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Functional Ito calculus and
stochastic integral representation of martingales

Rama Cont  David-Antoine Fournié

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Abstract

We develop a non-anticipative calculus for functionals of a continuous semimartingale, using
an extension of the Ito formula to path-dependent functionals which possess certain directional
derivatives. The construction is based on a pathwise derivative, introduced by B Dupire, for
functionals on the space of right-continuous functions with left limits. We show that this func-
tional derivative admits a suitable extension to the space of square-integrable martingales. This
extension defines a weak derivative which is shown to be the inverse of the Ito integral and
which may be viewed as a non-anticipative “lifting” of the Malliavin derivative.

These results lead to a constructive martingale representation formula for Ito processes.
By contrast with the Clark-Haussmann-Ocone formula, this representation only involves non-
anticipative quantities which may be computed pathwise.

Keywords: stochastic calculus, functional calculus, functional Ito formula, Malliavin derivative,
martingale representation, semimartingale, Wiener functionals, Clark-Ocone formula.

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Malliavin for encouraging this work.
1 Introduction

In the analysis of phenomena with stochastic dynamics, Itô’s stochastic calculus \[15, 16, 8, 23, 19, 28, 29\] has proven to be a powerful and useful tool. A central ingredient of this calculus is the Itô formula \[15, 16, 23\], a change of variable formula for functions \(f(X_t)\) of a semimartingale \(X\) which allows to represent such quantities in terms of a stochastic integral. Given that in many applications such as statistics of processes, physics or mathematical finance, one is led to consider path-dependent functionals of a semimartingale \(X\) and its quadratic variation process \([X]\) such as:

\[
Z_t = \int_0^t g(t, X_t) d[X](t), \quad G(t, X_t, [X]_t), \quad \text{or} \quad E[G(T, X(T), [X](T)) | \mathcal{F}_t]
\]

(where \(X(t)\) denotes the value at time \(t\) and \(X_t = (X(u), u \in [0, t])\) the path up to time \(t\)) there has been a sustained interest in extending the framework of stochastic calculus to such path-dependent functionals.

In this context, the Malliavin calculus \[3, 24, 22, 25, 30, 31, 32\] has proven to be a powerful tool for investigating various properties of Brownian functionals. Since the construction of Malliavin derivative does not refer to an underlying filtration \(\mathcal{F}_t\), it naturally leads to representations of functionals in terms of anticipative processes \[4, 14, 25\]. However, in most applications it is more natural to consider non-anticipative versions of such representations.
In a recent insightful work, B. Dupire [9] has proposed a method to extend the Ito formula to a functional setting in a non-anticipative manner, using a pathwise functional derivative which quantifies the sensitivity of a functional $F_t : D([0,t], \mathbb{R}) \rightarrow \mathbb{R}$ to a variation in the endpoint of a path $\omega \in D([0,t], \mathbb{R})$:

$$\nabla \omega F_t(\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon 1_t) - F_t(\omega)}{\epsilon}$$

Building on this insight, we develop hereafter a non-anticipative calculus [6] for a class of processes –including the above examples– which may be represented as

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t)$$

(2)

where $A$ is the local quadratic variation defined by $[X](t) = \int_0^t A(u) du$ and the functional

$F_t : D([0,t], \mathbb{R}^d) \times D([0,t], S_+^d) \rightarrow \mathbb{R}$

represents the dependence of $Y$ on the path $X_t = \{X(u), 0 \leq u \leq t\}$ of $X$ and its quadratic variation.

Our first result (Theorem 4.1) is a change of variable formula for path-dependent functionals of the form (2). Introducing $A_t$ as additional variable allows us to control the dependence of $Y$ with respect to the "quadratic variation" $[X]$ by requiring smoothness properties of $F_t$ with respect to the variable $A_t$ in the supremum norm, without resorting to $p$-variation norms as in "rough path" theory [20]. This allows our result to cover a wide range of functionals, including the examples in [1].

We then extend this notion of functional derivative to processes: we show that for $Y$ of the form (2) where $F$ satisfies some regularity conditions, the process $\nabla_X Y = \nabla \omega F(X_t, A_t)$ may be defined intrinsically, independently of the choice of $F$ in (2). The operator $\nabla_X$ is shown to admit an extension to the space of square-integrable martingales, which is the inverse of the Ito integral with respect to $X$: for $\phi \in L^2(X)$, $\nabla_X \left( \int \phi.dX \right) = \phi$ (Theorem 5.8). In particular, we obtain a constructive version of the martingale representation theorem (Theorem 5.9), which states that for any square-integrable $\mathcal{F}_t^X$-martingale $Y$,

$$Y(T) = Y(0) + \int_0^T \nabla_X Y.dX \quad \mathbb{P} - a.s.$$  

This formula can be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 13, 14, 18, 25]. The integrand $\nabla_X Y$ is an adapted process which may be computed pathwise, so this formula is more amenable to numerical computations than those based on Malliavin calculus.

Finally, we show that this functional derivative $\nabla_X$ may be viewed as a non-anticipative "lifting" of the Malliavin derivative (Theorem 6.1): for square-integrable martingales $Y$ whose terminal values is differentiable in the sense of Malliavin $Y(T) \in \mathcal{D}^{1,2}$, we show that $\nabla_X Y(t) = E[\partial_t H|\mathcal{F}_t]$. These results provide a rigorous mathematical framework for developing and extending the ideas proposed by B. Dupire [9] for a large class of functionals. In particular, unlike the results derived from the pathwise approach viewpoint presented in [5, 9], Theorems 5.8 and 5.9 do not require any pathwise regularity of the functionals and hold for non-anticipative square-integrable processes, including stochastic integrals and functionals which may depend on the quadratic variation of the process.
2 Functional representation of non-anticipative processes

Let \( X : [0, T] \times \Omega \to \mathbb{R}^d \) be a continuous, \( \mathbb{R}^d \)-valued semimartingale defined on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) assumed to satisfy the usual hypotheses [8]. Denote by \( \mathcal{P} \) (resp. \( \mathcal{O} \)) the associated predictable (resp. optional) sigma-algebra on \([0, T]\). \( \mathcal{F}_t^X \) denotes the \((\mathcal{P}-completed)\) natural filtration of \( X \). The paths of \( X \) then lie in \( C_0([0, T], \mathbb{R}^d) \), which we will view as a subspace of \( D([0, t], \mathbb{R}^d) \) the space of cadlag functions with values in \( \mathbb{R}^d \). We denote by \( [X] = ([X^i, X^j], i, j = 1..d) \) the quadratic \((\text{co-})\)variation process associated to \( X \), taking values in the set \( S^+_d \) of positive \( d \times d \) matrices. We assume that

\[
[X](t) = \int_0^t A(s)ds
\]

for some cadlag process \( A \) with values in \( S^+_d \). Note that \( A \) need not be a semimartingale. The paths of \( A \) lie in \( S_t = D([0, t], S^+_d) \), the space of cadlag functions with values \( S^+_d \).

2.1 Horizontal extension and vertical perturbation of a path

Consider a path \( x \in D([0, T]), \mathbb{R}^d \) and denote by \( x_t = (x(u), 0 \leq u \leq t) \in D([0, t], \mathbb{R}^d) \) its restriction to \([0, t]\) for \( t < T \). For a process \( X \) we shall similarly denote \( X(t) \) its value at \( t \) and \( X_t = (X(u), 0 \leq u \leq t) \) its path on \([0, t]\).

For \( h \geq 0 \), we define the horizontal extension \( x_{t,h} \in D([0, t+h], \mathbb{R}^d) \) of \( x_t \) to \([0, t+h]\) as

\[
x_{t,h}(u) = x(u) \quad u \in [0, t] \quad x_{t,h}(u) = x(t) \quad u \in [t, t+h]
\]

(4)

For \( h \in \mathbb{R}^d \), we define the vertical perturbation \( x_t^h \) of \( x_t \) as the cadlag path obtained by shifting the endpoint by \( h \):

\[
x_t^h(u) = x_t(u) \quad u \in [0, t] \quad x_t^h(t) = x(t) + h
\]

(5)

or in other words \( x_t^h(u) = x_t(u) + h1_{t-u} \).

2.2 Adapted processes as non-anticipative functionals

A process \( Y : [0, T] \times \Omega \to \mathbb{R}^d \) adapted to \( \mathcal{F}_t^X \) may be represented as

\[
Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t)
\]

(6)

where \( F = (F_t)_{t \in [0, T]} \) is a family of functionals

\[
F_t : D([0, t], \mathbb{R}^d) \times S_t \to \mathbb{R}
\]

representing the dependence of \( Y(t) \) on the underlying path of \( X \) and its quadratic variation.

