Detection and estimation of partially-observed dynamical systems: an outer-measure approach

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**Summary.** In real environments, it is seldom that physical dynamical systems can be observed without detection failures and without disturbances from the background. Yet, a vast majority of the literature regarding Bayesian inference for such systems ignore these undesired effects and assume that pre-processing can be applied to remove them. To some extent, this goes against the Bayesian philosophy which promotes the integration of the different aspects of the problem into a joint formulation. However, such a formulation usually involves a precise modelling of these adverse effects as well as the setting of the corresponding parameters, which is not always feasible or realistic. In this article, we propose to use outer measures of a certain form to allow for additional flexibility in the modelling of these effects within the Bayesian paradigm. It is shown that detection and estimation of partially-observed dynamical systems can be performed with little to no knowledge about the background disturbances and with only an upper bound on the probability of detection failure. It is confirmed in simulations that such an approach can compete with standard methods even when the latter are given the true parameter values.

**1. Introduction**

The concept of dynamical system is fundamental for the modelling of many different types of physical phenomenon and is used under one form or another throughout the range of modern scientific disciplines. We consider in particular dynamical systems that are observed through a partial and corrupted observation process: partial in the sense that some of the component of the state of the system are either not observed or observed with uncertainty as well as in the sense that the observation process might fail altogether, and corrupted in the sense that the true observation (if present) might be received together with false positives, originated from all sorts of background perturbations. Moreover, it might be the case that several dynamical systems of interest are present and that the observations they generate cannot be distinguished a priori. Estimation algorithm typically make the assumption that the system(s) of interest can be observed consistently and without interference from the background. However, these perturbations have to be dealt with in practice. A common approach is to pre-process the data in order to determine when the system(s) of interest are observed and which of the (possibly multiple) received observations are the true ones.

Principled solutions to this problem have been proposed in the context of multiple target tracking (Musicki et al., 1994) where the presence of false positives and detection failure is an inherent part of the problem. However, these methods require some modelling of the perturbations that is not always available. Learning the parameters of the corresponding models is possible when the perturbations are sufficiently consistent in time and when large amount of data is available. When these conditions are not met, no existing principled statistical methods are applicable (to the best of the authors knowledge).

The problem does not however seem to be unsolvable. One could argue that an adequate algorithm could retain the observations that correspond to the modelling of the considered
system(s) and discard the false positives regardless of their statistical properties. The objective in this article is to formalise this idea using a recently introduced framework for uncertainty modelling (Houssineau, 2015, 2017). In addition to model the lack of information about the false positives, this framework brings flexibility in the characterisation of the probability of detection as well as in the dynamics and observation of each dynamical system (Houssineau and Bishop, 2017). Although this article builds upon these previous work, it is the first to propose a full solution to a complex problem using this framework, making it more practical and showing its efficiency.

The representation of uncertainty considered here has strong connections with Dempster-Shafer theory (Dempster, 1967) and possibility theory (Dubois and Prade, 2015; Zadeh, 1978), but differs in some essential aspects: focusing on a particular class of outer measures enables the derivation of inference algorithms while preserving the probabilistic case as a special example. It is demonstrated in this article that such inference algorithms can even be derived for complex models describing a varying number of dynamical systems whose observation is corrupted in different ways.

The objective can then be formulated as follows: find a closed-form recursion inferring the presence and state(s) of the considered dynamical system(s) from partial and corrupted observations with little to no information about false positives and detection failures. In order to meet this objective, the following concepts have to be introduced: The specific class of outer measures that is considered as well as the analogues of the notions of conditioning, of independence and of Markov chains are presented in section 2, followed by an extension of the concept of point process as well as the description of the associated uncertainties in section 3. This leads in section 4 to the introduction of a recursion addressing the considered problem when no more than one dynamical system is present and to the extension of this solution to multiple systems in section 5.

2. Probability and possibility

Although the modelling capabilities provided by standard Bayesian estimation allow for representing a certain lack of information about the phenomena of interest, it is not generally possible to model a total absence of information (when ignoring improper priors that are not probability distributions.) This is a direct consequence of the fact that probability distributions represent randomness rather than (lack of) knowledge so that even uniform distributions, although the least informative, still hold some sort of information (the probability is the same everywhere). This can be a serious limitation in some cases.

Example 1. We might want to model that there is no knowledge about the number of false positives. If we want to model this using a uniform probability distribution, we have to decide on a maximum number \( N \) (which already requires to know something). Of course, one could argue that \( N \) can be taken arbitrary large, but this will also make the marginal likelihood arbitrary small. Both these drawbacks can be avoided with the approach presented in this section by modelling that any number of false positive is “possible”, which does not require to know a bound like \( N \) and which will be seen to have no harmful effect on the marginal likelihood in further sections.

Throughout this section, we consider a Polish space \( E \) equipped with its Borel \( \sigma \)-algebra \( \mathcal{B}(E) \). The set of non-negative bounded measurable functions on \( E \) is denoted \( L^\infty(E) \) and \( \| \varphi \|_\infty = \sup_{x \in E} |\varphi(x)| \) stands for the supremum norm of the function \( \varphi \) on \( E \). Note that the term “function” will be reserved for real-valued mappings. The point-wise product
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between two function \( f \) and \( f' \) is denoted \( f \cdot f' \) and omitted variables are denoted by a bold dot “·” to help distinguish the two notations. One of the central notions if the one of outer measure, which is defined in a similar way to a measure except that \( \sigma \)-additivity is replaced by sub-additivity. We are particularly interested in outer probability measures (o.p.m.s), that is any outer measure attributing a unit mass to the whole set on which it is defined. Note that the set of probability measures is a subset of the set of o.p.m.s. Most of the concepts presented in this section are originated from Houssineau (2015) and Houssineau and Bishop (2017), although a few notable extensions and modifications have been introduced here.

2.1. Outer probability measure and uncertain variable

We consider a set \( \Omega_u \) containing the possible states for the deterministic phenomena of interest. A mapping \( X \) from the set \( \Omega_u \) of outcomes to a set \( E \) is referred to as an \( E \)-valued deterministic uncertain variable. Uncertain variables with a random component are studied in Houssineau (2017). Since all considered uncertain variables will be deterministic in this article, we will just say “uncertain variable” when there is no ambiguity. Representing all forms of uncertainty as deterministic does no prevent from modelling random phenomena in the same way that the standard Bayesian approach models all forms of uncertainty as random and proceeds to the estimation of deterministic parameters.

Assume that there is some available information about \( X \) encoded into an o.p.m. \( \bar{P} \) on \( E \) of the form

\[
\bar{P}(\varphi) = \int \sup_{x \in E} (\varphi(x)f(x)) P(df) = \int \|\varphi \cdot f\|_{\infty} P(df),
\]

for any \( \varphi \in L^\infty(E) \), where \( P \) is assumed, without loss of generality, to be a probability distribution on \( L^\infty(E) \) supported by the set \( L(E) \) of non-negative measurable functions with supremum equal to one. The elements of the set \( L(E) \) are referred to as possibility functions on \( E \). For instance, the likelihood of the event \( X \in B \) is given by \( \bar{P}(1_B) \) where \( 1_B \) is the indicator function of \( B \). Note however that \( X \) does not characterise the o.p.m. \( \bar{P} \) since the latter only describes available information about \( X \). When proving results about o.p.m.s, we will assume that it is possible to have direct access to the integrand in the r.h.s. of eq. (1); this can be done formally by adding an argument to \( \bar{P} \) related to the possibility functions themselves, which would be cumbersome in general and is therefore not considered.

**Remark 1.** The formal definition of a probability measure on \( L^\infty(E) \) requires the careful construction of a suitably fine \( \sigma \)-algebra on this set. The approach considered in Houssineau (2015), based on (Fremlin, 2000, Section 245), involves the abstract concept of uniform space (Bourbaki, 1966) and makes difficult the verification of the measurability of the function \( f \mapsto \|\varphi \cdot f\|_{\infty} \). A more pragmatic approach is to consider that all probability measures on \( L^\infty(E) \) are supported by a set of the form \( F \cup \{1_x : x \in E\} \) with \( F \) countable.

Since the considered o.p.m.s represents the available knowledge about uncertain variables, they are primarily defined on the space in which these uncertain variables take values. However, it is possible to find the pull-back of the o.p.m. \( \bar{P} \) on \( \Omega \) which we denote \( \bar{P} \), and which is characterised by \( \bar{P}(\varphi(X)) = \bar{P}(\varphi) \) for every \( \varphi \in L^\infty(E) \). In particular, it is meaningful to denoted \( \bar{P}(X \in B) \) the scalar \( \bar{P}(1_B(X)) \) for some subset \( B \) in \( B(E) \), the quantity \( \bar{P}(X \in B) \in [0,1] \) being understood as the possibility for \( X \) to be in \( B \).
Remark 2. The mechanism here is the exact opposite of the one considered in standard probability theory where the probability measure on the sample space is the fundamental probability from which all others can be deduced. This is consistent with the idea that randomness is intrinsic to the consider phenomena whereas information is extrinsic.

It could be the case that $\bar{P}(X \in B) = \bar{P}(X \in B') = 1$ even if $B \cap B' = \emptyset$. For instance if a coin about which there is no information is tossed then both head and tail are “completely possible”, i.e. the two events have a possibility equal to 1. This is the intuitive reason for considering o.p.m.s rather than standard probability distributions since the sub-additivity of the former allows for this flexibility in modelling a lack of information. A simple example can be found when the outer measure $\bar{P}$ verifies $\bar{P}(\varphi) = \|\varphi \cdot f\|_{\infty}$ for some $f \in L(E)$, that is when $P = \delta_f$. In this case, it holds that

$$\bar{P}(X \in B) = \sup_{x \in B} f(x)$$

which justifies the name of possibility function for elements of $L(E)$. This case is at the intersection between the modelling proposed in Houssineau (2017) and the one considered here, and is referred to as the single-possibility case. Analogues of standard probability density functions (p.d.f.s) on $\mathbb{R}^d$ can be easily introduced as possibility functions. Two cases that will be relevant in further sections are: the uniform possibility function $\bar{U}(A) = \frac{1}{A}$ on a subset $A$ of $E$ and the Gaussian possibility function $\bar{N}(m, V)$ on $E = \mathbb{R}^d$ defined for some $m \in \mathbb{R}^d$ and some $d \times d$ positive-definite matrix $V$ as

$$\bar{N}(m, V) = \exp \left( -\frac{1}{2}(x - m)^t V^{-1} (x - m) \right)$$

where $\cdot^t$ denotes the matrix transposition. Note that many other possibility functions can be designed since the condition that the supremum of the function is equal to one is easy to satisfy. For instance, one could introduce possibility functions on $\mathbb{R}$ of the form

$$f_r(x) = \exp \left( -\frac{1}{s} |x - \mu|^r \right)$$

for any rational number $r \in \mathbb{Q}$, $\mu \in \mathbb{R}$ and $s > 0$ (the cases $r = 1$ and $r = 2$ being already covered by the analogues of the Laplace and Gaussian distributions). Remarkably, any of these possibility functions can be directly used on $\mathbb{N}_0$ as soon as the their maximum is reached at an integer value.

