Compositions inside a rectangle and unimodality

Bruce E. Sagan
Department of Mathematics, Michigan State University,
East Lansing, MI 48824-1027, USA, sagan@math.msu.edu

February 1, 2008

Key Words: composition, integer partition, unimodal
AMS subject classification (2000): Primary 05A20; Secondary 05A17.

Abstract

Let \( c_{k,l}(n) \) be the number of compositions (ordered partitions) of the integer \( n \) whose Ferrers diagram fits inside a \( k \times l \) rectangle. The purpose of this note is to give a simple, algebraic proof of a conjecture of Vatter that the sequence \( c_{k,l}(0), c_{k,l}(1), \ldots, c_{k,l}(kl) \) is unimodal. The problem of giving a combinatorial proof of this fact is discussed, but is still open.

1 Introduction

Let \( \mathbb{N} \) and \( \mathbb{P} \) denote the nonnegative and positive integers, respectively. A partition of \( n \in \mathbb{N} \) is a weakly decreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of positive integers called parts such that \( \sum_i \lambda_i = n \). We write \( \lambda \vdash n \) or \( |\lambda| = n \) if \( \lambda \) partitions \( n \). We will also use the notation \( \lambda = (n^{m_1}, \ldots, 1^{m_1}) \) where \( m_i \) is the number of times \( i \) appears as a part in \( \lambda \). If \( m_i = 1 \) then the exponent is suppressed and if \( m_i = 0 \) then so is the base. The Ferrers diagram of \( \lambda \), also denoted \( \lambda \), consists of left-justified rows of squares with \( \lambda_i \) squares in row \( i \). The Ferrers diagram of \( \lambda = (4, 3, 3, 1) = (4, 3^2, 1) \) is shown in Figure 1.

Partitions can be ordered by letting \( \lambda \leq \mu \) if the Ferrers diagram for \( \lambda \) is contained in the upper left corner of the one for \( \mu \). Equivalently, \( \lambda_i \leq \mu_i \) for all \( i \) where we set \( \lambda_i = 0 \) if \( i \) is greater than the number of parts of \( \lambda \) and similarly for \( \mu \). The set of partitions under this partial order is called Young’s lattice. More information about partitions and this lattice can be found in the books of Andrews [1], Sagan [10], or Stanley [15].
We say that $\lambda$ fits inside a $k \times l$ rectangle if $\lambda \leq (l^k)$. In other words, $\lambda$ has at most $k$ parts each of size at most $l$. Let $p^{k,l}(n)$ denote the number of such $\lambda$ where $\lambda \vdash n$. A sequence $a_0, a_1, \ldots, a_r$ of nonnegative integers is said to be unimodal if there is an index $m$ such that
\[ a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \geq a_r. \] (1)

Unimodal sequences arise in many aspects of combinatorics, geometry, and algebra. See the survey articles of Stanley [14] and Brenti [3] for details. Our interest is in the following well-known theorem.

**Theorem 1.1.** Given $k, l \in \mathbb{P}$ the sequence $p^{k,l}(0), p^{k,l}(1), \ldots, p^{k,l}(kl)$ is unimodal.

This result was first proved by Sylvester [17] using invariant theory. Since then, there have been a number of other proofs. In particular, Stanley [13] derived this and much more from the hard Lefschetz Theorem of algebraic geometry. Proctor [9] was able to reduce Stanley’s proof to pure linear algebra. And finally, Kathy O’Hara [8] gave a combinatorial proof of this theorem.

A composition $\kappa$ of $n$, written $\kappa \models n$, is any sequence $\kappa = (\kappa_1, \ldots, \kappa_r)$ of positive integers summing to $n$. Note that a composition need not be weakly decreasing. All of the definitions discussed so far have obvious analogues for compositions so we will not bother restating them. For example, the Ferrers diagram of the composition $(3, 1, 4, 1)$ is displayed in Figure 1. Although there is a large literature surrounding partitions, composition have only recently aroused interest due to their connection with quasi-symmetric functions [4, 5], the theory of patterns [6, 12], and the subword and factor partial orders [2, 7, 11].

