Log-logarithmic Time Pruned Polar Coding on Binary Erasure Channels

Hsin-Po Wang and Iwan Duursma
University of Illinois at Urbana–Champaign
{hpwang2, duursma}@illinois.edu

Abstract—A pruned variant of polar coding is reinvented for all binary erasure channels. For small \( \varepsilon > 0 \), we construct codes with block length \( \varepsilon^{-5} \), code rate \( \text{Capacity} - \varepsilon \), error probability \( \varepsilon \), and encoding and decoding time complexity \( O(N \log(\log \varepsilon)) \) per block, equivalently \( O(\log(\log \varepsilon)) \) per information bit (Propositions 5 to 8).

This result also follows if one applies systematic polar coding [Ari11] with simplified successive cancelation decoding [AYK11], and then analyzes the performance using [GX13] or [MHU16].

I. INTRODUCTION

In the theory of two-terminal error correcting codes, four of the most important parameters of block codes are block length \( N \), code rate \( R \), error probability \( P \), and per-bit time complexity \( bC \).

For instance, Shannon proves \((R, P) \rightarrow (\text{Capacity}, 0)\) by paying the price \( N \to \infty \) and exponential \( bC \). In the moderate deviations regime, \((N, R, P) \to (\infty, \text{Capacity}, 0)\) parametrically by paying exponential \( bC \) [AW10], [PV10], [AW14], [Ari15], [HT15]. LDPC codes and friends achieve acceptable \((N, R, P, bC)\)-tuples for practical use, but \( N, P, bC \) are difficult to parameterize [KRU13]. RA codes and friends enjoy bounded \( bC \) but \( N, P \) do not parameterize [PSU04], [PS05].

Polar codes have all four parameters parameterized. For instance, Lemma 3 implies

\[
(N, R, P, bC) = (N, \text{Capacity} - N^{-1/5}, 2^{-N^{1/24}}, O(\log N)).
\]

We provide a pruned variant of polar codes parameterized by

\[
(N, R, P, bC) = (N, \text{Capacity} - N^{-1/5}, N^{-1/5}, O(\log \log \varepsilon)),
\]

\[
(N, R, P, bC) = (\varepsilon^{-5}, \text{Capacity} - \varepsilon, \varepsilon, O(\log(\log \varepsilon))).
\]

over arbitrary binary erasure channels. That is, the per-bit complexity is log-logarithmic in \( N \), in \( P \), and in \( \text{Capacity} - R \). This justifies the title.

Section II introduces Arıkan’s idea of channel polarization and our generalization. Section III states and proves the main result. Section IV connects our work with others’.

II. PRELIMINARY

A. Channel polarization

Channel polarization [Ari09] is a method to synthesize some channels to form some extremely-unreliable channels and some extremely-reliable channels. The user then can transmit uncoded messages through extremely-reliable ones while transmitting predictable symbols through extremely-unreliable ones.

We summarize channel polarization as follows. Say we are going to communicate over this binary erasure channel

\[
W.
\]

We have two magic devices

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array}
\]

and

\[
\begin{array}{c}
\text{C} \\
\text{D}
\end{array}
\]

such that if we wire two i.i.d. instances of \( W \) as follows

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array},
\]

then pin \( A \) to \( B \) forms a less reliable synthetic channel \( W' \), while pin \( C \) to \( D \) forms a more reliable synthetic channel \( W'' \). Graphically, Formula (9) is equivalent to

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array}.
\]

Formula (9) being the base step, the next step is to duplicate Formula (9) and wire them as

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array} \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array},
\]

which is equivalent to four synthetic channels as

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array} \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array}.
\]

or simply

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array} \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \begin{array}{c}
\text{C} \\
\text{D}
\end{array}.
\]
Further wire Formula (11) as

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Similarly, draw

\[ W \xrightarrow{T_{An}} (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \]

which translates into the circuit

\[ \begin{array}{c}
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W}
\end{array} \]

That is, eight instances of \( W \) are transformed into four pairs of \( W^p, W^f \), into two quadruples of \( (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \).

