Complex WKB evolution of Markovian open systems

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Abstract

We derive a consistent complex WKB expression for the characteristic function, that is, the Fourier transform of the Wigner function, of a Markovian open system, defined by a generic Lindblad equation. The Hamiltonian can be a general function of positions and momenta, so the only restriction is that linear coupling to the environment is assumed. The propagation is shown to correspond to a classical evolution of the Liouville type in a complex double phase space, the imaginary part of the double Hamiltonian being responsible for decoherence, whereas dissipation is included in the real part. The theory, exact in the quadratic case, is designed to describe the decoherent and dissipative evolution of localized states, such as the interference terms of a Schrödinger cat state, as well as the evolution of extended states, which were the exclusive subjects of our previous real WKB approximation. The present rederivation allows us to interpret the real formalism as a first-order classical perturbation of the complex theory and to discuss its validity. This is also clarified by a simple cubic Hamiltonian that illustrates various levels of approximation derived from the complex WKB theory.

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1. Introduction

When a system is isolated, its evolution is unitary and preserves the purity of the state. When the same system interacts with an environment, which is always the case in realistic situations, it becomes necessary to consider the sum of the system plus its environment as a new quantum system so as to recover a unitary evolution. However, the state of the system itself undergoes a non-unitary dynamics, which is not described by any general theory, as it often depends on the state of environment, or the type of averaging performed on it. Three general phenomena are a characteristic of such open systems: decoherence corresponds to the vanishing of the
off-diagonal terms of the density operator, leading finally to a statistical distribution which can be interpreted classically. Thus, it can explain dynamically one aspect of the reduction of the wave packet, postulated for the measurement of a quantum state. Dissipation corresponds to a loss or gain of energy and is usually considered to be much slower than decoherence, especially for large systems. It is also present in classical open systems, as is diffusion, which accounts for the gradual spread of an initial phase space distribution.

The general evolution of the state of an open system is a practical issue in many experimental situations, already well established in quantum optics [1], also in atomic and nuclear physics, and more recently in the physics of quantum information where it is of crucial importance to control the interaction of the system with its environment, in order to avoid decoherence. To construct theoretical models with the maximum of generality, such an evolution should satisfy at least three basic requirements: preservation of Hermiticity (\( \text{Tr} \hat{\rho} A = \mathbb{R} \)), trace (\( \text{Tr} \hat{\rho} = 1 \)) and positivity (\( \text{Tr} \hat{\rho} |\psi\rangle \langle \psi| \geq 0 \)).

Two main approaches have been explored to obtain explicit dynamics of an open quantum system. The first is to consider that the open system is part of a bigger closed system, obeying a unitary evolution, and to trace over the environment part, that is, over all the degrees of freedom which are not directly concerned with the system under study. Such an approach leads to the Bloch–Redfield equations [2, 3]. The second is to assume that, beyond the above general requirements, the time evolution should obey a semigroup law, that is, a group law—as for unitary evolution—without time reversibility. Then the most general equation was derived by Lindblad [4]( see also [5]). The semigroup assumption can be related to the Markovian one, that the environment does not remember the past influence of the system.

The Lindblad equation can be written

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \frac{1}{2\hbar} \sum_k \left( 2\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \hat{L}_k^\dagger \hat{L}_k \hat{\rho} - \hat{\rho} \hat{L}_k^\dagger \hat{L}_k \right). \tag{1}
\]

The Hamiltonian \( \hat{H} \) describes the unitary evolution of the system without environment. There is no restriction on the operators \( \hat{L}_k \), which account for the coupling to an environment and are commonly dubbed Lindblad operators; they are not Hermitian when the coupling is dissipative. For instance, the master equation of quantum optics can be seen as a special case of the Lindblad equation, with the operator \( \hat{L}_1 = \hat{a} \), i.e. the annihilation operator, describing the emission process, and \( \hat{L}_2 = \hat{a}^\dagger \) the absorption.

Though the applicability of this formalism is not universal, as many realistic systems are not Markovian, it has the advantage of being relatively simple to deal with, while keeping a fairly general framework that is employed efficiently in quantum optics and in scattering problems. Its relation to unitary quantum mechanics is analogous to that between general (dissipative) dynamical systems and the restricted class of classical Hamiltonian systems.

For these reasons it is often taken as a theoretical basis for studying decoherence, and its consequences [6–8]. Even so, such investigations and discussions tend to be restricted to very simple systems, such as a Harmonic oscillator linearly coupled to a bath, for which the Lindblad equation can be solved exactly. Any more complex systems, such as a cubic potential, already require a tedious numerical simulation of the dynamical partial differential equation.

These simulations do not give access to scaling properties of the solutions. Individual computational experiments may lead to crucial insights into the delicate intertwined phenomena to be elucidated in open quantum systems, but this is not comparable to the achievements of the semiclassical theory of closed systems, with its unique power of unifying our view of phenomena displayed by a wide range of systems. By building a WKB framework for the Lindblad equation, our aim is to provide an analytical tool which could tackle, for
instance, the behavior of non-harmonic open systems, and give a more subtle picture of decoherence than the mere ‘exponential vanishing of non-diagonal terms’. In particular we here prepare the theoretical tools, so as to explain, on a more rigorous basis, the transition from quantum to classical evolution predicted by Zurek et al [8], the effect of chaos on this transition [9], or the competition between decoherence and nonlinearity observed by Toscano et al [10] which, up to now, are entirely grounded on numerical simulations.

In this paper, we will focus on building semiclassical approximate solutions of a Lindblad equation with a general Hamiltonian part and Lindblad operators that are linear functions of the momentum and position. This involves a considerable adaptation of the more familiar semiclassical approximations for pure states. In previous papers, we generalized the analysis of the evolution of extended states, the WKB theory based on a real classical phase space, that goes back to Van Vleck [11] (see also [12–14]). Here we develop the WKB theory on a complexified phase space, thus generalizing previous semiclassical theory of unitary evolution developed by Huber [15], Heller [16] and Littlejohn [17] and Maslov [18]. The advantage of the complex theory is that it allows us to include, within the same framework, density operators that are localized, such as the interference terms arising from a superposition of coherent states. Furthermore, the complex theory is more robust to the fast damping of quantum correlations induced by decoherence, which breaks down the validity of the real stationary phase method.

Just as with pure states, the evolution may in time extend the state beyond the range of validity for the theory and this intermediate stage also demands attention (see e.g. [19], for pure state evolution). However, this is a case where the treatment of open systems has an advantage: usually the process of decoherence is exponentially fast with respect to other relevant features of the motion.

The complex WKB theory is here employed to rederive its real counterpart. Both become exact in the special case of a quadratic Hamiltonian and Lindblad operators that are linear in the positions and momenta. However, we find that the complex theory is more accurate, even for extended states for which the real WKB theory should hold. If the action of the initial state has no imaginary part, which is fulfilled by a generic propagator introduced in [22], then the real theory coincides with a perturbative expansion of the complex one. Therefore, it is not necessary to abandon the more transparent intuitive content of real WKB for the evolution of extended states, since the Markovian approach assumes that the environment is weakly coupled to the system.

We represent the solution in the ‘chord’ space, that is the Fourier conjugate of the Weyl–Wigner representation of the state operator, which is defined in the ‘center’ space. Together they describe a double phase space, where the double Hamiltonian generates corresponding classical trajectories that support the semiclassical approximation. It is this double phase space that is now complexified. One should note that the imaginary part of the Hamiltonian accounts for the decoherence and diffusion induced by environment, whereas the dissipative process is entirely included in its real part. It is the doubling of phase space which allows us to combine a real symplectic Hamiltonian theory with a dissipative dynamics. This paper completes our general construction of semiclassical methods for the Lindblad equation, based on previous papers, [20, 21] and [22], but it can be understood separately, since all the approximations are quite transparent.

We first give a reminder for the Weyl–Wigner representation, and write the Lindblad equation in this formalism. Then we describe the general WKB procedure by introducing an adapted complex double phase space driven by a complex Hamiltonian, which leads to a Hamilton–Jacobi equation. After verifying the agreement of the theory with the quadratic case, we develop an effective real WKB dynamics, based on a perturbative approach, which is valid for states whose initial classical action is real. We apply this real WKB to a generic
mixed propagator $R_x(y, t)$ which can turn an initial Wigner function $W_0(x)$ into the evolving chord function $\chi(y, t)$.

Finally we construct a simple non-quadratic example. Unlike realistic systems of physical interest, for which our approximations can only be compared with detailed computations, we here relate them to exact solutions of the Lindblad equation.

One should note that all the following formulae are appropriate for a system with a single degree of freedom, in order to clarify the notation. Nonetheless, it is quite simple to generalize our treatment and results for a finite number of degrees of freedom.

