\( \mathcal{N} = 2 \) Instanton Calculus in Closed String Background

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In this contribution we describe how to obtain instanton effects in four dimensional gauge theories by computing string scattering amplitudes in D3/D(–1) brane systems. In particular we study a system of fractional D3/D(–1) branes in a \( \mathbb{Z}_2 \) orbifold and in a Ramond-Ramond closed string background, and show that it describes the gauge instantons of \( \mathcal{N} = 2 \) super Yang-Mills theory and their interactions with the graviphoton of \( \mathcal{N} = 2 \). Using string theory methods we compute the prepotential of the effective gauge theory exploiting the localization methods of the instanton calculus, showing that this leads to the same information given by the topological string.

\section*{\textsection 1. Introduction}

The relationship between string theory and perturbative field theories has been thoroughly investigated for many years. The study of non-perturbative effects in string theory and their comparison with field theory is instead much more recent. In particular, only after the introduction of D branes it has been possible to significantly improve our knowledge and address some non-perturbative issues using string theory. For instance, it has been shown\(^1\)–\(^3\) that the instanton sectors of \( \mathcal{N} = 4 \) supersymmetric Yang-Mills (SYM) gauge theory can be described in string theory by systems of D3 and D(–1) branes (or D-instantons) in flat space. In fact, the excitations of the open strings stretching between two D(–1) branes or between a D3 brane and a D-instanton, are in one-to-one correspondence with the moduli of the SYM instantons in the so-called ADHM construction (for comprehensive reviews on the subject see, for instance, Ref. 4)).

The above remarks have been further substantiated\(^5\) by showing that the tree-level string scattering amplitudes on disks with mixed boundary conditions for a D3/D(–1) system lead, in the infinite tension limit \( \alpha' \to 0 \), to the effective action on the instanton moduli space of the SYM theory. Furthermore, it has been proved that the same disk diagrams also yield the classical field profile of the instanton solution, and that these mixed disks effectively act as sources for the various components of the gauge supermultiplet. This approach can be easily adapted to describe SYM theories with \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) supersymmetry and their instantons: instead of

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considering D3/D(−1) systems in flat space, one has simply to place the branes at suitable orbifold singularities.

This description of gauge theories by means of systems of D-branes has proven to be very useful also to account for the deformations induced by non-trivial gravitational backgrounds. For instance, non-commutative gauge theories and their instanton sectors can be efficiently described by considering D3/D(−1) brane systems in a Kalb-Ramond background.6) Similarly, by placing the branes in a particular R-R background, non-anti-commutative gauge theories can be described and the corresponding non-anti-commutative instanton solutions can be determined.7) More recently, the string description of instantons has lead to new developments. In fact it has been shown in several different contexts that these stringy instantons may dynamically generate new types of terms in the low-energy effective action of the SYM theory with very interesting phenomenological implications.8)

In this contribution, which is mainly based on Ref. 9), we will concentrate on \( \mathcal{N} = 2 \) SYM theory in four dimensions and study the non-perturbative low energy effective action induced by instantons when a R-R graviphoton background is switched on. We show how to determine the non-perturbative gravitational F-terms in the \( \mathcal{N} = 2 \) prepotential exploiting localization methods.10,11)

The instanton sectors of \( \mathcal{N} = 2 \) SYM theory with gauge group \( SU(N) \) can be described by a system of \( N \) fractional D3 branes and \( k \) fractional D-instantons in the orbifold \( \mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2 \) where the orbifold group \( \mathbb{Z}_2 \) acts as the inversion of the last two complex coordinates. The gauge theory degrees of freedom are described by the massless excitations of the open strings stretching between two D3 branes, and can be assembled into a \( \mathcal{N} = 2 \) chiral superfield in the adjoint of \( SU(N) \)

\[
\Phi(x, \theta) = \phi(x) + \theta A(x) + \frac{1}{2} \theta \sigma^{\mu \nu} \theta F^+_{\mu \nu}(x) + \cdots .
\]  

(1.1)

String amplitudes for the string vertices corresponding to the SYM fields of Eq. (1.1) on disks attached to the D3 branes give rise, in the limit \( \alpha' \to 0 \) with gauge quantities fixed, to the tree level (microscopic) \( \mathcal{N} = 2 \) action for \( SU(N) \) SYM.

We are interested in studying the low energy effective action on the Coulomb branch parameterized by the v.e.v.’s \( \langle \Phi_{uv} \rangle = a_u \delta_{uv} \) of the adjoint chiral superfields breaking \( SU(N) \to U(1)^{N-1} \). From now on we will focus for simplicity on the \( SU(2) \) theory broken to \( U(1) \), and thus we will deal with a single chiral superfield \( \Phi \) and a single v.e.v. \( a \). Up to two-derivatives, \( \mathcal{N} = 2 \) supersymmetry constrains the effective action for \( \Phi \) to be of the form

\[
S_{\text{eff}}[\Phi] = \int d^4x d^4\theta F(\Phi) + \text{c.c.} ,
\]  

(1.2)

where the prepotential \( F(\Phi) \) is a holomorphic homogeneous function of degree two.

