\[ L^p - L^q \text{ estimates on the solutions to} \]
\[ u_{tt} - u_{x_1 x_1} = \Delta u_t \]

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\textbf{Abstract.} This paper focuses the study on the \(L^p - L^q\) estimates on the solutions to an asymmetric wave equation with dissipation which arises in the study for the magneto-hydrodynamics by using the method of Green function.

Keywords: \(L^p - L^q\) estimates; asymmetric wave equation; dissipation

\textit{MSC(2000):} 35L05; 35L15

\section{Introduction}

We consider the following Cauchy problem of the asymmetric dissipative wave equation,

\[
\begin{cases}
    u_{tt} - u_{x_1 x_1} = \Delta u_t, \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \quad t > 0, \\
    u(x, t)|_{t=0} = u_0(x), \\
    u_t(x, t)|_{t=0} = u_1(x),
\end{cases}
\]

(1.1)

where \(\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_n}, \quad n \geq 3\), and \(u_0, u_1\) are given functions.

Equation (1.1) is an asymmetric wave equation with dissipation and arises in the study for the magneto-hydrodynamics (see [2]). Due to the asymmetric structure of (1.1), the energy method fails in making the \(L^p - L^q\) estimates, and new difficulties arise in the study for this equation in contrast to that for the symmetric semilinear and nonlinear wave equation (see [1] \& [4-6]).

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In this paper, we obtain the $L^p - L^q$ estimates on the solutions to (1.1) by using the method of Green function combined with the technique of Fourier analysis.

Notations. We denote generic constants by $C$. $L^p (1 \leq p \leq \infty)$ is the usual Lebesgue space with the norm $\| \cdot \|_{L^p}$, $W^{m, p}$, $m \in \mathbb{Z}^+$, $p \in [1, \infty]$ denotes the usual Sobolev space with its norm

$$\| f \|_{W^{m, p}} := \left( \sum_{k=0}^{m} |\partial_x^k f|^p \right)^{\frac{1}{p}}.$$

2 Green function

The corresponding Green function to (1.1) satisfies the following equation,

$$\begin{align*}
G_{tt} - G_{x_1 x_1} &= \Delta G_t, \quad x \in \mathbb{R}^n, \ t > 0, \\
G(x, t)|_{t=0} &= 0, \\
G_t(x, t)|_{t=0} &= \delta(x).
\end{align*}$$

(2.2)

By Fourier transformation we get that

$$\begin{align*}
\hat{G}_{tt} + |\xi|^2 \hat{G}_t + \xi_1^2 \hat{G} &= 0, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, \ t > 0, \\
\hat{G}(\xi, t)|_{t=0} &= 0, \\
\hat{G}_t(\xi, t)|_{t=0} &= 1.
\end{align*}$$

(2.3)

It yields that

$$\hat{G}(\xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_0},$$

where $\lambda_{\pm} = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4 \xi_1^2}}{2}$, $\lambda_0 = \sqrt{|\xi|^4 - 4 \xi_1^2}$.

By Duhamel’s principle we know that the solution to (1.1) can be expressed as following,

$$u(x, t) = (\partial_t - \Delta) G \ast u_0(x, t) + G \ast u_1(x, t).$$

(2.4)

Denote $A = \{ \xi \in \mathbb{R}^n; |\xi|^2 \leq 2 |\xi_1| \}$, $B = \{ \xi \in \mathbb{R}^n; |\xi| < 1 \}$, $E = \{ \xi \in \mathbb{R}^n; |\xi| \leq |\xi'| \}$, $D_r = \{ \xi \in \mathbb{R}^n; |\xi|^2 \leq r |\xi_1| \}$, $r > 2$. Define $A^c$ the complete set of $A$, and $B^c, D_r^c, E^c$ can be defined similarly.

By direct calculation and induction we have the following lemma.
Lemma 2.1
\[
\partial_t (\partial_t + |\xi|^2) \hat{G}(\xi, t) = -\xi_t^2 \hat{G}(\xi, t), \\
\partial_t^2 (\partial_t + |\xi|^2) \hat{G}(\xi, t) = -\xi_t^2 \hat{G}_t(\xi, t).
\]
\(\forall h \geq 1, \text{if } \xi \in B, \text{then}\)
\[
\partial_t^{2h} \hat{G}(\xi, t) = O(|\xi|^{2h}) \hat{G}(\xi, t) + O(|\xi|^{2h}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+1} \hat{G}(\xi, t) = O(|\xi|^{2h+2}) \hat{G}(\xi, t) + O(|\xi|^{2h}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+1} (\partial_t + |\xi|^2) \hat{G}(\xi, t) = O(|\xi|^{2h+2}) \hat{G}(\xi, t) + O(|\xi|^{2h+2}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+2} (\partial_t + |\xi|^2) \hat{G}(\xi, t) = O(|\xi|^{2h+4}) \hat{G}(\xi, t) + O(|\xi|^{2h+2}) \hat{G}_t(\xi, t).
\]
\(\text{If } \xi \in B^c, \text{then}\)
\[
\partial_t^{2h} \hat{G}(\xi, t) = O(|\xi|^{4h-2}) \hat{G}(\xi, t) + O(|\xi|^{4h-2}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+1} \hat{G}(\xi, t) = O(|\xi|^{4h}) \hat{G}(\xi, t) + O(|\xi|^{4h}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+1} (\partial_t + |\xi|^2) \hat{G}(\xi, t) = O(|\xi|^{4h}) \hat{G}(\xi, t) + O(|\xi|^{4h}) \hat{G}_t(\xi, t), \\
\partial_t^{2h+2} (\partial_t + |\xi|^2) \hat{G}(\xi, t) = O(|\xi|^{4h+2}) \hat{G}(\xi, t) + O(|\xi|^{4h+2}) \hat{G}_t(\xi, t).
\]
By direct calculation we have the following estimates on \(\hat{G}(\xi, t)\) and \(\hat{G}_t(\xi, t)\).

