The Wilson Renormalisation Group Applied to the Potential in NN Scattering

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Abstract

Nonrelativistic two-body scattering by a short-ranged potential is studied using the renormalisation group. Two fixed points are identified: a trivial one and one describing systems with a bound state at zero energy. The eigenvalues of the linearised renormalisation group are used to assign a systematic power-counting to terms in the potential near each of these fixed points. The expansion around the nontrivial fixed point defines a power counting scheme which is equivalent to the effective-range expansion.

I. INTRODUCTION.

The suggestion by Weinberg [1] that effective field theory (EFT) techniques could be usefully applied to nuclear physics has generated much discussion. Since observed features in NN scattering such as bound and nearly-bound states indicate that naive perturbation theory will fail, it was proposed that power counting rules should instead be applied to the potential (or irreducible amplitude). This potential should then be iterated in a Schrödinger or Lippmann-Schwinger equation.

Immediately however there is a complication: the potential defined in this way is highly singular. Singular potentials are an unavoidable consequence of using local meson-nucleon and nucleon-nucleon couplings in the Lagrangian, and one must therefore specify a regularisation and renormalisation scheme in order to obtain finite physical quantities. Many such schemes have been proposed [2–13], but inconsistencies in the results obtained when the scattering length is large (as is the case in S-wave NN scattering) have led to the conclusion that Weinberg’s power counting fails in such systems. (For details and references see [14,15] and Section [III]).

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A different power counting has recently been suggested by Kaplan, Savage and Wise (KSW) \[10\] in the framework of the power divergence subtraction scheme (PDS). It has also been obtained by van Kolck \[12\] in a scheme independent approach. Although this power counting seems to avoid the problems just mentioned, its basis and relation to the other schemes remain unclear.

To try and understand the failure of Weinberg’s power counting and the apparent success of the new power counting, we have used the method of the Wilson (or “exact”) renormalisation group \[16\] applied to a potential which is allowed to have a cut-off dependence \[17\]. We implement this by imposing a momentum cut-off on the integrals encountered when iterating the full potential, and demanding that observables calculated in this way are cut-off independent.

The potential is rewritten in units of the cut-off \(\Lambda\). If \(\Lambda\) is allowed to vary, the potential must change to ensure that the theory still has the correct long range behaviour. The variation of \(V\) with \(\Lambda\) is given by a renormalisation group (RG) equation. As the cut-off is lowered to zero, all finite ranged physics is integrated out of the theory, and physical masses and energy scales no longer appear explicitly. In the limit, the (dimensionless) rescaled potential must therefore become independent of \(\Lambda\), the only remaining scale. The many possible limiting values of the potential are fixed points of the renormalisation group equation.

We find two kinds of fixed point: the first is trivial, corresponding to an identically zero potential (no scattering), and the second to potentials producing a bound state exactly at threshold.

By looking at perturbations around the single fixed point of the first type, we find that it is stable and corresponds to Weinberg’s power counting. This organisation of the potential is systematic in a natural situation where perturbation theory is applicable, but not in the case of interest (low energy NN scattering).

The simplest fixed point of the second type is unstable, and the power counting we find here closely matches that which was found by KSW. Perturbations around this fixed point generate terms in the effective range expansion, which has long been used to parameterise low energy NN data.

II. THE SCATTERING EQUATION

If one considers integrating all degrees of freedom out of the theory except nucleon fields, the resulting interaction Hamiltonian contains only four-nucleon contact terms. This Hamiltonian can be arranged as a series of terms containing derivatives of increasing order. The resulting potential has the following simple expansion:

\[
V(k', k, p) = C_{00} + C_{20}(k^2 + k'^2) + C_{02}p^2 \cdots,
\]

(1)

where \(k\) and \(k'\) are relative momenta and \(p = \sqrt{ME}\) is the corresponding on-shell value at energy \(E\).

We shall work with the reactance matrix \(K\) since it has a simpler relation to the effective range expansion than the scattering matrix \(T\). The Lippmann-Schwinger (LS) equation for the off-shell \(K\) matrix corresponding to the potential \([P]\) is given by
\[ K(k', k, p) = V(k', k, p) + \frac{M}{2\pi^2} \mathcal{P} \int_0^\infty q^2 dq \frac{V(k', q, p)K(q, k, p)}{p^2 - q^2}, \]  

where \( \mathcal{P} \) denotes a principal value integral. The reactance matrix consequently satisfies standing wave boundary conditions.