The main purpose of our discussion is to show how the instanton corrections to the prepotential arise in our string set-up, also in presence of a non-trivial supergravity background. In particular we will show that the low energy excitations of the D(−1)/D(−1) and D3/D(−1) open strings in the \( \mathbb{Z}_2 \) orbifold exactly account for the ADHM instanton moduli of the \( \mathcal{N} = 2 \) theory and that the integration measure
on the moduli space is recovered from disk amplitudes. Moreover, we show that the \( \mathcal{N} = 2 \) super-instanton solution for the fields appearing in Eq. (1.1) is obtained by computing one-point open string amplitudes on mixed disks. The fact that the instanton center and its super-partners decouple from the D3/D(–1) and D(–1)/D(–1) interaction show that these moduli actually play the role of the \( \mathcal{N} = 2 \) superspace coordinates; therefore the instanton induced prepotential \( F(\Phi) \) may be identified with the “centered partition function” of D-instantons coupled to \( \Phi \).\(^{12},13 \)

This identification opens the way to several generalizations. In particular, one may extend the above procedure to include also non-perturbative gravitational terms in the effective action by computing the instanton partition function in a non-trivial \( \mathcal{N} = 2 \) supergravity background. In our context, this amounts to compute the D(–1)/D(–1) and D3/D(–1) open string interactions in presence of a closed string background, and obtain the deformed integration measure on the instanton moduli space from mixed disk diagrams involving both open and closed string vertices. In particular we will consider a \( \mathcal{N} = 2 \) graviphoton background with self-dual field strength \( F^+ \) and determine the instanton induced prepotential \( F(\Phi, W) \), where \( W \) is the Weyl superfield whose first component is related to the graviphoton field strength \( F^+ \). In this way we may obtain from mixed disk amplitudes the gravitational F-terms of the form \( (R^+)^2 (F^+)^{2h-2} \), where \( R^+ \) is the self-dual Riemann curvature tensor, which have been previously computed\(^{14} \) from topological string amplitudes at genus \( h \). Therefore, the application of localization methods to the instanton calculus shows an interesting relation\(^{10},11 \) with the topological string which is worth further investigation.

§2. The D3/D(–1) system

As we mentioned above, the \( k \) instanton sector of a four-dimensional SYM theory with gauge group \( SU(N) \) can be described by a bound state of \( N \) D3 and \( k \) D(–1) branes\(^1 \) of fractional type in the space \( \mathbb{R}^4 \times M^6 \). The amount of supersymmetry possessed by the SYM theory depends on the features of the internal six-dimensional space \( M^6 \). If we want to describe \( \mathcal{N} = 2 \) gauge theories, we can choose \( M^6 \) to be the orbifold \( \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2 \).

In the D3/D(–1) system the string coordinates \( X^M \) and \( \psi^M \) (\( M = 1, \ldots, 10 \)) obey different boundary conditions on the two types of branes. Specifically, on the D(–1) branes we have Dirichlet boundary conditions in all directions, while on the D3 branes the longitudinal fields \( X^\mu \) and \( \psi^\mu \) (\( \mu = 1, 2, 3, 4 \)) satisfy Neumann boundary conditions, and the transverse fields \( X^a \) and \( \psi^a \) (\( a = 5, \ldots, 10 \)) obey Dirichlet boundary conditions.

The presence of the space-filling D3 branes and the orbifold projection break the 10-dimensional Euclidean invariance group \( SO(10) \) into \( SO(4) \times SO(2) \times SO(4)_{\text{int}} \) and therefore the 10-dimensional (anti-chiral) spin fields \( \hat{S}^A \) decompose as follows:

\[
S^A \rightarrow (S_\alpha S^- S_A, S_\alpha S^+ S_A, S^{\dot{\alpha}} S^- S_A, S^{\dot{\alpha}} S^+ S_A), \tag{2.1}
\]

where \( S_\alpha \) (\( S^{\dot{\alpha}} \)) are \( SO(4) \) Weyl spinors of positive (negative) chirality, \( S_A \) (\( S^{\dot{A}} \)) are
SO(4)\textsubscript{int} Weyl spinors of positive (negative) chirality and \(S^\pm\) are \(SO(2)\) spin fields with weight \(\pm \frac{1}{2}\). The chiral spin fields \(S_A\) are even under the orbifold generator, while the anti-chiral ones \(S^A\) are odd.

