Classical Hair in String Theory I: General Formulation

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ABSTRACT

We discuss why classical hair is desirable for the description of black holes, and show that it arises generically in a wide class of field theories involving extra dimensions. We develop the canonical formalism for theories with the matter content that arises in string theory. General covariance and duality are used to determine the form of surface terms. We derive an effective theory (reduced Hamiltonian) for the hair in terms of horizon variables. Solution of the constraints expresses these variables in terms of hair accessible to an observer at infinity. We exhibit some general properties of the resulting theory, including a formal identification of the temperature and entropy. The Cvetič-Youm dyon is described in some detail, as an important example.
1. Classical Hair

The appearance of $\hbar$ in the denominator of the Bekenstein-Hawking formula

$$S_{BH} = \frac{A}{4\hbar G_N},$$

(1.1)

suggests that, in a microscopic accounting, the entropy should be visible classically. Indeed, a similar appearance of $(\hbar)^{-1}$ is familiar in ordinary gas dynamics, where it provides a measure in classical phase space. In semiclassical quantization, one works with solutions of the classical equations, which are parametrized by classical phase space. One passes from these solutions to quantum states by requiring quantization conditions of the Bohr-Sommerfeld type, and the spacing of levels is therefore set by $\hbar^{-1}$. Classical structure would also be most welcome for another, related, reason. If black holes settle down to a unique (structureless) intermediate state, independent of how they formed, then it becomes impossible in principle to reconstruct the past state from the future state. Such a situation is difficult to reconcile with the unitary evolution of states one expects in quantum theory. The difficulty is especially acute if the hole subsequently evaporates, because one then appears to have an overall non-unitary evolution involving ordinary matter only. If black holes were sufficiently structured – if they had sufficient ‘hair’ – then these problems might be avoided.

Of course the obvious difficulty with this straightforward, attractive way to address these major issues in the quantum theory of black holes is the famous meta-theorem that black holes have no hair: that is, that there is a unique stationary classical solution for specified values of the conserved quantities (mass, angular momentum, and charges) at infinity. This theorem has been rigorously proved for Einstein-Maxwell theory, and for small perturbations a fairly general argument can be made [1]. It corresponds to the physical intuition that gravitational collapse rapidly carries all the participating matter through the horizon, leaving behind only those traces that correspond to surface integrals at infinity (e.g., charges, according to Gauss’ law) [2]. The arguments used to establish the no-hair theorems are not
entirely general, and some isolated counterexamples are known in spontaneously broken gauge theories [3]. There are also interesting possibilities for structure of an essentially quantum-mechanical nature (quantum hair) in theories with discrete gauge symmetries [4]. While these phenomena do serve to emphasize that the no hair meta-theorem can fail in theories with elaborate matter content, they do not appear to come close to providing the massive degeneracy implicit in (1.1).

Some recent developments, however, put this question in a new light [5–10]. In analyzing black hole solutions for low-energy field theories suggested by superstring theory, it is both interesting and technically simplest to focus on solutions that preserve some supersymmetry. The equations that ensure supersymmetry, and inter alia guarantee the equations of motion are obeyed, are first order equations, and their general solution may be found. When this is done, one discovers that the solution contains some freely specifiable functions. Our Appendix is devoted to reviewing a specific example, the Cvetič-Youm dyon. In this regular black hole solution — a generalization of the extremal Reissner–Nordström black hole — we exhibit hair of the form

\[ f(x_9, t) = f(x_9 - t) \]  

where \( t \) is Schwarzschild time and \( x_9 \) parametrizes a compactified dimension, specifically a circle of radius \( R \). More precisely, the hair is a product of (1.2) with a profile function, and also involves a change in the metric. \( f \) can be expanded

\[ f(x_9 - t) = f_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{2\pi n}{R}(x_9 - t)\right) + b_n \sum_{1}^{\infty} \sin\left(\frac{2\pi n}{R}(x_9 - t)\right). \]  

From the point of view of a four-dimensional observer who does not resolve the extra dimension, only \( f_0 \) is accessible. It is a quantity that must be specified as part of the macroscopic description of the hole. The remaining modes reduce to zero upon dimensional reduction, but from the perspective of the full theory they represent a potentially large number of hidden degrees of freedom, in principle discernable at infinity. We anticipate that the phenomenon of higher-dimensional
hair is quite general, once one has an appropriate field content. Indeed, the macro-
scopic fields at infinity do not specify the dependence on compactified coordinates.
Each of these microscopic configurations generates a regular solution throughout
space-time exterior to the hole by virtue of the no–hair theorem in the higher
dimensional space.

Our main goal in this paper is to derive an effective theory governing this
sort of higher-dimensional hair, and to set up the machinery for counting it (so
as eventually to compare with (1.1)). An important point of the analysis is the
derivation, from imposition of the requirement of vanishing source at the horizon,
of a matching condition which forces the existence of non-trivial hair. In the
companion paper [18], we will apply this machinery to the Cvetiĉ-Youm dyon.

It is appropriate now to outline the logic of the remainder of this paper.

We wish to exhibit and eventually count the effective low-energy degrees of
freedom for specified quantum numbers in a complicated field theory containing
many other real and fictitious (gauge) degrees of freedom. To do this it seems in-
evitable that we must cast the theory of interest – containing gravitational, dilaton,
gauge, and antisymmetric tensor fields – in canonical form. Issues involving the
choice of surface terms are particularly important for us. We find that general co-
variance and duality, important properties of the bulk theory, lead us to a definite
choice. As we formulate a dynamical theory based on fields entirely outside the
horizon the horizon appears formally as a spatial (null) boundary. Our complete
specification of surface terms guides us toward appropriate boundary conditions at
such a boundary.

Next we construct a reduced Hamiltonian, that depends only on the parameters
of the classical solution – the hair variables. There is a general procedure for
extracting such a reduced Hamiltonian, which we will discuss. In our context,
the result assumes a very characteristic form. The bulk Hamiltonian vanishes as
a consequence of reparametrization invariance, which imposes constraints. The
reduced Hamiltonian therefore consists entirely of surface terms and, since the
surface terms at infinity are of a very simple form, the dynamics is in this concrete sense localized to the horizon. However in solving the constraints one finds that some of the surface variables, when expressed in terms of the hair variables, involve integral expressions (over a fixed spatial profile), so that in another sense the dynamics extends outside. The form of the outer fields is quite restricted. After integrating it out one reaches, in the case of the Cvetič-Youm dyon, an effective string theory for the classical hair.

