Global hydrostatic approximation of the hyperbolic Navier-Stokes system with small Gevrey class 2 data

In memory of Professor Geneviève Raugel

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Abstract We investigate the hydrostatic approximation of a hyperbolic version of Navier-Stokes equations, which is obtained by using the Cattaneo type law instead of the Fourier law, evolving in a thin strip $\mathbb{R} \times (0, \varepsilon)$. The formal limit of these equations is a hyperbolic Prandtl type equation. We first prove the global existence of solutions to these equations under a uniform smallness assumption on the data in the Gevrey class 2. Then we justify the limit globally-in-time from the anisotropic hyperbolic Navier-Stokes system to the hyperbolic Prandtl system with such Gevrey class 2 data. Compared with Paicu et al. (2020) for the hydrostatic approximation of the 2-D classical Navier-Stokes system with analytic data, here the initial data belongs to the Gevrey class 2, which is very sophisticated even for the well-posedness of the classical Prandtl system (see Dietert and Gérard-Varet (2019) and Wang et al. (2021)); furthermore, the estimate of the pressure term in the hyperbolic Prandtl system gives rise to additional difficulties.

Keywords incompressible hyperbolic Navier-Stokes equations, hydrostatic approximation, hyperbolic Prandtl system

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1 Introduction

The Navier-Stokes system governing the evolution of Newtonian incompressible viscous fluids reads

\[
\text{(NS)} \begin{cases}
\partial_t U + U \cdot \nabla U - \nu \Delta U + \nabla P = 0, \\
\text{div} \ U = 0,
\end{cases}
\]

which is obtained from the momentum equations

\[
\partial_t U + U \cdot \nabla U = \text{div} \, \tau, \tag{1.1}
\]

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coupled with the Fourier law to describe the stress tensor
\[
\tau(t) = -P \text{Id} + \nu(\nabla U + (\nabla U)^T)(t),
\]
where \( U \) stands for the velocity field, \( P \) is the scalar pressure function and \( \nu \) is the viscosity coefficient of the fluid.

Nevertheless, the system \((\text{NS})\) gives rise to a physical paradox coming from the fact that the equation of the velocity has infinite propagation speed. In order to avoid this non-physical aspect, Cattaneo [7, 8] (see also Vernotte [31]) proposed modifying the heat equation to a hyperbolic version known as Cattaneo’s heat transfer law. More precisely, they proposed replacing the Fourier law (1.2) by the hyperbolic model \( \frac{1}{\alpha^2} \partial_t^2 \theta + \frac{1}{\alpha} \partial_t \theta - \Delta \theta = 0 \). The resulting system has finite propagation speed and constitutes a satisfactory physical model being compatible with the principle of relativity and with the second law of thermodynamics. It is therefore natural to consider a hyperbolic Navier-Stokes system by adding the term \( \eta \partial_t^2 u \) to the classical Navier-Stokes system \((\text{NS})\), where \( \eta = \frac{1}{\alpha^2} \) is a small parameter. The hyperbolic Navier-Stokes system (1.5) has been extensively studied in the literature and one may check [2, 6, 13, 18, 22, 28] and the references therein. The justification of this system by using the Cattaneo’s law follows by considering that the stress tensor is given by the solution to the retarded equation
\[
\tau(t + \eta) = -P \text{Id} + \nu(\nabla U + (\nabla U)^T)(t).
\]

By using the fact that \( \tau(t + \eta) \simeq \tau(t) + \eta \partial_t \tau(t) \) and applying the operator \((\eta \partial_t + \text{Id})\) to the momentum equations (1.1), we obtain the following quasilinear hyperbolic version of Navier-Stokes equations:
\[
\begin{aligned}
\eta \partial_t^2 U + \partial_t U + U \cdot \nabla U + \eta(U \cdot \nabla U) t - \nu \Delta U + \nabla P &= 0, \\
\text{div} U &= 0, \\
U|_{t=0} &= U_0, \quad \partial_t U|_{t=0} = U_1.
\end{aligned}
\]

For simplicity, we neglect the term \( \eta(U \cdot \nabla U) t \) in (1.4), and then we get the hyperbolic version of the Navier-Stokes system
\[
\begin{aligned}
\eta \partial_t^2 U + \partial_t U + U \cdot \nabla U - \nu \Delta U + \nabla P &= 0, \\
\text{div} U &= 0, \\
U|_{t=0} &= U_0, \quad \partial_t U|_{t=0} = U_1.
\end{aligned}
\]

We remark that the system (1.5) can also be obtained by the relaxation of the Euler’s equations under a diffusive scaling as is proposed in [6]. Indeed, Brenier et al. [6] proved the global existence and uniqueness of solutions to the following system:
\[
\begin{aligned}
\partial_t U + \nabla \cdot V &= \nabla P, \\
\sqrt{\eta} \partial_t V + \frac{\nu}{\sqrt{\eta}} \nabla U &= \frac{1}{\sqrt{\eta}} (U \otimes U - V), \\
\text{div} U &= 0, \\
U|_{t=0} &= U_0, \quad V|_{t=0} = V_0
\end{aligned}
\]
with initial data in \( H^2(T^2) \times H^1(T^2) \), where \( T^2 \) is the periodic square \( \mathbb{R}^2 / \mathbb{Z}^2 \). Moreover, they proved the convergence of such solutions towards a smooth solution to the classical Navier-Stokes system. In fact, one can show that as \( \eta \to 0^+ \), the equations governing the leading-order terms in (1.6) are again (1.5).

Paicu and Raugel [22] obtained the global existence and uniqueness result for the system (1.5) with significantly improved regularity for the initial data by using the Strichartz inequalities for this dispersive model. Hachicha [18] obtained the global existence and uniqueness result for the perturbed Navier-Stokes system (1.5) under suitable smallness assumptions on the initial data in the space \( H^{\frac{n}{2} + \delta} \times H^{\frac{n}{2} - 1 + \delta}(\mathbb{R}^n) \) for \( n = 2, 3 \). Moreover, she proved the local in-time convergence of such solutions towards solutions to

\[1\] Paicu M, Raugel G. A hyperbolic singular perturbation of the Navier-Stokes equations in \( \mathbb{R}^2 \). Manuscript, 2008
the classical Navier-Stokes system with initial data in $H^{\frac{n}{2}+s}(\mathbb{R}^n)$ for $s > 0$. Very recently, Coulaud et al. [13] investigated the well-posedness of a hyperbolic quasi-linear version of the Navier-Stokes equation in $\mathbb{R}^2$. In particular, under smallness assumptions on the data, they proved the global well-posedness of the system.

On the other hand, we recall that in order to describe geophysical flows on the earth, it is usually assumed that the vertical scale is much smaller than the horizontal one as the fluid layer depth is small compared with the radius of the earth. In order to take into account this anisotropy between horizontal and vertical directions, we shall suppose that the fluid is evolving in a thin strip domain and for a vanishing viscosity so that it is a good approximation to the atmospheric flow and the oceanic flow.

The purpose of this paper is first to show the global existence of solutions to (1.5) in the thin domain $S = \mathbb{R} \times (0, \epsilon)$ and for a vanishing viscosity $\nu = \epsilon^2$ (we denote the corresponding solutions by $(U^\epsilon, P^\epsilon)$), and then to justify the limit system when the parameter $\epsilon$ goes to zero. For simplicity, here we take $\eta = 1$ in (1.5). The case $\eta \to 0+$ also presents interest and will be considered separately.

We complement the system (1.5) with the non-slip boundary condition

$$U|_{y=0} = U|_{y=\epsilon} = 0$$

and the initial condition

$$U|_{t=0} = \left( u_0 \left( x, \frac{y}{\epsilon} \right), \frac{\epsilon v_0 \left( x, \frac{y}{\epsilon} \right)}{\epsilon} \right) = U_0^\epsilon \text{ in } S^\epsilon.$$

As in [4, 19, 25] for the classical Navier-Stokes system, we write

$$U(t, x, y) = \left( u^\epsilon \left( t, x, \frac{y}{\epsilon} \right), \frac{\epsilon v^\epsilon \left( t, x, \frac{y}{\epsilon} \right)}{\epsilon} \right) \text{ and } P(t, x, y) = p^\epsilon \left( t, x, \frac{y}{\epsilon} \right).$$

Let $S \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$. After a natural change of scale (see [25]), the system (1.5) becomes the following scaled anisotropic hyperbolic Navier-Stokes system:

$$\begin{align*}
\partial^2_t u^\epsilon + \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial^2_x u^\epsilon - \partial^2_y u^\epsilon + \partial_x p^\epsilon &= 0 \text{ in } S \times ]0, \infty[,
\epsilon^2 (\partial^2_t v^\epsilon + \partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial^2_x v^\epsilon - \partial^2_y v^\epsilon) + \partial_y p^\epsilon &= 0,
\partial_x u^\epsilon + \partial_y v^\epsilon &= 0,
(u^\epsilon, v^\epsilon)|_{t=0} = (u_0, v_0) \text{ and } (\partial_t u^\epsilon, \partial_t v^\epsilon)|_{t=0} = (u_1, v_1)
\end{align*}$$

(1.8)

together with the boundary condition

$$(u^\epsilon, v^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon)|_{y=1} = 0.$$ (1.9)

Formally taking $\epsilon \to 0$ in the system (1.8), we obtain the hyperbolic Prandtl equations

$$\begin{align*}
\partial^2_t u + \partial_t u + u \partial_x u + v \partial_y u - \partial^2_y u + \partial_x p &= 0 \text{ in } S \times ]0, \infty[,
\partial_y p &= 0,
\partial_x u + \partial_y v &= 0,
|u|_{t=0} = u_0, \quad |u_t|_{t=0} = u_1
\end{align*}$$

(1.10)

together with the boundary condition

$$(u, v)|_{y=0} = (u, v)|_{y=1} = 0.$$ (1.11)

The goal of this paper is to justify the limit from the system (1.8) to the system (1.10) with initial data in the Gevrey class 2. Before we present the main results of this paper, we remark that similar to the case of the classical Prandtl equation (P), the nonlinear term $v \partial_y u$ in (1.10) will lead to one derivative loss in the $x$ variable in the process of energy estimates. Thus, it is natural to work with analytic data in order to
overcome this difficulty if we do not impose extra structural assumptions on the initial data (see [15, 29]). Indeed, for the data which is analytic in $x$ and $y$ variables, Sammartino and Caflisch [30] established the local well-posedness result of the system (P) in the upper half space. Later, the analyticity in the $y$ variable was removed by Lombardo et al. [20]. Lately, Paicu and Zhang [24] proved the global well-posedness of the hyperbolic Prandtl system was obtained by Dietert and Gérard-Varet [14] (see also [16]), where the authors proved the local well-posedness of (P) with Gevrey class 2 data (while one may check [32] for the corresponding global result). We also mention that for a class of convex data, Gérard-Varet et al. [17] proved the well-posedness of the hydrostatic Navier-Stokes system in the Gevrey class.

Our first result in this paper is concerned with the global well-posedness of the hyperbolic Prandtl system (1.10) with small data in the Gevrey class.

**Theorem 1.1.** (1) Let $a > 0$. We assume that the initial data satisfies

$$S^0_{\frac{1}{2}}(u_0, u_1) \leq c_0$$

with

$$S^0_k(u_0, u_1) \equiv \|e^{\alpha |D_x|^{\frac{1}{2}} u}(u_0, u_1, \partial_y u_0)\|_{B^k} + \sqrt{aR}\|e^{\alpha |D_x|^{\frac{1}{2}} u_0}\|_{B^{k+rac{1}{4}}} + aR\|e^{\alpha |D_x|^{\frac{1}{2}} u_0}\|_{B^{k+rac{1}{2}}}$$

(1.12)

for some $c_0$ sufficiently small and the following compatibility condition holds:

$$\int_0^1 u_0 dy = \int_0^1 u_1 dy = 0.$$

Then the system (1.10) has a unique global solution $u$ so that for any $t > 0$, it holds that

$$E_{\frac{1}{2}}(u)(t) \leq CS^0_{\frac{1}{2}}(u_0, u_1),$$

(1.13)

where

$$E_s(u)(t) \equiv \|e^{\tilde{R}t}(u_\Phi, \partial_y u_\Phi, (\partial_t u_\Phi))\|_{L^\infty(B^s)} + \sqrt{aR}\|e^{\tilde{R}t} u_\Phi\|_{L^\infty(B^{s+rac{1}{4}})}$$

$$+ aR\|e^{\tilde{R}t} u_\Phi\|_{L^\infty(B^{s+rac{1}{2}})} + \|e^{\tilde{R}t}((\partial_t u_\Phi), \partial_y u_\Phi)\|_{L^2(B^s)}.$$  (1.14)

Here $u_\Phi$ will be determined by (2.1), the constant

$$\tilde{R} \equiv \min\left(\frac{1}{6}, \frac{1}{4(1+K)}\right)$$

with $K$ being determined by the Poincaré inequality on the strip $S$ (see (2.12)), and the functional spaces will be presented in Appendix A.

(2) If we assume moreover that for $s > -\frac{1}{2}$, $S^0_{\frac{s}{2}}(u_0, u_1) < \infty$, then one has

$$E_s(u)(t) \leq CS^0_{\frac{s}{2}}(u_0, u_1), \quad \forall t \in \mathbb{R}^+.$$  (1.15)

(3) If we assume in addition that

$$S^0_{\frac{1}{2}}(u_0, u_1) + S^0_{\frac{1}{4}}(u_0, u_1) + S^0_{\frac{1}{2}}(\partial_y u_0, \partial_y u_1) + \|\alpha |D_x|^{\frac{1}{2}} u_0\|_{B^2} \|\alpha |D_x|^{\frac{1}{2}} \partial_y u_0\|_{B^2} \|\alpha |D_x|^{\frac{1}{2}} u_0\|_{B^2} \leq c_1$$

(1.16)

for some $c_1$ sufficiently small, then for any $s > \frac{1}{2}$, one has

$$E_s(\partial_y u)(t) + \|e^{\tilde{R}t}(\partial_t^2 u_\Phi)\|_{L^2(B^s)}$$

$$\leq C(S^0_{\frac{s}{2}}(u_0, u_1) + S^0_{\frac{s+rac{1}{4}}{4}}(u_0, u_1) + S^0_{\frac{s}{2}}(\partial_y u_0, \partial_y u_1)$$

$$+ \|\alpha |D_x|^{\frac{1}{2}} u_0\|_{B^2} \|\alpha |D_x|^{\frac{1}{2}} \partial_y u_0\|_{B^2} \|\alpha |D_x|^{\frac{1}{2}} u_0\|_{B^2}^2).$$  (1.17)
Remark 1.1. Compared with the well-posedness result of the hydrostatic Navier-Stokes system with initial data in the Gevrey class in [17], here we do not need the convex assumption on the initial data. Compared with the proof of the well-posedness result of the Prandtl system in [14,32] with initial data in the Gevrey class 2, our proof here is much simplified. The main idea is to use multiple time-weighted norms like \( \|u\|_{L_x^t B^{s(i)}(\mathbb{R}^{d+1})}, \|u\|_{L_x^t \phi^{i}(\mathbb{R}^{d+1})} \) and \( \|u\|_{L_x^t \phi^{i}(\mathbb{R}^{d+1})} \) (see (2.25) and Definition A.3 below).

Remark 1.2. Here, we only prove Theorem 1.1 in two space dimensions. In order to extend the above result in higher space dimensions, we have difficulty in proving \( \int_0^1 u(t,x,y)dy = 0 \). One may check the proof at the beginning of Section 3 for the proof in two space dimensions.

The second result is about the global well-posedness of the system (1.8) with small Gevrey class 2 data in the \( x \) variable. The main interesting point is that the smallness of data is independent of \( \varepsilon \) and the global uniform estimate (1.20) holds with respect to the parameter \( \varepsilon \).

**Theorem 1.2.** Let \( a > 0 \). We assume that the initial data satisfies

\[
\|e^{\alpha D_x^{1/2}}(u_0, \varepsilon v_0, u_1, \varepsilon v_1, \partial_y(u_0, \varepsilon v_0))\|_{B^d_{\alpha}} + \|e^{\alpha D_x^{1/2}}(u_0, \varepsilon v_0)\|_{B^d_{\alpha}} \leq c_2 \tag{1.18}
\]

for some \( c_2 \) sufficiently small. Then the system (1.8) has a unique global solution \((u^\varepsilon, v^\varepsilon)\) so that for any \( t > 0 \), it holds that

\[
E_R^1(u^\varepsilon, v^\varepsilon)(t) \leq C\mathcal{S}_1^1(u_0, v_0), \tag{1.19}
\]

where

\[
E_R^1(u^\varepsilon, v^\varepsilon)(t) \overset{\text{def}}{=} \|e^{\alpha Rt}(u^\varepsilon, \varepsilon v^\varepsilon)\|_{L^\infty_t B^d_{\alpha}} + \sqrt{aR}\|e^{\alpha Rt}(u^\varepsilon, \varepsilon v^\varepsilon)\|_{L^1_t B^d_{\alpha}} + aR\|e^{\alpha Rt}(u^\varepsilon, \varepsilon v^\varepsilon)\|_{L^2_t B^d_{\alpha}} \tag{1.20}
\]

and

\[
\mathcal{S}_1^1(u_0, v_0) \overset{\text{def}}{=} \|e^{\alpha D_x^{1/2}}(u_0, \varepsilon v_0, u_1, \varepsilon v_1)\|_{B^d_{\alpha}^2} + \sqrt{aR}\|e^{\alpha D_x^{1/2}}(u_0, \varepsilon v_0, u_1, \varepsilon v_1)\|_{B^d_{\alpha}^2} + aR\|e^{\alpha D_x^{1/2}}(u_0, \varepsilon v_0, u_1, \varepsilon v_1)\|_{B^d_{\alpha}^2}, \tag{1.21}
\]

where \((u_0^\varepsilon, v_0^\varepsilon)\) will be given by (2.1).

The third result is concerned with the convergence of the solutions from the scaled anisotropic hyperbolic Navier-Stokes system (1.8) to the hyperbolic Prandtl system (1.10), which corresponds to [25] for hydrostatic approximation of the Navier-Stokes system in a thin strip with analytic data in the tangential variable. Compared with [25], here our initial data belongs to the Gevrey class 2.

