Non-SUSY heterotic string vacua of Gepner models with vanishing cosmological constant

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We study a natural generalization of the results given in K. Aoyama and Y. Sugawara, Prog. Theor. Exp. Phys. 2020, 103B01 (2020) to heterotic strings. Namely, starting from the generic Gepner models for Calabi–Yau three-folds, we construct non-SUSY heterotic string vacua with vanishing cosmological constant at the one-loop level. We especially focus on asymmetric orbifolding based on some discrete subgroup of the chiral $U(1)$ action which acts on both the Gepner model and the $SO(32)$ or $E_8 \times E_8$ sector. We present a classification of the relevant orbifold models leading to the string vacua with the properties mentioned above. In some cases, the desired vacua can be constructed in a manner quite similar to those given in the previous paper for the type II string, in which the orbifold groups contain two generators with discrete torsions. On the other hand, we also have simpler models that are just realized as asymmetric orbifolds of cyclic groups with only one generator.

1. Introduction and summary

Exploring non-supersymmetric vacua with the vanishing cosmological constant has been a subject of interest in superstring theory (at the level of one loop, at least), probably motivated by the cosmological constant problem. Consistent type II string vacua with such a non-trivial property were first constructed in Refs. [1–3] based on some non-Abelian orbifolds of higher-dimensional tori, followed by studies such as Refs. [4–9]. More recently, several non-SUSY vacua with this property have been constructed as asymmetric orbifolds [10] by simpler cyclic groups in Refs. [11,12].

On the other hand, there have been studies in heterotic string theory of string vacua with cosmological constants exponentially suppressed with respect to some continuous parameter expressing the “distance” from the SUSY vacua (for instance, the radii of tori of compactifications as given in Refs. [13,14]) in Refs. [4,15], and more recently, for example, Refs. [16–26] from the various viewpoints of model building in string phenomenology.

The purpose of the current study is to construct non-SUSY heterotic string vacua with the vanishing cosmological constant at the one-loop level based on non-toroidal models. The method we adopt is a natural generalization of those given in our previous work [27]. That is, we start from the generic Gepner models [28,29] for Calabi–Yau three-folds, and construct non-SUSY heterotic string vacua by implementing some asymmetric orbifolding. Since we have various $U(1)$ symmetries in the Gepner model, as well as $SO(32)$ or the $E_8 \times E_8$ sector in the left mover (which we assume bosonic), it would be quite natural to make the orbifolding associated with some cyclic subgroup of these $U(1)$
actions. Indeed, let us denote the generator of such a cyclic subgroup as $\delta_L$. Then, it is possible to construct non-SUSY string vacua by making the asymmetric orbifolding defined by the operator

$$\hat{\delta} \equiv \delta_L \otimes (-1)^F_R,$$

where $F_R$ denotes the spacetime fermion number (in other words, $(-1)^F_R$ acts as the sign flip on the right-moving Ramond sector). It is obvious that the orbifold projection generated by the $\hat{\delta}$ action completely breaks the Bose–Fermi cancellation in the untwisted Hilbert space. Moreover, any spacetime supercharges\(^1\) cannot be constructed even if incorporating the degrees of freedom in the twisted sectors, so far as we assume the chiral forms of supercharges, namely, the integrals of conserved world-sheet current $Q^\alpha = \oint d\bar{z} J^R_\alpha (\bar{z})$, as addressed in Ref. [27]. At this point it is crucial that the relevant twisted sectors are associated with the left-moving operator $\delta_L$, whereas the possible supercharges should originate from the right-moving degrees of freedom.

In the end, it is enough to ask whether or not the total partition function that contains all the twisted sectors vanishes. We will clarify the criterion for this aim, and present a classification of the relevant orbifold models leading to string vacua with the desired properties. In some cases, the desired vacua can be constructed in a manner similar to those given in Ref. [27] for the type II string, in which the orbifold groups contain two generators equipped with some discrete torsions [30–32]. On the other hand, we also find simpler models which are just realized as asymmetric orbifolds of cyclic groups with only one generator, in contrast to the type II string cases.

2. Preliminaries

We begin with a very brief review of heterotic string vacua, including the Gepner models for the CY\(_3\) compactifications, and describe the notation to be used in the main section.

2.1. Heterotic string vacua of Gepner models

The Gepner model [28,29] describing some CY\(_3\) compactifications is defined as the superconformal system

$$\left[ \mathcal{M}_{k_1} \otimes \cdots \otimes \mathcal{M}_{k_r} \right]_{\mathbb{Z}_N\text{-orbifold}} \sum_{i=1}^r \frac{k_i}{k_i + 2} = 3,$$  \hspace{1cm} (1)

where $\mathcal{M}_k$ denotes the $\mathcal{N} = 2$ minimal model of level $k$, $\hat{c} := \frac{c}{3} = \frac{k}{k + 2}$. We set

$$N := \text{lcm}\{k_i + 2 ; i = 1, \ldots, r\},$$  \hspace{1cm} (2)

where lcm means the least common multiplier. To describe the building blocks of the torus partition function, we start with the simple products of the characters of the $\mathcal{N} = 2$ minimal model [33–36]

\(^1\) In this paper, we shall regard spacetime supercharges as operators acting consistently on the whole Hilbert space of string states and made up of the local perturbative degrees of freedom on the world-sheet. We use the term “non-SUSY string vacuum” with the meaning of absence of supercharges under this definition. It is beyond our scope whether non-perturbative supercharges could exist. In addition, it may be possible that one would gain some “supercharges” after truncating massive degrees of freedom in the approximation by low-energy effective field theories.
in the NS sector:\textsuperscript{2}

\[ F_I^{(\text{NS})}(\tau, z) := \prod_{i=1}^{r} \text{ch}^{(\text{NS})}_{\ell_i, m_i}(\tau, z), \quad I \equiv \{ (\ell_i, m_i) \}, \quad \ell_i + m_i \in 2\mathbb{Z}, \quad \forall i. \]  

(3)

Those for other spin structures are defined by acting the half spectral flows \( z \mapsto z + \frac{r}{2} \tau + \frac{s}{2} \) \((r, s \in \mathbb{Z}_2)\):

\[ F_I^{(\text{NS})}(\tau, z) := F_I^{(\text{NS})}(\tau, z + \frac{1}{2}), \]  

(4)

\[ F_I^{(\text{NS})}(\tau, z) := q^{\frac{c}{2}} y^{\frac{c}{2}} F_I^{(\text{NS})}(\tau, z + \frac{\tau}{2}), \]  

(5)

\[ F_I^{(\text{NS})}(\tau, z) := q^{\frac{c}{2}} y^{\frac{c}{2}} F_I^{(\text{NS})}(\tau, z + \frac{\tau + 1}{2}), \]  

(6)

where we set \( \hat{c} = 3 \). Note that the label \( I \equiv \{ (\ell_i, m_i) \} \) of the building blocks (and the spectral flow orbits introduced below) expresses the quantum numbers for the NS sector even for \( F_I^{(r)} \) and \( F_I^{(R)} \).

