$N = 2$ and 4 Super Yang-Mills Theories on $M_4 \times Z_2 \times Z_2$ Geometry

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Abstract

We derive the $N = 2$ and 4 super Yang-Mills theories from the viewpoint of the $M_4 \times Z_2 \times Z_2$ gauge theory. Scalars and pseudoscalars appearing in the theories are regarded as gauge fields along the directions on $Z_2 \times Z_2$ discrete space.

PACS number(s):

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§1. Introduction

The non-commutative geometric construction of Connes[1, 2] has been successful in giving a geometrical interpretation of the standard model as well as some grand unification models. In this interpretation the Higgs fields are regarded as fields along directions in the discrete space. The bosonic parts of the actions are just the pure Yang-Mills actions containing gauge fields on both continuous and discrete spaces, and the Yukawa coupling is regarded as a kind of gauge interactions of fermions.

At the same time, applying non-commutative geometry(NCG) to SUSY theories has encountered many difficulties. A natural way is to introduce a non-commutative space which is a product of the superspace and a set of discrete points, similar to those which have been done in non-SUSY theories. However, such an extension of superspace has been proved to be rather difficult to accomplish. Chamseddine[3] then proposed an alternative approach in which SUSY theories were considered in their component form. He discussed how to derive from NCG the \( N = 2 \) and 4 SUSY Yang-Mills actions, and also the coupling of \( N = 1 \) and 2 super Yang-Mills fields to \( N = 1 \) and 2 matters.

Chamseddine’s paper is the first one in which a connection between space-time super-symmetry and NCG is discussed. However his approach is rather complicated, especially the geometric meaning of \( Z_2 \) he used is not so clear. In our paper we use \( Z_2 \times Z_2 \) rather than \( Z_2 \) only. Then we would like to discuss how to derive the \( N = 2 \) and 4 SUSY Yang-Mills theories from the viewpoint of the \( M_4 \times Z_2 \times Z_2 \) gauge theory, which was previously proposed by Konisi and Saito[4] without recourse to NCG. This approach appears to be geometrically very simple and clear. The scalar fields \( S^a(x), P^a(x) \) in \( N = 2 \) theory and \( A^{aI}(x), B^{aI}(x) \) \((I = 1, 2, 3)\) in \( N = 4 \) theory will be regarded as gauge fields along directions on \( Z_2 \times Z_2 \) discrete space.

This paper is scheduled as follows: In §2 we summarize the extended gauge theory on \( M_4 \times Z_2 \times Z_2 \) without recourse to NCG[4]. This will be applied to \( N = 2 \) theory in §3 and to \( N = 4 \) theory in §4, respectively. The final section is devoted to concluding remarks.

§2. Gauge theory on \( M_4 \times Z_2 \times Z_2 \)

In this section we summarize the gauge theory on \( M_4 \times Z_2 \times Z_2 \)[4]. Let us write the four elements of \( Z_2 \times Z_2 \) as

\[
g_0 = (e_1, e_2), \quad g_1 = (r_1, e_2), \quad g_2 = (e_1, r_2), \quad g_3 = (r_1, r_2). \tag{2.1}
\]
They are subject to the algebra

\[ g_0 + g_i = g_i, \quad g_i + g_i = g_0, \ (i = 1, 2, 3) \]
\[ g_1 + g_2 = g_3 \quad \text{and cyclic.} \tag{2.2} \]

To every point \((x, p)\) with \(x \in M_4\) and \(p = g_0, g_1, g_2, g_3\) we attach a complex \(n\)-dimensional internal vector space \(V_n[x, p]\). Generally, \(n\) may take different values with each other for different \(p\)'s. However, we confine ourselves here in the equal \(n\)-dimensional case. For any scalar field \(f(x, p)\) on \(V_n[x, p]\) we define the difference \(\delta_h f(x, p)\) by

\[ \delta_h f(x, p) = f(x, p) - f(x, p + h), \quad h = g_1, g_2, g_3. \tag{2.3} \]

It is easy to check the identity

\[ \delta_k \delta_h f(x, p) - C_{kh} \delta_l f(x, p) = 0, \quad C_{kh} = \delta_k^l + \delta_h^l - \delta_{k+h}^l. \tag{2.4} \]

Namely, the second-order difference can be written by the first-order differences.

