ANGULAR MOMENTUM REDISTRIBUTION BY WAVES IN THE SUN

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Received 1998 June 23; accepted 1999 March 5

ABSTRACT

We calculate the angular momentum transport by gravito-inertial-Alfvén waves and show that, so long as prograde and retrograde gravity waves are excited to roughly the same amplitude, the sign of angular momentum deposit in the radiative interior of the Sun is such as to lead to an exponential growth of any existing small radial gradient of rotation velocity just below the convection zone. This leads to formation of a strong thin shear layer (of thickness about 0.3% R⊙) near the top of the radiative zone of the Sun on a timescale of order 20 yr. When the magnitude of differential rotation across this layer reaches about 0.1 μHz, the layer becomes unstable to shear instability and undergoes mixing, and the excess angular momentum deposited in the layer is returned to the convection zone. The strong shear in this layer generates a toroidal magnetic field which is also deposited in the convection zone when the layer becomes unstable. This could possibly start a new magnetic activity cycle seen at the surface.

Subject headings: Sun: interior — Sun: rotation — waves

1. INTRODUCTION

The Sun is currently losing angular momentum at its surface via a wind at a rate of the order of 1031 g cm2 s−2, which is slowing down the surface layers. However, we know from the frequency splitting of p-modes that the radiative interior of the Sun is rotating as a solid body at a rate that is not very different from the mean surface rotation rate. Thus, the loss of angular momentum at the surface is communicated to the rest of the Sun on a timescale of the order of the solar age or less.

The convective motions in the outer third of the Sun can very efficiently redistribute angular momentum on a short convective timescale of about a month. However, processes responsible for the redistribution of angular momentum in the radiative interior are less certain. One possibility is that magnetic torques can extract angular momentum from the radiative interior. Charbonneau & MacGregor (1993) have carried out a detailed numerical analysis of this transport process. However, there are some drawbacks to that mechanism, which are discussed in Zahn (1997).

Another possibility is that the gravity waves generated in the convection zone can extract/deposit angular momentum in the radiative interior. The angular momentum flux carried by these waves is enormous and they are very efficient in redistributing angular momentum in the radiative interior (Schatzman 1993a, 1993b; Kumar & Quataert 1997; Zahn, Talon, & Matias 1997).

Here, we explore this second mechanism in some detail. In particular, we discuss the effects of the Coriolis force and of magnetic fields on the dispersion relation of gravity waves and on the deposition of angular momentum in the radiative interior on the Sun (§ 2). We describe how these waves lead to the formation of a strong and thin shear layer just below the convection zone which becomes unstable when the gradient exceeds a critical value leading to mixing of elements and angular momentum with the convection zone (§ 3). We further discuss how this shear layer could contribute to the generation of the magnetic field and perhaps to the magnetic cycle observed in the Sun (§ 4).

2. ANGULAR MOMENTUM TRANSPORT BY WAVES

We derive below a general expression for the angular momentum flux in waves. The dispersion relation for gravito-inertial waves in the presence of a magnetic field is calculated in § 2.2.

2.1. Angular Momentum Flux in Waves

Let us consider a fluid Lagrangian density, L, which is a function of the displacement field, ξ, and of its temporal and spatial derivatives. If L is not an explicit function of the azimuthal coordinate φ, the z-component of the angular momentum is conserved (this is a particular case of the Noether’s theorem; cf. Quigg 1983). The Lagrangian density thus obeys the relation

\[
\frac{dL}{dφ} = \frac{∂L}{∂φ} + \frac{∂L}{∂ξ_i} \frac{∂ξ_i}{∂φ} + \frac{∂L}{∂ξ_{i,t}} \frac{∂ξ_{i,t}}{∂φ} + \frac{∂L}{∂ξ_{i,j}} \frac{∂ξ_{i,j}}{∂φ},
\]

or

\[
\frac{dL}{dφ} = \frac{∂L}{∂t} \left( \frac{∂L}{∂ξ_i} \frac{∂ξ_i}{∂φ} \right) + \frac{∂L}{∂φ} \left( \frac{∂ξ_i}{∂t} \frac{∂ξ_i}{∂ξ_{i,j}} \frac{∂ξ_{i,j}}{∂φ} \right) - \frac{∂L}{∂ξ_i} \left( \frac{∂ξ_i}{∂t} \frac{∂ξ_i}{∂ξ_{i,j}} \frac{∂ξ_{i,j}}{∂φ} \right) = 0.
\]

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It follows from the Euler-Lagrange equation that the last term is zero leading to
\[
- \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \xi_i} \right) + \frac{\partial}{\partial x_i} \left( g_{ij} \frac{\partial L}{\partial \xi_j} \right) = 0 .
\] (3)
This equation expresses the conservation of wave angular momentum. The angular momentum density and flux are given by
\[
\mathcal{L}_{am} = - \frac{\partial L}{\partial \xi_i} \frac{\partial}{\partial \phi} = 0 ,
\] (4)
and
\[
\mathcal{F}_{am} = g_{ij} \frac{\partial L}{\partial \xi_j} \frac{\partial}{\partial \phi} .
\] (5)
We show below that the above expression for angular momentum density ($\mathcal{L}_{am}$) has the correct sign and magnitude. Consider a plane wave in a homogeneous medium traveling along $+x$ direction,
\[
\xi = \xi_0 \sin (\omega t - kx) .
\] (6)
The only term in the Lagrangian containing $\partial \xi_i \partial t$ is the kinetic energy term, so it is straightforward to calculate the angular momentum density for the wave using equation (4):
\[
\mathcal{L}_{am} = - \rho \frac{\partial \xi_i \partial \xi_i}{\partial t} = - \rho \xi_0^2 \omega \rho \sin \phi \cos^2 (\omega t - kx) = - \rho \left( \frac{k}{\omega} \right) \frac{\partial \xi_i \partial t}{\partial t} y ,
\] (7)
where $y = r \sin \phi$. Since the wave is traveling along the $+x$ direction, its linear momentum density is also along the $+x$ direction and has the magnitude $p = \rho (\omega \xi_0^2)(k/\omega)$. The angular momentum density of the wave should be $\mathcal{L}_{am} = x \times p = - \rho u^2 (k/\omega) y \hat{\xi}$, which is identical to the expression (7).

