On a weighted sum over multiplicative functions and its applications to the GPY sieve

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Abstract

In this paper, we investigate the asymptotics of a class of weighted sums over multiplicative functions and apply our results to deduce a stronger asymptotic form of Yitang Zhang’s smoothened GPY sieve with coefficient expressions that are friendly to numerical computations.

I. Introduction

In number theory, asymptotic behaviors of sums in the form of

\[ M_g(x, m, q) = \sum_{n \leq x \atop (n, q) = 1} g(n) \left( \frac{\log x}{n} \right)^m \]  

are often studied in order to investigate problems concerning primes. During the 19th and early 20th century, mathematicians such as Dirichlet [3], F. Mertens [10], and E. Landau [9] investigated the behavior of \( M_g(x, 0, q) \) when \( g(n) \) is chosen to be \( \tau_2(n)/n \), \( \mu^2(n)/n \), or \( \mu^2(n)/\phi(n) \), resulting in asymptotic formulas that are valid when \( x \to \infty \) and \( q \) is fixed. In the 1950s, Harold N. Shapiro & Jack Warga [12] and Yuan Wang [14] developed asymptotic estimates for sums of the type \( M_g(x, 0, q) \) that uniformly hold for all \( q \) where \( g(n) \) is chosen to be some multiplicative function supported on squarefree integers:

\[
\sum_{n \leq x \atop (n, q) = 1} \frac{\mu^2(n)}{n} = \frac{\phi(q)}{q} \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) \log x + O(\log \log q),
\]

\[
\sum_{n \leq x \atop (n, q) = 1} \frac{\mu^2(n)^2 \omega(n)}{n} = \frac{1}{2} \prod_{p|q} \left( \frac{1 - 1/p}{p} \right)^2 \left( 1 + \frac{2}{p} \right) \log^2 x + O(\log x \log \log xq) + O\{(\log \log q)^2\},
\]

\[
\sum_{n \leq x \atop (n, q) = 1} \frac{\mu^2(n)}{\phi(n)} = \frac{\phi(q)}{q} \log x + O(\log \log q).
\]

Although the asymptotic expansions of these formulas exhibit various similarities, the derivation of each of them relies heavily on the Dirichlet convolution properties of specific arithmetical functions. The first general study in this direction was done by Ankeny and Onishi [1] in 1964. Specifically, they applied a Tauberian argument* to prove that the

*Proofs without use of Tauberian theorems are found in Chapter 6 of [8] and Appendix A of [5].
general asymptotic relation

\[ M_g(x, 0, q) = \mathcal{G}(q) \frac{\log x}{k!} + O_q\{\log x^{k-1}\} \]  

holds for fixed \( q \) when \( g(p) \) is well approximated by \( k/p \) for some \( k \geq 1 \) and \( \mathcal{G}(q) \) denotes the convergent product

\[ \mathcal{G}(q) = \prod_{p \mid q} (1 + g(p)) \left( 1 - \frac{1}{p} \right)^k \prod_{p \nmid q} \left( 1 - \frac{1}{p} \right)^k. \]  

Applying partial summation to (2), we see that when \( m \geq 0 \) and \( q \) is fixed there is

\[ M_g(x, m, q) = \mathcal{G}(q) \frac{m!}{(k + m)!} (\log x)^{k+m} + O_q\{\log x^{k+m-1}\}. \]  

In this article, we propose and prove Theorem 1, which extends the validity of (4) to the range \( m \geq \max(0, -k) \) and offers an effective error term valid for all \( x \) and \( q \). In addition, we introduce and prove Theorem 2, a smoothend version of Theorem 1. After that, we apply these theorems to deduce a stronger form of Yitang Zhang’s GPY sieve, which will be discussed in detail in section VII.

II. Statement of results

**Theorem 1.** Let \( q \in \mathbb{Z}^+ \), \( k \in \mathbb{Z} \), \( \theta > 0 \), and \( g(n) \) be a multiplicative function supported on squarefree numbers such that

\[ g(p) = \frac{k}{p} + O\left( \frac{1}{p^{1+\theta}} \right) \quad \forall p \nmid q, p \rightarrow +\infty. \]  

Then for all integer \( m \geq \max(0, -k) \), there is

\[ M_g(x, m, q) = \mathcal{G}(q) \frac{m!}{(k + m)!} (\log x)^{k+m} + O\{\log x^{k+m-1}\log \log q \}, \]  

where \( M = M(k, m) \geq 0 \) is effectively computable and the implied constant in the \( O \)-term depends at most on \( k \) and \( m \).

