QUADRATIC IDEALS AND ROGERS-RAMANUJAN RECURSIONS

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Abstract. We give an explicit recursive description of the Hilbert series and Gröbner bases for the family of quadratic ideals defining the jet schemes of a double point. We relate these recursions to the Rogers-Ramanujan identity and prove a conjecture of the second author, Oblomkov and Rasmussen.

1. Introduction

In this paper, we study a family of quadratic ideals defining the jet schemes for the double point $D = \text{Spec} \ k[x]/x^2$. Here $k$ is a field of characteristic zero. Recall that the $(n-1)$-jet scheme of $X$ is defined as the space of formal maps $\text{Spec} \ k[t]/t^n \to X$. In the case of the double point, such a formal map is defined by a polynomial $x(t) = x_0 + x_1 t + \cdots + x_{n-1} t^{n-1}$, such that $x(t)^2 \equiv 0 \mod t^n$. By expanding this equation, we get a system of equations

$$f_1 = x_0^2, f_2 = 2x_0x_1, \ldots, f_n = \sum_{i=0}^{n-1} x_i x_{n-1-i}.$$  

We denote the defining ideal of Jet$^{n-1}$ $D \subseteq \mathbb{A}^n$ by

$$I_n := \langle f_1, \ldots, f_n \rangle \subseteq R_n := k[x_0, \ldots, x_{n-1}].$$

The ring $R_n$ is $\mathbb{Z}_{\geq 0}$-graded by assigning the grading $(i, 1)$ to $x_i$. It is then clear that the ideal $I_n$ is bihomogeneous. Let

$$H_n(q, t) = \sum_{i,j \geq 0} \text{dim}_k(R_n/I_n)_{i,j} q^i t^j \in \mathbb{Z}[[q, t]]$$

denote the bigraded Hilbert series for $R_n/I_n$. Our first main result is the following.

**Theorem 1.1.** The series $H_n(q, t)$ satisfies the recursion relation

$$H_n(q, t) = \frac{H_{n-2}(q, qt) + tH_{n-3}(q, q^2t)}{1 - q^{n-1}t}$$

with initial conditions

$$H_0(q, t) = 1, \ H_1(q, t) = 1 + t, \ H_2(q, t) = \frac{1}{1 - qt} + t.$$  

Using this recursion relation, we obtain explicit combinatorial formulas for $H_n(q, t)$:

**Theorem 1.2.** The Hilbert series $H_n(q, t)$ is given by the following explicit formula:

$$H_n(q, t) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)}t^p}{(1 - q^{n-h(n,p)}t) \cdots (1 - q^{n-1}t)},$$

where $h(n, p) = \left[ \frac{n-p}{2} \right]$. 

1
In the limit $n \to \infty$, we reprove the theorem of Bruschek, Mourtada and Schepers [4], which relates the Hilbert series of the arc space for the double point to the Rogers-Ramanujan identity. In fact, we refine their result by considering an additional grading, see equation (7.1). Similar results for $n = \infty$ were obtained by Feigin-Stoyanovsky [9] [10], Lepowsky et al. [3] [6], and the second author, Oblomkov and Rasmussen in [8].

Although our approach to the computation of the Hilbert series is inspired by [4], it is quite different. The key result in [4] shows that for $n = \infty$ the polynomials $f_i$ form a Gröbner basis of the ideal $I_{\infty}$. As we will see below, the Gröbner basis of the ideal $I_n$ for finite $n$ is larger and has a very subtle recursive structure. We completely describe such a basis in Theorems 4.2 and 4.6. In particular, we prove the following.

**Theorem 1.3.** Let $k > 2$. Then the reduced Gröbner basis for $I_n$ contains $\binom{n-k+1}{k-2}$ polynomials of degree $k$.

Our proof of Theorem 1.3 does not use Gröbner bases at all. First, by an explicit inductive argument in Theorem 2.2 we give a complete description of the first syzygy module for $f_i$. Then, we define a “shift operator” $S : R_n \to R_{n+1}$, which sends $x_i$ to $x_{i+1}$, and identify $I_n \cap x_0 R_n$ and $I_n/(I_n \cap x_0 R_n)$ with the images of $I_{n-3}$ and $I_{n-2}$ under appropriate powers of $S$. This implies the recursion relation in Theorem 1.1.

We also observe a recursive structure in the minimal free resolution of $R_n/I_n$. In particular, we prove the following:

**Theorem 1.4.** Let $b(i, n)$ denote the rank of the $i$-th term in the minimal free resolution for $R_n/I_n$, in other words the $i$-th Betti number. Then

$$b(i, n) = b(i, n-1) + b(i-1, n-3) + b(i-2, n-3).$$

As a consequence, we can compute the projective dimension of $R_n/I_n$.

**Corollary 1.5.** The projective dimension of $R_n/I_n$ equals $\left\lceil \frac{2n}{3} \right\rceil$.

**Remark 1.6.** It is easy to see that the reduced scheme $(\text{Jet}^{n-1} D)^{\text{red}}$ is a linear subspace given by the equations $x_0 = \ldots = x_{\left\lfloor \frac{n-1}{2} \right\rfloor} = 0$ and has dimension

$$\dim \text{Jet}^{n-1} D = n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor.$$  

A more careful analysis of the gradings in Theorem 1.3 implies another formula for the series $H_n(q, t)$ which was first conjectured in [8].

**Theorem 1.7.** The Hilbert series of $R_n/I_n$ has the following form:

$$H_n(q, t) = \frac{1}{\prod_{i=0}^{n-1} (1 - q^i t)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1} (1 - q^k t) \times$$

$$\left( q^{5p^2 - 2p} t^{2p} \binom{n-2p+1}{p} q - q^{5p^2 + 5p} t^{2p+2} \binom{n-2p-1}{p} q \right).$$

The paper is organized as follows. In Section 2 we introduce the shift operator $S$, describe its properties and prove Theorem 2.2 which explicitly describes all syzygies between the $f_i$. In Section 3 we use the shift operator to find a recursive relation for the Hilbert series and to prove Theorem 1.1. In Section 4 we use the recursive structure to describe a Gröbner basis for $I_n$. In Section 5 we give a recursive description of the minimal free resolution of $R_n/I_n$ and prove Theorem 1.4. In Section 6 we solve both of the above recursions explicitly (with the given initial conditions) and give two explicit combinatorial formulas for $H_n(q, t)$. Finally, in Section 7 we briefly discuss the limit of all these techniques at $n \to \infty$ and the connection to the Rogers-Ramanujan identity.
The shift operator.

The first syzygy module

Theorem 2.2.

Consider the map in \( R \) the RSF grant 16-11-10160. O.K. was also supported by the Ville, Kalle and Yrjö Väisälä foundation of the Finnish Academy of Science and Letters.

2. Ideals and syzygies

2.1. Ideals. Let \( R_n = k[x_0, \ldots, x_{n-1}] \) and \( f_k = \sum_{i=0}^{k-1} x_i x_{k-1-i} \). Define \( I_n \subseteq R_n \) to be the ideal generated by \( f_1, \ldots, f_n \). Let \( F_n \) be the free \( R_n \)-module with the basis \( e_1, \ldots, e_n \).

Consider the map \( \phi_n : F_n \to R_n \) given by the equation

\[
\phi_n(\alpha_1, \ldots, \alpha_n) = f_1 \alpha_1 + \ldots + f_n \alpha_n.
\]

The \( R_n \)-module \( \ker(\phi_n) \) is called the first syzygy module of \( I_n \).

