Birational geometry and deformations of nilpotent orbits

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Let $G$ and $\mathfrak{g}$ be a complex simple Lie group and its Lie algebra. An orbit $O \subset \mathfrak{g}$ for the adjoint action of $G$, is called a nilpotent orbit if it consists of nilpotent elements. For a parabolic subgroup $P$ of $G$, there exists a unique nilpotent orbit $O$ such that $O \cap n(p)$ is an open dense subset of $n(p)$, where $n(p)$ is the nilradical of $p$. Then this orbit $O$ is called the Richardson orbit for $P$. The closure $\bar{O}$ of $O$ admits symplectic singularities and it is an interesting object in the birational geometry and the singularity theory.

There is a generically finite, projective morphism (Springer map) from the cotangent bundle of $G/P$ to $\bar{O}$:

$$s: T^*(G/P) \rightarrow \bar{O}.$$ 

Let $T^*(G/P) \xrightarrow{s} \tilde{O} \rightarrow \bar{O}$ be the Stein factorization of $s$. When $\text{deg}(s) = 1$, $s$ is a crepant resolution of $\bar{O}$. But, in this case, $\bar{O}$ generally has other crepant resolutions. In [Na], we have shown that all such crepant resolutions are obtained as the Springer resolutions $s'$ for certain (finitely many) parabolic subgroups $P'$, and classified the conjugacy classes of such parabolic subgroups in terms of Dynkin diagrams. The (closed) movable cone $\text{Mov}(s) \subset \text{Pic}(T^*(G/P)) \otimes \mathbb{R}$ (cf. (P.2) below) is decomposed by the nef cones $\text{Amp}(s')$ of various Springer resolutions $s'$. In general, $\text{Mov}(s) \neq \text{Pic}(T^*(G/P)) \otimes \mathbb{R}$. The first purpose of this paper, is to understand all parts of $\text{Pic}(T^*(G/P)) \otimes \mathbb{R}$ in terms of ample cones of certain varieties. The second purpose is, to prove a similar result to [Na] when the Springer map $s$ has $\text{deg}(s) > 1$.

Our strategy is to use Brieskorn-Slodowy diagrams. Let us fix a maximal torus $T$ of $P$, and denote by $\mathfrak{h}$ its Lie algebra. Let $l(p)$ be the Levi factor of $p$ containing $\mathfrak{h}$, and $t(p)$ the centralizer of $l(p)$. Let $r(p)$ (resp. $n(p)$) be the solvable radical (resp. nilradical) of $p$. The Brieskorn-Slodowy diagram is the following commutative diagram (for details, see (P.1)):
\[ G \times^P r(p) \rightarrow Gr(p) \]
\[ \downarrow \quad \downarrow \]
\[ \mathfrak{t}(p) \rightarrow \mathfrak{h}/W. \]

The variety \( X_p := G \times^P r(p) \) is a flat deformation of the cotangent bundle \( T^*(G/P) = G \times^P n(p) \) over the parameter space \( \mathfrak{t}(p) \) with some universal property; this is indeed the universal Poisson deformation of the complex symplectic variety \( T^*(G/P) \). In turn, the fiber product \( \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p) \) has the projection map to \( \mathfrak{t}(p) \). This map is equi-dimensional. The central fiber is the closure \( \bar{O} \) of the Richardson orbit \( O \), and the fiber over a general point \( t \in \mathfrak{t}(p) \) is the semi-simple orbit containing \( t \). Lemma 1.1 claims that the induced map

\[ \mu'_p : X_p \to \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p) \]

is a birational projective morphism, and it is an isomorphism over a general point \( t \in \mathfrak{t}(p) \). Let

\[ X_p \xrightarrow{\mu_p} Y_{\mathfrak{t}(p)} \to \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p) \]

be the Stein factorization of \( \mu'_p \). In other words, \( Y_{\mathfrak{t}(p)} \) is the normalization of \( \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p) \). Here we must remark that \( Gr(p) \) actually depends only on \( \mathfrak{t}(p) \) (or \( l(p) \)) (cf. Theorem 0.1). So, \( Y_{\mathfrak{t}(p)} \) depends only on \( \mathfrak{t}(p) \) as this notation indicates. As we remark in Observation 1, the central fiber \( Y_{\mathfrak{t}(p),0} \) of \( Y_{\mathfrak{t}(p)} \to \mathfrak{t}(p) \), has \( \bar{O} \) as its normalization. This is an important point; when \( \text{deg}(s) > 1 \), the natural map \( Y_{\mathfrak{t}(p),0} \to \bar{O} \) has degree \( > 1 \). Since \( Y_{\mathfrak{t}(p),0} \) coincides with \( \bar{O} \) at least set theoretically (cf. Observation 1), \( Y_{\mathfrak{t}(p)} \to \mathfrak{t}(p) \) is almost a deformation of \( \bar{O} \). By Lemma 1.2, \( \mu_p \) is a crepant resolution which is an isomorphism in codimension one. Now let us fix one parabolic subalgebra \( p_0 \) containing \( \mathfrak{h} \), and put \( l_0 := l(p_0) \) and \( t_0 := \mathfrak{t}(p_0) \). Define \( S(l_0) \) as the set of all parabolic subalgebras \( p \) containing \( \mathfrak{h} \) such that their Levi factors \( l(p) \) coincide with \( l_0 \).

Our main theorem (= Theorem 1.3) states that any crepant resolution of \( Y_{t_0} \) has the form \( \mu_p : X_p \to Y_{t_0} \) with \( p \in S(l_0) \). Note that \( \mathfrak{t}(p) = t_0 \) for \( p \in S(l_0) \). Moreover, \( \mu_p \neq \mu_{p'} \) if \( p \neq p' \).

A main ingredient in the proof of Theorem 1.3 is the notion of a twist. Each parabolic subalgebra \( p \) corresponds to a marked Dynkin diagram when
a Borel subalgebra $b \subset p$ is fixed (cf. (P.1)). Let $D$ be such a marked Dynkin diagram determined by $p \in S(l_0)$. Some vertices of $D$ are marked with black. For each marked vertex $v$, we can make a new parabolic subalgebra $p' \in S(l_0)$. The new parabolic subalgebra $p'$ is called the twist of $p$ by $v$. Depending on $v$, the twists are classified into two classes (twists of the 1-st kind, and twists of the 2-nd kind). If the twist is of the 1-st kind, then $p'$ is not conjugate to $p$. But, for a twist of the 2-nd kind, $p'$ is conjugate to $p$. The two varieties $X_p$ and $X_{p'}$ are related by a flop. The proof of Theorem 1.3 goes as follows. Let $\mu : X \rightarrow Y_{l_0}$ be an arbitrary crepant resolution. Take a $\mu$-ample line bundle $L$ on $X$ and let $L^{(0)} \in \text{Pic}(X_p)$ be its proper transform. If $L^{(0)}$ is not $\mu_{p_0}$-nef, we have an extremal birational contraction map $g_0 : X_{p_0} \rightarrow \tilde{X}_{p_0}$ with respect to $L^{(0)}$. By the definition of $g_0$, $L^{(0)}$ is $g_0$-negative. Moreover, since $\mu_{p_0}$ is an isomorphism in codimension one (cf. Lemma 1.2), $g_0$ is an isomorphism in codimension one. A flop of $g_0$ is a diagram $X_{p_0} \xrightarrow{g_0} \tilde{X}_{p_0} \xleftarrow{g_0'} X_1$ such that the proper transform $L^{(1)} \in \text{Pic}(X_1)$ of $L^{(0)}$ is $g_0'$-ample (cf. (P.2)). In our case, $X_1$ coincides with $X_{p_1}$ for a twist $p_1$ of $p_0$. If $L^{(1)}$ is not $\mu_{p_1}$-nef, then we repeat this process. In this way, we continue the flops $X_{p_0} \rightarrow X_{p_1} \rightarrow \cdots$ as long as the proper transform of $L^{(0)}$ is not nef over $Y_{l_0}$. But, since $S(l_0)$ is a finite set, the sequence terminates at some $X_{p_k}$. Then the proper transform $L^{(k)} \in \text{Pic}(X_{p_k})$ of $L^{(0)}$ is $\mu_{p_k}$-nef, and $X = X_{p_k}$ (cf. Corollary 1.5).

For a parabolic subgroup $P$ with $p \in S(l_0)$, $\text{Pic}(G/P) \otimes \mathbb{R}$ is canonically identified with $M(L_0) \otimes \mathbb{R}$ (cf. (P.3)), where

$$M(L_0) := \text{Hom}_{\text{alg, gp}}(L_0, \mathbb{C}^*)$$

On the other hand, $\text{Pic}(G/P)$ and $\text{Pic}(X_p)$ are identified by the projection map $X_p \rightarrow G/P$. We therefore have an isomorphism

$$\Phi_P : \text{Pic}(X_p) \otimes \mathbb{R} \rightarrow M(L_0) \otimes \mathbb{R}.$$ 

Let us take another $P'$ with a fixed Levi part $L_0$. The isomorphism above is natural in the sense that $(\Phi_{P'})^{-1} \circ \Phi_P : \text{Pic}(X_p) \otimes \mathbb{R} \cong \text{Pic}(X_{p'}) \otimes \mathbb{R}$ coincides with the isomorphism defined by the proper transform by $X_p \rightarrow X_{p'}$ (Observation 2). Moreover, the image $\Phi_P(\text{Amp}(\mu_p))$ of the nef cone for $\mu_p$ by $\Phi_P$, can be described explicitly in terms of dominant characters. An important corollary to (the proof of) Theorem 1.3 is Remark 1.6, which says that

$$M(L_0) \otimes \mathbb{R} = \bigcup_{p \in S(l_0)} \Phi_P(\text{Amp}(\mu_p)).$$
On the $M(L_0)_{\mathbb{R}}$, a subgroup of the Weyl group $W$ of $\mathfrak{g}$ acts. More explicitly, this subgroup is the normalizer $N_W(L_0)$ of $L_0$. Let $W(L_0)$ be the Weyl group of $L_0$. Then $N_W(L_0)/W(L_0)$ acts effectively on $M(L_0)_{\mathbb{R}}$. In Section 2, we shall prove that the set $S(l_0)$ contains exactly $N \cdot \sharp(N_W(L_0)/W(L_0))$ elements, where $N$ is the number of the conjugacy classes of parabolic subalgebras contained in $S(l_0)$. The number $N$ can be calculated explicitly in terms of marked Dynkin diagrams $[Na]$, Definition 1 (see also [Ri]). Howlett [Ho] gives an explicit description of $N_W(L_0)/W(L_0)$. When the Springer map $s : T^*(G/P_0) \to \mathcal{O}$ is a resolution, the movable cone $\text{Mov}(\mu_p)$ for $\mu_p$ is a fundamental domain of the $N_W(L_0)/W(L_0)$-action (Proposition 2.3). In Section 3, we shall study the crepant resolutions of $\tilde{\mathcal{O}}$. The problem is actually in the case where $\text{deg}(s) > 1$. Now we can use the map $\mu_p : X_p \to Y_{\ell(p)}$ to study the map $\pi : T^*(G/P) \to \mathcal{O}$, because $\tilde{\mathcal{O}}$ is the normalization of $Y_{\ell(p),0}$ and $\pi$ is the Stein factorization of the birational map $\mu_{p,0}$. The main result in this section is Corollary 3.4. In order to state the result, we need to go back to the notion “twist” again. As remarked above, there are two kind of twists. In Section 3, we divide the twists of the 2-nd kind into two classes: small twists and divisorial twists. For $p \in S(l_0)$, let $p'$ be a twist of $p$. Let us compare $X_{p,0}$ and $X_{p',0}$, where $X_{p,0}$ (resp. $X_{p',0}$) is the central fiber, that is, the fiber of the map $X_p \to \mathfrak{t}_0$ (resp. $X_{p'} \to \mathfrak{t}_0$) over $0 \in \mathfrak{t}_0$. By Proposition 3.1, we see that $X_{p,0}$ and $X_{p',0}$ are related by a flop if and only if the twist is of the 1-st kind or is a small twist of the 2-nd kind. For a twist of the 1-st kind, this flop is one of those classified in $[Na]$ (Mukai flops of type A, D, $E_6$). But, for a small twist of the 2-nd kind, a new flop appears. Together with these new flops, we call them Mukai flops (cf. Definition 2). Let us fix a crepant resolution $\mu_{p,0} : X_{p,0} \to Y_{\ell_0,0}$. Take an arbitrary crepant resolution $\mu : X \to Y_{\ell_0,0}$. Corollary 3.4 states that $X$ is of the form $X_{p,0}$ for some $p \in S(l_0)$, and $X$ is related with $X_{p,0}$ by a sequence of Mukai flops. The proof of Corollary 3.4 goes as follows. Let $L' \in \text{Pic}(X)$ be $\mu$-ample and let $L \in \text{Pic}(X_{p,0})$ be its proper transform. Then $L$ is $\mu_{p,0}$-movable. Since $\text{Pic}(X_{p,0})$ is isomorphic to $\text{Pic}(X_{p_1})$, one can lift $L$ to $\mathcal{L} \in \text{Pic}(X_{p_1})$. Then $\mathcal{L}$ is $\mu_{p,0}$-movable. By the same argument as in Theorem 1.3, we can repeat the flops corresponding to the twists of $p_0$, and finally arrive at $X_{p_k}$ where the proper transform of $\mathcal{L}$ is $\mu_{p_k}$-nef. Since $L$ is $\mu_{p,0}$-movable, each flops corresponds to a twist of the