Figure 1: Ferrers diagrams for the partition $\lambda = (4, 3, 3, 1)$ and the composition $\kappa = (3, 1, 4, 1)$
Let \( c_{k,l}(n) \) be the number of compositions of \( n \) fitting inside a \( k \times l \) rectangle. In this note we will give a simple, algebraic proof of the following conjecture of Vatter [personal communication].

**Theorem 1.2.** Given \( k, l \in \mathbb{P} \) the sequence

\[
c_{k,l}(0), c_{k,l}(1), \ldots, c_{k,l}(kl)
\]

is unimodal.

In the next section, we will prove this result by passing to the generating function of the sequence. The final section will include some comments and an indication about how a combinatorial proof of Vatter’s conjecture might go.

## 2 Unimodality of the composition sequence

Let \( a_0, a_1, \ldots, a_r \) be a sequence of real numbers and let \( q \) be a variable. We consider the corresponding generating function \( f(q) = a_0 + a_1q + \cdots + a_rq^r \). By convention, we let \( a_i = 0 \) if \( i < 0 \) or \( i > r \). We will say that \( f(q) \) has a given property if the sequence itself does.

We will need the standard \( q \)-analogue of \( n \), namely

\[
[n] = 1 + q + q^2 + \cdots + q^{n-1}.
\]

It is well known that the generating function for the sequence \( p_{k,l}(n), 0 \leq n \leq kl \), is the \( q \)-binomial coefficient

\[
\left[ \begin{array}{c}
    k + l \\
    l
  \end{array} \right] = \frac{[k + l]!}{[k]![l]!},
\]

where \([k]! = [k][k-1] \cdots [1]\). So a restatement of Theorem 1.1 is that the \( q \)-binomial coefficients are unimodal.

To prove the analogous result about compositions, we will need a lemma. It is not true, in general, that the product of two unimodal polynomials is unimodal. For example, if \( f(q) = 1 + q + q^2 + 2.3q^3 + 2q^4 \) then

\[
f(q)^2 = 1 + 2q + 3q^2 + 6.6q^3 + 9.6q^4 + 8.6q^5 + 9.29q^6 + 9.2q^7 + 4q^8.
\]

But we do have the following more specialized result.

**Lemma 2.1.** Let \( f(q) \) be a unimodal polynomial and let \( l \in \mathbb{P} \). Then \([l]f(q)\) is also unimodal.
Proof. The lemma is clearly true if \( l = 1 \), so assume \( l \geq 2 \). Suppose that \( f(q) \) is the generating function for the sequence (1). Also let \([l]f(q) = \sum_n b_n q^n\). It follows immediately from the definitions that
\[
b_0 \leq b_1 \leq \ldots \leq b_m \quad \text{and} \quad b_{m+l-1} \geq b_{m+l} \geq \ldots \geq b_{r+l-1}.
\]
So the only way that \([l]f(q)\) could fail to be unimodal is if \( b_{i-1} > b_i < b_{i+1} \) for some \( i, m < i < m+l-1 \). We will show that the case \( i = m+1 \) leads to a contradiction as the other cases are similar.

Suppose \( b_m > b_{m+1} \) and \( b_{m+1} < b_{m+2} \). Expressing each \( b_i \) in terms of the \( a_j \) and then cancelling terms gives \( a_{m-l+1} > a_{m+1} \) and \( a_{m-l+2} < a_{m+2} \). Using these inequalities and (1), we have
\[
a_{m-l+1} > a_{m+1} \geq a_{m+2} > a_{m-l+2}.
\]
But this is a contradiction to (1) since \( l \geq 2 \). \( \square \)

Now let
\[
f^{k,l} = f^{k,l}(q) = \sum_{n \geq 0} c^{k,l}(n) q^n.
\]
Our main result is as follows.

**Theorem 2.2.** Let \( k, l \in \mathbb{P} \).

(a) If \( k \geq 2 \) then
\[
f^{k,l} = 1 + [l]f^{k-1,l}.
\]
(b) The polynomial \( f^{k,l} \) is unimodal.

