Non-equilibrium cumulants within model A near the QCD critical point

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We study the non-equilibrium cumulants of the \( \sigma \) field in the phase transition region via Langevin dynamics. Cumulants up to fourth-order have been calculated based on the spacetime-dependent \( \sigma \) configurations from the event-by-event numerical simulations. By limiting the cooling of the system in a Hubble-like way, the out-of-equilibrium cumulants illustrate clear memory effects during the evolution. Both the signs and the magnitudes of the high-order cumulants differ from the equilibrium ones below the phase transition temperature. At the same time, the dynamical cumulants grow more intensively from the first-order phase transition side than they do from the crossover side. In addition, analysis of the high-order off-equilibrium cumulants on the hypothetical freeze-out lines present non-monotonic curves in the critical region.

I. INTRODUCTION

The existence and the location of the critical point (CP) of quantum chromodynamics (QCD) matter is a fundamental topic in understanding the structure of the QCD phase diagram, which have been extensively studied for decades [1–12]. Induced by the divergent fluctuations, theoretical investigation predicts a dramatic increase of the high-order cumulants of the final state proton multiplicities at the CP, as a consequence of coupling with the slow mode of \( \sigma \) field (the order parameter of chiral phase transition). It is also found that the higher-order correlators are more sensitive to the existence of the CP since they are proportional to the higher power of the correlation length [13–17]. In addition, the quartic cumulant of baryon number is predicted to be negative as approaching the CP from the crossover side, but positive from the first-order phase transition side. Thus, the kurtosis is expected to present a non-monotonic behavior in the vicinity of the CP [18, 19].

The experimental exploration of the QCD phase diagram is performed at the BNL Relativistic Heavy Ion Collider (RHIC). The STAR collaboration has measured the high-order cumulants of net protons in \( \text{Au}+\text{Au} \) collisions, with energies ranging from 7.7 to 200 GeV [9, 20, 21]. The experimental data of \( \kappa \sigma^2 \left( \kappa \sigma^2 = C_4/C_2 \right) \) shows a large deviation from Poisson statistics at collision energies below 39 GeV and non-monotonic curves at medium collision energies [21, 22]. These measurements are qualitatively in accord with the theoretical predictions, but cannot be quantitatively explained by the model analysis. Meanwhile, more experimental measurements are executed by the HADES collaboration at GSI [23], and by the ALICE Collaboration at the LHC [24]. Future experimental facilities such as FAIR in Darmstadt, NICA in Dubna, HIAF in Huizhou and J-PARC in Tokai will further explore the QCD phase transition at high baryon density region.

Since the high-order cumulants of protons in experiments are measured after the hadrons chemically freeze out, to identify the unique signature of the CP in heavy ion collisions, the correlated fluctuations of net protons on the hydrodynamic freeze-out surface are investigated after introducing a proper treatment of the freeze-out scheme in the Refs [25–27]. In these studies, one of our authors and her collaborators have found that the model calculations of static critical fluctuations qualitatively described the acceptance dependence of the STAR data and roughly fitted the \( C_4 \) and \( \kappa \sigma^2 \) data by tuning the model parameters, but the magnitude of \( C_2 \) and \( C_3 \) were overestimated due to the positive contributions of equilibrium critical fluctuations. It is suggested that a fully dynamical model is necessary to interpret the experimental outcomes consistently [28].

Indeed, the off-equilibrium process plays an essential role in the dynamical evolution with phase transition to address the critical slowing down near a CP [29], and to reveal the domain formation at the first-order phase transition [30]. According to the classification scheme, the dynamic universality class of the QCD phase transition in heavy ion collisions is that of Model H, since it is governed by five conserved hydrodynamic modes [31–37]. Complete dynamical models such as chiral hydrodynamics [38–41] and Hydro+ [42–44] have been developed to describe the bulk evolution of the critical fluctuations of QCD matter in heavy ion collisions. However, numerical computations on these models are complicated and time-consuming, especially when it comes to examining the high-order fluctuations [45]. Thus, it’s still worth employing the simple relaxational Model A or diffusive Model B to study the dynamical phase transition [46–52]. As shown in the work of [46–49], the high-order cumulants vary significantly, not only in the magnitudes but also the signs, in the real-time evolution compared with the equilibrium hypothesis. Furthermore, in Ref. [50], it is pointed out that the early time fluctuations and the critical enhancements around the CP can be probed by the rapidity window dependent Gaussian cumulant. In Ref. [51], the effects of the nonlinear coupling and finite size are manifested through the reduction of the correlation length near the pseudo-critical temperature. In Ref. [52], the well separated equilibration time of nonhydrodynamic quasiregular modes in the different channels are investigated at the CP by a phenomenological realistic holographic model, likely to be model B.

