RESEARCH ARTICLE

Lie Bialgebroid of Pseudo-differential Operators

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Abstract
We associate a Lie bialgebroid structure to the algebra of formal Pseudo-differential operators, as the classical limit of a quantum groupoid. As a consequence, the non-commutative Kadomtsev–Petviashvili hierarchy is naturally obtained by an algebraic procedure.

Keywords Lie algebroid · Lie bialgebroid · Quantum groupoid · Deformation · Pseudo-differential operators · Kadomtsev–Petviashvili hierarchy

1 Introduction

Consider the formal pseudo-differential operator $L$ as

$$L = \frac{\partial}{\partial x} + u_1 \left( \frac{\partial}{\partial x} \right)^{-1} + u_2 \left( \frac{\partial}{\partial x} \right)^{-2} + \ldots,$$

where $u_1, u_2, \ldots$ are infinite number of variables which are dependent variables of systems of partial differential equations. The classical Kadomtsev–Petviashvili (KP) hierarchy is the infinite system of equations

$$\partial_{x_m} L = [(L^m)_+, L], \quad m = 1, 2, 3, \ldots,$$

where $(L^m)_+$ denotes the projection of the pseudo-differential operator $L^m$ into the space of differential operators, and $x_1, x_2, \ldots$ denote independent variables.

The non-commutative KP hierarchy usually has been defined (see e.g. [11]) using non-commutativity of the coordinates

$$[x_k, x_l] = i\theta^{kl},$$

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in Pseudo-differential operators, for the independent variables \( x_k \). Here \( \theta^{k\ell} \) are real constants and called the non-commutative parameters. Hence, the non-commutative KP hierarchy is described by

\[
\partial_{x_i} L = [(L^n)_+, L],
\]

where the star-product \( \star \) is represented by the Moyal product.

In this note, we are going to show that one can obtain naturally the non-commutative structure on KP equations from some algebraic data. To this end we explain the relation between Lie bialgebroids and quantum groupoids from work of Xu in [23], i.e. “ deformation of a bialgebroid ” (known as a quantum groupoid) induces a Lie bialgebroid as a classical limit. For this, during Sects. 2, 3, and 4 we review several algebraic structures such as bialgebroids, Lie bialgebras and Lie bialgebroids with their examples. Gathering together these concepts helps to understand the relation between them (especially for a non-familiar reader to these subjects) and provides the ingredients which we use later in the last sections. Then, in Sect. 5 we explain deformation of bialgebroids based on [23]. In Sect. 6 we use the result of Xu, to associate a Lie bialgebroid to the algebra of formal Pseudo-differential operators. In Sect. 7 we present non-commutative KP equations based on quantum groupoid associated to the algebra of formal Pseudo-differential operators. Solving KP equations, in particular showing existence and uniqueness of solutions of KP equations has been an interesting subject of study. For instance, in a recent work in [18] (also see references there in), the Cauchy problem of the KP hierarchy has been solved and formulated in several non-standard cases such as non-commutative case described by Moyal product. Now, with the algebraic construction in this paper, one can expect that the non-commutative KP equations can be solved more generally.

1.1 Preliminaries

Consider an associative algebra as a triple \((A, \mu, \eta)\) where \( A \) is a vector space over a field \( k \), the map \( \mu : A \otimes A \to A \) denotes the multiplication, and \( \eta : k \to A \) denotes the unit of the algebra. The maps \( \mu \) and \( \eta \) are linear and satisfy the following associative properties.

- \( \mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu) \)
- \( \mu \circ (\eta \otimes id) = \mu \circ (id \otimes \eta) = id. \)

If \( A \) is commutative then we also have \( \mu \circ \tau_{AA} = \mu \) where \( \tau_{AA} \) is the flip switching, that is \( \tau_{AA}(a \otimes a') = a' \otimes a \). A morphism between algebras \((A, \mu, \eta)\) and \((A', \mu, \eta')\) is a linear map \( f : A \to A' \) such that \( \mu' \circ (f \otimes f) = f \circ \mu \) and \( f \circ \eta = \eta' \).

The notion of coalgebra is dual to the notion of algebra in the following sense.

**Definition 1** A co-associative coalgebra is a triple \((C, \Delta, \varepsilon)\) where \( C \) is a vector space over a field \( k \) and the maps \( \Delta : C \to C \otimes C \) and \( \varepsilon : C \to k \) are linear satisfying the co-associative properties
• \((id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta\)
• \((id \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes id) \circ \Delta = id\).

If \(C\) is co-commutative then we also have \(\tau_{c,c} \circ \Delta = \Delta\). A morphism between coalgebras \((C, \Delta, \varepsilon)\) and \((C', \Delta', \varepsilon')\) is a linear map \(f : C \rightarrow C'\) such that \((f \otimes f) \circ \Delta = \Delta' \circ f\) and \(\varepsilon = \varepsilon' \circ f\).

**Definition 2** Suppose \(H\) is a vector space over the field \(k\) equipped with an algebra structure \((H, \mu, \eta)\) and a coalgebra structure \((H, \Delta, \varepsilon)\) then \((H, \mu, \eta, \Delta, \varepsilon)\) is a bialgebra verifying the compatibility conditions between these two structures i.e.,

1. The maps \(\mu\) and \(\eta\) are morphisms of coalgebras, or equivalently
2. the maps \(\Delta\) and \(\varepsilon\) are morphisms of algebras.

Given an algebra \((A, \mu, \eta)\) and a coalgebra \((C, \Delta, \varepsilon)\) one can define a bilinear map called the **convolution** on the vector space \(\text{Hom}(C, A)\), denoted by \(\star : \text{Hom}(C, A) \rightarrow \text{Hom}(C, A)\), of linear maps from \(C\) to \(A\). If \(f, g\) are such linear maps then the convolution \(f \star g\) is the composition of the following maps

\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.
\]

Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a bialgebra. An endomorphism \(S\) of \(H\) is called an **antipode** for the bialgebra \(H\) if \(S \star id_H = id_H \star S = \eta \varepsilon\).

**Remark 1** A bialgebra does not necessarily have an antipode, but if it does, it has only one. Because if \(S'\) is another antipode then we have

\[
S = S \star (\eta \varepsilon) = S \star (id_H \star S') = (S \star id_H) \star S' = (\eta \varepsilon) \star S' = S'.
\]

**Definition 3** A **Hopf algebra** is a bialgebra with antipode, and is denoted by \((H, \mu, \eta, \Delta, \varepsilon, S)\). A morphism of Hopf algebras is a morphism between the underlying bialgebras commuting with the antipode.

**Example 1** Any algebra \(A\) is the quotient of a free algebra \(K[X]\) where \(K\) is a field and \(X\) is a generating set. Since, it suffices to take any generating set \(X\) and \(A = K[X]/I\) where \(I\) is a two sided ideal of \(K[X]\).

Let \(X = \{x_1, \ldots, x_n\}\) and let \(I\) be the two sided ideal of \(K[x_1, \ldots, x_n]\) generated by elements of the form \(x_ix_j - x_jx_i\) where \(i, j\) run over all integers from 1 to \(n\). The quotient algebra \(K[x_1, \ldots, x_n]/I\) is isomorphic to the polynomial algebra \(K[x_1, \ldots, x_n]\) in \(n\) variables with coefficient in the ground field \(K\).

The structure \((K[x], \mu, \eta, \Delta, \varepsilon, S)\) is a Hopf algebra where the algebra \(K[x]\) is called **affine line** and \(\mu\) is the multiplication and \(\eta\) is the unit of \(K[x]\). The co-multiplication, co-unit and antipode are defined by
\[ \Delta : K[x] \to K[x', x'']; \quad \Delta(x) = x' + x'' \]
\[ \varepsilon : K[x] \to K; \quad \varepsilon(x) = 0 \]
\[ S : K[x] \to K[x]; \quad S(x) = -x. \]

If \( X = G \) is a finite group with commutative product, then \( K[G] \) is a bialgebra with co-multiplication
\[ \Delta : k[G] \to k[G] \otimes k[G] \]
and co-unit \( \varepsilon(x) = 1 \). Also, if \( X = G \) is a finite group with convolution product \( (f \star g)(x) = \sum_{i,j} f(x_i)g(x_j) \), where \( x_1 x_2 = x \), then \( K[G] \) is a bialgebra with \( \Delta(x) = x \otimes x \), and \( \varepsilon(x) = 1 \).

**Example 2** Let \( I \) be the two sided ideal of \( K \{x, y\} \) generated by \( xy - yx \), the quotient algebra \( K \{x, y\}/I \) is isomorphic to the polynomial algebra \( K[x, y] \) with two variables \( x, y \) with coefficients in the ground field \( K \). The algebra \( K \{x, y\} \) is called **affine plane**. In the affine plane if we define the two sided ideal \( I_q = \langle xy - qyx \rangle \) then we obtain a new algebra \( K \{x, y\}/I_q = K_q[x, y] \) which is clearly non-commutative and is called **quantum plane**. In this algebra instead of the relation \( xy = yx \) in commutative case we have \( xy = qyx \). This is a bialgebra equipped with
\[ \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes 1 + x \otimes y \]
\[ \varepsilon(x) = 1, \quad \varepsilon(y) = 0. \]

**Example 3** Let \( V \) be a vector space. Define \( T^0(V) = k, T^1(V) = V, T^n(V) = V^\otimes n \) (The tensor product of \( n \) copies of \( V \)) if \( n > 1 \).

The canonical isomorphisms \( T^n(V) \otimes T^m(V) \cong T^{n+m}(V) \) induce an associative product
\[ (x_1 \otimes \ldots \otimes x_n)(x_{n+1} \otimes \ldots \otimes x_{n+m}) = x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} \otimes \ldots \otimes x_{n+m} \]
on the vector space \( T(V) = \bigoplus_{n \geq 0} T^n(V) \). The unit for this product is the image of the unit element 1 in \( k = T^0(V) \). The vector space \( T(V) \) equipped with this algebra structure, is called the **tensor algebra** of \( V \). If \( V \) is a vector space, the **symmetric algebra** \( S(V) = T(V)/I(V) \) of the tensor algebra \( T(V) \) by the two sided ideal \( I(V) \) generated by all elements \( xy - yx \) where \( x, y \) run over \( V \).

