Preparing quasienergy states on demand: a parametric oscillator

Yaxing Zhang\textsuperscript{1} and M. I. Dykman\textsuperscript{2}

\textsuperscript{1}Department of Physics, Yale University, New Haven, CT 06511, USA
\textsuperscript{2}Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, USA

(Dated: November 7, 2018)

We study a parametrically driven nonlinear oscillator where the driving frequency is close to twice the oscillator eigenfrequency. By judiciously choosing the frequency detuning and adiabatically increasing the driving strength, one can prepare any even quasienergy state starting from the oscillator ground state. This is a consequence of a specific behavior of the quasienergy levels related to the oscillator symmetry. We find the Wigner distribution of the prepared states. We also discuss the Landau-Zener transitions in the Floquet dynamics and show that one can prepare on demand a superposition of quasienergy states via controlled nonadiabaticity. We find the spectrum of the transient radiation emitted by the oscillator after it has been prepared in a quasienergy state.

I. INTRODUCTION

Periodically driven quantum systems are described by quasienergy (Floquet) states, which are a time-domain analog of Bloch states in spatially periodic systems [1-4]. The new physics associated with quasienergy states has been attracting much interest recently. Examples include topological Floquet states, artificial gauge fields, and new many-body phases [5-15].

Preparation of Floquet states is often discussed in the adiabatic framework assuming that the periodic field is slowly turned on, cf. [16-19] and references therein. The analysis for many-body systems is complicated by the effect of heating, and much progress has been made by studying systems that display many-body localization, as it may alleviate the heating. Recently adiabatic state preparation was considered also for a parametrically driven nonlinear oscillator [20, 21]. In contrast to many-body systems, the energy spectrum here is discrete, which simplifies the problem. However, a potential complication, and also potentially new and interesting features stem from the fact that the quasienergy levels for weak driving can display degeneracy or what we call reduced-band (RB) degeneracy, see below. The analysis [20, 21] referred to the parameter range where there was no degeneracy.

The goal of this paper is to study preparation of quasienergy states in a small quantum system in the case where the quasienergy states can display degeneracy or RB degeneracy for weak driving. This case is of particular physical importance. In optics terms, it corresponds to either a multiphoton or a parametric resonance, where the distance between the energy levels of the system is close to either a multiple or a fraction of the radiation frequency multiplied by $\hbar$. Multiphoton resonance leads to Rabi oscillations described in Ref. [22] for a nonlinear oscillator using perturbation theory. In terms of the Floquet states, when the driving frequency is close to the oscillator eigenfrequency, the oscillator can display simultaneous multiple anticrossing of the quasienergy levels [23].

The model of a driven quantum oscillator is interesting as it describes a broad range of physical systems, from molecular vibrations [22] to the modes of nonlinear optical and microwave cavities to Josephson junctions [24]. Here we study the features of the Floquet dynamics that emerge when an oscillator is driven parametrically and the drive frequency $\omega_F$ is close to twice the oscillator eigenfrequency. The oscillator then can display resonance, if the distance between its energy levels $E_n - E_m$ is a multiple of $\hbar \omega_F/2$.

To explain the relation to the Floquet states, we note that quasienergies are defined modulo $\hbar \omega_F$. Therefore the resonance condition means that $E_n - E_m$ is either zero or $\hbar \omega_F/2$, where in the limit of zero driving the quasienergy is $E_n = E_n \mod (\hbar \omega_F)$. The case $E_n = E_m$ corresponds to the standard multiphoton resonance and the degeneracy of the quasienergy levels, whereas the case $|E_n - E_m| = \hbar \omega_F/2$ corresponds to a parametric resonance of an oscillator, $|E_n - E_m| = (k + 1/2)\hbar \omega_F$ with integer $k$. The latter case is the RB degeneracy, as the quasienergies would coincide if they were defined modulo $\hbar \omega_F/2$.

In what follows we show that, by slowly turning on resonant parametric drive, it is possible to prepare on demand various quasienergy states starting from the ground state of the oscillator ($n = 0$). We believe this is a general feature of Floquet systems. We also study preparation of a superposition of quasienergy states starting from the ground state. This can be accomplished using non-adiabatic transitions for the driving frequency $\omega_F$ tuned close to multiphoton resonance, so that $E_m - E_0$ is small for the targeted $m$. Our results for the driving protocol refer, but are not limited to a linear in time increase of the amplitude of the driving, a scenario that is easy to implement in the experiment and that brings the system to the regime of stationary driving in a finite time. The nonadiabatic dynamics in this case differs from the conventional Landau-Zener dynamics.

The paper is organized as follows. In Sec. II, we present the model of a parametric nonlinear oscillator and discuss its quasienergy spectrum. We show the evolution of the spectrum with the varying driving frequency in the limit of zero drive amplitude and the occurrence of the degeneracy and RB degeneracy of the quasienergy
levels as the system goes through multiphoton or subharmonic resonance. In Sec. III, we present the Wigner distribution for the quasienergy states prepared from the oscillator ground state by slowly ramping up the amplitude of the driving in the absence of degeneracy. We demonstrate the possibility to prepare a Floquet state “on demand” and the rich structure of its Wigner distribution. The only constraint is that the resulting Floquet states are “even” with respect to inversion in phase space. In Sec. IV, we consider preparation of a superposition of two quasienergy states via a non-adiabatic transition when the system is close to degeneracy for weak field. In Sec. V, we briefly discuss the adiabaticity in the presence of dissipation. In Sec. VI we study fluorescence of the oscillator driven into a Floquet state, and in particular the characteristic transient spectrum of the fluorescence. Sec. VI contains concluding remarks.

II. RWA HAMILTONIAN AND QUASIENERGY SPECTRUM

The Hamiltonian of a weakly nonlinear parametric oscillator with coordinate $q$ and momentum $p$ reads

$$H(t) = \frac{p^2}{2} + \frac{1}{2} q^2 [\omega_0^2 + F \cos(\omega_F t)] + \frac{\gamma}{4} q^4. \tag{1}$$

We assume that the driving amplitude $F$ and the nonlinearity are comparatively small, $F, \gamma (q^2) \ll \omega_0^2$, and the driving frequency $\omega_F$ is close to resonance, $|\omega_F - 2\omega_0| \ll \omega_0$; without loss of generality, we consider $F, \gamma > 0$. A quantum parametric oscillator described by Eq. (1) has been realized in various platforms, from optical and microwave cavities to nanomechanical systems, cf. [24–27].

Floquet theorem says that, for a periodically modulated quantum system, there exists a complete set of solutions to the Schrödinger equation called Floquet states, which are eigenfunctions of the operator $T_{rf}$ of time translation by the modulation period $t_F$,

$$\psi_F(t) = e^{-i\omega_F t/\hbar} u_F(t), \quad u_F(t + t_F) = u_F(t). \tag{2}$$

Parameter $\varepsilon$ is called quasienergy or Floquet eigenvalue. For the considered parametric oscillator in Eq. (1), $t_F = 2\pi/\omega_F$.

