COMPLETELY POSITIVE MAPS ON COXETER GROUPS, 
DEFORMED COMMUTATION RELATIONS, 
AND OPERATOR SPACES

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Abstract

In this article we prove that quasi-multiplicative (with respect to the usual length function) mappings on the permutation group $S_n$ (or, more generally, on arbitrary amenable Coxeter groups), determined by self-adjoint contractions fulfilling the braid or Yang-Baxter relations, are completely positive. We point out the connection of this result with the construction of a Fock representation of the deformed commutation relations $d_i d_j^* - \sum_{r,s} t_{ir}^{js} d_r^* d_s = \delta_{ij} 1$, where the matrix $t_{ir}^{js}$ is given by a self-adjoint contraction fulfilling the braid relation. Such deformed commutation relations give examples for operator spaces as considered by Effros, Ruan and Pisier. The corresponding von Neumann algebras, generated by $G_i = d_i + d_i^*$, are typically not injective.
1. Introduction

We will prove in this paper the following result.

**Theorem 1.1.** Consider for fixed $n \in \mathbb{N}$ the permutation group $S_n$ and denote by $\pi_i \in S_n$ ($i = 1, \ldots, n - 1$) the transposition between $i$ and $i + 1$. Furthermore, let operators $T_i \in B(H)$ ($i = 1, \ldots, n - 1$) on some Hilbert space $H$ be given with the properties:

i) $T_i^* = T_i$ for all $i = 1, \ldots, n - 1$

ii) $\|T_i\| \leq 1$ for all $i = 1, \ldots, n - 1$

iii) The $T_i$ satisfy the braid relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for all } i = 1, \ldots, n - 2$$

$$T_i T_j = T_j T_i \quad \text{for all } i, j = 1, \ldots, n - 1 \text{ with } |i - j| \geq 2$$

Define now a function

$$\varphi : S_n \to B(H)$$

by quasi-multiplicative extension of

$$\varphi(e) = 1, \quad \varphi(\pi_i) = T_i,$$

i.e. for a reduced word $S_n \ni \sigma = \pi_{i(1)} \ldots \pi_{i(k)}$ we put $\varphi(\sigma) = T_{i(1)} \ldots T_{i(k)}$.

Then $\varphi$ is a completely positive map, i.e. for all $l \in \mathbb{N}$, $f_i \in C S_n$, $x_i \in H$ ($i = 1, \ldots, l$) we have

$$\langle \sum_{i,j=1}^l \varphi(f_j^* f_i) x_i, x_j \rangle \geq 0.$$
the existence of the Fock representation of the $q$-relations, are now available, see [BJS,BSp1,BSp2,Spe2,Fiv,Gre,Zag].

In [Spe2], we considered, more generally, the relations

$$d_i d_j^* - q_{ij} d_j^* d_i = \delta_{ij} 1$$

for $-1 \leq q_{ij} = q_{ji} \leq 1$ and proved by central limit arguments the existence of a Fock representation. In Sect. 3, we will construct the Fock representation of these deformed commutation relations, now even for the most general case of complex $q_{ij}$ with $\bar{q}_{ij} = q_{ji}$. We will see that again the positivity of some map on $\mathbb{S}_n$ is the key point behind this construction. This positivity will then follow as a special case of our general Theorem 1.1.

Our construction of the $q_{ij}$-relations depends essentially on some operator $T$, which is a self-adjoint contraction and fulfills the braid or Yang-Baxter relation. Thus, our natural frame in Sects. 3 and 4 will be that we consider the general deformed commutation relations

$$d_i d_j^* - \sum_{r,s} t_{rs}^{ij} d_r^* d_s = \delta_{ij}.$$ 

Such general Wick ordering relations are also investigated by Jørgensen, Schmitt, and Werner [JSW2]. Whereas in the most general case, without any assumptions on $t_{rs}^{ij}$ apart from the necessary $t_{ab}^{dc} = t_{ab}^{cd}$, nothing can be said about the existence of a Fock representation, we get, by Theorem 1.1, a proof for the existence of this representation in the case where the matrix $t_{rs}^{ij}$ is given by a self-adjoint contraction $T$ fulfilling the braid relation.

In Sect. 4, we examine the deformed commutation relations from an operator space point of view, namely we extend a result of Haagerup and Pisier and show that the operator space generated by the $G_i := d_i + d_i^*$ is completely isomorphic to the canonic operator space $R \cap C$, which means

$$\| (a_1, \ldots, a_N) \|_{\max} \leq \| \sum_{i=1}^N a_i \otimes G_i \| \leq \frac{2}{\sqrt{1-q}} \| (a_1, \ldots, a_N) \|_{\max}$$

for all bounded operators $a_1, \ldots, a_N$ on some Hilbert space, where

$$\| (a_1, \ldots, a_N) \|_{\max} := \max(\| \sum_{i=1}^N a_i a_i^* \|^{1/2}, \| \sum_{i=1}^N a_i^* a_i \|^{1/2}).$$

We will also make some remarks on the von Neumann algebra generated by the $G_i$. In particular, we show that it is typically not injective.

Our main theorem, 1.1 and its general version 2.1, considers operator valued functions which are quasi-multiplicative with respect to the usual length function (=minimal number of generators). In Sect. 5, we replace this length function by another, also quite natural one (= minimal number of different generators) and prove the analogue of 2.1 for this case.
2. Completely positive maps on finite Coxeter groups

Let \((W, S)\) be a Coxeter system consisting of a Coxeter group \(W\) and a set \(S = \{s_1, \ldots, s_n\}\) of generators. This means that \(W\) is the group generated by the elements \(s_i = s_i^{-1} \in S\) and that for each two distinct generators \(s_i, s_j \in S\) \((i \neq j)\) there exists a natural number \(m_{ij} \geq 2\) such that we have the relation

\[(s_is_j)^{m_{ij}} = e,\]

where \(e\) is the unit element of \(W\). The fact \(s_i = s_i^{-1}\) can also be stated in this form as \(m_{ii} = 1\). In the following we will only consider finite Coxeter groups \(W\).

For each \(\sigma \in W\) we denote by \(|\sigma|\) the length of \(\sigma\) with respect to \(S\), i.e.

\[|\sigma| := \min \{k \in \mathbb{N} | \text{there exist } s_{i(1)}, \ldots, s_{i(k)} \in S \text{ with } \sigma = s_{i(1)} \ldots s_{i(k)}\},\]

and \(|e| = 0\).

The example of \(S_n\) fits into this frame by putting \(W = S_n, S = \{\pi_1, \ldots, \pi_n-1\}\).

The length function \(|\pi|\) is then given by the number of inversions and the relations are given by \(m_{ij} = 2\) for \(|i - j| \geq 2\), i.e.

\[\pi_i \pi_j \pi_i \pi_j = e \iff \pi_i \pi_j = \pi_j \pi_i \quad (|i - j| \geq 2)\]

and \(m_{i,i+1} = 3\), i.e.

\[\pi_i \pi_{i+1} \pi_i \pi_{i+1} \pi_i \pi_{i+1} = e \iff \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} .\]

For a general Coxeter group \(W\), we will also rewrite the defining relations \((s_i s_j)^{m_{ij}} = e\) in the braid like form

\[\underbrace{s_is_js_i \ldots}_{m_{ij} \text{ factors}} = \underbrace{s_js_is_j \ldots}_{m_{ij} \text{ factors}},\]

which means

\[(s_is_j)^{m_{ij}/2} = (s_js_i)^{m_{ij}/2} \quad \text{for } m_{ij} \text{ even}\]

and

\[(s_is_j)^{(m_{ij}-1)/2} s_i = (s_js_i)^{(m_{ij}-1)/2} s_j \quad \text{for } m_{ij} \text{ odd}.\]

Let now self-adjoint contractions \(T_i \in B(\mathcal{H})\) on some Hilbert space be given which fulfill the generalized braid relations

\[\underbrace{T_i T_j T_i \ldots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \ldots}_{m_{ij} \text{ factors}},\]

for all \(i, j = 1, \ldots, n\) with \(i \neq j\). Then we define the mapping

\[\varphi : W \to B(\mathcal{H})\]

by \(\varphi(e) = 1\) and

\[\varphi(\sigma) = T_{i_1} \ldots T_{i_{|\sigma|}} \quad \text{for } \sigma = s_{i_1} \ldots s_{i_{|\sigma|}} \quad \text{with } |\sigma| \leq k,\]
It is known [Bou] that the generalized braid relations for the $T_i$ ensure that this definition of $\varphi$ is well defined. We can also state our definition in the way that we put $\varphi(s_i) = T_i$ and extend $\varphi$ in a quasi-multiplicative way, which means

$$\varphi(\sigma_1 \sigma_2) = \varphi(\sigma_1) \varphi(\sigma_2) \quad \text{if} \quad |\sigma_1 \sigma_2| = |\sigma_1| + |\sigma_2|.$$ 

Note that the self-adjointness of the $T_i$ implies $\varphi(\sigma^{-1}) = \varphi(\sigma)^*$. 

