Proving the Herman-Protocol Conjecture

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Abstract. Herman’s self-stabilisation algorithm, introduced 25 years ago, is a well-studied synchronous randomised protocol for enabling a ring of \(N\) processes collectively holding any odd number of tokens to reach a stable state in which a single token remains. Determining the worst-case expected time to stabilisation is the central outstanding open problem about this protocol. It is known that there is a constant \(h\) such that any initial configuration has expected stabilisation time at most \(hN^2\). Ten years ago, McIver and Morgan established a lower bound of \(4/27 \approx 0.148\) for \(h\), achieved with three equally-spaced tokens, and conjectured this to be the optimal value of \(h\). A series of papers over the last decade gradually reduced the upper bound on \(h\), with the present record (achieved last year) standing at approximately 0.156. In this paper, we prove McIver and Morgan’s conjecture and establish that \(h = 4/27\) is indeed optimal.

* Stefan Kiefer is supported by a University Research Fellowship of the Royal Society.
1 Introduction

The concept of self-stabilisation was introduced in a seminal paper of Dijkstra [5], and has since become a central theme in distributed computing; see, e.g., [6]. Herman’s algorithm, proposed in [12], considers a ring of \(N\) processes (or nodes), where each process either holds or doesn’t hold a token. Starting from any initial configuration of \(K\) tokens, where \(K\) is required to be odd, Herman’s algorithm proceeds as follows: at each time step, every process that holds a token either keeps it or passes it to its clockwise neighbour with probability \(1/2\). All updates happen synchronously, and if a process finds itself with two tokens (having simultaneously kept one and received one from its counterclockwise neighbour) then both tokens are annihilated. It is straightforward to see that, starting from an odd number of tokens and following this procedure, almost surely only one token eventually remains, at which point the ring is said to have \emph{stabilised}.

In [12], Herman provided a concrete model to implement this algorithm, and showed that the expected time (number of synchronous steps) to stabilisation is \(O(N^2 \log N)\). In 2004, Fribourg \emph{et al.} improved this upper bound to \(2N^2\), and the following year Nakata achieved an upper bound \(0.936N^2\). At the same time and independently, McIver and Morgan showed in [14] that the initial configuration consisting of three equally-spaced tokens has an expected stabilisation time of exactly \(\frac{4}{27}N^2\), and conjectured that this value is an upper bound on the expected time to stabilisation starting from any initial configuration with \emph{any} (odd) number of tokens. The conjecture is intriguing since an increase in the initial number of tokens might be thought to entail a longer expected stabilisation time, due to the larger number of collisions required to achieve stabilisation.

Nevertheless, McIver and Morgan’s Herman-Protocol Conjecture is supported by considerable amount of experimental evidence [3], and in the intervening years a series of papers have gradually reduced the upper bound on the constant \(h\) such that stabilisation from any initial configuration takes expected time at most \(hN^2\): upper bounds of approximately 0.64, 0.521, 0.167, and 0.156 are given respectively in [13, 7, 8, 11], the last one provided last year by Haslegrave, and coming relatively close to McIver and Morgan’s lower bound of \(4/27 \approx 0.148\).

In this paper, we prove McIver and Morgan’s conjecture and establish that \(h = 4/27\) is indeed optimal. Writing \(T_z\) for the stabilisation time starting from an initial configuration \(z\), we seek to prove that \(\mathbb{E}T_z \leq \frac{4}{27}N^2\). To this end, one of the key ideas is to work with a Lyapunov function \(V(z)\) in lieu of the (more complicated) function \(\mathbb{E}T_z\). Combinatorial arguments exploiting the highly symmetrical structure of \(V(z)\) enable us to establish that, for an arbitrary configuration \(z\), we have \(\mathbb{E}T_z \leq V(z)\), with equality holding for all three-token configurations. Moreover, we use analytical techniques to show that \(V\) has no local maxima greater than \(\frac{4}{27}N^2\) on configurations of five tokens or more other than on the boundary. Combined together, these facts entail the Herman-Protocol Conjecture.

It is interesting to note that Herman’s ring is closely related to widely-studied models of random walks and Brownian motion in statistical physics. Observe that by a simple modification of the formalism, one may equivalently view Herman’s model as a ring in which tokens randomly move in discrete step in \emph{any} direction, with pairwise collisions leading to annihilation; this precisely corresponds to Fisher’s \emph{vicious drunks} model [9] (with periodic boundary conditions). Similar models have been studied in chemical physics [4, 2, 17] and statistical mechanics [10, 15, 16], among others.

Herman’s protocol can also be seen as an instance of the notion of \emph{coalescing random walks} as proposed by [11]. In this model, one considers independent random walks starting from each vertex of a given connected graph \(G\). When some of them meet, they coalesce into a new random walk, and the question considered is what is the expected time until all walks coalesce into a single random walk.
The rest of the paper is organized as follows. In Section 2 we review previous results in the literature that are relevant to our proof. In Section 3 we outline the structure of our proof, identifying two key lemmas, Lemma 8 and Lemma 9. Those are proved in Appendix A and Section 4, respectively.

2 Relevant Previous Results

For the rest of the paper we fix the number $N$ of processes. We assume that the number $K$ of tokens is odd, and both $N$ and $K$ are at least 3.

Processes are numbered from 1 to $N$, clockwise, according to their position in the ring. A configuration with $K$ tokens is formalised as a function $z : \{1, \ldots, K\} \to \{1, \ldots, N\}$ with $z(1) < \cdots < z(K)$, where the $i$th token ($i \in \{1, \ldots, K\}$) is held by the processor with the number $z(i)$. We write $Z_K$ for the set of configurations with $K$ tokens, and $Z$ for the set of all possible configurations, that is, $Z = Z_1 \cup Z_3 \cup \cdots \cup Z_{N-2} \cup Z_N$. For a fixed initial configuration $z = z_0$ we write $(z_t)_{t \geq 0}$ for the random sequence of configurations emanating from $z$. The stabilization time $T_z$ is the smallest $t \geq 0$ such that $z_t \in Z_1$, i.e., the time until only one token is left. In this paper we focus on the expectation $\mathbb{E}T_z$. It is shown in \cite{14} that if $N$ is odd and a multiple of 3, then there is a configuration $z \in Z_3$ (with the 3 tokens maximally separated in an equilateral triangle) such that $\mathbb{E}T_z = \frac{1}{27}N^2$.

In this paper we show:

**Theorem 1.** We have $\mathbb{E}T_z \leq \frac{4}{27}N^2$ for all $z \in Z$.

Equivalently, the Herman conjecture states that for all odd $K \geq 3$ and all $z \in Z_K$ we have $\mathbb{E}T_z \leq \frac{4}{27}N^2$. Only the case $K = 3$ was previously known \cite{14}.

