JACOB’S LADDERS AND NEW CLASS OF INTEGRALS
CONTAINING PRODUCT OF FACTORS $\zeta^2$

JAN MOSER

ABSTRACT. In this paper we obtain new properties of a signal generated by the Riemann zeta-function on the critical line. At the same time we obtain an asymptotic formula for a new class of transcendental integrals connected with the Riemann zeta-function

1. INTRODUCTION

1.1. In the paper [4] we have obtained the following formula

$$\frac{1}{U} \int_T^{T+U} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_k(t) \right) \right|^2 \sim \prod_{k=0}^{n} \frac{1}{\varphi_k(T+U) - \varphi_k(T)} \int_{\varphi_k(T)}^{\varphi_k(T+U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt,$$

(1.1)

$$U \in \left( 0, \frac{T}{\ln^2 T} \right], \ T \to \infty.$$  

A motivation for this formula was the well-known multiplicative formula

$$M \left( \prod_{k=1}^{n} X_k \right) = \prod_{k=1}^{n} M(X_k)$$

from the theory of probability where $X_k$ are the independent random variables and $M$ is the population mean. Some new art of the asymptotic independence of the partial functions

$$\left| \zeta \left( \frac{1}{2} + it \right) \right|^2, \ t \in \left[ \varphi_k(T), \varphi_k(T+U) \right], \ k = 0, 1, \ldots, n$$

is expressed by this formula.

1.2. For example, by using the mean-value theorem in (1.1) we obtain

$$\left| \zeta \left( \frac{1}{2} + i \varphi_k(t_n) \right) \right|^2 \frac{1}{U} \int_T^{T+U} \prod_{k=0}^{n-1} \left| \zeta \left( \frac{1}{2} + i \varphi_k(t) \right) \right|^2 dt \sim \frac{1}{\varphi_k(T+U) - \varphi_k(T)} \int_{\varphi_k(T)}^{\varphi_k(T+U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \times$$

$$\times \prod_{k=0}^{n-1} \frac{1}{\varphi_k(T+U) - \varphi_k(T)} \int_{\varphi_k(T)}^{\varphi_k(T+U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt$$

Key words and phrases. Riemann zeta-function.
Jan Moser

Jacob’s ladder . . .

i. e. (see (1.1), \( n \mapsto n - 1 \))

(1.2) \[
\left| \zeta \left( \frac{1}{2} + i \varphi_1^n (\bar{t}_n) \right) \right|^2 \sim \frac{1}{\varphi_1^n (T + U) - \varphi_1^n (T)} \int_{\varphi_1^n (T)}^{\varphi_1^n (T + U)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt.
\]

But

\[
\left| \zeta \left( \frac{1}{2} + i \varphi_1^n (\bar{t}_n) \right) \right|^2, \bar{t}_n \in (T, T + U)
\]

is the mean value with respect to the set of functions

(1.3) \[
\left\{ \left| \zeta \left( \frac{1}{2} + i \varphi_0^n (t) \right) \right|^2, \ldots, \left| \zeta \left( \frac{1}{2} + i \varphi_1^n (t) \right) \right|^2 \right\}
\]

i. e. \( \bar{t}_n \) is the nonlinear functional

(1.4) \[
\bar{t}_n = f_n[\varphi_1, \ldots, \varphi_1^{n-1}]; \ \varphi_1^0 (t) = t
\]

that is defined on the continuum set of points

(1.5) \[
(\varphi_1, \ldots, \varphi_1^{n-1}).
\]

Let us remind that there is the continuum set of the Jacob’s ladders (see \([1]\) generating the set of the iterations (1.5)). At the same time it follows from (1.2) that

(1.6) \[
\left| \zeta \left( \frac{1}{2} + i \varphi_1^n (\bar{t}_n) \right) \right|^2 \sim \left| \zeta \left( \frac{1}{2} + i \varphi_0^n (\tau) \right) \right|^2, \ \tau \in (T, T + U)
\]

where \( \tau \) is completely independent on the set of points (1.5).

Remark 1. Thus, the mean-value (1.2) with respect to the set of functions (1.3) is asymptotically independent on this set.

