DOUBLE INTEGRAL ESTIMATES FOR BESOV TYPE SPACES 
AND AN APPLICATION TO HANKEL TYPE OPERATORS

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ABSTRACT. For $0 < p < \infty$, we give a complete description of non-
negative radial weight functions $\omega$ on the open unit disk $D$ such that
\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2} \omega(z) dA(z) < \infty
\]
if and only if
\[
\int_D \int_D \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{1+\tau+\sigma}} (1 - |z|^2)^\tau (1 - |\zeta|^2)^\sigma \omega(\zeta) dA(z) A(\zeta) < \infty,
\]
where $f$ is analytic in $D$, $\tau$ and $\sigma$ are some real numbers. As an applica-
tion, we characterize the boundedness of Hankel type operators related
to Besov type spaces with radial Bekollé-Bonami weights.

1. INTRODUCTION

A classical topic in complex analysis is to study double integral estimates
for function spaces and applications. Recall that the Dirichlet integral of a
function $f \in L^2(T)$ is
\[
\mathcal{D}(f) = \int_T \int_T \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|} |d\zeta||d\eta|,
\]
where $T$ is the boundary of the open unit disk $D$ in the complex plane $\mathbb{C}$. By
this integral, in 1931 J. Douglas [13] studied the theory of minimal surfaces,
and in 1940 A. Beurling [11] proved that the Fourier series of a Dirichlet
function converges everywhere except on a set of logarithmic capacity zero.
Let $H(D)$ be the space of analytic functions in $D$. The Dirichlet space $\mathcal{D}$
consists of functions $f \in H(D)$ such that
\[
\int_D |f'(z)|^2 dA(z) < \infty,
\]
where $dA(z) = (1/\pi)dxdy$ is the normalized area measure on $D$. It is
known that every function $f$ in $\mathcal{D}$ has nontangential limit $f(\zeta)$ for almost

2010 Mathematics Subject Classification. 30H25; 30H20; 46E15; 47B35.
Key words and phrases. Besov type space; Dirichlet type space; Hankel type operator.
The work was supported by NNSF of China (No. 11720101003).
For \( \zeta \in \mathbb{T} \). For \( f \in H(\mathbb{D}) \), it is also well known that \( f \in \mathcal{Q} \) if and only if \( \mathcal{Q}(f) < \infty \).

For \( p > 1 \), the Besov space \( B_p \) is the space of functions \( f \in H(\mathbb{D}) \) satisfying
\[
\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2}dA(z) < \infty.
\]
Clearly, \( B_2 \) is equal to the Dirichlet space. In 1988 J. Arazy, S. Fisher and J. Peetre \[4, Theorem 6.4\] showed that a function \( f \in B_p \) if and only if the following double integral
\[
\int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f \circ \varphi_a(z) - f(a)|^2 (1 - |z|^2)^{\alpha}dA(z) \right)^{p/2} (1 - |a|^2)^{-2}dA(a)
\]
converges for \( f \in H(\mathbb{D}) \), where \( \alpha > -1 \) and \( \varphi_a(z) = (a - z)/(1 - \overline{a}z) \) is a Möbius map interchanging the points 0 and \( a \). In \[4\], this characterization was used to investigate the properties of Hankel operators on weighted Bergman spaces. In 1991 K. Zhu \[33\] gave another double integral estimates for Besov spaces; that is, if \( f \in H(\mathbb{D}) \), then \( f \in B_p \) if and only if
\[
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \overline{w}z|^{\sigma + \tau + 2\alpha}}dA(z)dA(w) < \infty.
\]
In 1993 R. Rochberg and Z. Wu \[27\] obtained a double integral characterization of Dirichlet type spaces \( \mathcal{D}_{\alpha} \) and applied this characterization to study Hankel type operators. In 2008 D. Blasi and J. Pau \[12\] generalized this characterization from \( \mathcal{D}_{\alpha} \) to Besov type spaces \( B^p(\alpha) \) by a different method. Recall that for \( p > 0 \) and \( \alpha < p/2 \), \( B^p(\alpha) \) consists of functions \( f \in H(\mathbb{D}) \) for which
\[
\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-2\alpha}dA(z) < \infty.
\]
If \( p = 2 \), then \( B^2_{\alpha} = \mathcal{D}_{\alpha} \). It is known from \[12, Theorem 2.2\] that for \( f \in H(\mathbb{D}) \), \( p > 1 \), \( \sigma, \tau > -1 \) and \( \alpha \leq 1/2 \) such that \( \min(\sigma, \tau) + 2\alpha > -1 \), \( f \in B^p_{\alpha} \) if and only if
\[
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \overline{w}z|^{3+\sigma+\tau+2\alpha}} (1 - |w|^2)^{\sigma}(1 - |z|^2)^{\tau}dA(z)dA(w) < \infty.
\]
This double integral characterization was also used to study Hankel type operators; see \[12, Section 4\]. We refer to \[6, Section 4\] and \[29\] for some recent results associated with double integral estimates for some analytic function spaces. See A. Reijonen \[26\] for recent results on Besov type spaces induced by some radial weights.

A function \( \omega : \mathbb{D} \to [0, \infty) \), integrable over \( \mathbb{D} \), is called a weight. If \( \omega(z) = \omega(|z|) \) for all \( z \in \mathbb{D} \), then we say that \( \omega \) is radial. Suppose \( 0 < p < \)
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∞ and \((1 - |z|^2)^{p-2}\omega(z)\) is a weight on \(\mathbb{D}\). Denote by \(B_p(\omega)\) the Besov type space consisting of those functions \(f \in H(\mathbb{D})\) such that

\[
\|f\|_{B_p(\omega)} = |f(0)| + \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2}\omega(z)dA(z) \right)^{1/p} < \infty.
\]

In this paper, we give a complete description of radial weights \(\omega\) such that an analytic function \(f\) belongs to \(B_p(\omega)\) if and only if \(f\) satisfies certain double integral estimates. We also apply this result to characterize the boundedness of Hankel type operators from \(B_p(\omega)\) with radial Bekollé-Bonami weights to the corresponding nonanalytic version of Besov type spaces.

Throughout this paper, we write \(a \lesssim b\) if there exists a positive constant \(C\) such that \(a \leq Cb\). If \(a \lesssim b \lesssim a\), then we write \(a \approx b\).

2. Double integral estimates for \(B_p(\omega)\) spaces

This section is devoted to investigate double integral estimates for \(B_p(\omega)\) spaces. In particular, for \(0 < p < \infty\) we characterize completely radial weights \(\omega\) such that

\[
\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2}\omega(z)dA(z)
\]

\[
\approx \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4+\tau+\sigma}} (1 - |z|^2)^{\tau} (1 - |\zeta|^2)^{\sigma}\omega(\zeta)dA(\zeta)A(\zeta)
\]

for all \(f \in H(\mathbb{D})\), where \(\sigma\) and \(\tau\) are real parameters with certain ranges. If we take \(\omega(z) = (1 - |z|^2)^{1-2\alpha}\), then we see that there exists double integral characterization of \(B_p(\alpha)\) for all possible \(p\) and \(\alpha\). If we take \(p = 2\), then our result also completes corresponding conclusions in [7, p. 1725] and [15, p. 210] respectively, where the radial weights need more restrictions.

2.1. Littlewood-Paley estimates for weighted Bergman spaces and a Forelli-Rudin type estimate. In this subsection, we recall some Littlewood-Paley estimates for weighted Bergman spaces. We also give an elementary proof of a well-known Forelli-Rudin type estimate. All of these estimates are tools to prove our main theorems in this paper.

Suppose \(0 < p < \infty\) and \(\omega\) is a weight on \(\mathbb{D}\). The Lebesgue space \(L^p_\omega\) consists of complex-valued measurable functions \(f\) on \(\mathbb{D}\) for which

\[
\|f\|_{L^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z)dA(z) < \infty.
\]

The weighed Bergman space \(A^p_\omega\) is the space of analytic functions in \(L^p_\omega\). If \(\omega(z) = (\alpha + 1)(1 - |z|^2)^\alpha\), \(\alpha > -1\), then \(A^p_\omega\) is the standard weighted Bergman space \(A^p_\alpha\).
For $f \in H(D)$, it is well known that the growth rate of the following functions are often comparable in some sense:

$$f(z), (1 - |z|^2)f'(z), (1 - |z|^2)^2f''(z), (1 - |z|^2)^3f'''(z), \ldots.$$ 

This kind of estimates is usually called Littlewood-Paley estimates. We refer to [1, 2, 3, 8, 18, 20] for the study of Littlewood-Paley estimates for $A^p_\omega$; that is, to characterize weight functions $\omega$ such that

$$\int_D |f(z) - f(0)|^p \omega(z) dA(z) \approx \int_D |f'(z)|^p (1 - |z|^2)^p \omega(z) dA(z)$$

for all $f \in H(D)$. When $\omega$ is radial, this question has been recently solved completely in [20].

The pseudo-hyperbolic metric on $D$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in D.$$ 

For $0 < r < 1$ and $a \in D$, denote by

$$\Delta(a, r) = \{z \in D : \rho(a, z) < r\}$$

the pseudo-hyperbolic disk of center $a$ and radius $r$.

The following Littlewood-Paley estimates for $A^p_\omega$ [8] will be useful in this paper. For a given weight $\omega$, the conditions appeared in the following theorems are easy to verify.

**Theorem A.** Suppose $p > 0$ and $\omega$ is a weight. If there exist two constants $r \in (0, 1)$ and $C > 0$ such that

$$C^{-1}\omega(\zeta) \leq \omega(z) \leq C\omega(\zeta)$$

for all $z$ and $\zeta$ satisfying $\rho(z, \zeta) < r$, then there exists another positive constant $C$ such that

$$\int_D (1 - |z|^2)^p |f'(z)|^p \omega(z) dA(z) \leq C \int_D |f(z) - f(0)|^p \omega(z) dA(z)$$

for all $f \in H(D)$.

**Theorem B.** Suppose $p > 0$, $\omega$ is a weight, and there exist $t_0 \geq 0$ and $s_0 \in [-1, 0)$ with the following property: for any $t > t_0$ and $s > s_0$ there is a positive constant $C = C(t, s)$ such that

$$\int_D \frac{\omega(\zeta)(1 - |\zeta|^2)^s dA(\zeta)}{|1 - \overline{\zeta}z|^{2+s+t}} \leq \frac{C\omega(z)}{(1 - |z|^2)^t}$$

for all $z \in D$. Then there exists another positive constant $C$ such that

$$\int_D |f(z) - f(0)|^p \omega(z) dA(z) \leq C \int_D (1 - |z|^2)^p |f'(z)|^p \omega(z) dA(z)$$

for all $f \in H(D)$. 
We recall some well-known estimates as follows (see \[32, \text{Lemma 3.10}\]).

**Lemma C.** Let $\beta$ be any real number. Then

\[
\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}} \approx \begin{cases} 
1 & \text{if } \beta < 0, \\
\log \frac{2}{1 - |z|^2} & \text{if } \beta = 0, \\
\frac{1}{(1 - |z|^2)^\beta} & \text{if } \beta > 0,
\end{cases}
\]

for all $z \in \mathbb{D}$. Also, suppose $c$ is real and $t > -1$. Then

\[
\int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - zw|^{2+t+c}} dA(w) \approx \begin{cases} 
1 & \text{if } c < 0, \\
\log \frac{2}{1 - |z|^2} & \text{if } c = 0, \\
\frac{1}{(1 - |z|^2)^c} & \text{if } c > 0,
\end{cases}
\]

for all $z \in \mathbb{D}$.

