$L^2$ SCHRODINGER MAXIMAL ESTIMATES ASSOCIATED WITH FINITE TYPE PHASES IN $\mathbb{R}^2$

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ABSTRACT. In this paper, we establish Schrödinger maximal estimates associated with the finite type phases

$$\phi(\xi_1, \xi_2) := \xi_1^m + \xi_2^m, \quad (\xi_1, \xi_2) \in [0, 1]^2,$$

where $m \geq 4$ is an even number. Following [12], we prove an $L^2$ fractal restriction estimate associated with the surfaces

$$F_{m}^2 := \{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in [0, 1]^2\}$$

as the main result, which also gives results on the average Fourier decay of fractal measures associated with these surfaces. The key ingredients of the proof include the rescaling technique from [16], Bourgain-Demeter’s $\ell^2$ decoupling inequality, the reduction of dimension arguments from [17] and induction on scales.

1. Introduction

The solution to the Cauchy problem of the free Schrödinger equation

$$\begin{cases}
    i\partial_t u - \Delta u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
    u(x, 0) = f(x)
\end{cases}$$

is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(x \cdot \xi + t|\xi|^2)} d\xi,$$

where $e(t) := e^{it}$.

Carleson in [6] proposed the problem of determining the optimal $s$ for which

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x)$$

almost everywhere, whenever $f \in H^s(\mathbb{R}^n)$. The problem has attracted many authors. In particular, Lee [15] proved that almost everywhere convergence holds if $s > \frac{3}{8}$ when $n = 2$. For general $n$, Bourgain [4] gave surprising counterexamples showing that the convergence fails if $s < \frac{n}{2(n+1)}$. Later, Du, Guth and Li [9] proved that the convergence holds for $s > \frac{1}{3}$ when $n = 2$, which combining with Bourgain’s necessary condition solves the Carleson’s problem when $n = 2$, except at the endpoint $s = \frac{1}{3}$. In higher dimensions ($n \geq 3$), Du and Zhang [12] proved that almost everywhere convergence holds in the sharp range $s > \frac{n}{2(n+1)}$.

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

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2010 Mathematics Subject Classification. Primary 42B10; Secondary 42B37.

Key words and phrases. pointwise convergence, finite type, decoupling, reduction of dimension arguments.
Thus, Carleson’s problem is completely solved for any dimension $n$, except at the endpoint $\frac{n}{2(n+1)}$. For the study of pointwise convergence problem on 2D fractional order Schrödinger operators, we refer to Miao, Yang and Zheng [20]. See also C. Cho and H. Ko [7].

Let $m \geq 4$ be an even number. The solution to the Cauchy problem of the following generalized Schrödinger equation

$$\begin{cases}
i \partial_t u - (\partial_1^m u + \partial_2^m u) = 0, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ u(x,0) = f(x)
\end{cases}$$

is given by

$$e^{it\phi(D)} f(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \hat{f}(\xi) e^{[x_1 \xi_1 + x_2 \xi_2 + t\phi(\xi)]} d\xi,$$

where $\phi(\xi_1, \xi_2) := \xi_1^m + \xi_2^m$.

We note that the Gaussian curvature of the surfaces associated with $\phi(\xi)$ vanishes when $\xi_1 = 0$ or $\xi_2 = 0$, which is different from those non-degenerate cases in the literature. To our knowledge, there are few works on the sufficient conditions for the almost everywhere convergence problem associated with such degenerate phases. For the study of necessary conditions, we refer to [1, 13].

In this article, we establish the following result.

**Theorem 1.1.** For every $f \in H^s(\mathbb{R}^2)$ with $s > \frac{1}{2} - \frac{1}{3m}$, $\lim_{t \to 0} e^{it\phi(D)} f(x) = f(x)$ almost everywhere.

We use $B^d(x,r)$ to denote the ball centered at $x$ with radius $r$ in $\mathbb{R}^d$. By a standard approximation argument, Theorem 1.1 is a consequence of the following Schrödinger maximal estimate associated with $\phi$:

**Theorem 1.2.** For any $s > \frac{1}{2} - \frac{1}{3m}$, there holds

$$\| \sup_{0 < t \leq 1} |e^{it\phi(D)} f| \|_{L^2(B^2(0,1))} \leq C_s \| f \|_{H^s(\mathbb{R}^2)}$$

for any function $f \in H^s(\mathbb{R}^2)$.

By Lemma 2.1 in [8], the Littlewood-Paley decomposition and a rescaling argument, Theorem 1.2 is a consequence of the following theorem.

**Theorem 1.3.** For any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$\| \sup_{0 < t \leq R} |e^{it\phi(D)} f| \|_{L^2(B^2(0,R))} \leq C_\varepsilon R^{\frac{1}{2} - \frac{1}{3m} + \varepsilon} \| f \|_2$$

holds for all $R \geq 1$ and all $f$ with $\text{supp} \hat{f} \subset A(1) := [0,1]^2 - [0,\frac{1}{2}]^2$.

**Remark 1.4.** If $m$ is odd, we can not reduce $\text{supp} \hat{f} \subset [-1,1]^2 - [-\frac{1}{2},\frac{1}{2}]^2$ to $\text{supp} \hat{f} \subset [0,1]^2 - [0,\frac{1}{2}]^2$. To see it, let us consider the case $\text{supp} \hat{f} \subset [\frac{1}{2},1] \times [-1,-\frac{1}{2}] =: A(1')$. Taking the change of variables

$$\xi_1 = \eta_1, \xi_2 = -\eta_2,$$
the phase function \( \phi(\eta_1, -\eta_2) := \eta_1^m - \eta_2^m \) can be viewed as small perturbations of \( \phi_0(\eta_1, \eta_2) := \eta_1^2 - \eta_2^2 \), where \((\eta_1, \eta_2) \in [0, 1]^2\). But we know from [21] that for any \( f \in H^s(\mathbb{R}^2) \)
\[
\lim_{t \to 0} e^{it\phi_0(D)}f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^2
\]
fails for \( s < \frac{1}{2} \). For this reason we only consider that \( m \) is even in the current paper.

Cubes of the form \( m+[0, M]^3 \) with \( m \in (MZ)^3 \) are called lattice \( M \)-cubes. We will prove the following \( L^2 \) restriction estimate, from which Theorem 1.3 follows.

**Theorem 1.5.** Suppose that \( X = \bigcup_k B_k \) is a union of lattice unit cubes in \( Q_R := [0, R]^3 \) and each \( R^{\frac{1}{2}} \)-cube intersecting \( X \) contains \( \sim \nu \) many unit cubes in \( X \). Let \( 1 \leq \alpha \leq 3 \) and \( \lambda \) be
\[
\lambda := \max_{B^3(x',r) \subset Q_R, x' \in \mathbb{R}^3, r \geq 1} \frac{r^{\alpha}}{r^{\alpha}} \{ B_k : B_k \subset B^3(x', r) \}.
\]

For any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) such that the following inequalities hold for all \( R \geq 1 \):
\[
\|e^{it\phi(D)}f\|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{1}{2}} \nu^{\frac{1}{2}} R^{\frac{1}{2} + \frac{\alpha}{6m} + \frac{\varepsilon}{6m}} \|f\|_2
\]
for all \( f \) with \( \text{supp} \hat{f} \subset A(1) \), and
\[
\|e^{it\phi(D)}f\|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{1}{2}} \nu\mu R^{\frac{1}{2} + \frac{\alpha}{6m} + \frac{\varepsilon}{6m}} \|f\|_2
\]
for all \( f \) with \( \text{supp} \hat{f} \subset [0, 1]^2 \).

**Remark 1.6.** To deduce Theorem 1.3, (1.1) is sufficient. We only use (1.2) in Section 4. The two different behaviors in (1.1) and (1.2) depend on whether the support of \( \hat{f} \) contains the origin or not. When the support of \( \hat{f} \) does not contain the origin, at most one principal curvature of the surface vanishes. If the support of \( \hat{f} \) contains the origin, both principal curvatures of the surface studied vanish. Therefore, we deduce a better result in (1.1) than in (1.2).

Clearly, one has \( \nu \leq \lambda R^{\alpha/2} \) in Theorem 1.5. As a direct result of Theorem 1.5, there holds a weaker \( L^2 \) restriction estimate:

**Corollary 1.7.** Suppose that \( X = \bigcup_k B_k \) is a union of lattice unit cubes in \( Q_R \). Let \( 1 \leq \alpha \leq 3 \) and \( \lambda \) be
\[
\lambda := \max_{B^3(x',r) \subset Q_R, x' \in \mathbb{R}^3, r \geq 1} \frac{r^{\alpha}}{r^{\alpha}} \{ B_k : B_k \subset B^3(x', r) \}.
\]

For any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) such that the following inequalities hold for all \( R \geq 1 \)
\[
\|e^{it\phi(D)}f\|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{1}{2}} R^{\frac{1}{2} + \frac{\alpha}{12} + \frac{4-\alpha}{6m} + \varepsilon} \|f\|_2
\]
for all \( f \) with \( \text{supp} \hat{f} \subset A(1) \), and
\[
\|e^{it\phi(D)}f\|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{1}{2}} R^{\frac{1}{2} + \frac{\alpha}{12} + \frac{3-\alpha}{6m} + \varepsilon} \|f\|_2
\]
for all \( f \) with \( \text{supp} \hat{f} \subset [0, 1]^2 \).
Corollary 1.7 is sufficient to derive the $L^2$ Schrödinger maximal estimate (Theorem 1.3). For the details, we refer to the proof of Theorem 1.3 by Corollary 1.7 in Du-Zhang [12]. Theorem 1.5 is based on Proposition 1.8 below, which will be proved by an induction.

A collection of quantities are said to be essentially constant provided that all the quantities lie in the same interval of the form $[2^n, 2^{n+1}]$, where $n \in \mathbb{Z}$.

**Proposition 1.8.** Suppose that $Y = \bigcup_{k=1}^{N} B_k$ is a union of lattice $K$-cubes in $Q_R$ and each $R^{1/3}$-cube intersecting $Y$ contains $\sim \nu$ many $K$-cubes in $Y$, where $K = R^d$. Suppose that $\|e^{it\phi(D)}f\|_{L^6(B_k)}$ is essentially constant in $k = 1, 2, \ldots, N$. Let $1 \leq \alpha \leq 3$ and $\lambda$ be

$$\lambda := \max_{B^3(x', r) \subset Q_R, x' \in \mathbb{R}^3, r \geq K} \frac{|\{B_k : B_k \subset B^3(x', r)\}|}{r^\alpha}.$$ 

For any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ and $\delta = \varepsilon^{100}$ such that the following inequalities hold for all $R \geq 1$:

$$\|e^{it\phi(D)}f\|_{L^6(Y)} \leq C_\varepsilon N^{-\frac{1}{3}} \lambda^{\frac{1}{2}} \nu^{\frac{1}{2}} R^{\frac{1}{3}} \cdot \frac{\delta - \alpha}{6\alpha} \|f\|_2$$

for all $f$ with $\text{supp } \hat{f} \subset A(1)$, and

$$\|e^{it\phi(D)}f\|_{L^6(Y)} \leq C_\varepsilon N^{-\frac{1}{3}} \lambda^{\frac{1}{2}} \nu^{\frac{1}{2}} R^{\frac{1}{3}} \cdot \frac{\delta - \alpha}{6\alpha} \|f\|_2$$

for all $f$ with $\text{supp } \hat{f} \subset [0, 1]^2$.

**Remark 1.9.** In the proof of Proposition 1.8, we will use the $\ell^2$ decoupling parabola and for the curve

$$\{(s, s^m) : s \in [0, 1]\}.$$ 

This is already known due to [3, 23]. To establish Proposition 1.8 in higher dimensions ($n \geq 3$) with the methods of Du-Zhang [12] and the current paper, one needs the $\ell^2$ decoupling inequalities for the hypersurfaces

$$\left\{(\xi_1, \ldots, \xi_{n-1}, \phi_1(\xi_1) + \ldots + \phi_s(\xi_s) + \xi_{n+1}^m + \ldots + \xi_{n-1}^m) : (\xi_1, \ldots, \xi_{n-1}) \in [0, 1]^{n-1}\right\},$$

$0 \leq s \leq n - 1$, with $\phi_1, \ldots, \phi_s$ being non-degenerate. With these in hand, one can obtain pointwise convergence results associated with corresponding finite type phases in higher dimensions. This is achieved in a recent preprint [14].

From [12], we know that Proposition 1.8 implies Theorem 1.5, and therefore it suffices to prove Proposition 1.8. Now we outline the proof of Proposition 1.8.

**Outline of the proof of Proposition 1.8.** We use $F^2_m$ to denote the surface

$$\{(\xi_2, \xi_2^m + \xi_2^m) : (\xi_1, \xi_2) \in [0, 1]^2\}.$$ 

The surface $F^2_m$ is badly behaved when $\xi_1 = 0$ or $\xi_2 = 0$. The strategy is to single out small neighborhoods of these two lines where we apply reduction of dimension arguments. More precisely, we first divide $[0, 1]^2$ into $J_0 \Omega_j$, where

$$\Omega_0 := [K^{-\frac{1}{3}}, 1] \times [K^{-\frac{1}{3}}, 1], \quad \Omega_1 := [K^{-\frac{1}{3}}, 1] \times [0, K^{-\frac{1}{3}}].$$
\( \Omega_2 := [0, K^{-\frac{1}{m}}] \times [K^{-\frac{1}{m}}, 1], \ \Omega_3 := [0, K^{-\frac{1}{m}}] \times [0, K^{-\frac{1}{m}}]. \)

Here \( K = R^t \) is a large number. Denote \( \hat{f} |_{\Omega_j} \) by \( \hat{f}_{\Omega_j} \) for \( j = 0, 1, 2, 3 \). Then, we have

\[
\| e^{it\phi(D)} f \|_{L^6(Y)} \leq \sum_{j=0}^{3} \| e^{it\phi(D)} f_{\Omega_j} \|_{L^6(Y)}. \tag{1.3}
\]

We will estimate \( \| e^{it\phi(D)} f \|_{L^6(Y)} \) in several cases. In (1.3) one of the \( \| e^{it\phi(D)} f_{\Omega_j} \|_{L^6(Y)} \) will dominate, and we call it \( \Omega_j \)-case.

