The paper is devoted to the analysis of relationships between principal objects of the spectral theory of dynamical systems (transfer and weighted shift operators) and basic characteristics of information theory and thermodynamic formalism (entropy and topological pressure). We present explicit formulae linking these objects with $t$-entropy and spectral potential. Herewith we uncover the role of inverse rami-rate, forward entropy along with essential set and the property of non-contractibility of a dynamical system.

**Keywords:** spectral potential, transfer operator, entropy, topological pressure, $t$-entropy, essential set, rami-rate, forward entropy

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1 Introduction

Transfer and weighted shift operators are the principal objects as in the theory of dynamical systems so also in numerous fields of analysis. One cannot also overestimate the role of entropy and topological pressure as in information theory so also in foundations of thermodynamic formalism.

The paper is devoted to the analysis of interrelations between the spectral radii of the mentioned operators and topological pressure, entropy, $t$-entropy and arising herewith dynamical and metric invariants.

Let $X$ be a Hausdorff compact space, $C(X)$ be the Banach space of continuous functions on $X$ equipped with the uniform norm, and $\alpha : X \to X$ be a continuous mapping. This mapping generates a dynamical system with discrete time which will be denoted by $(X, \alpha)$.

A linear operator $A : C(X) \to C(X)$ is called a transfer operator for the dynamical system $(X, \alpha)$ if

a) it is positive (i.e., it maps nonnegative functions to nonnegative ones) and

b) it satisfies the homological identity

$A((f \circ \alpha) \cdot g) = fAg, \quad f, g \in C(X).$ (1)

A typical (popular) example of a transfer operator is the classical Perron–Frobenius operator of the form

$A_f(x) := \sum_{y \in \alpha^{-1}(x)} a(y)f(y),$ (2)

where $a \in C(X)$ is a certain nonnegative function. This operator is well defined when $\alpha$ is a local homeomorphism and acts onto.

If $\alpha$ is a homeomorphism then transfer operator turns out to be a classical weighted shift (weighted composition) operator of the form

$Af(x) = a(x)f(\alpha^{-1}(x))$, (3)

where $a \in C(X)$ and $a \geq 0$.

Transfer and weighted shift operators have numerous applications in dynamical systems theory, mathematical physics and in particular in thermodynamics, stochastic processes, information theory, investigations of zeta functions, Fredholm determinants, and operator algebras theory. They serve as an inexhaustible source of important examples and counterexamples and so also as key constructive elements of the crossed product algebras, in the theory of solvability of functional differential equations, wavelet analysis, etc. We refer the reader to the books [Ant96, AL94, ABL98, Bal00, KS97, Pu90, Rue78, Rue91], recent papers [ABL11, ABL12, ABL03, BK10, Dd07, Ex03, Kit99, Kwa12, Kl13, Rou96], and the bibliography therein.

Given a transfer operator $A$ we define a family of operators $A_\psi : C(X) \to C(X)$ depending on the functional parameter $\psi \in C(X, \mathbb{R})$, where $C(X, \mathbb{R})$ is the space of continuous real-valued functions, by means of the formula

$A_\psi f := A(e^\psi f).$ (4)

Evidently, all the operators of this family are transfer operators as well. Let us denote by $\lambda(\psi)$ the logarithm of the spectral radius of $A_\psi$, that is

$\lambda(\psi) = \lim_{n \to \infty} \frac{1}{n} \ln \|A_\psi^n\|.$ (5)
The functional $\lambda(\psi)$ is called \textit{spectral potential} of $A$.

Spectral properties of weighted shift and transfer operators and especially the formulae and methods for calculation of their spectral radii are tightly related to the ergodic and entropy theory of dynamical systems via variational principles of thermodynamic and informational nature.

Namely, for the case when $\alpha$ is a homeomorphism and a transfer operator $A$ is a shift operator $Af(x) = f(\alpha^{-1}(x))$ the variational principle for spectral potential $\lambda(\psi)$ (i.e., variational principle for the spectral radius of weighted shift operator) has the form

$$\lambda(\psi) = \max_{\mu \in M_\alpha(X)} \mu[\psi] = \max_{\mu \in EM_\alpha(X)} \mu[\psi],$$

where $M_\alpha(X)$ is the set of all Borel $\alpha$-invariant probability measures and $EM_\alpha(X)$ is the set of all ergodic measures on $X$, and

$$\mu[\psi] := \int_X \psi \, d\mu.$$

This variational principle was established independently by Kitover [Kit79] and Lebedev [Leb79]. Its applications to elliptic theory of functional differential operators and spectral theory of operator algebras associated with automorphisms are presented in [Ant96], [AL94] and [ABL98]. A comprehensive analysis of the corresponding variational principles and their interrelations as with integrals so also with Lyapunov exponents for abstract weighted shift operators associated with endomorphisms of Banach algebras is presented in [KL20].

Spectral analysis of Perron–Frobenius operators (that is, transfer operators arising in the situation when $\alpha$ is a local homeomorphism) naturally involves an additional dynamical object, namely, topological pressure $P(\alpha, \psi), \psi \in C(X, \mathbb{R})$ (a detailed definition of topological pressure is given in Subsection 2.2). Topological pressure appears, in particular, in the analysis of complexity of dynamical systems $(X, \alpha)$, where $X$ is a compact metric space, and it is also a principal component of thermodynamic formalism.

Ruelle–Walters variational principle [Rue73, Rue89, Wal75, Wal82] expresses the topological pressure as

$$P(\alpha, \psi) = \sup_{\mu \in M_\alpha(X)} (\mu[\psi] + h_\alpha(\mu)),$$

where $h_\alpha(\mu)$ is Kolmogorov–Sinai entropy.

Let $X$ be a compact metric space, $\alpha : X \to X$ be a local homeomorphism, and

$$Af(x) = \sum_{y \in \alpha^{-1}(x)} f(y)$$

be the initial transfer operator in $C(X)$. Then one of the fundamental principles of thermodynamic formalism can be written as

$$\lambda(\psi) = P(\alpha, \psi) \text{ for an expanding map } \alpha.$$  \hfill (6)

A number of results in this direction is known, cf. [Bow75], [Rue78], [Wal78], [LS88], [Rue89], [LM98], [PJ01], [PU10]. However, none of these sources considers a general case: usually it is assumed that $\alpha$ is topologically mixing, $e^\psi$ is Holder continuous, and the space $X$ is a finite dimensional manifold or a shift space. Recently in [BK19] it is proven
that $\lambda(\psi) = P(\alpha, \psi)$ for an arbitrary open expanding map $\alpha : X \to X$ on a compact metric space and arbitrary function $\psi \in C(X, \mathbb{R})$.

Topological pressure itself is a weighted version of topological entropy $h(\alpha) = P(\alpha, 0)$, which is one of the principal ingredients of information theory. Here the corresponding variational principle was established by Dinaburg [Din70] and Goodman [Good71]:

$$h(\alpha) = \sup_{\mu \in M(\alpha)} h_\alpha(\mu).$$

The variational principle for spectral potential of a weighted shift and general transfer operators (i.e., in the situation when $X$ is not necessarily a metric space and $\alpha : X \to X$ is an arbitrary continuous mapping) was established in [B10], [ABL11] and has the form

$$\lambda(\psi) = \max_{\mu \in M_\alpha(X)} (\mu[\psi] + \tau(\mu)),$$

where $\tau(\mu)$ is a new entropy type object, called $t$-entropy, that depends not only on dynamical system $(X, \alpha)$ but on a generating transfer operator $A$ too (see [ABL11], [BL17]), and is a principal ingredient in entropy statistic theorem [BL19].

The latter variational principle means, in particular, that spectral potential $\lambda(\psi)$ is the Legendre transform of $-\tau(\mu)$ and this observation also naturally leads to its applications in thermodynamic formalism [ABLS03], [ABL12].

We have to emphasize that the aforementioned objects: spectral potential, topological pressure, entropy, $t$-entropy have different nature and origination and in general they can not be reduced to each other (see, in particular, Example 10). At the same time the phenomena when they do relate to each other play the principal role and clarify the internal structural basement of the corresponding fields of analysis (cf. (6)).

The goal of the paper is to uncover reasons for $\lambda(\psi), h(\alpha), P(\alpha, \psi), \tau(\mu)$, and $h_\alpha(\mu)$ to be related to each other and describe the arising relationships.

The article is organized as follows.

The starting Section 2 is devoted to the introductory overview of our principal operator, spectral, and dynamical heroes along with inevitably arising technical objects and instruments. Here we recall the notions of the spectral potential, $t$-entropy, topological entropy and pressure and the corresponding variational principles (Subsections 2.1, 2.2), introduce and examine in short the set $X_\alpha$ of essential points, i.e., the domain where all the variational principles in question live (see, in particular, Example 10). At the same time the phenomena when they do relate to each other play the principal role and clarify the internal structural basement of the corresponding fields of analysis (cf. (6)).

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The main part of the article starts with Section 3. Its goal is to bring to light the internal dynamical and metrical reasons for spectral potential, topological pressure and integrals with respect to invariant measures to be related to each other. The arising relations (estimates) are transparently presented in Theorems 9 and 16 by means of a subsidiary object — essential spectral potential and newly introduced dynamical invariants: rami-rate $\omega(\alpha)$ and forward entropy $\gamma(\alpha)$ that also serve as a convenient instrument for estimating and evaluating the topological entropy (Lemmas 7, 11 and Corollary 8). Moreover, we uncover here the dynamical-metric properties (Property (§) and Property (§§)) in the presence of which the estimates obtained become strict equalities (Theorems 12
In this context an important observation is revealing (in Lemma 14) of a wide class of dynamical systems (generated by open non-contracting mappings) possessing the mentioned properties.

In Section 4 we return back to the principal objects of analysis and highlight the situations when the spectral potential \( \lambda(\psi) \) is equal to topological pressure \( P(\alpha, \psi + \ln \rho) \) (Theorems 22 and 23). This is done on the base of the results obtained in the preceding section along with a thorough analysis of the properties of the cocycle \( \rho \) defining the ‘trace’ \( A_{X_\alpha} \) of the transfer operator \( A \) on the set \( X_\alpha \) of essential points (Lemma 18).

Up to Section 5 we consider mainly transfer operators \( A_\psi \) from (4) with positive weights \( e^\psi, \psi \in C(X, \mathbb{R}) \). After that in Section 5 we pursue the theme further and analyse the arising relationships between spectral radii, topological pressure and integrals for transfer operators with nonnegative (not necessarily positive) weights.

And finally in Section 6 we show how the results obtained can be used for explicit calculation of \( t \)-entropy by means of integrals and Kolmogorov–Sinai entropy.

2 Starters: spectral potential, topological pressure, entropy, \( t \)-entropy etc.

In this section we introduce and discuss a number of objects that will be inevitable in the analysis of problems in question.

2.1 \( T \)-entropy and variational principle for spectral potential

Let us start with the spectral potential \( \lambda(\psi) \) (5), i.e., the logarithm of the spectral radius of \( A_\psi \) (4).

The positivity of transfer operator implies that

\[
\lambda(\psi) = \lim_{n \to \infty} \frac{1}{n} \ln \| A_\psi^n \|, 
\]

where \( 1 \) is the unit function on \( X \), and \( \| \cdot \| \) denotes the uniform norm.

The principal object related to the spectral potential is \( t \)-entropy.

For an \( \alpha \)-invariant probability measure \( \mu \) (and only such measures will be essential in our considerations) \( t \)-entropy \( \tau(\mu) \) is defined in the following way [BL17]:

\[
\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{1}{n} \inf_G \sum_{g \in G} \mu[g] \ln \frac{\mu[A_\psi^n g]}{\mu[g]},
\]

where the infimum \( \inf_G \) is taken over the set of all continuous partitions of unity \( G \) in \( C(X) \) and we assume that if \( \mu[g] = 0 \) then the corresponding summand in the right hand part is equal to 0 independently of the value of \( \mu[A_\psi^n g] \).

Note parenthetically that if one identifies a Borel measure \( \mu \) on \( X \) with a linear functional \( \mu: C(X, \mathbb{R}) \to \mathbb{R} \) given by

\[
\mu[f] := \int_X f \, d\mu,
\]

then by Riesz’s theorem there exists a unique regular Borel measure on \( X \) defining the same functional. Thus, since in the foregoing definition of \( t \)-entropy only continuous
functions (forming partitions of unity) were exploited we can assume that \( t \)-entropy is defined namely for regular measures \( \mu \). By \( M_\alpha(X) \) we denote the set of regular \( \alpha \)-invariant probability measures.

