Computation of the Central Charge for the Leading Order of the $\mathcal{N} = 2$ Super-Yang-Mills Effective Action

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Abstract

The central charge in the $\mathcal{N} = 2$ Super-Yang-Mills theory plays an essential role in the work of Seiberg and Witten as it gives the mass spectrum of the BPS states of the quantum theory. Our aim in this note is to present a direct computation of this central charge for the leading order (in a momentum expansion) of the effective action. We will consider the $\mathcal{N} = 2$ Super-Yang-Mills theory with gauge group $\text{SU}(2)$. The leading order of the effective action is given by the same holomorphic function $F$ appearing in the low energy $U(1)$ effective action.

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Introduction

Tremendous progress has been made these last years in the understanding of the strong coupling regime of the $N = 2$ Super-Yang-Mills (SYM) theory, pioneered by the work of Seiberg and Witten for the case of a $SU(2)$ gauge group. They succeeded in determining a non-perturbative solution for the leading order of the low energy effective action, which is given, as explained in Ref. 2, by one single holomorphic function $F(\Psi)$, where $\Psi$ is the chiral $N = 2$ multiplet that contains the gauge field for the unbroken $U(1)$ gauge group. Later, it has been shown that this solution is in fact unique.

One essential tool used to obtain these results is the central charge $Z$ appearing in the $N = 2$ Super-Poincaré algebra, because it gives a lower bound for the mass $M$ of the states of the theory, known as the BPS bound:

$$M \geq |Z|.$$

States that saturate this bound are called BPS-states. In fact, it turns out that all the states of the classical theory saturate this bound and that this remains true at the quantum level.

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For the classical $N = 2$ SYM action with gauge group $SU(2)$, a direct computation of $Z$ leads to

$$Z = a \ (n_e + \tau n_m),$$

(2)

where $a$ is the Higgs field expectation value, $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ is the complex coupling constant, including the gauge coupling constant $g$ and the $\theta$-angle, and $n_e$ and $n_m$ are the electric and magnetic numbers of the state, respectively.

For the leading order of the low energy effective action, Seiberg and Witten argued that the central charge must be given by

$$Z = a n_e + a_D n_m,$$

(3)

where $a_D = \frac{\partial F}{\partial a}(a)$. A first check of this formula has been obtained in, where only the bosonic contributions to $Z$ have been considered. Recently, a direct and complete computation of $Z$ has been performed in and agrees with (3).

However it remains to show that the expression (3) is true for the high energy $SU(2)$ effective action. Indeed, even if the central charge is expected to be a boundary term and thus to receive contributions only from the massless $U(1)$ fields, this has not been proved so far. A first step in that direction has been achieved in, where (3) has been obtained by applying the BPS trick to the bosonic part of the leading order of the $SU(2)$ effective action.

In this note, we will also consider the leading order of the $SU(2)$ effective action, for which we will perform a direct computation of $Z$ by evaluating the Dirac bracket of two supersymmetry charges. The result shows that (3) is the correct expression.

The plan of this note is as follows. In the first section, we introduce the notations and expand the effective action defined by $F$ in components. In section 2, the supersymmetry current is constructed by performing a local supersymmetry transformation. In section 3 we implement the Dirac bracket, which is the natural canonical structure in the presence of fermions. Finally the central charge is obtained in section 4 by computing the Dirac bracket of two supersymmetry charges.

This note being only a sketchy presentation of the tools used to define the supercurrent and the canonical structure and of the computation itself, a more detailed publication is in preparation.

1 Field Content and Effective Action

The pure $N = 2$ SYM action with gauge group $SU(2)$ contains a triplet of $N = 2$ chiral multiplets $\Psi^a$ in the adjoint representation of $SU(2)$, labeled by the index $a$. The leading order of the effective Lagrangian is given in terms of $F(\Psi^a)$, with the same holomorphic function $F$ as in the low energy effective action, by:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \int d^4 \theta \text{Im} (F(\Psi)).$$

(4)

The classical Lagrangian can be recovered by letting $F(\Psi) = \frac{1}{2} \tau \Psi^a \Psi^a$, where $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$.

