PERFORMANCE BOUNDS FOR THE MEAN-FIELD LIMIT OF CONSTRAINED DYNAMICS

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Abstract. In this work we are interested in the mean-field formulation of kinetic models under control actions where the control is formulated through a model predictive control strategy (MPC) with varying horizon. The relation between the (usually hard to compute) optimal control and the MPC approach is investigated theoretically in the mean-field limit. We establish a computable and provable bound on the difference in the cost functional for MPC controlled and optimal controlled system dynamics in the mean-field limit. The result of the present work extends previous findings for systems of ordinary differential equations. Numerical results in the mean-field setting are given.

1. Introduction. In recent years many mathematical models of self-organized systems of interacting agents have been introduced in the literature, see for example [5, 6, 7, 8, 15, 17, 21, 22, 23, 24, 26, 32, 36, 38, 46, 48, 51] and the references therein. The general setting consists of a microscopic dynamics described by systems of ordinary differential equations where the evolution of the state of each agent is influenced by the collective behavior of all other agents. Examples in those microscopic interacting systems are frequently seen in the real world like: schools of fish, swarm of bees, herds of sheep, opinion formation in crowds and financial markets. Of interest is usually the case when the number of agents becomes very large. Here, the qualitative behavior is studied through a different level of description, i.e. through the introduction of distribution functions whose behavior is governed by kinetic (or fluid–dynamic) partial differential equations.

The control mechanisms of self–organized systems has been investigated recently as follow–up questions to the progress in mathematical modeling and simulation. The control of emergent behavior has been studied on the level of the microscopic agents [4, 11] as well as on the level of the kinetic [2, 3, 37] or fluid–dynamic equations [9, 20, 27, 31]. The contributions have to be further distinguished depending
on the type of applied control. Without intending to review all literature we give some references on certain classes of control, e.g., sparse control [30], Nash equilibrium control [11], control using linearized dynamics and Riccati equations [36, 37] or control driven by other external dynamics [8, 29].

Here, we focus on a general method to construct a control mechanism, called model predictive control (MPC). MPC utilizes the assumption that agents optimize their cost functional not necessarily over a large time horizon. Instead they determine their (locally best) action by minimizing their cost only over a short time interval which recedes as time evolves. The methodology of MPC is also called receding horizon control (or instantaneous control when the length of the horizon is equal to one). From the modeling point of view the fact that agents may be able to optimize strategically their trajectories over a small, but finite, interval of time opened several connections to socio–economic problems, where each agent, or a portion of them, is influenced in order to force the entire system toward specific patterns.

MPC has been used in the engineering community for over fifty years, see e.g. [42, 43, 44, 50] for an overview and further references. However, therein, only a small number of agents $M < \infty$ is considered and the optimization problems are then studied at the level of ODEs. The link between MPC on the level of agents and the MPC on the level of kinetic and fluid–dynamic equations has been subject to recent investigations [2, 27, 37], and also the relation between MPC and mean-field games [11] has been a subject to recent studies [29]. However, in all currently presented approaches on MPC in relation to mean-field limits the special case of a receding time horizon has been considered. While this is computationally advantageous, it is known to have some severe drawbacks: in the case of finitely many agents stability of the controlled system can expected only if the horizon is sufficiently large, the instability of the controlled system has been also observed numerically e.g. in [3]. Further, MPC leads to a control that is suboptimal compared with the theoretical optimal one, that is a control with infinite control horizon. Except for a very particular case [37] there is no result on the relation between the optimal control and the MPC approach in the mean-field limit.

In the case $M < \infty$ there has been recent progress on the relation between the time horizon for MPC and the stability as well as optimality estimates of MPC controls [33, 34, 35, 39]. In particular an estimate on the difference between MPC and optimal control has been given in [34, Corollary 4.5]. The theory therein covers finite and infinite dimensional phase spaces, but still requires the number $M$ of agents to be finite.

The main purposes of the present work is to extend the theory presented in [34] to the limit case of infinitely many agents. The goal is to derive the corresponding mean-field results for the optimality estimates under the same assumptions as in the case $M < \infty$. While the presentation will cover a general dynamics we exemplify the results on a first–order alignment model, as an extension to models recently presented [2, 8].

The rest of the manuscript is organized as follows. First, in Section 2, we introduce some notations and results for an exemplified constrained model deriving its mean-field formulation and highlighting the main features of the performance estimate for the MPC approach. In a more general setting, in Section 3, we define the objects of a mean-field optimal control problem subject to a given dynamics proving several estimates in relation to MPC. Here, an example is proposed with
numerical results, in Section 4 confirming the theoretical analysis. In A we recall technical details.

2. Notation and motivating example. In this section, in order to clarify the notations and exemplify the aims of this work, we introduce a stylized problem which has been extensively investigated in several recent works on constrained alignment dynamics [1 2 4]. At the microscopic level, the mathematical description of collective motion is given by a nonlinear system of ordinary differential equations, from which the mean-field level may be obtained through specific assumptions [13 21 28]. Let us assume that $M > 0$ agents fulfill the dynamics in discretized form

$$x_{i,n+1} = x_{i,n} + \frac{\Delta t}{M} \sum_{j=1}^{M} P(x_{j,n} - x_{i,n}) + u_n$$

(1)

where $P \geq 0$ is a general interaction function that may also depend on variables $(x_{j,n})_{j=1}^{M}$ and $x_{i,n} = x_i(t^n) \in \mathcal{X} \subset \mathbb{R}$ is the state of the $i$th agent at time $t^n \geq 0$ with $t^n = n \in \mathbb{N}$. We denote by

$$X_n = (x_{i,n})_{i=1}^{M}, \quad X_{-i,n} = (x_{j,n})_{j=1,j\neq i}^{M}$$

(2)

the state of the full system at time $t^n$ and the state of the all agents except the $i$th agent, respectively. In the following we will drop the dependence on the time variable whenever the intention is clear. Moreover we assume that initial conditions $X(0) = X_0$ are given.

The control sequence $(u_n)_n$ is to be determined in order to minimize a given cost functional

$$J^u_{\infty}(X_0) := \sum_{n=0}^{\infty} \ell(X_n, u_n)$$

(3)

where $X_n$ is the solution to (1) for the control $u_n \in U$, with $U \subset \mathbb{R}$ bounded, and initial datum $X(0) = X_0$. In [3] we introduce a general function $\ell : \mathbb{R}^M \times U \rightarrow \mathbb{R}$. Hence, the functional $J$ depends on the initial datum $X_0$ as well as the choice of the control sequence $u = (u_n)_n$. The dependence of $J$ on the time horizon is indicated by a subscript $+\infty$, whereas the dependence on the control by the superscript $u$. We assume here that there exists a solution $u^*$ of the optimal control problem

$$u^* = \arg \min_u J^u_{\infty}(X_0).$$

(4)

From the computational point of view this approach is generally too expensive, therefore, a suboptimal approach named model predictive control (MPC) has been proposed. The leading idea of the MPC approach is to avoid the solution of the dynamics on the whole time interval by considering a closed-loop control for the considered model, see [12 49] for an exhaustive introduction. In particular a control mechanism designed through a MPC might be interpreted as a strategy in the contest of the mean-field games [25].