Lemma 2.2 \(\text{If } \xi \in D_r, \ r > 2, \text{then}\)
\[
|\hat{G}(\xi, t)| \leq te^{-\frac{m}{2}|\xi|^2 |t|}, \ |\hat{G}_t(\xi, t)| \leq (1 + |\xi|^2) e^{-\frac{m}{2}|\xi|^2 |t|},
\]
here, \(m = 1 - \sqrt{1 - \frac{4}{r}}.\)

Proof. If \(\xi \in A, \text{then } |\xi|^4 \leq 4\xi^2_1. \text{In this case,}\)
\[
\hat{G}(\xi, t) = e^{-\frac{|\xi|^2}{2} t} \cdot \frac{e^{\sqrt{|\xi|^4 - 4\xi^2_1} t} - e^{-\sqrt{|\xi|^4 - 4\xi^2_1} t}}{\sqrt{|\xi|^4 - 4\xi^2_1}} = te^{-\frac{|\xi|^2}{2} t} e^{\theta(\xi) \sqrt{|\xi|^4 - 4\xi^2_1} t},
\]
for some \(\theta(\xi) \in (-1, 1). \text{It yields that } |\hat{G}(\xi, t)| \leq te^{-\frac{|\xi|^2}{2} t}.\)

Similarly, we have that
\[
|\hat{G}_t(\xi, t)| = \left| \frac{e^{\lambda_1 t} + e^{\lambda_2 t}}{2} - |\xi|^2 \hat{G}(\xi, t) \right| \leq (1 + |\xi|^2) e^{-\frac{|\xi|^2}{2} t}.
\]
If \(\xi \in A^c \cap D_r, \text{then } 4\xi^2_1 < |\xi|^4 \leq r^2 \xi^2_1. \text{In this case,}\)
\[
\hat{G}(\xi, t) = te^{-\frac{|\xi|^2}{2} t} e^{\theta(\xi) \sqrt{|\xi|^4 - 4\xi^2_1} t} \leq te^{-\frac{1}{2} \sqrt{1 - \frac{4}{r}} |\xi|^2 t}.
\]
Similarly, we have that
\[ |\hat{G}_t(\xi, t)| \leq (1 + |\xi|^2 t) e^{-\frac{1}{2} \chi(\xi^2 t)}. \]

So the lemma is proved. □

Denote \( \hat{G}_1(\xi, t) = \chi(\xi)\hat{G}(\xi, t) \), where \( \chi \in C^\infty_0(\mathbb{R}^n) \), supp\( \chi \subset D_{r+1} \) and \( \chi|_{D_r} = 1 \). Then we have the following estimates on \( \hat{G}_1(\xi, t) \).

**Lemma 2.3** For any multi-index \( \alpha \) and non-negative integer \( l \), we have that,

1. \( \| (\cdot)^\alpha \partial^l \hat{G}_1(\cdot, t) \|_{L^\infty} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{4}}, \)
2. \( \| (\cdot)^\alpha \partial^l (\partial_t + |\cdot|^2) \hat{G}_1(\cdot, t) \|_{L^\infty} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{2}}, \)
3. \( \| (\cdot)^\alpha \partial^l \hat{G}_1(\cdot, t) \|_{L^2} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{4}}, \)
4. \( \| (\cdot)^\alpha \partial^l (\partial_t + |\cdot|^2) \hat{G}_1(\cdot, t) \|_{L^2} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{2}}, \)
5. \( \| (\cdot)^\alpha \partial^l \hat{G}_1(\cdot, t) \|_{L^1} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{4}}, \)
6. \( \| (\cdot)^\alpha \partial^l (\partial_t + |\cdot|^2) \hat{G}_1(\cdot, t) \|_{L^1} \leq Ct^{-\frac{|\alpha|}{2} - \frac{|l|}{2}}, \)

where \([\cdot]\) is Gauss' symbol.

