FIXED-DIMENSIONAL ENERGY GAMES ARE IN PSEUDO-POLYNOMIAL TIME

MARCEL JURDZIŃSKI, RANKO LAZIĆ, AND SYLVAIN SCHMITZ

Abstract. We generalise the hyperplane separation technique (Chatterjee and Velner, 2013) from multi-dimensional mean-payoff to energy games, and achieve an algorithm for solving the latter whose running time is exponential only in the dimension, but not in the number of vertices of the game graph. This answers an open question whether energy games with arbitrary initial credit can be solved in pseudo-polynomial time for fixed dimensions 3 or larger (Chaloupka, 2013). It also improves the complexity of solving multi-dimensional energy games with given initial credit from non-elementary (Brázdil, Jančar, and Kučera, 2010) to $2 \text{EXPTIME}$, thus establishing their $2 \text{EXPTIME}$-completeness.

Key Words. Energy game, bounding game, first-cycle game, vector addition system with states

1. Introduction

Multi-Dimensional Energy Games are played turn-by-turn by two players on a finite multi-weighted game graph, whose edges are labelled with integer vectors modelling discrete energy consumption and refuelling. Player 1’s objective is to keep the accumulated energy non-negative in every component along infinite plays. This setting is relevant to the synthesis of resource-sensitive controllers balancing the usage of various resources like fuel, time, money, or items in stock, and finding optimal trade-offs; see [3, 10, 4, 12] for some examples. Maybe more importantly, energy games are the key ingredient in the study of several related resource-conscious games, notably multi-dimensional mean-payoff games [15] and games played on vector addition systems with states (VASS) [3, 1, 8].

The main open problem about these games has been to pinpoint the complexity of deciding whether Player 1 has a winning strategy when starting from a particular vertex and given an initial energy vector as part of the input. This particular given initial credit variant of energy games is also known as Z-reachability VASS games [3, 5]. The problem is also equivalent via logarithmic-space reductions to deciding single-sided VASS games with a non-termination objective [1], and to deciding whether a given VASS (or, equivalently, a Petri net) simulates a given finite state system [11, 13, 8, 2]. As shown by Brázdil, Jančar, and Kučera [3], all these problems can be solved in $(d - 1) \text{EXPTIME}$ where $d \geq 2$ is the number of energy components, i.e. a TOWER of exponentials when $d$ is part of the input. The best known lower bound for this problem is $2 \text{EXPTIME}$-hardness [8], leaving a substantial complexity gap. So far, the only tight complexity bounds are for $d = 2$.

Work partially supported by ANR grant 11-BS02-001-01 REACHARD, the Leverhulme Trust Visiting Professorship 1-2014-041, and the EPSRC grant EP/M011801/1.
Chaloupka [5] shows the problem to be PTIME-complete when using unit updates, i.e. when the energy levels can only vary by $-1$, $0$, or $1$. However, quoting Chaloupka, ‘since the presented results about 2-dimensional VASS are relatively complicated, we suspect this [general] problem is difficult.’

When inspecting the upper bound proof of Brázdil et al. [3], it turns out that the main obstacle to closing the gap and proving 2EXPTIME-completeness lies in the complexity upper bounds for energy games with an arbitrary initial credit—which is actually the variant commonly assumed when talking about energy games. Given a multi-weighted game graph and an initial vertex $v$, we now wish to decide whether there exists an initial energy vector $b$ such that Player 1 has a winning strategy starting from the pair $(v, b)$. As shown by Velner, Chatterjee, Doyen, Henzinger, Rabinovich, and Raskin [15], this variant is simpler: it is coNP-complete. However, the parameterised complexity bounds in the literature [3, 7] for this simpler problem involve an exponential dependency on the number $|V|$ of vertices in the input game graph, which translates into a tower of exponentials when solving the given initial credit variant.

**Contributions.** We show in this paper that the arbitrary initial credit problem for $d$-dimensional energy games can be solved in time $O(|V| \cdot \|E\|)^{O(d^4)}$ where $|V|$ is the number of vertices of the input multi-weighted game graph and $\|E\|$ the maximal value that labels its edges, and also deduce that the given initial credit problem is solvable in time $O(|V| \cdot \|E\|)^{O(d \cdot \log d)}$ (see Corollary 3.6). Both bounds are pseudo-polynomial when the dimension is fixed, and the latter establishes 2EXPTIME-completeness closing the gap left open in [3, 8]. Our parameterised bounds are of practical interest because typical instances of energy games would have small dimension but might have a large number of vertices.

By the results of Velner et al. [15], another consequence is that we can decide the existence of a finite-memory winning strategy for fixed-dimensional mean-payoff games in pseudo-polynomial time. The existence of a finite-memory winning strategy is the most relevant problem for controller synthesis, but until now, solving fixed-dimensional mean-payoff games in pseudo-polynomial time required infinite memory strategies [6].

**Overview.** We prove our upper bounds on the complexity of the arbitrary initial credit problem for $d$-dimensional energy games by reducing them to bounding games, where Player 1 additionally seeks to prevent arbitrarily high energy levels [Section 2.3]. We further show these games to be equivalent to first-cycle bounding games in [Section 6], where the total effect of the first simple cycle defined by the two players determines the winner. More precisely, first-cycle bounding games rely on a hierarchically-defined colouring of the game graph by perfect half-spaces (see [Section 5]), and the two players strive respectively to avoid or produce cycles in those perfect half-spaces.

First-cycle bounding games coloured with perfect half-spaces can be seen as generalising quite significantly both

- the ‘local strategy’ approach of Chaloupka [5] for 2-dimensional energy games, and
FIXED-DIMENSIONAL ENERGY GAMES ARE IN PSEUDO-POLYNOMIAL TIME

Figure 1. A 2-dimensional multi-weighted game graph.

- the ‘separating hyperplane technique’ of Chatterjee and Velner [6] for multi-dimensional mean-payoff games; see Section 4 for an overview of the latter approach.

The reduction to first-cycle bounding games has several important corollaries: the determinacy of bounding games, and the existence of a small hypercube property, which in turn allow to derive the announced complexity bounds on energy games (see Section 3). In fact, we found with first-cycle bounding games a highly versatile tool, which we use extensively in our proofs on energy games.

We start by presenting the necessary background on energy and bounding games in Section 2. Some omitted material on linear algebra can be found in Appendix A.

2. Multi-Weighted Games

We define in this section the various games we consider in this work. We start by defining multi-weighted game graphs, which provide a finite representation for the infinite arenas over which our games are played. We then define energy games in Section 2.2 and their generalisation as bounding games in Section 2.3.

2.1. Multi-Weighted Game Graphs. We consider game graphs whose edges are labelled by vectors of integers. They are tuples of the form $(V, E, d)$, where $d$ is the dimension in $\mathbb{N}$, $V \overset{\text{def}}{=} V_1 \uplus V_2$ is a finite set of vertices, which is partitioned into Player 1 vertices ($V_1$) and Player 2 vertices ($V_2$), and $E$ is a finite set of edges included in $V \times \mathbb{Z}^d \times V$, and such that every vertex has at least one outgoing edge; we call the edge labels in $\mathbb{Z}^d$ ‘weights’.

Example 2.1. Figure 1 shows an example of a 2-dimensional multi-weighted game graph on its left-hand-side. Throughout this paper, Player 1 vertices are depicted as triangles and Player 2 vertices as squares.

2.1.1. Norms. For a vector $a$, we denote the maximum absolute value of its entries by $\|a\| \overset{\text{def}}{=} \max_{1 \leq i \leq d} |a(i)|$, and we call it the norm of $a$. By extension, for a set of edges $E$, we let $\|E\| \overset{\text{def}}{=} \max_{(v, u, v') \in E} \|u\|$. We assume, without loss of generality, that $\|E\| > 0$ in our multi-weighted game graphs. Regarding complexity, we encode vectors of integers in binary, hence $\|E\|$ may be exponential in the size of the multi-weighted game graph.
2.1.2. Paths and Cycles. Given a multi-weighted game graph \((V, E, d)\), a configuration is a pair \((v, a)\) with \(v\) in \(V\) and \(a\) in \(\mathbb{Z}^d\). A path is a finite sequence of configurations \(\pi = (v_0, a_0)(v_1, a_1) \cdots (v_n, a_n)\) in \((V \times \mathbb{Z}^d)^*\) such that for every \(0 \leq j < n\) there exists an edge \((v_j, a_{j+1} - a_j, v_{j+1})\) in \(E\) (where addition is performed componentwise). The total weight of such a path \(\pi\) is 
\[
w(\pi) \overset{\text{def}}{=} \sum_{0 \leq j < n} a_{j+1} - a_j = a_n - a_0.
\]
A cycle is a path \((v_0, a_0)(v_1, a_1) \cdots (v_n, a_n)\) with \(v_0 = v_n\). Such a cycle is simple if \(v_j = v_k\) for some \(0 \leq j < k \leq n\) implies \(j = 0\) and \(k = n\). We assume, without loss of generality, that every cycle contains at least one Player 1 vertex. We often identify simple cycles with their respective vertices; the weights of the four simple cycles of the game graph in Figure 1 are displayed on its right-hand-side.

**Proposition 2.2.** In any game graph \((V, E, d)\), the total weight of any simple cycle has norm at most \(|V| \cdot ||E||\).

2.1.3. Plays and Strategies. Let \(v_0\) be a vertex from \(V\). A play from \(v_0\) is an infinite configuration sequence \(\rho = (v_0, a_0)(v_1, a_1) \cdots \) such that \(a_0 = 0\) is the null vector and every finite prefix \(\rho|_n \overset{\text{def}}{=} (v_0, a_0)(v_1, a_1) \cdots (v_n, a_n)\) is a path. Note that, because \(a_0 = 0\), the total weight of this prefix is \(w(\rho|_n) = a_n\). We define the norm of a play \(\rho\) as the supremum of the norms of total weights of its prefixes: \(\|\rho\| \overset{\text{def}}{=} \sup_n \|w(\rho|_n)\|\).

A strategy for Player \(p, p \in \{1, 2\}\), is a function \(\sigma_p\) taking as input a non-empty path \(\pi \cdot (v, a)\) ending in a Player \(p\) vertex \(v \in V_p\), and returning an edge \(\sigma_p(\pi \cdot (v, a)) = (v, u, v') \) from \(E\). A play \(\rho = (v_0, a_0)(v_1, a_1) \cdots \) is consistent with a strategy \(\sigma_p\) for Player \(p\) if whenever \(v_n\) is a Player \(p\) vertex in \(V_p\), then \(\sigma_p(\rho|_n) = (v_n, a_{n+1} - a_n, v_{n+1})\). Given strategies \(\sigma_1\) and \(\sigma_2\) for Player 1 and Player 2 respectively, and an initial vertex \(v_0\), observe that there is a unique play \(\rho_{v_0, \sigma_1, \sigma_2}\) from \(v_0\) consistent with both \(\sigma_1\) and \(\sigma_2\).

