Analysis of high-order velocity moments in a strained channel flow.

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Abstract

In the current study, model expressions for fifth-order velocity moments obtained from the truncated Gram-Charlier series expansions model for a turbulent flow field probability density function are validated using data from direct numerical simulation (DNS) of a planar turbulent flow in a strained channel. Simplicity of the model expressions, the lack of unknown coefficients, and their applicability to non-Gaussian turbulent flows make this approach attractive to use for closing turbulent models based on the Reynolds-averaged Navier-Stokes equations. The study confirms validity of the model expressions. It also shows that the imposed flow strain improves agreement between the model and DNS profiles for the fifth-order moments in the flow buffer zone including when the flow separates. The results of investigation of this phenomenon reveal sensitivity of odd velocity moments to the grid resolution. A new length scale is proposed as a criterion for the grid generation near walls and other areas of high velocity gradients when higher-order statistics are collected from DNS.
1. Introduction

The idea of using the Gram-Charlier series expansions (GCSE) for modeling a probability density function (PDF) of a turbulent flow field belongs to Kampé de Fériet [1]. His contribution is a model for a PDF for two variables: \( x_1 = u_i / \sigma_i \) and \( x_2 = u_2 / \sigma_2 \), in the form of a series in Hermite polynomials \( H_{l,k}(x_1, x_2) \) (commonly refer to as the Gram-Charlier series expansions) with respect to the Gaussian distribution, \( P_0(x_1, x_2) \):

\[
P(x_1, x_2) = P_0(x_1, x_2) \sum_{l+k=0}^{\infty} A_{l,k} H_{l,k}(x_1, x_2)
\]

Here, \( u_i \) are turbulent velocity fluctuations, \( \sigma_i = \sqrt{<u_i^2>} \) is the standard deviation, \(<u_i u_j>\) are the Reynolds stresses, and coefficients \( A_{l,k} \) are evaluated in terms of linear combinations of higher-order velocity correlations \( R^{m,n} = \frac{<u_i^m u_j^n>}{(\sigma_i^m \sigma_j^n)} \), where \( <u_i^m u_j^n> \) are statistical velocity moments of the \( m + n \) order. For the exact mathematical expressions of the coefficients \( A_{l,k} \) and the Hermite polynomials, we will refer to [1-3] as none of them will be directly used in the current paper.

By truncating the PDF model (1) to various orders, one can obtain different approximations to (1) varying by their accuracy. Validation of PDF models based on the truncated GCSE involves validation of algebraic expressions linking statistical velocity moments of higher and lower orders. Such expressions are obtained from the condition \( A_{l,k} = 0 \) for \( l + k > N \), where \( N \) is the truncation order [1-3]. For example, if \( N = 4 \), these expressions include those for \( l + k = 5 \):

\[
R^{5,0} = 10R^{3,0}, \quad R^{4,1} = 6R^{2,1} + 4R^{1,1}R^{3,0}, \quad \text{and} \quad R^{3,2} = 6R^{1,1}R^{2,1} + R^{3,0} + 3R^{1,2}
\]

and higher.
Experimental data collected in wall-bounded flows supported expressions (2) and similar ones for higher-order velocity moments [2-7], with the most complete validation study being conducted in a turbulent boundary layer over a flat plate at the zero pressure gradient [2]. In that study, the expressions for velocity moments through the 11th order obtained by truncating (1) at \( N = 4, 6, \) and 8 were validated.

However, even though the previous studies successfully validated expressions like (2) and for higher-order velocity moments, there will always be some expressions left untested due to the lack of data and because the upper limit for \( l + k \) is infinity. Thus, a proof that a truncated GCSE is an accurate representation of a turbulent flow PDF will likely remain incomplete. This issue is common for any PDF model involving series expansions, not only for those based on GCSE.

A more manageable problem of practical importance is validation of expressions (2) as a model for unknown terms describing turbulent diffusion in the Reynolds-averaged Navier-Stokes (RANS) equations for the fourth-order velocity moments [8]. Expressions (2) are particularly attractive for modeling purposes as they do not contain unknown coefficients and are applicable to non-Gaussian turbulent flows. Validation of expressions (2) in a wide range of flow geometries and flow conditions is the ultimate goal of our research.

After applying expressions (2) to the infinite set of RANS equations, one obtains a fourth-order (or FORANS) statistical closure [8]. Truncation of GCSE to \( N \) less than four with the purpose of developing lower than fourth-order statistical closures reduces (1) to the Gaussian distribution \( P_0(x_1,x_2) \) [1,2]. Multiple studies demonstrated that the Gaussian distribution is an inadequate PDF model in turbulent flows of practical interest (see e.g., Ch.6 in [9]). Thus, the lowest order for a statistical closure based on the truncated GCSE applicable in non-Gaussian turbulent flows is the fourth.
Measuring all moments required to validate (2) is still a complex matter. Direct numerical simulations expand a range of flows and flow conditions, where all expressions (2) can be tested. In the previous study conducted in our group [10], for example, all expressions (2) were successfully validated in a two-dimensional zero-pressure-gradient (ZPG) turbulent boundary layer (TBL) over a flat plate using DNS data from [11-13] at two different Reynolds numbers: $Re_\theta = 4101$ and 5200, where $Re_\theta$ is based on the free stream velocity and the boundary layer momentum thickness $\theta$.

In the current study, a DNS dataset of velocity moments through the fifth order collected in a strained planar turbulent channel flow [14] is used to test validity of expressions (2) in a presence of adverse pressure gradient, including flow separation. (The complete dataset is available in [15]).

Previously, dynamics of the velocity moments through the fourth order and the behavior of the budget terms in their transport equations in this flow were investigated in [14-19]. Here, we will explore in more detail their behavior in the near-wall area, where the flow separation initiates. The analysis will also be expanded to include all moments through the fifth order that are relevant to validation of expressions (2) and are useful to understanding effects of the flow separation on moments of higher orders.

Comparison of the model and DNS profiles of the fifth-order velocity moments revealed a beneficial role of the imposed flow strain on the profiles agreement in the near-wall flow area known as the buffer zone. The results of investigation of this phenomenon are also presented in the paper.

2. DNS data description

In DNS of planar turbulent flow in a strained channel [14], simulations were first conducted in a conventional (unstrained) planar fully-developed turbulent channel flow at $Re_{\tau_0} = 392$, where
$Re_{t_0}$ is based on the friction velocity $u_{t_0}$ and the channel half-height $h$. This corresponds to the Reynolds number $Re_{c_0} = U_{c_0} h / \nu = 7,910$ based on the centerline mean velocity $U_{c_0}$ (hereafter, $\nu$ is the kinematic viscosity of a fluid). The unstrained channel flow data were used then as the initial conditions in simulations of a strained channel flow.

In the strained channel flow simulations, uniform irrotational temporal deformations were applied to the unstrained channel flow domain by diverging the channel walls in the $y$-direction (normal to the channel walls) and simultaneously, by accelerating the channel walls in the streamwise direction $x$ to achieve the flow deceleration and reduction of the wall shear stress in this direction. The flow features obtained in such a manner replicate those of an adverse-pressure-gradient boundary layer including the flow separation [14-18].

The flow deformation is characterized by the rate of the applied strain, $A_{ij} = \partial U_i / \partial x_j, i = 1, 2, 3$, with its non-zero components $A_{11} = \partial U / \partial x$ and $A_{22} = \partial V / \partial y$ being of the same constant magnitude, but the opposite sign: $A_{11} < 0$ and $A_{22} = -A_{11} > 0$. Here, $\vec{x} = (x, y, z)$ and $\vec{U} = (U, V, 0)$. The uniform strain magnitude in both directions was set to 31% of the ratio of the initial friction velocity to the initial channel half-width, which gives $|A_{11}| = A_{22} = 0.0118$.

