On a conjecture by Lions-Perthame-Tadmor and non existence of the BV regularizing effect for scalar conservation laws

Shyam Sundar Ghoshal* and Animesh Jana†

Centre for Applicable Mathematics,
Tata Institute of Fundamental Research,
Post Bag No 6503, Sharadanagar,
Bangalore - 560065, India.

Abstract

We give an explicit generic construction for the entropy solution of scalar conservation laws in multi-dimension to prove non-existence of the regularity in Besov space for all time. We conclude that uniformly convexity and nondegenerate conditions on flux are not good enough to ensure the Besov regularity, in particular BV regularity of the entropy solution in multi-dimension.

1 Introduction

This paper deals with the aspects of regularity, in particular, we establish the failure of BV regularity for the following scalar conservation laws.

\[ \frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i(u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \]

\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}^d. \]  

In 1994, Lions-Perthame-Tadmor [42] conjectured the regularity of the entropy solution to be in \( W^{s,p}_{loc}(\mathbb{R}^d) \), for \( s < \alpha \), where \( \alpha \) as in (1.7). Another important question in the theory of conservation laws that has remained open is the presence of the regularizing effect from \( u_0 \in L^\infty \) to \( u(\cdot, t) \in BV_{loc} \) in multi-dimension (see Jabin-Perthame, [33, Page 6]). In the present article we settle the later question in a general setting (see Remark 1.2). We also establish that there exists an entropy solution \( u(\cdot, t) \notin W^{\alpha+\varepsilon,p}_{loc}(\mathbb{R}^d) \), which concludes the sharpness of the Lions-Perthame-Tadmor conjecture in higher dimension.

*ghoshal@tifrbng.res.in
†animesh@tifrbng.res.in
In general, the equation of the form \([1.1]\) has huge variety of applications in mathematical physics and fluid dynamics. It is well known that even with smooth initial data the solution may have discontinuities in finite time, hence there is no \(W^{1,1}\) or better Sobolev regularity for the solution, therefore, one has to define the solution in the sense of distribution. Lax-Oleinik \([39, 47]\) obtained an explicit formula when the flux is \(C^2\) and uniformly convex. Recently in \([4]\), an explicit formula for the solution has been obtained for a degenerate \(C^1\) convex flux. Wellposedness theory in multi-dimension with more general flux has been developed by Kruzkov \([38]\). For further studies on conservation laws we refer to \([6, 7, 8, 11, 12, 13, 17, 20, 21, 24, 32, 36, 38, 39, 40, 42, 50, 51, 52, 57]\) and the references therein.

Before we can state our main result we need to layout some definitions and elaborate the state of the art on the regularity aspects of the entropy solution of \([1.1]\). We use \(BV\), \(W^{s,p}\) and \(B^{s,p,\theta}\) as the standard notation of \(BV\) space, fractional Sobolev space and Besov space respectively. The detailed definitions are as follows:

**Definition 1.1.** Let \(\Omega \subset \mathbb{R}^d\) be an open set and \(u \in L^1(\Omega)\). We say \(u \in BV(\Omega)\) if for each \(i = 1, \ldots, d\) there exists a finite signed measure \(\mu_i : \mathcal{B}(\Omega) \to \mathbb{R}\) such that
\[
\int_{\Omega} u \frac{\partial \phi}{\partial x_i} \, dx = - \int_{\Omega} \phi \, d\mu_i,
\]
holds for all \(\phi \in C^\infty_0(\Omega)\), where \(\mathcal{B}(\Omega)\) denotes the Borel \(\sigma\)-algebra.

**Definition 1.2.** \(W^{s,p}(\mathbb{R}^d) = \left\{ u \in L^p(\mathbb{R}^d) \text{ such that } |u|_{W^{s,p}(\mathbb{R}^d)} < \infty \right\}\) for \(0 < s < 1\), \(1 \leq p < \infty\) where \(|u|_{W^{s,p}(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p}\).

**Definition 1.3.** \(B^{s,p,\theta}(\mathbb{R}^d) = \left\{ u \in L^p(\mathbb{R}^d) \text{ such that } |u|_{B^{s,p,\theta}(\mathbb{R}^d)} < \infty \right\}\) for \(0 < s < 1\), \(1 \leq p < \infty\), \(0 < \theta < \infty\), where \(|u|_{B^{s,p,\theta}(\mathbb{R}^d)} = \sum_{i=1}^d \left( \int_0^\infty \|\Delta_i^h u\|_{L^p(\mathbb{R}^d)} \, dh \right)^{\theta} \), and \(\Delta_i^h u(x,t) = u(x + he_i,t) - u(x,t)\) for \(i = 1, 2, \ldots, d\).

The following imbeddings are well known \([41, 50]\)
\[
B^{s,p,\theta_1}(\mathbb{R}^d) \subset B^{s,p,\theta_2}(\mathbb{R}^d) \text{ for } 1 \leq \theta_1 < \theta_2 < \infty, \quad (1.3)
\]
\[
BV(\mathbb{R}^d) \subset B^{s,1,\theta}(\mathbb{R}^d) \text{ for any } 1 \leq \theta < \infty, \quad (1.4)
\]
\[
W^{s,p}(\mathbb{R}^d) = B^{s,p,\theta}(\mathbb{R}^d) \text{ for } 1 \leq p < \infty. \quad (1.5)
\]
From \((1.4)\) and \((1.5)\), it is clear that if the entropy solution \(u(\cdot, t) \notin B^{s,p,\theta}\), for appropriate \(p\) and \(\theta\), then \(u(\cdot, t)\) is neither in \(W^{s,p}\) nor in \(BV\). Throughout the paper we assume that \(u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\), i.e. for some \(R_0 > 0\), \(|u_0|_{L^\infty} < R_0\) and the flux \(f = (f_1, f_2, \ldots, f_d)\), where \(f_i : \mathbb{R} \to \mathbb{R}\) are \(C^1\) functions. Now we state the following nondegeneracy conditions on the flux \([42]\).

1. **Nondegenerate condition on flux:**
\[
\text{meas}\{|v| < R_0, |\tau + f'(v) \cdot \xi| = 0\} = 0, \\
\text{for all } (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \text{ with } \tau^2 + |\xi|^2 = 1. \quad (1.6)
\]
2. Nondegenerate condition on flux of $\alpha$ order:

$$\exists \alpha \in (0, 1), \exists C \geq 0, \text{meas}\{|\nu| < R_0, |\tau + f'(v) \cdot \xi| < \delta\} < C\delta^\alpha,$$

for $\delta \in (0, 1), (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ with $\tau^2 + |\xi|^2 = 1$. (1.7)

Lions-Perthame-Tadmor \cite{34} introduced the mathematical notion of kinetic formulation for scalar conservation laws in several space dimension and also obtained the regularizing effect in $W^{s,p}$ by using the averaging lemma \cite{22} \cite{41}. They conjectured that the entropy solution $u(\cdot, t) \in W^{s,1}$ for all $s < \alpha$, where $\alpha$ is determined by the non degenerate flux condition (1.7). For a detailed study on kinetic formulation and its applications, we give an incomplete list of references \cite{34, 43, 45, 48, 50}.

