Abstract: A general technique for proving the irrationality of the zeta function constants $\zeta(2n+1)$, $n \geq 1$, from the known irrationality of the beta constants $L(2n+1, \chi)$ is developed in this note. The irrationality of the zeta constants $\zeta(2n)$, $n \geq 1$, and $\zeta(3)$ are well known, but the irrationality results for the zeta constants $\zeta(2n+1)$, $n \geq 2$, are new, and seem to show that these are irrational numbers. By symmetry, the irrationality of the beta constants $L(2n, \chi)$ are derived from the known irrationality of the zeta constants $\zeta(2n)$.

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1 Introduction
As the rationality or irrationality nature of a number is an arithmetic property, it is not surprising to encounter important constants, whose rationality or irrationality is linked to the properties of the integers and the distribution of the prime numbers, for example, the number $6\pi^{-1} = \prod_{p \geq 2} (1 - p^{-2})$. An estimate of the partial sum of the Dedekind zeta function of quadratic numbers fields will be utilized to develop a general technique for proving the irrationality of the zeta constants $\zeta(2n+1)$ from the known irrationality of the beta constants $L(2n+1, \chi)$, $1 \neq n \in \mathbb{N}$. This technique provides another proof of the first odd case $\zeta(3)$, which have well known proofs of irrationalities, see [1], [2], [13], et al, and an original proof for the other odd cases $\zeta(2n+1)$, $n \geq 2$, which seems to confirm the irrationality of these number.

Theorem 1. For each fixed odd integer $s = 2k + 1 \geq 3$, the zeta constant $\zeta(s)$ is an irrational number.

The current research literature on the zeta constants $\zeta(2n+1)$ states the following:

(i) The special zeta value $\zeta(3)$ is an irrational number, see [1], [2], [5], [13], et al.

(ii) At least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is an irrational number, see [16, p. 7] and [17].

(iii) The sequence $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$, … contains infinitely many irrational zeta constants, see [3]. Various advanced techniques for studying the zeta constants are surveyed in [9], and [14].

By the symmetry of the factorization of the Dedekind zeta function $\zeta_K(s) = \zeta(s)L(s, \chi)$ with respect to either $\zeta(s)$ or $L(s, \chi)$, almost the same analysis leads to a derivation of the irrationality of the beta constants $L(2n, \chi)$ from
the known irrationality of the zeta constants \( \zeta(2n) \) for \( n \geq 1 \).

**Theorem 2.** For each fixed even integer \( s = 2n \geq 2 \), the beta constant \( L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \) is an irrational number.

The second Section contains basic materials on the theory of irrationality, the transcendental properties of real numbers, and an estimate of the summatory function of the Dedekind zeta function. The proof of Theorem 1 is given in the third Section. Last but not least, a sketch of the proof of Theorem 2 is given in the fourth Section.

### 2 Fundamental Concepts and Background

The basic notations, concepts and results employed throughout this work are stated in this Section.

#### 2.1 Criteria for Rationality and for Irrationality

All the materials covered in this subsection are standard definitions and results in the literature, confer [6], [7], [10], [13], [16], et al.

A real number \( \xi \in \mathbb{R} \) is called *rational* if \( \xi = a/b \), where \( a, b \in \mathbb{Z} \) are integers. Otherwise, the number is *irrational*. The irrational numbers are further classified as algebraic if \( \xi \) is the root of an irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( \deg(f) > 1 \), otherwise it is transcendental.

**Lemma 3.** (Criterion of rationality) If \( \xi \in \mathbb{Q} \) is a rational number, then there exists a constant \( c = c(\xi) \) such that

\[
\frac{c}{q} \leq \left| \frac{\xi - p}{q} \right|
\]  

holds for any rational fraction \( p/q \neq \xi \). Specifically, \( c \geq 1/B \) if \( \xi = A/B \).

