The Existence of Global Solution for a Class of Semilinear Equations on Heisenberg Group ∗†

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Abstract Based on the concepts of a generalized critical point and the corresponding generalized P.S. condition introduced by Duong Minh Duc[1], we have proved a new $Z_2$ index theorem and get a result on multiplicity of generalized critical points. Using the result and a quite standard variational method, it is found that the equation

$$-\Delta_{H^n} u = |u|^{p-1} u \quad x \in H^n$$

has infinite positive solutions. Our approach can also be applied to study more general nonlinear problems.

Key Words and Phrases Subelliptic Operator, $Z_2$ index, Heisenberg Group, Generalized Critical Point.

1 Introduction

In this paper, we deal with the existence of the multiple global solutions to the following nonlinear equation

$$-\Delta_{H^n} u = |u|^{p-1} u \quad (1)$$
where $H^n$ is the Heisenberg group, $\Delta_{H^n} = \sum_{i=1}^{n} (X_i^2 + Y_i^2)$ is its subelliptic Laplacian operator. Under the real coordinate $(x_1, \cdots, x_n, y_1, \cdots, y_n, t)$, the vector field $X_i$ and $Y_i$ are defined by

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n.$$  
$$Y_i = \frac{\partial}{\partial x_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n.$$  

It is well known that $\{X_i, Y_i\}$ generate the real Lie algebra of Lie group $H^n$ and

$$[X_i, Y_i] = 4\delta_{ij} \frac{\partial}{\partial t}, \quad i, j = 1, \cdots, n.$$  

In this Lie group, there is a group of natural dilations defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda t^2), \quad \lambda > 0$$

where $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$. With this group of dilations, the Lie group $H^n$ is a two-step stratified nilpotent Lie group of homogeneous dimension $Q = 2n + 2$, and $\Delta_{H^n}$ is a homogeneous differential operator of degree 2.

Equation (1) comes from the CR-Yamabe problem (see [2]) and has been studied by several authors (see [4], and the references therein). In their works, they have got some results on the existence of the boundary value problem of equation (1) on bounded domain. In paper [2], when the domain is unbounded, they have defined a thin condition, and proved a compact Folland-Stein-Soblev type embedding theorem. By this compact theorem, they proved that the corresponding functional of equation (1) satisfies the P.S. condition, so by the normal variational methods they gave some results about the boundary problem of equation (1). If the domain is unbounded and does not satisfy the thin condition, to our knowledge, there exists no report of progress on this problem up to now.

The global space $R^n$ is the most simple domain which does not satisfy the thin condition. In the present paper, we study equation (1) on the global space $R^n$, and find that it has infinite positive solutions. The main result is the following

**Main theorem** If $1 < p < \frac{Q+2}{Q-2}$, where $Q = 2n + 2$, the homogeneous dimension of $H^n$, then equation (1) has infinite positive solutions belonging to $C^{2+\alpha}(H^n)$ with some $\alpha > 0$.

Our idea is based on the concept of generalized critical point and generalized P.S. condition introduced by Duong Minh Duc[1]. In his paper, he has obtained a deformation lemma and a generalized Mountain Pass Lemma. Using these lemmas, he has studied a class of nonlinear singular elliptic equations and obtained
some existent results. In our paper, based on the concept of generalized critical point and generalized P.S. condition, we proved a new deformation lemma. From our deformation lemma, we prove a multi-existence result on a class of even functional which lacks compact condition. Then using the lemma and quite standard variational method, we present a proof of our main theorem.

It should be pointed out that our method can be used to study more general problems, e.g. that related to a general unbounded domain which has $Z_2$ symmetry and that with a more complicated nonlinear term instead of $u|u|^{p-1}$ on the right hand of equation (1).

## 2 Preliminary results

In this section, we give some definitions and lemmas related to the generalized critical point, which come from the ideal of Duong Minh Duc (see [1]), and the definitions of $Z_2$ index. The theorem of $Z_2$ index related to generalized critical point is a new theorem and is our main tool in the proof of main theorem.

Let $X$ be a Banach space. Assume that there exist a family of closed vector subspaces $\{X_\rho\}_{\rho \in D}$, and a family of linear maps $\{\pi_\rho\}_{\rho \in D}$ such that $I = \text{Span}\{\bigcup_{\rho \in D} X_\rho\}$ is dense in $X$, and $\pi_\rho(X) = X_\rho$.

