Moduli Space Metric of $\mathcal{N} = 2$ Supersymmetric $SU(N)$ Gauge Theory and the Enhanccon

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We compute the moduli space metric of $SU(N)$ Yang-Mills theory with $\mathcal{N} = 2$ supersymmetry in the vicinity of the point where the classical moduli vanish. This gauge theory may be realized as a set of $N$ D7-branes wrapping a $K3$ surface, near the enhanccon locus. The moduli space metric determines the low-energy worldvolume dynamics of the D7 branes near this point, including stringy corrections. Non-abelian gauge symmetry is not restored on the worldvolume at the enhanccon point, but rather the gauge group remains $U(1)^{N-1}$ and light electric and magnetically charged particles coexist. We also study the moduli space metric for a single probe brane in the background of $N - 1$ branes near the enhanccon point. We find quantum corrections to the supergravity probe metric that are not suppressed at large separations, but are down by $1/N$ factors, due to the response of the $N - 1$ enhanccon branes to the probe. A singularity appears before the probe reaches the enhanccon point where a dyon becomes massless. We compute the masses of $W$-bosons and monopoles in a large $N$ limit near this critical point.

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1. Introduction

Some time ago, Argyres and Faraggi [1], and Klemm, Lerche, Theisen and Yankielowicz [2] obtained the exact solution of the low-energy effective action of $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory, generalizing the work of Seiberg and Witten for $SU(2)$ [3]. The answer is written as a prepotential for the $\mathcal{N} = 2$ theory, expressed as the period matrix of a certain hyperelliptic Riemann surface. The period matrix becomes the metric for the $N - 1$ dimensional moduli space of this gauge theory. This period matrix was evaluated in an instanton expansion, in the limit of well-separated moduli, by D’Hoker, Kirchever and Phong [4]. Picard-Fuchs equations were set up by [5]. Many of these results are described in the review article [6].

The large $N$ limit of this theory was considered by Douglas and Shenker [7]. This limit is rather subtle, because if the moduli have finite separations as $N \to \infty$ the instanton corrections vanish exponentially fast. To obtain a non-trivial limit, one must consider points on moduli space where separations vanish as $1/N$. Douglas and Shenker focused on the case where the moduli eigenvalues are located on the real axis. This is the case that is smoothly connected to vacua of the $\mathcal{N} = 1$ gauge theory with massive adjoint matter. In this paper we will generalize the analysis of [7], to the case when the moduli sit near a circle (or circles) in the complex plane, as well as to the case where a pair of eigenvalues are separated from such a circle. This limit has also been studied in [8,9].

One of the main motivations for studying this problem is given by recent attempts to understand the resolution of certain naked singularities in string theory. For the case of D-branes wrapping a $K3$ surface, it was argued [10] that the timelike repulson singularity present in the naive continuation of the geometry, should be replaced by a shell of D-brane source sitting outside this point – the so-called enhancon locus. Inside this locus, it was argued the geometry should be flat. This picture received further support from a detailed analysis of the junction conditions [11], and from black hole entropy considerations [12,13].

A detailed understanding of this phenomena requires a full string theory treatment of the problem. From the point of view of the supergravity analysis, the D-brane source boundary conditions are put in by hand. In this paper we present the nonperturbative moduli space metric that describes the low-energy worldvolume dynamics of D7-branes wrapping $K3$, near the enhancon locus. This is an important step toward a better string theory understanding of the enhancon phenomenon.

Our results support the picture advocated in [10]. A wrapped D7-brane probe approaching $N – 1$ branes at the enhancon point, smoothly melts into the other branes,
and then emerges from the other side. The probe brane does not see the interior of the geometry at all. The enhancon locus is simply a non-singular point from the viewpoint of the probe. Interestingly, we find nontrivial quantum corrections to the probe metric that do not fall off at large separations, however these appear at subleading order in $1/N$.

The plan of this paper is as follows: in section 2 we briefly review the solution of $SU(N)\,\mathcal{N} = 2$ supersymmetric gauge theory of $[1,2]$ to fix notation; in section 3 we compute the moduli space metric along a one-complex parameter slice of the space in two different situations: for a pair of enhancon-like shells, and for a probe brane in an enhancon background. An analytic form of the metric along this slice is presented for all values of $N$. The metric includes nontrivial quantum corrections. For the case of the probe brane, the moduli space metric has a singularity as the probe starts to get close to the enhancon shell, when two eigenvalues collide and a dyon becomes massless. We obtain analytic expressions for the masses of dyons and monopoles in the large $N$ limit, near this critical point. In section 4 we comment on the implications of these results for the resolution of spacetime singularities in string theory, and discuss future prospects.

2. Review and Notation

Recall that the solution to the low-energy effective action of $\mathcal{N} = 2$ supersymmetric
$SU(N)$ Yang-Mills theory is given in terms of an auxiliary Riemann surface $\mathcal{C}$, that consists of a genus $N - 1$ hyperelliptic curve $\mathcal{C}$ defined by

$$Y^2 = \prod_{j=1}^{N} (X - \phi_j)^2 - \Lambda^{2N} = (P(X))^2 - \Lambda^{2N}$$

along with the 1-form

$$\lambda = \frac{X dP}{2\pi i Y}$$

whose $\phi_j$ derivatives are holomorphic. The $\phi_i$ denote the classical expectation values of the gauge theory moduli. For SU(N) gauge theory the moduli space coordinates must satisfy the tracelessness condition

$$\sum_{j=1}^{N} \phi_j = 0.$$  

The periods $a_j$ and $a_{Dk}$ are then given by the integrals

$$a_j = \oint_{\beta_j} \lambda, \quad a_{Dm} = \oint_{\alpha_m} \lambda$$
for some choice of cycles $\alpha_m$ and $\beta_k$ where only $N - 1$ of each of these sets of cycles is independent.

