Analysis of synchronization in a supermarket refrigeration system

Rafael WISNIEWSKI 1†, John LETH 1, Jakob G. RASMUSSEN 2

1. Department of Electronic Systems, Aalborg University, Fredrik Bajers Vej 7C, 9220 Aalborg, Denmark;
2. Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, 9220 Aalborg, Denmark

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Abstract: In a supermarket refrigeration, the temperature in a display case, surprisingly, influences the temperature in other display cases. This leads to a synchronous operation of all display cases, in which the expansion valves in the display cases turn on and off at exactly the same time. This behavior increases both the energy consumption and the wear of components. Besides this practical importance, from the theoretical point of view, synchronization, likewise stability, Zeno phenomenon, and chaos, is an interesting dynamical phenomenon. The study of synchronization in the supermarket refrigeration systems is the subject matter of this work. For this purpose, we model it as a hybrid system, for which synchronization corresponds to a periodic trajectory. To examine whether it is stable, we transform the hybrid system to a single dynamical system defined on a torus. Consequently, we apply a Poincaré map to determine whether this periodic trajectory is asymptotically stable. To illustrate, this procedure is applied for a refrigeration system with two display-cases.

Keywords: Synchronization; Hybrid systems; Stability; Limit cycles; Stochastic approximation

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1 Introduction

The foodstuffs in supermarkets are typically stored in open display cases in the sales areas. By utilizing a refrigeration cycle, heat is transported from the display cases to the outdoor surroundings [1]. The refrigeration cycle is coordinated by a number of dedicated controllers, which are distributed within the refrigeration hardware such as display cases, compressors, and cold storage rooms. This concept has many practical advantages; for instance, it is flexible and yet simple. However, it neglects the cross-coupling between the subsystems, and the interplay between continuous and discrete dynamics. These effects may cause the degradation of refrigeration quality. A case in point is the synchronization of display cases. Each display case is equipped with a hysteresis controller that opens and closes an expan-
sion valve. It adjusts the flow of refrigerant such that the desired temperature is reached. Practical experience shows that the temperature in one display case influences the temperature in the neighboring ones. These interactions frequently lead to a synchronous operation of the display cases in which the expansion valves in the display cases turn on at the same time. As discussed in [1], this synchronization causes high wear of the compressors, inferior control performance, and increased energy consumption.

A number of inspiring publications have addressed the problem of synchronization in supermarket systems. In particular, references [2, 3] suggest a centralized controller based on hybrid model predictive control that by design avoids synchronization. Recently, a patent has been issued that proposes to adjust the cut-in and cut-out temperatures for the refrigeration entities to desynchronize them [4].

In this article, we leave the feedback design problem and focus entirely on the synchronization dynamics as a mathematically intricate phenomenon. We assume that the hysteresis control has been designed and ask whether synchronization takes place. Although, we do not consider any implementation issues at the current stage, we stress that the work is relevant for the reduction of energy consumption in supermarkets. Indeed, detecting a tendency to synchronize will eventually lessen the energy consumed. Our standpoint is that a deep insight into synchronization is important in general for understanding dynamical systems with discrete transitions.

The phenomenon of synchronization in dynamical systems has been studied before; the definitions of synchronizations have been formulated in [5], and numerous examples of synchronization have been analyzed in [6]. As for studying any phenomena in dynamical systems, the very first challenge is to establish a convenient definition of a state space and the notion of a trajectory. What immediately follows is the examination whether the existing methods from the theory of dynamical systems can be adapted for the analysis of synchronization. This approach has been taken in this work. In the study of synchronization, we model the supermarket refrigeration system as a hybrid system. The hybrid system in this work consists of several linear subsystems and a rule that orchestrates the discrete transitions among them [7]. The state space is a disjoint union of polyhedral sets, and the discrete transitions are realized by reset maps defined on the facets of the polyhedral sets. The reset maps are regarded as generators of an equivalence relation allowing ‘gluing’ the polyhedral sets together. The result of this construction is a quotient space. This idea has been used before in [8–10]. The original contribution of this work is to show that by the process of gluing, a hybrid system is transformed to a dynamical system that is defined on a single state space, a smooth manifold with boundary. If \( n \) is the number of display cases, then this manifold is the product of a \( 2n \)-torus and the non-negative reals. Consequently, the trajectories are continuous, piecewise smooth paths. A novelty of the current approach is that this construction allows the application of standard method from analysis of differential equations for the study of a refrigeration system. In particular, (asymptotic) synchronization corresponds to an asymptotically stable periodic trajectory.