To construct the Gepner models, we need to make the chiral \( \mathbb{Z}_N \times \mathbb{Z}_N \) orbifolding by \( g_I = e^{\frac{2\pi i}{2} J^{(a)}_I} \) and \( g_R = e^{\frac{2\pi i}{2} J^{(a)}_R} \), where \( J^{(a)} \) \((J^{(w)})\) expresses the total \( \mathcal{N} = 2 \) \((U(1))\) current in the left (right) mover acting over \( \otimes_i \mathcal{M}_k \). Recall that the zero-mode \( J^{(a)} \) takes the eigenvalues in \( \frac{1}{N} \mathbb{Z} \) for the NS sector. The chiral \( \mathbb{Z}_N \) orbifolding (in the left mover) is represented in a way respecting the modular covariance by considering the “spectral flow orbits” \[37\] defined as follows:

\[ \mathcal{F}_I^{(\text{NS})}(\tau, z) := \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} q^{\frac{c}{2} a^2 \hat{c} a} F_I^{(\text{NS})}(\tau, z + a \tau + b), \]  

(7)

\[ \mathcal{F}_I^{(\text{NS})}(\tau, z) := \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} (-1)^{\hat{c} a} q^{\frac{c}{2} a^2 \hat{c} a} F_I^{(\text{NS})}(\tau, z + a \tau + b), \]  

(8)

\[ \mathcal{F}_I^{(\text{NS})}(\tau, z) := q^{\frac{c}{2} y^{\frac{c}{2}}} \mathcal{F}_I^{(\text{NS})}(\tau, z + \frac{\tau}{2}), \]  

(9)

\[ \mathcal{F}_I^{(\text{NS})}(\tau, z) := q^{\frac{c}{2} y^{\frac{c}{2}}} \mathcal{F}_I^{(\text{NS})}(\tau, z + \frac{\tau + 1}{2}), \]  

(10)

We also use the abbreviated notation \( \mathcal{F}_I^{(a)}(\tau) \equiv \mathcal{F}_I^{(\text{NS})}(\tau, 0) \). See Appendix B for the explicit forms of \( \mathcal{F}_I^{(a)}(\tau, z) \) written in terms of the \( \mathcal{N} = 2 \) minimal characters.

Now we focus on the heterotic string. We take the convention

left mover : 26D bosonic; \quad right mover : 10D super.

\textsuperscript{2} We summarize the explicit character formulas as well as the conventions of theta functions in Appendix A. We set \( q := e^{2\pi i \tau}, y := e^{2\pi i z} \) throughout this paper.
Assuming the standard embedding of spin connection, the $SO(32)$ heterotic string vacuum compactified on CY$_3$ is described by the following modular invariant partition function:

$$Z_{SO(32)\text{ Het}}(\tau) = \left(\frac{1}{\sqrt{2\tau_2}}|\eta|^2\right)^2 \cdot \frac{1}{2N} \sum_{\sigma_L,\sigma_R} \epsilon(\sigma_R) \left(\frac{\theta_{|\sigma_L|}}{\eta}\right)^{13} \left(\frac{\theta_{|\sigma_R|}}{\eta}\right)$$

$$\times \sum_{I_L,I_R} N_{I_L,I_R} P_{I_L}(\sigma_L)(\tau)P_{I_R}^{\sigma_R}(\tau). \quad (11)$$

To avoid complexities, we shall assume the modular invariant coefficient $N_{I_L,I_R}$ to be diagonal throughout this paper:

$$N_{I_L,I_R} = \prod_{i=1}^r \frac{1}{2} \delta_{\ell_i,L_i} \delta_{m_i,L_i}, \quad I_L \equiv \{(\ell_i,L_i,m_i)\}, \quad I_R \equiv \{(\ell_i,M_i)\}. \quad (12)$$

Here, the summations of $\sigma_L$ and $\sigma_R$ are taken over the chiral spin structures. We also set $\epsilon(\text{NS}) = -\epsilon(\tilde{\text{NS}}) = -\epsilon(\text{R}) = 1$ in the standard fashion, and $\theta_{|\text{NS}|} \equiv \theta_3(\tau,0)$, $\theta_{|\text{NS}|} \equiv \theta_4(\tau,0)$, $\theta_{|\text{R}|} \equiv \theta_2(\tau,0)$ ($\theta_{|\tilde{\text{R}}|} \equiv -i\theta_1(\tau,0) \equiv 0$) to describe the free fermion contributions including the $SO(32)$ sector.

The $E_8 \times E_8$ heterotic string vacuum is likewise described as

$$Z_{E_8 \times E_8\text{ Het}}(\tau) = \left(\frac{1}{\sqrt{2\tau_2}}|\eta|^2\right)^2 \cdot \frac{1}{2N} \sum_{\sigma_L,\sigma_R} \epsilon(\sigma_R) \left(\frac{\theta_{|\sigma_L|}}{\eta}\right)^5 \chi_0^{E_8}(\tau)$$

$$\times \sum_{I_L,I_R} N_{I_L,I_R} P_{I_L}(\sigma_L)(\tau)P_{I_R}^{\sigma_R}(\tau), \quad (13)$$

where $\chi_0^{E_8}(\tau)$ denotes the character of the basic representation of affine $E_8$, written explicitly as

$$\chi_0^{E_8}(\tau) = \frac{1}{2} \left(\frac{\theta_3}{\eta}\right)^8 + \left(\frac{\theta_4}{\eta}\right)^8 + \left(\frac{\theta_5}{\eta}\right)^8 \right) \tau. \quad (14)$$

3. Constructions of non-SUSY heterotic string vacua

In this section we present our main analysis. Namely, we discuss how we can construct non-SUSY string vacua with the vanishing cosmological constant at one loop (or the vanishing torus partition function) based on the heterotic string compactified on CY$_3$ given in Eqs. (11) and (13). We start by specifying the relevant orbifold action.