Second perspective: in a tree like Formula (25), it could be that some synthetic channel, say \( (W^p)^f \), might not be polarized enough, i.e. it is neither extremely good nor extremely bad. Thus we further polarize it by applying an additional \( T_{An} \) as follows:

\[ W \xrightarrow{T_{An}} (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \]

which translates into the circuit

\[ \begin{array}{c}
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W} \\
\text{W}
\end{array} \]

That is, eight instances of \( W \) are transformed into four pairs of \( W^p, W^f \), into two quadruples of \( (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \).

In general, any rooted, full (each vertex has either zero or two children) binary tree of channels translates into a circuit of magic devices that transforms copies of the root channel to (copies of) leaf channels. See Appendix B for more examples.

Denote by \( T \) a tree of channels with root channel \( W \). The number of root channels that \( T \) consumes is \( 2^{\text{depth}(T)} \). The number of copies of a leaf channel \( w \) that \( T \) synthesizes is \( 2^{\text{depth}(T) - \text{depth}(w)} \). (Convention: the root has depth 0; the depth of a tree is the depth of the deepest leaf; and the tree with only one vertex has depth 0.)

C. Generalize to Unbalanced Tree Notation

The generalization comes in two perspectives, each motivated by an attempt to optimize polar coding.

First perspective: in a tree like Formula (26) or a larger tree, it could be the case that some synthetic channel, say \( (W^p)^f \), is so bad that applying further transformations sounds useless. If so, we may remove children of \( (W^p)^f \) to get

\[ W \xrightarrow{T_{An}} (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \]

That is, eight instances of \( W \) are transformed into four pairs of \( W^p, W^f \), into two quadruples of \( (W^p)^p, (W^p)^f, (W^f)^p, (W^f)^f \), and, notice another difference, only the two \( (W^p)^f \) are transformed into \( ((W^p)^p)^p, ((W^p)^p)^f, ((W^p)^f)^p, ((W^p)^f)^f \).

D. Bhattacharyya Parameter and Processes

The Bhattacharyya parameter \( Z(W) \) of a channel \( W \) measures the unreliability, the badness, of the channel. For binary erasure channels, \( Z(W) \) coincides with the erasure probability of \( W \). The Shannon capacity \( I(W) \) of \( W \) coincides with the complement \( 1 - Z(W) \).
Recall the processes \( K_i, Z_i, \) and \( I_i \) as defined in \cite[Section IV, third paragraph]{Ari09}. We now define their generalizations.

Given a channel tree \( T \) with root channel \( W \), define three discrete-time stochastic processes \( K_{i\wedge\tau}, Z_{i\wedge\tau}, I_{i\wedge\tau} \) and a stopping time \( \tau \) as follows: Start from the root channel \( K_{0\wedge\tau} := W \). For any \( i \geq 0 \), if \( K_{i\wedge\tau} \) is a leaf, let \( K_{i+1\wedge\tau} = K_{i\wedge\tau} \). If, otherwise, \( K_{i\wedge\tau} \) has two children, choose either child with equal probability as \( K_{i+1\wedge\tau} \). Let \( Z_{i\wedge\tau} \) be \( Z(K_{i\wedge\tau}) \). Let \( I_{i\wedge\tau} \) be \( I(K_{i\wedge\tau}) \). Let \( K_{\tau}, Z_{\tau}, I_{\tau} \) be the limits. Let \( \tau \) be depth\( (K_{\tau}) \).

By \cite[Proposition 8]{Ari09}, \( I_i \) is a martingale, hence \( I_{i\wedge\tau} \) is martingale by \cite[Theorem 5.2.6]{Dur10}. Since \( W \) is an erasure channel, \( Z_{i\wedge\tau} = 1 - I_{i\wedge\tau} \) is martingale as well. A charming consequence by \cite[Theorem 5.4.1]{Dur10} is

\[
I(W) = I_0 = \mathbb{E}[I_{\tau}]. \tag{31}
\]

For a tree \( T \) as in Formula (29), a possible instance of the process is

\[
K_{1\wedge\tau} \quad K_{2\wedge\tau} \quad K_{3\wedge\tau}
\]

\[
K_{0\wedge\tau}
\]

with \( K_{1\wedge\tau} = K_{4\wedge\tau} = K_{5\wedge\tau} = \cdots = K_{r} \) and \( \tau = 3 \). The probability measure of this path is \( 1/8 \). For another instance

\[
K_{0\wedge\tau} \quad K_{2\wedge\tau}
\]

\[
K_{1\wedge\tau}
\]

with \( K_{2\wedge\tau} = K_{3\wedge\tau} = K_{4\wedge\tau} = \cdots = K_{r} \) and \( \tau = 2 \), the probability measure is \( 1/4 \).

