(\square\)

Let \(T_V\) be an irreducible representation of \(U_q(\mathfrak{g})\), suppose that there exists a basis such that the corresponding \(R\)-matrix \(R_{VV}\) is upper triangular, and \(L\)-functionals \(L^\pm\) are upper (lower) triangular. Then from the structures of bialgebras \(U(L^\pm)\) and \(\overline{U(R_{VV})}\), we obtained easily \([HH]\) that the antipode \(S\) gives the morphism from the bialgebra \(U(L^\pm)\) generated by \(L^\pm\) to \(\overline{U(R_{VV})}\). On the other hand, we notice that the equality \(l^+_i l^-_j = l^-_j l^+_i = 1\) can be obtained by the definition of \(L\)-functionals, so the above morphism can be induced on Hopf algebra \(\overline{U(R_{VV})}\), namely,

\[
\begin{align*}
U(L^\pm) &\xrightarrow{S} U(R_{VV}) \\
\downarrow S &\quad \downarrow S \\
\overline{U(R_{VV})} &\xrightarrow{S} \overline{U(R_{VV})}
\end{align*}
\]

Thence, we obtain the following

**Lemma 3.3.** Under the antipode \(S\) of \(U_q(\mathfrak{g})\), the expression of generators \(m^\pm\) for \(\overline{U(R_{VV})}\) can be obtained by the \(L\)-functionals of \(U_q(\mathfrak{g})\) associated with representation \(T_V\).

From the explicit form \((7)\) of universal \(R\)-matrix \(R\) for \(U_q(\mathfrak{g})\), we know that there exist factors of form \(\exp(a_i h_i)\)’s in entries of \(m^\pm\), and these rational numbers \(a_i\)’s are not integers in general. When \(R_{VV}\) is one of standard \(R\)-matrices, \(\overline{U(R_{VV})}\) coincides with the extended Hopf algebra \(U_q^{ext}(\mathfrak{g})\) with \(K_i^a(= \exp(a_i h_i))\) adjoined (see \([KS]\)). In view of this fact, we still use notation \(U_q^{ext}(\mathfrak{g})\) instead of \(\overline{U(R_{VV})}\) for \(R_{VV}\) (not necessarily standard).
3.4. **The coquasitriangularity of Hopf algebra** \( H_{R_{VV}} \). After introducing Hopf algebra \( H_R \) in the preceding section, we will furthermore consider these Hopf algebras for the general \( R \)-matrices, not necessarily standard \( R \)-matrices. As we know, for the standard \( R \)-matrices, \( H_R \)'s are the coquasitriangular Hopf algebras. What about \( H_R \) for the general \( R \)-matrices? Firstly, we give some equalities about \( R \)-matrix under some conditions.

**Proposition 3.2.** Suppose \( R \) is an invertible \( R \)-matrix, and the matrices \( R^i, (R^{-1})^i, i = 1, 2 \) are also invertible, then we have the following equalities

\[
\begin{align*}
(26) & \quad (R^i)_{12}(R^i)_{13}(R^i)_{23} = (R^i)_{23}(R^i)_{13}(R^i)_{12}, \\
(27) & \quad R_{12}(R^i)_{13}^{-1}(R^i)_{23}^{-1} = (R^i)_{23}^{-1}(R^i)_{13}^{-1}R_{12}, \\
(28) & \quad (R^i)_{13}(R^i)_{23}R_{12} = R_{12}(R^i)_{23}(R^i)_{13}, \\
(29) & \quad R_{12}(R^i)_{13}^{-1}((R^{-1})^i)_{12}^{-1} = ((R^{-1})^i)_{13}^{-1}(R^i)_{12}^{-1}R_{13}, \\
(30) & \quad R_{23}((R^{-1})^i)_{12}^{-1}((R^{-1})^i)_{13}^{-1} = ((R^{-1})^i)_{13}^{-1}((R^{-1})^i)_{12}^{-1}R_{23}, \\
(31) & \quad (R^i)_{23}((R^i)_{13}^{-1}((R^{-1})^i)_{13}^{-1})_{23}^{-1} = (R^i)_{13}^{-1}((R^{-1})^i)_{23}^{-1}(R^i)_{12}, \\
(32) & \quad (R^i)_{12}((R^{-1})^i)_{23}^{-1}((R^{-1})^i)_{13}^{-1} = ((R^{-1})^i)_{13}^{-1}((R^{-1})^i)_{23}^{-1}(R^i)_{12}, \\
(33) & \quad (R^i)_{12}^{-1}((R^{-1})^i)_{13}^{-1}R_{12} = R_{12}(R^i)_{13}^{-1}((R^{-1})^i)_{23}^{-1}, \\
(34) & \quad (R^i)_{23}^{-1}((R^{-1})^i)_{13}^{-1}R_{13} = R_{13}^{-1}((R^{-1})^i)_{23}^{-1}(R^i)_{12}^{-1}, \\
(35) & \quad (R^i)_{23}^{-1}((R^i)_{13}((R^{-1})^i)_{13}^{-1})_{12}^{-1} = ((R^{-1})^i)_{12}^{-1}((R^{-1})^i)_{13}^{-1}R_{23}^{-1}.
\end{align*}
\]

**Proof.** All equalities follow mainly from the equality \( R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \) and some skills of matrices. \( \square \)

**Theorem 3.2.** Suppose \( R \) is an invertible \( R \)-matrix, the matrices \( R^i, (R^{-1})^i, i = 1, 2 \) are also invertible, if these matrices satisfy the condition (2), then \( H_R \) is a coquasitriangular Hopf algebra with its coquasitriangular structure given by

\[
\begin{align*}
(36) & \quad \tau(t^i_j \otimes \bar{t}^i_j) = R^k_{ji}, \quad \tau(t^i_j \otimes \bar{t}^i_j) = (R^i)^{jk}_{ji}, \\
(37) & \quad \tau(t^i_j \otimes \bar{t}^i_j) = ((R^{-1})^i)^{jk}_{ji}, \quad \tau(t^i_j \otimes \bar{t}^i_j) = ((R^i)^{-1})^{jk}_{ji}.
\end{align*}
\]

The equivalent statement in matrices is

\[
\begin{align*}
(38) & \quad \tau(T_1 \otimes T_2) = R_{12}, \quad \tau(T_1 \otimes T_2) = (R^i)_{12}, \\
(39) & \quad \tau(T_1 \otimes T_2) = ((R^{-1})^i)_{12}, \quad \tau(T_1 \otimes T_2) = ((R^i)^{-1})_{12}.
\end{align*}
\]

**Proof.** Let \( \mathbb{C}(t^i_j, \bar{t}^i_j) \) denote the free algebra generated by \( 1 \) and \( t^i_j, \bar{t}^i_j \). We first construct two linear forms \( \tau \) and \( \bar{\tau} \) on \( \mathbb{C}(t^i_j, \bar{t}^i_j) \otimes \mathbb{C}(t^i_j, \bar{t}^i_j) \). To do so, we define \( \tau \) and \( \bar{\tau} \) by (38), (39), (40)
Recall that (43) can obtain a well-defined linear form $\bar{\nu}(T_1 \otimes T_2) = (R^{1}_{VV})_{12}$, $\bar{\nu}(\tilde{T}_1 \otimes T_2) = (R^{1}_{VV})_{12}$, $\bar{\nu}(T_1 \otimes \tilde{T}_2) = [(R^{1}_{VV})_{12}]^{-1}_{12}$, $\bar{\nu}(T_1 \otimes T_2) = (R^{1}_{VV})_{12}$.

And we also define

$$\nu(1 \otimes t^i_j) = \nu(1 \otimes \tilde{t}^i_j) = \delta_{ij}, \quad \nu(t^i_j \otimes 1) = \nu(\tilde{t}^i_j \otimes 1) = \delta_{ij}. $$

Since $\mathbb{C}(t^i_j, \tilde{t}^i_j)$ is the free algebra with generators $t^i_j, \tilde{t}^i_j$ and a coalgebra structure (see $H_R$ in Section 2), there is a unique extension of $\nu$ to a linear form $\bar{\nu} : \mathbb{C}(t^i_j, \tilde{t}^i_j) \otimes \mathbb{C}(t^i_j, \tilde{t}^i_j) \rightarrow \mathbb{C}$ such that the following property holds

$$\nu(ab \otimes c) = \nu(a \otimes c)\nu(b \otimes c_{(1)}), \quad \nu(a \otimes bc) = \nu(a_{(1)} \otimes c)\nu(a_{(2)} \otimes b). $$

Similarly, $\bar{\nu}$ extends uniquely to a linear form $\bar{\nu} : \mathbb{C}(t^i_j, \tilde{t}^i_j) \otimes \mathbb{C}(t^i_j, \tilde{t}^i_j) \rightarrow \mathbb{C}$ such that (43) holds for the coalgebra $\mathbb{C}(t^i_j, \tilde{t}^i_j)^{op}$.

The crucial step of the proof is to show that $\nu$ and $\bar{\nu}$ pass to linear forms on $H_R \otimes H_R$. Recall that $H_R$ is the quotient of the algebra $\mathbb{C}(t^i_j, \tilde{t}^i_j)$ by relations in (1). By (43) and the definition of $\nu$, we compute

$$\nu(T_{12}T_1 T_2 \otimes T_3) = R_{12}\nu(T_1 T_2 \otimes T_3) = R_{12}\nu(T_1 \otimes T_3)\nu(T_2 \otimes T_3) = R_{12}R_{13}R_{23},$$

$$\nu(T_2 T_1 T_{12} \otimes T_3) = \nu(T_2 T_1 \otimes T_3)R_{12} = \nu(T_2 \otimes T_3)\nu(T_1 \otimes T_3)R_{12} = R_{23}R_{13}R_{12},$$

so we obtain

$$\nu((R_{12}^{-1}T_1 T_2 - T_2 T_1)R_{12} \otimes T_3) = 0.$$ 

Similarly, according to (27), (32), (26), (34), (31) respectively, we obtain

$$\nu((R_{12}^{-1}T_1 T_2 - T_2 T_1) \otimes T_3) = 0,$$

$$\nu((R_{12}^{-1}T_1 T_2 - T_2 T_1)R_{12} \otimes T_3) = 0,$$

$$\nu((R_{12}^{-1}T_1 T_2 - T_2 T_1R_{12}^{-1}) \otimes T_3) = 0,$$

$$\nu((R_{12}^{-1}T_1 T_2 - T_2 T_1(R_{12})_{12}^{-1}) \otimes T_3) = 0.$$ 

Again, applying (43), and equalities (30), (31), (26), (27), (35), we get that

$$\nu(T_1 \otimes ((R_{23}^{-1}T_2 T_3 - T_3 T_2)R_{23})) = 0,$$

$$\nu(T_1 \otimes ((R_{23}^{-1}T_2 T_3 - T_3 T_2)R_{23}^{-1})) = 0,$$

$$\nu(T_1 \otimes ((R_{23}^{-1}T_2 T_3 - T_3 T_2R_{23}^{-1})) = 0.$$ 

Furthermore, by (42), we have $\nu(1 \otimes *) = \nu(1 \otimes *) = 0$, where $*$ denotes the relations in (1). Therefore, $\nu$ induces a well-defined linear form, again denoted by $\nu$, on $H_R \otimes H_R$. Now we repeat the preceding reasoning with $\nu$ replaced by $\bar{\nu}$ and $\mathbb{C}(t^i_j, \tilde{t}^i_j)$ by $\mathbb{C}(t^i_j, \tilde{t}^i_j)^{op}$. Similarly, we can obtain a well-defined linear form $\bar{\nu}$ on $H_R \otimes H_R$. 
Next, we show that two linear forms $\tau$ and $\bar{\tau}$ on $H_R \otimes H_R$ satisfy the conditions of the Definition 2.3. By $\Delta(T) = T \otimes T$, $\Delta(\bar{T}) = \bar{T} \otimes \bar{T}$, we obtain that
\[
\Delta(T_1 \otimes T_2) = (T_1 \otimes T_2) \otimes (T_1 \otimes T_2), \quad \Delta(\bar{T}_1 \otimes T_2) = (\bar{T}_1 \otimes T_2) \otimes (\bar{T}_1 \otimes T_2),
\]
\[
\Delta(T_1 \otimes \bar{T}_2) = (T_1 \otimes \bar{T}_2) \otimes (T_1 \otimes T_2), \quad \Delta(\bar{T}_1 \otimes \bar{T}_2) = (\bar{T}_1 \otimes \bar{T}_2) \otimes (T_1 \otimes T_2).
\]

Then by definition, we can obtain easily that
\[
(47) \quad \tau \ast \bar{\tau} = \bar{\tau} \ast \tau = \epsilon \otimes \epsilon.
\]

It is obvious that the relations in (43) are equivalent to
\[
(48) \quad \tau \circ (\cdot \otimes \text{id}) = \tau_{13} \ast \tau_{23}, \quad \tau \circ (\text{id} \otimes \cdot) = \tau_{13} \ast \tau_{12}.
\]

To prove the relation $m^{\text{op}} = \tau \ast m \ast \bar{\tau}$ is equivalent to prove that
\[
(49) \quad \tau(a(1) \otimes b(1))a(2) _{b(2)} = \tau(a(2) \otimes b_{(2)})b(1)a(1)
\]
holds for any $a, b \in H_R$. If equation (49) holds for $a', b', c'$, then by (43), we have
\[
(50) \quad c'_{(1)}(a' b')_{(1)} \tau(a' b')_{(2)} \otimes c'_{(2)} = c'_{(1)}(a' b')_{(1)} \tau(a' b')_{(2)} \otimes c'_{(3)}
\]
\[
= \tau((a' b')_{(1)} \otimes c'_{(1)})a'_{(2)} b'_{(2)} \tau(b'_{(1)} \otimes c'_{(2)})
\]
\[
= \tau((a' b')_{(1)} \otimes c'_{(1)})a'_{(2)} b'_{(2)} \tau(b'_{(1)} \otimes c'_{(3)})
\]

Namely, according to (50) and (51), we verify that equation (49) holds for $a = a', b = b' c'$ and for $a = a' b', b = c'$ provided that they hold for $a = a', b = b'$, for $a = a', b = c'$ and for $a = b', b = c'$, respectively. Thus it suffices to show that equation (49) holds for 1 and the generators. By (42), we obtain
\[
\tau(1 \otimes t_u^i) t_r^j = \epsilon(t_u^i) r_j^i = t_u^i \epsilon(t_r^j) = \tau(1 \otimes t_u^i) t_r^j, \quad \tau(1 \otimes t_u^i) r_j^i = \epsilon(t_u^i) r_j^i = t_u^i \epsilon(t_r^j) = \tau(1 \otimes t_u^i) r_j^i.
\]

