Supersymmetric Domain Wall and RG Flow
from 4-Dimensional Gauged $\mathcal{N} = 8$ Supergravity

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abstract

By studying various, known extrema of 1) $SU(3)$ sectors, 2) $SO(5)$ sectors and 3) $SO(3) \times SO(3)$ sectors of gauged $\mathcal{N} = 8$ supergravity in four-dimensions, one finds that the deformation of seven sphere $S^7$ gives rise to non-trivial renormalization group(RG) flow in three-dimensional boundary conformal field theory from UV fixed point to IR fixed point. For $SU(3)$ sectors, this leads to four-parameter subspace of the supergravity scalar-gravity action and we identify one of the eigenvalues of $A_1$ tensor of the theory with a superpotential of scalar potential that governs RG flows on this subspace. We analyze some of the structure of the superpotential and discuss first-order BPS domain-wall solutions, using some algebraic relations between superpotential and derivatives of it with respect to fields, that determine a (super)symmetric kink solution in four-dimensional $\mathcal{N} = 8$ supergravity, which generalizes all the previous considerations. The BPS domain-wall solutions are equivalent to vanishing of variation of spin $1/2, 3/2$ fields in the supersymmetry preserving bosonic background of gauged $\mathcal{N} = 8$ supergravity. For $SO(5)$ sectors, there exist only nontrivial nonsupersymmetric critical points that are unstable and included in $SU(3)$ sectors. For $SO(3) \times SO(3)$ sectors, we construct the scalar potential(never been written) explicitly and study explicit construction of first-order domain-wall solutions.
1 Introduction

Few examples are known for three-dimensional interacting conformal field theories, mainly due to strong coupling dynamics in the infrared (IR) limit. In the previous papers [1, 2], three-dimensional (super)conformal field theories were classified by utilizing the AdS/CFT correspondence [3, 4, 5] and earlier, exhaustive study of the Kaluza-Klein supergravity [6].

In contrast to the Freund-Rubin compactifications, the symmetry of the vacuum of Englert type compactification is no longer given by the isometry group of seven-dimensional internal space but rather by the group which leaves invariant both the metric and four-form magnetic field strength. By generalizing compactification vacuum ansatz to the nonlinear level, solutions of the eleven-dimensional supergravity were obtained directly from the scalar and pseudo-scalar expectation values at various critical points of the $\mathcal{N} = 8$ supergravity potential [7]. They reproduced all known Kaluza-Klein solutions of the eleven-dimensional supergravity: round $S^7$ [8], $SO(7)^-$-invariant, parallelized $S^7$ [9], $SO(7)^+$-invariant vacuum [10], $SU(4)^-$-invariant vacuum [11], and a new one with $G_2$ invariance. Among them, round $S^7^-$ and $G_2$-invariant vacua are stable, while $SO(7)^\pm$-invariant ones are known to be unstable [12]. In [2], via AdS/CFT correspondence, deformation of $S^7$ was interpreted as renormalization group flow from $\mathcal{N} = 8$, $SO(8)$ invariant ultraviolet (UV) fixed point to $\mathcal{N} = 1$, $G_2$ invariant IR fixed point by analyzing de Wit-Nicolai potential.

Since embedding or consistent truncation of gauged supergravity is known for $S^7$ compactification of eleven-dimensional supergravity, we also are interested in domain-wall solution in four-dimensional supergravity. In [13], a renormalization group flow from $\mathcal{N} = 8$, $SO(8)$ invariant UV fixed point to $\mathcal{N} = 2$, $SU(3) \times U(1)$ invariant IR fixed point was found by studying de Wit-Nicolai potential which is invariant under $SU(3) \times U(1)$ group. For this interpretation it was crucial to know the form of superpotential that was encoded in the structure of T-tensor of a theory. Moreover, one can proceed this direction for $\mathcal{N} = 1$ $G_2$ fixed point [14]. It turned out that we found first-order BPS equations, by recognizing some algebraic and essential relation between the superpotential and derivative of it with respect to field, whose solutions constitute supersymmetric domain-walls both from direct minimization of energy-functional and from supersymmetry transformation rules.

It is natural and illuminating to ask whether one can construct the most general superpotential for so-far known any critical points in four-dimensional $\mathcal{N} = 8$ gauged supergravity: 1) $SU(3)$-invariant sectors, 2) $SO(5)$-invariant sectors and 3) $SO(3) \times SO(3)$-invariant sector. In order to find and study BPS domain-wall solutions by minimization of energy-functional, one has to reorganize it into sum of complete squares. Then one should expect that the scalar potential takes sum of square of physical quantities. One important feature of the de Wit-Nicolai $d = 4, \mathcal{N} = 8$ supergravity is that the scalar potential can be written as the difference
of two positive square terms. Together with kinetic terms this implies one may construct 
energy-functional in terms of sum of complete squares.

In this paper, we will continue to analyze various known vacua of four-dimensional \( \mathcal{N} = 8 \) supergravity, developed earlier by Warner [15] mainly. In section 2, after reviewing de Wit-Nicolai scalar potential and by explicitly constructing 28-beins \( u^{IJ}_{KL} \) and \( v_{IJKL} \) fields, which are elements of fundamental 56-dimensional representation of \( E_7 \), in terms of scalar, pseudo-scalar fields, and other two fields parametrizing \( SU(2) \times U(1) \) subgroup of \( SU(8) \) of \( \mathcal{N} = 8 \) supergravity, we get \( A_1^{IJ} \) and \( A_2^{IJK}_L \) tensors which are new findings and play an important role. Then we possess the full Lagrangian which consists of kinetic terms and scalar potential terms in terms of a restricted independent four-dimensional slice of scalar manifold. Moreover one also considers other two invariant sectors. In section 3, we identify one of the eigenvalues of \( A_1 \) tensor with “superpotential” of de Wit-Nicolai scalar potential. We describe and present some properties of all the critical points in this invariant subsector and discuss some of the implications of our results. We focus on the nontrivial supersymmetric critical points generalizing the previous results by [13, 14] and obtain the BPS domain-wall solutions from both direct extremization of energy-density and supersymmetry transformation rules. To arrive at this result, in particular, some algebraic relations of superpotential that are newly discovered results will play an important role because without them one can not cancel out the cross terms in energy-functional. We also present an analytic solution for domain-walls in \( SO(3) \times SO(3) \) invariant sector when we assume quadratic order in the fluctuation of field. Finally, in an appendix, there exist some details.

## 2 de Wit-Nicolai Potential

de Wit and Nicolai [16, 17] constructed a four-dimensional supergravity theory by gauging the \( SO(8) \) subgroup of \( E_7 \) in the global \( E_7 \times \text{local } SU(8) \) supergravity of Cremmer and Julia [18] by introducing the appropriate couplings by hand and then constructing the supersymmetry model by Noether procedure. In common with Cremmer-Julia theory, this theory contains self-interaction of a single massless \( \mathcal{N} = 8 \) supermultiplet of spins \((2, 3/2, 1, 1/2, 0^+, 0^-) \) but with local \( SO(8) \times \text{local } SU(8) \) invariance. There is a new parameter, the \( SO(8) \) gauge coupling constant \( g \) beside the gravitational constant. In order to preserve the \( \mathcal{N} = 8 \) supersymmetry, they modified the Cremmer-Julia Lagrangian and transformation rules by other \( g \)-dependent terms. In particular, there was a non-trivial effective potential for the scalars that is proportional to the square of the \( SO(8) \) gauge coupling. It is well known [19] that the 70 real, physical scalars of \( \mathcal{N} = 8 \) supergravity parametrize the coset space \( E_7/SU(8) \) (even though \( E_7 \) symmetry is broken in the gauged theory) since 63 fields \((133 - 63 = 70) \) may be gauged away by an \( SU(8) \) rotation (maximal compact subgroup of \( E_7 \)) and can be described by an element \( \mathcal{V}(x) \) of the
fundamental 56-dimensional representation of $E_7$:

$$\mathcal{V}(x) = \begin{pmatrix} u_{ij}^{IJ}(x) & v_{ijKL}(x) \\ v^{klij}(x) & u^{kl}_{KL}(x) \end{pmatrix},$$

(1)

where $SU(8)$ index pairs $[ij], \cdots$ and $SO(8)$ index pairs $[IJ], \cdots$ are antisymmetrized and therefore $u_{ij}^{IJ}$ and $v_{ijKL}$ fields are $28 \times 28$ matrices and $x$ is the coordinate on 4-dimensional spacetime. Complex conjugation can be done by raising or lowering those indices, for example, $(u_{ij}^{IJ})^* = u^{ij}_{IJ}$ and so on. Under local $SU(8)$ and local $SO(8)$, the matrix $\mathcal{V}(x)$ transforms as $\mathcal{V}(x) \rightarrow U(x)\mathcal{V}(x)O^{-1}(x)$ where $U(x) \in SU(8)$ and $O(x) \in SO(8)$ and matrices $U(x)$ and $O(x)$ are in the appropriate 56-dimensional representation. In the gauged supergravity theory, the 28-vectors transform in the adjoint of $SO(8)$ with resulting non-abelian field strength while, in ungauged supergravity theory, all the vector fields have abelian gauge symmetries and these gauge fields are not minimally coupled to the fermions. It is known that any ground state leaving the symmetry unbroken is necessarily $AdS_4$ space with a cosmological constant proportional to $g^2$. One cannot identify 70 scalars as the Goldstone bosons of $E_7$ breaking to $SU(8)$ because $E_7$ is no longer a symmetry.

Although the full gauged $\mathcal{N} = 8$ Lagrangian is rather complicated [17], the scalar and gravity part of the action is simple (we are considering a gravity coupled to scalar field theory since matter fields do not play a role in domain-wall solutions) and maybe written as

$$\int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{96} \left| A_{\mu}^{ijkl} \right|^2 - V \right),$$

(2)

where the scalar kinetic terms are completely antisymmetric and self-dual in their indices:

$$A_{\mu}^{ijkl} = -2\sqrt{2} \left( u_{ij}^{ij} \partial_{\mu} v^{klij} - v^{ijlij} \partial_{\mu} u^{kl}_{kl} \right),$$

(3)

where $SO(8)$ indices are contracted and $A_{\mu}^{ijkl}$ is a product of $A_{\mu}^{ijkl}$ and its complex conjugation, $A_{\mu,ijkl}$ as above and $\mu$ is the 4-dimensional space-time index. Note that the property of self-dual of $A_{\mu}^{ijkl}$ can not be obtained from directly (3) but from group theoretical arguments based on $E_7$ Lie algebra. Let us define $SU(8)$ so-called T-tensor which is cubic in the 28-beins $u_{ij}^{IJ}$ and $v_{ijKL}$ fields, manifestly antisymmetric in the indices $[ij]$ and $SU(8)$ covariant:

$$T_i^{kij} = \left( u_{ij}^{ij} + v_{ijlij}^{ijlij} \right) \left( u_{lm}^{JK} u_{KL}^{km} - v_{lm}^{JK} v_{lm}^{km} \right).$$

(4)

It is not $E_7$- but only $SO(8)$-invariant since the capital indices are contracted in (4). This comes naturally from introducing a local gauge coupling in the theory. Furthermore, other tensors coming from T-tensor play an important role in this paper and scalar structure is encoded in two $SU(8)$ tensors. These appear in the $g$-dependent interaction terms in addition to the
original Lagrangian. That is, $A_{ij}^{\dagger}$ tensor is symmetric in $(ij)$ and $A_{i j k}^{\dagger}$ tensor is antisymmetric in $[ijk]$:

$$A_{ij} = -\frac{4}{21} T_{ijm} m, \quad A_{ijk} = -\frac{4}{3} T_{[ijk]},$$

obtained by making use of some identities in T-tensor and projecting out the appropriate irreducible components.

Then de Wit-Nicolai effective nontrivial potential, which is invariant under the gauged subalgebra, $SO(8)$ of $E_7$, arising from $SO(8)$ gauging can be written as the difference of two positive definite terms:

$$V = -g^2 \left( \frac{3}{4} |A_{ij}^2| - \frac{1}{24} |A_{ijkl}^2| \right),$$

where $g$ is a $SO(8)$ gauge coupling constant and it is understood that the squares of absolute values of $A_{ij}^2$, $A_{ijkl}^2$ are nothing but products of those and their complex conjugations on 28-beins $u_{ij}^{IJ}$ and $v_{ijkl}^{KL}$ fields. The 56-bein $V(x)$ can be brought into the following form in the $SU(8)$ unitary gauge by the gauge freedom of $SU(8)$ rotation

$$V(x) = \exp \left( \begin{array}{cc} 0 & \phi_{ijkl}(x) \\ \phi^{ijkl}(x) & 0 \end{array} \right),$$

where $\phi^{ijkl}$ is a complex self-dual tensor describing the 35 scalars $35_v$(the real part of $\phi^{ijkl}$) and 35 pseudo-scalar fields $35_c$(the imaginary part of $\phi^{ijkl}$) of $\mathcal{N} = 8$ supergravity. After gauge fixing, one does not distinguish between $SO(8)$ and $SU(8)$ indices $[IJ]$ and $[ij]$, and they are on equal footing. The full supersymmetric solution where both $35_v$ scalars and $35_c$ pseudo-scalars vanish yields $SO(8)$ vacuum state with $\mathcal{N} = 8$ supersymmetry(Note that $SU(8)$ is not a symmetry of the vacuum). In this case, 70 scalars(and pseudo-scalars) are tachyonic.

### 2.1 $SU(3)$ Sectors of Gauged $\mathcal{N} = 8$ Supergravity

We will start with gauged $\mathcal{N} = 8$ supergravity in four-dimensions. The scalar potential is a function of 70 scalars and this number is too large to be managed practically and one should reduce the problem by looking at all critical points that reduce the gauge/R-symmetry to a group containing a particular $SU(3)$ subgroup of $SO(8)$. For one possible embedding of $SU(3)$ corresponding to the decomposition of three basic representations of $SO(8)$ into $SU(3)$ representations $8_v, 8_s, 8_c \rightarrow 3 + \overline{3} + 1 + 1$, all of the 35-dimensional representations of $SO(8)$ decompose into $8 + 6 + \overline{3} + 3 + \overline{3} + \overline{3} + 1 + 1 + 1$. Then the set of 70 scalars in $\mathcal{N} = 8$ supergravity contains 6 singlets of $SU(3)$ (three singlets for $35_v$ and three singlets for $35_c$).