**Theorem 1.3.** Let \( a > 0 \) and \((u_0, v_0)\) satisfy (1.18). Let \((u_0, u_1)\) verify (1.12), (1.16), \( \mathcal{S}_1^1(u_0, u_1) < \infty \), and the compatibility condition

\[
\int_0^1 u_0 dy = \int_0^1 u_1 dy = 0.
\]

Let \((u^\varepsilon, v^\varepsilon)\) and \( u \) be the global solutions to the system (1.8) and (1.10) obtained, respectively, in Theorems 1.1 and 1.2. Then we have

\[
E_R^1(u_0^\varepsilon, w_0^\varepsilon)(t) \leq C(\mathcal{S}_1^1(u_0^\varepsilon, w_0^\varepsilon) + M\varepsilon). \tag{1.22}
\]

Here, the energy functionals \( E_R^1(u_0^\varepsilon, w_0^\varepsilon)(t) \) and \( \mathcal{S}_1^1(u_0^\varepsilon, w_0^\varepsilon) \) are determined, respectively, by (1.20) and (1.21),

\[
w_0^\varepsilon \overset{\text{def}}{=} u^\varepsilon - u, \quad w_0^\varepsilon \overset{\text{def}}{=} v^\varepsilon - v,
\]

and \( v \) is determined from \( u \) via \( \partial_x u + \partial_y v = 0 \) and \( v|_{y=0} = v|_{y=1} = 0 \).
Remark 1.3. The similar convergence result from the hyperbolic Navier-Stokes system to hyperbolic hydrostatic equations has been rigorously justified in the analytic functional framework by Aarach [1].

The rest of this paper is organized as follows. In Section 2, we prove the global well-posedness of the system (1.10) with initial data in the Gevrey class 2 for the tangential variable. Let us point out that the multiple time-weighted Chemin-Lerner norms play an important role. These ideas are common to the proofs of the three theorems in this paper. Section 3 is devoted to the proof of the propagation of regularity for \( \partial_y u_\Phi \), which gives rise to additional difficulties in the estimate of the pressure function compared with that in [25]. Section 4 is devoted to the proof of the global well-posedness of the system (1.8), namely the proof of Theorem 1.2. Finally, we present the proof of Theorem 1.3 in Section 5. In Appendix A, we shall collect some basic tools on the functional framework used in this paper.

We end Section 1 by the notations that we shall use in this paper. For \( a \lesssim b \), we mean that there is a uniform constant \( C \), which may be different on different lines, such that \( a \leq Cb \). We denote by \( (a \mid b)_{L^2} \) the \( L^2(S) \) inner product of \( a \) and \( b \). We designate by \( L^p([0, T]; L^q_x; L^r_y) \) the space \( L^p([0, T]; L^q_x; L^r_y) \). Finally, we denote by \((d_k)_{k \in \mathbb{Z}}\) (resp. \((d_k(t))_{k \in \mathbb{Z}}\)) a generic element of \( l^1(\mathbb{Z}) \) so that \( \sum_{k \in \mathbb{Z}} d_k = 1 \) (resp. \( \sum_{k \in \mathbb{Z}} d_k(t) = 1 \)).

2 Global well-posedness of the system (1.10)

In this section, we study the global well-posedness of the hyperbolic Prandtl equations (1.10) with small Gevrey class 2 data, i.e., we are going to present the proof of the parts (1) and (2) of Theorem 1.1.

As in [9, 10, 24–27, 33], especially motivated by [32], we define

\[
u_\Phi(t, x, y) = F_{\xi \rightarrow x}^{-1}(e^{\Phi(t, \xi)} \tilde{u}(t, \xi, y)) \quad \text{with} \quad \Phi(t, \xi) \overset{\text{def}}{=} (a - \lambda \theta(t))|\xi|^\frac{1}{2},
\]

(2.1)

where the quantity \( \theta(t) \) describes the evolution of the loss of Gevrey radius of \( u \), which is determined by

\[
\dot{\theta}(t) = \delta^2 e^{-\frac{\theta}{2}t} \quad \text{with} \quad \theta|_{t=0} = 0 \quad \text{and} \quad \delta \overset{\text{def}}{=} \left(\frac{aR}{4\lambda}\right)^2,
\]

(2.2)

where \( \lambda \) is a large enough constant to be determined later on. Indeed, we observe from (2.2) that

\[
\theta(t) = \frac{2\delta^2}{\lambda}(1 - e^{-\frac{\theta}{2}t})
\]

so that for all \( t \in \mathbb{R}^+ \),

\[
a - \lambda \theta(t) = \frac{a}{2}(1 + e^{-\frac{\theta}{2}t}) > \frac{a}{2}.
\]

Hence by virtue of (2.1), \( \Phi(t, \xi) \) verifies the following convex inequality:

\[
\Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta), \quad \forall t \in \mathbb{R}^+ \quad \text{and} \quad \xi, \eta \in \mathbb{R}.
\]

(2.3)

Proof of Theorem 1.1. Part (1) For simplicity, we just present the a priori estimates for smooth enough solutions to (1.10). Indeed in view of (1.10) and (2.1), we observe that \( u_\Phi \) verifies

\[
\partial_t (\partial_t u)_\Phi + \lambda \dot{\theta}(t) |D_x|^\frac{1}{2} (\partial_t u)_\Phi + (\partial_t u)_\Phi + (\partial_y u)_\Phi + (\partial_y \partial_x u)_\Phi - \partial_y^2 u_\Phi + \partial_y p_\Phi = 0,
\]

(2.4)

where \( |D_x|^\alpha \) denotes the Fourier multiplier in the horizontal variable with symbol \( |\xi|^\alpha \).

By applying \( \Delta_k^b \) to (2.4) and taking the \( L^2 \) inner product of the resulting equation with \( \Delta_k^b (\partial_t u)_\Phi \), we find

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_k^b (\partial_t u)_\Phi(t)\|_{L^2}^2 + \lambda \dot{\theta} \| D_x |^\frac{1}{2} \Delta_k^b (\partial_t u)_\Phi \|_{L^2}^2 + \| \Delta_k^b (\partial_t u)_\Phi \|_{L^2}^2 - \| \Delta_k^b (\partial_y u)_\Phi \|_{L^2}^2 - \| \Delta_k^b (\partial_y \partial_x u)_\Phi \|_{L^2}^2 + \| \Delta_k^b (\partial_y \partial_y u)_\Phi \|_{L^2}^2
\]

\[
= -\langle \Delta_k^b (\partial_y \partial_x u)_\Phi \mid \Delta_k^b (\partial_t u)_\Phi \rangle_{L^2} - \langle \Delta_k^b (\partial_y \partial_y u)_\Phi \mid \Delta_k^b (\partial_t u)_\Phi \rangle_{L^2} - \langle \Delta_k^b (\partial_y \partial_y p)_\Phi \mid \Delta_k^b (\partial_t u)_\Phi \rangle_{L^2} - \langle \Delta_k^b (\partial_y p)_\Phi \mid \Delta_k^b (\partial_t u)_\Phi \rangle_{L^2}.
\]

(2.5)
Thanks to (1.11) and $\partial_t u + \partial_y v = 0$, by using integration by parts, we get
\[
(\Delta_h^k \partial_x p \phi | \Delta_h^k (\partial_t u) \phi)_{L^2} = - (\Delta_h^k p \phi | \Delta_h^k \partial_x (\partial_t u) \phi)_{L^2}
\]
\[
= (\Delta_h^k p \phi | \Delta_h^k (\partial_t v) \phi)_{L^2} = - (\Delta_h^k \partial_y p \phi | \Delta_h^k (\partial_t v) \phi)_{L^2} = 0.
\]

Whereas observing from (2.1) that
\[
(\partial_t u) \phi = \partial_x u \phi + \lambda \dot{\theta} |D_x|^\frac{1}{4} u \phi, \tag{2.6}
\]
on one has
\[
\dot{\theta} ||D_x|^\frac{1}{4} \Delta_h^k (\partial_t u) \phi||^2_{L^2} = \dot{\theta} ||D_x|^\frac{1}{4} \Delta_h^k \partial_x u \phi||^2_{L^2} + \lambda^2 \dot{\theta}^2 |||D_x||^\frac{1}{2} \Delta_h^k u \phi||^2_{L^2}
\]
\[
+ 2 \lambda^2 \dot{\theta}^2 (|D_x|^\frac{1}{4} \Delta_h^k \partial_x u \phi | |D_x|^\frac{3}{4} \Delta_h^k u \phi)_{L^2}
\]
and
\[
(|D_x|^\frac{1}{4} \Delta_h^k \partial_x u \phi(t) | |D_x|^\frac{3}{4} \Delta_h^k u \phi(t))_{L^2} = \frac{1}{2} \frac{d}{dt} ||D_x|^\frac{1}{4} \Delta_h^k u \phi(t) ||^2_{L^2}.
\]

Similarly, we find
\[
-(\Delta_h^k \partial_y^2 u \phi | \Delta_h^k (\partial_t u) \phi)_{L^2} = - (\Delta_h^k \partial_y^2 u \phi | \Delta_h^k (\partial_t u \phi + \lambda \dot{\theta} |D_x|^\frac{1}{4} u \phi))_{L^2}
\]
\[
= \frac{1}{2} \frac{d}{dt} ||\Delta_h^k \partial_y u \phi(t)||^2_{L^2} + \lambda \dot{\theta} ||D_x|^\frac{1}{4} \Delta_h^k \partial_y u \phi||^2_{L^2}.
\]

By inserting the above estimates into (2.5), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} ||\Delta_h^k (\partial_t u \phi(t))||^2_{L^2} + \frac{1}{2} ||\Delta_h^k \partial_y u \phi(t)||^2_{L^2} + \lambda^2 \dot{\theta}^2 |||D_x||^\frac{1}{2} \Delta_h^k u \phi(t)||^2_{L^2} \right)
\]
\[
+ \lambda \dot{\theta} (||D_x||^\frac{1}{4} \Delta_h^k \partial_x u \phi ||^2_{L^2} + ||D_x||^\frac{1}{4} \Delta_h^k \partial_y u \phi ||^2_{L^2})
\]
\[
= - \lambda^2 \dot{\theta}^2 ||D_x||^\frac{1}{4} \Delta_h^k u \phi ||^2_{L^2} + \lambda \dot{\theta} (\Delta_h^k (\partial_t u \phi)_{L^2} - (\Delta_h^k (\partial_y u \phi)_{L^2} - (\Delta_h^k (\partial_x p \phi) | \Delta_h^k u \phi)_{L^2}). \tag{2.7}
\]

On the other hand, by applying $\Delta_h^k$ to (2.4) and then taking the $L^2$ inner product of the resulting equation with $\Delta_h^k u \phi$, we get
\[
(\Delta_h^k \partial_t (\partial_t u) \phi | \Delta_h^k u \phi)_{L^2} + \lambda \dot{\theta} (||D_x||^\frac{1}{4} \Delta_h^k (\partial_t u) \phi | \Delta_h^k u \phi)_{L^2}
\]
\[
+ \lambda \dot{\theta} (\Delta_h^k (\partial_t u \phi) | \Delta_h^k u \phi)_{L^2} - (\Delta_h^k \partial_y^2 u \phi | \Delta_h^k u \phi)_{L^2}
\]
\[
= - \lambda^2 \dot{\theta}^2 ||D_x||^\frac{1}{4} \Delta_h^k u \phi ||^2_{L^2} - (\Delta_h^k (\partial_y u \phi) | \Delta_h^k u \phi)_{L^2} - (\Delta_h^k (\partial_x p \phi) | \Delta_h^k u \phi)_{L^2}. \tag{2.8}
\]

By using integration by parts, we find
\[
(\Delta_h^k \partial_t (\partial_t u) \phi | \Delta_h^k u \phi)_{L^2} = \frac{d}{dt} (\Delta_h^k (\partial_t u) \phi | \Delta_h^k u \phi)_{L^2} - (\Delta_h^k (\partial_t u \phi) | \Delta_h^k u \phi)_{L^2}
\]
\[
= \frac{d}{dt} (\Delta_h^k (\partial_t u \phi) | \Delta_h^k u \phi)_{L^2} - ||\Delta_h^k (\partial_t u \phi)||^2_{L^2} + \lambda \dot{\theta} (\Delta_h^k (\partial_t u \phi) | |D_x||^\frac{1}{4} \Delta_h^k u \phi)_{L^2}
\]
and
\[
(\Delta_h^k (\partial_t u) \phi | |D_x||^\frac{1}{4} \Delta_h^k u \phi)_{L^2} = \frac{1}{2} \frac{d}{dt} |||D_x||^\frac{1}{4} \Delta_h^k u \phi(t)||^2_{L^2} + \lambda \dot{\theta} ||D_x||^\frac{1}{4} \Delta_h^k u \phi ||^2_{L^2}.
\]

Similarly, one has
\[
(\Delta_h^k (\partial_t u) \phi | \Delta_h^k u \phi)_{L^2} = \frac{1}{2} \frac{d}{dt} ||\Delta_h^k u \phi(t)||^2_{L^2} + \lambda \dot{\theta} ||D_x||^\frac{1}{4} \Delta_h^k u \phi ||^2_{L^2}.
By substituting the above estimates into (2.8), we obtain
\[
\frac{d}{dt} \left( \Delta_h^k (\partial_t u \phi)(t) \right)_{L^2} + \lambda \dot{\theta}(t) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \frac{1}{2} \| \Delta_h u \phi(t) \|_{L^2}^2 \\
- \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \lambda (\partial_t u - \dot{\theta}) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \| \Delta_h^k \partial_t u \phi(t) \|_{L^2}^2 + 2 \lambda^2 \hat{\theta}^2 \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 \\
= - (\Delta_h^k (u \partial_x \phi) | \Delta_h^k u \phi(t) )_{L^2} - (\Delta_h^k (\partial_t u \phi) | \Delta_h^k u \phi(t) )_{L^2}.
\]  
(2.9)

By summing up (2.7) with \( \frac{1}{2} \times (2.9) \), we achieve
\[
\frac{d}{dt} \tilde{\mathcal{S}}_0(t) + \frac{\lambda}{2} \| \theta - \tilde{\theta} \| \| D_x \| \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \lambda \| \dot{\theta}(t) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \| \Delta_h^k \partial_t u \phi(t) \|_{L^2}^2 \\
+ \frac{1}{2} \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \| \Delta_h^k \partial_y u \phi(t) \|_{L^2}^2 \\
= - (\Delta_h^k (u \partial_x \phi) | \Delta_h^k (\partial_t u + \frac{1}{2} u) \phi(t) )_{L^2} - (\Delta_h^k (\partial_t u \phi) | \Delta_h^k (\partial_t u + \frac{1}{2} u) \phi(t) )_{L^2},
\]  
(2.10)

where \( \tilde{\mathcal{S}}_0(t) \) is determined by
\[
\tilde{\mathcal{S}}_0(t) = \frac{1}{2} \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \| \Delta_h^k \partial_y u \phi(t) \|_{L^2}^2 + (\Delta_h^k (\partial_t u \phi) | \Delta_h^k u \phi(t) )_{L^2} \\
+ \lambda \| \dot{\theta}(t) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \frac{1}{4} \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \lambda^2 \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2.
\]

Observing that
\[
\frac{1}{3} \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \frac{1}{2} (\Delta_h^k (\partial_t u \phi)(t) | \Delta_h^k u \phi(t) )_{L^2} + \frac{3}{16} \| \Delta_h^k u \phi(t) \|_{L^2}^2 \\
= \frac{1}{3} \left( \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \frac{3}{4} \| \Delta_h^k u \phi(t) \|_{L^2}^2 \right) \geq 0,
\]
we have
\[
\frac{1}{6} \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \frac{1}{2} \| \Delta_h^k \partial_y u \phi(t) \|_{L^2}^2 + \frac{1}{16} \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \frac{\lambda}{2} \| \dot{\theta}(t) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 \\
+ \lambda^2 \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 \\
\leq \tilde{\mathcal{S}}_0(t) \\
\leq \frac{3}{4} \| \Delta_h^k (\partial_t u \phi)(t) \|_{L^2}^2 + \frac{1}{2} \| \Delta_h^k \partial_y u \phi(t) \|_{L^2}^2 \\
+ \frac{1}{2} \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \frac{\lambda}{2} \| \dot{\theta}(t) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \lambda^2 \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2. 
\]  
(2.11)

Due to \( (u \phi, v \phi) \) \( |_{y=0} = (u \phi, v \phi) \) \( |_{y=1} = 0 \), by applying the Poincaré inequality, we get
\[
\| \Delta_h^k u \phi \|_{L^2}^2 \leq K \| \partial_y \Delta_h^k u \phi \|_{L^2}^2.
\]  
(2.12)

Moreover, by virtue of (2.2), one has \( \dot{\theta}(t) = -\frac{\lambda}{4} \dot{\theta}(t) \), and then by (2.12), if we take
\[
\tilde{\mathcal{S}} \triangleq \min \left( \frac{1}{6}, \frac{1}{4(1 + K)} \right),
\]
we find
\[
\lambda \left( \frac{\dot{\theta}(t)}{4} - \frac{\dot{\theta}(t)}{2} \right) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 + \lambda^2 \left( \frac{\dot{\theta}(t)^2}{2} - 2\dot{\theta}(t) \dot{\theta}(t) \right) \| D_x \| \Delta_h^k u \phi(t) \|_{L^2}^2 \\
+ \frac{1}{4} \left( \| \Delta_h^k (\partial_t u \phi \|_{L^2}^2 + \| \Delta_h^k \partial_y u \phi \|_{L^2}^2 - 2\tilde{\mathcal{S}} \tilde{\mathcal{S}}_0(t) \geq 0.
\]  
(2.13)
Then by multiplying (2.10) by $e^{2\lambda t}$ and integrating the resulting inequality over $[0, t]$, we achieve

$$
\mathcal{S}_k(u)(t) \triangleq \frac{1}{6} \left \| e^{2\lambda t} \Delta_k^h(\partial_t u)(0) \right \|_{L^2}^2 + \frac{1}{2} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^2}^2 + \frac{1}{2} \left \| e^{2\lambda t} \Delta_k^0 u_0 \right \|_{L^2}^2
$$

$$
+ \lambda^2 \| \theta^2(t) \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h u_0(t) \right \|_{L^2}^2 + \lambda^2 \| \theta^2(t) \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^0 u_0(t) \right \|_{L^2}^2
$$

$$
+ \lambda L^2 \int_0^t \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h (\partial_t u + 1/2 \dot{\lambda} u) \right \|_{L^2}^2 dt'
$$

$$
+ \lambda L^2 \int_0^t \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_{0,0} \right \|_{L^2}^2 dt'
$$

$$
(2.14)
$$

with $\mathcal{S}_k(u)(t)$ being determined by

$$
\mathcal{S}_k(u)(t) \overset{\text{def}}{=} \frac{1}{6} \left \| e^{2\lambda t} \Delta_k^h (\partial_t u)(0) \right \|_{L^2}^2 + \frac{1}{2} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^2}^2 + \frac{1}{2} \left \| e^{2\lambda t} \Delta_k^0 u_0 \right \|_{L^2}^2
$$

$$
+ \lambda^2 \| \theta^2(t) \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h u_0(t) \right \|_{L^2}^2 + \lambda^2 \| \theta^2(t) \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^0 u_0(t) \right \|_{L^2}^2
$$

$$
+ \lambda \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h \partial \phi u_{0,0} \right \|_{L^2}^2 + \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h \partial \phi u_{0,0} \right \|_{L^2}^2 dt'
$$

$$
+ \lambda L^2 \int_0^t \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h (\partial_t u + 1/2 \dot{\lambda} u) \right \|_{L^2}^2 dt'
$$

$$
+ \lambda L^2 \int_0^t \left \| e^{2\lambda t} \Delta_k^0 \partial \phi u_{0,0} \right \|_{L^2}^2 dt'
$$

(2.15)

In what follows, we shall always assume that $t < T_0^*$ with $T_0^*$ being determined by

$$
T_0^* \overset{\text{def}}{=} \sup \{ t > 0, \| \partial \phi u_{0,0} \|_{L^{B+1/2}} \leq \delta e^{-\lambda t} \}.
$$

(2.16)

The estimates of the last two terms in (2.14) rely on the following two lemmas.