2. Lindblad equation in the Weyl formalism

The Weyl representation maps every quantum operator onto a phase space function, that is, a function of the vector $x = (p, q)$ [23, 24]. For an operator $\hat{A}$ the Weyl symbol $A$ is defined as

$$A(x) = 2\int \exp\left(-\frac{i}{\hbar}pQ\right)\left\langle q + \frac{Q}{2}, \hat{A} \mid q - \frac{Q}{2}\right\rangle dQ.$$  \hspace{1cm} (2)

The Weyl symbol of the state operator $\hat{\rho}$ is the Wigner function

$$W(x) = \mathcal{N}\int \exp\left(-\frac{i}{\hbar}pQ\right)\left\langle q + \frac{Q}{2}, \hat{\rho} \mid q - \frac{Q}{2}\right\rangle dQ,$$  \hspace{1cm} (3)

with $\mathcal{N} = 1/(2\pi\hbar)$; its Fourier transform, the chord function $\chi(\xi)$, also called characteristic function, is

$$\chi(\xi) = \mathcal{N}\int \exp\left(-\frac{i}{\hbar}\xi \cdot x\right)W(x) dx,$$  \hspace{1cm} (4)

where the wedge product of two vectors $x = (p, q)$ and $x' = (p', q')$ is defined by $x \wedge x' = pq' - p'q = \parallel x \cdot x'\parallel$, which also defines the skew matrix $\parallel = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One can have the chord symbol directly from the quantum operator through the formula

$$\chi(\xi) = \mathcal{N}\int \exp\left(-\frac{i}{\hbar}\xi \cdot x\right)\left\langle q + \frac{\xi q}{2}, \hat{\rho} \mid q - \frac{\xi q}{2}\right\rangle dq.$$  \hspace{1cm} (5)

We call the space of all $x$ the center space, and the space of all $\xi$ the chord space.

In the chord space, by using product rules for the product of operators [24], the Lindblad equation is represented by a partial differential equation. This equation is actually simpler than in the Weyl (center) representation, and this justifies our choice. In the case where the Lindblad operators are the linear functions of $\hat{p}$ and $\hat{q}$, that is, $\hat{L} = l' \cdot \hat{x} + il'' \cdot \hat{x}$ with $l'$ and $l''$ real vectors, this equation can be written as

$$\frac{\partial \chi}{\partial t}(\xi, t) = -\frac{i}{\hbar}N^2\int \left[ H\left(x' + \frac{\xi}{2}, t\right) - H\left(x' - \frac{\xi}{2}, t\right) \right]$$

$$\times \exp\left(\frac{i}{\hbar}(\xi' - \xi) \wedge x'\right)\chi(\xi', t) d\xi' dx'$$

$$- \gamma \xi \cdot \frac{\partial \chi}{\partial \xi}(\xi, t) - \frac{1}{2\hbar}[(l' \cdot \xi)^2 + (l'' \cdot \xi)^2]\chi(\xi, t).$$  \hspace{1cm} (6)

The dissipation coefficient

$$\gamma = l'' \wedge l',$$  \hspace{1cm} (7)

is null for a Hermitian Lindblad operator ($l'' = 0$) and we then have a purely diffusive case. $H$ is the Weyl representation of the Hamiltonian of the isolated system and coincides with
the corresponding classical Hamiltonian, up to corrections coming from non-commutativity of $\hat{p}$ and $\hat{q}$. Its arguments in equation (6) are the pair of remarkable points $x_+ = x + \frac{\xi}{2}$ and $x_- = x - \frac{\xi}{2}$, which can be considered as both tips of a chord $\xi$. Although this chord $\xi = (\xi_\rho, \xi_q)$ can be interpreted as an auxiliary conjugate variable of $x$, in the current approach it is actually more convenient to write the solution in terms of $y = \frac{1}{2} \xi = (\xi_\rho, \xi_q)$. Indeed, the direct sum of these conjugate spaces can be interpreted as a double phase space, where $x$ formally plays the role of the position $q$, and $y$ the role of its Fourier conjugate, the momentum $p$. Then the above equation becomes

$$\frac{\partial \chi}{\partial t}(y, t) = -\frac{i}{\hbar} N^2 \int \left[H(x' - \frac{1}{2} \|y\|, t) - H(x' + \frac{1}{2} \|y\|, t)\right]$$

$$\times \exp \left(\frac{i}{\hbar} (y' - y) \cdot x'\right) \chi(y', t) \, dy' \, dx'$$

$$- \gamma y \cdot \frac{\partial \chi}{\partial y}(y, t) - \frac{1}{2\hbar} \left[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2\right]\chi(y, t).$$

(8)

The same name has been kept for the characteristic function $\chi(y, t)$, though strictly this should be $\chi(\xi, t) = \chi(-\|y\|, t)$, and we have set the complex vector

$$\lambda = \lambda' + i\lambda'' = \|l' + il''\|.$$  

(9)

The term $y \cdot \frac{\partial \chi}{\partial y}$ can actually be included in the integral term, thanks to an integration by parts of the exponential, and one finally has

$$\frac{\partial \chi}{\partial t}(y, t) = -\frac{i}{\hbar} N^2 \int H(x', y, t) \exp \left(\frac{i}{\hbar} (y' - y) \cdot x'\right) \chi(y', t) \, dy' \, dx'$$

$$- \frac{1}{2\hbar} \left[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2\right]\chi(y, t).$$

(10)

with

$$\mathcal{H}(x, y, t) = H \left(x - \frac{1}{2} \|y\|, t\right) - H \left(x + \frac{1}{2} \|y\|, t\right) - \gamma x \cdot y$$

$$= \mathcal{H}'(x, y, t) - \mathcal{H}''(x, y, t) - \gamma x \cdot y.$$

(11)

This is exactly the double Hamiltonian that generates the classical motion underlying the semiclassical approximations in [22]. Obviously, the double Hamiltonian will be time independent if it is obtained from a time-independent single Hamiltonian. Furthermore, in the absence of dissipation, both $\mathcal{H}'(x, y) = H(x - \frac{1}{2} \|y\|)$ and $\mathcal{H}''(x, y) = H(x + \frac{1}{2} \|y\|)$ will also be the constants which generate independent motions for both chord tips. The last hyperbolic term of $\mathcal{H}(x, y, t)$ is not present in unitary evolution and it accounts for all dissipation effects. In the usual case of a positive dissipation coefficient, a contraction of the $x$ coordinates is generated, while an expansion of the chords guarantees the overall volume conservation in double phase space, such that trajectories spiral into a stable equilibrium at origin.

Hence, doubling of phase space provides a description of dissipation which is imbedded in a symplectic Hamiltonian dynamics.

Note that (10) can also be written as

$$\frac{\partial \chi}{\partial t}(y, t) = -\frac{i}{\hbar} N \int \mathcal{H}(x', y, t) \exp \left(-\frac{i}{\hbar} y \cdot x'\right) W_t(x') \, dx'$$

$$- \frac{1}{2\hbar} \left[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2\right]\chi(y, t).$$

(12)
in terms of the evolving Wigner function. Alternatively, the definition of the complex double Hamiltonian

$$\mathcal{H}_c(x, y, t) = \mathcal{H}(x, y, t) - \frac{i}{2}[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2]$$

(13)

leads to

$$\frac{\partial \chi}{\partial t}(y, t) = -\frac{i}{\hbar} \mathcal{H}_c \left( \frac{-\hbar}{i} \frac{\partial}{\partial y} (^{(1)}y^{(2)}, y^{(2)}, t) \right) \chi(y, t),$$

(14)

where \(^{(1)}\) and \(^{(2)}\) mean that the derivatives are taken first and then the \(y\) terms are multiplied. Given this specific choice of ordering for the operators \(\hat{y} = y\) and \(\hat{x} = -\frac{\hbar}{i} \frac{\partial}{\partial y}\), this is completely analogous to the Schrödinger equation, which thus allows us to extend the various forms of WKB theory, once it is recalled that variations in operator ordering have effects that are semiclassically small.

The differential term on the RHS of (14) (or the integral term on the RHS of (12)) represents the unitary part of the evolution. In other words, in the chord representation, the commutator is specified by the real double Hamiltonian as

$$[\hat{H}, \hat{\rho}]_{\text{chord}} = \mathcal{H}(\frac{-\hbar}{i} \frac{\partial}{\partial y} (y^{(2)}, y^{(2)}, t)) \chi(y, t).$$

(15)

### 3. General complex dynamics

We here assume that the chord representation \(\chi(y, t)\) of the localized state has the usual semiclassical form:

$$\chi(y, t) = \exp \frac{1}{\hbar} S(y, t),$$

(16)

where \(S(y, t)\) is a function with complex values of order \(O(\hbar^0)\). One shall be aware that if we find the time evolution of such a state determined by the Lindblad equation, then we can also evolve any linear combination of such states, which can be coherent states for instance. This is a consequence of the linearity of the Lindblad equation.

