ABSTRACT

Observations suggest that relativistic particles play a fundamental role in the dynamics of jets emerging from active galactic nuclei as well as in their interaction with the intracluster medium. However, no general consensus exists concerning the acceleration mechanism of those high-energy particles. A gravitational acceleration mechanism is proposed here in which particles leaving precise regions within the ergosphere of a rotating supermassive black hole (BH) produce a highly collimated flow. These particles follow unbound geodesics which are asymptotically parallel to the spin axis of the BH and are characterized by the energy $E$, the Carter constant $Q$, and zero angular momentum of the component $L_z$. If environmental effects are neglected, the present model predicts the presence of electrons with energies around $9.4$ GeV at distances of about $140$ kpc from the ergosphere. The present mechanism can also accelerate protons up to the highest energies observed in cosmic rays by the present experiments.

Key words: black hole physics – galaxies: jets – gravitation

Online-only material: color figure

1. INTRODUCTION

Multiwavelength observations of different astrophysical objects indicate the presence of “jets” (Marscher 2005) that are probably driven by a compact object like a black hole (BH) or, in some cases, a highly magnetized rotating neutron star. Jets are observed at scales ranging from sub-parsec up to hundreds of kiloparsecs. Gamma-ray bursts are an example of small-scale jets, since they are supposed to be the consequence of shocks occurring in highly collimated relativistic flows (Mészáros et al. 1999; Frail et al. 2001; Rossi et al. 2002) originating either at the death of a massive star or when a neutron star merges with another neutron star or with a BH. Large-scale jets are, in general, observed in association with radio galaxies, quasars, blazars, or active galactic nuclei (AGNs) and their origin is probably the consequence of the twisting of magnetic fields anchored in the very inner region of an accreting disk around a supermassive black hole (SMBH; Blandford & Levinson 1995; Meier et al. 2000; Camenzind 2005).

Jets associated with AGNs have a complex structure with bulk motions characterized by Lorentz factors typically in the range 10–50 and total power ranging from $10^{44}$ up to $10^{47}$ erg $^{-1}$. The composition of these flows is still uncertain but, in general, is supposed to be constituted by electrons and protons and/or electron–positron pairs or even heavy nuclei. The composition of the jet is certainly related to the physical processes that create and energize the flow, representing an important key for the understanding of the launching mechanism. The large-scale acceleration required to explain the high velocities of the bulk motion cannot be purely hydrodynamic and are probably a manifestation of the presence of extended magnetic pressure gradients (Vlahakis & König 2004). Fully relativistic simulations of accretion disks around a rotating BH indicate that unbound flows can emerge self-consistently from the accretion flow (De Villiers et al. 2005). According to these simulations, the flow has two main components: a hot, fast, and tenuous outflow along the jet axis and a cold, slow, and dense flow along the funnel wall defining the jet geometry (see also Meliani & Keppens 2009). For slow rotating BHs, the flow energetics are dominated by the kinetic energy and the enthalpy of matter, whereas for fast rotating BHs the energetics are essentially given by a Poynting flux. Jets dominated by kinetic energy penetrate easily into the intracluster medium (ICM), forming low density cavities elongated in the radial direction (Guo & Mathews 2011). This is not the case if the energetics of the jet are dominated by highly relativistic particles. In this case, due to the low inertia, the jet decelerates rapidly in the ICM, producing large cavities due to the lateral expansion produced by the pressure of the relativistic particles (Guo & Mathews 2011).

The result of these simulations emphasizes the importance of the presence of relativistic particles in the jet, either to characterize the dynamics of the flow or the interaction with the ICM. However, the acceleration mechanism (or mechanisms) of these relativistic particles is not yet well understood although magnetohydrodynamic shocks and Fermi-like mechanisms are often invoked as possibilities. Here, we examine a completely different alternative, i.e., a purely gravitational acceleration process based on the presence of an ergosphere around a Kerr BH. In our scenario, we assume that an accreting rotating BH is present in the center of an active galaxy. Matter penetrating the ergosphere can undergo the Penrose process (Penrose 1969) and, under certain conditions, the emerging particles follow geodesics asymptotically parallel to the rotation axis, acquiring very high energies. Although the efficiency of the Penrose process is still a matter of debate, this important question will be not examined in the present paper.

