Differential Games of Pursuing in the Systems with Distributed Parameters and Geometrical Restrictions

M. Sh. Mamatov, E. B. Tashmanov, H. N. Alimov
Department “Geometry”, National University of Uzbekistan Named After M. Ulugbek, Tashkent, Uzbekistan
Email: mamatovms @ mail.ru

Received 2013

ABSTRACT

A problem of pursuit in the controlled systems of elliptic type without mixed derivatives with variable coefficients was considered. The model of the considered system is described by partial differential equations. The players (opponents) control parameters occur on the right-hand side of the equation and are subjected to various constraints. The first player’s goal is to bring the system from one state into another desired state; the second player’s goal is to prevent this from happening. We represent new sufficient conditions for bringing the system from one state into another. The finite-difference method is used to solve this problem.

Keywords: Pursuit; Pursuer; Evader; Terminal Set; Pursuit Control; Evasion control

1. Introduction

Some problem formulations in the theory of differential games may be illustrated by motion of two controlled objects, pursuer and evader. Let in the course of motion the objects continuously observe each other and at each time instant correct their motions depending on the information about the adversary. Depending on the pursuer’s aim, the problem of pursuit is then formulated as follows: using the information about the evader, at each time instant correct their motions depending on the information about the adversary. The operated distributed system described by partial differential equations (see, for example, [1,2]) is considered

\[ a(x,y)\frac{\partial^2 z}{\partial x^2} + b(x,y)\frac{\partial^2 z}{\partial y^2} = f(u(x,y),\nu(x,y)), \]

where \( z = z(x,y) \) – unknown function, \( a(x,y), b(x,y) \) – continuous functions in \( \Omega = \{(x,y); 0 < x < 1, 0 < y < 1\} \) with border \( \Gamma, \varphi(x,y) \) – smooth function on \( \Gamma, \xi \) – external normal. It is supposed that there is a positive constant \( \nu \) such that for any \( (x,y) \in \Omega \) the inequality, \( b(x,y) \geq \nu, u = u(x,y), \nu = \nu(x,y) \) – operating functions is executed from a class \( L_2(\Omega) \). The first (pursuing) player, (pursued or escaping) the player, \( u \in \bar{P}, \nu \in \bar{Q}, \bar{P} \) and \( \bar{Q} \) – nonempty compacts in \( \mathbb{R}^1 \) disposes of function \( \nu(x,y) \) second of function \( u(x,y) \). The terminal set \( \bar{M} \subset \mathbb{R} \) is allocated.

Definition 1. In a task (1) it is possible \( \epsilon \) – completion of \( (\epsilon > 0) \) prosecutions from “boundary” situation \( \varphi(\cdot,\cdot) \), if exist function \( u(x,y) \in \bar{P}, \nu \in \bar{Q}, (x,y) \in \Omega \), such that for any function \( \nu(x,y) \in \bar{Q}, (x,y) \in \Omega \) the solution \( z(x,y) \) of a task (1) where \( u = u(\nu(x,y),x,y), \nu = \nu(x,y) \), gets on a set \( cI + \bar{M} \), at some \( (\tilde{x},\tilde{y}), (\tilde{x},\tilde{y}) \in \Omega : z(\tilde{x},\tilde{y}) \in cI + \bar{M} \) where \( I = (-1,1) \).

Decompose the Euclidean space \( \mathbb{R}^l \) of variables \( (x,y) \) by the planes \( x_i = ih, h = 1/r, i = 0,1,..., \) and \( y_j = jl, l = 1/\theta, j = 0,1,2,..., \) into parallelepips...
\[ \Omega_{hht} = \{(x, y) : ih < x < (i + 1)h, jh < y < (j + 1)h\}, \quad r \text{ and } \theta \text{ being some natural numbers.} \] 

The points \((x_i, y_j)\) belonging to a set \(\Omega\) are the nodes of the grid \(\Omega_{hht}\). Each node has its neighbors. If all these neighbor nodes also belong to the grid \(\Omega_{hht}\), then the node \((x_i, y_j)\) is referred to as “internal”, otherwise, \((x_i, y_j)\) is called the “boundary” node. The set of all boundary nodes is called as border of net area and is designated through \(\Gamma_h\).

