THE TWO-DIMENSIONAL STRINGY BLACK-HOLE:
A NEW APPROACH AND A PATHOLOGY.

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ABSTRACT

The string propagation in the two-dimensional stringy black-hole is investigated from a new approach. We completely solve the classical and quantum string dynamics in the lorentzian and euclidean regimes. In the lorentzian case all the physics reduces to a massless scalar particle described by a Klein-Gordon type equation with a singular effective potential. The scattering matrix is found and it reproduces the results obtained by coset CFT techniques. It factorizes into two pieces: an elastic coulombian amplitude and an absorption part. In both parts, an infinite sequence of imaginary poles in the energy appear. The generic features of string propagation in curved D-dimensional backgrounds (string stretching, fall into spacetime singularities) are analyzed in the present case. A new physical phenomenon specific to the present black-hole is found: the quantum renormalization of the speed of light. We find $c_{\text{quantum}} = \sqrt{\frac{k}{k-2}} c_{\text{classical}}$, where $k$ is the integer in front of the WZW action. This feature is, however, a pathology. Only for $k \to \infty$ the pathology disappears (although the conformal anomaly is present). We analyze all the classical euclidean string solutions and exactly compute the quantum partition function. No critical Hagedorn temperature appears here.
1. Contextual Background

The space-time metric associated to the \( SL(2, R)/U(1) \) coset model (and its different versions), and interpreted as a two dimensional black hole \([1,2,3]\), arised enormous attention recently \([4,5,6]\).

In this paper, we look at this problem from our string gravity point of view, that we started and developed (see refs.[7-21]) for instance, independently and before the interest on two dimensional black holes flourished.

The two dimensional black hole model is interesting in the sense that it is an exact solution of the renormalization group equations of string theory. As it is known, these equations define the backgrounds in which strings propagate consistently.

Although two dimensional models have many attractive tractable aspects and can be used to test and get insights on particular features, \( D = 2 \) is not for string theory neither for gravity the most physically appealing dimension.

Strings in two dimensional black hole backgrounds have been extensively treated in the literature in the context of the conformal Field Theory (CFT) techniques. (See ref. [4] for a complete description). In this paper, we present a different view and a different approach to this problem which yields in a simple and physical way the full exact quantum result; giving new insights with respect the already known CFT descriptions.
2. Introduction

We study string propagation in the two-dimensional black hole background of refs.[1-3]

\[ dS^2 = k \left[ dr^2 - \tanh^2 r \, dt^2 \right], \quad \Phi = \log(\cosh^2 r), \quad 0 \leq r < \infty, \quad -\infty < t < \infty. \quad (2.1) \]

where \( k \) is a positive integer and \( \Phi \) is the dilaton field. (Classically, the string propagates in this metric, and \( \Phi \) is decoupled from them). In Kruskal null type coordinates \((u, v)\) the metric eq.(2.1) reads

\[ dS^2 = k \frac{dudv}{1-uv}, \quad 0 \leq u, v < \infty. \quad (2.2) \]

We completely solve the string equations of motion and constraints in the Lorentzian and on the Euclidean (i.e. \( t = i\Theta \)) regimes. The solutions fall into four types. In the Lorentzian regime in which there are no compact dimensions in the space-time, the \( \sigma \)-dependence can be completely gauged away, and all the solutions describe just the geodesics of a massless point particle. The only physical degree of freedom is the one associated to the center of mass of the string; we are left more with a point particle field theory rather than a string theory.

In the Euclidean regime, the Schwarzchild imaginary time is periodic, \((0 \leq \Theta \leq 2\pi)\), and then the \( \sigma \)-dependence of the string solutions remains. The solution must be also euclidean in the world sheet (\( \sigma \) becomes purely imaginary and identified to the imaginary time). There are two types of euclidean solutions: (i) a string winding \( n \) times along the \( \Theta \)-direction in the ”cigar” manifold \( dS^2 = k \left[ dr^2 + \tanh^2 r \, d\Theta^2 \right] \) from \( r = 0 \) to \( r = \infty \), and (ii) a string winding \( n \) times along \( \Theta \) in the ”trumpet” manifold \( dS^2 = k \left[ ds^2 + \coth^2 s \, d\Theta^2 \right], \quad 0 < s < \infty \). It must be noticed that these are rigid strings staying around the manifolds without any oscillation. In \( D = 2 \), \( e^{in\sigma} \) does not describes the string fluctuations, thus the name modes (”winding modes” or ”momentum modes”, as commonly refered (see for example ref.[4]), are misleading.
For these solutions, the metric on the world-sheet is

\[
\begin{align*}
    ds^2_{\text{cigar}} &= k \frac{n^2}{1 + e^{2n\tau}} (d\tau^2 + d\sigma^2) \\
    ds^2_{\text{trumpet}} &= k \frac{n^2}{1 - e^{-2n\tau}} (d\tau^2 + d\sigma^2)
\end{align*}
\] (2.3)

The proper string length (between two points \(\sigma\) and \(\sigma + d\sigma\) at fixed \(\tau\)) is

\[
\begin{align*}
    \Delta S_{\text{cigar}} &= \sqrt{k} \frac{n}{\sqrt{1 + e^{2n\tau}}} d\sigma \\
    \Delta S_{\text{trumpet}} &= \sqrt{k} \frac{n}{\sqrt{1 - e^{-2n\tau}}} d\sigma
\end{align*}
\] (2.4)

In \(D = 2\) the string world sheet and the physical space-time can be identified; the world sheet covers completely (generically, \(n\) times) the physical space. The trumpet manifold has a curvature singularity at \(s = 0\). Near \(s = 0\), i.e. \(\tau \to 0\), \(\Delta S_{\text{trumpet}}\) blows up, the string stretches infinitely. This typical feature of string instability [13-16,21], just corresponds here to the location of the string in the open extremity of the trumpet. The euclidean black hole manifold is non singular, and the string length (\(\Delta S_{\text{cigar}}\)), is always finite in it, since the cigar radius is finite. We also discuss the cosmological version of the black hole string solutions (in the euclidean regime); in this case the string solutions are the same as the trumpet solution.

We recall that in \(D = 2\) de Sitter space-time with Lorentzian signature, the general solution is a string wound \(n\) times around de Sitter space (the circle \(S^1\)) and evolving with it. For \(\tau \to 0\), \(\Delta S_{\text{deSitter}} \to \infty\); the string expands (or contracts, \(\Delta S_{\text{deSitter}} \to 0\)) with the universe itself.

It is interesting to point out the relation between \(\tau\) and the physical time: \(\tau\) interpolates between the Kruskal time (\(\tau \to 0\)) and the Schwarzschild time (\(\tau \to \infty\)). This is like strings in cosmological backgrounds in which \(\tau\) interpolates between the conformal time (\(\tau \to 0\)) and the cosmic time (\(\tau \to \infty\)). (And like strings in gravitational wave backgrounds). The logarithmic relation between the Schwarzschild time and \(\tau\) for \(\tau \to 0\) is exactly like the cosmic time and \(\tau\) in de Sitter space.
In this two dimensional black hole, the string takes an infinite time $\tau$ to approach the curvature singularity $uv = 1$, and then, never reaches it. This is to be contrasted with the higher dimensional black hole [8, 21]. For all $D > 2$, the string falls down into the physical singularity $uv = 1$ in a finite proper time $\tau$, and classically as well as quantum mechanically, the string is trapped by the singularity [21].

We study the string quantization in this background (in the lorentzian and in the euclidean regimes) and follow the spirit of ref.[7], in which one starts from the c.m. motion, and then takes the $\sigma$-dependence as fluctuations around. Here, in the lorentzian case, there are no $\sigma$-fluctuations, and the quantization of the c.m. Hamiltonian

$$\hat{H} = -\frac{1}{2k} \left( \frac{\partial^2}{\partial r^2} + \cotanh r \frac{\partial}{\partial r} - \coth^2 r \frac{\partial^2}{\partial r^2} \right) \quad (2.5)$$

yields the full and exact description of the system.

