Tuning and plateaux for the entropy of $\alpha$-continued fractions

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Abstract
The entropy $h(T_\alpha)$ of $\alpha$-continued fraction transformations is known to be locally monotone outside a closed, totally disconnected set $\mathcal{E}$. We will exploit the explicit description of the fractal structure of $\mathcal{E}$ to investigate the self-similarities displayed by the graph of the function $\alpha \mapsto h(T_\alpha)$. Finally, we completely characterize the plateaux occurring in this graph, and classify the local monotonic behaviour.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is a well-known fact that the continued fraction expansion of a real number can be analyzed in terms of the dynamics of the interval map $G(x) := \left\{ \frac{1}{x} \right\}$, known as the Gauss map. A generalization of this map is given by the family of $\alpha$-continued fraction transformations $T_\alpha$, which will be the object of study of this paper. For each $\alpha \in [0, 1]$, the map $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined as $T_\alpha(0) = 0$ and, for $x \neq 0$,

$$T_\alpha(x) := \frac{1}{|x|} - c_{\alpha,x}$$

where $c_{\alpha,x} = \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$ is a positive integer. Each of these maps is associated with a different continued fraction expansion algorithm, and the family $T_\alpha$ interpolates between maps associated with well-known expansions: $T_1 = G$ is the usual Gauss map which generates regular continued fractions, while $T_{1/2}$ is associated with the continued fraction to the nearest integer, and $T_0$ generates the by-excess continued fraction expansion. For more about $\alpha$-continued fraction expansions, their metric properties and their relations with other
Figure 1. An illustration of theorem 1.2 is given in the picture: on the left, you see the whole parameter space $[0, 1]$, and the graph of $h$. The coloured strips correspond to three maximal intervals. On the right, $x$ ranges on the tuning window $W_{1/3} = [\frac{1 - \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2}]$ relative to $r = 1/3$. Maximal intervals on the left are mapped via $\tau_r$ to maximal intervals of the same colour on the right. As prescribed by theorem 1.2, the monotonicity of $h$ on corresponding intervals is reversed. Note that in the white strips (even if barely visible on the right) there are infinitely many maximal quadratic intervals.

continued fraction expansions we refer to [Na, Sc, IK]. This family has also been studied in relation to the Brjuno function [MMY, MCM].

Every $T_\alpha$ has infinitely many branches, and, for $\alpha > 0$, all branches are expansive and $T_\alpha$ admits an invariant probability measure absolutely continuous with respect to Lebesgue measure. Moreover, the invariant densities have bounded variation and each $T_\alpha$ has a finite metric entropy $h(\alpha)$ [BDV]. The metric entropy of the map $T_\alpha$ determines the speed of convergence of the corresponding expansion algorithm (known as $\alpha$-euclidean algorithm) ([BDV, section 1.4]), and the exponential growth rate of the partial quotients in the $\alpha$-expansion of typical values ([NN, proposition 1]).

Nakada [Na], who first investigated the properties of this family of continued fraction algorithms, gave an explicit formula for $h(\alpha)$ for $\frac{1}{2} < \alpha \leq 1$, from which it is evident that entropy displays a phase transition phenomenon when the parameter equals the golden mean $g := \frac{\sqrt{5} - 1}{2}$ (see also figure 1, left):

$$h(\alpha) = \begin{cases} \frac{\pi^2}{6 \log(1 + \alpha)} & \text{for } g < \alpha \leq 1 \\ \frac{\pi^2}{6 \log(1 + g)} & \text{for } \frac{1}{2} \leq \alpha < g. \end{cases}$$ (1)

Several authors have studied the behaviour of the metric entropy of $T_\alpha$ as a function of the parameter $\alpha$ ([Ca, LM, NN, KSS]); in particular, Luzzi and Marmi [LM] first produced numerical evidence that the entropy is continuous, although it displays many more (even if less evident) phase transition points and it is not monotone on the interval $[0, 1/2]$. The entropy was later proven to be (Hölder) continuous [Ti, KSS]. Moreover, Nakada and Natsui [NN] identified a dynamical condition that forces the entropy to be, at least locally, monotone: in fact, they noted that for some parameters $\alpha$, the orbits under $T_\alpha$ of $\alpha$ and $\alpha - 1$ collide after a number of steps, i.e. there exist $N, M$ such that

$$T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1)$$ (2)

and they proved that, whenever the matching condition (2) holds, $h(\alpha)$ is monotone on a neighbourhood of $\alpha$. They also showed that $h$ has mixed monotonic behaviour near the
Tuning and plateaux for the entropy of \( \alpha \)-continued fractions

1051

origin: namely, for every \( \delta > 0 \), in the interval \((0, \delta)\) there are intervals on which \( h(\alpha) \) is monotone, others on which \( h(\alpha) \) is increasing and others on which \( h(\alpha) \) is decreasing.

In [CT] it is proven that the set of parameters for which (2) holds actually has full measure in parameter space. Moreover, such a set is the union of countably many open intervals, called maximal quadratic intervals (see section 3). Each maximal quadratic interval \( I_p \) is labelled by a rational number \( r \) and can be thought of as a stability domain in parameter space: in fact, the number of steps \( M, N \) it takes for the orbits to collide is the same for each \( \alpha \in I_p \), and even the symbolic orbit of \( \alpha \) and \( \alpha - 1 \) up to the collision is fixed. For this reason, the complement of the union of all \( I_p \) is called the bifurcation set or exceptional set \( E \).

Numerical experiments [LM, CMPT] show the entropy function \( h(\alpha) \) displays self-similar features: the main goal of this paper is to prove such a self-similar structure by exploiting the self-similarity of the bifurcation set \( E \).

The way to study the self-similar structure was suggested to us by the unexpected isomorphism between \( E \) and the real slice of the boundary of the Mandelbrot set [BCIT]. In the family of quadratic polynomials, Douady and Hubbard [DH] described the small copies of the Mandelbrot set which appear inside the large Mandelbrot set as images of tuning operators: we define a similar family of operators using the dictionary of [BCIT]. (We refer the reader to the appendix for more about this correspondence, even though knowledge of the complex-dynamical picture is strictly speaking not necessary in the rest of the paper).

Our construction is the following: we associate, with each rational number \( r \) indexing a maximal interval, a tuning map \( \tau_r \) (see section 4.2) from the whole parameter space of \( \alpha \)-continued fraction transformations to a subset \( W_r \), called tuning window. Note that \( \tau_r \) also maps the bifurcation set \( E \) into itself. More precisely, let \( r \) be a rational number such that \( I_p \) is a maximal quadratic interval, with continued fraction expansion \( r = \left[0; a_1, a_2, \ldots, a_n\right] \) and \( n \) even. Then the tuning window associated with \( r \) is the interval \( W_r := [\omega, a_0] \), where \( \omega := [0; a_1, \ldots, a_n] \) and \( \omega := [0; a_1, \ldots, a_n - 1, 1, a_1, \ldots, a_0] \) if \( a_n \geq 2 \), and \( \omega := [0; 1, a_1, \ldots, a_n - 2, a_n - 1, a_1, \ldots, a_0] \) if \( a_n = 1 \). The tuning window \( W_r \) is called neutral if the alternate sum \( a_1 - a_2 + \ldots - a_n = 0 \). Let us define a plateau of a real-valued function as a maximal, connected open set where the function is constant.

**Theorem 1.1.** The function \( h \) is constant on every neutral tuning window \( W_r \), and every plateau of \( h \) is the interior of some neutral tuning window \( W_r \).

Even more precisely, we will characterize the set of rational numbers \( r \) such that the interior of \( W_r \) is a plateau (see theorem 6.16). In particular, for \( r = \frac{1}{2} \), the interior of \( W_{1/2} = [g^2; g] \) is a plateau, and by formula (1) we recover the following recent result [KSS]:

\[
    h(\alpha) = \frac{\pi^2}{6\log(1 + g)} \quad \forall \alpha \in [g^2, g],
\]

and \( h \) is not constant on \([t, g]\) for any \( t < g^2 \).

On non-neutral tuning windows, instead, entropy is non-constant and \( h \) reproduces, on a smaller scale, its behaviour on the whole parameter space \([0, 1]\).

**Theorem 1.2.** If \( h \) is increasing on a maximal interval \( I_p \), then the monotonicity of \( h \) on the tuning window \( W_p \) reproduces the behaviour on the interval \([0, 1]\), but with reversed sign: more precisely, if \( I_p \) is another maximal interval, then

1. \( h \) is increasing on \( I_{\tau_r(p)} \) iff it is decreasing on \( I_p \);
2. \( h \) is decreasing on \( I_{\tau_r(p)} \) iff it is increasing on \( I_p \);
3. \( h \) is constant on \( I_{\tau_r(p)} \) iff it is constant on \( I_p \).

If, instead, \( h \) is decreasing on \( I_p \), then the monotonicity of \( I_p \) and \( I_{\tau_r(p)} \) is the same.
As a consequence, we can also completely classify the local monotonic behaviour of the entropy function \( \alpha \mapsto h(\alpha) \).