A standard procedure to find quasienergy states and quasienergies is to plug the solution Eq. (2) into the Schrödinger equation, and then solve the resulting equation for $u_F(t)$ using Fourier series expansion; see Appendix. Another much simpler way to find quasienergy is to go to the rotating frame at frequency $\omega_F/2$ by applying the standard unitary transformation $U(t) = \exp[-i\omega_F t/\hbar]$, where $a$ and $a^\dagger$ are the oscillator ladder operators. In the rotating wave approximation (RWA) we disregard fast oscillating terms in the transformed Hamiltonian $U^\dagger H U = -\hbar \delta \omega_F \hat{n} + \hbar V/2(\hat{n}^2 + \hat{n}) + \hbar \bar{F}/2(\hat{a}^2 + \hat{a}^2\dagger)$, which gives the RWA Hamiltonian

$$H_{\text{RWA}} = -\hbar \delta \omega_F \hat{n} + \frac{\hbar V}{2}(\hat{n}^2 + \hat{n}) + \frac{\hbar \bar{F}}{2}(\hat{a}^2 + \hat{a}^2\dagger) \tag{3}$$

where $\hat{n} = a^\dagger a$, $\delta \omega_F = \omega_F/2 - \omega_0$ is the detuning frequency, $\bar{F} = F/4\omega_0$, and $V = 3\gamma \hbar/4\omega_0^2$.

The Hamiltonian $H_{\text{RWA}}$ commutes with occupation number parity operator $P = \exp(-i\alpha a^\dagger a)\gamma \hbar/2$. Therefore, an eigenstate $\phi_E$ of $H_{\text{RWA}}$ has definite parity $P_E = \pm 1$; here $E$ is an eigenvalue of $H_{\text{RWA}}$, which can be called the RWA energy. As a consequence, the corresponding time dependent state in the lab frame $\Phi_E(t) \equiv \exp(-iEt/\hbar)U(t)\phi_E$ is a Floquet state of Eq. (2). The quasienergy $\varepsilon$ and the periodic factor in the Floquet wave functions $u_F$ are immediately expressed in terms of the RWA energy $E$ and the eigenfunction $\phi_E$,

$$\varepsilon = [E + (1 - P_E)\hbar \omega_F/4] \mod (\hbar \omega_F), \quad u_F(t) = \exp[i(1 - P_E)\omega_F t/4]U(t)\phi_E,$$

where $P_E = \{ -1, \quad \phi_E \text{ is odd} \}$

and

$\{ 1, \quad \phi_E \text{ is even} \} \tag{4}$

We note that if $H_{\text{RWA}}$ has degenerate states $\phi_E$ with the opposite parity, which is possible [29], the corresponding Floquet states have quasienergies differing by $\hbar \omega_F/2$. A state, which is a superposition of these states, is a period-two state; the expectation values of dynamical variables of the oscillator in this state oscillate with period $2t_F$.

![Fig. 1. The cross-section of the RWA Hamiltonian function $H_{\text{RWA}}(Q, P)$ given by Eq. (5) by the plane $P = 0$ and the RWA energy levels.](image)

The understanding of the spectrum of $H_{\text{RWA}}$ can be gained by looking at the Hamiltonian function $H_{\text{RWA}}$ in the phase space of the oscillator in the rotating frame, i.e., by writing $H_{\text{RWA}}$ in terms of the scaled quadratures $P$ and $Q$ defined as $Q = i(a - a^\dagger)\sqrt{\lambda/2}, P = (a^\dagger + a)\sqrt{\lambda/2}$. Here, $\lambda = V/2\bar{F}$ is the dimensionless Planck constant. In these variables

$$H_{\text{RWA}}(Q, P) = (F^2/6\gamma)g(Q, P), \quad g(Q, P) = \frac{1}{4}(P^2 + Q^2)^2 - \frac{1}{2}\mu(P^2 + Q^2) + \frac{1}{2}(P^2 - Q^2), \tag{5}$$

where $\mu = 2\omega_F(\delta \omega_F)/F$ [25]. The eigenstates of the Hamiltonian $H_{\text{RWA}}$ can be written in the $Q$-basis, $\phi_E \equiv \phi_E(Q)$. The parity operator $\hat{P}$ is then the inversion operator, $\hat{P}\phi_E(Q) = \phi_E(-Q)$.

For $\mu + 1 > 0$, function $H_{\text{RWA}}(Q, P)$ has two minima located at $P = 0, Q = \pm \sqrt{\mu + 1}$. Function $H_{\text{RWA}}(Q, P) = \begin{cases} H_{\text{RWA}}(Q, P) = (F^2/6\gamma)g(Q, P), & g(Q, P) = \frac{1}{4}(P^2 + Q^2)^2 - \frac{1}{2}\mu(P^2 + Q^2) + \frac{1}{2}(P^2 - Q^2), \end{cases}$

where $\mu = 2\omega_F(\delta \omega_F)/F$ [25]. The eigenstates of the Hamiltonian $H_{\text{RWA}}$ can be written in the $Q$-basis, $\phi_E \equiv \phi_E(Q)$. The parity operator $\hat{P}$ is then the inversion operator, $\hat{P}\phi_E(Q) = \phi_E(-Q)$.

For $\mu + 1 > 0$, function $H_{\text{RWA}}(Q, P)$ has two minima located at $P = 0, Q = \pm \sqrt{\mu + 1}$. Function $H_{\text{RWA}}(Q, P) = \begin{cases} H_{\text{RWA}}(Q, P) = (F^2/6\gamma)g(Q, P), & g(Q, P) = \frac{1}{4}(P^2 + Q^2)^2 - \frac{1}{2}\mu(P^2 + Q^2) + \frac{1}{2}(P^2 - Q^2), \end{cases}$
that, for certain values of $\delta \omega$ sition of Fock states $|\psi_n\rangle$ in $Q$
In this limit, $\tilde{H}_{\text{RWA}}$ is diagonalized in the basis of the oscillator Fock states. The order of the RWA eigenstates of symmetric superpositions of intra-well states.

Further insight into the structure of the spectrum of $H_{\text{RWA}}$ can be gained by considering the limit $F \to 0$. In this limit, $H_{\text{RWA}}$ is diagonalized in the basis of the oscillator Fock states. The order of the RWA eigenstates in the rotating frame can be changed compared to the order of the Fock states in the laboratory frame. From Eq. (3), for $F = 0$ the eigenvalues $E_n$ of $H_{\text{RWA}}$ are

$$E_n = E_n^0 - \frac{1}{2} \hbar \nu \left( n + \frac{1}{2} \right)^2. \quad (6)$$

From Eq. (6), $E_n$ is a parabolic function of $n$; see Fig. 2. For $\delta \omega_F / V < 1/2$, $E_n$ monotonically increases with the increasing $n$ (black dots in Fig. 2). As the ratio $\delta \omega_F / V$ increases, $E_n$, as a function of a continuously varying $n$ bends over and has a minimum at some positive $n$. When $\delta \omega_F / V > 1$, the state with the lowest RWA energy is no longer the Fock state $|0\rangle$. For instance, for $\delta \omega_F / V = 1.8$ (blue dots in Fig. 2), this state is $|1\rangle$.