Lemma 2.1. One has

\[
\sum_{i=0}^{n} (n-3i)x_i f_{n+1-i} = 0.
\]

Proof. Indeed,

\[
\sum_{i=0}^{n} (n-3i)x_i f_{n+1-i} = \sum_{i+k+l=n} (n-3i)x_i x_k x_l.
\]

The coefficient at each monomial \( x_i x_k x_l \) equals

\[
(n-3i) + (n-3k) + (n-3l) = 3n - 3(i + k + l) = 3n - 3n = 0.
\]

For \( 0 < k < n \), define

\[
\mu_k := (-2kx_k, (-2k + 3)x_{k-1}, \ldots, kx_0, 0, \ldots, 0) \in F_n.
\]

By (2.1), we have \( \phi_n(\mu_k) = 0 \). Denote also \( \nu_{ij} = f_i e_j - f_j e_i \) (for \( i \neq j \)). It is clear that \( \phi_n(\nu_{ij}) = 0 \). The main result of this section is the following.

Theorem 2.2. The first syzygy module \( \ker(\phi_n) \) is generated by \( \mu_k \) and \( \nu_{i,j} \) over \( R_n \).

We prove Theorem 2.2 in Section 2.4.

2.2. The shift operator. We define a ring homomorphism \( S : R_n \to R_{n+1} \) by the equation \( S(x_i) = x_{i+1} \). Note that \( S \) is injective and we can uniquely write any polynomial in \( R_n \) in the form

\[
f = x_0 f' + S(f''), \quad f', f'' \in R_n, f'' \in R_{n-1}.
\]

The following equation is clear from the definition and will be very useful below:

\[
f_n = 2x_0 x_{n-1} + S(f_{n-2}).
\]

By abuse of notation, denote also \( S : F_n \to F_{n+2} \) the map which is given by

\[
S(\alpha_1, \ldots, \alpha_n) = (0, 0, S(\alpha_1), \ldots, S(\alpha_n)).
\]

Lemma 2.3. Let \( \alpha \in F_n \). Then \( \phi_{n+2}(S(\alpha)) \) is divisible by \( x_0 \) if and only if \( \phi_n(\alpha) = 0 \).
Proof. By (2.2) we have
\[ \phi_n(S(\alpha)) = \sum_{i=1}^{n} S(\alpha_i)f_{i+2} \equiv S \left( \sum_{i=1}^{n} \alpha_i f_i \right) \mod x_0. \]
Therefore \( \phi_{n+2}(S(\alpha)) \) is divisible by \( x_0 \) if and only if \( S(\sum \alpha_i f_i) \) is divisible by \( x_0 \). But since no shift contains \( x_0 \), this happens if and only if
\[ S \left( \sum \alpha_i f_i \right) = 0 \iff \sum \alpha_i f_i = \phi_n(\alpha) = 0. \]

Since \( \phi_n(\mu_k) = \phi_n(\nu_{ij}) = 0 \), by Lemma 2.3 the images of \( S(\mu_k) \) and \( S(\nu_{ij}) \) under \( \phi_{n+2} \) are divisible by \( x_0 \). The following lemma describes these images explicitly.

**Lemma 2.4.** One has \( \phi_{n+2}(S(\mu_k)) = (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)x_0f_{k+4} \), \( \phi_{n+2}(S(\nu_{ij})) = 2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2} \).

**Proof.** By definition,
\[ S(\mu_k) = (0, 0, -2kx_{k+1}, -(2k+3)x_k, \ldots, kx_1, 0, \ldots, 0) = \]
\[ \mu_{k+3} + (2k+6)x_{k+3}e_1 + (2k+3)x_{k+2}e_2 - (k+3)x_0e_{k+4}, \]
so
\[ \phi_{n+2}(S(\mu_k)) = (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)x_0f_{k+4}. \]

Also, \( S(\nu_{ij}) = S(f_i)e_{j+2} - S(f_j)e_{i+2} \), so
\[ \phi_{n+2}(S(\nu_{ij})) = S(f_i)f_{j+2} - S(f_j)f_{i+2} = (f_{i+2} - 2x_0x_{i+1})f_{j+2} - (f_{j+2} - 2x_0x_{j+1})f_{i+2} = \]
\[ 2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2}. \]

**Corollary 2.5.** One has
\[ \phi_{n+2}(S(\mu_k)) = (2k+3)x_{k+2}f_2 - (k+3)x_0S(f_{k+2}) = kx_{k+2}f_2 - (k+3)x_0S^2(f_k). \]

**Proof.**
\[ \phi_{n+2}(S(\mu_k)) = (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)x_0f_{k+4} = \]
\[ (2k+6)x_{k+3}f_1 + (2k+3)x_{k+2}f_2 - (k+3)(2x_0^2x_{k+3} + 2x_0x_1x_{k+2} + x_0S(f_k)) = \]
\[ (2k+3)x_{k+2}f_2 - (k+3)x_0S(f_{k+2}) = kx_{k+2}f_2 - (k+3)x_0S^2(f_k). \]

**Example 2.6.** \( \mu_1 = (-2x_1, x_0) \), so \( S(\mu_1) = (0, 0, -2x_2, x_1) \), and
\[ \phi_4(S(\mu_1)) = -2x_2(2x_0x_2 + x_1^2) + x_1(2x_0x_3 + 2x_1x_2) = \]
\[ 2x_3x_0x_1 - 4x_0x_2^2 = x_3f_2 - 4x_0S^2(x_0^2). \]

**Lemma 2.7.** The polynomial \( x_1S(f_{n-2}) \) can be expressed via \( f_1, \ldots, f_{n-1} \) modulo \( x_0 \).

**Proof.** We have \( (n-3)x_0f_{n-2} + (n-6)x_1f_{n-3} + \ldots + 2(n-3)x_{n-2}f_0 = 0 \), so \( (n-3)x_1S(f_{n-2}) + (n-6)x_2S(f_{n-3}) + \ldots + 2(n-3)x_{n-1}S(f_0) = 0 \).

It remains to notice that \( S(f_i) \equiv f_{i+2} \mod x_0. \)

**Lemma 2.8.** Assume that \( \ker(\phi_{n-2}) \) is generated by \( \mu_k \) and \( \nu_{ij} \) and suppose that \( \phi_n(\alpha) \) is divisible by \( x_0 \). Then \( \alpha_n = Ax_0 + Bx_1 + \sum_{i=3}^{n-1} \gamma_if_i \) for some \( A, B \) and \( \gamma_i \).
Proof. As above, we can write \( \alpha_i = x_0 \alpha'_i + S(\alpha''_{i-2}) \) for \( i \geq 3 \). Since \( f_1 \) and \( f_2 \) are divisible by \( x_0 \), we get

\[
\phi_n(S(\alpha'')) = \sum_{i=3}^{n} S(\alpha''_{i-2}) f_i \equiv \sum_{i=1}^{n} \alpha_i f_i \equiv 0 \mod x_0.
\]

By Lemma 2.3 we get \( \phi_{n-2}(\alpha'') = 0 \). By the assumption, we can write

\[
\alpha'' = \sum_{k<n-2} \beta_k \mu_k + \sum_{i<j\leq n-2} \gamma_{i,j} \nu_{ij}.
\]

Therefore

\[
\alpha''_{n-2} = \beta_{n-1} x_0 + \sum_{j\leq n-3} \gamma_{j,n-2} f_j,
\]

and

\[
\alpha_n = x_0 \alpha'_1 + S(\alpha''_{n-2}) = x_0 \alpha'_0 + S(\beta_{n-1}) x_1 + \sum_{j\leq n-3} S(\gamma_{j,n-2})(f_{j+2} - 2x_0 x_{j+1}).
\]

\[ \qed \]

2.3. Examples. Before proving Theorem 2.2 we would like to present the proof for \( n \leq 4 \).

Example 2.9. For \( n = 2 \) we have \( f_1 = x_0^2 \) and \( f_2 = 2x_0 x_1 \), so the module of syzygies is clearly generated by \((-2x_1, x_0) = \mu_1 \).

