ACCELERATING UNIVERSE WITH A DYNAMIC COSMOLOGICAL TERM

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Keeping in mind the current picture of an accelerating and flat Universe, some specific dynamical models of the cosmological term $\Lambda$ have been selected for investigating the nature of dark energy. Connecting the free parameters of the models with the cosmic matter and vacuum energy density parameters, it is shown that the models are equivalent. Using the selected models, the present values of some of the physical parameters have been estimated, and a glimpse at the past decelerating universe has also been presented. It is observed that most of these cosmological parameters nicely agree with the values suggested by the Type Ia Supernovae and other experimental data.

1. Introduction

The observations on supernova by the High-z Supernova Search Team (HST) and the Supernova Cosmology Project (SCP) \cite{1,2} have revealed that, instead of slowing down, the expanding Universe is speeding up. An intense search is going on, in both theory and observations, to unveil the true nature of this acceleration. It is commonly believed by the cosmological community that a kind of repulsive force which acts as anti-gravity is responsible for gearing up the Universe some 7 billion years ago. This hitherto unknown exotic physical entity is termed as dark energy.

Now, there can be many variants of dark energy which can be responsible for this accelerated universe, and variation in the forms of dark energy also exhibit variation in expansion rates in different eras. So, there may be more than one candidate which can be stamped as dark energy. For example, one may select the so-called cosmological constant, introduced and later abandoned by Einstein, as dark energy. But selection of the cosmological constant as dark energy faces a serious fine-tuning problem which demands that the value of $\Lambda$ must be 123 orders of magnitude and 55 orders of magnitude larger on the Planck scale ($T \approx 10^{19}$ GeV) and the electroweak scale ($T \approx 10^{2}$ GeV), respectively, than its presently observed value. Moreover, the matter and radiation energy densities of the expanding Universe fall off as $a^{-3}$ and $a^{-4}$, respectively, where $a$ is the scale factor of the universe, while $\Lambda$ remains constant. This poses another disturbing fine-tuning problem. For these two reasons, at present $\Lambda$ with a dynamical character is preferred over a constant $\Lambda$, especially a time-dependent $\Lambda$ which has decreased slowly from its large initial value to reach its small value at present \cite{3}. A scalar field $\phi$ with a potential $V(\phi)$, which is known as quintessence and decreases slowly with time, may be another candidate for dark energy. Quintessence exerts negative pressure and is dynamic in nature (recall that $\Lambda_{\text{effective}} = 8\pi G \rho_\phi$). However, in the present article we have considered some phenomenological models of kinematical $\Lambda$ which is assumed to be one of the dark energy candidates to account for the accelerating expansion of the Universe.

Among the dynamical models of $\Lambda$ which are frequently used in the literature, we have particularly presented here three types, viz., $\Lambda \sim (\dot{a}/a)^2$, $\Lambda \sim \ddot{a}/a$ and $\Lambda \sim \rho$, where $a$ is the scale factor of the Robertson-Walker metric and $\rho$ is the matter energy density. The first type of $\Lambda$-model was proposed from dimensional arguments by Carvalho et al. \cite{4} and Waga \cite{5} and using another type of argument by Lima and Carvalho \cite{6}, and it was subsequently taken up by several workers \cite{7-11}. Using dimensional arguments, Vishwakarma \cite{12} suggested the $\Lambda \sim \rho$ model, whereas the second model mentioned above was dealt by Arbab \cite{13-15} and Overduin and Cooperstock \cite{3}.

Now, a key to catch up the nature of dark energy lies in $w$, the equation of state parameter which is nothing but the ratio of fluid pressure and energy density of dark energy, viz., $w = p/\rho$. This parameter $w$ has different forms in different models. In the present study, using the above three forms of $\Lambda$, general solutions of the field equations are obtained under the assumption that the Universe is flat. Also, particular solutions, wherever

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needed, are discussed for the specific cases of matter- and radiation-dominated universes related to three specific dynamic cosmological terms. It is possible to show the equivalence of the three models in terms of the solutions, obtained by connecting the free parameters α, β and γ of these models with Ω_m and Ω_Λ, the matter and vacuum energy density parameters of the Universe, respectively. This will enable us to establish a relationship between three parameters of the models in the pressureless dust and electromagnetic radiation cases \((w = 0, 1/3)\).

