Dispersive approach to QCD: $\tau$ lepton hadronic decay in vector and axial–vector channels

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Abstract

The dispersive approach to QCD, which extends the applicability range of perturbation theory towards the infrared domain, is developed. This approach properly accounts for the intrinsically nonperturbative constraints, which originate in the low–energy kinematic restrictions on pertinent strong interaction processes. The dispersive approach proves to be capable of describing OPAL (update 2012) and ALEPH (update 2014) experimental data on inclusive $\tau$ lepton hadronic decay in vector and axial–vector channels in a self–consistent way.

Keywords: nonperturbative methods, low–energy QCD, dispersion relations, $\tau$ lepton hadronic decay

Dispersion relations represent one of the sources of the nonperturbative information on the hadronic dynamics at low energies. In particular, the dispersion relations render the kinematic restrictions on the pertinent strong interaction processes. The dispersion relations for the functions on hand:

$$\Delta \Pi(q^2, q_0^2) = \Delta \Pi^{(0)}(q^2, q_0^2)$$

related $R(s)$ function

$$R(s) = \frac{1}{\pi} \text{Im} \lim_{\epsilon \to 0} \Pi(s + i\epsilon),$$

which is identified with the so–called $R$–ratio of electron–positron annihilation into hadrons, and Adler function $\Pi^{(0)}$

$$D(Q^2) = \frac{d \Pi(-Q^2)}{d \ln Q^2},$$

with $Q^2 = -q^2 = -s > 0$ being the spacelike kinematic variable.

The functions (1)–(3) play a crucial role in decisive self–consistency tests of Quantum Chromodynamics (QCD) and the entire Standard Model, that, in turn, puts strict restrictions on possible New Physics beyond the latter. In particular, the theoretical description of a number of the strong interaction processes, as well as of the hadronic contributions to precise electroweak observables is inherently based on these functions.

The aforementioned nonperturbative constraints are properly accounted for within dispersive approach to QCD (2) (its preliminary formulation was discussed in Ref. [4]), which provides unified integral representations for the functions on hand:

$$\Delta \Pi(q^2, q_0^2) = \Delta \Pi^{(0)}(q^2, q_0^2)$$

$$+ \int_{m^2}^{\infty} \rho(\sigma) \ln \left(\frac{\sigma - q^2 m^2 - q_0^2}{\sigma - q^2 m^2 + q_0^2}\right) \frac{d \sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{m^2}^{\infty} \rho(\sigma) \frac{d \sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2)$$

$$+ \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d \sigma}{\sigma}.$$

In these equations $\Delta \Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$, $m$ is the total mass of the pertinent lightest allowed hadronic final state, and $\theta(x)$ denotes the unit step–function [$\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise]. The leading–order
terms in Eqs. (4)–(6) read [7, 8]:

\[ \Delta \Pi^{(0)}(q^2, q_0^2) = 2 \frac{q^2 - \tan \varphi}{\tan^3 \varphi} - 2 \frac{\varphi_0 - \tan \varphi_0}{\tan^3 \varphi_0}, \] (7)

\[ R^{(0)}(s) = \theta(s - m^2) \left(1 - \frac{m^2 s^{3/2}}{s}\right), \] (8)

\[ D^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left[1 - \sqrt{1 + \xi^2 \sinh^{-2}(\xi^2/2)}\right]. \] (9)

where \( \rho(\sigma) \) denotes the spectral density

\[ \rho(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \text{Im} \lim_{\epsilon \to 0^+} \rho(\sigma - i \epsilon). \] (10)

Here \( \sin^2 \varphi = q^2/m^2 \), \( \sin^2 \varphi_0 = q_0^2/m^2 \), \( \xi = Q^2/m^2 \), and \( \rho(q^2), r(s), d(Q^2) \) stand for the strongly corrected functions to functions (1), (2), (3), respectively (see Refs. [2, 3] for the details).

It is worth mentioning that the derivation of representations (4)–(6) involves no phenomenological assumptions. The Adler function [6] agrees with corresponding experimental prediction in the entire energy range [3, 9] (the studies of \( D(Q^2) \)) can also be found in Refs. [10–17]), the functions (4)–(6) comply with the results obtained in Refs. [18–19], and the hadronic vacuum polarization function [4] conforms with relevant lattice simulation data [20, 21].

The unambiguous method to restore the complete expression for the spectral density \( \rho(\sigma) \) (10) is still far from being feasible (for a discussion of this issue see, e.g., Refs. [22, 23]). Nonetheless, the perturbative contribution to \( \rho(\sigma) \) can be calculated by making use of the perturbative expression for either of the strong corrections to the functions on hand (see, e.g., Ref. [24])

\[ \rho_{\text{pert}}(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \lim_{\epsilon \to 0^+} \rho_{\text{pert}}(\sigma - i \epsilon) \]

\[ = -\frac{d \rho_{\text{pert}}(\sigma)}{d \ln \sigma} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} d_{\text{pert}}(-\sigma - i \epsilon). \] (11)

In this paper the model [2] for the spectral density will be employed:

\[ \rho(\sigma) = \frac{4}{\beta_0} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} + \frac{\Lambda^2}{\sigma}. \] (12)

The first term on the right–hand side of Eq. (12) is the one–loop perturbative contribution, whereas the second term represents intrinsically nonperturbative part of the spectral density, see paper [2].