Then associated with $\tau(1 \otimes ab) = \tau(1 \otimes b) \tau(1 \otimes a)$, we have
\[
(52) \quad \tau(1 \otimes a(1)) a(2) = \tau(1 \otimes a(2)) a(1), \quad \text{for} \ a \in H_R.
\]
Similarly, we can prove
\begin{equation}
\tau(a_1 \otimes 1)a_2 = \tau(a_2 \otimes 1)a_1, \quad \text{for } a \in H_R.
\end{equation}
Namely, if \( a \) or \( b \) is 1, equation (49) is proved. Moreover, by the definition of \( \tau \), we can obtain that
\begin{equation}
m^\text{op} \ast \tau(T_1 \otimes T_2) = T_2T_1\tau(T_1 \otimes T_2) = T_2T_1R_{12}, \quad \tau \ast m(T_1 \otimes T_2) = \tau(T_1 \otimes T_2)T_1T_2 = R_{12}T_1T_2,
\end{equation}

namely, we have the following equivalent relation
\begin{equation}
m^\text{op} \ast \tau(T_1 \otimes T_2) = \tau \ast m(T_1 \otimes T_2) \iff R_{12}T_1T_2 = T_2T_1R_{12}.
\end{equation}

Hence, we can obtain other equivalent relations in a similar way.
\begin{equation}
m^\text{op} \ast \tau(T_1 \otimes \tilde{T}_2) = \tau \ast m(T_1 \otimes \tilde{T}_2) \iff \tilde{T}_2T_1(R^2)^{-1} = (R^2)^{-1}T_1\tilde{T}_2,
\end{equation}
\begin{equation}
m^\text{op} \ast \tau(\tilde{T}_1 \otimes T_2) = \tau \ast m(\tilde{T}_1 \otimes T_2) \iff T_2\tilde{T}_1(R^{-1})^\dagger = (R^{-1})^\dagger \tilde{T}_1T_2,
\end{equation}
\begin{equation}
m^\text{op} \ast \tau(\tilde{T}_1 \otimes \tilde{T}_2) = \tau \ast m(\tilde{T}_1 \otimes \tilde{T}_2) \iff \tilde{T}_2\tilde{T}_1R' = R'\tilde{T}_1\tilde{T}_2.
\end{equation}

Here, the relation \((R^{-1})^\dagger T_1T_2 = T_2\tilde{T}_1(R^{-1})^\dagger\) can be obtained by \( TT^i = T'^iT = I\) and Proposition 3.4, namely,
\begin{equation}
R'T_1T_2 = T_2\tilde{T}_1R' \iff RT_2^iT_1^i = T_1^iT_2^iR \iff T_2^iT_1^iR^{-1} = R^{-1}T_1^iT_2^i
\iff T_1^iR^{-1}T_2^i = T_2R^{-1}T_1^i \iff (R^{-1})^\dagger \tilde{T}_1T_2 = T_2\tilde{T}_1(R^{-1})^\dagger.
\end{equation}

Then equation (49) holds for generators \( a \) or \( b \) in virtue of (54)–(57). Up to now, with the conditions in the Theorem, we have proved that \( \tau \) is actually a universal \( r \)-form of \( H_R \), and \( H_R \) is coquasitriangular. The proof is complete. \( \square \)

We find that many conclusions result from some requirements for \( R \)-matrix, for instance, \( R^i, (R^{-1})^\dagger, i = 1, 2 \) are invertible, and \((R^{-1})^\dagger P(R^2)^{-1}PK_0 = \text{const} \cdot K_0\). However, it is uneasy to confirm these conditions. To our surprise, for the \( R \)-matrix \( R_{VV} \) associated to any finite-dimensional irreducible \( U_q(\mathfrak{gl}) \)-module, we can confirm this fact according to some features of the representation.

**Proposition 3.3.** Assume that \( V \) is an irreducible finite-dimensional \( U_q(\mathfrak{gl}) \)-module, then the matrices \( R_{VV}^1, (R_{VV}^{-1})^\dagger, i = 1, 2 \) are invertible, and they satisfy the FRT-condition
\begin{equation}
(R_{VV}^{-1})^\dagger P(R_{VV}^2)^{-1}PK_0 = \text{const} \cdot K_0.
\end{equation}

**Proof.** As well known, for any irreducible finite-dimensional module with the highest weight \( \lambda \), there exist uniquely crystal bases \((\mathcal{L}(\lambda), \mathcal{B}(\lambda)) (\mathcal{B}(\infty), \mathcal{L}(\infty))\), which are bases of \( U_q(\mathfrak{gl}) \)-modules at “\( q = 0 \)” (\( \infty \)), respectively. Crystal bases have a nice property, that is, for any \( b, b' \in \mathcal{B}(\lambda) \) and \( \alpha \) is a simple root: \( b = \hat{E}_\alpha b' \iff b' = \hat{F}_\alpha b \), where \( \hat{E}_\alpha, \hat{F}_\alpha \) are famous Kashiwara operators. Moreover, Kashiwara globalize this notion. Namely, with the aid
of a crystal base, Kashiwara constructed a unique base named the global crystal base of any finite-dimensional highest weight irreducible $U_q(g)$-module that has similar properties to crystal bases. The (global) crystal bases theory for quantum groups was introduced by Kashiwara in [Ka1], [Ka2], [Ka3].

Based on the global crystal base theory of quantum groups, for the finite-dimensional irreducible $U_q(g)$-module $V$, $V$ has a basis $\{v_i\}$ of $V$ with index-raising weights such that the actions of $T(V(E_i))$'s and $T(V(F_i))$'s raise and descend the indices of the basis, respectively. Moreover, they satisfy $E_\alpha(v_i) = f v_k \iff F_\alpha(v_k) = g v_i$ for some $f, g \in \mathbb{C}[q, q^{-1}]^*$. In order to analyze the indices of $R$-matrix conveniently, we substitute the expression (8) for $B_{VV} \circ (T_V \otimes T_V)(S \otimes \mathrm{id})(S \otimes \mathrm{id})$ and also obtain the upper triangular $R$-matrix $R_{VV}$ by with respect to the lexicographic order on the basis of $V^{\otimes 2}$ induced by the chosen basis $v_i$ of $V$. Since $(R_{VV}^{ij})_{kl} = R_{VV}^{ij}$, we have $i \leq k$ and $j \leq l$ by the defining formula of $R_{VV}$ if $(R_{VV}^{ij})_{kl} \neq 0$. More precisely, by its upper triangularity, we have $(ij) < (kl)$ if $i < k$; and if $i = k$, we have $l = j$, i.e., $(ij) = (kl)$. So, we prove that $(ij) \leq (kl)$ if $(R_{VV}^{ij})_{kl} \neq 0$. This means $R_{VV}^{ij}$ is also upper triangular. Furthermore, we obtain $\det(R_{VV}^{ij}) = \det(R_{VV})$ by $(R_{VV}^{ij})_{kl} = R_{VV}^{ij}$, so $R_{VV}^{ij}$ is invertible. We can also prove that $(R_{VV}^{-1})^{ij}$ is invertible owing to $R_{VV}^{-1} = B_{VV} \circ (T_V \otimes T_V)(S \otimes \mathrm{id})(S \otimes \mathrm{id})$. In the same way, we can prove that matrices $R_{VV}^{ij}$ and $(R_{VV}^{-1})^{ij}$ are invertible.

For the finite-dimensional irreducible $U_q(g)$-module $V$ involved, according to the equivalent relation $E_\alpha(v_i) = f v_k \iff F_\alpha(v_k) = g v_i$ for some $f, g \in \mathbb{C}[q, q^{-1}]^*$, we obtain that $E_1^{n_1} F_1^{n_2} \cdots E_i^{n_i}(v_j) = x v_i \iff F_i^{n_i} \cdots F_1^{n_2} E_1^{n_1}(v_i) = y v_j$ for some $x, y \in \mathbb{C}[q, q^{-1}]^*$. Thus $R_{VV}^{ai} = \text{const} \cdot \delta_{ij}^a$ by (8), and the equality $(R_{VV}^{-1})^{ja} = \text{const} \cdot \delta_{ij}^a$ can be obtained in the similar way, then $R_{VV}^{ai} = R_{VV}^{ja} = \text{const} \cdot \delta_{ij}^a$. With these, by the equality $(R_{VV}^{-1})^{ij} R_{VV}^{ia} = R_{VV}^{ij} = \text{const} \cdot \delta_{ij}^a$. With these analysis, we obtain

$$\left[(R_{VV}^{-1})^{ij} P(R_{VV}^{2})^{-1} P K_0^{ij}\right]_{kl} = \sum_{m, n, p, d} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} K_0^{pq}\right)_{kl}$$

$$= \sum_{m, n, p, d} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

$$= \sum_{m, n, p} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

$$= \sum_{m, n, p} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

$$= \sum_{m, n, p} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

$$= \sum_{m, n, p} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

$$= \sum_{m, n, p} \left(\left((R_{VV}^{ij})_{mn}^{pq} P(R_{VV}^{2})^{-1} P\right)_{pq} \delta_{pq} \delta_{kl}\right)$$

The proof is complete. □
By Theorem 3.2 and Proposition 3.3, we obtain an important conclusion as desired:

**Corollary 3.1.** Assume that \( V \) is an irreducible finite-dimensional \( U_q(\mathfrak{g}) \)-module, \( R_{VV} \) is the corresponding \( R \)-matrix, then \( H_{R_{VV}} \) is a coquasitriangular Hopf algebra, so that the categories of left (right) \( H_{R_{VV}} \)-comodules \( H_{R_{VV}}^\text{left} \) (\( H_{R_{VV}}^\text{right} \)) are braided.

### 3.5. The weakly quasitriangular dual pairs for general \( R \)-matrices.

Starting from any irreducible representation \( T_V \) of \( U_q(\mathfrak{g}) \) or \( U_q(\mathfrak{h}) \), we have the extended Hopf algebra \( U_q^\text{ext}(\mathfrak{g}) \) generated by \( m^\pm \) and the coquasitriangular Hopf algebra \( H_{R_{VV}} \) corresponding to the \( R \)-matrix \( R_{VV} \). In general, \( R_{VV} \) is no longer regular in the sense of Majid [M1] so that the Majid’s double-bosonization construction Theorem is invalid for the irregular \( R \)-matrix. In order to realize the inductive constructions of the exceptional quantum groups by double-bosonization procedure, we have to try to generalize the Majid’s double-bosonization construction Theorem to the irregular cases. To this end, we consider the dual pair of bialgebras \((\widehat{U}(\mathfrak{R}), H_{\mathfrak{R}})\) for the irregular \( R \)-matrices, instead of the original weakly quasitriangular dual pair of bialgebras \((\widehat{U}(\mathfrak{R}), A(\mathfrak{R}))\) for the regular \( R \)-matrices served as the starting point of the Majid’s framework. Moreover, we can establish a weakly quasitriangular dual pair of Hopf algebras \((U_q^\text{ext}(\mathfrak{g}), H_{R_{VV}})\) as we desired, where \( U_q^\text{ext}(\mathfrak{g}) = \widehat{U}(\mathfrak{R}) \) in subsection 3.3.

**Theorem 3.3.** Let \( T_V \) be an irreducible representation of \( U_q(\mathfrak{g}) \), \( R_{VV} \) the corresponding \( R \)-matrix, then there exists a weakly quasitriangular dual pair between \( U_q^\text{ext}(\mathfrak{g}) \) and \( H_{R_{VV}} \), given by

\[
\langle (m^+)_{ij}, \tilde{t}^i \rangle = R_{VV}^{ik} j, \quad \langle (m^-)_{ij}, \tilde{t}^i \rangle = (R_{VV}^{-1})^{ij} k,
\]

\[
\langle (m^+)_{ij}, \tilde{t}^i \rangle = (R_{VV}^{-1})^{ik} j, \quad \langle (m^-)_{ij}, \tilde{t}^i \rangle = [R_{VV}^{-1}]^{-1}^{ij} k.
\]

The convolution-invertible algebra/anti-coalgebra maps \( \mathcal{A}, \mathcal{A}^\text{c} \) in \( \text{Hom}(H_{\mathfrak{R}}, U_q^\text{ext}(\mathfrak{g})) \) are

\[
\mathcal{A}(\tilde{t}^i) = (m^+)_{ij}, \quad \mathcal{A}^\text{c}(t^i) = (m^-)^{-1} j, \quad \mathcal{A}(\tilde{t}^i) = (m^-)^{-1} j, \quad \mathcal{A}^\text{c}(t^i) = (m^+)_{ij}.
\]

Moreover, the following relations

\[
\langle \mathcal{A}(a), b \rangle = \langle \mathcal{A}^\text{c}(b), a \rangle, \quad \partial^R h = \mathcal{A} \ast (\partial^L h) \ast \mathcal{A}^{-1}, \quad \partial^L h = \mathcal{A}^\text{c} \ast (\partial^L h) \ast \mathcal{A}^{-1}
\]

hold for any \( a \in H_{R_{VV}}, h \in U_q^\text{ext}(\mathfrak{g}) \).

In order to prove the above theorem, we give the following Lemma 3.4, which can be proved in virtue of the definition of weakly quasitriangular dual pair.

