The generic limit set of cellular automata

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Abstract
In this article, we consider a topological dynamical system. The generic limit set is the smallest closed subset which has a comeager realm of attraction. We study some of its topological properties, and the links with equicontinuity and sensitivity. We emphasize the case of cellular automata, for which the generic limit set is included in all subshift attractors, and discuss directional dynamics, as well as the link with measure-theoretical similar notions.

Keywords: cellular automata, basin of attraction, limit set, attractor, directional dynamics, Baire category, symbolic dynamics.

1 Introduction
In topological dynamics, the limit set is the set of points that appear arbitrarily late during the evolution of the dynamical system (see [4]). But the latter may include points which look transient, because they do not appear arbitrarily late around any orbit. To capture only the points that can be approached by iteration of the dynamical system, one can study the asymptotic set, which consists of all limit points of orbits (see [8, 12]). Nevertheless, this set still includes some points that are, in some sense, very unlikely to be seen in the dynamics.

J. Milnor, interested in the dynamics on the space of measures, introduced in [22] the notion of likely limit set, that provides a useful tool for studying asymptotic behavior for almost all orbits. Following the former reference, the likely limit set is the smallest closed set that includes the asymptotic set of a set of full measure. More precisely, it is the smallest closed set that has a realm of attraction of measure one. Equivalently, the likely limit set is the unique maximal \(\mu\)-attractor (see [22, 15, 26]).

J. Milnor also implicitly defines a topological version of the same intuitive idea, that he calls the generic limit set. The goal of our article is to formalize this concept. In other words, we focus on the asymptotic behavior for almost all orbits in the sense of Baire category theory. We study some topological properties of the generic limit set, which is the smallest closed set that has a comeager realm of attraction.

For cellular automata, we know that all subshift attractors have a dense open realm of attraction (see [14, 9, 17, 16]), thus the generic limit set is included in all subshift attractors.

M. Delacourt et al. studied directional dynamics of cellular automata (see [7]): qualitative behaviour (equicontinuity, sensitivity, expansiveness) appear along arbitrary curves in space-time. A section of the present article is devoted to studying the generic limit set of cellular automata which have two directions of almost equicontinuity.
The paper is structured as follows: In Section 2, we provide the basic background on the subject of topological dynamical systems and cellular automata. In Section 3, we show some preliminary results about attractors, limit sets, realms. In Section 4, we define the generic limit set and prove the main results about it. In Section 5, we show some consequences on directional dynamics of cellular automata. In Section 6, we compare the generic limit set with similar measure-theoretical concepts. In Section 7, we give some simple examples.

2 Preliminaries

2.1 Topology

In this article, \((X,d)\) is a compact metric space. We put \(B_\delta(x) = \{ y \in X | d(x,y) < \delta \}\) and call it the open ball with center \(x \in X\) and radius \(\delta > 0\). For \(U \subseteq X\), we also write \(B_\delta(U) = \{ y \in X | d(U,y) < \delta \}\) and \(\overline{B}_\delta(U)\) its closure.

A subset \(U \subseteq X\) is called comeager in \(X\) if it includes a countable intersection of dense open sets. A subset \(U\) is meager if \(X \setminus U\) is comeager in \(X\). By the Baire category theorem, the intersection of countably many dense open sets in \(X\) is dense in \(X\). Hence, a comeager set is dense in \(X\).

A subset \(A\) of \(X\) is said to have the Baire property (in short BP) if there is an open set \(U\) such that the symmetric difference \(A \Delta U = (A\setminus U) \cup (U\setminus A)\) is meager in \(X\). Every Borel subset has the Baire property (for more details, see for instance [24]).

We shall use the following remark to show Lemma 2.4.

Remark 2.1. If \(W \subseteq X\) is not comeager and has the Baire property, then there exists a nonempty open set \(U\) in which \(W \cap U\) is meager.

Proof. Indeed, \(W^C \Delta U\) is meager for some open set \(U\), and \(W \cap U \subseteq W^C \Delta U\).

2.2 Topological dynamics

Now we introduce some key concepts of the topological dynamics. A (time-nonuniform) dynamical system (DS) is any sequence \(\mathcal{F} = (F_t)_{t \in \mathbb{N}}\) of continuous self-maps of some compact metric space \(X\). This general formalism will be useful when studying directional dynamics of cellular automata, but the reader should keep in mind the following specific, more classical, case. \(\mathcal{F}\) is uniform if \(F_t = G^t\), for all \(t \in \mathbb{N}\) and some continuous self-map \(G\) of \(X\). We may then write the uniform DS simply as \(G\).

We are interested in the orbits \(O_\mathcal{F}(x) = \{ F_t(x) | t \in \mathbb{N} \}\) of points \(x \in X\). We say that a set \(U\) is \(\mathcal{F}\)-invariant if \(F_{t+1}(U) \subseteq F_t(U)\), for every \(t \in \mathbb{N}\).

Equicontinuity. For \(\varepsilon > 0\), we define the set \(\mathcal{E}_{\mathcal{F},\varepsilon}\) of \(\varepsilon\)-stable points for \(\mathcal{F}\) as the set of \(x \in X\) for which there exists \(\delta > 0\) such that \(\forall y \in B_\delta(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon\). The set \(\mathcal{E}_{\mathcal{F}} \subseteq X\) of equicontinuous points for \(\mathcal{F}\) is the set of points which are \(\varepsilon\)-stable for every \(\varepsilon > 0\). If \(\mathcal{E}_{\mathcal{F}}\) is comeager, then we say that \(\mathcal{F}\) is almost equicontinuous. If \(\mathcal{E}_{\mathcal{F}} = X\), then we say that \(\mathcal{F}\) is equicontinuous. Equivalently by compactness, for every \(\varepsilon > 0\), there is a uniform \(\delta > 0\) such that \(\forall x \in X, \forall y \in B_\delta(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon\).

\(\mathcal{F}\) is sensitive if there exists \(\varepsilon > 0\) such that \(\forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists t \in \mathbb{N}, d(F_t(x), F_t(y)) \geq \varepsilon\).

This implies that \(\mathcal{E}_{\mathcal{F}} = \emptyset\).

Remark 2.2. \(\mathcal{E}_{\mathcal{F}}\) is comeager in some nonempty open set \(U \subseteq X\) if and only for every \(n \geq 1\), \(\mathcal{E}_{\mathcal{F},1/n}\) is.

Proof. \(\mathcal{E}_{\mathcal{F}}\) can be written as the decreasing countable intersection \(\bigcap_{n \geq 1} \mathcal{E}_{\mathcal{F},1/n}\) of open subsets. 

A class which is between nonuniform and uniform DS is the following: \(\mathcal{F}\) is semi-uniform if \(F_t = G_t \cdots G_1\), for all \(t \in \mathbb{N}\) and some equicontinuous sequence \((G_t)_{t \geq 1}\) of self-maps of \(X\). This is trivially satisfied when \(\{ G_t | t \geq 1 \}\) is finite; in particular, in the case of any uniform DS: \(G_t = G_1\) for every \(t \geq 1\).

We will sometimes denote \(G_{T+t}[1,\ell] F_T\) the composition \(G_{T+t} \cdots G_{T+2} G_{T+1}\), so that \(F_{t+T} = G_{T+[1,\ell] F_T}\).
2.3 Symbolic dynamics

Configurations. Let $A$ be a finite set called the alphabet. A word over $A$ is any finite sequence of elements of $A$. Denote $A^* = \bigcup_{n \in \mathbb{N}} A^n$ the set of all finite words $u = u_0 \ldots u_{n-1}; |u| = n$ is the length of $u$. We say that $v$ is a subword of $u$ and write $v \subseteq u$, if there are $k, l < |u|$ such that $v = u_{[k,l]}$. $A^n$ is the space of configurations, equipped with the following metric:

$$d(x, y) := 2^{-n}, \text{ where } n = \min \{ i \in \mathbb{N} \mid x_i \neq y_i \text{ or } x_{-i} \neq y_{-i} \}.$$ 

Any element of $A^n$ is called configuration. $A^n$ is a Cantor space. In $A^n$ the cylinder of $u \in A^*$ in position $i$ is $[u]_i = \{ x \in A^n \mid |x|_i + |u| = |u| \}$. Cylinders are clopen (closed and open set). The shift map $\sigma : A^n \to A^n$ defined by $\sigma(x)_i = x_{i+1}$ is continuous for $i \in \mathbb{Z}$. The dynamical system $(A^n, \sigma)$ is called the full shift.

The (spatially) periodic configuration $u^\infty_\infty$ is defined by $(u^\infty_\infty)_{|u|+i} = u_i$ for $k \in \mathbb{Z}, 0 \leq i < |u|$ and $u \in A^*$. A monochrome configuration is one with only one symbol: $\infty^\infty_0$, for some $0 \in A$.

Subshifts. A subshift is any subsystem $(\Sigma, \sigma)$ of a full shift; we usually simply write $\Sigma$, which is then simply a closed $\sigma$-invariant subset of $A^n$. Equivalently, there exists a forbidden language $F \subseteq A^*$ such that $\Sigma = \Sigma_F = \{ x \in A^n \mid \forall u \subseteq x, u \notin F \}$. If $F$ can be taken finite, then one says that $\Sigma_F$ is a subshift of finite type (SFT); in that case $F$ can be taken included in $A^k$ for some $k \in \mathbb{N}$, which is an order for the SFT. We write that $\Sigma_F$ is a k-SFT.

Let $\Sigma \subseteq A^n$ be a subshift. Then $L(\Sigma) = \{ u \in A^* \mid \exists x \in \Sigma, u \subseteq x \}$ is the language of $\Sigma$.

We shall use the following lemma to show results concerning the dynamics of cellular automata.

Lemma 2.3. Let $\varepsilon > 0$ and $V \subseteq X$.

1. $\sigma^j(B_{\varepsilon}(V)) \subseteq B_{2\varepsilon}(\sigma^j(V))$ for all $j$. If, moreover, $V$ is $\sigma$-invariant, and there exists integer $p > 0$ such that for all $n \in \mathbb{N}$, $\sigma^{pn}(V) \subseteq B_{\varepsilon}(V)$, then $\forall i \in \mathbb{Z}, \sigma^i(V) \subseteq B_{2p\varepsilon}(V)$.

2. If $\forall i \in \mathbb{Z}, \sigma^i(x) \subseteq B_{2-k}(V)$ and $V$ is a k-SFT, then $x \in V$.

Proof.

1. Let $x \in \sigma^j(B_{\varepsilon}(V))$. Then $\sigma^{-j}(x) \in B_{\varepsilon}(V) \iff d(\sigma^{-j}(x), V) < \varepsilon$. Moreover, $d(x, \sigma^j(V)) = d(\sigma^j \sigma^{-j}(x), \sigma^j(V)) < 2^j d(\sigma^{-j}(x), V) < 2^j \varepsilon$. So, $x \in B_{2\varepsilon}(\sigma^j(V))$.

Hence, $\forall i \in \mathbb{Z}, \sigma^i(V) = \sigma^{i \mod p} \sigma^{i/p}(V) \subseteq \sigma^{i \mod p} \sigma^{i \mod p}(V) \subseteq B_{2p\varepsilon}(V)$, where $V$ is $\sigma$-invariant, we have that $B_{2p\varepsilon}(\sigma^{i \mod p}(V)) \subseteq B_{2p\varepsilon}(V)$, hence, $\forall i \in \mathbb{Z}, \sigma^i(V) \subseteq B_{2p\varepsilon}(V)$.

2. Let $i \in \mathbb{Z}$.

$$\sigma^i(x) \subseteq B_{2-k}(V) \iff d(\sigma^i(x), V) < 2^{-k} \implies \exists y \in V, \sigma^i(x)[i-k,k] = y[i-k,k] \implies \exists y \in V, x[i, i+k] \subseteq L(V).$$

If this is true for every $i \in \mathbb{Z}$ and $V$ is k-SFT (or even $(2k+1)$-SFT), then $x \in V$.

Transitivity. A DS $\mathcal{G}$ is said to be transitive if for any nonempty open sets $U, V$ in $X$, there exists $t \in \mathbb{N}$ such that $F_t^{-1}(U) \cap V \neq \emptyset$.

It is clear that $(A^n, \sigma)$ is transitive, and the following lemma will be used later for this example: it states for example that subshifts are all either meager or full.

Lemma 2.4. Let $G$ be a transitive uniform DS, and let $W \subseteq X$ be $G$-invariant.

1. If $W$ is dense (resp. comeager) in some nonempty open set $U$, then $W$ is dense (resp. comeager).
2. If \( W \) has nonempty interior (resp. has the Baire property and is not meager), then \( W \) includes a dense open set (resp. is comeager).

3. If \( \bigcup_{i \in \mathbb{N}} W_i \) has nonempty interior (resp. is not meager), where each \( W_i \) is \( G \)-invariant (resp. and has the Baire property), then there exists \( i \in \mathbb{N} \) such that \( W_i \) includes a dense open set (resp. is comeager).

Proof.

