Kinky D-Strings

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Abstract

We study two-dimensional SQED viewed as the world-volume theory of a D-string in the presence of D5-branes with non-zero background fields that induce attractive forces between the branes. In various approximations, the low-energy dynamics is given by a hyperKähler, or hyperKähler with torsion, massive sigma-model. We demonstrate the existence of kink solutions corresponding to the string interpolating between different D5-branes. Bound states of the D-string with fundamental strings are identified with Q-kinks which, in turn, are identified with dyonic instanton strings on the D5-brane world-volume.
1 Introduction

Supersymmetric gauge theories with eight supercharges generically possess a classical moduli space of vacua. Moreover non-renormalisation theorems prohibit the dynamical generation of a potential on this space by either perturbative or non-perturbative effects and the moduli space survives in the full quantum theory, albeit possibly differing from the classical space in its metric and singularity structure. Indeed, the existence of such quantum moduli spaces has been of paramount importance in determining many properties of the low-energy dynamics of theories with eight supercharges in two, three and four dimensions.

However there are situations where, despite the existence of eight supercharges, the classical theory has only isolated vacua. Part of the motivation of the present paper is to investigate to what extent the low-energy dynamics of these theories can be described in terms of a potential on a quantum vacuum moduli space. We consider the simplest such model: two dimensional \( \mathcal{N} = (4, 4) \) SQED. As will be reviewed in section 2, the introduction of both Fayet-Iliopoulos (FI) parameters and hypermultiplet masses leads to a situation with only isolated vacua and, in different approximations, the low-energy dynamics is described as a massive sigma-model on different branches of the vacuum moduli space.

Further motivation comes from string theory where there exist brane configurations preserving eight supercharges that again have only isolated vacua. One such situation in type IIB theory has been described recently by Bergshoeff and Townsend [1]. These authors consider a \((1,1)\)-string (i.e. a bound state of a D-string with a fundamental (F-)string) lying parallel to \( k \) separated D5-branes. As is well known, the D1-D5 system preserves \( 1/4 \) of the spacetime supersymmetry, implying no force between a D-string and D5-brane. However, the F-string, and therefore the \((1,1)\)-string under consideration, is attracted to the D5-branes. The result is a situation with \( k \) isolated vacua corresponding to each of the possible \((1,1)\)-string/D5-brane bound states. Moreover, in each of these vacua eight supercharges are again preserved. Bergshoeff and Townsend further showed that there should exist stable, BPS, kink configurations in which the \((1,1)\)-string interpolates between two D5-branes as shown in figure 1. These solutions were identified with the T-duals of Q-kinks [2, 3].

There exist related scenarios which capture the same physics. For instance, consider the two-dimensional \( U(1) \) world-volume gauge theory of a D-string. The \((1,1)\)-string bound-state corresponds to the introduction of a single quantised unit of field strength [4]. From this perspective it is the non-zero world-volume electric field which leads to an attraction between the string and D5-branes. However one need not consider a full quantum of electric field. It was shown many years ago that the appearance
of an unquantised, constant, electric field in two-dimensions can be interpreted as the addition of a $\theta$-angle to the Lagrangian \([3]\). This in turn corresponds to the D1-D5 system in the presence of a constant background Ramond-Ramond (RR) scalar field, $C_0$, as can be seen by considering the Wess-Zumino terms which couple a flat Dp-brane to the bulk RR potential, $C = \sum_n C_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$

$$S_{W.Z.} = \int_{\mathbb{R}^{p+1}} C \wedge e^F. \tag{1}$$

Here $F = F - B/2\pi \alpha'$ where $F$ is the gauge field strength, $B$ is the pull back of the bulk NS-NS two-form potential, and $\alpha'$ is the inverse string tension. This situation will prove somewhat easier to discuss from the gauge theory point of view and we will show in section 3 that it does indeed lead to an attractive force between the string and D5-branes as claimed.

In fact there is a second way to attract a D-string and D5-brane: one may turn on a constant magnetic field on the D5-brane world-volume in directions orthogonal to the string. Let the D-string have world-volume $x^0, x^1$ and the D5-brane have world-volume $x^0, \cdots, x^5$. We will ultimately be interested in turning on a constant self-dual field strength $F_{ij}$, $i, j = 2, 3, 4, 5$, on the D5-brane (see next section) and vanishing field strength on the D-string. In order to see how this leads to a force between the string and five-brane, consider the bosonic terms in the action for an F-string
stretched between them,
\[ S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ \eta^{\alpha\beta} \eta_{\mu\nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \right\} + \int d\tau A_{\mu} \frac{dX^{\mu}}{d\tau} , \]  
(2)

where \( \sigma \in [0, \pi] \) is the spatial coordinate of the open string and the second term is evaluated at the boundary consisting of the two end points on the D-string (\( \sigma = 0 \)) and the D5-brane (\( \sigma = \pi \)). The standard free field equations of motion are obtained from varying the action (2). However the resulting boundary term for the bosons is modified to
\[ \frac{1}{2\pi\alpha'} \int d\tau \delta X^{\mu} \left( \partial_{\sigma} X^{\nu} + 2\pi\alpha' F^{\nu}_{\rho} \partial_{\tau} X^{\rho} \right) \eta_{\mu\nu} . \]
(3)

Thus while Dirichlet (D) boundary conditions (\( \delta X^{\mu} = 0 \)) are consistent, we no longer have pure Neumann (N) boundary conditions but rather
\[ \partial_{\sigma} X^{\mu} + 2\pi\alpha' F^{\mu}_{\rho} \partial_{\tau} X^{\rho} = 0 . \]
(4)

For our current situation, only fields with DN boundary conditions are affected, namely \( X^2, X^3, X^4, X^5 \) and their fermionic partners. It is a straightforward exercise to see that the modings of the fields are now shifted from the standard half-integer moding for the bosons (and the corresponding integer or half-integer moding for the fermions in the NS and R sectors respectively). We now find two bosonic fields with moding \(-\lambda + n\) and two bosonic fields with moding \(\lambda + n\). Here \( n \in \mathbb{Z} \) and

\[ \tan^2(\lambda\pi) = \frac{1}{\pi^2\alpha'^2 F^2} , \]
(5)

where \( 0 < \lambda \leq 1/2 \). Similarly the fermions in the NS and R sectors are shifted from these modings by \(1/2\) and \(0\) respectively. In particular, this leads to a tachyonic NS ground state from the DN sector, and the one loop open string amplitude for the potential between the D-branes [7] no longer enjoys the cancellation between the NS and R sectors arising from Jacobi’s abstruse identity, thus leading to an attractive force between the branes. In fact, this calculation is essentially the same as for the force between moving D-branes performed in [8]. Note that if the same field strength is introduced at both ends of an open string, then the moding is not altered and the various forces cancel. So in particular there is still no force between the D5-branes.

In summary, a force between the D-string and D5-branes may be generated by turning on either of the constant background fields, \( C_0 \) or \( B_{ij} \). As will be reviewed in the following section, both of these spacetime background fields have a simple interpretation as parameters of the D-string world-volume theory. In the remainder of the paper, we will examine the physics of this system. The following section reviews
the $\mathcal{N} = (4, 4)$ $U(1)$ gauge theory that describes the low-energy dynamics of the D-string. For certain parameters, the theory has only isolated vacua corresponding to D-string/D5-brane bound states described above. We further demonstrate the existence of BPS soliton solutions of the classical equations of motion, although unfortunately we are unable to solve the Bogomol’nyi equation in general for this case. In the remaining two sections we consider two different approximations in which the low-energy dynamics reduces to a massive supersymmetric sigma-model on the Coulomb and Higgs branches respectively. The former description is unfortunately rather sick as the D-string is forced down the throat of the five-brane metric where the approximation breaks down and the physics is badly understood. Nevertheless, we are able to solve for the kinky D-string solutions in this case. The Higgs branch description is better behaved. Moreover, in this approximation we find a three-way identification between D-string/F-string bound states, Q-kinks [2] and dyonic instanton strings [9]. We also give a T-dualised description of the Higgs branch where the Q-kink momenta are exchanged in favour of winding modes [3].