For the fermion field \(\psi^a(x, p)\) which is a vector on \(V_n[x, p]\), Eq.\((2.3)\) should be replaced by a covariant difference defined by

\[ \nabla_h \psi^a(x, p) = \psi^a(x, p) - (H(x, p, p + h))_b^a \psi^b(x, p + h), \tag{2.5} \]

where \(a, b = 1, 2, \cdots, n\). Since \(\psi^a(x, p)\) and \(\psi^b(x, p + h)\) are different vectors belonging to different internal spaces with each other, so the simple difference \(\psi^a(x, p) - \psi^a(x, p + h)\) has no meaning. However, if we give a scalar function \(H(x, p, p + h)^a_b\) of \(n \times n\) matrix, the \(\psi^a(x, p + h)\) is mapped to vector \(\psi^a_H(x, p)\) on \(V_n[x, p]\) by the product

\[ \psi^a_H(x, p) = H(x, p, p + h)^a_b \psi^b(x, p + h), \quad \text{where } H(x, p, p + h) \text{ is subject to a rule of the gauge transformation} \]

\[ H(x, p, p + h) \rightarrow H(x, p, p + h) = U^{-1}(x, p)H(x, p, p + h)U(x, p + h) \tag{2.6} \]

under a rotation \(U(x, p)\) of the \(V_n[x, p]\) frame

\[ \psi^a(x, p) \rightarrow \psi_H^a(x, p) = (U^{-1}(x, p))^a_{\alpha'} \psi^\alpha'(x, p). \tag{2.7} \]

Generally such a mapping function \(H(x, p, p + h)\) is called a connection, and it is, therefore, regarded as the gauge field associated with \(Z_2 \times Z_2\). Henceforth we refer to \(\psi_H^a(x, p) = H(x, p, p + h)^a_b \psi^b(x, p + h)\) as the parallel-transported vector of \(\psi^b(x, p + h)\) from \(p + h\) to \(p\).
In order to define a field strength (or curvature) for $H(x, p, p + h)$ we calculate the commutator

$$[\nabla_k, \nabla_h] \psi^a(x, p) = -(\tilde{F}_{kh}(x, p))^a_b \psi^b(x, p + k + h), \quad (h, k = g_1, g_2, g_3)$$

(2.8)

where

$$\tilde{F}_{kh}(x, p) = -[H(x, p, p + k)H(x, p + k, p + k + h) - H(x, p, p + h)H(x, p + h, p + k + h)].$$

(2.9)

This function $\tilde{F}_{kh}(x, p)$ can be regarded as such a field strength. The reason is as follows:

The first term $H(x, p, p + k)H(x, p + k, p + k + h)\psi(x, p + k + h)$ shows the parallel-transported vector of $\psi(x, p + k + h)$ from $p + k + h$ to $p$ through $p + k$ (see Fig.A), whereas the second term $H(x, p, p + h)H(x, p + h, p + k + h)\psi(x, p + k + h)$ shows the parallel-transported vector of $\psi(x, p + k + h)$ from $p + k + h$ to $p$ through another point $p + h$. The difference between both parallel transportations will, therefore, give the curvature.

However, we can consider another type of parallel transportations depicted in Fig.B, because on the discrete space any second-order difference can be written by the first-order one as was shown in (2.4). Actually, such a difference of the parallel transportations is given by

$$\big(\nabla_k \nabla_h - C^l_{kh} \nabla_l\big) \psi^a(x, p) = -(F_{kh}(x, p))^a_b \psi^b(x, p + k + h),$$

(2.10)

where

$$F_{kh}(x, p) = H(x, p, p + k + h) - H(x, p, p + k)H(x, p + k, p + k + h).$$

(2.11)

Namely, the first term $H(x, p, p + k + h)\psi(x, p + k + h)$ shows the parallel-transported vector of $\psi(x, p + k + h)$ from $p + k + h$ to $p$ directly, whereas the second term $H(x, p, p + k)H(x, p + k, p + k + h)\psi(x, p + k + h)$ shows the parallel-transported vector of $\psi(x, p + k + h)$ from $p + k + h$ to $p$ through $p + k$. The difference between such both parallel transportations will give another curvature $F_{kh}(x, p)$. Henceforth we call it the triangle curvature.

