Real multiplication and modular curves

Igor Nikolaev

Abstract

We construct an inverse of functor $F$, which maps isomorphism classes of elliptic curves with complex multiplication to the stable isomorphism classes of the so-called noncommutative tori with real multiplication. The construction allows to prove, that complex and real multiplication are mirror symmetric, i.e. $F$ maps each imaginary quadratic field of discriminant $-D$ to the real quadratic field of discriminant $D$.

Key words and phrases: complex and real multiplication

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1 Introduction

A. Real multiplication. Let $0 < \theta < 1$ be an irrational number; consider an $AF$-algebra, $\mathbb{A}_\theta$, given by the following Bratteli diagram:

$$
\begin{array}{c}
\cdots \\
\cdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
$$

, where $\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$

Figure 1: The noncommutative torus.

The $K$-theory of $\mathbb{A}_\theta$ is (essentially) the same as for noncommutative torus, i.e. the universal $C^*$-algebra generated by the unitaries $u$ and $v$ satisfying

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the commutation relation \( vu = e^{2\pi i \theta} uv \); for brevity, we call \( \mathbb{A}_\theta \) a noncommutative torus. Two such tori are stably isomorphic (Morita equivalent), whenever \( \mathbb{A}_\theta \otimes \mathbb{K} \cong \mathbb{A}_{\theta'} \otimes \mathbb{K} \), where \( \mathbb{K} \) is the \( C^* \)-algebra of compact operators; the isomorphism occurs, if and only if, \( \theta' = (a \theta + b)/(c \theta + d) \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \) \cite{2}. The \( \mathbb{A}_\theta \) is said to have real multiplication, if \( \theta \) is a quadratic irrationality \cite{4}; we shall denote such an algebra by \( \mathbb{A}_{RM} \). The real multiplication is equivalent to the fact, that the ring \( \text{End} \left( K_0(\mathbb{A}_\theta) \right) \) exceeds \( \mathbb{Z} \); here \( K_0(\mathbb{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z} \theta \) is called a pseudo-lattice, ibid.

B. The Teichmüller functor. Let \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) be the upper half-plane and for \( \tau \in \mathbb{H} \) let \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) be a complex torus; we routinely identify the latter with a non-singular elliptic curve via the Weierstrass \( \wp \) function. Two complex tori are isomorphic, whenever \( \tau' = (a \tau + b)/(c \tau + d) \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \). If \( \tau \) is imaginary and quadratic, the elliptic curve is said to have complex multiplication; the latter is equivalent to the condition, that the ring of endomorphisms of lattice \( L = \mathbb{Z} + \mathbb{Z} \tau \) exceeds \( \mathbb{Z} \). Such curves are fundamental in number theory; we shall denote them by \( E_{CM} \). There exists a continuous map from elliptic curves to noncommutative tori, which sends isomorphic curves to the stably isomorphic tori; an exact result is this. (We refer the reader to \cite{5} for the details.) Let \( \phi \) be a closed form on the topological torus, whose trajectories define a measured foliation; according to the Hubbard-Masur theorem, this foliation corresponds to a point \( \tau \in \mathbb{H} \). The map \( F : \mathbb{H} \to \partial \mathbb{H} \) is defined by the formula \( \tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi \), where \( \gamma_1 \) and \( \gamma_2 \) are generators of the first homology of the torus. The following is true: (i) \( \mathbb{H} = \partial \mathbb{H} \times (0, \infty) \) is a trivial fiber bundle, whose projection coincides with \( F \); (ii) \( F \) is a functor, which sends isomorphic complex tori to the stably isomorphic noncommutative tori. We shall refer to \( F \) as the Teichmüller functor; the restriction of \( F \) to elliptic curves with complex multiplication establishes a bijection between the isomorphism classes of \( E_{CM} \) and the stable isomorphism classes of \( \mathbb{A}_{RM} \), ibid.