Eq.(2.5) is the same as the $L_0$ operator of ref.[4] (hereafter refered as DVV). They only differ by additive and multiplicative constants, but as operators are identically the same). We solve the eigenfunction problem

$$\hat{H}\Psi = \frac{\lambda}{2k}\Psi$$

describing the scattering of the (massless) tachyon field by the black hole, which yields a Schrödinger type equation with an energy-dependent potential

$$V_{eff} = -\frac{E^2 + 1/4}{\sinh^2 r} + \frac{1}{4 \cosh^2 r} - E(E - 1) - \lambda + 1 \quad (2.6)$$

$$V_{eff} \left. r=0 \right. = - \frac{\gamma}{r^2} + O(1) \quad , \quad \gamma \equiv E^2 + 1/4 \ .$$

Since $\gamma \geq 1/4$, there exists absorption (fall into the event horizon $r = 0$) for all energy $E$. This is like the $D > 2$ black holes.
The constant $\lambda$ is determined from the $r \to \infty$ behaviour, by requiring the tachyon to be massless

$$\lambda - 1 = (1 - c^2)p^2$$

$p$ being the momentum, $p = E/c$.

The eigenfunctions $\Psi$ are given in terms of hypergeometric functions

$$\Psi_{E,c}^\pm (r,t) = e^{-iEt} (\sinh r)^{\mp iE} \left( 1 + \frac{1}{c}, \frac{1}{2} \mp \frac{iE}{2} (1 - \frac{1}{c}); 1 \mp iE; - \sinh^2 r \right)$$

They exhibit the typical behaviour of the wave functions near the black hole horizon

$$\Psi_{E,c}^\pm (r,t) \bigg|_{r \to 0} = e^{-iE(t \mp \log r)},$$

$\Psi^+$ describing purely incoming particles at the future event horizon. (\(\Psi^-\) describes outcoming particles from the past horizon). The tachyon field incident from spatial infinity is partially absorbed and reflected by the black hole

$$\Psi_{E,c}^+(r,t) \bigg|_{r \to \infty} = e^{-ipct} \left[ e^{-ipr} + S(p,c) e^{ipr} \right]$$

We find

$$S(p,c) = S_{coul}(p) \tilde{S}(p,c)$$

where

$$S_{coul}(p) = 2^{-2ip} \frac{\Gamma(ip)}{\Gamma(-ip)} = 2^{-2ip} e^{2i\text{arg} \Gamma(ip)}$$

$$\tilde{S}(p,c) = \left[ \frac{\Gamma \left( \frac{1}{2} - \frac{ip}{2} (c+1) \right)}{\Gamma \left( \frac{1}{2} - \frac{ip}{2} (c-1) \right)} \right]^2$$

$S_{coul}$ takes into account the large $r$ interaction and the purely elastic scattering; $\tilde{S}$ describes the genuine black hole features:

$$|S_{coul}(p)| = 1$$

$$|\tilde{S}(p,c)| = \left[ \frac{\cosh \frac{ip}{2} (c-1)}{\cosh \frac{ip}{2} (c+1)} \right]^2 < 1,$$

which describes the absorption by the black hole.
$S(p, c)$ exhibits an infinity sequence of imaginary poles at the values \( ip = n, (n = 0, 1, ...), \) which are like the Coulombian-type poles, and also an infinite sequence of imaginary poles at \( ip(c + 1) = 2n + 1, \ n = 0, 1, ... \) (in the \( \tilde{S} \) part).

It must be stressed that \( S(E, c) \) depends on two physical parameters: \( E \) and \( c \). For each energy \( E \), we have a monoparametric family of solutions depending on \( c \), each \( c \) yields a different \( S \)-matrix. \( c \) is a purely quantum mechanical parameter, which is not fixed by any special requirement, and can take any value; \( c \) accounts for a renormalization of the speed of light. Classically, we choose our units such that \( c = 1 \) (see eq.(2.1) for \( r \to \infty \)). But, quantum mechanically, \( c \) is no more unit in this problem. We find that \( c \) is related to the parameter \( k \) of the WZW model:

\[
c = \sqrt{\frac{k}{k - 2}} \tag{2.12}
\]

and we reproduce the results of ref.[4]. In ref.[4], the effect of the renormalization of the speed of light is also present (although it has not been noticed). Classically, before quantization, \( DVV \) have \( c = 1 \), but after quantization, since they choose \( k = 9/4 \), they have \( c = 3 \). This can be seen from the asymptotic behaviour of the metric for \( r \to \infty \). Classically, \( dS^2(r \to \infty) = dr^2 + d\theta^2 \), and after CFT quantization of the \( SL(2, R)/U(1) \) model:

\[
dS^2_{DVV} \xrightarrow{r=\infty} 2(k - 2) \left[ dr^2 + \frac{k}{k - 2} d\theta^2 \right] = \frac{4}{c^2 - 1} [dr^2 + c^2 d\theta^2]
\]

Notice that in the classical limit \( k \to \infty \), \( c \) takes its classical value. The \( S \)-matrix of ref.[4] is a particular case of our results for \( c = 3 \):

\[
S(p, c = 3) = S_{DVV}(p) = 2^{-2ip} \frac{\Gamma(ip)}{\Gamma(-ip)} \left[ \frac{\Gamma \left( \frac{1}{2} - 2ip \right)}{\Gamma \left( \frac{1}{2} - ip \right)} \right]^2 \tag{2.13}
\]

[The factor \( 2^{-2ip} \) is missing in ref. [4].] Notice that nothing special happens, however, at the conformal invariant point \( k = 9/4 \). The physical relevance of the
$k = 9/4$ point in the two dimensional black hole scattering matrix is not clear. Also notice that the presence or absence of conformal anomalies is totally irrelevant for the black-hole singularity $uv = 1$. [The conformal anomaly vanishes for $k = \frac{9}{4}$].

The computation of the Hawking radiation follows directly from the by now well known treatment of QFT in curved space-time and does not present any particular feature here. The vacuum spectrum is a Planckian distribution at the temperature $1/2\pi$.

We also quantize the system in the euclidean regime and exactly compute the partition function of the two dimensional stringy black hole at temperature $\beta^{-1}$. It is given by

$$Z(\beta) = \frac{1}{\pi} \sum_{m,n=-\infty}^{+\infty} \sqrt{m^2 + (kn)^2 + 1} K_1\left(\frac{\beta}{2k} \sqrt{m^2 + (kn)^2 + 1}\right)$$

where $K_1(z)$ is a modified Bessel function. $Z(\beta)$ is analytic for all positive temperatures and hence no critical Hagedorn temperature appears here.

In conclusion, the quantization of the center of mass of the string, i.e. of the classical solution, with the requirement that $m^2 = 0$, yields all the physics of the problem. We have obtained the full exact quantum result without introducing any correction $1/k$ to the space-time metric, neither to the dilaton.

The two dimensional CFT constructions can be avoided in a problem like the two dimensional stringy black hole. They introduce a lot of technicality in a problem in which all the physics can be described by the straightforward quantization of a two-dimensional massless scalar particle. Perhaps the CFT tools would be really necessary for the problem of interacting (higher genus) strings in the two dimensional black hole background, problem, which unfortunately, has not been treated until now.

Finally, let us comment about the pathological feature of the renormalization of the speed of light. Normally, $c$ is never affected by quantum corrections. Once
one chooses units such that $c = 1$ classically, this value remains true in the quantum theory (relativistic quantum mechanics, QFT, string theory, etc). In the two dimensional stringy black-hole, it turns out that $c = \sqrt{\frac{k}{k-2}}$ for the quantum massless particle described by the string, whereas the wave equation in the metric (2.2) gives unit speed of propagation. Only for $k = \infty$ both speeds coincide (but the conformal anomaly is present).

We think that this pathology is specific of the two dimensional string dynamics in such curved background. More precisely, this effect is to be traced back to the dilaton background which couples with the string only at the quantum level and as a surface effect (infinite distance).

