SINGULARITY FORMATION TO THE TWO-DIMENSIONAL NON-BAROTROPIC NON-RESISTIVE MAGNETOHYDRODYNAMIC EQUATIONS WITH ZERO HEAT CONDUCTION IN A BOUNDED DOMAIN

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Abstract. We are concerned with the singularity formation of strong solutions to the two-dimensional (2D) non-barotropic non-resistive compressible magnetohydrodynamic equations with zero heat conduction in a bounded domain. It is showed that the strong solution exists globally if the density and the magnetic field as well as the pressure are bounded from above. Our method relies on critical Sobolev inequalities of logarithmic type.

1. Introduction. Let \( \Omega \subset \mathbb{R}^n \) (\( n = 2, 3 \)) be a domain, the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in \( \Omega \) can be described by the non-barotropic compressible MHD equations

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div}u + \nabla P &= b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \\
c_v [(\rho \theta)_t + \text{div}(\rho u \theta)] + P \text{div} u - \kappa \Delta \theta &= 2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2 + \nu |\nabla \times b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div} u &= \nu \Delta b, \\
\text{div} b &= 0.
\end{aligned}
\]  

(1.1)

Here, \( t \geq 0 \) is the time, \( x \in \Omega \) is the spatial coordinate, and \( \rho, u, P = R \rho \theta \) (\( R > 0 \)), \( \theta, b \) are the fluid density, velocity, pressure, absolute temperature, and the magnetic field respectively; \( \mathcal{D}(u) \) denotes the deformation tensor given by

\[
\mathcal{D}(u) = \frac{1}{2} (\nabla u + (\nabla u)^{tr}).
\]

The constant viscosity coefficients \( \mu \) and \( \lambda \) satisfy the physical restrictions

\[
\mu > 0, \ 2\mu + n\lambda \geq 0. \tag{1.2}
\]
Positive constants $c_\nu$, $\kappa$, and $\nu$ are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient.

The study on the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for compressible MHD equations has grown enormously in recent years, refer to [4, 8–10, 15, 18, 23, 26, 27] and references therein. In particular, Kawashima [14] first obtained the global existence and uniqueness of classical solutions to the multi-dimensional compressible MHD equations when the initial data are close to a non-vacuum equilibrium in $H^3$-norm. For general large initial data, Hu-Wang [9, 10] proved the global existence of weak solutions with finite energy in Lions-Feireisl framework for compressible Navier-Stokes equations [5, 6, 16] provided the adiabatic exponent is suitably large, yet the uniqueness and regularity of these weak solutions remain unknown. Recently, Li-Xu-Zhang [15] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible MHD system in 3D with smooth initial data which are of small energy but possibly large oscillations and vacuum. Very recently, Hong-Hou-Peng-Zhu [8] improved the result in [15] to allow the initial energy large as long as the adiabatic exponent is close to 1 and $\nu$ is suitably large. Furthermore, Liu-Shi-Xu [18] established the global existence and uniqueness of strong solutions to the 2D MHD equations provided that the smooth initial data are of small total energy. Nevertheless, it is an outstanding challenging open problem to investigate the global well-posedness for general large strong solutions with vacuum.

Therefore, it is important to study the structure of possible singularities of strong (or classical) solutions to the compressible MHD equations. Recently, there are several results on the blow-up criteria of strong (or classical) solutions to the full compressible MHD equations (1.1). For the Cauchy problem and the IBVP of 3D full compressible MHD system, Huang-Li [11] established the following Serrin type criterion

$$\lim_{{T \to T^*}} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}) = \infty, \quad \text{for} \quad \frac{2}{s} + \frac{3}{r} \leq 1, \quad 3 < r \leq \infty. \quad (1.3)$$

On the other hand, for the IBVP of 2D full compressible MHD system, Fan-Li-Nakamura [2] proved that

$$\lim_{{T \to T^*}} (\|\text{div } u\|_{L^1(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^\infty)}) = \infty. \quad (1.4)$$

Later, Lu-Chen-Huang [17] extended (1.4) with a refiner form

$$\lim_{{T \to T^*}} \|\text{div } u\|_{L^1(0,T;L^\infty)} = \infty. \quad (1.5)$$

The criterion (1.5) is the same as [20] for 2D compressible full Navier-Stokes equations, which shows that the mechanism of blow-up is independent of the magnetic field.