E. Construct Codes and Communicate

In a given tree \( T \), non-leaf vertices represent channels that are consumed to obtain their children. They are not available to users. Leaves of \( T \), however, represent channels that are available to users.

A person who wants to send messages can (a) choose a subset \( A \) of leaves, (b) transmit uncoded messages through leaf channels in \( A \), and (c) transmit predictable symbols through the remaining leaf channels.

This tree-leaves pair \((T, A)\) determines a block code. A block code has block length \( N \), code rate \( R \), error probability \( P \), and time complexity. The following is how to read-off these parameters from the pair \((T, A)\).

The block length \( N \) of \( T \) is the number of instances of \( W \) in the circuit.

\[
N := 2^{\text{depth}(T)}. \tag{34}
\]

The code rate \( R \) of \((T, A)\) is the number of instances of synthetic channels generated by the circuit that are included in \( A \), divided by the number of root channels the circuit consumes. It is also the probability of \( K_{\tau} \) ending up in \( A \).

\[
R := P\{K_{\tau} \in A\}. \tag{35}
\]

The error probability \( P \) of \((T, A)\) is the probability that any leaf channel in \( A \) fails to transmit the message. For the usual polar codes, this quantity is less than the weighted sum given in \cite[Proposition 2]{Ari09}.

\[
P \leq \sum_{w \in A} NP\{K_{\tau} = w\} Z(w). \tag{36}
\]

This is still true in our case, proof omitted.

The per-block time complexity is the time the circuit generated by \( T \) takes to execute. It is bounded from above by the number of magic devices multiplied by the time each magic device spends. (No parallelism allowed).

The reader can find in Appendix \( C \) how magic devices work, verification omitted. The construction suggests that each magic device spends constant time. With the help of Appendix \( B \), the reader can also find that the total number of magic devices in the circuit is

\[
N\mathbb{E}[\tau]. \tag{37}
\]

(Hint: double-count the number of devices each wire passes.) Thus the per-block time complexity is proportional to

\[
N\mathbb{E}[\tau]. \tag{38}
\]

The per-bit time complexity is the amortized time each information bit should pay. Unless the rate vanishes, it is proportional to

\[
\mathbb{E}[\tau]. \tag{39}
\]

F. Grow a Tree

We have shown how to estimate the performance of a block code \((T, A)\) if \( T \) and \( A \) are explicitly given. Now we demonstrate how to grow a good tree of prescribed depth \( n \).

Begin with \( W \) as the only vertex of a new rooted tree. Let \( Y(w) \) be \( \min\{Z(w), 1 - Z(w)\} \) in the following framed rule:

\[
\text{Apply } T_{\Lambda n} \text{ to } w \text{ if and only if } \text{depth}(w) < n \text{ and } Y(w) > \varepsilon 2^{-n}. \tag{40}
\]

The rule says: for each leaf \( w \), if the criteria \( \text{depth}(w) < n \) and \( Y(w) > \varepsilon 2^{-n} \) are met, apply \( T_{\Lambda n} \) to \( w \) to obtain \( w^B \) and \( w^E \) (just like Formula (9) and (24)), and then append \( w^B \) and \( w^E \) as children of \( w \). If, otherwise, either criterion is not met, leave \( w \) as a leaf.