In one space dimension, due to the Lax-Oleinik formula, one can see that for an uniformly convex flux the solution is locally bounded variation even when the initial data $u_0 \in L^\infty$ and explicitly $BV_{loc}(u(\cdot, t)) \leq C$, for some constant $C > 0$. On the other hand, for $C^1$ strictly (non uniform) convex flux, by using the backward construction \cite{3}, a constructive counter example for BV blow up for all time $t > 0$ has been obtained in \cite{2}. For the one dimensional scalar conservation laws the regularity problem has been extensively studied in \cite{2, 5, 10, 14, 16, 17, 18, 19, 30, 31, 33, 35, 42, 46, 54} and see Remark 1.5 for general flux. In dimension one, De Lellis and Westdickenberg \cite{42} has given an example to conclude the sharpness of Lions-Perthame-Tadmor conjecture \cite{34}. Later Golse and Perthame \cite{31} proved the optimal regularity of $u(\cdot, t) \in B^{1/3,3}_{\infty, loc}(\mathbb{R})$.

On the other hand, for multi-dimensional scalar conservation laws the actual conjecture is still open. In \cite{42}, they proved $u(\cdot, t) \in W^{s,1}$ for all $s < \frac{\alpha}{\alpha+1}$. Subsequently it was improved by Jabin and Perthame \cite{33} to $W^{s,r}$ for $s < 1/3$, $r < 3/2$, when $\alpha = 1$ in (1.7). In a more general setting, Tadmor and Tao \cite{54} proved a new velocity-averaging lemma and raised the regularity of the Sobolev exponent up to $\frac{\alpha}{\alpha+1}$. Also see \cite{25} for recent improvement on the regularity for entropy solutions to (forced) scalar conservation laws. Also Jabin \cite{33} proved $u \in W^{s,1}(\mathbb{R}^d)$ for all $s < \alpha$, if

$$||t|\nabla_x \cdot (f'(u(\cdot, t))))||_{L^q_{t,loc}(\mathbb{R}^d)} \leq C(||u_0||_{L^\infty} + ||u_0||_{L^1})$$

holds along with (1.7). The study of regularizing effects and characterizing of non-linear flux in several space dimension by different approaches can be found in \cite{15, 16, 23, 29, 37, 49, 53, 55}.

We prove that uniform convexity and the nondegenearacy conditions (1.7), (1.6) are not good enough to capture BV regularity of the entropy solution. More importantly we offer an answer to the open question (posed in \cite{33} Page 6) whether $u_0 \in L^\infty$ induces $u(\cdot, t) \in BV$ regularity in multi-dimension in a general setting. Here we propose an explicit generic counter example for general non linear flux in multi-dimension to show that $u(\cdot, t) \notin B^{s,\theta,p}_{loc}$, for all $s > \alpha, \infty > \theta > 0$, $p \geq 1$, where $\alpha$ as in (1.7), which concludes $u(\cdot, t) \notin BV_{loc}$.

To elaborate this context, we answer the following questions:

1. Is it possible to construct an entropy solution of (1.1) in multi-dimension such that $||u||_{\alpha,+,p}(\cdot, t) = \infty$, for some $t > 0$? Where $\alpha$ is determined as in (1.7), which proves the sharpness of the conjecture in higher dimension.
2. Is it possible to construct an entropy solution for uniformly convex flux satisfying condition (1.6) in multi-dimension such that $||u||_{BV_{loc}}(\cdot, t) = \infty$, for some $t > 0$?

3. Is it possible to give a generic construction such that the counter example is true for general flux and for all time $t > 0$?

In Proposition 1.1, we have proved that $\exists u_0 \in L^\infty$ such that $|u(\cdot, t)|_{B^{p,\theta}} = \infty$, in particular $|u(\cdot, t)|_{W^{\alpha, p}} = \infty$, for all $t > 0$, for suitable $\theta$, with the fluxes like power laws. In Proposition 1.2, we construct a solution of (1.1) for uniformly convex fluxes $f, g, f \neq g$ in the sense of (1.6), $u_0 \in L^\infty$ such that $|u(\cdot, t)|_{B^{p,\theta}} = \infty$, in particular, $BV_{loc}(u(\cdot, t)) = \infty$, for all $t > 0$ and for suitable $\theta$. It is evident from the proof of Proposition 1.1 and Proposition 1.2 that blow up result can be elaborately extended to any flux $f$ (see Remark 1.2, Remark 1.3). Also see Remark 1.5 for general flux in one dimension situation.

In section 2, we have proved the following main results:

**Proposition 1.1.** Let $d > 1$. For a certain choice of flux function $f$ satisfying the nondegeracy condition (1.6), there exists an initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that the entropy solution $u(\cdot, t)$ of (1.1) is not in $\mathcal{B}^{p,\theta}_{loc}(\mathbb{R}^d)$ for all $t > 0$ and for any $s > \alpha, \theta > 0$, $p \geq 1$, where $\alpha$ is determined as in (1.7). Hence for all $t > 0$, $u(\cdot, t) \notin BV_{loc}$ (from (1.4)).

**Proposition 1.2.** For a certain choice of uniformly convex fluxes $f_1, f_2$ satisfying the nondegeracy condition (1.6), there exists an initial data $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ such that the entropy solution $u(\cdot, t)$ is not in $\mathcal{B}^{p,\theta}_{loc}(\mathbb{R}^2)$ for all $t > 0$ and for any $s > \alpha, \theta > 0$, $p \geq 1$, where the $\alpha$ is determined as in (1.7). Hence for all $t > 0$, $u(\cdot, t) \notin BV_{loc}$ (from (1.4)).

**Remark 1.1.** If the nondegeneracy condition (1.7) holds in an interval $I$, for some $\alpha$, then it is easy to see that there exists $\xi \in \mathbb{R}^d$ such that $\frac{\xi(\xi(f'(v)) - f'(w))}{|v - w|^d}$ is bounded for $v, w \in I$, then we say that “$\alpha$ is attained in $\xi$ direction”.

**Remark 1.2.** Let $d > 1$. Let $f$ satisfies nondegenerate condition on flux of order $\alpha$ (see (1.7)) and $\alpha$ is attained in $\xi_0 = (\xi_0, \ldots, \xi_0)$ direction (as in Remark 1.1). Let $M : \mathbb{R}^d \to \mathbb{R}$ be a linear function defined as

$$M(x) = \xi_0 \cdot x \text{ where } x \in \mathbb{R}^d.$$

Furthermore assume that $u \to M(f(u))$ is a real strictly convex function in a neighbourhood where (1.7) holds (in particular, an interval $I$ as in Remark 1.1). Then there exists an initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that the entropy solution $u(\cdot, t)$ of (1.1) is not in $\mathcal{B}^{p,\theta}_{loc}(\mathbb{R}^d)$ for all $t > 0$ and for any $s > \alpha, \theta > 0$, $p \geq 1$. Hence for all $t > 0$, $u(\cdot, t) \notin BV_{loc}$ (from (1.4)).

**Remark 1.3.** Let $d > 1$. In Remark 1.2, the convexity assumption of the map $u \to M(f(u))$ is required only in one neighbourhood. Also the result holds if we assume the strict concavity assumption of the map $u \to M(f(u))$ instead of the convexity. Hence we conclude that Remark 1.2 holds for any $C^2$ general flux.
Remark 1.4. Let \( d > 1 \). Similar results are also true for degenerate fluxes (in the sense that if the flux does not satisfy the condition \( (1.6) \)). For example, if we take (say in two dimension) \( f = \frac{u^2}{2}, g = u \) then for the initial data which are constant in the first \( x_1 \) variable, will allow us to construct the solution such that the desired \( B^{s,p,\theta} \) semi norm blows up.

