Yet another $GL_2$ subconvexity result

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Abstract

In this paper we establish a very flexible and explicit Voronoï summation formula. This is then used to prove an almost Weyl strength sub-convexity result for automorphic $L$-functions of degree two in the depth aspect. That is, looking at twists by characters of prime power conductor. This is the natural $p$-adic analogue to the well studied $t$-aspect.

Contents

1 Introduction 1
    1.1 Notation and prerequisites . . . . . . . . . . . . . . . . . . . . 3

2 A Voronoi summation formula 7
    2.1 Setting up the left hand side . . . . . . . . . . . . . . . . . . 3
    2.2 Computing the right hand side . . . . . . . . . . . . . . . . . 10
        2.2.1 The unramified places $p \nmid lN$ . . . . . . . . . . . . 10
        2.2.2 The ramified non-archimedean places $p \mid N, p \neq l$ . 11
        2.2.3 The special place $p = l$ . . . . . . . . . . . . . . . . . 12
        2.2.4 The archimedean places . . . . . . . . . . . . . . . . . 15
        2.2.5 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . 15

3 Application to the subconvexity problem 17
    3.1 $p$-Adic Farey dissection . . . . . . . . . . . . . . . . . . . . 17
    3.2 Applying the Voronoi formula . . . . . . . . . . . . . . . . . . 19

4 Extracting cancellation on average 21

A Tables for $c_{t,l}(\mu)$ 24
    A.1 Supercuspidal representations . . . . . . . . . . . . . . . . 24
    A.2 Twists of Steinberg . . . . . . . . . . . . . . . . . . . . . . 24
    A.3 Irreducible principal series . . . . . . . . . . . . . . . . . . 25

1 Introduction

This paper adds another result to the vast family of subconvex bounds for $L$-functions. However, we not only generalize a quite recent subconvexity result for degree two $L$-functions, we also work out a very versatile version of the Voronoi summation formula which hopefully has other applications in the future. Before
we state our results we will give a brief introduction to the subconvexity problem for automorphic \( L \)-functions.

Let \( L(s) \) be an \( L \)-function in the sense of \([9]\) and let \( C(s) \) be its analytic conductor. Then the Phagrmén-Lindelöf principle implies the bound

\[
L\left(\frac{1}{2} + it\right) \ll \epsilon C\left(\frac{1}{2} + it\right)^{\frac{1}{4} + \epsilon}.
\]  

(1.1)

Due to the nature of the Phagrmén-Lindelöf principle this bound is commonly referred to as convexity bound. The subconvexity problem for \( L(s) \), in its most general form, is the problem of improving upon (1.1) in the exponent. The best possible bound one may hope for is

\[
L\left(\frac{1}{2} + it\right) \ll \epsilon C\left(\frac{1}{2} + it\right)^{\epsilon}.
\]

This is known as the Lindelöf conjecture and is a corollary of the Riemann Hypothesis for \( L(s) \).

While there are very little results towards the subconvexity problem for general \( L \)-functions, there is a huge amount of literature dealing with special cases and special families. For example, the subconvexity problem for automorphic \( L \)-functions of \( GL_2 \) over number fields has been solved completely, with non-specific exponent, in the groundbreaking work \([11]\). On the other hand, it has become a big business to obtain best possible numerical values for the exponent. Establishing strong subconvex bounds in a single aspect of the analytic conductor or for special automorphic \( L \)-functions has become a benchmark for the tools in use. Examples for such developments are the following. In \([5]\) the bound

\[
\zeta\left(\frac{1}{2} + it\right) \ll \epsilon \left(1 + |t|\right)^{\frac{1}{13}p} + \epsilon
\]

demonstrates the strength of the decoupling method. This might be thought of as the \( t \)-aspect (or archimedean aspect) of the subconvexity problem for a very special \( L \)-function. A possible \( p \)-adic version of this has been considered in \([12]\). There it has been shown that

\[
L\left(\frac{1}{2}, \chi\right) \ll \epsilon, p q^{0.1645 + \epsilon}
\]

ger for a Dirichlet character \( \chi \) of level \( q = p^n \). This has been achieved by introducing an elaborate treatment of \( p \)-adic exponential pairs. The two bounds discussed so far are numerically very strong but work only for a very limited family of degree one \( L \)-functions. One out of many results concerning \( L \)-functions of \( GL_2 \) is

\[
L\left(\frac{1}{2} + it, f\right) \ll f, \epsilon \left(1 + |t|\right)^{\frac{1}{3} + \epsilon}
\]

for a holomorphic modular form \( f \) of full level. This is initially due too Good \([8]\). Another proof was later supplied by Jutila \([10]\). Recently the family to which this bound applies was enlarged by \([4]\). Indeed, the authors, relax the assumption on \( f \) in the sense that they allow arbitrary level and central character. The \( p \)-adic analogue of this problem was considered by Blomer and Miličević in \([3]\). They show that

\[
L\left(\frac{1}{2} + it, \chi \otimes f\right) \ll f, \epsilon \left(1 + |t|\right)^{\frac{1}{2}p^2 q^{\frac{1}{2}}},
\]
where \( f \) is a holomorphic or Maaß cuspidal newform of full level and \( \chi \) is a Dirichlet character modulo \( q = p^n \) for \( p > 2 \). Our contribution to the subconvexity problem, similarly to the one in [4], is to widen the family for which the above estimate holds. We will show the following.

**Theorem 1.1.** Let \( f \) be a cuspidal holomorphic or Maaß newform of level \( N \) and central character \( \omega \). Furthermore, let \( \chi \) be a Dirichlet character of conductor \( l^n \) for some prime \( l \) satisfying \((l, 2N) = 1\) and \( n \geq 5\). Then

\[
L\left(\frac{1}{2} + it, \chi \otimes f\right) \ll f, \epsilon \left(1 + |t|\right)^\frac{5}{2} l^\frac{7}{6} + \left(\frac{1}{3} + \epsilon\right)n^\frac{1}{2}.
\]

As in [3] this result will follow from a more general estimate for smooth sums of Hecke eigenvalues of automorphic forms. We will now state this result and refer to Subsection 1.1 below for notation that was not yet introduced.

**Theorem 1.2.** Let \( l \) be an odd prime, \( n_l \geq 10 \) even, and \( \pi \) be a cuspidal automorphic representation of conductor \( N_l n_l \) such that the \( l \)th component \( \pi_l \) of \( \pi \) is isomorphic to \( \chi_l \, |·|_{\kappa_l} \rtimes \chi_l \, |·|_{\kappa_l} \) for some character \( \chi_l : \mathbb{Q}_l \to \mathbb{C}^\times \) of conductor \( \frac{n_l}{2} \). Further, let \( W \) be a smooth function with support in \([1, 2]\) that satisfies \( W(j) \ll Z j^2 \) for some \( Z \geq 1 \). Then

\[
L := \sum_{n \in \mathbb{Z}} \lambda_{\pi}(m) F\left(\frac{n}{M}\right) \ll_{\pi, \epsilon} Z^\frac{5}{2} M^\frac{7}{6} (1 + \epsilon)n_l,
\]

for all \( M \geq 1 \) and all \( \epsilon > 0 \).

We will prove this in Section 3 below, following exactly the same strategy as in [3]. The novelty, which makes our generalization work, is a new version of the Voronoi summation formula. Such formulae play an important role in modern number theory, see [13] for a very nice introduction. Our approach to Voronoi summation is based on ideas outlined in [16]. The result is a very technical formula stated in Theorem 2.1 below. The upshot is that we do not need any coprimality conditions between the denominator of the additive twist and the level of the automorphic form. A similar summation formula, with a different proof, has been used in [4].

There are several natural generalizations of Theorem 1.2 that come to mind. Indeed, with a bit more work one should be able to relax the prescribed shape at the place \( l \). Indeed it seems possible to deal with \( \pi_l = \chi \pi_0 \) for some fixed twist-minimal representation \( \pi_0 \) of \( GL_2(\mathbb{Q}_l) \) and some non-trivial character \( \chi \).

Another interesting aspect would be to optimize the \( N \) dependence in Theorem 1.2. In our estimates we have been very wasteful in that aspect and thus included it in the absolute constant. However, one might be able to use explicit evaluations of ramified Whittaker new-vectors in order to get the \( N \)-dependence into a reasonable range.

Finally, it is clear how to adapt our approach to the Voronoi summation formula to the number field setting. It would certainly be interesting to see if it is possible to work out a version of Theorem 1.2 over number fields.

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1.1 Notation and prerequisites

Throughout this paper we will only consider the base field \( \mathbb{Q} \). Its places, including the archimedean place \( \infty \), are usually denoted by \( p \). Each place \( p \) comes with the local field \( \mathbb{Q}_p \). For \( p < \infty \) these fields are non-archimedean and we denote their ring of integers by \( \mathbb{Z}_p \), and the unique maximal ideal by \( \mathfrak{p} \). We choose uniformizers \( \varpi_p = p \) and normalize the absolute value by \( |\varpi_p|^p = p^{-1} \).

Further, we equip the local fields \( \mathbb{Q}_p \) with two measures. First of all, we consider the Haar measure \( \mu_p \) on \( (\mathbb{Q}_p, +) \). If \( p < \infty \), these measures will be normalized such that \( \mu_p(\mathbb{Z}_p) = 1 \). On \( \mathbb{Q}_\infty = \mathbb{R} \) we take \( \mu \) to be the standard Lebesgue measure. The second measure is the Haar measure \( \mu_p^\times \) on \( (\mathbb{Q}_p^\times, \times) \). If \( p < \infty \), it is explicitly given by \( \mu_p^\times = \zeta_p(1) \mu_p \), where \( \zeta_p(s) = (1 - p^{-s})^{-1} \) denotes the local Euler factor of the Riemann zeta function. In particular, one has \( \mu_p^\times(\mathbb{Z}_p^\times) = 1 \).

At the archimedean place \( \infty \) we simply choose \( \mu_\infty = \frac{1}{2\pi} \). The adele Ring (resp. idele Ring) over \( \mathbb{Q} \) will be denoted by \( \mathbb{A} \) (resp. \( \mathbb{A}^\times \)) and is equipped with the product measure \( \mu \) (resp. \( \mu^\times \)).