This is a possible execution of the rule with \( Z(W) = .5, n = 3, \) and \( \varepsilon = .5 \). Start with \( W \) and write down \( Z(W) \)

\[
.5 \tag{41}
\]

Both .5 and 1 − .5 are larger than .5 · 2^−3, so we append 1 − (1 − .5)^2 and .5^2

\[
.5 \quad .25 \quad .75 \tag{42}
\]
Both .75 and 1 − .75 are larger than .5 · 2^{−3}, so we append 1 − (1 − .75)^2 and .75^2

\begin{align*}
&\begin{array}{c}
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\end{array}
\end{align*}

Both .25 and 1 − .25 are larger than .5 · 2^{−3}, so we append 1 − (1 − .25)^2 and .25^2

\begin{align*}
&\begin{array}{c}
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\end{array}
\end{align*}

Among the four newcomers, the second and the third are such that \(Y(w) > .5 \cdot 2^{-3}\), so we grow them further

\begin{align*}
&\begin{array}{c}
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\cdot5 & \cdot25 & .75 & \cdot375 & .625 \\
\end{array}
\end{align*}

Now we reach depth \(n = 3\); terminate. See Appendix D for another visualization.

Having \(T\), we declare \(A\) by

\[
\begin{array}{c}
\text{w \in A if and only if} \\
\text{w is a leaf and } Z(w) \leq 2^{-n}. \\
\end{array}
\]

We show in the coming section how \((T, A)\) performs.

### III. MAIN RESULT

The following lemma is inspiring.

**Lemma 1.** \textbf{[GXi13 Theorem 1]} There exists \(\mu > 0\) such that

\[
\mathbb{P}\left\{Z_i \leq 2^{-2^{0.494}}\right\} \geq I(W) - O(2^{-i/\mu}).
\]

The following lemma generalizes the idea.

**Lemma 2.** \textbf{[MHUT16 Theorem 3 and Formula (56)]} For \(\mu = 3.627\) and \(\gamma\) such that \(1/(1 + \mu) < \gamma < 1\),

\[
\mathbb{P}\left\{Z_i \leq 2^{-2^{\gamma h_2^{-1}(\gamma \mu + \gamma - 1)/\gamma \mu}}\right\} \geq I(W) - O(2^{-i/(1-\gamma)/\mu}).
\]

Here \(h_2^{-1}\) is the inverse function of the binary entropy function. This lemma almost suffices for the choice of constants in this work. A stronger version of the lemma is in our previous work.

**Lemma 3.** \textbf{[WDI18 Theorem 6]} If for \(\pi \in [0, 1],

\[
1 - \pi \frac{1 - \pi \mu'}{\mu' - \mu \pi} + h_2\left(\frac{\beta' \mu'}{\mu' - \mu \pi}\right) < 1,
\]

then

\[
\mathbb{P}\left\{Z_i \leq 2^{-2^{\gamma h_2^{-1}(\gamma \mu + \gamma - 1)/\gamma \mu}}\right\} \geq I(W) - O(2^{-i/\mu}).
\]

For \(\mu = 3.627\) given by [FVI14] and \((\mu', \beta') = (4, 1/24)\) chosen by us, Formula (49) becomes

\[
\frac{4}{4 - 3.627\pi} + h_2\left(\frac{1/6}{4 - 3.627\pi}\right) < 1,
\]

which holds for all \(\pi \in [0, 1]\), as shown below.

Thus Formula (50) becomes

\[
\mathbb{P}\left\{Z_i \leq 2^{-2^{i/24}}\right\} \geq I(W) - O(2^{-i/4}).
\]

Since we are on erasure channels, the “flipped version”

\[
\mathbb{P}\left\{I_i \leq 2^{-2^{i/24}}\right\} \geq Z(W) - O(2^{-i/4})
\]

also holds.

We are now ready to state and prove the main theorem of this work. Recall that \(Y(w) := \min\{Z(w), 1 - Z(w)\}\). Let \(Y_i\) be \(\min\{Z_i, 1 - Z_i\}\).

**Theorem 4.** Given \(W\) and \(\varepsilon\). Assign \(n := -5 \log_2 \varepsilon\). The framed rule

\[
\text{Apply } T_n \text{ to } w \text{ if and only if } \text{depth}(w) < n \text{ and } Y(w) > 2^{-n}.
\]

generates a circuit with \(O(N \log[\log \varepsilon])\) uses of magic devices.

