Boundary Layer Solution of the Boltzmann Equation for Specular Boundary Condition

Fei-min HUANG¹,², Zai-hong JIANG³, Yong WANG¹,²,†

¹,² Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
(E-mail: yongwang@amss.ac.cn)
²School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
³Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Abstract In the paper, we establish the existence of steady boundary layer solution of Boltzmann equation with specular boundary condition in $L^2_{x,v} \cap L^\infty_{x,v}$ in half-space. The uniqueness, continuity and exponential decay of the solution are obtained, and such estimates are important to prove the Hilbert expansion of Boltzmann equation for half-space problem with specular boundary condition.

Keywords Boltzmann equation; boundary layer; steady problem; specular boundary condition; a priori estimate

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1 Introduction

In the present paper, we consider the steady Boltzmann equation

$$v_3 \cdot \partial_x F = Q(F, F) + S, \quad (x, v) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad (1.1)$$

with $\mathbb{R}_+ = (0, +\infty)$. The Boltzmann collision term $Q(F_1, F_2)$ on the right is defined in terms of the following bilinear form

$$Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u, \theta) F_1(u') F_2(v') \, d\omega du - \int_{\mathbb{R}^3} \int_{S^2} B(v-u, \theta) F_1(u) F_2(v) \, d\omega du$$

$$\quad := Q_+(F_1, F_2) - Q_-(F_1, F_2), \quad (1.2)$$

where the relationship between the post-collision velocity $(v', u')$ of two particles with the pre-collision velocity $(v, u)$ is given by

$$u' = u + [(v-u) \cdot \omega] \omega, \quad v' = v - [(v-u) \cdot \omega] \omega,$$

for $\omega \in S^2$, which can be determined by conservation laws of momentum and energy

$$u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2.$$

The Boltzmann collision kernel $B = B(v-u, \theta)$ in (1.2) depends only on $|v-u|$ and $\theta$ with $\cos \theta = (v-u) \cdot \omega/|v-u|$. Throughout this paper, we consider the hard sphere model, i.e.,

$$B(v-u, \theta) = |(v-u) \cdot \omega|.$$
We supplement the Boltzmann equation (1.1) with the perturbed specular reflection boundary condition

$$F(0, v)|_{v_3 > 0} = F(0, Rv) + F_b(Rv),$$

(1.3)

where $Rv = (v_1, v_2, -v_3)$, and $F_b(v)$ is a given function. We impose the condition at infinity

$$\lim_{x \to \infty} F(x, v) = \mu \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \exp \left\{ -\frac{|v-u|^2}{2} \right\},$$

(1.4)

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ is a given background macroscopic velocity which is independent of $x$. Throughout the paper, we always assume $u_3 = 0$ which insures that $\mu(v) = \mu(Rv)$, i.e., $\mu$ satisfies the specular boundary condition.

We look for solutions in the form

$$f(x, v) = \frac{F(x, v) - \mu}{\sqrt{\mu}},$$

then (1.1), (1.3) and (1.4) are rewritten as

$$\begin{cases}
v_3 \partial_x f + Lf = \Gamma(f, f) + S, \\
\left. f(v) \right|_{v_3 > 0} = f(0, Rv) + f_b(Rv), \\
\lim_{x \to \infty} f(x, v) = 0,
\end{cases}$$

(1.5)

where we have denoted

$$Lf = -\frac{1}{\sqrt{\mu}} \{ Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu) \},$$

$$\Gamma(f, f) = \frac{1}{\sqrt{\mu}} \{ Q(\sqrt{\mu}f, \sqrt{\mu}f) \},$$

$$f_b(v) = \frac{F_b(v)}{\sqrt{\mu}}, \quad S = \frac{S}{\sqrt{\mu}}.$$

It is noted that the function $f_b(v)$ is defined only for $v_3 < 0$, and we assume that it is extended to be 0 for $v_3 \geq 0$ throughout the paper.

There have been many studies on the half-space problem of the steady Boltzmann equation in the literature. The existence, uniqueness and properties of asymptotic behavior were proved in [1] for the linearized Boltzmann equation of a hard sphere gas for the Dirichlet type boundary condition, see [4] for a classification of well-posed kinetic boundary layer problem. Later, the existence of nonlinear Knudsen boundary layers with small magnitudes for the hard sphere model was established in [13], it was shown that the existence of a solution depends on the Mach number of the far field Maxwellian, and an implicit solvability conditions yielding the co-dimensions of the boundary data, see [6, 14] for the time-asymptotic stability of such boundary layer solution; we also refer [15] for the construction of a modified boundary layer solution in $L^\infty_{x,v}$ space which is used to prove the Hilbert expansion in a disk; recently, for the purpose of studying the transition from evaporation to condensation, Bernhoff-Golse[2] offered the existence and uniqueness of a uniformly decaying boundary layer type solution in the situation that gas is in contact with its condensed phase. For the diffuse boundary condition, the existence of steady Boltzmann solution is proved in [5, 7] in boundary domain, and the time-asymptotic stability of such steady solutions is also obtained. For the specular reflection condition and the solution tends to a global Maxwellian in the far field, Golse-Perthame-Sulem[9] proved
the existence, uniqueness and asymptotic behavior in the functional space (1.9). To prove
the Hilbert expansion of Boltzmann equation for half-space problem with specular boundary
condition, the continuity and uniform estimate in $L^\infty_{x,v}$ are needed, so in the present paper, we
will focus on the existence steady solution of (1.1) in the functional space $L^2_{x,v} \cap L^\infty_{x,v}$.

Now we list some notations that will be used in this paper. Throughout this paper, $C$
denotes a generic positive constant which may vary from line to line. And $C_a$, $C_b$, $\cdots$
denote the generic positive constants depending on $a, b, \cdots$, respectively, which also may vary from
line to line. $\| \cdot \|_{L^p}$ denotes the standard $L^p(\Omega \times \mathbb{R}^3_+)$-norm or $L^p(\mathbb{R}^3_+)$-norm, $\| \cdot \|_p = \| \nabla \cdot \|_{L^2}$.
When the norms need to be distinguished from each other, we write $\| \cdot \|_{L^p_a}$, $\| \cdot \|_{L^p_b}$ and $\| \cdot \|_{L^p_{v+}}$.
respectively. Moreover, for the phase boundary integration, we define $| \cdot |_{L^\infty(\gamma)}$ denotes the
$L^\infty(\gamma)$-norm, $| \cdot |_{L^2(\gamma)}$ denotes the $L^2(\gamma, |v|^3dv)$-norm, where $\gamma = \partial \Omega \times \mathbb{R}^3_+$ with $\partial \Omega = \mathbb{R}_+$ or $(0, d)$
for $d > 0$.

We define the weight function
\[
w(v) = (1 + |v|^2)^{\frac{\beta}{2}} e^{\zeta |v-u|^2},
\]
where $\beta, \zeta$ are two positive constants.

**Theorem 1.1.** Let $\beta \geq 3$ and $0 \leq \zeta < \frac{1}{4}$. We assume $S \in \mathbb{N}^+$ and $f_b(v)$ satisfies
\[
\int_{\mathbb{R}^3} f_b(v)v_3\sqrt{\mu}dv = \int_{\mathbb{R}^3} (v_1 - u_1)v_3 f_b(\sqrt{\mu}dv = \int_{\mathbb{R}^3} (v_2 - u_2)v_3 f_b(v)\sqrt{\mu}dv \\
= \int_{\mathbb{R}^3} v_3 |v-u|^2 f_b(v)\sqrt{\mu}dv = 0.
\]
There exists a small $\delta_0 > 0$ such that if $|w f_b|_{L^\infty} + \|\nu^{-1}w e^{\sigma x} S\|_{L^\infty_{x,v}} \leq \delta_0$, then the boundary
value problem (1.5) has a unique solution satisfying
\[
|e^{\sigma x}w f|_{L^\infty_{x,v}} \leq \frac{C}{\sigma_0 - \sigma} (|w f_b|_{L^\infty} + \|\nu^{-1}w e^{\sigma x} S\|_{L^\infty_{x,v}}),
\]
where $\sigma$ is a constant such that $0 < \sigma < \sigma_0$, and $C > 0$ is a constant independent of $\sigma$, and $\delta_0$
depends on $\frac{1}{\sigma_0 - \sigma}$ with $\delta_0 \to 0+$ as $\sigma \to \sigma_0$. Moreover, if $S$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^3$ and $f_b$
is continuous in $\{v \in \mathbb{R}^3 \}$, then $f(x,v)$ is continuous away from the grazing set $\{(0,v) : v \in \mathbb{R}^3, v_3 = 0 \}$.

**Remark 1.2.** Golse-Perthame-Sulem\cite{9} proved an existence result for (1.5) in the following
functional space
\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} e^{\sigma x} (1 + |v|)^2 f(x,v)^2 dvdx + \int_{\mathbb{R}^3} \sup_{x \in \mathbb{R}_+} \{e^{\sigma x} f(x,v)^2 \} dv < \infty.
\]
In the present paper, we are interested in the existence result of Boltzmann equation (1.1) in the
functional space $L^2_{x,v} \cap L^\infty_{x,v}$ with uniqueness, continuity; in particular, we shall use the linear
existence result in Theorem 3.1 below to prove the Hilbert expansion of Boltzmann equation
for half-space problem in [11].

This paper is organized as follows: in Section 2, we present some useful results which will be
used later. In Section 3, we study the existence theory of the linearized problem with a source
term. The proof of our main Theorem 1.1 is given in Section 4.
2 Preliminary

As in [8], \( L \) can be decomposed as \( L = \nu(v) - K \) where

\[
K_1 f(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - u) \cdot \omega| \sqrt{\mu(v) \mu(u)} f(u) \, d\omega \, du,
\]

\[
K_2 f(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - u) \cdot \omega| \sqrt{\mu(u) \mu'(u')} f(u') \, d\omega \, du,
\]

\[
K f(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \mu(u) \mu(v') f(u') \, d\omega \, du,
\]

\[
\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \mu(u) \, d\omega \, du \approx 1 + |v|.
\]

We list some properties of \( L, \Gamma \) for later use:

(I) The null space of the operator \( L \) is the 5-dimensional space of collision invariants:

\[
\mathbb{N} = \ker L = \text{span}\{\sqrt{\mu}, (v - u) \sqrt{\mu}, (|v - u|^2 - 3) \sqrt{\mu}\}.
\]

And let \( P \) denote the projection operator from \( L^2(\mathbb{R}^3) \) to \( \mathbb{N} \).

(II) \( \nu(v) \) satisfies

\[
0 < \nu_0 (1 + |v|) \leq \nu(v) \leq \nu_1 (1 + |v|).
\]

(III) The operator \( K \) satisfies the following Grad’s estimates

\[
K f(v) = \int_{\mathbb{R}^3} k(v, \eta) f(\eta) \, d\eta,
\]

where \( k(v, \eta) \)

\[
0 \leq |k(v, \eta)| \leq C \frac{|v - \eta|^2}{|v - \eta|^2} e^{-\frac{|v - \eta|^2}{8|v - \eta|^2}} + C |v - \eta| e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}}, \quad (2.1)
\]

where \( C > 0 \) is a given constant. Following (2.1), it is direct to have

\[
\int_{\mathbb{R}^3} |k(v, \eta) \cdot \frac{(1 + |v|)^{\alpha}}{(1 + |\eta|)^{\alpha}}| \, d\eta \leq C \alpha (1 + |v|)^{-1}.
\]

(IV) \( L \) satisfies

\[
\int_{\mathbb{R}^3} g L g dv \geq c_0 \| (I - P) g \|_{L^2},
\]

\[
\int_{\mathbb{R}^3} |L^{-1} h|^2 dv \leq c_0 \int_{\mathbb{R}^3} |h|^2 dv,
\]

for \( h \in \mathbb{N}^\perp \).

(V) The nonlinear term \( \Gamma(f, f) \in \mathbb{N}^\perp \) and

\[
\| \nu^{-1} w \Gamma(f, f) \|_{L^\infty \mathbb{N}} \leq C \| w f \|_{L^\infty \mathbb{N}}^2.
\]
We introduce a lemma which will be used to obtain the uniform $L_{x,v}^\infty$ of approximate solutions.

**Lemma 2.1** [5]. Consider a sequence $\{a_i\}_{i=0}^\infty$ with each $a_i \geq 0$. For any fixed $k \in \mathbb{N}_+$, we denote

$$A_i^k = \max\{a_i, a_{i+1}, \ldots, a_{i+k}\}.$$

1. Assume $D \geq 0$. If $a_{i+1+k} \leq \frac{1}{8}A_i^k + D$ for $i = 0, 1, \ldots$, then it holds that

$$A_i^k \leq \left(\frac{1}{8}\right)^{\frac{1}{\eta+1}} \cdot \max\{A_0^k, A_1^k, \ldots, A_k^k\} + \frac{8+k}{7}D, \quad \text{for } i \geq k+1.$$

2. Let $0 \leq \eta < 1$ with $\eta^{-k+1} \geq \frac{1}{4}$. If $a_{i+1+k} \leq \frac{1}{8}A_i^k + C_k \cdot \eta^i + \frac{8}{7}D$ for $i = 0, 1, \ldots$, then it holds that

$$A_i^k \leq \left(\frac{1}{8}\right)^{\frac{1}{\eta+1}} \cdot \max\{A_0^k, A_1^k, \ldots, A_k^k\} + 2C_k \cdot \frac{8+k}{7} \eta^i, \quad \text{for } i \geq k+1.$$

### 3 Existence for the Linearized Problem

This section is devoted to the existence result for the following linearized problem with a source term

$$\begin{cases}
  v_3 \partial_x f + Lf = \tilde{S}, \\
  \tilde{f}(0,v) = f_0(Rv) + f_b(Rv), \quad (x,v) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
  \lim_{x \to \infty} f(x,v) = 0,
\end{cases} \quad (3.1)$$

