Classification for Dynamical Systems: 
Model-based Approach and Support Vector Machines

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Abstract—We consider the problem of classifying trajectories generated by dynamical systems. We investigate a model-based approach, the common approach in control engineering, and a data-driven approach based on Support Vector Machines, a popular method in the area of machine learning. The analysis points out connections between the two approaches and their relative merits.

I. INTRODUCTION

Assume that we are given two dynamical systems, whose underlying dynamics might be unknown. We are interested in designing a classifier, that is a machine which, given an observed trajectory generated by either the systems, correctly identifies which of the candidate systems has generated such a trajectory. We will refer to this problem as the problem of classification for dynamical systems.

Problems of this type are ubiquitous within the community of systems and control. For instance, in fault detection one system may represent the behavior in nominal conditions and another a prescribed faulty behavior. Similarly, in network control one system may represent the closed-loop behavior when data are successfully transmitted, while another system may represent the open-loop behavior in the presence of packet dropouts. Problems of this type naturally arise also when dealing with systems that naturally exhibit multiple operating modes, for example switched circuits. On the other hand, in computer science, classification is the core of machine learning for pattern recognition [1], [2], which has found several applications in different fields of engineering, including applications that are intensively studied in control engineering like fault detection and diagnosis [3]. Yet, with regard to this problem, the interaction between these two communities has been low.

Within the systems and control community, classification for dynamical systems has been studied in connection with the analysis of switched/multi-mode systems [4]-[9], often under the term mode-identification, which is defined as the problem of reconstructing the active mode of a switched system from its output trajectories. The common approach is a model-based approach: assuming a correct model of the system for each operating mode, one can check whether or not mode-identification is feasible via dynamic-dependent conditions, and mode-identifiers (in fact, classifiers) can be obtained in terms of rank tests, least-square functions or dynamical systems. While the theory also generalizes to noisy observations [9], [10] and to some types of nonlinear dynamics [11], [12], little is known on how to approach model-based classification if one departs from the hypothesis that the dynamics of the system are known with perfect accuracy. Even in the simplest case where the dynamics are associated with parametric uncertainty, building a classifier is a non-trivial task. The difficulty is similar to the one encountered in adaptive control based on multiple models, where a main issue is indeed to guarantee that modelling inaccuracies do not destroy the learning capability of the control scheme [13].

In computer science, the main paradigm to classification is instead the data-driven paradigm. Classifiers are designed by choosing a function with adjustable parameters selected using a number of training data, called the examples. The resulting function (the classifier) is then evaluated according to its capability to generalize from the training dataset, that is to correctly map new examples. Popular methods are Neural Networks (NN) and Support Vector Machines (SVM), whose capability to generalize from a training dataset can be quantified via suitable loss functions such as the risk function [1]. Data-driven methods have the intrinsic potential to overcome issues related to model uncertainty, and have already proven their effectiveness in challenging applications such as the prediction of epileptic seizures from recording of EEG signals [14]. However, it is not obvious how to tailor the analysis and design of data-driven methods to the specific context where data come from dynamical systems. It is worth pointing out that classification of data generated from dynamical systems is not new in computer science. In fact, it can be regarded as classification of time series once we assume the existence of an underlying data-generating system. Yet, approaches which take this standpoint still try to incorporate models into the learning task, either to extract from data informative features [15] or to construct suitable kernel functions [16], [17]. While incorporating models is a natural step to take, it leads to the previous question of how to handle model uncertainty, and does not help to understand the performance achievable by model-free schemes. Similar issues related to model-based approaches have been pointed out also in the context of clustering [18].

In this paper, we consider autonomous linear systems and approach the classification problem from both model-based and data-driven perspectives, pointing out relative merits and establishing connections between the two. We first consider a model-based approach and derive a classifier assuming the knowledge of the system dynamics. This approach has two fundamental merits: i) to highlight necessary conditions for the existence of a correct classifier (problem feasibility); ii) to guide the design and analysis of a data-driven solution.
In connection with ii), the model-based approach shows that under problem feasibility one can design a correct classifier which can be interpreted in terms of polynomial kernels [2]. Building on this result, we consider a data-driven approach based on SVM. By using properties stemming from the model-based solution, we provide bounds on the margin of the classifier and quantify its generalization performance [19] as a function of the systems one wishes to classify.

The rest of this paper is as follows. Sections II and III formalize the problem of interest, and recall basic concepts regarding SVM. In Sections IV and V, we present the main results. Section VI discusses the results and open problems. Numerical simulations are reported in Section VII, while Section VIII provides concluding remarks.