The stability analysis of periodic trajectories can be completed by applying a Poincaré map. The Poincaré map has previously been used for the stability analysis of switched systems [11–14] and control synthesis of these [15, 16]. It was assumed in these works that switchings took place on a family of hyperplanes that are disjoint subsets of the state space. Thereby, small perturbations of the state contribute to small changes in system behavior. Whereas in this work, we generalize the discrete transitions to take also place at intersection points of hyperplanes. This is an important generalization as at a corner point small perturbations yield conceptually different system behavior. As mentioned above, this phenomenon cannot be captured in the concepts presented in previous works. Allowing mode switching to take place on the boundaries of polyhedral sets complicates the Poincaré map. In particular, we explicitly calculate the Poincaré map for a refrigeration system consisting of two display cases and a compressor.

## 2 Refrigeration system

In majority of supermarket refrigeration systems, the display cases and the compressors, which maintain the flow of refrigerant, are connected in parallel. The compressors compress refrigerant drained from the suction manifold. Subsequently, the refrigerant passes through the condenser and flows into the liquid manifold. Each display case is equipped with an expansion valve, through which the refrigerant flows into the evaporator in the display case. In the evaporator, the refrig-
erant absorbs heat from the foodstuffs. As a result, it changes its phase from liquid to gass. Finally, the vaporized refrigerant flows back into the suction manifold. The process described above is called a refrigeration cycle.

2.1 Motivation for the study

In a typical supermarket refrigeration system, the temperature in each display case is controlled by a hysteresis controller that opens the expansion valve when the air temperature \( T \) (measured near to the foodstuffs) reaches a predefined upper temperature limit \( T^u \). The valve stays open until \( T \) decreases to the lower temperature limit \( T^l \). At this point, the controller closes the valve again. Practice reveals that if the display cases are similar, the hysterers controllers have tendency to synchronize the display cases [1]. It means that the air temperatures \( T_i \) for \( i = \{1, \ldots, N\} \), where \( N \) is the number of display cases, tend to match as time progresses.

2.2 Model for synchronization

For simplicity of this exposition, the model of a refrigeration system consists of two identical display cases and a compressor.

The dynamics of the air temperature \( T_i \) for display case \( i \in \{1, 2\} \) and the suction pressure \( P \) for the system of two display cases are governed by the following system of equations, which is further discussed in the appendix,

\[
\begin{align*}
\dot{T}_i &= -a T_i + b - \delta_i(c T_i - dP - e), \\
\dot{P} &= -\alpha P + \beta + \delta_1 + \delta_2,
\end{align*}
\]

where \( a, b, c, d, e, \alpha \) and \( \beta \) are constants (their specific values are provided by equations (a3) in the appendix), and \( \delta_i \in \{0, 1\} \) is the switching parameter for the display case \( i \); it indicates whether the expansion valve is closed for \( \delta_i = 0 \) or open for \( \delta_i = 1 \). The switching law is given by the hysteresis control:

\[
\delta_i = \begin{cases} 
1, & \text{if } T_i \geq T^u_i, \\
0, & \text{if } T_i \leq T^l_i, \\
\delta_i, & \text{if } T^l_i < T_i < T^u_i,
\end{cases}
\]

where \( T^u_i \) and \( T^l_i \) are respectively the predefined upper and lower temperature limits for the display case \( i \). By convention, \( \delta_i = 0 \) for any initial condition \( T_i(t_0) = T^0_i \) of (1a) with \( T^0_i \in [T^l_i, T^u_i] \). Such an initial condition is assumed throughout this paper; hence, (2) is well defined.

We write equations (1) simply as

\[
\dot{x} = \xi_\delta(x) = A_\delta x + B_\delta,
\]

where

\[
A_\delta = \begin{bmatrix} -a - \delta_1 c & 0 & \delta_1 d \\ 0 & -a - \delta_2 c & \delta_2 d \end{bmatrix}, \quad B_\delta = \begin{bmatrix} b + e \delta_1 \\ b + e \delta_2 \end{bmatrix},
\]

\( x = (T_1, T_2, P) \), and \( \delta = (\delta_1, \delta_2) \in \mathbb{R}^2 \) with \( 2 = \{0, 1\} \).

Hence, for \( \delta \in \mathbb{R}^2 \), we study a Cauchy problem of the form \( \dot{x} = \xi_\delta(x), x(t_0) \in [T^l_1, T^u_1] \times [T^l_2, T^u_2] \times \mathbb{R}_+ \), and \( F_\delta = [T^l_1, T^u_1] \times [T^l_2, T^u_2] \times \mathbb{R}_+ \).