Taking into account the apparently different out-of-equilibrium influences, we continue the previous study [25–
...27] and simulate the dynamical processes of the critical field within the Langevin framework of Model A based on an event-by-event analysis. We present a detailed study of the dynamical cumulants in different phase transition scenarios and on the hypothetical freeze-out lines. In addition, we note that the method of Fokker-Plank equations [46,47], the ensemble dual of the Langevin equation, is equivalent to the Langevin dynamics for Markov processes. On the other hand, it is an open question for the equivalence between the two descriptions when considering the non-Markov effects in heavy ion collisions.

The paper is organized as follows: In Sec. II, we introduce the dynamical model in detail where the necessary input parameters are exhibited to solve the stochastic equation, including the initial profile, expansion routine of temperature, damping coefficient, and the magnitude of noise. In Sec. III, we present the numerical results of the field’s cumulants along the given evolution trajectories in both the crossover and the first-order phase transition scenario. Afterwards, we illuminate the results of non-equilibrium cumulants on the hypothetical freeze-out lines. In Sec. IV, we close the article by summarizing our main results and discussing future developments.

II. MODEL AND SET-UPS

Linear sigma model is a QCD-inspired low energy effective theory, plotting the phase structure of strongly interacting matter in the (μ, T) plane via order parameter σ = ⟨ϕ⟩ [53,54]. As the mass of the field approaches to zero near the CP, its correlation length grows to infinity. The corresponding equation of motion of the long wavelength mode is well described by the Langevin equation [39]:

\[ \partial_t^2 \delta \phi(t,x) + \eta \partial_t \phi(t,x) + \frac{\delta V_{\text{eff}}(\phi)}{\delta \phi} = \xi(t,x), \quad (1) \]

where the effective potential of the field is explicitly written as

\[ V_{\text{eff}}(\phi) = U(\phi) + \Omega_{\bar{q}q}(\phi). \quad (2) \]

\( U(\phi) \) denotes the vacuum contribution and takes the form of

\[ U(\phi) = \frac{\lambda^2}{4} (\phi^2 - v^2)^2 - h_\sigma \phi - U_0. \quad (3) \]

The parameters \( \lambda, \sigma, h_\sigma, \) and \( U_0 \) are set by the hadrons properties at zero temperature. As the chiral symmetry is spontaneously broken in the vacuum, the nonzero expectation of the field is \( \langle \phi \rangle = f_\pi = 93 \text{ MeV} \) with \( \langle \bar{\phi} \phi \rangle = 0 \). In reality, the chiral symmetry is explicitly broken by the light quark mass, then the linear term is included with \( h_\sigma = f_\pi m_\sigma^2 \) and \( m_\sigma = 138 \text{ MeV} \). \( v^2 = f_\pi^2 - m_\sigma^2 / \lambda^2 \) and the mass of the field is \( m_\sigma \sim 600 \text{ MeV} \) by setting \( \lambda^2 = 20 \). The zero-point energy \( U_0 = m_\sigma^4 / (4 \lambda^2) - f_\pi^2 m_\pi^2 \).

