TRIANGULAR SUBGROUPS OF $Sp(d, \mathbb{R})$ AND REPRODUCING FORMULAE

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ABSTRACT. We consider the (extended) metaplectic representation of the semidirect product $G = \mathbb{H}^d \rtimes Sp(d, \mathbb{R})$ between the Heisenberg group and the symplectic group. Subgroups $H = \Sigma \rtimes D$, with $\Sigma$ being a $d \times d$ symmetric matrix and $D$ a closed subgroup of $GL(d, \mathbb{R})$, are our main concern. We shall give a general setting for the reproducibility of such groups which include and assemble the ones for the single examples treated in [5]. As a byproduct, the extended metaplectic representation restricted to some classes of such subgroups is either the Schrödinger representation of $\mathbb{R}^{2d}$ or the wavelet representation of $\mathbb{R}^d \rtimes D$, with $D$ closed subgroup of $GL(d, \mathbb{R})$.

Finally, we shall provide new examples of reproducing groups of the type $H = \Sigma \rtimes D$, in dimension $d = 2$.

1. Introduction

Reproducing formulae appear almost everywhere in the literature, from coherent states in physics to group representations, Gabor analysis, wavelet analysis and its many generalizations. This theory has a wealth of applications in engineering, physics and numerical analysis (see, e.g., [1, 3, 4, 9, 10, 15, 25] and references therein). It is remarkable to observe that most existing reproducing formulae for functions $f \in L^2(\mathbb{R}^d)$ can be formulated in one and the same general form, that is, an integral formula of the type

\begin{equation}
    f = \int_H \langle f, \mu_e(h)\phi \rangle \mu_e(h)\phi \, dh, \quad \text{for all } f \in L^2(\mathbb{R}^d),
\end{equation}

in the following sense. First of all, the domain of integration $H$ is a (connected, closed, Lie) subgroup of the semidirect product $G = \mathbb{H}^d \rtimes Sp(d, \mathbb{R})$ between the Heisenberg group $\mathbb{H}^d$ and the symplectic group $Sp(d, \mathbb{R})$, $dh$ is a left Haar measure of $H$, and $\mu_e$ is the extended metaplectic representation of $G$, to be defined below in detail. In the current literature, the function $\phi \in L^2(\mathbb{R}^d)$ is usually referred as wavelet or admissible vector. As we shall show below, many standard reproducing formulae, such as those arising in Gabor analysis and wavelet theory, are of the above type: for some groups $H$, but not for all, one finds $\phi \in L^2(\mathbb{R}^d)$ in such a way that the above formula holds weakly for every $f \in L^2(\mathbb{R}^d)$. Thus, some groups $H$ give rise to interesting analysis and some
do not, like, for example, the full factors themselves, i.e. \( \mathbb{H}^d \subset \mathcal{G} \) or \( Sp(d, \mathbb{R}) \subset \mathcal{G} \). The question whether a given subgroup \( H \subset \mathcal{G} \) leads to a reproducing formula or not is both relevant and difficult, and is the main theme of our recent investigations [5, 6], together with the explicit description of the admissible vectors. The ongoing formulation of things may be simplified a bit because the reproducing formula (1) is equivalent to

\[
\|f\|^2 = \int_{H} |\langle f, \mu_e(h)\phi \rangle|^2 \, dh, \quad \text{for all } f \in L^2(\mathbb{R}^d),
\]

which is manifestly insensitive to phase multiplicative factors, that is, invariant under transformations of the type \( \phi \mapsto e^{i\alpha} \phi \). This allows a technical reduction, that is, one can factor out the center of the Heisenberg group, whose action through \( \mu_e \) is through phase factors, and one may safely pass from the whole group \( \mathcal{G} \) to the somewhat simpler group \( G = \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R}) \). We formalize this discussion in the following:

**Definition 1.** A connected Lie subgroup \( H \) of \( G = \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R}) \) is a reproducing group for \( \mu_e \) if there exists a function \( \phi \in L^2(\mathbb{R}^d) \) such that the identity (1) holds weakly for all \( f \in L^2(\mathbb{R}^d) \). Any such \( \phi \in L^2(\mathbb{R}^d) \) is called a reproducing function.

Notice that we do require formula (1) to hold for all functions in \( L^2(\mathbb{R}^d) \) for the same window \( \phi \), but we do not require the restriction of \( \mu_e \) to \( H \) to be irreducible.

In this paper we consider a class of triangular subgroups of the symplectic group, that we collectively denote as the class \( \mathcal{E} \). From the structural point of view, a group \( H \in \mathcal{E} \) is a semidirect product of the form \( H = \Sigma \rtimes D \), where \( \Sigma \) is a \( d \)-dimensional subspace of \( d \times d \) symmetric matrices and \( D \) is a closed subgroup of \( GL(d, \mathbb{R}) \) acting on \( \Sigma \); hence \( \Sigma \) is abelian and normal in \( H \). Evidently, when seen within \( G \), each of these groups is contained in the symplectic factor. We shall show that many examples used in the applications fall in this class. This motivates our study of the class \( \mathcal{E} \).

Our main result is provided by Theorem 13 below, which contains necessary and sufficient conditions for a wavelet \( \psi \) to be reproducing on a group \( H \) in \( \mathcal{E} \). This result is a far-reaching extension of the reproducing conditions for the special cases treated in [6]. We underline that this result is based on a deep study of the properties of a quadratic mapping \( \Phi \) on \( \mathbb{R}^d \), developed in Section 4.2 below. Finally, we shall provide new examples of reproducing groups in the class \( \mathcal{E} \) in dimension \( d = 2 \).

The paper also contains some additional remarks concerning an alternative formulation of the concept of **admissible group**, a formally stronger notion than the notion given in Definition 1. We also clarify in what sense our setup includes Gabor and wavelet analyses.

2. Preliminaries and notation

The symplectic group is defined by

\[
Sp(d, \mathbb{R}) = \{g \in GL(2d, \mathbb{R}) : \, ^t gJg = J\},
\]
where
\[
J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}
\]
is the standard symplectic form
\[(3) \quad \omega(x, y) = \langle x, Jy \rangle, \quad x, y \in \mathbb{R}^{2d}.\]

The metaplectic representation \(\mu\) of (the two-sheeted cover of) the symplectic group arises as intertwining operator between the standard Schrödinger representation \(\rho\) of the Heisenberg group \(\mathbb{H}^d\) and the representation that is obtained from it by composing \(\rho\) with the action of \(Sp(d, \mathbb{R})\) by automorphisms on \(\mathbb{H}^d\) (see, e.g., [12]). We briefly review its construction.

The Heisenberg group \(\mathbb{H}^d\) is the group obtained by defining on \(\mathbb{R}^{2d+1}\) the product
\[(z, t) \cdot (z', t') = (z + z', t + t' + \frac{1}{2} \omega(z, z')),\]
where \(\omega\) stands for the standard symplectic form in \(\mathbb{R}^{2d}\) given in (3). We denote the translation and modulation operators on \(L^2(\mathbb{R}^d)\) by
\[T_x f(t) = f(t - x) \quad \text{and} \quad M_{\xi} f(t) = e^{2\pi i \langle \xi, t \rangle} f(t).\]

The Schrödinger representation of the group \(\mathbb{H}^d\) on \(L^2(\mathbb{R}^d)\) is then defined by
\[\rho(x, \xi, t) f(y) = e^{2\pi i t} e^{-\pi i \langle x, \xi \rangle} e^{2\pi i \langle \xi, y \rangle} f(y - x) = e^{2\pi i t} e^{\pi i \langle x, \xi \rangle} T_x M_{\xi} f(y),\]
where we write \(z = (x, \xi)\) when we separate space components (that is \(x\)) from frequency components (that is \(\xi\)) in a point \(z\) in phase space \(\mathbb{R}^{2d}\). The symplectic group acts on \(\mathbb{H}^d\) via automorphisms that leave the center \(\{(0, t) : t \in \mathbb{R}\} \in \mathbb{H}^d \simeq \mathbb{R}\) of \(\mathbb{H}^d\) pointwise fixed:
\[A \cdot (z, t) = (Az, t).\]
Therefore, for any fixed \(A \in Sp(d, \mathbb{R})\) there is a representation
\[
\rho_A : \mathbb{H}^d \to \mathcal{U}(L^2(\mathbb{R}^d)), \quad (z, t) \mapsto \rho(A \cdot (z, t))
\]
whose restriction to the center is a multiple of the identity. By the Stone-von Neumann theorem, \(\rho_A\) is equivalent to \(\rho\). That is, there exists an intertwining unitary operator \(\mu(A) \in \mathcal{U}(L^2(\mathbb{R}^d))\) such that \(\rho_A(z, t) = \mu(A) \circ \rho(z, t) \circ \mu(A)^{-1}\), for all \((z, t) \in \mathbb{H}^d\). By Schur’s lemma, \(\mu\) is determined up to a phase factor \(e^{is}, s \in \mathbb{R}\). It turns out that the phase ambiguity is really a sign, so that \(\mu\) lifts to a representation of the (double cover of the) symplectic group. It is the famous metaplectic or Shale-Weil representation.

