RELATIVE BOUND AND ASYMPTOTIC COMPARISON OF EXPECTILE WITH RESPECT TO EXPECTED SHORTFALL

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ABSTRACT. Expectile bears some interesting properties in comparison to the industry wide expected shortfall in terms of assessment of tail risk. We study the relationship between expectile and expected shortfall using duality results and the link to optimized certainty equivalent. Lower and upper bounds of expectile are derived in terms of expected shortfall as well as a characterization of expectile in terms of expected shortfall. Further, we study the asymptotic behavior of expectile with respect to expected shortfall as the risk level goes to 0 in terms of extreme value distributions. Illustrating the formulation of expectile in terms of expected shortfall, we also provide explicit or semi-explicit expressions of expectile for some classical distributions.

I. INTRODUCTION

The expectile is a generalization of quantile introduced by Newey and Powell [27]. It is defined as the argmin of a quadratic loss

\[ e_\alpha(L) = \text{arg min} \left\{ (1 - \alpha)E \left[ (L - m)^+ \right]^2 + \alpha E \left[ (L - m)^- \right]^2 \right\}. \]

For \(0 < \alpha \leq 1/2\), the expectile is a coherent risk measure that corresponds to Föllmer and Schied [15]’s shortfall risk with loss function \(\ell(x) = (1 - \alpha)x^+ - \alpha x^-\). Widely used in insurance and statistics, it has recently gained some interest in finance as it bears some interesting features for the assessment of tail risk in comparison to the industry wide expected shortfall risk measure introduced by Artzner et al. [4]. From its definition, expectile is elicitable, which is a useful property in terms of backtesting, see Gneiting [17], Ziegel [35], Bellini and Bignozzi [6], Emmer et al. [14], and Chen [10] for a discussion about the financial relevance. In the seminal paper Weber [33], and later Ziegel [35], Bellini and Bignozzi [6], Delbaen et al. [13], it actually turns out that expectile is the only elicitable risk measure within the class of coherent and law invariant risk measures. Expectile is also invariant under randomization and robust to mixture distributions, while expected shortfall is not, see Weber [33] and Guo and Xu [18]. Finally, multivariate shortfall risk – expectile being an example of which – seems to be suitable in terms of systemic risk management and risk allocation, see Armenti et al. [5]. Due to these appealing properties, several authors suggest expectile as an alternative to expected shortfall at value at risk, see [5, 7, 10, 14] for instance.

The goal of this paper is to study the relationship between expectile and expected shortfall. More specifically, the objective is to provide lower and upper bounds of expectile in terms of expected shortfall, formulate explicitly expectile as a function of expected shortfall, and compare the asymptotic behavior of expectile with respect to expected shortfall as the risk level goes to 0. As for the bounds, our approach is based on duality results and the link between expectile and expected shortfall through optimized certainty equivalent. For loss

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profile $L$ with zero mean, our first result mainly focus on the bounds

\[(1.1) \quad \left(1 - \frac{\alpha}{(1 - 2\alpha)\beta + \alpha}\right) ES_{\beta}(L) \leq e_\alpha(L) \leq \left(1 - \frac{\alpha}{1 - \alpha}\right) ES_{\alpha}(L).\]

As shown in Proposition 3.2, the optimal lower bound is in fact an equality

\[e_\alpha(L) = \left(1 - \frac{\alpha}{(1 - 2\alpha)\beta* + \alpha}\right) ES_{\beta*}(L)\]

where $\beta^* = P[L > e_\alpha(L)]$. For continuous distribution, the expression of $\beta^*$ is mentioned in Taylor [32, Equation 7] based on results by Newey and Powell [27]. We generalized this result to any distribution using optimized certainty equivalent. As an application of this relation we can easily derive explicit or semi-explicit formulations of expectile for wide classes of distributions. As for the upper bound, Delbaen [12] and Ziegel [35] provide a comonotone least upper bound of expectile in terms of concave distortion risk measure. Using this result, we show that the upper bound given by Relation (1.1) is the smallest within the class of expected shortfalls dominating expectile.

According to these bounds, expected shortfall is more conservative than expectile. We therefore, address their comparative asymptotic behavior as the risk level goes to 0. In actuarial literature, asymptotic analysis is a subject of intensive research as it helps risk managers to model large losses with small amounts of data and to establish asymptotic relationships between risk measures, see Hua and Joe [19]. While Hua and Joe [19], Tang and Yang [30] and Mao and Hu [23] establish asymptotic relationship between expected shortfall and value at risk, Bellini and Bernardino [5] and Mao et al. [25] provides asymptotic analysis of expectile in terms of value at risk when the loss profile belongs to the maximum domain of attraction of extreme value distributions. Using these results, when the loss profile belongs to the domain of attraction of either Weibull type $MDA(\Psi_{\eta})$, Gumbel type $MDA(\Lambda)$ or Fréchet type $MDA(\Phi_{\eta})$, we establish asymptotic relationship between expectile and expected shortfall by providing both the first-order and second-order asymptotic expansion. For a Fréchet type tail distribution with $\eta > 1$, asymptotically the ratio of expectile to expected shortfall become strictly less than 1. In this case, it actually hold

\[e_\alpha(L) \sim \frac{(\eta - 1)}{\eta} ES_{\alpha}(L).\]

Our result also show that the upper bound provided by Relation (1.1) is not asymptotically equivalent to $e_\alpha(L)$ in general.

We also consider the asymptotic behavior of the parameter $\beta^*$. For loss profiles whose distribution belongs to Fréchet type $MDA(\Phi_{\eta})$ with $\eta > 1$, Bellini et al. [7] provide the asymptotic behavior of $\beta^*$ in terms of $\alpha$. For Weibull type $MDA(\Psi_{\eta})$ and Gumbel type $MDA(\Lambda)$, we show that $\alpha = o(\beta^*)$. For Fréchet case, we also provide a second-order asymptotic expansion for $\beta^*/\alpha$.

The paper is organized as follows. In Section 2, aside definitions and notations, we revisit the link between expectile and expected shortfall through optimized certainty equivalent. In Section 3, we address the lower and upper bounds of expectile in terms of expected shortfall as well as characterize expectile in terms of expected shortfall. Section 4 focuses on asymptotic behavior of expectile in terms of expected shortfall according to the maximum domain of attractions of extreme value distributions to which the loss profile belongs. Section 5 illustrate the results of Section 3 in terms of explicit or semi-explicit expression of expectile for commonly known distributions. It also provide an illustrations for the asymptotic expansion results of Section 4.
2. **Expectile Versus Expected Shortfall through Optimized Certainty Equivalent**

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(L^1\) be the set of integrable random variables identified in the almost sure sense. For \(a > 0\) and \(b \geq 0\) with \(1/a \geq b\), denote by

\[
Q_{a,b} = \left\{ Q \ll P : b \leq \frac{dQ}{dP} \leq \frac{1}{a} \right\}.
\]

Throughout, elements of \(L^1\) are generically denoted by \(L\) and considered as a loss profile. Given such an \(L\) in \(L^1\), we denote by \(F_L\) and \(q_L\) its cumulative distribution and left-quantile function, respectively, that is

\[
q_L(u) = \inf \{ m : F_L(m) := P[L \leq m] \geq u \}.
\]

We also denote the right-quantile function of \(L\) by \(q^+_L\), that is \(q^+_L(u) = \inf \{ m : F_L(m) > u \}\). A function \(R : L^1 \to \mathbb{R}\) is called a risk measure if it is

(I) quasi-convex: \(R(\lambda L_1 + (1-\lambda)L_2) \leq \max\{R(L_1), R(L_2)\}\) for every \(0 \leq \lambda \leq 1\).

(II) monotone: \(R(L_1) \leq R(L_2)\) whenever \(L_1 \leq L_2\) almost surely.

A risk measure is further called monetary if it is additionally

(III) cash-invariant: \(R(L - m) = R(L) - m\) for every \(m\) in \(\mathbb{R}\).