The existence of unbound geodesics leaving the ergosphere along the $z$-axis and focusing at infinity was already demonstrated by Gariel et al. (2010, hereafter GMMS10). In the present work, it is shown that test particles, independent of their electric charge, following these highly collimated geodesics, model a narrow energetic beam in precise regions of the ergosphere. The present investigation is addresses some particular solutions describing those geodesics as well as to the analysis of the initial
conditions (ICs) required for their existence. The paper is organized as follows. In Section 2, the constants of motion, the total particle energy $E$, and the Carter constant $Q$ are derived as functions of two real roots of the characteristic equation $R^2(r) = 0$, where the function $R$ governs the timelike geodesics in the Kerr’s metric. Section 3 is dedicated to the study of the particular case in which a double root of the equation $R^2(r) = 0$ exists. It is shown that high particle energy values are possible only for two narrow ranges of values of the considered double root; in Section 4, the two remaining roots of the characteristic equation are examined as well as the consequences for the allowed values of $E$ and for the asymptotes of the geodetic motion. In Section 5, an analysis of these various solutions is performed, permitting us to restrict one to the different possibilities. Finally, the main conclusions are given in Section 6.

2. THE CONSTANTS OF MOTION

First, it should be emphasized that our model is highly idealized since interactions with the ambient medium or with magnetic fields, which affect the motion and the energy budget of particles, are not included in the present approach and will be considered in a future paper. This investigation will be focused on the study of the motion of test particles following unbound geodesics along the $z$-axis, under the assumption that the central SMBH rotates steadily.

Assuming an axisymmetric geometry, the generalized cylindrical or Weyl generalized spherical coordinates $(\rho, z, \phi)$, related to Boyer–Lindquist coordinates $(r, \theta, \phi)$ by

$$\rho = [(r - 1)^2 - A]^{1/2} \sin \theta, \quad z = (r - 1) \cos \theta,$$

where

$$A = 1 - \left(\frac{a}{M}\right)^2,$$

are the most suitable for describing the system. As already mentioned, the existence of special unbound geodesics in this frame was recently demonstrated by GMMS10. These geodesics stem from the ergosphere and when $z \to \infty$, they are asymptotically parallel to the $z$-axis and positioned at a distance

$$\rho = \rho_1 = \left(\rho_0^2 + \frac{Q}{E^2 - 1}\right)^{1/2}$$

that depends on $\rho_0 \equiv a/M$ and on two constants of motion, the Carter constant $Q$ and the energy $E$, while the third constant of motion, the $z$ component of the angular momentum $L_z$, is necessarily null.

The function $R(r)$ (see, for instance, Chandrasekhar 1983) introduced in the expression of the Kerr timelike geodesics (test particle mass $\sqrt{\beta_1} = 1$) plays a fundamental role in the analysis of the jet collimation when the engine at the center of the accretion disk is supposed to be a stationary rotating SMBH. This function is a fourth-order polynomial, i.e.,

$$R^2(r) = a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0,$$

where the coefficients, except $a_3$, depend on the constants of motion and on the BH parameters (Chandrasekhar 1983) as

$$a_0 = -a^2 Q, \quad a_1 = 2(a^2 E^2 + Q),$$
$$a_2 = a^2 (E^2 - 1) - Q, \quad a_3 = 2, \quad a_4 = E^2 - 1.$$ 

Without loss of generality, we set $M = 1$ and $L_z = 0$ when considering the special two-dimensional geodesics given by Equation (3). Hence, with the spin $a$ of the SMBH fixed ($-1 \leq a \leq 1$), we have two independent parameters left, $Q$ and $E$, or, as from Equation (3), the position $\rho_1$ of the asymptote parallel to the $z$-axis and the energy $E$.

Let us consider the possible roots of the equation $R^2(r) = 0$ of the characteristics $i = 0$ of the autonomous system of geodesics equations (Chandrasekhar 1983), i.e.,

$$a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0. \quad (6)$$

The polynomial given by Equation (6) has four roots, labeled $r_i$, with $i = 1, 2, 3, 4$, which can be a priori real (positive or negative) or complex (contrary to the physical variable $r$, which is always real and defined in the interval $[1 + \sqrt{A}, \infty]$). The two equations $R^2(r_1) = 0$ and $R^2(r_2) = 0$ are linear in $Q$ and $(E^2 - 1)$. Thus, the solution of this linear system permits us to express the constants of motion as functions of the roots $r_1$ and $r_2$, namely,

$$Q = \frac{2r_1 r_2}{D} \left(a^4 + a^2 \left[r_1 (r_1 - 2) + r_2 (r_2 - 2)\right] + r_1^2 r_2^2\right) \quad (7)$$

and

$$(E^2 - 1) = -\frac{2}{D} \left(a^4 + a^2 \left[r_1^3 + r_1^2 r_2 + r_1 r_2 (r_2 - 2) - 4\right]\right) + r_1 r_2 \left[r_1^3 + r_1 r_2 (r_2 - 2) - 2r_2^2\right]. \quad (8)$$