### 3. Strings in two dimensional geometries and dilaton backgrounds

The action for a string in a two dimensional space-time can be written as

$$ S = -\frac{1}{2\pi} \int d\sigma d\tau \sqrt{h} \left[ \frac{h_{\mu\nu}(\sigma, \tau)}{2} G_{AB}(X) \partial^\mu X^A(\sigma, \tau) \partial^\nu X^B(\sigma, \tau) - \frac{R^{(2)}}{4} \Phi(X) \right] + \text{surface terms} \quad A, B = 1, 2; \quad \mu, \nu = 1, 2. $$

(3.1)

where $h_{\mu\nu}$ is the metric on the world sheet, $G_{AB}(X)$ is the space time metric, $R^{(2)}$ is the world sheet curvature and $\Phi(X)$ is the dilaton field. $\Phi(X)$ and $G_{AB}(X)$ are constrained by the vanishing of the beta functions. To one loop level this implies

$$ R_{AB} = D_A D_B \Phi $$

Classically, one has the string propagating in the background $G_{AB}$, and the dilaton is decoupled from them.
We can always choose the conformal gauge on the world sheet, such that

\[ h_{\mu\nu}(\sigma, \tau) = \exp[\varphi(\sigma, \tau)] \text{ diag}(-1, +1) \]

Then,

\[ \sqrt{h} R^{(2)} = (\partial_\tau^2 - \partial_\sigma^2)\varphi(\sigma, \tau) \] (3.2)

and

\[ S = -\frac{1}{2\pi} \int d\sigma d\tau \left[ \frac{1}{2} G_{AB}(X) \partial_\mu X^A(\sigma, \tau) \partial^\mu X^B(\sigma, \tau) - \frac{1}{4} \partial^\mu \Phi(X) \partial_\mu \varphi \right]. \] (3.3)

Here \( \partial^\mu \partial_\mu \beta = -\dot{\alpha} \dot{\beta} + \alpha' \beta' \) and we made a precise choice of the surface terms. Recall that under a Weyl transformation

\[ h_{\mu\nu}(\sigma, \tau) \rightarrow e^{\lambda(\sigma, \tau)} h_{\mu\nu}(\sigma, \tau) \] (3.4)

the first term in eq.(3.1) is (classically) invariant. This is not the case of the dilaton term that transforms as

\[ \sqrt{h} R^{(2)} \rightarrow \sqrt{h} R^{(2)} + \partial^2 \lambda \]

A way out to have the Weyl invariance arises when the dilaton term is a total divergence, and that needs \( \partial_\mu \Phi = \text{constant} \).

More precisely, for a genus zero euclidean world-sheet (dominant quantum level) we can start from a flat metric on the complex plane

\[ dS^2 = dz \, d\bar{z} \] (3.5)

and, map it into the euclidean world sheet \((\sigma, \tau')\) as

\[ z = e^{\tau' + i\sigma}, \quad 0 \leq \sigma < 2\pi, \quad -\infty < \tau' < +\infty \] (3.6)

The Lorentzian world sheet follows by Wick rotation, \( \tau' = i\tau \) where \( \tau \) is the (real)
world sheet time. It follows from eqs. (3.5) and (3.6) that

\[ dS^2 = e^{2\tau'} [ d\tau'^2 + d\sigma^2 ] = e^{2i\tau} [ -d\tau^2 + d\sigma^2 ] \]  

(3.7)

That is,

\[ \varphi(\sigma, \tau) = 2i\tau \]  

(3.8)

Therefore,

\[ \frac{1}{4} \partial^\mu \Phi(X) \partial_\mu \varphi = -i \frac{\partial \Phi(X)}{2} \frac{\partial \varphi}{\partial \tau} \]

and the last term in eq. (3.3) becomes a total derivative. It is therefore irrelevant for the classical solutions. Thanks to the Weyl invariance, the action eq. (3.3) is \( \varphi \)-independent and hence, real, except for the surface term.

As is known, since the space time is two dimensional, we can always find conformal coordinates \((U, V)\) where

\[ G_{AB}(X) = e^{2\omega(U, V)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]  

(3.9)

Then, the action (3.3) takes the form

\[ S = -\frac{1}{2\pi} \int d\sigma d\tau \left[ e^{2\omega(U, V)} \partial_\mu U(\sigma, \tau) \partial^\mu V(\sigma, \tau) + i \frac{\partial \Phi(X)}{2} \frac{\partial \varphi}{\partial \tau} \right] . \]  

(3.10)

The equations of motion result given by

\[ \partial_+ \partial_- V + 2 \frac{\partial \omega}{\partial V} \partial_+ V \partial_- V = 0 , \]  

(3.11)

\[ \partial_+ \partial_- U + 2 \frac{\partial \omega}{\partial U} \partial_+ U \partial_- U = 0 , \]  

(3.12)

and the constraints are given by

\[ T_{\pm \pm} = e^{2\omega(U, V)} \partial_{\pm} U \partial_{\pm} V = 0 \]  

(3.13)
where
\[ \partial_{\pm} = \frac{1}{2}(\partial_{\sigma} \pm \partial_{\tau}) \ , \ x_{\pm} = \sigma \pm \tau \ . \]

Eqs. (3.11) - (3.13) posses the following solutions

(i)
\[ U = f(\sigma - \tau) \ , \ V = g(\sigma + \tau) \quad (3.14) \]

(ii)
\[ U = f(\sigma + \tau) \ , \ V = g(\sigma - \tau) \quad (3.15) \]

where \( f \) and \( g \) are arbitrary functions of the indicated variables.

(iii)
\[ U = u_0 = \text{constant and } V = V(\sigma, \tau) \text{ is a solution of eq.}(3.11) \quad (3.16) \]

(iv)
\[ V = v_0 = \text{constant and } U = U(\sigma, \tau) \text{ is a solution of eq.}(3.12) \quad (3.17) \]

In the case (iii), in order to solve eq.(3.11) with \( U = u_0 \), we set \( W = F(V) \).

Then,
\[ \partial_{+}\partial_{-}W = F'(V) \left[ \partial_{+}\partial_{-}V + \frac{F''(V)}{F'(V)} \partial_{+}V \partial_{-}V \right] \quad (3.18) \]

By choosing
\[ \frac{F''(V)}{F'(V)} = 2 \frac{\partial \omega(u_0, V)}{\partial V} \quad (3.19) \]

eq.(3.11) is fulfilled with \( U = u_0 \). Now, by integrating eq.(3.19) we find
\[ F(V) = A \int_{0}^{V} dv e^{2\omega(u_0, v)} + B \ , \quad \text{ (3.20)} \]

\[ W = f(\sigma - \tau) + g(\sigma + \tau) \ , \ \text{and} \ V = F^{-1}(W) \]

where \( A \) and \( B \) are arbitrary constants.
Similarly, for the solutions of type (iv) eq.(3.17) we find

\[ U = G^{-1}(W) \text{ where } \partial_+ \partial_- W = 0 \text{ and } \]
\[ G(U) = C \int_0^U du e^{2\omega(u,u_0)} + D, \quad (3.21) \]

C and D being arbitrary constants.

Let us analyze the physical content of the solutions. The solutions of type (iii) and (iv) describe geodesics propagating on the characteristics \( U = u_0 \) and \( V = v_0 \) respectively. We can make on these solutions a conformal transformation \( \sigma \pm \tau \rightarrow \phi_{\pm}(\sigma \pm \tau) \) making \( W = \tau \). Then, we obtain \( \sigma \)-independent solutions

\[ U = u_0, \quad V = F^{-1}(\tau) \quad (3.22) \]

Or,

\[ U = G^{-1}(\tau), \quad V = v_0 \quad (3.23) \]

where \( F(V) \) and \( G(U) \) are defined by eqs. (3.20) - (3.21).

In other words, the solutions of type (iii) and (iv) reduce in an appropriate gauge to the geodesics for massless particles.

Let us discuss now the type (i) - solutions. In general, we can write

\[ U(\sigma - \tau) = q_U + (\sigma - \tau) p_U + \sum_{n \neq 0} \alpha_n e^{in(\sigma-\tau)} \]
\[ V(\sigma + \tau) = q_V + (\sigma + \tau) p_V + \sum_{n \neq 0} \tilde{\alpha}_n e^{-in(\sigma+\tau)} \quad (3.24) \]

Therefore,

\[ U(\sigma + 2\pi - \tau) - U(\sigma - \tau) = 2\pi p_U, \quad V(\sigma + 2\pi + \tau) - U(\sigma + \tau) = 2\pi p_V \]

If the physical space-time does not enjoy periodic directions, we must set \( p_U = p_V = 0 \). Then, we can locally gauge away completely the \( \alpha_n \) and \( \tilde{\alpha}_n \)-dependence leaving
no oscillator degrees of freedom. This is exactly like a string in two dimensional Minkowski space-time.

