Central extensions of classical and quantum $q$-Virasoro algebras

J. Avan

LPTHE, CNRS-URA 280, Universités Paris VI/VII, France

L. Frappat, M. Rossi, P. Sorba

Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, CNRS-URA 1436
LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France

Abstract

We investigate the central extensions of the $q$-deformed (classical and quantum) Virasoro algebras constructed from the elliptic quantum algebra $A_{q,p}(sl(N))_c$. After establishing the expressions of the cocycle conditions, we solve them, both in the classical and in the quantum case (for $sl(2)$). We find that the consistent central extensions are much more general that those found previously in the literature.
1 Introduction

In three previous papers [1, 2, 3] we have given a construction of \( q \)-deformed classical and quantum Virasoro and \( \mathcal{W}_N \) algebras. Such algebraic structures have been previously defined using explicit representations by \( q \)-deformed bosonic operators, associated to the collective variable representation of the relativistic Ruijsenaars-Schneider model [4, 5] or alternatively to the \( q \)-deformed version of the Miura transformation for the quantum group \( \mathcal{U}_q(sl(N))_c \) [6, 7, 8]. These explicit constructions yielded centrally extended \( q \)-deformed algebras due to particular representation-dependent relations arising between their generating operators. By contrast, our construction relied on abstract algebraic relations stemming from the newly proposed elliptic quantum algebra \( A_{q,p}(sl(N))_c \) [9, 10]. The essential feature of our construction was the understanding of the crucial role played by the supplementary parameter \( p \) (elliptic nome). It provided us with the intermediate step in the process of constructing a Poisson bracket on an extended center of \( A_{q,p}(sl(N))_c \), realizing immediate quantizations of this Poisson structure inside the original algebraic object \( A_{q,p}(sl(N))_c \). Elliptic algebras were thus shown to be a natural setting for generic construction of quantum \( q \)-\( \mathcal{W}_N \) algebras.

This procedure however does not naturally give rise to centrally extended algebras and the problem must therefore be considered separately. Starting from the abstract, non-centrally extended quadratic structure, we shall define the general cocycle condition for central extensions and systematically look for its solutions.

As a first step we wish to present here a number of results concerning the easiest case of \( q \)-deformed Virasoro algebra. We shall first of all describe the classical cocycle equations and give explicit sets of solutions to them. At this time, we only have explicit solutions for the \( k = 0 \) sector, equivalent to the original classical construction [3]. They are much more general than the central extension given in [3] and we give a scheme of realization for them.

We then describe the quantum cocycle equations and give the general solution for the sector \( k = 0 \). This sector corresponds to an exchange function which is the square of the original exchange function in [3, 4]. Interestingly enough, the cocycle condition is expressed as a residue formula for a particular meromorphic function!

We finally make some comments and conclusive remarks.

2 Classical case

2.1 General form of central extensions

We consider a general quadratic Poisson algebra

\[
\{s_n, s_m\} = \sum_{l \in \mathbb{Z}} c_l s_{n-2l} s_{m+2l} \tag{2.1}
\]

where \( n, m \in 2\mathbb{Z} \) and because of antisymmetry:

\[
c_l = -c_{-l} \tag{2.2}
\]

Associativity (Jacobi) property is implied by antisymmetry. Indeed, imposing:

\[
\{s_n, \{s_m, s_r\}\} + \{s_m, \{s_r, s_n\}\} + \{s_r, \{s_n, s_m\}\} = 0 \tag{2.3}
\]
one obtains the conditions (where \(\forall n, m, r \in 2\mathbb{Z}\)):
\[
\sum_{l,j \in \mathbb{Z}} c_l c_j \left(s_{n-2j}s_{m-2l+2j}s_{r+l} + s_{m-2j}s_{n-2j}s_{r+l+2j} + s_{n-2j}s_{n-2j}s_{r+2l+2j} + s_{m-2j}s_{r-2l+2j}s_{n+2l}ight) = 0,
\]
which can be rewritten as:
\[
\sum_{l,j \in \mathbb{Z}} s_{n-2j}s_{m-2l+2j}s_{r+l} (c_l c_j + c_{l-j} c_j + c_{j-l} c_{l-j} + c_{j-l} c_l + c_j c_{l-j}) = 0.
\]
Equations (2.4) are satisfied because of (2.2).

We now consider the general centrally extended version of algebra (2.1):
\[
\{s_n, s_m\} = \sum_{l \in \mathbb{Z}} c_l s_{n-2l}s_{m+2l} + h_{n,m}
\]
where \(h_{n,m}\) Poisson-commutes with \(s_r\) and satisfies the antisymmetry property:
\[
h_{n,m} = -h_{m,n}.
\]
The cocycle \(h_{n,m}\) can not be a trivial one (a coboundary). Indeed coboundary terms are generated by the redefinitions of the generators \(s_n\). Since the bracket is quadratic, a constant (non-dynamical) extension is generated specifically by shifting \(s_n\) by a constant \(\delta_n\). However, this would also simultaneously produce a linear term in (2.6), and no further redefinition of \(s_n\) by linear or higher terms may cancel such a linear term.