**Theorem 2.** Let \( M_g(x, m, q, z) \) be defined by

\[ M_g(x, m, q, z) = \sum_{\substack{n \leq x \\ (n, q) = 1, \, p|n \Rightarrow p < z}} g(n) \left( \frac{\log x}{n} \right)^m. \]  

If \( u \) lies in a fixed interval, then under the assumption of Theorem 1, we have

\[ M_g(x, m, q, z) = f(u; k, m) \mathcal{G}(q) \frac{m!}{(k + m)!} (\log x)^{k+m} + O\{(\log x)^{k+m-1}\log q \}, \]  

where \( f(u; k, m) \) is the solution to the differential-difference equation

\[ u^{k+m+1} f'(u) = -k(u - 1)^{k+m} f(u - 1) \]  

subjected to the initial condition that \( f(u; k, m) = 1 \) for \( 0 < u \leq 1 \).
Since the $m = 0$ case has already been proven by Ankeny and Onishi [1], $m > 0$ is assumed throughout the rest of this paper. It is possible to extend the validity of these results to situations when $k$ and $m$ are not integers using Hankel contour and Gamma function, but the current version is sufficient for our purpose.

### III. Lemmas

**Lemma 1.** For any $c > 0$ and $m \in \mathbb{Z}^+$, we have

$$
\frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s^{m+1}} ds = \begin{cases} t^m & t \geq 0 \\ 0 & t < 0 \end{cases}.
$$

**Proof.** Move the path of integration leftward when $t < 0$ and rightward otherwise. Q.E.D.

**Lemma 2.** Let $q$ be any positive integer and $0 \leq \delta < 1$. Then we have

$$
\sum_{p|q} \frac{\log p}{p^{1-\delta}} \ll (\log q)^\delta \log \log q,
$$

where the $\ll$ constant is absolute.

**Proof.** Let $1 < u < q$. Then we have

$$
\sum_{p|q} \frac{\log p}{p^{1-\delta}} \leq u^\delta \sum_{p\leq u} \frac{\log p}{p} + \sum_{p|q, p>u} \frac{\log p}{p^{1-\delta}}
\ll u^\delta \log u + u^{\delta-1} \log q.
$$

Setting $u = \log q$ completes the proof. Proof for the second relation is similar except that we use the bound $\sum_{p\leq u} p^{-1} \ll \log \log u$. Q.E.D.

**Lemma 3.** There exists some $c_0 > 0$ such that for all $|v| \geq 4$ and $u \geq 1 - \frac{c_0}{\log |v|}$ there is

$$
| \log \zeta(u + iv) | < \log \log |v| + O(1).
$$

**Proof.** See Theorem 6.7 of [11]. Q.E.D.

**Lemma 4.** Let $S_g(x, q, z)$ be defined as follows

$$
S_g(x, q, z) = \sum_{\substack{n \leq x \atop p|n \Rightarrow p < z \atop (n, q) = 1}} g(n) F \left( \log \frac{x}{n} \right).
$$

Then, for $2 \leq z \leq x$, we have

$$
S_g(x, q, z) = S_g(x, q, x) - \sum_{\substack{z \leq p < x \atop p|q}} g(p) S_g \left( \frac{x}{p}, q, p \right).
$$

**Proof.** This is a direct generalization of Buchstab’s identity in sieve theory. Q.E.D.
Lemma 5. Let \( a_n \) be some sequence of complex numbers such that
\[
\sum_{n \leq x} a_n = \frac{A(\log x)^k}{k!} + O\{(\log x)^k\}.
\]
for some \( A \in \mathbb{C} \) and \( k \in \mathbb{N} \). Then for any \( G \in C^1[0,1] \) we have
\[
\sum_{n \leq x} a_n G\left(\frac{\log x/n}{\log x}\right) = \frac{A(\log x)^k}{(k-1)!} \int_0^1 (1-t)^{k-1} G(t) dt + O\{(\log x)^k\},
\]
where the implied constant depends on \( A, k, \) and \( G \).

Proof. See Lemma 4 of [6]. Q.E.D.

IV. The Dirichlet series associated with \( g(n) \)

Let \( G(s) \) denote the Dirichlet series associated with \( g(n) \):
\[
G(s) = \sum_{n \geq 1} \frac{g(n)}{n^s}.
\]

In this section, we extract analytic properties of \( G(s) \) in order to perform contour integration. From the Euler product formula, we can factor \( G(s) \) into two parts:
\[
G(s) = \prod_{p \nmid q} \left(1 + \frac{g(p)}{p^s}\right) \left(1 - \frac{1}{p^{s+1}}\right)^k \prod_{p \mid q} \left(1 - \frac{1}{p^{s+1}}\right)^k \zeta^k(s+1).
\]

By Lemma 3, we see that \( \zeta^k(s+1) \ll (\log |t|)^{|k|} \) whenever \( \sigma \geq c_0/\log |t|, s = \sigma + it \). Thus, we only need to explore the properties of \( H(s) \) in the rest of this section.

