Multidimensional tunneling between potential wells at non degenerate minima

Anatoly ANIKIN
Moscow Institute of Physics and Technology, Moscow, Russia; e-mail: anikin83@inbox.ru

Michel ROULEUX
Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France, Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France; e-mail: rouleux@univ-tln.fr

We consider tunneling between symmetric wells for a 2-D semi-classical Schrödinger operator for energies close to the quadratic minimum of the potential $V$ in two cases: (1) excitations of the lowest frequency in the harmonic oscillator approximation of $V$; (2) more general excited states from Diophantine tori with comparable quantum numbers.

1 Tunneling between double wells: a short review

Tunneling for Schrödinger type operators involves various scenarios which depend on the details of the dynamics, ranging from integrable or quasi-integrable systems, to ergodic or chaotic ones.

Assume that $V$ is a smooth function, symmetric with respect to $\{x_1=0\}$, and $\{V(x) \leq E\}$ consists in 2 connected components $\{U_L\} \cup \{U_R\}$ (the potential wells), while $\limsup_{|x| \to \infty} V > E$. We are interested in the semi-classical spectrum of Schrödinger operator $P = -\hbar^2 \Delta + V$ on $L^2(\mathbb{R}^2)$ near energy $E$, which consists in pairs $E^\pm(h) = E^\pm_k(h)$ exponentially close to eigenvalues $E(h) = E_k(h)$ of the Dirichlet realization of $P$ in some neighborhood of a single well. We will always assume $\{U_L\} \cup \{U_R\}$ that $E(h)$ are simple (non degenerate) and asymptotically simple. As a general rule, the energy shift $\Delta E(h) = E^+(h) - E^-(h)$ (or splitting of eigenvalues) is related to so called Agmon distance $S(E)$ between the wells, associated with the degenerate, conformal metric $ds^2 = (V-E)^+ dx^2$ that measures the life-span of the particle in the classically forbidden region $V(x) \geq E$. Much is known in the 1-D case, even for excited states, or in several dimensions for the lowest eigenvalues.

At the higher level of generality, we only require that $V(x) \neq 0$ on $\{V = E\} = \partial U_L(E) \cup \partial U_R(E)$. In the 1-D case, Landau-Lifshitz formula reads

$$\Delta E(h) = 2\frac{\omega h}{\pi} e^{-(S(E)/\hbar)h(1+o(1))}$$  \hspace{1cm} (1)

where $\omega = \frac{\partial V}{\partial p}$ is the frequency of the periodic orbit at energy $E$, and $2S(E) = 1 = 2\pi^{-1} \frac{1}{2}(E-V)^+ dx$. In higher dimensions, the structure of the classical flow plays an essential rôle, so that we are left with the following equivalence (see [15] for a precise statement): Assume $V$ is analytic. Then the splitting $\Delta E(h)$ is non exponentially small with respect to Agmon distance (i.e. for all $\varepsilon > 0$, larger than a constant times $e^{-(S(E)+\varepsilon)/h}$, $0 < h \leq h_c$) iff the eigenfunctions of $P$, with eigenvalues $E^\pm(h)$, are non exponentially small (i.e. for all $\varepsilon > 0$, larger, in local $L^2$ norm, than a constant times $e^{-\varepsilon/h}$, $0 < h \leq h_c$) in an open set where minimal geodesics, connecting the 2 wells, meet their boundary.

Here are two propositions hold true. Let $V$ have non degenerate minima $a_{L/R}$ with $V(a_{L/R}) = 0$, and $V_0 = \sum \lambda_j^2\tilde{z}_j^2$, $\lambda_1 < \lambda_2$ be the harmonic approximation (in local coordinates $z$) around $a_{L/R}$ and $p_0(x,\xi) = \xi^2 + V_0$, the quadratic part of $p(x,\xi)$ near 0.

In 1-D the splitting between the lowest eigenvalues is found to be

$$\Delta E(h) = 2\sqrt{\frac{\pi \omega h}{\epsilon}} e^{-S_h/\hbar(1+o(1))} \hspace{1cm} (2)$$

$\omega = \lambda_1$ is the harmonic frequency, and $S_h$ half the action of the periodic orbit for the Hamiltonian with reversed potential $q = \xi^2 - V$ at energy $E = \omega h/2$. For higher energies we have

$$\Delta E_m(h) = 2b_m\frac{\omega h}{\pi} e^{-S(E)/\hbar(1+o(1))},$$

where

$$E = (2m+1)\omega h, \hspace{1cm} b_m = \sqrt{\frac{\pi(2m+1)^{m+1/2}}{2^{m}m!e^{m+1/2}}} \hspace{1cm} (3)$$
so long $mh \leq c$, $c > 0$ small enough, which somehow “interpolates” between (??) and (??) since $b_m \to 1$ as $m \to \infty$.

