CONGRUENCE PROPERTIES OF BORCHERDS PRODUCT EXponents

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Abstract. In his striking 1995 paper, Borcherds [2] found an infinite product expansion for certain modular forms with CM divisors. In particular, this applies to the Hilbert class polynomial of discriminant $-d$ evaluated at the modular $j$-function. Among a number of powerful generalizations of Borcherds’ work, Zagier made an analogous statement for twisted versions of this polynomial. He proves that the exponents of these product expansions, $A(n, d)$, are the coefficients of certain special half-integral weight modular forms. We study the congruence properties of $A(n, d)$ modulo a prime $\ell$ by relating it to a modular representation of the logarithmic derivative of the Hilbert class polynomial.

1. Introduction and statement of results

The modular $j$-invariant, one of the most important functions in modern number theory, has \(q\)-expansion

\[ j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \quad (q = e^{2\pi iz}) \]

Singular moduli are the values of $j(z)$ at imaginary quadratic arguments. To be more precise, let $Q_d$ be the set of positive definite binary quadratic forms $Q(x, y)$ of discriminant $-d < 0$, and let $\alpha Q$ be the unique root in the upper half-plane of some $Q(x, 1) \in Q_d$. Given one of these roots $\alpha Q$, the Hilbert class polynomial of discriminant $-d$, $H_d(z)$, is defined to be the minimal polynomial of $j(\alpha Q)$ (with minor modifications if $d/3$ or $d/4$ is a square). The roots of $H_d(z)$ generate the Hilbert class fields of imaginary quadratic fields.

The study of the form $H_d(j(z))$ has a long history. Until recently, it was notoriously difficult to compute Hilbert class polynomials. Borcherds [2] was the first to describe an infinite product expansion for $H_d(j(z))$,

\[ H_d(j(z)) = \prod_{Q \in Q_d/SL_2(\mathbb{Z})} (j(z) - j(\alpha Q))^{-\omega_Q} = q^{-h(d)} \prod_{n=1}^{\infty} (1 - q^n)^{A_d(n)} \]

where $h(d)$ is the Hurwitz-Kronecker class number and $\omega_Q$ is the weight of $Q$ (see [12]). Zagier [12] generalized these to a setting where Borcherds products and their twisted analogues are easier to compute. He defines, for all discriminants $D > 0$ and $-d < 0$,
a twisted Hilbert class polynomial by

\[
H_{D,d}(j(z)) = \prod_{Q \in Q_{D,d}/SL_2(\mathbb{Z})} (j(z) - j(\alpha_Q))^{\chi(Q)} = \prod_{n=1}^{\infty} \left( \prod_{k=1}^{D-1} \left( 1 - \zeta_D^k q^n \right) \right)^{A_{D,d}(n)}
\]

where \( \zeta_D = e^{2\pi i / D} \) and \( \chi(Q) = \left( \frac{-d}{p} \right) \) for some prime \( p \) represented by \( Q \) that does not divide \( dD \).

In light of this, we study the natural question of the distribution of the exponents \( A_{d}(n) \) among residue classes modulo \( \ell \) for various primes \( \ell \), and we show that these exponents often possess unexpected properties. To this end, let

\[
\delta_d(t, \ell; X) = \frac{\# \{ p < X : p \text{ is prime and } A_d(p) \equiv t \pmod{\ell} \}}{\pi(X)}
\]

where \( \pi(X) \) is the number of primes less than \( X \). For certain choices of \( d \) depending on \( \ell \), we will be able to compute the asymptotic value of \( \delta_d(t, \ell; X) \) for each \( t \in \mathbb{Z}/\ell\mathbb{Z} \). For example, for \( \ell = 11 \) and \( d = 4 \), we have the following:

**Table 1.**

| \( X \) | \( \delta_4(0, 11; X) \) | \( \delta_4(1, 11; X) \) | \( \delta_4(2, 11; X) \) | \( \delta_4(3, 11; X) \) | \( \delta_4(4, 11; X) \) | \( \delta_4(5, 11; X) \) |
|---|---|---|---|---|---|---|
| \( 10^4 \) | .0829 | .0928 | .0887 | .0911 | .0862 | .0903 |
| \( 10^5 \) | .0908 | .0927 | .0853 | .0898 | .0883 | .0888 |
| \( 10^6 \) | .0899 | .0897 | .0891 | .0915 | .0887 | .0894 |
| \( 10^7 \) | .0898 | .0909 | .0901 | .0903 | .0897 | .0895 |
| \( 2 \cdot 10^7 \) | .0899 | .0905 | .0901 | .0902 | .0896 | |
| \( 5 \cdot 10^7 \) | .0899 | .0901 | .0900 | .0902 | .0897 | |
| \( \cdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \infty \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) |

| \( X \) | \( \delta_4(6, 11; X) \) | \( \delta_4(7, 11; X) \) | \( \delta_4(8, 11; X) \) | \( \delta_4(9, 11; X) \) | \( \delta_4(10, 11; X) \) |
|---|---|---|---|---|---|
| \( 10^4 \) | .0846 | .0960 | .1009 | .1066 | .0797 |
| \( 10^5 \) | .0902 | .0925 | .0955 | .0948 | .0914 |
| \( 10^6 \) | .0893 | .0913 | .0976 | .0920 | .0914 |
| \( 10^7 \) | .0901 | .0906 | .0986 | .0898 | .0907 |
| \( 2 \cdot 10^7 \) | .0898 | .0902 | .0991 | .0897 | .0906 |
| \( 5 \cdot 10^7 \) | .0899 | .0901 | .0991 | .0899 | .0908 |
| \( \cdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \infty \) | \( \frac{9}{100} = .09 \) | \( \frac{9}{100} = .09 \) | \( \frac{119}{1200} \approx .0992 \) | \( \frac{9}{100} = .09 \) | \( \frac{109}{1200} \approx .0908 \) |