Finally, a monetary risk measure is called coherent if it is additionally

(IV) sub-additive: \(R(L_1 + L_2) \leq R(L_1) + R(L_2)\).

It is known that monetary risk measures are automatically convex, and, coherent monetary risk measures are positive-homogeneous. For \(L\) in \(L^1\), we define

- **Value at Risk**: for \(0 < \alpha < 1\),
  \[
  V\alpha R_\alpha(L) = q_L(1-\alpha) = \inf \{ m : P[L > m] \leq \alpha \}.
  \]

- **Expected Shortfall**: for \(0 < \alpha \leq 1\),
  \[
  ES_\alpha(L) = \frac{1}{\alpha} \int_0^\alpha V\alpha R_\alpha(L)du = \frac{1}{\alpha} \int_{1-\alpha}^1 q_L(u)du.
  \]

- **Expectile**: for \(0 < \alpha \leq 1/2\),
  \[
  e_\alpha(L) = \inf \{ m : (1-\alpha)E [(L - m)^+] \leq \alpha E [(L - m)^-], m \in \mathbb{R} \}.
  \]

The value at risk is cash invariant, monotone and positive-homogeneous, it is however not sub-additive, see \([4, 31]\). The expected shortfall is a special case of an optimized certainty equivalent, while the expectile corresponds to the shortfall risk with loss function \(\ell(x) = (1-\alpha)x^+ - \alpha x^-\) in the standard definition, \([15, 53, 55]\). Indeed, \(\ell\) is increasing, convex whenever \(\alpha \leq 1/2\) and such that \(\inf \ell(x) < 0\) whenever \(\alpha > 0\). Hence, the expectile can be seen as a scaled version of an optimized certainty equivalent, see \([8]\). In the literature, see for instance \([7, 27]\), expectile are also defined as

\[
\arg \min \left\{ (1-\alpha)E \left[ (L - m)^+ \right] + \alpha E \left[ (L - m)^- \right] \right\},
\]

for \(L\) in \(L^2\). However, due to the first order condition

\[
(1-\alpha)E[(L - e_\alpha(L))^+] = \alpha E[(L - e_\alpha(L))^-],
\]

they coincide with the present definition.

Let us recall the following known properties of expectile and expected shortfall.

\(^1\) \(R(\lambda L) = \lambda R(L)\) for every \(\lambda > 0\).
Proposition 2.1. The specteile and expected shortfall are law invariant monetary risk measures and it holds
\[
ES_a(L) = \min \left\{ m + \frac{1}{\alpha} E \left[ (L - m)^+ \right] : m \in \mathbb{R} \right\}
= q_L (1 - \alpha) + \frac{1}{\alpha} E \left[ (L - q_L (1 - \alpha))^+ \right]
= \max \left\{ E^Q [L] : Q \in \mathcal{Q}_{a,0} \right\}
\]
with optimal density
\[
\frac{dQ^*}{dP} = \frac{1}{\alpha} \left( 1 \{ L > q_L (1 - \alpha) \} + k L = q_L (1 - \alpha) \right)
\]
where \( k \) is a constant such that \( E[ dQ^* / dP] = 1 \) and
\begin{equation}
(2.1)
\end{equation}
\[
e_a (L) = \max_{a < \gamma < 1} \left\{ (1 - \gamma) ES_{a(1-\gamma)} (L) + \gamma E[L] \right\}
\begin{equation}
(2.2)
\end{equation}
\[
= \max_{a < \gamma < 1} \int_0^1 ES_a(L) \mu^\gamma (du)
\begin{equation}
(2.3)
\end{equation}
\[
= \max \left\{ E^Q[L] : Q \in \mathcal{Q}_{a/\gamma (1-\gamma), \gamma} \text{ for some } \gamma \in \left[ \frac{\alpha}{1 - \alpha}, 1 \right] \right\}
\]
with optimal density \( \frac{dQ^*}{dP} = \frac{(1 - \alpha) 1_{L > a_e} (L) + \alpha 1_{L \leq a_e} (L)}{(1 - 2\alpha) P[L > a_e (L)] + \alpha} \)
where \( \mu^\gamma = (1 - \gamma) \beta \left[ \frac{-\gamma}{1 - \gamma} \right] + \gamma \delta_1 \) is a parameterized family of distribution on \([0, 1]\).

These results can be found or derived from \([4, 7, 8, 15]\). Interestingly though, they are strongly connected through the optimized certainty equivalent from Ben-Tal and Teboulle \([8]\). For the sake of readability and further computation we expose briefly this connection.

Proposition 2.2. For a loss function \( \ell_{a,b}(x) := x^+ / a - bx^- \) where \( 0 < a < 1 \) and \( 0 \leq b \leq 1 \), the optimized certainty equivalent defined as
\begin{equation}
(2.4)
\end{equation}
\[
R_{a,b}(L) = \inf \{ m + E \left[ \ell_{a,b} (L - m) \right] : m \in \mathbb{R} \}, \ L \in L^1
\]
is a law invariant coherent risk measure such that
\begin{equation}
(2.5)
\end{equation}
\[
R_{a,b}(L) = q_L (1 - \lambda(a,b)) + E \left[ \ell_{a,b} (L - q_L (1 - \lambda(a,b))) \right]
\begin{equation}
(2.6)
\end{equation}
\[
= \frac{1}{a} \int \frac{q_L(u) du}{1 - \lambda(a,b)} + b \int_0^{1 - \lambda(a,b)} q_L(u) du
\begin{equation}
(2.7)
\end{equation}
\[
= (1 - b) ES_{\lambda(a,b)} (L) + b E[L]
\begin{equation}
(2.8)
\end{equation}
\[
= \sup \left\{ E^Q[L] : Q \in \mathcal{Q}_{a,b} \right\}
\]
where \( \lambda(a,b) = (a - ab) / (1 - ab) \). Furthermore, it holds
\begin{equation}
(2.9)
\end{equation}
\[
\inf \{ m : E \left[ \ell_{a,b} (L - m) \right] \leq 0 \} = \sup_{a \leq \gamma \leq 1/b} R_{a/\gamma, b\gamma} (L)
\begin{equation}
= \sup \left\{ E^Q[L] : Q \in \mathcal{Q}_{a/\gamma, b\gamma} \text{ for some } \gamma \in [a, 1/b] \right\}.
\]
\]

Proof. The proof in this special case can be found in \([8]\) with explicit first order conditions. The optimal \( m^* \) in \((2.4)\) satisfies
\[
\frac{1}{a} P[L > m^*] + b P[L \leq m^*] \leq 1 \leq \frac{1}{a} P[L \geq m^*] + b P[L < m^*].
\]
Rearranging, we get $P[L < m^*] \leq 1 - \ell(a, b) \leq P[L \leq m^*]$ showing that $m^* = q_L(1 - \ell(a, b))$. Plugging the optimizer into (2.4) yields (2.5). From (2.5) to (2.6) comes from the fact that $q_L \sim L$. As for (2.7)
\[
\frac{1}{a} \int_{1 - \ell(a, b)}^{1} q_L(u)du + b = \int_{0}^{1 - \ell(a, b)} q_L(u)du + bE[L] = (1 - b)ES_{\lambda(a, b)}(L) + bE[L].
\]
The Relation (2.6) implies that the optimized certainty equivalent is a law invariant and coherent risk measure. The Relation (2.8) follows from the general robust representation of optimized certainty equivalent in terms of divergences, that is
\[
\inf \{m + E[\ell_{a,b}(L - m)] : m \in \mathbb{R} \} = \sup \left\{ E^{Q} [L] - E \left[ \ell_{a,b} \left( \frac{dQ}{dP} \right) \right] : \frac{dQ}{dP} \in L^\infty \right\}
\]
see [8, Theorem 4.2], since the convex conjugate \(^2\) of optimized certainty equivalent in terms of divergences, that is coherent risk measure. The Relation (2.6) implies that the optimized certainty equivalent is a law invariant and shortfall risk \([8, \text{Section 5.2}]\) where
\[
\inf \{m : E[\ell_{a,b}(L - m)] \leq 0\} = \sup_{1/\gamma \in \text{dom}(\ell_{a,b}^*)} \inf \{m + \gamma E[\ell_{a,b}(L - m)]\}
\]
which gives the result.