\footnote{This condition can be translated to a combinatorial condition for $L_0$ when $G$ is classical $[He]$.}
1-st kind or a small twist of the 2-nd kind. This implies that the restriction of each flop to the central fibers is a Mukai flop. Then $X_{p_k,0} = X$ and we get the desired sequence of Mukai flops $X_{p_0,0} \to X_{p_1,0} \to \ldots \to X_{p_k,0}$.

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**Preliminaries**

We shall recall some basic facts on nilpotent orbits, algebraic groups and birational geometry. Notation defined here will be used in the next sections.

(P.1) **Parabolic subgroups and Brieskorn-Slodowy diagrams**: Let $G$ be a complex simple Lie group (or a simple algebraic group defined over $\mathbb{C}$) and let $\mathfrak{g}$ be its Lie algebra. We fix a maximal torus $T$ of $G$ and denote by $\mathfrak{h}$ its Lie algebra. Let $\Phi$ be the root system for $\mathfrak{g}$ determined by $\mathfrak{h}$. The root system $\Phi$ has a natural involution $-1$. With $-1$, one can associate an automorphism $\varphi_\mathfrak{g}$ of $\mathfrak{g}$ of order 2 (cf. [Hu], 14.3). This involution will play an important role in Section 1. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the root space decomposition. Let us choose a base $\Delta$ of $\Phi$ and denote by $\Phi^+$ the set of positive roots. Then $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is a Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Let $B$ be the corresponding Borel subgroup of $G$. Take a subset $I$ of $\Delta$. Let $\Phi_I$ be the root subsystem of $\Phi$ generated by $I$ and put $\Phi^-_I := \Phi_I \cap \Phi^-$, where $\Phi^-$ is the set of negative roots. Then

$$\mathfrak{p}_I := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-_I} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

is a parabolic subalgebra containing $\mathfrak{b}$. This parabolic subalgebra $\mathfrak{p}_I$ is called a standard parabolic subalgebra with respect to $I$. Let $P_I$ be the corresponding parabolic subgroup of $G$. By definition, $B \subset P_I$. Any parabolic subgroup $P$ of $G$ is conjugate to a standard parabolic subgroup $P_I$ for some $I$. Moreover, if two subsets $I, I'$ of $\Delta$ are different, $P_I$ and $P_{I'}$ are not conjugate. Thus, a conjugacy class of parabolic subgroups of $G$ is completely determined by $I \subset \Delta$. In this paper, to specify the subset $I$ of $\Delta$, we shall
use the marked Dynkin diagram. Recall that $\Delta \subset \Phi$ defines a Dynkin diagram; each vertex corresponds to a simple root (an element of $\Delta$). Now, if a subset $I$ of $\Delta$ is given, we indicate the vertices corresponding to $I$ by white vertices, and other vertices by black vertices. A black vertex is called a marked vertex. A Dynkin diagram with such a marking is called a marked Dynkin diagram, and a marked Dynkin diagram with only one marked vertex is called a single marked Dynkin diagram. Note that the standard parabolic subgroup corresponding to a single marked Dynkin diagram is a maximal parabolic subgroup. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Let $r(\mathfrak{p})$ (resp. $n(\mathfrak{p})$) be the solvable radical (resp. nilpotent radical) of $\mathfrak{p}$. We put $\mathfrak{k}(\mathfrak{p}) := r(\mathfrak{p}) \cap \mathfrak{h}$. Then

$$r(\mathfrak{p}) = \mathfrak{e}(\mathfrak{p}) \oplus n(\mathfrak{p}).$$

On the other hand, the Levi factor $l(\mathfrak{p})$ of $\mathfrak{p}$ is defined as $l(\mathfrak{p}) := \mathfrak{g}^{\mathfrak{e}(\mathfrak{p})}$. Here, $\mathfrak{g}^{\mathfrak{e}(\mathfrak{p})} := \{ x \in \mathfrak{g}; [x, y] = 0, \forall y \in \mathfrak{e}(\mathfrak{p}) \}$. Note that $\mathfrak{e}(\mathfrak{p})$ is the center of $l(\mathfrak{p})$. Then

$$\mathfrak{p} = l(\mathfrak{p}) \oplus n(\mathfrak{p}).$$

If $\mathfrak{p} = \mathfrak{p}_I$, we have

$$\mathfrak{e}(\mathfrak{p}_I) = \{ \mathfrak{h} \in \mathfrak{h}; \alpha(h) = 0, \forall \alpha \in I \}.$$

Moreover, we define

$$\mathfrak{e}(\mathfrak{p}_I)^{reg} = \{ \mathfrak{h} \in \mathfrak{e}(\mathfrak{p}_I); \alpha(h) \neq 0, \forall \alpha \in \Phi \setminus \Phi_I \}.$$

Note that $\mathfrak{e}(\mathfrak{p}_I)^{reg}$ is an open subset of $\mathfrak{e}(\mathfrak{p}_I)$.

Next let us consider the adjoint action of $G$ on $\mathfrak{g}$. An orbit $O \subset \mathfrak{g}$ for this action is called a nilpotent orbit if $O$ contains a nilpotent element of $\mathfrak{g}$. There is a unique nilpotent orbit $\mathcal{O}$ such that $n(\mathfrak{p}) \cap \mathcal{O}$ is a dense open subset of $n(\mathfrak{p})$. Such an orbit is called the Richardson orbit for $\mathfrak{p}$. Let $P \subset G$ be the parabolic subgroup such that $\text{Lie}(P) = \mathfrak{p}$. The cotangent bundle $T^*(G/P)$ of $G/P$ is isomorphic to the vector bundle $G \times^P n(\mathfrak{p})$, which is the quotient space of $G \times n(\mathfrak{p})$ by the equivalence relation $\sim$. Here $(g, x) \sim (g', x')$ if $g' = gp$ and $x' = Ad_{p^{-1}}(x)$ for some $p \in P$. Let us define a map $s : G \times^P n(\mathfrak{p}) \to \mathfrak{g}$ by $s([g, x]) := Ad_g(x)$. Then the image of $s$ coincides with the closure $\overline{\mathcal{O}}$ of the Richardson orbit for $\mathfrak{p}$. The map

$$s : T^*(G/P) \to \overline{\mathcal{O}}$$
is called the *Springer map* for \( P \). The Springer map is a generically finite, projective morphism. Thus, the Springer map \( s \) factorizes as

\[
T^* (G/P) \xrightarrow{s} \tilde{O} \to \bar{O}.
\]

The birational map \( \pi \) will be a main object in Section 3. In particular, when \( \deg(s) = 1 \), we call \( s \) the *Springer resolution* of \( \mathcal{O} \). The following theorem implies that \( Gr(p) \) depends only on \( \mathfrak{g}(p) \).

**Theorem 0.1.** \( Gr(p) = G\mathfrak{g}(p) \).

**Proof.** See [B-K], Satz 5.6.

Every element \( x \) of \( \mathfrak{g} \) can be uniquely written as \( x = x_n + x_s \) with \( x_n \) nilpotent and with \( x_s \) semi-simple such that \( [x_n, x_s] = 0 \). Let \( W \) be the Weyl group of \( G \) with respect to \( T \). The set of semi-simple adjoint orbits in \( \mathfrak{g} \) is identified with \( \mathfrak{h}/W \). Let \( \mathfrak{g} \to \mathfrak{h}/W \) be the map defined as \( x \to [\mathcal{O}_{x_s}] \). There is a direct sum decomposition

\[
r(p) = \mathfrak{g}(p) \oplus n(p), \quad (x \to x_1 + x_2)
\]

where \( n(p) \) is the nil-radical of \( p \) (cf. [Slo], 4.3). We have a well-defined map

\[
G \times P r(p) \to \mathfrak{g}(p)
\]

by sending \([g, x] \in G \times P r(p)\) to \( x_1 \in \mathfrak{g}(p)\) and there is a commutative diagram

\[
\begin{array}{ccc}
G \times P r(p) & \to & Gr(p) \\
\downarrow & & \downarrow \\
\mathfrak{g}(p) & \to & \mathfrak{h}/W,
\end{array}
\]

(cf. [Slo], 4.3). In this paper, we call this a **Brieskorn-Slodowy diagram**.

**Birational geometry and flops:** Let \( f : X \to S \) be a projective surjective morphism of normal varieties with connected fibers. For simplicity, we assume that \( S \) is an affine variety and \( R^1 f_* \mathcal{O}_X = 0 \). We are mainly interested in the two cases (1) \( X \) is a flag variety and \( S \) is a point, and (2) \( S \) is an affine variety with rational singularities and \( f \) is a resolution of \( S \). We put \( N^1(f) := \text{Pic}(X)/f^* \text{Pic}(S) \) modulo torsion, and \( N^1(f)_{\mathbb{R}} = N^1(f) \otimes \mathbb{R} \). We sometimes write \( N^1(X/S) \) (resp. \( N^1(X/S)_{\mathbb{R}} \)) for \( N^1(f) \)
(resp. $N^1(f)_{\mathbb{R}}$). Note that $N^1(f)_{\mathbb{R}}$ is a finite dimensional $\mathbb{R}$-vector space. In the following sections, we will only consider the case where $S$ is an affine cone (over an origin) with a $\mathbb{C}^*$-action with positive weights. Then $\text{Pic}(S) = 0$, and $N^1(f)_{\mathbb{R}} = \text{Pic}(X) \otimes \mathbb{R}$. We denote by $N_1(f)$ (or $N_1(X/S)$) the abelian group of numerical classes of curves contained in fibers of $f$. Here two curves $C$ and $C'$ (in some fibers) are numerically equivalent if $(L.C) = (L.C')$ for all $L \in \text{Pic}(X)$. We define $N_1(f)_{\mathbb{R}} := N_1(f) \otimes \mathbb{Z}_{\mathbb{R}}$. Note that $N_1(f)_{\mathbb{R}}$ and $N_1(f)$ are dual to each other by the intersection form. A line bundle $L$ on $X$ is called $f$-nef (or nef over $S$) if $(L.C) \geq 0$ for all irreducible proper curves $C$ contained in closed fibers of $f$. The open convex cone $\text{Amp}(f) \subset N^1(f)_{\mathbb{R}}$ (or $\text{Amp}(X/S)$) generated by $f$-ample line bundles is called the (relative) ample cone for $f$. Its closure $\overline{\text{Amp}(f)} \subset N^1(f)_{\mathbb{R}}$ coincides with the closure of the convex cone generated by $f$-nef line bundles, and it is called the nef cone for $f$. Note that, when $S$ is affine, $L$ is $f$-ample if and only if $L$ is ample in the absolute sense. We denote by $\text{NE}(f)$ the dual cone of $\overline{\text{Amp}(f)}$. In other words,

$$\text{NE}(f) = \{ z \in N_1(f)_{\mathbb{R}}; (L.z) \geq 0, \forall L \in \overline{\text{Amp}(f)} \}.$$ 