1. Since \( G \) is transitive, for every nonempty open set \( V \), there exists \( t \) such that \( G^{-t}(V) \cap U \) is a nonempty open set. Moreover, \( W \) is dense in \( U \), so that \( G^{-t}(V) \cap U \cap W \neq \emptyset \). Since \( G^t(W) \subseteq W \), \( V \cap G^t(U) \cap W \supseteq G^t(G^{-t}(V) \cap U \cap W) \neq \emptyset \). So, \( W \) is dense in \( X \).

One has \( W \cap U \supseteq \bigcap_{n \in \mathbb{N}} U_n \) where \( U_n \) is open and such that \( \overline{U}_n = U \). So, \( W \supseteq \bigcap_{n \in \mathbb{N}} G^t(W) \supseteq \bigcap_{n \in \mathbb{N}} G^t \big( W \cap U \big) \supseteq \bigcap_{n \in \mathbb{N}} G^t \big( \bigcap_{m \in \mathbb{N}} U_m \big) \). Note that each \( \bigcap_{n \in \mathbb{N}} G^t(U_n) \) is \( G \)-invariant, and dense in \( U \). It is open, and by the previous point, dense, so that \( W \) indeed includes an intersection of dense open sets.

2. Since \( W \) includes a nonempty open set and is \( G \)-invariant, \( W \) includes the (nonempty open) orbit of this open set. According to the previous point, \( W \) includes a dense open set.

By Remark 2.1 \( W \) is comeager in some nonempty open set \( U \). By the previous point, \( W \) is comeagern.

3. One of the \( W_i \) has to have nonempty interior (resp. to not be meager). We conclude by the previous point.

\[ \square \]

2.4 Cellular automata

A map \( F : A^\mathbb{Z} \to A^\mathbb{Z} \) is a cellular automaton (CA) if there exist integers \( r_- \leq r_+ \) (memory and anticipation) and a local rule \( f : A^{r_+ - r_- + 1} \to A \) such that for any \( x \in A^\mathbb{Z} \) and any \( i \in \mathbb{N} \), \( F(x)_i = f(x_{i+r_-}, \ldots, x_{i+r_+}) \). \( d = r_+ - r_- \in \mathbb{N} \) is sometimes called the diameter of \( F \). Sometimes we assume that \( -r_- = r_+ \), which is then called the radius of \( F \) (it is always possible to obtain this, by taking \( r = \max\{|r_-|, |r_+|\} \in \mathbb{N} \)). By Curtis, Hedlund and Lyndon [13], a map \( F : A^\mathbb{Z} \to A^\mathbb{Z} \) is a CA if and only if it is continuous and commutes with the shift. In particular, CA induce uniform DS over \( A^\mathbb{Z} \).

Directional dynamics. We call curve a map \( h : \mathbb{N} \to \mathbb{Z} \) with bounded variation, that is: \( M = \sup_{t \in \mathbb{Z}} |h(t + 1) - h(t)| \) is finite. The map is meant to give a position in space for each time step. Following [7], the CA \( F \) in direction \( h \) will refer to the sequence \((F^t\sigma^{h(t)})_{t \in \mathbb{N}}\). In a first reading, the reader can understand the next definitions and results by considering the classical case: \( h \) constantly 0. In general, the directional dynamics of a CA can be read on its space-time diagram, by following \( h \) as a curve when going in the time direction. The formalism includes that of [23]: a linear curve is \( h : t \mapsto |\alpha t| \), for some real number \( \alpha \).

Equicontinuity. A word \( u \in A^* \) is (strongly) blocking for a CA \( F \) along curve \( h \) if there exists an offset \( s \in \mathbb{Z} \) such that for every \( x, y \in [u]_s \), \( \forall t \in \mathbb{N} \), \( F^t(x)_{[0, M]} = F^t(y)_{[0, M]} \) where \( M = \max(-r_- + \max_t(h(t + 1) - h(t)), r_+ + \max_t(h(t) - h(t + 1))) \), and \( r_- \) and \( r_+ \) are the (minimal) memory and anticipation for \( F \). The terminology comes from the fact that in that case, \( u \) is both left- and right-blocking (with the same offset), which is taken as a definition in [7]: A word \( u \in A^* \) is right-blocking for a CA \( F \) in direction \( h \) if there exists an offset \( s \in \mathbb{Z} \) such that:

\[ \forall x, y \in [u]_{s, [x]_{-\infty, s]} = [y]_{-\infty, s} \implies \forall t \in \mathbb{N}, F^t(x)_{[\!-\infty, h(t)]} = F^t(y)_{[\!-\infty, h(t)]} \, . \]

We define left-blocking words similarly.

The following proposition explains how equicontinuity in cellular automata can be rephrased in terms of blocking words. The vertical case dates back from [17, 18], the linear directions from [23], the directions with bounded variations from [7], and the general case can be found in the proofs of [8] Prop 3.1.3, Cor 3.1.4].
Proposition 2.5 ([7]). Let $F$ be a CA and $h$ a curve.

1. If there is a left- and right-blocking word $u$ for $F$ in direction $h$, then $E_{F,h}$ includes the comeager set of configurations where $u$ appears infinitely many times on both sides.

2. Otherwise, $F$ is sensitive in direction $h$.

In other words, in the bounded-variation case, $E_{F,h}$ is either empty or comeager. The question is open whether this remains true in the unbounded-variation case (see [6, Remark 3.1.1]).

3 Attractors and realms

We will define notions that deal with asymptotic behavior of a DS $\mathfrak{F} = (F_t)_t$.

3.1 Limit sets

The $(\Omega)$ limit set of $U \subseteq X$ is the set $\Omega_{\mathfrak{F}}(U) = \bigcap_{T \in \mathbb{N}} \bigcup_{t \geq T} F_t(U)$, and the asymptotic set of $U \subseteq X$ is the set $\omega_{\mathfrak{F}}(U) = \bigcup_{t \in \mathbb{N}} \Omega_{\mathfrak{F}}(\{x\})$. By compactness, these sets are nonempty. $\Omega_{\mathfrak{F}}(U)$ is compact, but $\omega_{\mathfrak{F}}(U)$ may not be, even for $U = X$ (see Example [7,7]). Remark that $\Omega_{\mathfrak{F}}(U) \supseteq \bigcap_{t \in \mathbb{N}} F_t(U)$, and this is an equality if $U$ is a closed $\mathfrak{F}$-invariant set.

We note $\Omega_{\mathfrak{F}} = \Omega_{\mathfrak{F}}(X)$ and $\omega_{\mathfrak{F}} = \omega_{\mathfrak{F}}(X)$. For more about the asymptotic set of dynamical systems, one can refer to [12]. Note that it was called accessible set in [8], and ultimate set in [11].

Remark 3.1. By compactness of $X$, for every $U$ and $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that:

$$\forall t \geq T, d(F_t(U), \Omega_{\mathfrak{F}}(U)) < \varepsilon.$$ 

In the uniform case, it is clear that asymptotic sets are invariant. Here is a generalization of this fact.

Proposition 3.2. Let $\mathfrak{F} = (G_{t[1,\cdot]})_t$ be a semi-uniform DS, $U \subseteq X$, and $j \in \mathbb{N}$.

1. If $y \in \Omega_{\mathfrak{F}}(U)$, then $(G_{t+[1,j]}(y))_t$ admits a limit point in $\Omega_{\mathfrak{F}}(U)$.

2. Conversely, if $z \in \Omega_{\mathfrak{F}}(U)$, then it is a limit point of $(G_{t+[1,j]}(y))_t$ for some $y \in \Omega_{\mathfrak{F}}(U)$.

Of course, this remains true for the $\omega$, which is defined as a union of $\Omega$.

Proof.

1. By assumption, there are increasing times $(t_k)_{k \in \mathbb{N}}$ and points $(x_k)_{k \in \mathbb{N}} \in U^\mathbb{N}$ such that $\lim_{k \to \infty} F_{t_k}(x_k) = y$. Let $\varepsilon > 0$. By equicontinuity of the $(G_{t+[1,j]}(y))_t$, there exists $\delta > 0$ such that for all $z, z'$ with $d(z, z') < \delta$, we have $\forall t \in \mathbb{N}, d(G_{t+[1,j]}(z), G_{t+[1,j]}(z')) < \varepsilon/2$. Then there is $K \in \mathbb{N}$ such that for all $k \geq K$, $d(F_{t_k}(x_k), y) < \delta$, so that $d(F_{t_k+(j)}(x_k), G_{t_k+(j)}(y)) < \varepsilon/2$. If $z$ is a limit point for $(G_{t_k+(j)}(y))_{k \in \mathbb{N}}$, one sees that there exists $K' \in \mathbb{N}$ such that for all $k \geq K'$, $d(F_{t_k+(j)}(x_k), z) < \varepsilon$, so that $z$ is also in $\Omega_{\mathfrak{F}}(\{x_k | k \in \mathbb{N}\}) \subseteq \Omega_{\mathfrak{F}}(U)$.

2. Now let $z \in \Omega_{\mathfrak{F}}(U)$, so that it is the limit point of $(F_{t_k}(x_k))_{k \in \mathbb{N}}$ for some $(x_k)_{k \in \mathbb{N}} \in U^\mathbb{N}$ and $(t_k)$ an increasing, say nonzero, sequence. By compactness, $(F_{t_k+(j)}(x_k))_{k \in \mathbb{N}}$ admits a limit point $y \in \Omega_{\mathfrak{F}}(U)$. By triangular inequality, we have $d(G_{t_k+(j)}(y), z) \leq d(G_{t_k+(j)}(y), F_{t_k}(x_k)) + d(F_{t_k}(x_k), z)$. When $k$ goes to $\infty$, the second term of the sum converges to 0, and a subsequence of the first term converges to 0, thanks to equicontinuity of $(G_{t_k+(j)}(y))_{k \in \mathbb{N}}$. 

The following corollary is useless for the purpose of the present paper, but may help the reader to connect with the know, uniform case.

Corollary 3.3. If $\mathfrak{F}$ is a uniform DS, $U \subseteq X$ and $j \in \mathbb{N}$, then $\mathfrak{F}(\Omega_{\mathfrak{F}}(U)) = \Omega_{\mathfrak{F}}(U)$. 

5
3.2 Realms of attraction

The realm of attraction of \( V \) is:
\[
D_\emptyset(V) = \{ x \in X | \omega_\emptyset(x) \subseteq V \} = \left\{ x \in X \mid \lim_{t \to \infty} d(F_t(x), V) = 0 \right\}.
\]
It is nonempty if and only if \( V \) intersects \( \omega_\emptyset \).

The direct realm of \( V \) is the set \( D_\emptyset(V) = \bigcup_{T \in \mathbb{N}} \bigcap_{i \geq T} F^{-1}_i(V) \) of configurations whose orbits lie ultimately in \( V \). It is included in the realm of \( V \).

Just from the definition, the reader can be convinced of the following.

**Remark 3.4.** Let \( V_i \subseteq X \) be subsets, with \( i \) indexed within any set \( I \).

1. \( D_\emptyset(\bigcup_{i \in I} V_i) \supseteq \bigcup_{i \in I} D_\emptyset(V_i) \).
2. \( D_\emptyset(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} D_\emptyset(V_i) \).

3.3 Linked concepts

**Recurrence.** A point \( x \in X \) is **recurrent** if \( x \in \omega_\emptyset(x) \). We say that \( \emptyset \) is **nonwandering** if the set of recurrent points is comeager (see for instance [5] for equivalent definitions, for uniform DS).

**Nilpotence.** We say that \( \emptyset \) is **nilpotent** if there is a point \( z \in X \) such that \( \exists T \in \mathbb{N}, \forall x \in X, \forall t > T, F_t(x) = z \). \( \emptyset \) is **asymptotically nilpotent** if \( \omega_\emptyset \) is a singleton.

It is known that CA are nilpotent if and only if their limit set is finite (see for instance [4]). Also, it has been shown [12] that asymptotically nilpotent CA are actually nilpotent. In that case (see for instance [4, 12]), \( z = \sigma(z) \), so that the CA is actually nilpotent in every direction.

**Asymptotic pairs.** Two points \( x, y \in X \) are said to be **asymptotic** to each other whenever
\[
\lim_{t \to \infty} d(F_t(x), F_t(y)) = 0.
\]

Let us generalize the realm notations to every sequence \( (V_i)_{i \in \mathbb{N}} \) of subsets of \( X \), by defining: \( D_\emptyset((V_i)_i) = \{ x \in X \mid \lim_{t \to \infty} d(F_t(x), V_i) = 0 \} \) and \( D_\emptyset((y_i)_i) = \{ x \in X \mid \exists T \in \mathbb{N}, \forall t \geq T, F_t(x) \in V_i \} \). We may also note \( D_\emptyset((y_i)_i) \) if \( V_i \) is a singleton \( \{ y_i \} \). With this notation, \( A_\emptyset(y) = D_\emptyset(\omega_\emptyset(U)) \).

One can observe from the definition that \( U \subseteq A_\emptyset(U) \subseteq D_\emptyset(\omega_\emptyset(U)) \).

**Remark 3.5.** Let \( G \) be a uniform DS over a finite space \( X \), and \( x, y \in X \). If \( x \) and \( y \) are asymptotic, then \( \exists t \in \mathbb{N}, G^t(x) = G^t(y) \). In particular, if \( G \) is injective (or surjective), then \( x = y \).