The reordering of the quasienergy states described by Eq. (6) is essential for preparing quasienergy states on demand. Indeed, if the oscillator is initially in the ground state, then by tuning the driving frequency and increasing the driving strength, we make this state an arbitrary even in $Q$ quasienergy state, i.e., an arbitrary superposition of Fock states $|m\rangle$ with even $m$. We also note that, for certain values of $\delta \omega_F / V$, there can be degenerate RWA levels (the green and brown dots in Fig. 2). We will discuss such degeneracy later in details.

The driving mixes Fock states with the same parity. The evolution of the RWA spectrum with the increasing $F$ is shown in Fig. 3 for different values of the detuning $\delta \omega_F / V$. A common trend is that RWA energy levels of the same parity repel each other, whereas neighboring levels of opposite parity attract each other and form pairs for large $F / V$. As mentioned above, such pairs for large $F / V$ are even and odd superposition of “intra-well” states of $H_{\text{RWA}}$. The distance between the states within the pairs is determined by interwell tunneling [29].

We emphasize that the RWA levels do not cross each other as $F$ changes. Therefore any gaps that are present at $F \to 0$ will remain open for any finite $F$. For instance, Figs. 3a and b refer to the cases where the Fock state $|0\rangle$ is the first and the third lowest RWA eigenstate at $F \to 0$, respectively. As $F$ increases, it remains the first and the third lowest RWA eigenstate. Such non-crossing feature will be important for the preparation of quasienergy states by adiabatically turning on the driving.

A remarkable feature of the RWA spectrum is that, when the ratio $\delta \omega_F / V$ is a positive integer, there is a set of simultaneously doubly-degenerate levels of opposite parity regardless of the value of $F$. For $F \to 0$, this can be readily seen from Eq. (6) (cf. 23 where a similar feature was found in the case of the driving at frequency close to $\omega_0$). When $\delta \omega_F / V = k, k = 1, 2, 3...$, the minimum of $E_n$ as a continuous function of $n$ is reached at half odd integer $n = k - 1/2$. Because of the symmetry of the parabola with respect to the minimum, the levels separated by $\Delta n = 2m + 1$ are degenerate, that is,
$E_{k+m} = E_{k-(m+1)}$, for $m = 0, 1, ..., k-1$. The green curve in Fig. 2 refers to the case $k = 2$.

The degeneracy of the RWA energy levels persists for nonzero $F$, as can be seen in Fig. 3. At weak driving, this follows from the perturbation theory. To the second order in $F$, the correction $\delta E_n$ to $E_n$ is

$$\delta E_n = -\hbar V \frac{\hat{F}^2}{4V^2} \left[ 2E_n/\hbar V - (\delta \omega F/V)^2 - 3/4 \right] \frac{2E_n/\hbar V - 1}{2E_n/\hbar V} \tag{7}$$

The dependence of $\delta E_n$ on the level number $n$ is exactly the same as that of $E_n$, cf. Eq. (8). Therefore, if $E_n = E_{n'}$, then $\delta E_n = \delta E_{n'}$. Note that the perturbation theory still applies even if there are degenerate levels of opposite parity since there is no coupling between them. At strong driving, such degeneracy corresponds to the vanishing of tunnel splitting found in Ref. [29].

For the special case $\delta \omega F/V = 1$, $H_{\text{RWA}}$ can be factored [21],

$$H_{\text{RWA}} = \frac{\hbar V}{2} \left( a^{+2} + \frac{\hat{F}}{V} \right) \left( a^2 + \frac{\hat{F}}{V} \right) - \frac{\hbar \hat{F}^2}{2V}. \tag{8}$$

In this case the coherent states $| \pm \alpha \rangle$, $\alpha = \sqrt{-\hat{F}/V}$, are exact degenerate eigenstates of $H_{\text{RWA}}$ for arbitrary driving strength. However, no such eigenstates are known for other values of $\delta \omega F/V$.

If the ratio $\delta \omega F/V$ is a half-integer, $\delta \omega F/V = (2k + 1)/2, k = 1, 2, 3, ...$, the minimum of function $E_n$ for $\hat{F} \to 0$ is reached at integer $n = k$. Again, due to the symmetry of the parabola, levels $E_{k+m}$ are degenerate for $m = 1, 2, ..., k$. For instance, the brown curve in Fig. 2 refers to the case $k = 2$. The degeneracy of the levels of the same parity occurs when the driving frequency equals to one of the transition frequencies of the undriven oscillator. This can be seen by rewriting $E_n$ as $E_n = -n\hbar \omega_0/2 + E_n$, where $E_n = n\hbar \omega_0 + \hbar V n(n+1)/2$ is the $n$th energy level of the oscillator in the absence of driving. Clearly, the degeneracy condition $E_{k+m} = E_{k-m}$ is equivalent to $E_{k+m} - E_{k-m} = m\omega F$, which is the $m$-photon resonance condition for transition from $E_{k-m}$ to $E_{k+m}$. The degeneracy is lifted at finite $F$ due to level repulsion, as shown in Fig. 2.

### III. ADIABATIC PREPARATION OF QUASIENERGY STATES AND THE WIGNER DISTRIBUTION

The observation that the quasienergy levels of the same parity do not approach each other with the increasing field $F$ is critical for state preparation. It allows one to prepare a quasienergy state by adiabatically turning on the field, provided the states are non-degenerate for $F \to 0$.

We consider ramping up the driving amplitude $\hat{F}$ linearly with speed $s$ starting at $t = 0$, $\hat{F}(t) = s_0 t$. If $s_0$ is small compared to $\omega_0$, the time evolution of the oscillator wave function $\phi(t)$ can be described in the RWA,

$$i\hbar \partial_t \phi(t) = H_{\text{RWA}}(t) \phi(t). \tag{8}$$

We will solve this equation assuming that initially, for zero driving, the system is in the ground state of the oscillator, $\phi(Q, t = 0) = |0\rangle$.

The results of the numerical solution of Eq. (8) are illustrated in Fig. 4. The values of $\delta \omega F/V$ were chosen in such a way that, in one case ($\delta \omega F = 0$), the state remains close to the eigenstate of $H_{\text{RWA}}$ with the lowest eigenvalue $E_n$, whereas in the other case ($\delta \omega F/V = 1.8$) it is close to the third lowest-$E_n$ state, cf. Fig. 3(b). The quality of the adiabatic approximation for the chosen parameters can be characterized by the inner product of the state $\phi(Q)$ at the end of ramp-up and the corresponding stationary RWA eigenstate $\phi_F(Q)$ calculated.
for $\tilde{F} = \tilde{F}_{\text{final}}$. This inner product is 0.997 and 0.98 for the cases shown in Fig. 4a and Fig. 4b, respectively, which shows that the adiabatic approximation is very good.

The final value of the field amplitude $\tilde{F}_{\text{final}}$ in Fig. 4 refers to the case where the Hamiltonian function $H_{\text{RWA}}(Q,P)$, Eq. (5), has a pronounced double-well structure, cf. Fig. 1. For $\delta \omega_F = 0$, the state $\phi(Q)$ is well described by a symmetric superposition of the lowest intra-well states in Fig. 1, $\phi(Q) = (\phi_L + \phi_R)/\sqrt{2}$ where $\phi_L$ and $\phi_R$ refer to the left and right well, respectively. Near their maxima, functions $\phi_{L,R}$ are given by squeezed coherent states with equal amplitude and opposite phases, $\phi_{L,R} \propto \exp[-(Q \pm Q_0)^2/2\lambda \eta]$ where $Q_0 = \sqrt{\mu + 1}$ is the position of the right well and $\eta = 1/\sqrt{\mu + 1}$ characterizes the state squeezing, see Appendix B. The adiabatic preparation of such “cat” state has been discussed in Refs. (20, 21).

