DEFORMATIONS OF SYMPLECTIC COHOMOLOGY
AND EXACT LAGRANGIANS IN ALE SPACES

ALEXANDER F. RITTER

Abstract. ALE spaces are the simply connected hyperkähler manifolds which at infinity look like $\mathbb{C}^2 / G$, for any finite subgroup $G \subset SL_2(\mathbb{C})$. We prove that all exact Lagrangians inside ALE spaces must be spheres. The proof relies on showing the vanishing of a twisted version of symplectic cohomology. This application is a consequence of a general deformation technique. We construct the symplectic cohomology for non-exact symplectic manifolds, and we prove that if the non-exact symplectic form is sufficiently close to an exact one then the symplectic cohomology coincides with an appropriately twisted version of the symplectic cohomology for the exact form.

1. Introduction

An ALE hyperkähler manifold $M$ is a non-compact simply-connected hyperkähler 4–manifold which asymptotically looks like the standard Euclidean quotient $\mathbb{C}^2 / \Gamma$ by a finite subgroup $\Gamma \subset SU(2)$. These spaces can be explicitly described and classified by a hyperkähler quotient construction due to Kronheimer [8]. ALE spaces have been studied in a variety of contexts. In theoretical physics they arise as gravitational instantons in the work of Gibbons and Hawking. In singularity theory they arise as the minimal resolution of the simple singularity $\mathbb{C}^2 / \Gamma$. In symplectic geometry they arise as plumbings of cotangent bundles $T^*\mathbb{C}P^1$ according to ADE Dynkin diagrams:

```
A_n  (n = 6)  
```

```
D_n  (n = 7)  
```

Recall that the finite subgroups $\Gamma \subset SU(2)$ are the preimages under the double cover $SU(2) \to SO(3)$ of the cyclic group $\mathbb{Z}_n$, the dihedral group $D_{2n}$, or one of the groups $T_{12}, O_{24}, I_{60}$ of rigid motions of the Platonic solids. These choices of $\Gamma$ will make $\mathbb{C}^2 / \Gamma$ respectively a singularity of type $A_n-1, D_{n+2}, E_6, E_7, E_8$. The singularity is described as follows. The $\Gamma$–invariant complex polynomials in two variables are generated by three polynomials $x, y, z$ which satisfy precisely one polynomial relation $f(x, y, z) = 0$. The hypersurface $\{f = 0\} \subset \mathbb{C}^3$ has precisely one singularity at the origin. The minimal resolution of this singularity over the singular point 0 is a connected union of copies of $\mathbb{CP}^1$ with self-intersection $-2$, which
intersect each other transversely according to the corresponding ADE Dynkin diagram. Each vertex of the diagram corresponds to a $\mathbb{C}P^1$ and an edge between $C_i$ and $C_j$ means that $C_i \cdot C_j = 1$. We suggest Slodowy [14] or Arnol’d [1] for a survey of this construction.

In the symplectic world these spaces can be described as the plumbing of copies of $T^*\mathbb{C}P^1$ according to ADE Dynkin diagrams. Each vertex of the Dynkin diagram corresponds to a disc cotangent bundle $DT^*\mathbb{C}P^1$ and each edge of the Dynkin diagram corresponds to identifying the fibre directions of one bundle with the base directions of the other bundle over a small patch, and vice-versa. The boundary can be arranged to be a standard contact $S^3/T$, and along this boundary we attach an infinite symplectic cone $S^3/T \times [1, \infty)$ to form $M$ as an exact symplectic manifold.

Any hyperkähler manifold comes with three canonical symplectic forms $\omega_I, \omega_J, \omega_K$ which give rise to an $S^2-$worth of symplectic forms: $\omega_u = u_I \omega_I + u_J \omega_J + u_K \omega_K$, where $u = (u_I, u_J, u_K) \in S^2 \subset \mathbb{R}^3$.

An ALE space is an exact symplectic manifold with respect to $\omega_J$, $\omega_K$ or any non-zero combination $u_J \omega_J + u_K \omega_K$. Let $d\theta$ be one of these forms. The copies of $\mathbb{C}P^1$ described above are exact Lagrangian submanifolds with respect to $d\theta$, and a neighbourhood of this chain of $\mathbb{C}P^1$‘s is symplectomorphic to the plumbing of $T^*\mathbb{C}P^1$‘s by Weinstein’s Lagrangian neighbourhood theorem. The ALE space is not exact for $\omega_u$ if $u_I \neq 0$, in which case the $\mathbb{C}P^1$‘s are symplectic submanifolds.

**Question.** What are the exact Lagrangian submanifolds inside an ALE space?

Recall that a submanifold $j : L^n \rightarrow M^{2n}$ inside an exact symplectic manifold $(M, d\theta)$ is called exact Lagrangian if $j^* \theta$ is exact.

For example, the $A_1-$plumbing is $M = T^*S^2$ and the graph of any exact 1–form on $S^2$ is an exact Lagrangian sphere in $T^*S^2$. Viterbo [16] proved that there are no exact tori in $T^*S^2$. For homological reasons, the only orientable exact Lagrangians in $T^*S^2$ are spheres, and we proved in [10] that $L$ cannot be unorientable. Moreover, for exact spheres $L \subset T^*S^2$, it is known that $L$ is isotopic to the zero section (Eliashberg-Polterovich [2]), indeed it is Hamiltonian isotopic (Hind [3]). Thus the only exact Lagrangians in $T^*S^2$ are spheres isotopic to the zero section.

**Theorem.** The only exact Lagrangians inside the ALE space $(M, d\theta)$ are spheres, in particular there are no unorientable exact Lagrangians. For example, this holds for the plumbing of copies of $T^*\mathbb{C}P^1$ as prescribed by an ADE Dynkin diagram.

We approach this problem via symplectic cohomology, which is an invariant of symplectic manifolds with contact type boundary. It is constructed as a direct limit of Floer cohomology groups for Hamiltonians which become steep near the boundary. Symplectic cohomology can be thought of as an obstruction to the existence of exact Lagrangians in the following sense.

Viterbo [15] proved that an exact $j : L \rightarrow (M, d\theta)$ yields a commutative diagram

\[
\begin{array}{ccc}
H_{n-*}(LL) & \cong & SH^*(T^*L, d\theta) \\
\downarrow & & \downarrow \text{SH}^*(j) \\
H_{n-*}(L) & \cong & H^*(L)
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{SH}^*(M, d\theta) \\
\downarrow \text{SH}^* & & \downarrow \\
& & H^*(M)
\end{array}
\]
where $\mathcal{LL} = C^\infty(S^1, L)$ is the space of free loops in $L$ and the left vertical map is induced by the inclusion of constants $c : L \to \mathcal{LL}$. The element $c_*(j^*1)$ cannot vanish, and thus the vanishing of $SH^*(M, d\theta)$ would contradict the existence of $L$.

It is possible to describe the ALE space $(M, \omega)$ as a symplectic manifold with contact type boundary with a semi-infinite collar attached along the boundary, so that $SH^*(M, \omega_u)$ is well-defined. The symplectic cohomology $SH^*(M, d\theta)$ is never zero, indeed it contains a copy of the ordinary cohomology $H^*(M) \to SH^*(M, d\theta)$. However, we will show that if we make a generic infinitesimal perturbation of the closed form $d\theta$, then the symplectic cohomology will vanish. From this it will be easy to deduce that the only exact Lagrangians $L \subset M$ must be spheres.

We constructed the infinitesimally perturbed symplectic cohomology in \cite{10} as follows. For any $\alpha \in H^1(\mathcal{L}_0 N)$, we constructed the associated Novikov homology theory for $SH^*(M, d\theta)$. This involves introducing a local system of coefficients $\Lambda$, taking values in the ring of formal Laurent series $\Lambda = \mathbb{Z}((t))$. Let’s denote this twisted symplectic cohomology by $SH^*(M, d\theta; \Lambda)$.

We proved that the above functoriality diagram holds in this context – with the understanding that for unorientable $L$ we use $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients and the Novikov ring $\mathbb{Z}_2((t))$ instead.

Consider a transgressed form $\alpha = \tau \beta$, where $\tau : H^2(M) \to H^1(\mathcal{LM})$ is the transgression. The functoriality diagram simplifies to

$$
\begin{array}{ccc}
H_n(\mathcal{L}L; \Lambda, j^* \beta) & \xrightarrow{SH^*(j)} & SH^*(M, d\theta; \Lambda, \beta) \\
\downarrow c_* & & \downarrow c_* \\
H_n(L) \otimes \Lambda & \cong & H^*(L) \otimes \Lambda \xrightarrow{j^*} H^*(M) \otimes \Lambda
\end{array}
$$

For surfaces $L$ which aren’t spheres the transgression vanishes, so $H_*(\mathcal{L}L; \Lambda, j^* \beta)$ simplifies to $H_* (\mathcal{L}L) \otimes \Lambda$ and the left vertical arrow $c_*$ becomes injective. Thus $c_*(j^*1)$ cannot vanish, which contradicts the commutativity of the diagram if we can show that $SH^*(M, d\theta, \Lambda, \beta) = 0$ for some $\beta$.

**Theorem.** Let $M$ be an ALE space. Then for generic $\beta$,

$$SH^*(M, d\theta; \Lambda, \beta) = 0.$$ 

It turns out that there is a way to prove that the non-exact symplectic cohomology $SH^*(M, \omega)$ vanishes for a generic form $\omega$. So to prove the above vanishing result, we need to relate the twisted symplectic cohomology to the non-exact symplectic cohomology. We will prove the following general result.

**Theorem.** Let $(M, d\theta)$ be an exact symplectic manifold with contact type boundary and let $\beta$ be a closed two-form compactly supported in the interior of $M$. Then, at least for $\|\beta\| < 1$, there is an isomorphism

$$SH^*(M, d\theta + \beta) \to SH^*(M, d\theta; \Lambda, \beta).$$

For our ALE space $M$, we actually show that this result applies to a large non-compact deformation from $d\theta$ to the non-exact symplectic form $\omega_j$ which has a lot of symmetry. This symmetry will be the key to proving the vanishing of $SH^*(M, \omega_j)$ and therefore the vanishing of $SH^*(M, d\theta; \Lambda, \omega_j) = 0$, which concludes the proof of the non-existence of exact Lagrangians which aren’t spheres.
The key to the vanishing of $SH^* (M, \omega_I)$ is the existence of a global Hamiltonian $S^1$–action, which at infinity looks like the action $(a, b) \mapsto (e^{2\pi i a}, e^{2\pi i b})$ on $\mathbb{C}^2/T$. We will show that the grading of the 1–periodic orbits grows to negative infinity when we accelerate this Hamiltonian $S^1$–action, and this will imply that $SH^* (M, \omega_I) = 0$ because a generator would have to have arbitrarily negative grading. This concludes the argument.

The hyperkahler construction of $M$ depends on certain parameters, and the cohomology class of $\omega_I$ varies linearly with these parameters. Indeed, it turns out that the form $\omega_I$ can be chosen to represent a generic class in $H^2 (M; \mathbb{R})$.

**Theorem.** Let $M$ be an ALE space. Given a generic class in $H^2 (M; \mathbb{R})$, it is possible to choose a symplectic form on $M$ representing this class such that

$$SH^* (M, \omega) = 0.$$ 

The outline of the paper is as follows. In section 2 we recall the basic terminology of symplectic manifolds with contact type boundary and we define the moduli spaces used to define symplectic cohomology. In section 3 we construct the symplectic cohomology $SH^* (M, \omega)$ for a (possibly non-exact) symplectic form $\omega$, in particular in 3.1 we define the underlying Novikov ring $\Lambda$ that we use throughout. In section 4 we define the twisted symplectic cohomology $SH^* (M, d\theta; \Lambda_n)$, in particular the Novikov bundle $\Lambda_n$ is defined in 4.2 and the functoriality property is described in 4.7. In section 5 we define the grading on symplectic cohomology, which is a $\mathbb{Z}$–grading if $c_1 (M) = 0$. In section 6 we prove the deformation theorem which relates the twisted symplectic cohomology to the non-exact symplectic cohomology. In section 7 we recall Kronheimer’s hyperkahler quotient construction of ALE spaces, and we describe the details of the proof outlined above.

**Acknowledgements:** I would like to thank Paul Seidel for suggesting this project.

## 2. Symplectic manifolds with contact boundary

### 2.1. Symplectic manifolds with contact type boundary

Let $(M^{2n}, \omega)$ be a compact symplectic manifold with boundary. The contact type boundary condition requires that there is a Liouville vector field $Z$ defined near the boundary $\partial M$ which is strictly outwards pointing along $\partial M$. The Liouville condition is that near the boundary $\omega = d\theta$, where $\theta = i_Z \omega$. This definition is equivalent to requiring that $\alpha = \theta|_{\partial M}$ is a contact form on $\partial M$, that is $d\alpha = \omega|_{\partial M}$ and $\alpha \wedge (d\alpha)^{n-1} > 0$ with respect to the boundary orientation on $\partial M$.