Proof. (a) Let \( K^{k,l} \) be the set of compositions fitting inside a \( k \times l \) rectangle, and let \( K^{k,l}(r) \subseteq K^{k,l} \) be those compositions with first part equal to \( r \). So we have the disjoint union
\[
K^{k,l} = \{ \epsilon \} \uplus \left( \bigcup_{r=1}^l K^{k,l}(r) \right)
\]
where \( \epsilon \) denotes the empty composition. Removing the first part of any \( \kappa \in K^{k,l} \) leaves a composition in \( K^{k-1,l} \). So translating the union above into a generating function gives the desired result.

(b) We induct on \( k \). Clearly \( f^{1,l} = [l+1] \), so we are done in the base case. If \( k \geq 2 \) then using part (a) and the lemma finishes the proof. \( \square \)
3 Comments and open questions

3.1 Log concavity and symmetry

A sequence \(a_0, a_1, \ldots, a_r\) is log concave if \(a_i^2 \geq a_{i-1}a_{i+1}\) for all \(i\) with \(0 < i < r\). The following easily proved and well-known proposition gives a connection between log-concavity and unimodality.

**Proposition 3.1.** Let \(a_0, a_1, \ldots, a_r\) be a sequence of positive real numbers. If the sequence is log concave, then it is also unimodal.

Sometimes to prove a sequence is unimodal, it is actually easier to prove that it satisfies the stronger log-concavity condition. This is because proving unimodality directly may involve finding the index where the sequence is maximized, and that can be highly nontrivial. However, the sequence \(p^{k,l}(n), 0 \leq n \leq kl\), is not log concave in general. So it should come as no surprise that neither is \(c^{k,l}(n), 0 \leq n \leq kl\), and for much the same reason. In particular, if \(k, l \geq 2\) then both sequences start \(1, 1, 2\) which already violates log concavity.

Another common property of sequences is symmetry. Say that \(a_0, a_1, \ldots, a_r\) is symmetric if \(a_i = a_{r-i}\) for all \(i, 0 \leq i \leq r\). By taking complements in the rectangle, it is easy to see that \(p^{k,l}(n), 0 \leq n \leq kl\), is symmetric. In general, this property is not shared by compositions in a rectangle. For example, if \(k = l = 2\) then the corresponding sequence is \(1, 1, 2, 2, 1\).

3.2 Lower order ideals

Let \((P, \leq)\) be a poset (partially ordered set). Definitions for terms from the theory of posets which are not given here can be found in Stanley’s book [13, Chapter 3]. A lower order ideal is \(L \subseteq P\) such that \(x \in L\) and \(y \leq x\) implies \(y \in L\). The principal lower order ideal generated by \(x\) is the order ideal

\[L(x) = \{y \in P \mid y \leq x\}.
\]

Let \(Y\) and \(K\) denote Young’s lattice and the poset of all compositions, respectively. Then the set of partitions in a rectangle is the order ideal \(Y(l^k)\) and similarly for compositions.

If \(x, y \in P\) then \(x\) is covered by \(y\), written \(x < y\), if \(x < y\) and there is no \(z\) with \(x < z < y\). An \(x-y\) chain of length \(n\) in \(P\) is a subposet of the form \(x = x_0 < x_1 < \ldots < x_n = y\). This chain is saturated if each inequality is actually a cover. A poset is graded if it has a unique minimal element denoted \(\hat{0}\), a unique maximal element denoted \(\hat{1}\), and every saturated \(\hat{0}-\hat{1}\) chain has the same length.
If $P$ is graded and $x \in P$ then all $\hat{0}x$ chains have the same length, called the rank of $x$ and denoted $\text{rk} x$. In this case, the $n$th rank of $P$ is the subposet

$$P_n = \{x \in P \mid \text{rk} x = n\}.$$ 

We will say that a graded poset $P$ has a property if the sequence of cardinalities

$$|P_0|, |P_1|, \ldots, |P_r|$$

has that property, where $r = \text{rk} \hat{1}$. We will sometimes preface the property by “rank-” if clarification is needed. So Theorem 1.1 can be restated as saying that the poset $Y(l^k)$ is unimodal. It is natural to ask whether $Y(\lambda)$ is unimodal for all partitions $\lambda$. But this is too much to ask for, as demonstrated by the following theorem of Stanton [16].

