Adaptive Mode Selection and Power Allocation in Bidirectional Buffer-aided Relay Networks

Vahid Jamali†, Nikola Zlatanov‡, Aissa Ikhlef‡, and Robert Schober†
† Friedrich-Alexander University (FAU), Erlangen, Germany
‡ University of British Columbia (UBC), Vancouver, Canada

Abstract

In this paper, we consider the problem of sum rate maximization in a bidirectional relay network with fading. Hereby, user 1 and user 2 communicate with each other only through a relay, i.e., a direct link between user 1 and user 2 is not present. In this network, there exist six possible transmission modes: four point-to-point modes (user 1-to-relay, user 2-to-relay, relay-to-user 1, relay-to-user 2), a multiple access mode (both users to the relay), and a broadcast mode (the relay to both users). Most existing protocols assume a fixed schedule of using a subset of the aforementioned transmission modes, as a result, the sum rate is limited by the capacity of the weakest link associated with the relay in each time slot. Motivated by this limitation, we develop a protocol which is not restricted to adhere to a predefined schedule for using the transmission modes. Therefore, all transmission modes of the bidirectional relay network can be used adaptively based on the instantaneous channel state information (CSI) of the involved links. To this end, the relay has to be equipped with two buffers for the storage of the information received from users 1 and 2, respectively. For the considered network, given a total average power budget for all nodes, we jointly optimize the transmission mode selection and power allocation based on the instantaneous CSI in each time slot for sum rate maximization. Simulation results show that the proposed protocol outperforms existing protocols for all signal-to-noise ratios (SNRs). Specifically, we obtain a considerable gain at low SNRs due to the adaptive power allocation and at high SNRs due to the adaptive mode selection.

I. INTRODUCTION

In a bidirectional relay network, two users exchange information via a relay node [1]. Several protocols have been proposed for such a network under the practical half-duplex constraint, i.e., a node cannot transmit and receive at the same time and in the same frequency band. The simplest protocol is the traditional two-way relaying protocol in which the transmission is accomplished in four successive point-to-point phases: user 1-to-relay, relay-to-user 2, user 2-to-relay, and relay-to-user 1. In contrast, the time division broadcast (TDBC) protocol exploits the broadcast capability of the wireless medium and combines the relay-to-user 1 and relay-to-user 2 phases into one phase, the broadcast phase [2]. Thereby, the relay broadcasts a superimposed codeword, carrying information for both user 1 and user 2, such that each user is able to recover its intended information by self-interference cancellation. Another existing protocol is the multiple access broadcast (MABC) protocol in which the user 1-to-relay and user 2-to-relay phases are also combined into one phase, the multiple-access phase [3]. In the multiple-access phase,
both user 1 and user 2 simultaneously transmit to the relay which is able to decode both messages. Generally, for the bidirectional relay network without a direct link between user 1 and user 2, six transmission modes are possible: four point-to-point modes (user 1-to-relay, user 2-to-relay, relay-to-user 1, relay-to-user 2), a multiple access mode (both users to the relay), and a broadcast mode (the relay to both users), where the capacity region of each transmission mode is known [4], [5]. Using this knowledge, a significant research effort has been dedicated to obtaining the achievable rate region of the bidirectional relay network [11]- [8]. Specifically, the achievable rates of most existing protocols for two-hop relay transmission are limited by the instantaneous capacity of the weakest link associated with the relay. The reason for this is the fixed schedule of using the transmission modes which is adopted in all existing protocols, and does not exploit the instantaneous channel state information (CSI) of the involved links. For one-way relaying, an adaptive link selection protocol was proposed in [9] where based on the instantaneous CSI, in each time slot, either the source-relay or relay-destination links are selected for transmission. To this end, the relay has to have a buffer for data storage. This strategy was shown to achieve the capacity of the one-way relay channel with fading [10].

Moreover, in fading AWGN channels, power control is necessary for rate maximization. The highest degree of freedom that is offered by power control is obtained for a joint average power constraint for all nodes. Any other power constraint with the same total power budget is more restrictive than the joint power constraint and results in a lower sum rate. Therefore, motivated by the protocols in [9] and [10], our goal is to utilize all available degrees of freedom of the three-node half-duplex bidirectional relay network with fading, via an adaptive mode selection and power allocation policy. In particular, given a joint power budget for all nodes, we find a policy which in each time slot selects the optimal transmission mode from the six possible modes and allocates the optimal powers to the nodes transmitting in the selected mode, such that the sum rate is maximized.

Adaptive mode selection for bidirectional relaying was also considered in [8] and [11]. However, the selection policy in [8] does not use all possible modes, i.e., it only selects from two point-to-point modes and the broadcast mode, and assumes that the transmit powers of all three nodes are fixed and identical. Although the selection policy in [11] considers all possible transmission modes for adaptive mode selection, the transmit powers of the nodes are assumed to be fixed, i.e., power allocation is not possible. Interestingly, mode selection and power allocation are mutually coupled and the modes selected with the protocol in [11] for a given channel are different from the modes selected with the proposed protocol. Power allocation can considerably improve the sum rate by optimally allocating the powers to the nodes based on the instantaneous CSI especially when the total power budget in the network is low. Moreover, the proposed protocol achieves the maximum sum rate in the considered bidirectional network. Hence, the sum rate achieved with the proposed protocol can be used as a reference for other low complexity suboptimal protocols. Simulation results confirm that the proposed protocol outperforms existing protocols.

Finally, we note that the advantages of buffering come at the expense of an increased end-to-end delay. However, with some modifications to the optimal protocol, the average delay can be bounded, as shown in [9], which causes only a small loss in the achieved rate. The delay analysis of the proposed protocol is beyond the scope of the current work and is left for future research.
II. System Model

In this section, we first describe the channel model. Then, we provide the achievable rates for the six possible transmission modes.

A. Channel Model

We consider a simple network in which user 1 and user 2 exchange information with the help of a relay node as shown in Fig. 1. We assume that there is no direct link between user 1 and user 2, and thus, user 1 and user 2 communicate with each other only through the relay node. We assume that all three nodes in the network are half-duplex. Furthermore, we assume that time is divided into slots of equal length and that each node transmits codewords which span one time slot or a fraction of a time slot as will be explained later. We assume that the user-to-relay and relay-to-user channels are impaired by AWGN with unit variance and block fading, i.e., the channel coefficients are constant during one time slot and change from one time slot to the next. Moreover, in each time slot, the channel coefficients are assumed to be reciprocal such that the user 1-to-relay and the user 2-to-relay channels are identical to the relay-to-user 1 and relay-to-user 2 channels, respectively. Let $h_1(i)$ and $h_2(i)$ denote the channel coefficients between user 1 and the relay and between user 2 and the relay in the $i$-th time slot, respectively. Furthermore, let $S_1(i) = |h_1(i)|^2$ and $S_2(i) = |h_2(i)|^2$ denote the squares of the channel coefficient amplitudes in the $i$-th time slot. $S_1(i)$ and $S_2(i)$ are assumed to be ergodic and stationary random processes with means $\Omega_1 = E\{S_1\}$ and $\Omega_2 = E\{S_2\}$, respectively, where $E\{\cdot\}$ denotes expectation. Since the noise is AWGN, in order to achieve the capacity of each mode, nodes have to transmit Gaussian distributed codewords. Therefore, the transmitted codewords of user 1, user 2, and the relay are comprised of symbols which are Gaussian distributed random variables with variances $P_1(i)$, $P_2(i)$, and $P_r(i)$, respectively, where $P_j(i)$ is the transmit power of node $j \in \{1, 2, r\}$ in the $i$-th time slot. For ease of notation, we define $C(x) \triangleq \log_2(1+x)$. In the following, we describe the transmission modes and their achievable rates.