It should be noted that all the results mentioned above on the blow-up of strong (or classical) solutions of viscous, compressible, and heat conducting MHD flows are for $\kappa > 0$. Very recently, several results are devoted to the singularity formation of the full compressible MHD equations (1.1) with $\kappa = 0$. For 3D case, under the assumption

$$3\mu > \lambda, \quad (1.6)$$
Zhong [24] showed that
\[
\lim_{T \to T^*} \left( \|D(u)\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty.
\]
(1.7)

On the other hand, under the assumption
\[
2\mu > \lambda,
\]
(1.8)
Fan-Li-Nakamura [3] obtained
\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty,
\]
(1.9)
which was later improved by Zhong [25], where the author proved
\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^6)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty,
\]
(1.10)
provided that (1.6) holds true. It should be noted that the assumption (1.6) is weaker than (1.8) due to \(\mu > 0\). Very recently, for the Cauchy problem and the IBVP of 2D case, the author [28,29] established the following criterion
\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty.
\]
(1.11)
It is worth mentioning that the key idea in [28] is different from that of in [29]. Roughly speaking, weighted energy estimates and Hardy-type inequalities play a crucial role in [28], while the key ingredient of the analysis in [29] is critical Sobolev inequalities of logarithmic type. Furthermore, under the assumption (1.8), Huang-Wang [13] proved that
\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty
\]
(1.12)
for 3D full compressible MHD equations (1.1) with \(\kappa = \nu = 0\). It is worth mentioning that in a well-known paper [21], Xin considered non-isentropic compressible Navier-Stokes equations with \(\kappa = 0\) in multidimensional space, starting with a compactly supported initial density. He first proved that if the support of the density grows sublinearly in time and if the entropy is bounded from below then the solution cannot exist for all time. One key ingredient in the proof is a differential inequality on some integral functional (see [21, Proposition 2.1] for details). As an application, any smooth solution to the full compressible Navier-Stokes equations for polytropic fluids in the absence of heat conduction will blow up in finite time if the initial density is compactly supported. Recently, based on the key observation that if initially a positive mass is surrounded by a bounded vacuum region, then the time evolution remains uniformly bounded for all time, Xin-Yan [22] improved the blow-up results in [21] by removing the assumptions that the initial density has compact support and the smooth solution has finite energy, but the initial data only has an isolated mass group. Thus it seems very difficult to study globally smooth solutions of full compressible Navier-Stokes equations without heat conductivity in multi-dimension, the same difficulty also arises in multi-dimensional MHD equations.

Let \(\Omega \subset \mathbb{R}^2\) be a bounded smooth domain, when \(\kappa = \nu = 0\), and without loss of generality, take \(c_\nu = R = 1\), then the system (1.1) can be written as
\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u} + \nabla P &= \mathbf{b} \cdot \nabla \mathbf{b} - \frac{1}{2} \nabla |\mathbf{b}|^2, \\
P_t + \text{div}(P \mathbf{u}) + P \text{div} \mathbf{u} &= 2\mu |\nabla \mathbf{u}|^2 + \lambda (\text{div} \mathbf{u})^2, \\
\mathbf{b}_t - \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{b} + \mathbf{b} \text{div} \mathbf{u} &= 0, \\
\text{div} \mathbf{b} &= 0,
\end{align*}
\]

(1.13)

and the constant viscosity coefficients \(\mu\) and \(\lambda\) satisfy
\[
\mu > 0, \quad \mu + \lambda \geq 0.
\]

(1.14)

The present paper aims at giving a blow-up criterion of strong solutions to the system (1.13) with the initial condition
\[
(\rho, \rho \mathbf{u}, P, \mathbf{b})(x, 0) = (\rho_0, \rho_0 \mathbf{u}_0, P_0, \mathbf{b}_0)(x), \quad x \in \Omega,
\]
and the boundary condition
\[
\mathbf{u} = \mathbf{0}, \quad x \in \partial \Omega.
\]

(1.15)

(1.16)

Before stating our main result, we first explain the notations and conventions used throughout this paper. Set
\[
\int \cdot \, dx \triangleq \int_{\Omega} \cdot \, dx.
\]

For \(1 \leq p \leq \infty\) and integer \(k \geq 0\), the standard Sobolev spaces are denoted by:
\[
L^p(\Omega), \quad W^{k,p}(\Omega), \quad H^k = H^{k,2}(\Omega), \quad H^1_0 = \{u \in H^1 : u|_{\partial \Omega} = 0\}.
\]

Now we define precisely what we mean by strong solutions to the problem (1.13)–(1.16).

