Rigidity results for the $p$-Laplace type equations on compact Riemannian manifolds

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In this paper, we obtain two rigidity results for $p$-Laplace type equation and $n$-Laplace equation with exponential nonlinearity on $n$-dimensional compact Riemannian manifolds by using of nonlinear flow and the carré du champ methods, respectively, where rigidity means that the PDE has only constant solution when a parameter is in a certain range. Moreover, an interpolation inequality is derived as an application.

KEYWORDS
nonlinear flow, $p$-Laplacian, $p$-Bochner formula, rigidity, the carré du champ method

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1 | INTRODUCTION AND MAIN RESULTS

Throughout the paper, we assume that $(M, g)$ is an $n$-dimensional compact Riemannian manifold without boundary and $dV$ the volume element induced by the Riemannian metric $g$. In [1] and [2](or see [3]), Licois–Véron and Barky–Ledoux proved a rigidity result for Equation (1.1)

$$-\Delta u + \frac{\lambda}{q-2} (u - u^{q-1}) = 0, \quad (1.1)$$

on compact Riemannian manifolds with positive Ricci curvature; that is, if $\text{Ric} \geq Kg (K > 0)$ and $\lambda$ satisfies

$$\lambda \leq (1 - \theta)\lambda_0 + \theta \frac{n}{n-1}K, \quad \theta = \frac{(n - 1)(q - 1)}{n(n + 2) + q - 1}, \quad (1.2)$$

where $\lambda_0$ is the lowest positive eigenvalue of $-\Delta$, then Equation (1.1) has only constant solution 1. Both of them applied the Bochner formula or the carré du champ method.

Recently, Dolbeault et al. [4] established a rigidity result for Equation (1.1) by a nonlinear flow method. More precisely, if $\lambda \in (0, \lambda_0)$ and for any $q \in (1, 2) \cup (2, 2^*)$, Equation (1.1) has a positive constant solution, where $2^* = \frac{2n}{n-2}$ and

$$\Lambda := \inf_{u \in H^2 (M) \setminus \{0\}} \frac{(1 - \theta)\int_M (\Delta u)^2 dV + \frac{\theta n}{n-1} \int_M \| \nabla u \|^2 + \text{Ric}(\nabla u, \nabla u) |dV|}{\int_M |\nabla u|^2 dV}.$$
Here, the quantity $Q_{gu}$ is trace free and defined by

$$Q_{gu} := \nabla \nabla u - \frac{g}{n} \Delta u - \frac{(n-1)(q-1)}{\theta(n+3-q)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{n} \frac{\nabla |u|^2}{u} \right],$$

where $\theta$ is defined in (1.2). The advantage of this approach is that the value of $\theta$ can be directly calculated and does not require the assumption of positive Ricci curvature.

On the other hand, Dolbeault et al. [5] continue to study the rigidity for a semilinear elliptic equation with exponential nonlinearity (1.3) on two-dimensional compact Riemannian manifold

$$-\frac{1}{2} \Delta u + \lambda_\tau = e^u. \tag{1.3}$$

In other words, Equation (1.3) has only constant solution if $\lambda_\tau \in (0, \lambda_\tau^*)$, where $\lambda_\tau^*$ is defined by

$$\lambda_\tau^* := \inf_{u \in H^2(M) \setminus \{0\}} \frac{\int_M \left[ \|\nabla \nabla u - \frac{g}{2} \Delta u - \frac{1}{2} \left( \nabla u \otimes \nabla u - \frac{g}{2} |\nabla u|^2 \right) \right]^2 + \text{Ric}(\nabla u, \nabla u) e^{-\frac{u}{2}} dV}{\int_M |u|^2 e^{-\frac{u}{2}} dV}. \tag{1.4}$$

Equation (1.3) is related to a stationary version of the well-known classical Keller–Segel model describing chemotaxis. See the review article [6] for the latest progress.

Motivated by the above works, the purpose of this paper is to study the rigidity problems for $p$-Laplace type equations

$$-\Delta_p v + \frac{\lambda_p}{q-p} (v^{p-1} - v^{q-1}) = 0, \tag{1.5}$$

and

$$-\frac{1}{p} \Delta_p u + \lambda_\nu = e^u \tag{1.6}$$

on $n$-dimensional compact Riemannian manifolds, where $p$-Laplacian $\Delta_p$ is defined by $\Delta_p := \text{div}(\nabla |\nabla u|^{p-2} \nabla \cdot)$. Note that when $p = 2$, Equations (1.4) and (1.5) reduce to Equations (1.1) and (1.3), respectively.