**Proof of Proposition 2.7.** The relations for the expected shortfall follows directly from Proposition 2.2 by noticing that $ES_{\alpha}(L) = R_{a,b}(L)$ for $a = \alpha$ and $b = 0$. As for the relations for the expectile, they follow from (2.9) as $e_{\alpha}(L) = \inf \{m : E[\ell_{a,b}(L - m)] \leq 0\}$ for $a = 1/2(1 - \alpha)$ and $b = 2\alpha$ which fulfills the conditions of Proposition 2.2 as $0 < \alpha \leq 1/2$. As for the optimal density for expected shortfall, see Föllmer and Schied [16], McNeil [26] and for expectile it can be derived from [7, Proposition 8].

**Remark 2.3.** Relations (2.1)–(2.3) provide the link between expectile and expected shortfall. One sees in particular, that while expected shortfall is comonotone, the expectile is not. Indeed, Relation (2.2) is the Kusuoka representation which can not fulfill the assumptions of [16, Theorem 4.93, p. 260]. On the other hand, as showed in [33] while expectile is invariant under randomization, the expected shortfall is not.

3. **EXPECTILE AS A FUNCTION OF EXPECTED SHORTFALL**

Based on Relation (2.1) we provide bounds for the expectile in terms of expected shortfall in the spirit of [7, Proposition 9]. The upper bound $(1 - \alpha/(1 - \alpha))ES_{\alpha}$ is to our knowledge new, while the larger upper bound $ES_{\overline{\alpha}}$ is given in [12]. The present proof uses the relation between optimized certainty equivalent and expectile.

**Proposition 3.1.** Let $L$ be in $L^1$ with zero mean\(^3\) Then for each $0 < \beta < 1$, it holds
\[
\left(1 - \frac{\alpha}{(1 - 2\alpha)\beta + \alpha}\right) ES_{\beta}(L) \leq e_{\alpha}(L) \leq \left(1 - \frac{\alpha}{1 - \alpha}\right) ES_{\alpha}(L) \leq ES_{\overline{\alpha}}(L).
\]

\(^2\) $\ell_{a,b}^*(x) = \sup \{ x \cdot y - \ell_{a,b}(y) : y \in \mathbb{R}^d \}$.

\(^3\) Due to translation invariance, in the case where $E[L] \neq 0$, we get
\[
\left(1 - \frac{\alpha}{(1 - 2\alpha)\beta + \alpha}\right) ES_{\beta}(L) + \frac{\alpha}{(1 - 2\alpha)\beta + \alpha}E[L] \leq e_{\alpha}(L) \leq \left(1 - \frac{\alpha}{1 - \alpha}\right) ES_{\alpha}(L) + \frac{\alpha}{1 - \alpha}E[L] \leq ES_{\overline{\alpha}}(L).
\]
Conversely, suppose \( \beta \). If we set \( \beta \), let \( \lambda \). Since \( ab = \alpha/(1 - \alpha) \) and \( \lambda(ab, ab) = \alpha \), as a result of \( 2.7 \) the right hand side inequalities hold. On the other hand, \( e_\alpha(L) \geq R_{ab,ab}(L) \) and therefore from \( 2.7 \) it follows that \( e_\alpha(L) \geq (1 - \gamma)ES_{\lambda(\alpha/(\gamma(1-\alpha)), \gamma)}(L) \) for every \( \alpha < \gamma < 1 \). Solving \( \gamma \) for \( \lambda(\alpha/(\gamma(1-\alpha)), \gamma) = \beta \) yields the result. 

If we set \( \beta = \alpha \), the lower bound corresponds to the one stated in \( 7 \) Proposition 9, that is

\[
(1 - \frac{1}{2(1 - \alpha)}) ES_\alpha(L) \leq e_\alpha(L).
\]

As for the lower bound, from \( 2.4 \), it is immediate that there exists \( \beta^* \) satisfying the equality in the above proposition. Following related results \( 20, 32, 33 \), we characterize this optimal \( \beta^* \) in the general case and formulate expectile as a convex combinations of expected shortfalls. The following proposition also characterize equality \( 3.1 \) in terms of value at risk.

**Proposition 3.2.** Let \( L \) in \( L^1 \) be given. It holds

\[
e_\alpha(L) = \left(1 - \frac{\alpha}{(1 - 2\alpha)\beta^* + \alpha}\right) ES_{\beta^*}(L) + \frac{\alpha}{(1 - 2\alpha)\beta^* + \alpha} E[L]
\]

where \( \beta^* = P[L > e_\alpha(L)] \). In particular, if \( e_\alpha(L) \) is a \( 1 - \alpha \) quantile, that is \( q_L(1 - \alpha) \leq e_\alpha(L) \leq q^*_L(1 - \alpha) \), then

\[
e_\alpha(L) = \left(1 - \frac{1}{2(1 - \alpha)}\right) ES_\alpha(L) + \frac{1}{2(1 - \alpha)} E[L].
\]

If \( F_L \) is further strictly increasing, the converse also holds.

**Remark 3.3.** Note that Relation \( 3.2 \) shows that the inequality \( 3.1 \) provided in \( 7 \) is an equality if and only if the expectile is a value at risk, provided that \( F_L \) is strictly increasing. This is the case for instance when \( q_L(\alpha) = (2\alpha - 1)/\sqrt{\alpha(1 - \alpha)} \), see Koenker \( 27 \).

**Proof.** Let \( a = (\alpha + (1 - 2\alpha)\beta^*)/(1 - \alpha) \) and \( b = \alpha/(\alpha + (1 - 2\alpha)\beta^*) \). The Relations \( 2.7 \) and \( 2.8 \) in Proposition \( 2.2 \) gives

\[
\left(1 - \frac{\alpha}{(1 - 2\alpha)\beta^* + \alpha}\right) ES_{\beta^*}(L) + \frac{\alpha}{(1 - 2\alpha)\beta^* + \alpha} E[L] = R_{a,b}(L) = \sup_{Q \in Q_{a,b}} E^{Q}[L].
\]

Let

\[
dQ^* \over dP = \frac{(1 - \alpha)1_{L > e_\alpha(L)} + \alpha 1_{L \leq e_\alpha(L)}}{(1 - 2\alpha)P[L > e_\alpha(L)] + \alpha}
\]

be the optimal density of \( e_\alpha(L) \). From the choices of \( a \) and \( b \), it follows that \( Q^* \in Q_{a,b} \).

On the other hand by Proposition \( 3.1 \), \( e_\alpha(L) \geq R_{a,b}(L) \). It follows that \( R_{a,b}(L) = e_\alpha(L) \). When \( e_\alpha(L) \) is a \( 1 - \alpha \) quantile of \( L \), as a result of \( 11 \) Proposition 4.2, it follows that

\[
ES_\alpha(L) = e_\alpha(L) + \frac{E[(L - e_\alpha(L))^+]}{\alpha}.
\]

The first order condition can be written as

\[
\frac{E[(L - e_\alpha(L))^+]}{\alpha} = e_\alpha(L) - E[L] 1 - 2\alpha.
\]

Therefore, \( ES_\alpha(L) = e_\alpha(L) + (e_\alpha(L) - E[L])/(1 - 2\alpha) \). Solving for \( e_\alpha(L) \) yields Relation \( 3.2 \).