It is also useful to introduce a notion of independence for the proposed approach: let $F$ be another Polish space and let $X$ and $X'$ be uncertain variables on the spaces $E$ and $F$ respectively, then the o.p.m. $\bar{P}_{X,X'}$ associated with the joint uncertain variable $(X, X')$ is said to represent $X$ and $X'$ (weakly-)independently if there exist o.p.m.s $\bar{P}$ and $\bar{P}'$ such that

$$\bar{P}_{X,X'}(\varphi \times \varphi') = \bar{P}(\varphi) \bar{P}'(\varphi')$$

for any $\varphi \times \varphi' \in L^\infty(E \times F)$. Although weak-independence is a property of o.p.m.s, it is convenient to qualify the uncertain variables $X$ and $X'$ themselves as weakly-independent when the o.p.m. describing them has this property.

Henceforth, it will be assumed that a probability distribution $P$ is implicitly defined whenever an outer measure $\bar{P}$ is introduced. In order to perform inference, it is important to be able to express conditional information in order to model the dynamics and observation of the system(s) of interest. This aspect is the topic of the next section.
2.2. Conditional o.p.m. and possibility functions

The objective in this section is to express in an appropriate way the o.p.m. describing an uncertain variable on $F$ given the realisation $x$ of another random variable $X$ on $E$. In probability theory, a conditional probability measure $p(\cdot \mid X = x)$ on $F$ is assumed to verify that $x \mapsto p(B \mid X = x)$ is measurable for all $B \in \mathcal{B}(F)$ and that $p(\cdot \mid X = x)$ is a probability measure for all $x \in E$. We therefore consider an o.p.m. $\bar{P}(\cdot \mid X = x)$ on $L^\infty(F)$ for all $x \in E$ such that $\bar{P}(\varphi \mid X = x)$ is a measurable function for all $\varphi \in L^\infty(F)$. It is however necessary to define the measure underlying $\bar{P}(\cdot \mid X = x)$ in the same way as $P$ was characterising $\bar{P}$ earlier in this section. The form chosen in Houssineau and Bishop (2017), that is

$$\bar{P}(\varphi \mid X = x) = \int \|\varphi \cdot f\|_\infty P(df \mid X = x)$$

with $P(\cdot \mid X = x)$ a conditional probability measure on $L^\infty(F)$, showed some limitations in particular in the derivation of filtering equations where $\bar{P}(\cdot \mid X = x)$ was assumed to be based on a single possibility function. Alternatively, consider the set $L^\infty(F; E)$ of functions $f$ on $F \times E$ such that $f(\cdot \mid x)$ is non-negative bounded and measurable for all $x \in E$ and such that $f(x' \mid \cdot)$ is measurable for all $x' \in F$. Using this set, the conditional o.p.m. $P(\cdot \mid X = x)$ can be expressed as

$$\bar{P}(\varphi \mid X = x) = \int \|\varphi \cdot f(\cdot \mid x)\|_\infty P(df \mid X)$$

with $P(\cdot \mid X)$ a measure on $L^\infty(F; E)$. Note that the conditioning on $X$ in $P(\cdot \mid X)$ is purely notational and only indicates that this measure is defined on $L^\infty(F; E)$ rather than $L^\infty(F)$. Also, $P(\cdot \mid X)$ might not be a probability measure and it might not be supported by functions with supremum 1. This is because there is no possible normalisation with the dependence of $f$ on $x$. Yet, if $P(\cdot \mid X)$ is supported by a single function $f \in L^\infty(F; E)$ then it holds that $f(\cdot \mid x)$ is a possibility function for all $x \in E$. In this case, $f$ is referred to as a conditional possibility function.

Conditional possibility functions behaves like conditional p.d.f.s: if $f$ represents the knowledge about an uncertain variable $X$ on $E$ and $f'(\cdot \mid x)$ represents the knowledge about the transition from $X = x$ to another uncertain variable $X'$ on $F$, then, with obvious notations

$$f'(x') = \sup_{x \in E} f'(x' \mid x) f(x) = \|f'(x' \mid \cdot) \cdot f\|_\infty$$

for any $x' \in F$. This is the Chapman-Kolmogorov equation for possibility functions. Similarly, if $x'$ is a realisation of $X'$, then the analogue of Bayes’ theorem is expressed as

$$f(x \mid x') = \frac{f'(x' \mid x) f(x)}{f'(x')} = \frac{f'(x' \mid x) f(x)}{\|f'(x' \mid \cdot) \cdot f\|_\infty}$$

where $f(\cdot \mid x')$ is the possibility function describing what is known about $X$ after assimilating the information contained in the realisation $x'$ of $X'$. Note that, as opposed to random variables, uncertain variables do not change after conditioning, only the available information about them evolves. Finally, given that

$$\bar{P}(\varphi'(X')) = \|\varphi' \cdot f'\|_\infty \quad \text{and} \quad \bar{P}(\varphi(X) \mid X' = x') = \|\varphi \cdot f(\cdot \mid x')\|_\infty$$

for any $\varphi \in L^\infty(E)$ and any $\varphi' \in L^\infty(F)$, it follows easily that

$$\bar{P}(\bar{P}(\varphi(X) \mid X')) = \bar{P}(\varphi(X)),$$
which is the analogue of the law of total probability.

We show in the next theorem that the form given in eq. (2) enables the formulation of a Chapman-Kolmogorov equation for o.p.m.s.

**Theorem 1.** Let $\bar{P}$ be an o.p.m. describing an uncertain variable $X$ on $E$ and let $\bar{P}'(\cdot | X = x)$ be a conditional o.p.m. describing another uncertain variable $X'$ on $F$ conditioned on $X = x$ for any $x \in E$, then the marginal o.p.m. $\bar{P}'$ is characterised by

$$\bar{P}'(\varphi) = \int \| f'(x' | \cdot) \cdot f \|_\infty P(df) P'(df' | X)$$

for any $\varphi \in L^\infty(F)$.

Filtering and smoothing can then be performed based on theorem 1 without assumptions on $\bar{P}(\cdot | X = x)$, so that the expression given in eq. (2) is the one considered in this article. The introduction of conditional o.p.m.s motivates the study of independence properties of collections of uncertain variables as in the next section.

### 2.3. Markov process and outer Markov kernel

Since the objective is to study dynamical systems in a Bayesian formulation of the considered representation of uncertainty, it is natural to consider collections $\{X_n\}_{n \geq 0}$ of uncertain variables with $n$ a time-like index and with $X_n$ representing the state of the dynamical system(s) of interest in $E$ at time $n$. The collection $\{X_n\}$ can be referred to as a *(deterministic) uncertain process*. It is also convenient to assume some form of independence between uncertain variables at different times: if the uncertain variables in the uncertain process $\{X_n\}$ are pairwise weakly-independent and described by the same o.p.m. $\bar{P}$, then they are said to be weakly-independently identically described (weakly-i.i.d.) by $\bar{P}$. Alternatively, if for any $n \geq 0$ it holds that

$$\bar{P}_n(\varphi | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = \bar{P}_n(\varphi | X_{n-1} = x_{n-1}),$$

then $\{X_n\}$ is said to be a *(deterministic) uncertain Markov process*. In this case, the analogue of the concept of Markov kernel can be introduced: the conditional o.p.m. $\bar{P}_n(\cdot | X_n = x_n)$ is re-expressed as a function $\bar{Q}_n$ on $E \times L^\infty(E)$ such that

$$\bar{Q}_n : (x, \varphi) \mapsto \bar{Q}_n(x, \varphi) = \bar{P}_n(\varphi | X_n = x_n)$$

and $\bar{Q}_n$ is referred to as a *(outer) Markov kernel*. As is usual with Markov kernels in standard probability theory, we write $\bar{Q}(\varphi)$ for the function $x \mapsto \bar{Q}(x, \varphi)$. For consistency, any function $f$ in the support of the measure $P_n(\cdot | X_n = x_n)$ can be similarly re-expressed as a function $g$ on $E \times E$ defined as $g : (x, x') \mapsto f(x' | x)$. To sum up, the outer Markov kernel $\bar{Q}$ can be expressed as

$$\bar{Q}(x, \varphi) = \int \| \varphi \cdot g(x, \cdot) \|_\infty Q(dg)$$

for any $x \in E$ and any $\varphi \in L^\infty(E)$, where $Q$ is an appropriately defined measure. In some cases, we will need to introduce transitions which give a measure that is less than or equal to one to the whole space, these are simply referred to as *(outer) transition kernels*.

The last ingredient that is required to state the main result of this article is an analogue of the concept of point process.
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3. Uncertain counting measure

Whether a single or multiple systems are observed at the same time, there is an inherent need to model the false positives that comes in uncertain number and are often indistinguishable. It is therefore natural to introduce an analogue of the concept of point process (a.k.a. random counting measure) using a specific family of o.p.m.s.