**Definition 1.1** (Strong solutions). \((\rho, \mathbf{u}, P, \mathbf{b})\) is called a strong solution to (1.13)–(1.16) in \(\Omega \times (0, T)\), if for some \(q > 2\),
\[
\begin{align*}
\rho &\geq 0, \quad \rho \in C([0, T]; W^{1,q}), \quad \rho_t \in C([0, T]; L^q), \\
\nabla \mathbf{u} &\in L^\infty(0, T; H^1) \cap L^2(0, T; W^{1,q}), \quad \sqrt{\rho} \mathbf{u} \in L^\infty(0, T; L^2), \\
\mathbf{b} &\in C([0, T]; W^{1,q}), \quad \mathbf{b}_t \in C([0, T]; L^q), \\
P &\geq 0, \quad P \in C([0, T]; W^{1,q}), \quad P_t \in C([0, T]; L^q),
\end{align*}
\]

and \((\rho, \mathbf{u}, P, \mathbf{b})\) satisfies both (1.13) almost everywhere in \(\Omega \times (0, T)\) and (1.15) almost everywhere in \(\Omega\).

Our main result reads as follows:

**Theorem 1.1.** Assume that the initial data \((\rho_0 \geq 0, \mathbf{u}_0, P_0 \geq 0, \mathbf{b}_0)\) satisfies for any given \(q > 2\),
\[
(\rho_0, P_0, \mathbf{b}_0) \in W^{1,q}(\Omega), \quad \mathbf{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega), \quad \text{div} \mathbf{b}_0 = 0,
\]

(1.17)

and the compatibility conditions
\[
-\mu \Delta \mathbf{u}_0 - (\lambda + \mu) \nabla \text{div} \mathbf{u}_0 + \nabla P_0 - \mathbf{b}_0 \cdot \nabla \mathbf{b}_0 + \frac{1}{2} \nabla |\mathbf{b}_0|^2 = \sqrt{\rho_0} \mathbf{g}
\]

(1.18)

for some \(\mathbf{g} \in L^2(\Omega)\). Let \((\rho, \mathbf{u}, P, \mathbf{b})\) be the strong solution to the problem (1.13)–(1.16). If \(T^* < \infty\) is the maximal time of existence for that solution, then we have
\[
\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0, T; L^\infty)} + \|\mathbf{b}\|_{L^\infty(0, T; L^\infty)} + \|P\|_{L^\infty(0, T; L^\infty)}\right) = \infty.
\]

(1.19)
Remark 1.1. The local existence of a unique strong solution with initial data as in Theorem 1.1 can be established by similar arguments as in [4]. Hence, the maximal time $T^*$ is well-defined.

Remark 1.2. Compared with [13], where the authors showed the blow-up criterion (1.12) for the 3D full magnetohydrodynamic equations with $\kappa = \nu = 0$, there is no need to impose additional restrictions on the viscosity coefficients $\mu$ and $\lambda$ except the physical restrictions (1.14).

We now make some comments on the analysis of this paper. We mainly make use of continuation argument to prove Theorem 1.1. That is, suppose that (1.19) were false, i.e.,

$$\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|b\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) \leq M_0 < \infty.$$  

We want to show that

$$\sup_{0 \leq t \leq T^*} \left( \|\rho, P, b\|_{W^{1,q}} + \|\nabla u\|_{H^1} \right) \leq C < +\infty.$$  

We first obtain (see Lemma 3.2) a control on the $L^\infty_t L^2_x$-norm of $\nabla u$. To this end, the key ingredient of the analysis is a logarithmic Sobolev inequality (see Lemma 2.3). The inequality implies the uniform estimate of $\|u\|_{L^q(0,T;L^r)}$ due to the a priori estimate of $\|u\|_{L^2(0,T;H^1)}$ from the energy estimate (3.2). Then we derive the key a priori estimates on $L^\infty(0,T;L^q)$-norm of $(\nabla \rho, \nabla P, \nabla b)$ by solving a logarithmic Gronwall inequality based on a Brézis-Waigner type inequality (see Lemma 2.4) and the a priori estimates we have just derived.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries. In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gagliardo-Nirenberg inequality (see [7, Theorem 10.1, p. 27]).

**Lemma 2.1** (Gagliardo-Nirenberg). Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Assume that $1 \leq q, r \leq \infty$, and $j, m$ are arbitrary integers satisfying $0 \leq j < m$. If $v \in W^{m,r}(\Omega) \cap L^q(\Omega)$, then we have

$$\|D^j v\|_{L^p} \leq C \|v\|_{L^q}^{1-a} \|v\|_{W^{m,r}}^a,$$

where

$$-j + \frac{2}{p} = (1-a) \frac{2}{q} + a \left(-m + \frac{2}{r}\right),$$

and

$$a \in \begin{cases} \left[\frac{m}{m-j}, 1\right), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer,} \\
\left[\frac{m}{m-j}, 1\right], & \text{otherwise.} \end{cases}$$

The constant $C$ depends only on $m, j, q, r, a$, and $\Omega$.