Condition (2.3) now reads for all \(m, n, r \in 2\mathbb{Z}\) (terms involving products of two \(c_l\)'s give no contributions because of (2.4)):
\[
\sum_{l \in 2\mathbb{Z}} \left[c_{l-r}\left(h_{n,m-l+r} - h_{m,n-l+r}\right) + c_{l-m}\left(h_{r,n-l+m} - h_{n,r-l+m}\right) + c_{l-m}\left(h_{m,r-l+n} - h_{r,m-l+n}\right)\right] s_l = 0.
\]
Condition (2.8) must be satisfied for all \(s_l\), leading to the final cocycle equation for \(h_{n,m}\) (\(m, n, r, l \in 2\mathbb{Z}\)):
\[
c_{l-r}\left(h_{n,m-l+r} - h_{m,n-l+r}\right) + c_{l-m}\left(h_{r,n-l+m} - h_{n,r-l+m}\right) + c_{l-m}\left(h_{m,r-l+n} - h_{r,m-l+n}\right) = 0.
\]
To solve it, we note that in the particular case:
\[
m = l = 0
\]
this equation implies:
\[
h_{n,r} = h_{0,n+r} \frac{c_r}{c_r} - \frac{c_n}{c_r}.
\]
Inserting (2.11) in (2.3), one obtains, for all \(m, n, r, l \in 2\mathbb{Z}\):
\[
h_{0,n+m+l} = \left[c_{l-r}\left(\frac{c_{m+l-r}}{c_{m+l-r}} - \frac{c_{n+r-l}}{c_{n+r-l}}\right) + c_{l-m}\left(\frac{c_{n-m+l}}{c_{n-m+l}} + \frac{c_{n+r-l}}{c_{n+r-l}}\right) + c_{l-m}\left(\frac{c_{r-n+l}}{c_{r-n+l}} - \frac{c_{m-l+n}}{c_{m-l+n}}\right)\right] = 0.
\]
Equations (2.11, 2.12) are thus equivalent to (2.9). Let us concentrate on (2.12); two possibilities arise:

i) the \( c \)-dependent factor is different from zero for every value of \( m + n + r - l \); in this case \( h_{0,M} = 0 \) for all \( M \); using equation (2.11) we obtain \( h_{n,m} = 0 \) for all \( n, m \), then the algebra (2.1) cannot be centrally extended.

ii) the \( c \)-dependent factor is equal to zero for some values of \( m + n + r - l \); in this case \( h_{0,M} \) is left arbitrary for these values of \( M \); using (2.11) we then find the general non-zero central extensions of (2.1).

Turning to our particular case we consider now the family of quadratic Poisson algebras indexed by a non-negative integer \( k \):

\[
\{s_n, s_m\}_{(k)} = \sum_{l \in \mathbb{Z}} c_l^{(k)} s_{n-2l} s_{m+2l}, \quad (n, m \in 2\mathbb{Z}),
\]  
(2.13)

with

\[
c_l^{(k)} = (-1)^{k+1} 2 \ln q \frac{q^{(2k+1)l} - q^{-(2k+1)l}}{q^l + q^{-l}}.
\]  
(2.14)

It is easy to check that in the case \( k = 0 \) the \( c \)-dependent factor in (2.12) vanishes for every \( m, n, r, l \); then \( h_{0,m} \) can take arbitrary values for every \( m \) and equation (2.11) gives:

\[
h_{n,m} = h_{0,n+m} \frac{q^{\frac{m-n}{2}} - q^{\frac{n-m}{2}}}{q^{\frac{m+n}{2}} - q^{\frac{-m+n}{2}}},
\]  
(2.15)

Grouping all the \( n + m \) depending factors, one concludes that algebra (2.13) in the case \( k = 0 \) has central extensions of the form:

\[
h_{n,m} = \xi_{n+m} \left( q^{\frac{1}{2} (m-n)} - q^{\frac{1}{2} (n-m)} \right),
\]  
(2.16)

where \( \xi_n \) is a completely arbitrary function.

Remark that the particular choice \( \xi_n = -2 \ln q \delta_{n,0} \) gives the central extension found by Frenkel and Reshetikhin \[6\] for the Poisson bracket (2.13) at \( k = 0 \). However our calculations show that this Poisson bracket allows for more general non-local central extensions of the form (2.16).

The situation is not fully clarified for \( k \neq 0 \). Equation (2.12) implies in fact by numerical calculations \( h_{0,0} = h_{0,1} = h_{0,2} = 0 \), but we have not yet concluded for or against the existence of higher label \( h_{0,n} \) central extensions (\( n > 2 \)).

### 2.2 Explicit construction of central extensions

We have showed that algebra (2.13) in the case \( k = 0 \) can be centrally extended in the following way:

\[
\{s_n, s_m\} = -2 \ln q \sum_{l \in \mathbb{Z}} \frac{q^l - q^{-l}}{q^l + q^{-l}} s_{n-2l} s_{m+2l} + \xi_{m+n} \left( q^{\frac{m-n}{2}} - q^{\frac{n-m}{2}} \right), \quad m, n \in 2\mathbb{Z}.
\]  
(2.17)

We now give an explicit realization of (2.17) in terms of elements belonging to the center of \( \mathcal{A}_{q,p}(\widehat{sl}(2)_c) \) at \( c = -2 \).

