On the isomorphism classes of Legendre elliptic curves over finite fields *

Rongquan Feng¹, Hongfeng Wu² †

1 LMAM, School of Mathematical Sciences, Peking University,
Beijing 100871, P.R. China
2 Academy of Mathematics and Systems Science, Chinese Academy of Sciences,
Beijing 100190, P.R. China
fengrq@math.pku.edu.cn, whfmath@gmail.com

Abstract

In this paper the number of isomorphism classes of Legendre elliptic curves over finite fields is enumerated.

Keywords: elliptic curves, Legendre curves, isomorphism classes, cryptography

1 Introduction

A projective curve is a projective variety of dimension 1. Let \( K \) be an arbitrary field, an elliptic curve \( E \) over \( K \) is an absolutely irreducible smooth projective curve of genus 1 defined over \( K \) with a specified base point \( O \). The elliptic curve cryptosystem was proposed by Koblitz [3] and by Miller [5] which relies on the difficulty of discrete logarithmic problem on the group of rational points on an elliptic curve.

In order to study the elliptic curve cryptosystem, one need first to answer how many curves there are up to isomorphism, because two isomorphic elliptic curves are the same in the point of cryptographic view. So it is natural to count the isomorphism classes of some kinds of elliptic curves. Some formulae about counting the number of the isomorphism classes of general elliptic curves over a finite field can be found in literatures. For example, Schoof present the number of isomorphism classes of elliptic curves over the finite field \( \mathbb{F}_q \) in [7], Menezes present the number of isomorphism classes of elliptic curves of the forms \( y^2 = x^3 + ax + b \) over the finite

*Supported by NSF of China (No. 10990011)
†China Postdoctoral Science Foundation funded project.
field \( \mathbb{F}_q \) in [4], and Rezaeian Farashahi and Shparlinski [6] gave the exact formula for the number of distinct elliptic curves over a finite field (up to isomorphism over the algebraic closure of the ground field) in the family of Edwards curves.

In this paper the number of isomorphism classes of Legendre elliptic curves over the finite field \( \mathbb{F}_q \) is enumerated.

2 Background

It is well-known that every elliptic curve \( E \) over a field \( K \) can be written as a Weierstrass equation

\[
E : Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6
\]

with coefficients \( a_1, a_2, a_3, a_4, a_6 \in K \). The discriminant \( \Delta(E) \) and the \( j \)-invariant \( j(E) \) of \( E \) are defined as

\[
\Delta(E) = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6
\]

and

\[
j(E) = (b_2^2 - 24b_4)^3 / \Delta(E),
\]

where

\[
\begin{align*}
b_2 &= a_1^2 + 4a_2, \\
b_4 &= 2a_4 + a_1 a_3, \\
b_6 &= a_3^2 + 4a_6, \\
b_8 &= a_1^3 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2.
\end{align*}
\]

Two projective varieties \( V_1 \) and \( V_2 \) are isomorphic if there exist morphisms \( \phi : V_1 \to V_2 \) and \( \varphi : V_2 \to V_1 \), such that \( \varphi \circ \phi \) and \( \phi \circ \varphi \) are the identity maps on \( V_1 \) and \( V_2 \) respectively. Two elliptic curves are said to be isomorphic if they are isomorphic as projective varieties. Let

\[
E_1 : Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6
\]

and

\[
E_2 : Y^2 + a_1' XY + a_3' Y = X^3 + a_2' X^2 + a_4' X + a_6'
\]

be two elliptic curves defined over \( K \). It is known [3] that \( E_1 \) and \( E_2 \) are isomorphic over \( \overline{K} \), or \( E_1 \) is \( \overline{K} \)-isomorphic to \( E_2 \), if and only if \( j(E_1) = j(E_2) \), where \( \overline{K} \) is the algebraic closure of \( K \). However (see [3]), Let \( L \) be an extension of \( K \), then \( E_1 \) and \( E_2 \) are isomorphic over \( L \) if and only if there exist \( u, r, s, t \in L \) and \( u \neq 0 \) such that the change of variables

\[
(X, Y) \to (u^2 X + r, u^3 Y + u^2 sX + t)
\]
maps the equation of $E_1$ to the equation of $E_2$. Therefore, $E_1$ and $E_2$ are isomorphic over $L$ if and only if there exists $u, r, s, t \in L$ and $u \neq 0$ such that

$$\begin{align*}
ua_1' &= a_1 + 2s, \\
u^2a_2 &= a_2 - sa_1 + 3r - s^2, \\
u^3a_3 &= a_3 + ra_1 + 2t, \\
u^4a_4 &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st, \\
u^6a_6 &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1.
\end{align*}$$