**Theorem 3.2** (Stanton). The lower order ideal $Y(8, 8, 4, 4)$ is not unimodal. □

In view of Stanton’s result, it is perhaps surprising that all principal lower order ideals in the composition poset $K$ are unimodal. Given a graded poset $P$, we let $f^P = f^P(q)$ be the generating polynomial for the sequence (3). The proof of the following theorem is so much like that of Theorem 2.2 that we omit it.

**Theorem 3.3.** Consider a composition $\kappa \in K$.

(a) Suppose $\kappa = (\kappa_1, \ldots, \kappa_s) \neq \epsilon$, letting $l = \kappa_1$ and $\gamma = (k_2, \ldots, k_s)$. Then

$$f^{K(\kappa)} = 1 + q[l]f^{K(\gamma)}.$$ 

(b) The polynomial $f^{K(\kappa)}$ is unimodal. □

### 3.3 A combinatorial proof?

Theorem 2.2 is so easy to prove algebraically, one would think that there is also an easy combinatorial proof. But so far one has not been found. Here we present a possible inductive approach in the hopes that someone else may be able to push it through.

Let $P$ be poset. A chain decomposition of $P$ is a family of saturated chains $C_1, \ldots, C_a$ such that $P = \sqcup_i C_i$. If $P$ is graded then we say an $x$–$y$ chain in $P$ symmetric if $\text{rk} y = \text{rk} \hat{1} - \text{rk} x$. A symmetric chain decomposition or SCD is a chain decomposition where all the chains are symmetric. It is easy to see that if $P$ has an SCD then its rank sequence is symmetric and unimodal.

O’Hara [8] constructed her ground-breaking combinatorial proof of Theorem 1.1 as follows. Let $Z(\lambda)$ be the poset of all partitions in $Y(\lambda)$ ordered by $\mu \leq \nu$ if and only if $|\mu| \leq |\nu|$. So for any partition $\lambda$, $Z(\lambda)$ has the same set of ranks as does $Y(\lambda)$, but many more covering relations in general.
Theorem 3.4 (O’Hara). Given \( k, l \in \mathbb{P} \), the poset \( Z(l^k) \) has an SCD.

We note that it is still an open problem to give an SCD for \( Y(l^k) \).

As mentioned above, \( K(l^k) \) is not always rank-symmetric. But we can replace symmetry by another condition. If \( P \) is graded then we say that a chain decomposition is modal (an MCD) if there is some rank \( P_m \) such that every \( C_i \) contains an element of \( P_m \). We call \( P_m \) the modal rank. The proof of the following proposition is similar to the symmetric case, but we include it for completeness.

Proposition 3.5. Let \( P \) be a graded poset. If \( P \) has an MCD then \( P \) is rank-unimodal.

Proof. Let \( C_1, \ldots, C_a \) be an MCD and let \( P_m \) be its modal rank. We will show that \( |P_i| \leq |P_{i+1}| \) for \( i < m \) as the inequalities \( |P_i| \geq |P_{i+1}| \) for \( i \geq m \) are similar. Let \( P_i = \{x_1, \ldots, x_s\} \) and, since we have a cover, we can assume that \( C_j \) contains \( x_j \) for \( 1 \leq j \leq s \). But each \( C_j \) is saturated and goes through rank \( P_m \). So for \( 1 \leq j \leq s \), \( C_j \) must contain an element \( y_j \) in rank \( P_{i+1} \). By disjointness, the \( y_j \) are distinct and thus \( |P_i| \leq |P_{i+1}| \) as desired.

We now ask the obvious questions.

Question 3.6. Does \( K(l^k) \) have an MCD for all \( k, l \in \mathbb{P} \)? More generally, does \( K(\kappa) \) have an MCD for all compositions \( \kappa \)?

Note that the modal rank \( P_m \) for \( P = K(l^k) \) seems to occur when

\[ m = \lceil k(l + 1)/2 \rceil - 1 \]

where \( \lceil \cdot \rceil \) is the ceiling function. Also note that there are other partial orders on the set of compositions \( \{2, 7, 11\} \) and they have the same set of ranks as \( K(l^k) \) (but not for a general \( K(\kappa) \)). Of these, the partial order we are considering has the fewest covers. So in may be useful to consider one of the other orders instead.