We fix additive characters \( \psi_p \) on \( \mathbb{Q}_p \) such that the global additive character \( \psi = \otimes_p \psi_p \) is \( \mathbb{Q} \)-invariant. Furthermore, at \( p = \infty \) we take \( \psi_\infty(x) = e(x) = e^{2\pi i x} \) and we assume that \( \psi_p \) is trivial on \( \mathbb{Z}_p \) but non-trivial on \( p^{-1} \) for every \( p < \infty \). For a Schwartz-Bruhat function \( f \in \mathcal{S}(\mathbb{Q}_p) \) we define the \( p \)-adic Fourier transform by

\[
\hat{f}(y) = c_p \int_{\mathbb{Q}_p} f(x) \psi_p(xy) d\mu(x), \quad c_p = \begin{cases} 1 & \text{if } p < \infty, \\ \frac{1}{\sqrt{2\pi}} & \text{if } p = \infty. \end{cases}
\]

Note that our measures are normalized to be self-dual with respect to \( \psi_p \).

The set \( p\mathcal{X} \) denotes the set of all multiplicative characters \( \mu: \mathbb{Q}_p^\times \to \mathbb{S}^1 \) such that \( \mu(\varpi_p) = 1 \). If \( p < \infty \), we also write \( p\mathcal{X}_n \) (resp. \( p\mathcal{X}_n^\times \)) for the set of characters \( \mu \in p\mathcal{X} \) with exponent-conductor \( a(\mu) \leq n \) (resp \( a(\mu) = n \)). Note that \( X_\infty = \{ 1, \mathrm{sgn} \} \). Furthermore, every quasi-character \( \mu: \mathbb{Q}_p \to \mathbb{C}^\times \) can be decomposed as \( \mu = |t|^q \mu_0 \) for some \( t \in \mathbb{C} \) and some \( \mu_0 \in p\mathcal{X} \). A global homomorphism \( \chi: \mathbb{Q}_\infty^\times \times \mathbb{A}_\infty^\times \to \mathbb{C}^\times \) will be called a Hecke character. Note that each \( \mu \in \mathcal{X} \) induces a Hecke character \( \chi_\mu \) defined by \( \chi_\mu = \prod_p \chi_{\mu,p} \) with \( \chi_{\mu,\infty} = \mathrm{sgn} \frac{\mathrm{exp}(\log_p(1 + \varpi_p x))}{\varpi_p} \) and

\[
\chi_{\mu,p}(ap^k) = \begin{cases} \mu(a) & \text{if } p = l, \\ \mu(p)^{-k} & \text{if } p \neq l. \end{cases}
\]

for \( p < \infty \), \( a \in \mathbb{Z}_p^\times \).

A very useful tool is the \( p \)-adic logarithm \( \log_p \), which can be defined on the set \( 1 + p \subset \mathbb{Z}_p \) via the well known Taylor series of the logarithm. As in the archimedean setting the \( p \)-adic logarithm is useful in order to translate between multiplicative and additive oscillations. Indeed, for \( \mu_p \in p\mathcal{X}_n \), \( \kappa > 0 \) and \( x \in \mathbb{Z}_p \) we have

\[
\mu_p(1 + \varpi_p^\kappa x) = \psi_p\left( \frac{\alpha_{\mu_p}}{\varpi_p^\kappa} \log_p(1 + \varpi_p^\kappa x) \right) \tag{1.2}
\]

for some \( \alpha_{\mu_p} \in \mathbb{Z}_p^\times \). In particular, if \( \kappa \geq \frac{n}{2} \), one can safely truncate the logarithm.
after the first term and obtain

$$\mu_p(1 + \varpi_p^n x) = \psi_p \left( \frac{\alpha \mu_p x}{\varpi_p^n} \right).$$

Finally, it will be useful to have a shorthand notation to deal with several places at once. For every $M \in \mathbb{N}$ we define

$$\zeta_M(s) = \prod_{p | M} \zeta_p(s)$$

and

$$|\cdot|_M = \prod_{p | M} |\cdot|_p.$$

We also write $\mu$ for a $M$-tuple of characters $\mu_p \in \mathfrak{X}$. Since we can always complete the tuple to all $p$ by inserting the trivial character at the remaining places, we dropped $M$ from the notation. One evaluates these tuples as as follows:

$$\mu(x) = \prod_{p \leq \infty} \mu_p(x_p) = \prod_{p | M} \mu_p(x_p).$$

It is important not to confuse these tuples with Hecke characters. However, we can define the associated Hecke character

$$\chi_\mu = \prod_{p \leq \infty} \chi_{\mu_p}.$$

Let $R$ be a commutative ring with 1. In our case $R$ will be either $\mathbb{Q}$, $\mathbb{Q}_p$, or $\mathbb{A}$. We set $G(R) = GL_2(R)$ and define the subgroups

$$Z(R) = \left\{ z(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} : r \in R^\times \right\}, \quad A(R) = \left\{ a(r) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} : r \in R^\times \right\},$$

$$N(R) = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}$$

and $B(R) = Z(R)A(R)N(R)$.

We use the following compact subgroups of $G(R)$ which depend on the underlying ring $R$. Define

$$K_p = GL_2(\mathbb{Z}_p) \text{ for } p < \infty,$$

$$K_\infty = O_2(\mathbb{R}),$$

$$K = \prod_{p \leq \infty} K_p \subset G(\mathbb{A}).$$

At the non-archimedean places, $p < \infty$, we also need the congruence subgroups

$$K_{1,p}(n) = K_p \cap \left[ 1 + \varpi_p^n \mathbb{Z}_p \right].$$

Finally we denote the long Weyl element by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
on the local fields to the corresponding groups. Further, we take \( \mu_K \) to be the probability Haar measure on \( K_p \). Globally, we choose the product measure on \( K, N(A) \) and \( A(\mathbb{A}) \) coming from the previously defined local measures. The measure on \( G(\mathbb{A}) \), in Iwasawa coordinates, is given by

\[
\int_{\mathbb{Z}(\mathbb{A}) \backslash G(\mathbb{A})} f(g) d\mu(g) = \int_K \int_{A(\mathbb{A})} \int_{N(\mathbb{A})} f(na(y)k) d\mu_N(\mathbb{A})(n) \frac{d\mu_y^\times(\mathbb{A})}{|y|_\mathbb{A}} d\mu_K(k).
\]

In this work \( \pi \) will usually denote a cuspidal automorphic representation of \( G(\mathbb{A}) \) with central (Hecke) character \( \omega_\pi \). That is an irreducible constitute of the right regular representation on \( L^2_{\text{cusp}}(G(\mathbb{Q})/G(\mathbb{A}), \omega_\pi) \). It is well known that we can factor

\[
\pi = \bigotimes_{p \leq \infty} \pi_p,
\]

where \( \pi_p \) are irreducible, admissible, unitary representations of \( G(\mathbb{Q}_p) \). These local representations come with several invariants. For example, the log-conductor \( n_p = a(\pi_p) \) and the local central character \( \omega_{\pi,p} \). The contragredient representation will be denoted by \( \tilde{\pi}_p \). Note that \( \tilde{\pi}_p = \omega_{\pi,p}^{-1} \pi_p \). Attached to \( \pi_p \) there are the usual suspects \( \epsilon(\frac{1}{2}, \pi_p) \) and \( L(s, \pi_p) \). The representations of \( G \) over local fields are completely classified. More precisely, we know that each unitary irreducible admissible infinite dimensional representation \( \pi \) of \( G(\mathbb{A}) \) belongs to one of the following families.

1. **Twists of Steinberg**: \( \pi_p = \chi St, \) for some unitary character \( \chi \). In this case we have \( \omega_{\pi,p} = \chi^2 \) and \( a(\pi_p) = \max(1, 2a(\chi)) \). Furthermore, the \( L \)-factor as well as the \( \epsilon \)-factor are given by

\[
L(s, \pi_p) = \begin{cases} L(s, |\chi|^2) & \text{if } \chi = 1, \\ 1 & \text{if } \chi \neq 1, \end{cases} \quad \epsilon(\frac{1}{2}, \pi_p) = \begin{cases} -1 & \text{if } \chi = 1, \\ \epsilon(\frac{1}{2}, \chi^2) & \text{if } \chi \neq 1. \end{cases}
\]

2. **Principal series**: \( \pi_p = \chi_1 \boxplus \chi_2 \), for characters \( \chi_1 \) and \( \chi_2 \). In particular, \( a(\pi) = a(\chi_1) + a(\chi_2) \) and \( \omega_{\pi,p} = \chi_1 \chi_1 \). Concerning the \( L \)-factor we know

\[
L(s, \pi_p) = L(s, \chi_1)L(s, \chi_2) \quad \text{and} \quad \epsilon(\frac{1}{2}, \pi_p) = \epsilon(\frac{1}{2}, \chi_1)\epsilon(\frac{1}{2}, \chi_2).
\]

3. **Supercuspidal representations**: If \( \pi_p \) is supercuspidal then \( L(s, \pi_p) = 1 \). The other invariants are slightly more difficult to describe. Since it is not necessary for this work we will not go into further detail.

This list can be extracted from [7] and [15]. Note that the characters \( \chi_1, \chi_2 \) appearing in unitary principal series representations are usually unitary themselves. However, if \( \chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times} \) one might encounter situations where \( |\chi_1(\varpi_p)| \neq 1 \). In this case one is dealing with \( p \)-adic complementary series. Unfortunately we can not exclude these representations from our discussion as the Ramanujan conjecture for \( G(\mathbb{A}) \) is not yet known in full generality.

To any automorphic representation \( \pi \) we attach its (incomplete)-\( L \)-function

\[
L(s, \pi) = \prod_{p < \infty} L(s, \pi_p) = \sum_{n \in \mathbb{N}} \lambda_\pi(n) n^{-s} \text{ for } \Re(s) \gg 1.
\]
This function has a meromorphic continuation and satisfies the functional equation
\[ L(s, \pi_\infty)L(s, \pi) = \left( \prod_{p \leq \infty} \epsilon(s, \pi_p) \right) L(1 - s, \tilde{\pi}_\infty)L(1 - s, \tilde{\pi}). \]
The conductor of \( \pi \) is given by \( \prod_{p < \infty} p^{\rho(\pi_p)} \). This is not to be confused with the analytic conductor of \( \pi \) mentioned in the introduction.

It is well known that in our case each \( \pi \) is generic. Thus, there exists a (unique) \( \psi \)-Whittaker model \( W(\pi) \). This allows us, after fixing a suitable normalization, to associate to each \( \phi \) in the representation space of \( \pi \) a Whittaker function \( W_\phi \in W(\pi, \psi) \). If \( \phi \in L^2_{\text{cusp}}(G(\mathbb{Q})/G(\mathbb{A}), \omega_\pi) \) is a cuspidal function transforming according to \( \pi \) the associated Whittaker function is given by the well known Jacquet-integral.

The twist \( \chi_{\pi} \) of an automorphic representation \( \pi \) by a Hecke character \( \chi \) is also an automorphic representation. It has central character \( \chi^2 \omega_\pi \) and its local constituents are given by \( \chi_p \pi_p \).