\footnote{In this paper, we drop time index $i$ in expectations for notational simplicity.}
B. Transmission Modes and Their Achievable Rates

In the considered bidirectional relay network only six transmission modes are possible, cf. Fig. 2. The six possible transmission modes are denoted by $M_1, ..., M_6$, and $R_{j,j'}(i) \geq 0$, $j, j' \in \{1, 2, r\}$, denotes the transmission rate from node $j$ to node $j'$ in the $i$-th time slot. Let $B_1$ and $B_2$ denote two infinite-size buffers at the relay in which the received information from user 1 and user 2 is stored, respectively. Moreover, $Q_j(i)$, $j \in \{1, 2\}$, denotes the amount of normalized information in bits/symbol available in buffer $B_j$ in the $i$-th time slot. Using this notation, the transmission modes and their respective rates are presented in the following:

$M_1$: User 1 transmits to the relay and user 2 is silent. In this mode, the maximum rate from user 1 to the relay in the $i$-th time slot is given by $R_{1r}(i) = C_{1r}(i)$, where $C_{1r}(i) = C(P_1(i)S_1(i))$. The relay decodes this information and stores it in buffer $B_1$. Therefore, the amount of information in buffer $B_1$ increases to $Q_1(i) = Q_1(i-1) + R_{1r}(i)$.

$M_2$: User 2 transmits to the relay and user 1 is silent. In this mode, the maximum rate from user 2 to the relay in the $i$-th time slot is given by $R_{2r}(i) = C_{2r}(i)$, where $C_{2r}(i) = C(P_2(i)S_2(i))$. The relay decodes this information and stores it in buffer $B_2$. Therefore, the amount of information in buffer $B_2$ increases to $Q_2(i) = Q_2(i-1) + R_{2r}(i)$.

$M_3$: Both users 1 and 2 transmit to the relay simultaneously. For this mode, we assume that multiple access transmission is used, see [5]. Thereby, the maximum achievable sum rate in the $i$-th time slot is given by $R_{1r}(i) + R_{2r}(i) = C_r(i)$, where $C_r(i) = C(P_1(i)S_1(i) + P_2(i)S_2(i))$. Since user 1 and user 2 transmit independent messages, the sum rate, $C_r(i)$, can be decomposed into two rates, one from user 1 to the relay and the other one from user 2 to the relay. Moreover, these two capacity rates can be achieved via time sharing and successive interference cancelation. Thereby, in the first $0 \leq t(i) \leq 1$ fraction of the $i$-th time slot, the relay first decodes the codeword received from user 2 and considers the signal from user 1 as noise. Then, the relay subtracts the signal received from user 2 from the received signal and decodes the codeword received from user 1. A similar procedure is performed in the remaining $1 - t(i)$ fraction of the $i$-th time slot but now the relay first decodes the codeword received from user 1 and treats the signal of user 2 as noise, and then decodes the codeword received from user 2. Therefore, for a given $t(i)$, we decompose $C_r(i)$ as $C_r(i) = C_{12r}(i) + C_{21r}(i)$ and the maximum rates from users 1 and 2 to the relay in the $i$-th time slot are $R_{1r}(i) = C_{12r}(i)$ and $R_{2r}(i) = C_{21r}(i)$, respectively. $C_{12r}(i)$ and $C_{21r}(i)$ are given by

$$C_{12r}(i) = t(i)C(P_1(i)S_1(i)) + (1 - t(i))C\left(\frac{P_1(i)S_1(i)}{1 + P_2(i)S_2(i)}\right)$$

(1a)

$$C_{21r}(i) = (1 - t(i))C(P_2(i)S_2(i)) + t(i)C\left(\frac{P_2(i)S_2(i)}{1 + P_1(i)S_1(i)}\right)$$

(1b)

The relay decodes the information received from user 1 and user 2 and stores it in its buffers $B_1$ and $B_2$, respectively. Therefore, the amounts of information in buffers $B_1$ and $B_2$ increase to $Q_1(i) = Q_1(i-1) + R_{1r}(i)$ and $Q_2(i) = Q_2(i-1) + R_{2r}(i)$, respectively.

$M_4$: The relay transmits the information received from user 2 to user 1. Specifically, the relay extracts the information from buffer $B_2$, encodes it into a codeword, and transmits it to user 1. Therefore, the transmission rate from the relay to user 1 in the $i$-th time slot is limited by both the capacity of the relay-to-user 1 channel and
the amount of information stored in buffer $B_2$. Thus, the maximum transmission rate from the relay to user 1 is given by $R_{r1}(i) = \min\{C_{r1}(i), Q_2(i-1)\}$, where $C_{r1}(i) = C(P_r(i)S_1(i))$. Therefore, the amount of information in buffer $B_2$ decreases to $Q_2(i) = Q_2(i-1) - R_{r1}(i)$.

$\mathcal{M}_5$: This mode is identical to $\mathcal{M}_4$ with user 1 and 2 switching places. The maximum transmission rate from the relay to user 2 is given by $R_{r2}(i) = \min\{C_{r2}(i), Q_1(i-1)\}$, where $C_{r2}(i) = C(P_r(i)S_2(i))$ and the amount of information in buffer $B_1$ decreases to $Q_1(i) = Q_1(i-1) - R_{r2}(i)$.

$\mathcal{M}_6$: The relay broadcasts to both user 1 and user 2 the information received from user 2 and user 1, respectively. Specifically, the relay extracts the information intended for user 2 from buffer $B_1$ and the information intended for user 1 from buffer $B_2$. Then, based on the scheme in [4], it constructs a superimposed codeword which contains the information from both users and broadcasts it to both users. Thus, in the $i$-th time slot, the maximum rates from the relay to users 1 and 2 are given by $R_{r1}(i) = \min\{C_{r1}(i), Q_2(i-1)\}$ and $R_{r2}(i) = \min\{C_{r2}(i), Q_1(i-1)\}$, respectively. Therefore, the amounts of information in buffers $B_1$ and $B_2$ decrease to $Q_1(i) = Q_1(i-1) - R_{r2}(i)$ and $Q_2(i) = Q_2(i-1) - R_{r1}(i)$, respectively.

Our aim is to develop an optimal mode selection and power allocation policy which in each time slot selects one of the six transmission modes, $\mathcal{M}_1, \ldots, \mathcal{M}_6$, and allocates the optimal powers to the transmitting nodes of the selected mode such that the average sum rate of both users is maximized. To this end, we introduce six binary variables, $q_k(i) \in \{0, 1\}$, $k = 1, \ldots, 6$, where $q_k(i)$ indicates whether or not transmission mode $\mathcal{M}_k$ is selected in the $i$-th time slot. In particular, $q_k(i) = 1$ if mode $\mathcal{M}_k$ is selected and $q_k(i) = 0$ if it is not selected in the $i$-th time slot. Furthermore, since in each time slot only one of the six transmission modes can be selected, only one of the mode selection variables is equal to one and the others are zero, i.e., $\sum_{k=1}^{6} q_k(i) = 1$ holds.

In the proposed framework, we assume that all nodes have full knowledge of the CSI of both links. Thus, based on the CSI and the proposed protocol, cf. Theorem 2, each node is able to individually decide which transmission mode is selected and adapt its transmission strategy accordingly.

III. JOINT MODE SELECTION AND POWER ALLOCATION

In this section, we first investigate the achievable average sum rate of the network. Then, we formulate a maximization problem whose solution is the sum rate maximizing protocol.

A. Achievable Average Sum Rate

We assume that user 1 and user 2 always have enough information to send in all time slots and that the number of time slots, $N$, satisfies $N \to \infty$. Therefore, using $q_k(i)$, the user 1-to-relay, user 2-to-relay, relay-to-user 1, and relay-to-user 2 average transmission rates, denoted by $\bar{R}_{1r}$, $\bar{R}_{2r}$, $\bar{R}_{r1}$, and $\bar{R}_{r2}$, respectively, are obtained as

$$\bar{R}_{1r} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_1(i)C_{1r}(i) + q_3(i)C_{12r}(i)] \quad (2a)$$

$$\bar{R}_{2r} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_2(i)C_{2r}(i) + q_3(i)C_{21r}(i)] \quad (2b)$$
\[
\tilde{R}_{r_1} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_4(i) + q_6(i)] \min \{C_{r_1}(i), Q_2(i-1)\} 
\]

\[
\tilde{R}_{r_2} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_5(i) + q_6(i)] \min \{C_{r_2}(i), Q_1(i-1)\}. 
\]

The average rate from user 1 to user 2 is the average rate that user 2 receives from the relay, i.e., \( \tilde{R}_{r_2} \). Similarly, the average rate from user 2 to user 1 is the average rate that user 1 receives from the relay, i.e., \( \tilde{R}_{r_1} \). In the following theorem, we introduce a useful condition for the queues in the buffers of the relay leading to the optimal mode selection and power allocation policy.

