Relativistic field-theoretical formulation of the three-dimensional equations for the three fermion system.¹

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Abstract

A new kind of the relativistic three-body equations for the three fermion systems are suggested. These equations are derived in the framework of the standard field-theoretical S-matrix approach in the time-ordered three-dimensional form. Therefore, corresponding relativistic covariant equations are three-dimensional from the beginning and the considered formulation is free of the ambiguities which appear due to a three dimensional reduction of the four dimensional Bethe-Salpeter equations. The solutions of the considered equations satisfy automatically the unitarity condition and for the leptons these equations are exactly gauge invariant even after the truncation over the multiparticle \((n > 3)\) intermediate states. Moreover, the form of these three-body equations does not depend on the choice of the model Lagrangian and it is the same for the formulations with and without quark degrees of freedom. The effective potential of the suggested equations is defined by the vertex functions with two on-mass shell particles. It is emphasized that these INPUT vertex functions can be constructed from experimental data.

Special attention is given to the comparison with the three-body Faddeev equations. Unlike these equations, the suggested three-body equation have the form of the Lippmann-Schwinger-type equations with the connected potential. In addition, the microscopical potential of the suggested equations contains the contributions from the three-body forces and from the particle creation (annihilation) mechanism on the one external particle. The structure of the three-body forces, appearing in the considered field-theoretical formulation, is analyzed.

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1. INTRODUCTION

The purpose of my talk is to focus on the three-body equations with the particle creation and annihilation phenomena. The most popular tool for this investigation is the three-body generalization of the Bethe-Salpeter equations [1, 2]. Unfortunately in this four-dimensional formulation arise a set of complications which result in a serious approximations. For instance, by the three-dimensional reduction of the four-dimensional Bethe-Salpeter equations arise the ambiguities with the choice of the form of the three-dimensional Green functions and the three-dimensional effective potentials. Next, the difficulties with the unitarity and the gauge invariance force to use the tree approximation for the effective potentials and the one-particle propagators by the practical calculation. In addition the potential of the Bethe-Salpeter equation is constructed through the three-variable vertex functions, which are required as the "input" functions. Therefore, in the practical calculations based on the Bethe-Salpeter equations or their quasipotential reductions the off-shell variables in the vertex functions are usually neglected or a separable form for all three variables is introduced.

In my talk I shall consider the other way of derivation of the three-body field-theoretical equations which allows us to avoid the above difficulties and which can be solved with the considerable less number of approximations. The organization of this talk is as follows. At first I will consider the three-body spectral decomposition equations (which have the form of the off-shell unitarity conditions [3, 4]) for the amplitudes of the three fermion systems. These equation form a base for the derivation of the Lippmann-Schwinger type equation [4, 3]. After separation of the connected and disconnected parts in the amplitudes and effective potentials in these three-body spectral decomposition equations, one can separate the three-body equations for the connected and disconnected parts in the three-body amplitudes. Next, after linearization of these three-body equations we will get the three-body Lippmann-Schwinger equations for the connected part of the three-body amplitudes. The major difference between these equations and the Faddeev equations will be discussed. Afterwards, I shall consider the structure of the three-body potentials for the three-fermion (three electron or three nucleon, or for the ed scattering etc.) systems. Finally the short summary will be presented.

2. The three-body Lippmann-Schwinger type equations for the three-fermion scattering reactions

The problem of the relativistic description of an particle interactions in the framework of a potential picture is usually solved by relativistic generalization of the Lippmann-Schwinger type equation of the nonrelativistic collision theory [3, 4]. As for basis for the derivation of the Lippmann-Schwinger type equations in the collision theory, one can use the following quadratically nonlinear integral equations [4]

\[
T(E) = V + \sum_{d_3} X T(d_3) \frac{1}{E E_d_3} T_{d_3}(E) + \sum_{d_3} X T_{d_3}(E_{d_3}) \frac{1}{E_{d_3}} T(d_3) (E_d_3);
\]  

(2.1)
where \( T(E) \) is the transition amplitude between the channels and \( \); the \( ; \) denotes the pure three non-interacting fermion channels or or the one-fermion+two-body cluster states. For example the three nucleon systems = NNN or N d, for the two electron and nucleon = eeN or ed etc., d3 relates to the three-body bound states. It stands for the integration over the momenta and the summation over the quantum numbers of the complete set intermediate \( j > \)-channel states.

If we suppose, that there exists the full hermitian Hamiltonian \( \hat{H} \) which has the complete set of the eigenfunctions \( \hat{H}j > = Ej > \), then one can easily reduce eq. (2.1) to the Lippmann-Schwinger type equations

\[
T(E) = V + \frac{1}{E} \frac{1}{E + i} T(E); \quad (2.2)
\]

where we have used the decomposition formula of the full Green function \( G(E) = 1 = \hat{H} + i \) over the complete set of the functions \( j > \) and we have taken into account the connection formula between a amplitude, a multichannel potential and a wave function \( T(E) = \langle \psi | j > \) \([3,4]\).

