THE CONTROLLABILITY OF A THERMOELASTIC PLATE PROBLEM REVISITED

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Abstract. In this paper, the controllability for a thermoelastic plate problem
with a rotational inertia parameter is considered under two scenarios. In the
first case, we prove the exact and approximate controllability when the controls
act in the whole domain. In the second case, we prove the interior approximate
controllability when the controls act only on a subset of the domain. The
distributed controls are determined explicitly by the physical constants of the
plate in the first case, while this is no longer possible in the second case as
the relation (79) is no longer valid. In this case, we propose an approxima-
tion of the control function with an error that tends to zero. By means of a
powerful and systematic approach based on spectral analysis, we improve some
already existing results on the optimal rate of the exponential decay and on
the analyticity of the associated semigroup.

1. Introduction. We study the exact and the approximate controllability of the
following thermoelastic Kirchhoff-Love plate with controls acting in the whole do-
main Ω

\begin{align*}
  w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w + \nu \Delta \theta &= u_1, & \text{in } \Omega \times [0, \infty) \\
  \theta_t - \Delta \theta - \nu \Delta w_t &= u_2, & \text{in } \Omega \times [0, \infty) 
\end{align*}

(1)

with Dirichlet boundary conditions

\begin{align*}
  w(x, t) = \Delta w(x, t) = \theta(x, t) = 0, & \quad x \in \partial \Omega, \quad t \geq 0,
\end{align*}

(2)

and initial conditions

\begin{align*}
  w(x, 0) = w_0(x), \quad w_1(x, 0) = w_1(x), \quad \theta(x, 0) = \theta_0(x), & \quad x \in \Omega
\end{align*}

(3)

where Ω is a bounded domain in \( \mathbb{R}^d \) and \( \Delta = \sum_{i=1}^{d} \partial_{x_i}^2 \) is the Laplace operator.

Then we study the interior approximate controllability of the system (1)-(3) when
controls acting on a subset \( \omega \) of Ω

\begin{align*}
  w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w + \nu \Delta \theta &= 1_\omega u_1, & \text{in } \Omega \times [0, \infty) \\
  \theta_t - \Delta \theta - \nu \Delta w_t &= 1_\omega u_2, & \text{in } \Omega \times [0, \infty)
\end{align*}

(4)

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In (4) the distributed controls $u_1, u_2 \in L^2(0, \tau; L^2(\Omega))$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$.

The function $w$ represents the vertical displacement of the mid-plane of the plate from its equilibrium position, while $\theta$ the temperature at this middle plane. The vibrations of the plate are described by the Kirchhoff model which takes into account the rotational inertia through the second term in the first equation of system (1). The rotational inertia parameter $\gamma \geq 0$ related with the thickness of the plate and therefore it is usually small. The coupling term $\nu \Delta w_t$ ($\nu \geq 0$) takes into account the heat induced by the high frequency vibrations of the plate.

The above system is derived in [20] and references therein. Many authors discussed the exponential stability (with or without rotational inertia parameter) and analyticity of the system (1) under different types of boundary conditions (see, e.g. [19, 21, 30, 34, 35, 37]). If $\gamma > 0$ (Kirchhoff type), the system (1) has a hyperbolic character and hence the corresponding semigroup is not analytic, but the exponential stability of solutions is kept; see for example [35]. In fact, the rotational inertia parameter makes the plate more realistic physically but more difficult mathematically. Avalos and Lasiecka [6] proved the uniform exponential stability of system (1) with both clamped and simply supported boundary conditions, with Newton’s law of cooling applied to the thermal component on the boundary. Additional stability and analyticity results that apply to a class of abstract systems are given in Russell [36] and Ammar Khodja and Benabdallah [17].

In the particular case $\gamma = 0$ (Euler-Bernoulli type), which corresponds to neglecting rotational inertia in the vibration of the plate, Kim [19] proved that the energy of the plate decays exponentially fast with a certain dissipative boundary condition. Liu and Renardy [31] improved Kim’s result, showing that the semigroup associated to the elliptic part of the system is of analytic type, which in particular implies that the solution decays uniformly as time goes to infinity and reveals the parabolic character of the system. Chang and Triggiani [11] performed a spectral analysis of an abstract thermoelastic plate equations with different boundary conditions. They proved that the resulting semigroup of contractions is neither compact nor differentiable, which contradicts the case $\gamma = 0$. They also provided interesting spectral properties without giving the explicit expressions of the corresponding eigenvalues.

Control problems for system (1) have been studied intensively in the last years, since such systems combine the conservative effect of the motion equation with the dissipative effect of the heat equation.

In the case $\gamma = 0$, the null controllability of system (1) with hinged boundary conditions is proved by Lasiecka and Triggiani [22] with a single control either on the mechanical component or in the thermal one. These results were extended to other types of boundary conditions in [4, 5]. Some problems were studied by Benabdallah and Naso [8]. They proved that when the control functions ($u_2 = f$) act on the whole domain ($\omega \equiv \Omega$), for any $T > 0$ and for any initial data $(w_0, w_1, \theta_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0, T) \times \Omega)$ such that the problem (1) is null controllable. This results was found by Lasiecka and Triggiani [22] by a different method. The same result was obtained when the control function $f$ acts on a small subset $\omega$ of $\Omega$ [8]. The proof of this theorem follows from the procedure developed by Lebeau and Robbiano in [24] (see also [25, 29]).

In [38], de Terasa and Zuazua proved that system (1), is controllable in the following sense: If the control time is large enough and we act in the equation of
displacement by means of a control supported in a neighborhood of the boundary of the plate, then we may control exactly the displacement and simultaneously the temperature in an approximate way. The method of proof is inspired from techniques elaborated by Zuazua in [40]. Leiva [27] proved a necessary and sufficient algebraic condition for the approximate controllability for thermoelastic plate equations with Dirichlet boundary conditions. The controllability analysis is based on Kalman criterion.

In the case $\gamma > 0$, Lasiecka and Seidman [23] proved observability inequalities either for the thermal component or the elastic one with hinged boundary conditions. The observation is in the whole set $\Omega$ and it is valid for any $T > 0$. Avalos [4] proved the null controllability with a single control acting in the thermal under clamped boundary conditions. Another null controllability result was given by Castro and Teresa in [10] for system (1) when two different controls are written in series form. The result is then obtained by combining the null controllability of these new systems with the convergence of the series. Hansen and Zhang [16] proved that a linear thermoelastic beam may be controlled exactly to zero in a finite time by a single boundary control. Moreover, they showed that the optimal time of controllability becomes arbitrarily small if $\gamma = 0$, as in the case of Euler-Bernoulli beam.

Regularity, stability and controllability are the most interesting properties for the solutions to evolution equations that attract people’s attention. In spite of the importance of these results, the explicit expressions of the optimal decay rate and the distributed controls, have not yet been addressed in the literature for thermoelastic plates. So, we study this topic here to fill this gap.

In this paper, and through a simple, powerful and systematic approach based on the spectral analysis of semigroups, we prove the exponential stability, analyticity, approximate and exact controllability for a system (1)-(3) when the controls act on the whole domain ($\omega \equiv \Omega$) and the interior approximate controllability results when the controls act on an open subset of the domain ($\omega \Subset \Omega$). Moreover, we provide the explicit expressions of the optimal decay rate and the distributed controls ($\omega \Subset \Omega$) by the physical constants of the plate. In fact, these expressions have never been given explicitly in the literature. In the case ($\omega \Subset \Omega$), it is no possible to explicitly determine the control function since (79) is no longer valid. The analyticity of the thermoelastic plate was proved in [27, 28] by a spectral analysis also. But our approach is more detailed and complete since it based on the explicit expressions of the eigenvalues of the corresponding linear operator. The spirit of our approach is very different from the traditional methods used to study these topics for thermoelastic plates. In fact, our spectral analysis gives more precise bounds that allow us to provide the explicit expressions of the optimal decay rate and the distributed controls.

This paper is organized as follows. In Section 2, we perform the spectral analysis of the problem. In Section 3, we prove that the uncontrolled problem decays exponentially to zero at a rate determined explicitly. In the particular case $\gamma = 0$, we prove that the associated semigroup is analytic. In Section 4, we prove the approximate and the exact controllability of the problem and we provide the explicit expressions of the distributed controls, when ($\omega \equiv \Omega$). In the other case, when ($\omega \Subset \Omega$), we prove the interior approximate controllability and we propose an approximate expression of the control function with an error which tends to zero.
2. Spectral analysis of the problem. Before starting the analysis of the problem (1)-(3), let us summarize the main properties of some operators that will be useful to us. Consider the positive operators $A$ and $A^2$ on $X = L^2(\Omega)$ defined by $A\phi = -\Delta \phi$ and $A^2\phi = \Delta^2 \phi$ with domains $\mathcal{D}(A) = (H^2 \cap H^1_0)(\Omega)$, $\mathcal{D}(A^2) = \{w \in H^4(\Omega) \text{ with } w = \Delta w = 0 \text{ on } \partial \Omega\}$. The operator $A$ has the following very well-known properties (see [12]).

(a): The spectrum of $A = -\Delta$ consists of only eigenvalues
\[ 0 < \lambda_1 < \lambda_2 < ... < \lambda_n < \infty, \]
have finite multiplicity equal to the dimension of the corresponding eigenspace.
(b): The eigenfunctions of $A$ with Dirichlet boundary condition are real analytic functions.
(c): There exists a complete orthonormal set $\{\phi_n\}$ of eigenvector of $A$.
(d): For all $x \in \mathcal{D}(A)$ we have
\[ A x = \sum_{n=1}^{\infty} \lambda_n (x, \phi_n) \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x, \]
where $(\cdot, \cdot)$ is the inner product in $X = L^2(\Omega)$ and
\[ E_n x = (x, \phi_n) \phi_n. \]
So $\{E_n\}$ is a complete family of orthogonal projections in $X$ and
\[ x = \sum_{n=1}^{\infty} E_n x, \ x \in X. \]
(e): The operator $-A$ generates an analytic semigroup $\{e^{-At}\}_{t \geq 0}$ given by
\[ e^{-At} x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \]
(f): The fractional power spaces $X^r$ are given by
\[ X^r = \mathcal{D}(A^r) = \left\{ x \in X, \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \right\}, \ r \geq 0 \]
with the norm
\[ \|x\|_{X^r} = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \ x \in X^r \]
and
\[ A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \]
Setting $z = (z_0, z_1, z_2)$, we have
\[ z_0 = \Delta w, \ z_1 = \frac{\partial w}{\partial t}, \ z_2 = \theta. \]
Let the state space is
\[ Z_\gamma = X \times V_\gamma \times X, \text{ where } V_\gamma = \begin{cases} H^1_0(\Omega) & \text{if } \gamma > 0 \\ L^2(\Omega) & \text{if } \gamma = 0, \end{cases} \]
equipped with the norm
\[
\left\| \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right\|^2_{Z_\gamma} = \|z_0\|^2_X + \|z_1\|^2_{V_\gamma} + \|z_2\|^3_X. \tag{8}
\]
Let us introduce the inertia operator \((I - \gamma \Delta)\) with the domain \((H^2 \cap H^1_0)(\Omega)\). It is well known that the operator \((I - \gamma \Delta)\) is a positive and self-adjoint in \(L^2(\Omega)\). Then, one has
\[
V_\gamma = \mathcal{D}((I - \gamma \Delta)^{1/2}) = \{ z \in X : \sum_{n=1}^{\infty} (1 + \gamma \lambda_n)\|E_n z\|^2 < \infty \},
\]
endowed with the norm
\[
\|z\|^2_{V_\gamma} = \langle (I - \gamma \Delta)^{1/2} z, (I - \gamma \Delta)^{1/2} z \rangle_X = \sum_{n=1}^{\infty} (1 + \gamma \lambda_n)\|E_n z\|^2. \tag{9}
\]
The system (1)-(3) and (4) can be written respectively as a linear evolution equation of the form
\[
z' = Az + Bu, \quad z(0) = z_0, \quad u = (u_1, u_2). \tag{10}
\]
where
\[
B = \begin{cases}
B = \begin{pmatrix} 0 & 0 \\ J_\gamma & 0 \\ 0 & I \end{pmatrix} & \text{if } \omega \equiv \Omega \\
B_\omega = \begin{pmatrix} 0 & 0 \\ 1_\omega J_\gamma & 0 \\ 0 & 1_\omega I \end{pmatrix} & \text{if } \omega \subset \Omega,
\end{cases}
\]
\[
J_\gamma = (I - \gamma \Delta)^{-1} : X \to V_\gamma \text{ and } \mathcal{A} : \mathcal{D}(A) \subset Z_\gamma \to Z_\gamma \text{ is given by}
\]
\[
\mathcal{A} \left( \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right) = \begin{pmatrix} -A z_1 \\ J_\gamma A(z_0 + \nu z_2) \\ -A(\nu z_1 + z_2) \end{pmatrix}, \tag{12}
\]
with the domain
\[
\mathcal{D}(A) = \begin{cases}
H^1_0(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))^2 & \text{if } \gamma > 0 \\
(H^2(\Omega) \cap H^1_0(\Omega))^3 & \text{if } \gamma = 0.
\end{cases}
\]
Computing \(Az\) yields
\[
Az = \begin{pmatrix}
\sum_{n=1}^{\infty} \frac{\lambda_n}{1 + \gamma \lambda_n} E_n z_0 + \nu \sum_{n=1}^{\infty} \frac{\lambda_n}{1 + \gamma \lambda_n} E_n z_1 \\
-\nu \sum_{n=1}^{\infty} \lambda_n E_n z_2 - \sum_{n=1}^{\infty} \lambda_n E_n z_2
\end{pmatrix}
= \sum_{n=1}^{\infty} \begin{pmatrix}
\frac{\lambda_n}{1 + \gamma \lambda_n} & -\lambda_n & 0 \\
0 & 0 & 0 \\
-\nu \lambda_n & 0 & 0
\end{pmatrix} \begin{pmatrix} E_n \ 0 \ 0 \\ 0 \ E_n \ 0 \\ 0 \ 0 \ E_n \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \tag{13}
\]
where \(\{P_n\}_{n \geq 1}\) is a complete family of orthogonal projections in \(Z_\gamma\)
\[
P_n = \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix} \tag{14}
\]
satisfying
\[ P_j P_n = \begin{cases} P_n & \text{if } j = n, \\ 0 & \text{if } j \neq n, \end{cases} \quad \sum_{n \geq 1} P_n = I, \quad (15) \]
and
\[ A_n = R_n P_n = \begin{pmatrix} 0 & -\lambda_n & 0 \\ \frac{\lambda_n}{1+\gamma\lambda_n} & 0 & \frac{\nu\lambda_n}{1+\gamma\lambda_n} \\ 0 & -\nu\lambda_n & -\lambda_n \end{pmatrix} P_n, \quad n \geq 1. \quad (16) \]