**Definition 2.1** Let $f$ be a continuous functional on $X$, $x \in X$.

(I). We call $x$ a generalized critical point of $f$ if there exists a sequence $\{x_j\}$ in $X$ such that

$$\lim_{j \to \infty} f'(x_j) = \lim_{j \to \infty} \|\pi_\rho(x_j - x)\| = 0$$

for all $\rho \in D$. In this case $\{x_j\}$ is called an approximation sequence of $x$.

(II). Let $x$ be a generalized critical point of $f$. We say that $x$ is regular if we can find an approximation sequence $\{x_j\}$ of $x$ such that $\{f(x_j)\}$ is convergent in $\mathbb{R}$; denote by $\bar{f}(x)$ the set of such limits. If $c$ in $\bar{f}(x)$, then $c$ is called a generalized critical value of $f$.

Denote by $K$ the set of all generalized critical points of $f$. For any real number $c$ and positive number $\varepsilon$, we put

$$K_c = \{x \in K; c \in \bar{f}(x)\}$$

$$A(c, \varepsilon) = f^{-1}([c - \varepsilon, c + \varepsilon]).$$

The generalized P.S. condition is defined as follows.
Definition 2.2 Let $f$ be a continuous differential functional on $X$. We say that $f$ satisfies generalized P.S. condition, if for any sequence $\{x_j\}$ in $X$, along which $\{f(x_j)\}$ is bounded and $\{f'(x_j)\}$ is convergent to 0, there exists a point $x \in X$, and a subsequence $\{x_{j_k}\}$ of $\{x_j\}$, such that $\lim_{k \to \infty} \|\pi_\rho(x_{j_k} - x)\| = 0$ for all $\rho \in D$.

Next, we set $f$ to be a continuous differential functional, and $c \in \mathbb{R}$.

Lemma 2.1 Suppose that $K_c = \emptyset$. Then there exists positive real number $b$ and $\varepsilon$, such that for any $x$ in $A(c, \varepsilon)$, we have

$$\|f'(x)\| > b.$$  

Proof. If Lemma 2.1 is not true, there exist two sequences of positive numbers $\{b_j\}, \{\varepsilon_j\}, b_j \to 0, \varepsilon_j \to 0$, and $x_j \in A(c, \varepsilon_j)$, such that $\|f'(x_j)\| < b_j$. Since $f$ satisfies generalized P.S. condition, we have $c \in \bar{f}(x)$. That is $K_c \neq \emptyset$. This contradicts to the condition $K_c = \emptyset$. So Lemma 2.1 is true.

Lemma 2.2 Suppose that $K_c \neq \emptyset$. Let $b$ and $\varepsilon$ be defined as in Lemma 2.1. Then there exist an subset $u$ of $X$ containing $A(c, \varepsilon)$ and a locally Lipschitz continuous map $v$ from $u$ into $X$ such that for any $x$ in $u$

$$\|v(x)\| \leq 1$$  

$$f'(x)v(x) > \frac{1}{2}b.$$  

Proof. By Lemma 2.1, for any $x$ in $A(c, \varepsilon)$, there exists $h \in X, \|h\| = 1$, such that $f'(x)h > \frac{1}{2}b$. Since $f'(x)$ is continuous, there is a positive number $r_x$, such that for any $y \in B(x, r_x)$, we have

$$f'(y)h > \frac{1}{2}b$$  

So we get a cover $B = \{B(x, r_x)| x \in A(c, \varepsilon)\}$ of $A(c, \varepsilon)$. Let $\{B(x_i, r_{x_i})\}_{i \in \mathbb{Z}}$ be a locally finite subcover of $B$. Define the set $U = \bigcup_{i \in \mathbb{Z}} B(x_i, r_{x_i})$ and the functional $q_i(x)$ which is the distance from $x$ to $X \setminus B(x_i, r_{x_i})$. Then $q_i(x)$ is a Lipschitz continuous functional and $q_i|_{X \setminus B(x_i, r_{x_i})} \equiv 0$. Let

$$v(x) = \sum_i \frac{q_i(x)}{\sum_i q_j(x)} h_j$$

where $h_j$ is defined by formula (4). One can check that $v(x)$ is that we need.

Now, we can give our deformation lemma.
Lemma 2.3 Let $f$ be a continuous differential functional on $X$, and $f$ satisfy the generalized P.S. condition. For $c \in R, K_c = \emptyset$. $\varepsilon$ is the positive number given by Lemma 2.1. Then for every $q \in (0, \varepsilon)$, there is a homomorphism $w$ on $X$, such that

(i) $w(x) = x, \quad \forall x \in X \setminus A(c, \varepsilon)$,
(ii) $w(f^{-1}((\infty, c + q))) \subset f^{-1}((\infty, c - q))$.