C. A modular curve associated to real multiplication. One can attach a modular curve to \( \mathbb{A}_{RM} \) as follows; we refer the reader to Section 2.2 for details of the construction. Let \( \Lambda_{RM} \cong K_0(\mathbb{A}_{RM}) \) be the pseudo-lattice with real multiplication and consider \( \Lambda_{RM} \) to be a discrete subset of the boundary of the half-plane \( \mathbb{H} \). Let \( g \in SL_2(\mathbb{Z}) \) be a hyperbolic isometry of \( \mathbb{H} \); then \( g(x) = x \) and \( g(\bar{x}) = \bar{x} \) for a pair \((x, \bar{x})\) of conjugate quadratic irrationalities at the boundary of \( \mathbb{H} \). We shall denote by \( \Gamma(\mathbb{A}_{RM}) \) the maximal congruence subgroup of \( SL_2(\mathbb{Z}) \) whose hyperbolic fixed points belong to \( \Lambda_{RM} \); the cor-
responding modular curve $X(\mathbb{A}_{\text{RM}}) := \mathbb{H}/\Gamma(\mathbb{A}_{\text{RM}})$ will be called associated to the torus $\mathbb{A}_{\text{RM}}$. Lemma 1 describes the curve $X(\mathbb{A}_{\text{RM}})$ in terms of the discriminant and conductor of the real multiplication.

**D. The result.** For not a full square integer $D > 1$, we shall write $E_{CM}^{(-D,f)}$ to denote an elliptic curve with complex multiplication by an order of conductor $f \geq 1$ in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$ [7]. Likewise, we shall write $\mathbb{A}_{\text{RM}}^{(D,f)}$ to denote a noncommutative torus with real multiplication by an order of conductor $f \geq 1$ in the real quadratic number field $\mathbb{Q}(\sqrt{D})$ [4]. Our main result can be expressed as follows.

**Theorem 1** For every elliptic curve $E_{CM}^{(-D,f)}$ there exists a holomorphic map $F^{-1} : X(F(E_{CM}^{(-D,f)})) \to E_{CM}^{(-D,f)}$, such that $F(E_{CM}^{(-D,f)}) = \mathbb{A}_{\text{RM}}^{(D,f)}$.

The note is organized as follows. A brief introduction to the preliminary facts can be found in Section 2. Theorem 1 is proved in Section 3.

## 2 Preliminaries

### 2.1 AF-algebras and real multiplication

A $C^*$-algebra is an algebra $A$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| ||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $A$ represents a noncommutative topological space. The $C^*$-algebras $A$ and $A'$ are said to be stably isomorphic (Morita equivalent) if $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators; roughly speaking, stable isomorphism means that $A$ and $A'$ are homeomorphic as noncommutative topological spaces.

An AF-algebra (Approximately Finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \cdots \oplus M_{i_k}$ and
$M' = M_{i_1} \oplus \ldots \oplus M_{i_k}$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \to M'$ the homomorphism. One has two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i_1}', \ldots, V_{i_k}'$ joined by $b_{rs}$ edges whenever the summand $M_i$ contains $b_{rs}$ copies of the summand $M_{i_r}'$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix $B = (b_{rs})$ is called partial multiplicity matrix; an infinite sequence of $B_i$ defines a unique AF-algebra.

For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $A$. The functor $A \to K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a dimension group; an order-isomorphism class of the latter we denote by $(G, G^+)$. By $\mathcal{A}_\theta$ we denote an AF-algebra given by the Bratteli diagram of Fig. 1. It is known that $K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2$ and $K_0^+(\mathcal{A}_\theta) = \{(p, q) \in \mathbb{Z}^2 \mid p + \theta q \geq 0\}$. The AF-algebras $\mathcal{A}_\theta, \mathcal{A}_{\theta'}$ are stably isomorphic, i.e. $\mathcal{A}_\theta \otimes \mathbb{K} \cong \mathcal{A}_{\theta'} \otimes \mathbb{K}$, if and only if $\mathbb{Z} + \theta \mathbb{Z} = \mathbb{Z} + \theta' \mathbb{Z}$ as the subsets of $\mathbb{R}$. Usually the pseudo-lattice $\Lambda = \mathbb{Z} + \theta \mathbb{Z}$ has only trivial endomorphisms given by multiplication times integers $\mathbb{Z}$; the case when $\text{End} (\Lambda) > \mathbb{Z}$ happens if and only if $\theta$ is a quadratic irrationality. By analogy with the complex multiplication for lattices in $\mathbb{C}$, the pseudo-lattice is said to have a real multiplication if $\text{End} (\Lambda) > \mathbb{Z}$; the corresponding AF-algebra is denoted by $\mathcal{A}_{RM}$. The ring $\text{End} (\Lambda)$ is isomorphic to an order $R$ in the ring of integers of the real quadratic number field $K = \mathbb{Q}(\theta)$; any such order has the form $R = \mathbb{Z} + fO_K$, where $f \geq 1$ is an integer number called conductor and $O_K$ the ring of integers of $K$ \cite{4}.