3.1. Orbifold actions

Let us fix a subsystem of the minimal models $\otimes_{i \in S} M_{k_i}$, $S \subset \{1,2,\ldots,r\}$, on which the orbifold operators act non-trivially. We set

$$N' := \text{lcm} \{k_i + 2 : i \in S\}. \quad (15)$$

The total central charge of the subsystem $S$ is written in the form

$$\hat{c}_S = \sum_{i \in S} \frac{k_i}{k_i + 2} = \begin{cases} \frac{2M}{N'}, & (N' \in 2\mathbb{Z}, \ M \in \mathbb{Z}), \\ \frac{2M}{N'}, & (N' \in 2\mathbb{Z} + 1, \ M' \in \mathbb{Z}). \end{cases} \quad (16)$$
We fix a positive integer $L$ dividing $N'$, and set

$$4K := \text{lcm} \left\{ N'/L, 4 \right\}$$

(17)

for later convenience. We will shortly define the orbifold action $\hat{\delta}$ that satisfies $\hat{\delta}^{4K} = 1$ on the untwisted sector. We also define $S_1 \subset S$ by

$$S_1 := \left\{ i \in S : \frac{N'}{k_i + 2} \in 2\mathbb{Z} + 1 \right\}.$$  

(18)

Note that $S_1 \neq \emptyset$ since $N'$ is the lcm of $\{k_i + 2\}_{i \in S}$.

For the $SO(32)$ ($E_8 \times E_8$) heterotic string, we have the $SO(26)$ ($SO(10) \times E_8$) symmetry after making the standard embedding of the spin connection. We will adopt the relevant orbifold action as a cyclic subgroup of $U(1)^s$ or $U(1)^{s_1} \times U(1)^{s_2}$ given as

$$U(1)^s \times SO(26 - 2s) \subset SO(26)$$

(19)

for the $SO(32)$ case, and

$$\left[ U(1)^{s_1} \times SO(10 - 2s_1) \right] \times \left[ U(1)^{s_2} \times SO(16 - 2s_2) \right] \subset SO(10) \times E_8$$

(20)

for the $E_8 \times E_8$-case.

Now, let us specify the relevant orbifold action. For the $SO(32)$ heterotic string, we define

$$\hat{\delta} := (-1)^{F_R} \otimes \delta_L, \quad \delta_L := \exp \left\{ 2\pi i L \sum_{i \in S} J_L^{(i)} \right\} \exp \left\{ 2\pi i \frac{1}{2} \sum_{j=1}^{s} K_L^{(j)} \right\},$$

(21)

where $J_L^{(i)}$ is the left-moving $U(1)$ current in $M_{k_i}, i \in S$, and $K_L^{(j)}$ are those for the $U(1)^s$ factor in Eq. (19). In other words, $\delta_L$ acts on the left-moving characters of $\mathcal{M}_{k_i}, i \in S$, as the integral spectral flow $z \mapsto z + L(\alpha \tau + \beta)$:

$$\delta_{L,(\alpha, \beta)} \cdot \text{ch}_{\ell, m_i}^{(\sigma)} (\tau, z) := q^k y^{kL^2} e^{2\pi i \frac{kL^2}{2(\ell + 1)}} \chi_{\ell, m_i}^{(\sigma)} (\tau, z + L(\alpha \tau + \beta)),$$

(22)

$$(\alpha, \beta) \in \mathbb{Z}N'/L \times \mathbb{Z}N'/L,$$  

which yields the modular covariant actions on the spectral flow orbits $\mathcal{F}_{L}^{(\sigma_1)}(\tau)$. We summarize the explicit forms of $\delta_{L,(\alpha, \beta)} \cdot \mathcal{F}_{L}^{(\sigma_1)}(\tau)$ in Appendix B. $F_R$ denotes the spacetime fermion number of the right mover. In other words, the operator $(-1)^{F_R}$ acts as the sign flip of the right-moving R sector.

On the other hand, $\delta_L$ acts on the Jacobi theta functions associated with the $U(1)^s$ factor as follows:

$$\delta_{L,(\alpha, \beta)} \cdot \theta_i (\tau, z) \equiv \theta_i (\alpha, \beta) (\tau, z) := q^2 y^a e^{2\pi i \frac{a \beta}{2}} \theta_i \left( \tau, z + \frac{\alpha \tau + \beta}{2} \right), \quad i = 3, 4, 2.$$  

(24)

Here, the inclusion of the phase factor $e^{2\pi i \frac{a \beta}{2}}$ is necessary for the modular covariance as in the minimal sector, Eq. (23). The explicit forms of Eq. (24) are also summarized in Appendix B.

We similarly define the orbifold action $\hat{\delta}$ in the $E_8 \times E_8$ case, in which $\delta_L$ acts on

$$\bigotimes_{i \in \mathcal{S}} \mathcal{M}_{k_i} \bigotimes U(1)^{s_1} \otimes U(1)^{s_2},$$

in the same way as Eq. (21).
Since the \( \hat{\delta} \) orbifold action is defined so as to respect the modular covariance, it is easy to write down the modular invariant partition functions of our asymmetric orbifolds. For example, for the \( SO(32) \) heterotic string and in the cases of \( K \in 2\mathbb{Z} + 1 \), the \( \hat{\delta} \) orbifold is found to be of order \( 8K \), and the modular invariant partition function is written as

\[
Z_{\hat{\delta}-\text{orb}}(\tau) = \left( \frac{1}{\sqrt{2}} \right)^2 \frac{1}{2N} \sum_{\alpha, \beta \in \mathbb{Z}_{8K}} \epsilon(\sigma_R; \alpha, \beta) \left( \frac{\theta_{[\sigma_L],(\alpha,\beta)}}{\eta} \right)^s \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^{13-s} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)
\]

\[
\times \sum_{\sigma_L, \sigma_R} N_{\sigma_L, \sigma_R} \left[ \delta L(\alpha, \beta) \cdot \mathcal{F}_{\sigma_L}^{(\sigma_R)}(\tau) \right] \mathcal{F}_{\sigma_R}^{(\sigma_L)}(\tau)
\]

(25)

Here, we set

\[
\epsilon(\text{NS}; \alpha, \beta) := \begin{cases} 
-1, & \alpha, \beta \in 2\mathbb{Z} + 1, \\
1, & \text{otherwise}; 
\end{cases}
\]

\[
\epsilon(\text{NS}; \alpha, \beta) := \begin{cases} 
1, & \alpha \in 2\mathbb{Z} + 1, \beta \in 2\mathbb{Z}, \\
-1, & \text{otherwise};
\end{cases}
\]

\[
\epsilon(\text{R}; \alpha, \beta) := \begin{cases} 
1, & \alpha \in 2\mathbb{Z}, \beta \in 2\mathbb{Z} + 1, \\
-1, & \text{otherwise},
\end{cases}
\]

(26)

which originate from the GSO phases \( \epsilon(\sigma_R) \) modified by the \((-1)^F_R\) actions included in Eq. (21). Also, we again made use of the abbreviated notations \( \theta_{[\text{NS}],(\alpha,\beta)}(\tau) \equiv \theta_{3,(\alpha,\beta)}(\tau) \equiv \theta_{3,(\alpha,\beta)}(\tau,0) \), and so on.