Recently an essentially different formula for \( t \)-entropy was obtained. Namely, it is proven in [BL20] Theorem 21 that

\[
\tau(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_X \ln \left( \frac{d(A^n \mu)_{\alpha}}{d\mu} \right) d\mu,
\]

where \( A^*: C(X)^* \to C(X)^* \) is the operator adjoint to \( A \), and \( (A^n \mu)_{\alpha} \) is the absolutely continuous component of the measure \( A^n \mu \) in its decomposition into absolutely continuous and singular parts with respect to \( \mu \).

The relation between \( t \)-entropy and spectral potential is given by the next

**Theorem 1** (variational principle for spectral potential [ABL11] Theorem 5.6) Suppose \( A: C(X) \to C(X) \) is a transfer operator for a continuous mapping \( \alpha: X \to X \) of a Hausdorff compact space \( X \). Then its spectral potential \( \lambda(\psi) \) satisfies the variational principle

\[
\lambda(\psi) = \max_{\mu \in M_\alpha(X)} \left( \mu[\psi] + \tau(\mu) \right), \quad \psi \in C(X, \mathbb{R}).
\]

### 2.2 Entropy and topological pressure

Among the principal heroes in what follows will be the topological pressure and topological entropy. Therefore we recall the corresponding definitions.

These objects are defined for a dynamical system \((X, \alpha)\), where \( X \) is a compact metric space with a metric \( d \).

The definitions exploit the so-called \((n, \varepsilon)\)-spanning and \((n, \varepsilon)\)-separated subsets of \( X \). Let us describe them.

For every \( n \in \mathbb{N} \) we consider the metric \( d_n \) on \( X \) given by

\[
d_n(x, y) := \max \{ d(\alpha^i(x), \alpha^i(y)) \mid i = 0, 1, \ldots, n - 1 \}.
\]

For any \( \varepsilon > 0 \) a set \( E \subset X \) is called \((n, \varepsilon)\)-spanning if it is an \( \varepsilon \)-net for \( X \) with respect to metric \( d_n \), that is for any \( x \in X \) there exists \( y \in E \) such that \( d_n(x, y) < \varepsilon \).

A set \( F \subset X \) is called \((n, \varepsilon)\)-separated if for each pair of points \( x, y \in F, \ x \neq y \), one has \( d_n(x, y) > \varepsilon \).

Definition of the topological entropy is the following:

\[
h(\alpha) := \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \{ |E|^{1/n} : E \text{ is } (n, \varepsilon)\text{-spanning} \}
\]

\[
= \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \{ |E|^{1/n} : E \text{ is } (n, \varepsilon)\text{-separated} \}.
\]

Topological pressure is a (weighted) generalization of the notion of topological entropy. Namely, for each positive function \( a \in C(X, \mathbb{R}) \) the topological pressure \( P(\alpha, \ln a) \) is given by the formula

\[
P(\alpha, \ln a) := \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \left\{ \left( \sum_{y \in E} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n} \mid E \text{ is } (n, \varepsilon)\text{-spanning} \right\}
\]

\[
= \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \left\{ \left( \sum_{y \in E} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n} \mid E \text{ is } (n, \varepsilon)\text{-separated} \right\}.
\]
Clearly,
\[ h(\alpha) = P(\alpha, 0). \]

The well known Dinaburg–Goodman variational principle for topological entropy [Din70, Good71] states that
\[ h(\alpha) = \sup_{\mu \in M_\alpha(X)} h_\alpha(\mu), \] (15)
and the Ruelle–Walters variational principle for topological pressure [Rue73, Rue89, Wal75, Wal82] has the form
\[ P(\alpha, \ln a) = \sup_{\mu \in M_\alpha(X)} (\mu[\ln a] + h_\alpha(\mu)), \] (16)
where \( h_\alpha(\mu) \) is the metric (Kolmogorov–Sinaj) entropy.

### 2.3 Essential Set \( X_\alpha \)

In all the mentioned variational principles for topological entropy (15), spectral potential (10), and topological pressure (16) the invariant measures are of essential use. In this subsection we discuss the principal set associated with their supports.

A point \( x \in X \) will be called \textit{essential} (for the mapping \( \alpha \)), iff for every its neighborhood \( V(x) \) there exists an invariant measure \( \mu \in M_\alpha(X) \) such that \( \mu(V(x)) > 0 \). Clearly the set of all essential points is closed. We will call it the \textit{essential set} of the mapping \( \alpha \) and denote by \( X_\alpha \). The points \( x \in X \setminus X_\alpha \) will be called \textit{inessential}. For each \( x \in X \setminus X_\alpha \) there exists a neighborhood \( V(x) \) such that \( \mu(V(x)) = 0 \) for all \( \mu \in M_\alpha(X) \). This implies that for a compact metric space \( X \) the support of each invariant measure belongs to \( X_\alpha \), and for an arbitrary Hausdorff compact space \( X \) the support of each regular invariant measure belongs to \( X_\alpha \).

**Remark 1** Since for each \( \mu \in M_\alpha(X) \) one has \( \text{supp} \mu \subset X_\alpha \) the variational principles (15) and (16) imply that in the definitions of topological entropy (11), (12) and topological pressure (13), (14) it is enough to confine ourselves only to \((n,\varepsilon)\)-spanning and \((n,\varepsilon)\)-separated sets in \( X_\alpha \).

A subset \( Y \subset X \) is called \( \alpha \)-\textit{invariant} if \( \alpha^{-1}(Y) = Y \); and it is \textit{forward \( \alpha \)-invariant} if \( \alpha(Y) \subset Y \).

**Lemma 2** For the essential set \( X_\alpha \) one has
\[ \alpha(X_\alpha) = X_\alpha. \] (17)

**Proof.** Let us check first forward \( \alpha \)-invariance of \( X_\alpha \). Suppose on contrary that \( x \in X_\alpha \) while \( \alpha(x) \notin X_\alpha \). Take a neighborhood \( V(\alpha(x)) \) mentioned in the definition of inessential points, i.e., such that
\[ \mu(V(\alpha(x))) = 0 \text{ for each } \mu \in M_\alpha(X). \] (18)
Then \( \alpha^{-1}(V(\alpha(x))) \) is a neighborhood of \( x \) and \( \alpha \)-invariance of \( \mu \) along with (18) imply
\[ \mu(\alpha^{-1}(V(\alpha(x)))) = 0 \text{ for each } \mu \in M_\alpha(X), \]
that contradicts the assumption \( x \in X_\alpha \).
Now let us prove the equality $\alpha(X_\alpha) = X_\alpha$. Suppose on contrary, that
\[ X_\alpha \setminus \alpha(X_\alpha) \neq \emptyset. \tag{19} \]
Since $\alpha(X_\alpha)$ is compact the set
\[ W := X \setminus \alpha(X_\alpha) \]
is open, has nonempty intersection with $X_\alpha$, and
\[ \alpha^{-1}(W) \subset X \setminus X_\alpha. \]
As we have mentioned the support of each regular invariant measure belongs to $X_\alpha$. Thus
\[ \mu(\alpha^{-1}(W)) = 0 \quad \text{for each } \mu \in M_\alpha(X). \]
Therefore by $\alpha$-invariance of $\mu$ one has
\[ \mu(W) = \mu(\alpha^{-1}(W)) = 0 \quad \text{for each } \mu \in M_\alpha(X), \]
that contradicts (19). □

**Remark 2** In spite of the equality $X_\alpha = \alpha(X_\alpha)$, the set of essential points $X_\alpha$ in general is not $\alpha$-invariant, i.e., it may occur that $\alpha^{-1}(X_\alpha) \neq X_\alpha$. The next example demonstrates such a phenomenon.

**Example 1** Let $X = [0, 1] \subset \mathbb{R}$ and
\[ \alpha(x) = \begin{cases} 2x, & x \in [0, 1/2], \\ 1, & x \in [1/2, 1]. \end{cases} \]
Routine check shows that there are only two ergodic measures for this $(X, \alpha)$, namely, the Dirac measures $\delta_0$ and $\delta_1$. Thus $X_\alpha = \{0, 1\}$, while $\alpha^{-1}(X_\alpha) = \{0\} \cup [1/2, 1] \neq X_\alpha$.

The next result gives statistic criteria describing the essential set.
Recall that for every $x \in X$ the empirical measure $\delta_{x,n}$ is defined by the formula
\[ \delta_{x,n} = \frac{1}{n}(\delta_x + \delta_{\alpha(x)} + \cdots + \delta_{\alpha^{n-1}(x)}). \]

**Theorem 3** The following three conditions are equivalent:

a) a point $x \in X$ is essential;

b) for any neighborhood $V(x)$ of $x$ there exists $y \in X$ such that
\[ \liminf_{n \to \infty} \delta_{y,n}(V(x)) > 0; \tag{20} \]

c) for any neighborhood $V(x)$ of $x$ there exists $y \in X$ such that
\[ \limsup_{n \to \infty} \delta_{y,n}(V(x)) > 0. \tag{21} \]

**Proof.** For any point $x$ and any its neighborhood $V(x)$ there exists a neighborhood $U(x)$ such that its closure belongs to $V(x)$. By Urysohn’s lemma there exists a function $f \in C(X)$ that is equal to 1 on $U(x)$, equal to 0 outside $V(x)$, and takes values in $[0, 1]$ on $V(x) \setminus U(x)$.
a) \implies b). If a point \( x \) is essential then there is a measure \( \mu \in M_\alpha(X) \) such that 
\( \mu(U(x)) > 0 \). Therefore \( \mu[f] > 0 \). By the choice of \( f \) and the Ergodic theorem \cite[\S 1.6]{Wal82} there exists a measurable function \( \bar{f} : X \to [0, 1] \) satisfying the conditions 
\[
\bar{f} = \bar{f} \circ \alpha, \quad \mu[\bar{f}] = \mu[f], \quad \text{and} \quad \delta_{\bar{y},n}[f] \to \bar{f}(y) \text{ for } \mu\text{-almost all } y.
\]
This implies 
\[
\liminf_{n \to \infty} \delta_{\bar{y},n}(V(x)) \geq \liminf_{n \to \infty} \delta_{\bar{y},n}[f] = \bar{f}(y) \text{ a.e.} \tag{22}
\]
Since \( \mu[\bar{f}] = \mu[f] > 0 \) it follows that the function \( \bar{f}(y) \) is positive on a set of positive measure \( \mu \). And if \( \bar{f}(y) > 0 \) then (22) implies (21).

b) \implies c). Obvious.

c) \implies a). Let \( V(x), U(x) \) be the neighbourhoods and the function \( f \) mentioned above. By virtue of (21) with \( U(x) \) substituted for \( V(x) \) there exists a point \( y \in X \), a number \( \varepsilon > 0 \), and an infinite subset \( N_\varepsilon \subset \mathbb{N} \) such that
\[
\delta_{\bar{y},n}[f] \geq \delta_{\bar{y},n}(U(x)) > \varepsilon \text{ for all } n \in N_\varepsilon. \tag{23}
\]
By the Banach–Alaoglu theorem the set \( \{ \delta_{\bar{y},n} \mid n \in N_\varepsilon \} \) possesses a limit point \( \mu \) in the dual space \( C(X)^* \) equipped with the *-weak topology. This \( \mu \) is a linear functional on \( C(X) \), which by Riesz’s theorem is identified with a regular probability measure on \( X \). In a standard manner it can be verified that this measure is \( \alpha \)-invariant, i.e., \( \mu \in M_\alpha(X) \).

Passing to the limit in (23) we obtain \( \mu(V(x)) \geq \mu[f] \geq \varepsilon \). By the arbitrariness of \( \mu \) it follows that \( x \) is essential. \( \square \)

Recall that a point \( x \in X \) is non-wandering if for every open neighborhood \( V \) of \( x \) we have \( V \cap \alpha^n(V) \neq \emptyset \) for some \( n \in \mathbb{N} \). The set of non-wandering points is denoted by \( \Omega(\alpha) \). It is a closed forward \( \alpha \)-invariant set \cite[Theorem 5.6.]{Wal82}. We have \( \supp \mu \subseteq \Omega(\alpha) \) for every \( \mu \in M_\alpha(X) \).

Clearly wandering points are inessential and therefore \( X_\alpha \subset \Omega(\alpha) \). In reality the latter inclusion may be strict (i.e., not all the non-wandering points are essential, \( X_\alpha \neq \Omega(\alpha) \)). This phenomenon is demonstrated by the following example.