Next, we give some regularity results of the following Lamé system with Dirichlet boundary condition, the proof can be found in [19, Proposition 2.1].

$$\begin{cases} -\mu \Delta U - (\mu + \lambda) \nabla \text{div } U = F, & x \in \Omega, \\
U = 0, & x \in \partial \Omega. \end{cases}$$  

(2.1)
Lemma 2.2. Let \( q \geq 2 \) and \( U \) be a weak solution of (2.1). There exists a constant \( C \) depending only on \( q, \mu, \lambda, \) and \( \Omega \) such that the following estimates hold:

- If \( F \in L^q(\Omega) \), then
  \[
  \|U\|_{W^{2,q}} \leq C\|F\|_{L^q}.
  \]
- If \( F \in W^{-1,q}(\Omega) \) (i.e., \( F = \text{div} \ f \) with \( f = (f_{ij})_{3 \times 3}, f_{ij} \in L^q(\Omega) \)), then
  \[
  \|U\|_{W^{-1,q}} \leq C\|F\|_{L^q}.
  \]
- If \( F \in W^{-1,q}(\Omega) \) (i.e., \( F = \text{div} \ f \) with \( f = (f_{ij})_{3 \times 3}, f_{ij} \in L^\infty(\Omega) \)), then
  \[
  [\nabla \Omega]_{BMO} \leq C\|F\|_{L^\infty}.
  \]

Here \( BMO(\Omega) \) stands for the John-Nirenberg’s space of bounded mean oscillation whose norm is defined by
\[
\|f\|_{BMO} = \|f\|_{L^2} + [f]_{BMO}
\]
with
\[
[f]_{BMO} = \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r}(x)|dy,
\]
and
\[
f_{\Omega_r}(x) = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y)dy,
\]
where \( \Omega_r(x) = B_r(x) \cap \Omega, B_r(x) \) is the ball with center \( x \) and radius \( r \), and \( d \) is the diameter of \( \Omega \). \( |\Omega_r(x)| \) denotes the Lebesgue measure of \( \Omega_r(x) \). Moreover, in what follows, we denote
\[
L U = -\mu \Delta U - (\mu + \lambda) \nabla \text{div} U.
\]

Owing to the uniqueness of solutions, we denote \( U = L^{-1} F \).

Next, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brézis-Wainger [1]. The reader can refer to [12, Section 2] for the proof.

Lemma 2.3. Assume \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \) and \( f \in L^2(s,t; W^{1,q}(\Omega)) \) with some \( q > 2 \) and \( 0 \leq s < t \leq \infty \), then there is a constant \( C > 0 \) depending only on \( q \) and \( \Omega \) such that
\[
\|f\|_{L^2(s,t; L^\infty)} \leq C \left( 1 + \|f\|_{L^2(s,t; H^1)}^2 \log(e + \|f\|_{L^2(s,t; W^{1,q})}) \right). \tag{2.2}
\]

Finally, the following variant of the Brézis-Wainger inequality plays a crucial role in obtaining the estimate of \( \|\nabla \rho, \nabla P, \nabla b\|_{L^s} \). For its proof, please refer to [19, Lemma 2.3].

Lemma 2.4. Assume \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \) and \( f \in W^{1,q}(\Omega) \) with some \( q > 2 \), then there is a constant \( C > 0 \) depending only on \( q \) and \( \Omega \) such that
\[
\|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{BMO} \log(e + \|f\|_{W^{1,q}}) \right). \tag{2.3}
\]

3. Proof of Theorem 1.1. Let \( (\rho, u, P, b) \) be a strong solution described in Theorem 1.1. Suppose that (1.19) were false, that is, there exists a constant \( M_0 > 0 \) such that
\[
\lim_{T \to T_\ast} \left( \|\rho\|_{L^\infty(0,T; L^\infty)} + \|b\|_{L^\infty(0,T; L^\infty)} + \|P\|_{L^\infty(0,T; L^\infty)} \right) \leq M_0 < \infty. \tag{3.1}
\]

First of all, we have the following standard energy estimate.
Lemma 3.1. Under the condition (3.1), it holds that for any $T \in [0,T^*)$,
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C, \tag{3.2}
\]
where and in what follows, $C, C_1$ stand for generic positive constants depending only on $\Omega, M_0, \lambda, \mu, T^*,$ and the initial data.