For the other embeddings of $SU(3)$ $8_v, 8_s, 8_c \rightarrow 8$, all of the 35-dimensional representations of $SO(8)$ decompose into $27 + 8$ implying that there are no $SU(3)$ singlets in the scalar sectors. It
is known [13] that $SU(3)$ singlet space with a breaking of the $SO(8)$ gauge group into a group which contains $SU(3)$ may be written in terms of two real parameters $\lambda$ and $\lambda'$:

$$\phi_{ijkl} = S(\lambda G^+_1 + \lambda' G^+_2),$$

where the action $S$ is $SU(2) \times U(1)$ subgroup of $SU(8)$ on its 70-dimensional representation in the space of self-dual complex four-forms:

$$S = \text{diag}(w, w, w, w, w, w, w, w^{-3}P),$$

where $w = e^{i\alpha/4}$ is a pure phase, and $P$ is a general $SU(2)$ matrix:

$$P = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{pmatrix}
\begin{pmatrix}
e^{i\phi} & 0 \\
0 & e^{-i\phi} \\
\end{pmatrix}
\begin{pmatrix}
\cos \psi & -\sin \psi \\
\sin \psi & \cos \psi \\
\end{pmatrix}.$$}

In the notation of [20], $G^+_1 = X^+_1 + X^+_2 + X^+_3$ and $G^+_2 = X^+_4 + X^+_5 + X^+_6 + X^+_7$ where the self-dual and anti-self-dual four-forms are given by

$$X^\pm_1 = \frac{1}{2}(\delta^{1234}_{ijkl} \pm \delta^{5678}_{ijkl}), X^\pm_2 = \frac{1}{2}(\delta^{1256}_{ijkl} \pm \delta^{3478}_{ijkl}), X^\pm_3 = \frac{1}{2}(\delta^{1278}_{ijkl} \pm \delta^{3456}_{ijkl}),$$

$$X^\pm_4 = -\frac{1}{2}(\delta^{1357}_{ijkl} \pm \delta^{2468}_{ijkl}), X^\pm_5 = \frac{1}{2}(\delta^{1368}_{ijkl} \pm \delta^{2457}_{ijkl}), X^\pm_6 = \frac{1}{2}(\delta^{1458}_{ijkl} \pm \delta^{2367}_{ijkl}),$$

$$X^\pm_7 = \frac{1}{2}(\delta^{1467}_{ijkl} \pm \delta^{2358}_{ijkl}).$$

Then the parametrization of [13] for the $SU(3)$-singlet space that is invariant subspace under a particular $SU(3)$ subgroup of $SO(8)$ becomes

$$\phi_{ijkl} = \lambda \cos\alpha X^1 + \lambda \sin\alpha X^2 + \lambda' \cos\phi X^3 + \lambda' \sin\phi X^4,$$

where

$$Y^1_{ijkl} = \varepsilon_+ \left[(\delta^{1234}_{ijkl} \pm \delta^{5678}_{ijkl}) + (\delta^{1256}_{ijkl} \pm \delta^{3478}_{ijkl}) + (\delta^{1357}_{ijkl} \pm \delta^{2468}_{ijkl}) + (\delta^{1368}_{ijkl} \pm \delta^{2457}_{ijkl}) + (\delta^{1458}_{ijkl} \pm \delta^{2367}_{ijkl}) + (\delta^{1467}_{ijkl} \pm \delta^{2358}_{ijkl})\right],$$

$$Y^2_{ijkl} = \varepsilon_- \left[-(\delta^{1234}_{ijkl} \pm \delta^{5678}_{ijkl}) + (\delta^{1256}_{ijkl} \pm \delta^{3478}_{ijkl}) + (\delta^{1357}_{ijkl} \pm \delta^{2468}_{ijkl}) + (\delta^{1368}_{ijkl} \pm \delta^{2457}_{ijkl}) + (\delta^{1458}_{ijkl} \pm \delta^{2367}_{ijkl}) + (\delta^{1467}_{ijkl} \pm \delta^{2358}_{ijkl})\right],$$

where $\varepsilon_+ = 1$ and $\varepsilon_- = i$ and $+$ gives the scalars and $-$ the pseudo-scalars. Therefore 56-beins $\mathcal{V}(x)$ can be written as $56 \times 56$ matrix whose elements are some functions of scalar, pseudo-scalars, $\alpha$ and $\phi$ out of seventy fields by exponentiating the vacuum expectation value $\phi_{ijkl}$ through (3). On the other hand, 28-beins $u^{IJ}_{KL}$ and $v_{IJKL}$ are elements of this $\mathcal{V}(x)$ according to (4). One can construct 28-beins $u^{IJ}_{KL}$ and $v_{IJKL}$ in terms of these fields explicitly and they are given in the appendix (36). Now it is ready to get the complete expression for $A^I_1$ and $A^I_{2LJK}$ tensors in terms of $\lambda, \lambda', \alpha$ and $\phi$ using (4) and (5). This will be our first new findings and main ingredients that are necessary to proceed further.
It turns out from (3) that \( A_{1}^{IJ} \) tensor has three distinct complex eigenvalues, \( z_1, z_2 \) and \( z_3 \) with degeneracies 6, 1, and 1 respectively and has the following form

\[
A_{1}^{IJ} = \text{diag} (z_1, z_1, z_1, z_1, z_2, z_3),
\]

where the eigenvalues are functions of \( \lambda, \lambda', \alpha \) and \( \phi \):

\[
\begin{align*}
  z_1 &= e^{-2i(\alpha+\phi)} \left[ e^{3i\alpha} p^2 q r^2 t^2 + e^{i(3\alpha+4\phi)} p^2 q r^2 t^2 + e^{2i(\alpha+\phi)} p \left( 4q^2 r^2 t^2 + p^2 \left( r^4 + t^4 \right) \right) \\
  &+ pq^2 r^2 t^2 + e^{i\phi} pq^2 r^2 t^2 + e^{i(\alpha+2\phi)} q \left( 4p^2 r^2 t^2 + q^2 \left( r^4 + t^4 \right) \right) \right], \\
  z_2 &= e^{-4i\phi} \left( p + e^{i\alpha} q \right) \left( e^{4i\phi} p^2 r^4 - e^{i(\alpha+4\phi)} pq r^4 + e^{2i(\alpha+2\phi)} q^2 r^4 + 6e^{i(\alpha+2\phi)} pq r^2 t^2 \\
  &+ p^2 t^4 - e^{i\alpha} pq t^4 + e^{2i\alpha} q^2 t^4 \right), \\
  z_3 &= 6e^{i(\alpha+2\phi)} p^2 q r^2 t^2 + 6e^{2i(\alpha+\phi)} pq r^2 t^2 + p^3 \left( r^4 + e^{4i\phi} t^4 \right) + e^{3i\alpha} q^3 \left( r^4 + e^{4i\phi} t^4 \right),
\end{align*}
\]

and we denote hyperbolic functions of \( \lambda \) and \( \lambda' \) by the following quantities which will be used throughout this paper

\[
p \equiv \cosh \left( \frac{\lambda}{2\sqrt{2}} \right), \quad q \equiv \sinh \left( \frac{\lambda}{2\sqrt{2}} \right), \quad r \equiv \cosh \left( \frac{\lambda'}{2\sqrt{2}} \right), \quad t \equiv \sinh \left( \frac{\lambda'}{2\sqrt{2}} \right).
\]

One of the eigenvalues of \( A_{1}^{IJ} \) tensor, \( z_3 \), will provide a “superpotential” of scalar potential \( V \) and be crucial for analysis of domain-wall solutions later. First, the BPS domain-wall solutions are nothing but the gradient flow equations of this superpotential defined on a four-dimensional slice of the full scalar manifold. Second, the modified \( g \)-dependent supersymmetry transformation rule of gravitinos obtained by gauging \( SO(8) \) group contains the superpotential and it is very important to have this form of superpotential when we consider its properties under the supersymmetric bosonic background.

Similarly, \( A_{2,L}^{IJK} \) tensor can be obtained from the triple product of \( u^{IJ}_{KL} \) and \( v_{IJKL} \) fields, that is, from (3). It turns out that they are written as eight-kinds of fields \( y_i \) where \( i = 1, \cdots, 8 \) and are given in the appendix (37) where we stressed the fact that some of these are related to the derivatives of eigenvalues of \( A_{1}^{IJ} \) tensor with respect to \( \lambda \) and \( \lambda' \). As already mentioned before, the scalar potential consists of \( |A_1^{ij}|^2 \) and \( |A_2^{i,jkl}|^2 \). Since the former is made of squares of superpotential plus other terms and the latter is made of squares of derivatives of superpotential with respect to \( \lambda \) and \( \lambda' \) plus other terms, we will see that both other terms from \( A_1 \) and \( A_2 \) tensors are exactly cancelled out and lead to the sum of square of superpotential and square of derivatives of superpotential. Finally, the scalar potential (3) can be written, by combining all the components of \( A_{1}^{IJ}, A_{2,L}^{IJK} \) tensors, as

\[
V(\lambda, \lambda', \alpha, \phi) = -g^2 \left[ \frac{3}{4} \times \left( 6|z_1|^2 + |z_2|^2 + |z_3|^2 \right) \\
- \frac{1}{24} \times 6 \left( 12|y_1|^2 + 3|y_2|^2 + 3|y_3|^2 + 12|y_4|^2 + 12|y_5|^2 + 4|y_6|^2 + 4|y_7|^2 + 6|y_8|^2 \right) \right]
\]

6
\[
\frac{1}{2}g^2 \left( s'^4 \left[ (x^2 + 3)c^3 + 4x^2v^3s^3 - 3v(x^2 - 1)s^3 + 12xv^2cs^2 - 6(x - 1)cs^2 + 6(x + 1)c^2sv \right] \\
+ 2s'^2 \left[ 2(c^3 + v^3s^3) + 3(x + 1)vs^3 + 6xv^2cs^2 - 3(x - 1)cs^2 - 6c \right] - 12c \right),
\]

which is exactly the same form obtained by [13] using \( SU(8) \) coordinate as an alternative approach for which one has to know about kinetic terms explicitly as well as scalar potential terms in order to understand the supergravity domain-wall solutions and we introduce the following quantities for simplicity

\[
c \equiv \cosh \left( \frac{\lambda}{\sqrt{2}} \right), \quad s \equiv \sinh \left( \frac{\lambda}{\sqrt{2}} \right), \quad c' \equiv \cosh \left( \frac{\lambda'}{\sqrt{2}} \right), \quad s' \equiv \sinh \left( \frac{\lambda'}{\sqrt{2}} \right) \\
v \equiv \cos \alpha, \quad x \equiv \cos 2\phi.
\]

Although one gets the explicit form of scalar potential by exploiting the method given by [13], another task is to find out kinetic terms. This is one of the reasons why we took different route. The scalar potential does not depend on \( \theta \) and \( \psi \) of \( SU(2) \) matrix reflecting \( SO(8) \) invariance of the potential and a larger invariance of the \( SU(3) \)-singlet sector, respectively. The potential contains as special case the examples previously studied in the literature. One can easily see that by putting \( \lambda = \lambda' \) and \( \alpha = \phi \), (13) will reduce to the one studied in [14, 2] while by putting \( \alpha = 0 \) and \( \phi = \pi/2 \), one gets the one considered in [13].

### 2.2 \( SO(5) \) Sectors of Gauged \( \mathcal{N} = 8 \) Supergravity

The non-maximally symmetric example of the Freund-Rubin compactification to a product of \( AdS_4 \) space-time and an arbitrary compact seven-dimensional Einstein manifold is provided by squashed seven sphere \( S^7 \). The effective four-dimensional theory has \( SO(5) \times SO(3) \) gauge symmetry and \( \mathcal{N} = 1 \) or \( \mathcal{N} = 0 \) depending on the orientation of the \( S^7 \) [21]. The original motivation for studying all the critical points of gauged \( \mathcal{N} = 8 \) supergravity having \( SO(5) \) symmetry at least was to find some connections between Freund-Rubin type solution and de Wit-Nicolai theory. One must characterize the action of \( SO(5) \subset SO(8) \) on the physical fields. There are three ways of embedding \( SO(5) \) in \( SO(8) \). In the symmetric gauge, the seventy scalars of \( \mathcal{N} = 8 \) supergravity maybe described as elements of self-dual four forms and anti-self-dual four forms. The 35 scalars break into \( 35_v \rightarrow 10 + 10 + 10 + 5 \) of \( SO(5) \) while 35 pseudo-scalars \( 35_c \rightarrow 10 + 10 + 10 + 5 \). Therefore \( SO(5)^v \) embedding case gives no \( SO(5) \)-singlets among either true scalars or pseudo-scalars where \( 8_v \rightarrow 5 + 1 + 1 + 1 \). Then \( SO(5)^v \) embedding gives only trivial vacuum where all seventy scalars vanish with unbroken \( SO(8) \) symmetry. We restrict to ourselves for other two embeddings 1) \( SO(5)^+ \) embedding case where \( 8_s \rightarrow 5 + 1 + 1 + 1 + 1 \) and 2) \( SO(5)^- \) embedding case where \( 8_c \rightarrow 5 + 1 + 1 + 1 \).

- \( SO(5)^+ \) embedding
This case contains $SO(5)$-singlets among six scalars $(35 \rightarrow 14 + 5 + 5 + 1 + 1 + 1 + 1 + 1 + 1)$ which can be parametrized by
the following form after acting $SO(3)$ rotation of $SO(8)$ on its 70-dimensional representation in the space:

$$
\phi_{ijkl} = \lambda \left( X_1^+ + X_2^+ + X_3^+ \right) + \mu \left( X_1^+ + X_4^+ + X_5^+ \right) + \rho \left( X_1^+ - X_6^+ - X_7^+ \right),
$$

where $\lambda, \mu$ (not space-time index) and $\rho$ are real parameters and self-dual four-forms $X_\alpha'^+$’s are given in (8). We used the fact that as a consequence of $SO(8)$ symmetry of the theory, the potential does not depend on $SO(3)$ rotation parametrized by other three parameters. As we have done before, we can describe $28$-beins $u_{IJKL}^I$ and $v_{IJKL}^I$ in terms of $\lambda, \mu$ and $\rho$ and they are given in the appendix (39) with upper plus sign for each $4 \times 4$ submatrix $u_i^+$ and $v_i^+$. It turns out that $A_{IJ}^I$ tensor has a single eigenvalue $z_1$ with multiplicity 8 which will provide a superpotential of scalar potential and has the following expression

$$
A_{IJ}^I = \text{diag} \left( z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1 \right),
$$

where the eigenvalues are real and some combinations of $u, v$ and $w$ fields, and take the form

$$
z_1 = -\frac{1}{8\sqrt{uvw}} \left( 5 + u^2 v^2 + \text{two cyclic permutations} \right),
$$

and we introduce new fields $u, v$ and $w$ as

$$
u \equiv e^{\lambda/\sqrt{2}}, \quad v \equiv e^{\mu/\sqrt{2}}, \quad w \equiv e^{\rho/\sqrt{2}}.
$$

Also one can construct $A_{2L}^{IJK}$ tensor which are the combinations of triple product of $28$-beins $u_{KL}^{I}$ and $v_{IJKL}$, they are written as four-kinds of fields $y_{i,+}$ where $i = 1, \cdots, 4$ and are given in the appendix (31). Therefore one gets the scalar potential $V(\lambda, \mu, \rho)$ by summing all the components of $A_{IJ}^I, A_{2L}^{IJK}$ tensors and counting the degeneracies correctly:

$$
V = -g^2 \left[ \frac{3}{4} \times 8z_1^2 - \frac{1}{24} \times 48 \left( y_{1,+}^2 + 2y_{2,+}^2 + 2y_{3,+}^2 + 2y_{4,+}^2 \right) \right] \\
= \frac{1}{8} g^2 \left( u^3 v^3 / w - 10uv / w - 2uvw^3 + \text{two cyclic permutations} - 15 / uvw \right).
$$

This is exactly the same form obtained by Romans [22]. There exists one nontrivial extremum at $u = v = 1 / w = 5^{1/4}$ which has a $\mathcal{N} = 0$ nonsupersymmetric $SO(7)^+$ gauge symmetry besides trivial one which has $\mathcal{N} = 8$ maximal supersymmetric $SO(8)$ gauge symmetry for which $u = v = w = 1$.

