Subharmonic Shapiro steps and assisted tunneling in superconducting point contacts

J.C. Cuevas1, J. Heurich1, A. Martín-Rodero2, A. Levy Yeyati2 and G. Schön1,3
1 Institut für Theoretische Festkörperphysik, Universität Karlsruhe, 76128 Karlsruhe, Germany
2 Departamento de Física Teórica de la Materia Condensada C-V, Universidad Autónoma de Madrid, E-28049 Madrid, Spain
3 Forschungszentrum Karlsruhe, Institut für Nanotechnologie, D-76021 Karlsruhe, Germany

We analyze the current in a superconducting point contact of arbitrary transmission in the presence of a microwave radiation. The interplay between the ac Josephson current and the microwave signal gives rise to Shapiro steps at voltages \( V = (m/n)\hbar \omega_r/2e \), where \( n, m \) are integer numbers and \( \omega_r \) is the frequency of the radiation. The subharmonic steps \( (n \neq 1) \) are a consequence of multiple Andreev reflections (MAR) and provide a signature of the peculiar ac Josephson effect at high transmission. Moreover, the dc current exhibits a rich subgap structure due to photon-assisted MARs.

Introduction.— Our understanding of the electronic transport through superconducting nanostructures has experienced a notable development in last few years [1]. Partly, this has been due to the appearance on scene of the metallic atomic-size contacts, which can be produced by means of scanning tunneling microscope and break-junction techniques [3,4]. These nanowires have turned out to be ideal systems to test the modern transport theories in mesoscopic superconductors. Thus, for instance Scheer and coworkers [4] found a quantitative agreement between the measurements of the current-voltage characteristics of different atomic contacts and the predictions of the theory for a single-channel superconducting contact [5,6]. These experiments not only helped to clarify the structure of the subgap current in superconducting contacts, but also showed that the set of the transmission coefficients in an atomic-size contact is amenable to measurement. This possibility has recently allowed a set of experiments that confirm the theoretical predictions for transport properties like supercurrent [4] and noise [5]. From these combined theoretical and experimental efforts a coherent picture of transport in superconducting point contacts has emerged with multiple Andreev reflections (MAR) as a central concept. However, in spite of these recent successes, one of the most remarkable predictions of MAR theory remains to be confirmed, namely the ac Josephson effect. The theory says that in a constant voltage biased superconducting point contact, the time-dependent current is given by \( I(t) = \sum_n I_n e^{i\omega_n t} \). This means that the occurrence of MARs gives rise to the appearance of alternating currents that oscillate not only with the Josephson frequency \( \omega_J = 2eV/\hbar \), \( V \) being the voltage, as in the case of tunnel junctions, but also with all its harmonics. So far there is no experimental evidence of the existence of such components.

In this paper, we present a theoretical analysis of the current in a superconducting point contact under a microwave radiation. We show that the interplay between the ac Josephson current components and a microwave signal leads to the appearance of Shapiro steps at voltages \( V = (m/n)\hbar \omega_r/2e \), where \( n, m \) are integer numbers and \( \omega_r \) is the frequency of the radiation. This means that in addition to the usual steps \( (n = 1) \) found in tunnel junctions [4], there also appear subharmonic Shapiro steps \( (n \neq 1) \), which constitute an unambiguous signature of the ac Josephson effect in these contacts. Moreover, we also find that the dc background current, in which the Shapiro steps are superimposed, exhibits a rich subgap structure, which can be understood in terms of photon-assisted MARs and provides a natural explanation of experimental findings in the early seventies [10].

Theoretical model.— Our goal is to calculate the current in a voltage biased superconducting quantum point contact (SQPC) in the presence of a monochromatic radiation of frequency \( \omega_r \). We assume that the external radiation produces an effective time-dependent voltage \( V(t) = V + V_0 \sin \omega_r t \). Our task is to extend the MAR theory to the case of such a time-dependent voltage, for which the so-called Hamiltonian approach [10] is a convenient starting point. For the voltage range \( eV \sim \Delta \) one can neglect the energy dependence of the transmission coefficients and all transport properties can be expressed as a superposition of independent channel contributions. Thus, the problem reduces to the analysis of a single channel contact, which can be described by means of the following tight-binding-like Hamiltonian [7]

\[
\hat{H} = \hat{H}_L + \hat{H}_R + \sum_{\sigma} \left\{ v c_{L\sigma}^\dagger c_{R\sigma} + v^* c_{R\sigma}^\dagger c_{L\sigma} \right\},
\]

where \( H_{L,R} \) are the BCS Hamiltonians for the isolated electrodes. In the coupling term \( L \) and \( R \) stand for the outermost sites of each electrode, and \( v \) is a hopping parameter coupling these sites. This parameter determines the normal transmission coefficient of this model \( T \), which adopts the form \( T = 4(v/W)^2/[1 + (v/W)^2]^2 \), where \( W = 1/\pi \rho_F \), with \( \rho_F \) being the electrodes density of states at the Fermi energy [7].