As a by-product of our development we obtain in a canonical fashion formal expressions for the entropy that were previously deduced using Euclidean methods, which appears to us to be a conceptual advantage.

2. Canonical Formalism for Dilaton Gravity

In this section, we consider canonical treatment of the bosonic part of heterotic string theory compactified to $D$ dimensions on a torus with constant moduli. The Lagrangian density is

$$16\pi G_N \mathcal{L} = \sqrt{-G} e^{-2\Phi} R^{(D)} + 4(\nabla \Phi)^2 - \frac{1}{12} H^2 - \frac{1}{4} \alpha' F^{(i)} F^{(i)} + \partial_I V^I$$  \hspace{1cm} (2.1)

where

$$H_{IJK} = (\partial_I B_{JK} - \frac{1}{2} \alpha' A_I^{(i)} F_{JK}^{(i)}) + \text{(cyclic permutations)}$$ \hspace{1cm} (2.2)

$$F_{IK}^{(i)} = \partial_I A_K^{(i)} - \partial_K A_I^{(i)}$$

The total derivative

$$\partial_I V^I = 4 \frac{D - 1}{D - 2} \partial_I (\sqrt{-G} e^{-2\Phi} \partial^I \Phi)$$  \hspace{1cm} (2.3)

will be discussed in the following chapter.

* Index conventions are $I, J, \cdots = 0, \cdots, D - 1$ and $\alpha, \beta, \cdots = 1, \cdots, D - 1$ for the spacetime indices and $i = 1, \cdots, 16$ for the internal ones. In the following $\alpha' = 1$. 
The linchpin of a canonical formalism is division of the metric into spatial and temporal parts. We consider the explicit form

\[ dS^2 = -(Ndt)^2 + g_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) \quad (2.4) \]

General covariance implies that there is great arbitrariness in the choice of lapse and shift functions \( N \) and \( N^\alpha \).

One finds the field momenta by varying \( 16\pi G_N L \) with respect to \( \partial_t g_{\alpha\beta}, \partial_t \Phi, \partial_t A_\alpha, \) and \( \partial_t B_{\alpha\beta} \). By expanding the Lagrangian (2.1), assuming the metric (2.4), and performing the variations we find

\[ \Pi_{\alpha\beta} = \frac{1}{2}g^{\frac{1}{2}}e^{-2\Phi}[g^{\alpha\beta}\text{Tr}K - K_{\alpha\beta}] + \frac{2}{N}g^{\alpha\beta}g^{\frac{1}{2}}e^{-2\Phi}(\partial_t \Phi - N^\gamma \partial_\gamma \Phi) \]

\[ \Pi^\Phi = -\frac{8}{N}g^{\frac{1}{2}}e^{-2\Phi}[\partial_t \Phi - N^\alpha \partial_\alpha \Phi] - 4g^{\frac{1}{2}}e^{-2\Phi}\text{Tr}K \]

\[ \Pi^{(A)}_{\alpha} \equiv \mathcal{E}_\alpha = \frac{1}{N}g^{\frac{1}{2}}e^{-2\Phi}[(F_{t\alpha} - N^\beta F_{\beta\alpha}) + \frac{1}{2}(H_{t\alpha\beta} - N^\gamma H_{\gamma\alpha\beta})A^\beta] \]

\[ \Pi^{(B)}_{\alpha\beta} \equiv \mathcal{E}_{\alpha\beta} = \frac{1}{2N}g^{\frac{1}{2}}e^{-2\Phi}[H_{t\alpha\beta} - N^\gamma H_{\gamma\alpha\beta}] \]

where the extrinsic curvature is

\[ K_{\alpha\beta} = \frac{1}{2N}[N_{\alpha|\beta} + N_{\alpha|\beta} - \partial_t g_{\alpha\beta}] \quad (2.6) \]

The stroke denotes covariant derivative with respect to the spatial metric. The detailed calculations leading to (2.5) and to the equations below are rather involved. A good strategy for these calculations, and some useful identities, are presented in [12].

The Hamiltonian is defined by the Legendre transform

\[ 16\pi G_N H = \Pi^{\alpha\beta}\partial_t g_{\alpha\beta} + \Pi^\Phi \partial_t \Phi + \mathcal{E}^{\alpha} \partial_t A_\alpha + \mathcal{E}^{\alpha\beta} \partial_t B_{\alpha\beta} - 16\pi G_N \mathcal{L} \]

\[ = N\mathcal{H} + N_\alpha \mathcal{H}^\alpha - A_t C - B_{t\alpha} C^\alpha + 16\pi G_N \tilde{H} \quad (2.7) \]

Let us explain the general form indicated in the last line. The gauge fields \( A_t \) and \( B_{t\alpha} \) have no associated kinetic terms, so they act as Lagrange multipliers. The
corresponding constraint equations generalize Gauss’ law. They are $C = C^\beta = 0$ where

\[
C = \partial_\alpha \mathcal{E}^\alpha - \frac{1}{2} \mathcal{E}^{\alpha\beta} F_{\alpha\beta}
\]

\[
C^\beta = 2 \partial_\alpha \mathcal{E}^{\alpha\beta}
\]  

\[(2.8)\]

The lapse and shift functions $N$ and $N^\alpha$ are metric analogues of these non-dynamical variables. They enforce the constraints $\mathcal{H} = \mathcal{H}_\alpha = 0$ where

\[
\mathcal{H} = g^{-\frac{1}{2}} e^{2\Phi} [\text{Tr} \Pi^2 + \frac{1}{2} \Pi^\Phi \text{Tr} \Pi + \frac{D - 2}{16} \Pi^2_\Phi] - g^{\frac{1}{2}} e^{-2\Phi} R^{(D-1)}
\]

\[
+ 4g^{\frac{1}{2}} e^{-2\Phi} g^\alpha\beta \partial_\alpha \Phi \partial_\beta \Phi - 4e^{-2\Phi} \partial_\beta (g^{\frac{1}{2}} \partial^\beta \Phi)
\]

\[
+ \frac{1}{2} g^{-\frac{1}{2}} e^{2\Phi} g_{\alpha\beta} (\mathcal{E}^\alpha + \mathcal{E}^{\alpha\gamma} A_\gamma)(\mathcal{E}^\beta + \mathcal{E}^{\beta\delta} A_\delta) + \frac{1}{4} g^{\frac{1}{2}} e^{-2\Phi} F_{\alpha\beta} F^{\alpha\beta} +
\]

\[
+ g^{-\frac{1}{2}} e^{2\Phi} \mathcal{E}_{\alpha\beta} \mathcal{E}^{\alpha\beta} + \frac{1}{12} g^{\frac{1}{2}} e^{-2\Phi} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}
\]  

\[(2.9)\]

and

\[
\mathcal{H}_\alpha = -2\Pi^\beta_{\alpha |\beta} + \Pi^\Phi \partial_\alpha \Phi + F_{\alpha\beta}(\mathcal{E}^\beta + \mathcal{E}^{\beta\gamma} A_\gamma) + H^{\beta\gamma} \mathcal{E}_{\beta\gamma}
\]

\[(2.10)\]