Let us extend $\varphi$ from $W$ to its group algebra

$$\mathbb{C}W := \{ f = \sum_{\sigma \in W} f(\sigma) \delta_\sigma \}$$

(with the usual multiplication $(\delta_\sigma \delta_\pi = \delta_{\sigma \pi})$ and involution $(\delta_\sigma^* = \delta_{\sigma^{-1}})$ structure) in the canonical way

$$\varphi(\sum_{\sigma \in W} f(\sigma) \delta_\sigma) = \sum_{\sigma \in W} f(\sigma) \varphi(\sigma),$$

then we can state our main result in the following way.

**Theorem 2.1.** Let $T_i \in B(\mathcal{H})$ ($i = 1, \ldots, n$) be bounded operators on some Hilbert space $\mathcal{H}$, which fulfill the following assumptions:

i) $T_i^* = T_i$ for all $i = 1, \ldots, n$

ii) $\|T_i\| \leq 1$ for all $i = 1, \ldots, n$

iii) We have for all $i, j = 1, \ldots, n$ with $i \neq j$ the generalized braid relations

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots \underbrace{m_{ij}}_{m_{ij} \text{ factors}} \cdots \underbrace{m_{ij}}_{m_{ij} \text{ factors}}.$$

Then the quasi-multiplicative map

$$\varphi : \mathbb{C}W \rightarrow B(\mathcal{H})$$

given by

$$\varphi(e) = 1, \quad \varphi(s_i) = T_i \quad (i = 1, \ldots, n)$$

is completely positive, i.e. for all $l \in \mathbb{N}$, $f_i \in \mathbb{C}W$, $x_i \in \mathcal{H}$ ($i = 1, \ldots, l$) we have

$$\langle \sum_{i,j=1}^l \varphi(f_j^* f_i) x_i, x_j \rangle \geq 0.$$

**Remark.** Another equivalent characterization of complete positivity is the following (see, e.g., [Pau]): For arbitrary $\alpha : W \rightarrow \mathcal{H}$ (with finite support) we have

$$\sum_{\rho, \sigma \in W} \langle \varphi(\rho^{-1} \sigma) \alpha(\sigma), \alpha(\rho) \rangle \geq 0.$$

This formulation is the operator valued version of the definition of a positive definite function on $W$.

Theorem 2.1 follows essentially from the following theorem.
Theorem 2.2. Let $T_i$ and $\varphi$ be as in 2.1. Then the operator

$$P := \sum_{\sigma \in W} \varphi(\sigma)$$

is positive, i.e. $P \geq 0$.

By putting all $x_i \equiv x$ it is clear that complete positivity implies $P \geq 0$. Let us now see how we get, in the other direction, 2.1 from 2.2.

Proof of 2.2 $\Rightarrow$ 2.1. Let $\lambda$ be the left regular representation of $W$ acting on $l^2(W) \cong CW$ equipped with the scalar product $\langle f, g \rangle = \sum_{\sigma \in W} \bar{f}(\sigma) g(\sigma)$, i.e.

$$(\lambda(\rho)f)(\sigma) := f(\rho^{-1}\sigma) \quad \text{or} \quad \lambda(\rho)\delta_{\sigma} = \delta_{\rho\sigma} \quad \text{for } \rho, \sigma \in W, f \in l^2(W).$$

If we now define the operators

$$\hat{T}_i := \lambda(s_i) \otimes T_i \quad \text{on } l^2(W) \otimes \mathcal{H},$$

then they also satisfy the assumptions of Theorem 2.2, which yields

$$\hat{P} := \sum_{\sigma \in W} \hat{\varphi}(\sigma) \geq 0,$$

where $\hat{\varphi}$ is the quasi-multiplicative function given by the $\hat{T}_i$, clearly

$$\hat{\varphi}(\sigma) = \lambda(\sigma) \otimes \varphi(\sigma) \quad \text{for } \sigma \in W.$$

The positivity of $\hat{P}$ implies now

$$\langle \sum_{i,j=1}^l \varphi(f_j^* f_i)x_i, x_j \rangle \geq 0$$

in the following way: Put

$$\alpha := \sum_{\rho \in W} \delta_{\rho} \otimes \left( \sum_i f_i(\rho^{-1})x_i \right) \in l^2(W) \otimes \mathcal{H}.$$

Then

$$0 \leq \langle \hat{P}\alpha, \alpha \rangle$$

$$= \sum_{\sigma} \langle (\lambda(\sigma) \otimes \varphi(\sigma))\alpha, \alpha \rangle$$

$$= \sum_{\sigma, \rho, \tau, i,j} \langle (\lambda(\sigma) \otimes \varphi(\sigma))\delta_{\rho} \otimes f_i(\rho^{-1})x_i, \delta_{\tau} \otimes f_j(\tau^{-1})x_j \rangle$$

$$= \sum_{\sigma, \rho, \tau, i,j} \langle \delta_{\sigma\rho} \otimes f_i(\rho^{-1})\varphi(\sigma)x_i, \delta_{\tau} \otimes f_j(\tau^{-1})x_j \rangle$$

$$= \sum_{\sigma, \rho, \tau, i,j} \langle \delta_{\sigma\rho} \delta_{\tau} \langle \hat{f}_j(\tau^{-1})f_i(\rho^{-1})\varphi(\sigma)x_i, x_j \rangle \rangle$$

$$= \langle \sum \varphi(f_j^* f_i)x_i, x_j \rangle.$$
The last line follows from

\[ \langle \delta_\sigma, \delta_\tau \rangle = \begin{cases} 
1, & \sigma = \tau \rho^{-1} \\
0, & \text{else}
\end{cases} \]

and the fact that with

\[ f_i = \sum_\rho f_i(\rho^{-1}) \delta_{\rho^{-1}}, \quad f_j^* = \sum_\tau \bar{f}_j(\tau^{-1}) \delta_\tau \]

we have

\[ f_j^* f_i = \sum_{\rho, \tau} \bar{f}_j(\tau^{-1}) f_i(\rho^{-1}) \delta_{\tau \rho^{-1}}. \]

\[ \diamond \]

So we are left with the proof of 2.2. Note first that it suffices to treat the case of strict contractions.

**Theorem 2.3.** Let \( T_i \) and \( \varphi \) be as in 2.1, but with the stronger assumption of strict contractivity, i.e. \( \|T_i\| < 1 \) for all \( i = 1, \ldots, n \). Then the operator

\[ P := \sum_{\sigma \in W} \varphi(\sigma) \]

is strictly positive, i.e. \( P > 0 \).

Theorem 2.2 can be inferred from this version in the following way.

**Proof of 2.3 \( \Rightarrow \) 2.2.** Consider \( T_i^{(t)} := tT_i \) (\( i = 1, \ldots, n \)) for \( 0 \leq t < 1 \). If the \( T_i \) fulfill the assumptions of 2.2, then the \( T_i^{(t)} \) fulfill the assumptions of 2.3. Thus

\[ P^{(t)} = \sum_{\sigma \in W} \varphi(\sigma)t^{\|\sigma\|} > 0 \quad \text{for all } 0 \leq t < 1. \]

If now \( t \nearrow 1 \), then \( P^{(t)} \to P \) uniformly and we get the assertion.

\[ \diamond \]

To prove 2.3 we reduce the assertion about strict positivity to one about invertibility. Note that \( P \) is self-adjoint, since

\[ P^* = \sum_{\sigma \in W} \varphi(\sigma)^* = \sum_{\sigma \in W} \varphi(\sigma^{-1}) = P. \]

**Theorem 2.4.** Let \( T_i \), \( \varphi \), and \( P \) be as in 2.3. Then \( P \) is invertible.

For the reduction of 2.3 to 2.4 we need a fact on the norm-continuity of the smallest element in the spectrum of a self-adjoint operator. Define for a self-adjoint operator \( A \in B(H) \) the number

\[ m_0(A) := \inf\{ \langle Ax, x \rangle \mid x \in H, \|x\| = 1 \} \]

as the smallest element in the (compact!) spectrum of \( A \).
Lemma 2.5. We have for arbitrary self-adjoint operators $A, B \in B(\mathcal{H})$

$$|m_0(A) - m_0(B)| \leq \|A - B\|.$$  

Proof of 2.5. Assume $m_0(A) \geq m_0(B)$. Fix an arbitrary $\epsilon > 0$. Then there exists $x \in \mathcal{H}$ with $\|x\| = 1$ such that $m_0(B) \geq \langle Bx, x \rangle - \epsilon$. Since $m_0(A) \leq \langle Ax, x \rangle$ we have

$$|m_0(A) - m_0(B)| \leq \langle Ax, x \rangle - \langle Bx, x \rangle + \epsilon = \langle (A - B)x, x \rangle + \epsilon \leq \|A - B\| + \epsilon.$$  

For $\epsilon \to 0$ we get the assertion.  