This semiclassical form naturally induces an \(\hbar\) expansion of the unitary part of the equation, as is shown in appendix \(A\):

$$\mathcal{H} \left( \frac{-\hbar}{i} \frac{\partial}{\partial y} (y^{(2)}, y^{(2)}, t) \right) \chi(y, t) = \left[ \mathcal{H} \left( -\frac{\partial S}{\partial y}(y, t), y, t \right) + O(\hbar) \right] \chi(y, t).$$

(17)

Hence, by expanding (14) at leading order in \(\hbar\), one obtains the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t}(y, t) = -\mathcal{H} \left( -\frac{\partial S}{\partial y}(y, t), y, t \right) + \frac{i}{2}[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2] + O(\hbar)$$

$$= -\mathcal{H}_c \left( -\frac{\partial S}{\partial y}(y, t), y, t \right) + O(\hbar).$$

(18)

This leads to a double phase space generalization of the complex WKB theory in [15]. Note that the present Hamilton–Jacobi equation is defined by a Hamiltonian of an unusual form: not only does it not separate into the familiar kinetic and potential energy terms, but is complex. Therefore the action, \(S(y, t)\), becomes complex, even in the special case where the initial action, \(S_0(y) = S(y, 0)\) is real. Because of these unusual features, we will work directly with the Hamiltonian formalism in the full complex double phase space, rather than attempting to connect the evolving action to a Lagrangian.
In general the initial action specifies a Lagrangian surface of half the dimension of the complex double phase space:

\[ y_0 = z \]
\[ x_0 = -\frac{\partial S_0}{\partial y}(z). \]  

(19)

To obtain the formal solution of the Hamilton–Jacobi equation, it is convenient to define a family of classical trajectories \((x_t, y_t)\) in the complex double phase space with initial conditions on this initial surface and driven by the complex Hamiltonian (13) through Hamilton’s equations:

\[ \dot{y}_t = -\frac{\partial H_c}{\partial x}(x_t, y_t, \tau) = -\frac{\partial H_c}{\partial y}(x_t, y_t, \tau) \]
\[ \dot{x}_t = \frac{\partial H_c}{\partial y}(x_t, y_t, \tau) = \frac{\partial H_c}{\partial x}(x_t, y_t, \tau) - i(\lambda' \cdot y_t)\lambda' - i(\lambda'' \cdot y_t)\lambda''. \]  

(20)

In the present case of a single degree of freedom, this family is spanned by variables \(z\) (two complex dimensions) and \(\tau\) (one real dimension). Hence, considering the real and imaginary parts of each complex variable, this forms a real submanifold \(M\) with five dimensions, within a real phase space of eight dimensions, corresponding to the four complex dimensions of \((x, y)\). This submanifold, which will serve as a backbone to build our solution, is completely parametrized by \((y, \tau)\) through a function \(x_M(y, \tau)\), such that \((x_M(y, \tau), y)\) is the most general point of \(M\).

Our phase space can be considered as a hyperplane of a fully symplectic complex structure, which would also have complex times. That full structure would allow for Wick rotations and may be interesting for treating tunneling, but it also brings supplementary difficulties that we prefer to leave for now. Moreover, since the real time is stable throughout the dynamics, our treatment remains compatible with any future extension in that direction anyway.

In the real case, it is a well known fact, see for instance [25], that the solution \(S(y, t)\) of the Hamilton–Jacobi equation

\[ \frac{\partial S}{\partial t}(y, t) = -\mathcal{H}\left(-\frac{\partial S}{\partial y}(y, t), y, t\right) \]  

(21)

is the generating function of the submanifold \((x_M(y, t), y)\) for each \(t\). The complex generalization is straightforward and is proved in appendix B by showing that the differential form

\[ \delta s = -x_M(y, \tau) \cdot \delta y - \mathcal{H}_c(x_M(y, \tau), y, \tau) \, d\tau \]  

(22)

is a closed form on \((y, \tau)\). Hence, we can define the solution

\[ S(y, t) = S_0(y_i) + \int_{y_i, 0}^{y, t} \delta s, \]  

(23)

where \(y_i\) is arbitrary and the integral can be performed along any path of \(M\) joining \((y_i, 0)\) to \((y, t)\).

A natural choice of such a path is \((\bar{y}_t, \tau)\), such that \((\bar{x}_t, \bar{y}_t)\) is the classical trajectory of \(M\) with \(\bar{y}_t = y\), literally the history of \(y\) at time \(t\). We have in particular \(x_M(\bar{y}_t, \tau) = \bar{x}_t\) by construction. The choice of this trajectory sets the value of \(y_i = \bar{y}_0\). Then one can more explicitly write

\[ S(y, t) = S_0(\bar{y}_0) + \int_0^t \left[ -\bar{x}_t \cdot \frac{\partial \bar{y}_t}{\partial \tau} - \mathcal{H}_c(\bar{x}_t, \bar{y}_t, \tau) \right] d\tau. \]  

(24)
One should keep in mind that \( (\bar{x}_\tau, \bar{y}_\tau) \) implicitly depends on \((y, t)\), and in particular \(\bar{y}_0\) and \(\bar{x}_0 = -\frac{\partial S_0}{\partial y}(\bar{y}_0)\) are the complex functions of \((y, t)\). There may be many complex trajectories from the initial surface to that with fixed \(y\), just as in ordinary complex WKB for unitary evolution \[15\]. However, in the case that the initial action \(S_0(y)\) is real and the Lindblad terms are small, the relevant trajectory should lie close to a real trajectory of the unitary problem. This motivates a perturbative real WKB theory introduced in a further section. Tunneling might be included in the underlying classical description by allowing for complex time trajectories.

The final expression of the solution, in terms of the real part of the double Hamiltonian, is then

\[
\chi(y, t) = K \exp \left( \frac{i}{\hbar} S_0(\bar{y}_0) - \frac{i}{\hbar} \int_0^t \left[ \bar{x}_\tau \cdot \frac{\partial \bar{y}_\tau}{\partial \tau} + \mathcal{H}(\bar{x}_\tau, \bar{y}_\tau, \tau) \right] d\tau \right. \\
- \frac{1}{2\hbar} \int_0^t \left[ (\lambda \cdot \bar{y}_\tau)^2 + (\lambda'' \cdot \bar{y}_\tau)^2 \right] d\tau.
\] (25)

4. The quadratic case

Here we show that in the quadratic case, solution (25) of the complex WKB approximation coincides with the exact solution derived in \[20\]. We start from a quadratic Hamiltonian

\[
H(x) = \mathbb{H} \cdot x
\] (26)

with some symmetric matrix \(\mathbb{H}\). The complex double phase space Hamiltonian

\[
\mathcal{H}_c(x, y) = -2x \cdot \mathbb{H} x - \gamma x \cdot y - \frac{i}{2} \left[ (\lambda \cdot y)^2 + (\lambda'' \cdot y)^2 \right]
\] (27)

induces the following double phase space dynamics:

\[
\dot{x}_\tau = (2\mathbb{H} - \gamma) x_\tau - i(\lambda \cdot y_\tau) \lambda - i(\lambda'' \cdot y_\tau) \lambda''
\]

\[
\dot{y}_\tau = (2\mathbb{H} + \gamma) y_\tau.
\] (28)

Note that in this case, even though \(x_\tau\) is generally complex, its motion does not affect the trajectory \(y_\tau\), which remains real at all times. Then the ‘history’ of \((y, t)\) is

\[
\bar{x}_\tau = -e^{-\gamma T} R_t (e^{-\gamma T} R_t^T y) - \frac{1}{2} \int_0^T e^{i(2\gamma t - t - \gamma u)}
\]

\[
\times \left[ ((R_{t-u} \lambda \cdot y) R_{t-u} \lambda + (R_{t-u} \lambda'' \cdot y) R_{t-u} \lambda'') du \right] \\
\bar{y}_\tau = e^{i(\gamma(t - t - \gamma u)} R_{t-u}^T y,
\] (29)

in the notation of section 3. Here \(R_t\) defines the classical propagation operator generated by the Hamiltonian (26), that is

\[
R_t = \exp \left( 2\mathbb{H} t \right),
\] (30)

and \(R_t^T\) stands for its transpose. Because \(\mathcal{H}(x, y, t)\) is linear in \(x\), one has besides

\[
-\dot{x}_\tau \cdot \frac{\partial \bar{y}_\tau}{\partial \tau} = \mathcal{H}(\bar{x}_\tau, \bar{y}_\tau),
\] (31)

so that, the expression of \(S(y, t)\) given by (24) boils down to

\[
S(y, t) = S_0(e^{-\gamma T} R_t^T y) + \frac{i}{2} \int_0^T \left| \lambda \cdot e^{i(\gamma(t - t - \gamma u)} R_{t-u}^T y \right|^2 d\tau.
\] (32)
Hence, the full complex WKB solution can be written as
\[
χ(y, t) = χ₀(e^{-γtR^{\top}y}) \exp\left(-\frac{1}{2\hbar} \int_0^t e^{2γ(t'-t)} |\lambda \cdot R^{\top}_{t'-t} y|^2 dt'\right).
\] (33)