In each case the constraint is identified by varying with respect to the appropriate Lagrange multiplier.

Alternatively, the constraints can be viewed as a generators of continuous symmetries in the space of all field configurations. The symmetries are implemented by insisting that the generators act trivially. This explicit identification of symmetry generators is an important aspect of the canonical formalism. The Lagrangian formalism, in contrast, does not distinguish the constraints from other equations of motion.

The final term in the Hamiltonian, $\tilde{\mathcal{H}}$, differs qualitatively from all the preceding constraint terms. It is the total derivative

\[
16\pi G_N \tilde{\mathcal{H}} = \frac{2}{D - 2} \partial_t \text{Tr} \Pi + 2 \partial_\alpha (\Pi^\alpha \beta N_\beta - \frac{1}{D - 2} \text{Tr} \ Pi N^\alpha) +
\]

\[
+ 2 \partial_\alpha [N g^{\frac{1}{2}} e^{-2\Phi} g^{\alpha\beta} (\frac{1}{N} \partial_\beta N - \frac{2}{D - 2} \partial_\beta \Phi)] + \partial_\alpha (A_t \mathcal{E}^\alpha)
\]

\[+ 2 \partial_\alpha (B_{t\beta} \mathcal{E}^{\alpha\beta})
\]

\[(2.11)\]
This term includes both contributions from the explicit derivative term in the original Lagrangian and terms that follow from integrations by parts. It plays a crucial role in our considerations. To avoid any ambiguity let us emphasize that all terms have been retained.

The relations (2.5) are recovered by varying the Hamiltonian with respect to the momenta and simplifying the equations. This serves as an important check on the algebra. The remaining equations of motion are found by variation with respect to the fields. The resulting expressions are very lengthy and will not be displayed here (nor used below).

3. Duality and Surface Terms

The total derivative $\partial_I V^I$ in the Lagrangian (2.1) is not determined by classical macroscopic physics, because it does not affect the equations of motion. It must, however, be specified to define the action off shell, and can play a role in specifying the quantum theory. It is ordinarily fixed by choosing appropriate boundary conditions and requiring that the complete Hamiltonian is stationary under all variations which satisfy them [13, 16]. Here we proceed quite differently: we resolve the ambiguity by demanding invariance under a symmetry: duality. Under duality the metric and dilaton transform as

$$G_{IJ} \rightarrow e^{-2\gamma}G_{IJ} \quad \Phi \rightarrow -\Phi$$

(3.1)

where $\gamma = \frac{4}{D-2}$. Under this transformation

$$\sqrt{-G}e^{-2\phi}R^{(D)} \rightarrow \sqrt{-G}e^{-2\phi}R^{(D)} + 8\frac{D-1}{D-2}\partial_I(\sqrt{-G}e^{-2\phi}\partial^I \phi) .$$

(3.2)

The total derivative (2.3) was chosen such that it exactly cancels the inhomogeneous term in (3.2) , so that the graviton-dilaton portion of the Lagrangian is invariant. The corresponding condition in Einstein frame is the absence of an explicit surface term. In this manner the Lagrangian is specified uniquely by general covariance and duality, even off shell.
In the canonical formalism time and space are treated differently, and so general covariance is no longer manifest. Duality, on the other hand, can be expressed explicitly in terms of canonical variables. Indeed, the transformation

\[
\begin{align*}
g_{\alpha\beta} &\rightarrow e^{-2\gamma\Phi} g_{\alpha\beta} \\
\Pi^{\alpha\beta} &\rightarrow e^{2\gamma\Phi} \Pi^{\alpha\beta} \\
\Phi &\rightarrow -\Phi \\
\Pi^\Phi &\rightarrow -\Pi^\Phi - 2\gamma \text{Tr}\Pi \\
N &\rightarrow e^{-\gamma\Phi} N \\
N^\alpha &\rightarrow N^\alpha
\end{align*}
\]

leaves \( N\mathcal{H}, \mathcal{H}_\alpha, \) and \( \tilde{H} \) separately invariant \(^*\). It also preserves the Poisson brackets, so that it is a canonical transformation. As one consequence, the measure in classical phase space is invariant under duality.

Having determined the surface terms by duality it is meaningful to reverse the ordinary procedure, and use the variational principle to discover the appropriate boundary conditions. The equations of motion follow from the Hamiltonian after variation and integration by parts. The surface terms thus generated must cancel the variation of the explicit surface terms in order that the bulk equations truly represent the conditions for stationarity.

First consider the surface terms at the horizon. Let us choose an adapted coordinate system where the metric is of the form

\[
dS^2 = -(Ndt)^2 + d\rho^2 + \gamma_{mn}(dx^m + N^m dt)(dx^n + N^n dt) . \tag{3.4}
\]

Here \( \rho \) is the normal coordinate close to the horizon and the \( x^m \) are transverse coordinates. All black holes can be written in this form. For this metric the

\(^*\) We have not specified the transformation of the matter fields. In all cases known to us, useful duality transformations do not relate the graviton-dilaton to other sectors.
spatial curvature is

$$R^{(D-1)} = -2\gamma^{-\frac{1}{2}}\partial^2\gamma + \frac{1}{4}(\gamma_{mn}\partial\gamma_{mn})^2 - \gamma_{km}\partial\gamma_{mn}\gamma_{nl}\partial\gamma_{lk} + \cdots$$  \hspace{1cm} (3.5)$$

where $\gamma$ denotes the determinant of the transverse metric $\gamma_{mn}$ and the omitted terms contain no $\rho$-derivatives.