Since we consider a single type of fractional D3-branes, the Chan-Paton factors for the open string excitations are invariant under the orbifold. The orbifold projection therefore reduces the massless spectrum to the bosonic states which arise by acting on the vacuum with the even fields \(\Psi = (\psi_5 + i\psi_6)/\sqrt{2}\) and \(\Psi^\dagger = (\psi_5 - i\psi_6)/\sqrt{2}\) and to the fermionic states associated with the invariant spin fields \(S_\alpha S^- S_A\) or \(S^\alpha S^+ S_A\).

For our present purposes it is enough to specify the spectrum of excitations of the open strings with at least one end-point on the D-instantons, which, as explained in Ref. 5), describe the ADHM instanton moduli. Let us first consider the strings that have both ends on the \(D(-1)\) branes and therefore describe the neutral moduli: in the NS sector the physical excitations are \(a_\mu\), \(\chi\) and \(\chi^\dagger\), whose corresponding vertex operators are

\[
V_a(z) = a^\mu \psi_\mu(z) e^{-\phi(z)} , \quad V_\chi(z) = \chi^\dagger \Psi(z) e^{-\phi(z)} , \quad V_{\chi^\dagger}(z) = \chi \Psi^\dagger(z) e^{-\phi(z)},
\]

where \(\phi\) is the chiral boson of the superghost bosonization. In the R sector we find eight fermionic moduli which are conventionally denoted by \(M^{\alpha A}\) and \(\lambda_{\bar{\alpha} A}\), and correspond to the following vertex operators

\[
V_M(z) = M^{\alpha A} S_\alpha(z) S^-(z) S_A(z) e^{-\frac{i}{2} \phi(z)}, \quad V_{\bar{\lambda}}(z) = \lambda_{\bar{\alpha} A} S^\bar{\alpha}(z) S^+(z) S^A(z) e^{-\frac{i}{2} \phi(z)}.
\]

Let us now consider the open strings that start on a D3 and end on a \(D(-1)\) brane, or vice-versa and describe the charged moduli. These strings are characterized by the fact that the four directions along the D3 branes have mixed boundary conditions. Thus, in the NS sector the four fields \(\psi^\mu\) have integer-mode expansions with zero-modes that represent the \(SO(4)\) Clifford algebra and the corresponding physical excitations are organized in two bosonic Weyl spinors of \(SO(4)\). These are denoted by \(w_{\bar{\alpha}}\) and \(\bar{w}_{\bar{\alpha}}\) respectively, and are described by the following vertex operators

\[
V_w(z) = w_{\bar{\alpha}} \Delta(z) S^\bar{\alpha}(z) e^{-\phi(z)}, \quad V_{\bar{w}}(z) = \bar{w}_{\bar{\alpha}} \bar{\Delta}(z) S^\bar{\alpha}(z) e^{-\phi(z)}.
\]

Here \(\Delta(z)\) and \(\bar{\Delta}(z)\) are the bosonic twist and anti-twist fields with conformal dimension 1/4, that change the boundary conditions of the \(X^\mu\) coordinates from Neumann to Dirichlet and vice-versa by introducing a cut in the world-sheet. In the R sector the fields \(\psi^\mu\) have, instead, half-integer mode expansions so that there are fermionic zero-modes only in the common transverse directions. Thus, the physical excitations of this sector form two fermionic Weyl spinors of \(SO(4)\textsubscript{int}\) which are denoted by \(\mu^A\) and \(\bar{\mu}^A\) respectively, and correspond to the following vertex operators

\[
V_\mu(z) = \mu^A \Delta(z) S^-(z) S_A(z) e^{-\frac{i}{2} \phi(z)}, \quad V_{\bar{\mu}}(z) = \bar{\mu}^A \bar{\Delta}(z) S^-(z) S_A(z) e^{-\frac{i}{2} \phi(z)}.
\]

A systematic analysis\(^5\) shows that, in the limit \(\alpha' \to 0\), the scattering amplitudes involving the above vertex operators give rise to the following “action”

\[
S_\text{mod} = \text{tr} \left\{ -2 [\chi^\dagger, a_\mu^\dagger] [\chi, a^\mu] + \chi^\dagger \bar{w}_{\bar{\alpha}} w^{\bar{\alpha}} \chi + \chi \bar{w}_{\bar{\alpha}} w^{\bar{\alpha}} \chi^\dagger \right\}
\]
\[-i \frac{\sqrt{2}}{4} M^{\alpha A} \varepsilon_{AB} [\chi^\dagger, M^B_\alpha] + i \frac{\sqrt{2}}{2} \bar{\mu}^A \varepsilon_{AB} \mu^B \chi^\dagger \]
\[-i D_c (w_{\dot{a}} (\tau^c)_\beta^\dagger \bar{w}_\beta + i \bar{\eta}_{\mu \nu} [a^{\mu}, a^{\nu}]) + i \lambda_{\dot{A}}^\alpha (\bar{\mu}^A w_{\dot{a}} + \bar{w}_{\dot{a}} \mu^A + [a'_\alpha, M^{\alpha A}]) \].

