Equivalence of nonadditive entropies and nonadditive energies in long range interacting systems under macroscopic equilibrium

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We discuss that the thermodynamics of composite systems with non-additive entropies and additive energies can be equivalently derived considering additive entropies and non-additive energies. The general discussion is illustrated by a particular example.

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I. INTRODUCTION

Additivity of quantities such as entropy and energy, is an essential premise of classical thermodynamics and statistical mechanics of systems with short range interactions [1]. Arguments about positivity conditions of specific heat are based on this additivity postulate. On the other hand, systems where long range interactions are relevant or with statistical correlations are basically nonadditive in the energy and/or entropy. Due to this feature, such systems display unusual properties, like inequivalence of the different statistical mechanics ensembles, negative specific heat in microcanonical ensemble and possible temperature discontinuity at first order transitions [2]. For nonadditive systems, the non-concave entropy has been suggested to be the correct entropy of the system, with the result that the specific heat may be negative [3, 4, 5].

Similarly, the notion of macroscopic thermal equilibrium, which introduces the concept of temperature through the zeroth law, is based on additivity of entropy and energy. On the other hand, for large systems with long range interactions or even finite systems with short range interactions, the concept of equal temperatures over subsystems, does not hold at thermal equilibrium (defined as the maximum of total entropy). Clearly, if the additivity postulate is relaxed for either entropy or energy, we have a breakdown of the standard zeroth law.

This paper is divided into three sections. In section II, we present how within a statistical mechanical framework of correlated systems, one can interpret the fact that the standard Boltzmann entropy functional renders non-additive macroscopic entropies and thus a different entropy functional may lead to macroscopic entropy with the additive character. This analysis is done keeping in mind the examples recently presented by C.Tsallis [6].

which justify that for a particular kind of correlations, Tsallis entropy functional does behave additively. In section III, we analyze from a macroscopic thermodynamics point of view, how the equilibrium state (zero principle) of a composite system with non-additive entropy and additive energy functions can be equivalently described by an additive entropy but a non-additive energy. In section IV, we present a particular example of this equivalence. Finally in section V, we conclude that the description of the equilibrium state of a system with a non-Boltzmann entropy with additive properties requires the use of an energy functional which will lead to non-additive energies.

II. STATISTICAL FRAMEWORK

Let us consider two systems $A$ and $B$ with energy levels $\varepsilon_i^A \in \Omega^A$ and $\varepsilon_j^B \in \Omega^B$ whose probability distribution functions are $p_i^A$ and $p_j^B$. Macroscopically, thermodynamics of each of these systems can be developed, using the standard averages and Boltzmann entropy. This is:

$$E^{A(B)} = \langle \varepsilon^{A(B)} \rangle = \sum_{k \in \Omega^{A(B)}} p_k^{A(B)} \varepsilon_k^{A(B)}, \quad (1)$$

and

$$S^{A(B)} = \sum_k p_k^{A(B)} \ln p_k^{A(B)}. \quad (2)$$

Now let us consider the composite system $A + B$ with energy states in $\Omega^A \times \Omega^B$ and assume that its energy levels are given by

$$\varepsilon_{ij}^{A+B} = \varepsilon_i^A + \varepsilon_j^B, \quad (3)$$

but the probability distribution of the composite system exhibits some statistical correlations, so that

$$p_{ij}^{A+B} \neq p_i^A p_j^B. \quad (4)$$

It is clear that for the composite system the mean energy satisfies

$$E^{A+B} = \sum_{ij} p_{ij}^{A+B} (\varepsilon_i^A + \varepsilon_j^B) = E^A + E^B, \quad (5)$$
where we have used the fact that \( p_i^A \) and \( p_j^B \) are the marginal probabilities of \( p_{ij}^{A+B} \). In contrast, it is straightforward to see that the entropy of the composite system is non-additive, i.e.
\[
S^{A+B} \neq S^A + S^B.
\] (6)

Let us now suppose that we can introduce a non-Boltzmann entropy functional of the probability distribution \( S(p_i) \) such that it satisfies additivity for the composite system, i.e.
\[
\tilde{S}^{A+B} = \tilde{S}^A + \tilde{S}^B.
\] (7)

This idea is based on a recent paper by C. Tsallis [4], where for systems with specific correlations, the Tsallis entropy functional is shown to satisfy the additivity relation. The aim of the present paper is to show, using thermodynamic arguments (zero principle), that if one describes the composite system with an additive \( \tilde{S} \), then one should modify the average energies so that they become non-additive
\[
\tilde{E}^{A+B} \neq \tilde{E}^A + \tilde{E}^B.
\] (8)

This means that the “averages” cannot be the standard ones defined in Eq. 1. For instance, in the case of systems with specific correlations for which Tsallis entropy is additive, we conjecture that the new averages would correspond to escort-averages which will render non-additive average energies.