Lemma 6. There exists an absolute constant \( \delta > 0 \) such that for all \( \sigma \geq -\delta \), the infinite product
\[
\prod_{p \nmid q} \left(1 + \frac{g(p)}{p^s}\right) \left(1 - \frac{1}{p^{s+1}}\right)^k
\]
is absolutely convergent and uniformly bounded with respect to \( q \).

Proof. Plugging (28) into the product, we see that
\[
\log \left(1 + \frac{g(p)}{p^s}\right) \left(1 - \frac{1}{p^{s+1}}\right)^k = \frac{g(p) - kp^{-1}}{p^s} + O\left(\frac{1}{p^{2\sigma+2}}\right)
\ll \frac{1}{p^{\sigma+1+\theta}} + \frac{1}{p^{2\sigma+2}}.
\]
This indicates that the lemma’s condition is satisfied when \( 0 < \delta < \min(\theta, \frac{1}{2}) \). Q.E.D.
Lemma 7. There exists an absolute constant $C > 0$ such that for any $\sigma \geq -\delta$, we have

$$|H(s)| \ll (\log \log q)^C |k|^{|\log q|^\delta}.$$ 

Proof. Due to Lemma 6, it suffices to show that

$$\prod_{p|q} \left(1 - \frac{1}{p^{\sigma+1}} \right)^k \ll (\log \log q)^C |k|^{|k|} (\log q)^{\delta}.$$ 

Taking logarithm, we have

$$\left| \log \prod_{p|q} \left(1 - \frac{1}{p^{\sigma+1}} \right)^k \right| \leq |k| \sum_{p|q} \frac{1}{p^{1-\delta}} + O(1),$$

so plugging Lemma 2 into the right hand side completes the proof. Q.E.D.

V. Proof of Theorem 1

Setting $t = \log x/n$ in Lemma 1 and plugging it into (1), we have for $c > 0$ that

$$M_g(x, m, q) = \frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n \geq 1 \atop (n, q) = 1} g(n) \left(\frac{x}{n}\right)^s \frac{ds}{s^{m+1}} = \frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s) \frac{x^s}{s^{m+1}} ds.$$ 

To estimate the integral, we first truncate the path of integration so that it follows from Lemma 7 that for any large $T \leq x$ there is

$$M_g(x, m, q) = \frac{m!}{2\pi i} \int_{c-iT}^{c+iT} G(s) \frac{x^s}{s^{m+1}} ds + O \left\{ \frac{x^c (\log \log q)^C |c^{-|k|}|}{T^m} \right\}. \quad (9)$$

In order to estimate $M_g(x, m, q)$ using residue theorem, we apply Lemma 7 so that when $\delta_0 = c_1 (\log T)^{-1} (\log \log q)^{-1}$ for some $c_1 = \min(c_0, \log 2)$ there is

$$\int_{-\delta_0 - iT}^{-\delta_0 + iT} G(s) \frac{x^s}{s^{m+1}} ds \ll x^{-\delta_0} (\log \log q)^{2C|k|}. \quad (10)$$

$$\left( \int_{c-iT}^{c+iT} + \int_{-\delta_0 + iT}^{-\delta_0 - iT} \right) G(s) \frac{x^s}{s^{m+1}} ds \ll x^c (\log \log q)^{2C|k|} \frac{(\log T)^{|k|}}{T^m}. \quad (11)$$

Now, setting $c = 1/\log x$ and $log T = \sqrt{\log x}$ converts the above bounds of (9), (10), and (11) into

$$M_g(x, m, q) = \frac{m!}{2\pi i} \int_{(0+)} G(s) \frac{x^s}{s^{m+1}} ds + O \{ (\log \log q)^{2C|k|} \}. \quad (12)$$

Therefore, the remaining task is to calculate the residue of $G(s)x^s/s^{m+1}$ at $s = 0$. For convenience, let $I(s)$ be defined by

$$I(s) = s^k G(s) = H(s) [s \zeta(s+1)]^k. \quad (13)$$
Then it follows from (8) that \( I(s) \) is analytic near \( s = 0 \) and from (3) that \( I(0) = H(0) = \mathcal{G}(q) \). These facts allow us to transform the residue integral of (12) into
\[
\frac{1}{2\pi i} \oint \frac{x^s}{s^{k+m+1}} ds = \mathcal{G}(q) \frac{(\log x)^{k+m}}{(k+m)!} \quad \text{(14)}
\]
\[
+ \sum_{1 \leq n \leq k+m} \frac{(\log x)^{k+m-n}}{(k+m-n)!} \frac{1}{2\pi i} \oint \frac{I(s)}{s^{n+1}} ds. \quad \text{(15)}
\]