**Example 2** Let us consider the standard one-third Cantor set
\[
C := \left\{ x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \mid x_i \in \{0, 2\} \right\}.
\]
It is naturally identified with the set of all sequences of the form \( x = (x_1, x_2, x_3, \ldots) \in \{0, 2\}^\mathbb{N} \). Consider the mapping \( \alpha : C \to C, \quad \alpha(x) := 3x \mod 1 \). This mapping is the left shift on the set of sequences. Let \( X \subseteq C \) be the set consisting of all the sequences \( (x_1, x_2, \ldots) \in \{0, 2\}^\mathbb{N} \) such that each segment \( (x_{i}, \ldots, x_{i+2^n-1}) \) of length \( 2^n \) contains not more than \( n \) copies of digit \( 2 \) (for each \( i, n \in \mathbb{N} \)).

By routine check one sees that \( \alpha(X) = X \), the set \( X \) is closed and all its points are non-wandering. Note also that for the mapping \( \alpha : X \to X \) there is a unique essential point \( x^* = (0, 0, 0, \ldots) \), i.e., \( X_\alpha = \{x^*\} \). To prove this it is enough to verify that the dynamical system \( (X, \alpha) \) possesses a unique ergodic measure \( \delta_{x^*} \).

Let us consider an arbitrary ergodic measure \( \mu \) for \( (X, \alpha) \). Take a finite sequence \( y = (y_1, \ldots, y_m) \in \{0, 2\}^m \). It defines a (may be empty) cylinder
\[
Z_m(y) = \{x = (x_1, x_2, \ldots) \in X \mid x_1 = y_1, \ldots, x_m = y_m\}.
\]

\[ \sum_{i=1}^{\infty} \frac{x_i}{3^i} \]
By the ergodic theorem for $\mu$-almost all points $x \in X$ the relative number of points of the trajectory $x, \alpha(x), \ldots, \alpha^{2^n-1}(x)$ got into $Z_m(y)$ converges to $\mu(Z_m(y))$. On the other hand, if $y = (y_1, \ldots, y_m)$ contains at least one ‘two’, then by definition of $X$ this relative number does not exceed $m(n+1)/2^n$. Therefore, $\mu(Z_m(y)) = 0$ for each cylinder $Z_m(y)$ that do not contain $x^*$ and thus the measure $\mu$ is supported at $x^*$.

In the analysis of asymptotic properties of trajectories of dynamical systems it is often enough to consider the restriction of the initial mapping $\alpha$ onto the set of non-wandering points $\Omega(\alpha)$. In contrast, in our paper we consistently adhere to the point of view that in the analysis of invariant measures and characteristics associated with them (such as entropy and topological pressure) it is enough to consider the restriction of the initial mapping $\alpha$ onto the essential set $X_\alpha$.

**Remark 3** For the set $\Omega(\alpha)$ of non-wandering points one has $\alpha(\Omega(\alpha)) \subset \Omega(\alpha)$. Thus one can consider the non-wandering set $\Omega_2(\alpha) := \Omega(\alpha|_{\Omega(\alpha)}) \subset \Omega(\alpha)$. It can occur that $\Omega_2(\alpha) \neq \Omega(\alpha)$ (see, for example, [Wal82 § 5.3]). If we put $\Omega_1(\alpha) := \Omega(\alpha)$ and define inductively $\Omega_n(\alpha) := \Omega(\alpha|_{\Omega_{n-1}(\alpha)})$, $n = 2, 3, \ldots$ then $\Omega_1(\alpha) \supset \Omega_2(\alpha) \supset \cdots$ is a decreasing set of closed subsets of $X$. So one can put

$$\Omega_\infty(\alpha) := \bigcap_{n=1}^{\infty} \Omega_n(\alpha).$$

For the same reasons as for $\Omega(\alpha)$ we have that $\mu(\Omega_\infty(\alpha)) = 1$ for each $\mu \in M_\alpha(X)$.

One can also consider other subsets of $\Omega(\alpha)$ possessing the mentioned property. For example, we can take the set $R(\alpha) \subset \Omega(\alpha)$ of the so-called recurrent points (see, [Wal82 § 6.4]) that also possesses the property: $\mu(R(\alpha)) = 1$ for each $\mu \in M_\alpha(X)$.

Clearly, the set $X_\alpha$ is the minimal one possessing this property.

### 2.4 Transfer operators and positive functionals. Compatibility

In this subsection we present a procedure of ‘taking traces’ of transfer operators that will play an important technical role in the further analysis.

We start with a more explicit description of transfer operators linking them with special families of positive functionals.

Let, as above, $X$ be a Hausdorff compact space, $\alpha: X \to X$ be a continuous mapping, and $A: C(X) \to C(X)$ be a certain transfer operator.

For every point $x \in X$ define the functional $\phi_x$ according to the formula

$$\phi_x[f] := [Af](x), \quad f \in C(X).$$

(24)

In other words,

$$\phi_x := A^* \delta_x,$$

(25)

where $A^*: C(X)^* \to C(X)^*$ is the adjoint to $A$ operator and $\delta_x$ is the Dirac functional

$$\delta_x[f] = f(x), \quad f \in C(X).$$

Evidently, $\phi_x$ is a positive functional.

There are two possibilities for $x$.

1) $[A1](x) = 0$. This means that $\phi_x[1] = 0$ which implies $\phi_x = 0$ due to the positivity of $\phi_x$.
2) \([A1](x) \neq 0\). In this case \(\phi_x \neq 0\) and \(\phi_x\) defines a regular measure \(\nu_x = A^* \delta_x\) on \(X\). The homological identity (11) implies also that for any \(f \in C(X)\) we have
\[
\left[ A(f \circ \alpha) \right](x) = \left[ A \left( (f \circ \alpha) \cdot 1 \right) \right](x) = f(x) \cdot [A1](x),
\]
and therefore
\[
\frac{1}{[A1](x)} \phi_x(f \circ \alpha) = f(x),
\]
which means that
\[
\text{supp } \nu_x \subset \alpha^{-1}(x). \tag{26}
\]

Clearly, the mapping \(x \mapsto \phi_x\) is \(*\)-weakly continuous on \(X\).

The family \(\{\phi_x\}\) presented above in fact gives a complete description of transfer operators in \(C(X)\) since one can easily verify that every \(*\)-weakly continuous mapping \(x \mapsto \phi_x\), where \(\phi_x\) are positive functionals satisfying (26) (here \(\phi_x\) may be 0 as well), defines a certain transfer operator \(A : C(X) \to C(X)\) acting according to formula (24).

**Remark 4** The foregoing description of transfer operators implies, in particular, that in the situation when \(\alpha : X \to X\) is a local homeomorphism each transfer operator \(A\) acts as classical Perron–Frobenius operator (2) on \(\alpha(X)\) and \(Af|_{X \setminus \alpha(X)} \equiv 0\) for any \(f \in C(X)\); and in the situation when \(\alpha : X \to X\) is a homeomorphism each transfer operator is a weighted shift operator (3).

For any closed subset \(Y \subset X\) we can naturally define the ‘trace’ \(\phi_{Y,x}\) of the functional \(\phi_x\) from (24) on \(C(Y)\). Here is its definition. We can identify each function \(f \in C(Y)\) with the function \(\tilde{f}\) on \(X\) of the form
\[
\tilde{f}(x) := \begin{cases} f(x), & x \in Y, \\ 0, & x \not\in Y. \end{cases} \tag{27}
\]
Since each positive functional on \(C(X)\) is defined by a unique regular Borel measure we can uniquely extend its values onto the functions of the form (27) and in this way for the aforementioned functionals \(\phi_x\) we set
\[
\phi_{Y,x}(f) := \phi_x(\tilde{f}). \tag{28}
\]

The next notion is inevitable for taking ‘traces’ of transfer operators.

Let \(Y \subset X\) be a closed forward \(\alpha\)-invariant set. A transfer operator \(A : C(X) \to C(X)\) will be called \(Y\)-compatible (or compatible with \(Y\)) iff the family of functionals \(\phi_{Y,x}\) on \(C(Y)\) defined by (28) is \(*\)-weakly continuous on \(Y\).

To clarify the notion introduced we present examples as of \(Y\)-compatible so also not \(Y\)-compatible operators.

**Example 3** Let \(Y\) be any closed \(\alpha\)-invariant set, then any transfer operator \(A\) is \(Y\)-compatible. This follows from definition (28) of \(\phi_{Y,x}\) and property (26) of \(\phi_x\).

**Example 4** Even in the case when \(\alpha\) is a homeomorphism and \(Y\) is a closed forward \(\alpha\)-invariant set there can exist transfer operators that are not \(Y\)-compatible.

Let \(X = [0,1]\) and \(\alpha(x) = x^2\). If \(A : C(X) \to C(X)\) is a transfer operator then by the foregoing description we have \(Af(x) = \rho(\sqrt{x})f(\sqrt{x})\), where \(\rho \in C(X), \rho \geq 0\). That is
\( \phi_x = \rho(\sqrt{x})\delta_{\sqrt{x}} \). Set \( Y = [0, x_0] \), where \( 0 < x_0 < 1 \). This \( Y \) is a closed forward \( \alpha \)-invariant set. And we have that
\[
\phi_{Y,x} = \begin{cases} 
\phi_x, & \sqrt{x} \leq x_0, \\
0, & \sqrt{x} > x_0.
\end{cases}
\]

That is \( A \) is \( Y \)-compatible iff \( \rho(x_0) = 0 \).

**Example 5** Even in the situation when \( Y \) is closed and \( \alpha(Y) = Y \) there can exist transfer operators that are not \( Y \)-compatible.

Let
\[
X = [0, 1] \times [0, 1] = \Delta_1 \sqcup \Delta_2,
\]
where
\[
\Delta_1 := \left\{ (x_1, x_2) \in X \mid x_2 \leq 2 - \frac{x_1}{2} \right\},
\]
\[
\Delta_2 := \left\{ (x_1, x_2) \in X \mid x_2 > 2 - \frac{x_1}{2} \right\}.
\]

Set
\[
\alpha(x_1, x_2) = \begin{cases} 
(x_1, \sqrt{2x_2/(2-x_1)}), & (x_1, x_2) \in \Delta_1, \\
(x_1, 1), & (x_1, x_2) \in \Delta_2.
\end{cases}
\]

And let
\[
[Af](x_1, x_2) = f(\frac{x_1}{2}(2-x_1)/2), \quad (x_1, x_2) \in X.
\]

Thus we have
\[
\phi_{(x_1, x_2)} = \delta_{(x_1, x_2)(2-x_1)/2}, \quad (x_1, x_2) \in X.
\]

Take \( Y := [0, 1] \times \{0, 1\} \). We have \( \alpha(Y) = Y \) and
\[
\phi_{Y,(x_1, x_2)} = \begin{cases} 
\delta_{(x_1, x_2)}, & (x_1, x_2) \in [0, 1] \times \{0\}, \\
\delta_{(0,1)}, & (x_1, x_2) = (0, 1), \\
0, & (x_1, x_2) \in (0, 1] \times \{1\}.
\end{cases}
\]

Therefore \( A \) is not \( Y \)-compatible.

On the other hand one can verify in a routine way that for a given transfer operator \( A \) associated with the mapping \( (30) \) and the mentioned set \( Y \) the operator \( A \) is \( Y \)-compatible if \( \phi(0,1) = 0 \), where \( \phi_{(x_1, x_2)}, \quad (x_1, x_2) = x \in X \), is the family of functionals \( (25) \).

It is worth mentioning here that though in Example 1 \( X_\alpha \) is not \( \alpha \)-invariant it is a discrete set and therefore any transfer operator \( A \) is \( X_\alpha \)-compatible. However, in general a transfer operator \( A \) is not necessarily \( X_\alpha \)-compatible. This possibility is demonstrated by the next example.

**Example 6** Let us consider the objects mentioned in Example 5 i.e., \( X \) \( (29) \), \( \alpha \) \( (30) \) and \( A \) \( (31) \). One can check in a routine way that here we have \( X_\alpha = \Omega(\alpha) = [0, 1] \times \{0, 1\} \). And we have already verified in Example 5 that \( A \) is not compatible with this set.

The next two examples present popular situations when \( X_\alpha \) is such that any transfer operator \( A \) is \( X_\alpha \)-compatible.
Example 7 Let \( \alpha : X \to X \) be a homeomorphism. Since \( \alpha(X_\alpha) = X_\alpha \) it follows that in this case \( X_\alpha \) is \( \alpha \)-invariant and therefore any transfer operator \( A \) is \( X_\alpha \)-compatible.

Example 8 Let \( \alpha : X \to X \) be a local homeomorphism and \( Y \subset X \) be a closed subset such that \( \alpha(Y) = Y \). If \( \alpha \) is a local homeomorphism on \( Y \) then observation in Remark 4 implies that any transfer operator \( A \) is \( Y \)-compatible.