**Lemma 3.4.** \((H, A)\) is a weakly quasitriangular dual pair of Hopf algebras, \( \mathcal{A}, \mathcal{A}^\text{c} \) are the corresponding convolution-invertible algebra/anti-coalgebra maps, and \( \partial^L h, \partial^R h \) are left
and right “differentiation operators” for any fixed $h \in H$. Then for any $a, b, c \in A$, $g, h \in H$, we have

\begin{align}
\partial^L(gh) &= \partial^L g \ast \partial^L h, \\
\partial^R(gh) &= \partial^R g \ast \partial^R h,
\end{align}

\begin{align}
\langle \bar{\mathcal{R}}(ab), c \rangle &= \langle \mathcal{R}^{-1}(c), ab \rangle, \\
\langle \bar{\mathcal{R}}(a), bc \rangle &= \langle \mathcal{R}^{-1}(bc), a \rangle,
\end{align}

\begin{align}
\partial^R(gh) &= \bar{\mathcal{R}} \ast \partial^L(gh) \ast \mathcal{R}^{-1}, \\
\partial^L(gh) &= \bar{\mathcal{R}} \ast \partial^R(gh) \ast \mathcal{R}^{-1}.
\end{align}

**Proof of Theorem 3.3.** For convenience, we denote $R_{V V}$ by $R$ in the proof, and also give the equivalent expressions in matrix form for those relations in Theorem 3.3:

\begin{align}
\langle m^+_1, T_2 \rangle &= (R_{V V})_{12}, \quad \langle m^+_1, T_2 \rangle = (R^+_{12})_{21} = P \circ (R^+_{12})_{12} \circ P, \\
\langle m^+_1, \bar{T}_2 \rangle &= ((R^+_{12})^{-1})_{12}, \quad \langle m^+_1, \bar{T}_2 \rangle = (R^+_2)^{-1} = P \circ (R^+_2)^{-1} \circ P, \\
\bar{\mathcal{R}}(t) &= m^+, \quad \bar{\mathcal{R}}(\bar{t}) = ((m^+)^{-1})^j, \quad \bar{\mathcal{R}}(t) = m^-, \quad \bar{\mathcal{R}}(\bar{t}) = ((m^{-1})^{-1})^j.
\end{align}

**Step 1. The dual pairing.**

$\mathbb{C}((m^+)_{j}^j, (m^{-})_{j}^j)$ denotes the free algebra generated by $m^\pm$. Obviously, (59) and (60) define a dual pairing between $\mathbb{C}(t^j, \bar{t}^j)$ and $\mathbb{C}((m^+)_{j}^j, (m^{-})_{j}^j)$. By the QYBE, we have

\begin{align}
R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12} \iff R_{13}R_{12}R_{23}^{-1} = R_{23}^{-1}R_{12}R_{13}, \\
&\iff R_{23}R_{13}^{-1}R_{12} = R_{13}^{-1}R_{23}^{-1}R_{12} = R_{13}^{-1}R_{23}^{-1} = R_{13}^{-1}R_{23}^{-1}.
\end{align}

Associated with (27), (29) and (30) in Proposition 3.2, we obtain

\begin{align}
\langle R_{12}m^+_1m^+_2, T_3 \rangle &= R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} = \langle m^+_1m^+_2R_{12}, T_3 \rangle, \\
\langle R_{12}m^+_1m^+_2, \bar{T}_3 \rangle &= R_{12}(R^+_2)_{13}(R^+_2)_{23}^{-1} = \langle R^+_2(\bar{t}^j)^{13} R^+_2 \rangle R_{12}^{-1} = \langle m^+_1m^+_2R_{12}, \bar{T}_3 \rangle,
\end{align}

\begin{align}
\langle R_{23}m^+_2m^+_3, T_1 \rangle &= R_{23}R_{13}^{-1}R_{12}^{-1} = R_{13}^{-1}R_{12}^{-1}R_{23} = \langle m^+_2m^+_2R_{23}, T_1 \rangle, \\
\langle R_{23}m^+_2m^+_3, \bar{T}_1 \rangle &= R_{23}(R^{-1})_{13}^{-1}(R^{-1})_{13}^{-1} = \langle R^{-1}t^j \rangle^{-1}(R^{-1})_{13}^{-1}R_{23} = \langle m^+_2m^+_2R_{23}, \bar{T}_1 \rangle.
\end{align}

\begin{align}
\langle R_{13}m^+_1m^+_2, T_2 \rangle &= R_{13}R_{12}R_{23}^{-1} = R_{23}^{-1}R_{12}R_{13} = \langle m^+_1m^+_1R_{13}, T_2 \rangle, \\
\langle R_{13}m^+_1m^+_2, \bar{T}_2 \rangle &= R_{13}(R^+_2)_{12}^{-1}(R^{-1})_{13}^{-1} = \langle R^{-1}t^j \rangle^{-1}(R^+_2)_{12}^{-1}R_{13} = \langle m^+_1m^+_1R_{13}, \bar{T}_2 \rangle.
\end{align}

So the dual pairing between $\mathbb{C}(t^j, \bar{t}^j)$ and $\mathbb{C}((m^+)_{j}^j, (m^{-})_{j}^j)$ can pass to the dual pairing of $U_q^{\text{ext}}(g)$ and $\mathbb{C}(t^j, \bar{t}^j)$. On the other hand, in view of $\Delta^{op}(m^+) = m^+ \otimes m^+$ and the equalities (31), (32), (33), (34) in Proposition 3.2, we obtain

\begin{align}
\langle m^+_1, R_{23}T_2T_3 \rangle &= R_{23}R_{13}R_{23} = R_{12}R_{13}R_{23} = \langle m^+_1, T_3T_2R_{23} \rangle, \\
\langle m^+_1, R_{12}T_1T_2 \rangle R_{23}^{-1}R_{23}^{-1}R_{12} &= R_{12}R_{23}^{-1}R_{12} = \langle m^+_1, T_2T_1R_{12} \rangle.
\end{align}
\[ \langle m^+, R_{12}^i \tilde{T}_2 \tilde{T}_3 \rangle = (R')_3 \langle (R')_{12} (R')_{23} \rangle_{13}^{-1} \langle R'^{-1}_{23} \rangle_{12}^{-1} \]
\[ = (R'^{-1}_{23})_2 \langle (R')_{12} \rangle_{13}^{-1} \langle R'^{-1}_{23} \rangle_{12} = \langle m^+, \tilde{T}_2 R'_i \rangle, \]
\[ \langle m^-_3, R_{12}^i \tilde{T}_1 \tilde{T}_2 \rangle = (R')_{12} \langle (R')_{23} \rangle_{23}^{-1} \langle (R')_{12} \rangle_{13}^{-1} \]
\[ = (R'^{-1}_{23})_2 \langle (R')_{12} \rangle_{13}^{-1} \langle R'^{-1}_{23} \rangle_{12} = \langle m^-_3, \tilde{T}_2 R'_i \rangle. \]

Then we prove that (59) and (60) define a dual pairing of \( U^\text{ext}_q(\mathfrak{g}) \) and \( H_{R_{12}} \).

**Step 2.** \( \mathcal{R}, \mathcal{S} \) are algebra/anti-coalgebra maps.

We set \( R = A_i \otimes B_i \), and by Lemma 3.4, we obtain that
\[ Rm^+_1 m^+_2 = m^+_1 m^+_2 R \iff R (m^+_1 m^+_1)^i_j = (m^+_1 m^+_1)^i_j R \]
\[ \iff R ((m^+_1 m^+_1)^i_j R)_{12} = R (m^+_1 m^+_1)^i_j R_{12} \]
\[ \iff (m^+_1 m^+_1)^i_j R_{12} = (m^+_1 m^+_1)^i_j R_{12} \]
\[ \iff m^+_1 m^+_1 R = m^+_1 m^+_1 R \]
\[ \iff m^+_1 m^+_1 = m^+_1 m^+_1 \]
\[ \iff m^+_1 m^+_1 = m^+_1 m^+_1 \]

Then in virtue of (74) and (75), we obtain that \( \mathcal{R}, \mathcal{S} \) are algebra homomorphisms. On the other hand, since \( \Delta \) is an algebra homomorphism, we have
\[ \delta_{jk} \otimes 1 = \delta_{jk} \Delta(1) = \Delta(\delta_{jk}) = \Delta(\delta_{jk}) = \Delta((m^+_1 m^+_1)^i_j) = \Delta((m^+_1 m^+_1)^i_j) = \Delta ((m^+_1 m^+_1)^i_j), \]

Then we obtain that \( \Delta((m^+_1 m^+_1)^i_j) = ((m^+_1 m^+_1)^i_j) \otimes (m^+_1 m^+_1)^i_j \), so
\[ (\mathcal{R} \otimes \mathcal{R}) \Delta(j^i) = \mathcal{R} (j^i) \otimes \mathcal{R} (j^i) = (m^+_1 m^+_1)^i_j \otimes (m^+_1 m^+_1)^i_j \]
\[ = \Delta^\text{cop}((m^+_1 m^+_1)^i_j) = \Delta^\text{cop}((m^+_1 m^+_1)^i_j), \]
\[ (\mathcal{R} \otimes \mathcal{R}) \Delta(j^i) = \mathcal{R} (j^i) \otimes \mathcal{R} (j^i) = [(m^+_1 m^+_1)^i_j] \otimes [(m^+_1 m^+_1)^i_j] \]
\[ = \Delta^\text{cop}([(m^+_1 m^+_1)^i_j]) = \Delta^\text{cop}([(m^+_1 m^+_1)^i_j]). \]

We prove that \( \mathcal{R} \) is an anti-coalgebra map by (76) and (77). In a similar analysis, we can prove that \( \mathcal{S} \) is also an anti-coalgebra map.

**Step 3.** \( \mathcal{R}, \mathcal{S} \) are convolution-invertible.
Since the upper (lower) triangularity of \(m^+ (m^-)\), the matrices \((m^+)^{-1} (m^-)^{-1}\) are upper (lower) triangular, then matrices \(((m^+)^{-1})^t ((m^-)^{-1})^t\) are lower (upper) triangular, and \[1 = \left((m^+)^{-1}ight)^t_{ij} \left((m^-)^{-1}ight)^t_{ij} = \left((m^-)^{-1}ight)^t_{ij} \left((m^+)^{-1}ight)^t_{ij} = 1\] So the matrices \((m^+)^{-1})^t\) are invertible. With these, we define the following two maps \(\mathcal{R}^{-1}, \tilde{\mathcal{R}}^{-1}\) in the convolution algebra \(\text{Hom}(H_{U^q_{\text{ext}}(q)}))\),

\[
\mathcal{R}^{-1}(t^i_j) = ((m^+)^{-1})^t_{ij}, \quad \tilde{\mathcal{R}}^{-1}(\tilde{t}^i_j) = \left((m^-)^{-1}\right)^t_{ij}.
\]

By (78), for any \(t^i_j, \tilde{t}^i_j\), we have

\[
(\mathcal{R} \ast \mathcal{R}^{-1})(t^i_j) = \mathcal{R}(t^i_j) \mathcal{R}^{-1}(t^i_j) = (m^+)^t_{ij} \left((m^+)^{-1}\right)^t_{ij} = \delta_{ij} = \eta \circ \epsilon(t^i_j),
\]

\[
(\tilde{\mathcal{R}} \ast \tilde{\mathcal{R}}^{-1})(\tilde{t}^i_j) = \tilde{\mathcal{R}}(\tilde{t}^i_j) \tilde{\mathcal{R}}^{-1}(\tilde{t}^i_j) = \left((m^-)^{-1}\right)^t_{ij} \left((m^-)^{-1}\right)^t_{ij} = \delta_{ij} = \eta \circ \epsilon(\tilde{t}^i_j),
\]

\[
(\mathcal{R} \ast \tilde{\mathcal{R}}^{-1})(\tilde{t}^i_j) = \mathcal{R}(\tilde{t}^i_j) \tilde{\mathcal{R}}^{-1}(\tilde{t}^i_j) = \left((m^-)^{-1}\right)^t_{ij} \left((m^-)^{-1}\right)^t_{ij} = \delta_{ij} = \eta \circ \epsilon(\tilde{t}^i_j),
\]

\[
(\tilde{\mathcal{R}} \ast \mathcal{R}^{-1})(t^i_j) = \tilde{\mathcal{R}}(t^i_j) \mathcal{R}^{-1}(t^i_j) = \left((m^+)^{-1}\right)^t_{ij} \left((m^+)^{-1}\right)^t_{ij} = \delta_{ij} = \eta \circ \epsilon(t^i_j).
\]

So we obtain that \(\mathcal{R} \ast \mathcal{R}^{-1} = \tilde{\mathcal{R}} \ast \tilde{\mathcal{R}}^{-1} = \eta_{U^q_{\text{ext}}(q)} \circ \epsilon_{H^q_{U^q_{\text{ext}}(q)}}\), owing to (80), (81), (82), (83). \(\mathcal{R} \ast \mathcal{R}^{-1} = \tilde{\mathcal{R}} \ast \tilde{\mathcal{R}}^{-1} = \eta_{U^q_{\text{ext}}(q)} \circ \epsilon_{H^q_{U^q_{\text{ext}}(q)}}\) can be proved in a similar way.

**Step 4. The maps \(\mathcal{R}, \tilde{\mathcal{R}}\) satisfy the relations in (61).**

According to Lemma 3.4, we only need to prove that the relations in (61) hold for generators. In view of \(\mathcal{R}^{-1} \ast \mathcal{R} = \mathcal{R} \ast \mathcal{R}^{-1} = \eta \circ \epsilon\), then we have

\[
\{\mathcal{R}(t^i_{ij}), \mathcal{R}^{-1}(t^i_{ij})\} = \left(\mathcal{R}(t^i_{ij}), t^i_{ij}\right) \{\mathcal{R}^{-1}(t^i_{ij}), t^i_{ij}\} = \left((m^+)^t_{ij}, t^i_{ij}\right) \{\left((m^+)^{-1}\right)^t_{ij}, t^i_{ij}\} = (R^i_{ij} t^i_{ij}, R^i_{ij} t^i_{ij}) = \{R^i_{ij} t^i_{ij}, R^i_{ij} t^i_{ij}\} = (\delta_{ij}, \delta_{ij}) = \delta_{ij} \delta_{ij} = t^i_{ij} \tilde{t}^i_{ij} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\} = \left(\mathcal{R}^{-1}(t^i_{ij}), t^i_{ij} R^i_{ij}\right).
\]

So \(\{\mathcal{R}^{-1}(t^i_{ij}), t^i_{ij}\} = (R^i_{ij})^t_{ij}\), then we obtain that

\[
\{\mathcal{R}(t^i_{ij}), t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\}.
\]

We also obtain the following relations in a similar way

\[
\{\mathcal{R}(t^i_{ij}), \tilde{t}^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\},
\]

\[
\{\mathcal{R}(\tilde{t}^i_{ij}), t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\},
\]

\[
\{\mathcal{R}(\tilde{t}^i_{ij}), \tilde{t}^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} t^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\} = \{t^i_{ij} \tilde{t}^i_{ij}, \tilde{t}^i_{ij} R^i_{ij}\}.
\]
(87) \( \langle \mathcal{R}(t_1^j), \tilde{t}_1^j \rangle = \langle ((m^-)^{-1})_1^j, \tilde{t}_1^j \rangle = (R^{-1})_1^j_{ki} = \langle \mathcal{R}^{-1}(\tilde{t}_1^j), \tilde{t}_1^j \rangle. \)

According to (84)–(87), we prove that \( \langle \mathcal{R}a, b \rangle = \langle \mathcal{R}^{-1}b, a \rangle \) holds for any \( a, b \in H_{RV}. \)

Next we will prove the following relations for \( \mathcal{R}, \mathcal{R}. \)

(88) \( \partial^R(m^+)_{1}^j \ast \mathcal{R} = \mathcal{R} \ast \partial^R(m^+)_{1}^j, \)

(89) \( \partial^R(m^+)_{1}^j \ast \mathcal{R} = \mathcal{R} \ast \partial^R(m^+)_{1}^j. \)

With the convolution product and the coalgebra structure, we obtain

\[
(\partial^R(m^+)_{1}^j \ast \mathcal{R})(t_1^j) = \partial^R(m^+)_{1}^j(t_1^j) \mathcal{R}(t_1^j) = (m^+)_{1}^j \mathcal{R}(t_1^j) = [Rm_1^+ m_2^+]_{1}^j_{ki} = (m^+)_{1}^j(m^+)_{1}^j = (\mathcal{R}(t_1^j) \partial^R(m^+)_{1}^j)(t_1^j) \]

