Stochastic differential equations with path-independent solutions.

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Abstract

We present a condition for a stochastic differential equation \( dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dB_t \) to have a unique functional solution of the form \( Z(t, B_t) \). The condition expresses a relation between \( \mu \) and \( \sigma \). A generalization concerns solutions of the form \( Z(t, Y_t) \), where \( Y_t \) is an Ito-process satisfying a stochastic differential equation with coefficients only depending on time, to be determined from \( \mu \) and \( \sigma \). The solutions in question are obtained by solving a system of two partial differential equations, which may be reduced to two ordinary differential equations.

Keywords: Stochastic differential equations, systems of partial differential equations, Ito's Lemma.

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1 Introduction

Ito’s lemma gives conditions for a stochastic differential equation to have a solution in terms of a function \( Z(t, Y_t) \) of time and some Ito-process \( Y_t \): the coefficients of the stochastic differential equation should have a particular expression in terms of the partial derivatives of \( Z \). Ito’s lemma does not provide a direct method to find the function \( Z \).

For a definite class of stochastic differential equations we show that such a function may be obtained in the form of a solution of two subsequent ordinary differential equations. These equations may be reduced to some equations used in the differential method of H. Doss in [5]. This method solves autonomous stochastic differential equations path-by-path, formulating a particular ordinary differential equation along each individual path of the process \( Y \).

Here we give "integration conditions" on the coefficients of a given non-autonomous stochastic differential equation, for it to be solved by a global function \( Z \), through only one pair of ordinary differential equations. The integration conditions permit to determine an auxiliary process \( Y \), and the function \( Z \) will depend only on time \( t \) and the values taken by the process \( Y \) at the time \( t \); as
such it is independent of the paths of the process \( Y \). We consider in particular
the special case where \( Y \) may be taken equal to the Standard Brownian Motion.

The ordinary differential equations in question correspond to a system of
two first-order partial differential equations in three variables, solved along two
particular paths (in time and in space). In fact we show that the above men-
tioned integration conditions correspond to a well-known integration condition
for systems of partial differential equations to have a unique global solution.

This article has the following structure. In Section 2 we present our differen-
tial approach in more detail, and compare it with the differential method by H.
Doss. In Section 3 we recall some existing theory on the resolution of systems of
first-order partial differential equations. In Section 4 we state formal theorems
on the solution of stochastic differential equations with their respective proofs.
We comment on the role and form of the integration conditions, and end with
some examples.

Thorough treatments of stochastic differential equations can be found in for
example [1], [6] and [7]. The latter book also gives a presentation of the result
of [5].

The books [4] and [8] are books of reference for partial differential equations,
and a treatment of the background in differential geometry useful for the solution
of systems of first-order partial differential equations is given in [3].

The integration condition of a global solution of stochastic differential equa-
tions in terms of Brownian Motion has been stated as a sort of limit cas e for
the existence of a global solution of stochastic difference equations in terms of
the discrete Wiener Walk in [2].

2 Overview of the method

We consider stochastic differential equations of the form

\[
\begin{aligned}
    dx_t &= \mu(t, x_t) \, dt + \sigma(t, x_t) \, dB_t & \quad & 0 \leq t < T \\
    x_0 &= x_0,
\end{aligned}
\]

on some appropriate probability space \( \Omega \), where \( \mu \) and \( \sigma \) have some regularity
and \( x_0 \) is a constant.

We show that [1] has a functional solution of the form \( Z(t, B_t) \), where \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \), provided \( \mu \) and \( \sigma \) are related by the partial differential equation

\[
\sigma \frac{\partial \mu}{\partial X} - \mu \frac{\partial \sigma}{\partial X} - \frac{\partial \sigma}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 \sigma}{\partial X^2} = 0.
\]

In fact \( Z \) satisfies the system of two first-order partial differential equations

\[
\begin{aligned}
    \frac{\partial Z(t, x)}{\partial t} &= \mu(t, Z(t, x)) - \frac{1}{2} \sigma(t, Z(t, x)) \frac{\partial \sigma(t, Z(t, x))}{\partial Z} \\
    \frac{\partial Z(t, x)}{\partial x} &= \sigma(t, Z(t, x)) \\
    Z(0, 0) &= x_0,
\end{aligned}
\]

and (2) represents the integration condition for the system (3). Indeed, for a
well-defined two-times differentiable surface to be a solution, one must have
\[ \frac{\partial^2 Z}{\partial t \partial x} = \frac{\partial^2 Z}{\partial x \partial t}, \] which amounts to (2). Under an additional Lipschitz condition
the solution of (3) is also unique, meaning that it does not depend on the choice
of the path of integration. Hence any convenient path will do. In particular
one may integrate firstly horizontally from \((0, 0)\) to \((T, 0)\), and then vertically
from \((T, 0)\) to \((T, x)\), thus resolving subsequently the two ordinary differential
equations
\[
\left\{ \begin{array}{c}
\frac{dZ}{dt} = \mu(t, Z) - \frac{1}{2} \sigma(t, Z) \frac{\partial \sigma(t, Z)}{\partial Z} \\
Z(0) = x_0
\end{array} \right. \tag{4}
\]
and
\[
\left\{ \begin{array}{c}
\frac{dZ}{dx} = \sigma(T, Z_T) \\
Z_T(0) = Z_0(T),
\end{array} \right. \tag{5}
\]
with \(Z(T, x) = Z_T(x)\). In principle, the solution of the successive ordinary
differential equations (4) and (5) gives an a priori method to solve a class of
stochastic differential equations, in contrast to the usual a posteriori justifica-
tion by Ito’s Lemma, where one concludes that a given function satisfies the
stochastic differential equation by verifying some identities in terms of the co-
efficients of the equation and some partial derivatives of the function. As such,
the method is a sort of a converse to Ito’s Lemma.