Now, let us state the main results of this paper. The first rigidity result depends on the nonlinear flow of porous medium type

$$\partial_t u = u^{p-\beta} \left( \Delta_p u + \kappa |\nabla u|^p \right), \tag{1.7}$$

where

$$\beta = \frac{2(p-1)(n(p-1)+p)}{p[(2p-3)n+2p-1]-q[(p-2)n+p]} \tag{1.8}$$

and

$$\kappa = p-1 + \beta(q-p), \tag{1.9}$$

which can be view as a $p$-Laplacian version in [4]. Moreover, a nonlinear $p$-Bochner formula ([7, 8]) is required

$$\frac{1}{p} L(|\nabla u|^p) = |\nabla u|^{2p-4} \left( \|\nabla \nabla u\|_A^2 + \text{Ric}(\nabla u, \nabla u) \right) + |\nabla u|^{p-2} \langle \nabla (\Delta_p u), \nabla u \rangle, \tag{1.10}$$

where $L$ is the linearized operator of $\Delta_p$ at the point $u$, which is given by [9]

$$L(\psi) := \text{div} \left( |\nabla u|^{p-2} A(\nabla \psi) \right), \tag{1.11}$$
and $\nabla \nabla u$ denotes the Hessian of $u$, $\|\nabla \nabla u\|_A^2 = \sum_{i,j,k,l} A^{ik} A^{jl} \nabla_i \nabla_j u \nabla_k \nabla_l u$, where $A$ is a 2-tensor defined by [10]

$$A := \text{Id} + (p - 2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}, \quad a := \text{Id} - \left(\frac{p - 2}{p - 1}\right) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2},$$  \hspace{1cm} (1.11)

and $a$ is the inverse of $A$. Define two trace-free tensors with respect to $a$,

$$B_u := |\nabla u|^{p - 2} \nabla u - \frac{\Delta_p u}{n} a, \quad H_u := |\nabla u|^{p - 2} \nabla u - \frac{p - 1}{n} |\nabla u|^2 a$$  \hspace{1cm} (1.12)

and

$$Q_u := B_u - \frac{p(n - 1)(q - 1)}{\theta(n(p - 1) + p)(2p - 1 - q) + n(q - 1)} H_u$$  \hspace{1cm} (1.13)

where

$$\theta := \frac{p^2(n - 1)^2(q - 1)}{4n(p - 1)(n(p - 1) + 1) + (n(p - 2) + p)^2(q - 1)}.$$  \hspace{1cm} (1.14)

Set

$$\Lambda_* := \inf_{u \in H^2(M) \setminus \{0\}} \frac{(1 - \theta) \int_M (\Delta_p u)^2 dV + \frac{\theta n}{n - 1} \int_M [|Q_u u|_A^2 + |\nabla u|^{2p - 4} \text{Ric}(\nabla u, \nabla u)] dV}{\beta^{2-p} \int_M |u|^p dV},$$  \hspace{1cm} (1.15)

where $m = \max\{p, 2p - 2\}$. Moreover, if $0 < \theta < 1$, then

$$1 < q < p^* = \frac{np}{n - p}.$$  \hspace{1cm}

**Theorem 1.1.** Assume that $\Lambda_* > 0$. For any $q \in (1, p) \cup (p, p^*)$, if $\lambda_p \in (0, \Lambda_*)$, then Equation (1.4) on compact Riemannian manifolds has a unique constant solution, which equals to 1.

**Remark 1.2.** Taking $p = 2$, the rigidity in Theorem 1.1 turns to the result of Dolbeault et al. in [4].

As an application of the rigidity result, we can establish an $L^p$-interpolation inequality.

**Theorem 1.3.** Assume that $\Lambda_* > 0$. For any $q \in (1, p) \cup (p, p^*)$, if $\lambda_\Lambda < \Lambda_*$ and $v = u^\theta$, the following interpolation inequality holds:

$$\|\nabla v\|^p_{L^p(M)} \geq \frac{\lambda_p}{q - p} \left(\|v\|^p_{L^p(M)} - \|v\|^p_{L^p(M)}\right).$$  \hspace{1cm} (1.16)

For the second rigidity result, we define a quantity

$$\lambda_* := \inf_{a \in H^2(M) \setminus \{0\}} \frac{\int_M \left[\|B_u - \frac{1}{n} H_u\|_A^2 + |\nabla u|^{2p - 4} \text{Ric}(\nabla u, \nabla u)\right] e^{-\frac{n - 1}{2} u} dV}{(n - 1) \int_M |u|^n e^{-\frac{n - 1}{2} u} dV},$$  \hspace{1cm} (1.17)

where $B_u$ and $H_u$ are given in (1.12) and $a = (a_i)$ is the inverse of $A = (A^{ij})$ defined in (1.11).

**Theorem 1.4.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold without boundary. If $u$ is a smooth solution to equation

$$-\frac{1}{n} \Delta_n u + \lambda_n = e^u,$$  \hspace{1cm} (1.18)

then $u$ is a constant function if $\lambda_n \in (0, \lambda_*)$, where $\lambda_*$ is defined in (1.17).

