NORM-VARIATION OF CUBIC ERGODIC AVERAGES

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Abstract. In this paper we study cubic averages with respect to $d$ general commuting transformations and prove quantitative results on their convergence in the norm. The approach we are using is based on estimates for certain entangled multilinear singular integral forms, established recently by Durcik and Thiele.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a probability space and let $T_1, T_2, \ldots, T_d : X \to X$ be $d$ commuting, measure-preserving transformations on $X$. That means that for every $i \neq j$ we have $T_i T_j = T_j T_i$ and $\mu(T_i^{-1}(E)) = \mu(T_j^{-1}(E)) = \mu(E)$ for every $i$ and $E \in \mathcal{F}$. The study of multiple ergodic averages of the form

$$\frac{1}{n} \sum_{i=0}^{n-1} f_1(T_i^0 x) f_2(T_i^1 x) \cdots f_d(T_i^d x)$$

was motivated by the work of Furstenberg and others ([13], [14] and [15]) and it helped develop various tools in ergodic theory and topological dynamics. The norm convergence of such averages was shown by Tao [20], reproved by Austin [2] and Host [16] and once again by Walsh [21] in the more general case when the transformations generate a nilpotent group. Almost everywhere convergence of (1.1) is still an open problem when $d \geq 2$. For partial progress on that matter we refer the interested reader to [7] and [8]. Using an analytical approach, in [12] norm-variation estimates were established for (1.1) but only with respect to two commuting transformations; also see a related result in [19], discussing a simplified toy model.

Cubic ergodic averages are averages along certain cubical configurations. They appeared in the proof of the $L^2$ convergence of multiple ergodic averages (1.1) and can be found for instance in [16]. For any $2^d - 1$ measurable functions $f_j, j \in \{0, 1\}^d \setminus \{0\}$ on $X$ we define cubic ergodic averages with respect to $d$ commuting transformations as

$$M_n(\mathbf{f})(x) := \frac{1}{n^d} \sum_{i_1, \ldots, i_d=0}^{n-1} \prod_{j \in \{0, 1\}^d \setminus \{0\}} f_j \left( \prod_{l=1}^{d} T_j^{i_{j,l}} x \right).$$

In the above definition we use the notation $\mathbf{f}$ for the $(2^d - 1)$-tuple consisting of $2^d - 1$ functions $f_j, j \in \{0, 1\}^d \setminus \{0\}$, and $\mathbf{0}$ for the $d$-tuple $(0, \ldots, 0)$. In the ergodic theory literature one conveniently assumes that the functions $f_j$ are taken from the space $L^\infty(X)$.

The $L^2$ convergence of (1.2) was proved by Austin [2] and Host [16] using different methods. In the special case when $T_1 = T_2 = \cdots = T_d$, the pointwise convergence of (1.2) for almost every point $x \in X$ was proved by Assani [1], Chu and Frantzikinakis [8] and Huang, Shao and Ye [17]. The pointwise convergence in the case of general commuting
transformations was established by Donoso and Sun, first for two transformations in [6] and then in [7] for \(d\) transformations.

In this paper we are interested in proving a norm-variational estimate for (1.2), which reproves and quantifies their \(L^2\) convergence. In doing so we will not use the typical techniques from ergodic theory, but instead follow the harmonic analysis approach from [12]. In contrast with that paper, we are even able to prove norm-variation estimates for (1.2) with respect to \(d\) commuting transformations and not only two of them. The key ingredient in our proof is an estimate for a certain multilinear entangled singular integral form that appeared in a recent paper by Durcik and Thiele [11]. We also remark that the analytical techniques we are about to use are not sufficient to reprove almost everywhere convergence of the cubic ergodic averages. Quantifying their pointwise convergence is an interesting open problem.

Recall that for \(U \subseteq \mathbb{R}\) and \(1 \leq \varrho < \infty\) the \(\varrho\)-variation of an indexed collection \((a_n)_{n \in U}\) in a Banach space \(B\) is the quantity

\[
\|a_n\|_{V^\varrho(U,B)} := \sup_{m \in \mathbb{N}} \left( \sum_{j=1}^{m} \|a_{n_j} - a_{n_{j-1}}\|_B^{\varrho} \right)^{\frac{1}{\varrho}}.
\]

Usually \(U = \mathbb{N}\), in which case \((a_n)_{n \in U}\) is a sequence. Finiteness of the above quantity for any choice of \(\varrho < \infty\) certainly implies convergence of the sequence in question in the norm of the Banach space \(B\).