Since \( Y \) is non-anticipative, \( Y(t, \omega) \) only depends on the restriction \( \omega_t \) of \( \omega \) on \([0, t]\). This motivates the following definition:

**Definition 2.1** (Non-anticipative functional). A non-anticipative functional on \( \mathcal{Y} \) is a family of functionals \( F = (F_t)_{t \in [0, T]} \) where

\[
F_t : D([0, t], \mathbb{R}^d) \times D([0, t], S^+_d) \to \mathbb{R}
\]

\[
(x, v) \to F_t(x, v)
\]

is measurable with respect to \( \mathcal{B}_t \), the canonical filtration on \( D([0, t], \mathbb{R}^d) \times D([0, t], S^+_d) \).
We can also view $F = (F_t)_{t \in [0,T]}$ as a map defined on the space $\Upsilon$ of stopped paths:

$$\Upsilon = \{(t, \omega_{t,T-t}), (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d \times S^+_d)\} \quad (7)$$

Whenever the context is clear, we will denote a generic element $(t, \omega) \in \Upsilon$ simply by its second component, the path $\omega$ stopped at $t$. $\Upsilon$ can also be identified with the 'vector bundle'

$$\Lambda = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d) \times D([0, t], S^+_d). \quad (8)$$

A natural distance on the space $\Upsilon$ of stopped paths is given by

$$d_\infty((t, \omega), (t', \omega')) = |t - t'| + \sup_{u \in [0, T]} |\omega_{t,T-t}(u) - \omega'_{t',T-t'}(u)| \quad (9)$$

$(\Upsilon, d_\infty)$ is then a metric space, a closed subspace of $[0, T] \times D([0, T], \mathbb{R}^d \times S^+_d), ||.||_\infty)$ for the product topology.

Introducing the process $A$ as additional variable may seem redundant at this stage: indeed $A(t)$ is itself $\mathcal{F}_t$-measurable i.e. a functional of $X_t$. However, it is not a continuous functional on $(\Upsilon, d_\infty)$. Introducing $A_t$ as a second argument in the functional will allow us to control the regularity of $Y$ with respect to $[X]_t = \int_0^t A(u)du$ simply by requiring continuity of $F_t$ in supremum or $L^p$ norms with respect to the "lifted process" $(X, A)$ (see Section 2.3). This idea is analogous in some ways to the approach of rough path theory [20], although here we do not resort to $p$-variation norms.

If $Y$ is a $\mathcal{B}_t$-predictable process, then [8, Vol. I, Par. 97]

$$\forall t \in [0, T], \quad Y(t, \omega) = Y(t, \omega_{t-})$$

where $\omega_{t-}$ denotes the path defined on $[0, t]$ by

$$\omega_{t-}(u) = \omega(u) \quad u \in [0, t] \quad \omega_{t-}(t) = \omega(t-)$$

Note that $\omega_{t-}$ is caglad and should not be confused with the caglad path $u \mapsto \omega(u-)$. The functionals discussed in the introduction depend on the process $A$ via $[X] = \int_0^t A(t)dt$.

In particular, they satisfy the condition $F_t(X_t, A_t) = F_t(X_t, A_{t-})$. Accordingly, we will assume throughout the paper that all functionals $F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ considered have "predictable" dependence with respect to the second argument:

$$\forall t \in [0, T], \quad \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad F_t(x_t, v_t) = F_t(x_t, v_{t-}) \quad (10)$$

### 2.3 Continuity for non-anticipative functionals

We now define a notion of (left) continuity for non-anticipative functionals.

**Definition 2.2** (Continuity at fixed times). A functional $F$ defined on $\Upsilon$ is said to be continuous at fixed times for the $d_\infty$ metric if and only if:

$$\forall t \in [0, T], \quad \forall \varepsilon > 0, \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, (x', v') \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad d_\infty((x, v), (x', v')) < \eta \Rightarrow |F_t(x, v) - F_t(x', v')| < \varepsilon \quad (11)$$
We now define a notion of joint continuity with respect to time and the underlying path:

**Definition 2.3** (Continuous functionals). A non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is said to be continuous at $(x,v) \in D([0,t],\mathbb{R}^d) \times \mathcal{S}_t$ if
\[
\forall \epsilon > 0, \exists \eta > 0, \forall (x',v') \in \mathcal{Y}, \quad d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t'}(x',v')| < \epsilon
\] (12)

We denote $\mathbb{C}^{0,0}([0,T])$ the set of non-anticipative functionals continuous on $\mathcal{Y}$.

**Definition 2.4** (Left-continuous functionals). A non-anticipative functional $F = (F_t, t \in [0,T])$ is said to be left-continuous if for each $t \in [0,T]$, $F_t : D([0,t],\mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ in the sup norm and
\[
\forall \epsilon > 0, \forall (x,v) \in D([0,t],\mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, \forall \pi \in [0,0] \cap [0,t], \quad \forall (x',v') \in D([0,t-h],\mathbb{R}^d) \times \mathcal{S}_{t-h},
\]
\[
d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t-h}(x',v')| < \epsilon
\] (13)

We denote $\mathbb{C}_l^{0,0}([0,T])$ the set of left-continuous functionals.

We define analogously the class of right continuous functionals $\mathbb{C}_r^{0,0}([0,T])$.

We call a functional “boundedness preserving” if it is bounded on each bounded set of paths:

**Definition 2.5** (Boundedness-preserving functionals). Define $\mathbb{B}([0,T])$ as the set of non-anticipative functionals $F$ such that for every compact subset $K$ of $\mathbb{R}^d$, every $R > 0$ and $t_0 < T$:
\[
\exists C_{K,R,t_0} > 0, \quad \forall t \leq t_0, \forall (x,v) \in D([0,t],K) \times \mathcal{S}_t, \sup_{s \in [0,t]} |v(s)| < R \Rightarrow |F_t(x,v)| < C_{K,R,t_0}
\] (14)

### 2.4 Measurability properties

Composing a non-anticipative functional $F$ with the process $(X,A)$ yields an $\mathcal{F}_t$-adapted process $Y(t) = F_t(X_t,A_t)$. The results below link the measurability and pathwise regularity of $Y$ to the regularity of the functional $F$.

**Lemma 2.6** (Pathwise regularity). If $F \in \mathbb{C}_l^{0,0}$ then for any $(x,v) \in D([0,T],\mathbb{R}^d) \times \mathcal{S}_T$, the path $t \mapsto F_t(x_{t-},v_{t-})$ is left-continuous.

**Proof.** Let $F \in \mathbb{C}_l^{0,0}$ and $t \in [0,T]$. For $h > 0$ sufficiently small,
\[
d_\infty((x_{t-h},v_{t-h}),(x_{t-},v_{t-})) = \sup_{u \in (t-h,t]} |x(u) - x(t-h)| + \sup_{u \in (t-h,t]} |v(u) - v(t-h)| + h
\] (15)

Since $x$ and $v$ are cadlag, this quantity converges to 0 as $h \to 0^+$, so
\[
F_t(x_{t-h},v_{t-h}) - F_t(x_{t-},v_{t-}) \xrightarrow{h \to 0^+} 0
\]

so $t \mapsto F_t(x_{t-},v_{t-})$ is left-continuous. \hfill \square

**Theorem 2.7.** (i) If $F$ is continuous at fixed times, then the process $Y$ defined by $Y((x,v),t) = F_t(x_t,v_t)$ is adapted.

(ii) If $F \in \mathbb{C}_l^{0,0}([0,T])$, then the process $Z(t) = F_t(X_t,A_t)$ is optional.
(iii) If \( F \in \mathbb{C}_t^{0,0}([0,T]) \), and if either \( A \) is continuous or \( F \) verifies (10), then \( Z \) is a predictable process.

In particular, any \( F \in \mathbb{C}_t^{0,0} \) is a non-anticipative functional in the sense of Definition 2.1. We propose an easy-to-read proof of points (i) and (iii) in the case where \( A \) is continuous. The (more technical) proof for the cadlag case is given in the Appendix A.