Conversely, suppose \( F_L \) is strictly increasing and Relation \( 3.2 \) holds. From the first order condition, it follows that \( E[L] = e_\alpha(L) - (1 - 2\alpha)E[(L - e_\alpha(L))^+]/\alpha \). Hence,

\[
e_\alpha(L) = \left(1 - \frac{1}{2(1 - \alpha)}\right) ES_\alpha(L) + \frac{e_\alpha(L)}{2(1 - \alpha)} - \frac{(1 - 2\alpha)E[(L - e_\alpha(L))^+]}{2\alpha(1 - \alpha)}.
\]
There exist

Suppose within the class of expected shortfall in the sense stated in the following proposition. On the one hand, let \( \phi \) is the smallest risk measure of the form

\[
\alpha/\beta \geq \phi(L) = \min \left\{ t \mid \mathbb{P}(L > t) \leq \alpha \right\}
\]

However, the upper bound \( q_L \) is also coherent and comonotone, it follows

\[
s_{\alpha}(L) = q_L(1 - \beta^*) \text{ and } \beta^* \text{ solves}
\]

\[
q_L(1 - \beta^*) = \left( 1 - \frac{\alpha}{1 - \alpha} \right) ES_{\beta^*}(L) + \frac{\alpha}{1 - \alpha} E[L].
\]

If \( F_L \) is further continuous, then \( \beta^* \) is a unique solution for this equation.

We now turn to the question of the upper bound. \[12\] and \[35\] provide an upper bound for expectile in terms of distortion risk measure. The following theorem is a result of \[12\] Theorem 5 and 6.

**Theorem 3.5 (Delbaen \[12\]).** Suppose \( (\Omega, \mathcal{F}, P) \) is non-atomic. For each \( L \) in \( L^1 \), it holds

\[
e_{\alpha}(L) \leq R_{\phi}(L) := \int_{(0, 1]} \phi'(t) q_L(1 - t) dt
\]

where \( \phi : [0, 1] \to [0, 1] \), given by \( \phi(t) = (1 - \alpha)t / ((1 - 2\alpha)t + \alpha) \) is a concave distortion function. Moreover, \( R_{\phi} \) is the smallest law-invariant coherent and comonotonic risk measure dominating \( e_{\alpha} \). In particular, \( e_{\alpha}(1_A) = R_{\phi}(1_A) = \phi(P[A]) \) for each \( A \in \mathcal{F} \).

Since the upper bound given in Proposition\[3,4\] is also coherent and comonotone, it follows in particular that

\[
R_{\phi}(L) \leq \left( 1 - \frac{\alpha}{1 - \alpha} \right) ES_{\alpha}(L) + \frac{\alpha}{1 - \alpha} E[L].
\]

However, the upper bound \( (1 - \alpha/(1 - \alpha))ES_{\alpha}(L) + \alpha/(1 - \alpha)E[L] \) is the least one within the class of expected shortfall in the sense stated in the following proposition.

**Proposition 3.6.** Suppose \( (\Omega, \mathcal{F}, P) \) be non-atomic. Then

\[
(1 - \lambda) ES_{\beta}(L) + \lambda ES_{\delta}(L) = R_{\phi_{\lambda, \beta, \delta}}(L) := \int_{(0, 1]} \phi_{\lambda, \beta, \delta}(t) q_L(1 - t) dt
\]

is the smallest risk measure of the form \((1 - \lambda) ES_{\beta}(L) + \lambda ES_{\delta}(L) \) with \( 0 \leq \lambda \leq 1, 0 < \beta \leq 1 \) and \( 0 < \delta \leq 1 \) uniformly dominating \( e_{\alpha}(L) \) for \( L \) in \( L^1 \).

**Proof.** Note that

\[
(1 - \lambda) ES_{\beta}(L) + \lambda ES_{\delta}(L) = R_{\phi_{\lambda, \beta, \delta}}(L) := \int_{(0, 1]} \phi_{\lambda, \beta, \delta}(t) q_L(1 - t) dt
\]

for the concave distortion function \( \phi_{\lambda, \beta, \delta}(t) = (1 - \lambda)(t/\beta \land 1) + \lambda(t/\delta \land 1) \) for \( t \) in \([0, 1]\) which is continuous and strictly increasing with \( \phi_{\lambda, \beta, \delta}(0) = 0 \) and \( \phi_{\lambda, \beta, \delta}(1) = 1 \). For \( \beta = \alpha, \lambda = \alpha/(1 - \alpha) \) and \( \delta = 1 \), we have

\[
R_{\phi_{\lambda, \beta, \delta}}(L) = \left( 1 - \frac{\alpha}{1 - \alpha} \right) ES_{\alpha}(L) + \frac{\alpha}{1 - \alpha} E[L].
\]

On the one hand, let \( \phi(t^*) > \phi_{\lambda, \beta, \delta}(t^*) \) for some \( t^* \) in \((0, 1)\). Since \((\Omega, \mathcal{F})\) is non-atomic, there exist \( A \in \mathcal{F} \) such that \( P[A] = t^* \). Following \[16\] and \[29\], we get \( R_{\phi_{\lambda, \beta, \delta}}(1_A) < \phi(P[A]) = e_{\alpha}(1_A) \) and therefore \( R_{\phi_{\lambda, \beta, \delta}} \) can not dominate \( e_{\alpha} \). Hence, for every \( 0 \leq \lambda \leq 1, 0 < \beta \leq 1 \) and \( 0 < \delta \leq 1 \), we have \( R_{\phi_{\lambda, \beta, \delta}} \) dominate \( e_{\alpha} \) only if \( \phi \leq \phi_{\lambda, \beta, \delta} \).
On the other hand, for every $0 \leq \lambda \leq 1$, $0 < \beta \leq 1$ and $0 < \delta \leq 1$ such that $\varphi \leq \varphi_{\lambda,\beta,\delta}$, it holds
\[
R_{\varphi_{\lambda,\beta,\delta}}(L) \geq e_{\alpha}(L).
\]
In this case, $\varphi_{\alpha/(1-\alpha),\alpha,1} \leq \varphi_{\lambda,\beta,\delta}$. Indeed, since $\varphi_{\alpha/(1-\alpha),\alpha,1}$ is tangent to $\varphi$ at the point $(0,0)$ and $(1,1)$, and $\varphi$ is strictly concave, $\varphi_{\lambda,\beta,\delta}(t) < \varphi_{\alpha/(1-\alpha),\alpha,1}(t)$ for some $t$ in $(0,1)$ implies there exist $t^*$ in $(0,1)$ such that $\varphi(t^*) > \varphi_{\lambda,\beta,\delta}(t^*)$ for some $t^*$ in $(0,1)$, see Figure 1. By a similar argument, it follows that $R_{\varphi_{\lambda,\beta,\delta}}$ can not dominate $e_{\alpha}$. Therefore, $R_{\varphi_{\lambda,\beta,\delta}}$ is the better one when $\lambda = \alpha/(1-\alpha)$, $\beta = \alpha$ and $\delta = 1$. 

\[\text{Figure 1. Graph of } \varphi \text{ and optimal } \varphi_{\lambda,\beta,\delta} \text{ for } \alpha = 6\%.
\]

4. EXPECTILE VERSUS EXPECTED SHORTFALL: ASYMPTOTIC COMPARISON

For a given risk level $\alpha$, expectile and value at risk are less conservative than expected shortfall, that is, $e_{\alpha}(L) \leq ES_{\alpha}(L)$ and $q_{L}(1-\alpha) \leq ES_{\alpha}(L)$. Expectile can be less or more conservative than value at risk depending on the considered loss profile, see [5].

When $F_L$ is in the maximum domain of attraction of an extreme value distribution function, [5] and [25] give asymptotic comparison between value at risk and expectile. [30] also provides an asymptotic comparison between expected shortfall and value at risk. Following these results, we are considering asymptotic comparison of expectile and expected shortfall, and the asymptotic behavior of $\beta^*$ as the risk level $\alpha$ goes to 0. Throughout this section we consider a loss profile $L$ with zero mean. Furthermore, for ease of notations, throughout this section we use the notation
\[
e_{\alpha} := e_{\alpha}(L), \quad q_{\alpha} := q_{L}(1-\alpha) \quad ES_{\alpha} := ES_{\alpha}[L]
\]
The asymptotic comparison uses techniques from extreme value theory. We say $F_L$ is in the maximum domain of attraction of an extreme value distribution function $H$, denoted by $MDA(H)$, if
\[
\lim_{n \to \infty} F_L^n(c_n x + d_n) = H(x)
\]
for some constants $c_n > 0$ and $d_n \in \mathbb{R}, n \in \{1, 2, \ldots\}$. It is well known that extreme value distribution $H$ belongs to either one of the following three categories: Weibull$^{4}(\psi, \eta)$,

$^{4}\psi_{\eta}(x) = \exp(-(x)\eta)$ for $x < 0.$
Gumbel\(^5\)(\(\Lambda\)) or Fréchet\(^6\)(\(\Phi_\eta\)), where \(\eta > 0\), see \([5, 23, 26, 30]\) for more discussion in the present context.

