ON HAWKING MASS AND BARTNIK MASS OF CMC SURFACES

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Abstract. Given a constant mean curvature surface that bounds a compact manifold with nonnegative scalar curvature, we obtain intrinsic conditions on the surface that guarantee the positivity of its Hawking mass. We also obtain estimates of the Bartnik mass of such surfaces, without assumptions on the integral of the squared mean curvature. If the ambient manifold has negative scalar curvature, our method also applies and yields estimates on the hyperbolic Bartnik mass of these surfaces.

1. Introduction

Given a Riemannian 3-manifold $M$, let $\Sigma \subset M$ be a closed 2-surface with a unit normal vector field $\nu$. $\Sigma$ is called a CMC surface if its mean curvature with respect to $\nu$ is a constant. Throughout this paper, we assume $\Sigma$ is a CMC surface that is topologically a sphere.

When the ambient manifold $M$ has nonnegative scalar curvature, a classic result of Christodoulou and Yau [11] is the following:

**Theorem 1.1** ([11]). Suppose $\Sigma$ is a stable, CMC sphere in a 3-manifold $M$ with nonnegative scalar curvature, then $m_H(\Sigma) \geq 0$.

Here $m_H(\Sigma)$ is the Hawking quasi-local mass [13] of $\Sigma$ in $M$, given by

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \, d\sigma \right),$$

where $|\Sigma|$ is the area and $H$ is the mean curvature of $\Sigma$, respectively, and $d\sigma$ denotes the area form on $\Sigma$. A CMC surface $\Sigma$ is called stable if

$$\int_{\Sigma} |\nabla f|^2 - (|A|^2 + \text{Ric}(\nu, \nu)) f^2 \, d\sigma \geq 0$$

for any function $f$ on $\Sigma$ with $\int_{\Sigma} f \, d\sigma = 0$, where $\nabla$ denotes the gradient on $\Sigma$, $A$ is the second fundamental form of $\Sigma$ and $\text{Ric}(\nu, \nu)$ is the Ricci curvature of $M$ along $\nu$.

The stability condition (1.2) is a natural geometric condition and it plays a key role in the estimate of $m_H(\Sigma)$ in [11].

In this paper, one of the main questions that we consider is the non-negativity of $m_H(\Sigma)$ without imposing the stability condition on $\Sigma$. Instead, we assume $\Sigma$ bounds...
a finite region $\Omega$ with nonnegative scalar curvature. There are two reasons for making such a consideration:

i) First, from a quasi-local mass point of view, it is desirable to draw information on the quasi-local mass of $\Sigma$ purely from knowledge on the geometric data $(g, H)$, where $g$ is the intrinsic metric on $\Sigma$ and $H$ is the mean curvature;

ii) Second, in the special case when $g$ is a round metric on $\Sigma$, one indeed knows

\[ m_H(\Sigma) \geq 0 \]

for any CMC surface $\Sigma$ with positive constant mean curvature $H_o$. This is a consequence of the Riemannian positive mass theorem \[21, 28\]. To see this, suppose $\Sigma = \partial \Omega$ where $\Omega$ is compact and has nonnegative scalar curvature. Gluing $\Omega$ with an exterior Euclidean region $\mathbb{R}^3 \setminus B$, where $B$ is a round ball with boundary $\partial B$ isometric to $\Sigma$, one concludes $H_o \leq H_E$, where $H_E$ is the constant mean curvature of $\partial B$ in $\mathbb{R}^3$ (see \[13, 22\]). As a result, $m_H(\Sigma) \geq 0$.

In relation to ii) above, it is natural to ask if $m_H(\Sigma)$ has positivity property when the intrinsic metric on $\Sigma$ is not far from being round. As an application of our main result, Theorem 1.3 stated in a moment, we establish positivity of $m_H(\Sigma)$ for these surfaces.

To formulate our theorems, we make use of a scaling invariant number $\zeta_g$ that measures how far a metric $g$ is from a round metric. This $\zeta_g$ was introduced in \[20\] and we recall it here. Given any metric $g$ with positive Gauss curvature $K_g$ on the sphere $S^2$, let $r_o$ be the area radius of $(S^2, g)$, i.e., $|S^2|_g = 4 \pi r_o^2$. Let \( \{g(t)\}_{0 \leq t \leq 1} \) be a smooth path of metrics on $S^2$ such that $g(0) = g$, $g(1)$ is round, $g(t)$ has positive Gauss curvature $K_g(t)$ and $\text{tr}_g g'(t) = 0$ for all $t$. (Existence of such a path, for instance, follows from Mantoulidis and Schoen’s proof of \[17, \text{Lemma 1.2}].) Associated to this path $\{g(t)\}_{0 \leq t \leq 1}$, let $\alpha$ and $\beta$ be two constants given by

\begin{equation}
\alpha = \frac{1}{4} \max_{t \in [0,1]} \max_{S^2} |g'(t)|^2_{g(t)}, \quad \beta = r_o^2 \min_{t \in [0,1]} \min_{S^2} K_{g(t)}.
\end{equation}

It is clear $\beta \in (0, 1]$ by the Gauss-Bonnet Theorem, and $\alpha > 0$ if $g$ is not a round metric. With these notations, we let

\begin{equation}
\zeta_g = \inf_{\{g(t)\}} \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}},
\end{equation}

where the infimum is taken over all such paths $\{g(t)\}_{0 \leq t \leq 1}$. We point out that $\zeta_g$ in \[1.4\] satisfies $2\zeta_g^2 = \eta(g)^{-1}$, where $\eta(g)$ was defined in \[20, \text{Section 4}].

Evidently, $\zeta_g = 0$ if $g$ is a round metric; moreover, $\zeta_g$ is invariant under constant scaling of $g$. For any $\gamma \in (0, 1)$, it was shown in \[20, \text{Proposition 4.1} \] that, if $g$ is $C^{2,\gamma}$-close to a round metric $g_*$, normalized with area $4 \pi$, then $\zeta_g \leq C \|g - g_*\|_{C^{0,\gamma}(\Sigma)}$ where $C$ is an absolute constant.

The following theorem gives a sufficient condition on the intrinsic metric on $\Sigma$ that guarantees the positivity of $m_H(\Sigma)$. 
Theorem 1.2. Let $M$ be a Riemannian 3-manifold with nonnegative scalar curvature, with boundary $\partial M$, which is a minimal surface (possibly disconnected) minimizing area among all closed surfaces which bound a domain with $\partial M$. Suppose $\Sigma \subset M$ is a CMC surface bounding a domain $\Omega$ with $\partial M$ and $\Sigma$ has positive mean curvature with respect to the unit normal pointing out of $\Omega$. Let $g$ be the intrinsic metric on $\Sigma$. Suppose $g$ has positive Gauss curvature. If
\[
\zeta_g < C \sqrt{\frac{|\partial M|}{|\Sigma|}},
\]
then $m_h(\Sigma) > 0$. Here $C$ is some absolute constant (for instance $C$ can be $\frac{\sqrt{2}}{3}$).

Remark 1.1. A manifold $M$ in Theorem 1.2 can be taken as an asymptotically flat 3-manifold for which the Riemannian Penrose inequality [5, 14] applies.

We will deduce Theorem 1.2 from a general result which holds without assumptions on $\zeta_g$.

Theorem 1.3. Suppose $\Sigma$ is a CMC surface that bounds a compact 3-manifold $\Omega$ with nonnegative scalar curvature, which may have nonempty interior horizon. Precisely, this means that $\Sigma$ is a boundary component of $\partial \Omega$ and $\Sigma_h := \partial \Omega \setminus \Sigma$, if nonempty, is a minimal surface that minimizes area among surfaces enclosing $\Sigma_h$. Suppose the intrinsic metric $g$ on $\Sigma$ has positive Gauss curvature and the mean curvature of $\Sigma$ with respect to the outward normal $\nu$ is a positive constant $H_o$. Let $r_o = \sqrt{\frac{|\Sigma|}{4\pi}}$ and define $\tau = \frac{1}{2}r_o H_o$. Let $\theta$ be the unique root to
\[
(1.5) \quad \theta^3 - \frac{3\zeta_g \tau}{2} \theta^2 - 1 = 0.
\]
Then the following holds:

a) If $\Sigma_h = \emptyset$, i.e. $\Sigma = \partial \Omega$, then
\[
\tau \leq \theta.
\]
b) If $\Sigma_h \neq \emptyset$, then
\[
\tau^2 + \frac{r_h}{r_o} \leq \theta^2.
\]
Here $r_h = \sqrt{\frac{|\Sigma_h|}{4\pi}}$.

c) Let $m_B(\Sigma)$ denote the Bartnik quasi-local mass of $\Sigma$, then
\[
m_B(\Sigma) \leq \sqrt{\frac{|\Sigma|}{16\pi}} (\theta^2 - \tau^2)
\]
In particular, this shows $m_B(\Sigma) \leq Cr_o (1 + \zeta_g \tau) \zeta_g \tau + m_h(\Sigma)$, where $C$ is an absolute constant.
We defer the definition of the Bartnik mass $m_B(\cdot)$ to the next section. For the moment, we give a few remarks about Theorem 1.3.