3.1. General formulation

Following the definition of uncertain variables, the analogue of the notion of point process (Daley and Vere-Jones, 2003) can be referred to as (deterministic) uncertain counting measure (Houssineau, 2017) and defined as an uncertain variable on the set of counting measures on a Polish space $X$, that is as a mapping $\mathcal{X} : \Omega \rightarrow \mathbb{N}$ defined as

$$\mathcal{X}(\omega_u) = \sum_{i=1}^{N(\omega_u)} \delta_{X_i(\omega_u)}$$

where $N$ is an uncertain variable on $\mathbb{N}_0$ and where $\{X_i\}_{i=1}^N$ is a collection of uncertain variables on $X$. Our knowledge about the uncertain counting measure $\mathcal{X}$ is described by an o.p.m. $\bar{\mathcal{P}}$ which form is given in what follows. For any $n \in \mathbb{N}$, let $\bar{\mathcal{P}}(\cdot | N = n)$ be an o.p.m. on $X^n$ verifying

$$\bar{\mathcal{P}}(\varphi_1 \times \cdots \times \varphi_n | N = n) = \bar{\mathcal{P}}(\varphi_{\sigma(1)} \times \cdots \times \varphi_{\sigma(n)} | N = n),$$

for any $\varphi_1, \ldots, \varphi_n \in L^\infty(X)$ and any permutation $\sigma$ in the set $\text{Sym}(n)$ of permutation of $\{1, \ldots, n\}$. In the same way as with point processes, this assumption of symmetry corresponds to the indistinguishability of the uncertain counting measure. Also, let $\bar{p}$ be an o.p.m. on $\mathbb{N}_0$. The o.p.m. $\bar{\mathcal{P}}$ can then be expressed via the o.p.m.s $\bar{p}$ and $\bar{\mathcal{P}}(\cdot | N = n)$ respectively representing the information about the number of points in $\mathcal{X}$ and the location of each point conditioned on $N = n$, for any $n \geq 0$, that is

$$\bar{\mathcal{P}}(\varphi) = \int \max_{n \geq 0} \left( c(n) \| \varphi \cdot f_n \|_\infty \right) p(d\xi) \prod_{m \geq 0} P(df_m | N = m),$$

for any $\varphi \in L^\infty(\mathcal{X})$ with $\mathcal{X} = \bigcup_{n \geq 0} X^n$, where $P(\cdot | N = 0) = \delta_{f_0}$ with $f_0$ the unique possibility function on the singleton $X^0$ containing the empty sequence only, i.e.

$$f_0 : X^0 \rightarrow [0, 1]$$

$$(\cdot) \mapsto 1.$$

The expression eq. (5) of $\bar{\mathcal{P}}$ is well defined since the presence of the maximum has for consequence that only one of the probability measures in the product on the r.h.s. of eq. (5) is considered. The integral over a product of all the probability measures in the collection $\{P(\cdot | N = n)\}_{n \geq 0}$ in eq. (5) might be surprising; it is however consistent with the usual definition of point processes as shown in the following example.

Example 2. If the o.p.m. $\bar{p}$ is equivalent to a probability measure with probability mass function (p.m.f.) $w$ then the expression of $\bar{\mathcal{P}}$ given in eq. (5) simplifies to

$$\bar{\mathcal{P}}(\varphi) = \sum_{n \geq 0} w(n) \int \| \varphi \cdot f_n \|_\infty P(df_n | N = n),$$
in which the number of terms in the product is reduced to one. Moreover, if \( P(\cdot \mid N = n) \) is supported by indicators of singletons and if it gives mass \( p_n(dx) \) to the indicator \( 1_x \) for any \( x \in X^n \) then

\[
\bar{P}(\varphi) = w(0) + \sum_{n \geq 1} w(n) \int \varphi(x_1, \ldots, x_n) p_n(d(x_1, \ldots, x_n)).
\]

That is, in this case, the o.p.m. \( \bar{P} \) is the law of a point process.

The previous example shows how the o.p.m. can collapse to a point-process law by assuming specific forms for \( \bar{p} \) and \( \{P(\cdot \mid N = n)\}_n \). Alternatively, the following example introduce a simplification of \( \bar{P} \) that removes all the probabilistic part of it.

**Example 3.** If the o.p.m.s \( \bar{p} \) and \( P(\cdot \mid N = n) \), \( n \geq 1 \), are based on a single possibility function, that is, if there exists \( c \in L(N_0) \) such that \( \bar{p}(\varphi) = \|\varphi \cdot c\|_\infty \) and if for any \( n \in \mathbb{N} \) there exists \( f_n \in L(X^n) \) such that \( P(\varphi \mid N = n) = \|\varphi \cdot f_n\|_\infty \), then

\[
\bar{P}(\varphi) = \max_{n \geq 0} c(n) \|\varphi \cdot f_n\|_\infty.
\]

The o.p.m. \( \bar{P} \) can also be characterised by \( \bar{P}(\varphi) = \|\varphi \cdot f\|_\infty \) with \( f(x) = c(n)f_n(x) \) defined for any \( n \in \mathbb{N}_0 \) and any \( x \in X^n \). It is easy to verify that \( f \) is indeed a possibility function on \( X \).

Following the idea of example 3, note that it is also possible to define \( \bar{P} \) through a probability measure \( P \) on \( L^\infty(X) \) so that \( \bar{P}(\varphi) = \int \|\varphi \cdot f\|_\infty P(df) \) for any \( \varphi \in L^\infty(X) \). In spite of being more compact, this expression is less explicit than the one considered before.

Some of the simplest point processes are the ones for which all points are drawn independently from a single distribution; these are called i.i.d. point process. The analogue of this concept in the considered framework can be defined based on an o.p.m. \( \bar{P} \) on \( X \) by considering that\( \dagger \)

\[
P(\mathcal{d}(f_1 \times \cdots \times f_n) \mid N = n) \overset{\text{f}}{=} P(\mathcal{d}f_1) \cdots P(\mathcal{d}f_n)
\]

for any \( f_1, \ldots, f_n \in L(X) \), where the product of possibility functions is understood as

\[
f_1 \times \cdots \times f_n : x \mapsto f_1(x_1) \cdots f_n(x_n).
\]

In this way, the measure \( P(\cdot \mid N = n) \) automatically satisfies the symmetry condition eq. (4).

As mentioned above, uncertain counting measures will be useful for modelling the false positives. In section 5, we will also consider the case where multiple dynamical systems are also modelled in this way. Although the different systems could arguably be distinguished in many scenarios, it will appear that this modelling choice allows for deriving a simple inference algorithm.

### 3.2. Single-possibility formulation

In the spirit of example 3, we consider an o.p.m. \( \bar{P} \) for which the cardinality and spatial information are based on single possibility functions. If we additionally assume an i.i.d. structure, then

\[
\bar{P}(\varphi) = \max_{n \in \mathbb{N}_0} c(n) \|\varphi \cdot f^n\|_\infty
\]

\( \dagger \)for any two measures \( \mu \) and \( \mu' \) on a space \( E \), \( \mu(dx) \overset{\text{f}}{=} \mu'(dx) \) stands for \( \int \varphi(x) \mu(dx) = \int \varphi(x) \mu'(dx) \) for any bounded measurable function \( \varphi \) on \( E \).
for some \( f \in \mathcal{L}(\mathbf{X}) \), with \( f^{\times n} = f \times \cdots \times f \). The possibility function \( f \) can be a single term or a max-mixture, that is any
\[
    f(x) = \max_{i \in I} w_i f_i(x)
\]
for some collections \( \{w_i\}_{i \in I} \) of \([0, 1]\)-valued weights and some collection \( \{f_i\}_{i \in I} \) of possibility functions.

In order to better understand uncertain counting measures, a few aspects of it are briefly studied in the remainder of this section. We start with an important concept in point process theory, that is the concept of first-moment measure (a.k.a. intensity). One of the possible ways to generalise it is to study the possibility of having at least one point with a given state in \( x \in \mathbf{X} \) and to define a function \( F \) characterised by
\[
    F(x) = \widehat{\mathbb{P}}(\mathcal{X}(\{x\}) \geq 1) = \max_{n>0} \left( c(n) \| f_n(x, \cdot) \|_\infty \right)
\]
where \( f_n(x, \cdot) \) is the function \((x_2, \ldots, x_n) \mapsto f_n(x, x_2, \ldots, x_n)\). The function \( F \) is referred to as the intensity function of the uncertain counting measure \( \mathcal{X} \). Its form is justified in Houssineau (2017). In the general case, the intensity outer measure will not reduce to a single possibility function; although this generalisation does not bring any difficulties, it does not bring much insight either and is therefore not discussed here. The supremum \( \| F \|_\infty \) of the intensity function is the possibility that there is at least one point in the uncertain counting measure \( \mathcal{X} \); indeed, it holds that
\[
    \| F \|_\infty = \sup_{x \in \mathbf{X}} \left( \max_{n>0} (c(n) \| f_n(x, \cdot) \|_\infty) \right) = \max_{n>0} c(n) = \widehat{\mathbb{P}}(N > 0).
\]

We consider the two following propositions, proved in Houssineau (2017), will be useful when performing inference about multiple systems.

**Proposition 1.** Let \( \mathcal{X} \) and \( \mathcal{X}' \) be two weakly-independent single-possibility uncertain counting measures with intensity function \( F \) and \( F' \), then the intensity function \( F_+ \) of the sum \( \mathcal{X} + \mathcal{X}' \) is
\[
    F_+(x) = \max\{F(x), F'(x)\}.
\]

**Proposition 2.** Let \( \mathcal{X} \) be a single-possibility uncertain counting measure on \( \mathbf{X} \) with intensity function \( F \) and let \( g \) be a conditional possibility function from \( \mathbf{X} \) to \( \mathbf{Y} \), then the intensity function \( F' \) of the uncertain counting measure \( \mathcal{X}' \) resulting from the propagation of \( \mathcal{X} \) by \( g \) is expressed for \( y \in \mathbf{Y} \) as
\[
    F'(x') = \| F \cdot g(\cdot, x') \|_\infty = \sup_{x \in \mathbf{X}} F(x)g(x, x').
\]

Combining proposition 1 and proposition 2, one can obtain the predicted intensity function describing multiple dynamical systems that have been propagated from the last time step to the current time step plus some appearing systems.

In the weakly-i.i.d. case, the intensity function \( F \) simplifies to \( F(x) = f(x) \max_{n>0} c(n) \). Further assuming that \( c(0) < 1 \), i.e. that cardinality 0 is not fully possible, we obtain that the possibility functions \( F \) and \( f \) are equal, i.e. the analogue of the first-moment of \( \mathcal{X} \) is equal to the common spatial information about its points. If the only information about an uncertain counting measure \( \mathcal{X} \) is given by an intensity function \( F \) then one can define the possibility function \( c \) on \( \mathbb{N}_0 \) and the common spatial information \( f \in \mathcal{L}(\mathbf{X}) \) as
\[
    c(n) = \| F \|_\infty^n \quad \text{and} \quad f(x) = \begin{cases} \frac{F(x)}{\| F \|_\infty} & \text{if } \| F \|_\infty > 0 \\ 1 & \text{otherwise.} \end{cases}
\]
Indeed, a possibility function describes what is known about $X$ and since there is no information about the correlation between points, an i.i.d. representation is the best available. In the standard framework, only the law of Poisson i.i.d. point processes can be recovered from their first-moment measure.