II. FRAMEWORK

Consider two linear dynamical systems

\[
\Sigma_i \sim \begin{cases} 
  x_i(t+1) = A_i x_i(t) \\
  y_i(t) = C_i x_i(t) 
\end{cases}, \quad i = 1, 2 \tag{1}
\]

where \( t \in \mathbb{Z}_+ := \{0, 1, \ldots \} \) denotes time; \( x_i \in \mathbb{R}^{n_i} \) is the state; \( y_i \in \mathbb{R}^{m_i} \) is the output; \( A_i \) and \( C_i \) are state and output transition matrices. We will assume that each \( \Sigma_i \) is observable (in a control-theoretic sense). This entails no loss of generality in that if \( \Sigma_i \) is not observable, all subsequent developments apply to the observable subsystem obtained via a Kalman observability decomposition. Assume now that we are given a sequence

\[
Y := \text{col}(y(0), y(1), \ldots, y(N-1)) \tag{2}
\]

of \( N \) measurements generated by one of the two systems, that is \( Y = \text{col}(y_i(0), y_i(1), \ldots, y_i(N-1)) \) with \( i \in \{1, 2\} \), but we have no direct information on which of the two systems has generated \( Y \). We are interested in determining which of the two systems has generated \( Y \), referring to this problem as the problem of classification.

To make the problem definition precise, let

\[
\mathcal{O}_i := \begin{bmatrix} C_i \\ C_i A_i \\ \vdots \\ C_i (A_i)^{N-1} \end{bmatrix} \tag{3}
\]

be the observability matrix of order \( N \) of the pair \((C_i, A_i)\), and let

\[
\mathcal{L}_i := \{ \mathcal{O}_i x, x \in \mathbb{R}^{n_i} : \mathcal{O}_i x \neq 0 \} \tag{4}
\]

be the set of all possible nonzero trajectories of \( N \) samples that can be generated by the \( i \)-th system.

**Definition 1 (Classifiers and correctness):** A classifier is any function \( f : \mathcal{Y} \rightarrow \mathbb{R} \), where \( \mathcal{Y} \) is the space of the input data. A classifier for the dynamical systems in (1) is said to be correct if it satisfies:

\[
f(Y) \begin{cases} 
  > 0 & \text{if } Y \in \mathcal{L}_1 \\
  < 0 & \text{if } Y \in \mathcal{L}_2 
\end{cases} \tag{5}
\]

The problem of interest is to construct correct classifiers. We will investigate two approaches:

(i) **Model-based classification:** The classifier depends on the knowledge of the matrices \( A_i \) and \( C_i \), \( i = 1, 2 \).

(ii) **Data-driven (model-free) classification:** The classifier does not depend on the knowledge of the matrices \( A_i \) and \( C_i \), \( i = 1, 2 \), and has to be determined on the basis of a given number of sample trajectories, that is points in the sets \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

Case (i) reflects the situation where the dynamics of the systems are known and this information is exploited in the design of the classifier. On the contrary, case (ii) reflects the situation where the dynamics of the systems are unknown or this information is not directly exploited in the design of the classifier.

A. Limitations of the classification problem

Clearly, the knowledge of the system dynamics provides an extra degree of information that can be used to properly design a classifier. Yet, there are certain limitations which cannot be overcome even in the ideal situation where one has perfect knowledge of the dynamics. In particular, the following result holds true.

**Theorem 1 (Limitations of the classification problem):**
A correct classifier for the dynamical systems in (1) exists only if \( \text{rank} [\mathcal{O}_1 \mathcal{O}_2] = n_1 + n_2 \).

**Proof of Theorem** By definition, there exists no correct classifier whenever \( \mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset \), because this implies the existence of trajectories compatible with both the systems. This is equivalent to the fact that the dynamical system resulting from the parallel interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is observable, that is that the observability matrix of order \( N \) of the pair \((C, A)\) with

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{6}
\]

has column rank \( n_1 + n_2 \). This gives the result.

The condition in Theorem 1 can be satisfied only if the two systems do not share common eigenvalue-eigenvector pairs. This condition also requires \( Nm \geq n_1 + n_2 \), which means that a large enough observation window must be chosen to render classification feasible.

We will take this condition as a standing assumption.

**Assumption 1.** \( \text{rank} [\mathcal{O}_1 \mathcal{O}_2] = n_1 + n_2 \).

III. SUPPORT VECTOR MACHINES

In this section, we briefly recall some concepts on SVM focusing on the case of separable data. This material of this section is adapted from [2].

Assume we have \( L \) observations \((Y_k, \ell_k), k = 1, 2, \ldots, L\), each one consisting of a vector \( Y_k \in \mathbb{R}^d \) plus a label \( \ell_k \in \{-1, 1\} \) specifying the class to which \( Y_k \) belongs. In connection with the problem introduced in Section II, one can think of \((Y_k, \ell_k)\) as an observation collected from one of the two candidate systems \( \Sigma_i \), where \( Y_k \) is the measurement and \( \ell_k \) specifies which of the two systems has generated \( Y_k \).
Consider the problem of classifying the vectors \( Y_k \) using hyperplanes \( H(Y, \alpha) = \{ Y \in \mathbb{R}^d : w^T Y + b = 0 \} \), where \( \alpha = (w, b) \) is a vector of adjustable weights. If there exists a vector \( \alpha \) satisfying

\[
\begin{cases}
w^T Y_k + b > 0 & \text{if } \ell_k = 1 \\
w^T Y_k + b < 0 & \text{if } \ell_k = -1
\end{cases}
\] (7)

for \( k = 1, 2, \ldots, L \), then the vectors \( Y_k \) are called **linearly separable**, and the function

\[ f(Y, \alpha) = w^T Y + b, \quad \alpha = (w, b) \] (8)

defines a linear classifier which is correct with respect to the data \( (Y_k, \ell_k) \), \( k = 1, 2, \ldots, L \).