In several dimensions, the splitting between the two lowest eigenvalues $[6, 1, 2]$ is again of the form

$$\Delta E(h) = 2\frac{\sqrt{\pi} \lambda_1 h}{e} e^{-S_\lambda/h} (1 + o(1))$$

Further, such formulas hold between any low-lying eigenvalues, i.e. for any $N$, there is $h_N > 0$ such that for each principal quantum number $m \leq N$, the splitting $\Delta E_m(h)$ has an asymptotic of the form $\Delta E_m(h) \sim d_m(h) e^{-S_\lambda/h}$ provided $0 < h < h_N$ [11, 12]. See also [16] for degenerate minima.

In this report we restrict our attention to KAM states, i.e. supported near a Diophantine torus and with quantum numbers $(k_1, k_2)$ such that $|k| \leq c$, or semi-excited states in the limit $c \to 0$, i.e. when $|k| \to \infty$ and $h \to 0$ are related by $|k|h \leq h^\delta$, $0 < \delta < 1$. Further we shall only consider states (or approximate eigenfunctions) microlocalized on isotropic (generally Lagrangian) manifolds whose analytic continuation in the momentum space (i.e. in the classically forbidden region) are in a generic position. Lagrangian manifolds of 2 types are relevant to our analysis: (1) the flow-out of the boundary of the wells (2) the quasi-invariant tori making a local fibration of the energy surface inside the wells. They have a (singular) limit as $E \to 0$.

2 Energy surfaces and librations

The Lagrangian manifolds of the first type are the integral manifold of $q$ passing above $(\partial U_E)_{L/R}$. From now on we assume that in local coordinates near $a_{L/R}$, $p(x, \xi) = p_0(x, \xi) + O(|z|^2)$. Consider first a single well $U_E$ then locally

$$\Lambda^E = \{ \exp(iH_q(p) : p \in \partial U_E \times 0, q(p) = -E, t \in \mathbb{R} \}$$

is a smooth real Lagrangian submanifold of the form $\xi = \pm \nabla E(x)$, $x \notin U_E$, with a fold along $\partial U_E$. Here $d_E(x) = d_E(x, \partial U_E)$ is Agmon distance from $x$ to $\partial U_E$ and satisfies (locally) the eikonal equation $(\nabla E(x))^2 = V(x) - E$. As $E \to 0$, $\Lambda^E$ tends to the union of the outgoing/incoming Lagrangian manifolds $\Lambda^\pm$ (called separatrices in 1-D) with a conical intersection at the origin.

We shall assume that $(\Lambda^E)_{L/R}$, as integral manifolds of Hamiltonian flow, extend away from the wells as Lagrangian manifolds intersecting in the energy surface $\{q(p) = -E\}$ along a curve $\gamma_E$.

This curve projects onto $\mathbb{R}^2$ precisely as a libration $\text{Lib}_E$ between $U_L(E)$ and $U_R(E)$, i.e. a periodic orbit with end points at $\partial U_{L/R}(E)$ [3]. We assume for simplicity there is exactly one such family of curves. We call also $\text{Lib}_E$ a minimal geodesic between $U_L(E)$ and $U_R(E)$ for Agmon distance $ds^2 = \sqrt{V(x) - E} \, dx^2$. Assuming PT symmetry (i.e. $V$ symmetric with respect to $\{x_1 = 0\}$), we denote by $\{x_1 \} = \text{Lib}_E \cap \{x_1 = 0\}$.

Then $d_E(x_1, U_L^E) = d_E(x_1, U_R^E) = S_E/2$, and $\text{Lib}_E$ intersects $\{x_1 = 0\}$ at $x_1$ with a right angle. A neighborhood of $x_1$ in $\{x_1 = 0\}$ can be thought of as Poincaré section, intersecting $\gamma_E$ transversally. The $\gamma_E$ are (unstable) periodic orbits of hyperbolic type, with real Floquet exponent $\beta(E)$. Of course, because of focal points, $(\Lambda^E)_{L/R}$ doesn’t extend smoothly everywhere but only in a neighborhood of librations when the system is not integrable.