**Theorem 1 (Optimal Queue Condition):** The maximum average sum rate, \( \tilde{R}_{r_1} + \tilde{R}_{r_2} \), for the considered bidirectional relay network is obtained when the queues in the buffers \( B_1 \) and \( B_2 \) at the relay are at the edge of non-absorption. More precisely, the following conditions must hold for the maximum sum rate

\[
\tilde{R}_{1r} = \tilde{R}_{2r} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_5(i) + q_6(i)] C_{r_2}(i) 
\]

\[
\tilde{R}_{2r} = \tilde{R}_{r_1} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [q_4(i) + q_6(i)] C_{r_1}(i) 
\]

where \( \tilde{R}_{1r} \) and \( \tilde{R}_{2r} \) are given by (2a) and (2b), respectively.

**Proof** Please refer to [11, Appendix A].

Using this theorem, in the following, we derive the optimal transmission mode selection and power allocation policy.

**B. Optimal Protocol**

The available degrees of freedom in the considered network in each time slot are the mode selection variables, the transmit powers of the nodes, and the time sharing variable for multiple access. Herein, we formulate an optimization problem which gives the optimal values of \( q_k(i), P_j(i), t(i) \), for \( k = 1, ..., 6 \), \( j = 1, 2, r \), and \( \forall i \), such that the average sum rate of the users is maximized. The optimization problem is as follows

maximize \( \tilde{R}_{1r} + \tilde{R}_{2r} \)

subjected to \( \forall i, j, k \)

C1 : \( \tilde{R}_{1r} = \tilde{R}_{r_2} \)

C2 : \( \tilde{R}_{2r} = \tilde{R}_{r_1} \)

C3 : \( \bar{P}_1 + \bar{P}_2 + \bar{P}_r \leq P_t \)

C4 : \( \sum_{k=1}^{6} q_k(i) = 1, \forall i \)

C5 : \( q_k(i)[1 - q_k(i)] = 0, \forall i, k \)

C6 : \( P_j(i) \geq 0, \forall i, j \)

C7 : \( 0 \leq t(i) \leq 1, \forall i \)
where $P_t$ is the total average power constraint of the nodes and $P_1, P_2,$ and $P_r$ denote the average powers consumed by user 1, user 2, and the relay, respectively, and are given by

\begin{align}
\bar{P}_1 &= \frac{1}{N} \sum_{i=1}^{N} (q_1(i) + q_3(i)) P_1(i) \\
\bar{P}_2 &= \frac{1}{N} \sum_{i=1}^{N} (q_2(i) + q_3(i)) P_2(i) \\
\bar{P}_r &= \frac{1}{N} \sum_{i=1}^{N} (q_4(i) + q_5(i) + q_6(i)) P_r(i).
\end{align}

In the optimization problem given in (4), constraints C1 and C2 are the conditions for sum rate maximization introduced in Theorem 1. Constraints C3 and C6 are the average total transmit power constraint and the power non-negativity constraint, respectively. Moreover, constraints C4 and C5 guarantee that only one of the transmission modes is selected in each time slot, and constraint C7 specifies the acceptable interval for the time sharing variable $t(i)$. Furthermore, we maximize $\bar{R}_{1r} + \bar{R}_{2r}$ since, according to Theorem 1 (and constraints C1 and C2), $\bar{R}_{1r} = \bar{R}_{2r}$ and $\bar{R}_{2r} = \bar{R}_{r1}$ hold.

In the following Theorem, we introduce a protocol which achieves the maximum sum rate.

**Theorem 2 (Mode Selection and Power Allocation Policy):** Assuming $N \to \infty$, the optimal mode selection and power allocation policy which maximizes the sum rate of the considered three-node half-duplex bidirectional relay network with AWGN and block fading is given by

\[ q_{k^*}(i) = \begin{cases} 
1, & \text{if } k^* = \arg \max_{k=1,2,3,6} \Lambda_k(i) \\
0, & \text{otherwise} 
\end{cases} \] 

where $\Lambda_k(i)$ is referred to as selection metric and is given by

\begin{align}
\Lambda_1(i) &= (1 - \mu_1) C_{1r}(i) - \gamma P_1(i) \bigg|_{P_1(i)=P_1^{M_1}(i)} \\
\Lambda_2(i) &= (1 - \mu_2) C_{2r}(i) - \gamma P_2(i) \bigg|_{P_2(i)=P_2^{M_2}(i)} \\
\Lambda_3(i) &= (1 - \mu_1) C_{12r}(i) + (1 - \mu_2) C_{21r}(i) \\
&\quad - \gamma (P_1(i) + P_2(i)) \bigg|_{P_1(i)=P_1^{M_3}(i)} \bigg|_{P_2(i)=P_2^{M_3}(i)} \\
\Lambda_6(i) &= \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i) \bigg|_{P_r(i)=P_r^{M_6}(i)} 
\end{align}

where $P_j^{M_k}(i)$ denotes the optimal transmit power of node $j$ for transmission mode $M_k$ in the $i$-th time slot and is given by

\begin{align}
P_1^{M_1}(i) &= \left[ \frac{1 - \mu_1}{\gamma \ln 2} - \frac{1}{\mu_1 S_1(i)} \right]^+ \\
P_2^{M_2}(i) &= \left[ \frac{1 - \mu_2}{\gamma \ln 2} - \frac{1}{\mu_2 S_2(i)} \right]^+
\end{align}
where \([x]^+ = \max\{x, 0\}\), \(a = \gamma \ln 2 \times S_1(i)S_2(i)\), \(b = \gamma \ln 2 \times (S_1(i) + S_2(i)) - (\mu_1 + \mu_2)S_1(i)S_2(i)\), and \(c = \gamma \ln 2 - \mu_1 S_2(i) - \mu_2 S_1(i)\). The thresholds \(\mu_1\) and \(\mu_2\) are chosen such that constraints C1 and C2 in (4) hold and threshold \(\gamma\) is chosen such that the total average transmit power satisfies C3 in (4). The optimal value of \(t(i)\) in \(C_{12r}(i)\) and \(C_{21r}(i)\) is given by

\[
t^\ast(i) = \begin{cases} 
0, & \Omega_1 \geq \Omega_2 \\
1, & \Omega_1 < \Omega_2 
\end{cases}
\]  

(9)

**Proof** Please refer to Appendix A.

We note that the optimal solution utilizes neither modes \(\mathcal{M}_4\) and \(\mathcal{M}_5\) nor time sharing for any channel statistics and channel realizations.

**Remark 1:** The mode selection metric \(\Lambda_k(i)\) introduced in (7) has two parts. The first part is the instantaneous capacity of mode \(\mathcal{M}_k\), and the second part is the allocated power with negative sign. The capacity and the power terms are linked via thresholds \(\mu_1\) and \(\mu_2\) and \(\gamma\). We note that thresholds \(\mu_1\), \(\mu_2\), and \(\gamma\) depend only on the long term statistics of the channels. Hence, these thresholds can be obtained offline and used as long as the channel statistics remain unchanged. To find the optimal values for the thresholds \(\mu_1\), \(\mu_2\), and \(\gamma\), we need a three-dimensional search, where \(\mu_1, \mu_2 \in (0, 1)\) and \(\gamma > 0\).

**Remark 2:** Adaptive mode selection for bidirectional relay networks under the assumption that the powers of the nodes are fixed is considered in [11]. Based on the average and instantaneous qualities of the links, all of the six possible transmission modes are selected in the protocol in [11]. However, in the proposed protocol, modes \(\mathcal{M}_4\) and \(\mathcal{M}_5\) are not selected at all. Moreover, the protocol in [11] utilizes a coin flip for implementation. Therefore, a central node must decide which transmission mode is selected in the next time slot. However, in the proposed protocol, all nodes can find the optimal mode and powers based on the full CSI.