Proof. Let $w$ to be a solution of Cauchy problem

\[
\begin{cases}
\frac{dw}{dt} = -v(w) \\
w(x, 0) = x,
\end{cases}
\]

where $v$ is defined as in Lemma 2.2. Then one can check that $w$ satisfies

(a). For any $s \in (0, t), w(x, s) \in A(c, q)$ implies

\[|w(x, t) - x| \leq t, f(x) - f(w(x, t)) \geq \frac{1}{2}bt; \quad (5)\]

(b). For any $x \in X \setminus A(c, \varepsilon), w(x, t) = x$;

(c). The function $f_1(t) = f(w(x, t))$ is not a increasing function on $t$.

Let $b$ be given as in Lemma 2.1. It is clear that $w(x, t) = x$, for all $x \in X \setminus A(c, \varepsilon)$ and $t \geq 0$.

Let $t_0$ be $\frac{4\varepsilon}{b}$. Observe that the trajectory $w(x, t)$ emanate from $x$ in $f^{-1}((\infty, c + q))$, where $t \in [0, t_0]$. If $x$ belongs to $f^{-1}((\infty, -c - q)$, then $w(x, t)$ is in $f^{-1}((\infty, -c - \varepsilon))$.

Next it is proved that for any $x \in A(c, q), w(x, t_0)$ belongs to $f^{-1}((\infty, c - q))$. If this is not true, for any $s$ belongs to $[0, t_0]$, one can find a point $x \in A(c, q)$. Then by the formula (3) we have

\[f(w(x, 0)) - f(w(x, t_0)) = f(x) - f(w(x, t_0)) \geq \frac{1}{2}bt_0 = 2\varepsilon > 2q.\]

This is a contradiction. Therefore the lemma follows.  

When the set $K_c \neq \emptyset$, from the lemma 2.2 and the very similar argument, we have the following deformation lemma.

Lemma 2.4 Let $f$ be a continuous differential functional on $X$, and $f$ satisfy the generalized P.S. condition. For $c \in R, K_c \neq \emptyset$. Then there exists a positive $\varepsilon$, and $q \in (0, \varepsilon)$, and a neighbourhood $U$ of $K_c$, such that $f^{-1}((\infty, c + q) \setminus U$ is a deformation kernel of $f^{-1}((\infty, c - \varepsilon)$.
Proof of lemma 2.4 is very similar to those of lemma 2.3 and the corresponding lemma of deformation on the functional which satisfies the P.S. condition. And when the functional has a $Z_2$ symmetry, the deformation can be chosen to be odd.

As the usual variational method of Ljusternik-Schnirelmann type theory, after we have the above deformation lemma, we can give out a $Z_2$ index on the generalized critical points. To be self contained, we first give the definition of $Z_2$ index and some of its properties we shall use. Then we give a theorem which is used to compute the $Z_2$ index of the set of generalized critical points.

Let $\mathcal{A}$ denote the set

$$\mathcal{A} = \{ A \in \mathcal{A} | A \text{ is a symmetric close subset of } X \}$$

where symmetry means that $x \in A$ implies $-x \in A$. The $Z_2$ index is defined as follows.

**Definition 2.3** A function $i : \mathcal{A} \to \mathbb{Z}_+ \cup \{+\infty\}$ is called $Z_2$-index, if for $A \in \mathcal{A}$, $i(A)$ is defined by

(I). If $A = \emptyset$, $i(A) = 0$;

(II). if $A \neq \emptyset$, there exists a positive number $m$ and a continuous odd map $\varphi : A \to \mathbb{R}^m \setminus \{0\}$, then define $i(A)$ to be the minimum of this kind of $m$. i.e

$$i(A) = \min \{ m \in \mathbb{Z}_+ | \text{there is a continuous odd map } \varphi : A \to \mathbb{R}^m \setminus \{0\} \}$$

(III). If $A \neq \emptyset$, and there is none positive integer satisfies (II), define $i(A) = +\infty$.