A line bundle $L$ on $X$ is called $f$-movable if

$$\text{codimSupp}(\text{coker}(f^*f_*L \to L)) \geq 2,$$

where $\text{Supp}(F)$ means the support of a coherent sheaf $F$. We denote by $\overline{\text{Mov}(f)}$ (or $\overline{\text{Mov}(X/S)}$) the closure of the convex cone in $N^1(f)_{\mathbb{R}}$ generated by $f$-movable line bundles. Its interior $\text{Mov}(f)$ (or $\text{Mov}(X/S)$) is called the (relative) movable cone for $f$. Note that if $f$ is an isomorphism in codimension one, then $N^1(f)_{\mathbb{R}} = \overline{\text{Mov}(f)}$. In the next section, we will be concerned with the problem of finding all crepant resolutions of an affine variety $S$ with rational Gorenstein singularities, when one particular crepant resolution $f : X \to S$ is given. Here, a crepant resolution $f : X \to S$ means a (projective) resolution of singularities such that $K_X = f^*K_S$, where $K_X$ (resp. $K_S$) is a canonical divisor of $X$ (resp. $K_S$). Our strategy for describing other crepant resolutions is as follows. Take an arbitrary crepant resolution $f' : X' \to S$ and let $L'$ be an $f'$-ample line bundle. The natural birational map $X \dashrightarrow X'$ is an isomorphism in codimension one because $f$ and $f'$ are both crepant resolutions (cf. [K-M], Corollary 3.54). Then, one can consider the proper transform $L \in \text{Pic}(X)$ of $L'$. We want to recover $f' : X' \to S$ by using $L$. If $L$ is $f$-nef, then $X' = X$; so we assume that $L$ is not $f$-nef.
Then one can find a birational contraction map $g : X \to \bar{X}$ over $S$ such that every curve $C$ contracted by $g$ satisfies $(L.C) < 0$ and $g$ is an isomorphism in codimension one. We now need a flop of $g$. A flop of $g$ is a diagram

$$X \xrightarrow{g} \bar{X} \xleftarrow{g'} X_1$$

such that $g'$ is also crepant and the proper transform $L_1 \in \text{Pic}(X_1)$ of $L$ is $g'$-nef. In our situation, we can construct the flop explicitly (cf. Section 1); thus, we have another crepant resolution $X_1 \to S$. Now, replace $L$ by $L_1$ and continue the same process as above. Then we have a sequence of crepant resolutions of $S$:

$$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots$$

If this sequence terminates at some $X_k$, then $X_k$ is nothing but the original $X'$. For details on the birational geometry, see [Ka].

(P.3) The nef cone of a flag variety. Assume that $G$ and $P_I$ are the same as in (P.1). Then $G/P_I$ is a projective manifold. In this case, $\text{Pic}(G/P_I) \cong N^1(G/P_I)$. If $\chi : P_I \to \mathbb{C}^*$ is a character of $P_I$, then $P_I$ acts on $\mathbb{C}$ by $\chi$. Then $L_\chi := G \times_{P_I} \mathbb{C}$ is a $G$-equivariant line bundle on $G/P_I$. In this way, we have a natural injective homomorphism

$$\phi : \text{Hom}_{\text{alg}, \text{gp}}(P_I, \mathbb{C}^*) \to \text{Pic}(G/P_I),$$

where $\text{Im}(\phi)$ has finite index in $\text{Pic}(G/P_I)$. When $G$ is simply connected, this map is an isomorphism. Let $L(P_I)$ be the Levi-factor of $P_I$ corresponding to $l(p_I)$. Then

$$\text{Hom}_{\text{alg}, \text{gp}}(P_I, \mathbb{C}^*) \cong \text{Hom}_{\text{alg}, \text{gp}}(L(P_I), \mathbb{C}^*).$$

We define

$$M(L_{P_I}) := \text{Hom}_{\text{alg}, \text{gp}}(L(P_I), \mathbb{C}^*),$$

and put $M(L_{P_I})_R := M(L_{P_I}) \otimes \mathbb{R}$. Thus, we have

$$N^1(G/P_I)_R \cong M(L_{P_I})_R.$$
where $\alpha^\vee \in \mathfrak{h}$ is the coroot corresponding to $\alpha$, and $\chi$ is regarded as an element of $\mathfrak{h}^*$. Moreover, it is a simplicial cone and each extremal ray corresponds to the dominant characters $\chi$ such that $\langle \chi, \alpha^\vee \rangle = 0, \forall \alpha \in \Delta_i$. More geometrically, each extremal ray corresponds to the natural projection $p_i : G/P_i \to G/P_{\Delta_i}$ induced by the inclusion $P_i \subset P_{\Delta_i}$. Now let us consider the Springer map $s : T^*(G/P_i) \to \hat{O}$, where $O$ is the Richardson orbit for $P_i$ (cf. (P.1)). Let $T^*(G/P_i) \to \hat{O}$ be the Stein factorization of $s$. Then $x_0 := r^{-1}(0)$ consists of one point, and $\pi^{-1}(x_0) = G/P_I$. Let $C \subset T^*(G/P_I)$ be a proper curve contained in a fiber of $\pi$. Note that $\hat{O}$ admits a $\mathbb{C}^*$-action with positive weights, and this $\mathbb{C}^*$ action extends to an action on $T^*(G/P_I) : \rho : \mathbb{C}^* \times T^*(G/P_I) \to T^*(G/P_I)$. Then $C_0 := \lim_{t \to 0} \rho(t, C)$ is a curve in $\pi^{-1}(x_0)$ by the properness of the relative Hilbert scheme of $\pi$ (cf. [Kol], Claim 1.8.3, Step 5). Assume that $L$ is a line bundle on $T^*(G/P_I)$. Then, in order to check if $L$ is $\pi$-nef, it is enough to check that $(L.D) \geq 0$ only for proper curves $D$ inside $\pi^{-1}(x_0)$. Let $p : T^*(G/P_I) \to G/P_I$ be the natural projection map. Then we have an isomorphism $p^* : \text{Pic}(G/P_I) \cong \text{Pic}(T^*(G/P_I))$. In particular, any line bundle $L$ on $T^*(G/P_I)$ has the form $p^*M$ with some $M \in \text{Pic}(G/P_I)$ Then, $L$ is $\pi$-nef if and only if $M$ is nef on $G/P_I$. As a consequence, there is an identification map

$$p^* : N^1(G/P_I)_\mathbb{R} \cong N^1(\pi)_\mathbb{R}$$

such that

$$p^*(\text{Amp}(G/P_I)) = \text{Amp}(\pi).$$

In particular, the nef cone of $\pi$ is also a simplicial cone.

1

Let us consider a Brieskorn-Slodowy diagram as in (P.1).

Lemma 1.1. The induced map

$$\mu'_p : G \times^P r(p) \to \mathfrak{k}(p) \times_{h/W} Gr(p)$$

is a birational projective morphism. In particular, $\mathfrak{k}(p) \times_{h/W} Gr(p)$ is irreducible.
Proof. The map \[ G \times^P r(p) \to G \cdot r(p) \]
is projective. Indeed, it is factorized as \( G \times^P r(p) \to G/P \times G \cdot r(p) \to G \cdot r(p) \), the first map is a closed immersion and the second one is a projective map because \( G/P \) is projective. Hence \( \mu'_p \) is a projective morphism. Let \( h \in \mathfrak{t}(p)_{\text{reg}} \) (cf. (P.1)) and denote by \( \tilde{h} \in \mathfrak{h}/W \) its image by the map \( \mathfrak{t}(P) \to \mathfrak{h}/W \). Then the fiber \( f^{-1}(\tilde{h}) \) of the map \( f : Gr(p) \to \mathfrak{h}/W \) over \( \tilde{h} \) coincides with the semi-simple orbit \( G \cdot h \) of \( \mathfrak{g} \) containing \( h \). The proof goes as follows. The centralizer \( Z_G(h) \) of \( h \) is the Levi subgroup \( L(P) \) of \( P \). Indeed, since \( h \in \mathfrak{t}(p)_{\text{reg}}, \mathfrak{g}^h = l(p) \). This means that \( L(P) \) is a subgroup of \( Z_G(h) \) with finite index. By [Ko], 3.2, Lemma 5, \( Z_G(h) \) is connected; hence \( Z_G(h) = L(P) \).

Let \( U(P) \) be the unipotent radical of \( P \). Then \( P \cdot h = U(P) \cdot h \). \( U(P) \cdot h \) is closed, and its tangent space at \( h \) is \( h + n(p) \). On the other hand, \( h + n(p) \) is invariant under \( P \); hence \( P \cdot h \subset h + n(p) \). This implies that \( P \cdot h = h + n(p) \).

Therefore, \( G \cdot (h + n(p)) = G \cdot h \). By the (Brieskorn-Slodowy) diagram, we see that \( f^{-1}(\tilde{h}) = \cup G \cdot (h + n(p)) = \cup G \cdot h \), where \( h \) runs through all elements in the fiber of the map \( \mathfrak{t}(p) \to \mathfrak{h}/W \) over \( \tilde{h} \). If \( h, h' \in \mathfrak{t}(p) \) have the same image \( \tilde{h} \in \mathfrak{h}/W \), then \( G \cdot h = G \cdot h' \). Hence we have \( f^{-1}(\tilde{h}) = G \cdot h \).

Take a point \( (h, h') \in \mathfrak{t}(p)_{\text{reg}} \times_{\mathfrak{h}/W} Gr(p) \). Then \( h' \) is a semi-simple element \( G \)-conjugate to \( h \). Fix an element \( g_0 \in G \) such that \( h' = Ad_{g_0}(h) \). We have

\[
(\mu'_p)^{-1}(h, h') = \{[g, x] \in G \times^P r(p); x_1 = h, Ad_g(x) = h'\}.
\]

Since \( x = Ad_p(x_1) \) for some \( p \in P \) and conversely \( (Ad_p(x))_{1} = x_1 \) for any \( p \in P \) (cf. [Slo], Lemma 2, p.48), we have

\[
(\mu'_p)^{-1}(h, h') = \{[g, Ad_p(h)] \in G \times^P r(p); g \in G, p \in P, Ad_{g_0}(h) = h'\} = \{[g, h] \in G \times^P r(p); g \in G, p \in P, Ad_{g_0}(h) = h'\} = \{[g, h] \in G \times^P r(p); Ad_g(h) = h'\} = \{[g_0 g', h] \in G \times^P r(p); g' \in Z_G(h)\} = g_0(Z_G(h)/Z_P(h)).
\]

Since \( Z_G(h) = L(P), Z_G(h)/Z_P(h) = \{1\} \) and \( (\mu'_p)^{-1}(h, h') \) consists of one point. Hence, \( \mu'_p \) is a birational map onto its image. We shall prove that

\[
\text{Im}(\mu'_p) = \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p).
\]

Since \( G \times^P r(p) \) is an affine bundle over \( G/P \), it is irreducible. Then, the irreducibility of \( \mathfrak{t}(p) \times_{\mathfrak{h}/W} Gr(p) \) follows from this assertion. All non-empty
fibers of $Gr(p) \to h/W$ have the same dimension because $G \times^P \mathfrak{q}(p) \to \mathfrak{q}(p)$ is an affine bundle. So, the map $\mathfrak{q}(p) \times_{h/W} Gr(p) \to \mathfrak{q}(p)$ is equi-dimensional. Note that $\mathfrak{q}(p)^{reg} \times_{h/W} Gr(p)$ is contained in the closed subset $\text{Im}(\mu'_p)$ of $\mathfrak{q}(p) \times_{h/W} Gr(p)$. Hence, $\text{Im}(\mu'_p)$ is a unique irreducible component of $\mathfrak{q}(p) \times_{h/W} Gr(p)$ of maximal dimension. If $\mathfrak{q}(p) \times_{h/W} Gr(p)$ has an irreducible component different from $\text{Im}(\mu'_p)$, then its dimension is smaller than $\text{Im}(\mu'_p)$. Let $W'$ be the subgroup of $W$ which stabilizes $\mathfrak{q}(p)$ as a set. Then $W'$ acts on $\mathfrak{q}(p) \times_{h/W} Gr(p)$ in such a way that it acts naturally on the first factor and it acts trivially on the second factor. This $W'$-action must preserve $\text{Im}(\mu'_p)$. Assume that $(h, x) \in \mathfrak{q}(p) \times_{h/W} Gr(p)$ is not contained in $\text{Im}(\mu'_p)$. There is an element $h' \in \mathfrak{q}(p)$ such that $h' = w(h)$ with some $w \in W'$ and $x \in G(h' + n(p))$. But, $h' \times G(h' + n(p)) \subset \text{Im}(\mu'_p)$ and $w^{-1}$ sends $h' \times G(h' + n(p))$ to $h \times G(h' + n(p))$. As a consequence $(h, x) \in w^{-1}(\text{Im}(\mu'_p))$. This is a contradiction. Therefore,

$$\text{Im}(\mu'_p) = \mathfrak{q}(p) \times_{h/W} Gr(p).$$

Q.E.D.