Evidently, (14) will become the main term in the expansion of \( M_g(x, m, q) \), so the remaining task is to estimate (15). By Lemma 7, (13), and Cauchy’s inequality, we observe that when \( r = \log 2 / \log \log q \) there is
\[
\oint_{|s| = r} I(s) \frac{1}{s^{n+1}} ds \ll (\log \log q)^2 C |k|^n. \quad \text{(16)}
\]

Plugging this back into (15) we obtain
\[
\sum_{1 \leq n \leq k+m} \ll (\log x)^{k+m-n} (\log \log q)^2 C |k| + k + m. \quad \text{(16)}
\]

Finally, combining (16) and (12) completes the proof of Theorem 1 with \( M = 2C|k| + k + m \).

**VI. Proof of Theorem 2**

The case where \( 0 < u \leq 1 \) follows directly from Theorem 1, so it suffices to prove for \( u \geq 1 \).

Assume Theorem 2 is true when \( 0 < u \leq r - 1 \) for some integer \( r \geq 2 \), so when \( r - 1 < u \leq r \) it follows from Lemma 4 that
\[
M_g(x, m, q, x^{1/u}) = M_g(x, m, q) - \sum_{x^{1/u} \leq p < x} g(p) \frac{x}{p^u} \quad \text{(17)}
\]

Because \( \frac{\log x/p}{\log p} \leq u - 1 \leq r - 1 \), we apply the inductive hypothesis to deduce
\[
\Sigma = \mathcal{G}(q) \frac{m!}{(k+m)!} (\log x)^{k+m} \sum_{x^{1/u} \leq p < x} \frac{k}{p} + O \left\{ \mathcal{G}(q)(\log x)^{k+m} \sum_{x^{1/u} \leq p < x} \frac{1}{p^{1+\theta}} \right\} + O \left\{ \mathcal{G}(q)(\log x)^{k+m} \sum_{x^{1/u} \leq p < x} \frac{1}{p} \right\} + O \left\{ (\log \log q)^M (\log x)^{k+m-1} \sum_{x^{1/u} \leq p < x} \frac{1}{p} \right\}.
\]
By Lemma 2 and some standard estimates, we have

\[ \mathcal{S}(q) \ll (\log \log q)^{|k|}, \quad \sum_{x^{1/u} \leq p < x} \frac{1}{p} \ll 1, \quad \sum_{x^{1/u} \leq p < x} \frac{1}{p^{1+\theta}} \ll x^{-\theta/u}, \]

and

\[ \sum_{x^{1/u} \leq p < x} \frac{1}{p} \ll (\log x)^{-1} \sum_{p \mid q} \frac{\log p}{p} \ll (\log x)^{-1} \log \log q. \]

Moreover, by partial summation, we have

\[ \Gamma = k \int_{x^{1/u}}^x \frac{dt}{t \log t} \left( \frac{\log x}{\log t} - 1 \right) \left( 1 - \frac{\log t}{\log x} \right)^{k+m} + O \left( \frac{1}{\log x} \right) \]

\[ = k \int_1^u f(v-1; k, m)(1-v^{-1})^{k+m} \frac{dv}{v} + O \left( \frac{1}{\log x} \right). \]

Plugging all of these results back into (17), we obtain

\[ M_q(x, m, q, x^{1/u}) = \left\{ 1 - k \int_1^u f(v-1; k, m)(v-1)^{k+m} v^{-k-m-1} dv \right\} \]

\[ \times \mathcal{S}(q) \frac{m!}{(k+m)!} (\log x)^{k+m} + O\{ (\log \log q)^M (\log x)^{k+m-1} \}. \]

For \( r - 1 < u \leq r \), if we define \( f(u; k, m) \) by

\[ f(u; k, m) = 1 - k \int_1^u f(v-1; k, m)(v-1)^{k+m} v^{-k-m-1} dv, \]

the differentiating both side with respect to \( u \) yields the differential-difference equation stated in Theorem 2, thus completing the proof.

**VII. Yitang Zhang’s smoothened GPY sieve**

A positive real number \( \theta \) is called the level of distribution of primes in arithmetic progressions if for every \( \varepsilon > 0 \) and every \( A > 0 \),

\[ \sum_{d \leq x^{\theta - \varepsilon}} \frac{\max_{(a,d)=1} \left\{ \pi(x; q,a) - \frac{\pi(x)}{\varphi(d)} \right\}}{\varphi(d) \log^A x} \ll A \frac{x}{\log^A x} \quad (x \geq 2) \quad (18) \]

Under Bombieri–Vinogradov theorem [2], (18) holds unconditionally for \( \theta = \frac{1}{2} \). For the sake of brevity, we use \( \text{EH}(\theta) \) to denote the hypothesis that (18) holds for some given \( \theta \). It is conjectured by Elliott and Halberstam [4] that \( \text{EH}(1) \) holds, and Bombieri–Vinogradov theorem is equivalent to \( \text{EH}(\frac{1}{2}) \).