(90)

\[
(\partial^R(m^+)_{1}^j \ast \mathcal{R})(\tilde{t}_1^j) = \partial^R(m^+)_{1}^j(\tilde{t}_1^j) \mathcal{R}(\tilde{t}_1^j) = (m^+)_{1}^j(\tilde{t}_1^j) \mathcal{R}(\tilde{t}_1^j) = [Rm_1^+ m_2^+]_{1}^j_{ki} = (m^+)_{1}^j(m^+)_{1}^j = (\mathcal{R}(\tilde{t}_1^j) \partial^R(m^+)_{1}^j)(\tilde{t}_1^j). \]

(91)

On the other hand, we set \( R^{-1} = C_i \otimes D_i, \) then according to \( Rm_i^+ m_2^- = m_2^- m_i^+ R, \) we have

\[
Rm_i^+ m_2^- = m_2^- m_i^+ R \iff m_i^+ m_2^- R = R^{-1} m_2^- m_i^+ \\
\iff m_2^- R^{-1} (m_i^+) = (m_i^+) R^{-1} m_2^- \\
\iff (I \otimes m^-)(C_i \otimes D_i)((m^+)^{-1} \otimes I) = ((m^+)^{-1} \otimes I)(C_i \otimes D_i)(I \otimes m^-) \\
\iff C_i(m^+)^{-1} \otimes m^- D_i = (m^+)^{-1} C_i \otimes D_i m^- \\
\iff (m^+)^{-1} C_i \otimes m^- D_i = C_i((m^+)^{-1} \otimes D_i m^-) \\
\iff (I \otimes m^-)((m^+)^{-1} \otimes I)(C_i \otimes D_i) = (C_i \otimes D_i)((m^+)^{-1} \otimes I)(I \otimes m^-) \\
\iff m_2^- (m^+)^{-1} (R^{-1})^{1/2} = (R^{-1})^{1/2} (m^+)^{-1} m_2^- \\
\iff ((R^{-1})^{1/2} m_2^- (m^+)^{-1})_1^j = ((m^+)^{-1} m_2^- (R^{-1})^{1/2})_1^j. \]
\[
(\partial^R(m_+^y)_i^k * \mathcal{R})(\vec{p}_j) = \partial^R(m_+^y)_i^k(\vec{p}_a^a) \mathcal{R}(\vec{p}_j) = (m_+^y)_i^k((m_+^y)_b^a) \mathcal{R}(\vec{p}_j)
\]
\[
= ((R^{-1})^{\lambda i})^{k b}_a (m_+^y)_j^b ((m_+^y)^{-1})^{a i}_j
\]
\[
= [[(m_+^y)^{-1}]_i^j m_+^{-1} ((R^{-1})^{\lambda i})^{k b}_a
\]
\[
= [[((m_+^y)^{-1})^{k b}_a (m_+^y)_j^b]^{a i}_j
\]
\[
= [[((m_+^y)^{-1})^{k b}_a (m_+^y)_j^b]^{a i}_j (m_+^y)_i^k \mathcal{R}(\vec{p}_a^a) \partial^L (m_+^y)_i^k (\vec{p}_j) = (\mathcal{R} * \partial^L (m_+^y)_i^k)(\vec{p}_j).
\]

By (90), (91) and (93), we obtain that \(\partial^R(m_+^y)_i^k * \mathcal{R} = \mathcal{R} * \partial^L (m_+^y)_i^k\), thus prove the relation (88). (89) can also be proved in a similar way.

This completes the proof. \(\square\)

### 3.6. The generalized double-bosonization construction Theorem.

After building up the weakly quasitriangular dual pairs of Hopf algebras suitably for general \(R\)-matrices \(R_{VV}\)’s (especially for the irregular \(R\)-matrices), we are in a position to establish our generalized double-bosonization construction Theorem for any \(R_{VV}\). First of all, as we known, the two braided (co-)vector algebras involved in the double-bosonization construction were defined by Majid in the braided categories of left (right) \(A(R)\)-comodules, i.e., \(V(R', R) \in \mathcal{M}^{A(R)}\), \(V^+(R', R_{21}^{-1}) \in \mathcal{M}^{A(R)}\). However, our starting objects are the dual pairs of Hopf algebras \((U_{q_{\gamma}}^{\text{ext}}(g), R_{RVV}^v)\) for the irregular cases. Now by definition, observing that the FRT-bialgebra \(A(R)\) is a subbialgebra of \(H_{R}\), we can view \(V(R', R) \in \mathcal{M}^{H_{R}}\), \(V^+(R', R_{21}^{-1}) \in \mathcal{M}^{H_{R}}\).

Actually, in order to fit well into the application to the inductive construction of the exceptional quantum groups, we have to refine our framework. We not only need to work with the pairs of their central extensions: \((U_{q_{\gamma}}^{\text{ext}}(g), \widetilde{R}_{RVV})\), where \(U_{q_{\gamma}}^{\text{ext}}(g) = U_{q_{\gamma}}^{\text{ext}}(g) \otimes k[c, c^{-1}]\), \(\widetilde{R}_{RVV} = R_{RVV} \otimes k[g, g^{-1}]\), but also to make the normalization of \(R_{VV}\) at certain eigenvalue of the braiding \(PR_{VV}\) to obtain a quantum group normalization constant \(\lambda\) and the needed pair \((R, R')\) determined by the minimal polynomial of \(PR_{VV}\). Noticing the relation \(R_{VV} = \lambda R\), and \(H_{R_{VV}} \subset \widetilde{R}_{RVV}\), we have \(A(R) \cong A(R_{VV})\) as bialgebras, consequently, we get a bialgebra embedding \(A(R) \hookrightarrow \widetilde{R}_{RVV}\). This implies that our concerned braided objects \(V(R', R) \in \mathcal{M}^{H_{R_{VV}}}\), \(V^+(R', R_{21}^{-1}) \in \mathcal{M}^{H_{R_{VV}}}\). To be more precisely, we have

**Lemma 3.5.** Let \(R_{VV} = \lambda R\) and \((U_{q_{\gamma}}^{\text{ext}}(g), \widetilde{R}_{RVV})\) be the centrally extended weakly quasitriangular dual pair defined by

\[
\Delta(c) = c \otimes c, \quad \Delta(g) = g \otimes g, \quad \langle c, g \rangle = \lambda, \quad \partial(c) = c^{-1}, \quad \partial(g) = c.
\]

Then \(V(R', R)\) and \(V^+(R', R_{21}^{-1})\) are braided groups in the categories \(\mathcal{M}^{H_{R_{VV}}}\) and \(\mathcal{M}^{H_{R_{VV}}}\), respectively, where the coactions are given by \(e^\dagger \mapsto g e^\dagger \otimes e^\dagger\) and \(f_i \mapsto f_i \otimes g t_a^i\).
Proof. The duality pairing between \( Uq(V) \) and \( \overset{\text{ext}}{H_{RVV}} \) can be induced by \( \langle a \otimes c, b \otimes g \rangle = \langle a, b \rangle(c, g) \), for \( a \in Uq(V) \), \( b \in H_{RVV} \), and we have

\[
\begin{align*}
\langle c, g^{-1} \rangle &= \langle c^{-1}, g \rangle = \lambda^{-1}, \\
\langle c, t_j^i \rangle &= \langle 1 \otimes c, t_j^i \otimes 1 \rangle = \langle 1, t_j^i(c, 1) = e(t_j^i) e(c) = \delta_{ij}, \\
\langle c, \bar{t}_j^i \rangle &= \langle 1 \otimes c, \bar{t}_j^i \otimes 1 \rangle = \langle 1, \bar{t}_j^i(c, 1) = e(\bar{t}_j^i) e(c) = \delta_{ij}, \\
\langle (m^+)^j_i, g \rangle &= \langle (m^+)^j_i \otimes 1, 1 \otimes g \rangle = \langle (m^+)^j_i, 1 \rangle, e(g) = e((m^+)^j_i) = \delta_{ij}.
\end{align*}
\]

According to \( \langle ab, c \rangle = \langle a, c_{(1)} \rangle(b, c_{(2)}) \), we obtain

\[
\langle (m^+)^{-1}^j_i, g \rangle = \delta_{ij} = \langle (m^+)^{-1}^j_i \rangle(g).
\]

With these, we have

\[
\begin{align*}
\langle \bar{R}(g), t_j^i \rangle &= \langle \bar{R}^{-1}(t_j^i), g \rangle = \delta_{ij}, \quad \langle \bar{R}(g), \bar{t}_j^i \rangle = \langle \bar{R}^{-1}(\bar{t}_j^i), g \rangle = \delta_{ij}, \\
\langle \bar{R}(t_j^i), g \rangle &= \langle \bar{R}^{-1}(g), t_j^i \rangle = \delta_{ij}, \quad \langle \bar{R}(\bar{t}_j^i), g \rangle = \langle \bar{R}^{-1}(g), \bar{t}_j^i \rangle = \delta_{ij}.
\end{align*}
\]

In view of \( \bar{R}^{-1}(g) = c, \bar{R}^{-1}(g) = c^{-1} \), and (96), (97), we obtain that \( \langle \bar{R}(a), b \rangle = \langle \bar{R}^{-1}(b), a \rangle \) holds for any \( a, b \in \overset{\text{ext}}{H_{RVV}} \). Furthermore, by \( (m^+)^j_i c = c(m^+)^j_i \), we can prove easily that

\[
\partial^R C = \bar{R} \ast (\partial^L C) \ast \bar{R}^{-1}, \quad \partial^R C = \bar{R} \ast (\partial^L C) \ast \bar{R}^{-1}.
\]

Hence, (94) gives actually a weakly quasitriangular dual pair between \( Uq(V) \) and \( \overset{\text{ext}}{H_{RVV}} \).

Finally, we will check that our concerned objects \( V(R', R), V'(R', R_{21}^{-1}) \) are indeed the braided ones in the braided categories \( \overset{\text{ext}}{H_{RVV}} \mathcal{M} \) and \( \overset{\text{ext}}{H_{RVV}} \mathcal{M} \), respectively. To do so, we need to verify that the braiding induced by the left / right \( \overset{\text{ext}}{H_{RVV}} \mathcal{M} \)-comodule structures of \( V(R', R) \) and \( V'(R', R_{21}^{-1}) \), respectively, are consistent with those of their braided groups structure.

In fact, under the coaction \( e^i \mapsto g t^i_j \otimes e^b \) and \( e^j \mapsto g t^j_i \otimes e^a \), we have

\[
\Psi(e^i \otimes e^j) = e^i \otimes e^j \ast \bar{R}(g t^i_j) = e^i \otimes e^j \ast (m^+)^j_i = e^i \otimes e^j \langle c^{-1}(m^+)^j_i, g t^i_j \rangle = \langle c^{-1}, g \rangle \langle (m^+)^j_i, t^i_j \rangle e^a \otimes e^b = \lambda^{-1} R_{ab}^i e^a \otimes e^b
\]

for any \( e^i, e^j \in V(R', R) \). Using the right \( \overset{\text{ext}}{H_{RVV}} \mathcal{M} \)-comodule structure of \( V'(R', R_{21}^{-1}) \), we can also obtain that \( \Psi(f^i \otimes f^j) = (f^i \otimes f^j)(g t^i_j \otimes g d^j_i) = f^i \otimes f^j R_{ab}^i \). The proof is complete. \( \Box \)

By the above lemma, we obtain dually-paired braided groups \( V(R', R) \) and \( V'(R', R_{21}^{-1}) \) in the categories \( \overset{\text{ext}}{Uq(V)} \mathcal{M}, \overset{\text{ext}}{Uq(V)} \mathcal{M} \), respectively. With these, starting from general \( R \)-matrices \( R_{VV} \)'s (especially for the irregular \( R \)-matrices), we can arrive at our generalized version of double-bosonization construction Theorem.
Theorem 3.4. Let $R_{VV}$ be the $R$-matrix associated to an irreducible $U_q(\mathfrak{g})$-module $V$ with a minuscule highest weight. There exists a normalization constant $\lambda$ such that $\lambda R = R_{VV}$. Then the enlarged quantum group $U = U(V^\vee(R', R_{21}^{-1}), U^\text{ext}_q(\mathfrak{g}), V(R', R))$ has the relations
\[
c f_i = \lambda f_i c, \quad e^i c = \lambda c e^i, \quad [c, m^b] = 0, \quad [e^i, f_j] = \delta_{ij} \frac{(m^+)^i_j c^{-1} - c(m^-)^i_j}{q^* - q^{-1}_*};
\]
\[
e^i (m^+)^i_j c_j &= R_{VV}^{ji} (m^+)^j_k c_k, \quad (m^-)^i_j c_j = R_{VV}^{kj} (m^-)^i_k c_j;
\]
\[
(m^+)^i_j f_k &= f_a (m^+)^i_j R_{VV}^{ab} c_k, \quad f_i (m^-)^i_j c_k = (m^-)^i_j f_a R_{VV}^{ab} c_k,
\]
and the coproduct:
\[
\Delta c = c \otimes c, \quad \Delta e^i = e^a \otimes (m^+)^i_a c^{-1} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + c(m^-)^i_a \otimes f_a,
\]
and the counit $\epsilon e^i = \epsilon f_i = 0$.

Proof. The weakly quasitriangular dual pair $(\tilde{U}^\text{ext}_q(\mathfrak{g}), \tilde{H}_R^{VV})$ yields the cross relation, by Remark 2.2. The coalgebra structure follows from Theorem 2.1 due to Majid. □

4. Applications: Type-crossing Constructions of $U_q(F_4)$ and $U_q(G_2)$

This section is devoted to using Theorem 3.4 to give the type-crossing constructions of $U_q(F_4)$ and $U_q(G_2)$. Among which, the first crucial point is to choose an appropriate $U_q(\mathfrak{g}')$-module for a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of corank 1 to obtain a suitable $R$-matrix $R_{VV}$ (irregular in the exceptional cases).

4.1. Type-crossing of $U_q(F_4)$ via $U_q(B_3)$. Let us begin with $U_q(B_3)$ and its 8-dimensional spin module $V$, which is given by the following Figure 1 (see [HK]).

![Figure 1. spin module V for U_q(so_7)](image-url)
The representation space $V = \text{Span}_\mathbb{C}\{v_1, \ldots, v_8\}$. Let $(s_1, s_2, s_3)$ denote the basis vector of weight $\frac{1}{2}(s_1 e_1 + s_2 e_2 + s_3 e_3)$, $s_i = \pm$. Every basis vector $v_i$ is denoted by $v_1 = (-, -, -)$, $v_2 = (-, -, +)$, $v_3 = (-, +, -)$, $v_4 = (-, +, +)$, $v_5 = (+, -, -)$, $v_6 = (+, -, +)$, $v_7 = (+, +, -)$, $v_8 = (+, +, +)$. $E_i, F_i$’s actions can be read off directly from Figure 1, then $K_i, H_i$’s actions can be obtained by $[E_i, F_j] = \delta_j^{i} K_i^{-1}$ (or $\delta_i^{j} K_i^{-1}$), Moreover, we observe that the spin module $V$ is a minuscule weight module, and $E_\beta(v_m) = v_n \iff F_\beta(v_n) = v_m$, for some $m, n$ (see [HK]), as well as $E_i^k, F_i^k$ are 0 for all $k \geq 2$. Based on these, we can prove the following

**Lemma 4.1.** $\mu_m$ denote the weight of basis vector $v_m$ for any $m$. If there exist $v_i, v_j$ and $v_k, v_l$ such that $E_i^{r_i} E_j^{r_j} \cdots E_i^{r_i}(v_i) = v_k, F_i^{s_i} F_j^{s_j} \cdots F_j^{s_j}(v_j) = v_l$, where $\{i_1, j_1, \ldots, j_t\} \in \{i_0(t), \ell_0(t), \ldots, i_0(t)\}$, and $t_k = r_a$ when $j_k = i_a, r_m \in \{0, 1\}$ for any $m$, then we have $(\mu_i, \mu_j) = (\mu_k, \mu_l)$.