The three-body equations (2.1) are well defined after separation of the connected \( (T^c; V^c) \) and disconnected \( (V^{dc}; V^{dc}) \) parts of amplitudes and potentials. Therefore we split the complete amplitude and the complete potential in Eq. (2.1) into two corresponding parts

\[
V = V^c + V^{dc}; \quad T(E) = T^c(E) + T^{dc}(E); \quad (2.3)
\]

The disconnected parts of the two-body and three-body amplitudes are depicted in Fig. 1. The disconnected part of the three-body amplitudes are independent from the connected part of these amplitudes, because the two-body clusters with the asymptotic free third particle is independent on the three-particle interacted clusters. This is the requirement of the independency of the asymptotic clusters. The independence of the equations for the disconnected part of Eq. (2.1) can be easily demonstrated in the quantum-eld formulation \([8]\). As result equation (1) is split into two sets of independent equations

\[
T^{dc}(E) = V^{dc} + \frac{1}{E} \frac{1}{E + i} T^{dc}(E); \quad (2.4)
\]

\[
T^c(E) = W + \frac{1}{E} \frac{1}{E + i} T^c(E) + \frac{1}{E} \frac{1}{E + i} T_{d3}(E_{d3}) \quad (2.5)
\]

where

\[
W = V^c + \frac{1}{E} \frac{1}{E + i} T^c(E) + \frac{1}{E} \frac{1}{E + i} T^{dc}(E) + \frac{1}{E} \frac{1}{E + i} T^c(E); \quad (2.6)
\]
Figure 1: The disconnected parts of the two-fermion and the three-fermion S-matrix elements correspondingly in Fig.1A and Fig.1B, Fig.1C. The label d stands for the two-fermion bound state. The shaded circle corresponds to the two-body scattering amplitude.

The effective potential (2.6) in the eik-theoretical formulation with the particle propagators in the intermediate states. Nevertheless, we have shown in ref. [11, 8] that quadratically nonlinear equations (2.5) are equivalent to the following Lippman-Schwinger type equations

\[
T(E) = U(E) + \frac{X}{E} U(E) \left[ \frac{1}{E + i} \right] T(E); \quad (2.7)
\]

where for the sake of simplicity we have omitted the delta function for the total three-momentum conservation function (2.7) in the above equations.

The explicit form of the linear energy depending potential

\[
U(E) = A + E B
\]

with hermitian A and B matrices

\[
A = A; \quad B = B; \quad (2.8b)
\]

is considered in the next section. U(E) is simply connected with the W-potential (2.6)
\[ U (E) = W : \] (2.9a)

Therefore, for any potential \( W \) one can unambiguously construct \( U (E) \).

Solutions of the equations (2.5) and (2.7) coincide on energy shell

\[ T (E = E) = T^c \frac{i}{E - E} \] (2.9b)

and in the half on energy shell region these amplitudes are simply connected

\[ T = W + X W \frac{1}{E} \frac{T^c (E)}{E + 1} \] (2.10)

The Lippmann-Schwinger type equations (2.7) are our final equations for the three-fermion scattering amplitudes. From the other hand, Eq. (2.1) can be linearized without the separation of the connected and disconnected parts and from the Lippmann-Schwinger equation (2.2) one can derive the Faddeev type equations [3]. The advantage of Eq. (2.7) is that it does not need the splitting into four pieces \( V = \frac{3}{2} V^c + V^l + V^f \) in order to taken into account the disconnected parts in the perturbation series. Besides Eq. (2.7) are free from the overcounting problem which appear due to special disconnected diagrams and which generates the corresponding modification of the effective potentials [1, 2].

3. The three-dimensional three-body field-theoretical equations

In the standard formulation of the quantum field theory [5, 6, 7] the S-matrix element between the asymptotic three-body states \( = 1^0; 2^0; 3^0; f^0; d^0 \) and \( = 1; 2; 3; f; d \) is connected with the scattering amplitude \( f \),

\[ S = < \text{out}; j; \text{in} > = < \text{out}; eJ_{p_a} (\text{in}) f^0; \text{in} > (2) \frac{e^{i \phi}}{(P \ P \ P \ f)} \] (3.1)

where \( f \) denotes the one-fermion state, \( d \) stands for the bound state of two fermions, \( P (E; P) \) is the complete four-momentum of the asymptotic state, \( a \) and \( b \) corresponds to the one-fermion states extracted from the asymptotic and states

\[ a = a + e; \quad b = b + e \] (3.2)

and the four-momentum of the asymptotic one-fermion states \( a; b = 0 \) \( E_{p_a}; \ p_a \). The amplitude \( f \) has the form

\[ f = < \text{out}; eJ_{p_a} (0); j; \text{in} > \] (3.4)

where \( J_{p_a} (x) \) is the current operator of the fermion \( a \) which is determined by the Dirac equation \( J_{p_a} (x) = Z_a \frac{1}{\hbar^2} \gamma (p_a) (i \hbar \gamma \ m_a) a (x) \) with the renormalization constant \( Z_a \) and Dirac bispinor function \( u (p_a) [5, 6] \).
Using the well-known reduction formulas we obtain

$$f = < \text{out}^{n} e J_{p_{b}}^{+} (\text{out}) J_{p_{a}} (0) f_{p}^{o}; \text{in} >$$

$$< \text{out}^{n} e J_{p_{a}} (0); b_{p_{b}}^{+} (0) f_{p}^{o}; \text{in} > + i \frac{d^{4}G^{x}}{2^{x}} < \text{out}^{n} e J_{p_{a}} (0) \tilde{J}_{p_{b}} (x) f_{p}^{o}; \text{in} > ;$$

(3.5)

where

$$b_{p_{b}}^{+} (x_{0}) = Z_{p_{b}}^{1-2} \frac{d^{4}G^{x}}{2^{x}} P_{p_{b}}^{o} (p_{b}) o_{b} (x);$$

(3.6)

Here and afterwards we use the definitions and normalization conditions from the Itzykson and Zuber's book [6].