Recalling our use of the same symbol, \(χ(y, t)\), as a shorthand for \(χ(ξ, t) = χ(-Jy, t)\), together with (9), we find that the complex WKB approximation coincides exactly with the exact solution in the quadratic case. Indeed, the general solution derived in [20] was written in the following form:
\[
χ(ξ, t) = χ₀(e^{-γtR_{-t}ξ}) \exp\left(-\frac{1}{2\hbar} \int_0^t e^{2γ(t'-t)} |\lambda \cdot R_{t'-t} ξ|^2 dt'\right),
\] (34)
where \(χ₀(ξ)\) is any general initial chord function. To see that (34) is identical with (33) one just needs to note that
\[
R_{-t}ξ = -R_{-t}Jy = -J(R^{\top}_t y).
\] (35)

Thus, the general picture is that of the exact classical propagation of the chord function (the same as the unitary evolution generated by the quadratic Hamiltonian), multiplied by a Gaussian factor that progressively attenuates the contribution of long chords. Fourier transformation of this result then supplies a unitarily evolving Wigner function, convoluted with a Gaussian window that widens in time.

5. Real dynamics

5.1. General analysis

Complex WKB theory is more elegant than its real counterpart, in that the evolving complex action accounts explicitly for the entire evolution of the wavefunction, whereas the real WKB action has to be supplemented by an evolving amplitude. Even so, the advantage of working with a more intuitive real phase space may predominate, whenever both theories lead to similar results. It is important to note that, even though both sets of parameters \(λ'\) and \(λ''\) are real, the last integral in the general complex WKB solution (25) will not be real in the case of a complex trajectory. Thus, it is not immediately clear that this still has the role of attenuating the amplitude of long chords as in the exact quadratic theory. In contrast, the real WKB theory [22], which is also exact in the quadratic case, generalizes the quadratic exponent in (34) by a negative decoherence functional with monotonic increasing modulus.

The necessity of the complex formalism in the description of unitary evolution arises when the amplitudes, \(α_j(ξ) = α_j(-Jy)\), in
\[
χ(ξ) = \sum_j α_j(ξ) e^{i\mathcal{S}_j(ξ)/\hbar},
\] (36)
do not vary smoothly as compared to the complex oscillations. In this situation, simple (real) stationary phase approximations are not allowed, i.e. when the variation of the real part of the action is also divided by the small parameter, \(\hbar\), it is necessary to resort to full complex saddle point approximations. Where this difficulty is not present, it was shown that both WKB theories lead to equivalent descriptions in the case of unitary evolution [15]. In short, the evolution of the imaginary part of the complex action matches that of the logarithm of the evolving real WKB amplitude.

The formal similarity between the Schrödinger equation and the Lindblad equation (14) for the chord function would permit us to immediately incorporate this equivalence, were it not for the imaginary part of the double Hamiltonian (13). Note that the presence of the Lindblad
dissipation coefficient, $\gamma$, in the real part of the double Hamiltonian (11) is not a problem because it is naturally included in the real symplectic dynamics. This is a consequence of doubling phase space, and the shrinking of trajectories due to energy loss is compensated by an expansion of the conjugate variables. The difficulty is that an initial real action cannot continue to be real, as it evolves classically due to a general complex Hamiltonian. The correct description of this feature is an advantage of the complex theory in the present context, which will now be examined.

A simple redetermination of the real space theory developed in [22] is to treat the imaginary part of the action through classical perturbation theory. In the absence of the imaginary term, a real action defines a real Lagrangian manifold through (19), which evolves through part of the action through classical perturbation theory. In the absence of the imaginary complex) is to add a term, real trajectories. The first-order effect of a perturbation of the Hamiltonian (whether real or complex) is to add a term, integrated over the (real) unperturbed trajectory that we denote $(\bar{x}_\tau, \bar{y}_\tau)$ not to confuse it with the complex one $(\bar{x}_\tau, \bar{y}_\tau)$. In the present case,

$$\delta S_p = -i \int_0^t \delta \mathcal{H}(\bar{x}_\tau, \bar{y}_\tau, \tau) \, d\tau,$$

integrated over the (real) unperturbed trajectory that we denote $(\bar{x}_\tau, \bar{y}_\tau)$ not to confuse it with the complex one $(\bar{x}_\tau, \bar{y}_\tau)$. In the present case,

$$\delta \mathcal{H}(x, y) = -\frac{1}{2}[(\lambda' \cdot y)^2 + (\lambda'' \cdot y)^2] = -\frac{1}{2}[(\lambda \cdot y)^2],$$

so that $\exp(i\delta S_p/\hbar)$ is just the decoherence term in the real WKB theory [22]. It damps out the amplitude of long chords, thus canceling the fine oscillations of the Wigner function.

How crude is this approximation to the full complex theory? The way in which the classical perturbation is redetermined in appendix C allows us to analyze this question. There, we consider a general one-parameter family of Hamiltonians, so that henceforth we replace $\delta \mathcal{H}(x, y)$ by $\alpha \delta \mathcal{H}(x, y)$ such that, in the present case, the parameter in appendix C will be fixed as $\alpha = -i$. Applying this exact general treatment of one parameter family of Hamiltonians, the error in the perturbation theory, as compared to the exact action (C.4) can thus be expressed as

$$\Delta P = \delta S - \delta S_p = \int_0^{-i} d\alpha' \int_0^{-i} d\tau \left[ \delta \mathcal{H}(\bar{x}_\tau(\alpha'), \bar{y}_\tau(\alpha')) - \delta \mathcal{H}(\bar{x}_\tau(0), \bar{y}_\tau(0)) \right]$$

$$= \int_0^{-i} d\alpha' \int_0^{-i} d\tau \frac{1}{2}[(\lambda' \cdot \bar{y}_\tau(\alpha'))^2 - (\lambda' \cdot \bar{y}_\tau(0))^2] + (\lambda'' \cdot \bar{y}_\tau(\alpha'))^2 - (\lambda'' \cdot \bar{y}_\tau(0))^2].$$

(39)

But according to (20), $\bar{y}$ is not directly affected by the parameter ($\alpha = -i$), so that this error grows slowly as $t$ is increased. Indeed, at any point in phase space the trajectory $\bar{y}_\tau$ has a velocity $\ddot{y}$ that is independent of $\alpha$, but the curvature of the trajectory is affected, because

$$\ddot{y}(\alpha) = \ddot{y}(0) + \alpha (\lambda' \cdot y) \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda' + \alpha (\lambda'' \cdot y) \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda''.$$  

(40)

Thus, keeping constant the initial variable $\bar{y}_\tau(\alpha) = y$ leads to the approximate trajectory:

$$\bar{y}_\tau(\alpha) \approx y + \frac{\alpha^2}{2} \left[ (\lambda' \cdot y) \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda' + (\lambda'' \cdot y) \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda'' \right],$$

(41)

and the initial estimate of the error is

$$\Delta P \approx -\frac{\alpha^2 \lambda^2}{6} \left[ (\lambda' \cdot y)^2 \left( \lambda' \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda' \right) + (\lambda'' \cdot y)^2 \left( \lambda'' \frac{\partial^2 \mathcal{H}}{\partial x^2} \lambda'' \right) \right].$$

(42)

This first correction of the complex action is entirely real, because the final parameter is $\alpha^2 = i^2 = -1$. If $\lambda = J\lambda$ is considered small, as is usually assumed for Markovian evolution
(the small coupling limit), then the error is $O(|l|^4)$, while the decoherence functional is only $O(|l|^3)$. We also find that the decoherence functional grows linearly in time, while its error grows as $t^3$.

Once more, one shall perceive the advantage of doubling the phase space, for this allows one to include naturally the dissipation term $\gamma$ into the symplectic dynamics instead of having to treat it as a perturbation, which is known to be rather inefficient in that case. The only part to be treated perturbatively is the imaginary part which accounts for diffusion, that is, damping of large values of $y$. This peculiarity might play in favor of robustness of the approximation. Indeed, (16) implies that a large imaginary part of $S(y)$ corresponds automatically to a negligible contribution for that value of $y$; therefore, the solution is mainly built from the regions where $\text{Im}(S(y)) \leq \hbar$, where the perturbation theory has fair chances to work.