As $E \to 0$ the libration degenerates to an instanton $\gamma_0$. Parametrized as a biharmonic of $q(x, \xi)$ at $E = 0$, it takes an infinite time to reach the equilibria $a_L$ or $a_R$ along $\gamma_0$. We shall assume that the stable outgoing and incoming manifolds $\Lambda^E_{L/R}$ at 0 intersect transversally at $\gamma_0$.

3 Quasi-invariant Liouville tori

Lagrangian manifolds of the second type are the invariant tori foliating (locally) the energy surface in the integrable case, or KAM tori, or corresponding quasi-invariant tori in the quasi-integrable case. In the Section 6, we shall also allow these Lagrangian manifolds to shrink to periodic orbits.

We can have already a good insight into the problem in replacing $V$ by its quadratic approximation. This is what we call the model case. When frequencies $\lambda_j$ are rationally independent, we can essentially reduce to the model case by resorting to Birkhoff normal forms (or KAM theorem).

So assume for simplicity that $p = p_0$ near $a_{L/R}$. Then for small $E > 0$, the energy surfaces are foliated by invariant tori $\Lambda_E$, $E = 2\lambda_1t_1 + 2\lambda_2t_2$ which can be extended in the complex domain along complex times, e.g. as integral leaves $\Lambda_E$ of $q(x, \xi) = \lambda^2 - \lambda_1^2 z_1^2 - \lambda_2^2 z_2^2$, with purely imaginary time.

The caustics of $\Lambda_E$ can be viewed as a rectangle shaped fold line delimiting the zone of pure oscillations of the quasi-modes, and touching the boundary of the wells $\partial U_E, E = 2\lambda_1t_1 + 2\lambda_2t_2$ at 4 vertices, the hyperbolic umbilic points (HU) points, section of the torus by the plane $\xi = 0$ in $\mathbb{R}^3$. We
We say also that if there are tunnel bicharacteristic $q$ with clean intersection. A Diophantine condition on cycle $\partial U$ can identify $y$ with $\iota$. At the umbilic $y$, we have $T_y\tilde{\Lambda}_i = T_y\Lambda_i = T_y(\text{fiber})$, $T_y\Lambda_i \cap T_y\Lambda_\tilde{E} = RH_y$, where $E = 2\lambda_1\iota_1 + 2\lambda_2\iota_2$. More generally tori $\Lambda_i$ continue analytically in the $\xi$ variables as a multidimensional Riemann sheet structure, with a number of sheets corresponding to the choice of the sign of momentum, glued along the caustics, and all intersecting at the HU’s. On the other hand, $\Lambda_\tilde{E}$ has the fibre bundle structure $\Lambda_\tilde{E} = \bigcup_{\gamma \in \Omega_E} \gamma_y$ where $\gamma_y$ is the bicharacteristic of $q(x, \xi)$ at energy $-E$ issued from $\partial U_E$ at the point $y$. We have

$$\gamma_y = \tilde{\Lambda}_i \cap \Lambda_\tilde{E}, \quad E = 2\lambda_1\iota_1 + 2\lambda_2\iota_2$$

(4)

with clean intersection.

Of course, in the general case (not model case), tori $\Lambda_i$ or $\tilde{\Lambda}_i$ make only sense as asymptotic objects (via Birkhoff normal form) because they are not invariant under the Hamilton vector flow. Assuming a Diophantine condition on $\lambda_1/\lambda_2$ we can also select a dense family of such invariant tori.

4 THE TUNNEL CYCLE AND TUNNEL BICHARACTERISTICS

If the system were integrable near 0, because of PT symmetry, the extension of $(\tilde{\Lambda}_i)_L$ would usually coincide with $(\Lambda_i)_R$, the decaying branch of $(\Lambda_i)_R$. For a general, non integrable system, there is no reason for this holds and $\tilde{\Lambda}_L$ intersects $\Lambda_R$ along a one dimensional manifold.