**Proof.** The first statement comes from \( X \) being discrete. The second statement is clear because if \( X \) is finite, then injectivity or surjectivity of \( G \) are equivalent to bijectivity of any \( G^t \).

The following remark states that when an asymptotic class is big, then it should contain many equicontinuous points.

**Remark 3.6.** If \( A_\emptyset(y) \) is comeager in some nonempty open subset \( U \subseteq X \), for some \( y \in X \), then \( E_\emptyset \) is comeager in \( U \).

In particular, note that \( E_\emptyset \cap A_\emptyset(y) \) is also comeager in \( U \).

**Proof.** The assumption gives that for every \( n \geq 1 \), the union \( \bigcup_{T \in \mathbb{N}} \bigcap_{i \geq T} F^{-1}_i(B_{1/n}(F_t(y))) \) of closed sets is comeager in \( U \), as a superset of \( A_\emptyset(y) \). This means that there is \( T \in \mathbb{N} \) such that \( \bigcap_{i \geq T} F^{-1}_i(B_{1/n}(F_t(y))) \) includes a nonempty open subset \( W \subseteq U \). For every \( x \in W \), there exists \( \delta > 0 \) such that for every \( z \in B_{\delta}(x) \), \( z \in W \), which means that \( \forall t \geq T, F_t(z) \in B_{1/n}(F_t(y)) \subseteq B_{3/2n}(F_t(x)) \). We deduce that \( x \) is \( 3/2 \)-stable. In other words, the set of \( 3/2 \)-stable points includes nonempty open subsets of every nonempty open subset of \( U \), for every \( n \in \mathbb{N} \). We conclude by Remark 2.2.  \( \square \)
3.4 Decomposition of realms.

This proposition can be compared partly to [22 Lemma 3].

Proposition 3.7. Let $\mathfrak{F} = (G_{[1, t]})_{t \in \mathbb{N}}$ be a semi-uniform DS. Suppose $(V_i)_i$ is a finite collection of closed pairwise disjoint sets which are invariant by every $G_i$. Then $D_{\mathfrak{F}}(\bigcup_i V_i) = \bigcup_i D_{\mathfrak{F}}(V_i)$.

Proof. Since the $V_i$ are closed and pairwise disjoint, there are at positive pairwise distance. Let $\varepsilon = \min_{i \neq j} d(V_i, V_j)/2 > 0$. By equicontinuity of $(G_t)$, there exists $\delta > 0$ such that $\forall t \in \mathbb{N}, \forall x, y \in X, d(x, y) < \delta$ $\Longrightarrow d(G_t(x), G_t(y)) < \varepsilon$. Let $x \in D_{\mathfrak{F}}(\bigcup_i V_i)$, so that there exists $T \in \mathbb{N}$ such that $\forall t \geq T, d(F_t(x), \bigcup_i V_i) < \min(\delta, \varepsilon)$. In particular, there exists $i$ such that $d(F_T(x), V_i) < \min(\delta, \varepsilon)$. We let $y$ be the minimum such $y$ exists. Since $G_{t+1}(V_i) \subseteq V_i$, we have $d(F_{t+1}(x), V_i) \leq d(F_{t+1}(x), G_{t+1}(V_i))$. This is less than $\varepsilon$ by equicontinuity of $(G_t)$, using the recurrence hypothesis. By definition of $\varepsilon$, we have $\min_{j \neq i} d(F_{t+1}(x), V_j) \geq \min_{j \neq i} (d(F_{t+1}(x), V_i) - d(F_{t+1}(x), V_j)) \geq \varepsilon$. So $\min(\delta, \varepsilon) \geq \min_{j \neq i} (d(F_{t+1}(x), \bigcup_i V_j) = \min_{j} (d(F_{t+1}(x), V_j))$. It results that $d(F_{t+1}(x), V_i) \leq \min(\delta, \varepsilon)$, as wanted.

Since for every $t \geq T$ and $j \neq i$, we have $d(F_t(x), V_j) \geq \varepsilon$, we deduce $d(F_t(x), V_i) = \min_{i} d(F_t(x), V_i) = d(F_t(x), \bigcup_i V_i)$ converges to $0$.

The other inclusion comes from Point 1 of Remark 3.4.

3.5 Reals of finite sets.

Lemma 3.8. Let $\mathfrak{F} = (G_{[1, t]})_{t \in \mathbb{N}}$ be a semi-uniform DS and $V$ be finite. Then there exists $\delta > 0$ such that for all $x, x' \in D_{\mathfrak{F}}(V)$ and $T \in \mathbb{N}$ such that $d(F_T(x), F_T(x')) \leq \delta$, and $\forall t \geq T, d(F_t(x), V) \leq \delta$ and $d(F_t(x'), V) \leq \delta$, $(x, x')$ is an asymptotic pair.

Proposition 3.9. Let $\mathfrak{F} = (G_{[1, t]})_{t \in \mathbb{N}}$ be a semi-uniform DS and $V$ be finite. Then there are at most $|V|$ asymptotic classes in $D_{\mathfrak{F}}(V)$.

Proof: For $0 \leq i \leq |V|$, let $x_i \in D_{\mathfrak{F}}(V)$, and $\delta$ be as in Lemma 3.8. There exists $T \in \mathbb{N}$ such that for all $i$, $\forall t \geq T, d(F_t(x_i), V) < \delta/2$, and in particular, $\exists y_i \in V, d(F_T(x_i), y_i) < \delta/2$. By the pigeon-hole principle, there are distinct $i, j$ such that $y_i = y_j$, so that $d(F_T(x_i), F_T(x_j)) < \delta$ by the triangular inequality. By Lemma 3.8, we then know that $(x_i, x_j)$ is an asymptotic pair. Hence we can partition $D_{\mathfrak{F}}(V)$ into at most $|V|$ asymptotic classes.

Proof of Lemma 3.8. Let $\varepsilon = \frac{1}{2} \min \{ d(y, y') | y, y' \in V, y \neq y' \} > 0$. By equicontinuity of $(G_t)$, there exists $\delta > 0$ such that $\forall t \in \mathbb{N}, \forall x, x' \in X, d(x, x') \leq \delta$ $\Longrightarrow d(G_t(x), G_t(x')) < \varepsilon/2$. Without loss of generality, we can assume $\delta \leq \varepsilon$. Let $x, x'$ be as in the statement of the lemma, and for $t \in \mathbb{N}$, let $y(t) \in V$ be such that $d(F_t(x), y(t)) = d(F_t(x), V)$, and $y'(t)$ be defined similarly. Let us show by induction on $t \geq T$ that $y(t) = y'(t)$, which by definition of $\varepsilon$, is equivalent to $d(y(t), y'(t)) < 3\varepsilon$. First, by the triangular inequality, $d(y(T), y'(T)) \leq d(y(T), F_T(x)) + d(F_T(x), F_T(x')) + d(F_T(x'), y'(T)) < 3\delta \leq 3\varepsilon$.

Now suppose this is true for $t \geq T$, and let us prove it for $t + 1$. By the triangular inequality, we also have $d(y(t + 1), y'(t + 1)) \leq d(y(t + 1), F_{t+1}(x)) + d(F_{t+1}(x), F_{t+1}(x')) + d(F_{t+1}(x'), y'(t + 1))$. The first and third terms are at most $\delta$ by hypothesis, while the central one is at most $\varepsilon$ by definition of $\delta$. All in all, we get that $y(t + 1) = y'(t + 1)$. We can conclude with, once again, the triangular inequality: $d(F_t(x), F_t(x')) \leq d(F_t(x), y(t)) + d(y(t), y'(t)) + d(y'(t), F_t(x'))$. If $t \geq T$, this is $d(F_t(x), V) + 0 + d(F_t(x'), V) \rightarrow_{t \rightarrow \infty} 0$.

3.5 Reals for cellular automata

The following proposition is very important to show Proposition 5.10.

Proposition 3.10. Let $\mathfrak{F} = (F_i)$ be a sequence of CA over $X = A^2$, and $V \subseteq A^2$.

1. $D_{\mathfrak{F}}(\sigma(V)) = \sigma(D_{\mathfrak{F}}(V))$.

2. If $V$ is $\sigma$-invariant and $D_{\mathfrak{F}}(V)$ has nonempty interior, then $D_{\mathfrak{F}}(V)$ includes a dense open set.
3. If, moreover, \( V \) is a \( k \)-SFT, then \( \mathfrak{A}_\Theta(V) \) is dense.

4. If \( V \) is a subshift, then \( \mathfrak{A}_\Theta(V) \) is meager, unless \( F_T^{-1}(V) \) is full, for some \( T \in \mathbb{N} \).

**Proof.**

1. This is clear by definition, noting that \( \omega_\Theta(\sigma(x)) = \sigma(\omega_\Theta(x)) \).

2. \( \mathfrak{D}_\Theta(V) \) has nonempty interior and, from the previous point, is \( \sigma \)-invariant. By Point 2 of Proposition 3.10, we see that \( \mathfrak{D}_\Theta(V) \) then includes a dense open set.

3. Let us show that, for an arbitrary \( w \in A^* \), \( [w] \cap \mathfrak{D}_\Theta(V) \) is nonempty. Since \( \mathfrak{D}_\Theta(V) \) has nonempty interior, there exists \( u \in A^* \) such that \( [u] \subseteq \mathfrak{D}_\Theta(V) \).

   Let \( x = \infty (uw)^\infty \in [u] \) be the periodic configuration of period \( p = |uw| \) and such that \( x_{[0,p]} = uw \).

   Since \( x \in [u] \subseteq \mathfrak{D}_\Theta(V) \), there exists \( T \in \mathbb{N} \) such that \( \forall t > T, d(F_t(x), V) < 2^{-k-p} \). Since \( \forall n \in \mathbb{Z}, x = \sigma^{np}(x) \), we even have:

   \[ \forall t > T, \forall n \in \mathbb{Z}, d(F_t\sigma^{np}(x), V) < 2^{-k-p} \, . \]

   By Point 1 of Lemma 2.3, for all such \( t > T, \forall i \in \mathbb{Z}, \sigma^iF_t(x) \in B_{2T}(V) \). Since \( V \) is a \( k \)-SFT, \( F_t(x) \in V \), by Point 2 of Lemma 2.3, that is, \( x \in \mathfrak{D}_\Theta(V) \). By shift-invariance, we also have that \( \sigma^{|u|}(x) \in [w] \cap \mathfrak{D}_\Theta(V) \).

4. By definition, \( \mathfrak{D}_\Theta(V) \subseteq \bigcup_{T \in \mathbb{N}} F_T^{-1}(V) \). If for every \( T \in \mathbb{N} \), \( F_T^{-1}(V) \) is not full, Point 2 of Lemma 2.4 gives that it has empty interior (because it is closed and \( \sigma \)-invariant). In the end, \( \mathfrak{D}_\Theta(V) \) is meager.

**Proposition 3.11.** Let \( F \) be a CA, \( h \) a curve, and \( U \) be \( \sigma \)-invariant. Then \( F(\omega_{F,h}(U)) = \sigma(\omega_{F,h}(U)) = \omega_{F,h}(U) \).

**Proof.** By Point 1 of Proposition 3.10, we know that \( \omega_{F,h}(U) \) is \( \sigma \)-invariant, and since \( F \) commutes with \( \sigma \), we have that \( \forall t \geq 1, F(\sigma^{h(t)}(\omega_{F,h}(U))) = F(\omega_{F,h}(U)) \). By Proposition 3.2 (applied to \( j = 1 \) and \( G_t = F(\sigma^{h(t)}(\omega_{F,h}(U))) \)), so that \( \{G_t \mid t \in \mathbb{N}\} \) is finite), we obtain that for any \( y, z \in \omega_{F,h}(U) \), \( F(\sigma^k(y)) \in \omega_{F,h}(U) \), for some \( k \), and \( z = F(\sigma^k(y)) \) for some \( l \) and some \( x \in \omega_{F,h}(U) \).

**Corollary 3.12.** Let \( F \) be a CA, \( h \) a curve and \( U \) such that \( V = \omega_{F,h}(U) \) is finite and \( \sigma \)-invariant, and \( U = \mathfrak{D}_{F,h}(V) \). Then \( F \) induces a bijection of \( V \), and \( U = \bigcup_{y \in V} A_{F,h}(y) \).

**Proof.** By Proposition 3.11, we see that \( F(V) = V \), so that \( F \) induces a surjection, hence a bijection of \( V \). By Proposition 3.9, there are at most \( |V| \) asymptotic classes in \( U \). By the first point, \( V \subseteq U \), so that each \( y \in V \) should be in one of these classes. By Remark 3.5, they are all in distinct classes, so that we obtain the wanted result (the converse inclusion being trivial).

### 3.6 Attractors

We say that \( V \) is an **attractor** for \( \Theta \) if it is closed and equal to \( \Omega(U) \) for some open \( \Theta \)-invariant set \( U \) (in particular \( V \subseteq U \)). One can see that this notion is equivalent to some classical definitions, such as the one used in [19–9], which is focused on subshift attractors of CA: in that case the attractor enjoys a definition as the limit set of a so-called spreading cylinder. \( \Omega_\Theta = \Omega_{\Theta}(X) \) is then the (unique) maximal attractor. A **quasi-attractor** is a nonempty intersection of attractors.