5.2. Real dynamics of the mixed propagator

In [21] and [22] we introduced the mixed propagator $R_x(y, t)$, such that

$$\chi(y, t) = \int W_0(x) R_x(y, t) \, dx.$$  \hspace{1cm} (43)

This propagator is the Fourier transform of the Wigner function propagator $R_x(x', t)$ used by Dittrich et al [26]. It is a good candidate for applying the real dynamics because its initial expression is

$$R_x(y, 0) = \exp \left( -\frac{i}{\hbar} x \cdot y \right).$$  \hspace{1cm} (44)

which means that the initial action is purely real. Hence, the growth of the imaginary part of the action is initially perturbative, as will be verified in the following example. The important point is that this propagator can evolve any initial state, by using (43).

The previous perturbative approach gives the following expression for this propagator:

$$R_x(y, t) = \exp \left( -\frac{i}{\hbar} x \cdot \tilde{y}_0 + \frac{i}{\hbar} \int_0^t \left[ \tilde{x}_\tau \cdot \frac{\partial H}{\partial \tilde{x}}(\tilde{x}_\tau, \tilde{y}_\tau, \tau) - H(\tilde{x}_\tau, \tilde{y}_\tau, \tau) \right] d\tau \right)$$

$$- \frac{1}{2\hbar} \int_0^t [\lambda' \cdot \tilde{y}_\tau]^2 + (\lambda'' \cdot \tilde{y}_\tau)^2] d\tau \right).$$  \hspace{1cm} (45)

where $(\tilde{x}_\tau, \tilde{y}_\tau)$ is the real history of $y$ at time $t$ and with $\tilde{x}_0 = x$, governed by $H$ and not $H_c$. Hence $\tilde{x}_\tau$ and $\tilde{y}_\tau$ are real functions of $(x, y, t)$.

This expression has the advantage of separating the Lindbladian damping term from the oscillating phase, which is not the case for a general complex WKB. Therefore one can conclude that for short enough times, the chord function is fading exponentially fast for values of $y$ which are outside a disk centered in 0 and with radius $\sqrt{\hbar}$, as it does in the quadratic case [20]. Moreover, one can also conclude from (43) that it is also true for any initial state, although with possible modulation of velocity since the damping term depends on the initial $x$.

In the case without environment, the Lindbladian damping cancels out, and the action of (45) is a real function of $x$ and $y$. By expanding it up to third order around $y = 0$ and then taking the Fourier transform of (45), one obtains that the Wigner–Wigner propagator is an Airy function, thus recovering the propagator calculated by Dittrich [26]. The consequence of plugging in the environment is then to convolute this Airy propagator with a broadening Gaussian, which will fade out the oscillating fringes of the Wigner function. Hence, the approach of Dittrich can be interpreted as a cubic expansion around $y = 0$ of the action of the
present WKB theory. Furthermore, the latter provides a natural generalization of the former when there is a Markovian environment. In the example presented in a further section, this third-order $y$ expansion holds rigorously.

In the next section we treat the case of an initially Gaussian state, as an example where the above calculated propagator can be used to get a general formula. It relies on the fact that the contributing $x$, in $R_x(y, t)$, are then localized around the Gaussian’s center.

### 6. Dynamics of a Gaussian wave packet using the mixed propagator

An initially localized wave packet will restrict the range of the required trajectories of (25). For instance, if one starts with a Gaussian wave packet centered in $X$,

$$W_0(x) = \frac{1}{\pi \hbar} \exp \left( -\frac{1}{\hbar} (x - X)^2 \right),$$

then only trajectories whose starting point is within a distance $\sqrt{\hbar}$ from $X$ will contribute to the integral. By expanding the generic trajectory around the central one $(\hat{x}_0^X, \hat{y}_0^X) = (\bar{x}_0(X, y, t), \hat{y}_0(X, y, t))$, corresponding to the evolution of the maximum $X$ of the initial Gaussian, one can perform the explicit integration of (43).

If we call $S(X, y, t) = S'(X, y, t) + iS''(X, y, t)$ the action such that $R_X(y, t) = \exp iS(X, y, t)$, then $R_{X+x}(y, t)$ can be expanded as

$$R_{X+x}(y, t) = \exp \left( \frac{i}{\hbar} S(X, y, t) + \frac{i}{\hbar} \frac{\partial S}{\partial x} (X, y, t) \cdot x' + \frac{i}{2\hbar} x' \cdot \frac{\partial^2 S}{\partial x^2} (X, y, t) x' + \frac{i}{\hbar} \mathcal{O}(x^3) \right).$$

By interpreting $S$ as the generating function of the complex manifold driven by the complex Hamiltonian $\mathcal{H}_c$, we show in appendix D that

$$\frac{\partial S}{\partial x}(X, y, t) = -\hat{y}_0,$$

where $\hat{y}_0$ is the initial $y$ coordinate of the complex trajectory $(X, \hat{y}_0, 0) \to (x, \lambda(y, t), y, t)$, constrained to start at $x = X$ and to stop at $y$ in a time $t$.

Then (43) gives

$$\chi(y, t) \simeq \frac{1}{\pi \hbar} \int \exp \left( -\frac{1}{\hbar} x'^2 + \frac{i}{\hbar} S(X, y, t) - \frac{i}{\hbar} \hat{y}_0 \cdot x' + \frac{i}{2\hbar} x' \cdot \frac{\partial \hat{y}_0}{\partial x} x' \right) \, dx',$$

and then

$$\chi(y, t) \simeq \frac{1}{\sqrt{\det(1 - \frac{i}{2\hbar} \hat{y}_0)}} \exp \left( \frac{i}{\hbar} S(X, y, t) - \frac{1}{\hbar} \hat{y}_0 \cdot \left( 1 - \frac{i}{2} \frac{\partial \hat{y}_0}{\partial x} \right)^{-1} \hat{y}_0 \right).$$

where $\hat{y}_0$ is a complex valued function of $(X, y, t)$, hence a function of $(y, t)$ once the initial state is fixed.

The analysis holds as long as $S(x, y, t)$ can be approximated by its quadratic expansion in $x$, that is, when the dynamics is sufficiently regular, or time sufficiently small. For a chaotic dynamics one expects exponentially diverging trajectories which may make expansion (47) irrelevant for large times.

If one uses the real WKB theory instead of the complex WKB theory, the imaginary part of $\hat{y}_0$ can be interpreted as a perturbation of the origin $\hat{y}_0$ of the real trajectory $(\hat{x}_0^X, \hat{y}_0^X)$:

$$\frac{\partial S}{\partial x}(X, y, t) \simeq -\hat{y}_0 + i \int_0^t (\lambda \cdot \hat{y}_t) \frac{\partial \hat{y}_t}{\partial x} \lambda \, dt.$$

(51)
In the following example, both theories give the same result. Another way of propagating a Gaussian wave packet in Markovian open systems can be found in [27].

7. A cubic example

Let us consider the special case of the Hamiltonian \( \hat{H}(p, q) = \hat{p}^3 \) and a linear Lindblad operator, \( \hat{L} = \hat{l} \hat{p} \). Here, the exact solution can be obtained, which allows us to check on our different levels of approximation. Besides, we are in a case where the cubic \( y \) expansion of the action leading to the Wigner–Wigner propagator of Dittrich, as mentioned in section 5.2, is exact.

The Lindblad equation can be written as

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \frac{\hat{l}^2}{2\hbar} (\hat{p}^2 \hat{\rho} - \hat{\rho} \hat{p}^2).
\]

In the \( p \) representation we then have

\[
\frac{d}{dt} (p'|\hat{\rho}(t)|p'') = \frac{i}{\hbar} (p^3 - p'^3)(p'|\hat{\rho}(t)|p'') - \frac{\hat{l}^2}{2\hbar} (p' - p'')^2 (p'|\hat{\rho}(t)|p''),
\]

which gives the solution

\[
(p'|\hat{\rho}(t)|p'') = (p'|\hat{\rho}(0)|p'') \exp \left( -\frac{i}{\hbar} (p^3 - p'^3)t - \frac{\hat{l}^2}{2\hbar} (p' - p'')^2 t \right).
\]

The general expression for the chord function is then

\[
\chi(y, t) = \int dp \left\{ p + \frac{y_q}{2} |\hat{\rho}(t)| p - \frac{y_q}{2} \right\} \exp \left( -\frac{i}{\hbar} py_p \right)
\]

\[
= \exp \left( -\frac{it}{4\hbar} \frac{y_q}{2} \left( 3p^2 - \frac{y^3}{4} \right) \right) \int dp \left\{ p + \frac{y_q}{2} |\hat{\rho}(0)| p - \frac{y_q}{2} \right\}
\]

\[
\times \exp \left( -\frac{i}{\hbar} (yp + ty_q)3p^2 \right),
\]

i.e. the original chord function is convoluted with an evolving Gaussian and then multiplied by another time-dependent factor.

