A SURVEY ON REPRODUCING KERNEL KREĬN SPACES

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ABSTRACT. This is a survey on reproducing kernel Kreĭn spaces and their interplay with operator valued Hermitian kernels. Existence and uniqueness properties are carefully reviewed. The approach used in this survey involves the more abstract, but very useful, concept of linearisation or Kolmogorov decomposition, as well as the underlying concepts of Kreĭn space induced by a selfadjoint operator and that of Kreĭn space continuously embedded. The operator range feature of reproducing kernel spaces is emphasised. A careful presentation of Hermitian kernels on complex regions that points out a universality property of the Szegő kernels with respect to reproducing kernel Kreĭn spaces of holomorphic functions is included.

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1. INTRODUCTION

A kernel $K$ on a set $X$ and with operator entries is the analog of an abstract matrix double indexed on $X$ and for which each entry is a bounded linear operator acting between appropriate spaces. The adjoint $K^\sharp$ of the kernel $K$ can be defined by analogy with the adjoint of matrices. A Hermitian kernel $K$ is defined by the property $K = K^\sharp$. Roughly speaking, a reproducing kernel Kreĭn space on a set $X$ is a Kreĭn space $R$ of functions on $X$ for which there exists a Hermitian kernel with the property that the evaluations of the functions in $R$ can be calculated in terms of the kernel $K$. One of the main problems is that of associating a reproducing kernel Kreĭn space $R$ to a given Hermitian kernel $K$.

The classical theory says that for any reproducing kernel Hilbert space $R$ its reproducing kernel $K$ is positive semidefinite and uniquely determined by $R$. Conversely, to any positive semidefinite kernel $K$ there is associated a unique reproducing kernel Hilbert space such that $K$ is its reproducing kernel. These facts can be proven also for a slightly more general situation relating in a similar fashion reproducing kernel Pontryagin spaces to Hermitian kernels with finite negative (or positive) signatures.

For genuine indefinite Hermitian kernels the correspondence with reproducing kernel Kreĭn spaces is much more complicated. There is a variety of characterisations of those Hermitian kernels that produce reproducing kernel Kreĭn spaces in terms of boundedness with respect to positive semidefinite kernels and in terms of decomposability as a difference of two positive semidefinite kernels. Uniqueness of the reproducing kernel Kreĭn space, provided that it exists, is also problematic. Characterisations of uniqueness are available in terms of lateral spectral gaps as well as in terms of maximal uniformly positive subspaces.
When Hermitian kernels on regions in either a single complex variable or several complex variables have certain holomorphy properties, existence of reproducing kernel Kreǐn spaces is guaranteed. This fact is related to a certain universality property of the Szegő kernel that allows the construction of the corresponding reproducing kernel Kreǐn space inside the Hardy space or, respectively, the Drury-Arveson space.

The approach used in this survey involves a more abstract, but very useful, concept of linearisation or Kolmogorov decomposition, the underlying concept of Kreǐn spaces induced by selfadjoint operators, as well as the concept of Kreǐn spaces continuously embedded, pointing out the operator range feature of reproducing kernel spaces.

2. Kreǐn Spaces and their Linear Operators

A Kreǐn space $\mathcal{K}$ is a complex linear space on which it is defined an indefinite scalar product $\langle \cdot , \cdot \rangle$ such that $\mathcal{K}$ is decomposed in a direct sum

\begin{equation}
\mathcal{K} = \mathcal{K}_+ \cup \mathcal{K}_-
\end{equation}

in such a way that $\mathcal{K}_\pm$ are Hilbert spaces with scalar products $\pm \langle \cdot , \cdot \rangle$, and the direct sum in (2.1) is orthogonal with respect to the indefinite scalar product $\langle \cdot , \cdot \rangle$, i.e. $\mathcal{K}_+ \cap \mathcal{K}_- = \{0\}$ and $[x_+, x_-] = 0$ for all $x_\pm \in \mathcal{K}_\pm$. The decomposition (2.1) gives rise to a positive definite scalar product $\langle \cdot , \cdot \rangle_\pm$ by setting $\langle x, y \rangle_\pm := \langle x_+, y_+ \rangle - \langle x_-, y_- \rangle$, where $x = x_+ + x_-$, $y = y_+ + y_-$, and $x_\pm, y_\pm \in \mathcal{K}_\pm$. The scalar product $\langle \cdot , \cdot \rangle_\pm$ defines on $\mathcal{K}$ a structure of Hilbert space.

Subspaces $\mathcal{K}_\pm$ are orthogonal with respect to the scalar product $\langle \cdot , \cdot \rangle_\pm$, too. One denotes by $P_\pm$ the corresponding orthogonal projections onto $\mathcal{K}_\pm$, and let $J = P_+ - P_-$. The operator $J$ is a symmetry, i.e. a selfadjoint and unitary operator, $J^* J = J J^* = J^2 = I$. Unless otherwise specified, any Kreǐn space is considered as a Banach space with the norm given by an arbitrary norm induced by the positive definite inner product associated to a fundamental decomposition. For two Kreǐn spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, we denote by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ the Banach space of all bounded linear operators $T: \mathcal{K}_1 \to \mathcal{K}_2$.

Given a Kreǐn space $(\mathcal{K}, \langle \cdot , \cdot \rangle)$, the cardinal numbers

\begin{equation}
\kappa^+(\mathcal{K}) = \dim(\mathcal{K}_+), \quad \kappa^-(\mathcal{K}) = \dim(\mathcal{K}_-),
\end{equation}

do not depend on the fundamental decomposition and they are called, respectively, the geometric ranks of positivity/negativity of $\mathcal{K}$.

The operator $J$ is called a fundamental symmetry of the Kreǐn space $\mathcal{K}$. Note that $[x, y] = \langle J x, y \rangle$, $(x, y \in \mathcal{K})$. If $T$ is a densely defined operator from a Kreǐn space $\mathcal{K}_1$ to another Kreǐn space $\mathcal{K}_2$, the adjoint of $T$ can be defined as an operator $T^*$ defined on the set of all $y \in \mathcal{K}_2$ for which there exists $h_y \in \mathcal{K}_1$ such that $[T x, y] = [x, h_y]$, and $T^2 y = h_y$. In addition, $T^2 = J_1 T^* J_2$, where $T^*$ denotes the adjoint operator of $T$ with respect to the Hilbert spaces $(\mathcal{K}_1, \langle \cdot , \cdot \rangle_{\mathcal{J}_1})$ and $(\mathcal{K}_2, \langle \cdot , \cdot \rangle_{\mathcal{J}_2})$. The symbol $\mathcal{J}$ denotes the adjoint when at least of the spaces $\mathcal{K}_1$ or $\mathcal{K}_2$ is indefinite. In the case of an operator $T$ defined on the Kreǐn space $\mathcal{K}$, $T$ is called symmetric if $T \subseteq T^2$, i.e. if the relation $[T x, y] = [x, T y]$ holds for each $x, y \in \text{Dom}(T)$, and $T$ is called selfadjoint if $T = T^*$.

In this survey, a bit of geometry of Kreǐn spaces is used. Thus, a (closed) subspace $\mathcal{L}$ of a Kreǐn space $\mathcal{K}$ is called regular if $\mathcal{K} = \mathcal{L} + \mathcal{L}^\perp$, where $\mathcal{L}^\perp = \{x \in \mathcal{K} \mid [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$. Regular spaces of Kreǐn spaces are important since they are exactly the analogue of Kreǐn subspaces, that is, if we want $\mathcal{L}$ to be a Kreǐn space with the restricted indefinite inner product and the same strong topology, then it should be regular.
In addition, let us recall that, given a subspace $\mathcal{L}$ of a Kreın space, one calls $\mathcal{L}$ non-negative (positive) if the inequality $[x, x] \geq 0$ holds for $x \in \mathcal{L}$ (respectively, $[x, x] > 0$ for all $x \in \mathcal{L} \setminus \{0\}$). Similarly one defines non-positive and negative subspaces. A subspace $\mathcal{L}$ is called degenerate if $\mathcal{L} \cap \mathcal{L}^\perp \neq \{0\}$. Regular subspaces are non-degenerate. As a consequence of the Schwarz inequality, if a subspace $\mathcal{L}$ is either positive or negative it is nondegenerate. A remarkable class of subspaces are those regular spaces that are either positive or negative, for which the terms uniformly positive, respectively, uniformly negative are used. These notions can be defined for linear manifolds also, that is, without assuming closedness.

A linear operator $V$ defined from a subspace of a Kreın space $\mathcal{K}_1$ and valued into another Kreın space $\mathcal{K}_2$ is called isometric if $[Vx, Vy] = [x, y]$ for all $x, y$ in the domain of $V$. Note that isometric operators between genuine Kreın spaces may be unbounded and different inverse.

### 3. Hermitian Kernels

Let $X$ be a nonempty set and let $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X}$ be a family of Kreın spaces with inner products denoted by $[\cdot, \cdot]_{\mathcal{H}_x}$. A mapping $K$ defined on $X \times X$ such that $K(x, y) \in \mathcal{L}(\mathcal{H}_y, \mathcal{H}_x)$ for all $x, y \in X$ is called an $\mathcal{H}$-kernel on $X$. In case $\mathcal{H}_x = \mathcal{H}$ for all $x \in X$, where $\mathcal{H}$ is some fixed Kreın space, one talks about an $\mathcal{H}$-kernel on $X$, while, even more particularly, if $\mathcal{H} = \mathbb{C}$, one talks about a scalar kernel on $X$, or simply a kernel on $X$.

To any $\mathcal{H}$-kernel $K$ one associates its adjoint $K^\sharp$ defined by $K^\sharp(x, y) = K(y, x)^\sharp$, for all $x, y \in X$. The $\mathcal{H}$-kernel $K$ is called Hermitian if $K = K^\sharp$, that is,

$$K(x, y) = K(y, x)^\sharp, \quad x, y \in X. \tag{3.1}$$

Denote by $\mathcal{F}(\mathcal{H})$ the set of all $\mathcal{H}$-vector fields $f$ on $X$, that is, $f = \{f_x\}_{x \in X}$ such that $f_x \in \mathcal{H}_x$, for all $x \in X$, and let $\mathcal{F}_0(\mathcal{H})$ denote the set of all $f \in \mathcal{F}(\mathcal{H})$ of finite support, that is, the set $\text{supp}(f) = \{x \in X \mid f_x \neq 0\}$ is finite. Alternatively, one can view any $f \in \mathcal{F}(\mathcal{H})$ as a function $f : X \to \bigcup_{x \in X} \mathcal{H}_x$ such that $f(x) \in \mathcal{H}_x$ for all $x \in X$.

If $K$ is a Hermitian $\mathcal{H}$-kernel then one can introduce on $\mathcal{F}_0(\mathcal{H})$ an inner product $[\cdot, \cdot]_K$ defined by

$$[f, g]_K = \sum_{x, y \in X} [K(x, y)f(y), g(x)]_{\mathcal{H}_x}, \quad f, g \in \mathcal{F}_0(\mathcal{H}). \tag{3.2}$$

The $\mathcal{H}$-kernel $K$ is called positive semidefinite if

$$\sum_{x, y \in X} [K(x, y)h(y), h(x)]_{\mathcal{H}_x} \geq 0, \quad h \in \mathcal{F}_0(\mathcal{H}). \tag{3.3}$$

Every positive semidefinite $\mathcal{H}$-kernel is Hermitian. Also, a Hermitian $\mathcal{H}$-kernel is positive semidefinite if and only if the corresponding inner product in (3.2) is nonnegative.

Let us denote by $\mathcal{R}^h(\mathcal{H})$ the class of all Hermitian $\mathcal{H}$-kernels and by $\mathcal{R}^+(\mathcal{H})$ the subclass of all positive semidefinite $\mathcal{H}$-kernels. On $\mathcal{R}^h(\mathcal{H})$ one defines addition, subtraction and multiplication with real numbers in a natural way. Moreover, on $\mathcal{R}^h(\mathcal{H})$ one has a natural
partial order defined as follows: if \( H, K \in \mathfrak{R}^h(H) \) then \( H \leq K \) means \([f, f]_H \leq [f, f]_K\), for all \( f \in \mathcal{F}_0(H) \). With this definition one gets

\[
(3.4) \quad \mathfrak{R}^+(H) = \{ H \in \mathfrak{R}^h(H) \mid H \geq 0 \},
\]

and \( \mathfrak{R}^+(H) \) is a strict cone of \( \mathfrak{R}^h(H) \), that is, it is closed under addition and multiplication with nonnegative numbers, and \( \mathfrak{R}^+(H) \cap -\mathfrak{R}^+(H) = \{ 0 \} \), where 0 denotes the null kernel.

More generally, one can define the signatures of \( K \), denoted by \( \kappa_{\pm}(K) \), as the positive/negative signatures of the inner product \([\cdot, \cdot]_K\). Then, if \( \kappa(K) = \min\{ \kappa_-(K), \kappa_+(K) \} \) denotes the definiteness signature of \( K \), the Hermitian \( \mathcal{H} \)-kernel \( K \) is called quasi semi-definite if \( \kappa(K) < \infty \), that is, either \( \kappa_-(K) \) or \( \kappa_+(K) \) is finite.