In particular, if \((X,d)\) is a compact metric space, \( \alpha : X \to X \) is an expanding map (i.e., the map for which there exist \( r > 0 \) and \( \Lambda > 1 \) such that inequality \( d(\alpha(x),\alpha(y)) \geq \Lambda d(x,y) \), and if, additionally, \( \alpha \) is an open map then we have \( \Omega(\alpha) = \text{Per}(\alpha) \), where \( \text{Per}(\alpha) \) is the set of periodic points [PU10 Proposition 3.3.6]. Thus in this case we have \( X_\alpha = \Omega(\alpha) = \text{Per}(\alpha) \). Moreover, in this situation \( \alpha|_{\text{Per}(\alpha)} \) is an open map as well [PU10 Lemma 3.3.10]. Therefore observation in Remark 4 implies that any transfer operator \( A \) is \( X_\alpha \)-compatible.

Unfortunately, it can occur that the restriction of a local homeomorphism \( \alpha \) onto \( Y \) is not a local homeomorphism. In this case it can happen that a transfer operator \( A \) is not \( Y \)-compatible. Here is an example of such situation.

Example 9 Let \( X = [0,1] \times \Delta \), where \( \Delta = \{0\} \cup \{1/2^n \mid n = 0, 1, 2, \ldots \} \), and topology on \( X \) is induced from \( \mathbb{R}^2 \). Define \( \alpha : X \to X \) by the formulae

\[
\begin{align*}
\alpha(t,0) &= (t,0), \quad t \in [0,1]; \\
\alpha(t,1/2^n) &= (t,1/2^{n-1}), \quad t \in [0,1], \quad n = 1, 2, \ldots; \\
\alpha(t,1) &= (\sqrt{t},1), \quad t \in [0,1].
\end{align*}
\]

Clearly \( \alpha \) is a local homeomorphism.

Take \( Y = [1/2,1] \times \Delta \). We have that \( Y \) is a closed set, \( \alpha(Y) = Y \) while \( \alpha : Y \to Y \) is not a local homeomorphism at the point \((1/2,1)\).

According to observation in Remark 4 any transfer operator \( A : C(X) \to C(X) \) is of the form (2). Therefore \( A \) is \( Y \)-compatible iff \( a(1/2,1) = 0 \).

Given a dynamical system \((X,\alpha)\), a transfer operator \( A : C(X) \to C(X) \), and a set \( Y \) such that \( A \) is \( Y \)-compatible one can define a transfer operator \( A_Y : C(Y) \to C(Y) \) for the dynamical system \((Y,\alpha)\), that can be naturally considered as the ‘trace’ of \( A \) on \( C(Y) \). Namely, we set

\[
[A_Y f](x) := \phi_{Y,x}[f], \quad f \in C(Y), \quad x \in Y
\]  

(cf. (24) and (28)). The argument exploited for \( A \) and \( \phi_x \) proves also that \( A_Y \) is a transfer operator for \( \alpha : Y \to Y \).

Remark 5 Note that in the situation when \( Y \) is a closed and forward \( \alpha \)-invariant set the mapping \( A_Y \) given by (32) is defined on \( C(Y) \) and is ‘nearly’ a transfer operator: it is positive by positivity of \( \phi_{Y,x} \) and satisfies by construction the homological identity, but if \( A \) is not \( Y \)-compatible (i.e., the family \( \phi_{Y,x} \) is not *-weakly continuous on \( Y \)) then \( A_Y \) does not preserve \( C(Y) \). The foregoing Examples 4, 5 and 9 can also be considered as illustrations of such situations.

Once a dynamical system \((X,\alpha)\) and a transfer operator \( A \) are fixed then each pair \((Y,A_Y)\), consisting of a set \( Y \subset X \) such that \( A \) is \( Y \)-compatible and the above described transfer operator \( A_Y \) (32), defines a \( t \)-entropy \( \tau_Y(\mu) \) on the set \( M_\alpha(Y) \) of \( \alpha \)-invariant
probability measures for the dynamical system \((Y, \alpha)\). This \(\tau_Y(\mu)\) is given by formula (35) with \(A_Y\) substituted for \(A\). Note also that each measure \(\mu \in M_\alpha(Y)\) can be considered as \(\mu \in M_\alpha(X)\) by setting \(\mu(X \setminus Y) = 0\) and in this way we assume that \(M_\alpha(Y) \subset M_\alpha(X)\).

**Theorem 4** Let \(A\) be a transfer operator for a dynamical system \((X, \alpha)\) and \(Y \subset X\) be a subset such that \(A\) is \(Y\)-compatible. Then we have

\[
\tau(\mu) = \tau_Y(\mu), \quad \mu \in M_\alpha(Y),
\]

and, if \(A\) is \(X_\alpha\)-compatible, then

\[
\tau(\mu) = \tau_{X_\alpha}(\mu), \quad \mu \in M_\alpha(X). \tag{34}
\]

**Proof.** It is enough to prove (33), since as we have noted for each \(\mu \in M_\alpha(X)\) one has \(\text{supp} \mu \subset X_\alpha\) and so \(M_\alpha(X) = M_\alpha(X_\alpha)\).

Recall that we are identifying measures \(\mu \in M_\alpha(Y)\) with the measures \(\mu \in M_\alpha(X)\) such that \(\text{supp} \mu \subset Y\). By (9) we have

\[
\tau(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_X \ln \frac{d(A^n \mu)_a}{d\mu} d\mu, \tag{35}
\]

where \(A^* : C(X)^* \to C(X)^*\) is the operator adjoint to \(A\), and \((A^n \mu)_a\) is the absolutely continuous component of the measure \(A^n \mu\) in its decomposition into absolutely continuous and singular parts with respect to \(\mu\).

By the construction of \(A_Y\) (cf. (27), (28), (32)) and the already mentioned inclusion \(\text{supp} \mu \subset Y\) we have that

\[
(A^n \mu)_a = (A^n_Y \mu)_a.
\]

This along with (35) implies

\[
\tau(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_X \ln \frac{d(A^n_Y \mu)_a}{d\mu} d\mu = \tau_Y(\mu), \quad \mu \in M_\alpha(Y),
\]

that proves (33). \(\square\)

Let \(Y \subset X\) and \(A\) be any \(Y\)-compatible transfer operator. For the transfer operator \(A_Y : C(Y) \to C(Y)\) we denote by \(\lambda_Y(\psi)\) the spectral potential of operator \(A_Y\), i.e., given by formula (7) with \(A_Y\) substituted for \(A\) and exploiting restriction of \(\psi\) onto \(Y\). Inclusion \(M_\alpha(Y) \subset M_\alpha(X)\) along with Theorems 1 and 4 imply the next

**Theorem 5** Let \(A : C(X) \to C(X)\) be a transfer operator for a dynamical system \((X, \alpha)\) and \(Y \subset X\) be a subset such that \(A\) is \(Y\)-compatible. Then

\[
\lambda(\psi) \geq \lambda_Y(\psi) = \max_{\mu \in M_\alpha(Y)} \left( \mu[\psi] + \tau_Y(\mu) \right), \quad \psi \in C(X, \mathbb{R}); \tag{36}
\]

and, if \(A\) is \(X_\alpha\)-compatible, then

\[
\lambda(\psi) = \lambda_{X_\alpha}(\psi) = \max_{\mu \in M_\alpha(X)} \left( \mu[\psi] + \tau_{X_\alpha}(\mu) \right), \quad \psi \in C(X, \mathbb{R}). \tag{37}
\]

**Remark 6** Note also that the inequality \(\lambda(\psi) \geq \lambda_Y(\psi)\) follows directly from the relationship between \(A\) and \(A_Y\) (cf. (21) and (32)) and (7).
We finish the section with an example demonstrating that spectral potential \( \lambda(\psi) \) and \( t \)-entropy \( \tau(\mu) \) cannot be reduced to topological pressure \( P(\alpha, \psi) \) and entropy \( h_\alpha(\mu) \), respectively.

**Example 10** Let us consider once more the objects from Example 5, i.e., \( X \) and \( A \). Clearly here we have \( \Omega(\alpha) = [0, 1] \times \{0, 1\} \) and a measure \( \mu \) is \( \alpha \)-invariant iff \( \text{supp} \mu \subset \Omega(\alpha) \), i.e., \( X_\alpha = \Omega(\alpha) \).

For Kolmogorov–Sinai entropy one has
\[
h_\alpha(\mu) = 0, \quad \mu \in M_\alpha(X). \tag{38}
\]
And therefore according to variational principle (16) for a real-valued continuous function \( \psi \) the topological pressure \( P(\alpha, \psi) \) is equal to
\[
P(\alpha, \psi) = \sup_{\mu \in M_\alpha(X)} \mu[\psi] = \max_{x \in X_\alpha} \psi(x). \tag{39}
\]
On the other hand formula (35) implies that for \( t \)-entropy we have
\[
\tau(\mu) = \begin{cases} 
0, & \text{supp} \mu \subset \Delta, \\
-\infty, & \text{supp} \mu \cap (X_\alpha \setminus \Delta) \neq \emptyset,
\end{cases} \tag{40}
\]
where \( \Delta = [0, 1] \times \{0\} \cup \{(0, 1)\} \),
and therefore according to variational principle (10) for the spectral potential \( \lambda(\psi) \) one obtains
\[
\lambda(\psi) = \max_{x \in \Delta} \psi(x). \tag{41}
\]

Comparing formulae (38) and (40) one concludes that Kolmogorov–Sinai entropy and \( t \)-entropy are essentially different objects. And comparing formulae (39) and (41) one arrives at the same conclusion for topological pressure and spectral exponent.

Moreover, we have to emphasize that while Kolmogorov–Sinai entropy \( h_\alpha(\mu) \) is always non-negative the \( t \)-entropy \( \tau(\mu) \) can take negative and even infinite negative values. And, what is important, these infinite values of \( \tau(\mu) \) are quite natural: according to variational principle (10) they indicate the measures that do not play any role in the spectral potential calculation.

Now after we have introduced the key heroes of our study and established their principal difference we are going to uncover the analytic reasons for them to be related to each other and describe these relationships.

### 3 Essential spectral potential, rami-rate, forward entropy

We start with introduction of a number of characteristics of dynamical systems that will be of use in estimation of entropy, spectral potential and topological pressure.

Given a dynamical system \((X, \alpha)\), we put
\[
\tilde{\alpha}^{-n}(x) := \alpha^{-n}(x) \cap X_\alpha, \quad x \in X_\alpha. \tag{42}
\]
Recall that $\alpha(X_\alpha) = X_\alpha$ by Lemma [2]. Henceforth we will always assume that $\alpha$ is a finite-sheeted cover on $X_\alpha$, i.e., satisfies the condition

$$\sup_{x \in X_\alpha} |\alpha^{-1}(x)| < \infty.$$  \hspace{1cm} (43)

In what follows we need in two more notions.

The number

$$\omega(\alpha) := \ln \lim_{n \to \infty} \sup_{x \in X_\alpha} |\alpha^{-n}(x)|^{1/n}$$  \hspace{1cm} (44)

will be called the inverse rami-rate. It evaluates the ramification speed of $\alpha$ preimages.

In the case when $X$ is a compact metric space we put

$$\gamma(\alpha) := \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \{|\alpha^n(E)|^{1/n} : E \text{ is } (n, \varepsilon)\text{-spanning for } X_\alpha\}.$$  \hspace{1cm} (45)

Comparing formulae (11) and (45) we naturally call $\gamma(\alpha)$ forward entropy.

Note that since $|\alpha^n(E)| \leq |E|$ we have (by definitions (11), (15) along with Remark 1) that

$$\gamma(\alpha) \leq h(\alpha).$$  \hspace{1cm} (46)

There are examples when $\gamma(\alpha) < h(\alpha)$ (see, in particular, Lemma [11]) so in general forward entropy $\gamma(\alpha)$ and topological entropy $h(\alpha)$ are different characteristics of $\alpha$.

Relationship between inverse rami-rate $\omega(\alpha)$ and topological entropy $h(\alpha)$ is based on the following

**Lemma 6** Let $X$ be a compact metric space and $\alpha: X \to X$ be a local homeomorphism. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ and $x \in X$ the set $\alpha^{-n}(x)$ is $(n, \varepsilon)$-separated.

**Proof.** Since $\alpha$ is a local homeomorphism it follows that $|\alpha^{-1}(x)|$ is a continuous (locally constant) function. The set $\Delta := \{x \in X : |\alpha^{-1}(x)| \leq 1\}$ is clopen in $X$. Thus the set $X \setminus \Delta$ is compact.