**Proposition 3.13.** Consider some CA \( F \) in some direction \( h \).

1. The realm of a subshift attractor is a dense open set.

2. The realm of a subshift quasi-attractor is comeager.

Point 1 was already stated in [21].

**Proof.**
1. This is direct from Point 2 of Proposition 3.10.

2. Suppose \( V = \bigcap_{n \in \mathbb{N}} V_n \), where \( V_n \) is an attractor. Then \( O_\sigma(V) = \bigcap_{n \in \mathbb{N}} O_\sigma(V_n) \), so that \( D_{F,h}(O_\sigma(V)) = \bigcap_{n \in \mathbb{N}} D_{F,h}(O_\sigma(V_n)) \). For every \( n \in \mathbb{N} \), \( D_{F,h}(O_\sigma(V_n)) \) is shift-invariant, and includes the nonempty open subset \( D_{F,h}(V_n) \). From Point 2 of Proposition 3.10, it includes a dense open set. Globally, the realm of \( O_\sigma(V) \) is thus comeager.

This allows to answer a conjecture stated in [17].

Corollary 3.14. If a CA admits a minimal quasi-attractor in some direction, then:

1. it is a subshift;
2. its realm is comeager.

In particular, in that case, all (possibly not subshifts) attractors have dense open realms (see Example 7.3 in direction 0).

Proof.  
1. Just note that if \( V \) is a quasi-attractor, then it is closed, and, since \( \sigma \) is a conjugacy, any \( \sigma^n(V) \), with \( n \in \mathbb{Z} \), must also be a quasi-attractor. We deduce that a minimal quasi-attractor should also be included in \( \bigcap_{n \in \mathbb{N}} \sigma^n(V) \), which is a subshift. Applied to itself, we get that this is equal to \( V \), which is thus a subshift.

2. This now corresponds to Point 2 of Proposition 3.13.

This property of having a comeager realm motivates the next definition.

4 The generic limit set

Milnor [22] suggests the following definition, which is the purpose of the present section.

Definition 4.1 ([22]). Being given a DS \( \mathfrak{X} \), the generic limit set \( \tilde{\omega}_\mathfrak{X} \) is the intersection of all the closed subsets of \( X \) which have a comeager realm of attraction.

The generic limit set \( \tilde{\omega}_\mathfrak{X} \) can actually be defined as the smallest closed subset of \( X \) with a comeager realm, thanks to the following proposition.

Proposition 4.2. Let \( \mathfrak{X} \) be a DS. The realm of the generic limit set is comeager.

In other words, it is the smallest closed set which includes all limit points of all generic orbits.

Proof. Any compact metric space admits a countable basis: there exists a countable set \( \{ U_i | i \in \mathbb{N} \} \) of closed subsets such that every closed set \( U \) can be written as \( \bigcap_{i \in I_U} U_i \) for some \( I_U \subseteq \mathbb{N} \). In particular, \( \tilde{\omega}_\mathfrak{X} \) is the intersection \( \bigcap_U \bigcap_{i \in I_U} U_i \), where \( U \) ranges over closed sets with comeager realm; that is, \( \tilde{\omega}_\mathfrak{X} = \bigcap_{i \in I} U_i \), where \( I \) is the union of \( I_U \), for every closed \( U \) with comeager realm. If \( i \) is in \( I \), then it is in some \( I_U \), so that \( U \subseteq U_i \), where \( U \) has comeager realm, so that \( U_i \) has comeager realm, too.

By Point 2 of Remark 3.4, \( D_\mathfrak{X}(\tilde{\omega}_\mathfrak{X}) = D_\mathfrak{X}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} D_\mathfrak{X}(U_i) \). We know that an intersection of countably many comeager sets is comeager. Then \( D_\mathfrak{X}(\tilde{\omega}_\mathfrak{X}) \) is comeager.

Remark 4.3. For every DS \( \mathfrak{X} \) and subset \( V \), \( D_\mathfrak{X}(V^C)^C = \bigcap_{T \in \mathbb{N}} \bigcup_{t \geq T} F_t^{-1}(V) \) is the set of points whose orbits visit \( V \) infinitely many times, which includes \( D_\mathfrak{X}(V) \).

We deduce that if, for every \( \varepsilon > 0 \) and every \( T \in \mathbb{N} \), \( \bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(V)) \) is dense, then \( \nabla \) intersects \( \omega_\mathfrak{X}(x) \) for generic \( x \); in particular it intersects \( \tilde{\omega}_\mathfrak{X} \).
Proof. The first statement is direct. Now suppose that for every \( \varepsilon > 0 \) and every \( T \in \mathbb{N} \), \( \bigcup_{i \geq T} F^{-1}(B_\varepsilon(V)) \) is dense; by continuity it is also open, so that \( D_\overline{g}(B_\varepsilon(V)^C)^C \) is comeager, and so is \( D_\overline{g}(V^C)^C = \bigcap_{n \in \mathbb{N}} D_\overline{g}(B_{1/n}(V)^C)^C \). The orbits from this set visit \( B_{1/n}(V) \) infinitely many times for every \( n \in \mathbb{N} \), so that they all have a limit point in \( V \).

Since \( D_\overline{g}(V^C) \) is meager, \( V^C \) cannot include the generic limit set, which gives us the last statement. \( \square \)

4.1 First properties for CA

**Proposition 4.4.** Let \( \overline{g} = (F_i)_i \) be a sequence of CA. Then \( \omega_{\overline{g}} \) is a subshift. Its realm is \( \sigma \)-invariant.

Proof. By Definition, \( \omega_{\overline{g}} \) is closed. Let \( \omega_{\overline{g}} = \overline{\omega_{\overline{g}}(U)} \) where \( U \) is comeager. Since \( \sigma \) is a homeomorphism, \( \sigma^k(U) \) is also comeager for all \( k \in \mathbb{Z} \). Then \( W = \bigcap_{k \in \mathbb{Z}} \sigma^k(U) \) is still comeager, as an intersection of countably many comeager sets. One has \( W \subseteq U \), so that \( \omega_{\overline{g}}(W) \subseteq \omega_{\overline{g}}(U) \). Conversely, the definition of \( \omega_{\overline{g}} \) gives that it is included in \( \omega_{\overline{g}}(W) \). Overall, \( \omega_{\overline{g}}(W) = \omega_{\overline{g}} \). Since \( W \) is \( \sigma \)-invariant, \( \omega_{\overline{g}} = \omega_{\overline{g}}(W) \) is also \( \sigma \)-invariant. \( \square \)

Moreover, Proposition 3.13 directly gives that \( \omega_{\overline{g}} \) is included in all subshift attractors.

**Proposition 4.5.** Consider the CA \( F \) in some direction \( h \). Then \( \omega_{F,h} \) is an \( F \)-invariant subshift.

Proof. We just apply Proposition 3.11 to \( \omega_{F,h} = \overline{\omega_{F,h}(D(\omega_{F,h}))} \). \( \square \)

4.2 Nonwandering systems

For nonwandering dynamical systems, the generic limit set is the full space.

**Proposition 4.6.** Let \( \overline{g} \) be a nonwandering DS over space \( X \). Then \( \omega_{\overline{g}} = X \).

Proof. Consider the comeager set \( \mathcal{R} \) of recurrent points of \( \overline{g} \). The set \( U = D_\overline{g}(\omega_{\overline{g}}) \cap \mathcal{R} \) is comeager, and, by inclusion, \( \omega_{\overline{g}} \supseteq \overline{\omega_{\overline{g}}(U)} \). Since \( U \subseteq \mathcal{R} \), by definition, \( U \subseteq \omega_{\overline{g}}(U) \subseteq \omega_{\overline{g}}(\overline{\omega_{\overline{g}}(U)}) \). By the Baire category theorem, \( U \) is dense. So the closed superset \( \overline{\omega_{\overline{g}}(U)} \) is \( X \).

Since it is known that surjective CA are all nonwandering (see for instance 18), we get the following.

**Corollary 4.7.** A CA \( F \) over \( A^\mathbb{Z} \) is surjective if and only if \( \omega_F = A^\mathbb{Z} \).

4.3 Indecomposability

Now we prove that the generic limit set of a cellular automaton is indecomposable in some sense.

**Proposition 4.8.** Let \( V = \bigcup_{i=0}^{n-1} V_i \), where \( n \in \mathbb{N} \) and the \( V_i \) are closed subsets which are invariant by some CA \( F \) in some direction \( h \), and by \( \sigma^p \), for some \( p > 0 \). If \( D_{F,h}(V) \) has nonempty interior (resp. is not meager), then there exists \( i \in [0,n] \) such that \( D_{F,h}(V_i) \) is dense (resp. comeager).

Proof. One has \( D_{F,h}(V) = \bigcup_{i=0}^{n-1} D_{F,h}(V_i) \) by Proposition 3.7. By Point 3 of Lemma 2.4, there exists \( i \in [0,n] \) such that \( D_{F,h}(V_i) \) is dense (resp. comeager). \( \square \)

**Corollary 4.9.** Let \( F \) be a CA and \( h \) a curve. \( \omega_{F,h} \) cannot be decomposed as a disjoint union of non-trivial subshift subsystems (or even non-trivial \( \sigma^p \)-invariant subsystems, for some \( p > 0 \)).

In other words, we can say that \( \omega_{F,h} \) is connected, when considering the dynamical pseudo-metric related to the action of \( (F,\sigma) \): \( \overline{d}(x,y) = \inf_{i,j \in \mathbb{Z},i,t \in \mathbb{N}} d(F^i\sigma^j(x),F^i\sigma^j(y)) \).

Proof. We assume that \( \omega_{F,h} = \bigcup_{i=0}^{n-1} V_i \), where the \( V_i \) are closed, invariant, \( \sigma^p \)-invariant sets. By Proposition 4.8, there exists \( i \in [0,n] \) such that \( D_{F,h}(V_i) \) is comeager. By definition of \( \omega_{F,h} \), it is then included in \( V_i \), and hence equal. \( \square \)
4.4 Finite generic limit set

Proposition 4.10. Let \( \mathcal{F} \) be a semi-uniform DS and \( V = \omega(U) \) be finite, for some set \( U \) which is comeager in some nonempty open subset \( W \subseteq X \). Then \( \mathcal{E}_\mathcal{F} \) is comeager in \( W \).

Of course this means that \( \mathcal{E}_\mathcal{F} \cap U \) is comeager in \( W \).

We immediately deduce the following.

Corollary 4.11.

- If \( \omega_\mathcal{F} \) is finite, then \( \mathcal{F} \) is almost equicontinuous.
- If \( \mathcal{F} \) is sensitive (or has not equicontinuous point), then the asymptotic and limit sets of all non-meager sets are infinite.

Proof of Proposition 4.10. Let \( W' \subseteq W \) be a nonempty open subset. According to Proposition 3.9 \( \bigcup_{y \in I} A_\mathcal{F}(y) \supseteq U \), where \( I \) is finite, must also be comeager in \( W \). One such \( A_\mathcal{F}(y) \) should not be meager in \( W' \). By Remark 2.1, it is then comeager in some \( W'' \subseteq W' \). Remark 3.6 says that \( \mathcal{E}_\mathcal{F} \) is then comeager in \( W'' \). It results that \( \mathcal{E}_\mathcal{F} \) is not meager, in any nonempty open subset \( W' \subseteq W \), and again Remark 2.1 tells us that it is the comeager in \( W \).

In the case of cellular automata with finite generic limit set, we have the following.

Proposition 4.12. Let \( F \) be a CA and \( h \) a curve, such that \( \omega_{F,h} \) is finite. Then \( \omega_{F,h} \) contains one single (periodic) orbit, of a monochrome configuration \( y \), and \( \mathcal{A}_{F,h}(y) \) is comeager.

Note that the orbit of this monochrome configuration may be nontrivial (see Example 7.9), but still generic configurations are all asymptotic to the same configuration of that orbit. This could seem paradoxical, since it contrasts with the usual, uniform and synchronous aspect of dynamics of CA over the full set \( A^\mathcal{F} \), but here the genericity notion is not at all \( F \)-invariant.

Proof. By Proposition 4.15 \( \omega_{F,h} \) is a finite \( F \)-invariant subshift, so that all configurations are periodic (for the shift). Let \( p > 0 \) be a common period: \( \omega_{F,h} \) can be decomposed as \( \bigcup_{y \in V} O_{F,h}(y) \), where \( V \subseteq \omega_{F,h} \) is a set of orbit representatives.

We can apply Corollary 3.12: \( D_F(\omega_{F,h}) = \bigcup_{y \in \omega_{F,h}} A_{F,h}(y) \). Since every \( y \in \omega_{F,h} \) is \( \sigma^p \)-invariant, we can apply Point 3 of Lemma 2.4 (to \( G = \sigma^p \)), and get that there is \( y \in \omega_{F,h} \) such that \( A_{F,h}(y) \) is comeager. It results that \( \omega_{F,h} = O_{F,h}(y) \). Moreover, since \( \sigma \) is an automorphism of \( F \), \( \sigma(A_{F,h}(y)) = A_{F,h}(\sigma(y)) \) is also comeager. Then \( A_{F,h}(y) \cap A_{F,h}(\sigma(y)) \) is also comeager, and in particular nonempty. By transitivity of the asymptoticity relation, \( y \) is asymptotic to \( \sigma(y) \). Since they both lie in the bijective subsystem of \( F \) induced over the finite \( \omega_{F,h} \), Remark 3.5 gives that \( y = \sigma(y) \), which means that \( y \) is monochrome.