In this connection, we would like to point out that, concerning the cases \(\Lambda \sim (\dot{a}/a)^2\) and \(\Lambda \sim \rho\), it was already mentioned by Vishwakarma [16] that the estimates of the parameters for flat models are the same. Therefore, in view of this, the main purpose of the present paper is to reexamine the status of the phenomenological approach of the dynamical Λ-term and to provide more general result by including one more case \(\Lambda \sim \ddot{a}/a\) into a systematic analysis. However, though there are innumerable phenomenological Λ-decay laws available in the literature (see for exhaustive lists [3, 17]), this particular case, viz. \(\Lambda \sim \ddot{a}/a\), is not included there. This case has been so far, separately, taken up by Arbab [13–15] and also by Overduin and Cooperstock [3] with a different approach. We have, among other candidates of the list, purposely omitted the popular cases like \(\Lambda \sim t^{-2}\) and \(\Lambda \sim a^{-2}\) since the first one exactly coincides with that of the case \(\Lambda \sim (\dot{a}/a)^2\) as \(t \sim H^{-1}\) where \(H\) is the Hubble parameter, which is defined as \(H = \dot{a}/a\) and was extensively studied by several authors [14–16, 18–28]. The second case, \(\Lambda \sim a^{-2}\), which was first suggested through dimensional arguments related to quantum cosmology by Chen and Wu [29] and also results from a contracted Ricci collineation along the fluid-flow vector [30, 31], is dropped here since this case does not suit for our present scheme as will be clear from the field equations of the next section.

Based on all the available observational information, some physical features have been explored through the cosmological parameters, which are in good agreement with the observationally obtained present data of the Universe. These results are discussed in Sec. 6 considering both the present accelerating and the past decelerating Universe. Before this, we show the ranges of the parameters α, β and γ involved in different Λ-models in Sec. 5, whereas the equivalence of the Λ-models is established in Sec. 4. Sections 2 and 3 are related to the Einstein field equations and their general solutions for different Λ-dependent models. In the concluding Sec. 7, some discussion is presented.

2. **Einstein’s field equations**

Let us consider the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

\[
ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]
\]

where the curvature constant \(k = -1, 0, +1\) for hyperbolic, flat and closed models of the Universe, respectively.

The Einstein field equations are given by

\[
R^{ij} - \frac{1}{2} R g^{ij} = -8\pi G \left[ T^{ij} - \Lambda g^{ij} \right]
\]

where \(\Lambda\) is the so-called cosmological constant, assumed here to be time-dependent, viz., \(\Lambda = \Lambda(t)\), and \(c\), the velocity of light in vacuum, is assumed to be unity (we thus use relativistic units).

For the spherically symmetric metric considered above, the Einstein field equations with a time-dependent cosmological constant yield the following two equations, called the Friedmann equation and the Raychaudhuri equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}
\]

\[
\ddot{a}/a = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.
\]

The energy conservation law can be written as

\[
8\pi G(p + \rho) \frac{\dot{a}}{a} = -\frac{8\pi G}{3} \dot{\rho} - \frac{\dot{\Lambda}}{3}.
\]

Let us choose the barotropic equation of state

\[
p = wp
\]

where the parameter \(w\) can take the constant values 0, 1/3, -1 and +1 for dust, radiation, vacuum fluid and stiff fluid, respectively.

Using Eq. (6), Eq. (4) transforms to

\[
\ddot{a} = \frac{4\pi G}{3} \left( 1 + 3w \right) \rho = \frac{\Lambda}{3}.
\]

Differentiating Eq. (3) with respect to the time coordinate \(t\) and using Eqs. (4)–(7) to eliminate \(\rho\), we finally obtain the following equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \left[ 3 \left( \frac{1+w}{1+3w} \right) - 1 \right] \frac{\dot{a}}{a} + \frac{k}{a^2} = \left( \frac{1+w}{1+3w} \right) \Lambda.
\]

This is the dynamical equation relating the cosmic scale factor \(a\) to a known value of the dynamic cosmological term \(\Lambda\). It can readily be observed from the above equations (7) and (8) that \(\Lambda\) depends on the factors \(\dot{a}/a\), \(\rho\),...
\((\dot{a}/a)^2\) and \(a^{-2}\) in a specific way. However, the inflation theory of the Universe predicts and the CMB detectors such as BOOMERANG [32–34], MAXIMA [35–37], DASI [38], CBI [39] and WMAP [40,41] confirm that the Universe is spatially flat. Therefore, the \(\Lambda\) case, which is not suitable for \(k = 0\), is omitted here. However, for a detailed study of this case, viz. \(\Lambda \sim a^{-2}\), interested persons may consult the works done by Chen and Wu [29], Abdussattar and Vishwakarma [30] and Vishwakarma [31], mentioned earlier, as well as [4,5,42–48]. We shall therefore consider the phenomenological models related to the cases \((\dot{a}/a)^2\), \(\ddot{a}/a\) and \(\rho\) only for \(\Lambda\) and try to find solutions which will help us to model and explore the features of the Universe.

3. Cosmological models for an accelerating Universe

If we use \(\Lambda = 3\alpha(\dot{a}/a)^2 = 3\alpha H^2\), where \(\alpha\) is a constant and \(H\) is the Hubble parameter, then for the flat universe \((k = 0)\) Eq. (8) reduces to

\[
2\dot{a}a + (1 + 3w - 3w\alpha - 3\alpha)a^2 = 0. \tag{9}
\]

Solving Eq. (9), we get our general solution as

\[
a(t) = C_1 t^{2/\left(3(1-\alpha)-(1+w)\right)}, \tag{10}
\]

\[
\rho(t) = \frac{1}{6\pi G(1-\alpha)(1+w)^2} t^{-2}, \tag{11}
\]

\[
\Lambda(t) = \frac{4\alpha}{3(1-\alpha)^2(1+w)^2} t^{-2}, \tag{12}
\]

where \(C_1\) is an integration constant.