Proof of 2.4 $\Rightarrow$ 2.3. Consider again the collection of $T_i^{(t)} := tT_i$ ($i = 1, \ldots, n$) for all $t$ with $0 \leq t \leq 1$. Then, by 2.4, $P^{(t)} = \sum \varphi(\sigma)t^{|\sigma|}$ is invertible for all $0 \leq t \leq 1$ and we have $P^{(0)} = 1$ and $P^{(1)} = P$. Furthermore, the map $t \mapsto P^{(t)}$ is norm-continuous. Put $m_0(t) := m_0(P^{(t)})$ (note $P^{(t)*} = P^{(t)}$). Since

$$|m_0(t_1) - m_0(t_2)| \leq \|P^{(t_1)} - P^{(t_2)}\|,$$

the mapping $t \mapsto m_0(t)$ is continuous. But now invertibility of $P^{(t)}$ implies $m_0(t) \neq 0$. Because of $m_0(0) = m_0(1) = 1$, we have $m_0(t) > 0$ for all $0 \leq t \leq 1$, in particular $m_0(P) = m_0(1) > 0$, i.e. $P > 0$.  

Up to now we have only used very general arguments for the reduction of our theorem. This reduction has led us to a statement on invertibility of some operator $P \in CW$. This is now an algebraic problem which can be ‘calculated’ in our group algebra. Of course, now we need the special structure of Coxeter groups. The proof will be by induction on the cardinality of Coxeter generators of parabolic subgroups of $W$.

For $J \subseteq S$, let $W_J$ be the subgroup of $W$ generated by all $s \in J$. Such subgroups are called parabolic. They are also Coxeter groups, given by the system $(W_J, J)$. We need now the following known facts on these subgroups (see, e.g., [Bou,Car]): For $J \subseteq S$ we define

$$D_J := \{\sigma \in W \mid |\sigma s| = |\sigma| + 1 \text{ for all } s \in J\},$$

i.e. $\sigma \in D_J$ if and only if $\sigma$ is the element of smallest length in the coset $\sigma \cdot W_J$. Thus $D_J$ is a canonical representative system of the cosets of $W_J$. If we define for $\sigma \in W$ the set

$$J_\sigma := \{s \in S \mid |\sigma s| = |\sigma| + 1\},$$

then the definition of $D_J$ can also be put in the way

$$\sigma \in D_J \iff J \subseteq J_\sigma.$$  

This characterization gives at once the Euler-Solomon-formula [Sol] for all $\sigma \in W$

$$\sum_{J \subseteq S \quad \text{with} \quad |J| = |J_\sigma|} (-1)^{|J|} = \sum_{J \subseteq J_\sigma} (-1)^{|J|} = \begin{cases} 0, & \text{if } \sigma \neq \sigma_0 \\ 1, & \text{if } \sigma = \sigma_0 \end{cases},$$

where $\sigma_0$ is the longest element of $W$.
where \( \sigma_0 \) is the unique element in \( W \) with the greatest length, i.e. the unique element with the property \( J_{\sigma_0} = \emptyset \).

Furthermore, we have the nice property that each element \( \sigma \in W \) can uniquely be written in the form \( \tau_j \sigma_J \) with \( \tau_j \in D_J \) and \( \sigma_J \in W_J \), and with \( |\sigma| = |\tau_J| + |\sigma_J| \).

Note in particular that, for \( J = S \), we have \( W_S = W \) and hence \( D_S = \{e\} \). In the next section we will need the following special case: For \( W = S_{n+1} \) consider \( J := \{ \pi_2, \pi_3, \ldots, \pi_n \} \), i.e. \( W_J \cong S_n \). Then \( D_J = \{ e, \pi_1, \pi_2 \pi_1, \ldots, \pi_n \pi_{n-1} \ldots \pi_1 \} \).

As an example, take \( n = 2 \), then \( W = S_3 = \{ e, \pi_1, \pi_2, \pi_1 \pi_2, \pi_2 \pi_1, \pi_1 \pi_2 \pi_1 = \pi_2 \pi_1 \pi_2 \} \), \( J = \{ \pi_2 \} \), \( W_J = \{ e, \pi_2 \} \cong S_2 \), \( D_J = \{ e, \pi_1, \pi_2 \pi_1 \} \).

Now define for an arbitrary subset \( A \subseteq W \) the operator

\[
P(A) := \sum_{\sigma \in A} \varphi(\sigma).
\]

Then the uniqueness of the decomposition \( W = D_J W_J \) and the quasi-multiplicity of \( \varphi \) give for all \( J \subseteq S \)

\[
P = P(W) = P(D_J) P(W_J).
\]

For the above example of \( W = S_3 \) and \( J = \{ \pi_2 \} \) this decomposition is given by

\[
1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1 = (1 + T_1 + T_2)(1 + T_2).
\]

The crucial point for our induction is now the translation of the Euler-Solomon-formula to our operators \( P(A) \).

**Lemma 2.6.** Let \((W, S)\) be an arbitrary finite Coxeter group and \( \sigma_0 \) the unique longest element in \( W \). Then we have

\[
\sum_{J \subseteq S} (-1)^{|J|} P(D_J) = \varphi(\sigma_0).
\]

**Proof of 2.6.** We have

\[
\sum_{J \subseteq S} (-1)^{|J|} P(D_J) = \sum_{J \subseteq S} (-1)^{|J|} \sum_{\sigma \in D_J} \varphi(\sigma)
= \sum_{\sigma \in W} \left( \sum_{J \subseteq J_\sigma} (-1)^{|J|} \varphi(\sigma) \right)
= \varphi(\sigma_0).
\]

\(\Box\)

**Proof of 2.4.** We prove this by induction on the cardinality of \( S \). If \( |S| = 0 \), then \( W = \{ e \} \) and \( P = 1 \) is invertible. If \( |S| = 1 \), then \( W = \{ e, s_1 \} \) and \( P = 1 + T_1 \) is invertible because of \( \|T_1\| < 1 \).

Assume now we know the invertibility of \( P(W) \) for all finite Coxeter groups \((\tilde{W}, \tilde{S})\) with \( |\tilde{S}| \leq n - 1 \). Consider an arbitrary finite Coxeter group \((W, S)\) with \( |S| = n \). By \( J \subseteq S \) we will denote in the following the situation that \( J \subseteq S \), but \( J \neq S \).
Then we have by induction hypothesis the invertibility of \( P(W_J) \) for all \( J \subset S \), hence
\[
P(D_J) = P(W)P(W_J)^{-1}.  
\]
Lemma 2.6 yields then
\[
P(W) \left\{ \sum_{J \subset S} (-1)^{|J|} P(W_J)^{-1} \right\} = \sum_{J \subset S} (-1)^{|J|} P(D_J)
\]
\[
= \varphi(\sigma_0) - (-1)^{|S|} P(D_S)
\]
\[
= \varphi(\sigma_0) - (-1)^{|S|} 1.  
\]
Since \( \|\varphi(\sigma_0)\| < 1 \), the element \( \varphi(\sigma_0) - (-1)^{|S|} 1 \) is invertible and we get
\[
P(W) \left\{ \sum_{J \subset S} (-1)^{|J|} P(W_J)^{-1} \right\} \{\varphi(\sigma_0) - (-1)^{|S|} 1\}^{-1} = 1,  
\]
hence \( P = P(W) \) is right invertible. Because of \( P^* = P \) it is also left invertible, hence invertible.

\[\diamondsuit\]

Remarks. 1) If we specialize to \( W = S_n \) and \( S = \{\pi_1, \ldots, \pi_{n-1}\} \) then we recover Theorem 1.1 from the introduction. Note that even in this case our main step, namely the positivity of \( P \), is by no means trivial. E.g., for \( S_3 \) it states that the operator \( P = 1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1 \) is positive, whenever \( T_i^* = T_i, \|T_i\| < 1 \), and \( T_1T_2T_1 = T_2T_1T_2 \).