After substitution of the complete set of the asymptotic states $u_{n}^{o} in^{o}$ states $p_{n}^{o} j_{n}; \text{in} > < \text{in} ; n = \frac{b}{f}$ between the current operators in expression (3.5) and after integration over $x$ we get

$$f = W + (2) \frac{X}{3} X f < \text{out}^{n} e J_{p_{a}} (0); b_{p_{b}}^{+} (0) f_{p}^{o}; \text{in} >$$

$$+ (2) \frac{X}{3} X < \text{out}^{n} e J_{p_{a}} (0) j_{n}; \text{in} > < \frac{(P_{b}^{o} + P_{e}^{o} P_{n}^{o})}{E_{p_{b}} + P_{e}^{o} P_{n}^{o} + 1} < \text{in} ; n j_{p_{b}}^{0} (0) f_{p}^{o}; \text{in} > ;$$

(3.7)

$$+ (2) \frac{X}{3} X < \text{out}^{n} e \tilde{J}_{p_{b}} (0) j_{n}; \text{in} > < \frac{(P_{b}^{o} + P_{e}^{o} P_{l}^{o})}{E_{p_{b}} + P_{e}^{o} P_{l}^{o} + 1} < \text{in} ; l j_{p_{a}}^{0} (0) f_{p}^{o}; \text{in} > ;$$

(3.8)

where $n = 1^{o} 2^{o} 3^{o} 4^{o} 5^{o} 6^{o} 7^{o} 8^{o} 9^{o} \cdots$ denotes the four body states with the intermediate boson $b^{0}$ which denotes a photon for the three leptons system and $b^{0}$ stands for the intermediate meson for the three baryon system $s$. The third part of Eq.(3.8) describes the $u$-channel interaction terms which are obtained after crossing of the $a$ and $b$ particles. The intermediate states of this term contain one fem ion $l = f$ and one fem ion+ boson $l = f + b$ states. These diagram are depicted in Fig. 2A and in Fig. 2E.

Equation (3.7) contains the auxiliary amplitude

$$T = < \text{in} ; e J_{p_{a}} (0) j_{n}; \text{in} >$$

(3.9)
**Figure 2:** The on mass shell particle exchange diagrams which are included in the effective potential (3.11) for the three-fermion reactions \(1 + 2 + 3 \rightarrow 1^0 + 2^0 + 3^0\). The empty circle stands for the primary transition amplitude and the dashed circle corresponds to the following transition amplitude. Fermions with the index \(a\) are extracted from the asymptotic states in the expression (3.11) or (3.8) for \(a = 1^0\) and \(b = 1\). For any amplitude in the left side of Eq. (3.11) or (3.8) only one particle \((a\) or \(b\)) is considered on mass shell. All of diagrams have the three-dimensional time-ordered form with the \(0^0\) dressed \(0^0\) vertices. Therefore in all of the diagrams the initial empty circle is depicted in left-hand side and the following circle takes a place in the right-hand side.

In the transition matrix \(<e^j_p_a(0)j>\) with an arbitrary \(e\) and \(j\) states, all particle except of \(a\) are on mass shell. The four-momenta of particle \(a\) is expressed through the four-momenta of other on mass shell particles, i.e. \(p_a = P \quad P_e\). Therefore afterwards we shall consider particle \(a\) as on mass shell particle in the corresponding matrix element.

For the one-particle asymptotic state \(e = 1^0\) we have \(f_1^{(0)}_{1+a} = T_{1+a}^{(0)}\), because \(<\text{out}1^0_j < \text{in}1^0_j<\text{out}\). But for the three-particle asymptotic state \(<\text{out}\text{j}\text{the case is more complicated}\text{j}\text{the case is more complicated}\text{j}\text{the case is more complicated})\) \(T_{1+2+a}\); \(\text{we can obtain the analogous to (3.7) relation for}T(3.9)\text{using the S-matrix reduction formulas}\)

\[
T = w + (2)^3 X^3 T \frac{(P_{eb} + P_e \quad P_e)}{E_{P_{eb}} + P_e \quad P_e + 1} T
\]

(3.10)
Figure 3: Diagrams obtained after the two particle \((2;3)\) and \((2^0;3^0)\) transposition from the s-channel diagram in Fig. 2A and from the t-channel diagram in Fig. 2E.