Notice that most congruence classes modulo 11 asymptotically contain 9 percent of all \( A_d(p) \), but the classes of 8 and 10 have a slight excess, with 9.92\ldots percent and 9.08\ldots percent, respectively. As we shall see, this is a consequence of the fact that the coefficients
of the logarithmic derivative of $H_4(j(z))$ can be written modulo 11 as a function of the coefficients of a Hecke eigenform, and by a deep theorem of Deligne, the latter induces a Galois representation whose traces are the corresponding coefficients, which turn out to be dictated by group theory. Hence the given distribution of $A_d(p)$ modulo 11 is actually a statement about the distribution of the conjugacy classes in the image of the representation, which is given by the Chebotarev Density Theorem.

Let $S_{\ell+1}$ be the space of cusp forms of weight $\ell+1$ on $SL_2(\mathbb{Z})$. Then we have the following congruence for $A_d(n)$ modulo $\ell$.

**Theorem 1.1.** Suppose $\ell$ is prime and $-d < 0$ and $D > 0$ are fundamental discriminants such that $dD < \ell$ is fundamental, $\ell$ is inert in $\mathbb{Q}(\sqrt{-Dd})$, $(\frac{\ell}{Dd}) = 1$, and $\ell \nmid n$. Also let $\nu(m)$ be the Dirichlet inverse of $\sum_{k=1}^{D-1} (\frac{D}{k}) \zeta_D^k$. Then there exist constants $c_0, c_1 \cdots c_r$ depending only on $d$ and $\ell$ such that

$$A_{D,d}(n) \equiv \frac{1}{n} \sum_{m|n} \nu(n/m)(-24c_0\sigma_1(m) + c_1a_1(m) + \cdots + c_ra_r(m)) \pmod{\ell}$$

where $r$ is the dimension of $S_{\ell+1}$ and $f_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n$ are the normalized Hecke eigenforms in $S_{\ell+1}$.

**Remark.** When $D = 1$, we have $\nu(n/m) = \mu(n/m)$, so this formula simplifies. A more general version of this theorem can be obtained when the conditions $\ell > d$ and $(\frac{\ell}{D})$ are not necessarily met. The more general conditions are made clear in Section 3. Although at first glance, this strategy appears to give no information in the case $\ell | n$, there may be a similar result in such cases if we instead look modulo powers of $\ell$ exceeding the maximum power of $\ell$ dividing $n$.

When $\ell \in \{5, 7, 13\}$ (the primes for which $\dim_{\mathbb{C}} S_{\ell+1} = 0$) and $D = 1$ this implies the trivial congruences

$$A_d(n) \equiv -24h(d) \pmod{\ell}.$$ 

For $\ell \in \{11, 17, 19\}$, we have $\dim_{\mathbb{C}} S_{\ell+1} = 1$. If we let $D = 1$, and $n = p \neq \ell$, the formula simplifies to

$$A_d(p) \equiv -24h(d) + \frac{c_2}{p}(a_1(p) - 1) \pmod{\ell}.$$ 

These formulas relate Borcherds product exponents to explicit modular elliptic curves. For example, if $d = 4$ and $\ell = 11$, $S_{12}$ is spanned by the weight 12 cusp form with Fourier expansion

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \cdots = \sum_{n=1}^{\infty} \tau(n)q^n,$$

so in this case $a_1(m) = \tau(m)$. We have that

$$\Delta(z) \equiv \eta(z)^2 \eta(11z)^2 = \sum_{n=1}^{\infty} a_{X_0(11)}(n)q^n \pmod{11},$$
which is the modular form corresponding to the elliptic curve $X_0(11)$, which in turn implies that

$$a_{X_0(11)}(p) = p + 1 - \#(X_0(11)/\mathbb{F}_p).$$

Thus from Theorem 1.1 the values of $A_d(p)$ modulo 11 correspond to the number of points on $X_0(11)$ over $\mathbb{F}_p$, which has defining equation

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

Therefore, Table 1 reflects the distribution of the pairs $(p, a_{X_0(11)}(p))$ modulo 11. Serre proved (see Theorem 11 of [10]) that every modular form of weight $\ell + 1$ modulo $\ell$ on $SL_2(\mathbb{Z})$ corresponds to a weight 2 cusp form on $\Gamma_0(\ell)$. Thus we can see that the same phenomenon occurs for all pairs $(d, \ell)$ given here.

Table 2.

| $\ell$ | $d$          |
|--------|-------------|
| 11     | 3, 4, 11, 12, 15, 20, 67, 115, 148, 163, 267 |
| 17     | 3, 7, 11, 12, 24, 28, 88, 91, 163, 267, 403 |
| 19     | 4, 7, 11, 19, 20, 28, 35, 43, 163, 187, 235, 427 |

In these cases, the curves $X_0(17)$ and $X_0(19)$ are given by the defining equations

$$X_0(17) : y^2 + xy + y = x^3 - x^2 - 6x - 4$$

and

$$X_0(19) : y^2 + y = x^3 + x^2 - 9x - 15.$$  

To prove these congruences, we will consider the logarithmic derivative of Zagier’s twisted Hilbert class polynomials and use their relationship to supersingular polynomials to prove that they lie in $\tilde{M}_{\ell+1}$, the space of weight $\ell + 1$ modular forms modulo $\ell$. Dorman’s work on differences of singular moduli in [4] will be useful here. Using this, we find a formula for the exponents $A_{D,d}(n)$ in terms of the coefficients of Eisenstein series and cusp forms that form a basis for $S_{\ell+1}$.