For the simplified Weierstrass equations where $a_1 = a_3 = a_1' = a_3' = 0$, then $E_1$ is $L$-isomorphic to $E_2$ if and only if there exist $u, r \in L$ and $u \neq 0$ such that

$$\begin{align*}
u^2a_2 &= a_2 + 3r, \\
u^4a_4 &= a_4 + 2ra_2 + 3r^2, \\
u^6a_6 &= a_6 + ra_4 + r^2a_2 + r^3.
\end{align*}$$

(2.1)

The reader is referred to [8] for more results on the isomorphism of elliptic curves.

**Definition 2.1.** The Legendre elliptic curve is one whose Weierstrass equation can be written as

$$E_\lambda : y^2 = x(x-1)(x-\lambda).$$

It is clear that the Legendre elliptic curve $E_\lambda$ is nonsingular for $\lambda \neq 0, 1$. The points $O$, $(0,0)$, $(1,0)$, and $(\lambda,0)$ are all the 2-division points, that is, the points whose double are $O$'s. The $j$-invariant of $E_\lambda$ is $j(E_\lambda) = 2^8(\lambda^2-\lambda+1)^3/\lambda^2(\lambda-1)^2$.

### 3 Enumeration for Legendre curves

It is well known [8] that two Legendre curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ and $E_\mu : y^2 = x(x-1)(x-\mu)$ are isomorphic over $\mathbb{F}_q$ if and only if they have the same $j$-invariant, or

$$\mu \in \left\{ \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda} \right\}.$$

Therefore, the map $\lambda \mapsto j(L_\lambda)$ is exactly six-to-one unless when $\lambda \in \{-1,2,\frac{1}{2}\}$, the map is three-to-one, or when $\lambda^2 - \lambda + 1 = 0$, the map is two-to-one. Note that $\lambda^2 - \lambda + 1 = 0$ has a root in $\mathbb{F}_q$ if and only if $\mathbb{F}_q^*$ has an element of order 3, which is equivalent to $q \equiv 1$ or 7 (mod 12). Therefore, we have that the number of $\mathbb{F}_q$-isomorphism classes of Legendre elliptic curves is $\frac{q-2}{6} + 1 + 1 = \frac{q+5}{6}$ when $q \equiv 1, 7$ (mod 12), and is $\frac{q-2}{6} + 1 = \frac{q+1}{6}$ when $q \equiv 5, 11$ (mod 12) (see [11]).

In the following, the number of $\mathbb{F}_q$-isomorphism classes of Legendre elliptic curves will be counted.
Lemma 3.1. Let \( L_{a,b} : y^2 = x(x-a)(x-b) \) and \( L_{d,e} : y^2 = x(x-d)(x-e) \) be two elliptic curves defined over \( \mathbb{F}_q \), where \( ab(a-b) \neq 0 \) and \( de(d-e) \neq 0 \). Then \( L_{a,b} \) and \( L_{d,e} \) are \( \mathbb{F}_q \)-isomorphic if and only if there exists \( u \in \mathbb{F}_q^* \) such that the set \( \{d,e\} \) is equal to one of \( \{a-b, b\}, \{\frac{-a}{u^2}, \frac{b}{u^2}\} \) and \( \{a-b, -b\} \).

Proof. From Equation (2.1), we know that \( L_{a,b} \) and \( L_{d,e} \) are \( \mathbb{F}_q \)-isomorphic if and only if there exist \( u, r \in \mathbb{F}_q \) and \( u \neq 0 \) such that

\[
\begin{aligned}
-u^2(d+e) &= -(a+b) + 3r, \\
deu^4 &= ab - 2r(a+b) + 3r^2, \\
0 &= rab - r^2(a+b) + r^3.
\end{aligned}
\]

Thus \( r = 0 \) or \( r = a \) or \( r = b \). Therefore, \( L_{a,b} \) and \( L_{d,e} \) are \( \mathbb{F}_q \)-isomorphic if and only if there exists \( u \in \mathbb{F}_q^* \) such that

\[
\begin{aligned}
(d+e)u^2 &= a+b, \\
deu^4 &= ab, \quad \text{or} \quad (d+e)u^2 = b-2a, \\
deu^4 &= a^2 - ab, \quad \text{or} \quad (d+e)u^2 = a-2b, \\
deu^4 &= b^2 - ab.
\end{aligned}
\]

Solving these 3 equations, we get that the set \( \{d,e\} \) is \( \{a-b, b\} \), \( \{\frac{-a}{u^2}, \frac{b}{u^2}\} \) or \( \{a-b, -b\} \).