Continuous case. Assume that \( F \) is continuous at fixed times and that the paths of \((X,A)\) are almost-surely continuous. Let us prove that \( Y \) is \( \mathcal{F}_t \)-adapted: \( X(t) \) is \( \mathcal{F}_t \)-measurable. Introduce the partition \( t^n_i = \frac{iT}{2^n}, i = 0..2^n \) of \([0,T]\), as well as the following piecewise-constant approximations of \( X \) and \( A \):

\[
X^n(t) = \sum_{k=0}^{2^n} X(t^n_k) 1_{[t^n_k, t^n_{k+1})}(t) + X_T 1_{\{T\}}(t)
\]

\[
A^n(t) = \sum_{k=0}^{2^n} A(t^n_k) 1_{[t^n_k, t^n_{k+1})}(t) + A_T 1_{\{T\}}(t)
\]

The random variable \( Y^n(t) = F_t(X^n_t, A^n_t) \) is a continuous function of the random variables \( \{X(t^n_k), A(t^n_k), t^n_k \leq t\} \) hence is \( \mathcal{F}_t \)-measurable. The representation above shows in fact that \( Y^n(t) \) is \( \mathcal{F}_t \)-measurable. \( X^n_t \) and \( A^n_t \) converge respectively to \( X_t \) and \( A_t \) almost-surely so \( Y^n(t) \to Y(t) \) a.s., hence \( Y(t) \) is \( \mathcal{F}_t \)-measurable.

(i) implies point (iii) since the path of \( Z \) are left-continuous by Lemma 2.6.

3 Pathwise derivatives of non-anticipative functionals

3.1 Horizontal and vertical derivatives

We now define pathwise derivatives for a functional \( F = (F_t)_{t \in [0,T)} \in \mathbb{C}_{0,0} \), following Dupire [9].

**Definition 3.1** (Horizontal derivative). The horizontal derivative at \( (x,v) \in D([0,t], \mathbb{R}^d) \times S_t \) of non-anticipative functional \( F = (F_t)_{t \in [0,T)} \) is defined as

\[
\mathcal{D}_t F(x,v) = \lim_{h \to 0^+} \frac{F_{t+h}(x_{t+h}, v_{t+h}) - F_t(x_t, v_t)}{h}
\]

if the corresponding limit exists. If (17) is defined for all \( (x,v) \in T \) the map

\[
\mathcal{D}_t F : D([0,t], \mathbb{R}^d) \times S_t \to \mathbb{R}^d
\]

\[
(x,v) \to \mathcal{D}_t F(x,v)
\]

defines a non-anticipative functional \( \mathcal{D}F = (\mathcal{D}_t F)_{t \in [0,T]} \), the horizontal derivative of \( F \).

Note that our definition (17) is different from the one in [9] where the case \( F(x,v) = G(x) \) is considered.

Dupire [9] also introduced a pathwise spatial derivative for such functionals, which we now introduce. Denote \( (e_i, i = 1..d) \) the canonical basis in \( \mathbb{R}^d \).
Definition 3.2. A non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is said to be \textit{vertically differentiable} at $(x, v) \in D([0,t], \mathbb{R}^d) \times D([0,t], S^+_k)$ if

$$
\mathbb{R}^d \mapsto \mathbb{R} \\
e \mapsto F_t(x_t, v_t)
$$

is differentiable at 0. Its gradient at 0

$$
\nabla_x F_t(x,v) = (\partial_i F_t(x,v), i = 1..d) \quad \text{where} \quad \partial_i F_t(x,v) = \lim_{h \to 0} \frac{F_t(x^h_{ci}, v) - F_t(x,v)}{h}
$$

is called the \textit{vertical derivative} of $F_t$ at $(x,v)$. If (19) is defined for all $(x,v) \in \Upsilon$, the maps

$$
\nabla_x F : D([0,t], \mathbb{R}^d) \times S_t \mapsto \mathbb{R}^d \\
(x,v) \mapsto \nabla_x F_t(x,v)
$$

define a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t \in [0,T]}$, the \textit{vertical derivative} of $F$. $F$ is then said to be \textit{vertically differentiable} on $\Upsilon$.

Remark 3.3. $\partial_i F_t(x,v)$ is simply the directional derivative of $F_t$ in direction $(1_{(l)}, e_i)$. Note that this involves examining cadlag perturbations of the path $x$, even if $x$ is continuous.

Remark 3.4. If $F_t(x,v) = f(t,x(t))$ with $f \in C^{1,1}([0,T] \times \mathbb{R}^d)$ then we retrieve the usual partial derivatives:

$$
\partial_i F_t(x,v) = \partial_i f(t,x(t)) \quad \nabla_x F_t(x_t,A_t) = \nabla_x f(t,x(t)).
$$

Remark 3.5. Bismut [3] considered directional derivatives of functionals on $D([0,T], \mathbb{R}^d)$ in the direction of purely discontinuous (e.g. piecewise constant) functions with finite variation, which is similar to Def. 3.2. This notion, used in [3] to derive an integration by parts formula for pure-jump processes, is natural in the context of discontinuous semimartingales. We will show that the directional derivative [19] also intervenes naturally when the underlying process $X$ is \textit{continuous}, which is less obvious.

Definition 3.6 (Regular functionals). Define $\mathcal{C}^{1,k}([0,T])$ as the set of functionals $F \in \mathcal{C}^{0,0}_t$ which are

- horizontally differentiable with $D_i F$ continuous at fixed times,
- $k$ times vertically differentiable with $\nabla^j_x F \in \mathcal{C}^{0,0}_t([0,T])$ for $j = 1..k$.

Define $\mathcal{C}^{1,k}_k([0,T])$ as the set of functionals $F \in \mathcal{C}^{1,2}$ such that $D F, \nabla_x F, ..., \nabla^k_x F \in \mathcal{B}([0,T])$.

We denote $\mathcal{C}^{1,\infty}([0,T]) = \cap_{k \geq 1} \mathcal{C}^{1,k}([0,T])$.

Note that this notion of regularity only involves directional derivatives with respect to \textit{local} perturbations of paths, so $\nabla_x F$ and $D_i F$ seems to contain \textit{less} information on the behavior of $F$ than, say, the Fréchet derivative which consider perturbations in all directions in $C_0([0,T], \mathbb{R}^d)$ or the Malliavin derivative [21, 22] which examines perturbations in the direction of all absolutely continuous functions. Nevertheless we will show in Section 4 that knowledge of $D F, \nabla_x F, \nabla^2_x F$ along the paths of $X$ derivatives are sufficient to reconstitute the path of $Y(t) = F_t(X_t,A_t)$.
Example 1 (Smooth functions). In the case where $F$ reduces to a smooth function of $X(t)$,

$$F_t(x_t, v_t) = f(t, x(t))$$

where $f \in C^{1,k}([0, T] \times \mathbb{R}^d)$, the pathwise derivatives reduces to the usual ones: $F \in C^{1,k}_b$ with:

$$\mathcal{D}_t F(x_t, v_t) = \partial_x f(t, x(t)) \quad \nabla^2_x F_t(x_t, v_t) = \partial^2_x f(t, x(t))$$

(22)

In fact to have $F \in C^{1,k}$ we just need $f$ to be right-differentiable in the time variable, with right-derivative $\partial_t f(t, \cdot)$ which is continuous in the space variable and $f, \nabla f$ and $\nabla^2 f$ to be jointly left-continuous in $t$ and continuous in the space variable.

Example 2 (Cylindrical functionals). Let $g \in C^0(\mathbb{R}^d, \mathbb{R}), h \in C^k(\mathbb{R}^d, \mathbb{R})$ with $h(0) = 0$. Then

$$F_t(\omega) = h(\omega(t) - \omega(t_n^-)) \quad 1_{t \geq t_n} g(\omega(t_1^-), \omega(t_2^-), \ldots, \omega(t_n^-))$$

is in $C^{1,k}_b$ with $\mathcal{D}_t F(\omega) = 0$ and

$$\forall j = 1, \ldots, k, \quad \nabla^j_x F_t(\omega) = h^{(j)}(\omega(t) - \omega(t_n^-)) \quad 1_{t \geq t_n} g(\omega(t_1^-), \omega(t_2^-), \ldots, \omega(t_n^-))$$

Example 3 (Integrals with respect to quadratic variation). A process $Y(t) = \int_0^t g(X(u))d[X](u)$ where $g \in C_0(\mathbb{R}^d)$ may be represented by the functional

$$F_t(x_t, v_t) = \int_0^t g(x(u))v(u)du$$

(23)

It is readily observed that $F \in C^{1,\infty}_b$, with:

$$\mathcal{D}_t F(x_t, v_t) = g(x(t))v(t) \quad \nabla^2_x F_t(x_t, v_t) = 0$$

(24)