**Theorem 1.** Let \((X, \mathcal{F}, \mu)\) be a probability space and let \(T_1, \ldots, T_d\) be \(d\) measure-preserving transformations on that space. Then for any \(f_j \in L^\infty(X)\), \(j \in \{0, 1\}^d \setminus \{0\}\), \(1 \leq p < \infty\) and \(\varrho > \max\{2, p(2^d - 1)/2^{d-1}\}\) we have

\[
\|M_n(f)\|_{V^\varrho(N,L^p(X))} \leq C_{\varrho,p,d} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \|f_j\|_{L^\infty(X)}.
\]

In order to prove Theorem 1 we will first introduce the following **analytical averages** and establish certain estimates for them. For any function \(\varphi\) on \(\mathbb{R}\), for \(d\)-dimensional functions \(F_j \in L^{2^d}(\mathbb{R}^d)\), \(j \in \{0, 1\}^d \setminus \{0\}\), and \(r > 0\) we define the analytical averages as

\[
A^r_{\varphi}(\mathbb{F})(x) := \int_{\mathbb{R}^d} \prod_{j \in \{0, 1\}^d \setminus \{0\}} F_j(x + j \cdot s) \varphi_r(s_1) \cdots \varphi_r(s_d) \, ds,
\]

where \(\varphi_r(x) := r^{-1}\varphi(r^{-1}x)\) and \(j \cdot s = (j_1 s_1, \ldots, j_d s_d)\). Similarly as in (1.2) \(\mathbb{F}\) denotes the \((2^d - 1)\)-tuple consisting of \(2^d - 1\) functions \(F_j\), \(j \in \{0, 1\}^d \setminus \{0\}\). Bold letters will always denote \(d\)-tuples in \(\mathbb{R}^d\), for example \(j = (j_1, \ldots, j_d)\) etc. In particular, if \(\varphi = 1_{[0,1)}\), then we will write the corresponding analytical averages simply as \(A_r(\mathbb{F})\).

Theorem 1 will actually be a consequence of the following norm-variational estimate for (1.4).

**Theorem 2.** For any \(F_j \in L^{2^d}(\mathbb{R}^d)\), \(j \in \{0, 1\}^d \setminus \{0\}\), and \(\rho > 2\) we have

\[
\|A_n(\mathbb{F})\|_{V^\varrho(N,L^{2^d}(\mathbb{R}^d))} \leq C_{\varrho,d} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \|F_j\|_{L^{2^d}(\mathbb{R}^d)},
\]
where \((2^d)'\) is the conjugate exponent of \(2^d\).

Proof of Theorem 2 is presented in Section 2. The idea is to split the jumps into the long jumps (i.e. those corresponding to the scales \(r\) which are dyadic numbers \(2^k, k \in \mathbb{Z}\)) and the short jumps (i.e. those corresponding to \(r\) from a fixed interval of the form \([2^k, 2^{k+1}]\)). This is the usual approach to variational estimates, as can be seen, for instance, in [18] and [5]. The standard transition to Theorem 1 is given in Section 3.

2. Averages on \(\mathbb{R}^d\)

In this section we will prove certain estimates for the smooth analytical averages given by (1.4). Afterwards, to prove Theorem 2 we will compare \(A_r\) with \(A_r^\varphi\).

For two non-negative quantities \(A\) and \(B\) we write \(A \lesssim B\) if there exists an absolute constant \(C > 0\) such that \(A \leq CB\). More generally, if \(P\) is a set of parameters, we write \(A \lesssim P B\) if there exists a constant \(C_P\) depending only on the parameters in the set \(P\) such that \(A \leq C_P B\). For simplicity, we will denote by \(q\) the conjugate exponent of \(2^d\), i.e.

\[ q = (2^d)' = \frac{2^d}{2^d - 1}. \]

2.1. Deriving Theorem 2. First we will show how to derive Theorem 2 using the variational estimates for the smooth analytical averages defined by (1.4), given in the following proposition.

Proposition 3. For any Schwartz function \(\varphi\) and any \(F_j \in L^2(\mathbb{R}^d), j \in \{0, 1\}^d \setminus \{0\}\), we have

\[
\left( \sum_{j=1}^{J} \|A_{m_j}^\varphi(F) - A_{n_j}^\varphi(F)\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \lesssim_d J^{\frac{2-q}{2q}} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \|F_j\|_{L^2(\mathbb{R}^d)}
\]

for each choice of positive integers \(J\) and \(m_1 < n_1 < m_2 < n_2 \cdots < m_J < n_J\).