Corollary 3.2. Let \( E_\lambda : y^2 = x(x-1)(x-\lambda) \) and \( E_\mu : y^2 = x(x-1)(x-\mu) \) be two Legendre elliptic curves defined over \( \mathbb{F}_q \). Then \( E_\lambda \) and \( E_\mu \) are \( \mathbb{F}_q \)-isomorphic if and only if there exists \( u \in \mathbb{F}_q^* \) such that the set \( \{1, \mu\} \) is equal to one of \( \{\frac{1}{u^2}, \frac{a}{u^2}\} \), \( \{\frac{1}{u^2}, \frac{-1}{u^2}\} \) and \( \{\frac{1}{u^2}, \frac{\mu}{u^2}\} \).

Now let \( E_\lambda : y^2 = x(x-1)(x-\lambda) \) be a Legendre elliptic curve with \( \lambda(\lambda-1) \neq 0 \). We want to determine how many Legendre elliptic curves which are \( \mathbb{F}_q \)-isomorphic to \( E_\lambda \). Let \( E_\mu \) be one of them. Then \( j(E_\mu) = j(E_\lambda) \). Thus \( \mu \in \{\lambda, 1-\lambda, \frac{\lambda-1}{1-\lambda}, \frac{1}{\lambda-1}, \frac{1}{\lambda}\} \). For the different choices of \( \mu \), Table 1 gives what the \( u^2 \) and what the set \( \{1, \mu\} \) can be compared with the result in Corollary 3.2.

Throughout the paper, the characteristic of the finite field \( \mathbb{F}_q \) is assumed to be greater than 3. We distinguish the enumeration into the following two cases.

Case 1: \( q \equiv 3 \pmod{4} \)

In this case, \(-1\) is not a square in \( \mathbb{F}_q \). When \( j(E_\lambda) = 0 \), then \( \lambda^2 - \lambda + 1 = 0 \). The equation \( \lambda^2 - \lambda + 1 = 0 \) has roots in \( \mathbb{F}_q \) if and only if \( q \equiv 7 \pmod{12} \). The roots are \( \varepsilon_1 = \frac{1+\sqrt{-3}}{2} \) and \( \varepsilon_2 = \frac{1-\sqrt{-3}}{2} \). By Corollary 3.2, it is easy to check that \( E_{\varepsilon_1} \) is not \( \mathbb{F}_q \)-isomorphic to \( E_{\varepsilon_2} \). When \( j(E_\lambda) = 1728 \), we have \( \lambda \in \{-1, 2, \frac{1}{2}\} \). Since \( \{1, 2\} = \{1, \lambda, \frac{1}{\lambda}, \frac{\lambda-1}{1-\lambda}, \frac{\lambda}{1-\lambda}\} \), \( E_{-1} \) is \( \mathbb{F}_q \)-isomorphic to \( E_2 \). Furthermore, we know that exactly one of \(-2\) and \( 2 \) is a square. When \( 2 \) is a square in \( \mathbb{F}_q \), then \( \{1, \frac{1}{2}\} = \{\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\} \). When
Table 1: What the \( u^2 \) and what the set \( \{1, \mu\} \) can be

| \( \mu \)     | \( u^2 \)     | \( \{1, \mu\} \)     |
|--------------|--------------|-------------------|
| \( \lambda \) | 1            | \( \{\frac{1-\lambda}{u^2}, \frac{1}{u^2}\} \) |
| \( 1-\lambda \) | -1          | \( \{\frac{-1-\lambda}{u^2}, \frac{-1}{u^2}\} \) |
| \( \frac{1}{\lambda-1} \) | \( \lambda \) | \( \{\frac{1}{u^2}, \frac{-1}{u^2}\} \) |
| \( \frac{1}{\lambda} \) | \( \lambda - 1 \) | \( \{\frac{1-\lambda}{u^2}, \frac{1}{u^2}\} \) |
| \( \frac{1}{\lambda-1} \) | \( 1-\lambda \) | \( \{\frac{1-\lambda}{u^2}, \frac{-1}{u^2}\} \) |