It is evident from Eqs. (10)–(12) that \(\alpha \neq 1\) for physical validity. Moreover, a repulsive \(\Lambda\) demands positive \(\alpha\) via Eq. (12) while Eq. (11) shows that, for positive \(\rho\), the parameter \(\alpha\) must be less than 1 and imposes the constraint \(0 < \alpha < 1\). The case \(\alpha \geq 1\) is either nonphysical or incompatible with a time-dependent \(\Lambda\). This is because a solution with a variable \(\Lambda\) is possible in the presence of matter only when \(T_{ij}^\gamma \neq 0\) [49]. Again, since we are dealing with a non-zero \(\Lambda\), we have \(\dot{\Lambda} \neq 0\). This means that, when \(\Lambda \sim (\dot{a}/a)^2\), the expansion of the Universe never stops as long as \(\alpha \neq 0\).

Similarly, if we set \(\Lambda = \beta(\dot{a}/a)\) and \(\Lambda = 8\pi G\gamma \rho\), where \(\beta\) and \(\gamma\) are free parameters, then, for \(k = 0\), it can be very easily shown that in both cases the scale factor follows the same type of power laws as in Eq. (10) while, just as in Eqs. (11) and (12), \(\rho(t)\) and \(\Lambda(t)\) are inversely proportional to \(t\). It may be mentioned that for physical validity either \(\beta < 0\) or \(\beta > 3\) for the present model. On the other hand, for a non-negative, repulsive \(\Lambda\) one needs to impose the condition \(\gamma > 0\) while for positive \(\rho\) it should be \(\gamma > -1\). This means that \(\gamma\) is always a positive quantity.

4. Equivalence of three forms of dynamic \(\Lambda\)

Now, let us explore the interrelations between \(\alpha, \beta\) and \(\gamma\) and hence the equivalence of different forms of the dynamic cosmological terms, viz., \(\Lambda \sim (\dot{a}/a)^2\), \(\Lambda \sim \ddot{a}/a\) and \(\Lambda \sim \rho\).

From Eq. (10), differentiating it and then dividing by \(a\), we get

\[
t = \frac{2}{3(1-\alpha)(1+w)H} \tag{13}
\]

where \(H\) is the Hubble parameter, as mentioned earlier, and hence, for specific values of \(\alpha\) and \(w\), Eq. (13) shows that \(H \sim t^{-1}\). This point was indicated in the Introduction, and therefore we have omitted the case \(\Lambda \sim t^{-2}\) from the present investigation.

Using Eq. (13) in (11) and the definition of the cosmic matter density parameter \(\Omega_m(= 8\pi G\rho/3H^2)\), one gets

\[
\Omega_{ma} = 1 - \alpha \tag{14}
\]

where \(\Omega_{ma}\) is the cosmic energy density parameter for the \(\alpha\)-related dynamic \(\Lambda\)-model.

Again, using Eq. (13) in (12) and the definition of the cosmic vacuum energy density parameter \(\Omega_\Lambda = \Lambda/3H^2\), we have

\[
\Omega_{\Lambda a} = \alpha, \tag{15}
\]

where, in a similar fashion, \(\Omega_{\Lambda a}\) is the vacuum energy density parameter for the \(\alpha\)-related dynamic \(\Lambda\)-model.

Addition of Eqs. (14) and (15) yields

\[
\Omega_{ma} + \Omega_{\Lambda a} = 1 \tag{16}
\]

which is the relation between the cosmic matter- and vacuum density parameters for a flat \((k = 0)\) universe.

Equations similar to (16) can be obtained for \(\beta\)- and \(\gamma\)-related models as well. Thus, without loss of generality, we can write

\[
\Omega_{m0} = \Omega_{m\beta} = \Omega_{m\gamma} = \Omega_m, \tag{17}
\]

\[
\Omega_{\Lambda a} = \Omega_{\Lambda \beta} = \Omega_{\Lambda \gamma} = \Omega_\Lambda, \tag{18}
\]

where \(\Omega_m\) and \(\Omega_\Lambda\) are the cosmic matter and vacuum density parameters. Therefore, in the absence of curvature, one can obtain the general relation

\[
\Omega = \Omega_m + \Omega_\Lambda = 1. \tag{19}
\]