To construct the moduli space metric one requires derivatives of the periods with respect to the moduli space coordinates $\phi_k$. If the periods (2.4) can be evaluated as functions of the coordinates $\phi_j$ then this is easy to compute. Finding the periods at a generic point in moduli space however is difficult and it is easier instead to compute the derivatives at a fixed point in moduli space directly by noting that

$$\frac{\partial \lambda}{\partial \phi_k} = -\frac{1}{i2\pi Y} \frac{\partial P}{\partial \phi_k} dX + d \left( \frac{X \frac{\partial P}{\partial \phi_k}}{Y} \right).$$

(2.5)

The term inside the total derivative is periodic around a closed contour, so the $\phi$-derivatives of the period integrals can be evaluated using only the first term of (2.5).

Using this information one can then compute the period matrix $\tau_{mk}$ via the prescription

$$\tau_{mk} = \frac{\partial a_{Dm}}{\partial a_k} = \sum_{l=1}^{N-1} \frac{da_{Dm}}{d\phi_l} \left( \frac{da}{d\phi} \right)^{-1}_{lk}$$

(2.6)

where the derivatives on the far right-hand-side are total derivatives. That is, we have removed the $\phi_N$ dependence using the tracelessness property (2.3) so that

$$\frac{da_{Dm}}{d\phi_l} = \frac{\partial a_{Dm}}{\partial \phi_l} - \frac{\partial a_{Dm}}{\partial \phi_N},$$

(2.7)

It is important to note however that this prescription has an apparent singularity if $\phi_j = \phi_k$ for any indices $j \neq k$. The singularity can be easily seen from (2.3) by noting that $\partial a_l/\partial \phi_j = \partial a_l/\partial \phi_k$ when evaluated at the point $\phi_j = \phi_k$. The $(N - 1) \times (N - 1)$ matrix $[\partial a/\partial \phi]$ is then not invertible. If the branch points of the curve (2.1) are all distinct, then we do not expect this point in moduli space to be a real singularity, but rather just a coordinate singularity. In the cases that we discuss below we will indeed see that this is true.

3. Moduli Space Metric

It is rather difficult to obtain explicit expressions for the metric throughout the full moduli space. To simply matters we will focus on two distinct one complex parameter slices through the full $N - 1$-dimensional moduli space, each of which contains the enhancon point (all $\phi_i = 0$).
3.1. Two enhancon shells

The first slice we consider retains a $Z_N$ rotation symmetry in the curve $C$, and corresponds to a pair of concentric enhancon shells, which coalesce at the enhancon point. The corresponding brane configurations have been analyzed from the supergravity viewpoint in \cite{IIIB}. We parametrized the moduli space coordinates in terms of a complex parameter $v$ as

$$\phi_j = v e^{i2\pi j/N}$$

where $1 \leq j \leq N$. Note that this choice of moduli space coordinates does satisfy the tracelessness condition (2.3). The enhancon point sits at $v = 0$, which leads to a coordinate singularity in the period matrix (2.6) as discussed above. One can get around this problem as we shall see by computing the period matrix for nonzero $v$ and then taking the limit $v \to 0$, in which case a nonsingular answer is obtained. For nonzero $v$ this subspace corresponds to two separate enhancon shells.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plane.png}
\caption{Positions of branch cuts, and choice of 1-cycles for double enhancon configuration.}
\end{figure}

To see this in more detail we must solve for the branch points of the curve given by $Y = 0$. For the simple subspace in moduli space that we have taken these can be found
exactly. The point is that the product in (2.1) simplifies to
\[
\prod_{j=1}^{N} (X - \phi_j) = X^N - v^N, \tag{3.2}
\]
which follows easily from the identity
\[
\sum_{j=1}^{N} e^{i2\pi jk/N} = N \delta_{k,0} \mod N . \tag{3.3}
\]
The branch points are then found to be
\[
X_{2k} = (\Lambda^N + v^N)^{1/N} e^{i2\pi k/N} ,
\]
\[
X_{2k+1} = (\Lambda^N - v^N)^{1/N} e^{i2\pi (k+1/2)/N}
\tag{3.4}
\]
for 1 ≤ k ≤ N. For real v we get the structure shown in fig. 1 with a pair of concentric circles of branch points. We have also shown some of the cycles \(\alpha_m\) and \(\gamma_j\). We take the \(\alpha_m\) cycle to enclose the \(X_{2m}\) and \(X_{2m+1}\) branch points, with 1 ≤ m < N. The \(\gamma_j\) cycle encloses the \(X_{2j-1}\) and \(X_{2j}\) branch points, with 1 ≤ j ≤ N. The \(\beta_n\) cycles are then given in terms of the \(\gamma\) cycles by
\[
\beta_n = \sum_{j=1}^{n} \gamma_j \tag{3.5}
\]
and we take 1 ≤ n < N. The \(\alpha\) and \(\beta\) cycles comprise a canonical basis of homology 1-cycles.

The holomorphic 1-forms (2.5) on this slice of moduli space are given by
\[
\frac{\partial \lambda}{\partial \phi_k} = -\frac{1}{i2\pi} \frac{(X^N - v^N)}{(X - v e^{i2\pi k/N}) \sqrt{(X^N - v^N)^2 - \Lambda^2N}} \tag{3.6}
\]
While one could try to compute the period matrix using this basis of holomorphic 1-forms, it turns out that it is not the most convenient. The computation of the period matrix is simpler if we change basis to
\[
\frac{\partial \lambda}{\partial \theta_n} = \sum_{k=1}^{N} \frac{\partial \lambda}{\partial \phi_k} e^{i2\pi kn/N}
\]
\[
= \sum_{k=1}^{N-1} \left( \frac{\partial \lambda}{\partial \phi_k} - \frac{\partial \lambda}{\partial \phi_N} \right) e^{i2\pi kn/N} \tag{3.7}
\]
\[
= -\frac{1}{i2\pi N} \left( v \sqrt{X} \right)^{N-n} \frac{d X^N}{\sqrt{(X^N - v^N)^2 - \Lambda^2N}}
\]
where the second line follows from the identity (3.3).

