Symmetries of Heterotic String Effective Theory in Three and Two Dimensions

D.V. Gal’tsov

Department of Theoretical Physics
Moscow State University
Moscow 119899
Russia

Abstract. The four-dimensional bosonic effective action of the toroidally compactified heterotic string incorporating a dilaton, an axion and one $U(1)$ vector field is studied on curved space-time manifolds with one and two commuting Killing vectors. In the first case the theory is reduced to a three-dimensional sigma model possessing a symmetric pseudo-riemannian target space isomorphic to the coset $SO(2,3)/(SO(3) \times SO(2))$. The ten-parameter group $SO(2,3)$ of target space isometries contains embedded both $S$ and $T$ classical duality symmetries of the heterotic string. With one more ignorable coordinate, the theory reduces to a two-dimensional chiral model built on the above coset, and therefore belongs to the class of completely integrable systems. This entails infinite-dimensional symmetries of the Geroch–Kinnersley–Chitre type. Purely dilatonic theory is shown to be two-dimensionally integrable only for two particular values of the dilaton coupling constant. In the static case (diagonal metrics) both theories essentially coincide; in this case the integrability property holds for all values of the dilaton coupling.

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1. Introduction

A well-known property of the Einstein equations in General Relativity, crucial for
the opportunity to find many nontrivial exact solutions to this theory, is two-
dimensional integrability, which is manifest when the theory is restricted to space-
times possessing two commuting Killing vector fields [1]. Because of this property,
in principle all solutions with two ignorable coordinates can be found using various
solution-generating methods. Although the problem of constructing solutions hav-
ing desired physical properties remains rather complicated technically, with modern
mathematical tools it is becoming more and more tractable. As a matter of fact, the
existing knowledge of sufficiently general classes of solutions is essentially related to
this property; other methods to solve highly nonlinear Einstein equations are either
restricted to more symmetric cases, or able to produce only some particular types
of solutions [2].

It has been observed that a similar integrability property is shared by other theo-
ries related to General Relativity: Einstein–Maxwell (EM) theory [3], higher dimen-
sional vacuum Einstein equations [4], bosonic sectors of some dimensionally reduced
supergravity theories [5], [6], [7]. The crucial feature ensuring two-dimensional in-
tegrability is the symmetric-space property of the corresponding sigma model in
three dimensions (for a review see, e.g., Breitenlohner and Maison [6]). Both Ein-
stein and massless vector field equations, being reduced to three dimensions via an
imposition of a non-null space-time Killing symmetry, may be described in terms
of gravity coupled sigma models. If additional scalar fields present do not destroy
this property, the main question is whether the target space is a space of a constant
Riemann curvature. If yes, the equations of motion may be cast into the form of
modified chiral matrix equations. The corresponding Lax pair is readily given when
the theory is further reduced to two dimensions. The symmetric-space property
in its turn requires that the target space possess a sufficient number of isometries.
This convenient way to investigate the integrability property is basically due to the
idea of potential space introduced long ago by Neugebauer and Kramer [8].

It seems likely that the description of gravity at ultramicroscopic distances is in the
scope of string theory, the most promising being the heterotic string model. In the
low-energy (classical) limit of this theory with compactified extra dimensions one
gets in the bosonic sector an Einstein gravity coupled to massless vector and scalar
fields. If initially formulated on a $D$-dimensional curved space-time manifold pos-
sessing $d$ Abelian isometries with $p$ initial $U(1)$ vector fields present, such a theory is
known to possess global invariance under an $SO(d, d+p)$ group called $T$-duality [9],
[10]. In four dimensions there is another important symmetry, $S$-duality [11], whose
occurrence is related to the possibility of representing a Kalb–Ramond field by a
scalar Peccei–Quinn axion. Classical $S$-duality is the $SL(2, R)$ group containing electric-magnetic continuous rotations as well as the dilaton transformation ensuring a connection between weak and strong coupling regimes of the string theory. It is conjectured that discrete subgroups of these dualities are exact symmetries of the heterotic string.

The simplest model of this kind, often called dilaton-axion gravity, which incorporates basic features of the full effective action, can be formulated directly in $D = 4$ where it includes one $U(1)$ vector ($p = 1$) and two scalar fields coupled in a way prescribed by the heterotic string effective theory:

$$S = \frac{1}{16\pi} \int \left\{ -R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa + e^{-2\phi} F_{\mu\nu} F^{\mu\nu} + \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \sqrt{-g} \, d^4 x. \tag{1.1}$$

Here $F = dA$, $\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$, and $\phi$, $\kappa$ are dilaton and axion fields. This model may be regarded as the Einstein–Maxwell theory coupled to dilaton and axion (EMDA theory). The purpose of my talk is to show that this theory possesses the same integrability property as pure EM theory, while the underlying algebraic structure is rather different. The investigation has two goals. Firstly, the study of further dimensional reduction to three and two dimensions may reveal new hidden symmetries of the heterotic string. In fact, for $d = p = 1$, $T$-duality is $SO(1, 2)$, and together with $SL(2, R)$ $S$-duality this is insufficient to ensure the desired symmetric-space property of the target space. Hence, the question of existence of such a property is closely related to the possibility of embedding of $T$ and $S$ dualities into a larger group. Secondly, if it exists, the integrability property could be very helpful in constructing exact classical solutions to the action (1.1), which is important both for extracting gravitational predictions of the string theory, and for a search of new quantum four-dimensional string models on $D = 4$ curved space-time manifolds. Some of the results reported here were obtained in collaboration with O.V. Kechkin [12].

2. Potential space

Both three-dimensional (finite) and two-dimensional (infinite) symmetries are closely related for the class of two-dimensionally integrable systems, so we start with reducing (1.1) to three dimensions, imposing a non-null Killing symmetry on the $D = 4$ manifold. In the case of a timelike Killing vector, the four-dimensional metric can be presented in terms of a three-metric $h_{ij}$, a rotation one-form $\omega_i$ ($i, j = 1, 2, 3$), and a three-dimensional conformal factor $f$ depending only on the 3-space coordinates $x^i$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(dt - \omega_i dx^i)^2 - \frac{1}{f} h_{ij} dx^i dx^j. \tag{2.1}$$
The subsequent derivation of the three-dimensional $\sigma$ model is rather standard and mainly follows Israel and Wilson [13]. A four-dimensional vector field in three dimensions is represented by two scalars. In our case of timelike Killing symmetry they have the meaning of electric ($v$) and magnetic ($u$) potentials and can be introduced via relations

$$ F_{i0} = \frac{1}{\sqrt{2}} \partial_i v, \quad (2.2) $$

$$ e^{-2\phi} F^{ij} + \kappa \tilde{F}^{ij} = \frac{f}{\sqrt{2h}} \epsilon^{ijk} \partial_k u, \quad (2.3) $$

solving the spatial part of the modified Maxwell equations and Bianchi identity. The rotation one-form $\omega_i$ in (2.1) also reduces to a scalar — the twist potential $\chi$:

$$ \tau^i = -f^2 \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_j \omega_k, \quad \tau_i = \partial_i \chi + v \partial_i u - u \partial_i v. \quad (2.4) $$