**Example 2.1 (continued).** For instance, in the game graph depicted in Figure 1, a strategy for Player 1 could be to move to \(v_L\) whenever the current energy level on the first coordinate is non-negative, and to \(v_R\) otherwise—note that this is an infinite-memory strategy—:

\[
\sigma_1(\pi \cdot (v_0, a)) \overset{\text{def}}{=} \begin{cases} (v_0, (0, 0), v_L) & \text{if } a(1) \geq 0, \\ (v_0, (0, 0), v_R) & \text{otherwise}, \end{cases}
\]

and one for Player 2 could be to always select one particular edge in every vertex, regardless of the current energy vector—this is called a counterless strategy [3]—:

\[
\sigma_2(\pi \cdot (v, a)) \overset{\text{def}}{=} \begin{cases} (v_L, (-2, 2), v_0) & \text{if } v = v_L \\ (v_R, (2, -1), v_0) & \text{otherwise}. \end{cases}
\]

These strategies define a unique consistent play for \(v_0\), which starts with

\[
(v_0, 0, 0)(v_L, 0, 0)(v_0, -2, 2)(v_R, -2, 2)(v_0, 0, 1)(v_L, 0, 1)(v_R, -2, 3) \cdots
\]

In the following we consider several different winning conditions on plays, which define different games played on multi-weighted game graphs.
2.2. Multi-Dimensional Energy Games. Suppose \((V, E, d)\) is a multi-weighted game graph, \(v_0\) an initial vertex, and \(b\) is a vector from \(\mathbb{N}^d\). A play \(\rho\) from \(v_0\) is winning for Player 1 in the energy game \(\Delta_b(V, E, d)\) with initial credit \(b\) if, for all \(n\), \(b + w(\rho|_n) \geq 0\), using the product ordering over \(\mathbb{Z}^d\). Otherwise, Player 2 wins the play. As usual, this means that Player 1 wins the energy game \(\Delta_b(V, E, d)\) from \(v_0\) if there exists a winning strategy \(\sigma_1\) for Player 1, i.e. \(\sigma_1\) is such that for all strategies \(\sigma_2\) for Player 2 the play \(\rho_{v_0, \sigma_1, \sigma_2}\) is winning for Player 1. An immediate property of energy games is monotonicity: if \(\sigma_1\) is winning for Player 1 with some initial credit \(b\), and \(b' \geq b\), then it is also winning for Player 1 with initial credit \(b'\).

Example 2.1 (continued). For example, one may observe that the strategy (1) for Player 1 is winning for the game graph of Figure 1 with initial credit \((2, 2)\) (or larger).

A geometric intuition comes from the directions of the total weights of simple cycles in Figure 1: by choosing alternatively edges to \(v_L\) or \(v_R\), Player 1 is able to balance the energy levels above the ‘\(x + y = 0\)’ line. One way to see this more formally is to build the corresponding self-covering strategy tree up to the first time when a configuration is greater or equal to another configuration higher in the tree [3]. By monotonicity of the game, Player 1 can repeat the same actions from those leaves. See Figure 2 for our example.

Strategy \(\sigma_1\) uses the comparison of \(a(1)\) with 0 as a soft bound to trigger a change of strategy and attempt to forbid cycles with a negative effect on the first coordinate. Note that the energy level \(a(1)\) might nevertheless become less than 0, but will remain \(\geq -2\) at all times; we call this the hard bound. This follows the general scheme of Chaloupka [5]—and also ours—for Player 1 strategies.

2.3. Multi-Dimensional Bounding Games. A generalisation of energy games sometimes considered in the literature is to further impose a maximal capacity \(c \in \mathbb{N}^d\) (also called an upper bound) on the energy levels during the play [10, 12]. Player 1 then wins a play \(\rho\) if \(0 \leq b + w(\rho|_n) \leq c\) for all \(n\).

In the spirit of the arbitrary initial credit variant of energy games, we also quantify \(c\) existentially. This defines the bounding game \(\Gamma(V, E, d)\) over a multi-weighted game graph \((V, E, d)\), where a play \(\rho\) is winning for Player 1 if its norm \(||\rho||\) is finite, i.e. if the set \(\{||w(\rho|_n)|| : n \in \mathbb{N}\}\) of norms of total weights of all finite prefixes of \(\rho\) is bounded, and Player 2 wins otherwise, i.e. if the set is unbounded. In other words, Player 1 strives to contain the current vector within some \(d\)-dimensional hypercube, while Player 2 attempts to escape.

Example 2.1 (continued). Note that Player 2 is now winning the bounding game defined by the game graph of Figure 1 from any of the three vertices, for example using the strategy (2). Indeed, this strategy ensures that the only simple cycles that can be played have weights \((-2, 2)\) and \((2, -1)\). Because these vectors belong to an open half-plane, the total energy will drift deeper and deeper inside that open half-plane and its norm will grow unbounded.

Example 2.3. As a rather different example, consider the multi-weighted game graph of Figure 3. Although Player 2 does not control any vertex, and
Player 1 controls the ‘direction of divergence’, Player 2 wins the associated bounding game. Indeed, Player 1 can either eventually stay forever at one of the two vertices, or visit both vertices infinitely often. Anyway, she loses. □

3. Complexity Upper Bounds

Our main results are new parameterised complexity upper bounds for deciding whether Player 1 has a winning strategy in a given energy game. In turn, we rely for these results on a small hypercube property of bounding games, which we introduce next, and which will be a consequence of the study of first-cycle bounding games in Section 6.

3.1. Small Hypercube Property. In a bounding game, if Player 1 is winning, then by definition she has a winning strategy $\sigma_1$ such that for all
plays \( \rho \) consistent with \( \sigma_1 \) there exists some bound \( B_\rho \) with \( \| \rho \| \leq B_\rho \). We considerably strengthen this statement in Section 6 where we construct an explicit winning strategy, which yields an explicit uniform bound \( B \) for all consistent plays:

**Lemma 3.1.** Let \( (V, E, d) \) be a multi-weighted game graph. If Player 1 wins the bounding game \( \Gamma(V, E, d) \), then she has a winning strategy which ensures

\[
\| \rho \| \leq (4|V| \cdot \| E \|)^{2(d+2)^3}
\]

for all consistent plays \( \rho \).

Note that our bound is polynomial in \( |V| \) the number of vertices, unlike the bounds found in comparable statements by Brázdil et al. [3, Lemma 7] and Chatterjee et al. [7, Lemma 3], which incur an exponential dependence on \( |V| \). This entails pseudo-polynomial complexity bounds when \( d \) is fixed:

**Corollary 3.2.** Bounding games on multi-weighted graphs \( (V, E, d) \) are solvable in deterministic time \( O(|V| \cdot \| E \|)^{O(d^4)} \).

**Proof.** By Lemma 3.1 the bounding game is equivalent to a reachability game where Player 2 attempts to see the norm of the total weight exceed \( B \overset{\text{def}}{=} (4|V| \cdot \| E \|)^{2(d+2)^3} \). This can be played within a finite arena of size \((2B + 1)^d \) and solved in time linear in that size using the usual attractor computation algorithm. \( \Box \)

3.2. **Solving Energy Games.** We now show how energy games can be solved by solving bounding games on appropriately augmented game graphs. Given an energy game and an initial vertex, there are two standard decision problems:

- **arbitrary initial credit:** does Player 1 win for some initial energy vector;
- **given initial credit:** does Player 1 win for a given initial energy vector?

Another problem of interest is computing the Pareto limit, i.e. the set of all pointwise minimal initial energy vectors for which Player 1 wins.

It will turn out that solving the arbitrary initial credit problem is relatively easy since it amounts to solving the bounding game with self-loops added at all Player 1 vertices that give her power to prevent Player 2 from winning by diverging in a non-negative direction. Solving the given initial credit problem and computing the Pareto limit will require more work, involving reasoning à la Rackoff’s for the covering problem, where ‘very small’ vector components which may potentially become negative are tracked using graph vertices and the remaining ‘very large’ vector components are guaranteed to remain so by the small hypercube property of bounding games. Although the presentation of the latter work takes up most of this section, we remark that it is relatively uninteresting and essentially follows the pattern already seen in Brázdil et al.’s stepping up from the arbitrary to the given initial credit problem.

3.2.1. **Tracking Sets.** Supposing \( (V, E, d) \) is a multi-weighted game graph, let \( \lambda_i = (4|V| \cdot \| E \|)^{(3d)4i} \) for \( i = 1, \ldots, d \).

We write \( \mathbb{N}_\infty \) for the natural numbers extended by \( \infty \), where \( n < \infty, \infty + n = \infty, \) and \( \infty - n = \infty \), for all \( n \in \mathbb{N} \).
Inspired by [7, Section 8], we say that \( T \subseteq \{1, \ldots, d\} \) is a tracking set for \( \mathbf{a} \in \mathbb{N}_\infty^d \) iff \( a(i) \geq \lambda_{|T|+1} \) for all \( i \in \{1, \ldots, d\} \setminus T \). (We remark that if \( |T| = d \) then the quantification over \( i \) is empty, so it is not an issue that \( \lambda_{|T|+1} \) is undefined in that case.)

Suppose \( \mathbf{a} \in \mathbb{N}_\infty^d \). Observing that \( \{1, \ldots, d\} \) is a tracking set, and that the class of tracking sets is closed under intersection since the sequence \( \lambda_\cdot, \ldots, \lambda_1 \) is decreasing, we conclude that \( \mathbf{a} \) has a unique minimal tracking set \( \text{tr}(\mathbf{a}) \). We write \( \lambda(\mathbf{a}) \) for the abstraction \( \mathbf{a}|_{\text{tr}(\mathbf{a})} \in \mathbb{N}_\infty^d \) obtained from \( \mathbf{a} \) by setting to \( \infty \) all its components whose indices are outside of its unique minimal tracking set.

Intuitively, for every \( \mathbf{a} \in \mathbb{N}_\infty^d \), its unique minimal tracking set \( \text{tr}(\mathbf{a}) \) identifies the least collection of components of \( \mathbf{a} \) that are much smaller than the remaining components of \( \mathbf{a} \), in the sense that they are separated by the two corresponding consecutive elements of the increasing sequence \( \lambda_\cdot, \ldots, \lambda_d \).

Indeed, we have that \( a(i) \geq \lambda_{|\text{tr}(\mathbf{a})|+1} \) for all \( i \not\in \text{tr}(\mathbf{a}) \) since \( T(\mathbf{a}) \) is a tracking set, and that \( a(i) < \lambda_{|\text{tr}(\mathbf{a})|} \) for all \( i \in \text{tr}(\mathbf{a}) \) since \( \text{tr}(\mathbf{a}) \) is minimal. (We remark that if \( |\text{tr}(\mathbf{a})| = 0 \) then the latter quantification over \( i \) is empty, so it is not an issue that \( \lambda_{|\text{tr}(\mathbf{a})|} \) is undefined in that case.)

We have called those sets ‘tracking’ because, in some constructions to follow shortly, we shall use them to identify vector components whose values will be exactly tracked in vertices of game graphs.