2 The Model

We will be interested in the limit of infinite Planck mass to ensure the suppression of the kinetic terms for the bulk closed string fields. In addition, the limit of vanishing string length, $\alpha' \to 0$, allows us to ignore the higher order Born-Infeld interactions, and the D-brane dynamics reduces to a gauge theory. The configuration of D-string and D5-branes described in the introduction breaks ten-dimensional Lorentz invariance to,

$$Spin(1, 9) \to Spin(1, 1) \times Spin(4)_R \times SU(2)_R ,$$

where $Spin(1, 1)$ is the Lorentz group of the two-dimensional world-volume theory of the D-string, $Spin(4)_R$ describes the unbroken rotation group in $x^6, x^7, x^8, x^9$, transverse to the D5-branes, and $SU(2)_R$ describes self-dual rotations in the remaining directions tangent to the D5-brane, $x^2, x^3, x^4, x^5$. The full $Spin(4)$ symmetry rotating these directions is not realised due to the orientation of the D-branes; had we considered anti-D5-branes, then the anti-self-dual rotations would have been realised. The D1-D5 system breaks 1/4 of the spacetime supersymmetries, resulting in a $\mathcal{N} = (4, 4)$ theory in two dimensions.

The effective action for the D-string is determined by quantization of open strings with ends terminating on the D-string. Those that have both end points on the D-string yield a $\mathcal{N} = (4, 4)$ vector multiplet, also known as a twisted multiplet, and a neutral $\mathcal{N} = (4, 4)$ hypermultiplet. Two complex scalars in the latter parametrise the position of the D-string in the $x^2, x^3, x^4, x^5$ plane. For a single D-string, these decouple
and we shall ignore them for the remainder of the paper. The vector multiplet contains two further complex, neutral, scalars, $\sigma$ and $\phi$, parametrising the position of the D-string in the directions $x^6, x^7, x^8, x^9$ transverse to the D5-branes. The superpartners of these scalars are a two-dimensional gauge potential, $A_\mu$, together with two Dirac fermions, $\lambda$ and $\chi$, which are uncharged under the gauge group. The vector multiplet may be decomposed into an $N = (2, 2)$ gauge multiplet, $V$, and chiral multiplet, $\Phi$, with

$$\{A_\mu, \sigma, \lambda, D\} \in V \quad \text{and} \quad \{\phi, \chi, F\} \in \Phi,$$

where $D$ and $F$ are the usual real and complex auxiliary fields respectively. The field strength of $V$ is an $N = (2, 2)$ twisted chiral multiplet, $\Sigma = \bar{D} + D - V$, which has complex auxiliary field $D - iF_{01}$, where $F_{01}$ is the $U(1)$ field strength. Detailed conventions of $N = (2, 2)$ multiplets may be found in [10].

The presence of the D5-branes means that we must also consider open strings with one end point on the D-string and the other on one of the $k$ D5-branes. These give rise to $k$ charged hypermultiplets, with the gauge coupling constant given by $e^2 = g_s/\alpha'$. Each of these hypermultiplets is composed of two $N = (2, 2)$ chiral multiplets, $Q_i$ and $\tilde{Q}_i$, $i = 1, \cdots k$, each containing a complex scalar $q_i (\tilde{q}_i)$, a Dirac fermion, $\psi_i (\tilde{\psi}_i)$ and a complex auxiliary field $F_i (\tilde{F}_i)$. All fields in $Q_i$ transform with charge $+1$ under the $U(1)$ gauge group, while those in $\tilde{Q}_i$ transform with charge $-1$.

The Lagrangian for $k$ massless hypermultiplets coupled to a $U(1)$ vector multiplet is given by $\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F$, where

$$\mathcal{L}_D = \int d^4\theta \left\{ \frac{1}{4e^2} \left( \Phi^4 - \Sigma^4 \right) + k \sum_{i=1}^{k} \left( \bar{Q}_i \exp(2V) Q_i + \tilde{Q}_i \exp(-2V) \tilde{Q}_i \right) \right\}, \quad (7)$$

and

$$\mathcal{L}_F = \int d^2\theta \left\{ \sqrt{\frac{2}{k}} \sum_{i=1}^{k} Q_i \Phi \tilde{Q}_i \right\} + \text{h.c.} \quad (8)$$

The theory has a $H = Spin(4)_R \times SU(2)_R \times SU(k)$ global symmetry group, where the first two terms in the product are R-symmetries, and the latter is the flavour symmetry. The vector multiplet scalars, $\sigma$ and $\phi$, transform in the $(4, 1, 1)$ of $H$ while the hypermultiplet scalars, $q_i$ and $\tilde{q}_i$, transform as $(1, 3 + 1, k)$.

There are further parameters that we may add to the Lagrangian. The existence of two complex mass parameters consistent with supersymmetry follows from the existence of the two complex scalars in the vector multiplet, each of which may induce
a mass term for a hypermultiplet by the Higgs mechanism. From the string picture, the total mass (bare plus Higgs) of a hypermultiplet is determined by the distance from the D-string to the D5-brane, and the resulting mass parameters transform as \((4, 1, \mathbf{k} \otimes \mathbf{k})\) under \(H\). The complex matrix \(m_{ij}\) appears in the Lagrangian as a hypermultiplet dependent vacuum expectation value (VEV) for \(\phi\) and is referred to simply as the complex mass,

\[
\mathcal{L}_m = \int d^2\theta \left\{ \sqrt{2} \sum_{i,j=1}^{k} m_{ij} Q_i \bar{Q}_j \right\} + \text{h.c.} \quad (9)
\]

We will work in a flavour basis in which the complex mass matrix is diagonal, \(m_{ij} = m_i \delta_{ij}\) (no sum over \(i\)) and with \(\sum_i m_i = 0\). The second mass parameter is equivalent to a hypermultiplet dependent VEV for \(\sigma\), and is known as the twisted mass. In the diagonal flavour basis, it may be incorporated in the above Lagrangian by gauging the Cartan sub-algebra of the \(SU(k)\) flavour symmetry in a \(\mathcal{N} = (2,2)\) invariant fashion, thus introducing \(k - 1\) new gauge superfields, \(V_i, i = 1, \ldots, k\), with \(\sum_i V_i = 0\), and with corresponding field strengths \(\Sigma_i\). The hypermultiplet kinetic terms of (7) are now given by the substitution,

\[ V \to V + V_i \quad (10) \]

and a Lagrange multiplier is employed to restrict the complex scalar field that resides within \(V_i\) to equal the twisted mass, denoted \(\hat{m}_i\),

\[
\mathcal{L}_{\text{L.M.}} = \int d^2\vartheta \left\{ \frac{i}{2} \Lambda_i (\Sigma_i - \hat{m}_i) \right\} + \text{h.c.} \quad (11)
\]

where the measure \(d^2\vartheta\) denotes integration over the twisted half of superspace. Each Lagrange multiplier, \(\Lambda_i\), is a twisted chiral superfield. These will play a prominent role in the T-duality of the Higgs branch discussed in section four. By construction, we have \(\sum_i \hat{m}_i = 0\).

