A LOCAL QUANTUM VERSION OF THE KOLMOGOROV THEOREM

David Borthwick1, Sandro Graffi2

Abstract

Consider in $L^2(\mathbb{R}^l)$ the operator family $H(\epsilon) := P_0(\hbar, \omega) + \epsilon Q_0$. $P_0$ is the quantum harmonic oscillator with diophantine frequency vector $\omega$, $Q_0$ a bounded pseudodifferential operator with symbol holomorphic and decreasing to zero at infinity, and $\epsilon \in \mathbb{R}$. Then there exists $\epsilon^* > 0$ with the property that if $|\epsilon| < \epsilon^*$ there is a diophantine frequency $\omega(\epsilon)$ such that all eigenvalues $E_n(\hbar, \epsilon)$ of $H(\epsilon)$ near 0 are given by the quantization formula $E_{\alpha}(\hbar, \epsilon) = E(\hbar, \epsilon) + \langle \omega(\epsilon), \alpha \rangle \hbar + |\omega(\epsilon)| \hbar/2 + O(\alpha \hbar)^2$, where $\alpha$ is an $l$-multi-index.

1 Introduction and statement of the results

Denote by $F_{\rho, \sigma}$ the set of all functions $f(x, \xi) : \mathbb{R}^{2l} \to \mathbb{C}$ with finite $\|f\|_{\rho, \sigma}$ norm for some $\rho > 0$, $\sigma > 0$ (see Section 2 for the definition and examples). Any $f \in F_{\rho, \sigma}$ is analytic on $\mathbb{R}^l$ and extends to a complex analytic function in the region $|\Im z_i| \leq a_i |\Re z_i|$ for suitable $a_i > 0$; moreover $|f(z)| \to 0$ as $|z| \to +\infty$. Here $z := (x, \xi)$.

Let $\Phi_{\rho, \sigma}$ denote the class of semiclassical Weyl pseudodifferential operators $F$ in $L^2(\mathbb{R}^l)$ with symbol $f(x, \xi) \in F_{\rho, \sigma}$; namely, (notation as in [Ro])

\[(Fu)(x) := Op_W(f(x, \xi))u(x) = \frac{1}{\hbar^l} \int_{\mathbb{R}^l \times \mathbb{R}^l} e^{i((x-y),\xi)/\hbar} f((x+y)/2, \xi) u(y) dy d\xi, \ u \in \mathcal{S}(\mathbb{R}^l).\]

It follows directly from the definition of $\|f\|_{\rho, \sigma}$ in (2.5) that $F \in \Phi_{\rho, \sigma}$ extends to a continuous operator in $L^2(\mathbb{R}^l)$, with

\[\|F\|_{L^2 \to L^2} \leq \|f\|_{\rho, \sigma}. \tag{1.2}\]

Consider in $L^2(\mathbb{R}^l)$ the operator family $H(\epsilon) = P_0(\hbar, \omega) + \epsilon Q_0$ and assume:

\[\text{(A1) } P_0(\hbar, \omega) \text{ is the harmonic-oscillator Schrödinger operator with frequencies } \omega \in [0, 1]^l: \]

\[P_0(\hbar, \omega) u = -\frac{1}{2} \hbar^2 \Delta u + [\omega_1^2 x_1^2 + \ldots + \omega_l^2 x_l^2]u, \ D(P_0) = H^2(\mathbb{R}^l) \cap L^2_2(\mathbb{R}^l). \tag{1.3}\]
(A2) \( Q_0 \in \Phi_{\rho,\sigma}; \) its symbol \( q_0(x,\xi) = q_0(z) \) is real-valued for \( z = (x,\xi) \in \mathcal{R}^x \mathcal{R}^l, \) and \( q_0(z) = O(z^2) \) as \( z \to 0. \)

(A3) There exist \( \tau > l - 1, \gamma > 0 \) such that

\[
\langle \omega, k \rangle \geq \gamma |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^l \setminus \{0\}, \quad |k| := |k_1| + \ldots + |k_l|, \quad \omega := (\omega_1, \ldots, \omega_l). \quad (1.4)
\]

Denote \( \Omega_0 \) the set of all \( \omega \in [0,1]^l \) fulfilling (1.4), and \( |\Omega_0| \) its measure. It is well known that \( |\Omega_0| = 1. \)

Under the above assumptions the operator family \( H(\epsilon) \) defined on \( D(P_0) \) is self-adjoint with pure-point spectrum \( \forall \epsilon \in \mathcal{R}: \) \( \text{Spec}(H(\epsilon)) = \text{Spec}_p(H(\epsilon)). \) Moreover (1.4) entails in particular the rational independence of the components of \( \omega \) and hence the simplicity of \( \text{Spec}(P_0) \) and its density in \( \mathcal{R}^+ := \mathcal{R}_+ \cup \{0\}. \) Clearly, \( P_0 \) is a semiclassical pseudodifferential operator of order 2 with symbol

\[
p_0(x,\xi) = \frac{1}{2}(|\xi|^2 + |\omega x|^2) = \frac{1}{2} \sum_{k=1}^l \omega_k I_k(x,\xi), \quad I_k(x,\xi) := \frac{1}{2\omega_k} [\xi_k^2 + \omega_k^2 x_k^2], \quad k = 1, \ldots. \quad (1.5)
\]

**Theorem 1.1** Let (A1-A3) be verified; let \( h^* > 0. \) Then given \( \eta > 0 \) there exist \( \epsilon^* > 0 \) and, for all \( \epsilon \in [-\epsilon^*,\epsilon^*], \) \( \Omega^* \subset \Omega_0 \) independent of \( (\bar{h} \in [0,\bar{h}^*], \eta) \) and \( \omega(\bar{h},\epsilon) \in \Omega^*, \) such that if \( |\alpha h| < \eta \) the spectrum of \( H(\epsilon) \) is given by the quantization formula

\[
E_{\alpha}(\bar{h},\epsilon) = E(\bar{h};\epsilon) + \langle \omega(\bar{h},\epsilon),\alpha \rangle \bar{h} + \frac{1}{2} |\omega(\bar{h},\epsilon)| \bar{h} + \epsilon \mathcal{R}(\alpha h, \bar{h}; \epsilon). \quad (1.6)
\]

Here:

1. \( \mathcal{E}(x;\epsilon) : [0, h^*) \times [-\epsilon^*, \epsilon^*] \to \mathcal{R} \) is continuous in \( x \) and analytic in \( \epsilon, \) with \( \mathcal{E}(x,0) = 0, \mathcal{E}(0;\epsilon) = 0; \)
2. \( \omega(x;\epsilon) : [0, h^*) \times [-\epsilon^*, \epsilon^*] \to \mathcal{R} \) is continuous in \( x \) and analytic in \( \epsilon \) with \( \omega(x;0) = \omega. \)
3. \( \mathcal{R}(x,y;\epsilon) : \mathcal{R}^+ \times [0, h^*) \times [-\epsilon^*, \epsilon^*] \to \mathcal{R} \) is continuous in \( (x,y;\epsilon) \) and such that

\[
|\mathcal{R}(x,y;\epsilon)| = O(|x|^2), \quad (1.7)
\]
uniformly with respect to \( (y,\epsilon). \)
4. \( |\Omega^* - \Omega_0| \to 0 \) as \( \epsilon \to 0. \)

The uniformity in \( \bar{h} \) of the estimates needed to prove Theorem 1.1 yields in this particular setting a formulation of Kolmogorov’s theorem equivalent to that of [BGGS].
Corollary 1.1. Let $\epsilon^*, \Omega^\epsilon, \mathcal{E}(x; \epsilon), \omega(x; \epsilon)$ be as above. Then $\forall \epsilon$ there is an analytic canonical transformation $(x, \xi) = \psi_\epsilon(I, \phi)$ of $\mathbb{R}^2$ onto $\mathbb{R}_+ \times \mathcal{T}_l$ such that

$$(p_\epsilon \circ \psi)(I, \phi) = \mathcal{E}(\epsilon) + \langle \omega(\epsilon), I \rangle + \epsilon \tilde{R}(I, \phi; \epsilon) \quad (1.8)$$

Here $\mathcal{E}(\epsilon) := \mathcal{E}(0; \epsilon), \omega(\epsilon) := \omega(0; \epsilon) \in \Omega^\epsilon; \tilde{R}(I, \phi; \epsilon) = O(I^2)$ as $I \to 0$ uniformly in $\phi$.