The modular invariants in other cases are obtained similarly.

3.2. Criterion for the desired models

At this stage let us clarify the “criterion” to search for heterotic string vacua with the desired properties. To this end, we denote the contributions to the torus partition function from each twisted sector as \( Z(\alpha, \beta)(\tau) (\alpha, \beta \in \mathbb{Z}_{4K}) \). That is, we define

\[
Z(\alpha, \beta)(\tau) \equiv \text{Tr}_{\text{\hat{\delta} twisted}} \left[ \hat{\delta}^\beta q^{L_0 - \frac{c}{24}} \right]^{(\alpha, \beta)}
\]

(27)

for convenience. By our definition of the orbifold action \( \hat{\delta} \) presented above, the building blocks \( Z(\alpha, \beta)(\tau) \) behave covariantly under the modular transformations:

\[
Z(\alpha, \beta) \left( \frac{-1}{\tau} \right) = Z(\beta, -\alpha)(\tau), \quad Z(\alpha, \beta)(\tau + 1) = Z(\alpha, \alpha + \beta)(\tau).
\]

(28)

We require the following conditions:

\begin{itemize}
  \item For the “even sectors,” \( \alpha, \beta \in 2\mathbb{Z} \), each building block \( Z(\alpha, \beta)(\tau) \) separately vanishes:
    \[
    Z(\alpha, \beta)(\tau) \equiv 0.
    \]
    (29)
  \item The partition function for the untwisted sector does not vanish:
    \[
    Z(0)(\tau) \equiv \frac{1}{4K} \sum_{\beta \in \mathbb{Z}_{4K}} Z(0, \beta) \not\equiv 0.
    \]
    (30)
\end{itemize}
For all the twisted sectors of $\tilde{\delta}^\alpha$ with $\alpha \in 2\mathbb{Z} + 1$, we require
\[
\sum_{\beta'} Z(\alpha, 2\beta') (\tau) \equiv 0 \quad (\forall \alpha \in 2\mathbb{Z} + 1). \tag{31}
\]
Note that Eq. (31) just implies that
\[
\sum_{\alpha \in 2\mathbb{Z} + 1} Z(\alpha, \beta)(\tau) \equiv 0, \tag{32}
\]
due to the modular covariance, Eq. (28). Thus, combining it with the requirement in Eq. (29), we can conclude that the total partition function should vanish.

We also note that, in this situation, the Bose–Fermi cancellation can only occur among different twisted sectors because of the condition in Eq. (30). On the other hand, the possible spacetime supercharges should be of a form such as $Q^\alpha = \oint d\bar{z} J_R^\alpha (\bar{z})$, which is consistent with the conservation on the world-sheet. However, any operators of this form cannot induce the expected Bose–Fermi cancellation, because the relevant twisted sectors are associated with the left-moving operator $\delta_L$. In this way, we conclude that we do not have any spacetime supercharges as the operators consistently acting on the whole Hilbert space and conserved on the world-sheet. This is the reason why we claim that the heterotic string vacua that satisfy the above requirements are non-supersymmetric ones.

3.3. Classification of the models
We study here aspects of the orbifolds of heterotic string vacua in Eqs. (11) and (13) by the cyclic actions of $\tilde{\delta}$ given in Eq. (21). We classify the models according to the positive integer $N'/L$.

First of all, we note that
\[
Z(\alpha, \beta)(\tau) \equiv 0 \quad \forall \alpha, \beta \in 2\mathbb{Z} \quad \tag{33}
\]
for all the cases we will discuss below, since $\tilde{\delta}^2$ obviously preserves the spacetime supercharges.

One can also readily confirm that, for the untwisted sector $\alpha = 0$,
\[
Z_0(\tau) \left( \equiv \frac{1}{4K} \sum_{\beta \in \mathbb{Z}_{4K}} Z(0, \beta)(\tau) \right) \neq 0 \tag{34}
\]
in all cases.

Now, let us describe the classification.

(A) $N'/L \equiv 0 \pmod{4}$:
In this case, we have $N'/L = 4K$. We first focus on the $SO(32)$ case. The crucial point is as follows: for the product of left-moving minimal characters $\prod_{i \in S} \text{ch}_{i,L,m_i,-2n_L}(\tau)$ (where $n_L$ is the spectral flow momentum) as well as each “free fermion factor” $\left( \frac{\theta}{\pi} \right)^{13}$, the orbifold action $\tilde{\delta}$ picks up the next phase factor,
\[
\exp \left[ 2\pi i \left\{ \sum_{i \in S} d_i (m_i - 2n_L) L\beta + \frac{M}{N'} L^2 \alpha\beta + \frac{1}{8} c_{j\alpha\beta} \right\} \right] \equiv \exp \left[ 2\pi i \left\{ \sum_{i \in S} d_i (m_i - 2n_L) + ML\alpha + \frac{1}{2} K c_{j\alpha\beta} \right\} \right], \tag{35}
\]
In comparison with Ref. [27], the roles of left and right movers have been exchanged here. The arguments are almost the same for the same way as above. Thus, the constraint in Eq. (36) is replaced with projection imposing \( (\alpha \beta) \) for the gained, while the \( (\alpha \beta) \) is exchanged with \( (\beta \alpha) \). Fixing the value \( \alpha \in 2\mathbb{Z} + 1 \), let us evaluate the summation \( \sum_{\beta'} Z_{(\alpha \beta \gamma)}(\tau) \). It acts as the projection imposing

\[
\sum_{i \in S} d_i (m_{i,L} - 2n) + ML\alpha + \frac{1}{2} Kc_j \alpha \equiv 0 \quad (\text{mod } 2K).
\]

The arguments are almost the same for the \( E_8 \times E_8 \) case. We only have to replace the term \( \frac{1}{2} Kc_j \) in Eq. (35) with \( \frac{1}{2} K (c_1^{(r)} s_1 + c_2^{(r)} s_2) \) for the factor \( \left( \frac{\theta}{\eta} \right)^{s_1} \left( \frac{\theta}{\eta} \right)^{s_2} \), where \( c_j^{(r)} = \pm 1 \) is defined in the same way as above. Thus, the constraint in Eq. (36) is replaced with

\[
\sum_{i \in S} d_i (m_{i,L} - 2n) + ML\alpha + \frac{1}{2} K (c_1^{(r)} s_1 + c_2^{(r)} s_2) \alpha \equiv 0 \quad (\text{mod } 2K).
\]

Consequently, we obtain the next classification.