The Liouville flow of $Z$ is defined for small negative times $r$, and it parametrizes a collar $(-\epsilon, 0] \times \partial M$ of $\partial M$ inside $M$. So we may glue an infinite symplectic cone $([0, \infty) \times \partial M, d(e^r \alpha))$ onto $M$ along $\partial M$, so that $Z$ extends to $Z = \partial_r$ on the cone. This defines the completion $\widehat{M}$ of $M$,

$$\widehat{M} = M \cup_{\partial M} [0, \infty) \times \partial M.$$ 

We call $(-\epsilon, \infty) \times \partial M$ the collar of $\widehat{M}$. We extend $\theta$ and $\omega$ to the entire collar by $\theta = e^r \alpha$ and $\omega = d\theta$.

Let $J$ be an $\omega$–compatible almost complex structure on $\widehat{M}$ and denote by $g = \omega(\cdot, J \cdot)$ the $J$–invariant metric. We always assume that $J$ is of contact type on the collar, that is $J^* \theta = e^r dr$ or equivalently $J \partial_r = \mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field. This implies that $J$ restricts to an almost complex structure on the contact
distribution \( \text{ker} \alpha \). We will only need the contact type condition for \( J \) to hold for \( e^r \gg 0 \) so that a certain maximum principle applies there.

From now on, we make the change of coordinates \( R = e^r \) on the collar so that, redefining \( \epsilon \), the collar will be parametrized as the tubular neighbourhood \((\epsilon, \infty) \times \partial M \) of \( \partial M \) in \( \hat{M} \), so that the contact hypersurface \( \partial M \) corresponds to \( \{ R = 1 \} \).

In the exact setup, that is when \( \omega = d\theta \) on all of \( M \), we call \( (M, d\theta) \) a Liouville domain. In this case \( Z \) is defined on all of \( \hat{M} \) by \( i_Z \omega = \theta \), and \( \hat{M} \) is the union of the infinite symplectic collar \( ((-\infty, \infty) \times \partial M, d(R\alpha)) \) and the zero set of \( Z \).

### 2.2. Reeb and Hamiltonian dynamics.

The Reeb vector field \( R \in C^\infty(T\partial M) \) on \( \partial M \) is defined by \( i_R d\alpha = 0 \) and \( \alpha(R) = 1 \). The periods of the Reeb vector field form a countable closed subset of \([0,\infty)\), provided we choose \( \alpha \) generically.

For \( H \in C^\infty(\hat{M}, R) \) we define the Hamiltonian vector field \( X_H \) by

\[
\omega(\cdot, X_H) = dH.
\]

If inside \( M \) the Hamiltonian \( H \) is a \( C^2 \)-small generic perturbation of a constant, then the 1-periodic orbits of \( X_H \) inside \( M \) are constants corresponding precisely to the critical points of \( H \).

Suppose \( H = h(R) \) depends only on \( R = e^r \) on the collar. Then \( X_H = h'(R)R \).

It follows that every non-constant 1-periodic orbit \( x(t) \) of \( X_H \) which intersects the collar must lie in \( \{ R \} \times \partial M \) for some \( R \) and must correspond to a Reeb orbit \( z(t) = x(t/T) : [0,T] \to \partial M \) with period \( T = h'(R) \). Since the Reeb periods are countable, if we choose \( h \) to have a generic constant slope \( h'(R) \) for \( R \gg 0 \) then there will be no 1-periodic orbits of \( X_H \) outside of a compact set of \( M \).

### 2.3. Action 1-form.

Let \( \mathcal{L}\hat{M} = C^\infty(S^1, \hat{M}) \) be the space of free loops in \( \hat{M} \). Suppose for a moment that \( \omega = d\theta \) were exact on all of \( \hat{M} \), then one could define the \( H \)-perturbed action functional for \( x \in \mathcal{L}\hat{M} \) by

\[
A_H(x) = -\int x^*\theta + \int_0^1 H(x(t)) \, dt.
\]

If \( H = h(R) \) on the collar then this reduces to \( A_H(x) = -Rh'(R) + h(R) \) where \( x \) is a 1-periodic orbit of \( X_H \) in \( \{ R \} \times \partial M \). The differential of \( A_H \) at \( x \in \mathcal{L}\hat{M} \) in the direction \( \xi \in T_x \mathcal{L}\hat{M} = C^\infty(S^1, x^*T\hat{M}) \) is

\[
\frac{dA_H}{dt} \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X_H) \, dt.
\]

In the case when \( \omega \) is not exact on all of \( \hat{M} \), \( A_H \) is no longer well-defined, however the formula for \( dA_H \) still gives a well-defined 1–form on \( \mathcal{L}\hat{M} \). The zeros \( x \) of \( dA_H \) are precisely the 1-periodic Hamiltonian orbits \( \dot{x}(t) = X_H(x(t)) \).

It also meaningful to say how \( A_H \) varies along a smooth path \( u \) in \( \mathcal{L}\hat{M} \) by defining

\[
\partial_s A_H(u) = dA_H \cdot \partial_s u,
\]

but the total variation \( \int \partial_s A_H(u) \, ds \) will depend on \( u \), not just on the ends of \( u \).
2.4. Floer’s equation. With respect to the $L^2$-metric $\int_0^1 g(\cdot, \cdot) \, dt$ the gradient corresponding to $dA_H$ is $\nabla A_H = J(\dot{x} - X_H)$. For $u : \mathbb{R} \times S^1 \to M$, the negative $L^2$-gradient flow equation $\partial_s u = -\nabla A_H(u)$ in the coordinates $(s,t) \in \mathbb{R} \times S^1$ is

$$\partial_s u + J(\partial_t u - X_H) = 0 \quad \text{(Floer’s equation).}$$

Let $\mathcal{M}(x_-, x_+)$ denote the moduli space of solutions $u$ to Floer’s equation, which at the ends converge uniformly in $t$ to the 1-periodic orbits $x_\pm$:

$$\lim_{s \to \pm \infty} u(s, t) = x_\pm(t).$$

These solutions $u$ occur in $\mathbb{R}$--families because we may reparametrize the $\mathbb{R}$ coordinate by adding a constant. Denote the quotient by $\mathcal{M}(x_-, x_+) := \mathcal{M}(x_-, x_+) / \mathbb{R}$.

To emphasize the context, we may also write $\mathcal{M}^H(x_-, x_+) = \mathcal{M}(x_-, x_+; \omega)$.

The action $A_H$ decreases along $u$ since

$$\partial_s (A_H(u)) = dA_H \cdot \partial_s u = -\int_0^1 \omega(\partial_s u, \dot{\partial}_s u - X_H) \, dt = -\int_0^1 |\partial_s u|^2_{\mathbb{J}} \, dt \leq 0.$$

If $\omega$ is exact on $M$, the action decreases by $A_H(x_-) - A_H(x_+)$ independently of the choice of $u \in \mathcal{M}(x_-, x_+)$.  

2.5. Energy. For a Floer solution $u$ the energy is defined as

$$E(u) = \int |\partial_s u|^2 \, ds \wedge dt = \int \omega(\partial_s u, \partial_t u - X_H) \, ds \wedge dt = \int u^* \omega + \int H(x_-) \, dt - \int H(x_+) \, dt.$$

If $\omega$ is exact on $M$ then for $u \in \mathcal{M}(x_-, x_+)$ there is an a priori energy estimate, $E(u) = A_H(x_-) - A_H(x_+)$.  

2.6. Transversality and compactness. Standard Floer theory methods can be applied to show that for a generic time-dependent perturbation $(H_t, J_t)$ of $(H, J)$ there are only finitely many 1-periodic Hamiltonian orbits and the moduli spaces $\mathcal{M}(x_-, x_+)$ are smooth manifolds. We write $\mathcal{M}_k(x_-, x_+) = \mathcal{M}_k(x_-, x_+) / \mathbb{R}$ for the $k$-dimensional part of $\mathcal{M}(x_-, x_+)$.  

As explained in detail in Viterbo [15] and Seidel [13], there is a maximum principle which prevents Floer trajectories $u \in \mathcal{M}(x_-, x_+)$ from escaping to infinity.

**Lemma 1** (Maximum principle). *If on the collar $H = h(R)$ and $J$ is of contact type, then for any local Floer solution $u : \Omega \to [1, \infty) \times \partial M$ defined on a compact $\Omega \subset \mathbb{R} \times S^1$, the maxima of $R \circ u$ are attained on $\partial \Omega$. If $H_s = h_s(R)$ and $J = J_s$ depend on $s$, the result continues to hold provided that $\partial_s h^\prime_s \leq 0$. In particular, Floer solutions of $\partial_s u + J(\partial_t u - X_H) = 0$ or $\partial_s u + J_s(\partial_t u - X_H_s) = 0$ converging to $x_\pm$ at the ends are entirely contained in the region $R \leq \max R(x_{\pm})$.*

**Proof.** On the collar $u(s, t) = (R(s, t), m(s, t)) \in [1, \infty) \times \partial M$ and we can orthogonally decompose

$$T([1, \infty) \times \partial M) = \mathbb{R} \partial_t \oplus \mathbb{R} \mathcal{R} \oplus \xi$$

where $\xi = \ker \alpha$ is the contact distribution. By the contact type condition, $J \partial_t = \mathcal{R}$, $J \mathcal{R} = -\partial_t$ and $J$ restricts to an endomorphism of $\xi$. Since $X_H = h'(R) \mathcal{R}$, Floer’s equation in the first two summands $\mathbb{R} \partial_t \oplus \mathbb{R} \mathcal{R}$ after rescaling by $R$ is

$$\partial_s R - \theta(\partial_t u) + Rh' = 0 \quad \partial_t R + \theta(\partial_s u) = 0.$$

Adding $\partial_s$ of the first and $\partial_t$ of the second equation, yields $\partial_s^2 R + \partial_t^2 R + Rh'' = |\partial_s u|^2$. So $LR \geq 0$ for the elliptic operator $L = \partial_s^2 + \partial_t^2 + Rh''(R) \partial_s$, thus a standard result in PDE theory [3] Theorem 6.4.4] ensures the maximum principle for $R \circ u$. 
If $h_s$ depends on $s$ and $\partial_s h'_s \leq 0$, then we get $LR = |\partial_s u|^2 - R(\partial_s h'_s)(R) \geq 0$ which guarantees the maximum principle for $R$. □

\[\begin{array}{c}
\text{Figure 1. The } x, y', y'', y \text{ are 1-periodic orbits of } X_H, \text{ the lines are Floer solutions in } M. \text{ The } u_n \in M_1(x, y) \text{ are converging to the broken trajectory } (u'_1, u'_2) \in M_0(x, y') \times M_0(y', y).
\end{array}\]

If $\omega$ were exact on $M$, then the a priori energy estimate for $\mathcal{M}(x_-, x_+)$ described in 2.5 together with the maximum principle would ensure that the moduli spaces $\mathcal{M}(x_-, x_+)$ have compactifications $\overline{\mathcal{M}}(x_-, x_+)$ whose boundaries are defined in terms of broken Floer trajectories (Figure 1). In the proof of compactness, the exactness of $\omega$ excludes the possibility of bubbling-off of $J$–holomorphic spheres.

In the non-exact case if we assume that no bubbling-off of $J$-holomorphic spheres occurs, then the same techniques guarantee that, for any $K \in \mathbb{R}$, the moduli space

\[\mathcal{M}(x_-, x_+; K) = \{ u \in \mathcal{M}(x_-, x_+) : E(u) \leq K \}\]

of bounded energy solutions has a compactification by broken trajectories.

**Assumptions.** We assume henceforth that no bubbling occurs. If $c_1(M) = 0$ this will hold by Hofer-Salamon [7], as in our applications. To keep the notation under control we continue to write $(H, J)$ although one should use perturbed $(H_t, J_t)$.

3. *Symplectic Cohomology*

3.1. **Novikov Symplectic Chain Complex.** Let $\Lambda$ denote the Novikov ring,

\[\Lambda = \left\{ \sum_{j=0}^{\infty} n_j t^{a_j} : n_j \in \mathbb{Z}, a_j \in \mathbb{R}, \lim_{j \to \infty} a_j = \infty \right\}.\]

In [10] we allowed only integer values of $a_j$ because we were always using integral forms. In that setup $\Lambda$ was just the ring of formal integral Laurent series. In the present paper the $a_j$ will arise from integrating real forms so we use real $a_j$.

For an abelian group $G$ the Novikov completion $\hat{G}((t))$ is the $\Lambda$–module of formal sums $\sum_{j=0}^{\infty} g_j t^{a_j}$ where $g_j \in G$ and the real numbers $a_j \to \infty$.

Let $H \in C^\infty(\overline{M}, \mathbb{R})$ be a Hamiltonian which on the collar is of the form $H = h(R)$, where $h$ is linear at infinity. Define $CF^*$ to be the abelian group freely
generated by 1-periodic orbits of $X_H$,

$$CF^*(H) = \bigoplus \left\{ \mathbb{Z} \cdot x : x \in \mathcal{L}\widehat{M}, \dot{x}(t) = X_H(x(t)) \right\}.$$ 

It is always understood that we first make a generic $C^2$–small time-perturbation $H_t$ of $H$, so that there are only finitely many 1–periodic orbits of $X_{H_t}$, and therefore $CF^*(H)$ is finitely generated.