The wave displacement field in the spherical Sun can be written as
\[
\xi = \left( \xi_r, \xi_h \frac{\partial}{\partial \theta}, -\xi_h \frac{\partial}{\partial \phi} \right) Y_m(\theta, 0) \cos (\omega t - m\phi) .
\] (8)
Substituting this in the expression for $\mathcal{L}_{am}$ (eq. [4]), we obtain
\[
\mathcal{L}_{am} = \rho m \omega \left\{ \frac{\varepsilon_r^2}{\sigma_r} Y_m^2 + \xi_h^2 \left( \frac{\partial Y_m}{\partial \theta} \right)^2 \right\} \sin^2 (\omega t - m\phi) + \frac{m^2 \xi_h^2}{\sin^2 \theta} Y_m^2 \cos^2 (\omega t - m\phi) .
\] (9)
Integrating $\mathcal{L}_{am}$ over $\theta$ and $\phi$, we obtain
\[
\langle \mathcal{L}_{am} \rangle = \frac{\rho m \omega \rho}{2} \left[ \varepsilon_r^2 + \ell(\ell + 1)\xi_h^2 \right] = \frac{m \rho \varepsilon}{\omega} ,
\] (10)
where $\varepsilon$ is the energy density in the wave. Prograde waves ($m > 0$) thus carry positive angular momentum, whereas retrograde waves ($m < 0$) carry negative angular momentum.\(^6\)

Waves deposit their angular momentum in the star only when and where they are damped. For gravity waves, the main damping process is the photon diffusion, which is discussed in § 2.4. The angular momentum flux crossing a spherical shell may then be expressed as a function of depth:
\[
\mathcal{F}_{am}(\omega, \ell, r) = \mathcal{F}_{am}(\omega, \ell, r_c) \exp \left[ - \tau(\omega, \ell, r) \right] ,
\] (11)
where $\tau(\omega, \ell, r)$ is the damping “optical depth” for a wave of frequency $\omega$, and degree $\ell$, between the radius $r$ and $r_c$ (the radius of the bottom of the convection zone) cf. § 2.4.

2.2. Dispersion Relation for Gravito-Inertial-Alfvén Waves

The linearized momentum equation that includes both Coriolis and Lorentz forces is
\[
\rho \frac{\partial^2 \xi}{\partial t^2} + V p_1 - \rho_1 g + 2 \rho \Omega \times v = \frac{(V \times B_1)}{4\pi} \times B
\] (12)
where $\Omega$ is the rotational frequency, $g = - \hat{g} \omega$ is the gravitational acceleration, $\xi$ and $v$ are the fluid displacement and velocity associated with the wave, and $p_1$, $p_1$ and $B_1$ are the Eulerian perturbations of density, pressure and magnetic field associated with the wave. We assume for simplicity that the unperturbed magnetic field $B$ and the rotational speed $\Omega$ are constant. In the short-wavelength limit, we can take the spatial and the temporal dependence of all of the perturbed quantities to be

\(^6\) The sign of the angular momentum carried by the gravity waves was improperly defined in Kumar & Quataert (1997) and in Zahn et al. (1997), as was pointed out by Ringot (1998).
proportional to exp \((ik \cdot x - i\omega t)\). Substituting this in the above equation, we obtain
\[
-\rho \omega^2 \xi + ikp_1 - \rho_1 g - 2i\omega \rho \Omega \times \xi = \frac{i(k \times B_1)}{4\pi} \times B.
\] (13)

Making use of the linearized equation of entropy conservation
\[
\rho_1 = \frac{P_1}{c^2} + \frac{\rho}{g} N^2 \xi_r, \tag{14}
\]
where \(\xi_r\) is the radial component of the displacement vector, and taking \(k \gg g/c^2\), i.e., the wavelength is much smaller than the density scale height, we obtain
\[
-\rho \omega^2 \xi + ikp_1 + \rho N^2 \xi_r \hat{r} - 2i\omega \rho \Omega \times \xi = \frac{i(k \times B_1)}{4\pi} \times B.
\] (15)

Substituting for \(B_1\) from the linearized induction equation for a perfectly conducting fluid, and making use of the incompressibility condition, \(k \cdot \xi = 0\), we obtain
\[
-\rho \omega^2 \xi + ikp_1 + \rho N^2 \xi_r \hat{r} - 2i\omega \rho \Omega \times \xi + \frac{(k \cdot B)}{4\pi} \left[ (k \cdot B)\xi - (\xi \cdot B)k \right] = 0. \tag{16}
\]

The vector cross product of the above equation with \(k\) yields
\[
\omega^2 - \left( k \cdot \vec{B} \right)^2 V_A^2 \left[ (k \cdot \xi) - N^2 \xi_r \right] - 2i\omega(k \cdot \Omega)\xi = 0, \tag{17}
\]
where \(V_A^2 = B^2/4\pi \rho\) is the Alfvén velocity and \(\vec{B} = B/B\). The vector cross product of equation (17) with \(k\) gives \(k \times \xi\) in terms of \(\xi\), which, when substituted back into equation (17), results in
\[
\xi \left\{ \frac{\omega^2 - \left( k \cdot \vec{B} \right)^2 V_A^2}{2i\omega(k \cdot \Omega)} + 2i\omega(k \cdot \Omega) \right\} + \xi \left\{ N^2(k \times \hat{r}) + \frac{N^2\omega^2 - \left( k \cdot \vec{B} \right)^2 V_A^2}{2i\omega(k \cdot \Omega)} \right\} k \times (k \times \hat{r}) = 0. \tag{18}
\]