Definition 1 Assume again there is only one libration $\text{Lib}_E$. We call the lift $\gamma_E$ of $\text{Lib}_E$ the tunnel cycle. We call the bicharacteristic $\tilde{\gamma} \subset q^{-1}(-E)$ a tunnel bicharacteristic if there are $\rho_L, \rho_R \in \tilde{\gamma}$, with $E = 2\lambda_1\iota_1 + 2\lambda_2\iota_2$ and $\rho_L \in (\tilde{\Lambda}_i)_L$, $\rho_R \in (\tilde{\Lambda}_i)_R$. We say also that $\rho_L, \rho_R$ are in correspondence along $\tilde{\gamma}$.

The tunnel cycle is a tunnel bicharacteristic for which $\rho_L, \rho_R$ are umbilics, but it carries generally no interaction between wells, unless $\rho_L, \rho_R$ belong to quantized tori. But in a small, $h$-dependent neighborhood of $\gamma_E$ there are tunnel bicharacteristics that carry interaction between wells (but generally do not close). Non degeneracy of the tunnel cycle then implies the following:

**Proposition 1** Consider the model case. When $E = 2\lambda_1\iota_1 + 2\lambda_2\iota_2$ we have

$$\gamma_E = (\Lambda_\tilde{E})_L \cap (\Lambda_\tilde{E})_R = (\tilde{\Lambda}_i)_L \cap (\tilde{\Lambda}_i)_R$$

(5)

with a clean intersection.

It follows from (??) that along the tunnel cycle $\text{Lib}_E$ we have simultaneously $\gamma_E = (\Lambda_\tilde{E})_L \cap (\Lambda_\tilde{E})_R = (\tilde{\Lambda}_i)_L \cap (\tilde{\Lambda}_i)_R$ and $\gamma_E = (\Lambda_i)_L \cap (\Lambda_i)_R = (\tilde{\Lambda}_i)_L \cap (\tilde{\Lambda}_i)_R$ with clean intersections.

Unlike $\partial U_E$, the caustics of $\Lambda_\tilde{E}$ which is a smooth set, the caustics of $\tilde{\Lambda}_i$ issued from $y$ is a stratified set consisting of the umbilic $y$, and lines $C_1(y)$, $C_2(y)$ tangent at $y$ to the principal directions of $V''$. These caustics sets are the envelopes of Lissajous figures, whose lifts are (real) bicharacteristics of $q$.

Non degeneracy of the tunnel cycle $\gamma_E$ implies also the following splitting from (??):

**Claim 1** Let $\gamma_E$ be a minimal tunnel cycle, with end points $y_{L/R}^E$, intersecting $\{x_1 = 0\}$ at $\Omega_E$, with $x_E = \pi(\Omega_E)$. For $y \in \mathbb{R}^2$ close to $y_{L/R}^E$, let $E(y) = V(y)$ and $\tilde{\Lambda}_{(i)}(y)$ denote the Lagrangian manifold as above with $HU y$. Then for all $y$ close enough to $y_{L/R}^E$, we have:

1) $(\Lambda_{\tilde{E}})_{L} \cap (\Lambda_{\tilde{E}})_{R}$ is a curve $\gamma(y)$ whose projection is the libration $\text{Lib}_{E(y)}$, that intersects the caustics $\partial U_{E(y)}$ of $\Lambda_{\tilde{E}}$ at some $y'(y)$ (both for $L$ and $R$).

2) $(\tilde{\Lambda}_{(i)}(y))_{L} \cap (\tilde{\Lambda}_{(i)}(y))_{R}$ is a tunnel bicharacteristic $\tilde{\gamma}(y)$ transverse to $\pi^{-1}(\{x_1 = 0\})$, $\tilde{\gamma}(y) \cap \pi^{-1}(\{x_1 = 0\}) = \{\tilde{\sigma}(y)\}$, and $\pi(\tilde{\gamma}(y))$ intersects orthogonally $\{x_1 = 0\}$ at $\tilde{x}(y) = \pi(\tilde{\sigma}(y))$. Moreover $\tilde{\gamma}(y)$ projects at some $\tilde{\rho}(y) \in \Lambda_{(i)}(y)$ to $\tilde{y}(y)$ tangentially to the caustics $C(y)$ (both for $L$ and $R$).

