THE HEAT KERNEL AND SPECTRAL ZETA FUNCTION FOR THE QUANTUM RABI MODEL

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ABSTRACT. The quantum Rabi model (QRM) is widely recognized as a particularly important model in quantum optics. It is considered to be the simplest and most fundamental system describing quantum light-matter interaction. Its Hamiltonian is known to have a $\mathbb{Z}_2$-symmetry, called parity. The purpose of the present paper is twofold. Firstly, we describe the heat kernel of the Hamiltonian explicitly using the Trotter-Kato product formula. To the best knowledge of the authors, this is the first explicit derivation of a closed formula of the heat kernel for any non-trivial interacting quantum system. Further, the heat kernel for this model is given by a two-by-two matrix of operators and is expressed as a direct sum of two heat kernels representing the parity decomposition. Secondly, we investigate basic properties of the spectral zeta function for the QRM (and with each parity) via the Mellin transform of the partition function of the QRM, that is, the trace of the integral operator defined by the heat kernel. These properties show the meromorphic continuation, describe special values at negative integers of the spectral zeta function, and form the basis for advancing potential studies and further number theoretic investigation. We expect that the methods developed in this paper may be applicable to other quantum interaction systems.

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1. Introduction

The quantum Rabi model (QRM) is widely recognized as the simplest and most fundamental model describing quantum light-matter interactions, that is, the interaction between a two-level system and a bosonic field mode (see e.g. [4] for a recent collection of introductory, survey and original articles after Isidor Rabi’s seminal paper [42] on the semi-classical (Rabi) model in 1936 and the full quantization [23] established in 1963 by Jaynes and Cummings).

The Hamiltonian $H_{\text{Rabi}}$ is precisely given by

$$H_{\text{Rabi}} := \omega a^\dagger a + \Delta \sigma_z + g(a + a^\dagger)\sigma_x.$$  

Here, $a^\dagger$ and $a$ are the creation and annihilation operators of the single bosonic mode ($[a, a^\dagger] = 1$), $\sigma_x, \sigma_z$ are the Pauli matrices, $2\Delta$ is the energy difference between the two levels, $g$ denotes the coupling strength between the two-level system and the bosonic mode with frequency $\omega$ (subsequently, we set $\omega = 1$ without loss of generality). The Hamiltonian for the quantum Rabi model (QRM) possesses a $\mathbb{Z}_2$-symmetry, usually called parity. Using this $\mathbb{Z}_2$-symmetry, the QRM was shown to be integrable in [4]. The QRM actually appears ubiquitously in various quantum systems including cavity and circuit quantum electrodynamics, quantum dots and artificial atoms with potential applications in quantum information technologies (see e.g. [13, 23]) for a recent collection of introductory, survey and original articles after Isidor Rabi’s seminal paper [42] in 1936 and the full quantization [23] established in 1963 by Jaynes and Cummings).

The purpose of the present paper is first to obtain a closed explicit expression of the heat kernel and the partition function of the QRM. Let $K_{\text{Rabi}}(x, y, t)$ denote the heat kernel of $H_{\text{Rabi}}$, that is, the function satisfying $\frac{\partial}{\partial t} K_{\text{Rabi}}(x, y, t) = -H_{\text{Rabi}}K_{\text{Rabi}}(x, y, t)$ for all $t > 0$ and $\lim_{t \to 0} K_{\text{Rabi}}(x, y, t) = \delta_x(y)$ for $x, y \in \mathbb{R}$. The heat kernel is the integral kernel corresponding to the operator (one-parameter semigroup) $e^{-tH_{\text{Rabi}}}$, that is,

$$e^{-tH_{\text{Rabi}}} \phi(x) = \int_{-\infty}^{\infty} K_{\text{Rabi}}(x, y, t) \phi(y) dy$$

for a compactly supported smooth function $\phi$ on $\mathbb{R}$. We remark that the heat kernel for this model is given by a two-by-two matrix of operators. For the explicit computation of the heat kernel of the QRM, detailed calculations employing the Trotter-Kato product formula (or the exponential product formula for the semigroup) [24, 39] for the pair of (in general) self-adjoint unbounded operators are indispensable.

In statistical physics, the partition function of a system is of fundamental importance as it describes the statistical properties of the system in thermodynamic equilibrium. The partition function is a function of temperature and other parameters, such as the volume enclosing a gas. The partition function is indispensable. The computation of the partition function follows from the explicit formula of the heat kernel.

Then, as the second aim of the present paper, or rather the initial motivation of this study, we study the spectral zeta function $\zeta_{\text{QRM}}(s)$ of the QRM using the explicit formula of the heat kernel. Let $(-\infty) \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots (\nearrow \infty)$ be the eigenvalues of $H_{\text{Rabi}}$ (notice that in general the lowest eigenvalue can be negative, see e.g. [13, 23]). It is well-known that the multiplicity of $\lambda_j$ is less than or equal to 2 (see [4, 25, 13]). Therefore, for $\tau \in \mathbb{C}$ with $\tau + \lambda_j \neq 0$ for any $j \geq 1$, we may
define the (Hurwitz-type) spectral zeta function $\zeta_{\text{QRM}}(s; \tau)$ as

$$\zeta_{\text{QRM}}(s; \tau) := \sum_{j=1}^{\infty} (\lambda_j + \tau)^{-s}.$$  

The defining series converges absolutely for $\Re(s) > 1$ when $\tau > \Delta + g^2$, whence it defines a holomorphic function in this right half plane. Furthermore, the spectral zeta function $\zeta_{\text{QRM}}(s; \tau)$ is analytically continued to the whole complex plane as a meromorphic function with a unique simple pole at $s = 1$ having the residue $\text{Res}_{s=1}\zeta_{\text{QRM}}(s; \tau) = 2$ in [49] by a rather difficult computation on the asymptotic expansion of the (unknown) heat kernel followed by the technique developed in [20]. The explicit formula of the heat kernel simplifies the proof of the meromorphic continuation of $\zeta_{\text{QRM}}(s; \tau)$ to $\mathbb{C}$. Actually, the technique is an analogue of one of the standard ways to establish the meromorphic continuation for the Riemann zeta function and other Dirichlet series ([22, 50]).

In the present notation, we observe that $Z_{\text{Rabi}}(\beta) = \sum_{j=1}^{\infty} e^{-\beta \lambda_j}$. It is also expressed by the heat kernel as

$$Z_{\text{Rabi}}(\beta) = \int_{-\infty}^{\infty} \text{tr} K_{\text{Rabi}}(x, x, \beta) dx,$$

where $\text{tr}$ denotes the (two by two) matrix trace. Therefore, the complex power of the Hamiltonian can be naturally defined by the Mellin transform of the operator $e^{-i t H_{\text{Rabi}}}$ as

$$H_{\text{Rabi}}^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-i t H_{\text{Rabi}}} dt.$$  

Therefore, upon defining the constant shift of the Rabi Hamiltonian by $H_{\text{Rabi}, \tau} := H_{\text{Rabi}} + \tau$, the spectral zeta function is expressed by the Mellin transform of the partition function as follows.

$$\zeta_{\text{QRM}}(s; \tau) = \text{Tr}[H_{\text{Rabi}, \tau}^{-s}] = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \text{Tr}[e^{-i t H_{\text{Rabi}, \tau}}] dt = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} Z_{\text{Rabi}}(t) e^{-i \tau} dt.$$  

This expression also indicates that the partition function $Z_{\text{Rabi}}(t)$ is obtained by the inverse Mellin transform of $\zeta_{\text{QRM}}(s; \tau)$ (see e.g. [50]). On the other hand, in number theory, it is well-known that the Mellin transform sends modular forms to $L$-functions and this correspondence is fundamental for the study of $L$-functions because it gains the knowledge from representation theory. For instance, the functional equations follow from the invariance under a particular group action (e.g. a discrete subgroup of $SL_2(\mathbb{R})$). From this point of view, in addition to the representation theoretic understanding of eigenvalues ([25, 53]), the study of the spectral zeta function and its relation to the partition function may be helpful to deepen the understanding between the modularity (symmetry) and the physical interaction.

As an example to illustrate the discussion above, let us consider the quantum harmonic oscillator, mathematically given by the Hamiltonian $H := -\frac{d^2}{dx^2} + x^2 + \frac{1}{2}$ (we shift the Hamiltonian by $\frac{1}{2}$ for simplicity) acting on $L^2(\mathbb{R})$. The eigenvalues of $H$ are given by the set of all positive integers $\mathbb{N}$. Hence the partition function is given by the geometric sum $Z(t) := \sum_{n=1}^{\infty} e^{-n t} = \frac{e^{-t}}{1-e^{-t}}$ and the spectral zeta function is the Riemann zeta function $\zeta(s)$. Using the integral formula $a^{-s} = \Gamma(s) \int_{0}^{\infty} t^{s-1} e^{-a t} dt$ for $a > 0$ with $a = n \in \mathbb{N}$ we have the Mellin transform expression of $\zeta(s)$ by $Z(t)$, but taking $a = n^2 (n \in \mathbb{N})$ and replacing $s$ by $s/2$, we have the integral expression of $\zeta(s)$ by the Mellin transform of the Jacobi theta function $\theta(t)$ in place of $Z(t)$. The latter is a typical correspondence of modular forms and $L$-functions. We remark here that the special values of $\zeta(s)$ at the negative integers are obtained by the Mellin integral expression by $Z(t)$ through its expansion employing the Bernoulli numbers and by virtue of the functional equation between $s$ and $1-s$ of $\zeta(s)$, the special values at the positive even integers are obtained. Although, it is arguably difficult to expect any functional equation of a spectral zeta function in general, the collection of the special values (for a suitably chosen constant $\tau(\in \mathbb{R})$) may provide useful information on the partition function of a given quantum system similar.
to the way “moments” can describe certain properties of a distribution. Thus the study of the special values of the spectral zeta functions might be important in physics as well.

The paper is organized as follows. A large part of the paper (§2 through §4) is devoted to obtaining an explicit expression of the heat kernel of the QRM. In general, with the exception of trivial cases, an explicit derivation/computation of heat kernel is a difficult problem and often involves specific conditions on the operator or the use of sophisticated but mathematically transcendental techniques including the Feynman path integrals and Feynman-Kac formulas (cf. [8, 14] and also [16] for the relation between the path integral and the (Lie-)Trotter-Kato product formula). Actually, it is yet to be obtained for the non-commutative harmonic oscillator (NCHO), defined by a deformation of the tensor product of the quantum harmonic oscillator (cf. [15]) and trivial two dimensional representation of $\mathfrak{sl}_2$ (see [39, 37]), even though the NCHO (with generalized parameters) gives the QRM through a confluence process (for two regular singular points) at the Heun ODE picture of respective models [52]. In §2 we make preliminary calculations based on the Trotter-Kato product formula by employing newly defined two-by-two matrix-valued creation and annihilation operators $b$ and $b^\dagger$ depending on the parameter $g$. The resulting expression is a limit ($N \to \infty$) that, at a glance, resembles a Riemann sum where each of the summands contains a sum of exponential terms over subsets $C^{(t)}_{ij}$ of $\mathbb{Z}_2^N$ for $2 \leq \ell \leq N$ and $i,j \in \{0,1\}$ (see Definition 2.1 in §2.4). However, due to the presence of hard to control (infinitely many) changes of signature with non-trivial coefficients at the exponential terms, there does not appear to be a way to evaluate this expression directly.

In order to overcome the aforementioned difficulties in the evaluation of the limit, in §3 we make use of harmonic analysis on the finite groups $\mathbb{Z}_2^k$ ($k \in \mathbb{Z}_{\geq 0}$). Concretely, by noticing a natural bijection between $C^{(t)}_{ij}$ and $\mathbb{Z}_2^{-\ell}$, we transform the sum over $C^{(t)}_{ij}$ into an equivalent sum over the dual group of $\mathbb{Z}_2^{-\ell}$ using the Fourier transform. To simplify the resulting expressions, in addition to the standard theory of harmonic analysis, we also develop certain graph theoretical and combinatorial techniques in §3.2. The transformed sum is then seen to consist of a radial function part (for $\rho \in \mathbb{Z}_2^{-\ell}$) that is controlled by fixing $|\rho| = \lambda$, while the sum of remaining part is evaluated as a multiple integral in §3.4. We remark that transforming the computation into the dual stage (i.e. the Fourier image) is not only indispensable in order to evaluate the sum in practice, but it also reveals certain structural information that appears in the final expression of the heat kernel (cf. Lemma 3.14).

Finally, the limit is evaluated in §4, thus completing the computation of the heat kernel. The final result (Theorem 4.2) shows that the heat kernel of the QRM is expressed as an infinite sum of the terms given by $k$-iterated integrals ($k = 0, 1, 2, \ldots$). It is important to notice that the point-wise convergence of the iterated integral kernels to the heat kernel follows from the fact that the corresponding sequence of the Trotter-Kato approximation operators converges not only in the strong operator topology but in the operator norm topology as well (see §3 and references therein).

As we mentioned at the beginning, the Hamiltonian for the QRM possesses a parity ($\mathbb{Z}_2$-symmetry). From this fact, we see that the heat kernel for the model, given by two-by-two matrix of operators, is expressed as the direct sum of two heat kernels which represent the parity decomposition (Theorem 5.1). We then derive the explicit formula for the partition function for each parity (Corollary 5.2).

We note that in [20] the special values for the case of the NCHO were computed by knowing the trace of the iterated heat kernel directly (implicitly connected to the Trotter-Kato product formula used in the present paper) but without explicit knowledge (i.e. a closed formula) of the heat kernel itself. For this reason, the Mellin transform of the heat kernel of the NCHO (see also [21, 49]) could not be used. Precisely, the special value of the spectral zeta function $\zeta_{\text{NCHO}}(s)$ for the NCHO at $n \in \mathbb{N}$ is computed as

$$\zeta_{\text{NCHO}}(n) = \int_0^\infty dt \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty d\mathbf{x} \, \text{tr} \, K_{\text{NCHO}}(x,x_1,t)K_{\text{NCHO}}(x_1,x_2,t) \cdots K_{\text{NCHO}}(x_{n-1},x,t),$$

where $d\mathbf{x} = dx_1dx_2 \cdots dx_{n-1}$ and $K_{\text{NCHO}}(x,y,t)$ is the heat kernel of the NCHO. We remark that although the analytic continuation of $\zeta_{\text{NCHO}}(s)$ to the whole plane $\mathbb{C}$ has been obtained using the
asymptotic expansion of the partition function \[20, 38\], it has not been yet obtained by a contour integral representation for \(s \in \mathbb{C}\) like the one for the Riemann zeta function.

In this paper, we obtain explicitly a contour integral representation of the \(\zeta_{\text{QRM}}(s; \tau)\) (Theorem 6.3) using the formula of the partition function \(Z_{\text{Rabi}}(t)\) of the QRM (Corollary 6.3). From this expression, we define the Rabi-Bernoulli polynomials in the same way as the Bernoulli polynomials through a similar, but non-trivial, generating function for the classical Hurwitz zeta function (Definition 6.1). In this way, we immediately obtain the expression of the special values of \(\zeta_{\text{QRM}}(s; \tau)\) at negative integers in terms of the Rabi-Bernoulli polynomials. Note that the definition of the Rabi-Bernoulli polynomials defined here is seen to be identical with the one in \([49]\). Moreover, according to the parity decomposition of the heat kernel, we define also the parity spectral zeta functions \(\zeta_{\text{Rabi}}\) and give a contour integral expression by the respective partition function \(Z_{\text{Rabi}}(t)\) (Corollary 6.4). Thus, the Weyl law for the eigenvalue distribution for the positive (resp. negative) parity eigenvalues follows easily from the respective residue at \(s = 1\). Particularly, this shows that the positive and negative parity eigenstates are equally distributed (notice that this fact also follows from the relation \(G^-(x, g, \Delta) = G^+(x, g, -\Delta)\) between \(G^\pm\)-functions \([4]\) and the properties of the constraint functions/polynomials \([25]\) and supports also the original Braak conjecture concerning the number of eigenvalues (in each parity) in the consecutive intervals (see \([4]\)). Furthermore, we introduce the Rabi-Bernoulli polynomials corresponding to each parity in order to describe the special values of \(\zeta_{\text{QRM}}^\pm(s; \tau)\) at negative integer points. Like in \([20, 49]\), we show that the Rabi-Bernoulli polynomials have rational coefficients, that is, are in \(\mathbb{Q}[\tau, g^2, \Delta]\) (Theorem 6.9).

We note that despite having contour integral expression for \(\zeta_{\text{QRM}}(s; \tau)\), we can not get any information about the special values at positive integers (even for positive even integers points) directly from this expression. Since it is difficult to expect the existence of any reasonable functional equation for \(\zeta_{\text{QRM}}(s; \tau)\) or \(\zeta_{\text{QRM}}^\pm(s; \tau)\), it is necessary to explore another method in order to obtain the special values at positive integers as developed in \([21, 28]\) for the NeHO.

Moreover, as we have shown in \([25]\) for asymmetric quantum Rabi models, we see the \(G\)-functions for the parity Hamiltonians \(H_{\pm} \equiv H\) of \([4]\) are given, up to a non-zero entire function, by the corresponding spectral determinant (Corollary 6.14), i.e. zeta regularized product/determinant (cf. \([11, 20]\) defined through \(\zeta_{\text{QRM}}(s; \tau)\). This indicates (at least in the case of each parity of the QRM) that the integrability shown in \([4]\) of the QRM and the existence of an analytic continuation of the parity zeta functions \(\zeta_{\text{Rabi}}(s, \tau)\) to a neighborhood of \(s = 0\) from the right half plane \(\Re(s) > 1\) are intimately related. As a by-product of the explicit expression, we may also define an infinite family of hierarchical zeta functions \(\zeta_{\text{QRM}}^\pm(n)(s; g^2, \tau) (n = 0, 1, 2, \ldots)\) (which does not depend on \(\Delta\)) for the QRM. The 0-th hierarchical zeta function is essentially the Hurwitz zeta function and the 1st one is written in terms of a certain integral of the incomplete gamma function (see Remark 6.2). For the readers’ convenience, we put a brief introduction of necessary facts about the \(G\)-functions associated to the QRM in the Appendix.

In a forthcoming paper, we will focus the study on the special values of \(\zeta_{\text{QRM}}(s; \tau)\) and \(\zeta_{\text{QRM}}^\pm(s; \tau)\) (and \(\zeta_{\text{QRM}}^\pm(n)(s; g^2, \tau)\)) at positive integer points. Our expectation is that the special values of these zeta functions have a rich arithmetic structure similar to the one observed in the case of the NCHO (e.g. \([27, 28, 36, 33]\)). According to this line, the authors hope the study of special values of \(\zeta_{\text{QRM}}(s; \tau)\) may bridge an interesting relationship between number theory and physics through the partition function \(Z_{\text{Rabi}}(t)\) such as modular forms, congruence relations among Apéry-like numbers (\([28, 56, 29]\)), modular Mahler measures (\([15]\) and automorphic integrals’ (\([30]\), a slightly extended notion of Eichler’s forms) interpretation of the moments of the QRM from the study on special values of \(\zeta_{\text{QRM}}(s; \tau)\) and \(\zeta_{\text{QRM}}^\pm(s; \tau)\) as in \([27, 28]\).

To the best knowledge of the authors, this is the first non-trivial example of explicit computation of the heat kernel of an interacting quantum system using the Trotter-Kato product formula. The method developed in this paper using the Trotter-Kato formula may be generalized to other quantum systems, for instance to generalizations of the QRM like the asymmetric quantum Rabi model (AQRM).
or the Dicke model (see e.g. [2]). We believe that this method may play the role of a compass in the study of other Hamiltonians and their heat kernels. Therefore, the closed formula for the heat kernel and resulting time evolution operator $e^{-itH_{\text{Rabi}}} (t \in \mathbb{R})$ which is unitary and describes a strongly continuous one-parameter group of unitary transformations in the Hilbert space, because $H_{\text{Rabi}}$ is self-adjoint, according to Stone’s theorem, may have a great importance. Furthermore, we remark that although the QRM is in the scientific spotlight in theoretical and experimental physics, a full-fledged classification and consequent theoretical prediction of coupling regimes remains unclear (see, e.g. [16]). We expect the explicit closed formula of the heat kernel and partition function obtained in this paper may contribute to the investigation in this direction.

2. Preliminary calculations based on the Trotter-Kato product formula

The Hamiltonian of the quantum Rabi model (QRM) is given by

$$H_{\text{Rabi}} = a^\dagger a + \Delta \sigma_z + g(a + a^\dagger)\sigma_x,$$

where $\sigma_x, \sigma_z$ are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $a^\dagger, a$ are the creation and annihilation operators of the quantum harmonic oscillator satisfying the commutation relation $[a, a^\dagger] = 1$. In this paper we tacitly assume $\hbar = \omega = 1$ without loss of generality.

By setting $b = b(g) := a + g\sigma_x$ (and $b^\dagger = a^\dagger + g\sigma_x$), the Hamiltonian $H_{\text{Rabi}}$ is rewritten as

$$H_{\text{Rabi}} = (a^\dagger + g\sigma_x)(a + g\sigma_x) + \Delta \sigma_z - g^2 = b^\dagger b - g^2 + \Delta \sigma_z.$$

Notice that while the operators $b, b^\dagger$ satisfy the commutation relation $[b, b^\dagger] = I_2$, the operator $b$ does not commute with $\Delta \sigma_z$. In this sense we regard

$$b^\dagger b - g^2,$$

as a (two dimensional) non-commutative version of the quantum harmonic oscillator.

From the commutation relation $[b, b^\dagger] = I_2$, it is clear that the operator $b^\dagger b - g^2$ is self-adjoint and bounded below. The operator $\Delta \sigma_z$ is also self-adjoint and bounded for trivial reasons. We remark that it is a well-known fact (see for example [49]) that $H_{\text{Rabi}}$ is a self-adjoint bounded below operator. Therefore, the operators $b^\dagger b - g^2$ and $\Delta \sigma_z$ satisfy the conditions of the Trotter-Kato product formula (cf. [8, 24, 54]) and we have

$$e^{-tH_{\text{Rabi}}} = e^{-t(b^\dagger b - g^2 + \Delta \sigma_z)} = \lim_{N \to \infty} \left( e^{-t(b^\dagger b - g^2)/N} e^{-t(\Delta \sigma_z)/N} \right)^\!N,$$

in the strong operator topology. Moreover the sequence $\{(e^{-t(b^\dagger b - g^2)/N} e^{-t(\Delta \sigma_z)/N})^\!N \}_{N=1,2,...}$ of Trotter-Kato’s approximation operators converges in the operator norm topology when $N \to \infty$. In fact, we have

$$\|e^{-t(b^\dagger b - g^2 + \Delta \sigma_z)} - (e^{-t(b^\dagger b - g^2)/N} e^{-t(\Delta \sigma_z)/N})^\!N\|_{\text{op}} = O(N^{-1}),$$

where $\|A\|_{\text{op}} := \sup_{\|u\| \neq 0} \frac{\|Au\|}{\|u\|}$ denotes the operator norm (see the review paper [19] for the general theory leading to this fact). Moreover, pointwise uniformly convergence of the iterated integral kernels to the heat kernel follows from the convergence in operator norm topology ([8, 17, 19], cf. [18]).

The objective of this section is to compute the integral kernel $K^{(N)}(x, y, t)$ of the $N$-th power operator

$$(e^{-t(b^\dagger b - g^2)} e^{-t(\Delta \sigma_z)})^\!N,$$

explicitly. Concretely, in [2, 1] we compute the integral kernel $K(x, y, t)$ of the operator

$$e^{-t(b^\dagger b - g^2)} e^{-t(\Delta \sigma_z)},$$

following the standard procedure for the quantum harmonic oscillator. The computation of the $N$-th power kernel is divided into a scalar part in [2, 3] and a non-commutative part in [2, 4].
In this paper we consider the Hamiltonian $H_{\text{Rabi}}$ as an operator acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$. For convenience of the reader we recall that the creation and annihilation operators are realized by

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right)$$

as operators acting on the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ which has a basis consisting of Hermite functions, cf. [15, 1]).

2.1. Quantum Rabi model and quantum harmonic oscillators. As a first step in the computation of the heat kernel of the QRM, in this subsection we compute the integral kernel $K(x, y, t)$ of the operator

$$e^{-t(b^\dagger b - g^2)} e^{-t(\Delta_{x^2})}.$$  

First, we notice that by the elementary identity

$$e^{-t(\Delta_{x^2})} = \begin{bmatrix} e^{-t\Delta} & 0 \\ 0 & e^{t\Delta} \end{bmatrix},$$

the integral kernel for the operator $e^{-t(\Delta_{x^2})}$ is given by $K_2(x, y, t) = e^{-t(\Delta_{x^2})}\delta(x - y)$, where $\delta$ is the Dirac measure. Thus, the remainder of this subsection is dedicated to computing the integral kernel of $e^{-b^\dagger b - g^2}$. As we have remarked before, the commutation identity $[b, b^\dagger] = \mathbf{1}_2$ holds and thus the computation of the integral kernel follows the general procedure for the quantum harmonic oscillator.

In particular, if we find the ground state $\psi_0$ for $b$, that is, a solution of $b\psi_0 = 0$, then the spectrum of $b^\dagger b$ is equal to $\mathbb{Z}_{\geq 0}$ with each eigenvalue having multiplicity 2.

The general solution of the differential equation system $b\psi = 0$ with $\psi = T(\psi_1, \psi_2)$ is given by

$$
\begin{align*}
\psi_1 &= c_1 e^{-x^2/2 - \sqrt{2}g} - c_2 e^{-x^2/2 + \sqrt{2}g}, \\
\psi_2 &= c_1 e^{-x^2/2 - \sqrt{2}g} + c_2 e^{-x^2/2 + \sqrt{2}g}
\end{align*}
$$

for arbitrary constants $c_1, c_2 \in \mathbb{C}$. It is clear then that

$$
\begin{align*}
\psi_0^{(1)} &= e^{-x^2/2 - \sqrt{2}g} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\psi_0^{(2)} &= e^{-x^2/2 + \sqrt{2}g} \begin{bmatrix} -1 \\ 1 \end{bmatrix},
\end{align*}
$$

are two linearly independent eigenfunctions of $b^\dagger b$ corresponding to eigenvalue $\lambda = 0$. We obtain directly

$$
\begin{bmatrix} \psi_0^{(1)}, \psi_0^{(1)} \end{bmatrix}_{\mathcal{H}} = \begin{bmatrix} \psi_0^{(2)}, \psi_0^{(2)} \end{bmatrix}_{\mathcal{H}} = 2e^{2g^2\sqrt{\pi}},
$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the inner-product induced in $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ by the usual $L^2(\mathbb{R})$ inner-product.

For $\lambda = n$ the orthonormal eigenstates are given by

$$
\begin{align*}
\psi_n^{(1)} &= \frac{1}{\sqrt{2}} e^{g^2} H_n(x + \sqrt{2}g) \frac{\psi_0^{(1)}}{\sqrt{\pi n! 2^n} n^{1/2}}, \\
\psi_n^{(2)} &= \frac{1}{\sqrt{2}} e^{g^2} H_n(x - \sqrt{2}g) \frac{\psi_0^{(2)}}{\sqrt{\pi n! 2^n} n^{1/2}},
\end{align*}
$$

where $H_n$ is the $n$-th Hermite polynomial. Due to the normalization factor $e^{-g^2}$, we have

$$
\begin{align*}
\psi_n^{(1)} &= \frac{1}{\sqrt{2}} H_n(x + \sqrt{2}g) e^{-(x+\sqrt{2}g)^2/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\psi_n^{(2)} &= \frac{1}{\sqrt{2}} H_n(x - \sqrt{2}g) e^{-(x-\sqrt{2}g)^2/2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\end{align*}
$$

The heat kernel $K_1(x, y, t)$ of $b^\dagger b - g^2$ is given formally by the Schwartz kernel

$$K_1(x, y, t) := \sum_\lambda \psi_\lambda(x)^T \psi_\lambda(y) e^{-\lambda g^2 t},$$

where the sum is over the eigenvalues $\lambda$ of $b^\dagger b$ (counting multiplicities) and $\psi_\lambda(x)$ is the eigenfunction corresponding to the eigenvalue $\lambda$. It is left to verify the convergence and that $K_1(x, y, t) \to \delta(x - y)\mathbf{1}_2$ as $t \to 0$. 

Convergence follows component-wise by Mehler’s formula (Poisson kernel expression, cf. [1] [3]) for Hermite polynomials

\[ \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = (1 - r^2)^{-1/2} e^{(2xyr - (x^2 + y^2)r^2)/(1 - r^2)}, \]

valid for \(|r| < 1\).

For the second property, recall that, as \(r \to 1\) the following completeness identity holds

\[ \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)e^{-\frac{r^2}{2}}}{\sqrt{\pi} 2^n n!} r^n = \delta(x - y), \quad r \to 1 \]

in the sense of distributions.

Thus, applying the substitutions \(x \to x \pm \sqrt{2}y\), \(y \to y \pm \sqrt{2}g\) we see that

\[ \sum_{n=0}^{\infty} \psi_n^{(1)}(x) T \psi_n^{(1)}(y) r^n = \frac{\delta(x - y)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad r \to 1, \]

and

\[ \sum_{n=0}^{\infty} \psi_n^{(2)}(x) T \psi_n^{(2)}(y) r^n = \frac{\delta(x - y)}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad r \to 1, \]

giving the desired expression when \(r = e^{-t}\).

We write

\[ K_1(x, y, t) = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \psi_n^{(j)}(x) T \psi_n^{(j)}(y) e^{-t(n + g^2)}, \]

then, by Mehler’s formula we have

\[ \sum_{n=0}^{\infty} \psi_n^{(1)}(x) T \psi_n^{(1)}(y) u^n u^g^2 = \frac{1}{2 \sqrt{\pi}} \sum_{n} \frac{H_n(x + \sqrt{2}y)H_n(y + \sqrt{2}g)}{2^n n!} u^n e^{-\frac{1}{2}((x + \sqrt{2}y)^2 + (y + \sqrt{2}g)^2)} u^g^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ = \frac{1}{2 \sqrt{\pi}(1 - u^2)} \exp \left( \frac{1 - u (x + y + 2\sqrt{2}g)^2}{4} - \frac{1 + u (x - y)^2}{4} \right) u^g^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \]

with \(u := e^{-t}\), and similarly, we obtain

\[ \sum_{n=0}^{\infty} \psi_n^{(2)}(x) T \psi_n^{(2)}(y) u^n u^g^2 = \frac{1}{2 \sqrt{\pi}(1 - u^2)} \exp \left( \frac{1 - u (x + y - 2\sqrt{2}g)^2}{4} - \frac{1 + u (x - y)^2}{4} \right) u^g^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \]

By factoring out common terms we obtain

\[ K_1(x, y, t) = \frac{1}{\sqrt{\pi}(1 - u^2)} \exp \left( \frac{1 - u ((x + y)^2 + 8g^2)}{4} - \frac{1 + u (x - y)^2}{4} \right) u^g^2 \]

\[ \times \frac{1}{2} \left( \exp \left( \frac{1 - u \sqrt{2}g(x + y)}{1 + u} \right) M_{11} + \exp \left( \frac{1 - u \sqrt{2}g(x + y)}{1 + u} \right) M_{00} \right), \]

where

\[ M_{00} := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad M_{11} := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

The matrix terms (including the scalar factor \(\frac{1}{2}\)) are equal to

\[ \begin{bmatrix} \cosh \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \right) & -\sinh \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \right) \\ -\sinh \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \right) & \cosh \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \right) \end{bmatrix} = \exp \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \sigma_x \right), \]

and the final expression for the heat kernel of \(b^{1b} - g^2\) is

\[ K_1(x, y, t) = \frac{1}{\sqrt{\pi}(1 - u^2)} \exp \left( \frac{1 - u ((x + y)^2 + 8g^2)}{4} - \frac{1 + u (x - y)^2}{4} \right) u^g^2 \exp \left( \frac{1 - u}{1 + u} \sqrt{2}g(x + y) \sigma_x \right). \]
Summarizing the discussion above, we have the following explicit description for $K(x, y, t)$.