A pairing \((\cdot, \cdot)_\mathcal{F}\) can be defined for arbitrary \( f, g \in \mathcal{F}(H) \), provided at least one of \( f \) and \( g \) has finite support, by

\[
(3.5) \quad (f, g)_{\mathcal{F}} = \sum_{x \in X} [f(x), g(x)]_{\mathcal{H}_x}.
\]

When restricted to \( \mathcal{F}_0(H) \) this pairing becomes a nondegenerate inner product. To each \( \mathcal{H} \)-kernel \( K \) one associates the convolution operator, denoted also by \( K \), and defined by

\[
(3.6) \quad K : \mathcal{F}_0(H) \rightarrow \mathcal{F}(H), \quad (Kf)(x) = \sum_{y \in X} K(x, y)f(y), \quad f \in \mathcal{F}_0(H).
\]

Then

\[
(3.7) \quad [f, g]_K = (Kf, g)_{\mathcal{F}}, \quad f, g \in \mathcal{F}_0(H).
\]

Consequently, the kernel \( K \) is positive semidefinite (Hermitian) if and only if the corresponding convolution operator \( K \) is positive semidefinite (Hermitian), that is, \((Kf, g)_\mathcal{F} \geq 0 \) \((Kf, g)_{\mathcal{F}} = (f, Kg)_{\mathcal{F}}\), for all \( f, g \in \mathcal{F}_0(H) \). Similar assertions can be made about the signatures \( \kappa_{\pm}(K) \), with appropriate definitions of signatures of Hermitian operators on inner product spaces.

At this point, it is worth noting that there is no restriction of generality if one assumes that all spaces \( \mathcal{H}_x \) are Hilbert. To see this, fixing a fundamental symmetry \( J_x \) on each Krein space \( \mathcal{H} \), one can refer to the Hilbert spaces \((\mathcal{H}_x, \langle \cdot, \cdot \rangle_{\mathcal{H}_x})\) and, one considers the kernel \( H \) on \( X \) defined by \( H(x, y) = J_x K(x, y) J_y \), for all \( x, y \in X \). Taking into account that \( A^2 = J_x A^* J_y \) for any bounded linear operator \( A : \mathcal{H}_x \rightarrow \mathcal{H}_y \), it follows that all notions like Hermitian, positive semidefinite, signatures, etc. defined for the kernel \( K \), have word-for-word transcriptions for the kernel \( H \). However, since this survey is focusing on the indefinite case, allowing right from the beginning Krein spaces \( \mathcal{H}_x \) brings more symmetry and simpler formulas.

A reproducing kernel inner product space, with respect to the set \( X \), the collection of Krein spaces \( \mathbf{H} = \{ \mathcal{H}_x \}_{x \in X} \), and \( \mathbf{H} \)-kernel \( K \), is an inner product space \((\mathcal{R}, [\cdot, \cdot]_\mathcal{R})\) subject to the following conditions:

\[
\begin{align*}
(rk1) \quad & \mathcal{R} \subseteq \mathcal{F}(\mathbf{H}). \\
(rk2) \quad & \text{For all } y \in X \text{ and all } h \in \mathcal{H}_y \text{ the map } X \ni x \mapsto K(x, y)h \in \mathcal{H}_x \text{ belongs to } \mathcal{R}. \\
(rk3) \quad & [f(x), h]_{\mathcal{H}_x} = [f, K(\cdot, x)h]_{\mathcal{R}}, \text{ for all } f \in \mathcal{R}, x \in X, \text{ and } h \in \mathcal{H}_x.
\end{align*}
\]

The axiom (rk3) is usually called the reproducing property while the \( \mathbf{H} \)-kernel \( K \) is called the reproducing kernel of \( \mathcal{R} \). Note that \( K \) is necessarily a Hermitian \( \mathbf{H} \)-kernel.
According to the axiom (rk2), it is useful to consider the notation $K_y = K(\cdot, y) : X \to \mathcal{L}(\mathcal{H}_y, \mathcal{R})$, for $y \in X$. An immediate consequence of the axioms (rk1)–(rk3) is that the inner product $[\cdot, \cdot]_{\mathcal{R}}$ is nondegenerate. Also, recalling that on any nondegenerate inner product space the weak topology is separated, the following *minimality property* holds

(rk4) The span of $\{K_x \mathcal{H}_x \mid x \in X\}$ is weakly dense in $\mathcal{R}$.

Also, the reproducing kernel is uniquely determined by the reproducing kernel inner product space.

Given a Hermitian $\mathbf{H}$-kernel $K$ on $X$, consider the subspace $\mathcal{R}_0(\mathbf{H})$ of $\mathcal{F}(\mathbf{H})$ spanned by $K_x h$, for all $x \in X$ and all $h \in \mathcal{H}_x$, on which one can define the inner product

$$
(3.8) \quad \left[ \sum_{i=1}^m K_{x_i} h_i, \sum_{j=1}^n K_{y_j} k_j \right]_{\mathcal{R}_0} = \sum_{i=1}^m \sum_{j=1}^n [K(y_j, x_i) h_i, k_j]_{\mathcal{H}_{y_j}};
$$

This definition can be proven to be correct: vectors in $\mathcal{R}_0(\mathbf{H})$ may have different representations as linear combinations of $K_y k$ but the definition in (3.8) is independent of these. The subspace $\mathcal{R}_0(\mathbf{H})$ of $\mathcal{F}(\mathbf{H})$ is the range of the convolution operator $K$ defined at (3.6) and the inner product $[\cdot, \cdot]_{\mathcal{R}_0}$ is nondegenerate. In addition, $(\mathcal{R}_0(\mathbf{H}); [\cdot, \cdot]_{\mathcal{R}_0})$ is a reproducing kernel inner product space with reproducing kernel $K$.

In case the reproducing kernel inner product space $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is a Kreın space, one talks about a *reproducing kernel Kreın space*. The uniqueness of the reproducing kernel of a Kreın space has a stronger characterisation.

**Theorem 3.1.** Let $\mathcal{K}$ be a Kreın space of $\mathbf{H}$-valued vector fields on $X$, that is, $\mathcal{K} \subseteq \mathcal{F}(\mathbf{H})$. For each $x \in X$ consider the linear operator $E(x) : \mathcal{K} \to \mathcal{H}_x$ of evaluation at $x$, that is, $E(x)f = f(x)$ for all $f \in \mathcal{K}$. Then, $\mathcal{K}$ has a reproducing kernel if and only if $E(x)$ is bounded for all $x \in X$. In this case, the reproducing kernel of $\mathcal{K}$ is

$$
(3.9) \quad K(x, y) = E(x)E(y)^\sharp, \quad x, y \in X,
$$

hence uniquely determined by $\mathcal{K}$.

With the notation as in Theorem 3.1, it is useful to note that, if $\mathcal{K}$ is a reproducing kernel Kreın space with reproducing kernel $K$, then $K_x$ can be viewed as a linear operator $\mathcal{H}_y \to \mathcal{K}$ and $K_x = E(x)^\sharp$ for all $x \in X$. In particular,

$$
(3.10) \quad K(x, y) = K_y K_x, \quad x, y \in X.
$$

Conversely, given a Hermitian $\mathbf{H}$-kernel $K$ on $X$, the questions on existence and uniqueness of a reproducing kernel Kreın space $\mathcal{R} \subseteq \mathcal{F}(\mathbf{H})$ such that $K$ is its reproducing kernel are much more difficult. If $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is a reproducing kernel Kreın space with reproducing kernel $K$, in view of the axioms (rk1)–(rk3) and the minimality property (rk4), the inner products $[\cdot, \cdot]_{\mathcal{R}}$ and $[\cdot, \cdot]_{\mathcal{R}_0}$ coincide on $\mathcal{R}_0(\mathbf{H})$ and $\mathcal{R}_0(\mathbf{H})$ is dense in $\mathcal{R}$. Thus, existence of a reproducing kernel Kreın space associated to a given $\mathbf{H}$-kernel $K$ on $X$ depends heavily on the possibility of ”completing” the inner product space $(\mathcal{R}_0(\mathbf{H}); [\cdot, \cdot]_{\mathcal{R}_0})$ to a Kreın space $\mathcal{R}$ inside $\mathcal{F}(\mathbf{H})$, which is a core problem in the theory of indefinite inner product spaces.

In order to tackle these questions, let us note that if $\mathcal{R}$ is a reproducing kernel Hilbert space with reproducing kernel $K$, then $K$ should be positive semidefinite. Conversely, if $K$ is a positive semidefinite $\mathbf{H}$-kernel on $X$, then the inner product space $(\mathcal{R}_0, [\cdot, \cdot]_{\mathcal{R}_0})$
defined at (3.8) is a pre-Hilbert space and the existence of a reproducing kernel Hilbert space \( R \) with reproducing kernel \( K \) depends on whether \((R_0, [\cdot, \cdot]_{R_0})\) has a completion inside of \( \mathcal{F}(\mathcal{H}) \). Actually, this is always the case and, moreover, uniqueness holds as well, but these two facts are slightly more general, namely, they are true if \( K \) is quasi semidefinite, in which case it is obtained a unique reproducing kernel Pontryagin space, cf. Section 5.

In this survey, existence and uniqueness of reproducing kernel Kreši\'n spaces associated to Hermitian kernels are approached through the more abstract, but very useful, concept of linearisation or Kolmogorov decomposition. By definition, a linearisation, sometimes called Kolmogorov decomposition, of the \( \mathcal{H} \)-kernel \( K \) is a pair \((\mathcal{K}; V)\), subject to the following conditions:

\[(kd1) \ \mathcal{K} \text{ is a Kre\'in space and } V(x) \in \mathcal{L}(\mathcal{H}_x, \mathcal{K}) \text{ for all } x \in X.\]

\[(kd2) \ K(x, y) = V(x)^2 V(y) \text{ for all } x, y \in X.\]

The linearisation \((\mathcal{K}; V)\) is called minimal if the following condition holds as well:

\[(kd3) \ K = \bigvee_{x \in X} V(x) \mathcal{H}_x.\]

Two linearisations \((\mathcal{K}_j; V_j), j = 1, 2,\) of the same \( \mathcal{H} \)-kernel \( K \) are called unitary equivalent if there exists a Kre\'in space bounded unitary operator \( U : \mathcal{K}_1 \to \mathcal{K}_2 \) such that \( V_2(x) = UV_1(x) \) for all \( x \in X \).

There is a close connection between the notion of reproducing kernel Kre\'in space with reproducing \( \mathcal{H} \)-kernel \( K \) and that of minimal linearisation of \( K \).

**Proposition 3.2.** (1) Let \( \mathcal{R} \) be a reproducing kernel Kre\'in space with reproducing \( \mathcal{H} \)-kernel \( K \). Then, letting

\[(3.11) \ V(x) = K_x = K(\cdot, x) \in \mathcal{L}(\mathcal{H}_x, \mathcal{R}),\]

the pair \((\mathcal{R}; V)\) is a minimal linearisation of the \( \mathcal{H} \)-kernel \( K \).

Conversely, letting \((\mathcal{K}; V)\) be a minimal linearisation of the \( \mathcal{H} \)-kernel \( K \), then,

\[(3.12) \ \mathcal{R} = \{V(\cdot)^2 \mid k \in \mathcal{K}\},\]

is a vector subspace of \( \mathcal{F}(\mathcal{H}) \) which, with respect to the the inner product defined by

\[(3.13) \ [V(\cdot)^2 h, V(\cdot)^2 k]_\mathcal{R} = [h, k]_\mathcal{K}, \quad h, k \in \mathcal{K},\]

is a Kre\'in space with reproducing kernel \( K \).

(2) In the correspondence defined at (3.12) and (3.13), two unitary equivalent minimal realisations of the same \( \mathcal{H} \)-kernel \( K \) produce the same reproducing kernel Krein space and hence, the correspondence between reproducing kernel Krein spaces and minimal linearisations is one-to-one, provided that unitary equivalent minimal linearisations are identified.

### 4. Some Examples

**Example 4.1.** Matrices. Let \( X = \{1, 2, \ldots, n\} \) and \( \mathcal{H}_x = \mathbb{C} \) for each \( x \in X \). A kernel on \( X \) is simply a map \( K : X \times X \to \mathbb{C} \) and hence, it can be identified with the \( n \times n \) complex matrix \([K(i, j)]_{i,j=1}^n\). The kernel \( K \) is Hermitian if and only the matrix \([K(i, j)]_{i,j=1}^n\) is Hermitian. The vector space \( \mathcal{F} = \mathcal{F}_0 = \mathbb{C}^n \) hence, in view of the definition (3.6), the convolution operator \( K : \mathcal{F} \to \mathcal{F} \) is simply the linear operator \( K : \mathbb{C}^n \to \mathbb{C}^n \) associated to the matrix \( K \).
If $K$ is a Hermitian kernel on $X$ then the inner product $[\cdot, \cdot]_K$ is the familiar inner product associated to the Hermitian matrix $[K(i,j)]_{i,j=1}^n$. The signatures $\kappa_{\pm}(K)$ coincide, respectively, with the number of positive/negative eigenvalues of the matrix $[K(i,j)]_{i,j=1}^n$, counted with multiplicities. $K$ is positive semidefinite if and only if the matrix $[K(i,j)]_{i,j=1}^n$ is positive semidefinite.