In the third possible equation, $R^2(r_3) = 0$, the parameters $Q$ and $E^2 - 1$ can be replaced by Equations (7) and (8), leading to a relation between $r_3$, $r_1$, and $r_2$, allowing us, in principle, to determine the values of $r_3$ as a function of $r_1$ and $r_2$ only, since the spin parameter $a$ is fixed. The fourth possible equation, $R^2(r_4) = 0$, does not represent any new result since the roots $r_3$ and $r_4$ are the same.

In Equations (7) and (8), it is worth noting the symmetric role of $r_1$ and $r_2$, and that $Q$ and $E^2 - 1$ have the same denominator $D$. Thus, if and only if it cancels, then we have both $E \to \infty$ and $|Q| \to \infty$, whereas $\rho_1$, since it depends only on their ratio (see Equation (3)), tends toward a finite value. From Equations (3), (7), and (8), we obtain for the position of the asymptotes

$$\left(\frac{\rho_1}{\rho_0}\right)^2 = \frac{(a^2 + r_1^2)(a^2 - r_1 r_2)(a^2 + r_2^2)}{a^2 \left[a^4 + a^2 \left(r_1^3 + r_1^2 r_2 + r_1 r_2 (r_2 - 2) - 2r_2^2\right)\right]. \quad (10)$$

It should be emphasized that the roots $r_1$ and $r_2$, satisfying the condition $D = 0$ for a given value of the spin parameter $a$, permit us to define regions in the ergosphere from which unbound geodesics parallel to the $z$-axis emerge, along which move particles with “quasi”-infinite energies and which could eventually explain the very energetic particles observed in cosmic rays. We will return to this point later.

3. REAL ROOTS $r_1 = r_2$

In order to simplify the mathematical analysis, and without losing the physical insight, we assume in this paper the particular
and Equations (7) and (8) simplify as

These asymptotes can be evaluated numerically since they exist. As expected for unbound geodesics, they are clearly seen as well as the three values of $E$ for which $E^2 - 1$ tends (positively) to infinity.

case in which a double real root $r_1 = r_2 = Y$ exists. Under this assumption, Equation (4) can be recast as

$$R^2(r) = a_4(r - Y_1)^2(r - Y_2)^2$$

and Equations (7) and (8) simplify as

$$Q = \frac{[a^4 + 2a^2(Y - 2)Y + 4Y^2]^2}{a^4(1 + Y) + 2a^2(Y - 1)Y^2 + (Y - 3)Y^3}$$

and

$$E^2 - 1 = -\frac{a^4 + 2a^2Y^2 + (Y - 4)Y^3}{a^4(1 + Y) + 2a^2(Y - 1)Y^2 + (Y - 3)Y^3}.$$  

When $E \to \infty$, of course we also have $|Q| \to \infty$ but, as already mentioned, their ratio tends toward a finite value in order for the coordinate $\rho_1$ to remain finite, i.e.,

$$\left(\frac{\rho_1}{\rho_e}\right)^2 = \frac{(a^2 + Y_1)^2(a^2 + Y_2)^2}{a^4[a^2 + 2a^2Y^2 + (Y - 4)Y^3]}.$$  

In order to perform some numerical estimates, we will assume, unless otherwise stated, a “moderate” rotation for the SMBH, fixing $a = M/2$ (a value adopted already by GMMS10 in their investigation). Hence, the functions $E^2 - 1 = F(Y)$ and $(\rho_1/\rho_e)^2 = G(Y)$ can be plotted as shown in Figures 1 and 2.