Let \(U(t) := q_{1,t}\) for \(t > 1\). The condition that \(F_L\) belongs to the maximum domain of attraction can be equivalently given by the extended regular variation of \(U\). Recall that a measurable function \(f : \mathbb{R}_+ \to \mathbb{R}\) is said to be of extended regular variation with parameter \(\eta \in \mathbb{R}\), denoted by \(f \in ERV_\eta\), if there exist a function \(a : \mathbb{R}_+ \to \mathbb{R}_+\) such that for each \(x > 0\),

\[
\lim_{t \to \infty} \frac{f(tx) - f(t)}{a(t)} = \begin{cases} 
\frac{(n-1)}{\eta} \ln x, & \eta \neq 0 \\
\alpha, & \eta = 0.
\end{cases}
\]

It is known that \(F_L\) is in the maximum domain of attraction of Fréchet type MD\(A(\Phi_\eta)\), with \(\eta > 0\) if and only if \(U \in ERV_\eta\). \(F_L\) is in the maximum domain of attraction of Weibull type MD\(A(\Psi_\eta)\), with \(\eta > 0\) if and only if \(U \in ERV_{\frac{1}{\eta}}\). Finally, \(F_L\) is in the maximum domain of attraction of Gumbel type MD\(A(\Lambda)\) if and only if \(U \in ERV_0\), see \([11, Theorem 1.6]\) for instance.

A direct application of \([5, 7, 23]\) yields

**Proposition 4.1.** For \(F_L\) in the maximum domain of attraction of Fréchet type MD\(A(\Phi_\eta)\) with \(\eta > 1\), as the risk level \(\alpha\) goes to 0, we have that

\[
\beta^* \sim \alpha(\eta - 1), \quad \text{and} \quad e_\alpha \sim \frac{(\eta - 1)^{\frac{1}{\eta-1}}}{\eta} ES_\alpha.
\]

**Proof.** The relation between \(\beta^*\) and \(\alpha\) is given in \([7, Theorem 11]\) and from \([5, Proposition 2.3]\) and \([23, Theorem 3.4]\) we have

\[
(4.1) \quad e_\alpha \sim (\eta - 1)^{\frac{1}{\eta}} q_\alpha \quad \text{and} \quad ES_\alpha \sim \frac{\eta}{\eta - 1} q_\alpha,
\]

which yields the result. \(\square\)

Beyond this first order expansion, a second order one is useful to determine the rate of convergence. In order to do so, we impose a second-order regular variation condition on \(F_L\). A measurable function \(f : \mathbb{R} \to \mathbb{R}\) is said to be of regular variation with parameter \(\eta \in \mathbb{R}\), denoted by \(f \in RV_\eta\), if \(\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\eta\), for each \(x \in \mathbb{R}\). A regularly varying function \(f : \mathbb{R} \to \mathbb{R}\) which is eventually positive is said to be of second-order regular variation with first-order parameter \(\eta \in \mathbb{R}\) and second-order parameter \(\rho \leq 0\), denoted by \(f \in 2RV_{\eta,\rho}\), if \(f \in RV_\eta\) and there exists a measurable function \(A(t)\) which does not change sign eventually and converges to 0 as \(t\) goes to infinity such that, for each \(x > 0\)

\[
\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\eta \frac{\rho}{A(t)} = \begin{cases} 
x^\eta \frac{\rho - \frac{1}{\rho}}{x^{\frac{1}{\rho}}} & \text{if } \rho \neq 0 \\
x^\eta \ln x & \text{if } \rho = 0
\end{cases}
\]

see, \([11, 22]\) for further properties of regular variations. The function \(A\) is in \(RV_\rho\), see \([11, Theorem 2.3.3]\), and is called the auxiliary function for \(f\).

**Proposition 4.2.** For \(F_L\) in the maximum domain of attraction of Fréchet type MD\(A(\Phi_\eta)\) such that \(1 - F_L \in 2RV_{-\eta,\rho}\) with \(\eta > 1\), \(\rho \leq 0\) and auxiliary function \(A\), as the risk level \(\alpha\) goes to 0, it holds

\[
e_\alpha = \frac{(\eta - 1)^{\frac{1}{\eta-1}} (1 - 2\alpha)^{\frac{1}{2}}}{\eta} (1 - C_{\eta,\rho} A(q_\alpha) + o(A(q_\alpha)))]
\]

---

\(^5\)\(\Lambda(x) = \exp(-e^{-x})\) for \(x \in \mathbb{R}\).

\(^6\)\(\Phi_\eta(x) = \exp(-x^{-\eta})\) for \(x > 0\).
The first order condition given by Relation (3.3) can also be written as

\[
\frac{\beta^*}{\alpha} = \frac{\eta - 1}{1 - 2\alpha} \left[ 1 - \frac{(\eta - 1)^{-\frac{2}{\alpha}}}{\eta - \rho - 1} A(q_\alpha) + o(A(q_\alpha)) \right].
\]

Furthermore,

\[
\frac{\beta^*}{\alpha} = \frac{\eta - 1}{1 - 2\alpha} \left[ 1 - \frac{(\eta - 1)^{-\frac{2}{\alpha}}}{\eta - \rho - 1} A(q_\alpha) + o(A(q_\alpha)) \right].
\]

**Proof.** Let \(1 - F_L\) be in \(2RV_{-\eta, \rho}\) with \(\eta > 1, \rho \leq 0\) and auxiliary function \(A\). According to [23, Theorem 3.1], we get

\[
e_\alpha = \left( \frac{1 - 2\alpha}{\eta - 1} \right) \frac{1}{\eta} \left[ q_\alpha \left[ 1 + B_{\eta, \rho} A(q_\alpha) + o(A(q_\alpha)) \right] \right]
\]

where

\[
B_{\eta, \rho} = \left\{ \begin{array}{ll}
\frac{1}{\eta \rho} \left[ \frac{(\eta - 1)^{1 - \frac{2}{\alpha}}}{\eta - \rho - 1} - 1 \right] & \rho \neq 0 \\
-\frac{1}{\eta} \ln(\eta - 1) & \rho = 0
\end{array} \right.
\]

The condition \(1 - F_L \in 2RV_{-\eta, \rho}\) with auxiliary function \(A\) is equivalent to \(U\) is in \(2RV_{\frac{\eta}{\rho}, \frac{1}{\rho}}\) with auxiliary function \(\eta^{-2} A(U)\), see [11, Theorem 2.3.9]. Hence, by [23, Theorem 4.5], we get

\[
ES_\alpha = \frac{\eta}{\eta - 1} q_\alpha \left[ 1 + \frac{1}{\eta(\eta - \rho - 1)} A(q_\alpha) + o(A(q_\alpha)) \right].
\]

It follows that

\[
\frac{e_\alpha}{ES_\alpha} = \frac{(\eta - 1)^{\frac{n-1}{\alpha}} (1 - 2\alpha)^{\frac{2}{\alpha}}}{\eta} \left[ 1 + \frac{1}{\eta(\eta - \rho - 1)} A(q_\alpha) + o(A(q_\alpha)) \right]
\]

\[
= \frac{(\eta - 1)^{\frac{n-1}{\alpha}} (1 - 2\alpha)^{\frac{2}{\alpha}}}{\eta} \left[ 1 + \left( B_{\eta, \rho} - \frac{1}{\eta(\eta - \rho - 1)} \right) A(q_\alpha) + o(A(q_\alpha)) \right]
\]

which gives the required result for \(e_\alpha/ES_\alpha\).