In the remainder, we shall write $Y_{\mathfrak{q}(p)}$ for the normalization of $\mathfrak{q}(p) \times_{h/W} Gr(p)$, and write $X_p$ for $G \times^P r(p)$. Then we have a resolution

$$\mu_p : X_p \to Y_{\mathfrak{q}(p)}$$

of $Y_{\mathfrak{q}(p)}$. Let $X_{p,t}$ (resp. $Y_{\mathfrak{q}(p),t}$) be the fiber of the map $X_p \to \mathfrak{q}(p)$ (resp. $Y_{\mathfrak{q}(p)} \to \mathfrak{q}(p)$) over $t \in \mathfrak{q}(p)$. Let $\mu_{p,t} : X_{p,t} \to Y_{\mathfrak{q}(p),t}$ be the map induced by $\mu_p$.

**Lemma 1.2.** $\mu_p$ is a crepant resolution, which is an isomorphism in codimension one. For $t \in \mathfrak{q}(p)^{reg}$, $\mu_{p,t} : X_{p,t} \to Y_{\mathfrak{q}(p),t}$ is an isomorphism. Moreover, for the origin $0 \in \mathfrak{q}(p)$, $X_{p,0} = T^*(G/P)$, $\mu_{p,0}$ is birational, and the Springer map $s : T^*(G/P) \to \mathcal{O}$ factors through $\mu_{p,0}$ as

$$T^*(G/P) \xrightarrow{\mu_{p,0}} Y_{\mathfrak{q}(p),0} \to \mathcal{O}.$$ 

**Proof.** The second assertion follows from the proof of Lemma 1.1. By definition, the Springer map $s$ factors through $\mu_{p,0}$. On the other hand, since $\mu_p$ is a projective birational morphism and $Y_{\mathfrak{q}(p)}$ is a normal variety, $\mu_p$ has connected fibers by Zariski’s main theorem; hence $\mu_{p,0}$ also has connected fibers. Since $s$ is a generically finite map, we conclude that $\mu_{p,0}$ is birational.
These fact imply that, $\mu_{p,t}$ are birational for all $t \in \mathfrak{k}(p)$, and they are isomorphisms for general $t$. Therefore, $\mu_p$ is an isomorphism in codimension one. Since $X_{p,0} = T^*(G/P)$ has trivial canonical line bundle, $K_{X_p}$ is $\mu_p$-trivial. Then $K_{X_p}$ is the pull-back of a line bundle $M$ on $Y_{t(p)}$ by $\mu_p$ because $Y_{t(p)}$ has only rational singularities. But, as $Y_{t(p)}$ is an affine cone, $M$ is trivial; hence $K_{X_p}$ is trivial, and $K_{Y_{t(p)}} = (\mu_p)_* K_{X_p}$ is trivial. Q.E.D.

**Observation 1.** (1) If $\deg(s) > 1$, then $\mathfrak{k}(p) \times_{h/W} \text{Gr}(p)$ is non-normal. Indeed, the map $\mu'_{p,0}$ (restriction of $\mu'_p$ of Lemma (1.1) to the central fibers over $0 \in \mathfrak{k}(p)$), coincides with the Springer map $s : T^*(G/P) \to \bar{O}$, but $\mu_{p,0}$ is birational.

(2) Let $T^*(G/P) \to \bar{O} \to \bar{O}$ be the Stein factorization of $s$. Then $\bar{O}$ is the normalization of $Y_{t(p),0}$ and they coincide at least set-theoretically.

We fix a parabolic subgroup $P_0$ of $G$ containing $T$, and put $p_0 := \text{Lie}(P_0)$, $\mathfrak{k}_0 := \mathfrak{k}(p_0)$ and $l_0 := l(p)$. We define:

$$S(l_0) := \{ p \subset \mathfrak{g}; p : \text{parabolic s.t. } h \subset p, \mathfrak{k}(p) = \mathfrak{k}_0 \}.$$  

Note that

$$S(l_0) = \{ p \subset \mathfrak{g}; p : \text{parabolic s.t. } h \subset p, l(p) = l_0 \}.$$  

The set $S(l_0)$ is finite. Let $p \in S(l_0)$ and let $b$ be a Borel subalgebra such that $h \subset b \subset p$. Then $p$ corresponds to a marked Dynkin diagram $D$. Take a marked vertex $v$ of the Dynkin diagram $D$ and consider the maximal connected single marked Dynkin subdiagram $D_v$ of $D$ containing $v$. We call $D_v$ the single marked diagram associated with $v$. When $D_v$ is one of the following, we say that $D_v$ (or $v$) is of the first kind, and when $D_v$ does not coincide with any of them, we say that $D_v$ (or $v$) is of the second kind.

\[ A_{n-1} \ (k < n/2) \]

```
. .... k .... .... ....
. .... n-k .... .... ....
```

\[ D_n \ (n : \text{odd} \geq 5) \]

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\bullet
\bigtriangleup
. .... .... ....
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\bullet
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. .... .... ....
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In the single marked Dynkin diagrams above, two diagrams in each type (i.e. $A_{n-1}$, $D_n$, $E_{6,I}$, $E_{6,II}$) are called duals. Let $\bar{D}$ be the marked Dynkin diagram obtained from $D$ by making $v$ unmarked. Let $\bar{p}$ be the parabolic subalgebra containing $p$ corresponding to $\bar{D}$. Now let us define a new marked Dynkin diagram $D'$ as follows. If $D_v$ is of the first kind, we replace $D_v \subset D$ by its dual diagram $D^*_v$ to get a new marked Dynkin diagram $D'$. If $D_v$ is of the second kind, we define $D' := D$. As in (P.1), the set of unmarked vertices of $D$ (resp. $\bar{D}$) defines a subset $I \subset \Delta$ (resp. $\bar{I} \subset \Delta$). By definition, $v \in \bar{I}$. The unmarked vertices of $\bar{D}$ define a Dynkin subdiagram, which is decomposed into the disjoint sum of the connected component containing $v$ and the union of other components. Correspondingly, we have a decomposition $\bar{I} = I_v \cup I'_v$ with $v \in I_v$. The parabolic subalgebra $p$ (resp. $\bar{p}$) coincides with the standard parabolic subalgebra $p_I$ (resp. $p_{\bar{I}}$). Let $l_I$ be the (standard) Levi factor of $p_I$. Let $\mathfrak{z}(l_I)$ be the center of $l_I$. Then $l_I/\mathfrak{z}(l_I)$ is decomposed into the direct sum of simple factors. Now let $I_v$ be the simple factor corresponding to $I_v$ and let $I'_v$ be the direct sum of other simple factors. Then

$$l_I/\mathfrak{z}(l_I) = l_{I_v} \oplus l'_{I_v}.$$ 

\textsuperscript{2} The Weyl group $W$ of $\mathfrak{g}$ does not contain $-1$ exactly when $\mathfrak{g} = A_n(n \geq 2)$, $D_n$ (n: odd) or $E_6$ (cf. [Hu], p.71, Exercise 5). This property characterizes the Dynkin diagrams in the list. Moreover, the single marked Dynkin diagrams in the list are characterized by the following property. Let $p_D$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to $D$, and let $\varphi_{\mathfrak{g}}$ be an automorphism of $\mathfrak{g}$ determined by $-1$ (cf. (P.1)). Then $\varphi_{\mathfrak{g}}(p_D)$ is not conjugate to $p_D$. If $\varphi_{\mathfrak{g}}(p_D)$ corresponds to a single marked Dynkin diagram $D'$, then $D$ and $D'$ are mutually duals.
The marked Dynkin diagram $D_v$ defines a standard parabolic subalgebra $p_v$ of $I_v$. Here let us consider the involution $\varphi_{U_v} \in \text{Aut}(I_v)$ (cf. (P.1)). When $D_v$ is of the first kind, $\varphi_{U_v}(p_v)$ is conjugate to a standard parabolic subalgebra of $I_v$ with the dual marked Dynkin diagram $D_v^*$ of $D_v$. When $D_v$ is of the second kind, $\varphi_{U_v}(p_v)$ is conjugate to $p_v$ in $I_v$. Let $q : I_f \to I_f/\mathfrak{z}(I_f)$ be the quotient homomorphism. Note that

$$\bar{p} = I_f \oplus n(\mathfrak{p}),$$

$$p = q^{-1}(p_v \oplus I_{I_v')} \oplus n(\mathfrak{p}).$$

Here we define

$$p' = q^{-1}(\varphi_{U_v}(p_v) \oplus I_{I_v'}) \oplus n(\mathfrak{p}).$$

Then $p' \in S(l_0)$ and $p'$ is conjugate to a standard parabolic subalgebra with the marked Dynkin diagram $D'$. This $p'$ is said to be the parabolic subalgebra twisted by $v$.

We next define:

$$\mathcal{R}es(Y_{t_0}) := \{\text{the isomorphic classes of crepant projective resolutions of } Y_{t_0}\}.$$

Here we say that two resolutions $\mu : X \to Y_{t_0}$ and $\mu' : X' \to Y_{t_0}$ are isomorphic if there is an isomorphism $\phi : X \to X'$ such that $\mu = \mu' \circ \phi$.

**Theorem 1.3.** There is a one-to-one correspondence between the two sets $S(l_0)$ and $\mathcal{R}es(Y_{t_0})$. Any crepant resolution of $Y_{t_0}$ is an isomorphism in codimension one.