(Single-step) MPC with receding time horizon $N$ applies a control $u$ of the type

$$u_{\text{MPC}}(t) = \sum_{n=0}^{\infty} u^\text{MPC}_n \chi_{[t^n, t^{n+1})}(t).$$

(5)

The unknown control actions $u^\text{MPC}_n \in \mathbb{R}$ are determined at each time $t^n$ by

$$u^\text{MPC}_n = v_1$$

(6)
where \((v_k)_{k=1}^N\) are the solutions of the following auxiliary minimization problem

\[
(v_k)_{k=1,...,N} = \arg \min_{(v_k)_{k=1}^N} \Delta t \sum_{k=1}^N \ell(Y_k, v_k) \quad \text{subject to } 8),
\]

where the states \(Y_k, k = 1, \ldots, N\), are given by the dynamics \([8]\) for an initial value \(X_0\) and a time horizon \(N\), i.e., for each \(k = 1, \ldots, N\)

\[
y_{i,k+1} = y_{i,k} + \frac{\Delta t}{M} \sum_{j=1}^M P(y_{j,k} - y_{i,k}) + \Delta tv_k, \quad y_{i,1} = x_{i,n}.
\]

In the introduced notations, the case \(N = 2\) corresponds to instantaneous-type control, which has been extensively investigate in the literature, see \([2, 3, 25, 27]\).

A first obvious relation between the optimal control and the control introduced through a model predictive approach is the following

\[
J_{\infty}^{\text{MPC}}(X_0) \geq J_{\infty}^{u^*}(X_0).
\]

Part of the investigation in \([34]\) is related to a result to establish an upper bound on \(J_{\infty}^{\text{MPC}}\) by a multiple of \(J_{\infty}^{u^*}\), in particular the result \([34, \text{Theorem 4.2}]\) proves that such a multiplicative factor can be obtained and depends in particular on the optimization horizon \(N\) and on the decay rate of the function \(\ell(\cdot, \cdot)\). The result of the aforementioned work leads to an estimate at the ODE level of the type

\[
\alpha_N J_{\infty}^{u^*}(X_0) \leq \alpha_N J_{\infty}^{\text{MPC}}(X_0) \leq J_{\infty}^{u^*}(X_0),
\]

for some \(0 < \alpha_N \leq 1\). Where we indicated the dependence of \(u^{\text{MPC}}\) on the time horizon in problem \([7]\) by the subscript \(N\) on the control. Further, an estimate \(\alpha_N J_{\infty}^{\text{MPC}}(X_0) \leq J_{N}^{\text{MPC}}(X_0)\) has been established as an additional result in \([34, \text{Corollary 4.5}]\). Here, \(J_{N}^{\text{MPC}}\) is defined as in equation \([8]\) but for a finite time horizon \(T = N\Delta t\). An estimate on the crucial constant \(\alpha_N\) is provided e.g. in \([35]\).

We are interested in a corresponding result in the case of a large number of agents, that is in the limit \(M \to \infty\). We sketch here the derivation of the semi discrete mean-field formulation of the constrained problem \([1]\). Let us suppose that for each \(n \geq 0\) the introduced MPC control \(u^{\text{MPC}}_n\) is symmetric with respect to each position of the system of agents at time \(t^n\). We define the empirical measure

\[
f_M(t^n) = f_{M,n} = \frac{1}{M} \sum_{i=1}^M \delta(x - x_{i,n}),
\]

where \(\delta\) is the Dirac delta, or localizing function, defined in the space of probability measures of \(\mathbb{R}^d\), namely \(\mathcal{P}(\mathbb{R}^d)\). For any test function \(\phi \in C_0^1(\mathbb{R}^d)\) we have

\[
\int_{\mathbb{R}} \phi(x)f_{M,n}(x)dx = \frac{1}{M} \sum_{i=1}^M \phi(x_{i,n}),
\]

then through a first order Taylor expansion we obtain

\[
\phi(x_{i,n+1}) - \phi(x_{i,n}) = \phi'(x_{i,n})(x_{i,n+1} - x_{i,n}) + O(\Delta t^2)
\]

Now from the original dynamic \([1]\) we can replace the quantity \(x_{i,n+1} - x_{i,n}\) in \([13]\), we have

\[
\phi(x_{i,n+1}) - \phi(x_{i,n}) = \phi'(x_{i,n}) \left[\frac{\Delta t}{M} \sum_{j=1}^M P(x_{j,n} - x_{i,n}) + \Delta tu^{\text{MPC}}_n \right].
\]
If we consider not the sum over \( i = 1, \ldots, M \) equation (14) assumes the following form
\[
\frac{1}{M} \sum_{i=1}^{M} \phi(x_{i,n+1}) - \phi(x_{i,n}) = \frac{1}{M} \sum_{i=1}^{M} \phi'(x_{i,n}) \left[ \frac{\Delta t}{M} \sum_{j=1}^{M} P(x_{j,n} - x_{i,n}) + \Delta t u_{n}^{\text{MPC}} \right].
\] (15)

Given that \( f_{M,n} \) is a probability measure in the space \( \mathcal{P}(\mathbb{R}^d) \) with uniform support with respect to \( M \), Prokhorov’s theorem implies that the sequence \( (f_{M,n})_M \) is weakly-* relatively compact, i.e. there exists a subsequence \( (f_{M,n})_m \) and a probability measure \( f_n \in \mathcal{P}(\mathbb{R}^d) \) such that
\[
f_{M,n} \to w* f_n
\] (16)
in \( \mathcal{P}(\mathbb{R}^d) \) pointwise in time. Recall that for the Cucker-Smale model the tightness hypothesis is in general satisfied if the initial distribution \( f_{M,0} \) is compactly supported with respect to \( M \). For a rigorous proof we refer to [18, 28].

As an example, consider the special function \( \ell : \mathbb{R}^M \times U \to \mathbb{R} \)
\[
\ell(X, u) = \frac{1}{2} \left( \frac{1}{M} \sum_{j=1}^{M} x_j \right)^2 + \frac{\nu}{2} u^2,
\] (17)
for some regularization parameter \( \nu > 0 \). Then, the limit \( M \to \infty \) of \( \ell \) exists and is given by
\[
\tilde{\ell}(f, u) = \frac{1}{2} \left( \int_{\mathbb{R}} y f(y) dy \right)^2 + \frac{\nu}{2} u^2
\] (18)
with \( \tilde{\ell} : \mathcal{P}(\mathbb{R}) \times U \to \mathbb{R} \), where \( \mathcal{P}(\mathbb{R}) \) denotes the probability measures on \( \mathbb{R} \).