Example 2.10. Let \( n = 3 \), suppose that \( \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \). We can write \( \alpha_3 = \alpha'_3 x_0 + \alpha''_3 \), where \( \alpha''_3 \) does not contain \( x_0 \). Since \( f_1 \) and \( f_2 \) are divisible by \( x_0 \) and \( f_3 = 2x_0 x_2 + x_1^2 \), we get \( x_1^2 \alpha''_3 = 0 \), so \( \alpha''_3 = 0 \). Now \( \alpha = \frac{1}{2} \alpha'_3 \mu_2 + \gamma \), where \( \gamma \) is a syzygy between \( f_1 \) with \( \gamma_3 = 0 \). By the previous example, \( \gamma \) is a multiple of \( \mu_1 \), so the module of syzygies is actually generated by \( \mu_1 \) and \( \mu_2 \).

Example 2.11. Let \( n = 4 \), suppose that \( \alpha \) is a syzygy. We can write \( \alpha_3 = \alpha'_3 x_0 + \alpha''_3 \) and \( \alpha_4 = \alpha'_4 x_0 + \alpha''_4 \) where \( \alpha''_i \) do not contain \( x_0 \). Similarly to the previous case, we obtain

\[
(2.4) \quad \alpha''_3 x_1^2 + \alpha''_4 \cdot 2x_1 x_2 = 0.
\]

This means that there exists some \( \beta \) such that \( \alpha''_3 = -2x_2 \beta \) and \( \alpha''_4 = x_1 \beta \). Now

\[
\alpha_1 x_0^2 + 2 \cdot 2x_0 x_1 + (\alpha'_3 x_0 - 2x_2 \beta)(2x_0 x_2 + x_1^2) + (\alpha'_4 x_0 + x_1 \beta)(2x_0 x_3 + 2x_1 x_2) = 0.
\]

The terms without \( x_0 \) cancel, and the linear terms in \( x_0 \) are the following:

\[
x_0(2\alpha_2 x_1 + \alpha'_3 x_1^2 - 4x_2^2 \beta + 2\alpha'_4 x_1 x_2 + 2\beta x_1 x_3) = 0.
\]

Note that all terms but \(-4x_2^2 \beta \) are divisible by \( x_1 \), so \( \beta \) is divisible by \( x_1, \beta = mx_1 \). Then

\[
\alpha_4 = \alpha'_4 x_0 + mx_1^2 = (\alpha'_4 - 2x_2 m)x_0 + mf_3.
\]

By subtracting \( mv_{3,4} + \frac{1}{3}(\alpha'_4 - 2x_2 m) \mu_3 \) from \( \alpha \), we obtain a syzygy between \( f_1, f_2, f_3 \) and reduce to the previous case.
2.4. Syzygies. In this section, we prove Theorem 2.2 by induction on \( n \). The base cases were covered in Section 2.3. Suppose that \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \text{Ker}(\phi_n) \), i.e., is a linear relation between \( f_1, \ldots, f_n \). As above, write \( \alpha_i = \alpha'_i x_0 + S(\alpha''_{i-2}) \) for \( i \geq 3 \). Without loss of generality, we can assume that \( \alpha'_i \) do not contain \( x_0 \) (otherwise we can subtract a multiple of \( \nu_{1,i} \)). Since

\[
f_i = 2x_0x_{i-1} + S(f_{i-2}) ,
\]

by collecting terms without \( x_0 \) we get \( \sum_{i=3}^n S(\alpha''_{i-2})S(f_{i-2}) = 0 \). This means that \( \phi_{n-2}(\alpha'') = 0 \) and by the induction assumption we may then write

\[
\alpha'' = \sum_{i=3}^{n-1} \beta_{i+1} \mu_{i-2} + \sum_{3 \leq j < k \leq n, j \neq k} \beta_{j,k} \nu_{j-2,k-2}.
\]

Because

\[
S(\nu_{j-2,k-2}) = -S(f_{k-2})e_j + S(f_{j-2})e_k = \nu_{j,k} + 2x_0x_ke_j - 2x_0x_je_k,
\]

without loss of generality we can assume \( \alpha'' = S(\sum_{i=3}^{n-1} \beta_{i+1} \mu_{i-2}) \). By Corollary 2.5 we get

\[
\phi_n(S(\mu_{i-2})) = -(i + 1)x_0S(f_i) + (2i - 1)x_{i-1}f_2 ,
\]

hence

\[
\phi_n(\alpha) = \alpha_1 f_1 + (\alpha_2 + \sum_{i=3}^{n-1} (2i - 1) S(\beta_{i+1})x_{i-1})f_2 + \sum_{i=3}^n x_0 \alpha'_i f_i - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1})x_0S(f_i) = 0 .
\]

By collecting the terms linear in \( x_0 \), we get

\[
(\alpha_2 + \sum_{i=3}^{n-1} (2i - 1) S(\beta_{i+1})x_{i-1})2x_1 + \sum_{i=3}^n \alpha'_i S(f_{i-2}) - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1})S(f_i) = 0 ,
\]

so

\[
\sum_{i=3}^n \alpha'_i S(f_{i-2}) - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1})S(f_i)
\]

is divisible by \( x_1 \), and

\[
\sum_{i=3}^n \alpha''_i f_{i-2} - \sum_{i=3}^{n-1} (i + 1) \beta_{i+1} f_i
\]

is divisible by \( x_0 \), where \( \alpha'_i = S(\alpha''_{i}) \). By Lemma 2.8, this implies

\[
\beta_n = Bx_0 + Cx_1 + \sum_{i=3}^{n-2} \gamma_i f_i
\]

for some constants \( B, C \). Now we can rewrite

\[
\alpha_n = \alpha'_n x_0 + S(\beta_n x_0) = \alpha'_n x_0 + Bx_1^2 + Cx_1x_2 + \sum_{i=3}^{n-3} \gamma_i x_1(f_{i+2} - 2x_0x_{n-1}) + \gamma_{n-2} x_1 S(f_{n-2}) .
\]

Observe that \( x_1^2 = f_3 - 2x_0x_2, x_1x_2 = \frac{1}{2}(f_3 - 2x_0x_3) \) and by Lemma 2.4, \( x_1 S(f_{n-2}) \) can be expressed via \( f_1, \ldots, f_{n-1} \) modulo \( x_0 \). In other words,

\[
\alpha_n = \delta x_0 + \sum_{i=3}^{n-1} \delta_i f_i
\]

for some coefficients \( \delta_i \). Then \( \alpha - \frac{1}{n-1} \delta \mu_{n-1} - \sum_{i=3}^{n-1} \delta_i \nu_{i,j} \) is a syzygy between \( f_1, \ldots, f_{n-1} \), so by the induction assumption it can be expressed as an \( R_{n-1} \)-linear combination of the \( \mu_i \) and \( \nu_{i,j} \).
Lemma 3.1. One has
\[ R_n/(x_0R_n + I_n) \simeq S(R_{n-2}/I_{n-2})[x_{n-1}] \]
as $R_n$-modules, the module structure on the right coming from $S : R_{n-1} \to R_n$.