In this model the current evaluated at the interface between the two electrodes adopts the form

\[
I(t) = \frac{i e}{h} \sum_{\sigma} \left\{ v \langle c_{L\sigma}^\dagger(t)c_{R\sigma}(t) \rangle - v^* \langle c_{R\sigma}^\dagger(t)c_{L\sigma}(t) \rangle \right\}.
\]
The non-equilibrium expectation values in Eq. (2) can be expressed in terms of the Keldysh Green functions $\hat{G}_{i,j}^{\pm}$, which in the $2 \times 2$ Nambu representation read

$$\hat{G}_{i,j}^{\pm}(t,t') = i \left( \langle c_{i,j}^{\dagger}(t) c_{i,j}(t) \rangle - \langle c_{i,j}^{\dagger}(t') c_{i,j}^{\dagger}(t') \rangle \right).$$  \hfill (3)

Thus, the current can be now written as

$$I(t) = \frac{e}{\hbar} \text{Tr} \left[ \hat{\tau}_3 \left( \hat{v}(t) \hat{G}_{LL}^{-}(t,t) - \hat{v}^{\dagger}(t) \hat{G}_{LR}^{+}(t,t) \right) \right],$$  \hfill (4)

where $\hat{\tau}_3$ is the corresponding Pauli matrix, Tr denotes the trace in Nambu space and $\hat{v}$ is the hopping that in the Nambu matrix representation is written as

$$\hat{v}(t) = \begin{pmatrix} \psi e^{i\phi(t)/2} & 0 \\ 0 & -v^* e^{-i\phi(t)/2} \end{pmatrix}.$$  \hfill (5)

Here, $\phi(t) = \phi_0 + \omega_0 t + 2\alpha \cos \omega_0 t$ is the time-dependent superconducting phase difference. The constant $\alpha = eV_{dc} / (\hbar \omega_0)$ measures the strength of the coupling to the electromagnetic field, and is proportional to the square root of the radiation power.

In order to determine the Green functions we follow a perturbative scheme and treat the coupling term in Hamiltonian (1) as a perturbation. The retarded and advanced Green functions, $\hat{g}$, correspond to the uncoupled electrodes in equilibrium. Thus, the retarded and advanced components adopt the BCS form: $\hat{g}^{r,a}(\epsilon) = g^{r,a}(\epsilon) 1 + f^{r,a}(\epsilon) \hat{\tau}_1$, where $g^{r,a}(\epsilon) = - (\epsilon^{r,a} / \Delta) f(\epsilon) = -e^{r,a} / \sqrt{\Delta^2 - (\epsilon^{r,a})^2}$, where $\epsilon^{r,a} = \epsilon \mp i\eta$, with $\eta = 0^+$. Following Ref. [1], we express the current in terms of a T-matrix, rather than in terms of the Green functions. The T-matrix associated to the time-dependent perturbation of Eq. (5) is defined as $T^{r,a} = \hat{v} + \hat{v} \hat{g}^{r,a} \hat{g}^{r,a} \hat{v}$, where the $\circ$ product is a shorthand for integration over intermediate time arguments. As shown in Ref. [1], the current in terms of the T-matrix components reads

$$I(t) = \frac{e}{\hbar} \text{Tr} \left[ \hat{\tau}_3 \left( T_{RL}^{r} \circ \hat{g}^{r} \circ \hat{T}_{RL}^{a} \circ \hat{g}^{a} \circ \hat{T}_{RL}^{r} \circ \hat{g}^{r} \circ \hat{T}_{RL}^{a} \right) \right].$$  \hfill (6)

In order to solve the T-matrix integral equation it is convenient to Fourier transform with respect to the temporal arguments, $\hat{T}(t,t') = (1/2\pi) \int de \int de' e^{-i(e - e')t} \hat{T}(e,e')$. Due to time dependence of the coupling element (see Eq. (5)), one can show that $\hat{T}(e,e')$ admits the following solution: $\hat{T}(e,e') = \sum_{n,m} \hat{T}(\epsilon, \epsilon + neV + m\hbar \omega_0) \delta(e - \epsilon' + neV + m\hbar \omega_0)$. Thus, one can finally write down the current as $I(t) = \sum_{n,m} I_m^m \exp[i(n \phi_0 + n \omega_0 t + m \omega_0 t)]$, where the current amplitudes $I_m^m$ can be expressed in terms of the T-matrix Fourier components, $\hat{T}_{nm}^{kl} \equiv \hat{T}(\epsilon + neV + k\hbar \omega_0, \epsilon + meV + l\hbar \omega_0)$, in the following way