We can also easily verify that
\[ A_n P_n = P_n A_n, \quad n \geq 1. \]

The characteristic equation of \( R_n \) is given by
\[ \sigma^3 + \lambda_n \sigma^2 + \frac{\lambda^2}{c_n}(1 + \nu^2)\sigma + \frac{\lambda^3}{c_n} = 0, \quad (17) \]
where
\[ c_n = \sqrt{1 + \gamma\lambda_n}. \quad (18) \]

**Remark 2.1.** (i): The characteristic equation \( (17) \) is the same that obtained in [2, 16]. Moreover, Hansen and Zhang [16] proved that the roots of \( (17) \) are simple if \( 0 < \nu \leq 1/\sqrt{2} \). In Proposition 2.3, we shall improve this result.

(ii): The eigenvalues of \( R_n \) are given by
\[ \sigma_i(n) = c_n^{-1}\lambda_n \varrho_i(n), \quad i = 0, 1, 2, \quad (19) \]
where \( \varrho_i(n) \) are the roots of the following equation
\[ \varrho^3 + c_n\varrho^2 + (1 + \nu^2)\varrho + c_n = 0. \quad (20) \]

**Lemma 2.2.** Let \( \varrho_i(n), \quad i = 0, 1, 2, \quad n \geq 1, \) the roots of \( (20) \). Then we have \( \Re \varrho_i(n) < 0 \) for all \( i = 0, 1, 2, \quad n \geq 1. \quad (21) \)

**Proof.** Let
\[ p_c(\varrho) = \varrho^3 + c\varrho^2 + (1 + \nu^2)\varrho + c, \quad (22) \]
denotes the polynomial in \( (20) \). We now apply Routh and Hurwitz Theorem to prove that the polynomial \( (22) \) is stable, i.e., \( \Re \varrho < 0 \).

From Theorem 2.4 (iii) on p. 33 of [39], the polynomial \( p(\varrho) = \varrho^3 + a\varrho^2 + b\varrho + d \), with real coefficients, is stable if and only if \( a > 0, \quad b > 0, \quad d > 0 \) and \( ab > d \). We note that the coefficients of the polynomial \( (22) \) satisfy these inequalities, which gives \( (21) \). However, by \( (19) \) we obtain as well \( \Re \sigma_i(n) < 0, \quad i = 0, 1, 2, \quad n \geq 1. \)

Following Hansen and Zhang [16], we show that \( (17) \) has one real root and two non-real complex conjugate roots if \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \).

**Proposition 2.3.** All the roots of \( (17) \) are simple if \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \).

**Proof.** The discriminant of the characteristic polynomial in \( (20) \) reads as
\[ \Delta = 18c_n^2(1 + \nu^2) - 4c_n^4 + c_n^2(1 + \nu^2)^2 - 4(1 + \nu^2)^3 - 27c_n^2. \quad (23) \]

For \( \Delta < 0 \), we know that \( (20) \) possesses three distinct roots: one real and two (non-real) complex conjugate ones. Taking into account \( 1 + \nu^2 \geq 1 \) and using Young’s inequality
\[ 4c_n^2(1 + \nu^2) \leq 4c_n^4 + 4(1 + \nu^2)^2 \leq 4c_n^4 + 4(1 + \nu^2)^3. \]
Hence,
\[ \Delta \leq 14c_n^2(1 + \nu^2) + c_n^2(1 + \nu^2)^2 - 27c_n^2. \]
Therefore, for \( \Delta \) to be negative, it suffices that
\[ 14(1 + \nu^2) + (1 + \nu^2)^2 - 27 < 0 \]
or, solving for \( 1 + \nu^2 \), if
\[ -7 - 2\sqrt{19} < 0 < 1 + \nu^2 < -7 + 2\sqrt{19} \quad (24) \]
i.e.,
\[ 0 \leq \nu^2 < -8 + 2\sqrt{19} \]
or, equivalently,
\[ 0 \leq \nu < \sqrt{2\sqrt{19} - 8}. \quad (25) \]
Let \( q_1(c), q_2(c) = \overline{q}_1(c) \) and \( q_0(c) \) denote the roots of \( P_c(q) = 0 \) with \( q_0 \) real and \( q_1 \) in the lower half-plane for \( c > 0 \). Let
\[ (\sigma_0(c), \sigma_1(c), \sigma_2(c)) = M(c)(q_0(c), q_1(c), \overline{q}_1(c)), \]
where \( M(c) = \frac{c^2 - 1}{c\gamma} \).

Then
\[ (\sigma_0(n), \sigma_1(n), \sigma_2(n)) = (\sigma_0(c_n), \sigma_1(c_n), \sigma_2(c_n)), \]
for all \( n \geq 1 \).

Writing \( P_c(q) = 0 \) as \( (q - q_0)(q - q_1)(q - \overline{q}_1) = 0 \) leads to the system
\[ \begin{align*}
    c &= -q_0 - 2\Re q_1, \\
    1 + \nu^2 &= |q_1|^2 + 2q_0\Re q_1, \\
    c &= -q_0|q_1|^2.
\end{align*} \quad (26) \]

From this and Lemma 2.2 we deduce for \( c > 0 \) that
\begin{enumerate}
    \item \( c + q_0 > 0 \),
    \item \( |q_1|^2 > 1 \),
    \item \( \lim_{c \to \infty} |q_1| = 1 \),
    \item \( \lim_{c \to \infty} -\frac{q_0}{c} = 1 \),
    \item \( 3m\nu q_1 \neq 0 \).
\end{enumerate}

To prove (v) we used the fact that \( \nu < 2 \) and a contradiction argument.

By (v) we know that the branches \{\( \sigma_0(c_n) \), \( \sigma_1(c_n) \) and \( \sigma_2(c_n) \)\} are distinct. Thus we only need to show that the roots are distinct within each of these branches. It will therefore suffice to show that the functions \( |\sigma_i(c)|, \ i = 0, 1, 2, \) are monotone on \([1, \infty]\) (This is since \( c_n = \sqrt{1 + \gamma\lambda_n} \geq 1 \)).

First note that since \( q_0(c), q_1(c) \) and \( \overline{q}_1(c) \) are distinct roots of \( P_c(q) = 0 \), the Implicit Function Theorem implies that \( q_0(c), q_1(c) \) and \( \overline{q}_1(c) \) are analytic functions of \( c \) (locally) for each \( c \geq 1 \). In particular, these functions are differentiable for \( c \geq 1 \). Since \( M(c) \) is also differentiable for \( c > 0 \) it follows that \( |\sigma_i(c)|, \ i = 0, 1, 2, \) are differentiable for all \( c \geq 1 \).

(i) We shall prove that \( |\sigma_1(c)| \) is monotone on \([1, \infty]\). Let \( r = |q_1|^2 \). Eliminating \( \Re q_1 \) and \( q_0 \) from (26) gives
\[ r^3 - r^2(1 + \nu^2) + c^2r - c^2 = 0. \]

Implicitly solving for \( r'(c) \) we obtain
\[ r'(c) = \frac{2c(1-r)}{3r^2 - 2(1 + \nu^2)r + c^2} := \frac{N(c)}{D(c)} \]
\[ |\sigma_1(c)| = M^2(c)r(c) := R(c) \quad (27) \]
Note that by (ii), \( N(c) \) is negative for all \( c > 0 \).

On the other hand, the discriminant of \( D(c) = 3r^2 - 2(1 + \nu^2)r + c^2 \) is \( \Delta_D(c) = 4(1 + \nu^2 - 3c^2) \). As \( c = \sqrt{1 + \gamma \lambda} \geq 1 \) and \( 1 + \nu^2 < 2\sqrt{19} - 7 \), we obtain

\[
\frac{1 + \nu^2}{c^2} < \frac{2\sqrt{19} - 7}{c^2} < 3.
\]

Thus \( \Delta_D(c) = 4(1 + \nu^2 - 3c^2) < 0 \), therefore \( D(c) \) is positive for all \( c \geq 1 \). Then we conclude that \( r'(c) < 0 \) for all \( c \geq 1 \).

We are going to show now that \( R(c) \), given by (27)\textsubscript{2}, is monotone on \([1, \infty)\). We calculate

\[
R'(c) = M(c) \left( 2M'(c) r(c) + M(c) r'(c) \right).
\]

When \( c \geq 1 \) we have \( 0 < M(c) < \frac{c}{\gamma} \) and

\[
M'(c) = \frac{c^2 + 1}{c^2 \gamma} > \frac{1}{\gamma} > \frac{M(c)}{c}.
\]

Then

\[
R'(c) > M^2(c) \left( \frac{2}{c} r(c) + r'(c) \right) \quad \text{(use (27)\textsubscript{1})}
\]

\[
= \frac{M^2(c)}{c} D(c) \left( 2D(c) r(c) + cN(c) \right)
\]

\[
\geq 2 \frac{M^2(c)}{c} D(c) \left( r^2(3r + 2(7 - 2\sqrt{19})) + c^2 \right) \quad \text{(use } c \geq 1 \text{ and } r > 1 \text{)}
\]

\[
\geq 2 \frac{M^2(c)}{c} D(c) (18 - 4\sqrt{19}) > 0.
\]

We conclude that \( |\sigma_1|^2 = R(c) \) is increasing for \( c \geq 1 \) and consequently \( |\sigma_1| \) is increasing on \([1, \infty)\).

(ii) We shall prove that \( |\sigma_0(c)| = |M(c) \varrho_0(c)| \) is monotone on \([1, \infty)\). Since \( M(c) > 0 \) and \( \varrho_0(c) < 0 \) (see (21)), then

\[
|\sigma_0(c)| = -M(c) \varrho_0(c) = -\sigma_0(c).
\]

From (26)\textsubscript{3}, we have \( \varrho_0(c) = -\frac{c}{|\varrho_1|} = -\frac{c}{r(c)} \), and

\[
\varrho'_0(c) = -\frac{r(c) - cr'(c)}{r^2(c)}.
\]

Since \( r'(c) < 0 \) and \( r(c) = |\varrho_1(c)|^2 > 0 \), we conclude that \( \varrho'_0(c) < 0 \) and consequently \( \varrho_0(c) \) is decreasing on \([1, \infty)\).

Recalling that \( M(c) \) is positive and increasing on \([1, \infty)\) and \( \varrho_0(c) \) is negative and decreasing on \([1, \infty)\), one can conclude that \( \sigma_0(c) = M(c) \varrho_0(c) \) is decreasing on \([1, \infty)\) and consequently \( |\sigma_0(c)| = -\sigma_0(c) \) is increasing on \([1, \infty)\). The proof is complete.

Following Khusainov and Pokojovy \cite{18} (see also \cite{33}, page 179), one can get the roots of (20).