**Proposition 2.1.** The integral kernel $K(x, y, t)$ for $e^{-t(b^2 - g^2)}e^{-t(\Delta_x)}$ is given by

$$K(x, y, t) = \frac{v^g}{\sqrt{\pi(1 - u^2)}} \exp \left( -\frac{1 - u ((x + y)^2 + 8g^2)}{1 + u} - \frac{1 + u (x - y)^2}{1 - u} \right) \times \exp \left( -\frac{1 - u}{1 + u} \sqrt{2g(x + y)\sigma_x} \right) e^{-t(\Delta_x)}.$$

**Proof.** Since

$$K(x, y, t) = \int_{-\infty}^{\infty} K_1(x, z, t)K_2(z, y, t)dz$$

the desired expression follows from the definition of the Dirac distribution $\delta$. \qed

To simplify later computations we write

$$K(x, y, t) = K_0(x, y, g, u) \exp \left( -2g^2 \frac{1 - u}{1 + u} \right) \exp \left( -\frac{1 - u}{1 + u} \sqrt{2g(x + y)\sigma_x} \right) e^{-t(\Delta_x)},$$

with

$$K_0(x, y, g, u) := \frac{v^g}{\sqrt{\pi(1 - u^2)}} \exp \left( -\frac{1 + u^2}{2(1 - u^2)}(x^2 + y^2) + \frac{2uvy}{1 - u^2} \right).$$

2.2. The $N$-th power kernel $K^{(N)}(x, y, t)$. In the remainder of this section we compute explicitly the integral kernel $K^{(N)}(x, y, t)$ of the operator

$$\left(e^{-t(b^2 - g^2)}e^{-t(\Delta_x)}\right)^N$$

given by the integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K(x, v_1, t)K(v_1, v_2, t) \cdots K(v_{N-1}, y, t)dv_{N-1}dv_{N-2} \cdots dv_1. \leqno{(2)}$$

Writing $v_0 = x$ and $v_N = y$, we see that the integrand of (2) is given by the product of the scalar factor

$$\frac{u^Ng^2}{(\pi(1 - u^2))^{N/2}} \exp \left( \sum_{i=1}^{N} \left( -\frac{1 + u^2}{2(1 - u^2)}(v_i^2 + v_{i-1}^2) + \frac{2uv_i v_{i-1}}{1 - u^2} - 2g^2 \frac{1 - u}{1 + u} \right) \right)$$

and the matrix factor

$$\prod_{i=1}^{N} \left[ \cosh \left( \sqrt{2g} \frac{1 - u}{1 + u} (v_{i-1} + v_i) \right) I - \sinh \left( \sqrt{2g} \frac{1 - u}{1 + u} g(v_{i-1} + v_i) \right) J \right] \sigma_x \leqno{(3)}$$

where $\prod_{i=1}^{N} A_i$ denotes the (ordered) product $A_1 A_2 \cdots A_N$ of the matrices $A_i$'s and we write $J = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for simplicity.

Let us introduce some general notation. We write $\mathbb{Z}_2^k$ for $\{0, 1\}^k$ with $k \geq 1$, both as a set and as an abelian group (that is, for the group $(\mathbb{Z}/2\mathbb{Z})^k = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$ (k-times)), for $k = 0$ we define $\mathbb{Z}_2^0 = \{0\}$ both as a set and (trivial) group. To simplify the notation, at times we consider an element $s \in \mathbb{Z}_2^k$ as a function $s : \{0, 1, \cdots, k\} \rightarrow \{0, 1\}$ where $s(i)$ is the $i$-th component of $s \in \mathbb{Z}_2^k$.

Since, for $\alpha \in \mathbb{R}$, we have

$$\cosh(\alpha) I - \sinh(\alpha) J = \frac{1}{2} (I + J) e^{-\alpha} + \frac{1}{2} (I - J) e^\alpha,$$

the multiplication of matrices in (3) gives a linear combination of terms

$$G_N(u, \Delta, s) \prod_{i=1}^{N} \exp \left( (-1)^{s(i)} \sqrt{2g} \frac{1 - u}{1 + u} (v_i + v_{i-1}) \right),$$
where the choice of $s \in \mathbb{Z}_2^N$ depends on the factors sinh and cosh appearing in the expansion of $G_N(u, \Delta, s)$ is a matrix-valued function given by

$$G_N(u, \Delta, s) := \frac{1}{2^N} \prod_{i=1}^{N} [I + (-1)^{1-s(i)}] u^{\Delta s_i}.$$  

In addition, for $s \in \mathbb{Z}_2^N$, by defining

$$I_N(v_0, v_N, u, s) := \frac{u^{N s^2}}{(\pi (1-u^2))^{N/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( \sum_{i=1}^{N} \left( - \frac{1+u^2}{2 (1-u^2)} (v_i^2 + v_{i-1}^2) + \frac{2 u v_i v_{i-1}}{1-u^2} - 2 g^2 \frac{1-u}{1+u} \right) \right)$$  

$$(4) \quad \times \exp \left( \sqrt{2} g \frac{1-u}{1+u} \sum_{i=1}^{N} (-1)^{s(i)} (v_i + v_{i-1}) \right) dv_{N-1} dv_{N-2} \cdots dv_1,$$

we see that $K^{(N)}(x, y, t)$ is given by

$$K^{(N)}(x, y, t) = \sum_{s \in \mathbb{Z}_2^N} G_N(u, \Delta, s) I_N(x, y, u, s).$$  

2.3. Scalar part. The computation of $K^{(N)}(x, y, t)$ is, by (5), divided into a scalar part, given by $I_N(x, y, u, s)$ and a non-commutative part $G_N(u, \Delta, s)$. In this subsection we compute the integrals in the expression of $I_N(v_0, v_N, u, s)$ via multivariate Gaussian integration.

Notice that the variables $v_0$ and $v_N$ are not to be integrated in (4). Therefore, $I_N(v_0, v_N, u, s)$ can be rewritten as

$$\frac{u^{N s^2}}{(\pi (1-u^2))^{N/2}} \exp \left( - \frac{1+u^2}{2 (1-u^2)} (v_0^2 + v_N^2) + \sqrt{2} g \frac{1-u}{1+u} \left( (-1)^{s(1)} v_0 + (-1)^{s(N)} v_N \right) - 2 N g^2 \frac{1-u}{1+u} \right)$$  

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( - \frac{1+u^2}{2 (1-u^2)} \sum_{i=1}^{N-1} v_i^2 + \frac{2 u}{1-u^2} \sum_{i=1}^{N-2} v_i v_{i+1} + \frac{2 u}{1-u^2} (v_0 v_1 + v_{N-1} v_N) \right)$$  

$$\times \exp \left( \sqrt{2} g \frac{1-u}{1+u} \sum_{i=1}^{N-1} ((-1)^{s(i+1)} + (-1)^{s(i)}) v_i \right) dv_{N-1} dv_{N-2} \cdots dv_1.$$  

The quadratic form in variables $v_i (i = 1, 2, \ldots, N-1)$ inside of the exponential in the integrand above is equal to

$$(1 + u^2) \sum_{i=1}^{N-1} v_i^2 - 2 u \sum_{i=1}^{N-2} v_i v_{i+1} = T v A_{N-1} v,$$

for a vector $v \in \mathbb{R}^{N-1}$ and tridiagonal matrix $A_{N-1}$ given by

$$v = (v_1, v_2, \cdots, v_{N-1}), \quad A_{N-1} = \begin{pmatrix} 1 + u^2 & -u & 0 & \cdots & 0 & 0 \\ -u & 1 + u^2 & -u & \cdots & 0 & 0 \\ 0 & -u & 1 + u^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 + u^2 & -u \\ 0 & 0 & 0 & \cdots & -u & 1 + u^2 \end{pmatrix}.$$  

Moreover, by defining

$$B(s) := \frac{2 u}{1-u^2} (v_0 e_1 + v_N e_{N-1}) + \sqrt{2} g \frac{1-u}{1+u} C(s),$$  

where $e_i$ is the $i$-th standard basis vector of $\mathbb{R}^{N-1}$ and

$$C(s) := T \left[ (-1)^{s(1)} + (-1)^{s(2)} + (-1)^{s(3)} + \cdots + (-1)^{s(N-1)} + (-1)^{s(N)} \right],$$  

where $s \in \mathbb{Z}_2^N$. REYES-BUSTOS AND M. WAKAYAMA
we write \( I_N(v_0, v_N, u, s) \) as
\[
\frac{uN^2}{(\pi(1-u^2))^{N/2}} \exp\left(-\frac{1+u^2}{2(1-u^2)}(v_0^2 + v_N^2) + \sqrt{2}g \frac{1-u}{1+u}((-1)^{s(N)}v_0 + (-1)^s(N)v_N) - 2Ng \frac{1-u}{1+u}\right) \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{1-u^2}T^\top v A_{N-1}v + T^\top B(s)v\right) dv_{N-1} dv_{N-2} \cdots dv_1.
\]

Next, we obtain the expression for \( \det(A_{N-1}) \). For that, we need a lemma on Chebyshev polynomials of the second kind \( U_n(x) \), defined by the three-term recurrence relation
\[
U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),
\]
with initial values \( U_0(x) = 1 \) and \( U_1(x) = 2x \).

**Lemma 2.2.** For \( n \in \mathbb{Z}_{\geq 0} \), we have
\[
U_n\left(-\frac{1+u^2}{2u}\right) = (-1)^n \frac{1-u^{2(n+1)}}{u^n(1-u^2)}.
\]

**Proof.** Set \( z = -\frac{1+u^2}{2u} \). Then, the result clearly holds for \( U_0(z) \) and \( U_1(z) = 2z = -\frac{1+u^2}{u} = -\frac{1-u^2}{u(1-u^2)} \).

The recurrence relation for \( U_n(z) \) gives
\[
U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z) = (-1)^n\frac{1+u^2}{u} \left(\frac{1-u^{2(n+1)}}{u^n(1-u^2)}\right) - (-1)^{n-1} \frac{1-u^{2n}}{u^{n-1}(1-u^2)} = (-1)^{n+1} \frac{1}{u^{n+1}(1-u^2)} \left( (1+u^2)(1-u^{2(n+1)}) - u^2(1-u^{2n}) \right).
\]

as desired. \( \square \)

**Lemma 2.3.** For \( N \geq 2 \), the matrix \( A_{N-1} \) is positive definite and its determinant is given by
\[
\det(A_{N-1}) = (1 + u^2 + u^4 + \cdots + u^{2(N-1)}) = \frac{1-u^{2N}}{1-u^2}.
\]
Furthermore, the inverse of \( A_{N-1} \) is symmetric and given by
\[
(A_{N-1}^{-1})_{ij} = u^{j-i} \frac{(1-u^{2i})(1-u^{2(N-j)})}{(1-u^{2N})(1-u^2)}
\]
for \( i \leq j \).

**Proof.** The matrix \( A_{N-1} \) is symmetric and since \( 1 + u^2 > 2u^2 \) for \( 0 < u < 1 \), by the Gershgorin circle theorem (see [5]) all the eigenvalues of \( A_{N-1} \) are positive. Therefore, \( A_{N-1} \) is positive definite (see also [2]). The determinant expression is obtained by direct computation. From [11], it is known that the inverse \( A_{N-1} \) is given by
\[
(A_{N-1}^{-1})_{ij} = (-1)^{i+j+1} \frac{U_{i-1}(-\frac{1+u^2}{2u})U_{N-1-j}(\frac{1+u^2}{2u})}{U_{N-1}(-\frac{1+u^2}{2u})}
\]
for \( i \leq j \) and where \( U_n(x) \) is the Chebyshev polynomials of the second kind. The desired expression then follows from Lemma 2.2. \( \square \)
Let us introduce notation to simplify the expression of \( I_N(x, y, u, s) \). For \( s \in \mathbb{Z}_2^N \) and \( i, j \in \{1, 2, \cdots, N\} \), define

\[
\eta_i(s) := (-1)^{s(i)} + (-1)^{s(i+1)},
\]

\[
\Lambda^{(j)}(u) := u^{j-1} \left(1 - u^{2(N-j)+1}\right), \quad \Omega^{(i,j)}(u) = u^{j-i} \left(1 - u^{2i}\right) \left(1 - u^{2(N-j)}\right).
\]

**Theorem 2.4.** For \( N \in \mathbb{Z}_{\geq 1} \), we have

\[
I_N(x, y, u, s) = K_0(x, y, g, u^N) \exp \left( \sqrt{2g} \frac{\sum_{j=1}^{N} (-1)^{s(j)} \left(x \Lambda^{(j)}(u) + y \Lambda^{(N-j+1)}(u)\right)}{(1-u^2)^N} \sum_{i=1}^{N-1} \eta_i(s)^2 \Omega^{(i,i)}(u) + 2 \sum_{i < j} \eta_i(s)\eta_j(s) \Omega^{(i,j)}(u) \right) - 2Ng^2 \frac{1-u}{1+u}.
\]

**Proof.** Since \( A_{N-1} \) is positive definite by Lemma 2.3, by multivariate Gaussian integration (see e.g. 10) in (8) we obtain

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( -\frac{1}{1-u^2} T v A_{N-1} v + T B(s) v \right) dv_{N-1} dv_{N-2} \cdots dv_1
\]

\[
= \sqrt{(1-u^2)^{N-1} \pi^{N-1}} \frac{1}{\det(A_{N-1})} \exp \left( -\frac{1}{4} T B(s) (A_{N-1})^{-1} B(s) \right)
\]

\[
= \frac{\pi^{N-1}}{(1-u^2)^N} \exp \left( -\frac{1}{4} T B(s) (A_{N-1})^{-1} B(s) \right).
\]

Thus, we have

\[
I_N(v_0, v_N, u, s)
\]

\[
= \frac{u^N g^2}{\sqrt{\pi(1-u^2)}} \exp \left( -\frac{1+u^2}{2(1-u^2)} (v_0^2 + v_N^2) + \sqrt{2g} \frac{1-u}{1+u} \left((-1)^{s(1)} v_0 + (-1)^{s(N)} v_N\right) - 2Ng^2 \frac{1-u}{1+u}\right)
\]

\[
\times \exp \left( -\frac{1-u^2}{4} T B(s) A_{N-1}^{-1} B(s) \right).
\]

From the definitions, we see that

\[
T B(s) A_{N-1}^{-1} B(s) = \left( \frac{2u}{1-u^2} (v_0^T e_1 + v_N^T e_{N-1}) \right) A_{N-1}^{-1} \left( \frac{2u}{1-u^2} (v_0 e_1 + v_N e_{N-1}) \right)
\]

\[
+ 2 \left( \frac{2u}{1-u^2} (v_0^T e_1 + v_N^T e_{N-1}) \right) A_{N-1}^{-1} \left( \sqrt{2g} \frac{1-u}{1+u} C(s) \right)
\]

\[
+ \left( 2g^2 \frac{(1-u)^2}{(1+u)^2} \right) T C(s) A_{N-1}^{-1} C(s),
\]

the second line is justified by the symmetry of the inverse of the matrix \( A_{N-1}^{-1} \). By Lemma 2.3, we have

\[
\frac{1-u^2}{4} \left( \frac{2u}{1-u^2} (v_0^T e_1 + v_N^T e_{N-1}) \right) A_{N-1}^{-1} \left( \frac{2u}{1-u^2} (v_0 e_1 + v_N e_{N-1}) \right)
\]

\[
= \frac{u^2}{(1-u^2)} \left( \frac{1-u^{2(N-1)}}{1-u^2} (v_0^2 + v_N^2) + 2u^{N-2} \frac{1-u^2}{1-u^2} v_0 v_N \right),
\]
adding the term \(-\frac{1+u^2}{2(1-u^2)}(v_0^2 + v_N^2)\) we obtain
\[-\frac{1+u^{2N}}{2(1-u^{2N})}(v_0^2 + v_N^2) + \frac{2u^N}{1-u^{2N}}v_0v_N,
\]
giving the expression \(K_0(x,y,g,u^N)\) of \([S]\) by setting \(v_0 = x\) and \(v_N = y\).

For the second term, we have
\[
\frac{2(1-u^2)}{4} \left( \frac{2u}{1-u^2} (v_0^T e_1 + v_N^T e_{N-1}) \right) A_{N-1}^{-1} \left( \sqrt{2}g \frac{1-u}{1+u} C(s) \right) = \frac{\sqrt{2}g(1-u)}{(1-u^{2N})(1+u)}
\]
\[
\times \left( \sum_{j=1}^{N-1} u^j (1-u^{2(N-j)})((-1)^{s(j+1)} + (-1)^{s(j)}) + v_N \sum_{j=1}^{N-1} u^{N-j}(1-u^{2j})((-1)^{s(j+1)} + (-1)^{s(j)}) \right).
\]
By rewriting the first sum by using the identity
\[
u^j(1-u^{2(N-j)} + (1+u)u^{j-1}(1-u^{2(N-j)+1}),
\]
and adding the term \((-1)^{s(l)}(1-u^{2N})v_0\), we obtain the expression
\[
v_0 \sum_{j=1}^{N} (-1)^{s(j)}u^{j-1}(1-u^{2(N-j)+1}),
\]
and similarly for the second sum, giving the expression in the sum in the first line of \([S]\).

Finally, the term
\[
\frac{1-u^2}{4} \left( 2g^2 \frac{(1-u)^2}{(1+u)^2} \right) (T C(s) A_{N-1}^{-1} C(s)),
\]
is given by
\[
\frac{g^2(1-u)^2}{2(1+u)^2(1-u^{2N})} \left( \sum_{i=1}^{N-1} ((-1)^{s(i+1)} + (-1)^{s(i)})^2(1-u^{2i})(1-u^{2(N-i)})
\]
\[
+ 2 \sum_{i<j} ((-1)^{s(i+1)} + (-1)^{s(i)})((-1)^{s(j+1)} + (-1)^{s(j)})u^{j-i}(1-u^{2i})(1-u^{2(N-j)}) \right),
\]
yielding the expression in the second line of \([S]\). The proof is completed by setting \(v_0 = x\) and \(v_N = y\).

2.4. Non-commutative part. In this section we explicitly describe the matrix-valued function \(G_k(u,\Delta,s)\) for \(k \geq 1\), then by using the resulting expression and the expression for \(I_k(x,y,u,s)\) we describe the kernel \(K^{(N)}(x,y,t)\) and the limit expression for the heat kernel of QRM.

To simplify the notation, we denote by \(M_{ij}\), for \(i,j = 0,1\), the matrices
\[
M_{00} := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad M_{01} := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_{10} := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_{11} := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]
where we included the previously defined matrices \(M_{00}\) and \(M_{11}\) for reference.

Proposition 2.5. For \(s \in \mathbb{Z}_k\), we have
\[
G_k(u,\Delta,s) = \prod_{i=1}^{k-1} (1 + (-1)^{s(i)}s(i+1)u^{2\Delta}) M_k(s) \begin{bmatrix} u^\Delta & 0 \\ 0 & u^{-\Delta} \end{bmatrix},
\]
where the matrix \(M_k(s)\) is given by
\[
M_k(s) = \frac{1}{(-1)^{s(1)}(-1)^{s(1)}} \prod_{i=1}^{k-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -|s(i) - s(i+1)| & -(s(i+1) - s(i)) \\ (s(i+1) - s(i)) & -|s(i) - s(i+1)| \end{bmatrix}.
\]

Before proving Proposition 2.5, we observe that the matrix \(M_k(s)\) is only one of the matrices \(M_{00}\), \(M_{01}\), \(M_{10}\), and \(M_{11}\) (see (10)). In fact, \(M_k(s)\) only depends on the first and last entry of \(s \in \mathbb{Z}_k^2\).
Lemma 2.6. Let \( s \in \mathbb{Z}_2^k \). If \( s(1) = i \) and \( s(k) = j \), then
\[
M_k(s) = M_{ij},
\]
for \( i, j = 0, 1 \).

Proof. Let us consider only the case \( s(1) = 0 \), since the case \( s(1) = 1 \) is proved in a similar fashion. Notice that if \( s(i + 1) = s(i) \), the matrix inside the product in the definition of \( M_k(s) \) corresponding to the index \( i \in \{1, 2, \cdots, k-1\} \) is the identity. Let us consider the vector \( s \) as a word on the alphabet \( \mathbb{Z}_2 = \{0, 1\} \) in the standard way and \( s \) the word resulting of removing contiguous occurrences of ones or zeros. Then, if \( s(1) = 0 \), and \( s(k) = 0 \), \( s = 0(10)^k \) with \( k \in \mathbb{Z}_{\geq 0} \) and here exponentation means concatenation of words. From the definition of \( M_k(s) \) we see then that
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
\[
k = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix},
\]
since the expression in the parenthesis is equal to the identity matrix. Similarly, for \( s(1) = 0 \) and \( s(k) = 1 \), we have \( s = 0(10)^k1 \) for \( k \in \mathbb{Z}_{\geq 0} \). Therefore,
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
\[
k = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix},
\]
and the result follows.

\( \square \)

Proof of Proposition 2.3. The case \( k = 1 \) is trivial. Furthermore we easily by direct computation that
\[
G_2(u, \Delta, (0, 0)) = \frac{1 + u^2\Delta}{2u^2} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix}, \quad G_2(u, \Delta, (0, 1)) = \frac{1 - u^2\Delta}{2u^2} \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix}
\]
\[
G_2(u, \Delta, (1, 0)) = \frac{1 - u^2\Delta}{2u^2} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix}, \quad G_2(u, \Delta, (1, 1)) = \frac{1 + u^2\Delta}{2u^2} \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix}.
\]
Now, we suppose the result holds for \( k \in \mathbb{Z}_{\geq 2} \). Let \( s \in \mathbb{Z}_2^{k+1} \) and consider
\[
G_{k+1}(u, \Delta, s) = \frac{1}{2^{k+1}} \prod_{i=1}^{k+1} [I + (-1)^{1-s(i)}J]
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix},
\]
by the hypothesis, this is just
\[
\prod_{i=1}^{k-1} \frac{1 + (-1)^{s(i)-s(i-1)}u^2\Delta}{u(1-\Delta)2^{k+1}}M_k(s') \begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix} [I + (-1)^{1-s(k+1)}J]
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix},
\]
with \( s' \in \mathbb{Z}_2^k \).

Suppose that \( s(k+1) = 0 \), we consider the product
\[
M_k(s') \begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix} \begin{bmatrix}1 & -1 \\
-1 & 1
\end{bmatrix},
\]
we are going to verify the result for the possible combinations of \( s(1) \) and \( s(k) \).

First, if \( s(1) = 0 \) and \( s(k) = 0 \), by Lemma 2.6, the above product is
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix} \begin{bmatrix}1 & -1 \\
-1 & 1
\end{bmatrix} = \frac{1 + u^2\Delta}{u^\Delta}M_{k}(s') \begin{bmatrix}1 & 0 \\
0 & 1
\end{bmatrix},
\]
which is the desired expression. In the case \( s(1) = 0 \) and \( s(k) = 1 \), we have
\[
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix} \begin{bmatrix}1 & -1 \\
-1 & 1
\end{bmatrix} = \frac{1 - u^2\Delta}{u^\Delta}M_{k}(s') \begin{bmatrix}0 & 1 \\
1 & 0
\end{bmatrix},
\]
while in the case \( s(1) = 1 \) and \( s(k) = 0 \) we have
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix} \begin{bmatrix}1 & -1 \\
-1 & 1
\end{bmatrix} = \frac{1 + u^2\Delta}{u^\Delta}M_{k}(s') \begin{bmatrix}1 & 0 \\
0 & 1
\end{bmatrix}.
\]
and finally, the in the case \( s(1) = 1 \) and \( s(k) = 1 \), we have

\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
= \frac{1 - u^{2\Delta}}{u^\Delta}
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
= \frac{1 - u^{2\Delta}}{u^\Delta}
\mathbf{M}_k(s')
\begin{bmatrix}
0 & 1 \\
0 & -u^\Delta
\end{bmatrix}.
\]

The case of \( s(k + 1) = 1 \) is completely analogous. \( \square \)

By (5), the heat kernel of the QRM is given by the limit expression

\[
\tilde{K}(x, y, t) = \lim_{N \to \infty} \sum_{s \in \mathbb{Z}_2^N} G_N(u^{1/N}, \Delta, s) I_N(x, y, u^{1/N}, s).
\]

To deal with the sum over \( \mathbb{Z}_2^N \) in the expression above, we introduce a partition of \( \mathbb{Z}_2^N \).

**Definition 2.1.** Let \( N \in \mathbb{Z}_{\geq 1} \) and \( i, j \in \mathbb{Z}_2 \).

1. The subset \( C_{ij}^{(N)} \subset \mathbb{Z}_2^N \) is given by
   \[
   C_{ij}^{(N)} = \{s \in \mathbb{Z}_2^N | s(1) = i, s(N) = j\}.
   \]

2. For \( 3 \leq k \leq N \) the subset \( A_{ij}^{(k, N)} \subset \mathbb{Z}_2^N \) is given by
   \[
   A_{ij}^{(k, N)} = \{s \in \mathbb{Z}_2^N | s(1) = i, s(k - 1) = 1 - j, s(n) = j \text{ for } k \leq n \leq N\},
   \]

3. We have
   \[
   A_{00}^{(1, N)} = \{(0, 0, 0, 0, \cdots, 0)\}, \quad A_{01}^{(2, N)} = \{(0, 1, 1, 1, \cdots, 1)\},
   \]
   \[
   A_{11}^{(1, N)} = \{(1, 1, 1, 1, \cdots, 1)\}, \quad A_{10}^{(2, N)} = \{(1, 0, 0, 0, \cdots, 0)\},
   \]
   and \( A_{ij}^{(k, N)} = \emptyset \) for \( k = 1, 2 \) if it is not one of the four sets above.

For \( N \geq 2 \), the sets \( A_{ij}^{(k, N)} \subset \mathbb{Z}_2^N \) form a partition of \( \mathbb{Z}_2^N \), that is,

\[
\mathbb{Z}_2^N = \bigsqcup_{1 \leq k \leq N} \bigsqcup_{i, j \in \mathbb{Z}_2} A_{ij}^{(k, N)},
\]

from where it is clear that for \( i, j \in \mathbb{Z}_2 \), we have \( \#A_{ij}^{(k, N)} = 2^{k - 3} \) for \( k \geq 3 \) and \( \#A_{ij}^{(n, N)} = 1 \) if \( n = 1, 2 \).

We frequently use the constant elements \( 0_k = (0, 0, \cdots, 0) \) and \( 1_k = (1, 1, \cdots, 1) \in \mathbb{Z}_2^k \) for \( k \in \mathbb{Z}_{\geq 1} \). For \( \mathbf{r} \in \mathbb{Z}_2^k \) and \( \mathbf{s} \in \mathbb{Z}_2^\ell \) with \( k, \ell \in \mathbb{Z}_{\geq 1} \), we denote by \( \mathbf{r} \oplus \mathbf{s} \in \mathbb{Z}_2^{k + \ell} \) the element obtained by concatenation in the natural way.

We note that any element \( \mathbf{s} \in A_{00}^{(k, N)} \) for \( k \geq 3 \) can be expressed as

\[
\mathbf{s} = \tilde{s} \oplus 0_{N - k + 1}
\]

with \( \tilde{s} \in C_{01}^{(k - 1)} \). Similar expressions hold for elements of \( A_{01}^{(k, N)} \), \( A_{10}^{(k, N)} \) and \( A_{11}^{(k, N)} \).

Additionally, note that by Lemma 2.6, the matrix \( \mathbf{M}_k(s) \) only depends on the first and last entry of \( s \in \mathbb{Z}_2^k \) and thus it is fixed over any subset \( C_{ij}^{(k)} \subset \mathbb{Z}_2^k \) for \( i, j = 0, 1 \) (c.f. Definition 2.1). In practice, it is convenient to work with the scalar part of the function \( G_k(u, \Delta, s) \) above.

**Definition 2.2.** For \( k \geq 1 \), the function \( g_k(u, s) \) is given by

\[
g_k(u, s) = \prod_{i=1}^{k-1} \left( 1 + (-1)^{s(i) - s(i+1)} u^{2\Delta} \right).
\]

The result of Proposition 2.5 is then written as

\[
G_k(u, \Delta, s) = g_k(u, s) \mathbf{M}_k(s)
\begin{bmatrix}
u^\Delta & 0 \\
0 & u^{-\Delta}
\end{bmatrix},
\]

where we note that the degree of \( g_k(u, s) \) as a polynomial in \( u^\Delta \) is \( 2(k - 1) \).
With the foregoing notation, the heat kernel $K_{\text{Rabi}}(x, y, t)$ of the QRM is given by

$$K_{\text{Rabi}}(x, y, t) = \lim_{N \to \infty} K^{(N)}(x, y, u^{1/N}) = \lim_{N \to \infty} \sum_{s \in \mathbb{Z}_2^N} G_N(u^{1/N}, \Delta, s) \tilde{I}_N(x, y, u^{1/N}, s),$$

is, by (8) and (12), equal to

$$K_{\text{Rabi}}(x, y, t) = K_0(x, y, g, u) \lim_{N \to \infty} \sum_{k=1}^N \sum_{i,j=0}^1 \sum_{s \in \mathcal{A}_{ij}^{(k,N)}} G_N(u^{1/N}, \Delta, s) \tilde{I}_N(x, y, u^{1/N}, s),$$

with

$$\tilde{I}_N(x, y, u, s) = \exp \left( \frac{\sqrt{2}g(1-u)}{1-u^{2N}} \sum_{j=1}^N (-1)^s(j) \left( x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u) \right) \right)$$

$$\times \exp \left( \frac{g^2(1-u)^2}{2(1+u)^2(1-u^{2N})} \left( \sum_{i=1}^{N-1} \eta_i(s)^2 \Omega^{(i,i)}(u) + 2 \sum_{i<j} \eta_i(s)\eta_j(s)\Omega^{(i,j)}(u) \right) - \frac{2Ng^2(1-u)}{1+u} \right),$$

Note that for $k \geq 2$, by (13), the expression inside the limit in (14) is given by

$$\sum_{k=2}^N \left( \sum_{s=s\oplus 0_{N-k+1}}^N + \sum_{s=s\oplus 1_{N-k+1}}^N + \sum_{s=s\oplus 1_{N-k+1}}^N + \sum_{s=s\oplus 1_{N-k+1}}^N \right) G_N(u^{1/N}, \Delta, s) \tilde{I}_N(x, y, u^{1/N}, s),$$

note that $C_{ij}^{(1)} = 0$ with $i \neq j$.

Next, we describe how the term $\tilde{I}_N(x, y, u^{1/N}, s)$ factors in each of the sums. For $s \in \mathbb{Z}_2^{k-1}$ with $k \geq 1$, write

$$\tilde{I}_N(x, y, u, s \oplus 0_{N-k+1}) = J_{00}^{(k,N)}(x, y, u, g)R_0^{(k,N)}(u, g, s),$$

$$\tilde{I}_N(x, y, u, s \oplus 1_{N-k+1}) = J_{11}^{(k,N)}(x, y, u, g)R_1^{(k,N)}(u, g, s),$$

with functions $J_{\mu}^{(k,N)}(x, y, u, g)$ and $R_{\mu}^{(k,N)}(u, g, s)$ defined below for $\mu = 0, 1$. Here, we note that in the first line $s \in C_{(k-1)}$ and in the second line $s \in C_{(k-1)}$ for $i = 0, 1$.