where

\[
\begin{align*}
\mathbf{w} &= \langle \text{in}; \epsilon^j \bar{J}_{p_a} (0); \bar{J}_{p_{b}} (0) \rangle^o \langle \text{in} \rangle \\
+ (2)^3 & \prod_{n=1}^{3} \langle \text{in}; \epsilon^j \bar{J}_{p_a} (0); \bar{J}_{p_{b}} (0) \rangle^o \langle \text{in} \rangle \\
& \times \frac{\left( \frac{3}{E_{p_b} + \frac{P_{e}}{P_n}} \right)}{\text{in}; \epsilon^j \bar{J}_{p_a} (0); \bar{J}_{p_{b}} (0) \rangle^o \langle \text{in} \rangle} \\
& = \frac{(3) \left( \frac{p_{b}}{E_{p_b}} + \frac{P_{e}}{P_1} \right)}{\text{in}; \epsilon^j \bar{J}_{p_a} (0); \bar{J}_{p_{b}} (0) \rangle^o \langle \text{in} \rangle} ; \quad (3;11)
\end{align*}
\]
Using Eq. (3.7) and Eq.(3.9) we can find the connections formula between $f$ and $T$. If we suppose, that

$$T = wW^1f$$

and

$$f = Ww^1T ;$$

then after insertion of relation (3.12a) into Eq.(3.7) we obtain Eq.(3.9) for $T$. And vice versa, after insertion of relation (3.12b) into Eq. (3.9) we get Eq. (3.7). This is the justification of the relations (3.12a,b) for the nonsingular effective three-body potentials $W$ (3.8) and $W$ (3.11).

The consistent procedure of separation of the complete set of connected and disconnected parts in the three-dimensional equations (3.10) or (3.7) is well known as the theoretical cluster decomposition procedure[9, 10]. In ref. [8] this procedure is applied to the three-body equation for the $N$ system. For the three-body reactions \(1 + 2 + 3 \rightarrow 1^0 + 2^0 + 3^0\) the cluster decomposition procedure is the same as separation of the following connected and disconnected matrix elements

\[
T_{12}^{dc} = \begin{pmatrix} T_{123}^{dc} \\ T_{12}^{dc} \end{pmatrix}
\]

\[
T_{12}^{dc} = \begin{pmatrix} T_{123}^{dc} \\ T_{12}^{dc} \end{pmatrix}
\]

The suitable channel terms of the effective potential (3.11) are depicted in Fig.2A and in Fig.2E. As usual, mass shell one-particle states are taken $b = 1$ and $a = f$. These mass shell particles in the following figures are marked by . The diagrams in Fig.2A and in Fig.2E have different chronological sequences of absorption and emission of the particles $1$ and $3$. In particular, the s-channel diagram 2A corresponds to the chain of reactions, where firstly the initial three-body state $1 + 2 + 3$ transforms into intermediate mass shell one-particle states which afterwards produces the final $1^0 + 2^0 + 3^0$ state. In the diagram 2E at first the final fermion $1^0$ is generated with the intermediate state $1$ from the initial $2 + 3$ states and afterwards we obtain final $2^0 + 3^0$ state from the intermediate states $1 + 1$ states.

Using the cluster decomposition procedure for the s and u channel terms in Eq. (3.11) or in Eq. (3.16) one can change the chronological sequence of absorption of the initial mass shell fermions $2;3$ and emission of the mass shell particles $2;3$. In diagrams 2B, 2C and 2D are presented all possible transpositions of the particles $3$ and $3^0$ from the original s-channel diagram 2A which can be performed after transposition of
particles 3 and 3 using the disconnected structure (3.13) of the three-body amplitudes. In particular, Fig. 2B is obtained after transposition of fermion 3 or after substitution of the disconnected part of an amplitude in (0)h \psi \iota (m) in > . Fig. 2C is generated by transposition of fermion 3 and Fig. 2D is result of the permutation of both particles 3 and 3. Unlike the diagram 2A, in the diagram 2B the intermediate states arise together with the final state and next are creating the final two-fermion 1 + 2 states. The same procedure of transposition of particles 3 and 3 from the u-channel diagram in Fig. 2E generates the diagrams 2F, 2G and 2H. Another kind of permutations of the both particles 2 + 3 and 2 + 3 from s-channel diagram in Fig. 2A produces the diagrams 3A, 3B and 3C. In particular, diagrams 3G and 3I are obtained from diagram 2B and 2F after transposition of the 2 + 3 particle states. And transposition of 2 + 3 states in diagrams 2C and 2G produces 3H and 3J diagrams. The complete set of the diagrams which can be obtained after transpositions of the particles (2, 3); (2, 3) consists from the different disposition of these particles at the first vertex function and at the following vertex function. The first vertex function in Fig. 2 and in Fig. 3 is denoted with the empty circle in Fig. 2 and the dashed circle stands for the next vertex function. One has the following combinations of the dispositions of particles (2, 3); (2, 3) at the vertex functions: 1 + zero particles =) 10 + four particles, 1 + one particle =) 10 + three particles, 1 + two particles =) 10 + two particles, 1 + three particles =) 10 + one particle and 1 + four particles =) 10 + zero particles. For instance, we have four diagrams s 1 + 2 =) 10 + 3; 2; 2; 0 (Fig. 2C), 1 + 3 =) 10 + 2; 2; 2; 2 + 0 =) 10 + 2 + 2; 3; 0 and 1 + 3 =) 10 + 23; 23; 2 (Fig. 3G) for the disposition 1 + one particle =) 10 + three particles. The particle distribution 1 + three particles =) 10 + one particle have also the four diagrams s 1 + 3; 2; 2 =) 10 + 32; 3; 0 =) 10 + 3 (Fig. 3H), 1 + 23; 3 =) 10 + 20 (Fig. 2B), 1 + 3; 0 =) 10 + 3. The particle distribution 1 two particle =) 10 + two particles can be observed in the six diagrams s 1 + 23 =) 10 + 2; 2; 0 (Fig. 2A), 1 + 2; 3; 0 =) 10 + 3; 2; 0 (Fig. 2D), 1 + 2; 2; 2 =) 10 + 3; 3; 0, 1 + 3; 0 =) 10 + 2; 0; 3, 1 + 3; 0 =) 10 + 2; 3, 1 + 3; 0 =) 10 + 2; 0; 3, 1 + 2; 3; 0 =) 10 + 2; 3; 0 and 1 + 2; 3; 0 =) 10 + 2; 3; 0. And one diagram 1 =) 10 + 2; 3; 0 (Fig. 3A) and one diagram 1 + 23; 2; 3; 0 =) 10 (Fig. 3B) relates to the distributions 1 + zero particles =) 10 + four particles and 1 + four particles =) 10 + zero particles correspondingly. Thus the s-channel term in Eq. (3.16) generates the 2 4 + 6 + 2 = 16 connected terms after cluster decomposition. The other 16 connected terms produce the u-channel term in Eq. (3.16). Thus we get 32 independent skeleton diagram s after the cluster decomposition procedure performed in the second and in the third terms of Eq. (3.11) or Eq. (3.16). Diagrams s 3C, 3D, 3E, 3F, 3I and 3J contain the antiparticle intermediate states, because the time-ordered eikonal-theoretical formulation includes the complete set of the intermediate particle propagators with different time sequences. This means that for any diagram s with n; i; j; k; ... particle intermediate states appear corresponding diagram s with the antiparticle n; i; j; k; ... intermediate states.