The following expression describes the relation between the channel wall velocity $U_w$, the mean flow centerline velocity $U_c$, and $A_{11}$:

$$U_c(t) - U_w(t) = U_{c_0} \exp(A_{11} t),$$

where all velocity components are in the streamwise direction $x$ and $t$ is time.
DNS data were collected at seven non-dimensional times, \( t_i' = A_22 t_i, \ i = 1, \ldots, 7 \) (Table 1). The flow separation started around \( t_6' = 0.675 \) and continued at \( t_7' = 0.772 \). The unstrained channel data (\( t_0' = 0 \)) are also used in the study for comparison.

Figure 1a shows dynamics of the mean velocity \( U \) in the streamwise direction with time, with the separation area being enlarged in Fig. 1b. In the figure, all parameters are taken at a given time moment. Dashed and solid lines show velocity profiles before and after the flow separation, respectively. Labels from 0 to 7 correspond to times \( t_i', \ i = 0, \ldots, 7 \). At \( t_7' \), the separation region is below \( y/h = 0.0134 \ (y^+ \sim 5) \). The minimum velocity occurs at \( y/h \sim 0.005 \ (y^+ \sim 2) \).

Table 1. Characteristic velocities in DNS\(^{14} \) of a strained channel flow at different \( t_i' \).

| \( t' \) | 0  | 0.02 | 0.1 | 0.191 | 0.285 | 0.365 | 0.675 | 0.772 |
|---------|----|------|-----|-------|-------|-------|-------|-------|
| \( U_c/u_{r0} \) | 20.137 | 20.493 | 21.948 | 23.764 | 25.773 | 27.848 | 36.975 | 40.511 |
| \( U_w/u_{r0} \) | 0.755 | 3.727 | 7.128 | 10.630 | 13.869 | 26.722 | 31.206 |

3. Results and discussion

3.1 The strain effects on velocity moments of different orders

Expressions (2) model fifth-order velocity moments as sums of products of lower-order velocity moments. For this reason, it is useful to review responses of individual velocity moments of different orders to the imposed strain before conducting validation of models (2).

Previous studies investigated dynamics of the velocity moments through the fourth order in this flow and the behavior of budget terms in their transport equations (Appendix A) [14-19]. In this section, we will expand the analysis to include moments through the fifth order with the purpose of better understanding how the moment order and its other characteristics affect the moment response to the strain and what are implications for turbulence modeling. Of particular
interest are the flow separation effects. For this reason, all plots presented in this section highlight the near-wall area of the flow.

It was previously noticed that except for the initial short period of time, the strain imposed on the channel flow tends to reduce velocity moments everywhere in this flow, with the moments decay most rapidly occurring in the wall proximity. Figures 2-5 demonstrate that indeed all of the considered moments through the fifth order are affected in this way. In the figures, the shown moment profiles are those in the unstrained flow ($t_0'$), in pre-separated and separated flows ($t_5' - t_7'$), and at the time when a velocity moment reaches its maximum absolute value (this time varies for different moments).

A closer look reveals also differences in responses of different groups of velocity moments to the strain. In particular, all velocity moments $\langle u^k \rangle$, $k = 2, ..., 5$, where $u$ is a turbulent velocity fluctuation in the streamwise direction achieve their maximum values at $t_1'$ (Fig. 2) regardless the moment order, $k$, but the difference in the profiles at $t_1'$ and at $t_0'$ is minor for all $k$. On the other hand, all these moments experience strong suppression as time progresses: the higher the moment order, the stronger suppression. To illustrate the tendencies, Table 2 provides the ratio between the maximum absolute values of a velocity moment at $t_7'$ and $t_0'$:

$$r = \frac{\max|\langle u^k u_j \rangle| (t_0')} {\max|\langle u^k u_j \rangle| (t_7')}$$

for all $\langle u^k \rangle$ (and other considered moments).

The velocity moments $\langle v^k \rangle$ in the direction normal to the channel wall reach their maximum later in time than $\langle u^k \rangle$, with the odd moments $\langle v^3 \rangle$ and $\langle v^5 \rangle$ achieving their maximum
values at \( t_2' \) and the even moments \( \langle v^2 \rangle \) and \( \langle v^4 \rangle \) at \( t_3' \) (Fig. 3). With time increasing, the magnitude of all moments \( \langle v^k \rangle \) vary less than that of \( \langle u^k \rangle \), with the odd moments being affected stronger by the strain than the even ones (Table 2). When comparing only odd or only even moments, the effects are stronger for higher order moments.

Overall, changes in all \( \langle v^k \rangle \) with the time are less dramatic than in all \( \langle u^k \rangle \), which is due to the lack of direct contribution from the mean velocity gradients in the \( y \)-direction in the transport equations for the \( \langle v^k \rangle \) moments (Appendix A). The difference in dynamics between the odd and even moments can be linked to the fact that the odd moments of the \( k \) order are smaller than the even moments of the \( k - 1 \) and \( k + 1 \) order. As a result, a role of the turbulent diffusion terms is different in the budgets of the odd and even moments: the major contributor to the budget of \( \langle v^3 \rangle \) and the secondary one to the budgets of \( \langle v^2 \rangle \) and \( \langle v^4 \rangle \). In addition, new terms like

\[
\langle v^{k-1} \rangle \frac{\partial \langle v^2 \rangle}{\partial y} \quad \text{("production by turbulence")}
\]

appear in the transport equations of the velocity moments of the orders \( k > 2 \), and they are larger in the budgets of the odd moments tending to reduce them. On the other hand, the even moments have strong support from the velocity/pressure-gradient correlations at any time, whereas their contribution to the odd moments reduces strongly as time progresses. Other aspects of dynamics of the budget terms in the transport equations of various velocity moments under the strain are discussed in [14, 16].

When considering cross-correlations \( \langle u^m v^n \rangle \), all those with \( m > n \) reach their maximum at \( t_1' \). The others have maximum values at \( t_2' \), similar to the odd moments \( \langle v^k \rangle \).

As time progresses, all moments \( \langle u^m v^n \rangle \) are suppressed in a less degree than \( \langle u^k \rangle \), but more so than \( \langle v^k \rangle \) at the same order \( k = m + n \), with the even moments \( (k = 2,4) \) being
affected less than the odd ones (Table 2), similar to what was observed for $< v^k >$. The moments with $m > n$ are suppressed more than other cross-correlations.

Table 2. Ratio $r$ (Eq. (3)) for velocity moments in Figs. 2-5.

| $< u^5 >$ | $< u^4 >$ | $< u^3 >$ | $< u^2 >$ | $< uv >$ | $< v^2 >$ | $< v^3 >$ | $< v^4 >$ | $< v^5 >$ |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 15.41    | 7.20     | 6.48     | 3.02     | 1.28     | 1.03     | 1.39     | 1.15     | 1.71     |
| $< u^4v >$ | $< u^3v >$ | $< u^2v >$ | $< u^2v^2 >$ | $< uv >$ | $< uv^2 >$ | $< uv^3 >$ | $< uv^4 >$ | $< uv^5 >$ |
| 8.73     | 2.95     | 3.69     | 1.69     | 1.6      | 1.45     | 1.95     |          |          |
| $< u^4v^2 >$ |          |          |          |          |          |          |          |          |
| 4.07     |          |          |          |          |          |          |          |          |

Comparing all moments, the moment with the most reduced maximum value under the imposed strain is $< u^5 >$ and the moment with the least affected maximum value is $< v^2 >$ closely followed by $< v^4 >$.

In fact, of all considered moments only these two, $< v^2 >$ and $< v^4 >$, have their maximum values reduced to compare with those in the unstrained channel at the same time when the flow separation occurs, that is at $t_6'$ and not sooner than that. Since the transport equations for these two moments (Appendix A) do not explicitly contain terms with $\partial U/\partial y$, this is an indication that models for unknown terms in these equations have to provide a link between these moments and the mean velocity gradient in the direction normal to the wall.