The symplectic chain complex $SC^*(H)$ is the Novikov completion of $CF^*(H)$:

$$SC^* = CF^*((t)) = \left\{ \sum_{j=0}^\infty c_j t^{a_j} : c_j \in CF^*, \lim a_j = \infty \right\} = \left\{ \sum \lambda_i y_i : \lambda_i \in \Lambda, N \in \mathbb{N}, y_i \text{ is a 1–periodic orbit of } X_H \right\}.$$

The differential $\delta$ is defined by

$$\delta \left( \sum_{i=0}^N \lambda_i y_i \right) = \sum_{i=0}^N \sum \epsilon(u_i) t^{E(u)} \lambda_i x$$

where $\mathcal{M}_0(x, y_i)$ is the 0–dimensional component of the Floer trajectories connecting $x$ to $y_i$, and $\epsilon(u_i)$ are signs depending on orientations. The sum is well-defined because there are only finitely many generators $x$, and below any energy bound $E(u) \leq K$ the moduli space $\mathcal{M}_0(x, y_i)$ is compact and therefore finite.

**Lemma 2.** $SC^*(H)$ is a chain complex, i.e. $\delta \circ \delta = 0$. We denote the cohomology of $(SC^*(H), \delta)$ by $SH^*(H)$.

**Proof.** This involves a standard argument (see Salamon [11]). Observe Figure [1] The 1–dimensional moduli space $\mathcal{M}_1(x, y)$ has a compactification, such that the boundary consists of pairs of Floer trajectories joined at one end. Observe that $E(\cdot)$ is additive with respect to concatenation and $E(u)$ is invariant under homotoping $u$ relative ends. Therefore, in Figure [1] $E(u'_1) + E(u'_2) = E(u''_1) + E(u''_2)$. Since $\epsilon(u'_1) \epsilon(u'_2) = -\epsilon(u''_1) \epsilon(u''_2)$, we deduce

$$\epsilon(u'_1) t^{E(u'_1)} \epsilon(u'_2) t^{E(u'_2)} = -\epsilon(u''_1) t^{E(u''_1)} \epsilon(u''_2) t^{E(u''_2)}.$$

Thus the broken trajectories contribute opposite $\Lambda$–multiples of $x$ to $\delta(\delta y)$ for each connected component of $\mathcal{M}_1(x, y)$. Hence, summing over $x, y'$,

$$\delta(\delta y) = \sum_{(u'_1, u'_2) \in \mathcal{M}_0(x, y') \times \mathcal{M}_0(y', y)} \epsilon(u'_1) t^{E(u'_1)} \epsilon(u'_2) t^{E(u'_2)} x = 0. \quad \square$$

### 3.2. Continuation Maps.

Under suitable conditions on two Hamiltonians $H_{\pm}$, it is possible to define a continuation homomorphism

$$\varphi : SC^*(H_+) \to SC^*(H_-).$$

This involves counting parametrized Floer trajectories, the solutions of

$$\partial_x v + J_x(\partial_t v - X_{H_s}) = 0.$$

Here $J_s$ are $\omega$–compatible almost complex structures of contact type and $H_s$ is a homotopy from $H_-$ to $H_+$, such that $(H_s, J_s) = (H_-, J_-)$ for $s \ll 0$ and $(H_s, J_s) = (H_+, J_+)$ for $s \gg 0$. The conditions on $H_s$ will be described in Theorem [3].

If $x$ and $y$ are respectively 1–periodic orbits of $X_{H_+}$ and $X_{H_-}$, then let $\mathcal{M}(x, y)$ be the moduli space of such solutions $v$ which converge to $x$ and $y$ at the ends. This time there is no freedom to reparametrize $v$ in the $s$–variable.
The continuation map \( \varphi \) on a generator \( y \in \text{Zeros}(dA_{H_s}) \) is defined by

\[
\varphi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v) t^{E_0(v)} x
\]

where \( \mathcal{M}_0(x,y) \) is the 0–dimensional part of \( \mathcal{M}(x,y) \), \( \epsilon(v) \in \{ \pm 1 \} \) are orientation signs and the power of \( t \) in the above sum is

\[
E_0(v) = -\int_{-\infty}^{\infty} \partial_s A_{H_s}(v) \, ds
= \int [\partial_v v]_g^2 \, ds \wedge dt - \int (\partial_s H_s)(v) \, ds \wedge dt
= \int v^* (\omega - dK \wedge dt),
\]

where \( K(s,m) = H_s(m) \). The last expression shows that \( E_0(v) \) is invariant under homotoping \( v \) relative ends.

3.3. Energy of parametrized Floer trajectories. Let \( H_s \) be a homotopy of Hamiltonians. For an \( H_s \)–Floer trajectory the above weight \( E_0(v) \) will be positive if \( H_s \) is monotone decreasing, \( \partial_s H_s \leq 0 \). The energy is

\[
E(v) = E_0(v) + \int (\partial_s H_s)(v) \, ds \wedge dt.
\]

If \( \partial_s H_s \leq 0 \) outside of a compact subset of \( \hat{M} \), then a bound on \( E_0(v) \) imposes a bound on \( E(v) \). Note that \( E(v) \) is not invariant under homotoping \( v \) relative ends.

3.4. Properties of continuation maps.

**Theorem 3** (Monotone homotopies). Let \( H_s \) be a homotopy between \( H_\pm \) such that

1. on the collar \( H_s = h_s(R) \) for large \( R \);
2. \( \partial_s h_s \leq 0 \) for \( R \geq R_\infty \), some \( R_\infty \);
3. \( h_s \) is linear for \( R \geq R_\infty \) (the slope may be a Reeb period, but not for \( h_\pm \)).

Then, after a generic \( C^2 \)–small time-dependent perturbation of \( (H_s, J_s) \),

1. all parametrized Floer trajectories lie in the compact subset

\[
C = M \cup \{ R \leq R_\infty \} \subset \hat{M};
\]

2. \( \mathcal{M}(x; y) \) is a smooth manifold;
3. \( \mathcal{M}(x, y; K) = \{ v \in \mathcal{M}(x, y) : E_0(v) \leq K \} \) has a smooth compactification by broken trajectories, for any constant \( K \in \mathbb{R} \);
4. the continuation map \( \varphi : SC^*(H_+) \to SC^*(H_-) \) is well-defined;
5. \( \varphi \) is a chain map.

**Proof.** (1) is a consequence of the maximum principle, Lemma 1 and (2) is a standard transversality result. Let

\[
BC = \max_{x \in C} \{ \partial_s H_s(x), 0 \}.
\]

Suppose \( H_s \) varies in \( s \) precisely for \( s \in [s_0, s_1] \). Since all \( v \in \mathcal{M}(x, y; K) \) lie in \( C \), \( \int \partial_s H_s(v) \, ds \wedge dt \leq (s_1 - s_0)BC \), so there is an a priori energy bound

\[
E(v) \leq K + (s_1 - s_0)BC.
\]

From this the compactness of \( \mathcal{M}(x, y; K) \) follows by standard methods.

The continuation map \( \varphi \) involves a factor of \( t^{E_0(v)} \). The lower bound \( E_0(v) \geq E(v) - (s_1 - s_0)BC \) guarantees that as the energy \( E(v) \) increases also the powers \( t^{E_0(v)} \) increase, which proves (4).
Showing that \( \varphi \) is a chain map is a standard argument which involves investigating the boundaries of broken trajectories of the 1–dimensional moduli spaces \( \mathcal{M}_1(x, y; K) \). A sequence \( v_n \) in some 1–dimensional component of \( \mathcal{M}_1(x, y; K) \) will converge (after reparametrization) to a concatenation of two trajectories \( u^+ \# v \) or \( v \# u^- \), where \( u^+ \in \mathcal{M}^{H^+}_0(x, x') \), \( v \in \mathcal{M}_0(x', y) \), or respectively \( v \in \mathcal{M}_0(x, y') \), \( u^- \in \mathcal{M}^{H^-}_0(y', y) \). Such solutions get counted with the same weight
\[
E_0(v_n) = E_0(u^+ \# v) = E_0(v \# u^-)
\]
because \( E_0 \) is invariant under homotopies which fix the ends, and \( v_n, u^+ \# v, v \# u^- \) are homotopic since they belong to the compactification of the same 1–dimensional component of \( \mathcal{M}_1(x, y) \). Therefore, \( \partial_{H^-} \circ \varphi = \varphi \circ \partial_{H^+} \) as required. \( \square \)

3.5. Chain homotopies.

**Theorem 4.**

1. Given monotone homotopies \( H_s, K_s \) from \( H_- \) to \( H_+ \), there is a chain homotopy \( Y : SC^*(H_+) \to SC^*(H_-) \) between the respective continuation maps: \( \varphi - \psi = \partial_{H^-} Y + Y \partial_{H^+} \);
2. the chain map \( \varphi \) defines a map on cohomology,
\[
[\varphi] : SH^*(H_+) \to SH^*(H_-),
\]
which is independent of the choice of the homotopy \( H_s \);
3. the composite of the maps induced by homotoping \( H_- \) to \( K \) and \( K \) to \( H_+ \),
\[
SC^*(H_+) \to SC^*(K) \to SC^*(H_-),
\]
is chain homotopic to \( \varphi \) and equals \([\varphi]\) on \( SH^*(H_+) \);
4. the constant homotopy \( H_s = H \) induces the identity on \( SC^*(H) \);
5. if \( H_\pm \) have the same slope at infinity, then \([\varphi]\) is an isomorphism.

**Proof.** Let \((H_{s, \lambda})_{0 \leq \lambda \leq 1}\) be a linear interpolation of \( H_s \) and \( K_s \), so that \( H_{s, \lambda} \) is a monotone Hamiltonian for each \( \lambda \). Consider the moduli spaces \( \mathcal{M}(x, y; \lambda) \) of parametrized Floer solutions for \( H_{s, \lambda} \). Let \( Y \) be the oriented count of the pairs \((\lambda, v)\), counted with weight \( f^{E_0(v)} \), where \( 0 < \lambda < 1 \) and \( v \) is in a component of \( \mathcal{M}(x, y; \lambda) \) of virtual dimension \(-1\) (generically \( \mathcal{M}_{-1}(x, y; \lambda) \) is empty, but in the family \( \cup_{\lambda} \mathcal{M}_{-1}(x, y; \lambda) \) such isolated solutions \((\lambda, v)\) can arise).

Consider a sequence \((\lambda_n, v_n)\) inside a 1–dimensional component of \( \cup_{\lambda} \mathcal{M}(x, y; \lambda) \), such that \( \lambda_n \to \lambda \). If \( \lambda = 0 \) or \( 1 \), then the limit of the \( v_n \) can break by giving rise to an \( H_s \) or \( K_s \) Floer trajectory, and this breaking is counted by \( \varphi - \psi \). If \( 0 < \lambda < 1 \), then the \( v_n \) can break by giving rise to \( u^- \# v \) or \( v \# u^+ \), where \( u^\pm \) are \( H_{s, \lambda} \)-Floer trajectories and the \( v \) are as in the definition of \( Y \). This type of breaking is therefore counted by \( \partial_{H_-} Y + Y \partial_{H^+} \).

Both sides of the relation \( \varphi - \psi = \partial_{H_-} Y + Y \partial_{H^+} \) will count a (broken) trajectory with the same weight because \( E_0(\cdot) \) is a homotopy invariant relative ends and the broken trajectories are all homotopic, since they arise as the boundary of the same 1–dimensional component of \( \cup_{\lambda} \mathcal{M}(x, y; \lambda) \).

Claims (2) and (3) are standard consequences of (1) (see Salamon [11]). Claim (4) is a consequence of the fact that any non-constant Floer trajectory for \( H_s = H \) would come in a 1–dimensional family of solutions, due to the translational freedom in \( s \). Claim (5) follows from (3) and (4): we can choose \( H_s \) to have constant slope for large \( R \), therefore \( H_{s, \lambda} \) is also a monotone homotopy, and the composite of the chain maps induced by \( H_s \) and \( H_{s, \lambda} \) is chain homotopic to the identity. \( \square \)
3.6. **Hamiltonians linear at infinity.** Consider Hamiltonians $H^m$ which equal
\[ h^m(R) = mR + C \]
for $R \gg 0$, where the slope $m > 0$ is not the period of any Reeb orbit. Up to isomorphism, $SH^*(H)$ is independent of the choice of $C$ by Theorem [4].

For $m_+ < m_-$, a monotone homotopy $H_s$ defines a continuation map
\[ \phi^{m_+m_-} : SC^*(H^{m_+}) \to SC^*(H^{m_-}), \]
for example the homotopy $h_s(R) = m_s R + C_s$ for $R \gg 0$, with $\partial_s m_s \leq 0$.

By Theorem [4] the continuation map $[\phi^{m_+m_-}] : SH^*(H^{m_+}) \to SH^*(H^{m_-})$ on cohomology does not depend on the choice of homotopy $h_s$. Moreover, such continuation maps compose well: $\phi^{m_2m_3} \circ \phi^{m_1m_2}$ is chain homotopic to $\phi^{m_1m_3}$ where $m_1 < m_2 < m_3$, and so $[\phi^{m_2m_3}] \circ [\phi^{m_1m_2}] = [\phi^{m_1m_3}]$.