The \(r\)-component of the above equation yields the desired dispersion relation:
\[
\omega^2 - \left( k \cdot \vec{B} \right)^2 V_A^2 - (N \times \vec{k})^2 \left[ \omega^2 - \left( k \cdot \vec{B} \right)^2 V_A^2 \right] - 4\omega^2(k \cdot \Omega)^2 = 0, \tag{19}
\]
where \(\vec{k} = k/k\). This equation can be solved easily to determine the wave frequency
\[
\omega^2 = \left( k \cdot \vec{B} \right)^2 V_A^2 + \frac{1}{4}(N \times \vec{k})^2 + 4(k \cdot \Omega)^2 \pm \left[ \left( N \times \vec{k} \right)^2 + 4(k \cdot \Omega)^2 \right]^{1/2}. \tag{20}
\]

This dispersion relation has two branches. The lower frequency branch corresponds to Alfvén waves and the high-frequency branch, to gravito-inertial-Alfvén waves. The dispersion relation for the high-frequency branch can be written as
\[
\omega^2 \approx (k \cdot \vec{B}^2) V_A^2 + (N \times \vec{k})^2 + 4(k \cdot \Omega)^2. \tag{21}
\]

The radial wave number \(k_r\), when the magnetic field lies in the horizontal plane, can be obtained from the dispersion relation (eq. [19]) and is given by
\[
k_r = -\frac{2\omega^2 \Omega^2 k_\theta \sin \theta + \left( 4\omega^4 \Omega^4 k_\theta^2 \sin^2 \theta + \left( \omega^2 - 4\omega^2 \Omega^2 \cos^2 \theta \right) \left[ (N^2 - \omega^2) k_\theta^2 + 4\omega^2 \Omega^2 k_\theta \sin^2 \theta \right] \right)^{1/2}}{\omega_1^4 - 4\omega^2 \Omega^2 \cos^2 \theta}, \tag{22}
\]
where
\[
\omega_1^2 = \omega^2 - (k \cdot \vec{B})^2 V_A^2. \tag{23}
\]
\(\Omega = \Omega(\cos \theta \vec{r} - \sin \theta \vec{\theta})\), \(k_\theta\) is the horizontal wave number, \(k_\theta^2 = k^2 - k_\phi^2\), \(k_\phi = m/r \sin \theta\), and \(m\) is the azimuthal number. Figure 1 shows \(k_r\) as a function of \(\omega\) for a few cases of \(\theta\), \(\vec{B}\), and \(k_\theta\).

The frequency \(\omega\) in the expression for the wavenumber is the frequency as seen in the local rest frame of the fluid, i.e., \(\omega = \omega_1 - m \Omega\), where \(\omega_1\) is the wave frequency in an inertial frame. The component of the wavevector along the \(\phi\) direction being conserved along the ray path, the prograde waves moving inward in a region of increasing rotation speed are Doppler shifted to lower frequencies. From the above equation, we see that the wavenumber of these waves increases with decreasing frequency (see also Fig. 1) therefore enhancing their damping. Thus, in a region of increasing rotation speed, prograde waves are dissipated more strongly than retrograde waves.

For pure gravity waves, Zahn et al. (1997) discussed the existence of a critical layer determined by the condition \(\omega = m \delta \Omega\), where \(\delta \Omega\) is the difference between the rotation rate at radius \(r\) and the base of the convection zone. In that case, a wave is completely absorbed since its radial wave number diverges and its group speed goes to zero. The presence of nonzero magnetic field and/or rotation modifies the condition for the appearance of critical layers, which can occur even when \(\omega > 0\) in the local rest frame. However, it can be shown that \(\omega_1^2 = \pm 2\omega \Omega \cos \theta\), for which the denominator in equation (22) is zero, does not correspond to a critical layer since the numerator at this frequency is zero as well, unless either \(\Omega \cos \theta = 0\) or the
wave frequency in the local rest frame of the fluid ($\omega - m\delta\Omega$) is zero.\(^7\) For the low-frequency waves (which carry most of the angular momentum) radiative damping is very strong and the waves are damped on a distance scale much smaller than the solar radius even when waves do not encounter a critical layer. This is discussed in more detail in §2.4.

Wave propagation requires $\omega > (k \cdot B)V_A$. For horizontal magnetic field of strength less than about $2 \times 10^5/\ell$ G wave propagation in the radial direction requires the frequency to be greater than 0.5 \(\mu\)Hz, while for a radial magnetic field of strength less than about $10^3$ G the condition for wave propagation is $\omega > 0.5$ \(\mu\)Hz (angular momentum flux in waves peaks at the convective frequency of 0.3 \(\mu\)Hz and falls off as $v^{-5.5}$ at higher frequencies). Waves will undergo reflection when the radial component of the magnetic field is sufficiently large.

### 2.3. Excitation of Waves by Turbulent Convection

Gravity waves with periods of about 10 days are expected to be excited in the Sun by a number of different processes such as the Reynolds’s stress in the convection zone, plumes penetrating in the stably stratified radiative interior, etc. In Earth’s atmosphere, the latter process is known to be the most efficient, and it might be the dominant process in the Sun as well. However, the energy flux in gravity waves resulting from this process is subject to great uncertainty because we know little about the properties of plumes at the base of the solar convection zone. We will therefore consider wave excitation only by Reynolds’s stress. The results of this paper (the formation of shear layer at the top of the radiative interior of the Sun by gravity waves) can be easily modified to incorporate more efficient gravity waves generation by plumes or some other process. Generally, larger flux in gravity waves means more efficient formation of shear layer on a smaller timescale.