This analytical result is consistent with the observational constraint on the total energy density \(\Omega\) of the Universe, where \(\Omega = 1.00^{+0.25}_{-0.30}\) due to the MAXIMA-I flight and COBE-DMR experiment [37], \(\Omega = 1.05^{\pm 0.08}\) obtained from CBI-DMR observations [39], and \(\Omega = 1.01 \pm 0.03\) (68% CL) measured from the first acoustic peak in the angular power spectrum of CMB fluctuations [50].
Now, Eqs. (17) and (18) enable us to interrelate $\alpha$, $\beta$ and $\gamma$ with $\Omega_m$ and $\Omega_\Lambda$ as

$$\alpha = \Omega_\Lambda,$$

$$\beta = \frac{6\Omega_\Lambda}{2\Omega_\Lambda - \Omega_m(1 + 3w)},$$

$$\gamma = \frac{\Omega_\Lambda}{\Omega_m}.$$  

This result for the dust case of Eq. (21) corresponds to Arbab’s [14]. Thus, we find that the free parameter $\alpha$ here is nothing but the cosmic vacuum density parameter whereas $\gamma$ is the ratio of the cosmic vacuum and matter density parameters which, by virtue of Eq. (19), provides

$$\Omega = (1 + \gamma)\Omega_m = 1,$$

which is another relation for the total cosmic energy density in the case of a flat universe.

All the above general relations for $\alpha$, $\beta$ and $\gamma$ in terms of $\Omega_m$ and $\Omega_\Lambda$ also hold for the particular cases of dust ($\omega = 0$) and radiation ($\omega = 1/3$). It is interesting to note that, while the relations of $\alpha$ and $\gamma$ with the cosmic matter and vacuum density parameters are independent of $w$, the relations of $\beta$ with $\Omega_m$ and $\Omega_\Lambda$ are $w$-dependent.

It can easily be shown that the particular solutions of $\Lambda \sim (\dot{a}/a)^{2}$ model for dust and radiation cases become identical with their corresponding counterparts for the other models in terms of the time dependences of $a$, $\rho$ and $\Lambda$ when expressed in terms of $\Omega_m$ and $\Omega_\Lambda$. Therefore, these results imply that in $\Omega_m$ and $\Omega_\Lambda$ there are no distinctive features which could distinguish between the different forms of dynamic cosmological models, viz., $\Lambda \sim (\dot{a}/a)^{2}$, $\Lambda \sim \dot{a}/a$ and $\Lambda \sim \rho$. Thus, starting from any of our $\Lambda$-models, since they are equivalent, we can arrive at the other relations.

Now, from Eqs. (20)–(22), we find that the parameters involved in the three dynamical relations are connected by

$$\alpha = \frac{\beta(1 + 3w)}{3(\beta w + \beta - 2)} = \frac{\gamma}{1 + \gamma}.$$  

This again shows that the three forms $\Lambda = 3\alpha(\dot{a}/a)^2$, $\Lambda = \beta(\dot{a}/a)$ and $\Lambda = 8\pi G \gamma \rho$ are equivalent, and the three parameters $\alpha$, $\beta$ and $\gamma$ are connected by the relation (24). Thus it is possible to find out the identical physical features of others if any of the phenomenological $\Lambda$ relations is known. It can easily be seen that, for the dust case ($w = 0$), Eq. (24) relates $\alpha$ and $\beta$ as

$$\alpha = \frac{\beta}{3(\beta - 2)},$$

which is Arbab’s result [15]. Moreover, it can be observed that our $\gamma$ is identical to Majernik’s $\kappa$ [51,52], where

$$\kappa = \frac{1}{\Omega_m} - 1 = \frac{\Omega_\Lambda}{\Omega_m}$$

for the present situation in view of Eq. (22). He has also shown that this result, Eq. (26), is derivable from an ansatz by which $\Lambda$ is proportional to the stress-energy scalar $T = T_j^j$, the trace of the stress-energy tensor of ordinary matter $T_j^j$, and is Lorentz-invariant. In this regard, following Majernik [51], it can be mentioned here that determination of the parameter $\gamma$ entirely depends on the cosmic matter density parameter or both the matter and vacuum density parameters. Thus this relation constrains the value of $\gamma$ and will be discussed in the next section.

Another point to be mentioned here that Eq. (19) and hence (23), via (22), is nothing but another form of the Friedmann equation (3) for the flat Universe. Thus, it is interesting to note that Eq. (26) also represents the Friedmann equation. Therefore, starting from any of our $\Lambda$-models, since they are equivalent, we can arrive at the Friedmann field equation without any assumption.

5. Ranges of the parameters $\alpha$, $\beta$, $\gamma$

Recent measurements have given a wide range of values for $\Omega_{m0}$ and $\Omega_{\Lambda0}$. The first flight of the MAXIMA balloon-borne experiment (MAXIMA-I) combined with COBE-DMR resulted in $0.25 < \Omega_{m0} < 0.50$ and $0.45 < \Omega_{\Lambda0} < 0.75$ [37]. Observations of SNeIa combined with the total energy density constraints from CMB [50] and combined gravitational lens and stellar dynamical analysis [53] lead to $\Omega_{m0} \sim 0.3$ and $\Omega_{\Lambda0} \sim 0.7$. The pinpoint values of these parameters as obtained by Sivers et al. [39] and Spergel et al. [54] are $[\Omega_{m0}, \Omega_{\Lambda0}] = [0.34 \pm 0.12, 0.67\pm 0.13]$ and $[\Omega_{m0}, \Omega_{\Lambda0}] = [0.249^{+0.034}_{-0.024}, 0.719^{+0.021}_{-0.029}]$, respectively. These and other results are listed in Table 1.