Replace the internal nodes of the derivatives (1) differential second-order accuracy of approximation ratios with formulas

\[
\begin{align*}
\frac{\partial z}{\partial x}(x, y) &= \left(z(x_{i+1}, y) - z(x_i, y)\right) / h + O(h) \frac{\partial z}{\partial y}(x, y) = \left(z(x, y_{j+1}) - z(x, y_j)\right) / h + O(h), \\
\frac{\partial^2 z}{\partial x^2}(x, y) &= \left(z(x_{i+1}, y) - 2z(x_i, y) + z(x_{i-1}, y)\right) / h^2 + O(h^2), \\
\frac{\partial^2 z}{\partial y^2}(x, y) &= \left(z(x, y_{j+1}) - 2z(x, y_j) + z(x, y_{j-1})\right) / h^2 + O(h^2), \\
\frac{\partial^2 z}{\partial x \partial y}(x, y) &= \left(z(x_{i+1}, y_{j+1}) - z(x_{i+1}, y_{j-1}) - z(x_{i-1}, y_{j+1}) + z(x_{i-1}, y_{j-1})\right) / h^2 + O(h^2).
\end{align*}
\]

Substituting these ratios in (1), having rejected an error of approximation of derivatives, we will receive the differential equations for unknown

\[
\begin{align*}
a_{ij}(z_{i+1,j} - z_{i,j}) / h + b_{ij}(z_{i+1,j} - 2z_{i,j} + z_{i-1,j}) / h^2 &= f_{ij}, \\
&\quad i = 0, 1, \ldots, n - 1; \quad j = 0, 1, \ldots, r - 1.
\end{align*}
\]

where the following designations of values of coefficients and the right part in a hub \((x, y)\), \(a_{ij}, b_{ij}, c_{ij}, d_{ij}, g_{ij}, f_{ij}\), for example are entered

\[ f_{ij} = f(u(x_i, y_j), v(x_i, y_j)), \quad (x, y) \in \Omega_{hht}. \]

Ratios (2) contains except unknown \(z_{i,j}\) in internal nodes also unknown \(z_{i,j}\) on border of net area. For boundary nodes we will write down a ratio

\[
\begin{align*}
(z_{i,j} - z_{i,0}) / h + a_{i,0}(z_{i+1,j} - z_{i,j}) / 2 &= \phi_{i,0}, \quad (j = 0, 1, 2, \ldots, r - 1) \\
(z_{i,j} - z_{i-1,j}) / h + a_{i-1,j}(z_{i,j} - z_{i-1,j}) / 2 &= \phi_{i,j}, \\
(z_{i,0} - z_{i-1,0}) / h + a_{i-1,0}(z_{i,0} - z_{i-1,0}) / 2 &= \phi_{i,0}, \\
(z_{i,r} - z_{i,j}) / h + a_{i,j}(z_{i,r} - z_{i,j}) / 2 &= \phi_{i,r}.
\end{align*}
\]

Thus, we will receive system of \(r \theta + 2(r + \theta)\) equations with the same number of unknown \(z_{i,j}\).

Using boundary conditions (3), we will express \(z_{i,j}\) through \(z_{i,0}, z_{i-1,0}\). Let’s have

\[
\begin{align*}
z_{i,j} &= (2 - l)(z_{i,j} - z_{i,0}) - 2(2 + l \alpha_{i,j})z_{i,j} + 2(2 + l \alpha_{i,j})z_{i,j}; \\
z_{i,j} &= (2 - l)(z_{i,j} - z_{i,j}) - 2(2 + l \alpha_{i,j})z_{i,j} + 2(2 + l \alpha_{i,j})z_{i,j}.
\end{align*}
\]

Using these ratios, we will exclude in system (3) unknown \(z_{i,j}\), \(z_{i,0}\). If to enter designation \(\gamma = h^2 / l\), we will receive system

\[
\begin{align*}
z_{i,j} &= (2 + 2 \gamma - k_{i,j})z_{i,j} + z_{i,j} + z_{i,j} = F_{i,j}, \\
z_{i,j} &= (2 + 2 \gamma - k_{i,j})z_{i,j} + z_{i,j} + z_{i,j} = F_{i,j}, \\
z_{i,j} &= (2 + 2 \gamma - k_{i,j})z_{i,j} + z_{i,j} + z_{i,j} = F_{i,j}, \\
z_{i,j} &= (2 + 2 \gamma - k_{i,j})z_{i,j} + z_{i,j} + z_{i,j} = F_{i,j},
\end{align*}
\]

where

\[
\begin{align*}
F_{i,j} &= h^2 f_{i,j} - 2(2 + l \alpha_{i,j})f_{i,j}, \\
F_{i,j} &= h^2 f_{i,j} - 2(2 + l \alpha_{i,j})f_{i,j}.
\end{align*}
\]