The representations \(\rho\) and \(\mu\) can be combined and give rise to the extended metaplectic representation of the group \(G = \mathbb{H}^d \rtimes Sp(d, \mathbb{R})\), the semidirect product of \(\mathbb{H}^d\) and \(Sp(d, \mathbb{R})\). The group law on \(G\) is
\[(4) \quad ((z, t), A) \cdot ((z', t'), A') = ((z, t) \cdot (Az', t'), AA')\]
and the extended metaplectic representation \(\mu_e\) of \(G\) is
\[(5) \quad \mu_e((z, t), A) = \rho(z, t) \circ \mu(A).\]

Observe that the reproducing formula (1) is insensitive to phase factors: if we replace \(\mu_e(h) \phi\) with \(e^{is} \mu_e(h) \phi\) the formula is unchanged, for any \(s \in \mathbb{R}\). The role of the center
of the Heisenberg group is thus irrelevant, so that the “true” group under consideration is $\mathbb{R}^d \times Sp(d, \mathbb{R})$, which we denote again by $G$. Thus $G$ acts naturally by affine transformations on phase space, namely

$$g \cdot (x, \xi) = ((q, p), A) \cdot (x, \xi) = A^t(x, \xi) + t(q, p).$$

For elements $Sp(d, \mathbb{R})$ in special form, the metaplectic representation can be computed explicitly in a simple way. For $f \in L^2(\mathbb{R}^d)$, we have

$$\mu \left( \begin{bmatrix} A & 0 \\ 0 & tA^{-1} \end{bmatrix} \right) f(x) = (\det A)^{-1/2} f(A^{-1} x)$$

$$\mu \left( \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \right) f(x) = \pm e^{-i\pi \langle Cx, x \rangle} f(x)$$

$$\mu (J) = i^{d/2} \mathcal{F}^{-1},$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx, \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

In the above formula and elsewhere, $\langle x, \xi \rangle$ denotes the inner product of $x, \xi \in \mathbb{R}^d$. Similarly, for $f, g \in L^2(\mathbb{R}^d)$, $\langle f, g \rangle$ will denote their inner product in $L^2(\mathbb{R}^d)$. Other notation is as follows. We put $\mathbb{R} = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_\pm = (0, \pm \infty)$. For $1 \leq p \leq \infty$, $\| \cdot \|_p$ stands for the $L^p$-norm of measurable functions on $\mathbb{R}^d$ with respect to Lebesgue measure. The left Haar measure of a group $H$ will be written $dh$ and we always assume that the Haar measure of a compact group is normalized so that the total mass is one. Let $\Omega$ be an open set of $\mathbb{R}^d$. Then $C^\infty_0(\Omega)$ is the space of smooth functions with compact support contained in $\Omega$.

3. GABOR AND WAVELET ANALYSES

We show below that both Gabor and wavelet analyses can be viewed as particular cases of the general setup that we are considering. This fact is somehow known and is perhaps part of common knowledge, but on the one hand we could not locate it in the literature in a precise fashion and, on the other hand, we want to present some additional remarks that are of some independent interest. We thus start with a side observation.

3.1. WEAK ADMISSIBILITY. The notion of reproducing group admits an equivalent version that is obtained by weakening a property that has been investigated in [5] and that is formulated by means of the Wigner distribution. The cross-Wigner distribution $W_{f,g}$ of $f, g \in L^2(\mathbb{R}^d)$ is given by

$$W_{f,g}(x, \xi) = \int e^{-2\pi i \langle \xi, y \rangle} f \left( x + \frac{y}{2} \right) g \left( x - \frac{y}{2} \right) \, dy.$$

The quadratic expression $W_f := W_{f,f}$ is called the Wigner distribution of $f$. We collect below some of its well-known properties (see e.g. [12]).

**Proposition 2.** The Wigner distribution of $f \in L^2(\mathbb{R}^d)$ satisfies:
(i) $W_f$ is uniformly continuous on $\mathbb{R}^{2d}$, and $\|W_f\|_{\infty} \leq 2^d \|f\|_2^2$.

(ii) $W_f$ is real-valued.

(iii) Moyal’s identity: $\langle W_f, W_g \rangle_{L^2(\mathbb{R}^{2d})} = \langle f, g \rangle_{L^2(\mathbb{R}^d)} \overline{\langle f, g \rangle}_{L^2(\mathbb{R}^d)}$.

(iv) If $f \in \mathcal{S}(\mathbb{R}^d)$, then $W_f \in \mathcal{S}(\mathbb{R}^{2d})$.

(v) Marginal properties:

\begin{equation}
\int_{\mathbb{R}^d} W_f(x, \xi) \, d\xi = |f(x)|^2, \quad \int_{\mathbb{R}^d} W_f(x, \xi) \, dx = |\hat{f}(\xi)|^2
\end{equation}

for $\hat{f} \in L^1(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^d)$, respectively.

(vi) If both $f, \hat{f}$ are in $L^1(\mathbb{R}^d)$ (hence in $L^2(\mathbb{R}^d)$) then

\begin{equation}
\int_{\mathbb{R}^{2d}} W_f(w) \, dw = \|f\|_{L^2}^2.
\end{equation}

In [5] it is introduced the notion of admissible group, one for which there exists $\phi \in L^2(\mathbb{R}^d)$ such that (15) below holds. Together with some additional integrability and boundedness properties of $h \mapsto W_\psi(h^{-1} \cdot (x, \xi))$, it implies that a subgroup $H$ of $G = \mathbb{R}^{2d} \rtimes \text{Sp}(d, \mathbb{R})$ is reproducing. For the reader’s convenience, we recall the statement of [5, Thm.1], where the main point is made.

**Theorem 3.** Suppose that $\phi \in L^2(\mathbb{R}^d)$ is such that the mapping

\begin{equation}
h \mapsto W_{\mu_\psi(h)}(x, \xi) = W_\psi(h^{-1} \cdot (x, \xi))
\end{equation}

is in $L^1(H)$ for a.e. $(x, \xi) \in \mathbb{R}^{2d}$ and

\begin{equation}
\int_H |W_\psi(h^{-1} \cdot (x, \xi))| \, dh \leq M, \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}.
\end{equation}

Then condition (1) holds for all $f \in L^2(\mathbb{R}^d)$ if and only if the following admissibility condition is satisfied:

\begin{equation}
\int_H W_\psi(h^{-1} \cdot (x, \xi)) \, dh = 1, \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}.
\end{equation}

Assumptions (13) and (14) are sufficient but not necessary conditions for a subgroup to be reproducing, as illustrated below (see the example in the next Section 4).

The notion of weak admissible group is as follows. Consider first the vector space

\begin{equation}V := \text{span}\{W_f, \quad f \in \mathcal{S}(\mathbb{R}^d)\} = \left\{ \sum_{k=1}^N c_k W_{f_k}, \quad f_k \in \mathcal{S}(\mathbb{R}^d), c_k \in \mathbb{C} \right\}.
\end{equation}

For $f, g \in \mathcal{S}(\mathbb{R}^d)$, a straightforward computation gives

\begin{equation}
W_{f,g} = \frac{1}{2} [W_{f+g} + iW_{f+ig} - (1+i)(W_f + W_g)].
\end{equation}

Since $\text{span} \{W_{f,g} \mid f, g \in \mathcal{S}(\mathbb{R}^d)\}$ is dense in $\mathcal{S}(\mathbb{R}^{2d})$ (see [5]), it follows from (17) that also $V$ is dense in $\mathcal{S}(\mathbb{R}^{2d})$. 
Now, assume that the subgroup $H$ is reproducing and let $\phi \in L^2(\mathbb{R}^d)$ be a reproducing function. Define the conjugate-linear functional $\ell$ on $V$ by

$$\ell(F) = \int_H \langle W_{\mu_e(h)}\phi, F \rangle \, dh, \quad \forall F \in V.$$  

The functional $\ell$ is well-defined and continuous on $V$ with respect to the $L^1$ norm, as shown presently. Let $F = \sum_{k=1}^N c_k W_{f_k}, f_k \in \mathcal{S}(\mathbb{R}^d), c_k \in \mathbb{C}$, be an element of the space $V$. Using Moyal’s identity (Proposition 2) and the reproducing condition (2), we have

$$|\ell(F)| = |\int_H \langle W_{\mu_e(h)}\phi, \sum_{k=1}^N c_k W_{f_k} \rangle \, dh| = |\sum_{k=1}^N \bar{c}_k \int_H \langle W_{\mu_e(h)}\phi, W_{f_k} \rangle \, dh|$$

$$= |\sum_{k=1}^N \bar{c}_k \int_H |\langle f_k, \mu_e(h)\phi \rangle|^2 \, dh| = |\sum_{k=1}^N \bar{c}_k \|f_k\|_2^2| = |\sum_{k=1}^N \bar{c}_k \int_{\mathbb{R}^d} W_{f_k}(x, \xi) \, dx \, d\xi|$$

$$= |\int_{\mathbb{R}^d} \sum_{k=1}^N \bar{c}_k W_{f_k}(x, \xi) \, dx \, d\xi| = |\langle 1, \sum_{k=1}^N c_k W_{f_k} \rangle| = |\int_{\mathbb{R}^d} \bar{F}(x, \xi) \, dx \, d\xi| \leq \|F\|_1.$$  

Moreover, from the previous computations it is clear that $\ell(F) = \langle 1, F \rangle$, for every $F \in V$. Finally, since $V$ is dense in $L^1(\mathbb{R}^d)$, the functional $\ell$ can be uniquely extended to a continuous functional $\tilde{\ell}$ on $L^1(\mathbb{R}^d)$, which coincides with $1 \in L^\infty(\mathbb{R}^d)$.