Considering the values in Table 1, we particularly prefer the matter density parameter as $\Omega_{m0} = 0.330 \pm 0.035$ [49,50,55,56]. This gives us an opportunity to obtain ranges of $\alpha_0$, $\beta_0$ and $\gamma_0$ (the values of $\alpha$, $\beta$ and $\gamma$ at the present epoch) which can, using Eq. (24), be obtained as $0.635 \leq \alpha_0 \leq 0.705$, $3.417 \leq \beta_0 \leq 4.674$ and $1.739 \leq \gamma_0 \leq 2.389$ in the dust case. Thus we find that using our models we are able to obtain the range of $\beta$ smaller than Arbab’s [15] which was $3 < \beta < 4.5$ for dust.

Again, if we recall the quintessence equation of state $p_\Omega = w_\Omega \rho_\Omega$ where $w_\Omega = -\Omega_\Lambda$, we can easily obtain the relations between $\beta$ and $w_\Omega$ as $\beta = 6w_\Omega/(1 + 3w_\Omega)$ for $w = 0$. Using the above range of $\beta_0$, we can calculate the range of $w_\Omega$ in dust as $-0.705 \leq w_\Omega \leq -0.635$. It is interesting to note that the above result is consistent with the accepted range of $w_\Omega$ which is $-1 < w_\Omega < 0$. However, in the present investigation we are not concerned with the quintessence case and show the range of $w_\Omega$ as a check only.
Table 1: Values of $\Omega_m$ and $\Omega_\Lambda$ from various observational sources

| Source & Reference | Year | $\Omega_m$ | $\Omega_\Lambda$ |
|-------------------|------|-----------|-----------------|
| Efstathiou et al. [57] (SNeIa + CMB) | 1998 | 0.25$^{+0.18}_{-0.12}$ | 0.69$^{+0.17}_{-0.23}$ |
| Riess et al. [1] (SNeIa + MLCS) | 1998 | 0.24$^{+0.05}_{-0.24}$ | 0.70$^{+0.32}_{-0.48}$ |
| Perlmutter et al. [58] (SNLS) | 1999 | 0.4$\pm$ 0.1 | 0.7 |
| Balbi et al. [37] (MAXIMA-I+COBE) | 2001 | 0.25$\pm$ 0.50 | 0.45$\pm$ 0.75 |
| Reboi [50] (SNeIa + CMB) | 2003 | 0.30 | 0.70 |
| Koopmans et al. [53] (Lens + SD) | 2003 | 0.30 | 0.70 |
| Sievers et al. [39] (CBI) | 2003 | 0.34$\pm$ 0.12 | 0.67$^{+0.10}_{-0.13}$ |
| Barris et al. [59] (IA Deep Survey) | 2004 | 0.33 | 0.67 |
| Astier et al. [60] (SNLS) | 2006 | 0.31$\pm$ 0.21 | 0.80$\pm$ 0.31 |
| Spergel et al. [54] (WMAP + SNLS) | 2006 | 0.249$^{+0.024}_{-0.031}$ | 0.719$^{+0.021}_{-0.025}$ |

6. Features of the models

6.1. Physical parameters at the present accelerating epoch

Now, a search for the status of $\Lambda$ rests on some observational results from high-redshift type Ia Supernovae (SNeIa), the cosmic microwave background radiation (CMBR) and other sources which inform us that the present Universe is composed of about 30% of ordinary matter and 70% of dark energy. Thus, in the present Universe, the vacuum density parameter $\Omega_\Lambda$ is dominant over the matter density parameter $\Omega_m$. Determination of the Hubble parameter, a measure of the rate of cosmic expansion, has been done by several authors based on different values of density parameters as shown in Table 1. However, it is to be noted that there exists a certain amount of uncertainty in the value of $H_0$, as is obvious from Table 2.