Therefore in three dimensions we have three pairs of scalar variables: one ($f, \chi$) is inherited from the four-dimensional metric, the second ($v, u$) from the vector field, and the last is the dilaton-axion pair ($\phi, \kappa$). A typical feature of the four-dimensional gravitationally coupled system of $U(1)$ vector fields (possibly interacting with scalars representing some coset) is that in three dimensions they form a gravity coupled sigma model; well-known examples are provided by the Einstein–Maxwell system [8] and multidimensional vacuum Einstein equations compactified to four dimensions [4]. In our case it can be checked directly that the equations for $f, \chi, v, u, \phi, \kappa$ may be obtained by variation of the following three-dimensional action:

$$ S = \int \left\{ \mathcal{R} - \frac{1}{2f^2} [(\nabla f)^2 + (\nabla \chi + v \nabla u - u \nabla v)^2] - 2(\nabla \Phi)^2 \right. $$

$$ \left. - \frac{1}{2} e^{4\phi} (\nabla \kappa)^2 + \frac{1}{f} \left[ e^{2\phi} (\nabla u - \kappa \nabla v)^2 + e^{-2\phi} (\nabla v)^2 \right] \right\} \sqrt{h} \, d^3 x, \quad (2.5) $$

where $\mathcal{R} \equiv \mathcal{R}^i_i, \mathcal{R}^{ij}$ is the three-dimensional Ricci tensor, and $\nabla$ stands for the 3-dimensional covariant derivative. This action can be rewritten as the gravity-coupled three-dimensional $\sigma$ model

$$ S = \int (\mathcal{R} - \mathcal{G}_{AB} \partial_i \phi^A \partial_j \phi^B h^{ij}) \sqrt{h} \, d^3 x, \quad (2.6) $$
where \( \phi^A = (f, \chi, v, u, \kappa, \phi) \), \( A = 1, \ldots, 6 \). The corresponding target space metric reads

\[
dl^2 = G_{AB} d\phi^A d\phi^B = \frac{1}{2} f^{-2} [df^2 + (d\chi + vdu - udv)^2] - f^{-1} [e^{2\phi}(du - \kappa dv)^2 + e^{-2\phi} dv^2] + 2d\phi^2 + \frac{1}{2} e^{4\phi} d\kappa^2.
\]

(2.7)

This is a six-dimensional pseudoriemannian space with indefinite metric, which generalizes Neugebauer and Kramer’s potential space found for the stationary Einstein–Maxwell system in 1969 [8]. It is worth noting that the present \( \sigma \) model does not reduce to the Einstein–Maxwell one when \( \kappa = \phi = 0 \), because the equations for a dilaton and an axion generate constraints \( F^2 = F\tilde{F} = 0 \). Hence, generally the solutions of the Einstein–Maxwell theory with one Killing symmetry are not related to solutions of the present theory by target-space isometries (except for the case \( F^2 = F\tilde{F} = 0 \)). On the other hand, one can consistently set in the present \( \sigma \) model \( v = u = \phi = \kappa = 0 \), reducing it to the Einstein vacuum sigma model. Therefore all solutions to the vacuum Einstein equations with one non-null Killing symmetry are related to some solutions of the system in question by target space isometries. This fact was used in [12] to generate the most general family of black hole solutions to dilaton-axion gravity.

In connection with other symmetric gravity-coupled systems another model is worth discussing: Einstein–Maxwell–dilaton (EMD) theory described by the action

\[
S = \frac{1}{16\pi} \int \left\{ -R + 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2 \right\} \sqrt{-g} d^4x.
\]

(2.8)

It can be regarded as a truncated version of (2.1) without an axion field (although more general in the sense of arbitrariness of the dilaton coupling constant \( \alpha \)). The model is interesting both in the context of string theory, and also as an interpolation between two theories enjoying a two-dimensional integrability property. For \( \alpha = 0 \) the action (2.8) describes the EM system coupled to a scalar field, which is equivalent to the Brans–Dicke–Maxwell (BDM) theory with the Brans–Dicke constant \( \omega = -1 \). For \( \alpha = \sqrt{3} \) it corresponds to the five-dimensional Kaluza–Klein (KK) theory in the Einstein frame. In the context of string theory another particular value of \( \alpha \) is relevant: \( \alpha = 1 \). It was observed recently by Horowitz [14] that in this case some nontrivial transformations of the Harrison type [15] can be applied to the Schwarzschild metric to generate dilaton black holes [16] (see also [17]).
Repeating the above reasoning one finds the following potential-space metric (now five-dimensional):

\[ dl^2 = \frac{df^2 + (d\chi + vdu - udv)^2}{2f^2} - \frac{e^{-2\alpha\phi} dv^2 + e^{2\alpha\phi} du^2}{f} + 2d\phi^2. \]

(2.9)

Essentially the same potential space was studied long ago by Neugebauer [18].

An important distinction between these two target spaces is the nondiagonal structure of the electromagnetic sector in (2.7) in contrast to the diagonal one in (2.9). Physically this is due to the mixing of electric and magnetic components of the vector field by an axion; obviously the electric-magnetic duality associated with (2.7) should be non-Abelian. Similar duality for (2.9) turns out to hold only if the dilaton is decoupled (i.e., for \( \alpha = 0 \)), and it is Abelian in that case. Electric-magnetic duality for (2.7) is just the notable \( S \)-duality \( SL(2, R) \) [11] of the string theory. The question naturally arises, which isometries of (2.7) correspond to another symmetry of the toroidally compactified heterotic string — \( T \)-duality [10], [9], which in the present context (\( p = d = 1 \)) should be \( SO(1, 2) \) (i.e., have the same structure as \( S \)-duality). As we have shown recently [12], this symmetry is intimately related to the Ehlers–Harrison-type transformations associated with the action (1.1), which generalize in some nontrivial way the original Ehlers [19] and Harrison [15] transformations known in General Relativity.

As we will see [12], both \( T \) and \( S \) dualities are actually embedded into a larger ten-parameter isometry group of (2.7). This indicates that string duality symmetries are likely to be enhanced when the theory is reduced to three dimensions. Moreover, this embedding ensures that the target space (2.7) is a symmetric Riemannian space, and consequently the symmetries will be infinitely enhanced when the theory is further reduced to two dimensions [25]. Contrary to this, it turns out that the EMD target space (2.9) generally does not possess enough isometries to ensure such properties, unless \( \alpha = 0 \) (which is the BDM case) or \( \alpha = \sqrt{3} \) (KK case); in both these critical cases one has two-dimensional integrability. Thus the role of the axion field in the action (2.1) (which corresponds to \( \alpha = 1 \)) is very nontrivial in generating more hidden symmetries.

Between EMDA and EMD models there is another interesting link. If the space-time (2.2) is static, i.e., \( \omega_i = 0 \), then it is legitimate to consider in (2.9) purely electric \((u = 0)\) and purely magnetic \((v = 0)\) subspaces independently. At the same time, in this case it is consistent to set the axion field in (2.7) to zero provided we deal with either purely electric or purely magnetic fields. Then static truncation of the EMD action will be more general (because \( \alpha \) is arbitrary). Meanwhile, since static
no-axion truncation of (2.7) is inherited from a symmetric space and it corresponds to the noncritical \( \alpha \) in (2.9), this indicates that in the static case the EMD theory should be two-dimensionally integrable for (at least one) noncritical dilaton coupling constant value (as we will see, indeed for all \( \alpha \)). This explains the occurrence of the Harrison transformations in the diagonal dilaton gravity [14].

3. Unification of \( T \) and \( S \) dualities in three dimensions

Once the metric of the potential space is found, the isometries can be explored by solving the Killing equations

\[
K_{A:B} + K_{B:A} = 0, \tag{3.1}
\]

where covariant derivatives refer to the target space metric (2.7). It turns out that there are ten independent Killing vectors fields for this metric. We list them in the order of increasing complexity. The simplest symmetry is just \( \chi \)-translation, \( \chi = \chi_0 + g \), \( g \) being a real parameter, which physically is pure (gravitational) gauge (metric remains unchanged):

\[
K_g = \partial_\chi. \tag{3.2}
\]

Similarly, there are two translations of electric and magnetic potentials, accompanied by suitable transformations of the axion:

\[
v = v_0 + e, \quad \chi = \chi_0 - u_0 e, \tag{3.3}
\]

\[
u = u_0 + m, \quad \chi = \chi_0 + v_0 m, \tag{3.4}
\]

(with \( e \) and \( m \) real parameters), the corresponding generators being

\[
K_e = \partial_v - u \partial_\chi, \tag{3.5}
\]

\[
K_m = \partial_u + v \partial_\chi. \tag{3.6}
\]

They are also pure gauge (electric and magnetic). Also rather simple is the scale transformation which can be expected from power counting in (2.7):

\[
f = f_0 e^{2s}, \quad \chi = \chi_0 e^{2s}, \quad v = v_0 e^s, \quad u = u_0 e^s. \tag{3.7}
\]

It leaves \( \kappa \) and \( \phi \) unchanged so that

\[
K_s = 2f \partial_f + 2\chi \partial_\chi + v \partial_v + u \partial_u. \tag{3.8}
\]
Three Killing vectors correspond to the dilaton-axion S-duality subalgebra, including two rotations,

\[ K_{d1} = \partial_\kappa + v \partial_u , \]  
\[ K_{d2} = (e^{-4\phi} - \kappa^2)\partial_\kappa + \kappa \partial_\phi + u \partial_v , \]  

and a dilaton shift accompanied by suitable rescaling of other variables,

\[ K_{d3} = \partial_\phi - 2\kappa \partial_\kappa + v \partial_v - u \partial_u . \]  

These three satisfy \( sl(2, R) \) commutation relations

\[ [K_{d3}, K_{d1}] = 2K_{d1}, \quad [K_{d3}, K_{d2}] = -2K_{d2}, \quad [K_{d1}, K_{d2}] = K_{d3}. \]  

The corresponding finite transformations read

\[ v = v_0, \quad u = u_0 + v_0 d_1, \quad z = z_0 + d_1, \]  
\[ u = u_0, \quad v = v_0 + u_0 d_2, \quad z^{-1} = z_0^{-1} + d_2, \]  
\[ z = e^{-2d_3} z_0, \quad u = u_0 e^{-d_3}, \quad v = v_0 e^{d_3}, \]  

where \( z = \kappa + i e^{-2\phi} \) is the complex dilaton-axion field (\( s, d_1, d_2, d_3 \) are real parameters).