**Theorem 1.3.** Let \( \alpha \in (0, 1) \) be a parameter in the parameter space of \( \alpha \)-continued fractions, and \( E \) the bifurcation set (see equation (6) in section 3 for the definition). Then

1. if \( \alpha \notin E \), then \( h \) is monotone on a neighbourhood of \( \alpha \);
2. if \( \alpha \in E \), then either
   (i) \( \alpha \) is a phase transition: \( h \) is constant on the left of \( \alpha \) and strictly monotone (increasing or decreasing) on the right of \( \alpha \);
   (ii) \( \alpha \) lies in the interior of a neutral tuning window: then \( h \) is constant on a neighbourhood of \( \alpha \);
   (iii) otherwise, \( h \) has mixed monotonic behaviour at \( \alpha \), i.e. in every neighbourhood of \( \alpha \) there are infinitely many intervals on which \( h \) is increasing, infinitely many on which it is decreasing and infinitely many on which it is constant.

Note that all cases occur for infinitely many parameters: more precisely, (1) occurs for a set of parameters of full Lebesgue measure; (2)(i) for a countable set of parameters; (2)(ii) for a set of parameters whose Hausdorff dimension is positive, but smaller than \( \frac{1}{2} \); (2)(iii) for a set of parameters of Hausdorff dimension 1. Note also that all phase transitions are of the form \( \alpha = \tau_r(g) \), i.e. they are tuned images of the phase transition at \( \alpha = g \) which is described by formula (1). The largest parameter for which (2)(iii) occurs is indeed \( \alpha = g^2 \), which is the left endpoint of the neutral tuning window \( W_{1/2} \). Moreover, there is an explicit algorithm to decide, whenever \( \alpha \) is a quadratic irrational, which of these cases occurs.

The structure of the paper is as follows. In section 2, we introduce basic notation and definitions about continued fractions, and in section 3 we recall the construction and results from [CT] which are relevant in this paper. We then define the tuning operators and establish their basic properties (section 4), and discuss the behaviour of tuning with respect to monotonicity of entropy, thus proving theorem 1.2 (section 5). In section 6 we discuss untuned and dominant parameters, and use them to prove the characterization of plateaux (theorem 1.1 and theorem 6.16). Section 7 is devoted to the proof of theorem 1.3.

### 2. Background and definitions

#### 2.1. Continued fractions

The continued fraction expansion of a number

\[
x = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}
\]

will be denoted by \( x = [0; a_1, a_2, \ldots] \), and the \( n \)th convergent of \( x \) will be denoted by \( \frac{p_n}{q_n} := [0; a_1, \ldots, a_n] \). Often we will also use the compact notation \( x = [0; S] \) where \( S = (a_1, a_2, \ldots) \) is the (finite or infinite) string of partial quotients of \( x \).

If \( S \) is a finite string, its length will be denoted by \( |S| \). A string \( A \) is a prefix of \( S \) if there exists a (possibly empty) string \( B \) such that \( S = AB \); \( A \) is a suffix of \( S \) if there exists a (possibly empty) string \( B \) such that \( S = BA \); \( A \) is a proper suffix of \( S \) if there exists a non-empty string \( B \) such that \( S = BA \).

**Definition 2.1.** Let \( S = (s_1, \ldots, s_n) \), \( T = (t_1, \ldots, t_n) \) be two strings of positive integers of equal length. We say that \( S < T \) if there exists 0 \( \leq k < n \) such that

\[
s_i = t_i \quad \forall 1 \leq i \leq k \quad \text{and} \quad \begin{cases} s_{k+1} < t_{k+1} & \text{if } k \text{ is odd} \\ s_{k+1} > t_{k+1} & \text{if } k \text{ is even} \end{cases}
\]
This is a total order on the set of strings of given length, and it is defined so that \( S < T \) iff \([0; S] < [0; T]\). As an example, \((2, 1) < (1, 1) < (1, 2)\). Moreover, this order can be extended to a partial order on the set of all finite strings of positive integers in the following way:

**Definition 2.2.** If \( S = (s_1, \ldots, s_n) \) and \( T = (t_1, \ldots, t_m) \) are strings of finite (not necessarily equal) length, then we define \( S \ll T \) if there exists \( 0 \leq k < \min[n, m] \) such that

\[
s_i = t_i \quad \forall 1 \leq i \leq k \quad \text{and} \quad \begin{cases} s_{k+1} < t_{k+1} & \text{if } k \text{ is odd} \\ s_{k+1} > t_{k+1} & \text{if } k \text{ is even} \end{cases}
\]

As an example, \((2, 1) \ll (1)\), and \((2, 1, 2) \ll (2, 2)\). This order has the following properties:

1. If \(|S| = |T|\), then \( S < T \) if and only if \( S \ll T \);
2. If \( X, Y \) are infinite strings and \( S \ll T \), then \([0; SX] < [0; TY]\);
3. If \( A \leq B \) and \( B \ll C \), then \( A \ll C \).

### 2.2. Fractal sets defined by continued fractions

We can define an action of the semigroup of finite strings (with the operation of concatenation) on the unit interval. In fact, for each \( S \), we denote by \( S \cdot x \) the number obtained by appending the string \( S \) at the beginning of the continued fraction expansion of \( x \); by convention the empty string corresponds to the identity.

We shall also use the notation \( f_S(x) := S \cdot x \); let us point out that the Gauss map \( G(x) := \frac{1}{x} - \left[ \frac{1}{x} \right] \) acts as a shift on continued fraction expansions, hence \( f_S \) is a right inverse of \( G(S) \) (\( G(S) \circ f_S(x) = x \)). It is easy to check that concatenation of strings corresponds to composition \((ST) \cdot x = S \cdot (T \cdot x)\); moreover, the map \( f_S \) is increasing if \(|S| \) is even, decreasing if it is odd. Proceeding by induction one can prove that \( f_S \) is given by the formula

\[
f_S(x) = \frac{p_{n-1} x + p_n}{q_{n-1} x + q_n},
\]

where \( \frac{p_n}{q_n} = [0; a_1, \ldots, a_n] \) and \( \frac{p_{n-1}}{q_{n-1}} = [0; a_1, \ldots, a_{n-1}] \). The map \( f_S \) is a contraction of the unit interval: in fact, by taking the derivative in the previous formula and using the relation \( q_n p_{n-1} - p_n q_{n-1} = (-1)^n \) (see \([IK]\)), \( f_S'(x) = \frac{1}{q(S)^2} \), hence

\[
\frac{1}{4q(S)^2} \leq |f_S'(x)| \leq \frac{1}{q(S)^2} \quad \forall x \in [0, 1],
\]

where \( q(S) \) is the denominator of the rational number whose c.f. expansion is \( S \).

A common way of defining Cantor sets via continued fraction expansions is the following.

**Definition 2.3.** Given a finite set \( A \) of finite strings of positive integers, the regular Cantor set defined by \( A \) is the set

\[
K(A) := \{ x = [0; W_1, W_2, \ldots] : W_i \in A \ \forall i \geq 1 \}
\]

For instance, the case when the alphabet \( A \) consists of strings with a single digit gives rise to sets of continued fractions with restricted digits \([He]\).

An important geometric invariant associated with a fractal subset \( K \) of the real line is its Hausdorff dimension \( \text{H.dim } K \). In particular, a regular Cantor set is generated by an iterated function system, and its dimension can be estimated in a standard way (for basic properties about Hausdorff dimension we refer to Falconer’s book \([Fa]\), in particular chapter 9).
In fact, if the alphabet \( A = \{ S_1, \ldots, S_k \} \) is not redundant (in the sense that no \( S_i \) is prefix of any \( S_j \) with \( i \neq j \)), the dimension of \( K(\mathcal{A}) \) is bounded in terms of the smallest and largest contraction factors of the maps \( f'_{w} \) ([Fa, proposition 9.6]):

\[
\frac{\log N}{\log m_1} \leq H.\dim K(\mathcal{A}) \leq \frac{\log N}{\log m_2},
\]

where \( m_1 := \inf_{x \in [0,1]} |f'_{w}(x)| \), \( m_2 := \sup_{x \in [0,1]} |f'_{w}(x)| \), and \( N \) is the cardinality of \( \mathcal{A} \).

3. Matching intervals

Let us now briefly recall the main construction of [CT], which will be essential in the following.

Each irrational number has a unique infinite continued fraction expansions, while every rational number has exactly two finite expansions. In this way, one can associate with every rational \( r \in \mathbb{Q} \cap (0,1) \) two finite strings of positive integers: let \( S_0 \) be the string of even length, and \( S_1 \) be the one of odd length. For instance, since \( 3/10 = [0; 3, 3] = [0; 3, 2, 1] \), the two strings associated with \( 3/10 \) will be \( S_0 = (3, 3) \) and \( S_1 = (3, 2, 1) \). Let us remark that, if \( r = [0; S_0] = [0; S_1] \) then \( S_0 \ll S_1 \).