This is a statement about the lack of effective or good approximations of an arbitrary rational number \( \xi \in \mathbb{Q} \) by other rational numbers. On the other hand, irrational numbers \( \xi \in \mathbb{R} - \mathbb{Q} \) have effective approximations by rational numbers.

If the complementary inequality \( |\xi - p/q| < c/q \) holds for infinitely many rational approximations \( p/q \), then it already shows that the real number \( \xi \in \mathbb{R} \) is almost irrational, so it is almost sufficient to prove the irrationality of real numbers. More precise results for testing the irrationality of an arbitrary real number are stated below.

**Lemma 4.** (Criterion of irrationality) Let \( \xi \in \mathbb{R} \) be a real number. If there exists an infinite sequence of rational approximations \( p_n/q_n \) such that \( p_n/q_n \neq \xi \), and

\[
\frac{c}{q} \leq \left| \frac{\xi - p}{q} \right|
\]  

holds for infinitely many \( p/q \), then it already shows that the real number \( \xi \in \mathbb{R} \) is almost irrational, so it is almost sufficient to prove the irrationality of real numbers. More precise results for testing the irrationality of an arbitrary real number are stated below.
for all integers \( n \in \mathbb{N} \), and some \( \delta > 0 \), then \( \xi \) is an irrational number.

**Theorem 5.** (Dirichlet) Let \( \xi \in \mathbb{R} \) be a real number. If there exists an infinite sequence of rational approximations \( p_n / q_n \) such that \( p_n / q_n \neq \xi \), and

\[
\frac{c}{q} \leq \left| \frac{\xi - p}{q} \right|
\]

for all integers \( n \in \mathbb{N} \), then \( \xi \) is an irrational number.

### 2.2. Estimate of An Arithmetic Function

Let \( q \geq 2 \) be an integer, and let \( \chi \neq 1 \) be the quadratic character modulo \( q \). The Dedekind zeta function of a quadratic numbers field \( \mathbb{Q}(\sqrt{q}) \) is defined by \( \zeta_k(s) = \zeta(s) L(s, \chi) \). The factorization consists of the zeta function \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \), and the \( L \)-function \( L(s, \chi) = \prod_{p} \left( 1 - \chi(p) \right)^{-s} \). The product has the Dirichlet series expansion \( \zeta(s) \cdot L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \) and its summatory function is \( \sum_{n \leq x} r(n) = \sum_{n \leq x} \sum_{d|n} \chi(d) \). The counting function \( r(n) = \sum_{d|n} \chi(d) = \# \{ (a, b) : n = a^2 + b^2 \} \geq 0 \) tallies the number of representations of an integer \( n \geq 1 \) as sums of two squares.

**Lemma 6.** The average order of the summatory function of the Dedekind function is given by

\[
\sum_{n \leq x} \frac{r(n)}{n^s} = \zeta(s) L(s, \chi) + \frac{c_0}{x^{s-1}} + O\left( \frac{1}{x^{s-1/2}} \right)
\]

where \( c_0 > 0 \) is a constant, and \( x \geq 1 \) is a large number.

A proof appears in [8, p. 369], slightly different version, based on the hyperbola method, is computed in [12, p. 255].

### 2.3 The Irrationality of Some Constants

The different analytical techniques utilized to confirm the irrationality, transcendence, and irrationality measures of many constants are important in the development of other irrationality proofs. Some of these results will be used later on.

**Theorem 7.** The real numbers \( \pi \), \( \zeta(2) \), and \( \zeta(3) \) are irrational numbers.

The various irrationality proofs of these numbers are widely available in the open literature. These technique are
valuable tools in the theory of irrational numbers, refer to [1], [2], [5], [13], and others.

**Theorem 8.** For any fixed \( n \in \mathbb{N} \), the followings statements are valid.

(i) The real number \( \zeta(2n) = (-1)^{n+1}(2\pi)^{2n} B_{2n} / 2(2n)! \) is a transcendental number,

(ii) The real number \( L(2n + 1, \chi) = (-1)^n \pi^{2n+1} E_{2n} / 2^{2n+2} (2n)! \) is a transcendental number,

where \( B_{2n} \) and \( E_{2n} \) are the Bernoulli and Euler numbers respectively.