We first consider the \( \Omega_0 \)-case. The surface

\[
\{ (\xi_2, \xi_1^m + \xi_2^m) : (\xi_1, \xi_2) \in \Omega_0 \}
\]

has two positive principal curvatures with lower bound \( K^{-C_0} \) for some constant \( C_0 \). For \( (\xi_1, \xi_2) \in \Omega_0 \), our phase function \( \phi(\xi_1, \xi_2) \) can be viewed as perturbations of \( \xi_1^m + \xi_2^m \). By Proposition 3.1 of Du-Zhang [12], in the \( \Omega_0 \)-case we have

\[
\| e^{it\phi(D)} f \|_{L^6(Y)} \leq C_\epsilon K^{O(1)} N^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \nu^{\frac{1}{2}} R^{\frac{\alpha}{12} + \epsilon} \| f \|_2,
\]

where \( K^{O(1)} \) is a fixed power of \( K \), and the details will be given in Section 2. For simplicity, we assume that \( \text{supp} \hat{f} \subset A(1) \). Then, \( \hat{f}_{\Omega_3} = 0 \). It remains to discuss the \( \Omega_1 \)-case and the \( \Omega_2 \)-case. By symmetry, it suffices to treat the \( \Omega_1 \)-case. Since \( \text{supp} \hat{f} \subset A(1) \), we only need to consider the following subregion of \( \Omega_1 \):

\[
\tilde{\Omega}_1 := \left[ \frac{1}{2}, 1 \right] \times [0, K^{-1/m}].
\]

We will adapt Bourgain-Demeter’s \( \ell^2 \) decoupling inequality [5] to our needs and reduce the problem to each \( K^{-1/2} \times K^{-1/m} \)-rectangle \( \tau \). Now it is natural to employ certain rescaling technique. The main difficulty is that the phase function

\[
\phi(\xi_1, \xi_2) := \xi_1^m + \xi_2^m, \ (\xi_1, \xi_2) \in \tau \subset \tilde{\Omega}_1
\]

is not closed under the change of variables

\[
\begin{cases} 
\xi_1 = a + K^{-1/2} \eta_1, \\
\xi_2 = K^{-1/m} \eta_2.
\end{cases}
\]

In fact, it becomes a new phase function

\[
\phi(a + K^{-1/2} \eta_1, K^{-1/m} \eta_2) \sim \frac{1}{K} (\phi_1(\eta_1) + \eta_2^m) = \frac{1}{K} \psi(\eta_1, \eta_2),
\]

where

\[
\psi(\eta_1, \eta_2) := \phi_1(\eta_1) + \eta_2^m, \ (\eta_1, \eta_2) \in [0, 1]^2
\]

satisfying

\[
\phi_1^{(m)} \sim 1; \ |\phi_1^{(k)}| \lesssim 1, 3 \leq k \leq m; \ \phi_1^{(l)} = 0, l \geq m + 1
\]

on the interval \([0, 1]\). To overcome the difficulty, we establish a fractal \( L^2 \) restriction estimate associated with the new phase function \( \psi \) as an auxiliary proposition.
(Proposition 2.1), whose proof is given in Section 3. The key observation is that the new phase function 
\[ \psi(\eta_1, \eta_2), (\eta_1, \eta_2) \in [0, K^{-1/2}] \times [0, K^{-1/m}] \]
is closed under the change of variable
\[ \begin{cases} \xi_1 = K^{-\frac{1}{2}} \eta_1, \\ \xi_2 = K^{-\frac{1}{m}} \eta_2. \end{cases} \]
With this in hand, we can deduce the desired estimate in the \( \Omega_1 \) -case. This completes the outline of the proof of Proposition 1.8.

The paper is organized as follows. In Section 2 and Section 3, we give the proof of Proposition 1.8 by combining the methods in [12] and the reduction of dimension arguments from [17]. In section 4, we will give two applications of Corollary 1.7.

Notations: For nonnegative quantities \( X \) and \( Y \), we will write \( X \lesssim Y \) to denote the estimate \( X \leq CY \) for some large constant \( C \) which may vary from line to line and depend on various parameters. If \( X \lesssim Y \lesssim X \), we simply write \( X \sim Y \). Dependence of implicit constants on the power \( p \) or the dimension will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, \( X \lesssim u Y \) indicates \( X \leq CY \) for some \( C = C(u) \). For any set \( E \subset \mathbb{R}^d \), we use \( \chi_E \) to denote the characteristic function on \( E \). Usually, Fourier transform on \( \mathbb{R}^d \) is defined by
\[ \hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx. \]

2. The Proof of Proposition 1.8

Recall that \([0, 1]^2\) is divided into \( \bigcup_{j=0}^{3} \Omega_j \), where
\[ \begin{align*} \Omega_0 := [K^{-\frac{1}{2}}, 1] \times [K^{-\frac{1}{2}}, 1], & \quad \Omega_1 := [K^{-\frac{1}{2}}, 1] \times [0, K^{-\frac{1}{m}}], \\
\Omega_2 := [0, K^{-\frac{1}{2}}] \times [K^{-\frac{1}{m}}, 1], & \quad \Omega_3 := [0, K^{-\frac{1}{2}}] \times [0, K^{-\frac{1}{m}}]. \end{align*} \]
We denote by \( \Sigma_j := \{\xi_1, \xi_2, \phi(\xi_1, \xi_2) : (\xi_1, \xi_2) \in \Omega_j\} \) with \( j \in \{0, 1, 2, 3\} \). For technical reasons, \( K^{-\frac{1}{2}}, R^{-\frac{1}{2}} \) and \((R^{-1/m})^{-1/m}\) should be dyadic numbers. Therefore, we choose \( K = 2^m l \) and \( R = 2^m t \) \((l, t \in \mathbb{N})\) to be large numbers satisfying \( K \approx R^\delta \) with \( \delta \) being in Proposition 1.8. Recall that in (1.3) we have
\[ \|e^{it\phi(D)} f\|_{L^6(Y)} \leq \sum_{j=0}^{3} \|e^{it\phi(D)} f_{\Omega_j}\|_{L^6(Y)}. \]
We will estimate \( \|e^{it\phi(D)} f\|_{L^6(Y)} \) in several cases. If the contribution from the region \( \Omega_j \) dominates, we call it \( \Omega_j \)-case. By the triangle inequality, we have
\[ \|e^{it\phi(D)} f_{\Omega_j}\|_{L^6(B)} \geq \frac{1}{16} \|e^{it\phi(D)} f\|_{L^6(B)}. \]
for at least one of the \(j\)'s.

Now we sort the \(K\)-cubes in \(Y\). Denote
\[
\{ B \subset Y : \| e^{it\phi(D)} f_{\Omega j} \|_{L^6(B)} \geq \frac{1}{16} \| e^{it\phi(D)} f \|_{L^6(B)} \}
\]
by \(Y^j\). Clearly, one has \(Y = \bigcup_{j=0}^{3} Y^j\). If \(\sharp\{ B : B \subset Y^j \} \geq \frac{N}{4}\), we call it \(\Omega_j\)-case.

For the \(\Omega_0\)-case, we sort the \(K\)-cubes in \(Y^0\) as follows:

1. For any dyadic number \(A^{(0)}\), let \(Y^0_{A^{(0)}}\) be the union of the \(K\)-cubes in \(Y^0\) satisfying
\[
\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(B)} \approx A^{(0)}.
\]
2. Fix \(A^{(0)}\), for any dyadic number \(\nu^{(0)}\), let \(Y^0_{A^{(0)},\nu^{(0)}}\) be the union of the \(K\)-cubes in \(Y^0_{A^{(0)}}\) such that for each \(B \subset Y^0_{A^{(0)},\nu^{(0)}}\), the \(R^{1/2}\)-cube intersecting \(B\) contains \(\approx \nu^{(0)}\) cubes from \(Y^0_{A^{(0)}}\).

Without loss of generality, we may assume \(\| f \|_2 = 1\). Here \(\nu^{(0)}\) satisfies \(1 \leq \nu^{(0)} \lesssim R^3\), and the dyadic number \(A^{(0)}\) making significant contributions can be assumed to be between \(R^{-C}\) and \(R^C\) for a large constant \(C\). In fact, if we denote
\[
\{ B \subset Y^0 : \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(B)} \leq R^{-100} \}
\]
by \(Y^{-1}_0\), then
\[
\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^{-1}_0)} \lesssim R^{-90} \| f \|_2.
\]

Note that
\[
N^{(0)} \lesssim R^3, \lambda^{(0)} \geq R^{-3}, \nu^{(0)} \geq 1, \alpha > 0
\]
and \(K = R^\delta, \delta = \varepsilon^{100}\), where
\[
N^{(0)} := \sharp \{ B : B \subset Y^0_{A^{(0)},\nu^{(0)}} \}
\]
and
\[
\lambda^{(0)} := \max_{B^3(x',r) \subset Q_R} \sharp \{ B_k \subset Y^0 : B_k \subset B^3(x',r) \}^{r^\alpha}.
\]

So we have
\[
\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^{-1}_0)} \leq C_{\varepsilon} K^{O(1)} (N^{(0)})^{-\frac{1}{3}} (\lambda^{(0)})^{\frac{1}{6}} (\nu^{(0)})^{\frac{1}{6}} R^{(N^{(0)})^{\frac{5}{6}}} \| f \|_2.
\]
Therefore, there exist some dyadic numbers \(A^{(0)}, \nu^{(0)}\) such that
\[
N^{(0)} \gtrsim \frac{N}{(\log R)^2}.
\]

Fix that choice of \(A^{(0)}, \nu^{(0)}\) and denote \(Y^0_{A^{(0)},\nu^{(0)}}\) by \(Y^0\) for convenience. Since \(\| e^{it\phi(D)} f \|_{L^6(B_k)} \) is essentially constant in \(k = 1, 2, ..., N\) and \(\frac{N}{(\log R)^2} \lesssim N^{(0)}\), we have
\[
\| e^{it\phi(D)} f \|_{L^6(Y)} \lesssim (\log R)^{O(1)} \| e^{it\phi(D)} f \|_{L^6(Y^0)}, \tag{2.1}
\]

Note that
\[
\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(B)} \geq \frac{1}{16} \| e^{it\phi(D)} f \|_{L^6(B)}
\]
for \(B \subset Y^0\). It follows that
\[
\| e^{it\phi(D)} f \|_{L^6(Y^0)} \lesssim \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^0)}.
\]
This together with (2.1) gives
\[ \| e^{it\phi(D)} f \|_{L^6(Y)} \lesssim R^\varepsilon \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^0)} \] (2.2)
in the \( \Omega_0 \)-case. We also have
\[ \frac{N}{(\log R)^2} \lesssim N(0) \leq N, \quad \nu(0) \leq \nu, \quad \lambda(0) \leq \lambda. \] (2.3)

Using the argument from [12] as well as the rescaling trick from Case (c) in Subsection 2.2 of [16], we claim
\[ \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^0)} \leq C_{\varepsilon} K^{O(1)} (N(0))^{\frac{1}{4}} (\lambda(0))^{\frac{1}{4}} (\nu(0))^{\frac{1}{4}} R^{m_2 + \varepsilon} \| f \|_2. \] (2.4)

Note that \( \min\{ \frac{5}{12} - \frac{\alpha}{6m}, \frac{1}{3} - \frac{4-\alpha}{6m} \} \geq \frac{\alpha}{12} \) for any \( \alpha \in (0, 3] \) and each \( m \geq 2 \). The factor \( K^{O(1)} \) appears on the right-hand side of (2.4) because the principal curvature (in the direction of \( \xi_2 \)-axis) of the surface
\[ \Sigma_0 := \{ (\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in \Omega_0 \} \]
has a lower bound \( K^{-C} \). Inserting (2.3) into (2.4), we complete the proof of Proposition 1.8 in the \( \Omega_0 \)-case.

Now we turn to prove (2.4). We divide the region \( \Omega_0 \) into a family of subregions
\[ \Omega_0 = \bigcup_{\sigma_1, \sigma_2} \Omega_{\sigma_1, \sigma_2} \]
for dyadic numbers \( \sigma_i \in [K^{-1/m}, \frac{1}{2}] \) with \( i \in \{1, 2\} \). Here \( \Omega_{\sigma_1, \sigma_2} := I_{\sigma_1} \times I_{\sigma_2} \) with \( I_{\sigma_i} = [\sigma_i, 2\sigma_i] \). We divide \( I_{\sigma_i} \) further into
\[ I_{\sigma_i} = \bigcup_{j=1}^{K^{1/2}} I_{\sigma_i,j}, \]
Each \( I_{\sigma_i,j} \) has length of \( \sigma_i^{-\frac{m_2}{2}} K^{-1/2} \). So we can write
\[ \Omega_{\sigma_1, \sigma_2} = \bigcup_{\tau} \tau, \]
where each \( \tau \) in the above union has the form
\[ \tau = [a_1, a_1 + \sigma_1^{-\frac{m_2}{2}} K^{-1/2}] \times [a_2, a_2 + \sigma_2^{-\frac{m_2}{2}} K^{-1/2}] \]
with \( \sigma_i \leq a_i \leq 2\sigma_i \) (\( i = 1, 2 \)). Therefore, we have
\[ e^{it\phi(D)} f_{\Omega_0} = \sum_{\sigma_1, \sigma_2} e^{it\phi(D)} f_{\Omega_{\sigma_1, \sigma_2}}, \] (2.5)
and
\[ e^{it\phi(D)} f_{\Omega_{\sigma_1, \sigma_2}} = \sum_{\tau \subset \Omega_{\sigma_1, \sigma_2}} e^{it\phi(D)} f_{\tau}. \]
We will estimate \( \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^0)} \) in several cases. If the contribution from the region \( \Omega_{\sigma_1,\sigma_2} \) dominates, we call it \( \Omega_{\sigma_1,\sigma_2} \)-case. More precisely, given a \( K \)-cube \( B \), by the triangle inequality, we have

\[
\| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(B)} \gtrsim \frac{\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(B)}}{(\log K)^2}
\]

for at least one pair \((\sigma_1, \sigma_2)\).

We sort the \( K \)-cubes in \( Y^0 \) as follows. Denote by

\[
Y^{\sigma_1,\sigma_2} := \{ B \subset Y^0 : \| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(B)} \gtrsim \frac{1}{(\log K)^2} \| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(B)} \}.
\]

Clearly, one has \( Y = \bigcup_{\sigma_1,\sigma_2} Y^{\sigma_1,\sigma_2} \). If

\[
\sharp\{ B : B \subset Y^{\sigma_1,\sigma_2} \} \gtrsim \frac{N(0)}{(\log K)^2},
\]

we call it \( \Omega_{\sigma_1,\sigma_2} \)-case.

For the \( \Omega_{\sigma_1,\sigma_2} \)-case, we sort the \( K \)-cubes in \( Y^{\sigma_1,\sigma_2} \) as follows:

1. For any dyadic number \( A^{(\sigma_1,\sigma_2)} \), let \( Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2)} \) be the union of the \( K \)-cubes in \( Y^{\sigma_1,\sigma_2} \) satisfying

\[
\| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(B)} \approx \| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(B)} \approx A^{(\sigma_1,\sigma_2)}.
\]

2. Fix \( A^{(\sigma_1,\sigma_2)} \), for any dyadic numbers \( \nu^{(\sigma_1,\sigma_2)} \), let \( Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2),\nu^{(\sigma_1,\sigma_2)}} \) be the union of the \( K \)-cubes in \( Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2),\nu^{(\sigma_1,\sigma_2)}} \) such that for each \( B \subset Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2),\nu^{(\sigma_1,\sigma_2)}} \), the \( R^{1/2} \)-cubes intersecting \( B \) contains \( \approx \nu^{(\sigma_1,\sigma_2)} \) cubes from \( Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2)} \).

