Seiberg-like Dualities for 3d $\mathcal{N} = 2$ Theories with $SU(N)$ Gauge Group

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Abstract: We work out Seiberg-like dualities for 3d $\mathcal{N} = 2$ theories with $SU(N)$ gauge group. We use the $SL(2,\mathbb{Z})$ action on 3d conformal field theories with $U(1)$ global symmetry. One of generator $S$ of $SL(2,\mathbb{Z})$ acts as gauging of the $U(1)$ global symmetry. Utilizing $S = S^{-1}$ up to charge conjugation, we obtain Seiberg-like dual of $SU(N)$ theories by gauging topological $U(1)$ symmetry of the Seiberg-like dual of $U(N)$ theories with the same matter content. We work out the Aharony dualities for $SU(N)$ gauge theory with $N_f$ fundamental/anti-fundamental flavors, with/without one adjoint matter with the superpotential. We also work out the Giveon-Kutasov dualities for $SU(N)$ gauge theory with Chern-Simons term and with $N_f$ fundamental/anti-fundamental flavors. For all the proposed dualities, we give various evidences such as chiral ring matching and the superconformal index computations. We find the perfect matchings.
1. Introduction

Recently, there has been renewed interest in nonperturbative dualities between three dimensional theories such as mirror symmetry and Seiberg-like dualities. This is explained in part by the availability of sophisticated tools such as the partition function on $S^3$ and the superconformal index. Using these tools, one can give impressive evidence for various 3d dualities. Some of works in this area are [1]-[21]. One can also obtain the R-charge of the fields by maximizing the free energy of the theory of interest [22].

In this paper we continue this line of research and study Seiberg-like dualities [1, 23] for $\mathcal{N} = 2$ $d = 3$ gauge theories with $SU(N_c)$ group. In 3-d, Seiberg-like dualities are known for $U(N_c)$ theories but the duality for $SU(N_c)$ theories remains
elusive. If we knew the dual pairs of $SU(N_c)$ theories, we could obtain the dual pairs of $U(N_c)$ theories by gauging $SU(N_c)$ theories. On the other hand to obtain $SU(N_c)$ dual pairs out of $U(N_c)$ dual pairs, we need ungauging overall $U(1)$ of $U(N_c)$ gauge symmetry, which is not obvious how to do it.

The purpose of this paper is to propose such ungauging procedure to obtain Seiberg-like dualities for $SU(N_c)$ theories. The idea comes from the observation that there’s $SL(2,\mathbb{Z})$ transformation for 3d conformal field theories with $U(1)$ global symmetry [26]. In particular, one of the generator S of $SL(2,\mathbb{Z})$ involves the gauging of $U(1)$ global symmetry and introduces a new $U(1)$ flavor symmetry, often called ‘topological’ symmetry. This is the perfect setting to gauge $SU(N_c)$ theories since we can obtain $U(N_c)$ theories by gauging the overall $U(1)$ global symmetry of $SU(N_c)$ theories and new flavor symmetry of $U(N_c)$ theory is the topological symmetry $U(1)_T$, whose current is given by $*dTrA$ where $TrA$ denotes the overall $U(1)$ gauge field of $U(N_c)$ theories. Since $S^2 = C$ with $C$ being charge conjugation, by applying gauging the $U(1)_T$ we can obtain the same superconformal field theories as the original $SU(N_c)$ theories up to charge conjugation. With respect to $U(1)_T$, monopole operators are charged, which has the nonperturbative origin in the original theory. Thus we had better carry out the 2nd $S$-operation or gauging $U(1)_T$ in the Seiberg-like dual pair of $U(N_c)$ theory. If the matter contents are charge conjugation invariant, we obtain Seiberg-like dual pair of the original $SU(N_c)$ theory by this way. So we can obtain the Seiberg-like dual of any $SU(N_c)$ theory, if the Seiberg-like dualities are known for $U(N_c)$ theory with the same matter contents as the $SU(N_c)$ theory. All we have to is to gauge $U(1)_T$ of Seiberg-like dual pair of the given $U(N_c)$ theory. We apply this idea to various $SU(N_c)$ theories to obtain Aharony duals [24]. Giveon-Kutasov dualities [1] for $SU(N_c)$ gauge theories with Chern-Simons term can be obtained from Aharony dualities by giving axial mass to some of flavors. We subject such candidates of dual pairs to various tests such as the computation of the superconformal index to find the perfect matching.

The content of the paper is as follows; In section 2, we review the basics of the superconformal index in 3-dimensions. In section 3, we review the $SL(2,\mathbb{Z})$ transformation of 3d conformal field theories with $U(1)$ flavor symmetry and emphasize $S$-operation in relation to gauging $U(1)$ flavor symmetry. In section 4, we consider the simplest example of $U(1)$ gauge theory with one flavor and carry out gauging/ungauging operation. In section 5, we work out Aharony dualities of $SU(N_c)$ gauge theories with $N_f$ fundamental and anti-fundamental flavors and carry out various tests to find the nice agreements. In section 6, we obtain Giveon-Kutasov dualities for $SU(N_c)$ gauge theories with $N_f$ fundamental and anti-fundamental flavors and with Chern-Simons term. These dualities are obtained from Aharony dualities of the section 5 by giving axial mass to some flavors. In section 7, we obtain Aharony dual of $SU(N_c)$ gauge theories with $N_f$ fundamental and anti-fundamental flavors,
one adjoint $X$ and with the superpotential $W = \text{Tr} X^{n+1}$. Again we put the proposal to various tests and find the perfect matching.

As this work is finished, we receive the paper [24] which has overlap with ours.

2. 3d superconformal index

Let us discuss the superconformal index for $\mathcal{N} = 2$ $d = 3$ superconformal field theories (SCFT). The bosonic subgroup of the 3d $\mathcal{N} = 2$ superconformal group is $SO(2,3) \times SO(2)$. There are three Cartan elements denoted by $\epsilon, j_3$ and $R$ which come from three factors $SO(2)_\epsilon \times SO(3)_{j_3} \times SO(2)_R$ in the bosonic subgroup, respectively. The superconformal index for an $\mathcal{N} = 2$ $d = 3$ SCFT is defined as follows [32]:

$$I(x, t) = \text{Tr}(-1)^F \exp(-\beta' \{Q, S\}) x^{\epsilon + j_3} \prod_a t_a^{F_a}$$

where $Q$ is a supercharge with quantum numbers $\epsilon = \frac{1}{2}, j_3 = -\frac{1}{2}$ and $R = 1$, and $S = Q^\dagger$. The trace is taken over the Hilbert space in the SCFT on $\mathbb{R} \times S^2$ (or equivalently over the space of local gauge-invariant operators on $\mathbb{R}^3$). The operators $S$ and $Q$ satisfy the following anti-commutation relation:

$$\{Q, S\} = \epsilon - R - j_3 := \Delta.$$  (2.2)

As usual, only BPS states satisfying the bound $\Delta = 0$ contribute to the index, and therefore the index is independent of the parameter $\beta'$. If we have additional conserved charges $F_a$ commuting with the chosen supercharges $(Q, S)$, we can turn on the associated chemical potentials $t_a$, and then the index counts the number of BPS states weighted by their quantum numbers.

The superconformal index is exactly calculable using the localization technique [33, 34]. It can be written in the following form:

$$I(x, t) = \sum_{m \in \mathbb{Z}} \int da \frac{1}{|W_m|} e^{-S_{C/S}^{(0)}(a, m)} e^{ib_0(a, m)} \prod_a t_a^{q_a(m)} x^{x_a(m)} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_{\text{tot}}(e^{in}, t^n, x^n) \right].$$  (2.3)

The origin of this formula is as follows. To compute the trace over the Hilbert space on $S^2 \times \mathbb{R}$, we use path-integral on $S^2 \times S^1$ with suitable boundary conditions on the fields. The path-integral is evaluated using localization, which means that we have to sum or integrate over all BPS saddle points. The saddle points are spherically symmetric configurations on $S^2 \times S^1$ which are labeled by magnetic fluxes on $S^2$ and holonomy along $S^1$. The magnetic fluxes are denoted by $\{m\}$ and take values in the cocharacter lattice of $G$ (i.e. in $\text{Hom}(U(1), T)$, where $T$ is the maximal torus of $G$), while the eigenvalues of the holonomy are denoted $\{a\}$ and take values in
$S_{CS}^{(0)}(a, m)$ is the classical action for the (monopole-holonomy) configuration on $S^2 \times S^1$, $\epsilon_0(m)$ is the Casimir energy of the vacuum state on $S^2$ with magnetic flux $m$, $q_{0a}(m)$ is the $F_a$-charge of the vacuum state, and $b_0(a, m)$ represents the contribution coming from the electric charge of the vacuum state. The last factor comes from taking the trace over a Fock space built on a particular vacuum state. $|\mathcal{W}_m|$ is the order of the Weyl group of the part of $G$ which is left unbroken by the magnetic fluxes $m$. These ingredients in the formula for the index are given by the following explicit expressions:

$$S_{CS}^{(0)}(a, m) = i \sum_{\rho \in R_F} k\rho(m)\rho(a),$$

$$b_0(a, m) = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)|\rho(a),$$

$$q_{0a}(m) = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)|F_a(\Phi),$$

$$\epsilon_0(m) = \frac{1}{2} \sum_{\Phi} (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\alpha(m)|,$$

where $\sum_{\rho \in R_F} \sum_{\Phi} \sum_{\rho \in R_\Phi}$ and $\sum_{\alpha \in G}$ represent summations over all fundamental weights of $G$, all chiral multiplets, all weights of the representation $R_\Phi$, and all roots of $G$, respectively.

We will find it convenient to rewrite the integrand in (2.3) as a product of contributions from the different multiplets. First, note that the single particle index $f$ enters via the so-called Plethystic exponential:

$$P.E.(f(x, t, e^{ia}) = f_{\text{vector}}(x, e^{ia}) + f_{\text{chiral}}(x, t, e^{ia}),$$

$$f_{\text{vector}}(x, e^{ia}) = -\sum_{\alpha \in G} e^{ia(\alpha)|x|\alpha(m)|},$$

$$f_{\text{chiral}}(x, t, e^{ia}) = \sum_{\Phi} \sum_{\rho \in R_\Phi} \left[ e^{i\rho(a)(x|\rho(m)|+\Delta_\Phi/t_a} - e^{-i\rho(a)}t_a^{-F_a(\Phi)(x|\rho(m)|+2-\Delta_\Phi)} \right]$$

where $\sum_{\rho \in R_F} \sum_{\Phi} \sum_{\rho \in R_\Phi}$ and $\sum_{\alpha \in G}$ represent summations over all fundamental weights of $G$, all chiral multiplets, all weights of the representation $R_\Phi$, and all roots of $G$, respectively.