The inverse of the on-shell \( K \)-matrix differs from that of the on-shell \( T \)-matrix by a term \( iMp/4\pi \), which ensures that \( T \) is unitary if \( K \) is Hermitian. Observables are obtained from \( K \) by expanding the inverse of its on-shell value in powers of energy.

\[ \frac{1}{K(p, p, p)} = -\frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} re^2p^2 + \cdots \right). \]  

This is just the familiar effective range expansion where \( a \) is the scattering length and \( r_e \) is the effective range.

### III. Regularisation Schemes

A natural approach to regularising this theory is to impose a momentum space cut-off at a scale \( \Lambda < \Lambda_0 \) where \( \Lambda_0 \) is a scale corresponding to underlying physics which has been integrated out of the effective theory \([4,5]\). This can be done either by introducing a form factor in the potential reflecting the non-zero range of the interaction, or a cut-off on the momenta of intermediate virtual nucleon states. In the case of a sharp momentum cut-off, the divergent integrals which arise in solving the scattering equation are of the form

\[ I_n = \frac{M}{2\pi^2} \mathcal{P} \int_0^\Lambda dq \frac{q^{2n+2}}{p^2 - q^2}, \]  

where extra factors of \( q^2 \) in the numerator of the integrand occur in loop integrals with insertions of momentum dependent factors from the potential.

Expanding this integral in powers of energy, it is seen to have \( n+1 \) power law divergences,

\[ I_n = \frac{M}{2\pi^2} \left[ -\frac{\Lambda^{2n+1}}{2n+1} - \frac{\Lambda^{2n-1}}{2n-1}p^2 + \cdots - \Lambda p^{2n} + p^{2n} I(p) \right], \]  

where \( I(p) \) is a function of \( p \) which is finite as \( \Lambda \to \infty \);

\[ I(p) = \frac{p}{2} \ln \frac{\Lambda + p}{\Lambda - p}. \]  

A sharp cut-off is used here to avoid unnecessary complications. If a different choice was made, the divergences in the \( I_n \) would appear with different numerical coefficients, and the structure of \( I(p) \) would change. Neither of these modifications would affect the conclusions below.

If the potential \([4]\) is truncated at a given “order” in the energy/momentum expansion, it has an \( n \)-term separable form, and we can obtain an explicit expression for the \( K \)-matrix, and hence the observables in the effective range expansion. The undetermined coefficients
in the potential may then be fixed by demanding that they reproduce the observed effective range expansion up to the same order.

In this way, by including more and more terms in the potential, it is possible to determine whether the calculation is systematic. By systematic, we mean that the inclusion of extra terms in the potential should not result in large changes to the coefficients which have already been fitted. Satisfying this demand ensures that the resulting Hamiltonian is meaningful outside the process in which it is determined.

Two distinct cases have been identified. When the coefficients in the effective range expansion are natural, the cut-off $\Lambda$ can be chosen in such a way that the calculation is systematic. So long as

$$\Lambda << \frac{1}{a} \sim \frac{1}{r_e},$$

the requirements given above are satisfied. In this weak scattering regime, perturbation theory is valid, and Weinberg’s power counting rules apply.

The scattering length $a$ is unnaturally large in $S$-wave NN scattering however:

$$a \simeq 24 \text{ fm} \gg r_e \sim \frac{1}{m_\pi},$$

and in this case choosing $\Lambda << 1/a$ would lead to a (systematic) theory with an extremely limited range of validity. One might hope that the following choice for the cut-off could be useful:

$$\frac{1}{a} << \Lambda << \frac{1}{r_e},$$

but now corrections to the coefficients in the potential resulting from addition of extra terms will contain powers of both $1/\Lambda a$ and $\Lambda r_e$. This choice does not lead to a systematic expansion in either $\Lambda$ or $1/\Lambda$.

An alternative is to use dimensional regularisation (DR) in which the loop integrals are continued to $D$ dimensions. In the minimal subtraction scheme ($\overline{\text{MS}}$) [3], any logarithmic divergence (pole at $D = 4$) is subtracted and power law divergences do not appear. Since no logarithmic divergences appear in (5), the loop integrals $I_n$ are set to zero and the $K$-matrix is simply given by the first Born approximation,

$$K(k', k; p) = V(k', k, p).$$

From the on-shell version of this identity we can obtain the effective range expansion which is found to converge only in the region $p < \sqrt{2/|ar_e|}$. As in the case of a cut-off $\Lambda << 1/a$ this scheme is always systematic, but useless when $a$ is large.