Now, for each \( r \in \mathbb{Q} \cap (0,1) \) we define the \textit{quadratic interval} associated with \( r \) as the open interval

\[ I_r := (\alpha_1, \alpha_0) \]

whose endpoints are the two quadratic irrationals \( \alpha_0 = [0; \overline{S_0}] \) and \( \alpha_1 = [0; \overline{S_1}] \). It is easy to check that \( r \) always belongs to \( I_r \), and it is the unique element of \( \mathbb{Q} \cap I_r \) which is a convergent of both endpoints of \( I_r \). In fact, \( r \) is the rational with minimal denominator in \( I_r \), and it will be called the \textit{pseudocentre} of \( I_r \). Let us define the \textit{bifurcation set} (or \textit{exceptional set}) in the terminology of [CT]) as

\[ \mathcal{E} := [0, g] \setminus \bigcup_{r \in (0,1) \cap \mathbb{Q}} I_r. \]

The intervals \( I_r \) will often overlap; however, by ([CT, proposition 2.4 and lemma 2.6]), the connected components of \( [0, g] \setminus \mathcal{E} \) are themselves quadratic intervals, called \textit{maximal quadratic intervals}. That is to say, every quadratic interval is contained in a unique maximal quadratic interval, and two distinct maximal quadratic intervals do not intersect. This way, the set of pseudocentres of maximal quadratic intervals is a canonically defined subset of \( \mathbb{Q} \cap (0,1) \) and will be denoted by

\[ \mathbb{Q}_E := \{ r \in (0,1) : I_r \text{ is maximal} \}. \]

We shall sometimes refer to \( \mathbb{Q}_E \) as the set of \textit{extremal rational values}; this is motivated by the following characterization of \( \mathbb{Q}_E \):

**Proposition 3.1** ([CT, proposition 4.5]). A rational number \( r = [0; S] \) belongs to \( \mathbb{Q}_E \) if and only if, for any splitting \( S = AB \) of \( S \) into two strings \( A, B \) of positive length, either

\[ AB < BA \]

or \( A = B \) with \( |A| \) odd.

Using this criterion, for instance, one can check that \( [0; 3, 2] \) belongs to \( \mathbb{Q}_E \) (because \( (3, 2) < (2, 3) \)), and so does \( [0; 3, 3] \), while \( [0; 2, 2, 1, 1] \) does not (indeed, \( (2, 1, 1, 2) < (2, 2, 1, 1) \)). Related to the criterion is the following characterization of \( \mathcal{E} \) in terms of orbits of the Gauss map \( G \):
Proposition 3.2 ([BCIT, lemma 3.3]).

\[ \mathcal{E} = \{ x \in [0, 1] : G_k(x) \geq x \ \forall k \in \mathbb{N} \} \]

For \( t \in (0, 1) \) fixed, let us also define the closed set

\[ \mathcal{B}(t) := \{ x \in [0, 1] : G_k(x) \geq t \ \forall k \in \mathbb{N} \} \]

To get a rough idea of the meaning of the sets \( \mathcal{B}(t) \) let us mention that for \( t = 1/(N+1) \) one obtains the set of values whose continued fraction expansion is infinite and contains only the digits \( \{1, \ldots, N\} \) as partial quotients. A simple relation follows from the definitions:

Remark 3.3. For each \( t \in [0, 1], \mathcal{E} \cap [t, 1] \subseteq \mathcal{B}(t) \).

A thorough study of the sets \( \mathcal{B}(t) \) and their interesting connection with \( \mathcal{E} \) is contained in [CT2].

Note that, from remark 3.3 and ergodicity of the Gauss map, it follows that the Lebesgue measure of \( \mathcal{E} \) is zero.

### 3.1. Maximal intervals and matching

Let us now relate the previous construction to the dynamics of \( \alpha \)-continued fractions. The main result of [CT] is that for all parameters \( \alpha \) belonging to a maximal quadratic interval \( I_r \), the orbits of \( \alpha \) and \( \alpha - 1 \) under the \( \alpha \)-continued fraction transformation \( T_\alpha \) coincide after a finite number of steps, and this number of steps depends only on the usual continued fraction expansion of the pseudocentre \( r \):

Theorem 3.4 ([CT, theorem 3.1]). Let \( I_r \) be a maximal quadratic interval, and \( r = [0; a_1, \ldots, a_n] \) with \( n \) even. Let

\[ N = \sum_{i \text{ even}} a_i \quad M = \sum_{i \text{ odd}} a_i \]

Then for all \( \alpha \in I_r \),

\[ T_{\alpha}^{N+1}(\alpha) = T_{\alpha}^{M+1}(\alpha - 1). \]  

Equation (8) is called matching condition. Note that \( N \) and \( M \) are the same for all \( \alpha \) which belong to the open interval \( I_r \). In fact, even more is true, namely the symbolic orbits of \( \alpha \) and \( \alpha - 1 \) up to steps, respectively, \( N \) and \( M \) are constant over all the interval \( I_r \) ([CT, lemma 3.7]). Thus we can regard each maximal quadratic interval as a stability domain for the family of \( \alpha \)-continued fraction transformations, and the complement \( \mathcal{E} \) as the bifurcation locus.

One remarkable phenomenon, which was first discovered by Nakada and Natsui ([NN, theorem 2]), is that the matching condition locally determines the monotonic behaviour of \( h(\alpha) \):

Proposition 3.5 ([CT, proposition 3.8]). Let \( I_r \) be a maximal quadratic interval, and let \( N, M \) be as in theorem 3.4. Then

1. if \( N < M \), the entropy \( h(\alpha) \) is increasing for \( \alpha \in I_r \);
2. if \( N = M \) it is constant on \( I_r \);
3. if \( N > M \) it is decreasing on \( I_r \).

### 4. Tuning

Let us now define tuning operators acting on parameter space, inspired by the dictionary with complex dynamics (see the appendix). We will then see how such operators are responsible for the self-similar structure of the entropy.
4.1. Tuning windows

Let \( r \in \mathbb{Q}_E \) be the pseudocentre of the maximal interval \( I_r = (\alpha_1, \alpha_0) \); if \( r = [0; S_0] = [0; S_1] \) are the even and odd expansions of \( r \), then \( \alpha_i = [0; S_i^i] \) \((i = 0, 1)\). Let us also set \( \omega := [0; S_1, S_0] \) and define the tuning window generated by \( r \) as the interval

\[
W_r := [\omega, \alpha_0].
\]

The value \( \alpha_0 \) will be called the root of the tuning window. For instance, if \( r = \frac{1}{2} = [0; 2] = [0; 1, 1] \), then \( \omega = [0; 2, 1] = g^2 \) and the root \( \alpha_0 = [0; 1] = g \).

The following proposition describes in more detail the structure of the tuning windows: a value \( x \) belongs to \( B(\omega) \cap [\omega, \alpha_0] \) if and only if its continued fraction is an infinite concatenation of the strings \( S_0, S_1 \).

**Proposition 4.1.** Let \( r \in \mathbb{Q}_E \), and let \( W_r = [\omega, \alpha_0] \). Then

\[
B(\omega) \cap [\omega, \alpha_0] = K(\Sigma_1)
\]

where \( K(\Sigma) \) is the regular Cantor set on the alphabet \( \Sigma = \{S_0, S_1\} \).

For instance, if \( r = \frac{1}{2} \), then \( W_{1/2} = [g^2, g) \), and \( B(g^2) \cap [g^2, g] \) is the set of numbers whose continued fraction expansion is an infinite concatenation of the strings \( S_0 = (1, 1) \) and \( S_1 = (2) \).

4.2. Tuning operators

For each \( r \in \mathbb{Q}_E \) we can define the tuning map \( \tau_r : [0, 1] \to [0, r] \) as \( \tau_r(0) = \omega \) and

\[
\tau_r([0; a_1, a_2, \ldots]) = [0; S_1^{a_1-1} S_0 S_1^{a_2-1} \ldots]
\]

(9)

Note that this map is well defined even on rational values (where the continued fraction representation is not unique); for instance, \( \tau_{1/3}([0; 3, 1]) = [0; 3, 2, 1, 2, 1, 3] = [0; 3, 2, 1, 2, 1, 2, 1, 2, 1] = \tau_{1/3}([0; 4]). \)

It will be sometimes useful to consider the action that \( \tau_r \) induces on finite strings of positive integers: with a slight abuse of notation we shall denote this action by the same symbol \( \tau_r \).

**Lemma 4.2.** For each \( r \in \mathbb{Q}_E \), the map \( \tau_r \) is strictly increasing (hence injective). Moreover, \( \tau_r \) is continuous at all irrational points, and discontinuous at every positive rational number.

The first key feature of tuning operators is that they map the bifurcation set into a small copy of itself:

**Proposition 4.3.** Let \( r \in \mathbb{Q}_E \). Then

(i) \( \tau_r(\mathcal{E}) = \mathcal{E} \cap W_r \), and \( \tau_r \) is a homeomorphism of \( \mathcal{E} \) onto \( \mathcal{E} \cap W_r \);

(ii) \( \tau_r(\mathbb{Q}_E) = \mathbb{Q}_E \cap W_r \setminus \{r\} \).