Proof. 1. It follows from (1.13) that
\[
P_t + u \cdot \nabla P + 2P \text{ div } u = F = 2 \mu |\nabla u|^2 + \lambda (\text{div } u)^2 \geq 0. \tag{3.3}
\]
Define particle path before blowup time
\[
\begin{cases}
\frac{d}{dt} X(x,t) = u(X(x,t),t), \\
X(x,0) = x.
\end{cases}
\]
Thus, along particle path, we obtain from (3.3) that
\[
\frac{d}{dt} P(X(x,t),t) = -2 \text{ div } u + F,
\]
which implies
\[
P(X(x,t),t) = \exp \left(-2 \int_0^t \text{ div } u ds \right) \left[P_0 + \int_0^t \exp \left(2 \int_0^s \text{ div } u ds \right) F ds \right] \geq 0. \tag{3.4}
\]

2. Multiplying (1.13)$_2$ and (1.13)$_3$ by $u$ and $b$ respectively, then adding the two resulting equations together, and integrating over $\Omega$, we obtain after integrating by parts that
\[
\frac{1}{2} \frac{d}{dt} \int (\rho|u|^2 + |b|^2) \, dx + \int \left[ \mu |\nabla u|^2 + (\lambda + \mu)(\text{div } u)^2 \right] \, dx \\
= \int P \text{ div } u \, dx \leq \frac{\mu}{2} \int |\nabla u|^2 \, dx + \frac{1}{2\mu} \int P^2 \, dx,
\]
which combined with (3.1) implies that
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \mu \|\nabla u\|_{L^2}^2 \leq C. \tag{3.5}
\]
So the desired (3.2) follows from (3.5) integrated with respect to $t$. This completes the proof of Lemma 3.1.

The following lemma gives the estimate on the spatial gradient of the velocity, which is crucial for deriving the higher order estimates of the solution.

Lemma 3.2. Under the condition (3.1), it holds that for any $T \in [0,T^*)$,
\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} u\|_{L^2}^2 dt \leq C. \tag{3.6}
\]

Proof. 1. Multiplying (1.13)$_2$ by $u_t$ and integrating the resulting equation over $\Omega$ give rise to
\[
\frac{1}{2} \frac{d}{dt} \int \left( \rho|\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 \right) \, dx + \int \rho |\dot{u}|^2 \, dx \\
= \int \rho \dot{u} \cdot \nabla u \, dx + \int \left[ \left( P + \frac{1}{2} |b|^2 \right) \right] \nabla u_{t} \, dx. \tag{3.7}
\]
It follows from Cauchy-Schwarz inequality and (3.1) that
\[
\left| \int \rho \dot{u} \cdot u \cdot \nabla u dx \right| \leq \frac{1}{4} \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + C \| u \|_{L^\infty}^2 \| \nabla u \|_{L^2}^2.
\] (3.8)

To bound the second term of the right hand side of (3.7), we denote
\[
\Lambda \triangleq \left( P + \frac{1}{2} |b|^2 \right) I_2 + b \otimes b,
\] (3.9)
and set
\[
v \triangleq L^{-1} \text{div} \Lambda, \quad w \triangleq u - v.
\] (3.10)

Then it is easy to check that \( w \) satisfies
\[
\rho \dot{u} + Lw = 0,
\] (3.11)
and we infer from Lemma 2.2 that for any \( p \geq 2, \)
\[
\| \nabla v \|_{L^p} \leq C \| P \|_{L^p} + C \| |b|^2 \|_{L^p} \leq C,
\] (3.12)
\[
\| \nabla^2 v \|_{L^p} \leq C \| \nabla P \|_{L^p} + C \| \nabla b \|_{L^p},
\] (3.13)
and
\[
\| \nabla^2 w \|_{L^p} \leq C \| \rho \dot{u} \|_{L^p}.
\] (3.14)

Thus, we have
\[
\int \left[ \left( P + \frac{1}{2} |b|^2 \right) I_2 - b \otimes b \right] : \nabla u_t dx
= - \frac{d}{dt} \int \Lambda : \nabla u dx + \int \Lambda_t : \nabla u dx
= - \frac{d}{dt} \int \Lambda : \nabla u dx + \int \Lambda_t : \nabla v dx + \int \Lambda_t : \nabla w dx.
\] (3.15)

Integration by parts leads to
\[
\int \Lambda_t : \nabla v dx = - \int \text{div} \Lambda_t \cdot v dx = - \int (L v)_t \cdot v dx = - \frac{1}{2} \frac{d}{dt} \int |L^{1/2} v|^2 dx.
\] (3.16)