For \( T \in \{1, \ldots, d\} \), let \( \mathbb{N}_\infty^d|_T \) consist of all \( \mathbf{a} \in \mathbb{N}_\infty^d \) such that \( \text{tr}(\mathbf{a}) = T \) and \( \lambda(\mathbf{a}) = \mathbf{a} \). Informally, it is the collection of all vectors whose minimal set of ‘very small’ components is \( T \) and which have been abstracted by setting all their other components to \( \infty \). By what we observed above, the cardinality of \( \mathbb{N}_\infty^d|_T \) is at most \( (\lambda_{|T|})^{|T|} \). (We remark that if \( T \) is the empty set then this is \( 1 \).

3.2.2. Tracking Lossy Game Graphs. We now define \( \text{TrLo}_T(V, E, d) \) as game graphs which are obtained from the game graph \( (V, E, d) \) by extending the vertices so that they can exactly track vector components at indices in the set \( T \in \{1, \ldots, d\} \) as long as they are ‘much smaller’ than the remaining ones, and by inserting ‘lossy’ self-loops that enable Player 1 to prevent the latter vector components from becoming unboundedly large. These constructions will give us the means to reduce solving energy games to solving bounding games: intuitively, since bounding games have the small hypercube property (cf. [Lemma 3.1]), Player 1 in an energy game can treat ‘sufficiently large’ vector components as if she was playing a bounding game, but she has to be careful with keeping the others non-negative; and she should be given power artificially for preventing Player 2 from winning the bounding game by making values unboundedly large since that would not constitute a win for him in the energy game.

The definition is recursive, where we assume that \( \text{TrLo}_T(V, E, d) \) game graphs have been defined for all \( T \) of smaller cardinality (if any). We write \( \overline{T} \) for \( \{1, \ldots, d\} \setminus T \).

- The dimension is \( |T| = d - |T| \), where we regard edge weights as vectors in \( \mathbb{Z}^\overline{T} \).
The set of Player 1 vertices is \( (V_1 \times \mathbb{N}_+^{d_T}) \cup \{v_T, v_\perp\} \), and the set of Player 2 vertices is \( V_\perp \times \mathbb{N}_+^{d_T} \).

For every \((v, u, v') \in E\) and \(a \in \mathbb{N}_+^{d_T}\), there are four cases:

- \(\lambda(a + u)\) is in \(\mathbb{N}_+^{d_T}\): there is an edge \((v, a), u\_\perp, (v', a + u)\);
- \(\lambda(a + u)\) is in \(\mathbb{N}_+^{d_T}\) for a strict subset \(T'\) of \(T\) and Player 1 wins \(\Gamma(\text{TrLo}_{T'}(V, E, d))\) from \((v', \lambda(a + u))\): there is an edge \((v, a), u\_\perp, v_T\);
- \(\lambda(a + u)\) is in \(\mathbb{N}_+^{d_T}\) for a strict subset \(T'\) of \(T\) and Player 1 loses \(\Gamma(\text{TrLo}_{T'}(V, E, d))\) from \((v', \lambda(a + u))\): there is an edge \((v, a), u\_\perp, v_\perp\);
- \(a + u\) has a negative component: there is an edge \((v, a), u\_\perp, v_\perp\).

Every Player 1 vertex of \(\text{TrLo}_{T'}(V, E, d)\), for every \(i \in T\), has a self-loop labelled by the negative unit vector \(-e_i\).

The only other edge of \(\text{TrLo}_{T'}(V, E, d)\) is a self-loop on vertex \(v_T\) labelled by the \(|T|\)-dimensional zero vector.

The conditions that define the four cases above have the property that always exactly one of them is satisfied because the \(\Gamma(\text{TrLo}_{T'}(V, E, d))\) bounding game is determined, cf. Section 6.

When \(T = \{1, \ldots, d\}\), according to the definition above, \(\text{TrLo}_{T'}(V, E, d)\) has dimension 0, so it fails the assumption from Section 2.1 that the maximum norm of its edge weights is positive. In that case also, its vertex \(v_\perp\) fails the assumption from the same section that every vertex has at least one outgoing edge. Nevertheless, \(\text{TrLo}_{\{1, \ldots, d\}}(V, E, d)\) is otherwise well defined, and we shall regard \(\Gamma(\text{TrLo}_{\{1, \ldots, d\}}(V, E, d))\) as a reachability game in which the goal of Player 1 is to avoid the vertex \(v_\perp\) forever and the goal of Player 2 is to reach it. (We remark that in this game graph the vertex \(v_T\) has a self-loop and is hence winning for Player 1 in the reachability game.)

At the other extreme, when \(T\) is empty, \(\mathbb{N}_+^{d_T}\) is the singleton set consisting of the vector with \(\infty\) in every component. Moreover, the vertices \(v_T\) and \(v_\perp\) are not reachable from other vertices, and thus can be removed from the game graph. Therefore, \(\text{TrLo}_\emptyset(V, E, d)\) is essentially \((V, E, d)\) extended by the lossiness, i.e. the self-loops at every Player 1 vertex and with every negative unit weight.

### 3.2.3. Solving Energy Games Using Bounding Games

We now show that solving an energy game is reducible to solving a bounding game on a tracking and lossy game graph as defined above, where the set of ‘very small’ vector components to track is determined by the initial credit vector.

**Theorem 3.3.** For every \(p \in \{1, 2\}\), initial credit \(b \in \mathbb{N}_+^d\), and vertex \(v\), we have that Player \(p\) wins energy game \(\Delta_b(V, E, d)\) from \(v\) if Player \(p\) wins bounding game \(\Gamma(\text{TrLo}_{\text{tr}(b)}(V, E, d))\) from \((v, \lambda(b))\).

**Proof.** The proof is inductive, where the hypothesis is that the statement holds for every smaller cardinality of \(\text{tr}(b)\).

We first handle Player 1, so suppose she has a winning strategy \(\sigma\) in \(\Gamma(\text{TrLo}_{T}(V, E, d))\) from \((v, \lambda(b))\), where \(T\) is the unique minimal tracking set of vector \(b\). Intuitively, we shall obtain a winning strategy for Player 1 in the energy game by playing according to \(\sigma\) as long as the vector components
that have not been abstracted away in the initial credit $b$ remain ‘very small’, and by switching to a winning strategy provided from the inductive hypothesis as soon as one or more of those vector components become ‘too large’. The tracking of the former vector components in the bounding game will ensure that they remain non-negative, and the small hypercube property of bounding games will ensure that the abstracted vector components remain sufficiently large. All of the lossy self-loops that are performed by $\sigma$ can be skipped since doing so only results in larger values.

When $|T| < d$, the game is bounding of dimension $d - |T|$, so by Lemma 3.1 we can assume that $\sigma$ stays within a small hypercube. If also $|T| > 0$, this ensures that the norm of the total weight of every prefix of every consistent play is at most

$$
\left( 4 \left( |V| \cdot \|b\|_T + 2 \cdot \|E\| \right) \right)^{2(d - |T| + 2)^3} \leq
\left( 4 \left( |V| \cdot (\lambda|T|)^{|T|} + 2 \cdot \|E\| \right) \right)^{2(d - |T| + 2)^3} <
\left( 4 \left( |V| \cdot (4|V| \cdot \|E\|)^{d(3d)^{4|T|}} + 2 \cdot \|E\| \right) \right)^{2(d - |T| + 2)^3} <
\left( 8|V| \cdot \|E\| \cdot (4|V| \cdot \|E\|)^{d(3d)^{4|T|}} \right)^{2(d - |T| + 2)^3} \leq
\left( (4|V| \cdot \|E\|)^{d(3d)^{4|T|} + 1.5} \right)^{2(d - |T| + 2)^3} <
\left( (4|V| \cdot \|E\|)^{1.5d(3d)^{4|T|}} \right)^{2(d - |T| + 2)^3} \leq
\left( (4|V| \cdot \|E\|)^{3d(3d)^{4|T|} + 1(2d)^3} \right)^{2(d - |T| + 2)^3} < \lambda|T| + 1 - \lambda|T|.
$$

If $|T| = 0$, the bound simplifies to

$$
(4|V| \cdot \|E\|)^{2(d + 2)^3} < (4|V| \cdot \|E\|)^{(3d)^{4}} = \lambda_1.
$$

Let $\hat{\sigma}$ be the following strategy of Player 1 in the energy game $\Delta_b(V, E, d)$ from the vertex $v$.

- Strategy $\hat{\sigma}$ makes the same choices as the strategy $\sigma$ as long as playing by the latter does not reach the vertex $v_T$, except that it skips any self-loops with negative unit weights that were added to Player 1 vertices in the definition of $\text{TrLo}_T(V, E, d)$. This is well defined since every edge in $\text{TrLo}_T(V, E, d)$ of the form $\{(v', a), u|_{\lambda_T}, (v'', a + u)\}$ determines the corresponding edge $(v', a, v'')$ in $(V, E, d)$, and since $\sigma$ cannot keep choosing the lossy self-loops consecutively forever because it is winning. Note also that, for the same reason, playing by $\sigma$ cannot reach the vertex $v_{\perp}$.

- Observe that the portion of $\hat{\sigma}$ defined so far has the following property. For every vertex $(v', a)$ and total weight $w \in \mathbb{Z}^T$ which are reached by playing according to $\sigma$ in $\text{TrLo}_T(V, E, d)$ from $(v, \lambda(b))$, we have that $v'$ is the corresponding vertex reached by playing according to $\hat{\sigma}$ in $(V, E, d)$ from $v$, and that the corresponding energy vector $b'$ satisfies:
  \[ b'|_T = a|_T; \]
– for every $i \in \mathcal{T}$, we have that $b'(i) \geq b(i) + w(i)$, where the difference between the two sides is exactly the number of $-e_i$ added self-loops that have been taken by $\sigma$. The latter inequality, together with the small hypercube property of $\sigma$ and the fact that $b(i) \geq \lambda_{|T|+1}$ since $T = \text{tr}(b)$, implies that $b'(i) \geq 0$.

• If and as soon as playing by $\sigma$ reaches the vertex $v_T$, which can only be by and edge in $\text{TrLo}_T(V, E, d)$ of the form $((v', a), u|\mathcal{T}, v_T)$ where there exists an edge $(v'', a, v''')$ in $(V, E, d)$ such that $\lambda(a + u)$ is in $\mathbb{N}_\infty^d$ for a strict subset $T'$ of $T$ and Player 1 wins $\Gamma(\text{TrLo}_{T'}(V, E, d))$ from $(v''', \lambda(a + u))$, then $\hat{\sigma}$ follows such an edge $(v', u, v'')$ which results in some current energy vector $b''$ and subsequently plays according to some winning strategy of Player 1 in the energy game $\Delta_{b''}(V, E, d)$ from the vertex $v''$. Such a strategy exists by the inductive hypothesis because $\lambda(b'') = \lambda(a + u)$, which follows from the observations above and, in case $|T| < d$, from the small hypercube property of $\sigma$ (with recalling that now also $|T| > 0$).