\(-2\) is a square in \( \mathbb{F}_q \), then \( \{1, \frac{1}{2}\} = \{\frac{1-2}{\sqrt{2}u^2}, \frac{-2}{\sqrt{2}u^2}\} \). Hence \( E_{\frac{1}{2}} \) is \( \mathbb{F}_q \)-isomorphic to \( E_2 \). That means \( E_{-1}, E_2 \) and \( E_{\frac{1}{2}} \) are \( \mathbb{F}_q \)-isomorphic to each other. Now assume that \( j(E_{\lambda}) \neq 0, 1728 \), then the 6 possible values of \( \mu \) are different to each other. But now exactly one of 1 and \(-1\), one of \( \lambda \) and \( -\lambda \), one of \( \lambda - 1 \) and \( 1 - \lambda \) is a square in \( \mathbb{F}_q \). So there are exactly 3 values of \( \mu \) such that \( E_\mu \) is \( \mathbb{F}_q \)-isomorphic to \( E_{\lambda} \). Therefore the number of \( \mathbb{F}_q \)-isomorphism classes of Legendre curves is

\[
\begin{align*}
N_q = \left\{ \begin{array}{ll}
\frac{q + 2}{3}, & \text{if } q \equiv 7 \pmod{12}, \\
\frac{q - 2}{3}, & \text{if } q \equiv 11 \pmod{12}.
\end{array} \right.
\end{align*}
\]

This proves the following theorem.

**Theorem 3.3.** Suppose \( \mathbb{F}_q \) is a finite field with \( \text{char}(\mathbb{F}_q) > 3 \) and \( q \equiv 3 \pmod{4} \). Let \( N_q \) be the number of \( \mathbb{F}_q \)-isomorphism classes of Legendre curves \( E_{\lambda} : y^2 = x(x-1)(x-\lambda) \) defined over \( \mathbb{F}_q \) with \( \lambda(\lambda - 1) \neq 0 \). Then

\[
N_q = \left\{ \begin{array}{ll}
\frac{q + 2}{3}, & \text{if } q \equiv 7 \pmod{12}, \\
\frac{q - 2}{3}, & \text{if } q \equiv 11 \pmod{12}.
\end{array} \right.
\]

**Case 2: \( q \equiv 1 \pmod{4} \)**

For the finite field \( \mathbb{F}_q \), the Jacobi symbol \( \left( \frac{a}{q} \right) \) is defined as follows for all \( a \in \mathbb{F}_q \):

\[
\left( \frac{a}{q} \right) = \left\{ \begin{array}{ll}
1, & \text{if } a \neq 0 \text{ and } a \text{ is a square in } \mathbb{F}_q, \\
-1, & \text{if } a \text{ is not a square in } \mathbb{F}_q, \\
0, & \text{if } a = 0.
\end{array} \right.
\]

The following lemma can be got easily.
Lemma 3.4. Suppose that $\mathbb{F}_q$ is a finite field with char($\mathbb{F}_q$) > 3 and $q \equiv 1 \pmod{4}$.
Let $N(s, t)$ be the number of $a \in \mathbb{F}_q$ with $\left(\frac{a}{q}\right) = s$ and $\left(\frac{1-a}{q}\right) = t$. Then $N(1, 1) = \frac{q-5}{4}$, $N(1, -1) = N(-1, 1) = N(-1, -1) = \frac{q-1}{4}$.

Now we divide the Legendre elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ with $\lambda \neq 0, 1$, into the following 4 disjoint sets $H_1$, $H_2$, $H_3$ and $H_4$, where

\begin{align*}
H_1 &= \left\{ y^2 = x(x-1)(x-b) \left(\frac{b}{q}\right) = 1, \left(\frac{1-b}{q}\right) = 1 \right\}, \\
H_2 &= \left\{ y^2 = x(x-1)(x-b) \left(\frac{b}{q}\right) = 1, \left(\frac{1-b}{q}\right) = -1 \right\}, \\
H_3 &= \left\{ y^2 = x(x-1)(x-b) \left(\frac{b}{q}\right) = -1, \left(\frac{1-b}{q}\right) = 1 \right\}, \\
H_4 &= \left\{ y^2 = x(x-1)(x-b) \left(\frac{b}{q}\right) = -1, \left(\frac{1-b}{q}\right) = -1 \right\}.
\end{align*}

From Lemma 3.4, we get that $|H_1| = \frac{q-5}{4}$ and $|H_2| = |H_3| = |H_4| = \frac{q-1}{4}$.