2) Theorem 2.1 is also true for amenable Coxeter groups. Since we know by a result of de la Harpe [deH] that amenable Coxeter groups are either finite or affine Coxeter groups and hence the cardinality of the set \( \{\sigma \mid |\sigma| \leq k\} \) is at most of polynomial growth in \( k \) (see [Bou] for the structure of affine Coxeter groups), the operator \( P \) is also well defined in the case \( \|T_i\| < 1 \) for amenable Coxeter groups. In this case, all our arguments remain the same, only in Lemma 2.6 the value \( \varphi(\sigma_0) \) on the right side of the equation has to be replaced by 0 if the Coxeter group is infinite. Thus we get in the same manner as for finite groups the assertion of 2.1 also for amenable groups in the case \( \|T_i\| < 1 \). Since the statement on complete positivity involves only finite sums, we can now carry out the limit \( \|T_i\| \triangleright 1 \) and obtain in this way the validity of Theorem 2.1 for all amenable Coxeter groups.

3) It is an open question whether 2.1 is true for all infinite Coxeter groups. What can be proved in this general case is the validity of 2.1 for all Coxeter groups in the special case of scalar valued \( T_i \in \mathbb{C} \). This proof uses other methods and will be published elsewhere [Boz2]. In the special case, when \( W \) is the free product of 2-elements groups, i.e. when we have as only relations \( s_i^2 = e \) for all \( i \), then Theorem 2.1 was also proved for the general operator valued case, see [Boz1].

3. Fock Representation of Deformed Commutation Relations

We will now use our general result 1.1 for the construction of the Fock representation of the \( q_{ij} \)-relations
\[
d_i^* d_j - d_j^* d_i = \delta_{ij} 1,  
\]
for $\bar{q}_{ij} = q_{ji}$ and $|q_{ij}| \leq 1$, i.e. we are looking for operators $d_i, d_i^*$ on some Hilbert space $\mathcal{H}$ and some vector $\Omega \in \mathcal{H}$ (called vacuum), such that $d_i$ and $d_i^*$ are adjoints of each other and all annihilation operators $d_i$ annihilate the vacuum: $d_i\Omega = 0$. One should note that these requirements determine the structure of the Fock representation up to unitary equivalence, the only problem is to prove the existence of such a structure, in particular to show the positivity of the corresponding scalar product in $\tilde{\mathcal{H}}$.

We will treat in the following a more general case and specify this in the end to the above mentioned relations. Assume we are given some operator $T$ and a Hilbert space $\mathcal{H}$ such that $T \in B(\mathcal{H} \otimes \mathcal{H})$ is a self-adjoint contraction ($T^* = T$, $\|T\| \leq 1$) and such that it fulfills the braid relation

$$(\text{BR}) \quad (1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(T \otimes 1),$$

where $1 \otimes T$ and $T \otimes 1$ are the natural amplifications of $T$ to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. Then we define

$$T_i := 1 \otimes \cdots \otimes 1 \otimes T \quad \text{acting on } \mathcal{H}^{\otimes (i+1)},$$

and by amplification also on all $\mathcal{H}^{\otimes n}$ with $n \geq i + 1$. The $T_i$ fulfill the assumptions of Theorem 1.1.

The braid relation (BR) appears also in a lot of contexts under the name ‘Yang-Baxter equation’, see, e.g., [Man,Jim,Wen].

Now we define, for each $f \in \mathcal{H}$, a creation operator $d^*(f)$ and an annihilation operator $d(f)$ on a dense subset $\mathcal{F}$ of the full Fock space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, where $\mathcal{H}^0 := \mathbb{C}\Omega$ ($\|\Omega\| = 1$), $\mathcal{F}$ being the set of finite linear combinations of product vectors. On the full Fock space we have the canonic free creation and annihilation operators given by (see [Eva,Voi,Spe1])

$$l^*(f)\Omega = f$$
$$l^*(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$

and

$$l(f)\Omega = 0$$
$$l(f)f_1 \otimes \cdots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n.$$  

We define now our deformation by

$$d^*(f) := l^*(f)$$

and

$$d(f) := l(f)(1 + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}) \quad \text{on } \mathcal{H}^{\otimes n}.$$  

Of course, $d(f)$ and $d^*(f)$ are not adjoints of each other with respect to the usual scalar product $\langle \cdot , \cdot \rangle$. Thus we introduce a new one $\langle \cdot , \cdot \rangle_T$ given by

$$\langle f, \nu \rangle_T := \delta_{\nu, P^{(n)}_n f} \quad \text{for } f \in \mathcal{H}^{\otimes n}, \nu \in \mathcal{H}^{\otimes n}.$$  

where

$$P^{(n)} := \sum_{\sigma \in S_n} \varphi(\sigma)$$  \hspace{1cm} (note $P^{(0)} := 1$)

is the canonic operator corresponding to the quasi-multiplicative function $\varphi : S_n \to B(\mathcal{H}^{\otimes n})$ given by $\varphi(e) = 1$ and $\varphi(\pi_i) = T_i$ ($i = 1, \ldots, n - 1$). According to Theorem 2.2 the operators $P^{(n)}$ are positive, thus $\langle \ , \ \rangle_T$ is positive definite. If $\|T\| < 1$ then, by 2.3, we know that all $P^{(n)}$ are strictly positive and we can take as $\mathcal{F}_T$ the completion of $\mathcal{F}$ with respect to $\langle \ , \ \rangle_T$. In the case $\|T\| = 1$, we might get a kernel of $\langle \ , \ \rangle_T$ and we have to divide this out before taking the completion.

**Theorem 3.1.** i) For all $f \in \mathcal{H}$, $d(f)$ and $d^*(f)$ are adjoints of each other on $\mathcal{F}_T$, i.e. for all $k \in \mathbb{N}$ and all $\xi, \eta \in \bigoplus_{n=0}^{k} \mathcal{H}^{\otimes n}$ we have

$$\langle d^*(f)\xi, \eta \rangle_T = \langle \xi, d(f)\eta \rangle_T.$$

ii) If $\|T\| = q < 1$, then

$$\|d^*(f)\|_T \leq \|f\| \frac{1}{\sqrt{1 - q}}.$$

**Proof.** i) By definition of the $T_i$, we have

$$l^*(f)T_i = T_{i+1}l^*(f) \quad (i \geq 1),$$

which implies

$$l^*(f)P^{(n)} = (1 \otimes P^{(n)})l^*(f) \quad \text{or} \quad P^{(n)}l(f) = l(f)(1 \otimes P^{(n)}).$$

Furthermore, our general decomposition $P(W) = P(D_J)P(W_J)$ gives for the case

$$W = S_{n+1}, \quad J = \{\pi_2, \pi_3, \ldots, \pi_n\}, \quad D_J = \{e, \pi_1, \pi_2\pi_1, \ldots, \pi_{n-1}\pi_{n-2}\pi_1\}$$

the relation

$$P^{(n+1)} = P(S_{n+1}) = R^{(n+1)^*}(1 \otimes P^{(n)}) = (1 \otimes P^{(n)})R^{(n+1)},$$

where

$$R^{(n)} := 1 + T_1 + T_1T_2 + \cdots + T_1\cdots T_{n-2}T_{n-1}.$$  

Note that

$$d(f) = l(f)R^{(n)} \quad \text{on } \mathcal{H}^{\otimes n}.$$  

We have now for $\xi \in \mathcal{H}^{\otimes n}$ and $\eta \in \mathcal{H}^{\otimes(n+1)}$