For the linearly separable case, an SVM searches for the separating hyperplane with largest margin \( \rho \), that is it searches for the value of \( \alpha \) which maximizes

\[ \rho := \min_{k=1,2,\ldots,L} \frac{|w^T Y_k + b|}{\|w\|} \] (9)

This can be cast as a convex program:

\[
\begin{aligned}
& \min_{\alpha} \frac{1}{2} \|w\|^2 \\
& \text{subject to } \begin{cases}
w^T Y_k + b \geq 1 & \text{if } \ell_k = 1 \\
w^T Y_k + b \leq -1 & \text{if } \ell_k = -1
\end{cases}
\end{aligned}
\] (10)

The reason to search for the hyperplane with largest margin is related to the fact that \( f(Y, \alpha) \) is obtained from a finite set of observations, the so-called **training set**. On the other hand, one would like \( f(Y, \alpha) \) to be able to correctly classify also data which are not present in the training set. This property is usually called the **generalization** performance [2], and SVM can guarantee a good generalization performance. We will discuss this point in more detail in Section V.

Problem (10) involves \( L \) constraints and \( d + 1 \) unknowns. When \( d > L \) it can be more convenient to resort to a dual formulation of the problem, called **Wolfe dual**:

\[
\begin{aligned}
& \max_{\mu} \mu^T 1 - \frac{1}{2} \mu^T Z \mu \\
& \text{subject to } \mu \geq 0, \quad \sum_{k=1,2,\ldots,L} \mu_k \ell_k = 0
\end{aligned}
\] (11)

where \( \mu := \text{col}(\mu_1, \mu_2, \ldots, \mu_L) \) is the vector of Lagrange multipliers, \( Z = [Z_{kj}] \) is a symmetric \( L \times L \) matrix such that \( Z_{kj} = \ell_k \ell_j Y_k^T Y_j \), \( k, j = 1, 2, \ldots, L \), and where \( 1 \) is the vector of ones. Problem (11) involves \( L \) constraints and unknowns. The solution has the form

\[ w = \sum_{k=1,2,\ldots,L} \mu_k \ell_k Y_k \] (12)

and each vector \( Y_k \) associated to a positive multiplier \( \mu_k \) is a **support vector**. This means that for the optimal linear classifier resulting from (10) the parameter \( w \) is given by a linear combination of the support vectors, which can be then interpreted as the most representative points in the training dataset.

### A. Kernel functions

Finding a separating surface which is **linear** with respect to the space \( \mathcal{Y} \) of the input data is not always possible. One of the most important results about SVM is related to the possibility of finding **non-linear** separating surfaces in a very straightforward manner.

By looking at the optimization problem (11), one sees that the data appears only through the products \( Y_k^T Y_j \). One can think of mapping the input space \( \mathcal{Y} \) into a higher-dimensional space \( \mathcal{H} \) through a function \( \Phi : \mathcal{Y} \rightarrow \mathcal{H} \), and search for a function \( \kappa : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R} \) such that

\[ \kappa(Y, Z) = \langle \Phi(Y), \Phi(Z) \rangle, \quad \forall X, Z \in \mathcal{Y} \] (13)

The space \( \mathcal{H} \) is called the **feature space**, while \( \Phi(Y) \) is called the **feature vector**. Any function \( \kappa \) satisfying (13) is called a kernel function. Kernel functions define separating surfaces which are linear with respect to \( \mathcal{H} \). The remarkable feature of kernel functions is that there is no need to use or know the function \( \Phi \) in order to compute or use \( w \). In fact, in order to compute the solution of (11) with respect to \( \Phi \) one can simply use \( Z_{kj} = \ell_k \ell_j \kappa(Y_k, Y_j) \). Moreover,

\[ w^T \Phi(Y) = \sum_{k=1,2,\ldots,L} \mu_k \ell_k \kappa(Y_k, Y) \] (14)

Thus one can use \( \kappa \) instead of \( \Phi \) also for the classification task. Kernel functions are also advantageous from the point of view of computations since \( \kappa \) operates in the input space \( \mathcal{Y} \), which has usually lower dimension than \( \mathcal{H} \). Common kernel functions are polynomial, Gaussian and hyperbolic tangent functions [2].

In the sequel, we will show that for classifying dynamical systems polynomial kernels are good candidates.

### IV. Model-based Classification

We now consider a model-based approach to classification. The following example shows that no correct classifier exists which is linear in the input space.