The following Forelli-Rudin type estimate [17] is very useful in the analysis of some function spaces. A rather complicated proof of this estimate was given in [31]. Later, for $s > -1$, $r > 0$, $t > 0$, and $t + 1 < s + 2 < r$, a simple proof was given in [28, pp. 27-28], where the case of $t < s + 2 \leq t + 1 < r$ is missing.

**Lemma D.** Suppose $s > -1$, $r > 0$, $t > 0$, and $t < s + 2 < r$. Then there exists a positive constant $C$ such that

\[
\int_{\mathbb{D}} \frac{(1 - |w|^2)^s}{|1 - wz|^r |1 - w\zeta|^t} dA(w) \leq C \frac{(1 - |z|^2)^{2+s-r}}{|1 - \zeta z|^t}
\]

for all $z, \zeta \in \mathbb{D}$. 

We present an elementary proof of Lemma D here. For $z, \zeta \in \mathbb{D}$, by a change of variables $w = \varphi_z(a)$ and Lemma [C] we see that

$$
\int_{\mathbb{D}} \frac{(1 - |w|^2)^s}{|1 - \overline{w}z|^r |1 - \overline{w}\zeta|^t} dA(w) = (1 - |z|^2)^{s+2-r} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s dA(a)}{|1 - \overline{a}z|^{4+2s-r} |1 - \overline{a}\varphi_z(a)|^t} \leq (1 - |z|^2)^{s+2-r} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s dA(a)}{|1 - \overline{a}z|^{4+2s-r-t} |1 - \overline{a}\varphi_z(z)|^t} + \frac{(1 - |z|^2)^{s+2-r}}{|1 - \overline{z}\zeta|^t} \int_{\left\{ a \in \mathbb{D} : |1 - \overline{a}z| \geq |1 - a\varphi_z(z)| \right\}} \frac{(1 - |a|^2)^s dA(a)}{|1 - \overline{a}z|^{4+2s-r-t} |1 - a\varphi_z(z)|^t} + \frac{(1 - |z|^2)^{s+2-r}}{|1 - \overline{z}\zeta|^t} \left( \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s dA(a)}{|1 - \overline{a}b|^{4+2s-r}} dA(a) \right) \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^s dA(a)}{|1 - \overline{a}b|^t}
$$

which finishes the proof of Lemma D.

2.2. Weighted Bergman spaces and doubling weights. In this subsection, we recall characterizations of Carleson measures for $A^p_{\omega}$ induced by doubling weights and also give some estimates related to these weights.

Let $\mathring{D}$ be the class of radial weights $\omega$ on $\mathbb{D}$ for which $\hat{\omega}(r) = \int_r^1 \omega(s) ds$ admits the doubling property $\hat{\omega}(r) \leq C\hat{\omega}(\frac{1+r}{2})$ for all $r \in [0, 1)$, where $C = C(\omega) > 1$. A weight in $\mathring{D}$ is usually called a doubling weight. If there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that the radial weight $\omega$ satisfies

$$
\hat{\omega}(r) \geq C\hat{\omega}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1,
$$

then we say that $\omega \in \mathring{D}$. The intersection $\mathring{D} \cap \mathring{D}$ is denoted by $\mathring{D}$. The classes of these weights arise naturally in the study of some analytic function spaces and related operator theory. For instance, by [20], for a radial weight $\omega$, the Bergman projection $P_{\omega}$ is bounded from $L^\infty$ to the Bloch space $B$ if and only if $\omega \in \mathring{D}$; $P_{\omega} : L^\infty \to B$ is bounded and onto if and only if $\omega \in \mathring{D}$; the classes $\mathring{D}$ and $\mathring{D}$ also characterize completely Littlewood-Paley estimates for weighted Bergman spaces with radial weights. See [19, 21] for properties of these weights. Our investigation in this section will guide us to find new and more significance of the class $\mathring{D}$; that is, the
weight in $\hat{D}$ describes precisely certain double integral estimate for Besov type spaces.

For a space $X$ of analytic functions on $D$ and $0 < p < \infty$, a nonnegative Borel measures $\mu$ on $D$ is said to be a $p$-Carleson measure for $X$ if the identity operator $I_d : X \to L^p_\mu$ is bounded; that is,
\[
\left( \int_D |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \lesssim \|f\|_X
\]
for all $f \in X$.

For a doubling weight $\omega$ and $0 < p, q < \infty$, J. Peláez and J. Rättyä [23] characterized nonnegative Borel measures $\mu$ on $D$ such that the differentiation operator of order $n \in \mathbb{N} \cup \{0\}$ is bounded from $A^p_\omega$ into $L^q(\mu)$. In particular, they gave the following result.

**Theorem E.** Suppose $0 < p < \infty$, $\omega \in \hat{D}$ and $\mu$ is a nonnegative Borel measure on $D$. Then $\mu$ is a $p$-Carleson measure for $A^p_\omega$ if and only if the function
\[
a \mapsto \frac{\mu(S(a))}{\int_{S(a)} w dA}, \ a \in D \ \setminus \ {0},
\]
belongs to $L^\infty$, where
\[
S(a) = \left\{ z \in D : \left| \frac{\arg z - \arg a}{2\pi} \right| < \frac{1 - |a|}{2}, |z| \geq |a| \right\}
\]
is the Carleson box with vertex at $a$.

The following characterizations of doubling weights can be found in [19].

**Lemma F.** Suppose $\omega$ is a radial weight. Then the following conditions are equivalent:

(i) $\omega \in \hat{D}$;

(ii) there exists a positive constant $\beta$ depending only on $\omega$ such that
\[
\frac{\hat{\omega}(r)}{(1 - r)^\beta} \lesssim \frac{\hat{\omega}(t)}{(1 - t)^\beta}
\]
for all $0 \leq r \leq t < 1$;

(iii) there exists a positive constant $\gamma$ depending only on $\omega$ such that
\[
\int_0^t \frac{\omega(s)}{(1 - s)^\gamma} ds \lesssim \frac{\hat{\omega}(t)}{(1 - t)^\gamma}
\]
for all $0 \leq t < 1$;

(iv) there exists a nonnegative constant $\lambda$ depending only on $\omega$ such that
\[
\int_D \frac{\omega(z)dA(z)}{|1 - \xi z|^{\lambda+1}} \lesssim \frac{\hat{\omega}(\xi)}{(1 - |\xi|)^\lambda}
\]
for all $\xi \in \mathbb{D}$;
(v) there exists a positive constant $\eta$ depending only on $\omega$ such that
\[
\int_0^1 s^\xi \omega(s) ds \lesssim \left( \frac{y}{x} \right)^\eta \int_0^1 s^y \omega(s) ds
\]
for all $0 < x \leq y < \infty$.

Of course, constants $\beta$, $\gamma$, $\lambda$ and $\eta$ in Lemma F are not unique. In fact, if $\beta$ satisfies (ii) in Lemma F then any constant bigger than $\beta$ also satisfies (ii). The same phenomenon occurs for $\gamma$, $\lambda$ and $\eta$. Now we give the following relation among these parameters. The infimum below is useful for our results.

**Lemma 2.1.** Suppose $\omega \in \hat{\mathbb{D}}$. Then
\[
\inf \{ \beta : \beta \text{ satisfies (ii) in Lemma F} \} = \inf \{ \gamma : \gamma \text{ satisfies (iii) in Lemma F} \} = \inf \{ \lambda : \lambda \text{ satisfies (iv) in Lemma F} \} = \inf \{ \eta : \eta \text{ satisfies (v) in Lemma F} \}.
\]

**Proof.** Checking that proof of Lemma F [19], we get that
\[
\{ \gamma : \gamma \text{ satisfies (iii) in Lemma F} \} \subseteq \{ \lambda : \lambda \text{ satisfies (iv) in Lemma F} \},
\]
which yields that
\[
\inf \{ \lambda : \lambda \text{ satisfies (iv) in Lemma F} \} \leq \inf \{ \gamma : \gamma \text{ satisfies (iii) in Lemma F} \}.
\] (2.5)

Now suppose $\beta$ satisfies (ii) in Lemma F. Then for any $\epsilon > 0$, $\beta + \epsilon$ also satisfies the same property. Hence
\[
\hat{\omega}(0) \lesssim \frac{\hat{\omega}(t)}{(1 - t)^{\beta + \epsilon}}
\]
for all $0 \leq t < 1$. Consequently,
\[
\int_0^t \frac{\omega(s)}{(1 - s)^{\beta + \epsilon}} ds = \hat{\omega}(0) - \frac{\hat{\omega}(t)}{(1 - t)^{\beta + \epsilon}} + (\beta + \epsilon) \int_0^t \frac{\hat{\omega}(s)}{(1 - s)^{\beta + \epsilon + 1}} ds \lesssim \frac{\hat{\omega}(t)}{(1 - t)^{\beta + \epsilon}} + (\beta + \epsilon) \int_0^t \frac{1}{(1 - s)^{\epsilon + 1}} ds \lesssim \left( 1 + \frac{\beta + \epsilon}{\epsilon} \right) \frac{\hat{\omega}(t)}{(1 - t)^{\beta + \epsilon}}.
\] (2.6)

which means that when $\beta$ satisfies (ii) in Lemma F,
\[
\beta + \epsilon \in \{ \gamma : \gamma \text{ satisfies (iii) in Lemma F} \}.
\]
for any $\epsilon > 0$. Thus
\[
\inf \{ \gamma : \gamma \text{ satisfies (iii) in Lemma } F \} 
\leq \inf \{ \beta : \beta \text{ satisfies (ii) in Lemma } F \}. \tag{2.7}
\]

Next let $\lambda$ satisfy (iv) in Lemma $F$. Without loss of generality, we can assume $\lambda > 0$. In fact, if $\lambda = 0$, it is enough to consider the case of $\lambda + \epsilon$ for any $\epsilon > 0$. For $0 \leq r < t < 1$, take $\xi \in D$ such that $|\xi| = t$. It follows from (2.3) that
\[
\hat{\omega}(t) \geq \int_D \frac{\omega(z)dA(z)}{|1 - \xi z|^{\lambda+1}} \geq \int_r^1 \frac{\omega(s)}{(1 - st)^{\lambda}}ds \geq \frac{\hat{\omega}(r)}{(1 - r^2)^{\lambda}},
\]
which gives that $\lambda \in \{ \beta : \beta \text{ satisfies (ii) in Lemma } F \}$. Hence
\[
\inf \{ \beta : \beta \text{ satisfies (ii) in Lemma } F \} 
\leq \inf \{ \lambda : \lambda \text{ satisfies (iv) in Lemma } F \}. \tag{2.8}
\]

By (2.5), (2.7) and (2.8), we get
\[
\inf \{ \beta : \beta \text{ satisfies (ii) in Lemma } F \} 
= \inf \{ \gamma : \gamma \text{ satisfies (iii) in Lemma } F \} 
= \inf \{ \lambda : \lambda \text{ satisfies (iv) in Lemma } F \}. \tag{2.9}
\]

Let $\beta$ satisfy (ii) in Lemma $F$ so is $\beta + \epsilon$ for any $\epsilon > 0$. Consider $1 < x \leq y < \infty$ first. Note that for $0 < s < 1$ the function $f_{x, \beta + \epsilon}(s) = x^{\beta + \epsilon}s^x(1 - s)^{\beta + \epsilon}$ takes its maximum at $s = x/(x + \beta + \epsilon)$. Thus when $s \in (0, 1)$,
\[
x^{\beta + \epsilon}s^x(1 - s)^{\beta + \epsilon} \leq \left( \frac{x}{x + \beta + \epsilon} \right)^x \left( \frac{x(\beta + \epsilon)}{x + \beta + \epsilon} \right)^{\beta + \epsilon} \leq 2(\beta + \epsilon)^{\beta + \epsilon}.
\]