**Lemma 2.4.** Let \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \) and

\[
\Delta_0 = c_n^2 - 3(1 + \nu^2),
\]
\[ \Delta_1 = 2c_n^3 - 9c_n(1 + \nu^2) + 27c_n, \]
\[ C = \frac{3}{2} \left( \frac{1}{\sqrt{2}} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right) \right) \]  
where \( \sqrt{\cdot} \) and \( \sqrt[3]{\cdot} \) stand for the main branch of complex square and cubic roots. The three roots of \( (20) \) are given by
\[ \varrho_i(n) = -\frac{1}{3} \left( c_n + Ce^{\frac{2\pi i}{3} + \frac{\Delta_0}{C} e^{-\frac{2\pi i}{3}}} \right), \quad i = 0, 1, 2, \quad n \geq 1, \tag{29} \]
where \( i \) is the imaginary unit \( (i^2 = -1) \).

**Proof.** Let
\[ Q = \frac{\Delta_0}{9} \quad \text{and} \quad R = \frac{\Delta_1}{54}. \tag{30} \]
Otherwise, compute
\[ A = -(R + \sqrt{R^2 - Q^3})^{\frac{1}{3}}. \]
From (30), yields
\[ A = -\frac{1}{3} \sqrt[3]{\frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)}. \]
Next compute
\[ B = \begin{cases} Q & \text{if } A \neq 0, \\ 0 & \text{if } A = 0. \end{cases} \tag{31} \]
By (30), the expression of \( B \) becomes
\[ B = -\frac{1}{3} \sqrt[3]{\frac{1}{2} \left( \Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)}. \]
Following the same procedure as in [33] (page 179), one can get (29). \( \Box \)

**Remark 2.5.** (i): Note that \( C \) in (29) cannot be zero due to the simplicity of the roots. Moreover, we already know that \( \Delta \), defined by (23), is negative if \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \) and \( \Delta_1 - 4\Delta_0^3 = -27\Delta > 0 \). Also, it is easy to prove that \( \Delta_1 \), given by (28), is positive for \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \). Consequently, one can easily conclude that \( C > 0 \) for \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \).

(ii): Since \( \sqrt{2\sqrt{19} - 8} \approx 0.8472 > \sqrt[7]{\frac{1}{2}} \approx 0.7071 \), the condition in Proposition 2.3 is an improvement over Proposition 3.4 in [16]. Apparently, the condition \( 0 \leq \nu < \sqrt{2\sqrt{19} - 8} \) cannot be improved anymore (in general).

We shall prove that the supremum of \( n \mapsto \max_{i=0,1,2} \Re \sigma_i(n) \) over \( n \geq 1 \) is attained at \( n = 1 \).

**Lemma 2.6.** Let \( \sigma_i(n) \in \mathbb{C}, \quad i = 0, 1, 2, \quad n \geq 1, \) denote the roots of \( (17) \). Then we have
\[ \sup_{n \geq 1} \max_{i=0,1,2} \Re \sigma_i(n) = \max_{i=0,1,2} \Re \sigma_i(1) < 0. \]

**Proof.** We rewrite the equation \( (17) \) in the form
\[ \sigma^3 + \lambda \sigma^2 + \frac{\lambda^2}{c^2} (1 + \nu^2) \sigma + \frac{\lambda^3}{c^2} = 0. \]
Actually, \( \sigma = c^{-1}(\lambda) \lambda \varrho \) with \( c(\lambda) = \sqrt{1 + \gamma \lambda} \) for \( \lambda > 0 \), i.e., \( c \) is a function of \( \lambda \) with \( c'(\lambda) > 0 \).
Let $\sigma_i(\lambda) \in \mathbb{C}$, $i = 1, 2, 3$, denote the roots of (32) for arbitrary $\lambda > 0$. To prove the minimum of $\Re \sigma_i(\lambda)$ is attained at the smallest $\lambda$, for $i = 1, 2, 3$, we need to show \( \frac{\partial \Re \sigma_i(\lambda)}{\partial \lambda} = \Re \frac{\partial \sigma_i(\lambda)}{\partial \lambda} \leq 0 \). To compute the later derivative, we apply the Implicit Function Theorem to (32).

Since the dependence on $\lambda$ in equation (32) is quite nontrivial, instead of considering \( \frac{\partial \sigma_i(\lambda)}{\partial \lambda} \), one can adopt the equivalent equation (20) and prove \( \frac{\partial \sigma_i(c)}{\partial c} \leq 0 \) for

$$\text{Subtracting the above expressions, we obtain } g_1 = \frac{\partial \sigma_i(\lambda)}{\partial \lambda} \leq 0.$$  \hspace{1cm} (34)

One can easily verify that the denominator on the right-hand side of (34) can not be 0. Indeed, recalling (33), we obtain the algebraic system for “unknown” $g$

$$g^3 + cg^2 + (1 + \nu^2)g + c = 0.$$  \hspace{1cm} (35)

Equations (35) are equivalent to saying that the polynomial $P(g) = g^3 + cg^2 + (1 + \nu^2)g + c$ has a second-order zero (i.e., $P(g) = 0$, $P'(g) = 0$). But according to Proposition 2.3, we know it is impossible.

Hence, $g'(c)$ is smooth for any $c \geq 1$. Further, $g'(c)$ can only be zero if the numerator on the right-hand side of (34) turns zero which is equivalent with

$$1 + g^2 = 0, \text{ i.e., } g = \pm i$$

which is impossible according to Lemma 2.2 ($\Re g < 0$). Therefore, both $\Re g$ and $\Im g$ do not change their sign for $c \geq 1$. Hence, it suffices to prove $g'(1) \leq 0$. Plugging $c = 1$ in (33), solving it for $g = g(1)$ and plugging the result into (34), we will obtain $\Re g'(1) \leq 0$. Therefore, $\Re g(c)$ is monotonically decreasing for $c \geq 1$. Hence, $\Re \sigma(\lambda)$ is monotonically decreasing as well.

In the following we prove that $\max_{i=0,1,2} \Re \sigma_i(1)$ is attained at $i = 1$ (or $i = 2$).

**Lemma 2.7.** Let $\sigma_i(n) \in \mathbb{C}$, $i = 0, 1, 2, n \geq 1$, denote the roots of (17). Then, for $0 \leq \nu < \sqrt{2\sqrt{19} - 8}$, we have

$$\max_{i=0,1,2} \Re \sigma_i(1) = \Re \sigma_1(1).$$  \hspace{1cm} (36)

**Proof.** From (29) and Lemma 2.2, we have

$$\left\{ \begin{array}{l} \Re g_0(1) = \Re g_1(1) = -\frac{1}{2} \left( c_1 + C + \frac{\Delta_0}{C} \right) < 0, \\ \Re g_1(1) = \Re g_2(1) = -\frac{1}{2} \left( c_1 - \frac{1}{2} (C + \frac{\Delta_0}{C}) \right) < 0. \end{array} \right.$$  

Subtracting the above expressions, we obtain

$$g_0(1) - \Re g_1(1) = -\frac{1}{2} \left( C + \frac{\Delta_0}{C} \right).$$  \hspace{1cm} (37)

The sign of this difference depends on the sign of $C + \frac{\Delta_0}{C}$. In the following we study this issue.
Recall that, according to Remark 2.5, $\Delta_1 > 0$ and $C > 0$ for $0 \leq \nu < \sqrt{2\sqrt{19} - 8}$. From (31) and (30)₁, we have

$$\frac{\Delta_0}{C} = \sqrt{1/2 \left( \Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_3^3} \right)}$$

which leads together with (28) to

$$C^3 = \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_3^3} \right), \quad \left( \frac{\Delta_0}{C} \right)^3 = \frac{1}{2} \left( \Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_3^3} \right). \tag{38}$$

We distinguish now two cases:

**Case 1:** $\Delta_0 > 0$, then yields immediately that $C + \frac{\Delta_0}{C} > 0$ as $C > 0$ for $0 \leq \nu < \sqrt{2\sqrt{19} - 8}$.

**Case 2:** $\Delta_0 < 0$, then yields immediately that $C - \frac{\Delta_0}{C} > 0$ as $C > 0$ for $0 \leq \nu < \sqrt{2\sqrt{19} - 8}$. Adding both expressions of (38), we get

$$C^3 + \left( \frac{\Delta_0}{C} \right)^3 = (C + \frac{\Delta_0}{C})(C^2 - \Delta_0 + \left( \frac{\Delta_0}{C} \right)^2) = \Delta_1 > 0,$$

which implies that $C^3 > \left( \frac{-\Delta_0}{C} \right)^3$, i.e., $C^2 > -\Delta_0$. Moreover, we have

$$C^2 - \Delta_0 > C^2 + \Delta_0 > 0,$$

which implies that $C^2 - \Delta_0 + \left( \frac{\Delta_0}{C} \right)^2 > 0$. As $C + \frac{\Delta_0}{C}$ has the same sign that $C^2 - \Delta_0 + \left( \frac{\Delta_0}{C} \right)^2$, we deduce that $C + \frac{\Delta_0}{C} > 0$.

Since $C + \frac{\Delta_0}{C}$ is positive in both cases, (36) follows from (37) and (19). □

3. **Exponential decay and analyticity of the uncontrolled system.** In this section we study the exponential decay and analyticity of the uncontrolled system $(u_1 = u_2 = 0)$. Let us before recall some preliminaries concerning analytic semigroups on $\mathcal{X}$, a complex Banach space with dual space $\mathcal{X}'$. Let $\mathcal{L}(\mathcal{X})$ be the Banach space of all linear bounded operators on $\mathcal{X}$. For $0 < \varphi \leq \pi$, let $S(\varphi)$ be the sector

$$S(\varphi) = \{ z \in \mathbb{C} \setminus \{0\} : \arg(z) \leq \varphi \}.$$

**Definition 3.1.** (see [14] and [15]) Let $0 < \varphi \leq \frac{\pi}{2}$. We say that $\mathcal{T} = \{ T(t) : t \in S(\varphi) \cup \{0\} \} \subset \mathcal{L}(\mathcal{X}')$, is an analytic semigroup in $S(\varphi)$ if

(i) $T(t + s) = T(t)T(s)$ for all $t, s \in S(\varphi)$, $T(0) = I$,

(ii) $\langle T(t)f, \phi \rangle$ is analytic in $S(\varphi)$ for all $f \in \mathcal{X}$, $\phi \in \mathcal{X}'$,

(iii) $\lim_{t \to 0, t \in S(\varphi)} T(t)f = f$, for all $f \in \mathcal{X}$, $\alpha \in (0, \varphi)$.

**Remark 3.2.**

(i): Definition 3.1 is equivalently to the definition 3.7.1 in [3]. In fact, a semigroup is analytic if it has an analytic extension to a sector.

(ii): The sector $S(\varphi + \frac{\pi}{2})$ belongs to the resolvent set $\rho(\mathcal{A})$ (see [14, 15, 32]).

(iii): For an analytic semigroup $\mathcal{T} = \{ T(t) : t \in S(\varphi) \cup \{0\} \}$ on $\mathcal{X}$, the restriction $\{ T(t) \}_{t \geq 0}$, is a $C_0$-semigroup on $\mathcal{X}$.

(iv): From the Definition 3.1, we have $S(\varphi) \subset S(\varphi + \frac{\pi}{2})$.

Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ denote the generator of $\{ T(t) \}_{t \geq 0}$ and $\rho(\mathcal{A})$ be the resolvent set of $\mathcal{A}$. We will use the following well-known characterization of analytic semigroups (see [14, 15]).
Theorem 3.3. Let $0 < \varphi_0 < \frac{\pi}{2}$. The following assertions are equivalent.

(i) $(A, \mathfrak{D}(A))$ is the generator of an analytic semigroup in $S(\varphi_0)$ on $X$.

(ii) For any $\varphi \in (0, \varphi_0)$, there exist constants $M, R > 0$ such that for any $\lambda \in \mathbb{C}$, $|\lambda| > R$, $\lambda \in S(\varphi + \frac{\pi}{2})$ we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}.$$  

(iii) For any $\varphi \in (-\varphi_0, \varphi_0)$, the operator $(e^{i\varphi}A, \mathfrak{D}(A))$ generates a $C_0$-semigroup on $X$.

In the case of uncontrolled problem, it is well known that the operator $A$ defined by (12) generates a $C_0$-semigroup $\{T(t)\}_{t \geq 0} = \{e^{-tA}\}_{t \geq 0}$ (see [35]). In view of (13), the $C_0$-semigroup has the following representation

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad t \geq 0, \quad z \in Z_\gamma,$$  

where $P_n$ and $A_n$ are defined by (14) and (16), respectively.