Let us consider the dynamics [13] and denote by \( y \to f_k(y) \) the agent probability density at time \( t^k \) with \( f_k(\cdot) \in \mathcal{P}(\mathbb{R}) \) for \( k = 1, \ldots, N \). The limiting equation corresponding to the microscopic dynamics in [3] for \( M \to \infty \) and a.e. \( y \in \mathbb{R} \) reads
\[
f_{k+1}(y) = f_k(y) - \Delta t \partial_y \int_{X} P(z - y) f_k(z) f_k(y) dz - \Delta t v_k \partial_y f_k(y),
\] (19)
\[
f_1(y) = h_n(y).
\]
The probability distribution \( h_n(\cdot) \in \mathcal{P}(\mathbb{R}) \) is the distribution \( h(t_n) \) at time \( t^n \) obtained by propagation of the mean-field limit of the original dynamics [1], i.e. for each \( n \geq 0 \)
\[
h_{n+1}(x) - h_n(x) + \Delta t \partial_x \left( \int_{X} P(z - x) h_n(z) h_n(x) dz - u_n h_n(x) \right) = 0
\] (20)
In equation (20) \( u(\cdot) = u^{\text{MPC}}(\cdot) \) is the control obtained by the MPC approximation [15] and [16]. The initial state \( h(t^0, 0) = h_0(x) \) is obtained as the probability distribution corresponding to the mean-field limit of the initial data to [1]. The control \( (v_k)_{k=1}^{N} \) in equation (19) is determined by solving the corresponding mean-field optimization problem, i.e.,
\[
(v_k)_{k=1}^{N} = \arg \min_{(v_k)_{k=1}^{N}} \Delta t \sum_{k=1}^{N} \tilde{\ell}(f_k(\cdot), v_k), \text{ subject to } (19).
\] (21)
As usual, the constrained initial discrete dynamics in the case of the introduced MPC control is recovered by substituting the discrete measure \( f_{M,n} \), defined in (11), in the weak form of the equation (19). Again, we refer to [18, 28, 45] for rigorous results and more details on the mean-field limit. As an example, note that the mean-field limit \( J_n^u : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \) is

\[
J_n^u(h_0) = \sum_{n=0}^{\infty} \hat{J}(h_n(), u_n),
\]

where \( h \) is determined by equation (20) with initial condition \( h_0 \in \mathcal{P}(\mathbb{R}) \). As before a horizon of \( N = \infty \), corresponding to the optimal case, is desirable but computationally inefficient. In the sequel we want to establish the estimate (10) also for the mean-field cost functional \( J^u \). Except for the assumptions required to derive the mean-field limit we only enforce the assumptions of [34, Theorem 4.2] and we will show how those are sufficient to derive the corresponding estimates. Also, we will justify by obtaining the suitable mean-field limits the previously outlined recipe for MPC mean-field control for a broader class of agent dynamics.

3. Optimality estimate for the mean-field cost functional using MPC approach. We will follow the approach described in [34, 35] with applications to the infinite dimensional mean-field case taking first into account a discretized system of ordinary differential equations.

Let us consider a homogeneous time discretization for a general problem

\[
\dot{x}_i = g(x_i(t), X_{-i}(t)) + u(t)
\]

where \( g : \mathbb{R}^M \to \mathbb{R} \) is a general differentiable function that depends on the state of the \( i \)th agent and on the states of the other agents [2]. Also, \( \Delta t = t^{n+1} - t^n > 0 \) and for simplicity we assume \( \Delta t = 1 \). Let us suppose that \( g \) fulfills the assumptions of [10, Section 4], see also A. In order to pass to the mean-field limit we require that the trajectory of each agent \( x_{i,n} \) belongs to a compact subset \( \mathcal{X} \) of \( \mathbb{R} \) for all time steps \( n \geq 0 \). Let \( U \) be a compact subset of \( \mathbb{R} \). Then, we assume that for \( x_{i,0} \in \mathcal{X} \) and \( u_n \in U \) we have \( x_{i,n} \in \mathcal{X} \) for each \( i = 1, \ldots, M \). Then, according to the result [A.1] there exists a function

\[
G : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}
\]

such that the sequence

\[
G_M(x_{i,n}, m^M_{X_{-i,n}}) = g(x_{i,n}, X_{-i,n})
\]

converges toward \( G \) in the limit \( M \to \infty \). For the precise definition of \( (G_M)_M \) with \( G_M : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R} \) we refer to equation (69). This allows to obtain that the particle density function \( f_n \in \mathcal{P}(\mathcal{X}) \) satisfies the semi–discrete partial differential equation in strong form

\[
f_{n+1}(x) = f_n(x) - \partial_x[G(x, f_n(x))f_n(x)] - \partial_x[f_n(x) u_n],
\]

for a given initial distribution \( f_0 \in \mathcal{P}(\mathcal{X}) \). We denote with \( \mathcal{U} \) the set of admissible control sequences \( \{(u_n)_n\} \), where \( u_n \in U \subset \mathbb{R} \). In the following we will always assume that for any given initial distribution \( f_0 \in \mathcal{P}(\mathcal{X}) \) and control \( u = (u_n)_n \), there exists a sequence of sufficiently regular functions \( (f_n)_n \), \( f_n \in \mathcal{P}(\mathcal{X}) \), given by the dynamics described in [24]. This sequence depends on the initial distribution \( f_0 \) and on the choice of the control sequence \( u = (u_n)_n \). We observe how equation
is meaningful provided that $f_n$ are absolute continuous, in the sense of Radon-Nikodym, with sufficiently smooth densities. The assumption is rather strong for the introduced dynamics $G(\cdot, \cdot)$ and it is not trivial that, for a general $f_0$, the distribution $f_n$ for all $n > 0$ stays smooth.

The equation (24) is the discrete time counterpart of a non-local non-linear transport equation, see [19] and the references therein.

**Definition 3.1.** The infinite horizon mean-field cost $J^u_\infty : \mathcal{P}(\mathcal{X}) \to \mathbb{R}_0^+$ is denoted by

$$J^u_\infty(f_0) = \sum_{n=0}^{+\infty} \ell(f_n, u_n), \tag{25}$$

where $\ell : \mathcal{P}(\mathcal{X}) \times U \to \mathbb{R}_0^+$ is the running cost function and where $(f_n)_n, f_n : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$, is the solution to equation (24) with initial distribution $f_0 \in \mathcal{P}(\mathcal{X})$ and given control sequence $u = (u_n)_n$.

**Example 1.** Consider the discrete problem (8). Let the cost functional be given by a discretization of (3) with $\ell$ as in the previous section:

$$\ell(x, u_n) = \frac{1}{2} \left( \frac{1}{M} \sum_{i=1}^{M} x_{i,n} \right)^2 + \frac{\nu}{2} u_n^2$$

for some fixed parameter $\nu > 0$. The function $\ell$ is symmetric in $X_n$. Provided that $x_{i,n}, y_{i,n} \in \mathcal{X}, u_n \in U$, we obtain that $\ell$ is uniformly bounded independently on $M$, i.e. $\|\ell(\cdot, \cdot)\|_{\infty} \leq C_0$. Further, $\ell$ is locally Lipschitz-continuous in $\mathcal{X}_n$ as composition of locally Lipschitz continuous functions. In fact let $x_{i,n}, y_{i,n} \in \mathcal{X}$, then we can compute

$$\left| \frac{1}{2} \left( \frac{1}{M} \sum_{i=1}^{M} x_{i,n} \right)^2 + \frac{\nu}{2} u_n^2 - \frac{1}{2} \left( \frac{1}{M} \sum_{i=1}^{M} y_{i,n} \right)^2 - \frac{\nu}{2} u_n^2 \right| \leq \frac{2C_1}{M} \left| \sum_{i=1}^{M} (x_{i,n} - y_{i,n}) \right|,$$

with $C_1 \geq 0$ the Lipschitz constant. Therefore, $\ell(\cdot, \cdot)$ fulfills as function of $X$ the assumptions of Theorem [A.1] and its mean-field limit exists and is given by

$$\ell(f_n, u_n) = \frac{1}{2} \left( \int_{\mathcal{X}} x f_n(x) dx \right)^2 + \frac{\nu}{2} u_n^2. \tag{26}$$

The previous example shows that the cost functional (3) requires strong symmetry assumptions. This is fulfilled for example if it depends on functions of average quantities of the state of the particles. Under the symmetry assumption we expect to extend the results proposed in [24]. Therefore, we require in the following that the running cost $\ell$ is symmetric with respect to each agent, that the running costs are uniformly bounded and Lipschitz continuous with respect to the distance $d_1$, defined in [A].