The data of Table 2 indicate that the present value of the Hubble parameter is, in general, centralized at 72$\pm$8 kms$^{-1}$ Mpc$^{-1}$. Even the most recent value (73.4$^{+2.8}_{-3.8}$) as obtained from WMAP by Spergel et al. [54] lies well within this range. Assuming this value of $H_0$ and $\Omega_m = 0.330 \pm 0.035$ [49, 50, 55, 56], the present age ($t_0$), the present matter density ($\rho_0$), the present value of the cosmological term ($\Lambda_0$), and the value of the deceleration parameter at the present era ($q_0$) have been calculated using our equivalent models. All values of $\rho_0$ and $q_0$ are in nice agreement with the modern concept of an open, accelerating universe. Moreover, the values of $\Lambda_0$ support the idea of a small non-zero cosmological parameter which is slowly decreasing in time, and at present $\Lambda_0$ lies within $1 \times 10^{-35}$ s$^{-2} - 2 \times 10^{-35}$ s$^{-2}$, which agrees with the results of Carmeli [61] and Carmeli and Kuzmenko [62], where they obtain the value of 1.934 $\times 10^{-35}$ s$^{-2}$. All values of $\rho_0$ are one order of magnitude smaller than $10^{-29}$ g/cm$^3$, the critical density of the Universe. For the matter-dominated case, various results can be obtained for $t_0$ by finding $H_0$ for different $\Omega_m$ (see Table 3 for detail).

It is clear from Table 3 that, for the lower value of $\Omega_m$, the age becomes very high, whereas a higher value of the matter density parameter, say, $\Omega_m = 0.46$ provides a more realistic result for the age of the Universe with gradual increase of the Hubble parameter. The best result is therefore obtained for $\Omega_m = 0.46$ (the upper limit of Sievers et al. [39]) and $H_0 = 80$ km/s/Mpc.
Table 3: Age of the Universe from the present models

| $\Omega_{m0}$ | $H_0$ | $t_0$ |
|---------------|--------|--------|
| 0.33          | 64     | 27.79  |
|               | 72     | 24.70  |
|               | 80     | 22.23  |
| 0.365         | 64     | 25.13  |
|               | 72     | 22.33  |
|               | 80     | 20.10  |
| 0.40          | 64     | 22.93  |
|               | 72     | 20.38  |
|               | 80     | 18.34  |
| 0.46          | 64     | 19.94  |
|               | 72     | 17.72  |
|               | 80     | 15.95  |

$kms^{-1}Mpc^{-1}$ (the upper limit of Refs. [56, 69, 74, 75]) is 15.95 Gyr. This result exactly coincides with the upper limit of the value of Riess et al. [1] which is 14.2±1.7 Gyr as obtained from SNeIa observations and also very close to the values obtained by Sievers et al. [39] and Tegmark et al. [70] as predicted by the WMAP data and CMB observations.

6.2. Physical parameters in the past decelerating period

Using Eq. (10), one can obtain an expression for the deceleration parameter $q$ as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{3(1-\alpha)(1+w)}{2} - 1. \quad (27)$$

Thus, for an accelerating universe

$$\alpha > \frac{1+3w}{3(1+w)}. \quad (28)$$

From Eq. (28), it is evident that for dust ($w = 0$) an accelerating universe requires $\alpha > 1/3$. Now, $\alpha$ being the cosmic vacuum density parameter by virtue of Eq. (20), we find that our model fits an accelerating universe since the modern accepted value of $\Omega_0$ is about 0.7 (Refs. [72, 85–87] and see also Table 1) and is much larger than 1/3. Thus Eq. (19) implies that the value of $\Omega_m$ is 0.3, which provides $q_0 = -0.50 ± 0.05$ for the dust case which can nicely accommodate the currently accepted value related to the accelerating Universe [57, 88, 89]. Again, $q$ will be positive if $\alpha$ is less than 0.3. Thus, for a decelerating universe, the cosmic vacuum density parameter should be smaller than 0.3, which is also consistent with the modern ideas. Therefore, we find that, within our models, one can investigate accelerating as well as decelerating phases of the cosmic expansion since $q$ depends on $\alpha$.

Now, it has already been mentioned that the expanding Universe, which is about 14 Gyr old, entered into the present accelerating phase about 7 Gyr ago. Therefore, for about 8 Gyr earlier, i.e., when the Universe was about 6 Gyr old, it was passing through a period of deceleration. Let us try to estimate the values of some of the physical parameters when the age of the Universe was 6 Gyr. From Eq. (27) it is easy to obtain $q = (1.50\Omega_m - 1)$ for $w = 0$. We have already seen that $q$ will be positive for $\Omega_m > 0.66$. Putting $\Omega_m = 0.67$, we find that $q > 0$, i.e. the Universe was indeed in a decelerating phase. Assuming $\Omega_m = 0.67$ and $t = 6$ Gyr, we can estimate the values of $H$, $\rho$ and $\Lambda$, which are given by $H \sim 179$ km s$^{-1}$ Mpc$^{-1}$, $\rho = 3.3 \times 10^{-29}$ g cm$^{-3}$ and $\Lambda = 2.74 \times 10^{-35}$ s$^{-2}$, respectively. Similarly, for the radiation case, $\Omega_m$ and $q$ are related by $q = (2\Omega_m - 1)$. Thus $q$ will be positive for $\Omega_m > 0.5$. If we assume $\Omega_m = 0.6$ and $t = 6$ Gyr, then the estimated values of $H$, $\rho$ and $\Lambda$ are 135 km s$^{-1}$ Mpc$^{-1}$, $2.08 \times 10^{-29}$ g cm$^{-3}$ and $2.33 \times 10^{-35}$ s$^{-2}$, respectively. We thus find that in both cases ($w = 0, 1/3$) $\rho$ was above the critical density, which means that the expanding Universe with a decelerating mode had a closed geometry. The value of the Hubble parameter, a measure of the expansion rate of the Universe was slower in the radiation era than that of matter-dominated era. Also, we find that the value of $\Lambda$ was slightly above its present value, which justifies the idea of a dynamic $\Lambda$ which decreases very slowly with time. Finally, assuming the present value of the Hubble parameter as 72 km s$^{-1}$ Mpc$^{-1}$, we see that the rate of decrease of $H$ is about 13 km s$^{-1}$ Mpc$^{-1}$ Gyr$^{-1}$ for $w = 0$ and about 8 km s$^{-1}$ Mpc$^{-1}$ Gyr$^{-1}$ for $w = 1/3$.