To any Lie algebra \( L \) we can assign an (associative) algebra \( U(L) \) called **enveloping algebra** of \( L \), with a morphism of Lie algebras \( i_L : L \to \text{Lie}(U(L)) \). More precisely, let \( I(L) \) be the two sided ideal of the tensor algebra \( T(L) \) generated by all elements of the form \( xy - yx - [x, y] \) where \( x, y \in L \) and \( [\cdot, \cdot] \) is the Lie bracket. We define \( U(L) := T(L)/I(L) \). If \( L \) is a Lie algebra then the **universal enveloping algebra** of \( L \), denoted by \( U(L) \) is a Hopf algebra equipped with co-multiplication
\[ \Delta : U(L) \to U(L) \otimes U(L) \]
\[ \Delta(X) = X \otimes 1 + 1 \otimes X, \]
and with co-unit and antipode

\[ \epsilon : U(L) \to k, \quad \epsilon(X) = 0, \]
\[ S : U(L) \to U(L), \quad S(X) = -X, \]

for \( X \in L \). As an example, for the Lie algebra \( \mathfrak{sl}(2) \) of traceless \( 2 \times 2 \) matrices, the enveloping algebra \( U(\mathfrak{sl}(2)) \) is a Hopf algebra, see \cite{12}. Moreover, we can construct a Hopf algebra \( U_q(\mathfrak{sl}(2)) \) which is a one-parameter deformation of the enveloping algebra of the Lie algebra \( \mathfrak{sl}(2) \).

2 Bialgebroids

**Definition 4** A bialgebroid is a pair of algebras \( A \) and \( R \) together with an algebra homomorphism \( \alpha : R \to A \), an algebra anti-homomorphism \( \beta : R \to A \) such that the images of \( \alpha \) and \( \beta \) commute in \( A \). i.e. \( \alpha(r_1)\beta(r_2) = \beta(r_2)\alpha(r_1) \), for each \( r_1, r_2 \in R \). By these two morphisms we can assign an \( (R, R) \)-bimodule structure on \( A \) in natural way by \( r.a = \alpha(r)a \) and \( a.r = \beta(r)a, r \in R, a \in A \). With this bimodule structure on \( A \) we can also consider \( A \otimes_R A \) as an \( (R, R) \)-bimodule which is given as a left module by \( r.(a_1 \otimes a_2) = \alpha(r)a_1 \otimes a_2 \) and as a right module by \( (a_1 \otimes a_2)r = a_1 \otimes \beta(r)a_2, r \in R, a_1, a_2 \in A \).

We define the co-product \( \Delta : A \to A \otimes_R A \) and the co-unit \( \epsilon : A \to R \) as \( (R, R) \)-bimodule maps satisfying the co-associativity axiom for \( \Delta \) and co-unit as below

- \((\text{id}_A \otimes_R \Delta) \circ \Delta = (\Delta \otimes_R \text{id}_A) \circ \Delta \)
- \(\epsilon(1_A) = 1_R, \quad \epsilon \beta = \epsilon \alpha = \text{id}_R,\) and \((\text{id}_A \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}_A) \circ \Delta = \text{id}_A\).

The co-product \( \Delta \) and the algebra structure on \( A \) should be compatible in the sense that the kernel of the following map

\[ \Phi : A \otimes A \otimes A \to A \otimes_R A \]
\[ a_1 \otimes a_2 \otimes a_3 \mapsto (\Delta a_1)(a_1 \otimes a_2) \]

is a left ideal of \( A \otimes A^{\text{op}} \otimes A^{\text{op}} \). Here we are using the fact that \( A \otimes A \) acts on \( A \otimes_R A \) from the right by right multiplications. Also we need the co-unit map is compatible with the algebra structure on \( A \) in the sense that the kernel of \( \epsilon \) is a left ideal of \( A \). We denote a bialgebroid by \((A, R, \alpha, \beta, \Delta, \epsilon)\) where \( A \) is called total algebra, \( R \) is called the base algebra, and the maps \( \alpha, \beta \) are called source map and target map respectively.

Recall that a groupoid \( G \) over a set \( G_0 \) is a set together with a pair of structure maps \( s : G \to G_0 \) and \( t : G \to G_0 \), where \( s \) is called source and \( t \) is called target. Moreover, \( G \) is equipped with a product, identity, and an inversion, (for more details see for example \cite{4}). In fact, we can think of an element \( g \in G \) as an arrow (morphism) from \( x = s(g) \) to \( y = t(g) \) in \( G_0 \).
Example 4 If we consider a groupoid $\mathcal{G}$ over a set $X = \{x\}$ then the source map and target map are as constant maps $s, t : G \to \{x\}$, $s(g) = t(g) = x$ for each $g \in G$. The arrows or elements of $\mathcal{G}$ are as $G = s^{-1}(x) \to t^{-1}(x) = G$. We can consider them as $G \to G$ such that $h \mapsto gh$ for each $h, g \in G$. Also the unit for our groupoid is $\varepsilon : X \to G$ with $\varepsilon(x) = e$ ($e$ is the identity element of the group $G$). The inverse map is $i : G \to G$ with $g \mapsto g^{-1}$. The multiplication map

$$m : \mathcal{G}(2) = \mathcal{G} \times_{\{x\}} \mathcal{G} \to \mathcal{G}$$

where $\mathcal{G}(2) = \{(g, h) \in G \times G \mid s(g) = t(h)\} = G \times G$, is in fact the multiplication of the group $G$.

Now, we consider the space of smooth functions on $G$, i.e. $C^\infty(G)$ as the total algebra and $R$ the real numbers as the base algebra. We define $\alpha = s^*$ and $\beta = t^*$ such that

$$s^* = t^* : R = C^\infty(\{x\}) \to C^\infty(G).$$

It is clear that $(s^*f)(g) = f \circ s(g) = f(x) \in R$. In other words $s^*$ assigns to each $r \in R$ the constant function $r \in C^\infty(G)$. So $s^*(r) = r$ similarly $t^*(r) = r$ for $r \in R$.

Moreover,

$$\varepsilon^* : C^\infty(G) \to C^\infty(\{x\}) = R$$

with $(\varepsilon^*f)(x) = f \circ \varepsilon(x) = f(e)$. It is clear that $(\varepsilon^*id) = id(e) = 1 \in R$ hence $\varepsilon(1_{C^\infty(G)}) = 1$. Therefore $\varepsilon^*$ can be considered as the co-unit. Finally,

$$m^* : C^\infty(G) \to C^\infty(G) \otimes C^\infty(G)$$

defines the co-product, with $\Delta(f) = f \otimes f$. Clearly $C^\infty(G)$ is an $(R, R)$-bimodule. It is easy to check that $\varepsilon, \Delta$ are $(R, R)$-bimodule maps and they are compatible with the algebra structure on $C^\infty(G)$. Therefore, we constructed a bialgebroid $(C^\infty(G), R, s^*, t^*, \Delta, \varepsilon)$ by the groupoid $\mathcal{G}$ which is the group $G$.

Recall that a smooth map $f : M \to N$ between smooth manifolds $M$ and $N$ is a local diffeomorphism (or étale map) if $(df)_x$ is an isomorphism for any $x \in M$. An étale groupoid is a groupoid with the source map $s$ as a local diffeomorphism. For more details and examples see [19].

Example 5 Let $G_1 \xrightarrow{\Delta} G_0$ be an étale groupoid. We define

$$\alpha := s^*, \beta := t^* : C^\infty(G_0) \to C^\infty(G_1)$$

and we consider the co-unit as $\varepsilon^* : C^\infty(G_1) \to C^\infty(G_0)$. The multiplication of groupoid is

$$m : G_2 = G_1 \times_{G_0} G_1 \to G_1$$

where $G_2 = \{(g, h) \in G_1 \mid s(g) = t(h)\}$. So, the pull-back of this multiplication, i.e.
\[ \Delta = m^* : \mathcal{C}_c^\infty(G_1) \rightarrow \mathcal{C}_c^\infty(G_2) = \mathcal{C}_c^\infty(G_1 \times_{G_0} G_1) \cong C^\infty(G_1) \otimes_{\mathcal{C}_c^\infty(G_0)} C^\infty(G_1) \]
defines a co-product. Hence, \( C^\infty(G_1) \) over \( C^\infty(G_0) \) is a bialgebroid.

**Example 6** Again consider the étale groupoid \( \xymatrix{ G \ar[r] & G_0 } \). The Connes algebra \( C_c^\infty(G) \) of (smooth) complex (or real) functions with compact support on \( G \) (see [2, 6, 7]), together with the base algebra \( (C_c^\infty(G))_0 \) defines a bialgebroid, where \( (C_c^\infty(G))_0 \) is the subalgebra of \( C_c^\infty(G) \) of functions with support in \( G_0 \subset G \). This subalgebra is commutative and may be identified with the commutative algebra \( C_c^\infty(G_0) \).

Considering the inclusion \( i : C_c^\infty(G_0) \hookrightarrow C_c^\infty(G) \) we take \( \alpha = \beta = i \). The product is the convolution

\[ (aa')(g'') = \sum_{gg' = g''} a(g)a'(g'), \forall a, a' \in C_c^\infty(G), g'' \in G, \]

where the sum is over all possible decompositions of \( g'' \in G \). We take the co-unit

\[ \varepsilon : C_c^\infty(G) \rightarrow C_c^\infty(G_0) \]

by

\[ \varepsilon(a)(x) = \sum_{s(g) = x} a(g), \forall a \in C_c^\infty(G), x \in G_0. \]

This sum is over all the elements \( g \) of \( G \) which satisfy \( s(g) = x \). The coproduct \( \Delta : C_c^\infty(G) \rightarrow C_c^\infty(G) \otimes_{\mathcal{C}_c^\infty(G_0)} C_c^\infty(G) \) is defined as follows: Let \( d : G \rightarrow G' \times_{G_0} G \) be the diagonal open embedding, i.e. \( d(g) = (g, g) \). This map gives the inclusion \( C_c^\infty(G) \rightarrow C_c^\infty(G \times_{G_0} G) \). We define \( \Delta \) as the composition of this inclusion with the inverse of the isomorphism

\[ \Omega : C_c^\infty(G) \otimes_{\mathcal{C}_c^\infty(G_0)} C_c^\infty(G) \rightarrow C_c^\infty(G \times_{G_0} G) \]

\[ \Omega(a \otimes a')(g, g') = a(g)\xi(g') \]

(For the proof that \( \Omega \) is an isomorphism see [20]).