Remark 1.5. In one dimension, if \( f \) satisfies \( (1.7) \) with \( \alpha < 1 \), then similar construction of \( u_0 \in L^\infty(\mathbb{R}) \) will give us \( B^{s,p,\theta}_{\text{loc}} \) blow up for all \( t > 0 \), for \( s > \alpha, \theta > 0, p \geq 1 \).

2 Proof of the main result

In order to prove our main result we need the following two elementary lemmas and for the sake of completeness we give the proof in the Appendix. Also we have used some of the ideas as in [1, 26, 27].

Lemma 2.1. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a \( C^1 \) function. Assume \( M : \mathbb{R}^d \to \mathbb{R} \) is a linear function defined as

\[
M(x_1, \ldots, x_d) = a_1 x_1 + \cdots + a_d x_d \quad \text{where} \quad a_i \in \mathbb{R} \quad \text{for} \quad i = 1, \ldots, d.
\]

Define \( A_r \subset \mathbb{R}^d \) as \( A_r = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -r < x_i < r \quad \text{for} \quad i = 1, \ldots, d \} \) for any \( r > 0 \).

Consider

\[
u_0(x) = \begin{cases} 
  a & \text{if} \quad x \in \{M(x) < 0\} \cap A_r, \\
  b & \text{if} \quad x \in \{M(x) > 0\} \cap A_r, \\
  0 & \text{if} \quad x \in A^c_r,
\end{cases} \tag{2.9}
\]

where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( A^c_r \) denotes the complement of \( A_r \) in \( \mathbb{R}^d \). Let \( u(x,t) \) be the entropy solution of \( (1.1) \) with initial data \( (2.9) \) with the flux \( f \) having the following properties: \( u \mapsto M(f(u)) \) is a real strictly convex function and \( f'(0) = 0 \). Then for any \( 0 < r_1 < r \) there exists \( t_0 > 0 \) and \( r_2 > r \) such that the followings hold for all \( t \in (0, t_0) \) (see figure \ref{fig:figure}):

(i) when \( a > b \),

\[
u(x,t) = \begin{cases} 
  a & \text{if} \quad x \in \{M(x - \gamma t) < 0\} \cap A_{r_1}, \\
  b & \text{if} \quad x \in \{M(x - \gamma t) > 0\} \cap A_{r_1}, \\
  0 & \text{if} \quad x \in A^c_{r_2},
\end{cases} \tag{2.10}
\]

where \( \gamma = \frac{1}{a-b}(f(a) - f(b)) \in \mathbb{R}^d \).

(ii) when \( a < b \),

\[
u(x,t) = \begin{cases} 
  a & \text{if} \quad x \in \{M(x - f'(a)t) < 0\} \cap A_{r_1}, \\
  b & \text{if} \quad x \in \{M(x - f'(b)t) > 0\} \cap A_{r_1}, \\
  0 & \text{if} \quad x \in A^c_{r_2}. 
\end{cases} \tag{2.11}
\]

5
\[ u_0 = a, \quad u_0 = b \]

(i) case \( a > b \)

\[ A_r \]

\[ \{ M(x - \gamma t) = 0 \} \]

\[ A_{r_2} \]

\[ \{ M(x - f'(b)t) = 0 \} \]

\[ A_{r_1} \]

(ii) case \( a < b \)

\[ A_r \]

\[ \{ M(x - f'(a)t) = 0 \} \]

Figure 1: Illustration for the solutions of (1.1) with initial data (2.9) for both the cases and dotted lines represent the structure of \( u(x,t) \) at time \( t < t_0 \).

**Lemma 2.2.** Let \( x \geq 1 \) be a real number and \( 0 < \beta < 1 \) then following inequalities hold

\[
(x + 1)\beta > x^\beta + \frac{\beta}{x^{1-\beta}} - \frac{\beta(1-\beta)}{x^{2-\beta}},
\]

\[
(x + 1)^{1+\beta} > x^{1+\beta} + (1+\beta)x^\beta.
\]

**Proof of Proposition 1.1** Assume the flux \( f = (f_1, f_2, \ldots, f_d) \) satisfies

\[
f_k(u) = \frac{|u|^{\zeta+d+2-k}}{\zeta + d + 2 - k} \quad \text{for} \quad k = 1, 2, \ldots, d, \text{i.e.} \quad f_k'(u) = u^{\zeta+d+1-k} \quad \text{for} \quad k = 1, 2, \ldots, d,
\]

here \( \zeta \geq 1 \) and \( d \geq 1 \). Let \( a \in \mathbb{R} \), then define ‘box function’ \([a]\) to be the greatest integer \( \leq a \).

**Step 1.** First we will consider an initial data with compact support which will give us \( B^{s,p,\theta} \) blow up in finite time.

Now choose \( \tau = \frac{1}{\xi n} + \epsilon \) and two real number sequences \((l_m)_{m \geq 2}\) and \((\sigma_m)_{m \geq 2}\) such
that

\[ l_m = \frac{1}{n(\zeta+d)\tau} \quad \text{if} \ n = \left\lceil \frac{m}{2} \right\rceil, \]
\[ \sigma_m = \sigma_m(R) = \begin{cases} \frac{1}{n^{1+\tau}} & \text{for} \ m = 2n, \\ \frac{2}{n^{1+\tau}} & \text{for} \ m = 2n+1, \end{cases} \]

for \( R > 0 \), where this \( \epsilon(\sigma > 0) \) will be chosen later and \( \lceil \cdot \rceil \) is the box function. Define \( w_m = \sum_{k=N}^{m} l_k \) for \( m \geq N \) and \( X_1 = \sum_{k=N}^{\infty} l_k \). This \( N \) will be chosen later.

Consider

\[
u_0(x) = \begin{cases} 0 & \text{if} \ x_1 < w_N, \\ \sigma_m & \text{if} \ x \in A_R \text{ with } w_m < x_1 < w_{m+1} \text{ for } m \geq N, \\ \frac{1}{n^{1+\tau}} & \text{if} \ x_1 \in A_R \text{ with } x_1 > X_1, \\ 0 & \text{if} \ x \in A_R. \end{cases} \tag{2.12}
\]

with large enough \( R > 0 \) and \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) and \( A_R \) is as defined in lemma 2.1. Clearly \( u_0 \in L^\infty(\mathbb{R}^d) \) and since it has compact support in \( \mathbb{R}^d \), we get \( u_0 \in L^1(\mathbb{R}^d) \). Now we will use the lemma 2.1 to get the structure of the entropy solution of (1.1) with initial data \( u_0 \) as in (2.12).