Let us, moreover, note that tuning windows are nested:

**Lemma 4.4.** Let \( r, s \in \mathbb{Q}_E \). Then the following are equivalent:

(i) \( W_r \cap W_s \neq \emptyset \) with \( r < s \);

(ii) \( r = \tau_s(p) \) for some \( p \in \mathbb{Q}_E \);

(iii) \( W_r \subseteq W_s \).
4.3. Proofs

Proof of lemma 4.2. Let us first prove that $\tau_r$ preserves the order between irrational numbers. Pick $\alpha, \beta \in (0, 1) \setminus \mathbb{Q}, \alpha \neq \beta$. Then

$$\alpha := [0; P, a, a_2, a_3, \ldots], \quad \beta := [0; P, b, b_2, b_3, \ldots]$$

where $P$ is a finite string of positive integers (common prefix), and we may assume also that $a < b$. Then

$$\tau_r(\alpha) := [0; \tau_r(P), S_1, S_0^{a-1}, S_1, \ldots], \quad \tau_r(\beta) := [0; \tau_r(P), S_1, S_0^{b-1}, S_1, \ldots].$$

Since $|S_0^{a-1}|$ is even and $S_1 \ll S_0$, we obtain $S_0^{a-1}S_1 \ll S_0^{b-1}S_1$, whence $S_1S_0^{a-1}S_1 \ll S_1S_0^{b-1}S_1$. Therefore, since $|P| \equiv |\tau_r(P)| \mod 2$, we obtain that either $|P|$ is even, $\alpha > \beta$ and $\tau_r(\alpha) > \tau_r(\beta)$, or $|P|$ is odd, $\alpha < \beta$ and $\tau_r(\alpha) < \tau_r(\beta)$, so we are done. The continuity of $\tau_r$ at irrational points follows from the fact that if $\beta \in (0, 1) \setminus \mathbb{Q}$ and $x$ is close to $\beta$ then the continued fraction expansions of $x$ and $\beta$ have a long common prefix, and, by definition of $\tau_r$, their images will also have a long prefix in common, and will therefore be close to each other. Finally, let us check that the function is increasing at each rational number $c > 0$. This follows from the property:

$$\sup_{\alpha \in [0; P, a, a_2, a_3, \ldots]} \tau_r(\alpha) < \tau_r(c) < \inf_{\alpha \in [0; P, b, b_2, b_3, \ldots]} \tau_r(\alpha). \quad (10)$$

Let us prove the left-hand side inequality of (10) (the right-hand side one has essentially the same proof). Suppose $c = [0; S]$, with $|S| \equiv 1 \mod 2$. Then every irrational $\alpha < c$ has an expansion of the form $\alpha = [0; S, A]$ with $A$ an infinite string. Hence $\tau_r(\alpha) = [0; \tau_r(S), \tau_r(A)]$, and it is not hard to check that sup $\tau_r(\alpha) = [0; \tau_r(S), S_1, S_0^\infty] < [0; \tau_r(S)] = \tau_r(c)$. Discontinuity at positive rational points also follows from (10).

To prove propositions 4.1 and 4.3 we first need some lemmata.

**Lemma 4.5.** Let $r = [0; S_0] = [0; S_1] \in \mathbb{Q}_E$ and $y$ be an irrational number with c.f. expansion $y = [0; B, S_0, \ldots]$, where $B$ is a proper suffix of either $S_0$ or $S_1$, and $S_0$ equal to either $S_0$ or $S_1$. Then $y > [0; S_1]$.

**Proof.** If $B = (1)$ then there is hardly anything to prove (by proposition 3.1, the first digit of $S_1$ is strictly greater than 1). If not, then one of the following is true:

1. $S_0 = AB$ and $A$ is a prefix of $S_1$ as well;
2. $S_1 = AB$ and $A$ is a prefix of $S_0$ as well.

By proposition 3.1, in the first case we obtain that $BA \gg AB = S_0 \ll S_1$, while in the latter $BA \ll AB = S_1$; so in both cases $BA \ll S_1$ and the claim follows.

**Lemma 4.6.** Let $r \in \mathbb{Q}_E$, and $x, y \in [0, 1] \setminus \mathbb{Q}$. Then

$$G^k(x) \geq y \quad \forall k \geq 0$$

if and only if

$$G^k(\tau_r(x)) \geq \tau_r(y) \quad \forall k \geq 0$$

**Proof.** Since $\tau_r$ is increasing, $G^k(x) \geq y$ if and only if $G^k(y) \geq \tau_r(y)$ if and only if $G^{N_k}(\tau_r(\alpha)) \geq \tau_r(y)$ for $N_k = |S_0|(a_1 + \ldots + a_k) + (|S_1| - |S_0|)k$.
On the other hand, if $h$ is not of the form $N_k$, $G^h(\tau_r(x)) = [0; B, S_0, \ldots]$ with $B$ a proper suffix of either $S_0$ or $S_1$, and $S_n$ equal to either $S_0$ or $S_1$. By lemma 4.5 it follows immediately that

$$G^h(\tau_r(x)) > [0; S_1] \geq \tau_r(y)$$

\[\square\]

**Proof of proposition 4.1.** Let us first prove that, if $x \in B(\omega) \cap [\omega, \alpha_0]$ then $x = S \cdot y$ with $y \in B(\omega) \cap [\omega, \alpha_0]$ and $S \in \{S_0, S_1\}$; then the inclusion

$$B(\omega) \cap [\omega, \alpha_0] \subset K(\Sigma)$$

will follow by induction. If $x \in B(\omega) \cap [\omega, \alpha_0]$ then the following alternative holds

$(x > r)$ $x = S_0 \cdot y$ and $S_0 \cdot y = x < \alpha_0 = S_0 \cdot \alpha_0$, therefore $y \leq \alpha_0$;

$(x < r)$ $x = S_1 \cdot y$ and $S_1 \cdot y = x > \omega = S_1 \cdot \alpha_0$, therefore $y \leq \alpha_0$;

Note that, since the map $y \mapsto S \cdot y$ preserves or reverses the order depending on the parity of $|S|$, in both cases we obtain to the same conclusion. Moreover, since $B(\omega)$ is forward-invariant with respect to the Gauss map and $x \in B(\omega)$, then $y = G^k(x) \in B(\omega)$ as well, hence $y \in B(\omega) \cap [\omega, \alpha_0]$.

To prove the other inclusions, let us first remark that every $x \in K(\Sigma)$ satisfies $\omega \leq x \leq \alpha_0$. Now, let $k \in \mathbb{N}$; either $G^k(x) \in K(\Sigma)$, and hence $G^k(x) \geq \omega$, or $G^k(x) = [0; B, S_0, \ldots]$ satisfies the hypotheses of lemma 4.5, and hence we obtain that $y > [0; S_1] \geq \omega$. Since $G^k(x) \geq \omega$ holds for any $k$, then $x \in B(\omega)$.

**Proof of proposition 4.3.** (i) Recall the notation $W_r = [\omega, \alpha_0]$, and let $v \in E \cap W_r$. By remark 3.3, $E \cap W_r \subseteq B(\omega) \cap [\omega, \alpha_0]$, hence, by proposition 4.1, $v \in K(\Sigma)$. Moreover, $v < r$ because $E \cap [r, \alpha_0] = \emptyset$. As a consequence, the c.f. expansion of $v$ is an infinite concatenation of strings in the alphabet $\{S_0, S_1\}$ starting with $S_1$. Now, if the expansion of $v$ terminates with $S_0$, then $G^k(v) = \omega$ for some $k$, hence $v$ must coincide with $\omega = [0; S_1 S_0]$, so $v = \tau_r(0)$ and we are done. Otherwise, there exists some $x \in [0, 1)$ such that $v = \tau_r(x)$: then by lemma 4.6 we obtain that

$$G^k(v) \geq v \forall k \geq 0 \Rightarrow G^k(x) \geq x \forall k \geq 0$$

which means $x$ belongs to $E$.

Vice versa, let us pick $x := \tau_r(v)$ with $v \in E$. By definition of $\tau_r$, $x \in W_r$. Moreover, since $v$ belongs to $E$, $G^k(v) \geq v$ for any $n$, hence by lemma 4.6 also $\tau_r(v)$ belongs to $E$. The fact that $\tau_r$ is a homeomorphism follows from bijectivity and compactness.

(ii) Let $p \in \mathbb{Q}_E$ and $I_p = (\alpha_1, \alpha_0)$ the maximal quadratic interval generated by $p$; by point (i) above also the values $\beta_i := \tau_r(\alpha_i)$, $(i = 0, 1)$ belong to $\mathcal{E} \cap W_r$. Since $\tau_r$ is strictly increasing, no other point of $\mathcal{E}$ lies between $\beta_1$ and $\beta_0$, hence $(\beta_1, \beta_0) = I_s$ for some $s \in \mathbb{Q}_E \cap [\omega, r)$. Since $\tau_r(p)$ is a convergent to both $\tau_r(\alpha_0)$ and $\tau_r(\alpha_1)$, then $\tau_r(p) = s$.

To prove the converse, pick $s \in \mathbb{Q}_E \cap [\omega, r)$ and denote $I_p = (\beta_1, \beta_0)$. Again by point (i), $\beta_i := \tau_r(\alpha_i)$ for some $\alpha_0, \alpha_1 \in E$, and $(\alpha_0, \alpha_0)$ is a component of the complement of $E$, hence there exists $p \in \mathbb{Q}_E$ such that $I_p = (\alpha_1, \alpha_0)$. As a consequence, $s = \tau_r(p)$.