**Proof.** We will prove this Lemma by induction on $r_1 + r_2 + \cdots + r_s$.

1. When $r_1 + r_2 + \cdots + r_s = 1$, namely, the situation of simple root vectors.

According to the spin representation, suppose that there exist simple root vectors $E_\beta$ and $F_\beta$ such that $E_\beta(v_i) = v_k, F_\beta(v_j) = v_l$, which is equivalent to $E_\beta(v_l) = v_j, F_\beta(v_k) = v_i$, then $\mu_k = \mu_i + \beta, \mu_l = \mu_j - \beta$. Owing to $E_\beta^2, F_\beta^2$ are zero action, we obtain $E_\beta(v_i) = E_\beta(E_\beta(v_l)) = 0, F_\beta(v_l) = F_\beta(F_\beta(v_k)) = 0$. According to $E_\beta F_\beta - F_\beta E_\beta = \frac{K_\beta - K^{-1}_\beta}{q_\beta - \bar{q}_\beta}$, both sides acting on $v_i, v_j$, respectively, we obtain

\[
[E_\beta, F_\beta](v_i) = \frac{K_\beta - K^{-1}_\beta}{q_\beta - \bar{q}_\beta}(v_i) \implies (\mu_i, \beta) = \frac{(\beta, \beta)}{2},
\]

\[
[E_\beta, F_\beta](v_l) = \frac{K_\beta - K^{-1}_\beta}{q_\beta - \bar{q}_\beta}(v_l) \implies (\beta, \mu_l) = \frac{(\beta, \beta)}{2}.
\]

Then we have the following equalities

\[
(\mu_k, \mu_l) = (\mu_i + \beta, \mu_j - \beta) = (\mu_i, \mu_j) - (\mu_i, \beta) + (\beta, \mu_j) - (\beta, \beta) = (\mu_i, \mu_j) - (\frac{(q_\beta + q_\beta)}{2}) + (\frac{(q_\beta + q_\beta)}{2}) = (\mu_i, \mu_j).
\]

2. When $r_1 + r_2 + \cdots + r_s = 2$.

If there are basis vectors $v_k$ and $v_l$ such that $E_i E_j(v_k) = v_m$ and $F_i F_j(v_l) = v_n$(or $F_i F_j(v_l) = v_n$), then $\mu_m = \mu_k + \alpha_i + \alpha_j, \mu_n = \mu_l - \alpha_i - \alpha_j$. Let us consider the case of $E_i E_j(v_k) = v_m$ and $F_i F_j(v_l) = v_n$. From these, we have the following equivalent relations

(98) $E_i E_j(v_k) = v_m \iff E_j(v_k) = v_x$, $E_i(v_x) = v_m$,

(99) $F_i F_j(v_l) = v_n \iff F_j(v_l) = v_y$, $F_i(v_y) = v_n$.
for some basis vectors $v_s, v_y$. By (98) and (99), we obtain the following equalities for these weights based on the above analysis for simple root vectors.

\[
\begin{align*}
(\mu_k, \alpha_j) &= -\frac{(\alpha_j, \alpha_j)}{2}, \quad (\mu_y, \alpha_j) = \frac{(\alpha_j, \alpha_j)}{2}, \\
(\mu_s, \alpha_i) &= -\frac{(\alpha_i, \alpha_i)}{2}, \quad (\mu_t, \alpha_i) = \frac{(\alpha_i, \alpha_i)}{2}.
\end{align*}
\]

According to $\mu_s, \alpha_j) = \frac{(\alpha_j, \alpha_j)}{2}$ and $\mu_s, \alpha_i) = -\frac{(\alpha_i, \alpha_i)}{2}$, respectively, we obtain

\[
(\mu_t, \alpha_j) = (\alpha_j, \alpha_j) + \frac{(\alpha_j, \alpha_j)}{2}, \quad (\mu_t, \alpha_i) = -(\alpha_i, \alpha_j) - \frac{(\alpha_i, \alpha_j)}{2}.
\]

Then we have the following relations by (100),

\[
\begin{align*}
(\mu_t, \alpha_j + \alpha_j) &= (\alpha_j, \alpha_j) + \frac{(\alpha_j, \alpha_j) + (\alpha_j, \alpha_j)}{2}, \\
(\mu_t, \alpha_j + \alpha_i) &= -(\alpha_i, \alpha_j) - \frac{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_j)}{2}.
\end{align*}
\]

With these, we have

\[
(\mu_m, \mu_n) = (\mu_k + \alpha_i + \alpha_j, \mu_l - \alpha_i - \alpha_j) = (\mu_k, \mu_l) - (\mu_k, \alpha_i + \alpha_j) + (\mu_l, \alpha_i + \alpha_j) - (\alpha_i + \alpha_j, \alpha_i + \alpha_j) = 0.
\]

In a similar way, we also prove $(\mu_m, \mu_n) = (\mu_k, \mu_l)$ when $E_i E_j (v_k) = v_m$ and $F_i F_j (v_l) = v_n$.

(3) Suppose that the Proposition is correct for $r_1 + r_2 + \cdots + r_s = n - 1$ ($n \geq 2$), then we will consider the case of $r_1 + r_2 + \cdots + r_s = n$.

If there are $v_s, v_j, v_k, v_l$ such that $E_{i_1}^{j_1} E_{i_2}^{j_2} \cdots E_{i_s}^{j_s} (v_l) = v_k, F_{j_1}^{i_1} F_{j_2}^{i_2} \cdots F_{j_s}^{i_s} (v_l) = v_l$ and $r_1 \neq 0, t_1 \neq 0$, then we have the following equivalent relations:

\[
\begin{align*}
E_{i_1}^{j_1} E_{i_2}^{j_2} \cdots E_{i_s}^{j_s} (v_l) &= v_k \iff E_{i_1}^{j_1} E_{i_2}^{j_2} \cdots E_{i_s}^{j_s} (v_l) = v_s, E_{i_s} (v_s) = v_k, \\
F_{j_1}^{i_1} F_{j_2}^{i_2} \cdots F_{j_s}^{i_s} (v_l) &= v_l \iff F_{j_1}^{i_1} F_{j_2}^{i_2} \cdots F_{j_s}^{i_s} (v_l) = v_s, F_{j_s} (v_s) = v_l.
\end{align*}
\]

Thus we have $(\mu_i, \mu_j) = (\mu_s, \mu_l)$ by supposition, and $(\mu_s, \mu_i) = (\mu_k, \mu_l)$ by (1), then we prove $(\mu_i, \mu_j) = (\mu_k, \mu_l)$.) According to (1), (2), (3), the proof is complete.

For simplicity, we denote $E_{i_1}^{j_1} E_{i_2}^{j_2} \cdots E_{i_s}^{j_s}, F_{j_1}^{i_1} F_{j_2}^{i_2} \cdots F_{j_s}^{i_s}$ in the above Lemma 4.1 by $E_{L}^{j_2}, F_{L}^{j_2}$, respectively. Corresponding to this spin representation, we get the upper triangular $64 \times 64\overset{\pm}{R}$-matrix $R_{UV}$ associated with universal $R$-matrix and the above basis of representation space. The explicit expressions of root vectors in universal $R$-matrix (7) are obtained by Lusztig’s automorphisms $T_i$’s [L]. For example, we fix $w_0 = s_3 s_2 s_3 s_2 s_1 s_2$ a reduced decomposition of the longest element $w_0$ of Weyl group $W$ for $B_3$, then the corresponding sequence of all positive roots of $so_7$ is $\alpha_1, \alpha_2 + 2\alpha_3, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_3 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_1$. For composition root $\alpha_2 + \alpha_3$, we obtain root vectors $E_{\alpha_2 + \alpha_3} = -E_3 E_2 + q^{-1} E_2 E_3, F_{\alpha_2 + \alpha_3} = -F_2 F_3 + q F_3 F_2$. Moreover, by these root vectors
and the equality (8), we obtain \( R_{\theta V}^{53} = -q^{-1}(q^{\frac{3}{2}} - q^{-\frac{3}{2}})q^{-\frac{1}{2}} \) and \( R_{\theta V}^{17} = -q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})q^{-\frac{1}{2}} \), then we find \((PR_{\theta V})^{35} \neq (PR_{\theta V})^{53}\). That is, the matrix \( PR_{\theta V} \) isn’t symmetric. However, we have the following

**Proposition 4.1.** Corresponding to the 8-dimensional spin \( U_q(\mathfrak{so}_7) \)-module in Figure 1, the braiding matrix \( PR_{\theta V} \) is symmetrizable, that is, the braiding is of diagonal type.

**Proof.** We know that dual cases, obtained by Lusztig’s automorphisms \( T_i \)'s, is the standard choice of bases satisfying the condition (6) in Lemma 3.1. Naturally, if we can determine another dual base of \( U_q^+ \) and \( U_q^- \) such that they satisfy (6), namely, \( \langle E_i, F_i \rangle' = \langle E_i, F_i \rangle \) if a positive \( \beta \) and a simple root \( \alpha \) belong to the same \( W \)-orbit, then we derive another explicit formula of universal \( R \)-matrix of \( U_q(\mathfrak{g}) \). In fact, this is equivalent to make base transformation for vector spaces \( U_q^+ \) and \( U_q^- \). Specially, for \( U_q(\mathfrak{so}_7) \), we choose the following expressions for composition positive root vectors:

\[
E_{\alpha_1+\alpha_2} = -E_2 E_1 + q^{-1} E_1 E_2, \quad E_{\alpha_2+\alpha_3} = -E_3 E_2 + q^{-1} E_2 E_3,
\]

\[
E_{\alpha_2+2\alpha_3} = E_3^{(2)} E_2 - q^{-\frac{1}{2}} E_3 E_2 E_3 + q^{-1} E_2 E_3^{(2)}, \quad \text{where } E_3^{(2)} := \frac{(E_3)^2}{[2]_q!},
\]

\[
E_{\alpha_1+\alpha_2+\alpha_3} = -E_3 E_2 E_1 + q^{-1} E_2 E_3 E_1 + q^{-1} E_1 E_3 E_2 - q^{-2} E_1 E_2 E_3,
\]

\[
E_{\alpha_1+2\alpha_2+\alpha_3} = -E_2 E_{\alpha_2+2\alpha_3} E_1 + q^{-1} E_2 E_{\alpha_2+2\alpha_3} + q^{-1} E_{\alpha_2+2\alpha_3} E_1 E_2 - q^{-2} E_1 E_{\alpha_2+2\alpha_3} E_2.
\]

For negative root vectors:

\[
F_{\alpha_1+\alpha_2} = \frac{1}{2} q F_2 F_1 - \frac{1}{2} q^2 F_1 F_2, \quad F_{\alpha_2+\alpha_3} = (q - \frac{1}{2} q^2) F_3 F_2 - (q^3 - \frac{1}{2} q^2) F_2 F_3,
\]

\[
F_{\alpha_2+2\alpha_3} = F_2 F_3^{(2)} - q^{-\frac{1}{2}} F_3 F_2 F_3 + q F_3^{(2)} F_2, \quad \text{where } F_3^{(2)} := \frac{(F_3)^2}{[2]_q!},
\]

\[
F_{\alpha_1+\alpha_2+\alpha_3} = -q^{-2} F_3 F_2 F_1 + q^{-1} F_2 F_3 F_1 + q^{-1} F_1 F_3 F_2 - F_1 F_2 F_3,
\]

\[
F_{\alpha_1+2\alpha_2+\alpha_3} = -F_{\alpha_1+\alpha_2} F_3^{(2)} - q^{-1} F_3^{(2)} F_{\alpha_1+\alpha_2} + q^{-\frac{1}{2}} F_3 F_{\alpha_1+\alpha_2} F_3,
\]

\[
F_{\alpha_1+2\alpha_2+2\alpha_3} = -F_1 F_{\alpha_2+2\alpha_3} F_1 + q F_{\alpha_2+2\alpha_3} F_1 F_2 + q F_2 F_1 F_{\alpha_2+2\alpha_3} - q^2 F_2 F_{\alpha_2+2\alpha_3} F_1.
\]

According to

\[
\langle E_l E_k, F_m F_n \rangle' = \delta_{km}\delta_{ln} \frac{1}{q_m - q_m^{-1}} \frac{1}{q_m - q_m^{-1}} + \delta_{km}\delta_{ln} q^{-\langle \alpha_n, \alpha_l \rangle} \frac{1}{q_m - q_m^{-1}} \frac{1}{q_m - q_m^{-1}},
\]
where equality

\[ \langle E_i, E_j \rangle' = \langle E_i, F_j \rangle' \]

for these new root vectors if a positive \( \beta \) and a simple root \( \alpha \) belong to the same W-orbit, i.e., they are satisfy the condition (6). Starting from this new dual base, we can check that the corresponding matrix \( PR_{VV} \) is symmetric. In virtue of the actions of spin modules in Figure 1, and \( E_2^k, F_2^k \) is zero as operators, these new root vectors as operators have the following equalities:

\[
(101) \quad E_{a_1+a_2} = -E_2E_1 + q^{-1}E_1E_2, \quad E_{a_2+a_3} = -E_3E_2 + q^{-1}E_2E_3, \quad E_{a_1+2a_3} = -q^{-\frac{1}{2}}E_3E_2E_3,
E_{a_1+a_2+a_3} = -E_3E_2E_1 + q^{-1}E_2E_3E_1 + q^{-1}E_1E_2E_2 - q^{-2}E_1E_2E_3,
E_{a_1+a_2+2a_3} = q^{-\frac{1}{2}}E_3E_2E_3E_1 - q^{-\frac{1}{2}}E_3E_2E_3E_2 - E_3E_2E_3E_3,
F_{a_1+a_2} = \frac{1}{2}qF_2F_1 - \frac{1}{2}q^2F_1F_2, \quad F_{a_2+a_3} = (q - \frac{1}{4}q^2)F_3F_2 - (q^2 - \frac{1}{2}q^3)F_2F_3,
F_{a_1+2a_3} = -q^2F_3F_2F_3, \quad F_{a_1+a_2+a_3} = -qF_3F_2F_1 + q^{-1}F_2F_3F_1 + q^{-1}F_1F_3F_2 - F_1F_2F_3,
F_{a_1+a_2+2a_3} = q^{-\frac{1}{2}}F_3F_2F_1F_3 - \frac{1}{4}F_2F_1F_3F_3 - \frac{4}{q}q^2F_1F_2F_3F_3,
F_{a_1+2a_2+2a_3} = -q^2F_2F_1F_3F_2F_3 - q^2F_3F_2F_3F_1F_2.
\]