Thus $\gamma_E$, which was common to both $(\Lambda_{\tilde{E}})_{L} \cap (\Lambda_{\tilde{E}})_{R}$ and $(\tilde{\Lambda}_{(i)}(y))_{L} \cap (\tilde{\Lambda}_{(i)}(y))_{R}$, splits into 2 distinct curves: (1) the lift of the libration at energy $E(y)$, (2) a tunnel bicharacteristic passing through the regular part of $C(y)$. Because the action along $\tilde{\gamma}(y)$ gives the tunneling rate when $\Lambda_{(i)}(y)$ supports a quasi-mode we introduce the:
Definition 2 The action \( \int_{y(y_{\omega})} \xi \, dx \) computed on \( \gamma(y) \) is called the tunnel distance between \( (\Lambda_j(y))_L \) and \( (\Lambda_j(y))_R \) (it equals Agmon distance when \( \gamma(y) = \gamma_E \)).

Let \( y \in \partial E(y) \). Integrating \( \xi \, dx \) along \( \gamma_y \) gives (locally) Agmon distance to the well:

\[
d_E(x) = \int_y^x \xi \, dx = \sum_j \lambda_j \int_{y_j}^{x_j} \sqrt{t^2 - y_j^2} \, dt, \quad x \in \gamma_y
\]

Denote by \( F^E_y(x) \) the RHS of this equation; provided \( y \in \partial U_E \) is not too close to both \( z \)-axis, one can show that \( F^E_y(x) - d_E(x) \) is estimated by the square of the (Euclidean) distance of \( x \) to its orthogonal projection on \( \gamma_y \), for \( x \) in a neighborhood of \( \text{Lib}_E \). Similarly, we consider variations from the regular part of the caustics \( C(y) \) inf \( \{ \int_0^1 (V(\gamma(s)) - E)^{1/2} \xi | \gamma(s) \rangle ds \} \), with \( (\gamma(0), \gamma(1)) \in T \mathcal{C}(y), \gamma(1) = x \), and write the critical value as \( G^E_{C(y)}(x) = \int_{y(x)}^x \xi \, dx \), or simply \( G^E_{C(y)}(x) = \int_{y(x)}^x \xi \, dx \). Again \( G^E_{C(y)}(x) = d_E(x) + \int_{y(x)}^x \xi \, dx = F^E_y(x) - d_E(x), \) where \( \int_{y(x)}^x \xi \, dx, \xi \in C(y) \) is a small error term essentially independent of \( x \) in a neighborhood of \( \text{Lib}_E \).

The next step consists in constructing quasi-modes. First we construct quasi-modes microlocalized on the \( \Lambda_i \) selecting a sequence \( \iota = i_k(h) \) from Bohr-Sommerfeld-Maslov (or EBK) quantization rules. As a rule, these (oscillating) quasi-modes extend in the shadow zone near \( y_k(h) \) with exponential decay. They can further be extended to \( u_L \) and \( u_R \) along \( \gamma(y_k(h)) \) using WKB expansions, or the “Gaussian beams” method. The eigenvalue splitting is given by the usual formula

\[
\Delta E_k(h) \sim 4L^2 h^2 \int_{\Sigma} u_L(0, x_2) \frac{\partial u_R}{\partial x_1}(0, x_2) \, dx_2
\]

where \( \Sigma \) is a neighborhood of \( x_E \) in \( \{ x_1 = 0 \} \). We now treat some specific cases in more detail.