The main result is

**Theorem 3.1.** Recall the weight function $w(v)$ in (1.6), and let $\beta \geq 3$ and $0 \leq \varsigma < \frac{1}{4}$. We assume (1.7) and

$$\tilde{S} \in \mathbb{N}_+, \quad |w f_0|_{L^\infty_x} + \|\nu^{-1} w(v) e^{\sigma_0 x} \tilde{S}\|_{L^\infty_{x,v}} < \infty, \quad (3.2)$$

then there exists a unique solution $f$ to (3.1) such that

$$\|e^{\sigma x} f\|_{L^\infty_{x,v}} + |w f(0,\cdot)|_{L^\infty_v} \leq \frac{\hat{C}}{\sigma_0 - \sigma} \{\|e^{\sigma x} \nu^{-1} w \tilde{S}\|_{L^\infty_{x,v}} + |w f_0|_{L^\infty_v}\}, \quad (3.3)$$

and

$$\|e^{\sigma x} f\|_{L^2_{x,v}} \leq \frac{\hat{C}}{\sigma_0 - \sigma} \{\|e^{\sigma x} \tilde{S}\|_{L^2_{x,v}} + |(1 + |v|) f_b|_{L^2_v}\}, \quad (3.4)$$

where $\hat{C} > 0$ is a positive constant independent of $\sigma \in (0, \sigma_0)$. Moreover, if $\tilde{S}$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^3$ and $f_b(v)$ is continuous in $\{v \in \mathbb{R}^3\}$, then $f(x,v)$ is continuous away from the grazing set $\{(0,v) : v \in \mathbb{R}^3, v_3 = 0\}$. 
Let $\chi(x)$ be a monotonic smooth cut-off function

$$\chi(x) \equiv 1, \text{ for } x \in [0, 1] \quad \text{and} \quad \chi(x) \equiv 0, \text{ for } x \in [2, +\infty).$$

Similar as in [9], we define

$$f(x, v) := f(x, v) + \chi(x)f_b(v),$$

then (3.1) is equivalent

$$
\begin{align*}
\left\{
\begin{array}{ll}
v_3\partial_x f + Lf = g := \tilde{S} + v_3\partial_x \chi(x)f_b(v) + \chi(x)Lf_b, \\
f(0, v)|_{v_3>0} = f(0, Rv), \\
\lim_{x \to \infty} f(x, v) = 0.
\end{array}
\right.
\end{align*}
$$

To prove the existence of solution to (3.5), we first need to consider a truncated problem. We denote $\Omega := (0, d)$ with $d \geq 1$ and denote the phase boundary of $\Omega \times \mathbb{R}^3$ as $\gamma = \partial \Omega \times \mathbb{R}^3$. We split $\gamma$ into three disjoint parts, outgoing boundary $\gamma_+$, the incoming boundary $\gamma_-$, and the singular boundary $\gamma_0$ for grazing velocities:

$$\gamma_+ = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\},$$

$$\gamma_- = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\},$$

$$\gamma_0 = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\},$$

where $n(x)$ is the outward unit normal. It is direct to know that $\partial \Omega = \{0, d\}$, $\tilde{n}(0) = (0, 0, -1)$ and $\tilde{n}(d) = (0, 0, 1)$.

Now we consider the truncated problem with penalized term

$$
\begin{align*}
\left\{
\begin{array}{ll}
\varepsilon f^\varepsilon + v_3\partial_x f^\varepsilon + Lf^\varepsilon = g, & (x, v) \in \Omega \times \mathbb{R}^3, \\
f^\varepsilon(x, v)|_{\gamma_-} = f^\varepsilon(x, R_x v),
\end{array}
\right.
\end{align*}
$$

where $\varepsilon \in (0, 1]$ and $R_x v := v - 2(v \cdot \tilde{n}(x))\tilde{n}(x)$. We also define

$$h^\varepsilon(x, v) := w(v)f^\varepsilon(x, v),$$

then (3.6) can be rewritten as

$$
\begin{align*}
\left\{
\begin{array}{ll}
\varepsilon h^\varepsilon + v_3\partial_x h^\varepsilon + \nu(v)h^\varepsilon = K_w h^\varepsilon + wg, \\
h^\varepsilon(x, v)|_{\gamma_-} = h^\varepsilon(x, R_x v),
\end{array}
\right.
\end{align*}
$$

where $K_w h = wK(\frac{h}{w})$. It is direct to know that

$$K_w h(v) = \int_{\mathbb{R}^3} k_w(v, u)h(u)du \quad \text{with} \quad k_w(v, u) = w(v)k(v, u)w(u)^{-1}. \quad (3.8)$$

### 3.1 A Priori $L^\infty_{x,v}$ Estimate

For the approximate problem (3.7), the most difficult part is to obtain the $L^\infty_{x,v}$-bound.

**Definition 3.1.** Given $(t, x, v)$, let $[X(s), V(s)]$ be the backward characteristics for (3.7), which is determined by

$$
\begin{align*}
\left\{
\begin{array}{ll}
dX(s) \\
\frac{ds}{ds} = V_3(s), \\
\frac{dV(s)}{ds} = 0,
\end{array}
\right. \\
[X(t), V(t)] = [x, v].
\end{align*}
$$
The solution is then given by
\[ [X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v_3, v]. \]

Now for each \((x, v)\) with \(x \in \bar{\Omega}_d\) and \(v_3 \neq 0\), we define its backward exit time \(t_b(x, v) \geq 0\) to be the last moment at which the back-time straight line \([X(-\tau; 0, x, v), V(-\tau; 0, x, v)]\) remains in \(\Omega\):
\[ t_b(x, v) = \sup\{s \geq 0 : x - \tau v_3 \in \bar{\Omega}_d \text{ for } 0 \leq \tau \leq s\}. \]

We also define
\[ x_b(x, v) = x(t_b) = x - t_b(x, v) v_3 \in \partial\Omega_d. \]

We point out that \(X(s), t_b(x, v)\) and \(x_b(x, v)\) are independent of the horizontal velocity \(v_h := (v_1, v_2)\).

Let \(x \in \bar{\Omega}_d, (x, v) \notin \gamma_0 \cup \gamma_-\) and \((t_0, x_0, v_0) = (t, x, v)\), and inductively define for \(k \geq 1\)
\[ (t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_b(x_k, v_k), x_b(x_k, v_k), R_{x_{k+1}} v_k). \]

We define the back-time cycle as
\[
\begin{align*}
X_{cl}(s; t, x, v) &= \sum_k 1_{[t_{k+1}, t_k)}(s)\{x_k - v_{k,3} \cdot (t_k - s)\}, \\
V_{cl}(s; t, x, v) &= \sum_k 1_{[t_{k+1}, t_k)}(s)v_k.
\end{align*}
\] (3.9)

Clearly, for \(k \geq 1\) and \((x, v) \notin \gamma_0 \cup \gamma_-\), it holds that
\[
\begin{align*}
x_k &= \frac{1 - (-1)^k}{2}x_1 + \frac{1 + (-1)^k}{2}x_2, \quad v_{k,h} = v_{0,h}, \quad v_{k,3} = (-1)^k v_{0,3}, \\
t_k - t_{k+1} &= t_1 - t_2 = \frac{d}{|v_{0,3}|} > 0, \quad \nu(v) = \nu(v_k).
\end{align*}
\] (3.10)

We can represent the solution of (3.7) in a mild formulation which enables us to get the \(L^\infty\) bound of solutions. Indeed, for later use, we consider the following iterative linear problems involving a parameter \(\lambda \in [0, 1]::\)
\[
\begin{align*}
\varepsilon h^{i+1} + v_3 \partial_x h^{i+1} + \nu(v) h^{i+1} &= \lambda K_w h^i + w_g, \\
h^{i+1}(x, v)|_{\gamma_-} &= h^i(x, R_x v) + w_r r(x, v),
\end{align*}
\] (3.11)

for \(i = 0, 1, 2, \ldots\), where \(h^0 \equiv 0\) and \(w(v)r(x, v) \in L^\infty(\gamma_-)\) is any given function. For the mild formulation of (3.11), we have the following lemma whose proof is omitted for brevity as it is similar to that in [10].

**Lemma 3.2.** Let \(0 \leq \lambda \leq 1\). For each \((x, v) \in \bar{\Omega}_d \times \mathbb{R}^3 \setminus (\gamma_0 \cup \gamma_-)\), we have
\[
h^{i+1}(x, v) = e^{-\nu(v)(t-t_k)}h^{i-k+1}(x_k, v_k) + \sum_{l=0}^{k-1} e^{-\nu(v)(t-t_{l+1})}w_r(x_{l+1}, v_l)
\]
\[
+ \sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu(v)(t-s)}[\lambda K_w h^{i-l} + w_g](X_{cl}(s), v_l) ds,
\] (3.12)

where and whereafter denote \(\nu(v) = \varepsilon + \nu(v)\), and the parameter \(k \gg 1\) is the collision times of the particle with boundary \(\partial\Omega_d\).
Lemma 3.3. Let $h^i, i = 0, 1, 2, \cdots$, be the solutions to (3.11), satisfying
\[ \|h^i\|_{L^\infty_{t,v}} + |h^i|_{L^\infty_{(\gamma^+)}} < \infty. \]
Then there exists $k_0 > 0$ large enough such that for $i \geq 2k_0$, it holds that
\[
\frac{1}{8} \sup_{0 \leq t \leq 2k_0} \{ \|h^{i-1}\|_{L^\infty_{t,v}} + |h^{i-1}|_{L^\infty_{(\gamma^+)}} \}
+ C\{\|\nu^{-1}wg\|_{L^\infty_{t,v}} + |wr|_{L^\infty_{(\gamma^-)}}\} + C \sup_{0 \leq t \leq 2k_0} \left\{ \left\| \frac{h^{i-1}}{w} \right\|_{L^2_{t,v}} \right\}. \tag{3.13}
\]
Moreover, if $h^i \equiv h$ for $i = 1, 2, \cdots$, i.e., $h$ is a solution, then (3.13) is reduced to the following estimate
\[
\|h\|_{L^\infty_{t,v}} + |h|_{L^\infty_{(\gamma^+)}} \leq C\{\|\nu^{-1}wg\|_{L^\infty_{t,v}} + |wr|_{L^\infty_{(\gamma^-)}}\} + C \left\| \frac{h}{w} \right\|_{L^2_{t,v}}. \tag{3.14}
\]
We emphasize that the positive constant $C > 0$ depends on $k_0$, and is independent of $d$, $\lambda \in [0,1]$ and $\varepsilon > 0$.

Proof. For $(x,v) \notin \gamma_0 \cup \gamma^-$, it is noted that
\[
\frac{\nu_t(v)}{|v_0,3|} = \frac{\varepsilon + \nu(v)}{|v_0,3|} \geq \nu_0 \frac{1 + |v|}{|v|} \geq \nu_0 > 0,
\]
which, together with (3.10), yields that
\[
|e^{-\nu_t(v)(t-t_k)}h^{i-k+1}(x_k,v_k)| \leq |e^{-\nu_t(v)(t_1-t_k)}h^{i-k+1}(x_k,v_k)|
\leq C|h^{i-k+1}|_{L^\infty_{(\gamma^+)}} \exp \left\{ -\nu(v)(k-1) \frac{d}{|v_0,3|} \right\}
\leq C|h^{i-k+1}|_{L^\infty_{(\gamma^+)}} \exp \left\{ -\nu_0 d(k-1) \right\}
\leq C|h^{i-k+1}|_{L^\infty_{(\gamma^+)}} e^{-\frac{1}{2} \nu_0 d k}.
\]
For the second term on RHS of (3.12), one has that
\[
\left| \sum_{l=0}^{k-1} e^{-\nu_t(v)(t-t_{l+1})} wr(x_{l+1},v_l) \right| \leq C k |wr|_{L^\infty_{(\gamma^-)}}.
\]
The last term on RHS of (3.12) is bounded by
\[
\left| \sum_{l=0}^{k-1} \int_{t_{l+1}}^{t_l} e^{-\nu_t(v)(t-s)} wg(X_{cl}(s),v_l) ds \right| \leq C |\nu^{-1}wg|_{L^\infty_{t,v}} \sum_{l=0}^{k-1} \int_{t_{l+1}}^{t_l} e^{-\nu_t(v)(t-s)} |\nu(v)| ds
\leq C |\nu^{-1}wg|_{L^\infty_{t,v}}.
\]
For the third term on RHS of (3.12), we use (3.12) again to obtain
\[
\sum_{l=0}^{k-1} \int_{t_{l+1}}^{t_l} e^{-\nu_t(v)(t-s)} \lambda K w h^{i-l} (X_{cl}(s),v_{l,3}) ds
\]
\[
\begin{align*}
&= \lambda \sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_{s}(v)(t-s)} \int_{\mathbb{R}^3} k_w(v_l, v') h^{i-l}(X_{cl}(s), v')dv' ds \\
&\leq \lambda^2 \sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_{s}(v)(t-s)} \int_{\mathbb{R}^3} |k_w(v_l, v')|dv' ds \\
&\times \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} e^{-\nu_{s}(v')(s-s_j)} \int_{\mathbb{R}^3} |k_w(v'_j, v'') h^{i-l-j-1}(X_{cl}(s_1), v'')|dv'' ds_j \\
&+ C \left( e^{-\frac{1}{2} \nu_0 dk} \sup_{0 \leq l \leq 2k} \left| h^{i-l} \right|_{L^\infty(\gamma_+)} + \|\nu^{-1} w g\|_{L^\infty} + k \|w r\|_{L^\infty(\gamma_-)} \right),
\end{align*}
\] (3.15)

where we have denoted \( X'_{cl}(s_1) = X_{cl}(s_1; s, X_{cl}(s), v') \), and \( t'_j, v'_j \) are the corresponding times and velocities for specular cycles.

For the first term on RHS of (3.15), we divide the proof into several cases:

Case 1. For \(|v| \geq N\), the first term on RHS of (3.15) is bounded by

\[
\begin{align*}
&\sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_{s}(v)(t-s)} \int_{\mathbb{R}^3} |k_w(v_l, v')|dv' ds \cdot \sup_{0 \leq l \leq 2k} \left| h^{i-l} \right|_{L^\infty} \\
&\leq C \sup_{0 \leq l \leq 2k} \left| h^{i-l} \right|_{L^\infty} \sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_{s}(v)(t-s)} \frac{1}{1 + |v|} ds \\
&\leq C \frac{1}{1 + |v|} \sup_{0 \leq l \leq 2k} \left| h^{i-l} \right|_{L^\infty} \leq \frac{C}{N} \sup_{0 \leq l \leq 2k} \left| h^{i-l} \right|_{L^\infty},
\end{align*}
\]

where we have used the fact \(|v| \equiv |v_l| \) for \( l = 0, 1, \cdots \).