In Section 2 we rigorously revisit Borcherds product expansions and their generalizations by Zagier. In Section 3 we recall essential properties of Eisenstein series, Hilbert class polynomials, and supersingular polynomials. In Section 4 we use these properties to prove Theorem 1.1. In Section 5 we use the theory of Galois representations and the Chebotarev Density Theorem to determine the distribution of $p, a_1(p), \ldots, a_k(p)$ and we give two examples by applying this procedure to two specific cases, $d = 4, \ell = 11$ and $d = 20, \ell = 31$.

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2. A theorem of Zagier

Define $M_{1/2}^!(\Gamma(4))$ to be the space of all weight $1/2$ modular forms on $\Gamma(4)$ which are meromorphic at the cusps and holomorphic everywhere else. It turns out that the exponents in the infinite product expansions of $H_d(x)$ and $H_{D,d}(x)$ are the coefficients of special modular forms in the “plus space” $M_{1/2}^!(\Gamma(0))$ consisting of elements $\sum a(n)q^n$ of $M_{1/2}^!(\Gamma_0(4))$ such that $a(n)$ is nonzero only for $n \equiv 0$ or 1 (mod 4) and a finite number of $n < 0$. While we will not be using the modularity of these forms directly, we will be using their coefficients as a way to study Hilbert class polynomials. There is a unique basis of $M_{1/2}^!(\Gamma(4))$ consisting of the forms $f_d(z)$ whose coefficients are supported at 0, modulo 4. For every nonnegative integer $d$ congruent to 0 or 3 modulo 4, $f_d(z)$ has a Fourier expansion of the form $q^{-d} + \sum_{n=1}^\infty A(n, d)q^n$. The Fourier expansions of the first few $f_d$ are

- $f_0(z) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + 2q^{36} + 2q^{49} + 2q^{64} + \cdots$,
- $f_3(z) = q^{-3} - 248q + 26572q^4 - 85995q^5 + 1707264q^8 - 4096248q^9 + \cdots$,
- $f_4(z) = q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + 51180012q^9 + \cdots$,
- $f_7(z) = q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + 5734772736q^8 + \cdots$.

Borcherds’ infinite product expansion [2] then takes the specific form

$$H_d(j(z)) = q^{-h(d)} \prod_{n=1}^\infty (1 - q^n)^{A(n^2, d)}.$$

Just as Borcherds found a product expansion for $H_d(x)$, Zagier [12] extended this to the twisted functions $H_{D,d}(x)$ defined in (1.2).

**Theorem 2.1 (Zagier [12]).** Let $D > 1$ and $-d$ be a positive and negative discriminant, respectively, which are fundamental and relatively prime. Then we have a product expansion

$$H_{D,d}(j(z)) = \prod_{n=1}^\infty P_D(q^n)^{A(n^2, D, d)}$$

where, if $\zeta_D = e^{2\pi i/D}$,

$$P_D(t) = \prod_{k=1}^{D-1} (1 - \zeta_D^k t)^{\frac{\varphi(t)}{D}}.$$

**Remark.** When $D = 1$, we often have that $H_{D,d}(x) = H_d(x)$. Also, note that one can recover Borcherds’ result from Theorem 2.1 with only minor adjustments. Lastly, note that the value of $A_{D,d}(n)$ in the introduction is equivalent to $A(n^2 D, d)$.

3. Modular forms modulo $\ell$

To compute formulas for the exponents $A(n^2 D, d)$, we will take the logarithmic derivative of the twisted function $H_{D,d}(x)$ and prove that its reduction modulo $\ell$ is a modular form.
of weight $\ell + 1$. In order to do this, we must recall several preliminary facts about modular forms and their reductions modulo $\ell$.

We first recall that the Eisenstein series $E_{2k}(z)$ has $q$-series expansion

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where $B_m$ is the $m^{th}$ Bernoulli number and $\sigma_m(n) = \sum_{d|n} d^m$. For $k > 1$, $E_{2k}(z)$ is a modular form of weight $2k$. We will make use of the following fact about Eisenstein series which follows trivially from congruence properties of the Bernoulli numbers.

**Lemma 3.1** (See Lemma 1.22 in [S]). Let $\ell$ be an odd prime. Then $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$ and $E_{\ell+1}(z) \equiv E_2(z) \pmod{\ell}$.

We can use the supersingular polynomial modulo $\ell$ to obtain congruences for $\mathcal{H}_d(x)$, since for most discriminants $d$ the two polynomials share many roots. An elliptic curve $E$ over $\mathbb{F}_\ell$ is called *supersingular* if the group $E(\mathbb{F}_\ell)$ has no $p$-torsion. The supersingular polynomial $s_\ell(x)$ is given by

$$s_\ell(x) = \prod_{E \text{ supersingular}} (x - j(E))$$

where $j(E)$ is the $j$-invariant of the curve $E$.

Using this, we can give conditions for when we can write the logarithmic derivative of $\mathcal{H}_{D,d}(j(z))$ as an element of $\widetilde{M}_{\ell+1}$.