\( Ks \in 4\mathbb{Z} \) \( [SO(32)]; Ks_1, Ks_2 \in 4\mathbb{Z} \) or \( Ks_1, Ks_2 \in 4\mathbb{Z} + 2 \) \( [E_8 \times E_8] \):

In these cases the aspects are almost parallel to those of Ref. [27]. The constraint in Eqs. (36) or (37) implies

\[
\sum_{i \in S_1} m_{i,L} + LM \equiv 0 \quad (\text{mod } 2),
\]

where \( S_1 \) was defined in Eq. (18), that is, \( S_1 \equiv \{ i \in S : d_i \in 2\mathbb{Z} + 1 \} \). We then find that \( \sum_{\beta'} Z_{(\alpha \beta \gamma)}(\tau) \neq 0 \), since we generically possess many states satisfying the condition in Eq. (38). This means that \( \hat{\delta} \) orbifolding cannot satisfy Eq. (31) by itself. However, as shown in Ref. [27], we can make it possible by further introducing the \( \mathbb{Z}_2 \) orbifold action \( \gamma_R \), which commutes with \( \hat{\delta} \):

\[
\gamma_R := \prod_{i \in S_1} (-1)^{\ell_i, R}, \quad \gamma_R := \begin{cases} (-1)^{\ell_i} \otimes \gamma_R & (\#S_1 \in 2\mathbb{Z} + 1), \\ \gamma_R & (\#S_1 \in 2\mathbb{Z}), \end{cases}
\]

on the right-moving minimal characters \( \prod_{i \in S} \chi_{(\sigma)}^{(\ell_i, R, m_{i,R} - 2n)}(\tau) \), and \( (-1)^{\ell_i} \) denotes the sign flip of the left-moving R sector. We shall also introduce the discrete torsion [30–32] with respect to the \( \gamma_R \) and \( \hat{\delta} \) actions:

\[
\xi (a, \alpha ; b, \beta) := \begin{cases} (-1)^{(LM-1) (\alpha - \beta a)} & (a, b \in \mathbb{Z}_2, \quad \alpha, \beta \in \mathbb{Z}_{4K}), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( a, b \) label the spatial and temporal twistings by \( \gamma_R \), while \( \alpha, \beta \) are those associated with \( \hat{\delta} \) as above. Then, for any fixed \( \alpha \in 2\mathbb{Z} + 1 \), we readily obtain

\[
\sum_{\beta'} Z_{(\alpha \beta \gamma)}(\tau) \mid_{\text{\gamma_R orbifold}} = \frac{1}{2} \sum_{\beta'} \sum_{a, b \in \mathbb{Z}_2} \xi (a, \alpha ; b, 2\beta') Z_{(\alpha \alpha ; b, 2\beta')(\tau)} = 0.
\]

\[\text{In comparison with Ref. [27], the roles of left and right movers have been exchanged here.}\]
That is, the criterion in Eq. (31) is satisfied when combining the orbifolding by $\hat{\gamma}$ and $\hat{\delta}$. See Ref. [27] for more detailed arguments.

- $K_s \in 4\mathbb{Z} + 2 [SO(32)]; K_{s1} \in 4\mathbb{Z}, K_{s2} \in 4\mathbb{Z} + 2$, or $K_{s1} \in 4\mathbb{Z} + 2, K_{s2} \in 4\mathbb{Z} [E_8 \times E_8]$:
  In these cases, the constraint in Eqs. (36) or (37) yields
  \[
  \sum_{i \in S_1} m_i + LM + 1 \equiv 0 \quad (\text{mod} \ 2)
  \]  
  (42)

in place of Eq. (38). Therefore, we can make the criterion in Eq. (31) be satisfied by taking again the $\hat{\delta}$ and $\hat{\gamma}$ orbifolds but with the different discrete torsion $\xi$.

- $K_s \in 2\mathbb{Z} + 1 [SO(32)]; K (s_1 + s_2) \in 2\mathbb{Z} + 1 [E_8 \times E_8]$:
  In these cases, no state can satisfy the condition in Eq. (36), and thus the criterion in Eq. (31) is trivially achieved by only making the $\hat{\delta}$ orbifolding.

- $K_{s1}, K_{s2} \in 2\mathbb{Z} + 1 [E_8 \times E_8]$:
  In these remaining cases, both Eq. (38) and Eq. (42) are possible, depending on which theta function factors $(\theta^s_1)^{s_1} (\theta^s_2)^{s_2}$ the operator $\hat{\delta}$ acts. Thus, Eq. (31) cannot be satisfied even if incorporating the $\hat{\gamma}$ orbifolding. We conclude that string vacua with the desired properties are not constructed in these cases.

(B) $N'/L \not\equiv 0 \pmod{4}$:

In this case, $\frac{N'}{L} \in 4\mathbb{Z} + 2$ or $\frac{N'}{L} \in 2\mathbb{Z} + 1$, and $K \in 2\mathbb{Z} + 1$ for both cases.

