Limits on Counting Triangles using Bipartite Independent Set Queries

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Abstract

Beame et al. [ITCS 2018 & TALG 2021] introduced and used the Bipartite Independent Set (BIS) and Independent Set (IS) oracle access to an unknown, simple, unweighted and undirected graph and solved the edge estimation problem. The introduction of this oracle set forth a series of works in a short span of time that either solved open questions mentioned by Beame et al. or were generalizations of their work as in Dell and Lapinskas [STOC 2018], Dell, Lapinskas and Meeks [SODA 2020], Bhattacharya et al. [ISAAC 2019 & Theory Comput. Syst. 2021], and Chen et al. [SODA 2020]. Edge estimation using BIS can be done using polylogarithmic queries, while IS queries need sub-linear but more than polylogarithmic queries. Chen et al. improved Beame et al.’s upper bound result for edge estimation using IS and also showed an almost matching lower bound. Beame et al. in their introductory work asked a few open questions out of which one was on estimating structures of higher order than edges, like triangles and cliques, using BIS queries. Motivated by this question, we completely resolve the query complexity of estimating triangles using BIS oracle. While doing so, we prove a lower bound for an even stronger query oracle called Edge Emptiness (EE) oracle, recently introduced by Assadi, Chakrabarty and Khanna [ESA 2021] to test graph connectivity.
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1 Introduction

Motivated by connections to group testing, to emptiness versus counting questions in computational geometry, and to the complexity of decision versus counting problems, Beame et al. introduced the Bipartite Independent Set (shortened as BIS) and Independent Set (shortened as IS) oracles as a counterpoint to the local queries \[ GR02, Fei06, GR08 \]. The BIS query oracle can be seen in the lineage of the query oracles \[ Sto83, Sto85, RT16 \] that go beyond the local queries. Let us start by looking into the formal definitions of BIS and IS.

**Definition 1.1.** (Bipartite Independent Set) Given disjoint subsets \( U, U' \subseteq V(G) \), a BIS query answers whether there exists an edge between \( U \) and \( U' \) in \( G \).

**Definition 1.2.** (Independent Set) Given a subset \( U \subseteq V(G) \), a IS query answers whether there exists an edge between vertices of \( U \) in \( G \).

The introduction of this new type of oracle access to a graph spawned a series of works that either solved open questions \[ DL18, DLM20 \] mentioned in Beame et al. or were generalizations \[ DLM20, BBGM21, BBCM19a \]. Beame et al. used BIS and IS queries to estimate the number of edges in a graph \[ BHR+18, BHR+20 \]. One of their striking observations was that BIS queries were more effective than IS queries for estimating edges. This observation also fits in with the fact that IS queries can be simulated in a randomized fashion using polylogarithmic BIS queries \[ * \]. Edge estimation using BIS was also solved in \[ DL18 \] albeit in a higher query complexity than \[ BHR+18 \].

There were later generalizations of the BIS oracle to estimate higher order structures like triangles and hyperedges \[ DLM20, BBGM19b, BBCM19a \]. On the IS front, Beame et al.’s result for edge estimation using IS oracle was improved in \[ CLW20 \] with an almost matching lower bound and thereby resolving the query complexity of estimating edges using IS oracle. One can observe the interest generated in these (bipartite) independent set based oracles in a short span of time. The results are summarized in Table \[ 1 \] a cursory glance would tell us that commensurate higher order queries were needed for estimating higher order structures (Tripartite Independent Set (shortened as TIS) for counting triangles, Colorful Independence Oracle (shortened as CID) for hyperedges) if polylogarithmic number of queries is the benchmark. We provide the definitions of TIS and CID below.

**Definition 1.3** (Tripartite Independent Set \[ BBGM19b \]). Given three disjoint subsets \( A, B, C \) of the vertex set \( V \) of a graph \( G(V, E) \), the TIS oracle reports whether there exists a triangle having endpoints in \( A, B \) and \( C \).

**Definition 1.4** (Colorful Independence Oracle \[ BGK+18, DLM20 \]). Given \( d \) pairwise disjoint subsets of vertices \( A_1, \ldots, A_d \subseteq U(H) \) of a hypergraph \( H \) \((U(H) \) is the vertex set of the hypergraph \( H \)) as input, CID query oracle answers whether \( m(A_1, \ldots, A_d) \neq 0 \), where \( m(A_1, \ldots, A_d) \) denotes the number of hyperedges in \( H \) having exactly one vertex in each \( A_i, \forall i \in \{1, 2, \ldots, d\} \).

1.1 The open questions suggested by Beame et al.

For a work that has spawned many interesting results in such a short span of time, let us focus on the open problems and future research directions mentioned in \[ BHR+18, BHR+20 \].

*Let us consider an IS query with input \( U \). Let us partition \( U \) into two parts \( X \) and \( Y \) by putting each vertex in \( U \) to \( X \) or \( Y \) independently and uniformly at random. Then we make a BIS query with inputs \( X \) and \( Y \), and report \( U \) is an independent set if and only if BIS reports that there is no edge with one endpoint in each of \( X \) and \( Y \). Observe that we will be correct with at least probability 1/2. We can boost up the probability by repeating the above procedure suitable number of times.
| Work          | Oracle used | Structure estimated | Upper bound                                      | Lower bound                                      | Any other problem solved? |
|--------------|-------------|---------------------|-------------------------------------------------|-------------------------------------------------|---------------------------|
| [GR08]       | LOCAL       | edge                | $\bar{O}\left(\frac{n}{\sqrt{m}}\right)$       | $\Omega\left(\frac{n}{\sqrt{m}}\right)$       | Approximating average distance. |
| [CLW20]      | IS          | edge                | $O\left(\min\{\sqrt{m}, n/\sqrt{m}\}\right)$ | $\Omega\left(\min\{\sqrt{m}, n/\sqrt{m}\}\right)$ | –                          |
| [BHR+18]     | BIS         | edge                | $\text{poly}(\log n, 1/\varepsilon) = O(1)$    | Edge estimation using IS queries.               | –                          |
| [BBGM19b]    | TIS         | triangle            | $\text{poly}(\log n, \Delta, 1/\varepsilon)$   | –                                               | –                          |
| [DLM20],     | TIS         | hyperedge           | $\text{poly}(\log n, 1/\varepsilon)$           | –                                               | [DLM20] resolved Q2 in positive. |
| [BBGM19a]†   | CID         | hyperedge           | $\text{poly}(\log n, 1/\varepsilon) = O(1)$    | –                                               | –                          |
| [ELRS17]     | LOCAL       | triangle            | $\bar{O}\left(\frac{n^{1/3}}{T^{1/3}} + \min\left\{ m, \frac{m^{3/2}}{T} \right\}\right)$ | –                                               | –                          |
| [AKK19]      | LOCAL+ RANDOM EDGE | triangle | $O\left(\min\left\{ m, \frac{m^{3/2}}{T} \right\}\right)$ | Estimated number of arbitrary subgraphs. | –                          |
| This work    | BIS         | triangle            | $\bar{O}\left(\min\left\{ m, \frac{m^{3/2}}{T} \right\}\right)$ | –                                               | –                          |

Table 1: The whole gamut of results involving LOCAL queries [Go17], BIS, IS and its generalizations. † $\Delta$ is the maximum number of triangles on an edge. ‡ Both these results estimate the number of hyperedges in a $d$-uniform hypergraph, where $d$ is treated as a constant. Here $n$, $m$ and $T$ denote the number of vertices, edges and triangles in a graph $G$, respectively. $\bar{O}(\cdot)$ and $\Omega(\cdot)$ hide a multiplicative factor of $\text{poly}(\log n, 1/\varepsilon)$ and $1/\text{poly}(\log n, 1/\varepsilon)$, respectively.

**Q1** Can we estimate the number of cliques using polylogarithmic number of BIS queries?

**Q2** Can polylogarithmic number of BIS queries sample an edge uniformly at random?

**Q3** Can BIS or IS queries possibly be used in combination with local queries for graph parameter estimation problems?

**Q4** What other oracles, besides subset queries, allow estimating graph parameters with a polylogarithmic number of queries?

**Answers to the above questions and a discussion** Only Q2 has been resolved till now in the positive [DLM20] as can be observed from Table 1. At its core, Q1 asks if a query oracle can step up, that is, if it can estimate a structure that is of a higher order than what the oracle was designed for. The framing of Q1 seems that Beame et al. expected a polylogarithmic query complexity for estimation of the number of cliques using BIS. Pertinent to these questions, we also want to bring to focus a work [RWZ20] where the authors mention that it seems to them that estimation of higher order structures will require higher order queries (see the discussion after Proposition 23 of [RWZ20]). They showed that $\Omega(n^2/\log n)$ many BIS queries are required to separate triangle
free graph instances from graph instances having at least one triangle. This lower bound follows directly from the communication complexity of triangle freeness testing \cite{BKS02}. However, the full complexity of triangle estimation using emptiness queries like BIS remains elusive. It seems to us that the observations in \cite{BHR18,BHR20} and \cite{RWZ20} about the power of BIS in estimating higher order structures stand in contrast. In this backdrop, we completely resolve the lower bound for estimating triangles using BIS queries, and our upper bound result show that even if BIS can not estimate triangles using polylogarithmic many queries but it is still more powerful than LOCAL + RANDOM EDGE queries on graph for estimating triangles (see Table 1). BIS has an inherent asymmetry in its structure in the following sense – when BIS says that there exists no edge between two disjoint sets, then BIS stands as a witness to the existence of two sets of vertices having no interdependence, while a yes answer implies that there can be any number of edges, varying from one to the product of the cardinality of the two sets, going across the two sets. We feel that this property of BIS gives it its power, but on the other hand, also makes it difficult to analyze. That is probably the reason why works related to upper bound for BIS and its generalizations exist, whereas works on lower bound were not forthcoming. Though not on BIS, the work of Chen et al. using IS queries gave an interesting lower bound construction for IS oracle. Our work goes one step further in being able to prove a lower bound for the BIS oracle which is much stronger than IS oracle. We completely resolve the open question Q2 by proving matching lower and upper bounds involving BIS for estimating triangles.

1.2 A stronger oracle than BIS, our main result and its consequences

Now we define Edge Emptiness (shortened as EE) query oracle which is stronger than both BIS and IS. The Edge Emptiness query is a form of a subset query \cite{Sto83,Sto85,RT16} where a subset query with a subset $P \subseteq U$ asks whether $P \cap T$ is empty or not, where $T$ is also a subset of the universe $U$. The Edge Emptiness query operates with $U$ being the set of all vertex pairs in $G$, $T$ being the set of edges $E$ in $G$, and $P$ being a subset of pairs of vertices of $V$.

**Definition 1.5. (Edge Emptiness)** Given a subset $P \subseteq \binom{V(G)}{2}$, a EE query answers whether there exists an $\{u, v\} \in P$ such that $\{u, v\}$ is an edge in $G$.

EE query is recently introduce by Assadi et al. \cite{ACK21}. Note that BIS and IS queries can be simulated by an EE query \cite{ACK21}. We prove our lower bound in terms of the stronger EE queries that will directly imply the lower bound in terms of BIS. But we prove matching upper bound in terms of BIS. Our main results are stated below in an informal setting. The formal statements are given in Theorems 3.1 and 4.1.

**Theorem 1.6 (Main lower bound (informal statement)).** Let $m, n, t \in \mathbb{N}$. Any (randomized) algorithm that has EE query access to a graph $G(V, E)$ with $n$ vertices and $\Theta(m)$ edges, requires \(\tilde{\Omega}\left(\min \{\frac{m}{\sqrt{t}}, \frac{m^{3/2}}{t}\}\right)\) many BIS queries to decide whether the number of triangles in $G$ is at most $t$ or at least $2t$.

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1Assadi et al. \cite{ACK21} named EE query as OR query in their paper.

2Let us consider a BIS query with inputs $A$ and $B$. Let $P$ be the set of vertex pairs with one vertex from each of $A$ and $B$. We call EE oracle with input $P$, and report there is an edge having one vertex in each of $A$ and $B$ if and only if the EE oracle reports that there exists an $\{u, v\} \in P$ that forms an edge in $G$. Similarly, we can simulate an IS query with input $U$ by using an EE query with input $P = \binom{U}{2}$.

3For an example to show how EE query can be powerful than that of BIS or IS, $O(\log \log n)$ many EE queries \cite{Sto83} are enough to estimate the number of edges in a graph, as opposed to high query complexity (compared to $O(\log \log n)$) when we have BIS or IS queries (See Table 1).
Theorem 1.7 (Matching upper bound (informal statement)). There exists an algorithm, that has BIS query access to a graph $G(V,E)$, finds a $(1 \pm \varepsilon)$-approximation to the number of triangles in $G$ with high probability, and makes $\tilde{O} \left( \min \left\{ \frac{m}{\sqrt{T}}, \frac{m^{3/2}}{T} \right\} \right)$ many BIS queries in expectation. Here $m$, $n$, $T$ denote the number of vertices, edges and triangles in $G$.