• $SO(5)$ embedding

\footnote{Unfortunately we use $u, v$ letters here in order to keep the notation as in the literature [22]. We hope these are nothing to do with $28$-beins $u_{KL}^{I}, v_{IJKL}$ fields we have used before.}
In this case, there exist six $SO(5)$-singlets among the pseudo-scalars $(35 \rightarrow 14 + 5 + 5 + 1 + 1 + 1 + 1 + 1 + 1)$. By extra $SU(8)$ element transforming self-dual into anti-self-dual four-forms, one can parametrize as follows.

$$\phi_{ijkl} = i\lambda \left( X_1^- + X_2^- + X_3^- \right) + i\mu \left( X_4^- + X_5^- \right) + i\rho \left( X_6^- - X_7^- \right), \quad (18)$$

with (8). Similarly, it turns out that

$$A^{IJ}_1 = \text{diag} \left( z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1 \right),$$

where the eigenvalues are complex and are given by

$$z_1 = \frac{(1 + i)}{16} \frac{1}{(uvw)^{3/2}} \left( -iu^2 + u^3v^3w + \text{two cyclic permutations} + 5uvw - 5iu^2v^2w^2 \right), \quad (19)$$

with (17). Therefore one gets the scalar potential $V(\lambda, \mu, \rho)$ by summing over all the components of $A^{IJ}_1, A_{IL}^{IJK}$ tensors with (14):

$$V = -g^2 \left[ \frac{3}{4} \times 8|z_1|^2 - \frac{1}{24} \times 48 \left( |y_{1,-}|^2 + 2|y_{2,-}|^2 + 2|y_{3,-}|^2 + 2|y_{4,-}|^2 \right) \right]$$

$$= \frac{1}{16} g^2 \left[ u^3v^3/w + w/u^3v^3 - 2(uvw^3 + 1/uvw^3) - 10(w/w + w/uv) \right.$$

$$+ \text{two cyclic permutations} - 15(uvw + 1/uvw) \left. \right],$$

which was found in [22] and has two nontrivial extrema: one with $N = 0$ nonsupersymmetric $SO(7)$ symmetry at $u = v = 1/w = (1 + \sqrt{5})/2$ and the other with $N = 0$ nonsupersymmetric $SO(6) = SU(4)$ symmetry at $u = 1/w = \sqrt{2} + 1$ and $v = 1$. Since $SO(7)^\pm$ and $SO(6)^-$ contain $SU(3)$ as a subgroup, these critical points also appeared in the previous subsection for the $SU(3)$ sectors.

### 2.3 $SO(3) \times SO(3)$ Sectors of Gauged $\mathcal{N} = 8$ Supergravity

de Wit and Nicolai have constructed gauged supergravity theories for $\mathcal{N} = 5$ [23] and this form of $\mathcal{N} = 5$ scalar potential was obtained by natural truncation of $\mathcal{N} = 8$ scalar potential in [20] ($\mathcal{N} = 6$ scalar potential was obtained also). Moreover $\mathcal{N} = 4$ gauged $SO(4)$ supergravity was obtained by truncation of $\mathcal{N} = 8$ gauge $SO(8)$ supergravity [24]. It is known [20] that $SO(3) \times SO(3)$ singlet space with a breaking of the $SO(8)$ gauge group into $SO(3) \times SO(3)$ may be written as:

$$\phi_{ijkl} = S(\lambda^\alpha X^+_{\alpha}), \quad \alpha = 1, \cdots, 7 \quad (20)$$

\footnote{Our convention for $\lambda^\alpha$ field is different from that of Warner: $\lambda^\alpha_{\alpha r} = \lambda^\alpha_\alpha / \sqrt{2}$.}
where the action $S$ is $SO(3) \times SO(3)$ subgroup of $SU(8)$ on its 70-dimensional representation in the space of self-dual four-forms:

$$S = \text{diag}(1, 1, 1, P, 1, 1, 1), \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$  

Self-dual four forms $X^\alpha_\alpha$’s are the same as in \((8)\). From explicit form of 28-beins $u^{IJKL}_I$ and $v_{IJKL}$ (and are given in the appendix of original version of hep-th archive), those are functions of seven parameters $\lambda^\alpha$ and it turns out that $A_1^{IJ}$ tensor has eight distinct components, $z_i$ where $i = 1, \cdots, 8$ with degeneracies 2 and has the following form

$$A_1^{IJ} = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & z_2 \\ 0 & z_3 & 0 & 0 & 0 & 0 & z_4 & 0 \\ 0 & 0 & z_5 & 0 & 0 & z_6 & 0 & 0 \\ 0 & 0 & 0 & z_7 & z_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_8 & z_7 & 0 & 0 & 0 \\ 0 & 0 & z_6 & 0 & 0 & z_5 & 0 & 0 \\ 0 & z_4 & 0 & 0 & 0 & z_3 & 0 & 0 \\ z_2 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 \end{pmatrix},$$

where these components are some functions of $\lambda^\alpha(\alpha = 1, \cdots, 7)$ as follows:

\begin{align*}
z_1 &= q_1(p_4p_7q_2(p_5p_6q_3 + p_3q_5q_6) + q_4(p_2p_3p_6p_7q_5 + p_2q_5p_7q_3q_6 + p_3p_5p_6q_2q_7 + q_2q_3q_5q_6q_7)) + (p_i \leftrightarrow q_i), \\
z_2 &= -\frac{i}{4}(r_1 - r_2 - r_4 + r_7)(p_6q_3q_5 + p_3p_5q_6), \\
z_3 &= q_1(q_2q_4(p_3p_5p_6 + q_3q_5q_6)q_7 + p_4(p_3q_6p_7q_2q_3 + p_3q_7q_2q_5q_6 + p_2q_6q_3q_5q_6) + p_2p_6q_3q_5q_7 + p_2p_3p_5q_6q_7)) + (p_i \leftrightarrow q_i), \\
z_4 &= -\frac{i}{4}(r_1 - r_2 + r_4 - r_7)(p_6p_3q_5 + q_3p_5q_6), \\
z_5 &= q_1(q_2q_4(p_3p_5p_6 + q_3q_5q_6)q_7 + p_4(p_3q_6p_7q_4q_5 + p_5q_7q_3q_4q_6 + p_4p_6q_3q_5q_7)) + (p_i \leftrightarrow q_i), \\
z_6 &= -\frac{i}{4}(r_1 + r_2 - r_4 - r_7)(p_6q_3p_5 + p_3q_5q_6), \\
z_7 &= \frac{1}{4}(r_1 + r_2 + r_4 + r_7)(p_6q_3p_5 + q_3q_5q_6), \\
z_8 &= i(p_4(q_5(p_3q_7q_1q_2q_6 + p_1p_3p_6q_2q_7 + p_2q_6q_1q_3q_7) + p_5p_6q_7q_1q_2q_3 + p_2q_3q_1q_4q_7 + p_1q_2q_3q_6q_7)) + (p_i \leftrightarrow q_i),
\end{align*}

(21)

and a compact notation can be defined by setting (of course these are nothing to do with the one in \((12)\)):

$$p_i \equiv \cosh \left(\frac{\lambda_i}{2}\right), \quad q_i \equiv \sinh \left(\frac{\lambda_i}{2}\right), \quad r_i \equiv \cosh \lambda_i, \quad t_i \equiv \sinh \lambda_i, \quad i = 1, 2, \cdots, 7.$$  

(22)
One of the components of $A^{IJ}_1$ tensor, $z_7$, plays a role of superpotential of scalar potential. Similarly, $A^{IJK}_{2L}$ tensor can be obtained from the triple product of $u^{JK}_{KL}$ and $v^{IJKL}$ fields. It turns out that they are written as 54-kinds of fields $y_i, y_{i,+}$ and $y_{i,-}$ (and are given in the appendix of original version of hep-th archive). Finally, the scalar potential as a function of $y_i$ becomes

$$V = \frac{g^2}{32} \left[ \frac{3}{4} \times 2 \sum_{i=1}^{8} |z_i|^2 - \frac{1}{24} \times 12 \left( \sum_{i=1}^{4} |y_{i,+}|^2 + \sum_{i=1}^{4} |y_{i,-}|^2 + \sum_{i=5}^{8} |y_i|^2 + 2 \sum_{i=9}^{10} |y_i|^2 \right) \right]$$

$$= \frac{g^2}{32} \left[ -2r_r + r_r r_r + r_r r_r - 32 r_r r_r - 32 r_r r_r + r_r r_r - 2r_r r_r r_r - 16 t_t t_t - 16 t_t t_t + 2r_r r_r t_t + 4r_r t_t t_t + 16 r_r r_r t_t t_t \right.$$

$$- 16 t_t t_t - 4r_r t_t t_t - 32 t_t t_t + 2r_r^2 t_t t_t - r_r r_r t_t t_t + 2r_r^2 t_t t_t - r_r r_r r_r t_t t_t + 2r_r^2 t_t t_t$$

$$+ 24 r_r r_r r_r t_t t_t + 2r_r^2 t_t t_t + 2t_t t_t t_t + 2t_t t_t t_t + 2t_t t_t t_t - 2r_t t_t t_t - 16 t_t t_t + 2r_r t_t t_t + 4r_r t_t t_t + 16 r_r r_r t_t t_t$$

$$- 16 t_t t_t - 4r_r r_r t_t t_t + 24 t_t t_t t_t t_t + 2t_t t_t t_t t_t + r_5 (1 - 2r_t t_t t_t)$$

$$+ 16 r_r r_r r_r t_t t_t + 2t_t t_t t_t - 2r_t t_t t_t - 24 t_t t_t t_t + 2r_r t_t t_t + 4r_r t_t t_t + 16 r_r r_r t_t t_t$$

$$+ 16 r_r r_r r_r t_t t_t + t_t t_t t_t - 2r_t t_t t_t - 24 r_r t_t t_t + 4r_r t_t t_t + 2r_r^2 t_t t_t$$

$$+ 2r_r^2 t_t t_t + 2r_r^2 t_t t_t - 2r_t t_t t_t + 4r_r t_t t_t - 2r_t t_t t_t$$

$$+ 16 r_r r_r t_t t_t + t_t t_t t_t - 2r_t t_t t_t - 24 r_r t_t t_t + 4r_r t_t t_t - 2r_t t_t t_t$$

$$+ 2r_r^2 t_t t_t + r_5 (1 - 2r_t t_t t_t + 16 r_r r_r t_t t_t) - 4r_r t_t t_t (r_5 - 16 t_t t_t)$$

$$+ 2(-32 + 2r_r r_t + r_5 r_6 (1 + 24 r_r r_t) + 16 t_t t_t t_t)$$

$$+ 2(-2r_r r_7 (r_5 - 16 t_t t_t) + 4r_r r_6 (8 + 16 r_t t_t t_t)$$

with (22). Although the property of the scalar potential was discussed in [20], it was never written explicitly. One can easily calculate the derivatives of this scalar potential as a function of $y_i$, $y_{i,+}$ and $y_{i,-}$ which generalize $SO(3)$ gauge symmetry for which all parameters vanish and whose cosmological constant is $-6g^2$. However the cosmological constant becomes $-14g^2$ at this extremal surface [21] which generalizes $SO(5)$ model. The extremal structure of the $N = 5$ potential is exhibited by the $N = 8$ potential which breaks the $SO(8)$ down to $SO(3) \times SO(3)$. That is, the surface of stationary points was obtained by embedding the $N = 5$ stationary surface in the $N = 8$ theory. One finds that, at
this nontrivial surface, the $A_{1}^{IJ}$ tensor has the following form with degeneracies 6, 2

$$A_{1}^{IJ} = \text{diag} \left( \sqrt{5}, \sqrt{5}, \sqrt{5}, 3, 3, \sqrt{5}, \sqrt{5}, \sqrt{5} \right).$$

One can easily check that there is no supersymmetry because the eigenvalues of $A_{1}^{IJ}$ tensor are not equal to $\sqrt{-\Lambda/6g^{2}}$ at $\Lambda = -14g^{2}$. In other words, if there is a supersymmetry, then the cosmological constant must be either $-30g^{2}$ or $-54g^{2}$. But the gravitational field equations require that the $AdS_{4}$ vacuum on the extremal surface has $\Lambda = -14g^{2}$. Therefore there are no supersymmetries.

3 Supersymmetric Domain Wall and RG Flow

3.1 $SU(3)$ Sectors

In this subsection, we investigate domain walls \[25\] arising in supergravity theories with a nontrivial superpotential defined on a restricted independent four-dimensional slice of the scalar manifold. We analyze a particular $SU(3)$ invariant sector of the scalar manifold of gauged $\mathcal{N} = 8$ supergravity in four-dimensions and study all the critical points of the potential within this sector. The critical points give rise to $AdS_{4}$ vacua and preserve at least $SU(3)$ gauge symmetry in the supergravity(or R-symmetry of the dual field theory). The presence and exact knowledge of the supergravity potential implies a completely determined non-trivial operator algebra in dual field theory. Using Einstein’s equations and energy condition, it will be possible to show that monotonic function can be found in any kink geometry with Poincare symmetries of the boundary theory in flat space. On the subsector, one can write the supergravity potential describing RG flows through steepest descent in the canonical form. From the effective non-trivial scalar potential (6) which consists of two parts, one expects that the superpotential we are considering maybe encoded in either $A_{1}^{IJ}$ tensor or $A_{2,L}^{IJK}$ tensor. It turns out that one of the eigenvalues of $A_{1}^{IJ}$ tensor (11), $z_{3}$, provides a “superpotential” $W$ related to scalar potential $V$ by

$$V(\lambda, \lambda', \alpha, \phi) = g^{2} \left[ \frac{16}{3} \left| \frac{\partial z_{3}}{\partial \lambda} \right|^{2} + 4 \left| \frac{\partial z_{3}}{\partial \lambda'} \right|^{2} - 6 |z_{3}|^{2} \right],$$

(23)

where $z_{3}$ is a function of $\lambda, \lambda', \alpha$ and $\phi$:

$$z_{3}(\lambda, \lambda', \alpha, \phi) = 6e^{i(\alpha+2\phi)}p^{2}qr^{2}t^{2} + 6e^{2i(\alpha+\phi)}pq^{2}r^{2}t^{2} + p^{3}(r^{4} + e^{4i\phi}t^{4})$$

$$+ e^{3i\alpha}q^{3}(r^{4} + e^{4i\phi}t^{4}),$$

(24)

with (12). At first sight, there is no dependence on the derivatives of $z_{3}$ with respect to the fields $\alpha$ and $\phi$ in the (23). We have found that the complex-valued superpotential $z_{3}$ satisfies
the following algebraic relations:

\[
\begin{align*}
\partial_\alpha \log |z_3| &= 2\sqrt{2} pq \partial_\lambda \text{Arg} z_3^*, \\
\partial_\phi \log |z_3| &= 2\sqrt{2} rt \partial_{\lambda'} \text{Arg} z_3^*,
\end{align*}
\]

(25)

which relate the derivative of magnitude of \(z_3\) with respect to \(\alpha(\phi)\) to the one of angle of \(z_3^*\) with respect to \(\lambda(\lambda')\). Then it is elementary to show that one can express the scalar potential by exploiting the above relations as following form indicating the magnitude of \(z_3\) serves as the true superpotential:

\[
W(\lambda, \lambda', \alpha, \phi) = |z_3|,
\]

\[
V(\lambda, \lambda', \alpha, \phi) = g^2 \left[ \frac{16}{3} (\partial_\lambda W)^2 + \frac{2}{3pq^2} (\partial_\alpha W)^2 + 4 (\partial_{\lambda'} W)^2 + \frac{1}{2rt^2} (\partial_\phi W)^2 - 6W^2 \right].
\]