Finally, two dimensional abelian gauge theories with eight supercharges also allow for the possibility of a dimensionless theta angle, \(\theta\), and three dimensionless FI parameters, \(\zeta = (\zeta_1, \zeta_2, \zeta_3)\). The former is a singlet under \(H\), and we have already discussed its interpretation in the string theory: it corresponds to turning on a constant background RR scalar, as seen in (1). The FI parameters transform as \((1, 3 + 1, 1)\) under \(H\). The FI parameters and theta-angle may be considered as vacuum expectation values of a background hypermultiplet. This fact, together with their transformation under \(H\), is sufficient to identify their ten-dimensional spacetime interpretation and they correspond to a constant, background, self-dual NS-NS two form potential in the directions \(x^2, x^3, x^4, x^5\) [11],

\[
\zeta_a \sim \eta_{ai} \mathcal{F}_{ij} \quad (12)
\]
where $\eta_a$ are the self-dual 't Hooft matrices. Both FI parameters and the theta angle may be incorporated in the Lagrangian as (twisted) F-terms,

$$L_F = \int d^2 \theta \ W(\Phi) + \int d^2 \vartheta \ W(\Sigma) + \text{h.c.} .$$

(13)

The superpotential $W(\Phi) = \hat{\tau} \Phi / 2$ and the twisted superpotential $W = i \tau \Sigma / 2$ depend upon the complexified combinations $\tau = i \zeta_3 + \theta / 2 \pi$ and $\hat{\tau} = \zeta_1 + i \zeta_2$. The effect of FI-parameters and theta angle on the D1/D5-system were also considered yesterday in a slightly different context [12].

We turn now to the vacuum moduli space of the theory. The classical potential energy, obtained by eliminating all auxiliary fields, is given by

$$U = \frac{e^2}{2} \left( \sum_{i=1}^{k} (|q_i|^2 - |\tilde{q}_i|^2) - \zeta_3 \right)^2 + \frac{e^2}{2} \left( \sum_{i=1}^{k} (q_i \tilde{q}_i^\dagger + \tilde{q}_i q_i) - \zeta_1 \right)^2 + \frac{e^2}{2} \left( i \sum_{i=1}^{k} (q_i \tilde{q}_i^\dagger - \tilde{q}_i q_i) - \zeta_2 \right)^2 + 2 \sum_{i=1}^{k} \left( |\phi + m_i|^2 + |\sigma + \hat{m}_i|^2 \right) \left( |q_i|^2 + |\tilde{q}_i|^2 \right),$$

(14)

The structure of the classical vacuum moduli space, $U = 0$, is dependent upon the values of the FI and mass parameters. We deal with each case in turn.

i) $m_i = \hat{m}_i = \zeta = 0$:
This case corresponds to zero background NS-NS two form flux and coincident D5-branes. There exist two branches of vacua: the Coulomb branch and the Higgs branch. The Coulomb branch has $q_i = \tilde{q}_i = 0$, while the VEVs of $\sigma$ and $\phi$ are unconstrained, reflecting the fact that the D-string may roam the $x^6, x^7, x^8, x^9$ directions transverse to the D5-branes unimpeded. The metric on this space is the five-brane metric [13] and will be reviewed in the following section. On the Higgs branch however, $\sigma = \phi = 0$ while $q_i$ and $\tilde{q}_i$ are constrained only by the first three terms in (14). These constraints coincide with the ADHM equations for a single $U(k)$ instanton, resulting in a hyperKähler quotient construction of a $4(k-1)$ dimensional space of vacua which coincides with the 1 instanton moduli space. In the string theory interpretation, the D-string is absorbed by the D5-branes, where it appears as a single $U(k)$ instanton [6], as is apparent from (4).

ii) $m_i \neq m_j$ or $\hat{m}_i \neq \hat{m}_j$ for $i \neq j$, and $\zeta = 0$:
This corresponds to the separation of the D5-branes. The Higgs branch is now lifted as a single D5-brane is unable to absorb a D-string (there are no finite action $U(1)$ instantons). In section 4 we shall quantify the lifting of this moduli space for the simplest example of $k = 2$. In fact, the lifting in more complicated cases, including
multiple D5-branes and multiple D-strings, has been well understood for many years from the perspective of instanton calculus with spontaneously broken gauge groups. See for example [14].

iii) $m_i = \hat{m}_i = 0$, and $\zeta \neq 0$:
The D5-branes remain coincident, but a non-zero constant background NS-NS two form flux is turned on (12). The Higgs branch remains and coincides with the moduli space of a single $U(k)$ instanton on non-commutative $R^4$ [13]. The Coulomb branch is lifted, reflecting the attraction between the D-string and D5-branes as expected from the discussion in the introduction. In the following section, we quantify the lifting of the Coulomb branch.

iv) $m_i \neq m_j$ or $\hat{m}_i \neq \hat{m}_j$ for $i \neq j$, and $\zeta \neq 0$:
The D5-branes are separated, the NS-NS two form flux is turned on, and the D-string has only $k$ isolated vacuum states given by

$$\phi = -m_i \quad \sigma = -\hat{m}_i \quad ; \quad \eta_i \sigma_a \eta_i^\dagger = \zeta_a \quad \text{no sum over } i \quad (15)$$

where $\sigma_a$ are the Pauli matrices and we have introduced the $SU(2)_R$ covariant vectors $\eta_i = (q_i, \tilde{q}_i^\dagger)$. We see from the first two equations above that each vacuum state occurs at the position of a D5-brane, corresponding to a D-string/D5-brane bound state.

The theta angle has, of course, played no role in the above discussion. In the following section we shall integrate out all hypermultiplets, after which the $U(1)$ field strength, $F_{01}$, will play the role of an auxiliary field and we shall find that $\theta$ lifts the Coulomb branch in the same fashion as the FI parameters.

Let us now restrict attention to the fourth scenario above where, as discussed in the introduction, we may expect to find soliton solutions interpolating between two of the vacua (14), corresponding to the eponymous kinky D-string. In order to simplify the equations, we consider the case $k = 2$, and make full use of the $SU(2)_R \times Spin(4)_R$ R-symmetry to set $\zeta = (0, 0, \zeta_3)$ (with $\zeta_3 > 0$) and $m_1 = m_2 = 0$, $\hat{m}_1 = -\hat{m}_2 = i\mu$ (for real $\mu$). It is clear that a full $SU(2)_R \times Spin(4)_R$ multiplet of BPS solitons must exist in the complete theory. Our search for solitons begins by requiring half of the $(4, 4)$ supersymmetry to be preserved. An additional simplification resulting from the above $SU(2)_R \times Spin(4)_R$ rotation is that now it is sufficient to search for solutions which preserve half of the $(2, 2)$ supersymmetry. In terms of $\mathcal{N} = (2, 2)$ superfields, these supersymmetry transformations take the form

$$\delta V = (\epsilon_+ Q_+ + \epsilon_- Q_-) V ,$$

$$\delta \Phi = (\epsilon_+ Q_+ + \epsilon_- Q_-) \Phi ,$$

8
\[
\delta Q^i = (\epsilon_+ Q_- + \epsilon_- Q_+) Q^i, \\
\delta \tilde{Q}^i = (\epsilon_+ Q_- + \epsilon_- Q_+) \tilde{Q}^i.
\]

(16)