$$\langle d^*(f)\xi, \eta \rangle_T = \langle d^*(f)\xi, P^{(n+1)}\eta \rangle$$

$$= \langle \xi, l(f)P^{(n+1)}\eta \rangle$$

$$= \langle \xi, l(f)(1 \otimes P^{(n)})R^{(n+1)}\eta \rangle$$

$$= \langle \xi, P^{(n)}l(f)R^{(n+1)}\eta \rangle$$

$$= \langle \xi, P^{(n)}d(f)\eta \rangle$$

$$= \langle \xi, d(f)\eta \rangle.$$
ii) Since
\[ \| R^{(n)} \| \leq 1 + q + q^2 + \cdots + q^{n-1} \leq \frac{1}{1-q}, \]
we have
\[ P^{(n+1)} P^{(n+1)} = (1 \otimes P^{(n)}) R^{(n+1)} R^{(n+1)*} (1 \otimes P^{(n)}) \]
\[ \leq \frac{1}{(1-q)^2} (1 \otimes P^{(n)}) (1 \otimes P^{(n)}) \]
hence (because of the positivity of \( P^{(n)} \) and \( P^{(n+1)} \))
\[ P^{(n+1)} \leq \frac{1}{1-q} (1 \otimes P^{(n)}), \]
which gives for \( \xi \in \mathcal{H}^{\otimes n} \)
\[ \| d^*(f)\xi \|_T^2 = \langle d^*(f)\xi, d^*(f)\xi \rangle_T \]
\[ = \langle f \otimes \xi, f \otimes \xi \rangle_T \]
\[ = \langle f \otimes \xi, P^{(n+1)} f \otimes \xi \rangle \]
\[ \leq \frac{1}{1-q} \langle f \otimes \xi, (1 \otimes P^{(n)}) f \otimes \xi \rangle \]
\[ = \frac{1}{1-q} \langle f, \xi \rangle \langle \xi, P^{(n)} \xi \rangle \]
\[ = \frac{1}{1-q} \| f \|^2 \| \xi \|^2. \]

If we choose some basis \( \{ e_i \}_{i \in I} \) of \( \mathcal{H} \) and define the matrix \( t \) by
\[ T e_a \otimes e_b = \sum_{c,d \in I} t_{ab}^{cd} e_c \otimes e_d \quad (a, b \in I), \]
then, by using the definition of our creation and annihilation operators, it is easy to check that the operators \( d_i := d(e_i) \) \( (i \in I) \) fulfill the relations
\[ d_i d_j^* - \sum_{r,s \in I} t_{js}^{ir} d_j^* d_r = \delta_{ij} \quad (i, j \in I). \]
Since by construction \( d(f)\Omega = 0 \) for all \( f \in \mathcal{H} \), we have obtained the Fock representation of these relations.

Now we want to recover the \( q_{ij} \)-relations from our general construction. For this we consider the operator \( T = Q \pi_1 \), where \( Q \) is the multiplication operator
\[ Q(e_i \otimes e_j) = q_{ij} (e_i \otimes e_j) \]
and \( \pi_1 \) the natural action of the corresponding transposition
\[ \pi_1 (f \otimes e) = e \otimes f \quad (f, e \in \mathcal{H}). \]
This $T$ is self-adjoint (because of $q_{ij} = q_{ji}$), contractive ($\|T\| = \sup_{i,j \in I} |q_{ij}| \leq 1$), and fulfills the braid relation (BR). Thus the foregoing construction may be applied to it. In this case one gets the following concrete formula for the annihilation operator.

$$d(e_i) e_{i(1)} \otimes \cdots \otimes e_{i(n)} =$$

$$= \sum_{k=1}^{n} q_{i(k),i(k-1)} q_{i(k),i(k-2)} \cdots q_{i(1),i(1)} \delta_{i,i(k)} e_{i(1)} \otimes \cdots \otimes \hat{e}_{i(k)} \otimes \cdots \otimes e_{i(n)},$$

where $e_{i(k)}$ has to be deleted in the tensor.

Thus we get the following corollary on the existence of Fock representations out of our constructions.

**Corollary 3.2.** i) Let $T \in B(\mathcal{H} \otimes \mathcal{H})$ be a self-adjoint contraction fulfilling the braid relation and write

$$Te_{a} \otimes e_{b} = \sum_{c,d \in I} t_{ab}^{cd} e_{d} \otimes e_{c} \quad (a,b \in I)$$

for some basis $\{e_{i}\}_{i \in I}$ of $\mathcal{H}$. Then there exist operators $d_{i} (i \in I)$ on some Hilbert space $\tilde{\mathcal{H}}$ and a ‘vacuum vector’ $\Omega \in \tilde{\mathcal{H}}$ such that $d_{i} \Omega = 0$ for all $i \in I$ and

$$d_{i}d_{j} - \sum_{r,s \in I} t_{ij}^{rs} d_{r}^{s}d_{s} = \delta_{ij} \quad (i,j \in I).$$

If $\|T\| < 1$, then the $d_{i}$ are bounded.

ii) In particular, for given $q_{ij}$ $(i,j \in I)$ with $q_{ij} = q_{ji}$ for all $i,j \in I$ and $\sup_{i,j \in I} |q_{ij}| = q \leq 1$ there exist operators $d_{i}$ $(i \in I)$ on some Hilbert space $\tilde{\mathcal{H}}$ and a ‘vacuum vector’ $\Omega \in \tilde{\mathcal{H}}$ such that $d_{i} \Omega = 0$ for all $i \in I$ and

$$d_{i}d_{j} - q_{ij}d_{i}^{*}d_{j} = \delta_{ij} \quad (i,j \in I).$$

If $q < 1$, then the $d_{i}$ are bounded.

**Remarks.** 1) Consider the $q_{ij}$-relations. Let $q < 1$. Then, for $q_{ii} \geq 0$, it follows from $d_{i}d_{i}^{*} - q_{ii}d_{i}^{*}d_{i} = 1$ the estimate $\|d_{i}\|^{2} \leq 1 + q_{ii} \|d_{i}\|^{2}$, i.e. $\|d_{i}\| \leq 1/\sqrt{1-q_{ii}}$. For $q_{ii} < 0$, we even have $d_{i}d_{i}^{*} = 1 + q_{ii}d_{i}^{*}d_{i}$, i.e. $\|d_{i}\| \leq 1$. The restriction of our representation from $\mathcal{F}_{T}$ to the linear span of $\{e_{i}^{\otimes n} | n \in \mathbb{N}\}$ shows that these inequalities are indeed equalities, thus

$$\|d_{i}\|_{T} = \begin{cases} 1/\sqrt{1-q_{ii}}, & \text{if } q_{ii} \geq 0 \\ 1, & \text{if } q_{ii} < 0. \end{cases}$$

This is true for all $q < 1$, thus, by continuity, also for $q = 1$.

2) For the crucial step, namely the positivity of all $P^{(n)}$, in our construction of the $q_{ij}$-relations we do not need 2.2 in full generality but only for the special case of $T_{i} = Q_{i} \pi_{i}$, where the $Q_{i}$ commute. For this case a simpler proof of 2.2 (for $W = S_{n}$) is given in [JSW2] (but without any assertion on strict positivity of $P^{(n)}$ in the case $\|T\| < 1$).
4. Operator spaces

Now we want to consider the deformed commutation relations constructed in the last section from an operator space point of view. Operator spaces were introduced by Effros and Ruan [ER1,ER2] and further investigated by Blecher and Paulsen [BP] and Pisier [Pis2]. Operator spaces are closed linear subsets of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and have a lot of nice properties. The philosophy behind their introduction is that they quantize functional analysis in the sense that in the usual statements, e.g. in norm inequalities, numbers are replaced by operators. We refer to [ER2,BP,Pis2] for more details.

One canonical operator space is the Hilbert space $R \cap C \subset M_N \oplus M_N$ (where $M_N$ are the $N \times N$-matrices, for $N = \infty$ the compact operators) with basis $\{\delta_i\}_{i=1}^N$ given by

$$\delta_i = \begin{pmatrix}
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix},$$

where the 1 is appearing in the $i$-th position in the first column or first row, respectively. Operator spaces which are also Hilbert spaces are called Hilbertian operator spaces. The Hilbertian operator space $R \cap C$ has the following characterizing property: For all $a_i \in B(\tilde{\mathcal{H}})$ ($i = 1, \ldots, N$) on some Hilbert space $\tilde{\mathcal{H}}$ one has

$$\left\| \sum_{i=1}^N a_i \otimes \delta_i \right\|_{B(\tilde{\mathcal{H}}) \otimes M_N} = \left\| (a_1, \ldots, a_N) \right\|_{\max},$$

where

$$\left\| (a_1, \ldots, a_N) \right\|_{\max} := \max \left( \left\| \sum_{i=1}^N a_i a_i^* \right\|^{1/2}, \left\| \sum_{i=1}^N a_i^* a_i \right\|^{1/2} \right).$$

We consider now the operators $d(f)$ and $d^*(f)$ ($f \in \mathcal{H}$) on $\mathcal{F}_T$ as constructed, for a given self-adjoint contraction $T$ fulfilling the braid relation (BR), in the last section. Assume in the following $\|T\| = q < 1$. We choose a basis $\{e_i\}$ of $\mathcal{H}$ and put $d_i := d(e_i)$ and

$$G_i := d_i + d_i^*.$$ 

Then we claim that the operator space generated by the closure of the linear span of the $G_i$ is, as an operator space, isomorphic to $R \cap C$, where $N = \dim \mathcal{H}$. This means nothing else than the following norm estimate.