**Proof of lemma 4.4.** Let us denote $W_s = [\omega(s), \alpha_0(s)]$, $W_r = [\omega(r), \alpha_0(r)]$, $W_p = [\omega(p), \alpha_0(p)]$. Suppose (i): then, since the closures of $W_s$ and $W_r$ are not disjoint,
ω(s) ≤ α₀(r). Moreover, ω(s) ∈ E and E ∩ (r, α₀(r)] = {α₀(r)}, hence ω(s) ≤ r because ω(s) cannot coincide with α₀(r), not having a purely periodic c.f. expansion. Hence r ∈ Wᵢ and, by proposition 4.3, there exists p ∈ Qₛ such that r = τₛ(p).

Suppose now (ii). Then, since r = τₛ(p), also α₀(r) = τₛ(α₀(p)) ≤ s < α₀(s), and ω(r) = τₛ(ω(p)) ∈ Wₛ, which implies (iii).

(iii) ⇒ (i) is clear.

5. Tuning and monotonicity of entropy: proof of theorem 1.2

Definition 5.1. Let A = (a₁, ..., aₙ) be a string of positive integers. Then its matching index \([A]\) is the alternating sum of its digits:

\[ [A] := \sum_{j=1}^{n} (-1)^{|A|} a_j. \]  

Moreover, if r = [0; S₀] is a rational number between 0 and 1 and S₀ is its continued fraction expansion of even length, we define the matching index of r to be

\[ [r] := [S₀]. \]

The reason for this terminology is the following. Suppose r ∈ Qₛ is the pseudocentre of the maximal quadratic interval Iᵣ: then by theorem 3.4, a matching condition (8) holds, and by formula (7)

\[ [r] = \sum_{j=1}^{n} (-1)^{|A|} a_j = M - N \]  

where r = [0; S₀] and S₀ = (a₁, ..., aₙ). This means, by proposition (3.5), that the entropy function h(α) is increasing on Iᵣ iff \([r]\) > 0, decreasing on Iᵣ iff \([r]\) < 0, and constant on Iᵣ iff \([r]\) = 0.

Lemma 5.2. Let r, p ∈ Qₛ. Then

\[ [τₛ(p)] = -[r][p]. \]

Proof. The double bracket notation behaves well under concatenation, namely

\[ [AB] := \begin{cases} [A] + [B] & \text{if } |A| \text{ even} \\ [A] - [B] & \text{if } |A| \text{ odd.} \end{cases} \]

Let p = [0; a₁, ..., aₙ] and r = [0; S₀] be the continued fraction expansions of even length of p, r ∈ Qₛ; using the definition of τₛ we obtain

\[ [τₛ(p)] = \sum_{j=1}^{n} (-1)^{|A|} ([S₁] - (a_j - 1)[S₀]) \]

and, since n = |A| is even, the right-hand side becomes \([S₀] = \sum_{j=1}^{n} (-1)^{j} a_j\), whence the thesis. □

Definition 5.3. A quadratic interval Iᵣ is called neutral if \([r]\) = 0. Similarly, a tuning window Wₛ is called neutral if \([r]\) = 0.
As an example, the rational \( r = \frac{1}{2} = [0; 2] = [0; 1, 1] \) generates the neutral tuning window \( W_{1/2} = [g^2, g) \).

**Proof of theorem 1.2.** Let \( I_r \) be a maximal quadratic interval over which the entropy is increasing. Then, by theorem 3.4 and proposition 3.5, for \( \alpha \in I_r \), a matching condition (8) holds, with \( M - N > 0 \). This implies by (12) that \( \|r\| > 0 \). Let now \( I_p \) be another maximal quadratic interval. By proposition 4.3 (ii), \( I_{\tau_r(p)} \) is also a maximal quadratic interval, and by lemma 5.2

\[
\|\tau_r(p)\| = -\|r\| p.
\]

Since \( \|r\| > 0 \), then \( \|\tau_r(p)\| \) and \( \|p\| \) have opposite sign. In terms of the monotonicity of entropy, this means the following:

1. If the entropy is increasing on \( I_p \) then by (12) \( \|p\| > 0 \), hence \( \|\tau_r(p)\| < 0 \), which implies (again by (12)) that the entropy is decreasing on \( I_{\tau_r(p)} \);
2. If the entropy is decreasing on \( I_p \) then \( \|p\| < 0 \), hence \( \|\tau_r(p)\| > 0 \) and the entropy is increasing on \( I_{\tau_r(p)} \);
3. If the entropy is constant on \( I_p \) then \( \|p\| = 0 \), hence \( \|\tau_r(p)\| = 0 \) and the entropy is constant on \( I_{\tau_r(p)} \).

If instead, the entropy is decreasing on \( I_r \) then \( \|r\| > 0 \), hence \( \|\tau_r(p)\| \) and \( \|p\| \) have the same sign, which similarly to the previous case implies that the monotonicity of entropy on \( I_p \) and \( I_{\tau_r(p)} \) is the same.

\( \square \)

**Remark 5.4.** The same argument as in the proof of theorem 1.2 shows that, if \( r \in \mathbb{Q}_E \) with \( \|r\| = 0 \), then the entropy on \( I_{\tau_r(p)} \) is constant for each \( p \in \mathbb{Q}_E \) (no matter what the monotonicity is on \( I_p \)).

6. Plateaux: proof of theorem 1.1

The goal of this section is to prove theorem 6.16, which characterizes the plateaux of the entropy and has as a consequence theorem 1.1 in the introduction. Meanwhile, we introduce the sets of **untuned parameters** (section 6.2) and **dominant parameters** (section 6.3) which we will use in the proof of the theorem (section 6.4).

6.1. The importance of being Hölder

The first step in the proof of theorem 1.1 is proving that the entropy function \( h(\alpha) \) is indeed constant on neutral tuning windows:

**Proposition 6.1.** Let \( r \in \mathbb{Q}_E \) generate a neutral maximal interval, i.e. \( \|r\| = 0 \). Then the entropy function \( h(\alpha) \) is constant on \( W_r \).

By remark 5.4, we already know that the entropy is locally constant on all connected components of \( W_r \setminus \mathcal{C} \), which has full measure in \( W_r \). However, since \( W_r \cap \mathcal{C} \) has, in general, positive Hausdorff dimension, in order to prove that the entropy is actually constant on the whole \( W_r \) one needs to exclude a devil staircase behaviour. We shall exploit the following criterion:

**Lemma 6.2.** Let \( f : I \to \mathbb{R} \) be a Hölder-continuous function of exponent \( \eta \in (0, 1) \), and assume that there exists a closed set \( C \subseteq I \) such that \( f \) is locally constant at all \( x \notin C \). Suppose moreover \( \text{H.dim } C < \eta \). Then \( f \) is constant on \( I \).
Proof. Suppose $f$ is not constant: then by continuity $f(I)$ is an interval with non-empty interior, hence $H.\dim f(I) = 1$. On the other hand, we know $f$ is constant on the connected components of $I \setminus C$, so we obtain $f(I) = f(C)$, whence

$$H.\dim f(C) = H.\dim f(I) = 1.$$ 

But, since $f$ is $\eta$-Hölder continuous, we also obtain (e.g. by [Fa], proposition 2.3)

$$H.\dim f(C) \leq \frac{H.\dim C}{\eta}$$

and thus $\eta \leq H.\dim C$, contradiction. \hfill $\Box$

Let us now check that the hypotheses of lemma 6.2 hold in our case; the first one is given by the following.

**Theorem 6.3 ([Ti, theorem 1.1]).** For all fixed $0 < \eta < 1/2$, the function $\alpha \mapsto h(\alpha)$ is locally Hölder-continuous of exponent $\eta$ on $(0, 1]$.

We are now left with checking that the Hausdorff dimension of $E \cap W_r$ is small enough

**Lemma 6.4.** For all $r \in \mathbb{Q}_E$,

$$H.\dim E \cap W_r \leq \frac{\log 2}{\log 5} < 1/2.$$ 

**Proof.** Let $r \in \mathbb{Q}_E, r = [0; S_0] = [0; S_1]$ and $W_r = [\omega, \alpha]$. By remark 3.3 and proposition 4.1,

$$E \cap W_r \subset B(\omega) \cap [\omega, \alpha] = K(\Sigma),$$

with $\Sigma = \{S_0, S_1\}$.

Note we also have $K(\Sigma) = K(\Sigma_2)$ with $\Sigma_2 = \{S_0S_0, S_0S_1, S_1S_0, S_1S_1\}$ and, by virtue of (4) we have the estimate

$$|f'_{S_iS_j}(x)| \leq \frac{1}{q(S_iS_j)^2}, \quad i, j \in \{0, 1\}.$$ 

On the other hand, setting $Z_0 = (1, 1)$ and $Z_1 = (2)$ we can easily check that

$$q(S_iS_j) \geq q(Z_iZ_j) = 5 \quad \forall i, j \in \{0, 1\};$$

whence $|f'_{S_iS_j}(x)| \leq \frac{1}{5}$ and, by formula (5), we obtain our claim. \hfill $\Box$

Proposition 6.1 now follows from lemma 6.2, theorem 6.3 and lemma 6.4.