**Proposition 1** For any \( s \in [l, u] \), the vector field \( \xi_\delta, \delta \in \mathbb{R}^2 \), is transversal to \( F_\delta = \{T^l_1, T^u_1\} \times \{T^l_2, T^u_2\} \times \mathbb{R}_+ \), and \( F_\delta = \{T^l_1, T^u_1\} \times [T^l_2, T^u_2] \times \mathbb{R}_+ \), when using the parameter values from the appendix. This proves the proposition.

We note that the sign of (1b) determines whether the coordinate function \( T_i \) of \( x \) is increasing or decreasing.

For two display cases, this information is provided in Fig. 1.

![Fig. 1](image)

Fig. 1 The state space of the refrigeration system consisting of two display cases is illustrated. Here, the pressure axis is suppressed, and \( (T^l_1, T^u_1) = (0, 5) \). The direction of the vector field \( \xi_\delta \) is indicated by the dark shaded triangles.
3 Supermarket systems as a hybrid system

In order to represent (2) in a mathematically satisfactory way, we will use the modeling formalism of hybrid systems on polyhedral sets with state-dependent switching. It is seen as a subclass of hybrid systems of [7]; and yet, it is rich enough to model any system with multiple hysteresis control.

3.1 Hybrid systems

We write $F < P$ if $F$ is a face of the polyhedral set $P$. A map $f : P \to P'$ is polyhedral if

1) it is a continuous injection, and 
2) for any $F < P$ there is $F' < P'$ with $\dim(F) = \dim(F')$ such that $f(F) \subseteq F'$.

**Definition 1** (Hybrid systems on polyhedral sets with state-dependent switching) For finite index sets $I$ and $D$, a hybrid system on polyhedral sets with state-dependent switching (of dimension $n$) is a triple $(\mathcal{P}, \mathcal{S}, \mathcal{R}) = (\mathcal{P}_I, \mathcal{S}, \mathcal{R}_I)$, where

1) $\mathcal{P} = \{P_\delta \subseteq \mathbb{R}^n \mid P_\delta$ a polyhedral set, $\dim(P_\delta) = n$, $\delta \in D\}$ is a family of polyhedral sets.

2) $\mathcal{S} = \{\xi_\delta : P_\delta \to \mathbb{R}^n \mid P_\delta \in \mathcal{P}, \delta \in D\}$ is a family of smooth vector fields.

3) $\mathcal{R} = \{R_j : F : F' \mid F < P \in \mathcal{P}, F' < P' \in \mathcal{P}, \dim(F) = \dim(F') = n - 1, j \in J\}$ is a family of polyhedral maps, called reset maps.

Next, we shall refer to hybrid systems with state-dependent switching simply as hybrid systems.

**Remark 1** After identifying $D$ with a finite subset of $\mathbb{R}$, we can rewrite the hybrid system $(\mathcal{P}_I, \mathcal{S}, \mathcal{R}_I)$ as

$$\frac{d}{dt}(x, q) \in F(x, q) = \{\xi_\delta(x), 0\} \mid x \in P_\delta\}$$

for $(x, q) \in \bar{C}$, and

$$(x, q)^+ \in \{R_l(x, q) \mid x \in C(x, q) = \text{dom}(R_l) < P_q\}$$

for $(x, q) \in D$, where

$$\bar{C} = \bigcup_{\delta \in D} (P_\delta \times \{\delta\})$$

and

$$D = \bigcup_{\{(l_1, l_2) \in D \times \{\text{dom}(R_\delta) < P_q\}\}} (\text{dom}(R_\delta) \times \{\delta\}).$$

This is precisely the hybrid system in [7].

After this remark, we will show that a refrigeration system with two display cases furnished with a hysteresis control is a hybrid system. To begin with, we consider the following scenario. Let $x(t_0) \in [T_1, T_2] \times [T_3, T_4] \times \mathbb{R}^n$, and $\delta = (0, 0)$; thereby, both display cases are initially switched off. Suppose that at time $t$, the air temperature $T_i$ of the $i$th display case reaches the upper temperature limit $T_i^u$, then the $i$th display case is switched on, and $\delta_i = 1$. This scenario indicates that the refrigeration system comprises four dynamical systems $\dot{x} = \xi_\delta(x) = A_\delta x + B_\delta, \delta \in 2^2$ defined on the polyhedral set

$$Q = [T_1^u, T_1^u] \times [T_2^u, T_2^u] \times \mathbb{R}^n.$$  (5)

A discrete transition between these four systems takes place whenever a trajectory reaches the boundary of $Q$. The polyhedral set $Q$ has five facets; four of them will be instrumental in the sequel,

$$F_1^1 = F_1^1(Q) = \delta^1[T_1^u, T_1^u] \times [T_2^u, T_2^u] \times \mathbb{R}^n,$$

$$F_2^2 = F_2^2(Q) = [T_1^u, T_1^u] \times \delta^2[T_2^u, T_2^u] \times \mathbb{R}^n,$$

where $\alpha \in 2$ and $\delta^{[a, b]} = [a], \delta^1[a, b] = [b]$. To sum up, the set $\mathcal{P}$ consists of four copies of a polyhedral set $Q$ in (5)

$$\mathcal{P} = \{P_\delta \mid P_\delta = Q \times \{\delta\}, \delta \in 2^2\}.$$  (6)

Formally, in (6), we have separated (made disjoint) each of the copies of $Q$.