Let us now introduce the notion of optimal value-function, in the mean-field setting, and show a first result.

**Definition 3.2.** We denote by $V_\infty : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ the optimal value function of the mean-field control problem (24) associated with the infinite horizon cost $J^u_\infty(f_0)$:

$$V_\infty(f_0) = \inf_{u \in \mathcal{U}} J^u_\infty(f_0). \tag{27}$$
We define the approximate optimal cost $J_N^n : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ with optimization horizon $N$ as

$$J_N^n(f_0) = \sum_{n=0}^{N-1} \ell(f_n, u_n).$$

(28)

The approximate value function $V_N(f_0) : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ in the case of receding horizon strategy is defined by

$$V_N(f_0) = \inf_{u \in \mathcal{U}} J_N^n(f_0, u).$$

(29)

Further we introduce the notion of a feedback law. A feedback law for $M$ agents is a mapping $\mu_M : \mathcal{X}^M \to U$. A symmetric feedback law is a feedback law such that for all $X \in \mathcal{X}^M$ the symmetric group of degree $M$

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & M \\ \sigma(1) & \sigma(2) & \ldots & \sigma(M) \end{pmatrix}.$$  

(30)

As for the running cost $\ell$, we further assume that the feedback law $\mu_M$ is symmetric, uniformly bounded and Lipschitz continuous with respect to $d_1$.

We now establish an estimate of the type (10) in the mean-field case. Note that the result in [34] already covers the case of a cost functional (25) and (24). Therefore, our purpose is to derive the estimate (10) starting from the finite discrete dynamics (23) and in the mean-field limit case $M \to \infty$.

**Proposition 1.** Let us consider a set of $M$ agents which evolve according to the microscopic dynamics (23) with known initial data $(x_i, \mathcal{P}(x_i))_{i=1}^M$. Consider the functions $\ell_M : \mathcal{X}^M \to \mathbb{R}$ and $\tilde{V}_M : \mathcal{X}^M \times U \to \mathbb{R}$ and a symmetric feedback $\mu_M : \mathcal{X}^M \to U$, fulfilling the assertions of Theorem A.1 and Definition 3.2.

Assume furthermore that $\tilde{V}_M$ fulfills for all $X_0 \in \mathcal{X}^M$ the inequality

$$\tilde{V}_M(X_0) \geq \tilde{V}_M \left( (x_{0,i} + \Delta t (g(x_{i,n}, X_{-i,n}) + \mu_M(X_0))_{i=1}^M \right) + \alpha \ell_M(X_0, \mu_M(X_0))$$

(31)

with $\alpha \in (0, 1]$. Then, there exists a function $\tilde{V} : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ as mean-field limit of $\tilde{V}_M$ for $M \to \infty$ such that for all $f \in \mathcal{P}(\mathcal{X})$ we obtain

$$\alpha V_{\infty}(f) \leq \alpha J_N^n(f) \leq \tilde{V}(f).$$

(32)

for $u = (u_n)_n, u_n = \mu(f_n)$, where $\mu$ is the mean-field limit of $(\mu_M)_M$.

**Proof.** Due to the assertion of Theorem A.1 we have $\tilde{V}, \ell : \mathcal{P}(\mathcal{X}) \times U \to \mathbb{R}$ and $\mu : \mathcal{P}(\mathcal{X}) \to U$ exist. Further, we obtain for $f_0 \in \mathcal{P}(\mathcal{X})$ (as limit for $M \to \infty$ of the sequence $(m^n_M)_M$) the corresponding inequality for $\tilde{V}$

$$\tilde{V}(f_0) \geq \tilde{V}(f_0 - \partial_x [\mu(f_0)]) + \alpha \ell(f_0, \mu(f_0)).$$

In fact for all $i = 1, \ldots, M$ and all $M$

$$x_{1,i} = x_{0,i} + \Delta t g(x_{i,n}, X_{-i,n}) + \Delta t \mu_M(X_0),$$

(34)

which corresponds in the mean-field limit to

$$f_i = f_0 - \partial_x [\mu(f_0)] - \partial_x [\mu(f_0)].$$

(35)

The mean-field limit $\tilde{V}$ is obtained as limit of the sequence $V_M : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ where

$$V_M(f) = \inf_{X \in \mathcal{X}^M} \{\tilde{V}(X) + \alpha \omega(d_1(m_X^M, f))\},$$

(36)
see Theorem A.1 We therefore have $V_M(m_X^M) = V_M(X)$ and therefore for all $X_0 \in \mathcal{X}^M$

$$V_M(m_X^M) \geq V_M(m_X^M_0) + \alpha \ell_M(m_X^M_0, \mu_M(m_X^M_0)).$$

Further, $V_M$ has modulus of continuity $\omega$, i.e., $|V_M(f) - V_M(g)| \leq \omega(d_1(f, g))$. Let $f_0 \in \mathcal{P}(\mathcal{X})$ be the limit of $m_X^M_0$ for $M \to \infty$. Note that the limit exists for metric $d_1$ on the probability measures, since $\mathcal{X}$ is compact subset of $\mathbb{R}$ and therefore $m_X^M_0$ has finite 1–Wasserstein distance, i.e., $\int_X |x|dm_X^M_0 < C$ with $C$ independent of $M$ and $X_0$. Due to the dynamics in [34] we have $f_1$ is then the limit of $m_X^M_1$, $X_1$ given by [34]. Since $V_M$ has modulus of continuity $\omega$, we obtain

$$V_M(f_0) \geq V_M(f_1) + \alpha \ell(f_0, \mu(f_0)).$$

Hence, we have

$$\hat{V}(f_0) \geq \hat{V}(f_1) + \alpha \ell(f_0, \mu(f_0)).$$

Define now $u_n = \mu(f_n)$ and consider the solution to [34]. Since $X_0 \in \mathcal{X}^M$ is arbitrary we obtain that (33) holds for all $f_0 \in \mathcal{P}(\mathcal{X})$ and therefore

$$\hat{V}(f_n) \geq \hat{V}(f_{n+1}) + \alpha \ell(f_n, \mu(f_n)).$$

(37)

Summation over $n$ yields

$$\alpha \sum_{n=0}^{K-1} \ell(f_n, u_n) \leq \hat{V}(f_0) - \hat{V}(f_K) \leq \hat{V}(f_0).$$

(38)

Let now $K \to \infty$, then $\hat{V}(f_0)$ is an upper bound for $J_\infty^u = \sum_{n=0}^{\infty} \ell(f_n, u_n)$ and where $u_n = \mu(f_n)$. Since $u_n$ is an admissible control we obtain for all $f_0 \in \mathcal{P}(\mathcal{X})$

$$\alpha V_\infty(f_0) \leq \alpha J_\infty^u(f_0) \leq \hat{V}(f_0).$$

(39)

our assertion as limit of discrete measures.