As the next step of the analysis, we compare the largest and the smallest moments of the same order in an unstrained channel (at $t_0'$) and when the flow undergoes separation (at $t_7'$). The largest moment at any order $k$ is the one in the streamwise direction, $< u^k >$. The smallest moment varies depending on the moment order and time. At $k < 5$, the moments $< uv >$, $< v^3 >$, $< uv^3 >$ are the smallest at both times. At $k = 5$, the smallest moment at $t_0'$ is $< v^5 >$, but $< uv^4 >$ at $t_7'$. 

9
The ratio of the maximum absolute values of the largest and the smallest moments at a given \( k \):

\[
    r_1(t'_i; k) = \frac{\max|\langle u^k \rangle|}{\min(\max|\langle u^{m+n} \rangle|)}
\]  

(4)
is used to evaluate this effect. In (4), \( i = 0, 7 \) and \( m + n = k = 2, \ldots, 5 \). Results are presented in Table 3 for considered values of \( k \):

| Moment order, \( k \) | 2  | 3  | 4  | 5  |
|------------------------|----|----|----|----|
| \( t'_0 \)             | 8.87 | 20.40 | 52.10 | 84.75 |
| \( t'_7 \)             | 3.77 | 4.38 | 10.48 | 10.35 |

The table shows that the largest differences between the moments of the same orders occur in the unstrained channel, with the ratio \( r_1 \) growing with the moment order. It is on the order of magnitude less at \( k = 2 \) to compare with \( k = 5 \).

The imposed strain tends to reduce differences between the moments of the same order and more so for the moments with higher values of \( k \). As a result, the \( r_1 \)-value is still growing with \( k \), but more slowly than in the unstrained channel. At \( k = 5 \), it is of \( O(10) \) to compare with \( O(10^2) \) in the unstrained channel.

Based on the above data analysis, challenges of modeling fifth-order velocity moments in terms of lower-order ones become more apparent as different velocity moments respond differently to the imposed strain with the flowing complex evolution of the moment profiles in time. However, the strain (at least during the conducted simulation time) seems to have favorable
effects on the moment dynamics from the modeling perspective by smoothing differences rather than exaggerating them. Complexity of the moment transformation under the strain makes it unlikely to model relations between different moments in intuitive empirical fashion. This makes expressions (2) even more valuable as they are rigorously derived and do not contain unknown model parameters to vary. This is of course, if these expressions prove to be successful in this challenging flow. The following sub-section describes results of their validation.

3.2 The fifth-order moments modeling

A quick look at Figs. 2-5 encourages the idea of modeling higher-order moments in terms of lower-order ones based on the obvious similarity of some moment profiles, their dynamics with time, and times when the moments reach their maximum. Compare, for example, $<u^5>$ and $<u^3>$, $<v^5>$ and $<v^3>$. The figures also show that all moments have a range near the wall where their values are very small. This poses a potential problem when calculating $R_{mn}$, as two of the moments, $<u^2>$ and $<v^2>$, are in the denominator of all expressions $R_{mn}$. To avoid the issue of dividing by small numbers, expressions (2) can be re-written as

$$
<u_i^5> = 10 <u_i^2> \cdot <u_i^3>,
$$
$$
<u_i^4 u_j> = 6 <u_i^2> \cdot <u_i^2 u_j> + 4 <u_i^3> \cdot <u_i u_j>,
$$
$$
<u_i^2 u_j^3> = 6 <u_i u_j> \cdot <u_i u_j^2> + <u_i^2> \cdot <u_j^3> + 3 <u_i^2 u_j> \cdot <u_j^2>.
$$

In the strained channel flow that remains homogeneous in the stream- and spanwise directions under the imposed strain $[14,17]$, there are only six equations in the FORANS set for velocity
moments to solve (Appendix A). In these equations, there are only two fifth-order moments: $<uv^4>$ and $<v^5>$, to model. This is, however, before modeling unknown terms.

Model expressions (5) for $<uv^4>$ and $<v^5>$: $<uv^4> = 6 <v^2> \cdot <uv^2> + 4 <v^3> \cdot <uv>$ and $<v^5> = 10 <v^2> \cdot <v^3>$, do not bring new unknown parameters (and transport equations for them) into the set of equations (A1)-(A6). This is an additional benefit of the considered modeling approach.

On the other hand, models for velocity/pressure-gradient correlations and for components of the turbulent kinetic energy dissipation tensors are likely to add new parameters to the original set of equations (A1)-(A6) governing this flow. As a result, one may expect to have more fifth-order velocity moments to model. For this reason, we validate in this study expressions (5) for all fifth-order velocity moments that appear in the RANS equations for the fourth-order velocity moments in an incompressible planar turbulent flow in the $(x,y)$ plane (Appendix B).

Figure 6 compares profiles of the fifth-order moments $<u^m v^n>$, $m + n = 5$, obtained from DNS data [14,15] (dashed lines) with the model profiles $<u^m v^n>^{(M)}$ of the same moments obtained from expressions (5) using DNS data for the third- and second-order moments (solid lines). Comparison is made at different times to demonstrate that dynamics of the model profiles of the fifth-order moments is consistent with that of their DNS profiles. The logarithmic vertical scale is chosen for the moments $<u^m v^n>$ with $m = 4$ and 5, to better resolve the area close to the wall, where the magnitudes of these moments reach their maximum.

Overall, the agreement between the DNS and model profiles is very good for all moments everywhere in the flow. Dynamics of the model profiles with time is the same as that of the DNS profiles, with the agreement between the DNS and model profiles improving with time (as expected and discussed in the previous sub-section). In a presence of the flow separation, the
agreement is excellent. Locations of the maximum and minimum values as well as those of zeros are well reproduced at every time. These results are very encouraging from the modeling perspective.

However, there is a discrepancy between the model and DNS profiles, which appears to be sensitive to the turbulent velocity fluctuation in the streamwise direction, that is, the discrepancy grows with \( m \) in \( \langle u^m v^n \rangle \). The observed tendencies for this parameter are illustrated in Fig. 7 for \( \langle u^5 \rangle \) and \( \langle v^5 \rangle \) at \( t_0' \) and \( t_7' \) using the following ratio based on the \( L_{\infty} \)-norms:

\[
\Delta = \frac{\langle u^m v^n \rangle^{(M)} - \langle u^m v^n \rangle}{\langle u^m v^n \rangle_{L_{\infty}}}. \tag{6}
\]

This ratio was found to be more informative than others (not reported here) as it allows avoiding division by small numbers when a moment changes its sign (and in this flow, all moments do). Results for the other four moments (not shown here) fall between those in the figures. Profiles of velocity moments \( \langle u^5 \rangle \) and \( \langle v^5 \rangle \) are also shown in Fig. 7 at both times by dashed lines for comparison.

For \( \langle v^5 \rangle \), the discrepancy between the DNS and model profiles peaks near or at the locations of the velocity moment zero values at both times and can be explained by numerical errors associated with small numbers. The magnitude of this parameter does not change significantly with time (Figs. 7c and 7d, also Table 4).

For \( \langle u^5 \rangle \), on the other hand, the discrepancy between the DNS and model profiles tends to peak around the locations of the extremes in this moment profiles and more so at earlier times when this moment magnitudes are the largest. At \( t_0' \), for example, the \( \Delta \)-profile has two prominent
extremes (Fig. 7a) located at $y^+ = 8.8$ and 20. The $< u^5 >$-profile also has two extremes located at $y^+ = 7.5$ and 23 (see also Table 4 where locations of their maximum values are compared). As time progresses and this moment magnitude reduces, a connection between locations of the extremes in the profiles of $\Delta$ and $< u^5 >$ remains until the flow separates (Fig. 7b and Table 4). Still, even in the separated flow, the location of max $\Delta$ at $y^+ = 185$ is not that far from the location of max$|< u^5 >|$ at $y^+ = 235$.

Notice that before the flow separates (at $t < t_6'$), the extreme values of $\Delta$ and that of $< u^5 >$ are located within the buffer zone at $y^+ \lesssim 20$.