**Proof:** Formulæ 53 and 54 give the trichotomy and the corresponding probabilities:

\[
\begin{aligned}
0 \leq Z_i &\leq 2^{-2^{i/4}} \quad \text{w.p. } I(W) - O(2^{-i/4}); \\
2^{-2^{i/4}} < Z_i &< 1 - 2^{-2^{i/4}} \quad \text{w.p. } O(2^{-i/4}); \\
1 - 2^{-2^{i/4}} \leq Z_i &\leq 1 \quad \text{w.p. } Z(W) - O(2^{-i/4}).
\end{aligned}
\]

Here “w.p.” reads “with probability”. In terms of \(Y_i\), the second line becomes

\[
\mathbb{P}\left\{Y_i > 2^{-2^{i/4}}\right\} = O(2^{-i/4}).
\]

Let \(\tau\) be the stopping time

\[
\tau := \min\{i \text{ such that } Y_i \leq 2^{-n}\} \cup \{n\}.
\]
The last line breaks into two cases: (a) when $i$ is large, i.e., when the second disjunct is false, the first disjunct must happen so Formula (57) applies; (b) when $i$ is small, we expect no synthetic channel to be polarized so we apply the worst, yet educated bound, 1. That is,

$$\mathbb{P}\{\tau > i\} \leq \begin{cases} O(2^{-i/4}) & \text{when } i > O(\log \log \varepsilon) \\
1 & \text{otherwise}. \end{cases}$$

Therefore, we obtain an estimate by Durisi Lemma 2.2.8

$$\mathbb{E}[\tau] = \sum_{i=0}^{\infty} \mathbb{P}\{\tau > i\} = O(\log \log \varepsilon).$$

Now generate a channel tree $T$ with root $W$ by the framed rule. The criteria in the rule coincide with the stopping time $\tau$ in Formula (58). Thus $\tau$ coincides with depth($K_\tau$), and the number of magic devices is bounded by

$$N \mathbb{E}[\tau] = O(N \log \log \varepsilon).$$

**Proposition 5.** The tree $T$ defined above possesses block length $N = 2^n = \varepsilon^{-5}$.

**Proof:** By the fact that the framed rule stops applying $T_{An}$ at depth $n$.

**Proposition 6.** The tree $T$ defined above possesses per-bit time complexity $O(\log \log \varepsilon)$.

**Proof:** By Formula (55) and the discussion that leads to Formula (59).

**Proposition 7.** Given $T$ defined above, define $A$ by

$$w \in A \text{ if and only if } w \text{ is a leaf and } Z(w) \leq \varepsilon 2^{-n}.$$

Then $(T, A)$ possesses block error probability $\varepsilon$.

**Proof:** Compute the error probability

$$P \leq \sum_{w \in A} N \mathbb{P}\{K_\varepsilon = w\} Z(w)$$

$$\leq \sum_{w \in A} N \mathbb{P}\{K_\varepsilon = w\} \varepsilon 2^{-n}$$

$$= NR\epsilon 2^{-n}$$

$$= R\epsilon$$

$$\leq \varepsilon.$$

**Proposition 8.** The pair $(T, A)$ defined above possesses code rate $I(W) - \varepsilon$.

**Proof:** The sample space is partitioned into the following three events:

$$S := \{0 \leq Z_\varepsilon \leq \varepsilon 2^{-n}\}.$$  

$$M := \{\varepsilon 2^{-n} < Z_\varepsilon < 1 - \varepsilon 2^{-n} \text{ for all } i \leq n\}.$$  

$$L := \{1 - \varepsilon 2^{-n} \leq Z_\varepsilon \leq 1\}.$$  

Recall $n := -5 \log_2 \varepsilon$. The second event is

$$M = \{\varepsilon 2^{-n} < Z_\varepsilon < 1 - \varepsilon 2^{-n}\}$$

$$\leq \{\varepsilon 2^{-n} < Z_\varepsilon < \varepsilon 2^{-n} - 6/5\}$$

$$\leq \{\varepsilon 2^{-n} < Z_\varepsilon < \varepsilon 2^{-n} - 6\varepsilon 2^{-n}/5\}$$

$$\leq \{\varepsilon 2^{-n} < Z_\varepsilon < \varepsilon 2^{-n}\}.$$  

For $\varepsilon$ small enough ($n$ large enough), the case $n < O(\log n)$ does not happen. Thus whether $Y_n > 2^{-2n/3}$ happens dominates $M$. By Formula (57),