If \( a \in C_c^\infty(G) \) has the support in an open subset \( U \) of \( G \) which is so small that \( s|_U \) is injective, then \( \Delta(a) = a \otimes \xi = \xi \otimes a \) where \( \xi \) is any smooth function with compact support in \( U \) which constantly equals 1 on the support of \( a \). The functions \( a \in C_c^\infty(G) \) which satisfy the above condition generate the linear space \( C_c^\infty(G) \).

**Example 7** 1) If \( A \) is any algebra then there is a bialgebroid structure on \( H = A \otimes A^{op} \) over \( A \) with

1. \( \Delta : H \rightarrow H \otimes A H, \, a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b) \),
2. \( \varepsilon : H \rightarrow A, \, a \otimes b \mapsto ab \),
3. \( \alpha : A \rightarrow H, \, a \mapsto a \otimes 1, \, \beta : A \rightarrow H, \, a \mapsto 1 \otimes a \).
2) For any finite dimensional algebra over $k$, the algebra of $k$-linear maps from $A$ to itself, i.e. $H = \text{End}_k(A)$, has a bialgebroid structure over $A$, see [15].

**Definition 5** A morphism between two bialgebroids $(A, R, \alpha, \beta, \Delta, \epsilon)$ and $(A', R', \alpha', \beta', \Delta', \epsilon')$ consists of an algebra morphism $T: A \to A'$ and an algebra morphism $t: R \to R'$ which commute with all the structure maps.

As for Hopf algebras we expect that a Hopf algebroid is a bialgebroid with antipode.

**Definition 6** The antipode in a bialgebroid $(A, R, \alpha, \beta, \Delta, \epsilon)$ is a bijective map $\tau: A \to A$ which has the following properties

1. $\tau$ is an algebra anti-isomorphism for $A$.
2. $\tau\beta = \alpha$
3. $\mu(\tau \otimes \text{id})\Delta = \beta\epsilon\tau: A \to A$, (it is the same as the definition of antipode in section 1.)
4. There exists a linear map $\gamma : A \otimes_k A \to A \otimes A$ with:

   (a) $\gamma$ is a section for the natural projection $p : A \otimes A \to A \otimes_k A$
   (b) $\mu(\text{id} \otimes \tau)\gamma\Delta = \alpha\epsilon : A \to A$.

**Proposition 1** The maps $\tau: A \to A$ and $\text{id}: R \to R$ define a bialgebroid morphism.

**Proof** In the above definition, condition (1) means that for all $x, y \in A$, $\tau(xy) = \tau(y)\tau(x)$ and condition (2) means that $\tau(\beta(1_R)) = \alpha(1_R)$ or $\tau(1) = 1$. On the other hand $(\tau \otimes \tau)\Delta = \Delta^{\text{op}}\tau$, and $\epsilon\circ\tau = \epsilon$. Moreover by condition (2) $\tau$ commutes with structure maps. Clearly $\text{id}$ is a morphism of algebra and it commutes with the structure maps. $\square$

**Proposition 2** Bialgebras and bialgebroids are equivalent over a field $k$ as a base algebra.

**Proof** For any bialgebra $(A, \mu, \eta, \Delta, \epsilon)$ we can consider $R = k$, where $k$ is the ground field for the algebra $A$, and $\alpha, \beta : k \to A$ by $\alpha(r) = r$ and $\beta(r) = r$, $(r \in k)$. Hence, $(A, k, \alpha, \beta, \Delta, \epsilon)$ is a bialgebroid because for $r, s \in k$ we have $\alpha(rs) = rs = \alpha(r)\alpha(s)$ so $\alpha$ is an algebra homomorphism, also $\beta(rs) = rs = sr = \beta(s)\beta(r)$ so $\beta$ is an anti-homomorphism, and finally $\alpha(r)\beta(s) = rs = sr = \beta(s)\alpha(r)$ so the images of $\alpha$ and $\beta$ commute. Conversely, if $R$ is the field $k$ the definition of a bialgebroid over $k$ is reduced to a bialgebra over $k$, see [15]. $\square$

**Example 8** 1. If $L$ is a Lie algebra then the universal enveloping algebra of $L$, that is $U(L)$ can be considered as a bialgebroid by the above Proposition 2. Consider Hopf algebras $SL_q(2)$ and $U_q(sl(2))$ as mentioned before (there is a duality between
SL(2) and U($\mathfrak{sl}(2))$, see [12]). By the above proposition they are also examples of bialgebroids.

Let $(A, R, \alpha, \beta, \Delta, \tau)$ be a Hopf algebroid over the field $k$ of characteristic zero. And let $\text{End}_k R$ be the algebra of linear endomorphisms of $R$ over $k$. When $R$ acts on $\text{End}_k R$ from the left by left multiplication and acts from the right by right multiplication, we have $\text{End}_k R$ as an $(R, R)$-bimodule. Assume that $R$ is a left $A$-module and moreover the representation $\rho : A \to \text{End}_k R$ is an $(R, R)$-bimodule map. We define $\phi_\alpha, \phi_\beta : (A \otimes_R A) \otimes R \to A$ by

\[
\phi_\alpha(x \otimes_R y \otimes a) = \rho(x)(a)y \\
\phi_\beta(x \otimes_R y \otimes a) = x.\rho(y)(a)
\]

where $x, y \in A, a \in R$.

**Definition 7** Given a bialgebroid $(A, R, \alpha, \beta, \Delta, \epsilon)$, an anchor map is a representation $\rho : A \to \text{End}_k R$ which is an $(R, R)$-bimodule map such that

1. $\phi_\alpha(\Delta x \otimes a) = x\alpha(a)$ and $\phi_\beta(\Delta x \otimes a) = x\beta(a)$, $\forall x \in A, a \in R$,
2. $x(1_R) = \epsilon(x)$, $\forall x \in A$.

For a bialgebra $R = k$ and $\text{End}_k R \cong R$, clearly we can take the co-unit as the anchor. In general, for a bialgebroid, the existence of an anchor map is stronger than the existence of a co-unit.

**Example 9** Let $P$ be a smooth manifold and $D$ be the algebra of differential operators on $P$. Let $R$ be the algebra of smooth functions on $P$. Then $(D, R, \alpha, \beta, \Delta, \epsilon)$ is a bialgebroid where $\alpha = \beta$ is the embedding $R \to D$ and $\Delta : D \to D \otimes_R D$ is defined by

\[
\Delta(d)(f, g) = df(g), \forall d \in D
\]

and $f, g \in R$. Also, the usual action of differential operators on $C^\infty(P)$ defines an anchor $\mu : D \to \text{End}_k R$. The co-unit $\epsilon : D \to R$ is the natural projection to its 0-order part of a differential operator.

Below we see a construction of new bialgebroid from a given bialgebroid, which is called the twist construction. First we need the following proposition.

**Proposition 3** (Xu, Proposition 4.2 in [23]) Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid with anchor $\mu$. Let $F \in H \otimes_R H$ and define $\alpha_F, \beta_F : R \to H$ by

\[
\alpha_F(a) = \phi_\alpha(F \otimes a), \quad \beta_F(a) = \phi_\beta(F \otimes a), \quad \forall a \in R,
\]

and for any $a, b \in R$, set $a *_F b = \alpha_F(a)(b)$. Now assume that $F$ satisfies:

\[
(\Delta \otimes_R \text{id})FF^{12} = (\text{id} \otimes_R \Delta)FF^{23} \quad \text{in} \quad H \otimes_R H \otimes_R H,
\] (1)

and
\[(\varepsilon \otimes_R id)F = 1_H, (id \otimes_R \varepsilon)F = 1_H \quad (2)\]

where \(F^{12} = F \otimes 1 \in (H \otimes_R H) \otimes H\), \(F^{23} = 1 \otimes F \in H \otimes (H \otimes_R H)\). Then

- \((R, \ast_F)\) is an associative algebra, and \(1_R \ast_F a = a \ast_F 1_R = a, \forall a \in R\).
- \(\alpha_F : R_F \to H\) is an algebra homomorphism, and \(\beta_F : R_F \to H\) is an algebra anti-homomorphism. Here \(R_F\) stands for the algebra \((R, \ast_F)\).
- \((\alpha_F a)(\beta_F b) = (\beta_F b)(\alpha_F a), \forall a, b \in R\).

**Definition 8** Let \(M_1\) and \(M_2\) be given left \(H\)-modules. If the linear map

\[F^\#: M_1 \otimes_{R_F} M_2 \to M_1 \otimes_R M_2\]

(for \(m_1 \in M_1\) and \(m_2 \in M_2\)) is an isomorphism of vector spaces, we say that \(F\) is **invertible**. An element \(F \in H \otimes_R H\) is called a **twistor** if it is invertible and satisfies Eq. (1) and (2) in above proposition.

**Theorem 1** (Xu, Theorem 4.14 in [23]) Assume that \((H, R, \alpha, \beta, \Delta, \varepsilon)\) is a bialgebroid with anchor \(\mu\), and \(F \in H \otimes_R H\) is a twistor. Then \((H, R_F, \alpha_F, \beta_F, \Delta_F, \varepsilon)\) is a bialgebroid which still admits \(\mu\) as an anchor.

**Example 10** ([23]) Consider a smooth manifold \(P\) with the algebra \(D\) of differential operators on it, and \(R = C^\infty(P)\). We denote the space of formal power series in \(\hbar\) with coefficients in \(D\) by \(D[[\hbar]]\). The Hopf algebroid structure on \(D\) naturally can be extended to a Hopf algebroid structure on \(D[[\hbar]]\) over the base algebra \(R[[\hbar]]\), and it admits a natural anchor map. One can check that

\[F = 1 \otimes_R 1 + \hbar B_1 + \ldots \in D \otimes_R D[[\hbar]](\cong D[[\hbar]] \otimes_R D[[\hbar]] D[[\hbar]])\]

as a formal power series of bidifferential operators is a twistor if and only if the multiplication on \(R[[\hbar]]\) defined by

\[f \ast_h g = F(f, g), \forall f, g \in R[[\hbar]]\]

is associative with identity being the constant function 1, in other words, \(\ast_h\) is a star product on \(P\). Hence, the bracket

\[\{f, g\} = B_1(f, g) - B_1(g, f), \forall f, g \in C^\infty(P),\]

defines a Poisson structure on \(P\) and \(f \ast_h g = F(f, g)\) is a deformation quantization of this Poisson structure.