The method of solution by (4) and (5) is similar to the differential repr esen-
tation of the solution of autonomous stochastic differential equations
\[
\left\{ \begin{array}{c}
dX_t = \mu(X_t) dt + \sigma(X_t) dB_t \\
X_0 = x_0
\end{array} \right. \tag{6}
\]
of H. Doss in [5] (see also [7]), which is also an a priori method based on the
successive integration of two ordinary differential equations.

The representation in [5] of the solution of (1) is
\[ X_t = H(D_t, B_t), \]
where \(D_t(\omega)\) satisfies for nearly all \(\omega \in \Omega\)
\[
\begin{align*}
D_t(\omega) &= \exp \left( - \int_0^t \sigma'(H(D_s(\omega), B_s(\omega))) ds \right) \\
& \times \left( \mu(H(D_t(\omega), B_t(\omega))) - \frac{1}{2} \sigma(H(D_t(\omega), B_t(\omega))) \sigma'(H(D_t(\omega), B_t(\omega))) \right) \\
D_0(\omega) &= x_0,
\end{align*} \tag{7}
\]
and \(H\) satisfies
\[
\begin{align*}
\frac{\partial H(D, B)}{\partial B} &= \sigma(H(D, B)) \\
\frac{\partial H(D, B)}{\partial D} &= D. \tag{8}
\end{align*}
\]

To compare both integrations, let us begin by noting that in the latter
approach typically firstly the equation (8) is solved and then the equation in
time (7), while the former method starts with the equation in time (4) followed
by the equation in space (5).

The equation in time (4) corresponds to integrating (7) along the path, say
\(\omega_0\), of Brownian Motion which is everywhere 0; then the integration of (8) is
only over an interval of length 0. Indeed, identifying $D_t(\omega_0)$ with a real function $D$, equation (7) amounts to

$$\begin{align*}
\frac{dD}{dt}(0) &= \mu(H(D,0)) - \frac{1}{2}\sigma(H(D,0)) \frac{d\sigma(H(D,0))}{dH} = \mu(D) - \frac{1}{2}\sigma(D) \frac{d\sigma(D)}{dH} \\
D(0) &= x_0.
\end{align*}$$

(9)

On the other hand (8) becomes, identifying $H(D, B) = H_D(B)$ at the point $D = D$,

$$\begin{align*}
\frac{dH_D(B)}{dB}(0) &= \sigma(H_D(B)) \\
H_D(0) &= D.
\end{align*}$$

(10)

From the initial condition of (10), one obtains

$$X_t(\omega_0) = H(D_t(\omega_0), 0) = D_t(\omega_0),$$

as (trivial) solution of (11).

Suppose now that the stochastic differential equation (1) has indeed a solution $\tilde{H}(t, B_t)$ which depends only on time and the values taken by Brownian motion. We will again recognize (11) when integrating along a horizontal path, say $\omega$, and the “vertical equation” (5). One notes [5] that

$$\frac{\partial H(D, B)}{\partial D} = \exp \left( \int_0^{B_t(\omega)} \sigma(H(D_t(\omega), \xi)) d\xi \right).$$

Hence we derive from (7) indeed

$$\frac{\partial \tilde{H}(t, B_t)}{\partial t} = \mu(\tilde{H}(t, B_t)) - \frac{1}{2}\sigma(\tilde{H}(t, B_t)) \sigma'(\tilde{H}(t, B_t)),$$

with $\tilde{H}(t, B_t) = H(D_t(\omega), B_t(\omega))$ given by

$$\begin{align*}
\frac{\partial \tilde{H}(t, B_t)}{\partial B} &= \sigma(\tilde{H}(t, B_t)) \\
\tilde{H}(t, 0) &= D_t(\omega).
\end{align*}$$

We recall that (11) has a global solution of type $\tilde{H}(t, B_t)$, only if $\mu$ and $\sigma$ satisfy the integration condition (12).

The equation (7) shows that the approach of [5] is essentially path-dependent: each individual path of Brownian Motion generates a pair of ordinary differential equations which determines a particular solution. On the contrary, if the integration condition (12) holds, one couple of ordinary differential equations will yield a global solution $Z$, valid for all paths. The solution is path-independent in the sense that if $\omega, \omega' \in \Omega$ are such that $B_t(\omega) = B_t(\omega')$ at some time $t$, it holds that $Z(t, B_t(\omega)) = Z(t, B_t(\omega'))$. 