This system can shortly be written down in a look

\[
z_{i,j} = F_{i,j}, \quad (i = 0, 1, 2, \ldots, r - 1).
\]

where

\[
\begin{align*}
F_{i,j} &= h^2 f_{i,j} - 2(2 + l \alpha_{i,j})f_{i,j}, \\
F_{i,j} &= h^2 f_{i,j} - 2(2 + l \alpha_{i,j})f_{i,j}.
\end{align*}
\]

Boundary conditions (3) and (5) can be copied in a look

\[
z_{i,j} = (2 - h \alpha_{i,j})(z_{i+1,j} - 2h \alpha_{i,j})z_{i,j}; \\
&k_{i,j} = (2 - h \alpha_{i,j})(z_{i+1,j} - 2h \alpha_{i,j}),
\]

where

\[
\begin{align*}
k_{i,j} &= (2 - h \alpha_{i,j})(z_{i+1,j} - 2h \alpha_{i,j}); \\
y_{i,j} &= (2h \alpha_{i,j})(z_{i} - 2h \alpha_{i,j}); \\
k_{i,j} &= (2 - h \alpha_{i,j})(z_{i+1,j} - 2h \alpha_{i,j});
\end{align*}
\]

Having put

\[
\begin{align*}
h_0 &= (y_{i,j} - y_{i,j}), \\
C = \begin{pmatrix} k_{i,j} & 0 & 0 & \cdots & 0 \\
0 & k_{i,j} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & k_{i,j} \\
0 & 0 & 0 & \cdots & k_{i,j} \end{pmatrix},
\end{align*}
\]

it is possible to write down systems (9) in such a look:

\[
z_{i,j} = X_{i,j} + y_{i,j};
\]

Finally we have the following system of the equations:

\[
z_{i,j} = X_{i,j} + y_{i,j}; \\
z_{i,j} + A_{i,j}z_{i,j} = F_{i,j}, \quad (i = 1, 2, \ldots, r - 1)
\]

Instead of game (13) we will consider more the general game described by system of the equations

\[
\begin{align*}
C_{i,j}z_{i,j} - B_{i,j}z_{i,j} = f_{i,j}, \\
-A_{i,j}z_{i,j} + C_{i,j}z_{i,j} = f_{i,j}, \quad 1 \leq n \leq N - l,
\end{align*}
\]

where \(z_{i,j} \in R^n, \quad n = 0, N, \quad A_{i,j}, C_{i,j}, B_{i,j} - m \times m \text{ constant square matrices,} \quad u_i, v_i - \text{ operating parameters,} \quad u_i = \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial t}.\)
prosecution parameter, \( \nu \) – beanie parameter, \( u_n \in P \subset \mathbb{R}^r \), \( v_n \in Q \subset \mathbb{R}^r \), \( P \) and \( Q \) – nonempty sets; \( f_n \) – the set function displaying \( R \times \mathbb{R}^r \) in \( R^m \). Besides, in \( R^m \) the terminal set is \( M \) allocated.

**Definition 2.** We shall say that from “boundary” situation \((f_0, f_i)\) it is possible to complete pursuit for \( N \) steps if from any sequence \( \pi_0, \pi_1, \ldots, \pi_{n-1} \) of values of the evasion controls it is possible to construct a sequence \( x_0, x_1, \ldots, x_{n-1} \) of values of the pursuit control values such that the solution \( \{z_0, z_1, z_2, \ldots, z_{n-1}\} \) of the equation

\[
C_0 x_0 - B z_0 = f_0, \\
-A z_{n-1} + C_{n-1} x_{n-1} = f_i(\pi_{n-1}), 1 \leq n \leq N-1, \\
-A_{n-1} x_{n-1} + C_{n-1} z_{n-1} = f_{n-1}.
\]

(15)

Gets on \( M: \pi \in M \) for some \( i \). Thus for finding of value \( \pi \) it is allowed to use values \( x_{n-1} \) and \( z_{n-1} \).

Note that the type of systems (14) is difference schemes for elliptic equations of second order with variable coefficients in any field of any number of dimensions [3-14].