Case 2. For either \(|v| \leq N, |v'| \geq 2N \) or \(|v'| \leq 2N, |v''| \geq 3N \), noting \(|v_l| = |v| \) and \(|v'_j| = |v'| \) we get either \(|v_l - v'| \geq N\) or \(|v'_j - v''| \geq N\), then either one of the following is valid for some small positive constant \( 0 < c_1 < \frac{1}{32} \):

\[
\begin{align*}
|k_w(v_l, v')| &\leq e^{-c_1 N^2} |k_w(v_l, v')| \exp \left( c_1 |v_l - v'|^2 \right), \\
|k_w(v'_j, v'')| &\leq e^{-c_1 N^2} |k_w(v'_j, v'')| \exp \left( c_1 |v'_j - v''|^2 \right),
\end{align*}
\] (3.16)

which, together with (2.1), (3.8), yields that

\[
\begin{align*}
\int_{\mathbb{R}^3} |k_w(v_l, v') e^{c_1 |v_l - v'|^2}|dv' &\leq \frac{C}{1 + |v|}, \\
\int_{\mathbb{R}^3} |k_w(v'_j, v'') e^{c_1 |v'_j - v''|^2}|dv'' &\leq \frac{C}{1 + |v'|}.
\end{align*}
\] (3.17)

Using (3.16)–(3.17), one has

\[
\begin{align*}
&\sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_{s}(v)(t-s)} \left\{ \int_{|v| \leq N, |v'| \geq 2N} + \int_{|v'| \leq 2N, |v''| \geq 3N} \right\} (\cdots)dv'' ds_1 dv' ds \\
&\leq C e^{-c_1 N^2} \sup_{0 \leq l \leq 2k} \left| h^{i-1} \right|_{L^\infty} \leq \frac{C}{N} \sup_{0 \leq l \leq 2k} \left| h^{i-1} \right|_{L^\infty},
\end{align*}
\]
Case 3. For either $|v| \leq N, |v'| \leq 2N, |v''| \leq 3N$, this is the last remaining case. We denote $D = \{ |v'| \leq 2N, |v''| \leq 3N \}$. Noting $\nu_0(v) \geq \nu_0$, the corresponding part is bounded by

$$
\sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_0(t-s)} \int_D |k_w(v_l,v')k_w(v_j',v'')|dv'dv'\,ds
$$

$$
\times \sum_{j=0}^{k-1} \left( \int_{t_{j+1}}^{t_{j+1}'} e^{-\nu_0(s-s_1)}|h_i^{l-j-1}(X_{cl}(s_1),v'')|ds_1 \right) \leq \sum_{l=0}^{k-1} \int_{t_l}^{t_{l+1}} e^{-\nu_0(t-s)} \int_D |k_w(v_l,v')k_w(v_j',v'')|dv'dv'\,ds
$$

$$
\times \sum_{j=0}^{k-1} \left( \int_{t_{j+1}}^{t_{j+1}'} e^{-\nu_0(s-s_1)}|h_i^{l-j-1}(X_{cl}(s_1),v'')|ds_1 \right) + C \frac{k}{N} \sup_{0 \leq t \leq 2k} \|h^{l-1}\|_{L^\infty_v}.
$$

It follows from (2.1) that

$$
\int_D \sum_{j=0}^{k-1} \int_{t_{j+1}}^{t_{j+1}'} e^{-\nu_0(s-s_1)}|k_w(v_l,v')k_w(v_j',v'')|^2 ds_1 dv'dv'' \leq C.
$$

Define $y := x_j' - v_{j,3}'(t_j' - s_1)$. We have $x_j' = 0$ or $d$ and $v_{j,3}' = (-1)^j v_{0,3}'$. For $t_j' = t_{j+1}'(s_1; s_1, X_{cl}(s), v')$, it holds that

$$
s - t_j' = \begin{cases} 
\frac{X_{cl}(s)}{|v_{0,3}'|} + (j - 1) \frac{d}{|v_{0,3}'|}, & \text{for } v_{0,3}' > 0, \\
\frac{d - X_{cl}(s)}{|v_{0,3}'|} + (j - 1) \frac{d}{|v_{0,3}'|}, & \text{for } v_{0,3}' < 0,
\end{cases}
$$

which yields that

$$
y = \begin{cases} 
x_j' - (-1)^j \left\{ v_{0,3}'(s - s_1) - [X_{cl}(s) + (j - 1)d] \right\}, & \text{for } v_{0,3}' > 0, \\
x_j' - (-1)^j \left\{ v_{0,3}'(s - s_1) + [jd - X_{cl}(s)] \right\}, & \text{for } v_{0,3}' > 0.
\end{cases}
$$

Since $x_j' = 0$ or $d$, which is independent of $v_{0,3}'$, thus we have

$$
\left| \frac{dy}{dv_{0,3}'} \right| = (s - s_1) \geq \frac{1}{N}, \quad \text{for } s_1 \in \left[ t_{j+1}', t_j' - \frac{1}{N} \right],
$$

which yields that

$$
\left( \int_D \sum_{j=0}^{k-1} \int_{t_{j+1}}^{t_{j+1}'} e^{-\nu_0(s-s_1)}|h_i^{l-j-1}(x_j' - v_{j,3}'(t_j' - s_1), v'')|^2 ds_1 dv'dv'' \right)^{\frac{1}{2}}
$$
\[ \leq C_N k \sup_{0 \leq l \leq 2k} \| \frac{h^{i-l}}{w} \|_{L^2_x,v}. \]

Then the RHS of (3.18) is bounded by
\[ \frac{C}{N} k \sup_{0 \leq l \leq 2k} \| h^{i-l} \|_{L^\infty_x} + C_k, N \sup_{0 \leq l \leq 2k} \| \frac{h^{i-l}}{w} \|_{L^2_x,v}. \]

Combining above estimates, we obtain
\[ \| h^{i+1} \|_{L^\infty_x} + | h^{i+1} |_{L^\infty(\gamma_+)} \leq C \left( e^{-\frac{1}{2} v_0 dk} + \frac{k}{N} \right) \sup_{0 \leq l \leq 2k} \left\{ \| h^{i-l} \|_{L^\infty_x} + | h^{i-l} |_{L^\infty(\gamma_+)} \right\} \]
\[ + C_{N,k} \sup_{0 \leq l \leq 2k} \left\| \frac{h^{i-l}}{w} \right\|_{L^2_x,v} + C_k (\| \nu^{-1} w g \|_{L^\infty_x} + | w r |_{L^\infty(\gamma_-)}). \]

First taking \( k \) large, and then letting \( N \) suitably large so that
\[ C \left( e^{-\frac{1}{2} v_0 k} + \frac{k}{N} \right) \leq \frac{1}{8}, \]
which implies (3.13). This completes the proof of Lemma 3.3.

3.2 Approximate Solutions and Uniform Estimate

Now we are in a position to construct solutions to (3.6) or equivalently (3.7). First of all, we consider the following approximate problem
\[
\begin{cases}
\varepsilon f^n + v_3 \partial_x f^n + \nu(v) f^n - K f^n = g, \\
f^n(x,v)|_{\gamma_-} = (1 - \frac{1}{n}) f^n(x,R_x v),
\end{cases} \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \tag{3.19}
\]
where \( \varepsilon \in (0,1] \) is arbitrary and \( n > 1 \) is an integer. For later use, we choose \( n_0 > 1 \) large enough such that
\[ \frac{1}{8} \left( 1 - \frac{1}{n} \right)^{-2k_0 - 1} \leq \frac{1}{2} \]
for any \( n \geq n_0 \), where \( k_0 > 0 \) is the one fixed in Lemma 3.3.

**Lemma 3.4.** Let \( \varepsilon > 0 \), \( d \geq 1 \), \( n \geq n_0 \), and \( \beta \geq 3 \). Assume \( \| \nu^{-1} w g \|_{L^\infty_x} < \infty \). Then there exists a unique solution \( f^n \) to (3.19) satisfying
\[ \| w f^n \|_{L^\infty_x} + | w f^n |_{L^\infty(\gamma_+)} \leq C_{\varepsilon,n} \| \nu^{-1} w g \|_{L^\infty_x}, \]
where the positive constant \( C_{\varepsilon,n} > 0 \) depends only on \( \varepsilon \) and \( n \). Moreover, if \( g \) is continuous in \( \Omega_d \times \mathbb{R}^3 \), then \( f^n \) is continuous away from grazing set \( \gamma_0 \).

**Proof.** We consider the solvability of the following boundary value problem
\[
\begin{cases}
\mathcal{L}_\lambda f := \varepsilon f + v_3 \partial_x f + \nu(v) f - \lambda K f = g, \\
f(x,v)|_{\gamma_-} = (1 - \frac{1}{n}) f(x,R_x v),
\end{cases} \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \tag{3.20}
\]
for \( \lambda \in [0,1] \). For brevity we denote \( \mathcal{L}_\lambda^{-1} \) to be the solution operator associated with the problem, meaning that \( f := \mathcal{L}_\lambda^{-1} g \) is a solution to the BVP (3.20). Our idea is to prove the
existence of $L_{0}^{-1}$, and then extend to obtain the existence of $L_{1}^{-1}$ by a continuous argument on $\lambda$. Since the proof is very long, we split it into several steps.

**Step 1.** In this step, we prove the existence of $L_{0}^{-1}$. We consider the following approximate sequence

\[
\begin{aligned}
\begin{cases}
L_{0}f^{i+1} = \varepsilon f^{i+1} + v_{3} \partial_{x} f^{i+1} + \nu(v) f^{i+1} = g, \\
(f^{i+1}(x,v)|_{\gamma_{-}} = (1 - \frac{1}{n}) f^{i}(x,R_{x} v),
\end{cases}
\end{aligned}
\]  

(3.21)

for $i = 0, 1, 2, \cdots$, where we have set $f^{0} = 0$. We will construct $L_{\infty}$ solutions to (3.21) for $i = 0, 1, 2, \cdots$, and establish uniform $L_{\infty}$-estimates.

Firstly, we will solve inductively the linear equation (3.21) by the method of characteristics. Let $h^{i+1}(x,v) = w(v)f^{i+1}(x,v)$. For almost every $(x,v) \in \bar{\Omega}_{d} \times \mathbb{R}^{3}\setminus(\gamma_{0} \cup \gamma_{-})$, one can write

\[
h^{i+1}(x,v) = e^{-(\nu(x,v))t_{b}} \cdot (1 - \frac{1}{n}) w(v)f^{i}(x_{1}, R_{x_{1}} v)
\]

\[
+ \int_{t_{b}}^{t} e^{-(\nu(x,v))(t-s)} (w g)(x - v_{0,3}(t-s), v) ds,
\]

(3.22)

where $t_{b} = t - t_{b}$. We consider (3.22) with $i = 0$. Noting $h^{0} = 0$, then it is straightforward to see that

\[
\|h^{1}\|_{L_{w}^{\infty}} + \|h^{1}\|_{L_{\infty}}(\gamma_{+}) \leq C\|\nu^{-1} w g\|_{L_{w}^{\infty}} < \infty.
\]

Therefore we have obtained the solution to (3.21) with $i = 0$. Assume that we have already solved (3.21) for $i \leq l$ and obtained

\[
\|h^{l+1}\|_{L_{w}^{\infty}} + \|h^{l+1}\|_{L_{\infty}}(\gamma_{+}) \leq C_{l+1}\|\nu^{-1} w g\|_{L_{w}^{\infty}} < \infty.
\]

(3.23)

We now consider (3.21) for $i = l + 1$. Noting (3.23), then we can solve (3.21) by using (3.22) with $i = l + 1$. We still need to prove $h^{l+2} \in L_{\infty}$. Indeed, it follows from (3.22) that

\[
\|h^{l+2}\|_{L_{w}^{\infty}} + \|h^{l+2}\|_{L_{\infty}}(\gamma_{+}) \leq C\|h^{l+1}\|_{L_{\infty}}(\gamma_{+}) + C\|\nu^{-1} w g\|_{L_{w}^{\infty}}
\]

\[
\leq C_{l+2}\|\nu^{-1} w g\|_{L_{w}^{\infty}} < \infty.
\]

Therefore, inductively we have solved (3.21) for $i = 0, 1, 2, \cdots$ and obtained

\[
\|h^{i}\|_{L_{w}^{\infty}} + \|h^{i}\|_{L_{\infty}}(\gamma_{+}) \leq C_{i}\|\nu^{-1} w g\|_{L_{w}^{\infty}} < \infty,
\]

(3.24)

for $i = 0, 1, 2, \cdots$. The positive constant $C_{i}$ may increase to infinity as $i \to \infty$. Here, we emphasize that we first need to know the sequence $\{h^{i}\}_{i=0}^{\infty}$ is in $L_{w,v}^{\infty}$-space, otherwise one can not use Lemma 3.3 to get uniform $L_{w,v}$ estimates.

Since $\Omega_{d}$ is a convex domain, let $(x,v) \in \bar{\Omega}_{d} \times \mathbb{R}^{3}\setminus(\gamma_{0} \cup \gamma_{0})$, then it is easy to check that $t_{b}(x,v)$ and $x_{b}(x,v)$ are continuous. Therefore if $g$ and $r$ are continuous, we conclude that $f^{i}(x,v)$ is continuous away from grazing set.