The energy flux per unit frequency in waves just below the convective envelope of the Sun, as a result of excitation by the Reynolds’s stress, can be calculated using the following expression (cf. Goldreich et al. 1994):

$$F_E^{(\ell)} = \frac{\alpha^2}{4\pi} \int dr \frac{p^2}{r^2} \left[ \frac{\partial \xi^2}{\partial r} + \ell + 1 \right] \exp \left[ \frac{-h^2}{2} \right] \frac{v^3L^4}{1 + (\omega \tau_L)^{15/2}},$$  \hspace{1cm} (24)

where $\xi^r$ and $[\ell(\ell + 1)]^{1/2} \xi_h$ are the radial and horizontal displacement wavefunctions which are normalized to unit energy flux just below the convection zone, $v$ is the convective velocity, $L$ is the radial size of an energy bearing turbulent eddy, $\tau_L \approx L/v$ is the characteristic convective time, and $h$ is the radial size of the largest eddy at $r$ with characteristic frequency of $\omega$ or greater ($h_\omega = L \min \{1,(2\omega \tau_L)^{-\lambda/2}\}$). The gravity waves are evanescent in the convection zone, the region where they are excited. The above equation was derived under the assumption that the turbulence spectrum is Kolmogorov, and, as mentioned earlier, it ignores wave excitation resulting from convective overshooting.

The energy flux spectrum just below the convection zone is calculated numerically using equation (24) and the result is shown in Figure 2. Also shown in Figure 2 is the frequency-integrated wave action as a function of wave degree $\ell$; angular momentum flux for waves of a given $m$ is equal to $m$ times the wave action. We provide below a rough estimate of the energy flux in gravity waves in order to gain some understanding of the numerical results.

For the low-frequency waves of interest here, the WKB solution to the wave equation is quite accurate and can be used to calculate the energy flux. The displacement wavefunctions $\xi^r$ and $\xi_h$ in the short-wavelength limit are proportional to

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\(^7\) Barnes et al. (1998) discuss critical layers for gravito-Alfvén waves and their results can be obtained from ours by setting $\Omega = 0$. 
\( k_r^{-1/2} \exp \left( i \int \text{d}r k_r \right) \), and the energy flux in waves is \( \rho \omega^2 [\xi_r^2 + \ell (\ell + 1) \xi_k^2] v_g \), where \( k_r \) is radial wave number given by equation (22), and \( v_g \) is the wave group velocity.

Using the continuity of the radial velocity at the interface of the radiation and the convection zone, we find that the properly normalized radial displacement wave function in the convection zone is

\[
\xi \sim \left( \frac{k_h N_c}{\omega (\rho N N_t)} \right)^{1/2} \exp \left[ -\int \text{d}r k_r \right],
\]

(25)

where \( N_t \sim (g/H)^{1/2} \) is the Brunt-Väisälä frequency at the top of the radiation zone and \( N_c \) is the convective frequency at the bottom of the convection zone. Substituting this into equation (24) and making use of \( L \sim H \) and \( \rho v^3 \sim F_\odot \) (the energy flux carried by convection), we find the energy flux in waves to be

\[
F_E^\omega(\omega) \sim \frac{F_\odot k_h^3 H_c^2 \exp \left( -H_c^2 k_h^2 \right)}{r_c^2 N_t}.
\]

(26)

The subscript \( \omega \) on \( H \) emphasizes that it is the scale height at a location in the convection zone where the characteristic timescale of energy bearing eddies is \( \omega^{-1} \). For a polytropic atmosphere of index \( n \), the density scales with depth \( z \) (measured from the surface) as \( z^n \), and therefore the convective velocity scales as \( v \propto z^{-n/3} \). Thus, the convective frequency \( \omega \propto z^{-(n+3)/3} \), and the scale height \( H_c \propto H(\rho N_c/\omega)^{3(n+3)} \); \( H_c \) is the scale height at the bottom of the convection zone. Substituting this into the above equation we obtain

\[
F_E^\omega(\omega) \sim \frac{F_\odot k_h^3 H_c^2 \exp \left( -H_c^2 k_h^2 \right)}{r_c^2 N_t} \left( \frac{\omega}{N_c} \right)^{-15/(n+3)}.
\]

(27)

Since the polytropic index of the convection zone is approximately 1.5, we find the frequency dependence of \( F_E^\omega \) to be \( \omega^{-3.3} \). The numerical calculation of the energy flux, using equation (24), gives the frequency dependence to be \( \omega^{-4.5} \). The difference arises because of our neglect of the frequency dependence of \( k_r \) in equation (25) for \( \xi_r \).

Integrating \( F_E^\omega \) over the wave frequency and the horizontal wave number, we find the total energy flux in waves to be given by\(^8\)

\[
F_E^\omega \sim \frac{F_\odot N_c}{N_t} \sim F_\odot \mathcal{M}_t,
\]

(28)

\(^8\) Goldreich & Kumar (1990) had obtained the same result for the energy flux in surface gravity waves.
where \( \mathcal{M}_t \approx 3 \times 10^{-4} \) is the turbulent Mach number at the bottom of the convection zone. Thus, the total energy flux in the long-period gravity waves just below the solar convection zone is about 0.03% of the solar luminosity; more accurate numerical calculation based on the evaluation of equation (24) gives the energy flux to be 0.1% of the solar luminosity.