For $k \geq 1$, the function $J_{\mu}^{(k,N)}(x, y, u, g)$ is given by

$$J_{\mu}^{(k,N)}(x, y, u, g) = \exp \left( (-1)^{\mu} \frac{\sqrt{2}g(1-u)}{1-u^{2N}} \left( \sum_{j=k}^N (x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u)) \right) \right)$$

$$\times \exp \left( \frac{2g^2(1-u)^2}{(1+u)^2(1-u^{2N})} \left( \sum_{i=k}^{N-2} \Omega^{(i,i)}(u) + 2 \sum_{i=k}^{N-2} \sum_{j=i+1}^{N-1} \Omega^{(i,j)}(u) \right) - \frac{2Ng^2(1-u)}{1+u} \right),$$

while $R_{\mu}^{(k,N)}(u, g, s)$ is given, for $s \in \mathbb{Z}_2^{k-1}$, by

$$R_{\mu}^{(k,N)}(u, g, s) = \exp \left( \frac{\sqrt{2}g(1-u)}{1-u^{2N}} \sum_{j=1}^{k-1} (-1)^s(j) (x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u)) \right)$$

$$\times \exp \left( \frac{g^2(1-u)^2}{2(1+u)^2(1-u^{2N})} \left[ \sum_{i=1}^{k-2} \eta_i(s)^2 \Omega^{(i,i)}(u) + 2 \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-2} \eta_i(s)\eta_j(s)\Omega^{(i,j)}(u) \right. \right.$$

$$\left. + 4(-1)^{\mu} \sum_{i=1}^{k-2} \sum_{j=i}^{N-1} \eta_i(s)\Omega^{(i,j)}(u) \right).$$
Suppose that \( s = s_1 \oplus 0_{N-k+1} \) with \( s_1 \in C_{v_1}^{(k-1)} \) and \( v \in \{0,1\} \), then it is easy to see that

\[
G_N(u^{1/N}, s) = \left( \frac{1 - u^{2\Delta}}{2u^{\Delta}} \right) \left( \frac{1 + u^{2\Delta}}{2u^{\Delta}} \right)^{N-k} g_{k-1}(u^{1/N}, s_1)M_N(s) \begin{bmatrix} u^{\Delta/2} & 0 \\ 0 & u^{-\Delta/2} \end{bmatrix},
\]

with similar expressions for other cases. Therefore, the sum inside the limit (starting from \( k = 2 \)) is given by

\[
\left( \frac{1 - u^{2\Delta}}{2u^{\Delta}} \right) \sum_{k \geq 2} \left( \frac{1 + u^{2\Delta}}{2u^{\Delta}} \right)^{N-k} \left[ J_0^{(k,N)}(x,y,u^{1/N},g) \left( \sum_{s \in C_{v_1}^{(k-1)}} \sum_{s \in C_{v_1}^{(k-1)}} g_{k-1}(u^{1/N},s) R_{0}^{(k,N)}(u^{1/N},s) \right) \right] \begin{bmatrix} u^{\Delta/2} & 0 \\ 0 & u^{-\Delta/2} \end{bmatrix}.
\]

Next, we make some considerations to further simplify the expression of the heat kernel. First, we notice that the matrix factor \( \begin{bmatrix} u^{\Delta/2} & 0 \\ 0 & u^{-\Delta/2} \end{bmatrix} \) is the identity matrix at the limit \( N \to \infty \), so we omit it in the subsequent discussion. Similarly, without loss of generality, we drop the term corresponding to \( k = 2 \), since it vanishes due to the presence of the factor \( (1 - u^{2\Delta}/N) \). This is analogous to removing a finite number of terms from a Riemann sum.

Summing up, the expression for the heat kernel \( \hat{K}(x,y,t) \) is given by

\[
K_0(x,y,t) = \lim_{N \to \infty} \left( \frac{1}{2} \left( \frac{1 + u^{2\Delta}}{2u^{\Delta}} \right)^{N-1} \left( J_0^{(1,N)}(x,y,u^{1/N},g) \left[ 1 \phantom{1} -1 \right] + J_1^{(1,N)}(x,y,u^{1/N},g) \left[ 1 \phantom{1} 1 \right] \right) + \left( \frac{1 - u^{2\Delta}}{2u^{\Delta}} \right) \sum_{k \geq 2} \left( \frac{1 + u^{2\Delta}}{2u^{\Delta}} \right)^{N-k} \right) \begin{bmatrix} J_0^{(k,N)}(x,y,u^{1/N},g) & M_{00} \sum_{s \in C_{v_1}^{(k-1)}} g_{k-1}(u^{1/N},s) R_{0}^{(k,N)}(u^{1/N},s) + M_{10} \sum_{s \in C_{v_1}^{(k-1)}} g_{k-1}(u^{1/N},s) R_{1}^{(k,N)}(u^{1/N},s) \\ M_{01} \sum_{s \in C_{v_1}^{(k-1)}} g_{k-1}(u^{1/N},s) R_{0}^{(k,N)}(u^{1/N},s) + M_{11} \sum_{s \in C_{v_1}^{(k-1)}} g_{k-1}(u^{1/N},s) R_{1}^{(k,N)}(u^{1/N},s) \end{bmatrix}.
\]

Notice that the limit in the expression (17) resembles a Riemann sum of the type

\[
\lim_{N \to \infty} \sinh \left( \frac{t}{N} \right) \sum_{k=1}^{N} f \left( \frac{k t}{N} \right) = \int_0^t f(x)dx,
\]

for a Riemann integrable function \( f : [0, t] \to \mathbb{R} \). However, due to the presence of alternating sums depending of \( k \) in \( R_{\mu}^{(k,N)}(u,g,s) \) and in \( g_{k-1}(u^{1/N},s) \) it is not possible to interpret the limit directly as a Riemann sum.

3. Harmonic analysis in \( \mathbb{Z}_2^k \)

Denote by \( \mathbb{C}[\mathbb{Z}_2^k] \) the group algebra of the abelian group \( \mathbb{Z}_2^k \). For \( f, h \in \mathbb{C}[\mathbb{Z}_2^k] \) the elementary identity

\[
\sum_{s \in \mathbb{Z}_2^k} f(s)h(s) = (f \ast h)(0) = \frac{1}{2^k} \left( \hat{f} \cdot \hat{h} \right)(0) = \frac{1}{2^k} \sum_{\rho \in \mathbb{Z}_2^k} \hat{f}(\rho)\hat{h}(\rho),
\]

holds, where \( \hat{f} \) (resp. \( \hat{h} \)) is the Fourier transform of \( f \) (resp. \( h \)).

In this section we use the identity (18) to transform the sum appearing (17) into an expression that can be evaluated as a Riemann sum. First, we compute the Fourier transform of \( g_k(u^{1/N},s) \), then in
we describe the Fourier transform of $R_{q}^{(k,N)}(u^\rho, s)$. In \[3.2\] we collect a number of combinatorial results to simplify the expression of the Fourier transform of $R_{q}^{(k,N)}(u^\rho, s)$. In \[3.3\] we use identity \[13\] to simplify the expression \[17\] and in \[3.4\] we transform finite sums into definite integrals using the standard method with Riemann-Stieltjes integrations and estimate the order of the residual terms.

We begin by setting the notation and recalling the basic properties of the Fourier transform in $2^k$. We refer the reader to [9] for more details. For $\rho \in 2^k$, define the character $\chi_\rho(s) \in \hat{2^k}$ by

$$\chi_\rho(s) := (-1)^{|s|\rho},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $2^k$. It is known that all the characters in the dual group $\hat{2^k}$ are obtained in this way. Then, for $f \in \mathbb{C}[2^k]$, the Fourier transform $\hat{f}(\rho)$ is given by

$$\hat{f}(\rho) = \mathcal{F}(f) := \sum_{s \in 2^k} f(s) \chi_\rho(s),$$

for $\rho \in 2^k$. Since $\hat{f} \in \mathbb{C}[2^k]$, the Fourier inversion formula is given by

$$f = \frac{1}{2^k} \hat{\hat{f}}.$$

Next, we equip the set $\mathcal{C}_{vw}^{(k+2)}$ with an abelian group structure such that $\mathcal{C}_{vw}^{(k+2)} \simeq 2^k$. We naturally identify an element $s \in \mathcal{C}_{vw}^{(k+2)}$ via the projection $\overline{s} \in 2^k$ given by

$$s = (v, s_1, s_2, \cdots, s_k, w) \mapsto \overline{s} = (s_1, s_2, \cdots, s_k).$$

Clearly, the sum \[13\] may be regarded as a sum over $\mathcal{C}_{vw}^{(k+2)}$ by lifting an element $s \in 2^k$ to $\mathcal{C}_{vw}^{(k+2)}$ by using the inverse of the projection \[21\].

In the case of the function $g_k(u, s)$ we define a special notation.

**Definition 3.1.** Let $v, w \in \{0, 1\}$. Then, for $s \in 2^k$ with $k \geq 1$, define the function $g_k^{(v,w)}(u,s)$ by

$$g_k^{(v,w)}(u,s) := \frac{1}{2^k} (1 + (-1)^{v+s(1)} u^{2\Delta})(1 + (-1)^{w+s(k)} u^{2\Delta}) \prod_{i=1}^{k-1} (1 + (-1)^{s(i)+s(i+1)} u^{2\Delta}).$$

In addition, for $\rho \in 2^0$, define

$$g_0^{(v,w)}(u,s) = 1 + (-1)^{v+w} u^{2\Delta}.$$ 

For $s \in \mathcal{C}_{vw}^{(k+2)}$, we have

$$4u^{(k+1)\Delta} g_{k+2}(u,s) = g_k^{(v,w)}(u,s),$$

and in addition, we note that the degree of $g_k^{(v,w)}(u,s)$ as a polynomial in $u^{2\Delta}$ is $2(k+1)$.

For fixed $u, \Delta \in \mathbb{R}$, the function $g_k^{(v,w)}(u,s)$ is an element of the group algebra $\mathbb{C}[2^k]$ of the abelian group $2^k$. Since the parameters $g, \Delta > 0$ and $u \in \{0,1\}$ are assumed to be fixed, in the remainder of this section as it is obvious we may omit the dependence of $g, \Delta$ and $u$ from certain functions.

Next, we give an explicit expression for the Fourier transform $\hat{g}_k^{(v,w)}(\rho)$ for arbitrary character $\rho \in 2^k$.

**Definition 3.2.** Let $\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in 2^k$. The function $|\cdot| : 2^k \rightarrow \mathbb{C}$ is given by

$$|\rho| = |\rho|_1 := \sum_{i=1}^{k} \rho_i.$$
Let \( j_1 < j_2 < \cdots < j_{|\rho|} \) the position of the ones in \( \rho \), that is, \( \rho_{j_i} = 1 \) for all \( i \in \{1, 2, \cdots, |\rho|\} \) and \( \rho_i = 1 \) then \( i \in \{j_1, j_2, \cdots, j_{|\rho|}\} \). The function \( \varphi_k : \mathbb{Z}_2^k \to \mathbb{C} \) is given by

\[
\varphi_k(\rho) := \sum_{i=1}^{|\rho|} (-1)^{i-1} j_{|\rho|+1-i} = j_{|\rho|} - j_{|\rho|-1} + \cdots + (-1)^{|\rho|-1} j_1,
\]

and \( \varphi_k(0) = 0 \) where \( 0 \) is the identity element in \( \mathbb{Z}_2^k \). For \( k = 0 \), define \( \varphi_k(\rho) = |\rho| = 0 \) where \( \rho \) is the unique element of \( \mathbb{Z}_0^2 \).

Let \( \rho = (\rho_1, \rho_2, \cdots, \rho_k) \in \mathbb{Z}_2^k \) and \( \delta = (\rho_1, \rho_2, \cdots, \rho_{k-1}) \in \mathbb{Z}_2^{k-1} \). From the definition we obtain

\[
\varphi_k(\rho) = (-1)^{\rho_k} \varphi_{k-1}(\delta) + \rho_k \kappa.
\]

**Proposition 3.1.** For \( \rho \in \mathbb{Z}_2^k \), we have

\[
\hat{g}_k^{(v,w)}(\rho) = (-1)^{|v|\rho} \left( u^2 \varphi_k(\rho) \Delta + (-1)^{v+w} u^{2(k+1-\varphi_k(\rho))} \right)
\]

**Proof.** The identity is immediately verified for the cases \( k = 0 \), Next, suppose that \( \rho = (\rho_1, \rho_2, \cdots, \rho_k) \in \mathbb{Z}_2^{k+1} \) and let \( \delta = (\rho_1, \rho_2, \cdots, \rho_k) \in \mathbb{Z}_2^k \). Then we have

\[
\hat{g}_{k+1}^{(v,w)}(\rho) = \sum_{s \in \mathbb{Z}_2^{k+1}} g_{k+1}^{(v,w)}(s) \chi_\rho(s) = \sum_{s \in \mathbb{Z}_2^{k+1}} \sum_{s(k+1) = i} g_{k+1}^{(v,w)}(s) \chi_\rho(s)
\]

\[
= \frac{1}{2} \sum_{i=0}^1 (-1)^{\rho_{k+1}-i} \left(1 + (-1)^{w+i} u^{2\Delta}\right) g_k^{(v,i)}(s) \chi_\delta(s)
\]

\[
= \frac{1}{2} \sum_{i=0}^1 (-1)^{\rho_{k+1}-i} \left(1 + (-1)^{w+i} u^{2\Delta}\right) \hat{g}_k^{(v,i)}(\delta)
\]

\[
= \frac{1}{2} \sum_{i=0}^1 (-1)^{\rho_{k+1}-i} \left(1 + (-1)^{w+i} u^{2\Delta} \right) (-1)^{|v|^\delta} \left[ u^{2\varphi_k(\delta)\Delta} + (-1)^{v+i} u^{2(k+1-\varphi_k(\delta))} \right],
\]

the equality in the last line holding by the induction hypothesis. The expression above is equal to

\[
\frac{1}{2} (-1)^{|v|^\delta} \left[ (1 + (-1)^w u^{2\Delta}) \left( u^{2\varphi_k(\delta)\Delta} + (-1)^v u^{2(k+1-\varphi_k(\delta))} \right) \right.
\]

\[
+ (-1)^{\rho_{k+1}} \left(1 + (-1)^w u^{2\Delta}\right) \left( u^{2\varphi_k(\delta)\Delta} - (-1)^v u^{2(k+1-\varphi_k(\delta))} \right),
\]

the result then follows by considering the cases \( \rho_{k+1} \in \{0, 1\} \) by the identity (26), and the fact that \( |\rho| = |\delta| + \rho_{k+1}. \)

In the subsequent discussion of the heat kernel it is necessary to consider a generalization of the function \( \varphi_k \). We motivate the definition via the Fourier transform of \( \varphi_k \in \mathbb{C}[\mathbb{Z}_2^k] \).

**Proposition 3.2.** Let \( \varphi_k : \mathbb{Z}_2^k \to \mathbb{Z} \) be the function of Definition 3.2. We have

\[
\widehat{\varphi}_k(\rho) = \begin{cases} k^{2k-1} & \text{if } \rho = 0_k \\ -2^{k-1} & \text{if } \rho = 0_i \oplus 1_{k-i} (1 \leq i \leq k) \\ 0 & \text{in any other case} \end{cases}
\]
Proof. The case $k = 0$ is trivial. For $k \geq 1$, let $\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in \mathbb{Z}_2^k$ and $\delta = (\rho_1, \rho_2, \cdots, \rho_{k-1}) \in \mathbb{Z}_2^{k-1}$, then we have

$$\hat{\varphi}_k(\rho) = \sum_{\mathbf{s} \in \mathbb{Z}_2^k} \varphi_k(\mathbf{s})(-1)^{\langle s, \rho \rangle} = \sum_{\mathbf{s} \in \mathbb{Z}_2^k \atop s_k = 0} \varphi_k(\mathbf{s})(-1)^{\langle s, \rho \rangle} + \sum_{\mathbf{s} \in \mathbb{Z}_2^k \atop s_k = 1} \varphi_k(\mathbf{s})(-1)^{\langle s, \rho \rangle}$$

$$= \sum_{r \in \mathbb{Z}_2^{k-1}} \varphi_{k-1}(r)(-1)^{\langle r, \rho \rangle} + (-1)^{\rho_k} \sum_{r \in \mathbb{Z}_2^{k-1}} (k - \varphi_{k-1}(r))(-1)^{\langle r, \rho \rangle}$$

$$= (1 + (-1)^{\rho_k+1})\hat{\varphi}_{k-1}(\delta) + (-1)^{\rho_k} \sum_{r \in \mathbb{Z}_2^{k-1}} (-1)^{\langle r, \rho \rangle},$$

where the equality in the second line follows by (26). Next, suppose that $\rho_k = 0$, then

$$\hat{\varphi}_k(\rho) = k \sum_{\mathbf{s} \in \mathbb{Z}_2^{k-1}} (-1)^{\langle s, \rho \rangle} = \begin{cases} k2^{k-1} & \text{if } \delta = 0_{k-1} \\
0 & \text{if } \delta \neq 0_{k-1}. \end{cases}$$

On the other hand, if $\rho_k = 1$ we have

$$\hat{\varphi}_k(\rho) = 2\hat{\varphi}_{k-1}(\delta) - k \sum_{\mathbf{s} \in \mathbb{Z}_2^{k-1}} (-1)^{\langle s, \rho \rangle} = \begin{cases} (k - 1)2^{k-1} - k2^{k-1} & \text{if } \delta = 0_{k-1} \\
2\hat{\varphi}_{k-1}(\delta) & \text{if } \delta \neq 0_{k-1}, \end{cases}$$

and the result follows by induction.

By virtue of the proposition above, for $\rho = (\rho_1, \rho_2, \cdots, \rho_k)$ we can write

$$\varphi_k(\rho) = k - \frac{1}{2} \left( \sum_{i=1}^{k} (-1)^{\langle s, \rho \rangle} \right).$$

(27)

**Definition 3.3.** For $k \geq 1$ and $t \in \mathbb{C}$, the function $\varphi_k(\rho; t) : \mathbb{Z}_2^k \to \mathbb{C}$ is defined by

$$\varphi_k(\rho; t) := \frac{1}{2} \sum_{i=1}^{k} \left( 1 - (-1)^{\langle s, \rho \rangle} \right) t^{i-1}.$$  

In the following theorem we collect some properties and transformation formulas for $\varphi_k(\rho; t)$. For an integer $i \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{C}$, we write $[i]_t = \frac{1-t^i}{1-t}$. Notice that since

$$\lim_{t \to 1} \varphi_k(\rho; t) = \varphi_k(\rho),$$

the identities of the following theorem also apply to $\varphi_k(\rho)$.

**Theorem 3.3.** Let $\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in \mathbb{Z}_2^k$, $\hat{\rho} = (\rho_k, \rho_{k-1}, \cdots, \rho_1) \in \mathbb{Z}_2^k$ and $\nu \in \{0, 1\}$. Recall that for $\mathbf{r} \in \mathbb{Z}_2^k, \mathbf{s} \in \mathbb{Z}_2^\ell$, the vector $\mathbf{r} \oplus \mathbf{s} \in \mathbb{Z}_2^{k+\ell}$ denotes the concatenation of $\mathbf{r}$ and $\mathbf{s}$. Then

1. $\varphi_{k+1}(\rho \oplus (\nu); t) = \nu[k+1]_t + (-1)^\nu \varphi_k(\rho; t),$
2. $\varphi_{k+1}( (\nu) \oplus \rho; t) = \varphi_k(\rho; t) + \left( \frac{1 - (-1)^{\nu} \nu}{2} \right),$
3. $\varphi_k(\hat{\rho}; t) = (-1)^\nu t^k \varphi_k(\rho; t^{-1}) + \left( \frac{1 - (-1)^\nu \nu}{2} \right) [k+1]_t,$
4. $\sum_{i=1}^{k} (-1)^{\langle s, \rho \rangle} t^{i-1} = [k]_t - 2\varphi_k(\rho; t).$
The first claim is just the analog of (26), the second follows immediately from the expression of \( \varphi_k \) in (28). For the third one, we have

\[
\varphi_k(\rho; t) = \frac{[k]!}{2} - \frac{1}{2} \left( \sum_{i=1}^{k} (-1)^{i+1} \rho_i t^{-i} \right) = \frac{[k]!}{2} - \frac{1}{2} \left( \sum_{i=2}^{k} (-1)^{i+1} \rho_i t^{-i+1} \right) = \frac{1}{2} (-1)^{[\rho]} - \frac{1}{2} (-1)^{[\rho]}
\]

\[
= \frac{(-1)^{[\rho]} t^k}{2} [k]_{t^{-1}} - \frac{(-1)^{[\rho]} t^k}{2} \left( \sum_{i=1}^{k} (-1)^{i+1} \rho_i t^{-i+1} \right) = \frac{[k]!}{2} - \frac{(-1)^{[\rho]} t^k}{2} [k]_{t^{-1}} + \frac{t^k}{2} - \frac{1}{2} (-1)^{[\rho]}
\]

as desired. The last claim is obtained directly from the definition. \( \square \)

In addition, it is not difficult to see from the formulas in Theorem 3.3 that if 0 < \( j_1 < j_2 < \cdots < j_{[\rho]} \leq k \) are the position of the ones in \( \rho \in \mathbb{Z}_2^k \), we have

\[
(29) \quad \varphi_k(\rho; t) = \sum_{i=1}^{[\rho]} (-1)^{i-1} [j_{[\rho]+1-i}]_{t},
\]

so that \( \varphi_k(\rho; t) \) is seen to be a \( t \)-analogue of the function \( \varphi_k(\rho) \) of Definition 3.2.

**Remark 3.1.** The function \( \varphi_k(\rho; t) \) admits the following characterization. Denote by \( E(x; t) \) the generating function for the elementary symmetric functions (see e.g. [35])

\[
E(x; t) = \prod_{i=1}^{\infty} (1 + x_it).
\]

Let \( F(x; t) \) be a (formal) function defined in infinite vectors \( x = (x_1, x_2, x_3, \cdots) \) given by

\[
F(x; t) := E(x; t - 2) \sum_{i=1}^{\infty} \frac{[i]! x_i}{\prod_{j=1}^{i} (1 - 2x_j)}
\]

then we have the equality

\[
\varphi_k(\rho; t) = F((\rho_1, \rho_2, \cdots, \rho_k, 0, 0, 0, \cdots); t).
\]

Indeed, by successive application of the first transformation formula, we obtain

\[
(30) \quad \varphi_k(\rho; t) = \sum_{i=1}^{k} [i]! \rho_i \prod_{j=i+1}^{k} (1 - 2\rho_j)
\]

since \( 1 - 2\rho_j = (-1)^{\rho_j} \).

**Remark 3.2.** For \( k \geq 1 \), the function \( \varphi_k(\rho; t) \), with a small modification, may be interpreted as a morphism of abelian groups. To see this, we notice that by (30) we have

\[
(31) \quad \varphi_k(\rho + \theta; t) = \varphi_k(\rho; t) + \varphi_k(\theta; t) \quad (\text{mod } 2),
\]

for \( \rho, \theta \in \mathbb{Z}_2^k \). Next, by using equation (29) as the definition of \( \varphi_k(\rho; t) \) we can consider \( \mathbb{Z}_2[t]_k \), the vector space of polynomials of degree less than \( k \) over the ring \( \mathbb{Z}_2 \), as the codomain of \( \varphi_k(\rho; t) \), that is, \( \varphi_k(\cdot; t) : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2[t]_k \). Thus, the identity (31) exhibits \( \varphi_k(\rho; t) \) as an isomorphism of abelian groups and by linear extension, an isomorphism of vector spaces over \( \mathbb{Z}_2 \).
3.1. Fourier transform $R^{(k,N)}_{\mu}(y)$ of $R^{(k,N)}_{\mu}$. In this section we describe the Fourier transform of the function $R^{(k,N)}_{\mu}$. For convenience, we recall the definition

$$R^{(k,N)}_{\mu}(s) = \exp\left(\sqrt{2}g(u) \sum_{j=1}^{k-1} (-1)^{j} \left( x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u) \right) \right) \times \exp\left(\frac{g^{2}(1-u)^{2}}{2(1+u)^{2}(1-u^{2N})} \left( \sum_{i=1}^{k-2} \eta_{i}(s)^{2}\Omega^{(i,i)}(u) + 2 \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-2} \eta_{i}(s)\eta_{j}(s)\Omega^{(i,j)}(u) \right. \right.$$

$$\left. + 4(-1)^{\mu} \sum_{i=1}^{k-2} \sum_{j=k}^{N-1} \eta_{i}(s)\Omega^{(i,j)}(u) \right),$$

from where is it clear that $R^{(k,N)}_{\mu}(s) \in \mathbb{C}[\mathbb{Z}^{k-1}_{2}]$. As in the case of the function $g_{u}(u,s)$, the Fourier transform is computed in the abelian group $\mathbb{C}^{(k-1)}_{w_{u}} \simeq \mathbb{Z}^{k-3}_{2}$, with $v, w \in \{0, 1\}$, and we denote by $R^{(v,w)}_{\mu}(s) \in \mathbb{C}[\mathbb{Z}^{k-3}_{2}]$ the function resulting by applying the projection $\Pi$ to $R^{(k,N)}_{\mu}(s)$. We note that $R^{(v,w,k,N)}_{\mu}(s)$ would be a more appropriate notation for $R^{(v,w)}_{\mu}(s)$, but since $k, N \in \mathbb{Z}_{\geq 1}$ remain fixed in the computations of this section and there is no risk of confusion we have dropped the dependence of $k, N$ from the notation of $R^{(v,w)}_{\mu}(s)$.

We start with some general considerations. First, suppose $S$ is subset of characters $S \subset \mathbb{Z}^{k-3}_{2}$ and $f \in \mathbb{C}[\mathbb{Z}^{k-3}_{2}]$ is given by

$$(32) \quad f(s) := \exp\left(\sum_{\chi \in S} a_{\chi}\chi(s) \right) = \sum_{\xi \in \mathbb{Z}^{k-3}_{2}} C_{\xi}\xi(s),$$

for arbitrary $a_{\chi} \in \mathbb{C}$ with $\chi \in S$, and where $C_{\xi} \in \mathbb{C}$ is the Fourier coefficient corresponding to $\xi \in \mathbb{Z}^{k-3}_{2}$. The Fourier transform $\hat{f}$ is then given by

$$\hat{f}(\rho) = 2^{k-3} \sum_{\xi \in \mathbb{Z}^{k-3}_{2}} C_{\xi} \delta_{\xi,\chi_{\rho}}.$$

Therefore, in order to get the expression for the Fourier transform of $f(s)$, it is enough to describe the Fourier coefficients $C_{\xi} \in \mathbb{C}$ in terms of $a_{\chi} \in \mathbb{C}$. Let us consider the case $|S| = 1$, that is, $S = \{\chi\}$. In this case

$$f(s) = \cosh(a_{\chi}) + \sinh(a_{\chi})\chi(s),$$

since any character $\chi \in \mathbb{Z}^{k-3}_{2}$ is real.

To describe the general case, we introduce an ordering in $S = \{\chi_{1}, \chi_{2}, \cdots, \chi_{\ell}\}$ with $\ell = |S|$. Then, for $a \in \mathbb{C}^{\ell}$ and an index vector $r \in \{0, 1\}^{\ell}$ we define

$$T^{(r)}(a) := \prod_{i=1}^{\ell} \left[ \cosh(a_{i})^{1-r_{i}} \sinh(a_{i})^{r_{i}} \right],$$

where $a_{i}$ (resp. $r_{i}$) denotes the $i$-th component of $a$ (resp. $r$).

The Fourier coefficients of $f$ are given by

$$C_{\chi_{\rho}}(= C_{\rho}) = \sum_{r \in \{0,1\}^{\ell}} T^{(r)}(a),$$

where $a = \{a_{1}, a_{2}, \cdots, a_{\ell}\} \in \mathbb{C}^{\ell}$ is the vector of coefficients. In particular, note that $C_{\chi} \neq 0$ if and only if $\chi$ is generated by elements in the set $S$. 
Next, we specialise these considerations for the case of the function $R^{(v,w)}_{\mu}(s) \in \mathbb{C}[\mathbb{Z}_2^{k-3}]$. In this case, the set $S_{k-3}$ (corresponding to the set $S$ in the discussion above) is given by

$$S_{k-3} = \left\{ \chi = \chi_\rho \in \mathbb{Z}_2^{k-3} | \rho \in \mathbb{Z}_2^{k-3}, 0 < |\rho| \leq 2 \right\}.$$ 

In particular $|S_{k-3}| = \frac{(k-3)(k-2)}{2}$, and if $\chi_\rho \in S_{k-3}$, we have

$$\rho \in \{ \varepsilon_i + \varepsilon_j | 0 \leq i < j \leq k - 3 \}$$

where $\varepsilon_0 := 0$ is the zero vector. For $\rho = \varepsilon_i + \varepsilon_j$ we denote $\chi_\rho \in S_{k-3}$ by $\chi_{i,j}$. Similarly, we denote by $a_{i,j}$ (resp. $r_{i,j}$) the entries of the coefficient vector $a$ (resp. the vector $r \in \{0,1\}^k$) in lexicographical ordering.

Note that the trivial character is omitted from the set, since

$$\sum_{\chi \in S_{k-3}} \chi \chi(s) = \exp(a_0) \exp\left( \sum_{\chi \in S_{k-3}} a_\chi \chi(s) \right),$$

thus

$$\hat{R}^{(v,w)}_{\mu}(s) = 2^{k-3} \exp(a_0) \left( \sum_{\xi \in \mathbb{Z}_2^{k-3}} C_\xi \delta_{\xi,\chi_0} \right).$$

(33)

We note here that in the case $k = 3$, $R^{(v,w)}_{\mu}(s) = \hat{R}^{(v,w)}_{\mu}(s) = \exp(a_0)$ for $\rho \in \mathbb{Z}_2^3$.

The next lemma describes the coefficients $a_{i,j}$ for the case of the function $R^{(v,w)}_{\mu}(s)$.

**Lemma 3.4.** The trivial character coefficient $a_0$ is given by

$$a_0(u) = \left( \frac{1 - u}{1 - u^{2N}} \right) \left[ x\sqrt{2}\left( (-1)^v(1 - u^{2N-1}) + (-1)^w u^{k-2}(1 - u^{2(N-k+1)}) \right) 
+ y\sqrt{2}\left( (-1)^v u^{N-1}(1 - u) + (-1)^w u^{N-k+1}(1 - u^{2(k-1)-1}) \right) 
+ \frac{g^2}{1 + u} \left( \frac{1 - u}{1 + u} \left( (k - 2)(1 + u^{2N}) + (k - 3)u(1 + u^{2(N-1)}) \right) 
+ (1 + u^{2(N-k+1)})(1 - u^{2(k-1)-1}) + (-1)^{v+w} u^{k-3}(1 - u^2)(1 - u^{2N-2(k-2)/N}) 
+ \frac{2}{(1 + u)} u^{k-1}(1 - u^{N-k})(1 - u^{N-k+1}) \left( (-1)^v u(1 - u^2) + (-1)^w u^{k-1}(1 - u^{2(k-2)}) \right) \right).$$

For $1 \leq i \leq k - 3$, the Fourier coefficient $a_{0,i}$ is given by

$$a_{0,i}(u) = \left( \frac{1 - u}{1 - u^{2N}} \right) \left[ \sqrt{2}\left( xu^i(1 - u^{2(N-i)-1}) + yu^{N-i-1}(1 - u^{2i+1}) \right) 
+ \frac{g^2}{1 + u} \left( (1 - u)\left( (-1)^v u^{i-1}(1 - u^2)(1 - u^{2(N-i)-1}) + (-1)^w u^{k-2-i}(1 - u^{N-k+2})(1 - u^{2i-1}) \right) 
+ 2(-1)^i u^{k-1-i}(1 - u^{N-k})(1 - u^{N-k+1})(1 - u^{2i+1}) \right).$$

For $1 \leq i < j$, the Fourier coefficients $a_{i,j}$ are given by

$$a_{i,j}(u) = \frac{g^2(1 - u)^2}{1 - u^{2N}}(1 - u^{2i+1})(1 - u^{2(N-j)-1})u^{j-i-1}.$$
Proof. By factoring the terms in the definition of $R_k^{(k,N)}$, we see that the coefficient $a_0$ is given by
\[
a_0(u) = \frac{\sqrt{2}g(1 - u)}{1 - u^{2N}} \left[ (-1)^v \left( x\Lambda^{(1)}(u) + y\Lambda^{(N)}(u) \right) + (-1)^w \left( x\Lambda^{(k-1)}(u) + y\Lambda^{(N-k+2)}(u) \right) \right]
\]
\[+ \frac{g^2(1 - u)^2}{(1 + u)^2(1 - u^{2N})} \left[ (k - 2)(1 + u^{2N}) - (u^2 + u^{2(N-k+2)}) \right] \frac{1 - u^{2(k-2)}}{1 - u^2} + \]
\[+ \sum_{i=1}^{k-3} \Omega^{(i+1)}(u) + (-1)^{v+w}\Omega^{(1,k-2)}(u)
\]
\[+ 2(-1)^\mu \left( \frac{1 - u^{N-k}}{1 - u} \right) (u^k - u^{N+1}) \left( (-1)^v(u^{-1} + u) + (-1)^w(u^{2-k} - u^{k-2}) \right) \].