3.7. **Symplectic cohomology.**

**Definition 5.** The symplectic cohomology is defined to be the direct limit
\[ SH^*(M, \omega) = \lim_{\to} SH^*(H) \]
over the continuation maps between Hamiltonians linear at infinity.

Observe that $SH^*(M, \omega)$ can be calculated as the direct limit
\[ \lim_{k \to \infty} SH^*(H_k) \]
over the continuation maps $SH^*(H_k) \to SH^*(H_{k+1})$, where the slopes at infinity of the Hamiltonians $H_k$ increase to infinity as $k \to \infty$.

3.8. **The maps $c_*$ from ordinary cohomology.** The symplectic cohomology comes with a map from the ordinary cohomology of $M$ with coefficients in $\Lambda$,
\[ c_* : H^*(M; \Lambda) \to SH^*(M, \omega). \]

We sketch the construction here, and refer to [10] for a detailed construction. Fix a $\delta > 0$ which is smaller than all periods of the nonconstant Reeb orbits on $\partial M$. Consider Hamiltonians $H^\delta$ which are $C^2$-close to a constant on $M$ and such that on the collar $H^\delta = h(R)$ with constant slope $h'(R) = \delta$.

A standard result in Floer cohomology is that, after a generic $C^2$-small time-independent perturbation of $(H^\delta, J)$, the 1-periodic orbits of $X_{H^\delta}$ and the connecting Floer trajectories are both independent of $t \in S^1$. By the choice of $\delta$ there are no 1-periodic orbits on the collar, and by the maximum principle no Floer trajectory leaves $M$. The Floer complex $CF^*(H^\delta)$ is therefore canonically identified with the Morse complex $CM^*(H^\delta)$, which is generated by $\text{Crit}(H^\delta)$ and whose differential counts the negative gradient trajectories of $H^\delta$ with weights $t^{H^\delta(x_-)} - t^{H^\delta(x_+)}$. After the change of basis $x \mapsto t^{H^\delta(x)} x$, the differential reduces to the ordinary Morse complex defined over the ring $\Lambda$ which is isomorphic to the singular cochain complex of $M$ with coefficients in $\Lambda$. Thus
\[ SH^*(H^\delta) \cong HM^*(H^\delta; \Lambda) \cong H^*(M; \Lambda). \]

Since $SH^*(H^\delta)$ is part of the direct limit construction of $SH^*(M, \omega)$, this defines a map $c_* : H^*(M; \Lambda) \to SH^*(M, \omega)$ independently of the choice of $H^\delta$. 
3.9. Invariance under symplectomorphisms of contact type.

**Definition 6.** Let $M, N$ be symplectic manifolds with contact type boundary. A symplectomorphism $\varphi : \hat{M} \to \hat{N}$ is of contact type at infinity if on the collar $\varphi^* \theta_N = \theta_M + d(\text{compactly supported function})$.

This implies that at infinity $\varphi$ has the form

$$\varphi(e^r, y) = (e^{-f(y)}, \psi(y)),$$

with $f : \partial M \to \mathbb{R}$ smooth, $\psi : \partial M \to \partial N$ a contactomorphism with $\psi^* \alpha_N = e^f \alpha_M$.

Under such a map $\varphi : \hat{M} \to \hat{N}$, the Floer solutions on $\hat{N}$ for $(H, \omega_N, J_N)$ correspond precisely to the Floer solutions on $\hat{M}$ for $(\varphi^* H, \omega_M, \varphi^* J_N)$. However, for a Hamiltonian $H$ on $\hat{N}$ which is linear at infinity, the Hamiltonian $\varphi^* H(e^r, y) = h(e^{-f(y)})$ is not linear at infinity. Thus we want to show that for this new class of Hamiltonians on $\hat{M}$ we still obtain the usual symplectic cohomology.

In order to relate the two symplectic cohomologies, we need a maximum principle for homotopies of Hamiltonians which equal $H_s = h_s(R_s)$ on the collar, where

$$R_s(e^r, y) = e^{-f_s(y)},$$

and $f_s = f_-, h_s = h_-$ for $s \ll 0$ and $f_s = f_+, h_s = h_+$ for $s \gg 0$. We prove that if $h_- \gg h_+$, then one can choose $h_s$ so that the maximum principle applies. We denote by $X_s$ the Hamiltonian vector field for $h_s$ and we assume that the almost complex structures $J_s$ satisfy the contact type condition $J_s^* \theta = dR_s$ for $e^r \gg 0$.

**Lemma 7** (Maximum principle). There is a constant $K > 0$ depending only on $f_s$ such that if $h_- \geq Kh_+$ then it is possible to choose a homotopy $h_s$ from $h_-$ to $h_+$ in such a way that the maximum principle applies to the function

$$\rho(s, t) = R_s(u(s, t)) = e^{(u) - f_s(y(u))}$$

where $u$ is any local solution of Floer’s equation $\partial_s u + J_s(\partial_t u - X_s) = 0$ which lands in the collar $e^r \gg 0$, and where $J_s^* \theta = dR_s$ for $e^r \gg 0$. In particular, a continuation map $SH^*(h_-) \to SH^*(h_+)$ can then be defined.

**Proof.** We will seek an equation satisfied by $\Delta \rho$. Using $J_s^* dR_s = -\theta$ we obtain

$$\partial_s \rho = \partial_t R_s(u) + dR_s \cdot \partial_s u = -\rho \partial_s f_s + dR_s \cdot \partial_s u$$

$$= -\rho \partial_s f_s + dR_s \cdot (J_s(X_s - \partial_t u)) = -\rho \partial_s f_s - \theta(X_s) + \theta(\partial_t u),$$

$$\partial_t \rho = dR_s \cdot \partial_t u = dR_s \cdot (X_s + J_s \partial_t u) = J_s^* dR_s \cdot \partial_t u = -\partial_t h_s(u).$$

Since $X_s = h_s^*(\rho)\mathcal{R}$ and $\theta(\mathcal{R}(u)) = \rho$, we deduce $\theta(X_s) = \rho h_s^*(\rho)$ so

$$d^* \rho = d\rho \circ \mathcal{R} = -\partial_s \rho dt + \partial_t \rho ds = -u^* \omega + \rho h_s^*(\rho) dt + \rho \partial_s f_s dt.$$ 

Therefore $dd^* \rho = -\Delta \rho ds \wedge dt = -u^* \omega + F ds \wedge dt$ where

$$F = h_s^* \partial_s \rho + \rho \partial_s h_s^* + \rho h_s^* \partial_s \rho + \partial_t \rho \partial_s f_s + \rho \partial_s f_s^2 + \rho d(\partial_s f_s) \cdot \partial_s u.$$

We now try to relate $u^* \omega$ with $|\partial_s u|^2$:

$$|\partial_s u|^2 = \omega(\partial_s u, \partial_t u - X_s) = u^* \omega - dH_s \cdot \partial_s u = u^* \omega - h_s^* dR_s \cdot \partial_s u$$

$$= u^* \omega - h_s^* \partial_s \rho - \rho \partial_s f_s,$$

where we used that $\partial_s \rho = -\rho \partial_s f_s + dR_s \cdot \partial_s u$.

Thus, $\Delta \rho = u^* \omega - F$ equals

$$|\partial_s u|^2 + h_s^* \rho \partial_s f_s - \rho \partial_s h_s^* - \rho h_s^* \partial_s \rho - \partial_t \rho \partial_s f_s - \rho \partial_s f_s^2 - \rho d(\partial_s f_s) \cdot \partial_s u.$$
We may assume that $f$ is $C^2$-bounded by a constant $C > 0$. Then in particular $|d(\partial_s f_s) \cdot \partial_u u| \leq \|d(\partial_s f_s)\|_{\rho \times \partial M} \cdot |\partial_u u| \leq \rho^{-1}\|d(\partial_s f_s)\|_{1 \times \partial M} \cdot |\partial_u u| \leq \rho^{-1}C|\partial_s u|$.

We deduce an inequality for $\Delta \rho$:

$$\Delta \rho + \text{first order terms} \geq |\partial_u u|^2 - \rho \partial_u h'_n - \rho (h'_n C + C) - C|\partial_u u|$$

$$\geq (|\partial_u u| - \frac{1}{2}C)^2 - \rho (\partial_u h'_n + h'_n C + C) - \frac{1}{4}C^2.$$

Adding $-\rho h'_n \partial_s \rho$ to both sides and dropping the square bracket, we deduce

$$\Delta \rho + \text{first order terms} \geq -\rho e^{-Cs} [\partial_u (e^{Cs} h'_n) + e^{Cs}(C + C^2)].$$

Thus a maximum principle will apply for $\rho$ if the right hand side is non-negative. Now $f_s$ only depends on $s$ on a finite interval $I$ of $s$-values. On the complement of $I$, $\partial_s f_s \equiv 0$ so actually $\Delta \rho +$ first order $\geq -\rho \partial_u h'_n$, so we only require $\partial_u h'_n \leq 0$. On $I$ we need the last square-bracket to be negative. Integrate this condition over $I$:

$$h'_n e^{C \cdot \text{length}(I)} - h'_n \leq C'$$

where $C'$ is a constant depending only on $C$ and the length of $I$. Thus it is possible to choose $h_n$ satisfying the above conditions, provided that $h'_n \gg h'_n$.

Theorem 8. If $\varphi : \widehat{M} \to \widehat{N}$ is a symplectomorphism of contact type at infinity, then $\text{SH}^*(M) \cong \text{SH}^*(N)$.

Proof. By identifying the Floer solutions via $\varphi$, the claim reduces to showing that the symplectic cohomology $\text{SH}^*(M) = \lim_{n \to \infty} \text{SH}^*(h_n)$ is isomorphic to the symplectic cohomology $\text{SH}^*_\omega(M) = \lim_{n \to \infty} \text{SH}^*_\omega(h_n)$ which is calculated for Hamiltonians of the form $H(e^f, y) = h(e^{-f}(y))$, where the $h$ are linear at infinity and $f : \partial M \to \mathbb{R}$ is a fixed smooth function.

Pick an interpolation $f_s$ from $f$ to $0$, constant in $s$ for large $|s|$. We can inductively construct Hamiltonians $h_n$ and $k_n$ on $\widehat{M}$ with $h'_n \gg k'_n$ and $k'_n \gg h'_n$, which by Lemma 4 yield continuation maps

$$\phi_n : \text{SH}^*_\omega(k_n) \to \text{SH}^*(h_n), \quad \psi_n : \text{SH}^*(h_n) \to \text{SH}^*_\omega(k_{n+1}).$$

We can arrange that the slope at infinity of the $h_n$, $k_n$ grow to infinity as $n \to \infty$, so that $\text{SH}^*_\omega(M) = \lim_{n \to \infty} \text{SH}^*_\omega(k_n)$ and $\text{SH}^*(M) = \lim_{n \to \infty} \text{SH}^*(h_n)$.

The composites $\psi_n \circ \phi_n$ and $\phi_{n+1} \circ \psi_n$ are equal to the ordinary continuation maps $\text{SH}^*_\omega(k_n) \to \text{SH}^*_\omega(k_{n+1})$ and $\text{SH}^*(h_n) \to \text{SH}^*(h_{n+1})$.

Therefore the maps $\phi_n$ and $\psi_n$ form a compatible family of maps and so define

$$\phi : \text{SH}^*_\omega(M) \to \text{SH}^*(M), \quad \psi : \text{SH}^*(M) \to \text{SH}^*_\omega(M).$$

The composites $\psi \circ \phi$ and $\phi \circ \psi$ are induced by the families $\psi_n \circ \phi_n$ and $\phi_{n+1} \circ \psi_n$, which are the ordinary continuation maps defining the direct limits $\text{SH}^*_\omega(M)$ and $\text{SH}^*(M)$. Hence $\phi \circ \psi, \psi \circ \phi$ are identity maps, and so $\phi, \psi$ are isomorphisms. □

3.10. Independence from choice of cohomology representative.

Lemma 9. Let $\eta$ be a one-form supported in the interior of $M$. Suppose there is a homotopy $\omega_s$ through symplectic forms from $\omega$ to $\omega + d\eta$. By Moser’s lemma this yields an isomorphism $\varphi : (\widehat{M}, \omega + d\eta) \to (\widehat{M}, \omega)$, and therefore a chain isomorphism

$$\varphi : \text{SC}^*(H, \omega + d\eta) \to \text{SC}^*(\varphi^* H, \omega),$$

which is the identity on orbits outside $M$ and sends the orbits $x$ in $M$ to $\varphi^{-1}x$. 
4. Twisted symplectic cohomology

4.1. Transgressions. Let $ev : LM \times S^1 \to LM$ be the evaluation map. Define

$$\tau = \pi \circ ev^* : H^2(M; \mathbb{R}) \xrightarrow{ev^*} H^2(\mathcal{L}M \times S^1; \mathbb{R}) \xrightarrow{\pi} H^1(\mathcal{L}M; \mathbb{R}),$$

where $\pi$ is the projection to the Künneth summand. Explicitly, $\tau \beta$ evaluated on a smooth path $u$ in $\mathcal{L}M$ is given by

$$\tau \beta(u) = \int \beta(\partial_\nu u, \partial_t u) \, ds \wedge dt.$$ 

In particular, $\tau \beta$ vanishes on time-independent paths in $\mathcal{L}M$. If $M$ is simply connected, then $\tau$ is an isomorphism. After identifying

$$H^1(\mathcal{L}M; \mathbb{R}) \cong \text{Hom}_\mathbb{R}(H_1(\mathcal{L}M; \mathbb{R}), \mathbb{R}) \cong \text{Hom}(\pi_1(\mathcal{L}M), \mathbb{R})$$

and $\pi_1(\mathcal{L}M) = \pi_2(M) \times \pi_1(M)$, the $\tau \beta$ correspond precisely to homomorphisms $\pi_2(M) \to \mathbb{R}$. In particular, if $\beta$ is an integral class then this homomorphism is $f_* : \pi_2(M) \to \mathbb{Z}$ where $f : M \to \mathbb{C}P\infty$ is a classifying map for $\beta$.