Note that Edge Emptiness query is the strongest subset query on edges of the graph. Informally speaking, our lower bound states that no subset query on edges can estimate the number of triangles in a graph by using polylogarithmic many queries. However, the results of Bhattacharya et al. \cite{BBGM19} and Dell et al. \cite{DLM20} imply that polylogarithmic many TIS queries are enough to estimate the number of triangles in the graph. Note that TIS query is also a subset query on triangles in the graph. To complement our lower bound result, we also give an algorithm (see Theorem 1.7) for estimating the number of triangles in a graph with BIS queries that matches our lower bound. Here we would also like to mention that the number of BIS queries our algorithm uses is less than that of the number of local queries \cite{Go17} needed to estimate the number of triangles in a graph. This implies that we are resolving Q3 in positive in the sense that BIS queries are efficient queries for triangle estimation vis-a-vis local queries \cite{ELRS17} coupled with even random edge queries \cite{AKK19} (see Table 1).

1.3 Notations

Throughout the paper, the graphs are undirected and simple. For a graph $G(V,E)$, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively; $|V(G)| = n$, $|E(G)| = m$ and the number of triangles is $T$, unless otherwise specified. We use $\binom{V(G)}{2}$ to denote the set of vertex pairs in $G$. Note that $E(G) \subseteq \binom{V(G)}{2}$. For $P \subseteq \binom{V(G)}{2}$, $V(P)$ represents the set of vertices that belong to at least one pair in $P$. The neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and $|N_G(v)|$ is called the degree of vertex $v$ in $G$. $\Gamma\{x,y\}$ denotes the set $N_G(x) \cap N_G(y)$, that is, the set of common neighbors of $x$ and $y$ in $G$. If $e = \{x,y\} \in E(G)$, $\Gamma(e)$ denotes the set of vertices that forms triangles with $e$ as one of their edges. The induced degree of a vertex $v$ in $Z \subseteq V(G) \setminus \{v\}$ is the cardinality of $N_G(v) \cap Z$. For $X \subseteq V(G)$, the subgraph of $G$ induced by $X$ is denoted by $G[X]$. Note that $E(G[X]) = \{\{x,y\} : x, y \in X\}$. For two disjoint sets $A, B \subseteq V(G)$, the bipartite subgraph of $G$ induced by $A$ and $B$ is denoted by $G[A,B]$. Note that $E(G[A,B])$ is the set of edges having one vertex in $A$ and the other vertex in $B$.

Throughout the paper, $\varepsilon \in (0,1)$ is the approximation parameter. When we say $a$ is a $(1 \pm \varepsilon)$-approximation of $b$, then $(1 - \varepsilon)b \leq a \leq (1 + \varepsilon)b$. Polylogarithmic means $\log \left( \log n, 1/\varepsilon \right)$. $\tilde{O}(*)$ and $\tilde{\Omega}(*)$ hide a multiplicative factor of $\log \left( \log n, 1/\varepsilon \right)$ and $1/\log \left( \log n, 1/\varepsilon \right)$, respectively. We have avoided floor and ceiling for simplicity of presentation. The constants in this paper are not taken optimally. We have taken them to let the calculation work with clarity. However, those can be changed to other suitable and appropriate constants.

1.4 Paper organization

We start with the technical overview of our lower and upper bounds in Section 2.1 and Section 2.2 respectively. The detailed lower and upper bound proofs are in Section 3 and Section 4 respectively. The missing proofs are presented in Appendix A. In Appendix B we state some useful probability results.
2 Technical overview

2.1 Overview for the proof of our lower bound (Theorem 1.6)

Let us consider $m$, $n$, $t$ as in Theorem 1.6. We prove the desired bound for BIS (stated in Theorem 1.6) by proving the lower bound is $\tilde{\Omega}\left(\frac{m^{3/2}}{t}\right)$ when $t \geq km \log n$ and $\tilde{\Omega}\left(\frac{m}{\sqrt{t}}\right)$ when $t < km \log n$ for EE query access, where $k$ is a suitably chosen constant.

The idea for the lower bound of $\tilde{\Omega}\left(\frac{m^{3/2}}{t}\right)$ when $t \geq km \log n$:

We prove by using Yao’s method [MR99]. There are two distributions $\mathcal{D}_{Yes}$ and $\mathcal{D}_{No}$ (as described below) from which $G$ is sampled satisfying $\mathbb{P}(G \sim \mathcal{D}_{Yes}) = \mathbb{P}(G \sim \mathcal{D}_{No}) = 1/2$. Note that, for each $G \sim \mathcal{D}_{Yes} \cup \mathcal{D}_{No}$, $|V(G)| = n = \Theta(\sqrt{m})$ and $|E(G)| = \Theta(m)$ with a probability of at least $1 - o(1)$. But the number of triangles in each $G \sim \mathcal{D}_{No}$ is at least two factor more than that of the number of triangles in any $G \sim \mathcal{D}_{Yes}$, with a probability of at least $1 - o(1)$.

$\mathcal{D}_{Yes}$: The vertex set $V(G)$ (with $|V(G)| = \Theta(\sqrt{m})$) is partitioned into four parts $A$, $B$, $C$, $D$ uniformly at random. Vertex set $A$ forms a biclique with vertex set $B$ and vertex set $C$ forms a biclique with vertex set $D$. Then every vertex pair $\{x, y\}$, with $x \in A \cup B$ and $y \in C$, is added as an edge to graph $G$ with probability $\Theta\left(\sqrt{\frac{t}{m^{3/2}}}\right)$;

$\mathcal{D}_{No}$: The vertex set $V(G)$ (with $|V(G)| = \Theta(\sqrt{m})$) is partitioned into four parts $A$, $B$, $C$, $D$ uniformly at random. Vertex set $A$ forms a biclique with vertex set $B$ and vertex set $C$ forms a biclique with vertex set $D$. Then every vertex pair $\{x, y\}$, with $x \in A \cup B$ and $y \in C$, is added as an edge to graph $G$ with probability $\Theta\left(\sqrt{\frac{t}{m^{3/2}}}\right)$. Then each vertex of $C$ is sampled with probability $\Theta\left(\frac{t}{m^{3/2}}\right)$. Let $C'$ be the sampled set. Each vertex of $C'$ is connected to every vertex of $A \cup B$ with an edge;

\footnote{Without loss of generality, we assume that $\sqrt{m}$ is an integer. The proof can be extended to any graph having $n \geq \sqrt{m}$ vertices and $m$ edges by adding $n - \sqrt{m}$ isolated vertices.}
See Figure 1 for an illustration of the above construction. The constants, including $k$, in the order notations above are suitably set to have the following:

When $G \sim \mathcal{D}_{\text{Yes}}$: The number of triangles in each graph is at most $t$, with a probability of at least $1 - o(1)$; 

When $G \sim \mathcal{D}_{\text{No}}$: $|C'| = \Theta(\frac{t}{m})$ with a probability of at least $1 - o(1)$. Hence, the number of triangles in each $G \sim \mathcal{D}_{\text{No}}$ is at least $2t$, with a probability of at least $1 - o(1)$.

Now, consider a particular EE query with input $P \subseteq \binom{V(G)}{2}$. Here, we divide the discussion into two parts, based on $|P| \leq \tau$ and $|P| > \tau$, where $\tau = \Theta(\log^2 n)$ is a threshold. If we query with the number of vertex pairs more than the threshold, chances are more we will not be able to distinguish between $G \sim \mathcal{D}_{\text{Yes}}$ and $G \sim \mathcal{D}_{\text{No}}$. When $|P| > \tau$, we can show that there exists a vertex pair $\{x, y\} \in P$ such that $\{x, y\}$ is an edge in $G$, with a probability of at least $1 - o(1)$, irrespective of whether $G \sim \mathcal{D}_{\text{Yes}}$ or $G \sim \mathcal{D}_{\text{No}}$. Intuitively, this is because the number of vertices and edges in $G$ are $\Theta(\sqrt{m})$ and $\Theta(m)$, respectively. So, EE queries with input $P \subseteq \binom{V(G)}{2}$ such that $|P| > \tau$ will not be useful to distinguish whether $G \sim \mathcal{D}_{\text{Yes}}$ or $G \sim \mathcal{D}_{\text{No}}$.

We prove the desired lower bound by proving $\Omega\left(\frac{m^{3/2}}{t}\right)$ many EE queries are necessary to decide whether $G \sim \mathcal{D}_{\text{Yes}}$ or $G \sim \mathcal{D}_{\text{No}}$ with a probability of at least $2/3$. Note that $C' = \emptyset$ when $G \sim \mathcal{D}_{\text{Yes}}$. So, the number of EE queries needed to decide whether $G \sim \mathcal{D}_{\text{Yes}}$ or $G \sim \mathcal{D}_{\text{No}}$ is at least the number of EE queries needed to touch at least one vertex of $C'$ when $G \sim \mathcal{D}_{\text{No}}$. Here, by touching at least a vertex of $C'$, we mean $V(P) \cap C' \neq \emptyset$. As we have argued that only EE query with input $P \subseteq \binom{V(G)}{2}$ with $|P| \leq \tau$ can be useful, the probability that we touch a vertex in $C'$ with such a query is at most $p = O\left(\frac{C'}{\sqrt{m}} \cdot \tau\right) = O\left(\frac{t \log^2 n}{m^{3/2}}\right)$. Hence the number of EE queries to touch at least a vertex of $C'$, is at least $1/p$, that is, $\tilde{\Omega}\left(\frac{m^{3/2}}{t}\right)$.

To let the above discussion work, when $G \sim \mathcal{D}_{\text{No}}$, $|C'| = \Theta\left(\frac{t}{m}\right)$ must be at least $\Omega(\log n)$. But $|C'| = \Theta\left(\frac{t}{m}\right)$ with a probability of at least $1 - o(1)$. Because of this, we take $t \geq km \log n$ in the above discussion. The formal statement of the lower bound, when $t \geq km \log n$, is given in Lemma 3.2 in Section 3. What we have discussed here is just an overview, the formal proof of Lemma 3.2 is much more involved and delicate, which is presented in Section 3.1.

**The idea for the lower bound of $\tilde{\Omega}\left(\frac{m^{3/2}}{t}\right)$ when $t < km \log n$:**

Let us consider an algorithm $\mathcal{A}$, having EE query access to an unknown graph $G_2$, that decides whether the number of triangles in $G$ is at most $t$ or at least $2t$ with a probability of at least $2/3$, where the parameter $t$ satisfies $t < k|E(G_2)| \cdot \log |V(G_2)|$. We prove the desired lower bound by reducing from the case $t = km \log n$ in a graph $G$ to the case $t < k|E(G_2)| \cdot \log |V(G_2)|$ in a graph $G_2$. After the reduction, we get the lower bound for the case $t < k|E(G_2)| \cdot \log |V(G_2)|$ as we have already established the lower bound for any $t \geq km \log n$.

Let $G$ be the unknown graph to which we have EE query access and $t = km \log n$, where $|V(G)| = n$ and $|E(G)| = \Theta(\sqrt{m})$. The unknown graph $G_2$ (for algorithm $\mathcal{A}$) is $G \cup G_1$ such that $V(G_2) = V(G) \cup V(G_1)$ and $E(G_2) = E(G) \cup E(G_1)$, where $G_1$ is a triangle-free graph having $\Theta(\sqrt{t})$ many vertices (disjoint from $V(G)$), and $\Theta(t)$ many edges. We choose the constants in $\Theta(\sqrt{t})$ and $\Theta(t)$ such that $t < k|E(G_2)| \cdot \log |V(G_2)|$. Note that $|V(G_2)| = \Theta(n + \sqrt{t})$, $E(G_2) = \Theta(m + t)$, and the number of triangles in $G_2$ is same as that of $G$. Also, an EE query to graph $G_2$ can be answered.

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1This is to satisfy the requirement of algorithm $\mathcal{A}$.
by an EE query to graph $G$. Hence, because of our lower bound in the case of $t \geq km \log n$,

Lower bound for the number of EE queries made by algorithm $A$

to estimate the number of triangles in $G$

$\geq \Omega \left( \frac{m^{3/2}}{t} \right) = \Omega \left( \frac{|E(G_2)|}{\sqrt{t}} \right) \quad (\because |E(G_2)| = \Theta(m + t) \text{ and } t = km \log n)$

The formal statement of the lower bound, when $t < km \log n$, is given in Lemma 3.3 in Section 3, and the proof is presented in Section 3.2.

2.2 Overview for our upper bound (Theorem 1.7)

We establish the upper bound claimed in Theorem 1.7 by giving two algorithms that report a $(1 \pm \varepsilon)$-approximation to the number of triangles in the graph:

(i) $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ that makes $\tilde{O} \left( \frac{m^{3/2}}{T} \right)$ many BIS queries;

(ii) $\text{T}ri\text{angle-Est-Low}(G, \varepsilon)$ that makes $\tilde{O} \left( \frac{m + T}{\sqrt{T}} \right)$ many BIS queries.