Secondly, in order to take the limit $i \to \infty$, one has to get some uniform estimates. Multiplying (3.21) by $f^{i+1}$ and integrating the resultant equality over $\Omega_{d} \times \mathbb{R}^{3}$, one obtains that

\[
\varepsilon \|f^{i+1}\|_{L_{w}^{2}}^{2} + \|f^{i+1}\|_{L_{v}^{2}}^{2} + \|f^{i+1}\|_{\nu}^{2}
\]

\[
\leq \frac{1}{2} \left( 1 - \frac{1}{n} \right) ^{2} \|f^{i}\|_{L_{w}^{2}}^{2} + C\|g\|_{L_{w}^{2}}^{2} + \frac{1}{4} \|f^{i+1}\|_{\nu}^{2}.
\]

(3.25)
Then, from (3.25), we have

\[ 2\varepsilon \| f^{i+1} \|_{L^2_{\gamma^+}}^2 + \| f^{i+1} \|^2_{L^2_{\gamma^+}} + \| f^i \|_{L^2_{\gamma^+}}^2 \leq \left( 1 - \frac{1}{n} \right)^2 \| f^i \|^2_{L^2_{\gamma^+}} + C \| g \|^2_{L^2_{\gamma^+}}. \]

Now we take the difference \( f^{i+1} - f^i \) in (3.21), then by similar energy estimate as above, we obtain

\begin{align*}
2\varepsilon \| f^{i+1} - f^i \|_{L^2_{\gamma^+}}^2 + \| f^{i+1} - f^i \|^2_{L^2_{\gamma^+}} + 2\| f^{i+1} - f^i \|_{L^2_{\gamma^+}}^2 & \leq \left( 1 - \frac{1}{n} \right)^2 \| f^i \|^2_{L^2_{\gamma^+}} + C \| g \|^2_{L^2_{\gamma^+}} \leq C \left( 1 - \frac{1}{n} \right)^{2i} \| g \|^2_{L^2_{\gamma^+}} < \infty. \tag{3.26}
\end{align*}

Noting \( 1 - \frac{1}{n} < 1 \), thus \( \{ f^i \}_{i=0}^\infty \) is a Cauchy sequence in \( L^2 \), i.e.,

\[ \| f^i - f^j \|_{L^2_{\gamma^+}}^2 + \| f^i - f^j \|_{L^2_{\gamma^+}}^2 \to 0, \quad \text{as} \ i, j \to \infty. \]

And we also have, for \( i = 0, 1, 2, \cdots \), that

\[ \| f^i \|^2_{L^2_{\gamma^+}} + \| f^i \|^2_{L^2_{\gamma^+}} \leq C \| g \|^2_{L^2_{\gamma^+}}. \tag{3.27} \]

Next we consider the uniform \( L^\infty_{x;v} \) estimate. Here we point out that Lemma 3.3 still holds by replacing 1 with \( 1 - \frac{1}{n} \) in the boundary condition, and the constants in Lemma 3.3 do not depend on \( n \geq 1 \). Thus we apply Lemma 3.3 to obtain that

\begin{align*}
& \| h^{i+1} \|^2_{L^\infty_{x;v}} + | h^{i+1} |_{L^\infty_{\gamma^+}} \leq \frac{1}{8} \sup_{0 \leq t \leq 2k_0} \{ \| h^{i-1} \|^2_{L^\infty_{x;v}} + | h^{i-1} |_{L^\infty_{\gamma^+}} \} + C \| \nu^{-1} w g \|_{L^\infty_{x;v}} + C \sup_{0 \leq t \leq 2k_0} \| f^{i-1} \|^2_{L^2_{x;v}} \\
& \quad \leq \frac{1}{8} \sup_{0 \leq t \leq 2k_0} \{ \| h^{i-1} \|^2_{L^\infty_{x;v}} + | h^{i-1} |_{L^\infty_{\gamma^+}} \} + C_d \| \nu^{-1} w g \|_{L^\infty_{x;v}},
\end{align*}

where we have used (3.27) in the second inequality. Now we apply Lemma 2.1 to obtain that for \( i \geq 2k_0 + 1 \),

\begin{align*}
& \| h^i \|^2_{L^\infty_{x;v}} + | h^i |_{L^\infty_{\gamma^+}} \leq \frac{1}{8} \left( \frac{1}{8} \right)^{n-1} \sup_{0 \leq t \leq 2k_0} \{ \| h^1 \|^2_{L^\infty_{x;v}} + | h^1 |_{L^\infty_{\gamma^+}} , \cdots , \| h^{2k_0} \|^2_{L^\infty_{x;v}} + | h^{2k_0} |_{L^\infty_{\gamma^+}} \} + 8 + 2k_0 + \frac{C_d \| \nu^{-1} w g \|_{L^\infty_{x;v}}}{7} \\
& \quad \leq C_{k_0, d} \| \nu^{-1} w g \|_{L^\infty_{x;v}}, \tag{3.28}
\end{align*}

where we have used (3.24) in the second inequality. Hence it follows from (3.28) and (3.24) that

\[ \| h^i \|^2_{L^\infty_{x;v}} + | h^i |_{L^\infty_{\gamma^+}} \leq C_{k_0, d} \| \nu^{-1} w g \|_{L^\infty_{x;v}}, \quad \text{for} \ i \geq 1. \tag{3.29} \]

Taking the difference \( h^{i+1} - h^i \) and then applying Lemma 3.3 to \( h^{i+1} - h^i \), we have that for \( i \geq 2k_0 \),

\begin{align*}
& \| h^{i+2} - h^{i+1} \|^2_{L^2_{x;v}} + | h^{i+2} - h^{i+1} |_{L^\infty_{\gamma^+}} \leq \frac{1}{8} \max_{0 \leq t \leq 2k_0} \{ \| h^{i+1} - h^i \|^2_{L^\infty_{x;v}} + | h^{i+1} - h^i |_{L^\infty_{\gamma^+}} \}
\end{align*}
\[ + C \sup_{0 \leq t \leq 2k_0} \left\{ \| f^{i+1} - f^i \|_{L^2_{\gamma,v}} \right\} \]
\[ \leq \frac{1}{8} \max_{0 \leq i \leq 2k_0} \left\{ \| h^{i+1} - h^i \|_{L^\infty_{\gamma,v}} + \| h^{i+1} - h^i \|_{L^\infty(\gamma_+)} + C_{k_0,d} \| \nu^{-1} w g \|_{L^\infty_{\gamma,v}} \right\} \]
\[ \leq \frac{1}{8} \max_{0 \leq i \leq 2k_0} \left\{ \| h^{i+1} - h^i \|_{L^\infty_{\gamma,v}} + \| h^{i+1} - h^i \|_{L^\infty(\gamma_+)} + C_{k_0,d} \| \nu^{-1} w g \|_{L^\infty_{\gamma,v}} \right\} \]

for \( i \geq 2k_0 + 1 \). Then (3.31) implies immediately that \( \{ h^i \}_{i=0}^{\infty} \) is a Cauchy sequence in \( L^\infty_{\gamma,v} \), i.e., there exists a limit function \( h \in L^\infty_{\gamma,v} \) so that \( \| h^i - h \|_{L^\infty_{\gamma,v}} + \| h^i - h \|_{L^\infty(\gamma_+)} \to 0 \) as \( i \to \infty \).

Thus we obtained a function \( f := \frac{h}{\nu} \) solves

\[ \begin{cases} 
L_0 f = \varepsilon f + v_3 \partial_x f + \nu(v) f = g, \\
\left. f(x,v) \right|_{\gamma_-} = \left( 1 - \frac{1}{n} \right) f(x,R_x v),
\end{cases} \]

with \( n \geq n_0 \) large enough. Moreover, from (3.29), there exists a constant \( C_{k_0,d} \) such that

\[ \| h \|_{L^\infty_{\gamma,v}} + \| h \|_{L^\infty(\gamma)} \leq C_{k_0,d} \| \nu^{-1} w g \|_{L^\infty_{\gamma,v}}. \]

**Step 2. A priori estimates.** For any given \( \lambda \in [0,1] \), let \( f^n \) be the solution of (3.20), i.e.,

\[ \begin{cases} 
L_\lambda f^n = \varepsilon f^n + v_3 \partial_x f^n + \nu(v) f^n - \lambda K f^n = g, \\
\left. f^n(x,v) \right|_{\gamma_-} = \left( 1 - \frac{1}{n} \right) f^n(x,R_x v).
\end{cases} \] (3.32)

Moreover we also assume that \( \| w f^n \|_{L^\infty_{\gamma,v}} + \| w f^n \|_{L^\infty(\gamma)} < \infty \). Firstly, we shall consider a priori \( L^2 \)-estimates. Multiplying (3.32) by \( f^n \), one has that

\[ \varepsilon \| f^n \|_{L^2_{\gamma,v}}^2 + \frac{1}{2} \| f^n \|_{L^2(\gamma_+)}^2 + \frac{1}{2} \| f^n \|_{L^2(\gamma_-)}^2 + \| f^n \|_{L^2_{\gamma,v}}^2 \]
\[ \leq \lambda \langle K f^n, f^n \rangle + \frac{\varepsilon}{4} \| f^n \|_{L^2_{\gamma,v}}^2 + C \| g \|_{L^2_{\gamma,v}}^2. \] (3.33)

We note that \( \langle K f^n, f^n \rangle \geq 0 \), which implies that

\[ \langle K f^n, f^n \rangle \leq \| f^n \|_\nu. \] (3.34)

On the other hand, it follows from (3.32) that

\[ \frac{1}{2} \| f^n \|_{L^2(\gamma_+)}^2 + \frac{1}{2} \| f^n \|_{L^2(\gamma_-)}^2 = \frac{1}{2} \| f^n \|_{L^2(\gamma_+)}^2 \left[ 1 - \left( 1 - \frac{1}{n} \right)^2 \right] \geq 0. \] (3.35)
For any $f$ in $(\mathcal{L}^{-1})$ and $\nu^{-1}$, we define $L$ for us to extend $C_{\epsilon, k_o, d}$ to (3.38). We point out that the constant $\lambda$ which yields immediately that $L_{\epsilon, k_o, d}$ is a contraction mapping for $\nu^{-1} w g$ in (3.36), (3.37), (3.38) and (3.39) does not depend on $\lambda \in [0, 1]$. This property is crucial for us to extend $\mathcal{L}_0^{-1}$ to $\mathcal{L}_1^{-1}$ by a bootstrap argument.

**Step 3.** In this step, we shall prove the existence of solution $f^n$ to (3.20) for sufficiently small $0 < \lambda \ll 1$, i.e., to prove the existence of operator $L_{\epsilon, k_o, d}$ in (3.36). Firstly, we define the Banach space

$$X := \{ f = f(x, v) : \nu^{-1} w f \in L_{x,v}^{\infty}, w f \in L_{x,v}^{\infty}, \text{ and } f(x, v)|_{\gamma_+} = (1 - \frac{1}{n}) f(x, R_x v) \}.$$

Now we define

$$T_{\lambda} f = \mathcal{L}_0^{-1}(\lambda K f + g).$$

For any $f_1, f_2 \in X$, by using (3.39), we have that

$$\| w(T_{\lambda} f_1 - T_{\lambda} f_2)\|_{L_{x,v}^{\infty}} + \| w(T_{\lambda} f_1 - T_{\lambda} f_2)\|_{L_{x,v}^{\infty}} \leq C_{\epsilon, k_o, d} \| w(T_{\lambda} f_1 - T_{\lambda} f_2)\|_{L_{x,v}^{\infty}} + \| w(T_{\lambda} f_1 - T_{\lambda} f_2)\|_{L_{x,v}^{\infty}} \leq C_{\epsilon, k_o, d} \| w(T_{\lambda} f_1 - T_{\lambda} f_2)\|_{L_{x,v}^{\infty}}.$$

We take $\lambda > 0$ sufficiently small such that $\lambda C_{\epsilon, k_o, d} \leq 1/2$, then $T_{\lambda} : X \rightarrow X$ is a contraction mapping for $\lambda \in [0, \lambda_*]$. Thus $T_{\lambda}$ has a fixed point, i.e., $f^\lambda \in X$ such that

$$f^\lambda = T_{\lambda} f^\lambda = \mathcal{L}_0^{-1}(\lambda K f^\lambda + g),$$

which yields immediately that

$$\mathcal{L}_{\lambda} f^\lambda = \epsilon f^\lambda + {v_3 \partial_x} f^\lambda + \nu f^\lambda - \lambda K f^\lambda = g.$$
Hence, for any \( \lambda \in [0, \lambda_*] \), we have solved (3.20) with \( f^\lambda = \mathcal{L}_\chi^{-1}g \in X \). Therefore we have obtained the existence of \( \mathcal{L}_\chi^{-1} \) for \( \lambda \in [0, \lambda_*] \). Moreover the operator \( \mathcal{L}_\chi^{-1} \) has the properties (3.36), (3.37), (3.38) and (3.39).

Next we define

\[
T_{\lambda, + \lambda} f = \mathcal{L}_\chi^{-1}(\lambda K f + g).
\]

Noting the estimates for \( \mathcal{L}_\chi^{-1} \) are independent of \( \lambda_* \). By similar arguments, we can prove \( T_{\lambda, + \lambda} : X \to X \) is a contraction mapping for \( \lambda \in [0, \lambda_*] \). Then we obtain the existence of operator \( \mathcal{L}_{\lambda, + \lambda}^{-1} \), and (3.36), (3.37), (3.38) and (3.39). Step by step, we can finally obtain the existence of operator \( \mathcal{L}_1^{-1} \), and \( \mathcal{L}_1^{-1} \) satisfies the estimates in (3.36), (3.37), (3.38) and (3.39). The continuity is easy to obtain since the convergence of sequence under consideration is always in \( L^\infty \). Therefore we complete the proof of Lemma 3.4. □

**Lemma 3.5.** Let \( \varepsilon > 0, d \geq 1 \) and \( \beta \geq 3 \), and assume \( \|\nu^{-1}w g\|_{L^\infty} < \infty \). Then there exists a unique solution \( f^\varepsilon \) to solve the approximate linearized steady Boltzmann equation (3.6). Moreover, it satisfies

\[
\|w f^\varepsilon\|_{L^\infty_{x,v}} + |w f^\varepsilon|_{L^\infty(\gamma)} \leq C_{\varepsilon, d} \|\nu^{-1} w g\|_{L^\infty_{x,v}},
\]

(3.40)

where the positive constant \( C_{\varepsilon, d} > 0 \) depends only on \( \varepsilon \) and \( d \). Moreover, if \( g \) is continuous in \( \Omega_d \times \mathbb{R}^3 \), then \( f^\varepsilon \) is continuous away from the grazing set \( \gamma_0 \).