5 TUNNELING NEAR A PAIR OF DIOPHANTINE TORI

Assume \( c > 0 \) is so small that KAM theory ensures existence of a family invariant tori in the well \( U_E = U_L(E) \) for \( E \leq c \). We are interested in \( \Delta E_k(h) \) for \( E_k(h) \) near such fixed \( E > 0 \). Assume that \( \text{Lib}_E \) starts at umbilic \( y_{E} \) away from the \( z \)-axis, and for simplicity, that \( y_E \in \Lambda_i \) with \( i \) in the KAM set, i.e. such that the motion on \( \Lambda_i \) is quasi-periodic with Diophantine frequency vector \( \omega \) (this assumption seems to be generic, varying slightly \( E \)). In [8], we proved the following: Let \( 0 < \delta < 1 \). Then in a \( h^{\delta/2} \)-neighborhood of \( \Lambda_i \) in \( T^{\ast}M \), there is a family \( \tilde{\Lambda}_i \) of tori, labelled by their action variables \( J = J_k(h) \) for \( k \in \mathbb{Z}^d \) satisfying \( |kh - \iota| \leq h^\delta \), which verify Bohr-Sommerfeld-Maslov quantization condition, and are quasi-invariant under \( H_p \) with an accuracy \( \mathcal{O}(h^\infty) \). At first approximation, the umbilics \( y_k(h) \in \tilde{\Lambda}_j \) have the form \( y \sim (\lambda_1^{-1} \sqrt{2\lambda_1 k_1}, \lambda_2^{-1} \sqrt{2\lambda_2 k_2}) \) or \( y \sim (\lambda_1^{-1} \sqrt{2h\lambda_1 k_1}, \lambda_2^{-1} \sqrt{2h\lambda_2 k_2}) \), \( k = (k_1, k_2) = k(h) \in \mathbb{N}^2 \) so the typical neighboring distance between \( y_k(h) \) is \( h^{E - 1/2} \) when \( y_E \) stays away from the \( z \)-axis. Using Maslov canonical operator, we obtain from these tori a sequence of quasi-modes for \( P \) near \( E \). By complex contour integrals ([9, 12]) they extend in a \( |h \log h|^{2/3} \)-neighborhood of \( U_E \), as states microlocalized on \( \tilde{\Lambda}_j \), and decaying exponentially as \( \exp[-F^E_y(x)/h] \), or \( \exp[-G^E_{C(y)}(x)/h] \). This decay propagates all along \( \tilde{\gamma}(y_k(h)) \) and nearby bicharacteristics, which stay in the purely decaying branch \( \tilde{\Lambda}_j \) of \( \Lambda_j \).

Next we need to compare the tunnel distance with Agmon distance which coincide only on the tunnel cycle. Let \( S_L - S_R \) be the tunnel action between \( y_L \) and \( y_R \), we have at \( \{ x_1 = 0 \} \) (see Fig.1)

\[
S_L - S_R - 2S_0(E) = 2(F^E_y(\tilde{x}(y)) - d_E(\tilde{x}(y))) + 2(d_E(y)(\tilde{x}(y)) - d_E(\tilde{x}(y))) + 2(d_E(\tilde{x}(y)) - d_E(x_E))
\]

Evaluating each error term on the RHS, we arrive at \( S_L - S_R - 2S_0(E) = o(1), h \to 0 \). Then \( S_L - S_R \) has a non degenerate critical point at \( \tilde{x}(y_k(h)) \) belonging to the tunnel bicharacteristic \( \tilde{\gamma}(y_k(h)) \) common to \( (\tilde{\Lambda}_j(h))_L \) and \( (\tilde{\Lambda}_j(h))_R \). The integral can be computed by standard stationary phase expansion around \( x_k(h) \). Since the amplitude of \( u_R \) (and \( u_L \)) is non vanishing, we obtain eventually [5]

\[
\Delta E_k(h) \sim B_k(h) e^{-(S_L - S_R)/h}
\]

with \( B_k(h) \sim \frac{h^{3/2}}{\sqrt{\tau_L h_R}} \). Here \( H_L/R \) are Hamilton vector fields transverse to \( \gamma_E \), and \( \tau_L/R \) suitable Jacobians computed on \( (\tilde{\Lambda}_j(h))_L/R \).

6 THE QUASI 1-D CASE

In this section we shall assume that frequencies \( \lambda_1, \lambda_2 \) are non-resonant, with \( 2\lambda_1 < \lambda_2 \), and the
instanton $\gamma_0$ approaches the node singularity of the outgoing and incoming manifolds $\Lambda^\pm_{L/R}$ at $a_{L/R}$ in a regular direction (associated with $\lambda_1$). We consider eigenstates with quantum vector $(m,0)$ for $m \in \mathbb{N}$, i.e. $E_m = h(\lambda_1(2m + 1) + \lambda_2) + O(h^2)$, and compute asymptotics for the energy splitting $\Delta E_m$ (as $h \to 0$, while $m$ stays fixed, and probably also when $hm \leq h^3$, $0 < h < 1$.) This amounts to let $\Lambda$, shrink to an isotropic torus.

**Theorem 1** Under the assumptions above  
\[ \Delta E_m = 2b_m \omega_1 \hbar e^{-\frac{S_L}{h}} (1 + o(1)), \quad h \to 0, \]
where $b_m$ is found from (??), $S(\tilde{E})$ is half the action on $\text{Lib}_{\tilde{E}}$ at energy $\tilde{E} = \tilde{E}(h)$ which we determine as the solution of:  
\[ \tilde{E} + \hbar \beta(\tilde{E}) = h \left( \lambda_1(1 + 2m) + \lambda_2 \right). \]  
Here $\beta(\tilde{E})$ is positive Floquet exponent of $\text{Lib}_{\tilde{E}}$.