**Example 1.** Consider two systems as in (1), where \( A_1 = 1, A_2 = -1 \) and \( C_1 = C_2 = 1 \). Assumption 1 clearly holds true for \( N \geq 2 \). However, as depicted in Figure 1 there exists no linear classifier for the two candidate systems, and this is independent of the particular choice of \( N \). 

Example 1 indicates that for dynamical systems there is no correct classifier which is linear with respect to \( Y \). However, Figure 1 suggests that a correct classifier for Example 1 exists and is given by \( f(Y) = y(0)y(1) \), which can be rewritten as \( f(Y) = w^T \Phi \), where

\[ w^T = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \Phi = Y \otimes Y \] (15)

where \( \otimes \) stands for Kronecker product. We will see that \( \Phi \) defines a polynomial kernel. Before doing this, we show that the choice \( \Phi = Y \otimes Y \) is general in the sense that it applies to any linear dynamical system.
Let \( G_i := O_i^\top O_i \), \( i = 1, 2 \), be the observability Gramian corresponding to the \( i \)-th system. Note that \( G_i \) is nonsingular under Assumption 1. Also, let

\[
Q := Q_1 - Q_2
\]

where

\[
Q_i := O_i G_i^{-1} O_i^\top, \quad i = 1, 2
\]

The following result holds true.

**Theorem 2 (Model-based classifier):** Let \( \Phi = Y \otimes Y \) and let \( w_M = \text{vec}(Q) \), where \( \text{vec}(\cdot) \) is the vectorization operator. Under Assumption 1, \( f(Y) = w_M^\top \Phi \) is a correct classifier for the dynamical systems in (1).

**Proof of Theorem 2** The idea is to show that computing \( w_M^\top \Phi \) is equivalent to determining which of the sets \( L_i \) the vector \( Y \) belongs to. Consider the point-set distance

\[
\pi_i(Y) := \min_{x \in \mathbb{R}^n} \|O_i x - Y\|^2, \quad i = 1, 2
\]

Notice that if \( Y \in L_1 \) then \( \pi_1(Y) = 0 \) and \( \pi_2(Y) > 0 \) in view of Assumption 1. Likewise, if \( Y \in L_2 \) then \( \pi_1(Y) > 0 \) and \( \pi_2(Y) = 0 \). Hence, the function

\[
g(Y) := \pi_2(Y) - \pi_1(Y)
\]

defines a correct classifier for the dynamical systems in (1). Notice now that \( \pi_i(Y) = \|(I - Q_i^\top) Y\|^2 \) for all \( Y \). Hence, we get

\[
g(Y) = Y^\top (I - Q_2) Y - Y^\top (I - Q_1) Y
\]

\[
= Y^\top Q Y = \text{vec}(Q)^\top (Y \otimes Y)
\]

\[
= w_M^\top \Phi
\]

where the first equality follows because \( Q_i \) is idempotent. Thus \( g(\cdot) = f(\cdot) \), which concludes the proof.

**Remark 1 (Invariance to coordinate transformations):** Notice that \( w_M \) is independent of the particular state-space realization adopted for the dynamical systems since \( Q_1 \) and \( Q_2 \) are projection matrices.

### A. Form of the model-based classifier: Kernel function and support vectors interpretation

Theorem 2 could have been stated directly in terms of \( f(Y) = Y^\top Q Y \). Yet, the form \( f(Y) = w_M^\top \Phi \) turns out to be useful because it provides guidelines for the formulation of the data-driven approach. In fact, it guarantees the existence of a solution for the SVM formulation if we use \( \Phi = Y \otimes Y \) as input to the training algorithm. Moreover, it is immediate to verify that

\[
\Phi^\top \Phi = (Y^\top Y)^2
\]

that is \( \Phi \) defines a homogeneous polynomial kernel. This means that in the SVM formulation one can work directly in the space of \( Y \) by employing the kernel \( \kappa(Y, Z) = (Y^\top Z)^2 \).

This option is possible also for the model-based solution if we write \( w_M \) in terms of support vectors. This interpretation is simple and worth mentioning.

A (reduced) singular value decomposition of the matrix \( O_i \) yields \( O_i = U_i S_i V_i^\top \), where \( U_i \in \mathbb{R}^{N \times n_i} \), has orthonormal columns, \( S_i \in \mathbb{R}^{n_i \times n_i} \) is a diagonal matrix with positive entries (due to Assumption 1), and \( V_i \in \mathbb{R}^{n_i \times n_i} \) is unitary. Thus we have \( Q_i = U_i U_i^\top \). Let \( Y_{i,k} \) be the \( k \)-th column of \( U_i \) and let \( \Phi_{i,k} = Y_{i,k} \otimes Y_{i,k} \) be the corresponding feature vector. Hence,

\[
\sum_{k=1}^{n_i} \Phi_{i,k} = \sum_{k=1}^{n_i} Y_{i,k} \otimes Y_{i,k}
\]