It remains to handle Player 2, so suppose he has a winning strategy $\tau$ in $\Gamma(\text{TrLo}_T(V, E, d))$ from $(v, \lambda(b))$, where $T$ is the unique minimal tracking set of vector $b$. Intuitively, we shall obtain a winning strategy for Player 2 in the energy game by playing according to $\tau$ as long as the vector components that have not been abstracted away in the initial credit $b$ remain non-negative and ‘very small’. If that continues forever, Player 2 will win regardless of how large the other components of $b$ are, because the added lossy self-loops in the bounding game ensure that $\tau$ makes the total weight diverge in some negative direction. Otherwise, Player 2 will win either immediately or by switching to a strategy provided from the inductive hypothesis.

When $|T| < d$, we have that Lemma 6.3 and Lemma 6.6 apply to the bounding game of dimension $d - |T|$ and its first-cycle variant $G(\text{TrLo}_T(V, E, d))$, so we can assume that $\tau$ mimics a winning strategy of Player 2 in $G(\text{TrLo}_T(V, E, d))$ from $(v, \lambda(b))$ as in Section 6.2. Since every Player 1 vertex was augmented with every negative unit self-loop, the latter strategy chooses only perfect half-spaces that are disjoint from the non-negative orthant $\text{cone}(e_i : i \in \mathcal{T})$. Hence, from the proof of Lemma 6.3, for every infinite play $\rho$ consistent with $\tau$, letting $C_1, C_2, \ldots$ be its cycle decomposition, there exist an open half-subspace $H$ and a positive integer $N$ such that:

• $H$ is disjoint from the non-negative orthant;

• for each $n \geq N$, the total weight $w(C_n)$ belongs to $\mathcal{H}$ the topological closure of $H$;

• the set of all distances of $w(C_N) + \cdots + w(C_n)$ from the boundary of $H$ is unbounded.

Let $\hat{\tau}$ be the following strategy of Player 2 in the energy game $\Delta_{b}(V, E, d)$ from the vertex $v$.

• Strategy $\hat{\tau}$ makes the same choices as the strategy $\tau$ as long as playing by the latter does not reach the vertex $v_\perp$. This is well defined since every edge in $\text{TrLo}_T(V, E, d)$ of the form $((v', a), u|\mathcal{T}, (v'', a + u))$ determines the corresponding edge $(v', u, v'')$ in $(V, E, d)$. Note that the lossy self-loops that were added in the definition of $\text{TrLo}_T(V, E, d)$
Corollary 3.6. For energy games on multi-weighted game graphs all vectors in the Pareto limit are at most $\lambda$ the games $\Gamma(\TrLo(V, E, d))$. Every such $\hat{\rho}$ is then winning for Player 2 by the analysis above, regardless of how large the initial credit vector components $b|\TrLo$ are.

- One way in which playing by $\tau$ can reach the vertex $v_{\perp}$ is by and edge in $\TrLo(V, E, d)$ of the form $((v', a), u|\TrLo, v_{\perp})$ where there exists an edge $(v', u, v'')$ in $(V, E, d)$ such that $\lambda(a + u)$ is in $\N^{|T|}$, for a strict subset $T'$ of $T$ and Player 2 wins $\Gamma(\TrLo(V, E, d))$ from $(v'', \lambda(a + u))$. Then $\hat{\rho}$ follows such an edge $(v', u, v'')$ which results in some current energy vector $b''$ that coincides with $a + u$ on all the components indexed by $T$. Letting $b^1$ be some vector such that $b^1 \geq b''$ and $\lambda(b^1) = \lambda(a + u)$, subsequently $\hat{\rho}$ plays according to some winning strategy of Player 2 in the energy game $\Delta_{b^1}(V, E, d)$ from the vertex $v''$, which exists by the inductive hypothesis.

- The other way in which playing by $\tau$ can reach the vertex $v_{\perp}$ is by and edge in $\TrLo(V, E, d)$ of the form $((v', a), u|\TrLo, v_{\perp})$ where there exists an edge $(v', u, v'')$ in $(V, E, d)$ such that $a + u$ has a negative component. Then $\hat{\rho}$ follows such an edge $(v', u, v'')$ and at that point wins the energy game for Player 2, so subsequently may play arbitrarily.

3.2.4. Complexity Upper Bounds. Theorem 3.3 not only tells us how to solve energy games for given initial credits using bounding games, but it also has the following easy corollary saying that energy games with arbitrary initial credits can be solved just using the bounding game on the lossy game graph.

Corollary 3.4. The following are equivalent:

1. There exists $b \in \N^d$ such that Player 1 wins energy game $\Delta_b(V, E, d)$ from vertex $v$.
2. Player 1 wins energy game $\Delta_{(\lambda_1, \ldots, \lambda_1)}(V, E, d)$ from vertex $v$.
3. Player 1 wins bounding game $\Gamma(\TrLo(V, E, d))$ from vertex $(v, \infty, \ldots, \infty)$.

Example 3.5. By Corollary 3.4, because she was winning the energy game of Figure 1 with initial credit $(2, 2)$, Player 1 is now winning the bounding game played on the lossy multi-weighted game graph of Figure 4.

It is also straightforward to obtain the next corollary. Its first part follows from Corollary 3.4 and Corollary 3.2. Its second and third parts follow from Theorem 3.3, since $O(|V| \cdot |E|^{2d })$ deterministic time suffices for solving the games $\Gamma(\TrLo(V, E, d))$ for all $T \in \{1, \ldots, d\}$, and all components of all vectors in the Pareto limit are at most $\lambda_d = (4|V| \cdot |E|)^{(3d)^d}$.

Corollary 3.6. For energy games on multi-weighted game graphs $(V, E, d)$:

1. the arbitrary initial credit problem is solvable in $O(|V| \cdot |E|^{O(d^4)}$ deterministic time;
(2) the given initial credit problem is solvable in \( O(|V| \cdot \|E\|)^{2O(d \cdot \log d)} \) deterministic time;
(3) the Pareto limit, consisting of all pointwise minimal winning initial credits, is computable in \( O(|V| \cdot \|E\|)^{2O(d \cdot \log d)} \) deterministic time.

The upper bound for the given initial credit problem matches the \( \text{2EXPTIME} \) lower bound from [8], and encompasses Chaloupka’s \( \text{PTIME} \) upper bound in dimension \( d = 2 \) with unit updates, i.e. with \( \|E\| = 1 \). Because the given initial credit problem is \( \text{EXPTIME} \)-hard for fixed dimension \( d \geq 4 \) [8], the upper bound in terms of \( \|E\| \) cannot be improved.

4. Multi-Dimensional Mean-Payoff Games

This section summarises the technique for solving multi-dimensional mean-payoff games proposed by Chatterjee and Velner [6], which relies on open half-spaces. The rest of the paper does not rely formally on this section and it may be omitted by a reader eager to get on with our new ‘perfect half-spaces’ technique for solving multi-dimensional bounding games. We believe, however, that starting here helps put our work in context, appreciate similarities and differences between the two techniques, and understand the conceptual and some of the technical challenges we had to overcome.

4.1. Multi-Dimensional Mean-Payoff Games. Given a play \( \rho \) over a multi-weighted game graph \((V, E, d)\), we define its long-term average in \( \mathbb{Q}^d \) as

\[
\text{avg}(\rho) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{w(\rho|_n)}{n}.
\]

We say that \( \rho \) is winning for Player 1 in the mean-payoff game \( M(V, E, d) \) if \( \text{avg}(\rho) \geq 0 \). Otherwise, i.e. if there is a coordinate \( 1 \leq i \leq d \) such that \( \text{avg}(\rho)(i) < 0 \), the play \( \rho \) is winning for Player 2. As shown by Velner et al. [13], determining the winner in multi-dimensional mean-payoff games is \( \text{coNP} \)-complete, and in pseudo-polynomial time when the dimension is fixed [6, Theorem 1].

4.2. Energy Versus Mean-Payoff. In a one-dimensional arbitrary-initial-credit energy game, the goal of Player 1 is to keep the energy level bounded from below. It is folklore that Player 1 has a winning strategy in such a game if and only if she has a strategy in the mean-payoff game on the same game graph that guarantees a non-negative long-term average.

4.2.1. Infinite Memory Strategies for Player 1. This relationship between energy games and mean-payoff games does not generalise to multi-dimensional games. We illustrate this on the example of a 2-dimensional game graph from Figure 3. In Example 2.3 we have argued that Player 2 has a winning
strategy in the bounding game (and hence also in the arbitrary-initial-credit energy game). On the other hand, we argue that Player 1 has a strategy to guarantee that the long-term average in both dimensions is non-negative. Indeed, consider a strategy in which in stage $m$—for all $m = 1, 2, 3, \ldots$—Player 1 performs one of the self-loops $m$ times, then she moves to the other vertex where she performs the other self-loop $m$ times, and then finally returns to the starting vertex. After $m$ stages, the energy level in both dimensions is $-m$ and the number of steps performed is $\Theta(m^2)$, hence the long-term average in the infinite play is 0 in both dimensions, because $\lim_{m \to \infty} -\frac{m}{m^2} = 0$.

Note that this strategy for Player 1 in the game graph of Figure 3 is infinite-memory, since the actions depend on the stage $m$. Multi-dimensional mean-payoff games might require infinite memory in order to be won, as shown by Velner et al. [15, Lemma 4]—their proof can be used to show that the game of Figure 3 actually requires infinite memory.

4.2.2. Finite Memory Strategies for Player 1. In the multi-dimensional case, there is nevertheless a strong relation between energy and mean-payoff games. Call a strategy $\sigma$ finite memory if there exists an equivalence relation $\sim$ with finite index over $(V \times \mathbb{Z}^d)^+$ such that, whenever $\pi \sim \pi'$ for some non-empty paths $\pi$ and $\pi'$ in the domain of $\sigma$, then $\sigma(\pi) = \sigma(\pi')$ (such strategies are typically described using Moore machines).

**Fact 4.1** (Velner et al. [15]). Let $(V, E, d)$ be a multi-weighted game graph. There exists an initial credit $b$ such that Player 1 wins the energy game $\Delta_b (V, E, d)$ if and only if Player 1 has a finite memory winning strategy in the mean-payoff game $M(V, E, d)$.

Hence our complexity bounds in Corollary 3.6 on multi-dimensional energy games also yield a pseudo-polynomial time algorithm to find a winning finite-memory strategy for Player 1 in a given fixed-dimensional mean-payoff game.

4.3. The Open Half-Space Technique for Mean-Payoff Games. For technical convenience, we follow Chatterjee and Velner in considering mean-payoff games on lossy game graphs. In this context, the goal of Player 1 is to achieve a long-term average of 0 in all dimensions, and the goal of Player 2 is to achieve a negative long-term average in at least one dimension.

4.3.1. Winning Strategies for Player 2. The first key observation that underpins the solution of (lossy) multi-dimensional mean-payoff games by Chatterjee and Velner is the following sufficient condition for the existence of a winning strategy for Player 2 from some vertex in the game graph: there is a vertex $v_0$, an open half-space $H \subseteq \mathbb{R}^d$ and a strategy for Player 2 that guarantees all simple cycles formed along a play from $v_0$ to be in $H$. One can then argue that if Player 2 uses such a strategy indefinitely then the norms of the energy level vectors grow linearly in the number of steps performed, and hence the long-term average is non-zero in at least one dimension.