4.2. Novikov bundles of coefficients. We suggest [17] as a reference on local systems. Let $\alpha$ be a singular smooth real cocycle representing $\alpha \in H^1(\mathcal{L}M; \mathbb{R})$. The Novikov bundle $\Delta_\alpha$ is the local system of coefficients on $\mathcal{L}M$ defined by a copy $\Lambda_\gamma$ of $\Lambda$ over each loop $\gamma \in \mathcal{L}M$ and by the multiplication isomorphism

$$t^{-\alpha[u]} : \Lambda_\gamma \to \Lambda_{\gamma'},$$

for each path $u$ in $\mathcal{L}M$ connecting $\gamma$ to $\gamma'$. Here $\alpha[\cdot] : C_1(\mathcal{L}M; \mathbb{R}) \to \mathbb{R}$ denotes evaluation on smooth singular one-chains, which is given explicitly by

$$\alpha[u] = \int \alpha(\partial_\nu u) \, ds.$$ 

Changing $\alpha$ to $\alpha + df$ yields a change of basis isomorphism $x \mapsto t^{f(x)}x$ for the local systems, so by abuse of notation we write $\Delta_\alpha$ and $\alpha[u]$ instead of $\Delta_\alpha$ and $\alpha[u]$.

**Remark 10.** In [10] we used the opposite sign convention $t^{\alpha[u]}$. In this paper we changed it for the following reason. For a Liouville domain $(M, d\theta)$, the local system for the action 1-form $\alpha = dA_H$ acts on Floer solutions $u \in \mathcal{M}(x, y; d\theta)$ by

$$t^{-dA_H[u]} = t^{A_H(x) - A_H(y)}.$$

Therefore large energy Floer solutions will occur with high powers of $t$.

We will be considering the (co)homology of $M$ or $\mathcal{L}M$ with local coefficients in the Novikov bundles, and we now mention two recurrent examples. First consider a transgressed form $\alpha = \tau \beta$ (sec[14]). Since $\tau(\beta)$ vanishes on time-independent paths, $\Delta_{\tau \beta}$ pulls back to a trivial bundle via the inclusion of constant loops $c : M \to \mathcal{L}M$. So for the bundle $c^* \Delta_{\tau \beta}$ we just get ordinary cohomology with underlying ring $\Lambda$,

$$H^*(M; c^* \Delta_{\tau(\beta)}) \cong H^*(M; \Lambda).$$

Secondly, consider a map $j : L \to M$. This induces a map $\mathcal{L}j : \mathcal{L}L \to \mathcal{L}M$ which by the naturality of $\tau$ satisfies $(\mathcal{L}j)^* \Delta_{\tau(\beta)} \cong \Delta_{(\mathcal{L}j)^* \beta}$. For example if $\tau(j^* \beta) = 0 \in H^1(\mathcal{L}L; \mathbb{R})$ then this is a trivial bundle, so the corresponding Novikov homology is

$$H_*(\mathcal{L}L; (\mathcal{L}j)^* \Delta_{\tau(\beta)}) \cong H_*(\mathcal{L}L) \otimes \Lambda.$$
4.3. **Twisted Floer cohomology.** Let \((M^{2n}, \theta)\) be a Liouville domain. Let \(\alpha\) be a singular cocycle representing a class in \(H^1(\mathcal{L}M; \mathbb{R}) \cong H^1(\mathcal{L}\hat{M}; \mathbb{R})\). The Floer chain complex for \(H \in C^\infty(\hat{M}, \mathbb{R})\) with twisted coefficients in \(\Lambda_\alpha\) is the \(\Lambda\)-module freely generated by the \(1\)-periodic orbits of \(X_H\),

\[
CF^*(H; \Lambda_\alpha) = \bigoplus \left\{ \Lambda x : x \in \mathcal{L}\hat{M}, \dot{x}(t) = X_H(x(t)) \right\},
\]

and the differential \(\delta\) on a generator \(y \in \text{Crit}(A_H)\) is defined as

\[
\delta y = \sum_{u \in M_0(x,y)} \epsilon(u) t^{-\alpha[u]} x,
\]

where \(\epsilon(u) \in \{\pm 1\}\) are orientation signs and \(M_0(x,y)\) is the \(0\)-dimensional component of Floer trajectories connecting \(x\) to \(y\). It is always understood that we perturb \((H, J)\) as explained in [2.6].

The ordinary Floer complex (with underlying ring \(\Lambda\)) has no weights \(t^{-\alpha[u]}\) in \(\delta\). These appear in the twisted case because they are the multiplication isomorphisms \(\Lambda_x \to \Lambda_y\) of the local system \(\Lambda_\alpha\), which identify the \(\Lambda\)-fibres over \(x\) and \(y\) (see [12]).

**Proposition/Definition 11.** \([10]\) \(CF^*(H; \Lambda_\alpha)\) is a chain complex: \(\delta \circ \delta = 0\), and its cohomology \(HF^*(H; \Lambda_\alpha)\) is a \(\Lambda\)-module called twisted Floer cohomology.

4.4. **Twisted symplectic cohomology.**

**Proposition 12** (Twisted continuation maps, \([10]\)). For the twisted Floer cohomology of \((M, d\theta)\), Theorem [3] continues to hold for the continuation maps \(\phi : CF^*(H_+; \Lambda_\alpha) \to CF^*(H_-; \Lambda_\alpha)\) defined on generators \(y \in \text{Crit}(A_{H_+})\) by

\[
\phi(y) = \sum_{v \in M_0(x,y)} \epsilon(v) t^{-\alpha[v]} x.
\]

**Definition 13.** The twisted symplectic cohomology of \((M, d\theta; \alpha)\) is

\[
SH^*(M, d\theta; \Lambda_\alpha) = \lim_{\alpha \to \infty} HF^*(H, d\theta; \Lambda_\alpha),
\]

where the direct limit is over the twisted continuation maps between Hamiltonians \(H\) which are linear at infinity.

4.5. **Independence from choice of cohomology representative.**

**Lemma 14.** Let \(f_H \in C^\infty(\mathcal{L}\hat{M}, \mathbb{R})\) be an \(H\)-dependent function. Then the change of basis isomorphisms \(x \mapsto t^{f_H(x)} x\) of the local systems induce chain isomorphisms

\[
SC^*(H, d\theta; \Lambda_\alpha) \cong SC^*(H, d\theta; \Lambda_{\alpha + df_H})
\]

which commute with the twisted continuation maps.

4.6. **Twisted maps \(c_*\) from ordinary cohomology.** The twisted symplectic cohomology comes with a map from the induced Novikov cohomology of \(M\),

\[
c_* : H^*(M; c^*\Lambda_\alpha) \to SH^*(M; d\theta, \Lambda_\alpha).
\]

The construction is analogous to [10] and was carried out in detail in [10]. The map \(c_*\) comes automatically with the direct limit construction of \(SH^*(M; d\theta, \Lambda_\alpha)\), since for the Hamiltonian \(H^\delta\) described in [10] we have

\[
HF^*(H^\delta; \Lambda_\alpha) \cong HM^*(H^\delta; c^*\Lambda_\alpha) \cong H^*(M; c^*\Lambda_\alpha).
\]
4.7. Twisted Functoriality. In [10] we proved the following variant of Viterbo functoriality [15], which holds for Liouville subdomains \((W^{2n}, \theta') \subset (M^{2n}, \theta)\). These are Liouville domains for which \(\theta - e^\rho \theta'\) is exact for some \(\rho \in \mathbb{R}\). The standard example is the Weinstein embedding \(DT^*L \hookrightarrow DT^*N\) of a small disc cotangent bundle of an exact Lagrangian \(L \hookrightarrow DT^*N\) (see [10]).

**Theorem 15.** [10] Let \(i : (W^{2n}, \theta') \hookrightarrow (M^{2n}, \theta)\) be a Liouville embedded subdomain. Then there exists a map

\[
SH^*(i) : SH^*(W, d\theta'; \Delta_{(L_i)^*\alpha}) \leftarrow SH^*(M, d\theta; \Delta_\alpha)
\]

which fits into the commutative diagram

\[
\begin{cd}
SH^*(W, d\theta'; \Delta_{(L_i)^*\alpha}) & \arrow[d,e_\ast] & SH^*(M, d\theta; \Delta_\alpha) \\
H^*(W; c^*\Delta_{(L_i)^*\alpha}) & \arrow[d,e_\ast] & H^*(M; c^*\Delta_\alpha)
\end{cd}
\]

The map \(SH^*(i)\) is constructed using a “step-shaped” Hamiltonian, as in Figure 2, which grows near \(\partial W\) and reaches a slope \(a\), then becomes constant up to \(\partial M\) where it grows again up to slope \(b\). By a careful construction, with \(a \gg b\), one can arrange that all orbits in \(W\) have negative action with respect to \((d\theta, H)\), and for orbits outside of \(W\) they have positive actions. The map \(SH^*(i)\) is then the limit, as \(a \gg b \to \infty\), of the action restriction maps which quotient out by the generators of positive action.

![Figure 2](image-url)

**Figure 2.** The solid line is a diagonal-step shaped Hamiltonian with \(a \gg b\). The dashed line is the action \(A(R) = -Rh'(R) + h(R)\).

**Theorem 16.** [10] Let \((M, d\theta)\) be a Liouville domain and let \(L \subset M\) be an exact orientable Lagrangian submanifold. By Weinstein’s Theorem, this defines a Liouville embedding \(j : (DT^*L, d\theta) \to (M, d\theta)\) of a small disc cotangent bundle of \(L\). Then for all \(\alpha \in H^1(L\mathcal{M}; \mathbb{R})\) there exists a commutative diagram

\[
\begin{cd}
H_{n-\varepsilon}(\mathcal{L}L; \Delta_{(L_j)^*\alpha}) & \arrow[d,e_\ast] & SH^*(T^*L, d\theta; \Delta_{(L_j)^*\alpha}) & \arrow[d,e_\ast] & SH^*(M, d\theta; \Delta_\alpha) \\
H_{n-\varepsilon}(L; c^*\Delta_{(L_j)^*\alpha}) & \arrow[d,e_\ast] & H^*(L; c^*\Delta_{(L_j)^*\alpha}) & \arrow[d,e_\ast] & H^*(M; c^*\Delta_\alpha)
\end{cd}
\]
where the left vertical map is induced by the inclusion of constant loops \( c : L \to \mathcal{L}L \). If \( c^*\alpha = 0 \) then the bottom map is the pullback \( H^*(L) \otimes \Lambda \leftarrow H^*(M) \otimes \Lambda \).

**Corollary 17.** Let \((M, d\theta)\) be a Liouville domain and let \( L \subset M \) be an exact orientable Lagrangian. Suppose \( \beta \in H^2(\widehat{M}; \mathbb{R}) \) is such that \( \tau(j^*\beta) = 0 = H^1(\mathcal{L}L; \mathbb{R}) \). Then there is a commutative diagram

\[
\begin{array}{ccc}
H_n(L) \otimes \Lambda & \xrightarrow{c_*} & SH^*(M, d\theta; \Delta_{x,\beta}) \\
\downarrow & & \downarrow \\
H_n(L) \otimes \Lambda & \xrightarrow{c_*} & H^*(M) \otimes \Lambda
\end{array}
\]

Therefore \( SH^*(M, d\theta; \Delta_{x,\beta}) \) cannot vanish since \( c_*j^*1 = c_*1 \neq 0 \).

**Remark 18.** Unorientable exact Lagrangians. In Theorem 16 we assumed that the Lagrangian is orientable. However, the result easily extends to the unorientable case: instead of using \( \mathbb{Z} \) coefficients we use \( \mathbb{Z}_2 \) coefficients. This means that the moduli spaces do not need to be oriented and we can drop all orientation signs in the definitions of the differentials for the Floer complexes and the Morse complexes. The Novikov ring is now defined by

\[
\Lambda = \{ \sum_{n=0}^{\infty} a_n t^n : a_n \in \mathbb{Z}_2, r_n \in \mathbb{R}, r_n \to \infty \}.
\]

Note that the Novikov one-form \( \alpha \) is still chosen in \( H^1(\mathcal{L}M; \mathbb{R}) \).