Let us put $D: X \setminus \Delta \to (0, \infty)$ to be

$$D(x) := \min \{|d(u, v) : u, v \in \alpha^{-1}(x), u \neq v\}.$$  

Local homeomorphism of $\alpha$ implies that $D(x)$ is a continuous function. Therefore

$$\varepsilon := \frac{1}{2} \min \{D(x) : x \in X \setminus \Delta\} > 0.$$  

Routine verification shows that this $\varepsilon$ fits the statement of lemma. \hfill \Box

This lemma along with (12) and Remark 1 implies that in the situation when $\alpha$ is a local homeomorphism on $X_\alpha$ one has

$$\omega(\alpha) \leq h(\alpha).$$  \hspace{1cm} (47)

**Remark 7** Since, for example, for any invertible $\alpha$ we have $\omega(\alpha) = 0$, while (by choosing suitable invertible $\alpha$) $h(\alpha)$ could be any nonnegative number (see, for example, [Wal82, §7.3]), we conclude that $\omega(\alpha)$ and $h(\alpha)$ are different characteristics of $\alpha$.  

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Lemma 7 The characteristics \( h(\alpha), \gamma(\alpha), \) and \( \omega(\alpha) \) satisfy the inequality

\[
h(\alpha) \leq \gamma(\alpha) + \omega(\alpha).
\]

Proof. Recalling Remark 1 we obtain

\[
h(\alpha) = \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \left\{ |E|^{1/n} : E \text{ is } (n, \varepsilon)\text{-spanning for } X_{\alpha} \right\}
\leq \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \left\{ |\alpha^n(E)|^{1/n} : E \text{ is } (n, \varepsilon)\text{-spanning for } X_{\alpha} \right\} \times \sup_{x \in X_{\alpha}} |\tilde{\alpha}^{-n}(x)|^{1/n}
\leq \ln \left[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \left\{ |\alpha^n(E)|^{1/n} : E \text{ is } (n, \varepsilon)\text{-span. for } X_{\alpha} \right\} \times \lim_{n \to \infty} \sup_{x \in X_{\alpha}} |\tilde{\alpha}^{-n}(x)|^{1/n} \right]
\leq \gamma(\alpha) + \omega(\alpha).
\]

This lemma along with observation (46) implies

Corollary 8 If inequality (47) holds then

(i) if \( \omega(\alpha) = 0 \) then \( h(\alpha) = \gamma(\alpha) \),

(ii) if \( \gamma(\alpha) = 0 \) then \( h(\alpha) = \omega(\alpha) \).

Remark 8 Inequality (48) may be strict and equalities \( h(\alpha) = \gamma(\alpha) \) and \( h(\alpha) = \omega(\alpha) \) may take place not only in the case when the second summand (i.e., \( \omega(\alpha) \) or \( \gamma(\alpha) \), respectively) is zero (see Example 11).

In what follows we will make use of the next auxiliary spectral potential type object which will help us to present the results in a transparent way.

Let \((X, \alpha)\) be a dynamical system with \( \alpha \) being a finite-sheeted cover on \( X_{\alpha} \). For each nonnegative function \( a \in C(X) \) we put

\[
\ell(\alpha, a) := \ln \lim_{n \to \infty} \sup_{x \in X_{\alpha}} \left( \sum_{y \in \tilde{\alpha}^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n},
\]

where we set \( \ln 0 = -\infty \). The number \( \ell(\alpha, a) \) will be called essential spectral potential.

Remark 9 Note that \( \ell(\alpha, a) \) is the logarithm of the ‘spectral radius’ of Perron–Frobenius operator \( A \) associated with \((X_{\alpha}, \alpha)\), i.e., with the dynamical system on the essential set \( X_{\alpha} \) (and that is why we use the term essential spectral potential). We put here the ‘spectral radius’ in quotation marks since in general (when \( \alpha \) is not a local homeomorphism) formula (2) does not define an operator in \( C(X) \).

The next result links topological pressure with essential spectral potential via forward entropy \( \gamma(\alpha) \).

Theorem 9 Let \( X \) be a compact metric space, \( \alpha : X \to X \) be a local homeomorphism on \( X_{\alpha} \), and \( a \in C(X) \) be a positive function. Then

\[
P(\alpha, \ln a) - \gamma(\alpha) \leq \ell(\alpha, a) \leq P(\alpha, \ln a).
\]

Proof. The right-hand inequality follows from (49), (44) along with Lemma 6 and Remark 1.
To prove the left-hand inequality note that for each finite subset $E \subset X_\alpha$ one has

$$\sum_{y \in E} \prod_{i=0}^{n-1} a(\alpha^i(y)) \leq \sum_{x \in \alpha^n(y)} \sum_{y \in \alpha^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y))$$

$$\leq |\alpha^n(E)| \times \sup_{x \in X_\alpha} \sum_{y \in \alpha^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)).$$

Denoting for brevity

$$\Phi_n := \sup_{x \in X_\alpha} \sum_{y \in \alpha^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)),$$

we can rewrite formula (49) in the form

$$\ell(\alpha, a) = \lim_{n \to \infty} \frac{1}{n} \ln \Phi_n.$$

Now observation (50), (51) along with formula (13) for $P(\alpha, \ln a)$, Remark 1 and definition (14) of $\gamma(\alpha)$ implies

$$P(\alpha, \ln a) \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \left[ \inf \{ |\alpha^n(E)| : E \text{ is } (n, \epsilon)\text{-spanning for } X_\alpha \} \times \Phi_n \right]$$

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \inf \{ |\alpha^n(E)| : E \text{ is } (n, \epsilon)\text{-spanning for } X_\alpha \} + \lim_{n \to \infty} \frac{1}{n} \ln \Phi_n$$

$$= \gamma(\alpha) + \ell(\alpha, a). \quad \square$$

**Corollary 10** Under conditions of Theorem 9 if $\gamma(\alpha) = 0$ then $P(\alpha, \ln a) = \ell(\alpha, a)$.

**Remark 10** If $\gamma(\alpha) > 0$ then we can have

$$P(\alpha, \ln a) - \gamma(\alpha) = \ell(\alpha, a) < P(\alpha, \ln a).$$

Indeed, let $\alpha : X \to X$ be a homeomorphism. Then $\omega(\alpha) = 0$ and $h(\alpha) = \gamma(\alpha)$. By a suitable choice of $X$ and $\alpha$ one can assume that $h(\alpha) = \gamma(\alpha)$ is an arbitrary given nonnegative number. For this $X$ and $\alpha$ take also $a = 1$. Then $\ell(\alpha, 1) = 0$ and

$$P(\alpha, \ln 1) = P(\alpha, 0) = h(\alpha) = \gamma(\alpha) > 0.$$

Theorem 9 shows importance of forward entropy $\gamma(\alpha)$. This characteristic can be easily calculated in the presence of the next

**Property (**)** For each pair $(n, \epsilon)$, $n \in \mathbb{N}$, $\epsilon > 0$, there exists a finite set $F(n, \epsilon) \subset X_\alpha$ such that the set $\tilde{\alpha}^{-n}(F(n, \epsilon))$ is an $(n, \epsilon)$-spanning for $X_\alpha$ and $\lim_{n \to \infty} |F(n, \epsilon)|^{1/n} = 1$.

This property looks as being rather sophisticated. A particular (more convenient) variant is the next

**Property (**)** For each $\epsilon > 0$ there exists a finite set $F(\epsilon) \subset X_\alpha$ such that for each $n \in \mathbb{N}$ the set $\tilde{\alpha}^{-n}(F(\epsilon))$ is an $(n, \epsilon)$-spanning for $X_\alpha$.

Clearly Property (**) implies Property (**) since one can simply take $F(n, \epsilon) := F(\epsilon)$ for all $n \in \mathbb{N}$.
Lemma 11 If $\alpha$ possesses property (\*) then $\gamma(\alpha) = 0$ (and hence $h(\alpha) = \omega(\alpha)$).

Proof. By definition (15) of $\gamma(\alpha)$

$$\gamma(\alpha) \leq \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left| \alpha^n \left( \tilde{\alpha}^{-n}(F(n, \varepsilon)) \right) \right|^{1/n} = \ln \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left| F(n, \varepsilon) \right|^{1/n} = 0. \quad \Box$$

As a consequence of Lemma 11 and Theorem 9 we also obtain

Theorem 12 If $\alpha$ possesses property (\*) then $P(\alpha, \ln a) = l(\alpha, a)$.

Lemma 14 below presents a wide class of dynamical systems possessing Property (\**\) (and therefore Property (\*)�).

Recall that a mapping $\alpha : X \to X$ on a metric space $(X, d)$ is called non-contracting if there exists $r > 0$ such that inequality $d(x, y) \leq r$ implies $d(\alpha(x), \alpha(y)) \geq d(x, y)$.

In the proof of Lemma 14 we will use the following technical observation.

Lemma 13 Let $X$ be a compact metric space and $\alpha : X \to X$ be a non-contracting local homeomorphism. Then there exists an $\varepsilon > 0$ such that inequality $d(x, \alpha(y)) < \varepsilon$ implies existence of a point $z \in \alpha^{-1}(x)$ such that $d(z, y) \leq d(x, \alpha(y))$.

Proof. Let $r$ be the number from definition of non-contractiveness of $\alpha$. By the openness of $\alpha$ for each point $x \in X$ there exists $\varepsilon(x) > 0$ such that

$$\alpha(B(x, r/2)) \supset B(\alpha(x), 2\varepsilon(x)). \quad (54)$$

Now for each $x \in X$ let us take a (small) neighborhood $U(x)$ such that

$$U(x) \subset B(x, r/2) \quad \text{and} \quad \alpha(U(x)) \subset B(\alpha(x), \varepsilon(x)).$$

By the choice of $U(x)$ along with (54) for each point $y \in U(x)$ we have

$$\alpha(B(y, r)) \supset \alpha(B(x, r/2)) \supset B(\alpha(x), 2\varepsilon(x)) \supset B(\alpha(y), \varepsilon(x)). \quad (55)$$

For the family of the mentioned neighborhoods $U(x)$ there exists a finite subcover $U(x_1), \ldots, U(x_n)$ of the space $X$. Set $\varepsilon := \min \{ \varepsilon(x_i) \mid i = 1, \ldots, n \}$. Now (55) implies

$$\alpha(B(y, r)) \supset B(\alpha(y), \varepsilon), \quad y \in X. \quad (56)$$

Finally note that if $d(x, \alpha(y)) < \varepsilon$ then by (55) there exists $z \in \alpha^{-1}(x) \cap B(y, r)$. And since $\alpha$ is non-contracting one has $d(z, y) \leq d(x, \alpha(y)). \quad \Box$

Lemma 14 If the mapping $\alpha : X \to X$ is a non-contracting local homeomorphism on $X_\alpha$ then it possesses property (\**\).

Proof. It suffice to take $\varepsilon > 0$ for which the statement of Lemma 13 holds and as $F(\varepsilon)$ one can take any $\varepsilon$-net in $X_\alpha$. Indeed, for each $y \in X_\alpha$ there exists a point $x_n \in F(\varepsilon)$ such that $d(x_n, \alpha^n(y)) < \varepsilon$ and by means of Lemma 13 one can construct a sequence of points $x_{n-1}, x_{n-2}, \ldots, x_1, x_0$ in $X_\alpha$ such that for all $i = 1, \ldots, n$ the following conditions hold

$$x_{i-1} \in \alpha^{-1}(x_i), \quad d(x_{i-1}, \alpha^{i-1}(y)) \leq d(x_i, \alpha^i(y)) < \varepsilon.$$ 

These relations show that the set $\alpha^{-n}(F(\varepsilon))$ forms an $(n, \varepsilon)$-spanning in $X_\alpha$. \quad \Box

Summarising Lemma 11 Theorem 12 and Lemma 14 we obtain
Theorem 15 Suppose \( \alpha : X \to X \) is a non-contracting local homeomorphism on \( X_\alpha \). Then

(i) \( \gamma(\alpha) = 0 \) and thus \( h(\alpha) = \omega(\alpha) \);

(ii) \( P(\alpha, \ln a) = \ell(\alpha, a) \).

Remark 11 Properties (i) and (ii) for expanding diffeomorphisms of compact smooth manifolds where stated without proofs in \[\text{LM98}\].

The next example shows that inequality in (48) may be strict.