Solution of problem (14) will be sought in the form

\[
z_n = \alpha_n z_{n-1} + \beta_n, \quad n = N-1, N-2, \ldots, 0,
\]

(16)

where \( \alpha_n \) – uncertain while a square matrix of the sizes \( m \times m \), and \( \beta_n \) – a vector of dimension \( m \). From a formula (16) and the equations of system (14) for \( 1 \leq n \leq N-1 \) there are recurrent ratios for calculation of matrixes \( \alpha_n \) and vectors \( \beta_n \). Really from a formula (16) \( z_0 = \alpha_0 z_{-1} + \beta_0 \), substituting it in (14) we will receive

\[
-A_0 (\alpha_0 z_{-1} + \beta_0) + C_0 z_0 - B z_{-1} = f_0 (u_0, u_0), \\
1 \leq n \leq N-1; \\
(C_n - A_n \alpha_n) z_{n-1} + f_n (u_n, u_n) + A_n \beta_n, \\
z_n = (C_n - A_n \alpha_n)^{-1} B z_{n-1} + (C_n - A_n \alpha_n)^{-1} [f_n (u_n, u_n) + A_n \beta_n].
\]

Equating now the right parts of the last and (16) equalities we will receive

\[
\alpha_n = (C_n - A_n \alpha_n)^{-1} B_n, \quad n = 1, 2, \ldots, N-1; \\
\beta_n = (C_n - A_n \alpha_n)^{-1} [f_n (u_n, u_n) + A_n \beta_n], \\
n = 1, 2, \ldots, N.
\]

Further from (16) and the equations (14) for \( n = 0, N \), there are the initial values \( \alpha_0, \beta_0 \) and \( z_0 \), allowing beginning the account on recurrent ratios. From (14) and (16) for \( n = 0 \) we will have

\[
z_0 = C_0^i B_0 z_0 + C_0^i f_{n_0}, \quad z_0 = \alpha_0 z_{-1} + \beta_0.
\]

And, therefore

\[
\alpha = C_0^i B_0, \quad \beta = C_0^i f_{n_0}.
\]

In the same way for \( n = N \) we have

\[
-A_0 (\alpha_N z_N + \beta_N) + C_N z_N = f_N
\]

or

\[
z_N = (C_N - A_N \alpha_N)^{-1} [f_N + A_N \beta_N].
\]

Uniting, we will write out final formulas

\[
\alpha_n = (C_n - A_n \alpha_n)^{-1} B_n, \quad n = 1, 2, \ldots, N, \quad \alpha = C_0^i B_0
\]

(17)

\[
\beta_n = (C_n - A_n \alpha_n)^{-1} [f_n (u_n, u_n) + A_n \beta_n], \\
n = 1, 2, \ldots, N-1.
\]

(18)

\[
\beta_N = (C_N - A_N \alpha_N)^{-1} [f_N (u_N, u_N) + A_N \beta_N].
\]

(19)

It is clear, that if in game (17), (18), (19) \( z_n \in M \) that in game (14) too game comes to the end. Therefore further instead of game (14) we will consider discrete game described by system of the equations (17), (18), (19).

Before giving determination of stability of algorithm (17), (18), (19), we will provide some data from linear algebra.

Let \( A \) – any square matrix \( m \times m \) and \( \| x \|_w \) be norm of a vector in \( R^m \), then the norm \( A \) is defined by equality

\[
\| A \| = \sup_{x \neq 0} \| A x \| / \| x \|_w.
\]

For a case of Euclidean norms in \( R^m \) we have \( \| A \| = \sqrt{\rho} \), where \( \rho \) – maximum on the module own value of a matrix \( A A^T \).

Without the proof we will give the following known lemma (see [15]).

**Lemma 1.** Let for some matrix norm the square matrix meet a condition \( \| A \| < q < 1 \). Then there is a matrix \( (E + A)^{-1} \) and \( \| (E + A)^{-1} \| \leq 1 / (1 - q) \).

Let’s say that the algorithm is steady if the assessment \( \| \alpha \| < 1 \) for \( 1 \leq j \leq N \) is carried out.

**Lemma 2.** If \( C_j \) for \( 0 \leq j \leq N \) – no degenerate matrixes and \( A \) and \( B \) – nonzero matrixes for \( 1 \leq j \leq N-1 \) also are satisfied conditions

\[
\| C_j + B_j \| \leq 1, \quad \| C_j A_j \| \leq 1, \\
\| C_j A_j \| + \| C_j B_j \| \leq 1, \quad 1 \leq j \leq N-1.
\]

And at least in one of inequalities the strict inequality takes place, there are return to the \( C_j - A_j \alpha_j \) matrix and \( \| \alpha \| < 1 \), here \( \alpha = C^i B_0 \),

\[
\alpha = (C_j - A_j \alpha_j)^{-1} B_j, \quad 1 \leq j \leq N-1.
\]

**Proof.** Suppose \( \| \alpha \| < 1 \) for some \( \| \alpha \| \leq 1 \) also we will show \( \| \alpha \| < 1 \). After a course the proof of this fact we will receive existence of a matrix \( (C_j - A_j \alpha_j)^{-1} \).