This is particularly interesting in dimension four since \( H^2(\mathcal{L}L; \mathbb{R}) = 0 \) for unorientable \( L^2 \subset M^4 \), therefore the transgression vanishes. In particular the pullback of any transgression from \( M \) will vanish on \( L \). This immediately contradicts Corollary 17 if \( SH^*(M, d\theta; \Delta_{x,\beta}) = 0 \). For example in [10] we proved that \( SH^*(T^*S^2, d\theta; \Delta_{x,\beta}) = 0 \) for any non-zero \( \beta \in H^2(S^2; \mathbb{R}) \). Therefore there can be no unorientable exact Lagrangians in \( T^*S^2 \).

5. Grading of symplectic cohomology

5.1. **Maslov index and Conley-Zehnder grading.** We assume that \( c_1(M) = 0 \): this condition will ensure that the symplectic cohomology has a \( \mathbb{Z} \)-grading defined by the Conley-Zehnder index.

Since \( c_1(M) = 0 \), we can choose a trivialization of the canonical bundle \( \mathcal{K} = \Lambda^{n,0}TM \). Then over any 1-periodic Hamiltonian orbit \( \gamma \), trivialize \( \gamma \cdot TM \) so that it induces an isomorphic trivialization of \( \mathcal{K} \). Let \( \phi_t \) denote the linearization \( D\varphi^t(\gamma(0)) \) of the time \( t \) Hamiltonian flow written in a trivializing frame for \( \gamma \cdot TM \).

Let \( \text{sign}(t) \) denote the signature of the quadratic form

\[
\omega(\cdot, \partial_t \phi_t) : \ker(\phi_t - \text{id}) \to \mathbb{R},
\]

assuming we perturbed \( \phi_t \) relative endpoints to make the quadratic form non-degenerate and to make \( \ker(\phi_t - \text{id}) = 0 \) except at finitely many \( t \).

The Maslov index \( \mu(\gamma) \) of \( \gamma \) is

\[
\mu(\gamma) = \frac{1}{2} \text{sign}(0) + \sum_{0 < t < 1} \text{sign}(t) + \frac{1}{2} \text{sign}(1).
\]

The Maslov index is invariant under homotopy relative endpoints, and it is additive with respect to concatenations. If \( \phi_t \) is a loop of unitary transformations,
then its Maslov index is the winding number of the determinant, det $\phi_t : K \to K$. For example $\phi_t = e^{2\pi it} \in U(1)$ for $t \in [0, 1]$ has Maslov index 1.

In our applications, $\gamma$ will often not be an isolated orbit. It will typically lie in an $S^1$-worth or an $S^3$-worth of orbits. In this case it is possible to make a small time-dependent perturbation of $H$ so that $\gamma$ breaks up into two isolated orbits whose Maslov indices get shifted by $\pm \dim(S^1)/2$ or $\pm \dim(S^3)/2$ respectively.

The grading we use on $\text{SH}^*$ is the Conley-Zehnder index, defined by

$$|\gamma| = \frac{\dim(M)}{2} - \mu(\gamma).$$

This grading agrees with the Morse index when $H$ is a generic $C^2$-small Hamiltonian and $\gamma$ is a critical point of $H$.

6. Deformation of the Symplectic cohomology

Let $\beta$ be a compactly supported two-form representing a class in $H^2(M; \mathbb{R})$ such that $d\theta + \beta$ is symplectic. We want to construct an isomorphism between the non-exact symplectic cohomology and the twisted symplectic cohomology $\text{SH}^*(H, d\theta + \beta) \cong \text{SH}^*(H, d\theta; \Lambda_{r\beta})$.

We will show that this holds if $d\theta + s\beta$ is symplectic for $0 \leq s \leq 1$. For example, it will always hold if $\|\beta\| < 1$.

6.1. Outline of the argument. Let $H^m$ denote a Hamiltonian which only depends on $R$ on the collar and which has slope $m$ at infinity. Choosing $H^m$ generic and $C^2$-small inside $M$ ensures that the only 1-periodic Hamiltonian orbits inside $M$ are the critical points of $H^m$. We will prove that we may assume that the critical points lie outside the support of $\beta$. Therefore $SC^*(H^m, d\theta + \beta)$ and $SC^*(H^m, d\theta; \Lambda_{r\beta})$ have the same generators: the critical points of $H^m$ and the 1-periodic Hamiltonian orbits lying in the collar (we used that supp $\beta \subset M$).

We will build chain isomorphisms $\psi^m : SC^*(H^m, d\theta + \beta) \to SC^*(H^m, d\theta; \Lambda_{r\beta})$ which are defined for a sufficiently large parameter $\mu$; which are independent of $\mu$ on homology, say $\psi^m = \psi^m_\mu$; and which commute with the continuation maps

$$\text{SH}^*(H^m, d\theta + \beta) \overset{\psi_\mu^m}{\longrightarrow} \text{SH}^*(H^m, d\theta; \Lambda_{r\beta})$$

Therefore, by exactness of direct limits, $\psi = \lim \psi^m_\mu$ is the desired isomorphism $\psi : \text{SH}^*(M, d\theta + \beta) \to \text{SH}^*(M, d\theta, \Lambda_{r\beta})$.

The parameter $\mu$ arises in the construction of the maps $\psi^m_\mu$ because for large $\mu$ the identity map provides a natural chain isomorphism $\text{id} : SC^*(H^m, d\theta + \mu^{-1}\beta) \cong SC^*(H^m, d\theta; \Lambda_{\mu^{-1}r\beta})$.

This is proved by showing that the moduli spaces $\mathcal{M}(x, y; d\theta + \lambda \beta)$ form a 1-parameter family joining $\mathcal{M}(x, y; d\theta + \mu^{-1}\beta)$ to $\mathcal{M}(x, y; d\theta)$. 
To define the maps \( \psi^m_\mu \) we therefore just need to deform \( d\theta + \beta \) to \( d\theta + \mu^{-1}\beta \).

On the twisted side, there are no difficulties:

\[
SH^*(H^m, d\theta; \Lambda) \cong SH^*(H^m, d\theta; \Lambda_{\mu^{-1}\beta})
\]

is just a rescaling \( t \mapsto t^{(\mu^{-1})} \).

For the non-exact symplectic cohomology we first combine the Liouville flow \( \varphi_\mu \) for time \( \log \mu \) and a rescaling of the metric by \( \mu^{-1} \). This will change \( d\theta + \beta \) to \( d\theta + \mu^{-1}\varphi_\mu^*\beta \). Then we want to make a Moser deformation from \( d\theta + \mu^{-1}\varphi_\mu^*\beta \) to \( d\theta + \mu^{-1}\beta \), so we need a deformation through symplectic forms without changing the cohomology class. This is possible if \( d\theta + \beta \) is symplectic for \( 0 \leq s \leq 1 \).

**Lemma 19.** If \( d\theta + \beta \) is symplectic for \( 0 \leq s \leq 1 \), then it is possible to deform \( d\theta + \mu^{-1}\beta \) to \( d\theta + \mu^{-1}\varphi_\mu^*\beta \) through symplectic forms within its cohomology class.

**Proof.** Since \( d\theta + \beta \) are symplectic for \( 0 \leq s \leq 1 \), so are

\[
\omega_s = (s\mu)^{-1}\varphi_\mu^*(d\theta + \beta) = d\theta + \mu^{-1}\varphi_\mu^*\beta
\]

for \( \frac{1}{\mu} \leq s \leq 1 \). It remains to show that \( \partial_s\omega_s \) is exact. By Cartan’s formula,

\[
\partial_s\omega_s = \varphi_\mu^*\mathcal{L}_{Z/s\mu}\beta = \varphi_\mu^*(i_{Z/s\mu}d\beta + di_{Z/s\mu}\beta) = d\varphi_\mu^*(i_{Z/s\mu}\beta). \quad \square
\]

The argument hides a small technical challenge. The changes in symplectic forms will change the Hamiltonian \( H^m \) (without affecting the slope at infinity). Since the 1–parameter family argument heavily depends on \( H^m \), it is not clear that the same large \( \mu \) works for all Hamiltonians of a given slope. Therefore we first apply a continuation isomorphism to change the Hamiltonian back to the original \( H^m \).

Now it is no longer clear that \( \psi^m_\mu \) is independent of \( \mu \) on homology, and when we take the direct limit of continuation maps as \( m \to \infty \) it is not clear that the same choice of \( \mu \) will work for different Hamiltonians. Thus it is necessary to prove that the construction is independent of \( \mu \).

The 1–parameter family of moduli spaces argument is presented in 6.7. We will need several preliminary results: the Palais-Smale Lemma (6.3); the Lyapunov property for the action functional (6.4); an a priori energy estimate (6.3) and a transversality result (6.6). In section 6.9 we will construct the maps \( \psi^m_\mu \).

6.2. Metric rescaling.

**Lemma 20.** Let \( \mu > 0 \). There is a natural identification

\[
SC^*(H, \omega) \to SC^*(\mu H, \mu \omega),
\]

induced by the change of ring isomorphism \( \Lambda \to \Lambda, t \mapsto t^\mu \).

**Proof.** Under the rescaling, \( X_H \) does not change, so the Floer equations don’t change. The energy functional gets rescaled by \( \mu \), so a Floer trajectory contributes a factor \( t^{\mu E(u)} = (s^\mu)^{E(u)} \) to the differential instead of \( t^{E(u)} \). \( \square \)

6.3. Palais-Smale Lemma. Let \( X_t \) be a time-dependent vector field. Define

\[
F : \mathcal{L}M \to \bigcup_{x \in \mathcal{L}M} x^*TM, \quad F(x)(t) = \dot{x}(t) - X_t(x(t)).
\]

The solutions of \( F(x) = 0 \) are precisely the 1–periodic orbits of \( X_t \). The following standard result (see Salamon [11]) ensures that \( F \) is small only near such solutions.
Lemma 23. Let $M$ be a compact Riemannian manifold, and $X$ a time-dependent vector field on $M$ whose $1$–periodic orbits form a discrete set. Then

1. A sequence $x_n \in \mathcal{L} M$ with $\|F(x_n)\|_{L^2} \to 0$ has a subsequence converging in $C^0$ to a solution of $F(x) = 0$.

2. For any $\epsilon > 0$ there is a $\delta > 0$ such that $\|F(y)\|_{L^2} < \delta$ implies that there is some solution of $F(x) = 0$ close to $y$, $\sup_{t \in S^1} \text{dist}(x(t), y(t)) < \epsilon$.

Corollary 22. Let $(M, d\theta)$ be a Liouville domain, fix $J$ as in 2.1. Let $H_1$ be a time-dependent Hamiltonian on $\hat{M}$ such that $H_1 = h(R)$ is linear with generic slope for $R \gg 0$. Then for any $\delta > 0$ there is an $\epsilon > 0$ such that any smooth loop $x : S^1 \to \hat{M}$ with $\|F(x)\| < \delta$ will be within distance $\epsilon$ of some $1$-periodic orbit of $H_1$.

6.4. Lyapunov property of the action functional. Let $(M, d\theta)$ be a Liouville domain, and pick $J$ as in 2.1. The metric we use will be $d\theta(\cdot, J\cdot)$, and denote by $|\cdot|$ the norm and by $\|\cdot\|$ the $L^2$–norm integrating over time. Let $X$ be the Hamiltonian vector field for $(H, d\theta)$, where $H$ is linear at infinity, and recall $F(x) = \partial_t x - X(x)$.

Let $\beta$ be a closed two-form compactly supported in $M$ such that $d\theta + \beta$ is symplectic. Denote $X_\beta$ the Hamiltonian vector field for $(H, d\theta + \beta)$, and let

$$F_\beta(x) = \partial_t x - X_\beta(x).$$

Let $\|\beta\| = \sup |\beta(Y, Z)|$ taken over all vectors $Y, Z$ of norm 1. We will also use the notation $Y_{\text{supp } \beta}$ for a vector field $Y$, where

$$Y_{\text{supp } \beta}(m) = Y(m) \text{ if } m \in \text{supp } \beta, \text{ and } Y_{\text{supp } \beta}(m) = 0 \text{ otherwise.}$$

Lemma 23. Let $V$ be a neighbourhood containing the $1$–periodic orbits of $X$ in $M$, and let $\beta$ be a closed $2$-form compactly supported in $M$ and vanishing on $V$.

1. If $\|\beta\| < 1$ then $\|F_\beta(x) - F(x)\| \leq \frac{\|\beta\|}{1 - \|\beta\|} \|X_{\text{supp } \beta}\|$.

2. There is a $\delta > 0$ depending on $(M, H, d\theta, J, V)$, but not on $\beta$, such that

$$\|F(x)\| < \delta \implies x \text{ lies in } V \text{ or outside } M, \text{ so } F_\beta(x) = F(x).$$

3. If $\|\beta\|$ is sufficiently small, then $\|F_\beta - F\| \leq \frac{\|\beta\|}{1 - \|\beta\|} \|F\|$ and $\|F_\beta\| \leq 2\|F\|$.