In contrast, for the case in Fig. 4b, the driving brings the system to an excited state of $H_{\text{RWA}}$. The state $\phi(Q)$ for $t = \tilde{F}_{\text{final}}/s_0$ is no longer a superposition of the lowest intra-well states but, for the chosen $\delta \omega_F/V$, the superposition of the second lowest intra-well states, $\phi(Q) = (\phi'_L + \phi'_R)/\sqrt{2}$. Near their maxima, functions $\phi'_{L,R}$ are well described by a displaced and squeezed Fock state $|1\rangle$: $\phi'_{L,R} \propto (Q \pm Q_0) \exp[-(Q \pm Q_0)^2/2\lambda \eta]$.

IV. PREPARING A SUPERPOSITION OF QUASIENERGY STATES NONADIABATICALLY

As the driving amplitude $F$ is ramped up, the nonadiabaticity can mix quasienergy states of the same parity. The mixing is particularly strong if the quasienergy gap that separates the states is small. As shown in Sec. II, this gap is controlled by the driving frequency. In this section, we consider a situation where two nearest quasienergy states of the same parity have close quasienergies for $F \to 0$, whereas the quasienergies of other states are significantly different, so that mixing with these states can be disregarded for slowly varying $F(t)$. We show that, by ramping up the driving amplitude linearly in time, we can prepare a desired coherent superposition of the chosen two quasienergy states.

We assume that the states with close quasienergies for $F \to 0$ are $|n - 1\rangle$ and $|n + 1\rangle$, which means that $\delta \omega_F/V \approx n + 1/2$. As the drive is ramped up, these states are mixed with each other. Concurrently, they are mixed with other states of the same parity. However, this mixing is nonresonant and therefore is weaker.

The picture of the state evolution is as follows. The resonant mixing leads to a redistribution of the initial population between the resonating states and to a separation of their quasienergies already for a comparatively weak field, see Fig 5. The increase of the field afterwards does not change the state populations, even though it modifies the states by increasingly strongly admixing them to nonresonant states.

To describe the initial stage of the evolution we project the Hamiltonian $H_{\text{RWA}}$ onto the subspace formed by the functions $|n - 1\rangle$ and $|n + 1\rangle$, subtract the mean RWA energy $(E_{n+1} + E_{n-1})/2$, and disregard the coupling to other states. Then the Hamiltonian becomes

$$H_{\text{RWA}}(t) = \hbar \left( \frac{\Delta}{\nu(t)} - \Delta \right), \quad (9)$$

where $\Delta = (E_{n-1} - E_{n+1})/2\hbar$, $\nu(t) = \sqrt{n(n + 1)} \tilde{F}(t)$. For a field that linearly increases in time $\nu(t) = st$.

It is convenient to re-write the Hamiltonian (9) in the conventional form used in the analysis of the Landau-Zener tunneling. Making a unitary transformation $U_{\sigma} = (1/\sqrt{2})(\sigma_z + \sigma_x)$ ($\sigma_z, \sigma_x$ are Pauli matrices), we obtain

$$U_{\sigma}^\dagger H_{\text{RWA}} U_{\sigma} = H_{\text{LZ}} = \hbar \left( \frac{\nu(t)}{\Delta} \Delta - \nu(t) \right). \quad (10)$$

Note that the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the Hamiltonian (10) are, respectively, the wave functions $(|n - 1\rangle + |n + 1\rangle)/\sqrt{2}$ and $(|n - 1\rangle - |n + 1\rangle)/\sqrt{2}$.

The only difference of the evolution of the states we consider here from the standard Landau-Zener scenario is that the initial condition for the Schrödinger equation $i\hbar \phi(t) = H_{\text{LZ}} \phi(t)$ is set for $t = 0$ and the problem is considered on the semi-axis $t \geq 0$. It is convenient to seek the wave function as $\phi(t) = (1/\sqrt{2}) \sum_{n-\pm} C_n(t)|n - 1\rangle + \alpha|n + 1\rangle$. We will be interested in the solution that corresponds to the initial condition where the smaller-$n$ state is occupied while the larger-$n$ state is empty, $C_+(0) = C_-(0) = 1/\sqrt{2}$. As in the Landau-Zener problem, the solution to the Schrödinger equation can be expressed in terms of the parabolic cylinder functions, see Appendix C.

In Fig. 6 we show the result for the coefficient $C_+(t)$, which is equal to the projection $\langle \phi_{n+1}^{\text{ad}}(t)|\phi(t)\rangle$ of the wave function $\phi(t)$ on the upper branch (the higher energy branch in Fig. 5) of the adiabatic solutions $\phi_{n+1}^{\text{ad}}(t)$ of the Schrödinger equation, $H_{\text{LZ}} \phi_{n+1}^{\text{ad}}(t) = \pm [\nu^2(t) + \Delta^2]^{1/2} \phi_{n+1}^{\text{ad}}(t)$. The result is in full agreement with the numerical solution of the Schrödinger equation.

Of primary interest is the long time behavior of $C_+(t)$. It can be obtained from the asymptotic expansion of the
parabolic cylinder functions,
\[
C_\uparrow(t) \approx \alpha_\uparrow e^{i\theta(t)} + \beta_\uparrow e^{-i\theta(t)}(2st)^{-1/2},
\]
\[
C_\downarrow(t) \approx \alpha_\downarrow e^{-i\theta(t)} + \beta_\downarrow e^{i\theta(t)}(2st)^{-1/2},
\]
(11)
The expressions for \(\alpha_{\uparrow,\downarrow}, \beta_{\uparrow,\downarrow}\) in terms of \(\Delta^2/s\) follow from the general solution of the Schrödinger equation in Appendix C. Function \(\theta(t)\) is the dynamical phase of the Schrödinger equation
\[
\theta(t) = \frac{st^2}{2} + \frac{\Delta^2}{2s} \log \left(\frac{2st}{\Delta}\right) + \frac{\Delta^2}{4s},
\]
(12)
The coefficients \(C_{\uparrow,\downarrow}(t)\) approach their asymptotic values \(\propto \alpha_{\uparrow,\downarrow}\) as \(1/t\) and oscillate as \(\exp[\pm i\theta(t)]\). We note that \(C_\uparrow = C_\uparrow(t)\) and \(C_\downarrow = C_\downarrow(t)\) for \(t \rightarrow \infty\).

Figure 7 shows the asymptotic value \(|C_\uparrow(\infty)|^2 = |\alpha_\uparrow|^2\) of the quasienergy states. It reduces the fidelity of the state preparation. Here we consider the constraint on the dissipation in the case of adiabatic state preparation by ramping up the driving field. To achieve high fidelity, one needs to increase the field faster than the relaxation rate, but slower than the reciprocal of the relevant RWA energies divided by \(\hbar\). For a state \(\phi_E\), this means that the decay rate of this state \(\Gamma_E\) should be small compared to \(\Delta_E^{-1}\), where \(\hbar\Delta_E\) is the instantaneous energy difference with the nearest state of the same parity. The parity constraint here is the consequence of the fact that the field mixes only the same-parity states.