By (8), we know that

\[
B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = B_{VV} \circ (T_V \otimes T_V) \left( \sum_{[\mathcal{L}, \mathcal{J}], [\mathcal{Q}, \mathcal{J}]} \chi_{\mathcal{L}, \mathcal{J}}^{[\mathcal{Q}, \mathcal{J}]} \frac{E_{2,\mathcal{L}}^\mathcal{J}}{\chi_{\mathcal{L}, \mathcal{J}}^{[\mathcal{Q}, \mathcal{J}]} (v_k \otimes v_l) \cdots E_{i,\mathcal{J}}^\mathcal{J} (v_i) \cdot E_{n,\mathcal{J}}^\mathcal{J} (v_n)}{\chi_{\mathcal{L}, \mathcal{J}}^{[\mathcal{Q}, \mathcal{J}]} (v_k \otimes v_l) \cdots E_{i,\mathcal{J}}^\mathcal{J} (v_i) \cdot E_{n,\mathcal{J}}^\mathcal{J} (v_n)} \right) (v_i \otimes v_j) + \text{others},
\]

where coefficients \( x_{\mathcal{L}, \mathcal{J}}^{[\mathcal{Q}, \mathcal{J}]} \in \mathbb{C}[q, q^{-1}] \), and the set \( \Omega_{ij}^{\mathcal{L}, \mathcal{J}} \) is defined as

\[
\Omega_{ij}^{\mathcal{L}, \mathcal{J}} := \{(\mathcal{L}, \mathcal{J}), (\mathcal{Q}, \mathcal{J})|E_{2,\mathcal{L}}^\mathcal{J}(v_i) = E_{i,\mathcal{J}}^\mathcal{J}(v_i) = 0 \text{ for } \mu_i, \mu_j > 3\}.
\]

\( E_2^\mathcal{L}, F_2^\mathcal{J} \) can be divided into the following three cases:

1. Diagonal entries:

Firstly, the diagonal entries in the matrix \( R_{VV} \) satisfy \( R_{VV}^{ij} = R_{VV}^{ji} = q^{\mu_i \cdot \mu_j} \), where \( \mu_i, \mu_j \) denote the weight of \( v_i, v_j \), respectively. Next we will consider the non-diagonal entries in the matrix \( R_{VV} \). Since the operators \( E_i^k \)’s and \( F_i^k \)’s are 0 for all \( k \geq 2 \), we have the equality

\[
B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = B_{VV} \circ (T_V \otimes T_V) \left( 1 + \sum_{[\mathcal{L}, \mathcal{J}]} E_{\alpha, \mathcal{L}}^\mathcal{J} (v_l) \cdot E_{\alpha, \mathcal{J}}^\mathcal{L} (v_i) \cdot E_{\alpha, \mathcal{J}}^\mathcal{L} (v_j) \right) (v_i \otimes v_j).
\]

2. The entries related to simple root vectors:
According to the spin representation, suppose that there exist simple root vectors $E_\beta$ and $F_\beta$ such that $E_\beta(v_i) = v_k, F_\beta(v_j) = v_i,$ which is equivalent to $E_\beta(v_i) = v_j, F_\beta(v_k) = v_i.$ Namely, the entry $R_{VV}^{ij}_{kl} \neq 0,$ also is equivalent to $R_{VV}^{ik}_{jl} \neq 0.$ Moreover, by Lemma 4.1, the entry $R_{VV}^{ij}_{kl} = (q_\beta - q_\beta^{-1})q^{(\mu, \mu_0)} = (q_\beta - q_\beta^{-1})q^{(\mu, \mu_0)}.$ Correspondingly, $R_{VV}^{ik}_{jl} = (q_\beta - q_\beta^{-1})q^{(\mu, \mu_0)}.$ We obtain $R_{VV}^{ij}_{kl} = R_{VV}^{ik}_{jl}$, which is equivalent to $(PR_{VV})^{ij}_{kl} = (PR_{VV})^{ik}_{jl}.$

(3) The entries related to quantum composite root vectors:

The expressions of all composite vectors as operators are listed by (101) as above, which are $q$-commutators of simple root vectors. Thereby, the actions of these composite root vectors can be deduced by the actions of simple root vectors. Suppose that there are $v_i, v_j, v_k, v_l$ such that $E_{i_1}E_{i_2} \cdots E_{i_{n-1}}(v_i) = v_k, F_{j_1}F_{j_2} \cdots F_{j_{n-1}}(v_j) = v_l,$ which is equivalent to $F_{j_1}^s \cdots F_{j_{n-1}}^s v_k = v_l,$ and $(\mu, \mu_0) = (\mu_k, \mu_l)$ by Lemma 4.1. On the other hand, according to

$$R = \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}} E_{i_1}E_{i_2} \cdots E_{i_{n-1}} \otimes F_{j_1}F_{j_2} \cdots F_{j_{n-1}} + \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}} E_{i_1}E_{i_2} \cdots E_{i_{n-1}}F_{j_1}F_{j_2} \cdots F_{j_{n-1}} + \text{others},$$

then the corresponding entries are

$$R_{VV}^{ij}_{kl} = \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}}, \quad R_{VV}^{ik}_{jl} = \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}}.$$ 

Thence, we obtain the following equivalent relation

$$R_{VV}^{ij}_{kl} = R_{VV}^{ik}_{jl} \iff \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}} = \frac{\{q \}^{(\mu, \mu_0)}}{\{q \}^{(\mu, \mu_0)}}.$$ 

Furthermore, associated with the expressions (101) of composition root vectors as operators, it is easy to check that the equality $X_{i_1}^{(\mu, \mu_0)} = X_{i_2}^{(\mu, \mu_0)}$ in (102) is always true. Thereby, we have the equality $R_{VV}^{ij}_{kl} = R_{VV}^{ik}_{jl},$ namely, $(PR_{VV})^{ij}_{kl} = (PR_{VV})^{ik}_{jl}.$

Let us demonstrate this by an example. For example, starting from the composite root $\alpha_1 + \alpha_2,$ we know that the corresponding root vectors are $E_{\alpha_1 + \alpha_2} = -E_2E_1 + q^{-1}E_1E_2$ and $F_{\alpha_1 + \alpha_2} = \frac{1}{2}qF_2F_1 - \frac{1}{2}q^2F_1F_2$ by (101). Then we have

$$B_{VV} \circ (T_V \otimes T_V)(R)(v_i \otimes v_j)$$

$$= B_{VV} \circ (T_V \otimes T_V)(q_{\alpha_1 + \alpha_2}^{-1}E_{\alpha_1 + \alpha_2} \otimes F_{\alpha_1 + \alpha_2})(v_i \otimes v_j) + \text{others}$$

$$= B_{VV} \circ (q^{-1})(T_V \otimes T_V)((-E_2E_1 + q^{-1}E_1E_2) \otimes (\frac{1}{2}qF_2F_1 - \frac{1}{2}q^2F_1F_2))(v_i \otimes v_j) + \text{others}$$

$$= B_{VV} \circ (q^{-1})(T_V \otimes T_V)((-\frac{1}{2}E_2E_1F_2 + \frac{1}{2}E_1E_2F_1 + \frac{1}{2}E_1E_2F_2 + \frac{1}{2}E_1E_2F_1))(v_i \otimes v_j) + \text{others}.$$

Now, firstly, we analyze explicitly the entries obtained by the operator $T_V \otimes T_V(E_2E_1 \otimes F_2F_1).$

If $T_V(E_2E_1)(v_i) = v_k$ and $T_V(F_2F_1)(v_j) = v_l,$ then we obtain equivalently $T_V(F_1F_2)(v_k) = v_j$ and $T_V(E_2E_1)(v_l) = v_i.$ Thus $R_{VV}^{ij}_{kl} = \frac{1}{2}q^2(q - q^{-1})q^{(\mu, \mu_0)} = \frac{1}{2}q^2(q - q^{-1})q^{(\mu, \mu_0)} = R_{VV}^{ik}_{jl},$ and $(PR_{VV})^{ij}_{kl} = (PR_{VV})^{ik}_{jl}.$ Secondly, suppose that there exist $v_i, v_j$ and $v_k, v_l$ such that
$T_V(E_2 E_1)(v_i) = v_k, T_V(F_2 F_1)(v_j) = v_l$ by the action of spin module, which is equivalent to that $T_V(F_1 F_2)(v_k) = v_l, T_V(E_1 E_2)(v_i) = v_j$. That is,

$$(104) \quad (T_V \otimes T_V)(E_2 E_1 \otimes F_2 F_1)(v_i \otimes v_j) = v_k \otimes v_l \iff (T_V \otimes T_V)(E_1 E_2 \otimes F_1 F_2)(v_j \otimes v_k) = v_i \otimes v_l.$$  

Moreover, by (103), we know that the coefficients of $E_2 E_1 \otimes F_2 F_1$ and $E_1 E_2 \otimes F_1 F_2$ are equal, and both are $\frac{1}{2}q$ (i.e. satisfy the condition $x^{(c,d)}_{(i,j)} = x^{(c,d)}_{(k,l)}$ in (102)). So the corresponding entries have the equality $R_{VV}^{ij}_{kl} = \frac{1}{2}q(q - q^{-1})q^{\mu(i,j)} = \frac{1}{2}q(q - q^{-1})q^{\mu(i,j)} = R_{VV}^{lk}_{ji}$. For the situation of operator $\frac{1}{2}E_1 E_2 \otimes F_2 F_1$ can be analyzed in a similar way.

The proof is complete. □

According to Lemma 3.5, we need to seek the pair $(R, R')$ to determine the dually-paired braided groups $V(R', R)$ and $V'(R', R^{-1}_2)$. To this end, we have to figure out the minimal polynomial of the braiding $PR_{VV}$ in advance. Notice the size of the matrix $PR_{VV}$ is $64 \times 64$ due to $\dim V = 8$, it is not easy to obtain its minimal polynomial directly. In the course of demonstration of the following Proposition, we will give an ingenious method to capture the minimal polynomial by taking advantage of nice features of the representation involved. For simplicity, write $\hat{R}_{VV}$ for $PR_{VV}$.

**Proposition 4.2.** Associated to the spin $U_q(B_3)$-module $V$, the braiding matrix $\hat{R}_{VV}$ obeys the minimal polynomial equation

$$(\hat{R}_{VV} + q^\frac{1}{2}I)(\hat{R}_{VV} - q^\frac{1}{2}I)(\hat{R}_{VV} + q^{-\frac{1}{2}}I)(\hat{R}_{VV} - q^{-\frac{1}{2}}I) = 0.$$  

**Proof.** Note that $V^{\otimes 2} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ [FH], where $V_i$’s are the irreducible submodules with highest weights $e_1 + e_2 + e_3, e_1 + e_2, e_1, 0$, respectively. This means that $\hat{R}_{VV}$ has 4 different eigenvalues (since the braiding $\hat{R}_{VV}$ as module homomorphism is of diagonal type, by Proposition 4.1), denoted by $x_1, x_2, x_3, x_4$. Thus $f(t) = (t - x_1)(t - x_2)(t - x_3)(t - x_4)$ is the minimal polynomial of $\hat{R}_{VV}$, so $M = f(\hat{R}_{VV}) = 0$.

Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ denote the elementary symmetric polynomials: $x_1 + x_2 + x_3 + x_4, x_1 x_2 + x_1 x_3 + x_2 x_3 + (x_1 + x_2 + x_3)x_4, x_1 x_3 x_4 + x_1 x_2 x_4 + x_1 x_2 x_4 + x_2 x_3 x_4, x_1 x_2 x_3 x_4$, respectively.

Relative to the order of the fixed basis $v_l \otimes v_j$ ($1 \leq i, j \leq 8$), we will consider some special rows in matrix $\hat{R}_{VV} - x_l I$. Obviously, there is only one nonzero entry in the row (88), that is, $R_{VV_{88}} = q^{\frac{1}{2}}$. So we have

$$(105) \quad M_{88}^{88} = (q^\frac{1}{2} - x_1)(q^\frac{1}{2} - x_2)(q^\frac{1}{2} - x_3)(q^\frac{1}{2} - x_4).$$  

From it, then we obtain $x_l = q^\frac{1}{2}$ for a certain $x_l$. Next, we consider the row (12). Only two nonzero entries occur in row (12), which locate at columns (12) and (21), namely,
From the above equality, we need to further find those nonzero entries in row (21) of matrix $R_{VV} - x_j I$, listed as $(\hat{R}_{VV} - x_j I)_{21}^{1, 2}$ as follows:

$$[(\hat{R}_{VV} - x_j I)(\hat{R}_{VV} - x_j I)]_{21}^{1, 2} = (\hat{R}_{VV} - x_j I)_{12}^{1, 2}(\hat{R}_{VV} - x_j I)_{12}^{1, 2} + (\hat{R}_{VV} - x_j I)_{21}^{1, 2}(\hat{R}_{VV} - x_j I)_{21}^{1, 2},$$

$$= q^\frac{1}{2}(q^\frac{1}{2} - q^{-\frac{1}{2}}) - x_j.$$ 

Hence, the nonzero entries in row (21) of matrix $R_{VV} - x_j I$, listed as $(\hat{R}_{VV} - x_j I)_{21}^{1, 2}$, are:

$$[(\hat{R}_{VV} - x_j I)(\hat{R}_{VV} - x_j I)]_{21}^{1, 2} = q^\frac{1}{2}, (\hat{R}_{VV} - x_j I)_{21}^{2, 1} = q^\frac{1}{2}(q^\frac{1}{2} - q^{-\frac{1}{2}}) - x_j.$$ 

Namely, those nonzero entries still locate at columns (12) and (21) in row (12) of matrix $(\hat{R}_{VV} - x_j I)(\hat{R}_{VV} - x_j I)$, according to the same analysis, the nonzero entries in row (12) of $(\hat{R}_{VV} - x_j I)(\hat{R}_{VV} - x_j I)$ still lie at columns (12) and (21), given by

$$[(\hat{R}_{VV} - x_j I)(\hat{R}_{VV} - x_j I)]_{12}^{1, 2} = (\hat{R}_{VV} - x_j I)_{12}^{1, 2}(\hat{R}_{VV} - x_j I)_{12}^{1, 2} + (\hat{R}_{VV} - x_j I)_{21}^{1, 2}(\hat{R}_{VV} - x_j I)_{21}^{1, 2},$$

$$= q^\frac{1}{2}(x_1 x_2 + x_1 x_3 + x_2 x_3) - (q - 1)(x_1 + x_2 + x_3) + q^\frac{1}{2}(q - q^{-1})(q^\frac{1}{2} - q^{-\frac{1}{2}}) + q^\frac{1}{2}.$$ 