$$\mathbb{P}(M) \leq O(2^{-n/4}).$$

Rewrite the capacity; here $\mathbb{I}(\bullet)$ is the indicator function:

$$I(W) = \mathbb{E}[I_r]$$

$$= \mathbb{E}[I_r \mathbb{I}(S)] + \mathbb{E}[I_r \mathbb{I}(M)] + \mathbb{E}[I_r \mathbb{I}(L)]$$

$$\leq \mathbb{E}[I_r \mathbb{I}(S)] + \mathbb{E}[I_r \mathbb{I}(M)] + \mathbb{E}[I_r (1 - Z_r) \mathbb{I}(L)]$$

$$\leq \mathbb{E}[I_r \mathbb{I}(S)] + \mathbb{E}[I_r \mathbb{I}(M)] + \varepsilon 2^{-n} \mathbb{E}[I_r \mathbb{I}(L)]$$

$$= \mathbb{P}(S) + \mathbb{P}(M) + \varepsilon 2^{-n} \mathbb{P}(L)$$

$$\leq \mathbb{P}(S) + O(2^{-n/4}) + \varepsilon 2^{-n}.$$  

Use it to compute the code rate:

$$R = \mathbb{P}\{K_\varepsilon \in A\}$$

$$= \mathbb{P}(S)$$

$$\geq I(W) - O(2^{-n/4}) - \varepsilon 2^{-n}$$

$$= I(W) - O(\varepsilon^{n/4}) - \varepsilon^5$$

$$\geq I(W) - \varepsilon$$

for $\varepsilon$ small enough ($n$ large enough).

Combining Proposition 5, 6, 7, and 8 we certify that the constructed code $(T, A)$ satisfies the properties claimed in the abstract.

### IV. Connection to Other Works

#### A. In Terms of Deleting Vertices

[AYK11] introduces the so-called “simplified successive cancellation” decoder, working as below: During the construction of polar codes, some synthetic channel, for instance $(W^n)^b$, may find that all its descendants are frozen (potentially because $(W^n)^b$ is too bad). In such case, it is unnecessary to establish the part of en-decoder circuit that corresponds to its children.

Readers may find that the paragraph above coincides with the philosophy of Formula (27) and (28).

[AYK11] then calls the synthetic channel $(W^n)^b$ a “rate-zero node”. Similarly, a “rate-one node” is a synthetic channel that is so good, all of its descendants being utilized. In such case,
[AYKI1] argues that it could save some time by shortcutting the classical successive cancelation decoder of Arkani.

That said, [AYKI1] does not realize that by not applying $T_n$ in the first place it could have saved more, ultimately reducing the per-bit time complexity from $\log N$ to $\log \log N$. Frankly speaking, [AYKI1] is aiming for general channels while our result applies only to erasure channels.

[ZZW+15] does similar things to polar codes with other kernels. [ZZP+14] does similar things, but is based on belief propagation.

**B. In Terms of Adding Vertices**

[EKMF15, EKMF17] introduce the so called “relaxed polarization”. [WLZZ15] introduces the so-called “selective polarization”. They suggest that when some synthetic channel, say $(W^2)^2$, is not perfectly polarized, it should be further polarized by concatenating with an outer polar code.

Readers may find that the paragraph above coincides with the philosophy of Formula (29) and (30).

[EECB17] is another attempt, which they called “code augmentation”, to protect unpolarized channels by appending polar codes to them.

[WYYT15, WYYT18] do very similar things which they called “information-coupling”. The idea is: some information bit might not be well-protected by the synthetic channel, say $(W^2)^2$, that it goes through. For the sake of reliability, send the same bit again through the same synthetic channel, $(W^2)^2$, in the very next block. Doing so merges two consecutive blocks into one big block.

**C. In Terms of Special Treatment**

Recall the recursive definition

$$Z_i = \left\{ \begin{array}{ll} 1 - (1 - Z_i)^2 & \text{w.p. } 1/2 \text{ (head)}; \\ \epsilon^2 & \text{w.p. } 1/2 \text{ (tail)} \end{array} \right. \quad (92)$$

Assume for some $m \in [2n/5, n]$,

$$\epsilon^2 < Z_m < \epsilon^{2m-7n/5}. \quad (93)$$

It is clear that although this synthetic channel is quite good, it is not good enough to become a leaf. What can we say about its descendants?