Observe that, in general, this does not imply that $\tilde{\ell}(F) = \int_H \langle W_{\mu_e(h)}\phi, F \rangle \, dh, \quad \forall F \in L^1(\mathbb{R}^d)$. Indeed, one should know that the mapping $F \rightarrow \langle W_{\mu_e(h)}\phi, F \rangle$ is in $L^1(H)$ in order for the integral to make sense, and it should satisfy $\int_H \langle W_{\mu_e(h)}\phi, F \rangle = \langle 1, F \rangle, \quad \forall F \in L^1(\mathbb{R}^d)$.

Those observations yield to the following definition.

**Definition 4.** We say that the subgroup $H$ of $G$ is weakly admissible if there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that the functional (18) is well-defined on the set $V$ defined in (16) and verify

$$\ell(F) = \int_H \langle W_{\mu_e(h)}\phi, F \rangle \, dh = \langle 1, F \rangle, \quad \forall F \in V.$$  

We are now in a position to state and prove our first observation.

**Theorem 5.** The following are equivalent:

(i) $H$ is weakly admissible and (19) holds for $\phi \in L^2(\mathbb{R}^d)$; (ii) $H$ is reproducing and $\phi \in L^2(\mathbb{R}^d)$ is a reproducing function.

**Proof.** Observe that the reproducibility condition (2) can be checked on the dense subspace $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$.

(i) $\Rightarrow$ (ii). Take $f \in \mathcal{S}(\mathbb{R}^d)$ and use the admissibility condition with $F = W_f$. Then, Moyal’s identity and $\langle 1, W_f \rangle = \|f\|^2_2$ immediately provide the desired result.

(ii) $\Rightarrow$ (i). It follows by the previous observations. \[\square\]
Remark. If the assumptions (13) and (14) of Theorem 3 hold, then the mapping \( F \to \langle W_{\mu_c(h) \phi}, F \rangle \) is in \( L^1(H) \) (one can apply Fubini Theorem and exchange the integrals) and, moreover, \( \int_H \langle W_{\mu_c(h) \phi}, F \rangle dh = \langle \int_H W_{\mu_c(h) \phi} dh, F \rangle = \langle 1, F \rangle, \quad \forall F \in L^1(\mathbb{R}^{2d}). \) Hence the weak admissibility condition generalizes (15).

3.2. Gabor analysis. Gabor’s reproducing formula is given by

\[
f = \int_{\mathbb{R}^{2d}} \langle f, T_x M_{\xi} \hat{\psi} \rangle T_x M_{\xi} \hat{\psi} \, dx \, d\xi,
\]
which is (weakly) true for every \( \psi \in L^2(\mathbb{R}^d) \), with \( \|\psi\|_2 = 1 \). We shall refer to this basic fact as to Gabor’s theorem. The extended metaplectic representation \( \mu_e \) restricted to the subgroup \( H \cong \mathbb{R}^{2d} \) consisting of the first factor in \( G \) is, of course, the Schrödinger representation \( \rho \). If we consider a function \( \psi \in L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d) \), so that conditions (13) and (14) are satisfied, the admissibility condition (15) becomes

\[
\int_{\mathbb{R}^{2d}} W_\psi(x - q, \xi - p) \, dp \, dq = 1, \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}, \text{ that is}
\]

\[
(20) \quad \int_{\mathbb{R}^{2d}} W_\psi(q, p) \, dp \, dq = 1.
\]

We need the conditions (13) and (14) to have \( W_\psi \in L^1(\mathbb{R}^{2d}) \) for granted. However, if we drop the requirement \( W_\psi \in L^1(\mathbb{R}^{2d}) \), the reproducing window \( \psi \) can be rougher, as shown below.

**Proposition 6.** Let \( \psi \in L^2(\mathbb{R}^d) \) with \( \|\psi\|_{L^2} = 1 \).

(i) If \( \psi \in L^1(\mathbb{R}^d) \), then \( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} W_\psi(q, p) \, dq \right) \, dp = 1. \)

(ii) If \( \hat{\psi} \in L^1(\mathbb{R}^d) \), then \( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} W_\psi(q, p) \, dp \right) \, dq = 1. \)

**Proof.** If \( \psi \in L^1(\mathbb{R}^d) \) with \( \|\psi\|_2 = 1 \), then \( \psi \) is reproducing by the Gabor’s theorem. By the second marginal property in (11), the map \( q \to W_\psi(q, p) \) is integrable on \( \mathbb{R}^d \) for a.e. \( p \in \mathbb{R}^d \). Since \(|\hat{\psi}(p)|^2 = \int_{\mathbb{R}^d} W_\psi(q, p) \, dq \) and \( \int_{\mathbb{R}^d} |\hat{\psi}(p)|^2 \, dp = \|\hat{\psi}\|_2^2 = \|\psi\|_2^2 = 1 \), the claim is proved. If \( \hat{\psi} \in L^1(\mathbb{R}^d) \), we use the same arguments with the first marginal property in (11). \( \square \)

**Remark.** The previous proposition indicates that (20) may fail to be true even for a reproducing function \( \psi \in L^2(\mathbb{R}^d) \), but it is replaced by subtly weaker conditions for integrable reproducing functions (or for reproducing functions with integrable Fourier transform). Assumptions (13) and (14) are actually not necessary for \( \psi \) to be a reproducing function, as the simple example below illustrates.

**Example 7.** In dimension \( d = 1 \), consider the box function \( \psi(x) = \chi_{[-1/2,1/2]}(x) \). Then \( \psi \in L^1(\mathbb{R}) \) and \( \|\psi\|_{L^2} = 1 \), so that it is a reproducing function. On the other hand, conditions (13) and (14) are not fulfilled. This is seen by computing the Wigner
The left Haar measure of $K$ subgroup of $H$ is compact.

Clearly, $W_\psi \notin L^1(\mathbb{R}^2)$, however, observe that Proposition 6 is satisfied with the admissibility condition $\int_\mathbb{R} (\int_\mathbb{R} W_\psi(x, \xi) d\xi) \, dx = 1$.

The Gabor case is a particular example of a subgroup of the form $H = \mathbb{R}^{2d} \rtimes K$, with $K$ subgroup of $Sp(d, \mathbb{R})$ (here $K = \{I_{2d}\}$). The reproducibility of $H$ is equivalent to asking the compactness of $K$. Indeed, if arbitrary translations are allowed in the affine action of $H$, then the symplectic factor must be compact, as shown below.

**Proposition 8.** If $H = \mathbb{R}^{2d} \rtimes K$, with $K \subset Sp(d, \mathbb{R})$, then $H$ is reproducing if and only if $K$ is compact.

**Proof.** The left Haar measure of $H$ is given by $dh = dx d\xi dk$, where $dx d\xi$ is the Lebesgue measure on $\mathbb{R}^{2d}$ and $dk$ is the left Haar measure of $K$. In the computations below, we take $f \in S(\mathbb{R}^d)$. We first write the right-hand side of (2), we then apply Plancherel’s theorem, compute the Fourier transform of the time-shift $T_x$, then use Parseval Identity and, finally, the Fourier transform of the frequency-shift $M_\xi$:

$$\int_K \int_{\mathbb{R}^{2d}} |\langle f, T_x M_\xi \mu(k) \phi \rangle|^2 \, dx d\xi dk = \int_K \int_{\mathbb{R}^{2d}} |\langle \hat{f}, \mathcal{F}(T_x M_\xi \mu(k) \phi) \rangle|^2 \, dx d\xi dk$$

$$= \int_K \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^d} \hat{f}(t) e^{2\pi i xt} \overline{\mathcal{F}(M_\xi \mu(k) \phi)(t)} \, dt \right|^2 \, dx d\xi dk$$

$$= \int_K \int_{\mathbb{R}^d} \left| \mathcal{F}^{-1}(\hat{f} \overline{\mathcal{F}(M_\xi \mu(k) \phi)(x)}) \right|^2 \, dx d\xi dk$$

$$= \int_K \int_{\mathbb{R}^d} \left| \hat{f}(\eta) \right|^2 \left| \mathcal{F}(M_\xi \mu(k) \phi)(\eta) \right|^2 \, d\eta dk$$

$$= \int_K \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| \mathcal{F}(\mu(k) \phi)(\eta - \xi) \right|^2 \, d\xi \right) \left| \hat{f}(\eta) \right|^2 \, d\eta \right) dk$$

$$= \left( \int_{\mathbb{R}^d} \left| \hat{f}(\eta) \right|^2 \, d\eta \right) \int_K \|\mathcal{F}(\mu(k) \phi)\|_2^2 \, dk = \|f\|_2^2 \int_K \|\mu(k) \phi\|_2^2 \, dk$$

$$= \|f\|_2^2 \|\phi\|_2^2 \int_K dk = \|f\|_2^2 \|\phi\|_2^2 \text{mis}(K).$$

The interchange in the order of integration is justified by Fubini’s theorem, since

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{f}(\eta)|^2 \left| \mathcal{F}(M_\xi \mu(k) \phi)(\eta) \right|^2 \, d\xi d\eta \leq \|\phi\|_2^2 \|f\|_2^2.$$
Now, the Haar measure of the locally compact subgroup $K$ is finite if and only if $K$ is compact (see e.g., [11]). This concludes the proof.