Informally speaking, our final algorithm $\text{T}ri\text{angle-Est}$ calls $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ and $\text{T}ri\text{angle-Est-Low}(G, \varepsilon)$ when $T = \Omega(m)$ and $T = \mathcal{O}(m)$, respectively. Observe that, if $\text{T}ri\text{angle-Est}$ knows $T$ within a constant factor, then it can decide which one to use among $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ and $\text{T}ri\text{angle-Est-Low}(G, \varepsilon)$. If $\text{T}ri\text{angle-Est}$ does not know $T$ within a constant factor, then it starts from a guess $L = \frac{n^3}{2}$ and makes a geometric search on $L$ until the output of $\text{T}ri\text{angle-Est}$ is consistent with $L$. Depending on whether $L = \Omega(m)$ or $L = \mathcal{O}(m)$, $\text{T}ri\text{angle-Est}$ decides which one among $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ and $\text{T}ri\text{angle-Est-Low}(G, \varepsilon)$ to call. This guessing technique is very standard by now in property testing literature [GR08, ELRS17, ERS20, AKK19]. Another point to note is that we do not know $m$. However, we can estimate $m$ by using $\text{poly}(\log n)$ many BIS queries (see Table 1). An estimate of $m$ will perfectly work for us in this case.

**Algorithm** $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$  Algorithm $\text{T}ri\text{angle-Est-High}$ is inspired by the triangle estimation algorithm of Assadi et al. [AKK19], where we have ADJACENCY, DEGREE, RANDOM NEIGHBOR and RANDOM EDGE queries. Please see Section 4.2 for formal definitions of these queries. Note that the algorithm by Assadi et al. can be suitably modified even if we have approximate versions of DEGREE, RANDOM NEIGHBOR and RANDOM EDGE queries. Also refer Section 4.2 for formal definitions of approximate version of the above queries. By Corollary 4.8, $\tilde{O}(1)$ many BIS queries are enough to simulate the approximate versions of DEGREE and RANDOM NEIGHBOR, with a probability of at least $1 - o(1)$. By Proposition 4.7, approximate version of RANDOM EDGE queries can also be simulated by $\tilde{O}(1)$ many BIS queries, with a probability of at least $1 - o(1)$. Putting everything together, we get $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ for triangle estimation that makes $\tilde{O}(1)$ many BIS queries. The formal statement of the corresponding triangle estimation result is given in Lemma 4.2 and algorithm $\text{T}ri\text{angle-Est-High}(G, \varepsilon)$ is described in Section 4.2.
Algorithm Triangle-Est-Low\((G, \varepsilon)\) This algorithm is inspired by the two pass streaming algorithm for triangle estimation by McGregor et al. \cite{MVV16}. Basically, we show that the steps of McGregor et al.’s algorithm can be executed by using \( \tilde{O} \left( \frac{m + T}{\sqrt{t}} \right) \) BIS queries. To do so, we have used the fact that, given any \( X \subseteq V(G) \), all the edges of the subgraph induced by \( X \) can be enumerated by using \( \tilde{O} (|E(G[X])|) \) many BIS queries (see Proposition 4.4 for the formal statement). The formal statement of the corresponding triangle estimation result is given in Lemma 4.3 and algorithm Triangle-Est-Low\((G, \varepsilon)\) is described in Section 4.3 along with its correctness proof and query complexity analysis.

3 Lower bound for estimating triangles using Edge Emptiness queries

In this Section, we prove the main lower bound result as sketched in Theorem 1.6; the formal theorem statement is stated below. As mentioned earlier, the lower bound proofs will be for the stronger query oracle EE. This will imply the lower bound for BIS.

Theorem 3.1 (Main lower bound result). Let \( m, n, t \in \mathbb{N} \) be such that \( 1 \leq t \leq \frac{m^{3/2}}{2} \). Any (randomized) algorithm that has EE oracle access to a graph \( G(V, E) \) must make \( \Omega \left( \min \left\{ \frac{m}{\sqrt{t}}, \frac{m^{3/2}}{t} \right\} \right) \) many EE queries to decide whether the number of triangles in \( G \) is at most \( t \) or at least \( 2t \) with a probability of at least \( 2/3 \), where \( G \) has \( n \geq 4\sqrt{m} \) many vertices, \( \Theta(m) \) many edges.

We prove the above theorem by proving Lemmas 3.2 and 3.3 as stated below. Note that Lemmas 3.2 and 3.3 talk about the desired lower bound when the number of triangles in the graph is large (\( \Omega(m \log n) \)) and small (\( O(m \log n) \)), respectively.

Lemma 3.2 (Lower bound when there are large number of triangles). Let \( m, n, t \in \mathbb{N} \) be such that \( t \geq \frac{m \log n}{8} \). Any (randomized) algorithm that has EE oracle access to a graph \( G(V, E) \) must make \( \tilde{\Omega} \left( \frac{m \log n}{8} \right) \) many EE queries to decide whether the number of triangles in \( G \) is at most \( t \) or at least \( 2t \) with a probability of at least \( 2/3 \), where \( G \) has \( n \geq 4\sqrt{m} \) many vertices, \( \Theta(m) \) many edges.

Lemma 3.3 (Lower bound when there are small number of triangles). Let \( m, n, t \in \mathbb{N} \) be such that \( t < \frac{m \log n}{8} \). Any (randomized) algorithm that has EE oracle access to a graph \( G(V, E) \) must make \( \tilde{\Omega} \left( \frac{m}{\sqrt{t}} \right) \) many EE queries to decide whether the number of triangles in \( G \) is at most \( t \) or at least \( 2t \) with a probability of at least \( 2/3 \), where \( G \) has \( n \geq 4\sqrt{m} \) many vertices and \( \Theta(m) \) many edges.

We first show Lemma 3.2 in Section 3.1 and then Lemma 3.3 in Section 3.2. Note that the proof of Lemma 3.3 will use Lemma 3.2.

3.1 Proof of Lemma 3.2

Without loss of generality, assume that \( \sqrt{m} \) is an integer. We prove for the case when \( n = 4\sqrt{m} \). But, we can make the proof work for any \( n \geq 4\sqrt{m} \) by adding \( n - 4\sqrt{m} \) many isolated vertices. Note that \( t \geq \frac{m \log n}{8} \) here. We further assume that \( t \leq \frac{m^{3/2}}{128} \), and \( m = \Omega(\log^2 n) \). Otherwise, the stated lower bound of \( \tilde{\Omega} \left( \frac{m^{3/2}}{t} \right) \) trivially follows as \( \tilde{\Omega}(\cdot) \) hides a multiplicative factor of \( \frac{1}{\text{poly}(\log n)} \).

We use Yao’s min-max principle to prove the lower bound. To do so, we consider two distributions \( \mathcal{D}_{\text{Yes}} \) and \( \mathcal{D}_{\text{No}} \) on graphs where
• Any graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$ has $4\sqrt{m}$ many vertices;

• Any graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$ has $\Theta(m)$ many edges with a probability of at least $1 - o(1)$;

• The number of triangles in any graph $G \sim D_{\text{Yes}}$ is at most $t$ with a probability of at least $1 - o(1)$, and any graph $G \sim D_{\text{No}}$ has at least $2t$ many triangles with a probability of at least $1 - o(1)$.

Note that, if we can show that any deterministic algorithm that distinguishes graphs from $D_{\text{Yes}}$ and $D_{\text{No}}$, with a probability of at least $2/3$, must make $\tilde{\Omega}\left(\frac{m^{3/2}}{t}\right)$ many EE queries, then we are done with the proof of Lemma 3.2.

3.1.1 The (hard) distribution for the input, its properties, and the proof set up

$D_{\text{Yes}}$: A graph $G \sim D_{\text{Yes}}$ is sampled as follows:

• Partition the vertex set $V(G)$ into 4 parts $A, B, C, D$, by initializing $A, B, C, D$ as empty sets, and then putting each vertex in $V(G)$ into one of the parts uniformly at random and independent of other vertices;

• Connect each vertex of $A$ with every vertex of $B$ with an edge to form a biclique. Also, connect each vertex of $C$ with every vertex of $D$ with an edge to form another biclique;

• For every $\{x, y\}$ where $x \in A \cup B$ and $y \in C$, add edge $\{x, y\}$ to $G$ with probability $\sqrt{\frac{t}{16m^{3/2}}}$.

$D_{\text{No}}$: A graph $G \sim D_{\text{No}}$ is sampled as follows:

• Partition the vertex set $V(G)$ into 4 parts $A, B, C, D$, by initializing $A, B, C, D$ as empty sets, and then putting each vertex in $V(G)$ into one of the partitions uniformly at random and independent of other vertices;

• Connect each vertex of $A$ with every vertex of $B$ with an edge to form a biclique. Also, connect each vertex of $C$ with every vertex of $D$ with an edge to form another biclique;

• For every $\{x, y\}$ where $x \in A \cup B$ and $y \in C$, add edge $\{x, y\}$ to $G$ with probability $\sqrt{\frac{t}{16m^{3/2}}}$.

• Select $C' \subseteq C$ by putting each $x \in C$ into $C'$ with a probability of at least $\frac{324}{m^{3/2}}$, independently, and then, add each edge in $\{x, y : x \in A \cup B, y \in C' \}$ to $G$.

The following observation establishes the number of vertices, edges, and the number of triangles in the graphs that can be sampled from $D_{\text{Yes}} \cup D_{\text{No}}$. The proof uses large deviation inequalities (see Lemma B.1 and B.4 in Appendix B), and is presented in Appendix A.

**Observation 3.4 (Properties of the graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$).** (i) For $G \sim D_{\text{Yes}} \cup D_{\text{No}}$, the number of vertices in $G$ is $4\sqrt{m}$. Also, $\frac{\sqrt{m}}{2} \leq |A|, |B|, |C|, |D| \leq 2\sqrt{m}$ holds with a probability of at least $1 - o(1)$, and the number of edges in $G$ is $\Theta(m)$ with a probability of at least $1 - o(1)$;

(ii) If $G \sim D_{\text{Yes}}$, then there are at most $t$ triangles in $G$ with a probability of at least $1 - o(1)$,

(iii) If $G \sim D_{\text{No}}$, $\frac{8t}{m} \leq |C'| \leq \frac{64t}{m}$ with a probability of at least $1 - o(1)$, and there are at least $2t$ many triangles in $G$ with a probability of at least $1 - o(1)$. 


The following remark is regarding the connection between graphs in $D_{\text{Yes}}$ and that in $D_{\text{No}}$. This will be used later in our proof, particularly in the proof of Claim 3.12.

**Remark 1** (A graph $G' \sim D_{\text{No}}$ can be generated from a graph $G \sim D_{\text{Yes}}$). Let us first generate a graph $G \sim D_{\text{Yes}}$. Select $C' \subseteq C$ by putting each $x \in C$ into $C'$ with a probability of at least $\frac{32t}{m^{3/2}}$, and then, add each edge in $\{x, y : x \in A \cup B, y \in C'\}$ to $G$ to generate $G'$, then (the resulting graph) $G' \sim D_{\text{No}}$.

The following observation says that a $\{x, y\} \in \binom{V(G)}{2}$ (with some condition) forms an edge with a probability of at least a constant. It will be used while we prove Claim 3.11.

**Observation 3.5** (Any vertex pair $\{x, y\}$ is an edge in $G$ with constant probability). Let $\{x, y\} \in \binom{V(G)}{2}$, and we are in the process of generating $G \sim D_{\text{Yes}} \cup D_{\text{No}}$. Let at most one of $x$ and $y$ has been put into one of the parts out of $A, B, C$ and $D$. Then $\{x, y\}$ is an edge in $G$ with probability at least $\frac{1}{4}$.

The above observation follows from the description of $G \sim D_{\text{Yes}} \cup D_{\text{No}}$ -- each vertex in $V(G)$ is put into one of the parts out of $A, B, C, D$ uniformly at random, each vertex of $A$ is connected with every vertex in $B$, and each vertex of $C$ is connected with every vertex in $D$.

In order to prove Lemma 3.2 by contradiction, assume that there is a randomized algorithm that makes $q = o \left( \frac{m^{3/2}}{t} \frac{1}{\log^2 n} \right)$ many EE queries and decides whether the number of triangles in the input graph is at most $t$ or at least $2t$, with a probability of at least $2/3$. Then there exists a deterministic algorithm ALG that makes $q$ many EE queries and decides the following (when the input graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$ be such that both $G \sim D_{\text{Yes}}$ and $G \sim D_{\text{No}}$ holds with probability $1/2$) –

$$P_{G \sim D_{\text{No}}} (\text{ALG}(G) \text{ reports NO}) - P_{G \sim D_{\text{Yes}}} (\text{ALG}(G) \text{ reports NO}) \geq \frac{1}{3} - o(1).$$

(Here $P_{G \sim D_{\text{No}}} (E)$ and $P_{G \sim D_{\text{Yes}}} (E)$ denote the probability of the event $E$ under the conditional space $G \sim D_{\text{No}}$ and $G \sim D_{\text{Yes}}$, respectively.) Hence, we will be done with the proof of Lemma 3.2 by showing the following lemma.