Assuming that the kernel $K$ on $X$ is Hermitian, for each $j = 1, \ldots, n$, the "map" $K_j : X \to \mathbb{C}$ is simply the column vector $[K(i,j)]_{i=1}^n$ in $\mathbb{C}^n = \mathcal{F}$. The vector space $\mathcal{R}_0$, the range of the convolution operator $K$, is the vector subspace of $\mathbb{C}^n$ generated by all the column vectors $[K(i,j)]_{i=1}^n$. In this particular case, $\mathcal{R}_0$ is the reproducing kernel Krein space with reproducing kernel $K$. The inner product $[\cdot, \cdot]_{\mathcal{R}_0}$ is defined at (3.8).

Example 4.2. Operator Block Matrices. Let $X = \{1, 2, \ldots, n\}$ but, this time, for each $x \in X$ one denotes by $\mathcal{H}_x$ a Hilbert space and let $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X}$. An $\mathcal{H}$-kernel on $X$, originally defined as a map $K$ on $X \times X$ such that $K(x,y) \in \mathcal{L}(\mathcal{H}_y, \mathcal{H}_x)$, is naturally identified with the operator block matrix $[K(i,j)]_{i,j=1}^n$. The vector space $\mathcal{F}(\mathcal{H}) = \mathcal{F}_0(\mathcal{H})$ together with its natural inner product $(\cdot, \cdot)_\mathcal{F}$ defined at (3.3) is naturally identified with the Hilbert space $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. Then, the convolution operator $K$ associated to the $\mathcal{H}$-kernel $K$ is identified with the bounded linear operator $K : \mathcal{H} \to \mathcal{H}$ naturally associated to the operator block matrix $[K(i,j)]_{i,j=1}^n$. For each $j = 1, \ldots, n$, the map $K_j = K(\cdot, j)$ is the operator block column matrix $[K(i,j)]_{i=1}^n : \mathcal{H}_j \to \mathcal{H}$.

The $\mathcal{H}$-kernel $K$ on $X$ is Hermitian if and only if the corresponding operator block matrix $[K(i,j)]_{i,j=1}^n$ is Hermitian, if and only if the convolution operator $K \in \mathcal{L}(\mathcal{H})$ is selfadjoint (Hermitian). In this case, the vector space $\mathcal{R}_0(\mathcal{H})$ is identified with the range of the convolution operator $K$ as a subspace of $\mathcal{H}$ and is spanned by the ranges of the operator block column matrices $K_j = [K(i,j)]_{i=1}^n : \mathcal{H}_j \to \mathcal{H}$, for $j = 1, \ldots, n$. The inner product $[\cdot, \cdot]_{\mathcal{R}_0}$ is defined as in (3.8).

Example 4.3. The Hardy space $H^2(\mathbb{D})$. Let $\mathbb{D}$ denote the open unit ball in the complex field and consider the Szegö kernel

\begin{equation}
S(z, w) = \frac{1}{1 - z\overline{w}} = \sum_{n=0}^{\infty} \overline{w}^n z^n, \quad z, w \in \mathbb{D},
\end{equation}

the series converging absolutely and uniformly on any compact subset of $\mathbb{D} \times \mathbb{D}$. Clearly, $S$ is a Hermitian scalar kernel on $\mathbb{D}$ and for every $w \in \mathbb{D}$

$$S_w(z) = S(z, w) = \sum_{n=0}^{\infty} \overline{w}^n z^n, \quad z, w \in \mathbb{D},$$

the series converging absolutely and uniformly on any compact subset of $\mathbb{D}$, $S_w$ is a holomorphic function on $\mathbb{D}$, and its Taylor coefficients are $1, \overline{w}, \overline{w}^2, \ldots$. In order to describe the reproducing kernel inner product space associated to the Szegö kernel $S$, in view of (3.8), one has to consider the vector space generated by the functions $S_w$, $w \in \mathbb{D}$, and complete it with respect to the inner product

\begin{equation}
\langle S_u, S_w \rangle = S_u(w) = S(w, u) = \sum_{n=0}^{\infty} \overline{w}^n u^n, \quad u, v \in \mathbb{D}.
\end{equation}
This completion is called the Hardy space, denoted by $H^2(\mathbb{D})$, and in view of (4.2), is
\begin{equation}
H^2(\mathbb{D}) = \{ f \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{2} |a_n|^2 < \infty \}.
\end{equation}

Since $H^2(\mathbb{D})$ is a Hilbert space, it follows that the Szegö kernel is positive semidefinite.

The Hardy space $H^2(\mathbb{D})$ has some other special properties, among which, a distinguished role is played by the boundary values of its functions. More precisely, for each $f \in H^2(\mathbb{D})$, there exists $\tilde{f}$ a function defined on the unit circle $\mathbb{T} = \partial \mathbb{D}$, such that
\begin{equation}
\tilde{f}(e^{it}) = \lim_{r \to 1^{-}} f(re^{it}), \quad \text{a.e. } t \in [0,2\pi).
\end{equation}
The function $\tilde{f}$ is uniquely determined by $f$, a.e. on $\mathbb{T}$, and $\tilde{f} \in L^2(\mathbb{T})$. Usually, there is no distinction between a function $f$ in $H^2(\mathbb{D})$ and $\tilde{f}$, that is, the tilde sign is not used at all. In this way, an isometric embedding of $H^2(\mathbb{D})$ into $L^2(\mathbb{T})$ is defined, in particular,
\begin{equation}
\langle f,g \rangle_{H^2(\mathbb{D})} = \langle f,g \rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it})g(e^{it})dt, \quad f,g \in H^2(\mathbb{D}).
\end{equation}
With respect to this embedding, $H^2(\mathbb{D})$ is identified with the subspace of $L^2(\mathbb{T})$ of all functions $f$ whose Fourier coefficients of negative index vanish.

**Example 4.4. The Drury-Arveson Space.** Let $B_r(\xi)$ be the open ball of radius $r$ and center $\xi$ in the Hilbert space $\mathcal{G} = \mathbb{C}^N$, with inner product $\langle \xi, \eta \rangle_{\mathcal{G}} = \bar{\eta} \cdot \xi = \sum_{n=1}^{N} \bar{\eta}_n \xi_n$ for $\xi = (\xi_n)^{N}_{n=1}$ and $\eta = (\eta_n)^{N}_{n=1}$ in $\mathbb{C}^N$. We write $B_r$ instead of $B_r(0)$. The Szegö kernel is
\begin{equation}
S(\xi, \eta) = \frac{1}{1 - \langle \xi, \eta \rangle_{\mathcal{G}}}, \quad \xi, \eta \in B_1,
\end{equation}
and note that $S$ is a scalar Hermitian kernel on $B_1$. We now describe a minimal linearsisation of $S$. Let $F(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \mathcal{G}^\oslash n$ be the Fock space associated to $\mathcal{G} = \mathbb{C}^N$, where $\mathcal{G}^\oslash 0 = \mathbb{C}$ and $\mathcal{G}^\oslash n$ is the $n$-fold Hilbert space tensor product of $\mathcal{G}$ with itself, hence $F(\mathcal{G})$ is a Hilbert space. Let
\begin{equation}
P_n = \frac{1}{n!} \sum_{\pi \in S_n} \hat{\pi}
\end{equation}
be the orthogonal projection of $\mathcal{G}^\oslash n$ onto its symmetric part, where $\hat{\pi}(\xi_1 \oslash \ldots \oslash \xi_n) = \xi_{\pi^{-1}(1)} \oslash \ldots \oslash \xi_{\pi^{-1}(n)}$ for any element $\pi$ of the permutation group $S_n$ on $n$ symbols. Recall that the symmetric Fock space is $F^s(\mathcal{G}) =\bigoplus_{n=0}^{\infty} P_n \mathcal{G}$. For $\xi \in B_1$ set $\xi^\oslash 0 = 1$ and let $\xi^\oslash n$ denote the $n$-fold tensor product $\xi \otimes \ldots \otimes \xi$, $n \geq 1$. Note that
\begin{equation}
\| \bigoplus_{n \geq 0} \xi^\oslash n \|_{F(\mathcal{G})}^2 = \sum_{n \geq 0} \| \xi^\oslash n \|_{\mathcal{G}^\oslash n}^2 = \sum_{n \geq 0} \| \xi \|_{\mathcal{G}}^{2n} = \frac{1}{1 - \| \xi \|_{\mathcal{G}}^2}.
\end{equation}
Hence $\bigoplus_{n \geq 0} \xi^\oslash n \in F^s(\mathcal{G})$ and one can define the mapping $V_S$ from $B_1$ into $F^s(\mathcal{G})$,
\begin{equation}
V_S(\xi) = \bigoplus_{n \geq 0} \xi^\oslash n, \quad \xi \in \mathcal{G}.
\end{equation}
The pair \((F^*(G); V_S)\) is a minimal linearisation of the kernel \(S\). In order to see this, \(V_S(\xi)\) is also viewed as a bounded linear operator from \(\mathbb{C}\) into \(F^*(G)\) by \(V_S(\xi)\lambda = \lambda V_S(\xi)\), \(\lambda \in \mathbb{C}\), so that, for \(\xi, \eta \in \mathbb{B}_1\),

\[
V_S(\eta)^*V_S(\xi) = (V_S(\xi), V_S(\eta))_{F^*(G)} = \sum_{n \geq 0} \langle \xi_{\otimes n}, \eta_{\otimes n} \rangle_{G_{\otimes n}} = \sum_{n \geq 0} (\overline{\eta} \cdot \xi)^n = \frac{1}{1 - \overline{\eta} \cdot \xi} = S(\xi, \eta).
\]

In particular, this shows that the Szegö kernel is positive semidefinite. The set \(\{V_S(\xi) \mid \xi \in \mathbb{B}_1\}\) is total in \(F^*(G)\) since for \(n \geq 1\) and \(\xi \in G\) one has \(\frac{d^n}{dt^n} V(t\xi)|_{t=0} = n!\xi_{\otimes n}\).

The reproducing kernel Hilbert space associated to the Szegö kernel \(S\), called the Drury-Arveson space and denoted by \(H^2(\mathbb{B}_1)\), is given by the completion of the linear space generated by the functions \(S_{\eta} = S(\cdot, \eta), \eta \in \mathbb{B}_1\), with respect to the inner product defined by \(\langle S_\eta, S_\zeta \rangle_{H^2(\mathbb{B}_1)} = S(\zeta, \eta)\), see (3.8). We use the multiindex notation: for \(n = (n_1, \ldots, n_N)\) \(\in \mathbb{N}_0^N\), let \(|n| = n_1 + \cdots + n_N, n! = n_1! \cdots n_N!\), and \(\xi^n = \xi_1^{n_1} \cdots \xi_N^{n_N}\). Then, a function \(f\) holomorphic in \(\mathbb{B}_1\), with Taylor series representation \(f(z) = \sum_{n \in \mathbb{N}_0^N} a_n \xi^n\), belongs to \(H^2(\mathbb{B}_1)\) if and only if

\[
\|f\|_{H^2(\mathbb{B}_1)}^2 = \sum_{n \in \mathbb{N}_0^N} \frac{n!}{|n|!} |a_n|^2 < \infty.
\]

Note that there exists a unitary operator \(\Phi\) from the Drury-Arveson space \(H^2(\mathbb{B}_1)\) onto \(F^*(G)\) such that \(\Phi S_\zeta = V_S(\zeta), \zeta \in \mathbb{B}_1\).

If \(N = 1\), the Drury-Arveson space coincides with the Hardy space \(H^2(\mathbb{D})\) described at the previous example.

**Example 4.5. Holomorphic Kernels.** Given two Krein spaces \(G\) and \(H\) let \(\Omega\) be a sub-region of the open unit disc \(\mathbb{D}\) in the complex plane and \(\Theta : \Omega \rightarrow \mathcal{L}(G, H)\). One considers the following kernels

\[
K_\Theta(z, w) = \frac{I - \Theta(z)\Theta(w)^*}{1 - z\overline{w}},
\]

\[
K_{\tilde{\Theta}}(z, w) = \frac{I - \tilde{\Theta}(z)\tilde{\Theta}(w)^*}{1 - z\overline{w}},
\]

\[
D_\Theta(z, w) = \begin{bmatrix}
K_\Theta(z, w) & \frac{\Theta(z) - \Theta(w)}{z - \overline{w}} \\
\tilde{\Theta}(z) - \tilde{\Theta}(\overline{w}) & K_{\tilde{\Theta}}(z, w)
\end{bmatrix},
\]

where \(\tilde{\Theta}(z) = \Theta(\overline{z})^z\). These are operator valued Hermitian holomorphic kernels that are of interest in connection to Schur classes and their generalisations.