Following our procedure in [1], one can prove

Lemma 3.4. The operator $R_n$ defined by (16) can be written as

$$R_n = \sum_{i=0}^{2} \sigma_i(n) q_i^n,$$  

where $\{q_i^n\}_{i=0}^{2} \in \mathbb{R}^3$ is a complete family of complementarily projections defined by

$$q_i^n = \prod_{k=0 \atop k \neq i}^{2} \left( \frac{R_n - \sigma_k(n) I_3}{\sigma_i(n) - \sigma_k(n)} \right)$$

$$= \frac{1}{\Upsilon} \begin{pmatrix} g_i^2 + c_n q_i + \nu^2 & -c_n (c_n + q_i) & -\nu \\ c_n (c_n + q_i) & q_i^2 + c_n q_i & \nu c_n^{-1} q_i \\ -\nu & -\nu c_n q_i & q_i^2 + 1 \end{pmatrix}.$$  

where $c_n$ is defined by (18), $\sigma_i(n)$ is defined by (19) and

$$\Upsilon = 3g_i^2 + 2q_i c_n + (1 + \nu^2).$$  

Moreover, we have

$$e^{R_n t} = \sum_{i=0}^{2} e^{t\sigma_i(n)} q_i^n.$$  

Remark 3.5. (i): From (16) and (42), the family of linear operators given by (39) can be written as

$$e^{A_n t} = \sum_{i=0}^{2} e^{t\sigma_i(n)} P_{ni}, \quad T(t)z = \sum_{n=1}^{\infty} \sum_{i=0}^{2} e^{t\sigma_i(n)} P_{ni} z, \quad z \in Z_\gamma$$  

where $P_{ni} = q_i^n P_n$, $i = 0, 1, 2$

is a complete family of orthogonal projections in $Z_\gamma$. 
(ii): Since $R_n$ given by (16) is diagonalizable, it can be written as

$$R_n = K_n^{-1} J_n K_n,$$

where

$$K_n = \begin{pmatrix} 1 & 1 & 1 \\ -c_n^{-1} \varrho_0 & -c_n^{-1} \varrho_1 & -c_n^{-1} \varrho_2 \\ \frac{\varrho_0}{\varrho_0 + \varrho_n} & \frac{\varrho_1}{\varrho_1 + \varrho_n} & \frac{\varrho_2}{\varrho_2 + \varrho_n} \end{pmatrix}, \quad J_n = \begin{pmatrix} c_n^{-1} \lambda_n \varrho_0 & 0 & 0 \\ 0 & c_n^{-1} \lambda_n \varrho_1 & 0 \\ 0 & 0 & c_n^{-1} \lambda_n \varrho_2 \end{pmatrix}$$

and

$$K_n^{-1} = \frac{1}{\Lambda_n} \begin{pmatrix} a_{11} - a_{12} & a_{13} \\ -a_{21} & a_{22} - a_{23} \\ a_{31} & a_{32} - a_{33} \end{pmatrix}$$

where

$$a_{11} = \frac{\nu c_n^{-1} \varrho_2 \varrho_1 (\varrho_2 - \varrho_1)}{(\varrho_2 + c_n)(\varrho_2 + c_n)}, \quad a_{12} = \frac{\nu c_n^{-1} \varrho_2 \varrho_1 (\varrho_2 - \varrho_1)}{(\varrho_2 + c_n)(\varrho_2 + c_n)}, \quad a_{13} = c_n^{-1} (\varrho_1 - \varrho_2),$$

$$a_{21} = \frac{\nu c_n^{-1} \varrho_2 \varrho_0 (\varrho_2 - \varrho_0)}{(\varrho_2 + c_n)(\varrho_2 + c_n)}, \quad a_{22} = \frac{\nu c_n^{-1} \varrho_2 \varrho_0 (\varrho_2 - \varrho_0)}{(\varrho_2 + c_n)(\varrho_2 + c_n)}, \quad a_{23} = c_n^{-1} (\varrho_0 - \varrho_2),$$

$$a_{31} = \frac{\nu c_n^{-1} \varrho_1 \varrho_0 (\varrho_1 - \varrho_0)}{(\varrho_1 + c_n)(\varrho_1 + c_n)}, \quad a_{32} = \frac{\nu c_n^{-1} \varrho_1 \varrho_0 (\varrho_1 - \varrho_0)}{(\varrho_1 + c_n)(\varrho_1 + c_n)}, \quad a_{33} = c_n^{-1} (\varrho_0 - \varrho_1),$$

$$\Lambda_n = \frac{\nu c_n^{-1} \varrho_2 (\varrho_0 - \varrho_1)}{\varrho_2 + c_n} + \frac{\nu c_n^{-1} \varrho_0 (\varrho_1 - \varrho_2)}{\varrho_0 + c_n} + \frac{\nu c_n^{-1} \varrho_1 (\varrho_2 - \varrho_0)}{\varrho_1 + c_n}.$$

3.1. Optimal rate of decay. Here, we are interested in determining explicitly the optimal decay rate of (1)-(3) with $\gamma \geq 0$ and $u_1 = u_2 = 0$.

**Theorem 3.6.** For $0 \leq \nu < \sqrt{2 \nu_9 - 8}$, the semigroup $\{T(t)\}_{t \geq 0}$ given by (39) decays exponentially to zero,

$$\|T(t)\| \leq N e^{\mu_1 t}, \quad t \geq 0,$$

where $N$ is a positive constant and $\mu_1$ is the optimal decay rate given by

$$\mu_1 = -\frac{\lambda_1}{3\sqrt{1 + \gamma \lambda_1}} \left( \sqrt{1 + \gamma \lambda_1} + C + \frac{\Delta_0}{C} \right) < 0,$$

where

$$\Delta_0 = -2 + \gamma \lambda_1 - 3\nu^2,$$

$$\Delta_1 = \sqrt{1 + \gamma \lambda_1} \left( 20 + 2\gamma \lambda_1 - 9\nu^2 \right),$$

$$C = \sqrt{\frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4 \Delta_0^2} \right)}.$$

**Proof.** Since $J_n$ is diagonal, (39) can be written as

$$T(t)z = \sum_{n=1}^{\infty} K_n^{-1} e^{tJ_n} K_n P_n z \quad \text{(use $P_n^2 = P_n$)}$$

$$= \sum_{n=1}^{\infty} K_n^{-1} P_n e^{tJ_n} P_n K_n P_n z, \quad z \in Z, $$

which, written in norm, reads

$$\|T(t)z\| \leq \sum_{n=1}^{\infty} \|K_n^{-1} P_n\|_{L(\mathcal{Z})} \|e^{tJ_n} P_n\|_{L(\mathcal{U})} \|K_n P_n\|_{L(\mathcal{U},\mathcal{Z}_\gamma)} \|P_n z\|_{\mathcal{Z}_\gamma}^2.$$
We define the following two linear bounded operators

\[ K_n P_n : \mathcal{H} \to Z, \text{ and } K_n^{-1} P_n : Z \to \mathcal{H}, \]

where \( \mathcal{H} = X \times X \times X \), \( Z = X \times V \times X \) and \( X = L^2(\Omega) \). Let us find bounds for \( \|K_n^{-1} P_n\| \) and \( \|K_n P_n\| \). Consider \( z = (z_0, z_1, z_2) \in Z \), such that \( \|z\|_Z = 1 \), it follows from (9) that

\[
\|z_0\|_X^2 = \sum_{j=1}^{\infty} \|E_j z_0\|^2 \leq 1, \quad \|z_1\|_V^2 = \sum_{j=1}^{\infty} c_j^2 \|E_j z_1\|^2 \leq 1
\]

and \( \|z_2\|_X^2 = \sum_{j=1}^{\infty} \|E_j z_2\|^2 \leq 1 \), which immediately implies

\[
\|E_j z_0\| \leq 1, \quad c_j \|E_j z_1\| \leq 1, \quad \|E_j z_2\| \leq 1, \quad j \geq 1.
\] (52)

It follows from (14) and (47) that

\[
\|K_n^{-1} P_n z\|_H^2 = \frac{1}{|\Lambda_n|^2} \left\| \begin{array}{c}
  a_{11} E_n z_0 - a_{12} E_n z_1 + a_{13} E_n z_2 \\
  -a_{21} E_n z_0 + a_{22} E_n z_1 - a_{23} E_n z_2 \\
  a_{31} E_n z_0 - a_{32} E_n z_1 + a_{33} E_n z_2
\end{array} \right\|_H^2
\]

\[
= \frac{1}{|\Lambda_n|^2} \left\| \begin{array}{c}
  \|a_{11}\| E_n z_0 + \frac{|a_{12}|}{c_n} \|E_n z_1\| + |a_{13}| \|E_n z_2\| \\
  -\|a_{21}\| E_n z_0 + \frac{|a_{22}|}{c_n} \|E_n z_1\| + |a_{23}| \|E_n z_2\| \\
  \|a_{31}\| E_n z_0 + \frac{|a_{32}|}{c_n} \|E_n z_1\| + |a_{33}| \|E_n z_2\|
\end{array} \right\|_H^2
\]

\[
\leq \frac{1}{|\Lambda_n|^2} \left[ (\|a_{11}\| \|E_n z_0\| + |a_{12}| \|E_n z_1\| + |a_{13}| \|E_n z_2\|)^2 + (\|a_{21}\| \|E_n z_0\| + |a_{22}| \|E_n z_1\| + |a_{23}| \|E_n z_2\|)^2 + (\|a_{31}\| \|E_n z_0\| + |a_{32}| \|E_n z_1\| + |a_{33}| \|E_n z_2\|)^2 \right].
\]

Using (52) we infer that

\[
\|K_n^{-1} P_n z\|_H^2 \leq \frac{1}{|\Lambda_n|^2} \left[ (|a_{11}| + \frac{|a_{12}|}{c_n} + |a_{13}|)^2 + (|a_{21}| + \frac{|a_{22}|}{c_n} + |a_{23}|)^2 + (|a_{31}| + \frac{|a_{32}|}{c_n} + |a_{33}|)^2 \right] \]

From (5) and (29), we infer for \( i = 0, 1, 2 \) and \( n \geq 1 \), that

\[
|g_i| = (C_i(\gamma, \nu) + O(1)) \lambda_n, \quad |c_n + g_i| = (C'_i(\nu, \gamma) + O(1)) \lambda_n, \quad |3\sigma^2_2 + 2\sigma_2 c_n + 1 + \nu^2| = C''_i(\nu, \gamma) \lambda_n^2 + O(\lambda_n^2)
\] (53)

which implies the existence of a positive constant \( \Gamma_1 \) depending on \( \gamma \) and \( \nu \) such that

\[
\|K_n^{-1} P_n\|_{\mathcal{L}(Z, \mathcal{H})} \leq \frac{\Gamma_1(\gamma, \nu)}{\lambda_n}. \] (54)
To find a bound for $\|K_n P_n\|_{\mathcal{L}(\mathcal{H}, \mathcal{Z})}$, we consider $z = (z_0, z_1, z_2) \in \mathcal{H}$, such that $\|z\|_{\mathcal{H}} = 1$. As a result
\[
\|z_i\| = \sum_{j=1}^{\infty} \|E_j z_i\|^2 \leq 1, \quad i = 0, 1, 2,
\]
and
\[
\|E_j z_i\| \leq 1, \quad i = 0, 1, 2, \quad j \geq 1.
\]
Using (8), (14) and (46) we arrive at
\[
\|K_n P_n z\|_{\mathcal{Z}}^2 = \|K_n P_n z\|_{\mathcal{Z}}^2 = \left(\|E_n z_0 + E_n z_1 + E_n z_2\| + c_n^2\right) - c_n^{-1} \|\varrho_0 E_n z_0 - c_n^{-1} \|\varrho_1 E_n z_1 - c_n^{-1} \|\varrho_2 E_n z_2\|^2
\]
\[
+ \|\varrho_0 E_n z_0 + \varrho_1 E_n z_1 + \varrho_2 E_n z_2\|^2
\]
\[
\leq \left(\|E_n z_0\| + \|E_n z_1\| + \|E_n z_2\|\right) \leq \left(\|E_n z_0\| + \|E_n z_1\| + \|E_n z_2\|\right) \leq \left(\|E_n z_0\| + \|E_n z_1\| + \|E_n z_2\|\right).
\]
In light of (53) and (55), (56) becomes
\[
\|K_n P_n\|_{\mathcal{L}(\mathcal{H}, \mathcal{Z})} \leq \Gamma_2(\gamma, \nu) \lambda_n.
\]
By (14) and (46) we have
\[
\|e^{t J_n} P_n z\|_{\mathcal{H}}^2 = \left\|\begin{pmatrix} e^{t \sigma_0(n)} E_n & 0 & 0 \\ 0 & e^{t \sigma_1(n)} E_n & 0 \\ 0 & 0 & e^{t \sigma_2(n)} E_n \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right\|_{\mathcal{H}}^2
\]
\[
\leq \left\|e^{t \sigma_0(n)} E_n z_0\right\|^2 + \left\|e^{t \sigma_1(n)} E_n z_1\right\|^2 + \left\|e^{t \sigma_2(n)} E_n z_2\right\|^2
\]
\[
\leq \left[\|e^{t \sigma_0(n)}\|^2 + \|e^{t \sigma_1(n)}\|^2 + \|e^{t \sigma_2(n)}\|^2\right] \|z\|_{\mathcal{H}}^2
\]
\[
\leq e^{2 \mu_1 t} \|z\|_{\mathcal{H}}^2, \quad t \geq 0,
\]
where, from Lemma 2.6
\[
\mu_1 = \sup_{n \geq 1} \max_{i=0,1,2} \Re \sigma_i(n) = \max_{i=0,1,2} \Re \sigma_i(1) < 0.
\]
By (19) and (29), we get
\[
\mu_1 = - \frac{\lambda_1}{3(1 + \gamma \lambda_1)} \min_{i=0,1,2} \Re \left(\sqrt{1 + \gamma \lambda_1} + C e^{2 \lambda_1 t} + \Delta_0 + \Delta_1 e^{-2 \lambda_1 t}\right),
\]
where $\Delta_0$, $\Delta_1$ and $C$ are given by (50). According to Lemma 2.7, the minimum of the last expression is given by $i = 0$, and consequently (49) follows. Therefore (58) becomes
\[
\|e^{t J_n} P_n\|_{\mathcal{H}} \leq e^{\mu_1 t}.
\]
Finally, following [1] we get
\[
\sum_{n=1}^{\infty} \|P_n z\|^2 = \|z\|_{\mathcal{Z}}^2.
\]
Substituting (54), (57), (59) and (60) into (51) we get (48), where $N = \Gamma_1 \Gamma_2$ is a positive constant depending on $\gamma$ and $\nu$. \hfill \Box
Remark 3.7. Lemma 2.4 and Theorem 3.6 can be proved without the necessity of eigenvalues being simple, i.e., for any value of \( \nu \). In this case the supremum of \( n \mapsto \max_{i=0,1,2} \Re \sigma_i(n) \) will not necessarily be attained at \( n = 1 \).