\[
= \sum_{k=1}^{n_i} \text{vec}(Y_{i,k} Y_{i,k}^\top)
\]

\[
= \text{vec}(U_i U_i^\top)
\]

\[
= \text{vec}(Q_i)
\]

where the second equality follows from the vectorization rule \( \text{vec}(ABC) = (C^\top \otimes A) \text{vec}(B) \) for matrices \( A, B \) and \( C \) of appropriate dimension. Thus,

\[
w_M = \text{vec}(Q)
\]

\[
= \sum_{k=1}^{n_1} \Phi_{1,k} - \sum_{k=1}^{n_2} \Phi_{2,k}
\]

and the support vectors (cf. (12)) are the left singular vectors of the observability matrices \( O_1 \) and \( O_2 \).

### V. Data-driven classification based on Support Vector Machines

The model-based approach suggests an SVM formulation for the problem of classifying data generated by dynamical systems. We will first describe the SVM formulation and we will then make some considerations on the generalization performance of the solution. An interesting result is that the generalization performance of the data-driven classifier can be quantified as a function of the dynamics of the systems which generate the training dataset.
A. Data-driven classification based on SVM

Let \( \mathcal{L}_i^{tr} \subset \mathcal{L}_i, i = 1, 2, \) be a finite nonempty subset of \( \mathcal{L}_i \) consisting of all the nonzero trajectories recorded from \( \Sigma_i. \) Thus \( Y_k \in \mathcal{L}_i^{tr} := (\mathcal{L}_i^{tr} \cup \mathcal{L}_i^{tr}) \), \( k = 1, 2, \ldots, L, \) is the \( k \)-th training vector. Let \( \ell_k = 1 \) if \( Y_k \in \mathcal{L}_i^{tr} \), and \( \ell_k = -1 \) if \( Y_k \in \mathcal{L}_i^{tr}. \) Finally, let \( \Phi_k = Y_k \otimes Y_k \) be the feature vector associated to \( Y_k. \) Following Section III and Theorem 2 we formulate the data-driven approach as the problem of finding the hyperplane that contains the origin and separates the training datasets with maximum margin, that is:

\[
\begin{align*}
\min_w \frac{1}{2} \| w \|^2 \\
\text{subject to } \begin{cases} 
  w^\top \Phi_k \geq 1 & \text{if } \ell_k = 1 \\
  w^\top \Phi_k \leq -1 & \text{if } \ell_k = -1
\end{cases}
\end{align*}
\] (24)

The following result holds true.

Theorem 3 (Data-driven classifier): Let Assumption 1 be satisfied, and consider an arbitrary training dataset \( \mathcal{L}_i^{tr}. \) Then, the solution \( w_D \) to the optimization problem (24) exists and is unique. Hence, \( f(Y) = w_D^\top \Phi \) is a correct classifier with respect to \( \mathcal{L}_i^{tr}. \)

Proof of Theorem 2: The proof follows from Theorem 2

In fact, the model-based solution \( w_M \) guarantees that \( a_1 := \min_{\ell_k = 1} w_m^\top \Phi_k > 0 \) and \( a_2 := \max_{\ell_k = -1} w_m^\top \Phi_k < 0. \) Thus \( \overline{w}_M := w_M/a \) with \( a := \min\{a_1, -a_2\} \) guarantees the feasibility of the set of constraints. Uniqueness follows as the optimization problem is a convex program.

The constraint that the solution must contain the origin is simply to mimic the model-based solution. This constraint is actually not needed, and a standard formulation \( \| w \| \leq 1 \) would still guarantee existence and uniqueness of the solution. If we constrain the solution to contain the origin the Wolfe dual becomes

\[
\begin{align*}
\min_\mu \mu^\top 1 - \frac{1}{2} \mu^\top Z \mu \\
\text{subject to } \mu \geq 0
\end{align*}
\] (25)

which does not involve the constraint \( \sum_{k=1,2,\ldots,L} \mu_k \ell_k = 0. \) We notice that in this case the dimension of the feature space is \( (Nm)^2. \) Nonetheless, by considering a kernel-based implementation one can remain in the \( Nm \)-dimensional space of the sequences \( Y. \)

Theorem 3 indicates that one can find a surface separating the training dataset without information about the underlying systems except for their linearity, which suggests the feature space of choice. In the remaining part of this section, we will discuss on the capability of this SVM classifier to generalize to observations outside the training set.

B. Expected risk

Ideally, one would like to establish the correctness of the data-driven classifier in the same sense as Definition 1. This is a non-trivial problem which, to the best of our knowledge, has not yet been solved. In the sequel, we consider another way to characterize the generalization performance of the SVM classifier, based on the notion of expected risk. While this notion does not provide deterministic bounds, it has the merit to capture the situation where the training dataset is randomly chosen. Hence, it has the merit to describe cases in which one cannot perform dedicated experiments on the systems.