The classical Schur class corresponds to the case when \(G\) and \(H\) are Hilbert spaces and the Schur kernel \(K_\Theta\) is positive semidefinite. In case \(G\) and \(H\) are Krein or even genuine Pontryagin spaces, in order to define the Schur class, positive semidefiniteness has to be imposed on all \(K_\Theta, K_{\tilde{\Theta}}\) and \(D_\Theta\). Given \(\kappa \in \mathbb{N}\), the generalized Schur class corresponds to the requirement that the negative signatures of each kernel \(K_\Theta, K_{\tilde{\Theta}}\) and \(D_\Theta\) are \(\kappa\).
When \( \mathcal{G} = \mathcal{H} \) and for appropriate regions in the complex field, the \textit{Carathéodory kernel} \( C_\Theta \)

\[
C_\Theta(z, \zeta) = \frac{1}{2} \frac{\Theta(z) - \Theta(\zeta)\zeta}{1 - \zeta z},
\]

as well as the \textit{Nevanlinna kernel} \( N_\Theta \)

\[
N_\Theta(z, \zeta) = \frac{\Theta(z) - \Theta(\zeta)\zeta}{z - \zeta},
\]

are of special interest. As in the case of the Schur classes, positive semidefiniteness of the corresponding kernels define the \textit{Carathéodory class} and, respectively, the \textit{Nevanlinna class} of holomorphic functions. Generalized classes correspond to the case when the appropriate kernels have fixed negative signatures.

**Example 4.6. Töplitz Kernels.** Let \( \mathcal{H} \) be a Krein space and \( X = \mathbb{Z} \), the set of integer numbers. An \( \mathcal{H} \)-kernel on \( \mathbb{Z} \), \( K: \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H}) \), is called a Töplitz Kernel if \( K(i,j) = T(i-j) \) for some function \( T: \mathbb{Z} \to \mathcal{L}(\mathcal{H}) \), called the \textit{symbol} of \( K \).

One considers the set \( \mathfrak{T}(\mathcal{H}) \) of all Töplitz \( \mathcal{H} \)-kernels. In the following it is considered the subclass of Töplitz Hermitian \( \mathcal{H} \)-kernels \( \mathfrak{T}^h(\mathcal{H}) \) and the subclass of Töplitz positive semidefinite \( \mathcal{H} \)-kernels \( \mathfrak{T}^+(\mathcal{H}) \). One notes that \( \mathfrak{T}^h(\mathcal{H}) \) is closed under addition, subtraction and (left and right) multiplication with bounded operators on \( \mathcal{H} \). Also, \( \mathfrak{T}^+(\mathcal{H}) \) is a strict cone of \( \mathfrak{T}^h(\mathcal{H}) \).

Considering the complex vector space \( \mathcal{F}_0(\mathcal{H}) \) of all functions \( h: \mathbb{Z} \to \mathcal{H} \) with finite support and for an arbitrary Hermitian kernel \( H \in \mathfrak{S}^h(\mathcal{H}) \) one associates the inner product space \( (\mathcal{F}_0(\mathcal{H}), [\cdot, \cdot]_H) \) as in the previous sections. On the vector space \( \mathcal{F}_0(\mathcal{H}) \) one considers two operators, the \textit{forward shift} \( S_+ \) defined by \( (S_+ h)(n) = h(n+1) \), for all \( h \in \mathcal{F}_0(\mathcal{H}) \) and \( n \in \mathbb{Z} \), and the \textit{backward shift} \( S_- \) defined by \( (S_- h)(n) = h(n-1) \), for all \( h \in \mathcal{F}_0(\mathcal{H}) \) and all \( n \in \mathbb{Z} \). If \( H \in \mathfrak{S}^h(\mathcal{H}) \) then it is a Töplitz kernel if and only if

\[
[S_+ h, g]_H = [h, S_- g]_H, \quad f, g \in \mathcal{F}_0(\mathcal{H}).
\]

If \( H \) is a Hermitian Töplitz \( \mathcal{H} \)-kernel then both \( S_+ \) and \( S_- \) are isometric with respect to the inner product \([\cdot, \cdot]_H\), that is, for all \( h, g \in \mathcal{F}_0(\mathcal{H}) \) the equalities \([S_+ h, S_+ g]_H = [h, g]_H\) and \([S_- h, S_- g]_H = [h, g]_H\) hold. The converse is also true, if either \( S_+ \) or \( S_- \) is isometric with respect to the Hermitian \( \mathcal{H} \)-kernel \( H \) then \( H \) is Töplitz.

Let \( H \) be a Töplitz Hermitian \( \mathcal{H} \)-kernel. A \textit{Naïmark dilation} of \( H \) is, by definition, a triple \((U, Q; \mathcal{K})\) with the following properties:

\begin{enumerate}
\item[(nd1)] \( \mathcal{K} \) is Krein space, \( U \in \mathcal{L}(\mathcal{K}) \) is a unitary operator, and \( Q \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \).
\item[(nd2)] \( \mathcal{K} = \bigvee_{n \in \mathbb{Z}} U^n Q \mathcal{H} \).
\item[(nd3)] \( H(i,j) = Q^n U^{i-j} Q, \ i, j \in \mathbb{Z} \).
\end{enumerate}

If the Töplitz Hermitian \( \mathcal{H} \)-kernel \( H \) has a Naïmark dilation \((U, Q; \mathcal{K})\), letting \( V(n) = U^n Q \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) it is readily verified that the pair \((V; \mathcal{K})\) is a linearisation of \( H \).
5. Quasi Semidefinite Kernels

In the following, for simplicity and because most of the applications presented in this survey do not require the full generality as considered in Section 3, there is considered only the case of $\mathcal{H}$-kernels, that is, $\mathcal{H}_x = \mathcal{H}$ for some fixed Krein space $\mathcal{H}$. A Hermitian kernel $H: X \times X \to \mathcal{L}(\mathcal{H})$ is associated to an indefinite inner product space $[\cdot, \cdot]_H$ on $\mathcal{F}(\mathcal{H})$, the vector space of all complex valued functions $f: X \to \mathcal{H}$, as in (3.2). Recall that $\kappa_\pm(H)$, the signatures of $H$, are defined as the positive/negative signatures of the inner product $[\cdot, \cdot]_H$, while its rank of indefiniteness is defined by $\kappa(H) = \min\{\kappa_(H), \kappa_+(H)\}$. A Hermitian $\mathcal{H}$-kernel $H$ is quasi semidefinite if $\kappa(H) < \infty$, that is, either $\kappa_-(H)$ or $\kappa_+(H)$ is finite.

Theorem 5.1. Let $H$ be quasi semidefinite $\mathcal{H}$-kernel on $X$. Then:

(a) $H$ admits a minimal linearisation to a Pontryagin space $(\mathcal{K}; V)$ with $\kappa(H) = \kappa(\mathcal{K})$, unique up to a unitary equivalence.

(b) There exists a unique reproducing kernel Pontryagin space $\mathcal{R}$ on $X$ and with reproducing kernel $H$.

On the vector space $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$-fields on $X$, the inner product $[\cdot, \cdot]_H$, see (3.2), is defined. Since $\kappa(H)$ is finite, the inner product space $(\mathcal{F}(\mathcal{H}); [\cdot, \cdot]_H)$ is decomposable. To make a choice, assume that $\kappa_-(H) < \infty$, hence

$$\mathcal{F}(\mathcal{H}) = \mathcal{F}_- + \mathcal{F}_0[+] \mathcal{F}_+,$$

where $\mathcal{F}_0$ is the isotropic subspace, $\mathcal{F}_\pm$ are positive/negative subspaces, and $\dim(\mathcal{F}_-) = \kappa_-(H) < \infty$. Factoring out $\mathcal{F}_0$ one can assume, without loss of generality, that $\mathcal{F}_0 = 0$. Then, let $\mathcal{K}_+$ denote the completion of $(\mathcal{F}_+; [\cdot, \cdot]_H)$ to a Hilbert space and then $\mathcal{K} = \mathcal{F}_+ + \mathcal{K}_+$ is a Pontryagin space with $\kappa_-(\mathcal{K}) = \kappa_-(H)$. For arbitrary $x \in X$, the linear operator $V(x): \mathcal{H} \to \mathcal{K}$ is defined by assigning to each vector $h \in \mathcal{H}$ the $\mathcal{H}$-vector field $f: X \to \mathcal{H}$ given by $f(x) = h$ and $f(y) = 0$ for all $y \in X$, $y \neq x$. Then $(\mathcal{K}; V)$ is a minimal linearisation of $H$.

The uniqueness of minimal linearisations of $H$, modulo unitary equivalence, follows from the fact that any dense linear subspace of a Pontryagin space with finite negative/positive signature contains a maximal negative/positive subspace and the continuity of isometric densely defined operators between Pontryagin spaces with the same negative/positive signature.

From Proposition 3.2 (1), once one gets a minimal linearisation $(\mathcal{K}; V)$ of $H$ one immediately obtains a reproducing kernel Pontryagin space $\mathcal{R}$ with reproducing kernel $H$ as defined in (3.12) and (3.13). The uniqueness of the reproducing kernel Pontryagin space $\mathcal{R}$ follows from Proposition 3.2 (2) and the uniqueness, modulo unitary equivalence, of the minimal linearisation of $H$.

There is a more direct but longer way of constructing the reproducing kernel Krein space associated to a quasi positive semidefinite $\mathcal{H}$-kernel $H$, by considering the vector space $\mathcal{R}_0(H)$ linearly generated by $H_x h$, for $x \in X$ and $h \in \mathcal{H}$ on which the inner product $[\cdot, \cdot]_{\mathcal{R}_0}$ is defined as in (3.5) in such a way that $(\mathcal{R}_0(H); [\cdot, \cdot]_{\mathcal{R}_0})$ is a reproducing kernel inner product space with $H$ its reproducing kernel. The inner product space $(\mathcal{R}_0(H); [\cdot, \cdot]_{\mathcal{R}_0})$ is nondegenerate and its negative signature is the same with $\kappa_-(H)$, hence finite. Then $\mathcal{R}_0(H)$ is decomposable, hence $\mathcal{R}_0(\mathcal{H}) = \mathcal{R}_- [+\mathcal{R}_{0,+}]$, where $\dim(\mathcal{R}_-) = \kappa_-(H) < \infty$.
is negative definite, while $R_{0,+}$ is positive definite. It can be proven that $R_{0,+}$ has a completion to a Hilbert space $R_+$ inside $\mathcal{F}(\mathcal{H})$ and then $R = R_-([+)R_+$ is the reproducing kernel Kreǐn space of $H$.

The next theorem points out a property of propagation to arbitrarily large domains of holomorphy for quasi semidefinite holomorphic kernels.

**Theorem 5.2.** Let $K$ be a Hermitian holomorphic $\mathcal{H}$-kernel on some region $\Omega$ and let $\Omega_0$ be a subregion of $\Omega$. If $\kappa_-(K_0) < \infty$, where $K_0$ is the restriction of $K$ to $\Omega_0 \times \Omega_0$, then $\kappa_-(K) = \kappa_-(K_0)$.

The idea of proof is to use Cauchy’s Theorem in order to prove the propagation property for positive semidefinite kernels, first for disks around the origin of the complex plane, then for union of regions for which the kernel is positive definite, and finally to use the decomposition of $K = K_+ − K_-$, where $K_\pm$ are positive semidefinite kernels and $K_-$ has a reproducing Hilbert space of dimension $\kappa_-(K)$, that can be obtained from (5.1).

6. **Induced Kreǐn Spaces**

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})$ be a Hilbert space and consider a bounded selfadjoint operator $A$ on $\mathcal{H}$. A new inner product on $\mathcal{H}$ is defined by

\[
[h, k]_A = \langle Ah, k \rangle_\mathcal{H}, \quad h, k \in \mathcal{H}.
\]

In this section the properties of some Kreǐn spaces associated with this inner product are described.

By definition, a *Kreǐn space induced* by $A$ is a pair $(\mathcal{K}, \Pi)$, where

- (ik1) $\mathcal{K}$ is a Kreǐn space and $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has dense range.
- (ik2) $[\Pi x, \Pi y]_\mathcal{K} = \langle Ax, y \rangle_\mathcal{H}$, for all $x, y \in \mathcal{H}$.

There exist at least two main constructions of Kreǐn spaces induced by selfadjoint operators, and they are related by certain unitary equivalences. Two Kreǐn spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1, 2$, induced by the same $A$, are *unitary equivalent* if there exists a unitary operator $U \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\Pi_1 = \Pi_2$.

**Example 6.1.** The Induced Kreǐn Space $(\mathcal{K}_A, \Pi_A)$. Let $A$ denote the selfadjoint operator with respect to the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})$. Let $\mathcal{H}_-$ and $\mathcal{H}_+$ be the spectral subspaces corresponding to the semiaxis $(-\infty, 0)$ and, respectively, $(0, +\infty)$ and the operator $A$. Then one gets the decomposition

\[
\mathcal{H} = \mathcal{H}_- \oplus \text{Ker } A \oplus \mathcal{H}_+.
\]

Note that $(\mathcal{H}_-, [-, \cdot]_A)$ and $(\mathcal{H}_+, [\cdot, \cdot]_A)$ are positive definite inner product spaces and hence they can be completed to Hilbert spaces $\mathcal{K}^-$ and, respectively, $\mathcal{K}^+$. We can build the Kreǐn space $(\mathcal{K}_A, [\cdot, \cdot]_A)$ by letting

\[
\mathcal{K}_A = \mathcal{K}^- [+\mathcal{K}^+.
\]

The operator $\Pi_A \in \mathcal{L}(\mathcal{H}, \mathcal{K}_A)$ is, by definition, the composition of the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H} \ominus \text{Ker } A$ with the embedding of $\mathcal{H} \ominus \text{Ker } A$ into $\mathcal{K}_A$. With these definitions, it is readily verified that $(\mathcal{K}_A, \Pi_A)$ is a Kreǐn space induced by $A$. 
In order to take a closer look at the strong topology of the Kreĭn space $\mathcal{K}_A$, consider the seminorm $\mathcal{H} \ni x \mapsto |||A^{1/2}x||| = \langle |A|^{1/2}x, |A|^{1/2}x \rangle^{1/2}$. The kernel of this seminorm is exactly $\text{Ker } A$ and the completion of $\mathcal{H} \oplus \text{Ker } A$ with respect to this norm is exactly the space $\mathcal{K}_A$. Moreover, the strong topology of $\mathcal{K}_A$ is induced by the extension of this seminorm. The positive definite inner product on $\mathcal{K}_A$ associated with the norm $|||A^{1/2} \cdot|||$ is $\langle |A| \cdot, \cdot \rangle$. Hence, if $A = S_A |A|$ is the polar decomposition of $A$ and $S_A$ denotes the corresponding selfadjoint partial isometry, then it follows that $S_A$ can be extended by continuity to $\mathcal{K}_A$ and this extension is exactly the fundamental symmetry of $\mathcal{K}_A$ corresponding to $\langle |A| \cdot, \cdot \rangle$.