**Remark 1.5.** The result in Theorem 1.4 when $n = 2$ reduces to the rigidity in [5].
Recall two $A$-trace free tensors in (1.12), set
\begin{equation}
G_u := \frac{H_u}{u} = \frac{|\nabla u|^{p-2} \nabla u \otimes \nabla u}{u} - \frac{p-1}{n} \frac{|\nabla u|^p}{u} a,
\end{equation}
that is,
\begin{equation}
\text{tr}_A[B_u] := A^j[B_u]_{ij} = A^j\left(|\nabla u|^{p-2} \nabla_j u - \Delta_p u \frac{a_{ij}}{n}\right) = 0, \quad \text{tr}_A[G_u] := A^j[G_u]_{ij} = 0.
\end{equation}

If $T_{ij}$ and $S_{ij}$ are two tensors, we use the Einstein summation convention, then
\begin{equation}
(T,S)_A := A^{ik} A^{jl} T_{ik} S_{lj}, \quad \|T\|_A^2 = [T,T]_A,
\end{equation}
in particular, when $p = 2$, $a = g$, $A = g^{-1}$ and
\begin{equation}
[T,S]_g := g^{ik} g^{jl} T_{ik} S_{lj}, \quad \|T\|_g^2 = [T,T]_g.
\end{equation}

**Lemma 2.1.** Assume $n \geq 2$, the following identity holds:
\begin{equation}
\int_M (\Delta_p u)^2 dV = \frac{n}{n-1} \int_M \|B_u\|_A^2 dV + \frac{n}{n-1} \int_M |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) dV.
\end{equation}

**Proof.** Direct computation shows that
\begin{equation}
\|B_u\|_A^2 = \left| |\nabla u|^{p-2} \nabla u - \frac{a}{n} \Delta_p u \right|_A^2 = |\nabla u|^{2p-4} \|\nabla u\|_A^2 - \frac{1}{n} (\Delta_p u)^2.
\end{equation}

In fact,
\begin{equation}
|\nabla u|^{p-2} \text{tr}_A(\nabla \nabla u) = |\nabla u|^{p-2} A^j_i \nabla_i u = \Delta_p u.
\end{equation}

Integrating $p$-Bochner formula (1.9) on $M$ yields
\begin{equation}
\int_M (\Delta_p u)^2 dV = \int_M |\nabla u|^{2p-4} \|\nabla u\|_A^2 dV + \int_M |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) dV,
\end{equation}
where we used the integration by parts,
\begin{equation}
\int_M |\nabla u|^{p-2} (\Delta_p u, \nabla u) dV = - \int_M \nabla \cdot (|\nabla u|^{p-2} \nabla u) \Delta_p u dV = - \int_M (\Delta_p u)^2 dV.
\end{equation}
Combining (2.4) with (2.5), we get (2.3).

**Lemma 2.2.** If $u$ is a positive function, then we have
\begin{equation}
\int_M \Delta_p u \frac{|\nabla u|^p}{u} dV = \frac{n(p-1)}{n(p-1)+p} \int_M \frac{|\nabla u|^{2p}}{u^2} dV - \frac{np}{(p-1)(n(p-1)+p)} \int_M [B_u, G_u]_A dV.
\end{equation}
Proof. Integration by parts implies that

\[
\int_M \Delta_p u \frac{|\nabla u|^p}{u} dV = \int_M \frac{|\nabla u|^{2p}}{u^2} dV - p \int_M \left| \nabla u \right|^{2p-4} \left[ \nabla \nabla u, \frac{\nabla u \otimes u}{u} \right]_g dV
\]

\[
= \int_M \frac{|\nabla u|^{2p}}{u^2} dV - \frac{p}{(p-1)^2} \int_M [B u, G u]_A dV - \frac{p}{n(p-1)} \int_M \Delta_p u \frac{|\nabla u|^p}{u} dV,
\]

where

\[
[B u, G u]_A = A^{ik} A^{jl} [B u]_{lj} [G u]_{ik}
\]

\[
= A^{ik} A^{jl} \left[ |\nabla u|^{p-2} \nabla \nabla u - \frac{a}{n} \Delta_p u \right]_{lj} \left[ |\nabla u|^{p-2} \nabla \nabla u - \frac{p-1}{n} |\nabla u|^p u \right]_{ik}
\]

\[
= (p-1)^2 |\nabla u|^{2p-4} \left[ \nabla \nabla u, \frac{\nabla u \otimes u}{u} \right]_g - \frac{p-1}{n} \Delta_p u \frac{|\nabla u|^p}{u}.
\]

By collecting the same items in (2.7), we obtain (2.6).

Lemma 2.3. Let \( u \) be a smooth positive solution to Equation (1.6), then the functional \( u \mapsto \int_M u^\beta dV \) remains a constant with respect to \( t \).

Proof. By computing the derivative of \( t \) and the definition of \( \kappa \) in (1.8), we have

\[
\frac{d}{dt} \int_M u^\beta dV = \beta \int_M u^{\beta-1+\beta(q-p)} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) dV
\]

\[
= \beta \int_M u^{\beta-1+\beta(q-p)} \Delta_p u dV + \beta \kappa \int_M u^{\beta+\beta(q-p)} |\nabla u|^p dV
\]

\[
= -\beta q [p-1+\beta(q-p)] \int_M u^{\beta+\beta(q-p)} |\nabla u|^p dV + \beta \kappa \int_M u^{\beta+\beta(q-p)} |\nabla u|^p dV = 0.
\]

Lemma 2.4. If \( u \) is a smooth positive solution to Equation (1.6), then

\[
\frac{1}{p^\beta} \frac{d}{dt} F[u] = - \int_M \left( \Delta_p u \right)^2 + \beta(q-1) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa (p-1) \left| \nabla u \right|^{2p} \frac{u^2}{u^2} \right] dV + \beta^{2-p} \lambda_p \int_M u^{\beta-2} |\nabla u|^p dV,
\]

where \( \kappa = p-1+\beta(q-p) \) and the functional \( F[u] \) is defined by

\[
F[u] := \int_M \left( \nabla (u^\beta) \right)^p dV + \frac{\lambda_p}{q-p} \left[ \int_M u^{\beta q} dV - \left( \int_M u^\beta dV \right)^{p/q} \right].
\]