Now, set \(D_h = D[[\hbar]]\) equipped with the usual multiplication, \(R_h = R[[\hbar]]\) equipped with the \(\ast\)-product defined above. Consider \(\alpha_h : R_h \to D_h\) and \(\beta_h : R_h \to D_h\) given by

\[\alpha_h \quad \text{and} \quad \beta_h\]
Moreover, consider the co-product \( \Delta_h : D_h \to D_h \otimes_R D_h \) defined by \( \Delta_h = F^{-1} \Delta F \), and co-unit as the projection \( D_h \to R_h \). By Theorem 1 we obtain twisted Hopf algebroid \((D_h, R_h, \alpha_h, \beta_h, m, \Delta_h, \varepsilon)\). This twisted Hopf algebroid is called the quantum groupoid associated to the star product \(*_h\).

### 3 Lie Bialgebras

**Definition 9** Let \( G \) be a Lie group and \( \rho : G \to GL(V) \) be the representation of \( G \) on the vector space \( V \). Also, let \( d\rho : \mathfrak{g} \to End(V) \) be the infinitesimal representation of the group representation. then

- \( \varphi : G \to V \) is a 1-cocycle on \( G \) if \( \varphi(gh) = \varphi(g) + \rho(g)\varphi(h) \).
- \( \phi : \mathfrak{g} \to V \) is a 1-cocycle on \( \mathfrak{g} \) if \( \phi([u, v]) = d\rho(u)\phi(v) - d\rho(v)\phi(u) \).

Note that if \( \varphi \) is 1-cocycle on \( G \) then \( \phi = d_e\varphi \) is 1-cocycle on \( \mathfrak{g} \). Also, if \( G \) is simply connected then the 1-cocycle on \( \mathfrak{g} \) can be integrated to 1-cocycle on \( G \).

**Definition 10** Let \( \mathfrak{g} \) be a Lie algebra. A Lie bialgebra structure on \( \mathfrak{g} \) is a linear map \( \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \), called co-commutator, such that

1. \( \delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^* \) is a Lie bracket on \( \mathfrak{g}^* \)
2. \( \delta_{\mathfrak{g}} \) is a 1-cocycle of \( \mathfrak{g} \) with values in \( \mathfrak{g} \otimes \mathfrak{g} \).

A homomorphism of Lie bialgebras \( \varphi : \mathfrak{g} \to \mathfrak{h} \) is a homomorphism of Lie algebras such that \( (\varphi \otimes \varphi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \varphi \).

**Definition 11** Let \( G \) be a Lie group. A Poisson Lie group is \((G, \Pi)\) where \( \Pi \) is a Poisson structure such that \( m : G \times G \to G \) is Poisson map. In this case \( \Pi \) is called multiplicative.

**Remark 2** A Poisson structure \( \Pi \) is multiplicative if and only if \( \Pi_{gh} = (l_g)_*\Pi_h + (r_h)_*\Pi_g \), \( g, h \in G \), where \( l_g \) and \( r_h \) are left and right translations, see [5].

**Example 11** Any Lie group can be seen as a Poisson Lie group with \( \Pi = 0 \).

**Example 12** Let \( \mathfrak{g} \) be a finite dimensional Lie algebra with Lie bracket \([\cdot, \cdot]_{\mathfrak{g}}\). Each \( \mu \in \mathfrak{g}^* \) is a function on \( \mathfrak{g} \). Given functions \( f, g \in C^\infty(\mathfrak{g}^*) \) the new function \( \{f, g\} \in C^\infty(\mathfrak{g}^*) \) evaluated at \( \mu \) is \( \{f, g\}(\mu) = \mu([Df(\mu), Dg(\mu)]_{\mathfrak{g}}) \). Then \( \mathfrak{g}^* \) as an abelian Lie group (with addition) is a Poisson Lie group with this Poisson structure (Poisson bracket).
The following proposition gives an example of Lie bialgebras.

**Proposition 4** Let \((G, \Pi)\) be a Poisson Lie group with the Lie algebra \(\mathfrak{g}\). Also, let \(F : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}\) be 1-cocycle, and \(F^*\) Lie bracket then \((\mathfrak{g}, \mathfrak{g}^*)\) is a Lie bialgebra.

**Proof** Let \((G, \Pi)\) be a Poisson Lie group, and \(F = \Pi(1)\) the linear part of \(\Pi\) at the point \(e\). The map \(\Pi^{(1)} : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}\) is linear Poisson structure on \(\mathfrak{g}\) where \(\mathfrak{g} = T_eG\). Then \(F^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*\) is a Lie algebra structure on \(\mathfrak{g}^*\). In fact, for \(\Pi : G \to \bigwedge^2 TG\), if we consider \(\bar{\Pi} : G \to \bigwedge^2 \mathfrak{g}\) for which \(g \mapsto (r_g)_* \Pi_g\), then We can write \(\bar{\Pi}^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*\) is a Lie algebra structure on \(\mathfrak{g}^*\). In fact, for \(\Pi : G \to \bigwedge^2 TG\), if we consider \(\bar{\Pi} : G \to \bigwedge^2 \mathfrak{g}\) for which \(g \mapsto (r_g)_* \Pi_g\), then We can write \(\Pi^{(1)} = d_e \bar{\Pi}\). Note that, as we mentioned in the previous remark, \(\Pi\) is multiplicative if and only if \(\bar{\Pi}\) satisfies \(\bar{\Pi}(gh) = \bar{\Pi}(g) + Ad_g \bar{\Pi}(h)\). By above explanation, \(F\) satisfies \(F([u, v]) = ad_u F(v) - ad_v F(u)\). So \(F\) is 1-cocycle on \(\mathfrak{g}\) with values in \(\bigwedge^2 \mathfrak{g}\). That is if \((G, \Pi)\) is a Poisson Lie group then \(F = d_e \bar{\Pi} : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}\) such that

1. \(F^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*\) is Lie bracket. (equivalently, \(\Pi\) is Poisson.)
2. \(F\) is a 1-cocycle. (equivalently, \(\Pi\) is multiplicative.)

**Lemma 1** Let \(\Lambda\) be a bivector in the Lie algebra, i.e. \(\Lambda \in \bigwedge^2 \mathfrak{g}\) and \(\Pi_g = (l_g)_* \Lambda - (r_g)_* \Lambda\). Then \(\Pi\) is multiplicative.

**Proof** We have

\[
(l_g)_* \Pi_h + (r_h)_* \Pi_g = (l_g)_* ((l_h)_* \Lambda - (r_h)_* \Lambda) + (r_h)_* ((l_g)_* \Lambda - (r_g)_* \Lambda)
= (l_{gh})_* \Lambda - (l_g)_* (r_h)_* \Lambda + (r_h)_* (l_g)_* \Lambda - (r_g)_* \Lambda
= \Pi_{gh}.
\]

Hence, by Remark 2 in above, \(\Pi\) is multiplicative.

**Proposition 5** \(\Pi\) is Poisson structure if and only if \([\Lambda, \Lambda] \in \bigwedge^3 \mathfrak{g}\) is Ad-invariant.

**Proof** By the above lemma we write \(\Pi = \Lambda' - \Lambda''\). Moreover, a bivector field \(\Pi\) is a Poisson bivector field if and only if \([\Pi, \Pi] = 0\). So, we have

\[
[\Pi, \Pi] = 0 \iff [\Lambda' - \Lambda'', \Lambda' - \Lambda''] = [\Lambda', \Lambda'] + [\Lambda'', \Lambda''] = 0
\]

\[
\iff [\Lambda, \Lambda]' = [\Lambda, \Lambda]''
\]

\[
\iff Ad_g [\Lambda, \Lambda] = [\Lambda, \Lambda].
\]

For more details see [5] or [22].
4 Lie Bialgebroids

**Definition 13** A Lie algebroid over a manifold $P$ is a vector bundle $A$ over the manifold $P$ together with a Lie algebra structure $[.,.]_A$ on the space $\Gamma(A)$ of smooth sections of $A$, and a bundle map $\rho : A \to TP$ (called the anchor), such that

- The induced map $\Gamma(\rho) : \Gamma(A) \to \chi(P)$ is a Lie algebra homomorphism i.e., $\rho([X,Y]) = [\rho(X), \rho(Y)]$, $(X, Y \in \Gamma(A))$ and
- For any $f \in C^\infty(P)$ and for any smooth sections $X, Y$ of $A$ the Leibniz identity holds i.e., $[X, f Y]_A = f[X, Y]_A + (\rho(X).f)Y$.

**Remark 3**

1) The map $\Gamma(\rho)$ may be denoted simply by $\rho$ and also called the anchor.
2) For every $X \in \Gamma(A)$, we define $A$-Lie derivative operations on both $\Gamma(A)$ and $C^\infty(P)$ by $L_X f = \rho(X(f))$. Then the Leibniz identity is as a derivation $L_X(fY) = f(L_X Y) + (L_X f)Y$.

**Example 13** Let $P$ be a Poisson manifold with Poisson Tensor $\Pi$. Then $T^*P$ carries a natural Lie algebroid structure, called the cotangent bundle Lie algebroid of the Poisson manifold $P$. The anchor map $\Pi^# : T^*P \to TP$ is defined by $\Pi^#(\xi)(\eta) = \Pi(\xi, \eta)$ for each $\xi, \eta \in T^*_pP$ and the Lie bracket of 1-forms $\alpha$ and $\beta$ is given by $[\alpha, \beta] = L_{\Pi^#(\alpha)} \beta - L_{\Pi^#(\beta)} \alpha - d(\Pi(\alpha, \beta))$.

**Definition 14** Let $(A, \rho, [.,.]_A)$ be a Lie algebroid over the manifold $P$, and $\bigwedge^* A^*$ be the exterior algebra of its dual $A^*$. Sections of $\bigwedge^* A^*$ are called $A$-differential forms on $P$, or $A$-forms on $P$.

If $\omega \in \Gamma(\bigwedge^k A^*)$, we say that $\omega$ is homogeneous, and its degree is $|\omega| = k$, and we call it as an $A$-$k$-form.

**Definition 15** There is a differential operator which takes an $A$-$k$-form $\omega$ to an $A$-$(k+1)$-form $d_A \omega$ as below

$$d_A \omega(v_1, ..., v_{k+1}) = \sum_{i} (-1)^{i+1} \rho(v_i).\omega(v_1, ..., \hat{v}_i, ..., v_{k+1})$$

$$+ \sum_{i < j} (-1)^{i+j} \omega([v_i, v_j]_A, v_1, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_{k+1})$$

where $v_1, ..., v_{k+1} \in \Gamma(A)$ are $A$-vector fields.