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$$P.E.(f(x, t, e^{ia}, m)) \equiv \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} f(x^n, t^n, z^n = e^{ina}, m) \right)$$

while we define $z_j = e^{ia_j}$. Specifically, consider a single chiral field $\Phi$, whose single particle index is given by:

$$\sum_{\rho \in R_\Phi} \left( e^{i\rho(a)/F_a(\Phi)}\frac{e^{|\rho(m)|+\Delta_\Phi}}{1-x^2} - e^{-i\rho(a)}t_a^{-F_a(\Phi)}\frac{e^{|\rho(m)|+2-\Delta_\Phi}}{1-x^2} \right).$$

1Note that $a$ in $\rho(a)$ and the subscript $a$ in $t_a$ or $f_a$ denotes the different object.
whose conserved current is
\[ \ast \text{current to background gauge field we introduce } BF \]

where:

\[ \text{terms needed for supersymmetric completion. This introduces to the index} \]

the chemical potential for \( U \)

where

\[ \text{Note that by shifting } t_a \rightarrow t_a x^a, \text{ one can change the value of the R-charge } \Delta_\Phi. \]

Hence \( \Delta_\Phi \) remains the free parameter for generic cases.

The above is the ordinary superconformal index. We need two more generalizations for later purposes. The first one is the notion of the generalized index. When we turn on the chemical potential \( t_a \), which can be regarded as turning on a Wilson line for a fixed background gauge field. The generalized index is obtained when we turn on the nontrivial magnetic flux \( n_a \) for the corresponding background gauge field.

Only the contribution to the chiral multiplets are changed and this is given by the replacement \( \rho(m) \rightarrow \rho(m) + \sum_a f_a(\Phi) n_a \)

\[ Z_\Phi(x, t, z, m) = \prod_{\rho \in R_\Phi} \left( e^{(1-\Delta_\Phi)} e^{-i\rho(a) \sum_a x^a t_a - f_a(\Phi)} \right)^{|\rho(m)|/2 + \sum_a f_a(\Phi) n_a / 2} \]

\[ \text{P.E.} \left( e^{i\rho(a) t_a f_a(\Phi) x^{\rho(m)} + \sum_a f_a(\Phi) n_a + \Delta_\Phi} - e^{-i\rho(a) t_a - f_a(\Phi) x^{\rho(m)} + 2 - \Delta_\Phi} \right) (2.9) \]

Note that by shifting \( t_a \rightarrow t_a x^a \), one can change the value of the R-charge \( \Delta_\Phi \).

The full index will involve a product of such factors over all the chiral fields in the theory, as well as the contribution from the gauge multiplet. It is given by:

\[ I(x, t) = \sum_{m \in Z} \oint \prod_j \frac{dz_j}{2\pi i z_j} |W_m| e^{-S_{CS}(m,a)} Z_{\text{gauge}}(x, z, m) \prod_{\Phi} Z_\Phi(x, t, z, m) \]

where:

\[ Z_{\text{gauge}}(x, z = e^{i\alpha}, m) = \prod_{\alpha \in \text{ad}(G)} x^{-|\alpha(m)|} \left( 1 - e^{i\alpha(x) x^2|\alpha(m)|} \right), \]

\[ Z_\Phi(x, t, z, m) = \prod_{\rho \in R_\Phi} \left( e^{(1-\Delta_\Phi)} e^{-i\rho(a) \sum_a x^a t_a - f_a(\Phi)} \right)^{|\rho(m)|/2} \]

\[ \text{P.E.} \left( e^{i\rho(a) t_a f_a(\Phi) x^{\rho(m)} + \Delta_\Phi} - e^{-i\rho(a) t_a - f_a(\Phi) x^{\rho(m)} + 2 - \Delta_\Phi} \right) (2.9) \]

Here \( n_a \) should take integer value as does \( m_j \).

For every \( U(N) \) gauge group, we can define another abelian symmetry \( U(1)_T \) whose conserved current is \(*F\) of overall \( U(1) \) factor. To couple this topological current to background gauge field we introduce \( BF \) term \( \int A_{BG} \land \text{trd}A + \cdots \) and terms needed for supersymmetric completion. This introduces to the index

\[ z^n w^{\sum_j m_j} \]

where \( n \) is the new discrete parameter representing the topological charge while \( w \) is the chemical potential for \( U(1)_T \).
3. \textit{SL}(2,\mathbb{Z}) \textit{ action on the 3d CFTs with } U(1) \textit{ symmetry}

Gauging and ungauging of \( U(1) \) factor we adpoted in this paper is closely related to the \( S \)-operation for the 3d conformal field theories (CFTs) with \( U(1) \) flavor symmetry. It was found that there is a \( SL(2,\mathbb{Z}) \) action on the space of 3-dimensional conformal field theories with \( U(1) \) flavor symmetry. This action was first described in \cite{26} as a way to understand the meaning of different choices of boundary conditions for an abelian gauge field in \( AdS_4 \) in the context of \( AdS_4/CFT_3 \). And in \cite{27, 28} such \( SL(2,\mathbb{Z}) \) action on 3d abelian gauge theories with \( U(1) \) flavor symmetry was considered. We closely follow their explanation in the below.

\( SL(2,\mathbb{Z}) \) acts on the space of 3d theories equipped with a specific way to couple a \( U(1) \) flavor symmetry to a background \( U(1) \) gauge field. The \( SL(2,\mathbb{Z}) \) action has two generators \( S, T \) satisfying

\[
S^4 = (ST)^3 = I \tag{3.1}
\]

The \( T \)-operation on the 3d conformal theories only modifies the prescription of how to couple the theory to the background gauge field \( A \), by adding \( \star F = \star dA \) to the conserved current for the background \( U(1) \) symmetry. At the Lagrangian level, this means adding a background Chern-Simons interaction at level \( k = 1 \),

\[
T : \quad \mathcal{L} \to \mathcal{L} + \frac{1}{2}A \wedge dA. \tag{3.2}
\]

On the other hand, the \( S \)-operation changes the structure of the 3d theory by making the background gauge field \( A \) dynamical.\(^2\) Once the old \( U(1) \) flavor symmetry turns into gauge field, it has the new \( U(1) \) flavor current given by \( \star F \) of the gauged \( U(1) \). At the Lagrangian level

\[
S : \quad \mathcal{L} \to \mathcal{L} + A_{\text{new}} \wedge dA \quad (A \text{ dynamical}). \tag{3.3}
\]

Monopole operators for \( A \) are charged under the new \( U(1) \) flavor symmetry, hence this \( U(1) \) is sometimes called topological. Since \( S^2 = C \), if one carries out the gauging twice following the prescription eq. (3.3), the resulting theory is equivalent to the original theory up to charge conjugation. One can also say that gauging \( U(1) \) \( (S) \) is equivalent to ungauging \( U(1) \) up to charge conjugation \( (S^{-1}C) \). When we work out gauging/ungauging \( U(1) \), we are always intended to apply \( S / S^{-1} \) operation.

From the definitions of \( S \) and \( T \), one can prove that the relations \( S^2 = C \) and \( (ST)^3 = I \) hold, where the transformation \( C \) (charge conjugation) just inverts the sign of the background gauge field.

\(^2\) One can add a Yang-Mills kinetic term at intermediate stages in the calculation. But for \( S \) to have the correct properties, one must flow to the IR at the end, and then \( g_{YM} \to \infty \) and this term is removed.
This has a suitable $\mathcal{N} = 2$ generalization. Suppose that we have a theory with $U(1)$ global symmetry coupled to a background vector multiplet $V$. $V$ has a real scalar $\sigma$, two Majorana fermions $\lambda^\alpha$ and the gauge field $A$. This can be dualized to a linear multiplet $\Sigma$

$$V \leftrightarrow \Sigma = D^\alpha D_\alpha V$$

(3.4)

with the lowest component of $\Sigma$ being $\sigma$. Now in order to supersymmetrize $SL(2,\mathbb{Z})$ action, we simply have to substitute $V \Sigma'$ for $A \wedge dA'$. In particular, $S$-operation is given by

$$\mathcal{L}(V) \rightarrow \mathcal{L}(V) + \int d^4\theta \Sigma_{\text{new}} V$$

(3.5)

Now we can apply this idea to obtain Aharony dual of $SU(N)$ gauge theory. The basic idea is that we start from $SU(N)$ gauge theory with matters which are invariant under the charge conjugation so that if we apply $S$-operation twice we are back to the original theory. Given $SU(N)$ gauge group we have obvious global $U(1)$ symmetry and if we apply the $S$-operation we introduces the gauging of $U(1)$ theory with the BF type coupling to the background gauge field. Thus we now have the $U(N)$ gauge theory with the usual $U(1)_T$ topological symmetry for which the monopole operators are charged. This is the typical setting where Aharony duality of $U(N)$ gauge theory is discussed. If we apply $S$-operation again, then we gauge topological symmetry and introduce another $U(1)$ flavor symmetry. By this procedure we are back to our original theory of $SU(N)$ theory assuming the matter contents is charge-conjugation invariant. Thus gauging topological $U(1)$ corresponds to ungauging overall $U(1)$ gauge symmetry. On the other hand, by applying the same $S$-operation to the Aharony dual theory of $U(N)$ theory, we obtain Aharony-dual of $SU(N)$ theory.

For example, if we start from $U(N)$ gauge theory with $N_f$ fundamental flavors, Aharony dual is given by $U(N - N_f)$ gauge theory with $N_f$ flavors with the following superpotentials

$$W = v_+ V_- + v_- V_+ + M q \bar{q}$$

(3.6)

where $M, v_{\pm}$ is the singlet for $U(N - N_f)$ corresponding to the mesons and the monopole operators for $U(N)$ gauge theory. By applying the $S$-operation we obtain $U(1)_T \times U(N - N_f)$ gauge theory with the above superpotential and the monopole operators $v_\pm, V_\pm$ is charged under $U(1)_T$. Furthermore, we have the additional BF type coupling

$$A_T \wedge d\text{Tr}A$$

(3.7)

where $\text{Tr}A$ denotes the overall $U(1)$ gauge field of $U(N - N_f)$. Following the above logic this should be the Aharony dual of $SU(N)$ with $N_f$ flavors. We subject this claim to the various tests in the next section.

Furthermore this logic applies any $SU(N)$ gauge theory with charge-conjugation invaraint matter contents to obtain Aharony dual if the corresponding Aharony dual
of $U(N)$ theory is known. Thus we also consider the theory of $U(N)$ theory with $N_f$ flavors and adjoint matter $X$ with the superpotential $W = \text{Tr}X^{n+1}$ and work out its Aharony dual. One should note that starting from $SU(N)$ theory one can generate whole classes of SCFTs by $SL(2, \mathbb{Z})$ transformation. We are currently working out the details such SCFTs [31].