The following proposition has been used in a similar form in various papers on Herman’s protocol, for instance in \cite{14} Lemma 5. It bounds the stabilization time by a Lyapunov function $V$.

**Proposition 2 (Bound by a Lyapunov function).** Given $z \in Z$, denote by $z' \in Z$ the random successor configuration of $z$. Let $V : Z \to [0, \infty)$ be a function with

$$
\mathbb{E}(V(z') \mid z) \leq V(z) - 1 \quad \text{for all } z \in Z \setminus Z_1.
$$

Then $\mathbb{E}T_z \leq V(z)$ for all $z \in Z$.

Although this result is not new, we give a short proof based on a martingale argument. The proof is inspired by \cite{11}, and may provide some intuition.

**Proof.** Consider the random sequence $(z_t)_{t \geq 0}$ of configurations emanating from $z = z_0$. Define $W_t := V(z_t) + t$. By \cite{11} the process $(W_t)_{t \geq 0}$ is a supermartingale. The stabilization time $T_z = T_{z_0}$ is a stopping time with finite expectation, and the differences $|W_{t+1} - W_t|$ are bounded as the Markov chain reachable from $z$ has finitely many states. Hence, the optional stopping theorem applies, yielding $\mathbb{E}W_{T_z} \leq \mathbb{E}W_0 = V(z)$. By definition of $W_t$ we have $\mathbb{E}W_{T_z} = \mathbb{E}V(z_{T_z}) + \mathbb{E}T_z \geq \mathbb{E}T_z$. Hence $\mathbb{E}T_z \leq V(z)$.  

Following \cite{11} we associate to a configuration $z \in Z_K$ the gap vector $g(z) = (g_0, \ldots, g_{K-1}) \in \mathbb{N}^K$ by setting $g_0 := N + z(1) - z(K)$, and $g_i := z(i+1) - z(i)$ for $i \in \{1, \ldots, K-1\}$. Then $g(z)/N$ lives in the so-called standard $(K-1)$-simplex $D(K)$, defined by

$$
D(K) := \{ x = (x_0, \ldots, x_{K-1}) \in [0, 1]^K \mid x_0 + \cdots + x_{K-1} = 1 \}.
$$

2
Towards a suitable Lyapunov function $V$ we define the cubic polynomial $f_3^{(K)} : D(K) \to [0, \infty)$ by

$$f_3^{(K)}(x) := \sum_{0 \leq i_0 < i_1 < i_2 < K, \ i_2 - i_1, i_1 - i_0 \ odd} x_{i_0} x_{i_1} x_{i_2}.$$ 

For instance, we have $f_3^{(5)}(x) = x_0 x_1 x_2 + x_0 x_1 x_4 + x_0 x_3 x_4 + x_1 x_2 x_3 + x_2 x_3 x_4$.

The following lemma was implicitly proved in previous works:

**Lemma 3 (Lyapunov function $V_3$ [8, Proof of Theorem 1], [11, Theorem 4]).** Define $V_3 : Z \to [0, \infty)$ by $V_3(z) := 4N^2 f_3^{(K)}(g(z)/N)$. Let $z \in Z_K$ and denote by $z'$ the random successor configuration of $z$. Then $\mathbb{E}(V_3(z') \mid z) = V_3(z) - \frac{K-1}{2}$. Hence, by Proposition 2 $\mathbb{E}T_z \leq 4N^2 f_3^{(K)}(g(z)/N)$.

For $K = 3$ Lemma 3 gives $\mathbb{E}T_z \leq 4N^2 f_3^{(3)}(g(z)/N) = \frac{4}{7} g_0 g_1 g_2$. In fact, for $K = 3$ it was shown before in [11] that $\mathbb{E}T_z$ is identically equal to $\frac{4}{7} g_0 g_1 g_2$, providing an exact formula for the expected stabilization time of configurations with three tokens. Lemma 3 suggests analyzing $f_3$:

**Lemma 4 (Maximum of $f_3$ [8, Proof of Theorem 2], [11, Theorem 3]).** For all $K \geq 3$ odd we have

$$\max_{x \in D} f_3^{(K)}(x) = f_3^{(K)} \left( \frac{1}{K}, \ldots, \frac{1}{K} \right) = \frac{1}{24} \left( 1 - \frac{1}{K^2} \right).$$

By combining Lemmas 3 and 4 one obtains $\mathbb{E}T_z \leq \frac{N^2}{6} (1 - \frac{1}{K^2})$, which is the bound obtained in [8]. A slightly better bound is given in [11].

### 3 Proof of the Herman Conjecture

The function $V_3$ from Lemma 3 leaves room for improvement since $\mathbb{E}(V_3(z') \mid z) = V_3(z) - \frac{K-1}{2}$, which is strictly less than $V_3(z) - 1$ for $K > 3$. The idea for obtaining an optimal bound is to decrease the gap between $\frac{K-1}{2}$ and 1, by decreasing the Lyapunov function $V$. To this end we define a quintic polynomial $f_5^{(K)} : D(K) \to [0, \infty)$, similar to $f_3^{(K)}$:

$$f_5^{(K)}(x) := \sum_{0 \leq i_0 < i_1 < \cdots < i_4 < K, \ i_4 - i_3, \ldots, i_1 - i_0 \ odd} x_{i_0} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$$

For instance, $f_5^{(3)}(x) = 0$, $f_5^{(5)}(x) = x_0 x_1 x_2 x_3 x_4$, and $f_5^{(7)}(x) = x_0 x_1 x_2 x_3 x_4 + x_0 x_1 x_2 x_3 x_6 + x_0 x_1 x_2 x_5 x_6 + x_0 x_1 x_4 x_5 x_6 + x_0 x_3 x_4 x_5 x_6 + x_1 x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 x_6$. We also define a polynomial $f^{(K)} : D(K) \to [0, \infty)$:

$$f^{(K)}(x) := f_3^{(K)}(x) - \alpha f_5^{(K)}(x) \quad \text{with} \ \alpha := 24$$

(2)

For example, $f^{(5)}(x) = x_0 x_1 x_2 + x_0 x_1 x_4 + x_0 x_3 x_4 + x_1 x_2 x_3 + x_2 x_3 x_4 - \alpha x_0 x_1 x_2 x_3 x_4$. Throughout the paper we use $\alpha$ in the expression of $f^{(K)}$ for notational convenience. From now onwards we may drop the superscript $K$ from the domain $D^{(K)}$ or the functions $f_3^{(K)}$, $f_5^{(K)}$, and $f^{(K)}$ to avoid notational clutter when $K$ is understood.