For simplicity, let us first consider the situation when $\Phi = 0$. Then the variation of the Hamiltonian is

$$\delta H = \partial_{\rho}[2N\partial\delta\gamma^\frac{1}{2} - \frac{1}{2}N\gamma^\frac{1}{2}(\gamma_{kl}\partial\gamma_{kl}\gamma_{mn} - \gamma_{mk}\partial\gamma_{kl}\gamma_{ln})\delta\gamma_{mn}] + \text{bulk term}$$ \hspace{1cm} (3.6)$$

The bulk term gives the equation of motion. The condition for the boundary term to vanish is complicated. This feature reflects that dynamics at an arbitrarily specified surface must be highly non-trivial. We are however – not coincidentally – interested in the exceptional case where $N = 0$ on the surface, when things greatly simplify. $N = 0$ is a very strong condition in Minkowski space. It implies that $t$ is null\(^\dagger\).

The boundary is generated by $t$, so when $t$ is null the boundary is, according to a theorem of Penrose, a future event horizon. Moreover, Hawking’s theorem states that on the solution

$$\frac{\partial\gamma^\frac{1}{2}}{\partial\lambda} \geq 0$$ \hspace{1cm} (3.7)$$

where $\lambda = Nt$ is the affine parameter. (Sufficient positive energy conditions are satisfied in the classical field theory we are considering.) For reversible processes this is an equality that amounts to the condition

$$\Pi^\rho_\rho = 0$$ \hspace{1cm} (3.8)$$

by the definition (2.5). This requirement is a consequence of the causal structure of spacetime and of general positivity requirements, rather than a separate dynamical principle.

\(^\dagger\) Up to reparametrizations. Shift functions have been suppressed to simplify notation but should be retained as a matter of principle.
In summary, the boundary conditions $N = 0$ gives an off–shell definition of a black hole in Minkowski space. There is an additional condition $\Pi^\rho_\rho = 0$ for reversible processes. The boundary conditions on the metric $\gamma_{mn}$ were free, so there was no need to take $\partial_\rho \gamma^\rho_\gamma = 0$. This is the condition for an apparent horizon at the boundary. It may be satisfied dynamically for a specific solution, but in general it is not. We find it reasonable that the boundary must be a future event horizon for a good Cauchy problem to be posed, while the existence of an apparent horizon is a dynamical question. The boundary conditions we have obtained reflect this.

The remaining gravitational equations of motion are found by varying the momenta. The surface terms in this case lead to the requirement that the shifts $N^\alpha$ at the horizon must be kept fixed as the momenta are varied. On the other hand it is crucial for our later purposes to note that no restriction is placed on their value.

Finally, a general value of the dilaton $\Phi$ should be restored. This can be accomplished by transforming to the Einstein frame

$$
\begin{align*}
\gamma_{\alpha\beta}^E &= e^{-\frac{1}{D-2} \Phi} \gamma_{\alpha\beta} \\
\gamma_{\alpha\beta}^E &= e^{-\frac{1}{D-2} \Phi} \gamma_{\alpha\beta}^N
\end{align*}
$$

In this frame the preceding equations are valid without amendment. The variational principle for the dilaton can be derived at fixed Einstein metric. The boundary condition becomes $\Phi$ fixed on the horizon. This is curious because a fixed value of $\Phi$ is not duality invariant unless it is $\Phi = 0$. Therefore duality must be violated either at the horizon or, preferably from the present point of view, at infinity.

Having discussed the surface terms at the horizon, we now turn our attention to the surface terms at infinity. Again, there is a unique term that follows from general covariance and duality. Unfortunately the ensuing boundary conditions are so restrictive that only flat space is a solution! This is analogous to a situation that
arises in gauge theories: since charged states are gauge non-singlets, to specify a charged state – which is a very physical concept – one must formally violate gauge invariance [14]. Concretely, it is necessary to supplement the surface terms at infinity with non-invariant terms added by hand. For example, the term

\[ H_\infty = \mathcal{M}_{\text{ADM}} \tag{3.10} \]

is appropriate to allow solutions with ADM mass \( \mathcal{M}_{\text{ADM}} \). The explicit expression for the ADM Hamiltonian can be found in the literature [12]. The central issue is readily illustrated by considering a free gauge field: the boundary term that follows from integration by parts prior to variation, and the one that follows from variation, differ by a factor two. For gravity the details are more complicated but again the ADM Hamiltonian (3.10) turns out to be twice the value that follows from the unamended surface Hamiltonian. The term at infinity must always be chosen to be finite, and inequivalent choices are related by Legendre transforms of the generating functional. In the context of black hole thermodynamics it is possible to transform from the microcanonical to the canonical ensemble, for example. It is natural to retain this freedom by not committing to a specific term at infinity.

The breaking of general covariance by the boundary at infinity is a well-known subtlety, not unrelated to Mach’s principle [14]. Poincare invariance is a subgroup of the full symmetry that can be restored [12]. For our work the most important point is that the term at infinity can be chosen to be independent of our classical hair, so that general covariance in the compactified space remains unbroken.

Since general covariance is broken by the boundary at infinity it is a logical possibility that it is also broken by the boundary at the horizon, as has been explored by Teitelboim [16]*. We are proposing the principle that at a more microscopic level this does not happen. We find it an attractive hypothesis – and, as will appear, one with very non-trivial consequences – that sources of macroscopic charges can be generally covariant in this sense.

* see also [15] for a recent alternative approach.
4. The Reduced Hamiltonian

The full Hamiltonian contains the equations of motion for all physical degrees of freedom, as well as many redundant variables. In discussing the low-energy dynamics we propose for black holes we are primarily interested in a very small subset, namely the variables describing the hair. The reduced problem is still rather complicated, because we must take into account how the other degrees of freedom affect the hair. Hamiltonian reduction is the appropriate formalism for this problem [12].

Abstractly, Hamiltonian reduction works as follows. Denote the variables pertaining to the hair \((\pi_a, \phi_a)\), and the remaining variables which describe the background by \((\pi_A, \phi_A)\). The equations of motion for the hair are the canonical equations

\[
\dot{\phi}_a = \frac{\delta H(\phi_a, \pi_a, \phi_A, \pi_A)}{\delta \pi_a}; \quad \dot{\pi}_a = -\frac{\delta H(\phi_a, \pi_a, \phi_A, \pi_A)}{\delta \phi_a} \tag{4.1}
\]

In these equations the background variables \((\pi_A, \phi_A)\) should be treated as independent variables and therefore kept fixed.