V. ADIABATICITY IN THE PRESENCE OF DISSIPATION

Coupling to the environment leads to decoherence of the quasienergy states. It reduces the fidelity of the state preparation. Here we consider the constraint on the dissipation in the case of adiabatic state preparation by ramping up the driving field. To achieve high fidelity, one needs to increase the field faster than the relaxation rate, but slower than the reciprocal of the relevant RWA energies divided by \(\hbar\). For a state \(\phi_E\), this means that the decay rate of this state \(\Gamma_E\) should be small compared to \(\Delta_E^{-1}\), where \(\hbar\Delta_E\) is the instantaneous energy difference with the nearest state of the same parity. The parity constraint here is the consequence of the fact that the field mixes only the same-parity states.
The RWA level spacing $\hbar \Delta E$ can be estimated where the driving is weak, $F \ll V$, or strong, $F \gg V$. For weak driving, the RWA eigenstates are close to the Fock states. From the results of Sec. [1] $\Delta E \sim V$ and depends on the ratio $\delta \omega_F / V$, cf. Fig. [2] At strong driving, $\hbar \Delta E$ is given by the spacing of the intrawell energy levels of the Hamiltonian $H_{\text{RWA}}(Q,P)$, see Fig. [4] It is determined by the frequency $\omega_{\text{min}}$ of oscillations about the minima of $H_{\text{RWA}}(Q,P)$, which gives $\Delta E \approx 2[(\delta \omega_F + F)F]^{1/2}$, see Appendix [3].

We illustrate the effect of dissipation using the well-known model [31] where the kinetics in the rotating frame is described by the Markov master equation for the density matrix

$$\dot{\rho} = i\hbar^{-1}[\rho, H_{\text{RWA}}] - \hat{\Gamma} \rho,$$

$$\hat{\Gamma} \rho = \Gamma(\hat{a} \rho \hat{a}^\dagger - 2\hat{a}^\dagger \hat{a} \rho + \rho \hat{a}^\dagger \hat{a}).$$

(14)

Here, $\Gamma$ is the oscillator relaxation rate and we assume that the temperature of the environment is sufficiently low, $k_B T \ll \hbar \omega_0$.

The decay rate $\Gamma_E$ of an RWA eigenstate $\phi_E$ can be estimated as the decay rate of the diagonal matrix element of the density matrix $\langle \phi_E | \rho | \phi_E \rangle$. Assuming that the system is in state $\phi_E$, i.e., $\rho = |\phi_E \rangle \langle \phi_E |$, and taking into account that the matrix elements of the ladder operators on the states of the same parity are zero, we find from Eq. (14) $\Gamma_E = 2\Gamma \langle \phi_E | a^\dagger a | \phi_E \rangle$. At weak driving, $\Gamma_E \sim \Gamma$. At strong driving $\Gamma_E$ is determined by the rate of transitions between the intrawell states of $H_{\text{RWA}}$ [32], $\Gamma_E \sim \Gamma F / V$.

From the above estimates, the adiabaticity condition $\Gamma_E \ll \Delta E$ requires that $\Gamma \ll V |\delta \omega_F|$ for weak driving and $\Gamma \ll V$ for strong driving. Fig. 8 illustrates the evolution of $\Delta E$ and $\Gamma_E$ of an RWA eigenstate $\phi_E$ with the varying driving amplitude $F$. For the case shown in the figure, the state $\phi_E$ has the lowest RWA eigenenergy. At large $F/V$, both $\Delta E$ and $\Gamma_E$ increase linearly with $F$ as we expect from the analysis above. The slope of $\Gamma_E$ as a function of $F$ increases as $\Gamma / V$ increases. It coincides with the slope of $\Delta E$ for $F / V \approx 2$ as shown by the green curve. For the condition $\Gamma_E \ll \Delta E$ to be satisfied for any $F$, one needs to have $\Gamma / V \ll 2$. For $\Gamma / V \gtrsim 2$, in the considered case $\Gamma_E$ and $\Delta E$ as a function of $F$ can cross each other.

VI. TRANSIENT RADIATION FROM QUASIENERGY STATES

Decay of a parametrically driven oscillator is accompanied by emitting excitations into the surrounding medium. The most familiar picture is decay of optical/microwave cavity modes into propagating electromagnetic waves. Detection of the radiation from the cavity provides a way of characterizing the cavity modes. Radiation from the modes in a non-steady state, such as a quasienergy state, is transient. After a time of the order of the mode relaxation time, the system relaxes to a steady state, the radiation becomes steady and does not depend on the quasienergy state the system had been staying in. To identify a quasienergy state from the radiation, one needs to collect the transient radiation.

![Fig. 8. Solid lines from top down: the instantaneous decay rate $\Gamma_E \approx 2\Gamma \langle \phi_E | a^\dagger a | \phi_E \rangle$ of the RWA eigenstate $\phi_E$ for $\Gamma / V = 2$ (green), 1 (blue) and 0.5 (red). Dashed line: the instantaneous level spacing $\hbar \Delta E$ between the states with the same parity. The scaled detuning is $\delta \omega_F / V = 0$. The state $\phi_E$ is chosen to be the lowest RWA state, $\phi_E = |0\rangle$ for $F = 0$. The inset shows the evolution of the RWA spectrum with increasing $F$.](image-url)
cies \( \omega_k \) and with Hamiltonian \( H_{\text{rad}} = \sum_k \hbar \omega_k b_k^\dagger b_k \). We assume that the coupling of the considered oscillator to this field is bilinear in the ladder operators of the oscillator and the radiation, \( H_i = \sum_k \mu_k (b_k + b_k^\dagger)(a + a^\dagger) \), where \( \mu_k \) are the coupling parameters. The total Hamiltonian is \( H_{\text{total}} = H_0 + H_{\text{rad}} + H_i \). Operator \( H_0 \) is the Hamiltonian of the oscillator and the non-radiative thermal reservoir to which the oscillator is coupled. We assume that this reservoir and the radiation field are at the same temperature, which in what follows we assume to be sufficiently low, \( k_B T \ll \hbar \omega_0 \). The coupling to the reservoir leads to relaxation of the oscillator, which we will characterize by the relaxation rate \( \Gamma \), cf. Eq. (14).

If the coupling to the radiation field is weak, it can be considered as a perturbation to the non-radiative dynamics. The power of the radiation emitted into a spectral range \( d\Omega \) around frequency \( \Omega \) is given by the change of the energy of the radiation field in this interval per unit time \( W(\Omega, t)d\Omega = \frac{d}{dt} \sum_k \delta(\omega_k - \Omega) d\Omega (\hbar \omega_k b_k^\dagger b_k) \). To the lowest order in the coupling strength \( \mu_k \), we have in the resonant region where \( \Omega \) is close to \( \omega_F/2 \)

\[
W(\Omega, t) = Q[\Omega, t - t_0, \rho_0(t_0)] \mu^2(\Omega),
\]

\[
Q[\Omega, t - t_0, \rho_0(t_0)] = 2\text{Re} \int_{t_0}^t dt' e^{i(\Omega - \omega_F/2)(t - t')} \times \text{Tr}[a^\dagger(t' - t_0) a(t - t_0) \rho_0(t_0)],
\]

where \( \mu^2(\Omega) = \hbar^{-1} \sum_k |\mu_k|^2 \delta(\Omega - \omega_k) \).