Consider a training dataset of \( L \) random i.i.d. observations drawn according to a probability distribution \( P(Y, \ell). \) Given a classifier \( f(Y), \) its expected risk can be defined as [1]:

\[
R := \int \frac{1}{2} \ell - \text{sgn}(f(Y)) \| dP(Y, \ell)
\] (26)

where \( \text{sgn} \) is the sign function. The expected risk quantifies the capability of a classifier to generalize from the training dataset. Several studies have been devoted to provide upper bounds on the expected risk for a given family of classifiers. An interesting bound for linear classifiers which serves our discussion is reported hereafter.

Theorem 4 ([19]): Let \( Y \in \mathbb{R}^d \) belong to the sphere of radius \( R, \) and consider the class \( \mathcal{F} \) of real-valued functions defined as \( \mathcal{F} := \{ Y \mapsto w^\top Y : \| w \| \leq 1, \| Y \| \leq R \}. \) There is a constant \( c \) such that, for all probability distributions, with probability at least \( 1 - \eta \) over \( L \) randomly i.d. vectors, if a classifier has margin at least \( \rho \) on all the examples then its expected error is not larger than

\[
\frac{c}{\sqrt{L}} \left( \frac{R^2}{2 \rho^2} \log^2 L + \log (1/\eta) \right)
\] (27)

The constant \( c \) is related to the so-called fat-shattering dimension of linear classifiers, and its explicit expression can be found in [19]. Theorem 4 shows that one can quantify the generalization performance of a linear classifier as a function of the margin \( \rho \) obtained for the training dataset. We now show that the margin of the SVM classifier can be related to the margin of the model-based solution. This permits to quantify the generalization performance of the SVM classifier in terms of the dynamics of the systems that one wishes to classify. The analysis which follows holds for normalized data. We will briefly comment later on the general case.

Consider normalized training data

\[
\overline{Y}_k := \frac{Y_k}{\| Y_k \|}
\] (28)

with feature vector \( \overline{\Phi}_k := \overline{Y}_k \otimes \overline{Y}_k. \) It holds that \( \| \overline{\Phi}_k \| = 1. \) Consider now the optimal solution \( w_D \) to (24) computed with respect to \( \overline{\Phi}_k, \) whose existence and uniqueness is again ensured by the model-based solution. We can assume without loss of generality that \( \| w_D \| \leq 1. \) Let now \( \rho_M \) and \( \rho_D \) represent the margin corresponding to the model-based and the data-driven solutions, respectively,

\[
\rho_i := \min_{k=1,2,\ldots,L} \frac{\| w_i^\top \overline{\Phi}_k \|}{\| w_i \|}, \quad i \in \{ M, D \}
\] (29)
It holds that
\[ \rho_D \geq \rho_M \tag{30} \]
irrespective of the training dataset, because the data-driven solution is the margin maximizer. The next result shows that, using normalized data, \( \rho_M \) is bounded from below by a positive quantity that depends solely on the dynamics of the systems one wishes to classify. We refer the reader to [20] for a definition of principal angles.

**Theorem 5 (Bound on the data-driven classifier margin):** Let Assumption 1 be satisfied. Consider an arbitrary training dataset of vectors \( Y_k \), and let \( \Phi_k := Y_k \otimes Y_k \) where \( Y_k \) is as in (28). Let \( w_D \) be the unique solution to the optimization problem (24) computed with respect to \( \Phi_k \). Then, it holds that
\[ \rho_D \geq \frac{\beta}{\sqrt{2(n_1 + n_2)}} \tag{31} \]
where \( n_1 \) and \( n_2 \) are the orders of the dynamical systems in (1), and \( \beta \) is the squared sine of the smallest principal angle between the subspaces spanned by the columns of the observability matrices \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \).

**Proof of Theorem 5**: Since \( \rho_D \geq \rho_M \) it is sufficient to bound \( \rho_M \). The term \( \|w_M\| \) satisfies
\[ \|w_M\|^2 = \text{vec}(Q)^\top \text{vec}(Q) = \|Q\|^2_F \leq 2(n_1 + n_2) \tag{32} \]
where \( \|\cdot\|_F \) denotes Frobenius norm. The inequality follows from \( \|Q_1\|^2_F \leq 2\|Q_1\|^2_F + 2\|Q_2\|^2_F \) and \( \|Q_i\|_F^2 = n_i \) because the \( Q_i \)'s are projection matrices. Consider now the term \( |w_M^\top \Phi_k| \). Assume without loss of generality that its minimum is attained for some \( Y_s \in \mathcal{L}_1^\ast \). It holds that
\[ \min_{k=1,2,\ldots,L} |w_M^\top \Phi_k| = |w_M^\top \Phi_s| = |\tau_2(Y_s) - \tau_1(Y_s)| = |\pi_2(Y_s) - \pi_1(Y_s)| \tag{33} \]
The third equality comes from the fact that \( Y_s \in \mathcal{L}_1^\ast \) implies \( \pi_2(Y_s) > 0 \) and \( \pi_1(Y_s) = 0 \) in view of Assumption 1. As shown in [9, Theorem 1.1], \( \pi_2(Y_s) \geq \beta \|Y_s\|^2 \). Hence, the proof follows from \( \|Y_s\| = 1 \).