Remarks

1. The form (1.8) of the Hamiltonian entails that a quasi periodic-motion with diophantine perturbed frequency $\omega(\epsilon) \in \Omega^\epsilon$ exists on the perturbed torus $I = 0$; equivalently, a quasi periodic motion with frequency $\omega(\epsilon) \in \Omega^\epsilon$ exists on the unperturbed torus with parametric equations $(x, \xi) = \psi_\epsilon(0, \phi)$. Making $I = \alpha \hbar$ (1.6) represents the quantization of the r.h.s. of (1.8). In the formulation of [BGGS] a quasi periodic motion with the unperturbed frequency $\omega \in \Omega$ exists on an unperturbed torus with parametric equations $(x, \xi) = \psi_\epsilon(0, \phi)$. The selection of the diophantine frequency within $\Omega$ depends here on $\epsilon$ because of the isochrony of the Hamiltonian flow generated by $p_0$.

2. KAM theory (see e.g. Ko, [AA], [Mo]) was first introduced in quantum mechanics in [DS] to deal with quasi-periodic Schrödinger operators. For its applications to the Floquet spectrum of non-autonomous Schrödinger operators see [BG] and references therein. Its first application to generate quantization formulas for $\hbar$ fixed goes back to [Be] for operators in $L^2(\mathcal{T}_l)$ and to [Co] for non-autonomous perturbations of the harmonic oscillators. A uniform quantum version of the Arnold version has been obtained by Popov [Po2], within a quantization different from the canonical one. The related method of the quantum normal forms also yields (much less explicit) quantization formulas with remainders of order $O(\hbar^\infty), O(e^{-1/\hbar^a}), 0 < a < 1, O(e^{-1/\hbar})$ (see [Sj], [BGP], [Po1] respectively). These formulas hold for a much more general class of symbols; however they apply only to perturbations of semi-excited levels ([Sj], [BGP]) or again require a quantization different from the canonical one [Po1].

Acknowledgment We thank Dario Bambusi for many useful comments and for pointing out an error in the first draft of this paper.
2 Proof of the results

Define an analytic action \( \Psi \) of \( T^l \) into \( \mathbb{R}^{2l} \) through the flow of \( p_0 \):

\[
\Psi : T^l \times \mathbb{R}^{2l} \rightarrow \mathbb{R}^{2l}, \quad \phi, (x, \xi) \mapsto (x', \xi') = \Psi_\phi(x, \xi),
\]

where

\[
x'_k := \frac{\xi_k}{\omega_k} \sin \phi_k + x_k \cos \phi_k, \quad \xi'_k := \xi_k \cos \phi_k - \omega_k x_k \sin \phi_k.
\]

(2.2)

If \( z := (x, \xi) \), the flow of initial datum \( z_0 \) is indeed

\[
z(t) = \Psi_{\omega t}(z_0), \quad \omega t := (\omega_1 t, \ldots, \omega_l t).
\]

If \( f \in L^1_{loc}(\mathbb{R}^{2l}) \), the angular Fourier coefficient of order \( k \) is defined by

\[
\hat{f}_k(z) := \frac{1}{(2\pi)^l} \int_{T^l} f(\Psi_\phi(z)) e^{-i(k, \phi)} \, d\phi, \quad k \in \mathbb{Z}^l.
\]

(2.3)

If \( f \in C^1 \) one has, as is well known

\[
f(\Psi_\phi(z)) = \sum_{k \in \mathbb{Z}^l} \hat{f}_k(z) e^{i(k, \phi)} = \hat{f}(z).
\]

Note furthermore that \( f \equiv \hat{f}_k \) for some fixed \( k \) if and only if

\[
f(\Psi_\phi(z)) = e^{i(k, \phi)} f(z).
\]

(2.4)

Taking \( f \in L^1(\mathbb{R}^{2l}) \), we will consider the space Fourier transform

\[
\hat{f}(s) := \frac{1}{(2\pi)^{2l}} \int_{\mathbb{R}^{2l}} f(z) e^{-i(s, z)} \, dz,
\]

(2.5)

as well the space Fourier transforms of the \( \hat{f}_k \)'s:

\[
\hat{\hat{f}}_k(s) := \frac{1}{(2\pi)^{3l}} \int_{\mathbb{R}^{2l}} \int_{T^l} f(\Psi_\phi(z)) e^{-i(k, \phi)} e^{-i(s, z)} \, d\phi \, dz.
\]

Given \( \rho > 0, \sigma > 0 \), define the norm

\[
\|f\|_{\rho, \sigma} := \sum_{k \in \mathbb{Z}^l} e^{\rho|k|} \int_{\mathbb{R}^{2l}} |\hat{\hat{f}}_k(s)| e^{\sigma|s|} \, ds.
\]

Definition 2.1 Let \( \rho > 0, \sigma > 0 \). Then \( \mathcal{F}_{\rho, \sigma} := \{ f : \mathbb{R}^{2l} \rightarrow \mathbb{C} \mid \| f \|_{\rho, \sigma} < +\infty \} \).

Remarks.

1. If \( f \in \mathcal{F}_{\rho, \sigma} \) then \( f \) is analytic on \( \mathbb{R}^{2l} \), and extends to a complex analytic function on a region \( B_{\rho, \sigma} \subset \mathbb{C}^{2l} \) of the form \( B_{\rho, \sigma} := |\Im z| \leq a_1|\Re z| \), with suitable \( a_i \).
2. $F := Op_h^W W(f)$ is a trace-class, self-adjoint $\hbar$-pseudodifferential operator in $L^2(\mathcal{R}^l)$ if $f \in \mathcal{F}_{\rho,\sigma}$. Let $\hat{f}(s)$ be the Fourier transform of $f$. Since $\|\hat{f}\|_{L^1} \leq \|f\|_{\rho,\sigma}$, we have

$$
\|F\|_{L^2 \to L^2} \leq \int_{\mathbb{R}^2i} |\hat{f}(s)| \, ds \equiv \|\hat{f}\|_{L^1}, \quad \|F\|_{L^2 \to L^2} \leq \|f\|_{\rho,\sigma}. \tag{2.6}
$$

3. We introduce also the space $\mathcal{F}_\sigma$ of all functions $f : \mathbb{R}^{2l} \to \mathcal{C}$ such that

$$
\|g\|_\sigma := \int_{\mathbb{R}^{2i}} |\hat{g}(s)| e^{\sigma|s|} \, ds < +\infty.
$$

Obviously if $f \in \mathcal{F}_\sigma$ then $f$ is analytic on $\mathbb{R}^{2l}$, and extends to a complex analytic function in the multi-strip $\mathcal{S} := \{z \in C^{2l} \mid |\Im z_1| < \sigma\}$.

4. Example of $f \in \mathcal{F}_{\rho,\sigma}$: $f(x, \xi) = P(x, \xi)e^{-(|x|^2+|\xi|^2)}$, $P(x, \xi)$ any polynomial.