**Theorem 3.2.** Let $-d$ and $D$ be fundamental discriminants such that $-Dd$ is fundamental, and let $\ell$ be prime such that $\mathcal{H}_{D,d}(x)|s_\ell(x)$ in $\mathbb{F}_\ell[x]$. Then we have

$$-\left(\frac{\mathcal{H}_{D,d}(j(z))}{\mathcal{H}_{D,d}(j(z))}\right)' \in \widetilde{M}_{\ell+1}.$$

**Proof.** It is easy to see that the denominator of $-\left(\frac{\mathcal{H}_{D,d}(j(z))}{\mathcal{H}_{D,d}(j(z))}\right)'$ is exactly $\mathcal{H}_{D,d}(j(z))$. We want to construct a modular form $M(z)$ of weight $\ell - 1$ that is congruent to 1 modulo $\ell$ such that every pole introduced by the denominator of $\mathcal{H}_{D,d}(j(z))$ is canceled by a zero of $M(z)$.

We write $\ell - 1$ uniquely as $12m + 4\delta + 6\epsilon$ where $m \in \mathbb{Z}_+$, $\delta \in \{0, 1, 2\}$, and $\epsilon \in \{0, 1\}$. Write

$$E_{\ell-1}(z) = \Delta^m(z) E_4^\delta(z) E_6^\epsilon(z) \tilde{E}_{\ell-1}(j(z))$$

where $\tilde{E}_{\ell-1}(j(z))$ is a polynomial in $j(z)$. From Theorem 1 of [G], we have that

$$s_\ell(j(z)) \equiv \pm j(z)^\delta (j(z) - 1728)^\epsilon \tilde{E}_{\ell-1}(j(z)) \pmod{\ell}.$$

Since we have assumed that $\mathcal{H}_{D,d}(x)|s_\ell(x)$ in $\mathbb{F}_\ell[x]$, we can construct $P(x) \in \mathbb{Q}[x]$ such that

$$\mathcal{H}_{D,d}(x) P(x) \equiv \pm s_\ell(x) \pmod{\ell},$$

where the sign matches that of $s_\ell(j(z))$ above.
Define \( M(z) \) to be the weight \( \ell - 1 \) modular form given by
\[
M(z) = \Delta^m(z)E_4^\delta(z)E_6^\varepsilon(z) \frac{\mathcal{H}_{Dd}(j(z))P(j(z))}{j(z)^{\delta}(j(z) - 1728)^\varepsilon}.
\]
Then we can conclude that
\[
M(z) \equiv \Delta^m(z)E_4^\delta(z)E_6^\varepsilon(z)\tilde{E}_{\ell - 1}(j(z)) = E_{\ell - 1}(z) \equiv 1 \pmod{\ell}
\]
by Lemma \[3.1\].

Since \( E_4(z) \) vanishes when \( j(z) = 0 \) and \( E_6(z) \) vanishes when \( j(z) = 1728 \), we see that
\[
M(z) = \Delta^m(z)E_4^\delta(z)E_6^\varepsilon(z)\frac{\mathcal{H}_{Dd}(j(z))P(j(z))}{j(z)^{\delta}(j(z) - 1728)^\varepsilon}
\]
vanes at all the roots of \( \mathcal{H}_{Dd}(j(z)) \) to at least the same order. Also, \( M(z) \) is holomorphic everywhere on the upper half-plane, including at infinity, since the factor \( \mathcal{H}_{Dd}(j(z))P(j(z)) \) has degree in \( j(z) \) equal to the degree of \( s_\ell \), which is \( m \). Thus
\[
-\frac{\left(\mathcal{H}_{D,d}(j(z))\right)'}{\mathcal{H}_{D,d}(j(z))}M(z)
\]
is a holomorphic modular form of weight \( \ell + 1 \) that is congruent to \(-\frac{\left(\mathcal{H}_{D,d}(j(z))\right)'}{\mathcal{H}_{D,d}(j(z))} \pmod{\ell}\), so we are done.

The necessary condition of \( \mathcal{H}_{D,d}(x)|s_\ell(x) \) in \( \mathbb{F}_\ell[x] \) will occur for infinitely many pairs \((\ell, d)\). In fact, we can prove that the following conditions also imply the same result.

**Corollary 3.3.** Let \(-d \) and \( D \) be fundamental discriminants such that \(-Dd \) is fundamental and let \( \ell > Dd \) be a prime that is inert in \( \mathbb{Q}(\sqrt{-Dd}) \) and \( \left( \frac{\ell}{Dd} \right) = 1 \). Then we have
\[
-\frac{\left(\mathcal{H}_{D,d}(j(z))\right)'}{\mathcal{H}_{D,d}(j(z))} \in \tilde{M}_{\ell+1}.
\]

**Proof.** From Theorem 7.25 in \[8\], we have that
\[
\mathcal{H}_{Dd}(x)|s_\ell(x)^{h(-Dd)}
\]
in \( \mathbb{F}_\ell[x] \), so every root of \( \mathcal{H}_{Dd}(x) \) is also a root of \( s_\ell(x) \). Dorman proves in Corollary 5.6 in \[4\] (see also \[3\]) that the only primes \( \ell \) for which \( \mathcal{H}_{Dd}(x) \) can have a repeated root modulo \( \ell \) are those where \( \left( \frac{1}{Dd} \right) \neq 1 \), which is not the case by assumption. Thus every root of \( \mathcal{H}_{Dd}(x) \) has multiplicity 1, implying
\[
\mathcal{H}_{Dd}(x)|s_\ell(x),
\]
and by Theorem \[3.2\] the result follows. \(\square\)

We wish to massage this result into a form similar to what we had mentioned in the introduction; namely, that we can write the logarithmic derivative of \( \mathcal{H}_d(j(z)) \) as a multiple of \( E_2(z) \) plus several cusp forms that form a basis of \( S_{\ell+1} \). There is a correspondence between elements of \( S_{\ell+1} \) and elements of \( S_2(\Gamma_0(\ell)) \) which we illustrate here.