**Lemma 2.5** The $Z_2$-index on $\mathcal{A}$ has the following properties.

(I). $i(A) = 0 \iff A = \emptyset$;

(II). If $A$ has only a pair of symmetric point, then $i(A) = 1$;

(III). For any $A < B \in \mathcal{D}$, $A \subset B$, we have

$$i(A) \leq i(B);$$

(IV). $i(A \cup B) \leq i(A) + i(B), \ \forall A, B \in \mathcal{D}$;

(V). For any continuous odd map $\rho : X \to X$, and $A \in \mathcal{D}$, $i(A) \leq i(\rho(A))$;

(VI). $A \in \mathcal{D}$, if $A$ is compact, then there exists a symmetric neighborhood $N$, such that $i(\bar{N}) = i(A)$, further more, if $A$ is compact, then $i(A) < +\infty$;

(VII). Suppose $X_1$ is a subspace with dimension on $m$, $S$ is the unit sphere, then

$$i(X_1 \cap S) = m.$$
For their proofs, see e.g ref[7].

Let $M$ be a $C^{1,1}$ submanifold of $X$, $f$ is an even continuous differential functional with generalized P.S. condition. If $k \leq i(M) \leq \infty$, by Lemma 2.5, the set

$$Z_k = \{ A \in \mathcal{D} : A \subset M, i(A) \geq k \}$$

is not empty and invariant under odd continuous map.

For any positive integer $k \leq r(M)$, define

$$C_k = \inf_{A \in Z_k} \sup_{u \in A} f(u). \quad (6)$$

Then we have the following theorem.

**Theorem 2.1** Suppose $k < r(M), k + l - 1 \leq r(M)$. If

$$-\infty < C_k = C_{k+1} = \cdots = C_{k+l-1} = c < +\infty$$

then $i(K_c) \geq l$.

Proof. $K_c$ is a compact subset of $M$ (In fact, for any $\{x_j\} \subset K_c$, since $f$ satisfies generalized P.S. condition, there exist $x \in X$ and a subsequence $\{x_{j_k}\} \subset \{x_j\}$, such that, for any $\rho \in \mathcal{D}$,

$$\lim_{k \to \infty} \|\pi_\rho(x_{j_k} - x)\| = 0.$$ 

Notice that the subspace $\text{Span} \bigcup_{\rho \in \mathcal{D}} X_\rho$ of $X$ is dense in $X$, so $\{x_{j_k}\}$ have a limit $x \in X$. From the continuity of the functional $f$, we have $f(x) = c$ and $f'(x) = 0$. Obviously $x \in M$. Hence $x \in K_c$, so $i(K_c)$ can be defined. By Lemma 2.5, there exists a symmetric neighborhood $N$ in $M$, such that $i(N) = i(K_c)$. If $i(K_c) < l$, by the deformation Lemma 2.4, there is a positive number $\varepsilon > 0$ and a homomorphism $w : X \to X$, such that

$$w(f_{c+\varepsilon} \setminus N) \subset f_{c-\varepsilon}$$

where $f_{c+\varepsilon}$ is defined by $f_{c+\varepsilon} = \{ x \in M | f(x) < c + \varepsilon \}$, $f_{c-\varepsilon}$ is defined by the same way. For the positive number $\varepsilon$ defined above, by the definition $c_{m+l-1}$, we can find a set $A \in \mathcal{A}_{m+l-1}$, such that

$$c \leq \sup_{x \in A} f(x) < c + \varepsilon. \quad (7)$$

So we have

$$\sup_{x \in w(A \setminus N)} f(x) < c - \varepsilon.$$
From Lemma 2.5, noting $A \in A_{k+l-1}$,

\[ k + l - 1 \leq i(A \setminus N) + i(\bar{N}) \leq i(w(A \setminus N)) + l - 1. \]

So

\[ r(A \setminus N) \geq k. \]

This implies $A \setminus N \subset A$, which means

\[ \sup_{x \in w(A \setminus N)} f(x) \geq c_m = c. \]

This conflict with formula (3). So $r(K_c) \geq l$. #

As a result, we have the following theorem.

**Theorem 2.2** Suppose $I$ is a continuous functional on $M$. $M$ is a $C^{1,1}$ symmetric submanifold of $X \setminus \{0\}$. Suppose further that $I$ satisfies the generalized P.S. condition and bounded from below. Then $I$ has at least $\tilde{r}(M)$ pairs of generalized critical points, where $\tilde{i}(M)$ is defined by

\[ \tilde{i}(M) = \max\{i(A)|A \subset M, A is symmetric\}. \]

### 3 Main theorem and its proof

In this section, we present our main theorem and its proof. First we define some space, some of which are not new. Then we give our main theorem and its proof.