(26)

Let us note that by differentiating this \(V\) with respect to one of fields among \(\lambda, \lambda', \alpha\) and \(\phi\), the scalar potential \(V\) has critical points at 1) critical points of \(W\) and at 2) points for which \(W\) satisfies some differential equation. In this sense, the role of superpotential \(W\) is important because the property of critical points of scalar potential is encoded in those of superpotential. At the three supersymmetric critical points(Table 1) in the supergravity context for which supersymmetric flows are generically simpler, more controllable and represent very stable fixed points than nonsupersymmetric ones, the absolute values of gradients of \(z_3\) with respect to \(\lambda, \lambda'\) vanish. That is \(\frac{\partial z_3}{\partial \lambda} = \frac{\partial z_3}{\partial \lambda'} = 0\). In other words, in terms of \(W\), they are equivalent to \(\partial_\lambda W = \partial_{\lambda'} W = \partial_\alpha W = \partial_\phi W = 0\). This implies that supersymmetry preserving vacua have negative cosmological constant: the scalar potential \(V\) at the three critical points becomes \(V = -6g^2|z_3|^2\) or \(W = \sqrt{-V/6g^2}\). The critical points of \(W\) yield supersymmetric stable \(AdS_4\) vacua in supergravity which will imply non-trivial conformal fixed points in the dual field theory under appropriate conditions. Supersymmetry ensures that there are no unstable modes in a supersymmetry preserving solution to the supergravity equations. The other critical points of \(V\) yield nonsupersymmetric(but usually \(AdS_4\)) vacua that may or may not be stable(in order to be stable, the small oscillations must satisfy the Breitenlohner-Freedman condition [27]). The superpotential \(W\) has the following values at the various supersymmetric or nonsupersymmetric critical points. There is well-known trivial critical point, corresponding to the \(S^7\) compactification of the 11-dimensional supergravity, at which all the supergravity scalar and pseudo-scalar fields vanish and whose cosmological constant is \(\Lambda = -6g^2\) and which preserves \(\mathcal{N} = 8\) supersymmetry.

\[\text{This observation was motivated by the result of [14] where only a single relation holds because } G_2 \text{ invariant sector satisfies } \lambda = \lambda' \text{ and } \alpha = \phi. \text{ Similar aspect happens for } SO(3) \text{-invariant } AdS_5 \text{ gauged supergravity in [26].}\]
| Gauge symmetry | \( s, s', \alpha, \phi \) | \( W \) | \( V \) |
|----------------|-----------------|------|------|
| \( SO(8) \) | \( s = 0 = s' \) | 1 | \(-6g^2\) |
| \( SO(7)^- \) | \( s = \pm \frac{1}{2}, s' = \frac{1}{2}, \alpha = \frac{\pi}{2} = \phi \) | \( \frac{3 \times 5^{3/4}}{8} \) | \(-25\sqrt{5}g^2\) |
| \( SO(7)^+ \) | \( s = \sqrt{\frac{2}{5}} (\sqrt{3} - 1), s' = \sqrt{\frac{2}{5}} (\sqrt{3} - 1), \alpha = 0 = \phi \) | \( \frac{3}{2} \times 5^{-1/8} \) | \(-2 \times 5^{3/4}g^2\) |
| \( G_2 \) | \( s = \pm \sqrt{\frac{2}{5}} (\sqrt{3} - 1), s' = \sqrt{\frac{2}{5}} (\sqrt{3} - 1), \alpha = \cos^{-1} \frac{\sqrt{3 - \sqrt{3}}}{2} = \phi \) | \( \sqrt{\frac{36 \times 2 \times 3^{1/4}}{25 \sqrt{5}}} \) | \(-216 \sqrt{2} \times 3^{1/4}g^2\) |
| \( SU(4)^- \) | \( s = 0, s' = 1, \phi = \frac{\pi}{2} \) | \( \frac{3}{2} \) | \(-8g^2\) |
| \( SU(3) \times U(1) \) | \( s = \frac{1}{\sqrt{3}}, s' = \frac{1}{\sqrt{2}}, \alpha = 0, \phi = \frac{\pi}{2} \) | \( \frac{3^{3/4}}{2} \) | \(-2\sqrt{3}g^2\) |

Table 1. Summary of various critical points in the context of superpotential: symmetry group, vacuum expectation values of fields, superpotential and cosmological constants. We have taken the first, second and fourth columns from [14].

- **\( SO(8) \) case: \( N = 8 \)**
  
  At this point, complex self-dual tensor \( \phi_{ijkl} \) vanishes from (9) because \( \lambda \) and \( \lambda' \) vanish. In the dual field theory, 70 scalars are mapped into relevant chiral primary operators. The 35\( v \) scalars correspond to conformal dimension of 1, \( \text{Tr}X^iX^j - \frac{1}{8}\delta^{ij}\text{Tr}X^2 \) where \( X^i \) is an eight scalars \( 8_v \) of \( SO(8) \) of 3-dimensional \( N = 8 \) \( SU(N_c) \) gauge theory, after dualizing the gauge field, while the 35\( c \) pseudo-scalars correspond to conformal dimension of 2, \( \text{Tr}\lambda^i\lambda^j - \frac{1}{2}\delta^{ij}\text{Tr}\lambda^2 \) where \( \lambda^i \) is an eight fermion fields \( 8_c \). The RG trajectories of the relevant operators will interpolate the \( N = 8 \) \( SO(8) \) fixed point to other new fixed points if the supergravity potential allows additional stable critical points besides the \( SO(8) \) invariant point at \( \phi_{ijkl} = 0 \).

- **\( SO(7)^- \) case: \( N = 0 \)**
  
  In this case, all the eigenvalues of \( A_1^{IJ} \) tensor [11], \( z_1, z_2 \) and \( z_3 \) are complex and equal and their magnitude is given in Table 1. It is known that the number of supersymmetries is equal to the number of eigenvalues of \( A_1^{IJ} \) tensor whose absolute value are the same as \( \sqrt{-\Lambda/6g^2} \). It is easy to check that there is no supersymmetry and the lack of supersymmetry makes it hard to verify the results in the dual field theory. This critical point was found in [9] and is unstable and this fact suggests that the IR field theory limit maybe non-unitary. Since \( \alpha = \pi/2 = \phi \), this corresponds to giving only the pseudo-scalars expectation values corresponding to non-zero internal magnetic four-form field strength [9] in \( d = 11 \).

- **\( SO(7)^+ \) case: \( N = 0 \)**
  
  In this case, also all the eigenvalues of \( A_1^{IJ} \) tensor are real and equal and given in Table 1. Since \( \alpha = 0 = \phi \), this corresponds to giving only the scalars expectation values corresponding to some perturbation of the metric tensor in a dimensional reduction by some twisted \( S^7 \). This critical point was found in [10] and is unstable and there is no corresponding \( SO(7)^+ \) invariant \( d = 3 \) CFT.
• $G_2$ case: $\mathcal{N} = 1$

The eigenvalue $z_2$ is equal to $z_1$ which is different from $z_3$. So there exist two eigenvalues with degeneracies 7, 1 [14]. Since the absolute value of $z_3$ (nothing but the superpotential) is the same as $\sqrt{-\Lambda/6g^2}$ (see Table 1), this gives rise to $\mathcal{N} = 1$ supersymmetry that is a degeneracy of $|z_3| = W$. Simultaneously turning on both scalars and pseudo-scalars, one gets this vacuum. Group theoretically it is impossible to break $SO(8)$ into $G_2$ by giving expectation values to fields in a single $35_v$ or $35_c$ of $SO(8)$. The analysis of superpotential and scalar potential in these three cases was already given in [14] and one can see, by putting $\lambda = \lambda'$ and $\alpha = \phi$ in the (24), that it will lead to the one given in [14]. Our $z_3$ corresponds to their $z_2$. The scalar potential reduces to

$$V(\alpha, \lambda)_{SO(7)\pm G_2} = 2g^2 \left( (7v^4 - 7v^2 + 3)c^3 s^4 + (4v^2 - 7)v^5 s^7 + c^5 s^2 + 7v^3 c^2 s^5 - 3c^3 \right),$$

together with (14). Although the complete spectrum at the IR fixed point is not known, the chiral operators may be followed since their dimensions are protected from quantum corrections.

• $SU(4)^-$ case: $\mathcal{N} = 0$

All the eigenvalues of $A_1^{IJ}$ tensor (1) are equal and given in Table 1. Since $\lambda = 0$ and $\phi = \pi/2$, this invariant critical point occurs at purely pseudo-scalar expectation values and was found in [11]. This can be seen by breaking $SO(7)^-$ of the first vacuum into $SO(6)^- = SU(4)^-$ which is contained in $SO(7)^-$. In this case, 28-beins $u^{IJ}_{KL}$ and $v^{IJKL}$ are expressed in compact form as

$$u^{IJ}_{KL} = \frac{1}{4} \left( (c' + 1) \delta^{IJ}_{KL} + (c' - 1) F^{-[I}_{[K} F^{-J]}_{L]} \right), \quad v^{IJKL} = -\frac{1}{2} s' \chi_{IJJK}^{2-}.$$

where $F^{-J} = \text{diag}(i f, i f, i f, -i f)$ and $f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with (11) and (14). Moreover, the scalar potential can be written as $V(\lambda')_{SU(4)^-} = 2g^2 \left( s^4 - 2s^2 - 3 \right)$. The stability of critical point is not known. In 11-dimensional supergravity theory [11], the metric on the $S^7$ is distorted by stretching the $U(1)$ fibers and four-form field strength is nonzero in the $S^7$ direction.

• $SU(3) \times U(1)$ case: $\mathcal{N} = 2$

The eigenvalue $z_2$ is equal to $z_3$ and given in Table 1 which is different from $z_1(=2/3^{1/4})$. So there are two eigenvalues with degeneracies $6, 2$ [13]. By putting $\alpha = 0$ and $\phi = \pi/2$ in (24), it will lead to the one given in [13]. Our $z_3$ corresponds to their $z_2$. In this case, the scalar potential can be expressed as $V(\alpha, \lambda')_{SU(3) \times U(1)} = 2g^2 c^2 \left( (s^3 + c^3) s^2 - 3c \right)$. The critical point may be thought of as IR fixed point of the dual field theories on the branes. Since all the cosmological constants are negative and admit $AdS_4$ metrics, the corresponding gauge theories are conformal. Note that superpotential $W$ becomes real and this fact made it easier to find a BPS domain-wall solutions. Existence of an algebraic identity (25) may reflect that the supersymmetry restricts the structure on the scalar sectors but this is too sufficient condition.
since $SU(3) \times U(1)$ critical point does not possess that kind of identity. Maybe the group theoretical structure of $G_2$ symmetry rather than supersymmetry alone restricts the behavior of superpotential.

Let us begin with the resulting Lagrangian of the scalar-gravity sector by explicitly finding out the scalar kinetic terms appearing in the action (2) in terms of $\lambda, \lambda', \alpha$ and $\phi$. By taking the product of $A_{IJKL}^\mu$ appearing (38) and its complex conjugation and taking into account the multiplicity four (for given index pairs, there are four possible choices), we arrive at the following expression

$$|A_{IJKL}^\mu|^2 = 36 \left( (\partial_\mu \lambda)^2 + 2s^2 (\partial_\mu \alpha)^2 \right) + 48 \left( (\partial_\mu \lambda')^2 + 2s'^2 (\partial_\mu \phi)^2 \right).$$

In old days, the significance of construction of kinetic terms was not emphasized because, at that time, they concerned about only the structure of extrema of scalar potential. As we mentioned earlier, our approach to get kinetic terms directly through 28-beins is more appropriate. Therefore the resulting Lagrangian of scalar-gravity sector takes the form:

$$\int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{3}{8} (\partial_\mu \lambda)^2 - \frac{3}{4} s^2 (\partial_\mu \alpha)^2 - \frac{1}{2} (\partial_\mu \lambda')^2 - s'^2 (\partial_\mu \phi)^2 - V(\lambda, \lambda', \alpha, \phi) \right),$$

(27)

together with (14), (12) and (13).

Having established the holographic duals of both supergravity critical points, and examined small perturbations around the corresponding fixed point field theories, one can proceed the supergravity description of the RG flow between the two fixed points. The supergravity scalars whose vacuum expectation values lead to the new critical point tell us what relevant operators in the dual field theory would drive a flow to the fixed point in the IR. To construct the superkink (providing for a geometric description of RG flows) corresponding to the supergravity description of the nonconformal RG flow from one scale to other two connecting critical points in $d = 3$ conformal field theories, the form of a 3d Poincaré invariant metric but breaking the full conformal group $SO(3, 2)$ invariance takes the form:

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2, \quad \eta_{\mu\nu} = (-, +, +),$$

(28)

characteristic of space-time with a domain wall where $r$ is the coordinate transverse to the wall (can be interpreted as an energy scale) and $A(r)$ is the scale factor in the four-dimensional metric.

By change of variable $U(r) = e^{A(r)}$ at the critical points, the geometry becomes $AdS_4$ space with a cosmological constant $\Lambda$ equal to the value of $V$ at the critical points: $\Lambda = -3(\partial_r A)^2$. In the dual theory, this corresponds to a superconformal fixed point of the RG flow (from one scale to another). Our interest in domain wall space-times comes from their connection to the RG flow of the dual field theories. The variable $U$, the distance from the horizon, can be
identified with RG scale and linearly proportional to the energy scale of the boundary theory which is an important aspect of the AdS/CFT correspondence. \( U = \infty \) corresponds to long distance in the bulk (UV in the dual field theory) and \( U = 0 \) (near AdS\(_4\) horizon corresponds to short distances in the bulk (IR in the dual field theory). This implies that the RG flow of the coupling constants of the field theory is encoded in the \( U \) dependence of the supergravity scalar fields. At a fixed point the scalar field is constant and therefore corresponding \( \beta \)-function vanishes. We are looking for “interpolating” solutions that are asymptotic to AdS\(_4\) space both for \( \lambda \rightarrow \lambda_{UV}, \lambda' \rightarrow \lambda'_{UV}, \alpha \rightarrow \alpha_{UV}, \phi \rightarrow \phi_{UV} \) for \( r \rightarrow \infty \) so that the background is asymptotic to the supersymmetric AdS\(_4\) background at infinity while \( \lambda \rightarrow \lambda_{IR}, \lambda' \rightarrow \lambda'_{IR}, \alpha \rightarrow \alpha_{IR}, \phi \rightarrow \phi_{IR} \) for \( r \rightarrow -\infty \) and so we approach a new conformal fixed point. The AdS\(_4\) geometries at the endpoints imply conformal symmetry in the UV and IR limits of the field theory and there exists \( OSp(8|4) \) symmetry at the UV fixed point while \( OSp(\mathcal{N}|4) \) symmetry at the IR end. We will show how supergravity can provide a description of the entire RG flow from the maximal supersymmetric UV theory to the lower IR fixed point. With the above ansatz \[(28)\] the equations of motion for the scalars and the metric from \[(27)\] read

\[
4 \partial^2 A + 6(\partial_r A)^2 + \frac{3}{4}(\partial_r \lambda)^2 + \frac{3}{2} s^2 (\partial_r \alpha)^2 + (\partial_r \lambda')^2 + 2s^2 (\partial_r \phi)^2 + 2V = 0,
\]

\[
\partial^2 \lambda + 3\partial_r A \partial_r \lambda - \sqrt{2} s c (\partial_r \alpha)^2 - \frac{4}{3} \partial_r V = 0,
\]

\[
\partial^2 \lambda' + 3\partial_r A \partial_r \lambda' - \sqrt{2} s' c' (\partial_r \phi)^2 - \partial_r V = 0,
\]

\[
s^2 \partial^2 \rho + 3s^2 (\partial_r A \partial_r \phi + \sqrt{2} s' c' \partial_r \phi \partial_r \lambda') - \frac{1}{2} \partial_r V = 0. \tag{29}
\]

