CORRECTION: ROOT EXTRACTION IN ONE-RELATOR GROUPS AND SLENDERNESS

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Abstract. This very short correction notes a gap in an argument of an earlier paper, and also provides a theorem of similar flavor to the main result of that paper.

I am indebted to Dawid Kielak [8] for pointing out a gap in the proof of Lemma 2.4 in [3], which was used to prove Theorems A and B of that paper. There are no known counterexamples to those results, so they may be regarded as open problems. The problem with that original argument is that if $L$ and $K$ are two Magnus subgroups of a one-relator group, then the expression of an element in the intersection $L \cap K$ as a word in the free generating set of $L$ can be different from that in the free generating set for $K$. I'll state and prove a result similar to Theorem B with stronger conclusion but an extra assumption (that the one-relator group includes no Baumslag-Solitar subgroups). The proof similarly utilizes root extraction.

The notation will be identical with that of the paper [3], with HEG, HEG"m, etc., as therein. We'll recall the following definitions from [2].

Definition. A group $G$ is cm-slender (respectively lcH-slender) if every abstract group homomorphism $\phi : H \to G$, where $H$ is a completely metrizable (resp. locally compact Hausdorff) topological group, has open kernel.

Theorem. Let $G$ be a (possibly uncountable) one-relator group which has no Baumslag-Solitar subgroup. The following hold.

(1) If $\phi : \text{HEG} \to G$ is an abstract homomorphism then for some $m \in \mathbb{N}$ the image $\phi(\text{HEG}^m)$ is finite.

(2) If $\phi : H \to G$ is an abstract homomorphism, with $H$ either a completely metrizable or locally compact Hausdorff topological group, then there is a normal open subgroup $V \leq H$ with $\phi(V)$ finite.

In particular, a torsion-free one-relator group without Baumslag-Solitar subgroups is n-slender, cm-slender, and lcH-slender.

Proof. We will first prove claim (1) and then give the quite analogous argument for (2). Assume that $\phi : \text{HEG} \to G$ is an abstract group homomorphism, where $G$ is a one-relator group without Baumslag-Solitar subgroups. If $G = \langle X \mid r \rangle$ we let $Y \subseteq X$ be the set of generators used in the word $r$. The group $G$ is isomorphic in the natural way to the free product $\langle Y \mid r \rangle * F(X \setminus Y)$ where $F(X \setminus Y)$ is the free group on generators $X \setminus Y$. Then $\phi : \text{HEG} \to \langle Y \mid r \rangle * F(X \setminus Y)$, so by [8].
Theorem 1.3] we know that for some $N \in \mathbb{N}$ the image $\phi(\text{HEG}^N)$ is included into a conjugate of either $(Y \mid r)$ or $F(X \setminus Y)$. Thus without loss of generality we compose $\phi$ with conjugation by an appropriate element in $G$ so that $\phi(\text{HEG}^N) \leq (Y \mid r)$ or $\phi(\text{HEG}^N) \leq F(X \setminus Y)$. In case $\phi(\text{HEG}^N) \leq F(X \setminus Y)$, since free groups are n-slender [5 Corollary 3.7], we can select $m > N$ such that $\text{HEG}^m \leq \ker(\phi \upharpoonright \text{HEG}^N)$ is trivial, hence finite.

Suppose now that $\phi(\text{HEG}^N) \leq (Y \mid r)$. In case $J = (Y \mid r)$ has torsion we know it is hyperbolic (by the Spelling Theorem of Newman [10]). Then there exists some $m > N$ such that $\phi(\text{HEG}^m)$ is finite [1 Theorem B]. Therefore we may assume that $J$ is torsion-free. As $G$ does not have Baumslag-Solitar subgroups, we know that $J$ is commutative transitive [7 Theorem 1.3] and so has unique root extraction (i.e. if $s_0 = s_1$, $t > 0$, then $s_0 = s_1$). Letting $p$ be a prime greater than the length of the relator $r$, we have that for each nontrivial $s \in J$ there is some $n_s \in \mathbb{N}$ such that the equation $x^{p^{n_s}} = s$ has no solution in $J$ [10 Theorem 1]. Then by unique root extraction we have that for nontrivial $s \in (Y \mid r)$, the set $\{x \in J: (\exists k \in \mathbb{N}) x^{p^k} = s\}$ has cardinality at most $n_s$. Then in the terminology of [2] the group $J$ has finite $p$-antecedents, and as $J$ is countable and torsion-free, we know that $J$ is n-slender [2 Theorems A, B(c)]. Then there exists some $m > N$ such that $\phi \upharpoonright \text{HEG}^m$ is trivial and we have considered the last case for (1).

Now we’ll prove (2). Letting $\phi: H \to G$ with $H$ completely metrizable (respectively locally compact Hausdorff) and $G = (Y \mid r) * F(X \setminus Y)$, with $Y$ finite, we have that either $\ker(\phi)$ is open or $\phi(H)$ lies entirely in a conjugate of $(Y \mid r)$ or of $F(X \setminus Y)$ by [11] (resp. [9]). In case $\ker(\phi)$ is open we are already done. Else we conjugate appropriately so that without loss of generality either $\phi(H) \leq (Y \mid r)$ or $\phi(H) \leq F(X \setminus Y)$. If $\phi(H) \leq F(X \setminus Y)$ then since free groups are cm-slender [4] (resp. lcH-slender, also [4]) we see again that $\ker(\phi)$ is open. We are left with the case where $\phi(H) \leq (Y \mid r)$. If the group $J = (Y \mid r)$ has torsion then it is hyperbolic and by [1 Theorem A] there is an open normal subgroup $V \leq H$ such that $\phi(V)$ is finite. If $J$ is torsion-free, then as in (1) $J$ has finite $p$-antecedents and we have $J$ is cm-slender and lcH-slender [2 Theorems A, B(c)]. Then $\ker(\phi)$ is open and the last case for (2) is complete.

It should be noted that Baumslag-Solitar groups are themselves known to be n-slender, cm-slender, and lcH-slender [2 Theorems A, B(i)]. Finally, we point out that a positive answer to the following question allows one to remove the requirement that the group has no Baumslag-Solitar subgroups.

**Question 1.** If $G$ is a torsion-free one-relator group and $p$ is a prime number greater than the length of the relator of $G$, then does $G$ have finite $p$-antecedents?

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