2.4. Radiative Damping of Waves

For the long-period or short-wavelength waves which dominate the transport of angular momentum, the radiative diffusion of photons from regions of positive and negative temperature fluctuation across a wavelength is the dominant damping process. The damping opacity between radii \( r \) and \( r_c \) (the base of the convection zone) is calculated using

\[
\tau(\omega, \ell, m, r) = \int_r^{r_c} \frac{\gamma(\omega, \ell, m, r)}{v_g(\omega, \ell, m, r),} \, dr,
\]

where

\[
\omega_m(r) = \omega + m[\Omega_c - \Omega(r)],
\]

is the wave frequency in the local rest frame of the fluid at radius \( r \), \( \Omega_c \) is the angular rotation speed at the base of the convection zone, the damping rate

\[
\gamma(\omega, \ell, m, r) \approx K k_r^2
\]

\[
K = \frac{16\sigma T^3}{3\rho^2 \kappa c_p} \approx \frac{2F_r H_T}{5p}
\]

is the thermal diffusivity, \( \sigma \) is the Stefan-Boltzmann constant, \( \kappa \) is opacity per unit mass, \( c_p \) is the specific heat per unit mass, \( F_r \) is the radiative energy flux, \( H_T \) is the temperature scale height, \( p \) is the gas pressure, and \( k_r \) is the radial wavenumber of the wave.

For gravity waves, \( k_r \approx k_N \Omega / \omega \) and \( v_g \approx \omega^2 / (k_N N) \). Therefore, their damping increases rapidly with decreasing frequency and increasing \( \ell \) as \( \omega^{-4} \ell^3 \). The wave damping length \( d_w \sim v_g / \gamma \approx 5p\omega(2F_r H_T k_r^2) \), and so the ratio of the damping length and the wavelength is

\[
d_w k_r \sim \frac{\rho \omega^3 H_T}{F_r k_N^2} \sim \left( \frac{\omega}{N_c} \right)^3 \left( \frac{r_c}{H_T} \right)^2 \ell^{-2},
\]

where \( N_c \) is the convective frequency at the base of the convection zone. We have used the relations \( F_r \sim \rho(N_c H_T)^3 \) and \( N^2 \sim g / H_T \) in deriving the above scaling. At the bottom of the solar convection zone \( r_c / H_T \sim 10 \), therefore for dipole waves of frequency \( N_c \) the damping length is about 15 wavelengths. For waves of frequency 0.6 \( \mu \)Hz \( (\omega / N_c \approx 4) \) the wave damping length is of order the wavelength for \( \ell \sim 20 \) (see Fig. 3 for a more accurate numerical result).

Numerical calculation of damping length as a function of wave frequency and \( \ell \) are shown in Figure 3. Note that, even in the absence of a critical layer wave damping is extremely efficient (see also Press 1981). Waves with frequency of 0.5 \( \mu \)Hz and \( \ell = 15 \) are damped on a length scale of about 10 wavelengths. Our numerical calculations for the evolution of shear layer below the convection zone, presented in § 3, include radiative damping of waves throughout the shear layer and not just at the critical surface.

![Fig. 3.—Damping length for gravity waves divided by the wavelength of the wave is shown as a function of wave frequency for \( \ell = 2 \) (solid line), 15 (dotted line), and 50 (dashed line).](image-url)
Radiative damping only affects the gravity wave part of the gravito-Alfvén-inertial waves. Thus, after traversing some distance, a gravito-Alfvén-inertial wave can be converted into an Alfvén-inertial wave as a result of radiative damping. We have, however, ignored this complication here because, for the waves of frequency greater than about 0.6 μHz considered in this paper, the buoyancy force is more important than magnetic tension and the Coriolis forces as long as the toroidal magnetic field below the convection zone is less than 10^5 G, and the latitude is less than 45° (see § 2.2). Thus, much of the angular momentum flux carried by gravito-Alfvén-inertial waves gets deposited in the fluid before they turn into Alfvén-inertial waves. Moreover, it can be shown that for parameters of interest to the solar problem, the finite plasma conductivity damps out Alfvén-inertial waves near the critical surface.

3. THE FORMATION AND EVOLUTION OF A SHEAR LAYER

Long-period gravity waves generated at the base of the solar convection zone are efficient at redistributing the angular momentum in the radiative interior (Zahn et al. 1997; Kumar & Quataert 1997). The angular momentum deposited by gravity waves just below the convection zone is such as to enhance any pre-existing gradient of angular velocity. The physical reason for this, although not intuitive, is easy to understand. Consider the case where the angular speed increases with depth below the convection zone. The frequencies of prograde waves moving inward in the radiative interior are decreased as seen in the local rest frame of the rotating fluid, whereas the frequencies of retrograde waves are increased. Thus, the damping of prograde waves is enhanced relative to retrograde waves and, as a result, positive angular momentum is deposited just below the convection zone enhancing the magnitude of the existing differential rotation. This enhancement of angular momentum gradient is confined to a thin layer whose thickness is of the order of the wave damping length. Initially, the angular speed increases exponentially with time, and when the gradient becomes sufficiently large so that much of the angular momentum flux of prograde waves is absorbed in the shear layer below the convection zone, further growth becomes linear in time. Retrograde waves, which were only partially absorbed in this layer, continue to propagate deeper in the radiative interior and deposit their negative angular momentum in a thin layer which is spun down and develops a strong differential rotation.