By substituting the domain-wall ansatz \[(28)\] into the Lagrangian \[(27)\], the Euler-Lagrange equations are the second, third, fourth and fifth equations of \[(29)\] for the functional \( E[A, \lambda, \lambda', \alpha, \phi] \) \[(28)\] with the integration by parts on the term of \( \partial_r^2 A \). The energy-density per unit area transverse to \( r \)-direction is given by

\[
E[A, \lambda, \lambda', \alpha, \phi] = \int_{-\infty}^{\infty} d^3 r e^{3A} \left[ -3 \left( 2(\partial_r A)^2 + \partial_r^2 A \right) - 3 \left( \frac{1}{8} (\partial_r \lambda)^2 + p^2 q^2 (\partial_r \alpha)^2 \right) - \frac{1}{2} \left( (\partial_r \lambda')^2 + 8r^2 t^2 (\partial_r \phi)^2 \right) - V(\lambda, \lambda', \alpha, \phi) \right].
\]

We are looking for a nontrivial configuration along \( r \)-direction and in order to find out the first-order differential equations the domain-wall satisfy, let us rewrite and reorganize the energy-density by sum of complete squares plus others due to usual squaring-procedure as follows:

\[
E[A, \lambda, \lambda', \alpha, \phi] = -\frac{1}{2} \int_{-\infty}^{\infty} d^3 r e^{3A} \left[ -6 \left( \partial_r A + \sqrt{2} g |z_3| \right)^2 + \frac{3}{4} \partial_r \lambda - i2\sqrt{2} p q \partial_r \alpha - \frac{8\sqrt{2}}{3} g e^{2i\beta} \partial_{\lambda z_3} \right]^2.
\]

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It is evident that the left hand sides of the first four relations vanish as one approaches the configuration that is a topological soliton with a nontrivial kink number along where \( z_3^* = |z_3|e^{2i\beta} \). Then one can easily check that the last eight cross-terms in the above can be expressed as \( 4\sqrt{2}ge^{3A}|z_3| \) by using previous remarkable identities \((25)\). Therefore one arrives at

\[
-\frac{1}{2} \int_{-\infty}^{\infty} d\epsilon e^{3A} \left[ -6 \left( \partial_r A + \sqrt{2}g|z_3| \right)^2 + \frac{3}{4} \left( \partial_r \lambda - i2\sqrt{2}pq\partial_r \alpha - \frac{8\sqrt{2}}{3} e^{2i\beta} \partial_\lambda z_3 \right) \right.
\]

\[
+ \left. \left| \partial_r \lambda - i2\sqrt{2}rt\partial_r \phi - 2\sqrt{2}ge^{2i\beta} \partial_\lambda z_3 \right|^2 \right] - 2\sqrt{2}g \left( e^{3A} |z_3| \right) |_{-\infty}^{\infty}.
\]

Finally, we find BPS bound, inequality of the energy-density

\[
E[A, \lambda, \lambda', \alpha, \phi] \geq -2\sqrt{2}g \left( e^{3A(\infty)} W(\infty) - e^{3A(-\infty)} W(-\infty) \right).
\]  

(30)

Then \( E[A, \lambda, \lambda', \alpha, \phi] \) is extremized by the following so-called BPS domain-wall solutions. The first order differential equations for the scalar fields are the gradient flow equations of a superpotential defined on a restricted four-dimensional slice of the scalar manifold and simply related to the potential of gauged supergravity on this slice via \((26)\).

\[
\partial_r \lambda = \pm \frac{4\sqrt{2}}{3} g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda} + e^{2i\beta} \frac{\partial z_3}{\partial \lambda} \right) = \pm \frac{8\sqrt{2}}{3} g \partial_\lambda W,
\]

\[
\partial_r \lambda' = \pm \sqrt{2} g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda'} + e^{2i\beta} \frac{\partial z_3}{\partial \lambda'} \right) = \pm 2\sqrt{2} g \partial_{\lambda'} W,
\]

\[
\partial_r \alpha = \mp \frac{2}{3pq} ig \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda} - e^{2i\beta} \frac{\partial z_3}{\partial \lambda} \right) = \mp \sqrt{2} \frac{1}{3pq^2 q^2} g \partial_\alpha W,
\]

\[
\partial_r \phi = \mp \frac{1}{2rt} ig \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda'} - e^{2i\beta} \frac{\partial z_3}{\partial \lambda'} \right) = \pm \sqrt{2} \frac{1}{3r^2 t^2} g \partial_\phi W,
\]

\[
\partial_r A = \mp \sqrt{2} g W.
\]

(31)

It is evident that the left hand sides of the first four relations vanish as one approaches the supersymmetric extrema, i.e. \( \partial_\lambda W = \partial_{\lambda'} W = \partial_\alpha W = \partial_\phi W = 0 \) thus indicating a domain-wall configuration that is a topological soliton with a nontrivial kink number along \( r \)-direction. The asymptotic behaviors of \( A(r) \) are \( A(r) \rightarrow r/r_{UV} + \text{const} \) for \( r \rightarrow \infty \) and \( A(r) \rightarrow r/r_{IR} + \text{const} \) for \( r \rightarrow -\infty \). Then by differentiating \( A(r) \) with respect to \( r \), those of \( \partial_r A \) become \( \partial_r A \rightarrow 1/r_{UV} \) for \( r \rightarrow \infty \) and \( \partial_r A \rightarrow 1/r_{IR} \) for \( r \rightarrow -\infty \). At the two critical points, since \( V = -6g^2 W^2 \), one can write the inverse radii of \( AdS_4 \) as cosmological constant or superpotential \( W \). Therefore we conclude that \( 1/r \) is equal to \( \pm \sqrt{2} g W \). This fact is encoded in the last equation of \((31)\).
is straightforward to verify that any solutions \( \{ \lambda(r), \lambda'(r), \alpha(r), \phi(r), A(r) \} \) of (31) satisfy the gravitational and scalar equations of motion given by the second order differential equations (29). Embedding or consistent truncation means that the flow is entirely determined by the equations of motion of supergravity in four-dimensions and any solution of the truncated theory can be lifted to a solution of untruncated theory [29]. Using (31), the monotonicity [30] of \( \partial_r A \) which is related to the local potential energy of the superkink leads to

\[
\partial_r^2 A = -2g^2 \left( \frac{8}{3} (\partial_\lambda W)^2 + 2 (\partial_\lambda W')^2 + \frac{1}{3p^2q^2} (\partial_\alpha W)^2 + \frac{1}{3r^2t^2} (\partial_\phi W)^2 \right) \leq 0.
\]

Note that the value of superpotential at either end of a kink may be thought of as determining the topological sector.

One can understand the above bound (30) as a consequence of supersymmetry preserving bosonic background. In order to find supersymmetric bosonic backgrounds, the variations of spin-1/2 and spin-3/2- fields should vanish. From [17], the gravitational and scalar parts of these variations are:

\[
\begin{align*}
\delta \psi^i_\mu &= 2D_\mu \epsilon^i - \sqrt{2}gA_{ij}^i \gamma_\mu \epsilon_j, \\
\delta \chi^{ijk} &= -\gamma^\mu A_{ijk}^\mu \epsilon_\lambda - 2gA_{ij}^{i\lambda k} \epsilon_\lambda,
\end{align*}
\]

(32)

where the covariant derivative acting on supersymmetry parameter is

\[
D_\mu \epsilon^i = \partial_\mu \epsilon^i - \frac{1}{2} \omega_{\mu ab} \sigma^{ab} \epsilon^i + \frac{1}{2} B_{\mu}^i j^j, \quad B_{\mu j}^i = \frac{2}{3} \left( u_{iJ}^\mu \partial_\mu u_{jk}^J - v_{iJ}^\mu \partial_\mu u_{jkIJ} \right).
\]

Here \( \epsilon_i \) and \( \epsilon^j \) are complex conjugates each other under the chiral basis\(^4\). The field \( B_{\mu j}^i \) is a \( SU(8) \) gauge field for a local \( SU(8) \) invariance, \( \omega \) a spin connection, \( \sigma \) a commutator of two \( \gamma \) matrices. Under the projection operators \( (1 \pm \gamma_5)/2 \) where \( \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), the supersymmetry parameter \( \epsilon_i \) has the four column components as \( (\eta_1, \eta_2, 0, 0) \) where \( \eta_1, \eta_2 \) are complex spinor fields. Moreover, complex conjugate \( \tilde{\epsilon}^i \) is the charge conjugate spinor of \( \epsilon_i \) and satisfies \( \tilde{\epsilon}^i = C \gamma^{\mu}_T \epsilon_i^* \) where \( \gamma_i = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \) we introduce has the 4 column components as \( (0, 0, \eta_3 = i\eta_2^*, \eta_4 = -i\eta_1^*) \).

The variation of 56 Majorana spinors \( \chi^{ijk} \) gives rise to the first order differential equation of \( \lambda, \lambda', \alpha \) and \( \phi \) by exploiting the explicit forms of \( A_{\mu}^{ijkl} \) (38) and \( A_{ij}^{i\lambda k} \) (37) in the appendix. Although there is a summation over the last index \( l \) appearing in \( A_{\mu}^{ijkl} \) and \( A_{ij}^{i\lambda k} \) in the right hand side of (22), the structure of them implies that summation runs over only one index. For example, when \( i = 1, j = 7, k = 2 \) and \( l = 8 \), the vanishing of variation of \( \chi^{ijk} \) leads to

\[
\left( \partial_r \lambda + i2\sqrt{2}pq\partial_r \alpha \right) \gamma^2 \epsilon_8 = \frac{8\sqrt{2}g}{3} \frac{\partial z^8}{\partial \lambda} \epsilon_8,
\]

\(^4\) In this basis, the \( \gamma \) matrices satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \) and \( \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \) where \( \sigma^i \) are Pauli matrices and \( \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \).
and its complex conjugation. We used the fact that $y_3$ is proportional to $e^{i\alpha \partial z_3^*}$ according to (17): this functional relation implies that the scalar potential can be expressed in terms of $z_3$ and is expressed in terms of $z_3^*$ and $\partial_\lambda z_3$. Recognizing that $\gamma^2 \epsilon_8 = e^{8^*}$, we arrive at

$$\left( \partial_\tau \lambda + i2\sqrt{2}pq\partial_\tau \alpha \right) = \frac{8\sqrt{2}g}{3} \frac{\partial z_3^*}{\partial \lambda} \left( \frac{e^{8}}{|e^{8}|} \right)^2.$$  

(34)

Therefore one obtains two relations for $\lambda$ and $\alpha$ fields from this and its complex conjugation:

$$\partial_\tau \lambda = \frac{4\sqrt{2}}{3} g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda} + e^{2i\beta} \frac{\partial z_3}{\partial \lambda} \right), \quad \partial_\tau \alpha = -\frac{2}{3pq} i g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda} - e^{2i\beta} \frac{\partial z_3}{\partial \lambda} \right),$$

where complex spinor field has a phase $\beta(r)$ and $\eta_3 = |\eta_3(r)| e^{i\beta(r)}$. This is nothing but the first and third equations of (31). It is straightforward to re-express them in terms of a derivative of $W$ with respect to $\lambda$ field by writing $z_3^* = W e^{-2i\beta}$. On the other hand, when $i = 6$, $j = 2$, $k = 4$ and $l = 8$, the vanishing of variation of $\chi^{ijk}$ leads to

$$\left( \partial_\tau \lambda' + i2\sqrt{2}rt\partial_\tau \phi \right) = \frac{8\sqrt{2}g}{3} \frac{\partial z_3^*}{\partial \lambda'} e^{8},$$

and its complex conjugation. Again, we used the fact that $y_7$ is proportional to $e^{i\phi \partial z_3^*}$ from (17). It also implies that the scalar potential can be expressed in terms of $z_3$ and is expressed in terms of $z_3^*$ and $\partial_\lambda' z_3$. Therefore one obtains the following relations for $\lambda'$ and $\phi$ fields

$$\partial_\tau \lambda' = \sqrt{2} g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda'} + e^{2i\beta} \frac{\partial z_3}{\partial \lambda'} \right), \quad \partial_\tau \phi = -\frac{1}{2rt} i g \left( e^{-2i\beta} \frac{\partial z_3^*}{\partial \lambda'} - e^{2i\beta} \frac{\partial z_3}{\partial \lambda'} \right),$$

which are exactly the same as the second and fourth equations of (31) leading to a derivative of superpotential with respect to $\lambda'$ field as before.

Putting it in another way, the cross terms for second equation of (2) can be simplified by using the identity of (10)

$$D_\mu A_1^{ij} = \frac{\sqrt{2}}{24} \left( A_2^{i \ klm} A_\mu^{j \ klm} + A_2^{j \ klm} A_\mu^{i \ klm} \right),$$

and by realizing the following identity

$$A_1^{ik} A_1^{kj} = \frac{1}{18} A_2^{i \ klm} A_2^{j \ klm} = -\frac{1}{6g^2} \delta^i_j,$$

the spin-1/2 variations vanish if and only if the steepest descent equations given by first four equations of (31) are satisfied.

Moreover, the variation of gravitinos $\psi^{i\mu=1}$ will leads to $i\eta_3 \left( \partial_\tau A + \sqrt{2}gz_3^* e^{-2i\beta} \right) = 0$. Similar relation for spinor field $\eta_4$ holds. By realizing $\partial_\tau A$ is real, one can conclude that $e^{-2i\beta} = -|z_3|/z_3^*$. Finally, we obtains

$$\partial_\tau A = -\sqrt{2}gW,$$
which is the same as the last equation of (31). Similar equation appears in the $\eta_4$ spinor component. There are no other additional equations for $\mu = 0, 3$ indices.

According to (33), $\omega_{\mu=2,a,b}$ term has nonvanishing $\omega_{\mu=2,1,1}$ but these are summed over $\sigma^{1,1} = [\gamma^1, \gamma^1]$ which is identically zero. Therefore there is no contribution on this part:

$$2\partial_r e^8 + \frac{i}{2} [3(-1 + c)\partial_r \alpha + 4 (-1 + c') \partial_r \phi] e^8 - \sqrt{2} g z_3 \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ 0 \\ 0 \end{pmatrix} = 0,$$

where we used the fact that the $SU(8)$ connection $B_{\mu J}^{I}$ defined by (33) obtained by plugging (10) has the following diagonal form:

$$B_{\mu J}^{I} = \text{diag} \left( -iq^2 \partial_\mu \alpha, -iq^2 \partial_\mu \alpha, -iq^2 \partial_\mu \alpha, -iq^2 \partial_\mu \alpha, -iq^2 \partial_\mu \alpha, -iq^2 \partial_\mu \alpha, \right)$$

$$\frac{i}{2} [3(-1 + c)\partial_r \alpha - 4 (-1 + c') \partial_r \phi], \frac{i}{2} [3(-1 + c)\partial_r \alpha + 4 (-1 + c') \partial_r \phi],$$

together with (12) and (14). Finally, one of the variation of gravitinos $\psi_{\mu=2}^i$ gives rise to

$$2\partial_r \eta_3 + \frac{i}{2} [3(-1 + c)\partial_r \alpha + 4 (-1 + c') \partial_r \phi] \eta_3 - \sqrt{2} g z_3^* \eta_3^* = 0.$$ 

From this, we get two relations for spinor field $\eta_3$, and using $\eta_3 = |\eta_3(r)| e^{i\beta(r)}$ and plugging back, we get

$$\partial_r \beta = \frac{1}{4} [3(-1 + c)\partial_r \alpha + 4 (-1 + c') \partial_r \phi], \quad \partial_r |\eta_3| = \sqrt{2} g W |\eta_3| e^{-2i\beta}.$$ 

One can show that there exists a supersymmetric flow if and only if the equations (31) are satisfied, that is, the flow is determined by the steepest descent of the superpotential and the cosmology $A(r)$ is determined directly from this steepest descent.

