Electric Circuit Realizations of Fracton Physics

Michael Pretko
Department of Physics and Center for Theory of Quantum Matter, University of Colorado, Boulder, CO 80309, USA
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We design a set of classical macroscopic electric circuits in which charge exhibits the mobility restrictions of fracton quasiparticles. The crucial ingredient in these circuits is a transformer, which induces currents between pairs of adjacent wires. For an appropriately designed geometry, this inductor serves to enforce conservation of dipole moment. We show that a network of capacitors connected via ideal transformers will forever remember the dipole moment of its initial charge configuration. Relaxation of the dipole moment in realistic systems can only occur via flux leakage in the transformers, which will lead to violations of fracton physics at the longest times. We propose a concrete diagnostic for these “fracton-like” circuits in the form of their characteristic equilibrium charge configurations, which we verify using simple circuit simulation software. These circuits not only provide an experimental testing ground for fracton physics, but also serve as DC filters. We outline extensions of these ideas to circuits featuring other types of higher moment conservation laws, as well as to higher-dimensional circuits which act as fracton “current-ice.” While our focus is on classical circuits, we discuss how these ideas can be straightforwardly extended to realize quantized fractons in superconducting circuits.

Introduction. Advances in the study of quantum phases of matter over the past several decades have demonstrated the existence of numerous phenomena, such as topologically protected edge modes and fractionalized quasiparticles, which appear quite exotic from the perspective of classical physics. However, some of these phenomena also have clear analogues in simpler classical systems. For example, the robust edge modes seen in topological insulators can be found in both mechanical systems [14;15] and ordinary electric circuits [9;13]. In the latter context, a special class of AC circuits with topological admittance bands featuring robust boundary modes have been both theoretically designed and practically implemented. Similar work has also taken place on realizing the corner modes associated with certain higher order topological insulators [14;16].

While edge modes can be realized in classical systems in straightforward fashion, fractionalized quasiparticles represent a more significant challenge. The most common types of fractionalized quasiparticles are characterized by fractionalized charge and braiding statistics, which do not have obvious analogues in mechanical or electrical systems (though recent progress has been made in this direction [17;18]). In contrast, recent years have uncovered the existence of a striking new type of fractionalized quasiparticle, the “fracton,” characterized by its unusual restricted mobility [19;24]. Specifically, an isolated fracton is strictly immobile, while certain bound states of fractons are free to move around the system. This mobility restriction is naturally encoded in the higher moment conservation laws of such systems, such as conservation of dipole moment [23;25;26]. Fractons are notable both for their potential applications to quantum information storage [20;27;29], as well as their prevalence across numerous domains of physics, including spin liquids [30;44], elasticity [45;51], localization [19;52;53], hole-doped antiferromagnets [54], gravity [55;57], Majorana systems [21;58;59], and deconfined quantum criticality [60].

While this type of fractionalization is in some ways more exotic than fractionalized statistics, it also has a much clearer path towards realization in classical systems. The key ingredient in a classical realization should be a mechanism for enforcing conservation of higher charge moments without fine-tuning. In this work, we provide precisely such a mechanism in the context of ordinary electric circuits. Specifically, we show how transformers can be utilized in circuits to enforce conservation of dipole moment. We then design a set of circuits featuring capacitors and transformers, which we term “fracton-like circuits,” in which electric charge inherits the mobility restrictions of fractons. The dipole conservation in these circuits is quite robust, insensitive to internal resistances in the circuit. The only effect which violates this constraint is transformer flux leakage, which causes the dipole to relax at the longest times in realistic systems.