Again, we first consider the $SO(32)$ case. In the case of $\frac{N'}{L} \in 4\mathbb{Z} + 2$, $\hat{\delta}$ picks up the phase factor

\[
\exp \left[ 2\pi i \left\{ \sum_{i \in S} \frac{d_i (m_i - 2n)}{N'} L \beta + \frac{M}{N'} L^2 \alpha \beta + \frac{1}{8} s \epsilon \right\} \right]
\]

\[
\equiv \exp \left[ 2\pi i \frac{\beta}{4K} \left\{ \sum_{i \in S} 2d_i (m_i - 2n) + 2M \alpha \beta + \frac{1}{2} K_{s} \alpha \beta \right\} \right]
\]  
(44)

instead of Eq. (35). In the case of $\frac{N'}{L} \in 2\mathbb{Z} + 1$, we similarly obtain

\[
\exp \left[ 2\pi i \left\{ \sum_{i \in S} \frac{d_i (m_i - 2n)}{N'} L \beta + \frac{M}{2N'} L^2 \alpha \beta + \frac{1}{8} s \epsilon \right\} \right]
\]

\[
\equiv \exp \left[ 2\pi i \frac{\beta}{4K} \left\{ \sum_{i \in S} 4d_i (m_i - 2n) + 2M' \alpha \beta + \frac{1}{2} K_{s} \alpha \beta \right\} \right].
\]  
(45)

In the case of $E_8 \times E_8$, the term $\frac{1}{2} K_{s}$ in Eqs. (44) and (45) is again replaced with $\frac{1}{2} K (c_1^{(1)} s_1 + c_k^{(2)} s_2)$.

Combining all these, we obtain the next classifications:

- $s \not\in 4\mathbb{Z} [SO(32)]; s_1 + s_2 \in 2\mathbb{Z} + 1, s_1 \in 4\mathbb{Z}, s_2 \in 4\mathbb{Z} + 2$, or $s_1 \in 4\mathbb{Z} + 2, s_2 \in 4\mathbb{Z} [E_8 \times E_8]$:
  In all these cases we simply obtain
  \[
  \sum_{\beta} Z_{(\alpha, 2\beta')} = 0, \quad (\forall \alpha \in 2\mathbb{Z} + 1),
  \]
because \( \frac{1}{2}K_{sc}\alpha \), or \( \frac{1}{2}K(c_1^{(1)}s_1 + c_2^{(2)}s_2)\alpha \) for \( E_8 \times E_8 \), takes values in \( \mathbb{Z} + \frac{1}{2} \) or \( 2\mathbb{Z} + 1 \), and thus the phase factors in Eqs. (44) and (45) never cancel out. Hence, the criterion in Eq. (31) is again achieved by making only the \( \hat{\delta} \) orbifolding.

\[ \int_{\hat{\delta}} \frac{d^2z}{4\pi} \frac{d^2\tau}{2\pi} Z(\tau) = 0 \]

Otherwise:

\( \sum_{\mu'} Z_{(\alpha, 2\mu')} \neq 0 \). Moreover, Eq. (31) cannot be satisfied even if the \( \hat{\tau} \) orbifolding is incorporated with any discrete torsion. The desired string vacua are not constructed in these cases.

To summarize, we have obtained non-SUSY heterotic string vacua with the property \( Z_{1\text{-loop}}(\tau) \equiv 0 \) based on orbifolding by \( \hat{\delta} \) (and \( \hat{\tau} \) in some cases) as follows:

1. \( K_s \in 2\mathbb{Z} + 1 \) [\( SO(32) \)]; \( K(s_1 + s_2) \in 2\mathbb{Z} + 1 \) \( [E_8 \times E_8] \):
   - The desired vacua can be constructed only by making the \( \hat{\delta} \) orbifolding. The order of orbifolding is 8\( K \), although \( \hat{\delta}^{4K} = 1 \) if restricting on the untwisted Hilbert space.

2. \( K_s \in 4\mathbb{Z} + 2 \) and \( N'/L \not\equiv 0 \) (mod 4) [\( SO(32) \)]; \( Ks_i \in 4\mathbb{Z} + 2 \), \( Ks_j \in 4\mathbb{Z} \) (\( i \neq j \)) and \( N'/L \not\equiv 0 \) (mod 4) \( [E_8 \times E_8] \):
   - The desired vacua are again constructed only by the \( \hat{\delta} \) action as in case (1). However, we obtain an order 4\( K \) orbifold in this case.

3. \( K_s \in 2\mathbb{Z} \) and \( N'/L \equiv 0 \) (mod 4) [\( SO(32) \)]; \( Ks_1, Ks_2 \in 2\mathbb{Z} \) and \( N'/L \equiv 0 \) (mod 4) \( [E_8 \times E_8] \):
   - The desired vacua are constructed as the \( \mathbb{Z}_4K \times \mathbb{Z}_2 \) orbifold defined by \( \hat{\delta} \) and \( \hat{\tau} \) actions with the next discrete torsion included \( (a, b \in \mathbb{Z}_2 \text{ for } \hat{\tau} \text{ twists, and } \alpha, \beta \in \mathbb{Z}_4K \text{ for } \hat{\delta} \text{ twists}) \):

\[
\xi (a, \alpha ; b, \beta) = \begin{cases} 
-1 & \text{if } \alpha\beta = 4k + 1, \\
-1 & \text{if } \alpha\beta = 4k + 3,
\end{cases}
\]

\[ \{ [SO(32)], [E_8 \times E_8] \} \]

4. Some comments

In this paper, as an extension of our previous work in Ref. [27], we have studied the construction of non-SUSY heterotic string vacua with the vanishing cosmological constant at the one-loop level, based on the asymmetric orbifolding of the Gepner models.

In the string vacua we constructed, we could not make up the spacetime supercharges that are conserved on the world-sheet and consistently realize the Bose–Fermi cancellation expected from the one-loop partition functions. We would like to emphasize here that \( Z_{1\text{-loop}}(\tau) \equiv 0 \) just implies Bose–Fermi cancellation under the free string limit. Therefore, even if they might induce some low-energy effective field theories with unbroken SUSY, the absence of supercharges in the above sense should imply that they could not be supersymmetric ones when turning on the string interactions described by general world-sheets with higher genera. It would thus be possible for them to generate small non-vanishing cosmological constants after incorporating the (perturbative or non-perturbative) stringy quantum corrections, although such analyses still look very hard to carry out due to the complexities of the spectra arising from various twisted sectors.

When being motivated by the cosmological constant problem, it would be more desirable, though much more non-trivial, to have the vanishing one-loop cosmological constant without the Bose–Fermi cancellation at each mass level (in other words, \( Z(\tau) \not\equiv 0 \), but \( \Lambda \equiv \int \frac{d^2z}{12} Z(\tau) = 0 \)). On the other hand, a characteristic feature of the string vacua given in the present paper (and those given in Ref. [27]) is that we have the Bose–Fermi cancellation among the different twisted sectors of the relevant orbifolding, as was emphasized several times. We would like to discuss elsewhere...
the possibility of realizing such “desirable situations,” at least in some point particle theories with infinite mass spectra (not necessarily string theories), by implementing this feature (Y. Satoh and Y. Sugawara, work in progress).