In fact we will present a somewhat more general method to find a global solution of stochastic differential equations, for a class of equations which not necessarily satisfies (2). Indeed, if

\[
\frac{\partial \mu}{\partial X} - \frac{\mu \partial \sigma}{\sigma \partial X} - \frac{1}{\sigma} \frac{\partial \sigma}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 \sigma}{\partial X^2} = \phi(t)
\]

for some real function \( \phi \), a global solution of (1) can be found in terms of a deterministic function \( Z(t, Y_t) \) of time \( t \) and an Ito-process \( Y_t \), given by a stochastic integral of the form

\[
Y(T, \omega) = y_0 + \int_0^T F(t) \, dt + \int_0^T G(t) \, dB_t.
\]

Here \( G \) satisfies

\[
G(t) = \exp(-\Phi(t))
\]

for some primitive of \( \Phi \) of \( \phi \). We observe that the integration condition (25) may be seen as a first-order linear differential equation for the auxiliary function \( G \) of the Ito-process (26), i.e.

\[
\frac{dG}{dt} = \left( \frac{\mu \partial \sigma}{\sigma \partial Z} + \frac{1}{\sigma} \frac{\partial \sigma}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \sigma}{\partial Z^2} - \frac{\partial \mu}{\partial Z} \right) G.
\]

The choice of the function \( F \) in (12) is free. The two successive ordinary differential equations leading to \( Z(T, Y_T) \) take now the form

\[
\left\{ \begin{array}{l}
\frac{dZ}{dt} = \mu(t, Z) - \sigma(t, Z) \left( \frac{1}{2} \frac{\partial \sigma(t, Z)}{\partial Z} + \frac{F(t)}{\sigma(t)} \right) \\
Z(0) = x_0
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
\frac{dZ_T}{dt} = \frac{\sigma(T, Z_T)}{\sigma(t)} \\
Z_T(0) = Z_0(T)
\end{array} \right.
\]

with \( Z(T, Y_T) = Z_T(Y_T) \). It is of course possible to choose \( F \equiv 0 \), and then (15) reduces to (1). However, it may be that a proper adjustment by \( F \) makes the ordinary differential equation (15) easier to solve than (1). This will be illustrated by the example of the Ornstein-Uhlenbeck process of Example 2 of Section 4.

If \( G \) satisfies (13) the formula (11) expresses the equality

\[
\frac{\partial^2 Z(t, Y)}{\partial t \partial Y} = \frac{\partial^2 Z(t, Y)}{\partial Y^2}.
\]

Observe that if the function \( G \) may be taken equal to 1, the integration condition (11) reduces to (2). Then the Ito-process \( Y_t \) reduces to Brownian Motion, if \( F \) is chosen to be identically 0.
Existence and uniqueness of solutions of a system of two first order partial differential equations

Our approach is based on the existence and uniqueness of a solution for the system of partial differential equations in three variables of the form

\[
\begin{align*}
\frac{\partial Z}{\partial x} &= f(x, y, Z) \\
\frac{\partial Z}{\partial y} &= g(x, y, Z) \\
Z(x_0, y_0) &= z_0,
\end{align*}
\]

(17)

with initial condition in one single point \((x_0, y_0) \in \mathbb{R}^2\). If \(Z = Z(x, y)\) is a solution of this system of class \(C^2\), it follows easily from the equality \(\frac{\partial^2 Z(x, y)}{\partial x \partial y} = \frac{\partial^2 Z(x, y)}{\partial y \partial x}\) that \(f\) and \(g\) satisfy

\[
\frac{\partial f}{\partial y} + g \frac{\partial f}{\partial Z} - f \frac{\partial g}{\partial x} - g \frac{\partial g}{\partial Z} = 0.
\]

(18)

This formula represents the "integration condition" or "compatibility condition" of the system.

We use the following notations.

Notation 3.1 Let \(f\) be a function of two variables \(x\) and \(y\). With some abuse of language we may write \(f(x, y) = f_x(y)\) if \(x\) is temporarily fixed and \(f(x, y) = f_y(x)\) if \(y\) is temporarily fixed. We adopt an analogous convention for functions of three variables.

Definition 3.2 Let \(z_0 \in \mathbb{R}\). Let \(f : \mathbb{R}^3 \rightarrow \mathbb{R}\) and \(g : \mathbb{R}^3 \rightarrow \mathbb{R}\) be of class \(C^1\), both uniformly Lipschitz in the third variable. Consider the system of first-order partial differential equations

\[
\begin{align*}
\frac{\partial Z}{\partial x} &= f(x, y, Z) \\
\frac{\partial Z}{\partial y} &= g(x, y, Z) \\
Z(0, 0) &= z_0.
\end{align*}
\]

(19)

We let \(\tilde{Z} : \mathbb{R}^2 \rightarrow \mathbb{R}\) be defined by \(\tilde{Z}(\mathbf{r}, \mathbf{v}) = \tilde{Z}_\mathbf{r}(\mathbf{v})\), where \(\tilde{Z}_\mathbf{r}\) satisfies the ordinary differential equation

\[
\begin{align*}
\frac{d\tilde{Z}_\mathbf{r}}{dy} &= g(y, \tilde{Z}_\mathbf{r}(y)) \\
\tilde{Z}_\mathbf{r}(0) &= \tilde{Z}_\mathbf{r}(0),
\end{align*}
\]

(20)

with \(\tilde{Z}_0\) given by the ordinary differential equation

\[
\begin{align*}
\frac{d\tilde{Z}_0}{dx} &= f_0(x, \tilde{Z}_0(x)) \\
\tilde{Z}_0(0) &= z_0.
\end{align*}
\]

(21)
The following theorem expresses conditions for the existence and uniqueness of solutions of (17).

**Theorem 3.3** (Exact solution of systems of partial differential equations of first order) Let \( z_0 \in \mathbb{R} \). Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be of class \( C^1 \) and \( g : \mathbb{R}^3 \to \mathbb{R} \) be of class \( C^1 \), both uniformly Lipschitz in the third variable. Assume (18) holds. Then \( \tilde{Z} \) is solution of the system of first-order partial differential equations (19). As such it is unique and of class \( C^2 \).