A nontrivial part of the isometry group includes a conjugate pair of “charging” transformations (identified in [12] as Harrison-type transformations by exploring their action on solutions of the vacuum Einstein’s equations):

\[ K_{H1} = 2v f \partial_f + v \partial_\phi + 2w \partial_\kappa + (v^2 + fe^{2\phi}) \partial_v \]  
\[ + (\chi + uv + \kappa fe^{2\phi}) \partial_u + (v \chi + w fe^{2\phi}) \partial_\chi , \]  
\[ K_{H2} = 2u f \partial_f + (\kappa v - w) \partial_\phi + 2(\kappa w + ve^{-4\phi}) \partial_\kappa + (uv - \chi + \kappa fe^{2\phi}) \partial_v \]  
\[ + (u^2 + fe^{-2\phi} + \kappa^2 fe^{2\phi}) \partial_u + (u \chi - v fe^{-2\phi} + \kappa w fe^{2\phi}) \partial_\chi , \]  

where \( w = u - \kappa v \). The corresponding finite transformations are more tricky. The first (electric) leaves invariant the quantities

\[ fe^{-2\phi} \equiv f_0 e^{-2\phi_0}, \quad \bar{\chi} = \chi - wv \equiv \chi_0 - w_0 v_0, \]  

while the other variables transform as

\[ w = w_0 + \bar{\chi}_0 h_1, \quad \kappa = \kappa_0 + 2w_0 h_1 + \bar{\chi}_0 h_1^2, \]
\[
(\sqrt{f}e^\phi \pm v)^{-1} = (\sqrt{f_0}e^{\phi_0} \pm v_0)^{-1} \mp h_1.
\] (3.19)

The second (magnetic) also leaves two combinations invariant:
\[
q = f^{-1/2}|z|e^\phi \equiv f_0^{-1/2}|z_0|e^{\phi_0}, \quad p = f^{-1}u^+u^- \equiv f_0^{-1}u_0^+u_0^-,
\] (3.20)

where
\[
u = f^{-1}v \equiv f_0^{-1}v_0.
\] (3.21)

while the other transformations read
\[
\chi = k_+u^+ + k_-u^- + kqf, \quad v = k_+\frac{u^+}{u^-} + k_-\frac{u^-}{u^+},
\] (3.22)

where \(u^\pm = u \pm qf \equiv \frac{u_0^\pm}{1 - h_2u_0^\pm},\)

\[
k_\pm = \frac{u_0^\pm}{2u_0^\pm} \left( v_0 \pm \frac{\kappa_0 f_0 e^{2\phi_0} - \chi_1}{2qf_0} \right), \quad k = \frac{\kappa_0 f_0 e^{2\phi_0} + \chi_1}{2qf_0},
\] (3.23)

The last, Ehlers-type [19] generator, which closes the full isometry algebra,
\[
[K_{H_1}, K_{H_2}] = 2K_E,
\] (3.24)

reads
\[
K_E = 2f \chi \partial_f + wv \partial_\phi + (w^2 - v^2 e^{-4\phi}) \partial_\kappa + (v\chi + w e^{2\phi}) \partial_\nu
\] (3.25)

\[
+ (u\chi - v e^{-2\phi} + k\nu f e^{2\phi}) \partial_u + (\chi^2 - f^2 + v^2 e^{-2\phi} + f^2 e^{2\phi}) \partial_\chi.
\]

The corresponding finite transformation has three real invariants,
\[
e^{2\phi} - v^2 f^{-1} \equiv e^{2\phi_0} - v_0^2 f_0^{-1}, \quad 1 - \beta = f^{-1} |\Phi|^2 e^{2\phi} \equiv f_0^{-1} |\Phi_0|^2 e^{2\phi_0},
\] (3.26)

\[
\gamma = f^{-1}(\chi^2 + \beta f^2) \equiv f_0^{-1}(\chi_0^2 + \beta f_0^2),
\]

and one complex one,
\[
\nu = v + (if - \chi) \Phi^{-1} \equiv v_0 + (if_0 - \chi_0) \Phi_0^{-1},
\] (3.27)

where \(\Phi = u - zv,\) and
\[
f = \chi \xi^{-1} = \gamma (\beta + \xi^2)^{-1}, \quad \xi = \chi_0 f_0^{-1} - h_3 \gamma,
\] (3.28)
\[ \Phi^{-1} = \Phi_0^{-1} + \nu h_3, \quad z = z_0 - \nu^{-1}(\Phi - \Phi_0). \] (3.27)

(In these formulas \( h_1, h_2, h_3 \) are the last real group parameters).

Altogether these Killing vectors form a closed ten-dimensional algebra. Introducing instead of \( K_s \) and \( K_{d_3} \) two linear combinations \( K_1 = (K_{d_5} - K_s)/2, \quad K_2 = -(K_{d_3} + K_s)/2 \), and enumerating vectors \( K_m, K_e, K_g, K_{d_1}, K_{d_2}, K_{H_1}, K_{H_2}, K_E \) as \( K_3, K_4, \ldots, K_9, K_0 \) respectively, we can write the commutation relations

\[ [K_a, K_b] = C_{ab}^c K_c, \] (3.28)

where \( a, b = 0, 1, \ldots, 9 \), with the following nonzero structure constants:

\[
\begin{align*}
C_{13}^3 &= C_{15}^5 = C_{16}^6 = C_{24}^4 = C_{25}^5 = C_{27}^7 = C_{37}^4 = \\
C_{30}^8 &= C_{46}^3 = C_{58}^3 = C_{69}^8 = C_{78}^9 = C_{67}^1 = 1, \\
C_{17}^7 &= C_{19}^0 = C_{10}^0 = C_{28}^8 = C_{26}^6 = C_{20}^0 = \\
C_{59}^4 &= C_{40}^9 = C_{50}^1 = C_{50}^2 = C_{67}^2 = -1, \\
C_{89}^0 &= C_{49}^7 = C_{38}^6 = 2, \quad C_{48}^2 = C_{39}^1 = C_{34}^5 = -2. \quad (3.29)
\end{align*}
\]

This algebra is isomorphic to \( so(2,3) \). To show this, consider a five-dimensional pseudoeuclidean space,

\[ d\sigma^2 = G_{\mu \nu} d\xi^\mu d\xi^\nu = -(d\xi^0)^2 - (d\xi^\theta)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2, \] (3.30)

and denote the corresponding \( so(2,3) \) generators as \( L_{\mu \nu} \), where \( \mu, \nu = 0, \theta, 1, 2, 3 \). Then the following correspondence between \( L_{\mu \nu} \) and the above Killing vectors can be found:

\[
\begin{align*}
L_{01} &= \frac{1}{2}(K_{d_3} - K_s), \quad L_{02} = -\frac{1}{2}(K_{d_3} + K_s), \quad L_{03} = \frac{1}{2}(K_{H_2} - K_m), \\
L_{31} &= \frac{1}{2}(K_{H_2} + K_m), \quad L_{32} = \frac{1}{2}(K_{H_1} + K_e), \quad L_{03} = \frac{1}{2}(K_{H_1} - K_e), \\
L_{02} &= \frac{1}{2}(K_g - K_{d_1} - K_{d_2} - K_E), \quad L_{12} = \frac{1}{2}(K_g + K_{d_1} + K_{d_2} - K_E), \\
L_{00} &= \frac{1}{2}(K_g + K_{d_1} - K_{d_2} + K_E), \quad L_{12} = \frac{1}{2}(K_g - K_{d_1} + K_{d_2} + K_E). \quad (3.31)
\end{align*}
\]