### 6.2. Untuned parameters

The set of untuned parameters is the complement of all tuning windows:

$$UT := [0, g] \setminus \bigcup_{r \in \mathbb{Q} \cap (0, 1)} W_r$$

Note that, since $I_r \subset W_r, UT \subset E$. Moreover, we say that a rational $a \in \mathbb{Q}_E$ is an untuned label if it cannot be written as $a = \tau_r(a_0)$ for some $r, a_0 \in \mathbb{Q}_E$. We shall denote by $\mathbb{Q}_U$ the set of untuned labels. Let us start out by seeing that each pseudocentre of a maximal quadratic interval admits an ‘untuned factorization’.
Lemma 6.5. Each $r \in \mathbb{Q}_E$ either is an untuned label or can be written as
\[
r = \tau_{r_0} \circ \ldots \circ \tau_{r_i}(r_0), \quad \text{with } r_i \in \mathcal{Q}_{UT} \forall i \in \{0, 1, \ldots, m\}.
\] (14)

Proof. A straightforward check shows that the tuning operator has the following associativity property:
\[
\tau_{\tau(p)}(r)(x) = \tau_p \circ \tau_r(x) \quad \forall p, r \in \mathbb{Q}_E, \ x \in (0, 1).
\] (15)

For $s = [0; a_1, \ldots, a_m] \in \mathbb{Q}_E$ we shall set $\|s\|_1 := \sum_1^m a_i$; this definition does not depend on the representation of $s$, moreover
\[
\|\tau_p(s)\|_1 = \|p\|_1 \|s\|_1 \quad \forall p, s \in \mathbb{Q}_E.
\]

The proof of (14) follows then easily by induction on $N = \|r\|_1$, using the fact that $\max(\|p\|_1, \|s\|_1) \leq \|\tau_p(s)\|_1/2$.

As a consequence of the following proposition, the connected components of the complement of $\mathcal{U}T$ are precisely the tuning windows generated by the elements of $\mathcal{Q}_{UT}$:

Proposition 6.6. The set $\overline{\mathcal{U}T}$ is a Cantor set: indeed,

(i) $\mathcal{U}T = [0, g1] \setminus \bigcup_{r \in \mathcal{Q}_{UT}} W_r$;

(ii) if $r, s \in \mathcal{Q}_{UT}$ with $r \neq s$, then $W_r$ and $W_s$ are disjoint;

(iii) if $x \in \overline{\mathcal{U}T} \setminus \mathcal{U}T$, then there exists $r \in \mathcal{Q}_{UT}$ such that $x = \tau_r(0)$.

Proof. Let us point out that $\overline{\mathcal{U}T}$ is totally disconnected because $\bigcup_r W_r$ is dense; the fact that $\overline{\mathcal{U}T}$ has no isolated points will be a consequence of (ii).

(i) It is enough to prove that every tuning window $W_r$ is contained in a tuning window $W_s$, with $s \in \mathcal{Q}_{UT}$. In fact, let $r \in \mathbb{Q}_E$; either $r \in \mathcal{Q}_{UT}$ or, by lemma 6.5, there exists $p \in \mathbb{Q}_E$ and $s \in \mathcal{Q}_{UT}$ such that $r = \tau_s(p)$, hence $W_r \subseteq W_s$.

(ii) By lemma 4.4, if the closures of $W_r$ and $W_s$ are not disjoint, then $r = \tau_s(p)$, which contradicts the fact $r \in \mathcal{Q}_{UT}$.

(iii) By (i) and (ii), $\overline{\mathcal{U}T}$ is a Cantor set, and each element $x$ which belongs to $\overline{\mathcal{U}T} \setminus \mathcal{U}T$ is the left endpoint of some tuning window $W_r$ with $r \in \mathcal{Q}_{UT}$, which is equivalent to say $x = \tau_r(0)$.

Lemma 6.7. The Hausdorff dimension of $\mathcal{U}T$ is full:

\[
\text{H.dim } \mathcal{U}T = 1
\]

Proof. By the properties of Hausdorff dimension,

\[
\text{H.dim } \mathcal{E} = \max \{\text{H.dim } \mathcal{U}T, \sup_{r \in \mathcal{Q}_{UT}} \text{H.dim } \mathcal{E} \cap W_r\}
\]

Now, by ([CT, theorem 1.2]) $\text{H.dim } \mathcal{E} = 1$, and, by lemma 6.4, $\text{H.dim } \mathcal{E} \cap W_r < \frac{1}{2}$, hence the claim.
6.3. Dominant parameters

**Definition 6.8.** A finite string $S$ of positive integers is dominant if it has even length and

$$S = AB \ll B$$

for any splitting $S = AB$ of $S$ into two non-empty strings $A, B$.

That is to say, dominant strings are smaller than all their proper suffixes. A related definition is the following

**Definition 6.9.** A quadratic irrational $\alpha \in [0, 1]$ is a dominant parameter if its c.f. expansion is of the form $\alpha = [0; S]$ with $S$ a dominant string.

For instance, $(2, 1, 1, 1)$ is dominant, while $(2, 1, 1, 2) \ll (2)$. In general, all strings whose first digit is strictly greater than the others are dominant, but there are even more dominant strings (for instance $(3, 1, 3, 2)$ is dominant).

**Remark 6.10.** By proposition 3.1, if $S$ is dominant then $[0; S] \in \mathbb{Q}_E$.

A very useful feature of dominant strings is that they can be easily used to produce other dominant strings:

**Lemma 6.11.** Let $S_0$ be a dominant string, and $B$ a proper suffix of $S_0$ of even length. Then, for any $m \geq 1$, $S_0^m B$ is a dominant string.

**Proof.** Let $Y$ be a proper suffix of $S_0^m B$. There are three possible cases:

1. $Y$ is a suffix of $B$, hence a proper suffix of $S_0$. Hence, since $S_0$ is dominant, $S_0 \ll Y$ and $S_0^m B \ll Y$.
2. $Y$ is of the form $S_0^k B$, with $1 \leq k < m$. Then by dominance $S_0 \ll B$, which implies $S_0^{m-k} B \ll B$, hence $S_0^m B \ll S_0 B$.
3. $Y$ is of the form $C S_0^k B$, with $0 \leq k < m$ and $C$ a proper suffix of $S_0$. Then again the claim follows by the fact that $S_0$ is dominant, hence $S_0 \ll C$.

**Lemma 6.12.** A dominant string $S_0$ cannot begin with two equal digits.

**Proof.** By definition of dominance, $S_0$ cannot consist of just $k \geq 2$ equal digits. Suppose instead it has the form $S_0 = (a)^k B$ with $k \geq 2$ and $B$ non-empty and which does not begin with $a$. Then by dominance $(a)^k B \ll B$, hence $a \ll B$ since $B$ does not begin with $a$. However, this implies $a B \ll a a$ and hence $a B \ll (a)^k B = S_0$, which contradicts the definition of dominance because $a B$ is a proper suffix of $S_0$.

The reason why dominant parameters turn out to be so useful is that they can approximate untuned parameters:

**Proposition 6.13 ([CT2, proposition 6]).** The set of dominant parameters is dense in $UT \setminus \{g\}$. More precisely, every parameter in $UT \setminus \{g\}$ is accumulated from the right by a dominant parameter.

**Proposition 6.14.** Every element $\beta \in \overline{UT} \setminus \{g\}$ is accumulated by non-neutral maximal quadratic intervals.
Proof. We shall prove that either $\beta \in UT \setminus \{g\}$, and $\beta$ is accumulated from the right by non-neutral maximal quadratic intervals, or $\beta = \tau_s(0)$ for some $s \in Q_{UT}$, and $\beta$ is accumulated from the left by non-neutral maximal quadratic intervals.

If $\beta \in UT$ then, by proposition 6.13, $\beta$ is the limit point from the right of a sequence $\alpha_n = [0; A_n]$ with $A_n$ dominant. If $[A_n] \neq 0$ for infinitely many $n$, the claim is proven. Otherwise, it is sufficient to prove that every dominant parameter $\alpha_n$ such that $[A_n] = 0$ is accumulated from the right by non-neutral maximal intervals. Let $S_0$ be a dominant string, with $[S_0] = 0$, and let $\alpha := [0; S_0]$. First of all, the length of $S_0$ is bigger than 2: in fact, if $S_0$ had length 2, then condition $[S_0] = 0$ would force it to be of the form $S_0 = (a, a)$ for some $a$, which contradicts the definition of dominant. Hence, we can write $S_0 = AB$ with $A$ of length 2 and $B$ of positive, even length. Then, by lemma 6.11, $S_0B$ is also dominant, hence $p_m := [0; S_0B] \in Q_E$ by remark 6.10. Moreover, $\alpha < p_m$ since $S_0 \ll B$. Furthermore, $S_0$ cannot begin with two equal digits (lemma 6.12), hence $[A] \neq 0$ and $[S_0B] = [B] = [S_0] - [A] \neq 0$. Thus the sequence $I_{pm}$ is a sequence of non-neutral maximal quadratic intervals which tends to $\beta$ from the right, and the claim is proven.

If $\beta \in UT \setminus UT$, then by proposition 6.6 (iii) there exists $s \in Q_{UT}$ such that $\beta = \tau_s(0)$. Since $\overline{UT}$ is a Cantor set and $\beta$ lies on its boundary, $\beta$ is the limit point (from the left) of a sequence of points of $UT$, hence the claim follows by the above discussion. □

6.4. Characterization of plateaux

Definition 6.15. A parameter $x \in E$ is finitely renormalizable if it belongs to a finite, non-zero number of tuning windows. A parameter $x \in E$ is infinitely renormalizable if it lies in infinitely many tuning windows $W_r$, with $r \in Q_E$. Untuned parameters are also referred to as non-renormalizable.