The starting point of the proof is represented by the first step of the Kolmogorov iteration, and is summarized in the following

**Proposition 2.1** Let $\omega \in \Omega_0$. Then, for any $0 < d < \rho$, $0 < \delta < \sigma$:

1. There exists a unitary transformation $U(\omega, \epsilon, \hbar) = e^{itW_1/h} : L^2 \leftrightarrow L^2$, $W_1 = W_1^*$ and $\omega_1(\epsilon) \in [0, 1]^l$ such that:

$$
UH(\epsilon)U^{-1} = P_0(h, \omega_1(\epsilon)) + \epsilon E_1 I + \epsilon^2 Q_1(\epsilon, h) + \epsilon R_1(\epsilon, h). \tag{2.7}
$$

Here: $E_1 = \tilde{q}_0$; $W_1 = Op_h^W(w_1) \in \Phi_{\rho-d,\sigma-\delta}$, $Q_1(\epsilon, h) = Op_h^W(q_1) \in \Phi_{\rho-d,\sigma-\delta}$ with

$$
\|w_1\|_{\rho-d,\sigma-\delta} \leq d^{-\tau}\|q_0\|_{\rho,\sigma} \quad \|q_1\|_{\rho-d,\sigma-\delta} \leq \delta^{-2}d^{-2\tau}\|q_0\|_{\rho,\sigma}^2. \tag{2.8}
$$

2. $R_1(\epsilon)$ is a self-adjoint semiclassical pseudodifferential operator of order 4 such that $[R_1(\epsilon), P_0] = 0$; $\exists D_1 > 0$ such that, for any eigenvector $\psi_\alpha$ of $P_0(\omega)$:

$$
|\langle \psi_\alpha, R_1(\epsilon)\psi_\alpha \rangle| \leq D_1(|\alpha|\hbar)^2. \tag{2.9}
$$

3. $\forall K > 0$ with $(1 + K^\tau) < \frac{\gamma}{\epsilon\|q_0\|_{\rho,\sigma}} \exists \Omega_1 \subset \Omega_0$ closed and $d_1 > 1$ independent of $K$ such that

$$
|\Omega_0 - \Omega_1| \leq \gamma(1 + 1/K^{d_1}). \tag{2.10}
$$

Moreover if $\omega_1 \in \Omega_1$ then (2.4) holds with $\gamma$ replaced by

$$
\gamma_1 := \gamma - \epsilon\|q_0\|_{\rho,\sigma}(1 + K^\tau). \tag{2.11}
$$

**Proof** To prove Assertion 1 we first recall some relevant results of [BGP].
Lemma 2.1 (Lemma 3.6 of [BGP]) \(\text{Let } g \in \mathcal{F}_{\rho,\sigma}. \text{ Then the homological equation,}\)

\[
\{p_0, w\} + \mathcal{N} = g, \quad \{p_0, \mathcal{N}\} = 0 \quad (2.12)
\]

admits the analytic solutions

\[
\mathcal{N} := \tilde{g}_0; \quad w := \sum_{k \neq 0} \frac{\tilde{g}_k}{i(\omega, k)}, \quad (2.13)
\]

with the property \(\mathcal{N} \circ \Psi = \mathcal{N}\). Equivalently, \(\mathcal{N}\) depends only on \(I_1, \ldots, I_l\). Moreover, for any \(d < \rho:\)

\[
\|\mathcal{N}\|_{\rho,\sigma} \leq \|g\|_{\rho,\sigma}; \quad \|w\|_{\rho-d,\sigma} \leq c_\Psi \frac{\|g\|_{\rho,\sigma}}{d_t}; \quad c_\Psi := \left(\frac{\tau}{e}\right)^{\frac{1}{\gamma}}. \quad (2.14)
\]

Given \((g, g') \in \mathcal{F}_{\rho,\sigma}\), let \(\{g, g'\}_M\) be their Moyal bracket, defined as

\[
\{g, g'\}_M = g \# g' - g' \# g,
\]

where \# is the composition of \(g, g'\) considered as Weyl symbols. We recall that in Fourier transform representation, used throughout the paper, the Moyal bracket is (see e.g. [Fo], 3.4):

\[
(\{g, g'\}_M)^\wedge(s) = \frac{2}{\hbar} \int_{\mathbb{R}^{2n}} \tilde{g}(s^1) \tilde{g}'(s - s^1) \sin \left[\hbar(s - s^1) \wedge s^1 / 2\right] ds^1, \quad (2.15)
\]

where, given two vectors \(s = (v, w)\) and \(s^1 = (v^1, w^1)\), \(s \wedge s^1 := \langle w, v_1 \rangle - \langle v, w_1 \rangle\).

We also recall that \(\{g, g'\}_M = \{g, g'\}\) if either \(g\) or \(g'\) is quadratic in \((x, \xi)\).

Lemma 2.2 (Lemmas 3.1 and 3.3 of [BGP]) \(\text{Let } g \in \mathcal{F}_{\sigma}, g' \in \mathcal{F}_{\sigma-\delta}. \text{ Then:}\)

1. \(\forall 0 < \delta' < \sigma - \delta:\)

\[
\|\{g, g'\}_M\|_{\sigma-\delta-\delta'} \leq \frac{1}{e^{2\delta'(\delta + \delta')}} \|g\|_{\sigma} \|g'\|_{\sigma-\delta}. \quad (2.16)
\]

2. \(\text{Let } g \in \mathcal{F}_{\rho,\sigma} \text{ and } g' \in \mathcal{F}_{\rho,\sigma-\delta}. \text{ Then, for any positive } \delta' < \sigma - \delta:\)

\[
\|\{g, g'\}_M\|_{\rho,\sigma-\delta-\delta'} \leq \frac{1}{e^{2\delta'(\delta + \delta')}} \|g\|_{\rho,\sigma} \|g'\|_{\rho,\sigma-\delta}. \quad (2.17)
\]

As a simple corollary of Lemmas 2.1 and 2.2, we find:

Lemma 2.3 (Lemma 3.4 of [BGP]) \(\text{Let } g \in \mathcal{F}_{\rho,\sigma}, w \in \mathcal{F}_{\rho,\sigma}. \text{ Then:}\)

1. \(\text{Define}\)

\[
g_r := \frac{1}{r} \{w, g_{r-1}\}_M, \quad r \geq 1; \quad g_0 := g.
\]

\(\text{Then } g_r \in \mathcal{F}_{\rho,\sigma-\delta} \text{ for any } 0 < \delta < \sigma, \text{ and the following estimate holds}\)

\[
\|g_r\|_{\rho,\sigma-\delta} \leq \left(\delta^{-2} \|w\|_{\rho,\sigma}\right)^r \|g\|_{\rho,\sigma}. \quad (2.18)
\]
2. Let \( g \in \mathcal{F}_{\rho,\sigma} \), and \( w \) be the solution of the homological equation (2.12). Define the sequence \( p_r : r = 0, 1, \ldots \) as follows:

\[
p_0 := p_0; \quad p_r := \frac{1}{r} \{ w, p_{r-1} \} M, \quad r \geq 1.
\]

Then, for any \( 0 < d < \rho, 0 < \delta < \sigma, p_r \in \mathcal{F}_{\rho-d,\sigma-\delta} \) and fulfills the following estimate

\[
\| p_r \|_{\rho-d,\sigma-\delta} \leq 2 \left( \delta^{-2} \| w \|_{\rho-d,\sigma} \right)^{r-1} \| g \|_{\rho-d,\sigma}, \quad k \geq 1.
\]