We then expand this Lagrangian into components choosing the Wess-Zumino gauge and eliminating the auxiliary fields. The remaining fields are the gauge field $A_\mu^a$, the $SU(2)_R$-doublet $\bar{\varphi}$ of Weyl spinors

2 We denote by $SU(2)_R$ the $SU(2)$ that rotates the two supersymmetry charges into each other. Our conventions about the $SU(2)_R$ indices are the following: $\bar{\varphi}_n = \epsilon_{nm} \bar{\varphi}_m$ with $\epsilon_{nn} = -\epsilon_{nm}$, $\epsilon_{21} = 1$; $\varphi^m = \epsilon^{mn} \varphi_n$ with $\epsilon^{mn} = -\epsilon^{nm}$, $\epsilon^{12} = 1$; Finally, $(\varphi^n)^* = \varphi_n$, and, as a consequence, $(\bar{\varphi}^n)^* = -\bar{\varphi}_n$. 2
\( \lambda^{a \alpha} \) (n is the SU(2)\(_R\) index whereas \( \alpha \) is the spinor index) and the complex scalar field \( \phi^a \), all in the adjoint representation of SU(2). \( A^a_\mu \) and \( \phi^a \) are SU(2)\(_R\)-singlets.

This gives

\[
\mathcal{L}^{\text{eff}} = \frac{1}{8\pi} \text{Im} \left( \mathcal{F}^{ab}(\phi) \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{8} \varepsilon^{\mu\nu\rho\lambda} F_{\mu\rho} F_{\nu\lambda} - i \lambda^{a n} \sigma^\mu \nabla_{\mu} \bar{\lambda}^b_n + \frac{1}{2} \nabla_{\mu} \phi^a \nabla_{\mu} \phi^b \right\} ight.
\]

\[
- D^{a b m n} D^{b}_{m n} + i \frac{z}{2} \varepsilon^{a b c} (\bar{\lambda}^a \sigma^\mu \phi^c) \phi^c - i \frac{z}{2} \varepsilon^{c a b} (\bar{\lambda}^a \sigma^\mu \phi^c) \phi^c + \frac{1}{8} (\varepsilon^{a b c} \phi^c \phi^d) (\varepsilon^{b c f} \phi^f \phi^d) \left\} \right.
\]

where \( \varepsilon^{abc} \) are the structure constants of SU(2) and \( \mathcal{F}^{a_1 \cdots a_n}(\phi) \) denotes the \( n \)-th derivative of \( \mathcal{F} \) with respect to \( \phi^{a_1} \cdots \phi^{a_n} \). For further use, let us introduce the notation \( \mathcal{I}^{a_1 \cdots a_n} \) and \( \mathcal{R}^{a_1 \cdots a_n} \) for the imaginary and real part of \( \mathcal{F}^{a_1 \cdots a_n} \), respectively. Finally, we use the condensed notation

\[
D^{a b m n} = - \frac{i}{4} (\mathcal{I}^{-1})^{a c} \left( \mathcal{I}^{b c d}(\bar{\lambda}^a \sigma^m \lambda^c) + \mathcal{R}^{b c d}(\bar{\lambda}^a \lambda^c) \right),
\]

where \((\mathcal{I}^{-1})^{a b}\) denotes the inverse of \(\mathcal{I}^{a b}\), i.e. \( (\mathcal{I}^{-1})^{a c} \mathcal{I}^{c b} = \delta^{a b} \).

From now on, it is implicitly understood that \( \mathcal{F} \) and its derivatives (as well as \( \mathcal{I}, \mathcal{R}, (\mathcal{I}^{-1}) \) and so on) are to be evaluated at \( \phi \), except when otherwise stated.