Let us consider mass, $\tilde{M}^2$ for the $\mathring{X}, \mathring{\bar{X}}, \mathring{\alpha}$ and $\mathring{\phi}$ at the critical points of superpotential $W$ where $\mathring{X} = \sqrt{\frac{2}{3}} \lambda, \mathring{\bar{X}} = \mathring{\lambda}, \mathring{\alpha} = \sqrt{\frac{2}{3}} \alpha$ and $\mathring{\phi} = \sqrt{2} \phi$. By differentiating (23) and putting $\partial_\lambda W = \partial_{\mathring{\alpha}} W = \partial_{\mathring{\phi}} W = 0$, we get

$$\tilde{M}_{ij}^2 = \partial_{\phi_i} \partial_{\phi_j} V = 2 g^2 W^2 U_k (U_{kj} - 3 \delta_{kj}), \quad \phi_i = (\mathring{X}, \mathring{\bar{X}}, \mathring{\alpha}, \mathring{\phi}),$$

where $U$ is related to the second derivatives of $W$ with respect to various fields. The mass scale is set by the inverse radius, $1/r$, of the AdS$_4$ space and this can be written as $1/r = 4 \sqrt{-V/3} = 2 g W$ where we used $V = -6 g^2 W^2$. Via AdS/CFT correspondence, $U$ is related to the conformal dimension $\Delta$ of the field theory operator dual to the fluctuation of the fields $\mathring{X}, \mathring{\bar{X}}, \mathring{\alpha}$ and $\mathring{\phi}$. Since the matrix $U$ is real and symmetric, it has real eigenvalues $\delta_k$ and the eigenvalues of $\tilde{M}^2 r^2$ are given by $\delta_k (\delta_k - 3)$. Since a new radial coordinate $U(r) = e^{A(r)}$
is the renormalization group scale on the flow, we should find the leading contributions to the $\beta$ functions of the couplings $\lambda, \lambda', \alpha$ and $\phi$ in the neighborhood of the end points of the flow. At a fixed point, the fields are constant and corresponding $\beta$ function vanishes. Since $\frac{d}{dt} = \frac{d}{dt} U \frac{d}{dt}$, (31) becomes $U \frac{d}{dt} \phi_i = -\frac{2}{W} \frac{dW}{\partial \phi_i} \approx -U \delta \phi_j$, where we expanded to the first order in the neighborhood of a critical point. Thus $U$ determines the behavior of the $\lambda, \lambda', \alpha$ and $\phi$ near the critical points.

The RG flow of the coupling constants of the field theory is encoded in the $U$ dependence of the fields. To depart the UV fixed point ($U = +\infty$), the flow must take place in directions in which the operators must be relevant, and to approach the IR fixed point ($U \rightarrow 0$), the corresponding operators must be irrelevant.

### 3.2 $SO(5)$ Sector

- $SO(5)^+$ embedding

  The superpotential $W$ is generically extracted as an eigenvalue of the $A_{IJKL}$ tensor from (16) and it is related to the scalar potential as follows:

  $$V(\lambda, \mu, \rho) = g^2 \left[ \frac{2}{5} (\partial_{\lambda} W)^2 + \frac{2}{5} (\partial_{\mu} W)^2 + \frac{2}{5} (\partial_{\rho} W)^2 - \frac{16}{5} \partial_{\lambda} W \partial_{\mu} W - \frac{16}{5} \partial_{\mu} W \partial_{\rho} W - \frac{16}{5} \partial_{\rho} W \partial_{\lambda} W - 6W^2 \right],$$

  where the superpotential is a real function of $\lambda, \mu$ and $\rho$

  $$W(\lambda, \mu, \rho) = -\frac{1}{8\sqrt{uvw}} \left( 5 + u^2 v^2 + \text{two cyclic permutations} \right).$$

There is a trivial critical point at which all the fields vanish and whose cosmological constant is $\Lambda = -6g^2$ preserving $\mathcal{N} = 8$ supersymmetry.

| Gauge symmetry | $\lambda, \mu, \rho$ | $W$ | $V$ |
|---------------|-------------------|-----|-----|
| $SO(8), \mathcal{N} = 8$ | $\lambda = \mu = \rho = 0$ | $-1$ | $-6g^2$ |
| $SO(7)^+, \mathcal{N} = 0$ | $\lambda = \mu = -\rho = -\sqrt{\frac{2}{5}} \log 5$ | $-\frac{3}{2 \times 5^{3/4}}$ | $-2 \times 5^{3/4} g^2$ |

Table 2. Summary of one critical point in the context of superpotential : symmetry group, vacuum expectation values of fields, superpotential and cosmological constants.

In this case, there exists an unstable nonsupersymmetric critical point with $SO(7)^+$ gauge symmetry. By taking the product of $A_{IJKL}$ and its complex conjugation, $A_{IJKL}^*$, and summing over all the indices with appropriate multiplicities, we arrive at the following expression

$$\left| A_{IJKL} \right|^2 = 144 \left( (\partial_{\mu} \lambda)^2 + (\partial_{\mu} \mu)^2 + (\partial_{\mu} \rho)^2 \right) + 96 (\partial_{\mu} \lambda \partial_{\mu} \mu + \partial_{\mu} \lambda \partial_{\mu} \rho + \partial_{\mu} \mu \partial_{\mu} \rho).$$

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By substituting the domain-wall ansatz (28) as before into the resulting Lagrangian of the scalar and gravity part we have not presented here explicitly and by plugging the above kinetic terms, one gets the Euler-Lagrangian equations. Along the flow between $SO(8)$ fixed point and $SO(7)^+$ fixed point we are considering, the relations $\mu = -\rho = \lambda$ hold. Therefore, after repeating the analysis of the energy-density, the first order differential equations for the $\lambda$ field are the gradient flow equations of corresponding superpotential defined on a single dimensional slice of the scalar manifold:

$$
\partial_r \lambda = \pm \frac{8\sqrt{2}}{7} g \partial_\lambda W, \quad \partial_r A = \mp \sqrt{2}g W,
$$

which is also obtained by putting $\alpha = 0 = \phi$ and $\lambda' = \lambda$ in the previous subsection for $SU(3)$ invariant sectors. Unlike as supersymmetric flow we have studied in $SU(3)$ invariant sectors, at both ends, the derivative of $W$ with respect to $\lambda$ does not vanish: at supersymmetric $SO(8)$ fixed point, it vanishes, while, at nonsupersymmetric $SO(7)^+$ fixed point, it does not.

- $SO(5)^-$ embedding

The scalar potential can be written in terms of superpotential as follows:

$$
V(\lambda, \mu, \rho) = g^2 \left[ \frac{32}{5} |\partial_\lambda W|^2 + \frac{32}{5} |\partial_\mu W|^2 + \frac{32}{5} |\partial_\rho W|^2 - \frac{8}{5} \partial_\lambda W \partial_\mu W^* - \frac{8}{5} \partial_\mu W \partial_\lambda W^* - \frac{8}{5} \partial_\rho W \partial_\lambda W^* - \frac{8}{5} \partial_\mu W \partial_\rho W^* - \frac{6}{5} |W|^2 \right],
$$

where the complex-superpotential from (19) takes the form:

$$
W(\lambda, \mu, \rho) = \frac{(1 + i)}{16(uvw)^{3/2}} \left( -iu^2 + u^3v^3w + \text{two cyclic permutations} + 5uvw - 5iu^2v^2w^2 \right).
$$

The superpotential $W$ has the following values at two nonsupersymmetric critical points besides the supersymmetric one. Like as nonsupersymmetric flow for $SO(5)^+$ embedding case, at supersymmetric fixed point, derivatives of superpotential with respect to fields vanish while, at nonsupersymmetric fixed points, they do not vanish.

| Gauge symmetry | $\lambda, \mu, \rho$ | $W$ | $V$ |
|----------------|---------------------|-----|-----|
| $SO(8), N = 8$ | $\lambda = \mu = \rho = 0$ | 1   | $-6g^2$ |
| $SO(7)^-, N = 0$ | $\lambda = \mu = -\rho = \sqrt{2} \log \frac{1 + \sqrt{5}}{2}$ | $\frac{3}{8} \sqrt{11 + 2i}$ | $-\frac{25\sqrt{5}}{8} g^2$ |
| $SO(6)^-, N = 0$ | $\lambda = \sqrt{2} \log(\sqrt{2} + 1), \mu = 0, \rho = \sqrt{2} \log(\sqrt{2} - 1)$ | $\frac{3}{2}$ | $-8g^2$ |

Table 3. Summary of various critical points in the context of superpotential: symmetry group, vacuum expectation values of fields, superpotential and cosmological constants.

Contrary to the previous $SO(5)^+$ embedding case, there are no such first order differential equations for either a flow between $SO(8)$ fixed point and $SO(7)^-$ fixed point or a flow between
SO(8) fixed point and $SO(6)^-$ fixed point. Furthermore, in [22] there were no extrema with gauge symmetry of $SO(5), SO(5) \times U(1)$ or $SO(5) \times SU(2)$ which leads to the fact that the effective four-dimensional theory of Awada et al [21] is not a sector of de Wit-Nicolai theory. It can be a sector of the full $d = 4$ theory resulting from compactification on $S^7$ in which some of the massive scalars are given expectation values.

3.3 $SO(3) \times SO(3)$ Sectors

During the flow connecting $SO(8)$ fixed point to $SO(3) \times SO(3)$ fixed point, the six fields $\lambda^\alpha$ vanish for $\alpha = 2, \cdots, 7$. We are considering domain-walls in supergravity with a nontrivial superpotential defined on a restricted one-dimensional slice of the scalar manifold. One of the eigenvalues of $A_{IJ}^{\alpha}$ tensor, $z_7$ [21] restricted on $\lambda^\alpha, \alpha = 2, \cdots, 7$ provides a “superpotential” $W$ related to scalar potential $V$ by

$$V(\lambda^1)|_{\lambda^2 \cdots = \lambda^7 = 0} = \frac{g^2}{16} \left( -61 - 36 \cosh \lambda^1 + \cosh 2\lambda^1 \right)$$

$$= g^2 \left[ 8 (\partial_{\lambda^1} W)^2 - 6W^2 \right],$$

where the superpotential is $W(\lambda^1) = \frac{1}{4} \left( 3 + \cosh \lambda^1 \right)$. The plots of $V$ and $W$ on the $\lambda^1$ parameter space are shown in Figure 1.

Figure 1: The plots of $V$ (on the left) and $W$ (on the right), with $\lambda^1$ on the horizontal axis. Scalar potential $V$, at $\lambda^1 = 0$, is the maximally supersymmetric and locally maximum while superpotential $W$ at that point is locally minimum. The cosmological constant is $-6$. We took $g^2$ as 1 for simplicity. At $\lambda^1 = 2 \sinh^{-1} 2 = 2.89$, $V$ has locally minimum and is nonsupersymmetric and the cosmological constant is $-14$. See also [37].
At the supersymmetric $SO(8)$ fixed point, the critical point of scalar potential $V$ is nothing but the one of superpotential $W$ while, at the nonsupersymmetric $SO(3) \times SO(3)$, the critical point of $V$ is not a critical point of $W$ (that is, $\partial_{\lambda_1} W$ does not vanish at a point) but at point for which $W$ satisfies $4\partial_{\lambda_1}^2 W - 3W = 0$ (note that, at a fixed point, $\cosh \lambda^1 = 9$) if we differentiate $V$ with respect to $\lambda^1$.

| Gauge symmetry            | $\lambda^\alpha$       | $W$      | $V$     |
|---------------------------|-------------------------|----------|---------|
| $SO(8), N = 8$            | $\lambda^\alpha = 0, 1 \leq \alpha \leq 7$ | 1        | $-6g^2$ |
| $SO(3) \times SO(3), N = 0$ | $\sinh \frac{\lambda^1}{2} = 2, \lambda^\alpha = 0, 2 \leq \alpha \leq 7$ | 3        | $-14g^2$ |

Table 4. Summary of one critical point in the context of superpotential : symmetry group, vacuum expectation values of fields, superpotential and cosmological constants.

The resulting Lagrangian of the scalar-gravity sector after finding out the kinetic terms according to (45) and by realizing from correct counting of multiplicities that $|A^{IJKL}_\mu|^2 = 6(1 + \cosh \lambda^1)^2(\partial_{1} \lambda^1)^2$ in terms of $\lambda_1$ takes the form with (35):

$$\int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{16} \left(1 + \cosh \lambda^1\right)^2 \left(\partial_{1} \lambda^1\right)^2 - V(\lambda^1) \right).$$

By substituting the domain-wall ansatz (28) into this Lagrangian, one obtains domain-wall solutions by direct minimization of energy-functional when we assume quadratic order in the fluctuation of $\lambda_1$

$$\partial_r \lambda^1 = \pm 4\sqrt{2}g \partial_{1} W, \quad \partial_r A = \mp \sqrt{2}g W.$$  

Although the right hand side of the first relation vanishes at the supersymmetric $SO(8)$ fixed point, for the nonsupersymmetric $SO(3) \times SO(3)$ fixed point, the “velocity” of $\lambda^1$ does not vanish because the right hand side $\partial_{1} W$ at that point has nonzero value. The analytic solutions for these become

$$\lambda^1(r) = \log \frac{1 + e^{\sqrt{2}g(c-r)}}{1 - e^{\sqrt{2}g(c-r)}}, \quad A(r) = \frac{1}{4} \left(3\sqrt{2}gr + \log[2 \sinh \sqrt{2}g(c-r)]\right).$$

Although it is not known whether the Breitlohner-Freedman condition is satisfied by the solution in the $N = 8$ theory, the solution is stable in the context of positive energy theorem without supersymmetry [32]. So, in the terminology of [28], this solution is “non-BPS” domain-wall solution interpolating between supersymmetric $SO(8)$ vacuum and nonsupersymmetric $SO(3) \times SO(3)$ one.