Example 11 Let \( X = S^1 \sqcup Y \), where \( S^1 \) is the unit circle \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) and \( Y \) is a certain compact metric space. Set \( \alpha := \alpha_1 \oplus \alpha_2 \), where \( \alpha_1(z) = z^N, \ z \in S^1 \), and \( \alpha_2 \) is a homeomorphism of \( Y \). By Theorem 15 (i) \( \gamma(\alpha_1) = 0 \) and \( h(\alpha_1) = \omega(\alpha_1) = \ln N \) (where the latter equality follows directly from (14)). On the other hand, since \( \alpha_2 \) is a homeomorphism we have that \( \omega(\alpha_2) = 0 \), and therefore \( \gamma(\alpha_2) = h(\alpha_2) \). By a suitable choice of \( Y \) and \( \alpha_2 \) one can assume that \( \gamma(\alpha_2) = h(\alpha_2) \) is an arbitrary given nonnegative number. Note also that \( \omega(\alpha) = \omega(\alpha_1) \) and \( \gamma(\alpha) = \gamma(\alpha_2) \). Now we have

\[
\gamma(\alpha) = \max\{h(\alpha_1), h(\alpha_2)\} = \max\{\omega(\alpha_1), \gamma(\alpha_2)\} = \max\{\omega(\alpha), \gamma(\alpha)\}.
\]

In particular, when \( \gamma(\alpha_2) > 0 \) we have

\[
h(\alpha) < \omega(\alpha) + \gamma(\alpha).
\]

We note in addition that one can have here \( h(\alpha) = \omega(\alpha) \) or \( h(\alpha) = \gamma(\alpha) \) along with \( \gamma(\alpha) \neq 0 \) and \( \omega(\alpha) \neq 0 \).

We finish this section with an estimate of the essential spectral potential by means of integrals and inverse rami-rate \( \omega(\alpha) \). Note that the next theorem is valid for an arbitrary compact space \( X \) and arbitrary \( \alpha \), i.e., not necessarily satisfying condition of Theorem 15.

Theorem 16 Let \( X \) be a compact space, \( \alpha \) be a finite-sheeted cover on \( X_\alpha \), and \( a \in C(X) \) be a nonnegative function. Then

\[
\max_{\mu \in M_\alpha(X)} \int_X \ln a \, d\mu = \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu \leq \ell(\alpha, a) \leq \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu + \omega(\alpha).
\]

Proof. Keeping in mind that for each \( \mu \in M_\alpha(X) \) we have \( \text{supp}\mu \subset X_\alpha \) and applying \( \limsup \) Variational Principle \[\text{KL20} \rightline{\text{Theorem 3.5}}\] one obtains

\[
\lim_{n \to \infty} \sup_{x \in X_\alpha} \frac{1}{n} \ln \left( \prod_{i=0}^{n-1} a(\alpha^i(x)) \right) = \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu = \max_{\mu \in M_\alpha(X)} \int_X \ln a \, d\mu. \tag{57}
\]

Therefore,

\[
\ell(\alpha, a) = \ln \lim_{n \to \infty} \sup_{x \in X_\alpha} \left( \sum_{y \in \alpha^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n}
\geq \lim_{n \to \infty} \sup_{y \in X_\alpha} \frac{1}{n} \ln \left( \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)
\geq \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu,
\]

which proves the left-hand inequality in question.

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On the other hand one has
\[
\ell(\alpha, a) = \ln \lim_{n \to \infty} \sup_{x \in X_\alpha} \left( \sum_{y \in \tilde{\alpha}^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n}
\leq \ln \lim_{n \to \infty} \left\{ \sup_{x \in \tilde{X}_\alpha} |\tilde{\alpha}^{-n}(x)|^{1/n} \sup_{y \in \tilde{X}_\alpha} \left( \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n} \right\}
= \ln \lim_{n \to \infty} \sup_{x \in \tilde{X}_\alpha} |\tilde{\alpha}^{-n}(x)|^{1/n} + \lim_{n \to \infty} \sup_{y \in \tilde{X}_\alpha} \frac{1}{n} \ln \left( \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)
= \omega(\alpha) + \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu.
\]
Here in the final equality we again exploited (57). □

As an immediate corollary we have

**Theorem 17** Let $X$ be a compact space, $\alpha$ be a homeomorphism on $X_\alpha$, and $a \in C(X)$ be a nonnegative function. Then
\[
\ell(\alpha, a) = \max_{\mu \in EM_\alpha(X)} \int_X \ln a \, d\mu.
\]

4 Spectral potential vs topological pressure

Now we return back to description of interrelation between spectral potential $\lambda(\psi)$ and topological pressure $P(\alpha, \psi)$. Throughout this section we assume that $A$ is a transfer operator for a dynamical system $(X, \alpha)$ and $A_\psi$ and $\lambda(\psi)$ are defined by (4) and (5) respectively.

From now on we adopt the following convention. Once we use a transfer operator $A$ and the essential set $X_\alpha$ we assume that $A$ is an $X_\alpha$-compatible and $\alpha$ is a finite-sheeted cover on $X_\alpha$, i.e., satisfies condition (13).

For this situation the description of transfer operator $A_{X_\alpha}$ is given in Subsection 2.4 (cf. (27), (28), (32)). It implies that
\[
[A_{X_\alpha}f](x) = \sum_{y \in \tilde{\alpha}^{-1}(x)} \rho(y) f(y), \quad f \in C(X_\alpha), \quad x \in X_\alpha,
\]
where $\rho$ is a certain nonnegative function on $X_\alpha$. This function $\rho$ is usually called a *cocycle* associated with the transfer operator $A_{X_\alpha}$.

In fact the cocycle $\rho$ has rather specific properties and henceforth we proceed to describe some of them.

Throughout all the discussion of these properties (up to Corollary 19) in order not to overload the notation we will simply write $X$ instead of $X_\alpha$, $A$ instead of $A_{X_\alpha}$, and $\alpha^{-1}$ instead of $\tilde{\alpha}^{-1}$. In this notation our setting looks as follows: we consider a continuous finite-sheeted cover $\alpha: X \to X$, i.e., satisfying the condition
\[
\sup_{x \in X} |\alpha^{-1}(x)| < \infty;
\]

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are pairs of disjoint points \( x, x' \) for these pairs we have

\[
Af = \sum_{y \in \alpha^{-1}(x)} \rho(y)f(y), \quad f \in C(X), \quad x \in X,
\]

where \( \rho \) is a certain nonnegative function (cocycle) on \( X \) (cf. (58)).

A point \( x \in X \) will be called

— a local injectivity point (LIP) if there exists a neighborhood \( U(x) \) such that the mapping \( \alpha: U(x) \to X \) is injective;

— a local openness point (LOP) if for any neighborhood \( U(x) \) its image \( \alpha(U(x)) \) contains some neighborhood of \( \alpha(x) \);

— a local homeomorphism point (LHP) if \( x \) and \( \alpha(x) \) have \( \alpha \)-homeomorphic neighborhoods.

Lemma 18 Under condition (59)

a) if \( \rho(x_0) = 0 \) then \( \rho \) is continuous at the point \( x_0 \);

b) if \( \rho(x_0) \neq 0 \) then \( \rho \) is continuous at \( x_0 \) iff \( x_0 \) is a LIP;

c) if \( \rho(x_0) \neq 0 \) then \( x_0 \) is a LOP;

d) if \( \rho(x_0) \neq 0 \) then \( x_0 \) is a LIP iff it is a LHP.

**Proof.** Using (59), choose a neighborhood \( O(x_0) \) such that

\[
\alpha^{-1}(\alpha(x_0)) \cap O(x_0) = \{x_0\}. \tag{61}
\]

Choose a nonnegative function \( f \in C(X) \) such that \( f(x_0) = 1 \) and \( f(x) = 0 \) outside \( O(x_0) \). Then by (60) we have

\[
[Af](\alpha(x_0)) = \rho(x_0), \tag{62}
\]

and

\[
[Af](\alpha(x)) \geq \rho(x)f(x) \quad \text{for all} \quad x \in X. \tag{63}
\]

a) If \( \rho(x_0) = 0 \) then \([Af](\alpha(x_0)) = 0\) by (62). Along with continuity of \([Af](\alpha(x))\) and (63) this implies

\[
0 \leq \rho(x) \leq \frac{[Af](\alpha(x))}{f(x)} \to 0 \quad \text{as} \quad x \to x_0,
\]

which means the continuity of \( \rho(x) \) at \( x_0 \).

b) Let \( x_0 \) be a LIP and \( U(x_0) \) be a neighborhood where \( \alpha \) is injective. Then

\[
\alpha^{-1}(\alpha(x)) \cap U(x_0) = \{x\} \quad \text{for any} \quad x \in U(x_0)
\]

and hence by (60)

\[
[Af](\alpha(x)) = \rho(x)f(x) \quad \text{for any} \quad x \in U(x_0).
\]

Since \([Af](\alpha(x))\) is continuous and \( f(x_0) = 1 \), this implies the continuity of \( \rho(x) \) at \( x_0 \).

On the other hand, assume that \( \rho(x) \) is continuous at \( x_0 \) but \( x_0 \) is not a LIP. Then there are pairs of disjoint points \( x, x' \) both arbitrarily close to \( x_0 \) and such that \( \alpha(x) = \alpha(x') \). For these pairs we have

\[
\limsup_{x \to x_0} [Af](\alpha(x)) \geq \limsup_{x \to x_0} (\rho(x)f(x) + \rho(x')f(x')) \geq 2\rho(x_0).
\]

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Remark 12 Lemma just proven demonstrates that for a cocycle $g$

Now, the choice of $x$ is positive in a certain neighborhood of $g$. Take a neighborhood $V$ of $x$ of the form

$$V := \{ y \in X \mid [Ag](y) > 0 \}.$$ 

Now, the choice of $g$ and definition (60) imply $V \subset \alpha(U(x_0))$.

$d)$ Suppose $x_0$ is a LIP. Then by $b)$ the function $\rho$ is continuous at $x_0$ and hence it is positive in a certain neighborhood of $x_0$. Take a neighborhood $U(x_0)$ such that $\alpha$ is injective and $\rho$ is positive on it. By $c)$ all points of $U(x_0)$ are LOPs. Consequently, $\alpha$ is open on $U(x_0)$ and maps it homeomorphically onto $\alpha(U(x_0))$. □

Remark 12 Lemma just proven demonstrates that for a cocycle $\rho$ the property of being continuous is valid only in rather specific situations. Fortunately, as assertion $c)$ tells, at the points where $\rho$ does not vanish the mapping $\alpha$ behaves not ‘too pathologically’.

In general LIP and LHP are different notions. For example, the point $(1/2, 1)$ of the set $Y$ in Example 9 is a LIP (for $\alpha : Y \to Y$) but is not a LHP. Lemma 18 shows, in addition, that a transfer operator $A$ from (60) is a ‘clever machine’ — it distinguishes LIP and LHP: when $x_0$ is a LIP but not a LHP it puts $\rho(x_0) = 0$.

As an immediate consequence of the foregoing Lemma we also obtain

**Corollary 19** If $\rho(x_0) \neq 0$ then $\rho$ is continuous at a point $x_0$ iff $x_0$ is a LHP.

And, in particular, for the objects mentioned in (58) one has the following

**Corollary 20** If $\alpha : X \to X$ is a local homeomorphism on $X_\alpha$ then the cocycle $\rho$ defined in (58) is a continuous function.

Recall that we assume throughout the rest of the article that $A$ is an $X_\alpha$-compatible transfer operator and $\alpha$ is a finite-sheeted cover on $X_\alpha$.

The next observation links spectral potential $\lambda(\psi)$ defined in (12) and essential spectral potential $\ell(\alpha, a)$ defined in (19).

**Theorem 21** Let the cocycle $\rho$ of $A$ be continuous on $X_\alpha$ (in particular, this is true when $\alpha : X_\alpha \to X_\alpha$ is a local homeomorphism). Then

$$\lambda(\psi) = \ell(\alpha, \rho e^\psi),$$

(recall that $\ell(\alpha, a)$ in (19) exploits only the values of $a$ on $X_\alpha$).

**Proof.** By (41) we have

$$A_{X_\alpha, \psi} f := A_{X_\alpha}(e^\psi f), \quad f \in C(X_\alpha).$$

Let us denote

$$a := \rho e^\psi.$$ (64)

Since $A_{X_\alpha, \psi}^n$ is a positive operator we have that $\|A_{X_\alpha, \psi}^n\| = \|A_{X_\alpha, \psi}^n 1\|$ and routine computation shows that

$$\|A_{X_\alpha, \psi}^n\| = \max_{x \in X_\alpha} \left( \sum_{y \in \alpha^{-1}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right),$$ (65)

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and therefore

\[
\lambda_{X_\alpha}(\psi) = \ln \lim_{n \to \infty} \left\| A_{X_\alpha,\psi}^n \right\|^{1/n} = \ln \lim_{n \to \infty} \max_{x \in X_\alpha} \left( \sum_{y \in \alpha^{-n}(x)} \prod_{i=0}^{n-1} a(\alpha^i(y)) \right)^{1/n} = \ell(\alpha, a),
\]

where the final equality follows from definition (49).