IV. SIMULATION RESULTS

In this section, we evaluate the average sum rate achievable with the proposed protocol in the considered bidirectional relay network in Rayleigh fading. Thus, channel gains \(S_1(i)\) and \(S_2(i)\) follow exponential distributions.
Fig. 3. Maximum sum rate versus $P_t$ for different protocols.

with means $\Omega_1$ and $\Omega_2$, respectively. All of the presented results were obtained for $\Omega_2 = 1$ and $N = 10^4$ time slots.

In Fig. 3 we illustrate the maximum achievable sum rate obtained with the proposed protocol as a function of the total average transmit power $P_t$. In this figure, to have a better resolution for the sum rate at low and high $P_t$, we show the sum rate for both log scale and linear scale $y$-axes, respectively. The lines without markers in Fig. 3 represent the achieved sum rates with the proposed protocol for $\Omega_1 = 1, 2, 5$. We observe that as the quality of the user 1-to-relay link increases (i.e., $\Omega_1$ increases), the sum rate increases too. However, for large $\Omega_1$, the bottleneck link is the relay-to-user 2 link, and since it is fixed, the sum rate saturates.

As performance benchmarks, we consider in Fig. 3 the sum rates of the TDBC protocol with and without power allocation [2] and the buffer-aided protocols presented in [8] and [11], respectively. For clarity, for the benchmark schemes, we only show the sum rates for $\Omega_1 = \Omega_2$. For the TDBC protocol without power allocation and the protocol in [8], all nodes transmit with equal powers, i.e., $P_1 = P_2 = P_r = P_t$. For the buffer-aided protocol in [11], we adopt $P_1 = P_2 = P_r = P$ and $P$ is chosen such that the average total power consumed by all nodes is $P_t$. We note that since $\Omega_1 = \Omega_2$ and $P_1 = P_2$, the protocol in [11] only selects modes $M_3$ and $M_6$. Moreover, since $\Omega_1 = \Omega_2$, we obtain $\mu_1 = \mu_2$ in the proposed protocol. Thus, considering the optimal power allocation in (8c) and (8d), we obtain that either $P_1^{M_3}(i)$ or $P_2^{M_3}(i)$ is zero. Therefore, for the chosen parameters, only modes $M_1$, $M_2$, and $M_6$ are selected, i.e., the same modes as used in [8]. Hence, we can see how much gain we obtain due to the adaptive power allocation by comparing our result with the results for the protocol in [8]. On the other hand, the gain due to the adaptive mode selection can be evaluated by comparing the sum rate of the proposed protocol with the result for the TDBC protocol with power allocation. From the comparison in Fig. 3, we observe that for high

\[ \text{We note that although the protocol in [8] outperforms the protocol in [11] if the sum rate is plotted as a function of total transmit power as is done in Fig. 3, the protocol in [11] is optimal for given fixed node transmit powers.}\]
$P_t$, a considerable gain is obtained by the protocols with adaptive mode selection (ours and that in [8]) compared to the TDBC protocol which does not apply adaptive mode selection (around 6 dB gain). However, for high $P_t$, power allocation is less beneficial, and therefore, the sum rates obtained with the proposed protocol and that in [8] converge. On the other hand, for low $P_t$, optimal power allocation is crucial and, therefore, a considerable gain is achieved by the protocols with adaptive power allocation (ours and TDBC with power allocation).

V. CONCLUSION

We have derived the maximum sum rate of the three-node half-duplex bidirectional buffer-aided relay network with fading links. The protocol which achieves the maximum sum rate jointly optimizes the selection of the transmission mode and the transmit powers of the nodes. The proposed optimal mode selection and power allocation protocol requires the instantaneous CSI of the involved links in each time slot and their long-term statistics. Simulation results confirmed that the proposed selection policy outperforms existing protocols in terms of average sum rate.

APPENDIX A

PROOF OF THEOREM 2 (MODE SELECTION PROTOCOL)

In this appendix, we solve the optimization problem given in (4). We first relax the binary condition for $q_k(i)$, i.e., $q_k(i)[1 - q_k(i)] = 0$, to $0 \leq q_k(i) \leq 1$, and later in Appendix B we prove that the binary relaxation does not affect the maximum average sum rate. In the following, we investigate the Karush-Kuhn-Tucker (KKT) necessary conditions [12] for the relaxed optimization problem and show that the necessary conditions result in a unique sum rate and thus the solution is optimal.

To simplify the usage of the KKT conditions, we formulate a minimization problem equivalent to the relaxed maximization problem in (4) as follows

\[
\begin{align*}
\text{minimize} \quad & - (\tilde{R}_{1r} + \tilde{R}_{2r}) \\
\text{subject to} \quad & C1: \quad \tilde{R}_{1r} - \tilde{R}_{r2} = 0 \\
& C2: \quad \tilde{R}_{2r} - \tilde{R}_{r1} = 0 \\
& C3: \quad \tilde{P}_1 + \tilde{P}_2 + \tilde{P}_r - P_t \leq 0 \\
& C4: \quad \sum_{k=1}^6 q_k(i) - 1 = 0, \forall i \\
& C5: \quad q_k(i) - 1 \leq 0, \forall i, k \\
& C6: \quad -q_k(i) \leq 0, \forall i, k \\
& C7: \quad -P_j(i) \leq 0, \forall i, k \\
& C8: \quad t(i) - 1 \leq 0, \forall i \\
& C9: \quad -t(i) \leq 0, \forall i.
\end{align*}
\]
\[ \mathcal{L}(q_k(i), P_j(i), t(i), \mu_l, \gamma, \lambda(i), \alpha_k(i), \beta_k(i), \phi_l(i)) = \]

\[ - (\bar{R}_1 + \bar{R}_2) + \bar{R}_1 - \bar{R}_2) + \mu_1(\bar{R}_1 - \bar{R}_2) + \gamma (\bar{P}_1 + \bar{P}_2 + \bar{P}_3 - P_1) \]

\[ + \sum_{i=1}^{N} \lambda(i) \left( \sum_{k=1}^{6} q_k(i) - 1 \right) + \sum_{i=1}^{N} \sum_{k=1}^{6} \alpha_k(i) (q_k(i) - 1) - \sum_{i=1}^{N} \sum_{k=1}^{6} \beta_k(i) q_k(i) \]

\[ - \sum_{i=1}^{N} [\nu_1(i)P_1(i) + \nu_2(i)P_2(i) + \nu_3(i)P_3(i)] + \sum_{i=1}^{N} \phi_1(i) (t(i) - 1) - \sum_{i=1}^{N} \phi_0(i) t(i) \]

(11)

The Lagrangian function for the above optimization problem is provided in (11) at the top of the next page where \( \mu_1, \mu_2, \gamma, \lambda(i), \alpha_k(i), \beta_k(i), \nu_j(i), \phi_1(i), \) and \( \phi_2(i) \) are the Lagrange multipliers corresponding to constraints \( C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, \) and \( C_9, \) respectively. The KKT conditions include the following:

1) Stationary condition: The differentiation of the Lagrangian function with respect to the primal variables, \( q_k(i), P_j(i), \) and \( t(i), \forall i, j, k, \) is zero for the optimal solution, i.e.,

\[ \frac{\partial \mathcal{L}}{\partial q_k(i)} = 0, \forall i, k \]

(12a)

\[ \frac{\partial \mathcal{L}}{\partial P_j(i)} = 0, \forall i, j \]

(12b)

\[ \frac{\partial \mathcal{L}}{\partial t(i)} = 0, \forall i. \]

(12c)

2) Primal feasibility condition: The optimal solution has to satisfy the constraints of the primal problem in (10).