Being equipped with a way of describing imprecise information about complex systems, we proceed to the modelling of their dynamics and observation in order to enable the derivation of filtering equations.

4. Recursion

We consider in this section the case of a single-dynamical system whose observation is not only partial but also corrupted by the presence of unrelated observations, referred to as false positives, caused for instance by the background noise.

Let the state space $X$ be the union of a subset $S$ of $\mathbb{R}^d$, for some $d > 0$, with an isolated point $\psi$ representing the case where the dynamical system of interest does not admit a state in $S$ (which relates to the absence of the system from $S$, e.g. because its position is out of the bounds of the considered area). Also, let the observation space $Y$ be the union of a subset $S'$ of $\mathbb{R}^{d'}$, for some $d' > 0$, with an isolated point $\phi$ representing the case where the dynamical system of interest does not produce any observation. Without loss of generality, the time is assumed to take integer values. At any given time $t \in \mathbb{N}_0$, the system of interest is modelled by an uncertain variable $X_t$ on $X$ and its observation is represented by an uncertain counting measure $Y_t$ including, in addition to the true observation, a varying number of false positives represented by an i.i.d. uncertain counting measure $Y_\flat$. It is assumed that the collection $\{X_t\}_t$ is an uncertain Markov process so that the corresponding state equation can be expressed as

$$X_t = F(X_{t-1}, V_t)$$

where $F$ is a function related to the dynamics and where $\{V_t\}_t$ is a collection of weakly-independent uncertain variables modelling the uncertainty in the dynamical model. It is also assumed that the true observation at time $t$ only depends on the state at the same time so that the observation equation is

$$Y_t = \mathcal{H}(X_t, D_t) + Y_\flat,$$  \hspace{1cm} (7)

where $\mathcal{H}$ is an uncertain function describing the observation of the system and where $\{D_t\}_t$ is a collection of weakly-independent uncertain variables in $\{0, 1\}$ modelling detection. The function $\mathcal{H}$ is characterised by $\mathcal{H}(\psi, 0) = \mathcal{H}(\psi, 1) = \delta_\phi$ and by

$$\mathcal{H}(x, 1) = \delta_{H(x, W_t)} \quad \text{and} \quad \mathcal{H}(x, 0) = \delta_\phi$$

for any $x \in S$, where $H$ is the (standard) observation function of the system and $\{W_t\}_t$ is a collection of weakly-independent uncertain variables. Assuming that $Y_t$ is simple, i.e. that it contains no repeating points almost-surely, we can write any realisation of it as a set $Y_t$ of points in $S'$.

4.1. Evolution

The dynamical process induced by $F$ and $\{V_t\}_t$ is modelled by an outer Markov kernel $\bar{Q}$ from $X$ to itself. More specifically, we consider that the modelling the evolution of the system results from three types of transitions:
\( \alpha \): from \( \psi \) to \( X \) (appearing in \( S \) or remaining out of \( S \))

\( \pi \): from \( S \) to \( S \) (propagating within \( S \))

\( \omega \): from \( S \) to \( \psi \) (disappearing from \( S \))

The symbols \( \alpha \) and \( \omega \) are selected to represent appearance and disappearance respectively since \( \alpha \) is the first letter of the Greek alphabet and \( \omega \) the last.

We first define two measures \( Q_\alpha \) and \( Q_{\pi,\omega} \) on \( L^\infty(X;X) \) and \( L^\infty(X;X)^2 \) respectively, such that

\[
\int \|g(\psi, \cdot)\|_\infty Q_\alpha(dg) = 1 \quad \text{and} \quad \int \|g(x, \cdot)\|_\infty \|g'(x', \cdot)\|_\infty Q_{\pi,\omega}(dg(g', g')) = 1
\]

for any \( x, x' \in S \), representing these three transitions, that is, \( Q_{\pi,\omega} \) jointly represents transitions of the types \( \pi \) and \( \omega \). The meaning of the modelling above is that any functions \( g_\alpha \), \( g_\pi \) and \( g_\omega \) such that \( g_\alpha \) and \( (g_\pi, g_\omega) \) are in the respective supports of \( Q_\alpha \) and \( Q_{\pi,\omega} \) verify, for any \( x \in S \),

\[
g_\alpha(x, \cdot) = g_\pi(\psi, \cdot) = g_\pi(\cdot, \psi) = g_\omega(\psi, \cdot) = 0,
\]

and \( g_\omega(x, x') = 0 \) for any \( x, x' \in S \). The following example illustrates a way of defining the measure \( Q_{\pi,\omega} \).

**Example 4.** If \( Q_{\pi,\omega} = p_\pi \delta(g_\psi, 0) + (1 - p_\pi) \delta(0, g_\omega) \) for given \( g_\pi \) and \( g_\omega \) then the system remain within \( S \) with probability \( p_\pi \) and disappears with probability \( 1 - p_\pi \). If, more generally, it holds that

\[
Q_{\pi,\omega} = \sum_i w^i_\pi \delta(g_\pi^i, 0) + \sum_i w^i_\omega \delta(0, g_\omega^i)
\]

for some collections of weights \( \{w^i_\pi\}_i \) and \( \{w^i_\omega\}_i \) in the interval \((0, 1]\) and some collections of conditional possibility functions \( \{g_\pi^i\}_i \) and \( \{g_\omega^i\}_i \) then the maximum in eq. (8) disappears and a sum over these indices appears instead, as in a standard multi-model formulation.

The measure \( Q_\alpha \) clearly induces an outer transition kernel \( \bar{Q}_\alpha \) from \( X \) to itself, however, \( Q_{\pi,\omega} \) induces an outer kernel from \( X^2 \) to itself, which is not the one that is required. Indeed, we want to model that the system of interest can either be propagated or disappear instead of representing these two events as a pair of states. The expression of the sought outer transition kernel \( \bar{Q}_{\pi,\omega} \) is given in the next lemma.

**Lemma 1.** The outer transition kernel \( \bar{Q}_{\pi,\omega} \) is characterised by

\[
\bar{Q}_{\pi,\omega}(x, \varphi) = \int \max \left\{ \|\varphi \cdot g_\pi(x, \cdot)\|_\infty, \|\varphi \cdot g_\omega(x, \cdot)\|_\infty \right\} Q_{\pi,\omega}(dg_\pi, g_\omega)
\]

for any \( x \in S \), for any \( \varphi \in L^\infty(X) \).

Noticing that \( \bar{Q}_\alpha \) and \( \bar{Q}_{\pi,\omega} \) are outer Markov kernels when restricted to \( \psi \) and \( S \) respectively, the outer Markov kernel \( \bar{Q} \) representing the overall evolution of the system can now be introduced as \( \bar{Q}(\psi, \cdot) = Q_\alpha(\psi, \cdot) \) and \( \bar{Q}(x, \cdot) = \bar{Q}_{\pi,\omega}(x, \cdot) \) for any \( x \in S \). An analogue of the Chapman-Kolmogorov equation can then be derived as in the following theorem, where \( \bar{P}_{t-1} \) and \( \bar{P}_{t(t-1)} \) shorthand notations for \( \bar{P}_t(\cdot | Y_0, Y_1, \ldots, Y_{t-1}) \) and \( \bar{P}_t(\cdot | Y_0, Y_1, \ldots, Y_{t-1}) \) respectively. This notational shortcut can be referred to as implicit conditioning.

**Theorem 2.** Let \( \bar{P}_{t-1} \) be the posterior o.p.m. on \( X \) at time \( t - 1 \geq 0 \), then the predicted o.p.m. \( \bar{P}_{t(t-1)} \) at time \( t \) given the observations up to time \( t - 1 \) and which characterised by

\[
\bar{P}_{t(t-1)}(\varphi) = \int \sup_{x \in X} \left( \max_{i \in \{\alpha, \pi, \omega\}} \|f \cdot g_i(\cdot, x)\|_\infty \varphi(x) \right) \bar{P}_{t-1}(df) Q_\alpha(dg_\alpha) Q_{\pi,\omega}(dg_\pi, g_\omega),
\]
for any $\varphi \in L^\infty(X)$.

The o.p.m. $\tilde{P}_{t|t-1}$ will sometimes be denoted $\tilde{P}_{t-1} \tilde{Q}$ to emphasize that it results from the prediction of the o.p.m. $\tilde{P}_{t-1}$ via $\tilde{Q}$. Note that the outer Markov kernel $\tilde{Q}$ can be made time-dependent without difficulties, at the cost of slightly heavier notations. The expression of $\tilde{P}_{t|t-1}$ being relatively sophisticated, a special case is considered in the next section for illustration purposes.

**Single-possibility case**

A simple example of illustrating theorem 2 can be found when all the outer measure involved are based on a single possibility function. With obvious notational choices, it holds that

$$f_{t|t-1}(x) = \|f_{t-1} \cdot g(\cdot, x)\|_\infty$$

for any $x \in X$, with

$$g : (x', x) \mapsto \begin{cases} g_\alpha(\psi, x) & \text{if } x' = \psi \\ g_\pi(x', x) & \text{if } (x', x) \in S^2 \\ g_\omega(x', \psi) & \text{otherwise.} \end{cases}$$

As in the general case, $f_{t-1}$ is a shorthand notation for $f_{t-1}(\cdot | Y_0, \ldots, Y_{t-1})$ and similarly for $f_{t|t-1}$. If the possibility of disappearance is constant over the state space, i.e. $g_\omega(x', \psi) = a_\omega$ for all $x' \in S$, and if the only thing known about appearance is that the probability to stay in $\psi$ is not higher than $a_\alpha$, i.e. $g_\alpha(\psi, x) = 1$ for all $x \in S$ and to $a_\alpha$ for $x = \psi$, then

$$f_{t|t-1}(x) = \begin{cases} \max (f_{t-1}(\psi), \|f_{t-1} \cdot g_\pi(\cdot, x)\|_\infty) & \text{if } x \in S \\ \max (a_\alpha f_{t-1}(\psi), a_\omega \|1_S \cdot f_{t-1}\|_\infty) & \text{if } x = \psi. \end{cases}$$

The only non-trivial operation is to determine the possibility function resulting from the marginalisation $\|f_{t-1} \cdot g_\pi(\cdot, x)\|_\infty$; this is however the basic single-possibility prediction which can be performed for both linear-Gaussian models (Houssineau and Bishop, 2017) and non-linear models via an analogue (Houssineau and Ristić, 2017) of the sequential Monte Carlo methodology (Doucet et al., 2000) or via grid-based methods.