Example 4. The martingale $Y(t) = X(t)^2 - [X](t)$ is represented by the functional

$$F_t(x_t, v_t) = x(t)^2 - \int_0^t v(u)du$$

(25)

Then $F \in C^{1,\infty}_b$ with:

$$\mathcal{D}_t F(x, v) = -v(t) \quad \nabla_x F_t(x_t, v_t) = 2x(t)$$

$$\nabla^2_x F_t(x_t, v_t) = 2 \quad \nabla^2_x F_t(x_t, v_t) = 0, j \geq 3$$

(26)

Example 5. $Y = \exp(X - [X]/2)$ may be represented as $Y(t) = F(X_t)$

$$F_t(x_t, v_t) = e^{x(t) - \frac{1}{2} \int_0^t v(u)du}$$

(27)

Elementary computations show that $F \in C^{1,\infty}_b$ with:

$$\mathcal{D}_t F(x, v) = -\frac{1}{2} v(t) F_t(x, v) \quad \nabla^2_x F_t(x_t, v_t) = F_t(x_t, v_t)$$

(28)

Note that, although $A_t$ may be expressed as a functional of $X_t$, this functional is not continuous and without introducing the second variable $v \in \mathcal{S}_t$, it is not possible to represent Examples 3, 4 and 5 as a left-continuous functional of $x$ alone.
3.2 Obstructions to regularity

It is instructive to observe what prevents a functional from being regular in the sense of Definition 3.2: The examples below illustrate the fundamental obstructions to regularity:

Example 6 (Delayed functionals). Let $\epsilon > 0$. $F_t(x_t, v_t) = x(t - \epsilon)$ defines a $C_{b}^{0,\infty}$ functional. All vertical derivatives are 0. However, $F$ fails to be horizontally differentiable.

Example 7 (Jump of $x$ at the current time). $F_t(x_t, v_t) = x(t) - x(t^-)$ defines a functional which is infinitely differentiable and has regular pathwise derivatives:

$$D_t F(x_t, v_t) = 0 \quad \nabla_x F_t(x_t, v_t) = 1$$

(29)

However, the functional itself fails to be $C_{t}^{0,0}$.

Example 8 (Jump of $x$ at a fixed time). $F_t(x_t, v_t) = 1_{\{t \geq t_0\}}(x(t) - x(t_0^-))$ defines a functional in $C_{t}^{0,0}$ which admits horizontal and vertical derivatives at any order at each point $(x, v)$. However, $\nabla_x F_t(x_t, v_t) = 1_{\{t = t_0\}}$ fails to be either right- or left-continuous so $F$ is not $C_{t}^{0,1}$ in the sense of Definition 3.2.

Example 9 (Maximum). $F_t(x_t, v_t) = \sup_{s \leq t} x(s)$ is $C_{t}^{0,0}$ but fails to be vertically differentiable on the set

$$\{(x_t, v_t) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \; x(t) = \sup_{s \leq t} x(s)\}.$$

4 Functional Ito calculus

4.1 Functional Ito formula

We are now ready to prove our first main result, which is a change of variable formula for non-anticipative functionals of a semimartingale [6, 9]:

Theorem 4.1. For any non-anticipative functional $F \in C_{b}^{1,2}$ verifying (10) and any $t \in [0, T)$,

$$F_t(X_t, A_t) - F_0(X_0, A_0) = \int_0^t D_u F(X_u, A_u) du + \int_0^t \nabla_x F_u(X_u, A_u) dX(u)$$

$$+ \int_0^t \frac{1}{2} \text{tr} (\nabla^2_x F_u(X_u, A_u) d[X](u))$$

a.s. (30)

In particular, for any $F \in C_{b}^{1,2}$, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

[30] shows that, for a regular functional $F \in C^{1,2}([0, T])$, the process $Y = F(X, A)$ may be reconstructed from the second-order jet $(DF, \nabla_x F, \nabla^2_x F)$ of $F$ along the paths of $X$.

Proof. Let us first assume that $X$ does not exit a compact set $K$ and that $\|A\|_{\infty} \leq R$ for some $R > 0$. Let us introduce a sequence of random partitions $(\tau_k^n, k = 0..k(n))$ of $[0, t]$, by adding the jump times of $A$ to the dyadic partition ($t_i^n = \frac{i}{2^n}$, $i = 0..2^n$):

$$\tau_0^n = 0 \quad \tau_k^n = \inf\{s > \tau_{k-1}^n \lvert 2^n s \in \mathbb{N} \text{ or } \|A(s) - A(s^-)\| > \frac{1}{n}\} \wedge t$$

(31)
The following arguments apply pathwise. Lemma 3 ensures that

\[ \eta_n = \sup \{|A(u) - A(\tau^n_i)| + |X(u) - X(\tau^n_i)| + \frac{t}{2^n}, i \leq 2^n, u \in [\tau^n_i, \tau^n_{i+1}) \} \to 0. \]

Denote \( nX = \sum_{i=0}^{\infty} X(\tau^n_{i+1})1_{[\tau^n_i, \tau^n_{i+1})} + X(t)1_{[t]} \) which is a cadlag piecewise constant approximation of \( X_t \), and \( nA = \sum_{i=0}^{\infty} A(\tau^n_i)1_{[\tau^n_i, \tau^n_{i+1})} + A(t)1_{[t]} \) which is an adapted cadlag piecewise constant approximation of \( A_t \). Denote \( h^n_i = \tau^n_{i+1} - \tau^n_i \). Start with the decomposition:

\[
F_{\tau^n_{i+1}}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) - F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) = F_{\tau^n_{i+1}}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) - F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) + F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) - F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-})
\]

where we have used the fact that \( F \) has predictable dependence in the second variable to have \( F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) = F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) \). The first term in (32) can be written \( \psi(h^n_i) - \psi(0) \) where:

\[
\psi(u) = F_{\tau^n_i + u}(nX_{\tau^n_i+u}, m A_{\tau^n_i+u})
\]

Since \( F \in C^{1,2}([0, T]) \), \( \psi \) is right-differentiable and left-continuous by Lemma 2.6 so:

\[
F_{\tau^n_{i+1}}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) - F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) = \int_0^{\tau^n_{i+1} - \tau^n_i} D_{\tau^n_{i+u}} F(nX_{\tau^n_{i+u}-m} A_{\tau^n_{i+u}-}) du
\]

The second term in (32) can be written \( \phi(X(\tau^n_{i+1}) - X(\tau^n_i)) - \phi(0) \) where \( \phi(u) = F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) \).

Since \( F \in C^{1,2}_b \), \( \phi \) is a \( C^2 \) function and \( \phi'(u) = \nabla_x F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}), \phi''(u) = \nabla_x^2 F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) \).

Applying the Ito formula to \( \phi \) between 0 and \( \tau^n_{i+1} - \tau^n_i \) and the \((F_{\tau^n_i + s}), s \geq 0 \) continuous semimartingale \((X(\tau^n_i + s)), s \geq 0 \), yields:

\[
\phi(X(\tau^n_{i+1}) - X(\tau^n_i)) - \phi(0) = \int_0^{\tau^n_{i+1}} \nabla_x F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) dX(s) + \frac{1}{2} \int_0^{\tau^n_{i+1}} \text{tr} \left[ \nabla_x^2 F_{\tau^n_i}(nX_{\tau^n_{i+1}-m} A_{\tau^n_{i+1}-}) d[X](s) \right]
\]

Summing over \( i \geq 0 \) and denoting \( i(s) \) the index such that \( s \in [\tau^n_{i(s)}, \tau^n_{i(s)+1}] \), we have shown:

\[
F_t(nX_{t-m} A_t) - F_0(X_0, A_0) = \int_0^t D_s F(nX_{\tau^n_{i(s)+1}-m} A_{\tau^n_{i(s)+1}-}) ds + \int_0^t \nabla_x F_{\tau^n_{i(s)+1}}(nX_{\tau^n_{i(s)+1}-m} A_{\tau^n_{i(s)+1}-}) dX(s) + \frac{1}{2} \int_0^t \text{tr} \left[ \nabla_x^2 F_{\tau^n_{i(s)+1}}(nX_{\tau^n_{i(s)+1}-m} A_{\tau^n_{i(s)+1}-}) d[X](s) \right]
\]

\( F_t(nX_{t-m} A_t) \) converges to \( F_t(X_t, A_t) \) almost surely. Since all approximations of \((X, A)\) appearing in the various integrals have a \( d_\infty \)-distance from \((X_s, A_s)\) less than \( \eta_n \to 0 \), the continuity at fixed times of \( DF \) and left-continuity \( \nabla_x F, \nabla_x^2 F \) imply that the integrands appearing in the above integrals converge respectively to \( D_s F(X_s, A_s), \nabla_x F(X_s, A_s), \nabla_x^2 F(X_s, A_s) \) as \( n \to \infty \). Since the derivatives are in \( \mathbb{B} \) the integrands in the various above integrals are bounded by a constant dependant only
on $F, K$ and $R$ and $t$ does not depend on $s$ nor on $\omega$. The dominated convergence and the dominated convergence theorem for the stochastic integrals [23 Ch.IV Theorem 32] then ensure that the Lebesgue-Stieltjes integrals converge almost surely, and the stochastic integral in probability, to the terms appearing in (30) as $n \to \infty$.