**Proof.** By using lemma 2.1 and lemma 2.2 we give the proof.

(1) As \( l = 0 \),
\[ |\xi^\alpha \chi(\xi) \hat{G}(\xi, t)| \leq C|\xi|^{\alpha}te^{-\frac{\alpha}{2} |\xi|^2 t} \leq Ct^{-\frac{|\alpha|}{2} + 1}. \]

As \( l = 1 \),
\[ |\xi^\alpha \chi(\xi) \partial_t \hat{G}(\xi, t)| \leq C|\xi|^{\alpha}(1 + |\xi|^2 t) e^{-\frac{\alpha}{2} |\xi|^2 t} \leq Ct^{-\frac{|\alpha|}{2}}. \]

For \( h \geq 1 \), as \( l = 2h \),
\[ |\xi^\alpha \chi(\xi) \partial_t^{2h} \hat{G}(\xi, t)|_B = |\xi^\alpha \chi(\xi) \{ O(|\xi|^{2h}) \hat{G}(\xi, t) + O(|\xi|^{2h}) \hat{G}_t(\xi, t) \}| \]
\[ \leq C|\xi|^{\alpha+2h}(t + 1 + |\xi|^2 t) e^{-\frac{\alpha}{2} |\xi|^2 t} \leq Ct^{-\frac{|\alpha|}{2} - h + 1}, \]
\[ |\xi^\alpha \chi(\xi) \partial_t^{2h} \hat{G}(\xi, t)|_{BC} = |\xi^\alpha \chi(\xi) \{ O(|\xi|^{4h-2}) \hat{G}(\xi, t) + O(|\xi|^{4h-2}) \hat{G}_t(\xi, t) \}| \]
\[ \leq Ct^{-\frac{|\alpha|}{2} - 2h + 2} \leq Ct^{-\frac{|\alpha|}{2} - h + 1}; \]
as \( l = 2h + 1 \),
\[
|\xi^\alpha \chi(\xi) \partial_t^{2h+1} \hat{G}(\xi, t)|_B = |\xi^\alpha \chi(\xi) \{O(\|\xi\|^{2h+2}) \hat{G}(\xi, t) + O(\|\xi\|^{2h}) \hat{G}_t(\xi, t)\}|
\leq C t^{-\frac{|\alpha|}{2} - h},
\]
\[
|\xi^\alpha \chi(\xi) \partial_t^{2h+1} \hat{G}(\xi, t)|_{B^c} = |\xi^\alpha \chi(\xi) \{O(\|\xi\|^{4h}) \hat{G}(\xi, t) + O(\|\xi\|^{4h}) \hat{G}_t(\xi, t)\}|
\leq C t^{-\frac{|\alpha|}{2} - 2h + 1} \leq C t^{-\frac{|\alpha|}{2} - h}.
\]
Thus (1) is proved.

(2). As \( l = 0 \),

\[
|\xi^\alpha \chi(\xi) (\partial_t + |\xi|^2) \hat{G}(\xi, t)| \leq C|\xi|^{|\alpha|}(1 + 2|\xi|^2) e^{-\frac{|\alpha|^2}{4}|\xi|^2 t} \leq C t^{-\frac{|\alpha|}{2}}.
\]

As \( l = 1 \),

\[
|\xi^\alpha \chi(\xi) \partial_t (\partial_t + |\xi|^2) \hat{G}(\xi, t)| = |\xi^\alpha \chi(\xi) \xi^2 \hat{G}_t(\xi, t)| \leq C t^{-\frac{|\alpha|}{2}}.
\]

As \( l = 2 \),

\[
|\xi^\alpha \chi(\xi) \partial_t^2 (\partial_t + |\xi|^2) \hat{G}(\xi, t)| = |\xi^\alpha \chi(\xi) \xi^2 \hat{G}_t(\xi, t)| \leq C t^{-\frac{|\alpha|}{2} - 1}.
\]

For \( h \geq 1 \), as \( l = 2h + 1 \),
\[
|\xi^\alpha \chi(\xi) \partial_t^{2h+1} (\partial_t + |\xi|^2) \hat{G}(\xi, t)|_B
= |\xi^\alpha \chi(\xi) \{O(\|\xi\|^{2h+2}) \hat{G}(\xi, t) + O(\|\xi\|^{2h+2}) \hat{G}_t(\xi, t)\}|
\leq C t^{-\frac{|\alpha|}{2} - h},
\]
\[
|\xi^\alpha \chi(\xi) \partial_t^{2h+1} (\partial_t + |\xi|^2) \hat{G}(\xi, t)|_{B^c}
= |\xi^\alpha \chi(\xi) \{O(\|\xi\|^{4h}) \hat{G}(\xi, t) + O(\|\xi\|^{4h}) \hat{G}_t(\xi, t)\}|
\leq C t^{-\frac{|\alpha|}{2} - 2h + 1} \leq C t^{-\frac{|\alpha|}{2} - h};
\]

as \( l = 2h + 2 \),
\[
|\xi^\alpha \chi(\xi) \partial_t^{2h+2} (\partial_t + |\xi|^2) \hat{G}(\xi, t)|_B
= |\xi^\alpha \chi(\xi) \{O(\|\xi\|^{2h+4}) \hat{G}(\xi, t) + O(\|\xi\|^{2h+2}) \hat{G}_t(\xi, t)\}|
\leq C t^{-\frac{|\alpha|}{2} - h - 1},
\]

\[5\]
\[ |\xi^\alpha \chi(\xi) \partial_t^{2h+2} (\partial_t + |\xi|^2) \hat{G}(\xi, t)|_{B^c} \]

\[ = |\xi^\alpha \chi(\xi) \{ O(|\xi|^{4h+2}) \hat{G}(\xi, t) + O(|\xi|^{4h}) \hat{G}_t(\xi, t) \}| \]

\[ \leq Ct^{-|\alpha| - 2h} \leq Ct^{-|\alpha|-h-1}. \]

Thus (2) is proved.

