Rotational Invariance in the M(atrix) Formulation of Type IIB Theory

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The matrix model formulation of M-theory can be generalized by compactification to ten-dimensional type II string theory, formulated in the infinite momentum frame. Both the type IIA and IIB string theories can be formulated in this way. In the M-theory and type IIA cases, the transverse rotational invariance is manifest, but in the IIB case, one of the transverse dimensions materializes in a completely different way from the other seven. The full O(8) rotational symmetry then follows in a surprising way from the electromagnetic duality of supersymmetric Yang-Mills field theory.

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1. Introduction

Whatever describes the fundamental degrees of freedom of string theory must be capable of combining into an incredible variety of different objects in different regions of moduli space. Shenker has used the word ‘protean,’ which according to our dictionary means “readily assuming different shapes or roles.” For example, by varying the moduli, the degrees of freedom have to rearrange themselves from type IIA strings to heterotic, type I, type IIB strings, a variety of p-branes, D-branes, and eleven-dimensional gravitons. Even more remarkable is the ability of the system to manufacture new space dimensions in limits where standard reasoning would lead one to think dimensions should disappear. An example of this phenomenon occurs when M-theory is compactified on a two-dimensional torus, and the area of the torus is shrunk to zero. Since $11 - 2 = 9$, conventional logic would lead to the conclusion that the theory becomes nine-dimensional. In fact, it becomes ten-dimensional type IIB string theory. It is clear from these examples that the constituent objects which underly the theory must be very unusual. Recently, evidence has accumulated that M(atrix) theory may have exactly the right “protean bits” to describe this rich variety of objects. In this note, we want to add one more example of the ‘protean’ nature of M(atrix) theory by showing how the tenth direction of type IIB theory emerges.

In the matrix formulation of M-theory \cite{1}, some of the symmetries of the system are manifest, while others are not. For example, symmetry under transverse rotations of the infinite momentum frame is manifest, but symmetry under rotations which mix the transverse and longitudinal directions is not. When some of the transverse dimensions of M-theory are toroidally compactified, the resulting theory can be described in terms of type IIA string theory \cite{2}, \cite{3}, \cite{4}. The transverse rotational symmetry in the noncompact subspace is still manifest. On the other hand, T-duality is not at all obvious in the nonperturbative M(atrix) theory. It was argued in \cite{5} and \cite{6} that T-duality of the type IIA theory is a consequence of the electric-magnetic duality (S-duality) of 3+1-dimensional supersymmetric Yang-Mills theory with four supersymmetries. In this note, we consider the matrix formulation of type IIB theory. In this case, the transverse rotational symmetry is nonmanifest, and provides a nontrivial consistency test for the M(atrix) description. In the following discussion, we will again invoke S-duality of the 3+1-dimensional supersymmetric Yang-Mills theory to prove that there is a full $O(8)$ transverse rotational symmetry. As a consequence of our result, we will be able to confirm Seiberg’s prediction of the existence of a superconformal fixed point with $O(8)$ symmetry in strongly-coupled 2+1-dimensional Yang-Mills with eight supersymmetries.
Let us first consider the origin of the extra dimension which appears when M-theory is compactified on a two-torus of vanishing area. To determine how the extra dimension appears is not too difficult. It is fairly clear that the momentum conjugate to this coordinate is the conserved wrapping number of two-branes on the two-torus [7]. This quantum number plays the role of the Kaluza-Klein momentum in the new direction, and in the limit of vanishing area, the energy gap for this excitation vanishes and the dimension becomes non-compact. Obviously a consistent interpretation requires that the physics, if not the formalism, be invariant under the rotation of all eight transverse dimensions. Since M(atrix) theory can be compactified on tori, this symmetry requirement provides an interesting test for the theory.

2. Rotational Invariance

We begin with a word on notation. The eleven dimensions of M-theory will be labelled by \((t, X^1, \ldots, X^9, X^{11})\), where \(t\) is time, \((X^1, \ldots, X^9)\) are the transverse space coordinates, and \(X^{11}\) is the longitudinal direction of the infinite momentum frame. To obtain type IIB string theory, we compactify two of the transverse coordinates on a two-torus. The torus is specified by its complex structure modulus, \(\tau\), and its area. The complex parameter, \(\tau\), maps to the complexified type IIB string coupling,

\[
\tau = \chi + i e^{-\phi},
\]

where \(\chi\) is the Ramond-Ramond scalar, and \(\phi\) is the dilaton [4]. In this way, M-theory geometrizes the conjectured \(\text{SL}(2,\mathbb{Z})\) symmetry of type IIB string theory. At this stage, the type IIB string is compactified on a circle of finite circumference. As we take the area of the torus to zero, the circle will decompactify. In this limit, we expect to see a new dimension grow in M-theory. Without loss of generality, we shall assume the torus is rectangular, and take the two transverse coordinates \(X^1, X^2\) to be compactified on circles of circumference \(L_1, L_2\):

\[
0 < X^1 < L_1 \\
0 < X^2 < L_2.
\]