**Theorem 4.1.** We have for arbitrary operators $a_i \in B(\tilde{\mathcal{H}})$ ($i = 1, \ldots, N$) with $N \leq \dim \mathcal{H}$ the estimate

$$\left\| (a_1, \ldots, a_N) \right\|_{\max} \leq \left\| \sum_{i=1}^N a_i \otimes G_i \right\|_{\tilde{\mathcal{H}} \otimes \mathcal{F}_T} \leq \frac{2}{\sqrt{1-q}} \left\| (a_1, \ldots, a_N) \right\|_{\max}.$$ 

The case $T = 0$ was treated by Haagerup and Pisier [HP]. Our proof will follow their ideas.
Proof. By definition \( d(f) = l(f)R^{(n)} \) on \( \mathcal{H}^\otimes n \), i.e. more generally \( d(f) = l(f)R \), where the infinite sum

\[
R := 1 + T_1 + T_1T_2 + T_1T_2T_3 + \ldots
\]

makes sense because of \( \|T\| < 1 \). The crucial step is now a norm estimate for this \( R \). Of course, we have with respect to the usual norm on the full Fock space \( \|R\| \leq 1/(1-q) \). We want to show that the same estimate is true for \( \|R\|_T \). Remember that \( P^{(n+1)} = (1 \otimes P^{(n)})R^{(n+1)} \) and \( P^{(n+1)} \leq 1/(1-q)(1 \otimes P^{(n)}) \).

Then we have for \( \xi \in \mathcal{H}^\otimes(n+1) \)

\[
\|R\xi\|^2_T = \langle R^{(n+1)}\xi, R^{(n+1)}\xi \rangle_T = \langle R^{(n+1)}\xi, P^{(n+1)}R^{(n+1)}\xi \rangle \\
\leq \frac{1}{1-q} \langle R^{(n+1)}\xi, (1 \otimes P^{(n)})R^{(n+1)}\xi \rangle \\
= \frac{1}{1-q} \langle R^{(n+1)}\xi, P^{(n+1)}\xi \rangle \\
= \frac{1}{1-q} \langle R^{(n+1)}\xi, \xi \rangle_T \\
\leq \frac{1}{1-q} \|R\xi\|_T \|\xi\|_T,
\]

which implies

\[
\|R\xi\|_T \leq \frac{1}{1-q} \|\xi\|_T, \quad \text{hence} \quad \|R\|_T \leq \frac{1}{1-q}.
\]

Since

\[
\| \sum_{i=1}^N b_i c_i \| \leq \| \sum_{i=1}^N b_i b_i^*\|^{1/2} \| \sum_{i=1}^N c_i^* c_i \|^{1/2}
\]

for arbitrary bounded operators \( b_1, \ldots, b_N, c_1, \ldots, c_N \) on some Hilbert space, we obtain now

\[
\| \sum_{i=1}^N a_i \otimes d_i^* \|_{\bar{\mathcal{H}}^\otimes F_T} = \| \sum_{i=1}^N (1 \otimes d_i^*)(a_i \otimes 1) \|_{\bar{\mathcal{H}}^\otimes F_T} \\
\leq \| \sum_{i=1}^N d_i^* d_i \|_T^{1/2} \| \sum_{i=1}^N a_i^* a_i \|_{\bar{\mathcal{H}}}^{1/2}.
\]

Because of

\[
\sum_{i=1}^N d_i^* d_i = (\sum_{i=1}^N l_i^* l_i) R = (1 - P_\Omega)R,
\]

where \( P_\Omega \) is the projection onto the vacuum \( \Omega \), we get

\[
\| \sum_{i=1}^N a_i \otimes d_i^* \|_{\bar{\mathcal{H}}^\otimes F_T} \leq \|R\|_T^{1/2} \| \sum_{i=1}^N a_i^* a_i \|_{\bar{\mathcal{H}}}^{1/2} \\
\leq \frac{1}{\sqrt{1-q}} \| \sum_{i=1}^N a_i^* a_i \|_{\bar{\mathcal{H}}}^{1/2}.
\]
and by taking adjoints
\[ \| \sum_{i=1}^{N} a_i \otimes d_i \|_{\tilde{H} \otimes \mathcal{F}_T} \leq \frac{1}{\sqrt{1-q}} \| \sum_{i=1}^{N} a_i a_i^* \|_{\tilde{H}}^{1/2}, \]
which yield together the right inequality of our assertion.

For the other inequality we use the vacuum expectation state
\[ \epsilon(T) := \langle \Omega, T \Omega \rangle \quad (T \in B(F_T)). \]
We only need
\[ \epsilon(G_i G_j) = \langle \Omega, d_i d_j^* \Omega \rangle = \delta_{ij} \]
to obtain
\[ \| \sum_{i=1}^{N} a_i \otimes G_i \|_{\tilde{H} \otimes \mathcal{F}_T}^2 \geq \sup_{\varphi \text{ state on } B(\tilde{H})} (\varphi \otimes \epsilon)[(\sum_{i=1}^{N} a_i \otimes G_i)^* (\sum_{j=1}^{N} a_j \otimes G_j)] = \sup_{\varphi \text{ state on } B(\tilde{H})} \varphi(\sum_{i=1}^{N} a_i^* a_i) = \sum_{i=1}^{N} a_i^* a_i, \]
and analogously
\[ \| \sum_{i=1}^{N} a_i \otimes G_i \|_{\tilde{H} \otimes \mathcal{F}_T}^2 \geq \sum_{i=1}^{N} a_i a_i^*. \]

Our theorem characterizes completely the operator space structure of our deformations, namely this structure does not depend on the deformation (at least as long as \( \|T\| < 1 \)). One may also ask about the \( C^* \)- or \( W^* \)-structure of our deformations. In this respect, the situation is not so clear. Let us make in the following some remarks in this direction.

For \( |q| \) sufficiently small, the method of [JSW1] should still work showing that the \( C^* \)-algebra generated by all \( d(f) \) (\( f \in \mathcal{H} \)) is equal to the extension of the Cuntz algebra \( O_n \) by the compact operators, where \( n = \dim \mathcal{H} \). See [JSW2] for investigations in this direction. It is conceivable that the \( C^* \)-structure of the \( q_{ij} \)-relations or even of our general deformations is the same for all \( \|T\| < 1 \).

Another interesting problem is the structure of the von Neumann algebra \( \mathcal{M}_T \) generated by all \( G_i \). Typically, these von Neumann algebras are not injective. Injectivity of a von Neumann algebra \( \mathcal{M} \subseteq B(\mathcal{H}) \) means that there exists a projection of norm 1 from \( B(\mathcal{H}) \) onto \( \mathcal{M} \).

**Theorem 4.2.** If the vacuum expectation \( \epsilon \) is a faithful trace on \( \mathcal{M}_T \) and \( \dim \mathcal{H} > 16/(1-q)^2 \), then \( \mathcal{M}_T \) is not injective.

**Proof.** If \( \mathcal{M}_T \) were injective we would have for all \( a_i, b_i \in \mathcal{M}_T \) (compare Corollary 2 of [Was])
\[ \| \sum_{i=1}^{m} a_i \otimes b_i \| \geq |\epsilon(\sum_{i=1}^{m} a_i b_i^*)|, \]
in particular, for \( a_i = b_i = G_i \),
\[
\| \sum_{i=1}^{m} G_i \otimes \bar{G}_i \| \geq \epsilon(\sum_{i=1}^{m} G_i G_i) = m.
\]

But on the other side, by putting \( a_i = \bar{G}_i \) in Theorem 4.1, we also have
\[
\| \sum_{i=1}^{m} G_i \otimes \bar{G}_i \| \leq \frac{4}{1-q}\sqrt{m},
\]
which leads, for \( m > 16/(1-q)^2 \), to a contradiction.