Proof of Theorem 2. Take a small \(\delta > 0\) (which will be chosen later) and a nonnegative compactly supported \(C^\infty\) function \(\varphi\) such that \(\int \varphi = 1\) and

\[
\|\varphi - \mathbb{1}_{[0,1]}\|_{L^1(\mathbb{R})} \leq \delta.
\]

Recall that

\[
A_r(F)(x) = A_r(F)\mathbb{1}_{[0,1]}(x) = \frac{1}{r^d} \int_{[0,r]^d} \prod_{j \in \{0, 1\}^d \setminus \{0\}} F_j(x + j \cdot s) ds.
\]

Our goal is to compare \(A_r\) with \(A_r^\varphi\). Observe that by (2.2) we have

\[
\int_{\mathbb{R}^d} r^{-d} |\varphi(r^{-1}s_1) \cdots \varphi(r^{-1}s_d) - \mathbb{1}_{[0,r)}(s_1) \cdots \mathbb{1}_{[0,r)}(s_d)| ds = \int_{\mathbb{R}^d} |\varphi(s_1) \cdots \varphi(s_d) - \mathbb{1}_{[0,1)}(s_1) \cdots \mathbb{1}_{[0,1)}(s_d)| ds \leq d\delta,
\]

for each choice of positive integers \(J\) and \(m_1 < n_1 < m_2 < n_2 \cdots < m_J < n_J\).
so that Minkowski’s integral inequality and Hölder’s inequality give

\begin{equation}
\|A^q_r(\mathcal{F}) - A_r(\mathcal{F})\|_{L^q(\mathbb{R}^d)} \leq d\delta \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2d}(\mathbb{R}^d)}.
\end{equation}

We now take a positive integer \(J\) and arbitrary indices \(m_1 < n_1 \leq m_2 < n_1 \leq \cdots \leq m_J < n_J\).

Combining Proposition 3 with (2.4) gives

\begin{equation}
\left( \sum_{j=1}^{J} \|A_{m_j}(\mathcal{F}) - A_{n_j}(\mathcal{F})\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \leq \left( C_d J \frac{2^q}{2q} + 2d\delta J^\frac{1}{q} \right) \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2d}(\mathbb{R}^d)},
\end{equation}

where \(C_d\) is the actual constant implied in (2.1), and we observed that it depends only on \(d\).

In order to show (1.5) we now normalize \(F_j\) so that

\begin{equation}
\|F_j\|_{L^{2d}(\mathbb{R}^d)} = 1, \text{ for every } j \in \{0,1\}^d \setminus \{0\},
\end{equation}

fix \(0 < \varepsilon < 1\) and choose

\[ \delta = \frac{\varepsilon}{4d}. \]

We assume that the sequence of analytical averages \((A_n(\mathcal{F}))_{n=1}^\infty\) has at least \(J\) \(\varepsilon\)-jumps in the \(L^q\) norm. Let those jumps correspond exactly to the indices \(m_j, n_j\) for \(j = 1, \ldots, J\), i.e.

\[ \|A_{m_j}(\mathcal{F}) - A_{n_j}(\mathcal{F})\|_{L^q(\mathbb{R}^d)} \geq \varepsilon, \quad j = 1, \ldots, J. \]

Applying (2.5) to those indices gives us a bootstrapping estimate

\[ \varepsilon J^\frac{1}{q} \leq C_d J \frac{2^q}{2q} + 2d\delta J^\frac{1}{q} \leq C_d J \frac{2^q}{2q} + \frac{\varepsilon}{2} J^\frac{1}{q}, \]

which implies \(J \lesssim d^{-2}\).

Since \(F_j\) are normalized as in (2.6), to finally prove (1.5) we will show that

\begin{equation}
\sum_{j=1}^{m} \|A_{n_j}(\mathcal{F}) - A_{n_{j-1}}(\mathcal{F})\|_{L^q(X)} \lesssim d \varepsilon d \quad 1
\end{equation}

holds for any choice of positive integers \(m\) and \(n_0 < n_1 < \cdots < n_m\). From the previous discussion we conclude that the number of indices \(j\) such that

\[ 2^{-k} < \|A_{n_j}(\mathcal{F}) - A_{n_{j-1}}(\mathcal{F})\|_{L^q(\mathbb{R}^d)} \leq 2^{-k+1} \]

is at most \(C_d 2^{2k}\) for any integer \(k \geq 0\), where \(C_d\) is a constant depending only on \(d\). The sum on the left-hand side of (2.7) is then bounded by

\[ C_d 2^\varepsilon \sum_{k=0}^{\infty} 2^{(2-\varepsilon)k}, \]

which is finite since \(\varepsilon > 2\). \(\Box\)
2.2. Proof of Proposition 3 The standard separation into long and short jumps reduces Proposition 3 to showing the following two estimates.

**Proposition 4.** For any Schwartz function $\varphi$ and any $F_j \in L^2(\mathbb{R}^d)$, $j \in \{0, 1\}^d \setminus \{0\}$, we have

$$
\left( \sum_{j=1}^{J} \left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \lesssim_d J^{\frac{2-q}{2q}} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \| F_j \|_{L^2(\mathbb{R}^d)}
$$

for each choice of positive integer $J$ and integers $k_1 < l_1 < k_2 < l_2 < \cdots < k_J < l_J$.