The dyadic numbers \( A^{(\sigma_1,\sigma_2)} \) and \( \nu^{(\sigma_1,\sigma_2)} \) making significant contribution can be assumed to be between \( R^{-C} \) and \( RC \). Therefore, there exist some dyadic numbers \( A^{(\sigma_1,\sigma_2)} \) and \( \nu^{(\sigma_1,\sigma_2)} \) such that

\[
\sharp\{ B : B \subset Y^{\sigma_1,\sigma_2} \} \approx \frac{N(0)}{(\log K)^2}.
\]

Fix that choice of \( A^{(\sigma_1,\sigma_2)} \), \( \nu^{(\sigma_1,\sigma_2)} \) and denote \( Y^{\sigma_1,\sigma_2} \), \( A^{(\sigma_1,\sigma_2)} \) by \( Y^{\sigma_1,\sigma_2} \) for convenience. Then, in the \( \Omega_{\sigma_1,\sigma_2} \)-case we have that

\[
\| e^{it\phi(D)} f_{\Omega_0} \|_{L^6(Y^0)} \lesssim R^\varepsilon \| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(Y^{\sigma_1,\sigma_2})},
\]

and

\[
\frac{N(0)}{(\log K)^2} \lesssim N^{(\sigma_1,\sigma_2)} \leq N^{(0)}, \quad \nu^{(\sigma_1,\sigma_2)} \leq \nu^{(0)}, \quad \lambda^{(\sigma_1,\sigma_2)} \leq \lambda^{(0)},
\]

where

\[
\lambda^{(\sigma_1,\sigma_2)} := \max_{B^3(x',r) \subset Q_{n,x} \subset \mathbb{R}^3, r \geq K} \frac{\sharp\{ B_k \subset Y^{\sigma_1,\sigma_2} : B_k \subset B^3(x',r) \}}{r^\alpha}.
\]

We are going to estimate \( \| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(Y^{\sigma_1,\sigma_2})} \). By the triangle inequality, one has

\[
\| e^{it\phi(D)} f_{\Omega_{\sigma_1,\sigma_2}} \|_{L^6(B)} \lesssim \sum_{\tau \subset \Omega_{\sigma_1,\sigma_2}} \| e^{it\phi(D)} f_{\tau} \|_{L^6(B)}. \tag{2.6}
\]
For each $\tau$, we deal with $e^{it\Phi(D)}f_\tau$ by the rescaling trick from Case (c) in Subsection 2.2 of [16]. To do it, we need to further decompose $f_\tau$ in physical space and perform dyadic pigeonholing several times. First we divide the physical square $[0,R]^2$ into $\frac{R}{K^2\sigma_2} \times \frac{R}{K^2\sigma_1}$-rectangles $D$. For each pair $(\tau,D)$, let $f_{\Box_{\tau,D}}$ be the function formed by cutting off $f$ on the rectangle $D$ (with a Schwartz tail) in physical space and the rectangle $\tau$ in Fourier space. Note that $e^{it\Phi(D)}f_{\Box_{\tau,D}}$ is essentially supported on an $\frac{R}{K^2\sigma_1} \times \frac{R}{K^2\sigma_2}$-rectangle $\tau$ in physical space. For each $\tau$, we regroup tubes $S$ to stand for the collection of tubes $\tau,D$ tile $Q_R$. We have

$$f_{\Omega_{\sigma_1,\sigma_2}} = \sum_{\tau \in \Omega_{\sigma_1,\sigma_2}} \sum_D f_{\Box_{\tau,D}},$$

and write

$$f_{\Omega_{\sigma_1,\sigma_2}} = \sum_{\Box} f_{\Box}$$

for simplicity. For each $\tau$, a given $K$-cube $B$ lies in exactly one box $\Box_{\tau,D}$. Recall that $K' = R^\delta$, where $\delta = \varepsilon^{100}$. Denote

$$R_1 := \frac{R}{K} = R^{1-\delta}, \quad K_1 = R_1^2 = R^{2\delta-\delta^2}.$$

Tile $\Box$ by $\frac{m-2}{\sigma_1} K^{1/2} K_1 \times \frac{m-2}{\sigma_2} K^{1/2} K_1 \times K K_1$-tube $S$, and also tile $\Box$ by $\frac{R_1/2}{\sigma_2} \times \frac{K_1^2}{\sigma_2} K_1$-tubes $S'$ (all running parallel to the long axis of $\Box$). After rescaling, the $\Box$ becomes an $R_1$-cube, the tubes $S'$ and $S$ become lattice $R_1^2$-cubes and $K_1$-cubes, respectively.

We regroup tubes $S$ and $S'$ inside each $\Box$ as follows:

1) Sort those tubes $S$ which intersect $Y^0$ according to the value $\|e^{it\Phi(D)}f_{\Box}\|_{L^6(S)}$ and the number of $K$-cubes contained in it. For dyadic numbers $\eta, \beta_1$, we use $S_{\Box,\eta,\beta_1}$ to stand for the collection of tubes $S \subset \Box$ each of which containing $\sim \eta$ $K$-cubes and $\|e^{it\Phi(D)}f_{\Box}\|_{L^6(S)} \sim \beta_1$.

2) For fixed $\eta, \beta_1$, we sort the tubes $S' \subset \Box$ according to the number of the tubes $S \in S_{\Box,\eta,\beta_1}$ contained in it. For dyadic number $\nu_1$, let $S_{\Box,\eta,\beta_1,\nu_1}$ be the subcollection of $S_{\Box,\eta,\beta_1}$ such that for each $S \in S_{\Box,\eta,\beta_1,\nu_1}$, the tube $S'$ containing $S$ contains $\sim \nu_1$ tubes from $S_{\Box,\eta,\beta_1}$.

3) For fixed $\eta, \beta_1, \nu_1$, we sort the boxes $\Box$ according to the value $\|f\|_2$, the number $\sharp S_{\Box,\eta,\beta_1,\nu_1}$ and the value $\lambda_1$ defined below. For dyadic numbers $\beta_2, N_1, \lambda_1$, let $\sharp S_{\Box,\eta,\beta_1,\nu_1}$ denote the collection of boxes $\Box$ satisfying that

$$\|f\|_2 \sim \beta_2, \quad \sharp S_{\Box,\eta,\beta_1,\nu_1} \sim N_1$$

and

$$\max_{T_r \subset \Box, r \geq K_1} \frac{\sharp \{S \in S_{\Box,\eta,\beta_1,\nu_1} : S \subset T_r\}}{r^2} \sim \lambda_1, \quad (2.7)$$
where $T_r$ are $\sigma_2 \frac{m-2}{2} K^{1/2} r \times \sigma_2 \frac{m-2}{2} K^{1/2} r \times Kr$-tubes in $\Box$ running parallel to the long axis of $\Box$.

Let $Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}$ denote \{$S : S \subset S_{\Box, \eta, \beta, \nu_1}$\} and $\chi_{Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}}$ be the corresponding characteristic function. Without loss of generality, we can assume $\|f\|_2 = 1$. Therefore, there are only $O(\log R)$ significant choices for each dyadic number. By pigeonholing, we can choose $\eta, \beta, \nu_1, \beta_2, N_1, \lambda_1$ such that

$$\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(B)} \lesssim (\log R)^{1/2} \left( \sum_{\Box \in \mathcal{B}, B \subset Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}} \|e^{it\phi(D)} f_{\Box}\|_{L^6(\omega_B)}^2 \right)^{1/2}$$

(2.8)

holds for a fraction $\gtrsim (\log R)^{-6}$ of all $K$-cubes $B$, where we have used the fact that

$$\sharp \{\tau : \tau \subset \Omega_{s_1, s_2}\} \lesssim K.$$ 

For brevity, we denote by

$$Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2} := Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}, \quad \mathcal{B} := \mathcal{B}_{\eta, \beta, \nu_1, \beta_2, N_1, \lambda_1}.$$ 

Finally, we sort the $K$-cubes $B$ satisfying (2.8) by $\sharp \{\Box \in \mathcal{B} : B \subset Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}\}$. Let $Y_{s_1, s_2}$ be a union of $K$-cubes $B$ obeying

$$\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(B)} \lesssim (\log R)^{1/2} \left( \sum_{\Box \in \mathcal{B}, B \subset Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}} \|e^{it\phi(D)} f_{\Box}\|_{L^6(\omega_B)}^2 \right)^{1/2}$$

(2.9)

and

$$\sharp \{\Box \in \mathcal{B} : B \subset Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2} \sim \mu\}$$

(2.10)

for some dyadic number $1 \leq \mu \leq K^{O(1)}$. Moreover, the number of $K$-cubes $B$ in $Y'$ is $\gtrsim (\log R)^{-7} N$.

By our assumption that $\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(B_k)}$ is essentially constant in $k = 1, 2, \cdots, N^0$, we have

$$\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(Y_{s_1, s_2})}^6 \lesssim K^3 (\log R)^7 \sum_{B \subset Y'} \|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(B)}^6.$$ 

(2.11)

For each $B \subset Y'$, it follows from (2.9), (2.10) and Hölder’s inequality that

$$\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(B)}^6 \lesssim (\log R)^{36} K^3 \mu^2 \left( \sum_{\Box \in \mathcal{B}, B \subset Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2}} \|e^{it\phi(D)} f_{\Box}\|_{L^6(\omega_B)}^6 \right)^{1/2}. $$ 

(2.12)

Combining (2.11) with (2.12), one has

$$\|e^{it\phi(D)} f_{\Omega_{s_1, s_2}}\|_{L^6(Y_{s_1, s_2})} \lesssim (\log R)^C K^{1/2} \mu^{1/3} \left( \sum_{\Box \in \mathcal{B}} \|e^{it\phi(D)} f_{\Box}\|_{L^6(Y_{\Box, \eta, \beta, \nu_1}^{s_1, s_2})}^6 \right)^{1/6}. $$ 

(2.13)
Next, we apply rescaling to each $\|e^{it\phi(D)} f\|_{L^6(\Omega^{\alpha_1, \alpha_2})}$. For each $\sigma_1 \frac{m-2}{2} K^{-1/2} \times \sigma_2 \frac{m-2}{2} K^{-1/2}$-rectangle $\tau = \tau_\square$ in $\Omega_{\alpha_1, \alpha_2}$, we write

$$
\begin{cases}
\xi_1 = a_1 + \sigma_1 \frac{m-2}{2} K^{-1/2} \eta_1, \\
\xi_2 = a_2 + \sigma_2 \frac{m-2}{2} K^{-1/2} \eta_2.
\end{cases}
$$

Then

$$
|e^{it\phi(D)} f\|_{L^6(\Omega^{\alpha_1, \alpha_2})} = \sigma_1 \frac{m-2}{2} \sigma_2 \frac{m-2}{2} K^{-1/2} |e^{it\tilde{\phi}(D)} g(\tilde{x})|
$$

with $\|g\|_2 = \|f\|_2$ and

$$
\hat{g}(\eta_1, \eta_2) := \sigma_1 \frac{m-2}{2} \sigma_2 \frac{m-2}{2} K^{-1/2} \hat{f}(a_1 + \sigma_1 \frac{m-2}{2} K^{-1/2} \eta_1, a_2 + \sigma_2 \frac{m-2}{2} K^{-1/2} \eta_2),
$$

where

$$
\tilde{\phi}(\eta_1, \eta_2) := \sum_{i=1}^{2} \left( \frac{m(m-1)}{2} a_i^{m-2} \sigma_i^{-(m-2)} \eta_i^2 \right) + \frac{m(m-1)(m-2)}{6} a_i^{m-3} \sigma_i^{3(m-2)} K^{-1/2} \eta_i^3 + \cdots + \sigma_i \frac{m(m-2)}{2} K^{-1/2} \eta_i^m,
$$

and

$$
\begin{cases}
\tilde{x}_1 := \sigma_1 \frac{m-2}{2} K^{-1/2} (x_2 + ma_1^{-1} t), \\
\tilde{x}_2 := \sigma_2 \frac{m-2}{2} K^{-1/2} (x_2 + ma_2^{-1} t), \\
\tilde{t} := K^{-1} t.
\end{cases}
$$

For brevity, we denote the above relation by $(\tilde{x}, \tilde{t}) = \mathcal{L}_0(x, t)$. Therefore, we have

$$
\|e^{it\phi(D)} f\|_{L^6(\Omega^{\alpha_1, \alpha_2})} = \rho \frac{m-2}{2} K^{-1/6} \|e^{it\tilde{\phi}(D)} g(\tilde{x})\|_{L^6(\Omega^{\alpha_1, \alpha_2})}, \quad (2.14)
$$

where $\Omega^{\alpha_1, \alpha_2} = \mathcal{L}_0(\Omega^{\alpha_1, \alpha_2})$. Since $a_i \sim \sigma_i$ and $\frac{1}{2} \geq \sigma_i \geq K^{-1/m}$, it is easy to check that $\tilde{\phi}$ is a phase function of elliptic type, namely,

$$
\partial_{\eta_i}^2 \tilde{\phi} \sim_{m} 1, \quad i = 1, 2,
$$

and

$$
|\partial_{\eta_i}^l \tilde{\phi}| \lesssim_{m} 1, \quad 3 \leq l \leq m.
$$

Applying Proposition 3.1 in [12] to the term on the right-hand side of (2.14) at scale $R_1$, we deduce

$$
\|e^{it\phi(D)} f\|_{L^6(\Omega^{\alpha_1, \alpha_2})} \lesssim \sigma_1 \frac{m-2}{6} \sigma_2 \frac{m-2}{6} K^{-1/6} N_1^{-\frac{1}{3}} \lambda_1^{\frac{1}{6}} \nu_1^{\frac{1}{6}} \left( \frac{R}{K} \right)^{\frac{1}{2}} \|f\|_2, \quad (2.15)
$$

Using a similar argument as in the proof of inequality (3.24) in [12], one has

$$
\frac{\mu}{\|f\|_2} \lesssim \frac{(log R)^7 N_1 \eta}{N_0}, \quad (2.16)
$$

and

$$
\lambda_1 \eta \lesssim \max_{T_r \subset \Omega, \eta \geq K_1} \# \{ B \subset \Omega^0 : B \subset T_r \} \leq \frac{\lambda^{(0)}(K^{O(1)} r)^{\alpha}}{r^\alpha}.
$$
where we have used the fact that one can cover a \( \frac{m-2}{\sigma_1^{1/2}} K^{1/2} r \times \frac{m-2}{\sigma_2^{1/2}} K^{1/2} r \times Kr \)-tube \( T_r \) by \( K^{O(1)} \) finitely overlapping (min\{\( \sigma_1, \sigma_2 \)\})\(^{m-2} \) \( K^{1/2} r \)-balls. Hence, we get

\[
\eta \lesssim \frac{\lambda^0 K^{O(1)}}{\lambda_1} .
\] (2.17)

Now we relate \( \nu_1 \) and \( \nu^{(0)} \) by considering the number of \( K \)-cubes in each relevant \( \frac{R^{1/2}}{\sigma_1} \times \frac{R^{1/2}}{\sigma_2} \times K^{1/2} r \)-tube \( S' \). Recall that each relevant \( S' \) contains \( \sim \nu_1 \) tubes \( S \) in \( Y^3_{\alpha/2} \) and each such \( S \) contains \( \sim \eta \) \( K \)-cubes. On the other hand, we can cover \( S' \) by \( K^{O(1)} \) finitely overlapping \( R^{1/2} \)-cubes, and each \( R^{1/2} \)-cube contains \( \lesssim \nu^{(0)} \) many \( K \)-cubes in \( Y^3 \). Thus, it follows that

\[
\nu_1 \lesssim \frac{K^{1/2} \nu^{(0)}}{\eta} .
\] (2.18)

By inserting (2.16), (2.17) and (2.18) into (2.13), we derive

\[
\| e^{i t \phi(D)} f_{\Omega_1, \sigma_2} \|_{L^6(Y^{\sigma_1, \sigma_2})} \leq C_\varepsilon K^{O(1)} \sigma_1^{-\frac{m-2}{6m}} \sigma_2^{-\frac{m-2}{6m}} (N^{(0)})^{-\frac{1}{3}} (\nu^{(0)})^{\frac{1}{3}} (\lambda^{(0)})^{\frac{1}{3}} R^{m+\varepsilon} \| f \|_2 .
\] (2.19)

Recall that \( \sigma_1 \geq K^{-1/m} \). Combining (2.19) with (2.5), we obtain

\[
\| e^{i t \phi(D)} f_{\Omega_0} \|_{L^6(Y^0)} \leq C_\varepsilon K^{O(1)} (N^{(0)})^{-\frac{1}{3}} (\nu^{(0)})^{\frac{1}{3}} (\lambda^{(0)})^{\frac{1}{3}} R^{m+\varepsilon} \| f \|_2,
\]

which verifies inequality (2.4).