Proof. We have $x_0R_n + I_n = \langle x_0, f_1, \ldots, f_n \rangle = \langle x_0, S(f_1), \ldots, S(f_{n-2}) \rangle$, so
\[ R_n/(x_0R_n + I_n) = R_n/\langle x_0, S(f_1), \ldots, S(f_{n-2}) \rangle = S(R_{n-2}/I_{n-2})[x_{n-1}]. \]

Lemma 3.2. The subspace $x_0S^2(I_{n-3})[x_{n-1}]$ does not intersect the ideal $\langle f_1, f_2 \rangle$ in $R_n$. Furthermore, $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$ is an ideal in $R_n$ which is contained in $I_n \cap x_0R_n$.

Proof. Given a nonzero polynomial $g \in I_{n-3}$, the iterated shift $S^2(g)$ does not contain $x_0$ or $x_1$, so that $x_0S^2(g)$ is not contained in $\langle f_1, f_2 \rangle$. Furthermore, $I_{n-3}$ is stable under multiplication by $x_0, \ldots, x_{n-4}$, so $S^2(I_{n-3})$ is stable under multiplication by $x_2, \ldots, x_{n-2}$, and $x_0S^2(I_{n-3})[x_{n-1}]$ is stable under multiplication by $x_2, \ldots, x_{n-1}$. Multiplication by $x_0$ or $x_1$ sends the latter subspace to $\langle f_1, f_2 \rangle$, so $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$ is an ideal in $R_n$.

Finally, to prove that this ideal is contained in $I_n$, it is sufficient to prove that $x_0S^2(f_k) \in I_n$ for $k \leq n - 3$. On the other hand, by Corollary 2.5
\[ x_0S^2(f_k) = \frac{1}{k+3}\phi_n(S(\mu_k)) \mod \langle f_1, f_2 \rangle. \]

\[ I_n \cap x_0R_n = x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle. \]

Proof. By Lemma 3.2, the right hand side is a submodule of the left hand side, so it remains to prove the reverse inclusion. We have
\[ f_i = 2x_0x_{i-1} + S(f_{i-2}) = 2x_0x_{i-1} + 2x_1x_{i-2} + S^2(f_{i-4}). \]

Suppose that $\sum_{i=1}^n \alpha_if_i \in I_n \cap x_0R_n$. Then by Lemma 2.8
\[ \alpha_n = A'x_0 + B'x_1 + \sum_j \gamma_jf_j = A'x_0 + B'x_1 + \sum_j \gamma_jS^2(f_{j-4}). \]

Now by (2.1) and Corollary 2.5, $x_0f_n$ and $x_1f_n$ can be expressed as $R_n$-linear combinations of $f_1, \ldots, f_{n-1}$ and elements of $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$, so $\sum_{i=1}^n \alpha_if_i$ can be expressed as such a combination as well. Induction on $n$ finishes the proof.

Corollary 3.4. One has
\[ x_0R_n/(I_n \cap x_0R_n) = x_0S^2(R_{n-3}/I_{n-3})[x_{n-1}]. \]
Proof. We have
\[ x_0R_n/(f_1, f_2) = x_0R_n/(x_0^2, x_0x_1) = x_0k[x_2, \ldots, x_{n-1}] = x_0S^2(R_{n-3})[x_{n-1}] \]

Therefore
\[ x_0R_n/(I_n \cap x_0R_n) = x_0R_n/(x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle) = x_0S^2(R_{n-3}/I_{n-3})[x_{n-1}] \]

\[ \square \]

**Theorem 3.5.** Let \( H_n(q, t) \) denote the bigraded Hilbert series of the quotient \( R_n/I_n \). Then one has the following recursion relation
\[
H_n(q, t) = \frac{H_{n-2}(q, qt) + tH_{n-3}(q, q^2t)}{1 - q^{n-1}t}
\]
with initial conditions
\[
H_0(q, t) = 1, \quad H_1(q, t) = 1 + t, \quad H_2(q, t) = \frac{1}{1 - qt} + t.
\]

**Remark 3.6.** This recursion is similar, but not identical to the various recursions considered by Andrews [1, 2, 3] in his proofs of the Rogers-Ramanujan identity. It is also similar to the recursions recently considered by Paramonov [12] in a different context.

**Proof.** We have an exact sequence
\[ 0 \rightarrow x_0R_n/(x_0R_n \cap I_n) \rightarrow R_n/I_n \rightarrow R_n/(x_0R_n + I_n) \rightarrow 0. \]

By Lemma 3.1 the Hilbert series of \( R_n/(x_0R_n + I_n) \) equals \( H_{n-2}(q, qt) \), and by Corollary 3.4 the Hilbert series of \( x_0R_n/(x_0R_n \cap I_n) \) equals \( \frac{tH_{n-3}(q, q^2t)}{1 - q^{n-1}t} \). \[ \square \]

4. Gröbner bases

We will now compute Gröbner bases for the ideals \( I_n \). Recall that a Gröbner basis for an ideal \( I \) is a subset \( G = \{g_1, \ldots, g_s\} \subseteq I \) such that, for a chosen monomial ordering \( < \),
\[
\langle \text{LT}_<(g_1), \ldots, \text{LT}_<(g_s) \rangle = \text{LT}_<(I),
\]
where \( \text{LT}_< \) denotes leading term.

Let us order the monomials in \( R_n \) in grevlex order, that is
\[ x^\alpha < x^\beta \]
if \( |\alpha| < |\beta| \) or \( |\alpha| = |\beta| \) and the rightmost entry of \( \alpha - \beta \) is negative.

**Remark 4.1.** In fact, any order refining the reverse lexicographic order will work, but for definiteness and its popularity in computer algebra systems we shall fix grevlex order throughout.

**Theorem 4.2.** Let
\[ G_1 = \{f_1\} \subseteq R_1, G_2 = \{f_1, f_2\} \subset R_2 \]
and recursively define the sets \( G_n, n \geq 3 \) as follows:
\[ G_n = x_0S^2(G_{n-3}) \cup \{f_1, f_2\} \cup \tilde{S}(G_{n-2}), \]
where \( \tilde{S} \) is a modified shift operator as explained below. Then \( G_n \) is a Gröbner basis for \( I_n \).
Remark 4.3. The notation requires explanation. Note that any \( G_m \) is naturally a subset of \( R_n \), \( n \geq m \) so we can and will identify \( G_m \) inside a larger polynomial ring without explicit mention. Furthermore, we denote by \( x_0S^2(G_{n-3}) \) the image of \( G_{n-3} \) under \( S^2 : R_{n-2} \to R_n \) multiplied by \( x_0 \). The “operator” \( \tilde{S} \) is defined on elements \( p \in I_{n-2} \) as follows: write \( p = \sum_{i=1}^n \varphi_i f_i \), and let

\[
\tilde{S}(p) = \sum_{i=1}^n S(\varphi_i)f_{i+2}.
\]

Note that by (2.2), we have \( \tilde{S}(p) = S(p) + \sum_{i=1}^n x_0x_{i+2}S(\varphi_i) \in I_{n+2} \). In particular, if \( p \neq 0 \) and \( p \) is homogeneous then \( \text{LT}_< (\tilde{S}(p)) = S(\text{LT}_<(p)) \). Therefore the construction of \( \tilde{S}(p) \) requires a choice if \( \varphi_i \), but the leading term of the result does not depend on this choice.