As for the ratio of \(\beta^*/\alpha\), from [24, Proof of Theorem 3.1], we have

\[
E[(L - e_\alpha)^+] = \frac{\beta^* e_\alpha}{\eta - 1} \left[ 1 + \frac{1}{\eta - \rho - 1} A(e_\alpha) + o(A(e_\alpha)) \right].
\]

From relation (3.1), we have \(e_\alpha \sim (\eta - 1)^{-1/\alpha} q_\alpha\). Since \(A \in RV_p\), a straightforward application of [11, Proposition B.1.10] yields \(A(e_\alpha) = (\eta - 1)^{-\frac{2}{\alpha}} A(q_\alpha) + o(A(q_\alpha))\) showing that

\[
E[(L - e_\alpha)^+] = \frac{\beta^* e_\alpha}{\eta - 1} \left[ 1 + \frac{(\eta - 1)^{-\frac{2}{\alpha}}}{\eta - \rho - 1} A(q_\alpha) + o(A(q_\alpha)) \right].
\]

The first order condition given by Relation (3.3) can also be written as

\[
1 = \frac{(1 - 2\alpha)}{\alpha} E[(L - e_\alpha)^+].
\]

Combining the last two equations gives the required result for \(\beta^*/\alpha\). \(\Box\)

**Remark 4.3.** When \(E[L] \neq 0\), for \(\tilde{L} = L - E[L]\), by [24], we have that \(1 - F_{\tilde{L}}\) is in \(2RV_{-\eta, (1 - 1/\alpha)}\) with auxiliary function \(A^*(x) = \eta E[L]/x + A(x)\).

We now turn to the asymptotic comparison between expectile and expected shortfall for \(F_L\) in the domain of attraction of Weibull and Gumbel type. A direct application of [23, 24] yields
Proposition 4.4. Let \( \hat{x} := \sup \{ x : F_L(x) < 1 \} \). For \( F_L \) in the maximum domain of attraction of Weibull type \( MDA(\Psi_\eta) \), as \( \alpha \) goes to 0, it holds
\[
\alpha = o(\beta^*) \quad \text{and} \quad \frac{\hat{x} - ES_\alpha}{x - e_\alpha} = o(1).
\]

Proof. The first order condition given by (3.3) can also be re-written as
\[
\frac{\hat{x} - E[L]}{x - e_\alpha} - 1 = \frac{(1 - 2\alpha) E[(L - e_\alpha)^+]}{\alpha (x - e_\alpha)}.
\]
Since \( e_\alpha \rightarrow \hat{x} \) the first order condition becomes
\[
\frac{\hat{x} - E[L]}{x - e_\alpha} \sim \frac{E[(L - e_\alpha)^+]}{\alpha (x - e_\alpha)}.
\]
From [23] Lemma 3.2 and Remark 3.3] we have
\[
\frac{E[(L - x)^+]}{(\hat{x} - x)(1 - F_L(x))} \sim \frac{1}{\eta + 1}.
\]
Hence, as \( \alpha \) goes to 0, if follows that
\[
\frac{E[(L - e_\alpha)^+]}{(\hat{x} - e_\alpha)} \sim \frac{\beta^*}{\eta + 1}
\]
implying \( \alpha = o(\beta^*) \). From [23] Theorem 3.4] and [23] Proposition 3.3] we have
\[
\hat{x} - ES_\alpha \sim \frac{\eta}{\eta + 1}(\hat{x} - q_\alpha) \quad \text{and} \quad \frac{\hat{x} - q_\alpha}{x - e_\alpha} = o(1),
\]
which yields the desired asymptotic relationship between expectile and expected shortfall. \( \square \)

Proposition 4.5. For \( F_L \) in the maximum domain of attraction of Weibull type \( MDA(\Psi_\eta) \) such that \( P[L > 0] > 0 \) and \( 1 - F_L(1/\cdot) \in 2RV_{-\eta,\rho} \) with \( \eta > 0, \rho < 0 \) and auxiliary function \( A \), as the risk level \( \alpha \) goes to 0, it holds
\[
\frac{\hat{x} - ES_\alpha}{x - e_\alpha} = \frac{\eta((1 - 2\alpha)(\hat{x} - q_\alpha))^{\frac{1}{\eta + 1}}}{(\eta + 1)C_\eta}
\]
\[
\left[ 1 + \left( \frac{C_\eta(\hat{x} - q_\alpha)^{\frac{1}{\eta + 1}}}{(\eta + 1)\hat{x}} + \frac{C_\eta^{\rho}}{\rho (\eta - \rho + 1)} A_0(q_\alpha) \right) (1 + o(1)) \right]
\]
where
\[
C_\eta = (\hat{x}(\eta + 1))^{\frac{1}{\eta + 1}} \quad \text{and} \quad A_0(q_\alpha) = A \left( (\hat{x} - q_\alpha)^{-\frac{\rho}{\eta + 1}} \right).
\]
Furthermore,
\[
\frac{\beta^*}{\alpha} \sim \left( \frac{\hat{x}(\eta + 1)}{\hat{x} - q_\alpha} \right)^{\frac{1}{\eta + 1}}.
\]
Proof. Let \( 1 - F_L(\hat{x} - 1/\cdot) \) be in \( 2RV_{-\eta,\rho} \) with \( \eta > 0, \rho < 0 \) and auxiliary function \( A \). According to [23] Proposition 2.4], as \( x \) goes to \( \infty \) we have \( 1 - F_L(\hat{x} - 1/x) \sim c x^{-\eta} \) for some \( c > 0 \). Hence, by [23] Proposition 3.3], it holds
\[
\hat{x} - e_\alpha \sim C_\eta(\hat{x} - q_\alpha)^{\frac{1}{\eta + 1}}.
\]
In particular, for the same reason as in the proof of Proposition 4.2, it follows that
\[
A \left( \frac{1}{\hat{x} - e_\alpha(L)} \right) \sim C_\eta^{\rho} A_0(q_\alpha) \quad \text{and} \quad A \left( \frac{1}{\hat{x} - q_\alpha} \right) = o \left( A \left( \frac{1}{\hat{x} - e_\alpha} \right) \right).
\]
The regularity condition on \(1 - F_L(\hat{x} - 1/\cdot)\) implies that \(\hat{x} - U \in 2RV_{\frac{1}{\hat{x}}, \eta}\) with auxiliary function asymptotically equivalent to \(-\eta^{-2}A(1/(\hat{x} - U))\) as \(t\) goes to \(\infty\), see [22]. Hence, using Relation (4.4), and [30] Theorem 4.5] gives

\[
\hat{x} - ES_\alpha = \frac{\eta(\hat{x} - q_\alpha)}{\eta + 1} \left[ 1 - \frac{A\left(\frac{1}{\xi q_\alpha}\right)}{\eta(\eta - \rho + 1)}(1 + o(1)) \right] = \frac{\eta(\hat{x} - q_\alpha)}{\eta + 1} \left[ 1 + o\left(A\left(\frac{1}{\hat{x} - e_\alpha}\right)\right)\right].
\]

Using Relation (4.4) and [25 Relation 3.14 and 3.17], we also get that

\[
e_\alpha = \left(1 - 2\alpha\right)(\hat{x} - e_\alpha)^{\eta + 1}\frac{\eta + 1}{\rho(\eta - \rho + 1)} A\left(\frac{1}{\hat{x} - e_\alpha}\right) + o\left(A\left(\frac{1}{\hat{x} - e_\alpha}\right)\right)\]

Substituting the left hand side of Equation (4.6) with (4.3) and solving for \(\hat{x} - e_\alpha\) gives

\[
(1 - 2\alpha)C^\eta(\hat{x} - q_\alpha)^{\eta + 1} = \left[ 1 - \left(\frac{C(\hat{x} - q_\alpha)^{\eta + 1}}{\hat{x} \eta + 1} \rho(\eta - \rho + 1) A\left(\frac{1}{\hat{x} - e_\alpha}\right)\right)(1 + o(1)) \right] \]

where \(C = ((\hat{x} - E[L])(\eta + 1))^{\frac{1}{\eta + 1}}\).