The complex WKB approximation for this evolution is based on the classical dynamics (20) of the complex double phase space Hamiltonian \( \mathcal{H}_c(p, q, y_p, y_q) \), which is here equal to

\[
\mathcal{H}_c(p, q, y_p, y_q) = 3p^2 y_q + \frac{y^3}{4} - \frac{i}{2} (ly_q)^2.
\]

It generates the following trajectories:

\[
(y_q)_t = (y_q)_0 = y_q
\]

\[
(y_p)_t = (y_p)_0 - 6p_0 y_q t
\]

\[
\tilde{q}_t = q_0 + \left( 3p_0^2 + \frac{3}{4} y_q^2 \right)t + il_q^2 y_q t
\]

\[
\tilde{p}_t = p_0.
\]

One should note that in this simple example the chord motion, \( \tilde{y}_t \), is independent of the trajectory of the center variable, \( \tilde{x}_t = (\tilde{q}_t, \tilde{p}_t) \), just as it occurs for quadratic systems. Hence, the evolved chord will remain real if it is real to start with.

Given the initial action, \( S_0(y) \), its evolution is obtained explicitly from (24) as

\[
S(y, t) = S_0(y_p + 6p_0 y_q t, y_q) + t \left( 3p_0^2 y_q - \frac{y^3}{4} + \frac{i}{2} (ly_q)^2 \right).
\]
leading to the complex WKB approximation

$$\chi_{\text{WKB}}(y, t) = \exp \left( \frac{i}{\hbar} \left( S_0(y_0) + t \ y_q \ 3p_0^2 - \frac{t y_q^3}{4} + \frac{i t}{2} (ty_q^3) \right) \right).$$  \hspace{1cm} (59)

This can now be compared with the approximate saddle point evaluation of (55). Note that the initial density operator in the \( p \)-representation, corresponding to (16) with the initial action \( S_0(y) \), will also have a standard semiclassical form. It is just the Fourier inverse of (55) for \( t = 0 \), with an action \( \sigma_0(p', p'') \) (a symmetrized Legendre transform of \( S_0(y) \)), such that

$$\frac{\partial \sigma_0}{\partial p'} = -q_0', \quad \frac{\partial \sigma_0}{\partial p''} = q_0''.$$ \hspace{1cm} (60)

Then the integral expression (55) for the evolving chord function can be performed by the saddle point method. Thus, the stationary point, \( p_0 \), of the exponent of (55) is selected by the equation

$$q_0(p_0 + y_q/2) - q_0(p_0 - y_q/2) = y_p + 6ty_q p_0 = (y_p)_0,$$ \hspace{1cm} (61)

where, in the last equality, we have followed the \( y_p \)-component of the classical trajectory in (57) backward in time. This trajectory depends on \( p_0 \), a constant, according to (57). Hence, the full approximation for the evolving chord function becomes

$$\chi_{\text{SP}}(y, t) = \frac{\pi \hbar}{\sqrt{\frac{\partial \sigma_0}{\partial p'}} \left( p_0 + \frac{y_q}{2}, p_0 - \frac{y_q}{2} \right) + 3i y_q t} \times \exp \left( -\frac{iy_q^3}{4\hbar} - \frac{t^2 y_q^2}{2\hbar} + \frac{i}{\hbar} \sigma_0(p_0 + \frac{y_q}{2}, p_0 - \frac{y_q}{2}) + \frac{i}{\hbar} ty_q^3 p_0' \right).$$ \hspace{1cm} (62)

Thus the phase of \( \chi_{\text{SP}}(y, t) \) coincides with that of \( \chi_{\text{WKB}}(y, t) \).

In some cases this phase is exact. Consider, the simple example of a ‘Schrödinger cat state’, composed of the superposition of a pair of coherent states, with the wavefunction

$$\psi(q) = \frac{1}{(\pi \hbar)^{1/4}} \exp \left( -\frac{(q - Q)^2}{2\hbar} + \frac{iPq}{\hbar} \right),$$ \hspace{1cm} (63)

and different parameters, \( X_a = (P_a, Q_a) \) and \( X_b = (P_b, Q_b) \). The density operator, corresponding to \( (|\psi_a\rangle + |\psi_b\rangle)/\sqrt{2} \), has two diagonal terms and two off-diagonal terms:

$$\hat{\rho} = \frac{1}{2}|\psi_a\rangle\langle\psi_a| + \frac{1}{2}|\psi_b\rangle\langle\psi_b| + \frac{1}{2}|\psi_a\rangle\langle\psi_b| + \frac{1}{2}|\psi_b\rangle\langle\psi_a|. \hspace{1cm} (64)$$

We are interested here in the non-local terms, \( \hat{\rho}_{ab} = |\psi_a\rangle\langle\psi_b| \) and its complex conjugate. One can note that the two first ‘classical’ coherent terms can be retrieved by setting \( a = b \).

Inserting the \( p \)-representation of the initial density matrix,

$$\langle p' | \hat{\rho}_{ab}(0) | p'' \rangle = \frac{1}{\sqrt{\pi \hbar}} \exp \left( \frac{i}{\hbar} (p' Q_a - p'' Q_b) - \frac{1}{2\hbar} (p' - P_a)^2 - \frac{1}{2\hbar} (p'' - P_b)^2 \right),$$ \hspace{1cm} (65)

into (55), the integral can be performed to give the exact chord function evolution:

$$\chi_{ab}(y, t) = \frac{1}{\sqrt{T + 3ty_q t}} \exp \left( \frac{4P^2 - (y_p + \Delta Q)^2 - 4iP(y_p + \Delta Q)}{1 + 3ty_q} \right) \left( y_q - \Delta P \right)^2 - \frac{1}{2\hbar} t^2 y_q^2 - \frac{i}{\hbar} P^2 - \frac{i}{\hbar} Qy_q - \frac{i}{4\hbar} t y_q^3. \hspace{1cm} (66)$$
where we have defined

\[ P = \frac{P_a + P_b}{2}, \]
\[ Q = \frac{Q_a + Q_b}{2}, \]
\[ \Delta P = P_b - P_a, \]
\[ \Delta Q = Q_b - Q_a. \tag{67} \]

The initial action obtained from setting \( t = 0 \) is

\[ S_0(y) = -Q y_q - P (y_p + \Delta Q) + \frac{i}{4} (y_p + \Delta Q)^2 + \frac{i}{4} (y_q - \Delta P)^2, \tag{68} \]

so that

\[ p_0 = -\frac{\partial S_0}{\partial y_p}(y_0) = P + \frac{i}{2} (y_p + 6 p_0 y_q t + \Delta Q) \]
\[ q_0 = -\frac{\partial S_0}{\partial y_q}(y_0) = Q + \frac{i}{2} (y_q - \Delta P). \tag{69} \]

Then equation (24) gives

\[ S(y, t) = S_0(y_p + 6 p_0 y_q t, y_q) + t \left( 3 p_0^2 y_q - \frac{y_q^3}{4} - \frac{i}{2} (l y_q)^2 \right) \]
\[ = -Q y_q - P (y_p + 6 p_0 t y_q) + \frac{i}{4} (y_p + 6 p_0 t y_q + \Delta Q)^2 + t \left( 3 p_0^2 y_q - \frac{y_q^3}{4} - \frac{i}{2} (l y_q)^2 \right). \tag{70} \]

Here,

\[ p_0 = \frac{P + \frac{i}{2} (y_p + \Delta Q)}{1 + 3 i y_q t} \tag{71} \]

corresponds to the explicit evaluation of the stationary point defined by (61). Therefore, both \( \chi_{SP}(y, t) \) and \( \chi_{WKB}(y, t) \) have the exact phase, being that the former also has the exact amplitude.

This example of an initially localized wave packet is a case where the real WKB theory becomes a crude approximation, because the initial action is then complex and, hence, the contributing trajectory is also complex.

However the simple classical non-quadratic evolution in this section does not lead to a complex trajectory in (57), if it is initially real, i.e. for an initial real action. In this case, the saddle point evaluation of (55) becomes equivalent to the stationary phase method, leading to the real WKB result. In other words, the first-order perturbation of the phase, which was studied in section 5, gives the full correction with respect to the complex WKB phase.

In particular, the real WKB theory can be used to propagate the real initial action of the mixed propagator \( R_x(y, t) \) presented in section 5.2, with its chord space representation (44).