Really conditions of a lemma we will have

\[
\| C_j A_j \| \leq \| C_j \| \| A_j \| \| \alpha \| \leq \| C_j \| \| A_j \| \leq 1 - \| C_j B_j \| < 1.
\]

As \( C_j A_j \) square matrix that owing to a lemma 1 there are return to \( E - C_j A_j \), and \( C_j A_j \) matrixes and \( \| (E - C_j A_j) \| \leq 1 / \| C_j B_j \| . \) From here and from (17) we will receive

\[
\| \alpha \| \leq \| (E - C_j A_j \alpha_j) \| \| C_j B_j \|
\]

\[
\leq \| (E - C_j A_j \alpha_j) \| \| C_j B_j \| < 1.
\]

The proof of the lemma is complete.
3. Main Results

Everywhere further it is supposed that $M=M_0+M_1$, where $M_0$ – linear subspace $R^n$, $M_1$ – a subset a subspace, $L$ – orthogonal complement of $M_0$ in $R^n$. Denote $\Pi$ we will designate a matrix of orthogonal design from $R^n$ on $L$.

Let $W(0)=\{0\}$, $W(k)=\sum_{0=0}^{k-1} \bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k}),$ $W_1(k)=-M_1+W(k)$, $1 \leq k \leq N.$

(20)

Theorem 1. Let $N$ be the smallest of the numbers $k$, such that $-\prod \alpha_{N-1} \alpha_{N-2} \ldots \alpha_{N-k} z_0 \in W(k).$ (21)

Then from “boundary” situation $(f_n, f_s)$ it is possible to complete pursuit for $N$ steps.

Let now $W(0)=-M_1$, $W(0)=\bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k})$, $W(k)=\bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k})$ (22)

Theorem 2. If $N$ be smallest of those numbers $k$, for each of which takes place inclusion $-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W_2(k)$ (23)

that of “boundary” situation $(f_n, f_s)$ it is possible to complete pursuit for $N$ steps.

Let $\gamma_i(\cdot)\equiv\{\gamma_{n}, \gamma_{i+1}, \ldots, \gamma_{i+z} \geq 0, \sum_{m=n}^{k-1} \gamma_{m} = 1\}$

and

$W(\gamma_i(\cdot)) = \sum_{m=n}^{k-1} \bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k}),$

$W_2(0)=-M_1, W_2(k) = \bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (\gamma_{i}(\cdot)), 0 \leq k \leq N.$

(24)

Theorem 3. If $M_1$ – a convex set and $N$ be smallest of those numbers $k$. For each of which inclusion takes place $-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W(k)$. (25)

That of “boundary” situation $(f_n, f_s)$ it is possible to complete pursuit for $N$ steps.

It is easy to be convinced [15] that the solution of $z_{ij}$ differential task (2) meets to the solution of $z$ of an initial task (1), the following assessment of speed of convergence takes place

$\|z_{ij}\|_{\infty} \leq K_1 h^2 + K_2 f_{1}$

(26)

where $(z_{ij})$ – values of the exact decision a task (1) in grid functions, $\Phi_{ij}$ – spaces of net functions, $|||_{\infty}$ – is its norm and, $K_1$ and $K_2$ constants.

Theorem 4. Let an inequality (26) $K_1 h^2 + K_2 f_{1} < \epsilon$, and in game (13) from “boundary” situation $(f_n, f_s) = (-y_n, -y_s)$ completion of prosecution that is definitions 2 be possible. Then in game (1) from “boundary” situation $\partial z/\partial x + \alpha(x,y)z = \varphi(x,y), (x,y) \in \Gamma$ it is possible to complete pursuit that are definitions 1.