**Proof of Proposition 2.1**

With \( U_1 = e^{iW_1/h} \), \( W_1 \) continuous and self-adjoint, we have in general:

\[
U_1 (P_0 + \epsilon Q_0) U_1^{-1} = P_0 + \epsilon P_1 + \epsilon^2 Q_1,
\]

\[
P_1 := Q_0 + [W_1, P_0]/i\hbar,
\]

\[
 Q_1 := \epsilon^{-2} \left( U_1 (P_0 + \epsilon Q_0) U_1^{-1} - P_0 - \epsilon (Q_0 + [W_1, P_0]/i\hbar) \right). \tag{2.21}
\]

We start by looking for \( W_1 \in \mathcal{F}_{\rho,\sigma} \) such that the first order term yields an operator \( N_1 \in \mathcal{F}_{\rho,\sigma} \) commuting with \( P_0 \):

\[
Q_0 + [W_1, P_0]/i\hbar = N_1, \quad [N_1, P_0] = 0. \tag{2.22}
\]

Denoting by \( w_1, N_1 \) the (Weyl) semiclassical symbols of \( W_1, N_1 \), respectively, eq. (2.22) is equivalent to a classical homological equation in \( \mathcal{F}_{\rho,\sigma} \)

\[
\{ p_0, w_1 \}_M + N_1 = q_0, \quad \{ p_0, N_1 \}_M = 0. \tag{2.23}
\]

However \( p_0 \) is quadratic in \( (x, \xi) \). Therefore the Moyal bracket \( \{ p_0, w_1 \}_M \) coincides with the Poisson bracket \( \{ p_0, w_1 \} \) and the above equation becomes

\[
\{ p_0, w_1 \} + N_1 = q_0, \quad \{ p_0, N_1 \} = 0. \tag{2.24}
\]

The existence of \( w_1 \in \mathcal{F}_{\rho-d,\sigma}, N_1 \in \mathcal{F}_{\rho,\sigma} \) with the stated properties now follows by direct application of Lemma 2.1.

We now prove the second estimate in (2.8). We have:

\[
Q_1 = \int_0^1 \int_0^s e^{is_1 W_1/h} [[P_0 + \epsilon Q_0, W_1], W_1] e^{-is_1 W_1/h} ds_1 ds,
\]

and we can estimate

\[
\| [[P_0 + \epsilon Q_0, W_1], W_1] \|_{L^2 \to L^2} \leq \| \{ p_0 + \epsilon q_0, w_1 \}_M, w_1 \}_M \|_{\rho-d,\sigma-\delta}.
\]
It follows, by Lemma 2.3 and Lemma 2.1, that
\[ \|Q_1\|_{L^2 \to L^2} \leq \|\{p_0 + \epsilon q_0, w_1\}_M, w_1\}_M \|_{\rho - d, \sigma - d} \leq \delta^{-2} d^{-2\tau} \|q_0\|_{\rho, \sigma}^2. \]

This proves the second estimate of (2.8).

To prove the Assertion 2 set:
\[ E_1 := N_1(0); \quad \omega_1(\epsilon) = \omega(\nabla I N_1)(0), \quad (2.25) \]
\[ R_1(I, \epsilon) = N_1(I) - \langle (\nabla I N_1)(0), I \rangle - E_1, \quad (2.26) \]
and define
\[ R_1(\epsilon) := O_{\rho, \sigma}^W(R_1(I, \epsilon)). \quad (2.27) \]

Then clearly \( R_1(\epsilon) \) is a self-adjoint semiclassical, tempered pseudodifferential operator of order 4, vanishing to 4-th order at the origin, and with the property \([R_1(\epsilon), P_0] = 0\). Therefore formula (2.9) follows directly by Proposition A.1.

As far as Assertion 3 is concerned, set:
\[ T_k(\alpha) := \{ \omega \in [0, 1]^k : |\langle \omega, k \rangle| \leq \alpha \}, \quad (2.28) \]
\[ \Omega_1 := \Omega_0 - \bigcup_{|k| \geq K} T_k \left( \frac{\gamma_1}{|k|^\tau} \right). \quad (2.29) \]

As in [BG], Lemma 5.6, we have:
\[ |T_\ell(\alpha)| \leq \frac{4}{K^{\alpha}}. \]

Hence if \( \tau > l - 1 \) we can write
\[ \left| \bigcup_{|k| \geq K} T_k \left( \frac{\gamma_1}{|k|^\tau} \right) \right| \leq \sum_{|k| \geq K} \frac{\gamma_1}{|k|^\tau + 1} < \frac{\gamma_1}{K^d_1}. \]

Since \( |\langle \omega_1(\epsilon), k \rangle| \geq \gamma_1/|k|^\tau \) by construction when \( |k| \leq K \), the proposition is proved.

3 Iteration

The above result represents the starting point for the iteration. To ensure convergence, we first preassign the values of the parameters involved in the iterative estimates. Keeping \( \epsilon, K, \gamma, \rho \) and \( \sigma \) fixed define, for \( p \geq 1 \):
\[ \sigma_p := \frac{\sigma}{4p^2}, \quad s_p := s_{p-1} - \sigma_p, \quad \rho_p := \frac{\rho}{4p^2}, \quad r_p := r_{p-1} - \rho_p, \quad \epsilon_p := \frac{4\epsilon_p}{1 + K_p^\tau}, \quad K_p := pK. \quad (3.1) \]
\[ \gamma_p := \gamma_{p-1} - \frac{4\epsilon_p}{1 + K_p^\tau}, \quad \gamma_p = pK. \quad (3.2) \]
where \( \epsilon_p \) is defined in (3.15) below. The initial values of the parameter sequences are chosen as follows:

\[
\gamma_0 := \gamma; \quad s_0 := \sigma; \quad r_0 := \rho, \quad \epsilon_0 = 0.
\]

We then have:

**Proposition 3.1** let \( \omega \in \Omega_0 \). There exist \( \epsilon^*(\gamma) > 0 \) and \( \forall p \geq 1 \), a closed set \( \Omega_p^\gamma \subset \Omega_0 \) such that, if \( |\epsilon| < \epsilon^*(\gamma) > 0 \) and \( \omega_p(h; \epsilon) \in \Omega_p^\gamma \):

1. One can construct two sequences of unitary transformations \( \{X_p\}, \{Y_p\} \) in \( L^2(\mathbb{R}^I) \) with the property

\[
X_p(P_0(\omega) + \epsilon Q_0)X_p^{-1} = P_0(\omega_p(h; \epsilon)) + \epsilon E_p(h; \epsilon)I + \epsilon^{2p} Q_p + \epsilon^{2p} R_p(h; \epsilon) + \epsilon \sum_{s=2}^{p} Y_s R_{s-1}(h) Y_{s-1}^\dagger \epsilon^{2s-2}.
\]

2. \( X_p \) and \( Y_p \) have the form

\[
X_p = U_1 U_2 \cdots U_p; \quad Y_s = U_p U_{p-1} \cdots U_s.
\]

Here \( U_p(\omega, \epsilon, h) = \exp[\text{ie}^{2p-1} W_p/\hbar] : L^2 \leftrightarrow L^2, \quad W_p = W^*_p \)

\[
W_p = Op^W_p(w_p) \in \Phi_{r_p,s_p}, \quad Q_p(\epsilon, h) = Op^W_p(q_p) \in \Phi_{r_p,s_p},
\]

\[
\|w_p\|_{r_p,s_p} \leq \rho_p^{-2r} \|q_{p-1}\|_{r_{p-1},s_{p-1}} \leq \rho_p^{-2r} \sigma_p^{-2} \|q_{p-1}\|_{r_{p-1},s_{p-1}}^2;
\]

\[
E_p(h; \epsilon) = \sum_{s=0}^{p} N_s(h) \epsilon^{2s}, \quad N_s(h) = (\tilde{q}_s)\alpha(h).
\]