**Proof.** By Lemma 1.2, $\mu_{p_0} : X_{p_0} \to Y_{t_0}$ is a crepant resolution which is an isomorphism in codimension one. Let $\mu : X \to Y_{t_0}$ be an arbitrary crepant resolution. Then $X$ and $X_{p_0}$ are isomorphic in codimension one. This implies that $\mu$ is an isomorphism in codimension one. This is nothing but the second claim of the theorem. Let us consider the first claim. Here, the correspondence is given by

$$p \mapsto [\mu_p : X_p \to Y_{t_0}].$$

In order to check that this correspondence is injective, we have to prove that, for $\mu_p : X_p \to Y_{t_0}$ and $\mu_{p'} : X_{p'} \to Y_{t_0}$ with $p \neq p'$, $\mu_{p'}^{-1} \circ \mu_p$ is not an
isomorphism. For \( t \in (\mathfrak{t}_0)^{\text{reg}} \), \( X_{p,t} \cong G \times^P (t + n(p)) \), \( Y_{b,t} \cong G \cdot t \times \{t\} \), and \( \mu_{p,t} \) is an isomorphism defined by
\[
[g, t + x] \in G \times^P (t + n(p)) \to (Ad_g(t + x), t) \in G \cdot t \times \{t\}.
\]
Since \( Z_G(t) = L_0 \), the map
\[
G \to G \times^P (t + n(p)); g \to [g, t]
\]
induces an isomorphism
\[
\rho_t : G/L_0 \to G \times^P (t + n(p))
\]
In a similar way, we have an isomorphism
\[
\rho_t' : G/L_0 \to G \times^{P'} (t + n(p'))
\]
for \( t \in (\mathfrak{t}_0)^{\text{reg}} \). The composition of the two maps \( \rho_t' \circ (\rho_t)^{-1} \) defines an isomorphism
\[
X_{p,t} \cong X_{p',t}
\]
for each \( t \in (\mathfrak{t}_0)^{\text{reg}} \) and it coincides with \( \mu_{p',t}^{-1} \circ \mu_{p,t} \). Assume that \( \mu_{p'}^{-1} \circ \mu_{p} \) is a morphism. Take \( g \in G \) and \( p \in P \). We put \( g' = gp \). Then
\[
\lim_{t \to 0} \rho_t([g]) = [g, 0] \in G/P,
\]
and
\[
\lim_{t \to 0} \rho_t([gp]) = [gp, 0] \in G/P.
\]
Note that \([g, 0] = [gp, 0]\) in \( G/P \). On the other hand,
\[
\lim_{t \to 0} \rho_t'([g]) = [g, 0] \in G/P',
\]
and
\[
\lim_{t \to 0} \rho_t'([gp]) = [gp, 0] \in G/P'.
\]
Since we assume that \( \mu_{p'}^{-1} \circ \mu_{p} \) is a morphism, \([g, 0] = [gp, 0]\) in \( G/P' \). But, this implies that \( P = P' \), which is a contradiction. We next prove that the correspondence is surjective. Let \( p \in S(p_0) \) and let \( P \) be the corresponding parabolic subgroup of \( G \). For another \( p' \in S(l_0) \), \( X_{p'} \) and \( X_p \) are isomorphic in codimension one; thus, \( N^1(\mu_{p'})_R \) and \( N^1(\mu_p)_R \) are naturally identified.
Lemma 1.4. (1) $N^1(\mu P)_R$ is identified with $N^1(G/P)_R$. The nef cone $\overline{\text{Amp}}(\mu P)$ is a simplicial polyhedral cone. Each codimension one face $F$ of $\overline{\text{Amp}}(\mu P)$ corresponds to a marked vertex of the marked Dynkin diagram $D$ attached to $P$.

(2) For each codimension one face $F$, there is an element $p' \in S(l_0)$ such that $\overline{\text{Amp}}(\mu P) \cap \overline{\text{Amp}}(\mu p') = F$.

Proof. (1): First of all, $\overline{\text{Amp}}(G/P)$ is simplicial by (P.3). Moreover, in (P.3) we proved that there is an identification of $N^1(\pi)_R$ with $N^1(G/P)_R$ which sends $\overline{\text{Amp}}(\pi)$ to $\overline{\text{Amp}}(G/P)$. The first statement and the second statement are proved in the same manner. Let us consider the third statement. Let $\bar{D}$ be the marked Dynkin diagram which is obtained from $D$ by making a marked vertex $v$ unmarked. We then have a parabolic subgroup $\bar{P}$ with $P \subset \bar{P}$ corresponding to $\bar{D}$. Each codimension one face $F$ of $\overline{\text{Amp}}(G/P)$ corresponds to the natural surjection $G/P \to G/\bar{P}$. We are now going to construct the birational contraction map of $X_p$ corresponding to $F$. First look at the $\bar{P}$-orbit $\bar{P}r(p)$ of $r(p)$. Then we can write

$$\bar{P}r(p) = r(\bar{p}) \times L(\bar{P}) \cdot r(l(\bar{p}) \cap p),$$

where $l(\bar{p})$ is the Levi factor of $\bar{p}$, $L(\bar{P})$ is the corresponding Levi subgroup, and $r(l(\bar{p}) \cap p)$ is the solvable radical of the parabolic subalgebra $l(\bar{p}) \cap p$ of $l(\bar{p})$. Then $\mathfrak{g}(p)$ is decomposed as

$$\mathfrak{g}(p) = \mathfrak{g}(\bar{p}) \oplus \mathfrak{g}(l(\bar{p}) \cap p).$$

Let $W'$ be the subgroup of the Weyl group $W(l(\bar{p}))$ which stabilizes $\mathfrak{g}(l(\bar{p}) \cap p)$ as a set. We then have a map $L(\bar{P}) \cdot r(l(\bar{p}) \cap p) \to \mathfrak{g}(l(\bar{p}) \cap p)/W'$, and hence have a map $\bar{P}r(p) \to \mathfrak{g}(l(\bar{p}) \cap p)/W'$. Now we have a map

$$\alpha : \bar{P} \times^{P} r(p) \to \bar{P}r(p) \times_{\mathfrak{g}(l(\bar{p}) \cap p)/W'} \mathfrak{g}(l(\bar{p}) \cap p).$$

Here, the left hand side is isomorphic to

$$r(\bar{p}) \times L(\bar{P}) \times^{L(P) \cap P} r(l(\bar{p}) \cap p)$$

and the right hand side is isomorphic to

$$r(\bar{p}) \times L(\bar{P}) \cdot r(l(\bar{p}) \cap p) \times_{\mathfrak{g}(l(\bar{p}) \cap p)/W'} \mathfrak{g}(l(\bar{p}) \cap p).$$
Then, by Lemma (1.1), we see that $\alpha$ is a birational map. Since $\alpha$ is a $\overline{P}$-equivariant map, we get a birational map:

$$X_p \to (G \times^\overline{P} \overline{P} \cdot r(p)) \times_{\mathfrak{k}(l(\overline{p}) \cap p)/W'} \mathfrak{k}(l(\overline{p}) \cap p).$$

This is the desired birational contraction map.

(2): Let $p'$ be the parabolic subalgebra twisted by $v$. By [Na], Proposition 6.4, we have $\overline{P} \cdot n(p) = \overline{P} \cdot n(p')$. Since $\mathfrak{k}(p) = \mathfrak{k}(p') = 0$, we see that $\overline{P} \cdot r(p) = \overline{P} \cdot r(p').$

Moreover,

$$\mathfrak{k}(l(\overline{p}) \cap p) = \mathfrak{k}(l(\overline{p}) \cap p').$$

Thus, there is a diagram of birational morphisms

$$X_p \to (G \times^\overline{P} \overline{P} \cdot r(p)) \times_{\mathfrak{k}(l(\overline{p}) \cap p)/W'} \mathfrak{k}(l(\overline{p}) \cap p) \leftarrow X_{p'},$$

and

$$\overline{\text{Amp}}(\mu_p) \cap \overline{\text{Amp}}(\mu_{p'}) = F.$$

The following corollary implies the surjectivity of the correspondence and the proof of Theorem 1.3 is completed.

**Corollary 1.5.** Any crepant projective resolution of $Y_{k_0}$ is given by $\mu_p : X_p \to Y_{k_0}$ for some $p \in S(l_0)$.

**Proof.** Take an arbitrary crepant projective resolution $\mu : X \to Y_{k_0}$. Fix an $\mu$-ample line bundle $L$ on $X$ and denote by $L^{(0)} \in \text{Pic}(X_{p_0})$ its proper transform. If $L^{(0)}$ is $\mu_{p_0}$-nef, then $X$ coincides with $X_{p_0}$. So let us assume that $L^{(0)}$ is not $\mu_{p_0}$-nef. Then there is an extremal ray $R_+ [z] \subset NE(\mu_{p_0})$ such that $(L^{(0)}, z) < 0$. Let $F \subset \overline{\text{Amp}}(\mu_{p_0})$ be the corresponding codimension one face. By the previous lemma, one can find $p_1 \in S(l_0)$ such that

$$\overline{\text{Amp}}(\mu_{p_0}) \cap \overline{\text{Amp}}(\mu_{p_1}) = F.$$

We let $L^{(1)} \in \text{Pic}(X_{p_1})$ be the proper transform of $L^{(0)}$ and repeat the same procedure. Thus, we get a sequence of flops

$$X_{p_0} \dashrightarrow X_{p_1} \dashrightarrow X_{p_2} \dashrightarrow \cdots .$$
But, since $S(l_0)$ is a finite set, this sequence must terminate by the same argument as [Na], Theorem 6.1. As a consequence, $X = X_{p_k}$ for some $k$. Q.E.D.

Let $P$ be a parabolic subgroup of $G$ which contains $L_0$ as the Levi factor (this is equivalent to that $p \in S^1(l_0)$). By Lemma (1.4), (1), $N^1(\mu_p)_R$ is identified with $N^1(G/P)_R$. By (P.3), $N^1(G/P)_R$ is identified with $M(L_0)_R$. Let

$$\Phi_P : N^1(\mu_p)_R \cong M(L_0)_R$$

be the composition of these identifications. There is a similar identification

$$\phi_P : N^1(\pi)_R \cong M(L_0)_R.$$

The nef cone $Amp(\mu_p)$ and the (closed) movable cone $Mov(\mu_p)$ are regarded as cones in $M(L_0)_R$ by $\Phi_P$. The nef cone $Amp(\pi)$ and the (closed) movable cone $Mov(\pi)$ are regarded as cones in $M(L_0)_R$ by $\phi_P$.

**Observation 2.** Assume $p, p' \in S(l_0)$.

1. $(\Phi_{p'})^{-1} \circ \Phi_P : N^1(\mu_p)_R \cong N^1(\mu_{p'})_R$ coincides with the natural isomorphism induced by the proper transform by the birational map $X_p \dasharrow X_{p'}$.

2. If $X_{p,0}$ and $X_{p',0}$ are isomorphic in codimension one, then $(\phi_{p'})^{-1} \circ \phi_P : N^1(\mu_{p,0})_R \cong N^1(\mu_{p',0})_R$ coincides with the natural isomorphism induced by the proper transform by the birational map $X_{p,0} \dasharrow X_{p',0}$

**Proof.** (1): For $t \in (k_0)^{re}$, we have an isomorphism $\rho_t : G/L_0 \cong X_{p,t}$ (cf. the first part of the proof of Theorem 1.3). Similarly, we have an isomorphism $\rho'_t : G/L_0 \cong X_{p',t}$. Let $L_\chi \in \text{Pic}(X_p)$ be the line bundle associated with the character $\chi : L_0 \rightarrow \mathbb{C}^\ast$. Then,

$$(\rho_t)_*^{L_\chi}_{|X_{p,t}} = G \times_{L_0} \mathbb{C}.$$  

Similarly, let $L'_{\chi} \in \text{Pic}(X_{p'})$ be the line bundle associated with $\chi$. Then

$$(\rho'_t)^{L'_{\chi}}_{|X_{p',t}} = G \times_{L_0} \mathbb{C}.$$  

Thus we see that $L'_{\chi}$ is the proper transform of $L_{\chi}$.

(2): In this case, the proper transform is compatible with the restriction of $X_p$ (resp. $X_{p'}$) to $X_{p,0}$ (resp. $X_{p',0}$). Then the result follows from (1).
Remark 1.6. The corollary above shows that, for any \( L \in \text{Pic}(X_{p_0}) \), there is \( p \in S(l_0) \) such that the proper transform \( L' \in \text{Pic}(X_p) \) of \( L \) is \( \mu_p \)-nef. Thus
\[
M(L_0)_R = \bigcup_{p \in S(l_0)} \overline{\text{Amp}(\mu_p)}.
\]

Let us consider the involution \( \varphi_0 \in \text{Aut}(g) \) (cf. (P.1)). For \( p \in S(l_0) \), we put \( p^* := \varphi_0(p) \). Then \( p^* \in S(l_0) \). Note that \( (p^*)^* = p \).

Proposition 1.7. Let \( L \in \text{Pic}(X_p) \) be a \( \mu_p \)-ample line bundle and let \( L' \in \text{Pic}(X_{p^*}) \) be its proper transform. Then \( -L' \) is \( \mu_{p^*} \)-ample. In other words,
\[
-\text{Amp}(\mu_p) = \overline{\text{Amp}(\mu_{p^*})}.
\]

Proof. Let \( P \) (resp. \( P^* \)) be the parabolic subgroup of \( G \) with \( \text{Lie}(P) = p \) (resp. \( \text{Lie}(P^*) = p^* \)). Note that \( P \) and \( P^* \) have the common Levi factor \( L_0 \).