When *periodic directions do exist*, the situation is different. Let us call $a$ and $b$ the periods in the coordinates $X = U - V$ and $T = U + V$ respectively. Then, periodicity requires

$$p_U = \frac{Mb + Na}{4\pi}, \quad p_V = \frac{Mb - Na}{4\pi}$$

(3.25)

where $M$ and $N$ are integers. We can call them winding numbers. If there is periodicity only on $X$, we should set $Mb = 0$ (and viceversa). That is, we can locally write the solutions as

$$T(\sigma, \tau) = q_X + \frac{Mb}{2\pi} \sigma - \frac{Na}{2\pi} \tau, \quad X(\sigma, \tau) = q_T + \frac{Na}{2\pi} \sigma - \frac{Mb}{2\pi} \tau$$

(3.26)

In conclusion, in the absence of periodical directions we are left with solutions of type (iii) and (iv). The $\sigma$-components can be completely gauged away and we are left merely with (massless) point particle geodesics. The only physical degree of freedom is the one associated to the center of mass. That is, in the absence of transverse dimensions in two dimensions, the strings has no excitations and hence, it only describes one massless scalar particle. This is more a point particle field theory than a string theory.

4. String propagation in two dimensional black hole geometries

Let us now discuss the string propagation in the two-dimensional black hole background of refs.[1-3]. In this case

$$dS^2 = k \left[dr^2 - \tanh^2 r \, dt^2\right], \quad 0 < r < \infty, \quad -\infty < t < +\infty.$$
where $k$ is a positive integer and the dilaton field takes the form

$$\phi = \log(\cosh^2 r) \quad (4.2)$$

$k$ must be an integer in order to have $e^{iS}$ univalued since it is introduced as the coefficient of a multivalued WZW term [3].

Here, $(r, t)$ are Schwarzschild type coordinates related to Kruskal coordinates $(u, v)$ by

$$u = \sinh r \ e^t, \ v = -\sinh r \ e^{-t}, \ 0 < r < \infty, \ -\infty < t < +\infty. \quad (4.3)$$

In terms of $(u, v)$, the length element eq.(4.1) takes the form

$$dS^2 = k \frac{du dv}{1 - uv}, \ 0 < u, v < +\infty. \quad (4.4)$$

The hyperbola $uv = 1$ is a singularity of the space-time. (The curvature is $R = \frac{4}{\cosh^2 r} = \frac{4}{1-uv}$). The null lines $uv = 0$ (at which $r = 0$ and $t = \pm \infty$) correspond to the event horizon.

It is convenient to define the variable $r^*$ by

$$\sinh r = e^{r^*}, \ -\infty < r^* < +\infty. \quad (4.5)$$

in terms of which the metric is written as

$$dS^2 = k \frac{e^{2r^*}}{1 + e^{2r^*}} [(dr^*)^2 - dt^2] \quad (4.6)$$

and we have simply

$$u = e^{r^*+t}, \ v = -e^{r^*-t}. \quad (4.7)$$

In the black hole background with Lorentzian signature, there are no periodic dimensions. Then, the solutions of the string equations of motion in the Lorentzian
black hole background reduce to the type (iii) and type (iv) solutions discussed in
the previous section. That is, they are just the geodesics of massless point
particles. Namely,

\[ u = u_0 = \text{const}. \quad v = \frac{1}{u_0} \left[ 1 - e^{u_0(a\tau + b)} \right] \quad (4.8) \]

Or,

\[ v = v_0 = \text{const}. \quad u = \frac{1}{v_0} \left[ 1 - e^{v_0(a\tau + b)} \right] \quad (4.9) \]

That is,

\[ r^* = \frac{1}{2} \log \left[ e^{u_0(a\tau + b)} - 1 \right], \quad t = -\frac{1}{2} \log \left[ \frac{e^{u_0(a\tau + b)} - 1}{u_0^2} \right] \quad (4.10) \]

It is interesting to discuss the relation between the world sheet time (\( \tau \)), and the
physical-Schwarzschild (\( t \)) or Kruskal (\( T = u + v \)) time. (Let us take the \( u = \text{const.} \) case). We see that :

\[ v(\tau \to 0) \to A + B\tau \quad , \quad t(\tau \to 0) \to -\frac{1}{2} \log \tau \quad (4.11) \]

and

\[ v(\tau \to \infty) \to e^{u_0a\tau} \quad , \quad t(\tau \to \infty) \to -\frac{1}{2}(a\tau + b) \quad (4.12) \]

We see that the world sheet time \( \tau \) interpolates between the Kruskal time (\( \tau \to 0 \)) and the Schwarzschild time (\( \tau \to \infty \)). This is like strings in cosmological backgrounds in which the world sheet time interpolates between conformal time (\( \tau \to 0 \)) and cosmic time (\( \tau \to \infty \)). A similar relation holds also in gravitational wave backgrounds [13]. The logarithmic relation between Schwarzschild time and \( \tau \) for \( \tau \to 0 \) in eq.(4.11) is exactly like in de Sitter space between cosmic time and \( \tau \).

More generally, \( \tau \to \infty \) characterizes the time with respect to which strings in \( D \)-
dimensional curved spacetime oscillate, then ingoing and outgoing scattering states
can be defined and the usual particle interpretation can be applied. On the other
hand, the \( \tau \to 0 \) regime characterizes the time governing the string evolution in
the strong gravitational field regime where the string does not oscillate in time.
It must be noticed that inside this two dimensional black hole, the string takes an infinite proper time $\tau$ to approach the hyperbola $uv = 1$ and then, never crosses it. In other words, the string never reaches the space-time singularity. This is to be contrasted to the higher dimensional case, in which for all $D > 2$, the string reaches the singularity in a finite proper time $\tau$. Depending on the singularity strength, the string crosses smoothly the space-time singularity, or, alternatively, it is trapped by it [13]. In the $D$-dimensional black hole the string falls down into the space-time $uv = 1$ singularity (in a finite time $\tau$), and classically, as well as quantum mechanically, the string is trapped by the singularity [21].

Another generic feature of string propagating in $D$-dimensional curved backgrounds which is lost in the $D = 2$ case with Lorentzian signature is the string stretching. As it is known [13-16,21], when strings propagate in inflationary cosmological backgrounds and near space-time singularities, the string proper length grows indefinitely. This is accompanied by a non oscillatory behaviour in time, typical of this string instability. This is not present in the $D = 2$ case with Lorentzian signature, since the transverse oscillators are absent, and the $\sigma$-dependence is gauged away, the string in $D = 2$ behaving like a point particle rather than an extended object.

5. Two dimensional Euclidean black hole.

Cigar solutions and Trumpet solutions

In the $D = 2$ black hole space-time with Lorentzian signature, in which compact directions are absent, the only string physical degree of freedom is the center of mass (point particle). In the Euclidean signature case, however, compact dimensions are present and the $\sigma$-dependence remains. Let us consider now the solutions in the euclidean regime. The Schwarzschild time is purely imaginary, namely

$$t = i\Theta \quad \text{i.e.} \quad dS^2 = k \left[dr^2 + \tanh^2 r \, d\Theta^2\right], \quad (5.1)$$
and the Kruskal coordinates are

\[ u = \sinh r \ e^{i\Theta}, \ v = -\sinh r \ e^{-i\Theta}, \quad 0 < r < \infty, \quad 0 < \Theta < 2\pi. \tag{5.2} \]

The "euclidean time" \( \Theta \) is periodic with period \( 2\pi \).

For solutions of the type (i) or (ii) [eqs. (3.14) or (3.15)] in the euclidean regime, periodicity in \( \sigma \) requires

\[ \sigma = i\hat{\sigma} = i\frac{\Theta}{n} \]

that is, the world sheet signature must also be euclidean with \( \hat{\sigma} \) playing the rôle of the euclidean time - (i.e., \( t = n\sigma \)).

The solution is given by

\[ u = e^{n(\tau + i\hat{\sigma})}, \quad v = -e^{n(\tau - i\hat{\sigma})} \tag{5.3} \]

that is,

\[ \sinh r = e^{n\tau}, \quad \text{i.e.} \quad r^* = n\tau; \quad \Theta = n\hat{\sigma}, \tag{5.4} \]

with the dilaton given by

\[ \Phi = \log[1 + e^{2n\tau}]. \tag{5.5} \]

This is a non singular real manifold with metric

\[ dS^2 = k \frac{n^2}{1 + e^{-2n\tau}} (d\tau^2 + d\hat{\sigma}^2) \tag{5.6} \]

For \( \tau \to \infty \) the metric is flat.