Since $Z_{m+i+1} < 2Z_m$, it turns out

$$Z_{m+i} < 2^i Z_m < \epsilon^{2i+m-7n/5} < \epsilon^{2-2n/5} \quad (94)$$

for all $i < n - m$. Thus if tail ever happens, say at time $m + i + 1$, then

$$Z_{m+i+1} = Z_m^2 < \epsilon^{2i+2m-4n/5} = \epsilon^{2-2n}, \quad (95)$$

which means a leaf. That is, the subtree rooted at $K_m$ is such that every down-child becomes a leaf, and every up-child has children, till depth $n$. Visually, the upper-child at depth $n$ is then frozen while all other leaves are utilized.

[SG13] recognizes that this subtree generates a single-parity-check subcode, which can be decoded more efficiently than the magic devices do.

Similarly, a $Z_m$ that is close enough to the top threshold $1 - \epsilon^{2-n}$ generates a subtree that mainly “grows downward” and every leaf except the very bottom one is frozen. Visually, the very next block. Doing so merges two consecutive blocks into one big block.

This either induces a trivial code (if the very bottom leaf is frozen) or a repetition code (if the very bottom leaf is utilized) and, again, can be efficiently decoded.

The simulation by [SG13], and subsequently by [SGV+14], suggests that this ad hoc treatment accelerates the real world performance. For our purpose, however, special treatment makes it difficult to bound the complexity as they are special.

**D. In Terms of Systematic Coding**

[Arı1] suggests systematic polar coding, where the receiver is not interested in $\hat{u}$ but wants to recover $x$ from $y$.

One consequence is that, if the two right pins of the magic device

$$\begin{array}{c} x \end{array} \quad (98)$$

correspond to two frozen channels, then this device can be dropped without affecting the overall decoding ability of the circuit. Similarly, if the two right pins correspond to two utilized channels, it could also be dropped.

The argument above gives another reason (or perspective) why the tree should be pruned. One may keep dropping magic devices (keep pruning the tree) till it stabilizes. It is easy to see that a device remains if and only if some of its children are frozen and some are utilized. Then it is not hard to estimate the number of remaining devices.

Our intuition suggests that the number of remaining devices is

$$O \left( N \log \left| \log \frac{\epsilon}{N} \right| \right) \quad (99)$$

where $\epsilon$ is the threshold of a channel being utilized (which is $\epsilon^{2-n}$ in our construction). When $N$ is polynomial in $\epsilon$, this reassures our result. This is the strategy we refer to in the second paragraph of the abstract.

**V. Future Works**

For more general binary channels such as BSC or BI-AWGN, Formula (54) is no longer true. Consequently a large portion of estimation done in this work does not apply. Potentially one can mimic [MHU16 Appendix A] to control $I(W) - R$.

From studies of random codes, $I(W) - R$ is polynomial in $N$ while $P$ is exponential in $N$. Thus it seems improper to
parametrize $I(W) - R$ and $P$ with a single variable $\epsilon$. It would be interesting if one could come up with a description of more general tradeoffs among $N$, $R$, $P$, and time complexity.

VI. CONCLUDING REMARKS

We propose a pruned variant of polar coding where the channel tree is pruned by closely looking at the Bhattacharyya parameters. Then we prove that the per-bit complexity is log-logarithmic in block length, in gap to capacity and in error probability.

This idea turns out to coincide with some existing works mentioned in Section [IV]. They found that doing this type of simplification reduces the empirical execution time significantly. But we could not find any statement about the log-logarithmic asymptote. (Arikan mentions time complexity $O(N \log N)$ in [Ari10] Section VII-C, last paragraph.)

Although the log-logarithmic asymptote is not record-breaking as other constructions with bounded per-bit complexity exist [PSU04], [PS05], our construction controls the block length while other works do not.