Proof. By Equation (1.6) and integration by parts,

\[
\frac{d}{dt} \int_M \left( \nabla (u^\beta) \right)^p dV = -p \beta \int_M \Delta_p (u^\beta) u^{\beta-1+\beta(1-p)} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) dV
\]

\[
= -p \beta \int_M \left( \Delta_p u + (\beta-1)(p-1) \frac{|\nabla u|^p}{u} \right) \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) dV
\]

\[
= -p \beta \int_M \left( (\Delta_p u)^2 + \beta(q-1) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa (p-1) \frac{|\nabla u|^p}{u^2} \right) dV.
\]
and
\[
\frac{d}{dt} \int_M u^{p\beta} \, dV = p \beta \int_M u^{p\beta-1} u \, dV
\]
\[
= p \beta \int_M u^{p\beta-1} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) \, dV
\]
\[
= p \beta (\kappa - p + 1) \int_M u^{p\beta-2} |\nabla u|^p \, dV
\]
\[
= p \beta^2 (q - p) \int_M u^{p\beta-2} |\nabla u|^p \, dV.
\]
(2.10)

Combining (2.9) and (2.10) with Lemma 2.3, we finish the proof of Lemma 2.4.

For any \( \theta \in (0, 1) \), we can rewrite (2.8) as
\[
\frac{1}{p \beta p} \frac{d}{dt} F[u] = -(1 - \theta) \int_M (\Delta_p u)^2 \, dV - G[u] + \beta^{2-p} \lambda_p \int_M u^{p-2} |\nabla u|^p \, dV,
\]
(2.11)
where
\[
G[u] := \int_M \left[ \theta (\Delta_p u)^2 + \beta (q - 1) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa (\beta - 1)(p - 1) \frac{|\nabla u|^{2p}}{u^2} \right] \, dV.
\]

Let us set
\[
Q^\beta_u := B_u - \frac{p(n-1)(q-1)\beta}{2\theta[p-1][n(p-1)+p]} G_u.
\]

**Lemma 2.5.** Assume that \( n \geq 2 \), we have
\[
G[u] = \frac{\theta n}{n-1} \int_M \left[ \|Q^\beta_u\|^2_A + |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) \right] \, dV - \mu \int_M \frac{|\nabla u|^{2p}}{u^2} \, dV,
\]
(2.12)
where
\[
\mu = \frac{[p(n-1)(q-1)\beta]^2}{4\theta[p-1][n(p-1)+p]^2} - \kappa (p - 1)(\beta - 1) - \frac{n(p - 1)(q - 1)\beta}{n(p - 1) + p}.
\]
(2.13)

**Proof.** Combining Lemma 2.1 with Lemma 2.2, we have
\[
G[u] = \frac{\theta n}{n-1} \left[ \int_M \|B_u\|^2_A \, dV - \frac{p(n-1)(q-1)\beta}{\theta[p-1][n(p-1)+p]} \int_M [B_u, G_u]_A \, dV \right]
\]
\[
+ \left[ \kappa (\beta - 1)(p - 1) + \frac{n(p - 1)(q - 1)\beta}{n(p - 1) + p} \right] \int_M \frac{|\nabla u|^{2p}}{u^2} \, dV
\]
\[
+ \frac{\theta n}{n-1} \int_M |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) \, dV.
\]
(2.14)

With the above notations, one has the identities
\[
\|Q^\beta_u\|^2_A = \|B_u\|^2_A - \frac{p(n-1)(q-1)\beta}{2\theta[p-1][n(p-1)+p]} \| [B_u, G_u]_A \|^2_A
\]
\[
= \|B_u\|^2_A - \frac{p(n-1)(q-1)\beta}{\theta[p-1][n(p-1)+p]} \| [B_u, G_u]_A \|^2_A + \frac{p^2(n-1)(q-1)\beta^2}{4\theta^2[n(p-1)+p]^2} \frac{|\nabla u|^{2p}}{u^2}
\]
and

\[ \|G_u\|_A^2 = \|\nabla u|^p \frac{\nabla u \otimes \nabla u}{u} - \frac{p-1}{n} \frac{|\nabla u|^p}{u} a \|_A^2 = \frac{(n-1)(p-1)^2}{n} \frac{|\nabla u|^{2p}}{u^2}. \]

The routine calculation implies that

\[ G[u] = \frac{\theta n}{n-1} \int_M \|Q_0^u\|^2_dV - \frac{p^2(n-1)(q-1)^2}{4\theta n(p-1)+p} \int_M |\nabla u|^{2p}dV \]
\[ + \left[ \kappa(\beta-1)(p-1) + \frac{n(p-1)(q-1)}{n(p-1)+p} \right] \int_M |\nabla u|^{2p}dV \]
\[ + \frac{\theta n}{n-1} \int_M |\nabla u|^{2p-4}Ric(\nabla u, \nabla u)dV \]
\[ = \frac{\theta n}{n-1} \int_M \left[ \|Q_0^u\|^2 \right] + \|\nabla u|^{2p-4}Ric(\nabla u, \nabla u) dV - \mu \int_M \|\nabla u|^{2p}dV, \]

where \( \mu \) is defined in (2.13). Thus, we finish the proof of Lemma 2.5.