In Eq. (14) we assumed that the coupling to the radiation is switched on at time \( t_0 \); \( \rho_0(t_0) \) is the density matrix of the oscillator and the non-radiative environment, it is independent of the state of the radiation. Equation (14) is written in the rotating frame; the transition to this frame is performed by the operator \( U(t - t_0) = \exp[-i\omega_F(t - t_0) a^\dagger a / 2] \).

The two-time correlation function in Eq. (14) can be found by solving the quantum kinetic equation. Such equation requires an initial condition. To obtain it we note that, physically, the coupling of the oscillator to the radiation field and to the non-radiative environment
should be switched on at the same time $t_0$; corrections to the dynamics due to the switching are well-understood, they are small in the considered case [33]. Respectively, $\rho_0(t_0)$ is the product of the oscillator density matrix $\rho(t_0)$ and the density matrix of the non-radiative environment in thermal equilibrium. The time evolution of the oscillator density matrix in the rotating frame is then often described by Eq. (14). To study transient radiation, we set $\rho(t_0) = |\phi_E\rangle\langle\phi_E|$, where $\phi_E$ is a RWA eigenstate in which the oscillator has been prepared.

For not very strong driving, $\tilde{F} \lesssim V$, function $\phi_E$ has a contribution of only a few Fock states. Respectively, the oscillator will radiate only a few photons as it comes to the stationary state. Thus, rather than measuring the radiation power $W(\Omega,t)$ it is more feasible to measure the total energy emitted over the transient time. The observation time should exceed the relaxation time to enable sufficient spectral resolution.

The energy of the transient radiation has to be separated from the energy that the oscillator emits in the stationary state. This can be done by noting that the latter energy is proportional to the observation time. The spectral power density (power per unit frequency) in the stationary regime is given by Eq. (15) written for $t \to \infty$ [24]. Therefore one can define the transient radiation spectral density as $E_{\text{rad}}(\Omega)\rho_{st} = \int_{t_0}^{\infty} dt Q[\Omega,t-t_0,\rho_0(t_0) - \rho_{st}]$. (16)

Here, $\rho_{st}$ is the stationary density matrix of the driven oscillator and the non-radiative environment.

The spectral density $E_{\text{rad}}(\Omega)$ can be positive or negative. As the oscillator decays from the initial state $\phi_E$, it emits radiation at frequencies $\omega_F/2 + (E - E')/\hbar$, where $E'$ is the RWA energy of a state $\phi_{E'}$ into which the oscillator can make a dipolar transition from $\phi_E$. In contrast, in the stationary state, the oscillator generally can be found in the both states $\phi_E, \phi_{E'}$, and therefore it radiates at both frequencies $\omega_F/2 \pm (E - E')/\hbar$. As a result, in the spectrum $E_{\text{rad}}(\Omega)$ one may expect a peak or a dip at $\omega_F/2 + (E - E')/\hbar$ and a dip at $\omega_F/2 - (E - E')/\hbar$.

Figure 9 (a) and (b) show the spectrum $E_{\text{rad}}(\Omega)$ when the oscillator is initially in a RWA eigenstate $\phi_E$ prepared from the vacuum $|0\rangle$ by adiabatically ramping up the driving field. The driving frequency is chosen so that $\phi_E$ has the second lowest RWA energy among even states; see the insets. The transient radiation is dominated by transitions from the state $\phi_E$ to the lowest odd state $\phi_{E'}$. As expected, the spectrum $E_{\text{rad}}(\Omega)$ displays a peak at $\omega_F/2 + (E - E')/\hbar$ for relatively strong driving and a dip at this frequency for weak driving, as well as a dip at $\omega_F/2 - (E - E')/\hbar$ in the both cases.

For weak driving, the large dip is located close to the frequency $\omega_0 + V$ of the transition from the first excited to the ground state of the oscillator in the absence of driving, whereas the small dip is located at its mirror frequency with respect to $\omega_F/2$. For strong driving, the peak/dip are located at the frequencies $\omega_F/2 \pm \omega_{\text{min}}$, corresponding to transition between the first excited and the ground intrawell states of $H_{RWA}$. For strong driving, there is also a negative narrow peak at $\omega_F/2$ due to the interwell transitions [23].

For a comparison, Fig. 9 (c) and (d) show the steady-state radiation power spectrum $Q_{st}(\Omega) = Q(\Omega, \infty, \rho_{st})$ for the same parameters as in Figure 9 (a) and (b), respectively. We notice that the steady state spectrum is symmetric with respect to $\omega_F/2$. This is a consequence of the detailed balance present in the system at zero temperature, see [32, 34].

VII. CONCLUSIONS

We studied preparation of quasienergy states of a nonlinear oscillator by varying the strength of the parametric driving. The driving frequency $\omega_F$ was chosen to be close to twice the oscillator eigenfrequency $\omega_0$. This allowed us to use the rotating wave approximation, as strong excitation of the oscillator could be achieved for a comparatively weak driving field. The strategy for the state preparation sensitively depends on the interrelation between the detuning of the driving frequency $\delta\omega_F = \omega_F/2 - \omega_0$ and the nonequidistance of the oscillator energy levels due to the nonlinearity. We found that, depending on this interrelation, the state evolution with slowly varying field can be adiabatic or nonadiabatic.

An important factor for state preparation is that the quasienergy states are separated into even and odd with respect to inversion in the phase space. The states of different parity are not coupled by the driving. A remarkable consequence of this symmetry is that the oscillator energy levels calculated in the RWA do not cross or anti-cross with varying driving amplitude. Rather the RWA energy levels of even and odd states approach each other pairwise with the increasing amplitude. At the same time, these levels for states of opposite parity can cross with varying $\delta\omega_F$. This crossing does not lead to crossing of the quasienergy levels. Where the RWA energy levels cross, the quasienergy levels are separated by $\hbar\omega_F/2$.

Another important factor for state preparation is that, in the limit of zero driving, the RWA energy spectrum displays simultaneous multiple degeneracy due to multiphoton resonance or the subharmonic resonance. By tuning the driving frequency, one can bring pairs of the RWA energy levels close to or away from degeneracy, or make them cross each other. Such tunability allows one to prepare on demand an arbitrary even quasienergy state just by slowing ramping up the driving, if the oscillator is initially in the ground state. The resulting quasienergy states can have a very different structure in phase space, as evidenced by the Wigner tomography.

We found that an effective way of preparing superpositions of quasienergy states is to use non-adiabatic transitions induced by the increasing driving amplitude. By
tuning the driving frequency, one can bring quasiener-
gies of two states of the same parity close to each other
for weak field. Then the field can lead to their mixing
even if it increases comparatively slowly. The problem
differs from the standard Landau-Zener problem, since
the initial state is prepared at a finite time rather than
at $t \rightarrow -\infty$. As a result, for a linearly increasing driv-
ing, the non-adiabatic transition rate falls off as a power
law, rather than exponentially, with the Landau-Zener
parameter $\Delta^2/s$, where $\Delta$ is the level spacing and $s$ is the ramping speed.