It corresponds to a density operator given by

\[ \left( p' + \frac{y_q}{2} \right| \rho \left| p' - \frac{y_q}{2} \right) = \delta(p' - p) \exp \left( -i \frac{h}{\hbar} q y_q \right), \tag{72} \]

with \( x = (p, q) \). Then, the integral in (55) can again be performed exactly to obtain

\[ R_x(y, t) = \exp \left( -i \frac{t}{4 \hbar} y_q^2 - i \frac{t^2}{2 \hbar} y_q^2 \right) \exp \left( -i \frac{h}{\hbar} y_q q - i \frac{h}{\hbar} y_p p - i \frac{h}{\hbar} l y_q^2 p' \right). \tag{73} \]
This expression coincides exactly with (45), which can be obtained explicitly from (57) by setting \( l = 0 \). Thus, in this case both the real and the complex WKB evolutions are exact.

This is all the more interesting, in that \( R_x(y, t) \) can then be used to propagated any initial state. For instance, with the ‘Schrödinger cat state’, one gets

\[
\chi_{ab}(y, t) = \int W_0(x) R_x(y, t) \, dx, \tag{74}
\]

with

\[
W_0(x) = \exp \left( \frac{p^2 + q^2}{\hbar} + \frac{i}{\hbar} 2 y_p \, p \right), \tag{75}
\]

which again gives (66). This shows that the real WKB method, although inaccurate for a general initial state, can still be used indirectly through \( R_x(y, t) \).

Note that expression (74) then also coincides with expression (50), which is not so surprising as the action of \( R_x(y, t) \), in (73), is actually quadratic in \( x \).

8. Conclusion and outlook

We have identified a representation of the density operator in which an \( \hbar \) expansion, generated from a WKB-like expression (25) can be performed, and is compatible with a non-unitary evolution induced by a Lindblad equation with linear Lindblad operators. Dynamics is basically determined by a generalized classical Hamiltonian dynamics in a complex double phase space \( (x, y) \), where the subspace \( y = 0 \) corresponds to the ordinary classical phase space dynamics, while non-zero \( y \) accounts for quantum (non-classical) correlations. This Hamiltonian has two remarkable features. On the one hand, a real hyperbolic term includes dissipation in a symplectic framework. Although it seems incompatible with the phase space shrinking which usually goes along with dissipation, it is actually made possible by doubling of phase space, which re-establishes the overall volume conservation by stretching the chord variables, \( y \) or \( \xi \). On the other hand, an imaginary term describes quantum dephasing, truly responsible for decoherence, and classical diffusion. Thus, even prior to any specific application, one is able to understand the general semiclassical structure of the evolution of quantum Markovian systems.

When dynamics is linear, that is, when the Hamiltonian is at most quadratic, and when Lindbladians are linear, then the dynamics of chords \( y \) is real and decoupled from the dynamics of centers \( x \). As a consequence, the Lindbladian decoherence factor leads to an exponential damping of the values of \( y \) that lie beyond the disk of radius \( \sqrt{\hbar} \) around 0, thus interpreted as the region containing all classical information about the state. This is derived for instance in [20] and it is then compatible with the picture of a ‘decoherence which drives the system back into classical dynamics’.

In the case of a nonlinear dynamics, while maintaining linear Lindbladians, this scheme is generally not applicable, since the dynamics of \( x \) and \( y \) are both coupled and complex. However, a similar Lindbladian damping factor can be recovered when the action of the initial chord function is real, and time short enough, so that the imaginary part of the classical action grows only as a small perturbation. Furthermore, it can be extended to any initial state by using the mixed propagator \( R_x(y, t) \), which satisfies the real initial action requirement. Hence, the overall picture of a—here nonlinear—classical motion in a double phase space associated with a decoherence damping factor is conserved in a perturbative regime. Recalling that, in a Fourier transform, the decoherence factor gives rise to the convolution with a widening Gaussian window, we see that decoherence and diffusion are respectively the early and the late stages of the same coarse graining of the Wigner function.
The extension of the WKB approximation for Markovian quantum evolution to a complex double phase space, that is presented here, allows for its future application to the localized states that are presently manipulated in experiments (see e.g. [28]). In contrast, the evolution of highly excited Bohr–Sommerfeld states, that were the subject of our previous real WKB theory, are only of theoretical interest so far. Concrete applications will require the integration of classical trajectories for specific Hamiltonians. Though this will require a much lighter computational effort than that required to integrate partial differential equations for highly oscillatory states, it is pleasing that here we have already been able to illuminate different levels of approximation in the theory with a simple non-quadratic example, while comparing it with an exact result.

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Appendix A. Asymptotic expansion of the Hamiltonian operator

One can verify easily that

\[
\left( \frac{1}{\lambda} \frac{\partial}{\partial p} \right)^m \left( \frac{1}{\lambda} \frac{\partial}{\partial q} \right)^n \exp \lambda S = \left[ \left( \frac{\partial S}{\partial p} \right)^m \left( \frac{\partial S}{\partial q} \right)^n + \frac{1}{\lambda} \left( \frac{m(m-1)}{2} \frac{\partial^2 S}{\partial p^2} \left( \frac{\partial S}{\partial p} \right)^{m-2} \left( \frac{\partial S}{\partial q} \right)^n \right) \right.
\]
\[
+ mn \frac{\partial^2 S}{\partial p \partial q} \left( \frac{\partial S}{\partial p} \right)^{m-1} \left( \frac{\partial S}{\partial q} \right)^{n-1} + \frac{n(n-1)}{2} \frac{\partial^2 S}{\partial q^2} \left( \frac{\partial S}{\partial q} \right)^{m} \left( \frac{\partial S}{\partial p} \right)^{n-2} \right)
\]
\[
+ O \left( \frac{1}{\lambda^2} \right) \right] \exp \lambda S.
\]  
(A.1)

From there one can generalize and write

\[
\mathcal{H} \left( - \frac{\hbar}{i} \frac{\partial}{\partial y}, y \right) \exp \left( \frac{i}{\hbar} S \right) = \left[ \mathcal{H} \left( - \frac{\partial S}{\partial y}, y \right) + \frac{\hbar}{2i} \text{Tr} \left\{ \frac{\partial^2 \mathcal{H}}{\partial x^2} \left( - \frac{\partial S}{\partial y}, y \right) \frac{\partial^2 S}{\partial y^2} \right\} + O(\hbar^2) \right] \exp \left( \frac{i}{\hbar} S \right). \quad (A.2)
\]

Appendix B. Proof of the closeness of the differential form \( \delta a \)

We prove that the form

\[
\delta a(y, t) = -x_M(y, t) \cdot \delta y - \mathcal{H}(x_M(y, t), y, t) \, dt
\]  
(B.1)

is a closed form by showing that it obeys Schwartz equalities

\[
\frac{\partial}{\partial t} x_M(y, t) = \frac{\partial}{\partial y} \mathcal{H}(x_M(y, t), y, t),
\]  
(B.2)

and

\[
\frac{\partial p_M(y, t)}{\partial y_q} = \frac{\partial q_M(y, t)}{\partial y_p}.
\]  
(B.3)
The point \((x_M(y, t), y)\) is on a particular trajectory \((\bar{x}_t, \bar{y}_t)\) of the manifold \(\mathcal{M}\) with
\[
\bar{x}_0 = x_M(y, t)
\]
\[
\bar{y}_0 = y
\]
and
\[
\dot{\bar{x}}_t = \frac{\partial H}{\partial y}(\bar{x}_t, \bar{y}_t, t + \tau)
\]
\[
\dot{\bar{y}}_t = -\frac{\partial H}{\partial x}(\bar{x}_t, \bar{y}_t, t + \tau).
\]
One also has, of course,
\[
x_M(\bar{y}_t, t + \tau) = \bar{x}_t.
\]
Now we can write the following sequence of equalities:
\[
x_M(y, t + d\tau) - x_M(y, t) = x_M(y, t + d\tau) - x_M(y + \dot{y}_0 d\tau, t + d\tau)
\]
\[
+ x_M(y + \dot{y}_0 d\tau, t + d\tau) - x_M(y, t)
\]
\[
= -\frac{\partial x_M}{\partial y}(y, t + d\tau)\dot{y}_0 d\tau + \bar{x}_t - \bar{x}_0 + \mathcal{O}(d\tau^2)
\]
\[
= \frac{\partial x_M}{\partial y}(y, t + d\tau)\frac{\partial H}{\partial x}(\bar{x}_0, \bar{y}_0, t) d\tau + \frac{\partial H}{\partial y}(\bar{x}_0, \bar{y}_0, t) d\tau + \mathcal{O}(d\tau^2)
\]
\[
= \frac{\partial x_M}{\partial y}(y, t + d\tau)\frac{\partial H}{\partial x}(x_M(y, t), y, t) d\tau + \mathcal{O}(d\tau^2)
\]
\[
= \frac{\partial}{\partial y}(x_M(y, t), y, t) d\tau + \mathcal{O}(d\tau^2).
\]
This last equality shows \((B.2)\).