### 2 Supersymmetry Transformations and Currents

The Lagrangian \( \mathcal{L} \) is invariant under the global on-shell \( N = 2 \) supersymmetry transformations \( \delta \) (with \( \delta = \varepsilon^{n \alpha} \delta_{n \alpha} - \varepsilon^{n \alpha} \delta_{n \alpha} \)):

\[
\left\{ \begin{array}{l}
\delta A^a_\mu = \varepsilon^n \sigma^\mu \bar{\lambda}^a_n + \lambda^m \sigma^\mu \bar{\varepsilon}^a_m, \\
\delta \phi^a = 2 \varepsilon^n \lambda^a_n, \\
\delta \lambda^{a n} = - \frac{1}{2} \varepsilon^n \sigma^\mu F^a_{\mu \nu} + i \nabla_{\mu} \phi^a \varepsilon^n \sigma^\mu - 2 D^{a b m n} \varepsilon_{m n} - \frac{i}{2} \varepsilon^{a b c} \phi^b \phi^c \varepsilon^n .
\end{array} \right.
\]

To compute the supersymmetry current, we perform a local supersymmetry transformation, defined by \( \delta_{\text{local}} = - \varepsilon^{n \alpha}(x) \delta_{n \alpha} \) (and similarly for \( \delta_{n \alpha} \)). Since the action is invariant under a global supersymmetry transformation (which means that the Lagrangian transforms as a total derivative), we have, under a local transformation:

\[
\delta_{\text{local}} \mathcal{L}^{\text{eff}} = - i \partial_\mu \varepsilon^{n \alpha} \bar{J}^\mu_{n \alpha} + \partial_\mu V^\mu,
\]

which defines the supersymmetry current \( \bar{J}^\mu_{n \alpha} \).

**Remark:** In fact, this relation defines the supersymmetry current \( \bar{J}^\mu_{n \alpha} \) and the total divergence \( V^\mu \) only up to an improvement term \( \bar{X}^\mu_{n \alpha} \) since \( \bar{J}^\mu_{n \alpha} = \bar{J}^\mu_{n \alpha} + i \partial_\nu \bar{X}^\mu_{n \alpha} \) and \( V^\mu = V^\mu - \varepsilon^{n \alpha} \partial_\nu \bar{X}^\mu_{n \alpha} \) satisfies \( \delta \) as well.

This computation leads to:

\[
\bar{J}^\mu_{n} = \frac{i}{8\pi} F^{ab}(\phi) \left( \bar{\sigma}_\nu \lambda^a_n (F^{b \mu \nu} - \frac{i}{4} \varepsilon^{\mu \nu \rho \lambda} F^{b \rho \lambda}) + \bar{\sigma}_\mu \lambda^a_n F^{b \mu \nu} - \frac{i}{2} \bar{\sigma}_\nu \lambda^a_n \varepsilon^{b \mu \nu} \phi^b \right)
\]

\[
+ \frac{i}{16\pi} \bar{F}^{abc} (\bar{\lambda}^{a m} (\lambda^b_n \sigma^\mu \lambda^c_m)) + \frac{i}{48\pi} F^{abc} (\bar{\sigma}_\mu \lambda^{a m} (\lambda^b_n \lambda^c_m))
\]

\[
+ i \partial_\nu \bar{X}^\mu_{n \alpha} .
\]
The improvement term $\tilde{X}_a^{[\mu\nu]}$ will be fixed explicitly in the next section, by imposing that the canonical supersymmetry charge generates the supersymmetry transformation laws \[1\] we started with.

3 Canonical Structure and Dirac Bracket

To compute the supersymmetry algebra satisfied by the supersymmetry charges, we first have to implement the canonical structure of the model. Let us start by defining the canonical conjugate momenta of the dynamical fields (all fields excepted $A_0^a$):

$$
\begin{align*}
\Pi^a &= \frac{\partial \mathcal{L}^{\text{eff}}}{\partial \dot{\phi}^a} = \frac{1}{16\pi} T^{ab} \nabla_0 \phi^b, \\
\Lambda^a_{m\alpha} &= \frac{\partial \mathcal{L}^{\text{eff}}}{\partial \dot{\lambda}^a_{\alpha m}} = -\frac{1}{16\pi} F^{ab} (\sigma^0 \tilde{\lambda}_m^b)_\alpha, \\
M^{aj} &= \frac{\partial \mathcal{L}^{\text{eff}}}{\partial \dot{A}^a_j} = \frac{1}{8\pi} \left( T^{ab} E^{b} - \mathcal{R}^{ab} B^{b} + \frac{i}{4} F^{abc} (\lambda^{bn} \sigma^0 j_n^c) + \frac{1}{4} F^{abc} (\lambda^{bn} \sigma^0 j_n^c) \right),
\end{align*}
$$