From the result of \[11\] we know that the operators:

\[
t(z) = \text{Tr} \left( L^+ (q^z z) L^- (z)^{-1} \right)
\]  
(2.18)
satisfy the Poisson algebra at $c = -2$:

$$\left\{ t(z), t(w) \right\} = \mathcal{Y}(z/w) t(z) t(w), \quad (2.19)$$

where

$$\mathcal{Y}(x) = - (2 \ln q) \left[ \sum_{n \geq 0} \left( \frac{2x^2q^{4n+2}}{1-x^2q^{4n+2}} - \frac{2x^{-2}q^{4n+2}}{1-x^{-2}q^{4n+2}} \right) + \sum_{n > 0} \left( -\frac{2x^2q^{4n}}{1-x^2q^{4n}} + \frac{2x^{-2}q^{4n}}{1-x^{-2}q^{4n}} \right) - \frac{x^2}{1-x^2} + \frac{x^{-2}}{1-x^{-2}} \right]. \quad (2.20)$$

Because of the property:

$$\mathcal{Y}(x) = -\mathcal{Y}(xq), \quad (2.21)$$

the field

$$h(z) = t(zq) \quad (2.22)$$

satisfies the Poisson brackets:

$$\left\{ h(z), t(w) \right\} = -\mathcal{Y}(z/w) h(z) t(w)$$

$$\left\{ h(z), h(w) \right\} = \mathcal{Y}(z/w) h(z) h(w). \quad (2.23)$$

Writing equations (2.19), (2.23) in terms of the modes:

$$t_n = \oint_C \frac{dz}{2\pi i z} z^{-n} t(z), \quad h_n = \oint_C \frac{dz}{2\pi i z} z^{-n} h(z), \quad (2.24)$$

and extending as previously done the validity of (2.19) to the level of analytic continuations of the structure function $\mathcal{Y}(z/w)$, we obtain three families of Poisson brackets depending on a non-negative integer $k$:

$$\left\{ t_n, t_m \right\} = \sum_{l \in \mathbb{Z}} c_l^{(k)} t_{n-2l} t_{m+2l}$$

$$\left\{ h_n, t_m \right\} = -\sum_{l \in \mathbb{Z}} c_l^{(k)} h_{n-2l} t_{m+2l} \quad (2.25)$$

$$\left\{ h_n, h_m \right\} = \sum_{l \in \mathbb{Z}} c_l^{(k)} h_{n-2l} h_{m+2l},$$

where $c_l^{(k)}$ is given by equation (2.14). We are interested in the case $k = 0$: we define the observable

$$s_n = q^{-\frac{n}{2}} t_n + q^{\frac{n}{2}} h_n \quad (2.26)$$

Using (2.23), the Poisson bracket now reads:

$$\left\{ s_n, s_m \right\} = -2 \ln q \left[ \sum_{l \in \mathbb{Z}} \frac{q^l - q^{-l}}{q^l + q^{-l}} s_{n-2l} s_{m+2l} - \sum_{l \in \mathbb{Z}} (1 - q^{-2l}) q^{\frac{n-m}{2}} h_{n-2l} t_{m+2l} + \sum_{l \in \mathbb{Z}} (1 - q^{2l}) q^{\frac{m-n}{2}} h_{m+2l} t_{n-2l} \right]. \quad (2.27)$$
Remark that the analytic simplifications leading from (2.25) to (2.27) are specific of the behaviour of the structure coefficients \( c_i^{(k)} \) at \( k = 0 \), which gives a technical justification for the (easily proved) existence of central extensions at \( k = 0 \) and the difficulty to obtain central extensions at \( k \neq 0 \).

Performing the shifts:

\[
l \to l - \frac{m}{2}, \quad l \to \frac{n}{2} - l,
\]

(2.28)

respectively in the penultimate and in the last term of (2.27), we can simplify them:

\[
\{s_n, s_m\} = -2 \ln q \left[ \sum_{l \in \mathbb{Z}} \frac{q^l - q^{-l}}{q^l + q^{-l}} s_{n-2l} s_{m+2l} + \left( q^\frac{m+n}{2} - q^{-\frac{m+n}{2}} \right) \sum_{l \in \mathbb{Z}} h_{n+m-2l} t_{2l} \right].
\]

(2.29)

We now prove that the last term is central in the algebra generated by \( t_n \) and \( h_n \), that is:

\[
\{t_r, \sum_{l \in \mathbb{Z}} h_{n+m-2l} t_{2l}\} = 0
\]

(2.30)

\[
\{h_r, \sum_{l \in \mathbb{Z}} h_{n+m-2l} t_{2l}\} = 0.
\]

Let us consider the first Poisson bracket. The use of (2.25) for \( k = 0 \) gives:

\[
\{t_r, \sum_{l \in \mathbb{Z}} h_{n+m-2l} t_{2l}\} = -2 \ln q \sum_{l,l' \in \mathbb{Z}} \left( \frac{q'^l - q'^{-l}}{q'^l + q'^{-l}} h_{n+m-2l+2r} t_{r+2l} + \frac{q'^{l'} - q'^{-l'}}{q'^{l'} + q'^{-l'}} h_{n+m-2l'} t_{r+2l'} t_{2l} \right)\).
\]

(2.31)

Making the changes of variables \( l' \to -l' \) and \( l \to l + l' \) in the last term of (2.31), one obtains immediately the first identity of (2.30). The second one is proved in an identical way.