**Lemma 3.6** (Lower bound on the number of EE queries when $G \sim D_{\text{Yes}} \cup D_{\text{No}}$). Let the unknown graph $G$ be such that $G \sim D_{\text{Yes}}$ and $G \sim D_{\text{No}}$ hold with equal probabilities. Consider any deterministic algorithm ALG that has EE access to $G$, and makes $q = o \left( \frac{m^{3/2}}{t} \frac{1}{\log^2 n} \right)$ many EE queries to $G$. Then

$$P_{G \sim D_{\text{No}}} (\text{ALG}(G) \text{ reports NO}) - P_{G \sim D_{\text{Yes}}} (\text{ALG}(G) \text{ reports NO}) \leq o(1).$$

Next, we define an augmented EE oracle (EE* oracle). EE* is tailor-made for the graphs coming from $D_{\text{Yes}} \cup D_{\text{No}}$. Moreover, it is stronger than EE, that is, any EE query can be simulated by a EE* query. We will prove the claimed lower bound in Lemma 3.6 when we have access to EE* oracle. Note that this will imply Lemma 3.6.

Before getting into the formal description of EE* oracle, note that the algorithm (with EE* access) maintains a four tuple data structure initialized with $\emptyset$. With each query to EE* oracle, the oracle updates the data structure and returns the updated data structure to the algorithm. Note that the updated data structure is a function of all previously made EE* queries, and it is enough to answer corresponding EE queries.

### 3.1.2 Augmented Edge Emptiness oracle (EE*)

Before describing the EE* query oracle and its interplay with the algorithm, first we present the data structure $(E_Q, V(E_Q), e, \ell_v)$ that the algorithm maintains with the help of EE* oracle. The data structure keeps track of the following information.
Information maintained by \((E_Q, V(E_Q), e, \ell_v)\):

- \(E_Q\) is a subset of \(\binom{V(G)}{2}\) that have been seen by the algorithm till now, \(V(E_Q)\) is the set of vertices present in any vertex pair in \(E_Q\).
- \(e : \binom{V(E_Q)}{2} \rightarrow \{0, 1\}\) such that \(e(\{x, y\}) = 1\) means the algorithm knows that \(\{x, y\}\) is an edge in \(G\), \(e(x, y) = 0\) means that \(\{x, y\}\) is not an edge in \(G\).
- \(\ell_v : V(E_Q) \rightarrow \{A, B, C, C', D\}\), where
  \[
  \ell_v(x) = \begin{cases} 
  A, & x \in A \\
  B, & x \in B \\
  C, & x \in C \setminus C' \\
  C', & x \in C' \\
  D, & x \in D
  \end{cases}
  \]

Intuitively speaking, unless the algorithm knows about the presence of some vertex in \(C'\), it cannot distinguish whether the unknown graph \(G \sim D_{\text{Yes}}\) or \(G \sim D_{\text{No}}\). So, we define the notion of good and bad vertices, along with good and bad data structures. This notion will be used later in our proof.

**Definition 3.7 (Bad vertex).** A vertex \(x \in V(E_Q)\) is said to be a bad vertex if \(\ell_v(x) = C'\). \((E_Q, V(E_Q), e, \ell_v)\) is said to be good if there does not exist any bad vertex in \(V(E_Q)\).

**EE* oracle and its interplay with the algorithm:**

The algorithm initializes the data structure \((E_Q, V(E_Q), e, \ell_v)\) with \(E_Q = \emptyset, V(E_Q) = \emptyset\). So, \(e\) and \(\ell_v\) are initialized with trivial functions with domain \(\emptyset\). At the beginning of each round, the algorithm queries the \(EE^*\) oracle with a subset \(P \subseteq \binom{V(G)}{2}\) deterministically. Note that the choice of \(P\) depends on the current status of the data structure. Now, we explain how \(EE^*\) oracle responds to the query and how the data structure is updated.

1. If \(|P| \leq \tau = 25 \log^2 n\), the oracle sets \(E_Q \leftarrow E_Q \cup P\), and changes \(V(E_Q)\) accordingly. The oracle also sets the function \(e\) and \(\ell_v\) as per their definitions, and then it sends the updated data structure to the algorithm.

2. Otherwise (if \(|P| > \tau\)), the oracle finds a random subset \(P' \subseteq P\) such that \(|P'| = \tau\). The oracle checks if there is a pair \(\{u, v\} \in P'\) such that \(\{u, v\}\) is an edge. If yes, then the oracle responds as in (1) with \(P\) being replaced by \(P'\). If no, the oracle sends the data structure corresponding to the entire graph along with a Failure signal \(\ast\ast\).

Owing to the way \(EE^*\) oracle updates the data structure after each \(EE^*\) query, we can make some assumptions on the inputs to the \(EE^*\) oracle, as described in Remark 2. It will actually be useful when we prove Claim 3.11.

**Remark 2 (Some assumptions on the EE* query).** Let \((E_Q, V(E_Q), e, \ell_v)\) be the data structure just before the algorithm makes \(EE\) query with input \(P\), and let \((E'_Q, V(E'_Q), e', \ell'_v)\) be the data structure updated by \(EE^*\) oracle after the algorithm makes \(EE^*\) query with input \(P\). Without loss of generality, we assume that

\[\ast\ast\text{We later argue that Failure signal is sent with a very low probability.}\]
(i) $P$ is disjoint from $\binom{V(E_Q)}{2}$. It is because $EE^*$ maintains whether $\{x, y\}$ is an edge or not for each $\{x, y\} \in \binom{V(E_Q)}{2}$.

(ii) When $x, z \in V(E_Q)$, there does not exist $\{x, y\}$ and $\{y, z\}$ in $P$. It is because the oracle updates the data structure in the same way in each of the following three cases when $x, z \in V(E_Q)$ – (i) $\{x, y\}$ and $\{y, z\}$ are in $P$, (ii) $\{x, y\} \in P$ and $\{y, z\} \notin P$, and (iii) $\{x, y\} \notin P$ and $\{y, z\} \in P$.

By the description of $EE^*$ oracle and its interplay with the algorithm, in all the three cases, the updated data structure $(E'_Q, V(E'_Q), e', \ell')$ contains labels of all the three vertices $x, y, z$ along with the information whether $\{x, y\}$ and $\{y, z\}$ form edges in $G$ or not. So, instead of having both $\{x, y\}$ and $\{y, z\}$ in $P$ with $x, z \in V(E_Q)$, it is equivalent to have exactly one among $\{x, y\}$ and $\{y, z\}$ in $P$.

In the following observation, we formally show that $EE^*$ oracle is stronger than that of $EE$. Then we prove Lemma 3.9 that says that $\Omega\left(\frac{m^{3/2}}{t \log^2 n}\right)$ $EE^*$ queries are necessary to distinguish between $G \sim D_{Yes}$ and $G \sim D_{No}$. Note that Lemma 3.9 will imply Lemma 3.6.

**Observation 3.8 (EE* is stronger than EE).** Let $G \sim D_{Yes} \cup D_{No}$. Each EE query to $G$ can be simulated by using an $EE^*$ query to $G$.

**Proof.** Let us consider an EE query with input $P \subseteq \binom{V(G)}{2}$. We make a EE* query with the same input $P$, and answer the EE query as follows depending on whether $|P| \leq \tau$ or $|P| > \tau$.

$|P| \leq \tau$: The EE* oracle updates the data structure and let $(E'_Q, V(E'_Q), e', \ell')$ be the updated data structure. It contains the the information about each $\{x, y\} \in P$ whether it forms an edge in $G$ or not. So, from $(E'_Q, V(E'_Q), e', \ell')$, the EE query with input $P$ can be answered as follows: there exists an edge $\{x, y\} \in P$ with $\{x, y\} \in E(G)$ if and only if $e'(\{x, y\}) = 1$.

$|P| > \tau$: In this case, the EE* oracle finds a random subset $P' \subseteq P$ such that $|P'| = \tau$. It checks if there is an $\{x, y\} \in P'$ such that $\{x, y\}$ is an edge. If yes, the updated data structure contains the the information about each $\{x, y\} \in P'$ whether it forms an edge. In this case, we can report that there exists an $\{x, y\} \in P$ such that $\{x, y\}$ is an edge in $G$. If there is no $\{x, y\} \in P'$ such that $\{x, y\}$ is an edge, then (by the description of EE oracle and its interplay with the algorithm) the EE* oracle sends the data structure corresponding to the entire graph. Obviously, we can report whether there exists an $\{x, y\} \in P$ such that $\{x, y\} \in E(G)$ or not.

Hence, in any case, we can report the answer to EE query with input $P$. \qed

We are left with proving the following technical lemma. As noted earlier, this will imply Lemma 3.6.

**Lemma 3.9 (Lower bound on the number of EE* queries when $G \sim D_{Yes} \cup D_{No}$).** Let the unknown graph $G$ be such that $G \sim D_{Yes}$ and $G \sim D_{No}$ hold with equal probabilities. Consider any deterministic algorithm $ALG^*$ that has $EE^*$ access to $G$, and makes $q = o\left(\frac{m^{3/2}}{t \log^2 n}\right)$ many $EE^*$ queries to $G$. Then

$$\mathbb{P}_{G \sim D_{No}}(ALG^* (G) reports NO) - \mathbb{P}_{G \sim D_{Yes}}(ALG^* (G) reports NO) \leq o(1).$$

### 3.1.3 Proof of Lemma 3.9

For clarity of explanation, we first describe $ALG^*$ as a decision tree. Then we will prove Lemma 3.9.
Decision tree view of ALG*:

- Each internal node of $T$ is labeled with a nonempty subset $\binom{V(G)}{2}$ and each leaf node is labeled with YES or NO;
- Each edge in the tree is labeled with a data structure $(E_Q, V(E_Q), e, \ell_v)$;
- The algorithm starts the execution from the root node $r$ by setting $r$ as the current node. Note that for the root node $r$, $E_Q = V(E_Q) = \emptyset$ and $e$ and $\ell_v$ are the trivial functions. As the algorithm ALG* is deterministic, the first $EE^*$ query is same irrespective of the graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$ that we are querying. By making that query, we get an updated data structure from the oracle and let $\{r, u\}$ be the edge that is labeled with the updated data structure. Then ALG* sets $u$ as the current node.
- If the current node $u$ is not a leaf node in $T$, ALG* makes a $EE^*$ query with a subset $P \subseteq \binom{V(G)}{2}$, where $P$ is determined by the label of the node $u$. Note that $P$ satisfies the condition described in Remark 2. The oracle updates the knowledge structure and ALG* moves to a child of $u$ depending on the updated data structure;
- If the current node $u$ is a leaf node in $T$, report YES or NO according to the label of $u$.

Now, we define the notion of good and bad nodes in $T$. The following definition is inspired from Definition 3.7.

**Definition 3.10 (Bad node in the decision tree).** Let $u$ be a node of $T$ and $(E_Q, V(E_Q), e, \ell_v)$ be the current data structure. $u$ is said to be good if there does not exist $x$ in $V(E_Q)$ such that $\ell_v(x) = C'$. Otherwise, $u$ is a bad node.

If $G \sim D_{\text{Yes}}$, then ALG* will never encounter a bad node. In other words, when ALG* reaches a bad node of the tree $T$, then it can (easily) decide $G \sim D_{\text{No}}$. However, the inverse is not true. From this fact, consider two claims (Claims 3.11 and 3.12) about the traversal of the decision tree $T$ when the graph $G \sim D_{\text{Yes}} \cup D_{\text{No}}$. These claims will be useful to show Lemma 3.9. Intuitively, Claim 3.11 says that the probability of reaching a bad node is very low when $G \sim D_{\text{No}}$. Claim 3.12 says that the probability to reach any particular good node is more when $G \sim D_{\text{Yes}}$ as compared to that of when $G \sim D_{\text{No}}$.

**Claim 3.11 (Probability of reaching a bad node is very low when $G \sim D_{\text{No}}$).** Let $G \sim D_{\text{No}}$. Then the probability that ALG* reaches a bad node of the decision tree is $o(1)$. That is,

$$P_{G \sim D_{\text{No}}}(\text{ALG* reaches a bad node}) = o(1).$$

**Claim 3.12 (A technical claim to prove Lemma 3.9).** For any good node in the decision tree $T$, the following holds:

$$P_{G \sim D_{\text{No}}}(\text{ALG* reaches } v) \leq P_{G \sim D_{\text{Yes}}}(\text{ALG* reaches } v).$$

In the next section, we first prove Claims 3.11 and 3.12 then (using them) we prove Lemma 3.9.
Proofs of Claims 3.11 and 3.12

To prove Claim 3.11 we need Observations 3.13 and 3.14. Informally speaking, these two observations help us to argue that enough randomness is left in the unknown part of the graph when we make \( q = o \left( \frac{m^{3/2}}{t \log^2 n} \right) \) many EE* queries. Particularly, Observation 3.13 says that, when \( \text{ALG}^* \) is at a good node \( u \) of the decision tree, then a graph \( G \sim D_{\text{Yes}} \) can be generated respecting the current data structure that we have at node \( u \). Observation 3.14 says that, when \( \text{ALG}^* \) is at some node \( u \) of the decision tree, then a graph \( G \sim D_{\text{No}} \) can be generated respecting the current data structure that we have at node \( u \). The proof of Claim 3.12 uses the connection between graphs in \( D_{\text{Yes}} \) and \( D_{\text{No}} \), as described in Remark 1.