The following properties of $f$ are fundamental:

**Lemma 5 (Symmetry and harmony properties).** The function $f$ has the following properties.
(a) It is symmetric with respect to rotation:
\[ f(x_0, \ldots, x_{K-1}) = f(x_1, \ldots, x_{K-1}, x_0) \]

(b) It is in harmony\(^1\) For \( K \geq 5 \) we have
\[ f^{(K)}(x_0, 0, x_2, x_3, \ldots, x_{K-1}) = f^{(K-2)}(x_0 + x_2, x_3, \ldots, x_{K-1}). \]

Analogous properties were shown for \( f_3 \) in [8]. Their proof carries over to \( f_5 \) and hence to \( f \). The following lemma uses \( f \) to define a tighter Lyapunov function.

**Lemma 6 (Lyapunov function \( V \)).** Define \( V : Z \to [0, \infty) \) by \( V(z) := 4N^2f(g(z)/N) \). Let \( z \in Z \) and denote by \( z' \) the random successor configuration of \( z \). Then \( \mathbb{E}(V(z') \mid z) \leq V(z) - 1 \).

Lemma 6 suggests analyzing \( f \):

**Lemma 7 (Maximum of \( f \)).** For all \( K \geq 3 \) odd we have
\[ \max_{x \in D} f^{(K)}(x) = \frac{1}{27}. \]

With this in hand our main result follows:

**Proof (of Theorem 1).** Immediate by combining Lemmas 6 and 7. \( \square \)

It remains to prove Lemmas 6 and 7.

### 3.1 Proof of Lemma 6

Towards Lemma 6 we show:

**Lemma 8 (Lyapunov function \( V_5 \)).** Define \( V_5 : Z \to [0, \infty) \) by \( V_5(z) := 4N^2f_5(g(z)/N) \). Let \( K \geq 5 \) and \( z \in Z \) and denote by \( z' \) the random successor configuration of \( z \). Then
\[ \mathbb{E}(V_5(z') \mid z) = V_5(z) + \frac{1}{32} \frac{(K-1)(K-3)}{N^2} - \frac{1}{2}(K-3)f_3 \left( \frac{g(z)}{N} \right). \]

The proof in Appendix A requires an analysis of correlations among the changes in gaps between tokens in each step of the protocol. Using Lemma 8 one can readily prove Lemma 6.

**Proof (of Lemma 7).** For \( K = 3 \) the statement follows from Lemma 3. For \( K \geq 5 \) we have:
\[
\begin{align*}
\mathbb{E}(V(z') \mid z) &= \mathbb{E}(V_3(z')) - 24V_5(z') \mid z) \\
&= \mathbb{E}(V_3(z') \mid z) - 24\mathbb{E}(V_5(z') \mid z) \\
&= V_3(z) - \frac{K-1}{2} - 24V_5(z) - \frac{3}{4} \frac{(K-1)(K-3)}{N^2} \\
&\quad + 12(K-3)f_3 \left( \frac{g(z)}{N} \right) \\
&\leq V(z) - \frac{K-1}{2} + 12(K-3)f_3 \left( \frac{g(z)}{N} \right) \\
&\leq V(z) - \frac{K-1}{2} + \frac{K-3}{2} \\
&= V(z) - 1
\end{align*}
\]

\( ^1 \) We adopt this term from [8].
3.2 Proof of Lemma 7
Towards Lemma 7 we show:

Lemma 9 (Local maxima of f). Let $K \geq 5$ and odd. There is no $v \in D^K$ in the interior of $D^K$ such that $v$ is a local maximum and $f^K(v) > \frac{1}{27}$.

The proof in Section 4 involves a combinatorial analysis of inequalities arising from conditions on the derivatives of $f^K$. Using Lemma 9 one can readily prove Lemma 7.

Proof (of Lemma 7). We proceed by induction on $K$. For the induction base we have $K = 3$. It is straightforward to check that the maximum of $f^{(3)}(x) = f^{(3)}(x) = x_0x_1x_2$ is $f^{(3)}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$.

For the induction step we have $K \geq 5$. Let $v \in D^K$ with $f^K(v) = \max_{x \in D^K} f^K(x)$. If $v$ is in the interior of $D^K$, then by Lemma 9 we have $f^K(v) \leq \frac{1}{27}$. If $v$ is at the boundary of $D^K$, then $v_i = 0$ for some $i$. By Lemma 5(a) we can assume that $v_1 = 0$. Using Lemma 5(b) the statement follows from the induction hypothesis.

4 Proof of Lemma 9
In this section we prove Lemma 9. In Section 4.1 we state several properties that an interior local maximum of $f^K$ would have to satisfy. In Section 4.2 we prove Lemma 9 for $K = 5$ for a first taste of the general argument. In Section 4.3 we prove Lemma 9 for $K = 7$ to illustrate some fine points that occur only for larger values of $K$. In Section 4.4 we state some combinatorial facts needed for the general case. Finally, in Section 4.5 we prove Lemma 9.

4.1 Properties of an Interior Local Maximum
The following lemma is obtained by considering first and second derivatives of $f$ evaluated at an interior local maximum.

Lemma 10. Let $v$ be a local maximum of $f^K$ in the interior of $D^K$ and define $c \in \mathbb{R}$ by

$$c = \sum_{1<i_2<K, i_2 \text{ even}} v_{i_2} - \alpha \sum_{1<i_2<i_3<i_4<K, i_2, i_4 \text{ even, } i_3 \text{ odd}} v_{i_2}v_{i_3}v_{i_4}. \quad (3)$$

This expression holds for the same value of $c$ if the indices are rotated by an arbitrary $k$: for all $j$ the index $i_j$ becomes $(i_j + k) \mod K$. Further, we have

$$\sum_{3 \leq i_3 < i_4 < K, i_3 \text{ odd, } i_4 \text{ even}} v_{i_3}v_{i_4} \leq \frac{1}{\alpha}. \quad (4)$$

Again, this inequality also holds when indices are rotated.

For example, for $K = 7$ we have $c = v_2 + v_4 + v_6 - \alpha(v_2v_3v_4 + v_2v_3v_6 + v_2v_5v_6 + v_4v_5v_6) = v_1 + v_3 + v_5 - \alpha(v_1v_2v_3 + v_1v_2v_5 + v_1v_4v_5 + v_3v_4v_5)$.