3.3. Wavelet analysis. We now examine the (generalized) continuous wavelet transform in higher dimensions, that is in $\mathbb{R}^d$, with $d \geq 1$. It arises from restriction of the metaplectic representation to semidirect products of the form $\mathbb{R}^d \rtimes D$, where $D$ is any closed subgroup of $GL(d,\mathbb{R})$. The product law is

$$ (q,a) \cdot (q',a') = (aq' + qa', a,a') , \quad q,q' \in \mathbb{R}^d, \, a,a' \in D. $$

When $D = GL(d,\mathbb{R})$ it is the so-called the Affine Group of Motions on $\mathbb{R}^d$ [23], and corresponds to the action $t \mapsto at + q$. The wavelet representation of $\mathbb{R}^d \rtimes D$ on $L^2(\mathbb{R}^d)$ associated with this action is given by

$$ (\nu(q,a)\psi)(t) = |\det a|^{-1/2} \psi(a^{-1}(t - q)) = (T_q D_a \psi)(t) , \quad t \in \mathbb{R}^d. $$

A function $\psi \in L^2(\mathbb{R}^d)$ is admissible if the Calderón condition is fulfilled:

$$ \int_D |\hat{\psi}(a\xi)|^2 \, da = 1 , \quad \text{for a.e. } \xi \in \mathbb{R}^d , $$

where $da$ is a left Haar measure on $D$. The subgroup $D \subset GL(d,\mathbb{R})$ may be identified with the subgroup of $Sp(d,\mathbb{R})$ given by

$$ \left\{ \begin{bmatrix} a & 0 \\ 0 & t^{-1}a^{-1} \end{bmatrix} , \quad a \in D \right\}, $$

and the metaplectic representation $\mu$ of $D$ is

$$ (\mu(a)f)(x) = (\det a)^{-1/2} f(a^{-1} x) = \pm |\det a|^{-1/2} f(a^{-1} x) , \quad f \in L^2(\mathbb{R}^d). $$

The group $\mathbb{R}^d \rtimes D$ is isomorphic to the subgroup $H$ of $\mathbb{R}^{2d} \rtimes Sp(d,\mathbb{R})$ given by

$$ H = \left\{ h(q,a) = \left[ \begin{bmatrix} q \\ 0 \end{bmatrix} , \begin{bmatrix} a & 0 \\ 0 & t^{-1}a^{-1} \end{bmatrix} \right] , \quad q \in \mathbb{R}^d, \, a \in D \right\}. $$

Observe that the group law within $Sp(d,\mathbb{R})$ is $h(q,a)h(q',a') = h(aq' + q, aa')$, in accordance with (21). The extended metaplectic representation restricted to $H$ is

$$ (\mu_e(h(q,a)f))(t) = (\rho(q,0)\mu(a)f)(t) = (T_q \mu(a)f)(t) $$

$$ = \pm |\det a|^{-1/2} f(a^{-1}(t - q)) = \pm (T_q D_a f)(t). $$

Thus, up to a sign, the extended metaplectic representation of $H$ coincides with the wavelet representation $\nu$, so that they give rise to the same reproducing formula

$$ f = \int_D \int_{\mathbb{R}^d} \langle f, T_q D_a \psi \rangle T_q D_a \psi \, dh(q,a), $$

where $dh$ is a left Haar measure on $H$. If $da$ is a left Haar measure of the group $D$ and $dq$ is the standard Lebesgue measure on $\mathbb{R}^d$, a left Haar measure $dh(q,a)$ is given by $dh(q,a) = dq |\det a|^{-1} \, da$.

One would expect that also the admissibility conditions related to the two representations coincide. The next result shows the direct correspondence between them.
Proposition 9. Let \( \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Then,
\[
\int_{H} W_{\psi}(h^{-1} \cdot (x, \xi)) \, dh = \int_{D} |\hat{\psi}(ta\xi)|^2 \, da.
\]
In particular, the wavelet admissibility condition (23) and the Wigner one (15) coincide.

Proof. If \( \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), we have \( W_{\psi} \in C(\mathbb{R}^{2d}) \) and \( W_{\psi}(\cdot, \xi) \in L^1(\mathbb{R}^d) \), for every \( \xi \in \mathbb{R}^d \), and the Wigner marginal property (11) holds. For \( h(q, a)^{-1} = h(-a^{-1}q, a^{-1}) \) the action of \( H \) on the phase space is given by
\[
h(q, a)^{-1} \cdot (x, \xi) = \begin{bmatrix} a^{-1} & 0 \\ 0 & t_a \end{bmatrix} [x] + \begin{bmatrix} -a^{-1}q \\ 0 \end{bmatrix} = \begin{bmatrix} a^{-1}(x - q) \\ t_a\xi \end{bmatrix}.
\]
The change of variables \( a^{-1}(x - q) = u, \ dq = |\det a| \, du \), yields
\[
\int_{H} W_{\psi}(h^{-1} \cdot (x, \xi)) \, dh = \int_{D} \int_{\mathbb{R}^d} W_{\psi}(a^{-1}(x - q), t_a\xi) \, dq \, da / |\det a| \]
\[
= \int_{D} \int_{\mathbb{R}^d} W_{\psi}(u, t_a\xi) \, du \, da = \int_{D} |\hat{\psi}(a\xi)|^2 \, da,
\]
that is the claim. \( \square \)

Alike the Gabor case, the assumptions (13) and (14) are not necessary for a reproducing function. Indeed, it is enough that \( W_{\psi}(\cdot, \xi) \in L^1(\mathbb{R}^d) \), for almost every \( \xi \in \mathbb{R}^d \).

It is worthwhile observing that the wavelet reproducing formula associated to a given subgroup \( D \) can be obtained from another subgroup of \( G \), namely
\[(27) \quad \tilde{H} = \left\{ \tilde{h}(q, a) = \begin{bmatrix} 0 & 0 \\ -q & 0 \end{bmatrix}, \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}, \ q \in \mathbb{R}^d, \ a \in D \right\}.
\]
Indeed, \( q = (0, J) \in G \), then \( gh(q, a)g^{-1} = \tilde{h}(q, a) \) and \( H \) in (24) and \( \tilde{H} \) are conjugate. This implies that one is reproducing if and only if the other is, and the corresponding reproducing formulas are equivalent. Observe that the extended metaplectic representation of \( \tilde{H} \) is nothing else but the wavelet representation on the frequency side:
\[(28) \quad (\mu_{\psi}(\tilde{h}(q, a)f)(t) = (\rho(0, -q)\mu(t^{-1}a^{-1})f)(t) = (M_{-q}\mu(t^{-1}a^{-1})f)(t)
\]
\[= \pm(M_{-q}D_{t^{-1}}f)(t) = \pm\mathcal{F}(T_qD_a f)(t).
\]

4. The class \( \mathcal{E} \)

In this section we introduce a class of (lower) triangular subgroups of \( \text{Sp}(d, \mathbb{R}) \) and derive general conditions for reproducing formulae to hold true. From the structural point of view, a group \( H \in \mathcal{E} \) is of the form \( H = \Sigma \times D \), where again \( D \) is a closed subgroup of \( GL(d, \mathbb{R}) \) that acts by automorphism on \( \mathbb{R}^d \). Thus, we are given a homomorphism \( \theta : D \to \text{Aut}(\mathbb{R}^d) \), that is, a \( d \)-dimensional real representation of \( D \), and we define the semidirect product
\[(29) \quad h(q, a)h(q', a') = h(\theta(a)q' + q, aa').
\]
The connection with \( Sp(d, \mathbb{R}) \) is as follows. First of all, we shall identify \( D \) with
\[
\left\{ \begin{bmatrix} a & 0 \\ 0 & t_a^{-1} \end{bmatrix} : a \in D \right\} \subset Sp(d, \mathbb{R}).
\]

Secondly, we preliminarily observe that
\[
N = \left\{ \begin{bmatrix} I & 0 \\ \sigma & I \end{bmatrix} : \sigma \in \text{Sym}(d, \mathbb{R}) \right\} \subset Sp(d, \mathbb{R})
\]
is an abelian Lie subgroup of \( Sp(d, \mathbb{R}) \), whereby the matrix product amounts to \( \sigma + \sigma' \), the sum within the vector space of \( d \) by \( d \) symmetric matrices, denoted \( \text{Sym}(d, \mathbb{R}) \).

We then assume that we are given an injective homomorphism of abelian groups
\[
j : \mathbb{R}^d \rightarrow \text{Sym}(d, \mathbb{R}), \quad q \mapsto j(q) := \sigma_q
\]
which establishes a group isomorphism of \( \mathbb{R}^d \) with the image \( \Sigma = j(\mathbb{R}^d) \subset \text{Sym}(d, \mathbb{R}) \).