Similarly, the coefficients for the character of type $\chi_{0,i} = \chi_{i,0}$ for $1 \leq i \leq k - 3$ are given by
\[
a_{0,i}(u) = \frac{\sqrt{2}g(1 - u)}{1 - u^{2N}} \left( x\Lambda^{(i+1)}(u) + y\Lambda^{(N-i)}(u) \right)
\]
\[+ \frac{g^2(1 - u)^2}{(1 + u)^2(1 - u^{2N})} \left( (-1)^v \left( \Omega^{(i,j)}(u) + \Omega^{(1,i+1)}(u) \right) + (-1)^w \left( \Omega^{(i,k-2)}(u) + \Omega^{(i-1,k-2)}(u) \right) \right)
\]
\[+ 2 \left( \frac{1 - u^{N-k}}{1 - u} \right) (u^k - u^{N+1})(-1)^{v+w}(1 + u)(u^{i-1} - u^i) \],

and for the characters of the type $\chi_{i,j} = \chi_{i,j}$ for $1 \leq i < j \leq k - 3$, we have
\[
a_{i,j}(u) = \frac{g^2(1 - u)^2}{(1 + u)^2(1 - u^{2N})} \left( \Omega^{(i,j)}(u) + \Omega^{(i+1,j)}(u) + \Omega^{(i,j+1)}(u) \right).
\]
The result is then obtained by expanding the geometric sums involved in these expressions. \qed

3.2. Graph theoretical considerations for $R_k^{(k,N)}$. For $\rho \in \mathbb{Z}_2^{k-3}$, defining the set
\[V_\rho^{(k-3)} = \left\{ r \in \{0,1\}^{S_{k-3}} \left| \chi_\rho = \prod_{i=1}^{k-3} (\chi_{0,i})^{r_{0,i}} \prod_{1 \leq i < j} (\chi_{i,j})^{r_{i,j}} \right. \right\}, \]
the Fourier coefficients $C_{\chi_\rho}$ are given by
\[C_{\chi_\rho} = \sum_{\rho \in V_\rho^{(k-3)}} T^{(r)}(a). \]

In this subsection we describe several properties of the sets $V_\rho^{(k-3)}$ used in the Section 3.3 below. In particular, by the reduction procedure described in Proposition 3.7, we see that for our purposes it is enough to consider the case of $\rho = \emptyset$ (see Example 3.11 for case of $V_0^{(3)}$).

Note that by the choice of set $S_{k-3}$ it is clear that for any $\rho \in \mathbb{Z}_2^{k-3}$, the set $V_\rho^{(k-3)}$ is not empty. In fact, we see in Lemma 3.10 that the set $V_\rho^{(k-3)}$ has the same cardinality as $V_0^{(k-3)}$.

We now give an alternative description of the elements of the set $V_\rho^{(k-3)}$ as simple undirected graph allowing loops.

Definition 3.4. For $r \in \{0,1\}^{S_{k-3}} = \mathbb{Z}_2^{(k-2)(k-3)}$, the graph $G(r)$ is the undirected graph with the vertex set
\[V(G(r)) = \{1, \ldots, k-3\}, \]
and edges determined by
\[
(i, i) \in E(G(r)) \quad \text{if } r_{0,i} = 1, \quad \text{for } 0 < i
\]
\[
(i, j) \in E(G(r)) \quad \text{if } r_{i,j} = 1, \quad \text{for } 0 < i < j,
\]
where \( E(\mathcal{G}(r)) \) is the edge set of \( \mathcal{G}(r) \).

We denote by \( \text{deg}(\mathcal{G}(r)) \) the (ordered) list of degree of the vertices of \( \mathcal{G}(r) \).

Note that different to usual convention, when the graph \( \mathcal{G}(r) \) has a loop \((i, i) \in E(\mathcal{G}(r))\) we consider the loop to contribute 1 to the degree of the vertex \( i \).

**Example 3.5.** Let \( i \) the loop to contribute 1 to the degree of the vertex \( i \).

Actually, we easily verify that \( r \in V_\rho^{(4)} \) with \( \rho = (1, 1, 1, 1) \). Notice also that \( \text{deg}(\mathcal{G}(r)) = ((3, 1, 3, 3) \equiv (1, 1, 1, 1) \pmod{2}) \).

![Figure 1. Graph \( \mathcal{G}(r) \) associated to the vector \( r = (0, 0, 1, 1, 1, 1, 0, 0, 1) \)](image)

In fact, the last property of the example determines the set \( V_\rho^{(k-3)} \), as we can easily verify and state in the following lemma.

**Lemma 3.6.** For \( k \geq 3 \), we have

\[
V_\rho^{(k-3)} = \{ r \in \{0, 1\}^{S_{k-3}} \mid \text{deg}(\mathcal{G}(r)) \equiv \rho \}. \tag{34}
\]

**Proposition 3.7.** For \( \rho \in \mathbb{Z}^{k-3}_2 \), we have

\[
|V_\rho^{(k-3)}| = |V_0^{(k-3)}| = 2^{(k-3)(k-4)/2},
\]

and the bijection \( \sigma_\rho : V_0^{(k-3)} \rightarrow V_\rho^{(k-3)} \) is given explicitly by the map

\[
r \in V_0^{(k-3)} \mapsto r + (\rho \oplus 0^{(k-3)(k-4)/2}) \pmod{2} \in V_\rho^{(k-3)}.
\]

Furthermore, the map \( \sigma_\rho \) induces the relation

\[
T(\sigma_\rho(r))(a) = \prod_{i=1}^{k-3} (\tanh(a_i)^{1-2r_{0,i}})^{\rho_i} T(r)(a).
\]

**Proof.** From (34), we see that \( |V_0^{(k-3)}| \) is equal to the number of even graphs with \( k - 3 \) vertices. By §1.4 of [12], the number of such graphs is equal to \( 2^{(k-3)(k-4)/2} \).

Next, let us consider the effect of the map \( \sigma_\rho : V_0^{(k-3)} \rightarrow \{0, 1\}^{S_{k-3}} \) on the associated graphs \( \mathcal{G}(r) \), in particular on the degree of a given vertex \( i \in \{1, 2, 3, \ldots, k - 3\} \). First, it is clear that any edge \((i, j)\), with \( i \neq j \), in \( \mathcal{G}(r) \) is invariant under \( \sigma_\rho \), that is, if \((i, j)\) is an edge of \( \mathcal{G}(r) \) then it is also an edge of \( \mathcal{G}(\sigma_\rho(r)) \). Now, suppose that \( \rho_i = 1 \) and the vertex \( i \) does not have a loop in \( \mathcal{G}(r) \) (i.e. \( r_{0i} = 0 \)), then the vertex \( i \) has a loop in \( \mathcal{G}(\sigma_\rho(r)) \). On the other hand, if the vertex \( i \) has a loop in \( \mathcal{G}(r) \), then \( i \) does not have a loop in \( \mathcal{G}(\sigma_\rho(r)) \). Thus the degree of \( i \) in \( \mathcal{G}(\sigma_\rho(r)) \) is \( \pm 1 \) the degree of \( i \) in \( \mathcal{G}(r) \) (see Figure 2 for an example).

If \( \rho_i = 0 \), there is no change in the degree of the vertex \( i \). Consequently, we have

\[
\text{deg}(\mathcal{G}(\sigma_\rho(r))) \equiv \text{deg}(\mathcal{G}(r)) + \rho \equiv \rho \pmod{2},
\]

and thus \( \sigma_\rho(V_0^{(k-3)}) \subseteq V_\rho^{(k-3)} \). It is clear that the map \( \sigma_\rho \) is an involution, whence the second claim is proved. The third claim follows directly by the definition of \( T(r)(a) \). \qed
Example 3.8. Suppose \( k = 6 \), thus \(|S_3| = 6\). Let \( r = (0, 1, 1, 1, 0) \in V_0^{(3)} \) and \( \rho = (1, 0, 1) \). Then, let \( C = \sinh(a_1) \sinh(a_2) \cosh(a_6) \), we have

\[
T(\sigma_\rho(r))(a) = \sinh(a_1) \sinh(a_2) \cosh(a_3)C \\
= (\tanh(a_1) \coth(a_3)) \sinh(a_2) \sinh(a_3)C \\
= (\tanh(a_1) \coth(a_3)) T(r)(a).
\]

Moreover, by the foregoing discussion, the Fourier coefficients are given by

\[
C_\rho = \sum_{r \in V_0^{(k-3)}} T(\sigma_\rho(r))(a).
\]

Next, we define several projections on set \( V_0^{(k-3)} \).

**Definition 3.5.** Let \( k \geq 4 \), then

- \( p_1 : Z_2^{\frac{1}{2}(k-2)(k-3)} \rightarrow Z_2^{k-3} \) is the projection of the first \( k-3 \) components,
- \( p_2 : Z_2^{\frac{1}{2}(k-2)(k-3)} \rightarrow Z_2^{\frac{1}{2}(k-3)(k-4)} \) is the projection of the last \( (k-3)(k-4)/2 \) components.

Let \( k \geq 5 \), then:

- \( q_1 : Z_2^{\frac{1}{2}(k-2)(k-3)} \rightarrow Z_2^{k-4} \) is the projection of the \( k-4 \) components starting from \( k-2 \),
- \( q_2 : Z_2^{\frac{1}{2}(k-2)(k-3)} \rightarrow Z_2^{\frac{1}{2}(k-4)(k-5)} \) is the projection of the last \( (k-4)(k-5)/2 \) components.

Note that if the appropriate domain of the functions are considered, the relations

\[
q_1 = p_1 \circ p_2, \quad q_2 = p_2 \circ p_2.
\]

hold.

**Example 3.9.** Let \( k = 7 \) and \( r = (0, 0, 1, 1, 1, 1, 0, 0, 1) \in r \in V_\rho^{(4)} \), then

\[
p_1(r) = (0, 0, 1, 1), \quad p_2(r) = (1, 1, 1, 0, 0, 1),
\]

and \( r = p_1(r) \oplus p_2(r) \). Also,

\[
q_1(r) = (1, 1, 1), \quad q_2(r) = (0, 0, 1).
\]

The next results describes the structure of the set \( V_0^{(k-3)} \) used in \([3, 13]\) to evaluate the sums over the Fourier transforms of the functions \( R_\mu^{(k, N)} \) and \( g_k(u, s) \) (cf. Lemma 3.13).}

**Lemma 3.10.** Let \( r, s \in V_0^{(k-3)} \).

1. If \( r \neq s \), then \( p_2(r) \neq p_2(s) \). In other words, \( p_2 \) is a bijection of \( V_0^{(k-3)} \) onto \( Z_2^{(k-3)(k-4)/2} \).
2. If \( r \in V_0^{(k-3)} \), then \( |p_1(r)| \equiv 0 \pmod{2} \). Moreover,

\[
p_1(V_0^{(k-3)}) = \{ \rho \in Z_2^{k-3} \mid |\rho| \equiv 0 \pmod{2} \}.
\]

3. We have

\[
V_0^{(k-4)} = p_2(\{ v \in V_0^{(k-3)} : p_1(v) = 0_{k-3} \}),
\]
(4) For \( v \in \mathbb{Z}_2^{(k-4)(k-5)/2} \),
\[
p_1(\{ \sigma \in V_0^{(k-3)} : q_2(\sigma) = v \}) = \{ \sigma \in \mathbb{Z}_2^{k-3} : |\sigma| \equiv 0 \pmod{2} \},
\]
and
\[
q_1(\{ \sigma \in V_0^{(k-3)} : q_2(\sigma) = v \}) = \mathbb{Z}_2^{k-4}.
\]
Moreover, the restriction of \( p_1 \) and \( q_1 \) to the above sets are bijections.

(5) For \( v \in \mathbb{Z}_2^{(k-4)(k-5)/2} \), let \( r \in V_0^{(k-3)} \) such that \( q_2(r) = v \). Let \( r_0 \in V_0^{(k-3)} \) be the unique element such that \( p_1(r_0) = 0_{k-3} \) and \( q_2(r_0) = v \). Then,
\[
q_1(r) = q_1(r_0) + (r_{0,2}, r_{0,3}, \ldots, r_{0,k-3}) \pmod{2}.
\]

**Example 3.11.** We illustrate the statements of Lemma 3.10 with an example. For \( k = 6 \), the set \( V_0^{(k-3)} \) is given by
\[
V_0^{(3)} = \{(0,0,0,0,0,0), (0,0,0,1,1,1), (0,1,1,0,0,1), (0,1,1,1,1,0), (1,0,1,0,1,0), (1,1,0,1,0,1), (1,1,0,0,1,1), (1,1,0,1,0,0)\}.
\]

Then, we see directly that
\[
p_2(V_0^{(3)}) = \mathbb{Z}_2^3,
\]
and
\[
p_1(V_0^{(3)}) = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}.
\]
Moreover,
\[
p_2(\{ v \in V_0^{(3)} : p_1(v) = 0_3 \}) = \{(0,0,0), (1,1,1)\} = V_0^{(2)},
\]
and if \( v = (0) \)
\[
p_1(\{ \sigma \in V_0^{(3)} : q_2(\sigma) = v \}) = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\},
\]
and
\[
q_1(\{ \sigma \in V_0^{(3)} : q_2(\sigma) = v \}) = \{(0,0), (1,1), (1,0), (0,1)\} = \mathbb{Z}_2^2.
\]

Finally, with the notation of (4) in the lemma, let \( v = (1) \) and \( r = (1,0,1,1,0,1) \). Then \( r_0 = (0,0,0,1,1,1) \), \( q_1(r_0) = (1,1),(r_{0,2}, r_{0,3}) = (0,1) \) and
\[
q_1(r) = (1,0) \equiv (1,1) + (0,1) \pmod{2}.
\]

**Proof of Lemma 3.10.** In this proof we use repeatedly the well-known (elementary) fact from graph theory that the number of vertices in an undirected simple graph with odd degree is even.

Suppose \( v = (v_1, v_2, \ldots, v_\ell) \in \mathbb{Z}_2^\ell \), where \( \ell = \frac{(k-3)(k-4)}{2} \). Then, suppose \( r = (0,0, \ldots, 0, v_1, v_2, \ldots, v_\ell) \in \mathbb{Z}_2^{(k-3)(k-2)/2} \). The associated graph \( G(r) \) is a undirected simple graph on \( k-3 \) vertices. Then, there is a unique element \( r' \in V_0^{(k-3)} \) such that \( p_2(r') = v \), that is, the one corresponding to the graph obtained by adding loops to the vertices of \( G(r) \) with odd degree. The correspondence establishes an injection of \( \mathbb{Z}_2^{(k-3)/2} \) into \( V_0^{(k-3)} \), which is actually seen to be a bijection by comparing the cardinality of the sets (cf. Proposition 3.7), proving (1). Moreover, this argument also shows that for \( r \in V_0^{(k-3)} \) we have \( |p_1(r)| \equiv 0 \pmod{2} \).

Conversely, let \( v \in \mathbb{Z}_2^{k-3} \) with \( |v| \equiv 0 \pmod{2} \). Set \( r = (v_1, v_2, \ldots, v_{k-3}, 0, 0, \ldots, 0) \in \mathbb{Z}_2^{(k-3)(k-2)/2} \), then the graph \( G(r) \) is a graph with exactly an even number of loops. Moreover, the graph \( G_1 \) with all vertices of even degree obtained by joining pair of vertices with loops with exactly one edge corresponds to a vector \( r_1 \in \mathbb{Z}_2^{(k-3)(k-2)/2} \) such that \( p_1(r_1) = v \), proving (2).

Let \( v \in V_0^{(k-4)} \), and \( v = 0_{k-3} \oplus v \), we are to prove that \( \tilde{v} \in V_0^{(k-3)} \), in other words, that
\[
\chi_0 = \prod_{i=1}^{k-3} (\chi_{0,i})^{\tilde{v}_i} \prod_{i<j} (\chi_{i,j})^{\tilde{v}_{i,j}},
\]
equivalently, 
\[
0_{k-3} = \sum_{i=1}^{k-3} \bar{v}_{0,i} e_i + \sum_{i<j} \bar{v}_{i,j} (e_i + e_j).
\]

Notice that by the definitions, we have
\[
\bar{v}_{0,i} = 0, \quad \bar{v}_{1,j} = v_{0,j-1}, \quad \bar{v}_{n,m} = v_{n-1,m-1},
\]
where \(i \in \{1, 2, \ldots, k-3\}\), \(j \in \{2, 3, \ldots, k-3\}\), and \(2 \leq n < m \leq k-3\). Thus
\[
\sum_{i=1}^{k-3} \bar{v}_{0,i} e_i + \sum_{i<j} \bar{v}_{i,j} (e_i + e_j) = \sum_{j=2}^{k-3} \bar{v}_{1,j} (e_1 + e_j) + \sum_{2 \leq i<j} \bar{v}_{i,j} (e_i + e_j)
\]
\[
= |p_1(v)|e_1 + \sum_{i=1}^{k-4} a_i e_{i+1} + \sum_{i<j} v_{i,j} (e_{i+1} + e_{j+1})
\]
and this is equal to \(0_{k-3}\) since \(v \in V^{(k-4)}_0\) (and \(|p_1(v)| \equiv 0 \pmod{2}\) by (2)). The converse follows in the same way, proving (3).

Next, let \(v \in Z_{(k-4)(k-5)/2}^2\). By (2), we have
\[
p_1(\{\sigma \in V^{(k-3)}_0 : q_2(\sigma) = v\}) \subset \{\sigma \in Z_{k-3}^2 : |\sigma| \equiv 0 \pmod{2}\}.
\]
and by (1), we have
\[
|\{\sigma \in V^{(k-3)}_0 : q_2(\sigma) = v\}| = 2^{k-4},
\]
it suffices to show that for \(\sigma \in Z_{k-3}^2\) with \(|\sigma| \equiv 0 \pmod{2}\), there is an element \(v \in \{\sigma \in V^{(k-3)}_0 : q_2(\sigma) = \sigma\}\) such that \(p_1(v) = \sigma\). Let \(\sigma \in Z_{k-3}^2\) with \(|\sigma| \equiv 0 \pmod{2}\). Let \(v = v \oplus 0_{k-4} \oplus \sigma\), the associated graph \(G(v)\), it is a graph on \(k-3\) vertices with an even number \(|v|\) of loops where the vertex 1 has degree 1 (if it has a loop) or 0. We consider the two cases separately.

Suppose that degree of the vertex 1 is 0. In this case, the subgraph \(G_1\) of \(G(v)\) obtained by removing the vertex 1 is a graph on \(k-4\) vertices with \(n = |v| \equiv 0 \pmod{2}\) loops, let \(G_0\) be \(G_1\) without the loops. As in (1), we know that \(G_0\) has an even number of vertices with odd degree. Let \(a\) (resp. \(b\)) be the number of vertices with odd degree (resp. even degree) in \(G_0\) that have a loop in \(G_1\). Let \(m_1\) (resp. \(m_0\)) be the number of vertices of odd degree in \(G_1\) (resp. \(G_0\)), then we have \(m_1 = m_0 - a + b\). Since \(a + b = n \equiv 0 \pmod{2}\) and \(m_0 \equiv 0 \pmod{2}\), then \(a, b\) have the same parity and therefore \(m_1 = 0 \pmod{2}\). Let \(G\) be the graph obtained from \(G(v)\) by adding edges from 1 to each of the (even number of) vertices with odd degree. Then, \(G\) is a graph where all vertices have even degree. It corresponds to a vector \(v = V^{(k-3)}_0(\text{with } p_1(v) = v\text{ and } q_2(v) = \sigma)\). The case where the vertex 1 has a loop is dealt in a similar way. This proves the first part of (4). The second part follows directly from (1).

Finally, let \(v \in Z_{(k-4)(k-5)/2}^2\). By (1) and (3) the existence of a unique \(r_0 \in V^{(k-3)}_0\) with \(p_1(r_0) = 0_{k-3}\) and \(q_2(r_0) = v\) is guaranteed. First, we consider the case \(v = 0_{(k-4)(k-5)/2}^2\), where we have \(r_0 = 0_{(k-3)(k-2)/2}\). Let \(r \in V^{(k-3)}_0\) with \(q_2(r) = 0_{(k-4)(k-5)/2}\). Since \(p_1(r) = (r_{0,1}, r_{0,2}, \ldots, r_{0,k-3})\), we are to prove \((r_{1,2}, r_{1,3}, \ldots, r_{1,k-3}) = q_1(r) = (r_{0,1}, r_{0,2}, \ldots, r_{0,k-3})\). The graph \(G(r)\) is a graph with \(k-3\) vertices of even degree, with an even number of loops and edges only of the form \((1, j)\) for \(r_{1,j} = 1\). If \(r_{0,j} = 1\) for \(j \geq 2\) there is a loop in the vertex \(j\) and there must be a vertex \((1, j)\) to make the degree of \(j\) even, thus \(r_{1,j} = r_{0,j} = 1\). Similarly, if there is no loop in \(j\), then there is no vertex \((1, j)\) in the graph. This proves (5) for the case \(r = 0_{(k-4)(k-5)/2}\).

Next, for general \(v \in Z_{(k-4)(k-5)/2}^2\), let \(v \in V^{(k-3)}_0\) the unique vector with \(p_1(v) = p_1(r)\) and \(q_2(v) = 0_{(k-4)(k-5)/2}\). By our argument above, we have \((\bar{r}_{1,2}, \bar{r}_{1,3}, \ldots, \bar{r}_{1,k-3}) = (\bar{r}_{0,2}, \bar{r}_{0,3}, \ldots, \bar{r}_{0,k-3})\). The graph \(G(r)\) is a simple graph with no loops and the graph \(G(v)\) is a graph where the edges that are not loops are of the form \((1, j)\) for \(j \geq 2\) and where if such an edge appear then there is loop in \(j\). Both graphs have all vertices with even degree. From this, it is easy to see that the graph \(G(s)\) corresponding to \(s = r_0 + \bar{r} \pmod{2}\) has all even vertices and therefore \(s \in V^{(k-3)}_0\). Moreover,
\( p_1(s) = p_1(r) \) and \( q_2(s) = q_2(r) \), therefore, by (4), we have \( s = r \), proving (5). This completes the proof of Lemma \( \text{3.10} \).

3.3. Summation via Fourier transforms. With the preparations of the previous sections, we proceed to compute the innermost sum appearing in (17). By \( \text{(35)} \), we have

\[
\sum_{s \in \mathbb{C}_{v,w}} g_{k-1}(s) R^{(k,N)}_{\mu}(s) = \frac{1}{4u(k-2)\Delta} \sum_{s \in \mathbb{Z}_2^{k-1}} g^{(v,w)}_{k-3}(s) R^{(v,w)}_{\mu}(s) = \frac{1}{2k-1u(k-2)\Delta} \sum_{\rho \in \mathbb{Z}_2^{k-3}} g^{(v,w)}_{k-3}(\rho) R^{(v,w)}_{\mu}(\rho)
\]

By \( \text{Proposition 3.11} \), the sum in the last line can be written as

\[
\sum_{\rho \in \mathbb{Z}_2^{k-3}} g^{(v,w)}_{k-3}(\rho) \sum_{r \in \mathbb{V}_0^{(k-3)}} T(\sigma_{r}(r)) (a)
\]

\[
= \sum_{r \in \mathbb{V}_0^{(k-3)}} T(r)(a) \sum_{\rho \in \mathbb{Z}_2^{k-3}} \left( u^{2r_{k-3}(\rho)\Delta} + (-1)^{v+w} u^{2(k-2-r_{k-3}(\rho))\Delta} \right) \prod_{i=1}^{k-3} \left( \tanh(a_i)^{1-r_{\mu}} \right)^{\rho_i}.
\]

Setting \( A^{(r)}_i = (-1)^v \tanh(a_i)^{1-r_{\mu}} \), we obtain

\[
\sum_{r \in \mathbb{V}_0^{(k-3)}} T(r)(a) \sum_{\rho \in \mathbb{Z}_2^{k-3}} \left( u^{2r_{k-3}(\rho)\Delta} + (-1)^{v+w} u^{2(k-2-r_{k-3}(\rho))\Delta} \right) \prod_{i=1}^{k-3} \left( A^{(r)}_i \right)^{\rho_i},
\]

or equivalently

\[
\sum_{r \in \mathbb{V}_0^{(k-3)}} T(r)(a) \left( f^{(r)}_{k-3}(u^{2\Delta}) + (-1)^{v+w} g^{(r)}_{k-3}(u^{2\Delta}) \right),
\]

where the functions \( f^{(r)}_k(\tau) \) and \( g^{(r)}_k(\tau) \) are given by

\[
f^{(r)}_k(\tau) = \sum_{\rho \in \mathbb{Z}_2^{k}} \tau^{r_{k}(\rho)} \prod_{i=1}^{k} (A^{(r)}_i)^{\rho_i}, \quad g^{(r)}_k(\tau) = \sum_{\rho \in \mathbb{Z}_2^{k}} \tau^{k+r_{k}(\rho)} \prod_{i=1}^{k} (A^{(r)}_i)^{\rho_i}.
\]

Next, we compute explicitly the functions \( f^{(r)}_k(\tau) \) and \( g^{(r)}_k(\tau) \). For simplicity, we consider the general case

\[
f_k(\tau) = \sum_{\rho \in \mathbb{Z}_2^{k}} \tau^{r_{k}(\rho)} \prod_{i=1}^{k} A^{\rho_i}_i, \quad g_k(\tau) = \sum_{\rho \in \mathbb{Z}_2^{k}} \tau^{k+r_{k}(\rho)} \prod_{i=1}^{k} A^{\rho_i}_i,
\]

where \( A_i \in \mathbb{C} \) for \( i \in \{1, 2, \cdots, k\} \). Note also that \( g_k(\tau) = \tau^{k+1} f_k(\tau^{-1}) \).

Proposition 3.12. For \( k \in \mathbb{Z}_{\geq 1} \), we have

\[
f_k(\tau) + (-1)^{v+w} g_k(\tau) = \frac{1}{2k} \sum_{\ell=0}^{k} (1 + \tau)^{k-\ell} (1 - \tau)^{\ell} \left( 1 + (-1)^{v+w+\ell} \right) \prod_{j_1 < j_2 < \ldots < j_\ell \atop j_1 + j_2 + \ldots + j_\ell} \left( \prod_{n=j_1+1}^{j_\ell+1} (1 + (-1)^{v+w+\ell-i} A_n) \right),
\]

where in the innermost product we have \( j_0 = 0 \) and \( j_{r+1} = k \).
Proof. By property (1) of Theorem 3.3, we obtain the system of simultaneous recurrence relations

\[ f_k(\tau) = f_{k-1}(\tau) + A_k g_{k-1}(\tau), \quad g_k(\tau) = \tau (g_{k-1}(\tau) + A_k f_{k-1}(\tau)) \]

with initial conditions \( f_0(\tau) = 1 \) and \( g_0(\tau) = \tau \). The recurrence (38) can be written as

\[
\begin{bmatrix}
 f_k \\
 g_k
\end{bmatrix} = \prod_{j=1}^{k} \begin{bmatrix}
 1 & A_j \\
 \tau A_j & \tau
\end{bmatrix} \begin{bmatrix}
 f_0 \\
 g_0
\end{bmatrix},
\]

where \( f_0(\tau) = 1 \) and \( g_0(\tau) = \tau \). Notice that

\[
\begin{bmatrix}
 1 & A_j \\
 \tau A_j & \tau
\end{bmatrix} = \begin{bmatrix}
 1 & 0 \\
 0 & \tau
\end{bmatrix} \begin{bmatrix}
 1 & A_j \\
 \delta
\end{bmatrix}.
\]

Actually, we have

\[
(39) \quad \begin{bmatrix}
 f_k \\
 g_k
\end{bmatrix} = C \prod_{j=1}^{k} [aI + bJ] D(A_j) C \begin{bmatrix}
 f_0 \\
 g_0
\end{bmatrix},
\]

where \( a = \frac{1+\tau}{2}, b = \frac{1-\tau}{2}, C \) is the Cayley transform

\[
C = \frac{1}{\sqrt{2}} \begin{bmatrix}
 1 & 1 \\
 1 & -1
\end{bmatrix},
\]

and \( D(x) \) is a two-by-two matrix-valued function given by

\[
D(x) = \begin{bmatrix}
 1 + x & 0 \\
 0 & 1 - x
\end{bmatrix}.
\]

Indeed, (39) follows immediately from the facts

\[
C \begin{bmatrix}
 1 & A_j \\
 A_j & 1
\end{bmatrix} C = D(A_j), \quad C \begin{bmatrix}
 1 & 0 \\
 0 & \tau
\end{bmatrix} C = \frac{1}{2} (1 + \tau) I + (1 - \tau) J.
\]

Obviously, we have

\[
\prod_{j=1}^{k} [aI + bJ] D(A_j) = \sum_{\ell=0}^{k} a^{k-\ell} b^{\ell} \sum_{\delta \in Z^1_2} \prod_{j=1}^{k} D^{\delta_j} (A_j),
\]

where \( D^{\delta_j} (A_j) := D(A_j) \) when \( \delta_j = 0 \) and \( D^{\delta_j} (A_j) := JD(A_j) \) when \( \delta_j = 1 \).

For \( \delta \in Z^1_2 \) with \(|\delta| = \ell \), define \( j_i \) by enumerating

\[
\{ j \mid \delta_j = 1 \} = \{ (1 \leq j_1 < j_2 < \ldots < j_\ell \leq k) \},
\]

then

\[
\prod_{j=1}^{k} D^{\delta_j} (A_j) = D(A_k) \cdots JD(A_{j_\ell}) \cdots D(A_{j_{\ell-1}+1}) JD(A_{j_{\ell-1}}) \cdots JD(A_{j_1}) \cdots D(A_1)
\]

\[
= D(A_k) \cdots D(-A_{j_\ell}) \cdots D(-A_{j_{\ell-1}+1}) D((-1)^2 A_{j_{\ell-1}}) \cdots D((-1)^\ell A_{j_1}) \cdots D((-1)^\ell A_1) J^\ell.
\]

It follows that

\[
\prod_{j=1}^{k} [aI + bJ] D(A_j) = \sum_{\ell=0}^{k} a^{k-\ell} b^{\ell} \sum_{j_1 < j_2 < \ldots < j_\ell} D(A_k) \cdots D(-A_{j_\ell}) \cdots D(-A_{j_{\ell-1}+1}) D((-1)^2 A_{j_{\ell-1}}) \cdots D((-1)^\ell A_{j_1}) \cdots D((-1)^\ell A_1)
\]

\[
= \sum_{\ell=0}^{k} a^{k-\ell} b^{\ell} \sum_{j_1 < j_2 < \ldots < j_\ell} D(A_k) \cdots \prod_{i=1}^{k} D((-1)^{j_i} A_i) J^\ell.
\]
Define, for a vector \( j = \{ j_1, j_2, \cdots, j_\ell \} \in \mathbb{Z}_2^{\ell} \) with \( 1 \leq j_1 < j_2 < \cdots < j_\ell \leq k \), the expressions

\[
S(j) = \prod_{i=0}^{\ell} \left( \prod_{n=j_i+1}^{j_{i+1}} (1 + (-1)^{\ell-i} A_n) \right), \quad \overline{S}(j) = \prod_{i=0}^{\ell} \left( \prod_{n=j_i+1}^{j_{i+1}} (1 - (-1)^{\ell-i} A_n) \right),
\]

where \( j_0 := 0 \) and \( j_{\ell+1} := k \). Then, for \( j \) as above we can write

\[
\prod_{i=j_\ell+1}^{k} D(A_i) \cdots \prod_{i=j_{\ell-1}+1}^{j_\ell} D(-A_i) \cdots \prod_{i=1}^{j_1} D((-1)^{\ell} A_i) = \begin{pmatrix} S(j) & 0 \\ \overline{S}(j) \end{pmatrix}.
\]

Noticing that the factor \( J^\ell \) depends only on the parity of \( \ell \), we obtain

\[
f_k(\tau) = \frac{1}{2^{k+1}} \sum_{\ell=0}^{[k/2]} (1 + \tau)^{k-2\ell}(1 - \tau)^{2\ell} \sum_{j_1 < j_2 < \cdots < j_\ell} ((1 + \tau)S(j) + (1 - \tau)\overline{S}(j))
\]

\[
+ \frac{1}{2^{k+1}} \sum_{\ell=1}^{[k/2]} (1 + \tau)^{k-(2\ell-1)}(1 - \tau)^{2\ell-1} \sum_{j_1 < j_2 < \cdots < j_{2\ell-1}} ((1 - \tau)S(j) + (1 + t)\overline{S}(j)),
\]

and

\[
g_k(\tau) = \frac{1}{2^{k+1}} \sum_{\ell=0}^{[k/2]} (1 + \tau)^{k-2\ell}(1 - \tau)^{2\ell} \sum_{j_1 < j_2 < \cdots < j_\ell} ((1 + t)S(j) - (1 - \tau)\overline{S}(j))
\]

\[
+ \frac{1}{2^{k+1}} \sum_{\ell=1}^{[k/2]} (1 + \tau)^{k-(2\ell-1)}(1 - \tau)^{2\ell-1} \sum_{j_1 < j_2 < \cdots < j_{2\ell-1}} ((1 - \tau)S(j) - (1 + \tau)\overline{S}(j)).
\]

Hence the results follows. \( \square \)

We remark here that for \( 1 \leq \ell \leq k \), each set of \( \ell \) numbers \( j_i \ (1 \leq j_1 < j_2 < \cdots < j_\ell \leq k) \) determine a unique vector \( \rho \in \mathbb{Z}_2^{\ell} \) such that \( |\rho| = \ell \) and where \( j_i \) is the position of the \( i \)-th one in \( \rho \). Likewise, each vector \( \rho \in \mathbb{Z}_2^{\ell} \) determines a unique set of \( \ell = |\rho| \) integers such that \( 1 \leq j_1 < j_2 < \cdots < j_\ell \leq k \) by setting \( j_1 \) as the position of the \( i \)-th one in \( \rho \).