Similarly, Associated with nonzero entries in rows (58), (67), (76), (85):

$$(\hat{R}_{VV} - x_i)_{58}^{65} = -x_i, \ (\hat{R}_{VV} - x_i)_{85}^{65} = q^\frac{1}{2} = (\hat{R}_{VV} - x_i)_{65}^{85}. \ (\hat{R}_{VV} - x_i)_{67}^{76} = -x_i,$$

$$(\hat{R}_{VV} - x_i)_{76}^{67} = q^\frac{1}{2} = (\hat{R}_{VV} - x_i)_{76}^{67}. \ (\hat{R}_{VV} - x_i)_{67}^{85} = q^\frac{1}{2}(q^\frac{1}{2} - q^{-\frac{1}{2}}) = (\hat{R}_{VV} - x_i)_{67}^{85},$$

$$(\hat{R}_{VV} - x_i)_{76}^{85} = q^\frac{1}{2}(q - q^{-1}) - x_i, \ (\hat{R}_{VV} - x_i)_{85}^{76} = -q^\frac{1}{2}(q^\frac{1}{2} - q^{-\frac{1}{2}}) = (\hat{R}_{VV} - x_i)_{76}^{85},$$

$$(\hat{R}_{VV} - x_i)_{85}^{85} = q^\frac{1}{2}(q - 1)(1 + q^{-2}) - x_i.$$
we obtain
\begin{equation}
\mathcal{M}_{SR}^{\mathfrak{g}} = \Delta_4 + q^{-\frac{1}{2}}\Delta_2 - q^{-\frac{3}{2}}(q-1)(1+q^{-2})\Delta_1 + q^{-1} + (1-q^{-1})^2(1+q^{-2})(1+q+q^{-1}).
\end{equation}

Then according to (105)–(108), and \( \mathcal{M} = 0 \), we obtain the following equations:
\begin{equation}
\begin{cases}
\Delta_4 - q^\frac{1}{2}\Delta_3 + q^\frac{3}{2}\Delta_2 - q^\frac{5}{2}\Delta_1 + q^4 = 0, \\
\Delta_4 + q^\frac{1}{2}\Delta_2 - q^\frac{3}{2}(q-1)\Delta_1 + (q-1)^2 + q = 0, \\
-q^\frac{1}{2}\Delta_3 + (q-1)\Delta_2 - [q^\frac{1}{2}(q-1)(q^\frac{5}{2} - q^{-\frac{3}{2}}) + q^\frac{5}{2}]\Delta_1 + (q-1)(q^\frac{3}{2} + q^{-\frac{1}{2}}) = 0, \\
\Delta_4 + q^{-\frac{3}{2}}\Delta_2 - q^{-\frac{5}{2}}(q-1)(1 + q^{-2})\Delta_1 + q^{-1} + (1-q^{-1})^2(1+q^{-2})(1+q+q^{-1}) = 0.
\end{cases}
\end{equation}

Solving the system of equations, we get
\begin{equation}
\begin{cases}
\Delta_1 = q^\frac{5}{2} - q^{-\frac{1}{2}} - q^{-\frac{3}{2}} - q^{-\frac{7}{2}}, \\
\Delta_2 = q^{-\frac{3}{2}} - q^{-\frac{1}{2}} + q^\frac{3}{2} - q^{-\frac{7}{2}}, \\
\Delta_3 = q^{-\frac{1}{2}} + q^\frac{3}{2} - q^{-\frac{3}{2}} - q^{-\frac{7}{2}}, \\
\Delta_4 = -q^{-7},
\end{cases}
\end{equation}
and then \( x_1 = q^{\frac{3}{2}}, x_2 = -q^{-\frac{1}{2}}, x_3 = -q^{-\frac{3}{2}}, x_4 = -q^{-\frac{7}{2}} \). So the proof is complete. \( \square \)

Having the minimal polynomial equation of the braiding \( \hat{R}_{VV} \) in hand, it is necessary to consider at which eigenvalue of \( \hat{R}_{VV} \) we should make its normalization in order to construct the quantum group of higher-one rank we expect? Corresponding to the \( n \)-dimensional representation \( T_V \) with basis \( v_i \) we choose, we know that the new simple root vectors \( E_\alpha \), \( F_\alpha \) and the group-like element \( K_\alpha \) are always identified respectively as \( e^\alpha, f^\alpha, (m^\alpha)^n c^{-1} \) from those double-bosonization constructions in [HH]. Denote by \( \mu_n \) the weight of \( v_n \), we obtain \( R_{VV}^{nm} = q^\mu \). In order to construct the larger quantum group we want, the most important cross relation is \( e^\alpha((m^\alpha)^n c^{-1}) = R_m^{nm}(((m^\alpha)^n c^{-1}))e^\alpha \) because it gives us the length of the new additional simple root \( \alpha \), namely, \( e^\alpha((m^\alpha)^n c^{-1}) = q^{(\rho(\alpha,\alpha))}(((m^\alpha)^n c^{-1}))e^\alpha \). From these, we know at which one of eigenvalues we should take its normalization in virtue of \( R_{VV}^{nm} \) and \( q^{(\rho(\alpha,\alpha))} \) such that the value of \( R_m^{nm} \) satisfy our requirement. Thus we obtain the following

**Proposition 4.3.** Starting from an appropriate \( U_q(\mathfrak{g}') \)-module with highest weight \( \mu \) for a Lie subalgebra \( \mathfrak{g}' \subset \mathfrak{g} \) of corank 1, if we can construct inductively quantum group \( U_q(\mathfrak{g}) \) from \( U_q(\mathfrak{g}') \) by double-bosonization procedure, then the quantum group normalization constant is \( \lambda = \frac{d^{(\alpha,\alpha)}}{q^{\rho(m)}} \), where \( \alpha \) is the additional simple root for \( \mathfrak{g} \).

By Proposition 4.3, we choose the normalization constant \( \lambda = q^{-\frac{3}{2}} \) for the construction of \( U_q(F_4) \) from this spin representation. Then setting
\begin{equation}
R = q^{\frac{1}{2}} R_{VV}, \quad R' = R(\hat{R})^2 + (q^{-2} - q + q^{-5}) R \hat{R} + (q^{-7} - q^{-4} - q^{-1}) R - (q^{-6} - 1) P,
\end{equation}
we obtain \((PR + I)(PR' - I) = 0\). The entries in \( m^\alpha \)-matrices are given by the following Lemma.
Lemma 4.2. Corresponding to the 8-dimensional spin representation of $U_h(B_3)$, the entries in the matrices $m^i$ we need are listed as follows.

\[
(m^+)_{3}^5 = -(q - q^{-1})E_1K_1^{-1}K_2^{-1}K_3^{-1}, \quad (m^+)_{5}^3 = K_1^{-1}K_3^{-1},
\]
\[
(m^+)_{5}^5 = -(q - q^{-1})E_2K_1^{-1}K_2^{-1}K_3^{-1}K_4^{-1}, \quad (m^+)_{7}^3 = K_1^{-1}K_2^{-1}K_3^{-1}K_4^{-1},
\]
\[
(m^+)_{7}^5 = -(q^{-1} - q)E_3K_1^{-1}K_2^{-1}K_3^{-1}K_4^{-1}, \quad (m^+)_{8}^8 = K_1^{-1}K_2^{-1}K_3^{-1}K_4^{-1}.
\]
\[
(m^-)_{3}^5 = q(q - q^{-1})K_1^{-1}K_2^{-1}F_1, \quad (m^-)_{5}^3 = K_1^{-1}K_3^{-1},
\]
\[
(m^-)_{5}^5 = q(q - q^{-1})K_1^{-1}K_2^{-1}F_2, \quad (m^-)_{7}^3 = K_1^{-1}K_2^{-1},
\]
\[
(m^-)_{7}^5 = q(q - q^{-1})K_1^{-1}K_2^{-1}K_3^{-1}F_3, \quad (m^-)_{8}^8 = K_1^{-1}K_2^{-1}K_3^{-1}F_3.
\]

With these data in hand, we have the following

Theorem 4.1. Identify elements $e^8, f_8, (m^+)_{8}^8c^{-1}$ with the additional simple root vectors $E_4, F_4$ and the group-like $K_4$, then the quantum group $U(V^\lambda(R', R_+^{-1}), U_q^{\text{gr}}(B_3), V(R', R))$ is exactly the $U_q(F_4)$ with $K_i^{1/2}$ adjoined, $1 \leq i \leq 3$.

Proof. According to the cross relations in Theorem 3.4, we obtain

\[
e^8(m^+)_{8}^8c^{-1} = \lambda R_{88}^{88}(m^+)_{8}^8e^8c^{-1} = R_{88}^{88}(m^+)_{8}^8c^{-1}e^8 = q^3R_{V88}^{88}(m^+)_{8}^8c^{-1}e^8 = q(m^+)_{8}^8c^{-1}e^8.
\]

Then with the above identification, we have $E_4K_4 = qK_4E_4$. The relations between $E_4$ and $K_i$ can be obtained by the following relations

\[
(m^+)_{8}^8K_3 = (m^+)_{7}^5, \quad e^8(m^+)_{8}^8 = \lambda R_{88}^{88}(m^+)_{8}^8e^8 = q^{1/2}(m^+)_{5}^5e^8,
\]
\[
e^8(m^+)_{7}^7 = \lambda R_{78}^{78}(m^+)_{7}^7e^8 = q^{1/2}(m^+)_{5}^5e^8, \quad e^8(m^+)_{8}^8 = \lambda R_{88}^{88}(m^+)_{8}^8e^8 = q^{1/2}(m^+)_{5}^5e^8.
\]

From the above relations, we obtain that $e^8K_1 = K_1e^8$, $e^8K_2 = K_2e^8$ and $e^8K_3 = q^{1/2}K_3e^8$. Namely,

\[
E_4K_1 = K_1E_4, \quad E_4K_2 = K_2E_4, \quad E_4K_3 = q^{1/2}K_3E_4.
\]

The relations between $K_4 = K_1^{-1}K_2^{-1}K_3^{-1}c^{-1}$ and $E_i$ are given by

\[
E_1K_4 = E_1K_1^{-1}K_2^{-1}K_3^{-1}c^{-1} = q^{-1}qK_1^{-1}K_2^{-1}K_3^{-1}c^{-1}E_1 = K_4E_1,
\]
\[
E_2K_4 = E_2K_1^{-1}K_2^{-1}K_3^{-1}c^{-1} = q^{1/2}q^{-2}q^2K_1^{-1}K_2^{-1}K_3^{-1}c^{-1}E_2 = K_4E_2,
\]
\[
E_3K_4 = E_3K_1^{-1}K_2^{-1}K_3^{-1}c^{-1} = q^{-3/2}K_1^{-1}K_2^{-1}K_3^{-1}c^{-1}E_3 = q^{-1}K_4E_3.
\]

We will check the $q$-Serre relations related to $E_i$, which also can be obtained by the cross relations in Theorem 3.4. In fact, by Lemma 4.2 and Theorem 3.4, we have

\[
(m^+)_{3}^3 = -(q - q^{-1})E_1(m^+)_{3}^3, \quad e^8(m^+)_{8}^8 = \lambda R_{38}^{38}(m^+)_{8}^8e^8 = q^{1/2}(m^+)_{3}^3e^8.
\]
These yield $e^8E_1 = E_1e^8$, namely, $E_4E_1 = E_1E_4$. On the other hand, starting from the following cross relations (followed by Lemma 4.2 and Theorem 3.4, respectively)

$$
(m^+)_7 = -(q - q^{-1})E_2(m^+)_{17}, \quad e^8(m^+)_{67} = \lambda R^{68}_{68}(m^+)_{67}e^8 = q^4(m^+)_{67}e^8,
$$

we obtain $e^8E_2 = E_2e^8$, namely, $E_4E_2 = E_2E_4$. In order to check the relation between $E_4(e^8)$ and $E_3$, we consider the cross relations from 4.2 and Theorem 3.4,

$$
(m^+)_{87} = -(q^4 - q^{-2})E_3(m^+)_{87}, \quad e^7(m^+)_{87} = \lambda R^{87}_{87}(m^+)_{87}e^7 = q^4(m^+)_{87}e^7,
$$

$$
e^8(m^+)_{87} = q^4(m^+)_{87}e^8 + (q^4 - q^{-2})q^3(m^+)_{87}e^7,
$$

Thus we obtain that $e^7 = q^{-2}E_3e^8 - e^8E_3$. Further more, we need the relation $e^7E_3 = q^3E_3e^7$, which can be obtained by $e^7(m^+)_{87} = \lambda R^{87}_{87}(m^+)_{87}e^7 = q^4(m^+)_{87}e^7$ (by Theorem 3.4). Combining with the above equality $e^7 = q^{-2}E_3e^8 - e^8E_3$, we obtain $e^8(E_3)^2 - (q^4 + q^{-2})E_3e^8E_3 + (E_3)^2e^8 = 0$, namely,

$$
E_4(E_3)^2 - (q^4 + q^{-2})E_3E_4E_3 + (E_3)^2E_4 = 0.
$$

On the other hand, in order to check another $q$-Serre relation, we need to figure out the relation between $e^7$ and $e^8$, that is, $e^8e^7 = R^{78}_{ab}e^ae^b$. Since the nonzero entries in rows (78) and (87) of $R$ are $R^{78}_{78} = q^4$, $R^{78}_{77} = q - 1$, $R^{78}_{76} = q^4$, then we have

$$
(RR')_{78}^{78} = R^{78}_{78}R^{78}_{78}R^{78}_{78} + R^{78}_{76}R^{78}_{76}R^{78}_{78} = q^2 + q^4(q - 1)^2,
$$

$$
(RR')_{78}^{78} = R^{78}_{78}R^{78}_{78}R^{78}_{78} + R^{78}_{76}R^{78}_{76}R^{78}_{78} = q^3 - q^2 + q - 1,
$$

$$
(RR')_{78}^{78} = R^{78}_{78}R^{78}_{78} = q^2(q - 1), \quad (RR')_{78}^{78} = R^{78}_{78}R^{78}_{78} + R^{78}_{76}R^{78}_{76} = q^2 - q + 1.
$$

From the formula (109) of $R, R'$ we get, nonzero entries in rows (78) of $R'$ are

$$
R^{78}_{78} = q^2 + q^4(q - 1)^2 + (q^{-2} - q + q^{-5})q^3(q - 1) + (q^4 - q^{-4} - q^3 - 1)q^4,
$$

$$
R^{78}_{77} = q^3 - q^2 + q - 1 + (q^{-2} - q + q^{-5})(q^2 - q + 1) + (q^4 - q^{-4} - q^3 - 1)(q - 1) - (q^{-6} - 1).
$$