7. Discussion

In the present investigation, choosing some specific forms of dynamical $\Lambda$, we were able to show the equivalence of those forms in terms of the solutions obtained. While Arbab [14] has shown the equivalence of the same three $\Lambda$-models in the context of a built-in cosmological constant of Rastall [18] and Al-Rawaf-Taha [26] type models of modified general relativity, we have shown the equivalence of the models with respect to their characteristic solutions in the framework of Einstein’s general relativity. It has already been mentioned that $\Lambda \sim H^2$ and $\Lambda \sim t^{-2}$ are identical. In this context, it is interesting to note that since $\ddot{a}/a$ is equal to $\dot{H} + H^2$ which is again $t^{-2}$, then $\Lambda \sim \ddot{a}/a$ is also identical with the above-mentioned cases. This is reflected in our solution sets via Eq. (24). Moreover, since $\ddot{a}/a = \dot{H} + H^2$, the $\Lambda \sim \ddot{a}/a$ model can be thought of as a combination of two models, viz., $\Lambda \sim \dot{H}$ and $\Lambda \sim H^2$. Thus the $\Lambda \sim \ddot{a}/a$ and $\Lambda \sim \dot{H}^2/a$ models become identical when $\dot{H} = 0$. Now, $\dot{H} = 0$ implies a constant $H$, which in turn implies exponential expansion and hence an inflationary scenario. Thus the idea of inflation is inherent in the phenomenological model $\Lambda \sim \ddot{a}/a$. Moreover, the $\Lambda \sim \dot{a}^2/a$ and $\Lambda \sim \dot{a}/a$ models cannot exist as separate
entities during inflation.

We have also established a relation between $\alpha$, $\beta$ and $\gamma$, the three parameters of the three forms of $\Lambda$ which ultimately yields $\Omega_m + \Omega_\Lambda = 1$, the relation between the cosmic matter and vacuum density parameters for a flat universe. It can be shown that this particular relation between the density parameters also holds in the radiation case. On the other hand, since $\Omega_\Lambda = \rho_\Lambda / \rho_c$ and $\Omega_m = \rho_m / \rho_c$, it is clear from Eqs. (20), (21) and (22) that $\beta$ and $\gamma$ are independent of $\rho_c$, the critical density of the Universe, whereas $\alpha$ depends on the critical density. It can also be shown that, while $\beta$ and $\gamma$ decrease with the age of the Universe, $\alpha$ increases as time passes.

Moreover, the present models represent a flat, accelerating Universe and do not suffer from the low-age problem like many FRW models. Also, since the present Universe is dark energy dominated, and the closest approximation to $t_0 \sim 20.10$ Gyr for the matter-dominated case can be obtained for the specific choice of $\Omega_m = 0.330 \pm 0.035$ and $H_0 = 72 \pm 8$ km s$^{-1}$ Mpc$^{-1}$, our models point at the upper accepted limits of $\Omega_m$ and $H_0$. As has been shown earlier in Table 2 that there is a certain amount of uncertainty in the value of $H_0$, the lower bound being 50 km s$^{-1}$ Mpc$^{-1}$ (see also Table 2 of Ref. [64] for some more cases of a lower bound), whereas the upper bound is 97 km s$^{-1}$ Mpc$^{-1}$. The related values of the age of the Universe for our models with these two extreme Hubble parameters are 25.52 Gyr and 13.15 Gyr, respectively, when the matter density parameter is 0.46. Therefore, our calculated value of $t_0$, which seems to be a bit over-aged and also favours a $\Lambda$-dominated universe, can be accommodated to the accepted age of the Universe within the error bar. Even though the values of $t_0$ in case $w = 0$ show an excess with respect to the presently accepted age of the Universe but, anyway, there is no low-age problem. Indeed, these values are much higher than the age of the globular clusters which is 12.5 ± 1.2 Gyr [90–95]. In this connection, we can also note that examples of higher age are not available in the contemporary literature [49, 96–102]. For example, Vishwakarma [49] obtained, for $\Omega_m = 0.330 \pm 0.035$ and $H_0 = 72 \pm 7$ km s$^{-1}$ Mpc$^{-1}$, a remarkably high age of the Universe, $t_0 \approx 27.4 \pm 5.6$ Gyr! But it is evident from Table 2 that, whatever the values of the Hubble parameter, the experimental results for the age of the Universe lie around 14 Gyr. For the present phenomenological models (including Vishwakarma’s case [49]), the age of the Universe is inversely proportional to the Hubble parameter. This provides a reasonable age of the Universe only for a higher value of the Hubble parameter, which is also clear from the above discussion. This is obviously a drawback of the present models unless a higher value of $H_0$ is observationally established in the future.