Combining this with (2.6) and (2.1), we see that
\[
x^{\beta + \epsilon} \int_0^1 s^x \omega(s)ds 
= x^{\beta + \epsilon} \int_{1-\frac{x}{y}}^1 s^x \omega(s)ds + x^{\beta + \epsilon} \int_0^{1-\frac{1}{x}} s^x \omega(s)ds 
\approx \hat{\omega}(1 - \frac{1}{x}) \left( \frac{1}{1 - (1 - \frac{1}{x})^{\beta + \epsilon}} \right) + \int_0^{1-\frac{1}{x}} x^{\beta + \epsilon}s^x(1 - s)^{\beta + \epsilon} \frac{\omega(s)}{(1 - s)^{\beta + \epsilon}}ds 
\leq \hat{\omega}(1 - \frac{1}{x}) \left( \frac{1}{1 - (1 - \frac{1}{x})^{\beta + \epsilon}} \right) + \int_0^{1-\frac{1}{x}} \frac{\omega(s)}{(1 - s)^{\beta + \epsilon}}ds 
\leq \hat{\omega}(1 - \frac{1}{x}) \left( \frac{1}{1 - (1 - \frac{1}{y})^{\beta + \epsilon}} \right). 
\]
Also,
\[
\frac{\hat{\omega}(1 - \frac{1}{y})}{(1 - (1 - \frac{1}{y}))^{\beta+\varepsilon}} \approx y^{\beta+\varepsilon} \int_{1-\frac{1}{y}}^{1} s^y \omega(s) \, ds \leq y^{\beta+\varepsilon} \int_{0}^{1} s^y \omega(s) \, ds.
\]
Thus for \(1 < x \leq y < \infty\), one gets
\[
\int_{0}^{1} s^x \omega(s) \, ds \lesssim \left( \frac{y}{x} \right)^{\beta+\varepsilon} \int_{0}^{1} s^y \omega(s) \, ds.
\] (2.10)
When \(0 < x \leq y \leq 1\), (2.10) holds due to
\[
\int_{0}^{1} s^x \omega(s) \, ds \approx \int_{0}^{1} s^y \omega(s) \, ds \approx \int_{0}^{1} \omega(s) \, ds.
\]
When \(0 < x \leq 1 < y < \infty\), there is a small enough positive constant \(c\) with \(1 < 1 + c < y\). Then
\[
\int_{0}^{1} s^x \omega(s) \, ds \approx \int_{0}^{1} s^{1+c} \omega(s) \, ds \lesssim \left( \frac{y}{1+c} \right)^{\beta+\varepsilon} \int_{0}^{1} s^y \omega(s) \, ds
\]
\[
\lesssim \left( \frac{y}{x} \right)^{\beta+\varepsilon} \int_{0}^{1} s^y \omega(s) \, ds.
\]
Consequently, \(\beta + \varepsilon \in \{\eta : \eta \text{ satisfies (v) in Lemma F}\}\). Then
\[
\inf\{\eta : \eta \text{ satisfies (v) in Lemma F}\} \leq \inf\{\beta : \beta \text{ satisfies (ii) in Lemma F}\}. \tag{2.11}
\]
Conversely, suppose (v) in Lemma F holds for some positive constant \(\eta\). Then \(\omega \in \mathring{D}\) and hence there exits a positive constant \(C = C(\omega) > 1\) such that
\[
\hat{\omega}(r) \leq C\hat{\omega}\left( \frac{r + 1}{2} \right),
\]
for all \(r \in (0, 1)\). Let \(t \in \left( \frac{2}{3}, 1 \right)\) and \(t_n = 1 - 2^n(1-t)\), \(n = 0, 1, \ldots, N-1\). Here \(N = N(t)\) is the largest positive integer such that \(t_{N-1} > 0\). We set \(t_N = 0\). Therefore,
\[
\int_{0}^{t} s^\frac{1-t}{n} \omega(s) \, ds \approx \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} s^\frac{1-t}{n} \omega(s) \, ds \lesssim \sum_{n=0}^{N-1} \left( t_n^{\frac{1}{1-t}} C^n \hat{\omega}(t) \right)
\]
\[
\lesssim \left( \sum_{n=0}^{N-1} e^{-2n} C^n \right) \hat{\omega}(t) \lesssim \hat{\omega}(t),
\]
where we use
\[
t_n^{\frac{1}{1-t}} = \left( \left(1 - 2^n(1-t)\right)^{\frac{1}{1-t}} \right)^{2n} \lesssim e^{-2n}.
for all \( n = 0, 1, \cdots, N \) and \( t \in (\frac{3}{4}, 1) \). Hence,

\[
\int_0^1 s^{1-t} \omega(s)ds = \left( \int_0^t + \int_t^1 \right) s^{1-t} \omega(s)ds \lesssim \hat{\omega}(t).
\]

So, when \( \frac{4}{3} \leq x \leq y < \infty \), we have

\[
\frac{\hat{\omega}(1-\frac{1}{y})}{(1-(1-\frac{1}{y}))^\eta} \leq x^\eta \int_0^1 s^x \omega(s)ds \lesssim y^\eta \int_0^1 s^y \omega(s)ds
\]

\[
= \frac{\int_0^1 s^{1-(1-\frac{1}{y})} \omega(s)ds}{(1-(1-\frac{1}{y}))^\eta} \lesssim \frac{\hat{\omega}(1-\frac{1}{y})}{(1-(1-\frac{1}{y}))^\eta},
\]

which gives

\[
\frac{\hat{\omega}(r)}{(1-r)^\alpha} \lesssim \frac{\hat{\omega}(t)}{(1-t)^\alpha}
\]

for all \( 1/4 \leq r \leq t < 1 \). Joining this with some elementary estimates, we see that \( \eta \) satisfies (2.1) for all \( 0 \leq r \leq t < 1 \). This yields

\[
\inf \{ \beta : \beta \text{ satisfies (ii) in Lemma } 2.1 \} \leq \inf \{ \eta : \eta \text{ satisfies (v) in Lemma } 2.1 \}.
\]

By (2.9), (2.11) and (2.12), we get the desired result. 

For \( \omega \in \hat{D} \), denoted by \( U(\omega) \) the infimum in Lemma 2.1. For a radial weight \( \omega \), by [24, Lemma B], \( \omega \in \hat{\mathcal{D}} \) if and only if there exists \( \alpha = \alpha(\omega) > 0 \) such that

\[
\frac{\hat{\omega}(r)}{(1-r)^\alpha} \lesssim \frac{\hat{\omega}(t)}{(1-t)^\alpha}
\]

for all \( 0 \leq r \leq t < 1 \). Let \( L(\omega) \) be the supremum of the set of these parameters \( \alpha \). Due to \( D = \hat{D} \cap \tilde{D} \), both \( U(\omega) \) and \( L(\omega) \) are well defined for \( \omega \in D \). Clearly, \( 0 < L(\omega) \leq U(\omega) \) if \( \omega \in D \). In general (2.13) does not hold when \( \alpha = L(\omega) \). It is also possible that (2.1), (2.2), (2.3) and (2.4) are not true if we let \( \beta, \gamma, \lambda \) and \( \eta \) be equal to \( U(\omega) \). For example, write

\[
\psi(|z|) = (1 - |z|) \left( 2 \log \frac{e}{1 - |z|} - 1 \right), \quad z \in \mathbb{D}.
\]

Then \( \psi \) is a weight and \( \hat{\psi}(t) = (1 - t)^2 \log \frac{e}{1 - t} \) for all \( t \in [0, 1) \). A simple observation gives that \( L(\psi) = 2 \) and (2.13) is not true when \( \omega = \psi \) and \( \alpha = 2 \). Set

\[
\phi(|z|) = \frac{1}{1 - |z|} \left( \log \frac{2}{1 - |z|} \right)^{-2}, \quad z \in \mathbb{D}.
\]

Then \( \phi \) is a weight and \( \hat{\phi}(t) \approx (\log \frac{2}{1 - t})^{-1} \) for all \( t \in [0, 1) \). It is easy to check that \( U(\phi) = 0 \) and (2.1) does not hold when \( \omega = \phi \) and \( \beta = U(\phi) \).
Double integral estimates for $B_p(\omega)$ spaces. In this subsection, we give a complete description of radial weights $\omega$ such that double integral estimates for $B_p(\omega)$ spaces hold.

For real parameters $p$ and nonnegative functions $\omega$ on $\mathbb{D}$, we will write $\omega_p(z) = (1 - |z|^2)^p \omega(z)$ and $dA_p(z) = (1 - |z|^2)^p dA(z)$ for convenience. Now we give Theorem 2.2, one side of double integral estimates for $B_p(\omega)$ which always holds. It is worth mentioning that the weight function $\omega$ in the following theorem is not necessarily radial.

**Theorem 2.2.** Suppose $p > 0$, $\tau > -1$ and $\omega$ be a nonnegative function on $\mathbb{D}$. Let $\sigma$ be a real number such that $\omega_\sigma$ is a weight. Then

$$
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4 + \tau + \sigma}} \omega_\tau(z) dA_\tau(z) dA_\sigma(\zeta) \gtrsim \int_{\mathbb{D}} |f'(z)|^p \omega(z) dA_{p-2}(z) \quad (2.16)
$$

for all $f \in H(\mathbb{D})$.

**Proof.** For $f \in H(\mathbb{D})$, by a change of variable $z = \varphi_\zeta(u)$, we see that

$$
J_1(\zeta) := (1 - |\zeta|^2)^\sigma \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4 + \tau + \sigma}} dA_\tau(z)
$$

$$
= \frac{1}{(1 - |\zeta|^2)^2} \int_{\mathbb{D}} \frac{|f \circ \varphi_\zeta(u) - f \circ \varphi_\zeta(0)|^p}{|1 - \zeta u|^{\tau - \sigma}} dA_\tau(u) \quad (2.17)
$$

for every $\zeta \in \mathbb{D}$. From Proposition 4.5 and Lemma 4.30 in [12], for all $\zeta$, $u$ and $\eta$ in $\mathbb{D}$ such that $\rho(u, \eta) < r$ for any fixed $r \in (0, 1)$,

$$
|1 - \zeta u| \approx |1 - \zeta \eta|, |1 - |u|^2 \approx 1 - |\eta|^2 \approx |1 - \overline{\eta} u|, \quad (2.18)
$$

where the comparison constants are independent of $\zeta$, $u$ and $\eta$. These facts allow us to apply a Littlewood-Paley estimate for Bergman spaces with non-radial weights $\omega_\tau(u) = (1 - |u|^2)^\tau / |1 - \zeta u|^{\tau - \sigma}$ (see Theorem A and its proof in [8]). We get

$$
\int_{\mathbb{D}} \frac{|f \circ \varphi_\zeta(u) - f \circ \varphi_\zeta(0)|^p}{|1 - \zeta u|^{\tau - \sigma}} dA_\tau(u)
$$

$$
\geq C \int_{\mathbb{D}} \frac{|(f \circ \varphi_\zeta)'(u)|^p}{|1 - \zeta u|^{\tau - \sigma}} dA_{\tau + p}(u), \quad (2.19)
$$
where $C$ is a positive constant independent of $\zeta$ and $f$. Bearing in mind (2.17), (2.18), (2.19), the change of variable with $u = \varphi_\zeta(z)$ and the sub-mean value property for $|f|^p$, we deduce that

$$J_1(\zeta) \gtrsim \frac{1}{(1 - |\zeta|^2)^2} \int_D \frac{|(f \circ \varphi_\zeta)'(u)|^p}{|1 - \zeta u|^\tau} dA_{r + \tau}(u)$$

$$\approx \int_D |f'(z)|^p \frac{(1 - |\zeta|^2)^\sigma}{|1 - \zeta z|^\sigma + \tau} dA_{r + \tau}(z)$$

$$\gtrsim (1 - |\zeta|^2)^{p-4} \int_{\Delta(\zeta, 1/2)} |f'(z)|^p dA(z)$$

$$\gtrsim (1 - |\zeta|^2)^{p-2} |f'(\zeta)|^p$$

for all $\zeta \in \mathbb{D}$. This implies (2.16). The proof is complete. □

The conditions of $\tau$ and $\sigma$ in Theorem 2.2 are only used to ensure the convergence of the integrals in the left-hand of (2.16).