Using (3.31) one can rewrite commutation relations (3.28) as

\[
[L_{\mu \nu}, L_{\lambda \tau}] = G_{\mu \tau} L_{\nu \lambda} - G_{\mu \lambda} L_{\nu \tau} + G_{\nu \lambda} L_{\mu \tau} - G_{\nu \tau} L_{\mu \lambda} = C_{\alpha \beta}^{\mu \nu \lambda \tau} L_{\alpha \beta}. \] (3.32)
Different $so(1,2)$ subalgebras should correspond to $S$ and $T$ dualities. They can be chosen in various ways; physically this may be traced to the possibility of gauge and scale transformations. Indeed, one can see that both electric-magnetic duality and Ehlers–Harrison generators enter into genuine $SO(2,3)$ generators mixed with gauge and scale ones. Also it should be noted that the standard procedure of Kaluza–Klein reduction does not involve dualization of nondiagonal metric components and magnetic parts of vectors, while our procedure does. Here we deal with the potential space, as opposed to the metric space in the standard reduction. Both should be related by a kind of Neugebauer–Kramer map, but we do not enter into details of these matters here.

Now we are in a position to formulate the main result of the above analysis: the target space (2.7) is a symmetric pseudoriemannian space isomorphic to the coset $SO(2,3)/(SO(3) \times SO(2))$. Writing a decomposition of the full algebra

$$\mathcal{L} = (L_{0\theta}, L_{12}, L_{23}, L_{13}),$$
$$\mathcal{B} = (L_{01}, L_{02}, L_{03}, L_{\theta 1}, L_{\theta 2}, L_{\theta 3}),$$

one finds that

$$[\mathcal{B} \mathcal{B}] = \mathcal{L}, \quad [\mathcal{B} \mathcal{L}] = \mathcal{B}, \quad [\mathcal{L} \mathcal{L}] = \mathcal{L}. \quad (3.34)$$

This means that the homogeneous space generated by $\mathcal{B}$ is a symmetric space. The equations of motion for $\varphi^A$ then are equivalent to the set of conservation laws for Noether currents

$$\partial_i (h^{ij} \sqrt{\text{det} h} J^a_j) = 0, \quad J^a_i = \tau^a_A \frac{\partial \varphi^A}{\partial x^i}, \quad (3.35)$$

built using the corresponding Killing one-forms

$$\tau^a = \eta^{ab} K^A_b G_{AB} d\varphi^B, \quad (3.36)$$

where $\eta^{ab}$ is an inverse to the Killing–Cartan metric $\eta_{ab} = k C^c_{ad} C^d_{bc}$. This is a convenient form for further reduction to two dimensions.

4. Stringy Kerr–NUT dyon

Here we discuss an application of the target space isometries to derive dilaton-axion counterparts to solutions of the vacuum Einstein equations. Any solution to the vacuum Einstein equations is a solution of the present theory with $v = u = \kappa = \phi = 0$. Therefore using the above transformations an axion-dilaton counterpart can be
found to any stationary vacuum solution. In this case the above formulas simplify considerably. The first Harrison transformation will read

\[
\frac{f}{f_0} = \frac{\chi}{\chi_0} = e^{2\phi} = \frac{1}{1 - h_1^2 f_0}, \quad v = h_1 f, \quad u = h_1 \chi, \quad \kappa = h_1^2 \chi_0. \tag{4.1}
\]

If the seed solution is asymptotically flat, and one wishes to preserve this property, it has to be accompanied by the scale transformation (3.7) with the parameter \(e^{2s} = 1 - h_1^2\). The result can be concisely expressed in terms of the Ernst potential \(E = f + i\chi\):

\[
E = \frac{\sqrt{1 - h_1^2}}{h_1} (v + iu) = \frac{(1 - h_1^2)\mathcal{E}_0}{1 - h_1^2 \text{Re} \mathcal{E}_0}, \quad z = i \left(1 - h_1^2 \mathcal{E}_0\right). \tag{4.2}
\]

A similar combined transformation via (3.22)–(3.7) reads

\[
E = \frac{\sqrt{1 - h_2^2}}{h_2} (u - iv) = \frac{(1 - h_2^2)\mathcal{E}_0}{1 - h_2^2 \text{Re} \mathcal{E}_0}, \quad z = \frac{i}{(1 - h_2^2 \mathcal{E}_0)}. \tag{4.3}
\]

In both cases the metric rotation function is simply rescaled: \(\omega_i = (1 - h^2)^{-1} \omega_{0i}\), where \(h\) is either \(h_1\) or \(h_2\).

When applied to vacuum solutions, our Ehlers-type transformation reduces exactly to the original Ehlers transformation [19]. Indeed, in this case \(\beta = 1\) and from (3.27) we get \(E = \mathcal{E}_0(1 + ih_3\mathcal{E}_0)^{-1}\), while \(v, u, \phi, \kappa\) remain zero.

Starting with the vacuum Kerr–NUT solution we have obtained [12] a seven-parameter family of axion-dilaton black holes using Harrison transformations in suitable combinations with other isometries. The resulting metric may be written in the same form as the vacuum Kerr–NUT solution,

\[
 ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi)^2 - \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\varphi^2\right), \tag{4.4}
\]

with somewhat modified characteristic functions

\[
\begin{align*}
\Delta &= (r - r_-) (r - 2M) + a^2 - (N - N_-)^2, \\
\Sigma &= r (r - r_-) + (a \cos \theta + N)^2 - N_-^2, \\
\omega &= \frac{2}{a^2 \sin^2 \theta - \Delta} \left\{ N \Delta \cos \theta + a \sin^2 \theta [M (r - r_-) + N (N - N_-)] \right\}.
\end{align*} \tag{4.5}
\]
The corresponding electric and magnetic potentials and the axidilaton field are

\[ v = \sqrt{2e^{\phi_\infty}} \Sigma \Re \left[ Q (r - r_+ + i\delta) \right], \quad u = \sqrt{2e^{\phi_\infty}} \Sigma \Re \left[ Q z_\infty (r - r_+ + i\delta) \right], \quad (4.6) \]

\[ z = \frac{z_\infty \rho + D z_\infty^*}{\rho + D}, \quad r = r - \frac{M^* r_- + i\delta}{2M} + i\delta, \quad \delta = a \cos \theta + N - N_-. \quad (4.7) \]

Here \( M = M + iN \) is the complex mass, \( Q = Q - iP \) is the complex charge. A complex dilaton-axion charge \( D = D + iA \), defined asymptotically as

\[ z = z_\infty - 2ie^{-2\phi_\infty} \frac{D}{r} + O \left( \frac{1}{r^2} \right), \quad (4.8) \]

is related to the other parameters as follows:

\[ D = -\frac{Q^* r_-}{2M}. \quad (4.9) \]

Note that this relation does not involve the rotation parameter \( a \). Separately dilaton and axion charges are

\[ D = \frac{M(P^2 - Q^2) - 2PQN}{2(M^2 + N^2)}, \quad A = \frac{N(Q^2 - P^2) - 2PQM}{2(M^2 + N^2)}. \quad (4.10) \]

Two additional parameters in the metric functions read

\[ r_- = \frac{M|Q|^2}{|M|^2}, \quad N_- = \frac{N|Q|^2}{2|M|^2}. \quad (4.11) \]

Hence the solution contains seven independent real parameters: a mass \( M \), a rotation parameter \( a \), a NUT–parameter \( N \), electric \( Q \) and magnetic \( P \) charges (defined as in [20] to have the standard asymptotic normalization of the Coulomb energy), and asymptotic values of the axion \( \kappa_\infty \) and the dilaton \( \phi_\infty \) (combined in \( z_\infty \)). The complex axidilaton charge introduced through an asymptotic expansion is determined by the electromagnetic charge and the complex mass:

This family contains as particular cases many previously known solutions to dilaton-axion gravity. For \( N = P = 0 \) it corresponds to Sen’s solution [21] up to some coordinate transformation (in this case the axion charge \( A = 0 \)). For \( a = 0 \) it coincides (up to a transformation of the radial coordinate) with the 6-parameter solution reported recently by Kallosh et al. [20]; its 3-parameter subfamily was also
found by Johnson and Myers [22]. For \( N = 0 \), \( a = 0 \) we recover the 5-parameter solution presented by Kallosh and Ortin [23], and if in addition one of the charges \( Q, P \) is zero, the solution reduces to the Gibbons–Maeda–Garfinkle–Horowitz–Strominger black hole [16]. Finally, when \( P = Q = 0 \) we come back to the Kerr–NUT metric.