Combining these Relations together with Relation (4.3) yields the result. 

**Remark 4.6.** When \(E[L] \neq 0\), the proof of Proposition 4.5 allow to derive the expression

\[
\frac{\hat{x} - ES_\alpha}{\hat{x} - e_\alpha} = \frac{\eta((1 - 2\alpha)(\hat{x} - q_\alpha))^{\eta + 1}}{(\eta + 1)\tilde{C}_\eta} \left[ 1 + \left(\frac{\tilde{C}_\eta(\hat{x} - q_\alpha)^{\eta + 1}}{\eta + 1}(\hat{x} - E[L]) + \frac{\tilde{C}_\eta - \rho}{\rho(\eta - \rho + 1)} A_0(q_\alpha)\right)(1 + o(1)) \right]
\]

where \(\tilde{C}_\eta = ((\hat{x} - E[L])(\eta + 1))^{\frac{1}{\eta + 1}}\).

A direct combination of results by [23][30] and [5] yields the following proposition.

**Proposition 4.7.** For \(F_L\) in the domain of attraction of Gumbel type \(MDA(\Lambda)\), as the risk level \(\alpha\) goes to 0, it holds \(\alpha = o(\beta^*)\). If further \(F_L(x) = 1 - \exp(-x^\tau L(x))\) with \(L \in RV_0\) and \(\tau > 0\), then

\[
\ln(e_\alpha) \sim \ln(ES_\alpha).
\]

Moreover, if

\[
\lim_{x \to \infty} \left(\frac{L(cx)}{L(x)} - 1\right) = 0
\]

for some constant \(c > 0\), then

\[
e_\alpha \sim ES_\alpha.
\]
Proof. From (23) Proposition 3.6], we have $1 - F_L(q_\alpha) = o(\beta^*)$. For $F_L$ in $MDA(\Lambda)$, it is known that $\alpha \sim 1 - F_L(q_\alpha)$, see (30) for instance. Therefore, $\alpha = o(\beta^*)$. As for the relationship between expectile and expected shortfall it is a direct consequence of (23) [30] and [5] Proposition 2.4]

For $F_L$ in the domain of attraction of Fréchet type $MDA(\Phi_\eta)$ with $\eta > 2$, it holds that

$$
\lim_{\alpha \downarrow 0} \frac{e_\alpha}{ES_\alpha} < 1 \quad \text{and} \quad \left(1 - \frac{\alpha}{1 - \alpha}\right) ES_\alpha + \frac{\alpha}{1 - \alpha} E[L] \sim ES_\alpha.
$$

In this particular case, the bound is not asymptotically equivalent to $e_\alpha$. However, for $F_L$ in the Gumbel type $MDA(\Lambda)$ with condition (4.3), the bound becomes asymptotically equivalent to $e_\alpha$. See figure [3] for graphical illustrations.

5. Examples

For many common distributions explicit or semi-explicit expressions for both the quantile and expected shortfall are known. Taking advantage of relation between expectile, quantile and expected shortfall, first we compute $\beta^*$ using Relation (3.2) and then provide an expression of expectile as a function of expected shortfall. We provide an explicit expression for uniform and exponential distributions and semi explicit expression for normal, logistic, Pareto, generalized Pareto and Student $t$.

Example 5.1 (Uniform). Let $F_L(x) = x$ for $x$ in $[0, 1]$. Then $q_L(1 - \beta^*) = 1 - \beta^*$ and $ES_{\beta^*}(L) = 1 - \beta^*/2$. The Relation (3.4) is simplified to $(1 - 2\alpha)\beta^2 + 2\alpha\beta^* - \alpha = 0$. Solving for $\beta^*$ gives $\beta^* = (\sqrt{\alpha(1 - \alpha)} - \alpha)/(1 - 2\alpha)$. Hence,

$$
e_\alpha(L) = 1 - \frac{\sqrt{\alpha(1 - \alpha)} - \alpha}{1 - 2\alpha}.
$$

A similar expression for $e_\alpha$ can be found in (5). $F_L$ belongs to Weibull type $MDA(\Psi_\eta)$, with $\eta = 1$, see (19) for instance. Moreover, $1 - ES_\alpha(L) = o(1 - e_\alpha(L))$.

Example 5.2 (Beta). For $a > 0$, let $F_L(x) = x^a$ with $x$ in $[0, 1]$. Then

$$
q_L(1 - \beta^*) = (1 - \beta^*)^{1/a}, \quad E[L] = \frac{a}{a + 1} \quad \text{and} \quad ES_{\beta^*}(L) = \frac{a}{\beta^*(a + 1)} \left(1 - (1 - \beta^*)^{2/a + 1}\right)
$$

Relation (3.4) gives the optimal $\beta^*$ solving

$$
\frac{1 - \alpha}{1 - 2\alpha} = (1 - \beta^*)^{1/2} \left(1 + \frac{\beta^*}{a} + \frac{\alpha(a + 1)}{a(1 - 2\alpha)}\right).
$$

Hence,

$$
e_\alpha(L) = (1 - \beta^*)^{1/a}.
$$

Furthermore, $x = 1$ and $F_L$ belongs to Weibull type $MDA(\Psi_\eta)$, with $\eta = 1$. If $a \neq 1$, then $1 - F_L(1/x)\in 2RV_{-1, -1}$ with auxiliary function $A(x) = (a - 1)x^{-1/2}$, see (23) [25] for instance. By Remark 4.6 we have

$$
\frac{1 - ES_\alpha(L)}{1 - e_\alpha(L)} = \frac{\sqrt{(a + 1)(1 - 2\alpha)(1 - (1 - \alpha)^{1/a})}}{2\sqrt{2}} \left(1 + \frac{a + 2}{3} \sqrt{\frac{2(1 - (1 - \alpha)^{1/a})}{a + 1}(1 + o(1))}\right).
$$
Example 5.3 (Exponential). Let $F_L(x) = 1 - \exp(-x)$ for $x \geq 0$. Then $ES_{\beta^*}(L) = 1 - \ln \beta^*$ and $q_L(1 - \beta^*) = -\ln \beta^*$, see [2][28] for instance. The Relation (3.4) becomes $(1 - 2\alpha)\beta^* = \alpha (-1 - \ln \beta^*)$. For $x := -1 - \ln \beta^*$, it holds $xe^x = (1 - 2\alpha)/(-\alpha e)$. Thus, $x = W((1 - 2\alpha)/(-\alpha e))$ and $\beta^* = \exp(-x - 1)$ where, $W$ is Lambert function. Therefore, 

$$e_{\alpha}(L) = 1 + W\left(\frac{1 - 2\alpha}{\alpha e}\right).$$

A similar expression for $e_{\alpha}$ can also be found in [5]. It is also known that $F_L$ belongs to Gumbel type MDA($\Lambda$) and satisfy condition (4.8). Hence, $e_{\alpha}(L) \sim ES_{\alpha}(L)$.

Example 5.4 (Normal). Let $L \sim \text{Normal}(0, 1)$. Following [26][28], we get $ES_{\beta^*}(L) = (1/\beta^*)\phi(\Phi^{-1}(1 - \beta^*))$, where $\phi$ and $\Phi$ are the standard normal density and cumulative distribution, respectively. Relation (3.4) yields an optimal $\beta^*$ solving

$$\Phi^{-1}(1 - \beta^*) = \frac{1 - 2\alpha}{(1 - 2\alpha)\beta^* + \alpha} \phi(\Phi^{-1}(1 - \beta^*)).$$

Therefore,

$$e_{\alpha}(L) = \frac{1 - 2\alpha}{(1 - 2\alpha)\beta^* + \alpha} \phi(\Phi^{-1}(1 - \beta^*)).$$

$F_L$ belongs to Gumbel type MDA($\Lambda$), see [11] Example 1.1.7. Moreover, $e_{\alpha}(L) \sim \sqrt{-2\ln \alpha} \sim q_L(1 - \alpha) \sim ES_{\alpha}(L)$, see [5].