In other words, we assume that we are given a parametrization of a \( d \)-dimensional subspace \( \Sigma \) of \( \text{Sym}(d, \mathbb{R}) \). Explicitly:
\[
(31) \quad \Sigma = \left\{ \begin{bmatrix} I & 0 \\ \sigma & I \end{bmatrix} : q \in \mathbb{R}^d \right\} \subset Sp(d, \mathbb{R}).
\]

Finally, we consider the products \( h = h(q, a) \) defined by
\[
h(q, a) = \begin{bmatrix} I & 0 \\ \sigma_q & I \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & t_a^{-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ \sigma_q a & t_a^{-1} \end{bmatrix}.
\]

Since
\[
\begin{align*}
h(q, a)h(q', a') &= \begin{bmatrix} a & 0 \\ \sigma_q a & t_a^{-1} \end{bmatrix} \begin{bmatrix} a' & 0 \\ \sigma_q a' & (t_a')^{-1} \end{bmatrix} \\
&= \begin{bmatrix} aa' & 0 \\ \sigma_q aa' + t_a^{-1} \sigma_q a' & (t_a')^{-1} \end{bmatrix} \\
&= \begin{bmatrix} aa' & 0 \\ (\sigma_q + t_a^{-1} \sigma_q a') aa' & (t_a')^{-1} \end{bmatrix}
\end{align*}
\]
and
\[
h(\theta(a)q' + q, aa') = \begin{bmatrix} aa' & 0 \\ \sigma_{\theta(a)q' + q}(aa') & (aa')^{-1} \end{bmatrix},
\]
the semidirect product law (29) holds true if and only if the parametrization \( j \) and the representation \( \theta \) satisfy
\[
(33) \quad \sigma_{\theta(a)q} = t_a^{-1} \sigma_q a^{-1}, \quad a \in D, \ q \in \mathbb{R}^d.
\]

Formally, a group \( \mathcal{E} \) can thus be described by a triple \((D, j, \theta)\), where \( D \) is a closed subgroup of \( GL(d, \mathbb{R}) \), \( \theta : D \rightarrow \text{Aut}(\mathbb{R}^d) \) is a representation, and \( j : \mathbb{R}^d \rightarrow \text{Sym}(d, \mathbb{R}) \), is an injective homomorphism of abelian groups; the data must satisfy the compatibility equation (33). We avoid this excess of notation and write directly \( H = \Sigma \ltimes D \). If \( H \in \mathcal{E} \)
is chosen, and hence $D$, we assume that a left Haar measure $da$ on $D$ has been fixed
and consequently the left Haar measure on $H$ that will be fixed is

$$dh = dq \frac{da}{|\det \theta(a)|},$$

where $dq$ is the Lebesgue measure on $\mathbb{R}^d$. Finally, the metaplectic representation
restricted to $H \in \mathcal{E}$ is given by

$$\mu(h(q, a)) f(x) = \mu \left( \begin{bmatrix} I & 0 \\ \sigma_q & I \end{bmatrix} \right) \mu \left( \begin{bmatrix} a & 0 \\ 0 & 'a^{-1} \end{bmatrix} \right) f(x)$$

$$= \pm e^{\pi i (\sigma_q x, x)} \mu \left( \begin{bmatrix} a & 0 \\ 0 & 'a^{-1} \end{bmatrix} \right) f(x)$$

$$= \pm e^{\pi i (\sigma_q x, x)} (\det a)^{-1/2} f(a^{-1} x).$$

(34)

4.1. Examples.

(1) Let $d = 2$. The Translation-Dilation-Sheering group (TDS) is the 4-dimensional
triangular reproducing group introduced and studied in [5]. It is defined by

$$TDS = \left\{ \begin{bmatrix} tS_\ell & 0 \\ tB_y S_\ell & t^{-1} S_\ell^{-1} \end{bmatrix} : t > 0, \ell \in \mathbb{R}, y \in \mathbb{R}^2 \right\}$$

where if $y = (y_1, y_2) \in \mathbb{R}^2$ and $\ell \in \mathbb{R},$

$$B_y = \begin{bmatrix} 0 & y_1 \\ y_1 & y_2 \end{bmatrix}, \quad S_\ell = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}. $$

It is isomorphic to the semidirect product $\Sigma \rtimes D$, with $\Sigma = \{\sigma_y, y \in \mathbb{R}^2\}$
$D = \{tS_\ell, t > 0, \ell \in \mathbb{R}\}$. A simple computation shows that

$$\theta(tS_\ell)y = t'(tS_\ell)^{-2}y.$$

Thus $TDS \in \mathcal{E}$. The group $TDS$ is important because it is the group under-
ing the shearing theory (see e.g. [7, 16]). This terminology stems from the
geometric action of $S_\ell$ on $\mathbb{R}^2$, known as shearing transformation.

(2) In dimension $d = 2$, consider the group $SIM(2)$ given by

$$h(t, y, \varphi) = \begin{bmatrix} tR_\varphi & 0 \\ t\Sigma_y R_\varphi & t^{-1} R_\varphi \end{bmatrix}, \quad t > 0, y \in \mathbb{R}^2, \varphi \in [0, 2\pi) \right\},$$

where $R_\varphi = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}$ and $\Sigma_y = \begin{bmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{bmatrix}$. This subgroup of $Sp(2, \mathbb{R})$
is also reproducing [1, 5] and its semidirect structure is $\Sigma \rtimes D$, with $\Sigma = \{\sigma_y, y \in \mathbb{R}^2\}$ and $D = \{tR_\varphi, t > 0, \varphi \in [0, 2\pi) \in \mathbb{R}\}$. In this case, we have

$$\theta(tR_\varphi)y = t'(tR_\varphi)^{-2}y, \quad t > 0, \varphi \in [0, 2\pi), y \in \mathbb{R}^2.$$

The $SIM(2)$ group is is named so because it is isomorphic to the group of similitude transformations of the plane. It is one in the family of groups $\tilde{H}_{\alpha, \beta}$ introduced and studied in [5, Section 6]. They display an analogous semidirect
product structure, and they all belong to $\mathcal{E}$. 

4.2. The mapping $\Phi$. Much of the analysis of the metaplectic representation on a group in the class $\mathcal{E}$ originates from the properties of a fundamental quadratic mapping of $\mathbb{R}^d$, whose basic properties are described in the next proposition.

**Proposition 10.** There exists a quadratic mapping $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, that satisfies

$$(37) \quad \langle \sigma_q x, x \rangle = -2 \langle q, \Phi(x) \rangle,$$

$$(38) \quad \Phi(a^{-1} x) = t^\theta(a) \Phi(x),$$

for every $x \in \mathbb{R}^d$ and every $a \in D$. The mapping $\theta$ is defined in (33).

**Proof.** We shall compute explicitly the mapping $\Phi$ and prove (37) and (38). Select a basis $\{e_i\}$ of $\mathbb{R}^d$ and put $\sigma^i = j(e_i)$. Thus, if $q = \sum_{i=1}^d q_i e_i$, then $\sigma_q = \sum_{i=1}^d q_i \sigma^i$ and

$$\langle \sigma_q x, x \rangle = \langle \sum_{i=1}^d q_i \sigma^i x, x \rangle = \sum_{i=1}^d q_i \langle \sigma^i x, x \rangle = -2 \langle q, \Phi(x) \rangle,$$

where

$$(39) \quad \Phi(x) = (\Phi_1(x), \ldots, \Phi_d(x)), \quad \text{with} \quad \Phi_j(x) = -\frac{1}{2} \langle \sigma^j x, x \rangle.$$

This establishes (37). Finally, using (33) and (37), we obtain

$$-2 \langle q, \Phi(a^{-1} x) \rangle = \langle \sigma_q a^{-1} x, a^{-1} x \rangle = \langle t^\theta(a) q, a^{-1} x \rangle$$

$$= \langle \sigma_{\theta(a) q} x, a a^{-1} x \rangle = \langle \sigma_{\theta(a) q} x, x \rangle$$

$$= -2 \langle \theta(a) q, \Phi(x) \rangle = -2 \langle q, t^\theta(a) \Phi(x) \rangle,$$

for every $q \in \mathbb{R}^d$, hence equality (38). \qed

Now, we would expect that the reproducing formula coming from the metaplectic representation of $H$ in (34) is somehow equivalent to the wavelet case (25). This is what we are going to exhibit.

Our approach gives a general criterion for reproducing which contains those in [5, 6].

Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a quadratic mapping and assume that $J_\Phi$, the Jacobian of $\Phi$, does not vanish identically. Set

$$S = \{x \in \mathbb{R}^d : J_\Phi(x) = 0\}.$$

Notice that, if $y_0 \in \mathbb{R}^d \setminus \Phi(S)$ and $x_0 \in \Phi^{-1}(y_0)$, then $x_0 \notin S$. Hence, by the local invertibility Theorem there exists an open neighborhood $A$ of $x_0$ and an open neighborhood $B$ of $y_0$ such that $\Phi_A : A \to B$ is a diffeomorphism. The local invertibility theorem in $C^d$ also tells us that in a neighborhood of $x_0$ in $C^d$ there are no other solutions of $\Phi(x) = y_0$. Hence, being the solutions of $\Phi(x) = y_0$ isolated, by Bezout Theorem they are at most $2^d$ (see ([13, Section 2.3, page 10], or also [18]).