Define \( \tilde{M}_m : \mathbb{R}^d \to \mathbb{R} \) as \( \tilde{M}_m(x) = X_1 - w_m \) for \( m \geq N \). Since for \( n \geq N \), \( \sigma_{2n} > \sigma_{2n+1} \) by lemma 2.1 we get that the solution \( u(x,t) \) has discontinuity along the plane \( \tilde{S}_n = \{ \tilde{M}_{2n+1}(x - \rho_n t) = 0 \} \). Where \( \rho_n = (\rho_{n1}, \rho_{n2}, \ldots, \rho_{nd}) \) and

\[
\rho_{nk} = \frac{1}{R^{\zeta+d+2-k}} \left( \frac{1}{\zeta+d+2-k} \frac{2^{\zeta+d+2-k}}{n^{(\zeta+d+2-k)\tau}} - \frac{1}{\zeta+d+2-k} \frac{1}{n^{(\zeta+d+2-k)\tau}} \right)^{-1} \left( \frac{1}{n^{1+\tau}} - \frac{1}{n^{\tau}} \right)^{-1} \left( \zeta + d + 2 - k \right) R^{\zeta+d+2-k} \frac{1}{n^{(\zeta+d+2-k)\tau}}.
\]

By lemma 2.1 for \( 0 < r < R \) characteristic planes emitting from the sets \( \{ x_1 = w_{2n} \} \) \( \cap A_r \) and \( \{ x_1 = w_{2n+2} \} \) \( \cap A_r \) are \( \tilde{P}_{2n} = \{ \tilde{M}_{2n}(x - f'(\sigma_{2n})t) = 0 \} \) and \( \tilde{P}_{2n+2} = \{ \tilde{M}_{2n+2}(x - f'(\sigma_{2n+1})t) = 0 \} \), respectively. Suppose \( \tilde{P}_{2n} \) and \( \tilde{P}_{2n+2} \) meet \( \tilde{S}_n \) at time \( t_n \) and \( \tilde{t}_n \), respectively. As a consequence the points \( x \in \tilde{S}_n \cap \tilde{P}_{2n} \) and \( \tilde{x} \in \tilde{S}_n \cap \tilde{P}_{2n+2} \) will satisfy

\[
x_1 - \rho_{n1} t_n - w_{2n} = x_1 - f'_1(\sigma_{2n}) t_n - w_{2n}, \tag{2.13}
\]
\[
\tilde{x}_1 - \rho_{n1} \tilde{t}_n - w_{2n+1} = \tilde{x}_1 - f'_1(\sigma_{2n+1}) \tilde{t}_n - w_{2n+2}; \tag{2.14}
\]

7
Figure 2: Illustration for the solution of (1.1) with the initial data (2.12) and the dotted lines represent the structure of the solution $u(x,t)$ at time $t < t_0$. Respectively. From equation (2.13), we get

$$-\frac{2^{c+d+1} - 1}{R^{c+d}} n^{c+d} \frac{1}{(\zeta + d + 1)} \frac{1}{t_n} - w_{2n+1} = -\frac{2^{c+d} - 1}{R^{c+d}} n^{c+d} \frac{1}{(\zeta + d + 1)} \frac{1}{t_n} = w_{2n+1} - w_{2n}$$

i.e.

$$\frac{1}{R^{c+d}} \left( -\frac{2^{c+d+1} - 1}{(\zeta + d + 1)} + \frac{2^{c+d}}{n^{c+d}} \right) n^{c+d} \frac{1}{(\zeta + d + 1)} \frac{1}{t_n} = \frac{1}{n^{c+d}}$$

i.e.

$$t_n = \frac{R^{c+d} (\zeta + d + 1)}{2^{c+d} (\zeta + d - 1) + 1}.$$
Also from equation (2.14), we get
\[
- \frac{2^{\zeta+d+1} - 1}{(\zeta + d + 1) R^{\zeta+d} n^{(\zeta+d)^{\gamma}}} \tilde{t}_n - w_{2n+1} = - \sigma_{2n+1} \tilde{t}_n - w_{2n+2}
\]
i.e.
\[
\frac{1}{R^{\zeta+d}} \left( \frac{1}{n^{(\zeta+d)^{\gamma}}} - \frac{2^{\zeta+d+1} - 1}{(\zeta + d + 1) n^{(\zeta+d)^{\gamma}}} \right) \tilde{t}_n = w_{2n+1} - w_{2n+2}
\]
i.e.
\[
- \frac{1}{R^{\zeta+d}} \left( \frac{2^{\zeta+d+1} - (\zeta + d + 2)}{\zeta + d + 1} \right) \frac{1}{n^{(\zeta+d)^{\gamma}}} \tilde{t}_n = \frac{1}{n^{(\zeta+d)^{\gamma}}}
\]
i.e. \[\tilde{t}_n = \frac{R^{\zeta+d} (\zeta + d + 1)}{2^{\zeta+d+1} - (\zeta + d + 2)}.\]

Choose \(N \in \mathbb{N}\) large enough so that \((1 + \frac{1}{n})^\gamma < 2\) and \(\frac{1}{n} < \frac{1}{R^{\zeta+d}}\) hold for \(n \geq N\). Now by lemma 2.1, for \(0 < r < R\) we get a \(t'_1 > 0\). Let \(t_0 = \min\{t'_1, \frac{2^{\zeta+d+1} - (\zeta + d + 2)}{2^{\zeta+d}(\zeta+d+1)}\}\).

Now for \(t \in (0, t_0)\), we have the following structure of the entropy solution (see figure 2):
\[
u(x, t) = \begin{cases} \frac{2}{Rn^\gamma} & \text{if } x \in \left(\frac{w_{2n} + \frac{2^{\zeta+d}}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}} < x_1 < \frac{w_{2n+1} + \frac{2^{\zeta+d+1} - 1}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}}\right) \cap A_r, \\ \frac{1}{Rn^\gamma} & \text{if } x \in \left(\frac{w_{2n} + \frac{2^{\zeta+d}}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}} < x_1 < \frac{w_{2n+1} + \frac{2^{\zeta+d+1} - 1}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}}\right) \cap A_r, \\ 0 & \text{if } x \in A_{R_1}, \end{cases}
\]
where \(R_1 > R\) is large enough and \(x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d\).

Now we are interested to check the following semi-norm of \(u(x, t)\)
\[
|u|_{B_r^p, \sigma}(\mathbb{R}^d) = \sum_{i=1}^d \left( \int_0^\infty \frac{\|\Delta^h u\|_{p}(\mathbb{R}^d)}{h^{1+p}} dh \right).
\]
where \(\Delta^h u(x, t) = u(x + he_i, t) - u(x, t)\) for \(i = 1, 2, \ldots, d\).

Note that from (2.15), we can say
\[
\Delta^h_1 u(x, t) = \frac{1}{Rn^\gamma} \text{ if } x \in \left(\frac{w_{2n} + \frac{2^{\zeta+d}}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}} < x_1 < \frac{w_{2n+1} + \frac{2^{\zeta+d+1} - 1}{n^{(\zeta+d)^{\gamma}}} R^{\zeta+d}}{R^{\zeta+d}}\right) \cap A_r,
\]
for \(n \leq \left(\frac{1}{h} \left(1 - \frac{2^{\zeta+d}(\zeta+d-1)+1}{\zeta+d+1}\right) \right)^{1/(\zeta+d^\gamma)}\), hence
\[
\|\Delta^h_1 u(\cdot, t)\|_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |u(x + he_1, t) - u(x, t)|^p dx_1 dx_2 \cdots dx_d
\]
\[
\geq \sum_{n=N}^{M(h)} \int_{-r}^{r} \cdots \int_{-r}^{r} \int_{-r}^{r} \cdots \int_{-r}^{r} \frac{1}{n^{p\gamma}} dx_1 dx_2 \cdots dx_d
\]
\[
= (2r)^d - 1 (M(h) - N) \left(1 - \frac{2^{\zeta+d}(\zeta+d-1)+1}{\zeta+d+1} \right) \frac{1}{n^{p\gamma}} \frac{1}{h^{1+p}}
\]
where \(M(h) = \left[\left(\frac{1}{h} \left(1 - \frac{2^{\zeta+d}(\zeta+d-1)+1}{\zeta+d+1}\right) \right)^{1/(\zeta+d^\gamma)}\right].\) Here \([\cdot]\) is the box function.
First we will do the analysis for the case \( s > \frac{1}{\zeta + d}, \theta > 0 \) satisfying \((s - \frac{1}{\zeta + d})\theta < 1 \) with \( s\theta < 1 \). Then there exists small enough \( \epsilon \) such that \( 0 < s(1 + (\zeta + d)\epsilon) < 1 \) holds.