In 2005, Goldston, Yıldırım, and Pintz [7] proved that if \( \text{EH}(\theta) \) holds for some \( \theta > \frac{1}{2} \), then there will be infinitely many pairs of primes such that their absolute difference is bounded by some constant that only depends on \( \theta \). In other words, there exists some \( C(\theta) \geq 2 \) such that

\[ \lim_{n \to \infty} \inf_{n} (p_{n+1} - p_n) \leq C(\theta) \quad (19) \]
Let $\chi_p$ denote the characteristic function for primes. Then the GPY sieve is expressed as follows:

$$S = \sum_{N < n \leq 2N} \left( \sum_{1 \leq i \leq k} \chi_p(n + h_i) - 1 \right) w_n^2,$$

(20)

where $h_1 < h_2 < \cdots < h_k$ is chosen such that $Q(n) = \prod_{1 \leq i \leq k}(n + h_i)$ is not always divisible by a fixed prime divisor, and $w_n$ is some well chosen sequence of real numbers. If $S > 0$ for some $N$, then there exists some $N < n \leq 2N$ and some $1 \leq r, s \leq k$ such that $n + h_r$ and $n + h_s$ are both primes, so $p_{n+1} - p_n \leq h_k - h_1$ for infinitely many $n$.

In the original work of Goldston, Yıldırım, and Pintz, the shape of $w_n$ is as follows:

$$w_n = \sum_{d \in D_n} \lambda_d, \quad D_n = \{d \leq R : d|Q(n)\}, \quad \lambda_d = \mu(d)P\left(\frac{\log R/d}{\log R}\right),$$

(21)

In their original paper, the authors plugged in $P(x) = x^m$ for $m > k$ and obtained

$$S = [1 + o(1)]C_1(k, m, \theta) \frac{N}{\log^k N}.$$  

In addition, they showed that if EH($\theta$) holds for some $\theta > \frac{1}{2}$, then there exists some $k$ and $m$ such that $C_1(k, m, \theta) > 0$. In 2007, Soundararajan [13] showed that even when the choice of $P(x)$ is loosened to $x^{-k}P(x) \in C[0, 1]$, it is still necessary to assume EH($\theta$) for some $\theta > \frac{1}{2}$ in order that $C_1(k, m, \theta)$ is positive. Therefore, it is sufficient and necessary to assume EH($\theta$) for some $\theta > \frac{1}{2}$ for the prototypical GPY sieve to produce bounded gaps between primes.

In 2014, Yitang Zhang [15] introduced smoothing to the GPY sieve. While still using $P(x) = x^m$ for $\lambda_d$, he restricted the size of prime factors of integers in $D_n$:

$$D_n = \{d \leq R : d|Q(n) \land p|d \Rightarrow d < z\}, \quad z = N^\delta$$

(22)

This choice of $D_n$ corresponds to a smoothened variant of hypothesis EH($\theta$). Let EH($\theta, \delta$) denote the hypothesis that for every $A > 0$,

$$\sum_{d \leq x^\theta} \max_{p|d \Rightarrow p < N^\delta} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(d)} \right| \leq A \frac{x}{\log^4 x} \quad (x \geq 2).$$

(23)

Instead of looking for exact asymptotic, Zhang estimated the difference between smoothened GPY sieve and the prototypical GPY sieve to deduce a lower bound:

$$S \geq C_2(k, m, \theta, \delta) \frac{N}{\log^k N}.$$  

(24)

Zhang proved EH($\theta, \delta$) when $\delta = 1/1168$ and $\theta = 1/2 + 2\delta$ and that under this choice of parameters, $C_2(k, m, \theta, \delta) > 0$ for $k = 3.5 \times 10^6$, $m = k + 180$, and sufficiently large $N$, thereby deducing the existence of infinitely many pairs of primes that are $< 7 \times 10^7$ apart.

Zhang did not derive an asymptotic formula in [15] thus unable to utilize the full capacity of the smoothened GPY sieve. In this section, we apply Theorem 2 to deduce

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†These numbers are called an admissible $k$-tuple. A trivial example is to let $h_1, h_2, \ldots, h_k$ be the first $k$ primes greater than $k$. 

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an asymptotic form of Zhang’s original smoothened GPY sieve so that it would be possible for one to analyze its strengths and limits. Moreover, this would allow one to apply this sieve to number theory questions other than prime gaps. Apart from Theorem 2 itself, the derivation is purely elementary. The asymptotic coefficients obtained using this approach is also friendly to numerical computations as they are defined using iteration formulas.