Consider now the general case where $X$ and $A$ may be unbounded. Let $K_n$ be an increasing sequence of compact sets with $\bigcup_{n \geq 0} K_n = \mathbb{R}^d$ and denote the optional stopping times

$$
\tau_n = \inf\{s < t | X_s \notin K^n \text{ or } |A_s| > n\} \land t.
$$

Applying the previous result to the stopped process $(X_{t \land \tau_n}, A_{t \land \tau_n})$ and noting that, by (10), $F_t(X_t, A_t) = F_t(X_t, A_{t-})$ leads to:

$$
F_t(X_{t \land \tau_n}, A_{t \land \tau_n}) - F_0(Z_0, A_0) = \int_0^{t \wedge \tau_n} D_u F_u(X_u, A_u) du + \frac{1}{2} \int_0^{t \wedge \tau_n} \text{tr} \left( \nabla^2 F_u(X_u, A_u) d[X](u) \right) + \int_0^{t \wedge \tau_n} \nabla_x F_u(X_u, A_u) dX_u + \int_0^{t \wedge \tau_n} D_u F(X_{u \land \tau_n}, A_{u \land \tau_n}) du.
$$

The terms in the first line converge almost surely to the integral up to time $t$ since $t \wedge \tau_n = t$ almost surely for $n$ sufficiently large. For the same reason the last term converges almost surely to 0. □

Remark 4.2. The above proof is probabilistic and makes use of the (classical) Itô formula [15]. In the companion paper [5] we give a non-probabilistic proof of Theorem 4.1, using the analytical approach of Föllmer [12], which allows $X$ to have discontinuous (càdlàg) trajectories.

Example 10. If $F_t(x, v_t) = f(t, x(t))$ where $f \in C^{1,2}([0,t] \times \mathbb{R}^d)$, (30) reduces to the standard Itô formula.

Example 11. For the functional in Example 5 $F_t(x, v_t) = e^{x(t) - \frac{1}{2} \int_0^t v(u)^2 du}$, the formula (30) yields the well-known integral representation

$$
\exp(X(t) - \frac{1}{2} [X](t)) = \int_0^t e^{X(u) - \frac{1}{2} [X](u)} dX(u)
$$

(37)

An immediate corollary of Theorem 4.1 is that, if $X$ is a local martingale, any $C^{1,2}_b$ functional of $X$ which has finite variation is equal to the integral of its horizontal derivative:

Corollary 4.3. If $X$ is a local martingale and $F \in C^{1,2}_b$, the process $Y(t) = F_t(X_t, A_t)$ has finite variation if only if $\nabla_x F_t(X_t, A_t) = 0 \ d[X] \times d\mathbb{P}$-almost everywhere.

Proof. $Y(t)$ is a continuous semimartingale by Theorem 4.1 with semimartingale decomposition given by (30). If $Y$ has finite variation, then by formula (30), its continuous martingale component should be zero i.e. $\int_0^T \nabla_x F_t(X_t, A_t).dX(t) = 0$ a.s. Computing its quadratic variation, we obtain

$$
\int_0^T \text{tr} \left( \nabla^2 F_t(X_t, A_t).\nabla_x F_t(X_t, A_t).d[X] \right) = 0
$$

which implies in particular that $\|\partial_i F_t(X_t, A_t)\|^2 = 0 \ d[X] \times d\mathbb{P}$-almost everywhere for $i = 1..d$. Thus, $\nabla_x F_t(X_t, A_t) = 0$ for $(t, \omega) \notin A \subset [0, T] \times \Omega$ where $\int_A d[X] \times d\mathbb{P} = 0$ for $i = 1..d$. □
4.2 Vertical derivative of an adapted process

For a \((\mathcal{F}_t-\text{adapted})\) process \(Y\), the functional representation \((42)\) is not unique, and the vertical \(\nabla_x F\) depends on the choice of representation \(F\). However, Theorem \(4.1\) implies that the process \(\nabla_x F_t(X_t, A_t)\) has an intrinsic character i.e. independent of the chosen representation:

**Corollary 4.4.** Let \(F^1, F^2 \in \mathbb{C}^{1,2}_b([0,T])\), such that:

\[
\forall t \in [0,T), \quad F^1_t(X_t, A_t) = F^2_t(X_t, A_t) \quad \mathbb{P} - \text{a.s.} \tag{38}
\]

Then, outside an evanescent set:

\[
\int [\nabla_x F^1_t(X_t, A_t) - \nabla_x F^2_t(X_t, A_t)] A(t-)[\nabla_x F^1_t(X_t, A_t) - \nabla_x F^2_t(X_t, A_t)] = 0 \tag{39}
\]

**Proof.** Let \(X(t) = B(t) + M(t)\) where \(B\) is a continuous process with finite variation and \(M\) is a continuous local martingale. There exists \(\Omega_1 \subset \Omega\) such that \(\mathbb{P}(\Omega_1) = 1\) and for \(\omega \in \Omega\) the path of \(t \mapsto X(t, \omega)\) is continuous and \(t \mapsto A(t, \omega)\) is c\(\text{c}d\)c. Theorem \(4.1\) implies that the local martingale part of \(0 = F^1(X_t, A_t) - F^2(X_t, A_t)\) can be written:

\[
0 = \int_0^t [\nabla_x F^1_u(X_u, A_u) - \nabla_x F^2_u(X_u, A_u)] \, dM(u) \tag{40}
\]

Considering its quadratic variation, we have, on \(\Omega_1\)

\[
0 = \int_0^t \frac{1}{2} [\nabla_x F^1_u(X_u, A_u) - \nabla_x F^2_u(X_u, A_u)] A(u-)[\nabla_x F^1_u(X_u, A_u) - \nabla_x F^2_u(X_u, A_u)] \, du \tag{41}
\]

By Lemma \(2.6\) \(\nabla_x F^1(X_t, A_t) = \nabla_x F^1(X_{t-}, A_{t-})\) since \(X\) is continuous and \(F\) verifies \(10\). So on \(\Omega_1\) the integrand in \(41\) is left-continuous; therefore \(41\) implies that for \(t < T\) and \(\omega \in \Omega_1\),

\[
\int [\nabla_x F^1_u(X_u, A_u) - \nabla_x F^2_u(X_u, A_u)] A(u-)[\nabla_x F^1_u(X_u, A_u) - \nabla_x F^2_u(X_u, A_u)] = 0
\]

In the case where for all \(t < T\), \(A(t-)\) is almost surely positive definite, Corollary \(4.4\) allows to define intrinsically the pathwise derivative of a process \(Y\) which admits a functional representation \(Y(t) = F_t(X_t, A_t)\):

**Definition 4.5** (Vertical derivative of a process). Define \(\mathcal{C}^{1,2}_b(X)\) the set of \(\mathcal{F}_t\)-adapted processes \(Y\) which admit a functional representation in \(\mathbb{C}^{1,2}_b\):

\[
\mathcal{C}^{1,2}_b(X) = \{Y, \quad \exists F \in \mathbb{C}^{1,2}_b \quad Y(t) = F_t(X_t, A_t) \quad \mathbb{P} - \text{a.s.} \} \tag{42}
\]

If \(A(t)\) is non-singular i.e. \(\text{det}(A(t)) \neq 0\) \(dt\times d\mathbb{P}\) almost-everywhere then for any \(Y \in \mathcal{C}^{1,2}_b(X)\), the predictable process:

\[
\nabla_x Y(t) = \nabla_x F_t(X_t, A_t)
\]

is uniquely defined up to an evanescent set, independently of the choice of \(F \in \mathbb{C}^{1,2}_b\) in the representation \(42\). We will call \(\nabla_x Y\) the **vertical derivative** of \(Y\) with respect to \(X\).
In particular this construction applies to the case where $X$ is a standard Brownian motion, where $A = I_d$, so we obtain the existence of a vertical derivative process for $\mathbb{C}^{1,2}_b$ Brownian functionals:

**Definition 4.6** (Vertical derivative of non-anticipative Brownian functionals). Let $W$ be a standard $d$-dimensional Brownian motion. For any $Y \in \mathbb{C}^{1,2}_b(W)$ with representation $Y(t) = F_t(W_t)$, the predictable process

$$\nabla_W Y(t) = \nabla_x F_t(W_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}^{1,2}_b$.