Any centrally extended algebra (2.6)-(2.10) may thus be generated by the combination of the abstract generators \( t_n, h_m \) in (2.23), after setting each constant quantity \(-2 \ln q \sum_{l \in \mathbb{Z}} h_{n-2l} t_{2l}\) to a fixed value \( \xi_n \). Notice that the Frenkel-Reshetikhin construction uses a representation of \( t \) and \( h \) such that \( h(z) \equiv t(z)^{-1} \), hence indeed \( \xi_n = -2 \ln q \delta_{n,0} \).

3 Quantum case

We start from the general symmetrized exchange relation [3, 4]

\[
\sum_{l \in \mathbb{Z}} f_l (t_{n-l} t_{m+l} - t_{m-l} t_{n+l}) = 0.
\]

(3.1)

The structure constants \( f_l \) are given by [4] (hence the denomination “symmetrized”):

\[
f_l = \frac{1}{2} \left( f_l^{(0)} + f_l^{(1)} \right),
\]

(3.2)

where \( f_l^{(0)} \) and \( f_l^{(1)} \) are respectively the coefficients of the series expansion of the meromorphic function \( f(z) \) inside the convergence disk \(|z| < 1\) and in the convergence ring \( 1 < |z| < |p^{-\frac{1}{2}} q^{-1}|\) with

\[
f(z) = \frac{1}{(1 - z^2)^2} \prod_{n \geq 0} \frac{(1 - z^2 p q^{4n})^2 (1 - z^2 p^{-1} q^{4n+2})^2}{(1 - z^2 p q^{4n+2})^2 (1 - z^2 p^{-1} q^{4n+4})^2}.
\]

(3.3)
Equation (3.1) can be written as:
\[
[t_n, t_m] = \sum_{l \in 2\mathbb{Z}^*} -f_l (t_{n-l}t_{m+l} - t_{m-l}t_{n+l}).
\]
(3.4)

Hence a centrally extended version of (3.4) must take the form
\[
[t_n, t_m] = \sum_{l \in 2\mathbb{Z}^*} -f_l (t_{n-l}t_{m+l} - t_{m-l}t_{n+l}) + h_{n,m}.
\]
(3.5)

In order to determine the solutions of (3.5), one writes the Jacobi identity:
\[
[t_n, [t_m, t_r]] + [t_m, [t_r, t_n]] + [t_r, [t_n, t_m]] = 0.
\]
(3.6)

Inserting (3.5) into (3.6), one obtains a condition involving trilinear terms of the form \(t_{m-k}h_{r,n+k}f_k\) and quartic terms of the form \(t_{m-k} - l h_{r+n,k}f_kf_l\), summed over \(k, l \in 2\mathbb{Z}^*\). These sums can be rewritten by summing over the indices \(k, l \in 2\mathbb{Z}\), \(f_k\) being replaced by \(f'_k = f_k - \delta_{k,0}\). Plugging the explicit expression of \(f'_k\), the sum with the trilinear terms vanishes and one is left with the cocycle condition:
\[
\sum_{k \in 2\mathbb{Z}} (f_{k-n}f_{s-r-k} - f_{n-k}f_{k+r-s} + f_{k-m}f_{s-n-k} - f_{m-k}f_{k+n-s} + f_{k-r}f_{s-m-k} - f_{r-k}f_{k+m-s}) h_{s-k,k} = 0.
\]
(3.7)

which has to be satisfied for all possible triplets \(n, m, r \in 2\mathbb{Z}\) and for all values of \(s \in 2\mathbb{Z}\).

We will first focus on solutions of the form \(h_{n,m} = \lambda_m \delta_{n+m,0}\), i.e. the subset of cocycle equations for \(s = 0\). The cocycle condition (3.7) becomes:
\[
\sum_{k \in 2\mathbb{Z}} (f_{k-n}f_{r-k} - f_{n-k}f_{k+r} + f_{k-m}f_{n-k} - f_{m-k}f_{k+n} + f_{k-r}f_{m-k} - f_{r-k}f_{k+m}) \lambda_k = 0.
\]
(3.8)

The resolution of the cocycle condition will be done in four steps. In the first step, we show that if the solution of (3.8) exists, it is necessary of the form \(\lambda_k = q^k - q^{-k}\). In the second step, we prove that the series expansions that appears in (3.8) can then be resummed consistently. In the third step, we transform the derived equation into a contour integral that allows to achieve the proof. In the last step, we establish the immediate generalization of the form of the cocycle term for \(s \neq 0\) in (3.8).