By Proposition 3.12 applied to \( A_i^{(r)} \), (37) is equal to

\[
\sum_{j_1 < j_2 < \cdots < j_\ell} \sum_{r \in V_0^{(k-3)}} T(r)(a) \prod_{i=0}^{\ell} \left( \prod_{n=j_i+1}^{j_{i+1}} (1 + (-1)^{x+\ell-i} \tanh(a_n)^{1-2r_\rho}) \right).
\]

(40)

Next we deal with the innermost sum over the set \( V_0^{(k-3)} \). Let \( x \in \mathbb{Z}, \rho = \{ \rho_1, \rho_2, \cdots, \rho_{k-3} \} \in \mathbb{Z}_2^{k-3} \) with \( |\rho| = \ell \), and \( 1 \leq j_1 < j_2 < \cdots < j_\ell \) be the position of the ones in \( \rho \). We have

\[
\sum_{r \in V_0^{(k-3)}} T(r)(a) \prod_{i=0}^{\ell} \left( \prod_{n=j_i+1}^{j_{i+1}} (1 + (-1)^{x+\ell-i} \tanh(a_n)^{1-2r_\rho}) \right)
\]

\[
= \sum_{r \in V_0^{(k-3)}} T(r)(a) \prod_{i=1}^{k-3} (1 + (-1)^{v_0+v_i} \tanh(a_i)^{1-2r_\rho}).
\]

with \( v_0 = x \) and \( v_i = v_i(\rho) = \sum_{j=i}^{k} \rho_j \), for \( i = 1, 2, \ldots, k-3 \).
Lemma 3.13. Let $v_i \in \mathbb{C}$ for $i \in \{0, 1, 2, \cdots, k-3\}$. We have

$$
\sum_{r \in V_0^{(k-3)}} T^{(r)}(a) \prod_{i=1}^{k-3} \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)^{1-2r_{0i}}\right) = \exp \left( \sum_{m=0}^{k-4} \sum_{j=1}^{k-3-m} (-1)^{r_{m+j}^{v_i+b_{m+j}} a_{m+j}} \right).
$$

Proof. The proof is by induction. It is immediate to verify the result for the cases $k - 3 = 1, 2$. For $v \in V_0^{(k-3)}$, the single summand of (11) corresponding to $r$ is

$$
\prod_{i=1}^{k-3} \cosh(a_i)^{1-r_{0i}} \sinh(a_i)^{r_{0i}} \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)^{1-2r_{0i}}\right) T^{(p_2(r))}(p_2(a)),
$$

it is not difficult to see that it can be written as

$$
(-1)^{p_1(r)} 1^{\sum_{i=1}^{k-3} r_{0i} v_i} \prod_{i=1}^{k-3} \cosh(a_i) \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)\right) T^{(p_2(r))}(p_2(a))
$$

$$
= (-1)^{\sum_{i=1}^{k-3} r_{0i} v_i} \prod_{i=1}^{k-3} \cosh(a_i) \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)\right) T^{(p_2(r))}(p_2(a)),
$$

since $|p_1(r)| \equiv 0 \pmod{2}$. Next, observe that

$$
\prod_{i=1}^{k-3} \cosh(a_i) \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)\right) = \exp \left( \sum_{i=1}^{k-3} (-1)^{r_0 + v_i} a_{0i} \right),
$$

thus, the expression above is given by

$$
(-1)^{\sum_{i=1}^{k-3} r_{0i} v_i} \exp \left( \sum_{i=1}^{k-3} (-1)^{r_0 + v_i} a_{0i} \right) T^{(p_2(r))}(p_2(a)).
$$

Next, for $v \in \mathbb{Z}_2^{(k-4)(k-5)/2}$, we define the set $S(v) \subset V_0^{(k-3)}$ as $S(v) = \{ \sigma \in V_0^{(k-3)} : q_2(\sigma) = v \}$. By Lemma 3.10(4), we have $|S(v)| = 2^{k-4}$. For $v \in \mathbb{Z}_2^{(k-4)(k-5)/2}$, we have

$$
\sum_{r \in S(v)} T^{(r)}(a) \prod_{i=1}^{k-3} \left(1 + (-1)^{r_0 + v_i} \tanh(a_i)^{1-2r_{0i}}\right)
$$

$$
= \exp \left( \sum_{i=1}^{k-3} (-1)^{r_0 + v_i} a_{0i} \right) T^{(v)}(q_2(a)) \sum_{r \in S(v)} (-1)^{\sum_{i=1}^{k-3} r_{0i} v_i} T^{(q_1(r))}(q_1(a)),
$$

Let $\tilde{r} \in V_0^{(k-3)}$ be the unique element such that $p_1(\tilde{r}) = 0_{k-3}$ and $q_2(\tilde{r}) = v$. Then, by the proof of Lemma 3.10(5), we can write $r \in S(v)$ as

$$
r = r + \tilde{r},
$$

where $p_1(\tilde{r}) = p_1(r)$ and $q_2(\tilde{r}) = 0_{(k-4)(k-5)/2}$. Moreover, also by Lemma 3.10(5), we have

$$
\tilde{r} = (r_{0,1}, r_{0,2}, \cdots, r_{0,k-3}, r_{0,2}, \cdots, r_{0,k-3}) \oplus 0_{(k-4)(k-5)/2}.
Therefore, by \(3.10(4)\) we have
\[
\sum_{\mathbf{r} \in S(\mathbf{v})} (-1)^{\sum_{i=1}^{k-3} r_{i0} v_{i}} T^{(q_{1}(\mathbf{r}))}(p_{1}(\mathbf{a})) = \sum_{\mathbf{r} \in \mathbb{Z}^{k-3}_{2}} (-1)^{\sum_{i=1}^{k-3} r_{i0} v_{i}} T^{((r_{0}, 2, r_{0}, 3, \ldots, r_{0}, k-3)+\mathbf{a}(\mathbf{r}))}(q_{1}(\mathbf{a}))
\]
where each element \(\mathbf{r} \in \mathbb{Z}^{k-3}_{2}\) has entries \(r_{i0}\). Namely, we have, by Lemma \(3.10(3)\), that \(\sum_{m=0}^{k-3} (-1)^{v_{m}+v_{m+j}} a_{m, m+j}\) where \(v_{0} = x\). In terms of the entries of the vector \(\rho\), the expression for \(f(\mathbf{a}, \rho, x)\) in \(41\) is
\[
f(\mathbf{a}, \rho, x) = \sum_{m=0}^{k-4} \sum_{j=1}^{k-3-m} (-1)^{v_{m}+v_{m+j}} a_{m, m+j}.
\]
where \(\rho_{0} = 0\) and where \(\delta_{y}(x)\) is the Kronecker delta function. Using Lemma \(3.13\) with \(x = w\) in \(40\), we obtain
\[
\sum_{s \in \mathbb{C}^{(k-1)}} g_{k-1}(s) R_{\rho}(s) = \frac{1}{2^{k-1} u(k-2)\Delta} \sum_{\ell=0}^{k-3} (1+u^{2\Delta})^{k-3-\ell} (1-u^{2\Delta})^{\ell} (1+(-1)^{v_{m}+v_{m+j}} a_{m, m+j}) \sum_{\rho \in \mathbb{Z}^{k-3}_{2}} \exp(f(\mathbf{a}, \rho, w))
\]
where each element \(\mathbf{r} \in \mathbb{Z}^{k-3}_{2}\) has entries \(r_{i0}\) and \(v_{0} = x\). In terms of the entries of the vector \(\rho\), the expression for \(f(\mathbf{a}, \rho, x)\) in \(41\) is
\[
f(\mathbf{a}, \rho, x) = \sum_{m=0}^{k-4} \sum_{j=1}^{k-3-m} (-1)^{v_{m}+v_{m+j}} a_{m, m+j}.
\]
We note here that the right-hand side of the above equation consists of a sum of a radial function on \( \rho \) multiplied by an exponential factor. This is an essential fact for the evaluation of the limit appearing in the heat kernel of the QRM as a Riemann sum in \( \mathbb{R} \) In the remainder of this section we describe how to evaluate the expression on the right-hand side, concretely, the exponential factor, when the sum is restricted to a fixed value \( |\rho| = \lambda \in \mathbb{Z}_{\geq 0} \).

3.4. Riemann sums and residual terms. In this subsection we compute the sums given in the previous section 3.3 by changing sums to integrals with residual terms with explicitly given order.

First, by the discussion of 3.3 we see that the main limit in the expression of the heat kernel is given by

\[
\lim_{N \to \infty} \left( \frac{1 - u^{2/N}}{2u^{N}} \right)^{N-3} \sum_{k \geq 3} \left( \frac{1 + u^{2/N}}{2u^{N}} \right)^{N-3} \times \exp \left( a_0(u^{1/N}) + \sum_{m=0}^{k-4} \sum_{j=1}^{k-3-m} (-1)^{|\rho|+1} d_0(m) + \sum_{i=m}^{m+j-1} \rho_i a_{m,m+j}(u^{1/N}) \right) \]

Note that by Lemma 3.3 we have \( a_0(u^{1/N}) = O \left( \frac{1}{N} \right) \), and also for \( \rho \in \mathbb{Z}^{k-3} \) we have

\[
\sum_{j=1}^{k-3} (-1)^{\sum_{i=1}^{j-1} \rho_i} a_{0,j}(u^{1/N}) = \frac{\sqrt{2}(1 - u^{1/N})}{1 - u^2} \left( x \sum_{j=1}^{k-3} (-1)^{\sum_{i=1}^{j-1} \rho_i} \left( u^{1/N} - u^{2 \cdot \frac{k-1}{j}} \right) \right) + yu \sum_{j=1}^{k-3} (-1)^{\sum_{i=1}^{j-1} \rho_i} \left( u^{\frac{k-1}{j}} - u^{\frac{k-1}{j}} \right) + R_1(u^{1/N}),
\]

where the residual term is \( R_1(u^{1/N}) = O(\frac{1}{N}) \). Using identity (4) of Theorem 3.3 we obtain

\[
(-1)^{|\rho|} \sum_{j=1}^{k-3} (-1)^{\sum_{i=1}^{j-1} \rho_i} a_{0,j}(u^{1/N}) = \frac{\sqrt{2} g u^{1/N}}{1 - u^2} \left[ x (1 - u^{k-1/N})(1 - u^2 - u^{1/N}) - uy(1 - u^{k-1/N})(1 - u^{k-1/N}) \right] + \frac{2\sqrt{2} g (1 - u^{1/N})}{1 - u^2} \left[ x \left( u^{2 \cdot \frac{k-1}{N}} \varphi_{k-3}(\rho; u^{1/N}) - u^{\frac{k-1}{N}} \varphi_{k-3}(\rho; u^{1/N}) \right) + uy \left( u^{\frac{k-1}{N}} \varphi_{k-3}(\rho; u^{1/N}) - \varphi_{k-3}(\rho; u^{1/N}) \right) \right] + R_1(u^{1/N}),
\]

(42)
On the other hand, the sum of the Fourier coefficients \( a_{i,j}(u^{\frac{k}{N}}) \) with \( 1 \leq i < j \) is given by
\[
\sum_{m=1}^{k-4} \sum_{j=1}^{k-3-m} (-1)^{\sum_{i=m}^{j-1} \rho_i} a_{m,m+j}(u^{\frac{k}{N}}) = \sum_{m=1}^{k-4} \sum_{j=m}^{k-4} (-1)^{\sum_{i=m}^{j} \rho_i} a_{m,j+1}(u^{\frac{k}{N}})
\]
\[
= \frac{g^2(1-u^{\frac{k}{N}})^2}{1-u^2} \sum_{m=1}^{k-4} (u^{-\frac{m}{N}} - u^{-\frac{m+1}{N}}) \sum_{j=m}^{k-4} (u^{\frac{j}{N}} - u^{\frac{j+1}{N}})(-1)^{\sum_{i=m}^{j} \rho_i}
\]
\[
= \frac{g^2(1-u^{\frac{k}{N}})^2}{1-u^2} \sum_{j=1}^{k-4} (u^{\frac{j}{N}} - u^{2-\frac{j+3}{N}}) \sum_{m=1}^{j} (u^{-\frac{m}{N}} - u^{-\frac{m+1}{N}})(-1)^{\sum_{i=m}^{j} \rho_i},
\]
and by using Theorem 3.3(4) once more, we see that this is equal to
\[
\frac{g^2}{1-u^2} \left( (1-u^{\frac{k}{N}})(1-u^{2-\frac{k}{N}})(k-4) - u^{\frac{k}{N}}(1-u^{\frac{k}{N}})(1-u^{2-\frac{k}{N}}) + \frac{u^{\frac{k}{N}}(1-u^{2k/N})(1-u^{2-2k/N})}{1+u^{\frac{k}{N}}} \right)
\]
\[
+ \frac{2g^2(1-u^{\frac{k}{N}})^2}{(1-u^2)} \sum_{j=1}^{k-4} (u^{\frac{j}{N}} - u^{2-\frac{j+3}{N}}) \left( u^{\frac{j}{N}} \varphi_j(\text{Pre}_j(\rho);u^{\frac{k}{N}}) - u^{-\frac{j}{N}} \varphi_j(\text{Pre}_j(\rho);u^{-\frac{k}{N}}) \right),
\]
where \( \text{Pre}_j(\rho) \) is the prefix of length \( j \) of \( \rho \). Concretely, if \( \rho \in \mathbb{Z}_2^k \) and \( j \leq k \), then \( \text{Pre}_j(\rho) : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2^j \) is the projection into \( \mathbb{Z}_2^j \) of the first \( j \) elements of \( \rho \).

Next, for \( \rho \in \mathbb{Z}_2^{k-4} \) and \( \eta \in \{0,1\} \), define the auxiliary functions
\[
H_{\eta}^{(k,N)}(x,y,u^{\frac{1}{N}},\rho) = \exp \left( (-1)^{\eta} \frac{g\sqrt{2}g(1-u^{\frac{k}{N}})}{1-u^2} \left[ x \left( u^{\frac{k}{N}} \varphi_{k-3}(\rho;u^{-\frac{k}{N}}) - u^{\frac{k}{N}} \varphi_{k-3}(\rho;u^{\frac{k}{N}}) \right) + uy \left( u^{\frac{k}{N}} \varphi_{k-3}(\rho;u^{\frac{k}{N}}) - \varphi_{k-3}(\rho;u^{-\frac{k}{N}}) \right) \right) \right]
\]
\[
P^{(k,N)}(u^{\frac{k}{N}},\rho) = \exp \left( \frac{2g^2(1-u^{\frac{k}{N}})^2}{(1-u^2)} \sum_{j=1}^{k-4} (u^{\frac{j}{N}} - u^{2-\frac{j+3}{N}}) \left( u^{\frac{j}{N}} \varphi_j(\text{Pre}_j(\rho);u^{\frac{k}{N}}) - u^{-\frac{j}{N}} \varphi_j(\text{Pre}_j(\rho);u^{-\frac{k}{N}}) \right) \right).
\]

Next, for \( \lambda \geq 1 \) we proceed to rewrite the sum
\[
(44) \quad \sum_{\rho \in \mathbb{Z}_2^{k-3} \atop |\rho| = \lambda} H_{\eta}^{(k,N)}(u^{\frac{1}{N}},\rho) P^{(k,N)}(u^{\frac{k}{N}},\rho)
\]
into an expression that can be interpreted as multiple iterated integrals. We start with a lemma used to deal with the sums including terms \( \varphi_j(\text{Pre}_j(\rho);s) \).

**Lemma 3.14.** For \( k \geq 1 \) and \( 1 \leq \lambda \leq k \), for the indeterminates \( t, s \) we have
\[
\sum_{\rho \in \mathbb{Z}_2^k \atop |\rho| = \lambda} \exp \left( \sum_{j=1}^{k} t^j \varphi_j(\text{Pre}_j(\rho);s) \right) = \sum_{i_1 < i_2 < \ldots < i_{\lambda}} \exp \left( \sum_{0 \leq \alpha < \beta \atop \beta - \alpha \equiv 1 \mod 2} t^{i_\alpha}s^{i_\beta}[i_{\alpha+1} - i_\alpha]_s[i_{\beta+1} - i_\beta]_t \right),
\]
where \( i_0 := 0, i_{\lambda+1} := k + 1 \). Moreover, for \( \rho \in \mathbb{Z}_2^k \) with \( |\rho| = \lambda \), we have
\[
\exp \left( \sum_{j=1}^{k} t^j \varphi_j(\text{Pre}_j(\rho);s) \right) = \exp \left( \sum_{0 \leq \alpha < \beta \atop \beta - \alpha \equiv 1 \mod 2} t^{i_\alpha}s^{i_\beta}[i_{\alpha+1} - i_\alpha]_s[i_{\beta+1} - i_\beta]_t \right),
\]
where \( i_0 := 0, i_{\lambda+1} := k + 1 \) and \( i_j \), for \( 1 \leq j \leq \lambda \), is the position of the \( j \)-th one in \( \rho \).
Proof. Let us first consider the case $|\rho| = \lambda = 1$. In this case $\rho = e_i$ for $1 \leq i \leq k$. From the definition of $\varphi_j$, we verify that

$$
\varphi_j(\text{Pre}_j(\rho); s) = \begin{cases} 
0 & \text{if } j < i \\
[i]_s & \text{if } i \leq j
\end{cases},
$$

thus

$$
\sum_{j=1}^{k} t^j \varphi_j(\text{Pre}_j(\rho); s) = [i]_s \sum_{j=i}^{k} t^j = t^i[k + 1 - i][i]_s.
$$

Next, let us assume the result holds for all $|\rho| = \lambda - 1$ and consider the case $|\rho| = \lambda$. Set $\omega := \text{Pre}_{i\lambda - 1}(\rho)$, then we have

$$
\sum_{j=1}^{k} t^j \varphi_j(\text{Pre}_j(\rho); s) = \sum_{j=1}^{i\lambda - 1} t^j \varphi_j(\text{Pre}_j(\omega); s) + \sum_{j=i\lambda}^{k} t^j \varphi_j(\omega; s),
$$

since $\varphi_j(\text{Pre}_j(\rho); s) = \varphi_k(\rho; s)$ for $j \geq i\lambda$. On the one hand, we have

$$
\sum_{j=1}^{i\lambda - 1} t^j \varphi_j(\text{Pre}_j(\omega); s) = \sum_{0 < \alpha < \beta < \lambda \equiv 1 \pmod{2}} t^j [i_{\alpha + 1} - i_{\alpha}] [i_{\beta + 1} - i_{\beta}] t,
$$

by induction since $|\omega| = \lambda - 1$. On the other hand, we have

$$
\varphi_k(\rho; s) = \begin{cases} 
\sum_{n=1}^{\lambda - 1} [i_{2n}]_s - [i_{2n-1}]_s & \text{if } \lambda \equiv 0 \pmod{2} \\
\sum_{n=0}^{\lambda - 1} [i_{2n+1}]_s - [i_{2n}]_s & \text{if } \lambda \equiv 1 \pmod{2}
\end{cases}.
$$

Let us consider the case $\lambda \equiv 0 \pmod{2}$ since the alternative case is completely analogous. We immediately verify that

$$
\sum_{n=1}^{\lambda - 1} [i_{2n}]_s - [i_{2n-1}]_s = \sum_{n=1}^{\lambda} s^{i_{2n-1}} [i_{2n} - i_{2n-1}] s,
$$

and substituting in the second sum of the right-hand side we obtain

$$
\sum_{j=1}^{k} t^j \varphi_k(\rho; s) = t^{i\lambda} \sum_{n=1}^{\lambda} s^{i_{2n-1}} [i_{2n} - i_{2n-1}] s \sum_{j=0}^{k-i\lambda} t^j = t^{i\lambda} [k + 1 - i\lambda] t \sum_{n=1}^{\lambda} s^{i_{2n-1}} [i_{2n} - i_{2n-1}] s,
$$

finally, notice that since $\lambda$ is even and $2n - 1$ for $1 \leq n \leq \frac{\lambda}{2}$ runs over all odd integers smaller than $\lambda$ we see that the above is equal to

$$
\sum_{0 \leq \alpha < \lambda \equiv 1 \pmod{2}} t^{i\lambda} s^{i_{\alpha + 1} - i_{\alpha}} [i_{\lambda + 1} - i_{\lambda}] t,
$$

with $i_{\lambda + 1} := k + 1$, as desired. $\square$

Let us consider a fixed $\lambda \geq 1$ and $\rho \in \mathbb{Z}_2^{k-3}$ with $|\rho| = \lambda$. As usual, we denote by $1 \leq i_1 < i_2 < \cdots < i_\lambda$ the position of $1$ in $\rho$. By Lemma 3.14 and (29), we see that $H_\eta^{(k,N)}(u^1/N, \rho) P(k,N)(u^1/N, \rho)$ is...
given by
\[
\exp \left( -1 \eta \frac{2 \sqrt{2} g (1 - u^{1/N})}{1 - u^2} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} \left[ x \left( u^{2-\frac{\beta}{N}} [i_{\gamma}]_{u^{1/N}} - u^{\frac{\beta}{N}} [i_{\gamma}]_{u^{1/N}} \right) + uy \left( u^{\frac{\beta}{N}} [i_{\gamma}]_{u^{1/N}} - [i_{\gamma}]_{u^{1/N}} \right) \right] + 2g^2 \frac{(1 - u^{1/N})^2}{(1 - u^2)} \sum_{\beta - \alpha \equiv \gamma \pmod{2}}^{\lambda} \left[ u^{\frac{\beta}{N}} [i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} - u^{2} \frac{1+\gamma}{N} [i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} \right] \right)
\]
\[
- u^{\frac{\beta}{N}} [i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} \left( u^{\frac{\beta}{N}} [i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} - u^{2} \frac{1+\gamma}{N} [i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} \right) \right).
\]

Next, for \( 1 \leq \gamma \leq \lambda \), we immediately see that
\[
u^{2-\frac{\beta}{N}} [i_{\gamma}]_{u^{1/N}} - u^{\frac{\beta}{N}} [i_{\gamma}]_{u^{1/N}} = - \frac{u^{2-\frac{\beta}{N}}}{1 - u^{1/N}} (1 - u^{-2 + \frac{\gamma}{N}})(1 - u^{-\gamma}) - \frac{u^{1/N}}{1 - u^{1/N}} (1 - u^{-\gamma})(1 - u^{-\gamma}),
\]
and similarly, for \( 1 \leq \beta \leq \lambda \), we have
\[
u^{2+\gamma}[i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} - u^{2+\gamma}[i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}} = \frac{u^{2+\gamma}[i_{\gamma} + i_{\alpha - i_{\beta}}]_{u^{1/N}}}{1 - u^{1/N}} (1 - u^{-2 + \frac{\gamma}{N}})(1 - u^{-\gamma}) - \frac{u^{1/N}}{1 - u^{1/N}} (1 - u^{-\gamma})(1 - u^{-\gamma}).
\]

Therefore, the sum (44) is given by
\[
\sum_{\rho \in \mathbb{Z}_2^{k-3} \atop \|\rho\|=\lambda} H_{\eta}^{(k,N)}(u^{1/N}, \rho) P^{(k,N)}(u^{1/N}, \rho)
\]
\[
= \sum_{i_1 < i_2 < \cdots < i_{k-3}} \exp \left( -1 \frac{2 \sqrt{2} g N}{1 - u^2} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} \left[ xu^{2} (1 - u^{-2 + \frac{\gamma}{N}})(1 - u^{-\gamma}) - yu(1 - u^{-\gamma})(1 - u^{-\gamma}) \right] \right)
\]
\[
- \sum_{\beta - \alpha \equiv \gamma \pmod{2}}^{\lambda} \frac{2g^2 u^{\frac{\gamma+1}{N}}}{1 - u^2} (1 - u^{-2 + \frac{\gamma+1}{N}})(1 - u^{-\gamma})(1 - u^{-\gamma})(1 - u^{-\gamma})(1 - u^{-\gamma}) + O\left( \frac{1}{N} \right).
\]

For \( \lambda \geq 1 \), define the function
\[
f_{\lambda}^{(\eta)}(z_1, z_2, \cdots, z_{\lambda}; u^{1/N}) = (-1)^{\eta+1} \frac{2 \sqrt{2} g N}{1 - u^2} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} \left[ xu^{2} (1 - u^{-2 + \frac{\gamma+1}{N}})(1 - u^{-\gamma}) - yu(1 - u^{-\gamma})(1 - u^{-\gamma}) \right] \]
\[
- \sum_{\beta - \alpha \equiv \gamma \pmod{2}}^{\lambda} \frac{2g^2 u^{\frac{\gamma+1}{N}}}{1 - u^2} (1 - u^{-2 + \frac{\gamma+1}{N}})(1 - u^{-\gamma})(1 - u^{-\gamma})(1 - u^{-\gamma})(1 - u^{-\gamma}).
\]

where as before, we set \( z_0 := 0 \) and \( z_{\lambda+1} := k - 2 \). Notice that for fixed \( \lambda \), \( f_{\lambda}^{(\eta)}(z; u^{1/N}) \) is a smooth function on \( z_i \), with \( i = 1, 2, \cdots, \lambda \), for any \( u \in (0, 1) \).
With this notation, equation (45) is written as

\[
\sum_{\rho \in \mathbb{Z}_2^{k-3}} H_{\eta}^{(k,N)}(u^{1/N}, \rho) P^{(k,N)}(u^{1/N}, \rho) = \sum_{i_1 < i_2 < \ldots < i_\lambda} e^{f^{(n)}_{\lambda}(i_1,i_2,\ldots,i_\lambda; u^{1/N})}.
\]

We are now in the position to write this sum as an iterated integral and a residual term with explicit order. We first describe the behavior of the function with respect to \( u \in (0,1) \).

**Lemma 3.15.** Let \( \lambda \geq 1 \) be fixed. The (real valued) function \( e^{f_{\lambda}(z,u^{1/N})} \), where \( z = \{z_1, z_2, \ldots, z_\lambda\} \), is uniformly bounded with respect to \( u (0 < u < 1) \) for \( 0 \leq z_1 \leq z_2 \leq \cdots \leq z_\lambda \leq k - 3 \).

**Proof.** It is enough to observe the behavior when \( u \to 0 \) and \( u \to 1 \). When \( u \to 1 \) there is a limit for \( f_{\lambda}(z,u^{1/N}) \) which is bounded for any \( 0 \leq z_1 \leq z_2 \leq \cdots \leq z_\lambda \leq k - 3 \).

When \( u = e^{-t} \) approaches 0, let us observe the major contribution in \( f_{\lambda}(z,u^{1/N}) \). Let \( 1 \leq j \leq \lambda \), then it is easy to see the leading part in \( f_{\lambda}(z,u^{1/N}) \) as \( u \to 0^+ \) of the term involving \( x,y \) and \( z_j \) is given by

\[
(1)^{\beta+\gamma-1} u^{\frac{\beta}{N}} (2\sqrt{2g})(yu^2 - yu).
\]

For \( 0 \leq \alpha < \beta \) with \( \beta - \alpha \equiv 1 \pmod{2} \) and since \( 0 \leq z_\alpha \leq z_{\alpha+1} \) and \( z_\beta \leq z_{\beta+1} \), the leading part in \( f_{\lambda}(z,u^{1/N}) \) as \( u \to 0^+ \) of the term involving \( g^2, \alpha \) and \( \beta \) is given by

\[
-2g^2 u^2 u^{-\frac{\alpha+1}{N}} u^{-\frac{\beta+\gamma+1}{N}}.
\]

Summing up, the leading part of \( f_{\lambda}(z,u^{1/N}) \) as \( u \to 0^+ \) is given by

\[
-\sum_{\lambda} \frac{u^{-\frac{\alpha+1}{N}}}{\lambda} \left( (1)^{\gamma+\alpha-1} (2\sqrt{2g})(yu^2 - yu) + 2g^2 u^2 \sum_{\beta-i \equiv 0 \pmod{2}} \lambda u^{-\frac{\beta+\gamma+1}{N}} \right) \to -\infty.
\]

It follows that \( e^{f_{\lambda}(z,u^{1/N})} \to 0 \) when \( u \to 0^+ \). \( \square \)

In order to deal with the multiple summation over the \( i_1, i_2, \ldots, i_\lambda \), we need the following simple lemma.

**Lemma 3.16.** For fixed \( \lambda \geq 1 \) and \( a \in \mathbb{Z}_{\geq 0} \) with \( a \leq N \), we have

\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_\lambda} e^{f^{(n)}_{\lambda}(i_1,i_2,\ldots,i_\lambda; u^{1/N})} = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_\lambda} e^{f^{(n)}_{\lambda}(i_1,i_2,\ldots,i_\lambda; u^{1/N})} + O(a^{\lambda-1}).
\]

**Proof.** Since \( \exp \left( f^{(n)}_{\lambda}(i_1,i_2,\ldots,i_\lambda; u^{1/N}) \right) \) is uniformly bounded for \( 0 \leq i_j \leq a \) and \( 0 \leq u \leq 1 \) (this is verified in the same way as Lemma 3.15), we see that the difference between the number of summands of the two sums is given by

\[
\left( \binom{a+\lambda}{\lambda} \right) - \left( \binom{a}{\lambda} \right) = O(a^{\lambda-1}).
\]

\( \square \)

Finally, we transform the sum into integrals using Riemann-Stieltjes integration. We start by considering the case \( \lambda = 1 \) as it constitutes the basis for the proof of the general case.

