CHANG PALAIS-SMALE CONDITION AND GLOBAL INVERSION

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Abstract. Let $f : X \to Y$ be a local $C^1$-diffeomorphism between real Banach spaces. We prove that if the locally Lipschitz functional $x \mapsto \frac{1}{2}[f(x) - y]^2$ satisfies the Chang Palais-Smale condition for all $y \in Y$, then $f$ is a norm-coercive global $C^1$-diffeomorphism. We also give a version of this fact for a weighted Chan Palais-Smale condition. Finally, we study the relationship of this criterion to some classical global inversion conditions.

1. Introduction

Let $(X, |\cdot|)$ and $(Y, |\cdot|)$ be real Banach spaces. A $C^1$ map $f : X \to Y$ is said to be a global $C^1$-diffeomorphism provided it is bijective and its inverse is also a $C^1$-map. Recall, by the Inverse Mapping Theorem, $f$ is a $C^1$-map such that $df(x)$ is a linear isomorphism for all $x \in X$ if and only if it is a local $C^1$-diffeomorphism. An old question motivated by the study of existence and uniqueness of the solutions of a nonlinear equations is: Under which conditions a local $C^1$-diffeomorphism $f$ is a global one? An old answer, and one of the most important, was given by Banach and Mazur [1], and independently by Caccioppoli [3]: $f$ is a global $C^1$-diffeomorphism if and only if $f$ is a proper map. Recall, $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is compact whenever $K$ is compact. For example, if $X = Y = \mathbb{R}^n$, then $f$ is a proper map if and only if it is norm-coercive, namely, $\lim_{|x| \to +\infty} |f(x)| = +\infty$. So we get the following theorem, which goes back to Hadamard [7]: A local $C^1$-diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is a global $C^1$-diffeomorphism if and only if it is norm-coercive. The Hadamard Theorem supports the false idea that every continuous bijection between vector normed spaces must be norm-coercive. Actually, in the infinite-dimensional setting, every global homeomorphism is a proper map but not necessarily it is norm-coercive [21]. The main propose of this paper is give a criterion for a local $C^1$-diffeomorphism between real Banach spaces to be a norm-coercive global diffeomorphism.

The proof of the Banach-Mazur theorem and other classical global inversion theorems have exploited the use of covering space techniques via the homotopy-lifting property or some different approach, but always through some variant of the same argument which includes a sort of monodromy process [17, 10, 9, 2, 20]. Nevertheless, since the early nineties, some global inversion conditions that test the effectiveness of the monodromy argument have emerged. The crucial hypothesis involves the Palais-Smale condition of a suitable functional $F$ in terms of $f$, see e.g. [18, 15]. Recall, a $C^1$-functional $F : X \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{x_n\}$ in $X$ such that $\{F(x_n)\}$ is bounded and $\|\nabla F(x_n)\| \to 0$ contains

Date: February 16, 2018.

Key words and phrases. Palais-Smale condition, global inversion.
a convergent subsequence, whose limit is then a critical point of $F$. Some ideas on the matter have appeared long time before. For example, Gordon [5] relates the original condition (C) of Palais and Smale [16] to the Hadamard-Caccioppoli Theorem.  

A remarkable work in this line correspond to Katriel who gave a global inverse result for maps $f : X \rightarrow Y$ between certain metric spaces by means of an abstract mountain-pass theorem and an Ekeland variational principle, see Theorem 6.1 in [11]. Unlike approach followed in [18] or [15], Katriel technique works in infinite-dimensional setting as well. Looking closely, from the proof of Theorem 6.1 in [11] we can conjecture that: If $(Y, |\cdot|)$ is a normed space, $f$ is a local homeomorphism, and the map $x \mapsto |f(x) - y|$ satisfies a sort of Palais-Smale condition for all $y \in Y$, then $f$ must be bijective. In this spirit, Idczak et al. [8] adapted the ideas of Katriel [11] to get a global inversion theorem for a $C^1$ map: let $f : X \rightarrow Y$ be a local $C^1$-diffeomorphism such that $X$ is a real Banach space and $Y$ is a real Hilbert space, then $f$ is a global diffeomorphism if, for all $y \in Y$, the functional $F_y(x) = \frac{1}{2}|f(x) - y|^2$ satisfies the Palais-Smale condition —in such case, $\frac{1}{2}|\cdot|^2$ is of class $C^1$.