**Lemma 3.4.** An element \( M(z) \) of \( \tilde{M}_{\ell+1} \) can be written uniquely modulo \( \ell \) as the sum of a constant multiple of \( E_2(z) \) and an element of \( S_2(\Gamma_0(\ell)) \).
Proof. Let $c$ be the constant coefficient of $M(z)$. Then $c$ is the unique element of $\mathbb{F}_\ell$ so that $M(z) - cE_{\ell+1}(z) \in \tilde{S}_{\ell+1}$. We have from Lemma 3.1 that $E_{\ell+1}(z) \equiv E_2(z) \pmod{\ell}$, and by Theorem 11 of [10] that $M(z) - cE_{\ell+1}(z)$ corresponds modulo $\ell$ to an element of $S_2(\Gamma_0(\ell))$. \hfill \Box

We are now prepared to prove the types of congruences that we see in Theorem 1.1.

4. Proofs of Congruences

We have shown that the logarithmic derivative of $H_{D,d}(j(z))$ modulo $\ell$, which we can compute using Theorem 2.1, is an element of $\tilde{M}_{\ell+1}$, the reduction modulo $\ell$ of weight $\ell+1$ holomorphic modular forms with integer coefficients. In order to prove Theorem 1.1, we just need to solve for the exponents of $H_{D,d}(j(z))$.

Lemma 4.1. Let $\ell$ be prime, $D > 0$ and $-d < 0$ be fundamental discriminants, and $g(n)$ be defined by

$$-rac{(H_{D,d}(j(z)))'}{H_{D,d}(j(z))} \equiv \sum_{n=0}^{\infty} g(n)q^n \pmod{\ell}.$$ 

Then for all $n$ not divisible by $\ell$,

$$A(n^2D,d) \equiv \frac{1}{n} \sum_{m|n} \nu(m)g\left(\frac{n}{m}\right) \pmod{\ell}$$

where $\nu(m)$ is the Dirichlet inverse of $\sum_{k=1}^{D-1} \left(\frac{D}{k}\right) \zeta_D^{km}$.

Proof. From Theorem 2.1 we can compute

$$-rac{(H_{D,d}(j(z)))'}{H_{D,d}(j(z))} = -q \sum_{n=1}^{\infty} A(n^2D,d) \frac{d}{dq} \log P_D(q^n)$$

$$= -q \sum_{n=1}^{\infty} A(n^2D,d) \sum_{k=1}^{D-1} \left(\frac{D}{k}\right) \frac{-\zeta_D^{knq^{n-1}}}{1 - \zeta_D^{kq^n}}$$

$$= \sum_{n=1}^{\infty} nA(n^2D,d) \sum_{k=1}^{D-1} \left(\frac{D}{k}\right) (\zeta_D^k q^n + \zeta_D^{2k} q^{2n} + \zeta_D^{3k} q^{3n} + \cdots)$$

$$= \sum_{n=1}^{\infty} \sum_{m|n} mA(m^2D,d) \sum_{k=1}^{D-1} \left(\frac{D}{k}\right) \zeta_D^{kn/m} q^n.$$

Write $f_1(m) = mA(m^2D,d)$ and $f_2(m) = \sum_{k=1}^{D-1} \left(\frac{D}{k}\right) \zeta_D^{km}$. Then

$$f_1 * f_2 = \sum_{m|n} mA(m^2D,d) \sum_{k=1}^{D-1} \left(\frac{D}{k}\right) \zeta_D^{kn/m},$$
where $\ast$ is Dirichlet convolution. From the theory of Gauss sums (see [5]) we know that $f_2(1) \neq 0$, and so $f_2$ has a Dirichlet inverse $\nu$. Hence, since $\ell \nmid n$, given a congruence of the form $-\left( \frac{H_{D,d}(j(z))'}{H_{D,d}(j(z))} \right) \equiv \sum_{n=0}^{\infty} q(n)g^n \pmod{\ell}$ we can convolve $\nu$ with both sides to compute

$$\sum_{m|n} mA(m^2D, d) \sum_{k=1}^{D-1} \left( \frac{D}{k} \right) \zeta_{D}^{kn/m} \equiv g(n) \pmod{\ell},$$

$$nA(n^2D, d) \equiv \sum_{m|n} \nu(m)g\left( \frac{n}{m} \right) \pmod{\ell},$$

$$A(n^2D, d) \equiv \frac{1}{n} \sum_{m|n} \nu(m)g\left( \frac{n}{m} \right) \pmod{\ell}.$$

This allows us to prove, among other things, the trivial congruence s of $A(n^2, d)$ modulo 5, 7, and 13 mentioned in the introduction.