The S-matrix reduction formulas for the 1d =) 10 + d process gives us the following equation.
\[ T_{1d;ld} = \langle \text{out}; P_d^0 J_p^{(0)} P_1; P_{d;} \text{in} > = w_{1d;ld} + (2) \sum_{n=3}^{3} \frac{X T_{1d;ld}^{(3)} (P_1 + P_d P_n)}{E_{p_1} + P_{d;}^0 P_n^0 + i} \]

where

\[ w_{1d;ld} = \langle \text{out}; P_d^0 j J_p^{(0)} P_1; P_{d;} \text{in} > \]

\[ + (2)^3 \sum_{n=1}^{3} \langle \text{out}; P_d^0 J_p^{(0)} P_1; P_{d;} \text{in} > \]

\[ + (2)^3 \sum_{l=3}^{3} \langle \text{out}; P_d^0 J_p^{(0)} P_1; P_{d;} \text{in} > \]

\[ + (2)^3 \sum_{l=3}^{3} \langle \text{out}; P_d^0 J_p^{(0)} P_1; P_{d;} \text{in} > \]

\[ (3.15) \]

\[ (3.16) \]

**Figure 4:** The graphical representation of the on mass shell particle exchange potential (3.16) for the \( 1 + d = 1^0 + d^0 \) amplitude after cluster decomposition. The double line denotes a two-fermion bound state, \( n \) stands for the three-fermion ion + boson states, \( m = f + b; \cdots \) and \( l \) corresponds to the intermediate fermion \( f; f + b; \cdots \) states.

Equations (3.15) and (3.16) for the amplitude and for the effective potential of the 1d scattering reaction have the same form as the two-body equations. After cluster decomposition of \( w_{1d;ld} \) we obtain only 6 terms (see Fig.4) with a transposition \( d ( \text{ } ) d^0 \) and with crossing transformation \( 1^0 \rightarrow 1 \). Eq.(3.15) contains the 1d ! \( 1^0 \rightarrow 3^0 \) transition matrix which is connected with the 123 ! \( 1^0 \rightarrow 3^0 \) transition amplitude according to the Eq.(3.10), where \( ; = 3f, \text{ but } = 3f; fd \). Thus the 123 ! \( 1^0 \rightarrow 3^0, 1d ! 1^0 \rightarrow 3^0 \) and
1d ! 1^3_0 transition amplitudes are the solutions of the coupled equations (3.10) and (3.15). Using the linearization procedure of such type equations [11, 8], one can obtain the equivalent set of Lippman–Schwinger-type equations

$$T_{\mu \nu} (E_{rd}) = U_{\mu \nu} (E_{rd}) + \frac{X}{E_{rd}} \frac{1}{E + 1} T_{\mu \nu} (E_{rd});$$

(3.17)

where $E_{rd} = E_d + E_r$ is the energy of the asymptotic fermion and two-fermion bound state $d_1^1 = 3f; fd$ and $U (E)$ is unambiguously determined from the connected part of potentials $W$ according to the relation (2.9a). Note that the solution of the three-body equations (i.e., the 123 ! $1^2_0^2_0^0$ and 123 ! $3^2_0^0$ transition amplitudes) participate in the $w^c$ potential in the diagrams 3B and 3C. One can rid the three-fermion potential of such type nonlinearities after introduction of a new amplitude $f_1 = F_1 + A_1$ in Eq.(3.10) and in Eq.(3.15), where the choice of $A_1$ is conditioned by cancellation of the terms in Fig. 2B and in Fig. 2C which have the form $f g A^+$ and $A g f^+$. Afterwards we get the linear Lippman–Schwinger-type equation for $F_1$ amplitudes with the disconnected terms. Therefore this linearization procedure generates necessity to use the Faddeev-type equations for the three-fermion ion scattering problem. Certainly, in the intermediate energy region, i.e., up to 2 GeV of the energy of the incoming proton for the N d 3N systems, one can neglect the diagrams 3I and 3J with the $2^N$ states and the diagram 3F with the $3^N$ intermediate state.