For later purpose we need to work out how the index would transform under the S-transformation. Suppose that $\tilde{I}(z, s)$ denotes the generalized index with $U(1)$ global symmetry and $z, s$ are respectively chemical potential and magnetic flux associated with the $U(1)$ global symmetry. Let us denote the generalized index of S-transformed theory by $I(u, m)$ with $u, m$ are respectively chemical potential and magnetic flux associated with new $U(1)$ global symmetry. Then $I$ is given by

$$I(w, m) = \sum_{s \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^s z^m \tilde{I}(z, s)$$

(3.8)

The relation between the index and the S-transformed index is well known. It’s convenient to use the charge basis [28].

$$\tilde{I}(z, s) = \sum_{e \in \mathbb{Z}} \tilde{I}(e, s) z^e, \ I(w, m) = \sum_{e \in \mathbb{Z}} I(e, m) w^e$$

(3.9)

Then the right hand side of (3.8) becomes

$$\sum_{s \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^s z^m \tilde{I}(z, s) = \sum_{s \in \mathbb{Z}} \sum_{e \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^s z^{m+e} \tilde{I}(e, s) = \sum_{e \in \mathbb{Z}} w^e \tilde{I}(-m, s)$$

(3.10)

which is equal to $I(w, m) = \sum_{e \in \mathbb{Z}} I(e, m) w^e$. Thus we have

$$I(e, m) = \tilde{I}(-m, e).$$

(3.11)

Note that in the charge basis S-operation takes the simple form

$$\begin{pmatrix} e \\ m \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e \\ m \end{pmatrix}$$

(3.12)

Thus we have

$$\tilde{I}(z, s) = \sum_{e \in \mathbb{Z}} e^e \int \frac{dw}{2\pi i w^{s+1}} I(w, -e).$$

(3.13)

We regard this formula as ungauging $U(1)$. We also have the inverse relation

$$I(u, -m) = \sum_{e \in \mathbb{Z}} u^e \int \frac{dz}{2\pi i z^{m+1}} \tilde{I}(z, e)$$

(3.14)

Let’s denote the generalized index of the theory obtained from S-operation on the theory with index being $I(u, m)$ by $I'(z, s)$. One can easily check that $\tilde{I}(z, s) =$
I'\(z^{-1}, -s\). Or in charge basis \(\bar{I}(e, s) = I'(-e, -s)\), which is the consequence of \(S^2 = C\). If we consider \(U(N)\) theory with \(N_f\) flavors, the theory obtained by S-transformation differs from \(SU(N)\) theory with \(N_f\) flavors by the sign of the charges of the matter. Thus the role of chiral multiplet and the anti chiral multiplet is exchanged. But since we are dealing with the same number of chiral multiplets and the anti chiral multiplets, we obtain the same theory. However when we interpret the index result, we should keep in mind of such sign flipping.

4. Ungauging \(\mathcal{N} = 2\) \(N_f = 1\) SQED and its Aharony dual

Let’s consider the simplest example of \(\mathcal{N} = 2\) Aharony dual pair. \(\mathcal{N} = 2\) SQED with \(N_f = 1\) flavor. Its dual is given by XYZ model. In using the convention of the section 3, this is the theory with no gauge group with the superpotential

\[
W = v_+ v_- M
\]  

(4.1)

where \(v_\pm\) is charged under \(U(1)_T\). If we ungauging \(U(1)\) gauge group for SQED, we are left with the free theory with \(N_f = 1\) flavor, since this corresponds to \(S^{-1}\)-operation. This is the \(\mathcal{N} = 4\) theory with one free hypermultiplet. On the other hand, if we gauge \(U(1)_T\) of XYZ model, we obtain \(U(1)\) theory with \(N_f = 1\) flavor \(v_\pm\) with additional neutral chiral multiplet \(M\) whose superpotential is given by eq. (4.1). This is \(\mathcal{N} = 4\) SQED with one hypermultiplet. Thus \(\mathcal{N} = 4\) theory with one free hypermultiplet and \(\mathcal{N} = 4\) SQED with one hypermultiplet are related by \(S^2 = C\). But since the matter content is \(C\)-invariant, we obtain the equivalent theory. This is nothing but the simplest mirror pair. Thus we can regard this mirror pair as a special case of Aharony duality of \(SU(N)\) theory with \(N_f\) flavors with \(N = N_f = 1\).

This simple example also shows that why S-operation involves the duality transformation. Starting from \(\mathcal{N} = 4\) one free hypermultiplet, we obtain SQED with \(N_f = 1\). Under the \(U(1)_T\) symmetry, monopole operators of SQED are charged. Thus in order to carry out the gauging of \(U(1)_T\) we had better go to the frame where the monopole operators are elementary fields. This is possible if we work in the Aharony dual of SQED with \(N_f = 1\). This is nothing but the XYZ model and \(U(1)_T\) is mapped to the usual \(U(1)\) global symmetry. Hence gauging \(U(1)_T\) is straightforward and we obtain \(\mathcal{N} = 4\) SQED with one hypermultiplet. Following this example, we carry out gauging of \(U(1)_T\) for the Seiberg-like dual of original \(U(N_c)\) theories to obtain Seiberg-like dual of \(SU(N_c)\) theories.

4.1 Index of Ungauged SQED \(N_f = 1\)

As a warmup exercise of the index gymnastics, we consider the ungauging of SQED with one flavor. Since our major concern is the gauge symmetry and the topological
$U(1)_T$ symmetry, we will turn off the chemical potential for $U(1)$ axial symmetry. Similar manipulation will be used for handling $U(N)$ theory with $N_f$ flavors. The index formula of the SQED with one flavor is given by

$$
I(w, m_w; x) = \sum_{m_1 \in \mathbb{Z}} \oint \frac{dz_1}{2\pi i z_1} w^{m_1} z_1^{m_w} Z_Q Z_{\tilde{Q}}
$$

$$
= \sum_{m_1 \in \mathbb{Z}} \oint \frac{dz_1}{2\pi i z_1} w^{m_1} z_1^{m_w} Z_\Phi(z_1, m_1; x) \tag{4.2}
$$

where $z_1$ is the holonomy of $U(1)_T$ and $w$ is the chemical potential for the background gauge field coupled to $U(1)_T$. $Z_Q$ and $Z_{\tilde{Q}}$ is some function of $x$, $z_1$ and $m_1$. If we ungauged $U(1)$, we expect to have the free theory with $N_f = 1$. Using the formula eq. (3.14), the ungauged index is

$$
\tilde{I}(z, s; x) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi i w^{s+1}} I(w, m_w; x)
$$

$$
= \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi i w^{s+1}} \left( \sum_{m_1 \in \mathbb{Z}} \oint \frac{dz_1}{2\pi i z_1} w^{m_1} z_1^{m_w} Z_\Phi(z_1, m_1; x) \right) \tag{4.3}
$$

We can expand the $Z_\Phi(z_1, m_1; x)$ in the integer power of $z_1$, such as $Z_\Phi(z_1, m_1; x) = \sum_{n \in \mathbb{Z}} z_1^n \tilde{Z}_\Phi(n, m_1; x)$. So eq. (4.3) is equal to

$$
\tilde{I}(z, s; x) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi i w^{s+1}} \left( \sum_{m_1 \in \mathbb{Z}} \oint \frac{dz_1}{2\pi i z_1} w^{m_1} z_1^{m_w} \sum_{n \in \mathbb{Z}} z_1^n \tilde{Z}_\Phi(n, m_1; x) \right) \tag{4.4}
$$

The above integral has very simple dependence on $w$ and $z_1$. Integration over $z_1$ gives simply the restriction that $n = -m_w$ and the integration over $w$ gives $m_1 = s$. So $\tilde{I}(z, s)$ becomes

$$
\tilde{I}(z, s; x) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \tilde{Z}_\Phi(-m_w, s; x) = Z_\Phi(z, s; x) \tag{4.5}
$$

Explicit form of this ungauged index is $\tilde{I}(z, s; x) = Z_Q Z_{\tilde{Q}}$, where $Z_Q$ and $Z_{\tilde{Q}}$ is

$$
Z_Q = (x^{1-r} z^{-1})^{\lfloor s/2 \rfloor} P.E. \left( \frac{z x^r - z^{-1} x^{2-r}}{1 - x^2} x^{|s|} \right)
$$

$$
Z_{\tilde{Q}} = (x^{1-r} z)^{\lfloor s/2 \rfloor} P.E. \left( \frac{z^{-1} x^r - z x^{2-r}}{1 - x^2} x^{|s|} \right). \tag{4.6}
$$

The resulting index is that of the free theory with a chiral and an anti-chiral field as expected. The chemical potential of the $U(1)$ global symmetry is $z$. 

---

$\end{document}$
4.2 Index of ungauged (or gauged) XYZ

Let’s do the same process to the dual XYZ theory. Then the index of ungauged XYZ theory becomes

\[
\tilde{I}(z, s; x) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi iw^{s+1}} I(w, m_w; x)
\]

\[
= \sum_{m_w \in \mathbb{Z}} \oint \frac{dw}{2\pi iw} w^{-s} z^{-m_w} Z_M Z_{v_+} Z_{v_-}
\]

(4.7)

where

\[
Z_{v_+} = \left( x^{(1-r')} w^{-1} \right)^{m_w/2} P.E. \left( \frac{w x^{2-r'} - w^{-1} x^{2-(1-r')}}{1-x^2} \right)
\]

\[
Z_{v_-} = \left( x^{(1-r')} w \right)^{m_w/2} P.E. \left( \frac{w^{-1} x^{2-r'} - w x^{2-(1-r')}}{1-x^2} \right)
\]

\[
Z_M = x^{1-2r} P.E. \left( \frac{x^{2r} - x^{2-2r}}{1-x^2} \right).
\]

(4.8)

where the R-charge of the \( v_\pm \) fields \( r'' = N_f(1-r) - N_c + 1 = 1-r \). Eq. (4.7) is the generalized index of \( U(1) \) theory with matter \( M, v_\pm \) where \( v_\pm \) has charge \( \pm 1 \) under \( U(1) \). And the original \( U(1) \) gauge symmetry becomes \( U(1) \) topological symmetry of this new theory. If we directly gauge XYZ model, the generalized index has \( w^s \) instead of \( w^{-s} \) in (4.7), which is again due to \( S^2 = C \).