Note that we have neglected the meson fluctuations of \( \bar{\phi} \), since the mass of the triplet is finite in the critical regime. \( \Omega_{\bar{q}q} \) represents the contributions from the thermal quarks, which is

\[ \Omega_{\bar{q}q}(\phi; T, \mu) = -d_q \int \frac{d^3 p}{(2\pi)^3} [E + T \ln[1 + e^{-(E-p)/T}]] + T \ln[1 + e^{-(E+\mu)/T}]. \quad (4) \]

\( d_q = 12 \) is the degeneracy factor of the quarks. The energy dispersion of the valence quark is \( E = \sqrt{p^2 + m_q^2(\phi)} \) with the dynamical quark mass \( m_q(\phi) = m_0 + g_\sigma \phi \) [19,25]. For \( g = 3.3 \), the mass of the constituent quark is approximately 310 MeV, and the corresponding proton mass \( m_p \sim 930 \text{ MeV} \).

According to the effective potential of Eq. (2), the phase diagram is plotted in Fig. 1 as the function of \( (\mu, T) \). The solid line denotes crossover at small \( \mu \), and the solid line represents the first-order phase transition at large \( \mu \). The critical point locates at \( (\mu_c, T_{c}) \sim (205,100.2) \text{ MeV} \).

In Eq. (1), the damping coefficient \( \eta \) and the white noise \( \xi(t,x) \) originate from the interaction between the field and the heat bath, which is consisted of the thermal quarks. In this work, \( \eta \) is treated as a free parameter, within the range of models allowing, whose values \( \eta = 1, 3, 7 \text{ fm}^{-1} \) are taken in the following calculations. In the zero-momentum mode limit, the correlation of the noise has the form \( \langle \xi(t) \xi(t') \rangle = \frac{1}{m_\sigma^2} \coth \left( \frac{m_\sigma}{T} \right) \delta(t-t') \) [39]. We remark that the zero-momentum approximation only suits the CP scenario at the thermodynamic limit. In the realistic case with finite correlation length, we set the spatial noise at different time steps as \( \xi(x) = \sqrt{\frac{1}{m_\sigma^2} \coth \left( \frac{m_\sigma}{T} \right)} G(x) \), where \( G(x) \) is a random number generator of the standard normal distribution.

For our numerical implementation, we first construct the initial profiles of the field using the probability function:

\[ P[\phi] \sim \exp(-V(\phi)/T), \quad V(\phi) = \int d^4 x \frac{1}{2} (\nabla \phi(x))^2 + V_{\text{eff}}(\phi(x)). \]

In order to solve Eq. (1), the time-space information of the local temperature, \( T(t,x,y,z) \), and baryon chemical potential, \( \mu(t,x,y,z) \), have to be known which in principle shall be extracted from the heat bath. For simplicity, we assume the system evolves along the constant baryon density trajectories (seen traj. I and traj. II in Fig. 1), while the spatial-uniform temperature decreases in a Hubble-like way [46]:

\[ \frac{T(t)}{T_0} = \left( \frac{t}{t_0} \right)^{-0.45}, \quad (5) \]
where $T_0$ is the whole initial temperature, and $t_0 = 1$ fm is the initial time. The whole simulation is run in a $V = 6.8^3$ fm$^3$ box. The space step size $dx = dy = dz = 0.2$ fm and the time step size is $dt = 0.1$ fm/c. Note that the system volume will affect the configurations of $\sigma$ field in each event, due to the change in the magnitude of the noise term, but has no influence on the results of the event-averaged cumulants.

With all the ingredients in hand, we complete the event-by-event simulations of the $\sigma$ field from Eq. (1). In the numerical calculation process, we record the configurations of the $\sigma$ field at every time step over $10^3$ events. The moments of the $\sigma$ field are then calculated by:

$$\mu'_{n} = \langle \sigma^n \rangle = \frac{\int d\sigma \sigma^n P[\sigma(x)]}{\int d\sigma P[\sigma(x)]}, \quad (6)$$

where $\sigma = \frac{1}{V} \int d^3x \sigma(x)$. The non-equilibrium cumulants of the $\sigma$ field are iteratively determined by the following formula:

$$C_1 = \mu'_1, \quad (7)$$
$$C_2 = \mu'_2 - \mu'_1^2, \quad (8)$$
$$C_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'_1^3, \quad (9)$$
$$C_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'_2^2 + 12\mu'_2\mu'_1^2 - 6\mu'_1^4. \quad (10)$$

III. NUMERICAL RESULTS AND DISCUSSIONS

A. dynamical evolution in the crossover scenario — critical slowing down

We first solve the Langevin equation in the crossover scenario for $\mu = 200$ MeV. The effective potential and the probability distribution function for the phase transition region are shown in Fig. 2. Both the effective potential and the distribution function each have a dip or peak at the given temperature region. The system smoothly transits from the symmetry restored phase to the symmetry broken phase as the temperature decreases.