4.5 Asymptotic set of equicontinuous points

We shall show that if the system is almost equicontinuous, then its generic limit set is exactly the closure of the asymptotic set of its equicontinuous points. Also, if the system is equicontinuous, then its generic limit set is its limit set.

Proposition 4.13. Let \( \mathcal{F} \) be a DS and \( (V_i)_{i \in \mathbb{N}} \) a sequence of subsets of \( X \). Then \( \mathcal{E}_\mathcal{F} \cap \overline{D_\mathcal{F}((V_i)_i)} \subseteq \overline{D_\mathcal{F}((V_i)_i)} \).

Proof. Let \( x \in \mathcal{E}_\mathcal{F} \cap \overline{D_\mathcal{F}((V_i)_i)} \), and \( \varepsilon > 0 \). Since \( x \in \mathcal{E}_\mathcal{F} \), there exists \( \delta > 0 \) such that

\[
\forall y \in B_\mathcal{F}(x), \forall t \in \mathbb{N}, d(F_t(x), F_t(y)) < \varepsilon/2 .
\]

Since \( x \in \overline{D_\mathcal{F}((V_i)_i)} \), there exists \( y \in B_\mathcal{F}(x) \cap \overline{D_\mathcal{F}((V_i)_i)} \). Hence,

\[
\exists T \in \mathbb{N}, \forall t > T, d(F_t(y), V_i) < \varepsilon/2 .
\]

For this \( T \), \( \forall t > T, d(F_t(x), V_i) < \varepsilon \). Since this is true for every \( \varepsilon > 0 \), we get that \( x \in \overline{D_\mathcal{F}((V_i)_i)} \).
In particular, for \( z \in X \), we have \( \mathcal{A}_\tilde{\mathcal{F}}(z) \supseteq \overline{\mathcal{A}_\mathcal{F}(z)} \cap \mathcal{E}_\tilde{\mathcal{F}} \). For almost equicontinuous DS, Proposition 4.13 means that it is enough to prove that some realm is dense to prove that it is comeager.

**Corollary 4.14.** If \( \mathcal{F} \) is an almost equicontinuous DS, then \( \tilde{\omega}_\mathcal{F} = \overline{\omega_\mathcal{F}(\mathcal{E}_\mathcal{F})} \).

**Proof.** Since \( D_\mathcal{F}(\tilde{\omega}_\mathcal{F}) \) is dense, \( \mathcal{E}_\mathcal{F} \subseteq D_\mathcal{F}(\tilde{\omega}_\mathcal{F}) \) by Proposition 4.13. Hence, \( \omega_\mathcal{F}(\mathcal{E}_\mathcal{F}) \subseteq \tilde{\omega}_\mathcal{F} \). Since \( \tilde{\omega}_\mathcal{F} \) is closed, \( \overline{\omega_\mathcal{F}(\mathcal{E}_\mathcal{F})} \subseteq \tilde{\omega}_\mathcal{F} \).

Conversely, \( \tilde{\omega}_\mathcal{F} \) is the intersection of all closed subsets with comeager realms, among which \( \omega_\mathcal{F}(\mathcal{E}_\mathcal{F}) \) (whose realm includes the comeager \( \mathcal{E}_\mathcal{F} \)). So, \( \tilde{\omega}_\mathcal{F} = \overline{\omega_\mathcal{F}(\mathcal{E}_\mathcal{F})} \).

**Proposition 4.15.** If \( \mathcal{F} \) is an equicontinuous DS over space \( X \), then \( \tilde{\omega}_\mathcal{F} = \overline{\omega_\mathcal{F}} = \Omega_\mathcal{F} \).

**Proof.** Let \( y \in \Omega_\mathcal{F} \) and \( \varepsilon > 0 \). We will show that \( B_\varepsilon(y) \) intersects \( \omega_\mathcal{F} \). There exists \( \delta \) such that for every \( x \in \mathcal{E}_\mathcal{F} = X \) and every \( t \in \mathbb{N} \), \( F_t(B_\delta(x)) \subseteq B_{\varepsilon/2}(F_t(x)) \). By compactness of \( X \), there exists a finite \( I \subseteq X \) such that \( X = \bigcup_{x \in I} B_\delta(x) \). Since \( y \in \Omega_\mathcal{F} \), there is an infinite \( J \subseteq \mathbb{N} \), and for all \( t \in J \), some \( x_t \in X \) such that \( F_t(x_t) \in B_{\varepsilon/2}(y) \).

This proves that \( \omega_\mathcal{F}(X) \) is dense in \( \Omega_\mathcal{F} \); by Corollary 4.14 we obtain \( \tilde{\omega}_\mathcal{F} = \overline{\omega_\mathcal{F}(X)} = \Omega_\mathcal{F} \).

Another remark: it is known that a cellular automaton is nilpotent if and only if its limit set is finite. Hence, it is nilpotent if and only if it is equicontinuous and its generic limit set is finite.

## 5 Directional dynamics

In this section, we study cellular automata while varying the directions.

**Definition 5.1 (\([\mathbb{R}]\text{ Def 2.5}\)).** The set of curves (and, abusively, of directions) is noted \( \mathcal{B} \).

For \( h, h' \in \mathbb{B} \), we put \( h \preceq h' \) if there exists \( M > 0 \) such that \( h(t) \leq h'(t) + M \) for all \( t \in \mathbb{N} \). We put \( h \sim h' \) if, besides \( h' \preceq h \), \( \leq \) is a preorder relation on \( \mathcal{B} \), and we note \( \sim \) the corresponding equivalence relation.

We also note \( h \ll h' \) if \( \lim_{t \to +\infty} h'(t) - h(t) = +\infty \). \( \ll \) is a transitive relation which is finer than \( \preceq \).

The preorder \( \preceq \) induces a notion of (closed, open, semi-open) curve interval, with some bounds \( h' \preceq h'' \), noted \([h', h'']\), \([h', h'')\), \([h', h'')\), \([h', h'')\). We say that the interval is nondegenerate if \( h' \sim h'' \). For an interval \( S \subseteq \mathcal{B} \) with bounds \( h' \) and \( h'' \), we also note \( \mathcal{I}(S) = \{ h \in \mathcal{B} | h \ll h' \} \subset[h', h''] \).

A direction will implicitly refer to an equivalence class for \( \sim \) (sometimes abusively confused with one representative). The reader will note that all dynamical properties (equicontinuous points, finiteness of the generic limit set, etc.) are invariant through this equivalence class.

An example of curve is given by the (possibly irrational) lines: \( \alpha \in \mathbb{R} \) will stand for the direction \( t \mapsto \lfloor t\alpha \rfloor \).

The dynamics along \( \alpha \) then corresponds to that studied in \([\mathbb{R}]\text{ III}\).

Here are simple remarks about direction-invariance.

**Remark 5.2.** Let \( F \) be a CA and \( U, V_t \subseteq \mathbb{A}^2 \) be shift-invariant, for \( t \in \mathbb{N} \).

1. \( \mathcal{D}_{F,h}(V_t) \) does not depend on \( h \in \mathcal{B} \).

2. \( \Omega_{F,h}(U) \) does not depend on \( h \in \mathcal{B} \).

**Proof.**

1. Let \( x \in \mathcal{D}_{F,h}(V_t) \), that is, there exists a time \( T \in \mathbb{N} \) above which for all times \( t > T \), \( F_t\sigma x(t) \in V_t \). By assumption, we get that \( F_{T'}(T') \sigma x(t') \in V_t \) for any other \( h' : \mathbb{N} \to \mathbb{Z} \).

2. \( \Omega_{F,h}(U) = \bigcap_{T \in \mathbb{N}} \bigcup_{t \geq T} F_t\sigma x(t) = \bigcap_{T \in \mathbb{N}} \bigcup_{t \geq T} F(t) = \Omega_F(U) \). 


5.1 Oblique directions

Let $F$ be a CA with memory $r_− \in \mathbb{Z}$ and anticipation $r_+ \in \mathbb{Z}$. Then for every $t \in \mathbb{N}$, $F^t$ can be defined by a rule of memory $r_-t$ and anticipation $r_+t$. But it could be that smaller parameters also fit. This motivates the following definition.

For a sequence $(F_t)_{t \in \mathbb{N}}$ of CA, let us denote $r_-(t)$ and $r_+(t)$ the minimum possible memory and anticipation for $F_t$. Formally, $r_-(t) = \sup \{i \in \mathbb{Z} \mid \forall x, y \in A^Z, x_{i+1,+,i} = y_{i+1,+,i} \Rightarrow F_t(x) = F_t(y)\}$ and $r_+(t) = \inf \{i \in \mathbb{Z} \mid \forall x, y \in A^Z, x_{-i,-i} = y_{-i,-i} \Rightarrow F_t(x) = F_t(y)\}$. Then for every $r_- \leq r_-(t) < r_+(t) < +\infty$, if and only if $F_t$ is not a constant function.

Remark 5.3.

1. For every $t \in \mathbb{N}$, $-\infty < r_-(t) \leq r_+(t) < +\infty$, if and only if $F_t$ is equicontinuous and the slopes being called the Lyapunov exponents (see for instance [25]).

2. $(F_t\sigma^{h(t)})_{t \in \mathbb{N}}$ is equicontinuous if and only if $r_+ \leq -h \leq r_-$ (in particular, if $F_t$ is never constant, $h \sim -r_+ \sim -r_-$).

Proof. The first statement is direct from continuity of $F_t$.

If $(F_t\sigma^{h(t)})_t$ is equicontinuous, then there exists $r \in \mathbb{N}$ such that for all $t \in \mathbb{N}$, $x, y \in A^Z$, $x_{[-r,r]} = y_{[-r,r]} \Rightarrow F_t\sigma^{h(t)}(x) = F_t\sigma^{h(t)}(y)$. Since $F$ commutes with $\sigma$, we get $x_{[-r-h(t),-r-h(t)]} = y_{[-r-h(t),-r-h(t)]} \Rightarrow F_t(x) = F_t(y)$. We get that $-r - h(t) \leq r_-(t)$ and $r_+(t) \leq r - h(t)$.

Conversely, assume that there exists $M \in \mathbb{N}$ such that $\forall t \in \mathbb{N}$, $r_+(t) \leq M - h(t)$ and $-h(t) \leq M + r_-(t)$, and let $l \in \mathbb{N}$. Then for every $t \in \mathbb{N}$ and $x, y \in A^Z$ such that $x_{[-l-M+l,M]} = y_{[-l-M+l,M]}$, consider the configuration $z \in A^Z$ such that $z_i = x_i$ for every $i \in [-l, l+M]$ and $z_i = y_i$ for every $i \in [-l-M, +\infty]$. By the assumed inequalities, $x_{[-l-M, l-M]} = y_{[-l-M, l-M]}$ and $z_{[-l-M, l+M]} = y_{[-l-M, l+M]}$, so that $F_t\sigma^{h(t)}(x)_{[-l,l]} = F_t\sigma^{h(t)}(z)_{[-l,l]} = F_t\sigma^{h(t)}(y)_{[-l,l]}$. This proves equicontinuity of $(F_t\sigma^{h(t)})_t$.

The consequence between parentheses comes from the first point.

In the case of a single CA, we have seen that $r_-(t) \geq r_-t$ and $r_+(t) \leq r_+t$. It is known that these sequences will be asymptotically linear, the slopes being called the Lyapunov exponents (see for instance [25]).

We say that a direction $h$ is oblique for CA sequence $(F_t)_t$ if $h \notin [-r_+, -r_-]$. We will show that the generic limit set in an oblique direction is equal to the limit set.

We say that a DS $\mathcal{X}$ over space $X$ is semi-transitive if for every open sets $U, V$ such that $V \cap \Omega_\mathcal{X} \neq \emptyset$, there exists $T \in \mathbb{N}$ such that $U \cap F_T^{-1}(V) \neq \emptyset$. Equivalently, $\bigcup_{t \geq T} F_t^{-1}(V)$ is dense. $\mathcal{X}$ is weakly semi-mixing if for every open sets $U, V, U', V'$ such that $V$ and $V'$ intersect $\Omega_\mathcal{X}$, there exists (a common) $T \in \mathbb{N}$ such that $U \cap F_T^{-1}(V) \neq \emptyset$ and $U' \cap F_T^{-1}(V') \neq \emptyset$. Clearly if a system is weakly semi-mixing or transitive, then it is semi-transitive. But weak semi-mixing has a stronger consequence.

Remark 5.4. Any weakly semi-mixing DS $\mathcal{X}$ is sensitive or admits a trivial limit set.