3. \( R_s(\epsilon) \) is a self-adjoint semiclassical pseudodifferential operator of order 4; \( [R_s(\epsilon), P_0] = 0 \); there exist \( D_p > 0, \overline{D}_p > 0 \) such that, for any eigenvector \( \psi_\alpha \) of \( P_0(\omega) \):

\[
|\langle \psi_\alpha, R_p(\epsilon)\psi_\alpha \rangle| \leq D_p(|\alpha| \hbar)^2,
\]

\[
|\langle \psi_\alpha, \sum_{s=2}^{p} Y_s R_{s-1} Y_{s-1}^\dagger \epsilon^{2s-2} \psi_\alpha \rangle| \leq \overline{D}_p(|\alpha| \hbar)^2.
\]

4. \( \forall K_{p-1} > 0 \) such that

\[
1 + K_{p-1}^r < \frac{\gamma_p^{-1}}{\epsilon \|q_{p-1}\|_{r_{p-1},s_{p-1}}},
\]

\( \exists \Omega_p \subset \Omega_{p-1} \) closed and \( d_p > 1 \) independent of \( K_p \) such that

\[
|\Omega_p - \Omega_{p-1}| \leq \frac{\gamma_p^{-1}}{1 + 1/(K_{p-1})d_p}.
\]
Moreover if $\omega_p(\epsilon) \in \Omega_p$ then \[ \text{(3.4)} \] holds with $\gamma$ replaced by

\[
\gamma_p := \gamma_{p-1} - \epsilon_p (1 + K_{p-1}^\tau) \quad (3.14)
\]

\[
\epsilon_p := \epsilon^{2p-1} \left\| q_{p-1} \right\|_{r_{p-1},s_{p-1}} \quad (3.15)
\]

Proof

We proceed by induction. For $p = 1$ the assertion is true because we can take $W_1, Q_1, R_1, \omega_1, \Omega_1, K_1$ as in Proposition 2.1. To go from step $p - 1$ to step $p$ we consider the operator

\[ X_{p-1}(P_0(\omega) + \epsilon Q_0)X_{p-1}^{-1} := \]

\[ P_0(\omega_{p-1}(h; \epsilon)) + \epsilon E_{p-1}(h; \epsilon) I + \epsilon^{2p-1} Q_{p-1} + \]

\[ + \epsilon^{2p-1} R_{p-1}(h; \epsilon) + \epsilon \sum_{s=2}^{p-1} Y_s R_{s-1}(h) Y_{s-1} \epsilon^{2s-2}. \]

We have to determine and estimate the unitary map $U_p$ transforming it into the form \( \text{(3.4)} \) via the definitions \( \text{(3.5)} \). With $U_p = e^{iW_p/h}, W_p$ continuous and self-adjoint, we have at the $p$-th iteration step

\[ U_p(P_0(\omega_{p-1} + \epsilon^{2p-1} Q_{p-1})U_p^{-1} = P_0(\omega_p) + \epsilon^{2p-1} P_p + \epsilon^{2p} Q_p, \]

\[ P_p := Q_{p-1} + [W_p, P_0]/i\hbar, \]

\[ Q_p := \epsilon^{-2} \left( U_p(P_0(\omega_{p-1} + \epsilon Q_0)U_1^{-1} - P_0(\omega_{p-1}) - \epsilon(Q_{p-1} + [W_p, P_0]/i\hbar) \right). \]

(the explicit dependence of the frequencies on $(\hbar, \epsilon)$ has been omitted). We will look therefore for $W_p \in \Phi_{r_p,s_p}$ and an operator $N_p \in \Phi_{r_p,s_p}$ such that

\[ Q_p + [W_p, P_0]/i\hbar = N_p, \]

\[ [N_p, P_0] = 0. \quad \text{(3.16)} \]

Denoting $w_p, N_p$ the (Weyl) semiclassical symbols of $W_p, N_p$, respectively, eq. \( \text{(3.16)} \) is again equivalent to the classical homological equation in $F_{r_p,\sigma}$

\[ \{p_0, w_p\}_M + N_p = q_p, \quad \{p_0, N_p\}_M = 0 \]

which once more becomes

\[ \{p_0, w_p\} + N_p = q_p, \quad \{p_0, N_p\} = 0. \]

The existence of $w_p \in F_{r_p,s_p}, N_p \in F_{r_p,s_p}$ with the stated properties now follows by direct application of Lemma 2.1. Expanding $N_p$ as in the proof of Proposition 2.1 and taking into account the definitions \( \text{(3.5)} \) we immediately check that $X_p X_{p-1}(P_0(\omega) + \epsilon Q_0)X_{p-1}^{-1}X_p$ has the form \( \text{(3.4)} \). The estimate of $Q_p$ and the small denominator estimates follow by
It is clearly enough to prove that the principal symbol of the Hamiltonian flow on $R$ to prove the existence of (2.9). It remains to prove the estimate (3.11). By the inductive assumption, it is enough to prove the existence of $D_p' > 0$ such that

$$\langle \psi_{\alpha}, U_p R_{p-1} U_p^{-1} \psi_{\alpha} \rangle \leq D_p'(|\alpha| \hbar)^2.$$ 

We only have to prove that the operator $U_p R_{p-1} U_p^{-1}$ is a $\hbar$-pseudodifferential operator of order 4 fulfilling the hypotheses of Proposition A.1, assuming by the inductive argument the validity of these properties for $R_{p-1}$. On the other hand, $U_p = \exp (i \epsilon^2 \rho W_p / \hbar)$, and $W_p$ is an $\hbar$-pseudodifferential operator of order 0. We can therefore apply the semiclassical Egorov theorem (see e.g. [Ro], Chapter 4) to assert that $U_p R_{p-1} U_p^{-1}$ is again an $\hbar$-pseudodifferential operator. Denote $\sigma(x, \xi; \epsilon; h)$ the Weyl symbol of $U_p R_{p-1} U_p^{-1}$, and consider its expansion

$$\sigma(x, \xi; \epsilon; h) = \sigma_0(x, \xi; \epsilon) + \sum_{j=2}^{M} h^j \sigma_j(x, \xi; \epsilon) + O(h^{M+1}).$$

It is clearly enough to prove that the principal symbol $\sigma_0(x, \xi; \epsilon)$ has order 4. Denote by

$$\phi(x, \xi; \epsilon) := \exp [\epsilon^2 \mathcal{L}_{w_p}](x, \xi)$$

the Hamiltonian flow on $R^{2\mathbb{I}}$ generated by the Hamiltonian vector field $\mathcal{L}_{w_p}$ at time $\epsilon^2 \rho$; here $w_p^0(x, \xi)$ is the principal symbol of $W_p$. Then $\sigma_0(x, \xi; \epsilon) = \mathcal{R}_{p-1}^0(\phi(x, \xi; \epsilon))$ where $\mathcal{R}_{p-1}^0(x, \xi)$ is in turn the principal symbol of $R_{p-1}$. Now

$$\phi(x, \xi; \epsilon) = (x + \int_0^{\epsilon^2 \rho} \nabla_{\xi} w_p(x, \xi; \eta) d\eta, \xi - \int_0^{\epsilon^2 \rho} \nabla_x w_p(x, \xi; \eta) d\eta).$$

By Assumption A2 and the inductive hypothesis we know that $w_p(z) = O(|z|^2)$ as $|z| \to 0$. Hence we can write $\phi(z) = z + \epsilon r(z)$ where $r(z) = O(z), z \to 0$. This concludes the proof of Proposition 3.1.