It is worthwhile to note that in the massless limit \( m = 0 \) for the case of perturbative spectral function \( \rho(\sigma) = \text{Im} \lim_{\epsilon \to 0^+} d_{\text{pert}}(-\sigma - i \epsilon)/\pi \) two representations (5) and (6) become identical to those of the so–called analytic perturbation theory [25] (see also Refs. [26–33]). However, it is essential to keep the value of the hadronic production threshold \( m \) nonvanishing, since the massless limit loses some of the nonperturbative constraints, which relevant dispersion relations impose on the functions on hand, see paper [2] and references therein for the details.

The dispersive approach has been successfully applied to the study of the inclusive \( \tau \) lepton hadronic decay in vector (V) and axial–vector (A) channels. Specifically, the theoretical expression for the pertinent experimentally measurable quantity reads

\[ R^{\text{J} = 1}_{\tau, V} = \frac{N_c}{2} |V_{ud}|^2 S_{\text{ew}} \left( \Delta^{\text{V/A}}_{\text{QCD}} + \delta_{\text{ew}}^V \right), \] (13)

where \( N_c = 3 \) is the number of colors, \( |V_{ud}| = 0.97425 \pm 0.00022 \) is Cabibbo–Kobayashi–Maskawa matrix element [34], \( S_{\text{ew}} = 1.0194 \pm 0.0050 \) and \( \delta_{\text{ew}}^V = 0.0010 \) stand for the electroweak corrections [35], and

\[ \Delta^{\text{V/A}}_{\text{QCD}} = \frac{2}{\pi} \int_{q^2_{\text{min}}}^{M^2_{\tau}} \left(1 - \frac{s}{M^2_{\tau}}\right) \left(1 + 2 \frac{s}{M^2_{\tau}}\right) \times \text{Im} \Pi^{\text{V/A}}(s + i0^+, d\sigma/M^2_{\tau}) \] (14)

denotes the hadronic contribution, see Ref. [36]. In Eq. (14) \( M_{\tau} \approx 1.777 \text{ GeV} [34] \) is the mass of \( \tau \) lepton and \( m_{\text{QCD}} \) stands for the total mass of the lightest allowed hadronic decay mode of \( \tau \) lepton in the corresponding channel.

Table 1: Values of the QCD scale parameter \( \Lambda \) (MeV) obtained within perturbative and dispersive approaches from updated OPAL [5] and ALEPH [6] experimental data on inclusive \( \tau \) lepton hadronic decay (one–loop level, \( n_f = 3 \) active flavors), see also Ref. [4].

| Perturbative approach | Dispersive approach |
|-----------------------|---------------------|
| OPAL [5] (update 2012) | ALEPH [6] (update 2014) |
| OPAL [5] (update 2012) | ALEPH [6] (update 2014) |
| Vector channel | 445(204)_{-230}^{+210} | 439(110)_{-119}^{+10} | 409 \pm 53 | 409 \pm 28 |
| Axial–vector channel | no solution | 409 \pm 61 | 419 \pm 33 |
It is worth noting that the description of the inclusive \( \tau \) lepton hadronic decay within perturbative approach completely leaves out the effects due to the nonvanishing hadronic production threshold. Additionally, this approach suffers from its inherent difficulties, such as the infrared unphysical singularities. These facts eventually lead to the identity of the perturbative predictions for functions (13) in vector and axial–vector channels \( (\Delta_{\rm QCD}^V = \Delta_{\rm QCD}^A) \) that contradicts experimental data \([5, 6] \) and the failure of the perturbative approach to describe the experimental data on the inclusive semileptonic branching ratio in axial–vector channel, see Table 1.

In the framework of the dispersive approach the hadronic contribution (14) to the inclusive semileptonic branching ratio can be represented in the following form:

\[
\Delta_{\rm QCD}^V = 3 g_1 \left( \frac{X_{1/2}}{2} \right) \sqrt{1 - X_{1/2}} \\
-3 g_2 \left( \frac{X_{1/2}}{4} \right) \ln \left( \sqrt{X_{1/2}} + \sqrt{X_{1/2} - 1} \right) \\
+ \int_{m_{\tau}^2}^{\infty} \frac{G \left( \frac{\sigma}{M_\tau^2} \right) \rho(\sigma) d\sigma}{\sigma},
\]

(15)

where \( G(x) = g(x) \theta(1-x) + g(1) \theta(x-1) - g(x_{1/2}) \), \( g(x) = x(2 - 2x^2 + x^3) \), \( X_{1/2} = m_{\tau}^2 / M_\tau^2 \), \( m_{\tau}^2 \approx 0.075 \text{ GeV}^2 \), \( m_{\tau}^2 \approx 0.288 \text{ GeV}^2 \), spectral density \( \rho(\sigma) \) is specified in Eq. (12), and

\[
g_1(x) = \frac{1}{3} + 4x - \frac{5}{6} x^2 + \frac{1}{2} x^3,
\]

(16)

\[
g_2(x) = 8x(1 + 2x^2 - 2x^3),
\]

(17)
see papers [2, 23, 39] and references therein. The juxtaposition of the obtained result [15] with recently updated OPAL [5] and ALEPH [6] experimental data is presented in Fig. 1 and the corresponding values of the QCD scale parameter $\Lambda$ are listed in Table 1. As one can infer from Fig. 1 the dispersive approach is capable of describing the experimental data [5, 6] on inclusive $\tau$ lepton hadronic decay in vector and axial–vector channels. The obtained values of the QCD scale parameter $\Lambda$ appear to be nearly identical in both channels, that testifies to the self–consistency of the developed approach.

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