**Lemma 2.1.** For any $s \in ] - \frac{3}{4}, \frac{3}{4} [ $ and $t \leq T_0^*$, it holds that

$$
\int_0^t \left \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h (\partial_t u)(0) \right \|_{L^2}^2 + \left \| e^{2\lambda t} |D_x|^2 \Delta_k^0 u_0 \right \|_{L^2}^2 + \left \| e^{2\lambda t} \Delta_k^0 u_0 \right \|_{L^2}^2 \right \|_{L^{B+1/2}} dt'
$$

$$
\lesssim d_k^2 2^{-2ks} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^{B+1/2}} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^{B+1/2}}
$$

$$
(2.17)
$$

where the time-weighted Chemin-Lerner type norms $\| \|_{L^{B+1/2}}$ and $\| \|_{L^{B+1/2}}$ will be given by Definition A.3. In particular, when $\alpha = u$ or $\alpha = \partial \phi u$, (2.17) holds for any $s > -\frac{1}{4}$. Here and in all that follows, we always denote by $(d_k)_{k \in \mathbb{Z}}$ a generic element of $l^1(\mathbb{Z})$ so that $\sum_{k \in \mathbb{Z}} d_k = 1$.

**Lemma 2.2.** Let us assume that $\| \alpha_{\phi}(t) \|_{B+1/2} \leq \delta e^{-\lambda t}$ for $t \leq T_0^*$. Then for any $s \in ] - \frac{1}{2}, \frac{1}{2} [ $ and $t \leq T_0^*$, it holds that

$$
\int_0^t \left \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h (\partial_t u) \right \|_{L^2}^2 + \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^2}^2 + \left \| e^{2\lambda t} \Delta_k^0 \partial \phi u_0 \right \|_{L^2}^2 \right \|_{L^{B+1/2}} dt'
$$

$$
\lesssim d_k^2 2^{-2ks} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_{0,0} \right \|_{L^{B+1/2}} \left \| e^{2\lambda t} \partial \phi u_{0,0} \right \|_{L^{B+1/2}}
$$

$$
(2.18)
$$

Furthermore, for $t \leq T_{1,a}^*$, $\| u_{\phi}(t) \|_{B1} \leq \delta \frac{\lambda}{4} e^{-\lambda t} \} and $s > -\frac{1}{4}$, one has

$$
\int_0^t \left \| \left \| e^{2\lambda t} |D_x|^2 \Delta_k^h (\partial_t u)(0) \right \|_{L^2}^2 + \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_0 \right \|_{L^2}^2 + \left \| e^{2\lambda t} \Delta_k^0 \partial \phi u_0 \right \|_{L^2}^2 \right \|_{L^{B+1/2}} dt'
$$

$$
\lesssim d_k^2 2^{-2ks} \left \| e^{2\lambda t} \Delta_k^h \partial \phi u_{0,0} \right \|_{L^{B+1/2}} \left \| e^{2\lambda t} \partial \phi u_{0,0} \right \|_{L^{B+1/2}}
$$

$$
(2.19)
$$
We admit the above lemmas for the time being and continue our proof of Theorem 1.1. Indeed, we first deduce from Lemma 2.1 that for \( t \leq T_0^* \) and \( s \in \left] -\frac{1}{4}, \frac{3}{4} \right] \),

\[
\int_0^t \left| (e^{it\Delta} \Delta_h^b u \partial_x u \phi)_{L^2} \right| dt' \leq d_k^2 2^{-2k} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.20}
\]

and

\[
\int_0^t \left| (e^{it\Delta} \Delta_h^b (v \partial_y u) \phi)_{L^2} \right| dt' \leq d_k^2 2^{-2k} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.21}
\]

However, it follows from Lemma 2.2 that for \( t \leq T_0^* \) and \( s \in \left] -\frac{1}{4}, \frac{3}{4} \right] \),

\[
\int_0^t \left| (e^{it\Delta} \Delta_h^b (v \partial_y u) \phi)_{L^2} \right| dt' \leq d_k^2 2^{-2k} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.22}
\]

and

\[
\int_0^t \left| (e^{it\Delta} \Delta_h^b (v \partial_y u) \phi)_{L^2} \right| dt' \leq d_k^2 2^{-2k} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.23}
\]

Without loss of generality, we may assume that \( \lambda \geq 1 \). Then by inserting the above estimates into (2.14), we find that for \( t \leq T_0^* \) and \( s \in \left] -\frac{4}{3}, \frac{1}{3} \right] \),

\[
\mathcal{G}_k(u)(t) \leq \frac{3}{4} \| e^{it\Delta} \frac{1}{2} \Delta_h^b u_1 \|_{L^2} + \frac{1}{2} \| e^{it\Delta} \frac{1}{2} \Delta_h^b u_0 \|_{L^2} + \frac{1}{2} \| e^{it\Delta} \frac{1}{2} \Delta_h^b u_0 \|^2_{L^2} + \lambda^2 \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} + \lambda^2 \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.24}
\]

Then thanks to (2.2), by multiplying the above inequality by \( 2^{2ks} \) and then taking the square root of the resulting inequality, and finally by summing up the resulting ones over \( \mathbb{Z} \), we obtain

\[
E_{s,\lambda}(u(t)) \leq C \| e^{it\Delta} \frac{1}{2} (u_0, u_1, \partial_y u_0) \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} + \sqrt{\lambda} \| e^{it\Delta} u \phi \|_{L^2_{t,x}(B^{s+\frac{3}{4}})} \tag{2.25}
\]

for \( t \leq T_0^* \) and \( s \in \left] -\frac{1}{4}, \frac{3}{4} \right] \), where we used (2.2) so that \( \delta \approx \frac{3}{4} \) and

\[
E_{s,\lambda}(u(t)) \leq C \| e^{it\Delta} (u \phi, \partial_y u \phi, \partial_t u \phi) \|_{L^\infty_{t,x}(B^{s+\frac{3}{4}})} + \sqrt{\lambda} \| e^{it\Delta} u \phi \|_{L^\infty_{t,x}(B^{s+\frac{3}{4}})} \tag{2.26}
\]

Taking \( \lambda = 4C^2 \) in (2.24) leads to

\[
E_{s,\lambda}(u(t)) \leq C \mathcal{G}^0(u_0, u_1) \tag{2.26}
\]
for \( t \leq T^*_0 \) and \( s \in [\frac{1}{4}, \frac{3}{4}] \) and \( \delta^0_\nu(u_0, u_1) \) being determined by (1.12). In particular, we deduce from (1.12) and (2.26) that for \( t \leq T^*_0 \),
\[
E_{\frac{1}{2}, \frac{1}{2}}(t) \leq C\delta^0_\nu(u_0, u_1) \leq Cc_0,
\]
which implies
\[
\|\partial_y u_\Phi(t)\|_{B^0} \leq Cc_0 e^{-\delta t} \leq \frac{\delta}{2} e^{-\delta t}, \quad \forall t \leq T^*_0,
\]
if we take \( c_0 \) in (1.10) to be so small that \( Cc_0 \leq \frac{\delta}{2} \). Then we deduce by a continuous argument that \( T^*_0 \) determined by (2.16) equals \(+\infty\) and (1.13) holds. This completes the proof of the part (1) of Theorem 1.1.

**Part (2)** We also deduce from (2.25) and (2.27) that
\[
a\Re\|e^{\frac{\theta t}{2}} u_\Phi\|_{L^\infty(R^+, B^1)} \leq Cc_0.
\]
By taking \( c_0 \) in (1.10) to be so small that \( Cc_0 \leq \delta \frac{\delta}{2} \), we deduce that
\[
\|u_\Phi(t)\|_{B^1} \leq \delta \frac{\delta}{2} e^{-\delta t}, \quad \forall t \in \mathbb{R}^+.
\]
Thanks to (2.29), by applying (2.19), for any \( s > -\frac{1}{4} \), we get
\[
\int_0^t \left( e^{Rt} \Delta^0_\nu(-\nu \partial_y u) a \cdot \left( e^{Rt} \Delta^0_\nu \left( \partial_y u + \frac{1}{2} u \right) \Phi \right) \right) dt' \leq d_s^2 2^{-2k(t)} \|e^{Rt'} u_\Phi\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})}^2 + \|e^{Rt'} \partial_y u_\Phi\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})}^2 \\
\times (\|e^{Rt'} \partial_t u_\Phi\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})}^2 + \lambda \|e^{Rt'} u_\Phi\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})}^2).
\]
Then along the same lines as the proof of (2.24), we deduce that for any \( s > -\frac{1}{4} \),
\[
E_{s, \lambda}(t) \leq C(\delta^0_\nu(u_0, u_1) + \|e^{Rt'}(u_\Phi, \partial_y u_\Phi, \partial_t u_\Phi)\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})} + \lambda \|e^{Rt'} u_\Phi\|_{L^2_{t, s, 1}(B^{s+\frac{4}{3}})}).
\]
Taking \( \lambda = 4C^2 \) in the above inequality gives rise to
\[
E_{s, \frac{1}{4}}(t) \leq C\delta^0_\nu(u_0, u_1) \quad \text{for} \quad s > -\frac{1}{4},
\]
which implies (1.15). This finishes the proof of the part (2) of Theorem 1.1.

Now let us present the proofs of Lemmas 2.1 and 2.2. Indeed, we observe that it amounts to proving these lemmas for \( \Re = 0 \). Without loss of generality, we may assume that \( \hat{a} \geq 0 \) and \( \hat{b} \geq 0 \) (and make the similar assumption for the proof of the product law in the rest of this paper; one may check [10] for details).

**Proof of Lemma 2.1.** By applying Bony’s decomposition (A.3) in the horizontal variable to \( u \partial_x a \), we first get that
\[
u \partial_x a = T^h_u \partial_x a + T^h_{\partial_x a} u + R^h(u, \partial_x a).
\]
Accordingly, we shall handle the following three terms:

- The estimate of \( \int_0^t (|\Delta^0_\nu(T^h_u \partial_x a)\Phi | \Delta^0_\nu \Phi)_{L^2} |dt'\). Considering the support properties of the Fourier transform of the terms in \( T^h_u \partial_x a \), we find
\[
\int_0^t (|\Delta^0_\nu(T^h_u \partial_x a)\Phi | \Delta^0_\nu \Phi)_{L^2} |dt' \leq \sum_{|k' - k| \leq 4} \int_0^t \|\Delta^0_k \partial_x a\Phi(t)\|_{L^2} \|\Delta^0_k \Phi(t)\|_{L^2} dt'.
\]
However, due to \( u \big|_{y=0} = 0 \), we write
\[
u(t, x, y) = \int_0^y \partial_y u(t, x, y') dy'
\]
so that we deduce from Lemma A.1 and the Poincaré inequality that
\[
\|\Delta_k^h u_\Phi(t)\|_{L^\infty} \lesssim 2^{\frac{k}{2}} \|\Delta_k^h u_\Phi(t)\|_{L^2_k(L^\infty_y)}
\]
\[
\lesssim 2^{\frac{k}{2}} \|\Delta_k^h u_\Phi(t)\|_{L^2_k} \|\partial_k u_\Phi(t)\|_{L^2_k}
\]
\[
\lesssim 2^{\frac{k}{2}} \|\Delta_k^h \partial_k u_\Phi(t)\|_{L^2} \lesssim d_j(t) \|\partial_j u_\Phi(t)\|_{L^2_k}
\]
which implies
\[
\|\Delta_{k-1}^h u_\Phi(t)\|_{L^\infty} \lesssim \|\partial_k u_\Phi(t)\|_{L^2_k}.
\]
This together with (2.2) and (2.16) ensures that for \( t \leq T_0^* \),
\[
\int_0^t |\langle \Delta_k^h (T_{\partial_x} a) \Phi, \Delta_k^h b_\Phi \rangle_{L^2}| dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \|\partial_{k'} u_\Phi(t')\|_{L^2_k} \|\Delta_k^h a_\Phi(t')\|_{L^2} \|\Delta_k^h b_\Phi(t')\|_{L^2} dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \|\partial_{k'} u_\Phi(t')\|_{L^2_k} \|\Delta_k^h a_\Phi(t')\|_{L^2} \|\Delta_k^h b_\Phi(t')\|_{L^2} dt'.
\]
Then by applying Hölder’s inequality and using Definition A.3, we get
\[
\int_0^t |\langle \Delta_k^h (T_{\partial_x} a) \Phi, \Delta_k^h b_\Phi \rangle_{L^2}| dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \left( \int_0^t \|\partial_{k'} u_\Phi(t')\|_{L^2_k}^2 dt' \right)^{\frac{1}{2}} \left( \int_0^t \|\partial_{k'} b_\Phi(t')\|_{L^2_k}^2 dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim d_k 2^{-2 k s} \|a_\Phi\|_{\bar{L}_{r, \delta, t}((B^{\frac{s}{2}} + \frac{1}{4}), 1)} \|b_\Phi\|_{\bar{L}_{r, \delta, t}((B^{\frac{s}{2}} + \frac{1}{4}), 1)} \left( \sum_{|k' - k| \leq 4} d_k 2^{2(k' - k)(s - \frac{1}{4})} \right)
\]
\[
\lesssim d_k^2 2^{-2 k s} \|a_\Phi\|_{\bar{L}_{r, \delta, t}((B^{\frac{s}{2}} + \frac{1}{4}), 1)} \|b_\Phi\|_{\bar{L}_{r, \delta, t}((B^{\frac{s}{2}} + \frac{1}{4}), 1)}.
\]

- The estimate of \( \int_0^t |\langle \Delta_k^h (T_{\partial_x} a) \Phi, \Delta_k^h b_\Phi \rangle_{L^2}| dt' \).

Again considering the support properties of the Fourier transform of the terms in \( T_{\partial_x} a u \) and thanks to (2.30), we deduce that for \( t \leq T_0^* \),
\[
\int_0^t |\langle \Delta_k^h (T_{\partial_x} a) \Phi, \Delta_k^h b_\Phi \rangle_{L^2}| dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_0^t \|\partial_{k'} (T_{\partial_x} a) \Phi(t')\|_{L^2_k(L^\infty_y)} \|\Delta_k^h u_\Phi(t')\|_{L^2_k(L^\infty_y)} \|\Delta_k^h b_\Phi(t')\|_{L^2} dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_0^t \|\partial_{k'} (T_{\partial_x} a) \Phi(t')\|_{L^2_k(L^\infty_y)} \|\partial_k u_\Phi(t')\|_{L^2} \|\Delta_k^h b_\Phi(t')\|_{L^2} dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \left( \int_0^t \|\partial_{k'} (T_{\partial_x} a) \Phi(t')\|_{L^2_k(L^\infty_y)}^2 \|\partial_k u_\Phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_0^t \|\Delta_k^h b_\Phi(t')\|_{L^2}^2 \|\partial_k u_\Phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\]
Yet we observe that from Definition A.3 and $s < \frac{3}{4}$,

$$
\left( \int_0^t \| \Delta_h^b \partial_s a(t') \|_{L^\infty(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \lesssim \sum_{k' \geq k-2} 2^{\frac{3}{2}k} \left( \int_0^t \| \Delta_h^b a(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \\
\lesssim \sum_{k' \geq k-2} d_k 2^{|\frac{3}{2}k|} \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \sum_{k' \geq k-3} d_k 2^{|k'|} \| a(t') \|_{L^2(L^2)}^2 \right)^{\frac{1}{2}} \\
\lesssim d_k 2^{k} \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \sum_{k' \geq k-3} d_k 2^{|k'-k|} (s+\frac{1}{4}) \right). \\
(2.33)
$$

As a result, for $s < \frac{3}{4}$, it comes out that

$$
\int_0^t \| (\Delta_h^b (T^b_{\partial_s a} u) \cdot \Delta_h^b b(t') \|_{L^2} |dt' | \lesssim d_k 2^{-2k} \| a(t') \|_{L^2(L^2)} \| b(t') \|_{L^2(L^2)} \left( \sum_{k' \geq k-3} d_k 2^{|k'-k|} (s+\frac{1}{4}) \right). \\
(2.34)
$$

On the other hand, by a similar derivation of (2.32), we get

$$
\int_0^t \| (\Delta_h^b (T^b_{\partial_s a} u) \cdot \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim \sum_{|k' - k| \leq 4} d_k 2^{-\frac{3}{4}k} \int_0^t \| \Delta_h^b \partial_s a(t') \|_{L^\infty(L^2)} \| \partial_s u(t') \|_{L^2} \| \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim \sum_{|k' - k| \leq 4} d_k 2^{-\frac{3}{4}k} \int_0^t \| \Delta_h^b \partial_s a(t') \|_{L^2(L^2)} \| \partial_s u(t') \|_{L^2} \| \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim \left( \int_0^t \| \partial_s a(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \lesssim \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \int_0^t \| \Delta_h^b b(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}}.
$$

Yet it follows from the derivation of (2.33) that

$$
\left( \int_0^t \| \partial_s a(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \lesssim \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \int_0^t \| \Delta_h^b b(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}}.
$$

Therefore, (2.34) holds for $s = \frac{3}{4}$.