In Fig. 3, we plot the dynamical evolution of $\sigma$’s cumulants along traj. I as marked on the cartoon phase diagram of Fig. 1. The corresponding horizontal axis is the inverted temperature as time increases. The dashed lines represent the equilibrium cumulants, and the colored solid lines represent the non-equilibrium ones at a given damping coefficient. In the left panel, we show the behavior of $\sigma$’s cumulants at different damping coefficients. The initial configurations of $\sigma$ fields are constructed to satisfy the equilibrium distribution at $t_0$, thus both the non-equilibrium and equilibrium cumulants have the same values at the starting point. In the evolution, the non-equilibrium cumulants present clear memory effects, going after the trends of equilibrium ones as temperature decreases and reaching their maxima (or minima) at later times. Finally, after the phase transition, the effective potential changes from a non-Gaussian shape to a Gaussian shape. As expected, the non-equilibrium $C_1$ goes to the equilibrium value and the high-order cumulants vanish at the broken phase. During the expansion, non-equilibrium $C_2$ decreases slightly at the earlier stage, and then grows due to the broadening of the effective potential in the critical region. We emphasize that the signs and values of the dynamical $C_{1,4}$ strongly differ from the equilibrium ones in a large $T$ region below $T_c$. For larger damping coefficients, the $\sigma$ field relaxes as slowly as it should and the system takes a longer time to approach equilibrium. Prolonging the duration to its out-of-equilibrium state, the maximum (or minimum) of high-order cumulants are enhanced as $\eta$ increases.

Besides the damping coefficients, the initial conditions also have significant influence on the magnitude of the non-equilibrium cumulants. In the right panel of Fig. 3, we exhibit the results of dynamical cumulants starting from three different initial temperature above $T_c$, with the damping coefficient fixed at $\eta = 3$ fm$^{-1}$. The initial $\sigma$ field configurations are again sampled according to the equilibrium distribution function and thus vary for different $T_0$. Again, the initial values of the $\sigma$’s cumulants are governed by the $\sigma$ field’s distribution at starting temperature. We find that the high-order cumulants are strongly enlarged while $T_0$ are close to $T_c$ and maintain substantial values during the later non-equilibrium development until below $T_c$. In addition, with the same damping coefficient, the cumulants reach their maxima (or minima) at approximately the same temperature, which is almost unrelated to their starting points.

B. dynamical evolution in the first-order phase transition scenario — supercooling effect

In this subsection, we discuss the dynamical evolution of $\sigma$’s cumulants in the first-order phase transition scenario, with the baryon chemical potential fixed at $\mu = 240$ MeV. As shown in the left panel of Fig. 4, the thermodynamic potentials are characterized by two co-existing phases near $T_c$. At the thermodynamic limit, the probability distribution function is double $\delta$-like only at the phase transition point. The $\sigma$ field stays at the global minimum of the effective potential, and there is a discontinuity at the phase transition temperature in the equilibrium cumulants. In turn, confining to a finite system volume ($V = 6.8^3$ fm$^3$), the probability distribution function presents two peaks with comparable probability in the phase transition region (as shown in the right panel of Fig. 4), which leads to significantly different behaviors of the high-order cumulants.

In Fig. 5, we present the numerical results of out-of-equilibrium cumulants as functions of decreasing temperature, as denoted by the traj. II in Fig. 1. The left panel represents the evolution of cumulants starting from a given set of $\sigma$’s configurations under different damping coefficients. For fixed $\eta = 3$ fm$^{-1}$, the right panel shows the cumulative behaviors starting at various initial temperature. Similar to the case in the crossover scenario, the diffusive dynamics render the same memory effects for the non-equilibrium cumulants. In addition, the non-equilibrium cumulants are continuous and much larger than that of the equilibrium ones.