Theorem 5 permits to bound the risk of the data-driven classifier based on the dynamics of the systems one wishes to classify, and formalizes the intuition that the risk bound becomes smaller as the dynamics of the systems to classify are more distant from one another. In fact, the higher \( \beta \) the larger the coefficient of inclination between the subspaces spanned by the columns of \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), which is maximal when the two spaces are orthogonal.

Data normalization ensures that \( \rho_M \) is bounded away from zero. This property does not hold in general since trajectories of dynamical systems can be arbitrarily close to the origin. Nonetheless, one can obtain a very similar bound by adding to (27) an extra term which accounts for training data below the margin \( \rho \) [19, Theorem 1.7].

VI. DISCUSSION

It is intuitive that incorporating models can be beneficial to the classification task. This fact is obvious also from the analysis shown in this paper since under Assumption 1 no classifier can outperform the model-based classifier when the models are exact and the data are noise-free. However, as mentioned before, the model-based approach introduces the non-trivial issue of how to quantify the effect of modelling inaccuracies. The data-driven bypasses the intermediate step of identification, and thus it has the potential to be applicable also when accurate models are difficult to obtain. Hereafter, we briefly elaborate on this point also in connection with a number of open problems.

**A. Linear systems with noisy observations**

When observations are corrupted by noise, identification may be difficult and require, even for linear systems, many careful provisions [21]. In contrast, an SVM formulation can address the problem in a rather straightforward manner. Consider a soft-margin SVM [2]:
\[ \min_{(w,\xi)} \frac{1}{2}\|w\|^2 + \sum_{k=1,2,\ldots,L} C \xi_k \tag{34} \]
subject to \[ w^\top Y_k \geq 1 - \xi_k \quad \text{if } \xi_k = 1 \]
\[ w^\top Y_k \leq -1 + \xi_k \quad \text{if } \xi_k = -1 \]
where \( C \) is a parameter and \( \xi := \text{col} (\xi_1, \xi_2, \ldots, \xi_L) \) is the vector of slack variables, which account for the fact that noise may render the data non-separable. On one hand, there exist many studies aimed at quantifying the generalization performance of SVM also for soft-margin formulations [22]. On the other hand, even with noisy data one can still give a separation measure between linear systems (the margin \( \rho_M \)) as a function of their dynamics and the signal-to-noise ratio [9, Theorem 2]. This means that even with noisy data one can quantify the generalization performance of an SVM classifier along the same lines as in Section V-B.

We point out that while this reinforces the idea that for linear systems polynomial kernels are good candidates, it remains unclear if better performance can be obtained with different kernel functions.

**B. Nonlinear systems**

Classification for nonlinear systems is another situation in which an SVM formulation can bypass difficulties related to system identification. This is related to the capability of SVM to find non-linear separating surfaces in a straightforward manner through the kernel trick. Interestingly, even in the nonlinear case one can define a separation measure between dynamics [12, Theorem 3]. However, in contrast with the linear case where this measure involves principal angles between observability subspaces, for nonlinear systems this measure involves \( K \)-functions, which are often difficult to relate to the underlying dynamics. Like for linear systems, a deeper understanding of this point would be beneficial to
figure out which types of kernel functions are most suitable for a given class of nonlinear systems.

In fact, theoretical studies on classification for nonlinear systems are recent also within computer science, and the approaches appear largely diversified; for example, see [23] for an interesting recent account. Yet, also in this context, the question of which kernel functions are most suitable for a given class of dynamics is unresolved.

C. Classifiers in-the-loop

Thanks to their simple form, classifiers have the potential to be used in real-time applications, thus for control purposes. This fact has been noted in [24], where the authors introduce the term classifier in-the-loop to describe a framework in which a classifier can modify online the control action by looking at the process data. A notion of generalization is considered, which characterizes the capability of a classifier to work under small perturbations of the system vector fields, which is a sensitivity-type analysis. While the results are promising, it remains unexplored how to handle more general forms of uncertainty. Ideally, one should provide bounds on the risk function of a classifier that hold for all the possible system trajectories, and relate such bounds with closed-loop stability properties. A non-trivial difficulty is that much of the theory on the generalization properties of classifiers have been developed in a probabilistic setting, while for robust stability it is desirable to guarantee worst-case deterministic bounds.

Interestingly, the architecture considered in [24] can be regarded as a supervisory control system [13]. In supervisory control, the supervisor selects based on process data which candidate control law (hypothesis) is most appropriate at any given time. This is done by assigning to each candidate law a score function (cost function) that quantifies the performance level achievable by the control law given the process data. In supervisory control, one often uses the term cost detectability [25], [26] to measure the capability of a supervisor to learn from data an appropriate control law even when the process does not match the models used to design the control laws. In fact, the supervisor is a classifier and cost detectability is a measure of its generalization performance. The idea of adaptive control as learning-from-data is indeed not new [27], but a firm theoretical link with the realm of machine learning has not yet been established.