Proof. We will proceed by induction. The base cases \( n = 1, 2 \) are clear because the ideals are monomial. Consider now the ideal \( \text{LT}_<(I_n) \) generated by all the leading terms of elements of \( I_n \). It is clear by Lemma [5.1] and the fact that \( S \) respects the reverse lexicographic order that if \( g \in I_n \) is not divisible by \( x_0 \), its leading term is the image of a leading term in \( I_{n-2} \) under \( \tilde{S} \). Since we assumed \( G_{n-2} \) to be a Gröbner basis, we must have \( \text{LT}_<(g) \) divisible by some monomial in \( S(\text{LT}_<(G_{n-2})) \).

Similarly, if \( g \) is divisible by \( x_0 \), we know by Lemma [5.2] and order preservation that its leading term is the image under \( x_0S^2 \) of a leading term in \( I_{n-3} \) or divisible by \( f_1, f_2 \). By the induction assumption \( \text{LT}_<(g) \) is then divisible by an element of \( x_0S^2(\text{LT}_<(G_{n-3})) \) \( \subseteq \langle f_1, f_2 \rangle \). In particular, \( \text{LT}_<(I_n) \subseteq \langle \text{LT}_<(G_n) \rangle \). But the reverse inclusion is clear, so we have

\[
\text{LT}_<(I_n) = \langle \text{LT}_<(G_n) \rangle
\]

as desired, and \( G_n \) is a Gröbner basis for \( I_n \). \( \Box \)

Example 4.4. We have

\[
\begin{align*}
G_3 &= \{f_1, f_2, f_3\} \\
G_4 &= \{f_1, f_2, f_3, f_4, x_0x_2^2\} \\
G_5 &= \{f_1, f_2, f_3, f_4, f_5, x_0x_2x_3\} \\
G_6 &= \{f_1, \ldots, f_6, x_0x_2^2 + 2x_0x_2x_4, 2x_1x_3^2 + 3x_0x_3x_4 - x_0x_2x_5\}.
\end{align*}
\]

Note that the last polynomial in \( G_6 \) can be identified with \( \tilde{S}(x_0x_2^2) \in \tilde{S}(G_4) \). Indeed,

\[
4x_0x_2^2 = 2x_2(2x_0x_2 + x_1^2) - x_1(2x_0x_3 + 2x_1x_2) + x_3(2x_0x_1) = 2x_2f_3 - x_1f_4 + x_3f_2,
\]

so

\[
\tilde{S}(4x_0x_2^2) = 2x_3f_5 - x_2f_6 + x_4f_4 = 2x_3(2x_0x_4 + 2x_1x_3 + x_2^2) - x_2(2x_0x_5 + 2x_1x_4 + 2x_2x_3) + x_4(2x_0x_3 + 2x_1x_2) = 4x_1x_3^2 + 6x_0x_3x_4 - 2x_0x_2x_5.
\]

Remark 4.5. The Gröbner basis constructed in Theorem 4.2 is far from being reduced. The following theorem describes the reduced basis implicitly.

Since all \( G_n \) contain \( \{f_1, \ldots, f_n\} \) and none of their leading terms divides one another, we can throw away other polynomials in \( G_n \) in a controlled manner to obtain a minimal Gröbner basis. That is to say, if the leading terms of \( G_n \setminus \{g\} \) still generate the leading ideal we are in business. Therefore after appropriate reduction [7, Proposition 6 on p. 92] we get a reduced Gröbner basis with the same leading terms.
Let us call a monomial $\prod x_i^{a_i}$ admissible if $a_i + a_{i+1} \leq 1$ for all $i$, that is, it is not divisible by $x_i^2$ or by $x_ix_{i+1}$.

**Theorem 4.6.** Fix $k > 2$. The leading terms of $(t)$-degree $k$ in a reduced Gröbner basis for $I_n$ have the form $m(x)\text{LT}_<(f_{n+k-2})$ where $m(x)$ is an admissible monomial of degree $k - 2$ in variables $x_0, \ldots, x_{\left\lfloor \frac{n-1}{k-2} \right\rfloor}$. The number of degree $k$ polynomials in the reduced Gröbner basis equals $\left( \frac{n-1}{k-2} \right)^t$.

**Remark 4.7.** It is easy to see that there are no linear polynomials in the Gröbner basis (or in the ideal $I_n$), and $f_1, \ldots, f_n$ are the only quadratic polynomials in the reduced Gröbner basis.

**Proof.** We prove the statement by induction in $n$. Suppose that it is true for $G_{n-2}$ and $G_{n-3}$. By Theorem 4.2, the leading monomials in the degree $k$ part of $G_n$ consist of shifted degree $k$ monomials in $G_{n-2}$, and twice shifted degree $(k-1)$ monomials in $G_{n-3}$, multiplied by $x_0$.

Consider first the case $k = 3$. We will prove that the leading terms in the reduced Gröbner basis have the form $x_j \text{LT}_<(f_{n+1})$ for $j \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Indeed, in the first case we get $S(x_j \text{LT}_<(f_{n+2})) = x_j+1 \text{LT}_<(f_{n+1})$. In the second case we have to consider the polynomials $x_0S^2(f_i)$ for all $i \leq n-3$. Observe that for $i \leq n-4$ we get $\text{LT}_<(x_0S^2(f_i)) = x_0 \text{LT}_<(f_{i+4})$ and hence divisible by the leading term of $f_{i+4}$ and can be eliminated. For $i = n-3$ we get $\text{LT}_<(x_0S^2(f_{n-3})) = x_0 \text{LT}_<(f_{n+1})$.

Assume now that $k > 3$. In the first case we get

$$S(m(x) \text{LT}_<(f_{(n-2)+k-2})) = S(m(x)) \text{LT}_<(f_{n+k-2}).$$

If $m(x)$ is an admissible monomial in $x_j$, $0 \leq j \leq \left\lfloor \frac{n-2+k-7}{2} \right\rfloor$ then $S(m(x))$ is an admissible monomial in $x_j$, $1 \leq j \leq \left\lfloor \frac{n-2+k-7}{2} \right\rfloor + 1 = \left\lfloor \frac{n+k-7}{2} \right\rfloor$.

In the second case we get

$$x_0S^2(m(x)) \text{LT}_<(f_{(n-3)+(k-1)-2}) = x_0S^2(m(x)) \text{LT}_<(f_{n+k-2}).$$

Now $S^2(m(x))$ is an admissible monomial in $x_j$, $2 \leq j \leq \left\lfloor \frac{n-3+(k-1)-7}{2} \right\rfloor + 2 = \left\lfloor \frac{n+k-7}{2} \right\rfloor$, so $x_0S^2(m(x))$ is also an admissible in a correct set of variables. In fact, all such monomials not divisible by $x_0$ appear from the first case, and the ones divisible by $x_0$ appear from the second case.

It is easy to see that none of these leading monomials are divisible by each other. Therefore after appropriate reduction we get a reduced Gröbner basis with the same leading terms.