**Proposition 3.17.** Let \( a \in \mathbb{Z}_{\geq 0} \) with \( a \leq N \). We have

\[
\sum_{i=0}^{a} e^{f^{(n)}_{\lambda}(i;u^{1/N})} = \int_{0}^{a} e^{f^{(n)}_{\lambda}(z;u^{1/N})} dz + O\left( \frac{1}{N} \right).
\]
Proof. We write the sum as a Riemann-Stieltjes integral in the standard way
\[
\sum_{i=0}^{a} e^{f_1(i,u^{1/N})} = \int_0^{a} e^{f_1(z,u^{1/N})} d\Xi(z),
\]
where \(\Xi(z) = \sum_{1 \leq n < z} 1 = z - \{z\}\). By partial integration, we see that
\[
\sum_{i=1}^{a} e^{f_1(i,u^{1/N})} = \int_0^{a} e^{f_1(z,u^{1/N})} dz + \int_0^{a} \{z\} f'_1(z,u^{1/N}) e^{f_1(z,u^{1/N})} dz
\]
(47)
\[
= \int_0^{a} e^{f_1(z,u^{1/N})} + 2 \sum_{n=1}^{\infty} \int_0^{a} \cos(2\pi nz) e^{f_1(z,u^{1/N})} dz + O(1),
\]
the last equality is obtained by using the Fourier series of \(\psi(x) = x - [x] - \frac{1}{2}\), that is \(\psi(x) = -\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n\pi}\) (see also [22], equation (A26)).
Setting
\[
g(z) = \sum_{j=0}^{a-1} e^{f_1(z+j,u^{1/N})},
\]
we have
\[
\sum_{n=1}^{\infty} \int_0^{a} \cos(2\pi nz) e^{f_1(z,u^{1/N})} dz = \sum_{n=1}^{\infty} \int_0^{1} \cos(2\pi nz) g(z) dz.
\]
Now, integration by parts twice yields
\[
\int_0^{1} \cos(2\pi nz) g(z) dz = \frac{1}{4\pi^2 n^2} (g'(1) - g'(0)) - \frac{1}{4\pi^2 n^2} \int_0^{1} \cos(2\pi nz) g''(z) dz.
\]
Hence
\[
|\sum_{n=1}^{\infty} \int_0^{1} \cos(2\pi nz) g(z) dz| \leq \sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2} \left[ |g'(1) - g'(0)| + \int_0^{1} \cos(2\pi nz) g''(z) dz \right]
\]
\[
\leq \frac{1}{4\pi^2} \zeta(2) \left[ |g'(1) - g'(0)| + \int_0^{1} |g''(z)| dz \right],
\]
where \(\zeta(s)\) is the Riemann zeta function.
Next, since
\[
g'(z) = \sum_{j=0}^{a-1} f'_1(z+j,u^{1/N}) e^{f_1(z+j,u^{1/N})},
\]
we have
\[
g'(1) - g'(0) = \sum_{j=0}^{a-1} f'_1(1+j,u^{1/N}) e^{f_1(1+j,u^{1/N})} - \sum_{j=0}^{a-1} f'_1(j,u^{1/N}) e^{f_1(j,u^{1/N})}
\]
\[
= f'_1(a,u^{1/N}) e^{f_1(a,u^{1/N})} - f'_1(0,u^{1/N}) e^{f_1(0,u^{1/N})}.
\]
Noticing that the summation on \(j\) (over \(a\)) disappear and that \(\frac{d}{dz} u^{\pm \frac{s}{N}} = \pm \frac{s}{N} (\log u) u^{\pm \frac{s}{N}}\), we immediately observe that \(g'(1) - g'(0) = O(\frac{1}{N})\). Furthermore,
\[
g''(z) = \sum_{j=0}^{a-1} \{ f''_1(z+j,u^{1/N}) + (f'_1(z+j,u^{1/N}))^2 \} e^{f_1(z+j,u^{1/N})}.
\]
By Lemma 3.15 there is a positive constant \(C\) such that
\[
|g''(z)| \leq C \sum_{j=0}^{a-1} \{|f''_1(z+j,u^{1/N})| + (f'_1(z+j,u^{1/N}))^2\}.\]
Since again \( \frac{d}{dz}u^\pm \frac{1}{z} = \pm \frac{1}{N}(\log u)u^\pm \frac{1}{z} \), there are positive uniform constants \( A \) and \( B \) with respect to \( u \) such that
\[
|f''_1(z, u^{1/N})| \leq \log(u)^2 \frac{A}{N^2}, \quad |f'_1(z, u^{1/N})| \leq \log(u)^2 \frac{B}{N}.
\]
It follows that
\[
|g''(z)| \leq C(A + B^2) \log(u)^2 \frac{k}{N^2}.
\]
Therefore we have
\[
\left| \sum_{n=1}^{\infty} \int_0^a \cos(2\pi nz)e^{f_1(z, u^{1/N})}dz \right| = O\left( \frac{1}{N} \right).
\]

Lemma 3.18. For fixed \( \lambda \geq 1 \) and \( a \in \mathbb{Z}_{\geq 1} \) with \( a \leq N \), we have
\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_\lambda} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})} = \int_0^a \int_0^{z_\lambda} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz + O(a^{\lambda-1}).
\]

Proof. The proof is by induction. The case \( \lambda = 1 \) is given by Lemma 3.16 and Proposition 3.17. Suppose the result holds for some \( \lambda - 1 \geq 1 \). Then, by Lemma 3.16, the sum in the left-hand side is, up to a factor of order \( O(a^{\lambda-1}) \), given by
\[
\sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_\lambda} e^{i_\lambda(z_1, i_2, \ldots, i_\lambda; u^{1/N})} = \sum_{i_1=0}^{a} \left( \int_0^{i_1} \int_0^{z_\lambda-1} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz + R^{(i_\lambda)}_{\lambda-1}(z) \right)
\]
where the equality is obtained by applying the induction hypothesis with \( a = i_\lambda \) for each \( i_\lambda \). The residual terms are of order
\[
R^{(i_\lambda)}_{\lambda-1}(z) = O\left( a^{\lambda-2} \right) = O\left( a^{\lambda-2} \right),
\]
and thus
\[
\sum_{i_\lambda=0}^{a} R^{(i_\lambda)}_{\lambda-1}(z) = O\left( a^{\lambda-1} \right).
\]
On the other hand, we observe as in the case of \( \lambda = 1 \) that
\[
\sum_{i_\lambda=0}^{a} \int_0^{i_1} \int_0^{z_\lambda-1} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz = \int_0^a \int_0^{z_\lambda} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz
\]
\[
+ 2 \sum_{n=0}^{\infty} \int_0^a \int_0^{z_\lambda} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz \cos(2\pi nz)dz = O(a^{\lambda-1}),
\]
It remains to show that
\[
\sum_{n=0}^{\infty} \int_0^a \left[ \int_0^{z_\lambda} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz \right] \cos(2\pi nz)dz = O(a^{\lambda-1}),
\]
the proof follows in the same way as that of that of Proposition 3.17 by setting
\[
g(z_\lambda) = \sum_{j=0}^{a-1} \int_0^{z_\lambda+j} \cdots \int_0^{z_2} e^{i_\lambda(z_1, z_2, \ldots, z_\lambda; u^{1/N})}dz,
\]
and noticing, by Leibniz’s rule, that
\[
g'(1) - g'(0) = O(a^{\lambda-1}), \quad g''(z_\lambda) = O(a^{\lambda-3}).
\]
\[\square\]
4. Heat Kernel of the QRM

In this section we complete the computation of the heat kernel of the QRM. Recall from [2.4] that the expression of the heat kernel $\tilde{K}(x,y,t)$ is the sum of two limits. The first limit is given by

$$
\frac{1}{2} \lim_{N \to \infty} \left( 1 + \frac{u^{2\lambda}}{2u^{\lambda}} \right)^{N-1} \left( J_{0}^{(1,N)}(x,y,u^{\lambda},g) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + J_{1}^{(1,N)}(x,y,u^{\lambda},g) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right),
$$

while the second limit, by the results of [3] is given by

$$
\lim_{N \to \infty} \left( 1 - \frac{u^{2\lambda}}{2u^{\lambda}} \right) \sum_{k \geq 3} \left[ \sum_{\rho \in \mathbb{Z}^{3}_{2}} \left\{ \begin{array}{c} (1)^{1\mid \rho \mid +1}u^{\lambda} \\ -u^{\frac{\lambda}{N}} \end{array} \right. \frac{1}{1 + u^{\frac{\lambda}{N}}} \right] \times H_{1}^{(k,N)}(x,y,u^{1/N},\rho) P^{(k,N)}(u^{\lambda},\rho)
+ \frac{\sqrt{2g}}{1 - u^{2}} \left[ \left( 1 - u^{\frac{1}{N}} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \left( 1 + u^{2} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \right]
+ \frac{u^{\lambda}}{1 + u^{\lambda}} \left( 1 - u^{2 - \frac{1}{N}} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \right] \times H_{0}^{(k,N)}(x,y,u^{1/N},\rho) P^{(k,N)}(u^{\lambda},\rho)
$$

where

$$
\tilde{J}_{0}^{(k,N)}(u^{\lambda}) = J_{0}^{(1,N)}(u^{\lambda}) \exp \left( \frac{\lambda - y}{1 - u^{2}} \left[ x(1 - u^{2}) - uy(1 - u^{2}) \right] \right) + \frac{\sqrt{2g}}{1 - u^{2}} \left[ \left( 1 - u^{\frac{1}{N}} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \left( 1 + u^{2} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \right]
+ \frac{u^{\lambda}}{1 + u^{\lambda}} \left( 1 - u^{2 - \frac{1}{N}} \right) \left( 1 - u^{2 - \frac{1}{N}} \right) \right] \times (-1)^{1\mid \rho \mid +1} R_{1}(u^{\lambda}) + a_{0}(u^{\lambda})
$$

By expanding the geometric series in $J_{i}^{(1,N)}$ for $i = 0, 1$, it is easy to verify that the limit [48] is equal to

$$
\tilde{K}_{0}(x,y,g,t) e^{-2g \frac{1 - e^{-t}}{1 + e^{-t}}} \begin{bmatrix} \cosh \gamma & -\sinh \gamma \\ -\sinh \gamma & \cosh \gamma \end{bmatrix} \left( \sqrt{2g(x+y)} \frac{1 - e^{-t}}{1 + e^{-t}} \right)
$$

Next, we turn our attention to the limit [49]. First, we notice that the matrix factor appearing in the sums is fixed for all $\rho$ with $\mid \rho \mid \equiv i \pmod{2}$, with $i = 0, 1$. Thus, by partitioning the sum appearing in [49] according to the norm $\lambda$ of the vectors $\rho$ and omitting the matrix factor for now, we obtain a sum of the type

$$
\lim_{N \to \infty} \left( 1 - \frac{u^{2\lambda}}{2u^{\lambda}} \right) \sum_{\lambda \equiv i \pmod{2}} \sum_{k = \lambda + 3}^{N} \tilde{J}_{0}^{(k,N)}(u^{\lambda}) \sum_{\rho \in \mathbb{Z}^{3}_{2}} \left( 1 - \frac{u^{2\lambda}}{2u^{\lambda}} \right) H_{1-\eta}^{(k,N)}(u^{1/N},\rho) P^{(k,N)}(u^{\lambda},\rho)
$$
with \(i, \eta \in \{0, 1\}\). Notice that if \(\lambda > N\), we have

\[
\sum_{\rho \in \mathbb{Z}_2} \left( \frac{1 - u^{\frac{\lambda}{N}}}{1 + u^{\frac{\lambda}{N}}} \right)^\lambda H^{(k,N)}_{1-\eta}(u^{1/N}, \rho) P^{(k,N)}(u^{\frac{k}{N}}, \rho) = 0,
\]

whence, (\ref{eq:sphere}) is equal to

\[
\lim_{N \to \infty} \left( \frac{1 - u^{\frac{\lambda}{N}}}{2u^{\frac{\lambda}{N}}} \right)^\infty \sum_{\lambda \equiv 1 \text{ (mod } 2)}^N \sum_{k=\lambda+3} H^{(k,N)}_{1-\eta}(u^{\frac{k}{N}}) \left( \frac{1 - u^{\frac{\lambda}{N}}}{1 + u^{\frac{\lambda}{N}}} \right)^\lambda \left\{ \sum_{\rho \in \mathbb{Z}_2} H^{(k,N)}_{1-\eta}(u^{1/N}, \rho) P^{(k,N)}(u^{\frac{k}{N}}, \rho) \right\},
\]

and since the \(H^{(k,N)}_{1-\eta}(u^{1/N}, \rho) P^{(k,N)}(u^{\frac{k}{N}}, \rho)\) is uniformly bounded (cf. the discussion at the beginning of (\ref{eq:4.32})), the dominated convergence theorem shows that

\[
\sum_{\lambda \equiv 1 \text{ (mod } 2)}^\infty \lim_{N \to \infty} \left( \frac{1 - u^{\frac{\lambda}{N}}}{2u^{\frac{\lambda}{N}}} \right)^\infty \sum_{k=\lambda+3} H^{(k,N)}_{1-\eta}(u^{\frac{k}{N}}) \left( \frac{1 - u^{\frac{\lambda}{N}}}{1 + u^{\frac{\lambda}{N}}} \right)^\lambda \left\{ \sum_{\rho \in \mathbb{Z}_2} H^{(k,N)}_{1-\eta}(u^{1/N}, \rho) P^{(k,N)}(u^{\frac{k}{N}}, \rho) \right\}.
\]

Thus, the limit (\ref{eq:lemma41}) may be computed termwise for each value of \(\lambda \geq 0\). The innermost sum in (\ref{eq:lemma41}) is computed as an iterated integral by the results of (\ref{eq:3.43}) and the next lemma gives the explicit computation of the factor \(J^{(k,N)}_\mu(x, y, u^{\frac{k}{N}}, g)\).

**Lemma 4.1.** For \(\mu = 0, 1\), we have

\[
J^{(k,N)}_\mu(x, y, u^{\frac{k}{N}}, g) = \exp \left( \left( -1 \right)^\mu \frac{2\sqrt{2}g}{1 - e^{-2t}} \left( xe^{-\frac{t}{N}} (1 + e^{-2t + \frac{2k}{N}}) - ye^{-\frac{t}{N}} (1 + e^{\frac{2k}{N}}) - \sqrt{2g} \frac{1 + e^{-t}}{1 - e^{-t}} (x - y) \right) \right) \times \exp \left( -4g^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + 2g^2 \frac{e^{-\frac{tk}{N}} (1 + e^{-\frac{tk}{N}}) (1 - e^{-2t + \frac{2k}{N}})}{1 - e^{-t}} + \frac{g^2 (1 - e^{-\frac{2k}{N}}) (1 - e^{-2t + \frac{2k}{N}})}{1 - e^{-2t}} + O \left( \frac{1}{N} \right) \right).
\]

**Proof.** Direct evaluation of the geometric series using the identity

\[
(1 - u^{\frac{k}{N}})(N - k - 1) = \left[ \frac{t}{N} + O \left( \frac{1}{N^2} \right) \right] (N - k - 1) = t(1 - k) + O \left( \frac{1}{N} \right) \quad (N \to \infty).
\]

gives

\[
J^{(k,N)}_\mu(x, y, u^{\frac{k}{N}}, g) = \exp \left( \left( -1 \right)^\mu \frac{2\sqrt{2}g}{1 - u^2} \left( xu^{\frac{k}{N}} (1 - u^{1-\frac{k}{N}}) + y (1 - u^{1+\frac{k}{N}}) \right) \right) \times \exp \left( -g^2 \frac{tk}{N} - 2g^2 \frac{(1 - u^{1-\frac{k}{N}})(1 - u^{1+\frac{k}{N}})}{1 - u^2} + \frac{g^2 (1 - u^{2-\frac{k}{N}})(1 - u^{2+\frac{k}{N}})}{2(1 - u^2)} + O \left( \frac{1}{N} \right) \right).
\]

Similarly, we see that the remaining factor of \(J^{(k,N)}_\mu\) is

\[
\exp \left( \left( -1 \right)^\mu \frac{2\sqrt{2}g}{1 - u^2} \left[ xu^{\frac{k}{N}} (1 - u^{2-\frac{k}{N}}) - uy (1 - u^{2+\frac{k}{N}}) \right] \right) \times \exp \left( g^2 \frac{tk}{N} - 2g^2 \frac{(1 - u^{2-\frac{k}{N}})(1 - u^{2+\frac{k}{N}})}{1 - u^2} + \frac{g^2 (1 - u^{3-\frac{k}{N}})(1 - u^{3+\frac{k}{N}})}{2(1 - u^2)} + O \left( \frac{1}{N} \right) \right).
\]

The result then follows by joining the two expression and setting \(u = e^{-t} \). \(\square\)

Next, we define some auxiliary functions. These functions correspond to the expressions appearing inside the exponentials in \(J^{(k,N)}_\mu\) (see Lemma 1.1) and in the function \(f^{(\eta)}_{\lambda}\) (defined in \(\ref{eq:3.43} \)).
Definition 4.1. Let $\lambda \geq 0$, the functions $\phi(s, t)$ and $\alpha_\lambda(x, y, t)$ are given by

$$
\phi(s, t) := -4g^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + 2g^2 \frac{e^{-st} (1 + e^{(2s-1)t})}{1 - e^{-t}} + g^2 \frac{(1 - e^{-2st})(1 - e^{2t(s-1)})}{1 - e^{-2t}},
$$

$$
\alpha_\lambda(x, y, t) := \frac{2\sqrt{2g} e^{-t}}{1 - e^{-2t}} (x(e^t + e^{-t}) - 2y) \left( \frac{1 - (-1)^\lambda}{2} \right) - \sqrt{2g} (x - y) \frac{1 + e^{-t}}{1 - e^{-t}},
$$

for $\mu_\lambda = (\mu_1, \mu_2, \ldots, \mu_\lambda) \in \mathbb{R}^\lambda$ (where $\mu_0 := 0 \in \{0\} \in \mathbb{R}^0$), we define

$$
\phi_\lambda(x, y, \mu_\lambda, t) := \frac{2\sqrt{2g} e^{-t}}{1 - e^{-2t}} (-1)^\lambda \sum_{\gamma=0}^\lambda (-1)^\gamma \left[ x(e^{t(1-\mu_\gamma)} + e^{t(\mu_\gamma - 1)}) - y(e^{-t\mu_\gamma} + e^{t\mu_\gamma}) \right]
$$

$$
\xi_\lambda(\mu_\lambda, t) := -\frac{2g^2 e^{-t}}{1 - e^{-2t}} \sum_{\gamma=0}^\lambda \sum_{\begin{subarray}{c}0 \leq \alpha < \beta \leq \lambda - 1 \\ \beta - \alpha \equiv 1 \pmod{2}\end{subarray}} (e^{t(1-\mu_{\beta + 1})} + e^{t(\mu_{\beta + 1} - 1)}) - (e^{t(1-\mu_{\beta})} + e^{t(\mu_{\beta} - 1)})
$$

\times \left( e^{t\mu_\alpha} + e^{-t\mu_\alpha} - (e^{t\mu_{\alpha + 1}} + e^{-t\mu_{\alpha + 1}}) \right),
$$

where we use the convention $\mu_0 = 0$ whenever it appears in the formulas above.

With these preparations, we proceed to the computation of the limit (49). We consider the cases $\lambda = 0$ and $\lambda \geq 1$ by separate.

For $\lambda = 0$, the limit (49) is given by

$$
\frac{1}{2} \lim_{N \to \infty} \left( 1 - \frac{2\alpha}{2u_\infty} \right) \sum_{k=\lambda+3}^N J_{\eta}(k, N)(u_\infty^{1/k})
$$

since $H_{(k, N)}(u_\infty^{1/k}, \rho) = P(k, N)(u_\infty^{1/k}, \rho) = 1$ for $\rho = \emptyset_n$ with $n \geq 1$. By Lemma 4.1 the limit is the Riemann sum corresponding to the integral

$$
\frac{t\Delta}{2} \int_0^1 e^{(-1)^\eta(\alpha_1(x, y, t) + \phi_1(x, y, \mu_1, t)) + \phi(\mu_1, t)} d\mu_1.
$$

Notice that since $\xi_1(\mu_1, t) = 0$, we can write

$$
\frac{t\Delta}{2} \int_0^1 e^{(-1)^\eta(\alpha_1(x, y, t) + \phi_1(x, y, \mu_1, t)) + \phi(\mu_1, t) + \xi_1(\mu_1, t)} d\mu_1.
$$

Next, we consider the case $\lambda \geq 1$. In this case, since $H_{\eta}(u_\infty^{1/k}, \rho), P(k, N)(u_\infty^{1/k}, \rho)$ are non-vanishing, multiple iterated integrals appear in the computation.

To simplify the notation we set $h_\lambda(x, y, t) = \frac{2\sqrt{2g} e^{-t}}{1 - e^{-2t}} (x(e^t + e^{-t}) - 2y) \left( \frac{1 - (-1)^\lambda}{2} \right)$. Then, by Lemma 3.18 the limit (49) is given by

$$
\frac{1}{2} \lim_{N \to \infty} \left( \frac{1 - \frac{2\alpha}{2u_\infty}}{2u_\infty^{1/k}} \right) \sum_{k=\lambda+3}^N J_{\eta}(k, N)(u_\infty^{1/k}) \left( \frac{1 - u_\infty^{1/k}}{1 + u_\infty^{1/k}} \right)^\lambda \sum_{i_1 < \cdots < i_\lambda} e^{(-1)^\eta(i_1, \ldots, i_\lambda, u_\infty^{1/k})}
$$

$$
\times \int_0^{k-3} \int_0^{\mu_\lambda} \cdots \int_0^{\mu_2} e^{(-1)^\eta(\phi_{\lambda+1} + \xi_{\lambda+1}(\mu, t))} d\mu_\lambda
$$
with \( \nu = (\frac{1}{\lambda} \mu_1, \frac{1}{\lambda} \mu_2, \ldots, \frac{1}{\lambda} \mu_\lambda, \frac{1}{\lambda} \) and \( d\mu_\lambda = d\mu_1 d\mu_2 \cdots d\mu_\lambda \). The change of variable \( \mu_i \mapsto (k-3)\mu_i \) for \( i \in \{1, 2, \ldots, \lambda\} \) yields

\[
\frac{1}{2} e^{-\nu^T H_{\lambda+1}(x,y,t)} \lim_{N \to \infty} \left( \frac{1 - e^{-t \frac{2\pi i}{N}}}{2 e^{-t \frac{2\pi i}{N}}} \right)^N \sum_{k=\lambda+3}^N \mathcal{F}_{k,N}^{(k, N)} \left( e^{-\nu^T} \right) \left( \frac{1 - e^{-t \frac{2\pi i}{N}}}{1 + e^{-t \frac{2\pi i}{N}}} \right)^k \int_0^\lambda \cdots \int_0^\lambda e^{\nu^T (k \lambda + 1)(\nu_\lambda + 1)(\nu_\lambda)(\nu_\lambda) + \nu_\lambda(\nu_\lambda)} d\mu_\lambda,
\]

where \( \nu_2 = (\frac{1}{\lambda} \mu_1, \frac{1}{\lambda} \mu_2, \ldots, \frac{1}{\lambda} \mu_\lambda, \frac{1}{\lambda} \) and where, for clarity, we omitted terms of order \( O(k^{-1}) \) that vanish when taking the limit.

The limit is then the Riemann sum corresponding to the integral

\[
\frac{(t \Delta)^{\lambda+1}}{2} e^{\nu^T H_{\lambda+1}(x,y,t)} \int_0^\lambda \phi(\mu_\lambda) \int_0^\lambda \cdots \int_0^\lambda e^{\nu^T (k \lambda + 1)(\nu_\lambda + 1)(\nu_\lambda)(\nu_\lambda) + \nu_\lambda(\nu_\lambda)} d\mu_\lambda,
\]

where \( \nu_3 = (\mu_\lambda + 1, \mu_\lambda + 1 \mu_\lambda, \ldots, \mu_\lambda + 1 \mu_\lambda, \mu_\lambda + 1) \). Finally, the change of variable \( \mu_i \mapsto \frac{\mu_i}{\mu_{\lambda+1}} \) for \( i \in \{1, 2, \ldots, \lambda\} \), gives

\[
\frac{(t \Delta)^{\lambda+1}}{2} \int_0^\lambda \cdots \int_0^\lambda e^{\nu^T (k \lambda + 1)(\nu_\lambda + 1)(\nu_\lambda)(\nu_\lambda) + \nu_\lambda(\nu_\lambda)} d\mu_\lambda+1,
\]

with \( \mu_{\lambda+1} = (\mu_1, \mu_2, \ldots, \mu_\lambda, \mu_{\lambda+1}) \). From (55) and (56), the limit (19) is given by

\[
\sum_{\lambda=1}^\lambda (t \Delta)^\lambda \int_0^\lambda \cdots \int_0^\lambda e^{\nu^T (k \lambda + 1)(\nu_\lambda + 1)(\nu_\lambda)(\nu_\lambda) + \nu_\lambda(\nu_\lambda)} \left[ (1 - \lambda) \lambda + 1 \lambda - \lambda + 1 \lambda \right] (\nu_\lambda + 1 \lambda) + \nu_\lambda(\nu_\lambda, t)) d\mu_\lambda.
\]

Notice that

\[
\phi(0, t) = -2g^2 \frac{1 - e^{-t}}{1 + e^{-t}}, \quad \alpha_0(x, y, t) + \nu_0(x, y, \mu_0, t) = \sqrt{2g(x + y)} \frac{1 - e^{-t}}{1 + e^{-t}},
\]

therefore the expression for the limit (18) can be written in a way consistent with the notation of the one of the limit (11). We summarize the discussion above in the following result, giving the explicit expression for the heat kernel of the QRM.

**Theorem 4.2.** The heat kernel \( K_{\text{Rab}}(x,y,t) \) of the QRM is given by

\[
K_{\text{Rab}}(x,y,t) = \tilde{K}_0(x,y,g,t) \left[ e^{\phi(0,t)} \cosh \left( \begin{array}{cc} -\sinh \theta_0(x,y,\mu_0,t) \\ \cosh \theta_0(x,y,\mu_0,t) \end{array} \right) \right] + \sum_{\lambda=1}^\lambda (t \Delta)^\lambda \int_0^\lambda \cdots \int_0^\lambda e^{\nu^T (k \lambda + 1)(\nu_\lambda + 1)(\nu_\lambda)(\nu_\lambda) + \nu_\lambda(\nu_\lambda, t)) d\mu_\lambda,
\]

where for \( \lambda \geq 1 \), \( \mu_\lambda = (\mu_1, \mu_2, \ldots, \mu_\lambda) \) and \( d\mu_\lambda = d\mu_1 d\mu_2 \cdots d\mu_\lambda \) and where we set \( \mu_0 := 0 \).

Here \( \tilde{K}_0(x,y,g,t) := K_0(x,y,g,e^{-t}) \) (cf. (11) ) is given by

\[
\tilde{K}_0(x,y,g,t) = \frac{e^{-g^2 t}}{\sqrt{\pi(1 - e^{-2t})}} \exp \left( -\frac{1 + e^{-2t}}{2(1 - e^{-2t})} (x^2 + y^2) + \frac{2e^{-t} xy}{1 - e^{-2t}} \right)
\]

and \( \theta_\lambda(x,y,\mu_\lambda, t) \) is given by

\[
\theta_\lambda(x,y,\mu_\lambda, t) := \alpha_\lambda(x,y,t) + \nu_\lambda(x,y,\mu_\lambda, t)
\]

with \( \phi(s,t), \alpha_\lambda(x,y,t) \) defined in (53), and \( \xi_\lambda(x,y,\mu_\lambda, t), \theta_\lambda(x,y,\mu_\lambda, t) \) are defined in (54).
Remark 4.1. Define, for \( \lambda = 0 \), the notation
\[
\int_{\mu_1 \leq \cdot \cdot \cdot \leq \mu_\lambda \leq 1} \cdots \int f(x) d\mu_0 = f(x),
\]
for any function \( f \). We can then write the expression of the heat kernel in the form
\[
K_{\text{Rabi}}(x, y, t) = K_0(x, y, g, t) \left[ \sum_{\lambda = 0}^{\infty} (t\Delta)^\lambda \int_{0 \leq \mu_1 \leq \cdot \cdot \cdot \leq \mu_\lambda \leq 1} \cdots \int e^{\phi(\mu_\lambda, t)+\xi_\lambda(\mu_\lambda, t)} \right. \\
\left. \times \left[ (-1)^\lambda \cosh (-1)^{\lambda+1} \sinh \right] \right] (\theta(\lambda, x, y, \mu_\lambda, t)) d\mu_\lambda.
\]

In the remainder of this section, we give the explicit expression for the partition function \( Z_{\text{Rabi}}(\beta) \) of the QRM using the expression for the heat kernel of Theorem 4.2. First, by Theorem 4.2, the trace of \( K_{\text{Rabi}}(x, y, t) \) is equal to
\[
2 \tilde{K}_0(x, y, g, t) \left\{ e^{\phi(0, t)} \cosh (\theta_0(x, y, \mu_0, t)) \\
+ \sum_{\lambda = 1}^{\infty} (t\Delta)^{2\lambda} \int_{0 \leq \mu_1 \leq \cdot \cdot \cdot \leq \mu_{2\lambda} \leq 1} \cdots \int e^{\phi(\mu_{2\lambda}, t)+\xi_{2\lambda}(\mu_{2\lambda}, t)} \cosh (\theta_{2\lambda}(x, y, \mu_{2\lambda}, t)) d\mu_{2\lambda} \right\}.
\]

Furthermore, notice that
\[
\tilde{K}_0(x, y, g, t) = \frac{e^{-g^2 t}}{\sqrt{\pi(1 - e^{-2t})}} \exp \left( \frac{1 - e^{-t}}{1 + e^{-t}} x^2 \right)
\]
and, that for \( \lambda \equiv 0 \mod 2 \), we have
\[
\theta_\lambda(x, y, \mu_\lambda, t) = \frac{2\sqrt{2} gx}{1 + e^{-t}} \sum_{\gamma = 0}^{\lambda} (-1)^\gamma \left( e^{-t\mu_\gamma} - e^{t(\mu_\gamma - 1)} \right)
\]
with \( \mu_0 = 0 \). Thus, we observe that
\[
\text{tr} K_{\text{Rabi}}(x, y, t) = \frac{2e^{-g^2 t}}{\sqrt{\pi(1 - e^{-2t})}} \cdot \left\{ e^{-2g^2 t} \frac{1 - e^{-t}}{1 + e^{-t}} \cosh \left( \frac{2\sqrt{2} gx}{1 + e^{-t}} \right) \right. \\
+ \sum_{\lambda = 1}^{\infty} (t\Delta)^{2\lambda} \int_{0 \leq \mu_1 \leq \cdot \cdot \cdot \leq \mu_{2\lambda} \leq 1} \cdots \int e^{\phi(\mu_{2\lambda}, t)+\xi_{2\lambda}(\mu_{2\lambda}, t)} \cosh \left( \frac{2\sqrt{2} gx}{1 + e^{-t}} \sum_{\gamma = 0}^{2\lambda} (-1)^\gamma \left( e^{-t\mu_\gamma} - e^{t(\mu_\gamma - 1)} \right) \right) d\mu_{2\lambda} \right\}.
\]

In order to give the closed expression for the partition function of the QRM, we introduce the function
\[
\psi_\lambda^{\pm}(\mu_\lambda, t) := \frac{2g^2}{1 - e^{-2t}} \left( \sum_{\gamma = 0}^{\lambda} (-1)^\gamma \left( e^{-t\mu_\gamma} \pm e^{t(\mu_\gamma - 1)} \right) \right)^2,
\]
for \( \lambda \geq 1 \) and \( \mu_\lambda = (\mu_1, \mu_2, \cdots, \mu_\lambda) \) and where \( \mu_0 = 0 \).

Corollary 4.3. The partition function \( Z_{\text{Rabi}}(\beta) \) of the QRM is given by
\[
Z_{\text{Rabi}}(\beta) = \frac{2e^{-g^2 \beta}}{1 - e^{-\beta}} \left[ 1 + \sum_{\lambda = 1}^{\infty} (\beta\Delta)^{2\lambda} \int_{0 \leq \mu_1 \leq \cdot \cdot \cdot \leq \mu_{2\lambda} \leq 1} \cdots \int e^{\phi(\mu_{2\lambda}, \beta)+\xi_{2\lambda}(\mu_{2\lambda}, \beta)+\psi_{2\lambda}(\mu_{2\lambda}, \beta)} d\mu_{2\lambda} \right].
\]
Remark 4.2. With the same notation of Remark 4.1, since
\[ \psi_0^-(\mu_0, t) = \frac{2g^2(1 - e^{-t})}{1 + e^{-t}}, \]
the partition function of the QRM can be written as
\[ Z_{Rabi}(\beta) = \frac{2e^{-g^2\beta}}{1 - e^{-\beta}} \sum_{\lambda=0}^{\infty} (\beta \Delta)^{2\lambda} \int \ldots \int e^{\phi(\mu_2\lambda, \beta) + \xi_2(\mu_2\lambda, \beta) + \psi_2(\mu_2\lambda, \beta)} d\mu_2\lambda. \]

Proof. Recall that for \( \alpha > 0 \) and \( \gamma, \eta \in \mathbb{R} \), we have the elementary identity
\[ \int_{-\infty}^{\infty} e^{-\alpha x^2} \cosh(x \eta) dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\eta^2}{4\alpha}}. \]
In particular,
\[ \int_{-\infty}^{\infty} e^{\frac{-1-x^2}{1-e^{-x}}} \cosh \left( \frac{2\sqrt{2}gx}{1+e^{-\beta}} \right) dx = \pi^\frac{1}{2} \sqrt{1+e^{-\beta}} e^{\frac{2g^2}{1+e^{-\beta}}} \]
and, for \( \lambda \geq 1 \), we have
\[ \int_{-\infty}^{\infty} e^{\frac{-1-x^2}{1-e^{-x}}} \cosh \left( \frac{2\sqrt{2}gx}{1+e^{-\beta}} \lambda \right) \sum_{\gamma=0}^{\lambda} (-1)^\gamma \left( e^{-\beta \mu_\gamma} - e^{-\beta (\mu_\gamma - 1)} \right) dx \]
\[ = \pi^\frac{1}{2} \sqrt{1+e^{-\beta}} e^{\frac{2g^2}{1+e^{-\beta}}} \lambda \sum_{\gamma=0}^{\lambda} (-1)^\gamma \left( e^{-\beta \mu_\gamma} - e^{-\beta (\mu_\gamma - 1)} \right)^2 = \pi^\frac{1}{2} \sqrt{1+e^{-\beta}} e^{\psi_2(\mu, s, \beta)}. \]
The result then follows from
\[ Z_{Rabi}(\beta) := \int_{-\infty}^{\infty} \text{tr} K_{Rabi}(x, x, \beta) dx, \]
and the expression for \( \text{tr} K_{Rabi}(x, x, t) \) above.