Remark 2. Now, let \( k : 1 < k < n \). Then we have the mean-value

(1.7) \[
\left| \zeta \left( \frac{1}{2} + i \varphi_1^k (\bar{t}_k) \right) \right|^2
\]

of the inner factor of the product in (1.1) with respect to the set (comp. (1.3))

\[
\left\{ \left| \zeta \left( \frac{1}{2} + i \varphi_0^k (t) \right) \right|^2, \ldots, \left| \zeta \left( \frac{1}{2} + i \varphi_1^{k-1} (t) \right) \right|^2, \left| \zeta \left( \frac{1}{2} + i \varphi_1^k (t) \right) \right|^2 \right\}
\]

(1.8) \[
\ldots, \left| \zeta \left( \frac{1}{2} + i \varphi_1^n (t) \right) \right|^2 \right\},
\]

where (comp. (1.4))

\( \bar{t}_k = g_k[\varphi_1, \ldots, \varphi_1^{k-1}, \varphi_1^{k+1}, \ldots, \varphi_1^n] \)

is the functional defined on the continuum set of points

\[
(\varphi_1, \ldots, \varphi_1^{k-1}, \varphi_1^{k+1}, \ldots, \varphi_1^n).
\]

In this case the mean-value (1.7) is not, probable, asymptotically independent on the set (1.8).
1.3. In this paper we obtain new properties of the signal
\[ Z(t) = e^{i\varphi(t)\left(\frac{1}{2} + it\right)} , \quad \varphi(t) = -\frac{t}{2}\ln\pi + \text{Im} \ln \Gamma \left(\frac{1}{4} + \frac{t}{2}\right) \]
generated by the Riemann zeta-function on the critical line. Namely, we obtain an asymptotic formula for a new class of transcendental integrals of the type
\[ \int_T^{T+U} F[\varphi^{n+1}_1(t)] \prod_{k=0}^n \left| \zeta \left(\frac{1}{2} + i\varphi_k(t)\right) \right|^2 \, dt , \quad U \in \left(0, \frac{T}{\ln^2 T}\right) , \]
where
\[ F(w) , \quad w \in [\varphi^{n+1}_1(T), \varphi^{n+1}_1(T+U)] , \quad F(w) \geq 0 (\leq 0) \]
is an arbitrary Lebesgue integrable function. In this direction, we obtain, for example, new asymptotic formulae generalizing our formulae containing the factors
\[ \left| \zeta \left(\frac{1}{2} + i\varphi(t)\right) \right|^4 , \quad \left\{ \arg \zeta \left(\frac{1}{2} + i\varphi(t)\right) \right\} \]
where
\[ \varphi(t) \text{ is the Jacob's ladder, i. e. the exact solution of the nonlinear integral equation} \]
\[ \varphi(t) = \frac{1}{2} \varphi(t) , \quad t \geq T_0[\varphi] \]
and
\[ \varphi(t) = \frac{Z^2(t)}{2\varphi'[\varphi(t)]} = \frac{\left| \zeta \left(\frac{1}{2} + it\right) \right|^2}{\{1 + O \left(\frac{\ln \ln t}{\ln t}\right)\} \ln t} \]
where \( \varphi(t) \) is the Jacob's ladder, i. e. the exact solution of the nonlinear integral equation
\[ \int_0^{\mu(x(T))} Z^2(t)e^{-\frac{\mu(y)}{2}t} \, dt = \int_0^T Z^2(t) \, dt , \quad \mu(y) \geq 7y \ln y , \quad \mu(y) \to y = \varphi_{\mu}(T) = \varphi(T) \]
Jan Moser

Jacob's ladder . . .

(see [1]). Next, we have (see [4], (2.1))

\[
y = \frac{1}{2} \varphi(t) = \varphi_1(t); \quad \varphi_0(t) = t, \quad \varphi_1(t) = \varphi_1(t),
\]

\[
\varphi_1^2(t) = \varphi_1(\varphi_1(t)), \ldots, \varphi_1^k(t) = \varphi_1(\varphi_1^{k-1}(t)), \ldots
\]

where \(\varphi_1^k(t)\) stands for the \(k\)th iteration of the Jacob's ladder

\[
y = \varphi_1(t)
\]

(OF course, \(\varphi_1^k(t), t \in [T, T + U]\) are the increasing functions.) The following theorem holds true.