**Theorem 3.** Let \( S \) be defined as in (20) such that

\[
w_n = \sum_{d \leq R \atop d \mid Q(n)} \lambda_d, \quad \lambda_d = \mu(d)P \left( \frac{\log R/d}{\log R} \right), \quad P(x) = x^m, \quad m > k
\]

Then under \( \text{EH}(\theta, \delta) \), we have for \( z = N^\delta \) and \( R = N^{\theta/2} \) that

\[
S \sim \frac{N}{(\log R)^k} \prod_p \left( 1 - \frac{\nu_d}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \left\{ \frac{k \theta}{2} I_{k-1}(1, u) - I_k(1, u) \right\}, \quad u = \frac{\theta}{2\delta}
\]

where \( \nu_d \) is the number of roots of \( Q(n) \) in \( \mathbb{Z}/d\mathbb{Z} \) and

\[
I_s(t, v) = \int_0^t (1 - x)^{s-1} \left[ f(uxt; -s, m) P(s)(xt) \right]^2 dx, \quad 0 < v \leq 1
\]

\[
I_s(t, v) = I_s(t, 1) - s \int_1^v I_s[(1 - x^{-1})t, x - 1](1 - x^{-1}) x \, dx, \quad v > 1
\]

in which \( P(s)(x) \) denotes the \( s \)'th derivative of \( P(x) \).

We deduce Theorem 3 from the following auxiliary result:

**Theorem 4.** Let \( k \) be some positive integer and \( g(n) \) be some multiplicative function supported on squarefree numbers such that

\[
g(p) = \frac{k}{p} + O \left( \frac{1}{p^2} \right).
\]

Then we have for \( z = R^{1/u} \) that

\[
Q_g = \sum_{d_1, d_2 \leq R \atop p \mid [d_1, d_2]\Rightarrow p < z} \lambda_{d_1} \lambda_{d_2} g([d_1, d_2]) \sim \frac{I_k(1, u)}{(\log R)^k} \prod_p (1 - g(p)) \left( 1 - \frac{1}{p} \right)^{-k}.
\]

**Proof of Theorem 3.** By (20), we see that \( S = \sum_{1 \leq i \leq k} S_i - S_0 \), where

\[
S_0 = \sum_{N < n \leq 2N} w_n^2, \quad S_i = \sum_{N < n \leq 2N} \chi_p(n + h_i) w_n^2.
\]

Thus, it suffices to evaluate \( S_0 \) and \( S_i \) separately. By a change of order of summation, we have

\[
S_0 = \sum_{d_1, d_2 \leq R \atop p \mid [d_1, d_2]\Rightarrow p < z} \lambda_{d_1} \lambda_{d_2} \sum_{N < n \leq 2N \atop Q(n) \equiv 0([d_1, d_2])} 1 = NQ_{g_0} + E_0.
\]
where \( g_0(d) = \nu_d/d \), which is multiplicative due to Chinese remainder theorem and

\[
E_0 \ll \sum_{d \leq R^2 \atop p | d \Rightarrow p < z} \mu^2(d_1) \mu^2(d_2) \nu_{|d_1,d_2|} \leq R^2 \sum_{d \leq R^2 \atop p | d \Rightarrow p < z} \mu^2(d) \tau_3(d) \nu_d/d
\]

\[
= R^2 \prod_{p < z} \left(1 + \frac{3 \nu_p}{p}\right) \leq R^2 \prod_{p < z} \left(1 + \frac{3k}{p}\right) \ll R^2 (\log z)^{3k}.
\]

On the other hand, it follows from the prime number theorem that

\[
S_i = \sum_{d_1,d_2 \leq R \atop p | d_1,d_2 \Rightarrow p < z} \lambda_{d_1} \lambda_{d_2} \sum_{N < n \leq 2N} \chi_{\nu}(n + h_i)
\]

\[
= Q_{g_1} \sum_{N < n \leq 2N} \chi_{\nu}(n + h_i) + E_i
\]

\[
= [1 + o(1)] \frac{N}{\log N} Q_{g_1} + E_i,
\]

where \( g_1 \) is multiplicative such that \( g_1(p) = (\nu_p - 1)/\varphi(p) \) and

\[
E_i \ll \sum_{d \leq R^2 \atop p | d \Rightarrow p < z} \mu^2(d) \tau_{3k-3}(d) \max_{(a,d)=1} \left| \sum_{N < n \leq 2N} \chi_{\nu}(n + h_i) - \frac{1}{\varphi(d)} \sum_{N < n \leq 2N} \chi_{\nu}(n + h_i) \right|,
\]

which is \( \ll_A N/\log^A N \) by an application of Cauchy–Schwarz inequality and \( EH(\theta, \delta) \).