(2.6)

where \( \text{tr}_k \) means trace over \( U(k) \), \( \bar{\eta}^c \) (\( c = 1, 2, 3 \)) are the anti-self dual ’t Hooft symbols, and \( \tau^c \) are the Pauli matrices. In (2.6) there appear also three auxiliary fields \( D_c \) whose string representation is provided by the following vertex operators:

\[ V_D (z) = \frac{1}{2} D_c \bar{\eta}_{\mu \nu} \psi^\nu (z) \psi^\mu (z) . \]

(2.7)

As is well known, by simply taking \( e^{-S_{\text{mod}}} \) one obtains the measure on the instanton moduli space, while by varying \( S_{\text{mod}} \) with respect to \( D_c \) and \( \lambda_{\dot{A}}^\alpha \) one finds the bosonic and fermionic ADHM constraints.

Notice that \( S_{\text{mod}} \) does not depend on the instanton center \( x_0^\mu \) nor on its superpartners \( \theta^{\alpha A} \), which are, respectively, the \( U(1) \) components of \( a^\mu \) and \( M^{\alpha A} \), namely

\[ a'^\mu = x_0^\mu \mathbf{1} |k] \times [k] + y_c^\mu T^c , \]
\[ M^{\alpha A} = \theta^{\alpha A} \mathbf{1} |k] \times [k] + \zeta_c^{\alpha A} T^c . \]

(2.8)

The moduli \( x_0^\mu \) and \( \theta^{\alpha A} \) actually play the role of the \( N = 2 \) superspace coordinates, while the remaining moduli, collectively denoted by \( \widehat{M}_{(k)} \) are the so-called “centered moduli”. Using this decomposition, the \( k \)-instanton partition function can then be written as

\[ Z_{(k)} = \int d^4 x_0 d^4 \theta \widehat{Z}_{(k)} \]

(2.9)

where

\[ \widehat{Z}_{(k)} = \int d \widehat{M}_{(k)} e^{-\frac{8 \pi^2 k}{g^2} S_{\text{mod}} (\widehat{M}_{(k)})} \]

(2.10)

is the centered partition function. It is worth pointing out that among the “centered moduli” \( \widehat{M}_{(k)} \) there is the singlet part of the anti-chiral fermions \( \lambda_{\dot{A}}^\alpha \) which is associated to the supersymmetries that are preserved both by the D3 and by the D(−1) branes. Thus, despite the suggestive notation of Eq. (2.9), one may naively think that the full D-instanton partition function cannot yield an F-term in the effective action, i.e. an integral on half superspace, due to the presence of the anti-chiral \( \lambda_{\dot{A}}^\alpha \)’s among the integration variables. Actually, this is not true since the \( \lambda_{\dot{A}}^\alpha \)’s, including its singlet part, do couple to other instanton moduli (see the last terms in Eq. (2.6)) and their integration correctly enforces the fermionic ADHM constraints on the moduli space. Therefore, the instanton partition function (2.9) does indeed yield non-perturbative F-terms. Things are very different instead for the exotic instantons that have been recently considered in the literature. In this case, due to the different structure of the charged moduli, the \( \lambda_{\dot{A}}^\alpha \)’s do not couple to anything and in order to get a non-vanishing result, they have to be removed from the spectrum, for example with an orientifold projection.

The moduli \( x_0^\mu \) and \( \theta^{\alpha A} \) are the so-called “centered moduli” \( \widehat{M}_{(k)} \) there is the singlet part of the anti-chiral fermions \( \lambda_{\dot{A}}^\alpha \) which is associated to the supersymmetries that are preserved both by the D3 and by the D(−1) branes. Thus, despite the suggestive notation of Eq. (2.9), one may naively think that the full D-instanton partition function cannot yield an F-term in the effective action, i.e. an integral on half superspace, due to the presence of the anti-chiral \( \lambda_{\dot{A}}^\alpha \)’s among the integration variables. Actually, this is not true since the \( \lambda_{\dot{A}}^\alpha \)’s, including its singlet part, do couple to other instanton moduli (see the last terms in Eq. (2.6)) and their integration correctly enforces the fermionic ADHM constraints on the moduli space. Therefore, the instanton partition function (2.9) does indeed yield non-perturbative F-terms. Things are very different instead for the exotic instantons that have been recently considered in the literature. In this case, due to the different structure of the charged moduli, the \( \lambda_{\dot{A}}^\alpha \)’s do not couple to anything and in order to get a non-vanishing result, they have to be removed from the spectrum, for example with an orientifold projection.