**Theorem.** Let

(2.4) \[ U \in \left(0, \frac{T}{\ln^2 T}\right). \]

Then for every fixed \(n \in \mathbb{N}\) and for every Lebesgue-integrable function

\[ F(t), t \in [\varphi_1^{n+1}(T), \varphi_1^{n+1}(T + U)], \quad F(t) \geq 0 (\leq 0) \]

we have

(2.5) \[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 dt \sim \\
\sim \left\{ \int_{\varphi_1^{n+1}(T+U)}^{\varphi_1^{n+1}(T)} F(t) dt \right\} \ln^{n+1} T, \quad T \to \infty
\]

where

(2.6) \[
\varphi_1^k(T + U) - \varphi_1^k(T) < \frac{1}{2n + 3} \frac{T}{\ln T}, \quad k = 1, \ldots, n + 1,
\]

(2.7) \[
\varphi_1^k(T + U) - \varphi_1^k(T) > 0.2 \times \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n.
\]

Next, in the macroscopic case (comp. (2.4))

(2.8) \[ U \in \left[T^{\frac{1}{3} + \epsilon}, \frac{T}{\ln^2 T}\right], \]

we have more exact information

(2.9) \[
\|[\varphi_1^k(T), \varphi_1^k(T + U)]\| = \varphi_1^k(T + U) - \varphi_1^k(T) \sim U, \quad k = 1, \ldots, n,
\]

(2.10) \[
\varphi_1^k(T) - \varphi_1^{k+1}(T + U) \sim (1 - c) \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n,
\]

(2.11) \[
\rho \{[\varphi_1^{k-1}(T), \varphi_1^{k-1}(T + U)]; [\varphi_1^k(T), \varphi_1^k(T + U)] \} \sim (1 - c) \frac{T}{\ln T},
\]

\[ k = 1, \ldots, n + 1 \]

where \(\rho\) denotes the distance of the corresponding segments.

**Remark 3.** In the macroscopic case (2.8) the following is true. The system of iterated segments

(2.12) \[
[\varphi_1^{n+1}(T), \varphi_1^{n+1}(T + U)], [\varphi_1^n(T), \varphi_1^n(T + U)], \ldots, [T, T + U]
\]

is the disconnected set and its components are:

(a) asymptotically equal (see (2.9)),

Page 4 of 8
(b) distributed with the asymptotic regularity from the right to the left (see \((2.10) - (2.12)\)).

**Remark 4.** Every Jacob’s ladder

\[
\varphi_1(t) = \frac{1}{2} \varphi(t)
\]

(see \((2.11)\)) where \(\varphi(t)\) is the exact solution of the nonlinear integral equation \((2.3)\) is the asymptotic solution of the following nonlinear integro-iteration equation

\[
\frac{1}{U} \int_T^{T+U} F[x^{n+1}(t)] \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 dt =
\]

\[
= \left\{ \int_{x^{n+1}(T)}^{x^{n+1}(T+U)} F(t) dt \right\} \ln^{n+1} T
\]

(comp. \((2.5)\)) where

\[x_0(t) = t, \ x^1(t) = x(t), \ x^2(t) = x(x(t)), \ldots\]

i. e. the function \(x^k(t)\) is the \(k\)th iteration of the function \(x(t)\), (comp. \((2.13)\) with \([3], (11.1), (11.4), (11.6), (11.8), [4], (2.5)\) and \([5], (2.6)\)).

**Remark 5.** Similar remarks like Remark 1 – Remark 2 hold true also when speaking on the independence of the mean-value.