\( \Theta = n\hat{\sigma} \) are winding configurations. They correspond to the so-called "momentum modes", in the terminology of the literature (see ref.[4]), since here, \( \hat{\sigma} \) is an imaginary time. The physical interpretation of these modes is a string winding
$n$ times along the angular $\Theta$ direction, lying in the semi-infinite cigar from $r = 0$ to $r = \infty$. At $\tau = -\infty$, the string is at $r = 0$ (the horizon is shrinked to a point in the euclidean space). Notice, that this is a rigid string configuration staying around the cigar without any oscillations. In $D = 2$, $e^{in\tilde{\sigma}}$ does not describes string fluctuations as in the $D > 2$ cases, thus the name modes ("winding modes" or "momentum modes") referring to strings in $D = 2$ are misleading.

Also notice that in the $D = 2$ euclidean regime, the string world sheet covers completely the target space; world sheet and physical space became the same; (for $n = 1$, both are identified). This is characteristic of strings in $D = 2$ compact spaces. Recall that in $D = 2$ de Sitter space [14], the general solution of the string equations of motion describes a string winding $n$ times around de Sitter universe (the circle $S^1$) and evolving with it. In that case, the winding string solution naturally exists in the Lorentzian signature regime in both target and world sheet space-times; (it can be also extended to the Euclidean regime). The winding string de Sitter solution is given by [14]

$$dS^2_{\text{deSitter}} = \frac{1}{H^2} \left[ -du^2 + \cosh^2 u \ dv^2 \right]$$ (5.7)

with $v = n\sigma$, $u = \log(\tan \frac{n\tau}{2})$, $0 < \sigma \leq 2\pi$, $0 < \tau \leq \frac{2\pi}{n}$.

That is,

$$dS^2_{\text{deSitter}} = \frac{n^2}{H^2 \sin^2 n\tau} (d\sigma^2 - d\tau^2)$$ (5.8)

[here $(u, v)$ are the coordinates on the de Sitter hyperboloid].

In the black hole background, winding string solutions are only possible in the Euclidean signature regime.

It must be noticed that the transformation

$$r = s + \frac{i\pi}{2}, \quad \Theta = \Theta$$ (5.9)
maps the manifold eq. (5.1) into the dual or "trumpet" manifold

\[ dS^2 = k \left[ ds^2 + \coth^2 s \, d\Theta^2 \right] \] (5.10)

This is a real manifold with curvature singularity at \( s = 0 \), \((uv = 1)\). The Kruskal coordinates are complex conjugate one to another, i.e.

\[ u = \cosh s \, e^{i(\Theta + \frac{\pi}{2})} \quad , \quad v = \cosh s \, e^{-i(\Theta + \frac{\pi}{2})} \] (5.11)

The string solution eqs.(5.3) - (5.4) transforms under eq.(5.9) into the different real solution

\[ \cosh s = e^{n\tau} \quad , \quad \Theta = n\hat{\sigma} \] (5.12)

(the dilaton here is \( \Phi = \log[1 - e^{2n\tau}] \)). That is,

\[ u = e^{n(\tau + i\hat{\sigma}) + i\pi/2} \quad , \quad v = e^{n(\tau - i\hat{\sigma}) - i\pi/2} \] (5.13)

For this solution, the metric is

\[ dS^2 = k \frac{n^2}{1 - e^{-2n\tau}} \left( d\tau^2 + d\hat{\sigma}^2 \right) \] (5.14)

The proper string length (between two points \( \hat{\sigma} \) and \( \hat{\sigma} + d\hat{\sigma} \) at fixed \( \tau \)) is given by

\[ \Delta S_{\text{trumpet}} = \sqrt{k} \frac{n}{\sqrt{1 - e^{2n\tau}}} \, d\hat{\sigma} \] (5.15)

Near the curvature singularity \( s \rightarrow 0 \) (i.e. \( \tau \rightarrow 0 \)), \( \Delta S \) blows up, the string stretches indefinitely, corresponding to the location of the string in the open extremity of the trumpet.
5.1. Cosmological solutions

The cosmological version of the black hole solution eq.(4.1) is just obtained by setting
\[ r = iT , \quad t = x \] (5.16)
in the original metric. The cosmological background is described by
\[ dS^2 = k \left[ -dT^2 + \tan^2 T dx^2 \right] , \]
\[ \Phi = -\log \sin T , \quad 0 < T < \pi/2 , \quad -\infty < x < +\infty . \] (5.17)
which is singular in \( uv = 1 \), i.e. \( T = \pi/2 \). Here
\[ u = i \sin T \ e^x , \quad v = -i \sin T \ e^{-x} \] (5.18)
It is conveniently written in the variable \( T^* \)
\[ \sin T = e^{T^*} , \quad -\infty < T^* < 0 \]
i.e.
\[ dS^2 = \frac{1}{1 - e^{-2T^*}} \left[ (dT^*)^2 + dx^2 \right] \] (5.19)
The analysis of the string solutions done in section IV for the black hole, holds here with the corresponding mapping eq.(5.16). There are no solutions describing a string winding \( n \) times in the Lorentzian regime. In the Euclidean regime
\[ x = i \Theta , \quad 0 \leq \Theta < 2\pi , \]
\[ -dS^2 = \frac{1}{1 - e^{-2T^*}} \left[ (dT^*)^2 + d\Theta^2 \right] . \] (5.20)
We have the solution
\[ T^* = n\tau , \quad \Theta = n\hat{\sigma} , \quad -\infty < \tau \leq 0 , \quad 0 \leq \hat{\sigma} < \frac{2\pi}{n} \]
That is
\[ u = i e^{T^* + i\Theta} = i e^{n(\tau + i\hat{\sigma})}, \]
\[ v = -i e^{T^* - i\Theta} = -i e^{n(\tau - i\hat{\sigma})}, \] (5.21)
describes a string winding \( n \) times in the euclidean cosmology. This solution yields
\[ dS^2 = k \frac{n^2}{1 - e^{-2n\tau}} (d\tau^2 + d\hat{\sigma}^2). \] (5.22)
The string length blows up at \( \tau \rightarrow 0 \).

The cosmological version of the trumpet manifold is obtained by setting
\[ T = -iU + \pi/2, \] i.e.
\[ dS^2 = k \left[ ds^2 + \coth^2 U d\Theta^2 \right] = k \frac{1}{1 - e^{-2U\tau}} \left[ (dU^*)^2 + d\Theta^2 \right] \] (5.23)
with
\[ u = \cosh U \ e^{i\theta} = e^{U^* + i\theta}, \quad v = \cosh U \ e^{-i\theta} = e^{U^* - i\theta} \]

The winding string solution here is given by
\[ U^* = n\tau, \quad \Theta = n\hat{\sigma}, \quad -\infty < \tau < 0, \quad 0 < \hat{\sigma} < \frac{2\pi}{n} \] (5.24)
The solutions found in sections (IV) and (V) exhaust all the real string solutions in the lorentzian and euclidean regimes. The \( \sigma \)-dependent solutions exist only in the euclidean regime. These solutions are of the type (i) and (ii) [eqs.(3.14) and (3.15)]. There are no real \( \sigma \)-dependent euclidean solutions of the type (iii) and (iv) [eqs.(3.16) and (3.17)].

There are two types of euclidean string solutions: cigar solutions and trumpet solutions. These correspond to a string winding \( n \) times in the cigar and in the trumpet manifolds respectively. In the cosmological version of the black hole manifold (in the euclidean regime) the string solutions are the same as the trumpet solutions.
6. String quantization in the two dimensional black-hole

In the Lorentzian two dimensional black hole background eqs.(4.1) - (4.2) the action eq.(3.3) takes the form

\[ S = -\frac{1}{2\pi} \int d\sigma d\tau \left\{ \frac{k}{2} \left[ \dot{r}^2 - r'^2 - \tanh^2 r \left( \dot{t}^2 - t'^2 \right) \right] - i \dot{r} \tanh r \right\} \]  

(6.1)

where we also used eq.(3.7) for the world sheet metric. Notice that the dilaton field leaves only a surface term analogous to a background charge in conformal field theory [22].