Since $F(Y)$ and $G(Y)$ must be simultaneously positive, the only possible solutions correspond to the two intervals

$$Y \in [-0.5, Y_{0a}]$$

and

$$Y \in [Y_{0b}, 3.8697],$$

with $Y_{0a}$ and $Y_{0b}$ being asymptotes of $F(Y)$ for which $E \to \infty$. These asymptotes can be evaluated numerically since they correspond to the roots of the equation $D = 0$. For the assumed BH parameters, it results in $Y_{0a} \approx -0.2418$ and $Y_{0b} \approx 2.8832$.

Hence, there are only two possible values of $\rho_1$ for which $E \to \infty$, corresponding to the two intervals defined above. For these limits, $Y = Y_{0a} - \varepsilon$ and $Y = Y_{0b} + \varepsilon$, when $\varepsilon \to 0$, we obtain, respectively, for the coordinate $\rho_1$

$$\frac{\rho_1}{\rho_e} \approx 0.6932 \quad \text{and} \quad \frac{\rho_1}{\rho_e} \approx 10.2411.$$  

For the upper bound of the interval (16), i.e., for $Y = 3.8697$ where $\rho_1 \to \infty$, we have $E^2 - 1 = 0$ and, for the lower bound of the interval (15), i.e., for $Y = -0.5$ where $E^2 - 1 = 2$, we have $\rho_1 = 0$.

When $r_1 \neq r_2$, the problem is quite difficult and a more careful study is necessary. This is presently underway and will be reported in a future paper. Nevertheless, the following aspects may be anticipated. As we have seen, in the case of a double root, geodesics followed by particles of sufficiently high energy ($E > 3$) remain clustered around values of $\rho_1$ corresponding to $E \to \infty$. Moreover, in the general case, the condition $D(r_1, r_2) = 0$ defines the regions from which high-energy geodesics emerge. The latter condition is given by a third-order polynomial in $r_2$, which has a unique real root that can be explicitly stated as a function of $r_1$. This can be inserted into Equation (10), defining the position of the asymptote $\rho_1$ as a function only of $r_1$. That leads again to very restricted ranges of possible values of $\rho_1$.

For the sake of completeness and in order to investigate also the possible influence of the spin parameter in our analysis, we also considered the case of an extreme Kerr BH ($a/M = 1$). The results are qualitatively the same. The only physical solution leading to $E \to \infty$ corresponds to the double root $Y = -0.4142$ and to an asymptote whose position is $\rho_1 \approx 0.8284$.

4. ROOTS $r_3$ AND $r_4$

Comparing Equation (11) with Equation (4), rewritten in terms of the parameters $E^2 - 1 = F(Y)$ and $(\rho_1/\rho_e)^2 = G(Y)$, without the explicit form of these functions of $Y$ (see
Equations (13) and (14), yields the four relations,
\[ B - 2Y = \frac{2}{F}, \quad G + \frac{1}{F} = 2Y(BY - 2C), \]
\[ 1 - G = 16CY^2, \quad 2 - G = 4(C - 2YB + Y^2), \] (18)
which are linear in $1/F$, $G$, $B$, and $C$. After eliminating $1/F$ and $G$, we obtain from the equations above
\[ B = -\frac{2(4Y^2 - 1)[Y(4Y - 1) + Y - 1]}{(4Y^2 - 1)^2 - 16Y^2(4Y - 1)}, \] (19)
\[ C = \frac{(4Y^2 - 1)^2 + 16Y(Y - 1)}{4[(4Y^2 - 1)^2 - 16Y^2(4Y - 1)]}. \] (20)

Hence, Equation (11) can be recast as
\[ R^2 = a_3(r - Y)^2(r^2 - Sr + P) \]
\[ = a_4(r - Y)^2(r - r_1)(r - r_4), \] (21)
where $r_1$ and $r_4$ are the remaining roots, which are generally distinct, and we have introduced
\[ S \equiv r_3 + r_4 = -B, \quad P \equiv r_3r_4 = C, \] (22)
or
\[ r_3 = -\frac{1}{2}[B + (B^2 - 4C)^{1/2}], \] (23)
\[ r_4 = -\frac{1}{2}[B - (B^2 - 4C)^{1/2}], \] (24)
where $B(Y)$ and $C(Y)$ are given, respectively, by Equations (19) and (20).