**Proof:** Apply the Lindemann-Weierstrass theorem to the transcendental number. \( \blacksquare \)

3. Irrationality of the Zeta Constants \( \zeta(2n + 1) \)

For any integer \( 1 < s \in \mathbb{N} \), the zeta constant \( \zeta(s) \) is a real number classified as a period since it has a representation as an absolutely convergent integral of a rational function:

\[
\zeta(s) = \int_{1 > x_1 > x_2 > \ldots > x_s} \frac{d x_1}{x_1} \frac{d x_2}{x_2} \ldots \frac{d x_s}{1 - x_s} = \sum_{n \geq 1} \frac{1}{n^s}, \tag{5}
\]

where \( s > 1 \). A few related integral representations are devised in [2] to prove the irrationality \( \zeta(2) \) and \( \zeta(3) \). The general idea of a rational or nonrational integral proof of the zeta constant \( \zeta(s) \) for any integer \( s \geq 2 \) is probably feasible.

3.1 The Main Result

A different technique using two independent infinite sequences of rational approximations of the two constants \( \zeta_K(s) \), and \( 1 / L(s, \chi) \), which are linearly independent over the rational numbers, will be used to construct an infinite sequence of rational approximations of the zeta constant \( \zeta(2n + 1) \), \( n \geq 1 \). The properties of these sequences, such as sufficiently fast rates of convergence, are then used to derive the irrationality of any zeta constant \( \zeta(2n + 1) \), \( n \geq 1 \).
**Theorem 1.** For each fixed odd integer $s = 2k + 1 \geq 3$, the zeta constant $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is an irrational number.

**Proof:** Let $\chi \neq 1$ be the quadratic character modulo $q > 1$, and fix an odd integer $s = 2k + 1 \geq 3$. By Lemma 6, the summatory function of the Dedekind zeta function satisfies the expression

$$\zeta(s) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{1}{L(s, \chi)} = \frac{c_0}{x^{s-1}} + O \left( \frac{1}{x^{s-1/2}} \right)$$

for every $s > 1$, and a constant $c_0 > 0$. By Theorem 8, the real number number $L(2k + 1, \chi) = a \pi^{2k+1}$, where $a \in \mathbb{Q}$, is an irrational number, so there exists an infinite sequence of rational approximations \{ $p_n/q_n$ : $n \in \mathbb{N}$ \} such that

$$\left| \frac{1}{L(s, \chi)} - \frac{p_n}{q_n} \right| < \frac{c_1}{q_n^2},$$

where $c_1 > 0$ is a constant, see Lemmas 4 to 5. Combining these data, taking absolute value, and using the triangle inequality $\| x - y \| \geq \| x \| - \| y \|$, return

$$\left| \frac{c_0}{x^{s-1}} + O \left( \frac{1}{x^{s-1/2}} \right) \right| = \left| \zeta(s) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{1}{L(s, \chi)} \right|$$

$$\geq \left| \zeta(s) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{1}{L(s, \chi)} + \frac{p_n}{q_n} - \frac{p_n}{q_n} \right|$$

$$\geq \left| \zeta(s) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{p_n}{q_n} - \frac{1}{L(s, \chi)} - \frac{p_n}{q_n} \right|$$

$$\geq \left| \zeta(s) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{p_n}{q_n} \right| - \frac{c_1}{q_n^2}.$$.

Rewrite it as

$$0 < \left| \zeta(s) - \frac{p_n}{q_n} \sum_{n \leq x} \frac{r(n)}{n^s} \right| < \frac{c_1}{q_n^2} + \left( \frac{c_0}{x^{s-1}} + O \left( \frac{1}{x^{s-1/2}} \right) \right) \sum_{n \leq x} \frac{r(n)}{n^s}.$$

Now, taking $x \geq 1$ to infinity yields

$$0 < \left| \zeta(s) - \frac{p_n}{q_n} \sum_{n \leq x} \frac{r(n)}{n^s} \right| < \frac{c_2}{q_n^2},$$

where $c_2 > 0$ is a constant.
where \( c_2 > 0 \) is a constant, and \( s > 2 \).