**Remark 1.2.** The constant $\tau$ satisfies $\tau^2 = \frac{1}{16\pi} \int_\Sigma H_o^2 d\sigma$. Thus, $m_H(\Sigma) > 0 \iff \tau < 1$.

**Remark 1.3.** When $g$ is a round metric, $\zeta_g = 0$ and $\theta = 1$. In this case, it is easily seen Theorem 1.3 is true. For instance, a) follows from ii) above; b) is a special case of the result in [19]; and c) follows from the fact that one can attach a spatial Schwarzschild manifold with mass $m = m_H(\Sigma)$ to $\Omega$ at $\Sigma$.

**Remark 1.4.** Conclusions in a), b) of Theorem 1.3 concern how nonnegative scalar curvature and interior horizon may affect the geometry of a boundary CMC surface. Such a question was studied in [20]. Under smallness assumptions on $\tau$, results weaker than a) and b) were derived in [20].

**Remark 1.5.** An upper bound of the Bartnik mass for CMC surfaces was first derived by Lin and Sormani [15] for an arbitrary metric $g$ on $S^2$. If $H_o = 0$ and the first eigenvalue of $-\Delta_g + K_g$ is positive, Mantoulidis and Schoen proved $m_B(\Sigma) = m_H(\Sigma)$ in [17]. Assuming $K_g > 0$ and imposing a smallness assumption on $\tau$ used in [20], an upper bound on $m_B(\cdot)$ of $(g, H_o)$ was derived by Cabrera Pacheco, Cederbaum, McCormick and the first author in [9]. A comparison of the estimates in [15] and [9] can be found in [9, Remark 1.2]. Our estimate in c) of Theorem 1.3 shares the same feature as that in [9], but holds without assumptions on $\tau$.

**Remark 1.6.** If one does not assume $\Sigma$ bounds a manifold with nonnegative scalar curvature, the estimate of $m_B(\Sigma)$ in c) of Theorem 1.3 is still valid provided the pair $(g, H_o)$ satisfies $m_H(\Sigma) \geq 0$. See Remark 2.4 for detailed reasons.

As a corollary of Remark 1.6 and the theorem of Christodoulou and Yau, we have

**Corollary 1.1.** The Bartnik mass of any stable CMC surface $\Sigma$ with positive Gauss curvature in a 3-manifold with nonnegative scalar curvature satisfies the estimate in c) of Theorem 1.3.

We have an analogue of Theorem 1.2 with $\sqrt{\frac{\partial M}{|\Sigma|}}$ replaced by $\frac{2m_B(\Sigma)}{r_o}$.

**Theorem 1.4.** Let $\Sigma$ be a CMC surface, with positive mean curvature $H_o$, bounding a compact 3-manifold $\Omega$ with nonnegative scalar curvature. Suppose $m_B(\Sigma) > 0$ and the intrinsic metric $g$ on $\Sigma$ has positive Gauss curvature. If

$$\zeta_g < C \left(1 + \frac{2m_B(\Sigma)}{r_o}\right)^{-1} \min\left\{\frac{2m_B(\Sigma)}{r_o}, 1\right\},$$

then $m_H(\Sigma) > 0$. Here $r_o = \sqrt{\frac{|\Sigma|}{4\pi}}$ and $C$ is some absolute constant (for instance $C$ can be $\frac{\sqrt{2}}{3}$).

**Remark 1.7.** In the setting of Theorem 1.4 one may also consider the Brown-York mass of $\Sigma$ [7, 8], given by $m_{BY}(\Sigma) = \frac{1}{8\pi} \int_\Sigma (H_E - H_o) d\sigma$, where $H_E$ is the mean...
curvature of the isometric embedding of \((\Sigma, g)\) in \(\mathbb{R}^3\). As \(H_o\) is a constant, one has
\[
m_{BY}(\Sigma) = m_H(\Sigma) + \left(\frac{1}{8\pi} \int_{\Sigma} H_E \, d\sigma - r_o\right) + \frac{r_o}{2} (1 - \tau)^2,
\]
where the second term in the bracket is nonnegative by the Minkowski inequality. In [22], Shi and Tam proved \(m_{BY}(\Sigma) \geq 0\). It would be interesting to know if the positivity of \(m_{BY}(\Sigma)\) can be used in the study of \(m_H(\Sigma)\).

Remark 1.8. In relation to the positivity of \(m_H(\Sigma)\), a natural question is its rigidity. Under the assumption \(\Sigma\) is stable, recent results concerning \(m_H(\Sigma) = 0\) were given by Sun [25] and by Shi, Sun, Tian and Wei [24].

Our proof of Theorem 1.3 is built on the previous work of the first and the third authors [20]. The techniques we use to prove Theorem 1.3 here can also be applied to the setting of manifolds with a negative scalar curvature lower bound. It is known in the literature the Hawking mass \(m_H(\Sigma)\) has a hyperbolic analogue, \(m_H^H(\Sigma)\) (see (4.1)). Recently, Cabrera Pacheco, Cederbaum and McCormick [10] formulated a hyperbolic analogue of the Bartnik mass and derived results analogous to those in [17] and [9]. Combining the techniques in proving Theorem 1.3 and a gluing tool from [10], we obtain estimates of the hyperbolic Bartnik mass, which we denote by \(m_B^H(\Sigma)\), for the boundary of a compact manifold with negative scalar curvature.

**Theorem 1.5.** Suppose \(\Sigma\) is a CMC surface bounding a compact 3-manifold \(\Omega\) with scalar curvature \(R \geq -6\kappa^2\) for some constant \(\kappa > 0\). Let \(g\) be the intrinsic metric on \(\Sigma\) and suppose its Gauss curvature satisfies \(K_g > -3\kappa^2\). Let \(\tau = \frac{1}{2} H_o r_o\), where \(r_o\) is the area radius of \(\Sigma\) and \(H_o\) is the positive constant mean curvature of \(\Sigma\) in \(\Omega\). Then the hyperbolic Bartnik mass \(m_B^H(\Sigma)\) satisfies
\[
m_B^H(\Sigma) - m_H^H(\Sigma) \leq \frac{r_o}{2} \left[\kappa^2 r_o^2 \left(1 + \frac{3}{2} \tau \xi\right)^2 + \left(1 + \frac{3}{2} \tau \xi\right)^2 - \kappa^2 r_o^2 - 1\right]
\]
\[
\leq \frac{r_o}{2} \left(3\kappa^2 r_o^2 + 1\right) \left(1 + \frac{3}{4} \tau \xi\right) \tau \xi.
\]
Here \(\xi \geq 0\) is a constant that is specified as follows.

(i) When \(\inf_\Sigma K_g \leq 0\), \(\xi = \zeta_{g,\kappa}\), where \(\zeta_{g,\kappa}\) is a constant determined by \(g\), given by \(\zeta_{g,\kappa} = \inf_{\{g(t)\}} \left(\frac{\alpha}{2\beta + 6\alpha + 2\beta r_o^2}\right)^{\frac{1}{2}}\). Here the infimum is taken over all paths of metrics \(\{g(t)\}_{0 \leq t \leq 1}\) with \(g(0) = g, g(1)\) is round, \(K_{g(t)} > -3\kappa^2\), and \(t r_{g(t)} g'(t) = 0\), and \(\alpha, \beta\) are two constants defined in (1.3).

(ii) When \(\inf_\Sigma K_g > 0\), \(\xi\) is a constant given in (4.38). In particular, \(\xi\) satisfies \(\xi \leq \zeta_g \theta^2 \leq \zeta_g \left(1 + \frac{3}{2} \tau \zeta_g\right)^2\). Here \(\zeta_g\) is given in (1.4) and \(\theta\) is the unique root to \(\theta^3 - \frac{3}{2} \tau \zeta_g \theta^2 - 1 = 0\).
The remainder of this paper is organized as follows. In Section 2, we consider manifolds with nonnegative scalar curvature and prove Theorem 1.3. In Section 3, we apply Theorem 1.3 to prove Theorems 1.2 and 1.4. In Section 4, we consider manifolds with negative scalar curvature and prove Theorem 1.5.