The curves $r_1(Y)$ and $r_2(Y)$ are real only for some well-defined ranges of $Y$. In particular, in the interval (15) defining $Y$, $r_3$ and $r_4$ are not real. In order for the expression $r^2 + Br + C$ in Equation (11) to be real, where $B$ and $C$ are real, $r_3$ and $r_4$ have to be complex conjugated, i.e., $r_3 = z$, $r_1 + iC$, and $r_4 = \overline{z}$. Hence, the sign of the expression $r^2 + Br + C = (r + B_1)^2 + C_1^2$ is always positive, and $P = C = B_1^2 + C_1^2 \geq 0$ and $S = -B = -2B_1 \leq 0$.

In the interval (16), the two roots $r_1$ and $r_2$ are real, $P = C$ is negative (which means two roots of opposite signs), and $B = -T$ is positive. The most precise value we have numerically obtained for the left limit (where, in principle, $E \to \infty$) of the range (16) allows us to reach the value $E \simeq 1.1 \times 10^{32}$. Then, the corresponding real values of the roots are, respectively, $r_3 = -5.8988$ and $r_4 = 0.1324$.

Also, it is worth observing that $E$ is steeply decreasing either for a weak variation $\varepsilon$ of $Y$ from $Y_{0b}$ ($\varepsilon > 0$) or from $Y_{0a}$ ($\varepsilon < 0$), while $\rho_1$ is weakly increasing for this small interval of $Y$. For example, when $Y$ goes from $Y_{0b}$ to 2.922, the energy $E$ is steeply decreasing from $10^{30}$ to 3, while the position of the asymptote $\rho_1/\rho_e$ increases by a small amount from 10.24 to 10.68, which means that there is a large concentration of the most energetic part (the “spine”) of the beam immediately near the right-hand side of $\rho_1/\rho_e = 10.24$. At its left-hand side, there is no beam produced.

Likewise, for the interval (15), the energy $E$ is very steeply decreasing from “infinity” to six. Within our numerical precision, when $Y \to Y_{0a}$, the highest value obtained for the energy was $E_{\text{max}} \simeq 2 \times 10^{30}$. The corresponding asymptote $\rho_1/\rho_e$ inside the ergosphere decreases very slightly from 0.6932 to 0.6764. Here, the jet is still more concentrated just at the left of coordinate value 0.6932, while beyond its right side there is no beam at all.

As a result, the present model predicts a radial structure of the “jet” with a well-defined profile for the energy (or velocity) distribution of the particles.

5. UNBOUND GEODESICS FOR HIGH-ENERGY PARTICLES

Now let us consider the set of possible energetic geodesics framing a jet and their corresponding asymptotes, satisfying the conditions considered in the previous section. In this case, the choice of admissible ICs depends strongly on their positions relative to the characteristics of the system of geodesic equations (see Equations (2) and (3) in GMMS10). Indeed, each characteristic separates the plane into two regions and a given geodesic cannot cross the borders defined by those curves. In the Boyer–Lindquist coordinates, the characteristics are defined by the equations
\[ \dot{r} = 0, \quad \dot{\theta} = 0, \] (25)
each of which is equivalent to the equations,
\[ R(r) = 0, \quad T(\theta) = 0, \] (26)
respectively, where $R(r)$ is given by Equation (4) and $T(\theta)$ by
\[ T(\theta) = -(b_4 \mu^4 + b_2 \mu^2 + b_0)^{1/2}, \] (27)
with $\mu = \cos \theta$ and
\[ b_0 = \frac{Q}{M^2}, \quad b_2 = a_2, \quad b_1 = -\left(\frac{a}{M}\right)^2 (E^2 - 1). \] (28)

The solutions of Equation (26), when they exist, are the roots $r_j$ and the roots $\theta_i$, which define, respectively, circles and straight lines from the origin.

In Weyl coordinates $\rho$ and $z$, each characteristic equation (25) is equivalent to the equations (see Equations (17) and (18) in GMMS10)
\[ \dot{\rho} = \frac{R(\alpha^2 - A)\rho}{\alpha \rho \Delta}, \quad \dot{z} = -\frac{R\rho}{\Delta}, \] (29)
and
\[ \dot{\rho} = \frac{T\alpha^3 \rho}{(\alpha^2 - A)\Delta}, \quad \dot{z} = \frac{T \alpha z}{\Delta}, \] (30)
respectively, where
\[ \Delta = (\alpha + 1)\alpha^2 + \left(\frac{a}{M}\right)^2 \alpha^2. \] (31)

Each set of Equations (29) and (30) leads to
\[ \frac{dz}{d\rho} = -\frac{\alpha^2 \rho}{(\alpha^2 - A)z}, \] (32)
and
\[ \frac{dz}{d\rho} = \frac{(\alpha^2 - A)z}{\alpha^2 \rho}, \] (33)
respectively, defining the two families of characteristics for the geodesics of the Equation (25) type in which we are
interested, namely, ellipses (corresponding to \( \dot{r} = 0 \)) and hyperbolae (corresponding to \( \dot{\theta} = 0 \)). Let us note that the product of the derivatives of Equations (32) and (33), of these two characteristics, is \(-1\), indicating that they are orthogonal.