Finally, we can count monomials of given degree $k$. The number of admissible monomials of degree $l$ in $s$ variables equals $\left( \frac{s-1}{l} \right)^t$, so the number of polynomials in $G_n$ of degree $k$ equals

$$\left(1 + \left\lfloor \frac{n+k-7}{2} \right\rfloor - (k-2) + 1 \right) \binom{n+k-1}{k-2}.$$ 

\qed

**Example 4.8.** Let $n = 12$. The reduced Gröbner basis for $I_{12}$ contains quadratic polynomials $f_1, \ldots, f_{12}$. It also contains 5 cubic polynomials with leading terms $x_0x_6^2, x_1x_6^2, x_2x_6^2, x_3x_6^2, x_4x_6^2$, 6 quartic polynomials with leading terms $x_0x_2x_6x_7, x_0x_3x_6x_7, x_0x_4x_6x_7, x_1x_3x_6x_7, x_1x_4x_6x_7, x_2x_4x_6x_7$. 


and 4 quintic polynomials with leading terms
\[ x_0x_2x_4x_5^2, x_0x_2x_5x_7^2, x_0x_3x_5x_7^2, x_1x_3x_5x_7^2. \]
Observe that \( \text{LT}_<(f_{13}) = x_0^2, \text{LT}_<(f_{14}) = x_0x_7 \) and \( \text{LT}_<(f_{15}) = x_7^2. \)

5. Minimal resolution

In this section we describe the bigraded minimal free resolutions of \( I_n \) and \( R_n/I_n \). We write them as follows:

\[
0 \leftarrow I_n \leftarrow F(1, n) \leftarrow F(2, n) \leftarrow F(3, n) \cdots \]

and

\[
0 \leftarrow R_n/I_n \leftarrow R_n = F(0, n) \leftarrow F(1, n) \leftarrow F(2, n) \leftarrow F(3, n) \cdots \]

**Theorem 5.1.** Let \( F(i, n) \) be the \( i \)-th term in the minimal free resolution for \( I_n \). Then there is an injection \( F(i, n - 1) \hookrightarrow F(i, n) \), and

\[
F(i, n)/F(i, n - 1) \cong S(F(i - 1, n - 3)) \oplus x_0S(F(i - 2, n - 3))
\]
as \( R_n \)-modules, and the shift of a free \( R_n \)-module is as in (2.3). Note that the gradings in the right hand side are shifted by the bidegree of \( f_n \) (which equals \( q^{n-1}t^2 \)).

**Proof.** Observe that the ideal generated by \( f_1, \ldots, f_{n-1} \) in \( R_n \) is isomorphic to \( I_{n-1}[x_{n-1}] \), so its minimal resolution over \( R_n \) is identical to the one for \( I_{n-1} \) over \( R_{n-1} \) tensored over \( R_n \). Moreover, since \( I_n = \langle f_1, \ldots, f_n \rangle \), the minimal free \( R_n \)-resolution of \( I_{n-1}[x_{n-1}] \) is naturally a subcomplex of the minimal free resolution for \( I_n \). In other words, \( F(i, n - 1) \otimes_{R_{n-1}} R_n \) can be identified with a subspace in \( F(i, n) \), which we will by abuse of notation also denote \( F(i, n - 1) \). We have a short exact sequence

\[
0 \to F(i, n - 1) \to F(i, n) \to F(i, n)/F(i, n - 1) \to 0.
\]

From the long exact sequence in cohomology, it is easy to see that \( F(i, n)/F(i, n - 1) \) is acyclic in positive degrees. Now \( I_n = \langle f_1, \ldots, f_n \rangle \), so \( F(1, n)/F(1, n - 1) \cong R_n \) is generated by a single vector corresponding to \( f_n \). Furthermore, by Theorem 2.2, \( F(2, n) \) has generators corresponding to \( \mu_1, \ldots, \mu_{n-1} \) and \( \nu_i, j \) for \( 3 \leq i < j \leq n \), so \( F(2, n)/F(2, n - 1) \cong R^{n-2}_n \) is spanned by the basis elements corresponding to \( \mu_{n-1} \) and \( \nu_{i, n} \) for \( 3 \leq i \leq n - 1 \). The differential \( d : F(2, n) \to F(1, n) \) descends to \( d : F(2, n)/F(2, n - 1) \to F(1, n)/F(1, n - 1) \), sending \( \mu_{n-1} \) to \( x_0f_n \) and \( \nu_{i, n} \) to \( f_i \cdot f_n \).

Therefore, the quotient complex with terms \( F(i, n)/F(i, n - 1) \) is isomorphic to the minimal resolution of \( R_n/\langle x_0, f_3, \ldots, f_{n-1} \rangle = R_n/\langle x_0, S(f_1), \ldots, S(f_{n-3}) \rangle \). The latter is nothing but the (shifted) minimal resolution for \( I_{n-3} \) tensored with the two-term complex \( R_n/\mathfrak{x}_n R_n \). \( \square \)

**Corollary 5.2.** Let \( b(i, n) \) denote the rank of \( F(i, n) \). Then

\[
b(i, n) = b(i, n - 1) + b(i - 1, n - 3) + b(i - 2, n - 3).
\]

**Corollary 5.3.** Let \( H_n(q, t) \) denote the Hilbert series for \( R_n/I_n \), and let \( \tilde{H}_n(q, t) = H_n(q, t) \prod_{i=0}^{n-1} (1 - q^it) \). Then \( \tilde{H}_n(q, t) \) satisfies the following recursion relation:

\[
\tilde{H}_n(q, t) = \tilde{H}_{n-1}(q, t) - q^{n-1}t^2(1 - t^2)\tilde{H}_{n-3}(q, qt).
\]

**Corollary 5.4.** The projective dimension of \( I_n \) equals \( \lceil \frac{2n}{3} \rceil - 1 \). The projective dimension of \( R_n/I_n \) equals \( \lceil \frac{2n}{3} \rceil \).
Proof. By definition, the projective dimension pd($I_n$) is equal to the length of the minimal free (or projective) resolution. By (5.1) we have pd($I_n$) = pd($I_{n-3}$) + 2. The minimal free resolutions for $I_1$, $I_2$ and $I_3$ are easy to compute:

$$I_1 \xleftarrow{(f_1)} R_1$$

$$I_2 \xleftarrow{(f_1, f_2)} R_2$$

$$I_3 \xleftarrow{(f_1, f_2, f_3)} R_3$$

The minimal resolution of $R_n/I_n$ is one step longer than the one for $I_n$. □

6. COMBINATORIAL IDENTITIES

We define

$$\binom{a}{b}_q = \frac{(1 - q) \cdots (1 - q^a)}{(1 - q) \cdots (1 - q^b)}.$$  

If $a < b$, we set $\binom{a}{b}_q = 0$. The following lemma is well known.

Lemma 6.1. The following identities hold:

$$\binom{a}{b}_q + q^{b+1} \binom{a}{b+1}_q = \binom{a+1}{b+1}_q = q^{a-b} \binom{a}{b}_q.$$  

Proof. One has

$$\binom{a}{b+1}_q = \frac{(1 - q^{a-b})}{(1 - q^{b+1})} \binom{a}{b}_q,$$

hence

$$\binom{a}{b}_q + q^{b+1} \binom{a}{b+1}_q = \binom{a}{b}_q \left(1 + q^{b+1} \frac{1 - q^{a-b}}{1 - q^{b+1}}\right) = \binom{a}{b}_q \frac{1 - q^{a+1}}{1 - q^{b+1}} = \binom{a+1}{b+1}_q.$$  

□

Theorem 6.2. The Hilbert series $H_n(q, t)$ is given by the following explicit formula:

$$H_n(q, t) = \sum_{p=0}^{\infty} \binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)t^p} \cdot \left(1 - q^{n-h(n,p)t}\right) \cdots \left(1 - q^{n-1t}\right).$$  

where $h(n, p) = \left\lfloor \frac{n-p}{2} \right\rfloor$.