The set $\mathcal{S}$ consists of four dynamical systems given by (3) for $\delta \in 2^2$. To characterize the set $\mathcal{R}$, we define

$$l(i, \delta) = (l_1(i, \delta), l_2(i, \delta)) = \begin{cases} (\delta_1 + 1, \delta_2), & \text{if } i = 1, \\ (\delta_1, \delta_2 + 1), & \text{if } i = 2, \end{cases}$$

where the results of the summation are computed modulo 2. Intuitively, the map $l$ takes a polyhedral set enumerated by $\delta$ to the future polyhedral set. The variable $i$ indicates that the discrete transition takes place when the temperature $T_i$ reaches its upper or lower boundary. The set $\mathcal{R}$ consists of eight reset maps $\mathcal{R} = \{R_i(\delta) \mid (i, \delta) \in [1, 2] \times 2^2\}$, where the maps $R_i(\delta) : F_{l(i, \delta)}(x, \delta) \to F_{l(i, \delta)}(x, \delta)$ are defined by $R_i(\delta)(x, \delta) = (x, l(i, \delta))$. Observe that the reset maps are identities in the first argument. By abuse of notation, we frequently identify $(x, \delta)$ with $x$.

**Remark 2** For a supermarket refrigeration system with $N$ display cases, the definition of the facets on
the polyhedral set \( Q' = [T_1^1, T_2^1] \times \cdots \times [T_1^a, T_2^a] \times \cdots \times [T_N^1, T_N^a] \times \mathbb{R}_+ \) is generalized to

\[
F_i^a(Q') = [T_1^1, T_1^a] \times \cdots \times [T_i^1, T_i^a] \times \cdots \times [T_N^1, T_N^a] \times \mathbb{R}_+
\]

for \( i \in \{1, \ldots, N\} \) and \( a \in 2 \). The facet operators commute in the following sense

\[
F_i^a \circ F_j^b(Q) = F_j^b \circ F_i^a(Q), \quad i < j.
\]

As a consequence, the set \( \mathcal{R} \) of the reset maps is

\[
\mathcal{R} = \{ R_i(\delta) \mid (i, \delta) \in \{1, \ldots, N\} \times 2^2 \}.
\]

For the map \( l \) given by

\[
l(i, \delta) = (\delta_1, \ldots, \delta_{i-1}, \delta_i + 1, \delta_{i+1}, \ldots, \delta_N),
\]

the maps \( R_i(\delta) : F_i^a(l(\delta)) \rightarrow R(l(\delta)) \times \{(i, \delta)\} \) are defined by \( R_i(\delta)(x, \delta) = (x, l(i, \delta)) \).

The hybrid refrigeration system with two display cases is illustrated in Fig. 2. Here, each element of \( \mathcal{P}_0 \) has been (orthogonally) projected onto the \( (0, 1) \)-space. The polyhedral set \( P_0 \) is illustrated by a square. By abuse of notation, the facets of \( P_0 \) are denoted by \( F_0^1 \) (instead of \( F_0^1 \times \{0\} \)).

\[\text{Fig. 2 The } T_1T_2\text{-state space of the refrigeration system consists of two display cases. The reset maps are indicated by the stippled lines (see Fig. 1 and its caption for further explanation). The pressure axis has been suppressed; thus, each } P_0 = Q \times \{0\} \text{ is illustrated by a square. By abuse of notation, the facets of } P_0 \text{ are denoted by } F_0^1 \text{ (instead of } F_0^1 \times \{0\} \text{).}\]

\subsection{3.2 Trajectories of a refrigeration system}

We bring in a concept of a (hybrid) time domain \([7]\). Let \( k \in \mathbb{N} \cup \{\infty\} \); a subset \( T_k \subset \mathbb{R}_+ \times \mathbb{Z}_+ \) will be called a time domain if there exists an increasing sequence \( \{t_i\}_{i \in [0, \ldots, k]} \) in \( \mathbb{R}_+ \cup \{\infty\} \) such that

\[
T_k = \bigcup_{i \in [0, \ldots, k]} T_i \times [i],
\]

where \( T_i = [t_{i-1}, t_i] \) if \( i \in [1, \ldots, k-1] \), and

\[
T_k = \begin{cases} [t_{k-1}, t_k], & \text{if } t_k < \infty, \\ [t_{k-1}, \infty], & \text{if } t_k = \infty. \end{cases}
\]

Note that \( T_i = [t_{i-1}, t_i] \) for all \( i \) if \( k = \infty \). We say that the time domain is infinite if \( k = \infty \) or \( t_k = \infty \). The sequence \( \{t_i\}_{i \in [0, \ldots, k]} \) corresponding to a time domain will be called a switching sequence.