At last, we introduce two more notions. First, by \( \pi^b \) we denote the automorphic representation obtained from \( \pi \) by passing (essentially) to the contragredient at the places \( p \mid b \). More precisely,
\[ \pi^b = \left( \prod_{p \mid b} \chi_{\pi_p}^{-1} \right) \pi. \]
Second we define
\[ (\pi)_{\mu} = \chi_{\mu} \pi. \]
These constructions may seem quite artificial. However, they will prove useful later on. Even more, the first construction is closely related to the theory of Atkin-Lehner involutions for classical newforms.

## 2 A Voronoi summation formula

The goal of this section is to turn the machinery of automorphic representations to produce a very flexible Voronoi-type formula. In particular we want to produce a summation formula which relates a smoothed sum of Hecke eigenvalues to a dual sum which involves Hecke eigenvalues of twisted automorphic forms. To this end let \( \pi \) be a cuspidal automorphic representation with conductor \( N_l \) and central character \( \omega_\pi \). The \( L \)-function of the associated contragredient representation is given by
\[ L(s, \tilde{\pi}) = \sum_{n \in \mathbb{N}} \lambda_\pi(n) n^{-s}. \]
Our summation formula will feature the following ingredients. The main objects of interest are the Hecke eigenvalues \( \lambda_\pi(n) \). Furthermore, we will allow additive twists \( \psi_\infty(\zeta_0 m) \) for \( \zeta_0 \in \mathbb{Q} \) satisfying \( v_l(\zeta_0) > 0 \). Finally, we fix smooth, compactly supported test functions \( W_\infty: \mathbb{R} \to \mathbb{C} \) and \( W_l: \mathbb{Q}_l \to \mathbb{C} \).

In the following we will build on the ideas described in [16] to derive an explicit Voronoi summation formula which is well suited for our application to the subconvexity problem. On the way we will use results from [4] to treat the
places dividing $N$. Our method to implement the $l$-adic test function $W_l$ owes a great deal to the work [6].

The main theorems of this section are stated at its very end. The reason for this is, that one should view this chapter as a recipe for generating explicit Voronoi formulae. We start of with the following fundamental identity.

Lemma 2.1. Let $\zeta \in A$ and let $\phi$ be a cuspidal function transforming according to $\tilde{\pi}$. Then we have

$$\sum_{\gamma \in Q^*} \psi(\gamma \zeta) W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} \tilde{W}_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \right).$$

(2.1)

for $\tilde{W}_\phi(g) = W_\phi(w_1g^{-1})$.

This is essentially [16, Theorem 3.1].

Proof. We start by writing down the Whittaker expansion for $\phi$ with respect to $\psi$:

$$\phi \left( \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} W_\phi \left( \begin{pmatrix} \gamma & \gamma \zeta \\ 0 & 1 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} W_\phi \left( \begin{pmatrix} 1 & \gamma \zeta \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} \psi(\gamma \zeta) W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Then we observe that

$$\phi \left( \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \right) = [\iota \phi] \left( \begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \right).$$

where $\iota \phi(g) = \phi(tg^{-1})$.

We finish the proof by writing down the Whittaker expansion of $\iota \phi$ with respect to $\psi$:

$$[\iota \phi] \left( \begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} W_{\iota \phi} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \right) = \sum_{\gamma \in Q^*} \tilde{W}_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \right).$$

It is an easy calculation to check $\tilde{W}_\phi = W_{\iota \phi}$. Indeed,

$$\tilde{W}_\phi(g) = W_{\iota \phi}(w_1g^{-1}) = \int_{N(Q)\backslash N(A)} \phi(ww_1g^{-1})\overline{\psi(n)}dn = \int_{N(Q)\backslash N(A)} \phi(ww_1^{-1}g^{-1})\overline{\psi(n)}dn = \int_{N(Q)\backslash N(A)} \phi(\iota (ng)^{-1})\overline{\psi(n)}dn = W_{\iota \phi}(g).$$

We will now proceed by choosing $\zeta$ and $\phi$ such that the left hand side takes the desired shape. In our case this choice is motivated by our application to the subconvexity problem. The next step will be to compute the right hand side as explicit as possible.
2.1 Setting up the left hand side

We choose $\phi$ such that

$$W_\phi = \prod_{p \leq \infty} W_{\phi,p} \quad (2.2)$$

is a pure tensor. Thus, we can treat each place on its own.

Since the Kirillov model of $\tilde{\pi}_\infty$ contains the space of Schwartz functions, we can choose

$$W_{\phi,\infty} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) = \omega_{\tilde{\pi},\infty}(\gamma) \mid_\mathbb{F} W_\infty(\gamma).$$

for $W_\infty \in C_c^\infty(\mathbb{R}_+)$.

At all the finite places $p | lN$ we choose $\phi$ such that $W_{\phi,p}$ is the spherical $\psi_p$-Whittaker new-vector $W_{\tilde{\pi},p}$ of $\tilde{\pi}_p$ normalized such that $W_{\tilde{\pi},p}(1) = 1$. Indeed,

$$W_{\phi,p}(a(\gamma)) = W_{\tilde{\pi},p}(a(\gamma)) = \begin{cases} 
\lambda_p((\gamma, p^\infty)) \mid_\mathbb{F} & \text{if } v_p(\gamma) \geq 0, \\
0 & \text{else}, 
\end{cases}$$

If $p \neq l$ divides the level $N$, we will consider three cases. Recall from [14, Lemma 2.5] that

$$W_{\tilde{\pi},p}(a(\xi^k)) = \begin{cases} 
\xi (a(\xi)^k p^{-k}) & \text{if } k \geq 0 \text{ and } \tilde{\pi}_p = \xi \otimes St \text{ with } a(\xi) = 0, \\
\omega_{\tilde{\pi},p}(v) \chi_1(a(\xi^k) p^{-k}) & \text{if } k \geq 0 \text{ and } \tilde{\pi}_p = \chi_1 \boxplus \chi_2 \text{ for } a(\chi_1) > a(\chi_2) = 0, \\
\omega_{\tilde{\pi},p}(v) & \text{if } k = 0 \text{ and } L(\tilde{\pi}_p, s) = 1, \\
0 & \text{else}, 
\end{cases}$$

where $W_{\tilde{\pi},p}$ is the normalized $\psi_p$-Whittaker new-vector of $\tilde{\pi}_p$. We set

$$W_{\phi,p}(g) = W_{\tilde{\pi},p}(g) \text{ if } p \neq l \text{ divides } N.$$

At the place $l$ we choose $\phi$ so that

$$W_{\phi,l}(a(\gamma)) = \omega_{\tilde{\pi},l}(\gamma) \mid_\mathbb{F} W_l(\gamma).$$

As in the archimedean case this is possible because the Kirillov model of $\tilde{\pi}_l$ contains the space of Schwartz-Bruhat functions, which in this case are exactly the smooth (i.e. locally constant) compactly supported functions on $\mathbb{Q}_l^\times$.

We still have to pick $\zeta$. We define $\zeta_\infty = 0$ and set

$$\zeta_{\text{fin}} = (\zeta_0, \zeta_0, \zeta_0, \cdots) \in \mathbb{Q}.$$

With this choice we have

$$\psi(\zeta m) = \psi_{\text{fin}}(\zeta_{\text{fin}} m) \psi_\infty(0) = \psi_{\text{fin}}(\zeta_{\text{fin}} m) = \psi_\infty(-\zeta_0 m).$$

for every $m \in \mathbb{Q}$.

We conclude that the left hand side of (2.1) (with our choice of $\phi$) equals

$$\sum_{\gamma \in \mathbb{Q}_l^\times} \psi(\zeta \gamma) W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{m \in \mathbb{N}} \lambda_{\pi} \left( \frac{m}{(m, l^\infty)} \right) \psi_\infty(-\zeta_0 m) W_\infty(m) W_l(m).$$
2.2 Computing the right hand side

With the choices made above Lemma 2.1 yields the identity
\[
\sum_{m \in \mathbb{N}} \lambda \left( \frac{m}{(m, \infty)} \right) \psi_\infty(-\zeta m)W_\infty(m)W_1(m) = \sum_{\gamma \in \mathbb{Q}^*} \tilde{W}_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .
\]

We want to compute the right hand side as explicit as possible. To this end we observe that
\[
\tilde{W}_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) = W_\phi \left( w \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \right)
\]
\[
= W_\phi \left( \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \right) = \prod_{p \leq \infty} W_{\phi, p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right)
\]

(2.3)

The first equality follows directly from the definition and the second one uses $\mathbb{Q}$-invariance of the central character. The upshot is that we can do the remaining computations place by place.

2.2.1 The unramified places $p \nmid lN$

In this case we have $W_{\phi, p}(a(\gamma)) = W_{\tilde{\pi}, p}(a(\gamma))$ and $\tilde{\pi}_p$ is unramified. Thus, if $v_p(\zeta_p) \geq 0$, we obtain
\[
W_{\phi, p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} \right) = W_{\tilde{\pi}, p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
\[
= \begin{cases} \gamma \pi_p^{\frac{1}{l}} \lambda_p(p^{v_p(\gamma)}) & \text{if } \gamma \in \mathbb{Z}_p, \\
0 & \text{else.} \end{cases}
\]

If $v_p(\zeta_p) < 0$, the simple computation
\[
w \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\zeta_p & 1 \end{pmatrix} = \begin{pmatrix} 1 & \zeta_p^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -\zeta_p \end{pmatrix} w
\]
implies
\[
\left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} = \left( \begin{pmatrix} 1 & \gamma \zeta_p^{-1} \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -1 & -\zeta_p \end{pmatrix} \right).
\]

Thus, we arrive at
\[
W_{\phi, p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} \right) = \psi_p(-\gamma \zeta_p^{-1})W_{\phi, p} \left( a(\gamma \zeta_p^{-1}) \begin{pmatrix} -1 & 0 \\ -1 & -\zeta_p \end{pmatrix} \right)
\]
\[
= \psi_p(-\gamma \zeta_p^{-1})\omega_{\tilde{\pi}, p}(-\zeta_p)W_{\phi, p} \left( a(\gamma \zeta_p^{-2}) \begin{pmatrix} 1 & 0 \\ -1 & -\zeta_p \end{pmatrix} \right). \tag{2.4}
\]

By right-$K_p$-invariance, the expression above simplifies to
\[
W_{\phi, p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} \right) = \psi_p(-\gamma \zeta_p^{-1})\omega_{\tilde{\pi}, p}(-\zeta_p)W_{\tilde{\pi}, p} \left( \begin{pmatrix} \gamma \zeta_p^{-2} & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
\[
= \begin{cases} \psi_p(-\gamma \zeta_p^{-1})\omega_{\tilde{\pi}, p}(-\zeta_p) \frac{1}{\zeta_p} \lambda_p(p^{v_p(\gamma \zeta_p^{-2})}) & \text{if } v_p(\gamma \zeta_p^{-2}) \geq 0, \\
0 & \text{else.} \end{cases}
\]
2.2.2 The ramified non-archimedean places \( p \mid N, p \neq l \)

We define

\[
g_{t,l,v} = \begin{pmatrix} \varpi_p^t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi_p^{-t} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \varpi_p^{-t} \\ -1 & -\varpi_p^{-t} \end{pmatrix}.
\]

Then we observe that the matrix at which we want to evaluate \( W_{\phi,p} \) is

\[
\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} = \begin{cases} 
\begin{pmatrix} 1 & \zeta_p - 1 \\ 0 & u \end{pmatrix} & \text{if } \nu_p(\zeta_p) \geq 0 \text{ and } \gamma = u\varpi_p^t, \\
\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} & \text{if } \zeta_p = v\varpi_p^{-t} \text{ and } \gamma = u\varpi_p^t.
\end{cases}
\]

Since the matrices on the right are always in \( K_{l,p}(\infty) \) we can use the finite Fourier expansion (better known as \( c_{l,l}(\mu) \)-expansion) to calculate the value of \( W_{\phi,p} \) explicitly. This has been studied extensively in [1].