By applying Rouche’s Theorem to (20), we can prove \( \Re \sigma_i(n) \to -\infty \) as \( n \to +\infty \) for \( i = 0,1,2 \). Since \( \sigma_i(n) \) is always negative and continuously depends on \( \lambda_n \) (roughly speaking), \( \sup_{n \in \mathbb{N}} \sigma_i(n) \) must be attained at some finite \( n \).

3.2. Analyticity. In this subsection, we consider \( \gamma = 0 \). We recall that \( S(\varphi) \subset S(\varphi + \frac{\pi}{2}) \subset \rho(A) \) (see Remark 3.2 (ii)). Now we give a useful form of the resolvent of \( A \) needed in the sequel.

Lemma 3.8. The resolvent of the operator \( A \) with \( \gamma = 0 \), satisfies

\[
R(\lambda, A)z = \sum_{n \geq 1} K_n^{-1}(\lambda I - J_n P_n)^{-1} K_n P_n z
\] (61)

where \( z \in Z_0 = X^3 = (L^2(\Omega))^3 \) and \( \lambda \in S(\varphi + \frac{\pi}{2}) \).

Proof. In view of (13), the resolvent of \( A \) becomes

\[
R(\lambda, A)z = (\lambda I - \sum_{n=1}^{\infty} A_n P_n z)^{-1}, \quad z \in Z_0, \quad \lambda \in S(\varphi + \frac{\pi}{2}).
\] (62)

On the other hand, direct calculations give

\[
\left( \lambda I - \sum_{k=1}^{n} A_k P_k \right) \left( \sum_{k=1}^{n} (\lambda I - A_k P_k)^{-1} P_k \right) = \sum_{k=1}^{n} P_k
\]

which readily yields

\[
\left( \lambda I - \sum_{n \geq 1} A_n P_n \right) \left( \sum_{n \geq 1} (\lambda I - A_n P_n)^{-1} P_n \right) = \sum_{n \geq 1} P_n = I.
\]

Similarly, we can obtain that

\[
\left( \sum_{n \geq 1} (\lambda I - A_n P_n)^{-1} P_n \right) \left( \lambda I - \sum_{n \geq 1} A_n P_n \right) = \sum_{n \geq 1} P_n = I.
\]

Combining the two last identities, we conclude that

\[
\left( \sum_{n \geq 1} (\lambda I - A_n P_n)^{-1} P_n \right) = \left( \lambda I - \sum_{n \geq 1} A_n P_n \right)^{-1}.
\] (63)

Taking into consideration Eqs. (45), (62) and (63), we get (61).

Now we are ready to state the main result of this subsection.

Theorem 3.9. Let \( 0 < \varphi_0 < \frac{\pi}{2} \) and \( \gamma = 0 \), \( (A, D(A)) \) is the generator of an analytic semigroup \( \{T(t)\}_{t \geq 0} \) in \( S(\varphi_0) \) on \( Z_0 \).

Proof. Thanks to Theorem 3.3, it suffices to prove the assertion (ii). To this end, we define the following two linear bounded operators

\[ K_n P_n : Z_0 \to Z_0 \] and \[ K_n^{-1} P_n : Z_0 \to Z_0. \]
Using (61) and \( P_n^2 = P_n \), we get
\[
\| R(\lambda, A)z \|^2 \leq \sum_{n \geq 1} \| K_n^{-1} P_n (\lambda I - J_n P_n)^{-1} K_n P_n \|_{\mathcal{L}(Z_0)}^2 \| P_n z \|^2_{Z_0}
\]
\[
\leq \sum_{n \geq 1} \| K_n^{-1} P_n \|_{\mathcal{L}(Z_0)}^2 \| (\lambda I - J_n P_n)^{-1} \|_{\mathcal{L}(Z_0)}^2 \| K_n P_n \|_{\mathcal{L}(Z_0)}^2 \| P_n z \|^2_{Z_0}.
\]
\[
(64)
\]
Using the same argument to get the estimations (54) and (57), one can prove that
\[
\| K_n^{-1} P_n \|_{\mathcal{L}(Z_0)} \leq \Gamma_1(\nu) \text{ and } \| K_n P_n \|_{\mathcal{L}(Z_0)} \leq \Gamma_2(\nu).
\]
Let \( \lambda \in S(\varphi + \frac{\pi}{2}) \), to find a bound for \( \| (\lambda I - J_n P_n)^{-1} \|_{\mathcal{L}(Z_0)} \), we consider \( w = (w_0, w_1, w_2) \in Z_0 \) and \( z = (z_0, z_1, z_2) \in Z_0 \) such that
\[
z = (\lambda I - J_n P_n) w,
\]
which immediately implies
\[
\| (\lambda I - J_n P_n)^{-1} z \|^2_{Z_0} = \| w \|^2_{Z_0}.
\]
Moreover, it is not difficult to check that
\[
z_i = \lambda w_i - \lambda \varrho_i E_n w_i \text{ (use (6))}
\]
\[
= \lambda \sum_{j \geq 0} E_j w_i - \lambda \varrho_i E_n w_i
\]
\[
= \lambda \sum_{j \neq n} E_j w_i + E_n w_i - \lambda \varrho_i E_n w_i
\]
\[
= \lambda \sum_{j \neq n} E_j w_i + (\lambda - \lambda \varrho_i) E_n w_i.
\]
(67)
which, written in norm, reads
\[
\| z_i \|^2 = |\lambda|^2 \sum_{j \neq n} \| E_j w_i \|^2 + |\lambda - \lambda \varrho_i|^2 \| E_n w_i \|^2 \text{ (use (6))}
\]
\[
= |\lambda|^2 \| w_i \|^2 + (|\lambda|^2 - |\lambda - \lambda \varrho_i|^2) \| E_n w_i \|^2
\]
\[
(68)
\]
Since \( \{ E_n \} \) is a complete family of orthogonal projections in \( Z_0 \) satisfying
\[
E_i E_j = \begin{cases} E_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}
\]
(69)
it follows from (67) and (69) that
\[
E_n z_i = (\lambda - \lambda \varrho_i) E_n w_i.
\]
(70)
Since \( S(\varphi + \frac{\pi}{2}) \subset \rho(A) \), by substituting (70) into (68) we arrive at
\[
\| z_i \|^2 = |\lambda|^2 \| w_i \|^2 + (1 - \frac{|\lambda|^2}{|\lambda - \lambda \varrho_i|^2}) \| E_n z_i \|^2
\]
which implies
\[
\| w_i \|^2 = \frac{1}{|\lambda|^2} \| z_i \|^2 + \frac{1}{|\lambda|^2} \left( \frac{|\lambda|^2}{|\lambda - \lambda \varrho_i|^2} - 1 \right) \| E_n z_i \|^2.
\]
Let $\lambda_0 > 0$, the function $\eta(\lambda) := |\lambda|^2 / |\lambda - \lambda_n g_i|^2$ is continuous and bounded on the set 
$\{ \lambda \in \mathbb{C} : |\lambda| \leq \lambda_0, |\text{arg}(\lambda)| < \varphi + \frac{\pi}{2} \}$ where $g_i$’s are given by (29). On the other 
hand, it is clear that $\eta$ is bounded for $|\lambda| > \lambda_0$ and bounded on $S(\varphi + \frac{\pi}{2})$. Setting

$$R = \sup_{i \in \{0, 1, 2\}} \left\{ \frac{|\lambda|}{|\lambda - \lambda_n g_i|} : \lambda \in S(\varphi + \frac{\pi}{2}), n \geq 1 \right\},$$

we deduce that $|\lambda| > R$, and it turns out from the continuity of $E_n : X \to X$ 
$(\|E_n z_i\| \leq \|z_i\|$ obtained from (6)) that

$$\|w_i\|^2 \leq \frac{R^2}{|\lambda|^2} \|z_i\|^2; \quad i = 0, 1, 2.$$

We infer from (66) that

$$\|(\lambda - J_n P_n)^{-1}\|_{\mathcal{L}(\mathcal{H}_0)} \leq \frac{R}{|\lambda|}, \quad \lambda \in S(\varphi + \frac{\pi}{2}), \quad (71)$$

Now, substituting (60), (65) and (71) into (64) we obtain

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S(\varphi + \frac{\pi}{2}),$$

where $M = \Gamma_1(\nu) \Gamma_2(\nu) R$ is a positive constant depending on $\nu$. This finishes the proof. \hfill \Box

4. **Controllability.** In this section we show the controllability of the problem (1)-(3) with $\gamma \geq 0$ where the controls $u = (u_1, u_2) \in L^2(0, \tau; L^2(\Omega))$ act on the whole domain or an open subset of the domain. In this case, Eq. (10) has a unique mild solution given by

$$z(t) = T(t) z_0 + \int_0^t T(t - s) B u(s) ds, \quad 0 \leq t \leq \tau \quad (72)$$

where $B$ is given by (11).

Now, we shall give the definitions of approximate and exact controllability in terms of system (10) (see [12, 13]).

**Definition 4.1.** (i) We say that system (10) is approximately controllable on $[0, \tau]$ 
if for all $z_0, z_1 \in Z_\gamma$ and $\epsilon > 0$, there exists a control $u \in L^2(0, \tau; L^2(\Omega))$ such that 
the solution $z(t)$ given by (72) satisfies

$$\|z(\tau) - z_1\|_{Z_\gamma} \leq \epsilon.$$

(ii) We say that system (10) is exactly controllable on $[0, \tau]$, if for all $z_0, z_1 \in Z_\gamma$ 
there exists a control $u \in L^2(0, \tau; L^2(\Omega))$ such that the solution $z(t)$ given by (72) satisfies

$$z(\tau) = z_1.$$

Following the standard approach (see, e.g., Curtain and Zwart [13]), we define the following concepts:

**a:** The controllability map $G : L^2(0, \tau; L^2(\Omega)) \to Z_\gamma$

$$Gu = \int_0^\tau T(\tau - s) B u(s) ds, \quad (73)$$

whose adjoint operator $G^* : Z_\gamma \to L^2(0, \tau; L^2(\Omega))$, is given by

$$(G^* z)(s) = B^* T^*(\tau - s) z, \quad \forall s \in [0, \tau], \quad z \in Z_\gamma, \quad (74)$$
(b): The Gramian mapping $W : Z_\gamma \to Z_\gamma$ is given by $W = GG^*$ that is to say
\[
W(\tau)z = (GG^*)z(\tau) = \int_0^\tau T(s)BB^*(s)zds
\]  
(75)
where $T(t)$ is the $C_0$-semigroup defined by (39).

**Remark 4.2.** For $\tau \geq 0$, the operator $W$ is nonnegative ($W \geq 0$) and, hence
\[
\mathcal{R}(\alpha, -W) = (\alpha I + W)^{-1}
\]  
(76
is well-defined bounded linear operator for all $\tau \geq 0$ and $\alpha > 0$. $\mathcal{R}(\alpha, -W)$ is called the resolvent of $-W$. If $W > 0$, then $\mathcal{R}(\alpha, -W)$ is defined for $\alpha = 0$ as well. Then, for all $z \in Z_\gamma$ and $\alpha \geq 0$,
\[
\langle z, (\alpha I + W)z \rangle \geq (\alpha + k)\|z\|^2
\]
where $k > 0$ is a constant. Therefore,
\[
\|\mathcal{R}(\alpha, -W)\| = \|(\alpha I + W)^{-1}\| \leq \frac{1}{\alpha + k} \leq \frac{1}{k}
\]
We obtain that $\|\mathcal{R}(\alpha, -W)\|$ is bounded with respect to $\alpha$. Furthermore
\[
\|\mathcal{R}(\alpha, -W) - W^{-1}\| = \|(\alpha I + W)^{-1} - W^{-1}\|
\]
\[
= \|W^{-1}(W - \alpha I - W)(\alpha I + W)^{-1}\|
\]
\[
\leq \alpha \|W^{-1}\|\|(\alpha I + W)^{-1}\| \leq \frac{\alpha}{k^2}
\]
Thus, the operator $\mathcal{R}(\alpha, -W)$ converges to the operator $W^{-1}$ as $\alpha \to 0$.