Fig.A.  
Fig.B.
Two kinds of curvature $\bar{F}_{kh}(x,p)$ and $F_{kh}(x,p)$ have a relation

$$\bar{F}_{kh}(x,p) = F_{kh}(x,p) - F_{hk}(x,p),$$

(2.12)

namely, $\bar{F}_{kh}(x,p)$ corresponds to an antisymmetric part of $F_{kh}(x,p)$.

The ordinary Yang-Mills field $\omega_{\mu}(x,p)$ is introduced by the covariant derivative

$$\nabla_{\mu}\psi^{a}(x,p) = (\partial_{\mu} + i\omega_{\mu}(x,p))^{a}_{\nu}\psi^{\nu}(x,p). (\mu = 0, 1, 2, 3)$$

(2.13)

We assume that $\omega_{\mu}(x,p)$ is independent of $p$ and is set to be

$$\omega_{\mu}(x,p) = A_{\mu}(x).$$

(2.14)

Its curvature is given by

$$[\nabla_{\mu}, \nabla_{\nu}]\psi^{a}(x,p) = i (F_{\mu\nu}(x))^{a}_{\nu}\psi^{\nu}(x,p),$$

(2.15)

where

$$F_{\mu\nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x) + i [A_{\mu}(x), A_{\nu}(x)].$$

(2.16)

The other curvature component $F_{\mu h}(x,p)$ is calculated to be

$$[\nabla_{\mu}, \nabla_{h}]\psi^{a}(x,p) = -F_{\mu h}(x,p)^{a}_{b}\psi^{b}(x,p + h),$$

(2.17)

where

$$F_{\mu h}(x,p) = \partial_{\mu}H_{p,h}(x,p + h) + i [A_{\mu}(x), H_{p,h}(x,p + h)] = \nabla_{\mu}H_{p,h}(x,p).$$

(2.18)

Here, we need no accounting for a triangle-like curvature, since $[\partial_{\mu}, \delta_{h}] = 0$ for any function.

By taking into account of four kinds of curvatures the bosonic Lagrangian is now given by

$$\mathcal{L}_{B} = \mathcal{L}_{1} + \mathcal{L}_{2} + \mathcal{L}_{3},$$

(2.19)

with

$$\mathcal{L}_{1} = -\frac{1}{4}F^{a}_{\mu\nu}(x)F^{\mu\nu}_{a}(x),$$

(2.19a)

$$\mathcal{L}_{2} = \xi \sum_{p} \text{tr}[F_{\mu h}^{\dagger}(x,p)F^{\mu h}(x,p)]$$

$$= \xi \sum_{p,h} \text{tr}[(\nabla_{\mu}{H}(x,p,p + h))^{\dagger}(\nabla^{\mu}{H}(x,p,p + h))],$$

(2.19b)

$$\mathcal{L}_{3} = \eta \sum_{p} \text{tr}[F_{kh}^{(S)^{\dagger}}(x,p)F^{(S)kh}(x,p)] + \zeta \sum_{p} \text{tr}[F_{kh}^{(A)^{\dagger}}(x,p)F^{(A)kh}(x,p)],$$

(2.19c)
where $\xi$, $\eta$ and $\zeta$ are real normalization constants, $F_{kh}^{(S)}$ and $F_{kh}^{(A)}$ are symmetric and antisymmetric parts of the triangle curvature $F_{kh}$, respectively.

A fermionic Lagrangian may be written as

$$\mathcal{L}_F = i \sum_p \bar{\psi}_a(x,p)(\Gamma^\mu \nabla_\mu + \Gamma^h \nabla_h)\psi^a(x,p),$$  \hspace{1cm} (2.20)

where

$$\Gamma^\mu = \gamma^\mu \times \frac{\tau^0}{2}, \quad \Gamma^h = \gamma^5 \times \frac{\tau^h}{2}, \quad (h = g_1, g_2, g_3 \text{ or simply } 1, 2, 3)$$ \hspace{1cm} (2.21)

$\tau^0$ being a $(2 \times 2)$ unit matrix and $\tau^h$ the Pauli matrix.