The rest of the proof is broken up into two cases.

**Case I.** Assume that the constant \( \sum_{n=1}^{\infty} r(n) n^{-s} \in Q \) is a rational number, and that the constant \( \zeta(2k+1) = A/B \) is a rational number.

Consider a sequence of rational approximations \( \{ a_m/b_m : m \in \mathbb{N} \} \) for which \( \sum_{n=1}^{\infty} r(n) n^{-s} = a_m/b_m \) for some \( m \geq 1 \). By Lemma 3, there exists a constant \( c_3 \geq 1/B \) such that

\[
0 < \frac{c_3}{b_m q_n} \leq \left| \zeta(s) - \frac{a_m}{b_m} \frac{p_n}{q_n} \right| < \frac{c_2}{q_n^2}.
\]

(11)

Since the sequence \( \{ a_m/b_m : m \in \mathbb{N} \} \) is a finite sequence, (i.e., \( b_m \leq c \) is bounded by a constant \( c > 0 \)), it is clear that this is a contradiction for all sufficiently large \( q_n \geq 1 \). Ergo, the constant \( \zeta(2k+1) \) is not a rational number.

**Note 1.** The nonvanishing condition \( 0 < \left| \zeta(s) - \frac{a_m}{b_m} \frac{p_n}{q_n} \right| \) in inequalities (11) and (15), follows from the fact that the sequence \( \{ a_m p_n / b_m q_n : m, n \in \mathbb{N} \} \) has infinitely many distinct terms, and the fact that \( \zeta(s) \) is a constant for any fixed odd integer \( s = 2k+1 \geq 3 \). On the contrary, \( \zeta(s) = \frac{a_m}{b_m} \frac{p_n}{q_n} \) for all \( m, n \geq 1 \), which implies that is not a constant as \( m, n \geq 1 \) vary over the set of integers \( \mathbb{N} \).

**Case II.** Assume the constant \( \sum_{n=1}^{\infty} r(n) n^{-s} \in R - Q \) is an irrational number, and that the constant \( \zeta(2k+1) = A/B \) is a rational number.

By Lemma 4 or 5, there exists an infinite sequence of rational approximations \( \{ a_m/b_m : m \in \mathbb{N} \} \) such that

\[
0 < \left| \sum_{n \geq 1} \frac{r(n)}{n^s} - \frac{a_m}{b_m} \right| < \frac{c_4}{b_m^2}.
\]

(12)

for some constant \( c_4 > 0 \). For example, \( c_4 = 1 \) works for all \( n \geq 1 \), this is an important topic in the Markoff spectrum, see [16, p. 28] for related discussions. This inequality is equivalent to

\[
\frac{a_m}{b_m} - \frac{c_4}{b_m^2} \leq \sum_{n \geq 1} \frac{r(n)}{n^s} \leq \frac{a_m}{b_m} + \frac{c_4}{b_m^2}.
\]

(13)

Replacing this approximation into inequality (10) returns
where \( c_4 p_n = p_n \) is a suitable choice. Furthermore, since the constant \( \zeta(2k + 1) = A/B \) is a rational number, there exists a constant \( c_5 \geq 1/B \) such that

\[
\frac{c_2}{q_n^2} > \left| \zeta(s) - \frac{p_n}{q_n} \sum_{n \geq 1} \frac{r(n)}{n^t} \right| \\
\geq \left| \zeta(s) - \frac{p_n}{q_n} \left( \frac{a_m}{b_m} + \frac{c_4}{b_m^2} \right) \right| \\
\geq \left| \zeta(s) - \frac{a_m b_m p_n + c_4 p_n}{b_m^2 q_n} \right|,
\]

this follows from Lemma 3.