- The estimate of $\int_0^t \| (\Delta_h^b (R^b(u, \partial_s a)) \cdot \Delta_h^b b(t') \|_{L^2} |dt' |$.

Again considering the support properties of the Fourier transform of the terms in $R^b(u, \partial_s a)$, by applying Lemma A.1 and (2.30), we get that for $t \leq T_b^*$,

$$
\int_0^t \| (\Delta_h^b (R^b(u, \partial_s a)) \cdot \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim 2^{\frac{3}{4}} \sum_{k' \geq k-3} \int_0^t \| \Delta_h^b u_a(t') \|_{L^\infty(L^2)} \| \Delta_h^b \partial_s a(t') \|_{L^2} \| \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim 2^{\frac{3}{4}} \sum_{k' \geq k-3} 2^{\frac{3}{4}} \int_0^t \| \partial_s u(t') \|_{L^2} \| \Delta_h^b a(t') \|_{L^2} \| \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim 2^{\frac{3}{4}} \sum_{k' \geq k-3} 2^{\frac{3}{4}} \int_0^t \| \partial_s u(t') \|_{L^2} \| \Delta_h^b a(t') \|_{L^2} \| \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim \left( \int_0^t \| \partial_s a(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \lesssim \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \int_0^t \| \Delta_h^b b(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}}.
$$

Due to $s > -\frac{1}{4}$, by applying Hölder’s inequality and using Definition A.3, we obtain

$$
\int_0^t \| (\Delta_h^b (R^b(u, \partial_s a)) \cdot \Delta_h^b b(t') \|_{L^2} |dt' |
\lesssim 2^{\frac{3}{4}} \sum_{k' \geq k-3} 2^{\frac{3}{4}} \left( \int_0^t \| \Delta_h^b a(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}} \left( \int_0^t \| \Delta_h^b b(t') \|_{L^2(L^2)}^2 \hat{\partial}^3(t') dt' \right)^{\frac{1}{2}}
\lesssim d_k 2^{-2k} \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \sum_{k' \geq k-3} d_k 2^{|k'-k|} (s+\frac{1}{4}) \right)
\lesssim d_k 2^{-2k} \| a(t') \|_{L^2(L^2)} \| a(t') \|_{L^2(L^2)}^2 \left( \sum_{k' \geq k-3} d_k 2^{|k'-k|} (s+\frac{1}{4}) \right).
(2.35)
$$
By summing up the estimates (2.31), (2.34) and (2.35), we conclude the proof of (2.17) for \( s \in \left[ -\frac{1}{4}, \frac{3}{4} \right] \).

On the other hand, we observe from (2.30) that

\[
\| S_{k-1}^h \partial_x u_\Phi(t) \|_{L^\infty} \lesssim 2^k \| \partial_y u_\Phi(t) \|_{B_{\frac{3}{2}}^s},
\]

so that by a similar derivation of (2.32), we get that for any \( s \in \mathbb{R} \),

\[
\int_0^t \left( \| \Delta_k^h (T_{\partial_x^h, \partial_y^h} u_\Phi) \|_{L^2} + \| \Delta_k^h b_\Phi \|_{L^2} \right) dt',
\]

which together with (2.31) and (2.35) ensures that (2.17) holds for any \( s > -\frac{1}{4} \) in the case \( a = u \).

Finally, when \( a = \partial_y u \), by applying (2.30), for any \( s \in \mathbb{R} \) we get

\[
\int_0^t \left( \| \Delta_k^h (T_{\partial_x^h, \partial_y^h} u_\Phi) \|_{L^2} + \| \Delta_k^h b_\Phi \|_{L^2} \right) dt',
\]

which together with (2.31) and (2.35) ensures that (2.17) holds for any \( s > -\frac{1}{4} \) in the case \( a = \partial_y u \). This finishes the proof of Lemma 2.1.

**Proof of Lemma 2.2.** Once again, by applying Bony’s decomposition (A.3) for the horizontal variable to \( va \), we obtain

\[
va = T^h_v a + T^h_v u + R^h (v, a).
\]

Accordingly, we shall handle the following three terms:

- The estimate of \( \int_0^t \| \Delta_k^h (T_{\partial_x^h} a_\Phi) \|_{L^2} dt' \).

We first observe from (2.2) and the assumption \( \| a_\Phi(t) \|_{B_{\frac{3}{2}}^s} \leq \delta e^{-\alpha t} \) for \( t \leq T^* \) that

\[
\int_0^t \left( \| \Delta_k^h (T_{\partial_x^h} a_\Phi) \|_{L^2} + \| \Delta_k^h b_\Phi \|_{L^2} \right) dt',
\]

\[
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \left( \| S_{k-1}^h v_\Phi(t') \|_{L^\infty} + \| \Delta_k^h a_\Phi(t) \|_{L^2} + \| \Delta_k^h b_\Phi(t') \|_{L^2} \right) dt',
\]

\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-k'} \int_0^t \left( \| \Delta_k^h a_\Phi(t') \|_{B_{\frac{3}{2}}^s} \right) dt',
\]

\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-k'} \int_0^t \left( \| \Delta_k^h b_\Phi(t') \|_{L^2} \right) dt'.
\]
Due to $\partial_x u + \partial_y v = 0$ and (1.11), we write

$$v(t, x, y) = -\int_0^t \partial_x u(t, x, y') dy'.$$

Then we deduce from Lemma A.1 that

$$\|\Delta_k^{b} v_0(t)\|_{L^\infty} \leq \int_0^t \|\Delta_k^{b} \partial_x u_0(t, \cdot, y')\|_{L^\infty} dy'$$

$$\lesssim 2^{\frac k 2} \int_0^t \|\Delta_k^{b} u_0(t, \cdot, y')\|_{L^2} dy' \lesssim 2^{\frac k 2} \|\Delta_k^{b} u_0(t)\|_{L^2}, \tag{2.36}$$

from which and $s < \frac 3 4$, we infer

$$\left( \int_0^t \|S_{k-1}^{b} v_0(t')\|_{L^2}^\frac 2 s \hat \theta^3(t') dt' \right)^\frac 1 2 \lesssim \sum_{\ell \leq k - 2} 2^{\frac k 2} \left( \int_0^t \|\Delta_\ell^{b} u_0(t)\|_{L^2}^\frac 2 s \hat \theta^3(t') dt' \right)^\frac 1 2$$

$$\lesssim \sum_{\ell \leq k - 2} d_\ell 2^{\frac \ell 2} \|u_0\|_{L^2_{t, \Delta^\frac 3 4} (B^{s+\frac 3 4})}$$

$$\lesssim d_\ell 2^{k' (\frac 3 4 - s)} \|u_0\|_{L^2_{t, \Delta^\frac 3 4} (B^{s+\frac 3 4})}. \tag{2.37}$$

Consequently, by virtue of Definition A.3, we obtain

$$\int_0^t \|\Delta_k^{b} (T^a_0 v) \|_{L^2} |dt'$$

$$\lesssim \sum_{|k' - k| \leq 4} 2^{-k'} \left( \int_0^t \|S_{k-1}^{b} v_0(t')\|_{L^2} \hat \theta^3(t') dt' \right)^\frac 1 2 \left( \int_0^t \|\Delta_\ell^{b} u_0(t)\|_{L^2} \hat \theta(t') dt' \right)^\frac 1 2$$

$$\lesssim d_\ell 2^{-2ks} \|u_0\|_{L^2_{t, \Delta^\frac 3 4} (B^{s+\frac 3 4})} \|b_0\|_{L^2_{t, \Delta^\frac 3 4} (B^{s+\frac 3 4})}.$$
By applying Lemma A.1 and (2.36), we get
\[
\int_0^t |(\Delta_k^b(R^b(v, a))_\phi | \Delta_k^b b_\phi)_{L^2}| dt' \\
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k-3} 0^t \int \| \Delta_k^b v_\phi(t') \|_{L^2_k(L^2)} \| \Delta_k^b a_\phi(t') \|_{L^2} \| \Delta_k^b b_\phi(t') \|_{L^2} dt'
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k-3} 2^{\frac{k'}{2}} \int_0^t \| \Delta_k^b u_\phi(t') \|_{L^2} \| a_\phi(t') \|_{B^\frac{1}{2}} \| \Delta_k^b b_\phi(t') \|_{L^2} dt'
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k-3} 2^{\frac{k'}{2}} \left( \int_0^t \| \Delta_k^b u_\phi(t') \|_{L^2} \| \dot{\theta}^3(t') dt' \right)^{\frac{1}{2}} \left( \int_0^t \| \Delta_k^b b_\phi(t') \|_{L^2} \| \dot{\theta}(t') dt' \right)^{\frac{1}{2}},
\]
which together with Definition A.3 and \( s > -\frac{1}{2} \) ensures that
\[
\int_0^t |(\Delta_k^b(T^b(v, a))_\phi | \Delta_k^b b_\phi)_{L^2}| dt' \\
\lesssim d_k 2^{-2k\lambda} \| u_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})} \| b_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})} \left( \sum_{k' \geq k-3} d_k 2^{(k-k')(\frac{1}{2}+s)} \right)
\lesssim d_k^2 2^{-2k\lambda} \| u_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})} \| b_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})}. \tag{2.39}
\]
By summing up the above estimates, we arrive at (2.18) for \( s \in [\frac{1}{4}, \frac{3}{2}] \).
On the other hand, we deduce from (2.36) that for \( t \leq T_{k', a} \),
\[
\| S_{k'-1}^b v_\phi(t') \|_{L^\infty} \lesssim 2^{\frac{k'}{2}} \| u_\phi(t') \|_{B^1} \lesssim 2^{\frac{k'}{2}} \| \dot{\theta}(t') \|
\]
from which we deduce that
\[
\int_0^t |(\Delta_k^b(T^b a)_\phi | \Delta_k^b b_\phi)_{L^2}| dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S_{k'-1}^b v_\phi(t') \|_{L^\infty} \| \Delta_k^b a_\phi(t') \|_{L^2} \| \Delta_k^b b_\phi(t') \|_{L^2} dt'
\lesssim \sum_{|k' - k| \leq 4} 2^{\frac{k'}{2}} \left( \int_0^t \| \dot{\theta}(t') \| \| \Delta_k^b a_\phi(t') \|_{L^2} dt' \right)^{\frac{1}{2}} \left( \int_0^t \| \dot{\theta}(t') \| \| \Delta_k^b b_\phi(t') \|_{L^2} dt' \right)^{\frac{1}{2}}
\lesssim d_k^2 2^{-2k\lambda} \| a_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})} \| b_\phi \|_{L^2_{k, \phi}(\mathbb{R}^+ + \frac{2}{3})},
\]
which together with (2.38) and (2.39) ensures (2.19). This completes the proof of Lemma 2.2.

3 Propagation of regularities for \( \partial_y u_\phi \)

The purpose of this section is to present the proof of the part (3) of Theorem 1.1. Due to the boundary conditions for \( v : v|_{y=0} = 0 = v|_{y=1} \), by integrating \( \partial_x u + \partial_y v = 0 \) over \([0, 1] \) with respect to the \( y \) variable, we get
\[
\partial_x \int_0^1 u(t, x, y) dy = 0 \Rightarrow \int_0^1 u(t, x, y) dy \overset{\text{def}}{=} f(t).
\]
Let us determine the pressure \( p \) via
\[
\partial_x p = \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \frac{1}{2} \partial_x \int_0^1 u^2(t, x, y) dy.
\tag{3.1}
\]
So by integrating the equation
\[ \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_x^2 u + \partial_x p = 0 \]
for \( y \in [0, 1] \), we obtain
\[ f''(t) + f(t) = 0, \]
which together with the compatibility conditions that
\[ \int_0^1 u_0(x, y) dy = 0 = \int_0^1 u_1(x, y) dy, \]
i.e., \( f(0) = f'(0) = 0 \) ensures that \( f(t) = 0 \).

**Proof of (1.17).** In what follows, we shall always define
\[ \omega \equiv \partial_x u \]
and assume that \( t < T_t^* \) with \( T_t^* \) being determined by
\[ T_t^* \equiv \sup \{ t > 0, \| \partial_x \omega(t) \|_{L^2} \leq \delta e^{-rt} \}. \tag{3.2} \]

Thanks to \( \partial_x u + \partial_y v = 0 \), by applying \( \partial_y \) to (1.10), we get
\[ \partial_t^2 \omega + \partial_t \omega + u \partial_x \omega + v \partial_y \omega - \partial_x^2 \omega + \partial_x \partial_y p = 0. \]

In view of (2.1), by applying the Fourier multiplier \( e^{\Phi(t, D)} \) to the above equation, we get
\[ \partial_t (\partial_t \omega) + \lambda \tilde{\phi}_x |D_x|^3 (\partial_t \omega) + (\partial_t \omega) \Phi + (u \partial_x \omega) \Phi + (v \partial_y \omega) \Phi - \partial_x \omega + \partial_x \partial_y p = 0. \]

By taking the \( L^2 \) inner product of the above equation with \( (\partial_t \omega) \Phi + \frac{1}{2} \omega \Phi \) and using the fact that \( -\partial_x \omega + \partial_x p \) vanishes on the boundary of \( S \), we get, by using a similar derivation of (2.14), that
\[ \mathcal{G}_k(\omega)(t) \leq \frac{3}{4} \left[ (\partial_t \omega) \| \Delta_k^b (\partial_t \omega) \|_{L^2}^2 + \frac{1}{2} \| \partial_t \Delta_k^b (\partial_t \omega) \|_{L^2}^2 + \frac{1}{2} \| \Delta_k^b \partial_t \omega \|_{L^2}^2 \right] + \left\{ \begin{array}{l}
\left( \partial_t \omega \right) \| \partial_t \omega \|_{L^2}^2, \\
\left( \partial_t \omega \right) \| \partial_t \omega \|_{L^2}^2, \\
\end{array} \right. \tag{3.3} \]
where the functional \( \mathcal{G}_k(\omega)(t) \) is determined by (2.15) and
\[ I_k(t) \equiv \int_0^t \left( e^{\frac{1}{2} \omega} \right) \left. \left( e^{\frac{1}{2} \omega} \right) \right| \left. \left( e^{\frac{1}{2} \omega} \right) \right| dt'. \tag{3.4} \]

It follows from Lemma 2.1 that for any \( s > -\frac{1}{4} \),
\[ \int_0^t \left( e^{\frac{1}{2} \omega} \right) \left. \left( e^{\frac{1}{2} \omega} \right) \right| \left. \left( e^{\frac{1}{2} \omega} \right) \right| dt' \leq d_k^2 2^{-2k} \| e^{\frac{1}{2} \omega} \|_{L^2} \| e^{\frac{1}{2} \omega} \|_{L^2} \| e^{\frac{1}{2} \omega} \|_{L^2} + \lambda \| e^{\frac{1}{2} \omega} \|_{L^2} \| e^{\frac{1}{2} \omega} \|_{L^2}. \]

However, it follows from (2.29), Lemma 2.2 and (3.2) that for \( t \leq T_t^* \) and for any \( s > -\frac{1}{4} \),
\[ \int_0^t \left( e^{\frac{1}{2} \omega} \right) \left. \left( e^{\frac{1}{2} \omega} \right) \right| \left. \left( e^{\frac{1}{2} \omega} \right) \right| dt' \]
\[ \leq d_k 2^{-2ks} \left( \| e^{|R^t}|u \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} + \| e^{R^t}g_\omega |\Phi| \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} \right) \]
\[ \times \left( \| e^{R^t} (\omega_\theta, \partial_\omega \theta) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} + \lambda \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} \right). \]

We claim that
\[ |I_k(t)| \leq C d_k 2^{-2ks} \left( \mathcal{N}_k + \| e^{\frac{3}{2}t}u \|_{L^2_t(B^{s+\frac{1}{2}})}^2 + \| e^{R^t}(\partial_t u) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 \right) \]
\[ + \| e^{R^t} \omega \|_{L^2_t(B^s)}^2 \| e^{R_t}(\partial_\theta \omega) \|_{L^2_{t,\theta^3(t)}(B^s)} + \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^s)} \| e^{R^t}(\partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^s)} \]
\[ + \frac{aRt}{4C} \| e^{\frac{3}{2}t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 + \| e^{R^t} (\omega, \partial_\theta \omega, \partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 \]
\[ + \lambda \| e^{R^t} (u, \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 + \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 , \]
(3.5)

where
\[ \mathcal{N}_k \overset{\text{def}}{=} \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} \omega \|_{B^s} \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} \partial_\omega \omega \|_{B^s} \]
\[ + \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} u \|_{B^s} + \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} \partial_\omega \omega \|_{B^s} \]
(3.6)

The proof of (3.5) involves technicality, which we postpone to Appendix B.