Proof. If $\Omega_\mathcal{X}$ is not trivial, then there are two open subsets $V$ and $V'$ which are at positive distance $\varepsilon > 0$. Then for every $x \in X$ and $\delta > 0$, $B_\delta(x)$ intersects both $F_T^{-1}(V)$ and $F_T^{-1}(V')$, for some $T \in \mathbb{N}$, so that there are points $y$ and $y'$ in it, for which $d(F_T(y), F_T(y')) > \varepsilon$. By the triangular inequality, $F_T(x)$ should be at distance at least $\varepsilon/2$ of one of the two, which means that $x$ is not $\varepsilon/2$-stable.

Proposition 5.5. For every semi-transitive DS $\mathcal{X}$, $\hat{\omega}_\mathcal{X} = \Omega_\mathcal{X}$.

Proof. Let us consider $x \in \Omega_\mathcal{X}$. By assumption, for every $\varepsilon > 0$ and $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_\varepsilon(x))$ is dense so that we can apply Remark 4.3 \{ $x = \overline{x}$ \} intersects all generic orbits, hence is in $\hat{\omega}_\mathcal{X}$.

The other inclusion is always true.

Proposition 5.6. If $(F_t)_t$ is a CA sequence and $h$ an oblique curve, then the DS $(F_t\sigma^{h(t)})_t$ is weakly semi-mixing.
Lemma 5.8. Let \( \Omega \times \mathbb{Z} \) be a CA over \( \Omega \), and \( h \) an oblique curve. Then \( \omega_{\Omega,h} = \Omega_{\Omega,h} \); it is neither sensitive nor nilpotent.

Proof. The equality is direct from Propositions 5.6 and 5.5. Sensitivity comes from Proposition 5.6. Remark 5.4 and the known fact that the limit set of a CA is trivial if and only if it is nilpotent.

5.2 Almost equicontinuity in two directions

The purpose of this subsection is to show that if the CA is almost equicontinuous in two distinct directions, then its generic limit set is finite. We essentially reprove [6] Prop 3.2.3, Prop 3.3.4] (or the corresponding result for linear directions from [23]), but additionally discuss the generic limit set.

Here is the main lemma for understanding directional dynamics. It is based on the fact that if a word \( u \) is blocking along \( h' \in \mathcal{B} \), and \( s' \) is the minimal corresponding offset, then in particular for every \( t \in \mathbb{N} \) there exists \( a_{u,h'(t)}(t) = F^t(z)_{h'(t)} \).

Lemma 5.8. Let \( \Omega \times \mathbb{Z} \) be a CA over \( \mathbb{Z} \), with blocking words \( u \) and \( u \) along \( h' \in \mathcal{B} \) with offset \( s' \in \mathbb{Z} \) and \( v \) along \( h'' \in \mathcal{B} \) with offset \( s'' \in \mathbb{Z} \), and \( q' = h' + |v| - s' \) and \( q'' = h'' - s'' \). Then for every \( z \in [v]_0 \) and \( j \in [q'(t), q''(t)] \), \( F^t(z)_j = a_{u,h'}(t) \).

Of course the same is true for \( [u] \). The following statement is direct from the definition. Of course the symmetric statement holds for right-blocking words.

Remark 5.9.

1. From the definition, one can see that if \( u \) is blocking for CA \( F \) along curve \( h \), then any word containing \( u \) also.

2. If two directions are almost equicontinuous, Proposition 2.3 states that they admit blocking words \( u \) and \( v \). From the previous point, they admit a common blocking word \( uv \).

3. From the definition, one can see that: if \( u \) is a right-blocking word for CA \( F \) along directions \( h' \) and \( h'' \), then also along any direction \( h \succ \min(h', h'') \).

4. From the previous point and the symmetric statement, if \( u \) is right- and left-blocking along directions \( h' \) and \( h'' \), then also along any direction \( h \in [\min(h', h''), \max(h', h'')] \).

5. In particular, right- and left-blockingness are preserved by \( \sim \) (which is not the case for strong blockingness).

Proof of Lemma 5.8. Since they are left-blocking and right-blocking, respectively, for every \( t \in \mathbb{N} \):

\[
\forall x, y \in [u]_{s'}, x_{[s', +\infty]} = y_{[s', +\infty]} \implies F^t(x)_{[h'(t), +\infty]} = F^t(y)_{[h'(t), +\infty]} \\
\forall x, y \in [v]_{s''}, x_{[-\infty, s'']} = y_{[-\infty, s'']} \implies F^t(x)_{[-\infty, h''(t)]} = F^t(y)_{[-\infty, h''(t)]}.
\]

By definition, \( \forall z \in [u]_{s'}, a_{u,h'}(t) = F^t(z)_{h'(t)} \).
Now let \( j \in \llbracket q'(t), q''(t) \rrbracket \), so that there is a configuration \( y \in [v]_0 \cap [u]_{-h'(t)+s'} \) such that \( y_{-\infty,0} = z_{-\infty,0} \) and \( \sigma^{-h'(t)}(y)_{s'+\infty} = x_{s'+\infty} \). Since \( \sigma^{-h'(t)}(y) \) is in \([u]_{s'}\) and \( u \) is left-blocking along \( h' \), we get: \( F^t(y)_j = a_{u,h'}(t) \). On the other hand, since \( v \) is right-blocking along \( h'' \), \( \forall t \in \mathbb{N}, F^t(z)_{-\infty,h''(t)-s''} = F^t(z)_{-\infty,h''(t)-s''} \). In particular, we get that \( F^t(z)_j = F^t(y)_j = a_{u,h'}(t) \).

**Proposition 5.10.** Let \( F \) be a CA with a blocking word \( u \) along two distinct directions \( h' \) and \( h'' \), with \( h'' \not\preceq h' \). Along any direction \( h \in \mathcal{B} \), the direct realm \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \) includes all \( \sigma \)-periodic configurations where \( u \) appears; Moreover, the realm \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \) includes:

1. All configurations where \( u \) appears infinitely many times on the left and on the right, if \( h' \preceq h \preceq h'' \);
2. All configurations where \( u \) appears infinitely many times on the right, if \( h' \ll h \preceq h'' \);
3. All configurations where \( u \) appears infinitely many times on the left, if \( h' \ll h \preceq h'' \);
4. All configurations where \( u \) appears, if \( h' \ll h \ll h'' \).

**Proof.** Consider a configuration \( z \in [u]_i \), for some \( i \in \mathbb{Z} \), with some \( \sigma \)-period \( p \geq 1 \). Let \( q' \) and \( q'' \) be as in Lemma 5.8. Since \( q' \sim h'' \not\preceq h' \sim q'' \), there exists \( T \in \mathbb{N} \) such that \( q'(T) \leq q''(T) + p \). Lemma 5.8 says that then \( F^T(z)_j = a_{u,h'}(T) \) for all \( j \in \llbracket t + q'(T), t + q''(T) \rrbracket \), and, by periodicity, for all \( j \in \mathbb{Z} \). This means that \( F^T(z) \) is monochrome, and it is clear that it stays monochrome for \( t \geq T \). Since, by definition of \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \), it appears in \( F^T(z) \), we deduce that the latter is equal to \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \).

1. From Point 1 of Remark 5.9, \( u \) is right- and left-blocking along every curve \( h \in [\min(h',h''),\max(h',h'')] \). So from Proposition 2.5, configurations with infinitely many occurrences of \( u \) on the left and on the right are equicontinuous. From Proposition 4.13 and the previous point that \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \supseteq \mathcal{D}_{F,h}((\llbracket a_{u,h'}(t) \rrbracket) \) is dense, these equicontinuous configurations must also be in \( \mathcal{D}_{F,h}(\llbracket a_{u,h'}(t) \rrbracket) \).

2. Let \( z \in \bigcap_{i \in \mathbb{Z}} \bigcup_{j \geq i} [u]_j \) and \( n \in \mathbb{N} \). If \( h \preceq h'' \preceq q'' \), then there exists \( j \geq \max_{i \in \mathbb{N}} h(t) - q''(t) + n \) such that \( z \in [u]_j \). If \( q' \sim h' \ll h \), then there exists \( T \in \mathbb{N} \) such that \( \forall t \geq T, q'(t) + j \leq h(t) - n \). From Lemma 5.8 and these inequalities, for all \( t \geq T \), the pattern \( F^T(z)_{[h(t)-n,h(t)+n]} \) is monochrome. Since this is true for every \( n \), every limit point of the orbit must be monochrome.

3. This case is symmetric to the previous one.

4. Let \( z \in [u]_j \) for some \( j \in \mathbb{Z} \) and \( n \in \mathbb{N} \). If \( q' \sim h' \ll h \ll h'' \ll q'' \), then there exists \( T \in \mathbb{N} \) such that for all \( t \geq T \), \( h(t) + n \leq q''(t) + j \) and \( q'(t) + j \leq h(t) - n \). From Lemma 5.8 and these inequalities, for all \( t \geq T \), the pattern \( F^T(z)_{[h(t)-n,h(t)+n]} \) is monochrome.

**5.3 Classification of generic limit sets up to directions**

We recall the classifications of CA up to shift from [23] and emphasize the properties of each class in terms of generic limit set. By strictly almost equicontinuous, we mean almost equicontinuous but not equicontinuous.

**Theorem 5.11.** Every CA \( F \) with memory \( r- \) and anticipation \( r+ \), satisfies exactly one of the following statements:

1. \( F \) is nilpotent; there is a symbol \( a \in A \) such that for all \( h \in \mathcal{B} \), \( \hat{\omega}_{F,h} = \{\infty a^{\infty}\} \).
2. \( F \) is equicontinuous along a single direction \( h' \in [-r,+,-r] \), and sensitive along other directions; for every \( h \in \mathcal{B} \), \( \hat{\omega}_{F,h} = \Omega_F \) is infinite.
3. \( F \) is strictly almost equicontinuous along a nondegenerate interval \( S \subseteq [-r,+,-r] \) and sensitive along other directions; there exists \( a \in A \) such that \( \hat{\omega}_{F,h} = \Omega_F(\infty a^{\infty}) \) for every \( h \in S \), and \( \hat{\omega}_{F,h} \) is infinite for every \( h \notin S \); moreover, \( \varepsilon_{F,h} \subseteq \mathcal{D}_{F,h}(\hat{\omega}_{F,h}) = \mathcal{A}_{F,h}(\infty a^{\infty}) \), and if \( h \in \mathcal{T}(S) \), it includes a dense open set.
4. $F$ is strictly almost equicontinuous along a single direction $h' \in [-r_+, -r_-]$ and sensitive along other directions; for every $h \in \mathcal{B}$, $\hat{\omega}_{F,h}$ is infinite.

4'. $F$ is strictly almost equicontinuous along a single direction $h' \in [-r_+, -r_-]$ and sensitive along other directions; $\hat{\omega}_{F,h}$ is finite if and only if $h = h'$.

5. $F$ is sensitive in every direction; $\hat{\omega}_{F,h}$ is infinite along all $h \in \mathcal{B}$. 

Compared to [7] Theorem 2.9], we have merged the last two classes, because expansiveness is not relevant in terms of generic limit set, except that it implies surjectivity. Surjective CA have their generic limit set equal to the full shift of configurations in every direction: they are either in Class 2, Class 4 or Class 5. In Class 3 the bounds of the interval of almost equicontinuity can be closed or not; actually all four cases can happen: see [6] for some examples (the bound would be included if one allows directions with unbounded variation).

See Section 7 for examples of all classes (in particular, both subclasses of Class 4 can occur).

Proof. 1. Nilpotent CA have a trivial limit set (in every direction), which includes the generic limit set.

2. Now suppose that $F$ is not nilpotent, which is equivalent to $\Omega_F$ being infinite. Assume also that there is at least one direction of equicontinuity. By Remark 5.3, all other directions are oblique, hence sensitive by Corollary 5.7. By Proposition 4.15 and Corollary 5.7, $\hat{\omega}_{F,h} = \Omega_F$, for all $h \in \mathcal{B}$.

3. Now suppose that $F$ admits no direction of equicontinuity, but two distinct directions $h'$ and $h''$ of almost equicontinuity. By Point 2 of Remark 5.9, these two directions have a common blocking word $u$, so that we can apply Proposition 5.10: $u$ is also blocking for directions in the interval $[\min(h', h''), \max(h', h'')]$. Since this is true for every $h', h''$, we deduce that the set $S$ of almost equicontinuous directions is convex: it is a nondegenerate interval. By Corollary 5.7 it is included in $[-r_+, -r_-].$

Proposition 5.10 also states that $\mathcal{D}_{F,h}(\langle \infty a_{u,h}(t)^\infty \rangle_t)$ is dense for every $h \in \mathcal{B}$, so that it includes all equicontinuous points, by Proposition 4.13. If $h \in S$, Corollary 4.14 gives that the generic limit set is the closure of the asymptotic set of $\mathcal{E}_{F,h}$, that is then the asymptotic set of $\{ \langle \infty a_{u,h}(t)^\infty \rangle_t | t \in \mathbb{N} \}$, which is a set of monochrome configurations; in particular, it does not depend on $h$. By Proposition 4.12 it is the orbit by $F$ of a monochrome configuration.

For other directions, the generic limit set of a sensitive DS is infinite, by Corollary 4.11. Finally, if $h \in \mathcal{I}(S)$, it is easy to find $q', q'' \in S$ such that $q' \prec h \prec q''$. Point 4 of Proposition 5.10 gives that $\mathcal{D}_{F,h}(\langle \infty a_{u,q'}(t)^\infty \rangle_t)$ contains a dense open set. The same argument as above gives that this is in the realm of the generic limit set.