The previous results hold for any family of functions $\hat{V}_M$ and any symmetric feedback law. The idea is now to establish the inequality in [34] for a general MPC strategy and a family of functions $V_M$ given by the optimal running costs $V_N$ as in Definition 3.2. In order to establish equation [34] for a broad class of running costs $\ell$, the functions $\rho, \beta$ have been introduced in Section 3 in [34]. We recall their definition and assertions in Definition 3.3 below. Under Assumption [1] we prove that $\mu = u_N^{\text{MPC}}$ and $V_N$ fulfill the assertions of Proposition 1. The Assumption [1] is the mean-field analogous to the assumption imposed in [34] Assumption 3.1).

**Definition 3.3.** We say that a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}_\infty$ if

(i) $\rho(0) = 0$,
(ii) $\rho(\cdot)$ is strictly increasing
(iii) $\rho(\cdot)$ is unbounded.

Moreover a continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}_0$, if $\forall r > 0$ we have $\lim_{r \to +\infty} \beta(r, t) = 0$ and for each $t \geq 0$ we either have $\beta(\cdot, t) \in \mathcal{K}_\infty$ or (b) $\beta(\cdot, t) \equiv 0$.

We will denote by $\ell^*(f)$ the minimum of the mean-field running cost $\ell$ and as in [34] we assume it exists

$$\ell^*(f) = \min_{u \in \mathcal{U}} \ell(f, u).$$

(40)
**Assumption 1.** We assume that \( \ell^*(f) \) is well-defined for all \( f \in \mathcal{P}(X) \). Further, for given \( \beta \in KL_0 \) and each \( f_0 \in \mathcal{P}(X) \), there exists a sequence of controls \((u_n)_n, u_n \in \mathcal{U}\) depending only on \( f_0 \) such that for each \( n \) we have
\[
\ell(f_n, u_n) \leq \beta(\ell^*(f_0), n).
\]

In the following Lemma we prove that Assumption [1] is fulfilled provided that the finite-dimensional problem fulfills the corresponding assumption [34, Assumption 3.1]. We establish the proof in the special case of \( \beta \) given by
\[
\beta(r, n) = C\sigma^n r,
\]
where \( C \geq 1 \) is the overshoot constant and \( \sigma \in (0, 1) \) the decay rate. Clearly, the particular choice \( \beta(r, n) \in KL_0 \).

**Lemma 3.4.** Let \( \beta \) be given by equation \[42\]. Consider a dynamics with \( M \) agents given by the dynamics of equation \[29\] with a control sequence \((u_n)_n\) and \( u_n \in \mathcal{U} \) and initial conditions \( X_0 \in X^M \). Assume \( \ell_M : X^M \times \mathcal{U} \to \mathbb{R} \) and \( \ell^*_M : X^M \to \mathbb{R} \) fulfill the assumptions of Proposition [7] for all \( M \). Further, we assume that [34, Assumption 3.1] holds, that is for all \( M \) we have
\[
\ell_M(X_n, u_n) \leq \beta(\ell^*_M(X_0), n).
\]
Then, the mean-field limit \( (\ell_M)_M \) and \( (\ell^*_M)_M \) exist and the limit \( \ell : \mathcal{P}(X) \times \mathcal{U} \to \mathbb{R} \) and \( \ell^* : \mathcal{P}(X) \to \mathbb{R} \), fulfills Assumption [7].
\[
\ell(f_n, u_n) \leq \beta(\ell^*(f_0), n).
\]

**Proof.** Due to the assumptions on the family \((\ell_M)_M\) given in Proposition [1] we have the existence of the mean-field limit \( \ell \) according to Theorem [A.1]. Consider the family of functions
\[
\beta_M(X, n) := \beta(\ell^*_M(X), n).
\]
Clearly, the function \( \beta_M \) is symmetric in \( X \in X^M \). Using the definition of \( \beta \) by equation \[42\] and the properties of \( \ell^*_M \) we have that \( \beta_M(X, n) \) is uniformly bounded with respect to \( X \) on the compact subset \( X^M \) by \( C\sigma^n \|\ell^*_M(X)\| \). For each \( r_1, r_2 \) such that \( |r_1 - r_2| < \delta \) we have
\[
|\beta(r_1, n) - \beta(r_2, n)| \leq C\sigma^n |r_1 - r_2|.
\]
Hence, for \( \epsilon = C\sigma^n \delta \), we have uniform continuity of \( \beta_M \) due to the uniform continuity of \( \ell^*_M \). If \( \omega(\cdot) \) is the modulus of continuity of \( \ell^*_M \), then \( C\sigma^n |\omega(\cdot)| \) is the modulus of continuity of \( \beta_M \). Hence, for each fixed \( n \) there exists the mean-field limit \( \beta \) of \( (\beta_M)_M \). Also, there exists the mean-field limit \( \ell^* \) of \( (\ell^*_M)_M \). Due to the Lipschitz continuity of \( \beta \) we also have that \( \sup_X |\beta(\ell^*_M(X)) - \beta(\ell^*(m^M_X))| \to 0 \) for \((M_k)_k \to \infty\).

Therefore, the mean-field limit \( \beta(f, n) = \beta(\ell^*(f), n) \). Similarly to what we have proven in Proposition [1], it follows that the inequality [43] implies then [44].

**Example 2.** Consider the example of Section 2. The running cost has been given by
\[
\ell(f_n, u_n) = \frac{1}{2} \left( \int_X x f_n(x) dx \right)^2 + \frac{\nu}{2} u_n^2.
\]

The optimal running cost \( \ell^* \) can be computed explicitly and is given by
\[
\ell^*(f_n) = \frac{1}{2} \left( \int_X x f_n(x) dx \right)^2.
\]
From the mean-field dynamics for \( f_n \) are given by (19). Upon integration on \( \mathcal{X} \) we obtain
\[
\int_{\mathcal{X}} x f_{n+1}(x) dx = \int_{\mathcal{X}} x f_n(x) dx + \Delta t \, u_n.
\] (47)

In [2] the following feedback law \( \mu : \mathcal{P}(\mathcal{X}) \to \mathcal{U} \) has been proposed as instantaneous MPC:
\[
\mu(f_n) = \frac{1}{1 + \nu} \int_{\mathcal{X}} x f_n(x) dx.
\] (48)

Using \( \Delta t u_n := \mu(f_n) \) the optimal running cost \( \ell^*(f_n) \) is expressed in terms of the initial cost \( \ell^*(f_0) \) as
\[
\ell^*(f_n, u_n) = \left(1 + \frac{\nu}{(1 + \nu)^2}\right) \left(1 - \frac{1}{1 + \nu}\right)^2 \ell^*(f_0) = C\sigma^n \ell^*(f_0). \tag{50}
\]