**Step 1:**
The form of the coefficients \(\lambda_k\) can be guessed as follows. Consider all possible triplets \((n, m, r)\) such that \(\max(n, m, r) \leq M\) where \(M\) is a fixed integer and write all cocycle conditions (3.5) corresponding to these triplets. One assumes of course that (3.8) are absolutely convergent series which can be manipulated consistently. This will then be proved in Step 2 in order to guarantee the consistency of the overall derivation.

It appears that the obtained relations differ only by finite numbers of terms. One gets in this way a finite set of equations, each one having a finite number of terms, involving the coefficients \(\lambda_k\) with \(|k| \leq M\). It is then possible to solve these equations step by step. Numerical calculations for \(M = 4, 6, 8\) show that the solution is necessarily of the form \(\lambda_k = q^k - q^{-k}\) (up to an overall factor).
Therefore, we only need to show that the cocycle condition (3.8) with the choice \( \lambda_k = q^k - q^{-k} \) for all \( k \in 2\mathbb{Z} \) is satisfied, that is:

\[
\begin{align*}
\sum_{k \in 2\mathbb{Z}} (f_{k-n}^0 f_{-r-k}^0 - f_{n-k}^0 f_{k+r}^0 + f_{-m-k}^0 f_{k+n}^0 - f_{m-k}^0 f_{k+m}^0 - f_{r-k}^0 f_{k-m}^0) (q^k - q^{-k}) + \\
\sum_{k \in 2\mathbb{Z}} \alpha (m - n)(f_{k+r}^0 + f_{-r-k}^0) + (r - m)(f_{k+n}^0 + f_{-k-n}^0) + (n - r)(f_{k+m}^0 + f_{-k-m}^0) (q^k - q^{-k}) = 0
\end{align*}
\]

(3.9)

where \( \alpha = -\frac{1}{4}(1 - x^2)^2 f(x) \bigg|_{x=1} \).

For convenience, we denote by \( S_1 \) and \( S_2 \) respectively the first and second sums in (3.9). Remark that \( S_1 \) is in fact a finite sum since \( f_l^0 = 0 \) for \( l < 0 \).

**Step 2:**

**Lemma 1** The sum \( S_2 \) is a convergent series expansion, the sum of which is equal to

\[
S_2 = -\alpha \left( f(q) + f(q^{-1}) \right) \left( (m - n)(q^r - q^{-r}) + (n - r)(q^m - q^{-m}) + (r - m)(q^n - q^{-n}) \right).
\]

(3.10)

**Proof:**

One has

\[
S_2 = \sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} q^k f_{k+r}^0 - \sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} q^{-k} f_{-k-r}^0 + \sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} (q^k f_{-k-r}^0 - q^{-k} f_{k+r}^0).
\]

(3.11)

The first two sums are convergent and their sum is equal to \( \sum_{cyclic(m,n,r)} -\alpha(m - n)(q^r - q^{-r}) f(q) \). To study the last sum, let \( f_l^1 \) be the coefficients of the series expansion of \( f(x) \) in the convergence ring \( 1 < |x| < |p^{\frac{1}{2}} q^{-1}| \). One has \( f_l^1 = f_l^0 + 2\alpha l + \beta \) where \( \beta \) is a constant, \( \alpha \) is defined above and \( l \in \mathbb{Z} \).

The last sum can then be rewritten as

\[
\begin{align*}
\sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} \left( q^k (f_{k-r}^1 + 2\alpha(k + r) - \beta) - q^{-k} (f_{k-r}^1 - 2\alpha(k + r) - \beta) \right) \\
= \sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} (q^k f_{k-r}^1 - q^{-k} f_{k+r}^1) \\
+ \sum_{cyclic(m,n,r)} \alpha(m - n) \sum_{k \in 2\mathbb{Z}} \left( 2\alpha(k + r)(q^k + q^{-k}) - \beta(q^k - q^{-k}) \right).
\end{align*}
\]

(3.12)

The first sum is convergent and is equal to \( \sum_{cyclic(m,n,r)} -\alpha(m - n)(q^r - q^{-r}) f(q^{-1}) \), while the second sum is identically zero term by term due to cyclicity over the indices \( m, n, r \) since \( \sum_{cyclic(m,n,r)} (m - n) = 0 \) and \( \sum_{cyclic(m,n,r)} (m - n)r = 0 \).
It is understood at every step of the derivation that the cyclization must be done before the summation over $k \in 2\mathbb{Z}$.

This achieves the proof of Lemma 1.