The Hamiltonian associated to the action (6.1) has the form

\[ H = \frac{1}{2k} \left[ \left( p_r - i \tanh r \right)^2 - p_t^2 \coth^2 r \right] + \frac{k}{2} \left( r'^2 - \tanh^2 r \ t'^2 \right) \]  

(6.2)

As previously discussed, in absence of periodic directions in the space-time, the parameter \( \sigma \) can be classically gauged away. Since this is the case for black holes with Lorentzian signature, we can drop the second term in eq.(6.2) and take

\[ H = \frac{1}{2k} \left[ \left( p_r - i \tanh r \right)^2 - p_t^2 \coth^2 r \right] \]  

(6.3)

Let us now consider the quantum theory. We will follow the spirit of ref.[7] where one starts from the center of mass motion and then, takes into account the \( \sigma \)-dependence as fluctuations around. Here, in \( D = 2 \), there are no physical fluctuations, so we can expect that the quantum version of the center of mass hamiltonian eq.(6.3) should give the full description of the model.

In order to quantize this system, we use the canonical prescription

\[ p_r \rightarrow \hat{p}_r = -i \frac{\partial}{\partial r} , \quad p_t \rightarrow \hat{p}_t = -i \frac{\partial}{\partial t} \]  

(6.4)

Now, eq.(6.3) suffers from ordering ambiguities at the quantum level. We choose the following ordering prescription. The classical piece \( p_r^2 - p_t^2 \coth^2 r \) in eq. (6.3)
is replaced by the D’Alambertian in this space-time:

\[ \partial^2 = \frac{1}{\tanh r} \frac{\partial}{\partial r} \left( \tanh r \frac{\partial}{\partial r} \right) - \coth^2 r \frac{\partial^2}{\partial t^2} \]  

\( \text{(6.5)} \)

The operator \( \partial^2 \) is self adjoint with respect to the integration measure \( \sqrt{G} = k \tanh r \).

The double product \(-2ip_r \tanh r\) in eq.(6.3) is ordered symmetrically

\[ -2ip_r \tanh r \rightarrow -i(\tanh r \, \hat{p}_r + \hat{p}_r \tanh r) \]  

\( \text{(6.6)} \)

The last term \(-\tanh^2 r\) does not present ordering problem. We find in this way

\[ \mathcal{H} \rightarrow \hat{\mathcal{H}} = -\frac{1}{2k} \left( \frac{\partial^2}{\partial r^2} + (\coth r + \tanh r) \frac{\partial}{\partial r} - \coth^2 r \frac{\partial^2}{\partial t^2} \right) \]  

\( \text{(6.7)} \)

Let us write

\[ \mathcal{H} \rightarrow \hat{\mathcal{H}} = \frac{1}{2k} \hbar \quad , \quad \partial_t = -iE \]

then,

\[ h = \left[ \frac{d^2}{dr^2} + 2 \coth(2r) \frac{d}{dr} + E^2 \coth^2 r \right], \]  

\( \text{(6.8)} \)

and the eigenvalue problem

\[ h\Psi = \lambda\Psi \]  

\( \text{(6.9)} \)

yields

\[ \Psi'' + 2 \coth(2r)\Psi' + (E^2 \coth^2 r + \lambda)\Psi = 0 \]  

\( \text{(6.10)} \)

The constant \( \lambda \) describes the zero point quantum fluctuations and it will be determined below; (the prime ’ stands for \( \frac{d}{dr} \)). Eq.(6.10) describes the interaction of
the tachyon with the black hole geometry. It is convenient to set here

$$\Psi = \frac{\chi}{\sqrt{\sinh 2r}}$$  \hspace{1cm} (6.11)

then, eq.(6.10) reads

$$\chi'' + \left[ \frac{E^2 + \frac{1}{4}}{\sinh^2 r} - \frac{1}{4 \cosh^2 r} + E^2 + \lambda - 1 \right] \chi = 0$$  \hspace{1cm} (6.12)

which is a Schrödinger type equation with the energy-dependent effective potential

$$V_{eff} = -\frac{E^2 + \frac{1}{4}}{\sinh^2 r} + \frac{1}{4 \cosh^2 r} + E^2 - \lambda + 1 + E$$  \hspace{1cm} (6.13)

$$V_{eff \rightarrow 0} = -\frac{E^2 + 1/4}{r^2} + O(1)$$

For $r \rightarrow 0$, this is a strongly attractive singular potential $-\frac{\gamma}{r^2}$. Since $\gamma > \frac{1}{4}$ there is absorption (fall into the event horizon $r = 0$) for all energy $E$, even $E = 0$. This is exactly like the horizon behaviour of the effective potential describing the interaction of the black hole with a free scalar field in $D = 4$, where $\gamma = E^2 + 1/4$ too.

The Hamiltonian eq.(6.8), is the same Hamiltonian $L_0$ as in ref.[4]. In order to compare with ref.[4], (hereafter referred as DVV), please notice that

$$r_{DVV} = 2r$$ \hspace{1cm} $E = 2\omega_{DVV}$

Then, eq.(4.11) in DVV reads

$$L_0^{DVV} = -\frac{1}{k-2} \left[ \frac{d^2}{dr^2} + (\coth r + \tanh r) \frac{d}{dr} - \left( \coth^2 r - \frac{2}{k} \right) \frac{\partial^2}{\partial t^2} \right]$$

(6.15)

and the condition $(L_0 - 1)\chi$ yields

$$\left[ \frac{d^2}{dr^2} + \frac{E^2 + \frac{1}{4}}{\sinh^2 r} - \frac{1}{4 \cosh^2 r} + E^2 + 1 + 4(k - 2) \right] \chi = 0$$

(6.16)

$\hat{H}$ in eq. (6.7) and the $L_0^{DVV}$ only differ by additive and multiplicative constants. As operators they are identically the same.
Let us now determine the constant $\lambda$. It can be determined from the behaviour at \( r \to \infty \) of the solution. For \( r \to \infty \), from eq.(6.10) we have

\[
\Psi'' + 2\Psi' + (E^2 + \lambda)\Psi = 0 \quad (6.17)
\]

And then, (including the time dependence of the total solution):

\[
\Psi(r \to \infty, t) = e^{i(p r - E t)} \quad (6.18)
\]

with the momentum \( p = \sqrt{E^2 + \lambda - 1} \). The parameter $\lambda$ becomes determined by requiring the tachyon to be massless. In order to describe a relativistic and massless particle, (the string ground state particle is massless in $D = 2$ due to the absence of transverse dimensions), it must be $E = cp$, that is $\lambda - 1 = (1 - c^2)p^2$.

For comparison with DVV, notice that for \( r \to \infty \), DVV have

\[
\Psi'' + 2\Psi' + E^2(1 - \frac{2}{k})\Psi = 0 \quad (6.19)
\]

which corresponds to have $\lambda = 1 - 2E^2/k$. That is, eqs.(6.17) and (6.19) yield the relation

\[
c = \sqrt{\frac{k}{k - 2}} \quad (6.20)
\]

We will comment about the interpretation of $\lambda$ and this relation below.