(3). As \( l = 0 \),

\[ \| (\cdot)^{\alpha} \hat{G}_1(\cdot, t) \|_{L^2} = (\int_{\mathbb{R}^n} |\xi^{\alpha} \chi(\xi) \hat{G}(\xi, t)|^2 d\xi)^{\frac{1}{2}} \]

\[ \leq C (\int_{\mathbb{R}^n} |\xi|^{2|\alpha| t^2 e^{-m|\xi|^2 t}} d\xi)^{\frac{1}{2}} \]

\[ \leq Ct^{-|\alpha|+1-\frac{n}{4}}. \]

As \( l = 1 \),

\[ \| (\cdot)^{\alpha} \partial_1 \hat{G}_1(\cdot, t) \|_{L^2} = (\int_{\mathbb{R}^n} |\xi^{\alpha} \chi(\xi) \partial_1 \hat{G}(\xi, t)|^2 d\xi)^{\frac{1}{2}} \]

\[ \leq C (\int_{\mathbb{R}^n} |\xi|^{2|\alpha| t^2 e^{-m|\xi|^2 t}} |\xi|^{2|\alpha| t^2 e^{-m|\xi|^2 t}} d\xi)^{\frac{1}{2}} \]

\[ \leq Ct^{-1-\frac{n}{4}}. \]

For \( h \geq 1 \), as \( l = 2h \),

\[ \| (\cdot)^{\alpha} \partial_t^{2h} \hat{G}_1(\cdot, t) \|_{L^2} \]

\[ = (\int_B |\xi^{\alpha} \chi(\xi) \{ O(|\xi|^{2h}) \hat{G}(\xi, t) + O(|\xi|^{2h}) \hat{G}_t(\xi, t) \}|^2 d\xi \]

\[ + \int_{B^c} |\xi^{\alpha} \chi(\xi) \{ O(|\xi|^{4h-2}) \hat{G}(\xi, t) + O(|\xi|^{4h-2}) \hat{G}_t(\xi, t) \}|^2 d\xi)^{\frac{1}{2}} \]

\[ \leq Ct^{-|\alpha|-h+1-\frac{n}{4}}; \]

as \( l = 2h + 1 \),

\[ \| (\cdot)^{\alpha} \partial_t^{2h+1} \hat{G}_1(\cdot, t) \|_{L^2} \]

\[ = (\int_B |\xi^{\alpha} \chi(\xi) \{ O(|\xi|^{2h+2}) \hat{G}(\xi, t) + O(|\xi|^{2h}) \hat{G}_t(\xi, t) \}|^2 d\xi \]

\[ + \int_{B^c} |\xi^{\alpha} \chi(\xi) \{ O(|\xi|^{4h}) \hat{G}(\xi, t) + O(|\xi|^{4h}) \hat{G}_t(\xi, t) \}|^2 d\xi)^{\frac{1}{2}} \]

\[ \leq Ct^{-|\alpha|-h-\frac{n}{4}}. \]
Thus (3) is proved.

(4). As \( l = 0 \),
\[
\| (\cdot)^{\alpha} (\partial_t + \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^2} = \left( \int_{\mathbb{R}^n} |\xi^\alpha \chi (\xi) (\partial_t + |\xi|^2) \hat{G} (\xi, t) |^2 d\xi \right)^{\frac{1}{2}} \\
\leq C t^{-\frac{|\alpha|}{2} - \frac{n}{4}}.
\]
As \( l = 1 \),
\[
\| (\cdot)^{\alpha} \partial_\xi (\partial_t + \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^2} = \left( \int_{\mathbb{R}^n} |\xi^\alpha \chi (\xi) \partial_\xi (\partial_t + |\xi|^2) \hat{G} (\xi, t) |^2 d\xi \right)^{\frac{1}{2}} \\
= \left( \int_{\mathbb{R}^n} |\xi^\alpha \chi (\xi) \xi^1 \hat{G} (\xi, t) |^2 d\xi \right)^{\frac{1}{2}} \\
\leq C t^{-\frac{|\alpha|}{2} - \frac{n}{4}}.
\]
As \( l = 2 \),
\[
\| (\cdot)^{\alpha} \partial^2_\xi (\partial_t + \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^2} = \left( \int_{\mathbb{R}^n} |\xi^\alpha \chi (\xi) \partial^2_\xi (\partial_t + |\xi|^2) \hat{G} (\xi, t) |^2 d\xi \right)^{\frac{1}{2}} \\
= \left( \int_{\mathbb{R}^n} |\xi^\alpha \chi (\xi) \xi^2 \hat{G}_t (\xi, t) |^2 d\xi \right)^{\frac{1}{2}} \\
\leq C t^{-\frac{|\alpha|}{2} - \frac{n}{4}}.
\]
For \( h \geq 1 \), as \( l = 2h + 1 \),
\[
\| (\cdot)^{\alpha} \partial^{2h+1}_t (\partial_t + \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^2} \\
= \left( \int_{B} |\xi^\alpha \chi (\xi) \{ O(|\xi|^{2h+2}) \hat{G} (\xi, t) + O(|\xi|^{2h+2}) \hat{G}_t (\xi, t) \} |^2 d\xi \\
+ \int_{B^c} |\xi^\alpha \chi (\xi) \{ O(|\xi|^{4h}) \hat{G} (\xi, t) + O(|\xi|^{4h}) \hat{G}_t (\xi, t) \} |^2 d\xi \right)^{\frac{1}{2}} \\
\leq C t^{-\frac{|\alpha|}{2} - h - \frac{n}{4}};
\]
as \( l = 2h + 2 \),
\[
\| (\cdot)^{\alpha} \partial^{2h+2}_t (\partial_t + \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^2} \\
= \left( \int_{B} |\xi^\alpha \chi (\xi) \{ O(|\xi|^{2h+4}) \hat{G} (\xi, t) + O(|\xi|^{2h+4}) \hat{G}_t (\xi, t) \} |^2 d\xi \\
+ \int_{B^c} |\xi^\alpha \chi (\xi) \{ O(|\xi|^{4h+2}) \hat{G} (\xi, t) + O(|\xi|^{4h+2}) \hat{G}_t (\xi, t) \} |^2 d\xi \right)^{\frac{1}{2}} \\
\leq C t^{-\frac{|\alpha|}{2} - h - \frac{n}{4}}.
\]
Thus (4) is proved.