To keep the IIB coupling fixed, we will hold the ratio \(L_1/L_2\) constant, and consider the limit,
\[ L_1 \to 0 \]
\[ L_2 \to 0, \]  
(2.1)

which should yield ten-dimensional type IIB theory. The extra spatial coordinate conjugate to the two-brane wrapping number will be called \( Y \). Thus the eight transverse spatial coordinates of IIB theory are \((X^3, \ldots, X^9, Y)\).

Before taking the limit (2.1), the quantum of energy associated with a wrapped two-brane is given by,

\[ E_{\text{wrap}} = \frac{L_1 L_2}{(2\pi)^2 (l_{11}^p)^3}, \]  
(2.2)

where \( l_{11}^p \) is the eleven-dimensional planck length. If we identify this wrapping energy with the energy of the first Kaluza-Klein excitation of a massless particle, then the compactification circumference of the \( Y \) coordinate is

\[ L_Y = \frac{2\pi}{E_{\text{wrap}}} = \frac{(l_{11}^p)^3 (2\pi)^3}{L_1 L_2}. \]  
(2.3)

We will eventually take the limit (2.1) in which \( L_Y \to \infty \), but before doing so let us compactify one additional dimension \( X^3 \). Therefore, our starting point is M-theory on a three-torus.

Now, let us consider the M(atrix) formulation of toroidally compactified M-theory. One way of describing compactifications of M(atrix) theory is to begin with zero-branes on a d-dimensional space. Recall that the dynamics of zero-branes is governed by open strings connecting the various zero-branes. If the compactification space has a nontrivial fundamental group, we also have to take into consideration strings that wind around the various nontrivial cycles in the space. Nice compactifications will preserve a large degree of supersymmetry, and in these cases, we expect to be able to neglect higher string modes, and restrict our attention only to the massless excitations of the open strings. A toroidal compactification is certainly a nice choice since the amount of supersymmetry preserved by the compactification is maximal. So, we begin by considering zero-branes on a small d-dimensional torus. It is easy to see, either by T-duality or explicit construction \[{8}\], that the relevant dynamics is nicely described by \( U(N) \) supersymmetric Yang-Mills on the dual torus. We shall eventually take \( N \to \infty \), although the following argument is valid for finite \( N \). As we make our original torus smaller, we will be probing the infra-red dynamics of
the d+1-dimensional Yang-Mills theory. In this respect, compactification on the three-torus is very special since the model is conformally invariant. For $d < 3$, the Yang-Mills theory becomes strongly coupled in the infra-red, while for $d > 3$, the theory is free in the infra-red.

This way of describing M(atrix) compactifications on the three-torus was studied in [5] using the properties of 3+1-dimensional Yang-Mills with four supersymmetries. The parameters of the Yang-Mills theory were derived in [5], and in particular, the dimensionless Yang-Mills coupling constant is given by,

$$g^2 = \frac{(2\pi)^4 (l_{11}^p)^3}{L_1 L_2 L_3}.$$  \hspace{1cm} (2.4)

In this formulation, any symmetry between $L_Y$ and $L_3$ is hidden. To exhibit a symmetry, we shall choose the two lengths $L_Y$ and $L_3$ to be equal to one another. Since $Y$ and $X^3$ are compactified on identical circles, the rotation symmetry in the $Y, X^3$ plane is, for the moment, broken to a discrete subgroup generated by rotations by multiples of $\pi/2$. We will demonstrate this symmetry in the following way.

From (2.3), and the equality of $L_Y$ and $L_3$, we find

$$L_1 L_2 L_3 = (2\pi)^3 (l_{11}^p)^3.$$ \hspace{1cm} (2.5)

Combining this relation with (2.4), we obtain:

$$g^2 = 2\pi.$$ \hspace{1cm} (2.6)

Now this is a very special value of the coupling. Recall that S-duality implies that couplings related by,

$$\tilde{g} = \frac{2\pi}{g},$$ \hspace{1cm} (2.7)

describe identical theories related by electric-magnetic interchange. Eq (2.6) then implies that we are working at the self-dual point at which the theory is invariant under a rotation of electric charge into magnetic charge. In turn, this suggests that the hidden spatial rotation invariance is nothing more than the electric-magnetic self-duality.