**Remark 4** ([4]) The Lie algebroid axioms for $A$ implies that:
1. \( d_A \) is \( C^\infty(X) \)-multilinear,
2. \( d_A^2 = 0 \),
3. \( d_A(\omega_1 \wedge \omega_2) = d_A \omega_1 \wedge \omega_2 + (-1)^{\lvert \omega_1 \rvert} \omega_1 \wedge d_A \omega_2 \).

The triple \((\Gamma \bigwedge^* A^*, \wedge, d_A)\) forms a differential graded algebra, the same as the usual algebra of differential forms.

**Proposition 6** There is a one-to-one correspondence between Lie algebroid structures on \( A \) and differential operators on \( \Gamma(\bigwedge^* A^*) \) satisfying properties 1-3 in above.

**Proof** The anchor map is obtained from \( d_A \) on functions by

\[
\rho(v)f = (d_Af)(v), \quad v \in \Gamma(A), \quad f \in C^\infty(P).
\]

The Lie bracket \([., .]_A\) is determined by

\[
i_{[v,w]}(\omega) = \rho(v).\omega(w) - \rho(w).\omega(v) - d_A\omega(v, w) = i_v(d_A(i_w(\omega))) - i_w(d_A(i_v(\omega))) - i_{(v \wedge w)}d_A\omega,
\]

where \((v, w) \in \Gamma(A), \quad \omega \in \Gamma(A^*)\). \hfill \Box

**Definition 16** The exterior differential algebra \( \Gamma(\bigwedge^* A^*, \wedge, d_A) \) associated to a Lie algebroid \((A, \rho, [,.,.]_A)\) determines de Rham cohomology groups, called **Lie algebroid cohomology** of \( A \) or **A-cohomology**.

**Example 14** Let \( A = \mathfrak{g} \) be a Lie algebra i.e., a Lie algebroid over a one-point space, the cohomology of the differential complex

\[
\mathbb{R} \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow ...
\]

is the standard Lie algebra cohomology \((\bigwedge^* \mathfrak{g}^*, \wedge, d\mathfrak{g})\).

**Example 15** Let \( A = TP \) be a tangent bundle of a manifold \( P \), the cohomology computed by \((\Gamma(\bigwedge^* A^*), \wedge, d_A) = (\Omega^*(P), \wedge, d_{deRham})\) is the usual de Rham cohomology.

**Definition 17** **Lie algebroid multivector fields** or **A-multivector fields** are sections of the exterior algebra \( \bigwedge^* A \) of a Lie algebroid \((A, \rho, [,.,.]_A)\). If \( v \in \Gamma(\bigwedge^k A) \), then \( v \) is called homogeneous with degree \( \lvert v \rvert = k \).

**Remark 5** ([4])

1) The extension of\([.,.]_A\) to arbitrary A-multivector fields by setting it on homogeneous A-multivector fields \( v, w \) is

\[
i_{[v,w]}(\omega) = (-1)^{(\lvert v \rvert - 1)(\lvert w \rvert - 1)}i_v(d_A(i_w(\omega))) - i_w(d_A(i_v(\omega))) + (-1)^{\lvert v \rvert - 1}i_{(v \wedge w)}d_A\omega
\]

2) The bracket \([.,.]_A\) on A-multivector fields has the following properties.
1. $[\ldots]_A$ allows us to extend the $A$-Lie derivative operation defined for $A$-vector fields in definition of Lie algebroids to arbitrary elements of $v, w \in \Gamma(\Lambda^* A)$, i.e. $L_v w := [v, w]_A$.
2. $[\ldots]_A$ is a graded Lie algebra structure i.e. $[v, w]_A = (-1)^{|v||w|}[w, v]_A$
3. $[\ldots]_A$ satisfies a super-Jacobi identity:
   $[v, [w, y]]_A + (-1)^{|v||y|}[v, [w, y]]_A + (-1)^{|v||w||y|}[w, [v, y]]_A = 0$
4. $[\ldots]_A$ satisfies a super-Leibniz identity:
   $[v, w \wedge y]_A = [v, w]_A \wedge y + (-1)^{|w||y|}w \wedge [v, y]_A$.

**Definition 18** The triple $(\Gamma(\Lambda^* A), \wedge, [\ldots]_A)$ is called the **Gerstenhaber algebra** of the Lie algebroid $(A, \rho, [\ldots]_A)$, or just the **$A$-Gerstenhaber algebra**. We will refer to the bracket $[\ldots]_A$ on $\Gamma(\Lambda^* A)$ as the **$A$-Gerstenhaber bracket**.

In general,

**Definition 19** A Gerstenhaber algebra $(a, \wedge, [\ldots])$ is a graded vector space $a = a_0 \oplus a_1 \oplus \ldots$ together with a super commutative associative multiplication of degree 0, $a_i \cdot a_j \subseteq a_{i+j}$ and a super-Lie algebra structure of degree $-1$, $[a_i, a_j] \subseteq a_{i+j-1}$, satisfying the super-Leibniz identity

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{|a||b|}b \wedge [a, c].$$

**Remark 6** From a Lie algebroid structure on $A$, i.e. $(A, \rho, [\ldots]_A)$, we obtain a differential algebra structure on $\Gamma(\Lambda^* A^*)$, i.e. $(\Gamma(\Lambda^* A^*), \wedge, d_A)$, and from that we get a Gerstenhaber algebra structure on $\Gamma(\Lambda^* A)$, i.e. $(\Gamma(\Lambda^* A), \wedge, [\ldots]_A)$.

**Definition 20** For the tangent bundle Lie algebroid $(A, \rho, [\ldots]_A) = (TP, id, [\ldots])$, $d_A$ is the de Rham differential and $A$-Gerstenhaber bracket is usually called the **Schouten-Nijenhuis bracket** on multivector fields.

A bivector field $\Pi \in \Gamma(\Lambda^2 TP)$ is called a **Poisson bivector field** if and only if $[\Pi, \Pi] = 0$. This is equivalent to $d^2_\Pi = 0$ for the differential operator $d_\Pi := [\Pi, \cdot]$.

The notion of Poisson structure naturally generalizes to arbitrary Lie algebroid as follows:

Let $(A, \rho, [\ldots]_A)$ be a Lie algebroid over $P$. An element $\Pi \in \Gamma(\Lambda^2 A)$ is called an $A$-Poisson bivector field when $[\Pi, \Pi]_A = 0$, where $[\ldots]_A$ is the $A$-Gerstenhaber bracket.

**Example 16** Consider $A = \mathfrak{g}$ as a Lie algebra, and a $\mathfrak{g}$-Poisson bivector field $\Pi \in \mathfrak{g} \wedge \mathfrak{g}$ to a left invariant Poisson structure on the underlying Lie group $G$. The equation $[\Pi, \Pi]_\mathfrak{g} = 0$ is called the **classical Yang-Baxter equation**.
Remark 7 ([4]) The push-forward $\rho_* \Pi$ of an $\mathcal{A}$-Poisson bivector field $\Pi$ by the anchor $\rho : \Gamma(\bigwedge^2 \mathcal{A}) \to \Gamma(\bigwedge^2 TP)$ defines an ordinary Poisson structure on the manifold $P$. By the Jacobi identity, an arbitrary (not necessarily Poisson) element $\Theta \in \Gamma(\bigwedge^2 \mathcal{A})$ satisfies

$$d^2_{\Theta} + \left[ \frac{1}{2} [\Theta, \Theta]_{\mathcal{A}} \right]_{\mathcal{A}} = 0$$

which is similar to the equation for a flat connection.

In the first section we saw how one defines the universal enveloping algebra for a Lie algebra. Now we want to see the definition for Lie algebroids.

Definition 21 If $(\mathcal{A}, \rho, \ldots)$ is a Lie algebroid over the manifold $P$, then the $C^\infty(P)$-module $C^\infty(P) \oplus \Gamma(\mathcal{A})$ is a Lie algebra over $\mathbb{R}$ with the Lie bracket

$$[f + X, g + Y] = (\rho(X)g - \rho(Y)f) + [X, Y].$$

Considering the universal enveloping algebra of this Lie algebra as $U = U(C^\infty(P) \oplus \Gamma(\mathcal{A}))$, we denote $f'$ and $X'$ as the canonical images of $f \in C^\infty(P)$ and $X \in \Gamma(\mathcal{A})$ in $U$. If $I$ is the two-sided ideal of $U$ generated by all elements of the form $(fg)' - f'g'$ and $(fX)' - f'X'$ then the universal enveloping algebra of the Lie algebroid $\mathcal{A}$ is defined by $U(\mathcal{A}) = U/I$.

Theorem 2 The universal enveloping algebra $U(\mathcal{A})$ of a Lie algebroid $\mathcal{A}$ admits a co-commutative bialgebroid structure.

Proof Let $R = C^\infty(P)$, and $\alpha = \beta : R \to U(\mathcal{A})$ be the natural embedding. For the coproduct define

$$\Delta(f) = f \otimes_R 1, \quad \forall f \in R$$
$$\Delta(X) = X \otimes_R 1 + 1 \otimes_R X, \quad \forall X \in \Gamma(\mathcal{A}).$$

This formula extends to a co-product $\Delta : U(\mathcal{A}) \to U(\mathcal{A}) \otimes_R U(\mathcal{A})$ by the compatibility condition.