There is an alternative derivation of the same equations. Consider \(\phi_A\) to be given static functions and let the canonically conjugate variables \(\pi_A\) be solutions of the constraints given these \(\phi_A\). \(\pi_A\) will be a function of the prescribed functions \(\phi_A\) and of the hair variables. One has

\[
\frac{\delta H(\phi_a, \pi_a, \phi_A, \pi_A)}{\delta \pi_a} = \frac{\delta H(\phi_a, \pi_a, \phi_A, \pi_A)}{\delta \pi_a} \bigg|_{\pi_A} + \frac{\delta H(\phi_a, \pi_a, \phi_A, \pi_A)}{\delta \pi_A} \frac{\delta \pi_A}{\delta \pi_a} \tag{4.2}
\]

In the first term \(\pi_A\) is kept fixed, just as it was in (4.1). In the second term it is varied; however the canonical equations for the background fields equate this term to the time derivatives of \(\phi_A\), which were assumed to vanish. Thus the alternative

\[\dagger\] \(\delta\) denotes the variational derivative so here the symbol \(H\) denotes the spatial integral of the density given in the previous chapters. This should not cause any confusion.
way of performing the variation gives the same result as the original one. It is easy to show that this feature is shared by the variations with respect to the fields $\phi_a$.

In this way, we have identified a Hamiltonian that generates the correct equations of motion. It contains only the reduced (i.e. for us, hair) variables as dynamical degrees of freedom, and is known as the reduced Hamiltonian.

$\pi_A$ is chosen to be a solution of the constraints off-shell even when the hair coordinates and momenta do not satisfy the equation of motion. Since the constraints vanish off-shell, the reduced Hamiltonian like the full Hamiltonian simplifies to a the total derivative term $H_{\text{reduced}} = \tilde{H}$ where $\tilde{H}$ is the total derivative (2.11). This enormous reduction is a general feature of systems with general covariance.

The reduced Hamiltonian is a total derivative; thus formally it can be evaluated in terms of quantities living on the boundaries of spacetime. However the constraints are differential equations, so their integrals – and thereby the reduced Hamiltonian – can nevertheless contain information about all of space. Concretely in the present context, this means that the reduced Hamiltonian can include information about the radial profile of the hair.

However, because of the special form (1.3) of the hair, if it is specified at any radius, it will be uniquely determined throughout spacetime. This feature embodies, in our higher-dimensional context, the physics of the conventional no hair theorem. Thus the profile functions do not represent any independent dynamical variables and it is natural to regard the hair on the horizon as the proper reduced variables.

The reduced Hamiltonian can still be a non–local function of the transverse directions on the horizon. We expect that this non–locality is tightly restricted by the special properties of global horizons, but this point needs further investigation.

The effective surface Hamiltonian represents the low energy dynamics of space-time. There are equivalent representations that project on to any reasonable surface surrounding the black hole. This appears to implement something like
Susskind’s holographic principle in a natural manner [17]. Intricate interplay between constraints and apparent non-locality arises appears to be an inevitable consequence of reduced dynamics in a generally covariant system.

Before concluding this section we should mention the non–dynamical fields (e.g. the lapse and shift) that appear prominently in the metric, but appear to be quite secondary in the Hamiltonian framework. They do not enter the constraints explicitly, so they should not be specified in solving for the momenta. The equations of motion that follow from variation with respect to the background momenta were used crucially in the derivation of the reduced Hamiltonian. They require that the Lagrange multipliers are fixed during the variation, but left their values undetermined. The residual freedoms in the surface Lagrange multipliers implement symmetries of the effective surface theory.

5. Black Hole Statistical Mechanics

After these preparations, we are now prepared to discuss some specific features of the surface theory. For ease of reference we repeat the Hamiltonian (2.11)

\[ 16\pi G_N \hat{H} = \frac{2}{D-2} \partial_t \text{Tr} \Pi + 2 \partial_\alpha (\Pi^{\alpha\beta} N_\beta - \frac{1}{D-2} \text{Tr} \Pi N^\alpha) + \]
\[ + 2 \partial_\alpha [Ng^\frac{1}{2} e^{-2\Phi} g^{\alpha\beta} (\frac{1}{N} \partial_\beta N - \frac{2}{D-2} \partial_\beta \Phi)] + \partial_\alpha (A_\beta \mathcal{E}^\alpha) + 2 \partial_\alpha (B_\beta \mathcal{E}^{\alpha\beta}) \]

(5.1)

Each term should be considered, in the reduced description, as a function of the hair and the macroscopic quantum numbers that describe the background. The reduced Hamiltonian is the spatial integral of this expression.

The matter terms will not be considered here. For the rest, we find in the coordinates (3.4)

\[ 16\pi G_N (H_{\text{reduced}} - M_{\text{ADM}}) = -2 \int_{\rho=0} d^{D-2}x \left[ \Pi^\rho_\alpha N^\alpha + \frac{1}{2} \gamma^3 E_\kappa \right] . \]

(5.2)
Here

$$\kappa = \partial_\rho N^E$$  \hspace{1cm} \text{(5.3)}

is the surface acceleration in the Einstein frame, and we chose the surface term (3.10) at infinity for definiteness. According to the variational principle discussed in the previous chapter, $\kappa$ and $N^\alpha$ must be considered constants when varying other fields.

Each quantity in (5.2) is separately invariant under duality.

The lapse functions $N^\alpha$ are arbitrary and do not depend on the hair variables. This reflects reparametrization symmetry of the surface theory, as we have discussed. Variation with respect to $N^\alpha$ gives

$$\Pi^\rho_\alpha = 0 ; \ \alpha \neq \rho$$  \hspace{1cm} \text{(5.4)}

The momenta $\Pi^\rho_\alpha$ serve as generators of coordinate transformations in the surface theory, so these constraints are analogous to the constraints in the bulk theory. These equations take the form of matching conditions relating properties of the background to properties of the hair. For black hole solutions that derive from the fundamental string it agrees with the condition that can be derived by matching onto a source [8], or from cosmic censorship [7].