To compute the period matrix we now compute the derivatives of the periods

\[
\frac{\partial a_j}{\partial t_n} = \oint_{\beta_j} \frac{\partial \lambda}{\partial t_n}, \quad \frac{\partial a_{Dm}}{\partial t_n} = \oint_{\alpha_m} \frac{\partial \lambda}{\partial t_n}.
\] (3.8)

For the \(a_D\)-periods we make the change of variables

\[X = (\Lambda^N e^{i\varphi} + v^N)^{1/N} e^{i2\pi j/N}\] (3.9)

which reduces (3.7) to

\[
\frac{\partial \lambda}{\partial t_n} = -\frac{v^{N-n}}{2\pi N} e^{i2\pi j n/N} \frac{1}{(\Lambda^N e^{i\varphi} + v^N)^{1-n/N}} \frac{e^{i\varphi} d\varphi}{\sqrt{e^{2i\varphi} - 1}}.
\] (3.10)

Defining

\[f(v^N, n/N) \equiv -\frac{1}{\pi N} \int_0^\pi d\varphi \frac{e^{i\varphi}}{(\Lambda^N e^{i\varphi} + v^N)^{1-n/N}} \frac{e^{i\varphi}}{\sqrt{e^{2i\varphi} - 1}}\] (3.11)

we then have

\[
\frac{\partial a_{Dm}}{\partial t_n} = v^{N-n} f(v^N, n/N) e^{i2\pi mn/N}.
\] (3.12)

Similarly for the \(a\)-periods we find

\[
\frac{\partial \tilde{a}_k}{\partial t_n} = v^{N-n} f(-v^N, n/N) e^{i2\pi (k-1/2)n/N}.
\] (3.13)

The tilde notation here denotes a period evaluated along a \(\gamma\) contour. To obtain the \(a\)-period we must sum as in (3.5), or in matrix notation

\[
\frac{\partial a_j}{\partial t_n} = \sum_{k=1}^{N-1} q_{jk} \frac{\partial \tilde{a}_k}{\partial t_n}
\] (3.14)

where the matrix entries of \(q_{jk}\) are given by ones for \(j \leq k\) and zeroes everywhere else.

What makes the coordinate transformation in (3.7) so convenient is that (3.12) and (3.13) depend on the \(a_D\) and \(\tilde{a}\) indices respectively only through a phase factor. Consequently either matrix is easy to invert provided that \(f(v^N, n/N)\) does not vanish for any \(n\) and \(v\). Specifically the \((N-1) \times (N-1)\) matrix

\[m_{jn} \equiv e^{i2\pi j n/N}\] (3.15)

has the inverse

\[(m^{-1})_{nk} = \frac{1}{N} (e^{-i2\pi nk/N} - 1)\] (3.16)
from which it follows that
\[
\left( \frac{\partial a_D}{\partial t} \right)_{np}^{-1} = \frac{1}{v^{N-n} f(v^N, n/N)} (m^{-1})_{np},
\]
\[
\left( \frac{\partial \tilde{a}}{\partial t} \right)_{nj}^{-1} = \frac{e^{i\pi n/N}}{v^{N-n} f(-v^N, n/N)} (m^{-1})_{nj}.
\] (3.17)

Using (3.12) and (3.17) it is now straightforward to compute the period matrix (2.6), and we find
\[
\tau_{mk} = \frac{2i}{N} \sum_{n=1}^{N-1} \frac{f(v^N, n/N)}{f(-v^N, n/N)} \sin(\pi n/N) e^{i2\pi n(m-k)/N}.
\] (3.18)

In obtaining this result we have multiplied the inverse \( \tilde{a} \) matrix in (3.17) on the right by the matrix \( q^{-1}_{jk} = \delta_{j,k} - \delta_{j-1,k} \) which converts the \( \gamma \)-period \( \tilde{a} \) matrix to a \( \beta \)-period \( a \) matrix.

While we cannot simplify the period matrix any further for generic values of \( v \), at \( v = 0 \) the sum is straightforward to do,
\[
\tau_{mk} = \frac{i}{N} (\cot(\pi(m - k + 1/2)/N) - \cot(\pi(m - k - 1/2)/N)).
\] (3.19)

There is a theorem in Riemann surface theory that says that \( \tau_{mk} \) is symmetric and that its imaginary part is positive definite. By replacing the summation index \( n \) by \( N - n \) one easily shows that symmetry implies the condition that
\[
\frac{f(v^N, n/N)}{f(-v^N, n/N)} = \frac{f(v^N, 1 - n/N)}{f(-v^N, 1 - n/N)}.
\] (3.20)

For \( v = 0 \) this condition holds trivially as both sides are one, and moreover from the explicit expression given in (3.19). For generic values of \( v \) we have checked numerically that it does hold for various values of \( v \) and \( N \), which provides a check on our explicit expression for the period matrix.

For positivity we again do not have a direct analytic proof that this condition holds for the explicit form for \( \tau \) given above. Even for \( v = 0 \) we have not been able to show analytically that all eigenvalues are positive. Curiously though it is straightforward to check that \( \sin(2\pi kj/N) \) for \( 1 \leq j \leq \lfloor N/2 \rfloor \) are eigenvectors of \( \text{Im}[\tau_{mk}] \) with eigenvalues \( 2 \sin(\pi j/N) \), which are indeed positive. For the remaining eigenvalues and for more general values of \( v \), we have checked numerically that the eigenvalues are indeed positive.
3.2. Probe in Enhanccon Background

In this section we compute the period matrix in various limits along a one dimensional slice of moduli space corresponding to a wrapped D7-brane probe of the enhanccon background. For the most part we are able to derive explicit expressions for finite $N$, and consider the large $N$ expansion only when necessary.