Proof (of Lemma 7). The idea of the proof is as follows. We pick a particular direction in $D^K$, namely $d = (-1,0,1,0,0,\ldots,0)$, and consider the function $f(v+\epsilon d)$ as a univariate function of $\epsilon$. Since $v$ is a local maximum, the first derivative must be zero and the second derivative must be nonpositive. Exploiting the fact that $v_i > 0$ for all $i$ holds in the interior, we obtain (3) and (4), respectively. See Appendix B for the detailed proof.
Let $S_j^{(K)}(x)$ denote the scalar product of $x$ with a copy of itself rotated $j$ times:

$$S_j^{(K)}(x) := \sum_{i=0}^{K-1} x_i x_{i+j}$$

In all formulas it will be the case that the subscript of $S$ is odd. Also, the superscript will be omitted when unimportant or understood from context.

**Corollary 11.** Let $v$ be a local maximum of $f^{(K)}$ in the interior of $D^{(K)}$. Then the following inequality holds:

$$\sum_{1 \leq i < K-2}^{i \ odd} \frac{K-i-2}{2} S_i(v) \leq \frac{K}{\alpha}$$

For example, for $K = 11$ we have $4S_1(v) + 3S_3(v) + 2S_5(v) + S_7(v) \leq 11/\alpha$.

**Lemma 12 (Bound for $f_5$).** Suppose that $v \in D^{(K)}$ satisfies $f^{(K)}(v) > \frac{1}{27}$. Then $\alpha f_5(v) < \frac{1}{216}$.

*Proof.* By Lemma 11 we have $f_3(v) \leq \frac{1}{24}$ and hence $\alpha f_5(v) = f_3(v) - f(v) < \frac{1}{24} - \frac{1}{27} = \frac{1}{216}$. □

### 4.2 Proof of Lemma 9 for $K = 5$

Let $K = 5$. Then

$$f(x) = f_5(x) - \alpha f_5(x) = x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 - \alpha x_0x_1x_2x_3x_4$$

Towards a contradiction, suppose that there is a local maximum $v$ with $f(v) > \frac{1}{27}$ in the interior of $D$. By (3), the value $c = v_2 + v_4 - \alpha v_2v_3v_4$ (5) is invariant under rotations. Indeed, $v_{2+k} + v_{4+k} - \alpha v_{2+k}v_{3+k}v_{4+k} \equiv c$ for all $k$, but we shall avoid explicitly mentioning rotations, for notational simplicity. Summing (5) over all $K$ rotations we obtain:

$$5c = 2 - \alpha f_3(v) \quad (6)$$

By (3) we have $v_0v_1c = v_0v_1v_2 + v_0v_1v_4 - \alpha f_5(v)$ and, summing over all $K$ rotations,

$$cS_1(v) = 2f(v) - 3\alpha f_5(v) \quad (7)$$

Moreover,

$$cS_1(v) \leq \frac{11}{\alpha} \frac{5c}{\alpha} = \frac{2}{\alpha} - f_3(v) = \frac{2}{\alpha} - f(v) - \alpha f_5(v).$$

Combining this with (7) gives:

$$\frac{2}{\alpha} \geq 3f(v) - 2\alpha f_5(v) \quad \text{Lemma 12} \quad \frac{3}{\alpha} \geq \frac{3}{27} - 2 \cdot \frac{1}{216}$$

This implies $\alpha \leq 216/11 \approx 19.6$, which is a contraction as required (since $\alpha = 24$). □
4.3 Proof of Lemma 9 for $K = 7$

Let $K = 7$. Towards a contradiction, we suppose again that there is a local maximum $v$ with $f(v) > \frac{1}{27}$ in the interior of $D$. By (3), all $K$ rotations of the following hold with the same $c \in \mathbb{R}$:

$$c =  v_2 + v_4 + v_6 - \alpha (v_2v_3v_4 + v_2v_3v_6 + v_4v_5v_6)$$

(8)

Summing (8) over $K$ rotations we obtain:

$$7c = 3 - 2\alpha f_3(v)$$

(9)

By (8) we have

$$v_0v_1c = v_0v_1v_2 + v_0v_1v_4 + v_0v_1v_6 - \alpha (v_0v_1v_2v_3v_4 + v_0v_1v_2v_3v_6 + v_0v_1v_2v_5v_6 + v_0v_1v_4v_5v_6)$$

(10)

and

$$v_0v_3c = v_0v_3v_4 + v_0v_3v_6 - \alpha v_0v_3v_4v_5v_6 + v_0v_3v_3(1 - \alpha (v_3v_4 + v_3v_6 + v_5v_6))$$

(11)

where the last inequality is by (4). Summing (10) and (11) over $K$ rotations we obtain:

$$c(2S_1(v) + S_3(v)) \leq 4f_3(v) - 9\alpha f_5(v) = 4f(v) - 5\alpha f_5(v)$$

(12)

Further we have:

$$c(2S_1(v) + S_3(v)) \leq \frac{7c}{\alpha} \geq \frac{3}{\alpha} - 2f_3(v) = \frac{3}{\alpha} - 2f(v) - 2\alpha f_5(v)$$

Combining this with (12) gives:

$$\frac{3}{\alpha} \geq 6f(v) - 3\alpha f_5(v) \geq \frac{6}{27} - 3 \cdot \frac{1}{216}$$

This leads to $\alpha \leq 14.4$, which is a contradiction as desired.

4.4 Combinatorial Lemmas

In order to generalize the proofs from Sections 4.2 and 4.3 to any odd $K$, we state some combinatorial lemmas in this subsection. They are proved in Appendix C.

In order to generalize (9) and (12) we show the following lemma:

**Lemma 13.** We have:

$$\sum_{k=0}^{K-1} \sum_{\substack{0 \leq i_0 < i_1 < i_2 < K \\ i_0', i_2' \text{ even} \\ i_1' \text{ odd}}} x_{i_0+k}x_{i_1+k}x_{i_2+k} = \frac{K - 3}{2} \sum_{0 \leq i_0 < i_1 < i_2 < K \\ i_2 - i_1, i_1 - i_0 \text{ odd}} x_{i_0}x_{i_1}x_{i_2} = \frac{K - 3}{2} f_3(K)(x)$$

For example, if $K = 5$, then we obtain that summing the 5 rotations of $x_2x_3x_4$ gives $f_3^{(5)}(x)$. As another example, if $K = 7$, then we obtain that summing the 7 rotations of $x_2x_3x_4 + x_2x_3x_6 + x_2x_5x_6 + x_4x_5x_6$ gives $2f_3^{(7)}(x)$. These two instances of Lemma 13 help establish (9) and (12).