In the theory without hair the momenta $\Pi^\rho_\alpha$ with $\alpha = 4, \cdots, D - 1$ would be interpreted as a manifestation at the horizon of Kaluza–Klein charge at infinity. The hair modifies this, and our matching conditions relate the amplitudes of the hair to the background charges in such a way that the combined momenta vanish; this implements the ‘no-source’ boundary condition we discussed earlier.

The matching conditions have important consequences. First, they further reduce the number of independent variables. This is because they embody the requirement that hair variables that differ only by reparametrization are physically equivalent. Second and more profoundly, they delegitimize the original bald black
hole, which does not satisfy the constraint. In order that all constraints be satisfied, there must be hair.

Analogously, for a rotating black hole without hair the variables $\Pi_\theta^\rho$ and $\Pi_\phi^\rho$ are non-zero at the horizon, so the matching conditions (5.4) seem to rule out angular momentum. Actually what is excluded is pure rotation, which would violate general covariance at the boundary. The rotation should be matched with appropriate hair, so that the combined system is generally covariant. General covariance on the horizon appears to be so strong that, for black holes without hair, it excludes interesting physics and must be relaxed. In a complete theory including hair it appears it can be maintained, however. This provides an appealing raison d’être for the hair.

Now consider the second term in (5.2). In this paragraph we shall proceed schizophrenically, ignoring the presence of hair, in order to make contact with previous understanding. We have already met the surface acceleration $\kappa$ in (5.3). It is duality invariant and constant on the horizon, which suggests its independent physical significance. Let us introduce the formal temperature $\Theta = \frac{\kappa}{2\pi}$, which is of course also constant along the horizon. Then the integral over angular variables gives

$$H_{\text{reduced}} - M_{\text{ADM}} = -\frac{A_E}{4G_N} \Theta ; \quad A_E \equiv \int_{\rho=0} d^{D-2}x \frac{\gamma_E^i}{\gamma_E}$$

(5.5)

In this expression the area can be considered either the four dimensional area of the black hole or the higher dimensional one. They differ by a factor of the volume of the internal space, which should be absorbed in $G_N$ under dimensional reduction. The reduced Hamiltonian is the generator of equations of motion with the temperature $\Theta$ kept fixed i.e. the free energy $F = E - \Theta S$. The appropriate surface term at infinity, which keeps the temperature fixed, is the ADM mass that was already included above. Hence we recover the celebrated Bekenstein–Hawking formula

$$S = \frac{A_E}{4G_N}$$

(5.6)
The present derivation was made at a given spatial section. Therefore no Euclideanization was necessary, and no conical angle was introduced. There were also no infinite terms in need of regularization and subtraction. These are attractive features of this derivation but sever al heuristic elements remain: $\Theta$ as measured using surface acceleration was used as a macroscopic variable without formal justification. Its normalization was not fixed by any consideration internal to the calculation; this can almost certainly be remedied by inserting an appropriate thermometer, \textit{i.e.} by carefully quantizing the modes of a model field and detector at infinity.

Most importantly, the derivation had no explicit reference to the microscopic degrees of freedom that gave rise to the entropy. The classical hair is proposed to remedy this. In fact, the Bekenstein–Hawking formula has a somewhat unusual interpretation in the present framework. The area is dynamical and \textit{a priori} it has no independent physical meaning. It is a function of the hair variables that only reduces to the area when the hair is disregarded. Indeed, in the Cveti\v{c}–Youm example the nine-dimensional area vanishes at the origin by virtue of the matching condition (but the three-dimensional spatial metric is unaffected.) We suspect that, when fully spelled out, (5.6) will be appear directly as a phase space integral over the microscopic degrees of freedom. At present we do not know how to do this in generality \textit{i.e.} without considering the explicit form of the constraint $\Pi_\alpha = 0$. We shall carry this program out further in [18].

Finally, consider the terms that are total derivatives with respect to compactified coordinates. Upon integrating the Hamiltonian over a spatial section such terms vanish when the fields are periodic functions of the coordinates. But potentials, as opposed to field strengths, need not be periodic in general, and the total derivative terms can encode topological information. Concretely, assume that there is a string’s worth of hair, as occurs for the example discussed in the Appendix. In fundamental string theory the worldsheet coordinate can map into spacetime with a non-trivial winding number. In the present spacetime approach there is no world sheet and no fundamental string. Nevertheless there may be winding. This
would manifest itself through non-vanishing contributions from the total derivative terms. It is intriguing that the same quantity \( \text{Tr} \Pi (\propto P_1 P_2 \text{ for the Cvetič-Youm dyon}) \) sets the dynamical scale both for the compactified dimension \( x_9 \) and time \( t \) – or eventually, inverse temperature.

6. Comments

We have already made several interpretational remarks in the text; we wish only to add two brief comments.

Although most of the formalism in the main text is valid more generally, our working example involves extremal black holes. It is quite plausible that the physics underlying formulation of appropriate matching conditions and counting of states is more complicated for non-extremal holes; and specifically, that their non-zero temperature plays a crucial role. Indeed, the spirit of the matching condition is that the specified charges at infinity require, for a solution which is in a strong sense source-free at the horizon, a definite non-zero amount of hair outside. It has long seemed a striking coincidence that the energy of the classic extremal Reissner-Nordström hole can be regarded as being entirely in electrostatic fields outside the horizon, according to \( M = Q^2/R \text{ for } Q = R = M \). Thus there is at least one rough sense in which there’s ‘nothing inside’, as we have proposed in a somewhat different form. For non-extremal holes the qualitative picture is much less clear; one might speculate that the matching condition reflects the necessity not only to make up the requisite charges but also to provide the appropriate thermal excitations in a self-consistent manner.

Recently there have been some truly remarkable developments in the study of black holes closely related to ones we consider here generally and to the Cvetič-Youm dyon in particular [24]. (In its details most of the work has focussed on five dimensional versions, for reasons that are technical and presumably temporary.) The main thrust of this work is to use D-brane technology to count BPS saturated states with certain quantum numbers in the weak-coupling limit, and then to argue
that it is valid to extrapolate this counting to strong coupling, when the states become black holes. The approach suggested here in no way contradicts these developments, but attempts to deal more directly with the space-time aspects of the problem. Particularly when the states under consideration are macroscopic black holes, a classical or semi-classical approach ought to be appropriate and convenient. We have argued that classical hair exists in abundance, and have emphasized its potential for addressing the classic problems of microstate counting and information storage.