Example. If $\ell \in \{5, 7, 13\}$ and $-d$ is a negative fundamental discriminant satisfying the conditions of Theorem 3.2 for $D = 1$, then we have

$$A(n^2, d) \equiv -24h(d) \pmod{\ell}.$$

Proof. The assumed conditions let us use Theorem 3.2 to write $-\frac{(H_{D,d}(j(z)))'}{H_{D,d}(j(z))}$ as $h(d)E_{\ell+1}$ plus some element of $S_{\ell+1}$. However, for the primes 5, 7, and 13, the dimension of $S_{\ell+1}$ is 0. Thus using Lemma 3.1, we obtain

$$-\frac{(H_{D,d}(j(z)))'}{H_{D,d}(j(z))} \equiv h(d)E_{\ell+1} \pmod{\ell}$$

$$\equiv h(d)E_2 \pmod{\ell}$$

$$\equiv h(d) - \sum_{n=1}^{\infty} 24h(d)\sigma_1(n)q^n \pmod{\ell}.$$ 

Thus by Lemma 4.1 with $D = 1$ and the fact that $n = \sum_{m|n} \mu(m)\sigma_1\left( \frac{n}{m} \right)$, we can conclude the desired identity. 

Lemma 4.1 allows us to complete the proof of Theorem 3.1.

Proof of Theorem 1.1. Let $\nu(m)$ be the Dirichlet inverse of $\sum_{k=1}^{D-1} \left( \frac{D}{k} \right) \zeta_{D}^{kn} \pmod{\ell}$ as before. Since $\ell \nmid n$, Lemma 4.1 implies that

$$A(n^2, d) \equiv \frac{1}{n} \sum_{m|n} \nu\left( \frac{n}{m} \right) g(m) \pmod{\ell}.$$ 

To find $g(m)$, we use the assumption that $H_{D,d}(x)|s_\ell(x)$ (implied by the assumed conditions by the argument in Corollary 3.3) in $F_\ell[x]$ to invoke Theorem 3.2 implying that we can...
write $\sum_{n=0}^{\infty} g(n)q^n$ as an element of $\tilde{M}_{\ell+1}$. By Lemma 3.4, this element can be represented as $c_0E_{\ell+1}$ plus an element of $\tilde{S}_{\ell+1}$. For some choice of size $r$ basis of cusp forms for $S_{\ell+1}$ with expansions $\sum_{n=1}^{\infty} a_i(n)q^n$, we can use $E_{\ell+1} \equiv E_2 \pmod{\ell}$ by Lemma 3.1 to imply that

$$g(n) \equiv -24c_0\sigma_1(n) + c_1a_1(n) + c_2a_2(n) + \cdots + c_ra_r(n) \pmod{\ell}.$$ 

The desired result follows.

5. Applications of the Chebotarev Density Theorem

From [8], we have the following corollary of a theorem of Deligne regarding Galois representations associated to certain modular forms.

**Theorem 5.1.** Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\text{new}}(\Gamma_0(N), \chi)$ be a newform, and let $K_f$ be the number field obtained by adjoining the Fourier coefficients $a(n)$ and the values of $\chi$ to $\mathbb{Q}$. Let $\ell$ be any prime, $K$ any finite extension of $\mathbb{Q}$ containing $K_f$, and $p_{\ell,K}$ a prime ideal of $O_K$ dividing $\ell$. Then there is a continuous semisimple representation $\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(F_{\ell,K})$ for which the following are true:

1. We have that $\rho_{f,\ell}$ is unramified at all primes $p \nmid N\ell$.
2. For every prime $p \nmid N\ell$ we have
   $$\text{Tr}(\rho_{f,\ell}(\text{Frob}_p)) \equiv a(p) \pmod{p_{\ell,K}}.$$
3. For every prime $p \nmid N\ell$ we have
   $$\det(\rho_{f,\ell}(\text{Frob}_p)) \equiv \chi(p)p^{k-1} \pmod{p_{\ell,K}}.$$
4. For any complex conjugation $c$, we have
   $$\det \rho_{f,\ell}(c) = -1.$$

Here $\text{Frob}_p$ denotes any Frobenius element for the prime $p$.

We will only be concerned with the case of modular forms with rational integer Fourier coefficients and $\chi$ trivial. For $\ell = 11, 17, 19$ and $-d < 0$ satisfying the hypothesis of Corollary 3.3 in the case that the Galois representation for the cusp form furnished by Lemma 3.4 surjects onto $\text{GL}_2(F_{\ell})$, the Chebotarev Density Theorem tells us that every pair $(p, a(p))$ in $F_{\ell}^\times \times F_{\ell}$ occurs. Further, it yields the precise densities of each ordered pair $(p, a(p))$ in the space of possible ordered pairs. Our goal in the first subsection is to compute the relative sizes of unions of conjugacy class in $\text{GL}_2(F_{\ell})$ corresponding to each $(p, a(p))$ modulo $\ell$. Applying the Chebotarev Density Theorem yields our desired result. We then apply the result to the example where $d = 4$ and $\ell = 11$. 
5.1. **Characteristic polynomial frequencies in** \( \operatorname{GL}_2(\mathbb{F}_\ell) \). It will be useful to have a formula for the proportion of elements in \( \operatorname{GL}_2(\mathbb{F}_\ell) \) with a given trace and determinant.

**Lemma 5.2.** Of the \((\ell^2 - 1)(\ell^2 - \ell)\) elements of \( \operatorname{GL}_2(\mathbb{F}_\ell) \), the proportion with characteristic polynomial \( x^2 - ax + b \) (that is, trace \( a \) and determinant \( b \)), for \( a \in \mathbb{F}_\ell \) and \( b \in \mathbb{F}_\ell^\times \), are as follows:

\[
\begin{cases} 
\frac{1}{(\ell - 1)(\ell + 1)} & \text{if } \left( \frac{-b + a^2/4}{\ell} \right) = -1 \\
\frac{1}{(\ell - 1)^2} & \text{if } \left( \frac{-b + a^2/4}{\ell} \right) = 1 \\
\frac{\ell}{(\ell - 1)(\ell + 1)} & \text{if } \left( \frac{-b + a^2/4}{\ell} \right) = 0.
\end{cases}
\]

**Remark.** Note that for \( a = 0 \), the first two cases each occur \((\ell - 1)/2\) times; and for a fixed \( a \neq 0 \), the first, second, and third case respectively occur \((\ell - 1)/2\) times, \((\ell - 3)/2\) times, and once.