## 5 Martingale representation formulas

Consider now the case where $X$ is a Brownian martingale:

**Assumption 5.1.** $X(t) = X(0) + \int_0^t \sigma(u) \, dW(u)$ where $\sigma$ is a process adapted to $\mathcal{F}_t^W$ verifying

$$\det(\sigma(t)) \neq 0 \quad dt \times d\mathbb{P} - a.e. \quad (43)$$

The functional Ito formula (Theorem 4.1) then leads to an explicit martingale representation formula for $\mathcal{F}_t$-martingales in $\mathbb{C}^{1,2}_b(X)$. This result may be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 25, 14] and generalizes other constructive martingale representation formulas previously obtained using Markovian functionals [7, 10, 11, 17, 26], Malliavin calculus [2, 18, 14, 25, 24] or other techniques [1, 27].

Consider an $\mathcal{F}_T$ measurable random variable $H$ with $E|H| < \infty$ and consider the martingale $Y(t) = E[H \mid \mathcal{F}_t]$.

### 5.1 A martingale representation formula

If $Y$ admits a representation $Y(t) = F_t(X_t, A_t)$ where $F \in \mathbb{C}^{1,2}_b$, we obtain the following stochastic integral representation for $Y$ in terms of its derivative $\nabla_X Y$ with respect to $X$:

**Theorem 5.2.** If $Y(t) = F_t(X_t, A_t)$ for some functional $F \in \mathbb{C}^{1,2}_b$, then:

$$Y(T) = Y(0) + \int_0^T \nabla_x F_t(X_t, A_t) \, dX(t) = Y(0) + \int_0^T \nabla_X Y \, dX$$

(44)

Note that regularity assumptions are not on $H = Y(T)$ but on the functionals $Y(t) = E[H \mid \mathcal{F}_t]$, $t < T$, which is typically more regular than $H$ itself.

**Proof.** Theorem 4.1 implies that for $t \in [0, T)$:

$$Y(t) = \left[ \int_0^t \mathcal{D}_u F(X_u, A_u) \, du + \frac{1}{2} \int_0^t \text{tr} [\nabla^2_x F_u(X_u, A_u) d[X](u)] \right]$$

$$+ \int_0^t \nabla_x F_u(X_u, A_u) \, dX(u)$$

(45)

Given the regularity assumptions on $F$, the first term in this sum is a continuous process with finite variation while the second is a continuous local martingale. However, $Y$ is a martingale and its
decomposition as sum of a finite variation process and a local martingale is unique \[29\]. Hence the first term is 0 and: 

\[ Y_t = \int_0^t F_u(X_u, A_u) \, dX_u. \]

Since \( F \in \mathbb{C}_c^0([0, T]) \), \( Y(t) \) has limit \( F_T(X_T, A_T) \) as \( t \to T \), so the stochastic integral also converges.

**Example 12.**

If \( e^{X(t)} - \frac{1}{2}[X](t) \) is a martingale, applying Theorem \ref{thm:ito5} to the functional \( F_t(x_t, v_t) = e^{x(t)} - \int_0^t v_u \, du \) yields the familiar formula:

\[
e^{X(t)} - \frac{1}{2}[X](t) = 1 + \int_0^t e^{X(s)} - \frac{1}{2}[X](s) \, dX(s) \quad (46)
\]

### 5.2 Extension to square-integrable functionals

Let \( \mathcal{L}^2(X) \) be the Hilbert space of progressively-measurable processes \( \phi \) such that:

\[
\|\phi\|^2_{\mathcal{L}^2(X)} = E \left[ \int_0^t \phi_s^2 \, d[X](s) \right] < \infty
\]

and \( \mathcal{I}^2(X) \) be the space of square-integrable stochastic integrals with respect to \( X \):

\[
\mathcal{I}^2(X) = \{ \int_0^t \phi(t) \, dX(t), \phi \in \mathcal{L}^2(X) \}
\]

endowed with the norm \( \|Y\|_2^2 = E[Y(T)^2] \). The Itô integral \( I_X : \phi \mapsto \int_0^t \phi_s \, dX(s) \) is then a bijective isometry from \( \mathcal{L}^2(X) \) to \( \mathcal{I}^2(X) \).

We will now show that the operator \( \nabla_X : \mathcal{L}^2(X) \to \mathcal{L}^2(X) \) admits a suitable extension to \( \mathcal{I}^2(X) \) which verifies

\[
\forall \phi \in \mathcal{L}^2(X), \quad \nabla_X \left( \int_0^t \phi \, dX(t) \right) = \phi, \quad dt \times d\mathbb{P} - a.s. \quad (49)
\]

i.e. \( \nabla_X \) is the inverse of the Itô stochastic integral with respect to \( X \).

**Definition 5.3** (Space of test processes). The space of test processes \( D(X) \) is defined as

\[
D(X) = C_b^{1,2}(X) \cap \mathcal{I}^2(X)
\]

Theorem \ref{thm:ito5} allows to define intrinsically the vertical derivative of a process in \( D(X) \) as an element of \( \mathcal{L}^2(X) \).

**Definition 5.4.** Let \( Y \in D(X) \), define the process \( \nabla_X Y \in \mathcal{L}^2(X) \) as the equivalence class of \( \nabla_x F_t(X_t, A_t) \), which does not depend on the choice of the representation functional \( Y(t) = F_t(X_t, A_t) \)

**Proposition 5.5** (Integration by parts on \( D(X) \)). Let \( Y, Z \in D(X) \). Then:

\[
E[Y(T)Z(T)] = E \left[ \int_0^T \nabla_X Y(t) \nabla_X Z(t) \, d[X](t) \right]
\]

(51)
Applying the Ito isometry formula yields the result.

**Proof.** Let \( Y, Z \in D(X) \subset C_b^{1,2}(X) \). Then \( Y, Z \) are martingales with \( Y(0) = Z(0) = 0 \) and \( E[|Y(T)|^2] < \infty, E[|Z(T)|^2] < \infty \). Applying Theorem 5.2 to \( Y \) and \( Z \), we obtain

\[
E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y dX \int_0^T \nabla_X Z dX\right]
\]

Applying the Ito isometry formula yields the result.

Using this result, we can extend the operator \( \nabla_X \) in a weak sense to a suitable space of the space of (square-integrable) stochastic integrals, where \( \nabla_X Y \) is characterized by [51] being satisfied against all test processes.

The following definition introduces the Hilbert space \( W^{1,2}(X) \) of martingales on which \( \nabla_X \) acts as a weak derivative, characterized by integration-by-part formula [51]. This definition may be also viewed as a non-anticipative counterpart of Wiener-Sobolev spaces in the Malliavin calculus [22, 30].

**Definition 5.6** (Martingale Sobolev space). The Martingale Sobolev space \( W^{1,2}(X) \) is defined as the closure in \( \mathcal{I}^2(X) \) of \( D(X) \).

The Martingale Sobolev space \( W^{1,2}(X) \) is in fact none other than \( \mathcal{I}^2(X) \), the set of square-integrable stochastic integrals:

**Lemma 5.7.** \( \{\nabla_X Y, Y \in D(X)\} \) is dense in \( \mathcal{L}^2(X) \) and \( W^{1,2}(X) = \mathcal{I}^2(X) \).