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## 4 Appendix A: SU(3) Invariant Sectors

The 28-beins $u_{KL}^{IJ}$ and $v_{KL}^{IJK}$ fields which are elements of $56 \times 56 \mathcal{V}(x)$ of the fundamental 56-dimensional representation of $E_7$ can be obtained by exponentiating the vacuum expectation values $\phi_{ijkl}$ of SU(3) singlet space via (7). After tedious calculation, the nonzero components of the 28-beins have the following seven $4 \times 4$ block diagonal matrices respectively:

$$u_{KL}^{IJ} = \text{diag}(u_1, u_2, u_3, u_4, u_5, u_6, u_7),$$
$$v_{KL}^{IJK} = \text{diag}(v_1, v_2, v_3, v_4, v_5, v_6, v_7).$$

Each hermitian (for example, $(u_1)^{78}_{12} = ((u_1)^{12}_{78})^* = BK^2$) submatrix is $4 \times 4$ matrix and we denote antisymmetric index pairs $[IJ]$ and $[KL]$ explicitly for convenience. For simplicity, we make an empty space corresponding to lower triangle elements:

$$u_1 = \begin{pmatrix}
    [12] & [34] & [56] & [78] \\
    A & B & B & B \\
    34 & A & B & B \\
    56 & 78 & A & A \\
  \end{pmatrix},
  u_2 = \begin{pmatrix}
    [13] & [24] & [57] & [68] \\
    C & D & K_E & E \\
    24 & C & D & K_E \\
    57 & 68 & C & D \\
  \end{pmatrix},

u_3 = \begin{pmatrix}
    [14] & [23] & [58] & [67] \\
    C & D & E & L_E \\
    23 & C & D & E \\
    58 & 67 & C & D \\
  \end{pmatrix},
  u_4 = \begin{pmatrix}
    [15] & [26] & [37] & [48] \\
    C & D & L_E & K_E \\
    26 & C & D & L_E \\
    37 & 48 & C & D \\
  \end{pmatrix},

u_5 = \begin{pmatrix}
    [16] & [25] & [38] & [47] \\
    C & D & E & L_E \\
    25 & C & D & E \\
    38 & 47 & C & D \\
  \end{pmatrix},
  u_6 = \begin{pmatrix}
    [17] & [28] & [35] & [46] \\
    C & D & K_E & -K_E \\
    28 & C & D & K_E \\
    35 & 46 & C & D \\
  \end{pmatrix},

u_7 = \begin{pmatrix}
    [18] & [27] & [36] & [45] \\
    C & L_E & K_E & K_E \\
    27 & C & L_E & K_E \\
    36 & 45 & C & K_E \\
  \end{pmatrix},
  v_1 = \begin{pmatrix}
    [12] & [34] & [56] & [78] \\
    C & K_E & K_E & K_E \\
    34 & C & K_E & K_E \\
    56 & 78 & C & K_E \\
  \end{pmatrix},

v_2 = \begin{pmatrix}
    [13] & [24] & [57] & [68] \\
    H & L & \frac{L}{K} \frac{L}{K} \frac{L}{K} \\
    24 & H & L & \frac{L}{K} \frac{L}{K} \frac{L}{K} \\
    57 & 68 & H & L \\
  \end{pmatrix},
  v_3 = \begin{pmatrix}
    [14] & [23] & [58] & [67] \\
    H & L & K_L^2 & K_I \\
    23 & H & L & K_L^2 \\
    58 & 67 & H & K_I \\
  \end{pmatrix},

v_4 = \begin{pmatrix}
    [15] & [26] & [37] & [48] \\
    H & L & K_L^2 & K_I \\
    26 & H & L & K_L^2 \\
    37 & 48 & H & K_I \\
  \end{pmatrix},
  v_5 = \begin{pmatrix}
    [16] & [25] & [38] & [47] \\
    H & L & \frac{L}{K} \frac{L}{K} \frac{L}{K} \\
    25 & H & L & \frac{L}{K} \frac{L}{K} \frac{L}{K} \\
    38 & 47 & H & L \\
  \end{pmatrix}.$$
where simplified quantities are functions of $\lambda, \lambda', \alpha$ and $\phi$

\[ A \equiv p^3, \quad B \equiv pq^2, \quad C \equiv pr^2, \quad D \equiv pt^2, \quad E \equiv qr^2, \quad F \equiv q^3, \]

\[ G \equiv p^2q, \quad H \equiv q^2t, \quad I \equiv qr^2, \quad J \equiv prt, \quad K \equiv e^{i\alpha}, \quad L \equiv e^{i\phi} \]

and $p, q, r$ and $t$ are some functions of $\lambda$ and $\lambda'$ in (12). The lower triangle part can be read off from the upper triangle part by hermitian property. Also, 28-beins $u_{i,j}^{KL}$ and $v_{IJKL}$ are obtained by taking a complex conjugation of (36). The nonzero components of $A_2$ tensor, $A_{2L}^{IJK}$, are obtained from (11) and (37) by simply plugging (36) into there and they are classified by eight different fields with degeneracies 12, 3, 12, 4, 4, 4, 6 respectively and given by:

\[ A_{2,1}^{256} = A_{2,1}^{234} = A_{2,2}^{165} = A_{2,2}^{143} = A_{2,3}^{456} = A_{2,3}^{412} = A_{2,4}^{365} = A_{2,4}^{321} = \]
\[ A_{2,5}^{634} = A_{2,5}^{612} = A_{2,6}^{543} = A_{2,6}^{521} = y_1, \]
\[ A_{2,7}^{128} = A_{2,7}^{348} = A_{2,7}^{568} = y_2, \]
\[ A_{2,8}^{172} = A_{2,8}^{437} = A_{2,8}^{657} = y_3, \]
\[ A_{2,1}^{375} = A_{2,1}^{674} = A_{2,2}^{574} = A_{2,2}^{673} = A_{2,3}^{571} = A_{2,3}^{276} = A_{2,4}^{275} = A_{2,4}^{176} = \]
\[ A_{2,5}^{247} = A_{2,5}^{173} = A_{2,6}^{237} = A_{2,6}^{147} = y_4, \]
\[ A_{2,1}^{368} = A_{2,1}^{458} = A_{2,2}^{358} = A_{2,2}^{648} = A_{2,3}^{618} = A_{2,3}^{528} = A_{2,4}^{826} = A_{2,4}^{518} = \]
\[ A_{2,5}^{814} = A_{2,5}^{823} = A_{2,6}^{813} = A_{2,6}^{428} = y_5, \]
\[ A_{2,7}^{513} = A_{2,7}^{326} = A_{2,7}^{416} = A_{2,7}^{425} = y_6, \]
\[ A_{2,8}^{624} = A_{2,8}^{415} = A_{2,8}^{316} = A_{2,8}^{325} = y_7, \]
\[ A_{2,1}^{278} = A_{2,2}^{718} = A_{2,3}^{478} = A_{2,4}^{738} = A_{2,5}^{578} = A_{2,6}^{758} = y_8, \]

where their explicit forms are

\[ y_1 = -e^{-2i(\alpha+\phi)} \left( e^{3i\alpha} p^2qr^2t^2 + e^{i(3\alpha+4\phi)} p^2qr^2t^2 + pq^2r^2t^2 + e^{4i\alpha} pq^2r^2t^2 \right. \]
\[ + e^{i(\alpha+2\phi)} q(2q^2r^2t^2 + p^2(r^2 + t^2)^2) + e^{2i(\alpha+3\phi)} p(2p^2t^2r^2 + q^2(r^2 + t^2)^2) \right), \]

\[ y_2 = -e^{-2i(\alpha+\phi)} \left( e^{3i\alpha} p^2qr^4 + 2e^{i(\alpha+2\phi)} p(2p^2r^2t^2 + e^2t^2)(r^2 + t^2) + pq^2r^4 \right. \]
\[ + e^{i(3\alpha+4\phi)} p^2q^4t^4 + e^{4i\phi} pq^4t^4 \right) \]
\[ = \frac{2\sqrt{2}}{3} e^{i\alpha} \frac{\partial z_5^2}{\partial \lambda}, \]
\[
y_3 = -e^{-2i(\alpha+2\phi)} \left( e^{i(3\alpha+4\phi)} p^2 q r^4 + e^{4i\phi} pq^2 r^4 + 2e^{i(\alpha+2\phi)} q(2p^2 + q^2)r^2 t^2 \\
+ 2e^{2i(\alpha+\phi)} p(p^2 + 2q^2)r^2 t^2 + e^{3i\alpha} p^2 q t^4 + pq^4 t^4 \right) \\
= -\frac{2\sqrt{2}}{3} e^{i\alpha} \frac{\partial z_3^2}{\partial \lambda},
\]
\[
y_4 = e^{-i(\alpha+3\phi)} r t \left( e^{4i\phi} p^2 q r^2 + e^{i(3\alpha+4\phi)} pq^2 r^2 + e^{3i\alpha} pq^2 t^2 \\
+ 2e^{2i(\alpha+\phi)} q(2p^2 + q^2)(r^2 + t^2) + e^{(i+2\phi)} p(p^2 + 2q^2)(r^2 + t^2) \right),
\]
\[
y_5 = e^{-i(\alpha+\phi)} r t \left( p^2 q r^2 + e^{3i\alpha} pq^2 r^2 + e^{4i\phi} p^2 q t^2 + e^{i(3\alpha+4\phi)} pq^2 t^2 \\
+ 2e^{2i(\alpha+\phi)} q(2p^2 + q^2)(r^2 + t^2) + e^{(i+2\phi)} p(p^2 + 2q^2)(r^2 + t^2) \right),
\]
\[
y_6 = e^{-i(3\alpha+\phi)} \left( e^{i\alpha} p + q \right) r t \left( e^{2i(\alpha+2\phi)} p^2 q r^2 - e^{i(4\alpha+4\phi)} pq^2 r^2 \\
+ 2e^{2i(\alpha+\phi)} p q r^2 t^2 - e^{i\alpha} p q t^2 + q^2 r^2 t^2 + 3e^{i(\alpha+2\phi)} pq(r^2 + t^2) \right) \\
= -\frac{\sqrt{2}}{2} e^{i\phi} \frac{\partial z_2^4}{\partial \lambda},
\]
\[
y_7 = e^{-3i(\alpha+\phi)} \left( e^{i\alpha} p + q \right) \left( e^{2i(\alpha+2\phi)} p^2 r^2 q - e^{i(4\alpha+4\phi)} pq^2 r^2 \\
+ 2e^{2i(\alpha+\phi)} p^2 q^2 r^2 - e^{i\alpha} p q t^2 + q^2 r^2 t^2 + 3e^{i(\alpha+2\phi)} pq(r^2 + t^2) \right) \\
= -\frac{\sqrt{2}}{2} e^{i\phi} \frac{\partial z_2^4}{\partial \lambda},
\]
\[
y_8 = e^{-2i(\phi + e^{i\phi} q)} \left( p^2 q^2 t^2 + e^{4i\phi} p^2 q^2 t^2 - e^{i\alpha} pq^2 r^2 t^2 - e^{i(4\alpha+4\phi)} pq^2 t^2 \\
+ 2e^{2i(\alpha+\phi)} q^2 r^2 t^2 + e^{2i(\alpha+2\phi)} q^2 r^2 t^2 + e^{i(\alpha+2\phi)} pq(r^2 + 4t^2 + t^2) \right),
\]
\]

Notice that we did not write down the components of $A_2$ tensor which are interchanged between the second and third indices because it is manifest that $A_{2L}^{IJK} = -A_{2L}^{IKJ}$, by definition. Moreover there exists a symmetry between the upper indices: $A_{2L}^{IJK} = A_{2L}^{IKJ} = A_{2L}^{KIJ}$. Recall that in the supersymmetric transformation rules (32), $A_{2L}^{IJK}$ appears in the second equation given by spin-1/2 fields. It was inevitable to rewrite it in terms of superpotential in order to find out domain-wall solutions. Therefore we explicitly emphasize them here.

The kinetic terms (3) can be summarized as following seven $4 \times 4$ block diagonal hermitian matrices like as 28-beins $u^I_{KL}$ and $v^{IJKL}$:

\[
A_{\mu_I}^{JKL} = \text{diag}(A_{\mu,1}, A_{\mu,2}, A_{\mu,3}, A_{\mu,4}, A_{\mu,5}, A_{\mu,6}, A_{\mu,7}),
\]

where each hermitian submatrix can be expressed as with empty space for lower triangle parts

\[
A_{\mu,1} = \begin{pmatrix} [12] & [34] & [56] & [78] \\
[12] & 0 & -a^* & -a \\
[34] & 0 & -a^* & -a \\
[56] & 0 & -a \\
[78] & 0 & &
\end{pmatrix}, A_{\mu,2} = \begin{pmatrix} [13] & [24] & [57] & [68] \\
[13] & 0 & a^* & b^* & -b \\
[24] & 0 & -b^* & b \\
[57] & 0 & a \\
[68] & 0 & &
\end{pmatrix},
\]

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and the nonzero components are given by

\[
a \equiv \frac{1}{2} e^{i\alpha} \left( \partial_{\mu} \lambda + i \sqrt{2} s \partial_{\mu} \alpha \right), \quad b \equiv \frac{1}{2} e^{i\phi} \left( \partial_{\mu} \lambda' + i \sqrt{2} s' \partial_{\mu} \phi \right).
\]

5 Appendix B: $SO(5)$ Invariant Sectors

The 28-beins $u_{IKL}^I$ and $v_{IKL}^I$ fields which are elements of $56 \times 56 \mathcal{V}(x)$ (1) of the fundamental 56-dimensional representation of $E_7$ can be obtained by exponentiating the vacuum expectation values $\phi_{ijkl}$ (15) and (18) of $SO(5)$ singlet space via (7) simultaneously. After tedious calculation they have the following seven $4 \times 4$ block diagonal hermitian matrices respectively:

\[
A_{\mu,3} = \begin{pmatrix}
[14] & [23] & [58] & [67] \\
[23] & 0 & -a^* & -b - b^* \\
[58] & 0 & -b & -b^* \\
[67] & 0 & 0 & 0
\end{pmatrix},
\]

\[
A_{\mu,4} = \begin{pmatrix}
[15] & [26] & [37] & [48] \\
[26] & 0 & a^* & -b - b^* \\
[37] & 0 & -b & b^* \\
[48] & 0 & 0 & 0
\end{pmatrix},
\]

\[
A_{\mu,5} = \begin{pmatrix}
[16] & [25] & [38] & [47] \\
[25] & 0 & -a^* & b - b^* \\
[38] & 0 & b & b^* \\
[47] & 0 & 0 & 0
\end{pmatrix},
\]

\[
A_{\mu,6} = \begin{pmatrix}
[17] & [28] & [35] & [46] \\
[28] & 0 & a & b - b^* \\
[35] & 0 & -b & b^* \\
[46] & 0 & 0 & 0
\end{pmatrix},
\]

\[
A_{\mu,7} = \begin{pmatrix}
[18] & [27] & [36] & [45] \\
[27] & 0 & -a & -b - b^* \\
[36] & 0 & -b^* & -b \\
[45] & 0 & 0 & 0
\end{pmatrix},
\]

\[
(38)
\]
\[ u^\pm = \begin{pmatrix} u_{18} & u_{27} & u_{36} & u_{45} \\ qrs & \mp q\gamma\delta & \pm q\delta & \pm q\gamma s \\ \beta\gamma s & -\beta r\delta & qrs & -q\gamma \delta \end{pmatrix}, \]

\[ v_1^\pm = \varepsilon \begin{pmatrix} 12 & 34 & 56 & 78 \\ \alpha^2 & \alpha q^2 & \pm pq\beta & \pm pq\beta \\ \beta^2 & \beta q^2 & \pm p\gamma s & \pm p\gamma s \\ \gamma^2 & \gamma q^2 & \pm q\delta s & \pm q\delta s \end{pmatrix}, \]

\[ v_2^\pm = \varepsilon \begin{pmatrix} 13 & 24 & 57 & 68 \\ \alpha^2 & \alpha q^2 & \pm pq\beta & \pm pq\beta \\ \beta^2 & \beta q^2 & \pm p\gamma s & \pm p\gamma s \\ \gamma^2 & \gamma q^2 & \pm q\delta s & \pm q\delta s \end{pmatrix}, \]

\[ v_3^\pm = \varepsilon \begin{pmatrix} 14 & 23 & 58 & 67 \\ \alpha^2 & \alpha q^2 & \pm pq\beta & \pm pq\beta \\ \beta^2 & \beta q^2 & \pm p\gamma s & \pm p\gamma s \\ \gamma^2 & \gamma q^2 & \pm q\delta s & \pm q\delta s \end{pmatrix}, \]

\[ v_4^\pm = \varepsilon \begin{pmatrix} 15 & 26 & 37 & 48 \\ -\beta\gamma\delta & -\beta r s & -\beta s & \pm q\gamma s \pm q\delta \end{pmatrix}, \]

\[ v_5^\pm = \varepsilon \begin{pmatrix} 16 & 25 & 38 & 47 \\ -\beta\gamma\delta & -\beta r s & -\beta s & \pm q\gamma s \pm q\delta \end{pmatrix}, \]

\[ v_6^\pm = \varepsilon \begin{pmatrix} 17 & 28 & 35 & 46 \\ -\beta\gamma\delta & -\beta r s & -\beta s & \pm q\gamma s \pm q\delta \end{pmatrix}, \]

\[ v_7^\pm = \varepsilon \begin{pmatrix} 18 & 27 & 36 & 45 \\ -\beta\gamma\delta & -\beta r s & -\beta s & \pm q\gamma s \pm q\delta \end{pmatrix}, \]