where $i, j, k, \ldots$ denote the space indices (and thus run from 1 to 3) and we have set $E^{aj} = F^{aj0}$ and $B^{aj} = \frac{1}{2} \varepsilon^{0jkl} F^{a}_{kl}$ for the electric and magnetic non-abelian fields, respectively. As usual, $A_0^a$ plays the role of a Lagrange multiplier and the associated constraint is the Gauss law

$$
0 = \frac{\partial \mathcal{L}^{\text{eff}}}{\partial A_0^a} = \nabla_i M^{ai} + J^{a0}, \quad \text{with} \quad J^{a0} = \text{Im} \left( \frac{i}{8\pi} \varepsilon^{abc} F^{bd} (\lambda^{dn} \sigma^0 j_n^c) + 2i \varepsilon^{abc} \phi^b \Pi^a \right).
$$

The canonical Poisson bracket \[2\] is then defined by

$$
\{ \pi_m(x), \varphi^n(y) \} = \delta^m_n \delta^{(3)}(x - y),
$$

where the $\pi_m$ denote the conjugate momenta to $\varphi^m$. In fact, the second line of (11) defines two sets of constraints $f^a_{m\alpha} = 0$ and $f^a_{m\alpha} = 0$, where $f^a_{m\alpha} = \Lambda^a_{m\alpha} + \frac{1}{16\pi} F^{ab}(\sigma^0 \tilde{\lambda}_m^b)_\alpha$ (and similarly for $f^a_{m\alpha}$). We thus apply the usual Dirac approach for constrained systems \[3\]: first, these constraints are easily recognized to be of the second class type; then, we construct the Dirac bracket, which ensures that the constraints commute with any functions of $\varphi^m$ and $\pi_m$.

More precisely, the procedure to construct the Dirac bracket is the following: let us denote by $f_i = 0$ the constraints and by $m_{ij}$ the matrix $\{ f_i, f_j \} = \delta^{ij}$, which is invertible for second class constraints. The Dirac bracket of two functions $A$ and $B$ is then defined by

$$
\{ A, B \} = \{ A, B \} = \{ A, f_i \} (m^{-1})^{ij} \{ f_j, B \},
$$

and it can be indeed checked that it satisfies $\{ A, f_i \} = 0$ for all $A$ and $i$.

Applying this construction to the case under study means that we can indifferently use the spinor

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We do not write explicitly the time argument of the fields, but it is always understood that both expressions appearing in a Poisson or a Dirac bracket are evaluated at the same time.
fields or their conjugate momenta and leads to the following Dirac bracket:

\[
\{M^{\alpha i}(x), A^b_j(y)\} = \delta^{ab}\delta_{ij} \delta^{(3)}(x - y),
\]

\[
\{\Pi^a(x), \phi^b(y)\} = \delta^{ab}\delta^{(3)}(x - y),
\]

\[
\{\Pi^a(x), \lambda^{\alpha \beta \gamma}(y)\} = \frac{i}{2} \left( (I^{-1})^{bc} \mathcal{F}^{acdf} \lambda^{\alpha \beta \gamma}(x) \right) \delta^{(3)}(x - y),
\]

\[
\{\lambda^{\alpha \beta \gamma}(x), \bar{\lambda}^{\alpha \beta \gamma}(y)\} = 8\pi i (I^{-1})^{\alpha \beta}(x)\delta^{m^\alpha}_n \bar{\sigma}^{0\alpha\beta} \delta^{(3)}(x - y),
\]

\[
\{\Pi^a(x), \bar{\Pi}^b(y)\} = -\frac{i}{8\pi} \left( (I^{-1})^{cd} \mathcal{F}^{ace} \mathcal{F}^{bdf} (\lambda^{em} \bar{\sigma}^{00}(\bar{\lambda}^m_n)) (x) \right) \delta^{(3)}(x - y).
\]

Note that the first two lines are the same as for the canonical bracket, but we see in the next three lines how the Dirac construction modifies the relation between the fields and the conjugate momenta for the fields implied in the constraints.