The component expansions for these expressions can be found in [10]. For the case in hand we find that, setting the fermion fields to zero, supersymmetry is preserved if \( \epsilon_+ = \epsilon_- \), \( \eta_+ = \eta_- \) and the bosonic fields satisfy the first order equations

\[
\begin{align*}
\partial_x \sigma &= \frac{i}{\sqrt{2}} (D - i F_{01}) , \quad \partial_t \sigma = 0 , \\
\partial_x \phi &= \partial_t \phi = F = 0 , \\
D_x q^i &= \frac{i}{\sqrt{2}} (\sigma - \bar{\sigma} + 2 \hat{m}^i) q^i , \quad D_t q^i = -\frac{i}{\sqrt{2}} (\sigma + \bar{\sigma}) q^i , \\
D_x \tilde{q}^i &= -\frac{i}{\sqrt{2}} (\sigma - \bar{\sigma} + 2 \hat{m}^i) \tilde{q}^i , \quad D_t \tilde{q}^i = \frac{i}{\sqrt{2}} (\sigma + \bar{\sigma}) \tilde{q}^i ,
\end{align*}
\]

(17)

For \( \zeta_3 > 0 \), we see from (14) that \( \tilde{q}_i \) has vanishing VEV in both vacua and we may therefore trivially satisfy the last of these equations by \( \tilde{q}_i = 0 \). We also find that, although the Bogomol'nyi conditions (17) admit solutions with a background electric field \( F_{01} = \partial_x (\sigma + \bar{\sigma})/\sqrt{2} \), the equations of motion require \( \sigma = -\bar{\sigma} \). Therefore we set \( A_\mu = 0 \) and, after eliminating the auxiliary field \( D \), we find the remaining coupled Bogomol'nyi equations

\[
\begin{align*}
\partial_x \sigma &= -i e^2 \frac{\sqrt{\zeta_3}}{\sqrt{2}} \left( |q_1|^2 + |q_2|^2 - \zeta_3 \right) , \\
\partial_x q_1 &= i \sqrt{2} q_1 (\sigma + i \mu) , \\
\partial_x q_2 &= i \sqrt{2} q_2 (\sigma - i \mu) .
\end{align*}
\]

The solutions for the three functions \( \sigma, q_1, q_2 \) can now be written in terms of single function \( \varphi = \varphi(x - x_0) \)

\[
\begin{align*}
\sigma &= -i \mu \frac{d\varphi}{dx} , \\
q_1 &= \sqrt{\zeta_3} \exp \left( \sqrt{2} \mu [\varphi - (x - x_0)] \right) , \\
q_2 &= \sqrt{\zeta_3} \exp \left( i \omega \right) \exp \left( \sqrt{2} \mu [\varphi + (x - x_0)] \right) .
\end{align*}
\]

(18)

where \( \varphi(x) \) itself satisfies the differential equation

\[
\frac{d^2 \varphi}{dx^2} = \frac{e^2 \zeta_3}{\sqrt{2} \mu} \left[ \exp \left( 2 \sqrt{2} \mu (\varphi - x) \right) \right] + \exp \left( 2 \sqrt{2} \mu (\varphi + x) \right) - 1 .
\]

(19)
This indeed describes a soliton solution interpolating between the first and second vacua as \( x \) ranges from \(-\infty\) to \(+\infty\) provided \( \phi \) is assigned the boundary conditions

\[
\phi(x) \to \mp x \quad \text{as} \quad x \to \pm \infty
\]

Given these boundary conditions, there exists a unique solution for \( \phi \) and the soliton \((19)\) possesses two collective coordinates. The first, \( x_0 \), describes the centre of mass of the kink. The second, \( \omega \), has period \( 2\pi \) and describes the relative phase between the two vacua \( 1 \).

We have been unable to solve equation \((19)\) explicitly, although for the special case \( \mu^2 = e^2 \zeta_3 / 4 \) we find

\[
\phi(x) = -\frac{1}{\sqrt{2\mu}} \log(1 + e^{2\sqrt{2\mu}x}) + x.
\]

In general however, since the boundary conditions select a unique solution for \( \phi \), we expect to find that the soliton solution has only the two zero modes \( x_0 \) and \( \omega \). The low energy effective dynamics of the soliton are then described by \( N = 4 \) quantum mechanics with two bosonic fields.

Finally we consider the mass of the kink. The bosonic energy density for the fields \( \sigma, q_1 \) and \( q_2 \) is given by

\[
E = \frac{1}{e^2} |\partial_x \sigma - \frac{i}{\sqrt{2}} D|^2 + \sum_{i=1}^2 |\partial_i q^i - i\sqrt{2} q^i(\sigma + \hat{m}^i)|^2 + T, \quad (20)
\]

where

\[
T = \frac{i}{\sqrt{2} e^2} D \partial_x \bar{\sigma} + i\sqrt{2} \sum_{i=1}^2 q^i(\sigma + \hat{m}^i) \partial_x q^i + c.c. . \quad (21)
\]

Notice that the first two terms are each positive definite and attain zero when the Bogomol’nyi equations are satisfied. The mass, \( E \), of a Bogomol’nyi kink is therefore given by

\[
E = \int_{-\infty}^{\infty} T dx . \quad (22)
\]

Substituting in the form \((18)\) for the solutions we find that

\[
T = \sqrt{2} \zeta_3 \mu \left[ \left( \frac{d^2 \varphi}{dx^2} + 2\sqrt{2} \mu \left( \frac{d\varphi}{dx} + 1 \right)^2 \right) e^{2\sqrt{2}\mu(\varphi+x)} \right. \\
+ \left. \left( \frac{d^2 \varphi}{dx^2} + 2\sqrt{2} \mu \left( \frac{d\varphi}{dx} - 1 \right)^2 \right) e^{2\sqrt{2}\mu(\varphi-x)} - \frac{d^2 \varphi}{dx^2} \right],
\]

\[
= \frac{\zeta_3}{2} \frac{d^2}{dx^2} \left[ e^{2\sqrt{2}\mu(\varphi+x)} + e^{2\sqrt{2}\mu(\varphi-x)} + 2\sqrt{2} \mu \varphi \right], \quad (23)
\]

\(^1\)The overall phase may be set to zero by a gauge rotation.
and we find $T$ to be a total derivative, with the rest mass of the kink given by $E = 2\sqrt{2}\zeta\mu$. This expression has a simple $SU(2)_R \times Spin(4)_R$ invariant extension, namely

$$E = 2M|\zeta|$$

where $M = \frac{1}{\sqrt{2}}(|m_1 - m_2|^2 + |\hat{m}_1 - \hat{m}_2|^2)^{1/2}$ is the $Spin(4)_R$ invariant mass.

### 3 On the Coulomb Branch

In this and the following section, we consider the low-energy dynamics of the theory in different regions of the parameter space and discuss three further avatars of the kink solitons. One expects that at low-energies the physics is correctly described by a sigma-model on the classical vacuum moduli space. For $m_i = \hat{m}_i = \zeta = \theta = 0$ where we have both Coulomb and Higgs branches, consideration of the action of the R-symmetries on the scalars suggests that, despite strong coupling fluctuations, the Higgs and Coulomb branches decouple in the infra-red [16, 17]. Here we review the description of the Coulomb branch [17, 18] and describe its lifting by the theta angle and FI parameters.