Moreover, the operator $\mathcal{B}$ is given by (11) and its adjoint is given by the following

**Proposition 4.3.** The operator $J_\gamma : X \to V_\gamma$ is a continuous (bounded) in $X$. Moreover, the operator $\mathcal{B}$ is bounded in $Z_\gamma$ and
\[
\mathcal{B}^* = \begin{cases} 
B^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I \\ 0 & 1_\omega & 0 \end{pmatrix} & \text{if } \omega \equiv \Omega \\
B^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix} & \text{if } \omega \subset \Omega,
\end{cases}
\]  
(77)

**Proof.** First we will show that $J_\gamma : X \to V_\gamma$ is bounded. From (9), we have
\[
\|J_\gamma z\|_{V_\gamma}^2 = \langle (I - \gamma \Delta)^{1/2}(J_\gamma z), (I - \gamma \Delta)^{1/2}(J_\gamma z) \rangle_X
\]
\[
= \langle (I - \gamma \Delta)^{-1/2}z, (I - \gamma \Delta)^{-1/2}z \rangle_X = \langle z, J_\gamma z \rangle_X.
\]
Since
\[
J_\gamma z = (I - \gamma \Delta)^{-1}z = \sum_{n=1}^{+\infty} \frac{1}{1 + \gamma \lambda_n} E_n z,
\]
we have
\[
\|J_\gamma z\|_{V_\gamma}^2 = \langle z, \sum_{n=1}^{+\infty} \frac{1}{1 + \gamma \lambda_n} E_n z \rangle_X \text{ (use (6))}
\]
\[
= \langle \sum_{n=1}^{+\infty} E_n z, \sum_{n=1}^{+\infty} \frac{1}{1 + \gamma \lambda_n} E_n z \rangle_X
\]
\[
= \sum_{n=1}^{+\infty} \frac{1}{1 + \gamma \lambda_n} \|E_n z\|^2 \leq \sum_{n=1}^{+\infty} \|E_n z\|^2 = \|z\|^2.
\]
Therefore $J_\gamma$ is bounded and consequently the operator $\mathcal{B}$ is bounded. Now, we shall give $\mathcal{B}^* : \gamma \to X$ for $\omega \equiv \Omega$,

$$\langle Bu, z \rangle_{Z_\gamma} = \left\langle \begin{pmatrix} 0 \\ J_\gamma u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right\rangle_{Z_\gamma}$$

$$= \langle 0, z_0 \rangle_X + \langle J_\gamma^{-1/2} J_\gamma u_1, J_\gamma^{-1/2} z_1 \rangle_X + \langle J_\omega u_2, z_2 \rangle_X$$

$$= \langle u_1, z_1 \rangle_X + \langle u_2, z_2 \rangle_X$$

$$= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right\rangle$$

$$= \langle u, B^* z \rangle_{Z_\gamma}.$$ 

Analogously we obtain $\mathcal{B}^*$ for $\omega \in \Omega$. This completes the proof. \hfill \Box

**Remark 4.4.** (i): Thanks to Proposition 4.3, we get

$$BB^* = \begin{cases} 
BB^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_\gamma & 0 \\ 0 & 0 & I \end{pmatrix} & \text{if } \omega \equiv \Omega \\
B_\omega B^*_\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_\omega J_\gamma & 0 \\ 0 & 0 & 1_\omega I \end{pmatrix} & \text{if } \omega \in \Omega,
\end{cases} \quad (78)$$

(ii): In the case $\omega \equiv \Omega$, we have

$$P_n BB^* = BB^* P_n, \quad n \geq 1, \quad (79)$$

whereas in the case $\omega \in \Omega$, the relation (79) does not hold,

$$P_n B_\omega B^*_\omega \neq B_\omega B^*_\omega P_n, \quad n \geq 1.$$

4.1. **Approximate and exact controllability when $\omega \equiv \Omega$.** In the following we show that the operator $W$ can be written in the form of series thanks to (79).

From condition (75) and the representation (39) of $T(t)$ we obtain

$$W(\tau) = \int_0^\tau \left( \sum_{n=1}^\infty e^{A_n t} P_n \right) BB^* \left( \sum_{k=1}^\infty e^{A_k t} P_k \right) ds \quad \text{(use (15))}$$

$$= \int_0^\tau \sum_{n=1}^\infty e^{A_n t} P_n BB^* e^{A_n^* t} P_n ds.$$

Now using (79), we obtain

$$W(\tau) = \int_0^\tau \sum_{n=1}^\infty e^{A_n t} BB^* P_n e^{A_n^* t} P_n ds$$

$$= \sum_{n=1}^\infty \int_0^\tau e^{A_n t} BB^* e^{A_n^* t} P_n ds.$$

Hence

$$W(\tau)z = \sum_{n=1}^\infty W_n(\tau) P_n z \quad (80)$$
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where \( W_n(\tau) : \mathcal{P}(P_n) \rightarrow \mathcal{P}(P_n) \) is defined by

\[
W_n(\tau) = \int_0^\tau e^{A_n s} B B^* e^{A_n^* s} ds = P_n B_n(\tau) P_n,
\]

while \( \mathcal{P}(P_n) = \text{Range}(P_n) \) and

\[
B_n(\tau) = \int_0^\tau e^{R_n s} B B^* e^{R_n^* s} ds.
\]

The following result is proved in [27].

**Theorem 4.5.** The system (10) is approximately controllable on \([0, \tau]\) if and only if the finite dimensional systems

\[
y' = A_n P_n y + P_n B u, \quad y \in \mathcal{P}(P_n); \ n \geq 1
\]

are controllable on \([0, \tau]\).

Our main tool is the following result proved in [26] (see Lemma 1) and in [27] (see Proposition 4.1).

**Proposition 4.6.** The following statements are equivalent:

- (a): Systems (83) are controllable on \([0, \tau]\) for any \( n \in \mathbb{N} \).
- (b): \( B^* P_n e^{A_n^* t} z = 0 \), for all \( t \in [0, \tau] \), implies \( z = 0 \).
- (c): The operator \( W_n \) given by (81) is invertible.

Proposition 4.6 holds for general operators \( B^* \), \( P_n \), \( W_n \) and \( A_n^* \) (see [21] for more details). We remark that the assertion (b) means the approximate controllability of (10).

Now we prove the approximate controllability of (1)-(3).

**Theorem 4.7.** When \( \omega \equiv \Omega \) and for any \( \tau > 0 \), the system (10) is approximately controllable on \([0, \tau]\).

**Proof.** According to Theorem 4.5 and Proposition 4.6, it suffices to show that the assertion (b) holds. In view of (16), we observe that \( A_n^* = R_n^* P_n \) where \( A_n^* \) and \( R_n^* \) are the adjoint operators of \( A_n \) and \( R_n \), respectively

\[
R_n^* = \begin{pmatrix}
0 & \lambda_n & 0 \\
-\frac{\nu \lambda_n}{1 + \gamma \lambda_n} & 0 & -\nu \lambda_n \\
\frac{\nu \lambda_n}{1 + \gamma \lambda_n} & \lambda_n & -\nu \lambda_n \\
\end{pmatrix}.
\]

Following the same argument of our previous work (see Lemma 3.4 of [1]), one can prove that there exists a family of complete complementary projections \( \{(q^n_i)^*\}_{i=0,1,2} \), such that

\[
e^{R_n^* t} = \sum_{i=0}^2 e^{t \sigma_i} (q^n_i)^* \]

with

\[
(q^n_i)^* = \begin{pmatrix}
\Theta_{i1} & -c_n \Theta_{i2} & \Theta_{i3} \\
-c_n \Theta_{i2} & \Theta_{i4} & -c_n^{-1} \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6} \\
\end{pmatrix}
\]

where

\[
\Theta_{i1} = \frac{1}{T} (\vartheta_i^2 + c_n \varrho_i + \nu^2), \quad \Theta_{i2} = \frac{1}{T} (c_n + \varrho_i), \quad \Theta_{i3} = -\nu, \\
\Theta_{i4} = \frac{1}{T} (\vartheta_i^2 + c_n \varrho_i), \quad \Theta_{i5} = \frac{\nu \varrho_i}{T}, \quad \Theta_{i6} = \frac{1}{T} (\vartheta_i^2 + 1).
\]
Lemma 4.9. Technical results. Open Mapping Theorem. To prove Lemma 4.10, we need to perform the following.

When Lemma 4.8.

Combining (84) and (85) we get

\[ e^{R_n t} = \sum_{i=0}^{2} e^{t \sigma_i} \begin{pmatrix} \Theta_{i1} & c_n \Theta_{i2} & c_n^{-1} \Theta_{i3} \\ -c_n^{-1} \Theta_{i2} & \Theta_{i4} & -c_n \Theta_{i5} \\ \Theta_{i3} & -c_n \Theta_{i5} & \Theta_{i6} \end{pmatrix}, \]

(87)

From (44), we infer that

\[ e^{A_n t z} = \sum_{i=0}^{2} e^{t \sigma_i} P^*_n z \]

\[ = \sum_{i=0}^{2} e^{t \sigma_i} \begin{pmatrix} \Theta_{i1} & -c_n \Theta_{i2} & \Theta_{i3} \\ c_n^{-1} \Theta_{i2} & \Theta_{i4} & -c_n \Theta_{i5} \\ \Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6} \end{pmatrix} \begin{pmatrix} E_n z_0 \\ E_n z_1 \\ E_n z_2 \end{pmatrix}. \]

(88)

Multiplying (88) by \( B^* P_n \) and using (69), we arrive at

\[ B^* P_n e^{A_n t z} = \sum_{i=0}^{2} e^{t \sigma_i} \begin{pmatrix} c_n^{-1} \Theta_{i2} & \Theta_{i4} & -c_n \Theta_{i5} \\ \Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6} \end{pmatrix} \begin{pmatrix} E_n z_0 \\ E_n z_1 \\ E_n z_2 \end{pmatrix}. \]

(89)

Now, suppose for \( z \in Z \), that \( B^* P_n e^{A_n t z} = 0 \), for all \( t \in [0, \tau] \). We need to show that \( z = 0 \). From (89), for \( i = 0, 1, 2 \) and \( n \geq 1 \), we obtain

\[ e^{t \sigma_i} \begin{pmatrix} c_n^{-1} \Theta_{i2} & \Theta_{i4} & -c_n \Theta_{i5} \\ \Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6} \end{pmatrix} \begin{pmatrix} E_n z_0 \\ E_n z_1 \\ E_n z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Solving this last equation, we easily obtain that \( E_n z_i = 0 \) which readily implies that \( z_i = 0 \).

The following result (see [9, 13]) proves to be helpful in showing the exact controllability of the problem (1)-(3) when \( \omega \equiv \Omega \).

Theorem 4.8. When \( \omega \equiv \Omega \), the system (10) is exactly controllable on \([0, \tau]\) if and only if the operator \( W \) given by (80) is continuous invertible.

The continuous invertibility of the operator \( W \) follows from Lemma 4.10 and the Open Mapping Theorem. To proof Lemma 4.10, we need to perform the following technical results.