As in vacuum and electrovacuum cases, for \( N \neq 0 \) our solution cannot be properly interpreted as a black hole because of time periodicity, which is to be imposed in the presence of the wire singularity. We can still use the notation \( r^\pm_H \) for the values of the radial coordinate marking positions of the surfaces where \( \Delta = 0 \):

\[
r^\pm_H = M + r_-/2 \pm \sqrt{|\mathcal{M}|^2 (1 - r_-/2M)^2 - a^2}.
\] (4.12)

For \( N = 0 \) the upper value \( r^+_H \) corresponds to the event horizon of a black hole. The timelike Killing vector \( \partial_t \) becomes null at the surface \( r = r_e(\theta) \),

\[
r^\pm_e = M + r_-/2 \pm \sqrt{|\mathcal{M}|^2 (1 - r_-/2M)^2 - a^2 \cos^2 \theta},
\] (4.13)

which marks the boundary of a black hole ergosphere in the case \( N = 0 \). Inside the 2-surface \( r = r_e(\theta) \) the Killing vector \( \partial_t - \Omega \partial_\phi \) with some \( \Omega = \text{const} \) may still be timelike, the boundary value of \( \Omega \) at \( r = r^+_H \) where it becomes null being

\[
\Omega_H = \frac{a}{2} \left\{ |\mathcal{M}|^2 (1 - r_-/2M) + M \sqrt{|\mathcal{M}|^2 (1 - r_-/2M)^2 - a^2} \right\}^{-1}.
\] (4.14)

For \( N = 0 \) this quantity has a meaning of the angular velocity of the horizon. The area of the two–surface \( r = r^+_H \) is

\[
A = 4\pi a / \Omega_H.
\] (4.15)

The square root in (4.12) becomes zero for the family of extremal solutions. This corresponds to the relation between the parameters

\[
|\mathcal{D}| = |\mathcal{M}| - a,
\] (4.16)

which defines a 4-dimensional hypersurface in the 5-dimensional space of parameters \( Q, P, M, N, a \). For extremal solutions we have

\[
r^\text{ext}_H = 2M - \frac{aM}{|\mathcal{M}|}, \quad \Delta^\text{ext} = (r - r^\text{ext}_H)^2,
\]

\[
\omega^\text{ext} = \frac{2 \left\{ N \Delta^\text{ext} \cos \theta + a \sin^2 \theta [M (r - r^\text{ext}_H) + a|\mathcal{M}|] \right\}}{a^2 \sin^2 \theta - \Delta^\text{ext}},
\]

\[
\Sigma^\text{ext} = 2M (r - r^\text{ext}_H) + \Delta^\text{ext} - a^2 \sin^2 \theta + 2a (|\mathcal{M}| + N \cos \theta).
\] (4.17)
The metric for the non-rotating extremal dilaton-axion Taub-NUT family reads
\[ ds^2 = (1 - 2M/r)\left(dt + 2N \cos \theta d\varphi\right)^2 - (1 - 2M/r)^{-1} dr^2 - r(r - 2M)\left(d\theta^2 + \sin^2 \theta d\varphi^2\right). \] (4.18)

In this case \( r^\text{ext}_H = 2M \), coinciding with the curvature singularity. (Note that this is not so if \( a \neq 0 \), since then \( \Sigma^\text{ext}(r^\text{ext}_H) \neq 0 \).) For the dilaton we get from (4.7)
\[ e^{2(\phi - \phi_\infty)} = \frac{1}{r(r - 2M)} \left|r - 2M + \frac{i(QN - PM)Q^*}{|M|^2}\right|^2. \] (4.19)

Comparing (4.18) and (4.19) one can see that generally the string metric \( ds^2_{\text{string}} = e^{2\phi} ds^2 \) has nonsingular throat structure. However, if
\[ QN = PM, \] (4.20)
the dilaton factor (4.19) vanishes as \( r \to 2M \) and the string metric will have the same behavior as in the case of the static dilaton electrically charged black hole [16] (to which our solution reduces if \( N = P = 0 \)). Hence, regular Taub-NUT string throats form a 3-parameter family corresponding to the hypersurface \( |D| = |M| \) in the parameter space of \( M, N, P, Q \), from which a 2-dimensional subspace (4.20) has to be excluded. As was shown recently by Johnson [24], some of the family of extremal Taub-NUT solutions have exact gauged WZW model counterparts.

5. Two-dimensional integrability

If the target space of a three-dimensional sigma model is a symmetric riemannian space \( G/H \) with \( N \)–parameter isometry group \( G \) acting transitively on it (\( H \) being an isotropy subgroup), generated by the set of \( N \) Killing vectors forming the Lie algebra of \( G \), \( [K_a, K_b] = C^c_{\ ab} K_c \), \( a, b, c = 1, \ldots, N \), the current conservation equations (3.35) may be cast into the null-curvature form. With a proper normalization of the Killing–Cartan metric on the isometry group the one-forms dual to the Killing vectors will satisfy the Maurer–Cartan equation with the same structure constants,
\[ d\tau^a + \frac{1}{2} C^a_{\ bc} \tau^b \wedge \tau^c = 0. \] (5.1)

Let \( e_a \) denote some matrix representation of the Lie algebra of \( G \), \( [e_a, e_b] = C^c_{\ ab} e_c \). Define the following matrix-valued connection one-form: \( A = A_B d\varphi^B = e_a \tau^a \). In view of (5.1), the corresponding curvature vanishes,
\[ F_{BC} = A_{C,B} - A_{B,C} + [A_B, A_C] = 0, \] (5.2)
and thus $A_B$ is pure gauge,

$$A_B = -\left(\partial_B g\right)g^{-1}, \quad g \in G. \quad (5.3)$$

Because of the gauge invariance under $H$, the matrix $g$ actually belongs to the coset $G/H$ [6]. The pullback of $A$ onto the configuration space $x^i$ is equivalent to (3.35) and, hence, to the equations of motion of the sigma model. In terms of $g$ the equations (3.35) read

$$d\{\ast dg g^{-1}\} = 0, \quad (5.4)$$

where a star stands for a 3-dimensional Hodge dual.

Now reduce the system to two dimensions, imposing an axial symmetry condition. In order to ensure regularity on the polar axis, hypersurface orthogonality has to be supposed in addition. Then the 3-metric may be conveniently written in the Lewis–Papapetrou gauge:

$$h_{ij}dx^i dx^j = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2, \quad (5.5)$$

and (5.4) becomes equivalent to a modified chiral equation

$$(\rho g, \rho g^{-1})_\rho + (\rho g, z g^{-1})_z = 0. \quad (5.6)$$

A variety of techniques is available now to deal with such systems. A Lax pair with the complex spectral parameter $\lambda$ can be readily given (we use the Belinskii–Zakharov form [1]):

$$D_1 \Psi = \frac{\rho U - \lambda V}{\rho^2 + \lambda^2} \Psi, \quad D_2 \Psi = \frac{\rho V + \lambda U}{\rho^2 + \lambda^2} \Psi, \quad (5.7)$$

where $V = \rho g, \rho g^{-1}, U = \rho g, z g^{-1}, \Psi$ is a matrix “wave function”, and

$$D_1 = \partial_\rho - \frac{2\lambda^2}{\rho^2 + \lambda^2} \partial_\lambda, \quad D_2 = \partial_\rho + \frac{2\lambda\rho}{\rho^2 + \lambda^2} \partial_\lambda \quad (5.8)$$

are commuting operators. A chiral system (5.6) follows from the compatibility condition $[D_1, D_2] \Psi = 0$. This linearization is sufficient to establish a desired integrability property. The inverse scattering transform method can be directly applied to (5.7) to generate multisoliton solutions, and an infinite-dimensional Kac–Moody algebra can be derived [6].