**Proposition 2.6.** Assume \( n \geq 2, q \in (1, p) \cup (p, p^*), \beta \) and \( \theta \) are given in (2.19) and (2.20). If \( u \) is a smooth positive solution to Equation (1.6), we get

\[ \frac{d}{dt} F[u] \leq p\beta^2(\lambda_p - \Lambda^*) \int_M u^q \nabla u^p dV, \quad (2.15) \]

where \( \Lambda^* \) is defined in (1.15).

**Proof.** According to (2.11) and (2.12), we obtain

\[ \frac{1}{p\beta^p} \frac{d}{dt} F[u] = - (1 - \theta) \int_M (\Delta_p u)^2 dV - \frac{\theta n}{n-1} \int_M \|Q_0^u\|^2 dV + \int_M |\nabla u|^{2p-4}Ric(\nabla u, \nabla u)dV \]
\[ + \mu \int_M \frac{|\nabla u|^{2p}}{u^2} dV + \beta^2 \sum p \int_M u^{p-2} |\nabla u|^p dV. \]

Now, using \( \kappa = p - 1 + \beta(q - p), \mu \) can be rewritten as

\[ \mu = \left[ \frac{2p(n-1)(q-1)}{4\theta n(p-1)+p} \right] \beta^2 - (p-1) \left[ 2p - q + \frac{n(q-1)}{n(p-1)+p} \right] \beta + (p-1)^2. \]

Note that, unless

\[ \frac{2p(n-1)(q-1)}{4\theta n(p-1)+p} = (p-1)(q-p), \]

the coefficient \( \mu \) is quadratic in terms of \( \beta \). Thus, \( \mu \) takes the extremum value when

\[ \beta = \frac{2\theta n(p-1)+p(p-1)[(n(p-1)+p)(2p-1-q)+n(q-1)]}{(p-1)^2 q^2 - 4\theta q(p-1)(n(p-1)+p)^2}. \]

Our goal is to choose proper constants \( \theta \) and \( \beta \) such that \( \mu = 0. \) Set \( a \) and \( b \) to be the coefficients of \( \mu \) in (2.17) and \( b = 2\sqrt{a}(p-1), \) then

\[ \mu(\beta) = a\beta^2 - b\beta + (p-1)^2 = \left( \sqrt{a}\beta - (p-1) \right)^2 = 0. \]

Thus,

\[ \beta = \frac{p-1}{\sqrt{a}} = \frac{2(p-1)^2}{b} = \frac{2(p-1)(n(p-1)+p)}{(p-1)(2p-3)n + 2p-1 - q[(p-2)n + p]}, \quad (2.19) \]
where
\[ q \neq \frac{p[(2p - 3)n + 2p - 1]}{(p - 2)n + p}. \]

Based on the discriminant \( \Delta = b^2 - 4a(p - 1)^2 = 0 \), we obtain
\[ \theta = \frac{p^2(n - 1)^2(q - 1)}{4n(p - 1)(n(p - 1) + p) + (n(p - 2) + p)^2(q - 1)}. \] (2.20)

Therefore, if we choose \( \beta \) in (2.19) and \( \theta \) in (2.20), then \( \mu = 0 \) and \( Q_\alpha u = Q_\alpha^{\mu} u \). According to the definition of \( \Lambda_* \) in (1.15) and the fact \( \mu = 0 \), we obtain the desired inequality (2.15). \( \square \)

**Proof of Theorem 1.1.** If \( v = u^\beta \) is a solution to (1.4), then \( u \) satisfies the equation
\[ \Delta_\mu u + (\beta - 1)(p - 1) \frac{|\nabla u|^p}{u} + \frac{\lambda_p}{q - p} \beta^{1-p} (u^\beta - u^{p-1}) = 0. \] (2.21)

Multiplying (2.21) by \((\Delta_\mu u + \kappa \frac{|\nabla u|^p}{u})\) and applying the fact that \( \int_M u^\kappa (\Delta_\mu u + \kappa \frac{|\nabla u|^p}{u}) \, dV = 0 \), we have
\[
0 = -\int_M \left[ \Delta_\mu u + (\beta - 1)(p - 1) \frac{|\nabla u|^p}{u} + \frac{\lambda_p}{q - p} \beta^{1-p} (u^\beta - u^{p-1}) \right] \cdot \left[ \Delta_\mu u + \kappa \frac{|\nabla u|^p}{u} \right] \, dV
\leq \frac{1}{p_\beta} \frac{d}{dt} F[u]
\leq \beta^{2-p} (\lambda_p - \Lambda_*) \int_M |\nabla u|^p \, dV,
\]
where we use the results of Lemma 2.4 and Proposition 2.6. Thus, \( u \) is a constant if \( \lambda_p < \Lambda_* \). \( \square \)

So far, we have only obtained a rigidity result with the power law nonlinearity; in fact, we can extend our result to the case of general nonlinearities.