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2. Manifolds with nonnegative scalar curvature

Let $\Omega$, $\Sigma$, $r_0$, $H_o$ and $\tau$ be given in Theorem 1.3. By Remark 1.3, it suffices to assume that the intrinsic metric $g$ on $\Sigma$ is not round. We divide the proof of Theorem 1.3 into a few steps:

Step 1. We review the construction of a suitable metric on $N := [0,1] \times \Sigma$ from [20]. Let $\{g(t)\}_{t \in [0,1]}$ be any given smooth path of metrics on $\Sigma$, satisfying $g(0) = g$, $g(1)$ is round, $K_{g(t)} > 0$ and $\text{tr}_{g(t)} g'(t) = 0$, $\forall t$. Given any parameter $m \in (-\infty, \frac{1}{2} r_0)$, consider part of a spatial Schwarzschild metric

$$\gamma_m = \frac{1}{1 - 2m} dr^2 + r^2 \sigma, \ r \geq r_o,$$

where $\sigma$ is the standard metric with area $4\pi$ on the sphere $S^2$. Rewriting $\gamma_m$ as $\gamma_m = ds^2 + u_m^2(s) \sigma$, $s \geq 0$, one has $u_m(0) = r_0$ and

$$(2.1) \quad u_m'(s) = \left( 1 - \frac{2m}{u_m(s)} \right)^\frac{1}{2}.$$

Let $k > 0$ be a constant given by

$$(2.2) \quad k = \tau \left( 1 - \frac{2m}{r_o} \right)^{-\frac{1}{2}}.$$

Define a metric

$$\gamma^{(m)} = A^2 dt^2 + r_o^{-2} u_m^2(Akt) g(t).$$

Here $A > 0$ is some constant which will be chosen later. The following properties of $(N, \gamma^{(m)})$ follow from direct calculation (see (2.1) – (2.16) in [20]):

- each $\Sigma_t := \{t\} \times \Sigma$ has positive constant mean curvature w.r.t $\partial_t$;
- the induced metric on $\Sigma_0$ is $g$, and the mean curvature of $\Sigma_0$ w.r.t $\partial_t$ is $H_o$;
- the Hawking mass of each $\Sigma_t$ is

$$(2.3) \quad m_H(\Sigma_t) = \frac{1}{2} (u_m(Akt) - r_o) (1 - k^2) + m_H(\Sigma);$$
• the scalar curvature $R(\gamma^{(m)})$ of $\gamma^{(m)}$ satisfies

\[
R(\gamma^{(m)}) = 2u_m^{-2} \left[ r_o^2 K_{g(t)} - k^2 - \frac{1}{8} |g'(t)|_g^2 A^{-2} u_m^2 \right] \geq 2u_m^{-2} \left[ \beta - k^2 - \frac{1}{2} \alpha A^{-2} u_m^2 (Ak) \right].
\]

(2.4)

By (2.4), a sufficient condition to have $R(\gamma^{(m)}) \geq 0$ is that there exists an $A > 0$ such that

\[
\beta - k^2 - \frac{1}{2} \alpha A^{-2} u_m^2 (Ak) \geq 0.
\]

If this is the case, then necessarily $k^2 < \beta \leq 1$. As $k^2 < 1$ is equivalent to $m < m_o$, where $m_o = \frac{r_o}{2}(1 - \tau^2)$ is the Hawking mass of $\Sigma$, such an $A$ exists only if $m < m_o$.

Step 2. For any suitably given $m < m_o$, we choose an optimal $A = A_o$ such that $\gamma^{(m)}$ has nonnegative scalar curvature.

**Lemma 2.1.** For each $m \in (-\infty, m_o)$ satisfying

(2.5) \[ \beta > \left( 1 + \frac{\alpha}{2} \right) k^2, \]

there exists a constant $A_o > 0$ such that

(2.6) \[ \beta - k^2 - \frac{\alpha}{2} A_o^{-2} u_m^2 (A_o k) = 0. \]

Moreover, the set of all such $A_o$ is bounded from above and away from zero as $m$ tends to $-\infty$. That is, there are constants $B_2 > B_1 > 0$ and $\tilde{m} < 0$ such that $B_1 < A_o < B_2$ whenever $m < \tilde{m}$.

**Proof.** Since $\alpha > 0$, (2.6) is equivalent to

(2.7) \[ k^{-2} \alpha^{-1} (\beta - k^2) = (A_o k)^{-2} u_m^2 (A_o k). \]

Consider the function $f_m(s) = s^{-1} u_m(s)$. One has $\lim_{s \to 0^+} f_m(s) = \infty$ and

(2.8) \[ \lim_{s \to \infty} f_m(s) = \lim_{s \to \infty} u_m'(s) = \lim_{s \to \infty} \left( 1 - \frac{2m}{u_m(s)} \right)^{\frac{1}{2}} = 1. \]

Thus, the range of $f_m$ includes $(1, \infty)$. Since (2.5) implies

\[ k^{-2} \alpha^{-1} (\beta - k^2) > 1, \]

the existence of such an $A_o$ follows.

Now, by (2.6) and the fact $u_m(s) \geq r_o$, one has

(2.9) \[ \beta - k^2 = \frac{\alpha}{2} A_o^{-2} u_m^2 (A_o k) \geq \frac{\alpha}{2} A_o^{-2} r_o^2, \]

which gives

(2.10) \[ A_o^2 \geq \frac{\alpha}{2} r_o^2 (\beta - k^2)^{-1}. \]

As $\lim_{m \to -\infty} k = 0$, this shows $A_o$ is bounded away from 0 as $m \to -\infty$. 
Next, suppose \( m < 0 \). By (2.1),
\[
 u_m'(s) \leq \left( 1 - \frac{2m}{r_0} \right)^{\frac{1}{2}} = \tau k^{-1},
\]
which implies
\[
 u_m(s) \leq r_0 + \tau k^{-1}s.
\]
Thus, for \( 0 \leq s \leq A_o k \),
\[
 u_m'(s) \leq \sqrt{1 - \frac{2m(r_0 + A_o \tau)}{u_m^2(s)}},
\]
or equivalently
\[
 u_m(s)u_m'(s) \leq \sqrt{u_m^2(s) - 2m(r_0 + A_o \tau)}.
\]
Upon integration, (2.12) shows
\[
 u_m^2(A_0 k) \leq r_o^2 + A_o^2 k^2 + 2A_o k \sqrt{r_o^2 - 2m(r_0 + A_o \tau)}.
\]
Combined with (2.6) and (2.2), this implies
\[
 \beta - k^2 \leq \frac{\alpha}{2} A_o^{-2} \left( r_o^2 + A_o^2 k^2 + 2A_o k \sqrt{r_o^2 - 2m(r_0 + A_o \tau)} \right),
\]
i.e.
\[
 \beta - k^2 - \frac{\alpha}{2} k^2 \leq \frac{\alpha}{2} \left( r_o^2 A_o^{-2} + 2A_o^{-1} \sqrt{\tau^2 r_o^2 + (\tau^2 - k^2)r_o A_o \tau} \right).
\]
Since \( \beta > 0 \) and \( \lim_{m \to -\infty} k = 0 \), it follows from (2.13) that \( A_o \) is bounded from above as \( m \to -\infty \).

In what follows, for each \( m \) satisfying (2.5), we choose \( A \) to be the smallest root \( A_o \) to equation (2.6). By (2.4), the metric
\[
 \gamma^{(m)} = A_o^2 dt^2 + r_o^{-2} u_m^2(A_o k) g(t)
\]
has nonnegative scalar curvature. For each \( m \), we glue \((N, \gamma^{(m)})\) to \( \Omega \) by identifying \( \Sigma_0 \) with \( \Sigma \). The argument in [20, Section 3] leading to (3.9) therein then gives
\[
 m_H(\Sigma_1) \geq \sqrt{\frac{\abs{\Sigma_h}}{16\pi}}, \quad \text{if } \Sigma_h \neq \emptyset,
\]
and
\[
 m_H(\Sigma_1) \geq 0, \quad \text{if } \Sigma_h = \emptyset.
\]
Here, by (2.3),
\[
 m_H(\Sigma_1) = \frac{1}{2} (u_m(A_o k) - r_o) (1 - k^2) + m_H(\Sigma).
\]

Step 3. We follow the idea in [20] by letting \( m \to -\infty \) in (2.14) and (2.15). Since \( \lim_{m \to -\infty} k = 0 \), (2.5) is satisfied for every sufficiently negative \( m \). By Lemma 2.1, there exists a sequence \( \{m_i\} \) with \( \lim_{i \to \infty} m_i = -\infty \) such that the corresponding
sequence \( \{A_o^{(i)}\} \), where \( A_o^{(i)} \) is the \( A_o \) associated with \( m_i \), has a finite limit. Consequently, by (2.6), the sequence \( \{u_{m_i}(A_o^{(i)} k^{(i)})\} \) has a finite limit as well. Here \( k^{(i)} \) is the \( k \) associated with \( m_i \).