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This is to be contrasted with the \( \theta^{\alpha A} \) defined in (2.8), which are associated to the supersymmetries preserved by the D3 but broken by the D(−1) branes.
§3. The instanton solution from mixed disks

The construction of the ADHM instanton moduli and of their integration measure in terms of open strings given in the previous section clearly shows that gauge theory instantons are described by systems of D3/D(−1) branes. There is however an even more convincing evidence in favor of this identification, namely the fact that the mixed disks of the D3/D(−1) brane system are the source for the instanton background of the super Yang-Mills theory, and that the classical instanton profile can be obtained from open string amplitudes.

For simplicity we will discuss only the case of instanton number \( k = 1 \) in a \( SU(2) \) gauge theory, but no substantial changes occur in our analysis if one considers higher values of \( k \) and other gauge groups (see Ref. 5 for these extensions). Let us then consider the emission of the \( SU(2) \) gauge vector field \( A^c_\mu \) from a mixed disk. The simplest diagram which can contribute to this process contains two boundary changing operators \( \bar{V} \) and \( V \) and no other moduli, and is shown in Fig. 1.

The amplitude associated to this diagram is

\[
\left\langle V_{A^c_\mu}(-p) \right\rangle \text{mixed disk} \equiv \left\langle \bar{V} V_{A^c_\mu}(-p) V \right\rangle = A^c_\mu(p; \bar{w}, w),
\]

where \( V_{A^c_\mu}(-p) \) is the gluon vertex operator with outgoing momentum \( p \) and without polarization, namely

\[
V_{A^c_\mu}(-p) = 2i T^c (\partial X_\mu - i p \cdot \psi \psi_\mu) e^{-ip \cdot X(z)},
\]

where \( T^c \) is the adjoint \( SU(2) \) generator. Note that the amplitude (3.1) carries the structure and the quantum numbers of an emitted gauge vector field of \( SU(2) \). The evaluation of the amplitude (3.1) is quite straightforward and the result is

\[
A^c_\mu(p; \bar{w}, w) = i \rho^2 p^n \tilde{\eta}^c_{\mu\nu} e^{-ip \cdot x_0},
\]

where we have defined \( \tilde{w}^{\dot{a}} w_{\dot{a}} = 2\rho^2 \). By taking the Fourier transform of (3.3), after inserting a free propagator, we obtain

\[
A^c_\mu(x) = \int \frac{d^4p}{(2\pi)^2} \frac{1}{p^2} A^c_\mu(p; \bar{w}, w) e^{ip \cdot x} = \frac{2\rho^2 \tilde{\eta}^c_{\mu\nu} (x - x_0)^\nu}{(x - x_0)^4}.
\]
Equation (3.4) is the leading term in the large distance expansion (i.e. $|x - x_0| \gg \rho$) of the $SU(2)$ instanton with center $x_0$ and size $\rho$ in the singular gauge, namely

$$A_{\mu}^c(x) = \frac{2\rho^2 \eta_{\mu\nu} (x - x_0)^\nu}{(x - x_0)^2 (x - x_0^2 + \rho^2)} \approx \frac{2\rho^2 \eta_{\mu\nu} (x - x_0)^\nu}{(x - x_0)^4} \left(1 - \frac{\rho^2}{(x - x_0)^2} + \ldots\right).$$

(3.5)

This result explicitly shows that mixed disk diagrams, like that of Fig. 1, are the source for the classical gauge instanton. Note that the amplitude (3.1) is a 3-point function from the point of view of the two dimensional conformal field theory on the string world sheet, but is a 1-point function from the point of view of the four-dimensional gauge theory on the D3 branes. In fact, the two boundary changing operators $V_{\bar{w}}$ and $V_w$ that appear in (3.1) just describe non-dynamical parameters on which the background depends. Furthermore, the fact that the gluon field (3.4) is in the singular gauge is not surprising, because in our set-up the gauge instanton is produced by a $D(-1)$ brane which is a point-like object inside the D3 brane world-volume. Thus it is natural that the gauge connection exhibits a singularity at the location $x_0$ of the D-instanton.

An obvious question at this point is whether also the subleading terms in the large distance expansion (3.5) have a direct interpretation in string theory. Since such terms contain higher powers of $\rho^2 \sim \bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}$, one expects that they are associated to mixed disks with more insertions of boundary changing operators. This expectation has been explicitly confirmed in Ref. 5), so that one can conclude that mixed disks with the emission of a gauge vector field do indeed reproduce the complete $k = 1$ instanton solution.