Let us define
\[ \mathcal{L}_k \overset{\text{def}}{=} \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} (u, \partial_\theta \omega_\theta, \partial_\omega \omega_\omega) \|_{B^{s+\frac{1}{2}}}^2 + aRt \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} \omega \|_{B^{s+\frac{1}{2}}}^2 \]
\[ + a^2 \lambda^2 \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} u \|_{B^{s+\frac{1}{2}}}^2 + \| e^{\frac{3}{2}t}D_\theta \frac{1}{2} \omega \|_{B^{s+\frac{1}{2}}}^2 \]
(3.7)

Then by substituting the above estimates into (3.3), we find
\[ \mathcal{O}_k(\omega)(t) \leq C d_k 2^{-2ks} \left( \mathcal{L}_k + \| e^{\frac{3}{2}t}u \|_{L^2_t(B^{s+\frac{1}{2}})}^2 + \| e^{R^t}(\partial_t u) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 \right) \]
\[ + \| e^{R^t} \omega \|_{L^2_t(B^s)}^2 \| e^{R_t}(\partial_\theta \omega) \|_{L^2_{t,\theta^3(t)}(B^s)} + \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^s)} \| e^{R^t}(\partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^s)} \]
\[ + \frac{aRt}{4C} \| e^{\frac{3}{2}t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 + \| e^{R^t} (\omega, \partial_\theta \omega, \partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 \]
\[ + \lambda \| e^{R^t} (u, \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 + \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 , \]
(3.5)

Then thanks to (2.2) and (2.25), by multiplying the above inequality by $2^{2ks}$ and then taking the square root of the resulting inequality, and finally by summing up the inequalities for $k$ over $\mathbb{Z}$, we achieve
\[ E_{s,\lambda}(t)(L) \leq \frac{1}{2} \| e^{R^t}(\partial_\theta \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} + \| e^{R^t}(\partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} \]
\[ + C \| e^{\frac{3}{2}t}u \|_{L^2_t(B^{s+\frac{1}{2}})}^2 + \| e^{R^t}(\partial_t u) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 \]
\[ + \| e^{R^t} (u, \partial_\theta \omega, \partial_\omega \omega) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})}^2 + \sqrt{\lambda} \| e^{R^t} u \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} \]
\[ + \sqrt{\lambda} \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} + \lambda \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} , \]
(3.5)

where we used (2.2) so that $\lambda \frac{1}{2} = \frac{aR}{4}$. By taking $\lambda = 4C^2$, we obtain
\[ E_{s,\lambda}(t)(L) \leq C \| e^{\frac{3}{2}t}u \|_{L^2_t(B^{s+\frac{1}{2}})} + \| e^{R^t}(\partial_t u) \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} \]
\[ + \sqrt{\lambda} \| e^{R^t} u \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \| e^{R^t} \omega \|_{L^2_{t,\theta^3(t)}(B^{s+\frac{1}{2}})} , \]
which together with (2.26) ensures that for $t \leq T^*_1$,
\[ E_{s, \frac{3}{2}}(\omega)(t) \leq C(\mathcal{L}_S + \mathcal{F}_S(u_0, u_1) + \mathcal{F}_{s+\frac{3}{2}}(u_0, u_1)). \]  
(3.8)

In particular, we deduce from (1.16) and (3.8) that
\[ \|e^{rt} \partial_y \omega \Phi(t)\|_{\mathcal{B}_2} \leq E_{s, \frac{3}{2}}(\omega)(t) \leq C(\mathcal{L}_S + \mathcal{F}_S(u_0, u_1) + \mathcal{F}_{s+\frac{3}{2}}(u_0, u_1)) \leq Cc_1 \]
so that for $t \leq T^*_1$, it holds that
\[ \|\partial_y \omega \Phi(t)\|_{\mathcal{B}_2} \leq Cc_1 e^{-rt} \leq \frac{\delta}{2} e^{-rt} \]
for $c_1$ sufficiently small. Then a continuous argument shows that $T^*_1$ defined by (3.2) equals $\infty$. Moreover, (1.17) holds for $E_{\delta}(\partial_y u)(t)$.

Let us turn to the estimate of $\|e^{rt}(\partial_t^2 u)_\Phi\|_{L^2(B^*)}$. Indeed, by applying $\Delta^h_k$ to (2.4) and taking the $L^2$ inner product of the resulting equation with $\Delta^h_k(\partial_t^2 u)_\Phi$, we obtain
\[ \|\Delta^h_k(\partial_t^2 u)_\Phi\|_{L^2} \leq \|\Delta^h_k \partial_y u\|^2_{L^2} + \|\Delta^h_k \partial_t u\|_{L^2} + \|\Delta^h_k u\|_{L^2} + \|\Delta^h_k v\|_{L^2}. \]
By multiplying the above inequality by $2^{2k+2}e^{2rt}$ and then integrating the resulting inequality over $[0, t]$, and finally summing up $k$ over $\mathbb{Z}$, we achieve
\[ \|e^{rt}(\partial_t^2 u)_\Phi\|_{L^2(B^*)} \leq \|e^{rt}(\partial_t u, \partial_y u)\|_{L^2(B^*)} + \|e^{rt}(u \partial_t u, v \partial_t u)\|_{L^2(B^*)}. \]  
(3.9)
Yet it follows from the proofs of Lemmas 2.1 and 2.2 that
\[ \|e^{rt}(u \partial_t u, v \partial_t u)\|_{L^2(B^*)} \lesssim \|e^{rt} u\|_{L^2_{x,y}([0, T])}. \]
By substituting the above estimates, (2.27) and (3.8) into (3.9), we obtain the estimate (1.17) for $\|e^{rt}(\partial_t^2 u)_\Phi\|_{L^2(B^*)}$. This completes the proofs of (1.17) and thus Theorem 1.1. 

4 Global well-posedness of the system (1.8)

In this section, we prove the global well-posedness of the scaled anisotropic hyperbolic Navier-Stokes system (1.8) with small Gevrey class 2 data and establish uniform estimates for such solutions, i.e., we are going to present the proof of Theorem 1.2.

Proof of Theorem 1.2. In the rest of this section, we shall prove that under the assumption of (1.18), the a priori estimate (1.20) holds for smooth enough solutions to (1.8), and neglect the regularization procedure. For simplicity, we shall neglect the superscript $\varepsilon$. Then in view of (1.8) and (2.1), we observe that $(u_\Phi, v_\Phi)$ verifies
\begin{align}
&\partial_t(\partial_t u)_\Phi + \lambda \partial_t D_x \frac{1}{2}(\partial_t u)_\Phi + (\partial_t u)_\Phi + (u \partial_x u)_\Phi + (v \partial_y u)_\Phi \\
&- e^2 \partial_t^2 u_\Phi - \partial_y^2 u_\Phi + \partial_x p_\Phi = 0, \\
&- e^2 (\partial_t(\partial_t v)_\Phi + \lambda \partial_t D_x \frac{1}{2}(\partial_t v)_\Phi + (\partial_t v)_\Phi + (u \partial_x v)_\Phi + (v \partial_y v)_\Phi \\
&- e^2 \partial_t^2 v_\Phi - \partial_y^2 v_\Phi + \partial_x p_\Phi = 0, \\
&\partial_u u_\Phi + \partial_y v_\Phi = 0 \quad \text{for} \quad (t, x, y) \in \mathbb{R}_+ \times S, \\
&(u_\Phi, v_\Phi)|_{y=0} = (u_\Phi, v_\Phi)|_{y=1} = 0. \end{align}  
(4.1)

By applying the dyadic operator $\Delta^h_k$ to (4.1) and then taking the $L^2$ inner product of the resulting equations with $(\Delta^h_k(\partial_t u)_\Phi, \Delta^h_k(\partial_t v)_\Phi)$, we find
\[ \frac{1}{2} \int e^{\frac{1}{2}t} \|\Delta^h_k(\partial_t u, \varepsilon \partial_t v)_\Phi(t)\|^2_{L^2} + \lambda \|D_x \frac{1}{2} \Delta^h_k(\partial_t u, \varepsilon \partial_t v)_\Phi\|^2_{L^2} + \|\Delta^h_k(\partial_t u, \varepsilon \partial_t v)_\Phi\|^2_{L^2} \]
\[-v^2(\Delta_k^h \partial_x^2(u, \varepsilon)v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - (\Delta_k^h \partial_x^2(u, \varepsilon)v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2}
\]
\[-(\Delta_k^h (u \partial_x u) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - (\Delta_k^h (u \partial_x u) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2}
\]
\[-v^2(\Delta_k^h (u \partial_x u) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - \varepsilon^2(\Delta_k^h (v \partial_y v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2}.
\]

where we used the fact that \( \partial_x u \partial_t + \partial_y v \partial_t = 0 \) so that
\[ (\Delta_k^h \nabla p \partial_t | \Delta_k^h (\partial_t u, \partial_t v) \rangle_{L^2} = 0. \]

Here and in all that follows, we always have
\[ (\vec{a}, \vec{b})_{L^2} \overset{\text{def}}{=} \int_S (a_1(x) b_1(x) + a_2(x) b_2(x)) dx \]
for \( \vec{a}(x) = (a_1(x), a_2(x)) \) and \( \vec{b}(x) = (b_1(x), b_2(x)). \)

By similar derivations of those equalities above (2.7), we first get
\[ \lambda^2 \| D_x \|^{\frac{3}{2}} \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 = \lambda^2 \| D_x \|^{\frac{3}{2}} \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^4 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2
\]
and
\[ \lambda^2 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 = \lambda^2 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^4 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2. \]

By inserting the above equalities into (4.2), we obtain
\[ \frac{1}{2} \frac{d}{dt} (\| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + 2 \lambda^2 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^4 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2)
\]
\[ + \varepsilon^2 (\| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2)
\]
\[ + \lambda^2 (\| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^4 (\| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2)
\]
\[ \frac{1}{2} \frac{d}{dt} (\| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^2 (\| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 + \lambda^4 (\| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2)
\]
\[ = - \varepsilon^2 (\| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2 - \varepsilon^2 (\| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2).
\]

On the other hand, by taking the \( L^2 \) inner product of (4.1) with \( \Delta_k^h (u, \varepsilon) \), we get
\[ \frac{d}{dt} \langle \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} + \lambda^2 (\| D_x \|^{\frac{3}{2}} \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2)
\]
\[ + (\Delta_k^h (\partial_t u, \varepsilon \partial_t v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - \varepsilon^2 (\Delta_k^h (\partial_t u, \varepsilon \partial_t v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2}
\]
\[ - (\Delta_k^h (\partial_t u, \varepsilon \partial_t v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - \varepsilon^2 (\Delta_k^h (\partial_t u, \varepsilon \partial_t v) | \partial_t (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2}.
\]

By using integration by parts and (2.6), we find
\[ \langle \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} = \frac{d}{dt} \langle \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \rangle_{L^2} - \| \Delta_k^h (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2
\]
\[ + \lambda^2 \frac{d}{dt} (\| D_x \|^{\frac{3}{2}} \Delta_k^h (u, \varepsilon) \|_{L^2}^2 + \lambda^2 \| \partial_x (\partial_t u, \varepsilon \partial_t v) \|_{L^2}^2).
\]
Let and By substituting the above equalities into (4.4), we write
\[ \frac{d}{dt} \left( \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right)_{L^2} + \frac{1}{2} \left\| \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right\|_{L^2}^2 + \lambda \dot{\theta} \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2.
\]
By substituting the above equalities into (4.4), we write
\[ \frac{d}{dt} \left( \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right)_{L^2} + \frac{1}{2} \left\| \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right\|_{L^2}^2 + \lambda \dot{\theta} \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2.
\]
Then by virtue of (2.11), we get, by using Lemma A.1 and by multiplying (4.6) by \( e^{2\theta t} \) and then
\[ \frac{d}{dt} \left( \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right)_{L^2} + \frac{1}{2} \left\| \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right\|_{L^2}^2 + \lambda \dot{\theta} \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2.
\]
Let us define \( \mathcal{G}_1 (t) \) by
\[ \mathcal{G}_1 (t) \overset{\text{def}}{=} \frac{1}{2} \left\| \frac{d}{dt} \left( \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right) \right\|_{L^2}^2 + \frac{1}{2} \left\| \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right\|_{L^2}^2 + \lambda \dot{\theta} \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2.
\]
Then by summing up (3.4) with \( \frac{1}{2} \times (4.5) \), we get
\[ \frac{d}{dt} \mathcal{G}_1 (t) + \frac{\lambda}{2} (\dot{\theta} - \ddot{\theta}) \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2 + \frac{1}{2} \left\| \Delta_h^k (\partial_t u, \varepsilon \partial_t v) \phi (t) \right\|_{L^2}^2 + \lambda \dot{\theta} \left\| D_x \right\|_1^2 \Delta_h^k (u, \varepsilon v) \phi \left\|_{L^2}^2.
\]
Due to \( (u, v) \vert_{\gamma = 0} = (u, v) \vert_{\gamma = 1} = 0 \), by applying the Poincaré inequality, we have
\[ \left\| \Delta_h^k (u, v) \phi \left\|_{L^2}^2 \leq K \left\| \partial_y \Delta_h^k (u, v) \phi \right\|_{L^2}^2.
\]
Let
\[ \mathcal{R} \overset{\text{def}}{=} \min \left( \frac{1}{6}, \frac{1}{4(1 + K)} \right).
\]
Then by virtue of (2.11), we get, by using Lemma A.1 and by multiplying (4.6) by \( e^{2\mathcal{R} t} \) and then integrating the resulting inequality over \( [0, t] \), that
\[ G_k (u, v) (t) \leq \frac{3}{4} \left\| e^{[D_x] \mathcal{R} t} \Delta_h^k (u, v_1) (0) \right\|_{L^2}^2 + \frac{e^2}{2} \left\| e^{[D_x] \mathcal{R} t} \Delta_h^k (u_0, v_0) \right\|_{L^2}^2.
\]
\[ + \varepsilon^2 \left| \left( e^{\alpha t} \Delta^h_k(u \partial_x v + v \partial_y v) \right) \left( \partial_t v + \frac{1}{2} v \right) \right|_{L^2} \right) \, dt', \]  

(4.8)

where \( G_k(u, v)(t) \) is defined by

\[
G_k(u, v)(t) \overset{\text{def}}{=} \frac{1}{6} \left\| e^{\alpha t} \Delta^h_k(\partial_t u, \varepsilon \partial_t v) \right\|_{L^2}^2 + \frac{2}{2} \left\| e^{\alpha t} \Delta^h_k(\partial_x u, \varepsilon \partial_x v) \right\|_{L^2}^2 + \frac{1}{4} \left\| e^{\alpha t} \Delta^h_k(\partial_y u, \varepsilon \partial_y v) \right\|_{L^2}^2.
\]

Then it follows from Lemma 2.1 that for \( t \leq T_2^* \),

\[ \varepsilon^2 \int_0^t \left| (e^{\alpha t} \Delta^h_k(u \partial_x v) \Phi) + e^{\alpha t} \Delta^h_k(v \partial_y v) \Phi \right|_{L^2} \, dt' \leq d_k^2 2^{-k} \| e^{\alpha t} \varepsilon v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \| e^{\alpha t} \varepsilon v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)}\]

and

\[ \varepsilon^2 \int_0^t \left| (e^{\alpha t} \Delta^h_k(u \partial_x v) \Phi) + e^{\alpha t} \Delta^h_k(v \partial_y v) \Phi \right|_{L^2} \, dt' \leq d_k^2 2^{-k} \| e^{\alpha t} \varepsilon v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \left( \| e^{\alpha t} \varepsilon \partial_t v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} + \lambda \| e^{\alpha t} \varepsilon v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \right).
\]

The estimate of the remaining terms in (4.8) relies on the following lemma, which we admit for the time being.

**Lemma 4.1.** For \( t \leq T_2^* \), it holds that

\[ \varepsilon \int_0^t \left| (e^{\alpha t} \Delta^h_k(v \partial_y v) \Phi) + e^{\alpha t} \Delta^h_k(b \Phi) \right|_{L^2} \, dt' \leq d_k^2 2^{-k} \| e^{\alpha t} (u, v) \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \| e^{\alpha t} b \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)}. \]

(4.11)

We deduce from Lemma 4.1 that for \( t \leq T_2^* \),

\[ \varepsilon^2 \int_0^t \left| \left( e^{\alpha t} \Delta^h_k(v \partial_y v) \Phi \right) + e^{\alpha t} \Delta^h_k \left( \partial_t v + \frac{1}{2} v \right) \right|_{L^2} \, dt' \leq d_k^2 2^{-k} \| e^{\alpha t} (u, v) \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \left( \| e^{\alpha t} \varepsilon (v, \partial_t v) \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} + \lambda \| e^{\alpha t} \varepsilon v \Phi \|_{\tilde{L}^2_{t,t/2}((B_t^*)^2)} \right). \]

Without loss of generality, we may assume that \( \lambda \geq 1 \). Then by inserting the above estimates and
By multiplying the above inequality by $2^k$ and then taking the square root of the resulting inequality, and summing up the inequalities over $\mathbb{Z}$, we find that for $t \leq T_2^*$,

$$
\mathcal{E}_{R,L}^k(u,v) \leq C((\varepsilon \partial_x u, \varepsilon \partial_y u, \varepsilon \partial_t u, \varepsilon \partial_t v)_{\Phi, \mathcal{L}^*_{t,\delta^2(t)}} + \lambda \|e^{Rt'(u,v)\Phi}\|_{\mathcal{L}^*_{t,\delta^2(t)}}),
$$

(4.12)

where we used (2.2) so that $\lambda \delta^2 = \frac{aR}{4}$ and

$$
\mathcal{E}_{R,L}^k(u,v) \overset{\text{def}}{=} \|e^{Rt'(u,v)\Phi}\|_{\mathcal{L}^*_{t,\delta^2(t)}} + \lambda \|e^{Rt'(u,v)\Phi}\|_{\mathcal{L}^*_{t,\delta^2(t)}}.
$$

Taking $\lambda = \max(C^2, 1)$ in the above inequality leads to (1.19) for $t \leq T_2^*$.

In particular, we deduce from (1.18) and (1.19) that

$$
\|\varepsilon \partial_x u, \varepsilon \partial_y u\|_{\mathcal{L}^*_{t,\delta^2(t)}} \leq C(1 + \sqrt{aR} + aR)C_2e^{-Rt} \leq \frac{\delta^2}{2} e^{-Rt} \quad \text{for any } t \leq T_2^*,
$$

(4.13)

if we take $C_1$ in (1.18) to be so small that $C(1 + \sqrt{aR} + aR)C_2 \leq \frac{\delta^2}{2}$. Then we deduce by a continuous argument that $T_2^*$ determined by (4.10) equals $+\infty$ and (1.19) holds for any $t > 0$. This completes the proof of Theorem 1.2. 