For the \( \Omega_3 \)-case, we sort the \( K \)-cubes in \( Y^3 \) as follows:

1. For any dyadic number \( A^{(3)} \), let \( Y^3_{A^{(3)}} \) be the union of the \( K \)-cubes in \( Y^3 \) satisfying

\[
\| e^{i t \phi(D)} f_{\Omega_3} \|_{L^6(B)} \approx A^{(3)} .
\] (2.20)

2. Fix \( A^{(3)} \), for any dyadic number \( \nu^{(3)} \), let \( Y^3_{A^{(3)}, \nu^{(3)}} \) be the union of the \( K \)-cubes in \( Y^3_{A^{(3)}} \) such that for each \( B \subset Y^3_{A^{(3)}, \nu^{(3)}} \), the \( R^{1/2} \)-cube intersecting \( B \) contains \( \approx \nu^{(3)} \) cubes from \( Y^3_{A^{(3)}} \).

The dyadic numbers \( A^{(3)} \) and \( \nu^{(3)} \) making significant contribution can be assumed to be between \( R^{-C} \) and \( R^C \). Therefore, there exist some dyadic numbers \( A^{(3)} \) and \( \nu^{(3)} \) such that

\[
\#\{B : B \subset Y^3_{A^{(3)}, \nu^{(3)}} \} =: N^{(3)} \gtrsim \frac{N}{(\log R)^2} .
\]

Fix that choice of \( A^{(3)}, \nu^{(3)} \) and denote \( Y^3_{A^{(3)}, \nu^{(3)}} \) by \( Y^3 \) for convenience. Then in the \( \Omega_3 \)-case, we have

\[
\| e^{i t \phi(D)} f \|_{L^6(Y)} \lesssim R^\varepsilon \| e^{i t \phi(D)} f_{\Omega_3} \|_{L^6(Y^3)} ,
\] (2.21)

and

\[
\frac{N}{(\log R)^2} \lesssim N^{(3)} \leq N, \quad \nu^{(3)} \leq \nu, \quad \lambda^{(3)} \leq \lambda .
\]
where

$$\lambda^{(3)} := \max_{B^3(x',r) \subseteq Q_R, x' \in \mathbb{R}^3, r \geq 1} \sharp \{B_k \subset Y^3 : B_k \subset B^3(x', r)\}.$$  

We are going to estimate $$\|e^{it\phi(D)} f_{\Omega_3}\|_{L^6(Y^3)}$$. We deal with $$e^{it\phi(D)} f_{\Omega_3}$$ by rescaling and induction on scales. To do it, we further decompose $$f_{\Omega_3}$$ in physical space and perform dyadic pigeonholing several times.

Firstly, we divide the physical square $$[0, R]^2$$ into $$\frac{R}{K^{1-\frac{\varepsilon}{10}}} \times \frac{R}{K^{1-\frac{\varepsilon}{10}}}$$-rectangles $$D$$. For each pair $$D$$, let $$f_{\Omega_3, D}$$ be the function formed by cutting off $$f_{\Omega_3}$$ on the rectangle $$D$$ (with a Schwartz tail) in physical space. Thus, we have

$$f_{\Omega_3} = \sum_D f_{\Omega_3, D},$$

and write $$f_{\Omega_3} = \sum_{\square} f_\square$$ for abbreviation. Note that $$e^{it\phi(D)} f_{\Omega_3, D}$$ is essentially supported on an $$\frac{R}{K^{1-\frac{\varepsilon}{10}}} \times \frac{R}{K^{1-\frac{\varepsilon}{10}}} \times R$$-box, which is denoted by $$\square_{\Omega_3, D}$$. The long axis of $$\square_{\Omega_3, D}$$ is parallel to the vector $$(0, 0, 1)$$. The different boxes $$\square_{\Omega_3, D}$$ tile $$Q_R$$. In particular, a given $$K$$-cube $$B$$ lies in exactly one box $$\square_{\Omega_3, D}$$. For each $$K$$-cube $$B$$, it holds trivially that

$$\|e^{it\phi(D)} f_{\Omega_3}\|_{L^6(B)} \lesssim \left( \sum_{\square} \|e^{it\phi(D)} f_{\square}\|_{L^6(B)}^6 \right)^{1/6}, \quad (2.22)$$

and

$$\sharp \{\square \in \mathbb{B} : B \subset Y^3\} \sim 1. \quad (2.23)$$

Recall that $$K = R^K$$, where $$\delta = \varepsilon^{100}$$. Denote

$$R_1 := \frac{R}{K} = R^{1-\delta}, \quad K_1 = R_1^K = R^{\delta-\delta^2}.$$  

Tile $$\square$$ by $$K^{\frac{1}{3}} K_1^1 \times K^{\frac{1}{3}} K_1^1 \times K K_1$$-cube $$S$$, and also tile $$\square$$ by $$\frac{R^{1/2}}{K^{1/2}} \times \frac{R^{1/2}}{K^{1/2}} \times K^{\frac{1}{2}} R^{\frac{1}{2}}$$-tubes $$S'$$ (all running parallel to the long axis of $$\square$$). After rescaling the $$\square$$ becomes an $$R_1$$-cube, the tubes $$S'$$ and $$S$$ becomes lattice $$R_1^3$$-cubes and $$K_1$$-cubes, respectively. We regroup the tubes $$S$$ and $$S'$$ inside each $$\square$$ as follows:

1. Sort those tubes $$S$$ which intersect $$Y^3$$ according to the value $$\|e^{it\phi(D)} f_{\square}\|_{L^6(S)}$$ and the number of $$K$$-cubes contained in it. For dyadic numbers $$\eta, \beta_1$$, we use $$S_{\square, \eta, \beta_1}$$ to stand for the collection of tubes $$S \subset \square$$ each of which contains $$\sim \eta$$ $$K$$-cubes and $$\|e^{it\phi(D)} f_{\square}\|_{L^6(S)} \sim \beta_1$$.
2. For fixed $$\eta, \beta_1$$, we sort the tubes $$S' \subset \square$$ according to the number of the tubes $$S \in S_{\square, \eta, \beta_1}$$ contained in it. For each dyadic number $$\nu_3$$, let $$S_{\square, \eta, \beta_1, \nu_3}$$ be the subcollection of $$S_{\square, \eta, \beta_1}$$ such that for each $$S \in S_{\square, \eta, \beta_1, \nu_3}$$, the tube $$S'$$ containing $$S$$ contains $$\sim \nu_3$$ $$K$$-cubes from $$S_{\square, \eta, \beta_1}$$.
3. For fixed $$\eta, \beta_1, \nu_3$$, we sort the boxes $$\square$$ according to the value $$\|f_{\square}\|_2$$, the number $$\sharp S_{\square, \eta, \beta_1, \nu_3}$$ and the value $$\lambda_3$$ defined below. For dyadic numbers $$\beta_2, N_3, \lambda_3$$, let $$\mathbb{B}_{\eta, \beta_1, \nu_3, \beta_2, N_3, \lambda_3}$$ denote the collection of boxes $$\square$$ each of which satisfies

$$\|f_{\square}\|_2 \sim \beta_2, \quad \sharp S_{\square, \eta, \beta_1, \nu_3} \sim N_3.$$
and
\[
\max_{T_r, \eta, \beta, \nu, \lambda} \{ \| f_{\Omega} \|_{L^6(B)} : S \subset T_r \} \sim \lambda^3, \tag{2.24}
\]
where \( T_r \) are \( K^{1/4} r \times K^{1/4} r \times K r \)-tubes in \( \square \) running parallel to the long axis of \( \square \).

Let \( Y^3_{\square, \eta, \beta, \nu} \) denote \( \{ S : S \subset S_{\square, \eta, \beta, \nu} \} \). By (2.22), we have
\[
\| e^{i t \phi(D)} f_{\Omega} \|_{L^6(B)} \lesssim \sum_{\eta, \beta, \nu, \lambda} \| e^{i t \phi(D)} f_{\square} \|_{L^6(\omega_B)}.
\]
Thus, for each \( B \subset Y^3 \), we can choose \( \eta, \beta, \nu, \lambda \) depending on \( B \) such that
\[
\| e^{i t \phi(D)} f_{\Omega} \|_{L^6(B)} \lesssim (\log R)^6 \sum_{\eta, \beta, \nu, \lambda} \| e^{i t \phi(D)} f_{\square} \|_{L^6(\omega_B)} \tag{2.25}
\]
holds for a fraction \( \gtrsim (\log R)^{-6} \) of all \( K \)-cubes \( B \) in \( Y^3 \). For brevity, we denote by
\[
Y^3_{\square} := Y^3_{\square, \eta, \beta, \nu}, \quad B := B_{\eta, \beta, \nu}.
\]
Let \( Y' \) be the union of all \( K \)-cubes \( B \) each of which obeys
\[
\| e^{i t \phi(D)} f_{\Omega} \|_{L^6(B)} \lesssim (\log R)^6 \left( \sum_{\square \in B} \| e^{i t \phi(D)} f_{\square} \|_{L^6(\omega_B)} \right)^{1/6}. \tag{2.26}
\]
The number of \( K \)-cubes \( B \) in \( Y' \) is \( \gtrsim (\log R)^{-6} N^3 \). This together with (2.20) yields
\[
\| e^{i t \phi(D)} f_{\Omega} \|_{L^6(Y^3)} \lesssim (\log R)^{C} \sum_{B \subset Y'} \| e^{i t \phi(D)} f_{\Omega} \|_{L^6(B)}. \tag{2.27}
\]
Putting (2.27) and (2.26) together, one has
\[
\| e^{i t \phi(D)} f_{\Omega} \|_{L^6(Y^3)} \lesssim (\log R)^{C} \left( \sum_{\square \in B} \| e^{i t \phi(D)} f_{\square} \|_{L^6(\omega_B)} \right)^{1/6}. \tag{2.28}
\]
Next, we apply rescaling to each \( \| e^{i t \phi(D)} f_{\square} \|_{L^6(\omega_B)} \) and induction on scales. Taking the change of variables
\[
\begin{align*}
\xi_1 &= K^{-\frac{1}{m}} \eta_1, \\
\xi_2 &= K^{-\frac{1}{m}} \eta_2,
\end{align*}
\]
we have
\[
\| e^{i t \phi(D)} f_{\square}(x) \| = K^{-\frac{1}{m}} | e^{i t \phi(D)} g(\tilde{x}) |,
\]
where

\[ \hat{g}(\eta_1, \eta_2) := K^{-\frac{1}{m}} \hat{f}(K^{-\frac{1}{m}} \eta_1, K^{-\frac{1}{m}} \eta_2), \quad \|g\|_2 = \|f\|_2, \]

and

\[
\begin{aligned}
\bar{x}_1 &:= K^{-\frac{1}{m}} x_1, \\
\bar{x}_2 &:= K^{-\frac{1}{m}} x_2, \\
\bar{t} &:= K^{-1} t.
\end{aligned}
\]

For brevity, we denote the above relation by \((\bar{x}, \bar{t}) = L_3(x, t)\). Therefore, we have

\[ \|e^{it\phi(D)} f\|_{L^6(Y^3_\eta)} = K\frac{1}{\sqrt{m}} \|e^{it\phi(D)} g(\bar{x})\|_{L^6(\bar{Y}^3)}, \tag{2.29} \]

where \(\bar{Y}^3 = L_3(Y^3_\eta)\). Hence, by (2.29) and the inductive hypothesis at scale \(R_1\), we deduce

\[ \|e^{it\phi(D)} f\|_{L^6(Y^3)} \lesssim (N(3))^{-\frac{1}{2}} (\mu(3))^{\frac{1}{2}} (\lambda(3))^{\frac{1}{2}} \left( R K \right)^{\frac{5}{2} - \frac{5}{6m} + \epsilon} \||\Phi|\|^6_2. \tag{2.30} \]

By inserting (2.30) into (2.28), we get

\[ \|e^{it\phi(D)} f_{\Omega_3}\|_{L^6(Y^3)} \lesssim (\eta(3))^{-\frac{1}{2}} (\mu(3))^{\frac{1}{2}} (\lambda(3))^{\frac{1}{2}} R K^{\frac{5}{2} - \frac{5}{6m} + \epsilon} \|f\|_2. \tag{2.31} \]

Note that \(\|f\|_2\) is essentially constant for \(\Box \in \mathbb{B}\). It follows that

\[ \|e^{it\phi(D)} f_{\Omega_3}\|_{L^6(Y^3)} \lesssim (\eta(3))^{-\frac{1}{2}} (\mu(3))^{\frac{1}{2}} (\lambda(3))^{\frac{1}{2}} R K^{\frac{5}{2} - \frac{5}{6m} + \epsilon} \|f\|_2. \tag{2.32} \]

Consider the cardinality of the set \(\{(\Box, B) : \Box \in \mathbb{B}, B \subseteq Y^3_\eta \cap Y^r\}\). It is easy to see that there is a lower bound

\[ \#\{(\Box, B) : \Box \in \mathbb{B}, B \subseteq Y^3_\eta \cap Y^r \} \approx (\eta(3))^6 N(3). \]

On the other hand, by our choices of \(N_3\) and \(\eta\), for each \(\Box \in \mathbb{B}\), \(Y^3_\eta\) contains \(\sim N_3\) tubes \(S\) and each \(S\) contains \(\sim \eta\) cubes in \(Y^r\), so

\[ \#\{(\Box, B) : \Box \in \mathbb{B}, B \subseteq Y^3_\eta \cap Y^r \} \lesssim (\eta(3))^6 N_3(3). \]

Therefore, we get

\[ \frac{1}{\eta(3)} \lesssim (\log R)^6 N_3(3). \tag{2.33} \]

Then by our choices of \(\lambda(3)\) as in (2.24) and \(\eta\), we have

\[
\lambda(3) \eta \sim \max_{T \subseteq \Box \cap r \geq K_1} \frac{\#\{S : S \subseteq Y_3^\eta \cap T\}}{r^\alpha} \cdot \#\{B : B \subseteq S \cap Y^3\} \\
\leq \max_{T \subseteq \Box \cap r \geq K_1} \frac{\#\{B \subseteq Y^3 : B \subseteq T\}}{r^\alpha} \\
\leq \frac{K^{1 - \frac{1}{m}} (\lambda(3))^{\frac{1}{2}}}{r^\alpha},
\]

where the last inequality follows from the fact that we can cover a \(K^{\frac{1}{m}} r \times K^{\frac{1}{m}} r \times Kr\)-tube \(T\) by \(\sim K^{1 - \frac{1}{m}}\) finitely overlapping \(K^{\frac{1}{m}} r\)-balls. Hence, we obtain

\[ \eta \lesssim \frac{\lambda(3) K^{1 + \frac{\alpha - 1}{m}}}{\lambda(3)}. \tag{2.34} \]
Finally, we relate $\nu_3$ and $\nu^{(3)}$ by considering the number of $K$-cubes in each relevant $K^{1/2} \times \frac{R_1}{\delta} \times \frac{R_2}{\delta}$-tube $S'$. Recall that each relevant $S'$ contains $\sim \nu_3$ tubes $S$ in $Y$ and each such $S$ contains $\sim \eta$ $K$-cubes. On the other hand, we can cover $S'$ by $\sim K^{1/2}$ finitely overlapping $R_2$-cubes, and each $R_2$-cube contains $\lesssim \nu^{(3)}$ many $K$-cubes in $Y^3$. Thus, it follows that

$$
\nu_3 \lesssim \frac{K^{1/2} \nu^{(3)}}{\eta}.
$$

(2.35)

By inserting (2.33), (2.34) and (2.35) into (2.32), we derive

$$
\| e^{it\phi(D)} f_{\Omega_3} \|_{L^6(Y^3)} \lesssim K^{-\varepsilon} (N^{(3)})^{-\frac{1}{6}} (\nu^{(3)})^{\frac{1}{3}} \frac{1}{\delta} R^{\frac{5}{12} - \frac{\lambda - \alpha}{6\eta} + \varepsilon} \| f \|_2.
$$

(2.36)

Since $K = R^\delta$ and $R$ can be assumed to be sufficiently large compared to any constant depending on $\varepsilon$, we have $K^{-\varepsilon} \ll 1$, and the induction closes. Recall that $\frac{N}{\log K} \lesssim N^{(3)} \leq N$, $\nu^{(3)} \leq \nu$ and $\lambda^{(3)} \leq \lambda$. It follows that

$$
\| e^{it\phi(D)} f_{\Omega_3} \|_{L^6(Y^3)} \lesssim \delta^{-\frac{1}{3}} \frac{1}{\varepsilon} R^{\frac{5}{12} - \frac{\lambda - \alpha}{6\eta} + \varepsilon} \| f \|_2.
$$

For the $\Omega_1$-case, we will employ the following proposition, whose proof is given in Section 4. Let $\psi(\xi_1, \xi_2) := \phi_1(\xi_1) + \xi_2^m$ be a class of smooth phase functions satisfying

$$
\phi_1^{(m)} \sim 1; \quad |\phi_1^{(k)}| \lesssim 1, 3 \leq k \leq m; \quad \phi_1^{(l)} = 0, l \geq m + 1
$$

(2.37)

on $[0, 1]$.