For \( t \in (t_0)^{\text{reg}} \), we define an isomorphism
\[
\rho_t : G/L_0 \to G \times^P (t + n(p)),
\]
as in the first part of the proof of Theorem 1.3. Denote by the same letter \( \varphi \) the automorphism of \( G \) induced by \( \varphi \in \text{Aut}(g) \). Note that \( \varphi(P) = P^* \) and \( P \cap P^* = L_0 \). The automorphism \( \varphi \) induces an automorphism
\[
\bar{\varphi} : G/L_0 \to G/L_0.
\]

Then
\[
\rho_t \circ \bar{\varphi} \circ (\rho_t)^{-1} : G \times^P (t + n(p)) \to G \times^P (t + n(p))
\]
induces a birational involution
\[
\sigma : X_p \dashrightarrow X_p.
\]

As in the proof of Theorem 1.3, \( \sigma \) is not a morphism because \( P \neq P^* \). On the other hand, there is an isomorphism
\[
\tau : X_p \to X_{p^*}
\]
defined by \( \tau([g, x]) = [\varphi(g), -\varphi(x)] \). Then
\[
\mu_p^{-1} \circ \mu_{p^*} : X_{p^*} \dashrightarrow X_p
\]
coincides with the composition \( \sigma \circ \tau^{-1} \). \( \text{Pic}(G/L_0) \otimes R \) is identified with \( \text{Hom}_{\text{alg.gp.}}(L_0, C^*) \otimes R \). \( \text{Hom}_{\text{alg.gp.}}(L_0, C^*) \) is contained in \( \text{Hom}_{\text{alg.gp.}}(T, C^*) \), and \( \varphi \) acts on the latter space by \( -1 \). Therefore, \( \varphi^* \in \text{Aut}(\text{Pic}(G/L_0) \otimes R) \) is \( -1 \). This means that the proper transform of \( L \in \text{Pic}(X_p) \otimes R \) by \( \sigma \) is \( -L \); hence, \( -L' \) is \( \mu_{p^*} \)-ample.
Remark 1.8. (1) Assume \( p \in S(l_0) \). Assume that the Springer map \( T^*(G/P) \to \bar{O} \) is a resolution. Then, \( \bar{O} \) is the normalization of \( \bar{O} \). Let \( S^1(l_0; p) \) be the subset of \( S(l_0) \) consisting of the parabolic subalgebras \( q \) which are obtained from \( p \) by the sequence of twists by marked vertices \( v \) of the first kind. By Observation 1, \( \bar{O} \) is the normalization of \( Y_{b_0,0} \). By abuse of notation, we denote by \( \mu_q \) the Stein factorization \( X_q \to \bar{O} \) of \( \mu_q : X_q \to Y_{b_0,0} \). Then, by [Na], Theorem 6.1, \( \{ \mu_q \} \) is nothing but the set of all crepant resolutions of \( \bar{O} \). This implies that
\[
\overline{\text{Mov}(\pi)} = \bigcup_{q \in S^1(l_0; p)} \overline{\text{Amp}(\mu_q)}.
\]

(2) When \( p_0 \) is a Borel subalgebra \( b_0 \) of \( g \), \( l_0 = h \). In this case, \( S(h) \) is the set of all Borel subalgebras \( b \) such that \( h \subset b \). The cardinality of \( S(h) \) coincides with the order of the Weyl group \( W \). In this case, \( X_b = G \times^B b \), and \( Y_b = g \times h/W \).

Example 1.9. Let \( O \subset g \) be the Richardson orbit for \( P_0 \). Assume that the Springer map \( T^*(G/P_0) \to \bar{O} \) is not birational. Let \( T^*(G/P_0) \to \bar{O} \) be the Stein factorization of the Springer map. Theorem 1.3 is useful to describe crepant resolutions of \( \bar{O} \). For example, put \( g = \text{so}(8, \mathbb{C}) \) and fix a Cartan subalgebra \( h \) and a Borel subalgebra \( b \) containing \( h \). Consider the standard parabolic subalgebra \( p_0 \) corresponding to the marked Dynkin diagram
\[
\begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

In this case, the Springer map \( T^*(G/P_0) \to \bar{O} \) has degree 2. In the marked Dynkin diagram above, there are two marked vertices. For each marked vertex, one can twist \( p_0 \) to get a new parabolic subalgebra \( p_1 \) or \( p_2 \). The marked Dynkin diagrams of \( p_1 \) and \( p_2 \) are respectively given by:
\[
\begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

Since \( \bar{O} \) is the normalization of \( Y_{b_0,0} \), the three varieties \( X_{p_0,0} \), \( X_{p_1,0} \) and \( X_{p_2,0} \) are crepant resolutions of \( \bar{O} \). But these are not enough; actually, there are three more crepant resolutions. In fact,
\[
S(l_0) = \{ p_0, p_1, p_2, (p_0)^*, (p_1)^*, (p_2)^* \},
\]
and so three other crepant resolutions of $\tilde{O}$ are $X_{(p_0)^*},0$, $X_{(p_1)^*},0$ and $X_{(p_2)^*},0$. Moreover, $M(L_0)_R = \overline{\text{Mov}(\pi_0)}$ and it is divided into the union of six ample cones:

$$
\begin{array}{c}
X_{(p_0)^*},0 \\
X_{(p_1)^*},0 \\
X_{(p_2)^*},0
\end{array}
$$

Let $W$ be the Weyl group for $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $N_W(L_0) \subset W$ be the normalizer of $L_0$. Note that $N_W(L_0) = \{ w \in W; w(\mathfrak{k}_0) = \mathfrak{k}_0 \}$. For $w \in N_W(L_0)$ and $\chi \in M(L_0)$, we define $w\chi \in M(L_0)$ as $w\chi(g) = \chi(w^{-1}gw)$, $g \in L_0$. In this way, $N_W(L_0)$ acts on $M(L_0)_R$. Note that the Weyl group $W(L_0)$ of $L_0$ is a subgroup of $N_W(L_0)$ and it consists of the elements acting trivially on $M(L_0)_R$.

On the other hand, for $\mathfrak{p} \in S(l_0)$, $N_W(L_0)$ acts on $N^1(\mu_\mathfrak{p})_R$ in the following way. Since $N_W(L_0)$ acts on $\mathfrak{t}_0$, an element $w \in N_W(L_0)$ acts on $G \cdot r(\mathfrak{p}) \times_{h/W} \mathfrak{t}_0$ by $id \times w$; hence it acts on the normalization $Y_{\mathfrak{k}_0}$ of $G \cdot r(\mathfrak{p}) \times_{h/W} \mathfrak{t}_0$. The biregular automorphism of $Y_{\mathfrak{k}_0}$ thus defined, induces a birational automorphism $w_\mathfrak{p} : X_{\mathfrak{p}} \dasharrow X_\mathfrak{p}$. Then we have an isomorphism $w_\mathfrak{p} : N^1(\mu_\mathfrak{p})_R \cong N^1(\mu_\mathfrak{p}_\mathfrak{w})_R$ as the push-forward $(w_\mathfrak{p})_*$. The following lemma tells us that these two actions are the same under the identification $N^1(\mu_\mathfrak{p})_R \cong M(L_0)_R$.

**Lemma 2.1.** (i) $w_\mathfrak{p}$ coincides with $w$ under the identification $N^1(\mu_\mathfrak{p})_R \cong M(L_0)_R$.

(ii) $w(\overline{\text{Amp}(\mu_\mathfrak{p})}) = \overline{\text{Amp}(\mu_{w(\mathfrak{p})})}$. 


Proof. Note that $W$ acts on the set of parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$. Moreover, $N_W(L_0)$ acts on the set $S(l_0)$. In particular, $w(p) \in S(l_0)$ for $w \in N_W(L_0)$. We define an isomorphism

$$\phi_w : X_p \to X_{w(p)}$$

by

$$\phi_w([g, x]) := [gw^{-1}, Ad_w(x)],$$

where $[g, x] \in G \times P_r(p)$ and $[gw^{-1}, Ad_w(x)] \in G \times w(P) \cdot r(w(p))$. The isomorphism $\phi_w$ fits into the commutative diagram

$$X_p \xrightarrow{\phi_w} X_{w(p)}$$

$$\downarrow \quad \downarrow$$

$$Y_{k_0} \xrightarrow{w} Y_{k_0}.$$

Restrict this diagram over $t \in (\xi_0)^{reg}$. Then the vertical maps both induce isomorphisms and the horizontal maps induce an isomorphism $G/L_0(= X_{p,t}) \to G/L_0(= X_{p,t})$ given by $gL_0 \to gw^{-1}L_0$. Let us choose $\chi \in M(L_0)$. Then we get a commutative diagram of line bundles over $G/L_0$:

$$G \times_{X_p}^{L_0} C \to G \times_{X_{w(p)}}^{L_0} C$$

$$\downarrow \quad \downarrow$$

$$G/L_0 \longrightarrow G/L_0.$$

The left hand side of the horizontal map on the top is $L_{X_p,t}^{\chi}$, and the right hand side is $L_{X_{w(p)},t}^{\chi}$. This shows that

$$w_p([L_{\chi}]) = [L_{w\chi}].$$

Since $\Phi_p([L_{\chi}]) = \chi$ (resp, $\Phi_p([L_{w\chi}]) = w\chi$) by Observation 2, we have proved (i). The second claim (ii) is clear from the fact that $(\phi_w)_* \text{ sends } \text{Amp}(\mu_p) \text{ to Amp}(\mu_{w(p)})$.

**Proposition 2.2.** The set $S(l_0)$ contains exactly $N \cdot \sharp(N_W(L_0)/W(L_0))$ elements, where $N$ is the number of the conjugacy classes of parabolic subalgebras contained in $S(l_0)$. 

Proposition 2.3. Assume that the Springer map $N\rightarrow \text{subalgebras}$. Define $S_{\pi}$

which says that, the movable cone $\text{Mov}(\pi)$ is a fundamental domain for the $N_{W}(L_{0})/W(L_{0})$-action.

Proposition 2.3. Assume that the Springer map $T^{*}G/P_{0} \rightarrow \bar{O}$ is birational. Define $S^{1}(l_{0})$ to be the subset of $S(l_{0})$ consisting of the parabolic subalgebras $p$ which are obtained from $p_{0}$ by the sequence of twists of the 1-st kind.

(i) The (closed) movable cone $\overline{\text{Mov}(\pi_{0})}$ coincides with $\bigcup_{q \in S^{1}(l_{0})} \overline{\text{Amp}(\mu_{q})}$.

(ii) For any $p \in S(l_{0})$, there is an element $w \in N_{W}(L_{0})$ such that $w(p) \in S^{1}(l_{0})$.

(iii) For any non-zero element $w \in N_{W}(L_{0})/W(L_{0})$,

$$w(\text{Mov}(\pi_{0})) \cap \overline{\text{Mov}(\pi_{0})} = \emptyset.$$

Proof. (i) is nothing but Remark 1.8, (1).

(ii) By the definition of $S^{1}(l_{0})$, there is an element $p' \in S^{1}(l_{0})$ such that $p'$ is conjugate to $p$. Then (ii) follows from the first argument of the proof of Proposition 2.2.