**Proof.** Let \( f^n \) be the solution of (3.19) constructed in Lemma 3.4 for \( n \geq n_0 \) with \( n_0 \) large enough. Multiplying (3.19) by \( f^n \), one obtains that

\[
\varepsilon \|f^n\|_{L^2_{x,v}}^2 + c_0 \|(I - P)f^n\|_V^2 \leq C_{\varepsilon} \|g\|_{L^2_{x,v}}^2.
\]

(3.41)

We apply (3.14) and use (3.41) to obtain

\[
\|w f^n\|_{L^\infty_{x,v}} + |w f^n|_{L^\infty(\gamma)} \leq C \left\{ \|\nu^{-1} w g\|_{L^\infty_{x,v}} + \|f^n\|_{L^2_{x,v}} \right\} \leq C_{\varepsilon, d} \|\nu^{-1} w g\|_{L^\infty_{x,v}}.
\]

(3.42)

Taking the difference \( f^{n1} - f^{n2} \) with \( n_1, n_2 \geq n_0 \), we know that

\[
\begin{aligned}
\varepsilon (f^{n1} - f^{n2}) + v_3 \partial_3 (f^{n1} - f^{n2}) + L (f^{n1} - f^{n2}) &= 0, \\
(f^{n1} - f^{n2})(x,v)\big|_{\gamma_+} &= \left(1 - \frac{1}{n_1}\right)(f^{n1} - f^{n2})(x,R_x v) + \left(\frac{1}{n_1} - \frac{1}{n_2}\right)f^{n2}(x,R_x v).
\end{aligned}
\]

(3.43)

Multiplying (3.43) by \( f^{n1} - f^{n2} \), and integrating it over \( \Omega_d \times \mathbb{R}^3 \), we can obtain

\[
\begin{aligned}
\varepsilon \|f^{n1} - f^{n2}\|_{L^2_{x,v}}^2 + c_0 \|(I - P)(f^{n1} - f^{n2})\|_V^2 &\leq C \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \int_{\gamma_-} |v_3 (|f^{n1}| + |f^{n2}|) f^{n2}| dv \\
&\leq C \left( \frac{1}{n_1} + \frac{1}{n_2} \right) (|f^{n1}|_{L^2(\gamma_+)} + |f^{n2}|_{L^2(\gamma_+)}) \\
&\leq C_{\varepsilon, d} \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \|\nu^{-1} w g\|_{L^\infty_{x,v}} \to 0
\end{aligned}
\]

(3.44)

as \( n_1, n_2 \to \infty \), where we have used the uniform estimate (3.42) in the last inequality. Applying (3.14) to \( f^{n1} - f^{n2} \) and using (3.44), then one has

\[
\|w(f^{n1} - f^{n2})\|_{L^\infty_{x,v}} + |w(f^{n1} - f^{n2})|_{L^\infty(\gamma)}
\]
Lemma 3.6. Let \( n_1, n_2 \to \infty \), which yields that \( w^n \) is a Cauchy sequence in \( L^\infty \). We denote \( f^\varepsilon = \lim_{n \to \infty} f^n \), then it is direct to check that \( f^\varepsilon \) is a solution to (3.6), and (3.40) holds. The continuity of \( f^\varepsilon \) is easy to obtain since the convergence of sequences is always in \( L^\infty \), and \( f^n \) is continuous away from the grazing set. Therefore we have completed the proof of Lemma 3.5. \( \square \)

From (3.2), (1.7), it is direct to check that the source term \( g \) in (3.6) satisfies

\[
\int_{\mathbb{R}^3} \left( 1, v_1 - u_1, v_2 - u_2, |v - u|^2 - 3 \right) \sqrt{\mu} g(x,v) dv = (0, 0, 0, 0). \tag{3.45}
\]

Multiplying (3.6) by \( \sqrt{\mu} \) and integrating over \([0, d] \times \mathbb{R}^3\) to obtain

\[
\int_0^d \int_{\mathbb{R}^3} \sqrt{\mu} f^\varepsilon(x,v) dv dx = - \int_{\mathbb{R}^3} v_3 \sqrt{\mu} f^\varepsilon(d,v) dv + \int_{\mathbb{R}^3} v_3 \sqrt{\mu} f^\varepsilon(0,v) dv = 0, \tag{3.46}
\]

where we have used the property of specular boundary condition in the second equality.

Similarly, multiplying (3.6) by \((v_1 - u_1, v_2 - u_2, |v - u|^2 - 3)\sqrt{\mu}, \) respectively, one gets

\[
\int_0^d \int_{\mathbb{R}^3} (v_1 - u_1, v_2 - u_2, |v - u|^2 - 3) \sqrt{\mu} f^\varepsilon(x,v) dv dx = 0. \tag{3.47}
\]

We denote

\[
P f^\varepsilon(x,v) = \{ a^\varepsilon(x) + b^\varepsilon \cdot (v - u) + c^\varepsilon(x)(|v - u|^2 - 3) \} \sqrt{\mu},
\]

then it follows from (3.46) and (3.47) that

\[
\int_0^d a^\varepsilon(x) dx = \int_0^d b_1^\varepsilon(x) dx = \int_0^d b_2^\varepsilon(x) dx = \int_0^d c^\varepsilon(x) dx = 0. \tag{3.48}
\]

Lemma 3.6. Let \( d \geq 1 \). Assume (3.45) and let \( f^\varepsilon \) be the solution of (3.6) constructed in Lemma 3.4, then it holds that

\[
\| Pf^\varepsilon \|^2_{L^2_{x,v}} \leq Cd^6 \left\{ \| (I - P) f^\varepsilon \|^2_{L^2_{x,v}} + \| g \|^2_{L^2_{x,v}} \right\}. \tag{3.49}
\]

Remark 3.7. By choosing suitable test function, Yin-Zhao\[16\] and Guo-Hwang-Jang-Ouyang\[12\] proved similar result for the time-dependent Boltzmann equation and Landau equation in three dimensional case, respectively. The Lemma 3.6 can be regarded as a one dimensional version of \([12, 16]\). We point out that the condition (3.45) or (3.48) is necessary when we construct the test function in one dimensional cases.

Proof. The weak formulation of (3.6) is

\[
\varepsilon \int_0^d \int_{\mathbb{R}^3} f^\varepsilon(x,v) \psi(x,v) dv dx - \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon(x,v) \partial_x \psi(x,v) dv dx \\
= - \int_{\mathbb{R}^3} v_3 f^\varepsilon(d,v) \psi(d,v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon(0,v) \psi(0,v) dv
\]
Hence the RHS of (3.50) is bounded by

\[- \int_0^d \int_{\mathbb{R}^3} Lf^\varepsilon(x, v) \psi(x, v) dv dx + \int_0^d \int_{\mathbb{R}^3} g(x, v) \psi(x, v) dv dx. \tag{3.50}\]

Similar as in [16] we choose some special test function \( \psi \) to calculate the macroscopic part of \( f^\varepsilon \).

**Step 1.** Estimate on \( c^\varepsilon \). Define

\[ \zeta_c(x) = -\int_0^x c^\varepsilon(z) dz. \]

Noting (3.48), it is easy to check that

\[ \zeta_c(x)|_{x=0,d} = 0 \quad \text{and} \quad \| \zeta_c \|_{L^2} \leq d\| c^\varepsilon \|_{L^2}. \tag{3.51} \]

We define the test function \( \psi \) in (3.50) to be

\[ \psi = \psi_c(x, v) = v_3(|v - u| - 5)\sqrt{\mu}\zeta_c(x). \]

Then the second term on LHS of (3.50) is estimated as

\[- \int_0^d \int_{\mathbb{R}^3} v_3 f(x, v) \partial_x \psi_c(x, v) dv dx
\]

\[= \int_0^d \int_{\mathbb{R}^3} [\mu \varepsilon + b^\varepsilon \cdot (v - u) + c^\varepsilon(x)(|v - u|^2 - 3)]v_3^2(|v - u|^2 - 5)\mu(v)c^\varepsilon(x) dv dx
\]

\[+ \int_0^d \int_{\mathbb{R}^3} (I - P) f^\varepsilon(x, v) v_3^2(|v - u|^2 - 5)\sqrt{\mu}c^\varepsilon(x) dv dx
\]

\[\geq 10\|c^\varepsilon\|^2_{L^2} - C\| (I - P) f^\varepsilon \|_{L^2} \geq 5\|c^\varepsilon\|^2_{L^2} - C\| (I - P) f^\varepsilon \|_{L^2}, \tag{3.52} \]

where we have used

\[ \int_{\mathbb{R}^3} (|v - u|^2 - 3)v_3^2(|v - u|^2 - 5)\mu(v) dv = 10, \quad \int_{\mathbb{R}^3} v_3^2(|v - u|^2 - 5)\mu(v) dv = 0. \tag{3.53} \]

By using (3.53), the first term on LHS of (3.50) is bounded as

\[ \varepsilon \left| \int_0^d \int_{\mathbb{R}^3} f^\varepsilon(x, v) \psi_c(x, v) dv dx \right| \leq C\varepsilon\| (I - P) f^\varepsilon \|_{L^2} \| \zeta_c \|_{L^2} \leq C\varepsilon d \| (I - P) f^\varepsilon \|_{L^2} \| c^\varepsilon \|_{L^2}. \]

Noting (3.51), it is direct to have that

\[- \int_{\mathbb{R}^3} v_3 f^\varepsilon(d, v) \psi_c(d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon(0, v) \psi_c(0, v) dv = 0. \]

Hence the RHS of (3.50) is bounded by

\[ \text{RHS of (3.50)} \leq C d \| (I - P) f^\varepsilon \|_{L^2} \| c^\varepsilon \|_{L^2}. \tag{3.54} \]

Combining (3.52)–(3.54), one obtains

\[ \| c^\varepsilon \|^2_{L^2} \leq C d^2 (\| (I - P) f^\varepsilon \|_{L^2}^2 + \| g \|^2_{L^2}). \tag{3.55} \]

**Step 2.** Estimate on \( b^\varepsilon \). We define

\[ \zeta_{b,i}(x) = -\int_0^x b_{i}^\varepsilon(z) dz, \quad i = 1, 2, 3. \]
By using (3.48), it holds that
\[
\zeta_{b,i}(x)|_{x=0,d} = 0, \quad i = 1, 2, \quad \zeta_{b,3}(0) = 0,
\]
\[
||\zeta_{b,i}||_{L^2} \leq d ||b_i||_{L^2}, \quad i = 1, 2, 3. \tag{3.56}
\]

Now we take the test function \(\psi\) in (3.50) to be
\[
\psi = \psi_b(x, v) = \sum_{i=1}^{3} \zeta_{b,i}(x)(v_i - u_i)v_3\sqrt{\mu} - \frac{1}{2} \zeta_{b,3}(x)(|v| - 1)\sqrt{\mu}.
\]

Then the second term on LHS of (3.50) is controlled as
\[
- \int_0^d \int_{\mathbb{R}^3} v_3f^\varepsilon(x, v)\partial_x\psi_b(x, v) dv dx
\]
\[
= \int_0^d \int_{\mathbb{R}^3} [a^\varepsilon + b^\varepsilon \cdot (v - u) + c^\varepsilon(x)(|v - u|^2 - 3)]
\]
\[
\times \left[ \sum_{i=1}^{3} b_i^\varepsilon(x)(v_i - u_i)v_3^2 - \frac{1}{2} b_3^\varepsilon(x)(|v - u|^2 - 1)v_3 \right] \mu(v) dv dx
\]
\[
- \int_0^d \int_{\mathbb{R}^3} v_3(I - P)f^\varepsilon(x, v)\partial_x\psi_b(x, v) dv dx
\]
\[
= \int_0^d \int_{\mathbb{R}^3} \sum_{i=1}^{3} |b_i^\varepsilon(x)(v_i - u_i)|^2 v_3^2 \mu(v) - \frac{1}{2} |b_3^\varepsilon(x)|^2 (|v - u|^2 - 1)v_3^2 \mu(v) dv dx
\]
\[
- \int_0^d \int_{\mathbb{R}^3} v_3(I - P)f^\varepsilon(x, v)\partial_x\psi_b(x, v) dv dx
\]
\[
\geq ||(b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)||_{L^2}^2 - C||Pf^\varepsilon\nu||,(b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)\|_{L^2}
\]
\[
\geq \frac{3}{4}||((b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)||_{L^2}^2 - C||(I - P)f^\varepsilon\nu||_{L^2}^2, \tag{3.57}
\]

where we have used
\[
\int_{\mathbb{R}^3} (v_i - u_i)^2 v_3^2 \mu(v) dv = 1, \quad i = 1, 2, \quad \int_{\mathbb{R}^3} v_3^4 \mu(v) - \frac{1}{2} v_3^2 (|v - u|^2 - 1) \mu(v) dv = 1.
\]

The first term on LHS of (3.50) is bounded as
\[
\varepsilon \left| \int_0^d \int_{\mathbb{R}^3} f^\varepsilon(x, v)\psi_b(x, v) dv dx \right|
\]
\[
\leq C \varepsilon d ||c^\varepsilon||_{L^2} ||(b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)||_{L^2} + C d ||(I - P)f^\varepsilon\nu||,(b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)\|_{L^2}
\]
\[
\leq \frac{1}{4} ||(b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon)||_{L^2}^2 + C \varepsilon^2 d^2 (||g^\varepsilon\|_{L^2}^2 + ||c^\varepsilon||_{L^2}^2).
\]

For the boundary terms on RHS of (3.50), it follows from (3.56) that
\[
- \int_{\mathbb{R}^3} v_3 f^\varepsilon (d, v) \psi_b (d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon (0, v) \psi_b (0, v) dv
\]
\[
= - \zeta_{b,3}(d) \cdot \int_{\mathbb{R}^3} v_3 f^\varepsilon (d, v) \cdot \left[ v_3^2 - \frac{1}{2} (|v - u|^2 - 1) \right] \sqrt{\mu} dv = 0,
\]
due to the fact that $f(d, v)$ is an even function with respect to $v_3$. Thus the terms on RHS of (3.50), we have
\[
\text{RHS of (3.50)} \leq Cd\|\langle b_1^*, b_2^*, b_3^* \rangle\|_{L^2_\nu} \{ \| (I - P) f^\varepsilon \|_\nu + \| g \|_{L^2_\nu, \varepsilon} \}. \tag{3.58}
\]
Then combining (3.57)–(3.58), one obtains that
\[
\|\langle b_1^*, b_2^*, b_3^* \rangle\|_{L^2_\nu} \leq Cd\{ \| (I - P) f^\varepsilon \|_\nu^2 + \| g \|_{L^2_\nu, \varepsilon}^2 \}. \tag{3.59}
\]