Denote
\[
E \triangleq \theta + \frac{|u|^2}{2},
\]
then it follows from (1.13) that \( E \) satisfies
\[
\left( \rho E + \frac{1}{2} |b|^2 \right)_t + \text{div}(\rho u E + Pu + |b|^2 u)
= \frac{1}{2} \mu \Delta |u|^2 + \mu \text{div}(u \cdot \nabla u) + \lambda \text{div}(u \text{div} u) + \text{div}(u \cdot b) b.
\]

Hence, we get
\[
\int \Lambda_t : \nabla w dx
= - \int \left( \rho E + \frac{1}{2} |b|^2 \right)_t \text{div} w dx + \frac{1}{2} \int (\rho |u|^2)_t \text{div} w dx + \int (b \otimes b)_t : \nabla w dx
= - \int (\rho E u + P u + |b|^2 u - \mu \nabla u \cdot u - \mu u \cdot \nabla u - \lambda u \text{div} u - (u \cdot b) b) \cdot \nabla \text{div} w dx
- \frac{1}{2} \int \text{div}(\rho u) |u|^2 \text{div} w dx + \int \rho u \cdot u_t \text{div} w dx + \int (b \otimes b)_t : \nabla w dx
= - \int (2P u + |b|^2 u - \mu \nabla u \cdot u - \mu u \cdot \nabla u - \lambda u \text{div} u - (u \cdot b) b) \cdot \nabla \text{div} w dx
\[
+ \int \rho \dot{u} \cdot \nabla u \, dx + \int (b \otimes b)_t : \nabla w \, dx.
\]

Observing that

\[(b \otimes b)_t = b_t \otimes b + b \otimes b_t \]

which together with (3.17), (3.1), and (3.14) yields

\[
\left| \int \Lambda_t : \nabla w \, dx \right| \leq C \|u\|_{L^2}^2 \|\nabla^2 w\|_{L^2} + C \|u\|_{L^\infty} \|\nabla^2 w\|_{L^2}^2
\]

and

\[
\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} \|u\|_{L^\infty} \|\nabla w\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla w\|_{L^2}
\]

Putting (3.8) and (3.19) into (3.7), we get

\[
B'(t) + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \leq C \left(1 + \|u\|_{L^\infty}^2 \right) (1 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2),
\]

where

\[
B(t) \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda + \mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\mu}{2} \|\nabla v\|_{L^2}^2 + \frac{\lambda + \mu}{2} \|\nabla v\|_{L^2}^2
\]

+ \int \Lambda : \nabla u \, dx + \frac{1}{2} \int |\nabla^2 v|^2 \, dx.
\]

2. Let

\[
\Phi(t) \triangleq e + \sup_{0 \leq \tau \leq t} \left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right) + \int_0^t \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \, d\tau.
\]

Then we obtain from (3.20) and Gronwall’s inequality that for every 0 \leq s \leq T < T*,

\[
\Phi(T) \leq C \Phi(s) \exp \left\{ C \int_s^T \|u\|_{L^\infty}^2 \, d\tau \right\}.
\]

From Lemma 2.3, we get

\[
\|u\|_{L^2(s,T;L^\infty)}^2 \leq C \left[1 + \|u\|_{L^2(s,T;H^1)}^2 \log \left(e + \|u\|_{L^2(s,T;W^{1,1})}\right)\right]
\]

\[
\leq C_t \left[1 + \|\nabla u\|_{L^2(s,T;L^2)}^2 \log(C \Phi(T))\right],
\]

where one has used the Poincaré inequality, (3.2), and the following fact

\[
\|u\|_{W^{1,3}} \leq \|w\|_{W^{1,3}} + \|v\|_{W^{1,3}} \leq C \|w\|_{W^{2,2}} + C \|\nabla v\|_{L^3} \leq C \sqrt{\rho} \dot{u} + C.
\]
The combination (3.21) and (3.22) gives rise to
\[
\Phi(T) \leq C\Phi(s)(C\Phi(T))C_2\|\nabla u\|_{L^2(s,T;L^2)}^2.
\]
(3.23)
Recalling (3.2), one can choose \(s\) close enough to \(T^*\) such that
\[
\lim_{T \to T^*} -C_2\|\nabla u\|_{L^2(s,T;L^2)}^2 \leq \frac{1}{2}.
\]
Hence, for \(s < T < T^*\), we have
\[
\Phi(T) \leq C\Phi^2(s) < \infty.
\]
This completes the proof of Lemma 3.2.