We consider first the situation of arbitrary masses, but with $\zeta = 0$ and $\theta = 0$, ensuring the survival of the Coulomb branch. The classically massless superfields are the chiral field $\Phi$ and the twisted chiral field $\Sigma$. Up to two derivatives, the most general theory one can write down consistent with $\mathcal{N} = (2, 2)$ supersymmetry is

$$\mathcal{L} = \int d^4\theta \ K(\Phi, \Phi^\dagger, \Sigma, \Sigma^\dagger).$$

($25$)

$K$ is known as a generalised Kähler potential. In component form, the bosonic part of $(25)$ is given by a sigma-model with torsion

$$\mathcal{L}_{\text{bose}} = K_{\Phi^\dagger \Phi}(\partial_\mu \phi^\dagger \partial^\mu \phi - F^\dagger \Phi) - K_{\Sigma^\dagger \Sigma}(\partial_\mu \sigma^\dagger \partial^\mu \sigma - \frac{1}{2} D^2 - \frac{1}{2} F_{01}^2) + K_{\Phi \Sigma^\dagger}(\partial_\mu \phi^\dagger \partial_\nu \sigma)\epsilon^{\mu\nu} + K_{\Phi^\dagger \Sigma}(\partial_\mu \phi \partial_\nu \sigma^\dagger)\epsilon^{\mu\nu}.$$ 

($26$)

While $(25)$ is, by construction, invariant under $\mathcal{N} = (2, 2)$ supersymmetry, further restrictions on $K$ are required in order for the Lagrangian to respect the full $\mathcal{N} = (4, 4)$ algebra. If $K$ were a function of only chiral superfields, it is well known that it must give rise to a hyperKähler metric. If however, as in the present case, $K$ is a function of both chiral and twisted chiral superfields, the condition on $K$ is [19]

$$K_{\Phi^\dagger \Phi} = -K_{\Sigma \Sigma^\dagger}.$$ 

($27$)
The resulting metric is not hyper-Kähler but, rather, hyper-Kähler with torsion. The constraint (27), together with the requirement of $\text{Spin}(4)_R$ R-symmetry acting on the scalars $\sigma$ and $\phi$ is very restrictive and is sufficient to fix $K$ up to two constants, which are determined at tree level and one-loop [20, 18]. The resulting generalised Kähler potential is

$$K = \frac{1}{e^2} (\Phi^\dagger \Phi - \Sigma^\dagger \Sigma) + \sum_{i=1}^{k} \left\{ \log(\Phi + m_i) \log(\Phi^\dagger + m_i^\dagger) - \int_{X_i} \frac{dx}{x} \log(x + 1) \right\}, \quad (28)$$

where the limit of the integral is given by the ratio

$$X_i = \frac{(\Sigma + \hat{m}_i)(\Sigma^\dagger + \hat{m}_i^\dagger)}{(\Phi + m_i)(\Phi^\dagger + m_i^\dagger)}.$$

In the absence of a superpotential, the auxiliary fields, as well as the field strength, are set to zero by their equations of motion, and the bosonic action (26) has the target space metric and torsion of $k$ five-branes [13], with positions at $\sigma = -\hat{m}_i$ and $\phi = -m_i$, reflecting the fact that on the Coulomb branch the D-string probes the directions transverse to the D5-branes. The metric is given by

$$d^2s = H(\phi, \phi^\dagger, \sigma, \sigma^\dagger) \left( d\phi^\dagger d\phi + d\sigma^\dagger d\sigma \right),$$

with

$$H = \frac{1}{e^2} + \sum_{i=1}^{k} \frac{1}{|\phi + m_i|^2 + |\sigma + \hat{m}_i|^2}.$$

As is well known, the five-brane metric has singularities at $\phi = -m_i$, $\sigma = -\hat{m}_i$, near which the metric has the form of an infinitely long tube. The existence of singularities on the Coulomb branch is of course familiar from examples in three and four dimensions and is usually indicative of a dual description of the physics. In the present situation, no dual description is known - see [17] for a discussion of the meaning of the singularity.

We turn now to the fate of the Coulomb branch with non-zero FI and theta parameters. These appear in the classical action as a superpotential term (13). The generalised Kähler potential is again fully determined at tree-level and one-loop to be (28). Note that there are no longer any sources for the gauge field $A_\mu$ so we may treat $F_{01}$ as an auxiliary field. Now when we eliminate the auxiliary fields by their equations of motion we obtain a sigma-model on the five-brane background with a potential, $V$, given by

$$V(\phi, \phi^\dagger, \sigma, \sigma^\dagger) = \frac{1}{2} (\zeta \cdot \zeta + \theta^2/4\pi^2) H^{-1}.$$

(29)
As expected, the potential has $k$ zeroes at the points $\phi = -m_i$ and $\sigma = -\hat{m}_i$, for each value of $i$, each corresponding to a D-string/D5-bound state. Moreover, the massive sigma-model with five-brane target space and potential (29) is invariant under the full $\mathcal{N} = (4,4)$ supersymmetry algebra and all eight supercharges are preserved in each of the vacua. To see this, note that the five-brane metric admits two sets of complex structures $(I_j^{\pm i}, J_j^{\pm i}, K_j^{\pm i})$ which obey the algebra of the quaternions and are covariantly constant with respect to the connection with torsion $\Gamma_{ij}^{\pm k}$ [13]. Furthermore these complex structures are in fact constant. From this, and using the criteria of [21], it can be readily verified that this potential does indeed preserve the $(4,4)$ supersymmetry of the sigma-model.

Notice that the FI parameters and theta angle are present in (29) in a $\text{Spin}(4)$ invariant fashion. Indeed, in [17] it is argued that the $SU(2)_R$ R-symmetry under which $\zeta$ transforms as a 3 is enhanced in the infra-red to a second $\text{Spin}(4)$ R-symmetry. This effect was demonstrated in [22] using IIA intersecting brane constructions of this theory, the extra dimension arising upon lifting to M-theory.

Before progressing, it is important to determine in which limit of the theory the above description of massive vector multiplet fields is valid. In order to derive the potential (29), we have performed a one-loop calculation, expanding around configurations which are vacua only when the FI parameters vanish. One must check that the resulting description of the low-energy dynamics is consistent. Naively, we expect this to be the case for small FI parameters, $|\zeta| \ll 1$. More quantitatively, we require the potential energy of the Coulomb branch sigma-model to be less than the mass of each of the hypermultiplet fields that have been integrated out

$$\frac{1}{2}(\zeta \cdot \zeta + \theta^2/4\pi^2)H^{-1} \ll |\phi + m_i|^2 + |\sigma + \hat{m}_i|^2 - e^2|\zeta|,$$

where the last term on the right hand side arises from the triplet of D-terms in the scalar potential (43). In fact, certain hypermultiplet fields that have been integrated out actually become tachyonic at a radius $e^2|\zeta|$ from the five-brane singularity. Our conclusion is therefore that the massive Coulomb branch description of the low-energy dynamics is valid except in a region close to the vacua! However, despite this problem, we continue in our search for the kinky D-string soliton and are vindicated to some extent by the existence of a well behaved solution.