VII. A Numerical Example

Consider a system with transfer function

\[ G(s) = \frac{s + 1}{(s + 10)(s^2 + s + 1)} \] (35)

where \( s \) is the Laplace variable. To improve performance, the system is controlled with a proportional controller \( K = 30 \) under negative feedback. The goal is to design a classifier which can detect the loss of control effectiveness. We denote by \( \Sigma_1 \) the open-loop system and by \( \Sigma_2 \) the closed-loop system. Hence, \( \Sigma_1 \) and \( \Sigma_2 \) have transfer functions \( G(s) \) and \( W(s) := K G(s)/(1 + K G(s)) \), respectively. Finally, we denote by \( \Sigma_1 \) and \( \Sigma_2 \) the corresponding sampled-data systems under sampling time \( T_s \). The systems are as in [1] with \( n_1 = n_2 = 3 \) and \( m = 1 \). Using the previous notation, we let \( N \) be the length of the observation sequences, and \( L \) the number of training data. We let \( Q \) be the number of data used for validation. In order for Assumption 1 to be satisfied one needs \( N \geq 6 \). Under such condition, Assumption 1 holds for a generic choice of \( T_s \).

We focus on the SVM classifier because the model-based classifier is always correct under Assumption 1. We note that classifying the two systems is non-trivial, as one can observe from Figure 2. For instance, for \( T_s = 0.1 \), in the ideal case of \( N = \infty \) one has \( \beta = 0.004 \) in Theorem 5 and a cepstral distance [20] equal to 1.045.

We report in Table I simulations results for various choices of \( N, L \) and \( T_s \), with \( Q = 1000 \) validation data. The SVM classifier is computed as in Theorem 5. For the training and the validation test, trajectories are generated from random initial conditions with zero mean and variance \( \sigma^2 = 100 \). The error in the validation test is defined as:

\[ R_{\text{test}} := \sum_{k=1,2,...,Q} \frac{1}{2Q} | \hat{e}_k - \text{sgn}(w^T \Phi_k) | \] (36)

| Variation of the parameter | \( N \) (\( T_s = 0.1 \), \( L = 50 \)) |
|----------------------------|-------------------------------------|
| \( N = 2 \)                | 0.4720                              |
| \( N = 5 \)                | 0.0760                              |
| \( N = 10 \)               | 0.0685                              |
| \( N = 50 \)               | 0.0150                              |
| \( N = 100 \)              | 0.0150                              |

| Variation of the parameter | \( L \) (\( T_s = 0.1 \), \( N = 10 \)) |
|----------------------------|-------------------------------------|
| \( L = 3 \)                | 0.1480                              |
| \( L = 5 \)                | 0.1480                              |
| \( L = 10 \)               | 0.0720                              |
| \( L = 50 \)               | 0.0685                              |
| \( L = 100 \)              | 0.0620                              |

| Variation of the parameter | \( T_s \) (\( N = 10 \), \( L = 50 \)) |
|----------------------------|-------------------------------------|
| \( T_s = 0.01 \)           | 0.4850                              |
| \( T_s = 0.05 \)           | 0.0730                              |
| \( T_s = 0.1 \)            | 0.0685                              |
| \( T_s = 0.5 \)            | 0.0385                              |
| \( T_s = 1 \)              | 0.0835                              |

TABLE I: Numerical results for the SVM classifier with \( Q = 1000 \).

One sees that the classifier performs well for reasonable choices of the parameters. In particular:

(i) Dependence on \( N \). As \( N \) goes to zero, the performance is clearly that of a random guess. One the other hand, remarkably, classification becomes accurate exactly as soon as one approaches the theoretical bound \( N \geq 6 \). The performance saturates after \( N = 50 \). To further decrease the error we need to increase \( L \) (with \( L > 300 \) one can achieve an error below 1\%).

(ii) Dependence on \( L \). The performance variations are less evident in this case. This suggest that \( L \) is less critical than \( N \). The intuition is that random initial conditions generally ensure the excitation of all system dynamics so that even few examples may suffice.

(iii) Dependence on \( T_s \). The sampling time does not play a major role as long as we avoid over-sampling, in which case both \( A_1 \) and \( A_2 \) tend to the identity matrix, or under-sampling, in which case both \( A_1 \) and \( A_2 \) tend to the zero matrix.
VIII. CONCLUDING REMARKS

We have considered the problem of classifying trajectories generated by dynamical systems, looking at a model-based approach, the common approach in control engineering, as well as at a data-driven approach based on Support Vector Machines, a popular method in computer science. The present discussion suggests that both the approaches have distinct merits. A deeper understanding of the interplay between these two approaches would help to establish a sound theory for dynamical systems more general than those considered in this paper.

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