In the first part of this paper, we give a generalization of the Idczak et al. result for functions between real Banach spaces in terms of a weighted version of the Chang Palais-Smale condition; see Theorem 3. The Chang Palais-Smale condition was introduced by Chang [4] in order to extend the variational methods for locally Lipschitz functionals, as it is the mapping $x \mapsto \frac{1}{2}|f(x) - y|^2$, for any real Banach space $(Y, |\cdot|)$. The proposed weighted version is a particular case of the Zhong’s Palais-Smale condition given by Motreanu et al. [14] for a general class of nonsmooth functionals. Also, we split the hypothesis of Theorem 3 as an injectivity and surjectivity criteria in order to clarify the relationship of Theorem 3 with some classical global inversion theorems; see Theorem 8 and Theorem 9. For example, we prove that if $\int_0^{\infty} \inf_{|x| \leq \rho} \frac{1}{|dF(x)|} d\rho = \infty$ (hypothesis of the Hadamard-Levy-Plastock Theorem [7, 10, 17]) then, for every $y \in Y$, then the map $F_y(x) = \frac{1}{2}|f(x) - y|^2$ satisfies the weighted Chang Palais-Smale condition for some weight; see Example 10. In the second part, we present the whole picture of Theorem 3 with respect to other global inversion theorems, including the Theorem 6.1 of Katriel in [11] and the Banach-Mazur-Caccioppoli Theorem.

2. Global inversion theorem

Let $(X, |\cdot|)$ be a real Banach space and let $X^*$ be its dual space, where $(x^*, x)$ denotes the duality for all $x \in X$ and $x^* \in X^*$. Let $F : X \rightarrow \mathbb{R}$ be a locally Lipschitz functional. For each $x \in X$, the generalized directional derivative in direction $v$ is defined as:

$$F^0(x; v) = \limsup_{h \rightarrow 0} \frac{1}{t} (F(x + h + tv) - F(x + h)).$$

The map $v \mapsto F^0(x; v)$ is convex and continuous. Let $\partial F(x)$ be the Clarke generalized gradient of $F$ at $x$. That is, $w^* \in \partial F(x) \subset X^*$ if and only if $(w^*, v) \leq F^0(x; v)$ for all $v \in X$. Recall, $\partial F(x)$ is a non-empty $w^*$-weakly compact convex set. In

\[\text{defined as:} \]

\[
F^0(x; v) = \limsup_{h \rightarrow 0} \frac{1}{t} (F(x + h + tv) - F(x + h)).
\]

\[\text{The map } v \mapsto F^0(x; v) \text{ is convex and continuous. Let } \partial F(x) \text{ be the Clarke generalized gradient of } F \text{ at } x. \text{ That is, } w^* \in \partial F(x) \subset X^* \text{ if and only if } (w^*, v) \leq F^0(x; v) \text{ for all } v \in X. \text{ Recall, } \partial F(x) \text{ is a non-empty } w^*-\text{weakly compact convex set. In}
\]
order to extend the classical Palais-Smale condition to locally Lipschitz functions, Chang [4] defined the lower semi-continuous function:

$$\lambda_F(x) = \min_{w^* \in \partial F(x)} |w^*|_{X^*}$$

**Chang PS-condition.** A locally Lipschitz function $F$ satisfies the Chang PS-condition if any sequence $\{x_n\}$ in $X$ such that $\{F(x_n)\}$ is bounded and

$$\lim_{n \to \infty} \lambda_F(x_n) = 0$$

contains a (strongly) convergent subsequence.

Of course, if $F$ is continuously Fréchet differentiable map, then the above definition reduces to the usual Palais-Smale condition, since in this case $\partial F(x) = \{dF(x)\}$ and therefore $\lambda_F(x) = \|dF(x)\|$. A typical extension for Palais-Smale conditions involves a continuous non-decreasing function $h : [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^\infty \frac{1}{1 + h(\rho)} d\rho = +\infty.$$ (1)

We define:

**Weighted Chang PS-condition.** A locally Lipschitz function $F$ satisfies the weighted Chang PS-condition if any sequence $\{x_n\}$ in $X$ such that $\{F(x_n)\}$ is bounded and

$$\lim_{n \to \infty} \lambda_F(x_n)(1 + h(|x_n|)) = 0$$ (2)

contains a (strongly) convergent subsequence.