**Proof of Theorem 1.1**

Applying the estimates on $q_p$ in Propositions 2.1 and 3.1 iteratively, we have

$$\|q_p\|_{r_p, s_p} \leq \left(\frac{4p^2}{\rho}\right)^{2\tau_p} \left(\frac{4p^2}{\sigma}\right)^{2p} \|q_0\|^{2p},$$

whence

$$|\epsilon|^{2p} \|Q_p\|_{L^2 \to L^2} \leq |\epsilon|^{2p} (4p^2)^{2p(\tau_p+1)} \rho^{-2\tau_p \sigma^{-2p}} \|q_0\|^{2p} \to 0 \quad \text{as } p \to \infty,$$ (3.18)

for all $|\epsilon| \leq \epsilon^*$ provided $\epsilon^* > 0$ is small enough. At the $p$-th iteration the frequency is given by

$$\omega_p(h; \epsilon) = \omega + \sum_{s=1}^{p} \nabla I N_s(h) \epsilon^{2s}.$$ (3.19)
Since \( \| \nabla_z f(z) \|_{p-d,\sigma-\delta} \leq \frac{1}{d^0} \| f(z) \|_{p,\sigma} \), by (3.17) we have

\[
\sum_{s=1}^{p} |\nabla f_s(h)\epsilon| \leq \sum_{s=1}^{p} \epsilon^2 (4s^2)^{2s(\tau+1)} \rho^{-2\tau s} \sigma^{-2s} \| q_0 \|^2 \cdot (3.20)
\]

Hence the series (3.19) converges as \( p \to \infty \) for \( |\epsilon| < \epsilon^* \) if \( \epsilon^* \) is small enough, uniformly with respect to \( \hbar \) by (3.17). In the same way, the estimate (3.17) entails, by the definition (3.14), the existence of \( \lim \gamma_p := \gamma_\infty \). Let \( \omega(\hbar; \epsilon) := \lim_{p \to \infty} \omega_p(\hbar; \epsilon) \). Then \( \omega(\hbar; \epsilon) \) is diophantine with constant \( \gamma_\infty \) by Proposition 3.1. In the same way:

\[
\mathcal{E}(\hbar; \epsilon) = \sum_{s=1}^{\infty} N_s(h) \epsilon^2 \cdot |\epsilon| < \epsilon^*.
\]

Finally, let \( \mathcal{R}(\alpha \hbar, \epsilon) \) be an asymptotic sum of the power series \( \sum_{s=2}^{\infty} Y_s R_s^{-1} Y_s^{-1} \epsilon^{2s-2} \). Then the validity of (1.7) follows by its validity term by term. This concludes the proof of Theorem 1.1.

**Proof of Corollary 1.1**

It is enough to illustrate the specialization of the argument of Propositions 2.1 and 3.1 to the \( h = 0 \) case. Denoting by \( e^{\epsilon L_{w_1}} \) the canonical flow at time \( \epsilon \) generated by the Hamiltonian vector field generated by the symbol \( w_1 \), we have:

\[
e^{\epsilon L_{w_1}}(p_0 + \epsilon q_0)(x, \xi) = (p_0 + \epsilon p_1 + \epsilon^2 q_1^0)(x, \xi), \quad (3.21)
\]

\[
p_1 := q_0 + \{ w_1, p_0 \}, \quad (3.22)
\]

\[
q_1^0 := -2 \left( e^{\epsilon L_{w_1}}(p_0 + \epsilon q_0)(x, \xi) - p_0 - \epsilon(q_0 + \{ w_1, p_0 \}) \right). \quad (3.23)
\]

Remark that \( e^{\epsilon L_{w_1}}(p_0 + \epsilon q_0)(x, \xi) \) is the principal symbol of \( U_1(P_0 + \epsilon Q_0)U_1^{-1} \) by the semiclassical Egorov theorem; \( p_1 \) is the full, and hence principal, symbol of \( P_1 \) because \( p_0 \) is quadratic. Likewise, \( q_1^0 \) is the principal symbol of \( Q_1 \). Hence the classical definitions (3.21, 3.22, 3.23) correspond to the principal symbols of the semiclassical pseudodifferential operators \( U_1(P_0 + \epsilon Q_0)U_1^{-1} \), \( P_1, Q_1 \) defined in (2.21, 2.22, 2.23). Therefore we can take over the homological equation (2.24) and apply Lemma 2.1 once more. This yields the same \( w_1 \) and \( N_1 \) of Proposition 2.1. To prove the estimate (2.8) for \( q_1^0 \) we write

\[
q_1^0 = \int_0^1 e^{s \epsilon L_{w_1}} \{ \{ p_0 + \epsilon q_0, w_1 \}, w_1 \} ds.
\]

Now as in (BGGS), Lemma 1, note that if \( |\epsilon| < \epsilon^* \) and \( z = (x, \xi) \in \mathcal{B}_{p-d,\sigma-\delta} \) then \( e^{s \epsilon L_{w_1}} z \in \mathcal{B}_{p,\sigma} \) for \( 0 \leq s \leq 1 \) because (Lemma 2.1) \( \epsilon \| \nabla w_1 \|_{p-d,\sigma} \leq \epsilon (\tau/\epsilon) c \| d^{-\tau} \|_{p,\sigma} \). Therefore we can apply Lemma 2.3, valid a fortiori for the Poisson bracket, and, as in the
proof of Proposition 2.1, get the estimate corresponding to the second one of (2.8):

$$\|q_0\|_{p-d,\sigma-\delta} \leq \|\{p_0 + \epsilon q_0, w_1\}, w_1\|_{p-d,\sigma-\delta} \leq \delta^{-2} d^{-2}\|q_0\|_{\rho,\sigma}^2.$$  

(3.24)

Now, writing:

$$\psi_1 \epsilon(x, \xi) = e^{\epsilon L w_0} (x, \xi), \quad E_1 := N_1(0);$$

(3.25)

$$\omega_1(\epsilon) = \omega + \epsilon (\nabla I N_1)(0),$$

(3.26)

$$\tilde{R}_1(I, \epsilon) = N_1(0) - \langle (\nabla I N_1)(0), I \rangle - E_1,$$

(3.27)

we can sum up the above argument by writing (compare with (2.7))

$$\psi_1 \epsilon \circ (p_0 + \epsilon q_0) = E_1 + \langle \omega_1(0; \epsilon), I \rangle + \epsilon^2 q_1(I, \phi) + \epsilon R^0_1(I, \epsilon)$$

(3.28)

where \(R^0_1\) is the principal symbol of \(R_1\). Moreover, Assertion 3 of Proposition 2.1 holds without change.