\textbf{Definition 2 (Trajectory)} A trajectory of the hybrid system \((\mathcal{P}_D, \mathcal{S}, \mathcal{R})\) is a pair \((T_k, \gamma)\) where \( k \in \mathbb{N} \cup \{\infty\} \) is fixed, and

\[
\gamma : T_k \rightarrow X = \bigcup_{\delta \in D} P_0 \times \{\delta\}
\]

is continuous (\( X \) has the disjoint union topology) and satisfies:

1) For each \( i \in [1, \ldots, k-1] \), there exist \( \delta \neq \delta' \in D \) such that \( \gamma(t_i; i) \in \text{bd}(P_0) \), and \( \gamma(t_{i+1}; i) \in \text{bd}(P_0) \), where \( \text{bd}(P) \) is the boundary of \( P \).

2) For each \( i \in [1, \ldots, k] \), there exists \( \delta \in D \) such that the Cauchy problem

\[
\frac{d}{dt}\gamma(t; i) = \gamma(t; i) = \xi_{\delta}(\gamma(t; i)), \quad \gamma(t_{i-1}; i) = x_{i-1} \in P_0,
\]

has a solution on \( T_i \subset T_k \).

3) For each \( i \in [1, \ldots, k-1] \), there exists \( j \in \mathcal{J} \) such that \( R_j(\gamma(t; i)) = \gamma(t; i+1) \).

A trajectory at \( x \) is a trajectory \((T_k, \gamma)\) with \( \gamma(t_0; 1) = x \).

The next definition formalizes the notion of a periodic trajectory, which will be used in defining synchronization of the refrigeration system.

\textbf{Definition 3 ((T, l))-periodic trajectory} Let \((T, l) \in \mathbb{R}_+ \times \mathbb{Z}_+ \). A trajectory \((T_k, \gamma)\) is \((T, l)\)-periodic (or just periodic) if 1) \( T_k \) is an infinite time domain, and 2) for any \( i \in [1, \ldots, k] \) and \( l \in p(T_k) \), where \( p : T_k \rightarrow [0, \infty] \) is the projection \( p(t) = t \), we have \( \gamma(t + T; i + l) = \gamma(t; i) \).

In particular, if \((T_k, \gamma)\) is a \((T, l)\)-periodic trajectory, and \( T \) is nonzero then \( p : T_k \rightarrow [0, \infty] \) is surjective.
3.3 State space of the refrigeration system

To study any dynamical system, the starting point is a convenient definition of the state space. It was suggested in [9, 10] to glue the state spaces of respective subsystems of a hybrid system together along the subsets identified by the reset maps.

For a hybrid system \((P, S, R)\), we define the state space as the union of polyhedral sets \(X = \bigcup_{s \in \Sigma} P_s\). Let \(~\leq X \times X\) be the equivalence relation [17], generated by the relations \(x \sim R(x)\) for all reset maps \(R \in R\) and all points \(x\) in the domain of \(R\). A quotient space of the hybrid system is the quotient space \(X^* = X/\sim\).

Let \(R^{-1} = \{R^{-1} \in R\}\). Then, the equivalence class of \(x \in X\) is denoted by

\[ [x] = \{y \in X \mid \exists R_1, \ldots, R_l \in R \cup R^{-1} \text{ such that } y = R_1 \times \ldots \times R_l (x) \}. \]

In particular, for a refrigeration system with two display cases, let \(x = (T_1, T_2, P) \in X\). If \(x\) is in the interior of \(P_0\) for some \(\delta \in \mathbb{R}_+\), the equivalence class \([x] = \{(T_1, T_2, P, \delta)\}\). If \(x\) is in the interior of \(F_1^1 \times \{0,0\}\), then \([x] = \{(T_1, T_2, P; 0,0), (T_1, T_2, P; 1,0)\}\), and if \(x \in (F_1^1 \cap F_2^1) \times \{0,0\}\), we have \([x] = \{(T_1, T_2, P; 0,0), (T_1, T_2, P; 1,0), (T_1, T_2, P; 0,1), (T_1, T_2, P; 1,1)\}\). Furthermore, \(X^*\) is the product of a 2-torus with the non-negative reals, \(X^* = T^2 \times \mathbb{R}_+\), i.e., a smooth manifold with boundary. We note that the system consisting of \(N\) display cases will give rise to \(X^* = T^N \times \mathbb{R}_+\). In [18], we have explicitly constructed a differentiable structure on the state space \(X^*\) of a system with \(N\) hysterons, thereby of a refrigeration system with \(N\) display cases. Whereas, in [19], we have studied local stability of such a system.