In order to describe the solutions of eq.(6.10), it is useful to define the variable

\[ z \equiv \cosh^2 r \]

and to express $\Psi$, as

\[
\Psi_E(z) = (z - 1)^{iE/2} \psi_E(z) \quad (6.21)
\]

Then, eq.(6.10) reads

\[
z(1 - z)\psi_E(z)'' + [1 - z(iE + 2)]\psi_E(z)' - \left(\frac{iE}{2} + \frac{\lambda}{4}\right)\psi_E(z) = 0 \quad (6.22)
\]

whose solutions are the standard hypergeometric functions.
The general solution is given by

$$\Psi(r,t) = \left[ A U_{E,c}(r) + B U_{E,c}(r)^* \right] e^{-iEt} \quad (6.23)$$

where $A$ and $B$ are arbitrary constants and

$$U_{E,c}(r) = (\sinh r)^{-iEF} \left( \frac{1}{2} - \frac{iE}{2}(1 + \frac{1}{c}), \frac{1}{2} - \frac{iE}{2}(1 - \frac{1}{c}); 1 - iE ; - \sinh^2 r \right) \quad (6.24)$$

These solutions describe the scattering of the (massless) tachyon by the black hole geometry. The solutions $U_{E,c}$ and $U_{E,c}^*$ exhibit the typical behaviour of the wave functions near the black hole horizon, namely

$$U_{E,c}(r) e^{-iEt} \left. \right|_{r=0} = e^{-iE(t+\log r)} \quad , \quad U_{E,c}(r)^* e^{-iEt} \left. \right|_{r=0} = e^{-iE(t-\log r)} \quad (6.25)$$

$U_{E,c}(r,t)$ corresponds to have purely incoming particles at the (future) horizon. $U_{E,c}^*$ describes purely outcoming particles from the (past) horizon. The solution $U_{E,c}$ describes the physical process in which the tachyon field coming in from spatial infinity is partially reflected and partially absorbed by the black hole. The asymptotic behaviour of $U_{E,c}(r,t)$ for $r \to \infty$ is given by

$$U_{E,c}(r) e^{-iEt} \left. \right|_{r=\infty} = e^{-iEt} \left[ e^{-iEr/c} + S(E,c)e^{iEr/c} \right]. \quad (6.26)$$

From eq.(6.24) we obtain for the scattering matrix $S(E,c)$:

$$S(E,c) = 2^{-2iE/c} \frac{\Gamma(iE/c)}{\Gamma(-iE/c)} \left[ \frac{\Gamma \left( \frac{1}{2} - \frac{iE}{2}(1 + \frac{1}{c}) \right)}{\Gamma \left( \frac{1}{2} - \frac{iE}{2}(1 - \frac{1}{c}) \right)} \right]^2 \quad (6.27)$$

or, in terms of $p$ eqs.(2.10) . Notice that the S-matrix factorizes into two parts, one which is a pure phase, equal to the coulombian S-matrix, and another one which
describes both absorption and reflexion, namely

\[ S(p, c) = S_{coul}(p) \tilde{S}(p, c) \]

where

\[ S_{coul}(p) = 2^{-2ip} \frac{\Gamma(ip)}{\Gamma(-ip)} = 2^{-2ip} e^{2i\arg(\Gamma(ip))} \]

\[ \tilde{S}(p, c) = \left[ \frac{\Gamma\left(\frac{1}{2} - \frac{ip}{2}(c+1)\right)}{\Gamma\left(\frac{1}{2} - \frac{ip}{2}(c-1)\right)} \right]^2 \]

(6.28)

\( S_{coul} \) takes into account the large \( r \) interaction while \( \tilde{S} \) describes the genuine black hole features. Notice that

\[ |S_{coul}(p)| = 1 \]

\[ |\tilde{S}(p, c)| = \left[ \frac{\cosh \frac{\pi p}{2}(c-1)}{\cosh \frac{\pi p}{2}(c+1)} \right]^2 < 1 , \]

(6.29)

which describes the absorption by the black hole. Also notice that \( S(E, c) \) exhibits imaginary poles. An infinite sequence of purely imaginary poles occur for the values \( ip = n \) \( (n = 0, 1, ...) \) which are like the Coulombian-type poles (describing the relativistic hydrogen bound-state spectrum). In addition, the part \( \tilde{S}(E, c) \) exhibits an infinity sequence of purely imaginary poles at \( ip(c+1) = 2n + 1, n = 0, 1, 2, ... \).

It must be stressed that the scattering matrix \( S(E, c) \) depends on two parameters \( E \) and \( c \). That is, for each energy we have a monoparametric family of well defined solutions depending on the parameter \( c \). For each value of \( c \), we have a different S-matrix. It must be noticed that \( c \) is a purely quantum mechanical parameter which is not fixed at this level by any special requirement, \( c \) is a physical parameter which accounts for a renormalization of the speed of light. At the classical level, \( c \) can take any value, (we have taken \( c = 1 \)), the physics does not depend on it. But, quantum mechanically, \( c \) is not more equal to 1 in this problem.
The parameter \(c\) is related to the parameter \(k\) of the conformal field theory construction, \(c = \sqrt{\frac{k}{k-2}}\) [see eq.(6.20)]. In particular, for \(k = 9/4\), i.e. \(c = 3\) we reproduce the results of ref.[4]. In this respect, notice that in ref.[4], the effect of the renormalization of the speed of light is also present, (although it has not been noticed there). Classically, before quantization, DVV have \(c = 1\), and after quantization (since they choose \(k = 9/4\)), they have \(c = 3\). This can be seen from the asymptotic behaviour of the metric for \(r \to \infty\).

Classically,

\[
ds^2 = k \left[ dr^2 + \tanh^2 r d\Theta^2 \right] \quad \overset{r \to \infty}{\longrightarrow} \quad dr^2 + d\Theta^2
\]

After quantization of the euclidean black hole CFT coset \(SL(2,R)/U(1)\) model:

\[
dS^2_{DVV} = \frac{1}{2}(k-2) \left[ dr^2_{DVV} + \beta^2(r_{DVV}) \ d\Theta^2 \right] ,
\]

where \(\beta(r) = 2 \left( \coth^2 \frac{r}{2} - \frac{2}{k} \right)^{-1/2}\) i.e. \(\beta(\infty) = \frac{2}{\sqrt{1 - 2/k}}\), \(6.30\)

\[
dS^2_{DVV} \overset{r \to \infty}{\longrightarrow} \frac{1}{2}(k-2) \left[ dr^2_{DVV} + \frac{4k}{k-2} d\Theta^2 \right] ,
\]

Recall that \(r_{DVV} = 2r\), i.e.

\[
dS^2_{DVV} \overset{r \to \infty}{\longrightarrow} 2(k-2) \left[ dr^2 + \frac{k}{k-2} d\Theta^2 \right] = \frac{4}{c^2 - 1} \left[ dr^2 + c^2 d\Theta^2 \right] \quad 6.31
\]

We see that in ref. [4], it also appears that \(c^2 = \frac{k}{k-2}\) in particular there, \(k_{DVV} = 9/4\). Notice that nothing special happens at the conformal invariant point \(k = 9/4\). The physical relevance of the \(k = 9/4\) point in the two-dimensional black hole scattering matrix is not clear. The S-matrix of ref.[4] is a particular case of our results for \(c = 3\). The S-matrix, eq.(5.22) of ref.[4] is given by

\[
S_{DVV}(p) = 2^{-2ip} \frac{\Gamma(ip)}{\Gamma(-ip)} \left[ \frac{\Gamma\left(\frac{1}{2} - 2ip\right)}{\Gamma\left(\frac{1}{2} - ip\right)} \right]^2 \quad 6.32
\]
[The factor $2^{-2i\nu}$ is missing in ref. [4]. From eq.(6.28), we see that

$$S(p, c = 3) = S_{D\nu V}(p)$$

(6.33)

In conclusion, in this problem, the quantization of the center of mass of the string, that is the quantization of the classical solution, together with the requirement that $m^2 = 0$, yields all the physics of the problem. [In this sense, this is like the light cone gauge for strings in flat spacetime for D=26]. We have obtained the complete quantum result without introducing any correction $1/k$ to the space-time metric, neither to the dilaton field.

The 2-dimensional conformal field theory construction can be avoided for such a simple problem like the two-dimensional stringy black hole. They introduce a lot of technicality in a problem in which all the physics is described by the standard quantization of a 2 dimensional massless scalar particle. Perhaps, the CFT tools would be really necessary for the problem of interacting (higher genus) strings in the two dimensional black hole background, a problem which, unfortunately, has not been treated until now.