Let’s consider the sequence \( c_n = \frac{1}{n^{\gamma + \sigma \theta}} (1 - \frac{2^{\gamma + \sigma (\zeta + d - 1) + 1}}{\zeta + d + 1} \frac{t}{R^{\gamma + \sigma}}) \). Then for \( h \leq c_n \) we get \( M(h) \geq n \). Hence for \( n \geq N_1 \) we get \( ||\Delta_t^h u(., t)||_{L^p(\mathbb{R}^d)} \geq C \frac{n^{\frac{p}{\sigma}}}{n^{\gamma + \sigma}} \). This implies

\[
|u|_{B^{s,p,\theta}(\mathbb{R}^d)} \geq C \sum_{n \geq N_1} \frac{n^{\frac{p}{\sigma}}}{n^{\gamma + \sigma}} c_n \int_0^{c_n} \frac{1}{h^{1 + s\theta}} dh
\]

\[
\geq C \sum_{n \geq N_1} \frac{n^{\frac{p}{\sigma}}}{n^{\gamma + \sigma}} ((n + 1)^{(\zeta + d)s\theta} - n^{(\zeta + d)s\theta})
\]

\[
\geq C \sum_{n \geq N_1} \frac{n^{\frac{p}{\sigma}}}{n^{\gamma + \sigma}} \left( \frac{1}{n^{1-(\zeta + d)s\theta}} - \frac{1 - (\zeta + ds)^s\theta}{n^{1-(\zeta + d)s\theta}} \right)
\]

\[
\geq C \sum_{n \geq N_1} \frac{n^{\frac{p}{\sigma}}}{n^{\gamma + \sigma}} \left( \frac{1}{n^{2-(\zeta + d)s\theta + \tau + \frac{(\zeta + d)s\theta}{p}}} - \frac{1}{n^{2-(\zeta + d)s\theta + \tau + \frac{(\zeta + d)s\theta}{p}}} \right).
\]

Here we used lemma 2.2. Since \( \tau (\zeta + d) = 1 + (\zeta + d)\epsilon \) we get

\[
\frac{\theta}{p} + (\zeta + d)s\theta - \tau \theta - \frac{(\zeta + d)s\theta}{p} = (s - \frac{1}{\zeta + d})\theta (1 + (\zeta + d)\epsilon) - \zeta + d \frac{\epsilon}{p}.
\]

By our hypothesis \( s \in (\frac{1}{\zeta + d}, 1) \) and \( \theta > 0 \) holds. Then we can choose \( \epsilon > 0 \) small enough so that \( \frac{\theta}{p} + (\zeta + d)s\theta - \tau \theta - \frac{(\zeta + d)s\theta}{p} \in (0, 1) \). This implies

\[
\sum_{n \geq N_1} \frac{1}{n^{2-(\zeta + d)s\theta + \tau + \frac{(\zeta + d)s\theta}{p}}} < \infty \quad \text{and} \quad \sum_{n \geq N_1} \frac{1}{n^{2-(\zeta + d)s\theta + \tau + \frac{(\zeta + d)s\theta}{p}}} = \infty.
\]

Hence we get \( |u|_{B^{s,p,\theta}(\mathbb{R}^d)} = \infty \). For large \( \theta \), similar calculation holds and to handle this case we have to use second inequality of lemma 2.2. We know that \( B^{s_1,p,\theta_1}(\mathbb{R}^d) \subset B^{s_2,p,\theta_2}(\mathbb{R}^d) \) and \( B^{s,\theta_1,p}(\mathbb{R}^d) \subset B^{s,\theta_2,p}(\mathbb{R}^d) \) for \( s_1 > s_2 \) and \( \theta_2 > \theta_1 \). This completes the proof of step 1.

Note that by similar calculation we can show that the initial data \( u_0 \notin B^{s,p,\theta}(\mathbb{R}^d) \) for \( s > \frac{1}{\zeta + d}, \theta > 0, 1 \leq p < \infty \).

**Step 2.** Note that in step 1, we get for any \( 0 < r < R \) there exists a \( \tau_0 > 0 \) depending on \( R, r \) and \( ||u_0||_{L^\infty} \). Also observe that \( \tau_0 \) is increasing in \( R \). Since \( \tau'_1 \) is increasing in \( r \), so is \( \tau_0 \).

Define \( u_{m}^{1} = Y_{1} + \sum_{k=N_{1}}^{m} l_{k} \) for \( m \geq N_{1} \) and \( X_{2} = Y_{1} + \sum_{k=N_{1}}^{\infty} l_{k} \) and \( A_{2} = Y_{1} \epsilon_{1} + A_{R_{2}} \), where \( Y_{1} > 0, R_{2} > R_{1} > 0, N_{1} > 0 \) are large numbers which will be chosen later. Denote...
$A_1 = A_{R_1}$. Now consider the following data

$$u_0(x) = \begin{cases} 
0 & \text{if } x \in A_1 \cap \{ x_1 < w_N \}, \\
\sigma_n(R_1) & \text{if } x \in A_1 \cap \{ w_m < x_1 < w_{m+1} \}, \\
\frac{1}{R_1} & \text{if } x \in A_R \cap \{ x_1 > X_1 \}, \\
0 & \text{if } x \in A_2 \cap \{ x_1 < w_N^{\frac{1}{2}} \}, \\
\sigma_m(R_3) & \text{if } x \in A_2 \cap \{ w_m^{1} < x_1 < w_{m+1}^{1} \}, \\
\frac{1}{R_2} & \text{if } x \in A_2 \cap \{ x_1 > X_2 \}, \\
0 & \text{if } x \in (A_1 \cup A_2)^C 
\end{cases} \hspace{1cm} (2.16)$$

where $R_2 > R_1 > 0$. Let $t_0^1, t_0^2$ are the times we get if we consider the initial data like in step 1 for $R = R_1$ and $R = R_2$ respectively. By our previous observation $t_0^1 > t_0^2$. Due to finite speed of propagation we can choose $Y_1$ large enough so that for the initial data the characteristics from $A_1$ and characteristics from $A_2$ will not intersect for any $t \in (0, t_0^2)$. Hence we get the following structure of the solution