Now we give the other side of double integral estimates for $B_p(\omega)$ spaces as follows.

**Theorem 2.3.** Suppose $p > 0$ and $\omega_{[p-2]}$ is a radial weight. Then the following conditions are equivalent:

(i) there exist real numbers $\sigma$ and $\tau$ such that

$$\int_D \int_D \frac{|f(z) - f(\zeta)|^p}{|1 - z|^4 + |1 - \zeta|^4} \omega(\zeta) dA_{r}(z) A_{\sigma}(\zeta) \lesssim \int_D |f'(z)|^p \omega(z) dA_{p-2}(z) \tag{2.20}$$

for all $f \in H(\mathbb{D})$;

(ii) $\omega_{[p-2]} \in \tilde{D}$.

To understand well the existence of parameters $\sigma$ and $\tau$ in Theorem 2.3, we prove the following result which implies Theorem 2.3.

**Theorem 2.4.** Suppose $p > 0$ and $\omega$ is a radial nonnegative function on $\mathbb{D}$. Then the following statements hold:

(i) if $\omega_{[p-2]}$ is a weight and there exist real numbers $\sigma$ and $\tau$ such that (2.20) holds for all $f \in H(\mathbb{D})$, then $\omega_{[p-2]} \in \tilde{D}$;

(ii) if $\omega_{[p-2]} \in \tilde{D}$, then (2.20) holds for all $f \in H(\mathbb{D})$ when $\min\{\sigma, \tau\} > p - 2$ and $\tau > \max\{U(\omega_{[p-2]}) - p - 1, -1\}$.

**Proof.** (i) Let $\omega_{[p-2]}$ be a weight and there exist real numbers $\sigma$ and $\tau$ such that (2.20) holds for all $f \in H(\mathbb{D})$. This forces $\tau > -1$ and $\omega_{[\sigma]}$ is a weight.
If $\sigma > \tau$, then
\[
\frac{(1 - |z|)^\sigma(1 - |\zeta|)^\omega(\zeta)}{|1 - \bar{\zeta}z|^{4+2\sigma}} \leq \frac{(1 - |z|)^\tau(1 - |\zeta|)^\omega(\zeta)}{|1 - \bar{\zeta}z|^{4+\tau+\sigma}} \leq \frac{(1 - |z|)^\tau(1 - |\zeta|)^\tau\omega(\zeta)}{|1 - \bar{\zeta}z|^{4+2\tau}}
\]
for all $z, \zeta \in \mathbb{D}$. Thus, without loss of generality, we can assume that $\sigma = \tau$.

Checking the proof of Theorem 2.2, we get that
\[
J_1(\zeta) = (1 - |\zeta|^2)^\sigma \int_D \frac{|f(z) - f(\zeta)|^p}{|1 - \bar{\zeta}z|^{4+\tau+\sigma}} dA_T(z)
\]
\[
\geq \int_D |f'(z)|^p \frac{(1 - |\zeta|^2)^\tau dA_T(\zeta)}{|1 - \bar{\zeta}z|^{2\tau+4}}
\]
(2.21)
for all $\zeta \in \mathbb{D}$. It follows from (2.20), (2.21) and the Fubini theorem that
\[
\int_D |f'(z)|^p \int_D \frac{\omega(\zeta)}{|1 - \bar{\zeta}z|^{2\tau+4}} dA_T(z) dA_{T+p}(z)
\]
\[
\leq \int_D |f'(z)|^p \omega(z) dA_{T+2}(z)
\]
(2.22)
for all $f \in H(\mathbb{D})$. Take $f'(z) = z^n$ in (2.22), where $n = 1, 2, \cdots$. Then
\[
\int_0^1 r^{np+1} (1 - r)^{p-2} \omega(r) dr
\]
\[
\geq \int_0^1 t^{np+1} (1 - t)^{p+\tau} \int_0^1 \frac{\omega(r)(1 - r)^\tau r}{(1 - tr)^{2\tau+3}} dr dt
\]
\[
\approx \int_0^1 \omega(r)(1 - r)^\tau r \left( \int_0^1 t^{np+1} (1 - t)^{p+\tau} (1 - tr)^{2\tau+3} dt \right) dr.
\]
(2.23)
Also,
\[
\int_0^1 t^{np+1} (1 - t)^{p+\tau} (1 - tr)^{2\tau+3} dt \geq \int_0^{\sqrt{\tau}} t^{np+1} (1 - t)^{p+\tau} (1 - tr)^{2\tau+3} dt
\]
\[
\geq r^{np+1} (1 - r)^{p-\tau-2}
\]
for all $r \in [0, 1)$ and for all positive integers $n$. By this and (2.23), there exists a constant $M \in (1, \infty)$ such that
\[
\int_0^1 r^{np+1} \omega_{[p-2]}(r) dr \leq M \int_0^1 r^{np+1} \omega_{[p-2]}(r) dr
\]
(2.24)
for all positive integers $n$. For $b \in \mathbb{R}$, let $E(b)$ be the integer with $E(b) \leq b < E(b) + 1$. Next we use (v) in Lemma 2.3 to show $\omega_{[p-2]} \in \mathcal{D}$. 
Let $\frac{p+1}{2} \leq x \leq y < \infty$. Write $n_0 = E(\frac{2x-1}{p})$ and $k = E(\log_2 \frac{2y}{n_0 p+1}) + 1$ for convenience. Then

$$\frac{n_0 p + 1}{2} \leq x < \frac{(n_0 + 1)p + 1}{2} \quad \text{and} \quad y < 2^{k-1}(n_0 p + 1).$$

Combining these with (2.24), we deduce that

$$\int_0^1 r^x \omega_{[p-2]}(r) dr \lesssim \int_0^1 r^y \omega_{[p-2]}(r) dr$$

where $C_M = M \sup_{n_0 \geq 1} (\frac{n_0 + 1}{n_0 p+1}) \log_2 M$ is independent of $x$ and $y$.

Let $0 < x \leq \frac{p+1}{2}$. Clearly,

$$\int_0^1 r^x \omega_{[p-2]}(r) dr \approx \int_0^1 r^y \omega_{[p-2]}(r) dr$$

$$\lesssim \left( \frac{y}{x} \right)^{\log_2 M} \int_0^1 r^y \omega_{[p-2]}(r) dr.$$

Let $0 < x < \frac{p+1}{2} < y < \infty$. Using (2.25), we get

$$\int_0^1 r^x \omega_{[p-2]}(r) dr \approx \int_0^1 r^{\frac{p+1}{2}} \omega_{[p-2]}(r) dr$$

$$\lesssim C_M \left( \frac{y}{x} \right)^{\log_2 M} \int_0^1 r^y \omega_{[p-2]}(r) dr.$$

Consequently,

$$\int_0^1 r^x \omega_{[p-2]}(r) dr \lesssim \left( \frac{y}{x} \right)^{\log_2 M} \int_0^1 r^y \omega_{[p-2]}(r) dr$$

for all $0 < x \leq y < \infty$. Hence Lemma [F] yields $\omega_{[p-2]} \in \mathcal{D}$.

(ii) Suppose $\omega_{[p-2]} \in \mathcal{D}$,

$$\min \{ \sigma, \tau \} > p - 2, \text{ and } \tau > \max \{ U(\omega_{[p-2]}) - p - 1, -1 \}. \quad (2.26)$$

Of course, $\omega_{[p-2]}$ is a weight. Then $\omega_{[\sigma]}$ is also a weight. We need to show that (2.20) holds for all $f \in H(\mathcal{D})$. 
Let $\sigma \geq \tau$. Following the proof of Theorem 2.2 and using Theorem B, we deduce that

$$
\int_D \int_D \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4+\tau+\sigma}} \omega(\zeta) dA_\tau(z) A_\sigma(\zeta)
$$

$$
= \int_D \frac{\omega(\zeta)}{(1 - |\zeta|^2)^2} dA(\zeta) \int_D \frac{|f \circ \varphi(\zeta) - f \circ \varphi(0)|^p}{|1 - \zeta u|^\tau} dA_\tau(u)
$$

$$
\leq 2^{\sigma - \tau} \int_D \frac{\omega(\zeta)}{(1 - |\zeta|^2)^2} dA(\zeta) \int_D |f \circ \varphi(\zeta) - f \circ \varphi(0)|^p dA_\tau(u)
$$

$$
\leq \int_D |f'(z)|^p \left( \int_D \frac{(1 - |\zeta|^2)^\sigma \omega(\zeta) dA(\zeta)}{|1 - \zeta z|^{2\tau+4}} \right) dA_{\tau+p}(z). \quad (2.27)
$$

Let $\sigma < \tau$. For $\zeta \in D$, write $\eta(\zeta) = |1 - \zeta u|^\sigma - (1 - |u|^2)^\tau$. Note that $\sigma > p - 2 > -2$. Clearly, there exists $s_0 \in [-1, 0)$ and $t_0 > 0$ such that if $s > s_0$ and $t > t_0$, then

$$
s + \tau > -1, \quad 2 + s + t > 0, \quad t > \tau, \quad \sigma + s > -2.
$$

It follows from Lemma [14] that

$$
\int_D \frac{\eta(\zeta)(1 - |u|^2)^s}{|1 - z u|^{2+s+t}} dA(u) = \int_D \frac{(1 - |u|^2)^s}{|1 - z u|^s + t} dA(u)
$$

$$
\leq C \frac{\eta(z)}{(1 - |z|^2)^t},
$$

where $C$ is a positive constant independent of $\zeta$. Using Theorem [B] and checking its proof in [8], we get that there exits another positive constant $C$ independent of $\zeta$ such that

$$
\int_D \frac{|f \circ \varphi(\zeta) - f \circ \varphi(0)|^p}{|1 - \zeta u|^\tau} dA_\tau(u)
$$

$$
\leq C \int_D \frac{|(f \circ \varphi(\zeta)(u))|d(1 - |u|^2)^p}{|1 - \zeta u|^\tau} dA_\tau(u)
$$

$$
= C \int_D |f'(z)|^p \frac{(1 - |\zeta|^2)^{\sigma+2}}{|1 - \zeta z|^{\sigma+\tau+4}} dA_{\tau+p}(z).
$$

This yields

$$
\int_D \int_D \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4+\tau+\sigma}} \omega(\zeta) dA_\tau(z) A_\sigma(\zeta)
$$

$$
\leq \int_D |f'(z)|^p \left( \int_D \frac{(1 - |\zeta|^2)^\sigma \omega(\zeta) dA(\zeta)}{|1 - \zeta z|^{\sigma+\tau+4}} \right) dA_{\tau+p}(z). \quad (2.28)
$$
Write \( x = \min\{\sigma, \tau\} \). It follows from (2.27) and (2.28) that

\[
\int_{D} \int_{\mathcal{D}} |f(z) - f(\zeta)|^p \left| \frac{\omega(\zeta)}{1 - \zeta z} \right|^{4+\tau+\sigma}\,dA_\tau(z)\,dA_\sigma(\zeta) \\
\lesssim \int_{\mathcal{D}} |f'(z)|^p \left( \int_{\mathcal{D}} \frac{(1-|\zeta|^2)^x \omega(\zeta)}{|1 - \zeta z|^{x+4}} \,dA(\zeta) \right) \,dA_{\tau+p}(z) \\
:= \int_{\mathcal{D}} |f'(z)|^p \,d\mu(z). \tag{2.29}
\]