Example 5.5 (Logistic). Let $F_L(x) = (1 + \exp(-x))^{-1}$ for $x$ in $\mathbb{R}$. From [28], we have $q_L(1 - \beta^*) = \ln((1 - \beta^*)/\beta^*)$ and $ES_{\beta^*}(L) = -((1 - \beta^*)/\beta^*)\ln(1 - \beta^*) - \ln \beta^*$. Relation (3.4) provides an optimal $\beta^*$ solving $(1 - \alpha)\ln(1 - \beta^*) = \alpha \ln(\beta^*)$. Hence,

$$e_{\alpha}(L) = \frac{-(1 - 2\alpha)\ln \beta^*}{1 - \alpha}.$$

Since $U(t) = \ln (t - 1)$, it follows that $U \in ERV_0$. Hence, $F_L$ belongs to Gumbel type MDA($\Lambda$). From [5], we also have $e_{\alpha}(L) \sim \ln \alpha$. Since $ES_{\alpha}(L) \sim -\ln \alpha$, it holds $e_{\alpha}(L) \sim ES_{\alpha}(L)$.

---

\footnote{W is a function such that $xe^x = y$ if and only if $x = W(y)$.}
FIGURE 3. Graph of the ratio $e^\alpha / ES^\alpha$ for uniform, generalized Pareto with $\xi = -0.5$, exponential and standard normal distributions.

Example 5.6 (Pareto). For $a > 1$ and $x \geq 0$, let $F_L(x) = 1 - (1/(x + 1))^a$. It follows that $q_L(1 - \beta^*) = \beta^{* -1/a} - 1$, $E[L] = 1/(a - 1)$ and $ES_{\beta^*}(L) = aE[L]\beta^{* -1/a} - 1$. Relation (3.4) gives the optimal $\beta^*$ solving

$$a\alpha\beta^* + (1 - 2\alpha)\beta^* = \alpha(a - 1).$$

Hence,

$$e^\alpha(L) = \beta^{* -1/a} - 1.$$  

In particular, for $a = 2$,

$$\beta^* = \frac{\alpha}{1 + 2\sqrt{\alpha(1 - \alpha)}}$$

and

$$e^\alpha(L) = \frac{\sqrt{\alpha(1 - \alpha)}}{\alpha}.$$  

Furthermore, $F_L$ belongs to Fréchet type MDA($\Phi_\eta$) with $\eta = a$ and $1 - F_L \in 2\mathcal{RV}_{a, -1}$ with auxiliary function $A(x) = a/x$, see [19, 25] for instance. By Remark 4.3, it follows that $1 - F_L \in 2\mathcal{RV}_{a, -1}$ with auxiliary function $A^*(x) = a^2 x^{1/(a - 1)}$. Hence, by
Proposition 4.2, it holds that

\[
\frac{e_\alpha(\tilde{L})}{ES_\alpha(\tilde{L})} = \frac{(a - 1)^{\frac{1}{\alpha}}(1 - 2\alpha)^{\frac{1}{\alpha}}}{\alpha} \left( 1 + \frac{1 - (a - 1)^{\frac{1}{\alpha}}}{\alpha - \frac{a}{a-1}}(1 + o(1)) \right).
\]

The cash-invariant property gives

\[
e_\alpha(L) = \frac{1}{a - 1} + \left( \frac{1 - 2\alpha}{a - 1} \right)^{\frac{1}{\alpha}} \left( \alpha - \frac{a}{a-1} \right) \left( 1 + \frac{1 - (a - 1)^{\frac{1}{\alpha}}}{\alpha - \frac{a}{a-1}}(1 + o(1)) \right).
\]

FIGURE 4. Graph of \(e_\alpha(\tilde{L})/ES_\alpha(\tilde{L})\) for Pareto distribution with \(a = 1.5\).

Example 5.7 (Generalized Pareto). Let \(-1 < \xi < 1\) be given. Let \(F_L(x) = 1 - e^{-\xi}\) for \(\xi = 0\), and \(1 - (1 + \xi x)^{-1/\xi}\) for \(|\xi| \neq 0\) with \(x \geq 0\) if \(\xi \geq 0\), and \(0 \leq x \leq -1/\xi\) if \(\xi < 0\). Then due to the choice the parameter \(\xi\), it holds \(L \in L^1\) and \(F_L\) is continuous and strictly increasing. If \(\xi = 0\), and \(\xi > 0\), \(F_L\) become exponential, and Pareto with \(a = 1/\xi\), respectively.

As for \(\xi < 0\), from [28], we get \(E[L] = 1/(1 - \xi)\), \(q_L(1 - \beta^*) = (\beta^* - \xi - 1)/\xi\) and \(ES_{\beta^*}(L) = \beta^{*-\xi}/(1 - \xi) + (\beta^{*-\xi} - 1)/\xi\). Relation (3.4) yields the optimal \(\beta^*\) solving \(\alpha \beta^* + (1 - 2\alpha)\xi \beta^* + \alpha(\xi - 1) = 0\). Therefore,

\[
e_\alpha(L) = \begin{cases} 
(\beta^{*-\xi} - 1)/\xi & \text{for } \xi \neq 0 \\
1 + \mathcal{W}(\frac{1-2\alpha}{\alpha \xi}) & \text{for } \xi = 0
\end{cases} 
\]

As a result of Examples 5.3 and 5.6 it holds

\[
e_\alpha(L) \sim ES_\alpha(L) \quad \text{and} \quad e_\alpha(L) \sim \xi \left( \frac{1 - \xi}{\xi} \right)^{1-\xi} ES_\alpha(L)
\]

for \(\xi = 0\) and \(\xi > 0\), respectively. As for \(\xi < 0\), it holds that

\[
\frac{e_\alpha(L)}{ES_\alpha(L)} = \frac{\beta^{*-\xi} - 1(1 - \xi)}{\alpha^{-\xi} - 1 + \xi} \sim 1.
\]

For \(\xi < 0\), \(F_L\) belongs to the Weibull type \(MDA(\Psi_n)\), with \(\eta = 1/|\xi|\), see [26].
Example 5.8 (Standard Student t). Let \( L \) be a standard student \( t \) with degree of freedom \( v > 1 \). From [9, 26, 28], we get

\[
ES_{\beta^*}(L) = \frac{1}{\beta^*(v - 1)} \psi(\Psi^{-1}(1 - \beta^*))(v + (\Psi^{-1}(1 - \beta^*))^2)
\]

where \( \Psi \) and \( \psi \) are the cumulative distribution and probability density function of the standard Student \( t \) distribution with \( v \) degree of freedom, respectively. Relation (3.4), yields the optimal \( \beta^* \) solving

\[
\Psi^{-1}(1 - \beta^*) = \frac{(1 - 2\alpha)\psi(\Psi^{-1}(1 - \beta^*))}{(v - 1)((1 - 2\alpha)\beta^* + \alpha)}
\]

Hence,

\[
e_{\alpha}(L) = \Psi^{-1}(1 - \beta^*)
\]

Furthermore, \( F_L \) belongs to Fréchet type \( MDA(\Phi_{\eta}) \) with \( \eta = \nu \) such that \( 1 - F_L \in 2RV_{\nu, -2} \) with auxiliary function \( A(x) = \nu^2(\nu + 1)x^{-2}/(\nu + 2) \), see [19, 25] for instance. By Proposition 4.2, it holds that

\[
\frac{e_{\alpha}(L)}{ES_{\alpha}(L)} = \frac{1}{\nu} \left( \frac{1}{\nu} \frac{(\nu - 1)}{(1 - 2\alpha)\beta^*} \right)\left( 1 + \frac{(\nu - 1)(1 - (\nu - 1)\frac{1}{2})}{2(\nu + 2)(q_L^2(1 - \alpha))} \right) (1 + o(1)) 
\]

Figure 5. Graph of \( e_{\alpha}/ES_{\alpha} \) for Student \( t \) distribution with \( \nu = 1.5 \).

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