The co-unit map is the projection $\varepsilon : U(\mathcal{A}) \to R$. Also, the map $\mu : U(\mathcal{A}) \to \text{End}_R R$ defined by

$$(\mu x)(f) = (\rho x)(f), \quad \forall x \in UA, f \in R$$

is the anchor for our bialgebroid where $\rho : U(\mathcal{A}) \to TP$ is the algebra homomorphism extending the anchor of the Lie algebroid $\mathcal{A}$. Hence, $(U(\mathcal{A}), R, \alpha, \beta, m, \Delta, \varepsilon)$ is a co-commutative bialgebroid with anchor $\mu$. $\square$

The notion of Lie bialgebroids is a natural generalization of Lie bialgebras.
**Definition 22** A **Lie bialgebroid** is a dual pair \((A, A^*)\) of vector bundles equipped with Lie algebroid structures such that the differential \(d\) on \(\bigoplus_k \Gamma(\wedge^k A^*)\) is defined by

\[
d : \Gamma\left(\bigwedge^k A^*\right) \to \Gamma\left(\bigwedge^{k+1} A^*\right)
\]

and the differential \(d_*\) on \(\bigoplus_k \Gamma(\wedge^k (A^*)^*) \cong \bigoplus_k \Gamma(\wedge^k A)\) is defined by

\[
d_* : \Gamma\left(\bigwedge^k A\right) \to \Gamma\left(\bigwedge^{k+1} A\right)
\]

coming from the structure on \(A^*\) is a derivation of the Schouten bracket on \(\bigoplus_k \Gamma(\wedge^k A)\), equivalently, \(d_*\) is a derivation for sections of \(A\) i.e.,

\[
d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A).
\]

In other words, \((\bigoplus_k \Gamma(\wedge^k A), \Lambda, [., .], d_*)\) is a differential Gerstenhaber algebra.

**Example 17** For a Poisson manifold \((P, \Pi)\), the dual pair \((TP, T^*P)\) where \(TP\) is the standard tangent bundle and \(T^*P\) is the cotangent Lie algebroid, together with

\[
d_{\Pi}[X, Y] = [d_{\Pi}X, Y] + [X, d_{\Pi}Y]
\]

is a Lie bialgebroid. As before we saw, \(d_{\Pi} = [\Pi, .]\) which is obtained from the graded Jacobi identity, see [16].

**Proposition 7** In fact, a Lie bialgebroid is equivalent to a strong differential Gerstenhaber algebra structure on \(\bigoplus_k \Gamma(\wedge^k A)\). (See Proposition 2.3 in [24]).

**Remark 8** ([23]) In a Lie bialgebroid \((A, A^*)\), the base \(P\) has a natural Poisson structure as

\[
\{f, g\} = <df, d_*g>, \quad \forall f, g \in C^\infty(P) \text{ which satisfies } [df, dg] = d[f, g].
\]

The same as Lie bialgebras, a useful method of constructing Lie bialgebroids is using \(r\)-matrices. Recall that an \(r\)-matrix is a section \(\Lambda \in \Gamma(\wedge^2 A)\) satisfying

\[
L_X[\Lambda, \Lambda] = [X, [\Lambda, \Lambda]] = 0, \quad \forall X \in \Gamma(A).
\]

An \(r\)-matrix \(\Lambda\) defines a Lie bialgebroid, where the differential \(d_* : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)\) is \(d_* = [., \Lambda]\). The bracket on \(\Gamma(A^*)\) is

\[
[\xi, \eta] = L_{\Lambda^\#} \xi \eta - L_{\Lambda^\#} \xi \eta - d[\Lambda(\xi, \eta)].
\]

The anchor is \(\rho \Lambda^\# : A^* \to TP\), where \(\Lambda^\#\) is the bundle map \(A^* \to A\) with

\[
\Lambda^\#(\xi)(\eta) = \Lambda(\xi, \eta), \quad \forall \xi, \eta \in \Gamma(A^*).
\]
This Lie bialgebroid is called a **coboundary Lie bialgebroid**, and it is called a **triangular Lie bialgebroid** if \([\Lambda, \Lambda] = 0\). If \(\Lambda\) is also of constant rank then we call it as a **regular triangular Lie bialgebroid**.

**Remark 9** ([(23)]) When \(P\) has only one point, that means \(A\) is a Lie algebra, then \(L_X[\Lambda, \Lambda] = [X, [\Lambda, \Lambda]] = 0\) is equivalent to that \([\Lambda, \Lambda]\) is Ad-invariant, so \(\Lambda\) is an \(r\)-matrix.

If \(A\) is the tangent bundle \(TP\) with the standard Lie algebroid structure, \(L_X[\Lambda, \Lambda] = [X, [\Lambda, \Lambda]] = 0\) is equivalent to \([\Lambda, \Lambda] = 0\) i.e., \(\Lambda\) is a Poisson tensor.

## 5 Deformation of Bialgebroids

We consider a **topological bialgebra** \((A, \mu, \eta, \Delta, \epsilon)\) where \(A\) is a module over the ring \(k[[\hbar]]\) and \(\mu : A \otimes A \to A\), \(\eta : k[[\hbar]] \to A\), \(\Delta : A \to A \otimes A\) and \(\epsilon : A \to k[[\hbar]]\) are \(k[[\hbar]]\)-linear maps. Similarly, a **topological bialgebroid** is a bialgebroid which is a module over \(k[[\hbar]]\) and all structure maps are \(k[[h]]\)-linear maps.

Remember from Sect. 2, using twist construction, in Theorem 1 one obtains a new bialgebroid from a given bialgebroid. Also remember the Example 10, in which using twisted construction we obtained twisted Hopf algebroid \((D_h, R_h, \alpha_h, \beta_h, m, \Delta_h, \epsilon_h)\) which is called the **quantum groupoid** associated to the star product \(*_h\).

**Definition 23** A **deformation** of a bialgebroid \((A, R, \alpha, \beta, m, \Delta, \epsilon)\) over a field \(k\) is a topological bialgebroid \((A_h, R_h, \alpha_h, \beta_h, m_h, \Delta_h, \epsilon_h)\) over the ring \(k[[\hbar]]\) of formal power series in \(\hbar\) such that

1. \(A_h\) is isomorphic to \(A[[\hbar]]\) as \(k[[\hbar]]\)-module with identity \(1_A\), and \(R_h\) is isomorphic to \(R[[\hbar]]\) as \(k[[\hbar]]\)-module with identity \(1_R\).
2. \(\alpha_h = \alpha(\text{mod} \hbar)\), \(\beta_h = \beta(\text{mod} \hbar)\), \(m_h = m(\text{mod} \hbar)\), \(\epsilon_h = \epsilon(\text{mod} \hbar)\).
3. \(\Delta_h = \Delta(\text{mod} \hbar)\).

In this case, we simply say that the quotient \(A_h/hA_h\) is isomorphic to \(A\) as a bialgebroid.

**Lemma 2** (Xu, [(23)]) By the conditions (1), (2) in the above definition, \(A_h \otimes_{R_h} A_h/h(A_h \otimes_{R_h} A_h)\) is isomorphic to \(A \otimes R\) as \(k\)-module.

**Definition 24** A **quantum universal enveloping algebroid** (or **QUE algebroid**), also called a **quantum groupoid** is a deformation of the standard bialgebroid \((UA, R, \alpha, \beta, m, \Delta, \epsilon)\) of a Lie algebroid \(A\).

Suppose \((U_hA, R_h, \alpha_h, \beta_h, m_h, \Delta_h, \epsilon_h)\) is a quantum groupoid. Note that \(U_hA = UA[[\hbar]]\) and \(R_h = R[[\hbar]]\). Then \(R_h\) defines a star product on the base manifold \(P\) for the Lie algebroid \(A\), so that
\[ \{f, g\} = \lim_{\hbar \to 0} \frac{1}{\hbar} (f *_{\hbar} g - g *_{\hbar} f) \]

(with \( f, g \in R \)) is a Poisson structure on the base \( P \). Define

- \( \delta f := \lim_{\hbar \to 0} \frac{1}{\hbar} (\alpha_{\hbar} f - \beta_{\hbar} f) \in UA, \ \forall f \in R, \)
- \( \Delta^1 X := \lim_{\hbar \to 0} \frac{1}{\hbar} (\Delta_{\hbar} X - (1 \otimes_{R_{\hbar}} X + X \otimes_{R_{\hbar}} 1)) \in UA \otimes_R UA, \ \forall X \in \Gamma(A), \) and
- \( \delta X := \Delta^1 X - (\Delta^1 X)_{21} \in UA \otimes_R UA. \)

For any \( f, g \in R, x, y \in UA \)

\[
\begin{align*}
\alpha_{\hbar} f &= f + h\alpha_f + h^2\alpha_f + O(h^3), \\
\beta_{\hbar} f &= f + h\beta_f + h^2\beta_f + O(h^3), \\
f *_{\hbar} g &= fg + hB_1(f, g) + O(h^2), \\
x *_{\hbar} y &= xy + h\mu_1(x, y) + O(h^2)
\end{align*}
\]

where \( \alpha_f, \beta_f, \alpha_f, \beta_f, m_1(x, y) \) are elements in \( U(A) \). So, \( \{ f, g \} = B_1(f, g) - B_1(g, f) \) and \( \delta f = \alpha_f - \beta_f \).

**Proposition 8** (Xu, [23]) For any \( f, g \in R \) and \( X \in \Gamma(A) \) we obtain

1. \( \delta f \in \Gamma(A) \) and \( \delta X \in \Gamma(\wedge^2 A) \),
2. \( \delta(fg) = f\delta g + g\delta f \),
3. \( \delta(fX) = f\delta X + \delta f \wedge X \),
4. \( [\delta f, g] = \{ f, g \} \),
5. \( \delta^2 f = 0 \).

By properties (1)–(3) we can extend \( \delta \) to a well-defined degree 1 derivation \( \delta : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A) \). Then the algebra \( (\bigoplus \Gamma(\wedge^* A), \wedge, [\ldots, \ldots], \delta) \) is a strong differential Gerstenhaber algebra. Since, Considering the above proposition, it suffices to show that \( \delta \) is a derivation with respect to \([\ldots, \ldots]\), and \( \delta^2 = 0 \). The proof of these two are in Propositions 5.13 and 5.14 in [23].

By above proposition the dual pair \( (A, A^*) \) is a Lie bialgebroid, which is called the classical limit of the quantum groupoid \( U_{h}A \).

Therefore, the following theorem is obtained.

**Theorem 3** (Xu, [23]) A quantum groupoid \( (U_{h}A, R_{h}, \alpha_{h}, \beta_{h}, m_{h}, \Delta_{h}, \varepsilon_{h}) \) naturally induces a Lie bialgebroid \( (A, A^*) \) as a classical limit. The induced Poisson structure of this Lie bialgebroid on the base manifold \( P \) coincides with the one obtained as the classical limit of the base \( * \)-algebra \( R_{h} \).

In [10] it was shown that every Lie bialgebra is quantizable. Now, one can ask if every Lie bialgebroid is quantizable.
Definition 25 A Quantization of a Lie bialgebroid \((A, A^*)\) is a quantum groupoid 
\((U_hA, R_h, \alpha_h, \beta_h, m_h, \Delta_h, \varepsilon_h)\) whose classical limit is \((A, A^*)\).