By (2.26), \( \tau + x + 3 > p > 0 \). For any \( a \in \mathbb{D} \setminus \{0\} \), Lemma C yields

\[
\mu(S(a)) = \int_{S(a)} \left( \int_{\mathcal{D}} \frac{(1-|\zeta|^2)^x \omega(\zeta)}{|1 - \zeta z|^{x+4}} \,dA(\zeta) \right) \,dA_{\tau+p}(z) \\
\approx (1 - |a|) \int_{|a|}^1 (1-r)^{\tau+p} \int_0^{|a|} \frac{(1-t)^x \omega(t)\,dt}{(1-rt)^{\tau+x+3}} \,dr \\
+ (1 - |a|) \int_{|a|}^1 (1-r)^{\tau+p} \int_0^1 \frac{(1-t)^x \omega(t)\,dt}{(1-rt)^{\tau+x+3}} \,dr \\
:= (1 - |a|) (I_1(a) + I_2(a)). \tag{2.30}
\]

If \(|a| < r < 1 \) and \( 0 < t < |a| \), then

\[
1 > \frac{1-t}{1-rt} > \frac{1-|a|}{1-r|a|} > \frac{1-|a|}{1-|a|^2} > \frac{1}{2}.
\]

Note that \( \tau > U(\omega_{[p-2]}) - p - 1 \). By Lemma F, we have

\[
I_1(a) \approx \int_{|a|}^1 (1-r)^{p+\tau} \int_0^{|a|} \frac{\omega(t)(1-t)^{p-2}}{(1-t)^{\tau+p+1}} \,dt \,dr \\
\lesssim \frac{\omega_{[p-2]}(a)}{(1-|a|)^{\tau+p+1}} \int_{|a|}^1 (1-r)^{p+\tau} \,dr \\
\approx \omega_{[p-2]}(a). \tag{2.31}
\]

Due to \( x > p - 2 \) and \( \tau + p > -1 \),

\[
\int_0^1 \frac{(1-r)^{\tau+p}}{(1-rt)^{\tau+x+3}} \,dr \approx \frac{1}{(1-t)^{x+2-p}}
\]
for all \( t \in (0, 1) \). By this and the Fubini theorem, one gets
\[
I_2(a) = \int_{|a|}^1 \omega(t)(1 - t)^x \int_{|a|}^1 \frac{(1 - r)^{\tau + p}}{(1 - rt)^{\tau + x + 3}} dr dt
\]
\[
\lesssim \int_{|a|}^1 \omega(t)(1 - t)^{p - 2} \int_{0}^1 \frac{(1 - r)^{\tau + p}(1 - t)^{x + 2 - p}}{(1 - rt)^{\tau + x + 3}} dr dt
\]
\[
\approx \omega_{[p - 2]}(a).
\]

Note that
\[
\int_{S(a)} \omega_{[p - 2]}(z) dA(z) \approx (1 - |a|) \omega_{[p - 2]}(a)
\]
for all \( a \in \mathbb{D} \setminus \{0\} \). Consequently,
\[
\sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{\int_{S(a)} \omega_{[p - 2]}(z) dA(z)} < \infty.
\]

By (2.29) and Theorem 2.4 one gets
\[
\left( \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{\tau + x + 4}} \omega(\zeta) dA_\tau(\zeta) A_{\sigma}(\zeta) \right)^{\frac{1}{p}} dA_{\tau + p}(z) \right)^{\frac{1}{p}} \leq \sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{\int_{S(a)} \omega_{[p - 2]}(z) dA(z)} < \infty
\]

for all \( f \in H(\mathbb{D}) \). The proof is complete. \( \Box \)

Note that \( \mathcal{D} \not\subset \mathcal{D} \). For \( 0 < p < \infty \) and a radial nonnegative function \( \omega \) on \( \mathbb{D} \), if we assume \( \omega_{[p - 2]} \in \mathcal{D} \) then the range of parameter \( \sigma \) or \( \tau \) in (ii) of Theorem 2.4 should be larger. Because of this observation, we give the following result.

Proposition 2.5. Suppose \( p > 0 \) and \( \omega \) is a radial nonnegative function on \( \mathbb{D} \). If \( \omega_{[p - 2]} \in \mathcal{D} \), then (2.20) holds for all \( f \in H(\mathbb{D}) \) when \( \min\{\sigma, \tau\} > p - 2 - L(\omega_{[p - 2]}) \), \( \tau > \max\{U(\omega_{[p - 2]}) - p - 1, -1\} \), and \( \omega_{[\sigma]} \) is a weight.

Proof. Suppose \( \omega_{[p - 2]} \in \mathcal{D} \), \( \min\{\sigma, \tau\} > p - 2 - L(\omega_{[p - 2]}) \),
\[
\tau > \max\{U(\omega_{[p - 2]}) - p - 1, -1\},
\]
and \( \omega_{[\sigma]} \) is a weight. Recall that \( x = \min\{\sigma, \tau\} \). Then
\[
\tau + x + 3 > \tau + 1 + p - L(\omega_{[p - 2]}) \geq \tau + 1 + p - U(\omega_{[p - 2]}) > 0
\]
and hence (2.30) holds. Also, (2.31) is true. Checking the proof of (ii) of Theorem 2.4 step by step, it suffices to prove that \( I_2(a) \lesssim \omega_{[p - 2]}(a) \) for all
\( a \in \mathbb{D} \setminus \{0\} \) when \( p - 2 - L(\omega_{[p-2]}) < \min\{\sigma, \tau\} \leq p - 2 \). We write

\[
I_2(a) = \int_{|a|}^1 \omega(t) \int_{|a|}^t \frac{(1-r)^{\tau+p}(1-t)^x}{(1-rt)^{\tau+x+3}} dr dt \\
+ \int_{|a|}^1 \omega(t) \int_{t}^1 \frac{(1-r)^{\tau+p}(1-t)^x}{(1-rt)^{\tau+x+3}} dr dt \\
:= I_{21}(a) + I_{22}(a). \tag{2.32}
\]

Clearly,

\[
I_{22}(a) \lesssim \int_{|a|}^1 \omega(t)(1-t)^{-\tau-3} \int_{t}^1 (1-r)^{\tau+p} dr dt \approx \hat{\omega}_{[p-2]}(a). \tag{2.33}
\]

The Fubini theorem yields

\[
I_{21}(a) = \int_{|a|}^1 \int_r^1 \frac{(1-r)^{\tau+p}(1-t)^x \omega(t)}{(1-rt)^{\tau+x+3}} dt dr \\
\lesssim \int_{|a|}^1 \int_r^1 (1-r)^{p-x-3}(1-t)^x \omega(t) dt dr.
\]

If \( x = p - 2 \), due to \( \omega_{[p-2]} \in \check{D} \), it follows from (2.13) that

\[
I_{21}(a) \lesssim \int_{|a|}^1 (1-r)^{-1} \hat{\omega}_{[p-2]}(r) dr \lesssim \hat{\omega}_{[p-2]}(a). \tag{2.34}
\]

If \( p - 2 - L(\omega_{[p-2]}) < x < p - 2 \), we can choose a small enough positive number \( \varepsilon \) such that \( p - 2 - L(\omega_{[p-2]}) + \varepsilon < x \) and \( \frac{\omega_{[p-2]}(r)}{(1-t)^{\tau}(x-\omega_{[p-2]})} \) is essentially decreasing. An integration by parts gives that

\[
\int_{|a|}^1 \omega(t)(1-t^2)^x dt = \int_{|a|}^1 (1-|a|^2)^{x-p+2} \hat{\omega}_{[p-2]}(a) \\
+ 2(x - p + 2) \int_{|a|}^1 \hat{\omega}_{[p-2]}(t)(1-t^2)^{x-p+1} dt.
\]
Consequently,

\[ I_{21}(a) \lesssim \int_{[a]} \int_{r} (1 - r)^{p-x-3}(1-t)^x \omega(t) dtdr \]

\[ \approx \int_{[a]} \omega(t)(1-t)^x dt \int_{r} (1 - r)^{p-x-3} dr \]

\[ \lesssim (1 - |a|)^{p-x-2} \int_{[a]} \omega(t)(1-t)^x dt \]

\[ \lesssim \omega_{[p-2]}(a) + (1 - |a|)^{p-x-2} \int_{[a]} \frac{\omega_{[p-2]}(t)(1-t)^L(\omega_{[p-2]})^{-\varepsilon}}{(1-t)^L(\omega_{[p-2]})^{-\varepsilon}} dtdr \]

\[ \lesssim \omega_{[p-2]}(a). \quad (2.35) \]

Joining (2.32), (2.33), (2.34) and (2.35), we see that \( I_2(a) \lesssim \omega_{[p-2]}(a) \) for all \( a \in \mathbb{D}\setminus\{0\} \) when \( p - 2 - L(\omega_{[p-2]}) < \min\{\sigma, \tau\} \leq p - 2 \). The proof is finished. \( \square \)

Theorem 2.2 and Theorem 2.5 give complete descriptions of nonnegative radial weight functions \( \omega \) such that the double integral estimates for \( B_p(\omega) \) hold. As stated in Section 1 (cf. [12]), when \( p > 1 \) and \( \alpha \leq 1/2 \), there is double integral characterization for \( B_p(\alpha) \). Applying Theorem 2.2 and Proposition 2.5 we see that double integral characterization for \( B_p(\alpha) \) exists for all possible \( p \) and \( \alpha \). For \( p > 1 \) and \( \alpha \leq 1/2 \), the range of parameter \( \beta \) here is also better. Indeed, \( \max\{-1, -2\alpha - 1, -2 + 2\alpha\} = \max\{-1, -2\alpha - 1\} \) when \( \alpha \leq 1/2 \).

**Theorem 2.6.** Suppose \( p > 0, \alpha < \frac{p}{2}, \tau > \max\{-1, -2\alpha - 1, -2 + 2\alpha\} \) and \( \beta > -1 \). Then

\[ \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{3+\beta+\tau+2\alpha}} (1 - |\zeta|^2)^{\beta}(1 - |z|^2)^{\tau} dA(z) dA(\zeta) \]

\[ \approx \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-2\alpha} dA(z) \]

for all \( f \in H(\mathbb{D}) \).

**Proof.** Take \( \omega(z) = (1 - |z|^2)^{1-2\alpha} \). Then \( U(\omega_{[p-2]}) = L(\omega_{[p-2]}) = p - 2\alpha \). Apply Theorem 2.2 and Proposition 2.5 to this \( \omega \). If \( p > 0, p - 2\alpha > 0, \tau > \max\{-1, -2\alpha - 1\} \) and \( \min\{\sigma, \tau\} > -2 + 2\alpha \), then

\[ \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)|^p}{|1 - \zeta z|^{4+\tau+\sigma}} (1 - |\zeta|^2)^{1-2\alpha+\sigma}(1 - |z|^2)^{\tau} dA(z) dA(\zeta) \]

\[ \approx \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-2\alpha} dA(z) \]
for all $f \in H(\mathbb{D})$. Set $\beta = 1 - 2\alpha + \sigma$. The desired result follows. \qed

3. Hankel Type Operators Related to $B_p(\omega)$ Spaces with Radial Bekollé-Bonami Weights

In this section, applying double integral estimates for Besov type spaces $B_p(\omega)$, we characterize the boundedness of Hankel type operators related to $B_p(\omega)$ with radial Bekollé-Bonami weights.