The magnitude of $\Delta$ can also be linked to that of $< u^5 >$. As discussed in the previous subsection, $< u^5 >$ is the moment suppressed the most by the imposed strain. In the separated flow, the $< u^5 >$-magnitude reduces at least ten times to compare with its level at $t_0'$. The same trend is observed for the $\Delta$-values (Fig. 7b and Table 5).

| $y/h$ | $t_0'$ | $t_1'$ | $t_2'$ | $t_3'$ | $t_4'$ | $t_6'$ | $t_7'$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| max $\Delta$ | 0.05 | 0.05 | 0.05 | 0.02 | 0.02 | 0.6 | 0.5 |
| max $< u^5 >$ | 0.06 | 0.06 | 0.02 | 0.02 | 0.02 | 0.55 | 0.64 |

Table 4. Locations of max $\Delta$ and max $|< u^5 >|$ at different times.

| $t'$ | $< u^5 >$ | $< u^4v >$ | $< u^3v^2 >$ | $< u^2v^3 >$ | $< uv^4 >$ | $< v^5 >$ |
|------|-----------|------------|------------|------------|-----------|--------|
| $t_0'$ | 0.68 | 0.57 | 0.43 | 0.25 | 0.12 | 0.06 |
| $t_1'$ | 0.67 | 0.56 | 0.42 | 0.25 | 0.12 | 0.06 |
| $t_2'$ | 0.62 | 0.54 | 0.43 | 0.27 | 0.14 | 0.06 |
| $t_3'$ | 0.32 | 0.49 | 0.41 | 0.24 | 0.17 | 0.06 |
| $t_4'$ | 0.26 | 0.24 | 0.18 | 0.19 | 0.18 | 0.11 |
| $t_6'$ | 0.12 | 0.17 | 0.14 | 0.16 | 0.14 | 0.12 |
| $t_7'$ | 0.13 | 0.17 | 0.15 | 0.17 | 0.15 | 0.11 |
Finally, comparing the maximum values of $\Delta$ for all fifth-order moments at different times, one can notice that its largest value happens for $<u^5>$ in the unstrained channel (Table 5). Also, all moments $<u^m v^n>$ with $m > n$ have larger maximum $\Delta$-values than those with $m < n$ prior the flow separation. The least affected by the flow evolution is the maximum $\Delta$-value for $<v^5>$.

To summarize, the discrepancy between the DNS and model profiles can be linked to the magnitude of velocity fluctuations in the streamwise direction and to a specific location within the flow: the buffer zone. The difference is the largest in the unstrained channel, where the velocity moment $<u^5>$ is the largest. The imposed strain reduces $<u^5>$, which leads to improvement of the agreement between the DNS and model profiles for this moment. Similar tendencies are observed for other moments $<u^m v^n>$ with $m > n$. The discrepancy between the DNS and model profiles for $<v^5>$ and for other moments $<u^m v^n>$ with $m < n$ remains small and changes little with time.

For this reason, the following sub-section further scrutinizes DNS data in the unstrained channel to better understand how velocity fluctuations in the streamwise direction contribute into the observed discrepancy between the model and DNS profiles.

3.3 Analysis of discrepancy between the model and DNS profiles

The discussion in the previous sub-section points towards the turbulence production by the mean velocity gradient $\partial U/\partial y$ as a major culprit of the discrepancy between the DNS and model profiles obtained using (5). Indeed, these are the only terms that are absent in the transport equations for moments $<v^n>$. The contribution of these terms grows with $m$ in the transport equations for moments $<u^m v^n>$: $(-m <u^{m-1} v^{n+1}> \partial U/\partial y)$ (see Appendices A and B and also [20]). Contribution of these terms also reduces with the imposed strain.
To better understand how the production terms may contribute in the discrepancy between the DNS and model profiles obtained from (5), the following discussion will be limited to the moment $< u^5 >$ in the unstrained channel, where these effects are pronounced the most.

In the model for $< u^5 >$ (Eq. (5)), two lower-order moments are involved: $< u^2 >$ and $< u^3 >$. Figure 8 shows the turbulence production terms by $\partial U / \partial y$ (absolute values) in the transport equations for the three moments (dashed and dotted lines) as well as the $\Delta$-profile for $< u^5 >$ (solid line). The production terms are not to scale as the purpose of this plot is to demonstrate a connection between the locations of the extreme values in the $\Delta$-profile and those in the profiles of production terms.

There are two largest extremes in the $\Delta$-profile located at $y^+ = 8.8$ and 20.8. The production term for $< u^2 >$ (dotted line in Fig. 8) has only one maximum located at $y^+ = 15.1$ between the two extremes in the $\Delta$-profile. Thus, this is not the leading term contributing to $\Delta$. On the other hand, both profiles for the productions terms in the transport equations for the odd moments $< u^3 >$ and $< u^5 >$ (shown by the dashed lines in Fig. 8) have two largest extremes closely aligned between the profiles and with the extremes in the $\Delta$-profile. Specifically, in the $< u^3 >$-profile, their locations are $y^+ = 7.5$ and 20.1 and in the $< u^5 >$-profile, the extremes are located at $y^+ = 8.8$ and 20.8.

Observation that the odd moments contribute more in the discrepancy between the DNS and model profiles for $< u^5 >$ is consistent with our previous results [20], where RANS-DNS simulations were used to demonstrate that the errors in the DNS $< u^3 >$-profile are significantly larger than those in the profiles of even moments $< u^2 >$ and $< u^4 >$. Results were obtained with the same data [14,19] as used in the current study. No comments on the accuracy of $< u^5 >$ can
currently be made using the RANS-DNS simulations due to the lack of the complete dataset for
the budget terms in the transport equation for this moment.

Another interesting feature to notice in Fig. 8 is that alignment of the extremes in the profiles
of $< u^3 >$, $< u^5 >$, and $\Delta$ is “reverse”: the smallest of the two extremes in the $\Delta$-profile
corresponds to the maximum production values. This may be an indication that the odd moments
are “under-produced” in this flow area. This can happen, for example, if the collecting time of raw
DNS data was not sufficient for the profiles of odd moments to statistically converge or the grid
resolution may not be sufficient to resolve rapid variations in the moment values, or both.

Insufficient statistical convergence of DNS data is habitually brought into discussion whenever
there is an issue associated with such data. However, convergence of statistics including high-
order moments was tested, discussed at length and found adequate in [14]. In addition, the same
DNS in the unstrained channel were run about ten times longer in our separate study [21]. In [21],
RANS-DNS simulations were used to analyze the data convergence (a different method than used
in [14]). They confirmed that statistical convergence of the second-order moments and of the
budget terms in their transport equations occurs early in the simulations, close to the simulation
time in [14].

On the other hand, that study also revealed a systematic error present in the second-order
statistics. It can be seen, for example, in the balance error profiles in the budgets of the Reynolds
stresses that converged to non-zero values at the end of simulations, with the maximum values
being located in the flow buffer zone. As an example, Fig. 9a shows the non-dimensional balance
errors $Err_{\text{ax}}$ in the $< u^2 >$-budget obtained at the end of simulations at $t = 1.22 \cdot 10^5$ (compare
with the simulation time of $t = 5.06 \cdot 10^4$ in [14]). The simulation times in [21] and [14]
correspond to 1000 and 159 instantaneous flow field realizations, respectively. In this and
following figures, the balance errors and other budget terms in the Reynolds-stress transport equations are normalized by $u_{i0}^4/\nu$ [19].

Higher-order statistics were not collected in [21], but as has already been mentioned above, their statistical convergence was found acceptable in [14] using standard approaches to the statistical error estimation. As for the systematic error discovered in [21] in the second-order statistics, it is unlikely to expect higher-order statistics to be free from this error. Moreover, one would expect such error to be larger in odd moments as they are sensitive to the direction of velocity fluctuations in contrast to even moments.