### III. THERMODYNAMIC FRAMEWORK

We consider two thermodynamic systems \( A \) and \( B \) with individual equations of state \( S^A = S^A(E^A) \) and \( S^B = S^B(E^B) \) for which the energy of the composite system \( (A + B) \) is additive
\[
E^{A+B} = E^A + E^B,
\] (9)
but the entropy of the system \( (A + B) \) is non-additive and can be written as:
\[
S^{A+B} = S^A + S^B + F(S^A, S^B).
\] (10)

When the two systems \( A \) and \( B \) are in “thermal contact”, the macroscopic state of the composite system is obtained upon maximization of the total entropy under the constraint of constant energy. Therefore
\[
dE^{A+B} = 0 \Rightarrow dE^A = -dE^B.
\] (11)

The maximum entropy condition \( (dS^{A+B} = 0) \) yields:
\[
0 = \frac{dS^A}{dE^A} dE^A + \frac{dS^B}{dE^B} dE^B + \frac{\partial F}{\partial S^A} \frac{dS^A}{dE^A} dE^A + \frac{\partial F}{\partial S^B} \frac{dS^B}{dE^B} dE^B.
\] (12)

Using (11) one gets the zero principle (or equilibrium) condition for this system:
\[
\left(1 + \frac{\partial F}{\partial S^A}\right) \frac{dS^A}{dE^A} = \left(1 + \frac{\partial F}{\partial S^B}\right) \frac{dS^B}{dE^B}.
\] (13)

Note that if one considers the usual definition of the inverse local temperature of each system \( \alpha (\alpha = A, B) \) as \( \frac{1}{\alpha} = \frac{dS}{dE} \), the two temperatures are not equal in equilibrium but the zero principle still fixes the ratio between them, which will depend on the function \( F(S^A, S^B) \).

Let us now assume that the same two systems can be described assuming that the entropy is additive, while the non-additivity is only on the energy, through a function \( G(E^A, E^B) \) so that:
\[
E^{A+B} = E^A + E^B + G(E^A, E^B),
\] (14)
\[
S^{A+B} = S^A + S^B.
\] (15)

The question is, whether such a description of the joint system will lead to the same zero principle. Imposing again maximization of the composite entropy \( (S^{A+B}) \) under the constraint of constant composite energy \( (E^{A+B}) \), we obtain:
\[
\left(1 + \frac{\partial G}{\partial E^B}\right) \frac{dS^A}{dE^A} = \left(1 + \frac{\partial G}{\partial E^A}\right) \frac{dS^B}{dE^B}.
\] (16)

Now, if we demand that the same zero principle given by (13) is satisfied in both descriptions, then in general, the following conditions should hold:
\[
\left(1 + \frac{\partial G}{\partial E^B}\right) = I(E^A, E^B) \left(1 + \frac{\partial F}{\partial S^A}\right),
\] (17)
\[
1 + \frac{\partial G}{\partial E^A} = I(E^A, E^B) \left(1 + \frac{\partial F}{\partial S^B}\right),
\] (18)
where \( I(E^A, E^B) \) is an unknown differentiable function of its arguments. Clearly, the above conditions imply that Eq. (16) is the same condition as Eq. (13). In the following, we fix the function \( I \) to be a constant equal to unity, by noting the fact that for the case of standard thermodynamics, we have \( F = 0 \), implying that \( G = 0 \). With this choice, Eqs. (17) and (18) lead to two independent conditions:
\[
\frac{\partial G}{\partial E^B} = \frac{\partial F}{\partial S^A},
\] (19)
and
\[
\frac{\partial G}{\partial E^A} = \frac{\partial F}{\partial S^B}.
\] (20)

Are these two conditions enough to find the function \( G \), given a certain function \( F \)? The answer is yes, provided
that \( dG \) is an exact differential. This means that the second cross derivatives of \( G \) must be equal. This renders a condition on the original function \( F(S^A, S^B) \):

\[
\frac{\partial^2 F}{\partial S^A \partial E^A} \frac{dS^A}{dE^A} = \frac{\partial^2 F}{\partial S^B \partial E^B} \frac{dS^B}{dE^B}. \tag{21}
\]