For $p > 1$ and $s > -1$, the Bekollé-Bonami class $B_{p,s}$ consists of non-negative and integrable functions $\eta$ on $\mathbb{D}$ with the property that there exists a positive constant $C$ satisfying

$$
\left( \int_{S(a)} \eta(z)dA(z) \right) \left( \int_{S(a)} \left( \frac{\eta(z)}{(1 - |z|^2)^{s}} \right)^{\frac{p}{p'}} dA_s(z) \right)^{\frac{p'}{p}} \leq C(A_s(S(a)))^p
$$

for all Carleson boxes $S(a)$, where $p'$ satisfies $1/p + 1/p' = 1$. D. Békollé and A. Bonami [10] proved that $\eta \in B_{p,s}$ if and only if the Bergman projection

$$P_s f(z) = (s + 1) \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - \overline{\zeta} z)^{s+2}} dA_s(\zeta)
$$
is bounded from $L_{\eta}^p$ to $A_{\eta}^p$. Note that weights in $B_{p,s}$ are not necessarily radial. From Proposition 6 in [22], any radial weight in $B_{p,s}$ belongs to $\mathcal{D}$. See [9][10] for these weights.

Let $p > 0$ and let $\omega_{[p-2]}$ be a weight. Denote by $S_p(\omega)$ the Sobolev type space of smooth functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$
\|u\|_{S_p(\omega)} = |g(0)| + \left( \int_{\mathbb{D}} |\nabla u(z)|^p(1 - |z|^2)^{p-2}\omega(z)dA(z) \right)^{1/p} < \infty.
$$

Also, if $\omega(z) = (1 - |z|^2)^{1-2\alpha}$, then we write $S_p(\omega)$ as $S_p(\alpha)$. Clearly, $B_p(\omega)$ is a subset of all analytic functions in $S_p(\omega)$. Denote by $\mathcal{P}$ the class of polynomials on $\mathbb{D}$. If $w$ is radial, it is well known that $\mathcal{P}$ is dense in $B_p(\omega)$ (cf. [16]). For $s > -1$, consider the operator $\mathbb{P}_s$ given by

$$
\mathbb{P}_s u(z) = u(0) + \int_{\mathbb{D}} \frac{\partial u}{\partial w}(w) \frac{1 - (1 - \overline{w} z)^{1+s}}{(1 - \overline{w} z)^{1+s}} (1 - |w|^2)^s dA(w).
$$

Then one can define a (small) Hankel type operator on $\mathcal{P}$ with certain symbol $f$ by

$$
h^s_f g = \mathbb{P}_s(fg), \quad g \in \mathcal{P}.
$$

As explained by D. Blasi and J. Pau [12] p. 402], $\mathbb{P}_{1-2\alpha}$ defines a projection from $S_p(\alpha)$ to $B_p(\alpha)$. For $f$ analytic in $\mathbb{D}$, the boundedness of $h^{1-2\alpha}_f$ from $B_p(\alpha)$ to $S_p(\alpha)$ was also studied in [12]. Considering the same topic with weights $\omega$, one should define a Hankel type operator by the projection from $S_p(\omega)$ to $B_p(\omega)$. From this way, the results in Section 2 will not be used
in the study. Our purpose of this section is to present how to apply double integral estimates for $B_p(\omega)$ obtained in Section 2. Thus we still focus on the operator $h_s^f$. Let $f$ be analytic on $D$. We say that $h_s^f$ is bounded from $B_p(\omega)$ to $S_p(\omega)$ if there is a positive constant $C$ such that $\|h_s^f g\|_{S_p(\omega)} \leq C\|g\|_{B_p(\omega)}$ for all $g \in \mathcal{P}$. Also, by the Bergman projection $P_s$, it is also possible to define a (small) Hankel type operator $h_{s,f}$ on $\mathcal{P}$ by

$$h_{s,f}g = P_s(fg), \quad g \in \mathcal{P}.$$  

We refer to [23, 32] for more results on Hankel type operators.

We first recall Proposition 5 in [24] as follows.

**Proposition G.** Let $0 < p < \infty$, $\omega \in \hat{D}$ and write $\tilde{\omega}(r) = \hat{\omega}(r)/(1-r)$ for all $0 \leq r < 1$. Then $\omega \in \hat{D}$ if and only if $\|f\|_{A_p^\omega} \approx \|f\|_{A_p^{\tilde{\omega}}}$ for all $f \in H(D)$.

**Remark.** In fact, by the proof of the proposition above in [24], if $\omega \in D$, then $\tilde{\omega} \in R$. Here $R$ is the regular class of weights consisting of all $\omega \in D$ such that

$$\hat{\tilde{\omega}}(r) \approx (1-r)^{\omega(r)}, \quad r \in (0,1).$$

Also, if $\omega \in D$, then $\hat{\tilde{\omega}}(r) \approx \hat{\omega}(r)$ for all $0 \leq r < 1$. Then it is also clear that $U(\omega) = U(\tilde{\omega})$ and $L(\omega) = L(\tilde{\omega})$.

Now we give the following lemma.

**Lemma 3.1.** Suppose $1 < p < \infty$, $-1 < s < \infty$, and $\eta$ is a radial weight in $B_{p,s}$. Let $f \in A_{\eta}^p$ and let $h_{s,f}$ be a bounded operator from $B_p(\eta[2-p])$ to $L^p_{\eta}$. Then

$$\sup_{a \in D}(1 - |a|^2)|f(a)| < \infty.$$  

**Proof.** Bear in mind that any radial weight in $B_{p,s}$ belongs to $\mathcal{D}$. By Proposition[1] we have

$$\|h_{s,f}g\|_{L^p_{\eta}} = \|P_s(f \overline{g})\|_{L^p_{\eta}} = \|P_s(f \overline{g})\|_{A_{\eta}^p} \approx \|P_s(g)\|_{A_{\eta}^p} = \|h_{s,f}g\|_{L^p_{\eta}}$$

for any $g \in B_p(\eta[2-p])$. For any $a \in D$, set

$$g_a(z) = \frac{z^{n+1}}{(1-\overline{a}z)^{n+1}}, \quad z \in D,$

where $n$ is a positive integer. By [19] Lemma 3.1], if $n$ is large enough,

$$\|g_a\|_{B_p(\eta[2-p])} \leq \frac{(1 - |a|^{r} \eta(a))^{\frac{1}{r}}}{(1 - |a|)^{n+2}}$$

$$\leq \frac{(1 - |a|^{r} \eta(a))^{\frac{1}{r}}}{(1 - |a|)^{n+2}}$$

(3.2)
for all $a \in \mathbb{D}$. Since $f \in H(\mathbb{D})$ and $h_{s,f}$ is bounded, $P_s f = f$. For any fixed $\lambda > -1$,

\[
\begin{align*}
&f^{(n+1)}(a) \\
= & (s + 1) \left( \int_{\mathbb{D}} \frac{f(z)(1 - |z|^2)^s dA(z)}{(1 - az)^{s+2}} \right)^{(n+1)} \\
= & c_{n,s}(s + 1) \int_{\mathbb{D}} \frac{z^{n+1} f(z)}{(1 - az)^{n+1} (1 - a z)^{2+s}} (1 - |z|^2)^s dA(z) \\
= & c_{n,s,\lambda}(s + 1) \int_{\mathbb{D}} \frac{z^{n+1} f(z)}{(1 - az)^{n+1}} \left( \int_{\mathbb{D}} \frac{dA_\lambda(w)}{(1 - w z)^{2+s} (1 - aw)^{\lambda+2}} \right) dA_s(z) \\
= & c_{n,s,\lambda} \int_{\mathbb{D}} \frac{h_{s,f}(g_\lambda(w)) dA_\lambda(w)}{(1 - aw)^{\lambda+2}}.
\end{align*}
\]

Then, it follows from Hölder’s inequality, (3.1) and (3.2) that

\[
(1 - |a|^2)^{n+2} |f^{(n+1)}(a)| \lesssim (1 - |a|^2)^{n+2} \|h_{s,f}g_\lambda\|_{L^p_\eta} \left( \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\nu' \lambda} dA(w)}{(1 - aw)^{(\lambda+2)\nu'} \tilde{\eta}(w)} \right)^{\frac{1}{\nu'}} \\
\lesssim \|h_{s,f}\|_{B_p(\eta_{2-p'; 1}) \to L^p_\eta} (1 - |a|)^{\frac{1}{p'} \tilde{\eta}(a)} \left( \int_0^1 \frac{(1 - r)^{\nu' \lambda} dr}{(1 - |a|r)^{\lambda+2 - 1 \tilde{\eta}(r) \nu'}} \right)^{\frac{1}{\nu'}}
\]

for all $a \in \mathbb{D}$. Since $\tilde{\eta} \in \mathcal{R}$, there exist $-1 < \alpha, \beta < \infty$ such that $\frac{\tilde{\eta}(t)}{(1-t)^{\alpha}}$ and $\frac{\tilde{\eta}(t)}{(1-t)^{\beta}}$ are essential increasing and essential decreasing respectively (cf. [21] p. 11). Note that

\[-\frac{p' \beta}{p} - 2p' + 2 < 0 \iff \beta + 2 > 0.\]
Therefor, if \( \lambda \) is large enough,
\[
\int_0^{[a]} \frac{(1 - r)^{\mu' \lambda}}{1 - |a|r)^{(\lambda+2)p'-1} \tilde{\eta}(r)^{p'/p}} \leq \left( \frac{(1 - |a|)^{\alpha}}{\tilde{\eta}(|a|)} \right)^{p'/p} \int_0^{[a]} \frac{(1 - r)^{\mu' \lambda - \frac{\mu' \alpha}{p'}}}{1 - |a|r)^{(\lambda+2)p'-1} a} \int_0^{[a]} (1 - r)^{-\frac{\mu' \alpha}{p'} - 2p' + 1} dr
\]
\[
\frac{(1 - |a|)^{-2p' + 2}}{\tilde{\eta}(a)^{p'/p}}.
\]

and
\[
\int_0^1 \frac{(1 - r)^{\mu' \lambda}}{1 - |a|r)^{(\lambda+2)p'-1} \tilde{\eta}(r)^{p'/p}} \leq \left( \frac{(1 - |a|)^{\alpha}}{\tilde{\eta}(|a|)} \right)^{p'/p} \int_0^1 \frac{(1 - r)^{\mu' \lambda - \frac{\mu' \alpha}{p'}}}{1 - |a|r)^{(\lambda+2)p'-1} a} \int_0^1 (1 - r)^{-\frac{\mu' \alpha}{p'} - 2p' + 1} dr
\]
\[
\frac{(1 - |a|)^{-2p' + 2}}{\tilde{\eta}(a)^{p'/p}}.
\]

This and (3.3) yield that
\[
\sup_{a \in \mathbb{D}} (1 - |a|^2)^{n+2} |f^{(n+1)}(a)| < \infty,
\]
which is equivalent to
\[
\sup_{a \in \mathbb{D}} (1 - |a|^2) |f(a)| < \infty.
\]
The proof is complete. \( \Box \)

The following theorem is the main result in this section and the double integral estimates for Besov type spaces will be used in its proof.