If $h = 0$, the weighted Chang PS-condition reduces to PS-condition given by Chang. If $h(t) = t$, for all $t \geq 0$ the weighted Chang PS-condition expresses the extension of the PS-condition in the sense of Cerami to the locally Lipschitz functionals. In [22], Zhong established a weighted PS-condition but for Gâteaux differentiable functionals, in terms of $\|dF(x)\|$ instead of $\lambda_F(x)$. Our weighted Chang PS-condition is a particular case of a version of Zhong’s Palais-Smale condition given by Motreanu et al.; see Definition 1.3 of [14].

As we may expected, if we have a converging sequence $\{x_n\}$ in $X$ that satisfies (2), then the limit is a critical point of $F$. That is:

**Lemma 1.** Let $(X, | \cdot |)$ be a real Banach space and let $F : X \to \mathbb{R}$ be a locally Lipschitz functional. Let $\{x_n\} \subset X$ be a sequence such that $\lim_{n \to \infty} x_n = \hat{x}$ and satisfying (2). Then, for a continuous non-decreasing function $h : [0, +\infty) \to [0, +\infty)$ such that (1) is fulfilled, we have $\lambda_F(\hat{x}) = 0$.

**Proof.** Since $\lambda_F$ is lower semi-continuous, the map $x \mapsto \lambda_F(x)(1 + h(|x|))$ is lower semi-continuous. By (2), $\lambda_F(\hat{x}) = 0$, since:

$$\lim_{n \to \infty} \lambda_F(x_n)(1 + h(|x_n|)) = \liminf_{n \to \infty} \lambda_F(x_n)(1 + h(|x_n|)) \geq \lambda_F(\hat{x})(1 + h(|\hat{x}|)) \geq \lambda_F(\hat{x}).$$

□

Let $f$ be map between real Banach spaces. We are interested in establish a criterion in terms of the weighted Chang PS-condition in order to ensure the existence
and uniqueness of the solutions of a nonlinear equation:

$$f(x) = y.$$  \hspace{1cm} (3)

Our main result is the following:

**Theorem 2.** Let \((X, |\cdot|)\) and \((Y, |\cdot|)\) be real Banach spaces and let \(f : X \to Y\) be a local \(C^1\)-diffeomorphism. Let \(y \in Y\). If the locally Lipschitz functional

$$F_y(x) = \frac{1}{2}|f(x) - y|^2$$

satisfies the weighted Chang PS-condition for some continuous nondecreasing function \(h : [0, +\infty) \to [0, +\infty)\) such that \((1)\) is fulfilled, then there is a unique solution of the nonlinear equation \((3)\).

**Corollary 3** (Global Inverse Theorem). Let \((X, |\cdot|)\) and \((Y, |\cdot|)\) be real Banach spaces and let \(f : X \to Y\) be a local \(C^1\)-diffeomorphism. Let \(h : [0, +\infty) \to [0, +\infty)\) be a continuous nondecreasing function such that \((1)\) is fulfilled. Suppose that:

(*) For every \(y \in Y\), the map \(F_y(x) = \frac{1}{2}|f(x) - y|^2\) satisfies the weighted Chang PS-condition for \(h\).

Then \(f\) is a norm-coercive diffeomorphism onto \(Y\).

**Proof.** By Theorem 2, \(f\) is a global diffeomorphism. Furthermore, for every \(y \in Y\), the functional \(F_y\) is bounded from below and satisfies the weighted Chang PS-condition, then \(\lim_{|x| \to +\infty} F_y(x) = +\infty\); see Corollary 2.4 of [14]. In particular, for \(y = 0\), we have that \(f\) is norm-coercive.

We have Theorem 3.1 of [8] is a particular case of Corollary 3 of Theorem 2, with \(Y\) a Hilbert space and \(h = 0\). The technique of proof of Theorem 2 is basically the same of the proof of Theorem 6.1 of Katriel [11], given in an more abstract setting: a proper mountain-pass lemma provides the uniqueness of the solution; and the existence of a minimizing sequence converging to a critical point — given by the Ekeland Variational Principle — guarantees the existence of the solution. This approach is also the same as in the proof of Theorem 3.1 in [8]. In order to get a simpler and direct proof, we shall use the ‘made to size’ Lemma 4 (Theorem 7.2 of [11]) instead of the classical mountain-pass theorem of Ambrosetti and Rabinowitz considered in [8].