Note that the definition of $\alpha$ (see above) and the expression of $f(z)$ (see (3.3)) implies that

$$- \alpha \left( f(q) + f(q^{-1}) \right) = \frac{1}{2} \frac{(p-1)^2(p-q^2)^2}{p^2(1-q^2)^2} \equiv \frac{1}{2} \mathcal{E}. \quad (3.13)$$

**Step 3:**

**Lemma 2** The cocycle condition (3.7) is equivalent to the following contour integral, the contour enclosing all the poles (including the pole at infinity) of the integrand (that is $0, \pm q, \pm 1, \pm q^{-1}, \infty$):

$$\oint \frac{dz}{2i\pi z} \left[ q^{-r} z^{r+n} f(z) f(zq^{-1}) - q^{r} z^{r+n} f(z) f(zq) + q^{r} z^{-r-n} f(z) f(zq^{-1}) ight.$$

$$- q^{-r} z^{-r-n} f(z) f(zq) + q^{-m} z^{m+r} f(z) f(zq^{-1}) - q^{m} z^{m+r} f(z) f(zq)$$

$$+ q^{m} z^{-m-r} f(z) f(zq^{-1}) - q^{-m} z^{-m-r} f(z) f(zq) + q^{-n} z^{n+m} f(z) f(zq^{-1}) - q^{n} z^{n-m} f(z) f(zq)$$

$$\left. - q^{n} z^{n+m} f(z) f(zq) + q^{-n} z^{-n-m} f(z) f(zq) \right] = 0. \quad (3.14)$$

**Proof:** Let us introduce the function $g_{\pm}(z)$ defined by:

$$g_{\pm}(z) \equiv \frac{(1 - p^{\pm1} z)^2(1 - p^{\mp1} q^{\pm2} z)^2}{(1 - z)^2(1 - q^{\pm2} z)^2}, \quad (3.15)$$

which satisfies

$$f(z) f(zq^{\pm1}) = g_{\pm}(z^2). \quad (3.16)$$

By virtue of (3.15)-(3.16), the integrand in (3.14) has double poles at $z = 0, z = \pm 1, z = \pm q^{\pm1}$, $z = \infty$. One has

$$\oint_{C_0} \frac{dz}{2i\pi z} z^{r+n} (q^{-r} f(z) f(zq^{-1}) - q^{r} f(z) f(zq)) = \oint \frac{dz}{2i\pi z} \sum_{l \geq 0} \sum_{l' \geq 0} f_l^0 f_{l'}^0 (q^{r'-r} - q^{r'+r}) z^{r+n+l+l'}$$

$$= \sum_{l \geq 0} \sum_{l' \geq 0} f_l^0 f_{l'}^0 (q^{r'-r} - q^{r'+r}) \delta_{n+r+l+l',0}$$

$$= \sum_{l \geq 0} f_l^0 f_{-l-n-r}^0 (q^{n+l} - q^{-n-l})$$

$$= \sum_{k \in 2\mathbb{Z}} f_{-k-n}^0 f_{-r-k}^0 (q^{k} - q^{-k}), \quad (3.17)$$

where the contour $C_0$ is a small circle around the origin such that $f(z)$ be analytic inside the contour $C_0$. Summing all the terms of the form (3.17) occuring in (3.14), one sees that the contribution of the pole at $z = 0$ in (3.14) is exactly equal to the first sum of (3.9).

Moreover, the integral (3.14) being invariant under the change $z \rightarrow z^{-1}$, the contribution of the pole at infinity is equal to the contribution to the pole at the origin.
It remains to compute the contributions of the poles at \( z = \pm 1 \), and \( z = \pm q^{\pm 1} \). The calculation is greatly simplified if one performs the change of variables \( w = z^2 \). One obtains:

\[
\oint_{C_{w_0}} \frac{dz}{2i\pi z} \left[ q^{-r}z^{r+n}f(z)f(z^{-1}) - q^r z^{r+n}f(z)f(zq^{-1}) + q^r z^{-r-n}f(z)f(zq) + q^r z^{r-n}f(z)f(zq^{-1}) \right]
\]

\[
= \oint_{C_{w_0}} \frac{dw}{2i\pi w} \left[ g_-(w) \left( q^{-r}w^{\frac{1}{2}(r+n)} + q^r w^{\frac{1}{2}(-r-n)} + q^{-m}w^{\frac{1}{2}(m+r)} + q^m w^{\frac{1}{2}(-m-r)} \right) + g_+(w) \left( -q^r w^{\frac{1}{2}(r+n)} - q^{-r} w^{\frac{1}{2}(-r-n)} \right) \right], \tag{3.18}
\]

where the contours \( C_{\pm \omega} \) are small circles around the points \( z = \pm \omega_0 \) and the contour \( C_{w_0} \) is a small circle around the point \( w = w_0 = z_0^2 \). Notice that the factor \( \frac{1}{2} \) coming from the integration measure \( dz/2i\pi z \to \frac{1}{2} dw/2i\pi w \) is compensated by the fact that when the variable \( z \) winds around \( z_0 \) and \( -z_0 \) once, the variable \( w \) winds around \( w_0 \) twice.