**Proof.** Let \( g \in \operatorname{GL}_2(\mathbb{F}_\ell) \) have trace \( a \), so that \( g \) is of the form \( \begin{pmatrix} a + c & m \\ n & -c \end{pmatrix} \). Then \(-c(a + c) - mn \equiv b \mod \ell\), or \( mn \equiv -b - c(a + c) \equiv -b + a^2/4 - (c + a/2)^2 \mod \ell\). If \(-b + a^2/4\) is not a quadratic residue modulo \( \ell \), then for each of the \( \ell \) choices of \( c \) we have \( \ell - 1 \) choices of \( m \), after which \( n \) is fixed, giving a count of \( \ell(\ell - 1) \) ordered triples \((c, m, n)\). If \(-b + a^2/4\) is a nonzero quadratic residue modulo \( \ell \), then for each of the \( \ell - 2 \) choices of \( c \) so that \(-b + a^2/4 - (c + a/2)^2 \equiv 0 \mod \ell\) we still have \( \ell - 1 \) choices of \( m \) after which \( n \) is fixed, but if \(-b + a^2/4 - (c + a/2)^2 \equiv 0 \mod \ell\) then we have \( 2\ell - 1 \) choices of the ordered pair \((m, n)\). This gives a count of \( (\ell - 2)(\ell - 1) + 2(2\ell - 1) = \ell(\ell + 1) \) triples \((c, m, n)\). Finally, if \(-b + a^2/4 \equiv 0 \mod \ell\), by similar reasoning, we have \((\ell - 1)^2 + 2\ell - 1 = \ell^2 \) triples \((c, m, n)\). \( \square \)

Using Theorem 1.34 in [3], it is straightforward to show that the primes \( \ell \) for which \( \dim_C S_2(\Gamma_0(\ell)) = 1 \) are exactly 11, 17, and 19. The spaces \( S_2(\Gamma_0(11)) \), \( S_2(\Gamma_0(17)) \), and \( S_2(\Gamma_0(19)) \) are generated by the modular form associated with the elliptic curves \( X_0(11) \), \( X_0(17) \), and \( X_0(19) \), respectively. Because the dimensions of these spaces is 1, the associated modular forms are Hecke eigenforms, and hence their Dirichlet series each possess an Euler product by Theorem 6.19 of [11]. Thus, the result of Deligne applies. In Proposition 19 of [13], Serre provides sufficient criteria for when a subgroup \( G \) of \( \operatorname{GL}_2(\mathbb{F}_\ell) \) is all of \( \operatorname{GL}_2(\mathbb{F}_\ell) \). This criteria requires the existence of merely three elements of \( G \) with trace and determinant satisfying certain conditions, along with the hypothesis that \( \det : G \to \mathbb{F}_\ell^\times \) should be surjective. Of course, since \( \det \rho_\ell(\text{Frob}_p) = p \) in this case, the surjectivity of \( \det \) is trivially guaranteed. Using this result, we only need to check the image of \( \text{Frob}_p \) for primes \( p \leq 7 \) in order to show that \( \rho_{11}, \rho_{17}, \) and \( \rho_{19} \) are surjective. Thus, given \( \rho_\ell, \ell \in \{11, 17, 19\} \), and an appropriate \( d \) from Table 2, we can use Lemma 5.2, Theorem 1.14, and the Chebotarev Density Theorem to compute the densities in the limit of each congruence class modulo \( \ell \) for \( A(p^2, d) \) where \( p \) is prime.
Theorem 5.3 (The Chebotarev Density Theorem). Let $L/K$ be Galois and let $C \subset \text{Gal}(L/K)$ be a conjugacy class. Then

$$\{p : p \text{ a prime of } K, p \nmid \Delta_{L/K}, \text{Frob}_p \in C\}$$

has arithmetic density $\frac{|C|}{|G|}$.

We apply the Chebotarev Density Theorem with $K = \mathbb{Q}$ and $L$ the fixed field of the kernel of $\rho_\ell$, so that $\text{Gal}(L/K) \cong GL_2(\mathbb{F}_\ell)$. Because the cycle type of $\text{Frob}_p$ corresponds exactly to the splitting type of $p$ when $p$ lies above $\ell$, the Chebotarev Density Theorem gives us that the $p \neq \ell$ for which $a(p) \equiv a \pmod{\ell}$ and $p \equiv b \pmod{\ell}$ has density the relative size of the union of those conjugacy classes in $GL_2(\mathbb{F}_\ell)$ with characteristic polynomial $X^2 - aX + b$.