Proof. Observe that $F_\beta - F = X - X_\beta$ and that $d\theta(X - X_\beta, \cdot) = \beta(X, \cdot)$, so

$$|X - X_\beta|^2 = \beta(X_\beta, J(X - X_\beta)) = \|\beta\| \cdot |X_\beta| \cdot |X - X_\beta|.$$ 

Dividing out by $|X - X_\beta|$ gives $|X - X_\beta| \leq \|\beta\| \cdot |X_\beta|$. From $|X_\beta| \leq |X| + |X - X_\beta|$ we deduce that $|X_\beta| \leq \frac{1}{1 - \|\beta\|} |X|$. Therefore

$$\|F_\beta(x) - F(x)\| \leq \frac{\|\beta\|}{1 - \|\beta\|} \|X_{\text{supp } \beta}(x)\|,$$

since $F_\beta - F = X - X_\beta$ vanishes at $(x, t)$ if the loop $x$ lies outside the support of $\beta$ at time $t$. The first claim follows, and the second follows by Corollary 22.

Let $C = \sup |X_{\text{supp } \beta}|$. Then, whenever $\|F\| \geq \delta$,

$$\|F_\beta - F\| \leq \frac{\|\beta\|}{1 - \|\beta\|} C \leq \frac{\|\beta\|}{1 - \|\beta\|} \frac{C}{\delta} \|F\| \leq \frac{1}{3} \|F\|$$

for small enough $\|\beta\|$. The last claim then follows from (2).
For $\|\beta\| < 1$, $d\theta + \beta$ is symplectic and $(d\theta + \beta)(\cdot, J\cdot)$ is positive definite but may not be symmetric. By symmetrizing, we obtain a metric

$$\tilde{g}_\beta(V, W) = \frac{1}{2}[(d\theta + \beta)(V, JW) + (d\theta + \beta)(W, JV)].$$

There is a unique endomorphism $B$ such that $\tilde{g}_\beta(BV, W) = (d\theta + \beta)(V, W)$, and this yields an almost complex structure $J_\beta = (-B^2)^{-1/2}B$ compatible with $d\theta + \beta$, inducing the metric

$$g_\beta(V, W) = (d\theta + \beta)(V, J_\beta W) = \tilde{g}_\beta((-B^2)^{1/2}V, W).$$

For sufficiently small $\|\beta\|$, $J_\beta$ is $C^0$-close to $J$ and is equal to $J$ outside the support of $\beta$, so in particular $g_\beta$ induces a norm $\|\cdot\|_\beta$ which is equivalent to the norm $\|\cdot\|$. Moreover, on the support of $\beta$ we may perturb $J_\beta$ among $(d\theta + \beta)$-compatible almost complex structures so that transversality holds for $(d\theta + \beta)$-Floer trajectories. For convenience, we use the abbreviations

$$\delta J = J_\beta - J \quad \delta F = F_\beta - F.$$

**Theorem 24.** Let $V$ be a neighbourhood containing the 1-periodic orbits of $X$ in $M$, and let $\beta$ be a closed 2-form compactly supported in $M$ and vanishing on $V$. Then for sufficiently small $\|\beta\|$, we have

$$\partial_u A(u) \leq -\frac{1}{2}\|F(u)\|^2$$

for all $u \in M(x, y; d\theta + \beta, H)$, where $A(x) = -\int x^*\theta + \int H(x)\,dt$ is the action functional for $(H, d\theta)$. In particular, $A$ is a Lyapunov function for the action 1-form for $(H, d\theta + \beta)$.

**Proof.** The action $A$ for $(d\theta, H)$ varies as follows on $u \in M(x, y; d\theta + \beta, H)$,

$$\begin{align*}
-\partial_u A(u) &= \int_0^1 d\theta(\partial_u F(u))\,dt \\
&= \int_0^1 d\theta(F(u), J_\beta F_\beta(u))\,dt \\
&= \int_0^1 d\theta(F(u), (J + \delta J)(F + \delta F)(u))\,dt \\
&\geq \|F(u)\|^2 - \|\delta J\| \|F(u)\|^2 - \|\delta F(u)\| \|F(u)\| - \|\delta J\| \|F(u)\| \\
&\geq (1 - \|\delta J\| - \frac{1}{2}\|\delta J\|) \|F(u)\|^2,
\end{align*}$$

using Lemma 23 in the last line. \qed

6.5. **A priori energy estimate.** We now want an a priori energy estimate for all $u \in M(x, y; d\theta + \beta, H)$ when $\|\beta\|$ is small. The key idea is to reparametrize the action $A$ by energy and then use the Lyapunov inequality $\partial_u A(u) \leq -\frac{1}{2}\|F(u)\|^2$ of Theorem 24. Let $e(s)$ denote the energy up to $s$ calculated with respect to $(d\theta + \beta, J_\beta)$,

$$e(s) = \int_{-\infty}^s \int_0^1 \|\partial_u u\|^2 \,dt\,ds = \int_{-\infty}^s \|\partial_u u\|_\beta^2 \,ds$$

where $\|\cdot\|_\beta$ is the norm corresponding to the metric $(d\theta + \beta)(\cdot, J_\beta\cdot)$, and $\|\cdot\|_\beta$ is the $L^2$ norm integrated over time.

**Theorem 25.** Let $\beta$ be as in Theorem 24. Then there is a constant $k > 1$ such that for all $u \in M(x, y; d\theta + \beta, H)$,

$$E(u) \leq k(A(x) - A(y)).$$
Lemma 26 applied to the operators $L$. The first claim essentially follows from the implicit function theorem and $\| \cdot \|$ by making $\beta$ section $u \in S$. Transversality for deformations.

Proof. Let $\varepsilon$ denote a topological parameter deformation $G$ of a map $F$ for which transversality holds. We need a preliminary lemma.

Lemma 26. Let $L : B_1 \to B_2$ be a surjective bounded operator of Banach spaces, and consider a perturbation $L + P_\varepsilon : B_1 \to B_2$ where $P_\varepsilon$ is a bounded operator which depends on a topological parameter $\varepsilon$, with $P_0 = 0$ and $\|P_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

1. If $L$ is Fredholm then so is $L + P_\varepsilon$ for small $\varepsilon$.
2. If $L$ is Fredholm and surjective, then so is $L + P_\varepsilon$ for small $\varepsilon$.

Proof. The Fredholm property is a norm-open condition, hence (1). Recall some general results relating an operator $L : B_1 \to B_2$ to its Banach dual $L^\ast : B_2^* \to B_1^*$:

i) $L$ is surjective if and only if $L^\ast$ is injective and $L$ is closed;
ii) if $L$ is Fredholm then $L^\ast$ is Fredholm;
iii) a Fredholm operator is injective if and only if it is bounded below.

In (2), $L^\ast$ is bounded below, say $\|L^\ast v\| \geq \delta_L \|v\|$ for all $v \in B_2^*$, so

$$\|(L + P_\varepsilon)^\ast v\| \geq \|L^\ast v\| - \|P_\varepsilon^\ast v\| \geq (\delta_L - \|P_\varepsilon^\ast\|) \|v\|.$$ 

If $\varepsilon$ is so small that $\delta_L > \|P_\varepsilon^\ast\| = \|P_\varepsilon\|$, then $(L + P_\varepsilon)^\ast$ is bounded below and so $L + P_\varepsilon$ is surjective. □

Theorem 27. Let $Y \to X$ be a Banach vector bundle. Suppose that a differentiable section $F : X \to Y$ is transverse to the zero section with Fredholm differential $D_u F$ at all $u \in F^{-1}(0)$. Let $S : \mathbb{R} \times X \to Y$ be a differentiable parameter-valued section with $S(0, \cdot) = 0$. Then for the deformation $G = F + S : \mathbb{R} \times X \to Y$,

1. $G^{-1}(0)$ is a smooth submanifold near $\{0\} \times F^{-1}(0)$;
2. $G^{-1}(0)$ is transverse to $\{\lambda = 0\} = \{0\} \times X$ (where $\lambda$ is the $\mathbb{R}$-coordinate);
3. If $0$ is an index zero regular value of $F$ and $G^{-1}(0)$ is compact near $\lambda = 0$, then the deformation $G^{-1}(0)$ of $F^{-1}(0)$ is trivial near $\lambda = 0$.

Proof. The first claim essentially follows from the implicit function theorem and Lemma 26 applied to the operators $L = D_u F$ and $P_\varepsilon = D_{\lambda,u} S$ with parameter
More precisely, we reduce to the local setup by choosing an open
neighbourhood $U$ of $u$ so that $T_U X \cong U \times B_1$, $T_U Y \cong U \times B_2$,
so locally $D_u F : B_1 \to B_2$ and $D_{\lambda,u} S : \mathbb{R} \times B_1 \to B_2$. Suppose $F(u_0) = 0$, then
lema \ref{lem:lemmaparam} to $L = D_{u_0} F$ and $P_{(\lambda,u)} = D_u F - D_{u_0} F + D_{\lambda,u} S$. Therefore
$D_{\lambda,u} G = L + P_{(\lambda,u)}$ is Fredholm and surjective, so by the implicit function theorem
$G^{-1}(0)$ is a smooth submanifold for $u$ close to $u_0$. Thus claim (1) follows.

Observe that at $(\eta, \xi) \in T \mathbb{R} \oplus TX$,
$$
D_{0,u} G \cdot (\eta, \xi) = D_u F \cdot \xi + D_{0,u} S \cdot \xi + \partial_{\lambda}|_{\lambda = 0} S \cdot \eta
$$
Therefore, $D_{0,u} G \cdot (0, \xi) = D_u F \cdot \xi$. We deduce that $\text{im} D_u F \subset \text{im} D_{0,u} G$ and
$\text{ker} D_u F \subset \text{ker} D_{0,u} G$. Since $D_u F$ is surjective whenever $F(u) = 0 \; (= G(0,u))$,
also $D_{0,u} G$ is surjective and therefore $T_{0,u} G^{-1}(0) \cong \text{ker} D_{0,u} G$ must be 1 dimension larger
than $\text{ker} D_u F$, so it contains some vector $(1, \xi)$, which implies claim (2).

This also relates the indices at solutions of $F(u) = 0$:
$$
\text{ind} D_{0,u} G = \text{dim} \ker D_{0,u} G = \text{dim} \ker D_u F + 1 = \text{ind} D_u F + 1.
$$

If 0 is an index zero regular value of $F$, then $F^{-1}(0)$ is 0–dimensional and
$G^{-1}(0)$ is a 1–dimensional submanifold near $0 \times F^{-1}(0)$ diffeomorphic to a product
$[-\lambda_0, \lambda_0] \times F^{-1}(0)$, for some small $\lambda_0$. If $G^{-1}(0)$ is compact near $\lambda = 0$ then for
sufficiently small $\lambda_0$ all solutions of $G(\lambda,u) = 0$ with $|\lambda| \leq \lambda_0$ will be close to
$0 \times F^{-1}(0)$, proving claim (3). $\square$

6.7. The 1–parameter family of moduli spaces. Let $H$ be a Hamiltonian
which is linear at infinity. In this section we will prove

**Theorem 28.** For $\beta$ as in Theorem \ref{thm:thm24} the family of moduli spaces
$$
M_\lambda(x,y) = M(x,y; d\theta + \lambda \beta, H)
$$
is smoothly trivial near $\lambda = 0$,
$$
\bigcup_{-\lambda_0 < \lambda < \lambda_0} M_\lambda(x,y) \cong M(x,y; d\theta, H) \times (-\lambda_0, \lambda_0).
$$
In particular, the identity map
$$
id : SC^\ast(H, d\theta + \lambda \beta) \to SC^\ast(H, d\theta; \Delta H + \lambda^2 \beta)
$$
is a chain isomorphism for all small $\lambda$, where $A(x) = -\int x^2 \theta + \int H(x) dt$ is the
action functional for $(H, d\theta)$.

**Proof.** Let $X_{\lambda,\beta}$ be the Hamiltonian vector field determined by $(H, d\theta + \lambda \beta)$. We
want to compare the following two maps,
$$
F(u) = \partial_u u + J(\partial_u u - X) \quad \text{and} \quad G(u) = \partial_u u + J_{\lambda,\beta}(\partial_u u - X_{\lambda,\beta}),
$$
since $F^{-1}(0) = M(x,y)$ and $G^{-1}(0) = \bigcup_\lambda M_\lambda(x,y)$.

These maps can be extended to sections $X \to Y$ of an appropriate Banach
vector bundle and generically $F$ is a Fredholm map (its linearizations are Fredholm
operators). Indeed for $k \geq 1$ and $p > 2$, we can take $Y$ to be the $W_k^{1,p}$ completion
of the space of smooth sections of $u^* TM$ with suitable exponential decay at the
ends. The base $X$ is the space of $W_k^{1,p}$ maps $u : \mathbb{R} \times S^1 \to M$ connecting two fixed
1–periodic Hamiltonian orbits. We refer to Salamon [11] and McDuff-Salamon [12] for a precise description.

For convenience, denote $\delta J = J_{\lambda \beta} - J$ and $\delta X = X - X_{\lambda \beta}$. We may assume that $\delta J$ is $C^2$–small, and we showed in Lemma 23 that

$$|\delta X| \leq \frac{|\lambda| \|\beta\|}{1 - |\lambda| \|\beta\|} |X_{\text{supp } \beta}|.$$ 

So $\delta J, \delta X$ are small for small $\lambda$. We can rewrite $G(\lambda, u) = F(u) + S(\lambda, u)$, where

$$S(\lambda, u) = \delta J \cdot (F(u) + \delta X) + J \delta X,$$

where $F(u) = \partial_t u - X(u)$. $S$ is supported at those $(u, s, t)$ with $u(s, t) \in \text{supp } \beta$, and $S : X \to Y$ is a differentiable parameter-valued section vanishing at $\lambda = 0$.