The significant enhancements of the non-equilibrium cu-
mulants are explained in the following. In the first-order phase transition scenario, the existence of a barrier between the two minima in the thermodynamic potential prevents the $\sigma$’s configurations from shrinking to the global minimum even when the temperature is lower than $T_c$ (known as supercooling effects in thermodynamics). Since the $\sigma$ field is trapped in the original minima during the cooling down process, only the events with intense thermal fluctuations would overcome the potential barrier. Then as the broadening of the probability distribution function in a finite-size system, different events occupy both of the states with comparable weights, which leads to a strong departure of $C_{3,4}$ from the equilibrium cumulants. Such enhancements of cumulants at the boundary side of the first-order phase transition have the potential to address the large deviations of BES data $\kappa \sigma^2$ from the statistical baselines at low collision energies.

C. cumulants on the hypothetical freeze-out lines

The possible signals of phase transition are measured after the particles chemically freeze out, and in this subsection, we discuss the non-equilibrium cumulants’ behaviors on the hypothetical freeze-out lines as functions of baryon chemical potential. Within the present model setting, the phase transition line described by the linear sigma model is far away from the freeze-out line determined by the statistical model via fitting experimental data [55]. Thus, we are not able to directly borrow the freeze-out information from the experiments. In the following, we artificially choose the freeze-out lines and assume the dynamical evolution of $\sigma$ field along different trajectories starting at $T_0 = (T_c + 4)$ MeV, and freezing out at either $T_{f1} = (T_0 - 10)$ MeV or $T_{f2} = (T_0 - 15)$ MeV. In the duration for each trajectory, the baryon chemical potential is fixed.

In Fig. 6, we draw the $\sigma$’s cumulants as a function of the baryon chemical potential, adopting $T_{f1}$ and $T_{f2}$ as the freeze-out temperature individually. The high-order equilibrium cumulants decay to zero as the system is away from the phase transition region. Rather, the non-equilibrium cumulants exhibit large deviations from the equilibrium ones and significant non-monotonic structures on the hypothetical freeze-out line. Below $T_c$, the equilibrium $C_3$ is negative and limited to zero as temperature decreases. However, the non-equilibrium $C_3$ is positive in most phase space of $\mu$ for both applied freeze-out lines. The flipping of signs could address the sign problem based on the prediction of equilibrium critical fluctuations [25]. Last but not least, the non-equilibrium $C_4$ is oscil-
lating near the critical baryon chemical potential (∼ 205 MeV) and tends to zero around µ ∼ 270 MeV.

By comparing the vanishing equilibrium cumulants of C_{3,4}, the dynamical processes provide us abundant nontrivial behaviors. In our model calculation, the appearance of the non-monotonic curves of C_{4} originates from a combined effect of η and the hypothetical freeze-out line. At the first freeze-out temperature T_{f1}, the evolution of C_{4} at smaller η (red and green lines) show the non-monotonic structure, but they oscillate under the influence of larger η (green and blue lines) at freeze-out temperature T_{f2}. This means that C_{4} evolves following its own dynamical processes and its behavior is non-universal. Furthermore, the deviation of C_{4} not only comes from the development of itself but also from the non-equilibrium features of other cumulants since the higher order cumulants are coupled to the lower order ones as shown in Eq.(7-10).

The peaks of C_{3} and C_{4} are worth paying attention to, as well. Both of these maximums take place at a critical value around µ ∼ 240 MeV. Such maximums are induced by the maximization of the supercooling effect in the current model. With µ slightly larger than µ_{c}, the barrier between the global minima and the false minima prevents a critical number of the events from developing to the global minimum. The σ field is approximately evenly distributed in both minima for different events, and induces a dramatically large peak of the high-order cumulants. At the high baryon chemical potential, µ ∼ 270 MeV, the barrier is so strong that the σ field is trapped in the original minima and can not escape even at a temperature much lower than T_{c}. Without the fluctuations manifesting the phase transition, the high-order cumulants are suppressed, and their magnitude approaches the equilibrium limit.