Remark 4.3. In physics, the unitary operator \( e^{-itH_{Rabi}} \) (associated with the Schrödinger equation of to \( H_{Rabi} \)) is of fundamental importance. In our case, the operator can be obtained from \( e^{-\beta H} \) with \( \beta > 0 \) by meromorphic continuation to imaginary \( \beta \) (with a fixed branch for each \( \beta \in 2\pi i \mathbb{Z} \)). We leave the details for another occasion.

5. Parity decomposition of the heat kernel

As we already mentioned in the Introduction, the Hamiltonian \( H_{Rabi} \) possesses a \( \mathbb{Z}/2\mathbb{Z} \)-symmetry indicated by the existence of a parity operator \( \Pi = -\sigma_x e^{-i\pi a^1} \) satisfying \( [\Pi, H_{Rabi}] = 0 \) and with eigenvalues \( p = \pm 1 \). Consequently, the direct decomposition of the full space \( L^2(\mathbb{R}) \otimes \mathbb{C}^2 \) into the invariant subspaces (corresponding to the positive and negative parity) is
\[ L^2(\mathbb{R}) \otimes \mathbb{C}^2 = \mathcal{H}_+ \oplus \mathcal{H}_-. \]
In Appendix A we give a brief introduction to the parity decomposition in the Bargmann space. In this section we consider the equivalent \( L^2(\mathbb{R}) \) realization.

Let \( (\hat{T}\psi)(z) := \psi(-z) \) \( (\psi \in L^2(\mathbb{R})) \) be the reflection operator acting on \( L^2(\mathbb{R}) \), \( U \) be the unitary operator on \( L^2(\mathbb{R}) \otimes \mathbb{C}^2 \) by
\[ U := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \hat{T} & -\hat{T} \end{bmatrix}, \]
and \( C \) the Cayley transform
\[ C := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \]
The parity decomposition of the $H_{\text{Rabi}}$ is given by
\begin{equation}
(CU)^\dagger H_{\text{Rabi}} CU = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix},
\end{equation}
where the operators $H_{\pm}$ are given by
\begin{equation}
H_{\pm} = a^\dagger a + g(a + a^\dagger) \pm \Delta \hat{T}.
\end{equation}

Clearly, the subspaces
\begin{equation}
\mathcal{H}_+ = L^2(\mathbb{R}) \otimes \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{H}_- = L^2(\mathbb{R}) \otimes \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}
\end{equation}
are invariant subspaces of the operator $(CU)^\dagger H_{\text{Rabi}} CU$. Accordingly, we write $H_{\pm} = (CU)^\dagger H_{\text{Rabi}} CU|_{\mathcal{H}_\pm}$.

Now, we proceed to compute the heat kernel of the parity Hamiltonians $H_{\pm}$.

Recall $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Notice that
\begin{equation}
\begin{bmatrix} \cosh & -\sinh \\ -\sinh & \cosh \end{bmatrix}(t) = e^{-t\sigma_x} \quad \text{and} \quad \begin{bmatrix} -\cosh & \sinh \\ -\sinh & \cosh \end{bmatrix}(t) = -\sigma_x e^{-t\sigma_x}.
\end{equation}

For $\epsilon, \delta \in \{+, -, \}$, let us define four operators (infinite dimensional matrices) $K_{\epsilon\delta} = K_{\epsilon\delta}(x, y, t, \Delta) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by
\begin{equation}
(CU)^\dagger K_{\text{Rabi}}(x, y, t)CU = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}.
\end{equation}
It is not difficult to see that
\begin{equation}
\frac{\partial}{\partial t} \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} = -\begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix},
\end{equation}
and thus $(CU)^\dagger K_{\text{Rabi}}(x, y, t)CU$ is the heat kernel of the operator $(CU)^\dagger H_{\text{Rabi}} CU$. Similarly, from this we see that $K_{++}$ (resp. $K_{--}$) is the heat kernel of $H_+$ (resp. $H_-$. One knows from the general discussion for the G-function and constraint polynomials (see e.g. [4] and [25]) that $K_{--}(x, y, t, -\Delta) = K_{++}(x, y, t, \Delta)$. We will see this again below.

Recall that the action of the (semigroup) operator $e^{-tH_{\text{Rabi}}}$ is given by
\begin{equation}
e^{-tH_{\text{Rabi}}} \phi(x) = \int_{-\infty}^{\infty} K_{\text{Rabi}}(x, y, t)\phi(y)dy
\end{equation}
for any compactly supported smooth function $\phi \in C_0^\infty(\mathbb{R}) \otimes \mathbb{C}^2$. From this expression, we have
\begin{equation}
(CU)^\dagger e^{-tH_{\text{Rabi}}} CU((CU)^\dagger \phi)(x) = \begin{bmatrix} e^{-tH_+} & 0 \\ 0 & e^{-tH_-} \end{bmatrix}((CU)^\dagger \phi)(x) = \int_{-\infty}^{\infty} \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}(x, y, t)((CU)^\dagger \phi)(y)dy.
\end{equation}

From this expression, we observe the heat kernel is splitting along the two parities and in Theorem 5.1 we give the explicit expression of the heat kernel by taking $\phi \in H_{\pm}$ in the expression above.

For $\lambda \geq 1$, define
\begin{equation}
\Phi_{\lambda}^\pm(x, y, t) := \int_{0 \leq \mu_1 \leq \cdots \leq \mu_\lambda \leq 1} e^{\phi(\mu_1, t) + \xi(\mu_\lambda, t) \pm \theta_\lambda(x, y, \mu_\lambda, t)} d\mu_n
\end{equation}
and
\begin{equation}
\Phi_0^\pm(x, y, t) := e^{-2g^2 \tanh(\frac{t}{\lambda})} e^{\pm \sqrt{2g}(x+y) \tanh(\frac{t}{\lambda})} = e^{\phi(0, t) \pm \theta_0(x, y, \mu_0, t)}.
\end{equation}
Since $\theta_\lambda(x, y, \mu_\lambda, t)$, for $\lambda \geq 0$, is linear on $x$ and $y$, it is clear that
\begin{equation}
\Phi_{\lambda}^\mp(-x, -y, t) = \Phi_{\lambda}^\pm(x, y, t).
\end{equation}
We now observe that
\[
(CU)^\dagger K_0(x, y, g, t) = U^\dagger K_0(x, y, g, t)Ce^{-\theta_2 \lambda(x, y, \mu_{2\lambda}, t)\sigma_x} CU
\]
\[
= U^\dagger K_0(x, y, g, t)Ce^{-\theta_2 \lambda(x, y, \mu_{2\lambda}, t)} 0 e^{\theta_2 \lambda(x, y, \mu_{2\lambda}, t)} U
\]
\[
= \frac{1}{2} \begin{bmatrix} 1 & \frac{T}{T} \\ 1 & -T \end{bmatrix} K_0(x, y, g, t) \begin{bmatrix} e^{-\theta_2 \lambda(x, y, \mu_{2\lambda}, t)} & 0 \\ 0 & e^{\theta_2 \lambda(x, y, \mu_{2\lambda}, t)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix}
\]
\[
= \frac{1}{2} \begin{bmatrix} K_0 e^{-\theta_2 \lambda} + T \hat{K}_0 e^{\theta_2 \lambda} T & \hat{K}_0 e^{-\theta_2 \lambda} - T \hat{K}_0 e^{\theta_2 \lambda} T \\ \hat{K}_0 e^{-\theta_2 \lambda} + T \hat{K}_0 e^{\theta_2 \lambda} T & K_0 e^{-\theta_2 \lambda} - T \hat{K}_0 e^{\theta_2 \lambda} \end{bmatrix}(x, y).
\]

Similarly
\[
(CU)^\dagger \tilde{K}_0(x, y, g, t) = U^\dagger \tilde{K}_0(x, y, g, t)C(-\sigma_x)e^{-\theta_{2\lambda+1}(x, y, \mu_{2\lambda+1}, t)\sigma_x} CU
\]
\[
= -U^\dagger \tilde{K}_0(x, y, g, t) \begin{bmatrix} 0 & e^{\theta_{2\lambda+1}(x, y, \mu_{2\lambda+1}, t)} \\ e^{-\theta_{2\lambda+1}(x, y, \mu_{2\lambda+1}, t)} & 0 \end{bmatrix} U
\]
\[
= -\frac{1}{2} \begin{bmatrix} 1 & \frac{T}{T} \\ 1 & -T \end{bmatrix} \tilde{K}_0(x, y, g, t) \begin{bmatrix} 0 & e^{\theta_{2\lambda+1}(x, y, \mu_{2\lambda+1}, t)} \\ e^{-\theta_{2\lambda+1}(x, y, \mu_{2\lambda+1}, t)} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix}
\]
\[
= -\frac{1}{2} \begin{bmatrix} K_0 e^{\theta_{2\lambda+1} T} + \hat{T} \hat{K}_0 e^{-\theta_{2\lambda+1}} - \hat{K}_0 e^{-\theta_{2\lambda+1} T} + \hat{T} \hat{K}_0 e^{\theta_{2\lambda+1}} & \hat{K}_0 e^{\theta_{2\lambda+1} T} - \hat{K}_0 e^{-\theta_{2\lambda+1}} - \hat{T} \hat{K}_0 e^{\theta_{2\lambda+1}} \end{bmatrix}(x, y).
\]

From the discussion, we obtain the heat kernel for the parity Hamiltonians \(H_\pm\).

**Theorem 5.1.** The heat kernel \(K_\pm = K_\pm(x, y, t, \Delta)\) of \(H_\pm = H_{\text{Rabi}}|_{\mathcal{H}_\pm}\) is given by
\[
K_\pm(x, y, t, \Delta) = K_0(x, y, g, t) \sum_{\lambda=0}^{\infty} (t\Delta)^{2\lambda} \Phi_{2\lambda}(x, y, t) + \hat{K}_0(x, y, g, t) \sum_{\lambda=0}^{\infty} (t\Delta)^{2\lambda+1} \Phi^+_{2\lambda+1}(x, y, t)
\]

Moreover, \(K_{\pm}(x, y, t, \Delta) = 0\). In other words,
\[
K_{\text{Rabi}}(x, y, t, \Delta) = K_{++}(x, y, t, \Delta) \oplus K_{--}(x, y, t, \Delta).
\]

**Proof.** For \(\epsilon, \delta \in \{+, -\}\), we define operators \(k_{\epsilon\delta} = k_{\epsilon\delta}(x, y, t, \Delta) \in \text{End}_C(\mathcal{H}_\epsilon, \mathcal{H}_\delta)\) by
\[
(k_{\epsilon\delta}v_\epsilon)(x) = \int_{-\infty}^{\infty} K_{\epsilon\delta}(x, y, t, \Delta)v_\epsilon(y)dy
\]
for \(v_\epsilon \in \mathcal{H}_\epsilon\). Further, we write \(k_{\epsilon\delta}\) as
\[
(k_{\epsilon\delta}v_\epsilon)(x) = \sum_{\lambda=0}^{\infty} (t\Delta)^{\lambda} k_{\epsilon\delta}^{\lambda} v_\epsilon(x).
\]
By (68), we see that \( \tilde{\mathcal{H}} \simeq L^2(\mathbb{R}) \). First, we verify that \( K_{\pm \pi}(x, y, t, \Delta) = 0 \). Let \( v \in L^2(\mathbb{R}) \) be a function with appropriate decay at \( \pm \infty \) (e.g. \( v \) is a compactly supported function), then

\[
(k_{\lambda}^{2\lambda} v)(x) = \left( k_{\lambda}^{2\lambda} + v \right)(x) = \frac{1}{2} \int_{-\infty}^{\infty} [\tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) - \tilde{T} \tilde{K}_0(x, y) \Phi_{2\lambda}^+(x, y) \tilde{T}] v(y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda}^+(x, y) v(-y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda}^+(x, y) v(-y) dy
\]

and

\[
(k_{\lambda}^{2\lambda+1} v)(x) = - (k_{\lambda}^{2\lambda+1} v)(x) = \frac{1}{2} \int_{-\infty}^{\infty} [\tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) - \tilde{T} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^+(x, y) \tilde{T}] v(y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda+1}^+(x, y) v(-y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda+1}^+(x, y) v(-y) dy
\]

Thus, we see that \( (k_{\lambda}^{2\lambda} v)(x) = 0 \) for \( v \) with appropriate decay and \( \lambda \geq 0 \).

On the other hand, we have

\[
(k_{\lambda}^{2\lambda} v)(x) = \left( k_{\lambda}^{2\lambda} - v \right)(x) = \frac{1}{2} \int_{-\infty}^{\infty} [\tilde{K}_0(x, y) \Phi_{2\lambda}^+(x, y) + \tilde{T} \tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) \tilde{T}] v(y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) v(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda}^+(x, y) v(-y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda}^-(x, y) v(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda}^+(x, y) v(y) dy
\]

and

\[
-(k_{\lambda}^{2\lambda+1} v)(x) = \left( k_{\lambda}^{2\lambda+1} - v \right)(x) = \frac{1}{2} \int_{-\infty}^{\infty} [\tilde{K}_0(x, y) \Phi_{2\lambda+1}^+(x, y) + \tilde{T} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) \tilde{T}] v(y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda+1}^+(x, y) v(-y) dy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}^-(x, y) v(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{K}_0(-x, y) \Phi_{2\lambda+1}^+(x, y) v(y) dy
\]
Hence, we have
\[
(k_{\pm v})(x) = \sum_{\lambda=0}^{\infty} \Delta^{2\lambda} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda}(x, y)v(y)dy + \sum_{\lambda=0}^{\infty} \Delta^{2\lambda+1} \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}(x, y)v(y)dy.
\]
Thus we have the desired conclusion for $K_{i\delta}$ as a distribution, whence the result follows as functions in the standard way.

\[\Box\]

**Remark 5.1.** Note that for $v \in L^2(\mathbb{R})$, we can write
\[
\int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}(x, y)v(-y)dy = \int_{-\infty}^{\infty} \tilde{K}_0(x, y) \Phi_{2\lambda+1}(x, y)(\tilde{T}v)(y)dy.
\]

To conclude this section, we compute the partition function $Z_{Rabi}^\pm(\beta)$ of the parity Hamiltonian $H_\pm$.

**Corollary 5.2.** The partition function $Z_{Rabi}^\pm(\beta)$ for the parity Hamiltonian $H_\pm$ is given by
\[
Z_{Rabi}^\pm(\beta) = \frac{e^{-g^2\beta}}{1-e^{-\beta}} \left[ 1 + \sum_{\lambda=1}^{\infty} (\beta \Delta)^{2\lambda} \prod_{0 \leq \mu_1 \leq \cdots \leq \mu_{2\lambda} \leq 1} e^{\phi(\mu_{2\lambda}, \beta) + \xi_{2\lambda}(\mu_{2\lambda}, \beta) + \psi_{2\lambda}(\mu_{2\lambda}, \beta)} d\mu_{2\lambda} \right]
\]
\[
+ \frac{e^{-g^2\beta}}{1+e^{-\beta}} \sum_{\lambda=0}^{\infty} (\beta \Delta)^{2\lambda+1} \prod_{0 \leq \mu_1 \leq \cdots \leq \mu_{2\lambda+1} \leq 1} e^{\phi(\mu_{2\lambda+1}, \beta) + \xi_{2\lambda+1}(\mu_{2\lambda+1}, \beta) + \psi_{2\lambda+1}(\mu_{2\lambda+1}, \beta)} d\mu_{2\lambda+1}.
\]

**Proof.** The first part is computed in the same way as in the case of $Z_{Rabi}(\beta)$ (cf. Corollary 4.3) by noticing that
\[
\int_{-\infty}^{\infty} e^{-\alpha x^2} \cosh(x \eta)dx = \int_{-\infty}^{\infty} e^{-\alpha x^2 \pm \eta^2} dx = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\eta^2}{\alpha}}.
\]
For the second part, it is enough to observe that
\[
\tilde{K}_0(x, -x, g, t) = \frac{e^{-g^2t}}{\sqrt{\pi(1-e^{-2t})}} \exp \left( \frac{1 + e^{-t}}{1 - e^{-t}} x^2 \right),
\]
and, for $\lambda \equiv 1 \pmod{2}$,
\[
\theta_{\lambda}(x, -x, \mu_{\lambda}, t) = \frac{2\sqrt{2}gx}{1-e^{-t}} \sum_{\gamma=0}^{\lambda} (-1)\gamma \left( e^{-\mu_{\gamma}} + e^{t(\mu_{\gamma}-1)} \right),
\]
and then proceed as in the case of $Z_{Rabi}(\beta)$.

\[\Box\]

**Remark 5.2.** Though $Z_{Rabi}(t) = Z_{Rabi}^+(\beta) + Z_{Rabi}^-(\beta)$, we notice that $Z_{Rabi}^+(\beta) \neq Z_{Rabi}^-(\beta)$. Therefore we may define the spectral zeta functions of $Z_{Rabi}^\pm(s; \tau) = \zeta_{alt}(s; \tau)$ (see the next section). Moreover, from the parity decomposition viewpoint, as well as the study on $Z_{Rabi}(\beta)$, it is important to study the “virtual partition function”
\[
Z_{alt}(\beta) := Z_{Rabi}^-(\beta) - Z_{Rabi}^+(\beta),
\]
and the associated “zeta function”
\[
(59) \quad \zeta_{alt}(s; \tau) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} Z_{alt}(t)e^{-t\tau} dt.
\]

The analytical and number theoretical properties of $\zeta_{alt}(s; \tau)$, in particular the holomorphic extension to the complex plane (with no poles, that is, similarly to Dirichlet $L$-functions) and special values are of interest.
6. Spectral zeta function for the QRM

In this section, as an application of the computation of the closed expression of the heat kernel and partition function of the QRM we study basic properties of the spectral zeta function $\zeta_{\text{Rabi}}(s)$ for the QRM.

The meromorphic continuation of $\zeta_{\text{Rabi}}(s)$ to the complex plane was first obtained in [49]. In §5.1 we give another proof in a more direct way using the Mellin transform expression of $\zeta_{\text{QRM}}(s)$ by the partition function $Z_{\text{Rabi}}(t)$ like in one of Riemann's original proofs for the zeta function. In addition, we obtain the analytic continuation of the parity zeta function $\zeta^\pm_{\text{QRM}}(s; \tau)$ and the corresponding Weyl law for the distribution of the eigenvalues of $H_{\pm}$ using the same method. In §6.2 we see that the expression obtained for the analytic continuation of $\zeta_{\text{Rabi}}(s)$ allows to describe the special values at the negative integers (that is, $s = -n$) in terms of the so-called Rabi-Bernoulli polynomials, already considered in [49]. Using the explicit expressions for the partition function we give a proof of the rationality of the coefficients of the Rabi-Bernoulli polynomials in a similar manner than the proof given in [49]. Similar results hold for the spectral zeta functions of the parity Hamiltonians. Finally, since the parity zeta function $\zeta^\pm_{\text{QRM}}(s; \tau)$ is holomorphic in a neighborhood of the origin we can consider the associated zeta regularized product, that is, the spectral determinant of the Hamiltonian $H_{\pm}$. In §6.3 we extend a result of [25] and show that the spectral determinant is, up to a non-vanishing constant, equal to the completed G-function $G^\pm(\tau; g, \Delta)$ of $H_{\pm}$.

We recall the definition of the spectral zeta functions for §6.2 we see that the expression obtained for the analytic continuation of $\zeta_{\text{QRM}}(s)$ and $\zeta^\pm_{\text{QRM}}(s; \tau)$ are defined by

$$
\zeta_{\text{QRM}}(s; \tau) := \sum_{j=1}^{\infty} (\lambda_j + \tau)^{-s}, \quad \zeta^\pm_{\text{QRM}}(s; \tau) := \sum_{j=1}^{\infty} (\lambda_j^\pm + \tau)^{-s}.
$$

In both cases, it is seen (cf. [49]) that the zeta functions above are absolutely convergent for $\Re(s) > 1$ for $\tau \in \mathbb{C} - \text{Spec}(H_{\text{Rabi}})$. Furthermore, if $\tau > \Delta + g^2$ we have $\lambda_j + \tau > 0$ (resp. $\lambda_j^\pm + \tau > 0$) for any $j \in \mathbb{Z}_{\geq 1}$ and we have the following Mellin transform representation of the spectral zeta functions.

$$
\zeta_{\text{QRM}}(s; \tau) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} Z_{\text{Rabi}}(t) e^{-\tau t} dt,
$$

$$
\zeta^\pm_{\text{QRM}}(s; \tau) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} Z^\pm_{\text{Rabi}}(t) e^{-\tau t} dt.
$$

6.1. Meromorphic continuation. We define the function $\Omega(t) = \Omega(t; \Delta, g)$ implicitly by the equation $Z_{\text{Rabi}}(t) = \frac{\Omega(t)}{1 - e^{t\tau}}$. Concretely, $\Omega(t)$ is given by

$$
\Omega(t) := 2 e^{-g^2 t} \left[ 1 + \sum_{\lambda=1}^{\infty} (t\Delta)^{2\lambda} \int \cdots \int e^{\varphi(\mu_{2\lambda}, t) + \psi_{2\lambda}(\mu_{2\lambda}, t)} d\mu_{2\lambda} \right].
$$

Then, the standard argument for the Riemann zeta function (e.g. [60]) allows us to show the analytic continuation of the spectral zeta function $\zeta_{\text{QRM}}(s; \tau)$. As preparation, we study the holomorphicity of the function $\Omega(t)$.

**Lemma 6.1.** Suppose $\lambda \in \mathbb{Z}_{\geq 1}$ and let

$$
\mathcal{D}^\ast = \{ z \in \mathbb{C} \mid \Re(z) \geq 0, z \neq 0 \} \cup \{ z \in \mathbb{C} \mid |z| \leq 1, \Re(z) < 0 \}
$$

Then, for $t \in \mathcal{D}^\ast$ there is a real valued function $C(t) \geq 0$, bounded in compact subsets of $\mathcal{D}^\ast$, such that

$$
|\xi_{\lambda}(\mu_{\lambda}, t) + \psi_{\lambda}^\pm(\mu_{\lambda}, t)| \leq \left| \frac{2g^2}{1 - e^{-2t}} \right| C(t) \lambda
$$

uniformly for $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{\lambda} \leq 1$. 
Proof. Fix $\lambda \geq 1$ with $\lambda \equiv 1 \pmod{2}$ and $t = a + bi \in \mathbb{C}^*$. Then, immediately we see that

\begin{equation}
S(t) = \sum_{\gamma=0}^{\lambda} (-1)^\gamma \left( e^{-t^{(\mu_\gamma)}_+} \pm e^{t^{(\mu_\gamma)-1}} \right) = -t \sum_{\gamma=0}^{(\lambda-1)/2} \left( \int_{\mu_\gamma}^{\mu_{2\gamma+1}} e^{-tx} \, dx \pm e^{-t \int_{\mu_\gamma}^{\mu_{2\gamma+1}} e^{tx} \, dx} \right).
\end{equation}

Therefore,

\[ |S(t)| \leq |t| \sum_{\gamma=0}^{(\lambda-1)/2} \left( \int_{\mu_\gamma}^{\mu_{2\gamma+1}} e^{-a x} \, dx + e^{-a} \int_{\mu_\gamma}^{\mu_{2\gamma+1}} e^{a x} \, dx \right) \leq |t| \left( \int_0^1 e^{-ax} \, dx + e^{-a} \int_0^1 e^{ax} \, dx \right), \]

since $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_\lambda \leq 1$. Next, if $a \neq 0$, we have

\[ |S(t)| \leq \left| \frac{2t}{a} \right| |1 - e^{-a}|, \]

and if $a = 0$, we see that

\[ |S(t)| \leq 2|t|. \]

If $\lambda \equiv 0 \pmod{2}$, we apply the estimate above to the first $\lambda$ terms of the sum resulting in

\[ |S(t)| \leq \left| \frac{2t}{a} \right| |1 - e^{-a}| + |e^{-t^{(\mu_\lambda)}_+} + e^{t^{(\mu_\lambda)-1}}| \leq \frac{2|t|}{a} |1 - e^{-a}| + e + 1, \]

uniformly for $1 \leq a$. The discussion yields the estimate

\[ |\psi_\lambda^+(\mu_\lambda, t)| \leq \left| \frac{2g^2}{1 - e^{-2t}} \right| |S(t)|^2 \leq \left| \frac{2g^2}{1 - e^{-2t}} \right| C_1(t), \]

for $t \in D^*$ and with $C_1(t)$ defined by the estimate above.

Next, we fix $0 \leq n < \lambda - 1$ and consider the sum

\[ S_n(t) = \sum_{\substack{n < \beta \leq \lambda - 1 \mod{2} \beta - n \equiv 1 \mod{2}}} \left( (e^{-t^{(\mu_{\beta+1})}_+} + e^{t^{(\mu_{\beta+1})}-2t}) - (e^{-t^{(\mu_{\beta})}_+} + e^{t^{(\mu_{\beta})}-2t}) \right) \left( (e^{t^{(\mu_n)}_+} + e^{t^{(\mu_n)-1}}) - (e^{t^{(\mu_{n+1})}_+} + e^{t^{(\mu_{n+1})}-1}) \right) \sum_{\substack{n < \beta \leq \lambda - 1 \mod{2} \beta - n \equiv 1 \mod{2}}} \left( (e^{-t^{(\mu_{\beta+1})}_+} + e^{t^{(\mu_{\beta+1})}-2t}) - (e^{-t^{(\mu_{\beta})}_+} + e^{t^{(\mu_{\beta})}-2t}) \right). \]

Transforming the sums into definite integrals as in the case above we see that $S_n(t)$ is equal to

\[ -t^2 \left( \int_{\mu_n}^{\mu_{n+1}} (e^{-tx} + e^{tx}) \, dx \right) \sum_{\substack{n < \beta \leq \lambda - 1 \mod{2} \beta - n \equiv 1 \mod{2}}} \left( \int_{\mu_\beta}^{\mu_{\beta+1}} e^{-tx} \, dx + e^{-2t} \int_{\mu_\beta}^{\mu_{\beta+1}} e^{tx} \, dx \right). \]

It follows that

\[ |S_n(t)| \leq |t|^2 \left( \int_0^1 (e^{-ax} + e^{ax}) \, dx \right) \left( \int_0^1 e^{-ax} \, dx + e^{-2a} \int_0^1 e^{ax} \, dx \right) \leq \left| \frac{t}{a} \right|^2 \left( e^a - e^{-a} \right) (1 - e^{-2a}), \]

for $-1 < a$ and where with a limit interpretation for $a = 0$. It follows that

\[ |\xi_\lambda(\mu_\lambda, t)| \leq \left| \frac{2g^2}{1 - e^{-2t}} \right| \left( \sum_{n=0}^{\lambda-2} S_n(t) \right) \leq \left| \frac{2g^2}{1 - e^{-2t}} \right| C_2(t) \lambda, \]

where $C_2(t)$ is defined by the estimates above. The result then follows by taking $C(t) = \max(C_1(t), C_2(t))$ for $t \in D^*$. \qed

**Proposition 6.2.** The series defining the functions $\Omega(t)$ is uniformly convergent in compacts in the complex domain $D$ consisting a union of a half plane $\mathbb{R}^+ > 0$ and a disc centered at origin with radius $r = 1$. In particular, $\Omega(t)$ is a holomorphic function in the region $D$. 

\[ \square \]
Proof. Let $D^*$ be the region of Lemma 6.1 and $K$ a compact region contained in $D^*$. By Lemma 6.1 we see that there is a constant $C$ such that
\[ |\xi(t, \mu, t) + \psi(t, \mu, t)| \leq C, \]
uniformly in $K$. Then, for $t \in K$ we have
\[
|\Omega(t)| \leq c_1 \left| 1 + \sum_{\lambda=1}^{\infty} (t \Delta)^{2\lambda} \int_{0 \leq \mu_1 \leq \cdots \leq \mu_2 \leq 1} e^{\phi(t\mu_1, t) + \xi(t, \mu_1, t) + \psi(t, \mu_1, t)} d\mu_2 \right|
\]
\[
\leq c_1 \left( 1 + \sum_{\lambda=1}^{\infty} (t \Delta)^{2\lambda} e^{c_2 \lambda^3} \int_{0 \leq \mu_1 \leq \cdots \leq \mu_2 \leq 1} d\mu_2 \right)
\]
\[
\leq c_4 \left( 1 + \sum_{\lambda=0}^{\infty} (\Delta |c_5|)^{\lambda} e^{c_2 \lambda^3} \right)
\]
uniformly in the compact region $K$ for appropriate constants $c_4$. The result for the region $D^*$ follows from Weierstrass’ convergence theorem.

Next, we verify the behaviour of $\Omega(t)$ at the apparent singularity at $t = 0$. It is immediately to verify that
\[
\lim_{t \to 0} \phi(t, \mu, t) = 0, \quad \lim_{t \to 0} \xi(t, \mu, t) + \psi(t, \mu, t) = 0,
\]
for $\lambda \geq 1$ and uniformly for $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_\lambda \leq \mu \leq 1$. Thus we see that $\Omega(0) = 2$ and thus the result follows from Riemann continuation theorem.

We now proceed to prove the analytic continuation of $\zeta_{QRM}(s; \tau)$.

**Theorem 6.3.** We have
\[
(62) \quad \zeta_{QRM}(s; \tau) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{0+} \frac{(-w)^{s-1} \Omega(w)e^{-\tau w}}{1 - e^{-w}} dw.
\]
Here the contour integral is given by the path which starts at $\infty$ on the real axis, encircles the origin (with a radius smaller than $2\pi$) in the positive direction and returns to the starting point and it is assumed $|\arg(-w)| \leq \pi$. This gives a meromorphic continuation of $\zeta_{QRM}(s; \tau)$ to the whole plane where the only singularity is a simple pole with residue 2 at $s = 1$.

**Proof.** First, notice that $\lim_{w \to \infty} \Omega(t)e^{-\tau t} = 0$ for $\tau > g^2 + \Delta$. (this fact is known from [49] but it may be proved directly by the integral expression.) Therefore we see that $\zeta_{QRM}(s; \tau)$ is analytic when $\Re(s) > 1$. Now, suppose $\Re(s) \geq 1 + \delta$ for $\delta > 0$. Then it is legitimate to change the contour of the integral to get
\[
\int_{\infty}^{0+} \frac{(-w)^{s-1} \Omega(w)e^{-\tau w}}{1 - e^{-w}} dw = \left\{ e^{\pi(s-1)i} - e^{-\pi(s-1)i} \right\} \int_{0}^{\infty} \frac{\rho^{s-1} \Omega(\rho)e^{-\rho t}}{1 - e^{-\rho}} d\rho.
\]
Hence the formula (62) follows. Since $\Omega(w)$ is holomorphic everywhere in the path, the integral is a (single-valued) analytic function of $s \in \mathbb{C}$. The expression (62) shows that the only possible singularities of $\zeta_{QRM}(s; \tau)$ are at the singularities of $\Gamma(1-s)$, i.e. at the positive integer points. Since $\zeta_{QRM}(s; \tau)$ is analytic when $\Re(s) > 1$, only singularity of $\zeta_{QRM}(s; \tau)$ is at the point $s = 1$. Putting $s = 1$ in the integral (62), we obtain
\[
\frac{1}{2\pi i} \int_{\infty}^{0+} \frac{\Omega(w)e^{-\tau w}}{1 - e^{-w}} dw,
\]
which is the residue at $w = 0$ of the integrand, and this residue is $\Omega(0) = 2$. It follows that
\[
\lim_{s \to 1} \frac{\zeta_{QRM}(s; \tau)}{\Gamma(1-s)} = -2.
\]
Since $\Gamma(1-s)$ has a single pole at $s = 1$ with residue $-1$, we observe that the only singularity of $\zeta_{\text{QRM}}(s; \tau)$ is a simple pole with residue 2 at $s = 1$. This completes the proof of the theorem. $\square$

Since Lemma 6.1 applies for $\psi^+_{\lambda}$, from the proof of Proposition 6.2 it is immediate to see that $\Omega_{\text{odd}}(t)$ defined by the equation

$$Z^\pm_{\text{Rabi}}(t) = \frac{1}{2} \left( \frac{\Omega(t)}{1 - e^{-2it}} + \frac{\Omega_{\text{odd}}(t)}{1 + e^{-2it}} \right).$$

is holomorphic in the union of the right half plane and a disk of radius one centered at the origin.