**Lemma 4** (Schechter-Katriel Mountain-Pass Theorem). Let \((X, |\cdot|)\) be a Banach space and \(F : X \to \mathbb{R}\) be a locally Lipschitz functional. Suppose that, for some \(e \in X\), \(e \neq 0\), \(r > 0\) and \(\rho \in \mathbb{R}\):

- \(F(0) \leq \rho\) and \(F(e) \leq \rho\).
- \(|e| \geq r\).
- \(F(x) \geq \rho\) for \(|x| = r\).

Then, for a continuous non-decreasing function \(h : [0, +\infty) \to [0, +\infty)\) such that \((1)\) is fulfilled, there is a sequence \(\{x_n\} \subset X\) such that \(F(x_n) \to c\) for some \(c \geq \rho\) and satisfying \((2)\).

In [4] Chang proved, by means of a deformation lemma, that for every bounded from below locally Lipschitz functional \(F\) defined on a reflexive Banach space that satisfies the Chang PS-condition, the real number inf\(X F\) is a critical value of \(F\). The reflexivity assumption can be dropped out if we use a standard technique via a convenient variational principle, in this case, Zhong’s variational principle [22];
see also Theorem 2.1 in [14]. Actually, we have the following expected fact, which is a version of Corollary 3.3 of Zhong [22] for locally Lipschitz functionals.

**Lemma 5.** Let \((X, |·|)\) be a Banach space and let \(F : X \to \mathbb{R}\) be a locally Lipschitz functional and bounded from below. If \(h : [0, +\infty) \to [0, +\infty)\) is a non-decreasing function such that (1) is fulfilled, then there is a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} F(x_n) = \inf_X F\) and satisfying (2).

**Proof.** Let \(\{\epsilon_n\}\) be a sequence of positive numbers such that \(\{\epsilon_n\} \to 0^+\). Since \(F\) is bounded below, by the Ekeland Variational Principle with weight \((1 + h(r))^{-1}\), for every \(\epsilon_n > 0\) there exists \(x_n \in X\) such that, for all \(x \in X\), \(F(x_n) < \inf_X F + \epsilon_n\) and \(F(x) \geq F(x_n) - \epsilon_n(1 + h(|x_n|))^{-1}|x - x_n|\). Taking \(x = x_n + t(u - x_n)\), for an arbitrary \(u \in X\) and \(t > 0\) we have:

\[
\frac{F(x_n + t(u - x_n)) - F(x_n)}{t} \geq \frac{-\epsilon_n}{1 + h(|x_n|)}|u - x_n|.
\]

Passing to the upper limit as \(t \to 0^+\), we deduce that:

\[
F^0(x_n; u - x_n) \geq \frac{-\epsilon_n}{1 + h(|x_n|)}|u - x_n|, \quad \forall u \in X.
\]

Let \(n\) be a natural number. Then for \(x_n\) fixed, the sets \(A_n = \{(v, t) : t > F^0(x_n; v)\}\) and \(B_n = \{(v, t) : t < -\epsilon_n|v|(1 + h(|x_n|))^{-1}\}\) are both open and convex; and \(A_n \cap B_n = \emptyset\). Therefore, there is a hyperplane passing through zero, given by a linear functional \(\xi_n(v, t) = \omega_n(v) + \alpha t\) for some \(\alpha \neq 0\), separating \(A\) and \(B\). For every \(v \in X\), set \(\langle w_n^*, v \rangle = -\frac{\omega_n(v)}{\alpha}\). We have that \(\xi_n(v, \langle w_n^*, v \rangle) = 0\) for all \(v \in X\). Then \(\langle w_n^*, v \rangle \leq F^0(x_n; v)\) so \(w_n^* \in \partial f(x_n)\). Furthermore, \(|\langle w_n^*, v \rangle| (1 + h(|x_n|)) \leq \epsilon_n|v|\); therefore \(\lambda_F(x_n)(1 + h(|x_n|)) \leq |w_n^*|_{X^*}(1 + h(|x_n|)) \leq \epsilon_n\).