Step 3. Estimate on $a^\varepsilon$. Define
\[
\zeta_\alpha(x) = - \int_0^x a^\varepsilon(z)dz.
\]
Noting (3.48), it is easy to check that
\[
\zeta_\alpha(x)|_{x=0, d} = 0, \quad \|\zeta_\alpha\|_{L^2} \leq d\|a^\varepsilon\|_{L^2_\nu}.
\tag{3.60}
\]
We define the test function $\psi$ in (3.50) to be
\[
\psi = \psi_\alpha(x, v) = -((|v - u|^2 - 10)\sqrt{\mu}v_3\zeta_\alpha(x).
\]
Then the second term on LHS of (3.50) is estimated as
\[
- \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon(x, v) \partial_x \psi_\alpha(x, v) dv dx
\]
\[
= \int_0^d \int_{\mathbb{R}^3} \{ a^\varepsilon + b^* \cdot (v - u) + c^\varepsilon(x)(|v - u|^2 - 3) \} \cdot \alpha^\varepsilon(x)(|v - u|^2 - 10)v_3^2 \mu(v)
\]
\[
- \int_0^d \int_{\mathbb{R}^3} v_3(I - P)f^\varepsilon(x, v) \partial_x \psi_\alpha(x, v) dv dx
\]
\[
\geq 5\|a^\varepsilon\|_{L^2_\nu}^2 - C\|\langle I - P \rangle f^\varepsilon \|_{\nu} \|a^\varepsilon\|_{L^2} \geq 4\|a^\varepsilon\|_{L^2_\nu}^2 - C\|\langle I - P \rangle f^\varepsilon \|_{\nu}^2, \tag{3.61}
\]
where we have used
\[
\int_{\mathbb{R}^3} (|v - u|^2 - 3)v_3^2 \cdot (|v - u|^2 - 10)\mu(v) dv = 0.
\]
A direct calculation shows that
\[
\varepsilon \left| \frac{d}{dx} \int_{\mathbb{R}^3} f^\varepsilon(x, v) \psi_\alpha(x, v) dv dx \right|
\]
\[
\leq \varepsilon \left| \int_{\mathbb{R}^3} v_3(x) \zeta_\alpha(x) v_3^2 (|v - u|^2 - 10)\mu(v) dv dx \right| + C\varepsilon d\|\|\langle I - P \rangle f^\varepsilon \|_{\nu} \|a^\varepsilon\|_{L^2_\nu}
\]
\[
\leq \|a^\varepsilon\|_{L^2_\nu}^2 + Cd^2\varepsilon^2 \{ \|\langle I - P \rangle f^\varepsilon \|_{\nu} + \|b^\varepsilon\|_{L^2_\nu}^2 \}.
\]
By using (3.60), one has
\[
- \int_{\mathbb{R}^3} v_3 f^\varepsilon(d, v) \psi_\alpha(d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon(0, v) \psi_\alpha(0, v) dv = 0.
\]
Hence, for the RHS of (3.50), it holds that
\[
\text{RHS of (3.50)} \leq Cd\|a^\varepsilon\|_{L^2_\nu} \{ \|\langle I - P \rangle f^\varepsilon \|_{\nu} + \| g \|_{L^2_\nu, \varepsilon} \}. \tag{3.62}
\]
Combining (3.61)–(3.62) and using (3.59), we obtain
\[
\|a^\varepsilon\|_{L^2_\nu}^2 \leq Cd\{ \|\langle I - P \rangle f^\varepsilon \|_{\nu}^2 + \| g \|_{L^2_\nu}^2 \}. \tag{3.63}
\]
Therefore, (3.49) follows directly from (3.63), (3.59) and (3.55).
**Lemma 3.8.** Let $d \geq 1$, $\beta \geq 3$. Assume (3.45) and $\|\nu^{-1}wg\|_{L^\infty_{x,v}} < \infty$. Then there exists a unique solution $f = f(x,v)$ to the linearized steady Boltzmann equation

$$
\begin{aligned}
  v_3 \partial_x f + Lf &= g, \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \\
  f(x,v)|_{\gamma_-} &= f(x,R_x v),
\end{aligned}
$$

(3.64)

satisfying (3.48) and

$$
\|wf\|_{L^\infty_{x,v}} + |wf|_{L^\infty(\gamma)} \leq C_d\|\nu^{-1}wg\|_{L^\infty_{x,v}}.
$$

(3.65)

Moreover, if $g$ is continuous in $\Omega_d \times \mathbb{R}$, then $f$ is continuous away from the grazing set $\gamma_0$.

**Proof.** Let $f^\varepsilon$ be the solution of (3.6) constructed in Lemma 3.5 for $\varepsilon > 0$. Multiplying the first equation of (3.6) by $f^\varepsilon$ and integrating the resultant equation over $\Omega_d \times \mathbb{R}^3$, we have

$$
\varepsilon\|f^\varepsilon\|^2_{L^2_{x,v}} + c_0\| (I - P)f^\varepsilon\|^2_L \leq \delta\|f\|^2_{L^2_{x,v}} + C_\delta\|g\|^2_{L^2_{x,v}}.
$$

(3.66)

which, together with Lemma 3.6, yields that

$$
\|f^\varepsilon\|^2_{L^2_{x,v}} \leq C\delta\|f\|^2_{L^2_{x,v}} + C_\delta\|g\|^2_{L^2_{x,v}}.
$$

Taking $\delta$ small enough, we obtain

$$
\|f^\varepsilon\|^2_{L^2_{x,v}} \leq C_d\|g\|^2_{L^2_{x,v}}.
$$

(3.67)

Applying (3.14) to $f^\varepsilon$ and using (3.67), then we obtain

$$
\|wf^\varepsilon\|_{L^\infty_{x,v}} + |wf^\varepsilon|_{L^\infty(\gamma)} \leq C_d\|\nu^{-1}wg\|_{L^\infty_{x,v}}.
$$

(3.68)

Next we consider the convergence of $f^\varepsilon$ as $\varepsilon \to 0^+$. For any $\varepsilon_1, \varepsilon_2 > 0$, we consider the difference $f^{\varepsilon_2} - f^{\varepsilon_1}$ satisfying

$$
\begin{aligned}
  v_3 \partial_x (f^{\varepsilon_2} - f^{\varepsilon_1}) + L(f^{\varepsilon_2} - f^{\varepsilon_1}) &= -\varepsilon_2 f^{\varepsilon_2} + \varepsilon_1 f^{\varepsilon_1}, \\
  (f^{\varepsilon_2} - f^{\varepsilon_1})|_{\gamma_-} &= (f^{\varepsilon_2} - f^{\varepsilon_1})(x,R_x v).
\end{aligned}
$$

(3.69)

Multiplying (3.69) by $f^{\varepsilon_2} - f^{\varepsilon_1}$, integrating the resultant equation and by similar arguments as in (3.66)–(3.67), one gets

$$
\|f^{\varepsilon_2} - f^{\varepsilon_1}\|^2_{L^2_{x,v}} \leq C_d(\varepsilon_2^2 + \varepsilon_1^2)\|g\|^2_{L^2_{x,v}} \to 0,
$$

(3.70)

as $\varepsilon_1, \varepsilon_2 \to 0^+$. Finally, applying (3.14) to $f^{\varepsilon_2} - f^{\varepsilon_1}$ and using (3.70), then we obtain

$$
\begin{aligned}
  \|w(f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^\infty_{x,v}} + |w(f^{\varepsilon_2} - f^{\varepsilon_1})|_{L^\infty(\gamma)} \\
  \leq C\{\|\nu^{-1}w(\varepsilon_2 f^{\varepsilon_2} - \varepsilon_1 f^{\varepsilon_1})\|_{L^\infty_{x,v}} + \|f^{\varepsilon_2} - f^{\varepsilon_1}\|_{L^2_{x,v}}\} \\
  \leq C_d(\varepsilon_1 + \varepsilon_2)\|\nu^{-1}wg\|_{L^\infty_{x,v}} \to 0,
\end{aligned}
$$

(3.71)

as $\varepsilon_1, \varepsilon_2 \to 0^+$. With (3.71), we know that there exists a function $f$ so that $\|w(f^\varepsilon - f)\|_{L^\infty_{x,v}} \to 0$ as $\varepsilon \to 0^+$. And it is direct to see that $f$ solves (3.64). Also, (3.65) follows immediately from (3.68). The continuity of $f$ directly follows from the $L^\infty_{x,v}$-convergence and the continuity of $f^\varepsilon$. Therefore the proof of Lemma 3.8 is complete. \(\square\)
To obtain the solution for half-space problem, we need some uniform estimate independent of \( d \), then we can take the limit \( d \to \infty \). Let \( f \) be the solution of (3.64), we denote
\[
\mathbf{P}f(x, v) = [a(x) + b(x) \cdot (v - u) + c(x)(|v - u|^2 - 3)] \sqrt{\mu}.
\]

Multiplying (3.64) by \( \sqrt{\mu} \) and using (3.45), we have
\[
0 = \frac{d}{dx} \int_{\mathbb{R}^3} v_3 \sqrt{\mu} f(x, v) dv = \frac{d}{dx} b_3(x) \equiv 0. \tag{3.72}
\]

Since \( f \) satisfies the specular boundary, it holds that \( b_3(x)|_{x=0} = b_3(x)|_{x=d} = 0 \), which, together with (3.72) yields
\[
b_3(x) = 0, \quad \text{for} \ x \in [0, d]. \tag{3.73}
\]

Let \( (\phi_0, \phi_1, \phi_2, \phi_3) \) be some constants chosen later, we define
\[
\bar{f}(x, v) := f(x, v) + [\phi_0 + \phi_1 (v_1 - u_1) + \phi_2 (v_2 - u_2) + \phi_3 (|v - u|^2 - 3)] \sqrt{\mu}
\]
\[
= [\bar{a}(x) + \bar{b}_1(x) \cdot (v_1 - u_1) + \bar{b}_2(x) \cdot (v_2 - u_2) + \bar{c}(x)(|v - u|^2 - 3)] \sqrt{\mu}
\]
\[
+ (I - \mathbf{P}) \bar{f},
\]
where
\[
\begin{align*}
\bar{a}(x) &= a(x) + \phi_0, \\
\bar{b}_1(x) &= b_1(x) + \phi_1, \quad i = 1, 2, \\
\bar{c}(x) &= c(x) + \phi_3.
\end{align*}
\]

From (3.73), it is easy to check that
\[
\bar{b}_3(x) = 0 \quad \text{and} \quad (I - \mathbf{P}) \bar{f}(x, v) = (I - \mathbf{P}) f(x, v), \quad \forall x \in [0, d]. \tag{3.74}
\]

In fact, it is direct to check that \( \bar{f} \) still satisfies (3.64), i.e.,
\[
\begin{aligned}
& v_3 \partial_x \bar{f} + \mathbf{L} \bar{f} = g, \quad (x, v) \in \mathbb{R}^3, \\
& (\bar{f}(x, v))|_{\gamma_-} = \bar{f}(x, R_x v),
\end{aligned} \tag{3.75}
\]
with
\[
\|w f\|_{L_\infty^\gamma} + |w f|_{L_\infty} \leq C_d \|w g\|_{L_\infty^\gamma} + C_d (|\phi_0, \phi_1, \phi_2, \phi_3|).
\]

Multiplying (3.75) by \( (v_1 - u_1, v_2 - u_2, |v - u|^2 - 5) \sqrt{\mu} \) and using (3.45), we get
\[
\begin{aligned}
& v_3 (v_1 - u_1) \sqrt{\mu} \bar{f}(x, v) dv = 0, \quad \forall x \in [0, d], \quad i = 1, 2, \\
& v_3 (|v - u|^2 - 5) \sqrt{\mu} \bar{f}(x, v) dv = 0, \quad \forall x \in [0, d]. \tag{3.76}
\end{aligned}
\]

It follows from (3.74) and (3.76) that
\[
\begin{aligned}
\int_{\mathbb{R}^3} v_3 |\mathbf{P} \bar{f}(x, v)|^2 dv \\
= \int_{\mathbb{R}^3} v_3 [(\bar{a} + \bar{b}_1 \cdot (v_1 - u_1) + \bar{b}_2 \cdot (v_2 - u_2) + \bar{c}(|v - u|^2 - 3)]^2 \mu(v) dv \equiv 0, \tag{3.77}
\end{aligned}
\]
and
\[
\int_{\mathbb{R}^3} v_3 Pf(x, v) \cdot (I - P)f(x, v) dv = \int_{\mathbb{R}^3} v_3[\bar{a} + \bar{b}_1 \cdot (v_1 - u_1) + \bar{b}_2 \cdot (v_2 - u_2) + \bar{c}(|v - u|^2 - 3)]\sqrt{\mu(v)} \cdot (I - P)f dv \equiv 0. \tag{3.78}
\]

By utilizing (3.77) and (3.78), it holds that
\[
\int_{\mathbb{R}^3} v_3^2 f(x, v)^2 dv = \int_{\mathbb{R}^3} v_3^2 |(I - P)f(x, v)|^2 dv, \quad \forall x \in [0, d]. \tag{3.79}
\]

Multiplying (3.75) by \(f\) and using (3.79), (3.45), we obtain
\[
\frac{d}{dx} \int_{\mathbb{R}^3} v_3^2 |I - P|f|^2 dv + c_0 \|(I - P)f\|^2_{L^2} \leq C\|g\|^2_{L^2}, \tag{3.80}
\]
where we have used the fact
\[
\int_{\mathbb{R}^3} g_1 dv = \int_{\mathbb{R}^3} g(I - P)f dv \leq \frac{1}{2} c_0 \|(I - P)f\|^2_{L^2} + C\|g\|^2_{L^2}.
\]

Let \(0 \leq \sigma < \sigma_1 \leq \sigma_0\). Multiplying (3.80) by \(e^{2\sigma_1 x}\), one obtains that
\[
\frac{d}{dx} \left\{ e^{2\sigma_1 x} \int_{\mathbb{R}^3} v_3^2 |I - P|f|^2 dv \right\} + (c_0 - C\sigma_1) e^{2\sigma_1 x} \|(I - P)f\|^2_{L^2} \leq C e^{2\sigma_1 x} \|g\|^2_{L^2}.
\]
Taking \(\sigma_0 > 0\) small such that \(c_0 - C\sigma_1 \geq \frac{1}{2} c_0\), then we have
\[
\frac{d}{dx} \left\{ e^{2\sigma_1 x} \int_{\mathbb{R}^3} v_3^2 |I - P|f|^2 dv \right\} + \frac{1}{2} c_0 e^{2\sigma_1 x} \|(I - P)f\|^2_{L^2} \leq C e^{2\sigma_1 x} \|g\|^2_{L^2}. \tag{3.81}
\]

Integrating (3.81) over \([0, d]\) and noting (3.74) one has
\[
\int_0^d e^{2\sigma_1 x} \|(I - P)f\|^2_{L^2} dx \equiv \int_0^d e^{2\sigma_1 x} \|(I - P)f\|^2_{L^2} dx \leq C\|e^{\sigma_1 x} g\|^2_{L^2}. \tag{3.82}
\]
where we have used the fact \(\int_{\mathbb{R}^3} v_3^2 |(I - P)f(0, v)|^2 dv = \int_{\mathbb{R}^3} v_3^2 |(I - P)f(d, v)|^2 dv\) due to the specular boundary condition.

**Remark 3.9.** We point out that the estimations (3.76)–(3.82) are independent of the choice of \((\phi_0, \phi_1, \phi_2, \phi_3)\). However, to obtain uniform estimate independent of \(d\) for macroscopic part, we need to choose \((\phi_0, \phi_1, \phi_2, \phi_3)\) suitably.