The evolution of angular momentum and formation of the shear layer are determined by solving the following equation:

\[ \rho r^2 \frac{d\Omega}{dt} = \frac{dF_L}{dr} \]

where \( F_L \) is the angular momentum flux associated with waves

\[ F_L = \sum_{\ell, m} \int \frac{m F_{\ell m}(\omega, \ell)}{\omega} \cdot \exp \left( -\tau(\omega, \ell, m) \right). \]

To estimate the timescale for the initial exponential growth of the angular speed, we expand the damping optical depth \( \tau \)

\[ \tau(\omega, \ell, m) = \int_{r_c}^{r} \left( \gamma(r, \omega - m \delta \Omega, \ell) \right) \approx \int_{r_c}^{r} \frac{dr}{m \delta \Omega} \frac{dy}{d\omega} \equiv \tau_0(\omega, \ell) + m \tau_1(\omega, \ell). \]

Substituting this into equation (35), we obtain

\[ F_L = -\sum_{\ell} \int \int \frac{F_{\ell}^{(e)}}{\omega} \frac{d}{d\tau_1}^{(e)} \frac{1}{\sinh(\ell/2)} \exp(-\tau_0) \]

\[ = \sum_{\ell} \int \left[ \frac{\tau_1 F_{\ell}^{(e)}(\ell + 1/2)^3}{3\omega} \right] \exp(-\tau_0). \]

The last equality was obtained by assuming that \( \ell_1 \ll 1 \) and \( \ell_1 \gg 1 \). Finally, substituting this in equation (34), we obtain

\[ \frac{d}{dt} \delta \Omega \approx \frac{\delta \Omega}{\rho r^2} \sum_{\ell} \int \frac{dy}{d\omega} \exp(-\tau_0) \frac{F_{\ell}^{(e)}(\ell + 1/2)^3}{3\omega}, \]

and thus the characteristic timescale for the growth of angular speed is

\[ t_L = \left[ \frac{1}{\rho r^2} \sum_{\ell} \int \frac{dy}{d\omega} \exp(-\tau_0) \frac{F_{\ell}^{(e)}(\ell + 1/2)^3}{3\omega} \right]^{-1}. \]

Figure 4 shows the growth time as a function of \( r \). Note that the growth time near the top of the radiative zone is very short (of order a year). The growth time is inversely proportional to the energy flux in waves, and therefore the timescale for formation of the shear layer will be smaller if gravity waves are generated more efficiently than considered here (as should be the case if one considered also the waves excited by the convective overshooting); the magnitude of the differential rotation within the layer also increases with increase in the wave flux.

The evolution of the angular speed at a given latitude below the convection zone is obtained by solving equation (34), and the results are shown in Figure 5. The initial rotation speed was taken to increase with depth below the convection zone, and the magnitude of the differential rotation over a layer of thickness 0.01 \( R_\odot \) was considered to be \( \sim 0.05 \) mHz initially (the mean rotation speed of the interior is 0.4 μHz). We show results for several different colatitudes (60° and 90°) and magnetic field strength (10^5 and 8 \times 10^5 G). The seed magnetic field we have considered lies in the horizontal plane along the \( \Phi \)-axis, which is the expected configuration for fields generated by differential rotation. The initial seed toroidal magnetic field we start with...
gets amplified with time as the magnitude of the shear in the layer just below the convection zone increases with time as a result of angular momentum deposit. However, to determine the time evolution of the magnetic field strength in a self-consistent way is outside the scope of this work.

We see in Figure 5 that the timescale for the formation of a strong shear layer is about 20–30 years. The timescale for the growth of the layer is a factor of about two smaller when the initial value of the differential rotation across the layer is 1 nHz. Moreover, the growth timescale is inversely proportional to the energy flux of waves. Considering the uncertainty in the initial “seed” differential rotation rate and the energy flux in waves, the timescale for the formation of the shear layer we find should be considered to be of the same order as the solar cycle period.

In all of the cases shown in Figure 5, the rotation curve rises very sharply just below the convection zone, and this is followed by a more gently declining rotation curve. Below this layer, where positive angular momentum was deposited, lies another layer with negative differential rotation where retrograde gravity waves are absorbed. The sharp rise/decline of rotation in these two layers arises as a result of the rapid increase in wave damping rate with decreasing frequency in the local rest frame of the fluid, i.e., an increase in the rotation rate causes prograde waves to be damped over a smaller distance scale, which in turn increases the rotation rate, and the feedback continues to increase the gradient of angular velocity until much of the angular momentum carried by prograde waves is absorbed. Since the frequencies of retrograde waves traveling inward in the star are increased as they propagate through a layer with negative gradient of differential rotation, they deposit their negative angular momentum deeper in the Sun leading to the formation of another thin and strong shear layer. The thickness of the shear layer is related to the damping length of low-frequency gravity waves which carry bulk of the angular momentum flux. The distance from the top of the shear layer to the base of the convection zone is also about one damping length.

Shear layers at higher latitudes tend to be thinner and closer to the convection zone, and this is a result of the increase in the damping of waves. Moreover, waves of large $m$ and $r$ do not propagate to high latitudes. Thus, the formation of shear layer does not proceed at high latitudes.

For waves of a fixed frequency, an increase in the strength of the magnetic field generally leads to increase in the damping rate and, as a result, the shear layer is formed closer to the convection zone, is thinner, and grows more rapidly (see Fig. 5).

A self-consistent calculation must consider the building up of the shear layer as well as the generation of the magnetic field simultaneously. This is outside of the scope of this work. However, based on the physical understanding gained for the simpler problem considered here we expect that as the shear layer develops it twists poloidal magnetic field and generates toroidal magnetic field which in turn facilitates the damping of waves and prevents the redistribution of angular momentum in the layer by small scale turbulence (see § 3.1). Both of these effects speed up the growth of angular momentum gradient in the layer. Another effect evident from Figures 5 and 6 is that the increase in the strength of the field causes the layer to move closer to the convection zone. When the magnetic field becomes sufficiently strong it prevents the propagation of low-frequency waves, which are reflected at the top of the layer. These low-frequency waves bounce back and forth between the convection zone and the shear layer and are absorbed after a few bounces thereby depositing their angular momentum and extending the shear layer upward closer to the bottom of the convection zone. A combination of shear instability and magnetic buoyancy could then lead to the ejection of the toroidal field from the shear layer and its deposition in the convection zone.