Besides the above lemmas, we need to justify first some technical properties in order to give a tidier proof of Theorem 2. Let \((X, |·|)\) and \((Y, |·|)\) be real Banach spaces and let \(f : X \to Y\) be a local \(C^1\)-diffeomorphism. The classical Hadamard-Levy Theorem, as well as other classical global inverse results, involves the quantity \(|df(x)^{-1}|\). For local diffeomorphisms this quantity coincides with the so-called Banach constant:

\[
\text{Sur } df(x) = \inf_{|v|_{Y^*} = 1} |df(x)^*v^*|_{X^*}.
\]

As it is well known, \(df(x_0) : X \to Y\) is surjective if and only if \(\text{Sur } df(x_0) > 0\) if and only if \(f\) is open with linear rate around \(x_0\), namely, there exist a neighborhood \(V\) of \(x_0\) and a constant \(\alpha > 0\) such that for every \(x \in V\) and \(r > 0\) with \(B_r(x) \subset V\) we have \(B_{\alpha r}(f(x)) \subset f(B_r(x))\). We shall also considerate in the proof of Theorem 2, as well as in arguments below, the following facts.

**Claim 6.** \(\lambda_{F^0}(x) \geq \lambda_H(f(x) - y)\text{Sur } df(x)\) for all \(y \in Y\) and \(x \in X\).

**Proof.** Let \(y \in Y\) and \(x \in X\) be fixed. We have that \(\lambda_{F^0}(x) = \min_{w^* \in \partial F^0(x)} |w^*|_{X^*}\) and \(\partial F^0(x) = df(x)^*\partial |f(x) - y|\). Let \(w^* \in \partial F^0(x)\). Then there is \(v^* \in \partial |f(x) - y|\) such that \(w^* = df(x)^*v^*\). Note that \(|v^*| \geq \min_{v^* \in \partial |f(x) - y|} |v^*|_{Y^*} = \lambda_H(f(x) - y)|v^*|_{Y^*}\). Suppose that \(v^* \neq 0\). Therefore:

\[
|w^*|_{X^*} = \frac{|df(x)^*v^*|}{|v^*|_{Y^*}}|v^*|_{Y^*} \geq \text{Sur } df(x)\lambda_H(f(x) - y)
\]
If \( v^* = 0 \) then \( \lambda_H(f(x) - y) = |w^*|_{X^*} = 0 \) and (4) is satisfied trivially. Taking the minimum over the set \( \partial F_y(x) \) we have the desired inequality. \( \square \)

**Claim 7.** \( \lambda_H(f(x) - y) = 1 \) if \( f(x) \neq y \), otherwise \( \lambda_H(f(x) - y) = 0 \).

**Proof.** Since \( |v^*|_{Y^*} = 1 \) for \( v^* \in \partial f(0) \) with \( z \neq 0 \) we have that if \( f(x) \neq y \) then \( \lambda_H(f(x) - y) = 1 \). If \( f(x) = y \), since the zero functional in \( Y^* \) belongs to \( \partial f(0) \), then \( \lambda_H(f(x) - y) = 0 \). \( \square \)

**Proof of Theorem 2. Uniqueness.** Let \( y \in Y \) be fixed. Suppose that \( f \) is not injective. So, there are two different points \( u \) and \( e \) in \( X \) such that \( f(u) = f(e) = y \). Since \( f \) is open with linear rate around \( u \), there exists \( \epsilon > 0 \) such that:

\[
B_{\alpha \epsilon}(y) \subset f(B_{\epsilon}(u)), \text{ for all } 0 < r < \epsilon. \tag{5}
\]

Let \( r \in (0, \epsilon) \) be small enough such that \( f|_{B_{\epsilon}(u)} : B_{\epsilon}(u) \to f(B_{\epsilon}(u)) \) is a diffeomorphism. Set \( \rho = \frac{1}{2} \alpha^{-2} \epsilon^2 > 0 \). Suppose that \( u = 0 \). We have that:

- \( F_y(0) = 0 \leq \rho \) and \( F_y(e) = 0 \leq \rho \).
- \( |e| \geq \rho \), since \( f|_{B_{\epsilon}(0)} \) is injective.
- \( F_y(x) \geq \rho \) for \( |x| = r \), in view of (5).

By Lemma 4, there is a sequence \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} F_y(x_n) = c \) for some \( c \geq \rho \) and satisfying (2). Since \( F_y \) satisfies the weighted Chang PS-condition, the sequence \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \) with limit \( \hat{x} \). Therefore:

- \( \lambda_{F_{\hat{x}}}(\hat{x}) = 0 \), by Lemma 1.
- \( f(\hat{x}) \neq y \), due to \( \lim_{n \to \infty} F_y(x_{n_k}) = F_y(\hat{x}) = c \geq \rho > 0 \).