**Proposition 2.1.** For any $0 < \varepsilon < \frac{1}{100}$, there exist constants $C_\varepsilon$ and $\delta = \varepsilon^{100}$ such that the following holds for all $R \geq 1$ and all $f$ with supp $\hat{f} \subset [0, 1]^2$. Suppose that $Y = \bigcup_{k=1}^N B_k$ is a union of lattice $K$-cubes in $Q_R$ and each $R_2$-cube intersecting $Y$ contains $\sim \nu$ many $K$-cubes in $Y$, where $K = R^\delta$. Suppose that $\| e^{it\psi(D)} f \|_{L^6(Q_R)}$ is essentially constant in $k = 1, 2, ..., N$. Given $\lambda$ by

$$
\lambda := \max_{B^3(x', r) \subset Q_R} \left\{ \frac{2}{r^\alpha} \{ B_k : B_k \subset B^3(x', r) \} \right\},
$$

then

$$
\| e^{it\psi(D)} f \|_{L^6(Y)} \leq C_\varepsilon^{-\frac{1}{3}} \frac{1}{\lambda} \frac{1}{\delta} \frac{1}{\nu} \frac{1}{\mu} R^{\frac{5}{12} - \frac{\lambda - \alpha}{6\eta} + \varepsilon} \| f \|_2.
$$

Assume that Proposition 2.1 holds for a while, we estimate $\| e^{it\phi(D)} f_{\Omega_1} \|_{L^6(Y^1)}$ as follows. We divide the region $\Omega_1$ into a family of subregions

$$
\Omega_1 = \bigcup_{\rho} \Omega_{1, \rho},
$$

where $\Omega_{1, \rho} := I_\rho \times [0, K^{-\frac{1}{m}}]$ with $I_\rho = [\rho, 2\rho]$. Here $\rho$ is a dyadic number in $[K^{-\frac{1}{m}}, \frac{1}{2}]$. We abbreviate $\Omega_{1, \rho}$ by $\Omega_\rho$ and write

$$
e^{it\phi(D)} f_{\Omega_1} = \sum_{\rho} e^{it\phi(D)} f_{\Omega_\rho}.
We will estimate \( \| e^{it\phi(D)} f \|_{L^q(\Delta^j)} \) in several cases. Loosely speaking, if the contribution from the region \( \Omega_\rho \) dominates, we call it \( \Omega_\rho \) case. More precisely, given a \( K \)-cube \( B \), by the triangle inequality, we have

\[
\| e^{it\phi(D)} f_{\Omega_\rho} \|_{L^q(B)} \gtrsim \frac{\| e^{it\phi(D)} f_{\Omega_\text{L}} \|_{L^q(B)}}{\log K}
\]

for at least one of the \( \rho \)'s. We sort the \( K \)-cubes in \( Y^1 \) as follows: Denote

\[
\{ B \subset Y^1 : \| e^{it\phi(D)} f_{\Omega_\rho} \|_{L^q(B)} \gtrsim \frac{\| e^{it\phi(D)} f_{\Omega_\text{L}} \|_{L^q(B)}}{\log K} \}
\]

by \( Y^\rho \). Clearly, one has \( Y = \bigcup_\rho Y^\rho \). If \( \sharp \{ B : B \subset Y^\rho \} \gtrsim \frac{N}{\log K} \), we call it \( \Omega_\rho \)-case.

For the \( \Omega_\rho \)-case, we sort the \( K \)-cubes in \( Y^\rho \) as follows:

1. For any dyadic number \( A^{(\rho)} \), let \( Y^\rho_{A^{(\rho)}} \) be the union of the \( K \)-cubes in \( Y^\rho \) satisfying

\[
\| e^{it\phi(D)} f_{\Omega_\rho} \|_{L^q(B)} \approx A^{(\rho)}.
\]

2. Fix \( A^{(\rho)} \), for any dyadic numbers \( \nu^{(\rho)} \), let \( Y^\rho_{A^{(\rho)},\nu^{(\rho)}} \) be the union of the \( K \)-cubes in \( Y^\rho_{A^{(\rho)}} \) such that for each \( B \subset Y^\rho_{A^{(\rho)},\nu^{(\rho)}} \), the \( R^{1/2} \)-cube intersecting \( B \) contains \( \approx \nu^{(\rho)} \) cubes from \( Y^\rho_{A^{(\rho)}} \).

The dyadic numbers \( A^{(\rho)}, \nu^{(\rho)} \) making significant contributions can be assumed to be between \( R^{-C} \) and \( R^C \). Therefore, there exist some dyadic numbers \( A^{(\rho)}, \nu^{(\rho)} \) such that \( \sharp \{ B : B \subset Y^\rho_{A^{(\rho)},\nu^{(\rho)}} \} =: N^{(\rho)} \gtrsim \frac{N}{\log K \log R} \) cubes \( B \). Fix a choice of \( A^{(\rho)}, \nu^{(\rho)} \) and denote \( Y^\rho_{A^{(\rho)},\nu^{(\rho)}} \) by \( Y^\rho \) for convenience. Then, with a similar procedure as in the proof of (2.2), in the \( \Omega_\rho \)-case we have

\[
\| e^{it\phi(D)} f \|_{L^q(\Delta)} \lesssim R^\varepsilon \| e^{it\phi(D)} f_{\Omega_\rho} \|_{L^q(\Delta^j)} .
\]  

(2.38)

We also have

\[
\frac{N}{\log K \log R} \lesssim N^{(\rho)} \leq N, \quad \nu^{(\rho)} \leq \nu, \quad \lambda^{(\rho)} \leq \lambda,
\]

where

\[
\lambda^{(\rho)} := \max_{\rho^{1/2} K^{1/2} B^{(x', r)} \subset Q_{R,x' \in \mathbb{R}^3,r \geq K}} \sharp \{ B_k \subset Y^\rho : B_k \subset B^3(x', r) \}.
\]

We are going to estimate \( \| e^{it\phi(D)} f_{\Omega_\rho} \|_{L^q(\Delta^j)} \). For this purpose, we divide each \( I_\rho \) further into

\[
I_\rho = \bigcup_{j=1}^{\rho^{1/2} K^{1/2}} I_{\rho,j}.
\]

Each \( I_{\rho,j} \) has length of \( \rho^{-\frac{m-2}{2}} K^{-1/2} \). We write

\[
\Omega_\rho = \bigcup \tau,
\]

where each \( \tau \) in the above union has the form

\[
I_{\rho,j} \times [0, K^{-\frac{1}{\rho^2}]}.
\]
Therefore, we have
\[ e^{it\phi(D)} f_{\Omega_{\rho}} = \sum_{\tau \in \Omega_{\rho}} e^{it\phi(D)} f_{\tau}. \]
For each cube \( B \subset Y^\rho \), we employ the following decoupling inequality from [23], which has root in the original paper of Bourgain and Demeter [5].

**Lemma 2.2.** Suppose that \( B \) is a \( K \)-cube contained in \( Y^\rho \). Then for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that
\[
\| e^{it\phi(D)} f_{\Omega_{\rho}} \|_{L^6(B)} \leq C_\varepsilon K^{\varepsilon^2} \left( \sum_{\tau} \| e^{it\phi(D)} f_{\tau} \|_{L^6(\omega_B)}^2 \right)^{1/2},
\]
where \( \omega_B \) is a weight function essentially supported on \( B \).

**Remark 2.3.** The authors of [12] use a broad-narrow analysis to deal with the Schrödinger case. In particular, they apply Bourgain-Demeter’s decoupling inequality in [5] on each narrow ball. In the current paper, to estimate \( \| e^{it\phi(D)} f \|_{L^6(Y)} \), we sort the cubes in \( Y \) into \( Y = \bigcup_{j=0}^{\beta^j} Y^j \), which can be regarded as a “non-degenerate-degenerate” analysis. For the convenience of readers, in this remark we call the cubes in \( Y^0 \) and \( Y^1 \) non-degenerate cubes and degenerate cubes, respectively. Our strategy is to apply decoupling on each degenerate cube, which has a similar flavor as the narrow ball in [12].

For each \( \tau \), we will handle \( e^{it\phi(D)} f_{\tau} \) by rescaling and induction on scales. To do it, we further decompose \( f_{\Omega_{\rho}} \) in physical space and perform dyadic pigeonholing several times.

Firstly, we divide the physical square \([0, R]^2\) into \( \frac{R}{\rho^{N-2} K^{1/2}} \times \frac{R}{K^{1-\frac{m}{m-2}}} \)-rectangles \( D \). For each pair \( (\tau, D) \), let \( f_{\Box_{\tau, D}} \) be the function formed by cutting off \( f \) on the rectangle \( D \) (with a Schwartz tail) in physical space and the rectangle \( \tau \) in Fourier space. We have
\[
f_{\Omega_{\rho}} = \sum_{\tau} \sum_{D} f_{\Box_{\tau, D}},
\]
and write \( f_{\Box_{\tau}} = \sum_{\Box_{\tau}} f_{\Box} \) for abbreviation. Note that \( e^{it\phi(D)} f_{\Box_{\tau, D}} \) is essentially supported on an \( \frac{R}{\rho^{N-2} K^{1/2}} \times \frac{R}{K^{1-\frac{m}{m-2}}} \times R \)-box, which is denoted by \( \Box_{\tau, D} \). The box \( \Box_{\tau, D} \) is parallel to the normal direction of the surface \( \Sigma_\tau \) at the left bottom corner of \( \tau \). For a fixed \( \tau \), the different boxes \( \Box_{\tau, D} \) tile \( Q_R \). Note that a given \( K \)-cube \( B \) lies in exactly one box \( \Box_{\tau, D} \) for each \( \tau \). By Lemma 2.2, for each \( K \)-cube \( B \), there holds
\[
\| e^{it\phi(D)} f_{\Box_{\tau}} \|_{L^6(\omega_B)} \leq C_\varepsilon K^{\varepsilon^2} \left( \sum_{\Box} \| e^{it\phi(D)} f_{\Box} \|_{L^6(\omega_B)}^2 \right)^{1/2}.
\]
We denote
\[
R_1 := \frac{R}{K} = R^{1-\delta}, \quad K_1 = R_1^\delta = R^{\delta - \delta^2}.
\]
Tile \( \Box \) by \( \rho^{\frac{m-2}{2}} K^{\frac{1}{2}} K_1 \times K_1 \times KK_1 \)-tubes, and also tile \( \Box \) by \( \rho^{\frac{1}{2}} K^{\frac{1}{2}} \times K_1 \times KK_1 \times K^{\frac{1}{2}} R^{\frac{1}{2}} \)-tubes \( S' \) (all running parallel to the long axis of \( \Box \)). After rescaling the \( \Box \)
becomes an $R_1$-cube, the tubes $S'$ and $S$ become lattice $R_1^{1/3}$-cubes and $K_1$-cubes, respectively. We regroup the tubes $S$ and $S'$ inside each $\Box$ as follows:

1. Sort those tubes $S$ which intersect $Y^\rho$ according to the value $\|e^{it\phi(D)}f_\Box\|_{L^6(S)}$ and the number of $K$-cubes contained in it. For dyadic numbers $\eta, \beta_1$, we use $S_{\Box, \eta, \beta_1}$ to stand for the collection of tubes $S \subset \Box$ each of which containing $\sim \eta$ $K$-cubes and $\|e^{it\phi(D)}f_\Box\|_{L^6(S)} \sim \beta_1$.

2. For fixed $\eta, \beta_1$, we sort the tubes $S' \subset \Box$ according to the number of the tubes $S \in S_{\Box, \eta, \beta_1}$ contained in it. For dyadic number $\nu_\rho$, let $S_{\Box, \eta, \beta_1, \nu_\rho}$ be the sub-collection of $S_{\Box, \eta, \beta_1}$ such that for each $S \in S_{\Box, \eta, \beta_1, \nu_\rho}$, the tube $S'$ containing $S$ contains $\sim \nu_\rho$ tubes from $S_{\Box, \eta, \beta_1}$.

3. For fixed $\eta, \beta_1, \nu_\rho$, we sort the boxes $\Box$ according to the value $\|f_\Box\|_2$, the number $\sharp S_{\Box, \eta, \beta_1, \nu_\rho}$ and the value $\lambda_\rho$ defined below. For dyadic numbers $\beta_2, N_\rho, \lambda_\rho$, let $B_{\eta, \beta_1, \nu_\rho, \beta_2, N_\rho, \lambda_\rho}$ denote the collection of boxes $\Box$ satisfying

$$\|f_\Box\|_2 \sim \beta_2, \quad \sharp S_{\Box, \eta, \beta_1, \nu_\rho} \sim N_\rho$$

and

$$\max_{T_r \subset \Box, r \geq K_1} \frac{\sharp\{S \in S_{\Box, \eta, \beta_1, \nu_\rho} : S \subset T_r\}}{r^\alpha} \sim \lambda_\rho, \quad (2.39)$$

where $T_r$ are $K_1^{1/2} \rho \frac{m^2}{s^2} r \times K_1^{1/2} \rho r \times Kr$-tubes in $\Box$ running parallel to the long axis of $\Box$.