Proof. By Theorem 3.5 it is sufficient to prove that the right hand side of (6.1) satisfies the recursion relation (3.14). Let us denote the $p$-th term in (6.1) by $H_{n,p}(q, t)$ so that $H_n(q, t) = \sum_p H_{n,p}(q, t)$. We have $h(n - 2, p) = h(n - 3, p - 1) = h(n, p) - 1$, so

$$H_{n-2p}(q, t) = \binom{h(n,p)}{p}_q \cdot q^{p(p-1)t^p} \cdot q^{p} \cdot \left(1 - q^{n-h(n,p)t}\right) \cdots \left(1 - q^{n-2t}\right),$$

$$H_{n-3p-1}(q, q^2t) = \binom{h(n,p)}{p-1}_q \cdot q^{(p-1)(p-2)t^p-1} \cdot q^{2p-2} \cdot \left(1 - q^{n-h(n,p)t}\right) \cdots \left(1 - q^{n-2t}\right).$$
therefore

\begin{equation}
(6.2) \quad H_{n-2,p}(q, qt) + tH_{n-3,p-1}(q, q^2 t) = \frac{q^{p(p-1)p}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})} \left[ q^p \binom{h(n,p)}{p}_q + \binom{h(n,p)}{p-1}_q \right] = \frac{q^{p(p-1)p}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})} \binom{h(n,p) + 1}{p}_q = (1 - q^{n-1t})H_{n,p}(q, t).
\end{equation}

This proves (6.1), and the initial conditions are easy to check. \( \square \)

The free resolution of \( I_n \) gives another formula for the Hilbert series of \( R_n/I_n \).

**Proposition 6.3.** Let \( b(i, n) \), as above, denote the rank of \( i \)-th module in the free resolution of \( R_n/I_n \). Then

\[ b(i, n) = \sum_p \left[ \binom{n - 2p + 1}{p}_p \binom{p}{i - p} + \binom{n - 2p - 1}{p}_p \binom{p}{i - p - 1} \right] \]

**Remark 6.4.** The terms in the first sum are nonzero if \( p \leq (n + 1)/3 \) and \( i/2 \leq p \leq i \). The terms in the second sum are nonzero if \( p \leq (n - 1)/3 \) and \( (i - 1)/2 \leq p \leq (i - 1) \).

**Proof.** Let

\[ A(n, p, i) = \binom{n - 2p + 1}{p}_p \binom{p}{i - p}, B(n, p, i) = \binom{n - 2p - 1}{p}_p \binom{p}{i - p - 1}. \]

Then

\[ A(n - 1, p, i) + A(n - 3, p - 1, i - 1) + A(n - 3, p - 1, i - 2) = \binom{n - 2p}{p}_p \binom{p}{i - p} + \binom{n - 2p}{p - 1}_p \binom{p - 1}{i - p} + \binom{n - 2p}{p - 1}_p \binom{p - 1}{i - p - 1} = \binom{n - 2p}{p}_p \binom{p}{i - p} + \binom{n - 2p}{p - 1}_p \binom{p}{i - p} = \binom{n - 2p + 1}{p}_p \binom{p}{i - p} = A(n, p, i). \]

Similarly, \( B(n - 1, p, i) + B(n - 3, p - 1, i - 1) + B(n - 3, p - 1, i - 2) = B(n, p, i) \), so the right hand side satisfies the recursion relation (6.1). It remains to check the base cases:

\[ f(0, n) = 1 = \binom{n - 1}{0}, \]

\[ f(1, n) = n = \binom{n - 1}{1} + \binom{n - 3}{0}, \]

\[ f(2, n) = (n - 1) + \binom{n - 2}{2} = \binom{n - 1}{1} + \binom{n - 3}{1} + \binom{n - 3}{2}. \]

By Corollary 5.4 \( b(i, n) = 0 \) for \( i > 2 \) and \( n \leq 3 \). \( \square \)

We have the following \( (q, t) \)-analogue of Proposition 6.3.
Proposition 6.5. Let \( \hat{b}(i, n) \) denote the bigraded Hilbert polynomial for the generating set in \( F(i, n) \). Then

\[
\hat{b}(i, n) = \sum_{p>0} q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{n-2p+1}{p} \binom{p}{i-p}_{q} + q^{\frac{5p^2+5p+(i-p)(i-p-1)}{2}} t^{2p+2+(i-p)} \binom{n-2p-1}{p} \binom{p}{i-p-1}_{q}.
\]

Proof. The proof is completely analogous to the proof of Proposition 6.3, but we include it here for completeness. By Theorem 5.1, we have a recursion relation

\[
\hat{b}(i, n) = \hat{b}(i, n-1) + q^{n-1}t^2 \hat{b}(i-1, n-3)(q, qt) + q^{n-1}t^3 \hat{b}(i-2, n-3)(q, qt).
\]

We need to prove that the right hand side of (6.3) satisfies (6.4). Let

\[
\hat{A}(n, p, i) = q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{n-2p+1}{p} \binom{p}{i-p}_{q}.
\]

Then

\[
\hat{A}(n-3, p-1, i-1)(q, qt) = q^{\frac{5p^2-9p+4+(i-p)(i-p+1)}{2}} t^{2p-2+(i-p)} \binom{n-2p}{p-1} \binom{p-1}{i-p}_{q},
\]

\[
\hat{A}(n-3, p-1, i-2)(q, qt) = q^{\frac{5p^2-9p+4+(i-p)(i-p-1)}{2}} t^{2p-2+(i-p-1)} \binom{n-2p}{p-1} \binom{p-1}{i-p-1}_{q},
\]

so

\[
\hat{A}(n-3, p-1, i-1)(q, qt) + t\hat{A}(n-3, p-1, i-2)(q, qt) = q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+2+(i-p)} \binom{n-2p}{p-1} \binom{p}{i-p}_{q}.
\]

Now

\[
\hat{A}(n-1, p, i) + q^{n-1}t^2 \hat{A}(n-3, p-1, i-1)(q, qt) + q^{n-1}t^3 \hat{A}(n-3, p-1, i-2)(q, qt) = q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \left[ \binom{n-2p}{p} \binom{p}{i-p}_{q} + q^{n-3p+1} \binom{n-2p}{p-1} \binom{p}{i-p}_{q} \right] = q^{\frac{5p^2-3p+(i-p)(i-p-1)}{2}} t^{2p+(i-p)} \binom{n-2p+1}{p} \binom{p}{i-p}_{q} = \hat{A}(n, p, i)
\]