§4. Deformed $N = 2$ instanton calculus

In this section we analyze the instanton moduli space of $N = 2$ gauge theories in a non-trivial supergravity background. In particular we turn on a (self-dual) field strength for the graviphoton of the $N = 2$ supergravity multiplet and see how it modifies the instanton moduli action. This graviphoton background breaks Lorentz invariance in space-time (leaving the metric flat) but it allows to explicitly perform instanton calculations and establish a direct correspondence with the localization techniques that have been recently discussed in the literature.10),11),15)

In order to systematically incorporate the gravitational background in the instanton action, let us first discuss how to include the interactions among the instanton moduli and the gauge fields. Then, let us consider all correlators involving D3/D3 fields, and in particular the scalar $\phi$ in presence of $k$ D-instantons. It turns out 3),5),16) that the dominant contribution to the $n$-point function $\langle \phi_1 \ldots \phi_n \rangle$ is from $n$ one-point amplitudes on disks with moduli insertions. The result can therefore be encoded in extra moduli-dependent vertices for $\phi$’s, i.e. in extra terms in the moduli action containing such one-point functions

$$S_{\text{mod}}(\phi; \mathcal{M}_{(k)}) = \phi(x) J_\phi(\mathcal{M}_{(k)}) + S_{\text{mod}}(\mathcal{M}_{(k)}),$$

(4.1)

where $x$ is the instanton center (previously denoted by $x_0$) and $\phi(x) J_\phi(\mathcal{M}_{(k)})$ is given
Fig. 2. Mixed disk diagram describing the coupling of the scalar field of the $\mathcal{N} = 2$ vector multiplet with the instanton moduli.

by the disk diagrams with boundary (partly) on the D(–1)’s describing the emission of a $\phi$. To determine the complete action $S_{\text{mod}}(\phi; M)$ we have to systematically compute all mixed disks with a scalar $\phi$ emitted from the D3 boundary, such as the one represented in Fig. 2. Other non-zero diagrams involving the instanton moduli and the super-partners of $\phi$ can be obtained using the Ward identities of the supersymmetries that are broken by the D(–1) branes. Therefore, the complete superfield-dependent moduli action $S_{\text{mod}}(\Phi; M_k)$ can be obtained from (4.1) by simply letting $\phi(x) \rightarrow \Phi(x, \theta)$, with $\Phi(x, \theta)$ defined in Eq. (1.1).

Let us now extend this argument to the supergravity background we want to include. The field content of $\mathcal{N} = 2$ supergravity, namely the metric $h_{\mu \nu}$, the gravitini $\psi_{\alpha A}^\mu$ and the graviphoton $C^\mu$ can be organized in a chiral Weyl multiplet

$$W^\mu_{\mu \nu}(x, \theta) = F^\mu_{\mu \nu}(x) + \theta \chi^\mu_{\mu \nu}(x) + \frac{1}{2} \theta \sigma^{\lambda \rho} \theta R^\mu_{\mu \nu \lambda \rho}(x) + \cdots,$$

where the self-dual tensor $F^\mu_{\mu \nu}(x)$ can be identified on-shell with the graviphoton field strength, $R^\mu_{\mu \nu \lambda \rho}(x)$ is the self-dual Riemann curvature tensor and $\chi^\mu_{\mu \nu}$ is the gravitino field strength. All these fields belong to the massless sector of type IIB strings on $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2$. In particular, the graviphoton vertex is given by

$$V_F(z, \bar{z}) = \frac{1}{4\pi} \mathcal{F}^{\alpha \beta AB}(p) \left[ S_{\alpha}(z) S^{-}(z) S_{A}(z) e^{-\frac{i}{2} \varphi(z)} S_{\beta}(\bar{z}) S^{-}(z) S_{B}(\bar{z}) e^{-\frac{i}{2} \varphi(\bar{z})} \right] e^{ip \cdot X(z, \bar{z})}.$$  

where the left-right movers identification on disks has already been taken into account. The bi-spinor graviphoton polarization is given by

$$\mathcal{F}^{(\alpha \beta)[AB]} = \frac{\sqrt{2}}{4} \mathcal{F}^\mu_{\mu \nu} (\sigma^\mu_{\nu})^{\alpha \beta} \epsilon^{AB}$$

and corresponds to a R-R 3-form $\mathcal{F}_{\mu \nu z}$ with one index in the $\mathbb{C}$ internal direction. To determine the contribution of the graviphoton to the field-dependent moduli action we have to consider disk amplitudes with open string moduli vertices on the open string moduli vertices on the...
boundary and closed string graviphoton vertices in the interior, which survive in the field theory limit $\alpha' \to 0$.