**Example 6.2.** The Induced Kreĭn Space $(\mathcal{B}_A; \Pi_{\mathcal{B}_A})$. With notation as in Example 6.1 consider the polar decomposition of $A = S_A |A|$ as in Example 6.1. Define the space $\mathcal{B}_A = \text{Ran}(|A|^{1/2})$ endowed with the positive definite inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}_A}$ given by

$$
\langle |A|^{1/2}x, |A|^{1/2}y \rangle_{\mathcal{B}_A} = \langle P_{\mathcal{H} \oplus \text{Ker } A} x, y \rangle_{\mathcal{H}}, \quad x, y \in \mathcal{H}.
$$

This positive definite inner product is correctly defined and $(\mathcal{B}_A, \langle \cdot, \cdot \rangle_{\mathcal{B}_A})$ is a Hilbert space. To see this, just note that the operator $|A|^{1/2} : \mathcal{H} \oplus \text{Ker } A \to \mathcal{B}_A$ is a Hilbert space unitary operator.

On $\mathcal{B}_A$ one defines the inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}_A}$ by

$$
\langle |A|^{1/2}x, |A|^{1/2}y \rangle_{\mathcal{B}_A} = \langle S_A x, y \rangle_{\mathcal{H}}, \quad x, y \in \mathcal{H}.
$$

Since the operator $|A|^{1/2}$ and $S_A$ commute it follows that

$$
[a, b]_{\mathcal{B}_A} = \langle S_A a, b \rangle_{\mathcal{B}_A}, \quad a, b \in \mathcal{B}_A.
$$

The operator $S_A|\mathcal{B}_A$ is a symmetry. We define now a linear operator $\Pi_{\mathcal{B}_A} : \mathcal{H} \to \mathcal{B}_A$ by

$$
\Pi_{\mathcal{B}_A} h = |A| h, \quad h \in \mathcal{H}.
$$

It follows that $(\mathcal{B}_A, \langle \cdot, \cdot \rangle_{\mathcal{B}_A})$ is a Kreĭn space induced by $A$.

The Kreĭn space induced by a selfadjoint operator as in Example 6.2 is a genuine operator range subspace. More precisely, to any Kreĭn space $\mathcal{K}$ continuously embedded in the Hilbert space $\mathcal{H}$ one associates a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ in the following way: let $\iota : \mathcal{K} \to \mathcal{H}$ be the inclusion operator which is supposed bounded and take $A = \iota^* \in \mathcal{L}(\mathcal{H})$. Clearly $A$ is selfadjoint and $(\mathcal{K}, \iota^*)$ is a Kreĭn space induced by $A$. Conversely, from Example 6.2 it is easy to see that $\mathcal{B}_A$ is a Kreĭn space continuously embedded in $\mathcal{H}$.

The connection between the induced Kreĭn spaces $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ is explained by the following

**Proposition 6.3.** The induced Kreĭn spaces $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ are unitary equivalent, more precisely, the mapping

$$
\mathcal{K}_A \ni x \mapsto |A|x \in \mathcal{B}_A,
$$

extends uniquely to a unitary operator $V \in \mathcal{L}(\mathcal{K}_A, \mathcal{B}_A)$ such that $V \Pi_A = \Pi_{\mathcal{B}_A}$.

7. Uniqueness of Induced Kreĭn Spaces

The two examples of induced Kreĭn spaces described in Example 6.1 and Example 6.2 turned out to be unitary equivalent. However, in general, not all possible Kreĭn spaces induced by a fixed selfadjoint operator are unitary equivalent. We denote by $\rho(T)$ the resolvent set of the operator $T$. 
Theorem 7.1. Let $A$ be a bounded selfadjoint operator in the Hilbert space $\mathcal{H}$. The following statements are equivalent:

(i) The Krein space induced by $A$ is unique, modulo unitary equivalence.

(ii) There exists $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A)$ or $(-\epsilon, 0) \subset \rho(A)$.

(iii) For some (equivalently, for any) Krein space $\{\mathcal{K}, \Pi\}$ induced by $A$, the range of $\Pi$ contains a maximal uniformly definite subspace of $\mathcal{K}$.

The equivalence of (ii) and (iii) is a simple matter of spectral theory for bounded selfadjoint operators. The implication (iii)$\Rightarrow$(i) comes from the fact that any densely defined isometric operator whose domain contains a maximal uniformly definite subspace is bounded. So only the idea of the implication (i)$\Rightarrow$(ii) is clarified.

Assuming that the statement (ii) does not hold, there exists a decreasing sequence of numbers $(\mu_n)_{n \geq 1}$ with $\mu_n \in \sigma(A)$ and $0 < \mu_n < 1$ for all $n \geq 1$, such that $\mu_n \to 0$ ($n \to \infty$), and there exists a decreasing sequence of numbers $(\nu_n)_{n \geq 1}$ with $-\nu_n \in \sigma(-A)$ and $0 < \nu_n < 1$ for all $n \geq 1$, such that $\nu_n \to 0$ ($n \to \infty$). Then, letting $\mu_0 = \nu_0 = 1$, there exist sequences of orthonormal vectors $(e_n)_{n \geq 1}$ and $(f_n)_{n \geq 1}$ such that

\begin{equation}
(7.1) \quad e_n \in E((\mu_n, \mu_n-1))\mathcal{H}, \quad f_n \in E((-\nu_n-1, -\nu_n))\mathcal{H}, \quad n \geq 1.
\end{equation}

As a consequence, one also gets

\begin{equation}
(7.2) \quad [Ae_i, f_j] = 0, \quad i, j \geq 1.
\end{equation}

Define the sequence $(\lambda_n)_{n \geq 1}$ by

$$\lambda_n = \max \{\sqrt{1-\mu_n^2}, \sqrt{1-\nu_n^2}\}.$$ 

Then $0 < \lambda_n \leq 1$, $\lambda_n \uparrow 1$ ($n \to \infty$).

Consider the subspace $\mathcal{S}_n$ of the Krein space $\mathcal{K}_A$, defined by

$$\mathcal{S}_n = \mathbb{C}e_n + \mathbb{C}f_n, \quad n \geq 1,$$

and then define the operators $U_n \in \mathcal{L}(\mathcal{S}_n)$,

$$U_n = \frac{1}{\sqrt{1-\lambda_n^2}} \begin{bmatrix} 1 & -\lambda_n \\ \lambda_n & -1 \end{bmatrix}, \quad n \geq 1.$$ 

The operators $U_n$ are isometric in $\mathcal{S}_n$. Further, one defines the linear manifold $\mathcal{D}_0$ in $\mathcal{K}_A$ by

$$\mathcal{D}_0 = \bigcup_{k \geq 1} \mathcal{S}_k$$

and note that the closure of $\mathcal{D}_0 = \bigvee_{k \geq 1} \{e_k, f_k\}$ is a regular subspace in $\mathcal{K}_A$.

By construction, the linear manifold

$$\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- = \text{Ran}(\Pi)$$

is dense in $\mathcal{K}_A$, where $A = A_+ - A_-$ is the Jordan decomposition of $A$ and $\mathcal{D}_\pm = \text{Ran}(A_\pm)$. Also, $\mathcal{D}_0 \subseteq \mathcal{D}$ and the following decomposition holds

$$\mathcal{D} = \mathcal{D}_0 \oplus (\mathcal{D} \cap \mathcal{D}_0^\perp).$$

Then define a linear operator $U$ in $\mathcal{K}_A$, with domain $\mathcal{D}_0$ and the same range, by $U|\mathcal{S}_n = U_n$, $n \geq 1$, and $U|(\mathcal{D} \cap \mathcal{D}_0^\perp) = I|(\mathcal{D} \cap \mathcal{D}_0^\perp)$. The operator $U$ is isometric, it has dense range
as well as dense domain, and it is unbounded since it maps uniformly definite subspaces into subspaces that are not uniformly definite.

\((\mathcal{K}_A, \Pi)\) is a Kre˘ın space induced by \(A\). Indeed, \(\Pi \mathcal{H} = U\Pi A \mathcal{H} \supseteq \mathcal{D}\), and the latter is dense in \(\mathcal{K}_A\). Further,

\[
[\Pi x, \Pi y] = [U\Pi_A x, U\Pi_A y] = [\Pi_A x, \Pi_A y] = [Ax, y], \quad x, y \in \mathcal{H}.
\]

Since \(\Pi_A\) is bounded it follows that \(\Pi\) is closed. By the Closed Graph Principle, it follows that \(U\) is bounded. Since \(U\) is unbounded it follows that \((\mathcal{K}_A, \Pi_A)\) is not unitary equivalent with \((\mathcal{K}_A, \Pi)\). Thus, a contradiction with the assertion (i) is obtained.

Let \(\mathcal{K}_k\) be two Kre˘ın spaces continuously embedded into the Kre˘ın space \(\mathcal{H}\) and denote by \(\iota_k : \mathcal{K}_k \to \mathcal{H}\) the corresponding embedding operators, that is \(\iota_k h = h, h \in \mathcal{K}_k, k = 1, 2\). We say that the Kre˘ın spaces \(\mathcal{K}_1\) and \(\mathcal{K}_2\) correspond to the same selfadjoint operator \(A\) in \(\mathcal{H}\) if \(\iota_1\iota_1^* = \iota_2\iota_2^* = A\). If \(A\) is nonnegative, equivalently, the Kre˘ın spaces \(\mathcal{K}_k\) are actually Hilbert spaces, this implies that \(\mathcal{K}_1 = \mathcal{K}_2\). As a consequence of Theorem 8.1, the following corollary is obtained.

Corollary 7.2. Given a selfadjoint operator \(A \in \mathcal{L}(\mathcal{H})\), the following statements are mutually equivalent:

(a) There is a unique Kre˘ın space \(\mathcal{K}\) continuously embedded in \(\mathcal{H}\) and associated to \(A\).
(b) There exists \(\epsilon > 0\) such that either \((-\epsilon, 0) \subseteq \rho(A)\) or \((0, \epsilon) \subseteq \rho(A)\).
(c) There exists a Kre˘ın space \(\mathcal{K}\) continuously embedded in \(\mathcal{H}\), \(\iota : \mathcal{K} \to \mathcal{H}\) such that \(\iota^* = A\) and \(\text{Ran}(\iota^*)\) contains a maximal uniformly definite subspace of \(\mathcal{K}\).

Indeed, let \(\mathcal{K}_i\) be two Kre˘ın spaces continuously embedded in \(\mathcal{H}\) and let \(\iota_i : \mathcal{K} \to \mathcal{H}\) be the embedding operators, \(i = 1, 2\). Assume that the induced Kre˘ın spaces \((\mathcal{K}_i, \iota_i^*)\) are unitary equivalent, that is, there exists a unitary operator \(U \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)\) such that \(U\iota_2 = \iota_1\). Then \(\iota_1 = \iota_2 U^\sharp\) and taking into account that \(\iota_i\) are embeddings, it follows that

\[
U^\sharp x = \iota_2 U^\sharp x = \iota_1 x = x, \quad x \in \mathcal{K}_2,
\]

hence \(\mathcal{K}_1 = \mathcal{K}_2\) and \(\iota_1 = \iota_2\). This shows that the two Kre˘ın spaces coincide. We can now apply Theorem 8.1 and get the equivalence of the statements (a), (b), and (c).

8. Existence of Reproducing Kernel Kre˘ın Spaces

The next theorem clarifies the problem of existence of reproducing kernel Kre˘ın spaces associated to Hermitian \(\mathcal{H}\)-kernels. Notation is as in Section 3. In addition, two positive semidefinite \(\mathcal{H}\)-kernels are called independent if for any \(P \in \mathfrak{R}^+(\mathcal{H})\) such that \(P \leq \mathcal{H}, \mathcal{K}\), it follows \(P = 0\).