Therefore we have
\[
\ell(f_n, u_n) = \left(1 + \frac{\nu}{(1 + \nu)^2}\right) \left(1 - \frac{1}{1 + \nu}\right)^2 \ell^*(f_0) = C\sigma^n \ell^*(f_0). \tag{50}
\]

The overshoot constant \( C \) and the decay rate \( \sigma \) are computed explicitly for a given regularization \( \nu > 0 \) as
\[
C = 1 + \frac{\nu}{(1 + \nu)^2} \geq 1, \quad \sigma = \left(1 - \frac{1}{1 + \nu}\right)^2 \in (0, 1). \tag{51}
\]

Consider the receding horizon costs with length one as \( \tilde{V} : \mathcal{P}(\mathcal{X}) \to \mathbb{R} \) defined as
\[
\tilde{V}(f_0) := \sum_{n=0}^{1} \ell(f_n, \mu(f_n)). \tag{52}
\]

Due to equation (50) we obtain the assertion of Proposition 1 is true by simple computation
\[
\tilde{V}(f_0) \geq \tilde{V}(f_1) + \alpha \ell(f_0, \mu(f_0)) \tag{53}
\]
provided that \( \alpha := 1 - (C\sigma)^2 \) fulfills \( 0 < \alpha \). This yields a bound on the regularization parameter \( \nu \). This estimate for \( \alpha \) is only valid in the case of the feedback law (48).

The idea is to generalize the result to arbitrary symmetric running costs \( \ell \) and different control horizons. In the numerical results we then observe for large values of \( \nu \) also a decay in the receding horizon costs provided the control horizon is sufficiently large.

The following Lemma is the analog to [34, Theorem 4.2]. The main idea is to establish the inequality (31) using Lemma 3.4 for a function \( \tilde{V} \) given by the approximate value function \( \tilde{V}_n,\mathcal{M} \). The discrete approximate optimal cost \( J_u^{\mathcal{N}} : \mathcal{X}^M \times \mathcal{U}^N \to \mathbb{R} \) with running cost \( \ell_M : \mathcal{X}^M \times \mathcal{U} \to \mathbb{R} \) and corresponding approximate value function \( \tilde{V}_n,\mathcal{M} : \mathcal{X}^M \to \mathbb{R} \) are obtained by considering the discrete measure \( m_M^X \) for \( X \in \mathcal{X}^M \) and fixed \( M \):
\[
V_{n,\mathcal{M}}(X_0) := V_n(m_M^X, u_n)_{n=0}^{N-1} := \sum_{n=0}^{N-1} \ell_M(X_n, u_n). \tag{54}
\]

where
\[
\ell_M(X) = \ell(m_M^X, u_n). \tag{55}
\]
Here, \( X_n = (x_{i,n})_{i=0}^M \) fulfills the discrete dynamics \( (34) \) with initial data \( x_i(0) = x_{i,0} \). We assume that the discrete functions fulfill the corresponding relation \( (29) \) for all \( X \in \mathcal{X}^M \):

\[
V_{N,M}(X) = \min_{(u_n) \in \mathcal{U}^N} J^u_{N,M}(X, (u_n)_{n=0}^{N-1}).
\]

The symmetric feedback law \( \mu \) is the MPC feedback introduced on the discrete level by equation \( (6) \) and equation \( (7) \), respectively.

**Lemma 3.5.** Consider the discrete dynamics \( (23) \) with \( M \) agents and \( \beta \) given by equation \( (42) \) with \( C \geq 1 \) and \( \sigma \in (0,1) \). Consider a model predictive control horizon of \( N \). Assume the family \( (\ell_M)_M, \ell_M : \mathcal{X}^M \times \mathcal{U} \to \mathbb{R} \) fulfill the assertions of Proposition \( 4.1 \). Assume assumption \( 1 \) holds true. Let \( V_{N,M}, \ell_M \) and \( J^u_{N,M} \) be given by equation \( (54) \). Given are sequences \( \lambda_n > 0, n = 0, \ldots, N - 1 \) and \( \nu > 0 \) such that

\[
\sum_{n=k}^{N-1} \lambda_n \leq C \lambda_k \frac{1 - \sigma^{N-k}}{1 - \sigma}, \quad k = 0, \ldots, N - 2, \quad (56)
\]

\[
\nu \leq \sum_{n=0}^{j-1} \lambda_{n+1} + C \lambda_{j+1} \frac{1 - \sigma^{N-j}}{1 - \sigma}, \quad j = 0, \ldots, N. \quad (57)
\]

holds true. Assume that then also

\[
\sum_{n=0}^{N-1} \lambda_n - \nu \geq \lambda_0 \alpha, \quad (58)
\]

holds true for some \( \alpha \in (0,1) \). Then, for any \( M \) and any \( X_0 \in \mathcal{X}^M \) and any running cost \( \ell_M \) fulfilling \( (43) \) we obtain \( (31) \) for the MPC feedback law \( \mu_M \) given by \( (61) \) and for the value function

\[\hat{V}_M := V_{N,M}.\]

Provided that \( (\mu_M)_M \) is symmetric and fulfills the assertions of Theorem \( A.1 \) we obtain for each \( f \in \mathcal{P}(\mathcal{X}) \) as limit of \( (m^X_M)_M, M \to \infty \), the inequality

\[
\alpha V_{\infty}(f) \leq \alpha J^u_{\infty}(f) \leq V_N(f) \quad (59)
\]

where \( u = (u_n)_n, u_n = \mu(f_n) \) and where \( \mu \) is the mean-field limit of \( (\mu_M)_M \).

**Sketch of the proof.** The proof is analogous to the proof of \( 34 \) Theorem 4.2. We recall that condition \( (58) \) is equivalent to the assertion \( 34 \) (4.3). For \( \beta \) given by equation \( (42) \) the assertions \( 34 \) (4.1),(4.2) simplify to equation \( (57) \) and \( (56) \), respectively. Consider \( M \) agents with corresponding arbitrary initial condition \( X_0 \in \mathcal{X}^M \). Consider the finite horizon problem of length \( N \) given by

\[
(u^*_n)_{n=0}^{N-1} = \arg\min_{(u_n)_n \in \mathcal{U}^M} J^u_{N,M}(X_0, (u_n)_{n=0}^{N-1}). \quad (60)
\]

Then, we denote the corresponding optimal trajectory \( X^*_n \) obtained through the dynamics \( (23) \) for \( u_n = u^*_n \). We define

\[
\lambda_{n,M} = \ell_M(X^*_n, u^*_n), \quad n = 0, \ldots, N - 1
\]

and

\[
\nu_M = V_{N,M}(X^*_1).
\]

Similarly to \( 34 \) Proposition 4.1 the values \( \lambda_{n,M} \) and \( \nu_M \) defined in the proof above fulfill equation \( (57) \) and equation \( (56) \). This result has been established in the case
of finite number of agents in a sequence of auxiliary aftermaths that are not repeated here. Now, consider the MPC feedback law $\mu_M(X) = v_0$ where

$$(v_0)_{k=0,\ldots,N-1} = \arg \min_{(v_k),v_k \in U} \sum_{n=0}^{N-1} \ell_M(Y_n, v_k)$$