Note that by proposition 4.3, $x$ is finitely renormalizable iff $x = \tau_s(y)$, with $y \in UT$ and $r \in Q_E$. We are finally ready to prove theorem 1.1 stated in the introduction, and indeed the following stronger version:

Theorem 6.16. An open interval $U \subseteq [0, 1]$ of the parameter space of $\alpha$-continued fraction transformations is a plateau for the entropy function $h(\alpha)$ if and only if it is the interior of a neutral tuning window $U = \text{int}(W_r)$, with $r$ of either one of the following types:

| Type | Condition |
|------|-----------|
| (NR) | $r \in Q_{UT}$, $[r] = 0$ | (non-renormalizable case) |
| (FR) | $r = \tau_1(r_0)$ with $r_0 \in Q_{UT}$, $[r_0] = 0$, $r_1 \in Q_E$, $[r_1] \neq 0$ | (finitely renormalizable case) |
Proof. Let us pick \( r \) which satisfies (NR), and let \( W_r = [\omega, \alpha_0] \) be its tuning window. By proposition 6.1, since \([r]\] = 0, the entropy is constant on \( W_r \). Let us prove that it is not constant on any larger interval. Since \( r \in \mathbb{Q}_{UT} \), by proposition 6.6, \( \alpha_0 \) belongs to \( UT \). If \( \alpha_0 = g \), then by the explicit formula (1) the entropy is decreasing to the right of \( \alpha_0 \). Otherwise, by proposition 6.14, \( \alpha_0 \) is accumulated from the right by non-neutral maximal quadratic intervals, hence entropy is not constant to the right of \( \alpha_0 \). Moreover, by proposition 6.6, \( \omega \) belongs to the boundary of \( UT \), hence, by proposition 6.14, it is accumulated from the left by non-neutral intervals. This means that the interior of \( W_r \) is a maximal open interval of constance for the entropy \( h(\alpha) \), i.e. a plateau.

Now, suppose that \( r \) satisfies condition (FR), with \( r = \tau_r(r_0) \). By the (NR) case, the interior of \( W_r \) is a plateau, and \( W_\omega \) is accumulated from both sides by non-neutral intervals. Since \( \tau_r \) maps non-neutral intervals to non-neutral intervals and is continuous on \( E \), then \( W_r \) is accumulated from both sides by non-neutral intervals, hence its interior is a plateau.

Suppose now \( U \) is a plateau (see figure 2). Since \( E \) has no interior part (e.g. by formula (6)), there is \( r \in \mathbb{Q}_E \) such that \( I_r \) intersects \( U \), hence, by proposition 3.5, \([r]\] = 0 and actually \( I_r \subseteq U \). Then, by lemma 6.5 one has the factorization

\[ r = \tau_r \circ \ldots \circ \tau_r(r_0) \]

with each \( r_i \in \mathbb{Q}_{UT} \) an untuned label (recall \( n \) can possibly be zero, in which case \( r = r_0 \)). Since the matching index is multiplicative (equation (13)), there exists at least one \( r_i \) with zero matching index: let \( j \in \{0, \ldots, n\} \) be the largest index such that \([r_j]\] = 0. If \( j = n \), let \( s := r_n \); by the first part of the proof, the interior of \( W_s \) is a plateau, and it intersects \( U \) because they both contain \( r \) (by lemma 4.4, \( r \) belongs to the interior of \( W_s \)), hence \( U = W_s \), and we are in case (NR).

If, otherwise, \( j < n \), let \( s := \tau_{r_1} \circ \ldots \circ \tau_{r_{j+1}}(r_j) \). By associativity of tuning (equation (15)) we can write

\[ s = \tau_{s_1}(s_0), \]

with \( s_0 := r_j \) and \( s_1 := \tau_{r_n} \circ \ldots \circ \tau_{r_{j+1}}(r_{j+1}) \). Moreover, by multiplicativity of the matching index (equation (13)) \([s_i]\] \neq 0, hence \( s \) falls into the case (FR) and by the first part of the proof the interior of \( W_r \) is a plateau. Also, by construction, \( r \) belongs to the image of \( \tau_r \), hence it belongs to the interior of \( W_s \). As a consequence, \( U \) and \( W_s \) are intersecting plateaux, hence they must coincide. \( \square \)

7. Classification of local monotonic behaviour

Lemma 7.1. Any non-neutral tuning window \( W_r \) contains infinitely many intervals on which the entropy \( h(\alpha) \) is constant, infinitely many over which it is increasing, and infinitely many on which it is decreasing.

Proof. Let us consider the following sequences of rational numbers

\[ s_n := [0; n, 1] \]
\[ t_n := [0; n, n] \]
\[ u_n := [0; n + 1, 1, n]. \]

It is not hard to check (e.g. using proposition 3.1) that \( s_n, t_n, u_n \) belong to \( \mathbb{Q}_E \), so \( I_{s_n}, I_{t_n}, I_{u_n} \) are maximal quadratic intervals. Moreover, by computing the matching indices one finds that,
for $n > 2$, the entropy $h(\alpha)$ is increasing on the interval $I_{s_r}$, constant on $I_t$ and decreasing on $I_{u_r}$. Since $W_r$ is non-neutral, by theorem 1.2 $\tau_r$ either induces the same monotonicity or the opposite one, hence the sequences $I_{s_r(s_r)}$, $I_{t_r(s_r)}$ and $I_{u_r(a_r)}$ are sequences of maximal quadratic intervals which lie in $W_r$ and display all three types of monotonic behaviour.

\textbf{Proof of theorem 1.3.} Let $\alpha \in [0, 1]$ be a parameter. If $\alpha \in E$, then $\alpha$ belongs to some maximal quadratic interval $I_r$, hence $h(\alpha)$ is monotone on $I_r$ by proposition 3.5, and by formula (12) the monotonicity type depends on the sign of $[r]$. If $\alpha \in E$, there are the following cases:

(1) $\alpha = g$. Then $\alpha$ is a phase transition as described by formula (1);

(2) $\alpha \in UT \setminus \{g\}$. Then, by proposition 6.14, $\alpha$ is accumulated from the right by non-neutral tuning windows, and by lemma 7.1 each non-neutral tuning window contains infinitely many intervals where the entropy is constant, increasing or decreasing; the parameter $\alpha$ has therefore mixed monotonic behaviour.

(3) $\alpha$ is finitely renormalizable. Then one can write $\alpha = \tau_r(y)$, with $y \in UT$. There are three subcases:

(3a) $[r] \neq 0$, and $y = g$. Since $\tau_r$ maps neutral intervals to neutral intervals and non-neutral intervals to non-neutral intervals, the phase transition at $y = g$ gets mapped to a phase transition at $\alpha$.

(3b) $[r] \neq 0$, and $y \neq g$. Then, by case (2) $y$ is accumulated from the right by intervals with all types of monotonicity, hence so is $\alpha$.

(3c) If $[r] = 0$, then using the untuned factorization (lemma 6.5) one can write $\alpha = \tau_{r_0} \circ \ldots \circ \tau_{r_j}(y)$, with $r_j \in \mathbb{Q}_{UT}$.

Let now $j \in \{0, \ldots, m\}$ be the largest index such that $[r_j] = 0$. If $j = m$, then $\alpha$ belongs to the neutral tuning window $W_{r_m}$: thus, either $\alpha$ belongs to the interior of $W_{r_m}$ (which means by proposition 6.1 that the entropy is locally constant at $\alpha$), or $\alpha$ coincides with the left endpoint of $W_{r_m}$. In the latter case, $\alpha$ belongs to the boundary of $UT$, hence by proposition 6.14 and lemma 7.1 it has mixed behaviour. If $j < m$, then by the same reasoning as above $\tau_{r_j} \circ \ldots \circ \tau_{r_0}(y)$ either lies inside a plateau or has mixed behaviour, and since the operator $\tau_{r_{j+1}} \circ \ldots \circ \tau_{r_m}$ either respects the monotonicity or reverses it, also $\alpha$ either lies inside a plateau or has mixed behaviour.

(4) $\alpha$ is infinitely renormalizable, i.e. $\alpha$ lies in infinitely many tuning windows. If $\alpha$ lies in at least one neutral tuning window $W_r = [\omega, \alpha_0)$, then it must lie in its interior, because $\omega$ is not infinitely renormalizable. This means, by proposition 6.1, that $h$ must be constant on a neighbourhood of $\alpha$. Otherwise, $\alpha$ lies inside infinitely many nested non-neutral tuning windows $W_{r_n}$. Since the sequence of the denominators of the rational numbers $r_n$ must be unbounded, the size of $W_{r_n}$ must be arbitrarily small. By lemma 7.1, in each $W_{r_n}$ there are infinitely many intervals with any monotonicity type and $\alpha$ displays mixed behaviour.

Note that, as a consequence of the previous proof, $\alpha$ is a phase transition if and only if it is of the form $\alpha = \tau_r(g)$, with $r \in \mathbb{Q}_E$ and $[r] \neq 0$, hence the set of phase transitions is countable. Moreover, the set of points of $E$ which lie in the interior of a neutral tuning window has Hausdorff dimension less than 1/2 by lemma 6.4.