$$u(x,t) = \begin{cases} 
\frac{2}{R_1} & \text{if } x \in \left\{ w_{2n} + \frac{2^{\frac{d}{4}}}{n^{\frac{d}{4}}} \frac{t}{R_1^d} < x_1 < w_{2n+1} + \frac{2^{\frac{d}{4}+1-1}}{(\frac{d}{4}+1)n^{\frac{d}{4}}} \frac{t}{R_1^d} \right\} \cap A_{R_1}, \\
\frac{1}{R_2} & \text{if } x \in \left\{ w_{2n+1} + \frac{2^{\frac{d}{4}+1-1}}{(\frac{d}{4}+1)n^{\frac{d}{4}}} \frac{t}{R_1^d} < x_1 < w_{2n+2} + \frac{1}{n^{\frac{d}{4}}} \frac{t}{R_1^d} \right\} \cap A_{R_1}, \\
\frac{2}{R_2} & \text{if } x \in \left\{ w_{2n+1} + \frac{2^{\frac{d}{4}+1-1}}{(\frac{d}{4}+1)n^{\frac{d}{4}}} \frac{t}{R_1^d} < x_1 < w_{2n+2} + \frac{1}{n^{\frac{d}{4}}} \frac{t}{R_1^d} \right\} \cap \tilde{A}_2, \\
\frac{1}{R_2} & \text{if } x \in \left\{ w_{2n+1} + \frac{2^{\frac{d}{4}+1-1}}{(\frac{d}{4}+1)n^{\frac{d}{4}}} \frac{t}{R_1^d} < x_1 < w_{2n+2} + \frac{1}{n^{\frac{d}{4}}} \frac{t}{R_1^d} \right\} \cap \tilde{A}_2, \\
0 & \text{if } x \in (A_{R_1} \cap (Y_1 e_1 + A_{R_2}))^C \hspace{1cm}
\end{cases}$$

where $0 < r_1 < R_1 < \tilde{R}_1, 0 < r_2 < R_2 < \tilde{R}_2, \tilde{A}_2 = Y_1 e_1 + A_{R_2}$. Now we can choose $R_1, R_2$ large enough with $R_2 > R_1$ and corresponding $r_1, r_2$ such that $t_0^2 > t_0^1 + 1$ holds.

Now by similar calculation as in step 1 we can show the $B^{s,p,\theta}$ blow up in any time $t \in (0, t_0^2)$.

**Step 3.** Now with the help of step 1 and step 2 we will construct an initial data $u_0$ in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for which the corresponding entropy solution will give $B^{s,p,\theta}_{loc}$ blow up for all time.

For that purpose let’s define for each $k \in \mathbb{N}$, $w_m^k = Y_k + \sum_{j=N_k}^{m} l_j$ for $m \geq N_k$, $Y_0 = 0$ and $X_{k+1} = Y_k + \sum_{j=N_k}^{\infty} l_j$ and $A_{k+1} = Y_k e_1 + A_{R_{k+1}}$ where $R_k > X_k$ and $Y_k > X_k + R_k$ will be chosen later.
Now define

\[
u_0(x) = \begin{cases} 
0 & \text{if } x \in A_1 \cap \{x_1 < w_N\}, \\
\sigma_m(R_1) & \text{if } x \in A_1 \cap \{w_m < x_1 < w_{m+1}\}, \text{ for } m \geq N_1 \\
\frac{1}{R_1^{1/4}} & \text{if } x \in A_1 \cap \{x_1 > X_1\}, \\
0 & \text{if } x \in A_k \cap \{x_1 < w^{k}_{N_k}\} \text{ for } k \geq 2, \\
\sigma_m(R_k) & \text{if } x \in A_k \cap \{w^m_m < x_1 < w^m_{m+1}\} \text{ for } m \geq N_k, \ k \geq 2, \\
\frac{1}{R_k^{1/4}} & \text{if } x \in A_k \cap \{x_1 > X_k\} \text{ for } k \geq 2, \\
0 & \text{if } x \in \left( \bigcup_{k=1}^{\infty} A_k \right)^c. 
\end{cases}
\]

Note that we can choose \( R_k \in \mathbb{N} \) and \( R_{k+1} > R_k \). Since for each \( k \in \mathbb{N} \), \( \|u_0\|_{L^\infty(A_k)} \leq \frac{1}{R_k^{1/4}} \) and volume of \( A_k \) is \( R_k^d \) we get \( \|u_0\|_{L^1(\mathbb{R}^d)} \leq \sum_{k=1}^{\infty} \frac{R_k^d}{R_k^{1/4}} < \infty \). Hence \( u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \).

As we have already seen in step 2 that for a pair \((X_1, Y_0, R_1)\) we can choose another pair \((X_2, Y_1, R_2)\) such that there exists \( t_0^2 > t_0^1 + 1 \) and for \( t \in (0, t_0^1) \) no characteristic from \( A_1 \) meets any characteristic from \( A_2 \). Now in similar process we can choose \((X_{k+1}, Y_k, R_{k+1})\) for \( k \in \mathbb{N} \) such that for any \( t \in (0, t_0^k) \) no characteristic from the region \( \bigcup_{j=1}^{k} A_j \) will not...
intersect with any characteristic from the region \( A_{k+1} \) and \( t_{0}^{k+1} > t_{0}^{k} + 1 \) holds. This makes sure that for any \( k_{0} \in \mathbb{N} \) the entropy solution will look like (see figure 3).

\[
\frac{2}{R_{k} n^{m}} \quad \text{if} \quad x \in \left\{ w_{2n}^{k} + \frac{2^{k+1}}{n^{m+1} R_{k}^{m+1}} \frac{t}{R_{k}^{m+1}} < x < w_{2n+1}^{k} + \frac{2^{k+1}}{(n+1)^{m+1} R_{k}^{m+1}} \frac{t}{R_{k}^{m+1}} \right\} \cap \hat{A}_{k},
\]
\[
\frac{1}{R_{k} n^{m}} \quad \text{if} \quad x \in \left\{ w_{2n+1}^{k} + \frac{2^{k+1}}{(n+1)^{m+1} R_{k}^{m+1}} \frac{t}{R_{k}^{m+1}} < x < w_{2n+2}^{k} + \frac{1}{n^{m+1} R_{k}^{m+1}} \frac{t}{R_{k}^{m+1}} \right\} \cap \hat{A}_{k},
\]
\[
0 \quad \text{if} \quad x \in \left( \bigcup_{k=1}^{\infty} \text{Y}_{k-1} e_{1} + A_{R_{k}} \right)^{c},
\]

(2.17)

for \( t \in (0, t_{k_{0}}) \) and \( 0 < r_{k} < R_{k} < R \), \( \hat{A}_{k} = \text{Y}_{k-1} e_{1} + A_{r_{k}} \) for all \( k \geq k_{0} \).

Since \( t_{0}^{k+1} > t_{0}^{k} + 1 \) we have \( t_{0}^{k} \to \infty \) as \( k \to \infty \). Hence for any \( t > 0 \) there exists a \( k_{0} \in \mathbb{N} \) such that \( t < t_{0}^{k_{0}} \) holds. Hence at time \( t \) the entropy solution \( u(x, t) \) will look like (2.17). By the similar calculation that we have done in step 1 we can show that \( |u(., t)|_{B_{t}, p, R_{t}(A_{k_{0}})} = \infty \). This completes the proof of the proposition. \( \square \)

Now we will present the proof of Proposition 1.2. Idea is almost same as in Proposition 1.1 but choice of the linear function \( M \) and corresponding sequences are different. We describe the key steps and omit similar calculations.