In order to generalize the inequality in (11) we need the following lemma:
Lemma 14. Let \( \nu \) be a local maximum of \( f^{(K)} \) in the interior of \( D^{(K)} \). If \( i_1 \) is odd and \( 0 < i_1 < K \), then the following inequality holds:

\[
v_0v_{i_1}\left(\sum_{1<i_2<K \atop i_2 \text{ even}} v_{i_2} - \sum_{1<i_3<\nu \atop i_3 \text{ odd}} \alpha v_{i_2}v_{i_3}v_{i_4}\right) \geq v_0v_{i_1}\left(\sum_{1<i_2<K \atop i_2 \text{ even}} v_{i_2} - \sum_{1<i_3<\nu \atop i_3 \text{ odd}} \alpha v_{i_2}v_{i_3}v_{i_4}\right)
\]

The inequality says that if we drop those terms that do not occur in \( f_3^{(K)} \) or \( f_5^{(K)} \), then we obtain a lower bound. The proof groups those terms that are not in either of \( f_3^{(K)} \) or \( f_5^{(K)} \), and then invokes (4) to show that their sum is nonnegative.

In order to generalize (7) and (12) we need Corollary 16 below, which is a consequence of the following lemma:

Lemma 15. Let \( l \) be an odd, positive integer. Then:

\[
\sum_{k=0}^{K-1} \sum_{1 \leq i'_{1} < K-2 \atop i'_{1} \text{ odd}} \frac{K-i'_{1} - 2}{2} \sum_{i_{j} < i'_{j} < \cdots < i'_{l-1} < K \atop \forall j, i'_{j} \equiv j \text{ (mod 2)}} x_{i_{j} + k} \prod_{1 < j < l} x_{i'_{j} + k} =
\]

\[
= \left(\frac{l-1}{2} \right) (K-l) \sum_{0 \leq i_{j} < \cdots < i_{l-1} < K \atop i_{j} - i_{j-1} \text{ odd for } 0 < j < l} \prod_{j=0}^{l-1} x_{i_{j}}
\]

For example, if \( K = 5 \) and \( l = 3 \), then we have that summing 5 rotations of \( x_0x_1x_2 + x_0x_1x_4 \) gives \( 2f_3^{(5)}(x) \). As another example, if \( K = 9 \) and \( l = 3 \), then summing 9 rotations of \( 3x_0x_1(x_2 + x_4 + x_6 + x_8) + 2x_0x_3(x_1 + x_6 + x_8) + x_0x_5(x_6 + x_8) \) gives \( 6f_3^{(9)}(x) \).

Corollary 16. We have:

\[
\sum_{k=0}^{K-1} \sum_{1 \leq i_1 < K-2 \atop i_1 \text{ odd}} \frac{K-i_1 - 2}{2} \sum_{1 < i_2 < K \atop i_2 \text{ even}} x_{0+k}x_{i_1+k}x_{i_2+k} = (K-3)f_3^{(K)}(x)
\]

and also

\[
\sum_{k=0}^{K-1} \sum_{1 \leq i_1 < K-2 \atop i_1 \text{ odd}} \frac{K-i_1 - 2}{2} \sum_{1 < i_2 < i_3 < i_4 < K \atop i_2, i_4 \text{ even } \atop i_3 \text{ odd}} x_{0+k}x_{i_1+k}x_{i_2+k}x_{i_3+k}x_{i_4+k} = (2K-5)f_5^{(K)}(x)
\]

Proof. Instantiate Lemma 15 with \( l = 3 \) and, respectively, \( l = 5 \).

4.5 Proof of Lemma 9

Towards a contradiction, suppose that there is a local maximum \( \nu \) with \( f(\nu) > \frac{1}{27} \) in the interior of \( D \), i.e., \( v_i > 0 \) for all \( i \in \{0, \ldots, K-1\} \). Summing up the \( K \) rotations of (9) and using Lemma 15, we obtain:

\[
Kc = \frac{K-1}{2} - \frac{K-3}{2} \alpha f_3(\nu)
\]

(13)
Multiplying (3) on both sides by \( \sum_{1 \leq i_1 < K - 2} \frac{K - i_1 - 2}{2} v_0 v_{i_1} \) we obtain:

\[
c \sum_{1 \leq i_1 < K - 2} \frac{K - i_1 - 2}{2} v_0 v_{i_1} = \sum_{1 \leq i_1 < K - 2} \frac{K - i_1 - 2}{2} v_0 v_{i_1} \left( \sum_{1 < i_2 < K} \frac{v_{i_2} - \sum_{i_1 < i_3 < i_4 < K} \alpha v_{i_2} v_{i_3} v_{i_4}}{i_2, i_4 \text{ even}\ i_3 \text{ odd}} \right)
\]

using Lemma 14. Summing \( K \) rotations of this inequality yields:

\[
c \sum_{1 \leq i_1 < K - 2} \frac{K - i_1 - 2}{2} S_{i_1}(v) \geq (K - 3)f_3(v) - (2K - 5)\alpha f_5(v) = (K - 3)f(v) - (K - 2)\alpha f_5(v).
\]

using Corollary 16. Further we have:

\[
c \sum_{1 \leq i_1 < K - 2} \frac{K - i_1 - 2}{2} S_{i_1}(v) \geq \frac{M c}{\alpha} \leq \frac{K - 1}{2\alpha} - \frac{K - 3}{2} f_3(v)
\]

Combining this with (14) gives:

\[
\frac{K - 1}{2\alpha} \geq \frac{3K - 9}{2} f(v) - \frac{K - 1}{2} \alpha f_5(v) \geq \frac{K - 3}{2} \cdot \frac{1}{9} \cdot \frac{K - 1}{2} \cdot \frac{1}{216}
\]

This implies

\[
\alpha \leq \frac{216(K - 1)}{23K - 71} < 19.7
\]

Since \( \alpha = 24 \), this leads to a contradiction as desired. \( \square \)

5 Conclusions

In this paper we have proved the Herman-Protocol Conjecture posed by McIver and Morgan in [14] a decade ago, namely that the worst-case expected self-stabilization time of a ring of \( N \) processes is \( \frac{1}{27}N^2 \) regardless of the number of tokens \( K \). Interestingly, this value is equal to the expected self-stabilization time of a three-token configuration only, where the three tokens are initially equally spaced on the ring.

The proof uses a Lyapunov function approach. To do so, we first find a suitable Lyapunov function and then show that its maximum is \( \frac{1}{27}N^2 \). Then we show that this function gives an upper bound for the self-stabilization time for each possible configuration in Herman’s algorithm.

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Proof of Lemma 8

Let \( z : \{1, \ldots, K\} \to \{1, \ldots, N\} \) be a \( K \)-token configuration on a ring with \( N \) processes. Recall that the associated gap vector \( g(z) = (g_0, \ldots, g_{K-1}) \in \mathbb{N}^K \) is defined by \( g_0 := N + z(1) - z(K) \) and \( g_i := z(i+1) - z(i) \) for \( i = 1, \ldots, K-1 \).