Observation 3.13 (A graph \( G \sim D_{\text{Yes}} \) can be generated respecting any good node in the decision tree). Let \( u \) be the current node of the decision tree \( T \) that is good and \((E_Q, V(E_Q), e, \ell_v)\) be the current data structure. Then a graph \( G \sim D_{\text{Yes}} \) can be generated as follows conditioned on the fact that \( \text{ALG}^*(G) \) reaches \( u \).

- For each \( x \in V(E_Q) \), put \( x \) in the vertex partition indicated by \( \ell_v(x) \). Put each vertex in \( V(G) \setminus V(E_Q) \) to one of the parts out of \( A, B, C, D \) uniformly at random and independent of other vertices;

- For each \( \{x, y\} \in E_Q \), add an edge between \( x \) and \( y \) if and only if \( e(\{x, y\}) = 1 \). Then for each \( \{x, y\} \in (V(G)/E_Q) \), do the following:
  - Add an edge between \( x \) and \( y \) if one vertex is in \( A \) and the other is in \( B \);
  - Add an edge between \( x \) and \( y \) if one vertex is in \( C \) and the other is in \( D \);
  - If one vertex out of \( x \) and \( y \) is in \( A \cup B \) and the other in \( D \), then \( \{x, y\} \) does not form an edge.
  - Add an edge between \( x \) and \( y \) with probability \( \sqrt{\frac{t}{16m^{3/2}}} \) if one vertex in \( A \cup B \) and the other in \( C \).

Observation 3.14 (A graph \( G \sim D_{\text{No}} \) can be generated respecting any node in the decision tree). Let \( u \) be the current node of the decision tree \( T \) and \((E_Q, V(E_Q), e, \ell_v)\) be the current data structure. Then a graph \( G \sim D_{\text{No}} \) can be generated as follows conditioned on the fact that \( \text{ALG}^*(G) \) reaches \( u \).

- For each \( x \in V(E_Q) \), put \( x \) in the vertex partition indicated by \( \ell_v(x) \). Put each vertex in \( V(G) \setminus V(E_Q) \) to one of the parts out of \( A, B, C, D \) uniformly at random and independent of other vertices;

- For each \( \{x, y\} \in E_Q \), add an edge between \( x \) and \( y \) if and only if \( e(\{x, y\}) = 1 \). Then for each \( \{x, y\} \in (V(G)/E_Q) \), do the following:
  - Add an edge between \( x \) and \( y \) if one vertex is in \( A \) and the other is in \( B \);
  - Add an edge between \( x \) and \( y \) if one vertex is in \( C \) and the other is in \( D \);
  - If one vertex out of \( x \) and \( y \) is in \( A \cup B \) and the other is in \( D \), then \( \{x, y\} \) does not form an edge.
  - Add an edge between \( x \) and \( y \) with a probability of at least \( \sqrt{\frac{t}{16m^{3/2}}} \) if one vertex in \( A \cup B \) and the other is in \( C \);
For each vertex \( x \in C \setminus C' \), add \( x \) to \( C' \) with probability \( 32/m^{3/2} \). Add each edge of the form \( \{x, y : x \in A \cup B, y \in C' \} \) to the graph;

Observations \ref{claim:3.13} and \ref{claim:3.14} follow from the descriptions of our hard distributions \( \mathcal{D}_\text{Yes} \) and \( \mathcal{D}_\text{No} \) in Section \ref{sec:oracle}, along with the description of \( \text{EE}^* \) oracle and its interplay with algorithm \( \text{ALG}^* \).

Now we will prove Claims \ref{claim:3.11} and \ref{claim:3.12} by using Observations \ref{claim:3.13} and \ref{claim:3.14}.

**Claim 3.15 (Claim \ref{claim:3.11} restated).** Let \( G \sim \mathcal{D}_\text{No} \). Then the probability that \( \text{ALG}^* \) reaches a bad node of the decision tree is \( o(1) \). That is,

\[
\mathbb{P}_{G \sim \mathcal{D}_\text{No}}(\text{ALG}^* \text{ reaches a bad node}) = o(1).
\]

**Proof.** Recall that \( q = o \left( \frac{\sqrt{m^3}}{t \log^2 n} \right) \) and \( \text{ALG}^* \) makes (at most) \( q \) many \( \text{EE}^* \) queries. The execution of \( \text{ALG}^* \) starts from the root node of the decision tree \( T \), which is trivially a good node. For \( i \in [q] \), let \( u_i \) be the node of \( T \) that \( \text{ALG}^* \) reaches after making \( i \) many \( \text{EE}^* \) queries. Assume that \( u_0 \) is the root of the tree \( T \). Note that \( u_i \) is a child of \( u_{i-1} \) for each \( i \in [q] \). Claim \ref{claim:3.11} says that, there exists an \( i \in [q] \) such that \( u_i \) is a bad node, with a probability of at most \( o(1) \). Observe that we will be done with the proof by showing the following: if \( u_0, \ldots, u_k \) are good nodes with \( k \leq q - 1 \), then \( u_{k+1} \) is a bad node with a probability of at most \( O \left( \frac{t \log^2 n}{m^{3/2}} \right) \).

As \( k \leq q - 1, u_k \) is not a leaf node. Let the label \( u_k \) be \( P \). Note that \( P \) is a subset of \( \binom{V(G)}{2} \) with which \( \text{ALG}^* \) makes \( \text{EE}^* \) query. Let \( (E_Q, V(e), e, \ell_v) \) be the current data structure of \( \text{ALG}^* \) after making \( \text{EE}^* \) query with input \( P \). Recall the definition of a bad node (Definition \ref{def:3.10}). Observe that \( u_{k+1} \) is a bad node if and only if \( V(P) \cap C' \neq \emptyset \). Hence, when \( |P| \leq \tau \), we can deduce the following by Observations \ref{claim:3.13} and \ref{claim:3.14}.

\[
\begin{align*}
\mathbb{P}_{G \sim \mathcal{D}_\text{No}}(u_{k+1} \text{ is bad } | u_0, \ldots, u_k \text{ are good}) \\
\leq \mathbb{P}_{G \sim \mathcal{D}_\text{No}}(V(P) \cap C' \neq \emptyset | u_0, \ldots, u_k \text{ are good}) \\
\leq |V(P)| \times \frac{1}{4} \frac{32t}{m^{3/2}} \\
\leq 2\tau \cdot \frac{8t}{m^{3/2}} = O \left( \frac{t \log^2 n}{m^{3/2}} \right) \quad \left( \because |P| \leq \tau \text{ and } \tau = 25 \log^2 n. \right)
\end{align*}
\]

Now consider the case when \( |P| > \tau \). In this case, recall the behavior of \( \text{EE}^* \) oracle. Let \( P' \subseteq P \) be generated uniformly at random such that \( |P'| = \tau \). In this case, if there exists vertex pair in \( P' \) that forms an edge in \( G \), then \( u_{k+1} \) is a bad node if and only if \( V(P') \cap C' \neq \emptyset \). Hence, we can deduce the following by Observation \ref{claim:3.13} and \ref{claim:3.14}.

\[
\begin{align*}
\mathbb{P}_{G \sim \mathcal{D}_\text{No}}(u_{k+1} \text{ is bad } | u_0, \ldots, u_k \text{ are good}) \\
\leq \mathbb{P}_{G \sim \mathcal{D}_\text{No}}(V(P) \cap C' \neq \emptyset | u_0, \ldots, u_k \text{ are good}) + \mathbb{P} \left( \{x, y\} \notin E(G) \forall \{x, y\} \in P' \right)
\end{align*}
\]

The first term can be bounded by \( o \left( \frac{t \log^2 n}{m^{3/2}} \right) \) in the similar fashion as we did when \( |P| \leq \tau \). To finish the proof we need to show that

\[
\mathbb{P}_{G \sim \mathcal{D}_\text{No}}(\{x, y\} \notin E(G) \forall \{x, y\} \in V(P') | u_0, \ldots, u_k \text{ are good}) = o(1). \quad (1)
\]
Consider a particular \( \{x,y\} \in P' \). By Remark 2 (i), the labels of at most one vertex out of \( x \) and \( y \) is known till now. By Observation 3.5, the probability that \( \{x,y\} \) is an edge is at least 1/4. Note that \(|P'| = \tau = 25 \log^2 n \). So, there are either \( 5 \log n \) many pairwise disjoint vertex pairs in \( P' \) or \( 5 \log n \) many vertex pairs having a common vertex in \( P' \). If there are \( 5 \log n \) many pairwise disjoint vertex pairs in \( P' \), then each such vertex pair \( \{x,y\} \) forms an edge in \( G \) with probability at least 1/4, independent of other such vertex pairs. Now consider the case when there are \( 5 \log n \) many vertex pairs having a common vertex in \( P' \). Let \( o \) be the center vertex and \( \{o,z_1\}, \ldots, \{o,z_{5\log n}\} \) be the vertex pairs. By Remark 2 (i) and (ii), we can assume that either we do not know the label of any vertex in \( \{o,z_1, \ldots, z_{5\log n}\} \), or we know the label of only \( o \), or we know the label of exactly one \( z_i \). In any case, by Observation 3.5, the probability that \( \{o,z_i\} \) is an edge is at least 1/4, independent of all other \( \{o,z_j\}'s.\

Putting everything together, the probability that none of the pairs in \( P' \) form an edge holds with a probability of at most \( (3/4)^{5\log n} = o(1) \). So, we are done with the proof of Equation 1, and hence Claim 3.11.

**Claim 3.16 (Claim 3.12 restated).** For any good node in the decision tree \( T \), the following holds:

\[
P_{G \sim D_{\text{No}}} (\text{ALG}^* \text{ reaches } v) \leq P_{G \sim D_{\text{Yes}}} (\text{ALG}^* \text{ reaches } v).
\]

**Proof.** By Remark 1, we can generate a graph \( G \sim D_{\text{Yes}} \) first, and from that, we can generate \( G' \sim D_{\text{No}} \). Let \( G' \) be the set of all graphs in \( D_{\text{No}} \) that can be generated from \( G \sim D_{\text{Yes}} \). We refer the set \( G' \) as the corresponding set of graphs of \( G \sim D_{\text{Yes}} \) in \( D_{\text{No}} \). Recall that the unknown graph \( G \sim D_{\text{Yes}} \cup D_{\text{No}} \) such that both \( G \sim D_{\text{Yes}} \) and \( G \sim D_{\text{No}} \) hold with probability 1/2 each. So, we can consider generating the unknown graph \( G \) as follows:

First generate \( G \sim D_{\text{Yes}} \). The unknown graph is \( G \) itself with probability 1/2, and a graph in the correspondence set \( G' \) with probability 1/2, generated as described in Remark 1. This implies the following observation.

**Observation 3.17.** For any \( H \sim D_{\text{Yes}} \) and its corresponding set of graphs \( H' \) in \( D_{\text{No}} \),

\[
P(G = H \mid G \sim D_{\text{Yes}}) = P(G \in H' \mid G \sim D_{\text{No}}).
\]

In this claim, we are considering a good node \( v \). Now, first consider the term

\[
P_{G \sim D_{\text{Yes}}} (\text{ALG}^* (G) \text{ reaches } v).
\]

Let \( H_{v,\text{Yes}} \) be the set of graphs in \( D_{\text{Yes}} \) such that the algorithm reaches node \( v \) if the unknown graph \( G \in H_{v,\text{Yes}} \). So,

\[
P_{G \sim D_{\text{Yes}}} (\text{ALG}^* (G) \text{ reaches } v) = \sum_{H \in H_{v,\text{Yes}}} P(G = H \mid G \sim D_{\text{Yes}}) = \sum_{H \in H_{v,\text{Yes}}} P(G \in H' \mid G \sim D_{\text{No}}).
\]

The last equality follows from Observation 3.17. Taking \( H_v' = \bigcup_{H \in H_{v,\text{Yes}}} H' \),

\[
P_{G \sim D_{\text{Yes}}} (\text{ALG}^* (G) \text{ reaches } v) = \sum_{H_v' \in H_v'} P(G = H' \mid G \sim D_{\text{No}}).
\]

\[\text{[Jnk11]}\] says that either there is a matching of \( \sqrt{m} \) edges or a star of size \( \sqrt{m} \) in any graph with \( m \) edges. Note that, we are using analogous result in terms of vertex pairs.
Now, we consider the term
\[ \Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } u)]. \]
Let \( \mathcal{H}_{v, \text{No}} \) be the set of graphs in \( D_{\text{No}} \) such that the algorithm reaches node \( v \) if the unknown graph \( G \in \mathcal{H}_{v, \text{No}} \). Let \((E_Q, V(E_Q), e, \ell_v)\) be the data structure when the algorithm reaches node \( v \). By the definition of a good node (see Definition 3.10), the algorithm reaches node \( v \) if \( G \in \mathcal{H}_{v, \text{No}} \) and the corresponding \( C' \subseteq C \) does not intersect with \( V(E_Q) \). So,
\[
\Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } v)] \leq \sum_{H' \in \mathcal{H}_{v, \text{No}}} \Pr(G = H' \mid G \sim D_{\text{No}}).
\]
Now, observe that the algorithm does not reach node \( v \) (in the decision tree) if the unknown graph \( G \not\in H_v \cup H_v' \). That is, \( \mathcal{H}_{v, \text{No}} \subseteq \mathcal{H}_v' \). So,
\[
\Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } v)] \leq \sum_{H' \in \mathcal{H}_v'} \Pr(G = H' \mid G \sim D_{\text{No}}). \tag{3}
\]
By Equations 2 and 3 we are done with the proof of Claim 3.12.