Theorem 8.1. Let \(H \in \mathfrak{R}^+(\mathcal{H})\). The following assertions are equivalent:

1. There exists \(L \in \mathfrak{R}^+(\mathcal{H})\) such that \(-L \leq H \leq L\).
1'. There exists \(L \in \mathfrak{R}^+(\mathcal{H})\) such that

\[
|[f, g]_H| \leq [f, f]^{1/2}_L [g, g]^{1/2}_L, \quad f, g \in \mathcal{F}_0(\mathcal{H}).
\]

2. \(H = H_1 - H_2\) with \(H_1, H_2 \in \mathfrak{R}^+(\mathcal{H})\).
2'. \(H = H_+ - H_-\) with \(H_+ \in \mathfrak{R}^+(\mathcal{H})\) independent.
3. There exists a Kolmogorov decomposition \((V; \mathcal{K})\) of \(H\).
4. There exists a Kre˘ın space with reproducing kernel \(H\).
Note that letting \( f = g \) in (1)′ one obtains (1). Conversely, let \( L \in \mathfrak{R}^+(H) \) be such that \(-L \leq H \leq L\), that is \(|[f, f]_H| \leq |f, f|_L\), for all \( f \in \mathcal{F}_0(H)\). Let \( f, g \in \mathcal{F}_0(H)\). Since \( H \) is Hermitian, one gets \( 4 \Re[f, g]_H = |f + g, f + g|_H - |f - g, f - g|_H \) and hence

\[ 4 |\Re[f, g]_H| \leq |f + g, f + g|_L + |f - g, f - g|_L = 2|f, f|_L + 2|g, g|_L. \]

Let \( \lambda \in \mathbb{C} \) be chosen such that \(|\lambda| = 1\) and \( \Re[f, \lambda g] = [f, \lambda g] \). Then

\[ (8.1) \quad |[f, \lambda g]_H| \leq \frac{1}{2}|f, f|_L + \frac{1}{2}|g, g|_L. \]

We distinguish two possible cases. First, assume that either \( |f, f|_L = 0 \) or \( |g, g|_L = 0 \). To make a choice assume \( |f, f|_L = 0 \). Consider the inequality (8.1) with \( g \) replaced by \( t g \) for \( t > 0 \). Then \(|[f, g]_H| \leq \frac{1}{2}|g, g|_H\). Letting \( t \to 0 \) one gets \( |f, g|_H = 0 \).

Second case, assuming that both \( |f, f|_L \) and \( |g, g|_L \) are nontrivial, in (8.1) replace \( f \) by \( f^1/2 \) and \( g \) by \( g^1/2 \) to get \( |[f, g]_H| \leq |f, f|_L^2 |g, g|_L^{1/2} \). Thus, the assertions (1) and (1)′ are equivalent.

In order to describe the idea of the proof of the implication (1)′ ⇒ (2)′, let \( K_L \) be the quotient-completion of \((\mathcal{F}_0(H), [\cdot, \cdot]_L)\) to a Hilbert space. More precisely, letting \( N_L = \{ f \in \mathcal{F}_0 \mid |f, f|_L = 0 \} \) denote the isotropic subspace of the positive semidefinite inner product space \((\mathcal{F}_0(H), [\cdot, \cdot]_L)\), one considers the quotient \( \mathcal{F}_0(H)/N_L \) and complete it to a Hilbert space \( K_L \). The inequality (1)′ implies that the isotropic subspace \( N_L \) is contained into the isotropic subspace \( N_H \) of the inner product \((\mathcal{F}_0, [\cdot, \cdot]_H)\). Therefore, \([\cdot, \cdot]_H \) uniquely induces an inner product on \( K_L \), also denoted by \([\cdot, \cdot]_L \) such that the inequality in (1)′ still holds for all \( f, g \in K_L \). By the Riesz Representation Theorem one gets a selfadjoint contraction operator \( A \in \mathcal{L}(K_L) \), such that

\[ (8.2) \quad [f, g]_H = [Af, g]_L, \quad f, g \in K_L. \]

Let \( A = A_+ - A_- \) be the Jordan decomposition of \( A \) in \( K_L \). Then \( A_\pm \) are also contractions and hence

\[ (8.3) \quad [A_\pm f, f]_L \leq |f, f|_L, \quad f \in K_L. \]

It can be proven that the nonnegative inner products \([A_\pm \cdot, \cdot]\) uniquely induce kernels \( H_\pm \in \mathfrak{R}^+(H) \) such that \([f, f]_{H_\pm} \leq |f, f|_L, \quad f \in \mathcal{F}_0(H)\), and \( H = H_+ - H_- \).

Indeed, the inner product \([A_\pm \cdot, \cdot]\) restricted to \( \mathcal{F}_0(H)/N_H \) can be extended to an inner product \([\cdot, \cdot]_+ \) on \( \mathcal{F}_0(H) \) by letting it be null onto \( N_L \) and hence

\[ (8.4) \quad [f, f]_+ \leq |f, f|_L, \quad f \in \mathcal{F}_0(H). \]

Let \( x, y \in X \) be arbitrary and \( x \neq y \). Clearly, one can identify the Krőn space \( \mathcal{H}_x[+]\mathcal{H}_y \) with the subspace of all \( H \)-fields \( f \in \mathcal{F}_0(H) \) such that \( \text{supp} f \subseteq \{ i, j \} \). With this identification, one considers the restrictions of the inner products \([\cdot, \cdot]_+ \) and \([\cdot, \cdot]_L \) to \( \mathcal{H}_x[+]\mathcal{H}_y \). The inner product \([\cdot, \cdot]_L \) is jointly continuous with respect to the strong topology of \( \mathcal{H}_x[+]\mathcal{H}_y \). By (8.4) and the equivalence of (1) and (1)′ one concludes that the inner product \([\cdot, \cdot]_+ \) is also jointly continuous with respect to the strong topology of \( \mathcal{H}_x[+]\mathcal{H}_y \) and hence, by the Riesz Representation Theorem, there exists a selfadjoint operator \( S \in \mathcal{L}(\mathcal{H}_x[+]\mathcal{H}_y) \) such that

\[ [f, g]_+ = [Sf, g]_{\mathcal{H}_x[+]\mathcal{H}_y}, \quad f, g \in \mathcal{H}_x[+]\mathcal{H}_y. \]

Define \( H_+(x, y) = P_{\mathcal{H}_x}S|\mathcal{H}_y \), \( H_+(x, x) = P_{\mathcal{H}_x}S|\mathcal{H}_x \), \( H_+(j, j) = P_{\mathcal{H}_y}S|\mathcal{H}_x \) and \( H_+(j, i) = P_{\mathcal{H}_y}S|\mathcal{H}_x = H_+(x, y)^2 \).
In this way one obtains a kernel $H_+ \in \mathfrak{K}^+(\mathcal{H})$ such that $H_+ \leq L$ and

$$[f, g]_{H_+} = [f, g]_+, \quad f, g \in \mathcal{F}_0(\mathcal{H}).$$

Since the inner product $\langle \cdot, \cdot \rangle_+$ is nonnegative it follows that $H_+ \in \mathfrak{K}^+(\mathcal{H})$.

Similarly one constructs the kernel $H_- \in \mathfrak{K}^+(\mathcal{H})$ such that $H_- \leq L$ and

$$[f, g]_{H_-} = [f, g]_-, \quad f, g \in \mathcal{F}_0(\mathcal{H}),$$

where the inner product $[f, g]_-$ is the extension of the restriction of the inner product $[A_- f, g]$ to $\mathcal{F}_0(\mathcal{H})/\mathcal{N}_L$, by letting it be null onto $\mathcal{N}_H$.

From $A = A_+ - A_-$, (8.2) and the constructions of the kernels $H_+$ and $H_-$ one concludes that $H = H_+ - H_-$. Let $P \in \mathfrak{K}^+(\mathcal{H})$ be such that $P \leq H_\pm$. Then

$$(8.5) \quad [f, f]_P \leq [f, f]_L, \quad f \in \mathcal{F}_0(\mathcal{H}).$$

As before, $[\cdot, \cdot]_P$ induces a nonnegative inner product $[\cdot, \cdot]_P$ on $\mathcal{K}_L$ such that (8.3) holds for all $f \in \mathcal{K}_L$. From $P \leq H_\pm$ one concludes that

$$[f, f]_P \leq [A_\pm f, f]_L, \quad f \in \mathcal{K}_L,$$

and, since $A_+A_- = 0$ this implies $[f, f]_P = 0$ for all $f \in \mathcal{K}_L$. Since by (8.5) one gets $\mathcal{N}_L \subseteq \mathcal{N}_P$ this implies that the inner product $[\cdot, \cdot]_P$ is null onto the whole $\mathcal{F}_0(\mathcal{H})$ and hence $P = 0$.

The implications $(2)' \Rightarrow (2)$ and $(2) \Rightarrow (1)$ are clear.

The most interesting implication is $(1)' \Rightarrow (3)$. In a fashion similar to $(1)' \Rightarrow (2)'$, one considers the quotient-completion Hilbert space $\mathcal{K}_L$, the representation (8.2) and the Jordan decomposition $A = A_+ - A_-$. The latter yields in a canonical way a Krein space $(\mathcal{K}, [\cdot, \cdot]_H)$. We again consider $\mathcal{N}_L$ and $\mathcal{N}_H$, the isotropic spaces of the inner product spaces $(\mathcal{F}_0(\mathcal{H}), [\cdot, \cdot]_L)$ and, respectively, $(\mathcal{F}_0(\mathcal{H}), [\cdot, \cdot]_H)$. From the inequality $(1)'$ one gets $\mathcal{N}_L \subseteq \mathcal{N}_H$.

For every $x \in X$ and every vector $h \in \mathcal{H}_x$ one considers the function $h \in \mathcal{F}_0(\mathcal{H})$ defined by

$$h(y) = \begin{cases} h, & y = x, \\ 0, & y \neq x. \end{cases}$$

This identification of vectors with functions in $\mathcal{F}_0(\mathcal{H})$ yields a natural embedding $\mathcal{H}_x \hookrightarrow \mathcal{F}_0(\mathcal{H})$. With this embedding one defines linear operators $V(x) : \mathcal{H}_x \to \mathcal{K}$ by

$$V(x)h = h + \mathcal{N}_H \in \mathcal{F}_0(\mathcal{H})/\mathcal{N}_H \subseteq \mathcal{K}, \quad h \in \mathcal{H}_x.$$ 

It follows that the linear operators $V(x)$ are bounded, for all $x \in X$, and that $(\mathcal{K}; V)$ is a minimal linearisation of the $\mathcal{H}$-kernel $H$.

The idea of the proof of $(3) \Rightarrow (1)$ deserves an explanation as well. Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space and $\{V(x)\}_{x \in X}$ be a family of bounded linear operators $V(x) \in \mathcal{L}(\mathcal{H}_x, \mathcal{K})$, $x \in X$, such that

$$H(x, y) = V(y)^*V(x), \quad x, y \in X.$$ 

Fix on $\mathcal{K}$ a fundamental symmetry $J$ and for each $x \in X$ fix a fundamental symmetry $J_x$ on $\mathcal{H}_x$. Then, defining the kernel $L$ by

$$L(x, y) = J_yV(y)^*V(x), \quad x, y \in X,$$
it can be proven that $L \in \mathfrak{K}^+(H)$ and that $-L \leq H \leq L$.

**Example 8.2.** (1) Let $E$ be a reflexive real Banach space which is not a Hilbert space. For example, one can take $1 < p < 2$ and $E = l^p_2$. Let $E'$ denote its topological dual space. On $X = E \times E'$ consider the Hermitian form

$$H((e, \varphi), (f, \psi)) = \varphi(f) + \psi(e), \quad e, f \in E, \quad \varphi, \psi \in E'. \tag{8.7}$$

The Hermitian form $H$ can be viewed as a Hermitian scalar kernel on $X \times X$ and it can be proven that it cannot be written as a difference of two positive semidefinite scalar kernels. Briefly, the idea is that $X$ is a selfdual Banach space when given the norm $\|(e, \psi)\|_X^2 = \|e\|_E^2 + \|\psi\|_{E'}^2$, and $H$ is jointly continuous with respect to this norm, but the topological inner product space $(X; H; \| \cdot \|_X)$ is not decomposable.

9. **Uniqueness of Reproducing Kernel Krein Spaces**

Quasi semidefinite Hermitian kernels are associated with reproducing kernel Krein spaces which are unique, equivalently, they have linearisations having the uniqueness property modulo unitary equivalence. In the general Hermitian case, the existence of a reproducing kernel Krein space does not imply that it is unique, equivalently, the existence of a linearisation does not imply its uniqueness, modulo unitary equivalence. Recall that, two linearisations $(K;V)$ and $(H,U)$ of the same $H$-kernel $H$ are unitary equivalent if there exists a unitary operator $\Phi \in L(K, H)$ such that for all $x \in X$ one gets $U(x) = \Phi V(x)$.

Let $H$ be an $H$-kernel. If $L \in \mathfrak{K}^+(H)$ is such that $-L \leq H \leq L$ then one denotes by $K_L$ the quotient completion of $(F(H), [\cdot, \cdot])$ to a Hilbert space and by $A = A_L \in L(K_L)$ the Gram operator of the inner product $[\cdot, \cdot]_H$ with respect to the positive semidefinite inner product $[\cdot, \cdot]_H$, that is, $[h, k]_H = [A_L h, k]_L$ for all $h, k \in K_L$.