2.2. By \((2.9)\) and the formula (see \([4], (3.5)\))

\[
t - \varphi_1^{n+1}(t) \sim (1 - c)(n + 1) \frac{t}{\ln t}
\]

we obtain easily (for example) from \((2.5)\) the following

**Corollary.** In the macroscopic case \((2.8)\) we have

\[
\int_T^{T+U} \prod_{k=0}^{n} \left| \frac{1}{2} + i \varphi_1^k(t) \right|^2 dt \sim U \ln^{n+1} T, \ T \to \infty,
\]

\[
\sim \frac{1}{2\pi^2} U_1 \ln^{n+5} T, \ U_1 = T^{\frac{1}{2} + \epsilon},
\]

\[
\int_T^{T+U} \left\{ \arg \zeta \left( \frac{1}{2} + i \varphi_1^{n+1}(t) \right) \right\} \prod_{k=0}^{n} \left| \frac{1}{2} + i \varphi_1^k(t) \right|^2 dt \sim
\]

\[
\sim \frac{(2l)!}{l!^2} U \ln^{n+1} T(\ln T)' \left( U \in \left[ T^{\frac{1}{2} + \epsilon}, \frac{T}{\ln^2 T} \right], \right.
\]

\[
\int_T^{T+U} \left( S_1 \varphi_1^{n+1}(t) \right) \prod_{k=0}^{n} \left| \frac{1}{2} + i \varphi_1^k(t) \right|^2 dt \sim
\]

\[
\sim dt U \ln^{n+1} T, \ U \in \left[ T^{\frac{1}{2} + \epsilon}, \frac{T}{\ln^2 T} \right],
\]

for every fixed \(l, n \in \mathbb{N}\) where

\[
S_1(T) = \int_0^T S(t) dt, \ S(t) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + it \right)\
\]

Page 5 of
and the argument is defined by the usual way (comp. [6], p. 179).

Remark 6. The formulae (2.15) – (2.17) are can be understand as generalization of our formulae [3], (8.3), [5], Lemma 2, (5.4), (5.5). The formula (2.14) can be compared with the formula (2.3) from the paper of reference [4] in the macroscopic case. The small improvements of the Heath-Brown exponent \( \frac{7}{8} \) in (2.15) are irrelevant for our purpose.

3. Proof of Theorem

3.1. By the formula (see [4], (3.9))

\[
\prod_{k=0}^{n} \tilde{Z}^2[\varphi_k^1(t)] = \frac{d\varphi_1^{n+1}}{dt}
\]

we obtain

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_k^1(t)] dt = \int_T^{T+U} F[\varphi_1^{n+1}(t)] d\varphi_1^{n+1}(t) = \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt,
\]

i.e.

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_k^1(t)] dt = \int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)} F(t) dt.
\]

Since (see [4], (3.3), (3.6))

\[
t > \varphi_1^1(t) > \varphi_2^1(t) > \cdots > \varphi_1^{n+1}(t),
\]

\[
(1-\varepsilon)T < \varphi_1^{n+1}(T) < T
\]

then

\[
(1-\varepsilon)T < \varphi_1^{n+1}(T) < T + U, \quad U \in \left(0, \frac{T}{\ln^2 T}\right).
\]

Consequently,

\[
T' \in (\varphi_1^{n+1}(T), T + U) \Rightarrow \ln T' = \ln T + \mathcal{O}(1).
\]

Now, if we use the mean-value theorem on the left-hand side of (3.1) we obtain (see [2.2], (3.2))

\[
\int_T^{T+U} F[\varphi_1^{n+1}(t)] \prod_{k=0}^{n} \tilde{Z}^2[\varphi_k^1(t)] dt \sim \frac{1}{\ln^{n+1} T} \int_T^{T+U} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i\varphi_k^1(t) \right) \right|^2 dt.
\]

Hence, from (3.1) by (3.3) the asymptotic formula (2.5) follows.
3.2. Let us remind the estimates (see [4], (3.15))
\[
\varphi_k(T + U) - \varphi_k(T) < \frac{2k + 1}{2n + 1} \frac{T}{\ln T} \leq \frac{T}{\ln T}, \quad k = 1, \ldots, n.
\]
It is clear that by the substitution
\[
2n + 1 \rightarrow (2n + 3)^2
\]
(for example) in [4], part 3.4 we obtain the estimates
\[
(3.4) \quad \varphi_k(T + U) - \varphi_k(T) < \frac{2k + 1}{(2n + 3)^2} \frac{T}{\ln T} < \frac{1}{2n + 3} \frac{T}{\ln T}, \quad k = 1, \ldots, n + 1,
\]
i. e. the inequality (2.6) holds true.