Acknowledgements We are grateful to C. Teitelboim for important discussions.
REFERENCES

1. R. Wald, *General Relativity* University of Chicago press, Chicago (1984).

2. R. H. Price, Phys. Rev. **D5** (1972) 2419, 2439.

3. B. A. Campbell, N. Kaloper, and K. A. Olive, Phys. Lett. **B263** (1991), J. Ellis, N. Mavromatos, and D. V. Nanopoulos, Phys. Lett. **B294** 229 (1992).

4. S. Coleman, J. Preskill, and F. Wilczek, Nucl. Phys. **B378** 175 (1992).

5. A. Sen. Mod. Phys. Lett. **A10** 2081 (1995).

6. A. Dabholkar and J. Harvey, Phys. Rev. Lett. **63** (1989) 478; A. Dabholkar, G. Gibbons, J. Harvey, and F. Ruiz-Ruiz, Nucl. Phys. **B340** (1990) 33.

7. C. Callan, J. Maldacena, and A. Peet, hep-th/9510134.

8. A. Dabholkar, J. Gauntlett, J. Harvey, and D. Waldram, hep-th/9511053.

9. F. Larsen and F. Wilczek, hep-th/9511064.

10. M. Cvetič and A. Tseytlin, hep-th/9512031, A. Tseytlin, hep-th/9601177.

11. M. Cvetič and D. Youm, Phys. Rev. **D53** 584 (1996).

12. R. Arnowitt, S. Deser, and C. W. Misner, *Gravitation: An introduction to Current Research* (L. Witten, Ed.), Wiley, NY (1962); T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88** 286, A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems*, Acc. Nat. Dei Lincei (Roma 1976).

13. S. Carlip and C. Teitelboim, Class. Quant. Grav. **12** 1699 (1995).

14. L. F. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982); Phys. Lett. **B116**, 259 (1982).

15. A. Balachandran, L. Chandar, and A. Momen, hep-th/9512047.

16. C. Teitelboim, Phys. Rev. **D53** 2870 (1996).

17. L. Susskind, J. Math. Phys. **36** 6377 (1995).
18. F. Larsen and F. Wilczek, in preparation.

19. M. Cvetić and D. Youm, [hep-th/9512127].

20. M. Cvetić and A. Tseytlin, Phys. Lett. B366 95 (1996).

21. G. T. Horowitz and A. A. Tseytlin, Phys. Rev. D51 2896 (1995).

22. C. Callan, J. Harvey, and A. Strominger, Nucl. Phys. B367 (1991) 60.

23. G. Gibbons and M. Perry, Proc. R. Soc. A358 (1978) 467; G. Gibbons and S. Hawking, Comm. Math. Phys. 66 (1979) 291.

24. A. Strominger and C. Vafa, [hep-th 9601029], C. Callan and J. Maldacena, [hep-th/9602043], G. Horowitz and A. Strominger, [hep-th/9602051], J. Breckenridge, R. Myers, A. Peet, and C. Vafa, [hep-th/9602043], J. Breckenridge, D. Lowe, R. Myers, A. Peet, A. Strominger, and C. Vafa, [hep-th/9603078], C. Johnson, R. Khuri, and R. Myers, [hep-th/9603061], J. Maldacena and A. Strominger, [hep-th/9603060]. R. Dijkgraaf, E. Verlinde, H. Verlinde, [hep-th/9603120].
APPENDIX

A.1. The Cvetič–Youm dyon

In this appendix we explicitly construct hair on the Cvetič–Youm dyon [11]. This is a spherically symmetric four dimensional black hole solution to low energy heterotic string theory that has $N = 1$ supersymmetry. It is almost – but not quite – the most general solution with these properties [19]. It can be considered an exact conformal field theory [20]. The black hole exhibits remarkable features which strongly suggest that all its entropy can be accounted for by the mechanism pursued here [9].

The black hole is parametrized by 4 independent charges. The line element is, explicitly,

$$dS^2 = F du (dv + K du) + G_{ij} dx^i dx^j$$

$$G_{ij} dx^i dx^j = f[k(dx(4) + P(1)(1 - \cos \theta)d\phi)^2 + k^{-1}(dr^2 + r^2(d\theta^2 + \sin \theta^2 d\phi^2))]$$

$$+ \sum_{i,j=5}^{8} \delta_{ij} dx^i dx^j$$

in string metric. Here $u = x^{(9)} - t, v = x^{(9)} + t$, and

$$F^{-1} = 1 + \frac{Q(2)}{r} ; \quad K = \frac{Q(1)}{r} ; \quad f = 1 + \frac{P(2)}{r} ; \quad k^{-1} = 1 + \frac{P(1)}{r} \quad (A.2)$$

The other nonvanishing fields are

$$B_{uv} = F ; \quad B_{\phi 4} = P(2)(1 - \cos \theta) ; \quad e^\Phi = F f \quad (A.3)$$

We will also use the notation $\Phi|| = \ln F$ and $\Phi_\perp = \ln f$. Each compact dimension gives rise to two $U(1)$ gauge fields: one from the metric, and one from the antisymmetric tensor field. In the solution above, the $U(1)$’s from the 9th dimension are assigned electric charges, those from the 4th dimension are assigned magnetic
ones, and the remaining sectors are neutral. The standard extremal Reissner–
Nordström black hole is realized as the special case $Q^{(1)} = Q^{(2)} = P^{(1)} = P^{(2)}$, when the dilaton decouples. It appears, of course, in its Kaluza-Klein form.

The Cvetič–Youm dyon combines many of the known solutions of low energy string theory: $F \neq 1$ is characteristic of the fundamental string [6], $K \neq 0$ of the plane wave [21], and the combination of the two of the charged fundamental string [6, 7, 8]. $f \neq 1$ is the symmetric five–brane [22] and $k \neq 1$ is the self–dual taub-NUT gravitational instanton [23]. It is remarkable that all these solutions can coexist as they do here.