Similarly, the proof of analytic continuation extends to the spectral function for the parity Hamiltonians $H_\pm$. 

**Corollary 6.4.** With the notation of Theorem 6.3 we have

$$\zeta^\pm_{\text{QRM}}(s; \tau) = -\frac{\Gamma(1-s)}{4\pi i} \int_{\infty}^{(0+)} \left( \frac{(-w)^{s-1}\Omega(w)e^{-\tau w}}{1 - e^{-w}} + \frac{(-w)^{s-1}\Omega_{\text{odd}}(w)e^{-\tau w}}{1 + e^{-w}} \right) \, dw.$$ 

This gives a meromorphic continuation of $\zeta^\pm_{\text{QRM}}(s; \tau)$ to the whole plane where the only singularity is a simple pole with residue 1 at $s = 1$. $\square$

Introducing the spectral counting functions

$$N_{\text{Rabi}}(T) = \#\{\lambda \in \text{Spec}(H_{\text{Rabi}}) \mid \lambda \leq T\},$$

$$N_{\pm}(T) = \#\{\lambda \in \text{Spec}(H_{\pm}) \mid \lambda \leq T\},$$

for $T > 0$, we obtain the Weyl law for the distribution of the eigenvalues of the parity Hamiltonians $H_{\pm}$ in the usual way (cf. [20, 49]).

**Corollary 6.5.** We have

$$N_{\pm}(T) \sim \frac{1}{2} N_{\text{Rabi}}(T) \sim T,$$

as $T \to \infty$. $\square$

By Theorem 6.3 it is not difficult to obtain the following identity by differentiating $n$-times with respect to $\tau$ under the integral expression (62).

**Corollary 6.6.** We have

$$\frac{\partial^n}{\partial \tau^n} \zeta_{\text{QRM}}(s; \tau) = (-1)^n(s) \zeta_{\text{QRM}}(s+n; \tau),$$

where $(a)_n = a(a+1) \cdots (a+n-1)$ is the Pochhammer symbol. The same relation holds for $\zeta^\pm_{\text{QRM}}(s; \tau)$.

**Remark 6.1.** For the Hurwitz zeta function $\zeta(s; a)$, the identity $\frac{\partial^n}{\partial \tau^n} \zeta(s; \tau) = -(s)_n \zeta(s+n; \tau)$ also follows immediately from its very definition (series expression).

**Remark 6.2.** For $n \in \mathbb{Z}_{\geq 0}$ we define the $(n$-th) **hierarchical zeta function** $\zeta_{\text{QRM}}^{(n)}(s; g^2, \tau)$ as the coefficient of $\Delta^n$ in $\zeta^+_{\text{QRM}}(s; g^2, \tau)$ (or $\zeta_{\text{QRM}}(s; g^2, \tau)$ or $\zeta_{\text{alt}}(s; g^2, \tau)$ in [59]). Then, it is easy to see that $\zeta_{\text{QRM}}^{(0)}(s; g^2, \tau)$ is the Hurwitz function $\zeta(s; \tau - g^2)$. The case $n = 1$ is already non-trivial. The coefficient of $\Delta$ in $Z_-(t)$ is given by

$$\frac{e^{-g^2 t} t}{1 + e^{-t}} \int_0^1 e^{\phi(v, t) + \psi^+_1(v, \beta)} \, dv$$

$$= \frac{e^{-g^2 t} t}{1 + e^{-t}} \int_0^1 \exp \left( \frac{2g^2 e^{-t}}{1 - e^{-2t}} H(v, t) \right) \, dv.$$ 

$$H(v, t) = (4 - 2 \cosh(v - t) - 2 \cosh(v) + \cosh(2v - t) - \cosh(t)).$$
Thus, the 1st hierarchical zeta function is given by
\[ \zeta^{(1)}_{\text{QRM}}(s; g^2, \tau) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t(\tau+g^2)} \int_0^t \exp \left( \frac{2g^2 e^{-t}}{1 - e^{-2g} H(v, t)} \right) dvdt. \]
We note here that \( H(v, t) \) is invariant under the transformation \( v \to t - v \). It is easy to see that the integral with respect to \( v \) is expressed as the incomplete gamma function. In other words, each \( \zeta^{(n)}_{\text{QRM}}(s; g^2, \tau) \) is regarded as a generalization of the Riemann function and could be interesting to study special values at \( s = n \in \mathbb{Z} \). We might also expect a finer number theoretic property on \( \zeta^{(n)}_{\text{QRM}}(s; g^2, \tau) \).

To conclude the remark, we point out that the hierarchical zeta function are defined in terms of the \( \zeta^+_\text{QRM}(s; \tau) \) in order to have non-trivial hierarchical zeta functions for all \( n \geq 0 \) (with no alternating sign). If we use \( \zeta_{\text{QRM}}(s; \tau) \) for the definition, we see that the \( n \)-th hierarchical zeta functions is just the constant function 0 when \( n \) is an odd integer.

### 6.2. Special values at negative integers

In this subsection we describe the special values of the spectral zeta function \( \zeta_{\text{QRM}}(s; \tau) \) at the negative integers using the Mellin transform expression \( \|0\| \).

First, observe that in the special case \( s = n \in \mathbb{Z} \), the quotient \( \frac{(-w)^{n-1} \Omega(w)e^{-rw}}{1 - e^{-w}} \) is a single valued function of \( w \). Consequently, by the Cauchy integral formula, we see that \( \int_0^{0+} \frac{(-w)^{n-1} \Omega(w)e^{-rw}}{1 - e^{-w}} dw \) is the residue of the integrand at \( w = 0 \), that is, it is the coefficient of \( w^{-n} \) in \( \frac{(-w)^{n-1} \Omega(w)e^{-rw}}{1 - e^{-w}} \).

We now define the \( k \)-th Rabi-Bernoulli polynomials \( (RB)_k(\tau, g^2, \Delta) \) (according to the naming in \( \|49\| \), see Remark \( \|0.3\| \) below). Notice that when \( \Delta = 0 \), the \( k \)-th Rabi-Bernoulli polynomial is equal to the \( g^2 \)-shift \( B_k(\tau - g^2) \) of the \( k \)-th Bernoulli polynomial \( B_k(\tau) \).

**Definition 6.1.** The \( k \)-th Rabi-Bernoulli polynomial \( (RB)_k(\tau, g^2, \Delta) \in \mathbb{R}[\tau, g^2, \Delta] \) is defined through the equation
\[ (63) \quad \frac{w\Omega(w)e^{-rw}}{1 - e^{-w}} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k (RB)_k(\tau, g^2, \Delta)}{k!} w^k. \]

Similarly, the \( k \)-th positive (resp. negative) parity Rabi-Bernoulli polynomial \( (RB)^\pm_k(\tau, g^2, \Delta) \in \mathbb{R}[\tau, g^2, \Delta] \) is defined by the generating function
\[ (64) \quad \frac{1}{2} \left[ \frac{w\Omega(w)e^{-rw}}{1 - e^{-w}} \mp \frac{w\Omega_{\text{odd}}(w)e^{-rw}}{1 + e^{-w}} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k (RB)_k(\tau, g^2, \Delta)}{k!} w^k. \]

The special values of the spectral zeta functions at the negative integers are then obtained in the usual way.

**Lemma 6.7.** We have, for \( k \geq 1 \),
\[ \zeta_{\text{QRM}}(1 - k; \tau) = -\frac{2}{k} (RB)_k(\tau, g^2, \Delta^2). \]
and
\[ \zeta^+_\text{QRM}(1 - k; \tau) = -\frac{1}{k} (RB)^+_k(\tau, g^2, \Delta). \]

**Proof.** We have
\[
\zeta_{\text{QRM}}(1 - k; \tau) = \frac{(-1)^{k+1} \Gamma(k)}{2\pi i} \int_0^{0+} \frac{w\Omega(w)e^{-rw}}{w^{k+1} 1 - e^{-w}} dw = 2 \cdot \frac{(-1)^{k+1} \Gamma(k)}{k!} (-1)^k (RB)_k(\tau, g^2, \Delta^2) = -\frac{2}{k} (RB)_k(\tau, g^2, \Delta^2),
\]
as desired. The proof for \( \zeta^+_\text{QRM}(1 - k; \tau) \) is analogous. \( \square \)
From the lemma above, it is obvious that
\begin{equation}
2(RB)_k(\tau, g^2, \Delta^2) = (RB)_k^+(\tau, g^2, \Delta) + (RB)_k^-(\tau, g^2, \Delta).
\end{equation}

**Remark 6.3.** From the expression $\zeta_{QRM}(1-k; \tau)$ above, we find that $(RB)_k(\tau, g^2, \Delta)$ is identical with the Rabi-Bernoulli polynomials $R_k(g, \Delta, \tau)$ in (1.1) of [49]:

\[ R_k(g, \Delta, \tau) = (RB)_k(\tau, g^2, \Delta). \]

According to the result in [49], we have $(RB)_k(\tau, g^2, \Delta) \in \mathbb{Q}[g^2, \Delta^2, \tau]$. The $k$th Rabi-Bernoulli polynomial is monic and its degree with respect to the variable $\tau$ is $k$ (see also Theorem 6.9 below). Also, $(RB)_k(\tau, 0, 0)$ is equal to the Bernoulli polynomial $B_k(\tau)$. The coefficient 2 appearing at the definition of the Rabi-Bernoulli polynomials is considered to be the effect of the two-by-two system Hamiltonian.

The following simple difference-differential equation satisfied by the Rabi-Bernoulli polynomials is a consequence of Lemmas 6.7 and 6.8.

**Lemma 6.8.** We have
\begin{equation}
\frac{\partial}{\partial \tau} (RB)_{k+1}(\tau, g^2, \Delta) = -k(RB)_k(\tau, g^2, \Delta)
\end{equation}
for $k = 0, 1, 2, \ldots$ \hfill \Box

The explicit formula of $\Omega(t)$ allows us to give another proof to the rationality of the coefficients of the Rabi-Bernoulli polynomials $(RB)_k(\tau, g^2, \Delta^2)$ (proved originally in [49]) and to extend the result to the polynomials $(RB)_k^+(\tau, g^2, \Delta)$.

**Theorem 6.9.** The Rabi-Bernoulli polynomials $(RB)_k(\tau, g^2, \Delta^2)$ (resp. $(RB)_k^+(\tau, g^2, \Delta)$) as polynomials in $\Delta^2, g^2$ and $\tau$ (resp. in $\Delta, g^2$ and $\tau$) are rational numbers. That is, $(RB)_k(\tau, g^2, \Delta) \in \mathbb{Q}[g^2, \Delta^2, \tau]$. Similarly, we have $(RB)_k^+(\tau, g^2, \Delta) \in \mathbb{Q}[g^2, \Delta, \tau]$. Moreover, the degree of the Rabi-Bernoulli polynomials $(RB)_k(\tau, g^2, \Delta^2)$ (resp. $(RB)_k^+(\tau, g^2, \Delta)$) with respect to the variable $\tau$ is exactly equal to $k$.

**Proof.** We prove the theorem only for the polynomials $(RB)_k(\tau, g^2, \Delta^2)$ since the proof for the case of the polynomials $(RB)_k^+(\tau, g^2, \Delta)$ is completely analogous.

Let $\lambda \geq 1$ with $\lambda \equiv 0 \pmod{2}$. Expanding the exponentials as power series in the variable $t$ we see that
\[ e^{\phi(\mu_2, t)} + \xi_2^2(\mu_2, \lambda, t) + \psi_2^2(\lambda, t) \in \mathbb{Q}[g^2, \mu_1, \mu_2, \ldots, \mu_\lambda][[t]]. \]
Then, termwise integration yields
\[ (\Delta t)^{2\lambda} \int_0^{\Delta t} \cdots \int_0^{\Delta t} e^{\phi(\mu_2, t)} + \xi_2^2(\mu_2, \lambda, t) + \psi_2^2(\lambda, t) d\mu_2 \lambda \in \mathbb{Q}[g^2, \Delta^2][[t]], \]
and the minimum degree of any monomial appearing in the power series is at least $2\lambda$. It follows that $\Omega(t) \in \mathbb{Q}[g^2, \Delta^2][[t]]$ since the coefficient of any given degree $n$ is the sum of a finite number of elements of $\mathbb{Q}[g^2, \Delta^2]$ and similarly,
\[ \frac{w \Omega(w) e^{-\tau w}}{1 - e^{-\tau w}} \in \mathbb{Q}[g^2, \Delta^2, \tau][[w]], \]
and the result follows by comparing coefficients in the definition of $(RB)_k(\tau, g^2, \Delta^2)$. We notice that the only difference on the dependence of $\Delta$ from $(RB)_k(\tau, g^2, \Delta)$ is the contribution of $\Omega_{odd}(w)$ to the definition of $(RB)_k^+(\tau, g^2, \Delta)$.

Further, by Lemma 6.8 we have
\[ \frac{\partial^n}{\partial \tau^n} \zeta_{QRM}(1-k; \tau) = (-1)^n(1-k)n \zeta_{QRM}(1-k+n; \tau). \]
This shows that $\frac{\partial^{k+1}}{\partial \tau^{k+1}} \zeta_{QRM}(1-k; \tau) = 0$. Hence, the degree of $(RB)_k(\tau, g^2, \Delta^2)$ with respect to $\tau$ is at most $k$. Also, since $\zeta_{QRM}(s; \tau)$ has a simple pole at $s = 1$ (with non-zero residue), and looking at
the fact that radius of the circle at 1 (for the Laurent expansion) can be taken larger than 1 we see that \((k-n)\zeta_{\mathrm{QRM}}(1-k+n;\tau)|_{n=k} \neq 0\). It follows that \(\frac{\partial^k}{\partial \tau^k} \zeta_{\mathrm{QRM}}(1-k;\tau) \neq 0\). This proves the desired result for the degree with respect to \(\tau\).

\begin{proof}

Example 6.10. We give here for reader’s convenience the first and second Rabi-Bernoulli polynomials which are already given in \[49\] (Proposition 5.2 and 6.2). Since we define the Rabi-Bernoulli polynomials by the generating function \((6.1)\), we can compute these polynomials (at \(w=0\)) directly from the series expansion of the partition function \(Z_{\mathrm{Rabi}}(w) = \Omega(w)/(1-e^{-w})\) (the computation is essentially equivalent with the one in \[49\]). Actually, note first that

\[
 wZ_{\mathrm{Rabi}}(w) = \frac{w\Omega(w)}{1-e^{-w}} = \Omega(w)\left[1 + \left(\frac{w}{2} - \frac{w^2}{6} + \cdots\right) + \left(\frac{w}{2} - \frac{w^2}{6} + \cdots\right)^2 + \cdots \right]
\]

by taking small enough \(w\). Since \(\Omega(0) = 2\), we have \((RB)_1(\tau, g^2, \Delta) = 1\). Then, using integration (due to the relation \[(66)\]) and observing the first few terms’ expansion of \(\Omega(w)\) at \(w=0\) gives

\[
 (RB)_1(\tau, g^2, \Delta) = \tau - \frac{1}{2} + g^2 - \frac{1}{2} \frac{\partial}{\partial \tau} (RB)_2(\tau, g^2, \Delta)),
\]

\[
 (RB)_2(\tau, g^2, \Delta) = \tau^2 - (1+2g^2)\tau + \frac{1}{6} + g^2 + g^4 + \Delta^2.
\]

For the explicit formula for the third \((RB)_3(\tau, g^2, \Delta)\), see Proposition 6.6 in \[49\]. By means of this procedure, in principle, it is clear that we can compute the Rabi-Bernoulli polynomials explicitly but do not have a general formula in \(k\). Thus, apart from the equation \[(66)\], it is desirable to obtain a certain recursion formula among these Rabi-Bernoulli polynomials similarly to the Bernoulli one if any, e.g. from the Heun ODE \[48\] viewpoint. We will return this problem in the future.

6.3. Spectral determinant for the parity Hamiltonians and Braak’s \(G\)-functions. In this section we describe the spectral determinant of the parity Hamiltonian \(H_{\pm}\), corresponding to the zeta regularized product associated to \(\zeta_{\mathrm{QRM}}^\pm(s;\tau)\).

Fix the log-branch by \(-\pi \leq \arg(\tau - \lambda_i) < \pi\). The zeta regularized product (cf. \[41\]) associated to \(\zeta_{\mathrm{QRM}}^\pm(s;\tau)\) is defined by

\[
 \prod_{i=0}^{\infty}(\tau - \lambda_i^\pm) := \exp \left(-\frac{d}{ds} \zeta_{\mathrm{QRM}}^\pm(s;\tau)\right)|_{s=0},
\]

where the product is over the eigenvalues \(\lambda_i^\pm\) in the spectrum of \(H_{\pm}\). Now we define the spectral determinant of the Hamiltonians \(H_{\pm}\) as

\[
 \det(\tau - H_{\pm}) := \prod_{i=0}^{\infty}(\tau - \lambda_i^\pm).
\]

In \[41\], the authors proved that the zeta regularized product of \(\zeta_{\mathrm{Rabi}}(s;\tau)\), equivalently the spectral determinant of \(H_{\mathrm{Rabi}}\), is given (up to a non-vanishing entire function) by the complete \(G\)-function (called generalized \(G\)-function in \[41\]) given by

\[
 G(x;g,\Delta) = G_+(x;g,\Delta)G_-(x;g,\Delta)\Gamma(-x)^{-2},
\]

where \(G_\pm(x;g,\Delta)\) are the parity \(G\)-functions defined in \[42\] (see also Appendix \[A\] and cf. \[32\]).

To extend the result to the parity Hamiltonians we need some preparations.

Lemma 6.11. The residue of the \(G\)-function \(G_\pm(x;g,\Delta)\) at the (simple) pole at \(x = N \in \mathbb{Z}_{\geq 0}\) is given by

\[
 \text{Res}_{x=N} G_\pm(x;g,\Delta) = \frac{\Delta^2 g^N}{2(N+1)} K_N(N;g,\Delta) G_{\pm}(N)(g,\Delta).
\]

\begin{proof}

The result follows directly by computation and comparison with the definition of \(K_N(N;g,\Delta)\) and \(G_{\pm}(N)(g,\Delta)\) (see Appendix \[A\] and the proof of Proposition 6.8 of \[25\]).
\end{proof}
Finally, we show that the zeros of the complete $G$-function $G_{\pm}$ for each parity defined in the following captures the complete spectrum of $H_{\pm}$.

$$
G_{\pm}(x; g, \Delta) := G_{\pm}(x; g, \Delta)\Gamma(-x)^{-1}.
$$

**Theorem 6.12.** There is a one-to-one correspondence between eigenvalues $\lambda$ in $\text{Spec} H_{\pm}$ and zeros $x = \lambda + g^2$ of the generalized $G$-function $G_{\pm}(x; g, \Delta)$.

**Proof.** Let $\lambda \in \mathbb{R}$ be a regular eigenvalue of $H_{\pm}$, then by the definition $x = \lambda + g^2$ is a zero of $G_{\pm}(x; g, \Delta)$. Now, suppose $\lambda = N - g^2$ is an exceptional eigenvalue of $H_{\pm}$, then by Lemma 6.11 we see that at $x = \lambda - g^2 = N$ the function $G_{\pm}(x; g, \Delta)$ has a finite value, and then $G_{\pm}(x; g, \Delta)$ vanishes by the zero of $\Gamma(-x)^{-1}$. Conversely, let $x \in \mathbb{R}$ be a zero of $G_{\pm}(x; g, \Delta)$. If $x \notin \mathbb{Z}_{\geq 0}$ then $x$ is a zero of $G_{\pm}(x; g, \Delta)$ and $\lambda = x - g^2$ is a regular eigenvalue of $H_{\pm}$. If $x = N \in \mathbb{Z}_{\geq 0}$, then, since the zero of $\Gamma(-x)^{-1}$ at $x = N$ is canceled by the pole of $G_{\pm}(x; g, \Delta)$, $x = N$ must be a zero of the residue of $G_{\pm}(x; g, \Delta)$ at $x = N$, in other words, the tuple $(g, \Delta)$ must be a zero of $K_N(\tilde{N}; g, \Delta)$ or $G_{\pm}^{(s)}(g, \Delta)$ and thus $\lambda = N - g^2$ is an exceptional eigenvalue (the Juddian or non-Juddian exceptional, respectively). \qed

**Remark 6.4.** The spectrum of the QRM can be captured by irreducible representations of $\mathfrak{sl}_2$ (cf. [24, 33]). For instance, the Juddian (resp. non-Juddian [34]) exceptional solutions are obtained from the irreducible finite dimensional (resp. lowest weight) representations. The existence of these exceptional eigenvalues inherited from the quantum harmonic oscillator (or as its ruins) which are described by the oscillator representation of $\mathfrak{sl}_2$ is the reason for the presence of the gamma factor in $G_{\pm}(x; g, \Delta)$.

As a direct consequence of Theorem 6.12 we see that the complete parity $G$-function $G_{\pm}(x; g, \Delta)$ is, up to non-vanishing constant, the spectral determinant of the parity Hamiltonian $H_{\pm}$.

**Corollary 6.13.** There exists a non-vanishing entire function $c_{\pm}(\tau; g, \Delta)$ such that

$$
\det(\tau - H_{\pm}) = c_{\pm}(\tau; g, \Delta) G_{\pm}(\tau; g, \Delta). \quad \square
$$

**Remark 6.5.** In [1], Braak proved the integrability of the QRM by defining the $G$-function of the parity Hamiltonians $H_{\pm}$. In Corollary 6.13 above, it is shown that the $G$-function is, up to a non-vanishing constant, equal to the spectral determinant of $H_{\pm}$, in other words, the zeta regularized product of the spectral zeta function $\zeta_{\text{QRM}}^+(s; \tau)$. The zeta regularized product of a zeta function $\zeta(s)$ is defined when the function $\zeta(s)$ function is holomorphic in a neighborhood around $s = 0$ (in case $\zeta(s)$ has a pole at $s = 0$, a modified zeta regularized product may be used, cf. [20]). It would be interesting to investigate the relationship between the integrability (or exact solvability) of the Hamiltonian $H$ of a quantum interaction model, that is, the existence of entire solutions of the corresponding Fuchsian ODE (Bargmann model), and the existence of a zeta regularized product for its corresponding spectral zeta function $\zeta_H(s; t)$, or equivalently, the meromorphic continuation of the spectral zeta function to a region containing $s = 0$.

**Appendix A. Confluent Heun picture and $G$-functions of the QRM**

In this Appendix we give a brief introduction to the confluent picture of the QRM via the Bargmann space and to the $G$-functions used to prove its integrability in [1]. We follow the discussion in [1] and suggest the reader to consult [6, 25, 43] for more details.

We introduce first the Bargmann space (or Segal-Bargmann space). We refer the reader to [47] for an extended discussion on the application of Bargmann space to spectral problems. In this section we use the notation $\partial_z := \frac{1}{i} \partial_z$.

Denote by $\mathcal{V}(\mathbb{C})$ the space of entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$. In $\mathcal{V}(\mathbb{C})$ we have an inner-product defined for $f, g \in \mathcal{V}(\mathbb{C})$ by

$$
(f, g)_\mathcal{H} = \int_\mathbb{C} \overline{f(z)} g(z) d\mu(z)
$$

(69)
where \( d\mu(z) = \frac{1}{\pi}e^{-|z|^2} \, dx \, dy \) for \( z = x + iy \), and \( dx \, dy \) is the Lebesgue measure in \( \mathbb{C} \simeq \mathbb{R}^2 \).

The Bargmann space \( \mathcal{B} \) is the space of functions in \( \mathcal{V}(\mathbb{C}) \) satisfying

\[
\| f \|_{\mathcal{B}} = (f, f)^{1/2}_{\mathcal{B}} = \left( \int_{\mathbb{C}} |f(z)|^2 \, d\mu(z) \right)^{1/2} < \infty.
\]

It is known that the Bargmann space \( \mathcal{B} \) is a complete Hilbert space unitarily equivalent to the \( L^2(\mathbb{R}) \) Hilbert space by the Stone-von Neumann theorem (the inverse of the map is the Segal-Bargmann transform).

An important property of the Bargmann space is that it contains entire functions \( f \) having asymptotic expansion of the form

\[
f(z) = e^{\alpha_1 z^{-\alpha_0}}(c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots),
\]
as \( z \to \infty \). In particular, normal solutions of differential equations having and unramified singular point of rank 2 at infinity are included.

The creation and annihilation operators \( a \) and \( a^\dagger \) are realized in Bargmann space respectively as the differentiation and multiplication operators, that is

\[
a \to \partial_z, \quad a^\dagger \to z.
\]

The concrete realization of \( H_{\text{Rabi}} \) as an operator acting on \( \mathcal{H}_\mathcal{B} = \mathcal{B} \otimes \mathbb{C}^2 \) is given by

\[
H_{\text{Rabi}} = \begin{bmatrix} z\partial_z + \Delta & g(z + \partial_z) \\ g(z + \partial_z) & z\partial_z - \Delta \end{bmatrix},
\]

from this expression it is clear that the subspaces

\[
\mathcal{H}_+ = \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H}_\mathcal{B} \mid \phi_1 \text{ is an even function, } \phi_2 \text{ is an odd function} \right\},
\]

\[
\mathcal{H}_- = \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H}_\mathcal{B} \mid \phi_1 \text{ is an odd function, } \phi_2 \text{ is an even function} \right\},
\]

are \( H_{\text{Rabi}} \)-invariant subspaces of \( \mathcal{H}_\mathcal{B} \) and \( \mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}_\mathcal{B} \).

Let \( (\hat{T}\psi)(z) := \psi(-z) \ (\psi \in \mathcal{B}) \) be the reflection operator acting on \( \mathcal{B} \). Then, define the unitary operator \( U \) on \( \mathcal{H}_\mathcal{B} \) by

\[
U := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},
\]

and with \( C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), the Cayley transform, satisfying \( C^{-1} = C' = C \), i.e. \( C^2 = 1 \).

We obtain

\[
(CU)^\dagger H_{\text{Rabi}} CU = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix},
\]

with

\[
H_{\pm} = z\partial_z + g(z + \partial_z) \pm \Delta \hat{T},
\]

this is the parity decomposition of the QRM (see e.g. [5]).

Next, we describe the confluent Heun picture of the QRM and the \( G \)-function of the QRM. We refer the reader to [4, 5, 52] for more details.

From our discussion above, we consider \( H_{\pm} \) as operators acting on \( \mathcal{B} \). Consider a solution of the eigenvalue problem (time-independent Schrödinger equation) for \( H_+ \). Concretely, a real number \( \lambda \) is part of the spectrum of \( H_+ \) if and only if there is a function \( \psi \in \mathcal{B} \) such that

\[
z\partial_z \psi(z) + g(\partial_z + z)\psi(z) + \psi(-z) = \lambda \psi(z).
\]
Notice the presence of $\psi(-z)$ due to reflection operator. Therefore, by setting $\phi_1(z) = \psi(z)$ and $\phi_2(z) = \psi(-z)$ and applying the change of variable $z \to -z$ to the differential equation above we obtain the coupled system of differential equations
\begin{align}
(z + g)\partial_z \phi_1(z) + (gz - \lambda) + \Delta \phi_2(z) &= 0 \\
(z - g)\partial_z \phi_2(z) - (gz + \lambda) + \Delta \phi_1(z) &= 0.
\end{align}
(72)

This system of differential equations is equivalent to a second order confluent Heun differential equation with two regular singularities at $z = g, -g$ and one unramified singularity of rank 2 at $z = \infty$, we refer the reader to [48] for more details on confluent Heun differential equations and singularities. As mentioned already, entire solutions of this type of differential equation have asymptotic expansion (70) and are thus elements of the Bargmann space. Consequently, it is left to check only the holomorphicity in the complex plane of the solutions of (72).

Next, we consider the Frobenius solutions around the singularity $z = g$. The exponents of the equation (72) at the singularity are $\sigma_1 = 0, \lambda + g^2 + 1$ for $\phi_1$ and $\sigma_2 = 0, \lambda + g^2$ for $\phi_2$. Let us consider the case $\lambda + g^2 \notin \mathbb{Z}$, here the Frobenius solutions corresponding to the exponent 0 lead to the expressions
\begin{align}
\phi_1(z) &= e^{-gz}\Delta \sum_{n=0}^{\infty} K_n(x) \left( \frac{z + g}{x - n} \right) \\
\phi_2(z) &= e^{-gz}\sum_{n=0}^{\infty} K_n(x)(-z + g)^n,
\end{align}
where $x = \lambda + g^2$ and $K_n(x)$ are defined by the three term recurrence relation
\begin{equation}
nK_n(x) = f_{n-1}(x)K_{n-1}(x) - K_{n-2},
\end{equation}
with initial condition $K_0(x) = 1, K_1(x) = f_0(x)$ with
\begin{equation}
f_n(x) = 2g + \frac{1}{2g} \left( n - x + \frac{\Delta}{x - n} \right).
\end{equation}

The Frobenius solution $\phi_1(z)$ (resp. $\phi_2(-z)$) gives an expansions of $\psi(z)$ around $z = g$ (resp. $z = -g$) with radius of convergence $2g$. The condition for the solution $\psi(z)$ to be entire is then
\begin{equation}
G_+(x; z) = \phi_2(-z) - \phi_1(z) = 0,
\end{equation}
for all $z \in \mathbb{C}$. However, it is enough to check in the joint domain of $\phi_1(z)$ and $\phi_2(-z)$, with holomorphicity in rest of the plane following by analytic continuation.

In particular, taking $z = 0$, we obtain the $G$-function for the Hamiltonian $H_+$
\begin{equation}
G_+(x) = \phi_2(0) - \phi_1(0) = \sum_{n=0}^{\infty} K_n(x) \left( 1 - \frac{\Delta}{x - n} \right) g^n,
\end{equation}
and similarly
\begin{equation}
G_-(x) = \sum_{n=0}^{\infty} K_n(x) \left( 1 + \frac{\Delta}{x - n} \right) g^n.
\end{equation}

In this way, we see that solutions of the equation
\begin{equation}
G_\pm(x) = 0
\end{equation}
determine eigenvalues $\lambda = x - g^2$, with $x \notin \mathbb{Z}$. These eigenvalues constitute the \textit{regular spectrum} of the QRM and are known to be non-degenerate.

On the other hand, when the second exponent $\sigma_1 = \lambda + g^2 + 1$ of (72) at $z = g$ is an integer $N \in \mathbb{Z}_{\geq 0}$, that is when the eigenvalue is of the form $\lambda = N - g^2$, the Frobenius solutions corresponding to the exponent $\sigma_i = 0 (i = 1, 2)$ may develop a logarithmic singularity which forces the condition
\begin{equation}
K_N(N; g, \Delta) = 0,
\end{equation}
(74)
in order to obtain entire solutions. In fact, these solutions, known as Juddian solutions, have only a finite number of terms in the power series expansion. The condition (74) (usually expression in an equivalent polynomial form, see [25]) is known as constraint relation for Juddian eigenvalues of the QRM. It is known (cf. [31]) that Juddian eigenvalues are doubly degenerate, with one solution in each parity.

Even if the condition (74) does not hold, there may be entire solutions constructed from the Frobenius solutions with respect to the exponents \( \sigma_1 = N + 1, \sigma_2 = N \). The solutions are then constructed in a manner analogous to the case of regular solutions. In this case, the \( G \)-function for the non-Juddian exceptional eigenvalue \( \lambda = N - g^2 \) is given by

\[
G^{(N)}(g, \Delta) = -\frac{2(N + 1)}{\Delta} + \sum_{n=N+1}^{\infty} K_n(N; g, \Delta) \left( 1 \pm \frac{\Delta}{N - m} \right) g^{n-N-1},
\]

where \( K_n(N; g, \Delta) \) satisfies (73) with initial conditions \( K_{N+1}(N; g, \Delta) = 1 \) and \( K_n(N; g, \Delta) = 0 \) for \( n < N \). Similar to the case of regular eigenvalues, it is known that non-Juddian exceptional eigenvalues are non-degenerate.

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