### 2.2 Modular curve $X(\mathcal{A}_{RM})$

We shall denote by $\Lambda_{RM} = K_0(\mathcal{A}_{RM})$ a pseudo-lattice with real multiplication; it is a discrete subset of the boundary of the half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$. Consider a quadratic irrational number $\theta \in \Lambda_{RM}$ and let $\bar{\theta} \in \Lambda_{RM}$ be its algebraic conjugate; $\theta$ and $\bar{\theta}$ are fixed points of a hyperbolic isometry
g ∈ SL₂(ℤ). By Γ(ARM) we understand a congruence subgroup of SL₂(ℤ), such that its hyperbolic fixed points belong to the pseudo-lattice Λ₉; the modular curve $X(ARM) := \mathbb{H}/\Gamma(ARM)$ will be called associated to the non-commutative torus A₉. Let $N > 1$ be an integer. Recall, that $\Gamma_1(N) := \{(a,b,c,d) ∈ SL₂(ℤ) | a,d \equiv 1 \mod N, c \equiv 0 \mod N\}$ and $X_1(N) = \mathbb{H}/\Gamma_1(N)$; the following lemma characterizes the modular curves $X(ARM)$.

**Lemma 1** $X(ARM) \cong X_1(fD)$, where $D$ is the discriminant and $f$ the conductor of A₉.

**Proof.** Let $Λ₉$ be a pseudo-lattice with the real multiplication by an order $R$ in the real quadratic number field $Q(\sqrt{D})$; it is known ([1]), that $Λ₉ ⊆ R$ and $R = ℤ + (fω)ℤ$, where $f ≥ 1$ is the conductor of $R$ and

$$
ω = \begin{cases} 
\frac{1+\sqrt{D}}{\sqrt{D}} & \text{if } D \equiv 1 \mod 4, \\
\frac{1\sqrt{D}}{\sqrt{D}} & \text{if } D \equiv 2, 3 \mod 4.
\end{cases}
$$

(1)

Recall that matrix $(a,b,c,d) ∈ SL₂(ℤ)$ has a pair of real fixed points $x$ and $\bar{x}$ if and only if $|a+d| > 2$ (the hyperbolic matrix); the fixed points can be found from the equation $x = (ax+b)(cx+d)^{-1}$ by the formulas:

$$
x = \frac{a-d}{2c} + \frac{\sqrt{(a+d)^2 - 4}}{4c^2}, \quad \bar{x} = \frac{a-d}{2c} - \frac{\sqrt{(a+d)^2 - 4}}{4c^2}.
$$

(2)

**Case I.** If $D \equiv 1 \mod 4$, then formula (1) implies that $R = (1+\frac{f}{2})ℤ + \sqrt{\frac{f^2D}{2}ℤ}$. If $x ∈ Λ₉$ is the fixed point of transformation $(a,b,c,d) ∈ SL₂(ℤ)$, then formula (2) implies that:

$$
\begin{cases} 
\frac{a-d}{2c} = (1+\frac{f}{2})z_1, \\
\frac{(a+d)^2 - 4}{4c^2} = \frac{f^2D}{4}z_2^2.
\end{cases}
$$