(5). As $l = 0$,
\[
\| (\cdot)^{\alpha} \hat{G}_1 (\cdot, t) \|_{L^1} = \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \hat{G}(\xi, t)| \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} + 1 - \frac{n}{2}}.
\]

As $l = 1$,
\[
\| (\cdot)^{\alpha} \partial_\ell \hat{G}_1 (\cdot, t) \|_{L^1} = \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \partial_\ell \hat{G}(\xi, t)| \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} - \frac{n}{2}}.
\]

For $h \geq 1$, as $l = 2h$,
\[
\| (\cdot)^{\alpha} \partial_t^{2h} \hat{G}_1 (\cdot, t) \|_{L^1} \\
= \int_B |\xi^\alpha \chi(\xi) \{ O(\|\xi\|^{2h}) \hat{G}(\xi, t) + O(\|\xi\|^{2h}) \hat{G}_t(\xi, t) \} | \, d\xi \\
+ \int_{B^c} |\xi^\alpha \chi(\xi) \{ O(\|\xi\|^{4h-2}) \hat{G}(\xi, t) + O(\|\xi\|^{4h-2}) \hat{G}_t(\xi, t) \} | \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} - h + 1 - \frac{n}{2}};
\]
as $l = 2h + 1$,
\[
\| (\cdot)^{\alpha} \partial_t^{2h+1} \hat{G}_1 (\cdot, t) \|_{L^1} \\
= \int_B |\xi^\alpha \chi(\xi) \{ O(\|\xi\|^{2h+2}) \hat{G}(\xi, t) + O(\|\xi\|^{2h}) \hat{G}_t(\xi, t) \} | \, d\xi \\
+ \int_{B^c} |\xi^\alpha \chi(\xi) \{ O(\|\xi\|^{4h}) \hat{G}(\xi, t) + O(\|\xi\|^{4h}) \hat{G}_t(\xi, t) \} | \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} - h - \frac{n}{2}}.
\]

Thus (5) is proved.

(6). As $l = 0$,
\[
\| (\cdot)^{\alpha} (\partial_t + | \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^1} = \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) (\partial_t + |\xi|^2) \hat{G}(\xi, t)| \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} - \frac{n}{2}}.
\]

As $l = 1$,
\[
\| (\cdot)^{\alpha} \partial_\ell (\partial_t + | \cdot |^2) \hat{G}_1 (\cdot, t) \|_{L^1} = \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \partial_\ell (\partial_t + |\xi|^2) \hat{G}(\xi, t)| \, d\xi \\
= \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \xi_\ell^2 \hat{G}(\xi, t)| \, d\xi \\
\leq C t^{-\frac{\| \alpha \|}{2} - \frac{n}{2}}.
\]
As $l = 2$,
\[
\| \partial_t^l (\partial_t + | \cdot |^2) \hat{G}_1(\cdot, t) \|_{L^1} = \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \partial_t^l (\partial_t + | \xi |^2) \hat{G}(\xi, t)| d\xi \\
= \int_{\mathbb{R}^n} |\xi^\alpha \chi(\xi) \xi^2 \hat{G}_t(\xi, t)| d\xi \\
\leq C t^{-|\alpha|/2 - 1/2}.
\]

For $h \geq 1$, as $l = 2h + 1$,
\[
\| \partial_t^{2h+1} (\partial_t + | \cdot |^2) \hat{G}_1(\cdot, t) \|_{L^1} = \int_{\mathbb{B}} |\xi^\alpha \chi(\xi) \{ O(|\xi|^{2h+2}) \hat{G}(\xi, t) + O(|\xi|^{2h+2}) \hat{G}_t(\xi, t) \} | d\xi \\
+ \int_{\mathbb{B}^c} |\xi^\alpha \chi(\xi) \{ O(|\xi|^{4h}) \hat{G}(\xi, t) + O(|\xi|^{4h}) \hat{G}_t(\xi, t) \} | d\xi \\
\leq C t^{-|\alpha|/2 - h - \frac{n}{2}};
\]

as $l = 2h + 2$,
\[
\| \partial_t^{2h+2} (\partial_t + | \cdot |^2) \hat{G}_1(\cdot, t) \|_{L^1} = \int_{\mathbb{B}} |\xi^\alpha \chi(\xi) \{ O(|\xi|^{2h+4}) \hat{G}(\xi, t) + O(|\xi|^{2h+4}) \hat{G}_t(\xi, t) \} | d\xi \\
+ \int_{\mathbb{B}^c} |\xi^\alpha \chi(\xi) \{ O(|\xi|^{4h+2}) \hat{G}(\xi, t) + O(|\xi|^{4h+2}) \hat{G}_t(\xi, t) \} | d\xi \\
\leq C t^{-|\alpha|/2 - h - 1 - \frac{n}{2}}.
\]

Thus (6) is proved. \(\square\)

\section{3 L^p - L^q estimates}

The first result is the following theorem.

