Abstract. We give a cohomological criterion for a parabolic vector bundle on a curve to be semistable. It says that a parabolic vector bundle $E_*$ with rational parabolic weights is semistable if and only if there is another parabolic vector bundle $F_*$ with rational parabolic weights such that the cohomologies of the vector bundle underlying the parabolic tensor product $E_* \otimes F_*$ vanish. This criterion generalizes the known semistability criterion of Faltings for vector bundles on curves and significantly improves the result in [Bis07].

1. Introduction

We will work over an algebraically closed ground field of characteristic zero.

Let $X$ be an irreducible smooth projective curve. A theorem due to Faltings says that a vector bundle $E$ over $X$ is semistable if and only if there is a vector bundle $F$ over $X$ such that $H^0(X, E \otimes F) = 0 = H^1(X, E \otimes F)$ (see [Fal93, p. 514, Theorem 1.2] and [Fal93, p. 516, Remark]). Let $D$ be a reduced effective divisor on $X$. For a parabolic vector bundle $W_*$ on $X$ with parabolic divisor $D$, the underlying vector bundle will be denoted by $W_0$; see [MS80], [MY92] for parabolic vector bundles. Let $r$ be a positive integer. Denote by $\operatorname{Vect}(X, D, r)$ the category of parabolic vector bundles on $X$ with parabolic structure along $D$ and parabolic weights being integral multiples of $1/r$. In [Bis07] the following theorem was proved:

**Theorem 1.1.** There is a parabolic vector bundle $V_* \in \operatorname{Vect}(X, D, r)$ with the following property: A parabolic vector bundle $E_*$ is semistable if and only if there is a parabolic vector bundle $F_* \in \operatorname{Vect}(X, D, r)$ with $H^0(X, (E_* \otimes V_* \otimes F_*)_0) = 0 = H^1(X, (E_* \otimes V_* \otimes F_*)_0)$, where $(E_* \otimes V_* \otimes F_*)_*$ is the parabolic tensor product.

Theorem 1.1 was also proved in [Par10]. It should be mentioned that the vector bundle $V_*$ in Theorem 1.1 is not canonical; it depends upon the choice of a suitable ramified Galois covering $Y \to X$ that transforms parabolic bundles in $\operatorname{Vect}(X, D, r)$ into $G$-linearized vector bundles on $Y$, where $G$ is the Galois group for the covering. However, many different covers do this.

We prove that $V_*$ in Theorem 1.1 can be chosen to be the trivial line bundle $\mathcal{O}_X$ equipped with the trivial parabolic structure. More precisely, we prove the following theorem (see Theorem 6.1):

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Theorem 1.2. A parabolic vector bundle $E_* \in \text{Vect}(X, D, r)$ is semistable if and only if there is a parabolic vector bundle $F_* \in \text{Vect}(X, D, r)$ such that
$$H^0(X, (E_* \otimes F_*)_0) = 0 = H^1(X, (E_* \otimes F_*)_0).$$

Theorem 1.2 is proved by systematically working with stacks. Compare this method with the earlier attempts (cf. [Bis07], [Par10]) that landed in the weaker version given in Theorem 1.1.

2. PARABOLIC BUNDLES AND ROOT STACKS

Recall that to give a morphism $X \rightarrow \mathbb{A}^1/\text{G}_{m}$ is the same as giving a line bundle $L$ with section $s$ on $X$ (see [Cad07]). Given a positive integer $r$, there is a natural morphism
$$\theta_r : \mathbb{A}^1/\text{G}_{m} \rightarrow \mathbb{A}^1/\text{G}_{m}$$
defined by $t \mapsto t^r$, with $t \in \mathbb{A}^1$. We define the root stack $X(\mathcal{L}, s, r)$ to be the fibered product
$$X \times_{\mathbb{A}^1/\text{G}_{m}, \theta_r} \mathbb{A}^1/\text{G}_{m}.$$
When the section is non-zero, this root stack is an orbifold curve; see [Cad07, Example 2.4.6].

The data $(\mathcal{L}, s)$ corresponds to an effective divisor $D$ on $X$. We will henceforth assume that this divisor is reduced. Sometime we write $X_{D,r}$ instead of $X_{\mathcal{L},s,r}$.

We think of the ordered set $\frac{1}{r}\mathbb{Z}$ of rational numbers with denominator $r$ as a category. Let $j$ be an integer multiple of $1/r$. Given a functor from the opposite category
$$\mathcal{F}_* : (\frac{1}{r}\mathbb{Z})^{\text{op}} \rightarrow \text{Vect}(X),$$
we denote by $\mathcal{F}_*[j]$ its shift by $j$, so
$$\mathcal{F}_*[j] = \mathcal{F}_{i+j}.$$
There is a natural transformation $\mathcal{F}_*[j] \rightarrow \mathcal{F}_*$ when $j \geq 0$.