In view of the results of Sec. 3, the EMDA system in two dimensions will be described by the chiral model based on the coset $SO(3,2)/(SO(3) \times SO(2))$ [25].
It is worth noting that for $D = 4$, $p = 1$ theory $T$-duality in two dimensions is $SO(2,3)$—i.e., the same as an enhanced symmetry found here in three dimensions. However it is just because of existence of the same symmetry in three dimensions (ensuring the symmetric-space property of the target space) that the corresponding two-dimensional symmetry becomes infinite and is generated by the $SO(2,3)$ current algebra. A similar situation was recently described by Bakas [26] in the dilaton-axion gravity without vector field ($p = 0$), where an analogous role was played by the group $SO(2,2)$.

Let us describe the one-form representation of the present theory more explicitly [25]. Using the pair-index notation, the Killing–Cartan metric of $so(2,3)$ can be written as

$$\eta_{\mu\nu} \lambda_\tau = 1/12 \ C^{\alpha\beta} \ C_{\mu\nu} \ C^{\gamma\delta} \ C_{\lambda\tau} \ C_{\alpha\beta}, \quad (5.9)$$

and the target space metric in terms of the desired one–forms reads

$$G_{AB} = 1/2 \ \eta_{\mu\nu} \lambda_\tau A^\mu B^\nu. \quad (5.10)$$

Using Killing vectors $K_a$ and the correspondence rules (3.31) one can construct $\tau$–forms (3.36). The Abelian subalgebra will read:

$$\tau^{01} = - (\omega_1 + \omega_f), \quad \tau^{02} = \omega_1 - \omega_f + 2\omega_2, \quad (5.11)$$

where

$$\omega_1 = \kappa \omega_u - 2d\phi + u(v \omega_\chi - 2\omega_u),$$
$$\omega_f = f^{-1} df + \chi \omega_\chi, \quad \omega_2 = v \omega_v + u \omega_u,$$
$$\omega_\kappa = e^{4\phi} d\kappa, \quad \omega_\chi = f^{-2} (d\chi + v du - u dv),$$
$$\omega_v = f^{-1} e^{-2\phi} dv, \quad \omega_u = f^{-1} e^{2\phi} (du - \kappa dv). \quad (5.12)$$

The other components look more complicated:

$$2\tau^{00} = \omega + \omega_6 - \omega_7 - \omega_\chi, \quad 2\tau^{02} = \omega - \omega_6 - \omega_7 + \omega_\chi,$$
$$-2\tau^{01} = \omega + \omega_6 + \omega_7 + \omega_\chi, \quad 2\tau^{12} = \omega - \omega_6 + \omega_7 - \omega_\chi, \quad (5.13)$$
$$-\tau^{03} = \omega_5 + \omega_8, \quad \tau^{13} = \omega_5 - \omega_8, \quad \tau^{03} = \omega_4 - \omega_9, \quad -\tau^{23} = \omega_4 + \omega_9,$$

where the following recurrent sequence is used:

$$\omega_3 = \kappa \omega_u - \omega_v, \quad \omega_4 = u \omega_\chi - \omega_3, \quad \omega_5 = v \omega_\chi - \omega_u,$$
$$\omega_6 = d\kappa - \kappa^2 \omega_\kappa + 4 \kappa d\phi - u(\omega_4 - \omega_3), \quad \omega_7 = \omega_\kappa + v(\omega_5 - \omega_u),$$
$$\omega_8 = u \tau^{01} - v \omega_6 + \chi \omega_3 - u \omega_2 + du,$$
$$\omega_9 = v \tau^{02} - u \omega_7 + \chi \omega_u - v \omega_2 + dv,$$
$$\omega = u \omega_9 - v \omega_8 + \chi (\omega_\chi - \omega_2 + 2\omega_f) + d\chi. \quad (5.14)$$
Now, given an adjoint matrix representation of $so(3,2)$, one can build a $5 \times 5$ connection one-form $A$ and the corresponding matrix $g \in SO(3,2)/(SO(3) \times SO(2))$. Remarkably, the present theory also admits a more concise representation in terms of the symplectic $4 \times 4$ matrices (due to the isomorphism $SO(3,2) \sim Sp(4,R)$). The symplectic connection can be written in block form using three real $2 \times 2$ matrices, two of which ($B$, $D$) are symmetric,

\begin{align*}
B &= \frac{1}{2} \left\{ (\tau^{0\theta} - \tau^{3\theta})I_2 + (\tau^{23} - \tau^{02})\sigma_x + (\tau^{01} - \tau^{13})\sigma_z \right\}, \\
C &= \frac{1}{2} \left\{ \tau^{03}I_2 - \tau^{\theta2}\sigma_x - i\tau^{12}\sigma_y + \tau^{\theta1}\sigma_z \right\}, \\
D &= \frac{1}{2} \left\{ - (\tau^{0\theta} + \tau^{3\theta})I_2 - (\tau^{23} + \tau^{02})\sigma_x + (\tau^{01} + \tau^{13})\sigma_z \right\},
\end{align*}

as follows:

\begin{equation}
A = \begin{pmatrix} C & B \\ D & -C^T \end{pmatrix}.
\end{equation}

Here $I_2$ is a unit matrix and $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices with $\sigma_z$ diagonal. In view of (5.4), the equations of motion of the EMDA sigma model are equivalent to vanishing of the curvature (5.2) related to (5.16). This implies the existence of the symmetric symplectic $4 \times 4$ matrix $g \in Sp(4,R)/U(2)$ entering the Belinskii–Zakharov representation (an explicit form will be given elsewhere).

6. Dilaton gravity: critical couplings

Now consider the target space (2.9) of the purely dilatonic model. Of the ten Killing vectors of the dilaton-axion theory, four ($K_g$, $K_e$, $K_m$, $K_s$) are independent of the axion; they remain isometry generators for the dilatonic model too. However, $sl(2,R)$ duality is broken to a dilaton shift (inherited from $K_{d3}$):

\begin{equation}
K_\phi = v \partial_v - a \partial_a + \alpha^{-1} \partial_\phi.
\end{equation}

(When $\alpha \to 0$ it should be renormalized by multiplying by $\alpha$; in this limit $K_\phi$ becomes a pure dilaton shift.) The corresponding finite transformation is

\begin{equation}
v \to e^d v, \quad u \to e^{-d} u, \quad \phi \to \phi + \alpha^{-1} d,
\end{equation}

where $d$ is real parameter. These five generators form a closed algebra. Enumerating them $\xi$ from 1 to 5 as in Sec. 3 (with $K_\phi$ instead of $K_{d3}$) we get the following nonzero structure constants:

\begin{align*}
C^3_{13} = C^4_{24} = C^5_{15} = C^5_{25} = 1, \quad C^5_{34} = -2.
\end{align*}
This algebra is solvable. Denoting it as $\mathcal{K}$ one can see that its derivative $\mathcal{K}'$ contains as basis vectors $K_3, K_4, K_5$; the second derivative is one-dimensional: $\mathcal{K}'' = K_5$; and we have the chain of subalgebras

$$0 = \mathcal{K}''' \subset \mathcal{K}'' \subset \mathcal{K}' \subset \mathcal{K},$$

where each term is an ideal of the preceding one. It can also be shown that the Killing–Cartan bilinear form is degenerate. Since the target space is five-dimensional, this means that the above symmetries are insufficient for creating a symmetric-space structure.

Such algebras are known to admit a representation in terms of upper-triangular matrices. Consistently with (6.3) $K_1$ and $K_2$ can be chosen diagonal. Then the following $3 \times 3$ representation $K_a \rightarrow e_\mu$ holds:

$$e_1 = \frac{1}{3} \left( \begin{array}{ccc} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right), \quad e_2 = \frac{1}{3} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right), \quad e_3 = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

$$e_4 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right), \quad e_5 = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Obviously, this set constitutes a basis for the upper-triangular subalgebra of $sl(3, R)$.