**Proposition 5.** For any Schwartz function $\varphi$ and any $F_j \in L^2(\mathbb{R}^d)$, $j \in \{0, 1\}^d \setminus \{0\}$, and any finite set $\mathcal{J} \subset \mathbb{Z}$, we have

$$
\left( \sum_{j \in \mathcal{J}} \left\| A_j^\varphi (F) \right\|_{V^q(\{2j, 2j+1\}, L^q(\mathbb{R}^d))}^q \right)^{\frac{1}{q}} \lesssim_d \left| \mathcal{J} \right|^{\frac{2-q}{2q}} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \| F_j \|_{L^2(\mathbb{R}^d)},
$$

where $|\mathcal{J}|$ denotes the size of the set $\mathcal{J}$.

We will derive Proposition 3 from Propositions 4 and 5 following the standard approach described in [18]. Let $J$ and $m_1 < n_1 \leq m_2 < n_2 \leq \cdots \leq m_J < n_J$ be arbitrary positive integers. We will separate $j = 1, \ldots, J$ into two groups

$$
\mathcal{J}_S = \{ j : [m_j, n_j] \subset [2^k, 2^{k+1}) \text{ for some } k \in \mathbb{Z} \},
$$

$$
\mathcal{J}_L = \{ j : m_j < 2^k \leq n_j \text{ for some } k \in \mathbb{Z} \}.
$$

Since for every $p < 1$ and $a, b \geq 0$ we have $(a + b)^p \leq a^p + b^p$, observe that

$$
\left( \sum_{j=1}^{J} \left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \leq \left( \sum_{j \in \mathcal{J}_S} \left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}

+ \left( \sum_{j \in \mathcal{J}_L} \left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}.
$$

For the first term on the right hand side we get an estimate directly from (2.9)

$$
\left( \sum_{j \in \mathcal{J}_S} \left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \lesssim_d \left| \mathcal{J}_S \right|^{\frac{2-q}{2q}} \prod_{j \in \{0, 1\}^d \setminus \{0\}} \| F_j \|_{L^2(\mathbb{R}^d)}.
$$

Now for every $j \in \mathcal{J}_L$ choose $k_j < l_j$ such that

$$
2^{k_j} \leq m_j < 2^{k_j+1} \quad \text{and} \quad 2^{l_j} < n_j \leq 2^{l_j+1}.
$$

Then

$$
\left\| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)} \leq \left\| A_{m_j}^\varphi (F) - A_{2^{k_j}}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}

+ \left\| A_{2^{k_j}}^\varphi (F) - A_{2^{k_j+1}}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)} + \left\| A_{2^{k_j+1}}^\varphi (F) - A_{n_j}^\varphi (F) \right\|_{L^q(\mathbb{R}^d)}.
$$
so Minkowski’s inequality gives us

\[
\left( \sum_{j \in \mathcal{J}_L} \| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \leq \left( \sum_{j \in \mathcal{J}_L} \| A_{m_j}^\varphi (F) - A_{2^k_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}
\]

\[
+ \left( \sum_{j \in \mathcal{J}_L} \| A_{2^k_j}^\varphi (F) - A_{2^l_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}
\]

\[
+ \left( \sum_{j \in \mathcal{J}_L} \| A_{2^l_j}^\varphi (F) - A_{n_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}.
\]

Applying (2.8) to the second term on the right hand side and (2.9) to the first and the third terms implies

\[
(2.11) \quad \left( \sum_{j \in \mathcal{J}_L} \| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \lesssim d \left( |\mathcal{J}_L|^{\frac{2}{dq}} + 2 |\mathcal{J}_L|^{\frac{2}{dq}} \right) \prod_{j \in \{0,1\}^d \setminus \{0\}} \| F_j \|_{L^{2d}(\mathbb{R}^d)}.
\]

Applying (2.11) to the second term on the right hand side and (2.9) to the first and the third terms implies

\[
(2.11) \quad \left( \sum_{j \in \mathcal{J}_L} \| A_{m_j}^\varphi (F) - A_{n_j}^\varphi (F) \|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \lesssim d \left( |\mathcal{J}_L|^{\frac{2}{dq}} + 2 |\mathcal{J}_L|^{\frac{2}{dq}} \right) \prod_{j \in \{0,1\}^d \setminus \{0\}} \| F_j \|_{L^{2d}(\mathbb{R}^d)}.
\]

Finally, since $|\mathcal{J}_L|, |\mathcal{J}_S| \leq J$ combining (2.10) and (2.11) implies exactly (2.1).

### 2.3. Proof of Propositions 4 and 5.

The key ingredient in proving the propositions taking care of the long and short jumps will be the following special case of a result from [11], which shows $L^{2d}$ estimates for a certain type of entangled multilinear singular integral forms.