4. Proof of Theorem

Proof of Theorem 1. Let $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}$, $\sigma_i \in \Omega$, $1 \leq i \leq N-1$ – any sequence. Instead of inclusion (21) we will consider other inclusion equivalent to it $-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W(k-1)$

$\bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k})$

Means, exists $a_{N-k}$

$a_{N-k} \in \bigcap_{m=0}^{i=k-1} \prod_{j=k-1}^{i-j} (P_{N-j+k},u_{N-j+k}).$

Such that

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W(k-1) + a_{N-k}.$

(27)

Now control of the pursuing player $\pi_{N-k}$, the relevant control of the escaping player $\sigma_{N-k}$, we will construct as the solution of the following control

$\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} \beta_{1}(u_{N-k}, \sigma_{N-k}) = a_{N-k}.$

It is clear, that the equation has the decision. From here owing to (27) we have

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W(k-1) + a_{N-k}$

We write down this inclusion in other look.

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} \beta_{1}(u_{N-k}, \sigma_{N-k}) \in W(k-1).$

(28)

As a result from equalities (18) and (28) we will receive

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W(k-1)$

(29)

Done above a reasoning allow us to construct on the set control $\sigma_{N-2}$ providing inclusion (29). If now the control $\sigma_{N-2}$ becomes known that, we above can receive in the stated way control $\pi_{N-1}$ providing inclusion

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} \sigma_{N-k} \in W(k-2).$

(30)

Repeating this process, further we can construct step by step control by step control $\pi_{1}$, proceeding from becoming known controls $\sigma_{i}$, therefore, that in any step inclusion takes place

$-\prod \alpha_{N-k} \alpha_{N-k+1} \ldots \alpha_{N-z} z_0 \in W_i(0) = -M_1.$

(31)

It means that
\[ \prod z_{k+1} \in \mathbb{M} \]

As we set out to prove.

**Proof of Theorem 2.** Let \( \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N \), \( \bar{\sigma} \in Q \), \( 1 \leq i \leq N-1 \) – any sequence. For concrete \( \bar{\sigma}_i \), owing to (22) and (23) we will receive inclusion

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{k+2} = W_k(k-1) + \prod \alpha_{x_k} \alpha_{x_{k+1}} \beta(y_{k+1}, \bar{\sigma}_i) \tag{30}
\]

Now as \( \bar{\sigma}_{i+1} \) we take that element from \( P_{\bar{\sigma}} \) for which inclusion (30) remained. Then we will receive

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{k+2} \in W_k(k-1) + \prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \tag{31}
\]

From this it follows that

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{k+2} \in W_k(k-1) + \prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \tag{32}
\]

And therefore, owing to (19) we have

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{k+2} \in W_k(k-1) + \prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \tag{33}
\]

If now the control \( \bar{\sigma}_{i+2} \) becomes the stated way known that we above us can construct control \( \bar{\sigma}_{i+2} \) providing inclusion

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{k+2} \in W_k(k-2) + \prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \tag{34}
\]

Further arguing similarly in any step we will receive

\[
-\prod z_{k+1} \in W_k(0) = -M_i, \tag{35}
\]

that is

\[
\prod z_{N-1} \in \mathbb{M}. \tag{36}
\]

The theorem is proved completely.

**Proof of Theorem 3.** Instead of inclusion (25) meaning (26) we will consider inclusion equivalent to it

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{N+2} \in W(\mathcal{P}(\bar{\sigma})). \tag{37}
\]

Existence \( \mathcal{P}(\cdot) = \{ \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1} \geq 0, \mathcal{T}_N = 1 \} \) follows from (24). From here follows

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{N+2} \in \bigcup_{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1} \geq 0, \mathcal{T}_N = 1} [-\mathcal{F}_i + \prod \alpha_{x_k} \alpha_{x_{k+1}} \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}, \mathcal{T}_N = 1, \mathcal{T}_N = 1]. \tag{38}
\]

Let now \( \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N, \bar{\sigma} \in Q \), \( 1 \leq i \leq N-1 \) – any sequence. Owing to (31) exists such \( \nu_{k+1} \) that

\[
\nu_{k+1} \in \bigcup_{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1} \geq 0, \mathcal{T}_N = 1} [-\mathcal{F}_i + \prod \alpha_{x_k} \alpha_{x_{k+1}} \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}, \mathcal{T}_N = 1, \mathcal{T}_N = 1]. \tag{39}
\]

Therefore, controls \( \bar{\sigma}_{i+2} \) we will construct as the solution of the following equation

\[
-\mathcal{F}_i \mathcal{M} + \prod \alpha_{x_k} \alpha_{x_{k+1}} \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) = \mathcal{M}_i, \mathcal{M}_i \in \mathcal{M}. \tag{40}
\]