In order to exhibit the existence of kink solitons in the Coulomb branch we can form a bound on the energy of a solution. From the Hamiltonian we find the energy of any configuration is given by

$$E = \int dx \left\{ H(\partial_t \phi \partial_t \phi^\dagger + \partial_t \sigma \partial_t \sigma^\dagger) + H(\partial_x \phi \partial_x \phi^\dagger + \partial_x \sigma \partial_x \sigma^\dagger) + \frac{1}{2}(\zeta \cdot \zeta + \theta^2/4\pi^2)H^{-1} \right\},$$
\[
\begin{align*}
\geq & \int dx \left\{ H \left| \partial_x \phi - \frac{1}{\sqrt{2}} (\zeta \cdot \zeta + \theta^2/4\pi^2)^{1/2} H^{-1} \right|^2 \\
+ & \quad H \left| \partial_x \sigma - \frac{1}{\sqrt{2}} (\zeta \cdot \zeta + \theta^2/4\pi^2)^{1/2} H^{-1} \right|^2 \\
+ & \quad \frac{1}{\sqrt{2}} (\zeta \cdot \zeta + \theta^2/4\pi^2)^{1/2} \left( \gamma \partial_x \phi + \gamma^\dagger \partial_x \phi^\dagger + \hat{\gamma} \partial_x \sigma + \hat{\gamma}^\dagger \partial_x \sigma^\dagger \right) \right\}, \\
\geq & \quad \frac{1}{\sqrt{2}} (\zeta \cdot \zeta + \theta^2/4\pi^2)^{1/2} \left( \gamma \phi + \gamma^\dagger \phi^\dagger + \hat{\gamma} \sigma + \hat{\gamma}^\dagger \sigma^\dagger \right) \bigg|_{x=+\infty}^{x=-\infty},
\end{align*}
\]

where the first inequality is saturated by time independent configurations and where \(\gamma\) and \(\hat{\gamma}\) are both complex numbers satisfying \(|\gamma|^2 + |\hat{\gamma}|^2 = 1\). For solutions asymptoting to the two vacua

\[ \phi \to m_i \quad \text{and} \quad \sigma \to \hat{m}_i \quad \text{as} \quad x \to \infty, \]

\[ \phi \to m_j \quad \text{and} \quad \sigma \to \hat{m}_j \quad \text{as} \quad x \to -\infty, \]

the bound (30) is maximised by

\[ \gamma = \frac{m_i - m_j}{\sqrt{|m_i - m_j|^2 + |\hat{m}_i - \hat{m}_j|^2}} \quad \text{and} \quad \hat{\gamma} = \frac{\hat{m}_i - \hat{m}_j}{\sqrt{|m_i - m_j|^2 + |\hat{m}_i - \hat{m}_j|^2}}, \]

while the bound is saturated by solutions to the first order Bogomol'nyi equations

\[ \partial_t \left( \frac{\phi}{\sigma} \right) = 0, \quad \partial_x \left( \frac{\phi}{\sigma} \right) = \frac{1}{\sqrt{2}} \left( \frac{\gamma}{\hat{\gamma}} \right) (\zeta \cdot \zeta + \theta^2/4\pi^2)^{1/2} H^{-1}. \]

For the simplest situation of two D5-branes, the kink solution describing a D-string interpolating between the five-branes is found to be given by \(\phi = m_1 \Pi(x)\) and \(\sigma = \hat{m}_1 \Pi(x)\), where \(\Pi(x)\) satisfies the simple algebraic equation

\[ \left( \frac{1}{e^2} - \frac{4}{M^2} \frac{1}{\Pi(x)^2 - 1} \right) \Pi(x) = \frac{\zeta \cdot \zeta + \theta^2/4\pi^2}{M} (x - x_0), \]

where, as in the previous section, the integration constant \(x_0\) is the centre of mass of the kink. It is easy to check that the function on the left hand side is monotonically increasing and can be inverted over the range \(\Pi \in (-1, 1), x \in (-\infty, \infty)\), yielding a previously unknown kink solution preserving 1/2 of the supersymmetry. The energy of these solitons is easily determined from (30) to be

\[ E = 2M \left( \zeta \cdot \zeta + \theta^2/4\pi^2 \right)^{1/2}. \]

Notice that, with the exception of the contribution from the \(\theta\)-angle, the mass of this kink is the same as that of the classical soliton (24). The above Coulomb branch description includes quantum corrections however and the conclusion is that the masses
of these states are not renormalised. This is in contrast to similar states in the $\mathcal{N} = (2, 2)$ theories \[23, 24\]. Curiously however, and unlike the solutions to the classical equations of motion described in the previous section, these solitons depend on only a single bosonic collective coordinate. The periodic collective coordinate describing the relative phase of the two vacua is missing from the above description. It appears that this situation has arisen because of the sickness of the model near the vacua. As a check, one could consider the above $U(1)$ gauge theory with eight supercharges in three dimensions where the problems associated with the tube metric do not arise. In this case, the Coulomb branch is the $k$-centered Taub-NUT metric \[25\] with potential given by the length of the tri-holomorphic Killing vector. Solitons interpolating between two vacua are now strings in three dimensions and are in fact Q-kinks \[2\]. Similar solitons will appear in the following section and they do indeed have a second, periodic collective coordinate.

4 On the Higgs Branch

Our starting point in this section is the $4(k-1)$ dimensional Higgs branch which exists for coincident D5-branes. Fluctuations transverse to this space acquire a mass of order $e$ (the gauge coupling constant) and in the infra-red limit $e \to \infty$, the low-energy dynamics is well described a sigma-model on the Higgs branch. The metric on this branch receives no quantum corrections and arises as a hyperKähler quotient of the $4k$ dimensional space parametrised by the hypermultiplet scalars, $q_i$ and $\tilde{q}_i$, with momentum maps given by the first three equations of the scalar potential (29). The three FI parameters correspond to the blow-up modes of the singularities of this space. In the simplest case of $k = 2$ with the theta angle set to zero, the four-dimensional Higgs branch is the Eguchi-Hanson metric (for a derivation of this well-known fact see, for example, \[26\])

$$ds^2 = G(r)dr \cdot dr + G(r)^{-1}(d\psi - \omega \cdot dr)^2 ,$$  

with

$$G(r) = \frac{1}{|r - \zeta|} + \frac{1}{|r|} \quad \text{and} \quad \nabla \times \omega = \nabla G .$$

In terms of the hypermultiplet scalars, the coordinates on the metric (32) are given by $r = \eta_1 \sigma \eta_1^{\dagger} - \zeta = \eta_2 \sigma \eta_2^{\dagger}$ (where $\sigma$ denotes the triplet of Pauli matrices and $\eta_i$ were defined after equation (15)) and $\psi = 2 \arg (\tilde{q}_1^\dagger q_2^\dagger)$.

We now consider turning on mass terms for the hypermultiplets inducing a potential on the Higgs branch. As in the previous section, the dictates of supersymmetry are strong enough to determine the form of the potential: it must be proportional to the
length of a tri-holomorphic Killing vector. An explicit derivation of the potential will be given later in this section. In the case of Eguchi-Hanson, the Killing vector is simply \( \partial/\partial \psi \), and the potential is given by

\[
V(\mathbf{r}, \psi) = M^2 G^{-1}.
\]  

(34)