Let \( n_p = \nu_p(N) \). Then we treat several subcases which feature different behavior. We set

\[
N_0 = \prod_{\nu_p(\zeta_p) \leq 0} p^{n_p}, \quad N_1 = \prod_{0 < \nu_p(\zeta_p) < n_p} p^{n_p} \quad \text{and} \quad N_2 = \prod_{n_p \leq -\nu_p(\zeta_p)} p^{n_p}.
\]

In order to use the results from [1] we have to re-normalize our representation \( \pi_p \). To do so we fix an unramified character \( \xi_p \) such that \( \omega_{\xi_p^{-1}(\pi_p)}(\varpi_p) = 1 \).

If \( p \mid N_0 \), we have

\[
W_{\phi,p} \left( a(\gamma)w \begin{pmatrix} 1 & \zeta_p \\ 0 & 1 \end{pmatrix} \right) = \xi_p(\gamma)W_{\xi_p^{-1}(\pi_p)}(g_{\nu_p(\gamma),0,u-1}) = \xi_p(\gamma)c_{\nu_p(\gamma),0}(1)
\]

\[
= \begin{cases} 
\begin{pmatrix} \lambda_p(\nu_p(\gamma)+n_p) & 1 \\ 0 & 1 \end{pmatrix} & \text{if } \nu_p(\gamma) + n_p \geq 0, \\
0 & 1 \end{cases}
\]

This follows from the explicit evaluation of \( c_{l,0}(1) \) given in [1]. For a complete classification of the constants \( c_{l,l}(\mu) \) see Appendix A.

**Remark 2.1.** The case \( \nu_p(\zeta_p) \geq 0 \) at ramified places can be treated in general using the theory of Atkin-Lehner operators. This leads the same result as our \( c_{l,0}(1) \) approach. See [12, Section 6] for details.

If \( p \mid N_1 \) the situation is slightly more complicated. We define

\[
\mathcal{E}_p(\gamma, \zeta_p) = |\gamma|_{p}^{1/2}W_{\xi_p^{-1}(\pi_p)}(g_{t,l,u-1,v}).
\]

We have the following result towards the support of these coefficients.

**Lemma 2.2.** For \( \nu_p(\gamma) < \min(2\nu_p(\zeta_p), -n_p + \nu_p(\zeta_p)) \) we have

\[
\mathcal{E}_p(\gamma, \zeta_p) = 0.
\]

**Proof.** This follows directly from the explicit formulas given in [1] Lemma 3.1, 3.3, 3.4, 3.5 and 3.6.
Thus, we can write

\[
W_{\phi,p} \left( \begin{pmatrix} a(\gamma)w & 1 \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} |\gamma|^{\frac{1}{2}} \xi_p(\gamma) E_p(\gamma, \zeta_p), & \text{if } v_p(\gamma) \geq -n_p + v_p(\zeta_p), \\ |\gamma|^2 \xi_p(\gamma), & \text{else.} \end{cases}
\]

If, for the global application, it is not necessary to keep track of \(N_2\) dependence it can be useful to expand \(E_p(\gamma, \zeta_p)\) in terms of \(c_{l,t}(\mu)\). We will follow this path later on. However, if one is interested in keeping track of possible cancellation coming from these places, one has to work more carefully. In this scenario one can obtain completely explicit formulas involving \(p\)-adic oscillations if one evaluates \(W_{\xi_p^{-1} \pi_p}\). Such evaluations have been given in [1] in several special cases.

Finally, if \(p \mid N_2\), we make the following observation.

**Lemma 2.3.** Let \(-\nu(\zeta_p) \geq n_p\) then

\[
W_{\tilde{\pi},p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & \zeta_p \end{pmatrix} \right) = \begin{cases} \omega_{\tilde{\pi},p}(-\zeta_p \gamma^{-1}) \psi_p(-\gamma \zeta_p^{-1}) |\gamma \zeta_p^{-2}|^{\frac{1}{2}} \lambda_p(p^{\nu_s(\gamma \zeta_p^{-2})}), & \text{if } v_p(\gamma \zeta_p^{-2}) \geq 0, \\ 0 & \text{else.} \end{cases}
\]

**Proof.** The proof has three steps. First, if \(l \geq n\) one can relate \(g_{l,t,v}\) to \(g_{l+n,0}\). Then one uses [13] Lemma 2.18, Proposition 2.28 to relate \(W_{\pi}(g_{l+n,0})\) to \(W_{\tilde{\pi}}(g_{l+n,0,-s})\). The latter can be evaluated using \(c_{l+n,0,1}(1)\) which we evaluated in the appendix.

This completes the treatment for ramified non-archimedean places away from \(l\) for now.

**2.2.3 The special place \(p = l\)**

At this place we are dealing with a Whittaker function which is not necessarily a new-vector. To evaluate this function away from the diagonal we will use the local functional equation.

We define

\[
Z(W, s, \mu) = \int_{\mathbb{Q}_p^\times} W(a(y)) \mu(y) \left| y_p \right|^{s - \frac{1}{2}} dy,
\]

for a multiplicative character \(\mu \in \mathbb{X}\), a Schwartz-Bruhat function \(W\), and some complex number \(s\) with sufficiently large real part. Then the local functional equation is

\[
\frac{Z(W, s, \mu \pi_p)}{L(s, \mu \pi_p)} \epsilon(s, \mu \pi_p) = \frac{Z(\pi_p W, 1 - s, \mu^{-1}\omega_{\pi_p}^{-1})}{L(1 - s, \mu^{-1}\pi_p)}
\]

Recall that, since \(\psi_p\) is unramified, we have

\[
\epsilon(s, \mu \pi_p) = q^{(s - \frac{1}{2})a(\mu \pi_p)} \epsilon(\frac{1}{2}, \mu \pi_p).
\]

The upshot is, that the latter \(\epsilon\)-factors are well behaved. In particular they have absolute value 1.
Recall that we want to evaluate

\[ W_{\phi,p} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} \zeta_p & 0 \\ 0 & 1 \end{pmatrix} \right). \]

Thus, we define \( W = \tilde{\pi}_p(n(\zeta_p))W_{\phi,p} \) so that the local functional equation reads

\[
\int_{\mathbb{Q}_p^*} W_{\phi,p}(a(y)wn(\zeta_p))|\omega_{\tilde{\pi},p}\mu|^{-1}(y) |y|_p^{\frac{1}{2} - s} dy = \epsilon(s, \mu_{\tilde{\pi}}) \frac{L(1 - s, \mu^{-1} \pi_p)}{L(s, \mu \pi_p)} Z(W, s, \mu). 
\]

The latter \( Z \)-integral can be computed, because on the diagonal \( W_{\phi,p} \) is given by \( W_1 \). To do so we will apply \( p \)-adic Mellin inversion to this formula. Recall that the Mellin transform is given by

\[
[M] f(\mu|\pi_p) = \int_{\mathbb{Q}_p^*} f(y)\mu(y) |y|^\mu p^{-\mu} dy.
\]

The inverse Mellin transform is given by

\[
[M^{-1}] f(y) = \frac{\log(p)}{2\pi} \sum_{\mu \in \mathcal{X}} \mu(y)^{-1} \int_{\mathbb{R}} \frac{1}{\pi(p)} \tilde{f}(\mu, it) |y|^{-\mu} dt.
\]

Indeed, see \([6, \text{Proposition 7.1.4}]\), these transforms satisfy

\[ M^{-1} \circ M = M \circ M^{-1} = 1. \]

It will be useful for us to split the inverse transform into two pieces. We define the \textit{pre-Mellin-inversion} by

\[
[M_{\text{pre}}^{-1}] f(\mu, y) = \frac{\log(p)}{2\pi} \int_{\mathbb{R}} \frac{1}{\pi(p)} \tilde{f}(\mu, it) |y|^{-\mu} dt.
\]

This leads to the definition

\[
B_{p,\kappa}(y) = \frac{\log(p)}{2\pi} \int_{\mathbb{R}_{\mathbb{Q}_p}(y)} \frac{L(1 - \kappa - it, \pi_p)}{L(\kappa + it, \pi_p)} |y|^{-\mu} dt.
\]

Indeed, \( B_{p,\kappa} \) turns out to be a very valuable \( p \)-adic special function in this context.