$$I_m^m = \frac{e^2}{\hbar^2} \int dc \sum_{n,k} \text{Tr} \left[ \hat{\tau}_3 \times \\ \left( \hat{T}_{RL,0}^{r} \circ \hat{g}^{r} \circ \hat{T}_{RL,0}^{a} \circ \hat{g}^{a} \circ \hat{T}_{RL,0}^{r} \circ \hat{g}^{r} \circ \hat{T}_{RL,0}^{a} \circ \hat{g}^{a} \circ \hat{T}_{RL,0}^{r} \circ \hat{g}^{r} \circ \hat{T}_{RL,0}^{a} \circ \hat{g}^{a} \right) \right],$$  \hfill (7)

At this point, the calculation of the current has been reduced to determination of the Fourier components of the T-matrix. In the case of a symmetric contact considered here, one can show that the dc current can be expressed only in terms of $\hat{T}_{n}^{kl} \equiv \hat{T}_{RL,n}^{0}$, which fulfill the following set of linear algebraic equations

$$\hat{T}_n^{kl} = \hat{v}_n^{k} + \sum_{l} \left\{ \hat{\epsilon}_n^{kl} \hat{T}_n^{l} + \hat{\epsilon}_n^{kl} \hat{T}_n^{l} \hat{\epsilon}_n^{kl} + \hat{\epsilon}_n^{kl} \hat{T}_n^{l} + \hat{\epsilon}_n^{kl} \hat{T}_n^{l} \hat{\epsilon}_n^{kl} \right\},$$  \hfill (8)

where the different matrix coefficients adopt the following form in terms of the unperturbed Green functions

$$\hat{v}_n^{k} = \frac{V}{2} \hat{J}_k(\alpha_0) \left[ \hat{k} (1 + \hat{\tau}_3) \delta_{n,-1} - (-i)^k (1 - \hat{\tau}_3) \delta_{n,1} \right],$$

$$\hat{\epsilon}_n^{kl} = (e^{k,l} + e^{-k,l}) \sum_{j} \hat{J}_{k-j}(\alpha) \hat{J}_{j-l}(\alpha) \hat{g}_{j,-1}^{l} \hat{g}_{j-1}^{l} \hat{g}_{j,-1}^{l} \hat{g}_{j-1}^{l},$$

where we have used the shorthand notation $\hat{g}_n^{k} = \hat{g}^{k}(\epsilon + i\epsilon V + k\hbar \omega_0)$ and $J_n(\alpha)$ is the Bessel function of order $n$. In some limits one can find an analytical solution of these systems, but in general a numerical calculation is needed.

Results and discussions.— Let us concentrate in the dc current, $I_{dc}$. This current is the sum of two contributions: $I_{dc} = I_B + I_{Shapiro}$, where $I_B \equiv I_0^0$ is a background current and $I_{Shapiro} = \sum_{n,m} I_m^m e^{in\phi_0} \delta(V - V_m^{n})$ is the Shapiro steps contribution at discrete voltages $V_m^{n} = (m/n) \hbar \omega_0 / 2e$. Notice that several ac current amplitudes can give a dc contribution at the same voltage. Notice also that the Shapiro step contribution depends on the average value of the phase, $\phi_0$. We shall concentrate in the height of the Shapiro steps, which will be denoted as $S_m^n$. Let us remark that in the tunneling regime we recover the well-known results for both the background current and Shapiro step heights [12].

In order to illustrate the general results, in Fig. 1 we show the dc current, background current plus Shapiro steps, for different values of $\alpha$ and a frequency $\omega_0 = 0.5 \Delta$.
We can see the two main features that will be the subject of the rest of the paper: (i) the subharmonic Shapiro steps $S_n^m$, with $n \neq 1$, are clearly visible at high transmissions, and (ii) the background current exhibits a subharmonic gap structure at voltages $eV = (2\Delta + k\hbar \omega_r)/n$, with $n, k$ integers, which is specially pronounced at low transmissions.

FIG. 1. Zero temperature dc current, $I_{dc}$, as a function of voltage for a frequency $\omega = 0.5\Delta$ and several values of $\alpha$. The different curves in each panel correspond to different transmissions, as indicated in panel (a). In panels (b) and (c) the curves have been vertically displaced. Panel (d) shows in detail the curve $T = 0.95$ of panel (c). The current is normalized by the normal conductance $G_N = (2e^2/h)T$.