We evaluate \( \lim_{i \to \infty} u_{m_i}(A_o^{(i)} k^{(i)}) \). One way to achieve this is to implicitly solve (2.1). Suppose \( m < 0 \). Let \( v_m(s) > 0 \) be the function such that

\[
(2.17) \quad \frac{-2m}{u_m(s)} = \sinh^{-2}(v_m(s)).
\]

In term of \( v_m(s) \), (2.1) becomes

\[
-4m \sinh^2(v_m(s))v'_m(s) = 1,
\]

or equivalently

\[
(2.18) \quad (-m)[\sinh(2v_m(s)) - 2v_m(s)]' = 1.
\]

Plugging in

\[
\sinh(2v_m(s)) = 2 \left( \frac{-u_m(s)}{2m} \right)^{\frac{1}{2}} \left( 1 - \frac{u_m(s)}{2m} \right)^{\frac{1}{2}}
\]

and

\[
v_m(s) = \ln \left( \left( \frac{-u_m(s)}{2m} \right)^{\frac{1}{2}} + \left( 1 - \frac{u_m(s)}{2m} \right)^{\frac{1}{2}} \right),
\]

we get

\[
2m \left[ \ln \left( \left( \frac{-u_m(s)}{2m} \right)^{\frac{1}{2}} + \left( 1 - \frac{u_m(s)}{2m} \right)^{\frac{1}{2}} \right) - \left( \frac{-u_m(s)}{2m} \right)^{\frac{1}{2}} \left( 1 - \frac{u_m(s)}{2m} \right)^{\frac{1}{2}} \right] - 2m \left[ \ln \left( \left( \frac{-r_o}{2m} \right)^{\frac{1}{2}} + \left( 1 - \frac{r_o}{2m} \right)^{\frac{1}{2}} \right) - \left( \frac{-r_o}{2m} \right)^{\frac{1}{2}} \left( 1 - \frac{r_o}{2m} \right)^{\frac{1}{2}} \right] = s.
\]

Taking \( m = m_i, k = k^{(i)}, A_o = A_o^{(i)}, s = A_o^{(i)} k^{(i)} \), and let \( u_{m_i}^{(i)} := u_{m_i}(A_o^{(i)} k^{(i)}) \), we have

\[
(2.19) \quad 2m_i \left[ \ln \left( \left( \frac{-u_{m_i}^{(i)}}{2m_i} \right)^{\frac{1}{2}} + \left( 1 - \frac{u_{m_i}^{(i)}}{2m_i} \right)^{\frac{1}{2}} \right) - \left( \frac{-u_{m_i}^{(i)}}{2m_i} \right)^{\frac{1}{2}} \left( 1 - \frac{u_{m_i}^{(i)}}{2m_i} \right)^{\frac{1}{2}} \right] - 2m_i \left[ \ln \left( \left( \frac{-r_o}{2m_i} \right)^{\frac{1}{2}} + \left( 1 - \frac{r_o}{2m_i} \right)^{\frac{1}{2}} \right) - \left( \frac{-r_o}{2m_i} \right)^{\frac{1}{2}} \left( 1 - \frac{r_o}{2m_i} \right)^{\frac{1}{2}} \right] = A_o^{(i)} k^{(i)}.
\]
By Lemma 2.1, $\lim_{m_i \to \infty}^{(i)} u_{m_i} = O(|m_i|^{-1})$ as $i \to \infty$. Hence,

$$
\ln \left(\left(\frac{-u_{m_i}}{2m_i}\right)^{\frac{1}{2}} + \left(1 - \frac{u_{m_i}}{2m_i}\right)^{\frac{1}{2}}\right) - \left(\frac{-u_{m_i}}{2m_i}\right)^{\frac{1}{2}} \left(1 - \frac{u_{m_i}}{2m_i}\right)^{\frac{1}{2}}
$$

$$
= - \frac{2}{3} \left(\frac{-u_{m_i}}{2m_i}\right)^{\frac{3}{2}} + O(|m_i|^{-2}).
$$

Combined with (2.2), this gives

$$
\lim_{i \to \infty} \frac{2m_i}{k^{(i)}} \left[\ln \left(\left(\frac{-u_{m_i}}{2m_i}\right)^{\frac{1}{2}} + \left(1 - \frac{u_{m_i}}{2m_i}\right)^{\frac{1}{2}}\right) - \left(\frac{-u_{m_i}}{2m_i}\right)^{\frac{1}{2}} \left(1 - \frac{u_{m_i}}{2m_i}\right)^{\frac{1}{2}}\right]
$$

$$
= \frac{2}{3} r_o^{-\frac{1}{2}} \tau^{-1} \lim_{i \to \infty} u_{m_i}^{\frac{3}{2}}.
$$

Similarly,

$$
\lim_{i \to \infty} \frac{2m_i}{k^{(i)}} \left[\ln \left(\left(-\frac{r_o}{2m_i}\right)^{\frac{1}{2}} + \left(1 - \frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right) - \left(-\frac{r_o}{2m_i}\right)^{\frac{1}{2}} \left(1 - \frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right]
$$

$$
= \frac{2}{3} r_o^{-\frac{1}{2}} \tau^{-1} \lim_{i \to \infty} u_{m_i}^{\frac{3}{2}}.
$$

Hence, by (2.19), we have

$$
\lim_{i \to \infty} u_{m_i}^{(i)} = r_o \left(1 + \frac{3}{2} \tau r_o^{-1} \lim_{i \to \infty} A^{(i)}_o\right)^{\frac{3}{2}}.
$$

Now let $\bar{A}_o := \lim_{i \to \infty} A^{(i)}_o$. By (2.10),

$$
\bar{A}_o \geq \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} r_o > 0.
$$

Taking limit in (2.20), we have

$$
\beta = \frac{\alpha}{2} \bar{A}_o^{-2} \left(\lim_{i \to \infty} u_{m_i}^{(i)}\right)^{\frac{2}{3}}.
$$

Therefore, it follows from (2.20) and (2.21) that

$$
\left(\frac{r_o}{\bar{A}_o}\right)^{\frac{3}{2}} + \frac{3\tau}{2} \left(\frac{r_o}{\bar{A}_o}\right)^{\frac{1}{2}} = \left(\frac{2\beta}{\alpha}\right)^{\frac{4}{3}}.
$$

We now define $\theta > 0$ such that

$$
\frac{\bar{A}_o}{r_o} = \theta^2 \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}.
$$

Then (2.22) shows

$$
\theta^3 - \frac{3\tau}{2} \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} \theta^2 - 1 = 0.
$$
By (2.20), (2.23) and (2.24), we have

\[
\lim_{i \to \infty} u_{m_i}^{(i)} = r_o \left( 1 + \frac{3\tau}{2} \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} \right) = r_o \theta^2.
\]

From this and (2.16), we conclude

\[
\lim_{i \to \infty} m_H(\Sigma_1) = \lim_{i \to \infty} \frac{1}{2} \left( u_{m_i}^{(i)} - r_o \right) (1 - k^{(i)}) + m_H(\Sigma)
\]

\[
\frac{r_o}{2} (\theta^2 - 1) + m_H(\Sigma_o)
\]

\[
= \frac{r_o}{2} (\theta^2 - \tau^2).
\]

Here \(m_H(\Sigma_1)\) denotes the Hawking mass of \(\Sigma_1\) in \((N, \gamma^{(m_i)})\).

**Remark 2.1.** Since \(\{A^{(i)}\}\) can be taken to be any converging sequence, the argument above indeed shows

\[
\lim_{m \to -\infty} u_m(A_o) = r_o \theta^2 \quad \text{and} \quad \lim_{m \to -\infty} A_o = r_o \theta^2 \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3}.
\]

The following theorem follows directly from (2.14), (2.15) and (2.26).

**Theorem 2.1.** Let \(\Omega, \Sigma, g, r_o, H_o, \alpha, \beta\) be given in Theorem 1.3. Let \(\{g(t)\}_{t \in [0,1]}\) be a smooth path of metrics on \(\Sigma\) satisfying \(g(0) = \gamma, g(1)\) is round, \(K_{g(t)} > 0\) and \(tr_{g(t)}g'(t) = 0\). Let \(\alpha\) and \(\beta\) be the constants associated to \(\{g(t)\}_{t \in [0,1]}\), given by (1.3).

Let \(\theta > 0\) be the number that is the unique root to

\[
\theta^3 - \frac{3\tau}{2} \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} \theta^2 - 1 = 0.
\]

Then

\[
\tau \leq \theta \quad \text{if} \quad \Sigma_h = \emptyset,
\]

and

\[
\tau^2 + \frac{r_h}{r_o} \leq \theta^2 \quad \text{if} \quad \Sigma_h \neq \emptyset.
\]

Here \(r_h = \sqrt{\frac{\left|\Sigma_h\right|}{4\pi}}\) is the area radius of \(\Sigma_h\).