Every open half-space can be determined by a non-zero vector $n \in \mathbb{R}^d$ that is normal to the hyperplane on the boundary of the half-space:

$$H_n = \{ v \in \mathbb{R}^d : n \cdot v < 0 \}.$$
Chatterjee and Velner [6, Lemma 1] crucially point out that for every vector \( n \in \mathbb{R}^d \), one can check whether Player 2 has a strategy that guarantees all simple cycles formed to be in \( H_n \) from \( v_0 \) by solving a one-dimensional mean-payoff game on the multi-weighted game graph with every weight \( u \) replaced by the dot-product \( n \cdot u \).

4.3.2. Winning Strategies for Player 1. The second key insight of Chatterjee and Velner is that the above sufficient condition for the existence of a winning strategy for Player 2 in a lossy multi-dimensional mean-payoff game is necessary. Indeed, by (positional) determinacy of mean-payoff games [9], it follows that, if the sufficient condition described above does not hold for any open half-space \( H_n \) and any initial vertex \( v_0 \), then for all non-zero vectors \( n \) and all vertices \( v_0 \), Player 1 has a (positional) strategy to block simple cycles in \( H_n \) along any play from \( v_0 \), or in other words to force all simple cycles formed to be in \( \mathbb{R}^d \setminus H_n = H_{\overline{n}} = \{ v \in \mathbb{R}^d : n \cdot v \geq 0 \} \).

In such a case, Chatterjee and Velner [6, Lemma 2] show that such strategies of Player 1, which force simple cycles formed to be in any closed half-space, can be carefully combined to ensure that the long-term average is 0 in every dimension. The main idea in the construction of the strategy for Player 1 is to proceed in stages \( m = 1, 2, 3, \ldots \), to monitor the energy-level vector at the beginning of stage \( m \) of the game, say \( g_m \), and to ‘counteract’ its further growth in the direction of \( g_m \) throughout stage \( m \) by using the strategy that blocks simple cycles in the open half-space \( H_{g_m} \), i.e., that forces all the formed simple cycles to be in the closed half-space \( H_{\overline{g_m}} \). In the winning strategy we described for Player 1 for the mean-payoff game over the graph of Figure 3, Player 1 can avoid cycles in \( H_{(-1,1)} \) by playing the self-loop on \( v_L \), and she can avoid cycles in \( H_{(1,-1)} \) by playing the self-loop on \( v_R \).

Moving from one stage to another, and hence switching between such counteracting strategies to force simple cycles in different half-spaces, cannot be done too often because as a result of switching from one strategy to another a bounded number of unfavourable simple cycles may be formed. This is the case in our example, since switching between \( v_L \) and \( v_R \) closes a cycle with effect \((-1, -1)\) resulting in a drift away from the non-negative orthant.

The strategy for Player 1 proposed by Chatterjee and Velner overcomes this complication by increasing the number of steps made in every stage; in particular, they proposed making \( s(m) \overset{\text{def}}{=} m \) steps in stage \( m \) before proceeding to stage \( m + 1 \). The purpose is to make the drift grow slower than the number of steps in the play. This, as can be deduced from their analysis, gives a bound of \( O(n^{3/4}) \) for the norm of the energy-level vector after \( n \) steps, and hence the long-term average is 0 in all dimensions because \( \lim_{n \to \infty} \frac{n^{3/4}}{n} = 0 \). One may observe that more generally, if we set \( s(m) \overset{\text{def}}{=} m^\varepsilon \), for any \( \varepsilon > 0 \), then the norm of the energy-level vector after \( n \) steps is \( O(n^{1/2+\varepsilon}/2^{(1+\varepsilon)}) \). Hence, the best upper bounds on the norm of the energy-level vectors after \( m \) steps that can be guaranteed by Player 1—when using a strategy similar to that constructed by Chatterjee and Velner—are in \( \omega(\sqrt{n}) \).
Let us point out that such strategies require infinite memory because they need to ‘keep the count’ of the stage they are in and of the number of steps they need to perform in the current stage, both of which are unbounded.

5. Perfect Half-Spaces

We recall in this section the definition of subsets of $\mathbb{Q}^d$ called perfect half-spaces, which can also be characterised as the maximal salient blunt cones in $\mathbb{Q}^d$. They will be used next in Section 6 to define a condition for Player 2 to win bounding games, which relies on Player 2’s ability to force cycles inside perfect half-spaces. This can be understood as a generalisation of Chatterjee and Velner’s approach for solving multi-dimensional mean-payoff games, which relies on a similar ability to force cycles inside open half-spaces. We employ perfect half-spaces in Section 6 to colour the edges in first-cycle bounding games, which determine the winner using both the colours and the weight of the first cycle formed along a play.

5.1. Definitions from Linear Algebra. Given a subset $A$ of $\mathbb{Q}^d$, we write $\text{span}(A)$ (resp., $\text{cone}(A)$) for the vector space (resp., the cone) generated by $A$, i.e., the closure of $A$ under addition and under multiplication by all (resp., nonnegative) rationals.

Observe that the sufficient condition for existence of a winning strategy for Player 2 in a lossy multi-dimensional mean-payoff game is also a sufficient condition for him to have a winning strategy in a bounding game. Unlike for multi-dimensional mean-payoff games and as witnessed with the game on Figure 3, however, this condition is not necessary. In order to formulate a new more powerful sufficient condition, we use instead perfect half-spaces: a $k$-perfect half-space $H$ of $\mathbb{Q}^d$, where $k \in \{1, 2, \ldots, d\}$, is a (necessarily disjoint) union $H_d \cup \cdots \cup H_k$ such that:

- $H_d$ is an open half-space of $\mathbb{Q}^d$;
- for all $j \in \{k, \ldots, d - 1\}$, $H_j \subseteq \mathbb{Q}^d$ is an open half-space of the boundary of $H_{j+1}$.

Whenever we write a $k$-perfect half-space in form $H_d \cup \cdots \cup H_k$, we assume that each $H_j$ is $j$-dimensional. We additionally define the $(d + 1)$-perfect half-space as the empty set; a partially-perfect half-space is then a $k$-perfect half-space for some $k$ in $\{1, \ldots, d+1\}$. A perfect half-space is a 1-perfect half-space. Observe that a partially-perfect half-space is always a cone, which is blunt, i.e., does not contain 0, and salient, i.e., if it contains a vector $v$ then it does not contain its opposite $-v$. Moreover, a perfect half-space is a maximal blunt and salient cone.

5.2. Generated Perfect Half-Spaces. In order to pursue effective and parsimonious strategy constructions, we consider perfect half-spaces generated by particular sets of vectors, which will correspond to the total weights of simple cycles in multi-weighted game graphs. Given a norm $M$ in $\mathbb{N}$, we say that an open half-space $H$ is $M$-generated if its boundary equals $\text{span}(B)$ for some set $B$ of vectors of norm at most $M$. By extension, a partially-perfect half-space is $M$-generated if each of its open half-spaces is $M$-generated.
Proposition 5.1. Any $k$-dimensional vector space of $\mathbb{Q}^d$ has at most $L(k) \equiv 2(2M + 1)^d(k-1)$ open half-spaces that are $M$-generated.

Example 5.2. In the game graph of Figure 3 there are three 1-generated open half-spaces of interest: the half-plane $H_2 = \{(x, y) : x + y < 0\}$ with boundary $\text{span}((-1, 1), (1, -1))$ and containing $(-1, -1)$, and the two halflines $H_1 = \{(x, y) : x + y = 0 \wedge x < 0\}$ and $H_1' = \{(x, y) : x + y = 0 \wedge x > 0\}$ with boundary $\text{span}(0)$ and containing, respectively, $(-1, 1)$ and $(1, -1)$. In turn, those three open half-spaces define two perfect half-spaces: $H_2 \cup H_1$ and $H_2 \cup H_1'$.

5.3. Hierarchy of Perfect Half-Spaces. Finally, we fix a ranked tree-like structure on all $M$-generated partially-perfect half-spaces, which provide a scaffolding on which we will build strategies in multi-dimensional bounding games. Observe that an $M$-generated partially-perfect half-space $H_k \cup \cdots \cup H_k$ for $k > 1$ can be extended using any of the $M$-generated open half-spaces $H$ of the boundary of $H_k$; note that this boundary then equals $\text{span}(H)$. In Example 5.2 $H_2$ can be extended using $H_1$ or $H_1'$, and $\text{span}(H_1) = \text{span}(H_1') = \{(x, y) : x + y = 0\}$.

The set of $M$-generated perfect half-spaces can be totally ordered by positing a linear ordering $<$ between all $M$-generated open half-spaces. We write $<$ for the lexicographically induced linear ordering between all $M$-generated perfect half-spaces of $\mathbb{Q}^d$: if $\mathcal{H} = H_d \cup \cdots \cup H_1$ and $\mathcal{H}' = H'_d \cup \cdots \cup H'_1$, we define $\mathcal{H} < \mathcal{H}'$ to hold iff $H_j = H'_j$ for all $j \in \{k + 1, \ldots, d\}$ and $H_k < H'_k$ for some $k \in \{1, 2, \ldots, d\}$.

6. First Cycle Bounding Games

We define in this section first-cycle bounding games, which provide the key technical arguments for most of our results. Such games end as soon as a cycle is formed along a play, and the weight of this cycle determines the winner, along with a colouring information chosen by Player 2. In sections 6.2 and 6.3 we are going to show that first-cycle bounding games and infinite bounding games are equivalent, by translating winning strategies for each Player $p$, $p \in \{2, 1\}$, from first-cycle bounding games to bounding games. This yields in particular the small hypercube property of Lemma 3.1.

6.1. Definition. We define the first-cycle bounding game $G(V, E, d)$ on a multi-weighted game graph $(V, E, d)$:

- at any Player-1 vertex, Player 2 chooses a $|V| \cdot |E|$-generated perfect half-space $\mathcal{H}$ of $\mathbb{Q}^d$, and then Player 1 chooses an outgoing edge, whose occurrence in the play becomes coloured by $\mathcal{H}$;
- at any Player-2 vertex, he chooses an outgoing edge;
- the game finishes as soon as a vertex is visited twice, which produces a simple cycle $C$ with coloured Player-1 edges;
- Player 2 wins if $w(C)$, the total weight of the cycle, is in the largest partially-perfect half-space of $\mathbb{Q}^d$ that is contained in all the colours in $C$, i.e. the least common ancestor of all the colours in $C$; Player 1 wins otherwise.
6.1. Player 2 wins the first-cycle bounding game played in Figure 1 (but loses in its lossy version). E.g. strategy \(2\) is winning for Player 2 if he colours the edges outgoing from \(v_0\) by the perfect half-space \(H'_2 \cup H_1\) where \(H'_2 \overset{\text{def}}{=} \{(x, y) : x + y > 0\}\) and \(H_1 \overset{\text{def}}{=} \{(x, y) : x + y = 0 \land x < 0\}\). □

Example 6.2. Player 2 wins the first-cycle bounding game played in Figure 3. Indeed, he can choose the colour \(H_2 \cup H_1\) in \(v_L\) and the colour \(H_2 \cup H'_1\) in \(v_R\). Then Player 1 cannot avoid forming a simple cycle in either \(H_2 \cup H_1\) (if cycling on \(v_L\)), in \(H_2 \cup H'_1\) (if cycling on \(v_R\)), or in \(H_2\) (if cycling between \(v_L\) and \(v_R\)). □

Observe that first-cycle bounding games are finite perfect information games, and are thus determined: from any vertex, either Player 1 wins or Player 2 wins.