The moduli in this case are given by

$$\phi_j = -\frac{v}{N}, \quad 1 \leq j \leq (N-1)$$

$$\phi_N = v - \frac{v}{N}$$

where $v$ is the complex coordinate which parametrizes the subspace that we will be investigating. The associated curve $C$ is given by

$$Y^2 = (X - v + v/N)^2(X + v/N)^{2(N-1)} - \Lambda^{2N}. \quad (3.22)$$

$v/\Lambda \ll 1$ Limit

For $v/\Lambda \ll 1$ the branch points of the curve are given by the series expansion in $v$

$$X_k = \Lambda e^{i\pi k/N} \left(1 + \frac{1}{2N} \left(\frac{v}{\Lambda} e^{-i2\pi k/N} \left(1 - \frac{1}{N}\right) + O\left(\left(\frac{v}{\Lambda}\right)^3\right)\right)\right). \quad (3.23)$$

This form of the branch points suggests a more convenient basis for the holomorphic 1-forms than those given in (2.5) (which, as discussed above, are not independent when more than one of the moduli space coordinates $\phi_k$ are equal). Specifically we shall take the basis of holomorphic forms given by

$$\omega_n = -\frac{1}{2\pi i} \frac{X^{n-1} dX}{Y}, \quad 1 \leq n \leq (N-1). \quad (3.24)$$

The advantage of this basis is that the linear in $v$ correction to the period matrix vanishes identically, as we now explain in more detail. We shall take the same basis of cycles as in the previous example for $v = 0$. That is, the $\alpha_k$ cycle encircles the $X_{2k}$ and $X_{2k+1}$ branch points, while the $\gamma_j$ cycle encircles the $X_{2j-1}$ and $X_{2j}$ branch points. The $\beta_k$ cycles are then defined as in (3.5).
To evaluate a period integral of any of the holomorphic forms $\omega_n$ over an $\alpha_k$ cycle, it is convenient to parametrize the integral in terms of an angular variable $\varphi$ as

$$X = \Lambda e^{i2k/N} e^{i\varphi/N} \left(1 + \frac{1}{2N}(\frac{v}{\Lambda})^2 e^{-i4\pi k/N} e^{-i2\varphi/N} (1 - \frac{1}{N}) + O \left(\left(\frac{v}{\Lambda}\right)^3\right)\right),$$  

(3.25)
a result which follows naturally from the form of the branch points (3.23). Similarly for a period integral about a $\gamma_k$ cycle, one simply replaces $k \rightarrow (k - 1/2)$ in (3.25). In this parametrization $X^{n-1} dX$ clearly has no linear in $v$ piece. Furthermore $Y$ has no linear in $v$ piece as follows by substituting (3.25) into (3.22) and expanding, or more simply by noting that $dX(v)/dv$ vanishes at $v = 0$ for $X(v)$ a branch point of the curve which follows from differentiating (3.22) with respect to $v$.

The periods are now straightforward to write down in an expansion in $v$. Specifically we find

$$\frac{\partial a_{Dm}}{\partial \xi_n} = \oint_{\alpha_m} \omega_n = \frac{1}{2\pi N} \Lambda^{-(N-n)} \left(F(n)m_{mn} + \left(\frac{v}{\Lambda}\right)^2 \frac{(N-1)(n-2)}{2N^2} F(n-2)m_{m(n-2)} + O \left(\left(\frac{v}{\Lambda}\right)^3\right)\right),$$

(3.26)

$$\frac{\partial \tilde{a}_k}{\partial \xi_n} = \oint_{\gamma_k} \omega_n \bigg|_{m=k-1/2} = \frac{\partial a_{Dm}}{\partial \xi_n}$$

where we have defined

$$F(n) \equiv \sqrt{2} e^{-i\pi/4} \int_0^\pi d\varphi \frac{e^{i(n/N-1/2)\varphi}}{\sqrt{\sin(\varphi)}}$$

(3.27)

$$= -i4\sqrt{\pi} \frac{N}{n} \frac{\Gamma(1 + n/(2N))}{\Gamma(1/2 + n/(2N))} e^{in\pi/(2N)} \sin(n\pi/(2N))$$

and the matrix

$$m_{kn} \equiv e^{i2\pi kn/N}.$$  

(3.28)

The $\partial a_k/\partial \xi_n$ periods are integrals over $\beta_k$-cycles defined in terms of the $\gamma$ cycles in (3.5).

To compute the period matrix we need the inverse

$$\left(\frac{\partial \tilde{a}}{\partial \xi}\right)^{-1}_{nj} = -iNA^{-n}e^{i\pi n/N} F(n) \left(\frac{v}{\Lambda}\right)^2 \sum_{l,k=1}^{N-1} (m^{-1})_{nl} \alpha_2(l,k)m_{lk}(m^{-1})_{kj}$$

(3.29)

$$+ O \left(\left(\frac{v}{\Lambda}\right)^3\right)$$

where we have defined

$$\alpha_2(k,n) \equiv \frac{(N-1)(n-2)}{2N^2} F(n-2) F(n) e^{-i4\pi (k+1/2)/N}.$$  

(3.30)
The period matrix is then given by

\[ \tau_{mk} = \tau_{mk}^{\text{double}}|_{v=0} + \left( \frac{v}{\Lambda} \right)^2 \sum_{n,j=1}^{N-1} (\alpha_{D_2}(m,n)m_{mn}(m^{-1})_{nj} \right. \\
- \sum_{l,p=1}^{N-1} m_{mn}(m^{-1})_{nl} \alpha_2(l,p)m_{lp}(m^{-1})_{pj} (\delta_{j,k} - \delta_{j-1,k}) + \mathcal{O}(\left( \frac{v}{\Lambda} \right)^3) \]  

where the first term is the period matrix of the double enhancon evaluated at \( v = 0 \) found in the previous subsection and we have defined

\[ \alpha_{D_2}(m,n) \equiv \frac{(N - 1)(n - 2) F(n - 2)}{2N^2} \frac{F(n)}{F(n - 1)} e^{-i4\pi m/N}. \]

The last two Kronecker delta functions convert the \( \tilde{a} \) periods to \( a \) periods as discussed above.