By the properties of polynomials, we know that \( Q(n) \) has exactly \( k \) roots in \( \mathbb{Z}/p\mathbb{Z} \) if and only if \( p \) does not divide its discriminant, a finite number. Therefore, when \( p \to +\infty \) we have

\[
\nu_p = k, \quad g_0(p) = \frac{k}{p}, \quad g_1(p) = \frac{k-1}{p} + O \left( \frac{1}{p^2} \right).
\]

(25)

Finally, plugging (25) into Theorem 4, so we have

\[
S_0 \sim N \cdot \frac{I_k(1,u)}{(\log R)^k} \prod_{p} \left(1 - \frac{\nu_p}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}
\]

(26)

and for \( 1 \leq i \leq k \)

\[
S_i \sim \frac{N}{\log N} \cdot \frac{I_{k-1}(1,u)}{(\log R)^{k-1}} \prod_{p} \left(1 - \frac{\nu_p - 1}{p-1}\right) \left(1 - \frac{1}{p}\right)^{-(k-1)}.
\]

(27)

Notice that

\[
\left(1 - \frac{\nu_p - 1}{p-1}\right) = (p - \nu_p)(p - 1)^{-1} = \left(1 - \frac{\nu_p}{p}\right) \left(1 - \frac{1}{p}\right)^{-1},
\]

so combining (26) and (27) proves the asymptotic formula for \( S \).

Q.E.D.

Although in this paper we only use Theorem 2 to refine Zhang’s sieve, we expect that by modifying the methods applied to prove Theorem 4 to higher rank situation (i.e. Maynard–Tao sieve), thereby potentially improving current bounds for prime gaps.
VIII. Proof of Theorem 4

Let \( h(l) \) be a multiplicative function such that \( h(p) = g(p)/(1 - g(p)) \). Then by Möbius inversion, we have

\[
Q_g = \sum_{d_1, d_2 \leq R \atop p \mid |d_1, d_2| \Rightarrow p < z} \lambda_{d_1} g(d_1) \lambda_{d_2} g(d_2) \frac{1}{g((d_1, d_2))} = \sum_{d_1, d_2 \leq R \atop p \mid |d_1, d_2| \Rightarrow p < z} \lambda_{d_1} g(d_1) \lambda_{d_2} g(d_2) \sum_{l \mid (d_1, d_2)} \frac{1}{h(l)}
\]

\[
= \sum_{l \leq R \atop p \mid \Rightarrow p < z} \frac{\alpha_l^2}{h(l)}, \quad \alpha_l = \sum_{d \leq R \atop d \equiv \text{0}(l) \atop p \mid \Rightarrow p < z} g(d) \lambda_d = \mu(l) g(l) \sum_{t \leq R/l \atop (t, l) = 1 \atop p \mid \Rightarrow p < z} \mu(t) g(t) \left( \frac{\log R/lt}{\log R} \right)^m.
\]

For convenience, let \( A \) denote the following quantity

\[
A = \prod_p (1 - g(p)) \left( 1 - \frac{1}{p} \right)^k.
\]  \hspace{1cm} (28)

Then it follows that when \( q \) is squarefree,

\[
\prod_{p \mid q} (1 + \mu(p) g(p)) \left( 1 - \frac{1}{p} \right)^k \prod_{p \mid q} \left( 1 - \frac{1}{p} \right)^k = A \prod_{p \mid q} (1 - g(p))^{-1} = A \frac{h(q)}{g(q)}.
\]

Therefore, when we apply Theorem 2, we have

\[
\alpha_l = \frac{\mu(l) g(l)}{(\log R)^m} M_{pq} \left( \frac{R}{l}, m, l, z \right)
\]

\[
= \frac{A \mu(l) h(l)}{(\log R)^m} f \left( \frac{\log R/l}{\log z}; -k, m \right) \frac{m!}{(m-k)!} \left( \frac{\log R}{l} \right)^{m-k} + O \left\{ \frac{g(l)(\log R/l)^{m-k-1} \log \log l}{(\log R)^m} \right\}
\]

\[
= \frac{A \mu(l) h(l)}{(\log R)^k} f \left( u \cdot \frac{\log R/l}{\log R}; -k, m \right) P^{(k)} \left( \frac{\log R/l}{\log R} \right) + O \left\{ \frac{g(l)(\log R)^{m-k-1+\varepsilon}}{(\log R)^m} \right\}.
\]