The Theorem of Frobenius of Differential Geometry \([3]\) implies local existence and uniqueness of the solution. By the uniform Lipschitz property the ordinary differential equations (21) and (20) have existence and uniqueness of solutions on any interval. Hence \( \tilde{Z} \) is well-defined and unique everywhere.

Also, the value of \( \tilde{Z} \) at \((t, x)\) may be obtained by integrating along any simple continuously differentiable curve, say, \(\gamma\) going from \((0, 0)\) to \((t, x)\), i.e. by solving

\[
\begin{align*}
\frac{dZ}{d\tau} &= f((\gamma_1(\tau), \gamma_2(\tau), Z(\tau)))\gamma'_1(\tau) + g((\gamma_1(\tau), \gamma_2(\tau), Z(\tau)))\gamma'_2(\tau) \\
Z(0) &= z_0.
\end{align*}
\]

### 4 Functional solutions of stochastic differential equations.

Let \( T > 0 \). To fix ideas, we will always work within an appropriate probability space \((\Omega, \mathcal{F}, P)\), where \( \Omega \) is a sufficiently rich set, \( \mathcal{F}=(\mathcal{F}_t)_{t\in[0,T]} \) is the natural filtration to the Standard Brownian Motion \( B_t \) on \([0, T]\), and \( P \) the probability associated to this Standard Brownian Motion. Let \( \mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R} \) be measurable. For \( t \in T, \omega \in \Omega \) and \( x_0 : \Omega \to \mathbb{R} \) measurable and of class \( L^2 \) we use often the notation of stochastic differential equations

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = \mu(t, X_t) dt + \sigma(t, X_t) dB_t \\
X_0 = x_0
\end{array} \right. 
\end{align*}
\]

for the stochastic integral

\[
X(T, \omega) = x_0(\omega) + \int_0^T \mu(t, X(t, \omega)) dt + \int_0^T \sigma(t, X(t, \omega)) dB_t.
\]

We recall that such a stochastic process \( X_t \) is an *Ito-process* with respect to \( B_t \) if it is of the form

\[
X(T, \omega) = x_0(\omega) + \int_0^T F(t, \omega) dt + \int_0^T G(t, \omega) dB_t,
\]

where \( x_0 \) is \( \mathcal{F}_0 \)-measurable, \( F \) and \( G \) are at any time \( t \) adapted to \( \mathcal{F}_t \), and \( \int_0^T |F(t, \omega)| dt \) and \( \int_0^T |G(t, \omega)|^2 dt \) exist almost surely (later on, for reasons of simplicity, we will assume that \( x_0 \) is a constant).

For the sake of clarity we recall Ito’s Lemma for stochastic processes which are functions \( Z(t, B_t) \) of time and Brownian Motion and for stochastic processes which are functions \( Z(t, Y_t) \) of time and a general Ito process \( Y_t \).
Theorem 4.1 (Ito’s Lemma, processes of the form \( Z(t, B_t) \)) Let \( T > 0 \) and let \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^{12} \). The stochastic process \( Z(t, B_t) \) is an Ito-process with respect to \( B_t \) and satisfies for \( 0 \leq t \leq T \) the stochastic differential equation

\[
\begin{align*}
    dZ_t &= \left( \frac{\partial Z}{\partial t} + \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} \right) dt + \frac{\partial Z}{\partial x} dB_t \\
    Z_0 &= Z(0, x_0),
\end{align*}
\]

where \( x_0 = B_0 \).

Theorem 4.2 (Ito’s Lemma, processes of the form \( Z(t, Y_t) \)) Let \( T > 0 \) and let \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^{12} \). Let \( Y_t \) be an Ito-process of the form (23), with initial condition \( y_0 \). The stochastic process \( Z(t, Y_t) \) is an Ito-process and satisfies for \( 0 \leq t \leq T \) the stochastic differential equation

\[
\begin{align*}
    dZ_t &= \left( \frac{\partial Z}{\partial t} + F \frac{\partial Z}{\partial x} + \frac{1}{2} G^2 \frac{\partial^2 Z}{\partial x^2} \right) dt + G \frac{\partial Z}{\partial x} dB_t \\
    Z_0 &= Z(0, y_0),
\end{align*}
\]

(24)

The Main Theorem on the existence of global solutions of stochastic differential equations in the form of deterministic functions of Ito processes is as follows.

Theorem 4.3 (Main Theorem) Let \( T > 0 \) and \( x_0 \in \mathbb{R} \). Let \( \mu : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^1 \), and \( \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}^+ \setminus \{0\} \) be of class \( C^2 \), both uniformly Lipschitz in the second variable, and with \( \frac{\partial \sigma}{\partial x} \) bounded. Assume

\[
\frac{\partial \mu}{\partial X} - \frac{\mu \sigma}{\sigma} \frac{\partial \sigma}{\partial t} - \frac{1}{2} \frac{\sigma^2}{\sigma} \frac{\partial^2 \sigma}{\partial x^2} = \phi(t)
\]

for some real continuous function \( \phi \). Let \( \Phi(t) = \int_0^T \phi(t) dt \) and \( G(t) = \exp(-\Phi(t)) \). Let \( F \) be a real function of class \( C^1 \) and \( Y_t \) be the Ito process given by

\[
Y_t = y_0 + \int_0^t F(s) ds + \int_0^t G(s) dB_s,
\]