We denote
\[
A_{ij}(v) = \left\{ (v_i - u_i)(v_j - u_j) - \frac{\delta_{ij}}{3} |v - u|^2 \right\}\sqrt{\mu}, \quad i, j = 1, 2, 3. \tag{3.83}
\]
\[
B_i(v) = (v_i - u_i)(|v - u|^2 - 5)\sqrt{\mu},
\]
It is obviously that \(A_{ij}, B_i \in \mathbb{N}^+, \) and
\[
\kappa_1 := \int_{\mathbb{R}^3} A_{31} L^{-1} A_{31} dv \equiv \int_{\mathbb{R}^3} A_{ij} L^{-1} A_{ij} dv > 0, \quad \text{for } i \neq j,
\]
\[
\kappa_2 := \int_{\mathbb{R}^3} B_i L^{-1} B_i dv \equiv \int_{\mathbb{R}^3} B_i L^{-1} B_i dv > 0, \quad \text{for } i = 1, 2, 3. \tag{3.84}
\]
Lemma 3.10. There exist constants \((\phi_0, \phi_1, \phi_2, \phi_3)\) such that
\[
\begin{align*}
\int_{\mathbb{R}^3} v_3 \bar{f}(d, v) \cdot v_3 \sqrt{\mu} dv &= 0, \\
\int_{\mathbb{R}^3} v_3 \bar{f}(d, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv &= 0, \\
\int_{\mathbb{R}^3} v_3 \bar{f}(d, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv &= 0.
\end{align*}
\] (3.85)

Proof. A direct calculation shows that (3.85) is equivalent to the following
\[
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot v_3 \sqrt{\mu} dv = \bar{a}(x) + 2\bar{c}(x) + \int_{\mathbb{R}^3} A_{33}(v) \cdot (\mathbf{I} - \mathbf{P}) \bar{f}(x, v) dv
\]
\[
= \phi_0 + 2\phi_3 + a(x) + 2c(x) + \int_{\mathbb{R}^3} A_{33}(v) \cdot (\mathbf{I} - \mathbf{P}) f(x, v) dv, 
\] (3.86)

\[
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv = \kappa_1 \bar{b}_1(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv
\]
\[
= \kappa_1 \phi_1 + \kappa_1 b_1(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv, 
\] (3.87)

\[
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv = \kappa_2 \bar{b}_2(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv
\]
\[
= \kappa_2 \phi_2 + \kappa_2 b_2(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv, 
\] (3.88)

\[
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv = \kappa_2 \bar{c}(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv
\]
\[
= \kappa_2 \phi_3 + \kappa_2 c(x) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(x, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv, 
\] (3.89)

where we have used (3.84).

Using (3.86)–(3.89), then (3.85) is equivalent as
\[
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & \kappa_1 & 0 & 0 \\
0 & 0 & \kappa_1 & 0 \\
0 & 0 & 0 & \kappa_2
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
= -
\begin{pmatrix}
a(d) + 2c(d) + \int_{\mathbb{R}^3} A_{33}(v) \cdot (\mathbf{I} - \mathbf{P}) f(d, v) \cdot \mathbf{A}_{33}(v) dv \\
\kappa_1 b_1(d) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(d, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv \\
\kappa_2 b_2(d) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(d, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv \\
\kappa_2 c(d) + \int_{\mathbb{R}^3} v_3 (\mathbf{I} - \mathbf{P}) f(d, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv
\end{pmatrix}.
\]

Noting the matrix is non-singular, hence \((\phi_0, \phi_1, \phi_2, \phi_3)\) is found. Therefore the proof of Lemma 3.10 is completed.

\[\square\]

Lemma 3.11. Let \((\phi_0, \phi_1, \phi_2, \phi_3)\) be the ones determined in Lemma 3.10, then it holds that
\[
\|e^{\sigma x} \bar{f}\|_{L^2_{\alpha \beta}} \leq \frac{C}{\sigma_1 - \sigma} \|e^{\sigma_1 \cdot x} g\|_{L^2_{\alpha \beta}},
\] (3.90)

with \(0 < \sigma < \sigma_1 \leq \sigma_0\), and the constant \(C\) is independent of \(d\).

Proof. Multiplying (3.75) by \(\mathbf{L}^{-1}(\mathcal{A}_{31}), \mathbf{L}^{-1}(\mathcal{A}_{32})\) and \(\mathbf{L}^{-1}(\mathcal{B}_3)\), respectively, then we obtain
\[
\frac{d}{dx} \begin{pmatrix}
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv \\
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv \\
\int_{\mathbb{R}^3} v_3 \bar{f}(x, v) \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv
\end{pmatrix}
= \begin{pmatrix}
\int_{\mathbb{R}^3} [g - \mathbf{L}(\mathbf{I} - \mathbf{P}) \bar{f}] \cdot \mathbf{L}^{-1}(\mathcal{A}_{31}) dv \\
\int_{\mathbb{R}^3} [g - \mathbf{L}(\mathbf{I} - \mathbf{P}) \bar{f}] \cdot \mathbf{L}^{-1}(\mathcal{A}_{32}) dv \\
\int_{\mathbb{R}^3} [g - \mathbf{L}(\mathbf{I} - \mathbf{P}) \bar{f}] \cdot \mathbf{L}^{-1}(\mathcal{B}_3) dv
\end{pmatrix}.
\]
Integrating above system over \([x, d]\) and using (3.85)\(_{2,3,4}\) to get
\[
\begin{pmatrix}
\int_{\mathbb{R}^3} v_3 \tilde{f}(x, v) \cdot \mathbf{L}^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} v_3 \tilde{f}(x, v) \cdot \mathbf{L}^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} v_3 \tilde{f}(x, v) \cdot \mathbf{L}^{-1}(B_3) dv
\end{pmatrix}
= \int_x^d \begin{pmatrix}
\int_{\mathbb{R}^3} [\mathbf{L}(I - P) f - g] \cdot \mathbf{L}^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} [\mathbf{L}(I - P) f - g] \cdot \mathbf{L}^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} [\mathbf{L}(I - P) f - g] \cdot \mathbf{L}^{-1}(B_3) dv
\end{pmatrix}(z) dz,
\]
which, together with (3.87)–(3.89), yields that
\[
|(\kappa_1 \bar{b}_1, \kappa_1 \bar{b}_2, \kappa_2 \tilde{c})(x)| \leq C \| (I - P) f(x) \|_\nu + C \int_x^d \{ ||(I - P)f(z)||_\nu + ||g(z)||_{L_2^\nu} \} dz. \tag{3.91}
\]
Multiplying (3.91) by \(e^{\sigma x}\) with \(0 < \sigma < \sigma_1 \leq \sigma_0\) and using (3.82), then we can have
\[
\int_0^d e^{\sigma x} |(\bar{b}_1, \bar{b}_2, \tilde{c})(x)|^2 dx \leq C \int_0^d e^{\sigma x} ||(I - P)f(x)||_\nu^2 dx
\]
\[
+ C \int_0^d e^{\sigma x} \left[ \int_x^d \{ ||(I - P)f(z)||_\nu + ||g(z)||_{L_2^\nu} \} dz \right]^2 dx
\]
\[
\leq \frac{C}{\sigma_1 - \sigma} ||e^{\sigma x} g||_{L_2^\nu}^2. \tag{3.92}
\]
Finally we consider the case for \(\bar{a}\). In fact, multiplying (3.75) by \(v_3 \sqrt{\mu}\), we have that
\[
\frac{d}{dx} \int_{\mathbb{R}^3} \tilde{f}(x, v) \cdot v_3^2 \sqrt{\mu} dv = \int_{\mathbb{R}^3} g \cdot v_3 \sqrt{\mu} dv.
\]
Integrating above equation over \([x, d]\), and using (3.85)\(_1\), (3.74)\(_1\), one obtain
\[
\bar{a}(x) = -2\bar{c}(x) + \int_{\mathbb{R}^3} (I - P) \tilde{f}(x, v) \cdot v_3^2 \sqrt{\mu} dv - \int_x^d \int_{\mathbb{R}^3} g \cdot v_3 \sqrt{\mu} dv dx. \tag{3.93}
\]
Multiplying (3.93) by \(e^{\sigma x}\) with \(0 < \sigma < \sigma_1 \leq \sigma_0\) and using (3.82), (3.92), then we can get
\[
\int_0^d e^{\sigma x} |\bar{a}(x)|^2 dx \leq \frac{C}{\sigma_1 - \sigma} ||e^{\sigma x} g||_{L_2^\nu}^2. \tag{3.94}
\]
Combining (3.82), (3.92) and (3.94), we prove (3.90). Therefore the proof of Lemma 3.11 is completed. \(\square\)

**Lemma 3.12.** Let \(\beta \geq 3, d \geq 1\), and \(\tilde{f}\) to the solution of (3.75), it holds that
\[
||e^{\sigma x} w \tilde{f}||_{L_\infty^\nu} + ||e^{\sigma x} w \tilde{f}||_{L^\infty(\gamma)} \leq \frac{C}{\sigma_0 - \sigma} \|e^{\sigma_0 x} \nu^{-1} w g\|_{L_\infty^\nu}, \tag{3.95}
\]
where the constant \(C > 0\) is independent of \(d\).

**Proof.** Let \(\bar{h} := e^{\sigma x} w \tilde{f}\). Multiplying (3.75) by \(e^{\sigma x} w\) to have
\[
v_3 \partial_x \bar{h} + \nu_\sigma(v) \bar{h} = K_w \bar{h} + e^{\sigma x} w g, \quad \bar{h}(x, v)|_{\gamma_-} = \bar{h}(x, R_x v),
\]
where \(\nu_\sigma(v) := \nu(v) - \sigma v_3\). We further take \(\sigma_0 > 0\) small such that \(\nu_\sigma(v) \geq \frac{1}{2} \nu(v) > 0\). By the same arguments as in Lemma 3.3, we can obtain
\[
||\bar{h}||_{L_\infty^\nu} + ||\bar{h}||_{L^\infty(\gamma)} \leq C ||e^{\sigma x} \nu^{-1} w g||_{L_\infty^\nu} + C ||e^{\sigma x} \tilde{f}||_{L_2^\nu}.
\]
\[
\leq C\|e^{\sigma x} \nu^{-1} w g\|_{L^\infty_x} + \frac{C}{\sigma_1 - \sigma}\|e^{\sigma_1 x} g\|_{L^2_x},
\]

where we have used (3.90) and chosen \( \sigma_1 = \sigma + \frac{\sigma_0 - \sigma}{2} \) such that \( 0 < \sigma < \sigma_1 < \sigma_0 \). Hence the proof of Lemma 3.12 is completed.

We shall prove Theorem 3.1 by taking the limit \( d \to \infty \). From now on, we shall denote the solution \( \tilde{f}(x,v) \) of (3.75) to be \( \tilde{f}_d(x,v) \) to emphasize the dependence of parameter \( d \). We denote

\[
\tilde{f}(x,v) = \tilde{f}_{d_2}(x,v) - \tilde{f}_{d_1}(x,v), \quad 1 \leq d_1 \leq d_2.
\]

Then \( \tilde{f} \) satisfies the following equation

\[
\begin{cases}
  v_3 \partial_x \tilde{f} + L \tilde{f} = 0, & x \in [0,d_1], \ v \in \mathbb{R}^3, \\
  \tilde{f}(0,v)|_{v_3 > 0} = \tilde{f}(0,Rv).
\end{cases}
\] (3.96)

### 3.3 Proof of Theorem 3.1.

As previous, we divide the proof into two steps.

**Step 1.** Convergence in \( L^2 \)-norm. Multiplying (3.96) by \( \tilde{f} \) and using (3.95) to obtain

\[
\int_0^{d_1} \int_{\mathbb{R}^3} (1 + |v|)(I - P)\tilde{f}(x,v)|^2 dvdx \\
\leq C \int_{\mathbb{R}^3} |v_3| \cdot |\tilde{f}(d_1,v)|^2 dv \leq C\{\|w\tilde{f}_{d_2}(d_1)\|_{L^2}^2 + |w\tilde{f}_{d_1}(d_1)|_{L^\infty}^2\}
\]

\[
\leq \frac{C}{(\sigma_0 - \sigma)^2}\|e^{\sigma x} \nu^{-1} w g\|_{L^\infty_x} e^{-2\sigma d_2}. \quad (3.97)
\]

We still need to control the macroscopic part. We denote

\[
P\tilde{f} = [\tilde{a}(x) + \tilde{b}_1(x)(v_1 - u_1) + \tilde{b}_2(x)(v_2 - u_2) + \tilde{c}(x)(v - u)^2 - 3]\sqrt{\mu}.
\]

Similar as in Lemma 3.11, multiplying (3.96) by \( L^{-1}(A_{31}), L^{-1}(A_{32}) \) and \( L^{-1}(B_3) \), respectively, integrating the resultant equation over \([x,d_1]\) to have

\[
\begin{align*}
\left( \int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(A_{31})dv \right) + \\
\left( \int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(A_{32})dv \right) + \\
\left( \int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(B_3)dv \right)
\end{align*}
\]

\[
\left( \int_{\mathbb{R}^3} L(I - P) \tilde{f} \cdot L^{-1}(A_{31})dv \right) + \\
\left( \int_{\mathbb{R}^3} L(I - P) \tilde{f} \cdot L^{-1}(A_{32})dv \right) + \\
\left( \int_{\mathbb{R}^3} L(I - P) \tilde{f} \cdot L^{-1}(B_3)dv \right) (z)dz,
\]

which, together with (3.87)–(3.89), yields that

\[
|\langle \kappa_1 \tilde{b}_1, \kappa_2 \tilde{b}_2, \kappa_2 \tilde{c} \rangle(x)| \leq C\{\|w\tilde{f}_{d_2}(d_1)\|_{L^2} + |w\tilde{f}_{d_1}(d_1)|_{L^\infty} \} + C\|(I - P)\tilde{f}(x)\|_\nu
\]
Integrating (3.98) over \([0, d_1]\) and using (3.95), (3.97) to get

\[
\int_0^{d_1} |(\tilde{b}_1, \tilde{b}_2, \tilde{c})(x)|^2 dx \leq \frac{C}{(\sigma_0 - \sigma)^2} \|e^{\sigma_0 x} \nu^{-1} w g\|^2_{L^\infty, \nu} d_1^2 e^{-2\sigma d_1}. \tag{3.99}
\]