**Theorem 3.2.** Suppose \( 1 < p < \infty, -1 < s < \infty, f \in H(\mathbb{D}) \) and \( \eta \in B_{p,s} \) is a radial weight. Let \( \eta, p, s \) satisfy one of the following conditions:

(a) \( U(\eta) < p - 1, 1 < p \leq 2, \) and \( U(\eta) - \frac{L(\eta)}{p-1} < p - 1 + \frac{sp}{p-1}; \)

(b) \( U(\eta) < p - 1, p \geq 2, \) and \( U(\eta) - L(\eta) < ps + 1; \)

(c) \( p - 1 \leq U(\eta) < ps + p, p > 1, s > 0, \) and \( L(\eta) > p - 1 - ps. \)

Then the following conditions are equivalent:

(i) \( h_{s,f} : B_p(\eta[2-p]) \to L_p^p \) is a bounded operator;

(ii) \( h_{s,f}^* : B_p(\eta[2-p]) \to S_p(\eta[2-p]) \) is a bounded operator, where \( F \) satisfies \( F' = f \) on \( \mathbb{D}; \)

(iii) \( d\mu(z) = |f(z)|^p \eta(z) dA(z) \) is a \( p \)-Carleson measure for \( B_p(\eta[2-p]). \)
Proof. (i) \(\iff\) (ii). For \(g \in P\), it is easy to check that for any \(z \in \mathbb{D}\),
\[
\frac{\partial h_z g}{\partial z}(z) = 0
\]
and
\[
\frac{\partial h_z g}{\partial \overline{z}}(z) = (s + 1) \int_{\mathbb{D}} \frac{g(w)f(w)(1 - |w|^2)^s}{(1 - w\overline{z})^{s+2}}dA(w)
\]
Hence
\[
|\nabla h_z g(z)| \approx |h_z f g(z)|
\]
for all \(z \in \mathbb{D}\). The equivalence between (i) and (ii) follows.

(iii) \(\Rightarrow\) (i). Suppose \(\mu\) is a \(p\)-Carleson measure for \(B_p(\eta_{[2-p]}\)). Since \(\eta \in B_{p,s}\), the Bergman projection \(P_s : L^p_\eta \to L^p_\eta\) is bounded. So, for any \(g \in B_p(\eta_{[2-p]}\)), we have
\[
\|h_{s,f}g\|_{L^p_\eta}^p = \|P_s(fg)\|_{L^p_\eta}^p \lesssim \|fg\|_{L^p_\eta}^p \lesssim \|g\|_{B_p(\eta_{[2-p]}\)}^p.
\]
Thus \(h_{s,f}\) is bounded.

(i) \(\Rightarrow\) (iii). Suppose \(h_{s,f} : B_p(\eta_{[2-p]}\) \(\to L^p_\eta\) is bounded. Note that any radial weight in \(B_{p,s}\) belongs to \(D\). By (3.1), Proposition (3.1) and the remark after it, we have
\[
\tilde{\eta} \in R, \quad L(\tilde{\eta}) = L(\eta), \quad U(\tilde{\eta}) = U(\eta),
\]
and
\[
\|h_{s,f}\|_{B_p(\tilde{\eta}_{[2-p]}\) \(\to L^p_\eta\) \(\approx \) \|h_{s,f}\|_{B_p(\eta_{[2-p]}\) \(\to L^p_\eta\).}
\]
Thus \(h_{s,f} : B_p(\tilde{\eta}_{[2-p]}\) \(\to L^p_\eta\) is also a bounded operator. Here,
\[
\tilde{\eta}_{[2-p]}(z) = (1 - |z|^2)^2-p\tilde{\eta}(z) = (1 - |z|^2)^2-p\frac{\tilde{\eta}(z)}{1-|z|}, \quad z \in \mathbb{D}.
\]
We claim that the \(L^p_\eta\) norm of
\[
f(z)\overline{g(z)} - h_{s,f}(g)(z) = \int_{\mathbb{D}} \frac{f(w)(g(z) - g(w))}{(1 - w\overline{z})^{2+sp}}dA_s(w)
\]
is dominated by the \(B_p(\tilde{\eta}_{[2-p]}\) norm of \(g\). By this claim and the boundedness of \(h_{s,f} : B_p(\tilde{\eta}_{[2-p]}\) \(\to L^p_\eta\), \(\mu\) is a \(p\) Carleson measure for \(B_p(\tilde{\eta}_{[2-p]}\) and hence \(\mu\) is a \(p\)-Carleson measure for \(B_p(\eta_{[2-p]}\). It suffices to prove the claim for cases (a), (b) and (c).

Case (a). Suppose
\[
U(\eta) < p - 1, \quad 1 < p < 2, \quad U(\eta) - \frac{L(\eta)}{p-1} < p - 1 + \frac{sp}{p-1}.
\]
(3.5)
Since \( \tilde{\eta} \in \mathcal{R} \), for any given \( \varepsilon > 0 \), \( \frac{\tilde{\eta}(t)}{(1-t)^{\varepsilon}} \) is essentially increasing on \([0, 1]\). From the definition of \( h_{s, f} \) and Lemma 3.1

\[
f = h_{s, f}(1) \in A_{\tilde{\eta}}^p, \quad \text{and} \quad M_f := \sup_{a \in \mathbb{D}} (1 - |a|^2) |f(a)| < \infty. \tag{3.6}
\]

By (3.6), Hölder’s inequality and Lemma \( \mathfrak{F} \), we see that

\[
\left| \int_{\mathbb{D}} \frac{f(w)(g(z) - g(w))}{(1 - \overline{w}z)^{2+s}} dA_s(w) \right|^p \leq M_f^{2-p} \left( \int_{\mathbb{D}} \frac{|f(w)|^{p-1}|g(z) - g(w)| dA_{p-2+s}(w)}{|1 - \overline{w}z|^{2+s}} \right)^p \leq M_f^{2-p} \| f \|_{A_{\tilde{\eta}}^p}^{p-1} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p (1 - |w|^2)^{(p-2+s)} \eta(z)^{p-1}}{|1 - \overline{w}z|^{2+s} p} dA(w) \lesssim C(f) \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p (1 - |w|^2)^{(p-2+s)} \eta(z)^{(p-1)}}{|1 - \overline{w}z|^{2+s} p} dA(w).
\]

Consequently, the \( L_{\tilde{\eta}^p}^p \) norm of (3.4) is dominated by a positive constant times

\[
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p (1 - |w|^2)^{(p-2+s)} \eta(z) dA(w) \eta(z) dA(z)}{|1 - \overline{w}z|^{2+s} p} dA(w),
\]

by which denoted \( L_1 \). Since (3.5) holds, by \( L(\eta) \leq U(\eta) \) and \( U(\eta) - \frac{L(\eta)}{p-1} < p - 1 + \frac{sp}{p-1} \), we have \( U(\eta) > \frac{(p-1)^2 + sp}{p-2} \). Then, from \( U(\eta) < p - 1 \), we get \( \frac{(p-1)^2 + sp}{p-2} < p - 1 \). Therefore,

\[
U(\eta) < p - 1 + \frac{sp}{p-1}.
\]

So, by (3.5), we can choose \( \varepsilon > 0 \) such that

\[
(2 + s)p \leq 4 + (p - 2 + s)p - (U(\eta) - 1 + \varepsilon)(p - 1) + p - 2,
\]

\[
(p - 2 + s)p - (U(\eta) - 1 + \varepsilon)(p - 1) > \max \{ U(\eta) - p - 1, -1 \},
\]

and

\[
\min \{ (p - 2 + s)p - (U(\eta) - 1 + \varepsilon)(p - 1), p - 2 \} > p - 2 - L(\eta).
\]

Then, Proposition 2.5 yields \( L_1 \lesssim \| g \|_{B_p(\tilde{\eta}_{2-p})}^p \). Consequently, our claim holds.

When

\[
U(\eta) < p - 1, \quad p = 2, \quad U(\eta) - \frac{L(\eta)}{p-1} < p - 1 + \frac{sp}{p-1},
\]
we can prove that our claim holds in the same way and the process is much easier. Thus we omit it.

Case (b). Suppose

\[ U(\eta) < p - 1, \quad p > 2, \quad U(\eta) - L(\eta) < ps + 1. \]  \quad (3.7)

Let \( x, y > -1 \) be fixed later. It follows from Hölder’s inequality that

\[
\left| \int_{D} \frac{f(w)(g(z) - g(w))}{(1 - wz)^{2+s}} dA_s(w) \right|^p \\
\leq \left( \int_{D} \frac{|f(w)|^{p'}}{1 - wz^y} dA_{p'+x}(w) \right)^{p-1} \\
\times \left( \int_{D} \frac{|g(z) - g(w)|^p}{1 - wz^{(2+s)p-(p-1)y}} dA_{ps-(p-1)(p'+x)}(w) \right),
\]

and

\[
\left( \int_{D} \frac{|f(w)|^{p'}}{1 - wz^y} dA_{p'+x}(w) \right)^{p-1} \\
\leq \|f\|^p_{A_{\tilde{\eta}}} \left( \int_{D} \left( \frac{(1 - |w|^2)^{p'+x}}{|1 - wz|^y} \tilde{\eta}(w)^{-\frac{1}{p-2}} dA(w) \right)^{\frac{p-1}{p-2}} \right)^{p-2}.
\]

Note that \( \tilde{\eta} \in \mathcal{R} \). For any given \( \varepsilon > 0 \), \( \sup_{w \in D} \frac{(1 - |w|)^{U(\eta) - 1 + \varepsilon}}{\tilde{\eta}(w)} < \infty \). If there exists \( x' := p' + x > -1 \) such that

\[
\frac{x'(p - 1) - (U(\eta) - 1 + \varepsilon)}{p - 2} > -1 \quad \quad (3.8)
\]

and

\[
\frac{y(p - 1)}{p - 2} - \frac{x'(p - 1) - (U(\eta) - 1 + \varepsilon)}{p - 2} > 2, \quad (3.9)
\]

then

\[
\int_{D} \frac{|f(w)|^q}{1 - wz^y} dA_{q+x}(w)^{p-1} \\
\leq C(f) \left( \int_{D} \left( \frac{(1 - |w|^2)^{\frac{(p-1) - (U(\eta) - 1 + \varepsilon)}{p-2}}}{|1 - wz|^\frac{p-1}{p-2}} dA(w) \right)^{\frac{p-1}{p-2}} \right)^{p-2} \\
\leq C(f) \left( \int_{D} \frac{1 - |z|)^{(y-x')(p-1) + (U(\eta) - 1 + \varepsilon) - 2(p-2)}}{1 - |z|^{y-x'}} dA(w) \right).
Therefore,
\[
\int_{\mathbb{D}} \left| \int_{\mathbb{D}} \frac{f(w)(g(z) - g(w))}{1 - \overline{w}z} dA_s(w) \right|^p \eta(z) dA(z) \\
\lesssim C(f) \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p}{1 - \overline{w}z} |(2+s)p - (p-1)y| dA_{ps - (p-1)x'}(w) \\
\times \eta(z) dA_{2(p-2) - (y - x')(p-1) - (U(\eta) - 1 + \varepsilon)}(z).
\]

If \( x' > -1, \varepsilon > 0 \) and \( y \) also satisfy
\[
\begin{align*}
(2 + s)p - (p - 1)y & \\
\leq 4 + ps - (p - 1)x' + 3(p - 2) - (y - x')(p - 1) - (U(\eta) - 1 + \varepsilon),
\end{align*}
\]
(3.10)

\[
ps - (p - 1)x' > \max\{U(\eta) - p - 1, -1\},
\]
(3.11)

\[
ps - (p - 1)x' > p - 2 - L(\eta),
\]
(3.12)

and
\[
3(p - 2) - (y - x')(p - 1) - (U(\eta) - 1 + \varepsilon) > p - 2 - L(\eta),
\]
(3.13)

then Proposition 2.5 gives \( L_1 \lesssim \|g\|_{B_p(\eta_{[2-p]})}^p \) and hence our claim holds.