Indeed, using data from [14], Fig. 9b confirms that the balance errors in the transport equation for $<u^3>$ are significantly larger than those in the $<u^2>$ —equation. They are the largest in the buffer zone, where the systematic error is expected to be the largest, and their extreme values are closely aligned with those of the production term in the $<u^3>$-equation: $y^+ = 6.3$ and 18.8 vs. $y^+ = 7.5$ and 20.1 in the production profile. Similar to what was observed in Fig. 8, the highest peak from the two extreme values in the production term corresponds to the lower of the two extremes in the balance error profile.

To compare, there is no obvious relation between the balance errors in the transport equation for $<u^2>$ and the production term in that equation also shown in the figure. Profiles for the terms in the $<u^3>$-equation shown in Fig. 9b and the following figures are normalized by $u_{i0}^5/\nu$ [19].

To summarize the discussion above, the largest discrepancy between the DNS and model profiles (5) observed in the flow buffer zone can be linked to the production terms in the transport equations of the odd moments.

In this flow geometry, the buffer zone is the area where the considered velocity moments and the production terms in their transport equations reach their maximum values. This is also the
region where the balance errors in the DNS budgets of velocity moments are the largest, with the extreme values in their profiles being aligned with those in the production terms in the transport equations of the odd moments.

Since the statistical errors in DNS data cannot alone explain errors in DNS data in the buffer zone, this leaves the grid resolution as a possible source of the systematic error in DNS data [21]. However, the grid used in [14] was generated following the standard approach in which the Kolmogorov turbulence length scale [22]

\[ \eta = (\nu^{3}/\varepsilon)^{1/4}, \]  

(7)
is used as a reference to compare the grid resolution with. In (7), \( \varepsilon \) is the scalar dissipation, \( \varepsilon = 1/2 \sum_{l=1,3} \varepsilon_{ii} \), where \( \varepsilon_{ii} \) are the dissipation tensor components in the transport equations for Reynolds stresses \( <u_i^2> \).

The Kolmogorov length scale is considered to be the smallest turbulence scale to resolve based on the theory of locally isotropic turbulence (see [22] and also [23,24], where history of the theory development is discussed). The theory assumes, in particularly, a sufficiently large Reynolds number and small velocity fluctuations. Applicability of such assumptions to the buffer zone in wall-bounded flows, where the mean velocity gradients are high, local Reynolds numbers are small, and turbulence is highly anisotropic (Figs. 2-5) is questionable. In fact, it was explicitly stated in [22] that “the hypothesis of local isotropy is realized with good approximation in sufficiently small domains \( G \) of the four-dimensional space \( (x_1, x_2, x_3, t) \) not lying near the boundary of the flow or its other singularities.”
As an example, Fig. 10a compares local velocity scales defined as $\sqrt{\langle u^n \rangle}$ with the local mean velocity. Ratios $\sqrt{\langle u^n \rangle} / U$ are shown by blue and red lines for even and odd moments, respectively. Also shown are the balance errors in the transport equations for $\langle u^2 \rangle$ and $\langle u^3 \rangle$ (black lines). Solid color lines correspond to the lower-order moments ($n \leq 3$) and dashed color lines are for the moments with $n = 4$ and 5. Notations for the balance errors are the same as in Fig. 9b.

As the figure demonstrates, the ratio $\sqrt{\langle u^n \rangle} / U$ is not small for all considered $n$ and in the large flow area, but particularly, within the buffer zone at $y/h < 0.13$ ($y^+ < 50$). Moreover, the ratio grows with $n$, whereas the velocity scale extracted from $\sqrt{\langle u^n \rangle}$ should be independent from $n$ in accurate simulations.

The two odd moments have a dip in their profiles at the location where both of them change sign. The ratio value should be close to zero in this location, but instead, it is $O(0.1)$ for both moments, another evidence of unresolved values of the odd moments. The dip is located close to the locations of the extreme values in the profiles of the balance errors and of the production by $\partial U / \partial y$ in the $\langle u^3 \rangle$--budget. Notice that there is no apparent relation between the ratio $\sqrt{\langle u^n \rangle} / U$ and the balance errors in the flow area of $y^+ < 2$, but this is the region where all velocity moments diminish (Figs. 2-5) and so do the balance errors in the moment budgets.

Additional evidence that the Kolmogorov lengthscale may not be an adequate reference for determining the grid resolution in the buffer zone can be found in Fig. 11. In the figure, variation of the non-dimensional lengthscale $L_k' = \eta/h$ with $y$ is shown by the dash-dotted line. The lengthscale remains almost unchanged through this area, whereas the balance errors are at their
largest values. The non-dimensional grid cell size, $L_{14}^* = \Delta y/h$, used in [14] is also shown in the figure by the long-dashed line. This parameter remains below $L_k^*$ at $y^+ < 14$ ($y/h < 0.035$), but after that, exceeds it everywhere in the flow. Yet, the balance errors in the budgets of both moments $<u^2>$ and $<u^3>$ start to diminish regardless of the increase in $L_{14}^*$.

The main conclusion from the results shown in Fig. 11 is that the balance errors start to be noticeable when $L_{14}^*$ reaches a certain threshold of ~0.00134 at $y^+ \sim 1.6$ ($y/h \sim 0.004$). Outside the buffer zone, the simulation results are much less sensitive to the grid cell size, although fluctuations present in the balance errors of the $<u^3>$ - budget may be an indication that $L_{14}^*$ is rather large for the odd moments.

3.4 Comparison with the previous studies

Previously, we compared the DNS and model profiles obtained from (5) in a different flow geometry: a planar turbulent boundary layer over a flat plate under zero pressure gradient [10]. Raw DNS data from [13] at two different Reynolds numbers, $Re_\theta = 4101$ and 5200, were used to extract velocity moments. The different code and the grid were used in DNS in [13] to compare with those in [14]. Yet, there is similarity in the results obtained in the current work and in [10]: the model profiles for $<u^m v^n>$ obtained in [10] were higher than DNS profiles for the corresponding velocity moments, with the discrepancy between the DNS and model profiles being the largest at the locations of the velocity moment extreme values. The discrepancy increased with $m$ (Fig. 6 in [10]), with the largest discrepancy being between the DNS and model profiles of $<u^5>$. The agreement between the DNS and model profiles of $<v^5>$ were again excellent everywhere in the flow. For $<u^5>$, the extreme values and the largest deviation between the DNS and model profiles were found to be within the flow buffer zone ($y^+ < 50$).
Not much can be said about quality of the grid used in [13] though, as the budget terms are only available for the second-order moments in this flow. Data at $Re_\theta = 5200$ are used in the discussion below.

The non-dimensional grid cell size $L_{13}^* \equiv \Delta y/\delta_{99}$ is found to be smaller than the Kolmogorov lengthscale everywhere in this flow (Fig. 11b), but again, the balance errors in the $<u^2>$-budget are in no particular correlation with the grid cell size growth through the flow. The error absolute values start to increase right from the wall in contrast to what is seen in Fig. 11a with data from [14]. Interestingly, the Kolmogorov lengthscale is not a smooth function in simulations [13], as both the scalar dissipation profile and the dissipation profile in the $<u^2>$-budget change their sign on several occasions in the near wall area. Whether this is a physical phenomenon or a simulation artifact remains unclear as the balance errors are the largest in this area. (Notice that the Kolmogorov lengthscale is smooth when calculated using data from [14,19] (see Fig. 11a).) Extreme values of the balance errors are confined within $y^+ < 80$. The production term has its maximum at $y^+ = 10.4$ and its magnitude reduces ten times within $y^+ < 80$. The same occurs in the budget of $<u^2>$ from [14,19]. There is no particular alignment between the locations of the maximum production value and the balance error extreme values, another similarity between the data from [13] and [14,19].

Thus, although a relation between the balance errors and the production term in the $<u^2>$-budget can be detected, the relation is not that obvious to compare with what we saw when analyzing the budget of $<u^3>$ using data from [14,19]. That is, collecting budgets for odd moments should not be treated as optional, but necessary for the analysis of DNS accuracy. This is because odd moments are sensitive to the sign of velocity fluctuations in the given flow direction.
and therefore, reveal more information about the flow structure and development than even moments.