For convenience, let \( G(x) = [f(ux; -k, m) P^{(k)}(x)]^2 \), so we have

\[
\frac{\alpha_l^2}{h(l)} = \frac{h(l) A^2}{(\log R)^{2k}} G \left( \frac{\log R/l}{\log R} \right) + O \left\{ \frac{g(l)}{(\log R)^{2k+1+\varepsilon}} \right\}.
\]  \hspace{1cm} (29)

Using the multiplicative property of \( g \), we have

\[
\sum_{l \leq R \atop p \mid \Rightarrow p < z} g(l) \leq \sum_{p \mid \Rightarrow p < z} g(l) = \prod_{p < z} (1 + g(p)) \leq \exp \left\{ \sum_{p < z} \frac{k}{p} + O \left\{ \sum_p \frac{1}{p^2} \right\} \right\} = \exp \left\{ k \log \log z + O(1) \right\} \ll (\log z)^k \ll (\log R)^k.
\]

Therefore, plugging (29) into \( Q_g \) gives

\[
Q_g = \frac{A^2}{(\log R)^{2k}} \sum_{l \leq R \atop p \mid \Rightarrow l < z} h(l) G \left( \frac{\log R/l}{\log R} \right) + O \left\{ \frac{1}{(\log R)^{k+1+\varepsilon}} \right\}.
\]  \hspace{1cm} (30)
For convenience, we let \( H(x, z) \) denote the following quantity:

\[
H(x, z) = \sum_{l \leq x} h(l)G \left( \frac{\log x/l}{\log R} \right). \tag{31}
\]

Note that \( h(p) = g(p)/(1 - g(p)) = k/p + O(1/p^2) \) and

\[
\prod_p (1 + h(p)) \left( 1 - \frac{1}{p} \right)^k = \prod_p (1 - g(p))^{-1} \left( 1 - \frac{1}{p} \right)^k = \frac{1}{A}.
\]

so it follows from Theorem 1 that

\[
\sum_{l \leq R} h(l) = \frac{A^{-1}}{k!} (\log R)^k + O((\log R)^{k-1}).
\]

Plugging this into Lemma 5, we see that for \( 2 \leq x \leq R \) there is

\[
H(x, x) = \sum_{l \leq x} h(l)G \left( \frac{\log x/l}{\log x} \cdot \frac{\log x}{\log R} \right)
= A^{-1}(\log x)^k \int_0^1 \frac{(1 - t)^{k-1}}{(k - 1)!} \frac{t \log x}{G \left( \frac{\log R}{\log x} \right)} \, dt + O((\log x)^{k-1}).
= A^{-1}(\log x)^k I_k \left( \frac{\log x}{\log R}, 1 \right) + O((\log x)^{k-1})
\]

To generalize this result to \( z \leq x \), we apply Lemma 4 so that

\[
H(x, z) = A^{-1}(\log x)^k I_k \left( \frac{\log x}{\log R}, 1 \right) - \sum_{z \leq p < x} h(p)H \left( \frac{x}{p}, p \right) + O((\log x)^{k-1})
= A^{-1}(\log x)^k I_k \left\{ I_k \left( \frac{\log x}{\log R}, 1 \right) - \sum_{z \leq p < x} \frac{k}{p} I_k \left( \frac{\log x/p}{\log R} - \frac{\log x/p}{\log p} \right) \left( \frac{\log x/p}{\log x} \right) \right\}
+ O \left\{ \sum_{z \leq p < x} \frac{(\log x)^{k-1}}{p} \right\} + O \left\{ \sum_{z \leq p < x} \frac{(\log x)^k}{p^2} \right\} + O((\log x)^{k-1})
= A^{-1}(\log x)^k \left\{ I_k \left( \frac{\log x}{\log R}, 1 \right) - \int_z^x \frac{dt}{t \log t} I_k \left( \frac{\log x/t}{\log R} - \frac{\log x/t}{\log t} \right) \left( \frac{\log x/t}{\log x} \right) \right\}
+ O((\log x)^{k-1})
= A^{-1}(\log x)^k \left\{ I_k \left( \frac{\log x}{\log R}, 1 \right) - \int_1^{\log x/\log z} I_k \left[ (1 - \alpha^{-1}) \frac{\log x}{\log R}, \alpha - 1 \right] \left( 1 - \alpha^{-1} \right)^k \, d\alpha \right\}
+ O((\log x)^{k-1})
= A^{-1}(\log x)^k I_k \left( \frac{\log x}{\log R}, 1 \right) + O((\log x)^{k-1}).
\]

Plugging these results back into (31) and (30), we obtain

\[
Q_g \sim \frac{A^2}{(\log R)^2} \cdot A^{-1}(\log R)^k I_k \left( \frac{\log R}{\log R}, \frac{\log R}{\log z} \right) = A \cdot I_k(1, u) \left( \frac{\log R}{\log R} \right).
\]

Combining this result with (28), we complete the proof.
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