**Theorem 2.7.** Let \( f \) be a Lipschitz increasing function such that
\[ \frac{1}{q - p} \left[ f'(v) - (q - 1) \frac{f(v)}{v} \right] \leq 0, \quad \forall v > 0. \] (2.23)

Assume that \( \Lambda_* > 0 \). For any \( q \in (1, p) \cup (p, p^*) \), if \( \lambda_f \in (0, \Lambda_*) \), then the equation
\[ -\Delta_\mu v + \frac{\lambda_f}{q - p} (v^{p-1} - f(v)) = 0 \] (2.24)
has a unique positive constant solution \( c \), which satisfies \( f(c) = c^{p-1} \).

**Proof.** If \( v = u^\beta \) is a solution to (2.24), then \( u \) satisfies equation
\[ \Delta_\mu u + (\beta - 1)(p - 1) \frac{|\nabla u|^p}{u} - \frac{\beta^{1-p} \lambda_f}{q - p} \left( u^{p-1} - \frac{f(v)}{v} v^{p-1} + \beta^{2-p} \right) = 0. \]
Multiplying this equation by \( \left( \Delta_p u + \kappa \frac{|Vu|^p}{u} \right) \) and integrating by parts, using an analogous calculation in (2.22), we get

\[
0 = -\int_M \left[ \Delta_p u + (\beta - 1)(p - 1) \frac{|Vu|^p}{u} - \frac{\beta - 1}{q - p} \left( u^{p-1} - \frac{f(v)}{v} u^{p-1+\beta(2-p)} \right) \right] u \, dV \\
= -\int_M \left[ (\Delta_p u)^2 + \beta(q - 1) \Delta_p u \frac{|Vu|^p}{u} + \kappa(\beta - 1)(p - 1) |Vu|^{2p} \right] dV + \beta^2 - 1 \frac{\beta}{q - p} \int_M u^{p-2} |Vu|^p \, dV \\
+ \frac{\lambda_f}{q - p} \beta^2 - 1 \int_M \left( f'(v) - (q - 1) \frac{f(v)}{v} \right) u^{p-2} |Vu|^p \, dV \\
\leq (\lambda_f - \Lambda_\ast) \beta^2 - 1 \int_M u^{p-2} |Vu|^p \, dV,
\]

where we use (2.23) and Proposition 2.6 and the following computation,

\[
\int_M \frac{f(v)}{v} u^{p-1+\beta(2-p)} \left( \Delta_p u + \kappa \frac{|Vu|^p}{u} \right) dV \\
= \int_M f(v) u^{p-1(1-\beta)} |Vu| u dV + \kappa \int_M \frac{f(v)}{v} u^{p-2(1-\beta)} |Vu|^p dV \\
= -(p - 1)(1 - \beta) \int_M f(v) u^{p-1(1-\beta)} |Vu|^p dV - \beta \int_M f'(v) u^{p-2(1-\beta)} |Vu|^p dV + \kappa \int_M \frac{f(v)}{v} u^{p-2(1-\beta)} |Vu|^p dV \\
= -\beta \int_M \left( f'(v) - (q - 1) \frac{f(v)}{v} \right) u^{p-2(1-\beta)} |Vu|^p dV,
\]

where \( \kappa = p - 1 + \beta(q - p) \). Therefore, if \( \lambda_f < \Lambda_\ast \), \( u \) is a constant. \( \square \)

**Proof of Theorem 1.3.** According to (2.15) in Proposition 2.6, we know the functional \( F[u] \) is nonincreasing, if \( \lambda_p \leq \Lambda_\ast \).

Integrating on both sides of (2.15) implies that

\[
F[u(t, \cdot)] \leq F[u(t, \cdot)] \leq \int_0^\infty \left[ \int_M u^{p-2} |Vu|^p dV \right] dt.
\]

if \( \lambda_p < \Lambda_\ast \), we obtain \( \int_0^\infty \int_M u^{p-2} |Vu|^p dV dt \) is finite. Therefore, when \( t \to \infty \), \( \nabla u \) converges to 0 and \( u(t, \cdot) \) converges to a constant. Thus, for ant \( t \geq 0 \), there holds

\[
F[u(t = 0, \cdot)] \geq F[u(t, \cdot)] \geq \lim_{t \to \infty} F[u(t, \cdot)] = 0.
\]

Hence, we have

\[
\int_M |\nabla u|^p dV + \frac{\lambda_p}{q - p} \left[ \int_M u^p dV - \left( \int_M u^p dV \right)^{p/q} \right] \geq 0.
\]

Taking \( v = u^p \), we obtain the interpolation inequality (1.16)

\[
\|\nabla v\|_{L^p(M)}^p \geq \frac{\lambda_p}{q - p} \left( \|v\|_{L^p(M)}^p - \|v\|_{L^p(M)}^p \right).
\]

When \( q \to p \), by L’Hopital’s rule, we can get a \( L^p \)-Log-Sobolev inequality,

\[
\int_M |\nabla v|^p dV \geq \lambda_p \left[ \int_M v^p \log v dV - \frac{1}{p} \int_M v^p dV \log \int_M v^p dV \right].
\]