Next, construct an infinite subsequence of the sequence \( \{ r_{m,n} / s_{m,n} : m, n \in \mathbb{N} \} \), see (17), of distinct rational approximations of the zeta function \( \zeta(s) \) for odd \( s \geq 3 \) that contradict (15). To achieve this objective, fix a large integer \( m_0 \geq 1 \). Then, it follows that

\[
\frac{c_5}{b_m^2 q_n} \leq \left| \zeta(s) - \frac{a_m b_m p_n + c_4 p_n}{b_m^2 q_n} \right| < \frac{c_2}{q_n^2},
\]

Now, since the integer \( b_m \leq c \) is bounded by a constant \( c > 0 \), it is clear that this is a contradiction for all sufficiently large \( q_n \geq 1 \). Ergo, the constant \( \zeta(2k + 1) \) is not a rational number. ■

From a different point of view, the result in Theorem 1 shows that the assumed rational number \( \zeta(2k + 1) = A/B \) is well approximated by rational numbers: there are infinitely many rational approximations

\[
\frac{r_{m,n}}{r_{m,n}} = \frac{a_m b_m p_n + c_4 p_n}{b_m^2 q_n} \rightarrow \zeta(s) \quad \text{as } n, m \rightarrow \infty,
\]

where \( r_{m,n} = a_m b_m p_n + c_4 p_n \geq 1, s_{m,n} = b_m^2 q_n \geq 1 \) are integers. The sequences \( \{ a_m / b_m : m \in \mathbb{N} \} \) and \( \{ p_n / q_n : n \in \mathbb{N} \} \) are the convergents of the irrational numbers \( \sum_{n=1}^{\infty} n^{-s} \) and \( \sum_{n=1}^{\infty} n^{-s} s = 2k + 1 \geq 3 \), see (7) and (11) for other details. The existence of this sequence implies that \( \zeta(2k + 1) \) is not a rational number.

4. Irrationality of the Beta Constants \( L(2n, \chi) \)

For \( q = 4 \) the quadratic symbol is defined by \( \chi(n) = (-1)^{(n-1)/2} \) if \( n \in \mathbb{N} \) is odd, else \( \chi(n) = 0 \). The corresponding Dedekind zeta function is given by
\[ \zeta_K(s) = \zeta(s) L(s, \chi) = \frac{1}{4} \sum_{n \geq 1} \frac{r(n)}{n^s}, \]

where the counting function \( r(n) = \sum_{d \mid n} \chi(d) = \# \{ (a, b) : n = a^2 + b^2 \} \geq 0 \) tallies the number of representations of an integer \( n \geq 1 \) as sums of two squares, and \( s \in \mathbb{C} \) is a complex number. This is the zeta function of the Gaussian quadratic numbers field \( \mathbb{Q}(\sqrt{-1}) \). The corresponding L-series is

\[ L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots. \]

The evaluation at \( s = 2 \) is known as Catalan constant \( L(2, \chi) = 1 - 3^{-2} + 5^{-2} - 7^{-2} + \cdots \approx 0.915965594177 \ldots \).

**Theorem 2.** For each fixed even integer \( s = 2k \geq 2 \), the beta constant \( L(s, \chi) \) is an irrational number.

**Proof:** Use the symmetry of the factorization of the Dedekind zeta function \( \zeta_K(s) = \zeta(s) L(s, \chi) \) with respect to the zeta function \( \zeta(s) \) and the L-function \( L(s, \chi) \) to arrive at the asymptotic formula

\[ L(s, \chi) \left( \sum_{n \leq x} \frac{r(n)}{n^s} \right)^{-1} - \frac{1}{\zeta(s)} = \frac{c_0}{x^{s-1}} + O\left( \frac{1}{x^{s-1/2}} \right) \]

compare this to (6). Now, proceed as before in the proof of Theorem 1 for the verification of the irrationality of the zeta constant \( \zeta(2k + 1) \), mutatis mutandis. ■
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