Equations (3.10) and (3.15) represent the spectral decomposition formulas (or shell unitarity conditions) for the three-body amplitudes in the standard quantum field theory. Such three-dimensional time-ordered relations were considered in the textbooks in the quantum field theory [5, 6, 9] and in the nonrelativistic collision theory [3, 4] for the two-body reactions. Therefore, one can treat Eq. (3.10) and Eq. (3.15) as the three-body generalization of the field-theoretical spectral decomposition formulas (or shell unitarity conditions) for the two-body amplitudes. The field-theoretical formulation allows us to obtain the analytical structure of the three-body amplitudes and the problem of detemination of the three-body forces in the nonrelativistic Faddeev equations does not arise in the considered formulation.

4. Equal-time anticommutators as a mass shell particle exchange potential.

The important part of the effective potential $w$ is the equal-time commutator in the effective potentials (3.11) and (3.16). The equal-time anticommutators in the effective potential of the 1d ! 1^2_0^2_0^0, 1d ! 1^2_0^2_0^0 and 123 ! 1^2_0^2_0^0 reactions are

$$Y_{1d1^2_0^2_0^0} = \langle P_1^0; J_1^1; 0, P_1^1, 0 \rangle \_{\mu \nu}; \in >;$$

(4.1a)

$$Y_{1d1^2_0^2_0^0} = \langle P_3^0; P_3^0; J_1^1; 0, P_1^1, 0 \rangle \_{\mu \nu}; \in >;$$

(4.1b)
Figure 5: The graphical representation of the equal-time anticommutators (4.1a,b,c). These terms are depicted separately for the binary reactions $A;B$, for the process $2 =) 3^0 C;D$ and for the three-body process $3 () 3^0 E;F;G$. Diagrams A;C;E correspond to the one o mass shell particle exchange interactions which are appearing from in the $^3$-theory i.e. for the QED or for the Yukawa-type interactions. The triangle denotes the vertex functions in the tree approximation. Diagrams B;D;F describe the contact (overlapping) interaction which does not contain the internal ediathe hadron propagation between hadron states. Diagram 5G corresponds to the simplest one o mass shell boson fermion and two o mass shell boson exchange interaction which is obtained from the equal-time commutators in Eq. (4.1c) (diagram 5F) in the framework of the $^3$ theory.

\[
\chi_{123;0230} = \left< p_2^0; p_3^0 j \right| J_{p_1}^0 (0); b_{p_1}^0 (0) \right| p_2; p_3; i n > : \quad (4.1c)
\]

The explicit form of expression (4.1abc) can be determined using the a priori given Lagrangian and equal-time anticommutations relation between the Heisenberg field operators. In the case of renormalizable Lagrangian models or for nonrenormalizable simple phenomenological Lagrangians the equal-time anticommutation terms are easy to calculate [11, 8]. In that case expressions (4.1abc) which often are called as the seagull term, consists of the o mass shell internal one particle exchange potentials (see diagram s 5A, 5C and 5E) and of the contact (overlapping) terms s (Fig. 5B, Fig. 5D and Fig. 5F) which does not contain any particle propagator in the internal ediathe states between asymptotic $j^-$ and $< j^- j$ states. The equal-time commutators are the only part of effective potentials (3.11) or (3.16) which contains explicitly the internal o mass shell particle exchange diagram s, since other terms in the effective potential (3.11) or (3.16) consists of the on mass shell particle exchange terms, where o mass shell are external fermions. In or-
order to clarify the structure of the equal-time terms, we will consider Lagrangian of the simplest model for the electromagnetic fields and for the pseudoscalar N interactions

\[ L_{em} = e^{-} A; \quad L_{ps} = ig^{-} s \]  \hspace{1cm} (42)

The current operator and the equation of motion for this Lagrangians are

\[ \partial \partial A = J = e^{-}; \quad (\partial \partial + m^2)^{1} i j^{i} = ig^{-} s^{i} \]  \hspace{1cm} (43)

Using the equal-time anticommutation relations between the Heisenberg field operators for the expressions (4.1abc) we get

\[ Y = \frac{e}{Z_{1} Z_{1}^{
u}} u (\varphi_{1}^{0}) u (\varphi_{1}) < in; e \bar{\alpha}(0) j^{0}; in > = \frac{e}{Z_{1} Z_{1}^{
u}} (\varphi_{1}^{0}) (\varphi_{1}) < in; e j^{0}; in >; \]  \hspace{1cm} (4.4a)

or for the NNN system

\[ Y = \frac{ig}{Z_{1} Z_{1}^{
u}} u (\varphi_{1}^{0}) 5 i u (\varphi_{1}) < in; e j^{i} (0) j^{0}; in > = \frac{ig}{Z_{1} Z_{1}^{
u}} (\varphi_{1}^{0}) 5 i u (\varphi_{1}) < in; e j^{i} (0) j^{0}; in >; \]  \hspace{1cm} (4.4b)

where \( e; \varphi = (d; c^0); (d; 2c^0) \) and \( (2c; 2c^0) \) for (4.1a), (4.1b) and (4.1c) correspondingly. Expressions (4.4a) or (4.4b) relates to the one mass shell boson exchange diagrams 5A, 5C and 5E for the Lagrangian (4.2). Using more complete Lagrangian models one can obtain also heavy meson exchange diagrams [11, 8]. Moreover, in the ref.[11] the one Boson Exchange (OBE) Bonn model of NN potential was exactly reproduced from the equal-time anticommutation relations. There was also numerically estimated the contributions from the contact (overlapping) terms for NN phase shifts. These contact (overlapping) terms arise from the 4 (four-point) part of Lagrangian or from the nonrenormalizable Lagrangian and they play an important role for the NN scattering.