4 Periodic trajectory and stability

We direct our attention to the subject matter - the synchronization of refrigeration systems. A refrigeration system is said to exhibit asymptotic synchronization if there exists a \((T, \bar{P})\)-periodic trajectory which is asymptotically stable in \(X^*\) [20, Definition 13.3]. We remark that asymptotic synchronization is described in a more general setting in [5], and that the above definition is local as only trajectories sufficiently close to the periodic trajectory are considered. Using the Poincaré map [20, Theorem 13.1], we prove the following theorem.

Theorem 1 The refrigeration system with two display cases exhibits asymptotic synchronization.

Theorem 1 is a direct consequence of Lemmas 1 and 2 below.

Lemma 1 There exist initial temperatures on the diagonal, and initial pressure which give rise to a \((T_p, 2)\)-periodic trajectory.

Proof For \(i \in \mathbb{N}\), the analytic expression for a trajectory

\[ \gamma_x(t) = \begin{cases} \gamma_x(t), & t \leq t_i \\ \gamma_x(t_i), & t \geq t_i \end{cases} \]

at \(x_0 = (T_0, P_0)\) of the system (3) is

\[ P^\alpha(t) = C^0_0 e^{-\alpha(t)} + \beta + \delta_i + \delta_2, \]

\[ T^\alpha_i(t) = B_i \frac{A_i - e^{-\alpha(t)}}{A_i} + e^{-\alpha(t)} \]

for \(i = 1, 2\), where

\[ A_i = a + \delta_i c, \quad B_i = b + \delta_i d, \quad C^0_0 = \frac{\beta + \delta_t + \delta_2}{\alpha}, \]

\[ C_i = T^\alpha_i + \frac{B_i}{A_i} = \frac{D_i}{A_i} = \frac{P_{12} - \delta_i}{\alpha} \]

with all constants given in the appendix. We observe from equation (9) that the projection of a trajectory on the \(T_1 T_2\)-plane is \((T_p, 2)\)-periodic if its initial condition \(x_0\) is of the form \((T_0, T_0, P_0)\) with \(P_0 \in \mathbb{R}_+\), \(0 \leq T_0 \leq 5\), and initial \(\delta\) is \((0,0)\). As a consequence, the trajectory is on the diagonal \(T_1 = T_2\) of the polyhedral sets \(P_{0,0}\) and \(P_{1,1}\). Using the above \(T_1 T_2\)-plan analysis as a guideline, we find an initial condition \((T_0, P_0, P_0)\) of a \((T_p, 2)\)-periodic trajectory in the state space \(X\). Indeed, by choosing \(T_0 = 0\), the initial pressure \(P_0\) is determined by solving, for \(P_{12}^0\), \(T_1\) and \(T_2\), the system of equations

\[ P_0^0 = P_1^2(t_2) = (P_1^2(t_1) - \frac{\beta + 2}{\alpha} e^{-\alpha(t_1)}) + \frac{\beta + 2}{\alpha} \]

with \(P_1^2(t_1) = (P_1^2(t_1) - \frac{\beta}{\alpha} e^{-\alpha(t_1)}) + \beta\), and

\[ 5 = T^2(t_1) \]

with the notation as in the proof above, let \(x_0 = (0,0, P_0) \in P_{0,0}\) and \((T^\infty, \gamma_x)\) denote the \((T_p, 2)\)-periodic trajectory at \(x_0\); thus, \(T^\infty = \{0, t_1\} \cup \{T_1, 2T_1\} \cup \{T_2 + t_1, 2T_2 + t_1\} \cup \ldots\)

Lemma 2 The \((T_p, 2)\)-periodic trajectory at \(x_0\) is asymptotically stable in \(X^*\).
Proof We remark that stability of a periodic trajectory can be determined by a Poincaré map. Indeed, asymptotic stability of a periodic trajectory (in $X$) is equivalent to asymptotic stability of the fixed point of a corresponding Poincaré map [20, Theorem 13.1].

Next, we describe the Poincaré map. For this purpose, the evolution of a trajectory starting at a point nearby $(T_p, 2)$-periodic trajectory $\gamma_{x_0}$ is outlined.