On the other hand, at \(t = 0\) we have
\[
\frac{\partial p_M}{\partial y_q}(y, 0) = -\frac{\partial}{\partial y_q} A_0(y) = -\frac{\partial}{\partial y_p} \frac{\partial A_0}{\partial y_q}(y) = \frac{\partial q_M}{\partial y_p}(y, 0),
\]
which sets \((B.3)\) at \(t = 0\). Now one just has to note that \(\frac{\partial p_M}{\partial y_q}(\bar{y}_t, t + \tau) - \frac{\partial p_M}{\partial y_q}(\bar{y}_t, t + \tau)\) is time independent:
\[
\frac{\partial}{\partial \tau} \left( \frac{\partial p_M}{\partial y_q}(\bar{y}_t, t + \tau) - \frac{\partial q_M}{\partial y_p}(\bar{y}_t, t + \tau) \right) = \frac{\partial}{\partial y_q} \frac{\partial p}{\partial \tau} - \frac{\partial}{\partial y_p} \frac{\partial q}{\partial \tau}
\]
\[
= \frac{\partial}{\partial y_p} (\bar{y}_t, \bar{y}_t, t + \tau) - \frac{\partial}{\partial y_q} (\bar{y}_t, \bar{y}_t, t + \tau)
\]
\[
= 0,
\]
which shows the statement. We used \((B.5)\) and \((B.6)\).

Therefore, since \(\mathcal{M}\) is simply connected, \(\delta a(y, \tau)\) is the exact differential of a function \(a(y, \tau)\) such that
\[
\frac{\partial a}{\partial y}(y, \tau) = -x_M(y, \tau)
\]
\[
\frac{\partial a}{\partial \tau}(y, \tau) = -H(x_M(y, \tau), y, \tau).
\]
Hence, one has
\[
\frac{\partial a}{\partial \tau}(y, \tau) = -H \left( -\frac{\partial a}{\partial y}(y, \tau), y, \tau \right),
\]
so that \(A(y, t) = a(y, t)\) is the required solution.
Appendix C. Real WKB theory as a perturbation of the complex one

Let us consider a family of Hamiltonians, $\mathcal{H}(x, y; \alpha)$, where the parameter $\alpha$ may be real or complex. (It can also have more than one dimension.) It is then possible to reinterpret $\mathcal{H}$ as a single Hamiltonian in an expanded phase space: $(x, y; \beta, \alpha)$ in which $\beta$ is a hidden variable. It follows that $\alpha$ is a constant of the motion and the projection of each trajectory onto the $(x, y)$ hyperplane coincides exactly with the trajectory of the original family of Hamiltonians for its specific value of $\alpha$.

Consider now the root search problem for finding, given any value of $\alpha$, a trajectory that starts at the Lagrangian surface $x = \partial S_0 / \partial y$ and arrives at specified value of $y$. Varying the parameter continuously in the interval $[0, \alpha]$, we obtain an one parameter family of trajectory segments which forms a finite 2D strip in the enlarged phase space. This can be fully specified by choosing $\beta = 0$ as the arrival point for the extra coordinate (for all $\alpha$) and its initial point to be

$$\beta_0(\alpha) = \int_0^{-\tau} d\tau \beta_0 = \int_0^{-\tau} d\tau \frac{\partial \mathcal{H}}{\partial \alpha}(\hat{x}_t(\alpha), \hat{y}_t(\alpha); \alpha).$$  \hspace{1cm} (C.1)

According to the Poincare–Cartan theorem [25],

$$\oint [x \cdot d\mathbf{y} + \beta \cdot d\alpha - \mathcal{H} d\tau] = 0$$  \hspace{1cm} (C.2)

for any reducible circuit on a 2D surface that is spanned by trajectories. So we can build this circuit from four segments: (i) the trajectory that travels from the original Lagrangian surface and $\beta_0(\alpha)$ to $(y = \text{const}, \beta = 0)$ in the time $\tau$; (ii) a path $\alpha \rightarrow 0$, with all other extended phase space variables and the time held fixed; (iii) the trajectory that reverses trajectory (i), but with $\alpha = 0$; (iv) the circuit is closed (in zero time) by path along the original Lagrangian surface, while increasing $\alpha$ back to its original value, such that $\beta = \beta_0(\alpha)$, as defined by (C.1). Then (C.2) is rewritten as

$$S_0(\hat{y}_t(\alpha)) + \int_0^\tau [\hat{x}_t(\alpha) \cdot \hat{y}_t(\alpha) - \mathcal{H}(\hat{x}_t(\alpha), \hat{y}_t(\alpha))] d\tau = S_0(\hat{y}_t(0)) + \int_0^\tau [\hat{x}_t(0) \cdot \hat{y}_t(0) - \mathcal{H}(\hat{x}_t(0), \hat{y}_t(0))] d\tau - \int_0^\alpha \beta_0(\alpha) d\alpha.$$  \hspace{1cm} (C.3)

Hence, the difference in the final action is exactly

$$\delta S(\alpha) = S(y, t; \alpha) - S(y, t; 0) = \int_0^\alpha d\alpha \int_0^{-\tau} d\tau \frac{\partial \mathcal{H}}{\partial \alpha}(\hat{x}_t(\alpha'), \hat{y}_t(\alpha'); \alpha').$$  \hspace{1cm} (C.4)

For small enough $\alpha$, this may be approximated as

$$\Delta S(\alpha) = \int_0^{\tau} d\tau \mathcal{H}(\hat{x}_t(0), \hat{y}_t(0); \alpha) = \int_0^{\tau} d\tau \mathcal{H}(\hat{x}_t(0), \hat{y}_t(0); 0) = \alpha \int_0^{\tau} d\tau \mathcal{H}(\hat{x}_t(0), \hat{y}_t(0)), \hspace{1cm} (C.5)$$

if $\mathcal{H}(x, y; \alpha) = \mathcal{H}(x, y) + \alpha \delta \mathcal{H}(x, y)$. This is just the first-order classical perturbation. One integrates the change in the Hamiltonian along the unperturbed trajectory.

Appendix D. Derivative of the action of propagator $R_x(y, t)$ as regard to $x$

For every $x_0$ we define a manifold $\mathcal{M}(x_0)$ by propagating the manifold $x = x_0$ in the double phase space driven by the complex Hamiltonian $\mathcal{H}_c$. The coordinates of the manifold are $(x, y, t)$, thus parametrized by $(y, t)$. We want to evaluate the infinitesimal
difference \( S(x_0 + dx_0, y, t) - S(x_0, y, t) \), where the variable \( x_0 \) corresponds to the initial manifold, the variable \( y \) to the final observation point, and \( t \) to the time of evolution. We have

\[
S(x_0, y, t) \equiv -x_0 \cdot \dot{y}_0 - \int_{\tilde{y}_0}^{y} (\dot{x} \cdot \dot{y} - H_c(x, \dot{x}, y, \tau)) d\tau,
\]

where \((\dot{x}, \dot{y}) = (x_M(x_0, \dot{y}_0, \tau), \dot{y}_0)\) is the Hamiltonian trajectory which starts at \((x_0, \dot{y}_0)\) at time \( \tau = 0 \) and ends at \( y \) at time \( \tau = t \). On the other hand, one has

\[
S(x_0 + dx_0, y, t) = -(x_0 + dx_0) \cdot \dot{y}_0 - \int_{\dot{y}_0}^{y} x_M(x_0, y, \tau) \cdot dy + H_c(x_M(x_0, y, \tau), y, y, \tau) d\tau,
\]

where we used the lagrangian nature of \( M(x_0 + dx_0) \) which gives us the freedom to start from the point \((x_0 + dx_0, \dot{y}_0)\) at \( \tau = 0 \), which is indeed a point of the manifold, and follow any path which arrives at \( y \) on time \( t \). In particular, one can chose the path \((x_M(x_0 + dx_0, \dot{y}_0, \tau), \dot{y}_0, \tau)\), which follows ‘vertically’ the \( M(x_0) \) trajectory, but which is not a Hamiltonian trajectory on \( M(x_0 + dx_0) \). Then one can see that

\[
S(x_0 + dx_0, y, t) = S(x_0, y, t) - dx_0 \cdot \dot{y}_0 - \int_{\dot{y}_0}^{y} \left( \frac{\partial x_M}{\partial x_0}(x_0, \dot{y}_0, \tau) dx_0 \cdot \dot{y}_0 + \frac{\partial H_c}{\partial x}(x_M(x_0, y, \tau), y, y, \tau) \right) d\tau
\]

so that

\[
\frac{\partial S}{\partial x}(x_0, y, t) = -\dot{y}_0.
\]
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