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9 The spectrum of waves generated by turbulent convection has been taken to be independent of latitude, which is not strictly valid and could modify this dependence.
3.1. The Effect of Shear Instability on the Buildup of Shear Layer

So far, we have ignored the removal of angular momentum in the shear layer due to magnetic torque and the redistribution resulting from hydrodynamical instabilities.

Magnetic torque acting on the shear layer removes angular momentum and therefore weakens the effectiveness of the gravity waves to build up the shear. The magnetic torque per unit volume $T_B = r \times (V \times B) \times B/4\pi \approx rB_\theta \bar{B}_\phi / (dr) / 4\pi$. As long as the magnetic torque per unit area of the shear layer, $rB_\theta \bar{B}_\phi / 4\pi$, is small compared to the rate of angular momentum deposit in the layer we can ignore the effect of the magnetic field. Since the angular momentum flux in prograde/retrograde waves is of order $2 \times 10^{15}$ g s$^{-2}$, magnetic torque can be neglected so long as the product $B_\theta \bar{B}_\phi$ is less than about $5 \times 10^5$ G$^2$, which we assume is the case. It should be noted that if gravity waves are excited by some process which is more efficient than the Reynolds stress considered in this paper, such as convective overshooting (cf. Fritts, Vadas, & Andreassen 1998), then the limit on $B_\theta \bar{B}_\phi$ will increase correspondingly. The limit on $B_\theta \bar{B}_\phi$ is, however, independent of the thickness of the shear layer since both the magnetic torque and the torque applied by gravity waves increase as the inverse of the thickness of the shear layer.

The shear layer is expected to become unstable when the gradient of rotational velocity is sufficiently large, leading to the redistribution of angular momentum within the layer and possible mixing with the overlying convection zone. A shear layer in a stratified medium is unstable when the so-called Richardson criterion is satisfied (cf. Chandrasekhar 1961),

$$ \frac{g}{\rho} \frac{d\Delta \rho}{dr} < \frac{R_i}{R_i} ,$$

(40)
Evolution of the shear layer. The top left panel shows the difference between a calculation including turbulent diffusion (thin continuous line) and excluding turbulent diffusion (thick dashed line) at the equator and for $B = 10^3$ G at 20, 30, and 40 years. The other three panels show the shear layer including turbulent diffusion at 0º and 30º and $B = 10^3$ and $8 \times 10^3$ G. The initial rotation profile is as in Fig. 5. Ages are indicated on the plot.

where $U$ is the horizontal velocity, $\Delta \rho$ is the density difference between the perturbed fluid and the ambient medium, and $R_i_c = 1/4$ is the critical Richardson number for instability to set in. For an adiabatic perturbation, the above condition can be written as

$$\frac{N^2}{(dU/dz)^2} < R_i_c,$$

(41)

where $N^2 = -g(d\ln \rho/dr + g/c_s^2)$ is the Brunt-Väisälä frequency (here, we will not consider the part of the Brunt-Väisälä frequency related to the mean molecular weight stratification as it is very small is the Sun’s outer layers). Near the top of the radiative zone of the Sun $N = 2 \pi \approx 50 \mu Hz$. Thus, a shear layer of thickness $\delta r$ becomes unstable when the differential rotation across this layer $\delta \Omega \approx 2N \delta r/r_c$, or when $\delta \Omega/2\pi \approx 0.1 \mu Hz$ for $\delta r/r_c$ of $10^{-3}$.

Even when the Richardson criterion for instability as described above is not satisfied, instability can still set in on small length scales due to the weakening of the buoyancy force as a result of thermal diffusion. The first elements to become turbulent are of size such that the timescale for thermal exchange with the surrounding fluid is small and the stabilizing effect of positive entropy gradient is reduced (see Zahn 1992; Maeder 1995). The modified Richardson criterion which includes heat diffusion is as follows

$$\left(\frac{\eta}{\eta + 1}\right)N^2 \leq R_i \left(\frac{dU}{dr}\right)^2,$$

(42)

where $\eta = \nu_l/6K$, $\nu$ is the velocity of turbulent elements, $l_i$ is their size, $K$ is the thermal diffusivity defined in equation (32), $\kappa$ is opacity per unit mass, and $c_p$ is specific heat per unit mass, and $\sigma$ is the Stefan-Boltzmann constant. The turbulent diffusivity
(\(D_\circ\)) is given by those eddies which have the largest value of \(vl\) and which satisfy equation (42), and it takes the following simple form:

\[
D_\circ \approx \frac{vl}{3} \approx 2R_i K \frac{(dU/dr)^2}{N^2}
\]

(cf. Maeder 1995). This expression remains valid provided \(N^2(dU/dr)^2 < Ri_c\).

We include the effect of turbulent diffusivity on the evolution of the shear layer and the results are shown in Figure 6. Note that the main effect of the turbulence is to reduce the sharp velocity gradient at the top of the shear layer and to increase the thickness of the layer somewhat (compare Figs. 5 and 6).\(^{10}\)

When the velocity gradient in the shear layer reaches the critical value so that \(|dU/dr| = N R_i^{1/2}\), the nature of the instability changes. The thermal diffusion is no longer required to destabilize the fluid and a dynamical instability sets in whose growth rate is of order of the rotation period (see, e.g., Mac Donald 1983). The shear layer, which lies just below the convection zone, is likely to merge with the convection zone depositing the magnetic field that was build up in the layer as a result of differential rotation. This could be the start of a new magnetic cycle. That situation was never reached in our calculations since the diffusive shear instability was always large enough to redistribute angular momentum before condition (41) was satisfied. However, due to the lack of a reliable model, we have ignored the stabilizing effect of magnetic fields on the shear instability. We expect that magnetic fields will reduce the value for the turbulent diffusion coefficients we have used here, and therefore radial differential rotation will build up to larger values than shown in Figure 6, resulting in a more efficient generation of a toroidal magnetic field. In this case the dynamical instability condition is likely to be satisfied. The shear layer is also more likely to become dynamically unstable if the angular momentum flux in gravity waves is larger than estimated in § 2.3.