On the one hand, ellipses exist when there are solutions \( r = r_i = \text{constant of Equation (32)} \) for any \( r \), with \( r_i \geq 1 + \sqrt{A} \) or equivalently \( \alpha = \alpha_i = \text{constant} \) (because \( \alpha = r - 1 \)) with \( \alpha_i \geq \sqrt{A} \). Then, Equation (32) can be integrated yielding

\[
\left( \frac{z}{\alpha_i} \right)^2 + \frac{\rho^2}{\alpha_i^2 - A} = K_1, \tag{34}
\]

where \( K_1 \) is an integration constant. The comparison of Equation (34) with Equation (1), valid for any \( \theta \), implies \( K_1 = 1 \).

On the other hand, hyperbolae exist when there are solutions of \( \cos \theta \equiv \mu = \mu_i = \text{constant of Equation (33)} \) for any \( r \), with \( \mu_i^2 \leq 1 \). These are solutions of the equation \( T = 0 \), when \( L_z = 0 \), with

\[
T^2 = \left( \frac{a}{M} \right)^2 (E^2 - 1)\alpha^4 \left[ 1 - \left( \frac{z}{\alpha} \right)^2 \right] \Lambda, \tag{35}
\]

where we have defined \( \Lambda = \left[ \frac{Q}{a^2(E^2 - 1)} + (z/\alpha)^2 \right] \). There are two possible cases; namely, if \( \mu_i^2 = 1 \), then \( T = 0 \) for any \( Q \), or if \( \mu_i^2 = -Q/a^2(E^2 - 1) = 1 - (\rho_i/\rho_e)^2 \leq 1 \), which is positively defined only if \( Q \leq 0 \), or equivalently, if \( \rho_1 \leq \rho_e \).

Then, for \( \mu_i^2 = 1 \), we have \( z = r - 1 = \alpha \) and \( \rho = 0 \) for any \( r \) and Equation (33) reduces to \( \rightarrow \infty \), and the characteristics that are along the semi-axis \( z \geq \sqrt{A} \). For \( \mu_i^2 = -Q/a^2(E^2 - 1) \), we have from Equation (33)

\[
\frac{dz}{d\rho} = \frac{z^2 - A\mu_i^2}{z\rho}, \tag{36}
\]

which can be integrated leading to

\[
\rho = K_2 \left[ \left( \frac{z}{\mu_i} \right)^2 - A \right]^{1/2}, \tag{37}
\]

where \( K_2 \) is an integration constant. Comparing Equation (37) with Equation (1), valid for any \( r \), or equivalently for any \( \alpha \), yields \( K_2 + \mu_i^2 = 1 \).

Relation (37) represents a family of hyperbolae parameterized by

\[
\frac{\rho_i}{\rho_e} = (1 - \mu_i^2)^{1/2} \tag{38}
\]

yielding

\[
\frac{1}{A} \left[ 1 - \left( \frac{\rho_i}{\rho_e} \right)^2 \right]^{-1} \left( 1 - \frac{\rho_i}{\rho_e} \right)^{-2} \rho^2 = 1. \tag{39}
\]

An ellipse, the solution of Equation (34), when it exists (i.e., when \( r_i \in [1 + \sqrt{A}, \infty] \)), can intersect the ergosphere only if its semiminor axis \( b_i = (\alpha_i^2 - A)^{1/2} \) is smaller than \( \rho_e = a/M \), i.e., if \( r_i < 2 \).

As an example, let us consider again the special case of a double root \( Y \) studied previously.