To see the connection between $Y, X^3$ rotations and electric-magnetic rotations, let us consider the conjugate momenta $P_Y$ and $P_3$. As we have mentioned, $P_Y$ is proportional to the wrapping number of membranes wrapped on the $X^1 - X^2$ torus. Let us return
momentarily to M-theory on this two-torus. The wrapping number of the membranes is easily identified with a flux in the M(atrix) formulation of M-theory on a two-torus in the following way. Instead of starting with a theory of pure zero-branes on the two-torus, we can consider a theory of $N$ zero-branes and some number of two-branes. T-dualizing both cycles of the torus exchanges the number of zero-branes and two-branes. The zero brane charge is proportional to,

$$\int_{T^2} \text{tr} F,$$

where $F$ is the two-form field strength for the $U(N)$ gauge theory describing the $N$ two-branes. Hence, this wrapping number is identified with the abelian magnetic flux on this torus, as also mentioned in [9]. For compactification on a three-torus, similar arguments given in [8], identify the number of membranes wrapped on the $1-2$ plane with the amount of abelian magnetic flux in the $1-2$ plane. On the other hand, the electric flux along the $X^3$ cycle was identified with the momentum conjugate to $X^3$ [3]. These two fluxes are rotated into one another by electric-magnetic duality!

Now consider the limit in which $L_3 \to \infty$. In this limit, the rotational invariance relating $X^3$ to the other transverse coordinates is restored. Since we have demonstrated a discrete symmetry relating $X^3$ and $Y$, we must also have continuous rotations between $Y$ and the remaining transverse coordinates. In fact, the entire rotational $O(8)$ symmetry then follows.

What does this argument imply about the resulting 2+1-dimensional Yang-Mills theory? Let us begin by considering the abelian 2+1-dimensional theory that describes a single two-brane of type IIA string theory wrapped on a two-torus. This theory has a manifest $O(7)$ symmetry rotating the seven scalar fields into one another. In this case, it is easy to see that we can replace the vector-field by a scalar parametrizing a compact direction using vector-scalar duality in three dimensions. This argument has been used in [10], [11] and [12] to understand the eleven-dimensional origin of the type IIA two-brane. As we shrink the area of the torus to zero, the compact direction corresponding to this dual scalar decompactifies, and we recover the full $O(8)$ symmetry. Our interest is actually with the non-abelian generalization corresponding to $N$ two-branes wrapped on the two-torus. In this case, we cannot simply dualize the non-abelian vector-field. Nevertheless, from our preceeding discussion, we know that there should exist a superconformal theory in the infra-red with global symmetry enhanced from $O(7)$ to $O(8)$. It would interesting to show that this fixed point exists directly in the 2+1-dimensional Yang-Mills theory, and indeed,
Seiberg has argued for the existence of such a fixed point directly in the three dimensional theory \cite{Seiberg}. Furthermore, Banks and Seiberg have also studied type IIB strings in an alternate version of compactified M(atrix) theory and come to similar conclusions to our own \cite{BanksSeiberg}.

In light of our previous discussion, we can now establish a dictionary between type IIB \((p,q)\) strings, and backgrounds in the 3+1-dimensional Yang-Mills theory. Note that in the type IIB theory, there are two compact directions corresponding to \(X^3\) and \(Y\). If we take a membrane wrapped around the \(X^1\) cycle to correspond to a fundamental type IIB string, then a D-string will correspond to a membrane wrapped around the \(X^2\) cycle. Wrapping the membrane \(p\) times around \(X^1\), and \(q\) times around \(X^2\), which we will call a \((p,q)\) cycle, will give a \((p,q)\) string. Wrapping the remaining leg of the membrane around the \(X^3\) direction will correspond to a magnetic flux in the plane determined by \(X^3\) and the \((p,q)\) cycle. On the type IIB side, this membrane configuration corresponds to wrapping a \((p,q)\) string around the \(X^3\) direction. What corresponds to wrapping a \((p,q)\) string around the \(Y\) direction? We simply need to apply electric-magnetic duality, which will exchange the \(X^3\) and \(Y\) directions. Therefore a \((p,q)\) string wound around the \(Y\) direction will be realized in the Yang-Mills theory as an electric flux along the direction orthogonal to the plane determined by \(X^3\) and the \((p,q)\) cycle. Configurations corresponding to higher dimensional branes in type IIB will correspond to nontrivial configurations of the scalar fields in the Yang-Mills theory. Unlike the wrapped \((p,q)\) strings, understanding these configurations will require a study of the Yang-Mills theory in the \(N \to \infty\) limit.

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