This observation along with Theorem 5 implies

\[
\lambda(\psi) = \lambda_{X_\alpha}(\psi) = \ell(\alpha, \rho e^\psi). \quad \square
\]

The foregoing theorem along with results of Section 3 gives us a possibility to relate spectral potential and topological pressure. This is the theme of the next theorem.

**Theorem 22** Let \( X \) be a compact metric space and \( \alpha : X \to X \) be a local homeomorphism on \( X_\alpha \). If \( \alpha \) possesses property (\( \ast \)), and the cocycle \( \rho \) is strictly positive on \( X_\alpha \) then for each strictly positive continuous extension of \( \rho \) onto \( X \) we have

\[
\lambda(\psi) = P(\alpha, \psi + \ln \rho).
\]

**Proof.** Define \( a \) by (64). By Theorem 21 along with Theorem 9 we have

\[
\lambda(\psi) = \ell(\alpha, a) = P(\alpha, \ln a) = P(\alpha, \psi + \ln \rho). \quad \square
\]

Exploiting in the foregoing proof Theorem 15 in place of Theorem 9 one gets

**Theorem 23** Let \( X \) be a compact metric space, \( \alpha : X \to X \) be a non-contracting local homeomorphism on \( X_\alpha \), and the cocycle \( \rho \) be strictly positive on \( X_\alpha \). Then for each strictly positive continuous extension of \( \rho \) onto \( X \) we have

\[
\lambda(\psi) = P(\alpha, \psi + \ln \rho).
\]

5 Spectral radii of transfer operators with non-negative weights, topological pressure and integrals

In the preceding sections we analysed transfer operators \( A_\psi = A(e^{\psi \cdot \cdot}) \), \( \psi \in C(X) \). Here the weight (i.e., the function \( e^{\psi \cdot \cdot} \)) is always positive. In this section we extend the results obtained above onto transfer operators with non-negative (not necessarily positive) weights.

Let \( A \) be a fixed transfer operator for \( (X, \alpha) \). We define the family of operators \( Ag : C(X) \to C(X) \), where \( g \in C(X) \), as

\[
Ag := A(g \cdot \cdot).
\]  \hspace{1cm} (66)

Clearly, if \( g \geq 0 \), then \( Ag \) is a transfer operator.

For \( g \in C(X), \ g \geq 0 \), we denote by \( \ell(g) \) the logarithm of the spectral radius of \( Ag \).

**Remark 13** If \( g > 0 \), then \( Ag = A_{\ln g} \). In addition,

\[
\ell(g) = \lambda(\ln g), \quad g > 0.
\]  \hspace{1cm} (67)
Recall once more that whenever we use a transfer operator $A$ and the essential set $X_\alpha$ we assume that $A$ is $X_\alpha$-compatible and $\alpha$ is a finite-sheeted cover on $X_\alpha$.

The extension of Theorem 21 on the situation in question is

**Theorem 24** Let the cocycle $\rho$ be continuous on $X_\alpha$ (in particular, this is true when $\alpha: X_\alpha \to X_\alpha$ is a local homeomorphism). Then for each $g \in C(X)$, $g \geq 0$,

$$\ell(g) = \ell(\alpha, \rho g),$$

(here we recall that $\ell(\alpha, a)$ in (19) exploits only the values of $a$ on $X_\alpha$).

**Proof.** Choose a sequence of strictly positive continuous functions $g_n \downarrow g$. By upper semicontinuity of the spectral radius we have

$$\ell(g_n) \downarrow \ell(g). \quad (68)$$

By Theorem 21 and (67) one obtains

$$\ell(g_n) = \ell(\alpha, \rho g_n). \quad (69)$$

And from the explicit form of $\ell(\alpha, a)$ in (19) we conclude that

$$\ell(\alpha, \rho g_n) \downarrow \ell(\alpha, \rho g). \quad (70)$$

Now (68), (69), and (70) imply

$$\ell(g) = \ell(\alpha, \rho g). \quad \Box$$

The extension of Theorem 1 on the situation in question is

**Theorem 25** (variational principle for transfer operators with nonnegative weights, see [ABL11, Theorem 11.2.]) Let $Ag$ be a transfer operator defined in equation (66), where $g \in C(X)$ and $g \geq 0$. Then the following variational principle holds true:

$$\ell(g) = \max_{\mu \in M_\alpha(X)} \left( \int_X \ln g \, d\mu + \tau(\mu) \right). \quad (71)$$

Recalling Ruelle–Walters variational principle for topological pressure (16) one can set for $a \in C(X)$, $a \geq 0$,

$$P(\alpha, \ln a) := \sup_{\mu \in M_\alpha(X)} \left( \int_X \ln a \, d\mu + h_\alpha(\mu) \right). \quad (72)$$

Now the extension of Theorem 23 on the situation in question is

**Theorem 26** Let $X$ be a compact metric space, $\alpha: X \to X$ be a non-contracting local homeomorphism on $X_\alpha$, and $g \in C(X)$, $g \geq 0$. Then for any nonnegative continuous extension of the cocycle $\rho$ from $X_\alpha$ onto $X$ we have

$$\ell(g) = P(\alpha, \ln(g\rho)), \quad \text{where the topological pressure is defined by (12)}$$

(we recall that $\rho$ is continuous on $X_\alpha$ by Corollary 20).
Proof. Let a transfer operator $A_{X_n} : C(X_n) \to C(X_n)$ be given by (32) with $Y = X_n$ and denote by $\ell_{X_n}(g)$ the logarithm of the spectral radius of $A_{X_n}g$. Recalling Theorems 4 and 5 and exploiting Theorem 25 we conclude that

$$\ell(g) = \ell_{X_n}(g) = \max_{\mu \in M_{X_n}} \left( \int_{X_n} \ln g d\mu + \tau_{X_n}(\mu) \right).$$

(73)

Choose strictly positive continuous functions $g_n \searrow g$ and strictly positive continuous functions $\rho_n \searrow \rho$. Let $A_n : C(X_n) \to C(X_n)$ be transfer operators associated with cocycles $\rho_n$ and consider the arising transfer operators $A_n g_n : C(X_n) \to C(X_n)$, and set $\ell_{X_n}(g_n)$ to be the logarithm of the spectral radius of $A_n g_n$. By Theorem 23 one has

$$\ell_{X_n}(g_n) = P(\alpha, \ln(g_n \rho_n)).$$

In addition, by upper semicontinuitiy of the spectral radius we obtain

$$\ell_{X_n}(g_n) \searrow \ell_{X_n}(g),$$

(74)

and we also have

$$P(\alpha, \ln(g_n \rho_n)) \searrow P(\alpha, \ln(g \rho)),$$

(75)

where the topological pressure is defined by (72).

□

For convenience of further reasoning we mention the next observation which naturally should be considered as a folklore.

Lemma 27 Let the entropy map $M_\alpha(X) \ni \mu \mapsto h_\alpha(\mu) \in [0, \infty)$ be upper semicontinuous (in $^*$-weak topology) and $g \in C(X)$, $g \geq 0$, then variational principle (72) reduces to

$$P(\alpha, \ln g) = \max_{\mu \in M_\alpha(X)} \left( \int_{X_n} \ln g d\mu + h_\alpha(\mu) \right).$$

(76)

Proof. We have already mentioned that for $\mu \in M_\alpha(X)$ one has supp $\mu \subset X_n$. Also the map $M_\alpha(X) \ni \mu \mapsto \int_{X_n} \ln g d\mu$ is always upper semicontinuous. Therefore supremum in (72) can be replaced by maximum in (76).

□

Combining this lemma with Theorem 26 one gets

Corollary 28 Let $X$ be a compact metric space and $\alpha : X \to X$ be a non-contracting local homeomorphism of $X_\alpha$ such that the entropy map $M_\alpha(X) \ni \mu \mapsto h_\alpha(\mu) \in [0, \infty)$ is upper semicontinuous. Then for each $g \in C(X)$, $g \geq 0$ we have

$$\ell(g) = \max_{\mu \in M_\alpha(X)} \left( \int_{X_n} \ln(g \rho) d\mu + h_\alpha(\mu) \right).$$

(77)

There is quite a number of dynamical systems on a metric space $(X, d)$ for which upper semicontinuity of entropy map holds. Among them are, for example, expanding maps. Thus we also get the next

Corollary 29 Let $X$ be a compact metric space and $\alpha : X \to X$ be an expanding local homeomorphism on $X_\alpha$. Then for each $g \in C(X)$, $g \geq 0$, we have equality (77).
As is known $h_\alpha(\mu)$ is a concave function and therefore one can ‘close’ this function to make it upper semicontinuous. Namely, set
\[ \tilde{h}_\alpha(\mu) := \limsup_{\nu \to \mu} h_\alpha(\nu), \]
where $\nu \to \mu$ is taken in $^\ast$-weak topology. This function $\tilde{h}_\alpha(\mu)$ is concave and upper semicontinuous on $M_\alpha(X)$. Replacing entropy $h_\alpha(\mu)$ by $\tilde{h}_\alpha(\mu)$ in the proof of Lemma 27 one obtains

**Lemma 30** Let $X$ be a compact metric space, $\alpha : X \to X$ be a continuous mapping and $g \in C(X)$, $g \geq 0$, then along with variational principle (72) we have
\[ P(\alpha, \ln g) = \max_{\mu \in M_\alpha(X)} \left( \int_X \ln g \, d\mu + \tilde{h}_\alpha(\mu) \right). \]

And as an analogue of Corollary 28 for the ‘closed’ entropy $\tilde{h}_\alpha(\mu)$ one has

**Corollary 31** Let $X$ be a compact metric space and $\alpha : X \to X$ be a non-contracting local homeomorphism on $X_\alpha$. Then for each $g \in C(X)$, $g \geq 0$, we have
\[ \ell(g) = \max_{\mu \in M_\alpha(X)} \left( \int_X \ln(g \rho) \, d\mu + \tilde{h}_\alpha(\mu) \right). \]

We finish this section with relating $\ell(g)$ with integrals. As an immediate consequence of Theorems 16, 17, 24, and Corollary 20 one obtains

**Theorem 32** Let the inverse rami-rate be zero ($\omega(\alpha) = 0$), and the cocyle $\rho$ on $X_\alpha$ be continuous (in particular, this takes place when $\alpha$ is a homeomorphism on $X_\alpha$). Then
\[ \ell(g) = \max_{\mu \in EM_\alpha(X)} \int_{X_\alpha} \ln(g \rho) \, d\mu = \max_{\mu \in M_\alpha(X)} \int_{X_\alpha} \ln(g \rho) \, d\mu. \]

**Remark 14** If $\alpha : X \to X$ is a homeomorphism then the description of transfer operators given in Subsection 2.4 implies
\[ [Af](x) = \rho(\alpha^{-1}(x)) f(\alpha^{-1}(x)), \]
where $\rho \in C(X)$ is a nonnegative function. That is, $A$ is a weighted shift operator and so also is the operator
\[ [(Ag)f](x) = [\rho gf](\alpha^{-1}(x)). \]

Variational principles of (80) type for abstract weighted shift operators associated with commutative Banach algebras automorphisms generated by isometries where worked out in [Kit79] and [Leb79] (see also [AL94, 4] and [Ant96, 5]). A comprehensive analysis of the corresponding variational principles and their interrelations as with integrals so also with Lyapunov exponents for abstract weighted shift operators associated with endomorphisms of Banach algebras is presented in [KL20].
6 T-entropy vs Kolmogorov–Sinaj entropy and integrals

In the previous sections we analyzed relationships between spectral potential, topological pressure and integrals with respect to invariant measures. The results obtained naturally give us an opportunity to analyze relationships between t-entropy, entropy and integrals. This is the theme of the present section.

Henceforth we assume that $A$ is a given transfer operator for a dynamical system $(X, \alpha)$, provided $A$ is $X_\alpha$-compatible and $\alpha$ is a finite-sheeted cover on $X_\alpha$; and $\rho$ is a cocycle on $X_\alpha$ defined by (58).

We recall one more observation that will be exploited in sequel.