### §3. $N = 2$ super Yang-Mills theory

The $N = 2$ super Yang-Mills action is known to be

$$I_2 = \int d^4x \left[ -\frac{1}{4} F^{\mu\nu}_a F_{a}^{\mu\nu} + \frac{1}{2} \nabla_v S^a \nabla^v S_a + \frac{1}{2} \nabla_\mu P^a \nabla^\mu P_a + i \bar{\chi}^a \gamma^\mu \nabla_\mu \chi^a - i f_{abc} \bar{\chi}^a (S^b + i \gamma_5 P^b) \chi^c - \frac{1}{2} (f_{abc} S^b P^c)^2 \right],$$  \hspace{1cm} (3.1)

where $S^a$ and $P^a$ are scalar and pseudoscalar fields, respectively, and $\chi^a$ is a Dirac spinor, all in the adjoint representation of the gauge group $G$ with the structure constant $f_{abc}$. The action is invariant under the $N = 2$ super transformations. Our purpose is to consider a relationship between the above theory and the $M_4 \times Z_2 \times Z_2$ gauge theory.

In the fermionic Lagrangian (2.20) we require

$$\nabla_3 \psi^a(x,p) = \psi^a(x,p) - (H(x,p,p + g_3))^a_b \psi^b(x,p + g_3) = 0$$ \hspace{1cm} (3.2)

and

$$(H(x,p,p + g_3))^a_b = \delta^a_b.$$ \hspace{1cm} (3.3)

From these it follows that

$$\psi(x,p) = \psi(x,p + g_3),$$ \hspace{1cm} (3.4)

hence

$$\psi(x,g_0) = \psi(x,g_3) \text{ and } \psi(x,g_1) = \psi(x,g_2).$$ \hspace{1cm} (3.5)
From (3.3) we have
\[ H(x, p, p + g_3) = H(x, p + g_1, p + g_2) = H(x, p + g_2, p + g_1) = H(x, p + g_3, p) = 1 \] (3.6)

For other covariant differences
\[ \nabla_1 \psi(x, p) = \psi(x, p) - H(x, p, p + g_1)\psi(x, p + g_1), \]  
\[ \nabla_2 \psi(x, p) = \psi(x, p) - H(x, p, p + g_2)\psi(x, p + g_2), \]  
we set
\[ H(x, p, p + g_1) = P(x) = T_a \alpha^a(x), \]  
\[ H(x, p, p + g_2) = S(x) = T_a \alpha^a(x), \]  
where \( T^a \) is the generator of \( G \) subject to algebra
\[ [T_a, T_b] = i f_{abc} T^c. \] (3.11)

By substituting \( p + g_i \) into \( p \) in (3.9) and (3.10) we find
\[ H(p, p + g_1) = H(p + g_1, p) = H(p + g_2, p + g_3) = H(p + g_3, p + g_2) = P(x), \]  
\[ H(p, p + g_2) = H(p + g_1, p + g_3) = H(p + g_2, p) = H(p + g_3, p + g_1) = S(x). \] (3.12)

If we put in (3.5)
\[ \psi^a(x, g_0) = \psi^a(x, g_3) = \begin{pmatrix} L \alpha^a \\ 0 \end{pmatrix} \quad \text{and} \quad \psi^a(x, g_1) = \psi^a(x, g_2) = \begin{pmatrix} 0 \\ R \alpha^a \end{pmatrix}, \] (3.14)
where \( L \) and \( R \) are left-handed and right-handed projection operators, respectively, \( i.e., \)
\[ L = \frac{1 - \gamma_5}{2} \quad \text{and} \quad R = \frac{1 + \gamma_5}{2}, \] (3.15)
then the fermionic Lagrangian (2.20) is reduced to
\[ \mathcal{L}_F = i \bar{\chi}^a(x) \gamma^\mu \nabla_\mu \chi^a(x) - \bar{\chi}^a(x) [S(x) + i \gamma_5 P(x)]_a^b \chi^b(x). \] (3.16)

This is equivalent to that in (3.1) in the adjoint representation \((T_a)_{bc} = -i f_{abc}\).