Proof of Lemma 4.1. Once again we shall prove (4.11) for $\mathcal{R} = 0$. By applying Bony’s decomposition (A.3) to $v\partial_y v$, we write

$$
v\partial_y v = T^h_v \partial_y v + T^h_{\partial_y v} + R^h(v, \partial_y v).
$$

Let us handle the following three terms:

- The estimate of $\varepsilon \int_0^t \|\Delta_t^h(T^h_{\partial_y v}\Phi) | \Delta_2^h \Phi\|_{\mathcal{L}^2}[dt']$.

Due to $\partial_y v = -\partial_x u$, for $t \leq T_2^*$, one deduces from (2.2) and (4.10) that

$$
\varepsilon \int_0^t \|\Delta_t^h(T^h_{\partial_y v}\Phi) | \Delta_2^h \Phi\|_{\mathcal{L}^2}[dt'] \lesssim \varepsilon \sum_{|k' - k| \leq 4} \int_0^t \|S^h_{k-1} \vphi\|_{\mathcal{L}^\infty} \|\Delta_t^h \partial_y v\|_{\mathcal{L}^2} \|\Delta_2^h \Phi\|_{\mathcal{L}^2}[dt'] \lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{4}} \int_0^t \|S^h_{k-1} \vphi\|_{\mathcal{L}^\infty} \|\varepsilon \partial_x \Phi\|_{\mathcal{L}^2} \|\Delta_2^h \Phi\|_{\mathcal{L}^2}[dt'] \lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{4}} \left( \int_0^t \|S^h_{k-1} \vphi\|_{\mathcal{L}^\infty} \|\partial_x \Phi\|_{\mathcal{L}^\infty} [dt'] \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta_2^h \Phi\|_{\mathcal{L}^2} [dt'] \right)^{\frac{1}{2}}.
$$
from which, (2.37) and Definition A.3, we infer

\[ \varepsilon \int_0^t |(\Delta^h_b(T^h_v \partial_y v) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim d_k^2 2^{-k} \|u_\phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)} \|\Delta^h_b \Delta^h_b \phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)}. \]

- The estimate of \( \int_0^t |(\Delta^h_b(T^h_{\partial_y v}) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \).

Observe that

\[ \int_0^t |(\Delta^h_b(T^h_{\partial_y v}) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1} \partial_x u_\phi(t')\|_{L^\infty} \|\Delta^h_b v_\phi(t)\|_{L^2} \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt', \]

from which, Lemma A.1, (2.30) and (4.10), we deduce that

\[ \int_0^t |(\Delta^h_b(T^h_{\partial_y v}) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \|S_{k'-1} \partial_x u_\phi(t')\|_{L^\infty} \|\Delta^h_b v_\phi(t)\|_{L^2} \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \]

\[ \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \|\partial_y u_\phi(t')\|_{B^{1/2}} \|\Delta^h_b v_\phi(t)\|_{L^2} \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \]

\[ \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \left( \int_0^t \|\Delta^h_b v_\phi(t')\|_{L^2} \, dt' \right)^{1/2} \left( \int_0^t \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \right)^{1/2}. \]

Then thanks to Definition A.3, we achieve

\[ \int_0^t |(\Delta^h_b(T^h_{\partial_y v}) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim d_k^2 2^{-k} \|v_\phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)} \|\Delta^h_b \Delta^h_b \phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)}. \]

- The estimate of \( \int_0^t |(\Delta^h_b(R^h(v, \partial_y v)) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \).

Due to \( \partial_x u + \partial_y v = 0 \), by applying Lemma A.1 and (2.30), we get

\[ \int_0^t |(\Delta^h_b(R^h(v, \partial_y v)) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim 2^{k'} \sum_{k' \geq k-3} \int_0^t \|\Delta^h_b v_\phi(t')\|_{L^2} \|\Delta^h_b \partial_x u_\phi(t')\|_{L^2} \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \]

\[ \lesssim 2^{k'} \sum_{k' \geq k-3} 2^{k'} \int_0^t \|\Delta^h_b v_\phi(t')\|_{L^2} \|\partial_y u_\phi(t')\|_{B^{1/2}} \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \]

\[ \lesssim 2^{k'} \sum_{k' \geq k-3} 2^{k'} \left( \int_0^t \|\Delta^h_b v_\phi(t')\|_{L^2} \, dt' \right)^{1/2} \left( \int_0^t \|\Delta^h_b \Delta^h_b \phi(t')\|_{L^2} \, dt' \right)^{1/2}, \]

which together with Definition A.3 ensures that

\[ \int_0^t |(\Delta^h_b(R^h(v, \partial_y u)) \phi) - \Delta^h_b \Delta^h_b \phi|_{L^2} \, dt' \lesssim d_k^2 2^{-k} \|v_\phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)} \|\Delta^h_b \Delta^h_b \phi\|_{L^2_{\theta \varphi_1 \varphi_2} (B_g^2)}. \]

By summing up the above estimates, we obtain (4.11). This concludes the proof of Lemma 4.1.
5 The convergence to the hyperbolic Prandtl system

In this section, we justify the limit from the scaled anisotropic hyperbolic Navier-Stokes system (1.8) to the hydrostatic hyperbolic Navier-Stokes system (1.10) in a 2-D strip domain. To this end, we introduce

\[ w^1_v \equiv u^v - u, \quad w^2_v \equiv v^v - v, \quad q_v \equiv p^v - p, \]

where \((u^v, v^v, p^v)\) (resp. \((u, v, p)\)) are solutions to the system (1.8) (resp. (1.10)) obtained in Theorem 1.2 (resp. Theorem 1.1). Then \((w^1_v, w^2_v, q_v)\) verifies

\[
\begin{align*}
\partial_t^2 w^1_v + \partial_t w^2_v - \varepsilon^2 \partial_y^2 w^1_v - \partial_y q_v &= R^1_v \quad \text{in} \ S \times ]0, \infty[,

\varepsilon^2 (\partial_t^2 w^2_v + \partial_t w^2_v) - \varepsilon^2 \partial_y^2 w^2_v + \partial_y q_v &= R^2_v,

\partial_x w^1_v + \partial_y w^2_v &= 0,

(w^1_v, w^2_v)|_{y=0} = (w^1_v, w^2_v)|_{y=1} = 0,

(w^1_v, w^2_v)|_{t=0} = (u^0_v - u_0, v^0_v - v_0, \partial_t w^1_v, \partial_t w^2_v)|_{t=0} = (u^1_v - u_1, v^1_v - v_1),
\end{align*}
\]

where \(v_i (i = 0, 1)\) is determined from \(u_i\) via \(\partial_x u_i + \partial_y v_i = 0\) and \(v_i|_{y=0} = v_i|_{y=1} = 0\), and

\[
\begin{align*}
R^1_v &= \varepsilon^2 \partial_y u - (u^v \partial_x u^v - u \partial_x u) - (v^v \partial_y u^v - v \partial_y u),

R^2_v &= -\varepsilon^2 (\partial_t^2 v + \partial_t v - \varepsilon^2 \partial_y^2 v - \partial_y \partial_x v^v + \partial_y \partial_v v^v),
\end{align*}
\]

We now present the proof of Theorem 1.3. In what follows, we shall neglect the subscript \(\varepsilon\) in \((w^1_v, w^2_v)\).

**Proof of Theorem 1.3.** In view of (5.1), we get, by using a similar derivation of (4.8), that

\[
G_k(w^1, w^2)(t) \leq \frac{3}{4} \|\varepsilon\|_{0, D^2_k}^2 \Delta_k^h(w^0_\varepsilon, \varepsilon u_0^\delta)_{L^2_k} + \varepsilon^2 \frac{1}{2} \|\varepsilon\|_{0, D^2_k}^2 \Delta_k^h\partial_x u^0_\varepsilon, \varepsilon u_0^\delta)_{L^2_k} + \frac{1}{2} \|\varepsilon\|_{0, D^2_k}^2 \Delta_k^h(u^0_\varepsilon, \varepsilon u_0^\delta)_{L^2_k} + \frac{1}{2} \|\varepsilon\|_{0, D^2_k}^2 \Delta_k^h(u^0_\varepsilon, \varepsilon u_0^\delta)_{L^2_k} \right)
\]

where the functional \(G_k(w^1, w^2)(t)\) is defined by (4.9).

We now claim that

\[
\begin{align*}
\int_0^t \left( \Delta_k^h R^1_k \right) \left( \partial_t u^1 + \frac{1}{2} w^1 \right)_{L^2_k} \right) dt' &
\end{align*}
\]

and

\[
\int_0^t \left( \Delta_k^h R^2_k \right) \left( \partial_t u^2 + \frac{1}{2} w^2 \right)_{L^2_k} \right) dt' &
\end{align*}
\]

the proof of which will be postponed until we finish the proof of Theorem 1.3.

On the other hand, due to $\mathcal{G}_0(u_0, u_1) < \infty$, we deduce from (1.13) and (1.15) that there exists some positive constant $M$ so that

$$E_2(t) + E_3(u) \leq M \tag{5.6}$$

for the energy functional $E_3(u)$ being determined by (1.14).

Moreover, it follows from (1.17) and Theorem 1.2 that

$$E_2(\partial_p u)(t) + \|e^{At}(\partial_p^2 u)\|_{L^2(B^{\frac{3}{2}})} + E_3^1(w^\varepsilon, v^\varepsilon) \leq M. \tag{5.7}$$

In particular, we deduce from (5.6) that

$$\|u_\Phi\|_{L^2(B^{\frac{3}{2}})} + \|u_\Phi\|_{L^2(B^{\frac{3}{2}})} \leq CM. \tag{5.8}$$

Similarly, we have

$$\|u_\Phi\|_{L^2(B^{\frac{3}{2}})} + \|u_\Phi\|_{L^2(B^{\frac{3}{2}})} \leq CM. \tag{5.9}$$

Let $E_{0, \lambda}(w^1, w^2)$ be given by (4.12). Thanks to (5.6)–(5.9), by substituting (5.4) and (5.5) into (5.3), and taking the square root of the resulting inequality, and then multiplying it by $2^{\frac{1}{2}}$ and summing up the inequalities for $k$ over $\mathbb{Z}$, we obtain

$$E_{0, \lambda}^0(w^1, w^2) \leq C(\mathcal{G}_0(w_0^0, w_0^0) + (M\varepsilon)^{\frac{1}{2}} (1 + \|w_1^1, \partial_w^0 w_1^1, \varepsilon w_1^2, \varepsilon \partial_w^0 w_1^2, \varepsilon \partial_t w_1^2\|_{L^2(B^{\frac{3}{2}})} + M\varepsilon \|w_1^2, \partial_w^0 w_1^2\|_{L^2(B^{\frac{3}{2}})} + \lambda\|w_1^2, \partial_w^0 w_1^2\|_{L^2(B^{\frac{3}{2}})} + \lambda\|w_1^2, \partial_w^0 w_1^2\|_{L^2(B^{\frac{3}{2}})})^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} (M\varepsilon + \|w_1^1, \partial_w^0 w_1^1, \varepsilon w_1^2, \varepsilon \partial_w^0 w_1^2, \varepsilon \partial_t w_1^2\|_{L^2(B^{\frac{3}{2}})} + \lambda\|w_1^2, \partial_w^0 w_1^2\|_{L^2(B^{\frac{3}{2}})})^{\frac{1}{2}} \times (\|w_1^1, \partial_w^0 w_1^1, \varepsilon w_1^2, \varepsilon \partial_w^0 w_1^2, \varepsilon \partial_t w_1^2\|_{L^2(B^{\frac{3}{2}})} + \lambda\|w_1^2, \partial_w^0 w_1^2\|_{L^2(B^{\frac{3}{2}})}),$$

Applying Young’s inequality leads to

$$E_{0, \lambda}^0(w^1, w^2) \leq \frac{1}{2} \|w_1^1, \partial_w^0 w_1^1, \partial_t w_1^1, \varepsilon w_1^2, \varepsilon \partial_w^0 w_1^2, \varepsilon \partial_t w_1^2\|_{L^2(B^{\frac{3}{2}})} + C(\mathcal{G}_0(w_0^0, w_0^0) + M\varepsilon + \|w_1^1, \partial_w^0 w_1^1, \varepsilon w_1^2, \varepsilon \partial_w^0 w_1^2, \varepsilon \partial_t w_1^2\|_{L^2(B^{\frac{3}{2}})},$$

By taking $\lambda = 4C^2$ in (5.10), we achieve

$$E_{0, \lambda}^0(w^1, w^2) \leq C(\mathcal{G}_0(w_0^0, w_0^0) + M\varepsilon),$$

which leads to (1.22). This completes the proof of Theorem 1.3. \hfill $\Box$

Let us now present the proofs of (5.4) and (5.5).

**Proof of (5.4)**. According to (5.2), we write

$$R_1^\varepsilon = \varepsilon^2 \nabla^2 u - (u^\varepsilon \partial_x w^1 + w^1 \partial_x u) - (v^\varepsilon \partial_y w^1 + w^2 \partial_y u).$$

We first observe that

$$\varepsilon^2 \int_0^t \|\Delta^k \nabla^2 u_\Phi \cdot \Delta^k I^1(w^1, \partial_t w^1)\|_{L^2} |dt| \lesssim \varepsilon^2 \psi \|u_\Phi\|_{L^2(B^{\frac{3}{2}})} \|w^1, \partial_t w^1\|_{L^2(B^{\frac{3}{2}})}. \tag{5.11}$$
The estimate of \( \int_0^t \frac{1}{|\Delta_k^h (w^1 \partial_x u^1 + w^1 \partial_x u)|_F} | \Delta_k^h (\partial_t w^1 + \frac{1}{2} w^1)|_F \) \( dt' \).

It follows from (1.19) and Lemma 2.1 that

\[
\int_0^t \left| \phi \left( \Delta_k^h \left( \partial_t w^1 + \frac{1}{2} w^1 \right) \right) \right|_{L^2} \leq d_2^2 2^{-k} \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \| \partial_t w^1 \|_{B^\frac{3}{4}} + \lambda \right) \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \right).
\] (5.12)

While applying Bony’s decomposition (A.3) to \( w^1 \partial_x u \), we get

\( w^1 \partial_x u = T_{\partial_x u}^h \partial_x u + T_{\partial_x u}^h w^1 + R^h (w^1, \partial_x u) \).

Notice that

\[ \| \Delta_k^h \partial_x u \phi (t') \|_{L^\infty (L^2)} \leq d_2 (t) 2^{-\frac{k}{2}} \| \phi \| \| \partial_x u \phi (t') \|_{L^2} \frac{1}{2}. \]

We deduce from (1.13) and (2.2) that

\[
\int_0^t \left| \phi \left( \Delta_k^h \left( T_{\partial_x u}^h \partial_x u \right) \right) \right|_{L^2} \leq \sum_{k' \geq k-3} \int_0^t \| S_{k'-1} \|_{L^\infty (L^2)} \| \Delta_k^h \partial_x u \phi (t') \|_{L^\infty (L^2)} \| \Delta_k^h \partial_x u \phi (t') \|_{L^2} \| \Delta_k^h \partial_x u \phi (t') \|_{L^2} \leq \sum_{k' \geq k-3} \int_0^t \| \Delta_k^h \partial_x u \phi (t') \|_{L^\infty (L^2)} \| \Delta_k^h \partial_x u \phi (t') \|_{L^2} \| \Delta_k^h \partial_x u \phi (t') \|_{L^2} \leq d_2^2 2^{-k} \| \phi \|_{L^\infty (L^2)} \| \partial_x u \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \| \partial_t w^1 \|_{B^\frac{3}{4}} + \lambda \right) \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \right).
\]

While applying Lemma A.1, we get

\[
\int_0^t \left| \phi \left( \Delta_k^h \left( R^h (w^1, \partial_x u) \right) \right) \right|_{L^2} \leq d_2^2 2^{-k} \| \phi \|_{L^\infty (L^2)} \| \partial_x u \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \| \partial_t w^1 \|_{B^\frac{3}{4}} + \lambda \right) \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \right).
\]

The same estimate holds for

\[
\int_0^t \left| \phi \left( \Delta_k^h \left( T_{\partial_x u}^h w^1 \right) \right) \right|_{L^2} \leq d_2^2 2^{-k} \| \phi \|_{L^\infty (L^2)} \| \partial_x w^1 \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \| \partial_t w^1 \|_{B^\frac{3}{4}} + \lambda \right) \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \right).
\]

As a result, it comes out that

\[
\int_0^t \left| \phi \left( \Delta_k^h \left( \partial_t w^1 + \frac{1}{2} w^1 \right) \right) \right|_{L^2} \leq d_2^2 2^{-k} \| \phi \|_{L^\infty (L^2)} \| \partial_x w^1 \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \| \partial_t w^1 \|_{B^\frac{3}{4}} + \lambda \right) \| \phi \|_{L^2} \left( \| w^1 \|_{B^\frac{3}{4}} \right).
\] (5.13)
The same estimate holds for $\mathrm{I}_{136}^\mathrm{P} \Phi$.