### 4.2. Data assimilation

The observation process corresponding to the observation eq. (7) is modelled by a conditional o.p.m. $\tilde{L}$ from $X$ to $Y$, which act as a likelihood. By construction of $\mathcal{H}$, there is no observation originated from the system when this one is not in the space $S$, it follows that $\tilde{L}$ verifies

$$\tilde{L}(\varphi | \psi) = \|\varphi \cdot 1_\phi\|_\infty,$$

for any $\varphi \in L^\infty(Y)$. In order to model the false positives, we introduce a weakly-i.i.d. o.p.m. $\tilde{P}_\flat$ representing the uncertain counting measure $Y_\flat$ on $Y$ and specified as follows: the number of points $N_\flat$ in $Y_\flat$ is controlled by an o.p.m. $\tilde{P}_\flat$ on $\mathbb{N}_0$ while the spatial information about each point is represented by another o.p.m. $\tilde{P}_\flat$ on $Y$ verifying $\tilde{P}_\flat(\phi) = 0$. The following theorem is one of the main results in this article and provides an expression of the o.p.m. $\tilde{P}_t$ corresponding to the update of $\tilde{P}_{t|t-1}$ with the observation set $Y_t$. 
The posterior o.p.m. \( \bar{P}_t \) on \( X \) representing the knowledge about the system at time \( t \) including the observation set \( Y_t \) is characterised by

\[
\bar{P}_t(\varphi) \propto \int_{y \in \bar{Y}_t} \max_{y \in \bar{Y}_t} \left( \left\| \varphi \cdot f(y \mid \cdot) \right\|_{\infty} f(Y_t \setminus \{y\}) \right) \bar{P}_{t|t-1}(df) L(df) P_t(df),
\]

where \( \bar{Y}_t = Y_t \cup \{\phi\} \).

Note that the symmetric nature of the o.p.m. \( \bar{P}_t \) on \( Y \) allows for evaluating the possibility at any finite subset \( Y \) of \( Y \) by ordering the elements in the subset arbitrarily (the order being irrelevant).

**Single-possibility case**

With obvious notations, eq. (10) can be simplified in the single-possibility case to

\[
f_t(x) \propto \max_{y \in \bar{Y}_t} f_{t|t-1}(x) h(y \mid x) f_y(Y_t \setminus \{y\})
\]

for any \( x \in X \), with, for any finite subset \( Y \) of \( X \),

\[
f_y(Y) = c_y(|Y|) \prod_{y \in Y} f_y(y).
\]

In particular, if there is no spatial information about the false positives and if \( h(\phi \mid x) = a_{df} \) is the possibility of detection failure for any \( x \in S \), then

\[
f_t(x) \propto \begin{cases} 
\max \left\{ a_{df} c_y(|Y_t|) f_{t|t-1}(x), \max_{y \in \bar{Y}_t} c_y(|Y_t| - 1) f_{t|t-1}(x) h(y \mid x) \right\} & \text{if } x \in S \\
\max_{y \in \bar{Y}_t} c_y(|Y_t|) f_{t|t-1}(y) & \text{if } x = \psi
\end{cases}
\]

since, as a consequence of eq. (9), it holds that \( h(\cdot \mid \psi) = 1_\phi \), i.e. the detection fails almost-surely when the system is not in \( S \). If the cardinality of the false positives is also unknown then \( f_t(\psi) = C_t^{-1} f_{t|t-1}(\psi) \) and

\[
f_t(x) = C_t^{-1} \max \left\{ a_{df} f_{t|t-1}(x), \max_{y \in \bar{Y}_t} f_{t|t-1}(x) h(y \mid x) \right\}
\]

for any \( x \in S \), with

\[
C_t = \max \left\{ f_{t|t-1}(\psi), a_{df} \| 1_S \cdot f_{t|t-1} \|_{\infty}, \max_{y \in \bar{Y}_t} \| f_{t|t-1} : h(y \mid \cdot) \|_{\infty} \right\}
\]

Note that, the product \( f_{t|t-1}(x) h(y \mid x) \) only has to be evaluated for the observation \( y \) that is the closest to \( x \) with respect to (w.r.t.) the likelihood \( h \), which is particularly useful when approximating \( f_t \) via a sequential Monte Carlo implementation (Houssineau and Ristić, 2017).

### 4.3. Implementation in the single-possibility case

In order to implement the above recursion efficiently, we consider the analogue of Gaussian mixtures as follows: for some \( N \in \mathbb{N} \), some collection of scalars \( \{w_i\}_{i=1}^N \) in the interval \((0, 1]\) such that \( \max_i w_i = 1 \) and some collections \( \{m_i\}_{i=1}^N \) and \( \{V_i\}_{i=1}^N \) of points in \( S \) and of
S-valued positive-definite matrices respectively, we define the corresponding \((\text{max-})\)mixture of Gaussian possibilities \(f \in \mathbf{L}(S)\) as

\[
f(x) = \max_{1 \leq i \leq N} w_i \mathcal{N}(x; m_i, V_i).
\] (11)

Standard Gaussian mixture reduction techniques (Salmond, 1990; Williams and Maybeck, 2003; Runnalls, 2007) such as pruning can be straightforwardly applied. However, merging would need to be redefined since there is no guarantee that the usual approach which relies on sums will give a meaningful approximation of a max-mixture like eq. (11). However, devising meaningful and efficient merging procedures for max-mixtures is not straightforward and will be the topic of future research. Additionally, the possibility function \(f\) can be simplified without approximation whenever some index \(i \in \{1, \ldots, N\}\) verifies

\[
w_i \mathcal{N}(x; m_i, V_i) \leq \max_{j \neq i} w_j \mathcal{N}(x; m_j, V_j)
\]

for all \(x \in S\), since such a term will never contribute to the max in \(f\). It will also be necessary to devise efficient ways of verifying such a condition numerically.

The following single-possibility recursion is considered: for a given possibility function \(f_{t-1} \in \mathbf{L}(X)\) representing the posterior knowledge at time \(t-1\), the predicted possibility function at time \(t\) is characterised by

\[
f_{t|t-1}(x) = \begin{cases} 
\max \left\{ f_{t-1}(\psi) g_\alpha(\psi, x), \|f_{t-1} \cdot g_\pi(\cdot, x)\|_\infty \right\} & \text{if } x \in S \\
\max \left\{ f_{t-1}(\psi) g_\alpha(\psi, \psi), \|f_{t-1} \cdot g_\omega(\cdot, \psi)\|_\infty \right\} & \text{if } x = \psi.
\end{cases}
\]

and the updated function \(f_t\) is characterised for all \(x \in X\) by

\[
f_t(x) \propto \max_{y \in Y_t} f_{t|t-1}(x) h(y \mid x) f_y(\psi) f_y(Y_t \setminus \{y\}).
\]

A few assumptions are needed to ensure that the recursion is closed-form, i.e. that if the restriction of \(f_{t-1}\) to \(S\) is a mixture of Gaussian possibilities then \(f_t\) will also have this form. These assumptions are as follows:

**A.1** \(g_\pi\) is a linear-Gaussian conditional possibility function of the form

\[
g_\pi(x', x) = a_\pi \mathcal{N}(x; Fx', Q).
\]

**A.2** The restriction of \(g_\alpha(\psi, \cdot)\) to \(S\) is a mixture of Gaussian possibilities of the form

\[
g_\alpha(\psi, x) = \max_{1 \leq i \leq N_\alpha} w_\alpha^i \mathcal{N}(x; m_\alpha^i, V_\alpha^i).
\]

**A.3** For any \(x \in S\), the conditional possibility function \(h(\cdot \mid x)\) is of the form

\[
h(y \mid x) = \begin{cases} 
\mathcal{N}(y; Hx', R) & \text{if } y \in Y_t \\
a_{\text{def}} & \text{if } y = \phi.
\end{cases}
\]

Assumption **A.1** could be formulated more generally by allowing \(g_\pi\) to be a max-mixture of Gaussian possibility functions, this generalisation is however not considered for the sake of simplicity. Although there is no requirement on \(g_\omega\) for the prediction to be closed-form, the choice \(g_\omega(x, \psi) = a_\omega\) for any \(x \in S\) is convenient in terms of calculations (note that \(\max\{a_\omega, a_\pi\} = 1\) by construction in this case). If the possibility of disappearance is state-dependent, another choice that yields a simple implementation is

\[
g_\omega(x, \psi) = \max_i a_\omega^i 1_{A_\omega^i},
\]
where \( \{a^i\}_i \) and \( \{A^i\}_i \) are collections of scalars in the interval \((0, 1]\) and of subsets of \( S \) respectively.

We consider that the posterior possibility function \( f_{t-1} \) at time \( t - 1 \) takes the form

\[
f_{t-1}(x) = \max_{1 \leq i \leq N_{t-1}} \ w^i_{t-1} \mathcal{N}(x; m^i_{t-1}, V^i_{t-1}),
\]

for any \( x \in S \), where \( \{(w^i_{t-1}, m^i_{t-1}, V^i_{t-1})\}_{i=1}^{N_{t-1}} \) is an appropriate collection of scalars, points of \( S \) and \( S \)-valued positive-definite matrices. The following corollary shows that the prediction is closed-form under the given assumptions and details the obtained mixture of Gaussian possibilities.