If this condition holds, \( G \) can be obtained by integration of the Eqs. (19) and (20). The formal result is

\[
G(E^A, E^B) = G(E_0^A, E_0^B) + \int_{E_0^A}^{E^A} \frac{\partial F}{\partial S^A} dE^A + \int_{E_0^B}^{E^B} \frac{\partial F}{\partial S^B} dE^B, \tag{22}
\]

which will be independent of the path chosen for integration. In particular, note that for a composite system for which the entropy is \( S^{A+B} = S^A + S^B + \lambda S^A S^B \), the condition (21) is satisfied independently of the state equations of the two subsystems. (The state equations for individual systems are required for integrations in Eq. (22).) Thus, there exists a function \( G \) that will allow a description of the same thermal contact in terms of an additive entropy and a non-additive energy.

**IV. PARTICULAR EXAMPLE**

In this section, we take a specific model for a non-additive entropy and use the general treatment presented so far in order to translate it to the corresponding non-additive energy framework.

Let us assume a general polynomial function \( F(S^A, S^B) \), including terms up to second order

\[
F(S^A, S^B) = a_A S^A + a_B S^B + b_A S^A S^B + b_A (S^A)^2 + b_B (S^B)^2, \tag{23}
\]

and the following equations of state for the individual systems:

\[
S^\alpha(E^\alpha) = c_\alpha E^\alpha + d_\alpha. \tag{24}
\]

Note that from the definition of \( F \), the condition (21) is immediately fulfilled if we require

\[
b_A c_A = b_B c_B \equiv K. \tag{25}\]

To determine \( G \), we shall integrate the partial derivatives:

\[
\frac{\partial G}{\partial E^A} = a_B + b_A S^A(E^A) + 2b_B S^B(E^B), \tag{26}
\]

\[
\frac{\partial G}{\partial E^B} = a_A + b_A S^B(E^B) + 2b_A S^A(E^A). \tag{27}
\]

Integration yields:

\[
G(E^A, E^B) = (h_A - 1) E^A + (h_B - 1) E^B + J_{AB} E^A E^B + \frac{J_A}{2} (E^A)^2 + \frac{J_B}{2} (E^B)^2, \tag{28}
\]

where the constants \( h_A, h_B, J_{AB}, J_A \) and \( J_B \) are given by

\[
h_A = 1 + a_B + b_{AB} d_A + 2 b_B d_B + \sigma_A, \tag{29}
\]

\[
h_B = 1 + a_A + b_{AB} d_B + 2 b_A d_A + \sigma_B, \tag{30}
\]

\[
J_{AB} = 2 K, \tag{31}
\]

\[
J_A = \frac{1}{2} K \frac{b_{AB}}{b_A}, \tag{32}
\]

\[
J_B = \frac{1}{2} K \frac{b_{AB}}{b_B}, \tag{33}
\]

where \( \sigma_A \) and \( \sigma_B \) are integration constants. We note that the energy of the composite system is of the form:

\[
E^{A+B} = h_A E^A + h_B E^B + J_{AB} E^A E^B + \frac{J_A}{2} (E^A)^2 + \frac{J_B}{2} (E^B)^2, \tag{34}
\]

which corresponds to the energy of two coupled paramagnets with long range interactions. This model has been recently studied in depth by J. Oppenheim [8]. It is plausible that such a nonadditive behaviour of the macroscopic energy should arise from a non-standard averaging scheme of the microscopic Hamiltonian, different from Eq. (1).

**V. SUMMARY AND CONCLUSIONS**

In this paper, we have considered the problem of thermal contact between two thermodynamic systems, in order to discuss the interplay between non-additive entropies and non-additive energies to describe the same equilibrium state of the composite system. We show that an equilibrium condition between the two systems, described by set of quantities \( (E^A, S^A) \) and \( (E^B, S^B) \), can be realized in two equivalent ways: treating the entropy of the total system as non-additive, but with the constraint in the form of additive total energy, or alternately, treating the entropy as additive but the constraint in the form of nonadditive total energy. The thermodynamic analysis becomes simplified by assuming that the total entropy is a function of the entropies of the subsystems \( A \) and \( B \) only. Similarly, the total energy is completely determined from the system energies. Thus we get a reinterpretation of the nonadditive entropy rule in terms of a specific nonadditive energy rule. We illustrate this correspondence by taking a specific example and obtain a composite energy which represents the energy of two paramagnets interacting via long range interactions. As a special case, we also observe that Tsallis type nonadditivity [7] corresponds to the case when the individual paramagnetic systems experience long range interactions only within each system, but not between themselves.
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