The other source for the overlapping (contact) terms in the quantum field theory is the quark-gluon degrees of freedom. One can construct the hadron creation and annihilation operators as well as the Heisenberg field operators of hadrons from the quark-gluon fields in the framework of the Haag-Nishijima-Zimmermann treatment of the composed particles. In this case the contact term s (see diagrams 5B, 5D and 5F) contains the contributions from the quark-gluon exchange [11, 12]. But the equations (3.10), (3.11), (3.15) and (3.16) remain the same also for the formulation with the quark-gluon degrees of freedom.

The contact (overlapping) terms depicted in diagrams 5C, 5D and 5E, 5F can be treated as pure three-body forces. For these terms it is necessary to use an additional derivation of two-body and the three-body equations like the spectral decom position formulas (3.10). These extra auxiliary two-body and the three-body amplitudes are necessary for solution of the considered three-body equations. As an example we shall consider the amplitude \( < p_1^0; p_2^0; j, (0) p_2; p_3 > \) for the reaction \( 2p^2 \rightarrow 3^0 \). This amplitude participate

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in the diagram 5E and the corresponding on mass shell particle exchange diagrams are depicted in Fig.6. Diagrams 6B, 6C, 6D are obtained from the s-channel diagram 6A after transpositions 2;1$^{0\circ}$. The next four diagrams are produced by crossing permutation of the on mass shell particles 1 and b$^0$. The last four diagrams are obtained from diagrams 6A, 6C, 6E and 6D after transposition of the 1$^{0\circ}$ states to the first vertex.

The complete set of the diagrams which can be obtained after transpositions of the particles 2;1$^{0\circ}$ in the s-channel term of amplitude of the 1 + 2 =) b$^0$1$^{0\circ}$ reactions consists from the following dispositions of on mass shell particles 2;1$^{0\circ}$ at the first and the next vertices: 1 + zero particles =) b$^0$ + three particles, 1 + one particle =) b$^0$ + two particles, 1 + two particle =) b$^0$ + two particles and 1 + three particles =) b$^0$ + zero particles. For instance, we have three diagrams 1 + 2 =) b$^0$ + 1$^{0\circ}$ (Fig.6A), 1 + 1$^{0\circ}$ =) b$^0$ + 2$^{0\circ}$ (Fig.6D) and 1 + 1$^{0\circ}$ =) b$^0$ + 2$^{0\circ}$ for the distribution 1 + one particle =) b$^0$ + two particles. The particle dispositions 1 + two particles =) b$^0$ + one particle can be realized also with the three diagrams 1 + 2;1$^{0\circ}$ =) b$^0$ + 2$^{0\circ}$, 1 + 2;2$^{0\circ}$ =) b$^0$ + 1$^{0\circ}$ (Fig.6B) and 1 + 1$^{0\circ}$ =) b$^0$ + 2$^{0\circ}$ (Fig.6J). And the diagrams with the particle distribution 1 + 2;1$^{0\circ}$ =) b$^0$ (Fig.6C) and 1 =) b$^0$ + 2;1$^{0\circ}$ (Fig.6C) from the distributions 1 + three particles =) b$^0$ + zero particles and 1 + zero particles =) b$^0$ + three particles. Therefore the complete set of the connected s-channel terms are 2 + 3 = 8. Together with the u-channel terms we have 32 diagrams for the b-boson creation reaction on the two fermion system.

The simplest contact terms (4.1c) for the three-point Lagrangians (4.2) have the form

\[ Y_{1^{0\circ};1^{0\circ};1^{0\circ}} = e^2 \frac{\mathcal{U}(\rho_1)}{\mathcal{P}_{1^{0\circ};1^{0\circ};1^{0\circ}}} < p_1^0 \mathcal{J}_1^0 (0) p_2 > \frac{\mathcal{U}(\rho_3)}{Z_1 Z_3 (\rho_3^0) (p_3^0 + m_{e})} < p_2^0 \mathcal{J}_2^0 (0) p_3 > \frac{\mathcal{U}(\rho_2)}{Z_1 Z_3 (\rho_3^0) (p_3^0 + m_{e})} \] ;

(4.1a)

and for the NN system

\[ Y_{1^{0\circ};1^{0\circ};1^{0\circ}} = i g^3 \frac{\mathcal{U}(\rho_1)}{\mathcal{P}_{1^{0\circ};1^{0\circ};1^{0\circ}}} < p_1^0 \mathcal{J}_1^0 (0) p_2 > \frac{\mathcal{U}(\rho_3)}{Z_1 Z_3 (\rho_3^0) (p_3^0 + m_{e})} < p_2^0 \mathcal{J}_2^0 (0) p_3 > \frac{\mathcal{U}(\rho_2)}{Z_1 Z_3 (\rho_3^0) (p_3^0 + m_{e})} \] ;

(4.1b)

where \( Q = p_1 + p_2 + p_3^0 \) and this simplest two on mass shell boson exchange term is depicted in Fig.5G.