Let us now specialize the iterative argument of Proposition 3.1. First, the parameters defined in (3.1,3.2,3.3) remain unchanged. Then:

1. The construction of the two sequences of canonical transformations

$$\chi^p_\epsilon = \psi_1 \epsilon \circ \psi_2 \epsilon \cdots \circ \psi^p_\epsilon, \quad p = 1, 2, \ldots$$

(3.29)

$$\zeta^s_\epsilon = \psi_\epsilon^s \circ \psi_\epsilon^{p-1} \cdots \circ \psi_\epsilon^s, \quad p = 1, 2, \ldots$$

(3.30)

$$\psi_\epsilon^s(x, \xi) = e^{\epsilon L w^0_\epsilon} (x, \xi)$$

(3.31)

such that

$$\psi_\epsilon^p \circ I_0 \circ (p_0 + \epsilon q_0) =$$

(3.32)

$$\langle \omega_p(0, \epsilon), I \rangle + E_p(\epsilon) + \epsilon^2 q_p + \epsilon R^0_p + \epsilon \sum_{s=2}^p \psi_\epsilon^s \circ R^0_{s-1} \epsilon^{2s-2}.$$

follows as in the above argument valid for \(p = 1\). Here \(w^0_s, q^0_p, R^0_s\) are the principal symbols of the semiclassical pseudodifferential operators \(W_s, Q_p\) and \(R_s\) once reexpressed on the \((x, \xi)\) canonical variables via, with \(\omega_p\) in place of \(\omega_1\). Morover:

$$E_p(\epsilon) = \sum_{s=0}^p N_s(0) \epsilon^{2s}, \quad N_s(0) = \langle q^0_s \rangle_0(0).$$

(3.33)

$$\omega_p(\epsilon) = \omega + \sum_{s=0}^p \omega_s(0) \epsilon^{2s}, \quad \omega_s(0) = \nabla I N_s(0)$$

(3.34)

2. The estimates (3.8) are a fortiori valid with \(w^0_p, q^0_p, R^0_p\) in place of \(w_p, q_p, R_p\); as a consequence, (3.13) holds unchanged together with the definitions (3.12, 3.14, 3.15). Hence the uniform estimate (3.17) allows us to set \(h = 0\) in (3.19, 3.20).
3. Finally, remark that $\mathcal{R}_s^0(I) = O(I^2), s = 1, \ldots, p$. Now the estimate $\psi_s \mathcal{R}_s(I) = O(I^2)$ as $I \to 0$ follows by exactly the same argument of Proposition 3.1 after reexpression on the canonical variables $(x, \xi)$. 
Appendix

To establish the remainder estimate (1.7) the key fact is that vanishing of a symbol at the origin \((x, \xi) = 0\) implies bounds on harmonic oscillator matrix elements that are uniform in \(\hbar\). No analyticity of the symbol is required for this result, so we will state and prove it in somewhat greater generality, using the following semiclassical symbol class defined in Shubin [Sh]:

\[
\Sigma^{m,\mu} = \{ f \in C^\infty(\mathbb{R}^2 \times (0, \epsilon)) : |\partial_\gamma z f(z, \hbar)| \leq C_\gamma \langle z \rangle^{m-|\gamma|/\hbar^\mu} \},
\]

where \(z = (x, \xi)\), here considered a real variable, and \(\langle z \rangle = \sqrt{1 + |z|^2}\). For future reference we note that Proposition A.2.3 of [Sh] gives the result:

\[
\forall f \in \Sigma^{0,\mu}, \| Op_\hbar^W(f) \|_{L^2} \leq C(f) \hbar^\mu, \tag{A.1}
\]

for all \(\hbar \in (0, \epsilon]\).

The matrix elements in question are most easily computed in Bargmann space, with the remainder operator written as a Toeplitz operator. Since these are anti-Wick ordered, we first must consider the translation from Weyl symbols to anti-Wick (for these notions, see e.g. [BS]). Denoting by \(Op_\hbar^{AW}(f)\) the anti-Wick quantization of a symbol \(f \in \Sigma^{m,\mu}\), the correspondence is given by the action of the heat kernel on the symbol:

\[
Op_\hbar^{AW}(f) = Op_\hbar^W(e^{\hbar \Delta/4} f), \tag{A.2}
\]

where \(\Delta = \Delta_z = \partial_x \cdot \partial_x + \partial_\xi \cdot \partial_\xi\). To begin, we show that the Weyl symbol of an anti-Wick operator is given by formal expansion of the heat kernel up to a remainder.

**Lemma A1** For \(f, g \in \Sigma^{m,\mu}\), suppose that \(Op_\hbar^{AW}(g) = Op_\hbar^W(f)\). Then for all \(n \geq 1\),

\[
f = \sum_{k=0}^{n-1} \frac{1}{k!} \left( \frac{\hbar}{4 \Delta} \right)^k g \in \Sigma^{m-2n,\mu+n}.
\]

**Proof.** According to (A.2),

\[
f(z, \hbar) = \frac{1}{(\pi \hbar)^{l/2}} \int e^{-|z-w|^2/\hbar} g(w) dw.
\]

In this expression we will expand \(g(w)\) in a Taylor series centered at \(w = z\):

\[
g(w, \hbar) = \sum_{|\alpha| < 2n} \frac{1}{\alpha!} \partial^\alpha g(z, \hbar)(w-z)^\alpha + r(w, z, \hbar),
\]

where

\[
r(w, z) = \sum_{|\alpha| = 2n} c_\alpha'(w-z)^\alpha \int_0^1 (1-t)^{2n-1} \partial^\alpha g(z+t(w-z)) dt.
\]
Thus,

\[
f(z, h) = \sum_{|\alpha| < 2n} c_\alpha \partial^\alpha g(z, h) + r(z, h),
\]

where

\[
c_\alpha = \frac{1}{(\pi h)^{\frac{|\alpha|}{2}}} \int_0^1 w^{\alpha} e^{-|w|^2/h} \, dw,
\]

and

\[
r(z, h) = \sum_{|\alpha| = 2n} c''_\alpha h^{-l} \int_0^1 \int_0^1 (w - z)^{\alpha} e^{-|w|^2/h} (1 - t)^{2n-1} \partial^\alpha g(z + t(w - z)) \, dt \, dw.
\]

Note that \(c_\alpha = 0\) for \(|\alpha|\) odd, and for any integer \(k\)

\[
\sum_{|\alpha| = 2k} c_\alpha \partial^\alpha g = \frac{1}{k!} \left( \frac{h}{4\Delta} \right)^k g.
\]

The lemma is thus reduced to the claim that \(r(z, h) \in \Sigma^{m-2n, \mu+n}\).

To see this, we change variables by \(w' = (w - z)/\sqrt{h}\) to write

\[
r(z, h) = \sum_{|\alpha| = 2n} c''_\alpha h^{-l} \int_0^1 \int_0^1 w^{\alpha} e^{-|w|^2/(1 - t)^{2n-1}} \partial^\alpha g(z + tw\sqrt{h}) \, dt \, dw.
\]

We must estimate the derivatives:

\[
\partial^\gamma r(z, h) = \sum_{|\alpha| = 2n} c''_\alpha h^{-l} \int_0^1 \int_0^1 w^{\alpha} e^{-|w|^2/(1 - t)^{2n-1}} \partial^\beta g(z + tw\sqrt{h}) \, dt \, dw,
\]

where \(|\beta| = 2n + |\gamma|\). This integral for \(\partial^\gamma r\) we then split into two pieces according to the domain of the \(w\)-integral, \(I'_{\alpha,\beta} : |w| < |z|/2\) and \(I''_{\alpha,\beta} : |w| > |z|/2\). The assumption \(g \in \Sigma^{m,\mu}\) implies an estimate

\[
|I'_{\alpha,\beta}| \leq C \langle z \rangle^{m-2n-|\gamma|} h^{n+\mu}.
\]

(A.3)

The second term is taken care of by the exponential factor in \(|w|\):

\[
|I''_{\alpha,\beta}| < C t h^l \langle z \rangle^{-l}, \quad \forall l.
\]

Therefore \(\partial^\gamma r\) satisfies an estimate of the form (A.3) for any \(\gamma\), and hence \(r \in \Sigma^{m-2n, \mu+n}\).

\(\square\)

Our application of Lemma A.1 will be specifically to operators of order 4:

**Lemma A.2** For \(g \in \Sigma^{4,0}\),

\[
Op_h^W (g) = Op_h^W (g) - \frac{h}{4} Op_h^W (\Delta g) + R(h),
\]

16
where \( \|R(h)\|_{L^2} \leq Ch^2 \).