For example we have, if \( L(s, \pi_p) = 1 = L(s, \pi_{\tilde{\pi}}) \), then

\[
B_{p,\kappa}(y) = 1_{2\mathbb{Z}_p}(y^{-1}). \quad (2.5)
\]

The other extreme appears for \( \tilde{\pi}_p \) unramified. In this case we have

\[
B_{\pi,\frac{1}{2}}(y) = \begin{cases} 
|y|_p^{\frac{1}{2}} (\lambda \pi(p^{-v_p(y)}) \frac{1}{\pi(p^{1-v_p(y)})} + \omega_{\pi,p}(\omega_{p})p^{-2}\lambda \pi(p^{2-v_p(y)})) & \text{if } y^{-1} \in \mathbb{Z}_p, \\
\frac{1}{\pi(p^{-1} \omega_{p})(\omega_{\pi,p}(\omega_{p})p^{-1}\lambda \pi(p) - \lambda \pi(y))} & \text{if } y^{-1} \in \mathbb{Z}_p^{-1} \mathbb{Z}_p, \\
\omega_{\pi,p}(\omega_{p})p^{-1} & \text{if } y^{-1} \in \mathbb{Z}_p^{-2} \mathbb{Z}_p, \\
0 & \text{else.}
\end{cases}
\]
We are finally ready to evaluate $W_{\phi,p}$. The assumption $\text{supp}(W_l) \subset \mathbb{Z}_p^\times$ makes our life a lot easier. Indeed, $$[\mathfrak{M}W](\mu, s) = [\mathfrak{M}W_l](\mu, 0) \text{ for all } s \in \mathbb{C}.$$ We define $$W^{\omega,-1}_l(y) = \omega_{\pi,l}(y)W_l(y).$$ Then the local functional equation set up as above reads $$[\mathfrak{M}W_{\phi,p}(\cdot \cdot \cdot)](\mu, s) = [\mathfrak{M}W_{\omega,\pi,l}](\mu, 0) \text{ for all } s \in \mathbb{C}.$$ We define $$W^{\omega,-1}_l(y) = \omega_{\pi,l}(y)W_l(y).$$ Then the local functional equation set up as above reads $$[\mathfrak{M}W_{\phi,p}(\cdot \cdot \cdot)](\mu, s) = [\mathfrak{M}W_{\omega,\pi,l}](\mu, 0).$$ In this situation we can compute the pre-Mellin-inversion explicitly in terms of $B_{\mu,\sqrt{p}}$, $\mathbb{R}(s)$. After completing the process of Mellin-inversion we arrive at $$W_{\phi,p}(a(y)wn(\zeta_p)) = |y|^{\frac{1}{2}} W_l(y).$$ We will encounter a similar formula at the archimedean places. The $p$-adic Hankel-transform has the following properties.

**Lemma 2.4.** If for some $\kappa \geq 1$ $$W^{\omega,-1}_l(x + y^\kappa) = W^{\omega,-1}_l(x) \text{ for all } x \in \mathbb{Z}_p^\times, y \in \mathbb{Z}_p,$$ then one can restrict the $\mu$-sum in (2.6) to $\mu \in \mathfrak{X}_\kappa$. Furthermore, $$\text{supp}(\tilde{W}_l) \subset \omega_{l}^{\min(-2\kappa, -a(\pi_l))} \mathbb{Z}_q.$$

**Proof.** The first statement is a simple consequence of the following computation. For $\mu$ satisfying $a(\mu) > \kappa$ we have $$[\mathfrak{M}W^{\omega,-1}_l](\mu) = \sum_{x \in \mathbb{Z}_q^\times / (1 + t^\kappa \mathbb{Z}_q)} W^{\omega,-1}_l(x) \int_{1 + t^\kappa \mathbb{Z}_q} \mu(y)d^\times y = 0.$$ The second statement follows from the first one together with the support properties of $B_{\mu,\sqrt{p}}$. 

14
2.2.4 The archimedean places

At $\infty$ the action of the element $w$ in the archimedean Kirillov model is given by the Hankel-transform:

$$W_{\phi,\infty} \left( \left( \begin{array}{cc} \gamma & 0 \\ 0 & 1 \end{array} \right) w \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \zeta_\infty \right) = W_{\phi,\infty} \left( \left( \begin{array}{cc} \gamma & 0 \\ 0 & 1 \end{array} \right) w \right).$$

The function $j_{\hat{\pi},\infty}$ can be computed explicitly and it turns out that

$$j_{\hat{\pi},\infty}(y) = \begin{cases} 2\pi i^k \sqrt{y} J_{k-1}(4\pi \sqrt{y}) & \text{if } y > 0, \\ 0 & \text{if } y < 0. \end{cases}$$

if $\hat{\pi}_\infty$ is a discrete series representation of weight $k \geq 2$ with central character $\text{sgn}^k$. If $\hat{\pi}_\infty = |.|^r \boxplus |.|^{-r}$ then we have

$$j_{\hat{\pi},\infty}(y) = \begin{cases} i\pi \sqrt{y} J_{i\sqrt{y}}(4\pi \sqrt{y}) & \text{if } y > 0, \\ 4 \cosh(\pi t) \sqrt{|y|} K_{2t}(4\pi \sqrt{|y|}) & \text{if } y < 0. \end{cases}$$

These expressions also hold for complementary series $\hat{\pi}_\infty$, which appear when $r$ is imaginary. To shorten notation later on we write

$$\hat{W}_{\infty,\pm}(y) = \int_{\mathbb{R}_{>0}} \mathcal{J}_{\infty,\kappa}^\pm(4\pi \sqrt{xy}) W_\infty(x) dx,$$

for $y > 0$. Where we set

$$\mathcal{J}_{\infty,\kappa}^\pm(y) = \frac{4\pi}{y} j_{\hat{\pi},\infty} \left( \pm \frac{y^2}{16\pi^2} \right)$$

and $\kappa$ is $k-1$ in the case of discrete series and $2t$ for principal series or complementary series. We choose this notation to be compatible with [1]. In particular, at infinity, we have

$$W_{\phi,\infty} \left( \left( \begin{array}{cc} \gamma & 0 \\ 0 & 1 \end{array} \right) w \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \zeta_\infty \right) = |\gamma|^{\frac{\kappa}{2}} \hat{W}_{\infty,\text{sgn}(|\gamma|)}(|\gamma|).$$

2.2.5 Summary

The following proposition summarizes our findings from the previous subsections.

**Proposition 2.1.** Let $\pi$ be a cuspidal automorphic representation with conductor $N \ell^n$ and central character $\omega_\ell$. Furthermore, let $\frac{b}{a} \in \mathbb{Q}$, $W_\infty \in C_0^\infty(\mathbb{R}_+)$, and for some prime $l \mid b$ let $W_l \in \mathcal{S}(\mathbb{Q}_l)$ with support in $\mathbb{Z}_l$. We define $N_0$, $N_1$ and $N_2$ as above. Furthermore, we set

$$b_1 = (b, N_1), \quad b_2 = (b, N_2), \quad b_0 = \frac{b}{b_1 b_2},$$

$$\eta(\pi, a, b) = \prod_{p \mid N_0} \epsilon \left( \frac{1}{2}, \pi_p \right) \prod_{p \mid b_1 N_2} \omega_{\pi, p}(-ab),$$

and

$$\mathcal{E}(m, \frac{a}{b}) = \prod_{p \mid N_1} \xi_p \left( \frac{m}{b_1 N_1} \right) \xi_p \left( \frac{m}{b_0 b_2 N_0 b_1 N_1} \frac{a}{b} \right).$$
Then
\[
\sum_{m \in \mathbb{N}} e \left(-\frac{a}{b} m \right) \lambda_\pi \left(\frac{m}{(m, l^{\infty})}\right) W_\infty(m) W_1(m)
= \frac{\eta(\pi, a, b)}{b_0 b_2 \sqrt{N_0}} \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, (m, l^{\infty}) = 1} e \left(\frac{l^c m}{b_0 b_2 b_1 N_1 N_0}\right) \lambda_{\pi, \infty_2} \left(\frac{m}{(m, N_1^{\infty})}\right) \cdot \hat{W}_{\infty, \text{sgn}(m)} \left(\frac{l^c m}{b_0 b_2 b_1 N_1 N_0}\right) \hat{W}_1 \left(\frac{l^c m a N_0 N_1}{b_0 b_2}\right) E \left(l^c m, \frac{a}{b}\right).
\]

This proposition is already a very robust tool with many interesting features. However, it has the caveat that the contribution from the places \( p \mid N_1 \) is hidden in the mysterious term \( E \). In order to make our formula more suitable for applications we will now unfold this error using local Fourier analysis.

**Theorem 2.1.** Under the assumptions of Proposition 2.1 we have
\[
\sum_{m \in \mathbb{N}} e \left(-\frac{a}{b} m \right) \lambda_\pi \left(\frac{m}{(m, l^{\infty})}\right) W_\infty(m) W_1(m)
= \zeta_N(1) \frac{\eta(\pi, a, b)}{b_0 b_2 \sqrt{N_0}} \sum_{\mu \in \Pi_{p \mid b_1} \times \mathbb{R}_{p(v)}, \mu \neq \mu_0(N_0)} \frac{\mu(\delta \mu b_0 b_2 b_1 N_1)}{\sqrt{b_1 N_1^2(\mu)}} C(\pi, \mu, b_1, m_1) \cdot \hat{W}_{\infty, \text{sgn}(m)} \left(\frac{l^c m |m|}{b_0 b_2 b_1 N_1 N_0}\right) \hat{W}_1 \left(\frac{l^c m a N_0 N_1}{b_0 b_2}\right) E \left(l^c m, \frac{a}{b}\right).
\]

For some constants \( C(\pi, \mu, b_1, m_1) \in \mathbb{C} \) satisfying
\[
|C(\pi, \mu, b_1, m_1)| \ll_N m_1^{\frac{-1}{\pi^2 + c}}.
\]

**Proof.** The idea, taken from [13] (11), is to expand
\[
W_{\xi_1^{-1} (\mu), k, v} = \sum_{\mu \in p \mu \in \mathbb{Z}} c_{l, k}(\mu_p) \mu_p(v),
\]
for each \( p \mid N_1 \). The constants \( c_{l, k}(\mu_p) \) depend on the underlying representation \( \pi_p \) and have been described in Appendix A. Using these expansions we can write
\[
E(l^c m, \frac{a}{b}) = \zeta_N(1) \sum_{\mu \in \Pi_{p \mid b_1} \times \mathbb{R}_{p(v)}, \mu \neq \mu_0(N_0)} \frac{\mu(2\delta \mu b_0 b_2 b_1 N_1)}{\sqrt{b_1 N_1^2(\mu)}} \lambda_{\pi, \infty_2} \left(\frac{N_1(\mu)}{b_1 N_1}\right) \cdot C(\pi, \mu, b_1, m, N_1^{\infty})
\]
for
\[
N_1(\mu) = \prod_{p \mid N_1} p^{\delta \mu_p + \frac{\delta \mu_p}{p}},
\]
\[
C(\pi, \mu, b_1, m, N_1^{\infty}) = \prod_{p \mid N_1} c_p \left(\pi_p, v_p(b_1), v_p(b_1 N_0), \mu_p\right) \xi_1^{-1}(N_1(\mu)) p^{\frac{\delta \mu_p}{p}}.
\]

16
Inserting this expression in Proposition 2.1 completes the proof of the stated expression. The bound on the coefficients \( C(\pi, \mu, b_1, m_1) \in \mathbb{C} \) can be red of from (A.1) together with the current best possible results towards the Ramanujan conjecture. See for example (2).