1. The first admissible range is \( Y \in [Y_{0b}, 3.869] \) (see Figure 2). These roots, belonging to the domain of physical definition \( r \in [1 + \sqrt{A}, \infty] \), correspond to the existence of elliptic characteristics. The smallest ellipse has as its semiminor axis \( b_i = [(Y - 1)^2 - A]/[\left(1.882 - 0.75\right)1/2] = (2.788441/2) \) along \( \rho \), and as its semimajor axis \( a_i = \sqrt{A} = Y - 1 = 1.88 \) along \( z \), obtained for the smallest value \( Y_{0b} = 2.88 \), corresponding to \( \rho_i/\rho_e \approx 0.24 \). This ellipse contains the ergosphere, the limits of which are \( z_{\max} = \sqrt{A} = 0.866 \) and \( \rho_{\max} = 1/2 \). Hence, it is always impossible to have IC simultaneously inside the ergosphere and outside any ellipse. Thus, in this case, it is impossible to have unbound geodesics starting from the ergosphere.

2. For the second admissible range, we found that \( Y \in [-0.5, Y_{0b}] \) (see Figure 2). These roots do not belong to the domain of definition of the physical variable \( r \), which means that there is no corresponding elliptic characteristic. The only remaining possible limitation depends on the position of the hyperbolic characteristic equation (39). The hyperbola intersects the \( z \)-axis at the point with coordinates \( \rho = 0 \) and \( z_0 = \{A[1 - (\rho_i/\rho_e)^2]\}^{1/2} \) and tends asymptotically toward a straight line of equation \( \rho \approx z \tan \theta_1 \), with \( \sin \theta_1 = \rho_i/\rho_e \). The domain of the possible IC is located between the \( z \)-axis, the ergosphere, and the hyperbola. For example, for \( Y = -0.242 \), \( \rho_i/\rho_e = 0.693 \), \( \theta_1 = 21^\circ \), and \( z_0 = 0.624 \in \sqrt{A} = 0.866 \).

As we have seen, geodesics characterized by \( E \to \infty \) may exist. Indeed, trajectories defined by the differential Equation (21) in GMMS10 are well behaved when \( E \to \infty \) (but with \( \rho_i \) finite), as can be easily verified by factorizing \( E \) in the numerator and the denominator of the second member, leaving a finite quantity when \( E \to \infty \). The existence of such trajectories has been verified by the numerical solution of Equation (21) in GMMS10 for high-energy values and the corresponding values of \( \rho_i \) as described in Section 3. Of course, the value of the energy will always be finite and fixed by the particular Penrose process which takes place at the origin of these particular geodesics inside the ergosphere. The main point resulting from the present investigation is that if very energetic particles are produced as a consequence of such a process, then there are unbound geodesics that will be followed by those particles, which will permit their ejection in a collimated way along the spin axis of the BH.

In Figure 3 is plotted the geodesic which tends asymptotically toward the corresponding value \( \rho = \rho_i = 0.347 \), for which the test particle has a very high value (theoretically “infinite,” but for present calculations, we took values \( E = 10^6 \) and \( \rho_i/\rho_e = 0.693 \)). This plot corresponds to the IC \( \rho_i = 2.8 \times 10^{-6} \) and \( z_i = 0.8521 \), which correspond to a point just inside the ergosphere at its top near the \( z \)-axis, i.e., near the event horizon. For the other limit (\( Y = -0.5 \), \( \rho_i = 0 \), \( E = \sqrt{A} \), \( \theta_1 = 0 \), and \( z_0 = \sqrt{A} = 0.866 \).

In order to illustrate the considered geometry, we have plotted in Figure 4 the relevant curves and surfaces, such as the ergosphere, the critical hyperbola, a possible unbound geodesic, and its corresponding asymptote. Curves in black and with
Figure 3. Plot of the geodesic $\rho(z)$ for a Kerr BH characterized by $M = 1$ and a spin parameter $a/M = 0.5$. The constants of motion and initial conditions are those given in the text.

Figure 4. Ergosphere (labels Ia and Ib), the critical hyperbolae (labels IIa and IIb), an example of a unbound geodesics (labels IIIa and IIIb), and the respective asymptotes (labels IVa and IVb) are shown for black holes having a spin parameter $a/M = 0.5$ (black curves) or $a/M = 1$ (red curves). (A color version of this figure is available in the online journal.)

Figure 5. Lorentz factor as a function of the coordinate $z$ for the geodesic shown in Figure 3.
is essentially under the form of potential energy ($\Gamma \simeq 1$), while far from the ergosphere, the energy is essentially kinetic.