Next we have (see [4], (3.4))
\[
(3.5) \quad \varphi_k(T) - \varphi_k^{k+1}(T + U) + \varphi_k^{k+1}(T + U) - \varphi_k^{k+1}(T) > (1 - \epsilon)(1 - c) \frac{T}{\ln T}.
\]
Consequently we have (see (3.4))
\[
\varphi_k(T) - \varphi_k^{k+1}(T + U) > (1 - \epsilon)(1 - c) \frac{T}{\ln T} - \frac{1}{2n + 3} \frac{T}{\ln T},
\]
\[
\geq \left(1 - c - \frac{1}{5} - \epsilon\right) \frac{T}{\ln T} > (0.22 - \epsilon) \frac{T}{\ln T} > 0.2 \frac{T}{\ln T},
\]
since
\[
c < 0.58 \Rightarrow 1 - c > 0.42 > \frac{1}{5} = 0.2,
\]
i. e. the inequality (2.7) holds true.

3.3. We use the Hardy-Littlewood-Ingham formula
\[
(3.6) \quad \int_T^{T+U} Z^2(t) dt \sim U \ln T, \quad U \in \left[T^{1+\epsilon}, \frac{T}{\ln^2 T}\right]
\]
where \(\frac{1}{3}\) is the Balasubramanian exponent (the small improvements of the exponent \(\frac{1}{3}\) are irrelevant for our purpose), and our formula (see [3], (2.5))
\[
(3.7) \quad \int_T^{T+U} Z^2(t) dt \sim \{\varphi_1(T + U) - \varphi_1(T)\} \ln T.
\]
Comparing the formulae (3.6) and (3.7) we obtain
\[
\varphi_1(T + U) - \varphi_1(T) = \varphi_1(T + U) - \varphi_1(T) \sim U.
\]
Similarly, by comparison in the cases (see (2.6))
\[
T \rightarrow \varphi_1(T), \quad T + U \rightarrow \varphi_1(T + U);
\]
we obtain
\[
(3.8) \quad \varphi_k(T + U) - \varphi_k(T) \sim U, \quad k = 1, \ldots, n + 1
\]
i. e. the formula (2.9) holds true.
3.4. Next, we have (see (2.4), (3.5), (3.8))

\[
\varphi_k^1(T) - \varphi_{k+1}^1(T + U) + \varphi_{k+1}^k(T + U) - \varphi_{k+1}^1(T) \sim (1 - c) \frac{T}{\ln T},
\]

\[
\varphi_k^1(T) - \varphi_{k+1}^1(T + U) \sim (1 - c) \frac{T}{\ln T} - \{\varphi_{k+1}^k(T + U) - \varphi_{k+1}^1(T)\} \sim
\]

\[
(1 - c) \frac{T}{\ln T} - U \sim (1 - c) \frac{T}{\ln T}
\]

i. e. the formula (2.10) holds true. The proposition (2.11) follows from (2.10).

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

REFERENCES

[1] J. Moser, ‘Jacob’s ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 2010, 88, pp. 414-422, arXiv: 0901.3973.

[2] J. Moser, ‘Jacob’s ladders and the nonlocal interaction of the function \(|\zeta(\frac{1}{2} + it)|\) with the function arg \(\zeta(\frac{1}{2} + it)\) on the distance \((1 - c)\pi(t)\)’, arXiv: 1004.0169, (2010).

[3] J. Moser, ‘Jacob’s ladders, the structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations’, to be published in Trudy MIAN.

[4] J. Moser, ‘Jacob’s ladders and certain asymptotic multiplicative formula for the function \(|\zeta(\frac{1}{2} + it)|^2\)’, arXiv: 1201.3211, (2012).

[5] J. Moser, ‘Jacob’s ladders of the second order and the asymptotic formula for the integral of the eight order expression \(|\zeta(\frac{1}{2} + i\varphi_2(t))|^{14} \zeta(\frac{1}{2} + it)|^4\)’, to be published in FAOM.

[6] E.C. Titchmarsh, ‘The theory of the Riemann zeta-function’ Clarendon Press, Oxford, 1951.

Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynska Dolina M105, 842 48 Bratislava, SLOVAKIA

E-mail address: jan.moser@fmph.uniba.sk