Now, we will prove Lemma 3.9.

**Proof of Lemma 3.9.** Let \( \mathcal{L}_{\text{No}} \) denote the set of leaf nodes of the decision tree \( T \) that are labeled \( \text{No} \). Also, let \( \mathcal{L}_g \subseteq \mathcal{L}_{\text{No}} \) be the set of leaf nodes that are good and labeled as \( \text{No} \).

\[
\Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reports } \text{No})]
\leq \sum_{u \in \mathcal{L}_{\text{No}}} \Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } u)]
\leq \sum_{u \in \mathcal{L}_g} \Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } u)] + \sum_{u \in \mathcal{L}_{\text{No}} \setminus \mathcal{L}_g} \Pr[G \sim D_{\text{No}} (\text{ALG}^* (G) \text{ reaches } u)]
\leq \sum_{u \in \mathcal{L}_g} \Pr[G \sim D_{\text{Yes}} (\text{ALG}^* (G) \text{ reaches } u)] + o(1) \quad \text{(By Claims 3.11 and Claims 3.12)}
\leq \Pr[G \sim D_{\text{Yes}} (\text{ALG}^* (G) \text{ reports } \text{No})] + o(1)
\]

\[\blacksquare\]

### 3.2 Proof of Lemma 3.3

We assume that \( m = \omega(\log^6 n) \). Otherwise, as \( \tilde{\Omega}(\cdot) \) hides a multiplicative term of \( \frac{1}{\text{poly}(\log n)} \), the stated lower bound is trivial. Assume, for a contradiction, that there is an algorithm \( \mathcal{A} \) for \( t < \frac{m \log n}{8} \) such that

- it has EE oracle access to a graph \( G(V, E) \) with \( n \geq 4\sqrt{m} \) vertices and \( \Theta(m) \) edges;
- makes \( o\left(\frac{m}{\sqrt{t \log \frac{1}{\log n}} n}\right) \) many EE queries;
- decides whether the number of triangles in \( G \) is at most \( t \) or at least \( 2t \) with a probability of at least \( 2/3 \).
Now we give an algorithm $A'$ for $t = \frac{m \log n}{8}$ such that

- it has EE oracle access to a graph $G(V, E)$ with $n \geq 4\sqrt{m}$ vertices and $\Theta(m)$ edges;
- makes $o\left(\frac{\sqrt{m}}{\log^4 n}\right)$ many EE queries;
- decides whether the number of triangles in $G$ is at most $t$ or at least $2t$ with a probability of at least $2/3$.

**Description of $A'$ using $A$:**

- Let the unknown graph be $G_2 = G \cup G_1$ such that $V(G_2) = V(G) \sqcup V(G_1)$ and $E(G_2) = E(G) \sqcup E(G_1)$, where $G_1$ is a graph having $\Theta(\sqrt{t})$ many vertices (disjoint from $V(G)$), $\Theta(t)$ many edges, and no triangles. We choose the constants in $\Theta(\sqrt{t})$ and $\Theta(m)$ such that $t < \frac{|E(G_2)| \log |V(G_2)|}{8}$.
- the number of vertices and edges in $G_2$ are $n + \Theta(t)$ and $\Theta(m + t)$, respectively;
- the number of triangles in $G_2$ is same as in $G$;
- as an EE query to $G_2$ can be answered using one EE query to $G$, we can consider having EE query access to graph $G_2$;
- we run algorithm $A$ assuming $G_2$ as the unknown graph;
- we report the output of $A$ as the output of $A$.

The correctness of $A'$ follows from the correctness of $A$. The number of queries made by the algorithm $A'$ is $o\left(\frac{|E(G_2)| \log |V(G_2)|}{8}\right) = o\left(\frac{m + 4t}{\sqrt{t} \log^4 n}\right) = o\left(\frac{\sqrt{m}}{\log^4 n}\right)$, as $t = \frac{m \log n}{8}$. However, by Lemma 3.2, algorithm $A'$ does not exist as such an algorithm requires at least $\Omega\left(\frac{m^{3/2}}{t} \frac{1}{\log^3 n}\right)$ many EE queries.

Note that $\Omega\left(\frac{m^{3/2}}{t} \frac{1}{\log^3 n}\right) = \Omega\left(\frac{\sqrt{m}}{\log^3 n}\right)$, as here $t = \frac{m \log n}{8}$.

Hence, we are done with the proof of Lemma 3.3.

**4 Upper bound for estimating triangles using BIS**

In this Section, we prove Theorem 1.7, which is formally stated as follows:

**Theorem 4.1** (Upper bound matching the lower bound in Theorem 3.1). There exists an algorithm $\text{Triangle-Est}(G, \varepsilon)$ that has BIS query access to a graph $G(V, E)$ having $n$ vertices, takes a parameter $\varepsilon \in (0, 1)$ as input, and reports a $(1 \pm \varepsilon)$-approximation to the number of triangles in $G$ with a probability of at least $1 - o(1)$. Moreover, the expected number of BIS queries made by the algorithm is $\tilde{O}\left(\min\left\{\frac{m}{\sqrt{T}}, \frac{m^{3/2}}{T}\right\}\right)$, where $m$ and $T$ denote the number of edges and triangles in $G$, respectively.

In order to prove the above theorem, we first prove Lemmas 4.2 and 4.3.

\footnote{This is to satisfy the requirement of algorithm $A$.}
Lemma 4.2 (Upper bound when the number of triangles is large). There exists an algorithm Triangle-Est-High($G, \varepsilon$) that has BIS query access to a graph $G(V, E)$ having $n$ vertices, $\varepsilon \in (0, 1)$ as input, and reports a $(1 \pm \varepsilon)$-approximation to the number of triangles in $G$ with a probability of at least $1 - o(1)$. Moreover, the expected number of BIS queries made by the algorithm is $\tilde{O}\left(\frac{m^{3/2}}{\varepsilon T}\right)$, where $m$ and $T$ denote the number of edges and triangles in $G$, respectively.

Lemma 4.3 (Upper bound when the number of triangles is small). There exists an algorithm Triangle-Est-Low($G, \varepsilon$) that has BIS query access to a graph $G(V, E)$ having $n$ vertices, $\varepsilon \in (0, 1)$ as input, and reports a $(1 \pm \varepsilon)$-approximation to the number of triangles in $G$ with a probability of at least $1 - o(1)$. Moreover, the expected number of BIS queries made by the algorithm is $\tilde{O}\left(\frac{m+T}{\sqrt{T}}\right)$, where $m$ and $T$ denote the number of edges and triangles in $G$, respectively.

Our final algorithm Triangle-Est($G, \varepsilon$) (as stated in Theorem 4.1) is a combination of Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$). Informally speaking, algorithm Triangle-Est($G, \varepsilon$) calls Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$) when $T = \Omega(m)$ and $T = O(m)$, respectively. Observe that, if Triangle-Est knows $T$ within a constant factor, then it can decide which one to use among Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$). If Triangle-Est does not know $T$ within a constant factor, then it starts from a guess $L = \left(\frac{n}{3}\right)/2$ and updates $L$ by making a geometric search until the output of Triangle-Est is consistent with $L$. Depending on whether $L = \Omega(m)$ or $L = O(m)$, Triangle-Est decides which one among Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$) to call. This guessing technique is standard in the property testing literature. It has been used several times in the literature (for example in [GR08, ELRS17], generalized in [ERS20], and used directly in [AKK19]). So, we explain Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$) assuming a promised lower bound on $L$, and the respective query complexities will be in terms of $L$ instead of $T$.

Another important thing to observe is that to execute the above discussed steps of algorithm Triangle-Est($G, \varepsilon$) must know $m$. But we note that an estimate of $m$ will be good enough for our purpose, and that can be estimated by using $\tilde{O}(1)$ many BIS queries (see Table 1).

In Section 4.1 we discuss some properties of BIS and some tasks it can perform. These properties will be useful while describing our algorithms Triangle-Est-High($G, \varepsilon$) and Triangle-Est-Low($G, \varepsilon$), and proving Lemma 4.2 and Lemma 4.3 in Section 4.2 and Section 4.3 respectively.

4.1 Some preliminaries about BIS

Let $G(V, E)$ be the unknown graph to which we have BIS query access. One can compute the exact number of edges using $\tilde{O}(|E(G)|)$ queries [BHR+20] deterministically. Also, we can estimate the number of edges in graph $G$ [BHR+20, BBGM19, DLM20] and sample an edge from $G$ almost uniformly [DLM20], with a probability of at least $1 - o(1)$, and making $\tilde{O}(1)$ many BIS queries. Here, we would like to note that, all three results we mentioned above hold for induced subgraphs as well as induced bipartite subgraph, as formally described below. Those will be used when we design our upper bounds in Section 4.2 and 4.3.

Proposition 4.4 (Exact edge estimation using BIS [BHR+20]). There exists a deterministic algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes $X \subseteq V(G)$ (alternatively, two disjoint subsets $A, B$) as input, makes $\tilde{O}(|E(G[X])|)$ (alternatively, $\tilde{O}(|E(G[A, B])|)$) many BIS queries, and reports all the edges in $E(G)$ (alternatively, $E(G[A, B])$).

Proposition 4.5 (Approximate edge estimation using BIS [BBGM19, DLM20]). There exists an algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes
$X \subseteq V(G)$ (alternatively, two disjoint subsets $A, B$) and a parameter $\varepsilon \in (0, 1)$ as inputs, makes $\tilde{O}(1)$ many BIS queries, and reports a $(1 \pm \varepsilon)$-approximation to $|E(G[X])|$ (alternatively, $|E(G[A, B])|$), with a probability of at least $1-o(1)$.

To state the next proposition, we need the following definition.

**Definition 4.6 (Approximate uniform sample from a set).** For a nonempty set $X$ and $\varepsilon \in (0, 1)$, getting a $(1 \pm \varepsilon)$-approximate uniform sample from $X$ means getting a sample from a distribution on $X$ such that the probability of getting $x \in X$ lies in $[(1-\varepsilon)/|X|, (1+\varepsilon)/|X|]$.

**Proposition 4.7 (Approximate edge sampling using BIS [DLM20]).** There exists an algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes $X \subseteq V(G)$ (alternatively, two disjoint subsets $A, B$) and a parameter $\varepsilon \in (0, 1)$ as inputs, makes $\tilde{O}(1)$ many BIS queries, and reports a $(1 \pm \varepsilon)$-approximate uniform sample from $E(G[X])$ (alternatively, $E(G[A, B])$), with a probability of at least $1-o(1)$.

Observe that the following corollary follows from Propositions 4.4, 4.5 and 4.7, by taking $A = \{v\}$ and $B = Z$, where $v \in V(G)$ and $Z \subseteq V(G) \setminus \{v\}$.

**Corollary 4.8 (BIS query can extract useful information about the neighborhood of a given vertex).**

(i) **Entire neighborhood of a vertex using BIS:** There exists a deterministic algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes $v \in V(G)$ and $Z \subseteq V(G) \setminus \{v\}$ as input, makes $\tilde{O}(|N_G(v) \cap Z|)$ many BIS queries, and reports all the neighbors of $v$ in $Z$.

(ii) **Approximate degree using BIS:** There exists an algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes $v \in V(G)$, $Z \subseteq V(G) \setminus \{v\}$ and $\varepsilon \in (0, 1)$ as inputs, makes $\tilde{O}(1)$ many BIS queries, and reports a $(1 \pm \varepsilon)$-approximation to $|N_G(v) \cap Z|$, with a probability of at least $1-o(1)$.

(iii) **Finding an approximate random neighbor of a vertex using BIS:** There exists an algorithm that has BIS query access to an unknown graph $G(V, E)$ with $n$ vertices, takes $v \in V(G)$, $Z \subseteq V(G) \setminus \{v\}$ and $\varepsilon \in (0, 1)$ as inputs, makes $\tilde{O}(1)$ many BIS queries, and reports a $(1 \pm \varepsilon)$-approximate uniform sample from the set $N_G(v) \cap Z$, with a probability of at least $1-o(1)$.