4. The cases remain when there is at most one direction of almost equicontinuity; it cannot be oblique, and other directions have to all have infinite generic limit by Corollary 4.11. This settles the last three classes.

6 Links with the measure-theoretical approach

By a measure, we mean a Borel probability measure on $X$. The topological support $S_\mu$ of a measure $\mu$ is the smallest closed subset of measure 1. If $S_\mu = X$, we say that $\mu$ has full support.

We say that $\mathfrak{g} = (F_t)_{t \in \mathbb{N}}$ is $\mu$-equicontinuous if $\mu(\mathcal{E}_F) = 1$. Proposition 2.5 and Corollary 6.4 give that, if $\mu$ is $\sigma$-ergodic, a CA $F$ along some direction $h$ is $\mu$-equicontinuous unless $\mathcal{E}_{F,h} \cap S_\mu = \emptyset$. In the case of Bernoulli measure, this corresponds to [10] Prop 3.5.
6.1 $\mu$-likely limit set and $\mu$-limit set

The generic limit set is the topological variant of the $\mu$-likely limit set $\Lambda_{\mathfrak{g},\mu}$, which is the smallest closed subset of $X$ which has a realm of attraction of measure one. Examples 5, 6 point that there are no general inclusion relations between the two sets, but that they intersect. Here is a formalization of this argument.

Proposition 6.1. For every DS $\mathfrak{g}$ and support measure $\mu$, $\Lambda_{\mathfrak{g},\mu} \cap \omega_{\mathfrak{g}} \neq \emptyset$.

Proof. Remark that for every $\varepsilon > 0$ and every $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_{\varepsilon}(\Lambda_{\mathfrak{g},\mu})) \supseteq \mathcal{D}_{\mathfrak{g}}(\Lambda_{\mathfrak{g},\mu})$ has measure 1, so it is dense. We apply Remark 1.3 and get that $\Lambda_{\mathfrak{g},\mu}$ intersects every generic orbit, hence $\omega_{\mathfrak{g}}$.

The $\mu$-likely limit set should not be confused with the $\mu$-limit set $\Omega_{\mathfrak{g},\mu}$, from [20, 7], which is the intersection of all closed subsets $U$ such that $\lim_{t \to \infty} \mu(F_t^{-1}(U)) = 1$. We prove one general inclusion, though; it is a generalization of [26, Prop 1].

Proposition 6.2. For every DS $\mathfrak{g}$ and Borel probability measure $\mu$, $\Omega_{\mathfrak{g},\mu} \subseteq \Lambda_{\mathfrak{g},\mu}$.

Proof. It is enough to prove that, for every $\varepsilon > 0$ and every $T \in \mathbb{N}$, $\bigcup_{t \geq T} F_t^{-1}(B_{\varepsilon}(\Lambda_{\mathfrak{g},\mu})) \subseteq \mathcal{D}_{\mathfrak{g}}(\Lambda_{\mathfrak{g},\mu})$ has measure 1, so it is dense. We apply Remark 1.3 and get that $\Lambda_{\mathfrak{g},\mu}$ intersects every generic orbit, hence $\omega_{\mathfrak{g}}$.

The converse is in general false: Example 7.6 is a counter-example, as proved in [26]. However, we prove the converse in a specific case.

Proposition 6.3. Let $\mathfrak{g}$ be a $\mu$-equicontinuous DS for some measure $\mu$. Then $\Omega_{\mathfrak{g},\mu} = \Lambda_{\mathfrak{g},\mu} = \omega_{\mathfrak{g}}(\mathcal{E}_{\mathfrak{g}} \cap S_{\mu})$.

In particular, if $\mu$ has full support, then these sets are equal to $\omega_{\mathfrak{g}}(\mathcal{E}_{\mathfrak{g}})$.

Proof. Since $\mathcal{D}_{\mathfrak{F}}(\Lambda_{\mathfrak{g},\mu})$ is dense in $S_{\mu}$, it includes the full-measure set $\mathcal{E}_{\mathfrak{g}} \cap S_{\mu}$, thanks to Proposition 4.13. In the same way as in the proof of Corollary 4.14 we can quickly deduce the second equality.

Now let $y \in \omega_{\mathfrak{g}}(\mathcal{E}_{\mathfrak{g}} \cap S_{\mu})$ and $\varepsilon > 0$. There is a point $x \in \mathcal{E}_{\mathfrak{g}} \cap S_{\mu}$, and a subsequence $(t_n)_n$ such that $\forall n \in \mathbb{N}, F_{t_n}(x) \in B_{\varepsilon}(y)$. By equicontinuity of $x$, there exists $\delta > 0$ such that $\forall t \in \mathbb{N}, F_t(B_\delta(x)) \subseteq B_{\varepsilon/2}(F_t(x))$. In particular, for $n \in \mathbb{N}$, we get $F_{t_n}(B_\delta(x)) \subseteq B_{\varepsilon/2}(F_{t_n}(x))$, so that $\mu(F_{t_n}^{-1}(B_\delta(x))) \geq \mu(B_\delta(x)) > 0$.

Let $U$ be a closed subset of $B_{\varepsilon}(y)^C$. Then for every $n \in \mathbb{N}$, $\mu(F_{t_n}^{-1}(U)) \leq \mu(F_{t_n}^{-1}(B_{\varepsilon}(y)^C)) \leq 1 - \mu(B_{\delta}(x))$. Since this is positive and independent of $n$, we get that $\mu(F_{t_n}^{-1}(U))$ does not converge to 1.

By contraposite, we get that for every closed $U$ such that $\mu(F_{t_n}^{-1}(U))$ converges to 1 should intersect $B_{\varepsilon}(y)$, for every $\varepsilon > 0$. Hence it contains $y$, and we can conclude that $y \in \Omega_{\mathfrak{g},\mu}$.

The converse inclusion comes from Proposition 6.2.

6.2 $\mu$-likely limit set of cellular automata

If $G$ is a uniform DS, $\mu$ is $G$-invariant if $\mu(G^{-1}(U)) = \mu(U)$ for all measurable subsets $U$ of $X$. A $G$-invariant measure $\mu$ is $G$-ergodic if every $G$-invariant set $U$ has either zero or full measure. This is the measure-theoretical counterpart of the topological notion of transitivity, as emphasized by Lemma 2.4 which gives the following.

Corollary 6.4. Let $G$ be a transitive uniform DS, $\mu$ a full-support ergodic measure, and let $W \subseteq X$ be $G$-invariant. Then the following are equivalent.
1. $W$ is not meager.
2. $W$ is comeager.
3. $W$ has measure 1.
4. $W$ has positive measure.

Proof. The first two points are equivalent thanks to Lemma 2.4 and the last two by definition of ergodicity. Now if $W$ is comeager, it can be written as a countable intersection $\cap_{n \in \mathbb{N}} W_n$ of dense open sets $W_n$, and hence as a countable intersection $\cap_{n \in \mathbb{N}} \mathcal{O}_G(W_n)$ of dense open $G$-invariant sets. By full support, their measure should be positive, and by ergodicity, they should be 1, so that of $W$ also. Otherwise $W$ is meager, so that the previous argument holds for its complement.

In [22], J. Milnor asks for a good criterion for equality between the likely and generic limit sets. Here is at least a criterion, in the case of cellular automata.

**Corollary 6.5.** If $\overline{F}$ is a sequence of CA, and $\mu$ is a full-support $\sigma$-ergodic measure, then $\overline{\omega}_\overline{F} = \Lambda_{\overline{F},\mu}$.

**Proof.** Proposition 4.4 gives that the generic limit set is the intersection of all closed subsets with $\sigma$-invariant comeager realms. The same simple argument shows that the likely limit set is the intersection of all closed subsets with $\sigma$-invariant realms of measure 1. Thanks to Corollary 6.4, these realms are actually the same, so that the two sets are equal.

It results that all results from the previous sections hold for the $\mu$-likely limit set in that case. Actually, even when they are not equal, most results on the generic limit have a parallel result on the likely limit set, which can be proven with the same proof tools.

### 7 Examples

**Example 7.1 (Shift).** Let $\sigma$ be the CA over $A^\mathbb{Z}$ defined by $\sigma(x)_i = x_{i+1}$. This CA is reversible, hence surjective. By Corollary 4.7, $\overline{\omega}_{\sigma,h} = A^\mathbb{Z}$ along all $h \in \mathcal{B}$. Along direction $−1$, it corresponds to the identity CA. This CA has only one equicontinuous direction: it is in Class 3 of Theorem 5.11.

Let $F$ be a CA with alphabet $A$, memory $r_- \in \mathbb{Z}$, anticipation $r_+ > r_-$ and local rule $f$. A state $0 \in A$ is **spreading** if for all $u \in A^{r_+−r_-+1}$ such that $0 \sqsubseteq u$, one has $f(u) = 0$. Note that any CA is simulated, in a strong natural sense, by a CA with a spreading state (simply by artificially adding it to the alphabet). In particular, unlike the asymptotic set (which includes the nonwandering set), the generic limit set does not support the topological entropy (in the sense of [2]).

**Remark 7.2.** Let $F$ be a CA over $A^\mathbb{Z}$ with memory $r_- \in \mathbb{Z}$, anticipation $r_+ > r_-$, and spreading state $0 \in A$. Then it is in Class 4: $\overline{\omega}_{F,h} = \{0^\infty\}$ along all $h \in [-r_+,-r_-]$ and $\overline{\omega}_{F,h} = \Omega_F$ along all $h \notin [-r_+,-r_-]$.

**Proof.** We suppose that $F$ has a spreading state $0 \in A$. By definition, it is a (left- and right-) blocking word along all $h \in [-r_+,-r_-]$. By Proposition 5.10 there exists $a \in A$ such that $\overline{\omega}_{F,h} = \mathcal{O}_F(\infty a^\infty) = \overline{\omega}_{E_{F,h}}$ along all $h \in [-r_+,-r_-]$. In this case, $a$ is nothing else than 0. Moreover, since any $h \notin [-r_+,-r_-]$ is oblique, $\overline{\omega}_{F,h} = \Omega_F$ by Corollary 5.7.

The simplest example of spreading state is the following.

**Example 7.3 (Min).** Let $\text{Min}$ be defined over $\{0,1\}^\mathbb{Z}$ by $\text{Min}(x)_i = \min(x_i, x_{i+1})$. By Remark 7.2, we know that this CA is in Class 3: $\overline{\omega}_{\text{Min},h} = \{0^\infty\}$ along all $h \in [-1,0]$ and $\overline{\omega}_{\text{Min},h} = \Omega_{\text{Min}} = \{x \in \{0,1\}^\mathbb{Z} \mid \forall k > 0, 10^k1 \not\sqsubseteq x\}$ along all $h \notin [-1,0]$.

1. Along direction 0 (or, symmetrically, direction $−1$): the realm of $\{0^\infty\}$ is $\bigcap_{k \in \mathbb{Z}} \{x \in \{0,1\}^\mathbb{Z} \mid \exists i \geq k, x_i = 0\}$, it is a comeager set. $\{0^\infty\} = \bigcap_{k \geq 0} V_k$, where $V_k = \Omega_{\text{Min}}([0]_k) = \{x \in \Omega_{\text{Min}} \mid \forall i \leq k, x_i = 0\}$ is an attractor but not a subshift, for every $k \in \mathbb{Z}$ (see [18]).
Example 7.4 (Finite generic limit set). Consider the CA $\text{Min} \times \sigma^{-1}\text{Min}$ defined over $(\{0,1\}^\mathbb{Z})^2$ by $(\text{Min} \times \sigma^{-1}\text{Min})(x,y)_i = (\min(x_i,x_{i+1}),\min(y_{i-1},y_i))$. According to Example 7.3, $\tilde{\omega}_{\text{Min},h} = \{\infty 0^\infty\}$ along all $h \in [-1,0]$ and $\tilde{\omega}_{\text{Min},h} = \Omega_{\text{Min}}$ along all $h \notin [-1,0]$; of course $\tilde{\omega}_{\sigma^{-1}\text{Min},h} = \{\infty 0^\infty\}$ along all $h \in [0,1]$ and $\tilde{\omega}_{\sigma^{-1}\text{Min},h} = \Omega_{\text{Min}}$ along all $h \notin [0,1]$. Hence, $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},0} = \{\infty 0^\infty\}^2$, and $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},h} = \{\infty 0^\infty\} \times \Omega_{\text{Min}}$ (resp. $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},h} = \Omega_{\text{Min}} \times \{\infty 0^\infty\}$) along all $h \in [-1,0]$ (resp. $h \in [0,1]$), and $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},h} = \Omega_{\text{Min}}^2$ along all $h \notin [-1,1]$. In particular, $\text{Min} \times \sigma^{-1}\text{Min}$ has only one almost equicontinuous direction: $\{0\}$. Hence, it is in Class 5.