6.2. Winning Strategies for Player 2. Suppose \(\sigma\) is a strategy of Player 2 from a vertex \(v_0\) in a first-cycle bounding game \(G(V, E, d)\). Let \(\tilde{\sigma}\) be the following strategy of Player 2 in the infinite bounding game \(\Gamma(V, E, d)\):

- at any Player-2 vertex, \(\tilde{\sigma}\) chooses the edge specified by \(\sigma\);
- whenever a cycle is formed, \(\tilde{\sigma}\) cuts it out of its memory, and continues playing according to \(\sigma\).

Lemma 6.3. If \(\sigma\) is winning for Player 2 in \(G(V, E, d)\) from some vertex \(v_0\), then \(\tilde{\sigma}\) is winning for Player 2 in \(\Gamma(V, E, d)\) from the same vertex \(v_0\).

Proof. Consider any infinite play \(\tilde{\rho}\) consistent with \(\tilde{\sigma}\), and let:

- \(\rho\) be obtained from \(\tilde{\rho}\) by colouring all Player 1’s edges with the \(|V| \cdot |E|\)-generated perfect half-spaces of \(\mathbb{Q}^d\) as specified by \(\sigma\);
- \(C_1, C_2, \ldots\) be the cycle decomposition of \(\rho\), and for each \(n\), \(\rho_n\) be the simple path that remains after removing \(C_n\);
- \(\mathcal{H}_n\) be the largest partially-perfect half-space of \(\mathbb{Q}^d\) that is contained in all the colours in \(C_n\), for each \(n\).

Since \(\sigma\) is winning for Player 2 in the first-cycle game, each cycle weight \(w(C_n)\) belongs to the partially-perfect half-space \(\mathcal{H}_n\). The bulk of the proof consists in extracting a ‘direction of divergence’ of the total energy, notwithstanding that the \(\mathcal{H}_n\)’s may keep varying.

In short, by distinguishing those \(n\)’s for which the length of the simple path \(\rho_n\) is the smallest one that occurs infinitely often, we are going to show that the set of \(\mathcal{H}_n\)’s that occur infinitely often has a unique smallest element \(\mathcal{H} = H_d \cup \cdots \cup H_k\) with respect to inclusion. Further linear-algebraic reasoning in the upcoming Claim 6.4 then shows that one of the component half-spaces \(H'_k\) of \(\mathcal{H}\) provides the desired direction of divergence: after some \(N > 0\), all the sums of cycle weights \(w(C_N) + w(C_{N+1}) + \cdots + w(C_n)\) belong to the topological closure \(\overline{H'_k}\) and their distances from the boundary of \(H'_k\) diverge.

In more details now, along the infinite play \(\rho\), the prefixes \(\rho_n\) might get shorter or longer but are always of length bounded by \(|V|\). Those lengths are traced in blue in Figure 5. We let \(\ell\) be the minimal such length that occurs infinitely often.
Let us call a partially-perfect half-space that occurs infinitely often in the sequence $H_1, H_2, \ldots$ a recurring one. We want to show that, among the recurring partially-perfect half-spaces, there is one that is contained in all the others; the subsequent Claim 6.4 will then allow to conclude.

First observe that, for any $n < n'$ such that $|\rho_n| \geq |\rho_n'|$ and, for all $m \in \{n+1, n'+1\}$, $|\rho_n| \leq |\rho_m|$, we have that either $H_n$ contains $H_{n'}$ or vice-versa (since $|\rho_n|$ is bounded by $|V|$ this situation must occur infinitely often). Indeed, in this case $C_n$ starts with (and $\rho_n$ ends with) a vertex $v_1$ that occurs in $C_{n'}$. Consider the sequence of vertices $v_1, v_2, \ldots, v_{k-1}, v_k$ visited along the cycle $C_n$ until the first Player 1 vertex $v_k$ (recall that by assumption all the simple cycles must visit some Player 1 vertex). Because in step $n'$ Player 2 plays according to $\sigma$, he will choose the same actions in Player 2 vertices $v_1, v_2, \ldots, v_{k-1}$ as in step $n$. Hence $v_k$ is also visited inside $C_{n'}$, and Player 2 uses the same edge colouring at steps $n$ and $n'$ in $v_k$. Thus the two cycles $C_n$ and $C_{n'}$ share a colour, and the corresponding partially perfect half-spaces $H_n$ and $H_{n'}$ must be comparable for inclusion.

Assume now that there are two incomparable (for inclusion) recurring partially perfect half-spaces $H^\dagger$ and $H^\ddagger$ and let us show that some partially perfect half-space $H \subseteq H^\dagger, H^\ddagger$ must also be recurring. Consider the infinite suffix of the play where $\ell$ is the minimal observed length. By the previous observation, between any two occurrences of $H^\dagger$ and $H^\ddagger$ in this suffix, we can find a sequence $H^\dagger = H_{i_1}, H_{i_2}, \ldots, H_{i_k} = H^\ddagger$ of comparable partially perfect half-spaces connecting the two, i.e., with $H_{i_j} \subseteq H_{i_{j+1}}$ or $H_{i_j} \supseteq H_{i_{j+1}}$ for all $j$. This is because there will be a later occurrence of a simple path $\rho_{i_m}$ of length $\ell \leq \min(|\rho_{i_1}|, |\rho_{i_k}|)$ for some $i_m \geq i_1, i_k$; see the thick dashed violet line in Figure 5. Then some $H \subseteq H^\dagger, H^\ddagger$ occurs among those $H_{i_1}, \ldots, H_{i_k}$. Since there are infinitely many such pairs of occurrences of $H^\dagger$ and $H^\ddagger$ but finitely many $|V| \cdot ||E||$-generated partially-perfect half-spaces, there must be infinitely many occurrences of one such $H$.

Applying the previous reasoning to every pair of recurring partially perfect half-spaces $H^\dagger$ and $H^\ddagger$, we see that there must be a recurring partially perfect half-space that is contained in all the others. We conclude the proof using the following Claim 6.4.

\begin{claim}
Suppose $H = H_1 \cup \cdots \cup H_k$ is a partially-perfect half-space of $Q^d$ and $a_1, a_2, \ldots$ is an infinite sequence of vectors such that:

- the set $\{a_1, a_2, \ldots\}$ is finite;
\end{claim}
for each \( n \), there exists a partially-perfect half-space of \( \mathbb{Q}^d \) that contains \( \mathcal{H} \) and \( a_n \); we have \( a_n \in \mathcal{H} \) for infinitely many \( n \).

Then there exist \( k' \in \{d, \ldots, k\} \) and \( N > 0 \) such that

- for each \( n \geq N \), \( a_n \) belongs to \( \overline{H_{k'}} \) the topological closure of \( H_{k'} \), and
- the set of all distances of \( a_N + \cdots + a_n \) from the boundary of \( H_{k'} \) is unbounded.

In particular, the set of all norms \( \|a_1 + \cdots + a_n\| \) is unbounded.

**Proof.** We have that \( \mathcal{H} \) is of the form \( H_d \cup \cdots \cup H_{k'} \). Let \( k' \in \{d, \ldots, k\} \) be maximal such that \( a_n \in H_{k'} \) for infinitely many \( n \). Observe that \( \{1, 2, \ldots, k\} \), the set of all positive integers, can be partitioned into three:

1. The set of all \( n \) such that \( a_n \in H_{k'} \), which is finite by definition of \( k' \). We let \( N \) be larger than the index of the last such \( a_n \); then the vectors \( a_n \) for \( n \geq N \) belong to \( \overline{H_{k'}} \).
2. The set of all \( n \) such that \( a_n \in H_{k'} \), which is infinite by definition of \( k' \). Since \( H_{k'} \) is open and the set \( \{a_1, a_2, \ldots\} \) is finite, there is a positive minimal distance of those \( a_n \) from the boundary of \( H_{k'} \). These vectors bring the sums \( a_N + \cdots + a_n \) for \( n \geq N \) unboundedly close to the boundary of \( H_{k'} \).
3. The set of all \( n \) such that \( a_n \) is contained in the boundary of \( H_{k'} \). These vectors have no effect on the distance between the sums \( a_N + \cdots + a_n \) for \( n \geq N \) and the boundary of \( H_{k'} \). \( \square \)

### 6.3. Winning Strategies for Player 1

If there is no winning strategy for Player 2 in the first-cycle bounding game \( G(V, E, d) \) from a vertex \( v_0 \), then by determinacy of first-cycle bounding games, there is a winning strategy \( \sigma \) for Player 1 in \( G(V, E, d) \) from \( v_0 \).

**Example 6.5.** Recall the lossy game graph from Figure 4. Because Player 1 wins the energy game with initial credit \((2, 2)\), by Corollary 3.4 and Lemma 6.3, she wins the first-cycle bounding game. One winning strategy, whose moves depend only on the latest visited vertex (here only \( v_0 \)) and colour \( \mathcal{H} \) chosen by Player 2 in \( v_0 \), is as follows:

- (i) if \((-2, 2)\) and \((-1, 3)\) are both outside \( \mathcal{H} \), move to \( v_L \), and
- (ii) if \((2, -1)\) and \((3, -3)\) are both outside \( \mathcal{H} \), move to \( v_R \), and
- (iii) otherwise perform the self-loop labelled \((-1, 0)\).

Observe that the first two cases [i] and [ii] are disjoint. Since there is no perfect half-space that contains \((-1, 0)\) and intersects both \((-2, 2), (-1, 3)\) and \((2, -1), (3, -3)\), this strategy is indeed winning for Player 1—the same would apply if she were to choose the other self-loop \((0, -1)\) instead. \( \square \)

The proof of our main result consists in constructing from \( \sigma \) a finite-memory winning strategy \( \tilde{\sigma} \) for Player 1 in the infinite bounding game \( \Gamma(V, E, d) \) from \( v_0 \), which balances her various ‘perfect half-space avoidance strategies’ in order to ensure the small hypercube property stated in Lemma 3.1. Let us outline this construction. The memory of \( \tilde{\sigma} \) consists of:

- **a simple path:** \( \gamma \) from the initial vertex \( v_0 \) to the current vertex \( v \), in which Player 1’s edges are coloured by \( |V| \cdot |E| \)-generated perfect
half-spaces of $\mathbb{Q}^d$ (this can be represented concretely by a sequence of coloured edges from $E$);

**a colour:** I.e. a $|V| \cdot \|E\|$-generated perfect half-space $H = H_d \cup \cdots \cup H_1$ of $\mathbb{Q}^d$ (initially the $\prec$-minimal one);

**counters:** $c(k, W)$ for every $k \in \{1, 2, \ldots, d\}$ and for every nonzero total weight $W$ of a simple cycle, which are natural numbers (initially 0).