**Theorem 9.1.** Let $H$ be an $H$-kernel on a set $X$ which has a minimal linearisation, equivalently, it is associated to a reproducing kernel of a reproducing kernel Krein space on $X$. The following assertions are equivalent:

(i) The $H$-kernel $H$ has unique minimal linearisation, modulo unitary equivalence.

(ii) For any (equivalently, there exists a) positive semidefinite $H$-kernel $L$ such that $-L \leq H \leq L$ there exists $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A_L)$ or $(-\epsilon, 0) \subset \rho(A_L)$.

(iii) $H$ has a minimal linearisation (equivalently, any minimal linearisation) $(K;V)$ that has fundamental decomposition $K = K^+ [\pm] K^-$ such that either $K^+$ or $K^-$ is contained in the linear manifold generated by $V(x)H_x$, $x \in X$.

(iv) The reproducing kernel Krein space with reproducing kernel $H$ is unique.

In order to explain the implication (i) $\Rightarrow$ (ii), assume that there exists a positive semidefinite $H$-kernel $L$ such that $-L \leq H \leq L$ and for any $\epsilon > 0$ one gets $(0, \epsilon) \cap \sigma(A_L) \neq \emptyset$ and $(-\epsilon, 0) \cap \sigma(A_L) \neq \emptyset$. From Theorem 7.1 it follows that there exists two Krein spaces $(K, \Pi)$ and $(H, \Phi)$ induced by the same selfadjoint operator $A_L$, which are not unitary equivalent. It is easy to see that the operator $\Psi: \text{Ran}(\Pi) \rightarrow \text{Ran}(\Phi)$ defined by

$$\Psi \Pi f = \Phi f, \quad f \in K_L,$$
is isometric, densely defined with dense range, and it is unbounded due to the non-unitary equivalence of the two induced Kre˘ın spaces. As a consequence, Ψ is closable and its closure, denoted also by Ψ, shares the same properties.

Let \((V; \mathcal{K})\) be the minimal linearisation of \(H\) defined as in the proof of Theorem 8.1 (1)’ \(\Rightarrow\) (3). Define a new minimal linearisation \((\mathcal{H}; U)\) and prove that it is not unitary equivalent with \((V; \mathcal{H})\). More precisely, let \(U(x) = \Psi V(x)\) for all \(x \in X\). Since \(\text{Ran}(V(x)) \subseteq \mathcal{D}(\Psi)\) and \(\Psi\) is closed it follows, via the Closed Graph Principle, that \(U(x) \in \mathcal{L}(\mathcal{H}_x, \mathcal{K})\) for all \(x \in X\).

Let \(x, y \in X\) be arbitrary and fix vectors \(h \in \mathcal{H}_x\) and \(k \in \mathcal{H}_y\). Then

\[
[U(y)k, U(x)h] = [\Psi V(y)k, \Psi V(x)h] = [V(y)k, V(x)h] = [H(x, y)k, h],
\]

and hence \(U^*_y U_y = H(x, y)\). Also,

\[
\bigvee_{j \in X} U(y)\mathcal{H}_y = \bigvee_{j \in X} \Psi V(y)\mathcal{H}_y = \text{Clos Ran}(\Phi) = \mathcal{H}.
\]

Thus, \((\mathcal{H}; U)\) is a minimal linearisation of the \(H\)-kernel \(H\). On the other hand, since the operator \(\Psi\) is unbounded it follows that the two minimal linearisations \((\mathcal{K}; V)\) and \((\mathcal{H}; U)\) are not unitary equivalent.

For the implication (ii) \(\Rightarrow\) (i), let \((\mathcal{K}; V)\) and \((\mathcal{H}; U)\) be two minimal linearisations of \(H\). Let \(J\) and \(J_x\) be fundamental symmetries on \(\mathcal{K}\) and, respectively, \(\mathcal{H}_x, x \in X\). We consider the positive definite \(H\)-kernel \(L_V\) defined by

\[
L_V(x, y) = J_y V(y)^*V(x), \quad x, y \in X,
\]

and as in the proof of Theorem 8.1 it follows that \(-L_V \leq H \leq L_V\). We define a linear operator \(\Pi_V : \mathcal{F}_0(\mathcal{H}) \to \mathcal{K}\) by

\[
(9.1) \quad \Pi_V(h) = \sum_{x \in X} V(x)h_x, \quad h = (h_x)_{x \in X} \in \mathcal{F}_0(\mathcal{H}).
\]

Taking into account of the axiom (kd2) in the definition of a minimal linearisation one obtains

\[
(9.2) \quad [\Pi_V h, \Pi_V k]_H = [h, k]_{\mathcal{K}}, \quad h, k \in \mathcal{F}_0(\mathcal{H}),
\]

that is, the operator \(\Pi_V\) is isometric from \((\mathcal{F}_0(\mathcal{H}), [\cdot, \cdot]_H)\) into \((\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})\). In addition, \(\Pi_V\) is also isometric when considered as a linear operator from \((\mathcal{F}_0, [\cdot, \cdot]_{L_V})\).

Similarly, considering the positive semidefinite \(H\)-kernel \(L_U\) defined by

\[
L_U(x, y) = J_y U(y)^*U(x), \quad x, y \in X,
\]

one gets \(-L_U \leq H \leq L_U\) and, defining the linear operator \(\Pi_U : \mathcal{F}_0(\mathcal{H}) \to \mathcal{H}\) by

\[
(9.3) \quad \Pi_U(h) = \sum_{x \in X} U(x)h_x, \quad h = (h_x)_{x \in X} \in \mathcal{F}_0(\mathcal{H}),
\]

one obtains

\[
(9.4) \quad [\Pi_U h, \Pi_U k]_H = [h, k]_{\mathcal{K}}, \quad h, k \in \mathcal{F}_0(\mathcal{H}),
\]

that is, the operator \(\Pi_U\) is isometric from \((\mathcal{F}_0, [\cdot, \cdot]_{L_U})\) into \((\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})\) and \(\Pi_U\) is also isometric when considered as a linear operator from \((\mathcal{F}_0, [\cdot, \cdot]_{L_U})\) into \((\mathcal{K}, [\cdot, \cdot]_{L_U})\).
Let \( L = L_V + L_U \) and clearly \(-L \leq H \leq L\). Since \( L_V \leq L \) it follows that \( \mathcal{K}_L \) is contractively embedded into \( \mathcal{K}_{L_V} \) and hence \( \Pi_V \) induces a bounded operator \( \Pi_V : \mathcal{A}_L \to \mathcal{K} \). From (9.2) it follows

\[
(9.5) \quad \Pi_V^* J \Pi_V = A_L.
\]

By assumption, there exists \( \epsilon > 0 \) such that either \((-\epsilon, 0) \subset \rho(A_L)\) or \((0, \epsilon) \subset \rho(A_L)\) and hence from (9.5) and taking into account that by the minimality axiom (b) of the minimal linearisation the operator \( \Pi_V \) has dense range, it follows that there exists a uniquely determined unitary operator \( \Phi_V : \mathcal{K}_{A_L} \to \mathcal{K} \) such that

\[
(9.6) \quad \Phi_V h = \Pi_V h, \quad h \in \mathcal{A}_L,
\]

where \( \mathcal{K}_{A_L} \) is the Kreĭn space induced by the operator \( A_L \).

Similarly, performing the same operations with respect to the linearisation \((\mathcal{H}; V)\) one gets a uniquely determined unitary operator \( \Phi_U : \mathcal{K}_{A_L} \to \mathcal{H} \) such that

\[
(9.7) \quad \Phi_U h = \Pi_U h, \quad h \in \mathcal{A}_L.
\]

Define the unitary operator \( \Phi : \mathcal{K} \to \mathcal{H} \) by

\[
\Phi = \Phi_U \Phi_V^{-1}.
\]

Taking into account of (9.6) and (9.7), the definition of the operator \( \Pi_V \) as in (9.1), and the definition of \( \Pi_U \) as in (9.3), it follows that

\[
\Phi \left( \sum_{x \in X} V(x) h_x \right) = \sum_{x \in X} U(x) h_x, \quad (h_x)_{x \in X} \in \mathcal{F}_0(\mathcal{H}).
\]

This implies readily that for all \( x \in X \) one has \( \Phi V(x) = U(x) \) and hence the two Kolmogorov decompositions \((\mathcal{K}; V)\) and \((\mathcal{H}; U)\) are unitary equivalent.

As a consequence of Theorem 9.1 one can obtain a rather general sufficient condition of nonuniqueness. Let \( K \) and \( H \) be two positive semidefinite \( \mathcal{H} \)-kernel. Then one considers the Hilbert spaces \( \mathcal{K}_K \) and \( \mathcal{K}_H \), obtained by quotient completion of \((\mathcal{F}_0, [\cdot, \cdot]_K)\) and, respectively, of \((\mathcal{F}_0, [\cdot, \cdot]_H)\). If \( H \geq K \) then \( \mathcal{K}_H \) is contractively embedded into \( \mathcal{K}_K \). The kernel \( H \) is \( K \)-compact if the embedding of \( \mathcal{K}_H \) into \( \mathcal{K}_K \) is a compact operator.

**Corollary 9.2.** Let \( H_+, H_- \in \mathcal{K}^+(\mathcal{H}) \) be two independent kernels, both of them of infinite rank. If there exists a kernel \( K \in \mathcal{K}^+(\mathcal{H}) \) such that \( H_+ \) and \( H_- \) are \( K \)-compact, then the minimal linearisations of the kernel \( H_+ - H_- \) are not unique, modulo unitary equivalence.

Let \( H = H_1 - H_2 \). Clearly \(-K \leq H \leq K\). Let \( A_\pm \in \mathcal{L}(\mathcal{K}_K) \) denote the Gram operator of the kernel \( H_\pm \). Since \( H_+ \) and \( H_- \) are independent it follows that \( A = A_+ - A_- \) is the Gram operator of \( H \). Since \( H_\pm \) are of infinite rank and \( K \)-compact it follows that \( A_\pm \) are compact operators of infinite rank in \( \mathcal{L}(\mathcal{K}_K) \) and hence the spectra \( \sigma(A_\pm) \) are accumulating to 0. Then the spectrum \( \sigma(A) \) is accumulating to 0 from both sides. This clearly contradicts the condition (ii) in Theorem 9.1 and hence the kernel \( H \) has non-unique minimal linearisations. \( \square \)
10. Holomorphic Kernels: Single Variable Domains

Let $\Omega$ be a domain, a nonempty open subset, in the complex field $\mathbb{C}$, and let $\mathcal{H}$ be a Hilbert space. A kernel $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{H})$ is called holomorphic if it is holomorphic in the first variable, that is, for each $w \in \Omega$, the map $\Omega \ni z \mapsto K(z, w) \in \mathcal{L}(\mathcal{H})$ and conjugate holomorphic in the second variable, that is, for each $z \in \Omega$, the map $\Omega \ni w \mapsto K(z, w) \in \mathcal{L}(\mathcal{H})$ is holomorphic. Recall that, for Banach space valued functions of complex variable, strong holomorphy is the same with weak holomorphy. If $K$ is Hermitian, then $K$ is a holomorphic kernel if and only if the map $\Omega \ni z \mapsto K(z, w) \in \mathcal{L}(\mathcal{H})$ is holomorphic for all $w \in \Omega$.

Theorem 10.1. Let $\mathcal{H}$ be a Hilbert space and, for some $r > 0$, let $K$ be a holomorphic $\mathcal{H}$-kernel on $D_r = \{z \in \mathbb{C} \mid |z| < r\}$. Then, there exists $0 < r' \leq r$ and a reproducing kernel Kreĭn space on $D_{r'}$ with reproducing kernel $K|D_{r'} \times D_{r'}$.

The first step in the proof of this theorem is to observe that, without loss of generality, one can assume $r > 1$. Indeed, if $r \leq 1$ then, for some $0 < \rho < r$ small enough, the $\mathcal{H}$-kernel $K_{\rho}(z, w) = K(\rho z, \rho w)$ is holomorphic on $D_r/\rho$. If $K_{\rho}$ is the reproducing kernel Kreĭn space with reproducing kernel $K_{\rho}$ restricted to $D_{r''}$, for some $0 < r'' \leq r/\rho$, let $r' = r''\rho$ and let $K$ denote the vector space of functions $f: D_{r'} \rightarrow \mathcal{H}$ such that $f(z) = F(z/\rho)$ for some $F \in K_{\rho}$ and all $z \in D_{r'}$. On $K$ there is defined the inner product $[\cdot, \cdot]_K$

$$[f, g]_K = [F, G]_{K_{\rho}}, \quad f(z) = F(z/\rho), \quad g(z) = G(z/\rho), \quad F, G \in K_{\rho}.$$

Then $(K; [\cdot, \cdot]_K)$ is a Kreĭn space. For each $w \in D_{r'}$ the map $z \mapsto K(z, \rho w)$ belongs to $K$ and, for each $F \in K_{\rho}$, $f(z) = F(z/\rho)$, and $h \in \mathcal{H}$,

$$[f, K(\cdot, \rho w)h]_K = [F, K(\rho z, \rho w)h]_{K_{\rho}} = [F(\rho w), h]_{\mathcal{H}} = [f(w), h]_{\mathcal{H}},$$

hence $K$ is a reproducing kernel Kreĭn space with reproducing kernel $K|D_{r'} \times D_{r'}$.