By generalizing the calculation in the appendix of [21], one can show that the equations of motion for the ansatz (A.1) reduce to a number of Laplace equations with the solutions (A.2). For example, one equation is

$$\frac{1}{(r + P^{(1)})(r + P^{(2)})} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} K = 0 \quad (A.4)$$

If there were no magnetic charge, then this equation would be an ordinary Laplace equation with the usual $\delta$–function singularity at the origin. The magnetic charges regulate the solution, rendering it perfectly regular from the four dimensional point of view. Physically this means there is no source at the horizon. Despite the curved transverse space, it is the flat space Laplacian that appears in (A.4) . This is a remarkable property of the metric (A.1) , which arises because

$$e^{-\Phi_\perp} \sqrt{G^r r}$$

is equal to its flat space value.
A.2. Classical Hair

The classical solution is characterised by the conserved charges at infinity. In the present context these are the $U(1)$ charges and the ADM mass. There is a relation between these conserved charges

$$4G_NM_{ADM} = Q^{(1)} + Q^{(2)} + P^{(1)} + P^{(2)} \quad (A.6)$$

One finds that this saturation property is equivalent to a residual supersymmetry. Because it does not modify the conserved charges at infinity, hair must therefore be consistent with residual supersymmetry. This severely restricts the possible form of hair for this example, and makes it practical to find explicit expressions.

The following construction is inspired by the analogous problem of oscillations of the fundamental string [7, 8]. Technically, the main step in carrying over the previous results lies in showing that the transverse magnetic space does not obstruct the oscillations of the classical string. This turns out to follow from the special property (A.5).

First we consider adding fundamental gauge fields $F^{(i)}_{MN}$ that are non–trivial. The corresponding hair will be referred to as gauge hair. Supersymmetry leads to the ansatz $F^{(i)rv} \neq 0$. The equations of motion for the gauge fields are

$$\tilde{\nabla}_N [e^{-\Phi} F^{(i)Nv}] = 0 \quad (A.7)$$

Here $\tilde{\nabla}$ is the covariant derivative formed using the generalized connection $\tilde{\Gamma}^L_{MN} = \Gamma^L_{MN} + \frac{1}{2} H^L_{MN}$. In a gauge where only $A_u \neq 0$ this reduces to

$$\partial_r (F^{-2} r^2 \partial_r A^{(i)}_u) = 0 \quad (A.8)$$

The calculation that shows this relies on the special property of (A.5). The solution is $A^{(f)}_u \propto F$ because $F^{-1}$ is a solution to Laplace’s equation in flat space. The

* Gauge fields that are self–dual in the transverse space provide another possibility consistent with supersymmetry. We do not consider it here.
important qualitative point is that the proportionality constant can be any function of \( u \). Thus the explicit form of the gauge hair is

\[
F_{ru}^{(i)} = \frac{1}{Q^{(2)}} \partial_r F \ q^{(i)}(u) = \frac{1}{(r + Q^{(2)})^2} q^{(i)}(u) \tag{A.9}
\]

The internal index \( i = 1, \cdots, 16 \) so this hair amounts to 16 chiral fields in 1 + 1 dimensions.

The gauge hair generates energy–momentum, which must be included as source–terms in the Einstein equations. Due to the special form of the hair, only one component is affected. This is (A.4), which is modified to read

\[
-\nabla^2 K = \frac{1}{(r + Q^{(2)})^4} \sum_{i=1}^{16} q^{(i)}(u)^2 \tag{A.10}
\]

Here \( \nabla^2 \) is the Laplacian in a flat transverse space so all terms that depend on the magnetic charges dropped out again. The equation is easily integrated to give

\[
K = \frac{Q^{(1)}}{r} - \frac{1}{2} \frac{1}{r(r + Q^{(2)})} \sum_{i=1}^{16} q^{(i)}(u)^2 \tag{A.11}
\]

The first term enters as a constant of integration that could depend on \( u \). It is determined by the condition that the charges at infinity are unchanged. With this modification of the metric, the black hole with hair is an exact solution. The matching condition derived in the bulk of the paper turns out to amount to the condition that the function \( K \) be regular at \( r = 0 \). It relates the amplitudes of the hair to the background charge. Specifically, it requires the presence of hair.

If there are several types of hair, it is their total that enters into the matching condition.

In the original solution (A.1) the fundamental gauge fields \( A^{(i)}_J \) were chosen to be zero, but solutions with non–zero fields are related to the one given by T–duality. The radial profile of the fields obtained this way is the same as that of the
hair, so the black hole with $u$-independent ‘hair’ is formally related by T–duality to the one without hair. Unlike such a solution, however, the true gauge hair depends on the $u$ coordinate. The macroscopic charge is the average over the compactified dimension of the microscopic charge and can not change due to hair. We must therefore insist that the gauge hair has no constant mode, but all other modes constitute acceptable hair.

We should also consider the possibility of hair that reside solely in the metric. For hair in the longitudinal part of the metric the most general ansatz, consistent with supersymmetry*, can be shown to be

$$dS^2 = F du (dv + K du + 2V_i dx^i) + G_{ij} dx^i dx^j$$  \hspace{1cm} (A.12)

Here $K$ and $V_i$ may depend on $u$, but not on $v$. The appropriate components of the Einstein equations are

$$\hat{\nabla}^i e^{-\phi_\perp} (\partial_i V_k - \partial_k V_i) = 0$$  \hspace{1cm} (A.13)

This equation allows each $V_k ; k = 1, \cdots, 8$ to be an arbitrary function of $u$. Following the steps of [8], it can be shown that this solution is the most general regular one. It does not modify any of the other Einstein equations, so it is an exact solution.

With a non–vanishing $V_i$ the metric is no longer asymptotically Minkowskian. For large dimensions $V_i$ corresponds to momentum, while for compactified ones it corresponds to electric charge. Since hair is not allowed to contribute to the macroscopic quantities, the constant mode is forbidden, but the higher modes constitute legitimate hair. As a slightly different treatment of this hair, consider a change to asymptotically Minkowskian coordinates $x^i \rightarrow x^i - \delta^{ij} V_j$. In the new coordinate system the horizon is no longer at $r = 0$, but has $u$–dependent origin. We see that

* The transverse metric might also allow non–trivial excitations, but we will not consider that here.
the hair corresponds to the breaking of translational invariance — it is “Goldstone hair”. In fact, the gauge hair can also be understood in this way, because the fundamental gauge fields are related to the breaking of translational invariance on the internal 16-dimensional torus. Taken together all the hair amounts to 24 arbitrary chiral functions that respect the periodicity of the 9th coordinate. There is a chiral bosonic string’s worth of hair.