Once again, the description of the low-energy dynamics in terms of such a model is valid only if the surviving modes are of lower energy than those that have been integrated out. For the vector multiplet this requires

\[
ed^2(|\mathbf{r}| + |\mathbf{r} - \zeta|) \gg M^2 G^{-1},
\]

which is achievable at all points of the Higgs branch providing \( e \gg M \). This description therefore suffers from none of the sickness of the Coulomb branch. The low-energy dynamics is now described by a massive sigma-model with two isolated vacua at the fixed points of the isometry, \( \mathbf{r} = 0 \) and \( \mathbf{r} = \zeta \). Once again, we expect to find soliton solutions interpolating between these two vacua. In fact, the properties of these solitons have been previously explored by Abraham and Townsend [2], where they were christened Q-kinks. As we now review, they have a rather unusual property for kinks in 2-dimensions: they are dyonic. That is, they have an internal degree of freedom which may be excited, resulting in a tower of kink-states, analogous to the tower of dyons that arises when quantising four-dimensional monopole configurations. In fact, it has been shown that for such dyonic kinks in \( \mathcal{N} = (2,2) \) models, the similarity with four-dimensional dyons extends to the bound states and renormalised masses of these objects [23, 24]. The energy of any classical configuration is given by

\[
E = \int \mathcal{L} = \int \! dx \left\{ \frac{1}{2} (\mathbf{\dot{r}} \cdot \mathbf{\dot{r}} + \mathbf{r}' \cdot \mathbf{r}') G + \left( \dot{\psi} + \omega \cdot \mathbf{r} \right)^2 G^{-1} + M^2 G^{-1} \right\},
\]

where the dot and prime denote temporal and spatial derivatives respectively. Following [2], we further introduce a unit four-vector, \( (n_0, \mathbf{n}) \), such that \( n_0^2 + \mathbf{n} \cdot \mathbf{n} = 1 \), and rewrite the energy by completing the square

\[
E = \int \! dx \left\{ \frac{1}{2} G \left( \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} + G^{-1} \left( \dot{\psi} + \omega \cdot \mathbf{r} - M n_0 \right)^2 \right) 
+ G^{-1} \left( \phi' + \omega \cdot \mathbf{r}' \right)^2 + G \left( \mathbf{r}' - G^{-1} M \mathbf{n} \right)^2 
+ 2 \left( G^{-1} \dot{\psi} + \omega \cdot \mathbf{r} + \mathbf{r}' \cdot \mathbf{n} \right) M \right\} 
\geq 2M \left\{ Q_0 n_0 + \mathbf{Q} \cdot \mathbf{n} \right\}.
\]

The final expression for the energy bound contains only conserved quantities, namely the Noether charge, \( Q_0 \), and topological charge, \( \mathbf{Q} \), defined by

\[
Q_0 = \int \! dx \ G^{-1} \dot{\psi}, \quad \mathbf{Q} = \int \! dx \ \mathbf{r}'.
\]
The bound is clearly maximised by choosing \((n_0, n) \sim (Q_0, Q)\), in which case we have
\[
E^2 \geq 4M^2 (Q_0^2 + Q \cdot Q),
\]
where the inequality is saturated by the Bogomol’nyi configurations satisfying, \(\dot{r} = 0\) and \(\psi' = 0\), together with
\[
\begin{align*}
\dot{\psi} &= Mn_0, \\
\dot{r} &= MG^{-1}n.
\end{align*}
\]
The solutions to these equations, first found in [2], are given by
\[
\begin{align*}
\psi &= \psi_0 + Mn_0 t, \\
r &= \frac{1}{2} \zeta \tanh\left( \frac{1}{2} |n| M(x - x_0) \right) + \frac{1}{2} \zeta.
\end{align*}
\]
There is thus a family of soliton solutions parametrised by the angular velocity \(n_0\), each with the two centre of mass collective coordinates, \(\psi_0\) and \(x_0\). The periodicity of \(\psi_0\) ensures that upon quantisation the Noether charge \(Q_0\) will be integer valued [2]. What is the interpretation of the resulting tower of states in the ten-dimensional spacetime picture? We claim that they correspond to \((1, Q_0)\)-strings (that is a bound state of the D-string with \(Q_0\) F-strings) that interpolate between the two D5-branes in the manner described in the introduction. This clarifies the observation of [1] that such kinky string should have a description as Q-kinks.

In order to elucidate this point, let us examine various properties of the Q-kinks. Firstly, we may consider the limit of vanishing FI parameters, \(\zeta = 0\), in which the Eguchi-Hanson metric (32) becomes the singular one-instanton moduli space. The potential now has only a single zero at the singular point \(r = 0\) and the pure kink solution that carries no Noether charge shrinks to this point, reflecting the fact that the spontaneously broken gauge group that lives on the D5-branes cannot support a non-singular pure instanton solution. However, it was recently shown that non-singular “dyonic” instanton string solutions may exist in spontaneously broken gauge groups if the string also carries electric charge [9]. Moreover, the description of these strings in terms of the instanton moduli space sigma-model is as a solution to the sigma-model equations of motion which coincides with the \(\zeta \to 0\) limit of the Q-kink solution (35). Such strings break 1/4 of the supersymmetry of the six-dimensional theory on the D5-branes and therefore 1/8 of the 32 space-time supersymmetries. The only known string-like states with these properties are indeed the \((1, Q_0)\)-strings interpolating between the two D5-branes. We note in passing that, using the results of [13] and [9], the original Q-kink solution (35) with \(\zeta \neq 0\) describes a dyonic instanton string whose transverse space is non-commutative \(\mathbb{R}^4\). Moreover, the existence of a such a soliton with zero electric charge reflects the fact that there exist smooth, non-dyonic, Abelian instantons in non-commutative spaces [15].
So far our discussion of the Higgs branch has been limited to zero the theta angle. As shown in [2] and [23], the inclusion of $\theta$ induces a torsion on the Higgs branch sigma-model. To see this, consider the infra-red limit $e^2 \to \infty$, in which the gauge field kinetic terms vanish, $A_\mu$ satisfies an algebraic equation of motion which may be substituted in the theta term

$$S_\theta = \frac{\theta}{2\pi} \int d^2x F_{01} = \frac{\theta}{4\pi} \int d^2x e^{\mu\nu} b_{IJ} \partial_\mu X^I \partial_\nu X^J,$$

where $X^I = (r, \psi)$. By construction this term is a total derivative and therefore does not affect the sigma-model equations of motion. However, it does affect the theory through a shift in the Noether charge operator, in analogy with the Witten effect in four-dimensional gauge theories [27]. It is a straightforward task to adapt the analysis presented in [2] for the inclusion of $S_\theta$ to our case. We find that the charge operator $Q_0 = Q_\psi = -i\delta/\delta\psi$ is now shifted to

$$Q_0 = Q_\psi + \frac{\theta}{2\pi} \int_{-\infty}^{\infty} dx \ b_{\psi a} \partial_x r^a.$$

Using the relationship between the $(r, \psi)$ coordinates of Eguchi-Hanson and the hypermultiplet scalars $(q^i, \tilde{q}^i)$ (given after equation (33)), we may determine $A_\mu$ from the original action to find

$$b_{\psi a} \partial_x r^a = \partial_x \left( \frac{|r|}{|r| + |r - \zeta|} \right).$$

In this way we find for the Q-kink solitons that

$$Q_0 = Q_\psi + \frac{\theta}{2\pi}.$$

Including this effect we recover, for $Q_\psi = 0$, the same mass formula (31) that applies to the solitons on the Coulomb branch. The periodicity of $\psi$ ensures that upon quantisation $Q_\psi$ will be integer valued. The shift in the Noether charge of the Q-kinks that is induced by $\theta$ mirrors the effect of background the RR-scalar on the D-string/F-string bound states, where the allowed background electric field on the D-string is shifted from integral values [4]. This supports our interpretation of Q-kinks with $(1, Q_0)$-string bound states.