There are two main ideas of the proof. First, the Szegő kernel $S$, see Example 4.3, plays a distinguished role in holomorphy, and allows us to construct the convolution kernel of $K$ in the Hardy space $H^2(\mathbb{D})$. The second idea is that, once the convolution operator represented as a selfadjoint bounded operator on a Hilbert space of functions is defined, the construction of the induced Kreĭn space as in Example 6.2 will provide the reproducing kernel Kreĭn space with reproducing kernel $K$. Here are a few details.

Letting $r > 1$, there is considered the Hardy space $H^2(\mathbb{D}) \otimes \mathcal{H}$, identified with a space of $\mathcal{H}$-valued functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $(a_n)_{n \geq 0}$ is a sequence of vectors in $\mathcal{H}$ such that $\|f\|^2 = \sum_{n \geq 0} \|a_n\|^2_{\mathcal{H}} < \infty$. Also, the inner product on the Hilbert space $H^2(\mathbb{D}) \otimes \mathcal{H}$ is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_{\mathcal{H}} dt, \quad f, g \in H^2(\mathbb{D}) \otimes \mathcal{H}.$$

Thus, on $H^2(\mathbb{D}) \otimes \mathcal{H}$ one can define the analog of the convolution operator $K$

$$(Kf)(z) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) f(e^{it}) dt, \quad f \in H^2(\mathbb{D}) \otimes \mathcal{H}. \quad (10.1)$$

Letting $M = \sup_{|z|, |w| \leq 1} \|K(z, w)\| < \infty$, it follows that $\|Kf\| \leq M \|f\|_{H^2(\mathbb{D}) \otimes \mathcal{H}}$, hence the convolution operator $K$ is a bounded linear operator in $H^2(\mathbb{D}) \otimes \mathcal{H}$. On the other hand,

$$\langle Kf, g \rangle_{H^2(\mathbb{D}) \otimes \mathcal{H}} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \langle K(e^{it}, e^{is}) f(e^{is}), g(e^{is}) \rangle_{\mathcal{H}} dt ds,$$
hence $K$ is selfadjoint. Since, for any $w \in \mathbb{D}_1$ and any $h \in \mathcal{H}$, the function $f_w(z) = h/(1 - w\overline{z})$ belongs to $H^2(\mathbb{D}) \otimes \mathcal{H}$ and by the Cauchy formula, $(Kf_w)(z) = K(z, w)h$, the range of the convolution operator $K$ contains all the functions $K_w(\cdot)h$, for $w \in \mathbb{D}$ and $h \in \mathcal{H}$.

Then, one can use the construction of the induced Krein space $(\mathcal{B}_K; \Pi_{\mathcal{B}_K})$ inside of $H^2(\mathbb{D}) \otimes \mathcal{H}$, as in Example 6.2 but applied to the bounded selfadjoint operator $K$ in the Hilbert space $H^2(\mathbb{D}) \otimes \mathcal{H}$, in order to get the reproducing kernel Krein space $\mathcal{K}$ with reproducing kernel $K$.

11. Holomorphic Kernels: Several Variables Domains

In this section it is considered the analog of Theorem 10.1 in case the Hermitian kernels are defined on domains in $\mathbb{C}^N$ for $N \geq 2$. We use the notation as in Example 4.4 where the Drury-Arveson space was constructed as the reproducing kernel Hilbert space associated to the Szegö kernel. For simplicity, it is considered only scalar valued kernels.

Recall that a scalar-valued Hermitian kernel $K$, defined on a nonempty open subset $O$ of $\mathcal{G} = \mathbb{C}^N$, is holomorphic on $O$ if $K(\cdot, \eta)$ is holomorphic on $O$ for each fixed $\eta \in O$. Since $K$ is Hermitian, it follows that $K$ is conjugate holomorphic in the second variable.

**Theorem 11.1.** Let $r > 0$ and let $K$ be a Hermitian holomorphic kernel on the open ball $B_r$ in $\mathbb{C}^N$. Then, there exist $0 < r' \leq r$ and a reproducing kernel Krein space $\mathcal{K}$ on $B_r$ with reproducing kernel $K|_{B_r \times B_r}$.

To a certain extent, the proof of this theorem follows a pattern similar to that of the proof of Theorem 10.1, namely, first a scaling argument can be used in order to reduce the proof to the case $r > 1$, then the convolution kernel of $K$ can defined on the Drury-Arveson space $H^2(B_1)$ associated to the Szegö kernel $S$, see Example 4.4 and it can be proven that this convolution operator is bounded and selfadjoint. Finally, the construction of type $(\mathcal{B}_4; \Pi_{\mathcal{B}_4})$, see Example 6.2, can be used in order to produce a reproducing kernel space $\mathcal{K}$ with reproducing kernel $K$. This proof shows, once again, a certain universality property of the Szegö kernel with respect to holomorphic Hermitian kernels.

Let $K$ be a scalar Hermitian holomorphic kernel on $B_r$, $r > 0$. Since $K$ is Hermitian, it follows that $\zeta \mapsto K(\xi, \overline{\zeta})$ is holomorphic on $B_r$ for each $\xi \in B_r$, that is, letting $\{e_j\}_{j=1}^N$ denote the canonical orthonormal basis of $\mathcal{G} = \mathbb{C}^N$, the conjugation can be defined by

$$\xi = \sum_{j=1}^N \langle \xi, e_j \rangle e_j \rightarrow \sum_{j=1}^N \langle \xi, e_j \rangle e_j = \overline{\xi},$$

so that the function $f(\xi, \eta) = K(\xi, \overline{\eta})$ is separately holomorphic on $B_r \times B_r$. By Hartogs’ Theorem $f$ is holomorphic on $B_r \times B_r$, hence $f$ is locally bounded. Similar to the argument provided for Theorem 10.1, without loss of generality one can suppose that $r > 1$. Hence there exist $1 < \rho < r$ and $C > 0$ such that:

$$(11.1) \quad |K(\xi, \eta)| \leq C \quad \text{for all } \xi, \eta \in B_\rho$$

and

$$(11.2) \quad K(\xi, \overline{\eta}) = \sum_{m \geq 0} p_m(\xi, \eta)$$
uniformly on $B_\rho$, where each $p_m$, $m \geq 0$, is an $m$-homogeneous complex polynomial on $2N$ variables. There exists a continuous linear functional $A_m$ on $P_m(G \times G)^{\otimes m}$, see (4.7), such that
\begin{equation}
(11.3) \quad p_m(\xi, \eta) = A_m((\xi, \eta)^{\otimes m}), \quad \text{for all } \xi, \eta \in \mathbb{C}^N.
\end{equation}

Using Cauchy Inequalities, for $B_\rho$, one gets
\begin{equation}
(11.4) \quad \|A_m\| \leq C \frac{1}{\rho^m},
\end{equation}
hence
\begin{equation}
(11.5) \quad \sum_{m \geq 0} \|A_m\|^2 \leq C \sum_{m \geq 0} \frac{1}{\rho^{2m}} = C \frac{1}{1 - 1/\rho^2} = C' < \infty.
\end{equation}

By the Riesz Representation Theorem, there exist $a_m \in P_m(G \times G)^{\otimes m}$, $m \geq 0$, such that
\begin{equation}
(11.6) \quad A_m((\xi, \eta)^{\otimes m}) = \langle (\xi, \eta)^{\otimes m}, a_m \rangle_{(G \times G)^{\otimes m}},
\end{equation}
and
\begin{equation}
(11.7) \quad \|a_m\| = \|A_m\|,
\end{equation}
(with $a_0 = A_0 \in \mathbb{C}$). Since $P_m(G \times G)^{\otimes m}$ is isometrically isomorphic to $(P_m G)^{\otimes (m+1)}$, it is deduced that there are $a_m^k \in P_m G^{\otimes m}$, $k = 0, \ldots, m$, such that
\begin{equation}
(11.8) \quad \langle (\xi, \eta)^{\otimes m}, a_m \rangle_{(G \times G)^{\otimes m}} = \sum_{k=0}^m \langle b_m^k(\xi, \eta), a_m^k \rangle_{G^{\otimes m}},
\end{equation}
and
\begin{equation}
(11.9) \quad \sum_{k=0}^m \|a_m^k\|^2 = \|a_m\|^2,
\end{equation}
where $b_0^0 = 1$ and $b_m^k(\xi, \eta) = \xi^{\otimes (m-k)} \otimes \eta^{\otimes k}$, $m \geq 1$, $k = 0, \ldots, m$. By (11.2), (11.3), (11.6), and (11.8),
\begin{equation}
K(\xi, \eta) = \sum_{m \geq 0} \sum_{k=0}^m \langle b_m^k(\xi, \eta), a_m^k \rangle = \sum_{k \geq 0} \sum_{m \geq k} \langle b_m^k(\xi, \eta), a_m^k \rangle,
\end{equation}
where the series converge absolutely on $\eta$ by (11.4).

Using all these, it can be shown that $K_\eta \in H^2(B_1)$ for all $\eta \in B_1$, where $K_\eta(\xi) = K(\xi, \eta)$. Then letting $Ka_\eta = K_\eta$, $\eta \in B_1$, one gets a bounded linear operator in $H^2(B_1)$ such that
\begin{equation}
K(\xi, \eta) = \langle Ka_\xi, a_\eta \rangle_{H^2(B_1)}.
\end{equation}

This operator $K$ is selfadjoint and it is the analog of the convolution operator for which, applying the construction of type $(\mathcal{B}_A; \Pi_{\mathcal{B}_A})$ as in Example 6.2, one gets a reproducing kernel Krein space $\mathcal{K}$ with reproducing kernel $K|B_1 \times B_1$. 
12. Comments

The theory of reproducing kernel Hilbert spaces and their positive semidefinite kernels originates with the works of S. Zaremba [44], G. Szegő [42], S. Bergman [7], and S. Bochner [8]. E.H. Moore [33] also contributed significantly to this theory, but the first systematisation and abstract presentation belongs to N. Aronszajn [4]. A different but equivalent theory belongs to L. Schwartz [40], whose work remained almost unnoticed for a long time, although it was the first to consider reproducing kernel Kreĭn spaces (Hermitian spaces, as called there). So far, monographs on this subject have been written by T. Ando [3], S. Saitoh [38], and the forthcoming title of S. Saitoh and Y. Sawano [39], which are good sources for the large area of applications of the technique of reproducing kernel spaces in complex functions theory, ordinary and partial differential equations, integral equations, and approximation and numerical analysis.

So far, there are two monographs devoted to indefinite inner product spaces and their linear operators, J. Bognar [9] and T.Ya. Azizov and I.S. Iokhvidov [6], where proofs, examples, and counter-examples of the facts recalled in Sections:ks can be found.

The introductory material on Hermitian kernels in Section 3 follows T. Constantinescu and A. Gheondea [14]. Theorem 3.1 is classical. The concept of linearisation originates with J. Mercer [32], for the scalar case, and A.N. Kolmogorov [25], for the operator valued case.

The Hardy Space $H^2(D)$ originates with the G. Szegő Kernel [42]. For the theory of Hardy spaces there are monographs of P.L. Duren [19] and P. Koosis [20]. Bergman kernel is also another important example of a positive semidefinite kernel but it falls out of this chapter concern, see P.L. Duren and B. Schuster [20]. The Drury-Arveson space originates with S.W. Drury [21] and W.B. Arveson [5]. The holomorphic kernels considered at Example 4.5 make the main object of investigation of the monograph of D. Alpay et al. [2]. The investigations of A.V. Potapov [36], L. de Branges [10]–[12], M.G. Kreĭn and H. Langer [27]–[31], and H. Dym [22] highly motivate the interest for Hermitian kernels with or without finite negative signatures. The study of Töplitz type kernels is related to the investigations on operator dilations of M.A. Naimark [34] and B. Sz.-Nagy [43]. Our short presentation as in Example 4.6 follows [14]. We only mention that there is a more general and powerful theory of kernels invariant under actions of $\star$-semigroups presented in [15], motivated by problems in mathematical physics as in D.E. Evans and J.T. Lewis [23] and K.R. Parthasaraty, K. Schmidt [35], and many others.

Theorem 5.1 essentially belongs to P. Sorjonen [41] but this result follows from the more general theory of L. Schwartz [40] that has been obtained about ten years before. Theorem 5.2 can be found in D. Alpay et al. [2].

Examples 6.1, 6.2, and Theorem 7.1 on induced Kreĭn spaces can be found in [14], while Corollary 7.2 belongs to T. Hara [24]. Similar and, in a certain way, equivalent uniqueness conditions, can be found in [13] and M.A. Dritschel [18]. B. Ćurgus and H. Langer [17] proves that once non-equivalent induced Kreĭn spaces exist, there are infinitely many.

The characterisations of existence of reproducing Kreĭn spaces associated to given Hermitian kernels as in Theorem 8.1 belong essentially to L. Schwartz [40], but our presentation follows [14]. Example 8.2 is from [40] as well, cf. [2].

The uniqueness Theorem 9.1 is from [14] while Corollary 9.2 is from [40].
The result in Theorem 10.1 on single variable holomorphic kernels belongs to D. Alpay [1], while its several variables generalisation in Theorem 11.1 is from [16]: for the basics of several complex variables holomorphic functions facts used during the explanation of the ideas of the proof, see R.M. Range [37].

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