5. Aharony duality for \( SU(N_c) \) gauge theory with \( N_f \) fundamental flavors

| Fields | \( U(1)_R \) | \( U(1)_A \) | \( SU(N_f) \) | \( SU(N_f) \) |
|--------|---------------|---------------|----------------|----------------|
| \( Q \) | \( r \)       | 1             | \( N_f \)      | 1              |
| \( \bar{Q} \) | \( r \)       | 1             | 1              | \( \bar{N}_f \) |
| \( M \) | 2r            | 2             | \( N_f \)      | \( \bar{N}_f \) |
| \( Y \) | 2N_f(1-r) - 2N_c + 2 | -2N_f | 1           | 1              |
| \( q \) | 1 - r         | -1            | \( N_f \)      | 1              |
| \( \bar{q} \) | 1 - r        | -1            | 1              | \( N_f \)      |
| \( v_{\pm} \) | \( N_f(1-r) - N_c + 1 \) | -N_f | 1           | 1              |
| \( V_{\pm} \) | \( N_f(r-1) + N_c + 1 \) | \( N_f \) | 1           | 1              |
| \( u_{\pm} \) | \( N_f(r-1) + N_c \) | \( N_f \) | 1           | 1              |

Table 1: The global symmetry charges of the chiral fields.
Let’s consider the Aharony duality for $SU(N_c)$ gauge theory with $N_f$ flavors. Following the procedure of the previous section, we propose the following:

- **Electric theory:** $SU(N_c)$ gauge theory (without Chern-Simons term), $N_f$ pairs of fundamental/anti-fundamental chiral superfields $Q^a$, $\tilde{Q}^b$ (where $a$, $b$ denote flavor indices).

- **Magnetic theory:** $U(1) \times U(N_f - N_c)$ gauge theory with the BF coupling

$$A_{U(1)} \wedge d\text{Tr} A_{U(N_f-N_c)}$$

, with $N_f$ pairs of fundamental/anti-fundamental chiral superfields $q_a$, $\tilde{q}^a$ of $U(N_f - N_c)$, $N_f \times N_f$ singlet superfields $(M_j)_b^a$, $j = 0, \ldots, n - 1$. We have $v_\pm, V_\pm$ charged under $U(1)$ with charge $\pm 1$. The superpotential is given by

$$W = v_+V_- + v_-V_+ + Mq\tilde{q}$$

where $u_\pm$ is the monopole operator of $U(1)$.

Note that the gauged fields $v_\pm$ do not have the usual $U(1)_R$ charge compared with the elementary fields $Q, q$ since they have the nonperturbative origin. This will lead to interesting dynamics such as the nonperturbative truncation of the chiral ring.

Let’s compare the chiral ring elements of the both sides. For $SU(N_c)$ gauge theory with no superpotential, there are mesons $M_b^a = Q^a\tilde{Q}_b$, a monopole operator $Y$ which parametrizes the Coulomb branch, and baryons of the form

$$B_{a_1\ldots a_{N_f-N_c}} = \epsilon_{a_1\ldots a_{N_f-N_c}b_1\ldots b_{N_c}} \epsilon^{i_1\ldots i_{N_c}} Q^{b_1}_{i_1} \cdots Q^{b_{N_c}}_{i_{N_c}}.$$

And, similarly, there are $\tilde{B} \sim \tilde{Q}^{N_c}$ in the chiral ring. Where $a, b$ and $i$ are the flavor and gauge indices $a, b = 1, \ldots, N_f$ and $i = 1, \ldots, N_c$. In the magnetic theory $Y$ is mapped to $v_+v_-$, mesons are mapped to singlet fields $M$. Baryon fields $B, \tilde{B}$ are mapped to the monopole operators $\tilde{b} \sim u_+\tilde{q}^{a_1} \cdots \tilde{q}^{a_{N_c}}$ and $b \sim u_-q_{a_1} \cdots q_{a_{N_c}}$ where the flavor and gauge indices are totally anti-symmetric. The structure is a baryon operator of $SU(N_f - N_c)$ coupled to the basic monopole operator $u_\pm$. This structure is required due to the BF coupling. We can view the monopole operators as states on $S^2 \times R$ by operator-state correspondence of conformal field theories. When we turn on unit monopole of $A_{U(1)}$, due to the BF coupling (5.1), $N_f - N_c$ matters should couple.\(^3\) Due to the residual gauge invariance of $SU(N_f - N_c)$ the allowed operator should have the form $u_+\tilde{q}^{a_1} \cdots \tilde{q}^{a_{N_c}}$ and $u_-q_{a_1} \cdots q_{a_{N_c}}$. Note that number

\(^3\)In our convention, each $q_a$ has the charge $1/N_c$ with $\tilde{N}_c = N_f - N_c$ under the overall $U(1)$ of $U(\tilde{N}_c)$. 

of the baryons of both sides are the same $N_f C_{N_c} = N_f C_{N_f-N_c} = \frac{N_f!}{(N_f-N_c)!N_c!}$. $u_+ u_-$ is Q-exact since it has no BPS charge to protect. It is truncated nonperturbatively.$^4$

It’s worthwhile to consider the special cases. When $N_f = N_c \neq 1$ the dual is simply given by $U(1)$ gauge theory with singlet $M$, unit charged matter $v_\pm$ with the superpotential

$$W = v_+ v_- \det M$$

(5.4)

Note that the superpotential is inherited from the Aharony dual of $U(N_c)$ theory with $N_f = N_c$. The electric theory has two baryon operators $B, \tilde{B}$. The corresponding operator in the magnetic theory is given by $u_+, u_-$. For $N_f = N_c \neq 1$, the theory can also be described by mesons $M$, baryon fields $B, \tilde{B}$ and monopole field $Y$ with the superpotential $^5$

$$W = -Y (\det M - B \tilde{B})$$

(5.5)

One can check that both theories have the same chiral ring structure and the same superconformal index. Note that for the $U(1)$ theory with the superpotential (5.4), the corresponding chiral ring relation $\det M - u_+ u_- = 0$ should be generated by the $U(1)$ gauge dynamics.$^5$

When $N_f = N_c = 1$ the dual is $U(1)$ gauge theory with singlet $M$, charged matter $v_\pm$ with the superpotential

$$W = v_+ v_- \det M$$

(5.6)

This was already discussed in the previous section.

Also interesting case is $N_c = 1$ with arbitrary $N_f > 1$. The electric theory is free theory with $Q, \tilde{Q}$. The magnetic theory is given by $U(1) \times U(N_f-1)$ belonging to the generic case. The interesting thing is that we have $2N_f$ free fields $Q, \tilde{Q}$. These are matched by $u_+ \tilde{q}^{a_1} \cdots \tilde{q}^{a_{N_f-1}}$ and $u_- q_{a_1} \cdots q_{a_{N_f-1}}$. Apparently the chiral ring element $v_+ v_-$ exists in the magnetic side, which has no counterpart in the electric side. The truncation of this element occurs nonperturbatively. This is similar to what happens to $N = 2$ $U(1)$ theory with $N_f > 1$ flavors. The monopole operator $v'_+, v'_-$ in this theory has the same quantum number as the above $v_+, v_-$. Since the product of $v'_+ v'_-$ has no BPS charge to protect, it is truncated nonperturbatively. We think the similar thing happens in the case at hand as well. Indeed one can see this operator is canceled by a suitable fermionic operator in the index computation.

Of course when $N_f = 1$, $v_+, v_-$ has the quantum number of elementary fields. In this case due to the usual superpotential $W = v_+ v_- M$, $v_+ v_-$ is Q-exact.

$^4$This is the special for $U(1)$ theory. For $U(N_c > 1)$ theory, $u_+ \sim (1, 0, \cdots), u_- \sim (-1, 0, \cdots)$ where we denote the monopole charge for Cartans $U(1)^{N_c}$ of $U(N_c)$. $u_+ u_-$ is a BPS state.

$^5$This was pointed out by the authors of $^24$. after the first version of the draft was distributed. We thank them for making this point clear.
5.1 Index of $SU(N_c)$ theory obtained from ungauging $U(N_c)$

Now let’s consider the index of the electric $U(N_c)$ theory with $N_f$ flavors.

$$I(w, m_w; x) = \sum_{m_i \in \mathbb{Z}} \frac{1}{Sym} \oint \prod_{i=1}^{N_c} \frac{dz_i}{2\pi i z_i} w^{m_1 + \cdots + m_{N_c}} (z_1 \cdots z_{N_c})^{m_w} Z_{\text{gauge}} Z_Q^{N_f} Z_{\tilde{Q}}^{N_f}.$$ (5.7)

where $w$ is the fugacity for $U(1)_T$ and $z_i$ denotes the holonomy variable of the Cartans of $U(N)$. $z = z_1 z_2 \cdots z_{N_c}$ is the holonomy variables of the overall $U(1)$, $Z_Q$ and $Z_{\tilde{Q}}$ is given by

$$Z_Q = \prod_{i=1}^{N_c} (x^{1-r} z_i^{-1})^{m_i/2} P.E. \left( \frac{z_i x^r - z_i^{-1} x^{2-r}}{1 - x^2} \right)^{|m_i|},$$

$$Z_{\tilde{Q}} = \prod_{i=1}^{N_c} (x^{1-r} z_i^{-1})^{m_i/2} P.E. \left( \frac{z_i^{-1} x^r - z_i x^{2-r}}{1 - x^2} \right)^{|m_i|}. \quad (5.8)$$

And $Z_{\text{gauge}}$ is

$$Z_{\text{gauge}} = x^{-\sum_{i<j}^{N_c} m_i - m_j} \prod_{i<j}^{N_c} (1 - z_i z_j^{-1} x^{m_i - m_j}) (1 - z_i^{-1} z_j x^{-m_j - m_i}). \quad (5.9)$$

Equivalently this can be viewed as gauging overall $U(1)$ global symmetry of $SU(N_c)$ with additional BF term $A_{\text{new}} \wedge dA_{U(1)}$ where $U(1)$ symmetry acts as $Q_i \rightarrow e^{i\theta} Q_i$. Thus quark has charge 1 under this $U(1)$ symmetry. Thus the same index can be written as

$$I(w, m_w; x) = \sum_{m_i \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^s z^m I_{SU(N_c)}(z, s)$$ (5.10)

where $I_{SU(N_c)}(z, s)$ is the generalized index of $SU(N_c)$ theory. Comparing it with eq. (5.7) we find that

$$z = z_1 z_2 \cdots z_{N_c}, \quad s = m_1 + m_2 + \cdots m_{N_c}. \quad (5.11)$$