Strategy $\tilde{\sigma}$ copies its moves from strategy $\sigma$ for the first-cycle bounding game, based on the coloured simple path and the colour it has in its memory. Whenever a cycle is formed it is removed from the simple path, and provided its weight $W$ is nonzero, all the counters $c(k, W)$ are incremented.

Together with the current path, the counters provide the current energy level, which equals $w(\gamma) + \sum W c(d, W) \cdot W$ throughout the play, where $W$ ranges over all simple cycle weights. To keep the counters and thus the total energy bounded, $\tilde{\sigma}$ may perform one of the following operations after a counter increment:

- a $k$-shift to $H'_k > H_k$ changes the current colour $H$ to the $\prec$-minimal perfect half-space of the form $H_d \cup \cdots \cup H_{k+1} \cup H'_k \cup \cdots \cup H'_1$, and resets to 0 all the counters $c(k', W)$ with $k' < k$;

- a $k$-cancellation changes the current colour $H$ to the $\prec$-minimal perfect half-space of the form $H_d \cup \cdots \cup H_{k+1} \cup H'_k \cup \cdots \cup H'_1$. Simultaneously, given some simple cycle weights $W_1, \ldots, W_n$ and a positive integral solution $x$ to $\sum_{i=1}^n x(i) W_i = 0$ ,

\begin{equation}
\sum_{i=1}^n x(i) W_i = 0 ,
\end{equation}

it subtracts $x \cdot u(k)$ where

\begin{equation}
u(k) \overset{\text{def}}{=} (4|V| \cdot \|E\|)^{(2k-1)(d+2)}
\end{equation}

from all the tuples $(c(k', W_1), \ldots, c(k', W_n))$ with $k' \geq k$, and resets to 0 all the counters $c(k', W)$ with $k' < k$.

These two operations define the main phases of the strategy $\tilde{\sigma}$. A $k$-event is either a $k$-shift or a $k$-cancellation. By a $k$-month we mean a maximal period with only $<k$-events. By a $k$-year we mean a maximal period with only $<k$-cancellations and $\leq k$-shifts. This hierarchy of $k$-events mirrors in some sense the hierarchical structure of $|V| \cdot \|E\|$-generated perfect half-spaces.

These operations allow to maintain two main invariants, from which the small hypercube property of [Lemma 6.6](#) is derived (see [Claim 6.8](#)). For all $1 \leq k \leq d$ and simple path weights $W$ in the span of $H_k$:

- initially, after any $>k$-shift, and after any $\geq k$-cancellation,

\begin{equation}
c(k, W) < U(k) \overset{\text{def}}{=} (4|V| \cdot \|E\|)^{2k(d+2)}
\end{equation}

the so-called $k$-soft bound;

- at all times,

\begin{equation}
c(k, W) < U(k) + u(k)
\end{equation}

the so-called $k$-hard bound.

To ensure those invariants, strategy $\tilde{\sigma}$ further maintains that, whenever $c(k, W) \geq U(k)$ and $W$ is in $\text{span}(H_k)$, then $W$ is in $\overline{H}_k$. When this
Lemma 6.6. If games and the small hypercube property of Lemma 3.1: both the equivalence of infinite bounding games with first-cycle bounding games and the small hypercube property of Lemma 3.1.

Claim 6.7. Every $k$-cancellation is well-defined.

6.3.1. $\tilde{\sigma}$ Summarised. Let us first summarise the definition of $\tilde{\sigma}$. At any Player-1 vertex, $\tilde{\sigma}$ chooses the edge that $\sigma$ specifies for history $\gamma$ and perfect half-space $H$. After any move that leads to a vertex not occurring in $\gamma$, the memory of $\tilde{\sigma}$ is updated only by extending $\gamma$. Otherwise, a cycle $C$ is formed, and the memory is updated as follows:

- Cycle $C$ is cut out of $\gamma$. For all $k \in \{1, 2, \ldots, d\}$, counters $c(k, w(C))$ are incremented, unless $w(C) = 0$.
- If all soft upper bounds hold, that is if for all $k \in \{1, 2, \ldots, d\}$ and all simple-cycle weights $W \in \hat{H}_k$ we have $c(k, W) < U(k)$, then the memory update is finished.
- Otherwise, let $k \in \{1, 2, \ldots, d\}$ be the largest for which the $k$-soft upper bound $c(k, W) < U(k)$ fails for some $W \in \hat{H}_k$.
- (k-shift) If there is a $|V| \cdot \|E\|$-generated open half-space $H$ of $\text{span}(H_k)$ such that the $k$-soft upper bound holds for all simple-cycle weights $W \in \hat{H}$, then denoting by $H_k^i$ the $\prec$-minimal such $H$, $\mathcal{H}$ is replaced by the $\prec$-minimal perfect half-space of form $H_d \cup \cdots \cup H_{k+1} \cup H_k^i \cup \cdots \cup H_1^i$. All counters $c(k', W)$, where $k' \in \{1, 2, \ldots, k-1\}$ and $W$ is a simple-cycle weight, are reset to 0.
- (k-cancellation) Otherwise, let $W_1, W_2, \ldots, W_n$ be all the non-zero simple-cycle weights in $\text{span}(H_k)$ that fail the $k$-soft upper bound, and let $A$ be the matrix whose columns are the vectors $W_1, W_2, \ldots, W_n$. Then by duality and existence of small positive integer solutions of systems of linear equations [Lemma A.2, Lemma A.4 and Proposition 2.2], it follows that $Ax = 0$ has a solution in positive integers bounded by $S(k) \overset{\text{def}}{=} (2(|V| \cdot \|E\|+1))^{(k+2)^2}$. The perfect half-space $\mathcal{H}$ is replaced by the $\prec$-minimal $|V| \cdot \|E\|$-generated perfect half-space of the form $H_d \cup \cdots \cup H_{k+1} \cup H_k^i \cup \cdots \cup H_1^i$. For every $W_i$ and every $k' \in \{k, k+1, \ldots, d\}$, the value of $c(k', W_i)$ is replaced by

$$c(k', W_i) - u(k) \cdot x(i).$$

All counters $c(k', W)$, where $k' \in \{1, 2, \ldots, k-1\}$ and $W$ is a simple-cycle weight, are reset to 0. A $k$-cancellation is well-defined if all the differences in $[8]$ are non-negative.

We prove Lemma 6.6 through a sequence of claims. The first claim shows that $k$-cancellations are always well-defined:

Claim 6.7. Every $k$-cancellation is well-defined.
Claim 6.8. For all $Q$ is the whole space entails in particular that the second on the sequence of $k$-year, and for all cycle weights $W$ in $\text{span}(H_k)$, 

1. (k-soft bound) at the beginning of every k-year, $c(k, W) < U(k)$, and
2. (k-hard bound) $c(k, W) < U(k) + u(k)$.

Proof. We prove the two statements by nested induction, first on $k$ and second on the sequence of k-years seen so far.

Let us start with 1. For the initial $k$-year, and for $k$-years that begin just after a $>k$-shift or a $>k$-cancellation, since then $c(k, W) = 0$, 1 holds trivially. We are left with the case of a $k$-year that begins just after a $k$-cancellation. We can assume using the secondary induction hypothesis that 2 holds at the end of the previous $k$-year. Consider then some $W_i$ in $\text{span}(H_k)$ that fails the k-soft upper bound just before that $k$-cancellation. At that time, since $\text{span}(H_k)$ was not changed by the $k$-cancellation, 2 applies and $c(k, W_i) < U(k) + u(k)$. Therefore, at the beginning of the $k$-year, $c(k, W_i) < U(k) + u(k) - u(k) \cdot x(i)$, and thus $c(k, W_i) < U(k)$ since $x(i) > 0$.

By the secondary induction, it remains to establish 2 for every $k$-year such that 1 held at its beginning—this is the heart of the proof. Let $H_k^1, H_k^2, \ldots, H_k^N$ be the sequence of $k$-dimensional open half-spaces considered during the $N$ k-months spanned by the current $k$-year so far, where $N \leq L(k)$ by Proposition 5.1. We know that all these open half-spaces define the same vector space $\text{span}(H_k^1) = \text{span}(H_k^2) = \cdots = \text{span}(H_k^N)$; let $W$ belong to that space.

If $W$ satisfies the soft bound, there is nothing to be done. Otherwise, by the construction of $\tilde{c}$ and the assumption of 1 at the beginning of the $k$-year, there exists a first $k$-month in this sequence, say the $L$th for some $1 \leq L < N$, after which $W$ fails the soft bound onward. Then, during the $k$-months $1, \ldots, L$, $c(k, W) < U(k)$, and for all $n \in \{L + 1, \ldots, N\}$, we know that $w$ belongs to the closure $\overline{H_k^n}$.  

As explained before, the $k$-soft bound $U(k)$ in [6] is employed by $\tilde{c}$ to trigger a $k$-event and a change of strategy to avoid cycles with weight inside some perfect half-spaces. However, this change of strategy might allow a few more instances of those cycles to be formed—but, crucially, no more than $u(k)$ further instances. The $k$-hard bound in [7] is therefore enforced.

This informal argument is proven formally in the following Claim 6.8. It entails in particular that the $d$-hard bound is always enforced, since $\text{span}(H_d)$ is the whole space $\mathbb{Q}^d$.
Consider the $n$th $k$-month for $n \in \{L + 1, \ldots, N\}$ in the current $k$-year; we want to bound the increase on $c(k, W)$ during that $k$-month. There are two cases:

If $W \in H^n_k$: then the $k$-dimensional space in the current colour is left unchanged during the $n$th $k$-month. In turn, this means that no vertex of the game graph can be visited twice during that $k$-month while forming a cycle of weight $W$, as otherwise $\sigma$ would allow to form a cycle with effect inside $H^1_d \cup \cdots \cup H^n_k$ and Player 1 would lose. Therefore, cycles with weight $W$ that are closed during the $n$th $k$-month can only be formed by consuming edges from the simple path at the beginning of the $k$-month. Hence, $c(k, W)$ can be increased by at most $|V|$.

Otherwise: $W$ belongs to the boundary $\overline{H^n_k \setminus H^n_k}$ of $H^n_k$ and thus $k > 1$. In this case, during the $n$th $k$-month, $c(k, W)$ can only be increased by at most the maximal value of $c(k-1, W)$ during the same $k$-month. This is because $c(k-1, W)$ is 0 at the beginning of the $n$th $k$-month, and thereafter it can only decrease through $<k$-cancellations (or that $k$-month would have ended), which decrease $c(k, W)$ by the same value. By the main induction hypothesis for (1) with $W \in \text{span}(H^1_k) \subseteq (\overline{H^n_k \setminus H^n_k})$, $c(k-1, W)$ is less than $U(k-1) + u(k-1)$.