**Remark 2.2.** Let \(f(x) = x^3 - \frac{3\tau}{2} \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} x^2 - 1\). As \(f'(x) = 3x \left[ x - \tau \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} \right]\), it is easily seen that, given a number \(x\),

\[
x \leq \theta \iff f(x) \leq 0.
\]

Thus, the conclusion in Theorem 2.1 can be equivalently stated as

\[
\tau^3 \left[ 1 - \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} \right] \leq 1 \quad \text{if} \quad \Sigma_h = \emptyset,
\]

\[
(2.27)
\]

\[
\tau^3 \left[ 1 - \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^\frac{1}{3} \right] \leq 1 \quad \text{if} \quad \Sigma_h = \emptyset,
\]
and
\begin{equation}
\left(\tau^2 + \frac{r_h}{r_o}\right)^{\frac{3}{2}} - \frac{3\tau}{2} \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} \left(\tau^2 + \frac{r_h}{r_o}\right) \leq 1 \quad \text{if } \Sigma_h \neq \emptyset.
\end{equation}

Part a) and b) of Theorem 1.3 now follow from Theorem 2.1 by considering sequences of paths of metrics \(\{g(t)\}_{t\in[0,1]}\) with \(\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} \to \zeta_g\).

**Remark 2.3.** Because we have chosen \(A_o > 0\) to be the smallest root to (2.6), we have \(\beta - k^2 - \frac{\alpha}{2} A^{-2} u_m^2(Ak) < 0\), \(\forall A \in (0, A_o)\). Thus, if \(\tilde{A}_o > 0\) is any number such that \(\beta - k^2 - \frac{\alpha}{2} \tilde{A}_o^{-2} u_m^2(\tilde{A}_o k) \geq 0\), we must have \(A_o \leq \tilde{A}_o\), and hence \(u_m(A_o k) \leq u_m(\tilde{A}_o k)\). Thus, besides requiring no assumptions on \(\tau\), inequalities in a) and b) of Theorem 1.3 are stronger than those of Theorems 1.1 and 1.2 in [20].

In the remaining part of this section, we prove part c) of Theorem 1.3. First, we review the definition of \(m_B(\cdot)\). Given a metric \(g\) and a function \(H\) on a surface \(\Sigma\) that is topologically a sphere, the Bartnik mass \(m_B(\Sigma)\) associated to the triple \((\Sigma, g, H)\) can be defined as
\[
\inf \{m_{ADM}(M, \gamma) \mid (M, \gamma) \text{ is an admissible extension of } (\Sigma, g, H)\}.
\]

Here \(m_{ADM}(\cdot)\) denotes the ADM mass, and an asymptotically flat 3-manifold \((M, \gamma)\) with boundary is an admissible extension of \((\Sigma, g, H)\) if: \(\partial M\) is isometric to \((\Sigma, g)\); the mean curvature of \(\partial M\) in \((M, \gamma)\) equals \(H\); \((M, g)\) has nonnegative scalar curvature; and either \((M, \gamma)\) contains no closed minimal surfaces (except possibly \(\partial M\)), or \(\partial M\) is outer-minimizing in \((M, \gamma)\) (see [5, 6, 14, 29] for instance).

Working with this definition, one sees that part c) of Theorem 1.3 would be a natural consequence of the previous three steps. The reason is, because \(\Sigma_1\) has a round intrinsic metric and constant mean curvature in \((N, \gamma^{(m)})\), one can attach part of a spatial Schwarzschild manifold with mass \(m_H(\Sigma_1)\), outside a rotationally symmetric sphere isometric to \(\Sigma_1\), to \((N, \gamma^{(m)})\) at \(\Sigma_1\). The resulting manifold would be an admissible extension of \((\Sigma, g, H_o)\), except it may not be smooth across \(\Sigma_1\). If it were smooth across \(\Sigma_1\), then \(m_B(\Sigma) \leq m_H(\Sigma_1)\) by definition. Passing to the limit in Step 3, one would obtain the estimate in c).

To give a precise proof of c), we can make use of a gluing result in [9]. For this purpose, we return to the end of Step 2 to point out a few additional feature of \((N, \gamma^{(m)})\). By (2.14) and (2.15), the Hawking mass of \(\Sigma_1\) in \((N, \gamma^{(m)})\) satisfies
\begin{equation}
m_H(\Sigma_1) \geq 0.
\end{equation}

By (2.4) and (2.6), the scalar curvature of \(\gamma^{(m)}\) at any \((x, t) \in \Sigma \times [0,1] \subset N\) satisfies
\begin{equation}
R(\gamma^{(m)})(x, t) = 2u_m^{-2}(A_o k t) \left[ r_o^2 K_{g(t)}(x) - k^2 - \frac{1}{8} |g'(t)|^2 g(t)(x) A_o^{-2} u_m^2(A_o k t) \right]
\end{equation}
\begin{equation}
> 2u_m^{-2}(A_o k) \left[ \beta - k^2 - \frac{1}{2} \alpha A_o^{-2} u_m^2(Ak) \right] = 0.
\end{equation}
At $t = 1$, we also have

$$R(\gamma^m(x, 1)) = 2u_m^{-2}(A_o k) \left[ 1 - k^2 - \frac{1}{8} |g'(1)|^2 (x) A_o^{-2} u_m^2(A_o k) \right]$$

\begin{equation}
> 2u_m^{-2}(A_o k) \left[ \beta - k^2 - \frac{1}{2} \alpha A_o^{-2} u_m^2(A k) \right] = 0,
\end{equation}

because $\beta < 1$ (since $g(1)$ is round while $g(0) = g$ is not round). Thus, $R(\gamma^m(x, 1)) > 0$ everywhere on $N$.

Now we can apply [9, Proposition 2.1] to $(N, \gamma^m)$. We may first assume the path $\{g(t)\}_{t \in [0, 1]}$ has a property $g(t) = g(1)$ for $t$ in $(1 - \delta, 1]$ for some $\delta > 0$. In this case, a direct application of [9, Proposition 2.1] gives

\begin{equation}
m_b(S) \leq m_H(S).
\end{equation}

In general, by approximating $\{g(t)\}_{t \in [0, 1]}$ with paths satisfying such a property (see (3.9) – (3.13) in [9]), one knows (2.32) still holds.

Combining (2.26) and (2.32), we obtain

\begin{equation}
m_b(S) \leq \lim_{i \to \infty} m_H(S_i)
= \frac{r_o}{2} (\theta^2 - 1) + m_H(S_o).
\end{equation}

Elementary estimates show that the root $\theta$ to (1.5) satisfies $1 \leq \theta \leq 1 + \frac{3}{2} \tau \zeta_g$. Thus,

\begin{equation}
m_b(S) \leq \frac{3}{2} r_o \left( 1 + \frac{3}{4} \tau \zeta_g \right) \tau \zeta_g + m_H(S).
\end{equation}

This completes the proof of part c) of Theorem 1.3.

Remark 2.4. In Theorem 1.3, we assume $\Sigma$ bounds a compact 3-manifold with non-negative scalar curvature. If this assumption is dropped, the above proof is still valid to show (2.33), provided a sufficient condition $m_b(S) \geq 0$ is assumed on $(g, H_o)$. This is because, by (2.16), $m_H(S_1) > m_H(S)$ for each $(N, \gamma^m)$ used in the proof.

Remark 2.5. In [9], it was shown if $(g, H_o)$ on $\Sigma$ satisfies $\tau^2 < \frac{\beta}{1 + \alpha}$, then

\begin{equation}
m_b(S) \leq \left[ \frac{\alpha}{\beta - (1 + \alpha) \tau^2} \right]^{\frac{1}{2}} \tau m_H(S) + m_H(S).
\end{equation}

Comparing (2.33) and (2.35), we see (2.33) requires no assumptions on $\tau$ and it improves (2.35) when $\tau$ is small. For instance, as $\tau \to 0$,

$$\frac{\theta^2 - 1}{1 - \tau^2} = \left( \frac{\alpha}{2 \beta} \right) \sqrt{\tau + O(\tau^2)} \text{ and } \left[ \frac{\alpha}{\beta - (1 + \alpha) \tau^2} \right]^{\frac{1}{2}} \tau = \left( \frac{\alpha}{\beta} \right) \sqrt{\tau + O(\tau^2)}.$$
3. Applications of Theorem 1.3

We apply Theorem 1.3 to prove Theorems 1.2 and 1.4.

Lemma 3.1. Given two constants $b > 0$ and $\lambda > 0$, consider the function

\[(3.1) \Phi(\tau) = \left(\tau^2 + \lambda\right)\frac{3}{2} - \frac{b}{\tau} \left(\tau^2 + \lambda\right) - 1, \quad \tau \in (0, \infty).\]

If $b < \min\{\frac{\lambda}{\sqrt{1+\lambda}}, \frac{1}{\sqrt{1+\lambda}}\}$, then $\Phi(\tau) > 0$ for any $\tau \geq 1$.