In the case $m = 0$ Theorem 1 was proved, first, in [4] when $\gamma_0$ is a straight line $x_2 = 0$, and then in [1,2] in full generality (see also [6]). We want to show that passing to an arbitrary $m > 0$ is quite simple.

**Sketch of proof:** We express (6) with the instanton phase ($E = 0$). The tunnel WKB approximation for the normalized quasimodes reads  
\[ u_{L/R} = \hbar^{-\frac{m+1}{2}} A_{L/R} e^{-S_{L/R}} (1 + O(h)), \]
where $S_{L/R} = d_0(x, a_{L/R})$ (distance along the instanton), and the amplitudes $A_{L/R}$ are solution of the transport equation  
\[ A \left( \lambda_1(2m + 1) + \lambda_2 - D \right) + 2\nabla A \nabla S = 0. \]  
Inserting it into (??) and applying asymptotic stationary phase, we obtain:  
\[ \Delta E_m \sim 4h^{\frac{1}{2} - m} \sqrt{\pi} D^{-\frac{1}{2}} A_L^2(x_0) P_0 e^{-\frac{S_0}{h}}, \]
where $x_0 = x_E|_{E=0}$, $D = \frac{\partial S}{\partial x_2}(x_0)$, $P_0 = \frac{\partial S_{L/R}}{\partial x_1}(x_0)$, and $S_0 = 2S_{L}(x_0)$.

Thus, we arrived to the same formula as for $m = 0$, but for the numerical factor $b_m$. The rest of proof is similar to the case $m = 0$, its main ingredient is the following (see [6]):

**Proposition 2**  
\[ \beta(E) = \lambda_2 - \frac{4\log T}{T(E)}(1 + o(1)), \]
where $T(E)$ denote the period of $\text{Lib}_E$.

Note that proof of this Proposition uses assumption $2\lambda_1 < \lambda_2$. When the instanton $\gamma_0$ is not a straight line, we resort to special coordinates (proposed in [7,4]): $s$ denotes arclength along $\gamma_0$, while $q$ is a coordinate along a normal to $\gamma_0$. But these coordinates are ill-behaved when Euclidean curvature of $\gamma_0$ tends to infinity near $a_{L/R}$, which can happen, if $\frac{1}{\lambda_1} \leq 2$. On the other hand, we can use harmonic oscillator approximation for $b(t)$ as $t \to \infty$. Therefore  
\[ b(t) \sim \sqrt{\lambda_1 + 2m} \lambda_2 2^{\gamma/2} \left( \xi_1(t) \right)^m, \quad t \to +\infty \]
where $\xi_1(t)$ is a $\xi_1$-coordinate of $\gamma_0(t)$. Defining $\sigma = \lim_{t \to +\infty} e^{\lambda_1 t} \xi_1(t)$ and $J = J(\xi_1)$ we see that  
\[ \Delta E_m \sim \frac{2^{m+1} \hbar^{\frac{1}{2} - m} \sqrt{\lambda_1}}{m!} \rho e^{-\frac{S_0}{h}}. \]  

(10)

Let now $S_E$ be a half of the action along $\text{Lib}_E$. In [1] we proved:  
\[ S_E - S_0 = E(1 + \log 2) + ET_E + o(E), \]  
where $T_E$ stands for time to move along $\gamma_0$ between the intersections with $\partial U_E$. Inserting (??) with $E(h) = h(1 + 2m)\lambda_1$ into (??), we get  
\[ \Delta E_m \sim \frac{2^{1-m} \sqrt{\pi} \hbar \lambda_1}{m!} \rho \sqrt{D} e^{-\frac{S_0}{h}}, \]
where  
\[ \rho = \frac{\sigma \sqrt{\lambda_1}}{\sqrt{\hbar}} e^{-\lambda_1 T_E}. \]

One can easily see that $\rho \sim \sqrt{2^{m+1}}$, hence  
\[ \Delta E_m \sim b_m \frac{h \omega_1}{\rho} T e^{-\frac{S_0}{h}}. \]  
(12)
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