It might be hoped that one could come up with an inductive description of an MCD for \( K(l^k) \) analogous to the inductive proof given of Theorem 2.2. One possible way to do this is as follows. For simplicity, we will restrict ourselves to the case \( l = 2 \). Suppose we have an MCD \( C_1, \ldots, C_a \) for \( K(2^{k-1}) \). Then using \( (2) \) we can obtain a chain decomposition

\[ K(2^k) = \{\epsilon\} \uplus \{C'_1, \ldots, C'_a\} \uplus \{C''_1, \ldots, C''_a\} \]

where \( C'_i \) is gotten by prefixing every element of \( C_i \) by a one, and \( C''_i \) is obtained similarly using a two prefix.

Of course, this may be too many chains as not all of them will go through rank \( \lceil 3k/2 \rceil - 1 \). In particular, some of the \( C'_i \) may be too “low” and some of the
$C''_i$ too “high.” (Also, $\epsilon$ must be tacked onto some chain, but just use whichever $C'_i$ contains the composition (1).) To rectify this, note that if $\kappa'$ and $\kappa''$ are the top elements of $C'_i$ and $C''_i$, respectively, then by construction $\kappa' \preceq \kappa''$. So we can replace the pair $C'_i, C''_i$ by the pair $D'_i, D''_i$ where

$$D'_i = C'_i \uplus \{ \kappa'' \} \quad \text{and} \quad D''_i = C''_i - \{ \kappa'' \}. $$

Note that this may result in $D''_i = \emptyset$ in which case we throw away the chain. Unfortunately, even with this correction the construction breaks down when $k = 9$ as some of the chains do not go through the largest rank. So some other modification will be needed to obtain an MCD.

Acknowledgement. I would like to thank Adam Goyt and Vince Vatrer for interesting discussions.

References

[1] Andrews, G. E. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] Björner, A., and Sagan, B. E. Rationality of the Möbius function of a composition poset. *Theoret. Comput. Sci.* 359, 1-3 (2006), 282–298.

[3] Brenti, F. Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. In *Jerusalem combinatorics '93*, vol. 178 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1994, pp. 71–89.

[4] Ehrenborg, R. On posets and Hopf algebras. *Adv. Math.* 119, 1 (1996), 1–25.

[5] Gessel, I. M. Multipartite $P$-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, vol. 34 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1984, pp. 289–317.

[6] Heubach, S., and Mansour, T. Avoiding patterns of length three in compositions and multiset permutations. *Adv. in Appl. Math.* 36, 2 (2006), 156–174.

[7] Kitaev, S., Liese, J., Remmel, J., and Sagan, B. E. Rationality and irrationality in factor order on compositions. in preparation.

[8] O’Hara, K. M. Unimodality of Gaussian coefficients: a constructive proof. *J. Combin. Theory Ser. A* 53, 1 (1990), 29–52.

[9] Proctor, R. A. Solution of two difficult combinatorial problems with linear algebra. *Amer. Math. Monthly* 89, 10 (1982), 721–734.
[10] Sagan, B. E. *The symmetric group: Representations, combinatorial algorithms, and symmetric functions*, second ed., vol. 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[11] Sagan, B. E., and Vatter, V. The Möbius function of a composition poset. *J. Algebraic Combin.* 24, 2 (2006), 117–136.

[12] Savage, C. D., and Wilf, H. S. Pattern avoidance in compositions and multiset permutations. *Adv. in Appl. Math.* 36, 2 (2006), 194–201.

[13] Stanley, R. P. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods* 1, 2 (1980), 168–184.

[14] Stanley, R. P. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph theory and its applications: East and West (Jinan, 1986)*, vol. 576 of *Ann. New York Acad. Sci.* New York Acad. Sci., New York, 1989, pp. 500–535.

[15] Stanley, R. P. *Enumerative Combinatorics. Vol. 2*, vol. 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[16] Stanton, D. Unimodality and Young’s lattice. *J. Combin. Theory Ser. A* 54, 1 (1990), 41–53.

[17] Sylvester, J. J. Proof of the hitherto undemonstrated fundamental theorem of invariants. In *The collected mathematical papers of James Joseph Sylvester*, vol. 3. Cambridge University Press, Chelsea, NY, 1973, pp. 117–126.