Starting from any Lagrangian we always obtain one on mass shell boson exchange potentials (Fig.5A, Fig.5C and Fig.5E). The contact (overlapping) terms we use 4 terms in Lagrangian [8], or more complicated models of phenomenological Lagrangians [11], models of a nonrenormalizable Lagrangian, quark-gluon degrees of freedom [8] etc. These terms contains the other kind of a three-body amplitudes too and one must include these extra auxiliary two-body and three-body amplitudes in the set of coupled equations (3.10) and (3.15). Thus the number of the solved three-body equations and the form of the auxiliary amplitudes is depending on the form of the input Lagrangian. This means, that the unified description of the coupled three-fermion reactions can help us to determine the form of input Lagrangians which are sufficient and necessary for a description of the experimental observables.
Besides of the Lagrangians, as input" by construction of the effective three-body potentials are the amplitudes of the $2^+ 2^0$ and $2^+ 3^0$ reactions and the three-point vertex functions. In these vertex functions two particle are on mass shell and they are the function of a one variable. Therefore, one can determine these vertex functions from the experimental data using the quark counting rules, dispersion relations, the Regge trajectories theory, or inverse scattering method [13].

Figure 6: The graphical representation of the on mass shell particle exchange potential for the $1+ 2 = 1^0 + 2^0 + b^0$ amplitude with on mass shell boson $b^0$. This amplitude arise in the seagull term (4.1c) in the 3 theory after cluster decomposition. The curved line denotes the on mass shell boson $b^0$ which corresponds to the photon for leptons or meson for hadrons. $n = 2f^0 + b^0; 2f^0 + 2b^0; d^0 + b^0; ...$ stands for the intermediate on mass shell particle states in s channel, $m = f^0 + b^0; ...$ and $l = b^0; 2b^0; ...$.

5. Summary

In my talk I have considered the three-dimensional covariant scattering equations for
the amplitudes of the three-body scattering reactions. The basis of these three-body relativistic equations is the standard eik-theoretical S-matrix reduction formulas. After decomposition over the complete set of the asymptotic states the quadratically nonlinear three-dimensional equations (3.10) or (3.15) were derived. These equations have the same form as the off-shell unitary conditions (2.1) in the nonrelativistic collision theory. Afterwards the suggested nonlinear equations can be replaced by the equivalent Lippmann-Schwinger type equations (2.2) or (3.17) for the connected part of the three-body amplitudes. If one wants to rid the potential of these three-body equations from the nonlinear terms (see diagrams in Fig. 2B and in Fig. 2C with \( j = f \))

The effective potential of the suggested equations consists (i) from the on mass shell particle exchange diagrams in Fig. 2, Fig. 3 and Fig. 4 and (ii) from the equal-time commutators which contain the one on mass shell boson exchange diagrams (Fig. 5A, Fig. 5C and Fig. 5D) and overlapping (contact) terms (Fig. 5B, Fig. 5D and Fig. 5E). The form and the number of these equal-time potentials depend on the input Lagrangian model. For the three leptons interactions the overlapping (contact) terms do not appear and the diagrams in Fig. 5E are reduced to the simple on mass shell two-photon exchange diagram in Fig. 5F. In the case of the two-body reactions the equal-time commutators generate effective potential which can be constructed from the phenomenological one variable vertex functions if one use the simple on mass shell two-photon exchange diagrams in Fig. 5F.

In order to construct the three-body potential from the one-variable phenomenological vertices one need also to construct the two-body on mass shell scattering amplitudes, the two-off mass ion=) two-off mass ion+ boson transition amplitudes (see Fig. 5E and Fig. 6) and also the complicated overlapping (contact) terms (Fig. 5D and Fig. 5F). The main attractive feature of the considered eik-theoretical scheme of the three-body equation is that it allows us to estimate the importance of the overlapping (contact) terms. Therefore the unified description of a two-body and a three-body reactions in the considered formulation allows us to determine the form of the simplest Lagrangians which are necessary and enough for the unified description of the two-body and the three-body experimental data. In addition these calculations allow us to improve the accuracy of the calculations in the tree and in the Born approximations.

The considered eik-theoretical formulation is not less general as the four dimensional Bethe-Salpeter equations. The final form of the equations (3.11) or (3.15) are not depending on the choice of the Lagrangian and these equations are valid for any QCD motivated models with the quark-gluon degrees of freedom. But the suggested equations are much simpler than the analogous Bethe-Salpeter equations and they can be
numerically solved with the present computers. The only principal approximation, that is necessary to do in this approach is the truncation of the intermediate multi-particle states. But here, unlike to the Bethe-Salpeter equations, one use to cut down only the on mass shell intermediate states. In any case for the self-consistent calculation of the two-body and the three-body reactions in the low and intermediate energy region, it is advisable to work out the scheme of a suppression mechanism of the transition of one on mass shell fermion into on mass shell fermion+ on mass shell boson which arise together with the transition amplitude into the three-fermion+ boson states (see Fig.3A, Fig.3B, Fig.3D and Fig.3E).

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