3) Dual feasibility condition: The Lagrange multipliers for the inequality constraints have to be non-negative, i.e.,

\[ \alpha_k(i) \geq 0, \forall i, k \]

(13a)

\[ \beta_k(i) \geq 0, \forall i, k \]

(13b)

\[ \gamma \geq 0, \]

(13c)

\[ \nu_j(i) \geq 0, \forall i, j \]

(13d)

\[ \phi_l(i) \geq 0, \forall i, l. \]

(13e)

4) Complementary slackness: If an inequality is inactive, i.e., the optimal solution is in the interior of the corresponding set, the corresponding Lagrange multiplier is zero. Thus, we obtain

\[ \alpha_k(i) (q_k(i) - 1) = 0, \forall i, k \]

(14a)

\[ \beta_k(i) q_k(i) = 0, \forall i, k \]

(14b)

\[ \gamma(\bar{P}_1 + \bar{P}_2 + \bar{P}_3 - P_1) = 0 \]

(14c)

\[ \nu_j(i) P_j(i) = 0, \forall i, j \]

(14d)
Without loss of generality, we first obtain the necessary condition for obtaining one or more candidate solutions from the stationary conditions and the optimal solution is surely one of these candidates. In the following subsections, with this approach, we find the optimal values of \(q\).

\(\phi_1(i)(t(i) - 1) = 0, \quad \forall i\)  \hspace{2cm} (14e)

\(\phi_0(i)t(i) = 0, \quad \forall i.\)  \hspace{2cm} (14f)

A common approach to find a set of primal variables, i.e., \(q_k(i), P_j(i), t(i), \forall i, j, k\) and Lagrange multipliers, i.e., \(\mu_1, \mu_2, \gamma, \lambda(i), \alpha_k(i), \beta_k(i), \nu_j(i), \phi_1(i), \forall i, k, \ell\), which satisfy the KKT conditions is to start with the complementary slackness conditions and see if the inequalities are active or not. Combining these results with the primal feasibility and dual feasibility conditions, we obtain various possibilities. Then, from these possibilities, we obtain one or more candidate solutions from the stationary conditions and the optimal solution is surely one of these candidates. In the following subsections, with this approach, we find the optimal values of \(q_k^*(i), P^*_j(i),\) and \(t^*(i), \forall i, j, k.\)

A. Optimal \(q_k^*(i)\)

In order to determine the optimal selection policy, \(q_k^*(i)\), we must calculate the derivatives in \((12a)\). This leads to

\[
\frac{\partial \mathcal{L}}{\partial q_1(i)} = \frac{1}{N}(1 - \mu_1)C_{1r}(i) + \lambda(i) + \alpha_1(i) - \beta_1(i) + \frac{1}{N} \gamma P_t(i) = 0 \quad \text{(15a)}
\]

\[
\frac{\partial \mathcal{L}}{\partial q_2(i)} = \frac{1}{N}(1 - \mu_2)C_{2r}(i) + \lambda(i) + \alpha_2(i) - \beta_2(i) + \frac{1}{N} \gamma P_t(i) = 0 \quad \text{(15b)}
\]

\[
\frac{\partial \mathcal{L}}{\partial q_3(i)} = \frac{1}{N}(1 - \mu_1)C_{12r}(i) + (1 - \mu_2)C_{22r}(i) + \lambda(i) + \alpha_3(i) - \beta_3(i) + \frac{1}{N} \gamma (P_1(i) + P_2(i)) = 0 \quad \text{(15c)}
\]

\[
\frac{\partial \mathcal{L}}{\partial q_4(i)} = \frac{1}{N} \mu_2 C_{r1}(i) + \lambda(i) + \alpha_4(i) - \beta_4(i) + \frac{1}{N} \gamma P_t(i) = 0 \quad \text{(15d)}
\]

\[
\frac{\partial \mathcal{L}}{\partial q_5(i)} = \frac{1}{N} \mu_1 C_{r2}(i) + \lambda(i) + \alpha_5(i) - \beta_5(i) + \frac{1}{N} \gamma P_t(i) = 0 \quad \text{(15e)}
\]

\[
\frac{\partial \mathcal{L}}{\partial q_6(i)} = \frac{1}{N} \mu_1 [C_{r2}(i) + \mu_2 C_{r1}(i)] + \lambda(i) + \alpha_6(i) - \beta_6(i) + \frac{1}{N} \gamma P_t(i) = 0. \quad \text{(15f)}
\]

Without loss of generality, we first obtain the necessary condition for \(q_1^*(i) = 1\) and then generalize the result to \(q_k^*(i) = 1, k = 2, \ldots, 6.\) If \(q_k^*(i) = 1, \) from constraint \(C4\) in \((10), the other selection variables are zero, i.e., \(q_k^*(i) = 0, k = 2, \ldots, 6.\) Furthermore, from \((14), we obtain \(\alpha_k(i) = 0, k = 2, \ldots, 6 and \beta_1(i) = 0. By substituting these values into \((15), we obtain\)

\[
\lambda(i) + \alpha_1(i) = (1 - \mu_1)C_{1r}(i) - \gamma P_1(i) \triangleq \Lambda_1(i) \quad \text{(16a)}
\]

\[
\lambda(i) - \beta_2(i) = (1 - \mu_2)C_{2r}(i) - \gamma P_2(i) \triangleq \Lambda_2(i) \quad \text{(16b)}
\]

\[
\lambda(i) - \beta_3(i) = (1 - \mu_1)C_{12r}(i) + (1 - \mu_2)C_{22r}(i) - \gamma (P_1(i) + P_2(i)) \triangleq \Lambda_3(i) \quad \text{(16c)}
\]

\[
\lambda(i) - \beta_4(i) = \mu_2 C_{r1}(i) - \gamma P_r(i) \triangleq \Lambda_4(i) \quad \text{(16d)}
\]

\[
\lambda(i) - \beta_5(i) = \mu_1 C_{r2}(i) - \gamma P_r(i) \triangleq \Lambda_5(i) \quad \text{(16e)}
\]

\[
\lambda(i) - \beta_6(i) = \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i) \triangleq \Lambda_6(i), \quad \text{(16f)}
\]
where $\Lambda_k(i)$ is referred to as selection metric. By subtracting (16b) from the rest of the equations in (16), we obtain
\[
\Lambda_1(i) - \Lambda_k(i) = \alpha_1(i) + \beta_k(i), \quad k = 2, 3, 4, 5, 6.
\] (17)

From the dual feasibility conditions given in (13b) and (13b), we have $\alpha_k(i), \beta_k(i) \geq 0$. By inserting $\alpha_k(i), \beta_k(i) \geq 0$ in (17), we obtain the necessary condition for $q_1^*(i) = 1$ as
\[
\Lambda_1(i) \geq \max \{ \Lambda_2(i), \Lambda_3(i), \Lambda_4(i), \Lambda_5(i), \Lambda_6(i) \}.
\] (18)

Repeating the same procedure for $q_k^*(i) = 1, k = 2, \ldots, 6$, we obtain a necessary condition for selecting transmission mode $M_k^*$ in the $i$-th time slot as follows
\[
\Lambda_k^*(i) \geq \max_{k \in \{1, \ldots, 6\}} \{ \Lambda_k(i) \},
\] (19)

where the Lagrange multipliers $\mu_1, \mu_2, \gamma$ are chosen such that $C1, C2, C3$ in (10) hold and the optimal value of $t(i)$ in $C_{12r}(i)$ and $C_{21r}(i)$ is obtained in the next subsection. We note that if the selection metrics are not equal in the $i$-th time slot, only one of the modes satisfies (19). Therefore, the necessary conditions for the mode selection in (19) is sufficient. Moreover, in Appendix B we prove that the probability that two selection metrics are equal is zero due to the randomness of the time-continuous channel gains. Therefore, the necessary condition for selecting transmission mode $M_k$ in (19) is in fact sufficient and is the optimal selection policy.