To find further evidence for this identification, let us consider how these states transform under T-duality, an operation that one can perform on any two-dimensional sigma-model with a $U(1)$ isometry. We use the $\mathcal{N} = (2, 2)$ superfield duality transformations of Rocek and Verlinde [28]. T-dualisation of sigma-models with potentials was discussed in [23], while application of these transformations to Higgs branches of
theories with four supercharges were considered previously in [24]. Our starting point is the microscopic Lagrangian defined in equations (7)-(11). The plan is to exchange each hypermultiplet, containing two \( N = (2,2) \) chiral superfields of opposite charge, \( Q_i \) and \( \tilde{Q}_i \), for a \( N = (4,4) \) twisted multiplet, containing a single neutral \( N = (2,2) \) chiral multiplet \( \Gamma_i = Q_i \tilde{Q}_i \), together with a neutral \( N = (2,2) \) twisted chiral multiplet, \( \Lambda_i \), which is identified as the Lagrange multiplier introduced in (11). We will only consider \( \theta = 0 \) here and rewrite the FI parameter which appears as a twisted F-terms as the more usual D-term

\[
\frac{i}{2} \int d^2 \vartheta \, \Lambda_i \Sigma_i + \text{h.c.} = i \int d^4 \theta \, \Lambda_i V_i + \text{h.c.} .
\]

Using this trick, the full microscopic Lagrangian becomes \( \mathcal{L} = \mathcal{L}_D + \mathcal{L}_F \), where \( \mathcal{L}_D \) is given by (9) together with the replacement (10) and the FI D-term

\[
- \frac{i}{2} \int d^4 \theta \, \sum_{i=1}^{k} (\Lambda_i - \Lambda_i^\dagger)(V + V_i) ,
\]

while the F-terms are of the form (13) with the (twisted) superpotentials given by

\[
W(\Phi, \Gamma_i) = \frac{i}{2} \tilde{\tau} \Phi + \sum_{i=1}^{k} \Gamma_i (\Phi + m_i) ,
\]
\[
W(\Sigma, \Lambda_i) = \frac{i}{2} \tilde{\tau} \Sigma - \sum_{i=1}^{k} \frac{1}{2} \Lambda_i (\Sigma + \hat{m}_i) .
\]

The above manipulations have led us to a reformulation of the microscopic action. This form is particularly useful for describing the Higgs branch soliton solutions. To this end, we first integrate out the gauge superfields, \( V + V_i \). Moreover, in the strong coupling limit of the gauge theory, \( e^2 \to \infty \), the vector multiplet kinetic terms decouple and the fields \( \Sigma \) and \( \Phi \) become Lagrange multipliers and may also be integrated out, resulting in a \( N = (4,4) \) massive sigma-model, where the metric and torsion terms are given by \( \mathcal{L}_D = \int d^4 \theta \, \mathcal{K} \), with

\[
\mathcal{K} = \sum_{i=1}^{k} \left( -\frac{i}{4} (\Lambda_i - \Lambda_i^\dagger)^2 + \Gamma_i \Gamma_i^\dagger \right)^{1/2} + \frac{i}{2} \sum_{i=1}^{k} (\Lambda_i - \Lambda_i^\dagger) \log \left[ -\frac{i}{2} (\Lambda_i - \Lambda_i^\dagger) + \left( -\frac{i}{4} (\Lambda_i - \Lambda_i^\dagger)^2 + \Gamma_i \Gamma_i^\dagger \right)^{1/2} \right] , \quad (40)
\]

subject to the constraints arising from the elimination of the vector multiplet

\[
\sum_{i=1}^{k} \Lambda_i = \tau , \quad \sum_{i=1}^{k} \Gamma_i = \hat{\tau} . \quad (41)
\]
While (40) leads to a Lagrangian manifestly invariant under $\mathcal{N} = (2, 2)$ supersymmetry, full $\mathcal{N} = (4, 4)$ supersymmetry is preserved only if $\mathcal{K}$ satisfies
\[ \frac{\partial^2 \mathcal{K}}{\partial \Gamma_i \partial \Gamma_j} = - \frac{\partial^2 \mathcal{K}}{\partial \Lambda_i \partial \Lambda_j^\dagger}, \]
which indeed it does. The superpotentials are now simply $W = \sum \Gamma_i m_i$ and $\mathcal{W} = -i \sum \Lambda_i \hat{m}_i / 2$.

Finally, we restrict attention once more to the case of $k = 2$, where the constraints (11) may be easily solved, with $\Gamma = \Gamma_1 = \hat{\tau} - \Gamma_2$ and a similar expression for $\Lambda$. In order to exhibit the $SU(2)_R$ action on the Higgs branch, we introduce the 3-vector superfield, $\mathbf{R} = (\text{Re}(\Gamma), \text{Im}(\Gamma), \text{Im}(\Lambda))$, together with the $SU(2)_R$ singlet, $\Xi = \text{Re}(\Lambda)$. If the scalar components are denoted using lower case version of their parent superfield, the T-dualised description of the Eguchi-Hanson Higgs branch has metric
\[ ds^2 = G(r)(dr \cdot dr + d\xi d\xi), \]
where $G(r)$ is given once again by (33). The model further differs from the original Higgs branch (32) by a torsion term that may be easily derived from (40). The potential on the T-dualised Higgs branch now arises from the superpotentials and is given by $V(r) = M^2 G(r)^{-1}$. This is precisely the same as the potential in the un-T-dualised theory [3], thus providing an explicit derivation of (34). The kink solitons in this model are now simply found using the techniques of the previous sections. As on the Coulomb branch, we insist upon time independent solutions to ensure the vanishing of the torsion contribution to the action. Once more, introducing a unit 4-vector, $(n_0, \mathbf{n})$, the energy of a time independent configuration is given by
\[ E = \int dx \left\{ G(r) \left( r' \cdot r' + \xi' \xi' \right) + M^2 G(r)^{-1} \right\}, \]
\[ = \int dx \left\{ G \left( r' - MG^{-1} n \right)^2 + G \left( \xi' - MG^{-1} n_0 \right)^2 \right\} \]
\[ + 2M r' \cdot n + 2M \xi' n_0, \]
\[ \geq 2M (r \cdot n + \xi n_0)|_{-\infty}^{+\infty}. \]
In the familiar manner, the inequality is saturated for soliton solutions satisfying the Bogomol’nyi equations
\[ r' = MG^{-1} n \quad \text{and} \quad \xi' = MG^{-1} n_0, \]
Comparing with the Bogomol’nyi equations derived on the hyperKähler Higgs branch, we find that $\xi' = \dot{\psi}$. The Q-kinks with momentum in the T-dual direction are thus exchanged with winding configurations [3]. This provides further evidence for
the identification of the Q-kink time dependence as fundamental strings. Finally, imposing the boundary conditions

\[ r \to 0 \quad \text{and} \quad \xi \to 0 \quad \text{as} \quad x \to -\infty, \]
\[ r \to \zeta \quad \text{and} \quad \xi \to \Theta \quad \text{as} \quad x \to +\infty. \]

for arbitrary \( \Theta \). The energy bound is maximised by choosing \((n_0, n) \sim (\Theta, \zeta)\), and the Bogomol’nyi equations are solved by

\[ r = \frac{1}{2} \zeta \tanh \left( \frac{1}{2} |n| M(x - x_0) \right) + \frac{1}{2} \zeta, \]
\[ \xi = \frac{1}{2} \Theta \tanh \left( \frac{1}{2} n_0 M(x - x_0) \right) + \frac{1}{2} \Theta. \]

We again note that in the limit \( \zeta \to 0 \), there still exist non-trivial solutions to the sigma-model equations of motion corresponding to D-string/F-string bound state kinks.

Acknowledgements

D.T. is supported by an EPSRC fellowship. We would like to thank Bobby Acharya, Harm Jan Boonstra, Nick Dorey, Jerome Gauntlett, Sunil Mukhi and Paul Townsend for useful discussions.

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