**Corollary 1.** Under \( \text{A.1} \) and \( \text{A.2} \), if \( f_{t-1} \) takes the form of eq. (12) then

\[
f_{t|t-1}(x) = \begin{cases} \max_{1 \leq i \leq N_{t|t-1}} & w^i_{t|t-1} \mathcal{N}(x; m^i_{t|t-1}, V^i_{t|t-1}) & \text{if } x \in S \\ \max\{f_{t-1}(\psi)g_\alpha(\psi, \psi), \|f_{t-1} \cdot g_\alpha(\cdot, \psi)\|_\infty\} & \text{if } x = \psi \end{cases}
\]

where \( N_{t|t-1} = N_{t-1} + N_\alpha \), where for any \( i \in \{1, \ldots, N_{t-1}\} \)

\[
(w^i_{t|t-1}, m^i_{t|t-1}, V^i_{t|t-1}) = (w^i_{t-1}, Fm^i_{t-1}, FV^i_{t-1}F^T + Q)
\]

and where for any \( i \in \{1, \ldots, N_\alpha\} \)

\[
(w^{N_{t-1}+i}_{t|t-1}, m^{N_{t-1}+i}_{t|t-1}, V^{N_{t-1}+i}_{t|t-1}) = (w^i_\alpha, m^i_\alpha, V^i_\alpha).
\]

Note that for \( i \in \{1, \ldots, N_{t-1}\} \), the terms \( m^i_{t|t-1} \) and \( V^i_{t|t-1} \) are simply the Kalman-filter predicted mean and variance. In the case where there is no knowledge about the spatial information \( g_\alpha(\psi, \cdot) \), which often occurs in practice (Houssineau and Laneville, 2010; Ristić et al., 2010), one can consider \( g_\alpha(\psi, \psi) = a_\alpha \) and \( g_\alpha(\psi, x) = 1 \) for all \( x \in S \) so that the Gaussian approximation will be delayed to the first assumed observation of the system (this is a true observation-driven appearance). Note in particular that there is no need to spatially bound the locations in \( S \) at which the system might appear. Such an approach does not induce difficulties at later time steps since the possibility function 1 is invariant under prediction.

The following corollary demonstrates that the update step is also closed-form and that it mostly consists of the Kalman update of each of the components of the mixture.

**Corollary 2.** Under \( \text{A.3} \), if the predicted possibility function \( f_{t|t-1} \) takes the form of eq. (13) then \( f_t(\psi) = C_t^{-1} f_{t|t-1}(\psi) f_t(Y_t) \) and

\[
f_t(x) = \max_{1 \leq i \leq N_{t|t-1}} \left( \max_{y \in Y_t} w^{i,y}_{t} \mathcal{N}(x; m^{i,y}_{t}, V^{i,y}_{t}) \right)
\]

for any \( x \in S \), where

\[
(m^{i,y}_{t}, V^{i,y}_{t}) = \begin{cases} (m^{i,y}_{t|t-1} + K^i_t(y - Hm^{i}_{t|t-1}), (I - K^i_tH)V^{i}_{t|t-1}) & \text{if } y \in Y_t \\ (m^{i,y}_{t|t-1}, V^{i}_{t|t-1}) & \text{if } y = \phi \end{cases}
\]

with \( K^i_t = V^{i}_{t|t-1}H^T(S^i_{t})^{-1} \) and \( S^i_{t} = HV^{i}_{t|t-1}H^T + R \), where

\[
w^{i,y}_{t} = \begin{cases} C_t^{-1} w^{i,y}_{t|t-1} f_t(Y_t \setminus \{y\}) \mathcal{N}(y; Hm^{i}_{t|t-1}, S^i_{t}) & \text{when } y \in Y_t \\ C_t^{-1} w^{i,y}_{t|t-1} a_{df} f_t(Y_t) & \text{when } y = \phi.
\]
and where $C_t$ ensures that $\|f_t\|_{\infty} = 1$, that is

$$C_t = \max \left\{ f_{t|t-1}(\psi) f_s(Y_t), \quad w_{t|t-1}^i a_{df} f_s(Y_t), \quad \max_{1 \leq i \leq N_{t|t-1}} w_{t|t-1}^i \frac{f_s(Y_t \setminus \{y\}) \bar{N}(y; Hm_{t|t-1}^i, S_t^i)}{y \in Y_t} \right\}.$$ 

The expression eq. (14) can be easily rewritten under the same form as eq. (12) by relabelling the components by integers between 1 and $N_t = N_{t|t-1}|\bar{Y}_t|$. Note that $m_{t|y}^i$ and $V_{t|y}^i$ are simply the posterior Kalman-filter mean and variance when $y \in Y_t$. Corollary 2 provides a way of performing a data update with little to no knowledge of the statistics of false positives. This is useful in practice since these statistics might be highly variable in time and space and hence difficult to estimate.

Although we have obtained a closed-form recursion, the proposed algorithm would not be useful without a decision step about the state of the system in $X$. Direct analogues of standard techniques for Gaussian mixture such as choosing the mean of the Gaussian with highest weight as a state estimate (whenever this weight is above a given threshold) would not work for max-mixture of Gaussian possibility functions. Indeed, the fact that $f_t$ is equal to 1 at a given point or even in a given region of $S$ does not give sufficient guarantees for deciding that the true state is likely to be at that point or in that region. However, a lower bound for the probability for the true state to be in a region $B \subseteq S$ is given by $1 - \sup_{x \in B} f_t(x)$. An ad-hoc but simple way of selecting a state estimate is then as follows: if $\max_{i,y} w_{t|y}^i > f_t(\psi)$ and if the difference between the highest weight, say the term $(i,y)$, and the second highest weight is more than a given threshold $\tau_c$ then declare $m_{t|y}^i$ as the state estimate, otherwise declare $\psi$ as the state estimate. This state estimation procedure requires an appropriate merging procedure to perform well, but so does basic state estimation for standard Gaussian mixtures.

5. Extension to multiple dynamical systems

The objective is now to extend the previous results to the case where there are possibly several dynamical systems with different states in $S$, so that the state to be inferred is itself an uncertain counting measure. For the sake of simplicity, only the single-possibility case is considered in this section. The intensity of the uncertain counting measure $X_\alpha$ modelling the appearance of new systems in $S$ is denoted $\bar{F}_\alpha$. Throughout the section, $\max\{a, b\}$ will be denoted $a \lor b$ for any $a, b \in \mathbb{R}$ in order to improve readability. We also consider that $\lor$ has a lower precedence than the multiplication so that $a \lor bc = a \lor (bc)$ for any $a, b, c \in \mathbb{R}$. The other notations are consistent with the previous sections.

**Corollary 3.** Let $F_{t-1}$ be the updated intensity function representing the uncertain counting measure $X_{t-1}$ on $S$ given the observation up to time $t - 1$, then the predicted intensity function $F_{t|t-1}$ is

$$F_{t|t-1}(x) = F_\alpha(x) \lor \|F_{t-1} \cdot g_\alpha(\cdot, x)\|_{\infty}$$

for any $x \in S$.

Corollary 3 is a direct consequence of proposition 1 and proposition 2. Now consider the extended state space $\bar{S}$ where the isolated point $\psi_x$ has been added in order to give a state to the phenomena giving rise to false positives.
Theorem 4. Let \( F_{t|t-1} \) be the predicted intensity function representing the uncertain counting measure \( \mathcal{X}_t \) on \( S \), then the updated intensity function \( F_t \) based on the realisation \( \sum_{i=1}^n \delta_{y_i} \) of the observation uncertain counting measure \( \mathcal{Y}_t \) is
\[
F_t(x) = F_{t|t-1}(x)h(\phi \vert x) \vee \max_{1 \leq i \leq n} \frac{F_{t|t-1}(x)h(y_i \vert x)}{\|F_{t|t-1} \cdot h(y_i \cdot)\|_\infty \vee F_s(y_i)}
\]
for any \( x \in S \).

The proof of theorem 4, which can be found in appendix A.5, is adapted from Caron et al. (2011) which was dedicated to Poisson i.i.d. point processes, see also Del Moral and Houssineau (2015). It appears that proofs for this type of results regarding uncertain counting measures are of a similar complexity as their standard analogues.

The recursion defined by corollary 3 and theorem 4 is naturally reminiscent of the PHD filter (Mahler, 2003). The latter however assumes that the point process to be inferred as \( \mathcal{Y}_t \) is preferred. Note however that considering \( F_s(\phi) = 1 \) is natural when seeing false positives as observation generated by systems in \( S \) that are different from the ones of interest (sometimes referred to as false-positive generators). With this modelling, it is clear that potentially many of these undesired systems will actually fail to generate false-positives (so their observation is indeed \( \phi \)).

Remark 3. The expression of \( F_t \) can be simplified by setting \( F_s(\phi) \) to 1 and defining \( \bar{Y}_t \) as the extended set of observations \( \bar{Y}_t \cup \{\phi\} \), in which case
\[
F_t(x) = \max_{y \in \bar{Y}_t} \frac{F_{t|t-1}(x)h(y \mid x)}{\|F_{t|t-1} \cdot h(y \cdot)\|_\infty \vee F_s(y)}.
\]
However, this expression is less explicit than the one given in theorem 4 so that the latter is preferred. Note however that considering \( F_s(\phi) = 1 \) is natural when seeing false positives as observation generated by systems in \( S \) that are different from the ones of interest (sometimes referred to as false-positive generators). With this modelling, it is clear that potentially many of these undesired systems will actually fail to generate false-positives (so their observation is indeed \( \phi \)).

One of the strengths of the proposed method is that it requires very limited information about the different aspects of the problem. For instance, if \( F_\alpha(x) = a_\alpha \) and \( h(\phi \mid x) = a_{df} \) for any \( x \in S \) and if \( F_s(y) = a_y \) for any \( y \in S' \), then
\[
F_{t|t-1}(x) = a_\alpha \vee \|F_{t|t-1} \cdot g_\pi(\cdot, x)\|_\infty
\]
\[
F_t(x) = a_{df}F_{t|t-1}(x) \vee \max_{y \in \bar{Y}_t} \frac{F_{t|t-1}(x)h(y \mid x)}{\|F_{t|t-1} \cdot h(y \cdot)\|_\infty \vee a_y}.
\]
The second equation could be simplified further by assuming that there is no information at all about the false positive, that is \( a_y = 1 \), however that would cause \( \|F_t\|_\infty \) to inexorably decrease in time, which is not desired. Some properties of the proposed method can be highlighted:

(a) The intensity function \( F_t \) is upper-bounded by 1.

(b) If two observations are arbitrarily close to each other, then the proposed method gives essentially the same result as if only one of the two observations was present.