### 4 Supersymmetry and Central Charge

To define a supersymmetry charge, we have to choose one current in the class defined by (9). As already stated, the prescription is that we want that this charge reproduces the transformation laws (7) we started with. Clearly, we define

\[
\hat{Q}_{\dot{m}\dot{a}} = \int d^3x \, J^0_{\dot{m}\dot{a}},
\]

with \(J^0_{\dot{m}\dot{a}}\) given by (9) and impose that

\[
\{\hat{Q}_{\dot{m}\dot{a}}, \varphi\} = i\delta_{\dot{m}\dot{a}}\varphi,
\]

paying attention to the boundary terms contributions. This leads to the particular solution corresponding to \(X_n^{[\mu \nu]} = 0\) in (9), and thus to the following supersymmetry charge:

\[
\hat{Q}_m = i \int d^3x \left( \bar{\sigma}_j \lambda^m_j \left( M^{aj} + \frac{1}{8\pi} \mathcal{F}^{ab} B_{bj} \right) + 2\bar{\lambda}_m^a \bar{\Pi}^a + \frac{i}{8\pi} \mathcal{F}^{ab} \bar{\sigma}^{00} \bar{\lambda}^m_j \nabla_j \bar{\phi}^b - \frac{1}{16\pi} \mathcal{F}^{ab} \bar{\sigma}^{00} \lambda^m_j \bar{\phi}^b \right).
\]

We can now obtain the central charge, defined by \(Z = \frac{i}{8\epsilon} \epsilon^{nm\dot{a}\dot{b}} \{\hat{Q}_{n\dot{a}}, \hat{Q}_{m\dot{b}}\}\), by a tedious but straightforward computation which leads to

\[
Z = \int d^3x \left( \partial_i \left( \phi^a M^{ai} + \frac{1}{8\pi} \mathcal{F}^{ai} B_j  - \frac{i}{32\pi} \mathcal{F}^{ab} \lambda^{em} \bar{\phi}^b \right) - \phi^a (\nabla_i M^{ai} + J^{ai}) \right).
\]

Now, taking into account the fact that the spinors decrease faster than \(r^{3/2}\) as \(r \to \infty\) and that the last term is the Gauss law, we obtain:

\[
Z = \int d^3x \partial_i \left( \phi^a M^{ai} + \frac{1}{8\pi} \mathcal{F}^{ai} B_j \right).
\]

Let us then denote by \(a\) the expectation value of the Higgs field and by \(\bar{\phi}^a\) the direction (in the gauge space) where it points to \(\ddot{a}\). Thus, at spatial infinity, \(\bar{\phi}^a(x) = a \bar{\phi}^a(x)\). The direction defined by \(\bar{\phi}^a\) gives \(\bar{\phi}^a = \frac{1}{2}\).
in fact the direction of the unbroken $U(1)$ gauge symmetry that survives the Higgs phenomenon. The $U(1)$ electric and magnetic fields are thus defined by $E^i = \hat{\partial}^a E^{ai}$ and $B^i = \hat{\partial}^a B^{ai}$ respectively, and we define in the same way $M^i = \hat{\partial}^a M^{ai}$. Taking finally into account the fact that the integrals of $M^i$ and of $B^i$ on the sphere at spatial infinity define (up to numerical factor) the electric and magnetic numbers $n_e$ and $n_m$ of the state, we obtain the central charge in the form proposed by Seiberg and Witten:

$$Z = a n_e + a_D n_m,$$

(19)

where we have set $a_D = \frac{\partial F}{\partial \phi^a}$.

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\footnote{In fact, we have $a_D = \frac{\partial F}{\partial \phi^a} = \frac{\partial \hat{\phi}^a}{\partial \phi^a} \frac{\partial F}{\partial \hat{\phi}^a} = \hat{\phi}^a F^a$ and thus $F^a B^{ai} = a_D \hat{\phi}^a B^{ai} = a_D B^i$.}