Next we consider the bosonic Lagrangian. The triangle curvature (2.11) is given by
\[ F_{ij}(p) = H(p, p + g_i + g_j) - H(p, p + g_i)H(p + g_i, p + g_i + g_j), \] (3.17)
so that

\begin{align}
F_{12}(p) &= H(p, p + g_3) - H(p, p + g_1)H(p + g_1, p + g_3) = 1 - P(x)S(x), \\
F_{21}(p) &= H(p, p + g_3) - H(p, p + g_2)H(p + g_2, p + g_3) = 1 - S(x)P(x), \\
F_{23}(p) &= F_{32}(p) = -F_{31}(p) = -F_{13}(p) = -S(x) + P(x), \\
F_{11}(p) &= 1 - P^2(x), \\
F_{22}(p) &= 1 - S^2(x), \\
F_{33}(p) &= 0.
\end{align}

(3.17a)

(3.17b)

(3.17c)

(3.17d)

(3.17e)

(3.17f)

The antisymmetric part of $F_{ij}(p)$ is, therefore, given by

\begin{align}
F^{(A)}_{12}(p) &= \frac{1}{2}[S(x), P(x)] = \frac{1}{2}if_{abc}T_cS^a(x)P^b(x), \\
F^{(A)}_{23}(p) &= F^{(A)}_{31}(p) = 0.
\end{align}

(3.18a)

(3.18b)

The third bosonic Lagrangian $\mathcal{L}_3$ in (2.19c) then becomes

\begin{align}
\mathcal{L}_3 &= \eta \sum_p \text{tr}[F^{(S)\dagger}_{ij}(x, p)F^{(S)ij}(x, p)] + \zeta \sum_p \text{tr}[F^{(A)\dagger}_{ij}(x, p)F^{(A)ij}(x, p)] \\
&= \text{symmetric part} + \frac{1}{2}\zeta(f_{abc}S^a(x)P^b(x))^2.
\end{align}

(3.19)

The second bosonic Lagrangian $\mathcal{L}_2$ in (2.19b) is

\begin{align}
\mathcal{L}_2 &= \xi \sum_{p, i} \text{tr}[(\nabla_\mu H(x, p, p + g_i))\dagger(\nabla^\mu H(x, p, p + g_i)) \\
&= \xi \sum_p \text{tr}[\nabla_\mu H(x, p, p + g_1)\nabla^\mu H(x, p, p + g_1) + \nabla_\mu H(x, p, p + g_2)\nabla^\mu H(x, p, p + g_2)] \\
&= 2\xi[\nabla_\mu P^a(x)\nabla^\mu P_a(x) + \nabla_\mu S^a(x)\nabla^\mu S_a(x)]. \\
\end{align}

(3.20)

The first bosonic Lagrangian $\mathcal{L}_1$ is the same as (2.19a). The total bosonic Lagrangian $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is, therefore, identical with that in (3.1), only when $\eta = 0$, $\zeta = -1$ and $\xi = \frac{1}{4}$.

Thus we have obtained the $N = 2$ super Yang-Mills theory from the viewpoint of the $M_4 \times Z_2 \times Z_2$ gauge theory. In this interpretation the scalar field $S^a(x)$ and pseudoscalar field $P^a(x)$ have been regarded as gauge fields along two directions on $Z_2 \times Z_2$. The antisymmetric curvature for both scalar fields has been important in this construction.
§4. \( N = 4 \) super Yang-Mills theory

The \( N = 4 \) super Yang-Mills action is given by

\[
I_4 = \int d^4 x \left[ \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} i \bar{\chi}^{aj} \gamma^\mu \nabla_\mu \chi_{aj} + \frac{1}{2} (\nabla_\mu A^{aI})^2 + \frac{1}{2} (\nabla_\mu B^{aI})^2 
- i f_{abc} \chi^{aj} (\alpha_{jk}^I A^{bl} + i \gamma_5 \beta_{jk}^I B^{bl}) \chi^{ck}
- \frac{1}{4} \left( (f_{abc} A^{bl} A^{cI})^2 + (f_{abc} B^{bl} B^{cI})^2 + 2 (f_{abc} A^{bl} B^{cI})^2 \right) \right],
\]

where \( \nabla_\mu \) is the gauge covariant derivative with \( f_{abc} \) the structure constants of an arbitrary gauge group \( G \). All fields belong to the adjoint representation of \( G \), and there is a global \( SU(4) \) internal symmetry with a central \( SO(4) \) subgroup of scalar charges. The notation and classification of fields are given in Table. The \( 4 \times 4 \) matrices \( \alpha_{jk}^I \) and \( \beta_{jk}^I \) are coupling constant matrices of the \((1,0)\) or \((0,1)\) and \((\frac{1}{2}, \frac{1}{2})\) irreps of \( SO(4) \). They coincide with the \( \eta \) and \( \bar{\eta} \) matrices of instanton theory.