(c) As before, only the closest observations will have an impact on \( F_t \) at a given point \( x \in S \), so that it might be possible to devise a \textit{gating} procedure without introducing errors (especially in particle-based implementations).
Table 1: Parameters of the filters

|                       | Proposed method | Probabilistic method |
|-----------------------|-----------------|----------------------|
| detection             | $a_{df} = 0.2$  | $p_d = 0.8$          |
| disappearance         | $a_\omega = 0.01$ | $p_s = 0.99$        |
| appearance            | $a_\alpha = 0.5$ | $p_b = 0.5$         |
| pruning               | $\tau_p = 10^{-4}$ | $\tau'_p = 10^{-5}$ |
| merging               |                 | $\tau_m = 3.22$     |

(d) A true observation-driven appearance can be introduced in a simple way as in the case of a single system.

The implementation of the proposed intensity-function recursion can be performed using max-mixtures of Gaussian possibility functions or, alternatively, using an analogue of the sequential Monte Carlo methodology for models describing multiple dynamical systems (Vo et al., 2005).

6. Numerical study

We consider that the state of the system of interest is evolving in a 1-dimensional space and that both position and velocity are estimated, yielding a 2-dimensional state of the form $[x, v]^T$ with $x$ the position in meters and $v$ the velocity in meters per second. The system appears in $S$ at the 3rd time step and remains in $S$ until the 22nd time step while the scenario ends at the 25th time step. The state of the system follows a standard nearly-constant-velocity model with initial state $x_5 = [0, \nu]^T$ with $\nu \sim \mathcal{N}(0, 0.1^2)$. The time step $\Delta$ is of 0.1s and the dynamical noise has standard deviation $q = 1.5m/s^2$. Overall, we have for the dynamics

$$F = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = q^2 \begin{bmatrix} \frac{1}{2} \Delta^2 \\ \Delta \end{bmatrix}. $$

The observation consists of a linear observation of the position with an additive noise of standard deviation $r = 0.25m$ so that $H = [1\ 0]^T$ and $R = r^2$. The probability of detection $p_d$ is equal to 0.8 and there are a Poisson number of false positives with parameter $\lambda$ distributed uniformly on the interval $[-10, 10]$m. The implementation of the proposed method corresponding to corollary 1 and corollary 2 is compared to a Gaussian-mixture implementation of the probabilistic approach described in (Musicki et al., 1994). The parameters for both methods are given in table 1. They are mostly consistent with each others except for the confirmation and pruning thresholds, since the decision mechanisms are different and since probabilities tend to take smaller values than possibilities (the respective pruning threshold values have been selected so that the two methods propagate roughly the same number of terms in their respective Gaussian mixtures).

For both filters, the appearance is observation-oriented and the velocity upon appearance is Gaussian with mean 0 and standard deviation 1m/s. This is straightforward in the proposed method but requires some heuristics in a probabilistic context (Houssineau and Laneuville, 2010; Ristić et al., 2010).

The considered implementation of the proposed solution does not assume knowledge of the false positives, neither in terms of cardinality nor in terms of spatial distribution. However, in order to better assess the other aspects of the problem, the noise in the dynamics and observation are set to their simulated value (but interpreted in a non-probabilistic way.
by the proposed method). The probabilistic approach is given the true characteristics of the false positives. Simulations are run in three cases corresponding to three values of the false-positive parameter $\lambda \in \{1, 5, 10\}$. Since it is a 1-dimensional scenario that is considered, an average of 10 false positives per time step already makes the problem very challenging since at this rate, false positives become likely to form consistent trajectories over at least a few time steps as seen in fig. 1. In order to assess the performance of each approach, the error $e_t$ at time $t$ is calculated as follows:

$$e_t = \begin{cases} d_c(x_t, x_t^*) + c(1 - n_t) & \text{if } 3 \leq t \leq 22 \\ cn_t & \text{otherwise,} \end{cases}$$

where $c > 0$ is an arbitrary error, where $n_t$ equals 1 when the system is inferred to be present in $S$ and 0 otherwise and where

$$d_c(x_t, x_t^*) = \min\{d(x_t, x_t^*), c\}$$

with $d$ a distance function and with $x_t$ and $x_t^*$ the estimated and true state respectively. The results shown in fig. 2 are averaged over 1000 Monte Carlo simulations and report the error for different values of the confirmation threshold in two complementary ways for each value of $\lambda$: the top graph displays the average error $\frac{1}{T+1} \sum_{t=0}^{T} e_t$ with $T$ the final time while the bottom graph plots the evolution of the error $e_t$ over time with a transparency that is proportional to the error. The main common features that appear in fig. 2 are as follows: (a) the proposed method is less dependent on the confirmation threshold than the probabilistic approach, (b) the proposed method however takes longer to notice the disappearance of the system, and (c) its performance is similar to the probabilistic approach.

These results confirm that it is possible to detect and estimate the state of a dynamical system with little information about the characteristics of the disturbances that are affecting its observation. Intuitively, the proposed approach based on o.p.m.s can be understood to proceed by elimination since it is not directly the maximum weight in the considered max-mixture that is used for track extraction but the difference between the maximum weight and the other weights.

7. Conclusion

A variant of point processes adapted to outer-measure representation of uncertainty was introduced and briefly studied. This concept, referred to as deterministic uncertain counting measure, enables a greater versatility in the modelling of point patterns by, for instance, allowing for the representation of the absence of information on the number of points and/or on their spatial distribution. These uncertain counting measures were then used to represent false positives in a detection and estimation problem where there is uncertainty on the presence of the system of interest in the considered state space. The obtained filtering equations were subsequently implemented using a maximum-based mixture of Gaussian possibility functions and their performance was demonstrated on increasingly challenging simulated scenarios. An extension to the case where multiple dynamical systems are present was also presented.

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Fig. 1: Observations at all times for $\lambda = 10$. The true trajectory is indicated on the left for reference, the right plot being the raw information communicated to the filter.

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Fig. 2: Error vs. time for the proposed method and for the probabilistic approach for the different values of $\lambda$. Transparency in the bottom graphs is proportional to the time-averaged error shown in the top graph.
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A. Proofs

A.1. Proof of theorem 1
Let \( f \) be any possibility function in the support of \( P \) and let \( f'(\cdot | x) \) be any function in the support of \( P'(\cdot | X) \), that is \( f \in \mathbf{L}(E) \) and \( f' \in \mathbf{L}^\infty(F; E) \). Let \( f'(\cdot | x) \) be the conditional possibility function defined for any \( x \in E \) as

\[
\hat{f}'(x' | x) = \frac{f'(x' | x)}{\|f'(\cdot | x)\|_\infty}.
\]

We obtain from \( f \) and \( \hat{f}' \) the possibility function \( \hat{f} \) on \( E \times F \) describing the uncertain variable \((X, X')\) and defined as \( \hat{f}(x, x') = \hat{f}'(x' | x)f(x) \), from which we deduce that the function \( \hat{f} \) on \( E \times F \) corresponding to \( f' \) is

\[
\hat{f}(x, x') = \|f'(\cdot | x)\|_\infty \hat{f}(x, x') = f'(x' | x)f(x).
\]

Considering the integral w.r.t. \( P \) and \( P'(\cdot | X) \) of the supremum \( \|\varphi \cdot \hat{f}\|_\infty \) for any \( \varphi \in \mathbf{L}^\infty(F) \) completes the proof of the theorem.

A.2. Proof of lemma 1
Define \( \bar{Q}_{\pi,\omega} \) as the natural outer transition kernel from \( \mathbf{X}^2 \) to itself associated with \( Q_{\pi,\omega} \), expressed as

\[
Q_{\pi,\omega}(x, x'), \varphi_x \times \varphi_{x'} = \int \| (\varphi_x \times \varphi_{x'}) \cdot (g_{\pi}(x, \cdot) \times g_{\omega}(x', \cdot)) \|_\infty Q_{\pi,\omega}(d(g_{\pi}, g_{\omega})),
\]

for any \( \varphi_x, \varphi_{x'} \in \mathbf{L}^\infty(\mathbf{X}) \) and any \( x, x' \in \mathbf{X} \). The objective is to derive the expression of the outer transition kernel \( \bar{Q}_{\pi,\omega} \) from \( \mathbf{X} \) to itself corresponding to events of the form \( X \in B \times X \cup X \times B \) for some \( B \in \mathbf{B}(\mathbf{X}) \) with \( X \) the uncertain variable corresponding to the state of the system after applying the kernel. This type of events can be expressed as functions on \( \mathbf{X}^2 \) as

\[
(x, x') \mapsto 1_{B \times \mathbf{X} \cup \mathbf{X} \times B}(x, x') = \max \{ 1_B(x), 1_B(x') \},
\]

which translates more generally into functions of the form

\[
\varphi_{\pi \cup \omega}(x, x') = \max \{ \varphi(x), \varphi(x') \}
\]

for some \( \varphi \in \mathbf{L}^\infty(\mathbf{X}) \). Evaluating \( \bar{Q}_{\pi,\omega} \) at functions of this form, it follows that

\[
Q_{\pi,\omega}(x, x'), \varphi_{\pi \cup \omega} = \int \max \{ \| \varphi \cdot g_{\pi}(x, \cdot) \|_\infty, \| \varphi \cdot g_{\omega}(x', \cdot) \|_\infty \} Q_{\pi,\omega}(d(g_{\pi}, g_{\omega}))
\]

The result of the lemma follows by considering \( \bar{Q}_{\pi,\omega}(x, \varphi) = \bar{Q}_{\pi,\omega}(x, x'), \varphi_{\pi \cup \omega} \), which is justified by the fact that the system is at a unique state in \( \mathbf{X} \) before applying the kernel.
A.3. Proof of theorem 2
Temporarily extending the outer Markov kernel $\tilde{Q}$ to any pair of function $(\Phi, \varphi)$ in $L^\infty(X)^3 \times X$, this kernel can be expressed more generally as

$$\tilde{Q}(x, (\Phi, \varphi)) = \int \Phi(g_\alpha, g_\pi, g_\omega) \max_{i \in \{\alpha, \pi, \omega\}} (\|\varphi \cdot g_i(x, \cdot)\|_\infty) Q_\alpha(dg_\alpha) Q_\pi,d_\omega(dg_\pi, g_\omega)$$

for any $x \in X$, so that any function $g$ corresponding to the integrand can be extracted by considering $\Phi = \delta_{(g_\alpha, g_\pi, g_\omega)}$, that is

$$g : (x, x') \mapsto \tilde{Q}(x, (\delta_{(g_\alpha, g_\pi, g_\omega)}, 1_{x'})) = \max_{i \in \{\alpha, \pi, \omega\}} g_i(x, x').$$

The desired result follows from applying Chapman-Kolmogorov equation for possibility functions to any $f \in L(X)$ and integrating back w.r.t. $P_{t-1}$ and $Q_\alpha \times Q_\pi,d_\omega$.