(26)

where \( y_0 \) is some constant. Then there exists a unique function \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \) of class \( C^{21} \) such that \( Z(t, Y_t) \) is an Ito-process with respect to \( B_t \) satisfying (22). In fact, for all \( (\tau, \pi) \in [0, T] \times \mathbb{R} \) the value \( Z(\tau, \pi) \) may be determined by solving successively the ordinary differential equations

\[
\begin{align*}
    \frac{d\tilde{Y}}{dt} &= \mu(t, \tilde{Z}) - \sigma(t, \tilde{Z}) \left( \frac{1}{2} \frac{\partial \sigma(t, \tilde{Z})}{\partial \tilde{Z}} + \frac{F(t)}{\sigma(t)} \right) \\
    \tilde{Z}(0) &= x_0
\end{align*}
\]

(27)

and

\[
\begin{align*}
    \frac{d\tilde{Z}}{d\pi} &= \frac{\sigma(\tau, \tilde{Z})}{\sigma(\tilde{Z})} \\
    \tilde{Z}(\tau) &= Z(\tau)
\end{align*}
\]

(28)

8
with \( Z(\bar{t}, \bar{x}) = \bar{Z}_\bar{t}(\bar{x}) \).

Conversely, if \( \bar{Z}_0 \) has a solution of class \( C^{23} \) of the form \( Z(t, Y_t) \), where \( Y_t \) is given by \( \bar{Z}_t(\bar{x}) \), with \( F \) and \( G \neq 0 \) of class \( C^1 \), formula (28) holds with \( \phi(t) = -G'(t)/G(t) \).

Observe that the Main Theorem expresses path-independence of the process \( X = Z \) with respect to the process \( Y \): if \( \omega, \omega' \in \Omega \) are such that \( Y_t(\omega) = Y_t(\omega') \) at some time \( t \), it holds that \( X_t(\omega) = Z(t, Y_t(\omega)) = Z(t, Y_t(\omega')) = X_t(\omega') \). If \( G' \neq 0 \) the process does not have path-independence with respect to Brownian motion.

Functional dependence \( Z(t, B_t) \) on time and Brownian Motion is characterized by the following corollary, with a simpler integration condition and simpler ordinary differential equations for the function \( Z \).

**Theorem 4.4** Let \( T > 0 \) and \( x_0 \in \mathbb{R} \). Let \( \mu : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( \sigma : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^1 \), and \( \sigma \) of class \( C^{12} \), both uniformly Lipschitz in the third variable, and with \( \frac{\partial \sigma}{\partial X} \) bounded. Consider the stochastic differential equation (22). Assume

\[
\frac{\sigma}{\partial X} - \frac{\mu}{\partial X} \frac{\partial \sigma}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 \sigma}{\partial X^2} = 0. \tag{29}
\]

Then there exists a unique function \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \) of class \( C^{23} \) such that \( Z(t, B_t) \) is an Ito-process with respect to \( B_t \) satisfying (22). In fact, for all \( (t, x) \in [0, T] \times \mathbb{R} \) the value \( Z(t, x) \) may be determined by solving successively the ordinary differential equations

\[
\begin{aligned}
\frac{d\tilde{Z}}{dt} &= \mu(t, \tilde{Z}) - \frac{1}{2} \sigma(t, \tilde{Z}) \frac{\partial \sigma(t, \tilde{Z})}{\partial Z} \\
\frac{d\tilde{Z}}{dx} &= \sigma(t, \tilde{Z}) \\
\end{aligned} \tag{30}
\]

and

\[
\begin{aligned}
\frac{d\tilde{Z}}{dt} &= \sigma(\bar{t}, \bar{Z}) \\
\tilde{Z}(0) &= Z(\bar{t}), \tag{31}
\end{aligned}
\]

with \( Z(\bar{t}, \bar{x}) = \bar{Z}_\bar{t}(\bar{x}) \).

Conversely, if \( \bar{Z}_0 \) has a solution of class \( C^{23} \) of the form \( Z(t, B_t) \), formula (28) holds.

**Proof of the Main Theorem.** Let \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) be defined by

\[
\begin{aligned}
f(t, x, Z) &= \mu(t, Z) - \frac{1}{2} \sigma(t, Z) \frac{\partial \sigma(t, Z)}{\partial Z} - \frac{F(t)}{G(t)} \sigma(t, Z) \\
g(t, x, Z) &= \sigma(t, Z)/G(t). \tag{32}
\end{aligned}
\]

Then \( f \) is of class \( C^{1\infty \, 1} \) and \( g \) is of class \( C^{1\infty \, 2} \) and both are uniformly Lipschitz in the third variable. Consider the system of two partial differential equations

\[
\begin{aligned}
\frac{\partial Z}{\partial t} &= f(t, x, Z) \\
\frac{\partial Z}{\partial x} &= g(t, x, Z) \\
Z(0, 0) &= x_0. \tag{33}
\end{aligned}
\]
The conditions for the integration of the system (33) are satisfied, since

\[
\frac{\partial f}{\partial x} + g \frac{\partial f}{\partial Z} - \frac{\partial g}{\partial t} - f \frac{\partial g}{\partial Z} = \sigma \frac{G}{\mu - \frac{1}{2} \sigma \frac{\partial \sigma}{\partial Z} - \frac{F}{G} \frac{\partial (\sigma / G)}{\partial Z}} - \left( \mu - \frac{1}{2} \sigma \frac{\partial \sigma}{\partial Z} - \frac{F}{G} \frac{\partial (\sigma / G)}{\partial Z} \right) - \frac{1}{G} \left( \sigma \frac{\partial \mu}{\partial Z} - \frac{\partial \sigma}{\partial Z} - \frac{\sigma^2 \partial \sigma}{2 \partial Z^2} + \frac{G'}{G} \sigma \right)
\]

= 0.