(3)

for some integer numbers $z_1$ and $z_2$. The first equation yields $a-d = (f + 2)cz_1$; one can assume that $c$ is divisible by $fD$, since the equation of the fixed point $x = (ax+b)(cx+d)^{-1}$ will not change if we multiply the nominator and denominator of the fraction by a constant. Thus, $d ≡ a \mod (fD)$. The second equation gives us $(a+d)^2 - 4 = f^2Dc^2z_2^2$; therefore $(a+d)^2 - 4 ≡ 0 \mod (fD)$. Since $d ≡ a \mod (fD)$, we conclude that $a^2 - 1 ≡ 0 \mod (fD)$ and $a ≡ ±1 \mod (fD)$. We pick $a ≡ 1 \mod (fD)$ for otherwise matrix
\[(a, b, c, d) \pmod{(fD)} \text{ must be multiplied by } Const = -1. \text{ All together, we get:} \]
\[a \equiv 1 \pmod{(fD)}, \quad d \equiv 1 \pmod{(fD)}, \quad c \equiv 0 \pmod{(fD)}. \quad (4)\]

### Case II

If \(D \equiv 2 \text{ or } 3 \pmod{4}\), then formula \(\text{(1)}\) implies that \(R = \mathbb{Z} + (\sqrt{f^2D}) \mathbb{Z}\). If \(x \in \Lambda_{RM}\) is the fixed point of transformation \((a, b, c, d) \in SL_2(\mathbb{Z})\), then formula \(\text{(2)}\) implies that:

\[
\begin{cases}
\frac{a-d}{2c} = z_1 \\
\frac{(a+d)^2 - 4}{4c^2} = f^2Dz_2^2
\end{cases}
\quad (5)
\]

for some integer numbers \(z_1\) and \(z_2\). The first equation yields \(a - d = 2cz_1\); as explained, one can assume that \(c\) is divisible by \(fD\). Thus, \(d \equiv a \pmod{(fD)}\). The second equation gives us \((a+d)^2 - 4 = 4f^2Dc^2z_2^2\); therefore \((a+d)^2 - 4 \equiv 0 \pmod{(fD)}\). Since \(d \equiv a \pmod{(fD)}\), we conclude that \(a^2 - 1 \equiv 0 \pmod{(fD)}\) and \(a \equiv \pm 1 \pmod{(fD)}\). Again, we pick \(a \equiv 1 \pmod{(fD)}\) for otherwise matrix \((a, b, c, d) \pmod{(fD)}\) must be multiplied by \(Const = -1\). All together, we get equations \((4)\). Since all possible cases are exhausted, lemma 1 follows. \(\square\)

## 3 Proof of theorem \[1\]

Recall, that \(\Gamma(N) := \{(a, b, c, d) \in SL_2(\mathbb{Z}) \mid a, d \equiv 1 \pmod{N}, b, c \equiv 0 \pmod{N}\}\) is called a **principal congruence group** of level \(N\); the corresponding modular curve will be denoted by \(X(N) = \mathbb{H}/\Gamma(N)\).

**Lemma 2 (Hecke)** There exists a holomorphic map \(X(fD) \to E_{CM}^{(-D,f)}\).

**Proof.** We shall outline the proof referring the reader to the original work \cite{[3]}\. Let \(\mathfrak{A}\) be an order of conductor \(f \geq 1\) in the imaginary quadratic number field \(\mathbb{Q}(\sqrt{-D})\); consider an \(\mathbb{L}\)-function attached to \(\mathfrak{A}\):

\[
L(s, \psi) = \prod_{\mathfrak{P} \subset \mathfrak{A}} \frac{1}{1 - \frac{\psi(\mathfrak{P})}{N^s(\mathfrak{A})}}, \quad s \in \mathbb{C},
\quad (6)
\]