**B. Optimal $P_j^* (i)$**

In order to determine the optimal $P_j(i)$, we have to calculate the derivatives in (12b). This leads to
\[
\frac{\partial \mathcal{L}}{\partial P_1(i)} = -\frac{1}{N \ln 2}
\left[
\left\{(1 - \mu_1)q_1(i) - t(i)(\mu_1 - \mu_2)q_3(i)\right\}
\right.
\times
\frac{S_1(i)}{1 + P_1(i)S_1(i)}
\left. + \left\{t(i)(\mu_1 - \mu_2) + 1 - \mu_1\right\}q_3(i)
\right]
\times
\frac{S_1(i)}{1 + P_1(i)S_1(i) + P_2(i)S_2(i)}
\left. + \gamma \frac{1}{N}(q_1(i) + q_3(i)) - \nu_1(i) = 0
\right)
\] (20a)

\[
\frac{\partial \mathcal{L}}{\partial P_2(i)} = -\frac{1}{N \ln 2}
\left[
\left\{(1 - \mu_2)q_2(i) + (1 - t(i))(\mu_1 - \mu_2)q_3(i)\right\}
\right.
\times
\frac{S_2(i)}{1 + P_2(i)S_2(i)}
\left. + \left\{t(i)(\mu_1 - \mu_2) + 1 - \mu_1\right\}q_3(i)
\right]
\times
\frac{S_2(i)}{1 + P_1(i)S_1(i) + P_2(i)S_2(i)}
\left. + \gamma \frac{1}{N}(q_2(i) + q_3(i)) - \nu_2(i) = 0
\right)
\] (20b)

\[
\frac{\partial \mathcal{L}}{\partial P_r(i)} = -\frac{1}{N \ln 2}
\left[
\left\{\mu_2 (q_4(i) + g_6(i)) + \mu_1 (q_5(i) + g_6(i))\right\}
\right.
\times
\frac{S_1(i)}{1 + P_r(i)S_1(i)}
\left. + \mu_1 (g_5(i) + q_6(i))\right]
\times
\frac{S_2(i)}{1 + P_r(i)S_2(i)}
\left.ight]
\]
\[ + \gamma \frac{1}{N}(q_4(i) + q_5(i) + q_6(i)) - \nu_r(i) = 0 \]  

(20c)

The above conditions allow the derivation of the optimal powers for each transmission mode in each time slot. For instance, in order to determine the transmit power of user 1 in transmission mode \( M_1 \), we assume \( q_1^*(i) = 1 \). From constraint C4 in (10), we obtained that the other selection variables are zero and therefore \( q_2^*(i) = 0 \). Moreover, if \( M_1 \) is selected then \( P_1^*(i) \neq 0 \) and thus from (14d), we obtain \( \nu_1^*(i) = 0 \). Substituting these results in (20a), we obtain

\[ P_{1}^{M_1}(i) = \left[ \frac{1 - \mu_1}{\gamma \ln 2} - \frac{1}{S_1(i)} \right]^+ , \]

(21)

where \([x]^+ = \max\{0, x\}\). In a similar manner, we obtain the optimal powers for user 2 in mode \( M_2 \), and the optimal powers of the relay in modes \( M_4 \) and \( M_5 \) as follows:

\[ P_{2}^{M_2}(i) = \left[ \frac{1 - \mu_2}{\gamma \ln 2} - \frac{1}{S_2(i)} \right]^+ \]

(22a)

\[ P_{r}^{M_4}(i) = \left[ \frac{\mu_2}{\gamma \ln 2} - \frac{1}{S_1(i)} \right]^+ \]

(22b)

\[ P_{r}^{M_5}(i) = \left[ \frac{\mu_1}{\gamma \ln 2} - \frac{1}{S_2(i)} \right]^+ \]

(22c)

In order to obtain the optimal powers of user 1 and user 2 in mode \( M_3 \), we assume \( q_3^*(i) = 1 \). From C4 in (10), we obtain that the other selection variables are zero, and therefore \( q_1^*(i) = 0 \) and \( q_2^*(i) = 0 \). We note that if one of the powers of user 2 and user 1 is zero mode \( M_3 \) is identical to modes \( M_1 \) and \( M_2 \), respectively, and for that case the optimal powers are already given by (21) and (22b), respectively. For the case when \( P_1^*(i) \neq 0 \) and \( P_2^*(i) \neq 0 \), we obtain \( \nu_1^*(i) = 0 \) and \( \nu_2^*(i) = 0 \) from (14d). Furthermore, for \( q_3^*(i) = 1 \), we will show in Appendix A that \( t(i) \) can only take the boundary values, i.e., zero or one, and cannot be in between. Hence, if we assume \( t(i) = 0 \), from (22a) and (22b), we obtain

\[ - \frac{1 - \mu_1}{\ln 2} \frac{S_1(i)}{1 + P_1(i)S_1(i) + P_2(i)S_2(i)} + \gamma = 0 \]

(23a)

\[ - \frac{1}{\ln 2} \left[ \frac{\mu_1}{\gamma \ln 2} - \frac{1}{S_1(i)} \right] \frac{S_1(i)}{1 + P_1(i)S_1(i) + P_2(i)S_2(i)} + \gamma = 0 \]

(23b)

By substituting (23a) in (23b), we obtain \( P_{2}^{M_3}(i) \) and then we can derive \( P_{1}^{M_3}(i) \) from (23b). This leads to

\[ P_{1}^{M_3}(i) = \begin{cases} P_{1}^{M_1}(i), & \text{if } S_2 \leq \frac{S_1}{\hat{S}_1(i) + 1} \\ \left[ \frac{1 - \mu_1}{\gamma \ln 2} - \frac{\mu_1 - \mu_2}{\gamma \ln 2} \frac{1}{S_1(i)} \right]^+, & \text{otherwise} \end{cases} \]

(24a)

\[ P_{2}^{M_3}(i) = \begin{cases} P_{2}^{M_2}(i), & \text{if } S_2 \geq \frac{1 - \mu_1}{1 - \mu_2} S_1 \\ \left[ \frac{\mu_1 - \mu_2}{\gamma \ln 2} - \frac{1}{S_1(i)} \right]^+, & \text{otherwise} \end{cases} \]

(24b)
Similarly, if we assume \( t(i) = 1 \), we obtain

\[
P_1^{M_3}(i) = \begin{cases} 
\mu_2 \left( \frac{S_1(i) - \mu_1}{\sqrt{\ln 2} - \frac{S_1(i)}{S_2(i)}} \right) + 1, & \text{if } S_2 \leq 1 - \frac{\mu_1}{\mu_2} S_1 \\
\mu_1 \left( \frac{S_2(i) - \mu_2}{\sqrt{\ln 2} - \frac{S_2(i)}{S_1(i)}} \right) + 1, & \text{otherwise}
\end{cases}
\]

(25a)

\[
P_2^{M_3}(i) = \begin{cases} 
\frac{S_1(i)}{\sqrt{\ln 2} - \frac{S_1(i)}{S_2(i)}} + 1, & \text{if } S_2 \geq 1 - \frac{\mu_1}{\mu_2} S_1 \\
\frac{S_2(i)}{\sqrt{\ln 2} - \frac{S_2(i)}{S_1(i)}} + 1, & \text{otherwise}
\end{cases}
\]

(25b)

We note that when \( P_1^{M_3}(i) = P_1^{M_1}(i) \), we obtain \( P_2^{M_3}(i) = 0 \) which means that mode \( M_3 \) is identical to mode \( M_1 \). Thus, there is no difference between both modes so we select \( M_1 \). In Figs 4(a) and 4(b), the comparison of \( \Lambda_1(i), \Lambda_2(i), \) and \( \Lambda_3(i) \) is illustrated in the space of \( (S_1(i), S_2(i)) \). Moreover, the shaded area represents the region in which the powers of users 1 and 2 are zero for \( M_1, M_2, \) and \( M_3 \).

For mode \( M_6 \), we assume \( q_{6(i)}^* = 1 \). From constraint C4 in (10), we obtain that the other selection variables are zero and therefore \( q_1^*(i) = 0 \) and \( q_2^*(i) = 0 \). Moreover, if \( q_{6(i)}^* = 1 \) then \( P_r^*(i) \neq 0 \) and thus from (14d), we obtain \( \nu_r^* = 0 \). Using these results in (20d), we obtain

\[
\frac{S_1(i)}{1 + P_r(i)S_1(i)} + \mu_1 \frac{S_2(i)}{1 + P_r(i)S_2(i)} = \gamma \ln(2)
\]

(26)

The above equation is a quadratic equation and has two solutions for \( P_r(i) \). However, since we have \( P_r(i) \geq 0 \), we can conclude that the left hand side of (26) is monotonically decreasing in \( P_r(i) \). Thus, if \( \mu_2 S_1(i) + \mu_1 S_2(i) > \gamma \ln(2) \), we have a unique positive solution for \( P_r(i) \) which is the maximum of the two roots of (26). Thus, we obtain

\[
P_r^{M_6}(i) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}^+
\]

(27)
where \( a = \gamma \ln 2 S_1(i) S_2(i) \), \( b = \gamma \ln 2 (S_1(i) + S_2(i)) - (\mu_1 + \mu_2) S_1(i) S_2(i) \), and \( c = \gamma \ln 2 - \mu_1 S_2(i) - \mu_2 S_1(i) \).