In spite of the various cumulative behaviors found in the first-order phase transition scenario, we note that one should be cautious when comparing the current model simulation at high baryon chemical potential region to the experimental data, since a mismatch between the model and the experiments at small collision energies is likely to occur. In this work, we have presumed that system evolution occurs when the chiral symmetry is restored, but quark gluon plasma may not be created in heavy ion collisions at low collision energies. A sophisticated first-order phase transition with the proper degree of freedom should be further explored and studied in future work.

IV. SUMMARY AND OUTLOOK

To summarize, in this work, we study the dynamical evolution of the σ field based on the event-by-event simulations
of a single component Langevin equation. The temperature decrease for the system is set to be Hubble like and the computation is completed in a finite size system. We statistically weight the dynamical variable $\sigma$ over $10^5$ events during the real-time evolution to obtain its high-order cumulants.

With the current model setting, we find that the non-equilibrium cumulants express clear memory effects, and the magnitude of $C_2$ slowly increases as the system approaches its critical regions. The signs of $C_{3,4}$, as well as the magnitudes, can differ from the equilibrium ones below $T_\epsilon$. We also find that the high-order cumulants are significantly enhanced on the boundary side of the first-order phase transition. Finally, the spread of out-of-equilibrium cumulants along the hypothetical freeze-out lines has been presented. The non-equilibrium $C_3$ is positive at large baryon chemical potentials, in contrast to the negative sign of the equilibrium $C_3$. In the vicinity of CP, with certain parameter sets, the non-equilibrium $C_4$ expresses non-monotonic curves at large $\mu$ region. We conclude that the combination of supercooling effect and dynamical effects on the first-order phase transition side plays a dominant role in the nonmonotonicity of the high-order cumulants on the hypothetical freeze-out lines.

As a critical approach, we have adopted a relatively simple setup for the modeling and the parameters. The dynamics of the cumulants that we have studied can provide reference information for the analysis of the experimental measurements, but are far from fitting the data. In order to quantitatively describe the experimental observed cumulants of net proton multiplicities, more sophisticated and realistic dynamical modeling are required from the theoretical side [56, 57]. Besides, a number of effects play their role, including the subject of the proper equation of state [58–60], of the unknown parameters of the Ising-to-QCD mapping [61], of the critical transport coefficients [62–65], of the finite size, finite size scaling and global charge conservation in the vicinity of a CP [66–70], of the non-critical baseline for the cumulants of net-proton number fluctuations [71], and of the nonuniform temperature/chemical potential effects [72]. In addition, further connections between the criticality and the experimental observables have been established through theoretical efforts [73–78]. New research into the deep learning technique [79–81] is developed to help confirm the QCD phase transition signals in experiments, as well.

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**APPENDIX**

the comparison of dynamical results from diffusion equation and Langevin equation

In the critical region, the evolution of the critical mode is diffusion-like because $\omega \ll \eta$. Here we compare the dynamical evolution between the Langevin equation and the over-damped diffusion equation numerically. Neglecting the second-order derivative term with respect to time, we obtain the over-damped diffusion equation in the form of the following formula:

$$\eta \delta_{\sigma} (t, x) - \Psi^2 \sigma (t, x) + \frac{\delta V_{eff} (\sigma)}{\delta \sigma} = \xi (t, x).$$ (11)

In Fig. 7, we plot the Langevin (solid lines) dynamics of $\sigma$’s cumulants and the over-damped diffusion equation (dotted lines) with three different damping coefficients at $\mu = 200$ MeV. In the figure, it is shown that the differences of $\sigma$’s cumulants decrease with an increase in the damping coefficient. At $\eta = 7$ fm$^{-1}$, the evolution differences between the two kinds of equations can be safely ignored.
FIG. 7: $\sigma$'s cumulants as functions of temperature, rendered by either the Langevin equation (solid lines) or the over-damped diffusion equation (dotted lines).

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