A similar recursion holds for \( \hat{B}(n, p, i) \). It remains to check the initial conditions:

\[
\hat{b}(0, n) = 1,
\]

\[
\hat{b}(1, n) = (t^2 + qt^2 + \ldots + q^{n-1}t^2) = qt^2 \binom{n-1}{1}_{q} + t^2 \binom{n-3}{0}_{q},
\]

\[
\hat{b}(2, n) = qt^3[n-1]_{q} + q^5 t^4 \binom{n-2}{2}_q = qt^3 \binom{n-1}{1}_{q} + q^5 t^4 \binom{n-3}{1}_{q} + q^7 t^4 \binom{n-3}{2}_{q}.
\]

The following result was conjectured by the second author, Oblomkov and Rasmussen in [5] Conjecture 4.1.
Theorem 6.6. The Hilbert series of $R_n/I_n$ has the following form:

\begin{equation}
H_n(q, t) = \frac{1}{\prod_{i=0}^{n-1} (1 - q^i t)} \sum_{p=0}^\infty (-1)^p \prod_{k=0}^{p-1} (1 - q^k t) \times \left( q^{3p^2 - 3p + i} q^p \binom{n - 2p + 1}{p} q^{3p^2 + 5p} q^{2p + 2} \binom{n - 2p - 1}{p} q^2 \right).
\end{equation}

Proof. It is clear that $H_n(q, t) = \frac{1}{\prod_{i=0}^{n-1} (1 - q^i t)} \sum_{p=0}^\infty (-1)^p \hat{b}(i, n)$. The latter can be computed by (6.3), and it remains to use the identity

\[ p - 1 \prod_{k=0}^{p-1} (1 - q^k t) = \sum_{j=0}^{p} (-1)^j q^{j(j-1)/2} \binom{p}{j}. \]

\[ \square \]

7. LIMIT AT $n \to \infty$

In the limit $n \to \infty$ both formulas for the Hilbert series simplify. Indeed, for fixed $p$ we have

\[ \lim_{n \to \infty} \binom{n}{p}_q = \frac{1}{(1 - q) \cdots (1 - q^{p})}, \]

so we can take the limit of all the above results.

Proposition 7.1. The limit of the Hilbert series $H_n(q, t)$ has the following form:

\begin{equation}
H_\infty(q, t) = \sum_{p=0}^\infty \frac{q^{p(p-1)} q^p}{(1 - q)(1 - q^2) \cdots (1 - q^p)}.
\end{equation}

Proposition 7.2. The limit of the bigraded rank of the $i$-th syzygy module $F(i, n)$ equals

\begin{equation}
\hat{b}(i, \infty) = \sum_{p>0} \frac{q^{3p^2 - 3p + (i-p)(i-p-1)}}{2} q^{2p + (i-p)} \binom{p}{i-p}_q \frac{1}{(1 - q) \cdots (1 - q^p)} + q^{3p^2 + 5p + (i-p)(i-p-1)} q^{2p + 2 + (i-p)} \binom{p}{i-p-1}_q \frac{1}{(1 - q) \cdots (1 - q^p)}
\end{equation}

Proposition 7.3. The limit of the Hilbert series $H_n(q, t)$ has the following form:

\begin{equation}
H_n(q, t) = \frac{1}{\prod_{i=0}^{\infty} (1 - q^i t)} \sum_{p=0}^\infty (-1)^p \prod_{k=0}^{p-1} \frac{1 - q^k t}{1 - q^{k+1}} \left( q^{3p^2 - 3p} t^{2p} - q^{3p^2 + 5p} t^{2p+2} \right).
\end{equation}

The equality between the right hand sides of (7.3) and (7.1) was proved in [10, Theorem 3.3.2(b)]. At $t = 1$ and $t = q$ one recovers more familiar Rogers-Ramanujan identities.

The following proposition concerning Gröbner bases in the limit was proved first in [4], but we give an alternative proof here. In fact, [4] use a slightly different basis of Bell polynomials. Yet another proof can be obtained by taking the limit in Theorem 4.6.

Proposition 7.4. For $n \to \infty$ the polynomials $f_i$ form a Gröbner basis for the ideal $I_\infty$. 
Before embarking on the proof, we record the following lemmas concerning Gröbner bases here for the convenience of the reader.

**Lemma 7.5** ([7] Proposition 8 on p. 106). Given \((g_1, \ldots, g_s) \in F_s\), the \(S\)-pairs

\[
S_{ij} := \frac{\text{lcm}(\text{LT}_{<}(g_i), \text{LT}_{<}(g_j))}{\text{LT}_{<}(g_i)} e_i - \frac{\text{lcm}(\text{LT}_{<}(g_i), \text{LT}_{<}(g_j))}{\text{LT}_{<}(g_j)} e_j
\]

form a homogeneous basis for the syzygies on \(\{\text{LT}_{<}(g_1), \ldots, \text{LT}_{<}(g_s)\}\).

**Lemma 7.6** ([7] Proposition 9 on p. 107). Let \(I = \langle g_1, \ldots, g_s \rangle\). Then \(G = \{g_1, \ldots, g_s\}\) is a Gröbner basis for \(I\) if and only if every element of a homogeneous basis for the syzygies on \(\text{LT}_{<}(G)\) reduces to zero modulo \(G\).

**Lemma 7.7** ([7] Proposition 4 on p.103). \(G = \{g_1, \ldots, g_s\} \subset R_n\), and suppose \(g_i, g_j \in G\) have relatively prime leading monomials. Then the \(S\)-polynomial

\[
S(g_i, g_j) := \phi_n(S_{ij}) = \frac{\text{lcm}(\text{LT}_{<}(g_i), \text{LT}_{<}(g_j))}{\text{LT}_{<}(g_i)} g_i - \frac{\text{lcm}(\text{LT}_{<}(g_i), \text{LT}_{<}(g_j))}{\text{LT}_{<}(g_j)} g_j
\]

reduces to zero modulo \(G\).

**Proof of Proposition [7,4]**. Consider \(S(f_i, f_j)\). By Lemma 7.7, \(\text{gcd}(\text{LT}_{<}(f_i), \text{LT}_{<}(f_j)) = 1\) implies that \(S(f_i, f_j)\) reduces to zero modulo \(\{f_k\}_{k=1}^{\infty}\). Write \(i = 2q + r\), where \(r = 0, 1\). Then \(\text{LT}_{<}(f_i) = x_q^2\) if \(i\) is even and \(\text{LT}_{<}(f_i) = 2x_qx_{q+1}\) if \(i\) is odd. So the only case we need to consider is \(j = i + 1\). In this case, we have

\[
\text{lcm}(\text{LT}_{<}(f_i), \text{LT}_{<}(f_{i+1})) = \begin{cases} 2x_q^2x_{q+1}, & i \text{ even} \\ 2x_qx_{q+1}^2, & i \text{ odd.} \end{cases}
\]

Additionally

\[
S(f_i, f_{i+1}) = \begin{cases} 2x_{q+1}f_i - x_qf_{i+1}, & i \text{ even} \\ x_qf_i - 2x_{q+1}f_{i+1}, & i \text{ odd.} \end{cases}
\]

But from (2.1) it follows that these \(S\)-pairs appear in the relations \(\phi_n(\mu_{n-1}) = 0\) for \(n \gg 0\). Since \(n = \infty\), we always have these relations in \(I_\infty\). Additionally, moving the \(S\)-pair to the right-hand side we reduce \(S(f_i, f_{i+1}) \equiv 0\) modulo \(\{f_k\}_{k=1}^{\infty}\). In particular, Lemma 7.6 implies that \(\{f_k\}_{k=1}^{\infty}\) is a Gröbner basis for \(I_\infty\). \(\square\)

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