### 4.2 Algorithm Triangle-Est-High and proof of Lemma 4.2

Algorithm Triangle-Est-High is inspired by the triangle estimation algorithm of Assadi et al. [AKK19] when we have the following query access to the unknown graph.

**Adjacency Query:** Given vertices $u, v \in V(G)$ as input, the oracle reports whether $(u, v)$ is an edge or not;

**Degree Query:** Given a vertex $u \in V(G)$ as input, the oracle reports the degree of vertex $u$ in $G$;

**Random Neighbor Query:** Given a vertex $u \in V(G)$, the oracle reports a neighbor of $u$ uniformly at random if the degree of $u$ is nonzero. Otherwise, the oracle reports a special symbol $\perp$;

**Random Edge Query:** With this query, the oracle reports an edge from the graph $G$ uniformly at random.

---

§§Actually, they have given an algorithm for estimating the number of copies of any given subgraph of fixed size.
The number of queries to the oracle made by Assadi et al.’s algorithm \[\text{Apx Random Edge Query}\] is \(\tilde{O}\left(\frac{m^{3/2}}{L}\right)\), where \(m\) denotes the number of edges and \(L\) is a promised lower bound on the number of triangles in \(G\). Also, note that, the triangle estimation algorithm by Assadi et al. \[\text{Apx Random Neighbor Query}\] can be suitably modified even if we have approximate versions of Degree, Random Neighbor and Random Edge queries, as described below.

**Apx Degree Query:** Given a vertex \(u \in V(G)\) and \(\varepsilon \in (0, 1)\) as input, the oracle reports a \((1 \pm \varepsilon)\)-approximation to the degree of vertex \(u\) in \(G\);

**Apx Random Neighbor Query:** Given a vertex \(u \in V(G)\) and \(\varepsilon \in (0, 1)\) as input, the oracle reports a \((1 \pm \varepsilon)\)-approximate uniform sample from \(N_G(u)\) if the degree of \(u\) is nonzero. Otherwise, the oracle reports a special symbol \(\perp\);

**Apx Random Edge Query:** Given \(\varepsilon \in (0, 1)\), the oracle reports a \((1 \pm \varepsilon)\)-approximate uniform sample from \(E(G)\).

From Corollary 4.8 \(\tilde{O}(1)\) many BIS queries are enough to simulate Apx Degree Query and Apx Random Neighbor Query, with a probability of at least \(1 - o(1)\). Also, by Proposition 4.7 Apx Random Edge Query can be simulated by \(\tilde{O}(1)\) many BIS queries, with a probability of at least \(1 - o(1)\). Moreover, a BIS query can trivially simulate an Adjacency Query. Combining these facts with the fact that the triangle estimation algorithm by Assadi et al. \[\text{Apx Random Neighbor Query}\] can be suitably modified even if we have approximate versions of Degree, Random Neighbor and Random Edge queries, we are done with the proof of Lemma 4.3.

### 4.3 Algorithm Triangle-Est-Low and the proof of Lemma 4.3

Algorithm Triangle-Est-Low is inspired by the streaming algorithm for triangle counting by McGregor et al. \[\text{[MVV16]}\]. Algorithm Triangle-Est-Low extracts a subset of edges by making BIS queries in a specific way as explained below. Later, we discuss that those sets of edges will be enough to estimate the number of triangles in \(G\).

**Generating a random sample \(S \subseteq V(G)\) and exploring its neighborhood:**
Algorithm Triangle-Est-Low adds each vertex in \(V(G)\) to \(S\) with probability \(\tilde{O}\left(\frac{1}{\sqrt{m}}\right)\). Recall Corollary 4.8(i). For each \(v \in S\), we find all the neighbors of \(v\) (the set \(N_G(v)\)) by making \(\tilde{O}(|N_G(v)|)\) many BIS queries. Let \(E_S\) be the set of all the edges having at least one end point in \(S\), that is, \(E_S = \{(v, w) : v \in S \text{ and } w \in N_G(v)\}\). After finding \(E_S\), we do the following. For each \(v \in S\), we find all the edges in the subgraph induced by \(N_G(v)\) by using \(\tilde{O}(|E(G[N_G(v)]|))\) many BIS queries. This is again possible by Corollary 4.8(i). Note that \(|E(G[N_G(v)]|)|\) is the number of edges in the subgraph of \(G\) induced by \(N_G(v)\). Let \(E'_S\) be the set of all edges present in the subgraph induced by \(N_G(v)\) for some \(v \in S\), that is, \(E'_S = \bigcup_{v \in S} E(G[N_G(v)])\). Later, we argue that the number of BIS queries that we make to generate \(E_S\) and \(E'_S\) is bounded in expectation.

Apart from \(S, E_S\) and \(E'_S\), Triangle-Est-Low extracts some more required edges by making BIS queries, as explained below.

**Generating \(F\), a set of \((1 \pm O(\varepsilon))\)-approximate uniform sample from \(E(G)\), and exploring the subgraphs induced by sets \(N_G(v) \cap V(F)\) for each \(v \in V(F)\):**
Algorithm Triangle-Est-Low calls the algorithm corresponding to Proposition 4.7 for \(\tilde{O}\left(\frac{m}{\sqrt{L}}\right)\)
many times. By this process, we get a set \( F \) of \((1 \pm \mathcal{O}(\epsilon))\)-approximate uniform sample from \( E(G) \), with a probability of at least \( 1 - o(1) \). Note that \(|F| = \tilde{\mathcal{O}} \left( \frac{m}{\sqrt{\ell}} \right) \), and the number of BIS queries we make to generate \( F \) is \( \tilde{\mathcal{O}} \left( \frac{m}{\sqrt{\ell}} \right) \). Let \( V(F) \) be the set of vertices present in at least one edge in \( F \). For each vertex \( v \in V(F) \), we find all the edges in the subgraph of \( G \) induced by \( N_G(v) \cap V(F) \), by using \( \tilde{\mathcal{O}} \left( |E(G[N_G(v) \cap V(F)])| \right) \) many BIS queries (see Corollary 4.8(i)). Note that \(|E(G[N_G(v) \cap V(F)])|\) is the number of edges in the subgraph of \( G \) induced by \( N_G(v) \cap F \). Let \( E_F \) be the set of all the edges that are present in subgraphs induced by \( N_G(v) \cap V(F) \) for some \( v \in V(F) \), that is, \( E_F = \bigcup_{v \in V(F)} E(G[N_G(v) \cap V(F)]) \). Later we show that the expected number of BIS queries needed to find \( F \) and \( E_F \) is bounded.

In algorithm \textsc{Triangle-Est-Low}, BIS queries are made only to generate \( S, E_S, E'_S, F \) and \( E_F \). After these sets are generated, no more BIS queries are made by the algorithm. We formally prove the query complexity of \textsc{Triangle-Est-Low}. But, first, we show that \( S, E_S, E'_S, F \) and \( E_F \) can be carefully used to estimate \( T \), the number of triangles in \( G \).

**Connection with streaming algorithm for triangle counting by McGregor et al.** [MVV16]:

(Estimating the number of triangles from \( S, E_S, E'_S, F \) and \( E_F \))

McGregor et al. [MVV16] gave a two-pass algorithm that estimates the number of triangles in a graph \( G \) when the edges of \( G \) arrive in an arbitrary order. Moreover, the space complexity of their algorithm is \( \tilde{\mathcal{O}} \left( \frac{m}{\sqrt{\ell}} \right) \). Note that their algorithm assumes a lower bound \( L \) on the number of triangles in the graph. The high level sketch of their algorithm is as follows:

- Generate a subset \( X \) of \( V(G) \) by sampling each vertex in \( V(G) \) with probability \( \tilde{\mathcal{O}} \left( \frac{1}{\sqrt{\ell}} \right) \);

- In the first pass, the edges having at least one vertex in \( X \) is found, and let it be \( E_X \). Also, in the first pass, a subset of edges \( Y \) is generated by sampling each edge with probability \( \tilde{\mathcal{O}} \left( \frac{1}{\sqrt{\ell}} \right) \);

- In the second pass, for each edge \( e = \{x, y\} \) in the stream, their algorithm finds the vertices in \( X \) with which \( e \) forms a triangle. Also, for each edge, \( e = \{x, y\} \) in the stream, their algorithm finds the pairs of edges in \( Y \) that forms a triangle with \( e \). Let \( Z \) be the set of \textit{useful} edges in the second pass, that is, the set of edges that either forms a triangle with a vertex in \( X \) or forms a triangle with two edges in \( Y \).

Note that their algorithm does not talk about the set \( Z \). We are introducing it for our analysis. Executing the two passes described above is straightforward. They have proved that performing two passes as described is good enough to estimate the number of triangles in the graph.

Now, we compare the information maintained by our algorithm with the information maintained by McGregor et al.’s algorithm. \( S \) and \( E_S \) in our algorithm \textsc{Triangle-Est-Low} follows the same probability distribution as that of \( X \) and \( E_X \), respectively, in McGregor et al.’s algorithm. Recall that \( F \) is a set of \( \tilde{\mathcal{O}} \left( \frac{m}{\sqrt{\ell}} \right) \) many \((1 \pm \epsilon)\)-approximate sample from \( E(G) \) with a probability \( 1 - o(1) \).

But \( Y \) in McGregor et al.’s algorithm is generated by sampling each edge with probability \( \tilde{\mathcal{O}} \left( \frac{1}{\sqrt{\ell}} \right) \).

But observe that the total variation distance between the probability distributions of \( F \) and \( Y \) is \( o(1) \).

Before discussing about \( E'_S \) and \( E_F \) in our algorithm \textsc{Triangle-Est-Low}, consider the following observation about the set \( Z \). Note that we have defined \( Z \) while describing the second pass of McGregor et al.’s algorithm.
We touched upon two open questions of Beame et al. Let $E'_X$ be the edges in the subgraph induced by $N_G(v)$ for some $v \in X$, and let $E_Y$ be the set of edges in the subgraph induced by $N_G(v) \cap V(Y)$. Here $V(Y)$ denotes the set of vertices present in at least one vertex of $Y$. Then for each edge $e \not\in E'_X \cup E_Y$, there is neither a vertex in $X$ with which $e$ forms a triangle in $G$ nor there are two edges in $Y$ with which $e$ forms a triangle in $G$. Then the set of useful edges $Z$ is $E'_X \cup E_Y$.

By the above observation, $E'_S$ and $E_F$ in our algorithm Triangle-Est-Low are essentially enough for maintaining the information and executing the same steps as that of the second pass of McGregor et al.’s algorithm.

Putting everything together, algorithm Triangle-Est-Low outputs a $(1 \pm \varepsilon)$-approximation to the number of triangles in the graph.

Query complexity analysis:
The set $S$ can be generated without making any BIS queries. The number of BIS queries we make to find the set $E_S$ is at most $\sum_{v \in S} \tilde{O}(|N_G(v)|)$, which in expectation is

$$E \left[ \sum_{v \in S} \tilde{O}(|N_G(v)|) \right] = \sum_{v \in V(G)} \Pr(v \in S) \cdot \tilde{O}(|N_G(v)|) = \tilde{O}(m/\sqrt{L}).$$

The number of BIS queries we make to find the set $E'_S$ is at most $\sum_{v \in S} \tilde{O}(|E(G[N_G(v)])|)$. Note that $|E(G[N_G(v)])| = T_v$, that is, the number of triangles having $v$ as one of the vertex. So, the expected number of BIS queries we make to find $E'_S$ is at most

$$E \left[ \sum_{v \in S} \tilde{O}(T_v) \right] = \sum_{v \in V(G)} \Pr(v \in S) \cdot \tilde{O}(T_v) = \tilde{O}(m/\sqrt{L}).$$

The number of BIS queries we make to find the set $F$ is $\tilde{O}(|F|) = \tilde{O}(m/\sqrt{T_L})$.

The number of BIS queries to generate $E_F$ is at most $\sum_{e \in V(F)} \tilde{O}(|E(G[N_G(v) \cap V(F)])|)$. Observe that an edge $\{x, y\}$ is present in $E_F$ if there exists a $z \in V(G)$ such that $\{x, y, z\}$ forms a triangle in $G$ and $\{x, z\}$ and $\{y, z\}$ are in $F$. So, the probability that an edge $\{x, y\}$ is in $E(G[N_G(v) \cap V(F)]$ is at most $|\Gamma(\{x, y\})| \cdot \tilde{O}(1)$. Hence, the expected number of BIS queries made by the algorithm is $\tilde{O}(m + T_L)$. 