So $e^8e^7 = R^{78}_{78}e^7e^8 + R^{78}_{78}e^8e^7$, then we obtain $e^8e^7 = q^5e^7e^8$. Combining with the above equality $e^7 = q^{-2}E_3e^8 - e^8E_3$ again, we obtain $(e^8)^2E_3 - (q^4 + q^{-2})e^8E_3e^8 + E_3(e^8)^2 = 0$, namely,

$$
(E_4)^2E_3 - (q^4 + q^{-2})E_3E_4E_3 + E_3(E_4)^2 = 0.
$$

Thence, we obtain that the Cartan matrix of the new quantum group is

$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
$$

This completes the proof. □
4.2. **Type-crossing of $U_q(G_2)$ via $U_q(A_1)$**. From the classical cases, we know that $U_q(sl_3)$ was constructed by choosing braided groups corresponding to the vector representation of $U_q(sl_2)$. This inspires us to consider the spin $\frac{1}{2}$ representation $T_{sp}$ of $U_q(sl_2)$ when constructing $U_q(G_2)$, here $T_{sp}$ is taken from the 4-dimensional homogeneous subspace of degree 3 of braided group $\mathcal{O}(\mathbb{C}^2_2)$ generated by $x, y$ with $yx = qxy$ in the category of $U_q(sl_2)$-modules. In fact, this braided group $\mathcal{A}_q(2) = \mathbb{C}_q(x, y)$ is a left $U_q(sl_2)$-module algebra ([H]). Explicitly, starting from the 2-dimensional vector representation of $U_q(sl_2)$, we have the standard $4 \times 4$ $R$-matrix below

$$
R = \begin{pmatrix}
  q & 0 & 0 & 0 \\
  0 & 1 & q^{-1} & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & q
\end{pmatrix}, \quad T = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}.
$$

For such $R$, the FRT-bialgebra $A(R)$ has a quotient Hopf algebra generated by the entries $a, b, c, d$ in the above matrix $T$, denoted by $O_q(SL(2))$. Dually, there is a right $O_q(SL(2))$-comodule algebra structure on $\mathcal{A}_q(2)$, namely

$$(110) \quad \rho(y, x) = (y \otimes a + x \otimes c, y \otimes b + x \otimes d) = (y, x) \otimes T.$$

On the 4-dimensional homogeneous subspace of degree 3 consisting of $x^{i-1}y^{3-(i-1)}, i = 1, 2, 3, 4$, denoted by $\mathcal{A}_q(2)^{(3)}$, there exists a right $O_q(SL(2))$-subcomodule induced by $(110)$, its corepresentation matrix is still denoted by $T$, given by

$$(111) \quad T = \begin{pmatrix}
  r_1^1 & r_1^2 & r_1^3 & r_1^4 \\
  r_2^1 & r_2^2 & r_2^3 & r_2^4 \\
  r_3^1 & r_3^2 & r_3^3 & r_3^4 \\
  r_4^1 & r_4^2 & r_4^3 & r_4^4
\end{pmatrix} = \begin{pmatrix}
  a^3 & a^2b & ab^2 & b^3 \\
  [3]_q a^2 c d + (q^{-2}+1)cab & q^{-2}c^2b^2 + (q^{-2}+1)ad & [3]_q db^2 \\
  [3]_q c^2 a^{-1} b^{-1} + (q^{-2}+1)ad & ad^2 + (q^{-2}+1)cd & [3]_q d^2b
\end{pmatrix}.
$$

Since the dual pair of $(U^q_{SL}(sl_2), O_q(SL(2)))$, then this 4-dimensional right $O_q(SL(2))$-comodule induces a left $U_q(sl_2)$-module, called as spin $\frac{1}{2}$ representation $T_{sp}$, given by

$$(112) \quad K_1 \begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4
\end{pmatrix} = \begin{pmatrix}
  q^{-3}v_1 \\
  q^{-1}v_1 \\
  q^{-1}v_3 \\
  q^{3}v_4
\end{pmatrix}, \quad E_1 \begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4
\end{pmatrix} = \begin{pmatrix}
  q^{-2}[3]_q^2 v_2 \\
  q^{-2}[2] v_3 \\
  q^{-2}[3]_q^2 v_4 \\
  0
\end{pmatrix}, \quad F_1 \begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4
\end{pmatrix} = \begin{pmatrix}
  0 \\
  q^{-1}[3]_q^2 v_1 \\
  q^{-1} v_2 \\
  q^{-2}[3]_q^2 v_3
\end{pmatrix}.
$$

Where $v_i$ denote the basis of representation space of $T_{sp}$, with corresponding weights $-\frac{3}{2} \alpha_1, -\frac{1}{2} \alpha_1, \frac{1}{2} \alpha_1, \frac{3}{2} \alpha_1$. Moreover, we observe that $E^i_1, F^i_1$ are zero actions for all $i \geq 4$, so we
needn’t to consider every summand in the expression of universal $\mathcal{R}$-matrix. Then

$$B_{VV} \circ (T_V \otimes T_V)(\mathcal{R})(v_i \otimes v_j) = B_{VV} \circ (T_V \otimes T_V)(\sum_{i=0}^{3} c_i E_i \otimes F_i)(v_i \otimes v_j),$$

here, $c_n = \frac{(1-q^{-\frac{1}{2}})^n q^{-\frac{n(n+1)}{2}}}{\langle n \rangle_1}$. With these, we obtain that the following $16 \times 16$ $R$-matrix $R_{VV}$ corresponding to the spin representation (112).

$$
\begin{pmatrix}
q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^2 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-\frac{1}{2}} & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{1}{2}} \\
\end{pmatrix}
$$

Obviously, the matrix $PR_{VV}$ is symmetric. For this $R$-matrix $R_{VV}$, the existence of $R, R'$ is guaranteed by Remark 2.1. We can obtain that the quantum group normalization constant is $\lambda = \frac{q^{\frac{1}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} = \frac{q^{\frac{1}{2}}}{q^{-\frac{1}{2}}}$ by Proposition 4.3. For consistency, we denote the generators of braided groups $V^0(R', R_{21}^{-1})$ induced by the representation $T_{sp}$ by $f_i$, and identify $f_i$ as $x^{i-1}y^{3-(i-1)}, i = 1, 2, 3, 4$, then the algebra structure of $V^0(R', R_{21}^{-1})$ can be obtained directly by $yx = qxy$, not necessary deduced from the matrix $R'$, which is different from the case of construction for $U_q(F_4)$. On the other hand, in order to obtain the relations in the new quantum group by the generalized double-bosonization construction theorem, we need to know the matrix $m^\pm$ consisting of FRT-generators, which are listed in the following lemma.
Lemma 4.3. Corresponding to the spin $\frac{3}{2}$ representation, the matrices $m^\pm$ are given by

$$m^+ = \begin{pmatrix}
K_1^{\frac{1}{2}} & c_1[3]\frac{3}{2}q^2K_1^{\frac{1}{2}}E_1 & -c_2[3]q^2K_1^{\frac{1}{2}}E_1 & c_3[3][2]q^2K_1^{\frac{1}{2}}E_1 \\
0 & K_1^2 & c_1[2]q^2K_1^{\frac{1}{2}}E_1 & -c_2[3]q^2K_1^{\frac{1}{2}}E_1 \\
0 & 0 & K_1^{\frac{1}{2}} & c_1[3]\frac{3}{2}q^2K_1^{\frac{1}{2}}E_1 \\
0 & 0 & 0 & K_1^{\frac{1}{2}}
\end{pmatrix}$$

$$m^- = \begin{pmatrix}
K_1^{\frac{1}{2}} & 0 & 0 & 0 \\
-c_1[3]\frac{3}{2}q^2K_1^{\frac{1}{2}}F_1 & K_1^{\frac{1}{2}} & 0 & 0 \\
c_2[3][2]q^{-6}K_1^{\frac{1}{2}}F_1 & -c_1[2]q^{-6}K_1^{\frac{1}{2}}F_1 & K_1^{\frac{1}{2}} & 0 \\
-c_3[3][2]q^{-6}K_1^{\frac{1}{2}}F_1 & c_2[3]\frac{3}{2}K_1^{\frac{1}{2}}F_1 & -c_3[3]\frac{3}{2}q^2K_1^{\frac{1}{2}}F_1 & K_1^{\frac{1}{2}}
\end{pmatrix}$$

With these, we have the following

Theorem 4.2. Corresponding to the spin $\frac{3}{2}$ representation, the normalization constant $\lambda = q^{-\frac{3}{2}}$, and $R = q^{\frac{3}{2}}R_{VV}$. Identifying $e^4, f_4, m^+\lambda^{-1}c^{-1}$ with the new additional simple root vectors $E_2, F_2$ and the group-like element $K_2$, respectively, then the resulting new quantum group $U(V^\vee(R^D, R_{2}^{-1}), \hat{U}_q^{\text{ext}}(sl_2), V(R^D, R))$ is exactly the $U_q(G_2)$ with $K_2$ adjoined.

Proof. We just describe the cross relations and $q$-Serre relations in positive part, the corresponding relations in negative part can be obtained in a similar way. Under the identification of Theorem 4.2,

$$[E_2, F_2] = \frac{K_2 - K_2^{-1}}{q^3 - q^{-3}}, \Delta(E_2) = E_2 \otimes K_2 + 1 \otimes E_2, \text{and} \Delta(F_2) = F_2 \otimes 1 + K_2^{-1} \otimes F_2$$

can be obtained directly by Theorem 3.4. Moreover,

$$e^4(m^+)_1^{-\frac{1}{2}}c^{-1} = \lambda R_{ab}^{44}(m^+)_1^{-\frac{1}{2}}e^4c^{-1} = \lambda R_{ab}^{44}(m^+)_1^{-\frac{1}{2}}e^4c^{-1} = \lambda R_{ab}^{44}(m^+)\lambda^{-1}c^{-1}e^4 = q^{\frac{3}{2}}q^{-\frac{3}{2}}(m^+)_1^{-\frac{1}{2}}e^4 = q^3(m^+)\lambda^{-1}c^{-1}e^4.$$  

Namely, we obtain $E_2K_2 = q^6K_2E_2$. In view of the equality $(m^+)_1^{-\frac{1}{2}}K_2 = (m^+)\lambda$, obtained by Lemma 4.3, and the cross relation $e^4(m^+)_j = \lambda R_{ab}^{44}(m^+)_j\lambda^{-1}e^4$, we obtain that $e^4K_1 = q^{-3}K_1e^4$,

so $E_2K_1 = q^{-3}K_1E_2$. On the other hand, by $K_2 = K_1^{-\frac{3}{2}}c^{-1}$, we obtain that

$$E_1K_2 = E_1K_1^{-\frac{3}{2}}c^{-1} = q^{-3}K_1^{-\frac{3}{2}}e^{-1}E_1 = q^{-3}K_2E_1.$$  

Since $F_1$ is included in the entry $(m^-)_1^{\frac{1}{2}}$ of $m^-$, the relation between $E_2$ and $F_1$ can be obtained by the following cross relation

$$(m^-)_1^{\frac{1}{2}}e^4 = \lambda R_{ab}^{44}(m^-)^{-\frac{1}{2}}e^4 = q^{\frac{3}{2}}e^4(m^-)_1^{\frac{1}{2}}.$$
Then by the expression of \((m^-)_1^2\), we obtain
\[
K_1^{-\frac{1}{2}}F_1e^4 = q^{-\frac{1}{2}}e^4K_1^{-\frac{1}{2}}F_1 = q^{-\frac{1}{2}}q^{-\frac{1}{2}}K_1^{-\frac{1}{2}}e^4F_1 = K_1^{-\frac{1}{2}}e^4F_1,
\]
so \(F_1e^4 = e^4F_1\), namely, \([E_2, F_1] = 0\).

Let us check the \(q\)-Serre relations of \(E_i, i = 1, 2\). Firstly, by Theorem 3.4. we have
\[
e^4(m^+)_1^3 = q^{-\frac{1}{2}}(m^+)_2^3e^4 + c_1[3]q^{-\frac{1}{2}}(m^+)_2^3e^3 \quad \text{and} \quad e^4K_1 = q^{-3}K_1e^4.
\]
So we obtain
\[
e^4E_1 - q^{-3}E_1e^4 = [3]q^{-\frac{1}{2}}e^3.
\]
Next we need to know the relation of \(e^3, E_1\) and \(e^4\). \(f_4f_3 = x^3 x^2 y = q^{-3} x^2 y x^3 = q^{-3} f_3f_4\) by \(yx = qxy\), so \(R^{34}_{34} = q^3\), then \(e^4e^3 = R^{14}_{14}e^3e^3 = q^3e^3e^4\). Combining with (113), we have \((e^4)^2E_1 - (q^3 + q^{-3})e^4E_1e^4 + E_1(e^4)^2 = 0, \) namely,
\[
(E_2)^2 E_1 - \left[\begin{array}{c} 2 \\ 1 \end{array}\right] E_2 E_1 E_2 + E_1(E_2)^2 = 0.
\]
On the other hand, in order to obtain the relation of \(e^3, E_1\), starting from the following cross relations in the new quantum group
\[
e^3(m^+)_2^3 = c_1[3]^2[q^{-\frac{1}{2}}m_2^2 e^2 + q^{-\frac{1}{2}}(m^+)_2^3],
\]
\[
e^2(m^+)_2^3 = c_1[3]q^{-\frac{1}{2}}e^1 + q^{-\frac{1}{2}}e^3, \quad e^1(m^+)_2^3 = q^2(m^+)_2^3 e^1,
\]
we obtain \(E_1 e^3 - qe^3 E_1 = [3]^3q^{-\frac{1}{2}}e^2, \) \(E_1 e^2 - q^{-1}e^2 E_1 = [3]^3q^{-\frac{1}{2}}e^1, \) and \(e^1 E_1 = q^3 E_1 e^1\). Combining with (113) again, we obtain \((E_1)^4 e^4 - (q^3 + q^{-1} + q + q^3)E_1^3 e^4 E_1 + (q^2 + 1 + q^4 + 1 + q^{-4} + q^{-2})(E_1)^2 e^4 (E_1)^2 - (q + q^3 + q^{-1} + q^{-3})E_1 e^4 (E_1)^3 + e^4 (E_1)^4 = 0, \) namely,
\[
(E_1)^4 E_2 - \left[\begin{array}{c} 4 \\ 1 \end{array}\right] (E_1)^3 E_2 E_1 + \left[\begin{array}{c} 4 \\ 2 \end{array}\right] (E_1)^2 E_2 (E_1)^2 - \left[\begin{array}{c} 4 \\ 3 \end{array}\right] E_1 E_2 (E_1)^3 + E_2 (E_1)^4 = 0.
\]
So the Cartan matrix of the resulting quantum group is \(\left(\begin{array}{cc} 2 & -3 \\ -1 & 2 \end{array}\right)\), the proof is complete. \(\square\)

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