By Theorem 3.3 the system (33) has a solution \( Z : [0, T] \times \mathbb{R} \to \mathbb{R} \) at least of class \( C^2 \). It follows from the identities

\[
\frac{\partial^2 Z}{\partial x^2} = \frac{\sigma(t, Z(t, x)) \partial \sigma(t, Z(t, x))}{G^2(t)}
\]

and

\[
\frac{\partial^3 Z}{\partial x^3} = \frac{\sigma(t, Z(t, x)) \left( \frac{\partial \sigma(t, Z(t, x))}{\partial Z} \right)^2}{G^3(t)} + \frac{\sigma^2(t, Z(t, x)) \partial^2 \sigma(t, Z(t, x))}{G^3(t)} \frac{\partial \sigma(t, Z(t, x))}{\partial Z^2}
\]

that the solution \( Z \) is in fact of class \( C^{23} \). One verifies that \( Z \) satisfies

\[
\begin{cases}
\frac{\partial Z}{\partial t} + F \frac{\partial Z}{\partial x} + \frac{1}{2} G^2 \frac{\partial^2 Z}{\partial x^2} = \mu(t, Z) \\
\frac{\partial Z}{\partial x} = \frac{\sigma(t, Z)}{G(t)} \\
Z(0, 0) = x_0,
\end{cases}
\] (34)

and then by Ito's Lemma the Ito-process \( Z_t \equiv Z(t, Y_t) \) satisfies the stochastic differential equation

\[
\begin{cases}
dZ_t = \left( \frac{\partial Z}{\partial t} + F \frac{\partial Z}{\partial x} + \frac{1}{2} G^2 \frac{\partial^2 Z}{\partial x^2} \right) dt + G \frac{\partial Z}{\partial x} dB_t \\
Z_0 = x_0.
\end{cases}
\]

By (34) it satisfies also the stochastic differential equation (22).

As for uniqueness, we observe first that by Theorem 3.3 the function \( Z \) is the unique solution of class \( C^2 \) (in fact of class \( C^{23} \)) of the system (33). For fixed \( t > 0 \) let \( \zeta : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^{23} \) such that \( \zeta(t, Y_t) \) is an Ito-process satisfying (22). By the Existence-Uniqueness Theorem for stochastic differential equations [6], almost surely

\[
\sup_{0 \leq t \leq T} |Z_t - \zeta_t| = 0.
\]

Hence \( Z(t, Y_t(\omega)) = \zeta(t, Y_t(\omega)) \) almost surely. For \( t > 0 \) the range of the stochastic variable \( B_t \) is the whole of \( \mathbb{R} \), so because the positive and continuous function \( G \) is has a non-zero lower bound on \([0, t]\), the range of the stochastic variable \( \int_0^t G(s) dB_s \) is also the whole of \( \mathbb{R} \), hence because \( F \) is bounded on \([0, t]\)
the range of the stochastic variable \( Y_t = y_0 + \int_0^t F(s) \, ds + \int_0^t G(s) \, dB_s \) is also the whole of \( \mathbb{R} \). Hence \( Z(t, x) = \zeta(t, x) \) almost surely for \( x \in \mathbb{R} \) with respect to the measure on \( \mathbb{R} \) induced by \( F \) and \( P \). By continuity of \( Z \) and \( \zeta \) we have \( Z(t, x) = \zeta(t, x) \) for all \( (t, x) \in [0, T] \times \mathbb{R} \). Hence \( Z = \zeta \).

The converse follows from the equality \( \frac{\partial^2 Z(t, Y_t)}{\partial Y \partial t} = \frac{\partial^2 Z(t, Y_t)}{\partial t \partial Y} \).

**Theorem 4.5** Assume the conditions of Theorem 4.3 are satisfied. Let \( Y_t \) be an Ito-process such that \( Z(t, Y_t) \) solves the stochastic differential equation (22). Let \( 0 < t \leq T \) and \( D \) be the cumulative distribution function of \( Y_t \). Then for all \( x \in \mathbb{R} \)

\[
\Pr \{ X_t \leq x \} = D(Z_t^{-1}(x)).
\]

In particular, if \( Y_t = B_t \),

\[
\Pr \{ X_t \leq x \} = N(Z_t^{-1}(x)).
\]

The proof is obvious, noting that \( Z^{-1}(t, Y_t) \) is well-defined for fixed \( t \), since \( \frac{\partial Z}{\partial x} = \sigma(t, Z) \) is always positive.

**Remarks.**

1. In [2] the problem of path-independence was studied in a discrete setting from an asymptotic point-of-view. Roughly spoken, solutions of stochastic difference equations

\[
\delta X_t = \mu(t, X_t) \delta t + \sigma(t, X_t) \delta W_t,
\]

where \( \delta W_t = \pm \sqrt{\delta t} \) is the Wiener Walk and \( \delta t \to 0 \), happen to have in the limit the same probability distribution as a deterministic function \( Z(t, W_t) \) - i.e. some deformed Normal Distribution like in Theorem 4.3 - provided (29) holds. The condition (29) expresses a form of near path-dependence on microscopic level. Observe that an upward movement \( \delta W_t = +\sqrt{\delta t} \) followed by a downward movement \( \delta W_t = -\sqrt{\delta t} \) yields the same value as a downward movement \( \delta W_t = -\sqrt{\delta t} \) followed by an upward movement \( \delta W_t = +\sqrt{\delta t} \). This is not true for a general process \( X_t \) given by (35), but if (29) holds the values of an upward movement followed by a downward movement and a downward movement followed by an upward movement happen to be sufficiently close to permit the above limit property for its probability distribution. The property follows by applying Taylor-expansions to the increments \( \delta X_t \).