Note that $-1$ is a square in this case. As in Case 1, $j(E_\lambda) = 0$ if and only if $\lambda^2 - \lambda + 1 = 0$. The equation $\lambda^2 - \lambda + 1 = 0$ has roots in $\mathbb{F}_q$ if and only if $q \equiv 1, 13 \pmod{24}$. The roots are $\varepsilon_1 = \frac{1 + \sqrt{-3}}{2} = \left(\frac{\sqrt{3} + \sqrt{-1}}{2}\right)^2$ and $\varepsilon_2 = \frac{1 - \sqrt{-3}}{2} = \left(\frac{\sqrt{3} - \sqrt{-1}}{2}\right)^2 = 1 - \varepsilon_1$. Thus $E_{\varepsilon_1}, E_{\varepsilon_2} \in H_1$ and by Corollary 3.2 $E_{\varepsilon_1}$ is $\mathbb{F}_q$-isomorphic to $E_{\varepsilon_2}$ since $\{1, \varepsilon_2\} = \left\{\frac{-1}{(\sqrt{-1})}, \frac{\varepsilon_1 - 1}{(\sqrt{-1})}\right\}$. When $j(E_\lambda) = 1728$, we have $\lambda \in \{-1, 2, \frac{1}{2}\}$.

Now 2 is a square if and only if $q \equiv 1, 17 \pmod{24}$. So when $q \equiv 1, 17 \pmod{24}$, $E_{-1}, E_2, E_2^{1/2} \in H_1$ and they are $\mathbb{F}_q$-isomorphic to each other as in Case 1. When $q \equiv 5, 13 \pmod{24}$, $E_{-1} \in H_2$, $E_2 \in H_3$, while $E_2^{1/2} \in H_4$. It can be checked easily that $E_{-1}$ and $E_2$ are $\mathbb{F}_q$-isomorphic to each other but $E_2^{1/2}$ is not $\mathbb{F}_q$-isomorphic to them.

Now let $j(E_\lambda) \neq 0, 1728$. Then $\lambda, 1 - \lambda, \frac{1}{2}, \frac{\lambda-1}{\lambda}, 1 - \frac{\lambda}{\lambda-1}$, the 6 possible values for $\mu$, are distinct with each other. If $E_\lambda \in H_1$, then all the 6 values $1, -1, \lambda, -\lambda, 1 - \lambda$ and $\lambda - 1$ are squares in $\mathbb{F}_q$. Thus for any $\mu \in \{\lambda, 1 - \lambda, \frac{1}{2}, \frac{\lambda-1}{\lambda}, 1 - \frac{\lambda}{\lambda-1}\}$, $E_\mu$ is $\mathbb{F}_q$-isomorphic to $E_\lambda$ and $E_\mu \in H_1$. If $E_\lambda \in H_4$, then $E_\mu$ is $\mathbb{F}_q$-isomorphism to $E_\lambda$ only for $\mu \in \{\lambda, 1 - \lambda\}$ since only 1 and $-1$ are squares among the six numbers in the second column of Table 1. Now $E_{1 - \lambda} \in H_4$ too.

If $E_\lambda \in H_2$, then $1, -1, \lambda$ and $-\lambda$ are squares while $1 - \lambda$ and $\lambda - 1$ are non-squares. Thus $E_\mu$ is $\mathbb{F}_q$-isomorphic to $E_\lambda$ for $\mu \in \{\lambda, 1 - \lambda, \frac{1}{2}, \frac{\lambda-1}{\lambda},\}$. Note that now $E_{1/2} \in H_2$ but $E_{1 - \lambda}, E_{\frac{\lambda-1}{\lambda}} \in H_3$. Similarly, if $E_\lambda \in H_3$, then $E_\mu$ is $\mathbb{F}_q$-isomorphic to $E_\lambda$ for $\mu \in \{\lambda, 1 - \lambda, \frac{1}{2}, \frac{\lambda-1}{\lambda},\}$ and $E_{\frac{\lambda-1}{\lambda}} \in H_3$ but $E_{1 - \lambda}, E_{\frac{1}{2}} \in H_2$.