Proof. One has

\[(1 + \lambda)^{-1}\Phi(1) = \sqrt{1 + \lambda} - \frac{1}{1 + \lambda} - b\]

\[(3.2) = \frac{\lambda}{(1 + \lambda)(\sqrt{1 + \lambda} + 1)} + \frac{\lambda}{\sqrt{1 + \lambda}} - b > 0,
\]

and

\[(3.3) (3\tau^2 + \lambda)^{-1}\Phi'(\tau) = \frac{3(\tau^2 + \lambda)^{\frac{1}{2}}}{3\tau^2 + \lambda} \frac{\tau}{\sqrt{1 + \lambda}} - b \geq \frac{\tau}{\sqrt{1 + \lambda}} - b > 0\]

for $\tau \geq 1$. The lemma follows. \qed

Proof of Theorem 1.2. We take the constant $C = \frac{\sqrt{2}}{3}$. Suppose

\[(3.4) \zeta_g < \frac{\sqrt{2}}{3} \sqrt{\frac{\partial M}{|\Sigma|}}.\]

Applying b) of Theorem 1.3 to the domain $\Omega$, bounded by $\Sigma$ and $\partial M$, in $M$, we have

\[(3.5) \tau^2 + \frac{r_h}{r_o} \leq \theta^2,\]

where $r_o = \sqrt{\frac{\Sigma}{4\pi}}$, $r_h = \sqrt{\frac{\partial M}{4\pi}}$, $\tau = \frac{1}{2}r_o H_o$, $H_o$ is the positive constant mean curvature of $\Sigma$, and $\theta > 0$ is the unique root to (1.5). Similarly to Remark 2.2, we know (3.5) is equivalent to

\[(3.6) \left(\tau^2 + \frac{r_h}{r_o}\right)^{\frac{3}{2}} - \frac{3\tau\zeta_g}{2} \left(\tau^2 + \frac{r_h}{r_o}\right) - 1 \leq 0.\]

Let $b = \frac{3\zeta_g}{2}$ and $\lambda = \frac{r_h}{r_o}$. Condition (3.4) becomes $b < \frac{1}{\sqrt{2}}\lambda$. Since $|\partial M| \leq |\Sigma|$, $\lambda \leq 1$. Thus, by (3.6) and Lemma 3.1, we have $\tau < 1$, i.e. $m(H_o(\Sigma)) > 0$. \qed

Theorem 1.3 is proved in a similar way.
Proof of Theorem 1.4. Since $\Sigma$ bounds a compact $\Omega$ with nonnegative scalar curvature, $m_B(\Sigma)$ satisfies the estimate in c) of Theorem 1.3, i.e.

\begin{equation}
\tau^2 + \frac{2m_B(\Sigma)}{r_o} \leq \theta^2.
\end{equation}

Therefore,

\begin{equation}
\left(\tau^2 + \frac{2m_B(\Sigma)}{r_o}\right)^{\frac{3}{2}} - \frac{3r_o}{2} \left(\tau^2 + \frac{2m_B(\Sigma)}{r_o}\right) - 1 \leq 0.
\end{equation}

Now suppose

\begin{equation}
\zeta_g < \sqrt{\frac{2}{3}} \left(1 + \frac{2m_B(\Sigma)}{r_o}\right)^{-\frac{1}{2}} \min\left\{\frac{2m_B(\Sigma)}{r_o}, 1\right\}.
\end{equation}

Let $b = \frac{3\zeta_g}{2}$ and $\lambda = \frac{2m_B(\Sigma)}{r_o}$, (3.9) shows

\[b < (1 + \lambda)^{-\frac{1}{2}} \min\{\lambda, 1\}.
\]

By (3.8) and Lemma 3.1, we conclude $\tau < 1$, i.e $m_H(\Sigma) > 0$.

\[\square\]

4. Manifolds with negative scalar curvature

In the remaining of this paper, we turn attention to CMC surfaces in manifolds with negative scalar curvature. Let $M$ denote a Riemannian 3-manifold with scalar curvature $R \geq -6\kappa^2$, where $\kappa > 0$ is a constant. Let $\Sigma \subset M$ be a closed surface. In this context, the hyperbolic Hawking mass of $\Sigma$ is given by

\begin{equation}
m_H^\text{H}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \, d\sigma + \frac{1}{4\pi}\kappa^2|\Sigma|\right).
\end{equation}

A natural analogue of the Bartnik mass is

\[m_B^\text{H}(\Sigma) = \inf\{m(M^\text{H}, \gamma^\text{A})\},
\]

where $m(\cdot)$ is the mass of an asymptotically hyperbolic manifold and the infimum is taken over a space of “admissible asymptotically hyperbolic extensions” $(M^\text{H}, \gamma^\text{A})$ of $(\Sigma, g, H)$. We refer readers to the recent work of Cabrera Pacheco, Cederbaum and McCormick [10] for a detailed discussion of this definition.

We now let $\Omega, \Sigma, g, H_o, r_o$ and $\tau$ be given in Theorem 1.5. If $g$ is a round metric, then $\zeta = 0$ and $\xi = 0$. In this case, (1.7) reduces to $m_B^\text{H}(\Sigma) \leq m_H^\text{H}(\Sigma)$. This is true because a spatial AdS-Schwarzschild manifold with mass $m = m_H^\text{H}(\Sigma)$, lying outside a rotationally symmetric sphere isometric to $\Sigma$, can be attached to $\Omega$ at $\Sigma$.

In what follows, we assume that $g$ is not a round metric. Under the assumption $K_g > -3\kappa^2$, there exists a smooth path of metrics $\{g(t)\}_{0 \leq t \leq 1}$ on $\Sigma$ with $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $tr_{g(t)} g'(t) = 0$. (Existence of such a path can be provided by the solution to the normalized Ricci flow on $\Sigma$ starting at $g$. See [16, Lemma 4.2] and [10, Lemma 5.1] for instance.) We fix such a path $\{g(t)\}_{0 \leq t \leq 1}$ and let $\alpha, \beta$ be the constants given in (1.3). Then $\alpha > 0$ and $1 > \beta > -3\kappa^2 r_o^2$. 

Similar to Step 1 in the proof of Theorem 1.3, now one can consider a spatial AdS-Schwarzschild metric $\gamma_m = (1 - \frac{2m}{r} + \kappa^2 r^2)^{-1} dr^2 + r^2 \sigma$, $r \geq r_o$, where $m$ is any parameter such that $1 - \frac{2m}{r_o} + \kappa^2 r_o^2 > 0$. Rewriting $\gamma_m$ as $\gamma_m = ds^2 + u_m^2(s)\sigma$, $s \geq 0$, one has $u_m(0) = r_o$ and

$$u_m(s) = \left(1 - \frac{2m}{u_m(s)} + \kappa^2 u_m^2(s)\right)^{\frac{1}{2}}.$$  

Define a constant

$$k = \tau \left(1 - \frac{2m}{r_o} + \kappa^2 r_o^2\right)^{-\frac{1}{2}}$$

and a metric

$$\gamma^{(m)} = A^2 dt^2 + r_o^{-2} u_m^2(Akt)g(t)$$

on $N = [0, 1] \times \Sigma$, where $A > 0$ is a constant to be chosen. Direct calculation shows

- each $\Sigma_t := \{t\} \times \Sigma$ has positive constant mean curvature w.r.t $\partial_t$;
- the induced metric on $\Sigma_0$ is $g$, and the mean curvature of $\Sigma_0$ w.r.t $\partial_t$ is $H_o$;
- the hyperbolic Hawking mass of each $\Sigma_t$ is

$$m^m_H(\Sigma_t) = \frac{1}{2}(u_m(Akt) - r_o)(1 - k^2) + \frac{1}{2} \kappa^2 (1 - k^2)(u_m(Akt) - r_o^3) + m_H(\Sigma);$$

- the scalar curvature $R(\gamma^{(m)})$ of $\gamma^{(m)}$ satisfies

$$R(\gamma^{(m)}) = 2u_m^{-2}(r_o^2 K_{g(t)} - k^2) - \frac{1}{4}|g'(t)|^2 g(t) A^{-2} - 6k^2 \kappa^2$$

$$\geq 2u_m^{-2}(\beta - k^2) - \alpha A^{-2} - 6k^2 \kappa^2.$$  

**Remark 4.1.** The manifold $(N, \gamma^{(m)})$, constructed above via the warping function of an AdS-Schwarzschild metric, was also used in [10]. Estimates on $m^m_B(\cdot)$ for a pair $(g, H_o)$ were derived in [10] under suitable smallness conditions on $H_o$. In this section, by assuming $(g, H_o)$ arises from the boundary of $\Omega$ and by making an optimal choice of $A$, we obtain estimates on $m^m_B(\cdot)$ that require no assumption on $H_o$.