Furthertoing owing to (32) we have

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{N+2} \in \bigcup_{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1} \geq 0, \mathcal{T}_N = 1} [-\mathcal{F}_i + \prod \alpha_{x_k} \alpha_{x_{k+1}} \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}, \mathcal{T}_N = 1, \mathcal{T}_N = 1]. \tag{41}
\]

It is equivalent to the following

\[
-\prod \alpha_{x_k} \alpha_{x_{k+1}} \ldots \alpha_{x_{k+2}} z_{N+2} \in \bigcup_{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1} \geq 0, \mathcal{T}_N = 1} [-\mathcal{F}_i + \prod \alpha_{x_k} \alpha_{x_{k+1}} \alpha_{x_{k+2}} \beta(y_{k+1}, \bar{\sigma}_i) \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}, \mathcal{T}_N = 1, \mathcal{T}_N = 1]. \tag{42}
\]

The theorem is proved completely.

**Proof of Theorem 4.** Let in game (13) one be able to complete the pursuit from “boundary” situation \((f_0, f_k) = (y_0, -y_k)\) in \( N \) steps. Then, it follows from Definition 2 that from any sequence \( \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N, \bar{\sigma} \in Q \), \( 0 \leq k \leq N-1 \), of the evasion control it is possible to construct a sequence \( \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N, \bar{\sigma} \in P \), \( 0 \leq k \leq N-1 \), of pursuit control such that the solution \((z_0, z_1, \ldots, z_{N-1}, z_N)\) of the equation \( z_0 = X_0 + y_0 \), \( z_{n+1} = A z_n - z_{n+1} = F_n \), \( 1 \leq n \leq N-1 \), \( z_N = X_N + y_N \), for some \( d \leq N \) hits \( M = z_0, \ldots, z_N \). Let now in game (2) \( \nu = \mathcal{D}(x, y) \in \mathcal{Q}, (x, y) \in \mathcal{E} \), be an arbitrary control of an evader from the class \( L_0(\Omega) \).

With the knowledge of the evader control \( \nu = \mathcal{D}(x, y) \), it is possible to determine \( \mathcal{T}_k \) as the values of this function at the node points of the grid \( \Omega \), that is

\[
\nu_k = \mathcal{T}_k = (\mathcal{T}_{k,1}, \mathcal{T}_{k,2}, \ldots, \mathcal{T}_{k,N}). \tag{43}
\]

Whence it follows that in virtue of Theorem 4 we can construct the pursuer control in game (13) providing completion of pursuit

\[
u_k = \mathcal{T}_k = (\mathcal{T}_{k,1}, \mathcal{T}_{k,2}, \ldots, \mathcal{T}_{k,N}). \tag{44}
\]

Now in game (2) we construct the pursuer control \( u = \mathcal{D}(x, y) \) as follows: \( \mathcal{D}(x, y) = \mathcal{U}_k = h(x, y) \), \( i = 0, 1, \ldots, r-1 \), \( k = 0, 1, \ldots, \theta - 1 \). Obviously, \( u \in P \) and \( \mathcal{D}(x, y) \in \mathcal{Q} \). By substituting \( \nu = \mathcal{D}(x, y) \) and \( u = \mathcal{D}(x, y) \) in (2), we obtain a differen-
tial equation. Similarly, by substituting $\sigma_i^h$ and $\nu_i^h$ in (3), we obtain a grid equation approximating equation (2).

Let $(\Sigma)_h$ be the value of the exact solution corresponding to the controls $u=\sigma(x,y)$ and $u=\nu(x,y)$ of problem (2) at the nodes of the grid $\Omega_h$, $\Sigma_i^h$ be the solution corresponding to the controls $\sigma_i^h$ and $\nu_i^h$ of the difference problem (3). Then, we obtain from (13) and the condition of Theorem 4 that

$$||z_i^h-\Sigma_i^h||_{l_{\infty}} \leq Kl+KJh^\beta<\varepsilon.$$  

From this fact and $\Sigma_i^h \in \bar{M}_i$, we obtain $(z)_h-\Sigma_i^h \in \varepsilon S$, $(z)_h \in \varepsilon S+\Sigma_i^h$, $(z)_i \in \varepsilon S+\bar{M}_i$, which proves the theorem.

5. Conclusions

Thus, to solve the game problem of pursuit in the form (1) we pass to the discrete game (13) or (14), and Theorems 1-3 establish the sufficient condition for such problems. Theorem 4 establishes the sufficient conditions for solving the problem of pursuit (1). Here, the difference $(z)_h-\Sigma_i^h$ (see Section 3) plays the main part in the solution of problem and implies that the solutions of the grid equation (2) are stable.