**Lemma 6** (Durcik and Thiele [11]). Let $K$ be a $d$-dimensional Calderón-Zygmund kernel such that

\[
|\partial^\alpha \hat{K}(\xi)| \lesssim \| \xi \|_{L^2}^{-|\alpha|}
\]

for all multi-indices $\alpha$ up to some large finite order. Then for any $F_j \in L^{2d}(\mathbb{R}^d), j \in \{0,1\}^d$, the multilinear form defined as

\[
\Lambda((F_j)_{j \in \{0,1\}^d}) = \int_{\mathbb{R}^d} K(s) \prod_{j \in \{0,1\}^d} F_j(x + j \cdot s) ds dx
\]

satisfies

\[
(2.13) \quad |\Lambda((F_j)_{j \in \{0,1\}^d})| \lesssim d \prod_{j \in \{0,1\}^d} \| F_j \|_{L^{2d}(\mathbb{R}^d)}.
\]

The kernels we are going to use will satisfy (2.12) for all multi-indices $\alpha$. Note that the constant in (2.13) does not fully depend on the kernel $K$, but rather just on the implicit constants in (2.12).

Estimate (2.13) for quadrilinear forms, i.e., when $d = 2$, was already proven by Durcik in [7]. On the other hand, in [10] the previous lemma was generalized to higher dimensions, i.e., to the functions on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$.

**Proof of Proposition 4.** Let $J$ and

\[ k_1 < l_1 \leq k_2 < l_2 \leq \cdots \leq k_J < l_J \]
be arbitrary integers. Now take a sequence of real numbers \((\varepsilon_k)_{k \in \mathbb{Z}}\) such that \(|\varepsilon_k| \leq 1\) for every \(k \in \mathbb{Z}\) and define the kernel

\[
K_1(s) = \sum_{k=k_1+1}^{l_j} \varepsilon_k (\varphi_{2k-1}(s_1) \cdots \varphi_{2k-1}(s_d) - \varphi_{2k}(s_1) \cdots \varphi_{2k}(s_d)).
\]

By introducing the function

\[
\psi(t) := 2\varphi(2t) - \varphi(t)
\]

since

\[
\psi_{2k+1}(t) = \varphi_{2k}(t) - \varphi_{2k+1}(t),
\]

we can represent the kernel \(K_1(s)\) as

\[
K_1(s) = \sum_{j \in \{0,1\}^d \setminus \{0\}} \sum_{k=k_1+1}^{l_j} \varepsilon_k \varphi_{2k}^{(j_1)}(s_1) \cdots \varphi_{2k}^{(j_d)}(s_d),
\]

where

\[
\varphi_{2k}^{(0)} = \varphi_{2k} \quad \text{and} \quad \varphi_{2k}^{(1)} = \psi_{2k}.
\]

Since \(\int \psi = 0\), it is easy to see that \(K_1(s)\) satisfies the desired kernel estimates (2.12), even uniformly in the numbers \(\varepsilon_k\) (see for example [1]). Applying (2.13) to the corresponding multilinear form

\[
\Lambda_1((F_j)_{j \in \{0,1\}^d}) = \int_{\mathbb{R}^{2d}} \prod_{j \in \{0,1\}^d} F_j(x + j \cdot s) K_1(s) \, ds,
\]

gives us

\[(2.14) \quad |\Lambda_1((F_j)_{j \in \{0,1\}^d})| \lesssim_d \prod_{j \in \{0,1\}^d} \|F_j\|_{L^{2d}(\mathbb{R}^d)}.
\]

By expanding the kernel \(K_1\) and using the definition of the analytical averages (1.4) we see that

\[
\Lambda_1((F_j)_{j \in \{0,1\}^d}) = \int_{\mathbb{R}^d} \left( \sum_{k=k_1+1}^{l_j} \varepsilon_k (A^0_{2k-1}(F) - A^0_{2k}(F)) \right) F_0(x) \, dx,
\]

so (2.14) implies

\[
\left\| \sum_{k=k_1+1}^{l_j} \varepsilon_k (A^0_{2k-1}(F) - A^0_{2k}(F)) \right\|_{L^d(\mathbb{R}^d)} \lesssim_d \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2d}(\mathbb{R}^d)}.
\]

Now we can apply Khintchine’s inequality for random \(\pm\) signs \(\varepsilon_j\) to get

\[(2.15) \quad \left\| \left( \sum_{j=1}^J |A^0_{2j}(F) - A^0_{2j}(F)|^2 \right)^{\frac{1}{2}} \right\|_{L^d(\mathbb{R}^d)} \lesssim_d \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2d}(\mathbb{R}^d)}.
\]