(iii) We shall prove that all elements of $S^{1}(l_{0})$ are not conjugate to each other. Suppose that $p,p' \in S^{1}(l_{0})$ are conjugate to each other. Let us consider the diagram

$$X_{p} \xrightarrow{\mu_{p}} Y_{t_{0}} \xleftarrow{\mu_{p'}} X_{p'}.$$

Restrict the diagram over $0 \in t_{0}$ to get

$$X_{p,0} \xrightarrow{\mu_{p,0}} Y_{t_{0},0} \xleftarrow{\mu_{p',0}} X_{p',0}.$$
Since the Springer map (for $P_0$) has degree 1, $Y_{b,0}$ and $\mathcal{O}$ are birational and they have the same normalization, say $\mathcal{O}$. The Springer maps $T^*(G/P) \to \mathcal{O}$ and $T^*(G/P') \to \mathcal{O}$ also have degree 1. By assumption, $P$ and $P'$ are conjugate; hence $T^*(G/P) = T^*(G/P')$ and the two Springer maps (resolutions) are the same. Let $f_P : T^*(G/P) \to \mathcal{O}$ be the Stein factorization of the Springer map for $P$. Then $\mu_{p,0}$ coincides with the composition of $f_P$ and the normalization map $\mathcal{O} \to Y_{b,0}$. Similarly, $\mu_{p',0}$ is the composition of $f_{P'}$ and the normalization map $\mathcal{O} \to Y_{b,0}$. Since $f_P = f_{P'}$, $\mu_{p,0} = \mu_{p',0}$. In particular, the birational map $\mu_{p'}^{-1} \circ \mu_p$ is an isomorphism. We shall prove that $\mu_{p'}^{-1} \circ \mu_p$ is an isomorphism. Let $L$ be a $\mu$-ample line bundle on $X_p$ and let $L' \in \text{Pic}(X_{p'})$ be the proper transform of $L$ by $\mu_p^{-1} \circ \mu_p$. By Corollary 1.5, $X_p$ and $X_{p'}$ are connected by a sequence of birational transformations which are isomorphisms in codimension one. Since $p, p' \in S^1(l_0)$, these birational transformations all come from the twist of the first kind. This means that, there is a closed subset $F$ of $X_{p,0}$ with codimension $\geq 2$ such that $\mu_{p'}^{-1} \circ \mu_p$ is an isomorphism at each $x \in X_{p,0} \setminus F$. Hence we have

$$L'|_{X_{p',0}} \cong (\mu_{p'}^{-1} \circ \mu_{p,0})_*(L|_{X_p,0}).$$

But the right hand side is a $\mu_{p',0}$-ample line bundle. Hence $L'|_{X_{p',0}}$ is $\mu_{p',0}$-ample. This shows that $L'$ is $\mu_{p'}$-ample. Indeed, by the $C^*$-action of $X_{p'}$, every proper curve $D$ in a fiber of $\mu_{p'}$ is deformed to a curve inside $X_{p',0}$; hence $(L', D) > 0$ follows from the ampleness of $L'|_{X_{p',0}}$. Therefore, $\mu_{p'}^{-1} \circ \mu_p$ is an isomorphism. Then, by Theorem 1.3, $p = p'$.

**Remark 2.4.** When the Springer map $T^*(G/P_0) \to \mathcal{O}$ has degree $> 1$, this proposition is not true. In Example 1.9, the involution $-1$ of the root system $\Phi$ of $\mathfrak{g}$ can be realized as an element $w_0$ of the Weyl group $W$ (cf. the footnote in Section 1). Clearly, $w_0 \in N_W(L_0)$. But, $w_0$ acts non-trivially on

$$M(L_0)_{\mathbb{R}} = \bigcup_{p \in S^1(l_0)} \text{Amp}(\mu_p).$$

3

Recall that, in Section 1, we have defined operations called *twists* for the elements $p \in S(l_0)$. They are expressed in terms of corresponding marked Dynkin diagrams. Twists are divided into those of the 1-st kind and those
of the 2-nd kind. We shall further divide the twists of the 2-nd kind into two classes.

**Definition 1.** A single marked Dynkin diagram $D$ (of type $A,B,C,D,E,F$ or $G$) of the 2-nd kind is called small if $D$ is one of the following and, otherwise, $D$ is called divisorial. A twist by a small, single marked Dynkin diagram, is called of type (2-s), and a twist by a divisorial, single marked Dynkin diagram, is called of type (2-d).

- $B_n$ ($k : \text{even, } k > (2n + 1)/3$)
  
  
  \[ \bullet \quad \begin{array}{c} \cdots \\ k \end{array} \quad \begin{array}{c} \cdots \\ \Rightarrow \end{array} \]

- $C_n$ ($k : \text{odd, } k \leq 2n/3$)
  
  
  \[ \bullet \quad \begin{array}{c} \cdots \\ k \end{array} \quad \begin{array}{c} \cdots \\ \Leftarrow \end{array} \]

- $D_n$ ($k : \text{odd, } n - 2 \geq k > 2n/3$)
  
  
  \[ \bullet \quad \begin{array}{c} \cdots \\ k \end{array} \quad \begin{array}{c} \cdots \\ \Leftarrow \end{array} \]

**Proposition 3.1.** Assume that $P \subset G$ is a parabolic subgroup conjugate to the standard parabolic subgroup determined by a single marked Dynkin diagram $D$ of the 2-nd kind (with respect to some Borel subgroup). Let $\mathcal{O}$ be the Richardson orbit for $P$ and let $\pi : T^*(G/P) \to \mathcal{O}$ be the Stein factorization of the Springer map $T^*(G/P) \to \mathcal{O}$. Then the birational map $\pi$ is divisorial (i.e. $\text{codimExc}(\pi) = 1$) if $D$ is divisorial, and $\pi$ is small (i.e. $\text{codimExc}(\pi) \geq 2$) if $D$ is small.

**Proof.** When $D$ is divisorial, $\pi$ is divisorial by [Na], Proposition 5.1. We only have to check that, if $D$ is small, then $\text{codimExc}(\pi) \geq 2$. First assume that $g$ is of type $B_n$ (resp. $D_n$). Let $V$ be a $2n+1$ (resp. $2n$)-dimensional $\mathbb{C}$-vector space equipped with a non-degenerate symmetric bilinear form $\langle \ , \rangle$. We may assume that $G = SO(V)$. A parabolic subgroup $P$ of $G$ is described as the stabilizer group of an isotropic flag $F := \{F_i\}_{1 \leq i \leq s}$ of $V$. Here an isotropic flag means a flag such that $F_i^\perp = F_{s-i}$ for $1 \leq i \leq s$. The parabolic subgroups $P$ determined by small single marked Dynkin diagrams are stabilizer groups of the isotropic flags of (flag) type $(k, 2n - 2k + 1, k)$ (resp. $(k, 2n - 2k, k)$). When $k$ is even with $k > (2n + 1)/3$ (resp. $k$
is odd with $k > 2n/3$), the Richardson orbit $O$ for $P$ has Jordan type $[3^{2n-2k+1}, 2^{4k-2n-2}, 1^2]$ (resp. $[3^{2n-2k}, 2^{3k-2n-1}, 1^2]$). Moreover, the Springer map $s : T^*(G/P) \to \overline{O}$ has degree 2 (see for example, [Na], Section 4). When $\overline{O}$ has no codimension 2 orbits, $\text{codimExc}(\pi) \geq 2$ by [Na 1], Cor. 1.5. Thus, we only have to consider the case where $\overline{O}$ has a codimension 2 orbit, say $O'$. One can check, by using the dimension formula ([C-M], Cor. 6.1.4), that this is the case exactly when $k = (2n + 2)/3$ (resp. $k = (2n + 1)/3$). In both cases ($B_n$ and $D_n$), $O$ has Jordan type $[3^{k-1}, 1^2]$ and $O'$ has Jordan type $[3^{k-2}, 2^2, 1]$. We shall prove that, for $x \in O'$, $s^{-1}(x)$ consists of one point; here, $s$ is the Springer map. When we regard $x$ as an element of $\text{End}(V)$, an element of $s^{-1}(x)$ corresponds to the isotropic flag $F := \{F_i\}$ of type $(k, 2n - 2k + 1, k)$ (resp. $(k, 2n - 2k, k)$) such that $xF_i \subset F_{i-1}$ for all $i$. Let $d$ be the Young diagram defined by the Jordan type $[3^{k-2}, 2^2, 1]$. There is a basis $\{e(i, j)\}$ of $V$ indexed by $d$ with the following properties (cf. [S-S], p.259, see also [C-M], 5.1.)

(i) $\{e(i, j)\}$ is a Jordan basis of $x$, that is, $xe(i, j) = e(i-1, j)$ for $(i, j) \in d$.

(ii) $e(i, j), e(p, q) \neq 0$ if and only if $p = d_j - i + 1$ and $q = \beta(j)$, where $\beta$ is a permutation of $\{1, 2, ..., d^1\}$ which satisfies: $\beta^2 = id$, $d_\beta(j) = d_j$, and $\beta(j) \neq j \pmod{2}$ if $d_j \equiv 0 \pmod{2}$. One can choose an arbitrary $\beta$ within these restrictions.

Introducing such a basis, one can directly check that there is only one isotropic flag with the desired properties\footnote{Let $r : \overline{O} \to \overline{O}$ be the covering map of degree 2. Then, this observation shows that $r$ is ramified along $r^{-1}(O')$.}

Next assume that $g$ is of type $C_n$. Let $V$ be a $2n$-dimensional $C$-vector space equipped with a non-degenerate skew-symmetric bilinear form $\langle , \rangle$. We may assume that $G = \text{Sp}(V)$. Similarly to the cases of $B_n$ and $D_n$, a parabolic subgroup $P$ of $G$ is described as the stabilizer group of an isotropic flag $F := \{F_i\}_{1 \leq i \leq s}$ of $V$. The parabolic subgroups $P$ determined by single marked Dynkin diagrams are stabilizer groups of the isotropic flags of (flag) type $(k, 2n - 2k, k)$. When $k$ is odd with $k \geq 2n/3$, the Richardson orbit $O$ for $P$ has Jordan type $[3^{k-1}, 2^2, 1^{2n-3k-1}]$, and the Springer map $s$ has degree 2. As in the cases of $B_n$ and $D_n$, we only have to discuss the case where $\overline{O}$ has a codimension 2 orbit $O'$. This is the case exactly when $k = (2n - 1)/3$. In this case $O$ has Jordan type $[3^{k-1}, 2^2]$ and $O'$ has Jordan type $[3^{k-1}, 2, 1^2]$. Now, by the same argument as in the cases of $B_n$ and $D_n$, we can prove that $s^{-1}(x)$ consists of one point. Q.E.D.
Definition 2. Let \( P \subset G \) be a parabolic subgroup conjugate to a standard parabolic subgroup determined by a single marked Dynkin diagram \( D \). Assume that \( D \) is of the 2-nd kind and is small. Let \( p' \) be the twist of \( p := \text{Lie}(P) \). Consider the diagram

\[
X_p \xrightarrow{\mu_p} Y_{\mathfrak{t}(p)} \xleftarrow{\mu_{p'}} X_{p'}
\]

and restrict it to the fibers over \( 0 \in \mathfrak{k}(p) \). Then we have a diagram

\[
X_{p,0} \xrightarrow{\mu_{p,0}} Y_{\mathfrak{t}(p),0} \xleftarrow{\mu_{p',0}} X_{p',0}.
\]

Here \( X_{p,0} = T^*(G/P) \) and \( X_{p',0} = T^*(G/P') \). Let \( T^*(G/P) \to \mathring{O} \to \mathring{O} \) be the Stein factorization of the Springer map. Then \( \mathring{O} \) coincides with the normalization of \( Y_{\mathfrak{t}(p),0} \). Thus, the birational map \( \mu_{p,0} \) (resp. \( \mu_{p',0} \)) factors through \( \mathring{O} \) and we get a diagram

\[
T^*(G/P) \to \mathring{O} \leftarrow T^*(G/P').
\]

By Proposition 3.1, this diagram consists of small birational maps. So, this diagram is a flop because the original diagram

\[
X_p \xrightarrow{\mu_p} Y_{\mathfrak{t}(p)} \xleftarrow{\mu_{p'}} X_{p'}
\]

is a flop by Lemma 1.4. We call this the Mukai flop of type \( B_{n,k} \), \( C_{n,k} \) or \( D_{n,k} \) respectively when the single marked Dynkin diagram \( D \) is of type \( B_{n,k} \), \( C_{n,k} \) or \( D_{n,k} \). In [Na], we have defined Mukai flops of type \( A_{n,k} \), \( D_n \) (\( n: \text{odd} \)), \( E_6,1 \) and \( E_6,11 \). Together with these, we call them Mukai flops.