It is instructive to carry out a further geometric analysis of the situation. Computing the Riemann tensor, one can find that both the scalar curvature,

$$R = -(\alpha^2 + 12),$$

and the square of the Riemann tensor,

$$R_{ABCD}R^{ABCD} = 3(\alpha^4 + 16),$$

are coordinate-independent. However, for general $\alpha$ the target space is not a symmetric space. Direct computation shows that almost all covariant derivatives of the curvature tensor are zero, except for the following:

$$R_{\nu\nu\phi;\mu} = R_{\nu\nu\phi;\mu} = -\frac{\alpha(\alpha^2 - 3)}{f^2}.$$
in (2.8)) we conclude that only BMD and KK theories are exceptional within the class under consideration.

Also, it turns out that the target space is an “almost” Einstein space. Indeed,

\[ R_{AB} = -3G_{AB} \]  \hspace{1cm} (6.9)

for \( A, B = f, \chi, v, u, \phi \) (\( G_{AB} \) is the target space metric), but

\[ R_{\phi\phi} = -\alpha^2 G_{\phi\phi}. \]  \hspace{1cm} (6.10)

For \( \alpha^2 = 3 \) the target space is both a homogeneous symmetric space and an Einstein space.

Let us consider two critical cases in more detail. For \( \alpha = 0 \) the present theory coincides with the BDM model with the Brans–Dicke parameter \( \omega = -1 \). The presence of the scalar field just adds a trivial generator (constant dilaton shift) to the \( su(2,1) \) algebra of the corresponding EM theory. The latter includes four Killing vectors found above \( (K_1, K_2, K_3, K_5) \) and a continuous electric-magnetic duality rotation

\[ K_d = (v\partial_u - u\partial_v), \]  \hspace{1cm} (6.11)

which is broken if \( \alpha \neq 0 \).

The nontrivial part of the 8-parameter \( su(2,1) \) symmetry algebra consists of Harrison transformations [15]

\[ K_{H_1} = 2fv\partial_f + [v\chi + uf - u(u^2 + v^2)/2] \partial_\chi \\
+ [(u^2 - 3u^2)/2 + f] \partial_v + (2uv + \chi)\partial_u, \]  \hspace{1cm} (6.12)

\[ K_{H_2} = 2fu\partial_f + [u\chi - vf + v(u^2 + v^2)/2] \partial_\chi \\
+ [(u^2 - 3v^2)/2 + f] \partial_u + (2uv - \chi)\partial_v, \]  \hspace{1cm} (6.13)

and the EM version of the Ehlers transformation generated by the commutator of the Killing vectors (see (3.23))

\[ K_E = 2f\chi\partial_f + (\chi^2 - F^2)\partial_\chi + (\chi v + uF)\partial_v + (\chi u - vF)\partial_u, \]  \hspace{1cm} (6.14)

where \( F = f - (v^2 + u^2)/2 \). Already from the infinitesimal form it can be seen that Harrison transformations mix gravitational and electromagnetic potentials, while Ehlers transformations mix gravitational variables. Unfortunately, these symmetries turn out to be broken by the dilaton.
In the case $\alpha = \sqrt{3}$ (KK-theory) one encounters a remarkable counterpart to Har- rison transformations. The infinitesimal generators were found by Neugebauer in 1969 [18]. In our notation Neugebauer transformations read

\[
K_{N_1} = 2fv\partial_f + (v\chi + ufe^{2\alpha\phi} + uv^2)\partial_\chi
+ (2v^2 + fe^{2\alpha\phi})\partial_v - (uv - \chi)\partial_u + \sqrt{3}v\partial_\phi, \tag{6.15}
\]
\[
K_{N_2} = 2fu\partial_f + (u\chi - vfe^{-2\alpha\phi} - u^2v)\partial_\chi
+ (2u^2 + e^{-2\alpha\phi})\partial_u - (uv + \chi)\partial_v - \sqrt{3}u\partial_\phi. \tag{6.16}
\]

The commutator of two Neugebauer transformations gives a KK analog of the Ehlers transformation,

\[
[K_{N_1}, K_{N_2}] = 2K_{EN}, \tag{6.17}
\]

which explicitly reads

\[
K_{EN} = 2f\chi\partial_f + [\chi^2 - f^2 + f(v^2e^{-2\alpha\phi} + u^2e^{2\alpha\phi}) + u^2v^2]\partial_\chi
+ [v(uv + \chi) + ufe^{2\alpha\phi}]\partial_v + [u(\chi - uv) - vfe^{-2\alpha\phi} - u^2v]\partial_u + \sqrt{3}vu\partial_\phi. \tag{6.18}
\]

In spite of an apparent similarity of the Ehlers–Harrison transformations for $\alpha = 0$ and the Neugebauer transformations for $\alpha = \sqrt{3}$, no continuous $\alpha$-interpolation can be found between them within the EMD theory.

7. Diagonal metrics

If the rotation one-form $\omega_i$ in (2.1) is zero and, in addition, either the electric potential $v$ or the magnetic potential $u$ is set to zero too, the axion decouples in the EMDA system and hence can consistently be chosen zero. Both EMDA and EMD theories are then cast into electrostatics and magnetostatics, and in this situation the EMD theory is more general since the dilaton coupling constant is arbitrary. So we take as a starting point the target space (2.9). Setting $\chi = 0 = u$ we are left with the three-dimensional space

\[
dl_e^2 = \frac{df^2}{2f^2} - \frac{e^{-2\alpha\phi}dv^2}{f} + 2d\phi^2. \tag{7.1}
\]

It can be checked that all covariant derivatives of the corresponding Riemann tensor vanish, i.e., we are dealing with a symmetric space. It is, however, not an Einstein space. The nonzero components of the Ricci tensor read

\[
R_{ff} = -\frac{1}{4f^2}, \quad R_{f\phi} = \frac{\alpha}{2f}, \quad R_{vv} = -\frac{vfe^{-2\alpha\phi}}{f}, \quad R_{\phi\phi} = -\alpha^2, \tag{7.2}
\]
where $\nu = (\alpha^2 + 1)/2$. Clearly $R_{AB}$ is not proportional to the metric tensor in (7.1) except for $\alpha = 0$, if we set in addition $\phi \equiv 0$.

Solving the Killing equations for the metric (7.1) one finds four Killing vectors:

$$
K_0 = \frac{1}{2} \partial_\phi - \alpha f \partial_f, \quad K_1 = \partial_v,
$$

$$
K_2 = \frac{1}{2} \left( v^2 + \nu^{-1} f e^{2\alpha \phi} \right) \partial_v + \frac{v}{2\nu} \left( \alpha \partial_\phi + 2 f \partial_f \right),
$$

$$
K_3 = v \partial_v + \nu^{-1} \left( \frac{\alpha}{2} \partial_\phi + f \partial_f \right).
$$

The first one commutes with all others:

$$
[K_0, K_i] = 0, \quad i = 1, 2, 3,
$$

while the $K_i$ form an algebra $sl(2, \mathbb{R})$:

$$
[K_1, K_2] = K_3, \quad [K_1, K_3] = K_1, \quad [K_2, K_3] = -K_2.
$$

Here the Harrison-type generator is $K_2$, and it is worth noting that for $\alpha = \sqrt{3}$ the corresponding generator $K_{N_1}$ (6.15) does reduce to $K_2$ restricted to the static subspace.

Finite transformations corresponding to these infinitesimal symmetries can be found by a straightforward integration. Consider first an electrostatic case. The vector $K_0$ generates the transformation

$$
\phi \rightarrow \phi + \lambda_0, \quad f \rightarrow f e^{-\alpha \lambda_0}, \quad v \rightarrow v,
$$

(7.6)

where $\lambda_0$ is a real parameter. It can be interpreted as a constant dilaton shift, accompanied by rescaling of the three-dimensional conformal factor. Comparing with the results of the Sec. 3 one sees that it is a superposition of the stationary dilaton shift (6.1) and the scale transformation (3.8) restricted to $\chi \equiv 0, \ u \equiv 0$.