Let’s just denote that \( \frac{1}{Sym} Z_{\text{gauge}} Z_Q^{N_f} Z_{\tilde{Q}}^{N_f} = Z_{\Phi}(m_1, z_1, m_2, z_2, \cdots, m_{N_c}, z_{N_c}; x) \) for simplicity. Concentrating on an arbitrary $z_i$ out of $z_1, \cdots, z_{N_c}$, say $z_i = z_{N_c}$, we can expand $Z_{\Phi} = \sum_{n \in \mathbb{Z}} z_{N_c}^{-n} Z_{\Phi}(\cdots, m_{N_c}, n)$. Then by rewriting (5.7) using this expansion of $z_{N_c}$, and ungauging $U(1)$ of $U(N_c)$, it becomes

$$\tilde{I}(z, s) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi i w^{s+1}} I(w, m_w; x)$$

$$= \sum_{m_w, m_i \in \mathbb{Z}} \oint \frac{dw dz_i}{2\pi i w} \left( \prod_{i=1}^{N_c} \frac{dz_i}{2\pi i z_i} \right) w^{m_1 + \cdots + m_{N_c} - s} \left( \frac{z_1 \cdots z_{N_c}}{z} \right)^{m_w} \sum_{n \in \mathbb{Z}} z_{N_c}^{-n} \tilde{Z}_{\Phi}. \quad (5.12)$$
By integrating $w$ and $z_{N_c}$, we obtain

$$
\tilde{I}(z, s) = \sum_{m_w \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} \left( \prod_{i=1}^{N_c-1} \frac{dz_i}{2\pi i z_i} \right) \sum_{m_w \in \mathbb{Z}} \left( \frac{z}{z_1 \cdots z_{N_c-1}} \right)^{m_w} \tilde{Q}(\cdots, s - m_1 - \cdots - m_{N_c-1}, m_w).
$$

This means that the $z_{N_c}$ is changed by the $\left( \frac{z}{z_1 \cdots z_{N_c-1}} \right)$ and the $m_{N_c}$ is changed by the $s - m_1 - \cdots - m_{N_c-1}$ from the originial $U(N_c)$ gauge theory, which we already find at eq. (5.11). Thus we recover the generalized index of $SU(N_c)$ gauge theory with the global $U(1)$ with fugacity $z$ and charge $s$, as expected. By setting $z = 1$ and $s = 0$, we obtain the ungauged index $\tilde{I}(1, 0)$, which is equal to the index of usual $SU(N_c)$ theory with $N_f$ flavors.

### 5.2 Gauging $U(1)_T$ of magnetic $U(N_f - N_c)$ theory

Index of magnetic theory with $U(N_f - N_c = N_c)$ gauge theory is

$$
I = \sum_{m_w \in \mathbb{Z}} \frac{1}{\text{Sym}} \oint \prod_{i=1}^{N_c} \frac{dz_i}{2\pi i z_i} \left( z_{m_1} \cdots z_{m_{N_c}} \right)^{m_w} Z_{\text{gauge}} Z_q^{N_f} Z_{\tilde{q}}^{N_f} Z_M Z_{v_+} Z_{v_-.}
$$

Where

$$
Z_q = \prod_{i=1}^{N_c} \left( x^{(1-r')} z_i^{-1} \right)^{|m_i|/2} P.E. \left( \frac{x_i^{1-r'} - z_i^{-1} x_i^{2-r'}}{1 - x^2} \right),
$$

$$
Z_{\tilde{q}} = \prod_{i=1}^{N_c} \left( x^{(1-r')} z_i \right)^{|m_i|/2} P.E. \left( \frac{x_i^{-1} x^{2-r'} - z_i x^{2-r'}}{1 - x^2} \right),
$$

$$
Z_M = x^{1-2r} P.E. \left( \frac{x^{2r} - x^{2-2r}}{1 - x^2} \right),
$$

$$
Z_{v_+} = \left( x^{(1-r'')} w^{-1} \right)^{|m_w|/2} P.E. \left( \frac{w x^{r''} - w^{-1} x^{2-r''}}{1 - x^2} \right),
$$

$$
Z_{v_-} = \left( x^{(1-r'')} w \right)^{|m_w|/2} P.E. \left( \frac{w^{-1} x^{r''} - w x^{2-r''}}{1 - x^2} \right).
$$

The R-charge of chiral field $q$ is $r' = 1 - r$ and R-charge of $v_{\pm}$ is $r'' = N_f (1-r) - N_c + 1$.

Let's ungage $U(1)_T$ of the magnetic theory

$$
\tilde{I}(z, s) = \sum_{m_w \in \mathbb{Z}} z^{-m_w} \oint \frac{dw}{2\pi i w^{s+1}} I(w, m_w; x)
$$

$$
= \sum_{m_w \in \mathbb{Z}} \oint \frac{dw}{2\pi i w} w^{m_1 + \cdots + m_{N_c} - s} \left( \frac{z_1 \cdots z_{N_c}}{z} \right)^{m_w} Z_{v_+} Z_{v_-} \times (\cdots) \tag{5.14}
$$
Where the $\times(\cdots)$ term is independent of $w$ and $m_w$. Hence we can find $v_{\pm}$ are gauged in the index expression. Terms denoted above are just the form of the $U(1)$ gauge theory with fugacity $w$, charge $m_w$. Also from the functional form of $w, z$ one can see that there is a BF coupling between gauged $U(1)_T$ and overall $U(1)$ factor of $U(N_f - N_c)$.

5.3 Results of Indices

We can check the dualities by using superconformal indices. We expanded the indices about $x$ upto some orders and compared the dual pairs for $0 < N_c, N_f < 3$. We turned off the ungauged global $U(1)$ symmetry by setting fugacity variable $z = 1$ and charge $s = 0$.

| $(N_c, N_f)$ | Electric $SU(N_c)$ | Magnetic $U(1) \times U(N_f - N_c)$ | Index ($r$ is the IR $R$-charge of $Q$) |
|--------------|---------------------|-----------------------------------|--------------------------------------|
| (1, 1)       | free $SU(1)$        | $U(1)$                            | $1 - 4x^2 - 5x^4 + x^{4-2r} - 2x^{2-r} + 2x^r + 3x^{2r} + 4x^{3r} + 5x^{4r} - 4x^{2+2r} - 4x^{2+2r} + \cdots$ |
| (1, 2)       | free $SU(1)$        | $U(1) \times U(1)$               | $1 - 16x^2 + 6x^{4-2r} - 4x^{2-r} + 4x^r + 10x^{2r} + 20x^{3r} - 36x^{2+2r} + \cdots$ |
| (1, 3)       | free $SU(1)$        | $U(1) \times U(2)$               | $1 - 36x^2 + 15x^{4-2r} - 6x^{2-r} + 6x^r + 21x^{2r} + 56x^{3r} - 120x^{2+2r} + \cdots$ |
| (1, 4)       | free $SU(1)$        | $U(1) \times U(3)$               | $1 - 64x^2 + 28x^{4-2r} - 8x^{2-r} + 8x^r + 36x^{2r} + 120x^{3r} - 280x^{2+2r} + \cdots$ |
| (2, 2)       | $SU(2)$             | $U(1)$                            | $1 - 16x^2 + 88x^4 + x^{4-8r} + x^{2-4r} + 26x^{4-2r} + 6x^{2r} + 20x^{4r} + 50x^{6r} + 105x^{8r} - 64x^{2+2r} - 160x^{2+4r} + \cdots$ |
| (2, 3)       | $SU(2)$             | $U(1) \times U(1)$               | $1 - 36x^2 + 558x^4 + x^{8-12r} + x^{4-6r} + 21x^{4-2r} + 15x^{2r} + 105x^{4r} + 490x^{6r} - 384x^{2+2r} + \cdots$ |
| (2, 4)       | $SU(2)$             | $U(1) \times U(2)$               | $1 - 64x^2 + 188x^4 + x^{12-16r} + x^{6-8r} + 36x^{4-2r} + 28x^{2r} + 336x^{4r} - 1280x^{2+2r} + \cdots$ |
| (2, 5)       | $SU(2)$             | $U(1) \times U(3)$               | $1 - 100x^2 + 4750x^4 + x^{16-20r} + x^{8-10r} + 55x^{4-2r} + 45x^{2r} + 825x^{4r} - 3200x^{2+2r} + \cdots$ |
| (3, 3)       | $SU(3)$             | $U(1)$                            | $1 + 82x^2 + x^{2-6r} + 9x^{2-4r} + 36x^{2-2r} + 9x^{2r} + 2x^{3r} + 45x^{4r} + 18x^{5r} + 167x^{6r} + 90x^{7r} + 513x^{8r} + 332x^{9r} - 18x^{2+2r} + 81x^{2+2r} - 162x^{2+3r} + \cdots$ |
| (3, 4)       | $SU(3)$             | $U(1) \times U(1)$               | $1 - 32x^2 + x^{4-8r} + 16x^{4-6r} + 100x^{4-4r} + 16x^{2r} + 8x^{3r} + 136x^{4r} + 120x^{5r} + 836x^{6r} - 48x^{2+2r} - 480x^{2+2r} + \cdots$ |
| (3, 5)       | $SU(3)$             | $U(1) \times U(2)$               | $1 - 50x^2 + x^{6-10r} + 25x^{6-8r} + 25x^{2r} + 20x^{3r} + 325x^{4r} + 450x^{5r} - 100x^{2+2r} + \cdots$ |
For example, for $N_c$ we pick copies of $x$ and, for $N_f$ meson superfields give $N_f$, $\tilde{N}_f$ two chiral fields give $M, Y, B$, which gives constraint that $\det M$, $\det Y$, $\det B$, $\det \tilde{B}$ out the gauge invariant operators of the first few lower orders of the indices. For the case of the electric theory, $SU(N_c)$ gauge theory with no superpotential, there are mesons $M^a_b = Q^a \tilde{Q}_b$, a monopole operator $Y$, and baryons of the form $B \sim Q^{N_c}$, $\tilde{B} \sim \tilde{Q}^{N_c}$ with $a, b$ the flavor indices $a, b = 1, \ldots, N_f$. The lowest components of meson superfields give $N_f^2 x^{2r}$ to the index. From the bosonic components $B$ and $\tilde{B}$, we pick $N_c$ of chiral fields among $N_f$ so that it gives $N_f C_{N_c, x^{N_c r}}$ from each $B$ and $\tilde{B}$. For example, for $N_c = 2$, indices have the contribution

$$I = \cdots + \left( N_f^2 + N_f (N_f - 1) \right) x^{2r} + \cdots$$

and, for $N_c = 3$, indices have

$$I = \cdots + N_f^2 x^{2r} + \frac{1}{3} N_f (N_f - 1)(N_f - 2)x^{3r} + \cdots .$$

There is a monopole operator $Y$ which corresponds to $v_+ v_-$ of the dual theory. Since its R charge is just the twice of $v_\pm$, it gives $x^{2N_f(1-r)+2-2N_c}$ to the indices. For instance, when $N_c = 2$, $N_f = 3$ there is a term $x^{4-6r}$ for $Y$, $x^{8-12r}$ for $Y^2$.