We first deduce from (1.13), (1.19) and Lemma 2.2 that

$$
\int_0^t |(\Delta^h_k(w^2 \partial_y w^1))_\Phi | \Delta^h_k(\partial_t w^1 + \frac{1}{2} w^1)_\Phi| dt' \\
\lesssim d_k^2 2^{-k} \|w_k^1\|_{L^2_{\tilde{\tau}, \tilde{\tilde{\omega}}}(B^{\frac{3}{2}})} \left(\|\partial_t w_k^1\|_{L^2_{\tilde{\tau}, \tilde{\tilde{\omega}}}(B^{\frac{3}{2}})} + \lambda \|w_k^1\|_{L^2_{\tilde{\tau}, \tilde{\tilde{\omega}}}(B^{\frac{3}{2}})}\right). \tag{5.14}
$$

Whereas applying Bony’s decomposition (A.3) to $v \partial_y w^1$, we find

$$v \partial_y w^1 = T^h_{\partial_y} w^1 + T^h_{\partial_y} v + R^h(v, \partial_y w^1).$$

It follows from (2.36) that

$$\|S^h_{k-1} v_\Phi(t')\|_{L^\infty} \lesssim \sum_{k' \geq k-4} 2^k \|\Delta^h_{k'} u_\Phi(t')\|_{L^2} \|\Delta^h_{k'} \partial_y u_\Phi(t')\|_{L^2}^\frac{3}{2} \|\Delta^h_{k'} \partial_y u_\Phi(t')\|_{L^2}^\frac{3}{2},$$

from which we infer

$$\int_0^t \left| \Delta^h_k(T^h_{\partial_y} w^1)_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \\
\lesssim \sum_{k' \geq k-4} \int_0^t \|S^h_{k-1} v_\Phi(t')\|_{L^\infty} \|\partial_t w_k^1(t')\|_{L^2} \|\Delta^h_{k'} \partial_y w_k^1(t')\|_{L^2} \|\Delta^h_{k'} \partial_y w_k^1(t')\|_{L^2} \right\| dt' \\
\lesssim d_k^2 2^{-k} \|u_\Phi\|_{L^2(B^2)} \|\Delta^h_{k'} v_\Phi(t')\|_{L^2} \left(\int_0^t \|\partial_t(t')\|_{L^2} \|\Delta^h_{k'} w(t', \partial_t w_k^1(t'))\|_{L^2} \right)^\frac{3}{2} \|\partial_t(t')\|_{L^2} \right\| dt' \\
\lesssim \int_0^t \left| \Delta^h_k(R^h(v, \partial_y w^1))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt'. $$

Thanks to (2.36), by applying Lemma A.1, we get

$$\int_0^t \left| \Delta^h_k(R^h(v, \partial_y w^1))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \\
\lesssim d_k^2 \sum_{k' \geq k-3} \int_0^t \|\Delta^h_{k'} v_\Phi(t')\|_{L^2_{\tilde{\tau}, \tilde{\tilde{\omega}}}(B^{\frac{3}{2}})} \|\Delta^h_{k'} \partial_y w_k^1(t')\|_{L^2} \|\Delta^h_{k'} \partial_y w_k^1(t')\|_{L^2} \|\Delta^h_{k'} \partial_y w_k^1(t')\|_{L^2} \right\| dt' \\
\lesssim \int_0^t \left| \Delta^h_k(T^h_{\partial_y} v_\Phi(t'))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \\
\lesssim \int_0^t \left| \Delta^h_k(T^h_{\partial_y} v_\Phi(t'))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \tag{5.15}
$$

The same estimate holds for

$$\int_0^t \left| \Delta^h_k(T^h_{\partial_y} v_\Phi(t'))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \tag{5.15}$$

As a consequence, we arrive at

$$\int_0^t \left| \Delta^h_k(T^h_{\partial_y} v_\Phi(t'))_\Phi \right| \Delta^h_k \left(\partial_t w^1 + \frac{1}{2} w^1\right)_\Phi| dt' \tag{5.15}$$
The same estimate holds for

\[ \lesssim d_k^2 2^{-k} \| u \|_{L_\infty^2(B^2_\infty)} \| \partial_y w_k \|_{L_2^2(B^2_\infty)} \left( \left\| \left( \frac{1}{2} u_\phi, \partial_t w_k \right) \right\|_2 \right) + \lambda \| w_k \|_{L_2^2(B^{2}_\infty(k))}. \] (5.15)

- The estimate of \( \int_0^t \left( \left| \Delta_k^h \left( w^2 \partial_y u \right) \right|_2 + \left| \Delta_k^h \left( \partial_t w^1 + \frac{1}{2} w^1 \right) \right|_2 \right) dt' \).

By applying Bony’s decomposition (A.3) to \( w^2 \partial_y u \), we write

\[ w^2 \partial_y u = T_w^h \partial_y u + T_w^h w^2 + R^h \left( w^2, \partial_y u \right). \]

In view of (2.37), we have

\[ \left( \int_0^t \left\| \partial_t w_k \right\|_{L_2^2(B^{2}_\infty)} \left\| \partial_y u \right\|_{L_2^2(B^{2}_\infty)} \left\| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right\|_{L_2^2(B^{2}_\infty)} \right) \lesssim d_k^2 2^{-k} \| w_k \|_{L_2^2(B^{2}_\infty(k))}, \]

so that by applying Hölder’s inequality, we get

\[ \int_0^t \left| \Delta_k^h \left( T_w^h \partial_y u \right) \right| \left| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right| \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) dt' \]

\[ \lesssim \sum_{|k'-k| \leq 4} \left( \int_0^t \left\| \partial_t w_k \right\|_{L_2^2(B^{2}_\infty)} \left\| \partial_y u \right\|_{L_2^2(B^{2}_\infty)} \left\| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right\|_{L_2^2(B^{2}_\infty)} dt' \]

\[ \lesssim \sum_{|k'-k| \leq 4} \left( \int_0^t \left\| \partial_t w_k \right\|_{L_2^2(B^{2}_\infty)} \left\| \partial_y u \right\|_{L_2^2(B^{2}_\infty)} \left\| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right\|_{L_2^2(B^{2}_\infty)} dt' \]

Thanks to (2.36), by applying Lemma A.1, we get

\[ \int_0^t \left| \Delta_k^h \left( R^h \left( w^2, \partial_y u \right) \right) \right| \left| \Delta_k^h \left( \partial_t w^1, \partial_t w^1 \right) \right| \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) dt' \]

\[ \lesssim 2^k \sum_{|k' - k| < 3} \int_0^t \left\| \partial_t w_k \right\|_{L_2^2(B^{2}_\infty)} \left\| \partial_y u \right\|_{L_2^2(B^{2}_\infty)} \left\| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right\|_{L_2^2(B^{2}_\infty)} \left\| \Delta_k^h \left( \partial_t w^1, w^1 \right) \right\|_{L_2^2(B^{2}_\infty)} dt' \]

The same estimate holds for

\[ \int_0^t \left| \Delta_k^h \left( T_w^h \partial_y u \right) \right| \left| \Delta_k^h \left( \partial_t w^1, \partial_t w^1 \right) \right| \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) dt' \]

Hence we obtain

\[ \int_0^t \left| \Delta_k^h \left( w^2 \partial_y u \right) \right| \left| \Delta_k^h \left( \partial_t w^1 + \frac{1}{2} w^1 \right) \right| \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) \partial_y u \|_{L_2^2(B^{2}_\infty)} \right) dt' \]

\[ \lesssim d_k^2 2^{-k} \| w_k \|_{L_2^2(B^{2}_\infty(k))} \left\| \left( \left( \frac{1}{2} u_\phi, \partial_t w_k \right) \right\|_2 \right) + \lambda \| w_k \|_{L_2^2(B^{2}_\infty(k))}. \] (5.16)

By summing up (5.11)–(5.16), we conclude the proof of (5.4). \( \square \)
Proof of (5.5). We first observe from $\partial_x u + \partial_y v = 0$ and the Poincaré inequality that
\[
\int_0^t |(\Delta_k^h(\partial_t^2 v) + \Delta_k^h(\partial_t w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim d_k^2 2^{-k} \| (\partial_t^2 u)_{\Phi} \|_{\mathcal{E}_t^1(G^2)} \| (\partial_t w, \partial_t w)_{\Phi} \|_{\mathcal{E}_t^1(G^2)},
\]
\[
\int_0^t |(\Delta_k^h(\partial_t v) + \Delta_k^h(w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim d_k^2 2^{-k} \| (\partial_t u)_{\Phi} \|_{\mathcal{E}_t^1(G^2)} \| (\partial_t w, \partial_t w)_{\Phi} \|_{\mathcal{E}_t^1(G^2)},
\]
\[
\int_0^t |(\Delta_k^h(\partial_t^2 v) + \Delta_k^h(\partial_t w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim d_k^2 2^{-k} \| (\partial_t u)_{\Phi} \|_{\mathcal{E}_t^1(G^2)} \| (\partial_t w, \partial_t w)_{\Phi} \|_{\mathcal{E}_t^1(G^2)},
\]
\[
\int_0^t |(\Delta_k^h(\partial_t v) + \Delta_k^h(w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim d_k^2 2^{-k} \| (\partial_t u)_{\Phi} \|_{\mathcal{E}_t^1(G^2)} \| (\partial_t w, \partial_t w)_{\Phi} \|_{\mathcal{E}_t^1(G^2)}.
\]
\[
(5.17)
\]

\[\|S_{k-1}^h u_\Phi(t')\|_{L^\infty} \lesssim \|u_\Phi(t')\|_{L^2(G^2)} \|\partial_x u_\Phi(t')\|_{L^2(G^2)}\]
and (2.36), we have
\[
\int_0^t |(\Delta_k^h(T^h_{\partial_x}, u)_{\Phi} + \Delta_k^h(w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim \sum_{|k' - k| \leq 4} \int_0^t \|S_{k-1}^h u_\Phi(t')\|_{L^\infty} \|\Delta_k^h u_\Phi(t')\|_{L^2} \|\Delta_k^h(w, \partial_t w)_{\Phi}(t')\|_{L^2} |dt'.
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{2k'} \|u_\Phi\|_{L^2(G^2)} \|\Delta_k^h u_\Phi\|_{L^2(G^2)} \left( \int_0^t \|\Delta_k^h(w, \partial_t w)_{\Phi}(t')\|_{L^2} dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim d_k^2 2^{-k} \|u_\Phi\|_{L^2(G^2)} \|\Delta_k^h u_\Phi\|_{L^2(G^2)} \left( \int_0^t \|\Delta_k^h(w, \partial_t w)_{\Phi}(t')\|_{L^2} dt' \right)^{\frac{1}{2}}.
\]

Thanks to (2.36) again, we find
\[
\|S_{k-1}^h \partial_x v\|_{L^2(G^2)} \lesssim d_k^2 2^{-k} \|u_\Phi\|_{L^2(G^2)},
\]
from which we infer
\[
\int_0^t |(\Delta_k^h(T^h_{\partial_t}, u)_{\Phi} + \Delta_k^h(w, \partial_t w)_{\Phi})|_{L^2} |dt' \lesssim \sum_{|k' - k| \leq 4} \int_0^t \|S_{k-1}^h \partial_x v\|_{L^2(G^2)} \|u_\Phi\|_{L^2(G^2)} \left( \int_0^t \|\Delta_k^h(w, \partial_t w)_{\Phi}(t')\|_{L^2} dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim d_k^2 2^{-k} \|u_\Phi\|_{L^2(G^2)} \|\Delta_k^h u_\Phi\|_{L^2(G^2)} \left( \int_0^t \|\Delta_k^h(w, \partial_t w)_{\Phi}(t')\|_{L^2} dt' \right)^{\frac{1}{2}}.
\]
Along the same lines, we obtain
\[
\int_0^t \left| \left( \Delta^h_k \left( R^h (u^2, \partial_x v) \right)_\Phi - \Delta^h_k (w^2, \partial_t u^2)_\Phi \right)_{L^2} \right| dt' \\
\lesssim 2^{k} \sum_{k' \geq k-3} \int_0^t \left| \left( \Delta^h_{k'} u_{k'}^f (t') \right)_{L^2_{T\mapsto}(L^\infty)} \right| \left| \Delta^h_{k'} \partial_x v_{k'} (t') \right|_{L^2} \left| \left( \Delta^h_{k'} w_{k'}^f (t') \right) \right|_{L^2} dt'
\]
\[
\lesssim 2^{k} \sum_{k' \geq k-3} 2^{k' \epsilon} \left( \frac{\| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)}^\frac{7}{2} }{\Delta^h_{k'} \| \Delta^h_{k'} \phi \|_{L^2_{T\mapsto}}} \right) \left( \int_0^t \left| \dot{\theta} (t') \right| \| \Delta^h_{k'} w_{k'}^f (t') \|_{L^2_{T\mapsto}} dt' \right)^\frac{\frac{1}{2}}{2}
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| u_{k'}^f \|_{L^2_{T\mapsto}} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right).
\]

This gives rise to
\[
\int_0^t \left| \left( \Delta^h_k (v^2 \partial_y v^2)_\Phi - \Delta^h_k (\partial_t u^2 + \frac{1}{2} u^2)^2 \right)_{L^2} \right| dt'
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right),
\]

(5.19)

- The estimate of \( \int_0^t \left| \left( \Delta^h_k (v^2 \partial_y v^2)_\Phi - \Delta^h_k (\partial_t u^2 + \frac{1}{2} u^2)^2 \right)_{L^2} \right| dt' \)

We first note that
\[
v^2 \partial_y v^2 = v \partial_y w^2 + w^2 \partial_y w^2 + v \partial_y v + w^2 \partial_y v.
\]

We deduce from Lemma 4.1 that
\[
\epsilon \int_0^t \left| \left( \Delta^h_k (w^2 \partial_y w^2) \right)_{L^2} \right| dt'
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right)
\]

and
\[
\epsilon \int_0^t \left| \left( \Delta^h_k (v \partial_y v) \right)_{L^2} \right| dt'
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right).
\]

It follows from (5.15) that
\[
\int_0^t \left| \left( \Delta^h_k (v \partial_y w^2) \right)_{L^2} \right| dt'
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right).
\]

(5.13) ensures that
\[
\int_0^t \left| \left( \Delta^h_k (w^2 \partial_x u) \right)_{L^2} \right| dt'
\]
\[
\lesssim d_k^2 2^{-k} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right)^\frac{\frac{1}{2}}{2} \left( \| u_{k'}^f \|_{L^\infty_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_x u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) \left( \left( \| (u_{k'}^f, \partial_t u_{k'}^f) \|_{L^2_{T\mapsto} (B_+^3)} \right) + \lambda \| u_{k'}^f \|_{L^2_{T\mapsto} (B_+^3)} \right)
\]

As a result, it comes out that
\[
\epsilon^2 \int_0^t \left| \left( \Delta^h_k (v^2 \partial_y v^2) \right)_{L^2} \right| dt'
\]
\begin{align}
\lesssim \varepsilon d^2 2^{-k} (\| (u, \varepsilon v, w^1, \varepsilon w^2) \|_{L^2(\partial \Omega)} + \varepsilon \| u \|_{L^1(\Omega)} \| \partial_y w^2 \|_{L^2(\Omega^1)}) \\
\times (\| (w^2, \partial_t w^2) \|_{L^2(\Omega)} + \lambda \| w^2 \|_{L^2(\Omega)}).
\end{align}

(5.20)

Summing up (5.17)–(5.20) gives rise to (5.5). \qed

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Appendix A  Littlewood-Paley theory and the functional framework

In this appendix, we shall collect some basic facts on anisotropic Littlewood-Paley theory, which we have used in this paper. Let us first recall from [3] that

\[ \Delta_h^k a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}(|\xi|))\hat{a}), \quad S_h^k a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}(|\xi|))\hat{a}), \]  

(A.1)

where \( \mathcal{F}a \) and \( \hat{a} \) denote the partial Fourier transform of the distribution \( a \) with respect to the variable \( x \), i.e.,

\[ \hat{a}(\xi, y) = \mathcal{F}_{x \to \xi}(a)(\xi, y), \]

and \( \chi(\tau) \) and \( \varphi(\tau) \) are smooth functions such that

\begin{align*}
\text{Supp } \varphi &\subset \left\{ \tau \in \mathbb{R} \mid \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1, \\
\text{Supp } \chi &\subset \left\{ \tau \in \mathbb{R} \mid |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) \geq 0, \quad \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.
\end{align*}

Definition A.1. Let \( s \in \mathbb{R} \). For \( u \in S_h'(S) \), which means that \( u \) belongs to \( S'(S) \) and satisfies

\[ \lim_{k \to -\infty} \| S_h^k u \|_{L^\infty} = 0, \]

we set

\[ \| u \|_{B^s} \overset{\text{def}}{=} \left\| (2^k \| \Delta_h^k u \|_{L^2})_{k \in \mathbb{Z}} \right\|_{l^1(\mathbb{Z})}. \]

- For \( s \leq \frac{1}{2} \), we define

\[ B^s(S) \overset{\text{def}}{=} \{ u \in S_h'(S) \mid \| u \|_{B^s} < \infty \}. \]

- If \( k \) is a positive integer and \( -\frac{1}{2} + k < s \leq \frac{1}{2} + k \), then we define \( B^s(S) \) as the subset of distributions \( u \) in \( S_h'(S) \) such that \( \partial_x^k u \) belongs to \( B^{s-k}(S) \).

In order to obtain a better description of the regularizing effect of the diffusion equation, we need to use Chemin-Lerner type spaces \( \tilde{L}^1_h(B^s(S)) \) from [11].
In view of (3.1), we write

\[ C \]

Lemma A.1. \[12, 21].

With the following hold

\[ \text{Definition A.2.} \]

Proof of (3.5)

The goal of this appendix is to present the proof of (3.5).

Appendix B  Proof of (3.5)

For simplicity, we define

\[ T_f^h g = \sum_k s_k^h f \Delta_k^h g \quad \text{and} \quad R^h(f, g) = \sum_k \Delta_k^h f \tilde{\Delta}_k^h g \]

with

\[ \tilde{\Delta}_k^h g = \sum_{|k-k'| \leq 1} \Delta_k^h g. \]

Definition A.2. Let \( p \in [1, +\infty] \) and \( T \in [0, +\infty] \). We define \( \tilde{L}_p^\infty(B(S)) \) as the completion of \( C([0, T]; S(S)) \) by the norm

\[
\|a\|_{\tilde{L}_p^\infty(B(S))} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^T \|\Delta_k^h a(t)\|_{L_p^\infty}^p \, dt \right)^{\frac{1}{p}}
\]

with the usual change if \( p = \infty \).

In order to overcome the difficulty that one cannot use the Gronwall type argument in the framework of Chemin-Lerner type spaces, we need to use the time-weighted Chemin-Lerner norm, which was introduced by us in [23].