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Appendix A. Summary of conventions

We summarize the notations and conventions adopted in this paper. We set \( q = e^{2\pi ir} \), \( y = e^{2\pi iz} \).

A.1. Theta functions

\[
\theta_1(\tau, z) := i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} = 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \tag{A.1}
\]

\[
\theta_2(\tau, z) := \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} = 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \tag{A.2}
\]

\[
\theta_3(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-1/2})(1 + y^{-1}q^{m-1/2}), \tag{A.3}
\]

\[
\theta_4(\tau, z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1/2})(1 - y^{-1}q^{m-1/2}), \tag{A.4}
\]

\[
\Theta_{m,k}(\tau, z) := \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2})^2} y^{k(n+\frac{m}{2})}, \tag{A.5}
\]

\[
\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.6}
\]

Here, we have set \( q := e^{2\pi ir} \), \( y := e^{2\pi iz} \) (\( \forall r \in \mathbb{H}^+ \), \( z \in \mathbb{C} \)), and used the abbreviations, \( \theta_j(\tau) = \theta_j(\tau, 0) \) \( (\theta_1(\tau) \equiv 0, \Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0) \).

A.2. Character formulas for the \( \mathcal{N} = 2 \) minimal model

The character formulas for the level-\( k \) \( \mathcal{N} = 2 \) minimal model \( (\hat{c} = k/(k + 2)) \) [33–36] are described as the branching functions of the Kazama–Suzuki coset \( \frac{SU(2)_k \times U(1)_{1/2}}{U(1)_{k+2}} \) defined by

\[
\chi_{\ell}^{(k)}(\tau, w)\Theta_{s,2}(\tau, w - z) = \sum_{m \in \mathbb{Z}_{2(k+2)}} \chi_{m,s}^{\ell}(\tau, z)\Theta_{m,k+2}(\tau, w - 2z/(k + 2)),
\]

\[
\chi_{m,s}^{\ell}(\tau, z) \equiv 0, \quad \text{for } \ell + m + s \in 2\mathbb{Z} + 1, \tag{A.7}
\]
where $\chi^{(k)}_\ell(\tau, z)$ is the spin-$\ell/2$ character of $SU(2)_k$:

$$
\chi^{(k)}_\ell(\tau, z) = \frac{\Theta_{\ell+1, k} + 2 + \Theta_{-\ell-1, k} + 2}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} \equiv \sum_{m \in \mathbb{Z}_{2k}} c^{(k)}_{\ell,m}(\tau) \Theta_{m,k}(\tau, z).
$$

(A.8)

The branching function $\chi^{\ell_s}_m(\tau, z)$ is explicitly calculated as

$$
\chi^{\ell_s}_m(\tau, z) = \sum_{r \in \mathbb{Z}_k} c^{(k)}_{\ell,m+4r}(\tau) \Theta_{2m+(\ell_s + 4r)(k + 2)}(\tau, z/(k + 2)).
$$

(A.9)

Then, the character formulas of the unitary representations are written as

$$
\text{ch}_{\ell,m}^{(\text{NS})}(\tau, z) = \chi^{\ell,0}_m(\tau, z) + \chi^{\ell,2}_m(\tau, z),
$$

$$
\text{ch}_{\ell,m}^{(\text{5S})}(\tau, z) = \chi^{\ell,0}_m(\tau, z) - \chi^{\ell,2}_m(\tau, z),
$$

$$
\text{ch}_{\ell,m}^{(\text{F})}(\tau, z) = \chi^{\ell,1}_m(\tau, z) + \chi^{\ell,3}_m(\tau, z),
$$

$$
\text{ch}_{\ell,m}^{(\text{R})}(\tau, z) = \chi^{\ell,1}_m(\tau, z) - \chi^{\ell,3}_m(\tau, z).
$$

(A.10)

### Appendix B. Explicit forms of the building blocks and their orbifold twistings

We summarize here the explicit expressions for the spectral flow orbits in Eqs. (7)–(10) playing the role of building blocks of relevant modular invariants. We also describe the orbifold actions $\delta, \gamma$ on the spectral flow orbits, as well as the $\delta$ twistings on the theta function factor, denoted as $\theta_{i,(\alpha, \beta)}(\tau, z)$.

We make use of the abbreviated index $I \equiv \{(\ell_i, m_i)\} \{\ell_i + m_i \in 2\mathbb{Z}\}$ again, and set

$$
Q(I) \equiv Q \{\{(\ell_i, m_i)\} \mid \ell_i + m_i \in 2\mathbb{Z}\}.
$$

(B.1)

for convenience. $\mathcal{F}_I^{(\sigma)}(\tau, z)$ obviously vanishes for $Q(I) \notin \mathbb{Z}$ by the definitions in Eqs. (7)–(10), and we obtain the following expressions in the case of $Q(I) \in \mathbb{Z}$:

$$
\mathcal{F}_I^{(\text{NS})}(\tau, z) = \sum_{n \in \mathbb{Z}_N} \prod_{i=1}^r \text{ch}_{\ell_i m_i - 2n}^{(\text{NS})}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} F^{(\text{NS})}_{s_n(I)}(\tau, z),
$$

(B.2)

$$
\mathcal{F}_I^{(\text{5S})}(\tau, z) = (-1)^{Q(I)} \sum_{n \in \mathbb{Z}_N} (-1)^{(\ell + r)n} \prod_{i=1}^r \text{ch}_{\ell_i m_i - 2n}^{(\text{5S})}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} (-1)^{(\ell + r)n} F^{(\text{5S})}_{s_n(I)}(\tau, z),
$$

(B.3)

$$
\mathcal{F}_I^{(\text{F})}(\tau, z) = \sum_{n \in \mathbb{Z}_N} \prod_{i=1}^r \text{ch}_{\ell_i m_i - 2n - 1}^{(\text{F})}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} F^{(\text{F})}_{s_n(I)}(\tau, z),
$$

(B.4)

$$
\mathcal{F}_I^{(\text{R})}(\tau, z) = (-1)^{Q(I) + r} \sum_{n \in \mathbb{Z}_N} (-1)^{(\ell + r)n} \prod_{i=1}^r \text{ch}_{\ell_i m_i - 2n - 1}^{(\text{R})}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} (-1)^{(\ell + r)n} F^{(\text{R})}_{s_n(I)}(\tau, z),
$$

(B.5)

where we introduced the notation $s_n(I) := \{(\ell_i, m_i - 2n)\}$ for $I \equiv \{(\ell_i, m_i)\}$.