Finally, from (2.15) and the power mean inequality

\[
\left( \frac{1}{J} \sum_{j=1}^J |a_j|^q \right)^{\frac{1}{q}} \leq \left( \frac{1}{J} \sum_{j=1}^J |a_j|^2 \right)^{\frac{1}{2}},
\]

we have

\[
\left( \frac{1}{J} \sum_{j=1}^J |A^0_{2j}(F) - A^0_{2j}(F)|^q \right)^{\frac{1}{q}} \leq \left( \frac{1}{J} \sum_{j=1}^J |A^0_{2j}(F) - A^0_{2j}(F)|^2 \right)^{\frac{1}{2}}.
\]
we get
\[
\left( \sum_{j=1}^{J} \left\| A_{2,j}^{r} (F) - A_{2,j}^{r} (F) \right\|_{L^{q}(\mathbb{R}^{d})}^{q} \right)^{\frac{1}{q}} \leq d J^{2-q} \prod_{j \in \{0,1\}^{d} \setminus \{0\}} \left\| F_{j} \right\|_{L^{2q}(\mathbb{R}^{d})}. \]

Proof of Proposition \[\Box\] Define \( \vartheta(s) := (s \varphi(s))' \). This implies
\[
\vartheta_{r}(s) = -r \vartheta_{r} (\varphi_{r}(s)),
\]
so
\[
-r \vartheta_{r} (A_{r}^{p} (F)(x)) = \int_{\mathbb{R}^{d}} \prod_{j \in \{0,1\}^{d} \setminus \{0\}} F_{j}(x + j \cdot s) \left( \vartheta_{r}(s_{1}) \cdots \varphi_{r}(s_{d}) + \cdots \varphi_{r}(s_{1}) \cdots \vartheta_{r}(s_{d}) \right) ds
\]
\[
= \int_{\mathbb{R}^{d}} \prod_{j \in \{0,1\}^{d} \setminus \{0\}} F_{j}(x + j \cdot s) \left( \sum_{i=1}^{d} \left( \vartheta_{r}(s_{i}) \prod_{k=1}^{d} \varphi_{r}(s_{k}) \right) \right) ds
\]
\[
=: B_{r}^{\vartheta,\varphi}(F)(x).
\]
We claim that for any \( j \in \mathbb{Z} \) and \( 2^{j} \leq r_{0} < r_{1} < \cdots < r_{m} \leq 2^{j+1} \) we have
\[
(2.16) \quad \sum_{i=1}^{m} \left\| A_{r_{i}}^{p} (F) - A_{r_{i-1}}^{p} (F) \right\|_{L^{q}(\mathbb{R}^{d})}^{q} \leq \int_{\mathbb{R}^{d}} \int_{1}^{2} |B_{r}^{\vartheta,\varphi}(F)(x)|^{q} dr dx.
\]
To prove the claim it is enough to consider \( j = 0 \) since the left-hand side is invariant under simultaneously changing \( r_{i} \) to \( 2^{-j} r_{i} \), \( \varphi \) to \( \varphi \cdot 2 \), and \( \vartheta \) to \( \vartheta 2 \). We will denote \( I_{i} = [r_{i-1}, r_{i}] \subset [1, 2] \). Applying the fundamental theorem of calculus in variable \( r \) gives us
\[
\int_{\mathbb{R}^{d}} \left| A_{r}^{p} (F) - A_{r}^{p} (F) \right|^{q} dx = \int_{\mathbb{R}^{d}} \left| \int_{I_{i}} -r \vartheta_{r} (A_{r}^{p} (F)(x)) \frac{dr}{r} \right|^{q} dx
\]
\[
= \int_{\mathbb{R}^{d}} \left| \int_{I_{i}} (B_{r}^{\vartheta,\varphi}(F)(x)) \frac{dr}{r} \right|^{q} dx
\]
\[
\leq \int_{\mathbb{R}^{d}} \int_{I_{i}} \left| B_{r}^{\vartheta,\varphi}(F)(x) \right|^{q} dr dx,
\]
where the last inequality follows by using Jensen’s inequality and \( r \geq 1 \). Summing over all \( i = 1, 2, \ldots, m \) and using the disjointness of \( I_{i} \) we obtain the desired estimate \((2.16)\) for \( j = 0 \) and hence also for all \( j \in \mathbb{Z} \).

Now using \((2.16)\) we can estimate the left-hand side of \((2.1)\) as
\[
(2.17) \quad \sum_{j \in \mathcal{J}} \left\| A_{r}^{p} (F) \right\|_{L^{q}(\mathbb{R}^{d})}^{q} \leq \int_{1}^{2} \sum_{j \in \mathcal{J}} \int_{\mathbb{R}^{d}} \left| B_{r}^{\vartheta,\varphi}(F)(x) \right|^{q} dx dr
\]
\[
= \int_{1}^{2} \sum_{j \in \mathcal{J}} \left\| B_{r}^{\vartheta,\varphi}(F) \right\|_{L^{q}(\mathbb{R}^{d})}^{q} dr.
\]
The rest of the proof is now following the same outline as the proof of Proposition 4. For a fixed \( r \in [1, 2] \) define the kernel

\[
K_2(s) = \sum_{j \in J} \varepsilon_j \left( \sum_{i=1}^{d} (\vartheta_{2/r}(s_i) \prod_{k \neq i}^{d} \varphi_{2/r}(s_k)) \right),
\]

where \( \varepsilon_j \) are real numbers such that \( |\varepsilon_j| \leq 1 \).