We propose several concrete ways to characterize fracton-like circuits. For example, the steady-state charge distribution of such a circuit has a characteristic linear form, as we describe in detail below. We verify this prediction using the circuit simulator CircuitLab, in which we design one-dimensional circuits explicitly exhibiting dipole conservation. We also argue these circuits act as perfect DC filters. Such circuits could be readily built in tabletop experiments and would provide a new platform for testing the physics of fractons. We conclude by outlining a procedure for systematically imposing higher moment conservation laws beyond dipole moment into electric circuits, including those leading to other types of subdimensional particles besides fractons. We also consider higher-dimensional circuits which act as fracton “current-ice,” exhibiting pinch-point singularities in their current-current correlations. While we focus on macroscopic classical circuits, we briefly discuss extensions to superconducting quantum circuits realizing quantized fractons.
A transformer can be written as: the time derivative of current. The voltage across each
rents passing through the two wires, or more specifically,
primary. These voltages can then be related to the cur-
secondary voltage is simply inverted with respect to the
of windings, but with opposite orientation, so that the
−
physics, we will choose transformers designed such that
be negative if the coils are wound around the core in op-
around the central core,
the number of times their respective wires are wound
voltage to the secondary voltage is simply the ratio of
at different voltages. Specifically, the ratio of the primary
is the self-inductance of each of the two coils
where the spatial derivatives should be interpreted as
lattice differences. We then conclude that the change in
non-ideal transformers. By setting $V_1 = -V_2$, we can then conclude that:
$$\frac{dI_1}{dt} = -\frac{dI_2}{dt}$$
If we Fourier transform to the frequency domain, then
away from $\omega = 0$ (i.e. the DC component), we can con-
clude that $I_1(\omega) = -I_2(\omega)$. Alternatively, if we stipulate
that $I_1$ and $I_2$ are equal and opposite at $t = 0$, we can con-
clude that they remain equal and opposite for all times:
$$I_1(t) = -I_2(t)$$
We therefore conclude that, up to a constant DC offset,
the two currents remain equal and opposite at all times.
We could also have independently reached the same con-
clusion based on energy conservation. Neglecting internal
resistance, we can match the power input and output of
the two wires, yielding $I_1V_1 = I_2V_2$. If the voltages are
equal and opposite, then so too are the currents.

We now have a circuit element which enforces a perfect
“drag” effect, in the sense that current in one wire leads
equal and opposite current in some nearby wire. This
physics is highly reminiscent of the behavior of fractons,
for which motion of a charge is necessarily accompanied
by opposing motion of nearby charges in such a way as
to preserve the overall dipole moment. Indeed, by stor-
ing charge in an appropriate geometry, we can use these
transformers to construct a circuit which explicitly en-
forces conservation of dipole moment. For ease of nota-
tion, we represent transformers simply as an abstract box
with four connection points, as in Figure 1 (Note that
there is no ambiguity in this notation for the special case
of $N_1/N_2 = -1$.) We also use circles to represent one end
of a capacitor (with the other end implicitly grounded).

We consider a lattice of such capacitors, which carry
all charge in the system. We focus on a one-dimensional
chain, though the principal extends to arbitrary di-

dimension without difficulty. We connect the capacitors
of the chain using transformers, designed so that cur-
rent through any link between neighboring capacitors in-
duces an opposite current in a nearby link. An example
schematic of this type of circuit is shown in Figure 2a.
Note that two different current paths are available be-
tween any two neighboring capacitors in the bulk of the
chain, corresponding to the possibility of inducing an op-
posing current in either the leftward pair or rightward
pair of capacitors. This circuit explicitly exhibits conser-
vation of dipole moment, by design. One way to verify
this is by deriving the generalized continuity equation of
the circuit. If a transformer at location $x_n$ carries current
$I(x_n)$ (i.e. $I(x_n)$ in one wire and $-I(x_n)$ in the other),
it is straightforward to verify that, in the bulk of the
system, the charge $Q(x_n)$ obeys the following relation:
$$\partial_t Q + \partial_x^2 I = 0$$
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Circuit Design. A transformer is a component of elec-
tric circuits which is primarily used for transforming al-
ternating current from one voltage in one wire to a dif-
ferent voltage in another wire. The key physical principle
at work in a transformer is electromagnetic induction,
whereby an alternating current in the primary wire in-
duces an alternating current in the secondary wire.
A simple physical implementation of a transformer involves
both wires being coiled around a magnetic core, in gen-
eral with different numbers of windings (see Figure 1).
When the core material is chosen to have extremely high
magnetic permeability, essentially all of the magnetic flux
generated by a current in the primary wire will pass
through the coil of the secondary wire. When the pri-
mary current is alternating, the changing flux will induce
an alternating current in the secondary wire at the same
frequency.