To connect this result with the supergravity description, we compute the induced metric on the subspace of moduli space parametrized by \( v \). This induced metric corresponds to the moduli space metric of a probe \( D7 \) brane in the enhancon background, which may be computed using the Born-Infeld action plus Chern-Simons couplings. To construct this induced metric we need the periods \( a_k \) as functions of \( v \). These are given by the contour integrals (2.4). The computation is similar to what we have done before. Namely one parametrizes the integrals as (3.25) and expands the integrand in powers of \( v \). The end result is

\[ a_k(v) = -\frac{\Lambda}{i4\pi \sin(\pi/N)} \left( F(N + 1)(1 - e^{i2\pi k/N}) + \left( \frac{v}{\Lambda} \right)^2 \frac{(N - 1)}{2N^2} F(N - 1)(1 - e^{-i2\pi k/N}) \right. \\
+ \mathcal{O}\left( \left( \frac{v}{\Lambda} \right)^3 \right) \right). \]

(3.33)

The induced metric is now given by

\[ ds_{\text{probe}}^2 = \sum_{m,k=1}^{N-1} \text{Im}[\tau_{mk}] \left( \frac{da_m}{dv} \right) \frac{da_k}{dv} |dv|^2, \]

\[ = \frac{1}{8\pi^2 N \Lambda^2 \sin(\pi/N)} |F(N - 1)|^2 |vdv|^2 \]

\[ = \frac{1}{8\pi^2 N \Lambda^2 \sin(\pi/N)} (d\rho^2 + 4\rho^2 d\phi^2) \]

(3.34)
where the \((\rho, \phi)\) coordinates are defined as \(v \equiv \sqrt{2\rho} \exp(i\phi)\). The last form shows in particular that the induced metric has a conical singularity at \(\rho = 0\) with negative deficit angle.

One may use this result to compute the trajectory of a probe brane approaching the enhancon shell. At the enhancon point, the probe hits the conical singularity in the induced metric. The full metric is smooth at this point, and may be used to continue the motion of the probe past the conical singularity. On the subspace parametrized by \(v\), the \(\phi_i\) respect a \(Z_{N-1}\) symmetry. Since the periods \(a\) vary continuously (and with continuous first derivatives) through the enhancon point, the same symmetry will be present after passing through the enhancon point. The unique vacuum expectation value that preserves this symmetry corresponds to the subspace parametrized by \(v\), so the probe remains on the v-slice. From the point of view of the spacetime coordinates, the probe merges with the enhancon, and then smoothly reemerges from the other side.

\(v/\Lambda \gg 1\) Limit

In this section we consider the opposite extreme for the probe, \(v/\Lambda \gg 1\). In this limit the probe is located far from the remaining moduli as is clear from the polynomial (3.22). In particular one finds for the branch points in this limit

\[
\tilde{X}_\pm = \frac{v}{\Lambda} \left(1 \pm \left(\frac{\Lambda}{v}\right)^N\right),
\]

\[
\tilde{X}_j = x_j \left(1 + \frac{1}{N} \frac{\Lambda}{v} x_j - \frac{1}{2N} \left(\frac{\Lambda}{v}\right)^2 (x_j)^2 + \mathcal{O}\left(\frac{x_j \Lambda}{v}\right)^3\right)
\]

(3.35)

where

\[
x_j \equiv \left(\frac{\Lambda}{v}\right)^{1/(N-1)} e^{i\pi(j-1)/(N-1)}
\]

(3.36)

for \(1 \leq j \leq 2(N - 1)\). We have here introduced the scaled and shifted variable \(\tilde{X} \equiv (X + v/N)/\Lambda\).

So far our choice of electric versus magnetic cycles has been relatively arbitrary. However in this case we must be careful to identify the \(\gamma_N\)-cycle which encircles the branch cut connecting the \(\tilde{X}_\pm\) branch points, see fig. 2, with an electric contour and eg. \(\alpha_{N-1}\), the cycle encircling the \(\tilde{X}_{-1}\) and \(\tilde{X}_1\) branch points, with a magnetic contour. For the remaining
cycles we take $\gamma_j$ to encircle $X_{2j-1}$ and $X_{2j}$ for $1 \leq j \leq (N - 1)$ and $\alpha_k$ to encircle $X_{2k}$ and $X_{2k+1}$ for $1 \leq k \leq (N - 2)$. We further define $\beta$-cycles as

$$\beta_j \equiv \gamma_N + \sum_{k=1}^{N-1} \gamma_k = - \sum_{k=j+1}^{N-1} \gamma_k, \quad 1 \leq j \leq (N - 2)$$

(3.37)

$$\equiv \gamma_N = - \sum_{k=1}^{N-1} \gamma_k, \quad j = (N - 1).$$

The periods $a_j$ and $a_{Dm}$ are then defined as in (2.4).

To compute the period matrix we again need the matrices of integrals of some basis of holomorphic forms over the $\alpha$ and $\beta$-cycles. We shall take a slightly different basis for the holomorphic forms than in the previous two cases, namely let

$$\tilde{\omega}_n \equiv - \frac{1}{2\pi i} \frac{\tilde{X}^{n-1} d\tilde{X}}{Y}, \quad 1 \leq n \leq (N - 1).$$

(3.38)

The only difference as compared to (3.24) is to replace $X$ by $\tilde{X}$, so clearly the basis (3.38) is just a linear transformation of (3.24).

The periods encircling pairs of branch points on the circle are straightforward to compute in an expansion in powers of $(\Lambda/v)$ and basically follow from the previous two
sections. Specifically one finds

\[ \frac{\partial \tilde{a}_j}{\partial \tilde{\xi}_n} = \int_{\gamma_j} \tilde{\omega}_n = -\frac{1}{2\pi(N-1)\Lambda^N} (x_{2j-1})^n (\tilde{F}(n) + O(\frac{\Lambda}{v})), \quad 1 \leq j, n \leq (N-1) \]

\[ \frac{\partial a_{Dm}}{\partial \tilde{\xi}_n} = \int_{\alpha_m} \tilde{\omega}_n \frac{\partial \tilde{a}_k}{\partial \tilde{\xi}_n}|_{k=m+1/2}, \quad 1 \leq n \leq (N-1), \quad 1 \leq m \leq (N-2) \]

where we have defined

\[ \tilde{F}(n) \equiv F(n)|_{N \rightarrow (N-1)} \] (3.39)

where \( F(n) \) was given in (3.27).