We investigated the transient radiation of the oscilla-
tor prepared in a given quasienergy state. The transient
spectrum provides a tool to characterize the quantum
state of the system, which is complimentary to the com-
monly used Wigner tomography. This tool can be partic-
ularly useful for investigating quasienergy states of cavity
modes in microwave cavities, the area of much current in-
terest.

\section{VIII. ACKNOWLEDGEMENTS}

This work was supported in part by the National Sci-
ence Foundation (Grant No. DMR-1514591); YZ was
also partly supported by the U.S. Army Research Office
(W911NF1410011) and by the National Science Founda-
tion (DMR-1609326).

\appendix{Appendix A: Fourier series for quasienergy states}

The eigenvalue problem for the periodic part $u_x(t)$ of
the Floquet wave function defined in Eq. \ref{eq:2}
reads

$$\varepsilon u_x(t) = (H(t) - i\hbar \partial_t) u_x(t).$$

Since $u_x(t)$ and $H(t)$ are both periodic in time, it is
covenient to expand them in Fourier series. It is also
covenient to write $u_x(t)$ in the basis of the Fock states
$|n\rangle$ of the harmonic oscillator with frequency $\omega_0$. Then

$$u_x(t) = \sum_{k,n} u_{k,n} \exp(-i k \omega_0 t) |n\rangle$$

does the form of the standard eigenvalue problem

$$\varepsilon u_{k,n} = \sum_{k',n'} M_{k,n}^{k',n'} u_{k',n'},$$

$$M_{k,n}^{k',n'} = (\mathcal{E}_n - \hbar k \omega_0) \delta_{k,k'} \delta_{n,n'} + \frac{1}{4} \delta_{n,n'} \delta_{k,k+1} + \delta_{k,k-1}$$

where $\delta_{n,n'} = \langle n | q^2 | n' \rangle$ and $\mathcal{E}_n$ is the nth energy level of the
Duffing oscillator in the absence of driving; to the
leading order in the nonlinearity $\mathcal{E}_n = \hbar (\omega_0^2 + V(n^2 + n))/2$. The sum runs over $k = 0, \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots$

The matrix elements $\delta_{n,n'}$ are nonzero for $n = m$ and
for $n = m + 2$. Therefore the driving term $\propto F$ couples
$u_{k,n}$ to $u_{k\pm 1,n\pm 2}, u_{k\pm 1,n}$. However, only the coupling to
$u_{k+1,n+2}$ and $u_{k-1,n-2}$ is resonant, since the diagonal
elements of matrix $M$ for such $u$ are close; for example,
$(\mathcal{E}_n - \hbar k \omega_0) - (\mathcal{E}_{n+2} - (k+1) \hbar \omega_0) = 2\hbar \delta_{k+1} - \hbar V(2n+3)$
is small compared to $\hbar \omega_0$. Therefore, one can limit the
analysis to a set $G_{k,n}$ of the variables $u_{k,n}$ resonantly
coupled to $u_{k,n}$. It has the form $G_{k,n} = \{u_{k+k',n+2k',k' \in \mathbb{Z}} $ and $k' \geq -n/2\}$. This is the rotating wave approxima-
tion in the Floquet formulation \ref{eq:1}.

The sets $G_{k,n}$ with different $k$ but the same $n$ are equiv-
alent: indeed, changing $k \rightarrow k_1$ corresponds to changing
$\varepsilon \rightarrow \varepsilon + (k - k_1) \hbar \omega_0$ in Eq. \ref{eq:2}. Since $\varepsilon$ is defined
modulo $\hbar \omega_0$, such change makes no difference. We can
then simplify $G_{k,n}$ as follows. Consider first even $n$, i.e.,
$n = 2n'$, and set $k = n'$,

$$G_{k,n'} = G_{n',2n'} = \{u_{n' + k',2n' + 2k',k' = -n', -n' + 1, \ldots} \rangle = G_{0,0}. \quad (A3)$$

In the last equation, we simply redefine $k'$ to absorb $n'$
in the new definition.

Similarly, for odd $n$, where $n = 2n' + 1$,

$$G_{k,2n'+1} = G_{0,1}. \quad (A4)$$

The simplification described by Eqs. \ref{eq:3} and \ref{eq:4}
allows one to reduce Eq. \ref{eq:2} to two sets of equations,

$$(\varepsilon - \mathcal{E}_{2k} + \hbar k \omega_0) u_{2k} = \tilde{F} [k(2k - 1)u_{2k-2} + (k+1)(2k+1)u_{2k+2}]
\quad (\varepsilon - \mathcal{E}_{2k+1} + \hbar k \omega_0) u_{2k+1} = \tilde{F} [k(2k+1)u_{2k-1} + (k+1)(2k+3)u_{2k+3}]$$

where $\tilde{F} = F/(2 \omega_0) \approx F/2 \omega_F$ and we used the explicit
form of the matrix elements $(n|q^2|n')$. \par
Equation \ref{eq:5} coincides with the RWA Schrödinger
equation $E \phi_E = H_{RWA} \phi_E$ if one writes $\phi_E$ in the basis
of the Fock states and replaces $\varepsilon$ with $E$ using Eq. \ref{eq:1}.

\appendix{B: Semiclassical analysis of RWA Hamiltonian}

For completeness, here we present, following \cite{33},
the description of the scaled RWA Hamiltonian function
$g(Q,P)$ for large driving. For $\mu < -1$, function $g$ has
one minimum at $(Q,P) = (0,0)$. For $-1 < \mu < 1$, the
minimum at $(0,0)$ becomes a saddle point and there ap-
pears two minima located at $(Q,P) = (\pm Q_0,0), Q_0 = \sqrt{\mu + T}$. For $\mu > 1$, the saddle point at $(0,0)$ becomes
a minimum again and there appear two saddle points at
$(Q,P) = (0, \pm \sqrt{\mu - T})$.

Of primary interest in this paper is the regime $0 < \mu < 1$.
We expand $g$ about the minimum at $(Q_0,0)$ to second
order in $Q - Q_0$ and $P$,

$$g \approx (\mu + 1)(Q - Q_0)^2 + P^2 + g_{\text{min}}, \quad (B1)$$
where $g_{\text{min}} = -(\mu + 1)^2/4$.

Introducing ladder operators $b, b^\dagger$ defined as

$$Q - Q_0 = \sqrt{\frac{\lambda}{2}}(\mu + 1)^{-1/4}(b^\dagger + b),$$

$$P = i\sqrt{\frac{\lambda}{2}}(\mu + 1)^{1/4}(b^\dagger - b)$$

([b, b^\dagger] = 1), we write the Hamiltonian $g(Q, -i\lambda\partial_Q)$ for low-lying intrawell eigenstates in the form

$$g \approx \lambda\omega_{\text{min}}(b^\dagger b + 1/2) + g_{\text{min}},$$

$$\omega_{\text{min}} = 2\sqrt{\mu + 1}.$$  \hspace{1cm} (B2)

The eigenstates of operator $b^\dagger b$ give the intra-well states used in the main text.