Kaplan, Savage and Wise [10] have proposed using a different scheme called power-divergence subtraction (PDS) to avoid this problem. PDS subtracts linear divergences (corresponding to poles at $D = 3$), so that the $I_n$ do not vanish. The resulting $K$-matrix now has a dependence on the subtraction scale $\mu$, and choosing

$$\frac{1}{a} << \mu \sim p <= \frac{1}{r_e},$$

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where $p$ is the momentum scale of interest, a systematic power counting scheme emerges which appears to be useful. Note that the subtracted term, which is linear in $\Lambda$, is the only divergence which occurs when the potential is momentum independent ($C_{2i,0} = 0$ in (1)). PDS must therefore be interpreted as a momentum independent scheme. We shall return to this point later. Gegelia [11] has obtained a similar result by performing a momentum subtraction at the unphysical point $p = i\mu$. The results obtained using these schemes agree with those of van Kolck (in a scheme independent approach) [12] and more recently Cohen and Hansen [13] (in coordinate space). In the next section, we shall see that this new power counting can be understood in terms of an expansion around a fixed point of the renormalisation group (RG) equation which we derive for the potential. We shall also make clear that it is, at least to all orders considered so far, equivalent to an effective range expansion (as suggested by van Kolck [12]).

IV. THE RENORMALISATION GROUP

The starting point for the derivation of the RG equation is the LS equation (2) where all quantities are now allowed to depend on $\Lambda$. $V$ is the exact potential required to reproduce the observed $K$-matrix, and will therefore be $\Lambda$ dependent after renormalisation in a cut-off scheme. The $\Lambda$ dependence of the free Green’s function $G_0$ regulates divergent loop integrals. As above, we use a sharp cut-off $\Lambda$ on loop momenta. This choice will simplify the discussion but, as before, our results apply equally well to any reasonable choice of momentum space cut-off.

To proceed, it is helpful to demand that the entire off-shell $K$-matrix is independent of $\Lambda$. Note that this is a stronger requirement than is necessary simply to ensure cut-off independence of the resulting observables. After differentiating the LS-equation with respect to $\Lambda$ and setting $\partial K/\partial \Lambda = 0$, the RG-equation for $V(k', k, p, \Lambda)$ can be obtained by operating on the resulting expression from the right with $(1 + G_0 K)^{-1}$ to get

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda).$$

(12)

In the derivations to follow, we will impose the boundary conditions that $V$ should have an expansion in powers of $p^2$, $k^2$ and $k'^2$. This means that it can be obtained from the type of Hamiltonian proposed by Weinberg [1] and written in the form (1).

To bring out the interesting features of this approach, it is useful to introduce the dimensionless momentum variables, $k = k/\Lambda$ etc., and the scaled potential

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{MA}{2\pi^2} V(\Lambda \hat{k}', \Lambda \hat{k}, \Lambda \hat{p}, \Lambda),$$

(13)

where the overall factor of $M/2\pi^2$ comes from the LS-equation (2).

Finally, we rewrite the RG-equation in terms of these new variables,

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}, \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}, \Lambda).$$

(14)
In the following, the idea of a fixed point will be important. As \( \Lambda \) varies, the RG-equation (14) describes how the rescaled potential must flow to ensure that the long range behaviour of the theory does not change. As the cut-off is taken to zero, more and more physics is integrated out of the theory. In the limit, the cut-off is the only remaining energy scale and the rescaled potential, being dimensionless, must become independent of \( \Lambda \). The many possible limiting values of the potential are fixed points of the renormalisation group equation. To find fixed points, we look for solutions of (14) which satisfy
\[
\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = 0.
\] (15)
Near a fixed point, all physical energies and masses are large compared to the cut-off and power counting becomes possible.