Since \( \int \vartheta = 0 \), \( K_2(s) \) again satisfies (2.12) so we can apply (2.13) to the corresponding multilinear form

\[
\Lambda_2((F_j)_{j \in \{0,1\}^d}) = \int_{\mathbb{R}^d} \left( \sum_{j \in J} \varepsilon_j B_r^{\vartheta_j, \varphi_j}(\mathbb{F})(x) \right) F_0(x) \, dx
\]

and get

\[
\left\| \sum_{j \in J} \varepsilon_j B_r^{\vartheta_j, \varphi_j}(\mathbb{F}) \right\|_{L^q(\mathbb{R}^d)} \lesssim_d \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2q}(\mathbb{R}^d)}.
\]

If we proceed as in Proposition 4 by applying first Khintchine’s inequality and then the power mean inequality, we finally obtain

\[
(\sum_{j \in J} \left\| B_r^{\vartheta_j, \varphi_j}(\mathbb{F}) \right\|_{L^q(\mathbb{R}^d)} \right)^{\frac{1}{q}} \lesssim_d |J| \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2q}(\mathbb{R}^d)},
\]

where \( |J| \) denotes the size of the set \( J \). Combining (2.17) and (2.18) gives us (2.9). \( \square \)

3. TRANSITION TO ERGODIC AVERAGES

With Theorem 2 proved, the transition to ergodic averages is now standard and is described for instance in [20]. Here we are using a more straightforward approach, like the one explained in [12]. Therefore we only give the basic idea and omit the details.

Take \( J \in \mathbb{N} \), arbitrary positive integers \( n_0 < n_1 < \cdots < n_J \) and \( \rho > 2 \). For the functions \( F_j \in L^{2q}(\mathbb{R}^d) \), \( j \in \{0,1\}^d \setminus \{0\} \), Theorem 2 gives us

\[
\sum_{j=1}^{J} \left\| A_{n_j}(\mathbb{F}) - A_{n_{j-1}}(\mathbb{F}) \right\|_{L^q(\mathbb{R}^d)} \lesssim_{\rho,d} \prod_{j \in \{0,1\}^d \setminus \{0\}} \|F_j\|_{L^{2q}(\mathbb{R}^d)}.
\]

We will now transfer the estimate from \( \mathbb{R}^d \) to \( \mathbb{Z}^d \). For the functions \( \tilde{F}_j : \mathbb{Z}^2 \to \mathbb{C} \), \( j \in \{0,1\}^d \setminus \{0\} \), we define

\[
\tilde{A}_n(\tilde{\mathbb{F}})(k) := \frac{1}{n^d} \sum_{i_{1},...i_{d}:0}^{n-1} \prod_{j \in \{0,1\}^d \setminus \{0\}} \tilde{F}_j(k+j \cdot i),
\]

for \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}^d \).

In order to compare \( \tilde{A}_n(\tilde{\mathbb{F}}) \) to the averages \( A_n(\mathbb{F}) \) of some functions on \( \mathbb{R}^d \) we define \( F_j : \mathbb{R}^d \to \mathbb{R} \) to be

\[
F_j(x) := \tilde{F}_j([x_1], \ldots, [x_d]) = \sum_{i \in \mathbb{Z}^d} \tilde{F}_j(i) \mathbb{1}_{[i_{1},i_{1}+1]}(x_1) \cdots \mathbb{1}_{[i_{d},i_{d}+1]}(x_d),
\]

for every \( j \in \{0,1\}^d \setminus \{0\} \).
Since 
\[ \|\tilde{F}_j\|_{L^q(\mathbb{Z}^d)} = \|F_j\|_{L^q(\mathbb{R}^d)}, \]
it can easily be seen that 
\[ \left| \|A_{n_j}(F) - A_{n_{j-1}}(F)\|_{L^q(\mathbb{R}^d)} - \|\tilde{A}_{n_j}(F) - \tilde{A}_{n_{j-1}}(F)\|_{L^q(\mathbb{Z}^d)} \right| \lesssim \sum_{j=0}^{2^{d+1}} \prod_{J \in \{0,1\}^d \setminus \{0\}} \|\tilde{F}_j\|_{L^q(\mathbb{Z}^d)}. \]