While the primary and secondary wires carry alternat-
ing currents at the same frequency, they are generically
at different voltages. Specifically, the ratio of the primary
voltage to the secondary voltage is simply the ratio of
the number of times their respective wires are wound
around the central core, $V_1/V_2 = N_1/N_2$, as a simple con-
sequence of Faraday’s law. Importantly, these ratios can
be negative if the coils are wound around the core in op-
posite orientations. For the purpose of realizing fracton
physics, we will choose transformers designed such that
this ratio is $-1$, i.e. both coils have the same number
of windings, but with opposite orientation, so that the
secondary voltage is simply inverted with respect to the
primary. These voltages can then be related to the cur-
rents passing through the two wires, or more specifically,
the time derivative of current. The voltage across each
coil can be written as:
$$V_{1/2} = -L\partial_t I_{1/2} - M\partial_t I_{2/1}$$
where $L$ is the self-inductance of each of the two coils
(which is the same for both, since the wires are taken
to be identical up to orientation) and $M$ is the mutual
inductance of the two coils. We have neglected any inter-
nal resistance, which we discuss below in the context of

FIG. 1. A transformer functions via a central magnetic core which ideally allows all magnetic flux (green arrow) generated by an input current to pass through a secondary coil. The ratio of voltages equals the ratio of number of windings of the two coils, i.e. $V_1/V_2 = N_1/N_2$. We abstractly represent a transformer as a box with four contact points.
FIG. 2. a) Schematic of a dipole-conserving fractolectric circuit, with black circles representing capacitors and blue squares representing transformers. b) Implementation of a fractolectric circuit in CircuitLab. Each capacitor has $C = 1 \mu F$, the self and mutual inductances of coils in the transformers are 10 H, and external resistance and inductance of 20 $\Omega$ and 0.5 H have been added to regulate the circuit.

dipole moment, $\partial_t (\sum_n Q_n x_n) = - \sum_n x_n \partial^2 I_n$, is a boundary term, which vanishes with the boundary conditions chosen in Figure 2.

Diagnostics. Given that the circuit indicated by Figure 2a should exhibit conservation of dipole moment, what physical observable can we examine to test this? One particularly simple metric for dipole conservation is the steady-state charge distribution. We assume that the circuit contains a small internal resistance which eventually causes currents to relax and the charge to reach a steady state. (Importantly, dipole conservation, which follows from equality of flux on the two sides of a transformer, is not affected by equal resistances added to both sides.) Consider a chain of identical capacitors initialized with some non-uniform charge distribution. In the absence of dipole conservation, the charge would eventually spread out evenly in order to minimize energy, such that each capacitor carried equal charge. In the presence of dipole conservation, however, the chain can no longer relax to the true minimum energy configuration. Instead, the system will relax to the minimum energy configuration consistent with dipole conservation. To find this configuration, we minimize the following energy functional:

$$E = \frac{1}{2} C \sum_n Q_n^2 - \mu \sum_n Q_n - \lambda \sum_n x_n Q_n$$

(5)

where $Q_n$ is the charge on capacitor $n$, $x_n$ is its position, $C$ is the capacitance of each capacitor (assumed uniform), and $\mu$ and $\lambda$ are Lagrange multipliers which we will use to enforce particular values of charge and dipole moment. Varying the energy with respect to the $Q_n$, we obtain the minimum energy configuration as:

$$Q_n = \frac{2}{C} (\mu + \lambda x_n)$$

(6)

where $\mu$ and $\lambda$ are chosen such that this configuration has the same charge and dipole moment as the initial configuration. Note that, for a given total charge, $\lambda$ can be zero for only a single value of the dipole moment. Generically, $\lambda$ is nonzero. In other words, the steady state charge distribution is a linear function of $x$, instead of a uniform distribution.

We can test this prediction by directly simulating a fractolectric circuit using simple circuit simulation software. In Figure 2b, we display a fractolectric circuit built using CircuitLab which will allow us to put these ideas to the test. We can easily check the steady state charge distribution by reading off the voltage across each capacitor. We consider initializing the system with charge 1 on the leftmost capacitor, then we let the system relax to a steady state. In Figure 3, we plot the voltage across each capacitor as a function of time. After some initial oscillations, the system reaches a steady state in which charge behaves as a linear function of position, just as predicted, serving as a clear indication of conservation of dipole moment.