A vector bundle with parabolic structure over $D$ such that the parabolic weights are integral multiples of $1/r$ is a functor
$$\mathcal{F}_* : (\frac{1}{r}\mathbb{Z})^{\text{op}} \rightarrow \text{Vect}(X)$$
together with a natural isomorphism
$$j : \mathcal{F}_* \otimes \mathcal{O}_X(-D) \xrightarrow{\sim} \mathcal{F}[1]$$
such that the following diagram commutes
$$\xymatrix{\mathcal{F}_* \otimes \mathcal{O}_X(-D) \ar[r] \ar[d] & \mathcal{F}[1] \\
\mathcal{F}_* \ar[u]}$$
(see [MY92], [MS80]). The underlying vector bundle of a parabolic vector bundle is the value of this functor at 0. We have previously denoted this by $\mathcal{F}_0$. For a functor $\mathcal{F}_*$ defining a parabolic vector bundle, the value of $\mathcal{F}_*$ at $t \in \frac{1}{r}\mathbb{Z}$ will be denoted by $\mathcal{F}_t$. 

Denote by $\text{Vect}(X, D, r)$ the category of vector bundles on $X$ with parabolic structure along $D$ and parabolic weights integral multiples of $1/r$. It is a tensor category.

**Theorem 2.1.** There is an equivalence of tensor categories

$$F : \text{Vect}(X(L, s, r)) \xrightarrow{\sim} \text{Vect}(X, D, r).$$

The equivalence preserves parabolic degree and semistability (see §4 below).

The functor $F$ has the following explicit description. There is a natural root line bundle $\mathcal{N}$ on $X(L, s, r)$. Given a vector bundle $\mathcal{F}$ on the root stack, the corresponding parabolic bundle is the functor defined by

$$l/r \mapsto \pi^*(\mathcal{F} \otimes \mathcal{N}^l).$$

**Proof of Theorem 2.1** See [Bor07, Section 3] and [Bis97].

3. Root stacks as quotient stacks

For the map $z \mapsto z^n$ defined around $0 \in \mathbb{C}$, the ramification index at $0$ will be $n - 1$.

We will need the following theorem:

**Theorem 3.1.** Suppose $k = \mathbb{C}$. There is a finite Galois covering $Y \rightarrow X$ ramified over $D$ with ramification index $r - 1$ at each point in $D$ if and only if either $X \neq \mathbb{P}^1$ or $X = \mathbb{P}^1$ with $|D| \neq 1$.

**Proof.** See [Nam87, p. 29, Theorem 1.2.15].

**Corollary 3.2.** Theorem 3.1 holds over any algebraically closed ground field of characteristic zero.

**Proof.** This follows from [SGA1, Expose IX, Theorem 4.10]. See also Proposition 7.2.2 in [Mur67, p. 146].

**Proposition 3.3.** Suppose that either $X \neq \mathbb{P}^1$ or $|D| \neq 1$. Then $X(D, r)$ is a quotient stack.

**Proof.** Fix a covering $Y \rightarrow X$ as in Corollary 3.2. Let $G$ be the Galois group for this covering. Our goal is to show that $X(D, r) = [Y/G]$.

Let $R$ be the ramification divisor in $Y$. Then the reduced divisor $R_{\text{red}}$ produces a morphism

$$Y \rightarrow X(D, r)$$

via the universal property of root stacks. As $R_{\text{red}}$ is $G$-invariant so is the morphism in [1]. Hence we obtain a morphism

$$[Y/G] \rightarrow X(D, r).$$

To show that this morphism is an isomorphism is a local condition for the flat topology and follows from [Cad07, Example 2.4.6].
4. Semistability

Recall that the parabolic degree of a parabolic vector bundle $E_*$ over $X$ is defined to be
\[
\deg_{\text{par}}(E_*) := \text{rk}(E_0)(\deg D - \chi(O_X)) + \frac{1}{r}(\sum_{i=1}^{r} \chi(E_{i/r})) = \text{rk}(E_0) \deg D + \frac{1}{r} \sum_{i=1}^{r} \deg(E_{i/r})
\]
(see [MS80], [Bis97], [Bor07 § 4]). The slope is defined as usual:
\[
\mu(E_*) := \frac{\deg_{\text{par}}(E)}{\text{rk}(E)}.
\]
A parabolic vector bundle $E_*$ is said to be semistable if
\[
\mu(E_*) \geq \mu(F_*)
\]
for all parabolic subbundles $F_*$. 