Therefore, let $N_{q,H_1}$, $N_{q,H_2\cup H_3}$, and $N_{q,H_4}$ be the number of $\mathbb{F}_q$-isomorphism classes of Legendre elliptic curves $E_\lambda \in H_1$, $H_2 \cup H_3$ and $H_4$ respectively. Then
we have

\[
N_{q,H} = \begin{cases} 
\frac{q-5}{4} - 3 - 2 \right) /6 + 1 + 1 = \frac{q+23}{24}, & \text{if } q \equiv 1 \pmod{24}, \\
\frac{q-5}{4} /6 = \frac{q-5}{24}, & \text{if } q \equiv 5 \pmod{24}, \\
\frac{q-5}{4} - 2 \right) /6 + 1 = \frac{q+11}{24}, & \text{if } q \equiv 13 \pmod{24}, \\
\frac{q-5}{4} - 3 \right) /6 + 1 = \frac{q+7}{24}, & \text{if } q \equiv 17 \pmod{24}.
\end{cases}
\]

\[
N_{q,H_2 \cup H_3} = \begin{cases} 
\frac{q-1}{2} /4 = \frac{q-1}{8}, & \text{if } q \equiv 1,17 \pmod{24}, \\
\frac{q-1}{2} - 2 \right) /4 + 1 = \frac{q+3}{8}, & \text{if } q \equiv 5,13 \pmod{24},
\end{cases}
\]

and

\[
N_{q,H_4} = \begin{cases} 
\frac{q-1}{4} /2 = \frac{q-1}{8}, & \text{if } q \equiv 1,17 \pmod{24}, \\
\frac{q-1}{4} - 1 \right) /2 + 1 = \frac{q+3}{8}, & \text{if } q \equiv 5,13 \pmod{24}.
\end{cases}
\]

We know from above analysis that the Legendre curves from the 3 distinct sets $H_1$, $H_2 \cup H_3$ and $H_4$ can not be $\mathbb{F}_q$-isomorphic to each other. Summing up the above numbers, we have the following theorem:

**Theorem 3.5.** Suppose $\mathbb{F}_q$ is a finite field with $\text{char}(\mathbb{F}_q) > 3$ and $q \equiv 1 \pmod{4}$.

Let $N_q$ be the number of $\mathbb{F}_q$-isomorphism classes of Legendre curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ defined over $\mathbb{F}_q$ with $\lambda(\lambda-1) \neq 0$. Then

\[
N_q = \begin{cases} 
\frac{7q+17}{24}, & \text{if } q \equiv 1 \pmod{24}, \\
\frac{7q+13}{24}, & \text{if } q \equiv 5 \pmod{24}, \\
\frac{7q+29}{24}, & \text{if } q \equiv 13 \pmod{24}, \\
\frac{7q+1}{24}, & \text{if } q \equiv 17 \pmod{24}.
\end{cases}
\]

Combining Theorems 3.3 and 3.5 together, we have the following enumeration result.
Theorem 3.6. Suppose $\mathbb{F}_q$ is the finite field with $q$ elements and $\text{char}(\mathbb{F}_q) > 3$. Let $N_q$ be the number of $\mathbb{F}_q$-isomorphism classes of Legendre curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ defined over $\mathbb{F}_q$ with $\lambda(\lambda-1) \neq 0$. Then

$$N_q = \begin{cases} 
\frac{7q + 17}{24}, & \text{if } q \equiv 1 \pmod{24}, \\
\frac{7q + 13}{24}, & \text{if } q \equiv 5 \pmod{24}, \\
\frac{q + 2}{3}, & \text{if } q \equiv 7, 19 \pmod{24}, \\
\frac{q - 2}{3}, & \text{if } q \equiv 11, 23 \pmod{24}, \\
\frac{7q + 29}{24}, & \text{if } q \equiv 13 \pmod{24}, \\
\frac{7q + 1}{24}, & \text{if } q \equiv 17 \pmod{24}.
\end{cases}$$

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