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APPENDIX

A. Simulation

We wrote a python script to support our claims. The script chooses \( f(W) = .618 \), varies \( n \), calculates \( \varepsilon = 2^{-n/5} \), and profiles the process of 1 Mibit. The per-bit time is shown below.

![Graph](image1)

Notice how the per-bit time does not grow proportionally to \( n \), while it does in classical polar codes.

The plot does not really prove anything as there are a lot of factors. For instance: (a) it is python; (b) it tests a small amount of data; (c) the channel is simulated by the built-in PRNG, which might dominate the performance; (d) the tree traversal is implemented by function recursion, wherein a function call serves as few as one bit for leaves at the very bottom of the tree.

It is more obvious if we look directly at \( \mathbb{E}[\tau] \) (and believe that the real world performance is really proportional to \( \mathbb{E}[\tau] \)).

![Graph](image2)

Notice how the plot bends downward like \( \log \log \varepsilon \approx \log n \) does. At \( n = 25 \), our construction saves half of magic devices.

Starting from \( n = 26 \) it is difficult to calculate the exact value of \( \mathbb{E}[\tau] \). We instead sample the process \( Z_n \) a thousand times. The result is majestic.

![Graph](image3)

The plot stops at \( n = 63 \) because that is about the size of the internet, where the sample mean of \( \tau \) does not exceed 16. The sample mean of \( \tau \) exceeds 17 when \( n \approx 240 \). That is, when the block length is about the number of atoms in the known universe, and the error probability is \( 2^{-48} \).

B. Gallery of Trees and Circuits

For trees, the labels of transformations and channels are omitted.

For circuits, only the decoder component is shown; the encoder component is the reflection of the decoder component. Plus, we do not wire shallow channels to the right boundary as the order of shallow channels at a deeper layer is irrelevant.

![Graph](image4)
C. The Automata Model

1) The Sending Component: The person who sends owns two types of devices. The frozen bit sender corresponds to frozen bits in polar coding context. It sends out zero, once. The utilized bit sender, on the other hand, sends the information bit the person wants to send, once. It is important that these automata send out bits once and become “idle” thereafter, so the circuit is not flooded by repetitive bits.

2) The Encoding Component: The encoding component consists of many many \((u + v, v)\)-construction devices. The arithmetics is done in \(F_2\), i.e., \(\text{GF}(2)\). Every encoding device will be executed exactly once.

3) The Channel Component: We consider binary erasure channels. They are best described by the so called probabilistic automata.

4) The Decoding Component: The decoding component consists of devices that reverse the \((u + v, v)\)-construction. Each decoding device is executed for three things: Firstly it is activated by two incoming bits. It then tells the upper successor its best guess \(\hat{u} = y - z\), which is supposed to be the \(u\) given that inputs are \((u + v, v)\). Secondly it will receive feedback from the upper successor, saying that the correct bit is indeed \(\hat{u}\). Based on this information, it tells the lower successor its best guess of \(v\). Thirdly after the lower successor confirms the value of \(\hat{v}\), it forwards the information it collects, in the form of \((u + v, v)\), to its predecessors. Here the subtraction involving \(\hat{v}\) results in \(\hat{v}\). The binary operator \(\lor\) returns one of its non-\(\hat{v}\) operand(s), if any; otherwise it returns \(\hat{v}\).

5) The Receiving Component: The person who receives owns two type of devices. The frozen bit receiver receives any input symbol and reply 0. The utilized bit receiver receives any input symbol and blindly replies the exact same symbol. If it happens that the utilized bit receiver receives \(\hat{v}\), then there is no chance to recover this erasure anymore; the receiver may throw a BlockError exception that terminates the automata.

6) Assembled Automata: In the next page, notice that Formulae (146) and (149) are different codes, but provide the same protection.
D. Growing/Pruning Visualization

For each segment, the line width is $\sqrt{2}$ thinner than its predecessor, and is $\sqrt{2}$ more transparent than its predecessor. Thus the visual darkness indicates the probability that the process $K_0 \land \tau, K_1 \land \tau, \ldots$ passes there.

Consider this figure exaggerated as $\varepsilon 2^{-n}$ shall be smaller than $2^{-n}$, about 5 µm on this sheet of paper. A 300dpi printer prints in multiple of 85 µm.