\textbf{Theorem 3.1} Assume that $u_0, u_1 \in L^p(\mathbb{R}^n)$, $n \geq 3$, and there exists a constant $r > 2$, such that $\text{supp} \hat{u}_0, \text{supp} \hat{u}_1 \subset D_r$, then for any multi-index $\alpha$ and non-negative integer $l$, we have that,

(1). If $p \in [1, 2)$, it holds that, $\forall q \in [\frac{2p}{2-p}, +\infty]$,
\[
\| \partial_t^l \partial_t u(\cdot, t) \|_{L^q} \leq C t^{-\frac{|\alpha|}{2} - \frac{|l|}{2} - \frac{1}{2} \{ -\frac{n}{4} - \frac{1}{4} \}} u_0 \|_{L^p} + C t^{-\frac{|\alpha|}{2} - \frac{|l|}{2} - \frac{1}{2} \{ -\frac{n}{4} - \frac{1}{4} \}} u_1 \|_{L^p};
\]
Thus (1) is proved.

Young inequality and interpolation formula, we have that,

\[ \| \partial_t^\alpha \partial_x^\beta u(\cdot, t) \|_{L^q} \leq C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_0 \|_{L^2} + C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_1 \|_{L^2}. \]

where \( \| \cdot \|_{L^p} = \| \cdot \|_{L^p(\mathbb{R}^n)} \).

**Proof.** Since \( \text{supp} \hat{u}_0, \text{supp} \hat{u}_1 \subset D_r \), from (2.4) we have that

\[ \hat{u}(\xi, t) = (\partial_t + |\xi|^2) \hat{G}(\xi, t) \hat{u}_0(\xi) + \hat{G}(\xi, t) \hat{u}_1(\xi) \]

\[ = (\partial_t + |\xi|^2) \hat{G}_1(\xi, t) \hat{u}_0(\xi) + \hat{G}_1(\xi, t) \hat{u}_1(\xi), \]

it yields that

\[ u(x, t) = (\partial_t - \Delta) G_1 * u_0(x, t) + G_1 * u_1(x, t). \]

(1). Denote \( 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{\rho} \). If \( q \geq \frac{2p}{2-p} \), then \( \rho \geq 2 \). In view of lemma 2.3

Young inequality and interpolation formula, we have that,

\[ \| \partial_t^\alpha \partial_x^\beta u(\cdot, t) \|_{L^q} \]

\[ \leq \| \partial_x^\beta (\partial_t - \Delta) G_1 * u_0(\cdot, t) \|_{L^q} + \| \partial_x^\beta (\partial_t - \Delta) G_1 * u_1(\cdot, t) \|_{L^q} \]

\[ \leq \| \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \| u_0 \|_{L^p} + \| \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \| u_1 \|_{L^p} \]

\[ \leq \| \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \| \partial_t^\alpha \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \]

\[ + \| \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \| \partial_t^\alpha \partial_x^\beta (\partial_t - \Delta) G_1(t) \|_{L^p} \]

\[ \leq C \| (\cdot)^\alpha \partial_t^\beta (\partial_t + |\cdot|^2) \hat{G}_1(t) \|_{L^p} \| (\cdot)^\alpha \partial_t^\beta (\partial_t + |\cdot|^2) \hat{G}_1(t) \|_{L^p} \]

\[ + C \| (\cdot)^\alpha \partial_t^\beta (\partial_t + |\cdot|^2) \hat{G}_1(t) \|_{L^p} \| (\cdot)^\alpha \partial_t^\beta (\partial_t + |\cdot|^2) \hat{G}_1(t) \|_{L^p} \]

\[ \leq C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_0 \|_{L^p} + C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_1 \|_{L^p} \]

\[ = C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_0 \|_{L^p} + C t^{-|\alpha| \frac{1}{2} - \frac{|\beta|}{2} - \frac{\alpha - 1}{4}} \| u_1 \|_{L^p}. \]