Given \( z \), consider the gap-increment vector \( \Delta := g(z') - g(z) \), where \( z' \) is the random successor configuration of \( z \). This is a random variable taking values in \( \{-1,0,+1\}^K \) where, for each \( i \in \{0, \ldots, K-1\} \), \( \Delta_i = 0 \) with probability \( 1/2 \) (the two tokens adjacent to the \( i \)-th gap both stay or both move clockwise), and \( \Delta_i = \pm 1 \) with probability \( 1/4 \) (one token stays and the other moves clockwise).

We will need the following two properties (15) and (16) concerning the expectation of the random variable \( \Delta \). First, it is straightforward to verify by direct calculation that for \( 0 \leq k < K \),

\[
\mathbb{E}(\Delta_i \Delta_{i+1} \cdots \Delta_{i+k}) = \begin{cases} 0 & \text{if } k \text{ is even} \\ (-\frac{1}{4})^{(k+1)/2} & \text{if } k \text{ is odd} \end{cases} \tag{15}
\]

Secondly, suppose that \( 0 \leq i_1 < i_2 < i_3 < i_4 < K \), with \( i_3 \not\equiv i_2 + 1 \) and \( i_1 \not\equiv i_4 + 1 \) modulo \( K \), that is, \( \{i_1, \ldots, i_2\} \) and \( \{i_3, \ldots, i_4\} \) form two non-adjacent intervals (treating \( K-1 \) and \( 0 \) as adjacent). Then

\[
\mathbb{E}(\Delta_{i_1} \cdots \Delta_{i_2} \Delta_{i_3} \cdots \Delta_{i_4}) = \mathbb{E}(\Delta_{i_1} \cdots \Delta_{i_2}) \mathbb{E}(\Delta_{i_3} \cdots \Delta_{i_4}) \tag{16}
\]

because \( \Delta_{i_1}, \ldots, \Delta_{i_2} \) and \( \Delta_{i_3}, \ldots, \Delta_{i_4} \) are determined by the movements of disjoints sets of tokens, and hence are independent.

For a given configuration \( z \) we want to compute \( \mathbb{E}[f_5(g(z) + \Delta)] \). From the definition of \( f_5 \) and the linearity of expectation, this is a sum of expressions of the form

\[
\mathbb{E}(g_{i_0} + \Delta_{i_0})(g_{i_1} + \Delta_{i_1})(g_{i_2} + \Delta_{i_2})(g_{i_3} + \Delta_{i_3})(g_{i_4} + \Delta_{i_4}) \tag{17}
\]

over the set of indices \( 0 \leq i_0 < i_1 < i_2 < i_3 < i_4 < K \) of alternating parity.

Expression (17) evaluates to a degree-5 polynomial in the variables \( g \). Observe that this polynomial has no monomials of degree 4 or 2. Indeed, expanding (17) yields degree-4 monomials with coefficient \( \mathbb{E}(\Delta_i) \) and degree-2 monomials with coefficient \( \mathbb{E}(\Delta_i \Delta_j \Delta_k) \) with \( i < j < k \). But \( \mathbb{E}(\Delta_i) \) and \( \mathbb{E}(\Delta_i \Delta_j \Delta_k) \) are both zero by (15) and (16).

There is a single degree-5 monomial in (17)—namely \( g_{i_0} \ldots g_{i_4} \). Summing all such terms over indices \( 0 \leq i_0 < i_1 < i_2 < i_3 < i_4 < K \) of alternating parity yields \( f_5(g(z)) \).

Expanding the expression (17) yields degree-3 monomials of the form

\[
g_{j_0}g_{j_1}g_{j_2} \mathbb{E}(\Delta_{j_3} \Delta_{j_4})
\]

for distinct indices \( j_0 < j_1 < j_2 \). The coefficient of such a term is \( -1/4 \) if \( j_3 = j_3 + 1 \) or \( j_3 = j_4 + 1 \) and 0 otherwise. Moreover, if \( j_0, j_1, j_2 \) have alternating parity there are \( (K-3)/2 \) choices of \( j_3 \) such that \( g_{j_0}g_{j_1}g_{j_2} \mathbb{E}(\Delta_{j_3} \Delta_{j_3+1}) \) appears in (17). If \( j_0, j_1, j_2 \) do not have alternating parity then there are no such terms in (17). We conclude that the sum of all degree-3 monomials in \( \mathbb{E}(f_5(g(z) + \Delta)) \) is

\[
-\frac{(K-3)}{8} f_3(g(z)) .
\]

Finally, consider the degree-1 monomials. These have the form

\[
g_{j_0} \mathbb{E}(\Delta_{j_1} \Delta_{j_2} \Delta_{j_3} \Delta_{j_4})
\]
for distinct indices \( j_0 \) and \( j_1 < j_2 < j_3 < j_4 \). By Property [10], such terms are only non-zero if \( \{j_1, j_2, j_3, j_4\} \) comprises either a single block of adjacent indices or two non-adjacent blocks of length 2 (considering \( K - 1 \) and 0 to be adjacent). In this case \( \mathbb{E}(\Delta_{j_1}\Delta_{j_2}\Delta_{j_3}\Delta_{j_4}) = 1/16 \), and there are \( \binom{K-1}{2} = (K-1)(K-3)/8 \) such choices of \( \{j_1, j_2, j_3, j_4\} \) for each choice of \( j_0 \). Thus \( g_{j_0} \) has total coefficient \( (K-1)(K-3)/128 \) in \( \mathbb{E}(f_5(g(z) + \Delta)) \). Moreover, since \( g_0 + \ldots + g_{K-1} = N \), the degree-1 terms in \( \mathbb{E}(f_5(g(z) + \Delta)) \) sum to \( N(K-1)(K-3)/128 \).

In summary, we have proved:

**Proposition 17.** For each \( K \)-token configuration \( z \),

\[
\mathbb{E}(f_5(g(z) + \Delta)) = f_5(g(z)) - \frac{K-3}{8} f_5(g(z)) + \frac{(K-1)(K-3)N}{128}.
\]

Lemma [8] follows immediately from Proposition [17] by scaling, since \( V_5(z) = 4N^2 f_5(g(z)/N) = \frac{4}{\pi} f_5(g(z)) \) is a linear multiple of \( f_5(g(z)) \).