A.4. Proof of theorem 3
The outer measure $\tilde{P}_\beta$ is characterised by

$$\tilde{P}_\beta(\varphi) = \int \max_{n \in N_0} \left( c(n) \sup_{y \in Y^n} (\varphi(y)f_n(y)) \right) p_\beta(df) \prod_{m \geq 0} \left[ \prod_{i=1}^m P_i(df_i) \right],$$

for any $\varphi \in L^\infty(\bar{Y})$. The joint o.p.m. on $\bar{Y}$ describing the true observation and the false positives is denoted $\bar{L}$ and its expression is deduced to be

$$\bar{L}(\varphi | x) = \int \sup_{(y, y) \in Y \times Y} (\varphi(y \times y) h(y | x) f(y)) L(dh) P_p(df)$$

for any $\varphi \in L^\infty(\bar{Y})$, where $y \times y$ is the concatenation of $y \in Y$ with $y \in \bar{Y}$. Similarly, the joint o.p.m. on $X \times \bar{Y}$ describing the dynamical system as well as both its observation and the false positives is denoted $\bar{P} \cdot \bar{L}$ and expressed as

$$\bar{P} \cdot \bar{L}(\varphi) = \int \sup_{(x, y, y) \in X \times Y \times \bar{Y}} (\varphi(x, y \times y) f(x) h(y | x) f(y)) P_{t|t-1}(df) L(dh) P_p(df)$$

for any $\varphi \in L^\infty(X \times \bar{Y})$. The objective is then to prove the data assimilation equation eq. (10) by conditioning on the event that the considered dynamical system has generated one of the observations in $Y_t$ or no observation at all. A few additional notations are required: let $y_t$ and $\bar{y}_t$ be vectors respectively containing all the observations in $Y_t$ and $\bar{Y}_t$ in an arbitrary order. The information provided by the observation can then be expressed as a subset $A_t$ of $\bar{Y}$ as

$$A_t = \{ y_{t, \sigma} : \sigma \in Sym(n_t) \} \cup \{ \bar{y}_{t, \sigma} : \sigma \in Sym(n_t + 1) \}$$

where $n_t = |Y_t|$ and where $y_\sigma = (y_{\sigma(1)}, \ldots, y_{\sigma(n_t)})$ for any $y \in \bar{Y}$. Therefore, Bayes’ theorem can be simply expressed as

$$\bar{P}_t(\varphi) = \frac{\bar{P} \cdot \bar{L}(\varphi \times 1_{A_t})}{\bar{P}L(1_{A_t})}$$

where $\bar{P}L(1_{A_t})$ is the marginal likelihood, with $\bar{P}L$ the marginal o.p.m. describing the observations, characterised by $\bar{P}L(\varphi) = \bar{P} \cdot \bar{L}(1 \times \varphi)$ for $\varphi \in L^\infty(\bar{Y})$. The indicator function $1_{A_t}$ can however be expressed as a possibility function $f'_t$ on $\bar{Y}$ defined by

$$f'_t(y) = \max \left\{ \max_{\sigma \in Sym(n_t)} 1_{y_{t, \sigma}}(y), \max_{\sigma \in Sym(n_t + 1)} 1_{\bar{y}_{t, \sigma}}(y) \right\}.$$
The possibility function $f'_t$ represents the fact that all permutations of $Y_t$ are possible and we also ignore whether the system was observed or if all the observations are false positives. Using the fact that $\mathcal{Y}_t$ cannot generate the empty observation $\phi$ by construction, we obtain after marginalisation the o.p.m. $\bar{P}_t$ on $X$ as

$$
\bar{P}_t(\varphi) \propto \int \max_{\sigma \in \text{Sym}(n_t)} \left( \|\varphi \cdot f \cdot h(y_{t,\sigma(1)} | \cdot)\|_\infty f(y_{t,\sigma(2)}, \ldots, y_{t,\sigma(n_t)}) \right),
$$

$$
\max_{\sigma \in \text{Sym}(n_t)} \left( \|\varphi \cdot f \cdot h(\phi | \cdot)\|_\infty f(y_{t,\sigma(1)}, \ldots, y_{t,\sigma(n_t)}) \right) \bar{P}_t|t-1|df(d\mathcal{H})P_\phi(df).
$$

By the symmetry properties of uncertain counting measures, it follows that only the value of $\sigma(1)$ matters in the first term defining $\bar{P}_t$ and that the value does not depend on $\sigma$ in the second term, so that

$$
\bar{P}_t(\varphi) \propto \int \max_{1 \leq i \leq n_t} \left( \|\varphi \cdot f \cdot h(y_{t,i} | \cdot)\|_\infty f(y_{t,1}, \ldots, y_{t,i-1}, y_{t,i+1}, \ldots, y_{t,n_t}) \right),
$$

$$
\|\varphi \cdot f \cdot h(\phi | \cdot)\|_\infty f(y_{t}) \bar{P}_t|t-1|df(d\mathcal{H})P_\phi(df).
$$

The proof is concluded by noticing that $y_{t,i}$ and $(y_{t,1}, \ldots, y_{t,i-1}, y_{t,i+1}, \ldots, y_{t,n_t})$ can be respectively identified as $y$ and $Y_t \setminus \{y\}$ without ambiguity, and that the second term is of the same form as the first one with $y = \phi$.

A.5. Proof of theorem 4

Consider the uncertain counting measure $\bar{X}_{i|t-1}$ resulting from the superposition of $X_{i|t-1}$ with $X_{i}$ and define the following extended possibility function and likelihood

$$
\bar{f}(x) = \frac{F_{i|t-1}(x) \vee F_{\delta}}{\|F_{i|t-1}\|_\infty \vee \|F_{\delta}\|_\infty},
$$

$$
\bar{h}(y | x) = 1_S(x)h(y | x) \vee 1_{\delta}(x) \frac{F_{\delta}(y)}{\|F_{\delta}\|_\infty},
$$

where it is naturally assumed that $F_{i|t-1}(\delta) = 0$. Applying Bayes’ theorem, the common spatial information about this process given a possibly-empty observation $y \in Y$ is expressed, for any $x \in S$, as

$$
\bar{f}(x | y) = 1_{\phi}(y) \frac{F_{i|t-1}(x)h(\phi | x)}{\|F_{i|t-1} \cdot h(\phi | \cdot)\|_\infty} \vee 1_S(y) \left( \frac{F_{i|t-1}(x)h(y | x)}{\|F_{i|t-1} \cdot h(y | \cdot)\|_\infty \vee F_{\delta}(y)} \vee \frac{F_{\delta}(y)}{\|F_{i|t-1} \cdot h(y | \cdot)\|_\infty \vee F_{\delta}(y)} \right),
$$

so that, denoting $\bar{Y}_t = \sum_{i=1}^{N_t} \delta_{\bar{Y}_i} = \sum_{i=1}^{N_t} \delta_{\bar{Y}_i} + N_t^\delta \delta_{\phi}$ the extension of $\bar{Y}_t$ that includes the detection failures $\phi$,

$$
\bar{P}(\varphi(\bar{X}_{i|t-1}) | \bar{Y}_t) = \sup_{x \in S^{N_t}} \left( \varphi(x) \prod_{i=1}^{N_t} \bar{f}(x_i | \bar{Y}_i) \right),
$$
where $\bar{N}_t = \bar{Y}_t(1)$ is the number of observations (empty or not) in $\bar{Y}_t$. Denoting $c_{df}$ the possibility function on $\mathbb{N}_0$ induced by the intensity function $F_{t|t-1} \cdot h(\phi \mid \cdot)$, that is

$$c_{df}(n) = \|F_{t|t-1} \cdot h(\phi \mid \cdot)\|_\infty^n,$$

it follows that

$$\bar{P}(\varphi(\bar{Y}_t) \mid \mathcal{Y}_t) = \max_{n \geq 0} c_{df}(n) \varphi(\mathcal{Y}_t + n\delta_0).$$

Since it holds that $\bar{P}(\varphi(\bar{X}_{t|t-1}) \mid \mathcal{Y}_t) = \bar{P}(\bar{P}(\varphi(\bar{X}_{t|t-1}) \mid \bar{Y}_t) \mid \mathcal{Y}_t)$ by the law of total probability for single-possibility o.p.m.s, we obtain

$$\bar{P}(\varphi(\bar{X}_{t|t-1}) \mid \mathcal{Y}_t) = \max_{n \geq 0} \left( c_{df}(n) \sup_{\mathcal{X} \in \mathbb{S}_{N_t+n}} \left( \varphi(\mathcal{X}) \prod_{i=N_t+1}^{N_t+n} F_{t|t-1}(x_i) h(\phi \mid x_i) \prod_{i=1}^{N_t} \bar{f}(x_i \mid Y_i) \right) \right).$$

with $N_t = \mathcal{Y}_t(1)$. The proof is concluded by computing $F_t(x)$ for any $x \in \mathcal{X}$ as

$$F_t(x) = \bar{P}(\bar{X}_{t|t-1}(\{x\}) \geq 1 \mid \mathcal{Y}_t)
\begin{equation}
= \left( \max_{n \geq 1} c_{df}(n) \frac{F_{t|t-1}(x) h(\phi \mid x)}{\|F_{t|t-1} \cdot h(\phi \mid \cdot)\|_\infty} \right)
\vee \left( c_{df}(0) \max_{1 \leq i \leq N_t} \frac{F_{t|t-1}(x) h(y_i \mid x)}{\|F_{t|t-1} \cdot h(y_i \mid \cdot)\|_\infty} \vee \bar{f}(y_i) \right)
\end{equation}

= F_{t|t-1}(x) h(\phi \mid x) \vee \max_{1 \leq i \leq N_t} \frac{F_{t|t-1}(x) h(y_i \mid x)}{\|F_{t|t-1} \cdot h(y_i \mid \cdot)\|_\infty} \vee \bar{f}(y_i),$$

which terminates the proof of the theorem.