Appendix C: Non-adiabatic transition amplitude

The equations for $C_{\pm}(t)$ can be rescaled to the form of Weber differential equation,

$$\frac{d^2 C_{\pm}}{dz_{\pm}^2} + \left[-\frac{z_{\pm}^2}{4} \mp ip + \frac{1}{2}\right] C_{\pm} = 0,$$

$$p = \Delta^2/2s, \quad z_{\pm} = \sqrt{2se^{\pm iz/4t}}.$$ \hspace{1cm} (C1)

The general solution to this equation is a linear combination of two parabolic cylinder functions [35],

$$C_{\pm}(z) = A_{\pm}D_{\pm 1}(\mp iz_{\pm}) + B_{\pm}D_{\mp 1}(z_{\pm}).$$ \hspace{1cm} (C2)

Coefficients $A_{\pm}, B_{\pm}$ can be found from the initial values of $C_{\pm}(0)$ with account taken of the relation $i\hbar\dot{C}_{\pm}(0) = \Delta C_{\mp}(0)$.

Using the asymptotic expansion $D_q(z) \approx \exp(-z^2/4)z^q$ for $|z| \to \infty, |\arg z| < \frac{3}{4}\pi$, we find to the first order in $1/t$

$$C_{\pm}(t) \approx B_{\pm}\alpha_{\pm}e^{\mp i\theta(t)} + A_{\pm}\alpha_{\pm}^*e^{\mp i\theta(t) + iz/4(\hbar/2st^2)^{1/2}},$$

$$\alpha_{\pm} = \exp \left[\frac{p\pi}{4} - i\frac{p}{2}(\log p - 1)\right] = \alpha_{\pm}^*,$$ \hspace{1cm} (C3)

where $\theta(t)$ is given by Eq. (12).

For $|\Delta| \ll \nu(t)$ we have $C_{\pm} \approx (\Delta/2\nu)C_-$ and $C_{\pm} \approx C_-(\Delta/2\nu)C_+$. One can then immediately find the coefficients $\alpha_{\pm,\pm}, \beta_{\pm,\pm}$ in Eq. (11). In particular, $\alpha_{\pm} = B_{\pm,\pm}\alpha_{\pm,\pm}, \beta_{\pm,\pm} = B_{\pm,\pm}\beta_{\pm,\pm}$.

Of primary interest to us is the limiting value $C_{\pm,\pm}(\infty) \propto \alpha_{\pm,\pm}$. For the considered initial condition $C_{\pm}(0) = C_{-}(0) = 1/\sqrt{2}$, we find that

$$\alpha_{\pm,\pm} = \Lambda_{\pm} \left[\sqrt{\nu}\Gamma\left(\frac{\pm ip}{2}\right) + \text{sgn}(\Delta)(\pm 1 + i)\Gamma\left(\frac{1 \mp ip}{2}\right)\right],$$

$$\Lambda_{\pm} = \Lambda_{\pm}^* = \left(2p/e\right)^{-ip/2}(e^{3\pi p/4} - e^{-5\pi p/4}) \times \sqrt{\nu}\Gamma(ip)/\sqrt{2\pi},$$ \hspace{1cm} (C4)

where the upper sign refers to $\alpha_{\pm}$ and the lower sign refers to $\alpha_{\pm}^*$; $\Gamma(x)$ is the gamma-function.

The expressions for $\alpha_{\pm,\pm}$ in the adiabatic limit $p \to \infty$ can be obtained from Eqs. (C4) using the asymptotic form of the gamma function $\Gamma(z)$ for $|z| \to \infty$, cf. [36]. They were used in Eq. (13).

[1] J. H. Shirley, Phys. Rev. 138, B979 (1965)
[2] Y. B. Zel’dovich, J. Experimental And Theor. Phys. (U. S. S. R.) 51, 1492 (1966).
[3] V. I. Ritus, J. Exper. Theor. Phys. 51, 1544 (1966).
[4] H. Sambe, Phys. Rev. A 7, 2203 (1973).
[5] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B 82, 235114 (2010).
[6] N. H. Lindner, G. Refael, and V. Galitski, Nat Phys 7, 490 (2011).
[7] N. Goldman, G. Juzeliunas, P. Ohberg, and I. B. Spielman, Reports On Progress In Physics 77, 126401 (2014).
[8] M. Bukov, L. D’Alessio, and A. Polkovnikov, Adv. Phys. 64, 139 (2015).
[9] V. Peano, M. Hounde, C. Brendel, F. Marquardt, and A. A. Clerk, Nat. Comm. 7, 10779 (2016).
[10] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. 116, 250401 (2016).
[11] C. W. von Keyserlingk and S. L. Sondhi, Physical Review B 93, 245146 (2016).
[12] V. Khemani, C. W. von Keyserlingk, and S. L. Sondhi, arXiv: 1612.08758
[13] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, Nature 543, 217 (2017).
[14] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, Nature 543, 221 (2016).
[15] E. Bairey, G. Refael, and N. H. Lindner, arXiv: 1702.06208.
[16] L. D’Alessio and M. Rigol, Nat Commun 6, (2015).
[17] C. Heinisch and M. Holthaus, J. Mod. Opt., 1 (2016).
[18] P. Weinberg, M. Bukov, L. D’Alessio, A. Polkovnikov, S. Vajna, and M. Kolodrubetz, arXiv: 1606.02229.
[19] W. W. Ho and D. A. Abanin, arXiv: 1611.05024.
[20] H. Goto, Scientific Reports 6, 21686 (2016).
[21] S. Puri and A. Blais, arXiv: 1605.09408.
[22] D. M. Larsen and N. Bloembergen, Opt. Commun. 17,
M. I. Dykman and M. V. Fistul, Phys. Rev. B 71, 140508 (2005).

M. I. Dykman, in Fluctuating Nonlinear Oscillators: from Nanomechanics to Quantum Superconducting Circuits, edited by M. I. Dykman (OUP, Oxford, 2012) pp. 165–197.

C. D. Nabors, S. T. Yang, T. Day, and R. L. Byer, J. Opt. Soc. Am. B 7, 815 (1990).

C. M. Wilson, T. Duty, M. Sandberg, F. Persson, V. Shumeiko, and P. Delsing, Phys. Rev. Lett. 105, 233907 (2010).

Z. Lin, K. Inomata, K. Koshino, W. Oliver, Y. Nakamura, J. Tsai, and T. Yamamoto, Nat Commun 5, 4480 (2014).

S. Haroche and J. M. Raimond, Exploring the Quantum: Atoms, Cavities, and Photons (Oxford Univ. Press, Oxford, 2006).

M. Marthaler and M. I. Dykman, Phys. Rev. A 76, 010102R (2007).

Y. Zhang, J. Gosner, S. M. Girvin, J. Ankerhold, and M. Dykman, arXiv: 1702.07931.

L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, Cambridge, 1995).

M. Marthaler and M. I. Dykman, Phys. Rev. A 73, 042108 (2006).

M. I. Dykman, Zh. Eksp. Teor. Fiz. 68, 2082 (1975).

G. Y. Kryuchkyan and K. V. Kheruntsyan, Opt. Commun. 127, 230 (1996).

E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, 4th ed. (Cambridge University Press, 1990).

M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Table (Dover Publications, Inc., 1972).