**B Proofs of Properties of an Interior Local Maximum**

**Lemma 10.** Let \( v \) be a local maximum of \( f^{(K)} \) in the interior of \( D^{(K)} \) and define \( c \in \mathbb{R} \) by

\[
c = \sum_{1 \leq i_2 < K \atop i_2 \text{ even}} v_{i_2} - \alpha \sum_{1 \leq i_2 < i_3 < i_4 < K \atop i_2, i_4 \text{ even}, \ i_3 \text{ odd}} v_{i_2}v_{i_3}v_{i_4},
\]

(18)

This expression holds for the same value of \( c \) if the indices are rotated by an arbitrary \( k \): for all \( j \) the index \( i_j \) becomes \( (i_j + k) \mod K \). Further, we have

\[
\sum_{3 \leq i_3 < i_4 < K \atop i_3 \text{ odd}, \ i_4 \text{ even}} v_{i_3}v_{i_4} \leq \frac{1}{\alpha}
\]

(19)

Again, this inequality also holds when indices are rotated.

**Proof.** We consider the second-order Taylor expansion of \( f^{(K)} \) along the direction \( d = (-1, 0, 1, 0, \ldots, 0) \) (which is tangent to \( D^{(K)} \)):

\[
f^{(K)}(x + \epsilon d) = f^{(K)}(x) + \epsilon Q(x) + \epsilon^2 R(x) + O(\epsilon^3).
\]

Since \( v \) is a local maximum, we have \( Q(v) = 0 \) and \( R(v) \leq 0 \). Proving [18] boils down to calculating \( Q(x) \); proving [19] boils down to calculating \( R(x) \). In what follows, it helps to think of \( x_0, \ldots, x_{K-1} \) as symbolic variables used in formal manipulations. On the other hand, \( v \) is one particular point in \( D^{(K)} \), the assumed local maximum from the lemma statement.

First, we prove [18]. Let

\[
f^{(K)}(x_0 + \epsilon, x_1, \ldots, x_{K-1}) = f^{(K)}(x) + \epsilon P(x) + O(\epsilon^2).
\]

By the chain rule and using the rotational symmetry of \( f^{(K)} \) (Lemma [5(a)]), we find that

\[
Q(x_0, \ldots, x_{K-1}) = P(x_{0+2}, x_{1+2}, \ldots, x_{K-1+2}) - P(x_0, x_1, \ldots, x_{K-1}).
\]
Now recall the definition of $f^{(K)}$: 

$$f^{(K)}(x) := \sum_{0 \leq i_0 < i_1 < i_2 < K} x_{i_0} x_{i_1} x_{i_2} - \sum_{i_2 - i_1 \text{ and } i_4 - i_3, \ldots, i_1 - i_0 \text{ odd}} \alpha x_{i_0} x_{i_1} x_{i_2} x_{i_3} x_{i_4} \quad (20)$$

We differentiate $f^{(K)}$ with respect to $x_0$ to obtain

$$P(x) = \sum_{0 < i_1 < i_2 < K} x_{i_1} x_{i_2} - \sum_{0 < i_1 \cdots < i_4 < K} \alpha x_{i_0} x_{i_1} x_{i_2} x_{i_3} x_{i_4}. \quad (21)$$

Because $Q(v) = 0$, we have

$$\sum_{0 < i_1 < i_2 < K} v_{i_1} v_{i_2} - \sum_{0 < i_1 < i_2 < i_4 < K} \alpha v_{i_1} v_{i_2} v_{i_3} v_{i_4} = \sum_{0 < i_1 < i_2 < K} v_{i_1} + 2v_{i_2} + 2 - \sum_{0 < i_1 \cdots < i_4 < K} \alpha v_{i_1} + 2v_{i_2} + 2v_{i_3} + 2v_{i_4} + 2$$

Observe that the monomials not containing $v_1$ cancel each other out. Dividing by $v_1 = v_{1+K}$ (since $v_1 > 0$), we have

$$\sum_{1 < i_2 < K} v_{i_2} - \sum_{1 < i_2 < i_4 < K} \alpha v_{i_1} v_{i_2} v_{i_3} v_{i_4} = \sum_{1 < i_2 < K} v_{i_1} + 2 - \sum_{1 < i_2 < i_4 < K} \alpha v_{i_1} + 2v_{i_2} + 2v_{i_3} + 2v_{i_4} + 2$$

Now we observe that the right hand side can be obtained from the left hand side by changing each index $i$ into $i + 1$. Taking into account rotations of the above equality, we conclude \[ \text{(18)}. \]

Next, we prove \[ \text{(19)}. \] To do so, we first calculate the terms of order $\epsilon^2$ of $f^{(K)}(x + \epsilon d)$. Such terms occur only when $i_0 = 0$, $i_1 = 1$, and $i_2 = 2$. In this case, the first sum reduces to $(x_0 - \epsilon)x_1(x_2 + \epsilon)$, and the second sum reduces to

$$\alpha(x_0 - \epsilon)x_1(x_2 + \epsilon) \sum_{2 < i_3 < i_4 < K} x_{i_3} x_{i_4}.$$ 

Thus,

$$R(x) = -x_1 + \alpha x_1 \sum_{2 < i_3 < K} x_{i_3} x_{i_4}.$$ 

Since the assumed interior local maximum $v$ is in the interior of $D^{(K)}$, we have that $v_1 > 0$, and so the condition $R(v) \leq 0$ is equivalent to $R(v) / v_1 \leq 0$:

$$-1 + \sum_{2 < i_3 < K} \alpha v_{i_3} v_{i_4} \leq 0$$

Up to trivial rearrangement, we obtained \[ \text{(19)}. \]

**Corollary** \[ \text{(11)}. \] **Let** $v$ **be a local maximum of** $f^{(K)}$ **in the interior of** $D^{(K)}$. **Then the following inequality holds:**

$$\sum_{1 < i < K - 2} \frac{K - i - 2}{2} S_i(v) \leq \frac{K}{\alpha}$$

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Proof. We sum over all $K$ rotations:

\[
\frac{K}{\alpha} \geq \sum_{i=0}^{K-1} \sum_{\substack{3 \leq i_3 < i_4 < K \atop i_3 \text{ odd}}} v_{i_3+i} v_{i_4+i} = \sum_{3 \leq i_3 < i_4 < K \atop i_4 \text{ even}} \sum_{i=0}^{K-1} v_{i_3+i} v_{i_4+i} = \sum_{3 \leq i_3 < i_4 < K \atop i_3 \text{ odd}} \sum_{i \leq i_4 \text{ even}} S_{i_4-i_3}(v)
\]

\[
= \sum_{3 \leq i_3 < i_4 < K \atop i \text{ odd, } i_3 \text{ odd}} S_i(v) = \sum_{1 \leq i \text{ odd}} \sum_{3 \leq i_3 < K-i \atop i_3 \text{ odd}} S_i(v) = \sum_{1 \leq i < K-2 \atop i \text{ odd}} \frac{K-i-2}{2} S_i(v)
\]

\[\square\]

C Proofs of Combinatorial Lemmas

We repeat the combinatorial facts of Section 4.4 this time with proofs.