3 Application to the subconvexity problem

In this section we will prove Theorem 1.2. In doing so we will closely stick to (3) and assume some familiarity with the arguments within. From now on \( \pi \) will denote a cuspidal automorphic representation of conductor \( Nl \). We are interested in

\[
L := \sum_{m \in \mathbb{Z}} \lambda_\pi(m) F\left(\frac{m}{M}\right).
\]

We will restrict our attention to \( \pi_l = \chi_l \cdot |\cdot|^{\kappa_1} \otimes \chi_l \cdot |\cdot|^{\kappa_2} \) for some \( \chi_l \in \mathcal{X}_l \) and \( n_l \geq 10 \) even. In particular, there is a Hecke character \( \chi = \prod_{p \leq \infty} \chi_p \) such that \( \pi = \chi \otimes \pi_0 \) for some automorphic representation \( \pi_0 \) satisfying \( a(\pi_0, l) = 0 \). This implies that for all \((m, l) = 1\) we have

\[
\lambda_\pi(m) = \chi(l)^{-1} \lambda_{\pi_0}(m).
\]  

(3.1)

As shown in (3) Section 5.1] we can assume without loss of generality that \( Z^2 \leq l^{2n} \lesssim N M \sim (l^2 Z)^{1+\epsilon} \).

3.1 \( p \)-Adic Farey dissection

The first step is to apply (3) Theorem 3.1.

Theorem 3.1. Let \( \alpha \in \mathbb{Z}_l^2 \), \( q \in \mathbb{N} \) and an integer \(-q \leq r \leq q\) be given. Write \( r^+ = \max(r, 0) \) and \( r^- = \max(-r, 0) \), and let

\[
S = \{(a, b, k) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}_0 | b \leq t^{k+2r^-}, |a| \leq t^{k+2r^+}, (a, b) = (a, l) = (b, l) = 1\}.
\]

For \((a, b, k) \in S\), let

\[
Z_l^k[a, b, k] = \{m \in \mathbb{Z}_l^k | b\alpha/m - a \in l^{q+|r|+k} \mathbb{Z}_l\}.
\]

Then there exists a subset \( S^0 \subset S \) such that

\[
Z_l^k = \bigsqcup_{(a, b, k) \in S^0} Z_l^k[a, b, k]
\]

and in addition the following two properties hold: if \((a, b, k_1), (a, b, k_2) \in S^0\), then \( k_1 = k_2\), and for each \((a, b, k) \in S^0\) one has \( k \leq q - |r|\).

Applying this theorem with \( \alpha = \alpha_{\chi_1} \in \mathbb{Z}_l \) as defined in (1.2), \( q \leq \frac{n_l}{N} \) and some \( |r| \leq q \) yields

\[
L = \sum_{s = (a, b, k) \in S^0} \sum_{m \in \mathbb{Z} \cap Z_l^s[a, b, k]} \lambda_\pi(m) F\left(\frac{m}{M}\right).
\]

17
We estimate

\[ L \ll l^{\max_{0 \leq k \leq q - |r|, A \leq \frac{1}{2} + \varepsilon} \sum_{s=(a,b,k) \in S^0, A \leq |a| < 2A, B \leq b < 2B} L_{s} L_{A,B,k}} \].

Good bounds for \( L_{A,B,k} \) will suffice to establish good (non-trivial) bounds for \( L \). Thus, we fix \( A, B \) and \( k \) until otherwise stated.

One rewrites

\[ L_{s} = \sum_{m} \lambda_{\pi \omega}(m) \chi_{l}(m) \chi_{l}(m)^{-1} \psi_{l} \left( \frac{ab}{l^{2}} m \right) e \left( \frac{ab}{l^{2}} m \right) F \left( \frac{m}{M} \right) \]

Here we use the fact, that

\[ \psi_{l} \left( \frac{ab}{l^{2}} m \right) e \left( \frac{ab}{l^{2}} m \right) = \psi \left( \frac{ab}{l^{2}} m \right) = 1. \]

Furthermore, we use the reciprocity formula

\[ e \left( \frac{ab}{l^{2}} m \right) = e \left( \frac{ab}{l^{2}} m \right) e \left( \frac{a}{bl^{2}} m \right) \]

to obtain

\[ L_{s} = \sum_{m} \lambda_{\pi \omega}(m) e \left( \frac{ab}{l^{2}} m \right) W_{\infty} \left( \frac{m}{M} \right) W_{l}(s; m), \]

for

\[ W_{\infty}(x) = e \left( \frac{a}{bl^{2}} x \right) F \left( \frac{x}{M} \right). \]

This last formula for \( L_{s} \) suits the application of our Voronoi summation formula which we will apply in the next subsection. Before we will continue let us make the following observation.

**Lemma 3.1.** The function \( W_{l}^{\omega \pi \omega \omega}(s; \cdot) \) is periodic modulo \( l^{\frac{3}{2} - q - |r| - k} \mathbb{Z}_{l} \).

**Proof.** First, observe that for \( s = (a, b, k) \in S_{0} \) we have \( k \leq q - |r| \) and deduce

\[ \frac{nl}{2} - q - |r| - k \geq \frac{nl}{2} - 2q \geq \frac{nl}{4} > 0. \]

For \( m \in \mathbb{Z}_{l}^{\omega \pi \omega \omega}[a, b, k] \) and \( y \in \mathbb{Z}_{l} \) we argue as on [3, p. 582] to obtain

\[ W_{l}^{\omega \pi \omega \omega}(s; m + yl^{\frac{3}{2} - q - |r| - k}) = \chi_{l}(m) \psi_{l} \left( \frac{ab}{l^{2}} m \right) \psi_{l} \left( \frac{am^{-1} - ab}{lq + |r| + k} y \right), \]

here we used \( \omega \pi \omega \omega = \chi_{l}^{2} \). One concludes using the definition of \( \mathbb{Z}_{l}[a, b, k] \).
3.2 Applying the Voronoi formula

In this section we will apply Theorem 2.1 to the sum \( L_s \) and bring the resulting expression in a form which is suitable for extracting the necessary cancellation. Combining Theorem 2.1 with Lemma 2.4 and Lemma 3.1 yields

\[
L_s = \sum_{c \geq -n_1+2q+2|r|+2k} L_{s,c},
\]

for

\[
L_{s,c} = \zeta_N(1) \frac{\eta(\pi_0,a,b)}{b_0b_2\sqrt{N_0}} \sum_{\mu \in \mathcal{P} \mu_1 \in \mathcal{C}_{1\eta,\eta_0}(b_1)} \frac{\mu(ab\tau_1 b_0 N_0 N_1^m(\mu))}{\sqrt{b_1 N_1^m(\mu)}} \cdot \sum_{m_1 \in \mathbb{Z}^c} C(\pi_0 N_1, \mu, b_1, m_1) \lambda_{(\pi_0 N_1, \mu, b_1, m_1)} \left( \frac{m_1}{b_1 N_1} \right) \cdot \sum_{m \in \mathbb{Z}_{c} \setminus \{m, lN_1 = 1\}} e^{\left( \frac{1}{c} \sum_{m \in \mathbb{Z}^c} \right) \left( \frac{m}{b_0 b_2 b_1 N_1 N_0} \right)} \cdot W_{l,c,s}(\cdot).
\]

We define

\[
L_{s,c}(m) = \sum_{\mu_i \in \mathcal{P}} \epsilon\left( \frac{1}{2}, \frac{\mu_i}{\mu} \right)^2 \mu_i(b \mu^{-2} |\mathbb{P}| W_l_{l,c,s}(s; \cdot))(\mu),
\]

\[
T_{s,c}(m) = \int_0^\infty F\left( \frac{x}{M} \right) e\left( \frac{a}{bl^c x} \right) \left( \frac{4\pi \sqrt{mx}}{bl^c} \right) dx
\]

This notation is taken from [3]. However, the \( l \)-adic oscillatory function \( L_{s,c} \) differs slightly from the one given in [3] (5.11)). This is due to the fact that we are working in the adelic setting which makes our function purely local. However, it is a nice exercise in adelization of Dirichlet characters to relate the two formulations.

Then we make the following observations. For \( \mu_1 \neq 1 \) we have \( |\alpha(\mu, \pi_0, i) = 2\alpha(\mu) \), \( L(s, \mu, \pi_0, i) = 1 \), and \( \epsilon\left( \frac{1}{2}, \mu \pi_0, i \right) = \epsilon\left( \frac{1}{2}, \mu \right)^2 \). Thus, in view of (2.3) and (2.4) we find that \( L_{s,c} = 0 \) if \( -n_1+2q+2|r|+2k < -2 \). On the other hand, for \( 2 \leq c \leq \alpha - q - |r| - k \) we can rewrite

\[
L_{s,c} = \zeta_N(1) \frac{\eta(\pi, a, b)}{b_0b_2\sqrt{N_0}} \sum_{\mu \in \mathcal{P} \mu_1 \in \mathcal{C}_{1\eta,\eta_0}(b_1)} \frac{\mu(ab\tau_1 b_0 N_0 N_1^m(\mu))}{\sqrt{b_1 N_1^m(\mu)}} \cdot \sum_{m_1 \in \mathbb{Z}^c} C(\pi_0 N_1, \mu, b_1, m_1) \lambda_{(\pi_0 N_1, \mu, b_1, m_1)} \left( \frac{m_1}{b_1 N_1} \right) \cdot \sum_{m \in \mathbb{Z}_{c} \setminus \{m, lN_1 = 1\}} e^{\left( \frac{1}{c} \sum_{m \in \mathbb{Z}^c} \right) \left( \frac{m}{b_0 b_2 b_1 N_1 N_0} \right)} \cdot W_{l,c,s}(\cdot,s,m) \cdot T_{s,c}(m) \cdot \left( \frac{m_1 m}{b_1 N_0 N_1} \right)
\]

19
The oscillatory parts, $L_{s,c}$ and $I_{s,c}^\pm$, appearing in these sums have been evaluated in [3]. Since we only shifted the argument we can reuse these evaluations. Recall [3, Lemma 3, Lemma 4].