To sum up, for any open set $X \subset \mathbb{R}^n$ contained in $\Phi(\mathbb{R}^d) \setminus \Phi(S)$ such that the cardinality of $\Phi^{-1}(y)$ is locally constant for $y \in X$, the mapping $\Phi$ induces a surjective finite-sheeted covering

$$\Phi : \Phi^{-1}(X) \to X.$$
Proof. Since $\Phi : \Phi^{-1}(X)$ is also open. In other terms, every $y_0 \in X$ has an open neighborhood $B$ such that $\Phi^{-1}(B) = \cup_{j=1}^k A_j$, $k \leq 2^d$, $A_j \subset \mathbb{R}^d \setminus S$, $A_j$ open and $A_j \cap A_i = \emptyset$ if $i \neq j$, and $\Phi : A_j \to B$ is a diffeomorphism.

Remark It could be useful to observe that the assumption that the cardinality of $\Phi^{-1}(y)$ for $y \in X$ is locally constant (constant if $X$ is connected) is in fact equivalent to requiring the more easy to check hypothesis that the map $\Phi : \Phi^{-1}(X) \to X$ is proper, i.e. for every compact $K \subset X$, $\Phi^{-1}(K)$ is compact.

Indeed, assume that the cardinality of $\Phi^{-1}(y)$ for $y \in X$ is locally constant. Then, as we saw, $\Phi : \Phi^{-1}(X) \to X$ is a finite-sheeted covering, and therefore it is clear that $\Phi^{-1}(K)$ is a compact subset, when $K$ is contained in one of the open subset $B$ above. In the case of a general compact $K \subset X$, let $x_n \in \Phi^{-1}(K)$ be a sequence. Possibly after replacing $x_n$ with a subsequence, $\Phi(x_n)$ will converge to an element $\overline{y} \in K$, so that it must belong to a compact subset $K'$ contained in an a small neighbourhood $B$ of $\overline{y}$. Hence $x_n \in \Phi^{-1}(K')$, which is compact by what we have just observed, so that still a sequence of $x_n$ should converge to an element $\overline{x}$. By continuity, $\Phi(\overline{x}) = \overline{y}$, so that $\overline{x} \in \Phi^{-1}(K)$.

Viceversa, suppose that the map $\Phi : \Phi^{-1}(X) \to X$ is proper. Take $\overline{y} \in X$ and let $\Phi^{-1}(\overline{y}) = \{\overline{x}_1, \ldots, \overline{x}_k\}$. We already know that, for convenient pairwise disjoint open neighbourhoods $A_j$, $j = 1, \ldots, k$ of $x_j$ and for an open neighbourhood $B$ of $\overline{y}$, $\Phi : A_j \to B$ is a diffeomorphism, so that every $y$ sufficiently close to $\overline{y}$ has at least $k$ pre-images, each contained in one of the $A_j$’s. Now suppose, by contradiction, that there exists a sequence $y_n \to \overline{y}$, with each $y_n$ having a further pre-image $x_n$; hence $x_n \not\in \cup_{j=1}^k A_j$. By the hypothesis of properness, one subsequence of $x_n$ must converge to an element $\overline{x} \not\in \cup_{j=1}^k A_j$. By continuity we have $\Phi(\overline{x}) = \overline{y}$, so that $\overline{x} \in \Phi^{-1}(\overline{y}) \subset \cup_{j=1}^k A_j$, which is a contradiction.

With the notation above, we have the following result.

**Proposition 11.** Suppose $X \subset \mathbb{R}^d$ is an open, simply connected set, contained in $\Phi(\mathbb{R}^d) \setminus \Phi(S)$, such that the cardinality of $\Phi^{-1}(y)$ is constant for $y \in X$. Then there exists an integer $k \in \{1, \ldots, 2^d\}$ and there exist open, connected, and pair-wise disjoint sets $Y_1, \ldots, Y_k \subset \mathbb{R}^d \setminus S$ such that $\Phi^{-1}(X) = \cup_{j=1}^k Y_j$, and $\Phi|_{Y_j} : Y_j \to X$ is a diffeomorphism.

*Proof.* Since $\Phi : \Phi^{-1}(X) \to X$ is a covering and $X$ is open (hence locally path-connected) and simply connected, it follows (see, e.g., [14, Corollary 13.8]) that the covering is trivial: i.e., there exists an homeomomorphism $\Psi : X \times \Phi^{-1}(y_0) \to \Phi^{-1}(X)$, $y_0$ being any fixed point in $X$. Since the cardinality of $\Phi^{-1}(y_0)$ is at most $2^d$, we have $\Phi^{-1}(y_0) = \{x_1, \ldots, x_k\} \subset \mathbb{R}^d \setminus S$, $k \leq 2^d$, and the desired result follows by taking $Y_j = \Psi(X \times \{x_j\})$. 

Observe that, since $\Phi$ is even, if $\Phi$ is a diffeomorphism from $Y_j$ onto $X$, then it is a diffeomorphism from $-Y_j$ onto $X$. Hence the number $k$ of $Y_j$ is even.

**Examples.**
(1) In dimension $d = 2$, consider the mapping $\Phi$, related to the $TDS$ group in (35) and defined by (see [5, (5.15)])

$$\Phi(x_1, x_2) = \left(-x_1x_2, -\frac{x_2^2}{2}\right).$$

Here $S = \{(x_1, 0), x_1 \in \mathbb{R}\}$, $\Phi(S) = (0, 0)$ and $\Phi(\mathbb{R}^2) = \mathbb{R} \times \mathbb{R} - \{(0, 0)\}$. If we define $X = \mathbb{R} \times \mathbb{R} - \{(0, 0)\}$, then $X$ is open and simply connected and

$$\Phi^{-1}(X) = Y_1 \cup Y_2$$

where

$$Y_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}, \quad Y_2 = -Y_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}. $$

(2) In dimension $d = 2$, consider the mapping $\Phi$, related to the $SIM(2)$ group in (36) and defined by (see [5, (5.11)])

$$\Phi(x_1, x_2) = \left(\frac{x_2 - x_1^2}{2}, -x_1x_2\right).$$

Here $S = \{(0, 0)\}$, $\Phi(S) = (0, 0)$ and $\Phi(\mathbb{R}^2) = \mathbb{R}^2$. If we define $X = \mathbb{R}^d - \{(0, 0, x_1 \leq 0\}$, then $X$ is open and simply connected and

$$\Phi^{-1}(X) = Y_1 \cup Y_2$$

where

$$Y_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}, \quad Y_2 = -Y_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}. $$

(3) In dimension $d \geq 2$, consider the mappings $\Psi_p$, related to the group $F = \mathbb{H}^{d-1}_v \times U(d - 1) \subset Sp(d, \mathbb{R})$, with $\mathbb{H}^{d-1}_v$ the Heisenberg group extended by the usual 1-dimensional homogeneous dilations, studied in [6] and defined by (see [6, (20)])

$$\Psi_p(x', x_d) = \left(x'dx' - \frac{1}{2}x_d^2 p, \frac{1}{2}x_d^2\right), \quad (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

where $p \in \mathbb{R}^{d-1}$ is fixed. Here $S = \{(x', 0), x' \in \mathbb{R}^{d-1}\}$, $\Phi(S) = (0, 0)$ and $\Phi(\mathbb{R}^d) = \mathbb{R}^{d-1} \times \mathbb{R}_+ \cup \{(0, 0)\}$. If we define $X = \mathbb{R}^{d-1} \times \mathbb{R}_+$, then $X$ is open and simply connected and

$$\Phi^{-1}(X) = Y_1 \cup Y_2$$

where

$$Y_1 = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0\}, \quad Y_2 = -Y_1 = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d < 0\}. $$

(4) In dimension $d = 2$, consider the mapping $\Phi$, related to the TDH group, defined in the subsequent (49), given by (see (52))

$$\Phi(x_1, x_2) = \left(-\frac{1}{2}(x_1^2 + x_2^2), -x_1x_2\right).$$

We have

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \pm x_2\}, \quad \Phi(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \pm x_2, x_1 \leq 0\},$$
If we set

\[(40) \quad X = \{(u, v) \in \mathbb{R}_- \times \mathbb{R} : u^2 - v^2 > 0\},\]

then

\[\Phi(\mathbb{R}^2) = X \cup \{(0, 0)\}.\]

In this case \(k = 2^2 = 4\) and

\[\Phi^{-1}(X) = Y_1 \cup Y_2 \cup Y_3 \cup Y_4,\]

where

\[(41) \quad Y_1 = \{(x_1, x_2) \in \mathbb{R}_- \times \mathbb{R} : x_1^2 - x_2^2 > 0\}, \quad Y_2 = -Y_1 = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1^2 - x_2^2 > 0\},\]

\[(42) \quad Y_3 = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_- : x_1^2 - x_2^2 < 0\}, \quad Y_4 = -Y_3 = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : x_1^2 - x_2^2 < 0\};\]

(see Figure 1).