By (1.5), a sufficient condition to guarantee $R(\gamma^{(m)}) \geq -6\kappa^2$ on $N$ is

$$u_m^{-2}(Akt)(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2 (1 - k^2) \geq 0, \ \forall \ t \in [0, 1].$$

As $u_m(s)$ is monotone, (1.6) is equivalent to

$$u_m^{-2}(Ak)(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2 (1 - k^2) \geq 0, \ \text{if} \ \beta - k^2 > 0,$$

or

$$r_o^{-2}(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2 (1 - k^2) \geq 0, \ \text{if} \ \beta - k^2 \leq 0.$$
Next, as in Step 2 in the proof of Theorem 1.3, we choose an optimal $A = A_o$ so that (4.7) or (4.8) are met. If $\beta \leq 0$, using the fact $\beta + 3\kappa^2 r_o^2 > 0$, one easily sees an optimal $A$ satisfying (4.8) is

$$A_o = r_o \left( \frac{\frac{1}{2} \alpha A_o}{\beta + 3\kappa^2 r_o^2 - (1 + 3\kappa^2 r_o^2) k^2} \right)^{1/2},$$

provided $k$ is small.

If $\beta > 0$ (which occurs only if $\inf_{\Sigma} K_g > 0$), we choose an optimal $A_o$ satisfying (4.7) according to the following lemma.

**Lemma 4.1.** Suppose $\alpha > 0$ and $\beta > 0$. For every $m \in (-\infty, 0)$ satisfying $k^2 < \beta$, there exists a positive constant $A_o$ such that

$$u_m(A_o k) = 0.$$  

Moreover, the set of all such $A_o$ is bounded from above and away from zero as $m$ tends to $-\infty$.

**Proof.** For each fixed $m$, consider the function

$$f_m(A) = (\beta - k^2) + \left[ 3\kappa^2 (1 - k^2) - \frac{1}{2} \alpha A^{-2} \right] u_m(A k), \ A \in (0, \infty).$$

One has $\lim_{A \to 0^+} f_m(A) = -\infty$ since $\alpha > 0$, and $\lim_{A \to \infty} f_m(A) = \infty$ because $\lim_{s \to \infty} u_m(s) = \infty$ and $k^2 < \beta < 1$. Moreover, $f_m(A)$ is strictly increasing in $A$. Hence, there exists a unique root $A_o > 0$ to (4.10). For this $A_o$, one has

$$3\kappa^2 (1 - k^2) - \frac{1}{2} \alpha A_o^{-2} \leq 0,$$

for otherwise the left side of (4.10) would be positive. Thus,

$$A_o^2 \leq \frac{1}{6} \alpha \kappa^{-2} (1 - k^2)^{-1}.$$  

As $\lim_{m \to -\infty} k = 0$, this shows that $A_o$ is bounded from above as $m$ tends to $-\infty$. On the other hand, by (4.10), (4.11) and the fact $u_m(s) \geq r_o$, one has

$$0 \leq (\beta - k^2) + \left[ 3\kappa^2 (1 - k^2) - \frac{1}{2} \alpha A_o^{-2} \right] r_o^2,$$

i.e.

$$\alpha r_o^2 \left[ 2(\beta - k^2) + 6\kappa^2 (1 - k^2) r_o^2 \right]^{-1} \leq A_o^2.$$  

This shows $A_o$ is bounded away from 0 as $m$ tends to $-\infty$. \hfill $\square$

In what follows, we assume $m$ is sufficiently negatively large so that $k^2$ is small. We choose $A = A_o > 0$ so that $A_o = O(1)$ if $\beta > 0$; and $A_o$ is given by (4.9) if $\beta \leq 0$. In either case, $A_o = O(1)$, as $m \to -\infty$.

Before we compute $\lim_{m \to -\infty} A_o$ and $\lim_{m \to -\infty} u_m(A_o k)$, we point out the non-negativity of $m^\#(\Sigma_1)$ in our setting.
Proposition 4.1. Let $\Omega, \Sigma, (N, \gamma^{(m)}), A_o$, be given above. Then $m^\mu_H(\Sigma_1) \geq 0$.

**Proof.** This is essentially a consequence of the positive mass theorem on asymptotically hyperbolic manifolds (see [12, 27] for instance). More precisely, this follows from such a theorem on manifolds with corners along a hypersurface (see [4] and also [26, 23]). Consider three manifolds $\Omega, (N, \gamma^{(m)})$, and $M_m$, where $M_m$ is part of the spatial AdS-Schwarzschild manifold, with mass $m = m^\mu_H(\Sigma_1)$, lying outside a rotationally symmetric sphere $S$ isometric to $\Sigma_1$. One can glue $M_m$ to $(N, \gamma^{(m)})$ by identifying $S$ with $\Sigma_1$ and glue $\Omega$ to $(N, \gamma^{(m)})$ by identifying $\Sigma$ with $\Sigma_0$. Applying [4, Theorem 1.1] to the resulting manifold, one concludes $m \geq 0$. □

The above proof of Proposition 4.1 indeed indicates $m^B_H(\Sigma) \leq m^\mu_H(\Sigma_1)$ if the manifold obtained by gluing $M_m$ and $(N, \gamma^{(m)})$ along $\Sigma_1$ is smooth. By invoking a gluing result in [10, Proposition 3.3], one can verify this assertion.

Proposition 4.2. Let $\Omega, \Sigma, (N, \gamma^{(m)}), A_o$, be given above. Then $m^\mu_B(\Sigma) \leq m^\mu_H(\Sigma_1)$.

**Proof.** Since $\beta < 1$ and $\Sigma_1$ is round in $(N, \gamma^{(m)})$, an examination of (4.7) and (4.8) shows $R(\gamma^{(m)}) > -6\kappa^2$ near $\Sigma_1$ in $N$. The claim now follows from Proposition 4.1 and [10, Proposition 3.3] in the same way that (2.32) follows from (2.29) and [9, Proposition 2.1]. □

Next, we proceed to evaluate $\lim_{m \to -\infty} A_o$ and $\lim_{m \to -\infty} u_m(A_o k)$. First, as $m < 0$, (4.2) implies

$$u_m'(s) \leq \left(1 - \frac{2m}{r_o} + \kappa^2 u_m^2(s)\right)^{1/2},$$

which, upon integration, gives

$$\kappa u_m(A_o k) + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 u_m^2(A_o k)} \leq e^{\kappa A_o k} \left[\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right].$$

This yields

$$u_m(A_o k) \leq u^*_m,$$

where

$$u^*_m = \frac{e^{\kappa A_o k} \left(\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right) - e^{-\kappa A_o k} \left(1 - \frac{2m}{r_o}\right)}{2\kappa \left(\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right)}$$

$$= r_o \left[\frac{1}{2} \left(e^{\kappa A_o k} + e^{-\kappa A_o k}\right) + \frac{1}{2\kappa k} \left(e^{\kappa A_o k} - e^{-\kappa A_o k}\right) \frac{H_o}{2}\right].$$

For $0 \leq s \leq A_o k$, by (4.2), we have

$$u_m'(s) \leq \left(\frac{u^*_m - 2m + \kappa^2 u_m^3}{u_m(s)}\right)^{1/2},$$

(4.17)
which implies
\begin{equation}
  u_m^3(A_0k) \leq \frac{3}{2} A_0k \sqrt{u_m^* - 2m + \kappa^2 u_m^*} + r_o^3.
\end{equation}

On the other hand, by (4.2),
\begin{equation}
  u'_m(s) \geq \left( -\frac{2m}{u_m(s)} \right)^{\frac{1}{2}},
\end{equation}
which implies
\begin{equation}
  u_m^3(A_0k) \geq \frac{3}{2} A_0k \sqrt{2m} + r_o^3.
\end{equation}

Now, let \( \{m_i\} \) denote any sequence that tends to \(-\infty\) so that the corresponding sequence \( \{A_o^{(i)}\} \) has a finite limit, where \( A_o^{(i)} \) is the \( A_o \) associated with \( m_i \). Let \( \bar{A}_o := \lim_{m \to -\infty} A_o^{(i)} \). As \( \lim_{i \to \infty} k = 0 \), by (4.4) and (4.16),
\begin{equation}
  \lim_{i \to \infty} u_{m_i}^* = r_o \left( 1 + \frac{1}{2} \bar{A}_o H_o \right)
\end{equation}
and
\begin{equation}
  \lim_{i \to \infty} k \sqrt{-2m_i} = \tau r_o^{\frac{1}{2}} = \lim_{i \to \infty} k \sqrt{u_{m_i}^* - 2m_i + \kappa^2 u_{m_i}^3}.
\end{equation}

Hence, if we let
\[ \xi = \bar{A}_o r_o^{-1}, \]
then, by (4.18) and (4.20),
\begin{equation}
  \bar{u}_o := \lim_{i \to \infty} u_{m_i}(A_o^{(i)}k) = r_o \left( 1 + \frac{3}{2} \tau \xi \right)^{\frac{3}{2}}.
\end{equation}

As a result of (4.23) and (4.4), we see that the limit of the hyperbolic Hawking mass of \( \Sigma_1 \) in \( (N, \gamma^{(m_i)}) \) is given by
\begin{equation}
  \lim_{i \to \infty} m_H(\Sigma_1) = \frac{1}{2} (\bar{u}_o - r_o) + \frac{1}{2} \kappa^2 (\bar{u}_o^3 - r_o^3) + m_H(\Sigma)
\end{equation}
\begin{equation}
  = r_o \left[ \left( 1 + \frac{3}{2} \tau \xi \right)^{\frac{3}{2}} + \kappa^2 r_o^2 \left( 1 + \frac{3}{2} \tau \xi \right)^2 - 1 - \kappa^2 r_o^2 \right] + m_H(\Sigma).
\end{equation}