Finally we consider the case for \(\tilde{a}\). Multiplying (3.96) by \(v_3\sqrt{\mu}\), we have that

\[
\frac{d}{dx} \int_{\mathbb{R}^3} \tilde{f}(x, v) \cdot v_3^2 \sqrt{\mu} dv = 0.
\]

Integrating above equation over \([x, d]\) and using (3.86), one obtain

\[
\tilde{a}(x) = -2 \tilde{c}(x) + \int_{\mathbb{R}^3} (I - \mathbf{P}) \tilde{f}(x, v) \cdot v_3^2 \sqrt{\mu} dv - \int_{\mathbb{R}^3} \tilde{f}(d_1, v) \cdot v_3 \sqrt{\mu} dv. \tag{3.100}
\]

It follows from (3.100), (3.97), (3.99) and (3.95), one has

\[
\int_0^{d_1} |\tilde{a}(x)|^2 dx \leq \frac{C}{(\sigma_0 - \sigma)^2} \|e^{\sigma_0 x} \nu^{-1} w g\|^2_{L^\infty, \nu} d_1^2 e^{-2\sigma d_1},
\]

which, together with (3.97) and (3.99), yields that

\[
\int_0^{d_1} \int_{\mathbb{R}^3} |\tilde{f}(x, v)|^2 dv dx \leq \frac{C}{(\sigma_0 - \sigma)^2} \|e^{\sigma_0 x} \nu^{-1} w g\|^2_{L^\infty, \nu} d_1^2 e^{-2\sigma d_1}. \tag{3.101}
\]

**Step 2. Convergence in \(L^\infty\)-norm.** In the following, we shall use \(t_k = t_k(t, x, v), X_c(t; s, t, x, v), x_k = x_k(v, x)\) to be the back-time cycles defined for domain \([0, d_1] \times \mathbb{R}^3\). For later use, we denote \(\hat{h} := uf\). Let \((x, v) \in [0, d_1] \times \mathbb{R}^3\), it follows from (3.96) that

\[
\hat{h}(x, v) = e^{-\nu(v)(t-t_k)} \hat{h}(d_1, v_{k-1}) + \sum_{l=0}^{k-1} \int_{t_{l+1}}^{t_l} e^{-\nu(v)(t-s)} K_w \hat{h}(X_c(s), v_l) ds, \tag{3.102}
\]

with \(k = 1\) for \(v_{0,3} < 0\), and \(k = 2\) for \(v_{0,3} > 0\). We will use this summation convention in the following of this lemma.

From now on, we assume \(x \leq \frac{1}{2} d_1\). Then, if \(v_{0,3} < 0\), we have

\[
t - t_k = t - t_{k-1} = \frac{(d_1 - x)}{|v_{0,3}|} \geq \frac{d_1}{2|v_{0,3}|}. \tag{3.103}
\]

If \(v_{0,3} > 0\), we obtain

\[
t - t_k = t - t_2 \geq t_1 - t_2 = \frac{d_1}{|v_{0,3}|},
\]

which, together with (3.103), yields that

\[
\nu(v)(t - t_k) \geq \frac{1}{2} v_0 d_1. \tag{3.104}
\]

Hence, utilizing (3.104), we always have

\[
|e^{-\nu(v)(t-t_k)} \hat{h}(d_1, v_{k-1})|
\]
\begin{align}
&\leq e^{-\frac{1}{2}v_0d_1}\left(\left\|w_\tilde{f}_d, w_\tilde{f}_d\right\|_{L_{x,v}^\infty} + \left\|w_\tilde{f}_d(d_1)\right\|_{L^\infty(\gamma)} + \left\|w_\tilde{f}_d(d_1)\right\|_{L^\infty(\gamma)}\right) \\
&\leq \frac{C}{\sigma_0 - \sigma} e^{-\frac{1}{2}\nu_0d_1}||e^{\sigma_0x\nu^{-1}}wg||_{L_{x,v}^\infty}.
\end{align}

For the second term on RHS of (3.102), we use (3.102) again to obtain
\begin{align}
&\left|\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^{-\nu(t-s)} K_w \tilde{h}(X_{cl}(s), v_i) ds \right| \\
&= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^{-\nu(t-s)} \int_{\mathbb{R}^3} |k_w(v_i, v') \tilde{h}(X_{cl}(s), v')| dv' ds \\
&\leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^{-\nu(t-s)} \int_{\mathbb{R}^3} |k_w(v_i, v')| \\
&\times \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} e^{-\nu(s-s_1)} \int_{\mathbb{R}^3} |k_w(v_j, v'') \tilde{h}(X'_{cl}(s_1), v'')| dv'' ds_1 dv' ds \\
&+ \frac{C}{\sigma_0 - \sigma} e^{-\frac{1}{2}v_0d_1}||e^{\sigma_0x\nu^{-1}}wg||_{L_{x,v}^\infty},
\end{align}

where we have used (3.105) and denote $X'_{cl}(s_1) = X_{cl}(s_1; s, X_{cl}(s), v')$, $t'_j = t'_j(s_1; s, X_{cl}(s), v')$ and $v'_j$ to be the back-time cycle of $(s, X_{cl}(s), v')$. Then, by the same arguments as in Lemma 3.3, we get
\begin{align}
&\|\tilde{h}\|_{L^\infty([0, \frac{1}{2}d_1] \times \mathbb{R}^3)} + \|\tilde{h}(0)\|_{L^\infty(\gamma_+)} \\
&\leq \frac{1}{8} (\|\tilde{h}\|_{L^\infty([0, d_1] \times \mathbb{R}^3)} + \|\tilde{h}(0)\|_{L^\infty(\gamma_+)} + C (\|\tilde{h}_{d_2}(d_1)\|_{L^\infty} + \|\tilde{h}_{d_1}(d_1)\|_{L^\infty(\gamma)})) \\
&+ \frac{C}{\sigma_0 - \sigma} e^{-\frac{1}{2}v_0d_1}||e^{\sigma_0x\nu^{-1}}wg||_{L_{x,v}^\infty} + C ||\tilde{f}||_{L^2([0, d_1] \times \mathbb{R}^3)},
\end{align}

which, together with (3.95) and (3.101), yields that
\begin{align}
&\|\tilde{h}\|_{L^\infty([0, \frac{1}{2}d_1] \times \mathbb{R}^3)} + \|\tilde{h}(0)\|_{L^\infty(\gamma_+)} \\
&\leq \frac{C}{\sigma_0 - \sigma} \left(e^{-\frac{1}{2}v_0d_1} + d_1^2 e^{-\frac{1}{2}\sigma d_1}\right) ||e^{\sigma_0x\nu^{-1}}wg||_{L_{x,v}^\infty} \rightarrow 0 \quad \text{as} \quad d_1 \rightarrow \infty. \quad (3.106)
\end{align}

With the help of (3.106), there exists a function $f(x, v)$ with $(x, v) \in \mathbb{R}_+ \times \mathbb{R}^3$ so that $\|w(\tilde{f} - f)\|_{L^\infty([0, \frac{1}{2}d_1] \times \mathbb{R}^3)} \rightarrow 0$ as $d \rightarrow \infty$. The uniform bound (3.3), (3.4) follow from (3.95), (3.90) and the strong convergence in $L^\infty_{x,v}$. It is direct to see that $f(x, v)$ solves (3.5). The continuity of $f$ follows directly from the $L^\infty_{x,v}$-convergence and the continuity of $\tilde{f}_d$.

For the uniqueness, let $f_1, f_2$ be two solution of (3.1) with the bound (3.3) holds, then it holds that
\begin{align}
\begin{cases}
\nu_3 \partial_t f_i + L(f_i) + f_i = 0, \\
\|f_i(0, v)\|_{v_3 > 0} = f_i(0, Rv), \quad i = 1, 2, \\
\lim_{x \to \infty} f_i(x, v) = 0, \quad i = 1, 2.
\end{cases}
\end{align}

Multiplying (3.107) by $(f_1 - f_2)$, it is direct to prove that
\begin{align}
\int_0^\infty \|((I - P)(f_1 - f_2))\|^2 dx = 0.
\end{align}
That is, \((f_1 - f_2) = P(f_1 - f_2)\). Then by the same arguments as in Lemma 3.11 that
\[
\int_0^\infty \|P(f_1 - f_2)\|_{L^2}^2 \, dx = 0.
\]
Thus, we prove \(f_1 \equiv f_2\). Therefore the proof of Lemma 3.1 is completed. \(\square\)

4 Proof of Theorem 1.1

To prove the Theorem 1.1, we consider the following iterative sequence
\[
\begin{align*}
\{v_0 \partial_x f_j^{j+1} + L_j f_j^{j+1} &= \Gamma(f_j, f_j^1) + S, \\
f_j^{j+1}(0,v)|_{v_0>0} &= f_j^{j+1}(0, Rv) + f_b(Rv), \\
\lim_{x \to \infty} f_j^{j+1} &= 0,
\end{align*}
\]
for \(j = 0, 1, 2 \cdots\) with \(f^0 \equiv 0\). It is direct to note \(\Gamma(f_j, f_j^l) \in \mathbb{N}^+\) and
\[
\|\nu^{-1} w \Gamma(f_j, f_j^l)\|_{L^\infty_w} \leq C \|w f^j\|_{L^\infty_w}^2.
\]
Noting (42), and using Theorem 3.1, we can solve (4.1) inductively for \(j = 0, 1, 2, \cdots\). By taking \(\frac{3}{4} \sigma_0 < \sigma < \sigma_0\), it follows from (4.2) and (3.3) that
\[
\|e^{\sigma x} w f_j^{j+1}\|_{L^\infty_{w,v}} + \|w f_j^{j+1}(0)\|_{L^\infty(\gamma)} \\
\leq \frac{C_1}{\sigma_0 - \sigma} (|w f_b|_{L^\infty(\gamma)} + \|e^{\sigma_0 x} \Gamma^{-1} w S\|_{L^\infty_{w,v}}) + \frac{C_1}{\sigma_0 - \sigma} \|e^{\sigma x} w f^j\|_{L^\infty_{w,v}}^2.
\]
We denote
\[
\delta =: |w f_b|_{L^\infty(\gamma)} + \|e^{\sigma_0 x} \Gamma^{-1} w S\|_{L^\infty_{w,v}}.
\]
By induction, we shall prove that
\[
\|e^{\sigma x} w f^j\|_{L^\infty_{w,v}} + \|w f^j(0)\|_{L^\infty(\gamma)} \leq \frac{2C_1 \delta}{\sigma_0 - \sigma}, \quad \text{for } j = 1, 2, \cdots.
\]
Indeed, for \(j = 0\), it follows from \(f^0 \equiv 0\) and (4.3) that
\[
\|e^{\sigma x} w f^1\|_{L^\infty_{w,v}} + |w f^1(0)|_{L^\infty(\gamma)} \leq \frac{C_1 \delta}{\sigma_0 - \sigma}.
\]
Now we assume that (4.4) holds for \(j = 1, 2, \cdots, l\), then we consider the case for \(j = l + 1\). Indeed it follows from (4.3) that
\[
\|e^{\sigma x} w f^{l+1}\|_{L^\infty_{w,v}} + |w f^{l+1}(0)|_{L^\infty(\gamma)} \\
\leq \frac{C_1 \delta}{\sigma_0 - \sigma} + \frac{C_1}{\sigma_0 - \sigma} \|e^{\sigma x} w f^l\|_{L^\infty_{w,v}}^2 \\
\leq \frac{C_1 \delta}{\sigma_0 - \sigma} + \left(1 + 4 \left(\frac{C_1}{\sigma_0 - \sigma}\right)^2\right) \leq \frac{3C_1 \delta}{\sigma_0 - \sigma},
\]
where we have used (4.4) with \(j = l\), and chosen \(\delta \leq \delta_0\) with \(\delta_0\) small enough such that
\(4\left(\frac{C_1}{\sigma_0 - \sigma}\right)^2\delta_0 \leq 1/2\). Therefore we have proved (4.4) by induction.
Finally we consider the convergence of sequence $f^j$. For the difference $f^{j+1} - f^j$, we have
\[
\begin{aligned}
& (v_3 \Delta (f^{j+1} - f^j) + \mathbf{L}(f^{j+1} - f^j) = \Gamma(f^j - f^{j-1}, f^j) + \Gamma(f^{j-1}, f^j - f^{j-1}), \\
& (f^{j+1} - f^j)(0,v)|_{v_3 > 0} = (f^{j+1} - f^j)(0, Rv), \\
& \lim_{x \to -\infty} (f^{j+1} - f^j)(x,v) = 0.
\end{aligned}
\] (4.5)

Applying (3.3) to (4.5), we have
\[
\begin{aligned}
& \|e^{\sigma x} w \{f^{j+1} - f^j\}\|_{L^\infty_{x,v}} + \|w \{f^{j+1} - f^j\}(0)\|_{L^\infty} \\
& \leq C_{\sigma_0 - \sigma} \left\{ \|e^{\sigma_0 x} v^{-1} w \Gamma(f^j - f^{j-1}, f^j)\|_{L^\infty_{x,v}} + \|e^{\sigma_0 x} v^{-1} w \Gamma(f^{j-1}, f^j - f^{j-1})\|_{L^\infty_{x,v}} \right\} \\
& \leq C_{\sigma_0 - \sigma} \left\{ \|e^{\sigma_0 x} w f^j\|_{L^\infty_{x,v}} + \|e^{\sigma_0 x} w f^{j-1}\|_{L^\infty_{x,v}} \cdot \|e^{\sigma_0 x} w (f^j - f^{j-1})\|_{L^\infty_{x,v}} \right\} \\
& \leq \frac{C \delta_0}{(\sigma_0 - \sigma)^2} \|e^{\sigma_0 x} w (f^j - f^{j-1})\|_{L^\infty_{x,v}} \leq \frac{1}{2} \|e^{\sigma_0 x} w (f^j - f^{j-1})\|_{L^\infty_{x,v}},
\end{aligned}
\] (4.6)

where we have used (4.4) and further taken $\delta_0 > 0$ small such that $\frac{C}{(\sigma_0 - \sigma)^2} \delta_0 \leq 1/2$. Hence $f^j$ is a Cauchy sequence in $L^\infty_{x,v}$, then we obtain the solution by taking the limit $f = \lim_{j \to \infty} f^j$. The uniqueness can also be obtained by using the inequality as (4.6). The continuity of $f$ is a direct consequence of $L^\infty_{x,v}$-convergence. Therefore we complete the proof of Theorem 1.1. \hfill \Box

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