The second Killing vector $K_1$ obviously corresponds to a gauge transformation of the electrostatic potential,

$$
v \rightarrow v + \lambda_1, \quad f \rightarrow f, \quad \phi \rightarrow \phi,
$$

(7.7)

and is directly related to the stationary gauge transformation (3.3).
An essentially nontrivial (Harrison-type) transformation is generated by $K_2$. Integration gives an invariant:

$$fe^{-2\phi/\alpha} = f_0e^{-2\phi_0/\alpha},$$

(7.8)

and a linear transformation law for the quantities

$$s = \frac{ve^{-\alpha\phi}}{\sqrt{f}}, \quad t = \frac{\sqrt{f}e^{\alpha\phi}}{\nu} - \frac{v^2e^{-\alpha\phi}}{\sqrt{f}};$$

(7.9)

$$t = t_0, \quad s = s_0 + t_0\lambda_2/2,$$

(7.10)

where the index 0 corresponds to $\lambda_2 = 0$. Finally, equations generated by $K_3$ give the following last isometry of the target space (7.1):

$$v \rightarrow ve^{\lambda_3}, \quad f \rightarrow fe^{\lambda_3/\nu}, \quad \phi \rightarrow \phi + \frac{\alpha\lambda_3}{2\nu}.$$  

(7.11)

To select combinations of these transformations which preserve asymptotic flatness it is convenient to write down the general four-parameter isometry transformation obtained by successive application of the above transformations:

$$f = f_0\Lambda^{-1/\nu}e^{-\alpha\lambda_0 + \lambda_3/\nu},$$

$$\exp(\phi) = \Lambda^{-\alpha/(2\nu)}\exp(\phi_0 + \lambda_0/2 + \alpha\lambda_3/(2\nu)),$$

$$v = e^{\lambda_3}\left(v_0 + \frac{1}{2}t_0\lambda_2\sqrt{f_0e^{\alpha\phi_0}}\right)\Lambda^{-1} + \lambda_1,$$

(7.12)

where

$$\Lambda = \frac{1 - \nu s_0^2}{1 - \nu s_0^2}.$$  

(7.13)

It can be shown that the Harrison-like transformation in dilaton gravity recently used for generating purposes [14], [17] is a particular case of this four-parameter class.

Magnetostatics ($\chi = 0 = v$),

$$dl^2_m = \frac{df^2}{2f^2} - \frac{e^{2\alpha\phi}du^2}{f} + 2d\phi^2,$$

(7.14)

can be treated along the same lines. Moreover, by reparametrization

$$\xi = (\alpha\phi - 1/2 \ln f)/\mu, \quad \eta = (\phi + \alpha/2 \ln f)/\mu,$$

(7.15)
for the magnetic case, and
\[
\xi = -(\alpha \phi + 1/2 \ln f)/\mu, \quad \eta = (\phi - \alpha/2 \ln f)/\mu, \quad (7.16)
\]
for the electric one, where \(\mu^2 = \nu\), one gets a unified description of both cases. Denoting by \(u\) either the magnetic \((u)\) or the electric \((v)\) potential, one can represent the line element of the truncated target space as \(dl^2_3 = d\eta^2 + dl^2_2\), where
\[
dl^2_2 = d\xi^2 - e^{2\mu \xi} du^2.
\]
This 2-dimensional space can be easily shown to represent a coset \(SL(2, R)/U(1)\). Indeed, one can find three Killing vectors for (7.17):
\[
K_1 = \partial_u, \quad K_2 = p\partial_u - \mu^{-1} u \partial_\xi, \quad K_3 = u \partial_u - \mu^{-1} \partial_\xi, \quad (7.18)
\]
where \(p = (u^2 + \mu^{-2} e^{-2\mu \xi})/2\), with the \(sl(2, R)\) structure constants \(C^3_{12} = C^2_{32} = C^1_{13} = 1\). The corresponding Killing–Cartan one-forms, with the normalization \(k = (2\mu)^{-2}\), will satisfy (5.1), and we have \(dl^2_2 = 1/2 \eta_{ab} \tau^a \otimes \tau^b\), where \(\eta_{ab} = 2k \text{ diag}(1, 1, -1)\). Choosing as \(e_a\) a \(2 \times 2\) representation of \(sl(2, R)\), one can find using (5.3) the following matrix \(g \in SL(2, R)/U(1)\):
\[
g = \mu e^{\mu \xi} \sqrt{2} \begin{pmatrix} u^2 - p & -u/\sqrt{2} \\ -u/\sqrt{2} & 1 \end{pmatrix}
\]
which can be used in the Lax-pair (5.7) in the axisymmetric case.

Perhaps, all these considerations were unnecessary in order to reveal the two-dimensional integrability of the arbitrary-\(\alpha\) EMD theory in the diagonal case. Indeed, for \(\alpha = 0\) \((\nu = 1/2)\) the EMD theory reduces to the corresponding representation for the electrovacuum. Since, as we have shown, the underlying algebraic structure is \(\alpha\)-independent, this fact is already sufficient to reveal integrability of the static axisymmetric EMD system with arbitrary \(\alpha\). However, an explicit construction may be useful for practical generating purposes. Note that, contrary to the stationary EMDA case, here we can extend any diagonal two-Killing solution of the EM theory to the arbitrary-\(\alpha\) EMD theory.

8. Conclusions
We have shown that the EMDA theory reduced to three dimensions possesses a ten-parameter global symmetry group \(SO(2, 3)\) containing \(T\) and \(S\) dualities as different \(SO(1, 2)\) subgroups. This theory may be represented as a sigma model on the symmetric coset space \(SO(2, 3)/(SO(3) \times SO(2))\). Therefore, further two-dimensional
reduction gives an integrable system with associated infinite symmetries. These features were found in the $D = 4$, $p = 1$ model, but it is likely that they will persist as well for the full ten-dimensional bosonic heterotic string effective action $^1$.

An integrability property of the EMDA theory with two commuting Killing vectors opens the way for application of a variety of generating techniques to build (at least) as many classical solutions to this theory as one has in General Relativity. Construction of EMDA classical solutions seems to be useful in the search for the so-called exact four-dimensional string backgrounds. We hope that the possibility of getting (in principle) all classical solutions with two commuting Killing vectors will also be helpful in understanding more general aspects of this approach.

Acknowledgments

$^1$ Note added (July 3rd, 95): After this talk was given, a number of papers appeared closely related to the subject. An enhancement of the heterotic string symmetries in three dimensions was studied for the full effective action by A. Sen (Nucl. Phys. B434 (1995) 179); more general aspects were discussed by C. Hull and P. Townsend (Unity of Superstring Dualities, [hep-th/9410167]). Reduction to two dimensions was discussed by J. Maharana (Hidden Symmetries of Two Dimensional String Effective Action, [hep-th/9502001], Symmetries of the Dimensionally Reduced String Effective Action, [hep-th/9502002]), A. Sen (Duality Symmetry Group of Two Dimensional Heterotic String Theory, , TIFR-TH-95-10, [hep-th/95030157]) and J.H. Schwarz (Classical Symmetries of Some Two-Dimensional Models, preprint CALT-68-1978, [hep-th/9503078]; Classical Duality Symmetries in Two Dimensions, preprint CALT 68–1994, [hep-th/9505170]). Related discussion can be found also in A. Kumar, K. Ray, Ehlers Transformations and String Effective Action, preprint IP/BBSR/95-18, [hep-th/9503154], A.K. Biswas, A. Kumar, K. Ray, Symmetries of Heterotic String Theory, preprint IP/BBSR/95-51, [hep-th/9506037], I.R. Pinkstone, Structure of dualities in bosonic string theory, DAMTP R94-62, [hep-th/9505147]. The Kerr–NUT solution described above was uplifted by A. Sen to ten dimensions (Nucl. Phys. 440 (1995) 421), while C.V. Johnson and R.C. Myers have shown that the corresponding extremal cases may be interpreted as exact string backgrounds using the WZWN approach (A Conformal Theory of a Rotating Dyon, preprint PUPT–1524, McGill/95–01, [hep-th/9503027]). Further development within the present framework can be found in D.V. Gal’tsov, A.A. Garcia, and O.V. Kechkin, Symmetries of the Stationary Einstein–Maxwell Dilaton Theory, [hep-th/9504153], and D.V. Gal’tsov and O.V. Kechkin, U–duality and Symplectic Formulation of Dilatonic–Axion Gravity, [hep-th/9507003].
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