As pointed out in [25] for the case $N_f = N_c$, the theory is described by in terms of $M, Y, B, \tilde{B}$ with the superpotential

$$W = -Y (\det M - B \tilde{B})$$

which gives constraint that $\det M - B \tilde{B} = 0$. One state corresponding to them becomes Q-exact so it doesn’t appear in the index result. We can check it in the index expansion. From $\det M$ we have $N_f^2 H_{N_f, x^{2N_f r}}$, and from $B^2$, $\tilde{B}^2, B \tilde{B}, MB$ and $M \tilde{B}$ we have $(3 + 2N_f^2)x^{2N_f r}$. But actually the coefficient of the term is smaller by 1 than the naive counting.

There are terms like $x^2$. This is due to the mixed contribution of bosonic and fermionic operators. For example consider $N_c = 1$ case. Because these are free chiral theories so just $Q$ and $\tilde{Q}$ are chiral ring elements. $Q$ and $\tilde{Q}$ contribute $2N_f x^r$ to the index. Fermion components of chiral fields give $-2N_f x^{2-r}$, and to the next order, two chiral fields give $2N_f H_2 x^{2r}$. When fermions coupled to $Q$, $\tilde{Q}$ they gives $(2N_f)^2$ copies of $x^r \times x^{2-r}$ so there are $-(2N_f)^2 x^2$.

| $(3, 6)$ | $SU(3)$ | $U(1) \times U(3)$ | $1 - 72x^2 + x^{8-12r} + 36x^{8-10r} + 36x^{4-2r} + 36x^{2r} + 40x^{3r} + 666x^{4r} - 180x^{2+r} + \cdots$ |
|---|---|---|---|

**Table 2**: The results of the superconformal index computation.

\[ ^6_{N_f}C_{N_c} = \binom{N_f}{N_c} = \frac{N_f!}{N_c!(N_f-N_c)!} \]

\[ ^7_{n}H_{m,n} = \frac{(n+m-1)!}{(n-1)!m!} \]
As alluded before, the chiral ring elements of magnetic theory corresponding to $B$ and $\bar{B}$ are $\bar{b} \sim u_+ \tilde{q}^{a_1} \cdots \tilde{q}^{a_{s_{\tilde{c}}}}$ and $b \sim u_- q_{a_1} \cdots q_{a_{s_c}}$ where the flavor and gauge indices are totally anti-symmetric. Since $u_+$ and $q$, $\tilde{q}$ have R-charge $r_u = N_f(1-r) - N_c$ and $1-r$ respectively, $b$ and $\bar{b}$ have R-charge $N_f(1-r) - N_c + \bar{N}_c(1-r) = N_c r$. Since we pick $\bar{N}_c = N_f - N_c$ of $q$ and $\tilde{q}$ out of $N_f$, $b$ and $\bar{b}$ together give $2 \times \binom{N_f}{N_f-N_c} x^{N_c r}$ to the index.

For the case of $N_c = 1$, on the other hand, there is no monopole operator in electric theories because they are just free chiral field theories. But there are still $v_\pm$ exist in the dual theories which possibly can make the chiral ring element $v_+ v_-$. We propose that this chiral ring element is truncated nonperturbatively. $v_- v_+$ gives $x^{N_f(2-2r)}$ but it is canceled by the contribution of fermionic partner of det$M$ which gives $-x^{N_f(2-2r)}$.

When $N_f = N_c = 1$, the magnetic theory is $U(1)$ theory with 3 fields $v_\pm, M$ with superpotential $W = v_+ v_- M$. $v_\pm$ give $2x^r$ to the index. Due to the superpotential $W = v_+ v_- M$, $v_+ v_-$ are Q-exact. So the singlet $M$ gives $x^{2r}$ to the index and monopoles from charge sector of 2 and $-2$, $u_+^2$ and $u_-^2$ respectively, give $2x^{2r}$ to the index. In sum, we have $3x^{2r}$. One can go to higher orders if one wishes.

6. Chern-Simons theory of $SU(N_c)_k$ and it’s dual

In the standard fashion, the duality of Chern-Simons theory of $SU(N_c)$ gauge theory can be obtained from the Aharony duality of $SU(N_c)$ gauge theory. By giving some of the flavors axial mass, one can generate Chern-Simons term. We start with $SU(N_c)$ theory with $N_f + k$ flavors. Matters have axial charge +1 and integrating of $k$ matters gives CS level $k > 0$. The dual theory starts with $U(1) \times U(N_f + k - N_c)$ gauge theory with BF term $A_{U(1)} \wedge d\text{Tr} A$, where Tr$A$ denotes the overall $U(1)$ of $U(N_f + k - N_c)$. Integrating $k$ of $q$, $\tilde{q}$ gives CS $-k$ of $U(N_f + k - N_c)$, since $q$, $\tilde{q}$ have axial charge $-1$. Integrating out $v_\pm$ gives CS level $-1$ for $U(1)$ gauge field. Thus the dual theory is given by $U(1)_{-1} \times U(N_f + k - N_c)_{-k}$ with $N_f$ flavors only charged under $U(N_f + k - N_c)$ and the BF term $A_{U(1)} \wedge d\text{Tr} A$. Subscript of the gauge group denotes the Chern-Simons level. Note that the above theory is charge-conjugation invariant. Now no field is charged under $U(1)_{-1}$ so it can be integrated out. Thus we obtain the following duality;

- Electric theory: $SU(N_c)_k$ gauge theory, $N_f$ pairs of fundamental/anti-fundamental chiral superfields $Q^a$, $Q_b$ (where $a$, $b$ denote flavor indices) with Chern-Simons level $k$.

- Magnetic theory: $U(N_f + k - N_c)$ gauge theory, $N_f$ pairs of fundamental/anti-fundamental chiral superfields $q_a$, $\tilde{q}^a$ of $U(N_f + k - N_c)$, $N_f \times N_f$ singlet su-
perfields \((M_j)_j^g, j = 0, \ldots, n - 1\). The superpotential is given by

\[
W = Mq\bar{q}.
\]  

(6.1)

with the Chern-Simons term

\[
\tilde{A} \wedge d\tilde{A} - k(A_{Nc} \wedge dA_{Nc} - \frac{2i}{3} A_{Nc} \wedge A_{Nc} \wedge A_{Nc})
\]  

(6.2)

where \(A_{Nc}\) is the \(U(Nc) = U(Nf + k - Nc)\) gauge field and \(\tilde{A} = \text{Tr} A_{Nc}\) is the overall \(U(1)\) gauge field.

Again we can discuss the chiral ring elements. In the electric side we have mesons \(M\), and baryons \(B, \tilde{B}\). In the magnetic side, mesons are trivially matched. The baryon operators are mapped to the monopole operators of the form \(u_+ \tilde{q}^{a_1} \cdots \tilde{q}^{a_{Nf-Nc}}\) and \(u_- q_1 \cdots q_{Nf-Nc}\). If we turn on the unit flux of the overall \(U(1)\) of \(U(Nf+k-Nc)\), Gauss constraint dictates that we should turn on \(Nf-Nc\) matters. Using the residual gauge invariance, one can choose the color index running from 1 to \(Nf-Nc\).

The index of Chern-Simons \(SU(Nc)\) theory is

\[
I = \sum_{m_i \in \mathbb{Z}} \frac{1}{8ym} \oint \prod_{i=1}^{Nc} \frac{dz_i}{2\pi iz_i} (-z_i)^{km_i} Z_{gauge} Z_I^{Nf} Z_{\bar{Q}}^{Nf},
\]  

(6.3)

where \(Z_{Q}, Z_{\bar{Q}}, Z_{gauge}\) are given by eq. (5.8), (5.9). Here \(z_{Nc} = (z_1 \cdots z_{Nc-1})^{-1}\) and \(m_{Nc} = -(m_1 + \cdots + m_{Nc-1})\).

The index of dual Chern-Simons \(U(Nf+k-Nc)\) is

\[
I = \sum_{m_i \in \mathbb{Z}} \frac{1}{8ym} \oint \prod_{i=1}^{Nc} (-1)^{-km_i+m_i} z_i^{-km_i+m_i+\cdots+m_{Nc}} \frac{dz_i}{2\pi iz_i} Z_{gauge} Z_I^{Nf} Z_{\bar{Q}}^{Nf} Z_{M}^{Nf},
\]  

(6.4)

where the same expressions are adopted from (5.13).

Here the \(Nc\) is the rank of the gauge group \(Nc = Nf+k-Nc\). Note that there’s additional sign factor \((-1)^{km_i+m_i}\) at eq. (6.4). Such sign factor \((-1)^{km}\) is observed for every \(U(1)\) factor with CS level \(k\) where \(m\) is the magnetic flux associated with \(U(1)\) gauge group. The origin of such sign factor is described in [28].

We compute the index of the above CS theory for some cases and find the perfect matchings.

| \((Nc, Nf, k)\) | Electric | Magnetic | Index (\(r\) is the IR \(R\)-charge of \(Q\)) |
|-----------------|-----------|-----------|-------------------------------------------|
| \((2, 2, 1)\)   | \(SU(2)\) | \(U(1)\)  | \(1 - 16x^2 + 88x^4 - x^{4-4r} + 20x^{4-2r} + 6x^{2r} + 20x^{4r} - 64x^{2+2r} + \cdots\) |
| \((2, 1, 2)\)   | \(SU(2)\) | \(U(1)\)  | \(1 - 4x^2 - 5x^4 + 4x^6 + 3x^{4-2r} + 4x^{6-2r} + x^{2r} + x^{4r} + x^{6r} - 4x^{4+2r} + \cdots\) |
We examine how gauge invariant BPS operators appear on the index expression. On the electric theory side, since these are Chern-Simons $SU(N)$ gauge theories, chiral ring elements consist of mesons $M$ and baryons $B$, $\tilde{B}$. Mesons $M = Q\tilde{Q}$ contribute $N_f^2 x^{2r}$ to the index and baryons $B \sim Q^{N_c}$, $\tilde{B} \sim \tilde{Q}^{N_c}$ give $2 \times \binom{N_f}{N_c} x^{N_c r}$ to the index. But when $N_f < N_c$ there are no baryons. It is easily seen from the result we have here. For example, when $N_c = 2$, we should look for the coefficient of $x^{2r} = x^{N_c r}$ term. If $N_f = 2$ there is the baryon contribution $I = \cdots + (2^2 + 2(3/2)) x^{2r} + \cdots$ but if $N_f = 1$ there is no such contribution.

Looking for magnetic theory, mesons are trivially mapped to the singlet operator $M$ the magnetic theory always have. As explained before, baryon operators are matched by the monopole operators $u_+ q^a \tilde{q}_{N_f-N_c}$ and $u_- q_{a_1} \cdots q_{a_{N_f-N_c}}$. They give $\binom{N_f}{N_f-N_c}$ for each $\pm 1$ monopole flux, which give the term $2 \binom{N_f}{N_f-N_c} x^{N_c r}$.