**Example 4.1.** Let us describe all the parabolic semistable bundles on $\mathbb{P}^1$ with one parabolic point, meaning $D = x$, where $x$ is some point on $\mathbb{P}^1$. Let $E_*$ be a semistable parabolic vector bundle. Then we may write
\[
E_0 = \bigoplus_{k=1}^{m} O(n_k)^{s_k}
\]
[Gr057]. We may assume that the integers $n_i$ are strictly decreasing. A subbundle $F_*$ is defined by taking
\[
F_{i/r} = O(n_1)^{s_1} \cap E_{i/r}
\]
for $0 \leq i < r$. This extends to a parabolic subbundle of $E_*$. We see immediately that
\[
\mu(F_*) > \mu(E_*)
\]
when $m > 1$. Consequently, a parabolic vector bundle $E_*$ of rank $n$ over $\mathbb{P}^1$ with one parabolic point is semistable if and only if
\[
E_* = (L_*)^{\oplus m},
\]
where $L_*$ is a parabolic line bundle.

5. Grothendieck-Riemann-Roch theorem for Deligne-Mumford stacks

In this section we recall the pertinent results from [Tö99]. An excellent summary of this paper of Töen can be found in the appendix to [Bor07]. We denote by $\mathcal{X}$ a smooth Deligne-Mumford stack that is proper over our ground field $k$. We equip it with the étale topology. The category of vector bundles (respectively, coherent sheaves) on $\mathcal{X}$ is an exact category so we may form the groups
\[
K_i(\mathcal{X}) \quad \text{(respectively, } G_i(\mathcal{X})\text{).}
\]

Let $\mathcal{K}_i$ denote the sheaf in the étale topology on $\mathcal{X}$ associated to the presheaf
\[
(X \rightarrow \mathcal{X}) \mapsto K_i(X).
\]
Set
\[
H^i(\mathcal{X}, \mathbb{Q}) = H^i(\mathcal{X}, \mathcal{K}_i \otimes \mathbb{Q}).
\]
By \cite{Gil81} we have Chern classes and hence Chern characters and Todd classes
\[ c_i^{et}, \; ch^{et}, \; td^{et} : K_0(\mathcal{X}) \longrightarrow H^*(\mathcal{X}). \]

Let \( I_\mathcal{X} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \) be the inertia stack of \( \mathcal{X} \). Let \( \mu_\infty \) denote the group of roots of unity in \( \mathbb{Q} \), and set \( \Lambda := \mathbb{Q}(\mu_\infty) \). If \( \mathcal{G} \) is a locally free sheaf on \( I_\mathcal{X} \), the inertial action induces an eigenspace decomposition
\[ \mathcal{G} = \bigoplus_{\zeta \in \mu_\infty} \mathcal{G}^{(\zeta)}. \]

Let
\[ \rho_\mathcal{X} : K_0(I_\mathcal{X}) \otimes \mathbb{Z} \Lambda \longrightarrow K_0(I_\mathcal{X}) \otimes \mathbb{Z} \Lambda \]
be the morphism defined by
\[ G \mapsto \sum_{\zeta} [G^{(\zeta)}]. \]

We have a morphism, called the Frobenius character,
\[ \phi_\mathcal{X} : K_0(\mathcal{X}) \otimes \mathbb{Z} \Lambda \xrightarrow{\pi^*_X} K_0(I_\mathcal{X}) \otimes \mathbb{Z} \Lambda \xrightarrow{\rho_X} K_0(I_\mathcal{X}) \otimes \mathbb{Z} \Lambda \longrightarrow K_{0,et}(I_\mathcal{X}) \otimes \mathbb{Z} \Lambda. \]

The ring \( K_0 \) is a lambda ring and we write \( \lambda_{-1}(x) = \sum (-1)^i \lambda_i(x) \). Define
\[ \alpha_\mathcal{X} := \rho_X(\lambda_{-1}([\Omega^1_{I_\mathcal{X}/\mathcal{X}}])); \]

Finally define the characteristic classes
\[ ch^{rep}(x) := ch^{et}(\phi_\mathcal{X}(x)) \]
and
\[ td^{rep}(\mathcal{X}) := ch^{et}(\alpha^{-1}_\mathcal{X})td^{et}(T_{I_\mathcal{X}}). \]

**Theorem 5.1.** Denote by \( \int^{rep}_\mathcal{X} \) the push-forward \( p_* \) for \( p : I_\mathcal{X} \longrightarrow \text{Spec}(k) \). The following holds:
\[ \chi(\mathcal{X}, \mathcal{F}) = \int^{rep}_\mathcal{X} td^{rep}(\mathcal{X})ch^{rep}(\mathcal{F}). \]