Remark 10 In [13], Kontsevich has shown that given a Poisson structure \(\{\cdot,\cdot\}\) on a
Poisson manifold \(P\) one can find a *-product such that
\[ f \ast g = fg + \{f, g\} + \ldots \]
So, we can say that the Lie bialgebroid \((TP, T^*P)\) associated to a Poisson manifold \(P\)
is always quantizable.

In a special case, Xu in [23] showed that any regular triangular Lie bialgebroid
is quantizable. In general case, in [3] it was shown that every Lie bialgebroid is
quantizable.

6 The Lie Bialgebroid of Formal Pseudo-differential Operators

In this section we are going to indicate the Theorem 3 for formal pseudo-differential
operators. In other words, we see how naturally the algebra of formal pseudo-differential
operators can be considered as a Lie bialgebroid. First we recall some
definitions. Here we consider the formal pseudo-differential operators in a gen-
eral algebraic setting described for instance in [1].

Let \(A\) be a commutative \(k\)-algebra with unit 1, where \(k\) is a field of the zero-
characteristic. Assume that \(A\) is equipped with a derivation, i.e. there is a \(k\)-lin-
ear map \(D : A \to A\) satisfying the Leibnitz rule \(D(f \cdot g) = (Df) \cdot g + f \cdot (Dg)\) for all
\(f, g \in A\). We say that an element is constant if \(Df = 0\). Consider \(\xi\) as a formal
variable not in \(A\). The algebra of symbols over \(A\) is the vector space
\[ \Psi_\xi(A) = \left\{ P_\xi = \sum_{\nu \in \mathbb{Z}} a_\nu \xi^\nu \mid a_\nu \in A, a_\nu = 0 \text{ for } \nu \gg 0 \right\} \]
which is equipped with the associative multiplication \(\circ\) defined by
\[ P_\xi \circ Q_\xi = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} D^k Q_\xi. \]
Note that the algebra \(A\) is included in \(\Psi_\xi(A)\), and for \(f \in A\) and \(\nu \in \mathbb{Z}\) the multiplica-
tion \(\circ\) is given by \(\xi^\nu of = \sum_{k \geq 0} \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!} (D^k f) \xi^{\nu-k} \).

Definition 26 The algebra of formal pseudo-differential operators over \(A\) is the vec-
tor space
\[ \Psi(A) = \left\{ P = \sum_{\nu \in \mathbb{Z}} a_{\nu} D^\nu \mid a_{\nu} \in A, a_{\nu} = 0 \text{ for } \nu \gg 0 \right\} \]

which is equipped with the unique multiplication which makes the map \( \sum_{\nu \in \mathbb{Z}} a_{\nu} x^\nu \mapsto \sum_{\nu \in \mathbb{Z}} a_{\nu} D^\nu \) an algebra homomorphism. The algebra \( \Psi(A) \) is associative but not commutative. It is a Lie algebra over \( k \) with the Lie bracket \([P, Q] := PQ - QP\).

Now, consider \( A = \mathbb{R}^n \) as the commutative \( \mathbb{R} \)-algebra with unit \( 1 \), over the field \( \mathbb{R} \), equipped with the Hadamard product (entrywise product) which for \( n = 1 \) is the usual product on \( \mathbb{R} \).

**Theorem 4** There is a natural Lie bialgebroid structure on the algebra of formal pseudo-differential operators \( \Psi(\mathbb{R}^n) \) as the classical limit of the quantum groupoid associated to it.

**Proof** Remember from above definition that the algebra \( A \) is included in \( \Psi(A) \). Moreover, if we consider \( \mathbb{R}^n \) as a manifold, then we can define the bundle map \( \pi : \Psi(\mathbb{R}^n) \to \mathbb{R}^n \) as the natural projection. Hence, \( \Psi(\mathbb{R}^n) \) is a Lie algebroid with the Lie bracket defined in above definition. By Definition (21), \( U(\Psi(\mathbb{R}^n)) = U(\Gamma(\Psi(\mathbb{R}^n))) \oplus C^\infty(\mathbb{R}^n) \) is the universal enveloping algebra of our Lie algebroid. Therefore, by Theorem 2, we have the bialgebroid \((U(\Psi(\mathbb{R}^n)), C^\infty(\mathbb{R}^n), \alpha, \beta, \Delta, \varepsilon)\) in which \( \alpha = \beta = i : C^\infty(\mathbb{R}^n) \hookrightarrow U(\Psi(\mathbb{R}^n)) \) are the natural embeddings, and \( \varepsilon : U(\Psi(\mathbb{R}^n)) \to C^\infty(\mathbb{R}^n) \) is the projection. Moreover, we have

\[
\Delta(f) = f \otimes_{C^\infty(\mathbb{R}^n)} 1, \quad \forall f \in R = C^\infty(\mathbb{R}^n)
\]

\[
\Delta(X) = X \otimes_{C^\infty(\mathbb{R}^n)} 1 + 1 \otimes_{C^\infty(\mathbb{R}^n)} X, \quad \forall X \in \Gamma(U(\Psi(\mathbb{R}^n))).
\]

Now, consider \((U_h(\Psi(\mathbb{R}^n)), C^\infty(\mathbb{R}^n)_h, \alpha_h, \beta_h, \Delta_h, \varepsilon_h)\) as the quantum universal enveloping algebra of the above bialgebroid. Here we define \( \alpha_h, \beta_h : C^\infty(\mathbb{R}^n)_h \to U_h(\Psi(\mathbb{R}^n)) \) by

\[
\alpha_h f = f + \hbar \sum_{i=1}^{m} \partial_i f + \hbar^2 \alpha_2 f + O(\hbar^3)
\]

\[
\beta_h f = f + \hbar \left( - \sum_{i=1}^{n} \partial_i f \right) + \hbar^2 \beta_2 f + O(\hbar^3),
\]

where \( \partial_i \) means differentiation with respect to the \( i \)-th variable and \( m \) is fixed such that \( 1 \leq m \leq n \).

Also, if we consider the coordinates \((q_i, p_i)\) on \( T\mathbb{R}^n \) we define
\[
 f \ast_h g = fg + \hbar \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} + O(h^2), \\
 x \ast_h y = xy + \hbar m_1(x, y) + O(h^2),
\]

moreover, we consider \( \epsilon_h(x) = \epsilon(x) + \hbar \epsilon_1(x) + O(h^2), \) \( (\epsilon_1(x) \in C^\infty(\mathbb{R}^n)) \), and \( \Delta_h X \) with

\[
\Delta^1 X = \lim_{h \to 0} \frac{1}{h} (\Delta_h X - (1 \otimes_{C^\infty(\mathbb{R}^n)} X + X \otimes_{C^\infty(\mathbb{R}^n)} 1)) \\
\in U(\Psi(\mathbb{R}^n)) \otimes_{C^\infty(\mathbb{R}^n)} U(\Psi(\mathbb{R}^n)), \forall X \in \Gamma(U(\Psi(\mathbb{R}^n))).
\]

for any \( f, g \in C^\infty(\mathbb{R}^n), x, y \in U(\Psi(\mathbb{R}^n)), \) where \( \alpha_f, \beta_f, m_1(x, y) \) are elements in \( U(\Psi(\mathbb{R}^n)) \).

Therefore, we have

\[
\{f, g\} = \lim_{h \to 0} \frac{1}{h} (f \ast_h g - g \ast_h f) \\
= \lim_{h \to 0} \frac{1}{h} \left( fg + \hbar \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} + O(h^2) - gf - \hbar \sum_{i=1}^{n} \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} - O(h^2) \right) \\
= \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}
\]

which is the standard Poisson bracket on the base manifold \( \mathbb{R}^n \). Moreover,

\[
\delta f = \lim_{h \to 0} \frac{1}{h} (\alpha_h f - \beta_h f) \\
= \lim_{h \to 0} \frac{1}{h} \left( h \sum_{i=1}^{m} \partial f - h \left( - \sum_{i=m}^{n} \partial f \right) + h^2 \alpha_2 f - h^2 \beta_2 f + O(h^3) - O(h^3) \right) \\
= \sum_{i=1}^{n} \partial f - \left( - \sum_{i=m}^{n} \partial f \right) \\
= df.
\]

Thus we obtain the ordinary derivation which clearly satisfies all the properties of \( \delta \) in Proposition 8. Therefore, by Theorem 3 the quantum groupoid \( (U_\hbar(\Psi(\mathbb{R}^n), C^\infty(\mathbb{R}^n)_h), \alpha_h, \beta_h, \Delta_h, \epsilon_h) \) naturally induces the Lie bialgebroid \( (\Psi(\mathbb{R}^n), (\Psi(\mathbb{R}^n)_h)) \) as a classical limit.

With the bracket \([\cdot, \cdot]_{U(\Psi(\mathbb{R}^n))}\) and the exterior product \( \wedge \) we have \( \bigoplus \Gamma(\wedge^* U(\Psi(\mathbb{R}^n)), \wedge, [\cdot, \cdot], \delta) \) as a differential Gerstenhaber algebra, and

\[
\delta^2 = 0, \\
\delta[X, Y] = [\delta X, Y] + [X, \delta Y].
\]
Remark 11 Note that in the proof of above theorem, we chose the algebra $A$ as $\mathbb{R}^n$ which is also considered as a manifold. We would like to mention that the above proof is applicable to a diffeological $A$, that is a choice where $A$ has its algebraic operations which are smooth in the diffeological sense, along the lines of [9]. In this context, $\ast_h$ needs to be smooth.

Remark 12 In fact for any algebra $A$, if the algebra of pseudo-differential operators $\Psi(A)$ admits a Lie agebroid structure then by Definition (21), $U(\Psi(A)) = U(\Gamma(\Psi(A)) \oplus C^\infty(A))$ is the universal enveloping algebra of our Lie agebroid. Therefore, by Theorem 2, we have the bialgebroid $(U(\Psi(A)), C^\infty(A), \alpha, \beta, \Delta, \varepsilon)$. Now, consider $(U_h(\Psi(A)), C^\infty(A)_h, \alpha_h, \beta_h, \Delta_h, \varepsilon_h)$ as the quantum universal enveloping algebra of the above bialgebroid. By Theorem 3 and considering Remark 11, this quantum groupoid naturally induces a Lie bialgebroid $(\Psi(A), (\Psi(A))^*)$ as the classical limit. Then one can obtain non-commutative KP equations based on $U_h(\Psi(A))$ as we explain in the next section. The interesting application is the choice of $A$ as Connes-Kreimer Hopf algebra, i.e. the algebra $A$ is generated by rooted planar binary trees, as it is described in [8]. If $\Psi(A)$ admits a Lie agebroid structure then by above explanation one can obtain non-commutative KP equations in terms of Connes-Kreimer Hopf algebra.