For any domain $\Omega \subset H^n$, let $C_0^\infty(\Omega)$ denote the set of smooth functions with compact support. For $u \in C_0^\infty(\Omega)$, define

\[ \|u\|^2 = \int_\Omega |\nabla_{H^n} u(x)|^2 dx. \quad (8) \]

One can check that $\| \cdot \|$ is a norm (see [3]). Let $S^2_1(\Omega)$ denote the complete of $C_0^\infty(\Omega)$ under the norm $\| \cdot \|$. Then we have the following embedding theorem [3].

**Lemma 3.1** Let $D \subset H^n$ be a bounded open set, then the embedding

\[ \overset{\circ}{S}^2_1(D) \hookrightarrow L^p(D), \quad 1 \leq p < \frac{2Q}{Q - 2} \]

is compact.
In $H^n$, we define metric

$$d((x, y, t), (x', y', t')) = [t^2 + (|x|^2 + |y|^2)^2]^{\frac{1}{2}}$$

where $(\bar{x}, \bar{y}, \bar{t}) = (x, y, t) - (x', y', t')$. Let $B_k = \{x \in H^n | d((x, y, t), 0) < k\}$. For $k > 1$, define $\rho_k \in C_0^{\infty}(B_{k+1})$, and $\rho_k | B_k \equiv 1$. For $k > 1$, denote $H_k = \tilde{S}_1^2(B_k)$, and define a continuous map $\pi_k : \tilde{S}_1^2(\mathbb{R}^n) \to H_k$ by

$$\pi_k(u) = \rho_k u.$$

In $\tilde{S}_1^2(\mathbb{R}^n)$, we define

$$M = \{u \in \tilde{S}_1^2(\mathbb{R}^n) | \frac{1}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1} dx = 1\}.$$

It is obvious that $M$ is a $C^{1,1}$ submanifold of $\tilde{S}_1^2(\mathbb{R}^n)$, and is symmetric, and that $0 \not\in M$. We define a functional on $M$

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_{H^n} u|^2 dx, \quad \forall u \in M. \quad (9)$$

Then we have the following lemma.

**Lemma 3.2** If $1 \leq p < \frac{q+2}{q-2}$, the functional $I$ defined by formula (9) satisfies generalized P.S. condition.

Proof. Obviously, $I$ is a continuous differential functional defined on $M$. Along the $\{u_j\} \subset M, \{I(u)\}$ is bounded and $\{I'(u)\}$ converges to 0.

For any $k \in \mathbb{Z}^+, \text{ and } k \geq 2$, then $C_0^{\infty}(B_k)$ is dense in $\tilde{S}_1^2(B_k)$, and $\{I(\pi_k(u_j))\}$ is bounded. For any $u \in C_0^{\infty}(B_k)$, we have

$$< I'(\pi_k(u_j)), u > = \int_{B_k} \nabla_{H^n} \pi_k(u_j) \cdot \nabla_{H^n} u dx$$

$$= \int_{\mathbb{R}^n} \nabla_{H^n}(\rho_k u_j) \cdot \nabla_{H^n} u dx$$

$$= \int_{\mathbb{R}^n} \rho_k \nabla_{H^n} u_j \cdot \nabla_{H^n} u dx + \int_{\mathbb{R}^n} u_j \nabla \rho_k \cdot \nabla_{H^n} u dx$$

$$= \int_{B_k} \rho_k \nabla_{H^n} u_j \cdot \nabla_{H^n} u dx + \int_{B_k} u_j \nabla \rho_k \cdot \nabla_{H^n} u dx$$

$$\leq C \int_{B_k} \nabla_{H^n} u_j \cdot \nabla_{H^n} u dx \to 0,$$

where $C$ is a positive constant independent of $k$. So on $S_1^2(B_k), I^1(\pi_k(u_j)) \to 0$.