| Spin | Multiplicity | Fields | \( SO(4) \) irrep. |
|------|--------------|--------|-------------------|
| 1    | 1            | \( A^a_\mu \) | \((0,0)\)          |
| \( \frac{1}{2} \) | 4           | Majorana spinor \( \chi^{aj} \), \( j = 1, \cdots, 4 \) | \((\frac{1}{2}, \frac{1}{2})\) |
| 0\( \pm \) | 3 + 3       | \( A^{aI}, B^{aI} \), \( I = 1, 2, 3 \) | \((1,0)\) and \((0,1)\) |

Our purpose is to derive the above action from the viewpoint of the \( M_4 \times Z_2 \times Z_2 \) gauge theory. The procedure is quite the same as in the previous section. Only difference is to introduce the coupling constants \( \alpha_{jk}^I \) and \( \beta_{jk}^I \) into the covariant differences \((3.7)\) and \((3.8)\)

\[
\nabla_1 \psi^j(x, p) = \psi^j(x, p) - 2 \alpha_{jk}^I H^I(x, p, p + g_1) \psi^k(x, p + g_1),
\]

\[
\nabla_2 \psi^j(x, p) = \psi^j(x, p) - 2 \beta_{jk}^I H^I(x, p, p + g_2) \psi^k(x, p + g_2),
\]

where \( (H^I(x, p, p + g_i))^a_b \psi^{bk}(x, p + g_i) \) is a parallel-transported vector of \( \psi^{bk}(x, p + g_i) \) from \( p + g_i \) to \( p \). In the following we set

\[
H^I(x, p, p + g_1) = B^I(x) = T_a B^{aI}(x),
\]

\[
H^I(x, p, p + g_2) = A^I(x) = T_a A^{aI}(x),
\]
where $T_a$ is subject to the algebra (3.11). If we put fermionic fields

$$
\psi^{a j}(x, g_0) = \psi^{a j}(x, g_3) = \left( \frac{L \chi^{a j}}{\sqrt{2}} \right),
$$

(4.6)

$$
\psi^{a j}(x, g_1) = \psi^{a j}(x, g_2) = \left( \frac{R \chi^{a j}}{\sqrt{2}} \right),
$$

(4.7)

then the fermionic Lagrangian (2.20) becomes

$$
L_F = \frac{1}{2} i \bar{\chi}^{a j} \gamma^{\mu} \nabla_{\mu} \chi^{a j} - \bar{\chi}^{aj} (\alpha_{j k}^I A^I + i \gamma_5 \beta_{j k}^I B^I)_{ab} \chi^{bk},
$$

(4.8)

which is equivalent to that in (4.1) in the adjoint representation.

Next we consider the bosonic Lagrangian. In the same way as in the $N = 2$ case, we consider only antisymmetric curvatures for the discrete space. However, contrary to the $N = 2$ case, there are three kinds of antisymmetric curvatures here. One of them corresponds to (3.18a), i.e.,

$$
F_{g_1 g_2}(x, p) = [A^I(x), B^I(x)] = i f_{abc} T_c A^{a I}(x) B^{b J}(x).
$$

(4.9)

Geometrically, this is a difference between two routes of parallel transportations depicted in Fig.C1. This curvature $F_{g_1 g_2}(x, p)$ is the antisymmetric part of the triangle curvature corresponding to Fig.C2.

The other two are

$$
[A^I(x), A^J(x)] \quad \text{and} \quad [B^I(x), B^J(x)].
$$

(4.10)

Both curvatures vanish in the $N = 2$ case. However, in the $N = 4$ case they don’t vanish since they have $I, J$ components. Geometrically, $[A^I, A^J]$ corresponds to a difference between two routes of parallel transportations depicted in Fig.D1, i.e.,

$$
(p + g_3 \rightarrow A^I \rightarrow p + g_1 \rightarrow 1 \rightarrow p + g_2 \rightarrow A^I \rightarrow p) \\
- (p + g_3 \rightarrow A^I \rightarrow p + g_1 \rightarrow 1 \rightarrow p + g_2 \rightarrow A^I \rightarrow p),
$$

(4.11)
where the mapping function $H(p+g_1, p+g_2)$ is unity from (3.6). The curvature $[A^I(x), A^J(x)]$ is the antisymmetric part of the triangle-like curvature corresponding to Fig.D$_2$. The same is true for $[B^I(x), B^J(x)]$.