**Lemma 3.2.** The function $I_{s,c}^\pm(m)$ is $O((l^{\nu_1}m)^{-100})$ unless

$$m \ll l^{2c} \frac{B^2Z^2}{M} + \frac{A^2M}{l^m} l^{3\nu_1} = M l^{3\nu_1}.$$  \hfill (3.2)

In the range (3.2) one has

$$I_{s,c}^\pm(m) = \left( \frac{M l^{3\nu_1}}{m} \right)^{\frac{1}{2}} \min \left( M, \frac{BZl^{\mu}}{A} \right) e(\theta_{s,c}m) W_{s,c}(m) + O(l^{-100\nu_1}),$$

where $W_{s,c}$ is smooth and satisfies

$$x^j \frac{d^j}{dx^j} W_{s,c}(x) \ll_j l^{3\nu_1} (Z^2 l^{3\nu_1})^j$$

and where

$$\theta_{s,c} = \begin{cases} \frac{Z}{ab} & \text{if } \frac{AM}{BZl^{\mu}} \geq 1, \\ 0 & \text{else.} \end{cases}$$

**Lemma 3.3.** The function $L_{s,c}$ evaluates to

$$L_{s,c}(m) = \begin{cases} \gamma p \frac{n+c}{4} \chi_i(ab) \sum_{\pm} \Phi^{\pm}_c \left( \frac{m}{ab} \right) & \text{if } \frac{\alpha \beta m}{a} \in \mathbb{Z}_l^{2\times}, \\ 0 & \text{else.} \end{cases}$$

where $\gamma$ is a constant of absolute value $1$ which depends only on the parity of $\frac{n}{2}$, and

$$\Phi^{\pm}_c(x) = \epsilon(\pm(\alpha x)\frac{1}{2}, p^c) \chi_i(\alpha + \frac{1}{2}p^{2(n-c)}x \pm p^{n-c}(\alpha x + \frac{1}{4}p^{2(n-c)}x^2) \frac{1}{2})$$

$$\cdot \psi_p(-1) \psi_p(\frac{1}{p^c}(\frac{1}{2}p^{n-c}x \pm (\alpha x + \frac{1}{4}p^{2(n-c)}x^2) \frac{1}{2})).$$

**Proof.** The computations are essentially the same as in [3, Section 7.2]. Thus let us simply point out the key differences.

Taking the normalizations of our integrals into account one can adapt the proof of [3, Lemma 9] to our setting. Carrying out the necessary details reveals the same result up to the identity $\theta(x) = \psi_l(-x)$, where $\theta$ is the additive character used in [3]. Thus, it is straightforward to modify the proof of [3, Lemma 10].

Combining everything we have

$$L_s = \sum_{2 \leq c \leq \frac{B^2}{Z^2} - q - |\pi| - k} L_{s,c} + E.$$ 

Where $E$ collects the values of $c$ together that we neglected till so far. The following estimate for the error $E$ can be understood as a truncation in the $l$-aspect.
Lemma 3.4. Under our working assumptions we have
\[
E \ll_{\pi_0, \zeta} \zeta(1) M p^{-q-|r|-k} \left(1 + \frac{ll + l - r}{\sqrt{M}}\right)^{j + \frac{1}{2}} \left(\frac{Zl^{1+l-r}}{\sqrt{M}} + \frac{l^{1-l+r}M}{l^{\frac{1}{2}}}\right)^{j}.
\]

Proof. Estimating trivially yields
\[
E = \sum_{c \geq -2} L_{s,-c} \ll_{\pi_0} \frac{1}{b} \sum_{\mu \in \prod_{p \mid b_1} \mathbb{Z}_{p}(b_1)} \sum_{m \in \mathbb{Z}} (m_1, N_1^{\infty}) \left| \lambda_{\pi_0} \left( \frac{mN_1^{\prime}(\mu)}{(m, l^{\infty})b_1N_1} \right) \right| \cdot \left| \tilde{W}_{\infty, \mathrm{sgn}} \left( \frac{|m|}{l^2 b_0 b_2 b_1 N_1 N_0} \right) \right| \left| \tilde{W}_{1} \left( s; \frac{m}{l^2 b_0 b_2 b_1 N_1 N_0} \right) \right|.
\]

As described in Section 5.3, (2.10) we have the bound
\[
\tilde{W}_{\infty, \mathrm{sgn}} \left( \frac{|m|}{l^2 b_0 b_2 b_1 N_1 N_0} \right) \ll_{\pi_0, \zeta} M \left(1 + \frac{b}{\sqrt{M|m|}}\right)^{j + \frac{1}{2}} \left(\frac{Zl^{1+l-r}}{\sqrt{M|m|}}\right)^{j},
\]
for \(Z_0 = Z + ABM^{-1}l^{-\frac{1}{2}}\). From (2.10) we deduce the trivial estimate
\[
\tilde{W}_{1} \left( s; \frac{m}{l^2 b_0 b_2 b_1 N_1 N_0} \right) \ll (m, l^{\infty}) \sup_{\mu_2} \left| \mathbb{R} \tilde{W}_{l^2 b_2 b_1 N_1 N_0}^{(s)}(\mu) \right|.
\]

Using these estimates together with
\[
w \leq pb \leq 2pB \leq p^{k+2r+1}, \quad k \leq l - |r|, \quad -|r| + 2r^{\pm} = \pm r, \quad \frac{w}{B} \leq 2p
\]
yields
\[
E \ll_{\pi_0, \zeta} M \left(1 + \frac{ll + l - r}{\sqrt{M}}\right)^{j + \frac{1}{2}} \left(\frac{Zl^{1+l-r}}{\sqrt{M}} + \frac{l^{1-l+r}M}{l^{\frac{1}{2}}}\right)^{j} \cdot \sup_{\mu_2} \left| \mathbb{R} \tilde{W}_{l^2 b_2 b_1 N_1 N_0}^{(s)}(\mu) \right|.
\]

The result follows by estimating \(\left| \mathbb{R} \tilde{W}_{l^2 b_2 b_1 N_1 N_0}^{(s)}(\mu) \right|\) trivially.

\[\square\]

4 Extracting cancellation on average

We start this section by introducing some more notation. First, we set
\[
\mathcal{M}_{\tilde{s}} = \mathcal{M} N_0 N_1 b_1 \quad \mathrm{and}
\]
\[
\tilde{W}_{s,c} = \frac{B}{b} \chi(\tilde{m}\tilde{n}) N_0 N_1 b_1^{-1} b_2 N_0 N_1^{\prime}(\mu)^{2c} \eta(\pi_0, a, b) W_{s,c}.
\]

In particular we have \(|\tilde{W}_{s,c}| \geq |W_{s,c}|\). Furthermore, we define
\[
\kappa_s(x, \mu) = e \left( \left( l \frac{a N_0 N_1}{b_0 b_2} + \theta_{s,c} b_1 \right) x \right) \tilde{W}_{s,c} \left( \frac{x b_1}{N_0 N_1} \right) \sum_{\pm} \Phi_{\pm} \left( \frac{x b_1}{N_0 N_1 a b} \right).
\]
Inserting the results from the previous subsection and dealing with the error terms in the obvious way leads to

\[ L \ll \pi_0 \left( \prod_{0 \leq k \leq q - |r|, \quad 2 \leq \ell \leq \frac{q_2}{q} - \frac{|r|}{k} - k} \right) \left( \left| L_{A,B,b_2,k,c,\mu} \right| + (A B I)^{en_l} E + l^{-50 n_l} \right), \]

for

\[ L_{A,B,b_2,k,c,\mu} = \min \left( M, BZl^{\frac{n_2}{2}} B \right) \sum_{r \leq \ell \leq q_{11} \ell_{11}^t} \sum_{(m,\nu) = 1, \nu \leq q_{11} \ell_{11}^t} C(\pi_0,\mu_1,\nu,b_1,r) \]

\[ \lambda_{(\pi_0^{n_2})} \left( \frac{N_1(\mu)}{b_1 \nu^t} \right) \left( \frac{M_1^{3en_l}}{m |r|} \right)^{\frac{1}{q}} \sum_{s = (a,b,k) \in \mathbb{S}_1} \kappa_s (mr,\mu). \]

Finally, we define

\[ \Xi_{s_1,s_2,\mu} = \sum_{1 \leq m \leq M} \kappa_{s_1} (m,\mu) \kappa_{s_2} (m,\mu). \]

An application of the Cauchy-Schwarz inequality yields

\[ L_{A,B,b_2,k,c,\mu} \ll \pi_0 M^{\frac{1}{2}} Z^{2n_2} l^{2e-n_2} \left( \sum_{s_1,s_2 \in \mathbb{S}_1} \left| \Xi_{s_1,s_2,\mu} \right| \right)^{\frac{1}{2}}. \]

This is similar to [3 (5.17)]. The last hurdle is to adapt [3 Lemma 5] to our situation. This goes as follows.

**Lemma 4.1.** Under the usual assumptions we have

\[ \Xi_{s_1,s_2,\mu} = \sum_{\Omega \in \{0,\text{ord}_{(a_1,b_1)} - a_2(b_2)\}} \Xi_{s_1,s_2,\mu,\Omega}. \] (4.1)

Furthermore,

\[ \Xi_{s_1,s_2,\mu,\Omega} \ll \begin{cases} q^{\frac{1}{2} c} Z^{2p(p+1)} M + p^{a_3(b_3)} \quad & \text{if } \Omega \leq c - 2, \\ q^{a_5(b_5)} M \quad & \text{if } c - 1 \leq \Omega \leq \infty. \end{cases} \]
In the proof we closely follow [3, Section 9].

Proof. Let \( v = \frac{m}{2} - c, \epsilon = \pm 1 \) and \( x \in \alpha \mathbb{Z}_2^\times \). Then

\[
\tilde{\Phi}_v(x) = \chi_l(\alpha + \frac{1}{2} 2^v x + \epsilon l^v (\alpha x + \frac{1}{4} l^{2v} x^2)) \psi_l(-\frac{1}{\epsilon^2} (\frac{1}{2} l^v x + \epsilon (\alpha x + \frac{1}{4} l^{2v} x^2))).
\]

Further, we define

\[
\Phi_{s,c}(x) = \tilde{\Phi}_{\epsilon_1}(\frac{xb_1}{a(1)b_1N_0N_1}) \tilde{\Phi}_{\epsilon_2}(\frac{xb_1}{a(2)b_2N_0N_1})
\]

where \( s = (s_1, s_2) \) and \( \epsilon = (\epsilon_2, \epsilon_2) \). We also set

\[
W_{s,c}(x) = \tilde{W}_{s_1,c}(\frac{xb_1}{a(1)b_1N_0N_1}) \tilde{W}_{s_2,c}(\frac{xb_1}{a(2)b_2N_0N_1})
\]

\[
e_{s,c}(x) = \epsilon(\epsilon_1(\alpha b_1 N_0 N_1 a(1)b_1(\frac{1}{2}, p^\alpha)) \pi(\epsilon(\alpha b_1 N_0 N_1 a(2)b_2(\frac{1}{2}, p^\alpha)) \pi(\epsilon_2(\alpha b_1 N_0 N_1 a(1)b_1(\frac{1}{2}, p^\alpha)) \pi:\\)

\[
w_{s,c} = \left( l^c b_1 a N_0 N_1 \right)^{s_1,c} + \frac{\theta_{s_1,c} b_1}{N_0 N_1} - \left( l^c b_1 a N_0 N_1 \right)^{s_2,c} + \frac{\theta_{s_2,c} b_1}{N_0 N_1}.
\]

We can also assume that \( a(1)b_1 a(2)b_2 \in \mathbb{Z}_2^\times \) since otherwise the two conditions in the \( s_1, s_2 \) sum can not be satisfied simultaneously. Under this condition, and in the new notation we have

\[
\Xi_{s_1,s_2,\zeta} = \sum_{\epsilon \in \{\pm 1\}^2} \epsilon_{s_1,c}(m) \epsilon_{s_2,c}(m) \Phi_{s,c}(m) W_{s,c}(m).
\]

It is clear that [3, Lemma 13] holds also in our case. This is because our \( \Phi_{s,c} \) is simply a shift of the one considered in the reference. Furthermore, all the necessary assumptions are in place to make this work. The decomposition [4, 1] is as in [3] and is obvious from the result [3, Lemma 13].