This implies that, $\forall \varepsilon > 0, \exists N$, if $r, s > N$, we have, for any $k \in \mathcal{D}$,

$$\int \nabla_{H^n} \pi_k(u_r) \nabla_{H^n}(u_r - u_s) dx \leq \varepsilon \| \pi_k(u_r - u_s) \|_p, \quad (10)$$
\[
\int \nabla H \pi_k(u_s) \nabla H \pi_k(u_r - u_s) dx \leq \varepsilon \|\pi_k(u_r - u_s)\|.
\] (11)

By the inequalities (10),(11), we have

\[
\|\pi_k(u_r - u_s)\| < \varepsilon \quad \forall k \in \mathcal{D}.
\]

So \( \{\pi_k(u_i)\} \) converges in \( H_k \) and converges almost every where in \( B_k \). But for \( k + 1, \pi_{k+1}(u_i)|_{B_k} = \pi_k(u_i) \). So we can choose a function \( u \in S^2_1(\mathbb{R}^n) \), such that

\[
\pi_k(u_i) \to \pi_k(u) \quad \forall k \in \mathcal{D}.
\] (12)

The formula (12) implies that the functional satisfies generalized P.S. condition.

By Theorem 2.2 and Lemma 3.2, we have the following theorem

**Theorem 3.1** If \( 1 \leq p < \frac{Q+2}{Q-2} \), the functional (9) has infinite numbers of generalized critical point.

Next, we shall prove that a generalized critical point of \( I \) on \( M \) is its critical point in \( M \). This is our following lemma.

**Lemma 3.3** If \( u \) is a generalized critical point of the functional \( I \) on \( M \), it must be a critical point of \( I \) on \( M \).

**Proof.** Suppose that \( u \) is a generalized critical point of \( I \) on \( M \). By the definition of generalized critical point, there exists a sequence \( \{u_j\} \subset M \), such that, for any positive integer number \( k \),

\[
I'(\pi_k(u_j)) \to 0, \|\pi_k(u_j) - \pi_k(u)\| \to 0.
\]

Since \( \bigcup_{k=1}^{\infty} \pi_k(S^2_1(\mathbb{R}^n)) \cap M \) is a dense subset of \( M \) and

\[
\|u_j - u\| \leq \|\pi_k(u_j) - \pi_k(u_j)\| + \|\pi_k(u_j) - \pi_k(u)\| + \|\pi_k(u) - u\|
\]

we have

\[
u_j \to u.
\]

For \( I \) is a continuous differential functional, we have, as \( j \to \infty \),

\[
I'(\pi_k(u_j)) \to I'(u).
\]

So we have \( I'(u) = 0 \). This implies \( u \) to be a critical point of \( I \) on \( M \).
By Lemma 3.3 and Lemma 3.2 we know that the functional $I$ has infinitely many pairs of critical point on $M$; we denote them by $\{\tilde{u}_j\}$. So by Lagrangian multiplier rule, there exists parameter $\mu \in \mathbb{R}$, such that

$$
< v, (DI(\tilde{u}_j) - \mu \tilde{u}_j|\tilde{u}_j|^{p-2}) > = \frac{1}{2} \int_{\mathbb{R}^n} \nabla h \tilde{u}_j \cdot \nabla h v - \mu \tilde{u}_j|\tilde{u}_j|^{p-2}v dx = 0 \text{ for all } v \in S^2_1(H^n).
$$

Setting $v = \tilde{u}_j$ in this equation yields that

$$
\frac{1}{2} \int_{H^n} |\nabla h \tilde{u}_j|^2 dx = \mu \int_{H^n} |\tilde{u}_j|^{p+1} dx = (p+1)\mu.
$$

For $\tilde{u}_j \neq 0$ in $S^2_1(H^n)$, we have $\mu > 0$. Rescaling $\tilde{u}_j$ with a suitable power of $(p+1)\mu$, we obtain a weak solution $u_j = ((p+1)\mu)^{\frac{1}{p-2}}\tilde{u}_j \in S^2_1(H^n)$ of problem (1), in the sense of

$$
\int_{H^n} (\nabla h u_j \cdot \nabla v - |u_j|^{p-2}u_j v) dx = 0 \text{ for all } v \in S^2_1(H^n).
$$

By the regulation theorem of operator $\Delta_{H^n}$ (see [1], [2]), we know $u_j \in C^{2+\alpha}(\mathbb{R}^n)$. Hence they are strong solutions of problem (1). Further more, by the minimum theorem of $\Delta_{H^n}$, we know $u_j(p) \neq 0$ for all $p \in H^n$. So we get our main theorem.

**Theorem 3.2** If $1 < p < \frac{Q+2}{Q-2}$, where $Q = 2n + 2$ is the homogeneous dimension of $H^n$, the problem

$$
\left\{
\begin{array}{ll}
-\Delta_{H^n} u = |u|^{p-1}u & x \in \mathbb{R}^n \\
u(x) > 0
\end{array}
\right.
$$

has infinite $C^{2+\alpha}$ solutions.

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