**Lemma 3.3.** Under the condition (3.1), it holds that for any \(T \in [0,T^*)\),
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho}u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C.
\]
(3.24)

**Proof.** 1. Operating \(\partial_t + \text{div}(u\cdot)\to \) \(j\)-th component of (1.13) and multiplying the resulting equation by \(\dot{u}_j\), one gets by some calculations that
\[
\frac{1}{2} \frac{d}{dt} \int \rho|\dot{u}|^2 dx
\]
\[
= \mu \int \dot{u}_j(\partial_i\Delta u_j + \text{div}(u\Delta u_j))dx + (\lambda + \mu) \int \dot{u}_j(\partial_i\partial_j(\text{div} u) + \text{div}(u\partial_j(\text{div} u)))dx
\]
\[
- \int \dot{u}_j(\partial_j P_t + \text{div}(u\partial_j P))dx - \frac{1}{2} \int \dot{u}_j(\partial_i\partial_j|b|^2 + \text{div}(u\partial_j|b|^2))dx
\]
\[
+ \int \dot{u}_j(\partial_i(b \cdot \nabla b_j) + \text{div}(u(b\nabla b_j)))dx \triangleq \sum_{i=1}^5 J_i.
\]
(3.25)
Integration by parts leads to
\[
J_1 = -\mu \int (\partial_i\dot{u}_j\partial_i\partial_j u_{ij} + \Delta u_{ij} u \cdot \nabla \dot{u}_j)dx
\]
\[
= -\mu \int (\|\nabla \dot{u}\|^2 - \partial_i\dot{u}_j u_k \partial_k \partial_i u_{ij} - \partial_i\dot{u}_j \partial_i u_k \partial_k u_{ij} + \Delta u_{ij} u \cdot \nabla \dot{u}_j)dx
\]
\[
= -\mu \int (\|\nabla \dot{u}\|^2 + \partial_i\dot{u}_j \partial_k u_k \partial_i u_{ij} - \partial_i\dot{u}_j \partial_i u_k \partial_k u_{ij} - \partial_i u_j \partial_i u_k \partial_k u_{ij})dx
\]
\[
\leq -\frac{3\mu}{4} \|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4.
\]
(3.26)
Similarly, one has
\[
J_2 \leq -\frac{3(\lambda + \mu)}{4} \|\text{div} \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4.
\]
(3.27)
It follows from integration by parts, (1.13)\(_3\), (3.1), and (3.6) that
\[
J_3 = \int (\partial_j\dot{u}_j P_t + \partial_j Pu \cdot \nabla \dot{u}_j)dx
\]
\[
= \int \partial_j\dot{u}_j (2\mu|\mathcal{D}(u)|^2 + \lambda(\text{div} u)^2 - \text{div}(Pu) - P\text{div} u)dx - \int P\partial_j(u \cdot \nabla \dot{u}_j)dx
\]
\[
= \int \partial_j\dot{u}_j (2\mu|\mathcal{D}(u)|^2 + \lambda(\text{div} u)^2 - \text{div}(Pu) - P\text{div} u)dx - \int P\partial_j u \cdot \nabla \dot{u}_j dx
\]
Thus, putting (3.31) into (3.30), one has
\[ + \int \partial_j \hat{u}_j \text{div}(Pu) dx \]
\[ = \int \partial_j \hat{u}_j (2\mu |D(u)|^2 + \lambda (\text{div} u)^2 - P \text{div} u) \, dx - \int P \partial_j u \cdot \nabla \hat{u}_j dx \]
\[ \leq C \int |\nabla \hat{u}||\nabla u|^2 + 1) dx \]
\[ \leq \frac{\mu}{4} \|
abla \hat{u}\|_{L^2}^2 + C \|
abla u\|_{L^4}^4 + C. \] (3.28)

Now, we claim that
\[ J_4 + J_5 \leq \frac{\mu}{4} \|
abla \hat{u}\|_{L^2}^2 + C, \] (3.29)
whose proof is similar to (3.37) in [13, Lemma 3.5] and omit the details for simplicity. Inserting (3.26)–(3.29) into (3.25) yields
\[ \frac{d}{dt} \|
abla \hat{u}\|_{L^2}^2 + \mu \|
abla \hat{u}\|_{L^2}^2 \leq C \|
abla u\|_{L^4}^4 + C. \] (3.30)