2. The second-order integration condition (25) may be simplified and also be solved for \( \mu \). Firstly, put

\[
\nu(t, X) = \mu(t, X) - \frac{1}{2} \sigma(t, X) \frac{\partial \sigma(t, X)}{\partial X}.
\]
Then (29) becomes the first-order linear partial differential equation

$$\sigma \frac{\partial \nu}{\partial X} - \frac{\partial \sigma}{\partial X} \nu = \frac{\partial \sigma}{\partial t} + \phi(t).$$

(37)

When solved for $\nu$, with $\mu(t, X) = \nu(t, X) + \frac{1}{2} \sigma(t, X) \frac{\partial \sigma(t, X)}{\partial X}$ one finds

$$\mu(t, X) = \left( \frac{1}{2} \frac{\partial \sigma(t, X)}{\partial X} + \int_0^X \frac{\partial \sigma(t, \xi)}{\partial t} + \phi(t)}{\sigma^2(t, \xi)} d\xi + \gamma(t) \right) \sigma(t, X),$$

(38)

for some function $\gamma$ of class $C^1$.

3. Some special cases lead to simplifications of the integration condition (38). For (22) to have solutions of the form $Z(t, B_t)$ one has $\phi = 0$ and (38) reduces to

$$\mu(t, X) = \left( \frac{1}{2} \frac{\partial \sigma(t, X)}{\partial X} + \int_0^X \frac{\partial \sigma(t, \xi)}{\partial t} \sigma^2(t, \xi) d\xi + \gamma(t) \right) \sigma(t, X).$$

(39)

In the autonomous case one also has $\phi = 0$ and eliminating all dependence on $t$ in (39) one finds

$$\mu(X) = \left( \frac{1}{2} \sigma'(X) + c \right) \sigma(X)$$

(40)

for some constant $c$. Moreover, if $\sigma$ is linear, $\mu$ is must also be linear.

Examples.

1. **Autonomous case.** Consider the stochastic differential equation

$$\begin{cases}
    dX_t = \mu(X_t) dt + \sigma(X_t) dB_t \\
    X_0 = x_0,
\end{cases}$$

$0 \leq t < T$

with $x_0 \in \mathbb{R}$, and $\mu$ of class $C^1$ and $\sigma \neq 0$ of class $C^2$. We saw that for functional solutions of the form $Z(t, Y_t)$, where $Y_t$ is an Ito Process of the form (20), in fact $Y_t$ is equal to $B_t$, and that the trend $\mu$ must satisfy (40). If the remaining conditions of Theorem 4.4 are satisfied the function $Z$ may be determined by means of the equations (30) and (31), which become the differential equations with separable variables

$$\begin{cases}
    \frac{dZ}{dt} = c \sigma(Z) \\
    Z(0) = x_0
\end{cases}$$

(41)

and

$$\begin{cases}
    \frac{d\tilde{Z}}{d\tau} = \sigma(\tilde{Z}) \\
    \tilde{Z}(0) = Z(t).
\end{cases}$$

(42)
A well-known special case is the Geometric Brownian Motion. It satisfies the stochastic differential equation
\[ \begin{align*}
\{ & dX_t = \hat{\mu}X_t dt + \sigma X_t dB_t \quad 0 \leq t < T \\
X_0 & = x_0, 
\end{align*} \]
with \( \hat{\mu} \in \mathbb{R} \) and \( \sigma > 0 \). One has \( c = (\hat{\mu} - \sigma^2/2)/\hat{\sigma} \), hence the solution of (41) is \( Z(t) = x_0 \exp(\hat{\mu} - \sigma^2/2)t \) (this formula is perhaps most easily found applying (30) directly). Then the solution of (42) is \( Z(t, B_t) = x_0 \exp \left( (\hat{\mu} - \sigma^2/2)t + \sigma B_t \right) \) and one derives the well-known formula \( Z(t, B_t) = x_0 \exp \left( (\hat{\mu} - \sigma^2/2)t + \sigma B_t \right) \).