Lastly, the ratio $\sqrt{\langle u^n \rangle} / U$ for $n = 2$ and 3 is presented in this section for this flow (Fig. 10b). Once again, the ratio is not small in a large portion of the flow and particularly, near the wall demonstrating that the theory of locally isotropic turbulence is inapplicable in this area. The dip in the ratio value at $n = 3$ is resolved better than with the data from [14,19]. However, this ratio also shows that the moment $\langle u^3 \rangle$ from [13] is less accurate than in [14,19], because the ratios at $n = 2$ and 3 deviate from each other farther everywhere in the flow in Fig. 10b than in Fig. 10a. These results are in agreement with our previous observations [20], where the errors in both datasets were analyzed using the RANS-DNS framework.

3.5 Alternative lengthscales

Since the analysis of errors conducted in the previous section revealed a connection between the turbulence production by the mean velocity gradient and the balance errors in the flow buffer zone, and identified insufficient grid resolution as the main contributor to the balance errors, this section provides a discussion on possible alternatives to the Kolmogorov length scale that will help to improve the grid quality in this flow region. Such alternatives are expected to be smaller than the Kolmogorov scale in the areas with strong velocity gradients. The dimensional analysis is used to obtain the scales.

All considered scales involve viscosity, as relevant effects are high in the buffer zone. Other parameters considered in determining alternative length scales include the production terms in the transport equations of velocity moments $\langle u^n \rangle$, the mean velocity gradient $\partial U / \partial y$, mean velocity $U$, its the second derivative $\partial^2 U / \partial y^2$ and velocity moments $\langle u^n \rangle$ as the largest ones.
From all these options, results for those scales that are based on $< u^n >$ and the productions terms are discussed here, as other length scales turned out to be larger than the Kolmogorov length scale everywhere in the flow.

A length scale based on the production terms for arbitrary order $n$ of the corresponding velocity moment can be defined as:

$$L_{pn} = \left( \frac{v^{n+1}}{|P_n|} \right)^{\frac{1}{n+2}},$$

where $P_n$ is the production term by $\partial U / \partial y$ in the transport equation for $< u^n >$.

DNS data are often presented in a non-dimensional form. In [19], for example, viscosity, the channel half-height and the friction velocity in an unstrained channel were used for this purpose, with all budget terms being normalized by $u_{r0}^{n+2} / \nu$. For the production terms, it gives, $P_n = P_n^* \cdot u_{r0}^{n+2} / \nu$, where $P_n^*$ is the non-dimensional production term from the DNS dataset for a corresponding velocity moment. Then, the non-dimensional length scale $L^*_{pn} = L_{pn} / h$ can be derived from (8) as

$$L^*_{pn} = \frac{1}{Re_{r0}} \left( \frac{1}{|P_n^*|} \right)^{\frac{1}{n+2}}.$$  

Notice that the scale has the limit at $n \to \infty$:

$$L^*_{inf} = \frac{1}{Re_{r0}}.$$  

(10)
This is a useful limit when one concerns with collecting statistics of different orders. It is also easier to find an estimate for (10) prior conducting simulations, which is essential for the grid generation process. Moreover, as previously discussed, the production terms are likely underpredicted in DNS in the flow areas with high-velocity gradients due to insufficient grid resolution. Expression (10) will not have this issue.

Since errors in the production terms affect velocity moments, and because an estimate for a velocity moment is easier to obtain than that for the production terms prior simulations, a lengthscale based on viscosity and a velocity moment of the $n$-th order was also derived:

$$L_{un} = \frac{\nu}{(|\langle u^n \rangle|)^{1/n}}. \quad (11)$$

Using relations between dimensional and non-dimensional parameters: $\langle u^n \rangle \frac{1}{n} = \nu (\langle u^m \rangle)^{\frac{1}{m}}$, where $u^*$ is a non-dimensional velocity fluctuation, and $L_{un}^* = L_{un}/h$, expression (11) can be re-written as:

$$L_{un}^* = \frac{1}{Re_{\tau0}(|\langle u' \rangle|)^{1/n}}. \quad (12)$$

which at $n \to \infty$ becomes again expression (10).

Figure 12 shows how 5 lengths scales: $L^*_{inf}$, $L^*_{p2}$, $L^*_{p3}$, $L^*_{u2}$, and $L^*_{u3}$, vary with $y$. In the figure, the Kolmogorov scale and the balance errors in the $\langle u^3 \rangle$ - equation are shown for comparison by the black solid and dash-dotted lines, respectively. Red and blue lines are for the
scales at \( n = 2 \) and 3, respectively. The scales \( L^*_{un} \) and \( L^*_{pn} \) are shown by the color dashed and solid lines.

The Kolmogorov scale is larger than \( L^*_{inf} \) everywhere in the flow and larger than the other four scales at \( y^+ > 5 \). This makes \( L^*_{inf} \) the safe choice for the reference scale in the areas of high mean velocity gradients. Whether the scales \( L^*_{p2}, L^*_{p3}, L^*_{u2}, \) and \( L^*_{u3} \) being larger than the Kolmogorov scale at \( y^+ < 5 \) has an impact on the simulation results remains to be verified particularly in the light that the velocity moments are effectively suppressed in this flow area (Figs. 2-5).

The value of \( L^*_{inf}, 0.00255 \), is about twice larger than the threshold value of \( L^*_{14} \) above which the balance errors become noticeable. From that, \( L^*_{inf}/2 \) is suggested as a criterion for the grid cell size to meet in the buffer zone and other flow areas with large velocity gradients.

The scales \( L^*_{u2}, \) and \( L^*_{u3} \) remain smaller than the Kolmogorov scale outside the flow buffer zone, but the scales \( L^*_{p2} \) and \( L^*_{p3} \) became larger than this scale close to the channel axis, where the mean velocity gradient and the production terms became close to zero. Whether this makes \( L^*_{u2} \) and \( L^*_{u3} \) a preferable choice as the reference scales outside the areas of high mean velocity gradients also require conducting separate DNS. Notice, that in well-resolved simulations, the difference between \( L^*_{u2} \) and \( L^*_{u3} \) should disappear. In this simulation these scales are very close to each other except for the location where \( < u^3 > \) approaches the zero value. After that, the discrepancy between the two scales continue to grow, most likely because \( L^*_{14} > L^*_{k} \) in this area (Fig. 11a), and \( < u^3 > \) is more sensitive to the grid resolution than \( < u^2 > \).

In sum, the best reference lengthscale outside the areas of high mean velocity gradients has yet to be determined in separate studies. Whether it can be the Kolmogorov scale is not that clear,
because having the grid cell size less than the Kolmogorov lengthscale did not help with reducing the balance errors in [13] (see Fig. 11b).

4. Conclusion

In the paper, the model expressions for fifth-order velocity moments derived from the truncated Gram-Charlier series expansions model for a turbulent flow field probability density function were successfully validated in a planar incompressible flow in a strained channel with and without separation using DNS data for fifth and lower-order moments.

Strain was found to affect differently odd and even moments, and moments of different orders. In particular, the magnitude of the velocity moments which are the most and the least effected, are those in the streamwise direction, $< u^k >$, and in the direction normal to the wall, $< v^k >$, respectively. The odd moments are suppressed more than the even ones, and the higher the moment order, the stronger is the correlation with strain. The most sensitive moment to strain is $< u^5 >$, and the moment with the least affected maximum value is $< v^2 >$, closely followed by $< v^4 >$.

Strain was also found to be beneficial for suppressing a small discrepancy between the model and DNS profiles of the fifth-order velocity moments observed in the buffer zone of velocity moments containing velocity fluctuations in the streamwise direction. Accuracy of DNS profiles for odd velocity moments was identified as the cause of this phenomenon.