**Proof of Proposition 1.2** Assume the fluxes \( f_{1}, f_{2} \) satisfy

\[
f_{1}(u) = \frac{(u^{2} + 1)^{2}}{4} \quad \text{and} \quad f_{2}(u) = \frac{u^{2}}{2}, \quad \text{i.e.} \quad f_{1}'(u) = u^{3} + u \quad \text{and} \quad f_{2}'(u) = u.
\]

Let \( \gamma = \frac{1}{3} + \epsilon \) and \( (b_{m})_{m \geq 2}, (\lambda_{m})_{m \geq 2} \) be two real number sequences defined as

\[
b_{m} = \frac{1}{n^{3} \gamma} \quad \text{for} \quad n = \left\lfloor \frac{m}{2} \right\rfloor,
\]
\[
\lambda_{m} = \lambda_{m}(R) = \begin{cases} \frac{2}{R_{0}^{\gamma}} & \text{for} \quad m = 2n, \\ \frac{1}{R_{0}^{\gamma}} & \text{for} \quad m = 2n + 1 \\ \end{cases}
\]

and this \( \epsilon(> 0) \) will be chosen later and \( \left\lfloor . \right\rfloor \) is the box function.

Let’s define for each \( k \in \mathbb{N}, w_{m}^{k} = Y_{k} + \sum_{j=N_{k}}^{m} b_{j} \) for \( m \geq N_{k}, Y_{0} = 0 \) and \( X_{k+1} = Y_{k} + \sum_{j=N_{k}}^{\infty} b_{j} \) and \( B_{k+1} = Y_{k} e_{1} + A_{R_{k+1}} \) where \( R_{k} > X_{k} \) and \( Y_{k} > X_{k} + R_{k} \) for \( k \geq 1 \) will be
chosen later. Now define

\[
    u_0(x) = \begin{cases} 
        0 & \text{if } x \in B_1 \cap \{x_1 < w_N\}, \\
        \lambda_m(R_1) & \text{if } x \in B_1 \cap \{w_m < x_1 < w_{m+1}\}, \text{ for } m \geq N_1 \\
        \frac{1}{R_k} & \text{if } x \in B_1 \cap \{x_1 > X_1\}, \\
        0 & \text{if } x \in B_k \cap \{x_1 < w^k\} \text{ for } k \geq 2, \\
        \lambda_m(R_k) & \text{if } x \in B_k \cap \{w^m_m < x_1 < w^k_{m+1}\} \text{ for } m \geq N_k, k \geq 2, \\
        \frac{1}{R_k} & \text{if } x \in B_k \cap \{x_1 > X_k\} \text{ for } k \geq 2, \\
        0 & \text{if } x \in \bigcup_{k=1}^{\infty} B_k^c.
    \end{cases}
\]

Note that we can choose \( R_k \in \mathbb{N} \) and \( R_{k+1} > R_k \). By similar argument as given in step 3 of Proposition 1.1 we can show \( u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \). Now in similar process as done in step 3 of Proposition 1.2 we can choose \((X_{k+1}, Y_k, R_{k+1})\) for \( k \in \mathbb{N} \) such that for any \( t \in (0, t_0^k) \) no characteristic from the region \( \bigcup_{j=1}^k B_j \) will not intersect with any characteristic from the region \( B_{k+1} \) and \( t_0^{k+1} > t_k^k + 1 \). This makes sure that for any \( k_0 \in \mathbb{N} \) the entropy solution will have structure like

\[
    u(x, t) = \begin{cases} 
        \frac{2}{R_k \alpha} & \text{if } x \in \left\{ \frac{w^k_{2n} + \frac{84}{n^6 R_k}}{n^6 R_k} < x_1 < \frac{w^k_{2n+1} + \frac{15t}{n^3 R_k}}{n^3 R_k^2} \right\} \cap \tilde{B}_k \\
        \frac{1}{R_k \alpha} & \text{if } x \in \left\{ \frac{w^k_{2n+1} + \frac{15t}{n^3 R_k}}{n^3 R_k^2} < x_1 < \frac{w^k_{2n+2} + \frac{t}{n^3 R_k}}{n^3 R_k^2} \right\} \cap \tilde{B}_k \\
        0 & \text{if } x \in \bigcup_{k=1}^{\infty} Y_k, t_{k+1} + A_{R_k}^c,
    \end{cases}
\]

for \( t \in (0, t_0^k) \) and \( 0 < r_k < R_k < R, \tilde{B}_k = Y_{k-1}e_1 + A_{r_k} \) for all \( k \geq k_0 \). Again similar calculation shows that for any time \( t > 0 \), there exists \( k_0 \in \mathbb{N} \) such that \(|u(., t)|_{B^{s,p}(B_{k_0})} = \infty\). This completes the proof of the proposition.

**Proof of the Remark 1.2 (Sketch)**. Suppose \( f \) is a flux function satisfying

\[
    |\{w \in \mathbb{R}| \xi \cdot (a(w) - a(v))| < \delta\}| < \delta^\alpha,
\]

where \( a(u) = f'(u) \). By our assumption \( \alpha \) is attained at direction \( \xi_0 \) in an interval \( I \) and \( h(u) = \xi_0 \cdot a(u) \) is increasing in this interval \( I \). This is equivalent to the following

\[
    \inf_{w, w' \in I, w \neq w'} \frac{|h(w) - h(w')|}{|w - w'|^{\frac{1}{\alpha}}} > 0. \tag{2.18}
\]

Since \( \alpha \in (0, 1) \) is the smallest number such that (2.18) holds, for any \( \gamma > \alpha \) we can get an increasing sequence \((a_k)_{k \geq 1}\) in \( I \) such that \(|h(a_{k+1}) - h(a_{k-1})| < |a_{k+1} - a_{k-1}|^{\frac{1}{\alpha}}\) and \( a_{k+1} - a_{k-1} = \frac{1}{q_k} \) with another increasing real number sequence \( q_k \geq 2 \) hold for each \( k \geq 1 \). Let \( g : \mathbb{R} \to \mathbb{R} \) be defined as \( g(u) = \xi_0 \cdot f(u) \). Since \( g'(u) = h(u) \) holds, \( g \) is convex in that neighbourhood. We also get

\[
    g'(a_{k-1}) < \frac{g(a_{k+1}) - g(a_{k-1})}{a_{k+1} - a_{k-1}} < g'(a_{k+1}), \quad g'(a_{k+1}) - g'(a_{k-1}) \leq \frac{1}{a_{k+1} - a_{k-1}}.
\]


Define $N_k = \left\lceil \frac{1}{(a_{k+1} - a_{k-1})^{\frac{1}{2}} k^{1+\epsilon}} \right\rceil$ and $J_k = \sum_{j=1}^{k} N_j$ for all $k \in \mathbb{N}$, $J_0 = 0$ and $\epsilon(>0)$ will be chosen later. Let $(l_m)_{m \geq 2}$, $(\sigma_m)_{m \geq 2}$ be two real number sequences defined as

$$\sigma_m = \begin{cases} a_{k+1} & \text{if } m = 2n, \quad J_{k-1} < n \leq J_k, \quad k \geq 1 \\ a_{k-1} & \text{if } m = 2n + 1, \quad J_{k-1} < n < J_k, \quad k \geq 1 \end{cases}$$

$$l_m = (a_{k+1} - a_{k-1})^{\frac{1}{2}} \text{ if } J_{k-1} < \left[ \frac{m}{2} \right] \leq J_k, \quad k \geq 1.$$