Let $x_0$ denote any of the two points $(0, e, \bar{P}_0 \pm \epsilon)$ for some $\epsilon > e' > 0$. Thus, $|x_0 - x_{0\text{lim}}| = \epsilon$ where $|\cdot|_{\text{lim}}$ is the max-norm. We show that for sufficiently small $\epsilon$, $|x_1 - x_{0\text{lim}}| < \epsilon$, where $x_1 \in P_{(0,0)}$ is the point at which the trajectory $\gamma_{x_0}$ at $x_0$ meets the hyperplane $T_1 = 0$ in $P_{(0,0)}$ for the first time. From this, we will conclude that this Poincaré map is a contraction.

For $x = (x'; \delta)$ and $y = (y'; \delta')$, both in $X$, we write $x = y$ (resp. $x \leq y$) whenever $x' = y'$ (resp. $x' < y'$). Furthermore, let $\gamma_{x_0}(t; i) = (T_i^1(t), T_i^2(t), P(t))$. Below, we describe one period of the trajectory $\gamma_{x_0}$ in the following four steps.

1) From the choice of initial condition $x_0$, and from (8), it follows that $T_1^1(t_1^*) = 5$ for some $0 < t_1^* < t_1$, and $0 < T_2^1(t_1^*) < 5$ for $0 < t < t_1^*$ if $\epsilon$ is chosen sufficiently small. Note that $T_1^1(t) = T_1^1(\bar{t})$ for $0 < \bar{t} < t_1^*$.

2) Since $\gamma_{x_0}(t_1^*; 2) \in P_{(0,1)}$, we conclude that $5 = T_2^1(t_1^*)$ and $0 < T_2^1(t_2^*) < 5$ for $t_1^* < t < t_2^*$ if $\epsilon$ is chosen sufficiently small. Note that $T_2^1(t) = T_2^1(\bar{t})$ for $0 < \bar{t} < t_1^*$.

3) Since $\gamma_{x_0}(t_1^*; 3) \in P_{(1,1)}$, we conclude that there exists a $t_1^* \in (t_1; t_2)$ such that $T_1^1(t_1^*) = 0$, and $T_2^1(t_1^*) > 0$ for $t_1^* < t < t_1'$. Note that $T_2^1(t) = T_2^1(\bar{t})$ for $0 < \bar{t} < t_1'$. This approach has been carried out for a refrigeration system with two display cases.

4) Finally, since $\gamma_{x_0}(t_1^*; 4) \in P_{(1,0)}$, we use (8) to conclude that $T_2^1(t_2^*) = 0$; whereas, $0 < T_1^1(t_2^*) < 5$ for $t_2^* < t < t_2$. Note that $T_2^1(t) = T_2^1(\bar{t})$ for $0 < \bar{t} < t_1'$. This approach has been carried out for a refrigeration system with two display cases.

Having described one period of the trajectory $\gamma_{x_0}$, we are now ready to show that $|x_1 - x_{0\text{lim}}| < \epsilon$. Since $\delta = (1, 1)$ and $T_2^1(t_1^*) > T_2^1(t_1)$, from symmetry of the systems (3) for $\delta = (1, 0)$ and $\delta = (1, 1)$, it follows that $T_2^1(t_2) \leq 5 - T_1^1(t_2')$. Moreover, since $\gamma_{x_0}((1; 1)$ and $\gamma_{x_0}((1; 1)$ are solutions to the stable affine linear system (3), with $\delta = (0, 0)$, starting $(t = t_0)$ at a distance $|x_0 - x_{0\text{lim}}| = \epsilon$, we conclude that $|\gamma_{x_0}(t_1^*; 1) - \gamma_{x_0}(t_1^*; 1)|_{\text{lim}} < \epsilon$. Together with item 1 and 2), this implies that $5 - T_1^1(t_2') < \epsilon$; hence, $T_2^1(t_2) < \epsilon$. Therefore, we only need to show that $|P^2(t_2) - \bar{P}_0| < \epsilon$, which follows by straightforward computations involving the explicit expression (7) of $P$. The case with $x_0 = (e, 0, \bar{P}_0 \pm \epsilon')$ is conceptually the same as above and is, therefore, left to the reader.

As shown in Section 3.3, the dynamics of the refrigeration system carries over to the manifold $X'$; thus, the above procedure defines a standard Poincaré map. Using the fact that the periodic trajectory can be covered by charts, the results in [20] applies. The Poincaré map takes a point $x_0$ with $|x_0 - x_{0\text{lim}}| = \epsilon$ to the point $x_1$ with $|x_1 - x_{0\text{lim}}| < \epsilon$. Therefore, the Poincaré map is a contraction, and the $(T_p, 2)$-periodic trajectory $\gamma_{x_0}$ is asymptotically stable according to [20, Theorem 13.1].