Let $Y_{\Box, \eta, \beta_1, \nu_\rho}^\rho$ denote $\{S : S \subset S_{\Box, \eta, \beta_1, \nu_\rho}\}$. Thus, there are only $O(\log R)$ significant choices for each dyadic number. By pigeonholing, we can choose $\eta, \beta_1, \nu_\rho, \beta_2, N_\rho, \lambda_\rho$ so that

$$\|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(B)} \lesssim (\log R)^6 K^{2\gamma} \left( \sum_{\Box \in B_{\eta, \beta_1, \nu_\rho, \beta_2, N_\rho, \lambda_\rho} : B \subset Y_{\Box, \eta, \beta_1, \nu_\rho}^\rho} \right)^{1/2} \quad (2.40)$$

holds for a fraction $\gtrsim (\log R)^{-\delta}$ of all $K$-cubes $B$. For brevity, we denote by

$$Y_{\Box}^\rho := Y_{\Box, \eta, \beta_1, \nu_\rho}^\rho, \quad B := B_{\eta, \beta_1, \nu_\rho, \beta_2, N_\rho, \lambda_\rho}.$$

We sort the $K$-cubes $B$ satisfying (2.40) by $\sharp\{\Box \in \Box : B \subset Y_{\Box}^\rho\}$. Let $Y'_\mu \subset Y^\rho$ be a union of $K$-cubes $B$ that obey

$$\|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(B)} \lesssim (\log R)^6 K^{2\gamma} \left( \sum_{\Box \in B_{\Box, B \subset Y_{\Box}^\rho}^\rho} \|e^{it\phi(D)}f_{\Box}\|_{L^6(B)}^2 \right)^{1/2} \quad (2.41)$$

and

$$\sharp\{\Box \in \Box : B \subset Y_{\Box}^\rho\} \sim \mu. \quad (2.42)$$

Since the dyadic number $\mu$ must satisfy $1 \leq \mu \leq R^C$, there are $\sim \log R$ different choices for $\mu$, so by pigeonholing there exists one such $\mu$ such that the number of cubes in $Y_{\mu}^\rho$ is $\gtrsim \frac{\sharp\{B \subset Y_{\mu}^\rho\}}{\log R} \gtrsim \frac{N_{\mu}(\rho)}{\log R^7}$. And then for simplicity we rename $Y_{\mu}^\rho$ as $Y'$. We have

$$\|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(Y')}^6 \lesssim (\log R)^{C \mu} \|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(B)}^6. \quad (2.43)$$
For each $B \subset Y'$, it follows from (2.41), (2.42) and Hölder’s inequality that
\[
\|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(B)}^6 \lesssim (\log R)^C K^{6\nu^2} \rho^2 \sum_{\square \in B, B \subset Y'_\rho} \|e^{it\phi(D)}f\|_{L^6(\omega_B)}^6.
\] (2.44)

Putting (2.43) and (2.44) together, one has
\[
\|e^{it\phi(D)}f_{\Omega_\rho}\|_{L^6(Y')}^6 \lesssim (\log R)^C K^{6\nu^2} \rho^{1/3} \left( \sum_{\square \in B} \|e^{it\phi(D)}f\|_{L^6(Y'_\square)}^6 \right)^{1/6}.
\] (2.45)

Next, we apply rescaling to each $\|e^{it\phi(D)}f\|_{L^6(Y'_\rho)}$ and run induction on scales. For each $\rho^{-m/2} K^{-1/2} \times K^{-1/m}$-rectangle $\tau = \tau_{\square}$ in $\Omega_\rho$, we write
\[
\left\{ \begin{array}{l}
\quad \xi_1 = a + \rho^{-m/2} K^{-1/2} \eta_1, \\
\quad \xi_2 = K^{-1/2} m \eta_2,
\end{array} \right.
\]
for some $\rho \leq a \leq 2 \rho$. Then
\[
|e^{it\phi(D)}f_{\square}(x)| = \rho^{-m/2} K^{1/2 - \frac{1}{2m}} |e^{i\hat{\psi}(D)}g(\tilde{x})|,
\]
with $\|g\|_2 = \|f_{\square}\|_2$, and
\[
\hat{g}(\eta_1, \eta_2) := \rho^{-m/2} K^{-m/2m} \hat{f}_{\square}(a + K^{-1/2} \eta_1, K^{-1/2} \eta_2),
\]
where
\[
\psi(\eta_1, \eta_2) := \left( \frac{m(m-1)}{2} a^{m-2} \rho^{-(m-2)} \eta_1^2 + \frac{m(m-1)(m-2)}{6} a^{m-3} \rho^{-3(m-2)/2} K^{1/2} \eta_1^3 + \ldots + \rho^{-m/2} K^{-m/2} \eta_2^m \right) + \eta_2^m =: \phi_1(\eta_1) + \eta_2^m,
\]
and
\[
\left\{ \begin{array}{l}
\quad \tilde{x}_1 := \rho^{-m/2} K^{-1/2} (x_1 + ma^{m-1} t), \\
\quad \tilde{x}_2 := K^{-1/2} m x_2, \\
\quad \hat{t} := K^{-1} t.
\end{array} \right.
\]
For brevity, we denote the above relation by $(\tilde{x}, \hat{t}) = \mathcal{L}_1(x, t)$. We write
\[
\|e^{it\phi(D)}f_{\square}\|_{L^6(Y'_\rho)} = \rho^{-m/2} K^{-m/2} \|e^{i\hat{\psi}(D)}g(\tilde{x})\|_{L^6(\tilde{\Omega}_\rho)},
\] (2.46)
where $\tilde{\Omega}_\rho = \mathcal{L}_1(Y'_\rho)$. Note that $a \sim \rho$ and $K^{-1/m} \leq \rho \leq \frac{1}{2}$. It is easy to see that the new phase function $\psi(\eta_1, \eta_2) = \phi_1(\eta_1) + \eta_2^m$ satisfies
\[
\phi^{(m)}_1 \sim 1; \quad |\phi^{(k)}_1| \lesssim 1, 3 \leq k \leq m; \quad \phi^{(l)}_1 = 0, l \geq m + 1
\]
on $[0, 1]$. Therefore, applying Proposition 2.1 to the term
\[
\|e^{i\hat{\psi}(D)}g(\tilde{x})\|_{L^6(\tilde{\Omega}_\rho)}
\]
at scale $R_1$, we deduce
\[
\|e^{it\phi(D)}f_{\square}\|_{L^6(Y'_\rho)} \lesssim \rho^{-m/2} K^{-m/2} N_\rho^{-1/2} \lambda^{1/2} \nu^{1/2} \left( \frac{R}{R_1} \right)^{3/2 - 4/3m + \epsilon} \|f_{\square}\|_2.
\]
Note that \( \frac{1}{3} - \frac{4-\alpha}{6m} \leq \frac{5}{12} - \frac{5-\alpha}{6m} \) when \( m \geq 2 \). Consider the cardinality of the set \( \{ (\Box, B) : \Box \in \mathbb{B}, B \subseteq Y_\Box^{\rho} \cap Y' \} \). By our choice of \( \mu \) as in (2.42), there is a lower bound

\[
\sharp \{ (\Box, B) : \Box \in \mathbb{B}, B \subseteq Y_\Box^{\rho} \cap Y' \} \leq \frac{(\log R)^7 N_\rho \eta}{N(\rho)}.
\]  

(2.47)

Then by our choices of \( \lambda, \eta \), we have

\[
\lambda_\rho \eta \sim \max_{T \subseteq Y_\Box^{\rho}} \frac{\sharp \{ S : S \subseteq Y_\Box^{\rho} \cap T_r \}}{r^\alpha} \cdot \sharp \{ B : B \subseteq S \cap Y' \} \leq \frac{\rho^{\frac{m-2}{2} K^{\frac{3}{2}} - \frac{2}{m} \lambda(\rho) (K^{\frac{1}{2}}) r^\alpha}}{r^\alpha},
\]

where the last inequality follows from the fact that we can cover a \( \rho^{\frac{m-2}{2} K^{\frac{1}{2}} r} \times K^{\frac{1}{2}} r \times K r \)-tube \( T_r \) by \( \rho^{\frac{m-2}{2} K^{\frac{3}{2}} - \frac{2}{m} \lambda(\rho) (K^{\frac{1}{2}}) r^\alpha} \) finitely overlapping \( K^{\frac{1}{2}} r \)-balls. Hence, we obtain

\[
\eta \lesssim \frac{\lambda(\rho) \rho^{\frac{m-2}{2} K^{\frac{3}{2}} - \frac{2}{m} \lambda(\rho) (K^{\frac{1}{2}}) r^\alpha}}{\lambda}. \tag{2.48}
\]

Finally, we relate \( \upsilon_\rho \) and \( \upsilon^{(\rho)} \) by considering the number of \( K \)-cubes in each relevant \( \frac{R^{1/2}}{\rho^{1/2}} \times \frac{R^{1/2}}{K^{1/2} - \frac{1}{2}} \times K^{\frac{1}{2}} R^{\frac{1}{2}} \)-tube \( S' \). Recall that each relevant \( S' \) contains \( \sim \upsilon_\rho \) tubes \( S \) in \( Y_\Box \) and each such \( S \) contains \( \sim \eta \) \( K \)-cubes. On the other hand, we can cover \( S' \) by \( \sim K^{1/2} \) finitely overlapping \( R^{\frac{1}{2}} \)-cubes, and each \( R^{\frac{1}{2}} \)-cube contains \( \lesssim \upsilon^{(\rho)} \) many \( K \)-cubes in \( Y' \). Thus, it follows that

\[
\upsilon_\rho \lesssim \frac{K^{\frac{1}{2}} \upsilon^{(\rho)}}{\eta}. \tag{2.49}
\]

By inserting (2.47), (2.48) and (2.49) into (2.45), we derive

\[
\| e^{it \phi^{(D)}} f_{\Omega_\rho} \|_{L^6(Y')} \lesssim K^{2\varepsilon - \varepsilon} (N^{(\rho)})^{-\frac{1}{2}} (\upsilon^{(\rho)})^{\frac{1}{2}} \lambda^{\frac{1}{2}} (\upsilon^{(\rho)})^{\frac{1}{2}} R \frac{5}{12} - \frac{5-\alpha}{6m} + \varepsilon \| f \|_2.
\]

Since \( K = R^{\frac{1}{6}} \) and \( R \) can be assumed to be sufficiently large compared to any constant depending on \( \varepsilon \), we have \( K^{2\varepsilon - \varepsilon} \ll 1 \), and the induction closes. Recall that \( \frac{N}{\log K (\log R)^{\frac{1}{2}}} \lesssim N^{(\rho)} \leq N, \upsilon^{(\rho)} \leq \upsilon \) and \( \lambda^{(\rho)} \leq \lambda \). This yields

\[
\| e^{it \phi^{(D)}} f_{\Omega_\rho} \|_{L^6(Y')} \lesssim \varepsilon N^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \upsilon^{\frac{1}{2}} R \frac{5}{12} - \frac{5-\alpha}{6m} + \varepsilon \| f \|_2.
\]

Combining it with (2.38), we get

\[
\| e^{it \phi^{(D)}} f_{\Omega_1} \|_{L^6(Y')} \lesssim \varepsilon N^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \upsilon^{\frac{1}{2}} R \frac{5}{12} - \frac{5-\alpha}{6m} + \varepsilon \| f \|_2.
\]
where $\Omega_0 := [0, 1] \times [K^{-\frac{1}{n}}, 1]$, $\Omega_1 := [0, 1] \times [0, K^{-\frac{1}{n}}]$. We denote $\hat{f} |_{\Omega_0}$ and $\hat{f} |_{\Omega_1}$ by $\hat{f}_{\Omega_0}$ and $\hat{f}_{\Omega_1}$, respectively. Given a $K$-cube $B$, by the triangle inequality, we have either
\[
\|e^{it\psi(D)}f_{\Omega_0}\|_{L^6(B)} \geq \frac{1}{8}\|e^{it\psi(D)}f\|_{L^6(B)}
\]
or
\[
\|e^{it\psi(D)}f_{\Omega_1}\|_{L^6(B)} \geq \frac{1}{8}\|e^{it\psi(D)}f\|_{L^6(B)}.
\]
We sort the $K$-cubes in $Y$ as follows. Denote by
\[
\{B \subset Y : \|e^{it\psi(D)}f_{\Omega_0}\|_{L^6(B)} \geq \frac{1}{8}\|e^{it\psi(D)}f\|_{L^6(B)}\}
\]
and
\[
\{B \subset Y : \|e^{it\psi(D)}f_{\Omega_1}\|_{L^6(B)} \geq \frac{1}{8}\|e^{it\psi(D)}f\|_{L^6(B)}\}
\]
by $Y^0$ and $Y^1$, respectively. Clearly, one has $Y = Y^0 \cup Y^1$. If $\sharp\{B : B \subset Y^0\} \geq \frac{N}{2}$, we call it $\Omega_0$-case. If $\sharp\{B : B \subset Y^1\} \geq \frac{N}{2}$, we call it $\Omega_1$-case.

For the $\Omega_0$-case, we further sort the $K$-cubes in $Y^0$ as follows:

(1) For any dyadic number $A^{(0)}$, let $Y^0_{A^{(0)}}$ be the union of the $K$-cubes in $Y^0$ satisfying
\[
\|e^{it\psi(D)}f_{\Omega_0}\|_{L^6(B)} \approx A^{(0)}.
\]

(2) Fix $A^{(0)}$, for any dyadic number $\nu^{(0)}$, let $Y^0_{A^{(0)}, \nu^{(0)}}$ be the union of the $K$-cubes in $Y^0_{A^{(0)}}$ such that for each $B \subset Y^0_{A^{(0)}, \nu^{(0)}}$, the $R^{1/2}$-cube intersecting $B$ contains $\approx \nu^{(0)}$ cubes from $Y^0_{A^{(0)}}$.

With the assumption $\|f\|_2 = 1$ the dyadic numbers $A^{(0)}, \nu^{(0)}$ making significant contributions are between $R^{-C}$ and $R^C$. Therefore, there exist some dyadic numbers $A^{(0)}, \nu^{(0)}$ such that $\sharp\{B : B \subset Y^0_{A^{(0)}, \nu^{(0)}}\} =: N^{(0)} \gtrsim \frac{N}{(\log R)^2}$. Fix a choice of $A^{(0)}, \nu^{(0)}$ and denote $Y^0_{A^{(0)}, \nu^{(0)}}$ by $Y^0$ for convenience. Then, in the $\Omega_0$-case we have
\[
\|e^{it\psi(D)}f\|_{L^6(Y)} \lesssim R^\varepsilon\|e^{it\psi(D)}f_{\Omega_0}\|_{L^6(Y^0)},
\]
and
\[
\frac{N}{(\log R)^2} \lesssim N^{(0)} \leq N, \quad \nu^{(0)} \leq \nu, \quad \lambda^{(0)} \leq \lambda,
\]
where
\[
\lambda^{(0)} := \max_{B^3(x^{'}, r) \subset Q, r \leq 1, x^{'}, r \in \mathbb{R}^3} \sharp\{B_k \subset Y^0 : B_k \subset B^3(x^{'}, r)\}.
\]

By a similar argument as in the proof of (2.4), we have
\[
\|e^{it\psi(D)}f_{\Omega_0}\|_{L^6(Y^0)} \leq C_\varepsilon K^{O(1)}(N^{(0)})^{-\frac{1}{4}}(\lambda^{(0)})^{\frac{1}{2}}(\nu^{(0)})^{\frac{1}{8}} R^\frac{1}{2} \varepsilon^{\frac{1}{2}}\|f\|_2.
\]

For the $\Omega_1$-case, we sort the $K$-cubes in $Y^1$ as similar as in the $\Omega_0$-case:
Lemma 3.1. Suppose that

\[ \|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)} \approx A^{(1)}. \]

(2) Fix \( A^{(1)} \), for any dyadic number \( \nu^{(1)} \), let \( Y_{A^{(1)}, \nu^{(1)}} \) be the union of the \( K \)-cubes in \( Y_{A^{(1)}} \) such that for each \( B \subset Y_{A^{(1)}, \nu^{(1)}} \), the \( R^{1/2} \)-cube intersecting \( B \) contains \( \approx \nu^{(1)} \) cubes from \( Y_{A^{(1)}} \).