6. CONCLUSIONS

Taking the roots of the characteristic equation for unbound two-dimensional geodesics with $L_z = 0$ as free parameters, we have shown that the two remaining constants of motion, $E$ and $Q$, of a test particle following geodesics that asymptotically tend to a parallel line to the $z$-axis, can be expressed as a function of the aforementioned parameters. In the particular case of a double root and assuming a spin parameter to be equal to $a = M/2$, restricted domains of asymptotes corresponding to high particle energies are found. This means that the Kerr metric can generate powerful collimated jets of high-energy particles in some well-defined regions inside the ergosphere. Indeed, for this special case, only two possible ranges of $\rho_1$ are possible, namely, $\rho_1 \in [0.3382, 0.3466]$ and $\rho_1 \in [5.12, 5.34]$ for $E \in [\sqrt{6}, \infty]$ and $E \in [\infty, \sqrt{5}]$, respectively (see Figure 2).

It is worth mentioning that the present results are a strict consequence of the structure of the Kerr metric. The main approximation is the assumption of the existence of a double real root $Y$ for the characteristic equation. Although we have considered in some more detail the case of a moderately spinning BH ($a/M = 0.5$), the results are qualitatively the same for the case of an extreme Kerr BH. We hope that by relaxing the double root hypothesis, maybe it would be possible to obtain thicker beams of energetic particles. An investigation is currently in progress and preliminary results are encouraging.

Two positions of the asymptote corresponding to an “infinite” energy in our model are $\rho_1 \simeq 5 M$ and $\rho_1 \simeq 0.3466 M$. The latter is the only case compatible with the limitations imposed by the characteristics of the system of geodesic equations, according to our discussion in Section 5. The consequence of these mathematical constraints is that the resulting jet is very narrow. However, there is some observational evidence for the presence of radial flows in some jets (Giroletti et al. 2004) and, as already mentioned, they may have a two-component flow, i.e., a relativistic powerful inner jet and a slower, less powerful outer flow (De Villiers et al. 2005; Xie et al. 2012). The present model could be related to the inner flow which carries most of the power, being constituted mainly by relativistic particles.

Our results can also easily be extended to particles other than electrons, for example, protons or heavy nuclei. This does not modify the “geometry” that we obtained, i.e., the positions $\rho_1$ of the jets, but their energy only. For a proton ($c^2 \sqrt{\delta_1} \simeq 1$ GeV), the maximal energy we can here numerically calculate (which is theoretically as large as wanted) is about $E \simeq 5.6 \times 10^{25}$ eV = $5.6 \times 10^7$ EeV, which largely includes the highest energies of the current observed UHECR (Dermer et al. 2009; Hoover et al. 2010). Thus, our model could be relevant to explain not only a collimated flux of relativistic particles, but also the production of these very energetic particles that could be related to UHECR.

A detailed analysis of the Penrose process and of its efficiency is beyond the aim of this paper. Nevertheless, we would expect that the rate of emerging particles would be proportional to the accretion rate and to the ratio between the volume of the region in the ergosphere where unbound geodesics exist and the total volume of the ergosphere. A simple example of a Penrose event could be the ionization of the inner shells of an iron atom inside the ergosphere, with the nucleus being captured by the BH and the electron being ejected with a high energy. The accreted gas near the ergosphere has temperatures around $10^6$ K (Montesinos & de Freitas Pacheco 2011), high enough to ionize the $K$–$L$ shells of iron, providing a theoretical basis for such a possibility.

Recent results from the Pierre Auger Observatory (Roulet 2009) suggest a correlation between UHECR above $57$ EeV and nearby (<70 Mpc) AGNs. Since in the present model the collimation of the relativistic particles occurs along the spin axis, it would be interesting to investigate if the associated AGNs are of type I or II, since in the context of the “unified model” these classes differ only by the inclination of the jet axis with respect to the line of sight.

Let us recall briefly that since our model does not require magnetic fields, it can be applied to neutral particles such as, for instance, neutrinos. If they have a mass of $\sqrt{\delta_1} = 0.33$ eV (Steidl 2009), then according to our previous estimate, they could attain energies of the order of $E \simeq 2 \times 10^{-2}$ EeV.

Finally, different authors have recently discussed the possibility of producing high-energy particles by collisions near a rotating BH (Grib & Pavlov 2011), raising the interest for having further investigations on the Penrose process (see also Bañados et al. 2011). The present investigation can be seen as an additional contribution to this debate.

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