It turns out that very few diagrams (like the one represented in Fig. 3) contribute in this limit. They can be easily evaluated and for instance for the diagram in Fig. 3 one finds

$$\langle V_M V_M V_\bar{F} \rangle = \frac{1}{4\sqrt{2}} \text{tr}_k \left\{ M^{\alpha A} M^{\beta B} \bar{F}^+_{\mu\nu} \right\} (\sigma^\mu)_{\alpha\beta} \epsilon_{AB}.$$ (4.5)

Other diagrams, connected by the broken supersymmetries, have the effect of promoting the dependence of the moduli action to the full Weyl multiplet, i.e. $F^+_{\mu\nu} \to W^+_{\mu\nu}(x, \theta)$. In this way the superfield-dependent moduli action $S_{\text{mod}}(\Phi, W^+; \tilde{M}(k))$ is obtained from the previous results.

Integrating over the moduli, one gets the effective action, and hence the prepotential at instanton number $k$, namely

$$S_{\text{eff}}^{(k)}[\Phi, W^+] = \int d^4x \, d^4\theta \, d\tilde{M}(k) e^{-\frac{8\pi k}{\alpha'}} - S_{\text{mod}}(\Phi, W^+; \tilde{M}(k))$$

$$= \int d^4x \, d^4\theta \, F^{(k)}(\Phi, W^+).$$ (4.6)

Since $\Phi(x, \theta)$ and $W^+_{\mu\nu}(x, \theta)$ are constant with respect to the integration variables $\tilde{M}(k)$, we can actually compute $F^{(k)}$ by reducing them to a constant value, i.e. $\Phi(x, \theta) \to a$ and $W^+_{\mu\nu}(x, \theta) \to f_{\mu\nu}$. In this case the prepotential becomes just a function of the scalar and graviphoton v.e.v.’s and is determined by a “deformed” moduli action depending on $a, \bar{a}, f, \bar{f}$:

$$S_{\text{mod}}(a, \bar{a}; f, \bar{f}; \tilde{M}(k)) = -\text{tr}_k \left\{ \left( [\chi^\dagger, a'^{\alpha\beta}] + 2\bar{f}_c (\tau^c a'^{\alpha}) \right) \left( [\chi, a'^{\beta\alpha}] + 2f_c (a'^{c})^{\beta\alpha} \right) \right. \right.$$

$$- (\chi^\dagger \bar{w}_{\alpha} - \bar{w}_{\alpha} \bar{a}) (w^\alpha \chi - a \bar{w}_{\alpha}) - (\chi \bar{w}_{\alpha} - \bar{w}_{\alpha} a) (w^\alpha \chi^\dagger - \bar{a} \bar{w}_{\alpha}) \left. \right\}$$

$$+ i \frac{\sqrt{2}}{2} \text{tr}_k \left\{ \mu^A \epsilon_{AB} (\mu^B \chi^\dagger - \bar{a} \mu^B) - \frac{1}{2} M^{\alpha A} \epsilon_{AB} (\left[ \chi^\dagger, M^B_{\alpha} \right] + 2\bar{f}_c (\tau^c)_{\alpha\beta} M_{\beta}^B) \right. \right.$$

$$\left. \text{tr}_k \left\{ - iD_c (w_{\alpha} (\tau^c)^{\alpha}_{\beta} \bar{w}_{\beta} + i\epsilon_{\mu\nu} [a'^{\mu}, a'^{\nu}]) + i\lambda_{\alpha} (\bar{a}^A w_{\alpha} + \bar{w}_{\alpha} \mu^A + [a'^{\alpha}, M^A_{\alpha}]) \right\} \right. \right.$$

$$\left. \right\}.$$ (4.7)
Notice that the ADHM constraints, appearing in the last line of (4.7), are not modified by the graviphoton background.

In the action (4.7) the v.e.v.’s $a, f$ and $\bar{a}, \bar{f}$ are not on the same footing. Indeed, one can write

$$S_{\text{mod}}(a, \bar{a}; f, \bar{f}; \hat{M}(k)) = Q \Xi, \quad (4.8)$$

where $Q$ is the scalar component of the twisted supercharges, i.e.

$$Q \equiv \frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} Q^{\dot{\alpha}\dot{\beta}}, \quad (4.9)$$

where the topological twist acts as $Q^{\dot{\alpha}B} \xrightarrow{\text{top. twist}} Q^{\dot{\alpha}\dot{\beta}}$. It turns out that the parameters $\bar{a}, \bar{f}$ appear in $S_{\text{mod}}$ only through the gauge fermion $\Xi$, and thus the instanton partition function and the prepotential $F^{(k)}$ in Eq. (4.6) are in fact independent of $\bar{a}, \bar{f}$, because their variation with respect to these parameters is $Q$-exact.