A simple example which turns out to be important is the trivial fixed point \( \hat{V}(\hat{k'}, \hat{k}, \hat{p}, \Lambda) = 0 \). It is easy to see that the \( K \)-matrix calculated from this potential is also zero, corresponding to no scattering. It is necessary to know how perturbations around the fixed point potential scale if we are to use it as the basis of a power counting scheme. This can be done by linearising the RG-equation by looking for eigenfunctions which scale as an integer power of \( \Lambda \),
\[
\hat{V}(\hat{k'}, \hat{k}, \hat{p}, \Lambda) = C \Lambda^\nu \phi(\hat{k'}, \hat{k}, \hat{p}).
\] (16)
Substituting this back into (14), the linear eigenvalue equation for \( \phi \) is found to be
\[
\hat{k'} \frac{\partial \phi}{\partial \hat{k'}} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi = \nu \phi.
\] (17)
The solutions of this equation which satisfy the boundary conditions specified above are easily found to be
\[
\phi(\hat{k'}, \hat{k}, \hat{p}) = \hat{k'}^l \hat{k}^m \hat{p}^n,
\] (18)
with RG eigenvalues \( \nu = l + m + n + 1 \), where \( l, m \) and \( n \) are non-negative even integers. The momentum expansion of the rescaled potential around the trivial fixed point is therefore given by
\[
\hat{V}(\hat{k'}, \hat{k}, \hat{p}, \Lambda) = \sum_{l, m} \tilde{C}_{lmn} \left( \frac{\Lambda}{\Lambda_0} \right)^\nu \hat{k'}^l \hat{k}^m \hat{p}^n.
\] (19)
(For a hermitian potential one must take \( C_{lmn} = C_{mnl} \).) The coefficient \( \tilde{C}_{lmn} \) has been made dimensionless by taking out a factor \( 1/\Lambda_0^\nu \) where, as before, \( \Lambda_0 \) is a scale corresponding to underlying (integrated out) physics.

Since all of the allowed eigenvalues are positive, this potential vanishes as \( \Lambda/\Lambda_0 \to 0 \), and the trivial fixed point is stable. The power counting scheme associated with this fixed point can be made explicit by considering the unscaled potential
\[
V(k', k, p, \Lambda) = \frac{2\pi^2}{M\Lambda_0} \sum_{l, m, n} \tilde{C}_{lmn} \frac{k'^l k^m p^n}{\Lambda_0^{l+m+n}}.
\] (20)
So long as the $C_{lmn}$ are natural, the contributions of energy and momentum dependent terms to the potential are suppressed by powers of $1/\Lambda_0$. Assigning an order $d = \nu - 1$ to these perturbations leads to a scheme which is exactly equivalent to that originally suggested by Weinberg [1].

In section III, we saw that power counting in this way is useful in the case where scattering is weak at low energies, and either dimensional regularisation with minimal subtraction or a cut-off may be used. When the scattering length is unnaturally large however, we are forced to choose $\Lambda$ to be so small that the theory becomes essentially useless. To find a useful expansion of the potential in this case, it is necessary to look for a non-trivial fixed point which describes strong scattering at low energies.

The simplest fixed point of this kind can be found by looking for a momentum independent potential, $\hat{V} = \hat{V}(\hat{p})$, which satisfies the full RG-equation. This becomes

$$\hat{p} \frac{\partial \hat{V}_0}{\partial \hat{p}} + \hat{V}_0(\hat{p}) + \frac{\hat{V}_0(\hat{p})^2}{1 - \hat{p}^2} = 0.$$  \hspace{1cm} (21)

Solving this equation subject to the boundary condition that the potential be analytic in $\hat{p}^2$ as $\hat{p}^2 \to 0$ we obtain

$$\hat{V}_0(\hat{p}) = -\left[1 - \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}}\right]^{-1}. \hspace{1cm} (22)$$

The corresponding unscaled potential is

$$V_0(p, \Lambda) = -\frac{2\pi^2}{M} \left[\Lambda - \frac{p}{2} \ln \frac{\Lambda + p}{\Lambda - p}\right]^{-1} = -\frac{2\pi^2}{M} \left[\Lambda - I(p)\right]^{-1}, \hspace{1cm} (23)$$

where $I(p)$ was defined in Eq. (13). The RG-equation (14) was derived using a sharp cut-off, and this form for the fixed point potential is a consequence of that choice. Choosing another type of cut-off would change the second ($p$ dependent) term in the square brackets, but the $1/\Lambda$ behaviour as $p \to 0$ is independent of the form of cut-off used.

Inserting this potential in the LS-equation, we find $1/K = 0$. This fixed point therefore corresponds to a zero energy bound state (i.e. the scattering length $a$ is infinite). Not surprisingly, this fixed point has properties which are quite different from those we found in the trivial case. As before, we look at perturbations around this fixed point which scale as an integer power of $\Lambda$

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \hat{V}_0(\hat{p}) + C\Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}), \hspace{1cm} (24)$$

where $\phi$ is a function which is well behaved for small momenta and energy and satisfies the linearised RG-equation:

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi + \frac{\hat{V}_0(\hat{p})}{1 - \hat{p}^2} \left[\phi(\hat{k}', 1, \hat{p}) + \phi(1, \hat{k}, \hat{p})\right] = \nu \phi. \hspace{1cm} (25)$$