7 Non-Commutative Kadomtsev–Petviashvili Hierarchy

According to [11], the non-commutativity is arbitrarily introduced for the variables $(x_1, x_2, \ldots)$ as the following equation.

$$[x_k, x_l] = i\theta^{kl},$$

where the real constants $\theta^{kl}$ are called the non-commutative parameters.

Non-commutative field theories are obtained from given commutative field theories by using star-products instead of the ordinary products in the commutative field theories. For instance, on Euclidean spaces, a star-product is given by

$$f(x) \star g(x) := f(x)g(x) + \frac{i}{2} \theta^{kl} \partial_k f(x) \partial_l g(x) + O(\theta^2),$$

which is known as Moyal product, where $\partial_k = \frac{\partial}{\partial x_k}$. The star-product is associative, i.e. $f \star (g \star h) = (f \star g) \star h$, and it turns to the ordinary product in the commutative limit $\theta^{kl} \to 0$. Using the star product we obtain non-commutativity, i.e.

$$[x_k, x_l]_\star = x_k \star x_l - x_l \star x_k = i\theta^{kl}.$$

The usual definition of non-commutative KP hierarchy, i.e.

$$\partial_m L = [(L^n)_+, L]_\star,$$

based on this non-commutative condition is given in [11].
Now, assume that our KP equations are defined on the algebra of formal pseudo-differential operators $\Psi(\mathbb{R}^n)$. By Theorem 4, there is a Lie bialgebroid structure on $\Psi(\mathbb{R}^n)$ as the classical limit of the quantum groupoid $(U_\hbar(\Psi(\mathbb{R}^n)), C^\infty(\mathbb{R}^n)_\hbar, \alpha_\hbar, \beta_\hbar, \Delta_\hbar, \epsilon_\hbar)$. Therefore, we can naturally obtain non-commutative KP equations based on this quantum groupoid. More precisely, imitating the procedure given in [11], we obtain the non-commutative KP equations in terms of $*_\hbar$ defined on the mentioned quantum groupoid (see definition of $*_\hbar$ in the proof of Theorem 4) instead of Moyal product $\star$. Moreover, the Lie bracket on $U_\hbar(\Psi(\mathbb{R}^n))$ is defined by

$$[x_k, x_l]_\hbar = (x_k *_\hbar x_l - x_l *_\hbar x_k) + (x_k *_\hbar x_l - x_l *_\hbar x_k)\hbar + (x_k *_\hbar x_l - x_l *_\hbar x_k)\hbar^2 + \ldots,$$

where in the classical limit $\hbar \to 0$, it turns to the usual Lie bracket on $U(\Psi(\mathbb{R}^n))$.

Consider an $N$th order (monic) pseudo-differential operator $T$ as

$$T = \partial_x^N + a_{N-1}\partial_x^{N-1} + \cdots + a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \ldots,$$

and denote $T_{\geq} := \partial_x^N + a_{N-1}\partial_x^{N-1} + \cdots + a_r\partial_x^r$. A differential operator $\partial_x^n$ acts formally on a multiplicity operator $f$ as the following generalized Leibniz rule.

$$\partial_x^n f := \sum_{i \geq 0} \left(\begin{array}{c} n \\ i \end{array}\right) (\partial_x^i f) \partial_x^{n-i},$$

where the binomial coefficient

$$\left(\begin{array}{c} n \\ i \end{array}\right) := \frac{n(n-1) \ldots (n-i+1)}{i(i-1) \ldots 1}$$

is applicable for negative $n$. For example,

$$\partial_x^{-1} f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \ldots,$$

$$\partial_x^{-2} f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \ldots,$$

$$\partial_x^{-3} f = f\partial_x^{-3} - 3f'\partial_x^{-4} + 6f''\partial_x^{-5} - \ldots,$$

where $f' := \frac{\partial f}{\partial x}, f'' := \frac{\partial f}{\partial x^2}$. Now, consider the following first-order pseudo-differential operator.

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \ldots,$$

where the coefficients $u_k, (k = 1, 2, \ldots)$ are functions of infinite variables $(x_1, x_2, \ldots)$ with $x_1 = x$, i.e. $u_k = u_k(x, x_2, \ldots)$. We consider the non-commutativity given by the star-product $*_\hbar$, as described in the proof of Theorem 4. Then the Lax hierarchy is defined by

$$\partial_m L = [B_m, L]_{*_\hbar}, \ m = 1, 2, \ldots,$$

where applying $\partial_m$ on the pseudo-differential operator $L$ is considered coefficient-wise, i.e., $\partial_m L := [\partial_m, L]$ or $\partial_m \partial_x^k = 0$. Moreover, the operator $B_m$ is defined by
B_m := (L \ast_h \cdots \ast_h L)_{\ge r} \ (\text{for } m \text{ times}). \quad \text{When } u_1 = 0(r = 0) \text{ the Lax hierarchy is the non-commutative KP hierarchy which includes the non-commutative equations, see [14,21]. The coefficients of each power of pseudo-differential operators in above Lax hierarchy give a series of infinite non-commutative evolution equations. For instance, for } m = 1 \text{ the coefficient of } \partial_x^{1-k} \text{ gives the equations}

\begin{align*}
\partial_x u_k &= u'_k, \quad k = 2, 3, \ldots \Rightarrow x_1 = x.
\end{align*}

For } m = 2 \text{ the coefficient of } \partial_x^{-1} \text{ leads to}

\begin{align*}
\partial_x u_2 &= u''_2 + 2u'_3,
\end{align*}

and the coefficient of } \partial_x^{-2} \text{ gives}

\begin{align*}
\partial_x u_3 &= u'''_3 + 2u''_4 + 2u_2 \ast_h u'_2 + 2[u_2, u_3]_{\ast_h} \\
&= u'''_3 + 2u''_4 + \left(2u_2u'_2 + m_1(2u_2, u'_2)\hbar + O(h^2)\right) \\
&\quad + 2\left(u_2u_3 + m_1(u_2, u_3)\hbar - u_3u_2 - m_1(u_3, u_2)\hbar\right) \\
&\quad + (u_2u_3 + m_1(u_2, u_3)\hbar - u_3u_2 - m_1(u_3, u_2)\hbar)\hbar + O(h^2),
\end{align*}

where } m_1(u_2, u_3) \text{ and } m_1(u_3, u_2) \text{ are elements in } U(\Psi(\mathbb{R}^n)).

Next, the coefficient of } \partial_x^{-3} \text{ yields the equation}

\begin{align*}
\partial_x u_4 &= u''''_4 + 2u'''_5 + 4u_3 \ast_h u'_2 - 2u_2 \ast_h u'''_2 + 2[u_2, u_4]_{\ast_h},
\end{align*}

and the same as previous case we can expand it based on } \ast_h .

For } m = 3 \text{ the coefficients of } \partial_x^{-1}, \partial_x^{-2}, \partial_x^{-3}, \text{ and } \partial_x^{-4} \text{ respectively give the equations}

\begin{align*}
\partial_x u_2 &= u''''_2 + 3u'''_3 + 3u''_4 + 3u'_5 \ast_h u_2 + 3u_2 \ast_h u'_2, \\
\partial_x u_3 &= u'''''_3 + 3u'''_4 + 3u''_5 + 6u_2 \ast_h u'_3 + 3u'_5 \ast_h u_3 + 3u_3 \ast_h u'_2 + 3[u_2, u_4]_{\ast_h}, \\
\partial_x u_4 &= u''''''_4 + 3u'''_5 + 3u''_6 + 3u'_5 \ast_h u_4 + 3u_2 \ast_h u'_4 + 6u_4 \ast_h u'_2 - 3u_2 \ast_h u''_3 \\
&\quad - 3u_3 \ast_h u'_2 + 6u_3 \ast_h u'_3 + 3[u_2, u_5]_{\ast_h} + 3[u_3, u_4]_{\ast_h}, \\
\partial_x u_5 &= \ldots .
\end{align*}

If we set } 2u_2 \equiv u, \quad x_2 \equiv y, \quad x_3 \equiv t \text{ then we obtain the non-commutative KP equation

\begin{align*}
3u_{xy} = \partial(4u_t - 3(\partial u \ast_h u + u \ast_h \partial u) - \partial^3 u) + 3\partial[u, \partial^{-1} u]_{\ast_h}.
\end{align*}

Note that this equation is analogous to the non-commutative KP equation in [11]. The difference between them is that here the star product is defined by } \ast_h \text{ instead of Moyal product.}

Remark 13 To solve the above non-commutative KP equation, we consider the KP equation on } U_h(\Psi(\mathbb{R}^n)) \text{ with the coefficient algebra equipped with } \ast_h . \text{ In [9] we showed the well-posedness of the KP hierarchy with arbitrary (diffeological)}
coefficient algebra. Hence, considering diffeological algebraic structures, and Remark 11 we can conclude that the non-commutative KP hierarchy (equipped with $*_h$) is well-posed, see also [17].

8 Conclusion

The algebra of formal Pseudo-differential operators $\Psi(\mathbb{R}^n)$ can be naturally equipped with a Lie bialgebroid structure $(\Psi(\mathbb{R}^n), (\Psi(\mathbb{R}^n))^*)$ as the classical limit of the quantum groupoid $(U_h(\Psi(\mathbb{R}^n)), C^\infty(\mathbb{R}^n), \alpha_h, \beta_h, \Delta_h, \epsilon_h)$ associated to it. Then, the KP equations defined on Pseudo-differential operators $\Psi(\mathbb{R}^n)$ are “classical limit” of non-commutative KP equations defined by the above quantum grupoid. Hence, we obtain non-commutative KP equations in a more general sense and not by using non-commutativity of the coordinates i.e., $[x_k, x_l] = i\delta^{kl}$. Up to now, in special cases, non-commutative KP equations have been solved or it has been shown the existence and uniqueness of solutions of them using Moyal product, see [18] and references there in. The algebraic procedure described during this paper provides the possibility of solving non-commutative KP equations very generally.

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