where \(\mathfrak{P}\) is a prime ideal in \(\mathfrak{A}\), \(N(\mathfrak{P})\) its norm and \(\psi\) the Grössencharacter. Put it in a slightly different terms, it was observed by Hecke that \(L(s, \psi)\) coincides with a cusp form \(F(s)\) of the principal congruence group \(\Gamma(fD)\). On the other hand, the Deuring theorem says that \(L(E_{CM}^{(-D,f)}, s) = L(s, \psi)L(s, \bar{\psi})\),
where \( L(E_{CM}^{(-D,f)}, s) \) is the Hasse-Weil \( L \)-function of the elliptic curve and \( \bar{\psi} \) a conjugate of the Grössencharacter; thus, \( L(E_{CM}^{(-D,f)}, s) = L(F, s) \), where \( L(F, s) := \sum_{n=1}^{\infty} \frac{c_n}{n^s} \) and \( c_n \) the Fourier coefficients of the cusp form \( F \). In other words, the elliptic curve \( E_{CM}^{(-D,f)} \) is modular; if we denote by \( A_F \) an abelian variety given by the periods of holomorphic differential \( F'(s)ds \) and its conjugates on the Riemann surface \( X(fD) \), then the following diagram commutes:

\[
\begin{array}{ccc}
X(fD) & \xrightarrow{\text{canonical embedding}} & A_F \\
\downarrow & & \downarrow \text{holomorphic projection} \\
E_{CM}^{(-D,f)} & & 
\end{array}
\]

The holomorphic map \( X(fD) \rightarrow E_{CM}^{(-D,f)} \) is obtained by composition of the horizontal and the vertical arrows of the diagram. □

**Lemma 3** The Teichmüller functor \( F \) acts by the formula \( E_{CM}^{(-D,f)} \mapsto A_{RM}^{(D,f)} \).

**Proof.** Let \( L_{CM} \) be a lattice with complex multiplication by an order \( \mathfrak{A} = \mathbb{Z} + (f \omega)\mathbb{Z} \) in the imaginary quadratic field \( \mathbb{Q}(\sqrt{-D}) \); the multiplication by \( \alpha \in \mathfrak{A} \) generates an endomorphism \( (a, b, c, d) \in M_2(\mathbb{Z}) \) of the lattice \( L_{CM} \). We shall use an explicit formula for the Teichmüller functor \( F \) ([3], p.524):

\[
F : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{End} (L_{CM}) \mapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in \text{End} (\Lambda_{RM}),
\]

where \( \Lambda_{RM} \) is the pseudo-lattice with real multiplication corresponding to \( L_{CM} \); one can reduce to the case \( d = 0 \) by a proper choice of basis in \( L_{CM} \). We shall consider the following two cases.

**Case I.** If \( D \equiv 1 \mod 4 \), then formula (11) implies that \( \mathfrak{A} = \mathbb{Z} + (f + \sqrt{-f^2D})/2 \mathbb{Z} \); thus \( \alpha = \frac{2m+fn}{2} + \sqrt{-f^2Dn^2}/4 \) for some \( m, n \in \mathbb{Z} \). The multiplication by \( \alpha \) gives an endomorphism \( (a, b, c, 0) \in M_2(\mathbb{Z}) \) with

\[
\begin{cases}
    a = Tr(\alpha) = \alpha + \bar{\alpha} = 2m + fn \\
    b = -1 \\
    c = N(\alpha) = \alpha\bar{\alpha} = (\frac{2m+fn}{2})^2 + \frac{f^2Dn^2}{4}.
\end{cases}
\]
The norm $N(\alpha)$ attains non-trivial minimum $f^2D$ on $m = -f$ and $n = \pm 2$, i.e. on $\alpha_0 = f\sqrt{-D}$. To find $F(\alpha_0)$, one substitutes in (7) $a = 0, b = -1, c = f^2D, d = 0$:

\[
\begin{pmatrix}
0 & -1 \\
\end{pmatrix} \in \text{End} \ (L_{CM}) \mapsto \begin{pmatrix}
0 & -1 \\
\end{pmatrix} \in \text{End} \ (\Lambda_{RM}). \tag{9}
\]

Thus, $F(\alpha_0) = f\sqrt{D}$ and pseudo-lattice $\Lambda_{RM}$ has real multiplication by an order $R$ in the real quadratic field $\mathbb{Q}(\sqrt{D})$. To find $R$, notice that the all calculations above apply to the real field $\mathbb{Q}(\sqrt{D})$ if one replaces $D$ by $-D$ and $\alpha \in \Re$ by $F(\alpha) \in R$; therefore, $R = \mathbb{Z} + \left(\frac{f + \sqrt{-D}}{2}\right)\mathbb{Z}$. In other words, $F(E_{CM}^{(-D,f)}) = \kappa_{RM}^{(D,f)}$ in this case.