Further investigation revealed the DNS grid resolution in the buffer zone to be insufficient for such moments. A new length scale to improve the accuracy of higher-order statistics near walls and other areas of high velocity gradients is proposed.
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Appendix A

This section provides transport equations for velocity moments that appear in a fourth-order closure prior any modeling in the channel flow considered in the current study:

\[
\frac{\partial \langle v^2 \rangle}{\partial t} + V \frac{\partial \langle v^2 \rangle}{\partial y} = 2 \langle v^2 \rangle \frac{\partial U}{\partial x} - \frac{\partial \langle v^3 \rangle}{\partial y} + \nu \frac{\partial^2 \langle v^2 \rangle}{\partial y^2} + \Pi_{yy} - \varepsilon_{yy}, \tag{A1}
\]

\[
\frac{\partial \langle uv \rangle}{\partial t} + V \frac{\partial \langle uv \rangle}{\partial y} = -\langle v^2 \rangle \frac{\partial U}{\partial y} - \frac{\partial \langle uv^2 \rangle}{\partial y} + \nu \frac{\partial^2 \langle uv \rangle}{\partial y^2} + \Pi_{xy} - \varepsilon_{xy}, \tag{A2}
\]

\[
\frac{\partial \langle v^3 \rangle}{\partial t} + V \frac{\partial \langle v^3 \rangle}{\partial y} = 3 \langle v^3 \rangle \frac{\partial U}{\partial x} - \frac{\partial \langle v^4 \rangle}{\partial y} + \nu \frac{\partial^2 \langle v^3 \rangle}{\partial y^2} + 3 \langle v^2 \rangle \frac{\partial \langle v^2 \rangle}{\partial y} + \Pi_{yyy} - \varepsilon_{yyy}, \tag{A3}
\]

\[
\frac{\partial \langle uv^2 \rangle}{\partial t} + V \frac{\partial \langle uv^2 \rangle}{\partial y} = \langle uv^2 \rangle \frac{\partial U}{\partial x} - \langle v^3 \rangle \frac{\partial U}{\partial y} - \frac{\partial \langle uv^3 \rangle}{\partial y} + \nu \frac{\partial^2 \langle uv^2 \rangle}{\partial y^2} + \\
+ \langle v^2 \rangle \frac{\partial \langle uv^2 \rangle}{\partial y} + 2 \langle uv \rangle \frac{\partial \langle v^2 \rangle}{\partial y} + \Pi_{yyy} - \varepsilon_{yyy}, \tag{A4}
\]

\[
\frac{\partial \langle v^4 \rangle}{\partial t} + V \frac{\partial \langle v^4 \rangle}{\partial y} = 4 \langle v^4 \rangle \frac{\partial U}{\partial x} - \frac{\partial \langle v^5 \rangle}{\partial y} + \nu \frac{\partial^2 \langle v^4 \rangle}{\partial y^2} + 4 \langle v^3 \rangle \frac{\partial \langle v^2 \rangle}{\partial y} + \Pi_{yyyy} - \varepsilon_{yyyy}, \tag{A5}
\]
\[
\frac{\partial < uv^3 >}{\partial t} + V \frac{\partial < uv^3 >}{\partial y} = 2 < uv^3 > \frac{\partial U}{\partial x} - < v^4 > \frac{\partial U}{\partial y} - \frac{\partial < uv^4 >}{\partial y} + \frac{\partial^2 < uv^3 >}{\partial y^2} + \\
+ < v^3 > \frac{\partial < uv >}{\partial y} + 3 < uv^2 > \frac{\partial < v^2 >}{\partial y} + \Pi_{xyy} - \varepsilon_{xyy}, \quad (A6)
\]

The flow remains homogeneous in the \(x\)- and \(z\)-directions even when strained [17]. The \(z\)-direction is the spanwise one.

To close this set of six equations, one has to model two fifth-order velocity moments: \(< uv^4 >\) and \(< v^5 >\) and some of the components of the tensors \(\Pi\) and \(\varepsilon\) (six of each) describing interaction of the turbulent velocity and pressure fields and the turbulent kinetic energy dissipation, respectively.

Models (5) for \(< uv^4 >\) and \(< v^5 >\): \(< uv^4 > = 6 < v^2 > \cdot < uv^2 > + 4 < v^3 > \cdot < uv >\) and \(< v^5 > = 10 < v^2 > \cdot < v^3 >\), do not bring additional unknown parameters into the set of equations (A1)-(A6), which is another benefit of this modeling approach. However, models for unknown components of the tensors \(\Pi\) and \(\varepsilon\) (not considered here) usually rely on variables that are not originally present in (A1)-(A6). As a result, the original set of equations may expand to include equations for new unknown parameters. For this reason, Appendix B provides transport equations for all velocity moments of the fourth order that may be found in an incompressible planar turbulent flow.

Expressions for unknown components of the tensors \(\Pi\) and \(\varepsilon\) in (A1)-(A6) are the following:

\[
\Pi_{yy} = -\frac{2}{\rho} < v \frac{\partial p}{\partial y} >, \quad \Pi_{xy} = -\frac{1}{\rho} \left[ < v \frac{\partial p}{\partial x} > + < u \frac{\partial p}{\partial y} > \right], \\
\Pi_{xxy} = -\frac{1}{\rho} \left[ < v^2 \frac{\partial p}{\partial x} > + 2 < uv \frac{\partial p}{\partial y} > \right], \quad \Pi_{yyx} = -\frac{3}{\rho} < v^2 \frac{\partial p}{\partial y} >
\]
\[
\Pi_{yy} = -\frac{1}{\rho} \left[ < v^i \frac{\partial p}{\partial x} > + 3 < uv^2 \frac{\partial p}{\partial y} > \right], \quad \Pi_{yzy} = -\frac{4}{\rho} < v^i \frac{\partial p}{\partial y} >,
\]

\[
\varepsilon_{yy} = 2\nu < \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 >, \quad \varepsilon_{xy} = 2\nu < \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) >,
\]

\[
\varepsilon_{yy} = 4\nu < \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) > + 2\nu < \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 >,
\]

\[
\varepsilon_{yy} = 6\nu < \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 >, \quad \varepsilon_{yzy} = 12\nu < v^2 \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 >,
\]

\[
\varepsilon_{yzy} = 6\nu \left< v^2 \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] > + 6\nu < \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 >.
\]

Here, \( p \) is pressure fluctuation, \( \rho \) is the fluid density.
The set of RANS equations for the fourth-order velocity moments in an incompressible planar turbulent flow is the following:

\[
\frac{D < u^4 >}{Dt} = -\frac{\partial < u^5 >}{\partial x} - \frac{\partial < u^4 v >}{\partial y} - 4 < u^4 > \frac{\partial U}{\partial y} - 4 < u^3 v > \frac{\partial U}{\partial y} + 4 < u^3 > \left[ \frac{\partial < u^2 >}{\partial x} + \frac{\partial < uv >}{\partial y} \right] + \\
+ v \left[ \frac{\partial^2 < u^4 >}{\partial x^2} + \frac{\partial^2 < u^4 >}{\partial y^2} \right] + \Pi_{xxxx} - \varepsilon_{xxxx}, \tag{B1}
\]

\[
\frac{D < u^3 v >}{Dt} = -\frac{\partial < u^4 >}{\partial x} - \frac{\partial < u^3 v^2 >}{\partial y} - 2 < u^3 v > \frac{\partial U}{\partial y} - 3 < u^2 v^2 > \frac{\partial U}{\partial y} - < u^4 > \frac{\partial \bar{V}}{\partial x} + \\
+ 3 < u^2 v > \left[ \frac{\partial < u^2 >}{\partial x} + \frac{\partial < uv >}{\partial y} \right] + < u^3 > \left[ \frac{\partial < v^2 >}{\partial y} + \frac{\partial < uv >}{\partial x} \right] + \\
+ v \left[ \frac{\partial^2 < u^3 v >}{\partial x^2} + \frac{\partial^2 < u^3 v >}{\partial y^2} \right] + \Pi_{xxxy} - \varepsilon_{xxxy}, \tag{B2}
\]