(61)

where $Y_n \in X^M$ solves equation [23] with initial data $Y_0 = X$ and let $(X^\mu_n)_{n}$ be the trajectory obtained through [23] for initial data $X_0$ and for $u_n = \mu(X_n)$. We observe that $u^*_n = \mu(X_0)$ and $X^\mu_i = X^*_i$ for $i = 0$ and $i = 1$. Therefore, $\ell_M(X_0, u^*_0) = \ell_M(X_0, \mu(X_0))$. Therefore, we obtain for all $M$ and any $\alpha$ from equation (58)

$$V_{N,M}(X^\mu_n) + \alpha \ell_M(X_0, \mu(X_0)) = V_{N,M}(X^*_n) + \alpha \ell_M(X_0, u^*_0)$$

$$= \nu_M + \alpha \nu_{0,M} \leq \sum_{n=0}^{N-1} \lambda_{n,M} = \sum_{n=0}^{N-1} \ell_M(X^*_n, u^*_n) = V_{N,M}(X_0).$$

Therefore, $V_{N,M}$ fulfills the assertion on $V_M$ of Proposition [1]. The second assertion follows as a consequence of Proposition [1]. This finishes the outline of the proof.

The assumption on existence of an optimal control (60) for $J_{N,M}$ is also precisely as in the case of finitely many agents. Note that as in the finite dimensional case the optimal control might not exist. The previous result (59) gives a performance bound in the following sense: due to the definition of the approximate value function $V_N(f)$ and $V_\infty(f)$ we have

$$V_N(f) \leq V_\infty(f).$$

Therefore, we obtain the (usable) estimate on the suboptimality of the MPC $\mu$ as

$$J^\mu_\infty(f) \leq \frac{1}{\alpha} V_\infty(f).$$

(62)

This precisely tells the dependence of the MPC cost on the optimal expected cost $V_\infty$ provided that $\alpha$ is known. The value of $\alpha$ is the effective degree of $\mu$ with respect to the (unknown) infinite horizon control. Clearly, the computation of $\alpha$ fulfilling inequality (58) is in general a difficult task requiring estimates on the value function and running costs. However, for $\beta$ given by equation (42) we may estimate $\alpha$ solely based on the inequalities (56) and (57). This estimate is denoted by $\alpha_N$. The corresponding result is independent of the mean-field limit and has been established in [35] Theorem 5.4.

**Lemma 3.6.** Let $\beta$ be given by equation (42) for some $C \geq 1$ and $\sigma \in (0,1)$. Let $N$ be the prediction horizon $N$. Given is a sequence $\lambda_n$ and $\nu > 0$ such that equation (56) and (57) holds true. Assume that

$$\alpha_N = 1 - \frac{(\gamma_N - 1) \prod_{i=2}^{N} (\gamma_i - 1)}{\prod_{i=2}^{N} \gamma_i - \prod_{i=2}^{N} (\gamma_i - 1)} > 0$$

(63)

holds with $\gamma_i = C \sum_{n=0}^{i-1} \sigma^n$. Then, for $\alpha = \alpha_N$ the inequality (58) is fulfilled.

Equation (63) is therefore called performance bound and may be computed a priori to estimate the distance of the optimal cost towards the MPC controlled problem. It solely depends on $C$ and $\sigma$ being the estimates on a the running
cost $\ell$. As already noted in [34] this estimate might give not necessarily optimal performance bounds.

4. **Numerical results.** First, we investigate the performance bound (63). In the example 2 we have the following explicit values for $C$ and $\sigma$:

$$C = 1 + \frac{\nu}{(1 + \nu)^2}, \quad \sigma = \left(1 - \frac{1}{1 + \nu}\right)^2.$$

Estimations on the coefficient $\alpha_N$ allow to measure the quality of the MPC generated control sequence. We depict the value of $\alpha_N$ as a function of $N$ and $\nu$ in Figure 1. The performance bound can only be used if $\alpha_N > 0$ and we indicate the line $\alpha_N = 0$ by a black line. We observe that the performance bound increases with respect to the MPC horizon as expected. The best bound is $\alpha_N = 1/2$. For large values of the regularization parameter $\nu$ we have to consider a sufficiently large MPC horizon $N$ in order to use the theoretical results. Moreover, we observe that the result of Lemma 3.6 is consistent with the estimate derived in the special case of example 2 in the case $N = 2$. The numerical results below indicate that the bound is too pessimistic, similarly to what has been already observed in the finite dimensional case.

As a numerical example we propose the following discretization coherently with what we discussed in Section 3. This discretization reduces the $N$ step MPC problem to again a discrete problem of $M$ agents. We approximate the initial distribution $f_0 \in \mathcal{P}(\mathcal{X})$ by $f_{M,0}$ given by a sum of Dirac delta

$$f_{M,0} = \frac{1}{M} \sum_{i=1}^{M} \delta(x - x_{i,0}).$$

located at points $x_{i,0} \in \mathcal{X}$. Following this approach we recover the microscopic formulation of [34, 35] from which we started in Section 3. The continuous description is approximated in the large particle limit $M \to +\infty$. In particular the Example 2 we observe that if $f_0 = f_{M,0}$, then $f_n$ is also composed of a sum of Dirac delta. We assume in the following that $f_0$ as well as $f_n$ decays to zero at the boundaries.
We observe that if \( \int_{\mathcal{X}} f_0 \, dx = 1 \) then we have \( \int_{\mathcal{X}} f_n \, dx = 1 \). An approach based on Dirac delta converges toward a continuous distribution function in the limit \( M \to +\infty \), provided we have a considerably amount of particles centered in \( x_{i,n} \in \mathcal{X} \). Within the described discretization we also recover the setting of [34] as numerical scheme.

Thanks to the structure of example 2 further simplifications can be obtained. We recall the mean-field running cost \( \ell(f_n,u_n) = \frac{1}{2} \left( \int_{\mathcal{X}} f_n(x) \, dx \right)^2 + \frac{\nu}{2} u_n^2 \). We consider the mean-field equation equivalent to the discretized dynamics of (1) for \( P = 1 \) and initial data \( Y_0 \).

For a fixed time horizon \( T \) and initial data \( Y_0 \) the previous dynamics. We computed for a horizon \( N \) the MPC control at time \( n \) and integrating with respect \( dx \) we obtain

\[
\int_{\mathcal{X}} x f_{n+1}(x) \, dx = \int_{\mathcal{X}} x f_n(x) \, dx + u_n.
\]

(66)

If we introduce a new variable for the mean \( Y_n := \int_{\mathcal{X}} x f^n(x) \, dx \) the problem simplifies to the equation for the evolution of \( Y_n \). Further, the cost function is also expressed in terms of \( Y_n \) as \( \ell(Y_n, u_n) = \frac{1}{2} Y_n^2 + \frac{\nu}{2} u_n^2 \), and equation (66) is solved explicitly for \( Y_{n+1} = Y_n + u_n \).