![Figure 1: The sets X and Y_j, j = 1, \ldots, 4, for the TDH group.](image)

**Lemma 12.** Let \(\Phi, Y_j, X\) as in Proposition 11. If \(h \in C_0^\infty(Y_j)\), then

\[(43) \quad \int_{\mathbb{R}^d} \left| \int_{Y_j} h(x) e^{2\pi i \langle q, \Phi(x) \rangle} \, dx \right|^2 \, dq = \int_{Y_j} |h(x)|^2 \frac{dx}{|\mathcal{J}_\Phi(x)|},\]

where \(\mathcal{J}_\Phi(x)\) is the Jacobian of \(\Phi\) at \(x\).
Proof. Recall that \( Y_j \subset \mathbb{R}^d \setminus S \), so that \( J_\Phi(x) \neq 0 \) on \( Y_j \) and \( \Phi_j := \Phi|_{Y_j} \) is a diffeomorphism from \( Y_j \) onto \( X \). This let us make the change of variables \( \Phi(x) = u \) and use Plancherel’s formula:

\[
\left| \int_{\mathbb{R}^d} \left| \int_{Y_j} h(x) e^{2\pi i q, \Phi(x)} \, dx \right|^2 \, dq \right|^2 = \int_{\mathbb{R}^d} \left| \int_X h(\Phi_j^{-1}(u)) e^{2\pi i q, u} |J_{\Phi_j^{-1}}(u)| \, du \right|^2 \, dq \\
= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \chi_X(u) h(\Phi_j^{-1}(u)) e^{2\pi i q, u} |J_{\Phi_j^{-1}}(u)| \, du \right|^2 \, dq \\
= \int_{\mathbb{R}^d} \left| \chi_X(u) h(\Phi_j^{-1}(u)) |J_{\Phi_j^{-1}}(u)|^2 \, du \right| \, dq \\
= \int_X |h(\Phi_j^{-1}(u))| |J_{\Phi_j^{-1}}(u)|^2 \, du \\
= \int_{Y_j} |h(x)|^2 \, dx / |J_\Phi(x)|^2, \\
\]

where in the last row we have performed the change of variables \( \Phi_j^{-1}(u) = x \). Observe that, since the \( \text{supp} \ h \) is a compact set contained in the open set \( Y_j \), there exist two constants \( 0 < c < C \), such that \( c < |J_\Phi(x)| < C \) on \( \text{supp} \ h \) and the last integral is well defined. \( \square \)

Now we have all the pieces in place to provide the reproducing condition on the wavelet \( \psi \) which guarantees the reproducibility of the group \( H \).

**Theorem 13.** Let \( H = \Sigma \rtimes D \cong \mathbb{R}^d \rtimes D \) be as at the beginning of this section and let \( X, Y_j \) be as in Proposition 11. Then, the identity

\[
\|f\|^2_2 = \int_H |\langle f, \mu(h(q, a))\psi \rangle|^2 \, dh(q, a) \tag{44}
\]

holds for every \( f \in C^\infty_0(Y_j) \) if and only if \( \psi \) satisfies the condition

\[
\int_E |\psi(a^{-1}x)|^2 |\det \theta(a)|^{-1} \, da = |J_\Phi(x)|, \quad \text{a.e.} \ x \in Y_j. \tag{45}
\]

Moreover, \( (44) \) holds for every \( f \in C^\infty_0(Y_j \cup (-Y_j)) \) if and only if \( \psi \) satisfies the following two conditions:

\[
\int_E |\psi(a^{-1}x)|^2 |\det \theta(a)|^{-1} \, da = \int_E |\psi(-a^{-1}x)|^2 |\det \theta(a)|^{-1} \, da = |J_\Phi(x)|, \tag{46}
\]

for a.e. \( x \in Y_j \), and

\[
\int_E \psi(-a^{-1}x) \overline{\psi(a^{-1}x)} |\det \theta(a)|^{-1} \, da = 0, \quad \text{a.e.} \ x \in Y_j. \tag{47}
\]

**Proof.** The left Haar measure on \( H \) is given by

\[
dh(q, a) = dq \cdot |\det \theta(a)|^{-1} \, da.
\]
and the metaplectic representation on $H$ in (34), let us write, for every $f \in \mathcal{C}_0^\infty(Y_j \cup (-Y_j))$,

\begin{equation}
\int_H |\langle f, \mu(h(q, a))\rangle|^2 \,dh(q, a) = \int_D \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x)e^{-\pi i(aq,x)}(\det a)^{-1/2}\psi(a^{-1}x) \,dx \right|^2 \,dq \\
\quad \cdot \left| \det \theta(a) \right|^{-1} \,da
\end{equation}

\begin{align*}
&= \int_D \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x)e^{2\pi i(q,\Phi(x))}\psi(a^{-1}x) \,dx \right|^2 \,dq \left| \det \theta(a) \right|^{-1} \,da \\
&= \int_D \int_{\mathbb{R}^d} \int_{Y_j} \left[ f(x)\overline{\psi(a^{-1}x)} + f(-x)\overline{\psi(-a^{-1}x)} \right] \\
&\quad \cdot e^{2\pi i(q,\Phi(x))} \,dx \,dq \left| \det \theta(a) \right|^{-1} \,da
\end{align*}

where the last equality is due to the even property of $\Phi$.

Now, we set $h(x) := f(x)\overline{\psi(a^{-1}x)} + f(-x)\overline{\psi(-a^{-1}x)}$ and apply Lemma 12, so that

\begin{align*}
\int_H |\langle f, \mu(h(q, a))\rangle|^2 \,dh(q, a) &= \int_D \int_{Y_j} \left[ |f(x)\overline{\psi(a^{-1}x)}|^2 + |f(-x)\overline{\psi(-a^{-1}x)}|^2 \right] \\
&\quad + 2\Re \left( f(x)\overline{\psi(a^{-1}x)}f(-x)\overline{\psi(-a^{-1}x)} \right) \frac{dx}{|J_{\Phi(x)}|} \left| \det \theta(a) \right|^{-1} \,da.
\end{align*}

Suppose at first that $f$ satisfies the additional property: $f(x) = 0$ on $-Y_j$, then

\begin{align*}
\int_H |\langle f, \mu(h(q, a))\rangle|^2 \,dh(q, a) &= \int_{Y_j} |f(x)|^2 \left( \int_D |\psi(a^{-1}x)|^2 \left| \det \theta(a) \right|^{-1} \,da \right) \frac{dx}{|J_{\Phi(x)}|} \\
&\quad \text{so that } \int_H |\langle f, \mu(h(q, a))\phi \rangle|^2 \,dh(q, a) = \|f\|_2^2 \quad \text{if and only if (45) holds.}
\end{align*}

If, instead, $f(x) = 0$ on $Y_j$, the equality $\int_H |\langle f, \mu(h(q, a))\phi \rangle|^2 \,dh(q, a) = \|f\|_2^2$ holds true if and only if the second and the last expression in (46) are equal.

Finally, taking $f \in \mathcal{C}_0^\infty(Y_j \cup (-Y_j))$, such that $f(x)\overline{f(-x)}$ is real-valued, purely imaginary-valued, respectively, we have

\begin{align*}
\int_H |\langle f, \mu(h(q, a))\phi \rangle|^2 \,dh(q, a) &= \|f\|_2^2
\end{align*}

if and only if both conditions (46) and (47) are fulfilled.

**Corollary 14.** Theorem 13 still holds if the assumptions $f \in \mathcal{C}_0^\infty(Y_j)$ or $f \in \mathcal{C}_0^\infty(Y_j \cup (-Y_j))$ are replaced by $f \in L^2(Y_j)$, or $f \in L^2(Y_j \cup (-Y_j))$, respectively.

**Proof.** It follows by the density of $\mathcal{C}_0^\infty(Y_j)$ and $\mathcal{C}_0^\infty(Y_j \cup (-Y_j))$ in $L^2(Y_j)$ and $L^2(Y_j \cup (-Y_j))$, respectively. 

\[\square\]
4.3. The case $H = \mathbb{R}^d \rtimes N$. We shall exhibit that for subgroups of the kind $H = \mathbb{R}^d \rtimes N$, with $N$ defined in (30), formula (1) ever fails. The product law is given by $h(q,n)h(q',n') = h(q + nq', nn')$.

First of all, in the semidirect product above, we are dealing with the case $\mathbb{R}^d \cong \{0\} \times \mathbb{R}^d \subset \mathbb{R}^{2d}$. The choice $\mathbb{R}^d \cong \mathbb{R}^d \times \{0\}$ reduces $\mathbb{R}^d \rtimes N$ to $\mathbb{R}^d$, since it forces $N$ to be $I$. In our case, for $p, p' \in \mathbb{R}^d$, $n = \begin{bmatrix} I & 0 \\ c & I \end{bmatrix}$, with $c \in M(d, \mathbb{R})$, the action on $\mathbb{R}^d$ is $p + np' = \begin{bmatrix} 0 \\ p + np' \end{bmatrix}$, and the product law becomes $h(p, c)h(p', c') = h(p + p', c + c')$.

The extended metaplectic representation on $H$ is given by

$$
\mu_e(h \psi)(x) = \rho(0, p)\mu(n)f(t) = \pm Me^{i\pi \langle ct, t \rangle} f(t).
$$

The right-hand side of (1) has the form

$$
\int_H |\langle f, \mu_e(h)\psi \rangle|^2 dh = \int_N \int_{\mathbb{R}^d} |F(e^{i\pi \langle c \cdot \cdot \rangle} \bar{\psi} f)(p)|^2 dp d\mu(c)
$$

$$
= \int_N \int_{\mathbb{R}^d} |e^{i\pi \langle ct, t \rangle} f(t)\bar{\psi}(t)|^2 dt d\mu(c)
$$

$$
= \int_N d\mu(c) \int_{\mathbb{R}^d} |f(t)|^2 |\psi(t)|^2 dt,
$$

where we used Plancherel’s formula, so that the last integral either vanishes or diverges.