The timescale for the buildup of the shear layer and its distance from the base of the convection zone decreases with increasing latitude. Thus, dynamical instability and mixing of layer is expected to first occur at high latitudes and later in the equatorial region. However, a complete description of the latitude dependence would require a truly two-dimensional model. Indeed, as there is no stabilizing stratification along horizontal layers, horizontal shear instabilities are much stronger than their vertical analogs, and we expect important horizontal redistribution of angular momentum to occur.

4. DISCUSSION

The gravity waves generated by turbulent convection are potentially very efficient in redistributing angular momentum in the radiative interior of the Sun. Most of the angular momentum flux is carried by the lowest frequency waves, which have periods of the order of the convective timescale at the bottom of the convection zone (or \(\sim 0.3 \mu Hz\) for the present day Sun). The total energy flux in these low-period gravity waves is of the order of 0.03% of the solar luminosity and the angular momentum flux (\(F_L\)) in prograde waves is \(\sim 2 \times 10^{15} \text{ g s}^{-2}\). We assume that prograde waves, which carry positive angular momentum flux, and retrograde gravity waves, which have negative angular momentum flux, are excited to the same amplitude and thus, the total net angular momentum flux at the top of the radiative interior is zero; in this case, gravity waves alone merely redistribute angular momentum in the interior.

Kumar & Quataert (1997) and Zahn et al. (1997) suggested that gravity waves generated by turbulent convection could bring the radiative interior of the Sun in corotation with the convection zone. The work presented here corrects a sign error and adds one crucial element to their picture that was left out, i.e., the formation of a double shear layer below the convection zone filters out either prograde or retrograde waves depending on the sign of the initial velocity gradient below the convection zone.

As the angular velocity increases in the layer, it becomes unstable to small length scale perturbations which cause mixing of the layer, and thus it contributes to the mixing of elements near the top of the radiative interior of the Sun as seems to be required by the helioseismic inversion of the sound speed (Basu 1997).

If only “diffusive” turbulence is present (as in our calculations), the first shear layer lying at the top of the radiative zone will eventually merge with the surface convection zone and the second shear layer, of opposite sign, will replace it at the top of the radiative zone, reversing the structure, much as in the biennial oscillation, which is well known in atmospheric sciences (cf. Holton & Lindzen 1972). Taking into account the stabilizing effect of a toroidal magnetic field could change this situation as stronger shears would result, leading perhaps to the appearance of a more drastic “dynamical” instability which might destroy much of the double shear layer.

Most of the angular momentum deposited in these shear layers is transferred back into the convection zone once they become turbulent, but further work is needed to check whether there will be a net amount of angular momentum taken out of the radiation zone and how it will depend on the angular velocity gradient. This issue must also be settled in order to get a

\(^{10}\) In this work, meridional circulation was ignored since it is known that, when differential rotation is strong, the momentum transport is dominated by shear turbulence (cf. Zahn et al. 1997).
clear picture of the interaction between these short-lived shear layers and the quasi-stationary tachocline, which is the somewhat thicker transition layer that links the differential rotation of the convection zone to the almost uniform rotation below (see Spiegel & Zahn 1992).

A poloidal magnetic field threading the shear layer will get twisted as the differential rotation in the layer increases and, as a result, toroidal magnetic field will be generated. The increasing toroidal field in turn accelerates the rate of angular momentum deposit in the layer, and inhibits small length scale shear instability which reduces the rate of growth of the angular momentum gradient. Once the magnetic field in the shear layer becomes strong enough to reflect low-frequency or high azimuthal wavenumber waves, these waves travel back and forth between the shear layer and the convection zone and their angular momentum is absorbed in this intermediate layer in a few transit time. The result of this is to extend the shear layer closer to the convection zone. At some stage in its evolution, the shear layer becomes unstable and mixes with the convection zone and the magnetic field is the layer floats upward to start a new magnetic (half) cycle. Once the layer has relaxed and settled back to a stable configuration, after having given up much of the angular momentum and magnetic field that it had contained, the whole process starts all over again. Let us point out that this model predicts cyclic reversals of the toroidal field with a steady poloidal field at the base of the convection zone, which could well be a fossil field, whereas in classical dynamo theory the poloidal field too undergoes such reversals. We note also that waves of large azimuthal order \( (m) \) do not reach high latitude. Therefore, the formation of the shear layer and the generation of the magnetic field in not likely to occur at high latitudes, which is consistent with the lack of sunspots there.

Incorporating these various processes in a self-consistent manner in the evolution calculation is, however, very complicated and has not been considered here. But it is tempting to link the cyclic build-up of these shear layers, which produce toroidal field of alternating sign, with the solar magnetic cycle, since the timescales are of the same order. Future helioseismic observations and inversions with improved resolution should be able to determine the presence of a double shear layer at the top of the radiative interior predicted here.

P. K. thanks Peter Gilman for helpful discussion; S. T. and J. P. Z. are grateful to Olivier Ringot for helping them to clarify the issue of angular momentum transport by gravity waves. We thank Evry Schatzman and an anonymous referee for a careful reading of the manuscript and for suggestions for improvements. This work was supported in part by a NASA grant NAG5-7395.

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