Using the reformulation of the control of the mean the problem therefore reduces to a problem appearing in the existing theory [34]. In particular, the MPC subproblem to determine the optimal control for the horizon \( N \) is solved explicitly for the previous dynamics. We computed for a horizon \( N \) the MPC control at time \( n \) and initial data \( Y_0 \) as \( (u_{nMPC}^N)_n(Y_0) = v_1 \), where

\[
(v_j)_j := \arg \min_{\sum_{j=n}^{n+N}} \ell(Y_j, u_j), \quad Y_{j+1} = Y_j + u_j, \quad Y_n = Y_0.
\]

For a fixed time horizon \( T = 100 \), fixed initial datum \( Y_0 \) and fixed \( N \) we then compute the value of the cost functional for

\[
J_T^{MPC} = \sum_{n=0}^{T} \ell(Y_j, (u_{nMPC}^N)_n)
\]

where \( Y_{j+1} = Y_j + (u_{nMPC}^N)_n(Y_n) \). Further, we compute \( J_{100}^{MPC} \) to obtain the optimal cost \( V_{100}^* \).

According to Lemma 3.6 we obtain the behavior of the MPC cost \( J_T^{MPC} \) in relation to the optimal cost \( V_{100}^* \) in Figure 2. As expected for larger MPC horizons we observe convergence towards the optimal cost. The performance bound \( \alpha_N \) is negative for \( N \leq 4 \) and therefore the Theorem 3.5 cannot be applied. In the results we choose \( \nu = 10^2 \). We observe that the bound on \( \alpha_N \) is quite pessimistic and the distance of the estimated mismatch of the MPC controlled case to the optimal one is quite large for small horizons, i.e., of order \( 10^3 \) for the horizon \( N = 5 \).

We further investigate the behavior of the particle system 65 for controls with different MPC horizon. According to the behavior of the cost we expect that for increasing time horizon we are closer to the optimal cost. Defining

\[
E_n := \int_{\mathcal{R}} x^2 f(x) \, dx.
\]
we obtain from equation (65)

\[ E_{n+1} = -E_n + 2Y_n^2 + u_nY_n. \]

The running cost tries to minimizes a trade‐off of the mean of the distribution and the control action. If the mean \( Y_n \) tends to zero, then we observe that the energy \( E_n \) tends to zero exponentially fast. Therefore, we expect with longer time horizon a mean \( Y_n \) closer to zero and small variance of the solution to the kinetic equation.

We simulate using \( M = 10^5 \) discrete points randomly distributed on \( \mathcal{X} = [-1, 1] \) as initial condition \( f_{M,0} \) as in equation (64). The MPC control is computed according to the considerations above for \( \nu = 10^2 \) and \( \nu = 10^3 \) reported in Figure 3 and Figure 4. In both figures we show the computational results for the time evolution of the distribution \( f_n \) for \( n = 0, \ldots, 100 \). As expected longer optimization horizons leads to a faster decay in the variance of the distribution \( f_n \).

5. Conclusion. We have extended the estimates for the suboptimal MPC to the mean-field limit. The derived estimates yield performance bounds for general symmetric multi–agent dynamics. Except for the assumptions necessary to obtain the mean-field limit no additional requirements compared to the finite–dimensional theory are required. The results apply to common agent dynamics modeling for example swarming, alignment and economics. We exemplified the theoretical results as well as the estimates on a simple opinion formation model. The stability of the mean-field controller is still open and will be investigated in a forthcoming work. Further, the estimates on \( \alpha_N \) are pessimistic due to its generality. It is expected
that the bounds can be improved for specific problems as in the finite dimensional case.

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Appendix A. We collect some results of [16] for convenience; see also [10] Theorem 4.1. The Kantorowich–Rubenstein distance $d_1(\mu, \nu)$ for measures $\mu, \nu \in \mathcal{P}(Q)$ is defined as

$$d_1(\mu, \nu) := \sup \left\{ \int \phi \, d(\mu - \nu); \phi : Q \to \mathbb{R}, \phi \text{ is } 1 \text{- Lipschitz } \right\}.$$  \hspace{1cm} (67)

Theorem A.1 (Theorem 2.1[16]). Let $Q^M$ be a compact subset of $\mathbb{R}^M$. Consider a sequence of functions $(u_M)_{M=1}^\infty$ with $u_M : Q^M \to \mathbb{R}$. Assume each $u_M(X) = u_M(x_1, \ldots, x_M)$ is a symmetric function in all variables, i.e.,

$$u_M(X) = u_M(x_{\sigma(1)}, \ldots, x_{\sigma(M)})$$

for any permutation $\sigma$ on $\{1, \ldots, M\}$. Let $d_1$ be the Kantorowich–Rubenstein defined in (67) and let $\omega$ be a modulus of continuity independent of $M$. Assume that the
sequence is uniformly bounded $\|u_M\|_{L^\infty(Q^M)} \leq C$. Further assume that for all $X, Y \in Q^M$ and all $M$ we have

$$|u_M(X) - u_M(Y)| \leq \omega(d_1(m_X^M, m_Y^M))$$

where $m_k^M \in P(Q)$ is defined by $m_k^M(x) = \frac{1}{M} \sum_{i=1}^{M} \delta(x - \xi_i)$.

Then there exists a subsequence $(u_{M_k})_k$ of $(u_M)_M$ and a continuous map $U : P(Q) \rightarrow \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} \sup_{X \in \mathbb{R}^M} |u_{M_k}(X) - U(m_{M_k}^X)| = 0. \quad (68)$$

Theorem A.1 has been extended to the case of functions $g(x_i, X_{-i}) : \mathcal{X}^M \subset \mathbb{R}^M \rightarrow \mathbb{R}$ being symmetric only in $X_{-i}$. Here, $\mathcal{X}$ is a compact subset of $\mathbb{R}$. The corresponding result is given in [10] Section 4] and repeated here for convenience. For any permutation $\sigma$ of the set $\{1, \ldots, M\} \setminus \{i\}$ and all $x_i \in \mathbb{R}$ we have $g(x_i, X_{-i}) = g(x_i, (x_{\sigma(j)})_{j \neq i})$. Moreover, there exists a modulus of continuity $\omega$ such that for all $x_i, y_i \in \mathbb{R}$ and all $M$ we have

$$\|g(x_i, X_{-i}) - g(y_i, Y_{-i})\| \leq \omega(\|x_i - y_i\|) + \omega(d_1(m_{X_{-i}}^{M-1}, m_{Y_{-i}}^{M-1})).$$
Further assume that $\|g(X)\|_{L^\infty(R^M)} \leq C$. Then, $g(x_i, X_{-i}) : R^M \rightarrow R$ can be extended to a function $G_M : X \times P(X) \rightarrow R$ by

$$G_M(x, \nu) = \inf_{X_{-i} \in R^M-1} \{g(x, X_{-i}) + \omega(d_1(M^{-1}, \nu))\}. \quad (69)$$

It can be shown as before that $(G_M)_M$ is a sequence of uniformly equi-continuous functions on $X \times P(X)$. Therefore, $(G_M)_M$ converges to a function $G : X \times P(X)$, see also [10] Theorem 4.1.

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