From the explicit expression of $S_{\text{mod}}(a, 0; f, 0)$ the general form of the prepotential $F^{(k)}(a; f)$ can be easily deduced; reinstating the superfields it reads

$$F^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} c_{k,h} \Phi^2 \left( \frac{A}{\phi} \right)^{4k} \left( \frac{W^+}{\phi} \right)^{2h} \quad (4.10)$$

Summing over the instanton sectors we obtain the full non-perturbative prepotential

$$F_{\text{n.p.}}(\Phi, W^+) = \sum_{k=1}^{\infty} F^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} C_h(A, \phi)(W^+)^{2h} \quad (4.11)$$

where

$$C_h(A, \phi) = \sum_{k=1}^{\infty} c_{k,h} \frac{A^{4k}}{\phi^{4k+2h-2}} \quad (4.12)$$

This gives rise to many different terms in the effective action, which is obtained, see Eq. (4.6), by integrating the prepotential over $d^4x d^4\theta$. In particular, saturating the $\theta$ integration with four $\theta$’s all from $W^+$ we get gravitational F-terms in the $\mathcal{N} = 2$ effective action involving the curvature tensor and graviphoton field strength

$$\int d^4x \ C_h(A, \phi) (R^+)^2 (\mathcal{F}^+)^{2h-2}. \quad (4.13)$$

The stringy instanton calculus accounts thus for such F-terms, and it gives a way to compute them, because the coefficients $C_{k,h}$ can be explicitly determined by performing the integrals over the instanton moduli space.

This is a formidable task, that was finally performed in Refs. 10), 11) and 15), using a suitable “deformation” of the moduli action which localizes the integrals. This localization deformation exactly coincides with Eq. (4.7) if we set

$$f_c = \frac{\varepsilon}{2} \delta_{3c}, \quad \bar{f}_c = \frac{\varepsilon}{2} \delta_{3c} \quad (4.14)$$

(and moreover $\varepsilon = \bar{\varepsilon}$).
As we remarked above, \( Z^{(k)}(a, \varepsilon) \) does not depend on \( \bar{\varepsilon} \). However, \( \bar{\varepsilon} = 0 \) is a limiting case and some care is needed. In fact, while \( F^{(k)}(a; \varepsilon) \) is well-defined, the complete partition function \( Z^{(k)}(a; \varepsilon) \) diverges because of the (super)volume integral \( \int d^4 \theta d^4 x \). The presence of \( \bar{\varepsilon} \) regularizes the superspace integration by a Gaussian term, leading to the following effective rule:

\[
\int d^4 \theta d^4 x \rightarrow 1/\varepsilon^2 ;
\]

(4.15)

one can then work with the full instanton partition function. Moreover, the \( a \) and \( \varepsilon, \bar{\varepsilon} \) deformations localize completely the integration over moduli space which can then be evaluated explicitly.\(^{10,11,15}\)

With \( \bar{\varepsilon} \neq 0 \), i.e. with complete localization, the trivial superposition of several instantons of charges \( k_i \) contributes to the sector \( k = \sum k_i \); such disconnected configurations do not contribute instead when \( \bar{\varepsilon} = 0 \). The partition function computed by localization thus corresponds in this case to the exponential of the non-perturbative prepotential, namely

\[
Z(a; \varepsilon) = \sum_{k=1}^{\infty} Z^{(k)}(a; \varepsilon) = \exp \left( \frac{F_{n.p.}(a, \varepsilon)}{\varepsilon^2} \right) = \exp \left( \sum_{k=1}^{\infty} \frac{F^{(k)}(a, \varepsilon)}{\varepsilon^2} \right)
\]

\[
= \exp \left( \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} c_{k,h} \frac{\varepsilon^{2h-2}}{a^{2h-2}} \left( \frac{A}{a} \right)^{4k} \right) .
\]

(4.16)

In conclusion, the computation via localization techniques of the multi-instanton partition function \( Z(a; \varepsilon) \) determines the coefficients \( c_{k,h} \) which appear in the gravitational F-terms of the \( \mathcal{N} = 2 \) effective action Eq. (4.13) via the expression of \( C_{h}(A, \phi) \) given in Eq. (4.11).

The very same gravitational F-terms can be extracted in a completely different way by considering topological string amplitudes at genus \( h \) on suitable Calabi-Yau manifolds.\(^{14,17}\) In our computation the role of the genus \( h \) Riemann surface is played by a (degenerate) surface with the same Euler number made by \( 2h \) disconnected disks, instead of \( h \) handles. The two different roads to determine the \( F \)-couplings of Eq. (4.13) must lead to the same result. This is a very natural way to state the conjecture by Nekrasov\(^{10}\) that the coefficients arising in the \( \varepsilon \)-expansion of multi-instanton partition functions match those appearing in higher genus topological string amplitudes on Calabi-Yau manifolds.

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