\( \square \)
3 | PROOF OF THEOREM 1.4

Lemma 3.1. Let $p, q \geq 1$, then we have

$$\int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV = \frac{n(p-1)}{q(n(p-1)+1)} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV - \frac{pn}{(p-1)(n(p-1)+1)} \int_M [B_u, H_u]_A e^{-\frac{u}{q}} dV, \quad (3.1)$$

where

$$[B_u, H_u]_A := A^{ik}A^{jl}[B_u]_{ij}[H_u]_{kl} - \frac{p-1}{n} \Delta_p u |\nabla u|^p,$$

$$= (p-1)^2 \left[ w^{p-1} \nabla u, w^{p-1} \nabla u \otimes \nabla u \right]_A - \frac{p-1}{n} \Delta_p u |\nabla u|^p. \quad (3.2)$$

Proof. Integrating by parts shows that

$$- \int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV$$

$$= -\frac{1}{q} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV + \frac{p}{n} \int_M |\nabla u|^{p-2} |\nabla \nabla u, \nabla u \otimes \nabla u| |\nabla u|^{p-2} e^{-\frac{u}{q}} dV$$

$$= -\frac{1}{q} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV + \frac{p}{n} \int_M \left( B_u + \frac{\Delta_p u}{n} \right) |\nabla u|^{p-2} e^{-\frac{u}{q}} dV$$

$$= -\frac{1}{q} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV + \frac{p}{(p-1)^2} \int_M [B_u, H_u]_A e^{-\frac{u}{q}} dV + \frac{p}{n(p-1)} \int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV,$$

where $H_u$ is also a trace free quantity defined in (1.12). Thus,

$$\frac{n(p-1)}{n(p-1)} \int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV = \frac{1}{q} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV - \frac{p}{(p-1)^2} \int_M [B_u, H_u]_A e^{-\frac{u}{q}} dV.$$

By a direct computation, one can get (3.1).

□

Lemma 3.2. We have the following identity:

$$\int_M (\Delta_p u)^2 e^{-\frac{u}{q}} dV = \frac{2p-1}{pq} \int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV - \frac{p-1}{pq^2} \frac{n}{n-1} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV$$

$$+ \frac{n}{n-1} \int_M (\|B_u\|_A^2 + \text{Ric}(\nabla u, \nabla u)) e^{-\frac{u}{q}} dV. \quad (3.3)$$

where $\text{Ric}(\nabla u, \nabla u) := |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u)$.

Proof. By the definition of $\mathcal{L}(\psi)$, we have

$$\mathcal{L}(e^{-\frac{\psi}{q}}) = \frac{1}{q} \left[ \frac{p-1}{q} |\nabla u|^p - \mathcal{L}(u) \right] e^{-\frac{\psi}{q}} = \frac{p-1}{q} \left[ \frac{1}{q} |\nabla u|^p - \Delta_p u \right] e^{-\frac{\psi}{q}}, \quad (3.4)$$

where we use the fact $\mathcal{L}(u) = (p-1) \Delta_p u$. Multiplying $|\nabla u|^p$ on both sides of (3.4) and integrating on $M,$

$$\int_M |\nabla u|^p \mathcal{L}(e^{-\frac{\psi}{q}}) dV = \frac{p-1}{q^2} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV - \frac{p-1}{q} \int_M \Delta_p u |\nabla u|^p e^{-\frac{u}{q}} dV.$$
Using $p$-Bochner formula (1.9) and integrating by parts lead to

$$
\int_M \frac{p - 1}{q} \left\| \nabla u \right\|^2 \left( e^{-\frac{u}{q}} - \frac{p - 1}{q} e^{-\frac{u}{q}} \right) dV
$$

$$
= \int_M \left\| \nabla u \right\|^2 (\frac{p - 1}{q}) u e^{-\frac{u}{q}} dV + \int_M \frac{p - 1}{q} \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV
$$

$$
= \int_M \frac{p - 1}{q} \left\| B u \right\| e^{-\frac{u}{q}} dV - \frac{n - 1}{n} \int_M \left\| \Delta p u \right\|^2 e^{-\frac{u}{q}} dV + \int_M \frac{p - 1}{q} \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV.
$$

By observing, we can know that the lemma is a direct consequence of above identity.

**Lemma 3.3.** If $u$ is a solution to Equation (1.5), we have

$$
\int_M (\Delta p u)^2 e^{-\frac{u}{q}} dV + \frac{q - 1}{q} \int_M \Delta p u \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV - p \lambda_n \int_M \left\| \nabla u \right\|^p e^{-\frac{u}{q}} dV = 0.
$$

**Proof.** According to the previous calculations, multiplying (3.4) by $-\frac{1}{p} \Delta p u$ shows that

$$
- \frac{1}{p} \int_M (\Delta p u) \mathcal{L} \left( e^{-\frac{u}{q}} \right) dV = \frac{p - 1}{pq} \int_M (\Delta p u)^2 e^{-\frac{u}{q}} dV - \frac{p - 1}{pq^2} \int_M \Delta p u \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV.
$$