The dyadic numbers \( A^{(1)}, \nu^{(1)} \) making significant contributions can be assumed to be between \( R^{-C} \) and \( R^C \). Therefore, there exist some dyadic numbers \( A^{(1)}, \nu^{(1)} \) such that \( \sharp\{B : B \subset Y_{A^{(1)}, \nu^{(1)}}\} =: N^{(1)} \gtrsim N \frac{N}{(\log R)^2} \) many cubes \( B \). Fix that choice of \( A^{(1)}, \nu^{(1)} \) and denote \( Y_{A^{(1)}, \nu^{(1)}} \) by \( Y^{(1)} \) for convenience. Then, in the \( \Omega_1 \)-case we have

\[ \|e^{it\psi(D)} f\|_{L^6(Y)} \lesssim e^{R^\epsilon} \|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(Y^{(1)})}, \]

and

\[ \frac{N}{(\log R)^2} \lesssim N^{(1)} \leq N, \quad \nu^{(1)} \leq \nu, \quad \lambda^{(1)} \leq \lambda, \]

where

\[ \lambda^{(1)} := \max_{B^3(x', r) \subset Q_{R, x' \in \mathbb{R}^3, r \geq K}} \sharp\{B_k \subset Y^1 : B_k \subset B^3(x', r)\}. \]

We are going to estimate \( \|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(Y^{(1)})} \). To this end, we partition \( \Omega_1 \) into disjoint rectangles \( \tau \) of dimensions \( K^{-\frac{1}{2}} \times K^{-\frac{1}{m}} \), and write \( f_{\Omega_1} = \sum_{\tau} f_{\tau} \). For each \( K \)-cube \( B \), we have the following decoupling inequality due to Bourgain-Demeter [5] (see also Lemma 3.3 in [17]).

**Lemma 3.1.** Suppose that \( B \) is a \( K \)-cube in \( \mathbb{R}^3 \). Then for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that

\[ \|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)} \leq C_\varepsilon K^{\frac{\varepsilon}{2}} \left( \sum_{\tau} \|e^{it\psi(D)} f_{\tau}\|_{L^6(\omega_B)}^2 \right)^{1/2}, \]

where \( \omega_B \) is a weight function essentially supported on \( B \).

We decompose the function \( f_{\Omega_1} \) in physical space further and perform some dyadic pigeonholing like in Section 2.

First we divide the physical square \([0, R]^2\) into \( \frac{R}{K^{1/2}} \times \frac{R}{K^{1-\frac{1}{m}}} \)-rectangles \( D \). For each pair \((\tau, D)\), let \( f_{\square_{\tau, D}} \) be the function formed by cutting off \( f \) on the rectangle \( D \) (with a Schwartz tail) in physical space and the rectangle \( \tau \) in Fourier space. We see that \( e^{it\psi(D)} f_{\square_{\tau, D}} \) is essentially supported on an \( \frac{R}{K^{1/2}} \times \frac{R}{K^{1-\frac{1}{m}}} \times R \)-box, which is denoted by \( \square_{\tau, D} \).

**Remark 3.2.** In \( x_2 \) the wave packets follow the evolution of the phase \( \xi_2^m \), but in \( x_1 \) they essentially follow the Schrödinger evolution because the phase \( \phi_1(\xi_1) \) is basically like \( \xi_1^3 \). So the scaling in both directions is different.

The box \( \square_{\tau, D} \) is parallel to the normal direction of the surface \( \Sigma_\tau \) at the left bottom corner of \( \tau \). For any fixed \( \tau \), the different boxes \( \square_{\tau, D} \) tile \( Q_R \). In particular,
for each $\tau$, a given $K$-cube $B$ lies in exactly one box $\Box_{\tau,D}$. We have

$$f_{\Omega_1} = \sum_{\tau} \sum_{D} f_{\Box_{\tau,D}}$$

and write $f_{\Omega_1} = \sum_{\Box} f_{\Box}$ for abbreviation. By Lemma 3.1, for each $K$-cube $B$, there holds

$$||e^{it\psi(D)} f_{\Omega_1}||_{L^6(B)} \leq C_e K^{\varepsilon_2} \left( \sum_{\Box} ||e^{it\psi(D)} f_{\Box}||_{L^6(\omega_B)}^2 \right)^{1/2}.$$  

Recall that $K = R^\delta$, where $\delta = \varepsilon^{100}$. We denote by

$$R_1 := \frac{R}{K} = R^{1-\delta}, \quad K_1 = R_1^\delta = R^{\delta-\delta^2}.$$  

Tile $\Box$ by $K^{1/2} K \times K^{1/2} K \times K K_1$ -tube $S$, and also tile $\Box$ by $R_1^{1/2} \times \frac{R_1^{1/2}}{K^{1/2}} \times K^{1/2} R_1^{1/2}$-tubes $S'$ (all running parallel to the long axis of $\Box$). After rescaling the $\Box$ becomes an $R_1$-cube, the tubes $S'$ and $S$ become lattice $R_1^2$-cubes and $K_1$-cubes, respectively. We regroup tubes $S$ and $S'$ inside each $\Box$ as follows:

(1) Sort those tubes $S$ which intersect $Y^1$ according to the value $||e^{it\psi(D)} f_{\Box}||_{L^6(S)}$ and the number of $K$-cubes contained in it. For dyadic numbers $\eta, \beta_1$, we use $\mathbb{S}_{\Box, \eta, \beta_1}$ to stand for the collection of tubes $S \subset \Box$ each of which containing $\sim \eta K$-cubes and $||e^{it\psi(D)} f_{\Box}||_{L^6(S)} \sim \beta_1$.

(2) For fixed $\eta, \beta_1$, we sort the tubes $S' \subset \Box$ according to the number of the tubes $S \in \mathbb{S}_{\Box, \eta, \beta_1}$ contained in it. For dyadic number $\nu_1$, let $\mathbb{S}_{\Box, \eta, \beta_1, \nu_1}$ be the subcollection of $\mathbb{S}_{\Box, \eta, \beta_1}$ such that for each $S \in \mathbb{S}_{\Box, \eta, \beta_1, \nu_1}$, the tube $S'$ containing $S$ contains $\sim \nu_1$ tubes from $\mathbb{S}_{\Box, \eta, \beta_1}$.

(3) For fixed $\eta, \beta_1, \nu_1$, we sort the boxes $\Box$ according to the value $\|f_{\Box}\|_2$, the number $\# \mathbb{S}_{\Box, \eta, \beta_1, \nu_1}$ and the value $\lambda_1$ defined below. For dyadic numbers $\beta_2, N_1, \lambda_1$, let $\mathbb{B}_{\eta, \beta_1, \nu_1, \beta_2, N_1, \lambda_1}$ denote the collection of boxes $\Box$ each of which satisfying that

$$\|f_{\Box}\|_2 \sim \beta_2, \quad \# \mathbb{S}_{\Box, \eta, \beta_1, \nu_1} \sim N_1$$

and

$$\max_{T_r \subset \Box : r \geq K_1} \frac{\# \{ S \in \mathbb{S}_{\Box, \eta, \beta_1, \nu_1} : S \subset T_r \}}{r^a} \sim \lambda_1,$$

where $T_r$ are $K^{1/2} r \times K^{1/2} r \times K r$-tubes in $\Box$ running parallel to the long axis of $\Box$.

Let $Y^1_{\Box, \eta, \beta_1, \nu_1}$ denote $\{ S : S \subset \mathbb{S}_{\Box, \eta, \beta_1, \nu_1} \}$. By pigeonholing, we can choose $\eta, \beta_1, \nu_1, \beta_2, N_1, \lambda_1$ such that

$$\|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)} \lesssim (\log R)^6 K^{\varepsilon_1} \left( \sum_{\Box \in \mathbb{B}_{\eta, \beta_1, \nu_1, \beta_2, N_1, \lambda_1}} \|e^{it\psi(D)} f_{\Box}\|_{L^6(\omega_B)}^2 \right)^{1/2}$$

holds for a fraction $\gtrsim (\log R)^{-6}$ of all $K$-cubes $B$. For brevity, we denote by

$$Y^1_{\Box} := Y^1_{\Box, \eta, \beta_1, \nu_1}, \quad \mathbb{B} := \mathbb{B}_{\eta, \beta_1, \nu_1, \beta_2, N_1, \lambda_1}.$$
Finally, we sort the $K$-cubes $B$ satisfying (3.2) by $\#\{\square \in \mathbb{B} : B \subset Y'\}$. Let $Y'_\mu \subset Y'$ be a union of cubes that obey

$$\|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)} \lesssim (\log R)^6 K^{\varepsilon^2} \left( \sum_{\square \in \mathbb{B}, B \subset Y'_\mu} \|e^{it\psi(D)} f_{\square}\|_{L^6(\omega_B)}^2 \right)^{1/2}$$

(3.3)

and

$$\#\{\square \in \mathbb{B} : B \subset Y'_\mu\} \sim \mu.$$  

(3.4)

Since the dyadic number $\mu$ must satisfy $1 \leq \mu \leq R^C$, there are $\sim \log R$ different choices for $\mu$, so by pigeonholing there exists one such $\mu$ such that the number of cubes in $Y'_\mu$ is $\gtrsim \frac{N}{(\log R)^7}$. We have

$$\|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(Y')} \lesssim (\log R)^7 \sum_{B \subset Y'} \|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)}.$$  

(3.5)

For each $B \subset Y'$, it follows from (3.3), (3.4) and Hölder’s inequality that

$$\|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(B)} \lesssim (\log R)^{36} K^{6\varepsilon^2} \mu^2 \sum_{\square \in \mathbb{B}, B \subset \square} \|e^{it\psi(D)} f_{\square}\|_{L^6(\omega_B)}.$$  

(3.6)

Putting (3.5) and (3.6) together, one has

$$\|e^{it\psi(D)} f_{\Omega_1}\|_{L^6(Y')} \lesssim (\log R)^C K^{\varepsilon^2} \mu^{1/3} \left( \sum_{\square \in \mathbb{B}} \|e^{it\psi(D)} f_{\square}\|_{L^6(Y'_\mu)} \right)^{1/6}.$$  

(3.7)

The key point is that phase functions of the form $\psi(\xi_1, \xi_2)$ are closed over each region $\tau = \tau_\square$ in $\Omega_1$ under the change of variable

$$\begin{cases} 
\xi_1 = a + K^{-\frac{1}{2}} \eta_1, \ a \in [\frac{1}{2}, 1 - K^{-\frac{1}{2}}] \cap K^{-1/2} \mathbb{Z}, \\
\xi_2 = K^{-\frac{1}{m}} \eta_2.
\end{cases}$$

More precisely, under the change of variables $\psi(\xi_1, \xi_2)$ becomes

$$\tilde{\psi}(\eta_1, \eta_2) := K[\phi_1(a + K^{-1/2} \eta_1) - \phi_1(a) - K^{-1/2} \eta_1 \phi'_1(a)] + \eta_2^m,$$

which satisfies the condition (2.37). We write

$$|e^{it\psi(D)} f(\square)(x)| = K^{-\frac{m+2}{4m}} |e^{i\tilde{\psi}(D)} g(\tilde{x})|,$$

where

$$\tilde{g}(\eta_1, \eta_2) := K^{-\frac{m+2}{4m}} \tilde{f}(a + K^{-\frac{1}{2}} \eta_1, K^{-\frac{1}{m}} \eta_2), \quad \|\tilde{g}\|_2 = \|f\|_2,$$

and

$$\begin{cases} 
\tilde{x}_1 := K^{-\frac{1}{2}} (x_1 + \phi'_1(a) t), \\
\tilde{x}_2 := K^{-\frac{1}{m}} x_2, \\
\tilde{t} := K^{-1} t.
\end{cases}$$

We denote the above relation by $(\tilde{x}, \tilde{t}) = \mathcal{L}(x, t)$ and write

$$\|e^{it\psi(D)} f(\square)\|_{L^6(Y'_\mu)} = K^{-\frac{1}{3m}} \|e^{i\tilde{\psi}(D)} g(\tilde{x})\|_{L^6(\tilde{Y}')},$$

(3.8)

where $\tilde{Y}' = \mathcal{L}(Y'_\mu)$. By (3.8) and the inductive hypothesis at scale $R_1$, we derive

$$\|e^{it\psi(D)} f(\square)\|_{L^6(\tilde{Y}')_1} \lesssim K^{-\frac{1}{3m}} N_1^{-\frac{1}{6}} \lambda_1^{\frac{1}{6}} \nu_1^{\frac{1}{6}} (\frac{R}{R_1}) \frac{4m}{3m} + \varepsilon \|f(\square)\|_2.$$
We complete the proof of Lemma 2.1.

On the other hand, by the choice of \( N \), \( K \), \( \varepsilon \), and \( \delta \), one can cover a constant depending on \( \varepsilon \) by \( 2\)-balls. Thus, we obtain

\[
\mu \ll \frac{(\log R)^7 N_1 \eta}{N^{(1)}}.
\]

Next, by our choices of \( \lambda_1 \) as in (3.1) and \( \eta \), it holds that

\[
\lambda_1 \eta \sim \max_{T_r \in E} \#\{S : S \subseteq Y_1 \cap T_r\} \cdot \#\{B : B \subseteq S \cap Y^1\} \lesssim \max_{T_r \in E} \#\{B \in Y : B \subset T_r\} \lesssim K^{\frac{3}{2}} r^\alpha \lambda^{(1)} (K^{\frac{1}{2}} r) \eta,
\]

where the last inequality follows from the fact that one can cover a \( K^{-\frac{1}{2}} r \times K^{\frac{1}{2}} r \times Kr \)-tube \( T_r \) by \( K^{\frac{3}{2}} - K^{\frac{1}{2}} \) finitely overlapping \( K^{\frac{1}{2}} r \)-balls. Thus, we obtain

\[
\eta \lesssim \frac{\lambda^{(1)} K^{\frac{3}{2} + \frac{\alpha - 2}{m}}}{\lambda_1}.
\]

Finally, we relate \( \nu_1 \) and \( \nu^{(1)} \) by considering the number of \( K \)-cubes in each relevant \( R^{\frac{1}{2}} \times K^{R^{\frac{1}{2}}} \times K^{\frac{1}{2}} R^{\frac{1}{2}} \)-cube \( S' \). On one hand, each relevant \( S' \) contains \( \sim \nu_1 \) tubes \( S \) in \( Y \) and each such \( S \) contains \( \sim \eta \) \( K \)-cubes. On the other hand, a relevant \( S' \) can be covered by \( \sim K^{1/2} \) finitely overlapping \( R^{\frac{1}{2}} \)-cubes and each \( R^{\frac{1}{2}} \)-cube contains \( \lesssim \nu^{(1)} \) many \( K \)-cubes in \( Y^1 \). It follows that

\[
\nu_1 \lesssim \frac{K^{\frac{1}{2} \nu^{(1)}}}{\eta},
\]

Using the similar argument as in the proof of (2.36), we derive

\[
\|e^{it\psi(D)} f_{\Omega_1} \|_{L^6(Y^1)} \lesssim K^{2\varepsilon^2 - \varepsilon} (N^{(1)})^{-\frac{1}{2}} (\nu^{(1)})^{\frac{1}{2}} (\lambda^{(1)})^{\frac{1}{2}} R^{\frac{1}{2} - \frac{4 + \alpha}{6m} + \varepsilon} \|f\|_2.
\]

Since \( K = R^{\delta} \) and \( R \) can be assumed to be sufficiently large compared to any constant depending on \( \varepsilon \), we have \( K^{2\varepsilon^2 - \varepsilon} \ll 1 \) and the induction closes. Recall that \( \frac{N}{\log R} \lesssim N^{(1)} \leq N, \nu^{(1)} \leq \nu \) and \( \lambda^{(1)} \leq \lambda \). This yields

\[
\|e^{it\psi(D)} f_{\Omega_1} \|_{L^6(Y^1)} \lesssim \varepsilon \nu^{-\frac{1}{2}} K^\frac{1}{2} \nu^{\frac{1}{2}} R^{\frac{1}{2} - \frac{4 + \alpha}{6m} + \varepsilon} \|f\|_2.
\]

We complete the proof of Lemma 2.1.
4. Applications of Corollary 1.7

4.1. Application to the average Fourier decay of fractal measures associated with the surfaces $F^2_m$. We recall the definition of $\alpha$-dimensional measure as follows.