It is also useful to consider how such a circuit responds to an externally applied voltage. A direct current can easily pass through the circuit, which only places restrictions on changing currents. For an applied voltage difference $V$ across the two terminal points of the circuit, the system will exhibit a current $I = V/R_{eff}$, where $R_{eff}$ is the effective resistance generated by all internal components of the circuit. For a purely alternating applied voltage, however, the conservation of dipole moment will not allow any net flow of charge from one end of the circuit to the other, acting as an infinite impedance. More generally, for an applied voltage $V(t)$, the fractolectric circuit will only allow passage of the DC component, $\tilde{V}(\omega = 0)$. Thus, neglecting small losses due to flux leakage, the fractolectric circuit acts as a perfect DC filter.

Extensions. So far, we have considered classical circuits which use transformers to implement conservation of dipole moment. However, there are various ways in
we require \( \partial_i \partial_j I^{ij} = 0 \). We expect that such a system will generically have an energy functional of the form \( E = \sum_n I^{ij} I_{ij} \), where \( I_{ij} \) is subject to the constraint \( \partial_i \partial_j I^{ij} = 0 \). The current-current correlations in this system, \( \langle I_{ij}(x) I_{k\ell}(y) \rangle \), then ought to exhibit the "pinch-point" singularities characteristics of \( U(1) \) fracton systems [61]. In this way, these circuits serve as a fracton "current-ice," in analogy with the spin-ices found in frustrated magnets.

In addition to fractons, higher-dimensional systems can host particles which have mobility restrictions only along certain directions. For example, in certain "vector charge" models, an individual charge is restricted to move in a single direction, while perpendicular motion can only occur in bound states. Realizing this type of physics in circuits is not much more difficult than the simpler fracton case. Motivated by microscopic models for one-dimensional particles, where charges typically live on links of a lattice, we design a circuit with capacitors on each link of a square lattice, as depicted in Figure 4b. Current can flow normally between capacitors along a fixed line, while motion perpendicular to each line is governed by a set of transformers on all plaquettes of the square lattice. By grouping the charge on an \( x- \) and \( y- \)directed link touching site \( n \) into a vector \((Q_x, Q_y)_n\), it can readily be verified that the circuit of Figure 4b exhibits conservation of the angular charge moment, \( \sum_n \epsilon^{ij} Q_i x_j \), enforcing the one-dimensional nature of charge. By introducing more complicated geometries, we can further generalize this logic to endow charge with any desired type of mobility.

**Quantum Circuits.** While we have so far focused on classical circuits, our ideas can also be naturally implemented in superconducting quantum circuits, which provides at least three significant technical advantages. First, due to the direct relationship between the current in a superconducting loop and the magnetic flux through that loop, the use of superconducting wires would enforce Equation 4 even for direct currents, not just alternating currents. Second, a superconducting flux transformer [62, 63] could be used to achieve direct transfer of flux from one coil to the other, eliminating violations of fracton behavior arising from flux leakage out of an imperfect core material. Finally, in contrast to the macroscopic charges carried by classical capacitors, a quantum circuit could store quantized charges via the use of quantum dots. All of these features add up to a more robust realization of fractons in quantum circuits, as compared with their classical counterparts. Using these techniques, a quantum circuit consisting of only quantum dots and superconducting wires could be used to realize the physics of discrete charges exhibiting perfect fracton behavior, even in the DC limit.

**Conclusions.** In this work, we have established a design for realizing the constrained dynamics of fractons in ordinary classical macroscopic electric circuits. These circuits rely on the induction physics of transformers to naturally enforce conservation of dipole moment, allow-
ing for a simple classical realization of the fracton phenomenon. We have shown that the charge distribution on a network of capacitors connected through appropriate transformers will be forced to remember its initial dipole moment, instead of relaxing to its true minimum energy configuration. We have proposed various probes of the fractonic nature of these circuits, which we have verified using simple circuit simulation software. Finally, we have outlined extensions of this circuit design which conserve other higher multipole moments, such as those leading to subdimensional behavior. We have also considered higher-dimensional fractolectric circuits which behave as a fracton current-ice, exhibiting characteristic pinch-point singularities. Our work opens the door for simple table-top experiments on fracton physics. We have also discussed the extension of these ideas to superconducting quantum circuits, which allow for perfect realization of quantized fractons.

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