5. New 2-dimensional reproducing subgroups $H = \Sigma \rtimes D$

We shall construct two new examples of reproducing subgroups $H = \Sigma \rtimes D$, in dimension $d = 2$. To prove their reproducibility, we shall apply the theory developed in the previous section.

5.1. The TDH group. Consider the 4-dimensional group:

$$
TDH = \left\{ h((x, y), (s, t)) := \begin{bmatrix} e^{-s}H(t) & 0 \\ e^{s}T(x, y)H(t) & e^{s}H(-t) \end{bmatrix} : s, t, x, y \in \mathbb{R} \right\} \subset Sp(2, \mathbb{R}),
$$

with the hyperbolic matrix $H(t)$ given by

$$
H(t) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad t \in \mathbb{R}
$$
whereas the symmetric matrix $T(x, y)$ displays the entries

$$T(x, y) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \quad x, y \in \mathbb{R}.$$  

The semidirect structure $H = \mathbb{R}^2 \rtimes D$ is clear:

$$\begin{bmatrix} e^{-s}H(t) & 0 \\ e^{s}T(x, y)H(t) & e^{s}H(-t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ T(x, y) & I \end{bmatrix} \begin{bmatrix} e^{-s}H(t) & 0 \\ 0 & e^{s}H(-t) \end{bmatrix}.$$  

**Proposition 15.** The subgroups $TDH$ of $Sp(2, \mathbb{R})$ satisfy the following properties:

(a) The product law in $TDH$, is explicitly given by:

$$h(s, t, z)h(s', t', z') = h(s + s', t + t', z + e^{2s}H(-2t)z'), \quad z, z' \in \mathbb{R}^2, t, t', s, s' \in \mathbb{R}.$$  

(b) The left Haar measure on $TDH$ is $dh(s, t, z) = e^{-4s} ds dt dz$.

(c) The mapping $\Phi$ in (39) is explicitly given by

$$\Phi(x) = (-\frac{1}{2}(x^2 + y^2), -xy)$$

and has Jacobian $J_\Phi(x, y) = -(x^2 - y^2)$. Observe that $\Phi(\mathbb{R}^2) = X \cup \{(0, 0)\}$, where the open set $X$ is defined in (40).

(d) The restriction of the metaplectic representation to $TDH$ is given by:

$$\mu(h((s, t), (x, y)))f(u) = \pm e^{i\pi((T(x, y)u, u))} f(e^{s}H(-t)u), \quad f \in L^2(\mathbb{R}^2)$$

(e) The group homomorphism $\theta$ in (29) is

$$\theta(e^{-s}H(t)) = e^{2s}H(-2t) = (e^{-s}H(t))^{-2} = t(e^{-s}H(t))^{-2},$$

since the matrix $a = a(s, t) = e^{-s}H(t)$ is symmetric. Hence $\theta$ is the same homomorphism encountered in the $TDS(2)$ and $SIM(2)$ group cases.

Since the proof consists of easy calculations, we leave it to the interested reader.

The reproducibility of the group $TDH$ is then a mere application of Theorem 13, with $\Phi^{-1}(X) = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$, and $X$ defined in (40), $Y_1, Y_2 = -Y_1$ defined in (41) and $Y_3, Y_4 = -Y_3$ defined in (42).

**Theorem 16.** The subgroup $TDH$ is reproducing for $L^2(Y_1 \cup -(Y_1))$. Moreover, $\psi \in L^2(Y_1 \cup -(Y_1))$ is a reproducing function for $TDH$ if and only if

$$\int_{Y_1} |\psi(u, v)|^2 \frac{du \, dv}{(u^2 - v^2)^2} = \int_{Y_1} |\psi(-u, -v)|^2 \frac{du \, dv}{(u^2 - v^2)^2} = 1,$$

and

$$\int_{Y_1} \psi(-u, -v)\overline{\psi(u, v)} \frac{du \, dv}{(u^2 - v^2)^2} = 0.$$  

Similarly, $TDH$ is reproducing for $L^2(Y_3 \cup -(Y_3))$, and $\psi \in L^2(Y_3 \cup -(Y_3))$ is a reproducing function if fulfills (55) and (55) with $Y_3$ in place of $Y_1$.  

Proof. We use Theorem 13 and translate conditions (46) and (47) into this context. The automomorphism \( \theta \) is computed in (54), whence
\[
| \det \theta(a) | = | \det((e^{-s} H(t))^{-2} e^{-s} H(t)) | = | \det(e^{s} H(-t)) |^{-1} = e^{-2s}.
\]
We consider the case \( \Phi = \Phi_{\theta} \). The first condition in (46) reads in this framework as
\[
\int_{\mathbb{R}^2} |\psi(e^{s} H(-t)^t(x,y))|^2 e^{-2s} \, ds \, dt = x^2 - y^2 \quad \text{a.e.} \ (x,y) \in Y_1.
\]
Performing the change of variables \( e^s H(-t)^t(x,y) = t(u,v) \), we get \( e^{2s}(x^2 - y^2) = u^2 - v^2 \). Here \( ds \, dt = \frac{1}{u^2 - v^2} \, du \, dv \). Hence, the previous integral coincides con the first one in the left-hand side of (55). The other cases are analogous. \( \square \)

5.2. The TDW group. The TDW group arises by tensor-product of 1-dimensional wavelets (see the end of this section) and is defined as follows.
\[
H = \left\{ h((x,y),(s,t)) = \begin{bmatrix} e^{s} & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 \\ e^{sx} & 0 & e^{-s} & 0 \\ 0 & e^{ty} & 0 & e^{-t} \end{bmatrix} : s, t, x, y \in \mathbb{R} \right\} \subset Sp(2, \mathbb{R}).
\]
The TDW group enjoys the following properties:
(i) If we set \( a(s,t) = \begin{bmatrix} e^{s} & 0 \\ 0 & e^{t} \end{bmatrix} \), the product law in \( H \) is explicitly given by:
\[
h(s,t,z)h(s',t',z') = h(s + s', t + t', z + a(s,t)^{-2}z'), \quad z, z' \in \mathbb{R}^2, t, t', s, s' \in \mathbb{R}.
\]
Hence the automorphism \( \theta \) is \( \theta(a(s,t)) = a(s,t)^{-2} \) (notice that \( a(s,t) \) is symmetric).
(ii) The mapping \( \Phi \) in (39) is given by
\[
\Phi(x) = -\frac{1}{2}(x^2, y^2)
\]
and has Jacobian \( J\Phi(x,y) = xy \), so that \( J\Phi(x,y) = 0 \) on the set \( S = \{(x,y) : x = 0 \lor y = 0\} \) and \( \Phi(S) = \{(x,0), x \leq 0\} \cup \{(0,y), y \leq 0\} \). Moreover, \( \Phi(\mathbb{R}^2) = \{(x,y) : x \leq 0, y \leq 0\} \). In this case, we have
\[
X = \mathbb{R}_- \times \mathbb{R}_-, \quad \Phi^{-1}(X) = Y_1 \cup (-Y_1) \cup Y_2 \cup (-Y_2),
\]
where
\[
Y_1 = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\}, \quad Y_2 = \{(x,y) \in \mathbb{R}_- \times \mathbb{R}_+\}.
\]
(iii) The restriction of the metaplectic representation to \( H \) is given by:
\[
(57) \quad \mu(h((s,t),(x,y))) f(u,v) = \pm e^{-s/2} e^{ixu} e^{-t/2} e^{iyv} f(e^{-s}u, e^{-t}v).
\]
Theorem 13 rephrased for the TDW group is as follows.

Theorem 17. The subgroup TDW is reproducing on \( L^2(Y_1 \cup (-Y_1)) \). A function \( \psi \in L^2(Y_1 \cup (-Y_1)) \) is a reproducing function if and only if
\[
(58) \quad \int_{Y_1} |\psi(u,v)|^2 \frac{du \, dv}{u^2 v^2} = \int_{Y_1} |\psi(-u,-v)|^2 \frac{du \, dv}{u^2 v^2} = 1, \quad \int_{Y_1} |\psi(-u,-v)| \overline{\psi(u,v)} \frac{du \, dv}{u^2 v^2} = 0,
\]
and, similarly, TDW is reproducing on $L^2(Y_2 \cup (-Y_2))$.

Now, recall the reproducing subgroup $H_1 \subset \mathbb{R}^2 \rtimes SL(2, \mathbb{R})$ given by

$$H_1 = \left\{ \begin{pmatrix} 0 & 1 \\ b & 1 \end{pmatrix} \begin{pmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}.$$ 

A reproducing function $\psi_1$ is reproducing for $H_1$ if and only if $\psi_1 \in L^2(\mathbb{R})$ and

$$\int_0^\infty |\psi_1(x)|^2 \frac{dx}{x^2} = \int_0^\infty |\psi_1(-x)|^2 \frac{dx}{x^2} = \frac{1}{2}, \quad \int_0^\infty \psi_1(x)\overline{\psi_1(-x)} \frac{dx}{x^2} = 0.$$

It is then clear that every function $\psi(u, v) = 4\psi_1(u)\psi_1(v)$ fulfills (58), so that a reproducing function is obtained by a tensor product of two 1-dimensional wavelets.

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