Finally, let us comment about the only non trivial physical feature, perhaps, appearing in this two dimensional stringy black hole problem, namely the renormalization of the speed of light. Normally, the speed of light is never affected by quantum corrections. That is, once one chooses units such that $c = 1$ classically, this value remains true in the quantum theory (relativistic quantum mechanics, or quantum field theory or string theory, etc...). In the present case, $c = 1$, for the classical propagation in the background eq.(2.1)), while $c = \sqrt{\frac{k}{k^2 - 2}}$ for the quantum propagation of the massless particle described by the string. We think that this pathology is specific of the two dimensional string dynamics in such curved backgrounds. Only for $k \to \infty$, the pathology disappears (but the conformal anomaly
is present). In that case:

\[ S(p, c = 1) = S_{\text{coul}}(p) \frac{1}{\pi} \left[ \Gamma \left( \frac{1}{2} - ip \right) \right]^2, \]  
\[ |S(p, c = 1)| = \frac{1}{\cosh^2 \pi p}. \]  

6.1. Quantum Partition Function

Let us now consider the quantum statistical mechanics for this system by computing its partition function \( Z = \text{Tr} \left[ e^{-\beta \hat{H}_E} \right] \), where \( \hat{H}_E \) is the euclidean version of the Lagrangian in eq.(6.1). In the euclidean regime (\( t = i\Theta \)) the variable \( \Theta \) is periodic with period \( 2\pi \) and in an appropriate conformal gauge, generic configurations take the form

\[ \Theta(\sigma, \tau) = n\hat{\sigma} + \hat{\Theta}(\tau) \]  

(6.35)

The whole \( \sigma \)-dependence can be gauged away from \( r \). Thus, using eq.(6.35), we find an extra term in eq.(6.1):

\[ \frac{1}{2} kn^2 \tanh^2 r \]

This term plus the Wick rotated version of \( \hat{H} \) [eq.(6.7)] yields the euclidean hamiltonian

\[ \hat{H}_E = \frac{1}{2k} h_E \]

with

\[ h_E = -\frac{\partial^2}{\partial r^2} - 2 \coth(2r) \frac{\partial}{\partial r} - \coth^2 r \frac{\partial^2}{\partial \Theta^2} + (kn)^2 \tanh^2 r \]  

(6.36)

The eigenfunctions of \( h_E \) take the form

\[ W(r, \Theta) = e^{im\Theta} w_{m,n}(r) \]

where \( m \in \mathbb{Z} \) in order to have \( 2\pi \) periodicity and \( w_{m,n}(r) \) is a solution of the eigen-
value equation

\[
\left[ -\frac{d^2}{dr^2} - 2 \coth(2r) \frac{d}{dr} + m^2 \coth^2 r + (kn)^2 \tanh^2 r \right] w_{m,n}(r) = \Lambda w_{m,n}(r)
\]  

(6.37)

Regular solutions behave as

\[
w_{m,n}(r) \overset{r \to 0}{=} r |m| , \quad w_{m,n}(r) \overset{r \to \infty}{=} e^{-r} \left[ e^{ipr} + S_{m,n}(p) \right]
\]  

(6.38)

where \( p \in \mathcal{R} \) is the euclidean momentum. The eigenvalues \( \Lambda \) of \( h_E \) on such eigenfunctions are

\[
\Lambda_{p,m,n} = \sqrt{p^2 + m^2 + (kn)^2 + 1} \quad n, m \in \mathbb{Z} , \quad p \in \mathcal{R}.
\]

Then, the partition function results:

\[
Z(\beta) = Tr \left[ e^{-\beta \hat{H}_E} \right] = \sum_{m,n=-\infty}^{+\infty}
\]

The integral over \( p \) can be computed with the result

\[
Z(\beta) = \frac{1}{\pi} \sum_{m,n=-\infty}^{+\infty} \sqrt{m^2 + (kn)^2 + 1} K_1 \left( \frac{\beta}{2k} \sqrt{m^2 + (kn)^2 + 1} \right)
\]  

(6.39)

There is a manifest duality invariance \( m \leftrightarrow kn \).

For large \( \beta \) (low temperatures), \( Z(\beta) \) decreases exponentially as

\[
Z(\beta) \overset{\beta \to \infty}{=} \sqrt{\frac{k}{\pi \beta}} e^{-\frac{\beta}{2\pi}}
\]

due to the presence of a gap in the eigenvalues \( \Lambda_{p,m,n} \geq 1 \). For high temperatures, \( Z(\beta) \) grows as \( \beta^{-3} \):

\[
Z(\beta) \overset{\beta \to 0}{=} \frac{16k^2}{\beta^3}
\]

Notice that \( Z(\beta) \) is analytic for all \( \infty \geq \beta > 0 \) and hence no phase transition shows up here for finite temperatures. The absence of a Hagedorn critical temperature is due to the lack of transverse string modes in two dimensions.
7. Hawking radiation

The computation of the Hawking radiation follows directly from the standard treatment of the by now well known QFT in curved space time and does not present any particular problem here. For the sake of completeness we point out the essential points to be considered and which allow to avoid unnecessary complications. The formulation of quantum (point particle) field theory, as well as quantum string theory in non trivial spacetime exhibits new fundamental features with respect to the usual formulation in flat space time: (i) the possibility for a given field or string theory to have different alternative well defined ground states, and thus different Fock spaces or "sectors" of the theory. (ii) the presence of intrinsic statistical features (temperature, entropy) arising from the non trivial structure (geometry, topology) of the space time and not from a superimposed statistical description of the quantum matter fields or strings.

An important step in field as well as string quantization is the definition of positive frequency states and its associated ground state. Different possible choices of the physical time lead to different alternative positive frequency basis.

The modes $U_{E,c}(r,t)$ (see eq.(6.24)) define positive frequencies with respect to the Schwarschild time and are the analogous of the Rindler modes for an uniformly accelerated observer $r = \text{const}$. Creation and anhiliation operators $a_E$, $a_E^*$ are associated to these modes which define the ground state $|0_S>$ or "Schwarzschild" ground state.

The modes $U_{E,c}$ are however, a linear combination of positive and negative frequency modes with respect to the Kruskal time $T$. For a locally inertial observer near the horizon or Kruskal observer, the positive frequency basis is

$$\varphi_k \equiv 0 \left\{ \frac{1}{2\sqrt{\pi k}} e^{-iku}, \frac{1}{2\sqrt{\pi k}} e^{-ikv} \right\}, \quad k > 0,$$

with respect to which the global or Kruskal ground state is defined $\alpha_k |0_K> = 0$. 

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The Bogoliubov coefficients relating such two basis, namely,
\[ \varphi_k = A_{kE} U_{E,c}(r,t) + B_{kE} U^*_{E,c}(r,t) \]  
[ or \( \alpha_k = A_{kE} a_E + B_{kE} a^*_E \),
are defined by the scalar products
\[ A_{kE} = \langle U_E, \varphi^*_k \rangle, \quad B_{kE} = \langle U_E, \varphi_k \rangle, \]
and explicitly given by
\[ A_{kE} = e^{\pi E} B_{kE}, \quad B_{kE} \overset{k \to \infty}{=} -\frac{\sqrt{E}}{2\pi \sqrt{k}} \frac{k^iE}{\Gamma(-iE)} e^{-\pi E/2} \]
The behaviour of the basis \( \{U_E\} \) and \( \{\varphi_k\} \) near the horizon (and thus, the \( k \to \infty \) behaviour of the Bogoliubov coefficients ) is enough to determine the relevant quantities of interest. The expectation value of the Schwarzschild number operator in the Kruskal vacuum is determined by the function
\[ N(E,E') \equiv \langle 0_K | \alpha^*_E \alpha_{E'} | 0_K \rangle = \int_{-\infty}^\infty dk \ B^*_{kE} B_{kE'} \]  
(7.1)
The \( E \to E' \) limit is determined by the \( k \to \infty \) behavior of eq.(7.1), namely
\[ N(E,E') = N_V(E) \delta(E - E') , \quad N_V(E) = \frac{1}{e^{2\pi E} - 1} \]
\( N_V(E) \) is the number of created modes per unit frequency and per unit volume of the Schwarzschild manifold, at temperature \( T = 1/(2\pi) \).

This can be seen directly from the mapping eq.(4.7) \( u = f(u') \), i.e.,
\[ u = e^{r^*+t} , \quad v = -e^{r^*-t} \]
which defines the maximal extension of the manifold [23]. This mapping is holomorphic in the complex variable \( r^* - i\Theta \) and has a critical point defining the event horizon, i.e. \( f'(-\infty) = 0 \). The inverse of the periodicity in the imaginary time \( \Theta \) is the temperature characterizing the vacuum spectrum \( N_V(\lambda) \).
A complete classification of the vacuum spectra and thermal properties in a large class of curved and accelerated manifolds can be done directly in terms of the properties of the holomorphic mappings $u = f(u')$ defining the maximal analytic extension of the manifold, of which the exponential mapping eq. (4.7) is just a particular case, ($u$ being global or Kruskal-like coordinates, $u'$ being accelerated or Schwarzschild-like ones) [23].

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