**Proof.** We first observe that the set \( U \) of “cylindrical” processes of the form

\[
\phi_{n,f,(t_1,\ldots,t_n)}(t) = f(X(t_1),\ldots,X(t_n))1_{t>t_n}
\]

where \( n \geq 1, 0 \leq t_1 < \ldots < t_n \leq T \) and \( f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}) \) is a total set in \( \mathcal{L}^2(X) \) i.e. the linear span of \( U \) is dense in \( \mathcal{L}^2(X) \). For such an integrand \( \phi_{n,f,(t_1,\ldots,t_n)} \), the stochastic integral with respect to \( X \) is given by the martingale

\[
Y(t) = I_X(\phi_{n,f,(t_1,\ldots,t_n)})(t) = F_t(X_t, A_t)
\]

where the functional \( F \) is defined on \( Y \) as:

\[
F_t(x_1, v_t) = f(x(t_1),\ldots,x(t_n))1_{t>t_n}
\]

so that:

\[
\nabla_x F_t(x_1, v_t) = f(x(t_1),\ldots,x(t_n))1_{t>t_n}, \nabla_x^2 F_t(x_1, v_t) = 0, D_t F(x_1, v_t) = 0
\]

which shows that \( F \in C_b^{1,2} \) (see Example 2). Hence, \( Y \in C_b^{1,2}(X) \). Since \( f \) is bounded, \( Y \) is obviously square integrable so \( Y \in \mathcal{D}(X) \). Hence \( I_X(U) \subset D(X) \).

Since \( I_X \) is a bijective isometry from \( \mathcal{L}^2(X) \) to \( \mathcal{I}^2(X) \), the density of \( U \) in \( \mathcal{L}^2(X) \) entails the density of \( I_X(U) \) in \( \mathcal{I}^2(X) \), so \( W^{1,2}(X) = \mathcal{I}^2(X) \). 

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Theorem 5.8 (Extension of $\nabla_X$ to $W^{1,2}(X)$). The vertical derivative $\nabla_X : D(X) \to \mathcal{L}^2(X)$ is closable on $W^{1,2}(X)$. Its closure defines a bijective isometry

$$\nabla_X : \mathcal{W}^{1,2}(X) \to \mathcal{L}^2(X) \quad \int_0^\phi dX \mapsto \phi \quad (52)$$

classified by the following integration by parts formula: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique

$$\forall Z \in D(X), \quad E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y(t)\nabla_X Z(t)d[X][t]\right]. \quad (53)$$

In particular, $\nabla_X$ is the adjoint of the Ito stochastic integral

$$I_X : \mathcal{L}^2(X) \to \mathcal{W}^{1,2}(X) \quad \phi \mapsto \int_0^\phi dX \quad (54)$$

in the following sense:

$$\forall \phi \in \mathcal{L}^2(X), \quad \forall Y \in \mathcal{W}^{1,2}(X), \quad E[Y(T)\int_0^T \phi(t)dX] = E\left[\int_0^T \nabla_X Y \phi d[X]\right] \quad (55)$$

Proof. Any $Y \in \mathcal{W}^{1,2}(X)$ may be written as $Y(t) = \int_0^t \phi(s)dX(s)$ with $\phi \in \mathcal{L}^2(X)$, which is uniquely defined $d[X] \times d\mathbb{P}$ a.e. The Ito isometry formula then guarantees that (53) holds for $\phi$. To show that (53) uniquely characterizes $\phi$, consider $\psi \in \mathcal{L}^2(X)$ which also satisfies (53), then, denoting $I_X(\psi) = \int_0^\psi dX$ its stochastic integral with respect to $X$, (53) then implies that

$$\forall Z \in D(X), \quad <I_X(\psi) - Y, Z>_{\mathcal{W}^{1,2}(X)} = E[(Y(T) - \int_0^T \psi dX)Z(T)] = 0$$

which implies $I_X(\psi) = Y d[X] \times d\mathbb{P}$ a.e. since by construction $D(X)$ is dense in $\mathcal{W}^{1,2}(X)$. Hence, $\nabla_X : D(X) \to \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$.

This construction shows that $\nabla_X : \mathcal{W}^{1,2}(X) \to \mathcal{L}^2(X)$ is a bijective isometry which coincides

with the adjoint of the Ito integral on $\mathcal{W}^{1,2}(X)$.

Thus, the Ito integral $I_X$ with respect to $X$

$$I_X : \mathcal{L}^2(X) \to \mathcal{W}^{1,2}(X)$$

admits an inverse on $\mathcal{W}^{1,2}(X)$ which is an extension of the (pathwise) vertical derivative $\nabla_X$ operator introduced in Definition 3.2, and

$$\forall \phi \in \mathcal{L}^2(X), \quad \nabla_X \left(\int_0^\phi dX\right) = \phi \quad (56)$$

holds in the sense of equality in $\mathcal{L}^2(X)$.

The above results now allow us to state a general version of the martingale representation formula, valid for all square-integrable martingales:

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Theorem 5.9 (Martingale representation formula: general case). For any square-integrable $\mathcal{F}_t^X$-martingale $Y$,

$$Y(T) = Y(0) + \int_0^T \nabla_X Y dX \quad \mathbb{P} \text{-a.s.}$$

6 Relation with the Malliavin derivative

The above results hold in particular in the case where $X = W$ is a Brownian motion. In this case, the vertical derivative $\nabla_W$ may be related to the Malliavin derivative [22, 2, 3, 31] as follows.

Consider the canonical Wiener space $(\Omega_0 = C_0([0, T], \mathbb{R}^d), ||.||_\infty, \mathbb{P})$ endowed with its Borelian $\sigma$-algebra, the filtration of the canonical process. Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0, T]) = H(X_T)$ with $E[|H|^2] < \infty$. If $H$ is differentiable in the Malliavin sense [2, 22, 24, 31] e.g. $H \in \mathbf{D}^{1,2}$ with Malliavin derivative $\mathbb{D}_tH$, then the Clark-Haussmann-Ocone formula [25, 24] gives a stochastic integral representation of $H$ in terms of the Malliavin derivative of $H$:

$$H = E[H] + \int_0^T pE[\mathbb{D}_tH | \mathcal{F}_t] dW_t$$

(57)

where $pE[\mathbb{D}_tH | \mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. This yields a stochastic integral representation of the martingale $Y(t) = E[H | \mathcal{F}_t]$:

$$Y(t) = E[H | \mathcal{F}_t] = E[H] + \int_0^t pE[\mathbb{D}_uH | \mathcal{F}_u] dW_u$$

Related martingale representations have been obtained under a variety of conditions [2, 7, 11, 18, 26, 24].

Denote by

- $L^2([0, T] \times \Omega)$ the set of (anticipative) processes $\phi$ on $[0, T]$ with $E \int_0^T ||\phi(t)||^2 dt < \infty$.
- $\mathbb{D}$ the Malliavin derivative operator, which associates to a random variable $H \in \mathbf{D}^{1,2}(0, T)$ the (anticipative) process $(\mathbb{D}_tH)_{t \in [0, T]} \in L^2([0, T] \times \Omega)$.

Theorem 6.1 (Lifting theorem). The following diagram is commutative in the sense of $dt \times d\mathbb{P}$ equality:

$$\begin{array}{ccc}
\mathcal{I}^2(W) & \xrightarrow{\nabla_W} & \mathcal{L}^2(W) \\
\uparrow \{E[|\mathcal{F}_t]_{t \in [0, T]}\} & & \uparrow \{E[|\mathcal{F}_t]_{t \in [0, T]}\} \\
\mathbf{D}^{1,2} & \xrightarrow{\mathbb{D}} & L^2([0, T] \times \Omega)
\end{array}$$

In other words, the conditional expectation operator intertwines $\nabla_W$ with the Malliavin derivative:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \quad \nabla_W (E[H | \mathcal{F}_t]) = E[\mathbb{D}_tH | \mathcal{F}_t]$$

(58)

Proof. The Clark-Haussmann-Ocone formula [25] gives

$$\forall H \in \mathbf{D}^{1,2}, \quad H = E[H] + \int_0^T pE[\mathbb{D}_tH | \mathcal{F}_t] dW_t$$

(59)
where $^pE[D_tH|\mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. On other hand theorem 5.2 gives:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \quad H = E[H] + \int_0^T \nabla_W Y(t) \, dW(t)$$

(60)

where $Y(t) = E[H|\mathcal{F}_t]$. Hence $^pE[D_tH|\mathcal{F}_t] = \nabla_W E[H|\mathcal{F}_t], \, dt \times d\mathbb{P}$ almost everywhere. \qed

Thus, the conditional expectation operator (more precisely: the predictable projection on $\mathcal{F}_t$ [8, Vol. I]) can be viewed as a morphism which “lifts” relations obtained in the framework of Malliavin calculus into relations between non-anticipative quantities, where the Malliavin derivative and the Skorokhod integral are replaced, respectively, by the vertical derivative $\nabla_W$ and the Ito stochastic integral.

From a computational viewpoint, unlike the Clark-Haussmann-Ocone representation which requires to simulate the anticipative process $D_tH$ and compute conditional expectations, $\nabla_X Y$ only involves non-anticipative quantities which can be computed path by path. It is thus more amenable to numerical computations. This topic is further explored in a forthcoming work.