By Claim 6 and Claim 7, \( \text{Surdf}(\hat{x}) = 0 \). So we get a contradiction. If \( u \neq 0 \) then we can consider \( G_y(x) = F_y(u - x) \) instead of \( F_y(x) \) and carry on an analogous reasoning.

**Existence.** Let \( y \in Y \) be fixed. By Lemma 5, there is a sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} F_y(x_n) = \inf_X F_y \) and satisfying (2). Since \( F_y \) satisfies weighted Chang PS-condition, the sequence \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \) with limit \( \hat{x} \). By Lemma 1, we have that \( \hat{x} \) is a critical point of \( F_y \). By Claim 6 and Claim 7 we have that \( f(\hat{x}) = y \). \( \square \)

From the ideas of the proof of Theorem 2 we can also deduce the following injectivity criterion:

**Theorem 8 (Injectivity Criterion).** Let \( (X, \| \cdot \|) \) and \( (Y, \| \cdot \|) \) be real Banach spaces and let \( f : X \to Y \) be a local \( C^1 \)-diffeomorphism. Let \( h : [0, +\infty) \to [0, +\infty) \) be a continuous nondecreasing function such that (1) is fulfilled. If:

\[ \forall y \in Y, \text{ there is no sequence } \{x_n\} \text{ in } X \text{ such that } \]

\[
\lim_{n \to \infty} F_y(x_n) = c > 0 \quad \text{and} \quad \lim_{n \to \infty} \lambda_{F_{\hat{x}}}(x_n)(1 + h(x_n)) = 0.
\]

Then \( f \) is one-to-one.

**Proof.** First, suppose that \( f \) is not injective. Following step-by-step the first part of the proof of Theorem 2 we conclude that there is a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} F_y(x_n) = c > 0 \) and \( \lim_{n \to \infty} \lambda_{F_{\hat{x}}}(x_n)(1 + h(x_n)) = 0 \), so we get a contradiction. If \( u \neq 0 \) consider the sequence \( \{x_n'\} = \{u - x_n\} \). \( \square \)

Furthermore, by the second part of the proof of Theorem 2 we deduce the following result:
Theorem 9 (Surjectivity Criterion). Let \((X,|\cdot|)\) and \((Y,|\cdot|)\) be real Banach spaces and let \(f: X \to Y\) be a local \(C^1\)-diffeomorphism. Let \(h: [0, +\infty) \to [0, +\infty)\) be a continuous nondecreasing function such that (1) is fulfilled. Suppose that:

\(\dagger\) For each \(y \in Y\), every sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} F_y(x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \lambda_{F_y}(x_n)(1 + h(x_n)) = 0
\]

has a convergent subsequence.

Then \(f\) is onto.

Note that if \(f\) is a local \(C^1\)-diffeomorphism then \(\star\) is fulfilled if and only if both conditions, \(\dagger\) and \(\dagger\), are satisfied (with the same \(h\)). Indeed: suppose that \(\star\) is fulfilled but for a point \(y \in Y\) there is a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} F_y(x_n) = c\) and \(\lim_{n \to \infty} \lambda_{F_y}(x_n)(1 + h(x_n)) = 0\). As before, since \(F_y\) satisfies the weighted Chang PS-condition, the sequence \(\{x_n\}\) has a convergent subsequence \(\{x_{n_k}\}\) with limit \(\hat{x}\) such that \(\lambda_{F_y}(\hat{x}) = 0\) and \(f(\hat{x}) \neq y\). So we get the contradiction: \(\text{Sur}\{(\hat{x}) = 0\). Then we have \(\dagger\). Furthermore \(\dagger\) is also fulfilled by definition of the weighted Chang PS-condition. Since \(F_y\) is always nonnegative, the converse is trivial.

Example 10. Let \(f\) be as before. Suppose also that \(f\) satisfies the Hadamard integral condition, namely (see for instance [7, 10, 17]):

\(\star\star\) \(\lim_{r \to 0} g(r) = \infty\) where \(g(r) = \int_r^\infty \inf_{|z| \leq \rho} \frac{1}{|df(z)|} d\rho\).