Finally, the set of parameters for which there is mixed behaviour has zero Lebesgue measure because it is a subset of $E$. On the other hand, it has full Hausdorff dimension because such a set contains $UT \setminus \{g\}$, and by lemma 6.7 $UT$ has full Hausdorff dimension.
Appendix: Tuning for quadratic polynomials

The definition of tuning operators given in section 4 arises from the dictionary between \( \alpha \)-continued fractions and quadratic polynomials first discovered in [BCIT]. In this appendix, we will recall a few facts about complex dynamics and show how our construction is related to the combinatorial structure of the Mandelbrot set.

Tuning for quadratic polynomials. Let \( f_c(z) := z^2 + c \) be the family of quadratic polynomials, with \( c \in \mathbb{C} \). Recall the Mandelbrot set \( M \) is the set of parameters \( c \in \mathbb{C} \) such that the orbit of the critical point 0 is bounded under the action of \( f_c \).

The Mandelbrot set has the remarkable property that near every point of its boundary there are infinitely many copies of the whole \( M \), called baby Mandelbrot sets. A hyperbolic component \( W \) of the Mandelbrot set is an open, connected subset of \( M \) such that all \( c \in W \), the orbit of the critical point \( f^n(0) \) is attracted to a periodic cycle.

Douady and Hubbard [DH] related the presence of baby copies of \( M \) to renormalization in the family of quadratic polynomials. More precisely, they associated with any hyperbolic component \( W \) a tuning map \( \iota_W : M \to M \) which maps the main cardioid of \( M \) to \( W \), and such that the image of the whole \( M \) under \( \iota_W \) is a baby copy of \( M \).

External rays. A coordinate system on the boundary of \( M \) is given by external rays. In fact, the exterior of the Mandelbrot set is biholomorphic to the exterior of the unit disc \( \Phi : \{ z \in \mathbb{C} : |z| > 1 \} \to \mathbb{C} \setminus M \).

The external ray at angle \( \theta \) is the image of a ray in the complement of the unit disc:

\[
R(\theta) := \Phi(\{ re^{i\theta} : r > 1 \}).
\]

The ray at angle \( \theta \) is said to land at \( c \in \partial M \) if \( \lim_{r \to 1} \Phi(re^{i\theta}) = c \). This way angles in \( \mathbb{R}/\mathbb{Z} \) determine parameters on the boundary of the Mandelbrot set, and it turns out that the binary expansions of such angles are related to the dynamics of the map \( f_c \).

The tuning map can be described in terms of external angles in the following terms. Let \( W \) be a hyperbolic component, and \( \eta_0, \eta_1 \) the angles of the two external rays which land on the root of \( W \). Let \( \eta_0 = 0.\Sigma_0 \) and \( \eta_1 = 0.\Sigma_1 \) be the (purely periodic) binary expansions of the two angles which land at the root of \( W \). Let us define the map \( \tau_W : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) in the following way:

\[
\theta = 0.\theta_1\theta_2\theta_3 \ldots \mapsto \tau_W(\theta) = 0.\Sigma_0\Sigma_{\theta_1}\Sigma_{\theta_2} \ldots. \tag{16}
\]

where \( \theta = 0.\theta_1\theta_2 \ldots \) is the binary expansion of \( \theta \), and its image is given by substituting the binary string \( \Sigma_0 \) to every occurrence of 0 and \( \Sigma_1 \) to every occurrence of 1.

Proposition 7.2 ([Do, proposition 7]). The map \( \tau_W \) has the property that, if the ray of external angle \( \theta \) lands at \( c \in \partial M \), then the ray at external angle \( \tau_W(\theta) \) lands at \( \iota_W(c) \).

The real slice. If we now restrict ourselves to the case when \( c \in [-2, \frac{1}{4}] \) is real, each map \( f_c \) acts on the real line, and has a well-defined topological entropy \( h(f_c) \). A classical result is the following.

Theorem 7.3 ([MT, Douady–Hubbard]). The entropy of the quadratic family \( h(f_c) \) is a continuous, decreasing function of \( c \in [-2, \frac{1}{4}] \).
If $W$ is a real hyperbolic component, then $\iota_W$ preserves the real axis. We will call the tuning window relative to $c_0$ the intersection of the real axis with the baby Mandelbrot set generated by the hyperbolic component $W$ with root $c_0$. Also in the case of quadratic polynomials, plateaux of the entropy are tuning windows:

**Theorem 7.4 ([Do]).** $\text{h}(f_c) = \text{h}(f_{c'})$ if and only if $c$ and $c'$ lie in the same tuning window relative to some $c_0$, with $\text{h}(f_{c_0}) > 0$.

The set of angles $R$ corresponding to external rays which land on the real slice of the Mandelbrot set $\partial \mathcal{M} \cap \mathbb{R}$ has a nice combinatorial description: in fact it coincides, up to a set of Hausdorff dimension zero, with the set $R = \{ \theta \in \mathbb{R} / \mathbb{Z} : T^{k+1}(\theta) \leq T(\theta) \forall k \geq 0 \}$, where $T(x) = \min\{2x, 2-2x\}$ is the usual tent map (see [BCIT, proposition 3.4]). A more general introduction to the combinatorics of $\mathcal{M}$ and the real slice can be found in [Za].

**Dictionary.** Using the above combinatorial description, one can establish an isomorphism between the real slice of the boundary of the Mandelbrot set and the bifurcation set $\mathcal{E}$ for $\alpha$-continued fractions. In fact, the following is true:

**Proposition 7.5 ([BCIT, theorem 1.1 and proposition 5.1]).** The map $\varphi : [0, 1] \to [0, \frac{1}{2}]$

$$[0; a_1, a_2, \ldots] \mapsto \frac{1}{a_1} \ldots \frac{1}{a_2} \ldots$$

is a continuous bijection which maps the bifurcation set for $\alpha$-continued fractions $\mathcal{E}$ to the set of real rays $R \cap [0, \frac{1}{2}]$.

As a corollary,

$$I_r = (\alpha_1, \alpha_0) \quad \Leftrightarrow \quad (\varphi(\alpha_1), \varphi(\alpha_0))$$

is a maximal quadratic interval $\Leftrightarrow$ is a real hyperbolic component.

Proposition 7.6 will show that the tuning operators defined in section 4 and the Douady–Hubbard tuning correspond to each other via the dictionary defined by proposition 7.5. For instance $g = [0; \overline{1}] \mapsto \varphi([0; \overline{1}]) = 0.01 = 1/3$, while

$$\tau_{1/3}(g) = [0; \overline{3}] \mapsto \varphi([0; \overline{3}]) = 0.011100 = \tau_W(0.01)$$

where $\tau_W$ is the tuning operator defined by equation (16) with $\Sigma_0 = 011$, $\Sigma_1 = 100$. Note that the values 0.01 = 1/3 and 0.011100 = 28/63 are the external angles corresponding to the rays labelled by $\theta$ and $\tau_W(\theta)$ in figure 3.

**Proposition 7.6.** Suppose $W$ is a real hyperbolic component and let $c \in \partial \mathcal{M} \cap \mathbb{R}$ be its root. Moreover, let $\eta_0 \in [0, \frac{1}{2}]$ be the external angle of a ray which lands at $c$. Suppose $\eta_0$ has binary expansion

$$\eta_0 = 0.0\overline{1} \ldots 0 \overline{0} \ldots 0 \overline{0} \ldots 0 \overline{0}$$

Then, for each $x \in [0, 1]$,

$$\tau_W(\varphi(x)) = \varphi(\tau_r(x))$$

with $r = [0; b_1, \ldots, b_n]$. 

Tuning and plateaux for the entropy of \( \alpha \)-continued fractions

Figure 3. The Mandelbrot set \( \mathcal{M} \), and a baby copy of it along the real axis (inside the framed rectangle). The tuning homeomorphism maps the parameter at angle \( \theta \) to the parameter at angle \( \tau_W(\theta) \).

\textbf{Proof.} Suppose \( x = [0; a_1, a_2, \ldots] \) so that \( \tau_r(x) = [0; S_1 S_0^{a_1-1} S_1 S_0^{a_2-1} \ldots] \). Then by definition
\[
\varphi(\tau_r(x)) = 0.01^{b_1} \ldots 0^{b_{a_1-1}} (0^{b_1} \ldots 1^{b_1})^{a_1-1} 0^{b_1} \ldots 1^{b_1-1} 0 (1^{b_1} \ldots 0^{b_1})^{a_2-1} \ldots \\
= 0. (01^{b_1} \ldots 0^{b_{a_1-1}}) (10^{b_1} \ldots 1^{b_1-1})^{a_1} \ldots = 0. S_0 S_1 S_0^{a_2} S_1 S_0^{a_3} \ldots = \tau_W(\theta)
\]
where \( \theta = 0.01^{b_1} 0^{b_2} \ldots \). \( \square \)

Let us point out that thinking in terms of binary expansions often simplifies the combinatorial picture. For instance, it is easy from the definition to see that \( \tau_W \) is strictly monotone, so \( \tau_r = \varphi^{-1} \circ \tau_W \circ \varphi \) is monotonic as well, giving a simpler alternative proof of lemma 4.2.

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