Here, by (4.9),
\begin{equation}
  \xi = \left( \frac{\frac{1}{2} \alpha}{\beta + 3\kappa^2 r_o^2} \right)^{\frac{1}{2}}, \quad \text{if } \beta \leq 0.
\end{equation}

When \( \beta > 0 \), by (4.13),
\begin{equation}
  \xi \geq \left( \frac{\frac{1}{2} \alpha}{\beta + 3\kappa^2 r_o^2} \right)^{\frac{1}{2}} > 0.
\end{equation}
Hence, by (4.10) and (4.23),
\begin{equation}
\beta + \left(3\kappa^2 r_o^2 - \frac{\alpha}{2} \xi^{-2}\right) \left(1 + \frac{3}{2} \tau \xi\right)^\frac{4}{3} = 0,
\end{equation}
or equivalently
\begin{equation}
\left[\beta + 3\kappa^2 r_o^2 \left(1 + \frac{3}{2} \tau \xi\right)^\frac{4}{3}\right] \xi^2 - \frac{\alpha}{2} \left(1 + \frac{3}{2} \tau \xi\right)^\frac{4}{3} = 0.
\end{equation}

**Remark 4.2.** Consider
\begin{equation}
\Psi(x) = \beta + \left(3\kappa^2 r_o^2 - \frac{\alpha}{2} x^{-2}\right) \left(1 + \frac{3}{2} \tau x\right)^\frac{4}{3}, \quad x \in (0, \infty).
\end{equation}
Then
\begin{equation}
\Psi'(x) = \left(6\kappa^2 r_o^2 \tau + \alpha x^{-3} + \frac{1}{2} \alpha \tau x^{-2}\right) \left(1 + \frac{3}{2} \tau x\right)^\frac{4}{3} > 0.
\end{equation}
As \(\lim_{x \to 0^+} \Psi(x) = -\infty\) and \(\lim_{x \to \infty} \Psi(x) = \infty\), \(\Psi(x)\) has a unique root \(\xi > 0\).

**Remark 4.3.** Since \(\{A_o^{(i)}\}\) can be any converging sequence, the argument above shows
\begin{align*}
\lim_{m \to -\infty} A_o &= r_o \xi \quad \text{and} \quad \lim_{m \to -\infty} u_m(A_o k) = r_o \left(1 + \frac{3}{2} \tau \xi\right)^\frac{2}{3}.
\end{align*}

Suppose \(\beta > 0\), we want to estimate \(\xi > 0\) which is the solution to (4.28). Similar to (2.23), we make a change of variable by letting
\begin{equation}
\xi = \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \theta_\kappa^2
\end{equation}
for \(\theta_\kappa > 0\). Then (4.28) becomes
\begin{equation}
\left[1 + 3\kappa^2 r_o^2 \beta^{-1} \left(1 + \frac{3}{2} \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \theta_\kappa^2\right)^\frac{4}{3}\right] \theta_\kappa^3 - \frac{3}{2} \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \tau \theta_\kappa^2 - 1 = 0.
\end{equation}

For \(x \in (0, \infty)\), consider the function
\begin{equation}
f(x) = \left[1 + 3\kappa^2 r_o^2 \beta^{-1} \left(1 + \frac{3}{2} \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \tau x^2\right)^\frac{4}{3}\right] x^3 - \frac{3}{2} \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \tau x^2 - 1.
\end{equation}
By Remark 4.2, \(f(x)\) has a unique positive root \(\theta_\kappa\). As in Theorem 2.1, we let \(\theta > 0\) be the unique root to
\begin{equation}
\theta^3 - \frac{3}{2} \left(\frac{\alpha}{2\beta}\right)^\frac{1}{2} \tau \theta^2 - 1 = 0.
\end{equation}
Then $f(\theta) \geq 0$. Therefore, we conclude

(4.32) \[ \theta_\kappa \leq \theta. \]

In particular, using the fact $\theta \leq 1 + \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \tau$, we have

(4.33) \[ \theta_\kappa \leq 1 + \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \tau. \]

Our above discussion has established the following theorem.

**Theorem 4.1.** Let $\Omega, \Sigma, g, H_0, r_0$ and $\tau$ be given in Theorem 1.5. Suppose $g$ is not a round metric and its Gauss curvature satisfies $K_g > -3\kappa^2$. Let $\{g(t)\}_{0 \leq t \leq 1}$ be a given smooth path of metrics on $\Sigma$ satisfying $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $\text{tr}_{g(t)} g'(t) = 0$. Then the hyperbolic Hawking mass $m_B^H(\Sigma)$ satisfies

$$m_B^H(\Sigma) - m_B^H(\Sigma) \leq r_o \left[ \kappa^2 r_o^2 \left( 1 + \frac{3}{2} \tau \xi \right)^2 \right] - \kappa^2 r_o^2 - 1 \right).$$

(4.34) Here $\xi > 0$ is a constant given by

(4.35) \[ \xi = \left( \frac{\frac{1}{2} \alpha}{\beta + 3\kappa^2 r_o^2} \right)^{\frac{1}{2}}, \text{ if } \beta \leq 0; \]

and $\xi$ is the unique positive root to

(4.36) \[ \left[ \beta + 3\kappa^2 r_o^2 \left( 1 + \frac{3}{2} \tau \xi \right) \right] \xi^2 - \frac{\alpha}{2} \left( 1 + \frac{3}{2} \tau \xi \right)^{\frac{4}{3}} = 0, \text{ if } \beta > 0. \]

In the latter case, if one writes $\xi = \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \theta_\kappa^2$ for a positive $\theta_\kappa$, then $\theta_\kappa \leq \theta$ where $\theta > 0$ is the unique root to

$$\theta^3 - \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \tau \theta^2 - 1 = 0. \]

In particular, this shows

(4.37) \[ \xi \leq \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \left[ 1 + \frac{3}{2} \left( \frac{\alpha}{2\beta} \right)^{\frac{1}{2}} \tau \right]^2. \]

Theorem 1.5 is a corollary of Theorem 4.1.
Proof of Theorem 1.5. Note that the second inequality in (1.7) simply follows from
\[ \kappa^2 r_o^2 \left( 1 + \frac{3}{2} \tau x \right)^2 + \left( 1 + \frac{3}{2} \tau x \right)^{\frac{4}{3}} - \kappa^2 r_o^2 - 1 \]
\[ = \tau x \left( 1 + \frac{3}{4} \tau x \right) \left[ \frac{3 \kappa^2 r_o^2}{4} + \frac{3}{4} \left( 1 + \frac{3}{2} \tau x \right)^{\frac{4}{3}} + \frac{1}{4} \right], \ x \geq 0. \]

If \( \inf_{\Sigma} K_g \leq 0 \), the pair \((\alpha, \beta)\) associated to any path \( \{g(t)\}_{0 \leq t \leq 1} \) with \( g(0) = g, \ g(1) \) is round, \( K_{g(t)} > -3\kappa^2 \), and \( \text{tr}_{g(t)} g'(t) = 0 \), necessarily has \( \beta \leq 0 \). Thus, (i) follows from taking the infimum over such paths in (4.35) of Theorem 4.1.

Suppose \( \inf_{\Sigma} K_g > 0 \), moreover we assume \( g \) is not a round metric. In this case, we can restrict the attention to the paths \( \{g(t)\}_{0 \leq t \leq 1} \) with \( g(0) = g, \ g(1) \) is round, \( K_{g(t)} > 0 \), and \( \text{tr}_{g(t)} g'(t) = 0 \). A pair \((\alpha, \beta)\) associated to such a path has \( \beta > 0 \).

Applying Theorem 4.1, by (4.36), we see (1.7) holds for
\[ (4.38) \quad \xi = \inf_{\{g(t)\}} \left\{ \text{the root of} \left[ \beta + 3 \kappa^2 r_o^2 \left( 1 + \frac{3}{2} \tau x \right)^{\frac{4}{3}} \right] x^2 - \frac{\alpha}{2} \left( 1 + \frac{3}{2} \tau x \right)^{\frac{4}{3}} = 0 \right\}. \]
Furthermore, such an \( \xi \) satisfies
\[ \xi \leq \zeta_g \theta^2, \]
where \( \theta \) is the unique root to \( \theta^3 - \frac{3}{2} \zeta_g \tau \theta^2 - 1 = 0 \). Since \( \theta \leq 1 + \frac{3}{2} \zeta_g \tau \), we have \( \xi \leq \zeta_g \left( 1 + \frac{3}{2} \zeta_g \tau \right)^2 \). This completes the proof. \( \square \)

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