On Domination Number of Triangulated Discs

SHIN-ICHI TOKUNAGA\(^1,a)\)

Abstract: Let \( G \) be a 3-connected triangulated disc such that the boundary cycle \( C \) of the outer face is an induced cycle of \( G \) and \( G - C \) is a tree. In this paper we prove that \( \gamma(G) \leq \frac{3n}{4} \), which gives a partial solution for the conjecture that the same inequality holds for any 3-connected triangulated disc. We also show related conjectures.

Keywords: dominating set, domination number, planar graph, triangulated disc

1. Introduction

For a graph \( G = (V(G), E(G)) \) and \( v \in V(G) \), let \( N_G(v) \) denote the set of all the vertices which are adjacent to \( v \) in \( G \), and let \( \Delta(G) \) denote the maximum degree of \( G \). In 2013, Campos and Wakabayashi\(^1\) and Matheson and Tarjan\(^2\) holds for maximal planar graphs with a maximum degree 6. In 2016, Li, Zhu, Shao, and Xu improved the upper bound in Refs.\(^3\) by showing that there also exists a vertex outerplanar graph \( \gamma \) with \( \gamma(G) \leq \frac{4n}{5} \), where \( k \) is the number of pairs of consecutive 2-degree vertices with a distance of at least 3 on the outer cycle.

In 1996, Matheson and Tarjan\(^2\) proved that any triangulated disc \( G \) with \( n \) vertices satisfies \( \gamma(G) \leq \frac{k}{2} - 1 \). They also conjectured that \( \gamma(G) \leq \frac{k}{2} \) for every \( n \)-vertex maximal planar graph \( G \) with sufficiently large \( n \). Note that we need two vertices to dominate the six vertices of the octahedron graph, and there also exists a 11-vertex maximal planar graph with \( \gamma(G) = 3 > \frac{k}{2} \) (Fig. 1), therefore we cannot omit the condition that \( n \) is sufficiently large.

In 2010, King and Pelsmajer\(^7\) proved that the conjecture of Matheson and Tarjan holds for maximal planar graphs with a maximum degree 6. In 2013, Campos and Wakabayashi\(^1\) and Tokunaga\(^3\) independently proved \( \gamma(G) \leq \frac{4n}{5} \) for each \( n \)-vertex outerplanar graph \( G \) with \( n \geq 3 \) having \( k \) vertices of degree 2. In 2016, Li, Zhu, Shao, and Xu improved the upper bound in Refs.\(^1,3\) by showing \( \gamma(G) \leq \frac{4n}{5} \), where \( k \) is the number of pairs of consecutive 2-degree vertices with a distance of at least 3 on the outer cycle.

In Ref.\(^3\), the author gave a simple proof by showing that \( G \)

\[ \gamma(G) \leq \frac{3n}{4} \]

has a proper 4-coloring such that each vertex except those with degree two is dominated by all the four colors, and a similar method is also applied to other related problems\(^4,5\). Moreover, the author conjectured as follows.

Conjecture 1  Suppose \( G \) is a 3-connected \( n \)-vertex triangulated disc, then \( \gamma(G) \leq \frac{3n}{4} \).

Figure 2 shows that the upper bound in Conjecture 1 is sharp. Note that the inner subgraph of the graph in Fig. 2 is a path. There are many graphs satisfying the equality in Conjecture 1 whose inner subgraphs are trees. In this paper, we prove the following theorem.

Theorem 1  Suppose \( G \) is an \( n \)-vertex triangulated disc such that \( \text{In}(G) \) is a tree and \( C(G) \) is an induced cycle of \( G \), then \( \gamma(G) \leq \frac{3n}{4} \).

2. Proof of Theorem 1

To prove Theorem 1, we show the following lemma.

Lemma 1  Suppose \( G \) is an \( n \)-vertex triangulated disc such that \( \text{In}(G) \) is a tree and \( C(G) \) is an induced cycle of \( G \), and let \( v \) be a vertex of \( C(G) \) with \( \text{deg}_G(v) = 3 \). Then, \( G - v \) has a proper
In view of the assumption of Lemma 1. Therefore by induction hypothesis, the statement of Lemma 1 clearly holds for all cases except $x'$. We use induction on $n = |V(G)|$. Since the statement of Lemma 1 clearly holds for $K_4$, we assume $n \geq 5$. In view of $\deg_G(w) \geq 3$ and $\deg_G(u) \geq 3$, there are two cases as follows.

**Case 1.** $\deg_G(u) = 3$ or $\deg_G(w) = 3$.

We may assume $\deg_G(u) = 3$ without loss of generality. Let $u'$ be the vertex of $N_G(u) \setminus \{u, w, x\}$ and let $G' = G - u'w$. Since $\operatorname{Int}(G') = \{u, w, x\}$ and $\operatorname{C}(G') = C - v + uw$ is an induced cycle of $G'$, $G'$ satisfies the assumption of Lemma 1. Thus the induction hypothesis, $G'$ has proper 4-coloring $f'$ such that each vertex of $G' - u$ is dominated by all the four colors except $u', x, w$. Here we define $f'$ as follows: If $f''(u') \neq 0$, then let $f(u) = f''(u')$, and if $f''(u') = 0$, then let $f(u)$ be any value different from $f''(u')$ and $f(x)$. Furthermore, let $f(y) = f'(y)$ for $y \neq u$. Then, $f$ satisfies the conclusion of Lemma 1.