We note that \( \text{ord}_l(b_1((N_0N_1)^{-1}) = 0 \) so that we can continue exactly as in [3]. After discarding possible factors coming from the shift in the archimedean factor we obtain the desired bounds.

Next we note that the bounds for \( \Xi_{s_1,s_2,\zeta,\Omega} \) as well as the \( \Omega \)-decomposition are independent of \( \mu \) and \( b_1 \). Thus, we can follow exactly the argument from [3, Section 5.5]. Which ultimately yields too

\[
L \ll_\Psi M^{2} Z^{\frac{5}{2}} l^{\frac{1}{2} + \epsilon} + \epsilon l n_l
\]

and completes the proof of Theorem 1.2.

The statement of Theorem 1.1 follows from Theorem 1.2 using standard arguments including adelization, approximate functional equation and partitions of unity.
A Tables for $c_{t,l}(\mu)$

In this appendix we recall some results from [1]. We will state them in a notation which is suitable for applications in the setting of paper.

Throughout this section we are dealing with unitary admissible irreducible representations $\pi_p$ of $GL_2(\mathbb{Q}_p)$. To such a representation we attach the local Whittaker new-vector $W_{\pi_p}$, normalized by $W_{\pi_p}(1) = 1$. We have the expansion

$$W_{\pi_p}(g_{t,l,v}) = \sum_{\mu_p \in \mathbb{X}_t} c_{t,l}(\mu_p)\mu_p(v).$$

These constants have been computed in [1, Section 2]. In the following we define new constants via

$$c_{t,l}(\mu_p) = c_p(\pi_p, l, t, \mu_p)\zeta_p(1)^{-1} - 1,$$

for some $\delta_{\mu_p} \in \mathbb{N}$ which in most cases turn out to be the degree of the Euler-factor of $\mu_p$.

In the following subsections we give evaluations of the constants for each possible representation focusing on the non-zero cases. As a result we obtain the bound

$$|c_p(\pi_p, l, t, \mu_p)| \leq 5p^{\frac{1}{2}t} \max_{i=1,2}(|\alpha_i|), \quad \text{(A.1)}$$

for $\alpha_i = \chi_{i}(p_w)$ if $\pi_p = \chi_1 \boxplus \chi_2$ and $\alpha_i = 1$ otherwise. Note that, since we are dealing with admissible, unitary representations $\pi_p$, we have $|\alpha_i| = 1$ except for $\chi_1$ equals $\chi_2$ up to unramified twist. The latter can not be excluded without assuming the Ramanujan conjecture. Indeed, such representations might arise as components of twists of Maass forms failing the Ramanujan conjecture.

### A.1 Supercuspidal representations

Recall that in this case $\lambda_{\chi_{p} \otimes \pi_p}(p^m) = \delta_{m=0}$ and $\delta_{\mu_p} = 0$ for all $\mu_p$. Thus from [1, Section 2.1] we extract the following.

| $c_p(\pi_p, l, t, \mu_p)$ | $\mu_p = 1$ | $\mu_p \in p\mathbb{X}_t \setminus \{1\}$ |
|------------------------|-------------|------------------|
| $l = 0$                | $\epsilon(\frac{1}{2}, \pi_p)\zeta_p(1)^{-1}$ | $-$ |
| $l = 1$                | $-p^{\frac{1}{2}}\epsilon(\tfrac{1}{2}, \pi_p)$ | $\epsilon(\tfrac{1}{2}, \mu_p)\epsilon(\tfrac{1}{2}, \mu^{-1}_{p} \pi_p)$ |
| $l > 1$                | $0$         | $\epsilon(\tfrac{1}{2}, \mu_p)\epsilon(\tfrac{1}{2}, \mu^{-1}_{p} \pi_p)$ |

### A.2 Twists of Steinberg

Here we consider $\pi_p = \chi St$ for some ramified character $\chi$. We have

$$\lambda_{\chi_{p} \otimes \pi_p}(p^m) = \begin{cases} \delta_{m=0} & \text{if } \mu_p \neq \chi^{-1}, \\ q^{-\frac{1}{2}}\delta_{m \geq 0} & \text{if } \mu_p = \chi^{-1} \end{cases}$$

and $\delta_{\mu_p} = 1$ if $\mu_p = \chi^{-1}$ and 0 otherwise. As in [1, Lemma 2.1] one obtains the following evaluations.
Furthermore, if \( \chi \in \mathcal{O}^* \) we produce the following table.

The following table can be deduced from \([1, \text{Lemma 2.2}]\).

### A.3 Irreducible principal series

In this section we treat three cases. First, we look at \( \pi_p = \chi_1 \Box \chi_2 \) with \( \chi_1|\mathcal{O}^* \neq \chi_2|\mathcal{O}^* \). In this case \( \delta_{\mu_p \pi_p} = 1 \) if \( \mu_p|_\pi \neq \chi_1^{-1}|\pi \) and 0 otherwise. Furthermore,

\[
\lambda_{\mu_p \pi_p}(p^m) = \begin{cases} 
\delta_{m=0} & \text{if } \mu_p|_\pi \neq \chi_1^{-1}|\pi, \\
\chi(p^m) & \text{if } \mu_p|_\pi = \chi_1^{-1}|\pi.
\end{cases}
\]

The following table can be deduced from \([1, \text{Lemma 2.2}]\).

Next we look at \( \pi_p = \chi_1 \Box \chi_2 \) where \( \chi_1|_{\mathcal{O}^*} = \chi_2|_{\mathcal{O}^*} \). In this case \( \delta_{\mu_p \pi_p} = 2 \) if \( \mu|_\pi = \chi_1^{-1}|_\pi \) and 0 otherwise. Furthermore,

\[
\lambda_{\mu_p \pi_p}(p^m) = \begin{cases} 
\delta_{m=0} & \text{if } \mu_p|_\pi \neq \chi_1^{-1}|\pi, \\
\frac{\chi(p^m) - \chi_2(p^m)}{\chi_1(p) - \chi_2(p)} & \text{if } \mu_p|_\pi = \chi_1^{-1}|\pi.
\end{cases}
\]

Using \([1, \text{Lemma 2.2}]\) we produce the following table.
\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
\mu_p & \mu_p = 1 & \mu_p|_{\mathcal{L}} = \chi_1^{-1}|_{\mathcal{L}} & \mu_p \in p^{\mathcal{L}} \setminus \{1, \chi_1^{-1}\} \\
\hline
l = 0 & \epsilon(\frac{1}{2}, \pi_p) & \zeta_p(1)^{-1} & \epsilon(\frac{1}{2}, \pi_p) \\
\hline
l = 1 & -\epsilon(\frac{1}{2}, \pi_p)p^{-\frac{l}{2}} & \epsilon(\frac{1}{2}, \mu_p)p^{-2} & \epsilon(\frac{1}{2}, \mu_p) \\
& & \epsilon(\frac{1}{2}, \mu_p) \zeta_p^{-1}(1) & \epsilon(\frac{1}{2}, \mu_p) \\
& & \epsilon(\frac{1}{2}, \mu_p)(\frac{1+p^{-1}}{2} - \frac{1}{\zeta_p(1)^{2}}) & \epsilon(\frac{1}{2}, \mu_p)(1+p^{-1} - \frac{1}{\zeta_p(1)^{2}}) - \zeta_p(1)^{-1} \\
\hline
l > 1 & 0 & \epsilon(\frac{1}{2}, \mu_p)p^{-2} & \epsilon(\frac{1}{2}, \mu_p) \\
& & \epsilon(\frac{1}{2}, \mu_p) \zeta_p^{-1}(1) & \epsilon(\frac{1}{2}, \mu_p) \\
& & \epsilon(\frac{1}{2}, \mu_p)(\frac{1+p^{-1}}{2} - \frac{1}{\zeta_p(1)^{2}}) & \epsilon(\frac{1}{2}, \mu_p)(1+p^{-1} - \frac{1}{\zeta_p(1)^{2}}) - \zeta_p(1)^{-1} \\
\hline
\end{array}
\end{align*}

Finally, we need to look at \( \pi_p = \chi_1 \boxplus \chi_2 \) with \( a(\chi_1) > a(\chi_2) = 0 \). In this case we have
\[
\lambda_{\mu_p \pi_p}(p^m) = \begin{cases} 
\delta_m = 0 & \text{if } \mu_p \neq \omega_{\pi_p}^{-1}, \\
\chi_2(p^m) \delta_{m \geq 0} & \text{if } \mu_p = \omega_{\pi_p}^{-1}.
\end{cases}
\]
Also, \( \delta_{\mu_p \pi_p} = 1 \) if \( \mu_p = \omega_{\pi_p}^{-1} \) and 0 otherwise. For technical reasons we also put \( \delta_{\pi_p} = l \). From [3, Lemma 2.3] one gets the following results.

\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
\mu_p & \mu_p = 1 & \mu_p = \omega_{\pi_p}^{-1} & \mu_p \in p^{\mathcal{L}} \setminus \{1, \omega_{\pi_p}^{-1}\} \\
\hline
l = 0 & \epsilon(\frac{1}{2}, \pi_p) & \zeta_p(1)^{-1} & \epsilon(\frac{1}{2}, \pi_p) \\
\hline
l > 1 & \epsilon(\frac{1}{2}, \pi_p) \chi_2(p^l) & \omega_{\pi_p}(-1) \chi_2(p^{l-1}) & \epsilon(\frac{1}{2}, \mu_p) \\
& & \omega_{\pi_p}(-1) \chi_2(p^{l-1}) & \epsilon(\frac{1}{2}, \mu_p) \\
& & \omega_{\pi_p}(-1) \chi_2(p^{l-1}) & \epsilon(\frac{1}{2}, \mu_p) \\
\hline
\end{array}
\end{align*}

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