Lemma 4.9. (i) The operator \( W_n \) is invertible and its inverse is given by

\[ W_n^{-1}(\tau) = \frac{1}{\Delta_n} \sum_{i=0}^{2} \begin{pmatrix} b_{i1} & c_n^{-1} b_{i2} & b_{i3} \\ -c_n b_{i2} & b_{i4} & c_n b_{i5} \\ b_{i3} & -c_n^{-1} b_{i5} & b_{i6} \end{pmatrix} P_n \]

(90)

where

\[ b_{i1} = a_{i5}^2 + a_{i4} a_{i6}, \quad b_{i2} = a_{i2} a_{i6} + a_{i5} a_{i3}, \]

\[ b_{i3} = a_{i2} a_{i5} - a_{i3} a_{i4}, \quad b_{i4} = a_{i6} a_{i1} - a_{i3}^2, \]

\[ b_{i5} = a_{i1} a_{i5} + a_{i2} a_{i3}, \quad b_{i6} = a_{i2}^2 + a_{i1} a_{i4}, \]

(91)
and given by (93) is nonzero, then
\[ a_{i1} = \Theta_{12}^2 J_\gamma + \Theta_{13}^2, \quad a_{i2} = \Theta_{14} \Theta_{12} J_\gamma + \Theta_{15} \Theta_{13}, \]
\[ a_{i3} = -\Theta_{13} \Theta_{12} J_\gamma + \Theta_{10} \Theta_{13}, \quad a_{i4} = \Theta_{14}^2 J_\gamma + \Theta_{15}^2, \]
\[ a_{i5} = (\Theta_{14} \Theta_{15} J_\gamma - \Theta_{15} \Theta_{16}), \quad a_{i6} = (\Theta_{15}^2 J_\gamma + \Theta_{16}^2) \]  \hspace{1cm} (92)

and
\[ \Delta_n = \sum_{i=0}^{2} e^{2z_i(n)} - \frac{1}{2\sigma_i(n)} \left( a_{i1}a_{i4}a_{i6} - a_{i1}a_{i5}^2 - a_{i6}a_{i2}^2 - a_{i4}a_{i3}^2 - 2a_{i2}a_{i3}a_{i5} \right), \quad n \geq 1. \]  \hspace{1cm} (93)

(ii) Moreover, there exist a positive constant \( C_1 \) depending on \( \nu \) and \( \gamma \) such that
\[ \| W_n^{-1}(\tau) \|_{L_2} \leq C_1(\gamma, \nu), \quad n \geq 1. \]  \hspace{1cm} (94)

Proof. (i) Thanks to the equivalence between assertions (a) and (c) of Proposition 4.6, we obtain that the operator \( W_n \) is invertible. By (81), \( B_n(\tau) \) is then invertible on \( R(P_n) \) and we have
\[ W_n^{-1}(\tau) = B_n^{-1}(\tau) P_n. \]  \hspace{1cm} (95)

We have used (15) and the fact that \( P_n \) is invertible on \( R(P_n) \) and its inverse is equal to itself.

First we give the expression of \( B_n^{-1}(\tau) \). We infer from (78) and (87) that
\[ e^{R_j s} B_n e^{R_j s} = \sum_{i=0}^{2} e^{2s_i}, \]
\[ \left( \begin{array}{cccc}
\Theta_{12}^2 J_\gamma + \Theta_{13}^2 & c_n (\Theta_{14} \Theta_{12} J_\gamma + \Theta_{15} \Theta_{13}) & -\Theta_{15} \Theta_{12} J_\gamma + \Theta_{16} \Theta_{13} \\
-c_n^{-1} (\Theta_{14} \Theta_{12} J_\gamma + \Theta_{15} \Theta_{13}) & \Theta_{14}^2 J_\gamma + \Theta_{15}^2 & -c_n^{-1} (\Theta_{14} \Theta_{15} J_\gamma + \Theta_{16} \Theta_{15}) \\
-\Theta_{15} \Theta_{12} J_\gamma + \Theta_{16} \Theta_{13} & -c_n (\Theta_{14} \Theta_{15} J_\gamma + \Theta_{16} \Theta_{15}) & \Theta_{15}^2 J_\gamma + \Theta_{16}^2
\end{array} \right). \]

By (82) we get
\[ B_n(\tau) = \sum_{i=0}^{2} \int_{0}^{\tau} e^{2s_i(n)} ds \left( \begin{array}{cccc}
a_{i1} & c_n a_{i2} & a_{i3} \\
c_n^{-1} a_{i2} & a_{i4} & -c_n^{-1} a_{i5} \\
a_{i3} & -c_n a_{i5} & a_{i6}
\end{array} \right), \quad n \geq 1 \]
\[ \int_{0}^{\tau} e^{2s_i(n)} ds = \frac{e^{2z_i(n)} - 1}{2\sigma_i(n)}, \]
and \( a_{ij} \) are given by (92). Since \( B_n(\tau) \) is invertible, the determinant \( \Delta_n \) of \( B_n(\tau) \) given by (93) is nonzero, then
\[ B_n^{-1}(\tau) = \frac{1}{\Delta_n} \sum_{i=0}^{2} \left( \begin{array}{ccc}
b_{i1} & c_n^{-1} b_{i2} & b_{i3} \\
c_n b_{i2} & b_{i4} & c_n b_{i5} \\
-b_{i3} & -c_n^{-1} b_{i5} & b_{i6}
\end{array} \right), \]
where \( b_{ij} \) are given by (91). From (95), (90) follows.

(ii) Secondly, we show that \( W_n^{-1} \) is bounded. Thanks to Proposition 4.3, we have \( J_\gamma \) is bounded. By (53), (86) becomes
\[ |\Theta_{14}| = |\Theta_{16}| = |\Theta_{16}| = C_i(\gamma, \nu) + O(1), \]
\[ |\Theta_{15}| = (C_i(\gamma, \nu) + O(1)) \frac{1}{\lambda_n}, \]
\[ |\Theta_{31}| = (C_i(\gamma, \nu) + O(1)) \frac{1}{\lambda_n}. \]  \hspace{1cm} (96)
Then (92) becomes
\[ |a_{i1}| = |a_{i4}| = |a_{i6}| = C_i(\gamma, \nu) + O(1), \]
\[ |a_{i2}| = |a_{i5}| = (C_i(\gamma, \nu) + O(1)) \frac{1}{\sqrt{\lambda_n}}, \]
\[ |a_{i3}| = (C_i(\gamma, \nu) + O(1)) \frac{1}{\lambda_n}. \]

Since \( b_{ij} \) are bounded for \( i = 0, 1, 2; \ j = 1, 2, \ldots, 6 \), then \( B_n^{-1} \) is bounded. Since \( P_n \) is also bounded, we infer from (95) that (94) holds.

Now we show that \( W \) is bounded.

**Lemma 4.10.** There exist a positive constant \( C_2 \) depending on \( \nu \) and \( \gamma \) such that
\[ \|W(\tau)\| \leq C_2(\nu, \gamma). \]  

**Proof.** From (80), it suffices to show that \( W_n \) is bounded. For this end, we shall show that \( e^{A_n t} \) is bounded (see (81)). By using (43), we have
\[ e^{A_n t} = \sum_{i=0}^{2} e^{i\sigma_i q_i^n} P_n. \]  

From (14) and (40), we have for \( i = 0, 1, 2 \)
\[ \|e^{i\sigma_i q_i^n} P_n z\|_Z^2 = e^{2i\sigma_i} \left\| \Theta_{i1} E_n z_0 + c_n \Theta_{i2} E_n z_1 + \Theta_{i3} E_n z_2 \right\|_X^2 \]
\[ + e^{2i\sigma_i} \left\| -c_n^{-1} \Theta_{i2} E_n z_0 + \Theta_{i4} E_n z_1 + c_n^{-1} \Theta_{i5} E_n z_2 \right\|_Y^2 \]
\[ + e^{2i\sigma_i} \left\| \Theta_{i3} E_n z_0 - c_n \Theta_{i5} E_n z_1 + \Theta_{i6} E_n z_2 \right\|_Y^2 \]
\[ = e^{2i\sigma_i} \sum_{j=1}^{\infty} \left\| E_j \left( \Theta_{i1} E_n z_0 + c_n \Theta_{i2} E_n z_1 + \Theta_{i3} E_n z_2 \right) \right\|_X^2 \]
\[ + e^{2i\sigma_i} \sum_{j=1}^{\infty} c_n^2 \left\| E_j \left( -c_n^{-1} \Theta_{i2} E_n z_0 + \Theta_{i4} E_n z_1 + c_n^{-1} \Theta_{i5} E_n z_2 \right) \right\|_X^2 \]
\[ + e^{2i\sigma_i} \sum_{j=1}^{\infty} \left\| E_j \left( \Theta_{i3} E_n z_0 - c_n \Theta_{i5} E_n z_1 + \Theta_{i6} E_n z_2 \right) \right\|_Y^2 \]
\[ \leq \left( \|\Theta_{i1}\| E_n z_0 + \|\Theta_{i2}\| c_n E_n z_1 + \|\Theta_{i3}\| E_n z_2 \right)^2 e^{2i\sigma_i}. \]  

Using (69) we arrive at
\[ \|e^{i\sigma_i q_i^n} P_n z\|_Z^2 = \left( \|\Theta_{i1}\| E_n z_0 + \|\Theta_{i2}\| c_n E_n z_1 + \|\Theta_{i3}\| E_n z_2 \right)^2 e^{2i\sigma_i}. \]
\[
\begin{align*}
&+ \left( |\Theta_{12}| \| E_n z_0 \| + |\Theta_{14}| c_n \| E_n z_1 \| + |\Theta_{15}| \| E_n z_2 \| \right)^2 e^{2t\sigma_i} \\
&+ \left( |\Theta_{13}| \| E_n z_0 \| + |\Theta_{15}| c_n \| E_n z_1 \| + |\Theta_{16}| \| E_n z_2 \| \right)^2 e^{2t\sigma_i}.
\end{align*}
\]

From (52), we infer that
\[
\| e^{t\sigma_i} q^n P_n z \|_{Z_\gamma}^2 \leq \left\{ (|\Theta_{11}| + |\Theta_{12}| + |\Theta_{13}|)^2 + (|\Theta_{12}| + |\Theta_{14}| + |\Theta_{15}|)^2 \\
+ (|\Theta_{13}| + |\Theta_{15}| + |\Theta_{16}|)^2 \right\} e^{2t\sigma_i}.
\]

In light of (5) and (96), we have that
\[
|\Theta_{11}| + |\Theta_{12}| + |\Theta_{13}| \leq C_i'(\nu, \gamma) + O(1), \\
|\Theta_{12}| + |\Theta_{14}| + |\Theta_{15}| \leq C''(\nu, \gamma) + O(1), \\
|\Theta_{13}| + |\Theta_{15}| + |\Theta_{16}| \leq C'''(\nu, \gamma) + O(1).
\]

Finally, (100) becomes
\[
\| e^{t\sigma_i} q^n P_n z \|_{Z_\gamma}^2 \leq M_i(\nu, \gamma) e^{2t\sigma_i}, \quad t \geq 0, \quad i = 0, 1, 2, \quad n \geq 1.
\]

Then, from (98) we have
\[
\| e^{\lambda_n^1 t} \|_{Z_\gamma} \leq M e^{\mu_1 t}, \quad t \geq 0, \quad n \geq 1.
\]

where \( \mu_1 \) is given by (49) and \( M = \max_i M_i(\nu, \gamma) \geq 0 \). Similarly, one can prove that
\[
\| e^{\lambda_n^2 t} \|_{Z_\gamma} \leq M' e^{\mu_2 t}, \quad t \geq 0, \quad n \geq 1.
\]

Thus, from (101) and (102), we get
\[
\| e^{\lambda_n^1 t} BB^* e^{\lambda_n^1 t} \| \leq MM' \| BB^* \| e^{2\mu_1 t}
\]

when \( BB^* \) is given by (78). Moreover, by Proposition 4.3 we have \( J_\gamma : X \to V_\gamma \) is bounded, so \( BB^* \) is bounded. Then from (81), we infer that
\[
\| W_n(\tau) \| \leq C(\nu, \gamma), \quad n \geq 1.
\]

Since \( \{ P_n \}_{n \geq 1} \) is a complete family of orthogonal projections in \( Z_\gamma \) and \( P_n W_n(\tau) = W_n(\tau) P_n \), so \( \{ W_n(\tau) P_n z \}_{n \geq 1} \) is a family of orthogonal vector in \( Z_\gamma \). Then, from (80) we have
\[
\| W(\tau) z \|^2 = \sum_{n=1}^{\infty} \| W_n(\tau) \|^2 \| P_n z \|^2 \\
\leq C(\nu, \gamma) \sum_{n=1}^{\infty} \| P_n z \|^2.
\]

Therefore from (60), (97) follows immediately.

\[\square\]

**Remark 4.11.** We remark that the operator \( W^{-1} \) can not be obtained explicitly from the expression \( W \) because the projection \( P_n \) as a mapping from \( Z_\gamma \) to \( Z_\gamma \) is not invertible. In fact only the restriction of \( P_n \) on \( \mathcal{A}(P_n) \) is invertible from \( Z_\gamma \) to \( Z_\gamma \).

Now we are ready to prove that \( W \) is invertible in \( Z_\gamma \).
Lemma 4.12. The operator $W$ is invertible in $Z_\gamma$ and its inverse $W^{-1}(\tau) : Z_\gamma \to Z_\gamma$ is defined by

$$W^{-1}(\tau) z = \sum_{n=1}^{\infty} \frac{1}{\Delta_n} \sum_{i=0}^{2} \begin{pmatrix} b_{11} & c_n^{-1} b_{i2} & b_{i3} \\ -c_n b_{i2} & b_{14} & c_n b_{i5} \\ b_{i3} & -c_n^{-1} b_{i5} & b_{i6} \end{pmatrix} P_n z,$$ \hspace{1cm} (103)

where $\Delta_n$ and $b_{ij}$ are given by (93) and (91), respectively.