Set the nondecreasing map \(h\) given by \(h(x) = \sup_{|x| \leq \rho} \frac{1}{|df(x)|} \), where \(\alpha = \|df(0)^{-1}\|^{-1}\).

The map \(h\) is continuous since \(f\) is \(C^1\). Furthermore, condition \(\star\star\) implies that \(h\) satisfies (1). It is easy to see that for all \(x \in X\):

\[
0 < \alpha \leq \text{Sur} df(x)(1 + h(|x|)).
\]

Therefore, by Claim 6, for all \(y \in Y\) and \(x \in X\):

\[
\lambda_{F_y}(x)(1 + h(|x|)) \geq \lambda_{\|f\|}(f(x) - y) \alpha \geq 0.
\]

Suppose that there is a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} F_y(x_n) = c > 0\). Then, without loss of generality, we can assume that \(f(x_n) = y\) for all natural \(n\). Then, by Claim 7, \(\lim_{n \to \infty} \lambda_{F_y}(x_n)(1 + h(|x_n|))\) can’t be zero. Therefore, \(\dagger\) is satisfied and by Theorem 8 the map \(f\) is injective. Now, let \(\{x_n\} \subset X\) be such that \(\lim_{n \to \infty} F_y(x_n) = 0\) and \(\lim_{n \to \infty} \lambda_{F_y}(x_n)(1 + h(|x_n|)) = 0\). Then, by the above inequality, \(\lim_{n \to \infty} \lambda_{\|f\|}(f(x_n) - y) = 0\). Claim 7 implies that there exists \(m > 0\) such that \(f(x_n) = y\) for all \(n \geq m\). Since \(f\) is injective, this means that \(x_n = f^{-1}(y)\) for all \(n \geq m\), so \(\{x_n\}\) converges to \(f^{-1}(y)\). Therefore \(\dagger\) is fulfilled. So, we have shown that if \(f\) satisfies the Hadamard integral condition then, for every \(y \in Y\), then the map \(F_y = \frac{1}{\|f\|}(f(x) - y)^2\) satisfies the weighted Chang PS-condition for \(h\) given as above. We can conclude that \(f\) is norm-coercive and, as it is well known, \(f\) is a global diffeomorphism.

3. Relationship to the classical results

We have shown that Hadamard integral condition satisfies criterion \(\star\). We are now interested in locating condition \(\star\) with respect to other global inversion criteria. As it is known, by the Invariance of Domain Theorem, every mapping \(f: \mathbb{R}^n \to \mathbb{R}^n\) such that \(|f(x) - f(u)| = |x - u|\) for all \(u, x \in \mathbb{R}^n\) is a homeomorphism onto \(\mathbb{R}^n\). There is an infinite-dimensional version of this fact for Fredholm maps of
index zero between real Banach spaces \([20]\). In particular, if \(f : X \to Y\) is a local \(C^1\)-diffeomorphism and:

1) \(f\) is a distance preserving isometry

then \(f\) is a surjective map (and of course, an injective map). Therefore, \(f\) is a global diffeomorphism. Actually, in this case, by the Mazur-Ulam Theorem, \(f\) is an affine transformation. The distance-preserving hypothesis can be relaxed by the following one \([20]\):

2) \(f\) is an expansive map, that is, there is \(\alpha > 0\) such that for all \(x, u \in X\):

\[
|f(u) - f(x)| \geq \alpha|u - x|.
\]

Since if \(f\) is local diffeomorphism at \(x\), \(\inf_{|v|=1} |df(x)\,v| = \lim\inf_{u \to x} \frac{|f(u) - f(x)|}{|u - x|} \geq 10\), both criteria above are a particular case of the Hadamard-Levy hypothesis \([12]\):

3) There is \(\alpha > 0\) such that \(|df(x)\,v| \geq \alpha|v|\) for all \(x, v \in X\)

Which, in turn, is a specific case of the Hadamard integral condition \(\star\star\), since \(\|df(x)^{-1}\|^{-1} = \inf_{|v|=1} |df(x)\,v|\). Note that by Example \(10\), criterion \(\star\star\) implies that \(f\) is norm-coercive and \(\inf_{|x|<\rho} \|df(x)^{-1}\|^{-1} < 0\) for all \(\rho > 0\). Therefore, if \(\star\star\) is fulfilled then:

4) \(f\) is norm-coercive and:

\[
\sup_{|x|\leq\rho} \|df(x)^{-1}\| < \infty, \text{ for all } \rho > 0.
\]

The item 4) corresponds to another well-known criterion of global inversion \([17, 11]\).