Now consider $M(x) = \xi_0 \cdot x$ and define $w_m = \sum_{j=N}^{m} l_j$ for $m \geq N$, where this $N$ will be chosen later. Let $w_k \to w_0$ as $k \to \infty$. Now we are all set to define an initial data $u_0$ as

$$u_0(x) = \begin{cases} 0 & \text{if } x \in \{ M(x) < w_N \} \cap A_R, \\
\sigma_m & \text{if } x \in \{ w_m < M(x) < w_{m+1} \} \cap A_R, \\
\frac{1}{p_i} & \text{if } x \in \{ M(x) > w_0 \} \cap A_R, \\
0 & \text{if } x \in A_{R_1}^c, 
\end{cases}$$

where $R, q > 0$ are large numbers. Again by similar argument as in step 1 of Proposition 1.1 we can say there exists a time $t_0$ such that no characteristic from the set $\{ M(x) = w_{2n} \}$ and $\{ M(x) = w_{2n+2} \}$ will meet the discontinuity plane starting from $\{ M(x) = w_{2n+1} \}$ before time $t_0$. Hence the entropy solution will look like

$$u(x, t) = \begin{cases} a_{k+1} & \text{if } x \in \left\{ w_{2n} + g'(a_{k+1}) t < M(x) < w_{2n+1} + \frac{g(a_{k+1}) - g(a_{k-1})}{a_{k+1} - a_{k-1}} t \right\} \cap A_R, \\
\sigma_m & \text{if } x \in \left\{ w_{2n+1} + \frac{g(a_{k+1}) - g(a_{k-1})}{a_{k+1} - a_{k-1}} t < M(x) < w_{2n+2} + g'(a_{k-1}) t \right\} \cap A_R, \\
0 & \text{if } x \in A_{R_1}^c, 
\end{cases}$$

for $0 < r < R < R_1$. There exists $1 \leq i \leq d$ such that $M(e_i) \neq 0$. Then by similar calculation as done in Proposition 1.1 we can show for $0 < h < C_n(a_{n+1} - a_{n-1})^{\frac{1}{2}}$,

$$||\Delta_t^h u(\cdot, t)||_{L^p(\mathbb{R}^d)}^p = C \sum_{k=N}^{n} \sum_{j=1}^{N_k} (a_{k+1} - a_k)^{\frac{1}{2}} (a_{k+1} - a_{k-1})^p \geq C \sum_{k=N}^{n} N_k (a_{k+1} - a_k)^{\frac{1}{2}} (a_{k+1} - a_{k-1})^p \geq C \sum_{k=N}^{n} \frac{(a_{k+1} - a_{k-1})^p}{k^{1+\epsilon}} \geq C \frac{(n-N)(a_{n+1} - a_{n-1})^p}{n^{1+\epsilon}}.$$ 

where this $c_n$ depends on time $t$. 

15
Proof of Lemma 2.1 Let \( u(x, t) \) be the entropy solution of \((1.1)\) with the initial data

\[
u_0(x) = \begin{cases} & a \text{ if } M(x) < 0, \\ & b \text{ if } M(x) > 0 \end{cases}
\]

and flux having the same property as mentioned in this lemma. Now there are two possibilities

i) When \( a > b \), consider \( x_0, y_0 \in \mathbb{R}^d \) such that \( M(x_0) < 0 \) and \( M(y_0) > 0 \). Then equations of characteristics from \( x_0, y_0 \) are \( x = x_0 + f'(a)t \), \( y = y_0 + f'(b)t \) respectively. Note that for \( x_0, y_0 \) with the property \( x_0 - y_0 = (f'(b) - f'(a))t \) those two characteristics intersect at some time \( t_1 > 0 \). This means that the solution has discontinuities.
Let the discontinuity surface be \( S(t, x) = 0 \). Then \( S(t, x) \) is the solution to the following problem
\[
(b - a)S_t + (f(b) - f(a)) \cdot \nabla_x S = 0, \quad (3.19)
\]
\[
S(0, x) = M(x) \forall x \in \mathbb{R}^d. \quad (3.20)
\]
Notice that equation (3.19) comes from Rankine-Hugoniot condition. Also note that \( (t, x) \mapsto M(x - \gamma t) \) is the unique solution of (3.19), (3.20).

Let’s define \( u(x, t) \) as
\[
u(x, t) = \begin{cases} a & \text{if } M(x - \gamma t) < 0, \\ b & \text{if } M(x - \gamma t) > 0. \end{cases}
\]

From the fact that \( u \mapsto M(f(u)) \) is convex therefore it is clear that \( u(x, t) \) satisfies Kruzkov entropy condition
\[
\int_0^\infty \int_{\mathbb{R}^d} |u - k| \frac{\partial \phi}{\partial t} + \text{sgn}(u - k)(f(u) - f(k)) \cdot \nabla_x \phi \, dx dt \geq 0,
\]
for all \( \phi \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}_+) \) and for any \( k \in \mathbb{R} \). Hence \( u(x, t) \) is the entropy solution. This completes the proof of the lemma.

(ii) When \( a < b \), consider \( x_0, y_0 \in \mathbb{R}^d \) such that \( M(x_0) < 0 \) and \( M(y_0) > 0 \). Then equations of characteristics from \( x_0, y_0 \) are \( x = x_0 + f'(a)t, \quad x = y_0 + f'(b)t \) respectively. If they meet for some \( t_2 \) then \( x_0 - y_0 = (f'(b) - f'(a))t_2 \) holds. Now consider the images of both the sides under the map \( M \) and from the linearity of the map \( M \) we get
\[
M(x_0) - M(y_0) = M(f'(b)) - M(f'(a))
\]
which gives us contradiction since LHS is \(< 0 \) but RHS is \( \geq 0 \). Hence characteristics will not intersect and solution \( u(x, t) \) will behave like (see Figure I)
\[
u(x, t) = \begin{cases} a & \text{if } M(x - f'(a)t) < 0, \\ b & \text{if } M(x - f'(b)t) > 0. \end{cases}
\]

Now if we consider our initial data like as mentioned in the lemma due to the finite speed of propagation, for any \( 0 < r_1 < r \) there exists \( t_0 > 0 \) such that any characteristic from \( A_x \) will not intersect with any characteristic from \( A_r \) for \( t \in (0, t_0) \). Finite speed of propagation also ensures that there exists \( 0 < r_2 = r_2(t_0, f, \max(|a|, |b|)) \) such that \( u(x, t) = 0 \forall x \notin A_{r_2} \) and \( \forall t \in (0, t_0) \). This completes the proof of the lemma.

**Proof of Lemma 2.2** Consider the Taylor’s expansion for \( x > 0 \),
\[
(x + 1)^\beta = x^\beta + \beta \frac{(1 - \beta)}{x^{1-\beta}} + \beta \frac{(1 - \beta)(2 - \beta)}{x_0^{3-\beta}}, \quad \text{where } x_0 \in (x, x + 1).
\]
Since \( \beta \in (0, 1) \) we have
\[
(x + 1)^\beta \geq x^\beta + \beta \frac{(1 - \beta)}{x^{1-\beta}}.
\]
Similarly we can show the other inequality. \( \square \)
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