### 7. $SU(N)$ theory with $N_f$ fundamental flavors and adjoint matter

| $(2, 3, 1)$ | $SU(2)$ | $U(2)$ |
|------------|---------|--------|
| $1 - 36 x^2 + 558 x^4 - x^{6-6r} + 21 x^{4-2r} + 15 x^{2r} + 105 x^{4r} + 490 x^{6r} - 384 x^{2+2r} + \cdots$ |

| $(2, 2, 2)$ | $SU(2)$ | $U(2)$ |
|------------|---------|--------|
| $1 - 16 x^2 + 88 x^4 + 10 x^{4-2r} + 6 x^{2r} + 20 x^{4r} + 50 x^{6r} + 105 x^{8r} - 64 x^{2+2r} - 160 x^{4+4r} + \cdots$ |

| $(2, 4, 1)$ | $SU(2)$ | $U(3)$ |
|------------|---------|--------|
| $1 - 64 x^2 + 36 x^{4-2r} + 28 x^{2r} + 336 x^{4r} + \cdots$ |

| $(2, 3, 2)$ | $SU(2)$ | $U(3)$ |
|------------|---------|--------|
| $1 - 36 x^2 + 15 x^{2r} + 105 x^{4r} + \cdots$ |

| $(3, 3, 1)$ | $SU(3)$ | $U(1)$ |
|------------|---------|--------|
| $1 - 18 x^2 + 9 x^{2r} + 2 x^{3r} + 45 x^{4r} + 18 x^{5r} + 167 x^{6r} + 90 x^{7r} + 513 x^{8r} + 332 x^{9r} - 18 x^{2+2r} - 162 x^{2+3r} + \cdots$ |

| $(3, 2, 2)$ | $SU(3)$ | $U(1)$ |
|------------|---------|--------|
| $1 - 8 x^2 + 28 x^4 - x^{4-4r} + 8 x^{4-2r} + 16 x^{4-2r} + 4 x^{2r} + 10 x^{4r} - 4 x^{2+2r} - 24 x^{2+2r} + \cdots$ |

| $(3, 4, 1)$ | $SU(3)$ | $U(2)$ |
|------------|---------|--------|
| $1 - 32 x^2 + 16 x^{2r} + 8 x^{3r} + 136 x^{4r} + 120 x^{5r} + 836 x^{6r} - 48 x^{2+2r} - 480 x^{2+2r} + \cdots$ |

| $(3, 3, 2)$ | $SU(3)$ | $U(2)$ |
|------------|---------|--------|
| $1 - 18 x^2 + 9 x^{2r} + 2 x^{3r} + 45 x^{4r} + 18 x^{5r} + 167 x^{6r} + 90 x^{7r} + 513 x^{8r} + 332 x^{9r} - 18 x^{2+2r} - 162 x^{2+3r} + \cdots$ |

| $(3, 2, 3)$ | $SU(3)$ | $U(2)$ |
|------------|---------|--------|
| $1 - 8 x^2 + 4 x^{4-2r} + 4 x^{2r} - 4 x^{2+2r} + \cdots$ |

| $Q$ | $SU(N_f)$ | $SU(N_f)$ | $U(1)_A$ | $U(1)_J$ | $U(1)_R$ |
|-----|--------|--------|--------|--------|---------|
| $N_f$ | 1  | 1  | 1  | 0  | $r$  |
| $\bar{Q}$ | 1  | $\overline{N_f}$ | 1  | 0  | $r$  |
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$X, \tilde{X}$ & $1$ & $1$ & $0$ & $0$ & $\frac{2}{n+1}$ & $2r + \frac{2i}{n+1}$ \\
$M_j$ & $N_f$ & $\overline{N}_f$ & $2$ & $0$ & $-N_f r + N_f - \frac{2}{n+1} (N_c - 1) + \frac{2i}{n+1}$ & $r - \frac{2}{n+1}$ \\
$v_{j,\pm}$ & $1$ & $1$ & $-N_f$ & $\pm 1$ & $N_f r - N_f + \frac{2}{n+1} (N_c + 1) - \frac{2i}{n+1}$ & $nN_f(1-r) + \frac{2n}{n+1} (N_c - 1) + \frac{2n}{n+1}$ \\
$q$ & $\overline{N}_f$ & $1$ & $-1$ & $0$ & $-N_f r + N_f - \frac{2}{n+1} (N_c - 1) + \frac{2i}{n+1}$ & $r - \frac{2}{n+1}$ \\
$\tilde{q}$ & $1$ & $N_f$ & $-1$ & $0$ & $nN_f(1-r) + \frac{2n}{n+1} (N_c - 1) + \frac{2n}{n+1}$ & $r - \frac{2}{n+1}$ \\
$\tilde{v}_{j,\pm}$ & $1$ & $1$ & $N_f$ & $\pm 1$ & $N_f r - N_f + \frac{2}{n+1} (N_c + 1) + \frac{2i}{n+1}$ & $nN_f(1-r) + \frac{2n}{n+1} (N_c - 1) + \frac{2n}{n+1}$ \\
$u_{\pm}$ & $1$ & $1$ & $N_f$ & $0$ & $nN_f(1-r) + \frac{2n}{n+1} (N_c - 1) + \frac{2n}{n+1}$ & $r - \frac{2}{n+1}$ \\
\hline
\end{tabular}
\caption{Quantum numbers of various fields}
\end{table}

One can apply the same procedure to obtain Aharony duality for $N = 2$ $SU(N_c)$
gauge theory with $N_f$ fundamental and anti-fundamental flavors and with one adjoint
matter. The proposed duality is as follows;

- Electric theory: $SU(N_c)$ gauge theory (without Chern-Simons term), $N_f$ pairs
  of fundamental/anti-fundamental chiral superfields $Q^a$, $\tilde{Q}_b$ (where $a$, $b$
denote flavor indices), an adjoint superfield $X$, and the superpotential $W_e = \text{Tr} X^{n+1}$.

- Magnetic theory: $U(1) \times U(nN_f - N_c)$ gauge theory with BF coupling,
  \begin{equation}
  A_{U(1)} \wedge d\text{Tr} A_{U(nN_f - N_c)}
  \end{equation}

$N_f$ pairs of fundamental/anti-fundamental chiral superfields $q_a$, $\tilde{q}^a$ of $U(nN_f -$ $N_c)$, $N_f \times N_f$ singlet superfields $(M_j)^b$, $j = 0, \ldots, n-1$, an adjoint superfield
$\tilde{X}$ of $U(nN_f - N_c)$, $2n$ superfields $v_{0,\pm}, \ldots, v_{n-1,\pm}$ charged under $U(1)$, $2n$
superfields $\tilde{v}_{0,\pm}, \ldots, \tilde{v}_{n-1,\pm}$ charged under $U(1)$ and a superpotential
\begin{equation}
W_m = \text{Tr} \tilde{X}^{n+1} + \sum_{j=0}^{n-1} M_j \tilde{q}^a \tilde{X}^{n-1-j} q^a + \sum_{i=0}^{n-1} (v_{i+,\pm \tilde{v}_{n-1-i,-} + v_{i-,\pm \tilde{v}_{n-1-i,+}}}. \end{equation}

The chiral superfields of the theory have charges under various symmetries as we
specified at the table above.

$v_{0,\pm}$ and $\tilde{v}_{0,\pm}$ are minimal bare monopoles of electric theory and magnetic theory,
respectively. Those correspond to excitation of magnetic flux $(\pm 1, 0, \ldots, 0)$. For
the description of the monopole operators we had better use the operator state
representation to describe the operator as the corresponding state on $R \times S^2$. When
magnetic flux $(\pm 1, 0, \ldots, 0)$ is excited the gauge group $U(N_c)$ is broken to $U(1) \times$ $U(N_c - 1)$. Let
\begin{equation}
X = \begin{pmatrix}
X_{11} & 0 \\
0 & X'
\end{pmatrix}
\end{equation}

where $X'$ is an adjoint field of $U(N_c - 1)$ unbroken gauge group. We denote the
dressed monopole operator $v_{i,\pm} \equiv \text{Tr}(v_{i,\pm} X')$, $i = 1, \ldots, n - 1$ with the trace taken
over $U(1)$. For example $v_{1,\pm} = X_{11}|_{\pm 1, 0, \ldots}$. The details can be found in [29].
Let’s consider the chiral ring elements and how they are mapped. For the adjoint and mesons, we have the following correspondence

\[ \text{Tr} X^i \leftrightarrow \text{Tr} \tilde{X}^i \]
\[ Q^a X^j \tilde{Q}^b \leftrightarrow (M_j)^a_b \]

(7.4)

Where the \( a \) and \( b \) are the flavor indices. We have \( n \) independent monopole operators \( Y_i \) with \( i = 0 \cdots n - 1 \), which are mapped to \( v_{i,+}v_{i,-} \). Baryons can be constructed not only from \( Q, \tilde{Q} \) but also from some combinations of \( X \) and \( Q \) or \( X \) and \( \tilde{Q} \). For example \( X^i Q^i \text{ or } X^i X^i_k Q_k \) can replace one or many \( Q \) in the baryon operators, here \( i, j, k \) are gauge indices. Thus \( B' \sim Q^{N_c-1}(XQ) \) could be a chiral ring element.

On the dual side, baryon like operators \( b, \tilde{b} \) which is a coupled state of the monopole operators \( u_{\pm} \) of the gauged \( U(1) \) and (anti)fundamental and adjoint matters fields. Where baryons could be \( b \sim u_{-}q^{\tilde{N}_c} \text{ or } b' \sim u_{-}q^{\tilde{N}_c-1}(\tilde{X}q) \) where \( \tilde{N}_c = nN_f - N_c \). Anit-baryons \( \tilde{b} \) could be defined similarly, e.g., \( \tilde{b} \sim u_{+}q^{\tilde{N}_c} \text{ or } \tilde{b}' \sim u_{+}q^{\tilde{N}_c-1}(\tilde{X}q) \). The detailed matching of the chiral ring is quite delicate. For the Aharony duality of \( U(N_c) \) gauge theory with an adjoint \( X \), the chiral rings are constrained by characteristic equations of adjoint \( X \) and \( \tilde{X} \). Classically, there are \( N_c \) independent operators \( \text{Tr} X^i, i = 1, \ldots, N_c \) due to characteristic equation of \( X \) which is in \( U(N_c) \) adjoint representation. With a superpotential \( W = \text{Tr} X^{n+1} \) there are \( a \) independent operators \( \text{Tr} X^i, i = 0, \ldots, a \) where \( a = \min(n-1, N_c) \). As explained in [20, 29], if the gauge group of electric side is smaller than that of magnetic side, \( N_c \leq N'_c \), the number of (classical) chiral ring generators of electric side is less than magnetic side. The redundant chiral ring generators of magnetic side are cancelled by some monopole operators. On the other hand, if \( N_c > N'_c \), the electric theory seems to have more chiral ring generators than magnetic theory. But some non-trivial relation of monopole operators reduce the number of state so the chiral ring is again the same.