In [ABL11, Propositions 8.4, 8.6] it is proven that t-entropy map $\mu \mapsto \tau(\mu)$ is a concave and upper semicontinuous function (in $\ast$-weak topology). Therefore formula (10):

$$\lambda(\psi) = \max_{\mu \in \mathcal{M}_\alpha(X)} (\mu[\psi] + \tau(\mu))$$

means that the spectral potential $\lambda(\psi)$ is nothing else than the Fenchel–Legendre transform of $-\tau(\mu)$. Moreover, by the Fenchel–Legendre–Moreau duality upper semicontinuity of $\tau(\mu)$ implies the equality

$$-\tau(\mu) = \inf_{\psi \in C(X, \mathbb{R})} (\mu[\psi] - \lambda(\psi)),$$

which means that $-\tau(\mu)$ is the Fenchel–Legendre dual functional to $\lambda(\psi)$. Therefore $\tau(\mu)$ is uniquely defined by the spectral potential $\lambda(\psi)$. By the mentioned Fenchel–Legendre–Moreau duality we also conclude that if $S(\mu)$ is a certain concave and upper semicontinuous (in $\ast$-weak topology) function of $\mu$ such that

$$\lambda(\psi) = \sup_{\mu \in \mathcal{M}_\alpha(X)} (\mu[\psi] + S(\mu))$$

(i.e., $\lambda(\psi)$ is the Fenchel–Legendre transform of $-S(\mu)$), then

$$S(\mu) = \tau(\mu), \quad \mu \in \mathcal{M}_\alpha(X).$$

Recall once more that whenever we use a transfer operator $A$ and the essential set $X_\alpha$ we assume that $A$ is $X_\alpha$-compatible.

Our first observation is relationship between t-entropy and integrals.

**Theorem 33** Let the inverse rami-rate be zero ($\omega(\alpha) = 0$) and the cocycle $\rho$ on $X_\alpha$ be continuous (in particular, this takes place when $\alpha$ is a homeomorphism on $X_\alpha$). Then for $\mu \in \mathcal{M}_\alpha(X)$ we have

$$\tau(\mu) = \int_{X_\alpha} \ln \rho \, d\mu.$$  

**Proof.** By Theorem 32 for $\psi \in C(X)$ one has

$$\lambda(\psi) = \max_{\mu \in \mathcal{M}_\alpha(X)} (\mu[\psi] + \int_{X_\alpha} \ln \rho \, d\mu).$$

Note that the function

$$\mathcal{M}_\alpha(X) \ni \mu \mapsto \int_{X_\alpha} \ln \rho \, d\mu$$

is linear and upper semicontinuous. This observation along with (32) and (82) implies the assertion of the theorem. □
Remark 15 In the case when $\alpha : X \to X$ is a homeomorphism and $Af(x) = f(\alpha^{-1}(x))$ the corresponding formula for $t$-entropy was obtained in [BKKL19].

The next observation links $t$-entropy with Kolmogorov–Sinai entropy.

Theorem 34 Let $X$ be a compact metric space, $\alpha$ be an open and non-contracting on $X_\alpha$, and the entropy map $M_\alpha(X) \ni \mu \mapsto h_\alpha(\mu) \in [0, \infty)$ be upper semicontinuous (in particular, this takes place when $\alpha$ is open and expanding on $X_\alpha$). Then for $\mu \in M_\alpha(X)$ we have

$$\tau(\mu) = \int_{X_\alpha} \ln \rho d\mu + h_\alpha(\mu).$$

(86)

Proof. By Corollaries [28] [29] and [67], for $\psi \in C(X)$ one has

$$\lambda(\psi) = \max_{\mu \in M_\alpha(X)} \left( \mu[\psi] + \left[ \int_{X_\alpha} \ln \rho d\mu + h_\alpha(\mu) \right] \right).$$

And by the condition of the theorem the function

$$M_\alpha(X) \ni \mu \mapsto \int_{X_\alpha} \ln \rho d\mu + h_\alpha(\mu)$$

is upper semicontinuous. Now the assertion of the theorem follows from (82) and (83). □

Remark 16 In the situation when $\alpha : X \to X$ is open and expanding formula (86) for $t$-entropy was obtained in [BKKL19] and [BK19].

Replacing in the proof of the previous theorem entropy $h_\alpha(\mu)$ by its ‘closure’ $\bar{h}_\alpha(\mu)$ (78) and exploiting Corollary [31] one obtains

Theorem 35 Let $X$ be a compact metric space, and $\alpha$ be open and non-contracting on $X_\alpha$. Then for $\mu \in M_\alpha(X)$ we have

$$\tau(\mu) = \int_{X_\alpha} \ln \rho d\mu + \bar{h}_\alpha(\mu).$$

Remark 17 In all the statements the set $X_\alpha$ and the assumption that $A$ is $X_\alpha$-compatible are essential. In fact the properties of $X_\alpha$ that were exploited are the following:

1) each $\mu \in M_\alpha(X)$ is supported on $X_\alpha$,

2) the operator $A$ is compatible with this set.

Any set $Y \subset X$ possessing these two properties can be exploited in all the statements. For example, one can take the set $\Omega(\alpha)$ of non-wandering points when $A$ is $\Omega(\alpha)$-compatible. Or simply take the whole $X$.

In fact, the essential set $X_\alpha$ is the minimal one possessing the mentioned properties whenever $A$ is compatible with it.

References

[Ant96] A. B. Antonevich, Linear Functional Equations. Operator Approach, Birkhauser Verlag, Operator Theory Advances and Applications, V. 83, 1996.

[ABL11] A.B. Antonevich, V.I. Bakhtin, A.V. Lebedev, On t-entropy and variational principle for the spectral radii of transfer and weighted shift operators, Ergod. Th. & Dynam. Sys. 31 (2011), 995–1042.
[ABL11] A.B. Antonevich, V.I. Bakhtin and A.V. Lebedev, *Crossed product of a C*-algebra by an endomorphism, coefficient algebras and transfer operators*, Sbornik Mathematics 202 (9) (2011), 1253–1283.

[ABL12] A.B. Antonevich, V.I. Bakhtin, A.V. Lebedev, *A road to the spectral radius of transfer operators*, Contemp. Math. 567 (2012), 17–51.

[ABL03] A.B. Antonevich, V.I. Bakhtin, A.V. Lebedev, D.S. Sarzhinsky, *Legendre analysis, thermodynamic formalizm and spectra of Perron-Frobenius operators*, Dokl. Math. 67 (2003), 343–345.

[ABL98] A. Antonevich, M. Belousov, A. Lebedev, *Functional Differential Equations: II. C*-Applications: Part 1: Equations with Continuous Coefficients* (Pitman Monographs and Surveys in Pure and Applied Mathematics, 94). Addison-Wesley, Longman, Harlow, England, 1998.

[ABL98] A. Antonevich, M. Belousov, A. Lebedev, *Functional Differential Equations: II. C*-Applications: Part 2: Equations with Discontinuous Coefficients and Boundary Value Problems* (Pitman Monographs and Surveys in Pure and Applied Mathematics, 95). Addison-Wesley, Longman, Harlow, England, 1998.

[AL94] A. Antonevich, A. Lebedev, *Functional differential equations: I. C*-theory*, Longman Scientific & Technical, Pitman Monographs and Surveys in Pure and Applied Mathematics 70, 1994.

[Bal00] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, World Scientific, River Edge, NJ, 2000.

[B10] V.I. Bakhtin, *On t-entropy and variational principle for the spectral radius of weighted shift operators*, Ergod. Th. & Dynam. Sys. 30 (2010), 1331–1342.

[BK19] K. Bardadin, B.K. Kwasniewski, *Spectrum of weighted isometries: C*-algebras, transfer operators and topological pressure*, arXiv: 1911.04811v1 [mathFA] 12 Nov 2019, 40 pp.

[BKKL19] K. Bardadyn, B.K. Kwasniewski, K.S. Kurnosenko, A.V. Lebedev, *t-Entropy formulae for concrete classes of transfer operators*, Journal of the Belarusian State University. Mathematics and Informatics, (3) (2019), 122–128.

[BL17] V.I. Bakhtin, A.V. Lebedev, *A New Definition of t-Entropy for Transfer Operators*. Entropy, 573 (19) (2017), 1–6.

[BL19] V.I. Bakhtin, A.V. Lebedev, *Entropy statistic theorem and variational principle for t-entropy are equivalent*, J. Math.Anal.Appl. 474 (2019) 59–71.

[BL20] V.I. Bakhtin, A.V. Lebedev, *Sup-Sums Principles for F-Divergence and a New Definition for t-Entropy*, Journal of Theoretical Probability (2020). https://doi.org/10.1007/s10959-020-01046-5.

[Bow75] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, vol.470. Berlin, Heidelberg, New York: Springer 1975.
[Did07] V. Didenko, *Estimates of the spectral radius of refinement and subdivision operators with isotropic dilations*, J. Operator Theory 58 (2007), 3–22.

[Din70] E.I. Dinaburg, *A correlation between topological entropy and metric entropy*, Dokl. Akad. Nauk SSSR, 190 (1) (1970), 19–22.

[Ex03] R. Exel, *A new look at the crossed-product of a $C^*$-algebra by an endomorphism*, Ergod. Th. & Dynam. Sys. 23 (6) (2003), 1733–1750.

[FJ01] A. H. Fan, Y. P. Jiang, *On Ruelle–Perron–Frobenius Operators. I. Ruelle’s Theorem*, Commun. Math. Phys. 223 (2001), 125–141.

[Good71] Tim N. T. Goodman, *Relating topological entropy and measure entropy* Bull. London. Math. Soc. 3 (1971), 176–180.

[Kit99] A. Kitaev, *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness*, Nonlinearity 12 (1999), 141–179.

[Kit79] A. K. Kitover, *Spectrum of automorphisms with weight and the Kamowitz–Scheinberg theorem*, Funct. Anal. Appl., 13 (1) (1979), 57–58.

[KS97] A. Knauf, Ya. Sinai, *Classical Nonintegrability, Quantum Chaos*, Birkhauser, Basel, 1997.

[Kwa12] B.K. Kwasniewski, *On transfer operators for $C^*$-dynamical systems*, Rocky J. Math. 42 (3) (2012), 919–938.

[KL13] B.K. Kwasniewski, A.V. Lebedev, *Crossed products by endomorphisms and reduction of relations in relative Cuntz-Pimsner algebras*, J. Funct. Anal. 264 (8) (2013), 1806–1847.

[KL20] B.K. Kwasniewski, A.V. Lebedev, *Variational principles for spectral radius of weighted endomorphisms of $C(X, D)$*, Transactions of the American Mathematical Society, 373 (4) (2020), 2659–2698.

[Leb79] A. V. Lebedev, *On the invertibility of elements in $C^*$-algebras generated by dynamical systems*, Russian Math. Surveys, 34 (4) (1979), 174–175.

[LM98] A. Lebedev, O. Maslak, *The spectral radius of a weighted shift operator, variational principles, entropy and topological pressure*, Spectral and evolutionary problems. Proceedings of the Eighth Crimean Autumn Mathematical School Symposium (Simferopol, 1998). Tavria Publishers, Moscow, 26–34.

[LS88] Ju. D. Latushkin, A. M. Stepin, *Weighted shift operators on a topological Markov chain*, Funct. Anal. Its Appl., 22(4) (1988), 330–331.

[PP90] W. Parry, M. Pollicott, *Zeta functions and closed orbits for hyperbolic systems* Asterisque 187-188 (1990), 1–268.

[PU10] F. Przytycki, M. Urbanski, *Conformal Fractals: Ergodic Theory Methods*, London Mathematical Society Lecture Note Series 371, Cambridge University Press, 2010.

[Rou96] H. Rough, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergod. Th. & Dynam. Sys. 16 (1996), 805–819.
[Rue73] D. Ruelle, *Statistical mechanics on a compact set with \( Z^\alpha \) action satisfying expansiveness and specification*, Transactions of the American Mathematical Society, 185 (1973) 237–251.

[Rue78] D. Ruelle, *Thermodynamic formalism. Encyclopedia of Math. and its Appl.*, V.5, Reading, Mass: Addison-Wesley 1978.

[Rue89] D. Ruelle, *The thermodynamic formalism for expanding maps*, Comm. Math. Phys. 125 (1989), 239–262.

[Rue91] D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval* (CRM Monograph Series, 4). American Mathematical Society, Providence, RI, 1991.

[Wal75] P. Walters, *A variational principle for the pressure on continuous transformations*, American Journal of Mathematics. 97 (1975) 4, 937–971.

[Wal78] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances, Trans. Am. Math. Soc. 236 (1978), 121–153.

[Wal82] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York—Berlin, 1982.