**Proof.** Let \( \sigma(A) \) denote the Weyl symbol of the \( h \)-pseudodifferential operator \( A \). Applying Lemma A.1 with \( n = 2 \) gives

\[
\sigma(Op_h^{AW}(g)) = g + \frac{\hbar}{4} \Delta g + r_1,
\]

and

\[
\frac{\hbar}{4} \sigma(Op_h^{AW}(\Delta g)) = \frac{\hbar}{4} \Delta g + r_2,
\]

where \( r_1, r_2 \in \Sigma^{0,2} \). Noting that

\[
Op_h^{W}(g) - Op_h^{AW}(g) + \frac{\hbar}{4} \sigma(Op_h^{AW}(\Delta g)) = Op_h^{W}(r_1 - r_2),
\]

the bound on \( R(h) \) follows from (A.1).

The point of introducing anti-Wick symbols is to exploit the Bargmann space representation of the harmonic oscillator. The Bargmann space is (see e.g. [BS])

\[
\mathcal{H}_h = L^2_{hol}(\mathcal{C}^l, e^{-|z|^2/\hbar} \, dzd\bar{z}).
\]

The Bargmann transform is an isomorphism \( \mathcal{B} : L^2(\mathcal{R}^l) \to \mathcal{H}_h \), defined so as to intertwine anti-Wick operators with Toeplitz operators:

\[
\mathcal{B} \circ Op_h^{AW}(f) \circ \mathcal{B}^{-1} = T_h(f).
\]

The Toeplitz operator \( T_h(f) : \mathcal{H}_h \to \mathcal{H}_h \) is defined for \( f \in \Sigma^{m,\mu} \) by

\[
T_h(f) = \Pi_h M(f),
\]

where \( M(f) \) denotes the multiplication operator on \( L^2(\mathcal{C}^l, e^{-|z|^2/\hbar} \, dzd\bar{z}) \) (identifying \( \mathcal{R}^{2l} = \mathcal{C}^l \) by \( z = x + i\xi \)), and \( \Pi_h : L^2(\mathcal{C}^l, e^{-|z|^2/\hbar} \, dzd\bar{z}) \to \mathcal{H}_h \) is orthogonal projection onto the holomorphic subspace.

The main result of this Appendix is the following matrix element estimate:

**Proposition A.1** Let \( \{\psi_\alpha\} \) be the normalized eigenstates of the standard harmonic oscillator on \( L^2(\mathcal{R}^l) \). Suppose \( f \in \Sigma^{4,0} \) satisfies

\[
f(z, \hbar) = \sum_{|\gamma|=4} z^\gamma g_\gamma(z, \hbar),
\]

where \( \sup |\partial^\beta g_\gamma| \leq M \) for all \( |\beta| \leq 2 \). Then

\[
|\langle \psi_\alpha, Op_h^{W}(f)\psi_\alpha \rangle| \leq CM(|\alpha|\hbar)^2
\]
for all $\alpha, \bar{h}$, where $C$ depends only on the dimension.

**Proof.** Under the Bargmann transform the harmonic oscillator eigenstates have a particularly convenient form:

$$(B^{-1}\psi_\alpha)(z) = (\pi \hbar^{\frac{\alpha+1}{2}})^{-1} \cdot z^\alpha.$$ 

Using Lemma A.1 we write

$$Op^W(f) = Op^W_{\bar{h}}(f) - \frac{\hbar}{4} Op^W_{\bar{h}}(\Delta f) + R(\bar{h}), \quad (A.4)$$

where $|\langle R(\bar{h}) \rangle| \leq C\bar{h}^2$.

Consider the matrix element of the first term on the right-hand side of (A.4). In Bargmann space this becomes

$$\langle \psi_\alpha, Op^W_{\bar{h}}(f) \psi_\alpha \rangle = \frac{1}{\pi^{l+1/2}h^{\frac{\alpha+l}{2}}} \int z^\alpha f(z, \bar{h}) z^\alpha e^{-|z|^2/\bar{h}} \, dzd\bar{z}.$$ 

Writing $f$ as a sum over $z^\gamma g_\gamma$ with $|\gamma| = 4$, the estimate for a particular $\gamma$ is straightforward:

$$|\langle \psi_\alpha, Op^W_{\bar{h}}(z^\gamma g_\gamma) \psi_\alpha \rangle| \leq \frac{M}{\pi^{l+1/2}h^{\frac{\alpha+l}{2}}} \int |z^\alpha|^2 |z|^4 e^{-|z|^2/\bar{h}} \, dzd\bar{z} = M\bar{h}^2(|\alpha| + l)(|\alpha| + l + 1).$$

The second term on the right in (A.4) is handled in a similar way. By assumption we can write $\Delta f = \sum_{|\eta|=2} z^\eta h_\eta(z, \bar{h})$, where $\sup |h_\eta| \leq 12M$. The estimate then proceeds exactly as above (noting that there is an extra factor of $\bar{h}$ in front of this term).

---

**References**

[AA] V.I. Arnold, A. Avez, *Problèmes ergodiques de la mécanique classique*, Gauthier-Villars, Paris 1967

[BG] D. Bambusi, S. Graffi, *Nonautonomous Schrödinger operators with undounded quasiperiodic coefficients and KAM methods*, Commun.Math.Phys. 219 (2001), 465–480.

[BGP] D. Bambusi, S. Graffi, T. Paul, *Normal Forms and Quantization Formulae*, Commun.Math.Phys. 207, 173-195 (1999).

[BGGS] G. Benettin, L. Galgani, A. Giorgilli, J.M. Strelczyn, *A proof of Kolmogorov’s theorem*, Nuovo Cimento 29B (1984), 201-223.

[BS] F.A. Berezin and M.S. Shubin, *The Schrödinger Equation*, Kluwer 1991.

[Be] J. Bellissard, *Stability and Instability in Quantum Mechanics*. Trends and developments in the eighties (Bielefeld, 1982/1983), 1–106, World Sci. Publishing, Singapore, 1985.
[Co] M.COMBESCURE. The quantum stability problem for time-periodic perturbations of the harmonic oscillator. Ann. Inst. H. Poincaré Phys. Théor. 47 (1987), no. 1, 63–83.

[DS] E.I.DINABURG, YA.G.SINAI, The one-dimensional Schrödinger operator with a quasiperiodic potential, Functional Anal.Appl. 9, (1976), 279-289

[Fo] G.FOLLAND, Harmonic analysis in phase space, Princeton University Press 1988.

[Ko] A.N.KOLMOGOROV, On conservation of conditionally periodic motions for a small change in Hamilton’s function. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 98, (1954). 527–530. (English translation in: G.CASATI, J.FORD (Editors) Lecture Notes in Physics 91, Springer-Verlag 1979.

[Mo] J.Moser, Stable and random motions in Hamiltonian systems, Princeton University Press 1973

[Po1] G.POPOV. Invariant tori effective stability and quasimodes with exponentially small terms. I: Birkhoff normal form, Ann. Henri Poincaré 1 (2000), 223–248.

[Po2] G.POPOV. Invariant tori effective stability and quasimodes with exponentially small terms. II: Quantum Birkhoff normal form, Ann. Henri Poincaré 1 (2000), 249–279.

[Ro] D.ROBERT, Autour de l’approximation semiclassique, Birkhäuser, Basel 1987.

[Sh] M.S.SHUBIN, Pseudodifferential Operators and Spectral Theory, Springer-Verlag 1987.

[Sj] J.SjöSTRAND, Semi-excited levels in non-degenerate potential wells, Asymptotic Analysis 6 (1992), 29–43.