Remark 3.2. For the Mukai flops of type \( B_{n,k} \), \( C_{n,k} \) and \( D_{n,k} \), the parabolic subgroups \( P \) and \( P' \) are \( G \)-conjugate. Hence, there is a natural \( \mathring{O} \)-isomorphism \( T^*(G/P) \cong T^*(G/P') \). This isomorphism induces a non-trivial covering transformation of \( \mathring{O} \to \mathring{O} \).

Example 3.3. The Mukai flop of type \( B_{n,n} \) with \( n \) even, coincides with the Mukai flop of type \( D_{n+1} \). In fact, let \( G_{\text{iso}}^+(n+1,2n+2) \) and \( G_{\text{iso}}^-(n+1,2n+2) \) be the two connected components of the orthogonal Grassmann variety parameterising \( n+1 \)-dimensional isotropic subspaces of a \( 2n+2 \)-dimensional \( \mathbb{C} \)-vector space \( V \) with a non-degenerate symmetric form \( < , > \). Let \( V' \) be a \( 2n+1 \)-dimensional vector subspace of \( V \) such that the restriction
of \( < , > \) to \( V' \) is a non-degenerate symmetric form. Let \( G_{\text{iso}}(n, 2n + 1) \) be the orthogonal Grassmann variety parameterising \( n \)-dimensional isotropic subspaces of \( V' \). For an \( n + 1 \)-dimensional isotropic subspace \( W \) of \( V \), \( W \cap V' \) is an isotropic subspace of \( V' \) of dimension \( n \). This correspondence gives isomorphisms \( G_{\text{iso}}^+(n + 1, 2n + 2) \cong G_{\text{iso}}(n, 2n + 1) \) and \( G_{\text{iso}}^-(n + 1, 2n + 2) \cong G_{\text{iso}}(n, 2n + 1) \). These isomorphisms induce the isomorphisms of cotangent bundles

\[
T^*(G_{\text{iso}}^+(n + 1, 2n + 2)) \cong T^*(G_{\text{iso}}(n, 2n + 1))
\]

and

\[
T^*(G_{\text{iso}}^-(n + 1, 2n + 2)) \cong T^*(G_{\text{iso}}(n, 2n + 1)).
\]

Let \( T^*(G_{\text{iso}}(n, 2n + 1)) \to \mathcal{O} \) be the Springer map. As noticed in the proof of Proposition 3.1, this has degree 2. Then its Stein factorization \( T^*(G_{\text{iso}}(n, 2n + 1)) \to \tilde{\mathcal{O}} \) coincides with the Springer map for \( T^*(G_{\text{iso}}^+(n + 1, 2n + 2)) \) or \( T^*(G_{\text{iso}}^-(n + 1, 2n + 2)) \). Now the Mukai flop of type \( B_{n,n} \) with \( n \) even:

\[
T^*(G_{\text{iso}}(n, 2n + 1)) \to \tilde{\mathcal{O}} \leftarrow T^*(G_{\text{iso}}(n, 2n + 1))
\]

is identified with the Mukai flop of type \( D_{n+1} \):

\[
T^*(G_{\text{iso}}^+(n + 1, 2n + 2)) \to \tilde{\mathcal{O}} \leftarrow T^*(G_{\text{iso}}^-(n + 1, 2n + 2)).
\]

Let \( S^*(l_0) \) be the subset of \( S(l_0) \) consisting of the parabolic subalgebras \( p \) which are obtained from \( p_0 \) by a sequence of twists of the 1-st kind and twists of type (2-s).

**Corollary 3.4.** Let \( \pi_0 : T^*(G/P_0) \to \tilde{\mathcal{O}} \) be the Stein factorization of the Springer map \( T^*(G/P_0) \to \mathcal{O} \). Then any two (projective) symplectic resolutions of \( \mathcal{O} \) are connected by a sequence of Mukai flops (Definition 2). Moreover, as a cone in \( M(L_0)_R \), one has

\[
\overline{\text{Mov}}(\pi_0) = \bigcup_{p \in S^*(l_0)} \overline{\text{Amp}}(\mu_p).
\]

**Proof.** Take a \( \pi_0 \)-movable line bundle \( L \) on \( T^*(G/P_0) \). We shall prove that

\[
[L] \in \bigcup_{p \in S^*(l_0)} \overline{\text{Amp}}(\mu_p).
\]
We may assume that $L$ is not $\pi_0$-nef. By the identification
$$N^1(\mu_{p_0})_R \cong N^1(\pi_0)_R,$$
$\overline{\text{Amp}}(\mu_{p_0})$ and $\overline{\text{Amp}}(\pi_0)$ are identified. There is an extremal ray $R_+[z] \subset NE(\pi_0)$ such that $(L.z) < 0$. Let $F \subset \overline{\text{Amp}}(\pi_0)$ be the corresponding codimension one face. This $F$ can be regarded as a codimension one face of $\overline{\text{Amp}}(\mu_{p_0})$. By Lemma 1.4, $F \subset \overline{\text{Amp}}(\mu_{p_0})$ determines a marked vertex $v$ of the marked Dynkin diagram $D$ attached to $P_0$. Let $\bar{D}$ be the marked Dynkin diagram which is obtained from $D$ by making $v$ unmarked. We then have a parabolic subgroup $\bar{P}$ with $P_0 \subset \bar{P}$ corresponding to $\bar{D}$. Notation being the same as in the proof of Lemma 1.4, the birational morphism determined by $F$ is given by
$$\phi_F : X_{p_0} \to (G \times^P \bar{P} \cdot n(p_0)) \times_{\xi((\bar{p}) \cap p_0)}/W \cdot \xi((\bar{p}) \cap p_0).$$

Note that $\phi_F$ is a morphism over $k_0$. Restrict $\phi_F$ to the fibers over $0 \in k_0$. Then we have a morphism
$$\phi_{F,0} : T^*(G/P_0) \to G \times^P \bar{P} \cdot n(p_0).$$

The birational morphism $\tilde{\phi}_F$ determined by $F \subset \overline{\text{Amp}}(\pi_0)$ coincides with the Stein factorization of $\phi_{F,0}$. Since $L$ is $\pi_0$-movable, $\tilde{\phi}_F$ must be a small birational map, i.e., $\text{codim} (\text{Exc}(\tilde{\phi}_F)) \geq 2$. On the other hand, by Proposition 3.1 and by the argument of the proof of [Na], Proposition 6.4, (iii), we see that $\tilde{\phi}_F$ is a small birational map if and only if the single marked Dynkin diagram $D_v$ is of the 1-st kind or of the 2-nd kind and small. Let $p_i$ be the twist of $p_0$ by $v$. Then, by Lemma 1.4, we have a flop $X_{p_0} \to X_{p_1}$ over $k_0$. Restrict this flop to the fibers over $0 \in k_0$. Then we have a flop $T^*(G/P_0) \to T^*(G/P_1)$, which is a locally trivial family of Mukai flops (cf. [Na], Proposition 6.4, (iii)). Let $L_1$ be the proper transform of $L$ by this flop and replace $T^*(G/P_0)$ by $T^*(G/P_1)$. The same procedure above produces a sequence of flops
$$T^*(G/P_0) \to T^*(G/P_1) \to T^*(G/P_2) \to \cdots.$$ 

By the same argument as Corollary 1.5, this sequence terminates. As a consequence, for some $k$, the proper transform $L_k \in \text{Pic}(T^*(G/P_k))$ of $L$ is $\pi_k$-nef. This implies that
$$[L] \in \bigcup_{p \in S^* (l_0)} \overline{\text{Amp}}(\mu_p).$$
**Problem:**  
(1) Calculate $\sharp S^*(l_0)$ explicitly.  

(2) For $p \in S(l_0)$, characterize an element $q \in S^*(l_0)$ such that $\mu_{p,0} = \mu_{q,0}$.  

When deg$(s) = 1$, $S^*(l_0) = S^1(l_0)$ and any two different elements of the set are not conjugate. So, in this case, $q$ is characterized as a unique element of $S^*(l_0)$ which is conjugate to $p$. However, when deg$(s) > 1$, $S^*(l_0)$ may possibly contain conjugate, but different elements (cf. Example 1.9). The problem is how to characterize $q$ among the elements of $S^*(l_0)$ which are conjugate to $p$.

**Remark 3.5.** B. Fu [F, Cor. 5.9] proved that if $s_i : X_i := T^*(G/P_i) \to \hat{O}$ $(i = 1, 2)$ are two Springer maps of the same degree, then $X_1$ and $X_2$ are connected by a sequence of Mukai flops of type $A_{n,k}$, $D_n$ (n:odd), $E_{6,1}$ and $E_{6,II}$: $Y_j \to Y_{j+1}$ $(j = 0, ..., k - 1)$ with $Y_0 = X_1$ and $Y_k = X_2$. But, his result is in a different context from ours. In fact, let $\pi_1 : X_1 \to \hat{O}$ be the Stein factorization of $s_1$. Then $\phi_j$ are birational maps over $\hat{O}$, but not necessarily over $\hat{O}$. More exactly, each $\phi_j$ may possibly induce a non-trivial covering automorphism of $\hat{O} \to \hat{O}$. For example, in Example 1.9, $X_{p_i,0}$ and $X_{p_i}^*(0)$ are isomorphic as the varieties over $\hat{O}$, but not isomorphic as those over $\hat{O}$.

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**References**

[B-K] Borho, W., Kraft, H.: Über Bahnen und deren Deformationen bei linearen Aktionen reductiver Gruppen, Comment Math. Helv. 54 (1979), 61-104

[Bo] Borel, A.: Linear algebraic groups, second enlarged edition, GTM 126, Springer, 1991

[C-M] Collingwood, D., McGovern, W.: Nilpotent orbits in semi-simple Lie algebras, van Nostrand Reinhold, Math. Series, 1993

[F] Fu, B.: Extremal contractions, stratified Mukai flops and Springer maps, Adv. Math. 213 (2007), 165-182

[Fuj] Fujiki, A.: A generalization of an example of Nagata, in the proceeding of Algebraic Geometry Symposium, Jan. 2000, Kyushu University, 111-118
[He] Hesselink, W.: Polarizations in the classical groups, Math. Z. 160 (1978) 217-234

[H-L] Howlett, R., Lehrer, G.: Induced cuspidal representations and generalized Hecke rings, Invent. Math. 58 (1980) 37-64

[Ho] Howlett, R.: Normalizers of parabolic subgroups of reflection groups, J. London Math. Soc. 21 (1980), 62-80

[Hu] Humphreys, J.: Introduction to Lie algebras and representation theory, Graduate Texts in Math. 9, Springer (1972)

[Ka] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. Math. 127, 93-163 (1988)

[Ko] Kostant, B.: Lie group representations on polynomial rings, Amer. J. Math. 85 (1963) 327-404

[Kol] Kollár, J.: Rational curves on algebraic varieties, Springer, Berlin Heidelberg New York, 1996

[K-M] Kollár, J., Mori, S.: Birational geometry of algebraic varieties, Cambridge University Press, 1998

[Na] Namikawa, Y.: Birational geometry of symplectic resolutions of nilpotent orbits, to appear in Advances Studies in Pure Mathematics 45, (2006), Moduli Spaces and Arithmetic Geometry (Kyoto, 2004), pp. 75-116, see also math.AG/0404072, math.AG/0408274

[Na 1] Namikawa, Y.: Deformation theory of singular symplectic n-folds, Math. Ann. 319 (2001) 597-623

[Ri] Richardson, R.W.: Conjugacy classes of involutions in Coxeter groups. Bull. Austral. Math. Soc. 26 (1982), no. 1, 1-15

[Slo] Slodowy, P.: Simple singularities and simple algebraic groups. Lect. Note Math. 815, Springer-Verlag, 1980

[S-S] Springer, T.A., Steinberg, R.: Conjugacy classes, In: Borel, et. al.: seminar on algebraic groups and related finite groups, Lect. Note Math. 131 167-266, Springer Verlag 1970
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