In Fig. 4 c), the comparison between selection metrics \( \Lambda_4(i) \), \( \Lambda_5(i) \), and \( \Lambda_6(i) \) is illustrated in the space of \((S_1(i), S_2(i))\). We note that \( \Lambda_6(i) \geq \Lambda_4(i) \) and \( \Lambda_6(i) \geq \Lambda_5(i) \) hold and the inequalities hold with equality if \( S_2(i) = 0 \) and \( S_1(i) = 0 \), respectively, which happen with zero probability for time-continuous fading. To prove \( \Lambda_6(i) \geq \Lambda_4(i) \), from (16), we obtain

\[
\Lambda_6(i) = \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i) \bigg|_{\gamma P_r(i) = \mu_i \gamma P_{rM}(i)}
\]

\[
\geq \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i) \bigg|_{\gamma P_r(i) = \mu_i \gamma P_{rM}(i)}
\]

\[
\geq \mu_2 C_{r1}(i) - \gamma P_r(i) \bigg|_{\gamma P_r(i) = \mu_i \gamma P_{rM}(i)} = \Lambda_4(i),
\]

where \( (a) \) follows from the fact that \( P_{rM}(i) \) maximizes \( \Lambda_6(i) \) and \( (b) \) follows from \( \mu_1 C_{r2}(i) \geq 0 \). The two inequalities \( (a) \) and \( (b) \) hold with equality only if \( S_2(i) = 0 \) which happens with zero probability in time-continuous fading or if \( \mu_1 = 0 \). However, in Appendix C \( \mu_1 = 0 \) is shown to lead to a contradiction. Therefore, the optimal policy does not select \( M_4 \) and \( M_5 \) and selects only modes \( M_1, M_2, M_3, \) and \( M_6 \).

C. Optimal \( t^*(i) \)

To find the optimal \( t(i) \), we assume \( q^*_l(i) = 1 \) and calculate the stationary condition in (12b). This leads to

\[
\frac{\partial \mathcal{L}}{\partial t(i)} = \frac{1}{N} (\mu_1 - \mu_2) [C_r(i) - C_{1r}(i) - C_{2r}(i)]
\]

\[
\phi_1(i) - \phi_0(i) = 0
\]

(29)

Now, we investigate the following possible cases for \( t^*(i) \):

Case 1: If \( 0 < t^*(i) < 1 \) then from (14b) and (14c), we have \( \phi_1(i) = 0, \ l = 0, 1 \). Therefore, from (29) and \( C_r(i) - C_{1r}(i) - C_{2r}(i) \leq 0 \), we obtain \( \mu_1 = \mu_2 \). Then, from (20c) and (20d), we obtain

\[
\begin{cases}
- \frac{1}{\ln 2} (1 - \mu_1) \frac{S_1(i)}{1 + P_1(i) S_1(i) + P_2(i) S_2(i)} + \gamma = 0 \\
- \frac{1}{\ln 2} (1 - \mu_1) \frac{S_2(i)}{1 + P_1(i) S_1(i) + P_2(i) S_2(i)} + \gamma = 0
\end{cases}
\]

(30)

In Appendix C we show that \( \mu_1 \neq 1 \), therefore, the above conditions can be satisfied simultaneously only if \( S_1(i) = S_2(i) \), which, considering the randomness of the time-continuous channel gains, occurs with zero probability. Hence, the optimal \( t(i) \) takes the boundary values, i.e., zero or one, and not values in between.

Case 2: If \( t^*(i) = 0 \) then from (14b), we obtain \( \phi_1(i) = 0 \) and from (13b), we obtain \( \phi_0(i) \geq 0 \). Combining these results into (29), the necessary condition for \( t(i) = 0 \) is obtained as \( \mu_1 \geq \mu_2 \).

Case 3: If \( t^*(i) = 1 \), then from (14b), we obtain \( \phi_0(i) = 0 \) and from (13b), we obtain \( \phi_1(i) \geq 0 \). Combining these results into (29), the necessary condition for \( t(i) = 1 \) is obtained as \( \mu_1 \leq \mu_2 \).

We note that if \( \mu_1 = \mu_2 \), we obtain either \( P_{rM}(i) = 0 \) or \( P_{rM}(i) = 0, \ \forall i \). Therefore, mode \( M_3 \) is not selected and the value of \( t(i) \) does not affect the sum rate. Moreover, from the selection metrics in (16), we can conclude
that $\mu_1 > \mu_2$ and $\mu_1 < \mu_2$ correspond to $\Omega_1 > \Omega_2$ and $\Omega_1 < \Omega_2$, respectively. Therefore, the optimal value of $t(i)$ is given by

$$t^*(i) = \begin{cases} 
0, & \Omega_1 \geq \Omega_2 \\
1, & \Omega_1 < \Omega_2 
\end{cases} \tag{31}$$

Now, the optimal values of $q_k(i), P_j(i)$, and $t(i), \forall i, j, k$ are derived based on which Theorem 2 can be constructed. This completes the proof.

**APPENDIX B**

PROOF OF OPTIMALITY OF BINARY RELAXATION

In this appendix, we prove that the optimal solution of the problem with the relaxed constraint, $0 \leq q_k(i) \leq 1$, selects the boundary values of $q_k(i)$, i.e., zero or one. Therefore, the binary relaxation does not change the solution of the problem. If one of the $q_k(i), k = 1, \ldots, 6$, adopts a non-binary value in the optimal solution, then in order to satisfy constraint $C4$ in (42), there has to be at least one other non-binary selection variable in that time slot. Assuming that the mode indices of the non-binary selection variables are $k'$ and $k''$ in the $i$-th time slot, we obtain $\alpha_k(i) = 0, k = 1, \ldots, 6$ from (14a), and $\beta_k(i) = 0$ and $\beta_k(i) = 0$ from (14b). Then, by substituting these values into (15), we obtain

$$\lambda(i) = \Lambda_{k'}(i) \tag{32a}$$

$$\lambda(i) = \Lambda_{k''}(i) \tag{32b}$$

$$\lambda(i) - \beta_k(i) = \Lambda_k(i), \quad k \neq k', k'' \tag{32c}$$

From (32a) and (32b), we obtain $\Lambda_{k'}(i) = \Lambda_{k''}(i)$ and by subtracting (32a) and (32b) from (32c), we obtain

$$\Lambda_{k'}(i) - \Lambda_k(i) = \beta_k(i), \quad k \neq k', k'' \tag{33a}$$

$$\Lambda_{k''}(i) - \Lambda_k(i) = \beta_k(i), \quad k \neq k', k'' \tag{33b}$$

From the dual feasibility condition given in (13b), we have $\beta_k(i) \geq 0$ which leads to $\Lambda_{k'}(i) = \Lambda_{k''}(i) \geq \Lambda_k(i)$. However, as a result of the randomness of the time-continuous channel gains, $\Pr\{\Lambda_{k'}(i) = \Lambda_{k''}(i)\} > 0$ holds for some transmission modes $M_{k'}$ and $M_{k''}$, if and only if we obtain $\mu_1 = 0, 1$ or $\mu_2 = 0, 1$ which leads to a contradiction as shown in Appendix C. This completes the proof.

**APPENDIX C**

THRESHOLD REGIONS

In this appendix, we find the intervals which contain the optimal value of $\mu_1$ and $\mu_2$. We note that for different values of $\mu_1$ and $\mu_2$, some of the optimal powers derived in (21), (22), (24), (25), and (27) are zero for all channel realizations. For example, if $\mu_1 = 1$, we obtain $P_1^{M_1}(i) = 0, \forall i$ from (21). Fig. 5 illustrates the set of modes that
can take positive powers with non-zero probability in the space of \((\mu_1, \mu_2)\). In the following, we show that any values of \(\mu_1\) and \(\mu_2\) except \(0 < \mu_1 < 1\) and \(0 < \mu_2 < 1\) cannot lead to the optimal sum rate or violate constraints C1 or C2 in (10).