Computation of the remaining periods over the \( \alpha_{N-1} \) cycle is slightly more involved. As it turns out we have not been able to evaluate all of the remaining periods directly. However, as we discuss in the appendix, one can evaluate them indirectly and still compute the period matrix. We record the results here and relegate the details to the appendix.

For the remaining periods we find

\[ \frac{\partial a_{DN_{N-1}}}{\partial \tilde{\xi}_n} \approx -\frac{\tilde{F}(n)}{2\pi(N-1)\Lambda^N} e^{\frac{\pi n}{N-1}} - 1 \left( \frac{v}{\Lambda} \right)^{\frac{n-1}{N-1}} \] (3.41)

for \( 1 \leq n \leq (N-2) \) while the \( n = (N-1) \) case is given by

\[ \frac{\partial a_{DN_{N-1}}}{\partial \tilde{\xi}_{N-1}} \approx i \frac{(N+1)\Lambda}{\pi\Lambda^N} \frac{1}{v} \log \left( \frac{v}{\Lambda} \right). \] (3.42)

The period matrix can now be expressed as

\[ \tau_{mj} = \sum_{n,k=1}^{N-1} \frac{da_{Dm}}{d\tilde{\xi}_n} \left( \frac{d\tilde{a}}{d\tilde{\xi}} \right)^{-1}_{nk} (\tilde{q})^{-1}_{kj}. \] (3.43)

The \( \tilde{q}_{jk} \) matrix was essentially defined in (3.37) as the matrix which converts the \( \gamma \)-cycles (or \( \tilde{a} \) periods) into \( \beta \)-cycles (or \( a \) periods), analogously to the earlier discussion around (3.14). The inverse matrix does the reverse, and is given by

\[ (q^{-1})_{kj} \equiv (\delta_{k,j} - \delta_{k,j+1}), \quad 1 \leq k \leq (N-1), \quad 1 \leq j \leq (N-2) \]

\[ \equiv -\delta_{k,1}, \quad 1 \leq k \leq (N-1), \quad j = (N-1). \] (3.44)

The period matrix is now straightforward to evaluate and we find to leading order at large \( v \)

\[ \tau_{mj} = \frac{i}{(N-1)} \left( \cot(\pi \frac{m-j+1/2}{N-1}) - \cot(\pi \frac{m-j-1/2}{N-1}) \right), \quad 1 \leq m, j \leq (N-2) \]

\[ = \frac{-i}{(N-1) \sin(\pi (m-1/2)/(N-1))}, \quad 1 \leq m \leq (N-2), \quad j = (N-1) \]

\[ = \frac{i}{\pi} (N+1) \log \left( \frac{v}{\Lambda} \right), \quad m = (N-1), \quad j = (N-1). \] (3.45)
The row $\tau_{(N-1),j}$ for $1 \leq j \leq (N-2)$ then follows by the fact that the period matrix is symmetric.

The general form of this result was to be expected. Indeed, one might think that when $(v/\Lambda) \gg 1$ the probe is located far from the enhancon and the physics of the probe reduces to that of a weakly coupled $SU(2)$ gauge theory, which is given by the tree-level plus one-loop result. In the large $N$ limit, this accounts for the $\tau_{N-1,N-1}$ entry, which agrees with the supergravity probe brane metric [10]. However, already at order $1/N$ we see a shift in the coefficient of the log term. The one-loop beta function would give a coefficient of $N - 1$, rather than the $N + 1$ that emerges from our detailed calculation. This arises from the response of the $N - 1$ background branes to the position of the probe, which can generate such corrections due to the close spacing $\mathcal{O}(\Lambda/N)$ of the eigenvalues of the associated moduli. We also see instanton corrections in the other entries, which do not fall off as $v$ becomes large, but are also suppressed by $1/N$ factors. These have a similar interpretation.

\[ \frac{v}{\Lambda} \approx 1 \text{ Limit} \]

We have so far discussed the large and small $v/\Lambda$ limits. Starting from $v/\Lambda \gg 1$ and decreasing $v$, the probe branch points move toward the enhancon branch points distributed on (an approximately) unit circle. On the real axis there is a critical point

\[ \frac{v}{\Lambda} = \frac{N^{1/N}}{(1 - 1/N)^{(1-1/N)}} \approx 1 + \frac{\log N}{N} \quad (3.46) \]

at large $N$, where one of the probe branch points collides with an enhancon branch point. At this point the period matrix becomes singular, and a dyon becomes massless. For finite $N$, the singularity is of the same form as the singularities of the $SU(2)$ theory [3]. Near, but not at, this point however the period matrix is nonsingular. To fully understand the motion of the probe in this region we must compute the period matrix. We will obtain analytic expressions for the masses of charged particles near this point, in a large $N$ limit, but so far have been unable to obtain an analytic expression for the period matrix. Some of these periods have also been computed in [8], and it has been argued in [4], that the large $N$ limit can be taken in such a way as to define a four-dimensional non-critical string theory.

The roots of the Seiberg-Witten curve for $v/\Lambda < 1$ are given approximately in a large $N$ expansion by

\[ \tilde{X}_k = e^{i\pi k/N} \left( 1 - \frac{1}{N} \log(1 - \frac{v}{\Lambda} e^{-i\pi k/N}) - \frac{1}{2} \left( \frac{v}{\Lambda} e^{-i\pi k/N} \right)^2 + \mathcal{O}(1/N^3) \right) \quad (3.47) \]
for $1 \leq k \leq 2N$ where $\tilde{X}$ was defined after (3.36). This expansion is valid provided the log term is much smaller than $N$. We keep $v/\Lambda$ fixed as $N$ becomes large.