On the other hand, multiplying both sides of Equation (1.5) by $\mathcal{L} \left( e^{-\frac{u}{q}} \right) - \frac{p - 1}{q} \left\| \nabla u \right\|^2 e^{-\frac{u}{q}}$, we can get

$$
\frac{p - 1}{pq} \int_M (\Delta p u)^2 e^{-\frac{u}{q}} dV + \frac{(p - 1)(1 - q)}{q^2} \int_M (\Delta p u) \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV + \lambda_n \left[ \frac{p - 1}{pq} \int_M \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV - \frac{p - 1}{pq} \int_M \Delta p u e^{-\frac{u}{q}} dV \right]
$$

$$
= \frac{(p - 1)(1 - q)}{q^2} \int_M \left\| \nabla u \right\|^2 e^{-\frac{u}{q}} dV - \frac{p - 1}{q} \int_M \Delta p u e^{-\frac{u}{q}} dV.
$$

By integrating by parts, the formula (3.5) is proved.

**Proposition 3.4.** If $u$ is a solution to Equation (1.5), we get

$$
\int_M \left\| B u + \frac{b}{2} \nabla u \right\|^2 e^{-\frac{u}{q}} dV + \left( c - \frac{b^2}{4} \right) \int_M \left\| H u \right\|^2 e^{-\frac{u}{q}} dV + \int_M \operatorname{Ric}(\nabla p u, \nabla p u) e^{-\frac{u}{q}} dV = \frac{p - 1}{q} \lambda_n \int_M \left\| \nabla u \right\|^p e^{-\frac{u}{q}} dV.
$$

(3.6)

where $b$ and $c$ are constants defined by

$$
b = -\frac{n - 1}{p - 1} \left[ \frac{p}{n(p - 1) + p} + \frac{1}{q(n - 1)} \right], \quad c = \frac{n}{q(p - 1)(n(p - 1) + p)}.
$$

(3.7)

**Proof.** Combining (3.5) and (3.3), we have

$$
\int_M \left\| B u + \frac{b}{2} \nabla u \right\|^2 e^{-\frac{u}{q}} dV + \left( c - \frac{b^2}{4} \right) \int_M \left\| H u \right\|^2 e^{-\frac{u}{q}} dV + \int_M \operatorname{Ric}(\nabla p u, \nabla p u) e^{-\frac{u}{q}} dV = \frac{p - 1}{q} \lambda_n \int_M \left\| \nabla u \right\|^p e^{-\frac{u}{q}} dV.
$$

(3.8)
Plugging (3.1) into (3.8) implies
\[
\frac{n}{n-1} \int_M \left( |B_u|_A^2 + \text{Ric}(\nabla_p u, \nabla_p u) \right) e^{-\frac{u}{q}} dV + \frac{n(p-1)}{q(n-1)+p} \int_M |\nabla u|^{2p} e^{-\frac{u}{q}} dV
\]
\[
- \frac{n}{p-1} \left[ \frac{p}{n(p-1)+p} + \frac{1}{q(n-1)} \right] \int_M [B_u, H_u]_A e^{-\frac{u}{q}} dV = p \lambda_n \int_M |\nabla u|^{p} e^{-\frac{u}{q}} dV.
\]
(3.9)

Note that the definition in (1.12)
\[
\|H_u\|^2_A = (p-1)^2 \left( 1 - \frac{1}{n} \right) |\nabla u|^{2p}.
\]
(3.10)

Substituting (3.10) into (3.9), we get
\[
\int_M \left( |B_u|_A^2 + b[B_u, H_u]_A + c\|H_u\|^2_A + \text{Ric}(\nabla_p u, \nabla_p u) \right) e^{-\frac{u}{q}} dV = \frac{p(n-1)}{n} \lambda_n \int_M |\nabla u|^{p} e^{-\frac{u}{q}} dV.
\]
(3.11)

where \([B_u, H_u]_A\) is defined in (3.2), and \(b\) and \(c\) are given by (3.7). Rewriting (3.11), we can obtain (3.6). □

**Proof of Theorem 1.4.** According to the expressions of \(b\) and \(c\) in (3.7), when \(p = n\), a direct computation implies
\[
c - \frac{1}{4} b^2 = -\frac{1}{4} \left( 1 - \frac{1}{n} - \frac{1}{q(n-1)} \right)^2.
\]

In particular, if \(q = \frac{n}{n-1}\), it follows from (3.6) that
\[
0 = \int_M \left( \frac{1}{n} H_u \right)_A^2 e^{-\frac{u}{q}} dV + \int_M \text{Ric}(\nabla_p u, \nabla_p u) e^{-\frac{u}{q}} dV - \frac{p-1}{q} \lambda \int_M |\nabla u|^{p} e^{-\frac{u}{q}} dV
\]
\[
\geq (n-1)(\lambda_* - \lambda_n) \int_M |\nabla u|^{n} e^{-\frac{u}{q}} dV,
\]

where we use the definition of \(\lambda_*\) in (1.17). Hence, we have proved that \(\nabla u \equiv 0\) for any \(\lambda < \lambda_*\); that is, \(u\) is a constant. Thus, the proof of Theorem 1.4 is finished. □

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**CONFLICT OF INTEREST STATEMENT**

This work does not have any conflicts of interest.

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