**Case 2.** $\deg_G(u) \geq 4$ and $\deg_G(w) \geq 4$.

We divide this case into two subcases in view of $\deg_G(x)$.

**Subcase 2.1.** $\deg_G(x) = 1$

Let $x'$ be the unique vertex of $T$ which is adjacent to $x$, and let $G' = G - x$. By the assumption of Subcase 2 and Subcase 2.1, $\deg_{G'}(x) \geq 3$. Further, since $\operatorname{Int}(G') = \{u, w, x\}$ and $\operatorname{C}(G') = C - v + ux + xv$ is an induced cycle of $G'$, $G'$ satisfies the assumption of Lemma 1. Therefore by induction hypothesis, $G' - x$ has proper 4-coloring $f''$ such that each vertex of $G' - x$ is dominated by all the four colors except $u', x', w$. Here we define $4$-coloring $f'$ as follows: If $f''(x') \neq 0$, then let $f(x) = f''(x')$. If $f''(x') = 0$, then let $f(x)$ be any value different from $f''(u)', f''(w)'$, and $f''(w)$. Moreover, let $f(y) = f'(y)$ for $y \neq x$. Then, $f$ satisfies the conclusion of Lemma 1.

**Subcase 2.2.** $\deg_{G'}(x) \geq 2$.

Let $x_1$ be the unique vertex of $V(T) \cap N_G(u) \cap N_G(x)$, and let $u'$ be the vertex of $N_G(x_1) \cap N_G(x)$ satisfying $u' \neq u$. Let $T_1$ be a component of $T - x$ containing $x_1$ and let $T_2 = T - T_1$. Also, let $G_1 = (N_G[V(T_1)])_G$ and $G_2 = (N_G[V(T_2)])_G - u + x_1v$. Since $T_1, T_2$ are trees and $C(G_1), C(G_2)$ are induced cycles of $G_1, G_2$, respectively, both $G_1$ and $G_2$ satisfy the assumption of Lemma 1. Thus by induction hypothesis, $G_1 - x$ has proper 4-coloring $f_1$ such that each vertex of $G_1 - x$ is dominated by all the four colors except $u, x_1, u'$, and $G_2 - v$ has proper 4-coloring $f_2$ such that each vertex of $G_2 - v$ is dominated by all the four colors except $x_1, x, w$. Let $j \in \{1, 2, 3, 4\}$ such that $x \notin V(G_1)$. Hence $x \notin V(G_2)$. Let $j \in \{1, 2, 3, 4\} - \{f_1(u), f_1(x_1), f_1(u')\}$, and let

$$k = \begin{cases} j & \text{when } f_1(x_1) = 0 \\ f_1(x_1) & \text{when } f_1(x_1) \neq 0. \end{cases}$$

We can make $f_1(y) = f_2(y)$ for $y \in V(G_1 - x) \cap V(G_2)\setminus \{x_1, u', w\}$ and $f_2(x) = k$ by exchanging colors. Now let $f(y) = f_2(y)$ for $y \in V(G_1) - x$ and let $f(y) = f_2(y)$ for $y \in (V(G_2) - v)$, then $f$ satisfies the conclusion of Lemma 1.

**Proof of Theorem 1.** Let $G, v, f$ be as in Lemma 1 and let $u, x, w$ be as in the proof of Lemma 1. Let $G'$ be the $(n + 2)$-vertex graph such that $V(G') = V(G) \cup \{p, q\}$ and $E(G') = E(G) \cup \{pu, pe, pw, pq, qu, qw\}$. Further, we give a 4-coloring $f'$ of $G'$ satisfying $f'(v) = f(y)$ for $y \in V(G) - v$ and $f'(x), f'(u), f'(p), f'(q) = \{1, 2, 3, 4\}$. Then, each vertex of $V(G)$ is dominated by all the four colors, and hence we may assume $S = \{v \in V(G') \mid f'(v) = 1\}$ satisfies $|S| \leq \left[\frac{2n^2}{3}\right]$ without loss of generality. Finally, if we let

- $S' = \begin{cases} S & \text{when } S \cap \{p, q\} = \emptyset, \\ S - p + v & \text{when } p \in S, \\ S - q + v & \text{when } q \in S, \end{cases}$

then, $S'$ is a dominating set of $G$ satisfying $|S'| \leq \left[\frac{2n^2}{3}\right]$. □
3. Other Conjectures

If we weaken the assumption of 3-connectivity in Conjecture 1 to δ(G) ≥ 3, then the upper bound in Conjecture 1 appears to change as follows.

**Conjecture 2** Suppose G is an n-vertex triangulated disc satisfying δ(G) ≥ 3, then γ(G) ≤ ⌊3n/11⌋.

Figure 6 shows that the upper bound in Conjecture 2 cannot be improved.

Though there is still a gap between Conjecture 1 and Theorem 1, if the following conjecture is true, then Conjecture 1 holds for 4-connected maximal planar graphs.

**Conjecture 3** Suppose G is a 4-connected n-vertex maximal planar graph. Then V(G) can be divided into S₁, S₂ such that ⟨S₁⟩G, ⟨S₂⟩G are a maximal outerplanar graph and a tree, respectively.

Note that if we delete all the edges connecting two vertices of S₁ in the above conjecture, we get a graph satisfying the assumption of Theorem 1.

Acknowledgments The author would like to thank the referees for their many helpful comments and suggestions.

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