To compute the periods we use the same set of holomorphic one forms (3.38) as given in the previous section. Furthermore we shall take the same set of cycles as in the small $v/\Lambda$ case. Parameterizing the integrals in terms of an angular coordinate $\varphi$ as

$$
\tilde{X}(\varphi) \equiv \tilde{X}_k |_{k \to k + \varphi/\pi},
$$

the expansion of the holomorphic one forms (3.38) produces to leading order in $1/N$ the expression

$$
\tilde{\omega}_n = \frac{-d\varphi}{2\pi N} \frac{e^{i\varphi n/N}}{\sqrt{e^{i2\varphi} - 1}} \left(1 - \frac{(v/\Lambda)e^{-i2\pi k/N}e^{-i\varphi/n}}{e^{i2\pi k/N} - v/\Lambda}\right) + \cdots.
$$

(3.49)

Expanding the exponential $\exp(i\varphi/N)$ in powers of $N$, one is left with integrals that we have seen before, (3.27). In particular one finds

$$
\frac{\partial a_{D_k}}{\partial \xi_n} = -\frac{1}{2\pi N} \frac{e^{i2\pi k/N}}{\left(1 - (v/\Lambda)e^{-i2\pi k/N}(n-1)/N\right)N^2} \left(1 + 2F(n)\frac{v}{\Lambda} \frac{(n - 1) \log(1 - (v/\Lambda)e^{-i2\pi k/N})}{N} + \frac{e^{i2\pi k/N} - v/\Lambda}{N} \right)
$$

$$
- \frac{1}{N} \log(1 - (v/\Lambda)e^{-i2\pi k/N}) - \frac{v}{\Lambda^2} \frac{1}{e^{i2\pi k/N} - v/\Lambda}) + 2\frac{v}{\Lambda^2} \frac{n}{N} \frac{F(n) - F(n + 1)}{e^{i2\pi k/N} - v/\Lambda) + O\left(\frac{1}{N^2}\right)},
$$

$$
\frac{\partial \tilde{a}_m}{\partial \xi_n} = \frac{\partial a_{D_k}}{\partial \xi_n} |_{k \to (m - 1/2)}
$$

(3.50)

The periods are given in terms of the derivatives of the periods by

$$
a_{D_k} = -N\Lambda \left( \frac{\partial a_{D_k}}{\partial \xi_{N+1}} - \frac{v}{\Lambda} \frac{\partial a_{D_k}}{\partial \xi_N} + (1 - \frac{1}{N}) \left(\frac{v}{\Lambda}\right)^2 \frac{\partial a_{D_k}}{\partial \xi_{N-1}'} \right)
$$

(3.51)

with similar expressions for the $a$’s and $\tilde{a}$’s. As discussed in [8] the large $N$ limit of the period $a_{D_{N-1}}$ is non-uniform, in the sense that when $|v/\Lambda - 1|$ decreases to of order $e^{-N}$, the large $N$ expansion breaks down (this is because the log term that appears in (3.47) becomes of order $N$).

To compute the period matrix we must invert the matrix $\partial a/\partial \xi$. Unlike in our previous cases however the $k$-dependence of $\partial a_k/\partial \xi_n$ is no longer contained only in a phase factor, but rather something more complicated. We do not know of any nice analytic expression for the inverse of this matrix and therefore have not been able to compute an analytic form for the period matrix. Nevertheless the expressions for the periods given above generate the masses of charged particles including instanton corrections, at leading orders in the large $N$ expansion.
4. Conclusions

In this paper we have computed the moduli space metric for $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory along some distinguished one-complex dimensional slices of relevance for resolution of singularities in string theory via the enhanccon mechanism. One unexpected result of this analysis is the presence of corrections suppressed by powers of $1/N$ that do not fall off at large separations, in the regime where one of the classical moduli is taken to be large. When all moduli are well-separated, one expects the tree-level plus one-loop contributions to only receive corrections due to instantons, which should be exponentially suppressed in the large $N$ limit. The reason other corrections survive here is because the moduli associated with the branes on the enhanccon shell have separations of order $1/N$, so we are never in a purely semiclassical regime. In [8], similar effects have been attributed to fractional instantons.

From the supergravity perspective, this limit of moduli space corresponds to a probe brane moving in the background of $N-1$ branes on an enhanccon shell. The corrections, which are suppressed by powers of $1/N$, contain information about stringy and quantum corrections to the classical supergravity description of the enhanccon.

We have also explicitly computed the enhanccon metric near the point where all the classical moduli vanish. As expected, the moduli space metric is smooth in this limit. Gauge symmetry is not enhanced at this point – the gauge symmetry remains $U(1)^{N-1}$. The light particles include both electrically and magnetically charged states which are mutually non-local. The lightest masses go like $\Lambda/N$, which sets the cutoff scale for our description of the low-energy effective field theory.

We computed the induced metric on a one-complex dimensional subspace corresponding to a probe brane near the enhanccon locus, and find a conical singularity at the center. In [14,15] it was argued a similar conical singularity appears on the induced metric on a subspace of a higher dimensional analog of the Atiyah-Hitchin space, relevant for a probe brane in the background of an enhanccon shell arising from wrapped D6-branes. As mentioned above, the full moduli space metric at this point is smooth. This supports the picture of a probe brane approaching the enhanccon shell advocated in [10]. The probe brane melts into the enhanccon branes and becomes indistinguishable from them at the point where the classical moduli vanish. In particular, a probe brane corresponding to a wrapped D7-brane is unable to probe the region inside the enhanccon shell. We are currently completing a computation of the moduli space metric for $SU(N)$ with fundamental
matter. This will allow us to consider instead a probe D3-brane which is able to probe inside the enhancon shell. This will provide a useful check on the idea that the region inside the enhancon shell is simply flat space.

Finally, it is interesting to address the question of to what extent the enhancon mechanism is a general phenomenon. In the highly symmetric situations we have studied in this paper, the presence of the enhancon shell corresponds to the fact that the branch points of the Riemann curve are not coincident, but rather become closely spaced points around a circle. However nothing stops us considering other regions in moduli space where branch points collide. In particular, we could consider subspaces of moduli space where large numbers of branch points collide and give rise to generalizations of the Argyres-Douglas point \cite{16}. Related multi-critical points have been studied at large $N$ in \cite{9}. These presumably give rise to timelike singularities in the supergravity description for which the enhancon mechanism will not work. We hope to return to a study of this regime in the future.