The contribution of the poles at \( w = q^2 \) comes from the relevant factors of \( g_- (w) \). It is given by:

\[
\mathcal{E} \left[ \left( \frac{r+n}{2} - 1 \right) q^n + \left( \frac{m+r}{2} - 1 \right) q^r + \left( \frac{n+m}{2} - 1 \right) q^m - \left( \frac{m+r}{2} + 1 \right) q^{-r} \right] - \left( \frac{n+m}{2} + 1 \right) q^{-m} - \left( \frac{r+n}{2} + 1 \right) q^{-n} + \mathcal{E}' \left( q^n + q^m + q^r + q^{-n} + q^{-m} + q^{-r} \right), \tag{3.19}
\]

where \( \mathcal{E} \) is given by (3.13) and \( \mathcal{E}' = -2\mathcal{E} \left( \frac{1}{p^{-1} - 1} + \frac{1}{q^{-2}p^{-1} - 1} - \frac{1}{q^2 - 1} \right) \).

In the same way, the contribution of the poles at \( w = q^{-2} \) comes from the relevant factors of \( g_+ (w) \). It is given by:

\[
\mathcal{E} \left[ \left( \frac{m+r}{2} + 1 \right) q^r + \left( \frac{n+m}{2} + 1 \right) q^m + \left( \frac{r+n}{2} + 1 \right) q^n - \left( \frac{m+r}{2} - 1 \right) q^{-r} \right] - \left( \frac{n+m}{2} - 1 \right) q^{-m} - \left( \frac{r+n}{2} - 1 \right) q^{-n} + \mathcal{E}'' \left( q^n + q^m + q^r + q^{-n} + q^{-m} + q^{-r} \right), \tag{3.20}
\]

where \( \mathcal{E}'' = 2\mathcal{E} \left( \frac{1}{p^{-1} + 1} + \frac{1}{q^2p^{-1} - 1} - \frac{1}{q^2 - 1} \right) \).

Finally, the contribution of the poles at \( w = 1 \) comes from the relevant factors of both \( g_- (w) \) and \( g_+ (w) \). It is given by:

\[
\mathcal{E} \left[ \left( \frac{r+n}{2} - 1 \right) q^{-r} + \left( \frac{m+r}{2} - 1 \right) q^{-m} + \left( \frac{n+m}{2} - 1 \right) q^{-n} - \left( \frac{r+n}{2} + 1 \right) q^r \right] - \left( \frac{m+r}{2} + 1 \right) q^m - \left( \frac{n+m}{2} + 1 \right) q^n - \mathcal{E}'' \left( q^n + q^m + q^r + q^{-n} + q^{-m} + q^{-r} \right) + \mathcal{E} \left[ -\left( \frac{r+n}{2} - 1 \right) q^r - \left( \frac{r+m}{2} - 1 \right) q^m - \left( \frac{n+m}{2} - 1 \right) q^n + \left( \frac{r+n}{2} + 1 \right) q^{-r} \right] + \mathcal{E}'' \left( q^n + q^m + q^r + q^{-n} + q^{-m} + q^{-r} \right). \tag{3.21}
\]
Summing up all contributions, one is left with

\[ \mathcal{E} \left[ (m - n)(q^r - q^{-r}) + (n - r)(q^m - q^{-m}) + (r - m)(q^n - q^{-n}) \right], \tag{3.22} \]

which is exactly the expression given in eq. (3.10) up to a factor 2.

Finally, the contour integral (3.14) is equal to twice the cocycle condition (3.9). The function to be integrated in (3.14) being meromorphic and the contour of integration encircling all the poles of the function (including the pole at infinity), one concludes that the contour integral (3.14) is equal to zero. It follows that the cocycle condition (3.9) is identically satisfied.

**Step 4:**

We want now to relax the constraint \( h_{m,n} = \lambda_m \delta_{n+m,0} \). Let us return to the general cocycle condition (3.7) which has to be satisfied for any fixed value \( s \in 2\mathbb{Z} \).

- **case 1:** \( s \in 4\mathbb{Z} \)

  We perform the change of variable \( m' = m - \frac{1}{2}s, n' = n - \frac{1}{2}s, r' = r - \frac{1}{2}s \) and \( k' = k - \frac{1}{2}s \). (3.7) is then written as:

  \[
  \sum_{k' \in 2\mathbb{Z}} (f_{k' - n'} f_{r' - k'} - f_{n' - k'} f_{k' + r'} + f_{k' - m'} f_{n' - k'} - f_{m' - k'} f_{k' + n'} + f_{k' - r'} f_{m' - k'} - f_{r' - k'} f_{k' + m'}) h_{\frac{1}{2} s - k', \frac{1}{2} s + k'} = 0. \tag{3.23}
  \]

  From the previous results, this condition is satisfied, for each value \( s \in 4\mathbb{Z} \), by

  \[
  h_{\frac{1}{2} s - k', \frac{1}{2} s + k'} = \xi_s (q^{k'} - q^{-k'}) \tag{3.24}
  \]

- **case 2:** \( s \in 4\mathbb{Z} + 2 \)

  The previous shift now leads to the condition

  \[
  \sum_{k' \in 2\mathbb{Z} + 1} (f_{k' - n'} f_{r' - k'} - f_{n' - k'} f_{k' + r'} + f_{k' - m'} f_{n' - k'} - f_{m' - k'} f_{k' + n'} + f_{k' - r'} f_{m' - k'} - f_{r' - k'} f_{k' + m'}) h_{\frac{1}{2} s - k', \frac{1}{2} s + k'} = 0, \tag{3.25}
  \]

  where \( m', n', r', \frac{1}{2}s \in 2\mathbb{Z} + 1 \).