5.2. Example: Let $d = 4, \ell = 11$. To compute the congruence relation for $A(n^2, 4)$, we consider the logarithmic derivative of $H_4(j(z))$,

$$-(H_4(j(z)))' = -\frac{d}{dq} \log \left( q^{\frac{d}{2}} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2, 4)} \right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \sum_{m \mid n} mA(m^2, 4)q^n.$$

By Theorem 3.2 and Lemma 3.1, we know we can write this as a multiple of $E_{12}(z) \equiv E_2(z)$ and a cusp form modulo 11 of weight 12. Since $S_{12}$ is spanned by $\Delta(z)$, we can compare the coefficients of the constant, $q$, and $q^2$ terms on each side to obtain

$$-(H_4(j(z)))' \equiv 6E_2(z) + 9\Delta(z) \pmod{11}.$$

Thus by Möbius inversion, we have a formula of the form of Theorem 1.1:

$$A(n^2, 4) \equiv 10 + \frac{9}{n} \sum_{m \mid n} \mu\left(\frac{n}{m}\right) \tau(m) \pmod{11}.$$

In the special case of $n = p$, this simplifies to

$$A(p^2, 4) \equiv 10 + 9p^9(\tau(p) - 1) \pmod{11}.$$

Because the associated Galois representation $\rho_11$ is surjective, the densities in the limit of each congruence class modulo 11 for $A(p^2, 4)$ can be calculated from Lemma 5.2. They appear in the last row of Table 1. This implies that for any $c \in \mathbb{F}_{11}$ there exist infinitely many primes $p$ for which $A(p^2, 4) \equiv c \pmod{11}$.
5.3. Example: Let \( d = 20, \ell = 31 \). Here the dimension of \( S_{32} \) is 2, so this example is more complicated. To compute the congruence relation for \( A(n^2, 20) \), we consider the logarithmic derivative of \( \mathcal{H}_{20}(j(z)) \),

\[
-\frac{(\mathcal{H}_{20}(j(z)))'}{\mathcal{H}_{20}(j(z))} = -d \frac{d}{dq} \log \left( q^{-2} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2, 20)} \right) = 2 + \sum_{n=1}^{\infty} \sum_{m|n} mA(m^2, 20) q^n.
\]

Since it is straightforward to verify that \( \mathcal{H}_{20}(x)|_{S_{31}}(x) \) in \( \mathbb{F}_{31} \), by Theorem 3.2 and Lemma 3.1, we know we can write this as a multiple of \( E_{32}(z) \equiv E_2(z) \) and a cusp form modulo 31 of weight 32.

We choose the basis \( \Delta^2(z)E_4^2(z), \Delta(z)E_4^2(z)E_6^2(z) \) for \( S_{32} \), and so by comparing the coefficients of the constant, \( q \), and \( q^2 \) terms on each side, we obtain

\[
(5.2) \quad -\frac{(\mathcal{H}_{20}(j(z)))'}{\mathcal{H}_{20}(j(z))} \equiv 2E_2(z) + 14\Delta^2(z)E_4^2(z) + 23\Delta(z)E_4^2(z)E_6^2(z) \pmod{31}.
\]

We now rewrite this in terms of simultaneous normalized Hecke eigenforms:

\[
F_1(z) = \Delta(z)E_4^2(z)E_6^2(z) + \frac{1711 + \sqrt{18295489}}{184415616} \Delta^2(z)E_4^2(z) \quad \text{and}
\]

\[
F_2(z) = \Delta(z)E_4^2(z)E_6^2(z) + \frac{1711 - \sqrt{18295489}}{184415616} \Delta^2(z)E_4^2(z),
\]

which are Galois conjugates. Since they are already normalized, Theorem 5.1 applies. Now we conveniently chose the modulus 31 so that these eigenforms are defined over \( \mathbb{F}_{31} \). They are

\[
F_1(z) = \Delta(z)E_4^2(z)E_6^2(z) + 22\Delta^2(z)E_4^2(z) \quad \text{and}
\]

\[
F_2(z) = \Delta(z)E_4^2(z)E_6^2(z) + 19\Delta^2(z)E_4^2(z),
\]

in terms of which Equation 5.2 may be rewritten

\[
-\frac{(\mathcal{H}_{20}(j(z)))'}{\mathcal{H}_{20}(j(z))} \equiv 2E_2(z) + 14F_1(z) + 9F_2(z) \pmod{31}.
\]

Letting \( F_1(z) = \sum a_1(n)q^n \) and \( F_2(z) = \sum a_2(n)q^n \), we apply Möbius inversion to obtain

\[
A(n^2, 20) \equiv 14 + \frac{1}{n} \sum_{m|n} \mu \left( \frac{n}{m} \right) (14a_1(m) + 9a_2(m)) \pmod{31}
\]

as in Theorem 1.1.

If \( \rho_1 \) and \( \rho_2 \) are the Galois representations associated with \( F_1 \) and \( F_2 \) respectively, it is easily checked that

\[
\left\{ \begin{pmatrix} \rho_1(Frob_p) & 0 \\ 0 & \rho_2(Frob_p) \end{pmatrix} : p \neq 31 \right\} = \left\{ \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} : M, N \in GL_2(\mathbb{F}_p), \det N = \det M \right\}
\]
using the results of Serre \cite{Serre11} and Ribet \cite{Ribet9}.  

By a computation similar to the one in the $d = 4$, $\ell = 11$ case, we get

$$
\delta_{20}(t, 31; \infty) =
\begin{cases}
991 & \text{if } t = 0 \\
29760 & \text{if } t = 1, 2, 9, 14, 21, 29 \\
1199 & \text{if } t = 3, 4, 5, 11, 16, 19, 20, 23, 28 \\
37200 & \text{if } t = 4, 5, 10, 25, 30 \\
29 & \text{if } t = 6, 7, 10, 18, 25, 30 \\
900 & \text{if } t = 8 \\
991 & \text{if } t = 9 \\
22320 & \text{if } t = 10, 12, 13, 15, 17 \\
14399 & \text{if } t = 11, 12, 22, 24, 26, 27. \\
\end{cases}
$$

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