Thus (1) is proved.
(2). By using lemma 2.3 and Young inequality, we have that
\[
\|\partial_x^\alpha \partial_t^{\nu} u(\cdot, t)\|_{L^\infty}
\leq \|\partial_x^\alpha \partial_t^{\nu}(\partial_t - \Delta) G_1 * u_0(\cdot, t)\|_{L^\infty} + \|\partial_x^\alpha \partial_t^{\nu} G_1 * u_1(\cdot, t)\|_{L^\infty}
\leq \|\partial_x^\alpha \partial_t^{\nu}(\partial_t - \Delta) G_1(\cdot, t)\|_{L^2} \|u_0\|_{L^2} + \|\partial_x^\alpha \partial_t^{\nu} G_1(\cdot, t)\|_{L^2} \|u_1\|_{L^2}
= \|\partial_x^\alpha \partial_t^{\nu}(\partial_t + |\cdot|^{2}) \hat{G}_1(\cdot, t)\|_{L^2} \|\hat{u}_0\|_{L^2} + \|\partial_x^\alpha \partial_t^{\nu} \hat{G}_1(\cdot, t)\|_{L^2} \|\hat{u}_1\|_{L^2}
\leq C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_0\|_{L^2} + C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_1\|_{L^2}.
\]
\[
\|\partial_x^\alpha \partial_t^{\nu} u(\cdot, t)\|_{L^2}
\leq \|\partial_x^\alpha \partial_t^{\nu}(\partial_t - \Delta) G_1 * u_0(\cdot, t)\|_{L^2} + \|\partial_x^\alpha \partial_t^{\nu} G_1 * u_1(\cdot, t)\|_{L^2}
= \|\partial_x^\alpha \partial_t^{\nu}(\partial_t + |\cdot|^{2}) \hat{G}_1(\cdot, t)\|_{L^2} \|\hat{u}_0\|_{L^2} + \|\partial_x^\alpha \partial_t^{\nu} \hat{G}_1(\cdot, t)\|_{L^2} \|\hat{u}_1\|_{L^2}
\leq \|\partial_x^\alpha \partial_t^{\nu}(\partial_t + |\cdot|^{2}) \hat{G}_1(\cdot, t)\|_{L^\infty} \|u_0\|_{L^2} + \|\partial_x^\alpha \partial_t^{\nu} \hat{G}_1(\cdot, t)\|_{L^\infty} \|u_1\|_{L^2}
\leq C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_0\|_{L^2} + C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_1\|_{L^2}.
\]
By using interpolation formula, we have that, for \( q \in [2, \infty] \),
\[
\|\partial_x^\alpha \partial_t^{\nu} u(\cdot, t)\|_{L^q} \leq \|\partial_x^\alpha \partial_t^{\nu} u(\cdot, t)\|_{L^2}^{\frac{2}{q}} \|\partial_x^\alpha \partial_t^{\nu} u(\cdot, t)\|_{L^\infty}^{1 - \frac{2}{q}}
\leq C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_0\|_{L^2} + C t^{-\frac{|\alpha|}{2} - \frac{\nu}{2} - \frac{1}{4}} \|u_1\|_{L^2}.
\]
Thus (2) is proved and so the theorem is proved. \( \Box \)

4 \( L^1 - L^\infty \) estimates

In the last section we have some good results by assuming the Fourier transform of the initial data have special compact support. In this section we intend to obtain the \( L^1 - L^\infty \) decay estimates without the assumption of compact support.

Denote \( \hat{G}_2(\xi, t) = (1 - \chi(\xi)) \hat{G}(\xi, t) \), then \( \hat{G}(\xi, t) = \hat{G}_1(\xi, t) + \hat{G}_2(\xi, t) \). Denote \( u(x, t) = v(x, t) + w(x, t) \), where \( v(x, t) \) and \( w(x, t) \) satisfying
\[
\hat{v}(\xi, t) = (\partial_t + |\xi|^2) \hat{G}_1(\xi, t) \hat{u}_0(\xi) + \hat{G}_1(\xi, t) \hat{u}_1(\xi)
\]
\[
\dot{w}(\xi, t) = (\partial_t + |\xi|^2) \hat{G}_2(\xi, t) \hat{u}_0(\xi) + \hat{G}_2(\xi, t) \hat{u}_1(\xi)
\]

By direct calculation we have that
\[
\dot{w} = (1 - \chi(\xi)) \frac{1}{2} (e^{\lambda^+ t} + e^{\lambda^- t} + |\xi|^2 \hat{G}) \hat{u}_0 + \hat{G} \hat{u}_1.
\]

Since \( \chi \in C_0^\infty(\mathbb{R}^n) \), \( \text{supp} \chi \subset D_{r+1} \) and \( \chi|_{D_r} = 1 \), we have that
\[
\| \dot{w}(\cdot, t) \|_{L^1(\mathbb{R}^n)} \leq \int_{D_r} \left[ \frac{1}{2} (e^{\lambda^+ t} + e^{\lambda^- t} + |\xi|^2 \hat{G}(\xi, t)) |\hat{u}_0(\xi)| + \hat{G}(\xi, t) |\hat{u}_1(\xi)| \right] d\xi
\]

\[
\leq \frac{1}{2} \int_{D_r \cap E} e^{\lambda^+ t} |\hat{u}_0(\xi)| d\xi + \frac{1}{2} \int_{D_r \cap E^c} e^{\lambda^- t} |\hat{u}_0(\xi)| d\xi + \frac{1}{2} \int_{D_r \cap E} |\xi|^2 \hat{G}(\xi, t) |\hat{u}_0(\xi)| d\xi
\]

\[
+ \frac{1}{2} \int_{D_r \cap E^c} |\xi|^2 \hat{G}(\xi, t) |\hat{u}_0(\xi)| d\xi
\]

Next we come to estimate the six terms respectively, where we will use the facts that \( \lambda_+ \leq -\frac{\xi^2}{|\xi|^2} \), if \( \xi \in D_r \).