Lemma 13. We have:

\[
\sum_{k=0}^{K-1} \sum_{\substack{1 < i_0', i_1', i_2' < K \atop i_0', i_2' \text{ even}}} x_{i_0'+k} x_{i_1'+k} x_{i_2'+k} = \frac{K-3}{2} \sum_{0 \leq i_0 < i_1 < i_2 < K} x_{i_0} x_{i_1} x_{i_2} = \frac{K-3}{2} f_3^{(K)}(x)
\]

Proof. Let us fix $0 \leq i_0 < i_1 < i_2 < K$ such that both $i_2 - i_1$ and $i_1 - i_0$ are odd. We want to show that the term $x_{i_0} x_{i_1} x_{i_2}$ occurs $(K-3)/2$ times in each side of the equality. The middle and right sides are trivial; it remains to check the left side. Let us now fix an arbitrary $k$. For a term on the left hand side to equal $x_{i_0} x_{i_1} x_{i_2}$, it must be that the sets \{$(i_0'+k) \mod K, (i_1'+k) \mod K, (i_2'+k) \mod K$\} and \{$i_0, i_1, i_2$\} are equal. In other words, once $i_0, i_1, i_2, k$ are fixed, the set \{$i_0', i_1', i_2'$\} is uniquely determined. Since, $i_0' < i_1' < i_2'$, the potential values of $i_0', i_1', i_2'$ are also uniquely determined. The remaining question is for how many $k \in \{0, \ldots, K-1\}$ it is the case that the values $i_0', i_1', i_2'$ so determined obey the other constraints.

There are three disjoint cases: The smallest value in the set \{$i_0', i_1', i_2'$\}, namely $i_0'$, is \((i_0 - k) \mod K\) or \((i_1 - k) \mod K\) or \((i_2 - k) \mod K\). The case $i_0' = (i_1 - k) \mod K$ occurs exactly when (a) $i_0 < k < i_1$, and (b) $k$ has the same parity as $i_1$. Let $\delta_0$ denote the size of the gap between $i_0$ and $i_1$. Then, there are $(\delta_0 - 1)/2$ values of $k$ that obey both (a) and (b). The other two cases are similar, and so we conclude that the term $x_{i_0} x_{i_1} x_{i_2}$ occurs on the left hand side

\[
\frac{\delta_0 - 1}{2} + \frac{\delta_1 - 1}{2} + \frac{\delta_2 - 1}{2} = \frac{K-3}{2}
\]

times.

\[\square\]

Lemma 14. Let $v$ be a local maximum of $f^{(K)}$ in the interior of $D^{(K)}$. If $i_1$ is odd and $0 < i_1 < K$, then the following inequality holds:

\[
v_0 v_1 \left( \sum_{1 \leq i_2 < K} v_{i_2} - \sum_{1 \leq i_2 < i_3 < i_4 < K} \alpha v_{i_2} v_{i_3} v_{i_4} \right) \geq v_0 v_1 \left( \sum_{1 \leq i_2 < K} v_{i_2} - \sum_{1 \leq i_2 < i_3 < i_4 < K} \alpha v_{i_2} v_{i_3} v_{i_4} \right)
\]

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Proof. The proof below is a case analysis of where $i_1$ can be inserted in-between $0 < i_2 < i_3 < i_4 < K$. As noted before, the task is to show that retaining the terms that occur in $f_5$ gives a lower bound. In other words, we want to show that those terms not occurring in $f_5$ have a positive sum. We calculate this sum:

\[
\sum_{1<i_2<i_1 \atop i_2 \text{ even}} v_0 v_{i_2} v_{i_1} - \sum_{1<i_2<i_1 \atop i_2 < i_1} \alpha v_0 v_{i_2} v_{i_1} v_{i_3} v_{i_4} = \sum_{1<i_2<i_1 \atop i_2 \text{ even}} v_0 v_{i_2} v_{i_1} \left(1 - \sum_{i_3 < i_4 < K \atop i_3 \text{ odd, } i_4 \text{ even}} \alpha v_{i_3} v_{i_4}\right) \geq \sum_{1<i_2<i_1 \atop i_2 \text{ even}} v_0 v_{i_2} v_{i_1} \left(1 - \sum_{3<i_4 < K \atop i_3 \text{ odd, } i_4 \text{ even}} \alpha v_{i_3} v_{i_4}\right) \geq 0
\]

\square

Lemma 15. Let $l$ be an odd, positive integer. Then:

\[
\sum_{k=0}^{K-1} \sum_{1 \leq i'_1 \leq K-2 \atop i'_1 \text{ odd}} \frac{K-i'_1 - 2}{2} \sum_{i'_1 < i'_2 < \cdots < i'_{l-1} < K \atop \forall j, i'_j \equiv j \pmod{2}} x_k x_{i'_j+k} \prod_{1<j<l} x_{i'_j+k} = \left(\frac{l-1}{2} - K - l\right) \sum_{i_0 < \cdots < i_{l-1} < K \atop i_j - i_{j+1} \text{ odd for } 0 < j < l} \prod_{j=0}^{l-1} x_{i_j}
\]

The proof is similar to that of Lemma 13.

Proof. Let us fix $0 \leq i_0 < \cdots < i_{l-1} < K$ with odd gaps in-between. We want to show that the term $x_{i_0} \cdots x_{i_{l-1}}$ occurs $(l-1)K/2 - l$ times on each side of the equation. For the right side, it is trivial. The general form of a term on the left side is $x_k x_{i'_1+k} x_{i'_2+k} \cdots x_{i'_{l-1}+k}$. It must be that $k$ is one of $i_0, \ldots, i_{l-1}$. Let us consider the case $k = i_0$; the others are similar. If $k = i_0$, then, in fact,

\[(k, i'_1 + k, i'_2 + k, \ldots, i'_{l-1} + k) \mod K = (i_0, i_1, \ldots, i_{l-1}).\]

In particular, $i'_1$ equals the size of the gap between $i_0$ and $i_1$. Let us denote this gap by $\delta_0$. On the left hand side, the term is multiplied by $(K - i'_1 - 2)/2$, which is $(K - \delta_0 - 2)/2$. The cases $k = i_1, k = i_2, \ldots$ are similar. Because $\delta_0 + \cdots + \delta_{l-1} = K$, we conclude that the term $x_{i_0} \cdots x_{i_{l-1}}$ occurs

\[
\frac{K-\delta_0-2}{2} + \cdots + \frac{K-\delta_{l-1}-2}{2} = \frac{l-1}{2} K - l
\]
times on the left side. \square