Let us start by analyzing the background current. In Fig. 2(e-d) we show the background current for two different frequencies at a moderate power, $\alpha = 1.0$. The current in absence of radiation is also shown for comparison. As mentioned above, the most prominent feature in the background current is the appearance of a pronounced subgap structure at voltages $eV = (2\Delta + k\hbar \omega_r)/n$. This structure is specially clear at low transmissions (see Fig. 2d) and progressively disappears as the transparency is increased. Indeed, this peculiar subharmonic gap structure was already observed in several experiments in the early seventies in point contacts and thin-film microbridges. At that time no consistent explanation was given, but it is clear that this structure can be explained in terms of photon-assisted MARs. A step at $eV = (2\Delta + k\hbar \omega_r)/n$ is simply due to the opening of a MAR of order $n$ in which $k$ photons in total are absorbed ($k$ negative) or emitted ($k$ positive). This is illustrated in the upper panels of Fig. 2. In order to understand how this subharmonic structure evolves with the rf power, one can do a systematic perturbative expansion in the transmission. This analysis tells us that at low transparency the height of a current jump at $eV = (2\Delta + k\hbar \omega_r)/n$ is proportional to $J_r^2(n\alpha)$, which is valid as long as $\hbar \omega_r \ll 2\Delta/n$. This results coincides with the phenomenological functional form that was used to fit the experiments by Soerensen et al.

![Fig. 2](image_url)

Let us now discuss the Shapiro steps. In this case the most important aspect is the existence of subharmonic steps absent in tunnel junctions. These steps arise from the phase locking between the harmonics of the Josephson frequency and the harmonics of the ac radiation. Early experiments on the ac Josephson effect in weak links observed subharmonic steps in the I-V curves. More recently, there have been reported observations of non-integer Shapiro steps in high-$T_C$ contacts, S-semiconductor-S junctions and diffusive S-N-S systems. Although the Shapiro steps can be understood as a simple consequence of a non-sinusoidal current-phase relation, the present approach goes beyond a simple “adiabatic” approximation and provides the first microscopic theory of Shapiro steps in contacts of arbitrary transmission. The adiabatic approximation, which introduces the time-dependence into the zero bias supercurrent through the Josephson relation, gives rise to the well known Bessel-function-like behavior of the steps and gives a good description of the tunnel regime. However, as we show below, such a simple approach fails in the description of a highly transmissive contact.

As a rule of thumb, a Shapiro step $S_n^m$ is visible when the corresponding ac Josephson component, $I_n$, in ab-
rence of radiation gives a significant contribution. In particular, this means high transmissions (see Figs. 3-4 in Ref. [4]). One can show that the leading order in transmission of a Shapiro step $S_n^m$ goes like $\sim T^n$, which is a consequence of the fact that $J_n \sim T^n$, and the reason for the absence of the $n \neq 1$ steps in low transmissive contacts. However, near perfect transmission the subharmonic steps can be even higher than the integer ones. This behavior is illustrated in Fig. 3 where we show the Shapiro steps $S_n^1$ as a function of the transmission for two different frequencies.

![Fig. 3](image)

**FIG. 3.** Shapiro steps $S_n^1$ versus transmission for $\alpha = 0.25$.

Fig. 4 shows the power dependence of the Shapiro steps for a frequency $\omega_r = 0.5\Delta$. Notice that this dependence is rather complicated for both integer and subharmonic steps, and clearly deviates from the usual Bessel function behavior. This is due to the frequency-dependence of the Josephson components, which is specially pronounced at high transmissions. Neglecting this dependence, i.e. within an adiabatic approximation, one would get that $S_n^m$ evolves as $|J_m(2n\alpha)|$. However, as shown in Fig. 4, as the transmission increases the validity of this approximation is restricted to $\alpha \ll 1$. Notice also the complex oscillation pattern at high transmissions (see $T = 0.8$ curves in Fig 3), which is due to the fact that several ac components give a significant contribution to the same Shapiro step.

In summary, we have presented a theoretical analysis of the dc current in a superconducting point contact in the presence of a microwave radiation. We have shown that the microscopic theory of coherent multiple Andreev reflections provides an unified description of Shapiro steps and assisted tunneling, explaining in a natural way the observations of subharmonic steps [2-4] and the peculiar subharmonic gap structure under a microwave radiation [10]. Let us finally remark that the results presented in this work are amenable to a quantitative experimental test using atomic-size contacts [3,4].

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![Fig. 4](image)

**FIG. 4.** (a-c) Shapiro steps $S_n^1$ $(n = 1, 2, 3)$ as a function of $\alpha$ for $\omega_r = 0.5\Delta$. The dotted lines in panel (b) correspond to the adiabatic approximation: $\sim |J_1(4\alpha)|$.

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