(39)
where we denote the following combinations for simplicity
\[ p \equiv \cosh \left( \frac{\lambda + \mu + \rho}{2 \sqrt{2}} \right), \quad \alpha \equiv \sinh \left( \frac{\lambda + \mu + \rho}{2 \sqrt{2}} \right), \quad q \equiv \cosh \left( \frac{\lambda}{2 \sqrt{2}} \right), \]
\[ r \equiv \cosh \left( \frac{\mu}{2 \sqrt{2}} \right), \quad \gamma \equiv \sinh \left( \frac{\mu}{2 \sqrt{2}} \right), \quad s \equiv \cosh \left( \frac{\rho}{2 \sqrt{2}} \right), \quad \delta \equiv \sinh \left( \frac{\rho}{2 \sqrt{2}} \right), \]
and \( \varepsilon_+ = 1 \) and \( \varepsilon_- = i \). We hope these quantities have no relations with the one in \([12]\).

- \( SO(5)^+ \) embedding
  
  They have the following seven \( 4 \times 4 \) block diagonal matrices respectively:
  
  \[ u^{IJ}_{KL} = \begin{pmatrix} u_1^+ & u_2^+ & u_3^+ & u_4^+ & u_5^+ & u_6^+ & u_7^+ \end{pmatrix}, \]
  \[ v^{IJ}_{KL} = \begin{pmatrix} v_1^+ & v_2^+ & v_3^+ & v_4^+ & v_5^+ & v_6^+ & v_7^+ \end{pmatrix}, \]
  \( (40) \)

where the submatrices are in \([39]\). The nonzero components of \( A_2 \) tensor, \( A_{2,L}^{IJL} \), can be obtained from \([3]\) by simply plugging \([39]\) and \([40]\) they are classified by four different fields with degeneracies 8, 16, 16, 16 respectively and given by:

\[
A_{2,4}^{312} = A_{2,2}^{134} = A_{2,6}^{578} = A_{2,1}^{324} = A_{2,5}^{768} = A_{2,2}^{214} = A_{2,3}^{658} = A_{2,8}^{567} \equiv y_{1,+}; \\
A_{2,5}^{612} = A_{2,7}^{812} = A_{2,5}^{634} = A_{2,1}^{834} = A_{2,5}^{256} = A_{2,3}^{456} = A_{2,4}^{356} = A_{2,1}^{278} = \\
A_{2,3}^{478} = A_{2,5}^{215} = A_{2,8}^{437} = A_{2,2}^{516} = A_{2,4}^{738} = A_{2,8}^{217} = A_{2,6}^{435} = A_{2,2}^{718} \equiv y_{2,+}; \\
A_{2,5}^{814} = A_{2,6}^{714} = A_{2,5}^{823} = A_{2,6}^{723} = A_{2,1}^{458} = A_{2,2}^{358} = A_{2,1}^{467} = A_{2,2}^{367} = \\
A_{2,8}^{415} = A_{2,7}^{326} = A_{2,7}^{416} = A_{2,8}^{325} = A_{2,4}^{617} = A_{2,3}^{528} = A_{2,4}^{618} = A_{2,3}^{627} \equiv y_{3,+}; \\
A_{2,5}^{713} = A_{2,8}^{613} = A_{2,6}^{824} = A_{2,7}^{524} = A_{2,5}^{427} = A_{2,1}^{757} = A_{2,4}^{257} = A_{2,2}^{468} = \\
A_{2,7}^{315} = A_{2,8}^{426} = A_{2,3}^{816} = A_{2,1}^{638} = A_{2,2}^{547} = A_{2,3}^{517} = A_{2,4}^{628} = A_{2,6}^{318} \equiv y_{4,+}.
\]

Here they have simple form:

\[
y_{1,+} = \frac{1}{8 \sqrt{uvw}} \left( -3 + u^2 v^2 + u^2 w^2 + v^2 w^2 \right), \\
y_{2,+} = \frac{1}{8 \sqrt{uvw}} \left( 1 - u^2 v^2 - u^2 w^2 + v^2 w^2 \right), \\
y_{3,+} = \frac{1}{8 \sqrt{uvw}} \left( -1 - u^2 v^2 + u^2 w^2 + v^2 w^2 \right), \\
y_{4,+} = \frac{1}{8 \sqrt{uvw}} \left( -1 + u^2 v^2 - u^2 w^2 + v^2 w^2 \right). \quad (41)
\]

As before in \( SU(3) \) invariant sectors, there exists a symmetry between the upper indices: \( A_{2,L}^{IJL} = A_{2,L}^{JIK} = A_{2,L}^{KIJ} \). The kinetic terms can be summarized as following block diagonal
hermitian matrices:

\[
A_{\mu,1} = \begin{pmatrix}
[12] & 0 & -K & -A & -A \\
[34] & 0 & -A & -A \\
[56] & 0 & -K \\
[78] & 0 \\
\end{pmatrix},
A_{\mu,2} = \begin{pmatrix}
[13] & 0 & K & B & -B \\
[24] & 0 & -B & B \\
[57] & 0 & K \\
[68] & 0 \\
\end{pmatrix},
\]

\[
A_{\mu,3} = \begin{pmatrix}
[14] & 0 & -K & F & F \\
[23] & 0 & F & F \\
[58] & 0 & -K \\
[67] & 0 \\
\end{pmatrix},
A_{\mu,4} = \begin{pmatrix}
[15] & 0 & A & -B & -F \\
[26] & 0 & -F & -B \\
[37] & 0 & A \\
[48] & 0 \\
\end{pmatrix},
\]

\[
A_{\mu,5} = \begin{pmatrix}
[16] & 0 & -A & B & -F \\
[25] & 0 & -F & B \\
[38] & 0 & -A \\
[47] & 0 \\
\end{pmatrix},
A_{\mu,6} = \begin{pmatrix}
[17] & 0 & A & B & F \\
[28] & 0 & F & B \\
[35] & 0 & A \\
[46] & 0 \\
\end{pmatrix},
\]

\[
A_{\mu,7} = \begin{pmatrix}
[18] & 0 & -A & -B & F \\
[27] & 0 & F & -B \\
[36] & 0 & -A \\
[45] & 0 \\
\end{pmatrix},
\]

and nonzero components are

\[
A \equiv \frac{1}{2} \partial_\mu \lambda, \quad B \equiv \frac{1}{2} \partial_\nu \mu, \quad F \equiv \frac{1}{2} \partial_\nu \rho, \quad K \equiv \frac{1}{4} (\partial_\mu \lambda + \partial_\nu \mu + \partial_\rho \rho).
\]

• \(SO(5)^-\) embedding

The 28-beins \(u^{IJ}_{KL}\) and \(v^{IJKL}\) fields which are elements of \(56 \times 56\) \(V(x)\) \((\text{II})\) of the fundamental 56-dimensional representation of \(E_7\) can be obtained by exponentiating the vacuum expectation values \(\phi_{ijkl} (\text{II})\) of \(SO(5)\) singlet space via \((\text{I})\). After tedious calculation they have the following seven \(4 \times 4\) block diagonal hermitian matrices respectively:

\[
u^{IJ}_{KL} = \text{diag}(u_1^-, u_2^-, u_3^-, u_4^-, u_5^-, u_6^-, u_7^-),
\]

\[
u^{IJKL} = \text{diag}(v_1^-, v_2^-, v_3^-, v_4^-, v_5^-, v_6^-, v_7^-),
\]

where the submatrices are the same as those in \((\text{III})\). In this case, \(A^{IJKL}_{\mu}\) is \(-i\) times the one of \(SO(5)^+\) invariant case. Moreover, The nonzero components of \(A_2\) tensor, \(A^{IJKL}_{2\mu}\) can be obtained from \((\text{II})\) by simply plugging \((\text{III})\) and \((\text{III})\). They are classified by four different fields with degeneracies 8, 16, 16, 16 respectively and given by

\[
y_{1,-} = \frac{(1 + i)}{16(uvw)^{3/2}} (-iu^2 + u^3v^3w + \text{two cyclic permutations} - 3uvw + 3iu^2v^2w^2),
\]

32
\[ y_{2,-} = \frac{(1 + i)}{16(uvw)^{3/2}} \left( -iu^2 + iv^2 + iu^2 + u^3v^3 + u^3vw^3 + uvw - iu^2v^2w^2 \right), \]
\[ y_{3,-} = \frac{(1 + i)}{16(uvw)^{3/2}} \left( iu^2 - iv^2 + u^3v^3 + u^3vw^3 - uvw - iu^2v^2w^2 \right), \]
\[ y_{4,-} = \frac{(1 + i)}{16(uvw)^{3/2}} \left( -iu^2 + iv^2 - u^3v^3 + u^3vw^3 - uvw + iu^2v^2w^2 \right). \]

6 Appendix C: \( SO(3) \times SO(3) \) Invariant Sector

The 28-beins \( u^{IJ}_{KL} \) and \( v^{IJKL} \) fields which are elements of \( 56 \times 56 \mathcal{V}(x) \) \( \mathbb{I} \) of the fundamental 56-dimensional representation of \( E_7 \) can be obtained by exponentiating the vacuum expectation values \( \phi_{ijkl} \) \( \mathbb{II} \) of \( SO(3) \times SO(3) \) singlet space via \( \mathbb{II} \). After tedious calculation they have the following thirteen \( 4 \times 4 \) matrices respectively:

\[
u^{IJKL} = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & u_2 & 0 & 0 & u_3 & 0 & u_4 & 0 & 0 & 0 & u_5 & u_6 & 0 & 0 & 0 & 0 & u_7 & u_8 & 0 & 0 & 0 & 0 & u_9 & 0 & u_{10} & 0 & 0 & u_{11} & 0 & 0 & 0 & 0 & 0 & u_{12} & 0 & 0 & 0 & 0 & 0 & u_{13} \end{pmatrix},\]

Each submatrix is \( 4 \times 4 \) matrix and we denote antisymmetric index pairs. Since they are very complicated expressions in terms of hyperbolic functions of \( \lambda^\alpha \), we do not list them here. For explicit forms we refer to the original version in the hep-th archive.

The kinetic terms \( \mathbb{III} \) restricted to the scalar manifold satisfying the constraint \( \lambda^\alpha = 0, \alpha = 2, \cdots, 7 \) can be summarized as following nine \( 4 \times 4 \) matrices:

\[
A^{IJKL}_\mu = \begin{pmatrix} A_{\mu,1} & 0 & 0 & 0 & 0 & A_{\mu,2} & 0 & 0 & 0 & A_{\mu,3} & 0 & 0 & 0 & 0 & A_{\mu,4} & 0 & 0 & 0 & 0 & A_{\mu,5} & 0 & 0 & 0 & 0 & 0 & A_{\mu,6} & 0 & 0 & 0 & 0 & 0 & A_{\mu,7} & 0 & 0 & 0 & 0 & 0 & A_{\mu,8} & 0 & 0 & 0 & 0 & 0 & A_{\mu,9} & 0 & 0 & 0 & 0 & 0 & A_{\mu,10} & 0 & 0 & 0 & 0 & 0 & A_{\mu,11} & 0 & 0 & 0 & 0 & 0 & A_{\mu,12} & 0 & 0 & 0 & 0 & 0 & A_{\mu,13} & 0 & 0 & 0 & 0 & 0 & A_{\mu,14} \end{pmatrix}, \quad (45)
\]
where each submatrix has the following forms:

\[
A_{\mu,1} = \begin{pmatrix}
12 & 34 & 56 & 78 \\
12 & 0 & -a & 0 \\
34 & -a & 0 & 0 \\
56 & 0 & 0 & -a \\
78 & 0 & 0 & -a \\
\end{pmatrix},
A_{\mu,2} = \begin{pmatrix}
12 & 0 & 0 & -ia & 0 \\
34 & 0 & 0 & 0 & 0 \\
56 & 0 & 0 & 0 & 0 \\
78 & 0 & 0 & 0 & -ia \\
\end{pmatrix},
\]

\[
A_{\mu,3} = \begin{pmatrix}
13 & 24 & 57 & 68 \\
13 & 0 & a & 0 \\
24 & a & 0 & 0 \\
57 & 0 & 0 & a \\
68 & 0 & 0 & a \\
\end{pmatrix},
A_{\mu,4} = \begin{pmatrix}
13 & 0 & ia & 0 \\
24 & 0 & 0 & 0 \\
57 & 0 & 0 & 0 \\
68 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
A_{\mu,5} = \begin{pmatrix}
14 & 23 & 58 & 67 \\
14 & 0 & -a & 0 \\
23 & -a & 0 & 0 \\
58 & 0 & 0 & -a \\
67 & 0 & 0 & -a \\
\end{pmatrix},
A_{\mu,6} = \begin{pmatrix}
14 & 0 & 0 & 0 \\
23 & -ia & 0 & 0 \\
58 & 0 & 0 & 0 \\
67 & 0 & 0 & -ia \\
\end{pmatrix},
\]

\[
A_{\mu,8} = \begin{pmatrix}
15 & 26 & 37 & 48 \\
15 & 0 & 0 & 0 \\
26 & 0 & 0 & 0 \\
37 & 0 & 0 & 0 \\
48 & 0 & 0 & -ia \\
\end{pmatrix},
A_{\mu,10} = \begin{pmatrix}
15 & 0 & 0 & 0 \\
26 & 0 & 0 & 0 \\
37 & 0 & 0 & 0 \\
48 & 0 & 0 & -ia \\
\end{pmatrix},
\]

\[
A_{\mu,12} = \begin{pmatrix}
17 & 35 & 46 \\
17 & 0 & 0 \\
28 & 0 & 0 \\
35 & -ia & 0 \\
46 & 0 & 0 \\
\end{pmatrix},
\]

where nonzero component is

\[
a \equiv \frac{1}{4}(1 + \cosh \lambda_1)\partial_{\mu}\lambda_1.
\]

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