We conclude that, during the current $k$-year, since $N - L \leq L(k)$,

- if $k = 1$, $c(k, W) - U(k)$ is less than $L(1) \cdot |V| = 2|V| < u(1)$, and
- if $k > 1$,

\[
\begin{align*}
c(k, W) - U(k) &< L(k) \cdot \max(|V|, U(k-1) + u(k-1)) \\
&< L(k) \cdot 2U(k-1) \\
&< (4|V| \cdot ||E||)^{(d+2)^2} / 2 \cdot 2(4|V| \cdot ||E||)^{2(k-1)(d+2)^2} \\
&= u(k) .
\end{align*}
\]

Proof of Lemma 6.6. By the previous claims, $\tilde{\sigma}$ is winning for Player 1 in the infinite bounding game, and thanks to the $d$-hard bound, it ensures that the norm of the current energy is bounded by $\|w(\gamma)\| + \sum W(U(d) + u(d)) \cdot \|W\|$ where $W$ ranges over the total weights of simple cycles in the game graph, and $w(\gamma)$ is the weight of a simple path. Hence the norms $\|w(\gamma)\|$ and $\|W\|$ are bounded by $(|V| \cdot ||E||)^d$. Finally, there are at most $(2(|V| ||E||)^d + 1)^d$ different total weights of simple cycles $W$. □

7. Concluding Remarks

In this paper, we have shown in Corollary 3.6 that fixed-dimensional energy games can be solved in pseudo-polynomial time, regardless of whether the initial credit is arbitrary or given. For the variant with given initial credit, this closes a large complexity gap between the TOWER upper bounds of Brázdíl, Jančar, and Kučera [3] and the lower bounds of Courtois and Schmitz [8], and also settles the complexity of simulation problems between VASS and finite state systems [8]:
Corollary 7.1. The given initial credit problem for multi-dimensional energy games is $2\text{EXPTIME}$-complete, and $\text{EXPTIME}$-complete in fixed dimension $d \geq 4$.

The main direction for extending these results is to consider a parity condition on top of the energy condition. Abdulla, Mayr, Sangnier, and Sproston [1] show that multi-dimensional energy parity games with given initial credit are decidable. They do not provide any complexity upper bounds—although one might be able to show $\text{TOWER}$ upper bounds from the memory bounds on winning strategies shown by Chatterjee et al. [7, Lemma 3]—, leaving a large complexity gap with $2\text{EXPTIME}$-hardness. This gap also impacts the complexity of weak simulation games between VASS and finite state systems [1].

Acknowledgements. The authors thank Dmitry Chistikov for his assistance in proving Lemma 6.3, the anonymous reviewers for their insightful comments, and Christoph Haase, Jérôme Leroux, and Claudine Picaronny for helpful discussions on linear algebra.

Appendix A. Linear Algebra

An Alternatives Lemma. Given a norm $M$ in $\mathbb{N}$, we write $\mathbb{Z}^\pm_M$ for the set of integers $\{-M, \ldots, M\}$. We say that a vector space, cone, or half-space in $\mathbb{Q}^d$ is $M$-generated iff it can be generated by vectors in $(\mathbb{Z}^\pm_M)^d$. We use Weyl’s Theorem:

Theorem A.1 (Weyl [19], Theorem 1). Any $d$-dimensional cone generated by a finite set $A$ in $\mathbb{Q}^d$ is the intersection of a finite number of closed half-spaces, where the boundary of each half-space contains $d-1$ linearly independent vectors from $A$.

Lemma A.2. Suppose $A \subseteq (\mathbb{Z}^\pm_M)^d$ is contained in an $M$-generated subspace $S$ of $\mathbb{Q}^d$. Either $A$ is contained in some $M$-generated closed half-space of $S$, or $\sum_{a \in A} a$ contains the zero vector.

Proof. If the cone$(A)$ is not the whole space span$(A)$, then by Theorem A.1 it is contained in an $M$-generated closed half-space $H$ of span$(A)$. Since $S$ is $M$-generated, it is easy to obtain from $H$ an $M$-generated closed half-space of $S$ that contains cone$(A) \supseteq A$.

Otherwise, if cone$(A)$ is the whole space span$(A)$, it contains in particular the vectors $-\sum_{a \in A} a$ (from span$(A)$) and $\sum_{a \in A} a$ (from cone$(A)$), and thus $\sum_{a \in A} Q_{\geq 0} a$ contains the zero vector. \hfill \Box

Small Solutions. We also use a lemma that bounds the positive integral solutions on systems of linear equations. The lemma is a corollary of the following result of von zur Gathen and Sieveking [10]:

Theorem A.3 (von zur Gathen and Sieveking [10]). Let $A$, $b$, $C$, $d$ be $m \times n$, $m \times 1$, $p \times n$, $p \times 1$-matrices respectively with integer entries. The rank of $A$ is $r$, and $s$ is the rank of the $(m+p) \times n$-matrix $(\hat{A})$. Let $M$ be

\[\begin{align*}
1\text{In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.} & \text{ [18]}
\end{align*}\]
an upper bound on the absolute values of those \((s - 1) \times (s - 1)\)- or \(s \times s\)-subdeterminants of the \((m + p) \times (n + 1)\)-matrix \(\begin{pmatrix} A & b \\ d & 1 \end{pmatrix}\), which are formed with at least \(r\) rows from \(\begin{pmatrix} A & b \end{pmatrix}\). If \(Ax = b\) and \(Cx \geq d\) have a common integer solution, then they have one with coefficients bounded by \((n + 1)M\).

**Lemma A.4** (Small Solutions Lemma). Suppose \(A\) is a \(d \times n\)-matrix with entries from \(\mathbb{Z}_n^+\) and mutually distinct columns. If \(Ax = 0\) has a solution in positive rationals, then it has a solution in positive integers bounded by \((2(M + 1))^{(r+2)^2}\), where \(r\) is the rank of \(A\).

**Proof.** We can assume that \(d = r\). Apply Theorem A.3 with \(b\) the \(d\)-dimensional zero vector, \(C\) the \(n\)-dimensional identity matrix and \(d\) the \(n\)-dimensional vector of ones. The absolute value of any subdeterminant of \(\begin{pmatrix} A & 0 \end{pmatrix}\) is at most \(n^{d+1}M^d\). Since \(n \leq (2M + 1)^d\), we have that

\[
(n + 1)n^{d+1}M^d \leq 2(2(M + 1))^d(2(M + 1))^{d(d+1)}M^d = 2^{d(d+1)^2}(M + 1)^{d(d+3)} \leq (2(M + 1))^{(d+2)^2}. \tag*{□}
\]

**References**

[1] P. A. Abdulla, R. Mayr, A. Sangnier, and J. Sproston. Solving parity games on integer vectors. In *Concur 2013*, volume 8052 of LNCS, pages 106–120. Springer, 2013. doi:10.1007/978-3-642-40184-8_9

[2] P. A. Abdulla, M. F. Atig, P. Hofman, R. Mayr, K. N. Kumar, and P. Totzke. Infinite-state energy games. In *CSL-LICS 2014*, ACM, 2014. doi:10.1145/2603088.2603100

[3] T. Brazdil, P. Jančar, and A. Kučera. Reachability games on extended vector addition systems with states. In *ICALP 2010*, volume 6199 of LNCS, pages 478–489. Springer, 2010. doi:10.1007/978-3-642-14162-1_40

[4] T. Brazdil, K. Chatterjee, A. Kučera, and P. Novotný. Efficient controller synthesis for consumption games with multiple resource types. In *CAV 2012*, volume 7358 of LNCS, pages 23–38. Springer, 2012. doi:10.1007/978-3-642-31424-7_8

[5] J. Chaloupka. Z-reachability problem for games on 2-dimensional vector addition systems with states is in P. *Fund. Inform.*, 123(1):15–42, 2013. doi:10.3233/FI-2013-798

[6] K. Chatterjee and Y. Velner. Hyperplane separation technique for multidimensional mean-payoff games. In *Concur 2013*, volume 8052 of LNCS, pages 500–515. Springer, 2013. doi:10.1007/978-3-642-40184-8_35

[7] K. Chatterjee, M. Randour, and J.-F. Raskin. Strategy synthesis for multi-dimensional quantitative objectives. *Acta Inf.*, 51(3–4):129–163, 2014. doi:10.1007/s00165-013-0182-6

[8] J. Courtois and S. Schmitz. Alternating vector addition systems with states. In *MFCS 2014, Part I*, volume 8634 of LNCS, pages 220–231. Springer, 2014. doi:10.1007/978-3-662-45228-8_19

[9] A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *Int. J. Game Theor.*, 8(2):109–113, 1979. doi:10.1007/BF01768705

[10] U. Fahrenberg, L. Juhl, K. G. Larsen, and J. Srba. Energy games in multiweighted automata. In *ICTAC 2011*, volume 6916 of LNCS, pages 95–115. Springer, 2011. doi:10.1007/978-3-642-32383-1_9

[11] P. Jančar and F. Moller. Checking regular properties of Petri nets. In *Concur ’95*, volume 962 of LNCS, pages 348–362. Springer, 1995. doi:10.1007/3-540-60218-6_26

[12] L. Juhl, K. G. Larsen, and J. Raskin. Optimal bounds for multiweighted and parametrised energy games. In *Theories of Programming and Formal Methods—Essays Dedicated to Jifeng He on the Occasion of his 70th Birthday*, volume 8051 of LNCS, pages 244–255. Springer, 2013. doi:10.1007/978-3-642-39698-4_15

[13] S. Lasota. *EXPSPACE* lower bounds for the simulation preorder between a communication-free Petri net and a finite-state system. *Inf. Process. Lett.*, 109(15):850–855, 2009. doi:10.1016/j.ipl.2009.04.003
[14] C. Rackoff. The covering and boundedness problems for vector addition systems. Theor. Comput. Sci. 6(2):223–231, 1978. doi:10.1016/0304-3975(78)90036-1.

[15] Y. Velner, K. Chatterjee, L. Doyen, T. A. Henzinger, A. Rabinovich, and J.-F. Raskin. The complexity of multi-mean-payoff and multi-energy games. Inform. and Comput., 241:177–196, 2015. doi:10.1016/j.ic.2015.03.001.

[16] J. von zur Gathen and M. Sieveking. A bound on solutions of linear integer equalities and inequalities. Proc. Amer. Math. Soc., 72(1):155–158, 1978. doi:10.1090/S0002-9939-1978-0500555-0.

[17] H. Weyl. Elementare Theorie der konvexen Polyeder. Comm. Math. Helv., 7:290–306, 1934–35.

[18] H. Weyl. Invariants. Duke Math. J., 5(3):489–502, 1939. doi:10.1215/S0012-7094-39-00540-5.

[19] H. Weyl. The elementary theory of convex polyhedra. In Contributions to the Theory of Games, volume 24 of Ann. Math. Stud., pages 3–18. Princeton University Press, 1950. Translation from the German original [17].

DIMAP, Department of Computer Science, University of Warwick, UK

Email address: {mju,lazic}@dcs.warwick.ac.uk

LSV, ENS Cachan & CNRS & INRIA, France

Email address: schmitz@lsv.ens-cachan.fr