Finally, combining the above estimate with (3.1) and \( \sum_{j=1}^{J} n_j^{-q} \lesssim 1 \) gives 
\[ \sum_{j=1}^{J} \|\tilde{A}_{n_j}(F) - \tilde{A}_{n_{j-1}}(F)\|_{L^q(\mathbb{R}^d)} \lesssim_{\theta,d} \prod_{J \in \{0,1\}^d \setminus \{0\}} \|\tilde{F}_j\|_{L^q(\mathbb{Z}^d)}. \]

Now we transfer to the probability space \((X, \mathcal{F}, \mu)\). First let \( f_j \in L^\infty(X) \), \( j \in \{0,1\}^d \setminus \{0\} \), take a point \( x \in X \) and a positive integer \( N \geq n_j \). We define the functions \( \tilde{F}_j^{x,N} : \mathbb{Z}^d \to \mathbb{C} \), \( j \in \{0,1\}^d \setminus \{0\} \), along the forward trajectory of \( x \) by 
\[ \tilde{F}_j^{x,N}(k) := \begin{cases} f_j(T_1^{k_1} \cdots T_d^{k_d} x) & \text{if } k_1, \ldots, k_d \in \mathbb{Z}, 0 \leq k_1, \ldots, k_d \leq 2N - 1, \\ 0 & \text{otherwise}. \end{cases} \]

Since the transformations are commuting, we have 
\[ M_n(f)(T_1^{k_1} \cdots T_d^{k_d} x) = \tilde{A}_n(\tilde{F}_j^{x,N})(k), \]
for all integers \( 0 \leq i_1, \ldots, i_d < N \) and \( 0 < n \leq N \), where again \( \tilde{F}_j^{x,N} \) is the \((2d-1)\)-tuple consisting of functions \( \tilde{F}_j^{x,N} \), \( j \in \{0,1\}^d \setminus \{0\} \). Since the transformations are measure preserving it allows us to conclude 
\[ \|M_{n_j}(f) - M_{n_{j-1}}(f)\|_{L^q(X)}^q \leq \frac{1}{N^d} \int_X \|\tilde{A}_{n_j}(\tilde{F}_j^{x,N}) - \tilde{A}_{n_{j-1}}(\tilde{F}_j^{x,N})\|_{L^q(\mathbb{Z}^d)}^q d\mu(x) \]
and then by Jensen’s inequality since \( \varrho > q \) 
\[ \|M_{n_j}(f) - M_{n_{j-1}}(f)\|_{L^q(X)}^\varrho \leq \frac{1}{N^{d\varrho}} \int_X \|\tilde{A}_{n_j}(\tilde{F}_j^{x,N}) - \tilde{A}_{n_{j-1}}(\tilde{F}_j^{x,N})\|_{L^q(\mathbb{Z}^d)}^\varrho d\mu(x). \]

It can also easily be seen that 
\[ \|\tilde{F}_j^{x,N}\|_{L^q(\mathbb{Z}^d)} \lesssim (2N)^{d/2} \|f_j\|_{L^\infty(X)}. \]

We can now apply (3.3) to the functions \( \tilde{F}_j^{x,N} \), \( j \in \{0,1\}^d \setminus \{0\} \) to get 
\[ \sum_{j=1}^{J} \|M_{n_j}(f) - M_{n_{j-1}}(f)\|_{L^q(X)^\varrho} \lesssim_{\theta,d} N^{-d\varrho} \left( N^{d\varrho} \right)^{2d-1} \prod_{J \in \{0,1\}^d \setminus \{0\}} \|f_j\|_{L^\infty(X)}, \]
for any \( n_0 < n_1 \leq \cdots < n_J \). Since \( q = 2d/(2d-1) \), this of course implies 
\[ \|M_n(f)\|_{L^q(N(L^q(X)))} \leq C_{\theta,d} \prod_{J \in \{0,1\}^d \setminus \{0\}} \|f_j\|_{L^\infty(X)} \]
for arbitrary functions \( f_j \in L^\infty(X) \).
Finally, we are ready to prove (1.3). In the case $p \leq q$ we can use the monotonicity of $L^p$ norms on a probability space to get

$$
\left\| M_{n_j}(f) - M_{n_{j-1}}(f) \right\|_{L^p(X)} \leq \left\| M_{n_j}(f) - M_{n_{j-1}}(f) \right\|_{L^q(X)}.
$$

For $p > q$ by their log-convexity we have

$$
\left\| M_{n_j}(f) - M_{n_{j-1}}(f) \right\|_{L^p(X)} 
\leq \left(2 \prod_{j \in \{0,1\} \setminus \{0\}} \| f_j \|_{L^\infty(X)} \right)^{1 - \frac{2}{p}} \left\| M_{n_j}(f) - M_{n_{j-1}}(f) \right\|_{L^q(X)}^{\frac{2}{q}}.
$$

In both cases we apply (3.4). In the second one we need to replace $q$ with $qq/p$ and for that purpose we need the condition $qq/p > 2$.

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