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A Proof of Theorem 2.7

In order to prove theorem 2.7 in the general case where $A$ is only required to be cadlag, we need the following three lemmas. The first lemma states a property analogous to 'uniform continuity' for cadlag functions:

**Lemma A.1.** Let $f$ be a cadlag function on $[0,T]$ and define $\Delta f(t) = f(t) - f(t-)$. Then
\[ \forall \epsilon > 0, \quad \exists \eta(\epsilon) > 0, \quad |x - y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \epsilon + \sup_{t \in [x,y]} \{|\Delta f(t)|\} \] (61)

**Proof.** If (61) does not hold, then there exists a sequence $(x_n, y_n)_{n \geq 1}$ such that $x_n \leq y_n$, $y_n - x_n \to 0$ but $|f(x_n) - f(y_n)| > \epsilon + \sup_{t \in [x_n, y_n]} \{|\Delta f(t)|\}$. We can extract a convergent subsequence $(x_{\psi(n)})$ such that $x_{\psi(n)} \to x$. Noting that either an infinity of terms of the sequence are less than $x$ or an infinity are more than $x$, we can extract monotone subsequences $(u_n, v_n)_{n \geq 1}$ which converge to $x$. If $(u_n), (v_n)$ both converge to $x$ from above or from below, $|f(u_n) - f(v_n)| \to 0$ which yields a contradiction. If one converges from above and the other from below, $\sup_{t \in [u_n, v_n]} \{|\Delta f(t)|\} \geq |\Delta f(x)|$ but $|f(u_n) - f(v_n)| \to |\Delta f(x)|$, which results in a contradiction as well. Therefore (61) must hold. \(\square\)

**Lemma A.2.** If $\alpha \in \mathbb{R}$ and $V$ is an adapted cadlag process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $\sigma$ is a optional time, then:
\[ \tau = \inf\{t > \sigma, \quad |V(t) - V(t-)| > \alpha\} \] (62)
is a stopping time.

**Proof.** We can write that:
\[ \{\tau \leq t\} = \bigcup_{q \in \mathbb{Q} \cap (0,t)} \{\{\sigma \leq t - q\} \cap \{\sup_{t \in [t-q, t]} |V(u) - V(u-)| > \alpha\}\} \] (63)
and, using Lemma A.1,
\[ \sup_{u \in [t-q, t]} |V(u) - V(u-)| > \alpha = \bigcup_{n_0 > 1} \bigcup_{n > n_0} \bigcup_{m \geq 1} \{\sup_{1 \leq i \leq 2^n} |V(t-q\cdot \frac{i-1}{2^n}) - V(t-q\cdot \frac{i}{2^n})| > \alpha + \frac{1}{m}\}. \] (64)
\(\square\)
Lemma A.3 (Uniform approximation of cadlag functions by step functions). Let $f \in D([0,T],\mathbb{R}^d)$ and $\pi^n = (t^n_i)_{n \geq 1, i = 0..k_n}$ a sequence of partitions ($0 = t^n_0 < t^n_1 < ... < t^n_{k_n} = T$) of $[0,T]$ such that:

$$
\sup_{0 \leq i \leq k_n-1} |t^n_{i+1} - t^n_i| \xrightarrow{n \to \infty} 0 \quad \sup_{u \in [0,T] \setminus \pi^n} |\Delta f(u)| \xrightarrow{n \to \infty} 0
$$

then

$$
\sup_{u \in [0,T]} |f(u) - \sum_{i=0}^{k_n-1} f(t^n_i)1_{[t^n_i, t^n_{i+1})}(u) + f(t^n_{k_n})1_{(t^n_{k_n}, T]}(u)| \xrightarrow{n \to \infty} 0
$$

Proof. Denote $h^n = f - \sum_{i=0}^{k_n-1} f(t^n_i)1_{(t^n_i, t^n_{i+1}]} + f(t^n_{k_n})1_{(t^n_{k_n}, T]}$. Since $f - h^n$ is piecewise constant on $\pi^n$ and $h^n(t^n_{k_n}) = 0$ by definition,

$$
\sup_{t \in [0,T]} |h^n(t)| = \sup_{i=0..k_n-1} \sup_{t^n_i < t < t^n_{i+1}} |h^n(t)| = \sup_{t^n_i < t < t^n_{i+1}} |f(t) - f(t^n_i)|
$$

Let $\epsilon > 0$. For $n \geq N$ sufficiently large, $\sup_{u \in [0,T] \setminus \pi^n} |\Delta f(u)| \leq \epsilon/2$ and $\sup_i |t^n_{i+1} - t^n_i| \leq \eta(\epsilon/2)$ using the notation of Lemma A.1. Then, applying Lemma A.3 to $f$ we obtain, for $n \geq N$,

$$
\sup_{t \in [t^n_i, t^n_{i+1}]} |f(t) - f(t^n_i)| \leq \frac{\epsilon}{2} + \sup_{t^n_i < t < t^n_{i+1}} |\Delta f(u)| \leq \epsilon.
$$

We can now prove Theorem 2.7 in the case where $A$ is a cadlag adapted process. 

Proof of Theorem 2.7: Let us first show that $F_t(X_t, A_t)$ is adapted. Define:

$$
\tau^n_0 = 0 \quad \tau^n_k = \inf\{t > \tau^n_{k-1} | 2^N t \in \mathbb{N} \text{ or } |A(t) - A(t^-)| > \frac{1}{N}\} \wedge t
$$

From lemma A.2, $\tau^n_k$ are stopping times. Define the following piecewise constant approximations of $X_t$ and $A_t$ along the partition $(\tau^n_k, k \geq 0)$:

$$
X^n(s) = \sum_{k \geq 0} X^n_{\tau^n_k}1_{[\tau^n_k, \tau^n_{k+1})}(s) + X(t)1_{(t)}(s)
$$

$$
A^n(s) = \sum_{k = 0} A^n_{\tau^n_k}1_{[\tau^n_k, \tau^n_{k+1})}(t) + A(t)1_{(t)}(s)
$$

as well as their truncations of rank $K$:

$$
\kappa X^n(s) = \sum_{k = 0}^K X^n_{\tau^n_k}1_{[\tau^n_k, \tau^n_{k+1})}(s) \quad \kappa A^n(t) = \sum_{k = 0}^K A^n_{\tau^n_k}1_{[\tau^n_k, \tau^n_{k+1})}(t)
$$

Since $(\kappa X^n_t, \kappa A^n_t)$ coincides with $(X^n_t, A^n_t)$ for $K$ sufficiently large,

$$
F_t(X^n, A^n) = \lim_{K \to \infty} F_t(\kappa X^n_t, \kappa A^n_t).
$$

The approximations $F^n_t(\kappa X^n_t, \kappa A^n_t)$ are $F_t$-measurable as they are continuous functions of the random variables:

$$
\{(X(\tau^n_k)1_{\tau^n_k \leq t}, A(\tau^n_k)1_{\tau^n_k \leq t}), k \leq K\}
$$
so their limit $F_t(X_t^N, A_t^N)$ is also $F_t$-measurable. Thanks to Lemma A.3, $X_t^N$ and $A_t^N$ converge uniformly to $X_t$ and $A_t$, hence $F_t(X_t^N, A_t^N)$ converges to $F_t(X_t, A_t)$ since $F_t : (D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, ||\cdot||_\infty) \rightarrow \mathbb{R}$ is continuous.

To show the optionality of $Z$ in point (ii), we will show that $Z$ it as limit of right-continuous adapted processes. For $t \in [0, T]$, define $i^n(t)$ to be the integer such that $t \in \left[\frac{it}{T}, \frac{(i+1)t}{T}\right)$. Define the process: $Z_t^n = F_{\left(\frac{it}{T}, \frac{(i+1)t}{T}\right)}(X_{\left(\frac{it}{T}, \frac{(i+1)t}{T}\right)}, A_{\left(\frac{it}{T}, \frac{(i+1)t}{T}\right)})$, which is piecewise-constant and has right-continuous trajectories, and is also adapted by the first part of the theorem. Since $F \in \mathbb{C}_0^{0,0}$, $Z^n(t) \rightarrow Z(t)$ almost surely, which proves that $Z$ is optional. Point (iii) follows from (i) and lemma 2.6 since in both cases $F_t(X_t, A_t) = F_t(X_{t-}, A_{t-})$ hence $Z$ has left-continuous trajectories.