Example 7.5 (Sensitivity in every direction). Consider the CA $\sigma^{-1}\text{Min} \times \sigma$ defined over $(\{0,1\}^\mathbb{Z})^2$ by $(\sigma^{-1}\text{Min} \times \sigma)(x,y)_i = (\min(x_{i-1},x_i),\min(y_{i+1},y_i))$. According to Example 7.4, $\tilde{\omega}_{\sigma,h} = \{0,1\}^\mathbb{Z}$ along all $h \in B$. According to Example 7.3, $\tilde{\omega}_{\sigma^{-1}\text{Min},h} = \{\infty 0^\infty\}$ along all $h \in [0,1]$ and $\tilde{\omega}_{\sigma^{-1}\text{Min},h} = \Omega_{\text{Min}}$ along all $h \notin [0,1]$. Hence, $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},h} = \{\infty 0^\infty\} \times \{0,1\}^\mathbb{Z}$ along all $h \in [0,1]$, and $\tilde{\omega}_{\text{Min}\times\sigma^{-1}\text{Min},h} = \Omega_{\text{Min}} \times \{0,1\}^\mathbb{Z}$ along all $h \notin [0,1]$. Since there is no almost equicontinuous direction, this CA is in Class 5.

Example 7.6 (Just Gliders). Let $A = \{\leftarrow,0,\rightarrow\}$ and $F$ the CA defined by the following local rule:

$$f(x_{i-1},x_i,x_{i+1}) = \begin{cases} \rightarrow & \text{if } x_{i-1} = \rightarrow \text{ and } x_i \neq \leftarrow \text{ and } x_{i+1} \neq \leftarrow \text{ or } x_i = \rightarrow \\ \leftarrow & \text{if } x_{i-1} = \leftarrow \text{ and } x_i \neq \rightarrow \text{ and } x_{i+1} \neq \leftarrow \text{ or } x_i = \leftarrow \\ 0 & \text{otherwise} \end{cases}$$

A typical space-time diagram of this CA is shown in Figure 3. It is possible to interpret it as a background of 0’s where particles $\rightarrow$ and $\leftarrow$ go to the right and to the left, respectively. When two opposite particles meet they disappear.

This example is known to have a $\mu$-limit set which is strictly included in the $\mu$-likely limit set, when $\mu$ is the uniform Bernoulli measure (see [20] Ex 3 and [21] Ex 4]). One can see (or read in these references) that the limit set is $\Omega_F = \{ x \in A^{\mathbb{Z}} | \forall k \in \mathbb{N}, \exists 0^k \rightarrow \forall x \}$. We prove here that $F$ is weakly semi-mixing in every direction $h \notin \{-1,1\}$. Hence $\tilde{\omega}_{F,h} = \Omega_{F,h}$. Moreover, $\tilde{\omega}_{F,-1} = \{0,\rightarrow\}^\mathbb{Z}$, $\tilde{\omega}_{F,1} = \{0,\leftarrow\}^\mathbb{Z}$, and $F$ is sensitive in every direction; it is in Class 5.
Proof. By induction on \( t \in \mathbb{N} \), one can see that \( F^t(x)_k \to \) if and only if \( x_{t+k} \to \) and \( u = x_{[-t+k+1,t+k]} \) is a right-balanced pattern, that is it does not send any particle to the left, or more formally:

\[
\forall j \in \|0,|u|\|, \sum_{i=0}^j u_i \geq 0 \text{ , where } \to = +1 \text{ and } \leftarrow = -1 .
\]

Generalizing this induction, we can see that if \( k \in \mathbb{Z} \), \( t \in \mathbb{N} \), \( \leftarrow \in \mathcal{L} \) \( w \) and \( u \in A^2 \) is right-balanced, then \( F^t([wu]_k) \subseteq [w]_{k+t} \). We define left-balanced patterns symmetrically, and get that if \( \rightarrow \in \mathcal{L} \) \( z \) and \( u \) is left-balanced, then \( F^t([uz]_k) \subseteq [z]_{k+t} \).

- Let \([u]_m, [v]_n \) and \([v']_{n'} \) be three cylinders, the last two intersecting \( \Omega_F \). By the expression of \( \Omega_F \), note that we can decompose them as \( v = wz, v' = w'z' \), with \( \leftarrow \in \mathcal{L} \) \( w, w' \) and \( \rightarrow \in \mathcal{L} \) \( z, z' \). We prove that there is common time step \( t \in \mathbb{N} \) such that \( F^t\sigma^h(t)([u]_m) \cap [v]_n \neq \emptyset \) and \( F^t\sigma^h(t)([u]_m) \cap [v']_{n'} \neq \emptyset \). If \( h \) is oblique, the result follows from Proposition \ref{prop:oblique}, so we can assume that \( h \in \{-1, +1\} \).

We can assume that \( u \) is a left- and right-balanced pattern: just extend it with the suitable number of \( \rightarrow \) on the left, and the suitable number of \( \leftarrow \) on the right (the obtained cylinder is included in the original one).

Since \( h > -1 \), there exists \( t \in \mathbb{N} \) such that \( h(t) > -t + \max(n + |w|, n' + |w'|) - m \). Since \( h < 1 \), there exists \( t \in \mathbb{N} \) such that \( h(t) < t + \min(n + |w|, n' + |w'|) - m - |u| \). These \( t \) could be distinct, but it is not difficult to be convinced that, since \( h \) has bounded variation, there is a common \( t \in \mathbb{N} \) which satisfies both. In that case we can define \( \tilde{u} = 0^{t-n+h(t)}|w|^{t-m}-m+n-h(t)+|w| \). Clearly, it is still left- and right-balanced, so that \( F^t\sigma^h(t)([w]\tilde{u}^{-h(t)-t}) \subseteq [u]_n \) and \( F^t\sigma^h(t)([\tilde{u}z]_{n-h(t)+|w|-t}) \subseteq [z]_{n+|w|} \); taking the intersection we get \( F^t\sigma^h(t)([w]\tilde{u}z\tilde{v}^{-h(t)-t}) \subseteq [v]_n \). Moreover, \([w]\tilde{u}z\tilde{v}^{-h(t)-t} \subseteq [u]\tilde{u}^{h(t)-t+|w|} \subseteq [u]_m \); we get the wanted nonempty intersection. The exact same can be achieved for \([v']_{n'} \) for the same \( t \).

- Proposition \ref{prop:oblique} then gives that for every \( h \notin \{-1, +1\} \), \( \tilde{\omega}_{F,h} = \Omega_F \) and \( F \) is sensitive. Since the set of almost equicontinuous directions is an interval, then at least one direction in \( \{-1, +1\} \) should also be sensitive. Since the definition of the local rule is exactly symmetric, we get that both directions are also sensitive.

- Now consider direction \( h = +1 \). For \( i \in \mathbb{N} \), let \( W_i \) be the set of configurations \( x \in A^2 \) such that \( x_{[1,i]} \) is not right-balanced. If \( x \in W_i \), then by definition \( x_{[1,i]} \) is not right-balanced, for \( t \geq i \), so that, by the first claim of the proof, \( F^t\sigma^i(x)_0 \neq \rightarrow \). Since every pattern can be extended to the right...
in a pattern which is not right-balanced, we see that \( W = \bigcup_{i \in \mathbb{N}} W_i \) is a dense open set. We get that \( \omega_{F,1}(W) \cap [\rightarrow] = \emptyset \). Hence \( \omega_{F,1}(\bigcap_{n \in \mathbb{Z}} \sigma^n(W)) \subseteq \{0, \leftarrow\}^2 \), \( \bigcap_{n \in \mathbb{Z}} \sigma^n(W) \) being comeager, we get that \( \omega_{F,1} \subseteq \{0, \leftarrow\}^2 \). Conversely, for every cylinders \([u]_m\) and \([v]_n\), the latter intersecting \(\{0, \leftarrow\}^2\), the same argument above allows to find \( t \in \mathbb{N} \) such that \( F^t([u]_m) \) intersect \([v]_n\), so that Remark 4.3 states that \([v]_n\) intersects the generic limit set.

- The exact symmetric argument settles the case of \( h = -1 \).

\[ \begin{array}{c}
\text{Figure 3: Lonely Gliders (← (resp. >) are represented by black squares (resp. white squares) and → (resp. <) are represented by dark grey squares (resp. light grey squares)).}
\end{array} \]

**Example 7.7 (Lonely gliders).** Let \( A = \{>,<,\rightarrow,\leftarrow\} \), and \( F \) the CA, defined by the following local rule:

\[
f : (x_{-1}, x_0, x_1) \mapsto \\
\rightarrow \quad \text{if } x_{-1} = \rightarrow \text{ and } x_0 = < \\
\rightarrow \quad \text{if } x_{-1} \neq > \text{ and } x_0 = \leftarrow \\
< \quad \text{if } x_{-1} = > \text{ and } x_0 = \leftarrow \\
> \quad \text{if } x_0 = \rightarrow \text{ and } x_1 = < \\
\leftarrow \quad \text{if } x_0 = \rightarrow \text{ and } x_1 \neq < \\
\leftarrow \quad \text{if } x_0 = > \text{ and } x_1 = \leftarrow \\
x_0 \quad \text{otherwise}
\]

A typical space-time diagram of this CA is shown in Figure 3.

Intuitively, each configuration can be decomposed into valid zones, which contain at most one arrow, towards with chevrons \(<\) and \(>\) are supposed to point. The arrow moves in the direction to which it points, until it reaches the end of the zone, in which case it turns back. With this in mind, it is not difficult to understand that \( F \) is reversible (hence surjective), and that any invalid pattern \( ab \), where \( a \neq > \) and \( b \in \{>,\rightarrow,\leftarrow\} \), or symmetric, is a blocking word. Hence this CA is in Class 4'.

The reason to introduce this CA rather any simpler surjective almost equicontinuous one is the following feature, which answers a question left open in [II]: \( \omega_F \subseteq A^\mathbb{Z} = \hat{\omega}_F \), is comeager (like for every surjective CA) but not full.

**Proof.** Let us prove that some configuration \( x \) with an infinite valid zone which contains one arrow cannot be the limit point of an orbit. Indeed, any configuration whose orbit comes arbitrarily close to \( x \) should also have an infinite valid zone (because the zones are invariant), and hence at most one arrow in it. Any limit point of such an orbit has no arrow in its infinite valid zone (the arrow goes to infinity).

The last two examples are only cited, and not defined formally, because they can fit a whole article by themselves. The first one shows that it is relevant to study arbitrary curves rather than just linear directions.
Almost equicontinuous: \( \tilde{\omega}_F = \omega_F(E) \)
(Corollary 4.14).

Sensitive: \( \tilde{\omega}_F \) is infinite.
(Corollary 4.11).

Almost equicontinuous in two directions of opposite sign: \( \tilde{\omega}_F \) is finite.
(Proposition 5.10).

Surjective: \( \tilde{\omega}_F = A^\mathbb{Z} \).
(Corollary 4.7).

Oblique: \( \tilde{\omega}_F = \Omega_F \).
(Corollary 5.7).

Equicontinuous: \( \tilde{\omega}_F = \Omega_F \).
(Proposition 4.15).

\( \varnothing \)

Figure 4: Summary of results. Nilpotent CA are not included, for a better readability.

Example 7.8. In [4, Prop 3.3], a CA is built, which is almost equicontinuous along \( h \) if and only if \( 0 < h < p \), where \( p \) is an explicit function close to the square root function. This CA is in Class 3: there is a nondegenerate interval of almost equicontinuous directions, but only one of them is linear.

With some horizontal bulking operation and a product with the CA built in [4, Prop 3.1] which deals with the other side of a parabola, one can even obtain a CA which is still in Class 3, but which is sensitive along all linear directions.

Example 7.9. In [3, Thm 6.1], a CA \( F \) is built with a word \( u \) which is blocking in a nondegenerate interval of directions, such that \( \omega_F([u]) \) is the nontrivial orbit of a monochrome configuration (in particular, \( \tilde{\omega}_F = \Omega_{F,\mu} \subseteq \Omega_F, \mu = \tilde{\omega}_F \)). This shows that CA of Class 3 are not always generically nilpotent: they can converge to a nontrivial orbit, provided that the blocking words are Gardens of Eden.

8 Conclusion

We studied the generic limit set of dynamical systems, and we emphasized the example of cellular automata. Our main results are:

- The generic limit set of a nonwandering system (in particular, of a surjective CA) is full.
- The generic limit set of an almost equicontinuous system is exactly the closure of the asymptotic set of its set of equicontinuity points. In particular, the generic limit set of an equicontinuous dynamical system is its limit set.
- The generic limit set of a cellular automaton which has two distinct directions of almost equicontinuity is finite; it is the periodic orbit of a monochrome configuration.
• The generic limit set of a sensitive system is infinite.
• The generic limit set of a semi-transitive system (in particular, of an oblique CA) is its limit set.

A summary of these results, for non-nilpotent CA, is represented in Figure 4.

Among the interesting questions that the directional classification brings, one can wonder whether, fixing one CA and making the directions vary, we obtain only finitely many generic limit sets, or whether they should intersect, at least as the orbit of a monochrome configuration (which is not clear for the last two classes).

Of course, another natural question is about what happens for two-dimensional CA: in that case almost equicontinuity does not correspond to existence of blocking words, and neither to non-sensitivity, so that everything becomes much more complex.

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