If \( \xi \in E \), then \( |\xi|^2 \leq 2|\xi'|^2 \).

\[
\int_{D_r \cap E} e^{\lambda^+ t} |\hat{u}_0(\xi)| d\xi \leq \int_{D_r \cap E} e^{-\frac{\xi^2}{|\xi'|^2} t} |\hat{u}_0(\xi)| d\xi
\]

\[
\leq \int_{D_r \cap E} e^{-\frac{\xi^2}{|\xi'|^2} t} \frac{1}{|\xi'|^{n-1} (1 + |\xi'|^2)} d\xi \sup_{\xi \in \mathbb{R}^n} \{ |\xi'|^{n-1} (1 + |\xi'|^2) |\hat{u}_0(\xi)| \}
\]

\[
\leq Ct^{-\frac{n}{2}} \| u_0 \|_{W^{n+1,1}(\mathbb{R}^n)}.
\]

If \( \xi \in E^c \), then \( |\xi|^2 \leq 2|\xi'|^2 \).

\[
\int_{D_r \cap E^c} e^{\lambda^+ t} |\hat{u}_0(\xi)| d\xi \leq \int_{D_r \cap E^c} e^{-\frac{\xi^2}{|\xi'|^2} t} |\hat{u}_0(\xi)| d\xi
\]

\[
\leq e^{-\frac{\xi^2}{|\xi'|^2} t} \int_{D_r \cap E^c} \frac{1}{(1 + |\xi|^2)^{\frac{n+1}{2}}} d\xi \sup_{\xi \in \mathbb{R}^n} \{ (1 + |\xi|^2)^{\frac{n+1}{2}} |\hat{u}_0(\xi)| \}
\]

\[
\leq Ce^{-\frac{\xi^2}{|\xi'|^2} t} \| u_0 \|_{W^{n+1,1}(\mathbb{R}^n)}.
\]

Since \( \lambda_- \leq -\frac{|\xi'|^2}{2} \), \( \forall \xi \in D_r \),

\[
\int_{D_r} e^{\lambda^- t} |\hat{u}_0(\xi)| d\xi \leq \int_{D_r} e^{-\frac{|\xi'|^2}{2} t} |\hat{u}_0(\xi)| d\xi \leq Ct^{-\frac{n}{2}} \| u_0 \|_{L^1(\mathbb{R}^n)}.
\]
Since $|\xi|^2 \hat{G} \leq C e^{\lambda t}$, $\forall \xi \in D^c_r$, by the previous calculation we have that
\[
\int_{D^c_r} |\xi|^2 \hat{G}(\xi, t)|\hat{u}_0(\xi)|d\xi \leq C \int_{D^c_r} e^{\lambda t}|\hat{u}_0(\xi)|d\xi \leq C t^{-\frac{1}{2}} \|u_0\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
Since $\hat{G} \leq \frac{2\lambda}{\lambda_0}$, $\forall \xi \in D^c_r$,
\[
\int_{D^c \cap E} \hat{G}(\xi, t)|\hat{u}_1(\xi)|d\xi
\leq \int_{D^c \cap R^c} \frac{2e^{\lambda t}}{\lambda_0} |\hat{u}_1(\xi)|d\xi \leq C \int_{D^c \cap E} e^{-\frac{\xi'^2}{|\xi|^2} t} |\hat{u}_1(\xi)|d\xi
\leq C \int_{D^c \cap E} e^{-\frac{\xi'^2}{|\xi|^2} t} |\hat{u}_1(\xi)|d\xi \sup_{\xi \in \mathbb{R}^n} \|\xi\|^{n-3}(1 + |\xi'|^2)|\hat{u}_1(\xi)|
\leq C t^{-\frac{1}{2}} \|u_0\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
Since $\hat{G} \leq t e^{\lambda t}$, $\forall \xi \in D^c_r$,
\[
\int_{D^c \cap E} \hat{G}(\xi, t)|\hat{u}_1(\xi)|d\xi \leq \int_{D^c \cap E} t e^{\lambda t}|\hat{u}_1(\xi)|d\xi
\leq \int_{D^c \cap E} t e^{-\frac{\xi'^2}{|\xi|^2} t} |\hat{u}_1(\xi)|d\xi \leq C e^{-\frac{\xi'^2}{|\xi|^2} t} \|u_0\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
By combining the six inequalities, we obtain that
\[
\|\hat{w}(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|(u_0, u_1)\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
It yields that
\[
\|w(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|(u_0, u_1)\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
From theorem 3.1, we know that $v(x, t)$ satisfies,
\[
\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n}{2} - 1} \|(u_0, u_1)\|_{L^1(\mathbb{R}^n)}.
\]
Since $n \geq 3$, the solution $u(x, t)$ to the equation (1.1) satisfies,
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|(u_0, u_1)\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
Thus we obtain the following decay estimate.

**Theorem 4.1** Assume that $u_0, u_1 \in W^{n+1,1}(\mathbb{R}^n), n \geq 3$, then the solution $u(x, t)$ to the equation (1.1) satisfies,
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|(u_0, u_1)\|_{W^{n+1,1}(\mathbb{R}^n)}.
\]
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