The problem of stability of the grid equation (2) lies in determining the conditions under which the numerical error $p_i^h=(z)_h-\Sigma_i^h$ tends to zero with growing $j$ uniformly in all $i$, $0 \leq j \leq h$, or at least remains bounded.

Equation (2) is called stable if the round off errors generated in the course of calculations have tendency to decrease or at least not to increase. Otherwise, the accumulated errors may reach a value such that the numerical solution $(z)_h$ has nothing in common with the exact solution of the grid problem (2). It goes without saying that such unstable grid equations cannot be used for numerical solution of the differential games.

Theorems 1-4 are easily generalized to a wider class of differential equations, for example, when

$$\sum_{a=0}^{d} a_d(x, x_1, x_2, \ldots, x_n) \frac{\partial^{d} z}{\partial x^{d}_a} = f(t(x, x_1, x_2, \ldots, x_n), u(x, x_1, x_2, \ldots, x_n))$$

with discontinuous coefficients.

REFERENCES

[1] O. A. Ladyzhenskaya, “Kraevye Zadachi Matematicheskoi Fiziki,” (Boundary Problems of Mathematical Physics), Moscow, Nauka, 1973.

[2] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural’tseva, “Lineine i Kvazilineinye Uravneniya Parabolicheskogo Tipa,” (Linear and Quasilinear Equations of Parabolic Type), Moscow, Nauka, 1967.

[3] V. A. Il’in, “Boundary Control of String Oscillations at One End with Other End Fixed, Provided that Finite Energy Exists,” Dokl. Ross. Akad. Nauk, Vol. 378, No. 6, 2001, pp. 743-747.

[4] V. A. Il’in and V. V. Tikhomirov, “Wave Equation with Boundary Control at Two Ends and Problem of Complete Oscillation Damping,” Diff. Uravn., Vol. 35, No. 5, 1999, pp. 692-704.

[5] Yu. S. Osipov and S. P. Okhezin, “On the Theory of Differential Games in Parabolic Systems,” Dokl. Akad. Nauk SSSR, Vol. 226, No. 6, 1976, pp. 1267-1270.

[6] F. L. Chernous’ko, “Bounded Controls in Distributed-parameter Systems,” Prikl. Mat. Mekh., Vol. 56, No. 5, 1992, pp. 810-826.

[7] N. Satimov and M. Sh. Mamatov, “On a Class of Linear Differential and Discrete Games between Groups of Pursuers and Evaders,” Diff. Uravn., Vol. 26, No. 9, 1990, pp. 1541-1551.

[8] N. Satimov and M. Tukhtasinov, “On some Game Problems in the Distributed Controlled Systems,” Prikl. Mat. Mekh., Vol. 69, No. 6, 2005, pp. 997-1003.

[9] N. Satimov and M. Tukhtasinov, “On some Game Problems in Controlled First-order Evolutionary Equations,” Diff. Uravn., Vol. 41, No. 8, 2005, pp. 1114-1121.

[10] M. Sh. Mamatov, “On the Theory of Differential Pursuit Games in Distributed Parameter Systems,” Automatic Control and Computer Sciences, Vol. 43, No. 1, 2009, pp. 1-8. doi:10.3103/S0146411609010015

[11] M. Sh. Mamatov, “About Application of a Method of Final Differences to the Decision a Prosecution Problem in Systems with the Distributed Parameters,” Automation and Remote Control, Vol. 70, No. 8, 2009, pp. 1376-1384. doi:10.1134/S0005117909080104

[12] M. Tukhtasinov and M. Sh. Mamatov, “On Pursuit Problems in Controlled Distributed Systems,” Mathematical notes, Vol. 84, No. 2, 2008, pp. 273-280.

[13] M. Tukhtasinov and M. Sh. Mamatov, “About Transition Problems in Operated Systems,” Diff. Uravn., Vol. 45, No.3, 2009, pp. 1-6.

[14] M. Sh. Mamatov and M. Tukhtasinov, “Pursuit Problem in Distributed Control Systems,” Cybernetics and Systems Analysis, Vol. 45, No. 2, 2009, pp. 297-302. doi:10.1007/s10559-009-9100-x

[15] G. I. Marchuk, “Metody Vychislitel’noi MateMatiki,” (Methods of computational Mathematics), Moscow, Nauka, 1989.