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Appendix A. Period matrix for $v/\Lambda \gg 1$

We discuss the details of the computation of the periods $\partial a_{D_{N-1}}/\partial \tilde{\xi}_n$ for the $(v/\Lambda) \gg 1$ case in this appendix. Some of these integrals have essentially been done in \cite{8}. The basic point is the following. The integral is between the limits $\tilde{X}_1 \approx 1 + \log(\Lambda/v)/N$ and $\tilde{X}_- \approx (v/\Lambda)(1 - (\Lambda/v)^N)$. It follows that over most of the integration region the $\tilde{X}^{2(N-1)}$ term in $Y$ will dominate so that the 1 can be dropped and one has the simple integral

$$\frac{\partial a_{D_{N-1}}}{\partial \tilde{\xi}_n} = \int_{\alpha_{N-1}} \tilde{\omega}_n \approx \frac{1}{i\pi \Lambda^N} \int_{\tilde{X}_1}^{\tilde{X}_-} \frac{d\tilde{X}}{(\tilde{X} - v/\Lambda)\tilde{X}^{N-n}} \approx \frac{i}{\pi \Lambda^N} \frac{1}{N-1-n} \left(\frac{v}{\Lambda}\right)^{-n/(N-1)}, \quad 1 \leq n \leq (N-2) \quad \text{(A.1)}$$

Actually the integral on the far right-hand-side of the top line in (A.1) can be done exactly and one would get $O(N-n)$ terms. We have written only the dominant term above. However for this approximation to the full integral to be valid one needs to check if the
contribution of both the other $\mathcal{O}(N - n)$ terms as well as the contribution to the integral coming from the integration region near the limits where $Y$ vanishes is subdominant. Actually the former contribution can be removed easily by taking a different basis of holomorphic one-forms. For example, if we take the basis

$$\omega'_n \equiv (\tilde{\omega}_n - \frac{\Lambda}{v} \tilde{\omega}_{n+1})$$

for $1 \leq n < (N - 2)$ and $\omega'_{N-1} = \tilde{\omega}_{N-1}$ instead, the periods in the above approximation would reduce to just one term.

The real problem is that the contribution to the periods coming from the region of integration near the limits becomes important once $n = \mathcal{O}(1)$. To see this consider the following correction to the $\partial a_{D_{N-1}}/\partial \xi'_n$ period (we are working in the $\omega'$ basis instead of the $\tilde{\omega}$ basis here simply because it decouples the two effects mentioned above),

$$-\frac{1}{\pi i} \int^{\bar{X}_m}_{\bar{X}_1} \left[ (1 - (\Lambda/v)\bar{X}) \frac{\bar{X}^{n-1}}{Y} + \frac{v}{\Lambda} \bar{X}^{-N+n} \right] d\bar{X}, \quad (A.3)$$

where $\bar{X}_1 < \bar{X}_m < \bar{X}_-$, but otherwise $\bar{X}_m$ is arbitrary. The integrand here is just the difference of the exact holomorphic one-form (A.2) and its approximate form given by dropping the 1 in $Y$. Expanding near the lower limit as $\bar{X} = \bar{X}_1(1 + x/(N - 1))$ one obtains the integral

$$\approx \frac{1}{\pi i} \left( \frac{\Lambda}{v} \right)^{N/(N-1)} \frac{1}{N-1} \int^{x_m}_0 \left[ e^{x/(N-1)} - e^{-x(N-n)/(N-1)} \right] dx. \quad (A.4)$$

The upper limit $x_m$ can be taken to infinity at the cost of an exponentially small correction to the integral. This integral can now be evaluated and one finds

$$-\frac{1}{\pi i} \left( \frac{\Lambda}{v} \right)^{n/(N-1)} \frac{1}{N-1} \left[ -\frac{N-1}{N-n} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(N-n/2)}{\Gamma(N-n)} \right]. \quad (A.5)$$

This should be much less than the leading order contribution to the period integral, which in this case simply coincides with the contribution coming from the first term in brackets of the above expression. When $n = N - \mathcal{O}(1)$ then the term in brackets vanishes to leading order and indeed this correction is subdominant. However when $n = \mathcal{O}(1)$ the term in brackets does not vanish to leading order and the “correction” term is of the same order as the leading order piece, so the approximations made in this regime are not valid.
As we commented on in the text, we do not actually need to compute all of these period integrals directly to construct the period matrix. We only need to evaluate eg. \( \partial a_{D_{N-1}} / \partial \tilde{\xi}_{N-1} \) (we are working once again with the \( \tilde{\omega} \) basis) directly, for which the approximations described above are valid, and the remaining periods can be obtained indirectly as we now discuss. Basically one notes that given just the periods in (3.39), and the fact that the period matrix is symmetric, is enough to construct the entire period matrix except for the element \( \tau_{(N-1),(N-1)} \). To compute this element we need the values of the remaining periods. We can however solve for them by viewing the symmetry requirement \( \tau_{(N-1),k} = \tau_{k,(N-1)} \) for \( 1 \leq k \leq (N-2) \) as a set of equations to determine the periods \( \partial a_{D_{N-1}} / \partial \tilde{\xi}_k \) over the same range of \( k \), where one must remember that \( \partial a_{D_{N-1}} / \partial \tilde{\xi}_{N-1} \) has been given above. In more detail, in terms of the tilded period matrix defined as \( \tau_{m,j} = \sum_k \tilde{\tau}_{mk} (q^{-1})_{kj} \), the equations determining the remaining periods are

\[
\sum_{n=1}^{N-1} \frac{\partial a_{D_{N-1}}}{\partial \xi_n} \left( \frac{\partial \tilde{a}}{\partial \tilde{\xi}} \right)^{-1}_{nk} \left( \frac{\partial \tilde{a}}{\partial \tilde{\xi}} \right)^{-1}_{n,(k+1)} = -\tilde{\tau}_{k,1}. \tag{A.6}
\]

It is a simple exercise to solve these equations for the remaining periods to find (3.41).
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