Definition A.3. Let \( f(t) \in L_1^{\text{loc}}(\mathbb{R}^+) \) be a nonnegative function. We define

\[
\|a\|_{\tilde{L}_p^\infty(B(S))} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^T f(t') \|\Delta_k^h a(t')\|_{L_p^\infty}^p \, dt' \right)^{\frac{1}{p}}.
\]

(A.2)

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [12, 21].

Lemma A.1. Let \( B_h \) be a ball of \( \mathbb{R}_h \), \( C_h \) be a ring of \( \mathbb{R}_h \), \( 1 \leq p_2 \leq p_1 \leq \infty \) and \( 1 \leq q \leq \infty \). Then the following hold:

1. If the support of \( \tilde{a} \) is included in \( 2^kB_h \), then

\[
\|\partial_x^N a\|_{L_{p_1}^1(L^2)} \lesssim 2^{2kN + \frac{N}{p_1} - \frac{1}{p_1}} \|a\|_{L_{p_1}^1(L^2)}.
\]

2. If the support of \( \tilde{a} \) is included in \( 2^kC_h \), then

\[
\|a\|_{L_{q}^1(L^2)} \lesssim 2^{2kN} \|\partial_x^N a\|_{L_{q}^1(L^2)}.
\]

In this context, we constantly use Bony’s decomposition (see [5]) in the horizontal variable

\[
f g = T_f^h g + T_g^h f + R^h(f, g),
\]

(A.3)

Proof of (3.5). For simplicity, we define

\[
\mathfrak{M}(t, x) \overset{\text{def}}{=} \omega(t, x, 1) - \omega(t, x, 0).
\]

In view of (3.1), we write

\[
\begin{align*}
\int_\mathbb{R} e^{\alpha t} \Delta_k^h \partial_x p \cdot e^{\alpha t} \Delta_k^h \left( \partial_x \mathfrak{M} + \frac{1}{2} \mathfrak{M} \right) \phi \, dx
&= \left( \frac{1}{2} - R \right) \int_\mathbb{R} (e^{\alpha t} \Delta_k^h \mathfrak{M})^2 \phi \, dx \\
&\quad + \frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (e^{\alpha t} \Delta_k^h \mathfrak{M})^2 \phi \, dx + \lambda \int_\mathbb{R} (e^{\alpha t} \Delta_k^h |D_x| \mathfrak{M})^2 \phi \, dx
\end{align*}
\]
\[-\frac{1}{2} \int_{\mathbb{R}} \left( e^{rt} \int_0^1 \Delta^h_x \partial_x (u^2) dy \right) \cdot e^{rt} \Delta^h_t \left( \partial_t \Phi + \frac{1}{2} \Phi \right) \, dx.\]

Since we shall not handle the estimate of the term $\partial_t \partial_y \omega \Phi$ and $\partial_y p = 0$, by integrating the above equation over $[0, t]$ and using integration by parts, we get

\[I_k(t) = \left( \frac{1}{2} - \frac{\lambda}{2} \right) \int_0^t \int_{\mathbb{R}} \left( e^{r(t+t')/2} \Delta^h_t \omega \Phi \right)^2 \, dx \, dt' + \frac{\lambda}{2} \int_{\mathbb{R}} \left( \int_0^1 \Delta^h_t \partial_x (u^2) dy \right) \cdot e^{rt} \Delta^h_t \omega \Phi \, dx \]

\[+ \frac{\lambda}{2} \int_{\mathbb{R}} \left( \int_0^1 \Delta^h_t \partial_x (u^2) dy \right) \cdot e^{rt} \Delta^h_t \omega \Phi \, dx \]

\[+ \frac{\lambda}{2} \int_{\mathbb{R}} \left( \int_0^1 \Delta^h_t \partial_x (u^2) dy \right) \cdot e^{rt} \Delta^h_t \omega \Phi \, dx \]

\[= \sum_{i=1}^9 I^i_k. \quad (B.1)\]

Observing that

\[\int_0^1 \Delta^h_t \omega \Phi(t, x, y) \, dy = 0\]

for any fixed $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, thus there exists $Y_k(t, x)$ so that

\[\Delta^h_t \omega \Phi(t, x, Y_k(t, x)) = 0.\]

So we have

\[(\Delta^h_t \omega \Phi(t, x, y))^2 = (\Delta^h_t \omega \Phi(t, x, y))^2 - (\Delta^h_t \omega \Phi(t, x, Y_k(t, x)))^2\]

\[\leq \int_0^1 |\partial_y (\Delta^h_t \omega \Phi(t, x, y))^2 | \, dy \leq 2 \|

from which we infer

\[\|\Delta^h_t \omega \Phi(t)|^2_{L^2(L^\infty)} \leq 2 \|

Thanks to (B.2), we deduce that

\[|I^1_k| \leq \frac{1}{2} \|e^{rt} \Delta^h_t \omega \Phi\|^2_{L^2(L^\infty)} \cdot \|e^{rt} \Delta^h_t \partial_y \omega \Phi\|_{L^2(L^\infty)} \]

\[\lesssim \|e^{rt} \Delta^h_t \omega \Phi\|_{L^2(L^\infty)} \cdot \|e^{rt} \Delta^h_t \partial_y \omega \Phi\|_{L^2(L^\infty)} \]

Along the same lines, one has

\[|I^2_k| \leq \frac{1}{2} \|e^{rt} \Delta^h_t \omega \Phi\|^2_{L^\infty(L^\infty)} \cdot \|e^{rt} \Delta^h_t \partial_y \omega \Phi\|_{L^\infty(L^\infty)} \]

\[\lesssim \|e^{rt} \Delta^h_t \omega \Phi\|_{L^\infty(L^\infty)} \cdot \|e^{rt} \Delta^h_t \partial_y \omega \Phi\|_{L^\infty(L^\infty)} \]

\[\lesssim d^2 \|e^{rt} \omega \Phi\|_{L^\infty(B^* \times \mathbb{R})} \cdot \|e^{rt} \partial_y \omega \Phi\|_{L^\infty(B^* \times \mathbb{R})} \]
and
\[
|I_{2}^{h}| \lesssim d_{1}^{2}2^{-2k\theta}\|\varepsilon^{j[D_{1}]}\omega_{0}\|_{B^{\theta}}\|\varepsilon^{j[D_{1}]}\frac{1}{2}\partial_{y}\omega_{0}\|_{B^{\theta}}.
\]

Furthermore, by a similar derivation of (B.2), we get
\[
|I_{2}^{h}| \lesssim d_{1}^{2}2^{-2k\theta}\|\varepsilon^{R[-\theta]}\omega_{0}\|_{L_{2}(\mathbb{R}^{+}\frac{1}{2})}\|\varepsilon^{R[-\theta]}\partial_{y}\omega_{0}\|_{L_{2}(\mathbb{R}^{+}\frac{1}{2})}.
\]

To handle the estimates of $I_{2}^{\frac{h}{2}}$ to $I_{2}^{h}$, we use Bony’s decomposition (A.3) for $u^{2}$ in the horizontal variable to write
\[
u^{2} = 2T_{u}^{h}u + R^{h}(u, u).
\]

By applying (2.30) and (2.16), we first get
\[
\int_{\mathbb{R}}\left(e^{Rt}\int_{0}^{1}\Delta_{b}^{h}\partial_{y}(T_{u}^{0}u_{0})_{\theta}dy\right) : e^{Rt}\Delta_{b}^{h}\mathbb{M}_{\theta}dx\bigg| \lesssim 2^{k}\sum_{\theta > k' > \theta - 3}\|\tilde{\Delta}_{b}^{h}u_{0}(t)\|_{L^\infty}\|\varepsilon^{R}[\Delta_{b}^{h}u_{0}(t)]\|_{L^{2}}\|\varepsilon^{R}\Delta_{b}^{h}\omega_{0}(t)\|_{L^{2}(L^{2})}.
\]

The same estimate holds for
\[
\int_{\mathbb{R}}\left(e^{Rt}\int_{0}^{1}\Delta_{b}^{h}\partial_{y}(T_{u}^{0}u_{0})_{\theta}dy\right) : e^{Rt}\Delta_{b}^{h}\mathbb{M}_{\theta}dx.
\]

Hence, we obtain
\[
|I_{2}^{h}| \lesssim d_{1}^{2}2^{-2k\theta}\|\varepsilon^{R[-\theta]}u_{0}\|_{L^{2}(\mathbb{R}^{+}\frac{1}{2})}\|\varepsilon^{R[-\theta]}\omega_{0}\|_{L^{2}(\mathbb{R}^{+}\frac{1}{2})}\|\varepsilon^{R\theta}\partial_{y}\omega_{0}\|_{L^{2}(\mathbb{R}^{+})}.
\]

While applying the product of the local Besov spaces and (B.2), we have
\[
|I_{2}^{h}| \lesssim \|\varepsilon^{j[D_{1}]}\frac{1}{2}\partial_{y}\Delta_{b}^{h}(u_{0}^{0})\|_{L^{2}}\|\varepsilon^{j[D_{1}]}\frac{1}{2}\partial_{y}\Delta_{b}^{h}\omega_{0}\|_{L^{2}(L^{2})}.
\]

Similarly by applying Lemma A.1, (2.30) and (2.16), we find
\[
\int_{0}^{t}\hat{\theta}(t')\int_{\mathbb{R}}\left(e^{Rt}\int_{0}^{1}\Delta_{b}^{h}\partial_{y}(R^{h}(u, u))_{\theta}dy\right) : e^{Rt}|D_{x}^{\frac{1}{2}}\Delta_{b}^{h}\mathbb{M}_{\theta}dxdt'\bigg| \lesssim 2^{2k}\sum_{\theta > k' > \theta - 3}\int_{0}^{t}\hat{\theta}(t')\|\Delta_{b}^{h}u_{0}(t')\|_{L^{2}(L^{2})}\|\varepsilon^{R[t]}\Delta_{b}^{h}u_{0}(t')\|_{L^{2}}\|\varepsilon^{R[t]}|D_{x}^{\frac{1}{2}}\Delta_{b}^{h}\omega_{0}\|_{L^{2}(L^{2})}dt'.
\]

from which and (B.2), we deduce that
\[
\int_{0}^{t}\hat{\theta}(t')\int_{\mathbb{R}}\left(e^{Rt}\int_{0}^{1}\Delta_{b}^{h}\partial_{y}(R^{h}(u, u))_{\theta}dy\right) : e^{Rt}|D_{x}^{\frac{1}{2}}\Delta_{b}^{h}\mathbb{M}_{\theta}dxdt'\bigg| \lesssim 2^{2k}\sum_{\theta > k' > \theta - 3}\int_{0}^{t}\hat{\theta}(t')\|\varepsilon^{R[t]}\Delta_{b}^{h}u_{0}(t')\|_{L^{2}}\|\varepsilon^{R[t]}\Delta_{b}^{h}\omega_{0}\|_{L^{2}(L^{2})}dt'.
\]
\[\lesssim 2^{2k} \sum_{k' \geq k-3} 2^{-\frac{k'}{2}} \left( \int_0^t \dot{\theta} \left( t' \right) \| e^{Rt'} \Delta_k^h u_\Phi \left( t' \right) \|^2_{L^2 dt'} \right)^{\frac{1}{2}} \times \left( \int_0^t \dot{\theta} \left( t' \right) \| e^{Rt'} \Delta_k^h \partial_y \omega_\Phi \left( t' \right) \|^2_{L^2 dt'} \right)^{\frac{1}{2}} \]

\[\lesssim \delta \frac{2}{k} \sum_{k' \geq k-3} d_{k'} \left( \frac{-\frac{k}{2}}{2} \right) \| e^{Rt'} \|_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \partial_y \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}}.\]

The same estimate holds for

\[\int_0^t \dot{\theta} \left( t' \right) \int_R \left( e^{Rt'} \int_0^1 \Delta_k^h \partial_x \left( T^h_{u_\Phi} u \right) \phi \right) \cdot e^{Rt} \| D_x \|_{\frac{1}{2}} \Delta_k^h \| W_\phi \|_{dxdt'}.\]

Therefore, we obtain

\[|I_k^1| \lesssim \lambda \delta \frac{2}{k} \sum_{k' \geq k-3} \| e^{Rt'} u_\Phi \|^2_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \omega_\Phi \|_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \partial_y \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}}.\]

Along the same lines, we have

\[\left| \int_0^t \int_R \left( e^{Rt'} \int_0^1 \Delta_k^h \partial_x \left( R^h \left( u, u \right) \right) \phi \right) \cdot e^{Rt} \| D_x \|_{\frac{1}{2}} \Delta_k^h \| W_\phi \|_{dxdt'} \right| \]

\[\lesssim 2^k \sum_{k' \geq k-3} \int_0^t \left| \Delta_k^h u_\Phi \left( t' \right) \|_{L^\infty} \| e^{Rt'} \Delta_k^h u_\Phi \left( t' \right) \|_{L^2} \| e^{Rt'} \Delta_k^h \omega_\Phi \|_{L^2 \left( L^2 \right)} dt' \right| \]

\[\lesssim 2^k \sum_{k' \geq k-3} \left( \int_0^t \dot{\theta} \left( t' \right) \| e^{Rt'} \Delta_k^h u_\Phi \left( t' \right) \|^2_{L^2 dt'} \right)^{\frac{1}{2}} \times \left( \int_0^t \dot{\theta} \left( t' \right) \| e^{Rt'} \Delta_k^h \partial_y \omega_\Phi \left( t' \right) \|^2_{L^2 dt'} \right)^{\frac{1}{2}} \]

\[\lesssim d^2_{k'} \sum_{k' \geq k-3} \left| e^{Rt'} \right| \| e^{Rt'} u_\Phi \|^2_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \partial_y \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}}.\]

The same estimate holds for

\[\int_0^t \int_R \left( e^{Rt'} \int_0^1 \Delta_k^h \partial_x \left( T^h_{u_\Phi} u \right) \phi \right) \cdot e^{Rt} \| D_x \|_{\frac{1}{2}} \Delta_k^h \| W_\phi \|_{dxdt'} \right|.\]

So it holds that

\[|I_k^1| \lesssim d^2_{k'} \sum_{k' \geq k-3} \left| e^{Rt'} \right| \| e^{Rt'} u_\Phi \|^2_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}} \| e^{Rt'} \partial_y \omega_\Phi \|^\frac{1}{2}_{L^2_{t,\theta^3 \left( B^{s+\frac{1}{2}} \right)}}.\]

Finally to deal with the estimate of $I_k^1$, we use Bony’s decomposition (A.3) for $u_{t_\Phi}$ in the horizontal variable to write

\[u_{t_\Phi} = T^h_{u_{t_\Phi}} + T^h_{u_{t_\Phi}} + R^h \left( u, u_{t_\Phi} \right).\]

We first observe from (2.30) that

\[\left| \int_0^t \int_R \left( e^{Rt'} \int_0^1 \Delta_k^h \partial_x \left( T^h_{u_{t_\Phi}} u \right) \phi \right) \cdot e^{Rt} \| D_x \|_{\frac{1}{2}} \Delta_k^h \| W_\phi \|_{dxdt'} \right|.\]
\[
\leq 2^k \sum_{|k'-k| \leq 4} \int_0^t \| \tilde{S}_{k'}^{1} u \Phi (t') \|_{L^{\infty}} \| \tilde{e} R (\Delta^h (\partial_t u) \Phi) (t') \|_{L^2} \| \tilde{e} R (\Delta^h \omega \Phi) (t') \|_{L^\infty (L^h_{s} \to L^2)} dt' \\
\leq 2^k \sum_{|k'-k| \leq 4} \left( \int_0^t \| \tilde{\varrho}^2 (t') \|_{L^2} \| \tilde{e} R (\Delta^h (\partial_t u) \Phi) (t') \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \| \tilde{\varrho} (t') \|_{L^2} \| \tilde{e} R (\Delta^h \omega \Phi) (t') \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
\leq d_1^2 2^{-2k} \| \tilde{e} R (\partial_t u) \Phi \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\Delta^h \omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})}.
\]

From (1.13) and (2.2), notice that
\[
\left| \int_0^t \int_\Sigma \left( \tilde{e} R \int_0^1 \tilde{\Delta}^h \partial_x (R^h (u, u_t)) \Phi \right) \cdot \tilde{e} R (\Delta^h \omega \Phi) dxdy dt' \right|
\leq 2^k \sum_{|k'-k| \geq 3} \int_0^t \| \tilde{\Delta}^h (\partial_t u) \Phi (t') \|_{L^\infty (L^2)} \| \tilde{e} R (\Delta^h (\partial_t u) \Phi) (t') \|_{L^2} \| \tilde{e} R (\Delta^h \omega \Phi) (t') \|_{L^\infty (L^2)} dt'
\leq 2^k \sum_{k' \geq k-3} \int_0^t \| \tilde{\varrho} (t') \|_{L^2} \| \tilde{e} R (\Delta^h (\partial_t u) \Phi) (t') \|_{L^2} \| \tilde{e} R (\Delta^h \omega \Phi) (t') \|_{L^\infty (L^2)} dt',
\]
from which we deduce by a similar derivation of (B.3) that
\[
\left| \int_0^t \int_\Sigma \left( \tilde{e} R \int_0^1 \tilde{\Delta}^h \partial_x (R^h (u, u_t)) \Phi \right) \cdot \tilde{e} R (\Delta^h \omega \Phi) dxdy dt' \right|
\leq d_1^2 2^{-2k} \| \tilde{e} R (\partial_t u) \Phi \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\Delta^h \omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})}.
\]

The same estimate holds for
\[
\int_0^t \int_\Sigma \left( \tilde{e} R \int_0^1 \tilde{\Delta}^h \partial_x (T^h (u, u_t)) \Phi \right) \cdot \tilde{e} R (\Delta^h \omega \Phi) dxdy dt'.
\]
As a result, it comes out that
\[
\| \tilde{P}_k \| \leq d_1^2 2^{-2k} \| \tilde{e} R (\partial_t u) \Phi \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} + \| \tilde{e} R (\partial_t u) \Phi \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} + \| \tilde{e} R (\partial_t u) \Phi \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})} \| \tilde{e} R (\omega \Phi) \|_{L^2_{t, \theta}(t, \theta) (B^s + \frac{1}{2})}.
\]
By inserting the above estimates into (B.1), we arrive at (3.5).