We numerically investigated a subset $B$ of the basin of attraction of the stable limit cycle corresponding to synchronization that contains the point $\tilde{x}_0$. We determined that $B = \{(T_{i_1}, T_{i_2}, P) \mid 0 < i_1 < 1, 8 < P < 19\}$.

5 Conclusions

In this paper, we have studied the synchronization phenomenon in supermarket refrigeration systems. We have associated synchronization with a periodic trajectory in a hybrid system. To determine whether asymptotic synchronization occurs, i.e., the periodic trajectory is asymptotically stable, we have used a Poincaré map. This approach has been carried out for a refrigeration system with two display cases.

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For the present, a summary of the model developed in [21].

\[ T_{\text{wall},i} = \frac{Q_{\text{goods-air},i} + Q_{\text{load},i}}{UA_{\text{air-wall},i}}, \]

where the parameter processes are specified in Table a1, and \( \delta_i \in [0,1] \) is the switch parameter for the \( i \)th display case. When \( \delta_i = 0 \), the \( i \)th expansion valve is switched off, whereas when \( \delta_i = 1 \) it is switched on. The suction manifold dynamics is governed by the differential equation

\[ \frac{dP_{\text{suc}}}{dt} = \frac{\sum_{i=1}^{N} \delta_i n_{i0} + n_{i0,\text{const}} - V_{\text{comp}}(n_{i0} P_{\text{suc}} + b)}{V_{\text{suc}} - V_{\text{suc},0}}, \]

where \( N \) is the number of display cases, and \( n_{i0} = \theta_0 \) for \( i \in \{1, \ldots, N\} \).

We denote \( T_j = T_{\text{air},j} \) and \( P = P_{\text{suc}} \) and write the dynamics of the air temperature and suction pressure in the concise form (with the process constants in (a1a) collected in \( a, b, c, d, e, \alpha, \beta \) and then replaced by their numerical values)

\[ T_j = -aT_j + b - \delta_i(cT_j - dP - e) = -0.0019T_j + 0.0244 - \delta_i(-0.0012T_j + 0.0506P + 0.1065), \]

\[ P = -aP + b + \delta_1 + \delta_2 = -0.056P + 0.0038 + \delta_1 + \delta_2. \]

Table a1 Parameters for a simplified supermarket refrigeration system.

| Display cases |
|----------------|
| \( UA_{\text{wall-ref, max}} \) | \( UA_{\text{goods-air}} \) | \( UA_{\text{air-wall}} \) | \( n_{i0,\text{const}} \) | \( V_{\text{suc},0} \) |
| \( 500 \) J/(s·K) | \( 300 \) J/(s·K) | \( 500 \) J/(s·K) | \( 0.2 \) kg/s | \( 4.6 \) kg/(m\(^3\)·bar) |

\( T_{g0} \quad n_0 \quad Q_{\text{load}} \quad M_{\text{wall}} \quad C_{p,\text{wall}} \)

3.0 °C \quad 1.0 kg/s \quad 3000 J/s \quad 260 kg \quad 385 J/(kg·K)

Note: the same parameters has been used for all display cases.

| Compressor | Suction manifold | Air temperature control |
|------------|------------------|-------------------------|
| \( V_{\text{comp}} \) | \( V_{\text{suc}} \) | \( T_1 \) | \( T_1' \) |
| 0.28 m\(^3\)/s | 5.00 m\(^3\) | 0.00 °C | 5.00 °C |

Coefficients

\( a_1 = -16.2072 \quad b_1 = -41.9095 \quad \alpha = 4.6 \quad \delta_1 = 0.4 \)

Rafael WISNEWSKI is a professor in the Section of Automation & Control, Department of Electronic Systems, Aalborg University. He received his Ph.D. in Electrical Engineering in 1997, and Ph.D. in Mathematics in 2005. In 2007–2008, he was a control specialist at Danfoss A/S. His research interest is in system theory, particularly in hybrid systems. E-mail: raf@es.aau.dk.
John LETH received his M.S (2003) and Ph.D. (2007) degrees from the Department of Mathematical Sciences, Aalborg University, Denmark. Currently, he is employed as Assistant Professor at the Department of Electronic Systems, Aalborg University. His research interests include mathematical control theory and (stochastic) hybrid systems. E-mail: jjl@es.aau.dk.

Jakob Gulddahl RASMUSSEN is an associate professor at the Department of Mathematical Sciences, Aalborg University, Denmark. He received his Ph.D. in Statistics in 2006 also at the Department of Mathematical Sciences. His research interests include spatial statistics and stochastic processes (including stochastic hybrid systems). E-mail: jgr@math.aau.dk.