**Case 1:** Sets \(\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}\) and \(\{\mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6\}\) lead to selection of either the transmission from the users to the relay or the transmission from the relay to the users, respectively, for all time slots. This leads to violation of constraints C1 and C2 in (10) and thus the optimal values of \(\mu_1\) and \(\mu_2\) are not in this region.

**Case 2:** In set \(\{\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_6\}\), both modes \(\mathcal{M}_4\) and \(\mathcal{M}_6\) need the transmission from user 2 to the relay which cannot be realized in this set. Thus, this set leads to violation of constraint C2 in (10). Similarly, in set \(\{\mathcal{M}_2, \mathcal{M}_5, \mathcal{M}_6\}\), both modes \(\mathcal{M}_5\) and \(\mathcal{M}_6\) require the transmission from user 1 to the relay which cannot be selected in this set. Thus, this region of \(\mu_1\) and \(\mu_2\) leads to violation of constraint C1 in (10).

**Case 3:** In set \(\{\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6\}\), there is no transmission from user 2 to the relay. Therefore, the optimal values of \(\mu_1\) and \(\mu_2\) have to guarantee that modes \(\mathcal{M}_4\) and \(\mathcal{M}_6\) are not selected for any channel realization. However, from (10), we obtain

\[
\Lambda_6(i) = \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_6}(i)} \\
\geq a \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_6}(i)} \\
\geq b \mu_1 C_{r2}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_6}(i)} = \Lambda_5(i),
\]

where \((a)\) follows from the fact that \(P_r^{M_6}(i)\) maximizes \(\Lambda_6(i)\) and \((b)\) follows from \(\mu_2 C_{r1}(i) \geq 0\). The two inequalities \((a)\) and \((b)\) hold with equality only if \(S_1(i) = 0\) which happens with zero probability for time-continuous fading, or \(\mu_2 = 0\) which is not included in this region. Therefore, mode \(\mathcal{M}_6\) is selected in this region which leads to violation of constraint C2 in (10). A similar statement is true for set \(\{\mathcal{M}_2, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6\}\). Thus, the optimal values of \(\mu_1\) and \(\mu_2\) cannot be in these two regions.

**Case 4:** In set \(\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6\}\), we obtain

\[
\Lambda_6(i) = \mu_1 C_{r2}(i) + \mu_2 C_{r1}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_6}(i)} \\
\leq a \mu_2 C_{r1}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_6}(i)} \\
\leq b \mu_2 C_{r1}(i) - \gamma P_r(i)\bigg|_{P_r(i) = P_r^{M_4}(i)} = \Lambda_4(i),
\]

where inequality \((a)\) comes from the fact that \(\mu_1 C_{r2}(i) \leq 0\) and the equality holds when \(S_2(i) = 0\) which happens with zero probability, or \(\mu_1 = 0\). Inequality \((b)\) holds since \(P_r^{M_4}(i)\) maximizes \(\Lambda_4(i)\) and holds with equality only if \(P_r^{M_4}(i) = P_r^{M_6}(i)\) and consequently \(\mu_1 = 0\). If \(\mu_1 \neq 0\), mode \(\mathcal{M}_6\) is not selected and there is no transmission from the relay to user 2. Therefore, the optimal values of \(\mu_1\) and \(\mu_2\) have to guarantee that modes \(\mathcal{M}_1\) and \(\mathcal{M}_4\) are not selected for any channel realization. Thus, we obtain \(\mu_1 = 1\) which is not contained in this region. If \(\mu_1 = 0\),
from (16), we obtain

$$
\Lambda_6(i) = \Lambda_4(i) = \mu_2 C_{r1}(i) - \gamma P_r(i) \bigg|_{P_r(i) = P_{M4}(i)} \\
\leq C_{r1}(i) - \gamma P_r(i) \bigg|_{P_r(i) = P_{M4}(i)} \\
\leq C_{1r}(i) - \gamma P_1(i) \bigg|_{P_1(i) = P_{M4}(i)} = \Lambda_1(i)
$$

(36)

where both inequalities \((a)\) and \((b)\) hold with equality only if \(\mu_2 = 1\). If \(\mu_2 \neq 1\), modes \(M_4\) and \(M_6\) are not selected. Thus, there is no transmission from the relay to the users which leads to violation of \(C_1\) and \(C_2\) in (10). If \(\mu_2 = 1\), we obtain \(P_{M2}(i) = 0\), thus mode \(M_2\) cannot be selected and either \(P_{M3}(i) = 0\) or \(P_{M4}(i) = 0\), thus mode \(M_3\) cannot be selected either. Since both modes \(M_4\) and \(M_6\) require the transmission from user 2 to the relay, and both modes \(M_2\) and \(M_3\) are not selected, constraint \(C_2\) in (10) is violated and \(\mu_1 = 0\) and \(\mu_2 = 1\) cannot be optimal. A similar statement is true for set \(\{M_1, M_2, M_3, M_5, M_6\}\). Therefore, the optimal values of \(\mu_1\) and \(\mu_2\) are not in this region.

Hence, set \(\{M_1, M_2, M_3, M_4, M_5, M_6\}\) contains the optimal values of \(\mu_1\) and \(\mu_2\), i.e., \(0 < \mu_1 < 1\) and \(0 < \mu_2 < 1\). This completes the proof.

REFERENCES

[1] S. J. Kim, N. Devroye, P. Mitran, and V. Tarokh, “Achievable Rate Regions and Performance Comparison of Half Duplex Bi-Directional Relaying Protocols,” IEEE Trans. Inf. Theory, vol. 57, no. 10, pp. 6405–6418, Oct. 2011.

[2] Y. Wu, P. A. Chou, and S.-Y. Kung, “Information Exchange in Wireless Networks with Network Coding and Physical-Layer Broadcast,” in Proc. 39th Ann. Conf. Inf. Sci. Syst., March 2005.

[3] P. Popovski and H. Yomo, “Bi-directional Amplification of Throughput in a Wireless Multi-Hop Network,” in Proc. IEEE VTC, vol. 2, May 2006, pp. 588–593.
[4] T. Oechtering, C. Schnurr, I. Bjelakovic, and H. Boche, “Broadcast Capacity Region of Two-Phase Bidirectional Relaying,” *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 454–458, Jan. 2008.

[5] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley, John and Sons, Incorporated, 1991.

[6] R. F. Wyrembelski, T. J. Oechtering, and H. Boche, “Decode-and-Forward Strategies for Bidirectional Relaying,” in *Proc. IEEE PIMRC*, Sept. 2008, pp. 1–6.

[7] P. Popovski and H. Yomo, “Physical Network Coding in Two-Way Wireless Relay Channels,” in *Proc. IEEE ICC*, June 2007, pp. 707–712.

[8] H. Liu, P. Popovski, E. de Carvalho, and Y. Zhao, “Sum-Rate Optimization in a Two-Way Relay Network with Buffering,” *IEEE Commun. Lett.*, vol. 17, no. 1, pp. 95–98, Jan. 2013.

[9] N. Zlatanov, R. Schober, and P. Popovski, “Buffer-Aided Relaying with Adaptive Link Selection,” *IEEE J. Select. Areas Commun.*, vol. 31, no. 8, pp. 1–13, Aug. 2013.

[10] N. Zlatanov and R. Schober, “Capacity of the State-Dependent Half-Duplex Relay Channel Without Source-Destination Link,” *Submitted IEEE Transactions on Information Theory*, 2013. [Online]. Available: [http://arxiv.org/abs/1302.3777](http://arxiv.org/abs/1302.3777)

[11] V. Jamali, N. Zlatanov, A. Ikhlef, and R. Schober, “Adaptive Mode Selection in Bidirectional Buffer-aided Relay Networks with Fixed Transmit Powers,” *Submitted in part to EUSIPCO’13*, 2013. [Online]. Available: [http://arxiv.org/abs/1303.3732](http://arxiv.org/abs/1303.3732)

[12] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.