2. **Ornstein-Uhlenbeck process.** This process is given by the stochastic differential equation
\[ \begin{align*}
\{ & dR_t = \theta(\hat{\mu} - R_t) dt + \sigma dB_t \\
R_0 & = r_0, 
\end{align*} \]
with \( \theta, \hat{\mu}, \sigma \neq 0 \) and \( r_0 \) are all constants. With \( \mu(t, R_t) = \theta(\hat{\mu} - R_t) \) and \( \sigma(t, R_t) = \sigma \), the integration condition (25) becomes
\[ \frac{\partial \mu}{\partial R} - \frac{\mu \partial \sigma}{\sigma \partial R} - \frac{1}{\sigma} \frac{\partial^2 \sigma}{\partial R^2} = -\theta. \]
Hence \( G(t) = ce^{\theta t} \), for some \( c \in \mathbb{R} \). By Theorem 4.3 one has \( R_t = Z(t, Y_t) \), where \( Y_t \) is of the form
\[ Y_t = y_0 + \int_0^t F(s) ds + c \int_0^t e^{\theta s} dB_s; \]
here \( y_0 \in \mathbb{R} \), \( F \) is of class \( C^1 \) on \([0, T]\), and \( Z \) is of class \( C^{13} \) and is determined by solving successively the auxiliary differential equations
\[ \begin{align*}
\{ & \frac{d\tilde{Z}}{ds} = \theta(\hat{\mu} - \tilde{Z}) - \tilde{\sigma} \left( 0 + \frac{F(s)}{\sigma} \right) = \theta \hat{\mu} - \frac{\tilde{\sigma}}{\sigma} F(s) e^{-\theta s} - \theta \tilde{Z} \quad (43) \\
\tilde{Z}(0) & = r_0 
\end{align*} \]
and
\[ \begin{align*}
\{ & \frac{d\tilde{Z}}{dt} = \frac{\tilde{\sigma}}{\sigma} e^{-\theta t} \\
\tilde{Z}(0) & = \tilde{Z}(t). \quad (44) 
\end{align*} \]
We may choose \( y_0, c \) and \( F \) freely in order to obtain simplifications and assume that \( y_0 = 0, c = 1 \) and \( F(s) = \frac{\hat{\mu}}{\sigma} e^{\theta s} \). Then (43) becomes \( \frac{d\tilde{Z}}{ds} = -\theta \tilde{Z}, \) with \( \tilde{Z}(0) = r_0 \). Hence \( \tilde{Z}(t) = r_0 e^{-\theta t} \). Solving (44) we find
\[ Z(t, Y_t) = r_0 e^{-\theta t} + \tilde{\sigma} e^{-\theta t} Y_t. \]
With \( Y_t = \frac{\tilde{\sigma}}{\sigma} \int_0^t e^{\theta s} ds + \int_0^t e^{\theta s} dB_s = \frac{\tilde{\sigma}}{\sigma} (e^{\theta t} - 1) + \int_0^t e^{\theta s} dB_s \) we derive the well-known formula
\[ R_t = r_0 e^{-\theta t} + \hat{\mu}(1 - e^{-\theta t}) + \tilde{\sigma} \int_0^t e^{\theta(s-t)} dB_s. \]
3. **Homogeneous linear stochastic differential equations.** Consider the stochastic differential equation

\[
\begin{cases}
    dX_t = \alpha(t)X_t \, dt + \beta(t)X_t \, dB_t & 0 \leq t < T \\
    X_0 = x_0.
\end{cases}
\] (45)

We consider the non-degenerate case where \(\beta(t) \neq 0\). It is well-known [1] that the solution of (45) is given by

\[
X_t = \exp \left( \int_0^t \alpha(s) - \frac{1}{2} \beta^2(s) \, ds + \int_0^t \beta(s) \, dB_s \right). \quad (46)
\]

We give a direct derivation, not using verification by Ito’s lemma. The integration condition (14) becomes

\[
\frac{G'(t)}{G(t)} = \frac{\beta'(t)}{\beta(t)} \quad (47)
\]

Hence \(G(t) = c\beta(t)\) for some \(c \in \mathbb{R}\). One may put \(c = 1\) and \(F = 0\). Then (27) and (28) become

\[
\begin{cases}
    \frac{d\tilde{Z}}{dt} = (\alpha(t) - \frac{1}{2} \beta^2(t))\tilde{Z} \\
    \tilde{Z}(0) = x_0
\end{cases}
\] (48)

and

\[
\begin{cases}
    \frac{d\tilde{\beta}}{dt} = \tilde{Z}_T \\
    \tilde{Z}_T(0) = \tilde{Z}(T).
\end{cases}
\] (49)

The solution of (48) is \(X_t = Z(t, Y_t)\), with

\[Y_t = x_0 = \int_0^t \beta(s) \, dB_s,\]

Solving the equations (48) and (49), we derive (46).

4. **A non-autonomous nonlinear stochastic differential equation.** Consider the stochastic differential equation (22) with

\[
\begin{cases}
    \mu(t, X_t) = \exp \left( -\frac{X_t^2}{2} \right) \int_0^{X_t} \exp \left( \frac{X_t^2}{2} \right) d\xi - \frac{X_t(t+1)^2}{2} \exp \left( -\frac{X_t^2}{2} \right) + \exp \left( -\frac{X_t^2}{2} \right) \\
    \sigma(t, X_t) = (t+1) \exp \left( -\frac{X_t^2}{2} \right).
\end{cases}
\]

One verifies that \(\mu\) and \(\sigma\) satisfy the integration condition (29). On any time-interval \([0, T], T > 0\) the functions \(\sigma\) and \(\frac{\partial \sigma}{\partial x}\) are uniformly bounded and \(\mu(t, X)\) has an upper bound of the form \(K |X| + L\), where the constants \(K\) and
do not depend on $t$. This means that the remaining conditions of Theorem 4.4 are also satisfied. Hence (22) has a solution $Z(t, B_t)$, which may be found by solving (in principle) the ordinary differential equations (30), i.e.

$$\begin{align*}
\frac{d\tilde{Z}}{dt} &= \frac{1}{t+1} \exp \left(-\frac{\tilde{Z}^2}{2} \right) \int_0^{\tilde{Z}} \exp \frac{\xi^2}{2} d\xi + \exp \left(-\frac{\tilde{Z}^2}{2} \right) \\
\tilde{Z}(0) &= x_0,
\end{align*}$$

and (31), i.e.

$$\begin{align*}
\frac{d\tilde{Z}}{dx} &= (\tilde{t} + 1) \exp \left(-\frac{\tilde{Z}^2}{2} \right) \\
\tilde{Z}(0) &= \tilde{Z}_0(\tilde{t}).
\end{align*}$$

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