The general case $\phi = \phi(\hat{k}', \hat{k}, \hat{p})$ can be tackled by breaking it down into simpler types of perturbation. Perturbations which depend only on energy $\phi = \phi(\hat{p})$ turn out to be
particularly important, so we will consider these first. Equation (25) can be integrated subject to the same boundary conditions as before, and the result is simply

\[ \phi(\hat{p}) = \hat{p}^{\nu+1} \hat{V}_0(\hat{p})^2. \] (26)

with eigenvalues \( \nu = -1, 1, 3, 5, \ldots \). The first of these eigenvalues is negative, and is associated with an unstable perturbation since it gives rise to a term in the potential which scales like \( \Lambda_0 / \Lambda \) as \( \Lambda \to 0 \). This fixed point is therefore unstable; only potentials which initially lie exactly on the critical surface \( 1 / K = 0 \) approach it as the cut-off is taken to zero.

We must also consider perturbations which are momentum dependent. This time, we look for a function \( \phi \) of the form

\[ \phi(\hat{k}', \hat{k}, \hat{p}) = \hat{k}^n \phi_1(\hat{p}) + \phi_2(\hat{p}) \] (27)

which satisfies (25). The solutions are given by

\[ \phi(\hat{k}', \hat{k}, \hat{p}) = \left[ \hat{k}^n - \hat{p}^n + \left( \frac{1}{n+1} + \frac{\hat{p}^2}{n-1} + \cdots + \frac{\hat{p}^{n-2}}{3} \right) \hat{V}_0(\hat{p}) \right] \hat{V}_0(\hat{p}). \] (28)

If a Hermitian potential is required, a matching perturbation of this form should be added with \( k \) replaced by \( k' \). As usual, the eigenvalues are given by the condition that the potential be a well behaved function of energy and squared momenta; in this case they are \( \nu = n = 2, 4, 6 \ldots \).

In general, new solutions to the linearised RG equation (25) can be obtained by multiplying existing ones by \( p^m \) where \( m \) is a positive even integer. Applying this to the functions in equation (28) results in a new set of eigenfunctions with corresponding eigenvalues \( \nu = n+m \).

An important point to note is that the momentum-dependent eigenfunctions have different eigenvalues from the corresponding purely energy-dependent ones and so they scale differently with \( \Lambda \). This is quite unlike the more familiar case of perturbations around the trivial fixed point where, for example, the \( \hat{p}^2 \) and \( \hat{k}^2 \) terms in the potential are both of the same order, \( \nu = 3 \). It means that, in the vicinity of the nontrivial fixed point, one cannot use the usual equations of motion to eliminate energy dependence from the potential in favour of momentum dependence.

One more kind of perturbation is possible: a product of two factors of the type given in equation (28), where one is a function of \( \hat{k} \) and the other is a function of \( \hat{k}' \), is a solution of the linearised RG-equation with eigenvalues \( \nu = 5, 7, 9 \ldots \). These solutions can also be multiplied by powers of \( p^2 \) to give eigenfunctions with higher eigenvalues.

Any perturbation which solves equation (25) subject to the given boundary conditions can expressed as a sum of those given above, so we are in a position to expand a general potential around this fixed point. Near the fixed point we need only consider the perturbations with the lowest eigenvalues. Including the unstable perturbation, the first three give the following unscaled potential

\[ V(k', k, p, \Lambda) = V_0(p, \Lambda) + \frac{M \Lambda_0}{2\pi^2} \sum_{\nu=-1}^{3} \hat{C}_\nu \left( \frac{p}{\Lambda_0} \right)^{\nu+1} V_0(p, \Lambda)^2 \]

\[ + \hat{C}_2 \left[ k'^2 + k^2 - 2p^2 + \frac{2}{3} \frac{M \Lambda^3}{2\pi^2} V_0 \right] \frac{V_0(p, \Lambda)}{\Lambda_0^2}. \] (29)
Although this potential looks complicated, it has a two term separable structure, and we can obtain an exact expression for the $K$-matrix \[5,7\]. Expanding the inverse of the on-shell $K$-matrix, we find the following effective range expansion

\[
\frac{1}{K(p, p, p)} = -\frac{M\Lambda_0}{2\pi^2} \sum_{\nu=-1}^{3} \hat{C}_\nu \left( \frac{p}{\Lambda_0} \right)^{\nu+1} + \cdots,
\]

(30)

to first order in the $\hat{C}_\nu$.