**Proof.** See \cite{Tö99} Corollary 4.13. \( \square \)

**Corollary 5.2.** Suppose that \( \mathcal{X} \) is a proper orbifold curve. Then
\[ \mu(\mathcal{F}) = \chi(\mathcal{F}) - \int^{rep}_\mathcal{X} td^{rep}(\mathcal{X}). \]

**Proof.** We have that \( \pi^*_X(\mathcal{F}) \) is an eigensheaf with eigenvector 1 as the stack \( \mathcal{X} \) is generically a variety. There is a diagram
\[ \xymatrix{ I_\mathcal{X} \ar[d]_{\pi_X} \ar[r]^{p_!} & \text{Spec}(k). \\
\mathcal{X} \ar[ur]_p } \]
By the projection formula,
\[ p_!\ast(c^et_1(\pi^*_X \mathcal{F})) = p_*\ast(c^et_1(\mathcal{F})). \]
In view of Theorem 5.1 the result follows from the fact that \( \deg(\mathcal{F}) = p_\ast(c^et_1(\mathcal{F})) \) (\cite{Bor07} Theorem 4.3) and the usual expression for the Chern character. \( \square \)
Corollary 5.3. Suppose that there is a vector bundle $\mathcal{E}$ so that $H^i(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}) = 0$ for $i = 0, 1$. Then $\mathcal{F}$ is semistable.

Proof. Suppose there is a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ with

$$\mu(\mathcal{F}') > \mu(\mathcal{F}).$$

Then it follows from Corollary 5.2 that

$$\frac{\chi(\mathcal{E} \otimes \mathcal{F}')}{{\text{rank}}(\mathcal{E} \otimes \mathcal{F}')} - \frac{\chi(\mathcal{E} \otimes \mathcal{F})}{{\text{rank}}(\mathcal{E} \otimes \mathcal{F})} > 0.$$ 

Since $\chi(\mathcal{E} \otimes \mathcal{F}) = 0$, this implies that $H^0(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}') \neq 0$. But $\mathcal{E} \otimes \mathcal{F}' \subset \mathcal{E} \otimes \mathcal{F}$. Hence $H^0(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}) \neq 0$ which is a contradiction. □

6. Semistability criterion

Theorem 6.1. A vector bundle with parabolic structure $\mathcal{E}_* \in \text{Vect}(X, D, r)$ is semistable if and only if there is a parabolic vector bundle $\mathcal{F}_* \in \text{Vect}(X, D, r)$ with

$$H^i(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0$$

for all $i$, where $(\mathcal{E}_* \otimes \mathcal{F}_*)_*$ is the parabolic tensor product.

Proof. We have a morphism $\pi : X_{D, r} \longrightarrow X$, and $\pi_*$ is exact as $\text{char}(k) = 0$. Hence by the Leray spectral sequence,

$$H^i(X, \pi_*(\mathcal{F})) = H^i(X_{D, r}, \mathcal{F})$$

for all $i$.

Suppose that there is a parabolic vector bundle $\mathcal{F}_* \in \text{Vect}(X, D, r)$ with

$$H^0(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0 = H^1(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0).$$

Applying Theorem 2.1, we deduce from Corollary 5.3 that $\mathcal{E}_*$ is semistable.

To prove the converse, assume that $\mathcal{E}_*$ is semistable. We break up into two cases.

The case of $\mathbb{P}^1$ with exactly one parabolic point: Applying Example 4.1, we see that

$$\mathcal{E}_0 = \bigoplus \mathcal{O}(n)^m.$$ 

So tensoring with $\mathcal{O}(-n - 1)$ does the job.

All other cases: In view of Proposition 3.3 we may assume that we have a quotient stack, so $X_{D, r} = [Y/G]$. Then given a semistable parabolic bundle on $X$, we obtain a corresponding semistable $G$-linearized vector bundle $\mathcal{E}$ on $Y$. We note that this implies that the vector bundle $\mathcal{E}$ is semistable [Bis97, p. 308, Lemma 2.7]. By [Fal93, p. 514, Theorem 1.2], there is a vector bundle $\mathcal{F}$ on $Y$ such that all the cohomology groups of $\mathcal{F} \otimes \mathcal{E}$ vanish. Consider

$$\tilde{\mathcal{F}} = \bigoplus_{g \in G} g^* \mathcal{F}.$$ 

The vector bundle $\tilde{\mathcal{F}}$ has a natural $G$-action and

$$H^i(Y, \tilde{\mathcal{F}} \otimes \mathcal{E}) = 0$$
for all $i$. The vector bundle $\tilde{F}$ produces a vector bundle on $[Y/G]$, which will also be denoted by $\tilde{F}$. Finally

$$H^i([Y/G], \tilde{F} \otimes \mathcal{E}) = H^i(Y, \tilde{F} \otimes \mathcal{E})^G = 0 .$$

The theorem now follows. □

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