Then, the actions of $\delta$ twisting$^4$ for $\mathcal{F}_I^{(\sigma)}(\tau)$ are expressed explicitly as

$$
\delta_{i,(\alpha, \beta)} \cdot \mathcal{F}^{(\sigma)}_{\{(\ell_i, m_i)\}}(\tau) = \zeta_{i,\epsilon\delta} L(\sigma; \alpha, \beta) e^{2\pi i \frac{k_{\ell_m}^2 M_{\ell_m}}{N} \alpha \beta}
$$

$^4$Here we omit the subscripts $L$ and $R$ used in the main text.
\[
\sum_{n \in \mathbb{Z}} \exp \left\{ 2\pi i \sum_{i \in S} \frac{L(m_i - 2n)}{k_i + 2} \right\} F_{\left\{ (\ell_i, m'_i - 2n) \right\}}^{(\sigma)}(\tau) \quad (B.6)
\]

\[
\equiv \zeta_{2LM/(N')}^{2} \left( \sigma; \alpha, \beta \right) e^{2\pi i L^2 M N' \alpha \beta / N'}
\]

\[
\sum_{n \in \mathbb{Z}} \exp \left\{ 2\pi i \sum_{i \in S} \frac{d_i(m_i - 2n)}{k_i + 2} \right\} F_{\left\{ (\ell_i, m'_i - 2n) \right\}}^{(\sigma)}(\tau), \quad (B.7)
\]

\[d_i \equiv \frac{N'^2}{k_i + 2},\] where we introduced the phase factor

\[
\zeta_{\kappa}(NS; \alpha, \beta) = 1, \quad \zeta_{\kappa}(\tilde{NS}; \alpha, \beta) = e^{i\pi \kappa \alpha}, \\
\zeta_{\kappa}(R; \alpha, \beta) = e^{i\pi \kappa \beta}, \quad \zeta_{\kappa}(\tilde{R}; \alpha, \beta) = e^{i\pi \kappa (\alpha + \beta)},
\]

and set

\[m''_i := \begin{cases} m_i - 2L\alpha, & i \in S, \\ m_i, & \text{otherwise}. \end{cases}\]

On the other hand, the \(\gamma\) twisting of \(F_{\left\{ (\ell_i, m_i) \right\}}^{(\sigma)}(\tau)\) is expressed as

\[
\gamma_{(a,b)} : F_{\left\{ (\ell_i, m_i) \right\}}^{(\sigma)}(\tau) = \begin{cases} (-1)^b \sum_{i \in S_1} \ell_i F_{\left\{ (\ell_i, m_i) \right\}}^{(\sigma)}(\tau), & a = 0, \\ (-1)^b \sum_{i \in S_1} (\ell'_i + 1) F_{\left\{ (\ell'_i, m_i) \right\}}^{(\sigma)}(\tau), & a = 1, \end{cases} \quad (B.8)
\]

where we set

\[\ell'_i := \begin{cases} k_i - \ell_i, & i \in S_1, \\ \ell_i, & \text{otherwise}. \end{cases}\]

We next describe explicitly the Jacobi theta functions twisted by the \(\delta\) actions given in Eq. (24), that is,

\[
\theta_{i, (\alpha, \beta)}(\tau) := q^{\frac{\alpha^2}{2}} e^{2\pi i \frac{\alpha \beta}{2}} \theta_{i}(\tau, \frac{\alpha \tau + \beta}{2}), \quad i = 3, 4, 2, 1. \quad (B.9)
\]

They are explicitly written down as follows:

\(\alpha, \beta \in 2\mathbb{Z}\):

\[
\theta_{3, (\alpha, \beta)}(\tau) = (-1)^{\alpha \beta} \theta_{3}(\tau), \\
\theta_{4, (\alpha, \beta)}(\tau) = (-1)^{\alpha \beta + \frac{\alpha}{2}} \theta_{4}(\tau), \\
\theta_{2, (\alpha, \beta)}(\tau) = (-1)^{\alpha \beta + \frac{\beta}{2}} \theta_{2}(\tau), \\
\theta_{1, (\alpha, \beta)}(\tau) = (-1)^{\frac{\alpha \beta + \alpha + 1}{2}} \theta_{1}(\tau) \equiv 0; \quad (B.10)
\]

\(\alpha \in 2\mathbb{Z}, \beta \in 2\mathbb{Z} + 1:\)

\[
\theta_{3, (\alpha, \beta)}(\tau) = e^{i\pi \left( \frac{\alpha \beta + \alpha}{2} \right)} \theta_{3}(\tau), \\
\theta_{4, (\alpha, \beta)}(\tau) = e^{i\pi \frac{\alpha \beta}{2}} \theta_{3}(\tau), \\
\theta_{2, (\alpha, \beta)}(\tau) = e^{i\pi \left( \frac{\alpha \beta + \alpha + 1}{2} \right)} \theta_{1}(\tau) \equiv 0.
\]
\[ \theta_{1,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\beta - 1}{2} \right)} \theta_{2}(\tau); \quad (B.11) \]

\( \alpha \in 2\mathbb{Z} + 1, \beta \in 2\mathbb{Z}: \)

\[ \theta_{3,\alpha,\beta}(\tau) = e^{i\pi \frac{\alpha \beta}{4}} \theta_{2}(\tau), \]
\[ \theta_{4,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\beta}{2} \right)} \theta_{1}(\tau) \equiv 0, \]
\[ \theta_{2,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\alpha + \beta}{2} \right)} \theta_{3}(\tau), \]
\[ \theta_{1,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\beta - 1}{2} \right)} \theta_{4}(\tau); \quad (B.12) \]

\( \alpha, \beta \in 2\mathbb{Z} + 1: \)

\[ \theta_{3,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\beta}{2} \right)} \theta_{1}(\tau) \equiv 0, \]
\[ \theta_{4,\alpha,\beta}(\tau) = e^{i\pi \frac{\alpha \beta}{4}} \theta_{2}(\tau), \]
\[ \theta_{2,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\alpha + \beta + 1}{2} \right)} \theta_{4}(\tau), \]
\[ \theta_{1,\alpha,\beta}(\tau) = e^{i\pi \left( \frac{\alpha \beta}{4} + \frac{\beta - 1}{2} \right)} \theta_{3}(\tau). \quad (B.13) \]

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