Next we explain the existence of \( x', y \) and \( \varepsilon \) such that (3.8)-(3.13) hold. Clearly, (3.10) is equivalent to \( U(\eta) \leq p - 1 - \varepsilon \).
(3.8), (3.11), (3.12) and \( x' > -1 \) are equivalent to
\[
x'(p - 1) > \max\{U(\eta) - p + 1 + \varepsilon, 1 - p\}
\]
and
\[
x'(p - 1) < \min\{ps + p + 1 - U(\eta), ps + 1, ps - p + 2 + L(\eta)\}.
\]
(3.9) and (3.13) are equivalent to
\[
2p - 3 - U(\eta) - \varepsilon < (y - x')(p - 1) < 2p - 3 - U(\eta) + L(\eta) - \varepsilon.
\]
 Indeed (3.7) ensures the existence of \( x', y \) and \( \varepsilon \).

Case (c). Suppose
\[
p - 1 \leq U(\eta) < ps + p, \ p > 1, \ s > 0, \ L(\eta) > p - 1 - ps.
\]
(3.14)
By Hölder’s inequality and Lemma 3.1, we have
\[
\left| \int_{\mathbb{D}} \frac{f(w)(g(z) - g(w))}{(1 - \overline{w}z)^{2+s}} dA_s(w) \right|^p \leq \left( \int_{\mathbb{D}} \frac{|f(w)|^{p'}}{1 - \overline{w}z} dA_{p'+x}(w) \right)^{p-1} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p}{(2+s)^{p-(p-1)y}} dA_{ps-p-(p-1)x}(w)
\]
\[
\lesssim M_p^p \left( \int_{\mathbb{D}} \frac{dA_x(w)}{1 - |w|^p} \right)^{p-1} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p}{(2+s)^{p-(p-1)y}} dA_{ps-p-(p-1)x}(w).
\]
From (3.14), we can choose
\[
x > -1, y - x - 2 = \varepsilon > 0,
\]
such that
\[
(2 + s)p - (p - 1)y = 4 + ps - p - (p - 1)x + p - 2 - \varepsilon(p - 1),
\]
\[
ps - p - (p - 1)x > \max\{U(\eta) - p - 1, -1\},
\]
and
\[
\min\{ps - p - (p - 1)x, p - 2 - \varepsilon(p - 1)\} > p - 2 - L(\eta).
\]
Then, by Proposition 2.5
\[
\left| \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{f(w)(g(z) - g(w))}{(1 - \overline{w}z)^{2+s}} dA_s(w) \right|^p \lesssim M_p^p \left( \int_{\mathbb{D}} \frac{dA_x(w)}{1 - |w|^p} \right)^{p-1} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p}{(2+s)^{p-(p-1)y}} dA_{ps-p-(p-1)x}(w) \tilde{\eta}(z) dA(z)
\]
\[
\lesssim M_p^p \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^p}{(2+s)^{p-(p-1)y}} dA_{ps-p-(p-1)x}(w) \tilde{\eta}(z) dA_{-\varepsilon(p-1)}(z)
\]
\[
\lesssim M_p^p \int_{\mathbb{D}} |g(z)| |\tilde{\eta}(z)| dA(z).
\]
Hence our claim also holds for this case. The proof is complete. 

When \(\omega \in B_{p,s}\) satisfies one more condition, the description of Carleson measures for the spaces \(B_p(\omega)\) was obtained in [5].

The following result is Theorem 4.2 in [12].

**Theorem H.** Let \(1 < p < \infty\) and \(\alpha \leq 1/2\), and let \(s\) with \(s > \max\{\frac{-1}{p}, \frac{1-p}{p}\}\) if \(\alpha = 1/2\) and \(s > \max\{0, -2\alpha\}\) if \(\alpha < 1/2\). Let \(g\) be analytic on \(\mathbb{D}\). Then the operator \(h_{s,g}\) is bounded from \(B_p(\alpha)\) to \(L^p(1-|z|^2)^{p-1-2\alpha}\) if and only if the measure \(|g(z)|^p(1-|z|^2)^{p-1-2\alpha} dA(z)\) is a \(p\)-Carleson measure for \(B_p(\alpha)\).

Applying Theorem 3.3, we obtain a result similar to Theorem [14].

**Theorem 3.3.** Suppose \(g \in H(\mathbb{D})\), \(1 < p < \infty\), \(\alpha < \frac{k}{2}\), and \(s > -\frac{2\alpha}{p}\).

Furthermore, if \(p, \alpha\) and \(s\) satisfy one of the following conditions:

(a) \(1 < p \leq 2\), \(\alpha > \frac{1}{2}\) and \(s > \frac{(4-2p)\alpha-1}{p}\);
(b) $p > 2$, $\alpha > 1/2$ and $s > -\frac{1}{p}$;
(c) $p > 1$ and $s > \max\{0, \frac{2\alpha - 1}{p}\}$.

Then $h_{s,g}$ is bounded from $B_p(\alpha)$ to $L^p_{(1-|z|^2)^{p-1-2\alpha}}$ if and only if the measure $\mu(\alpha) = (1-|z|^2)^{p-1-2\alpha}d\ast(z)$ is a $p$-Carleson measure for $B_p(\alpha)$.

Proof. Set $\eta(z) = (1-|z|^2)^{p-1-2\alpha}$ with $p > 2\alpha$. Then $B_p(\eta_{2-p}) = B_p(\alpha)$. Clearly, $U(\eta) = L(\eta) = p - 2\alpha$. For $-1 < s < \infty$ and $a \in D \setminus \{0\}$, we deduce that

$$
\left(\int_{S(a)} \eta(z) d\ast(z) \right) \left(\int_{S(a)} \frac{\eta(z)}{(1-|z|^2)^s} d\ast(z) \right)^{\frac{p}{p'}} \approx (1-|a|)^{p-1-2\alpha} \left( \left(1-|a|\right) \int_{|a|}^1 (1-r)^{-1+s+\frac{2\alpha+s}{p-r}} dr \right)^{\frac{p}{p'}}
$$

where we use $s + \frac{2\alpha+s}{p-1} = \frac{2\alpha+ps}{p-1} > 0$. Hence $\eta \in B_{p,s}$ when $s > -\frac{2\alpha}{p}$. Note that $-\frac{2\alpha}{p} > -1$. Then the desired result follows from Theorem 3.2. \qed

Remark. Theorem 3.3 contains the case of $1 < p < \infty$ and $1/2 < \alpha < p/2$ missing in Theorem 3.3. For $1 < p < \infty$ and $\alpha < 1/2$, $\max\{0, (2\alpha - 1)/p, -2\alpha/p\} \leq \max\{0, -2\alpha\}$ and hence the range of $s$ in Theorem 3.3 is larger. Comparing with Theorem 3.3, Theorem 3.3 has shortcoming at the point $\alpha = 1/2$. Bear in mind examples in (2.14) and (2.15).

Z. Lou and R. Qian [15] investigate the boundedness of Hankel type operator $h_{s,f}$ related to a class of Dirichlet type spaces $D_\rho$. Let $\rho : [0, \infty) \to [0, \infty]$ be a right-continuous and nondecreasing function. $D_\rho$ is equal to $B_2(\omega_\rho)$ with $\omega_\rho(|z|) = \rho(1-|z|)$. $\rho$ is said to be upper (resp. lower) type if $\gamma \in (0, \infty)$ (cf. [14])

$$
\rho(xy) \leq C x^\gamma \rho(y), \quad x \geq 1 \ (\text{resp.} \ x \leq 1) \quad \text{and} \ \ 0 < y < \infty.
$$

Clearly, if $\rho$ is upper type $\gamma$ for some $\gamma > 0$, then $\rho(2y) \lesssim \rho(y)$ for all $y > 0$.

The following result is Theorem 1 in [15].

**Theorem I.** Let $0 < \gamma < 1$, $s > \frac{1+\gamma}{2}$ and $f \in H(D)$. Suppose $\rho : [0, \infty) \to [0, \infty)$ is a right-continuous and nondecreasing function of upper type $\gamma$. Then $h_{\rho,s} : B_2(\omega_\rho) \to L^2_{\omega_\rho}$ is bounded if and only if $|f(z)|^2 \rho(1-|z|^2)dA(z)$ is a 2-Carleson measure for $B_2(\omega_\rho)$.
In [15, p. 219, Remark 2], the authors mentioned that they failed to prove Theorem and other results in their paper without the condition \( s > \frac{1+\gamma}{2} \). Motivated by this remark, we apply Theorem 3.2 to give the following result which means that the condition in Theorem 3.2 can be improved.

**Theorem 3.4.** Let \( 0 < \gamma < \infty, s > \max\{0, \frac{2-1}{2}\} \) and \( f \in H(\mathbb{D}) \). Suppose \( \rho : [0, \infty) \to [0, \infty) \) is a right-continuous and nondecreasing function of upper type \( \gamma \). Then \( h_{s,f} : B_2(\omega_\rho) \to L^2_{\omega_\rho} \) is bounded if and only if \( |f(z)|^2 \rho(1-|z|^2)dA(z) \) is a \( 2 \)-Carleson measure for \( B_2(\omega_\rho) \).

**Proof.** Note that

\[
\hat{\omega}_\rho(r) = \int_r^1 \rho(1-s)ds \leq (1-r)\rho(1-r)
\]

and

\[
\hat{\omega}_\rho(r) \geq \int_r^{\frac{1}{1-r}} \rho(1-s)ds \geq (1-r)\rho(1-r)
\]

for all \( r \in [0,1) \). Hence \( \hat{\omega}_\rho(r) \approx (1-r)\rho(1-r) \) for all \( r \in [0,1) \). Since \( \rho \) is upper type \( \gamma \), \( \rho(1-x) \lesssim \frac{(1-r)^\gamma}{1-y} \rho(1-y) \) for all \( 0 < x \leq y < 1 \). Consequently, \( \hat{\omega}_\rho(r) \) is essentially increasing on \([0,1)\) and \( \frac{\hat{\omega}_\rho(r)}{1-r} \) is essentially decreasing on \([0,1)\). Thus \( 1 \leq L(\omega_\rho) \leq U(\omega_\rho) \leq \gamma + 1 \).

Since \( s > \frac{2-1}{2} \), we deduce that

\[
\int_{S(a)} \rho(1-|z|)dA(z) \int_{S(a)} \left( \frac{\rho(1-|z|)}{(1-|z|^2)^s} \right)^{-1} dA(z)
\]

\[
\approx (1-|a|)^2 \omega_\rho(|a|) \int_{|a|}^1 (1-r)^{2s+\gamma} dr
\]

\[
\leq (1-|a|)^{3+\gamma} \int_{|a|}^1 (1-r)^{2s-\gamma} dr \approx (1-|a|)^{4+2s} \approx (A_s(S(a)))^2
\]

for all \( a \in \mathbb{D} \setminus \{0\} \). This gives \( \omega_\rho \in B_{2,s} \) when \( s > \frac{2-1}{2} \).

Set \( p = 2 \) and \( \eta = \omega_\rho \) in (c) of Theorem 3.2. Since \( s > \max\{0, \frac{2-1}{2}\} \) and \( 1 \leq L(\eta) \leq U(\eta) \leq \gamma + 1 \), we see that (c) of Theorem 3.2 holds. Then we get the desired result. \( \square \)

**Remark.** Related to Theorem 3.2, for \( 0 < p < \infty \) and a weight \( \eta \), denote by \( W^p_\eta \) the space of those functions \( f \in H(\mathbb{D}) \) such that \( |f'(z)|^p \eta(z) dA(z) \) is a \( p \)-Carleson measure for \( B_p(\eta_2-\rho) \). Similar to results in [12, 15, 27], the interested reader can also establish an atomic decomposition theorem for \( W^p_\eta \).
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