5 Conclusion

We touched upon two open questions of Beame et al. in this paper. We resolved the query complexity of triangle estimation when we have a Bipartite Independent Set oracle access to the unknown graph. A natural open question is to study other graph parameter estimation problems through the lens of Bipartite Independent Set or Edge Emptiness query oracle.
References

[ACK21] Sepehr Assadi, Deeparnab Chakrabarty, and Sanjeev Khanna. Graph connectivity and single element recovery via linear and OR queries. In Petra Mutzel, Rasmus Pagh, and Grzegorz Herman, editors, 29th Annual European Symposium on Algorithms, ESA 2021, September 6-8, 2021, Lisbon, Portugal (Virtual Conference), volume bisis20 of LIPIcs, pages 7:1–7:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[AKK19] Sepehr Assadi, Michael Kapralov, and Sanjeev Khanna. A Simple Sublinear-Time Algorithm for Counting Arbitrary Subgraphs via Edge Sampling. In Proceedings of the 10th Innovations in Theoretical Computer Science Conference, ITCS, volume 124, pages 6:1–6:20, 2019.

[BBGM19a] Anup Bhattacharya, Arijit Bishnu, Arijit Ghosh, and Gopinath Mishra. Hyperedge Estimation using Polylogarithmic Subset Queries. CoRR, abs/1908.04196, 2019.

[BBGM19b] Anup Bhattacharya, Arijit Bishnu, Arijit Ghosh, and Gopinath Mishra. Triangle Estimation Using Tripartite Independent Set Queries. In Proceedings of the 30th International Symposium on Algorithms and Computation, ISAAC, volume 149, pages 19:1–19:17, 2019.

[BBGM21] Anup Bhattacharya, Arijit Bishnu, Arijit Ghosh, and Gopinath Mishra. On Triangle Estimation Using Tripartite Independent Set Queries. Theory Comput. Syst., 65(8):1165–1192, 2021.

[BGK+18] Arijit Bishnu, Arijit Ghosh, Sudeshna Kolay, Gopinath Mishra, and Saket Saurabh. Parameterized Query Complexity of Hitting Set Using Stability of Sunflowers. In Proceedings of the 29th International Symposium on Algorithms and Computation, ISAAC, volume 123, pages 25:1–25:12, 2018.

[BHR+18] Paul Beame, Sariel Har-Peled, Sivaramakrishnan Natarajan Ramamoorthy, Cyrus Rashtchian, and Makrand Sinha. Edge Estimation with Independent Set Oracles. In Proceedings of the 9th Innovations in Theoretical Computer Science Conference, ITCS, volume 94, pages 38:1–38:21, 2018.

[BHR+20] Paul Beame, Sariel Har-Peled, Sivaramakrishnan Natarajan Ramamoorthy, Cyrus Rashtchian, and Makrand Sinha. Edge Estimation with Independent Set Oracles. ACM Trans. Algorithms, 16(4):52:1–52:27, 2020.

[BKS02] Ziv Bar-Yossef, Ravi Kumar, and D. Sivakumar. Reductions in Streaming Algorithms, with an Application to Counting Triangles in Graphs. In Proceedings of the Thirteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA, pages 623–632, 2002.

[CLW20] Xi Chen, Amit Levi, and Erik Waingarten. Nearly Optimal Edge Estimation with Independent Set Queries. In Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA, pages 2916–2935, 2020.

[DL18] Holger Dell and John Lapinskas. Fine-Grained Reductions from Approximate Counting to Decision. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC, pages 281–288, 2018.
[DLM20] Holger Dell, John Lapinskas, and Kitty Meeks. Approximately Counting and Sampling Small Witnesses Using a Colourful Decision Oracle. In Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA, pages 2201–2211, 2020.

[DP09] Devdatt P. Dubhashi and Alessandro Panconesi. Concentration of Measure for the Analysis of Randomized Algorithms. Cambridge University Press, 2009.

[ELRS17] Talya Eden, Amit Levi, Dana Ron, and C. Seshadhri. Approximately Counting Triangles in Sublinear Time. SIAM J. Comput., 46(5):1603–1646, 2017.

[ERS20] Talya Eden, Dana Ron, and C. Seshadhri. On Approximating the Number of k-Cliques in Sublinear Time. SIAM J. Comput., 49(4):747–771, 2020.

[Fei06] Uriel Feige. On Sums of Independent Random Variables with Unbounded Variance and Estimating the Average Degree in a Graph. SIAM J. Comput., 35(4):964–984, 2006.

[Gold17] Oded Goldreich. Introduction to Property Testing. Cambridge University Press, 2017.

[GR02] Oded Goldreich and Dana Ron. Property Testing in Bounded Degree Graphs. Algorithmica, 32(2):302–343, 2002.

[GR08] Oded Goldreich and Dana Ron. Approximating Average Parameters of Graphs. Random Struct. Algorithms, 32(4):473–493, 2008.

[Jan04] Svante Janson. Large Deviations for Sums of Partly Dependent Random Variables. Random Struct. Algorithms, 24(3):234–248, 2004.

[Juk11] Stasys Jukna. Extremal Combinatorics - With Applications in Computer Science. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2011.

[MR99] Rajeev Motwani and Prabhakar Raghavan. Randomized Algorithms. In Algorithms and Theory of Computation Handbook, Chapman & Hall/CRC Applied Algorithms and Data Structures series. CRC Press, 1999.

[MVV16] Andrew McGregor, Sofya Vorotnikova, and Hoa T. Vu. Better Algorithms for Counting Triangles in Data Streams. In Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS, pages 401–411, 2016.

[RT16] Dana Ron and Gilad Tsur. The Power of an Example: Hidden Set Size Approximation Using Group Queries and Conditional Sampling. ACM Trans. Comput. Theory, 8(4):15:1–15:19, 2016.

[RWZ20] Cyrus Rashtchian, David P. Woodruff, and Hanlin Zhu. Vector-Matrix-Vector Queries for Solving Linear Algebra, Statistics, and Graph Problems. In Proceedings of the 24th International Conference on Randomization and Computation, RANDOM, volume 176, pages 26:1–26:20, 2020.

[Sto83] Larry J. Stockmeyer. The Complexity of Approximate Counting (Preliminary Version). In Proceedings of the 15th Annual ACM Symposium on Theory of Computing, STOC, pages 118–126, 1983.

[Sto85] Larry J. Stockmeyer. On Approximation Algorithms for #$P$. SIAM J. Comput., 14(4):849–861, 1985.
A Missing proofs from Section 3

Observation A.1 (Observation 3.4 restated). (i) For $G \sim D_{\text{Yes}} \cup D_{\text{No}}$, then the number of vertices in $G$ is $4\sqrt{m}$. Also, $\sqrt{m}/2 \leq |A|, |B|, |C|, |D| \leq 2\sqrt{m}$ with a probability of at least $1 - o(1)$, and the number of edges in $G$ is $\Theta(m)$ with a probability of at least $1 - o(1)$;

(ii) If $G \sim D_{\text{Yes}}$, then there are at most $t$ triangles in $G$ with a probability of at least $1 - o(1)$;

(iii) If $G \sim D_{\text{No}}$, $\frac{8t}{m} \leq |C'| \leq \frac{32t}{m}$ with a probability of at least $1 - o(1)$, and there are at least $2t$ many triangles in $G$ with a probability of at least $1 - o(1)$.

Proof. (i) By the construction of $G \sim D_{\text{Yes}} \cup D_{\text{No}}$, the number of vertices in any such $G$ is $4\sqrt{m}$.
For the cardinalities of $A, B, C, D$, we only argue for $|A|$. For $|B|, |C|, |D|$, the proofs are similar. Note that each vertex in $V(G)$ is put into one of $A, B, C, D$ uniformly at random, and independent of other vertices. So, $E[|A|] = \sqrt{m}$. By using Chernoff-Hoeffding bound (See Lemma B.1 in Appendix B), we have

$$P \left( |A| - \sqrt{m} \geq \frac{\sqrt{m}}{2} \right) \leq \exp \left( -\frac{\left(\frac{\sqrt{m}/2}{1/2}\right)^2}{3} \right) \leq o(1).$$

The last inequality holds as $m = \Omega(\log^2 n)$. Note that the above equation implies that

$$P \left( \frac{\sqrt{m}}{2} \leq |A| \leq 2\sqrt{m} \right) \geq 1 - o(1).$$

(ii) Here we work on the conditional space that $\frac{\sqrt{m}}{2} \leq |A|, |B|, |C|, |D| \leq 2\sqrt{m}$.

If we set a indicator random variable for each triple of vertices (one in each of $A, B, C$) such that it is set to 1 if the tree vertices forms a triangle in $G$.

Let $N$ be the number of indicator random variables. Observe that $\frac{m^{3/2}}{8} \leq N \leq 8m^{3/2}$. Due to our construction of $G \sim D_{\text{Yes}}$, the probability that each such indicator variable takes vale 1 is $p = \left( \sqrt{\frac{t}{16m^{3/2}}} \right)^2 = \frac{t}{16m^{3/2}}$, and each such indicator random variable may depend on at most $d \leq 6\sqrt{m}$ many other random variables. Taking $X$ as the sum of $N$ indicator random variables,

$$E[X] = Np \leq \frac{t}{2}.$$

Setting $\delta = \frac{t}{2}$ and applying Lemma B.4 we get

$$P(X \geq t) = P(X \geq E[X] + \delta) \leq \exp \left( -\frac{\delta^2 (1 - \frac{d+1}{4N})}{2(d+1) \left( Np + \frac{\delta}{3} \right)} \right) \leq \exp \left( -\frac{(t/2)^2 \left( 1 - \frac{6\sqrt{m}+1}{m^{3/2}} \right)}{2(6\sqrt{m} + 1) \left( t/2 + t/6 \right)} \right) = o(1).$$
Here we work on the conditional space that \( \frac{\sqrt{m}}{2} \leq |C| \leq 2\sqrt{m} \).

As we put each vertex in \( C \) into \( C' \) with probability \( \frac{32t}{m^{3/2}} \),

\[
\frac{16t}{m} \leq \mathbb{E}[C'] \leq \frac{64t}{m}.
\]

By using Chernoff-Hoeffding bound (See Lemma B.1 in Appendix B), we have

\[
P \left( \frac{8t}{m} \leq |C'| \leq \frac{32t}{m} \right) \geq 1 - o(1).
\]

By the construction of \( G \sim D_{\mathcal{N}_0} \), the number of triangles in \( G \) is at least \( |C'||A||B| \), which is at least \( 2t \) with a probability of at least \( 1 - o(1) \).
B Some probability results

Lemma B.1 (Chernoff-Hoeffding bound [DP09]). Let \( X_1, \ldots, X_n \) be independent random variables such that \( X_i \in [0, 1] \). For \( X = \sum_{i=1}^{n} X_i \) and \( \mu = \mathbb{E}[X] \), the followings hold for any \( 0 \leq \delta \leq 1 \).

\[
P(|X - \mu| \geq \delta \mu) \leq 2 \exp \left( -\frac{\mu \delta^2}{3} \right)
\]

Lemma B.2 (Chernoff-Hoeffding bound [DP09]). Let \( X_1, \ldots, X_n \) be independent random variables such that \( X_i \in [0, 1] \). For \( X = \sum_{i=1}^{n} X_i \) and \( \mu_l \leq \mathbb{E}[X] \leq \mu_h \), the followings hold for any \( \delta > 0 \).

(i) \( \mathbb{P}(X \geq \mu_h + \delta) \leq \exp \left( -\frac{2\delta^2}{n} \right) \).

(ii) \( \mathbb{P}(X \leq \mu_l - \delta) \leq \exp \left( -\frac{2\delta^2}{n} \right) \).

Lemma B.3 (Chernoff bound for bounded dependency, see Theorem 2.1 from [Jan04]). Let \( X_1, \ldots, X_n \) be random variables such that \( a_i \leq X_i \leq b_i \) and \( X = \sum_{i=1}^{n} X_i \). Let \( D \) be the dependent graph, where \( V(D) = \{X_1, \ldots, X_n\} \) and \( E(D) = \{(X_i, X_j) : X_i \text{ and } X_j \text{ are dependent}\} \). Then for any \( \delta > 0 \),

\[
P(|X - \mathbb{E}[X]| \geq \delta) \leq 2 \exp \left( -\frac{2\delta^2}{\chi^*(D) \sum_{i=1}^{n} (b_i - a_i)^2} \right),
\]

where \( \chi^*(D) \) denotes the fractional chromatic number of \( D \).

The following lemma is a special case of the above when \( X_i \)'s are indicator random variable.

Lemma B.4 (See Corollary 2.6 from [Jan04]). Let \( X_1, \ldots, X_N \) be indicator random variables such that \( \mathbb{P}(X_i = 1) = p \) for each \( i \) there are at most \( d \) many \( X_j \)'s on which an \( X_i \) depends, where \( 0 < p < 1 \). For \( X = \sum_{i=1}^{N} X_i \) and \( \mu = \mathbb{E}[X] \), the followings hold for any \( \delta > 0 \).

(i) \( \mathbb{P}(X \geq \mu + \delta) \leq \exp \left( -\frac{\delta^2 (1 - \frac{d+1}{N})}{2(\delta+1)(\mu+p+\frac{1}{3})} \right) \),

(ii) \( \mathbb{P}(X \leq \mu - \delta) \leq \exp \left( -\frac{\delta^2 (1 - \frac{d+1}{N})}{2(\delta+1)\mu p} \right) \),

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