By following the ideas of Theorem 2.9 in [Pis2], we see that faithfulness of \( \epsilon \) is not really needed. But according to the next theorem we do not need to make this distinction.

**Theorem 4.3.** Assume that the vacuum expectation \( \epsilon \) is tracial on \( \mathcal{M}_T \). Then the vacuum \( \Omega \) is cyclic and separating for \( \mathcal{M}_T \). In particular, \( \epsilon \) is faithful.

**Proof.** For cyclicity of \( \Omega \) it suffices to see that we can obtain all basis product vectors \( e_{i(1)} \otimes \cdots \otimes e_{i(k)} \) for all \( k \in \mathbb{N} \) and all \( i(1), \ldots, i(k) \in \{1, \ldots, n\} \) from \( \Omega \) by application of some polynomial in the \( G_i \). Since
\[
e_{i(1)} = G_{i(1)} \Omega
\]
and
\[
e_{i(1)} \otimes \cdots \otimes e_{i(k)} = G_{i(1)} \cdots G_{i(k)} \Omega - \eta \quad \text{with} \quad \eta \in \bigoplus_{l=0}^{k-1} \mathcal{H}^\otimes l,
\]
this follows by induction.

To show that \( \Omega \) is separating for \( \mathcal{M}_T \) is the same as showing that \( \Omega \) is cyclic for \( \mathcal{M}'_T \). Let us define the anti-linear conjugation operator \( J : \mathcal{F}_T \to \mathcal{F}_T \) by \( JA\Omega = A^* \Omega \) for \( A \in \mathcal{M}_T \). This is well-defined because the trace property of \( \epsilon \) implies \( \| A\Omega \|_{\mathcal{F}_T} = \| A^* \Omega \|_{\mathcal{F}_T} \). One can easily check that \( J\mathcal{M}_T J \subseteq \mathcal{M}'_T \). Since \( \Omega \) is cyclic for \( J\mathcal{M}_T J \), the assertion follows.

Note that we have shown that \( \mathcal{M}_T \) is in standard form and thus \( J\mathcal{M}_T J = \mathcal{M}'_T \). By the way, the conjugation operator \( J \) is explicitly given by
\[
J(e_{i(1)} \otimes \cdots \otimes e_{i(n)}) = e_{i(n)} \otimes \cdots \otimes e_{i(1)},
\]
i.e. the operators \( JG_iJ \) are like the \( G_i \), only action from the left is replaced by action from the right.

This raises the question whether \( \epsilon \) is a trace on \( \mathcal{M}_T \). This can be answered like follows.

**Theorem 4.4.** 1) The vacuum expectation \( \epsilon \) is a trace on \( \mathcal{M}_T \) if and only if \( T \) fulfills
\[
\langle e_a \otimes e_a, T e_b \otimes e_b \rangle = \langle e_a \otimes e_a, T e_b \otimes e_b \rangle
\]
for all $a, b, c, d \in \{1, \ldots, \dim \mathcal{H}\}$. 

2) In particular, in the case of the $q_{ij}$-relations, $\epsilon$ is a trace if and only if the $q_{ij}$ are symmetric and hence real.

Proof. 1) We will only give a sketch of the proof for the general case. Let us write 

$$Te_a \otimes e_b = \sum_{c, d} t_{ab}^{dc} e_d \otimes e_c.$$ 

Then the asserted condition for $T$ reads as 

$$t_{ab}^{dc} = t_{da}^{cb},$$

i.e. $t_{ab}^{dc}$ is cyclic in its four indices. (By the way, $T = T^*$ reads in this language as $t_{ab}^{dc} = t_{dc}^{ab}$.)

Necessity of this cyclicity condition follows from 

$$\epsilon(G_aG_bG_cG_d) = \epsilon(d_a d_b d_c d_d^*) + \epsilon(d_a^* d_b d_c d_d^*) = \langle e_a, e_d \rangle \langle e_b, e_c \rangle + t_{cd}^{ba} + \langle e_a, e_b \rangle \langle e_c, e_d \rangle.$$ 

To see that cyclicity is also a sufficient condition, one has to check, by using the very definition of $d_i$ and $d_i^*$, the following formula.

$$\epsilon(G_{i(1)} \ldots G_{i(m)}) = \begin{cases} 0, & m \text{ odd} \\ \sum_{\mathcal{V}} \mathcal{T}(\mathcal{V}), & m = 2r, \end{cases}$$

where the sum runs over all pairings $\mathcal{V} = \{(a_1, z_1), \ldots, (a_r, z_r)\}$ of the indices $1, \ldots, 2r$ (we always assume $a_k < z_k$ and $a_k < a_l$ for $k < l$) and where $\mathcal{T}(\mathcal{V})$ is a factor which is calculated from a given $\mathcal{V}$ in the following way: Put $2r$ points on a circle and denote them in clock-wise order by $1, \ldots, 2r$. Connect the points $a_k$ and $z_k$ for all $k = 1, \ldots, r$ by an arc inside the circle in such a way that at most two arcs cross in one point and such that the number of these crossing points is minimal. Thus we get a graph consisting of points, namely the outer points on the circle and the crossing points, and edges, namely the pieces of our arcs connecting two points. To each edge, we assign a variable $a, b, c, \ldots$. This graph determines now $\mathcal{T}(\mathcal{V})$ by the following rules: Each outer point $k$ with edge $a$ gives a factor $\delta_{i(k), a}$. Each crossing point with the four edges $a, b, c, d$ (in clock-wise order) gives a factor $t_{ab}^{dc}$. Take then the product over the factors corresponding to all points and sum this over all variables of the edges, each running from 1 to $\dim \mathcal{H}$. The result is $\mathcal{T}(\mathcal{V})$.

Examples: For $\mathcal{V} = \{(1, 4), (2, 4)\}$ we have 

$$\mathcal{T}(\mathcal{V}) = \sum_{a, b} \delta_{a, i(1)} \delta_{a, i(4)} \delta_{b, i(2)} \delta_{b, i(3)} = \delta_{i(1), i(4)} \delta_{i(2), i(3)}.$$ 

For $\mathcal{V} = \{(1, 3), (2, 5), (4, 6)\}$ we have 

$$\mathcal{T}(\mathcal{V}) = \sum_a \epsilon_i^{(6)(1)} \epsilon_i^{(2)(3)} \epsilon_i^{(4)(5)} t_{i(2)i(3)}^{a(6)} t_{i(4)i(5)}^{a(1)} t_{i(6)i(1)}^{a(2)},$$

whereas for $\mathcal{V} = \{(1, 4), (2, 5), (3, 6)\}$ we obtain 

$$\mathcal{T}(\mathcal{V}) = \sum_i \epsilon_i^{(6)(1)} \epsilon_i^{(2)(3)} \epsilon_i^{(4)(5)} t_{i(6)i(1)}^{a(1)} t_{i(2)i(3)}^{a(2)} t_{i(4)i(5)}^{a(3)}.$$
Note that the braid relation for \( T \) ensures that \( \mathcal{T}(\mathcal{V}) \) does not depend on the way we have drawn our graph, as long as we keep the number of crossing points minimal. If we do not assume cyclicity of \( t_{ab}^{dc} \), then a similar formula would be valid, one only has to take care how to arrange the variables at \( t_{ab}^{dc} \) at the crossing points. For this one has to distinguish between ingoing and outgoing edges.

Having the above formula for the calculation of \( \epsilon \), one sees quite easily that under a cyclic permutation of \( \mathcal{V} \) the clockwise order at the crossing points does not change, thus \( \mathcal{T}(\mathcal{V}) \) does not change under such a cyclic permutation (under the assumption \( t_{ab}^{dc} = t_{da}^{cb} \)) and hence \( \epsilon \) is a trace on \( \mathcal{M}_T \).