The bosonic Lagrangian is, therefore, given by

$$\mathcal{L}_B = -\frac{1}{4} (F^a_{\mu\nu})^2 + 2\xi[(\nabla_\mu A^a)^2 + (\nabla_\mu B^a)^2]
-2\zeta \text{tr}\{[A^I, A^J]^2 + [B^I, B^J]^2 + 2[A^I, B^J]^2\}.$$  \hspace{1cm} (4.12)

This is equivalent to the bosonic parts of (1.1) if $\xi = -\zeta = \frac{1}{4}$. Thus we have obtained the $N = 4$ super Yang-Mills theory from the viewpoint of the $M_4 \times Z_2 \times Z_2$ gauge theory, when we used antisymmetric curvatures for scalar fields. Both scalar fields $A^aI(x)$ and $B^aI(x)$ have been regarded as gauge fields along two directions on $Z_2 \times Z_2$.

§5. Concluding remarks

We have considered the $N = 2$ and 4 super Yang-Mills theories from the viewpoint of the $M_4 \times Z_2 \times Z_2$ gauge theory. The scalar fields $S^a(x)$, $P^a(x)$ in the $N = 2$ case and $A^aI$, $B^aI$, $I = 1, 2, 3$ in the $N = 4$ case have been introduced as gauge fields along directions on $Z_2 \times Z_2$ discrete space. The “covariant derivatives” on the discrete space have given the Yukawa couplings between fermions and scalar fields.
The kinetic terms of these scalar fields and the Higgs potentials have been determined by curvatures which come from scalar fields. Here, the important things are that there are symmetric and antisymmetric curvatures for scalar fields. We have seen that only antisymmetric curvatures for scalar fields are related to the $N = 2$ and 4 super Yang-Mills theories.

There is no antisymmetric curvature for $Z_2$ or $Z_3$ discrete space. The $Z_2 \times Z_2$ discrete space is the first space that includes such antisymmetric curvature. $Z_4$ is essentially the same as $Z_2 \times Z_2$. This is the reason why we use $Z_2 \times Z_2$. We have seen the geometrical meaning of symmetric and antisymmetric curvatures.

In the NCG formulation one can also define such symmetric and antisymmetric curvatures. To see this let us use Sitarz’s one-form $\chi^i$. The two-form curvature is

$$ F = F_{ij} \chi^i \wedge \chi^j. \tag{5.1} $$

where $F_{ij}$ corresponds to the triangle curvature $\chi^i$ and $F_{ij} \neq \pm F_{ji}$. In general $\chi^i \wedge \chi^j \neq \pm \chi^j \wedge \chi^i$. The Lagrangian is given by the inner product

$$ \langle F, F \rangle = \text{tr}(F_{ij}^\dagger F_{kl})(\chi^i \wedge \chi^j, \chi^k \wedge \chi^l) $$

$$ = \text{tr}(F_{ij}^\dagger F_{kl})[\langle \chi^i, \chi^k, \chi^j, \chi^l \rangle + \langle \chi^i, \chi^l, \chi^j, \chi^k \rangle] $$

$$ = \text{tr}(F_{ij}^\dagger F_{kl})(a \delta^{ik} \delta^{jl} + b \delta^{il} \delta^{jk}) $$

$$ = a \text{tr}(F_{ij}^\dagger F_{ij}) + b \text{tr}(F_{ij}^\dagger F_{ji}), \tag{5.2} $$

where $a, b$ are generally arbitrary real constants. In the framework of NCG one cannot fix them definitely. So we have the symmetric curvature Lagrangian for $a = b$, the antisymmetric curvature Lagrangian for $a = -b$, and generally both mixed. The result (5.2) corresponds to our $L_3$ in (2.19).

Finally we emphasize that a superspace formulation for $M_4 \times Z_2 \times Z_2$ gauge theory may be one of the most important problems left unsolved. We expect that from this formulation the relationship between the antisymmetric curvature and the supersymmetry becomes clearer.

**Acknowledgments:** We thank G. Konisi and K. Shigemoto for useful discussions and invaluable comments. One of us (K.U.) is grateful to the special research funds at Tezukayama University.
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