To this order, the potential is determined uniquely by comparing equation (30) with the effective range expansion (3).

\[
\hat{C}_{-1} = -\frac{\pi}{2\Lambda_0 a}, \quad \hat{C}_{1} = \frac{\pi \Lambda_0 r_e}{4} \quad \ldots
\]

(31)
The identification of the terms in the potential and the effective range expansion is straightforward at this order because only energy dependent perturbations contribute to the scattering. It is not clear that this equivalence persists to higher order in the $\hat{C}_\nu$. To illustrate this, we have calculated corrections to the unscaled potential (29) up to order $\hat{C}_1 \hat{C}_2$. We find the following extra contributions

\[
\frac{M}{2\pi^2} \hat{C}_3 p^4 \frac{V_0(p, \Lambda)^2}{\Lambda_0^3} + \frac{M}{2\pi^2} \hat{C}_1 \hat{C}_2 \left( k^2 + k^2 + A p^2 + \frac{4M\Lambda^3}{3} \frac{V_0(p, \Lambda)^2}{\Lambda_0^3} \right) p^2 \frac{V_0(p, \Lambda)^2}{\Lambda_0^3},
\]

(32)

where $A$ is a constant of integration which is not fixed by the boundary conditions. This unfixed term arises from the solution of the homogeneous part of the linearised RG equations, and has the same structure as the contribution proportional to $\hat{C}_3$. In the above expression we have arbitrarily chosen to associate it with the momentum dependent perturbation multiplying $\hat{C}_1 \hat{C}_2$.

Both of these perturbations contribute to the shape parameter $P$ which is the observable obtained from the coefficient of $p^4$ in the effective range expansion. To avoid spoiling the one to one correspondence between observables and terms in the potential, it is possible to choose $A = -2$ which ensures that the contribution of $\hat{C}_1 \hat{C}_2$ to $P$ vanishes.

So long as analogous procedures can be carried out to all orders, the effective theory defined by an expansion around this fixed point is systematic and completely equivalent to the effective range expansion. This corresponds to the fact that the parts of our potential which now contribute to observables act like a quasipotential: an energy-dependent boundary condition on the logarithmic derivative of the wave function at the origin. Such an equivalence has previously been suggested by van Kolck [12].

As has already been pointed out, the form of the fixed point potential \[23\] depends on the type of cut-off used, but for small $p$ it always behaves like $1/\Lambda$. We can therefore compare the terms that appear in the expansion of our potential (29) with those of KSW \[10\], since the subtraction scale $\mu$ in PDS plays a similar role to $\Lambda$ in a cut-off scheme. For example, in the case $1/\Lambda = 0$ (on the critical surface), the lowest order contribution to their potential, $C_0(\mu)$, scales like $1/\mu$. This is in agreement with the scaling behaviour of our fixed point potential. Similarly, the term which gives the leading contribution to their effective range, $C_2(\mu)$, scales like $1/\mu^2$ which is consistent with the $V_0^2$ multiplying $p^2$ in (29) and so
on. In fact, the power counting for perturbations around the nontrivial fixed point agrees with that of KSW if, as before, we assign them an order \( d = \nu - 1 \).

One detail remains: the fixed point is unstable and, since \( 1/a \neq 0 \), as \( \Lambda \to 0 \) the potential will either flow to the non-trivial fixed point or diverge to infinity. So long as it is possible to choose the cut-off so that \( 1/a \ll \Lambda \ll \Lambda_0 \) however, the behaviour of the potential is dominated by the flow towards the nontrivial fixed point, and the eigenfunctions found above still define a systematic expansion of the potential, as noted in Ref. [10].

V. SUMMARY

In the case of weak scattering, the effective field theory originally suggested by Weinberg [1] to describe NN scattering is systematic, and gives predictions which are independent of the regularisation scheme used (coordinate space or momentum space cut-off, dimensional regularisation (\( \overline{MS} \)), etc.). This can be understood in terms of an expansion of the potential around the trivial fixed point of the renormalisation group equation.

In the presence of a resonance or bound state close to threshold, conventional regularisation schemes do not lead to power counting schemes which are both systematic and useful. Recently, several equivalent power counting schemes based on new regularisation or subtraction schemes have been proposed which appear to avoid the problems mentioned above. These correspond to an expansion of the potential around the simplest non-trivial fixed point of the renormalisation group. Terms in this potential have a one to one correspondence with on-shell scattering observables in the effective range expansion to all orders considered so far. The success of the effective range expansion [18,19] can be therefore be understood in terms of an effective field theory based on this nontrivial fixed point [8–10].

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