\[
\frac{D < u^2 v^2 >}{Dt} = -\frac{\partial < u^3 v^2 >}{\partial x} - \frac{\partial < u^2 v^3 >}{\partial y} - 2 < uv^2 > \frac{\partial U}{\partial y} - 2 < u^2 v > \frac{\partial \bar{V}}{\partial x} + 2 < u^2 > \left[ \frac{\partial < u^2 >}{\partial x} + \frac{\partial < uv >}{\partial y} \right] + \\
+ 2 < u^2 v > \left[ \frac{\partial < v^2 >}{\partial y} + \frac{\partial < uv >}{\partial x} \right] + v \left[ \frac{\partial^2 < u^2 v^2 >}{\partial x^2} + \frac{\partial^2 < u^2 v^2 >}{\partial y^2} \right] + \Pi_{xxxy} - \varepsilon_{xxxy}, \tag{B3}
\]

\[
\frac{D < uv^3 >}{Dt} = -\frac{\partial < u^2 v^3 >}{\partial x} - \frac{\partial < uv^4 >}{\partial y} + 2 < uv^3 > \frac{\partial U}{\partial y} - < v^4 > \frac{\partial U}{\partial y} - 3 < u^2 v^2 > \frac{\partial \bar{V}}{\partial x} + \\
+ < v^3 > \left[ \frac{\partial < u^2 >}{\partial x} + \frac{\partial < uv >}{\partial y} \right] + 3 < uv^2 > \left[ \frac{\partial < v^2 >}{\partial y} + \frac{\partial < uv >}{\partial x} \right] + \\
+ v \left[ \frac{\partial^2 < uv^3 >}{\partial x^2} + \frac{\partial^2 < uv^3 >}{\partial y^2} \right] + \Pi_{xxxy} - \varepsilon_{xxxy}, \tag{B4}
\]
\[
\frac{\mathcal{A} <v^4>}{Dt} = -\frac{\partial <uv^4>}{\partial x} - \frac{\partial <v^5>}{\partial y} + 4 <v^4> \frac{\partial U}{\partial x} - 4 <uv^3> \frac{\partial V}{\partial x} + 4 <v^3> \left[ \frac{\partial <v^2>}{\partial y} + \frac{\partial <uv>}{\partial x} \right] + \\
+ V \left[ \frac{\partial^2 <v^4>}{\partial x \partial x} + \frac{\partial^2 <v^4>}{\partial y \partial y} \right] + \Pi_{yy} - \varepsilon_{yyy}.
\]

(B5)

Here, \( \frac{\mathcal{A}}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} \). Unknown components of the fourth-rank tensors \( \Pi \) and \( \varepsilon \) in equations (B1)-(B5) are expressed as:

\[
\Pi_{xxx} = -\frac{4}{\rho} <u^3 \frac{\partial p}{\partial x}>, \quad \Pi_{xyy} = -\frac{1}{\rho} \left[ 3 <u^3 \frac{\partial p}{\partial y}> + <u^3 \frac{\partial p}{\partial y}> \right], \quad \Pi_{yy} = -\frac{2}{\rho} \left[ <uv^3 \frac{\partial p}{\partial y}> + <u^3 \frac{\partial p}{\partial y}> \right],
\]

\[
\varepsilon_{xxx} = 12\nu <u^2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right]>, \quad \varepsilon_{yyy} = 12\nu <v^2 \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right]>,
\]

\[
\varepsilon_{xyy} = 6\nu \left[ <u^2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] + <uv \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] > \right],
\]

\[
\varepsilon_{vyy} = 2\nu \left[ <v^2 \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] + <uv \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] > \right] + 8\nu <uv \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] >.
\]

Other terms can be found in Appendix A.
FIG. 1. The mean velocity profiles in a time-evolving strained planar turbulent channel flow at different values of $t'$: 0, 0.02, 0.1, 0.191, 0.285, 0.365, 0.675, and 0.772 (shown from left to right). Labels 0, 3, 5, 6, and 7 correspond to $t'_i$, where $i = 0, 3, 5, 6, \text{ and } 7$. Line patterns: dotted – $t'_7$, solid – $t'_6$, dashed – pre-separated flow at earlier times indicated by the labels. Horizontal thin lines in Fig. 1b show approximate locations of the separation region and of the velocity minimum within the flow.
FIG. 2. Variation of $< u^k >, k = 2, ..., 5$ with time in a time-evolving strained planar turbulent channel flow.

Notations are the same as in Fig. 1.
FIG. 3. Variation of $\langle v^k \rangle$, $k = 2, \ldots, 5$ with time in a time-evolving strained planar turbulent channel flow. Notations are the same as in Fig. 1.
FIG. 4. Variation of \( \langle u^m v^n \rangle \) (\( 2 \leq m + n \leq 4, \ m, n \geq 1 \)) with time in a time-evolving strained planar turbulent channel flow. Notations are the same as in Fig. 1.
FIG. 5. Variation of $< u^m v^n >$ ($m = 1, ..., 4; n = 5 - m$) with time in a time-evolving strained planar turbulent channel flow. Notations are the same as in Fig. 1.
FIG. 6. The fifth-order moments: solid lines – profiles obtained from expressions (5) using DNS data [14,15] for the second- and third-order moments, dashed lines – DNS data for the fifth moments [14,15]. Labels are the same as in Fig. 1. Color scheme: black – profiles in the unstrained channel and in the channel flow with separation, red – the maximum profile for the given moment, blue – profiles after reaching the maximum and before the flow separation.
FIG. 7. The ratio $\Delta$ (Eq. (6)) for $<u^5>$ (a,b) and $<v^5>$ (c,d) at $t_6'$ (a,c) and $t_7'$ (b,d). Notation: solid lines – the ratios $\Delta$; dashed lines – velocity moments.
FIG. 8. Variation of the ratio $\Delta$ (Eq. (6)) for $< u^5 >$ and the turbulence production by $\partial U / \partial y$ (the production terms are not to scale). Notations: solid line – $< u^5 >$, dashed lines – production terms in the transport equations for $< u^5 >$ and $< u^3 >$, dotted line – production term in the transport equations for $< u^2 >$. 
FIG. 9. The balance errors in the DNS budgets of velocity moments $< u^n >$ in the unstrained channel flow: a) in the equation for $< u^2 >$ at $t = 1.22 \cdot 10^5$ from [21], b) in the equations for $< u^2 >$ and $< u^3 >$ at $t = 5.06 \cdot 10^4$ [14]. The production terms are also shown in b) by blue lines. Solid lines correspond to the terms in the $< u^2 >$ transport equation and dashed lines to the terms in the $< u^3 >$ transport equation.
FIG. 10. Ratios between the local mean velocity and local velocity scales obtained from velocity moments $<u^n>$, where $n = 2, ..., 5$ using DNS data in a) the unstrained channel [14,19], b) the zero-pressure-gradient boundary layer at $Re_\theta = 5200$ [13]. Balance errors in the budgets of $<u^2>$ and $<u^3>$ are also shown by black solid and dashed lines, respectively.
FIG. 11. Variation of the non-dimensional lengthscales with $y$: a) $L_k^*$ (dash-dotted line) and $L_1^*$ (long-dashed line) obtained with DNS data for the unstrained channel from [14,19], b) $L_k^*$ (dash-dotted line) and $L_1^*$ (long-dashed line) obtained with DNS data for the zero-pressure-gradient boundary layer at $Re_p = 5200$ [13]. Balance errors in the budgets of $<u^2>$ and $<u^3>$ are also shown by solid and dashed lines, respectively.
FIG. 12. Variation of the non-dimensional parameters with $y$. Notations: lengthscales $L'_{\text{inf}}$ (black dashed line), $L'_{p2}$ (red solid line), $L'_{p3}$ (blue solid line), $L'_{u2}$ (red dashed line), and $L'_{u3}$ (blue dashed line). Balance errors in the budget of $<u^3>$ are also shown by dash-dotted line.