  As before, the condition (3.25) is satisfied by \( h_{\frac{1}{2} s - k', \frac{1}{2} s + k'} = \xi_s (q^{k'} - q^{-k'}) \) for each value \( s \in 2\mathbb{Z} + 1 \). Indeed, the proof runs along the same lines as for (3.9), in particular the contour integral formulation also holds.

The results above can then be summarized in the following theorem:

**Theorem 1** The \( q \)-deformed Virasoro algebra \((m, n \in 2\mathbb{Z})\)

\[
\left[ t_n, t_m \right] = \sum_{l \in 2\mathbb{Z}} f_l (t_{n-l} t_{m+l} - t_{m-l} t_{n+l})
\]

where the structure constants \( f_l \) are given by (3.2)-(3.3), admits a central extension of the form

\[
h_{n,m} = \xi_{n+m} (q^{\frac{1}{2}(n-m)} - q^{\frac{1}{2}(m-n)})
\]

where \( \xi_n \) is a completely arbitrary function.
We must remark that the cocycle condition (3.7) cannot be fulfilled if one chooses \( f_l = f_l^0 \) instead of the symmetrized coefficient \( \frac{1}{2}(f_l^0 + f_l^1) \): the contour integral argument is not applicable in this case since the cocycle equation contains only the sum \( S_1 \) which does not vanish.

In addition, the replacement of the \((k = 0)\) symmetrized structure coefficients in (3.4) by higher label structure functions containing contributions from other poles of \( \mathcal{Y}(z/w) \) leads to serious difficulties in the evaluation of the cocycle condition (3.9) since the sum \( S_2 \) then becomes divergent owing to the presence of quadratic and higher-power terms in the structure coefficients \( f_l \) compared to the simple linear dependance in \( f_l^{(1)} \) which is canceled by permutation arguments.

It seems therefore that the symmetrized \((k = 0)\) sector for the structure function (3.3) is the only one admitting consistent central extensions.

4 Perspectives

The results which we have obtained here, and the limitations which we have encountered, clearly indicate two further directions of investigations. One is the study of more general central plus linear extensions, both classical and quantum. The cocycle conditions are necessarily much more complicated, involving two sets of unknowns \( c_{nm} \) (linear) and \( h_{nm} \) (constants); furthermore, there now exist coboundary-type extensions as follows from the comment in section 1.

On the other hand this increase in the number of degrees of freedom relaxes the constraints in the resolution of the cocycle equations, leading us to hope for the existence of solutions for all structure functions.

The other direction in which spirit this investigation has been pursued is the search for explicit operator representations of the centrally extended algebras. Based on the previous constructions \[6, 7, 8\] we indeed expect that such representations will contain central (or linear?) extensions, motivating our current study.

At this time, only for the simplest structure function originally obtained in [4] were such constructions proposed, using deformed bosons [7, 8]. Even our simplest structure function [1], essentially equal to the square of the structure function for [4] has yet no explicit simple operatorial representation. It actually seems [11] that mere simple extensions by “normalizations” of the vertex operator constructions in [7, 8] do not lead to realizations of this quantum algebra. The problem is thus fully open.
References

[1] J. Avan, L. Frappat, M. Rossi, P. Sorba, Poisson structures on the center of the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$, Phys. Lett. A 235 (1997) 323.

[2] J. Avan, L. Frappat, M. Rossi, P. Sorba, New $\mathcal{W}_{q,p}(sl(2))$ algebras from the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$, Phys. Lett. A 239 (1998) 27.

[3] J. Avan, L. Frappat, M. Rossi, P. Sorba, Deformed $\mathcal{W}_N$ algebras from elliptic $sl(N)$ algebras, math.QA/9801105.

[4] J. Shiraishi, H. Kubo, H. Awata, S. Odake, A Quantum Deformation of the Virasoro algebra and the Macdonald symmetric functions, Lett. Math. Phys. 38 (1996) 33.

[5] H. Awata, H. Kubo, S. Odake, J. Shiraishi, Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials, Commun. Math. Phys. 179 (1996) 401.

[6] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and $W$-algebras, Commun. Math. Phys. 178 (1996) 237.

[7] B. Feigin, E. Frenkel, Quantum $W$ algebras and elliptic algebras, Commun. Math. Phys. 178 (1996) 653.

[8] P. Bouwknegt, K. Pilch, The deformed Virasoro algebra at roots of unity, q-alg/9710026.

[9] M. Jimbo, H. Konno, T. Miwa, Massless XXZ model and degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$, hep-th/9610079, Proceedings of the conference “Deformation Theory, Symplectic Geometry and Applications”, Ascona, Switzerland, 1996.

[10] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, Quasi-Hopf twistors for elliptic quantum groups, q-alg/9712029.

[11] J. Avan, L. Frappat, M. Rossi, P. Sorba, unpublished preliminary computations.