In this regard, we would like to discuss the causal connection of our models. We know that the proper distance $L(t)$ to the horizon is given by

$$L(t) = a(t) \int_0^t d\tau / a(\tau),$$

and if the integral diverges, the model is causally connected. Since, for $\Lambda \sim (\dot{a}/a)^2$, the scale factor $a(t)$ is given by Eq. (10), the proper distance $L(t)$ diverges if $1/3 < \alpha < 1$ or, in other words, if $1/3 < \Omega_\Lambda < 1$ for the dust case and $\alpha < 0.5$, i.e., $\Omega_\Lambda < 0.5$ for radiation. Since the present Universe is matter-dominated and the observational results indicate that at present $\Omega_\Lambda \sim 0.7$, the Universe is causally connected in our $\Lambda \sim (\dot{a}/a)^2$ model. Besides, since it has already been shown that the present three phenomenological $\Lambda$-models are equivalent, the causal connection of the Universe indicated in the above model implies that the other models are also causally connected.

It should be mentioned that Arbab [14] has put his models to the neoclassical tests like luminosity distance, angular diameter distance and gravitational lensing, whereas we have tested the viability of our models through age determination and some other measurements. So, in that respect our work can be thought of as complementary to Arbab’s investigation [14]. Perspective, we are studying new forms of a dynamic cosmological term, such as those from the renormalization group, and its confrontation with astrophysical observational data sets [103, 104], expecting that this global description can help one to better understand the mysterious dark energy nature and to alleviate the long-standing cosmological constant problem.

Finally, it is to be noted here that, in general, $w$ is a function of time [79, 105, 106]. But the current observational data can hardly distinguish between time-varying and constant equations of state [107, 108], as demonstrated in some works ( [108] and references therein). For this reason, $w$ is usually assigned a constant value, as has been done by Caldwell et al. [109] while dealing with a relation between scalar field models and the XCDM parameterization. Likewise, in the present work related to phenomenological $\Lambda$ models, without showing a complete time evolution of $w$ (which is no doubt a better representation), some specific cases are highlighted corresponding to constant $w$. A more accurate analysis may be made in a later work by considering $w = w(t)$.

**Appendix**

A comparative analysis of models

$$\Lambda = \alpha(\dot{a}/a)^2, \quad \Lambda = \beta(\dot{a}/a), \quad \Lambda = \gamma \rho. \quad (A.1)$$

can be, alternatively, done as follows. It can be shown that all three models do not differ from one another from both mathematical and physical points of view.
Let us denote
\[ f_1 = \alpha(\dot{a}/a)^2, \quad f_2 = \beta(\dot{a}/a), \quad f_3 = \gamma \rho. \] (A.2)
Then Einstein’s equations for spatially flat space-time used in this work (Eqs. (3) and (7)) can be written in the form
\[ f_1 + 0 f_2 - \frac{8\pi G}{3} f_3 + \frac{1}{3} \Lambda = 0, \] (A.3)
\[ 0 f_1 + f_2 + \frac{4\pi G}{3} (1 + 3w) f_3 + \frac{1}{3} \Lambda = 0. \] (A.4)
This is a set of linear algebraic equations for \( f_1, f_2 \) and \( f_3 \) with constant coefficients. If we add to this set any linear equation of the form
\[ af_1 + bf_2 + cf_3 = d\Lambda \] (A.5)
with constant coefficients \( a, b, c, d \), then the Einstein equations and this new equation make a closed set of three linear algebraic equations for \( f_1, f_2, f_3 \). Its solution for any real \( a, b, c, d \) can be written as
\[ f_1 = \frac{1}{\alpha} \Lambda, \quad f_2 = \frac{1}{\beta} \Lambda, \quad f_3 = \frac{1}{\gamma} \Lambda. \] (A.6)
A comparison of Eqs. (A.2) and (A.6) clearly shows that they are equivalent to Eq. (A.1). However, an important point is the arbitrariness of \( a, b, c, d \). Therefore, using one of the relations in Eq. (A.1) automatically leads to one of the other two relations, and consequently there is no difference in the dynamic behaviour of these models.

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