2. We obtain from Hölder’s inequality, (3.6), (3.12), (3.14), and (3.1) that
\[ \|
abla u\|_{L^4}^4 \leq C \|
abla u\|_{L^2} \|
abla u\|_{L^6}^3 \]
\[ \leq C (\|
abla v\|_{L^6} + \|
abla w\|_{L^6})^3 \]
\[ \leq C (1 + \|
abla w\|_{L^2} + \|
abla^2 w\|_{L^2})^3 \]
\[ \leq C (1 + \|
abla \hat{u}\|_{L^2})^3 \]
\[ \leq C + C \|
abla \hat{u}\|_{L^2}^2. \] (3.31)

Thus, putting (3.31) into (3.30), one has
\[ \frac{d}{dt} \|
abla \hat{u}\|_{L^2}^2 + \mu \|
abla \hat{u}\|_{L^2}^2 \leq C \|
abla \hat{u}\|_{L^2}^2 \|
abla \hat{u}\|_{L^4}^2 + C, \] (3.32)
which together with Gronwall’s inequality and (3.6) implies the desired (3.24). The proof of Lemma 3.3 is completed. \(\square\)

The following lemma will treat the higher order derivatives of the solutions which are needed to guarantee the extension of local strong solution to be a global one.

**Lemma 3.4.** Under the condition (3.1), and let \( q > 2 \) be as in Theorem 1.1, then it holds that for any \( T \in [0, T^*) \),
\[ \sup_{0 \leq t \leq T} (\|\rho, P, b\|_{W^{1,q}} + \|\nabla u\|_{H^1}) \leq C. \] (3.33)

**Proof.** 1. For \( q > 2 \), it follows from the mass equation (1.13) that \( \nabla \rho \) satisfies
\[ \frac{d}{dt} \|\nabla \rho\|_{L^q} \]
\[ \leq C(q)(1 + \|\nabla u\|_{L^\infty}) \|
abla \rho\|_{L^q} + C(q) \|
abla^2 u\|_{L^q} \]
\[ \leq C(1 + \|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|
abla \rho\|_{L^q} + C(\|\nabla^2 w\|_{L^q} + \|\nabla^2 v\|_{L^q}) \]
\[ \leq C(1 + \|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|
abla \rho\|_{L^q} + C\|\nabla P\|_{L^q} + C\|\nabla \rho\|_{L^q} + C\|\nabla b\|_{L^q}, \] (3.34)
due to (3.13). From Lemma 2.4, (3.12), and (3.13), one gets
\[ \|\nabla v\|_{L^\infty} \]
\[ \leq C(1 + \|\nabla v\|_{BMO} \log(e + \|\nabla v\|_{W^{1,q}})) \]

Similarly, one has

\[ \text{This along with (3.24) and Gronwall's inequality gives that} \]

Moreover, we infer from (3.14), (3.1), and Sobolev's inequality that

Substituting (3.35)–(3.37) into (3.34), we derive that

Similarly, one has

and

2. Let

Then we derive from (3.38), (3.39), and (3.40) that

which, due to \( f(t) > 1 \), yields

This along with (3.24) and Gronwall’s inequality gives that

3. We infer from (3.13), (3.14), and (3.1) that

which combined with (3.43), Hölder’s inequality, (3.24), and (3.6) implies that

\[ \sup_{0 \leq t \leq T} \| \nabla^2 \mathbf{u} \|_{L^2} \leq C. \]
Thus the desired (3.33) follows from (3.43), (3.44), (3.6), and (3.1). The proof of Lemma 3.4 is finished.

With Lemmas 3.1–3.4 at hand, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We argue by contradiction. Suppose that (1.19) were false, that is, (3.1) holds. Note that the general constant $C$ in Lemmas 3.1–3.4 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.4 are uniformly bounded for any $t < T^*$. Hence, the function

$$(\rho, u, P, b)(x, T^*) \triangleq \lim_{t \to T^*} (\rho, u, P, b)(x, t)$$

satisfy the initial condition (1.17) at $t = T^*$.

Furthermore, standard arguments yield that $\rho \dot{u} \in C([0, T]; L^2)$, which implies $\rho \dot{u}(x, T^*) = \lim_{t \to T^*} \rho \dot{u} \in L^2$.

Hence,

$$-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P - b \cdot \nabla b + \frac{1}{2} \nabla |b|^2|_{t=T^*} = \sqrt{\rho(x, T^*)} g(x)$$

with

$$g(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(\rho \dot{u})(x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\}, \\ 0, & \text{for } x \in \{x|\rho(x, T^*) = 0\}, \end{cases}$$

satisfying $g \in L^2$ due to (3.33). Therefore, one can take $(\rho, u, P, b)(x, T^*)$ as the initial data and extend the local strong solution beyond $T^*$. This contradicts the assumption on $T^*$. Thus we finish the proof of Theorem 1.1.

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