Proof. Since $W_n^{-1}$ is bounded, we set $U(\tau) = \sum_{n=1}^{\infty} W_n^{-1}(\tau)P_n$. By (60) and (94) we infer that the operator $U$ is bounded. For all $z \in Z_\gamma$, from (80) we have

$$U(\tau)W(z) = \left(\sum_{k=1}^{\infty} W_k^{-1}(\tau)P_k\right)\left(\sum_{n=1}^{\infty} W_n(\tau)P_n\right)z$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} W_k^{-1}(\tau)P_k W_n(\tau)P_n z \quad \text{(use } W_n(\tau)P_n = P_n W_n(\tau))$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} W_k^{-1}(\tau)P_k P_n W_n(\tau)z \quad \text{(use (15))}$$

$$= \sum_{n=1}^{\infty} W_n^{-1}(\tau)P_n W_n(\tau)z$$

$$= \sum_{n=1}^{\infty} \underbrace{W_n(\tau)W_n^{-1}(\tau)}_{=I} P_n z$$

$$= \sum_{n=1}^{\infty} P_n z = z.$$

Similarly one can prove that $W(\tau)U(\tau)z = z$ for all $z \in Z_\gamma$. Then $W^{-1}(\tau) = U(\tau)$. By (90) our conclusion follows. \hspace{1cm} \Box

Now, we are ready to formulate the exact controllability of the problem (1)-(3) by a control $u = (u_1, u_2)$ acting on the whole domain $\Omega$ that will be determined explicitly by the physical coefficients of the plate.

Theorem 4.13. The problem (1)-(3) is exactly controllable on $[0, \tau]$ by the distributed controls

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \sum_{k=1}^{n} \sum_{i=0}^{2} \frac{e^{(\tau-t)s_i(k)}}{\Delta_k} \begin{pmatrix} d_{11}^k E_k z_0 + d_{12}^k E_k z_1 + d_{13}^k E_k z_2 \\ d_{14}^k E_k z_0 + d_{15}^k E_k z_1 + d_{16}^k E_k z_2 \end{pmatrix}, \quad t \in [0, \tau]$$

where

\begin{align*}
d_{11}^k &= c_k^{-1} \Theta_{i2} b_{i1} + c_k \Theta_{i4} b_{i2} - c_k^{-1} \Theta_{i5} b_{i3} \\
d_{12}^k &= c_k^{-2} \Theta_{i2} b_{i2} + \Theta_{i4} b_{i4} + c_k^{-2} \Theta_{i5} b_{i5} \\
d_{13}^k &= c_k^{-1} \Theta_{i2} b_{i3} + c_k \Theta_{i4} b_{i5} - c_k^{-1} \Theta_{i5} b_{i6} \\
d_{14}^k &= \Theta_{i3} b_{i1} - c_k^2 \Theta_{i5} b_{i2} + \Theta_{i6} b_{i3} \\
d_{15}^k &= c_k^{-1} \Theta_{i3} b_{i2} + c_k \Theta_{i5} b_{i4} - c_k^{-1} \Theta_{i6} b_{i5} \\
d_{16}^k &= \Theta_{i3} b_{i3} + c_k^2 \Theta_{i5} b_{i5} + \Theta_{i6} b_{i6}, \hspace{1cm} \text{(104)}
\end{align*}
Proof. By virtue of Lemma 4.12 and Theorem 4.8, the problem (1)-(3) is exactly controllable on \([0, \tau]\). Following \([9, 13]\), one can show that given \(z \in \mathcal{Z}\), there exists a control \(u \in L^2(0, \tau; L^2(\Omega))\) such that

\[
Gu = z,
\]

where \(G\) is the controllability map defined by (73). Hence, the control \(u\) can be expressed as

\[
 u(t) = G^*(G^*)^{-1}z,
\]

where \(G^*\) is defined by (74). Substituting (74) and (75) into (106), the control \(u\) becomes

\[
u(t) = B^*T^*(\tau - t)W^{-1}(\tau)z, \quad t \in [0, \tau].
\]

Multiplying the adjoint of (43) by \(B^*\), we arrive at

\[
B^*T^*(\tau - t)z = \sum_{n=1}^{\infty} \sum_{i=0}^{2} e^{(\tau-t)\sigma_i(n)}B^*(q_i^n)^*P_nz.
\]

Then, from (103) and (108), (107) becomes

\[
u(t) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{i=0}^{2} e^{(\tau-t)\sigma_i(k)} \Delta_{n-k} B^*(q_i^k)^* \left( \begin{array}{ccc}
 b_{i1} & c_{i1}^{-1} & b_{i3}
 -c_{i2}b_{i2} & b_{i4} & c_{i2}b_{i5}
 b_{i3} & -c_{i3}^{-1}b_{i5} & b_{i6}
 \end{array} \right) P_k P_{n-k} z.
\]
4.2. Interior approximate controllability when \( \omega \in \Omega \). The previous approach can not be applied here since the relation (79) does not hold when \( \omega \in \Omega \). Consequently the operator \( W \) can not be written in the form of series. However the following property of analytic functions proves to be helpful (see Theorem 1.23 from [7], pg. 20).

**Theorem 4.14.** Suppose \( \Omega \subset \mathbb{R}^d \) is an open, nonempty, and connected set. Let \( f \) be a real analytic function in \( \Omega \) with \( f = 0 \) on a non-empty open subset \( \omega \) of \( \Omega \). Then, \( f \equiv 0 \) in \( \Omega \).

Now we are ready to give the main result of this subsection.

**Theorem 4.15.** (i): For all nonempty open subsets \( \omega \) of \( \Omega \) and \( \tau > 0 \), the system (10) is approximately controllable on \([0, \tau]\).

(ii): Moreover, a sequence of controls steering the system (10) from initial state \( z_0 \) to an \( \varepsilon \)-neighborhood of the final state \( z_1 \) at time \( \tau > 0 \) is given by

\[
u_\alpha(t) = B_\ast^\tau T^\ast (\tau - t)R(\alpha, -W)(z_1 - T(\tau)z_0),
\]

where \( R(\alpha, W) \) is defined by (76). The error of the approximation of \( Gu(t) \)

\[
E_\alpha(t) = \alpha R(\alpha, -W)(z_1 - T(\tau)z_0)
\]

tends to zero as \( \alpha \) tends to zero.

**Proof.** (i): According to Theorem 4.5 and Proposition 4.6, it suffices to show that the assertion (b) holds. In view of (16), we observe that \( A_n^* = R_n^* P_n \) where \( A_n^* \) and \( R_n^* \) are the adjoint operators of \( A_n \) and \( R_n \), respectively

\[
R_n^* = \begin{pmatrix}
0 & \lambda_n & 0 & -\nu \lambda_n \\
-\lambda_n & 0 & -\nu \lambda_n & 1 + \gamma \lambda_n \\
0 & -\nu \lambda_n & 0 & 1 + \gamma \lambda_n \\
\nu \lambda_n & 1 + \gamma \lambda_n & 0 & 0
\end{pmatrix}.
\]

Following the same argument of our previous work (see Lemma 3.4 of [1]), one can prove that there exists a family of complete complementary projections \( \{q_i^n\} \}_{i=0,1,2} \) such that

\[
e^{R_n^* t} = \sum_{i=0}^2 e^{t \sigma_i (q_i^n)^*},
\]

with

\[
(q_i^n)^* = \begin{pmatrix}
\Theta_{i1} & -c_n \Theta_{i2} & \Theta_{i3} \\
c_n \Theta_{i2} & -c_n \Theta_{i3} & -c_n \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{pmatrix}.
\]

Combining (111) and (112), we get

\[
e^{R_n t} = \sum_{i=0}^2 e^{t \sigma_i} \begin{pmatrix}
\Theta_{i1} & -c_n \Theta_{i2} & \Theta_{i3} \\
c_n \Theta_{i2} & -c_n \Theta_{i3} & -c_n \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{pmatrix},
\]

\[
e^{R_n^* t} = \sum_{i=0}^2 e^{t \sigma_i} \begin{pmatrix}
\Theta_{i1} & -c_n \Theta_{i2} & \Theta_{i3} \\
c_n \Theta_{i2} & -c_n \Theta_{i3} & -c_n \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{pmatrix}.
\]
From (44), we infer that
\[ e^{A_n t} z = \sum_{i=0}^{2} e^{i \sigma_i} P_{ni} z \]
\[ = \sum_{i=0}^{2} e^{i \sigma_i} \left( \begin{array}{ccc}
\Theta_{i1} & \Theta_{i4} & \Theta_{i6} \\
-c_n \Theta_{i2} & \Theta_{i3} & -c_n^{-1} \Theta_{i5} \\
c_n \Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{array} \right) \left( \begin{array}{c}
E_n z_0 \\
E_n z_1 \\
E_n z_2
\end{array} \right). \]  
(113)

Multiplying (113) by \( B^* P_n \) and using (69), we arrive at
\[ B^* P_n e^{A_n t} z = \sum_{i=0}^{2} e^{i \sigma_i} \left( \begin{array}{ccc}
\Theta_{i1} & \Theta_{i4} & \Theta_{i6} \\
-c_n \Theta_{i2} & \Theta_{i3} & -c_n^{-1} \Theta_{i5} \\
c_n \Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{array} \right) \left( \begin{array}{c}
1 \omega E_n z_0 \\
1 \omega E_n z_1 \\
1 \omega E_n z_2
\end{array} \right). \]  
(114)

Now, suppose for \( z \in Z \) that \( B^* P_n e^{A_n t} z = 0 \), for all \( t \in [0, \tau] \). We need to show that \( z = 0 \). From (114), for \( i = 0, 1, 2 \) and \( n \geq 1 \), we obtain for all \( x \in \omega \)
\[ \left( \begin{array}{ccc}
-c_n^{-1} \Theta_{i2} & \Theta_{i4} & -c_n^{-1} \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{array} \right) \left( \begin{array}{c}
1 \omega E_n z_0 \\
1 \omega E_n z_1 \\
1 \omega E_n z_2
\end{array} \right) = \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right). \]  
(115)

On the other hand, from the assertion (b) of page 4, we know that \( \phi_n \) are analytic functions and therefore \( E_n z_i = (z_i, \phi_n) \phi_n \) is analytic. Thus, the left-hand side of (115) is analytic. Hence, by Theorem 4.14, Eq. (115) holds for all \( x \in \Omega \), i.e.,
\[ \left( \begin{array}{ccc}
\Theta_{i1} & \Theta_{i4} & c_n \Theta_{i5} \\
-c_n \Theta_{i2} & \Theta_{i3} & -c_n^{-1} \Theta_{i5} \\
\Theta_{i3} & c_n \Theta_{i5} & \Theta_{i6}
\end{array} \right) \begin{pmatrix} E_n z_0(x) \\ E_n z_1(x) \\ E_n z_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall x \in \Omega. \]

Solving this last equation, we easily obtain \( E_n z_i(x) = 0 \) which readily implies \( z_i(x) = 0 \) for all \( x \in \Omega \).

(ii): Since the operator \( W \) can be written in the form of series, we propose an approximation of (107),
\[ u_\alpha(t) = B^* T^*(\tau - t) R(\alpha, -W) z \]
where \( W^{-1} \) is replaced by the resolvent of \( -W \) as \( R(\alpha, -W) \) converges to \( W^{-1} \) as \( \alpha \to 0 \) (see Remark 4.2) and \( z = z_1 - T(\tau) z_0 \). Using (74) we have
\[ u_\alpha(t) = G^* R(\alpha, -W) z, \quad \alpha > 0. \]  
(116)

By multiplying (116) of this expression by \( G \), we get
\[ Gu_\alpha(t) = GG^* R(\alpha, -W) z \]
\[ = (\alpha I + GG^* - \alpha I) R(\alpha, -W) z \quad \text{(use (75))} \]
\[ = (\alpha I + W - \alpha I) R(\alpha, -W) z \quad \text{(use (76))} \]
\[ = z - \alpha R(\alpha, -W) z \quad \text{(use 105)} \]
\[ = Gu(t) - E_\alpha(t) z, \]
where \( E_\alpha(t) z \) is the error of the approximation of \( Gu(t) \),
\[ E_\alpha(t) = \alpha R(\alpha, -W). \]

Since \( R(\alpha, -W) \) is bounded (see Remark 4.2), we have
\[ E_\alpha(t) \to 0 \quad \text{as} \quad \alpha \to 0. \]  
(117)
This completes the proof of the theorem.

5. **Conclusion.** In the case $\omega \equiv \Omega$, the control $u$ is determined explicitly by the physical parameters by writing the expressions of $W$ and $W^{-1}$ in the form of series (see (109) where (103) is used) thanks to the relation (79). When $\omega \not\equiv \Omega$, the relation (79) is no longer possible and consequently the control $u$ can not be expressed explicitly. As $R(\alpha, -W)$ converges to $W^{-1}$ as $\alpha \to 0$, we propose an approximation of the control functions by replacing $W^{-1}$ in (107) by $R(\alpha, -W)$. Then we prove that the error of this approximation given by (117) tends to zero as $\alpha \to 0$.

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