**Definition 4.1.** Let $\alpha \in (0, d]$. We say that $\mu$ is an $\alpha$-dimensional measure in $\mathbb{R}^d$ if it is a probability measure supported in the unit ball $B_d(0, 1)$ and satisfies

$$\mu(B(x, r)) \leq C_\mu r^\alpha, \forall r > 0, \forall x \in \mathbb{R}^d.$$ 

We denote $d\mu_{R}(\cdot) := R^\alpha d\mu(\frac{\cdot}{R})$. Let $\gamma_d(\alpha)$ denote the supremum of the numbers $\gamma$ for which

$$\|\hat{\mu}(R \cdot)\|^2_{L^2(\mathbb{S}^{d-1})} \leq C_{\alpha, \mu} R^{-\gamma}$$

whenever $R > 1$ and $\mu$ is an $\alpha$-dimensional measure in $\mathbb{R}^d$. The problem of identifying the precise value of $\gamma_d(\alpha)$ was proposed by Mattila [18]. The lower bound of $\gamma_d(\alpha)$ has been studied by several authors. For instance, one can see [22, 11, 12].

Let $\beta_{2,m}(\alpha)$ denote the supremum of the numbers $\beta$ for which

$$\|\hat{\mu}(R \cdot)\|^2_{L^2(F^2_m)} \leq C_{\alpha, \mu} R^{-\beta}$$

whenever $R > 1$ and $\mu$ is an $\alpha$-dimensional measure in $\mathbb{R}^3$. We will study the lower bound of $\beta_{2,m}(\alpha)$. For $m = 2$, the surface $F^2_m$ is exactly the paraboloid $P^2$ over the region $[0, 1]^2$. Note that the Gaussian curvature of the surface $F^2_m$ vanishes when $\xi_1 = 0$ or $\xi_2 = 0$, which is different from the sphere or the paraboloid case studied in the literature.

We are going to derive the following result.

**Theorem 4.2.** Let $m \geq 4$ be an even number and $0 < \alpha \leq 3$. Then

$$\beta_{2,m}(\alpha) \geq \left(\frac{5}{6} - \frac{1}{3m}\right)\alpha + \frac{5}{3m} - \frac{5}{6}.$$ 

By the arguments in Remark 2.5 from [11], Theorem 4.2 can be reduced to the weighted restriction estimate in Theorem 4.3. In fact, one can obtain

$$\beta_{2,m}(\alpha) \geq 2\left(\frac{\alpha}{2} - \gamma\right),$$

if there holds that for any $\varepsilon > 0$,

$$\|e^{it\phi(D)}f\|_{L^2(B^3(0,R);d\mu_{R}(x,t))} \leq C_\varepsilon R^{\gamma + \varepsilon}\|f\|_2.$$ 

**Theorem 4.3.** Let $m \geq 4$ be an even number and $0 < \alpha \leq 3$. Suppose that $\mu$ is an $\alpha$-dimensional measure in $\mathbb{R}^3$. For all $R > 1$ and any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that

$$\|e^{it\phi(D)}f\|_{L^2(B^3(0,R);d\mu_{R}(x,t))} \leq C_\varepsilon R^{\frac{5}{12} - \frac{\alpha}{6m} + \left(\frac{1}{12} + \frac{1}{6m}\right)\alpha + \varepsilon}\|f\|_2,$$

holds for all $f$ with Fourier supports in $[0, 1]^2$.

Using Theorem 1.5 and the argument in the proof of Theorem 2.2 from [12] we may prove Theorem 4.3 as follows.
Denote $e^{it\phi(D)}f(x)$ by $\mathcal{E}f(x,t)$, and $(x,t)$ by $\tilde{x}$. Since $\text{supp} \ f \subseteq B^2(0,1)$, we have $\text{supp} \ \hat{\mathcal{E}}f \subseteq B^3(0,1)$. Thus there exists a Schwartz bump function $\psi$ on $\mathbb{R}^3$ such that $(\mathcal{E}f)^2 = (\mathcal{E}f)^2 \ast \psi$. The function $\max_{|\tilde{y}| \leq 300} |\psi(\tilde{y})|$, which we denote it by $\psi_1(\tilde{x})$, rapidly decays. Note also that any $(x,t)$ in $\mathbb{R}^3$ belongs to a unique integer lattice cube whose center we denote by $\tilde{m} = (m_1, m_2, m_3) = \tilde{m}(x,t)$. Then we have

$$\left\| e^{it\phi(D)}f \right\|_{L^2(B^3(0, R); d\mu_R(x,t))}^2 = \int_{B^3(0, R)} |\mathcal{E}f(x,t)|^2 d\mu_R(x,t)$$

$$\leq \int_{B^3(0, R)} (|\mathcal{E}f|^2 \ast \psi)(x,t) d\mu_R(x,t)$$

$$\leq \int_{B^3(0, R)} (|\mathcal{E}f|^2 \ast \psi_1)(\tilde{m}(x,t)) d\mu_R(x,t)$$

$$\leq \sum_{\tilde{m} \in \mathbb{Z}^3, |m| \leq R} \left( \int_{|(x,t) - \tilde{m}| \leq 10} d\mu_R(x,t) \right) \cdot (|\mathcal{E}f|^2 \ast \psi_1)(\tilde{m}).$$

Without loss of generality, we may assume $\|f\|_2 = 1$. For each $\tilde{m} \in \mathbb{Z}^3$ we define

$$\nu_{\tilde{m}} := \int_{|(x,t) - \tilde{m}| \leq 10} d\mu_R(x) \lesssim 1.$$

From the discussion above, we have $(|\mathcal{E}f|^2 \ast \psi_1)(\tilde{m}) \lesssim 1$, which implies

$$\left\| e^{it\phi(D)}f \right\|_{L^2(B^3(0, R); d\mu_R(x,t))}^2 \lesssim \sum_{\nu \in [R^{-300}, 1]} \sum_{\tilde{m}, \nu \sim \tilde{m}} \nu \cdot (|\mathcal{E}f|^2 \ast \psi_1)(\tilde{m}) + R^{-270},$$

where the first summation is over all dyadic numbers $\nu$. For each dyadic $\nu$, denote $A_\nu = \{\tilde{m} \in \mathbb{Z}^3 : |m| \leq R, \nu \sim \nu\}$. Performing a dyadic pigeonholing over $\nu$ we see that there exists a dyadic $\nu \in [R^{-300}, 1]$ such that for any small $\varepsilon > 0$,

$$\left\| e^{it\phi(D)}f \right\|_{L^2(B^3(0, R); d\mu_R(x,t))}^2 \lesssim \varepsilon R^\varepsilon \sum_{\tilde{m} \in A_\nu} \nu \left( \int_{B^3(\tilde{m}, R^\varepsilon)} |\mathcal{E}f|^2 \right) + R^{-240}$$

$$\lesssim \varepsilon R^\varepsilon \nu \cdot \int_{\bigcup_{\tilde{m} \in A_\nu} B^3(\tilde{m}, R^\varepsilon)} |\mathcal{E}f|^2 + R^{-240}.$$
Thus, the intersection of $X_\nu$ and any $r-$ball can be contained in no more than $\nu^{-1} r^\alpha$ disjoint $R^\epsilon$-balls. Hence, we can decompose the set $\bigcup_{\tilde{m} \in A_\nu} B^3(\tilde{m}, R^\epsilon)$ further into a union of unit cubes $X = \bigcup B_k$ satisfying that

$$\# \{B_k \subset X : B_k \subset B^3(\tilde{x}, r)\} \lesssim R^{30\epsilon} \nu^{-1} r^\alpha$$

for any $\tilde{x} \in \mathbb{R}^3$ and $r \geq 1$.

Now, we can apply Corollary 1.7 to $X$ with $\lambda \lesssim R^{30\epsilon} \nu^{-1}$ and $\alpha$, which yields

$$\| e^{it\phi(D)} f \|_{L^2(B^3(0,R) \cup B^2(x,t))} \lesssim C_{\epsilon} \lambda^{1/2} \| e^{it\phi(D)} f \|_{L^2(X)} \lesssim C_{\epsilon} R^{\frac{5}{8} + \frac{\alpha}{4} - \frac{\alpha}{3m} + 20\epsilon} \| f \|_{L^2} \lesssim C_{\epsilon} R^{\frac{5}{8} + \frac{\alpha}{4} - \frac{\alpha}{3m} + 20\epsilon} \| f \|_{L^2}.$$

Here, we used the fact $\nu \lesssim 1$ in the last inequality.

It completes the proof of Theorem 4.3.

4.2. Application to the almost everywhere convergence problem associated to fractal dimensional measures. One can also consider the best Sobolev exponent for the almost everywhere convergence problems associated to $\alpha$-dimensional Hausdorff measures. Results of this direction are widely studied by many authors, and one can refer to [2, 10, 12, 19, 13] for more details.

We denote

$$s_m(\alpha) = \inf \left\{ s \geq 0 : \lim_{t \to 0} e^{it\phi(D)} f = f \ \alpha \text{-a.e. for every } f \in H^s(\mathbb{R}^2) \right\}.$$

Combining Corollary 1.1 and results of Eceizabarrena and Ponce-Vanegas [13] and the trivial dispersive estimates

$$\| e^{it\phi(D)} \varphi(x) \|_{L^\infty(\mathbb{R}^2)} \leq C \varphi \ |t|^{-\frac{1}{2} - \frac{1}{m}},$$

for any Schwartz function $\varphi$ with support in $A(1)$, we have

**Corollary 4.4.** For $m \geq 4$ being an even number, we have

$$s_m(\alpha) \begin{cases}
\leq \frac{2 - \alpha}{2}, & 0 \leq \alpha \leq \frac{1}{2} + \frac{1}{m}; \\
\leq \frac{3}{4} - \frac{1}{2m}, & \frac{1}{2} + \frac{1}{m} \leq \alpha \leq \frac{7m - 2}{5m - 2}; \\
\leq \frac{4}{3} - \frac{2}{3m} - \frac{\alpha}{12} (5 - \frac{2}{m}), & \frac{7m - 2}{5m - 2} \leq \alpha \leq 2.
\end{cases}$$

**Remark 4.5.** For general $0 < \alpha \leq 2$, we collect the known results of upper bound and lower bound of $s_m(\alpha)$ and show them in Figure 1 (we take $m = 4$ for simplicity). Eceizabarrena and Ponce-Vanegas [13] proved that the light grey region is the divergence region and the dark grey region is the convergence region. Our results show that the yellow region is also a convergence region. To the authors’ best knowledge, the convergence of the remaining regions remains open.

\footnote{In fact, we need Theorem 1.1, Theorem 1.2, and Lemma 2.4 in [13].}
Acknowledgments. We thank Daniel Eceizabarrena for pointing out their results in [13] to us. J. Zhao is supported by National Natural Science Foundation of China (Grant No. 12101562) and Natural Science Foundation of Zhejiang (No. LQ20A010003). T. Zhao is supported by the Fundamental Research Funds for the Central Universities (FRF-TP-20-076A1).

References

1. Chen An, Rena Chu, and Lillian B. Pierce, Counterexamples for high-degree generalizations of the Schrödinger maximal operator, arXiv e-prints (2021), arXiv:2103.15003.
2. Juan Antonio Barceló, Jonathan Bennett, Anthony Carbery, and Keith M. Rogers, On the dimension of divergence sets of dispersive equations, Math. Ann. 349 (2011), no. 3, 599–622. MR 2754999
3. Chandan Biswas, Maxim Gilula, Linhan Li, Jeremy Schwend, and Yakun Xi, $l^2$ decoupling in $\mathbb{R}^2$ for curves with vanishing curvature, Proc. Amer. Math. Soc. 148 (2020), no. 5, 1987–1997. MR 4078083
4. Jean Bourgain, A note on the Schrödinger maximal function, J. Anal. Math. 130 (2016), 393–396. MR 3574661
5. Jean Bourgain and Ciprian Demeter, The proof of the $l^2$ decoupling conjecture, Ann. of Math. (2) 182 (2015), no. 1, 351–389. MR 3374964
6. Lennart Carleson, Some analytic problems related to statistical mechanics, Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), Lecture Notes in Math., vol. 779, Springer, Berlin, 1980, pp. 5–45. MR 576038
7. Chu-hee Cho and Hyerim Ko, Note on maximal estimates of generalized Schrödinger equation, arXiv e-prints (2018), arXiv:1809.03246.
8. Chu-Hee Cho, Sanghyuk Lee, and Ana Vargas, Problems on pointwise convergence of solutions to the Schrödinger equation, J. Fourier Anal. Appl. 18 (2012), no. 5, 972–994. MR 2970037
9. Xiumin Du, Larry Guth, and Xiaochun Li, *A sharp Schrödinger maximal estimate in $\mathbb{R}^2$*, Ann. of Math. (2) **186** (2017), no. 2, 607–640. MR 3702674
10. Xiumin Du, Larry Guth, Xiaochun Li, and Ruixiang Zhang, *Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates*, Forum Math. Sigma **6** (2018), Paper No. e14, 18. MR 3842310
11. Xiumin Du, Larry Guth, Yumeng Ou, Hong Wang, Bobby Wilson, and Ruixiang Zhang, *Weighted restriction estimates and application to Falconer distance set problem*, Amer. J. Math. **143** (2021), no. 1, 175–211. MR 4201782
12. Xiumin Du and Ruixiang Zhang, *Sharp $L^2$ estimates of the Schrödinger maximal function in higher dimensions*, Ann. of Math. (2) **189** (2019), no. 3, 837–861. MR 3961084
13. Daniel Eceizabarrena and Felipe Ponce-Vanegas, *Pointwise convergence over fractals for dispersive equations with homogeneous symbol*, arXiv e-prints (2021), arXiv:2108.10339.
14. Chuanwei Gao, Zhuoran Li, Tengfei Zhao, and Jiqiang Zheng, *Decoupling and Schrödinger maximal estimates for finite type phases in higher dimensions*, arXiv e-prints (2022), arXiv:2202.11326.
15. Sanghyuk Lee, *On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$*, Int. Math. Res. Not. (2006), Art. ID 32597, 21. MR 2264734
16. Zhuoran Li, Changxing Miao, and Jiqiang Zheng, *A restriction estimate for a certain surface of finite type in $\mathbb{R}^3$*, J. Fourier Anal. Appl. **27** (2021), no. 4, Paper No. 63, 24. MR 4287304
17. Zhuoran Li and Jiqiang Zheng, $\ell^2$ decoupling for certain surfaces of finite type in $\mathbb{R}^3$, arXiv e-prints (2021), arXiv:2109.11998.
18. Pertti Mattila, *Hausdorff dimension, projections, and the Fourier transform*, Publ. Mat. **48** (2004), no. 1, 3–48. MR 2044636
19. , *Fourier analysis and Hausdorff dimension*, Cambridge Studies in Advanced Mathematics, vol. 150, Cambridge University Press, Cambridge, 2015. MR 3617376
20. Changxing Miao, Jianwei Yang, and Jiqiang Zheng, *An improved maximal inequality for 2D fractional order Schrödinger operators*, Studia Math. **230** (2015), no. 2, 121–165. MR 3476484
21. Keith M. Rogers, Ana Vargas, and Luis Vega, *Pointwise convergence of solutions to the nonelliptic Schrödinger equation*, Indiana Univ. Math. J. **55** (2006), no. 6, 1893–1906. MR 2284549
22. Thomas Wolff, *Addendum to: “Decay of circular means of Fourier transforms of measures” [Internat. Math. Res. Notices 1999, no. 10, 547–567; MR1692851 (2000k:42016)],* vol. 88, 2002, Dedicated to the memory of Tom Wolff, pp. 35–39. MR 1979770
23. Tongou Yang, *Uniform $l^2$-decoupling in $\mathbb{R}^2$ for polynomials*, J. Geom. Anal. **31** (2021), no. 11, 10846–10867. MR 4310157

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