As Katriel noticed in \([11]\), if criterion 4) holds then:

5) For some —hence any— \(y \in Y \) \([11]\):

\[
\inf \{ \text{sur}(f, x) : |f(x) - y| < \rho \}, \text{ for all } \rho > 0.
\]

Above, \(\text{sur}(f, x)\) is the surjection constant of \(f\) at \(x\), originally introduced by Ioffe \([9]\) for non differentiable maps between Banach spaces, namely:

\[
\text{sur}(f, x) = \lim_{r \to 0} \inf \frac{1}{r} \sup \{ R \geq 0 : B_R(f(x)) \subset f(B_r(x)) \}.
\]

Actually, if \(f\) is a local diffeomorphism at \(x\) we have \(\text{sur}(f, x) = \text{Sur } df(x)\). This last point corresponds to the hypothesis of the Katriel global inversion theorem. So far we have:\[2\]

\[
1) \Rightarrow 2) \Rightarrow 3) \Rightarrow \star\star \Rightarrow 4) \Rightarrow 5)
\]

In order to join the above chain with \(\star\) we shall see that:

5) \(\text {implies } \star\). Let \(y \in Y\) be fixed. We will prove that \(F_y\) satisfies the Chang PS-condition. First, we shall see that \(\dagger\) is fulfilled for \(h = 0\): suppose that there is a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} F_y(x_n) = c > 0\) and \(\lim_{n \to \infty} \lambda_{F_y}(x_n) = 0\). Then there exists \(\rho > 0\) such that \(F_y(x_n) < \rho\) and, without loss of generality, we can suppose that \(f(x_n) \neq y\) for all \(n\). Since \(f\) satisfies the Katriel condition then

\[
\inf \{ \text{Sur } df(x) : F_y(x) < \rho \} > 0.
\]

\[2\]In \([0]\), the author presents an analogous relationship between these criteria for mappings between Banach-Finsler manifolds (e.g. Riemannian manifolds).
So, $\text{Sur } df(x_n) \geq \alpha$ for all $n$ and some $\alpha > 0$. Therefore, by Claim 6:

$$0 < \alpha < \text{Sur } df(x_n) \leq \lambda F_y(x_n).$$

Thus $F_y(x_n) \rightarrow 0$, then we get a contradiction.

Now, let $\{x_n\} \subset X$ be such that $\lim_{n \rightarrow \infty} F_y(x_n) = 0$ and $\lim_{n \rightarrow \infty} \lambda F_y(x_n) = 0$. Condition 5 implies that $\text{Sur } df(x_n) > \alpha > 0$ for all $x_n$ such that $f(x_n) \rightarrow y$ for some $\alpha > 0$. Then, as in Example 10, since $f$ is injective, this means that $\frac{1}{\lambda}$ is fulfilled.

In [19] Rabier gives a characterization of global diffeomorphisms in terms of sort of a generalized PS-condition which is satisfied trivially:

6) There is no sequence $\{x_n\}$ in $X$ with $f(x_n) \rightarrow y \in Y$ and $\text{Sur } df(x_n) \rightarrow 0$.

Rabier proved that a local $C^1$-diffeomorphism is a global one if and only if 6) is fulfilled. On the other hand, by Banach-Mazur-Caccipoli Theorem, the local $C^1$-diffeomorphism $f$ is a diffeomorphism onto $Y$ if and only if it is a proper map (equivalently a closed map). Summing up, if $f$ is a local $C^1$-diffeomorphism between Banach spaces then:

$$1) \Rightarrow 2) \Rightarrow \cdots 5) \Rightarrow \ast) \Rightarrow f \text{ is a norm-coercive}$$

$$\text{global diffeomorphism} \iff 6)$$

$$f \text{ is a global}$$

$$\text{diffeomorphism} \iff f \text{ is proper/closed}$$

map

Now, suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a global $C^1$-diffeomorphism, then $f$ is a proper map, and so is a norm-coercive map. Furthermore, since $x \mapsto \|df(x)^{-1}\|$ is continuous, (6) is fulfilled. Therefore, $f$ satisfies condition 4). So, 4), 5), $\ast$), and 6) are all equivalent in the finito-dimensional case.

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