2) Since \( t_{ab}^{dc} = q_{ba} \delta_{ba} \delta_{ca} \), we have \( t_{ab}^{dc} = t_{da}^{cb} \) if and only if \( q_{ab} = q_{ba} \). In this case, \( \mathcal{T}(\mathcal{V}) \) from part 1 can be written more explicitly as

\[
\mathcal{T}(\mathcal{V}) = \delta_{i(a_1),i(z_1)} \cdots \delta_{i(a_r),i(z_r)} \cdot t(\mathcal{V})
\]

for a given pairing \( \mathcal{V} = \{(a_1, z_1), \ldots, (a_r, z_r)\} \) of the indices \( 1, \ldots, 2r \). The number \( t(\mathcal{V}) \) denotes a weighting factor taking into account the number of inversions of \( \mathcal{V} \), namely with

\[
I(\mathcal{V}) := \{(k, l) \mid k, l = 1, \ldots, r \text{ such that } a_k < a_l < z_k < z_l\}
\]

it is given by

\[
t(\mathcal{V}) = \prod_{(k, l) \in I(\mathcal{V})} q_{i(a_k), i(a_l)}.
\]

In this case the formula for \( \epsilon(G_{i(1)} \cdots G_{i(m)}) \) can be proved quite easily from the identical one for \( \epsilon(d_{i(1)}^\# \cdots d_{i(m)}^\#) \), where \( d_{i(k)}^\# \) stands for \( d_{i(k)} \) or \( d_{i(k)}^* \). This latter formula follows by noticing that it is true for products of the form

\[
d_{i(1)}^* \cdots d_{i(k)}^* d_{i(k+1)} \cdots d_{i(k+l)}
\]

and that both sides of the formula change in the same way under application of the \( q_{ij} \)-relations, see [BSp1,Spe2].

\[\Diamond\]

We conjecture that \( \mathcal{M}_T \) is, at least for the \( q_{ij} \)-relations, a factor. This will be pursued further in forthcoming investigations.

5. Completely positive maps corresponding to block length

The completely positive maps on Coxeter groups considered in Sect. 2 were canonical generalizations of the basic example \( \varphi(\sigma) = q^{||\sigma||} \), where \( ||\sigma|| \) is the usual length function on our Coxeter group \( W \). This example appeared (for \( W = S_n \)) quite naturally in the course of our investigations on generalized Brownian motions in [BSp1]. In [BSp3] we considered another example of a Brownian motion which is intimately connected with Voiculescu’s concept of freeness [VDN]. We found that once more the positive definiteness of some function on \( S_n \) is the key point in our construction. This function is again of the form

\[
\varphi(\sigma) = q^{||\sigma||},
\]

but now \( ||\sigma|| \) is another length function on \( S_n \). Namely, whereas \( ||\sigma|| \) counts the number of generators in a reduced representation of \( \sigma \), the function \( ||\sigma|| \) gives
the number of different generators. This length function and the corresponding quasi-multiplicative \( \varphi \) can now again be extended in a canonical way to arbitrary Coxeter groups and to operator valued functions.

Let \((W, S)\) be an arbitrary Coxeter group. If \( \sigma = s_{i(1)} \ldots s_{i(k)} \) is a reduced representation of \( \sigma \), then we put

\[
b(\sigma) := \{s_{i(1)}, \ldots, s_{i(k)}\},
\]

the set of generators appearing in \( \sigma \). Although a reduced representation is not unique, \( b(\sigma) \) is well defined, see [Bou]. For example, in \( W = S_3 \), we have \( \pi_1 \pi_2 \pi_1 = \pi_2 \pi_1 \pi_2 \) and \( b(\pi_1 \pi_2 \pi_1) = b(\pi_2 \pi_1 \pi_2) = \{\pi_1, \pi_2\} \).

We call the corresponding length function

\[
\|\sigma\| := \#b(\sigma)
\]

block length function. As will follow from our Theorem 5.1, it is a positive definite function on \( W \). For \( W = S_n \), it has a nice graphical meaning, namely

\[
\|\sigma\| = n - \text{the number of connected components of the graph of } \sigma,
\]

e.g.

\[
\sigma = \pi_1 \pi_4 \pi_3 = \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5
\end{array} \quad \|\sigma\| = 5 - 2 = 3
\]

\[
\sigma = e = \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5
\end{array} \quad \|\sigma\| = 5 - 5 = 0.
\]

The analogue of 2.1 for this concept of block length is now the following.

**Theorem 5.1.** Let \( T_i \in B(\mathcal{H}) \) \((i = 1, \ldots, n)\) be bounded operators on some Hilbert space \( \mathcal{H} \), which satisfy:

i) \( 0 \leq T_i \leq 1 \) for all \( i = 1, \ldots, n \).

ii) The \( T_i \) commute: \( T_i T_j = T_j T_i \) for all \( i, j = 1, \ldots, n \).

Define now a quasi-multiplicative (with respect to \( \|\sigma\| \)) function

\[
\varphi : CW \rightarrow B(\mathcal{H}) \quad \text{by} \quad \varphi(\sigma) := \prod_{i \text{ with } s_i \in b(\sigma)} T_i \quad (\varphi(e) := 1).
\]

Then \( \varphi \) is completely positive.

**Remarks.** Note that our assumptions on the \( T_i \) are quite natural.

1) In the example \( \varphi_q(\sigma) = q^{\|\sigma\|} \) in the case \( W = S_n \) one can check that \( \varphi_q \) is positive definite only for \( 1 \geq q \geq \alpha_n \), where \( \alpha_n < 0 \), but \( \lim_{n \rightarrow \infty} \alpha_n = 0 \). Thus, in general, we have to assume \( T_i \geq 0 \).

2) Also commutativity of the \( T_i \) is necessary, otherwise the relations in \( W \) would conflict with a canonical definition of \( \varphi \), e.g. for \( W = S_3 \) and \( \pi := \pi_1 \pi_2 \pi_1 = \pi_2 \pi_1 \pi_2 \) there is no canonical preference for one of the two possibilities \( \varphi(\pi) = T_1 T_2 \) or \( \varphi(\pi) = T_2 T_1 \) thus they should coincide.
Proof. Since the pointwise product of two commuting completely positive maps is again completely positive [Boz3], it suffices to consider the special case where all $T_i$ but one are equal to 1, i.e. for arbitrary but fixed $k \in \{1, \ldots, n\}$ we consider

$$\varphi(\sigma) = \begin{cases} T, & \text{if } s_k \in b(\sigma) \\ 1, & \text{if } s_k \notin b(\sigma), \end{cases}$$

where $T := T_k$ fulfills $0 \leq T \leq 1$. Since $T$ can be diagonalized by the spectral theorem, the assertion can be reduced to the scalar valued case and we only have to treat the special cases

$$\varphi_q(\sigma) = \begin{cases} q, & \text{if } s_k \in b(\sigma) \\ 1, & \text{if } s_k \notin b(\sigma) \end{cases}$$

for all $k \in \{1, \ldots, n\}$ and all $q$ with $0 \leq q \leq 1$. Let $W_k$ be the parabolic subgroup of $(W, S)$ generated by $J := S \setminus \{s_k\}$. Then one knows [Bou] that

$$s_k \in b(\sigma) \iff \sigma \cdot W_k \neq W_k.$$  

Now consider the kernel $\delta$ on all subsets of $W$ given by

$$\delta(A, B) = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{if } A \neq B \end{cases} \text{ for } A, B \subseteq W.$$  

Then we have by putting $q = \exp(-t)$ ($0 < t < \infty$)

$$\varphi_q(\sigma) = q^{1-\delta(\sigma \cdot W_k, W_k)} = e^{-t} e^{t(\sigma \cdot W_k, W_k)}$$

or

$$\varphi_q(\tau^{-1}\sigma) = e^{-t} e^{t(\sigma \cdot W_k, \tau \cdot W_k)}.$$  

Since $\delta$ is positive definite on all subsets of $W$ we get, by the Schönberg theorem (see, e.g., [Boz3]), the positive definiteness of $\varphi_q$ for $t > 0$. The case $q = 1$ is trivial, and $q = 0$ follows by continuity from $q \searrow 0$.

Remarks. 1) Note that, contrary to the situation considered in Sect. 2, the scalar valued case contains all essential information, the operator valued version is a mere transcription to diagonal operators. Thus, in the spirit of the remarks 2 and 3 at the end of Sect. 2, we are not restricted to amenable Coxeter groups, but Theorem 5.1 is valid for all Coxeter groups.

2) Note that here there is no reduction to a positivity problem for some operator $P$ like the reduction from 2.1 to 2.2. The statement that $\sum_{\sigma \in W} \varphi(\sigma) \geq 0$, is trivially true because of $T_i \geq 0$ and $T_i T_j = T_j T_i$, but it is by far not sufficient for the complete positivity of $\varphi$. \[\Box\]
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