**Case II.** If $D \equiv 2$ or $3 \mod 4$, then formula (11) implies that $\Re = \mathbb{Z} + (\sqrt{f^2D})\mathbb{Z}$; thus $\alpha = m + \sqrt{-f^2Dn^2}$ for some $m, n \in \mathbb{Z}$. The multiplication by $\alpha$ gives an endomorphism $(a, b, c, 0) \in M_2(\mathbb{Z})$ with

\[
\begin{cases}
a = \text{Tr}(\alpha) = \alpha + \bar{\alpha} = 2m \\
b = -1 \\
c = N(\alpha) = \alpha\bar{\alpha} = m^2 + f^2Dn^2.
\end{cases} \tag{10}
\]

The norm $N(\alpha)$ attains non-trivial minimum $f^2D$ on $m = 0$ and $n = \pm 1$, i.e. on $\alpha_0 = f\sqrt{-D}$. To find $F(\alpha_0)$, one substitutes in (7) $a = 0, b = -1, c = f^2D, d = 0$:

\[
\begin{pmatrix}
0 & -1 \\
\end{pmatrix} \in \text{End} \ (L_{CM}) \mapsto \begin{pmatrix}
0 & -1 \\
\end{pmatrix} \in \text{End} \ (\Lambda_{RM}). \tag{11}
\]

Thus, $F(\alpha_0) = f\sqrt{D}$ and pseudo-lattice $\Lambda_{RM}$ has real multiplication by an order $R$ in the real quadratic field $\mathbb{Q}(\sqrt{D})$. We repeat the argument of part I and get $R = \mathbb{Z} + (\sqrt{f^2D})\mathbb{Z}$. In other words, $F(E_{CM}^{(-D,f)}) = \kappa_{RM}^{(D,f)}$ in this case. Since all possible cases are exhausted, lemma 3 is proved. □

**Lemma 4** For every $N \geq 1$ there exists a holomorphic map $X_1(N) \to X(N)$.

**Proof.** Indeed, $\Gamma(N)$ is a normal subgroup of index $N$ of the group $\Gamma_1(N)$; therefore, there exists a degree $N$ holomorphic map $X_1(N) \to X(N)$. □

Theorem 1 follows from lemmas 1-3 and lemma 4 for $N = fD$. □
References

[1] Z. I. Borevich and I. R. Shafarevich, Number Theory, Acad. Press, 1966.

[2] E. Effros and C.-L. Shen, Approximately finite $C^*$-algebras and continued fractions, Indiana J. Math. 29 (1980), 191-204.

[3] E. Hecke, Bestimmung der Perioden gewisser Integrale durch die Theorie der Klassenkörper, Math. Z. 28 (1928), 708-727.

[4] Yu. I. Manin, Real multiplication and noncommutative geometry, in “Legacy of Niels Hendrik Abel”, 685-727, Springer, 2004.

[5] I. Nikolaev, Remark on the rank of elliptic curves, Osaka J. Math. 46 (2009), 515-527.

[6] M. Pimsner and D. Voiculescu, Imbedding the irrational rotation $C^*$-algebra into an $AF$-algebra, J. Operator Theory 4 (1980), 201-210.

[7] J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, GTM 151, Springer 1994.

The Fields Institute for Mathematical Sciences, Toronto, ON, Canada, E-mail: igor.v.nikolaev@gmail.com

Current address: 101-315 Holmwood Ave., Ottawa, ON, Canada, K1S 2R2