We expect similar mechanism works here. Especially interesting case is baryon operators. For the electric case, such baryon operators exist for \( N_f \geq N_c \). However, for the magnetic side, the condition is \( N_f \geq \tilde{N}_c = nN_f - N_c \). Unless \( n = 2 \) and \( N_f = N_c \) two conditions are incompatible. Thus we expect the nonperturbative truncation of the chiral ring occurs. Here we examine simple cases of \( n = 2 \) and \( N_f = N_c \) in the below and find the perfect matching. We will explore more general cases in the future work [31].

We compute the indices for some values of \( n, N_c \) and \( N_f \). Results are listed in the table below.

| \( (n, N_c, N_f) \) | Electric \( U(N_c) \) | Magnetic \( U(1) \times U(nN_f - N_c) \) | Index (\( r \) is the IR \( R \)-charge of \( Q \)) |
| (2,1,1) | free | $U(1) \times U(1)$ | $1 + x^{2/3} - 4x^2 - 3x^{8/3} + (3 + 3x^{2/3})x^{2r} + (4 + 4x^{2/3})x^{3r} + 5x^{4r} + x^r(2 + 2x^{2/3} - 4x^2) + x^{-r}(-2x^2 - 2x^{8/3}) + \cdots$ |
| (2,2,1) | $SU(2)$ | $U(1) \times U(0)$ | $1 + x^{2/3} + x^{4/3} - 4x^2 + x^{12r} + x^{10r} - 10r + (1 + 4x^{2/3})x^{4r} + x^{6r} + x^{2r}(1 + 4x^{2/3} + x^{4/3}) + x^{-2r}(x^{2/3} + 2x^{4/3} + x^2) + x^{-4r}(x^{4/3} + 2x^2 + x^{8/3}) + x^{-6r}(x^2 + 2x^{8/3}) + x^{-8r}(x^{8/3} + 2x^{10/3}) + \cdots$ |
| (2,2,2) | $SU(2)$ | $U(1) \times U(2)$ | $1 + x^{2/3} + x^{4/3} + 16x^2 - 31x^{8/3} + x^{16/3} - 8r + 20x^{4r} + x^{2r}(6 + 16x^{2/3} + 6x^{1/3}) + x^{-4r}(x^{8/3} + 2x^{10/3} + x^4) + \cdots$ |
| (2,3,2) | $SU(3)$ | $U(1) \times U(1)$ | $1 + x^{2/3} + 10x^{4/3} + 20x^2 + x^{12r} + 4x^{8/3} - 6r + (10 + 26x^{2/3})x^{4r} + 20x^6r + 4x^5 - 3r + x^{2r}(4 + 8x^{2/3} + 24x^{4/3}) + x^{-2r}(4x^{4/3} + 12x^2) + x^{-4r}(x^{4/3} + 3x^2 + 12x^{8/3}) + x^{-8r}(x^{8/3} + 3x^{10/3}) + \cdots$ |
| (3,2,1) | $SU(2)$ | $U(1) \times U(1)$ | $1 + \sqrt{x} + 2x + x^{3/2} - 3x^2 + x^4 - 8r + x^{4r} + x^{2r}(1 + 4\sqrt{x} + 5x) + x^{-2r}(x + 2x^{3/2} + 3x^2 + 2x^{5/2}) + x^{-4r}(x^2 + 2x^{5/2} + 3x^3) + x^{-6r}(x^3 + 2x^{7/2}) + \cdots$ |
| (3,4,2) | $SU(4)$ | $U(1) \times U(2)$ | $1 + \sqrt{x} + 12x + 47x^{3/2} + 154x^2 + x^{16r} + (1 + 3\sqrt{x})x^{3 - 12r} + 4x^{3 - 10r} + (4 + 16\sqrt{x})x^{2 - 6r} + (20 + 60\sqrt{x})x^6r + 35x^8r + x^{4r}(10 + 26\sqrt{x} + 99x) + x^{2r}(4 + 8\sqrt{x} + 36 + 116\sqrt{x})x + x^{-2r}((4 + 16\sqrt{x})x + 60x^2) + x^{-4r}((1 + 3\sqrt{x})x + (17 + 55\sqrt{x})x^2) + x^{-8r}((1 + 3\sqrt{x})x^2 + 19x^3) + \cdots$ |
| (4,2,1) | $SU(2)$ | $U(1) \times U(2)$ | $1 + x^{2/5} + 2x^{4/5} + 2x^{6/5} + 2x^{8/5} - 3x^2 + (1 + 4x^{2/5})x^{4r} + x^{2r}(1 + 4x^{2/5} + 5x^{4/5} + 8x^{6/5}) + x^{-2r}(x^{6/5} + 2x^{8/5} + 3x^2 + 4x^{12/5}) + x^{-4r}(x^{12/5} + 2x^{14/5}) + \cdots$ |
Table 5: Indices of $SU(N_c)$ gauge theory with adjoint matter and its dual theory.

| $(4, 3, 1)$ | $SU(2)$ | $U(1) \times U(1)$ | $1 + 2x^{2/5} + 6x^{4/5} + x^{2-10r} + x^{5-8r} + (1 + 3x^{2/5}) x^{2r} + x^{4r} + x^{-4r} (x^{4/5} + 4x^{6/5}) + x^{-2r} (x^{2/5} + 4x^{4/5} + 10x^{6/5}) + x^{-6r} (x^{6/5} + 4x^{8/5}) + \cdots$ |

Let’s work out the operator contents for the few lower orders of the index results. Consider $n = N_c = N_f = 2$ case. The $x^{2/3}$ comes from $X$ of the electric side and $\tilde{X}$ of the magnetic theory. In the electric theory, $4x^{2r}$ term comes from the meson contribution $2x^{2r}$ and from baryon contribution $Q_1Q_2, \tilde{Q}_1\tilde{Q}_2$. On the dual side the singlet $M_0$ gives $4x^{2r}$ and $u_q q^1 q^2$ and $u_{q^1} q^2$ gives $2x^{2r}$.

We check another term $16x^{2r+2/3}$. On the electric side, $Tr XQ\tilde{Q}$ counts four, $QX\tilde{Q}$ counts four. Another four come from $Q^a(XQ^b)$ and one from $QQTrX$ but they are not completely independent of each other. They satisfies $Q^1(XQ^2) - Q^2(XQ^1) - Q^1Q^2TrX = 0$. So the baryon like operators give 4, and the similar anti-baryons operators (e.g., $\tilde{Q}^a(X\tilde{Q}^b)$) give 4, summing to $16x^{2r+2/3}$. On the dual magnetic theory, the counting is basically the same as the electric case, except the mesons are mapped to singlet $M_j$ and baryons $B'$ and $\tilde{B}'$ are mapped to $b', \tilde{b}'$. From $Y$, there are $x^{8/3-4r}$ and the same term comes from $v_{0,+} v_{0,-}$ of magnetic theory. There are also $x^{4-4r}$ which comes from $v_{1,+} v_{1,-}$.

We consider one more example. For the case of $N_c = N_f = 1$ and $n = 2$, the electric theory is a free theory with matter fields $Q, \tilde{Q}$ and $X$ and the dual magnetic theory has gauge group $U(1) \times U(1)$. The chiral ring elements of the electric theory is simply the free chiral matter fields $Q, \tilde{Q}$ and $X$. We can read off it from the index expression. For example, adjoint field $X$ gives $x^{2/3}$. Each $Q$ and $\tilde{Q}$ gives $x^r$, leading to $2x^r$ in the index. The term $3x^{2r}$ comes from $QQ, \tilde{Q}\tilde{Q}, Q\tilde{Q}$, the term $3x^{2r+2/3}$ comes from $XQQ, X\tilde{Q}\tilde{Q}, XQ\tilde{Q}$ and so on.

On the dual magnetic theory, the adjoint matter field $\tilde{X}$ correspond to the $X$ of electric theory and it gives $x^{2/3}$ as expected. The monopole operator $u_\pm$ of the gauged $U(1)$ coupled with the matter fields $q, \tilde{q}$, so $u_\pm \tilde{q}$ and $u_\mp q$ correspond to the chiral fields $Q, \tilde{Q}$. We also have a singlet field $M_0, (u_+ q)^2, (u_- q)^2$ giving $3x^{2r}$ which corresponds to $QQ, \tilde{Q}\tilde{Q}, Q\tilde{Q}$ of the electric theory. The index doesn’t have the term $x^{2-2r}$ which may correspond to $v_{0,+} v_{0,-}$ because the superpotential $W = M_0 \tilde{X} q\tilde{q} + M_1 q\tilde{q} + M_0 v_{0,+} v_{0,-}$ makes it Q-exact. The last term of the superpotential or similar terms can be generated for special values of $(n, N_c, N_f)$.

Let’s work out one more example where nonperturbative truncation of the chiral ring occurs. For terms in $x^{2r+2/3}$, naively there are four chiral ring element in the magnetic theory which are $M_1, \tilde{X}M_0, \tilde{X}(u_+ \tilde{q})^2$ and $\tilde{X}(u_- q)^2$. The existence of $M_1$ could be a problem here because the electric theory is a free theory, the cor-
responding operator $\bar{Q}X\bar{Q}$ is not different from $Q\bar{Q}X = XQ\bar{Q}$. This means that $M_1$ should not be a chiral ring element and canceled by some other terms. If we consider R-charges of the possible states, there are four candidates canceling $M_1$, $(\tilde{v}_0, +\tilde{v}_0, -\psi_{v_1}, +\tilde{v}_0, -\psi_{v_1}, -\psi_{v_1})$. But normally they are canceled against each other due to the superpotential $v_{1, \pm}\tilde{v}_0, +\pm\tilde{v}_0, -\pm\tilde{v}_0, -\psi_{v_1}$. But the state $\tilde{v}_0, +\tilde{v}_0, +$ does not exist for the case of $U(1)$ gauge theory because $\tilde{v}_0, +\tilde{v}_0, -$ have opposite charges for the same Cartan sector, so they are paired up. Rest of the candidate states have the same energy states, two of them fermionic and one of them bosonic. Thus the $M_1$ truncated nonperturbatively. Therefore we have the right value $3x^{2r}$ from $\bar{X}M_0$, $\bar{X}(u, q)^2$ and $\bar{X}(u, q)^2$.

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