By the a priori energy estimate of Theorem 25, $G^{-1}(0)$ is compact near $\lambda = 0$. Theorem 27 implies that if 0 is an index zero regular value of $F$ then $G^{-1}(0)$ is a trivial 1–dimensional family in the parameter $\lambda$, for small $\lambda$.

Thus, for sufficiently small $\lambda_0$, there is a natural bijection between the moduli spaces which define the differentials of $SC^*(H, d\theta + \lambda_0 \beta)$ and $SC^*(H, d\theta; \Delta_{\lambda A + \lambda_0 \tau \beta})$. Indeed, if $u_{\lambda_0} \in M_0(x, y; H, d\theta + \lambda_0 \beta)$ then there is a natural 1–parameter family

$$u_{\lambda} \in M_0(x, y; H, d\theta + \lambda \beta)$$

connecting $u_{\lambda_0}$ to some $u_0 \in M_0(x, y; H, d\theta)$. Since $u_{\lambda_0}$ is homotopic to $u_0$ relative endpoints via $u_{\lambda}$, the local system $\Delta_{\lambda A + \lambda_0 \tau \beta}$ yields the same isomorphism for $u_{\lambda_0}$ as for $u_0$, which is multiplication by

$$t - \int u^* d\theta + \int_0^1 (H(x) - H(y)) \ dt - \int u^* (\lambda_0 \beta) = t - \int u^* (d\theta + \lambda_0 \beta) + \int_0^1 (H(x) - H(y)) \ dt$$

and which is the same weight used in the definition of $\partial y$ for $SC^*(H, d\theta + \lambda_0 \beta)$. Therefore the two complexes have exactly the same generators and the same differential, and in particular the identity map between them is a chain isomorphism. □

6.8. **Continuation of the 1–parameter family.**

**Theorem 29.** Let $\beta$ be as in Theorem 27. Let $H_s$ be a monotone homotopy. Then the family of moduli spaces of parametrized Floer trajectories

$$M_\lambda(x, y; H_s) = M(x, y; d\theta + \lambda \beta, H_s)$$

is smoothly trivial near $\lambda = 0$. In particular, the following diagram commutes for all small enough $\lambda$,

$$\begin{array}{ccc}
SC^*(H_+, d\theta + \lambda \beta) & \xrightarrow{id} & SC^*(H_+, d\theta; \Delta_{\lambda A + \tau \lambda \beta}) \\
\text{continuation} & & \text{continuation}
\end{array} \quad \begin{array}{ccc}
SC^*(H_-, d\theta + \lambda \beta) & \xrightarrow{id} & SC^*(H_-, d\theta; \Delta_{\lambda A + \tau \lambda \beta}) \\
\text{continuation} & & \text{continuation}
\end{array}$$

**Proof.** Let $X_{s, \lambda \beta}$ be the Hamiltonian vector field determined by $(H_s, d\theta + \lambda \beta)$, and let $X_s = X_{s, 0}$. The claim follows by mimicking the proof of Theorem 28 for

$$F(u) = \partial_s u + J_s(\partial_t u - X_s) \quad \text{and} \quad G(u) = \partial_s u + J_{s, \lambda \beta}(\partial_t u - X_{s, \lambda \beta}). \quad \square$$

**Theorem 30.** Let $\beta$ be as in Theorem 27. Let $\lambda$ be so small that Theorem 28 holds for $H$. Let $\varphi_s$ be a smooth parameter-valued isotopy of $\tilde{M}$, with $\varphi_0 = \text{id}$, such that $\varphi_s^* H$ is a monotone homotopy in $\varepsilon$. Let $H_{s, \varepsilon} = \varphi_s^* H$ for $s \in [0, 1]$, be the homotopy from $H$ to $\varphi_1^* H$. Then the family of moduli spaces of parametrized Floer
trajec\(t\)ories \(M_\varepsilon (x, y; H_{s, \varepsilon}) = M(x, y; d\theta + \lambda \beta, H_{s, \varepsilon})\) is smoothly trivial near \(\varepsilon = 0\). So there is a commutative diagram of chain isomorphisms for all small \(\varepsilon\),

\[
\begin{array}{c}
SC^*(H, d\theta + \lambda \beta) \\
\xrightarrow{\text{continuation}}
\end{array}
\begin{array}{c}
SC^*(H, d\theta; \Delta_{dA + r \lambda \beta}) \\
\xrightarrow{\text{continuation}}
\end{array}
\begin{array}{c}
SC^*(\varphi^*_\varepsilon H, d\theta + \lambda \beta) \\
\xrightarrow{\text{continuation}}
\end{array}
\begin{array}{c}
SC^*(\varphi^*_\varepsilon H, d\theta; \Delta_{dA + r \lambda \beta})
\end{array}
\]

where the vertical maps send the generators \(x \mapsto \varphi_{-1}(x)\).

**Proof.** Let \(X_{s, \varepsilon}\) be the Hamiltonian vector field determined by \((H_{s, \varepsilon}, d\theta + \lambda \beta)\), and let \(X = X_s = X_{s, 0}\). The claim follows by mimicking the proof of Theorem 28 for \(F(u) = \partial_s u + J_s (\partial_t u - X)\) and \(G(u) = \partial_s u + J_{s, \varepsilon} (\partial_t u - X_{s, \varepsilon})\). □

### 6.9. Construction of the isomorphism.

We now give the proof outlined in 6.1.

Let \(\beta\) be a closed two-form compactly supported in the interior of \(M\), and suppose that \(d\theta + s \beta\) is symplectic for all \(0 \leq s \leq 1\) (so that Lemma 19 applies).

Let \(H_m\) be a Hamiltonian linear at infinity with slope \(m\). Up to a continuation isomorphism on symplectic cohomologies, we may assume that all critical points of \(H_m\) in the interior of \(M\) lie in a neighbourhood \(V\) contained in \(M \setminus \text{supp} \beta\). This technical remark is explained in section 6.10.

Define \(\psi^m_{\mu}\) by the diagram of isomorphisms

\[
\begin{array}{c}
SC^*(H^m, d\theta + \beta) \xrightarrow{(1)} \text{Liouville } \varphi_{\mu} \\
\xrightarrow{(2)} \text{Moser } \sigma_{\mu} \\
\xrightarrow{(3)} \text{continuation} \\
\xrightarrow{(4)} \text{rescale} \\
\xrightarrow{(5)} \text{id} \\
\xrightarrow{(6)} \text{rescale} \\
\xrightarrow{(7)} \text{change of basis}
\end{array}
\]

where the maps are defined as follows:

(1) apply \(\varphi_{\mu}\), the Liouville flow for time \(\log \mu\) (see 2.1 for the definition of the Liouville vector field);

(2) apply the Moser symplectomorphism \(\sigma_{\mu} : (\hat{M}, \mu d\theta + \beta) \rightarrow (\hat{M}, \mu d\theta + \varphi^*_\mu \beta)\) obtained by Lemmas 19 and 9 and denote \(\phi_{\mu} = \sigma_{\mu} \circ \varphi_{\mu}\);

(3) observe that \(\phi^*_\mu H^m\) has slope \(\mu m\) at infinity, so the linear interpolation from \(\mu H^m\) to \(\phi^*_\mu H^m\) is a compactly supported homotopy and therefore induces a continuation isomorphism;

(4) metric rescaling by \(\mu^{-1}\) (Lemma 20), which changes \(t\) to \(T = t(\mu^{-1})\);
(5) the identity map is a chain isomorphism by Theorem 28 provided \( \mu \) is sufficiently large (depending on \( m \));
(6) rescale \( \tau \beta \) to \( \mu^{-1} \tau \beta \), so change \( t \) to \( T = t (\mu^{-1}) \);
(7) adding an exact form \( dA \) to \( \mu^{-1} \tau \beta \), where \( A \) is the action 1-form for \((H^m, d\theta)\), corresponds to a change of basis \( x \mapsto T^{\lambda(x)}x \) by Lemma 14.

**Lemma 31.** The map \( \psi^m_\mu : SH^* (H^m, d\theta + \beta) \to SH^* (H^m, d\theta; \Delta_{\tau \beta}) \) on homology does not depend on the choice of large \( \mu \).

**Proof.** In this proof we abbreviate \( H^m \) by \( H \) and pullbacks \( \phi^* \) by \( \phi \). Consider \( \mu' \) close to \( \mu \), and write \( \phi = \phi_{\mu'} \phi_{\mu}^{-1} \) and \( \varphi = \phi_{\mu'} \varphi_{\mu}^{-1} \). Observe the following commutative diagram, in which the top row and bottom diagonal are part of the construction of the maps \( \psi^m_\mu \) and \( \psi^m_{\mu'} \), for \( \mu' > \mu \).

\[
\begin{array}{ccc}
SH^* (H, d\theta + \beta) & \xrightarrow{\phi_{\mu}} & SH^* (\phi_{\mu} H, \mu d\theta + \beta) \\
\downarrow{\phi_{\mu'}} & & \downarrow{\phi_{\mu'}^{-1}} \\
SH^* (\phi_{\mu'} H, \mu' d\theta + \beta) & & SH^* (\phi_{\mu'}^{-1} \mu' H, \mu d\theta + \beta) \\
\downarrow{\text{continuation}} & & \downarrow{\phi_{\mu'}^{-1}} \\
SH^* (\mu H, \mu d\theta + \beta) & & \phi_{\mu'}^{-1} \circ C : SH^* (H, d\theta + \mu^{-1} \beta) \to SH^* (H, d\theta + \mu'^{-1} \beta)
\end{array}
\]

The last vertical composite, after a metric rescaling, is the map

\[
\phi_{\mu'}^{-1} \circ C : SH^* (H, d\theta + \mu^{-1} \beta) \to SH^* (H, d\theta + \mu'^{-1} \beta)
\]

where \( C \) is the continuation map

\[
C : SH^* (H, d\theta + \mu^{-1} \beta) \to SH^* (\mu^{-1} \phi_{\mu'}^{-1} H, d\theta + \mu^{-1} \beta).
\]

For \( \mu' \) sufficiently close to \( \mu \), \( \phi_{\mu'}^{-1} \) is an isotopy of \( \hat{M} \) close to the identity, therefore by Theorem 20 \( C \) maps the generators by \( \phi \). Thus \( \phi_{\mu'}^{-1} \circ C = \text{id} \) for \( \mu' \) close to \( \mu \).

For the twisted symplectic cohomology we just apply changes of basis so we deduce the following commutative diagram (using abbreviated notation),

\[
\begin{array}{ccc}
SH^* (d\theta + \beta) & \xrightarrow{\text{id}} & SH^* (d\theta + \mu^{-1} \beta) \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
SH^* (d\theta + (\mu')^{-1} \beta) & \xrightarrow{\text{id}} & SH^* (\Delta_{dA+(\mu')^{-1} \tau \beta})
\end{array}
\]

We showed that this diagram holds for all \( \mu' \) close to \( \mu \). Suppose it holds for all \( \mu, \mu' \in [\mu_0, \mu_1] \), for some maximal such \( \mu_1 < \infty \). Apply the above result to \( \mu = \mu_1 \), then the diagram holds for all \( \mu, \mu' \in (\mu_1 - \epsilon, \mu_1 + \epsilon) \), for some \( \epsilon > 0 \). Thus it holds for all \( \mu, \mu' \in [\mu_0, \mu_1 + \epsilon] \). So there is no maximal such \( \mu_1 \) and the diagram must hold for all large enough \( \mu, \mu' \), and thus the map \( SH^* (H, d\theta + \beta) \to SH^* (H, d\theta; \Delta_{\tau \beta}) \) does not depend on the choice of (large) \( \mu \).

**Lemma 32.** The maps \( \psi^m : SH^* (H^m, d\theta + \beta) \to SH^* (H^m, d\theta; \Delta_{\tau \beta}) \) commute with the continuation maps induced by monotone homotopies. 

\[\square\]
Proof. Let $H_s$ be a monotone homotopy from $H^{m'}$ to $H^m$. By theorem 29 for sufficiently large $\mu$ the following diagram commutes

$$SC^*(H_+, d\theta + \mu^{-1}\beta) \xrightarrow{id} SC^*(H_+, d\theta; \Delta_{\mu\theta+\mu^{-1}\beta})$$

and by Lemma 14 we deduce the required commutative diagram

$$SC^*(H_+, d\theta + \beta) \xrightarrow{\psi^m} SC^*(H_+, d\theta; \Delta_{\beta})$$

$$SC^*(H_-, d\theta + \beta) \xrightarrow{\psi^{m'}} SC^*(H_-, d\theta; \Delta_{\beta})$$

Theorem 33. Let $\beta$ be a closed two-form compactly supported in the interior of $M$, and suppose that $d\theta + s\beta$ is symplectic for $0 \leq s \leq 1$. Then there is an isomorphism

$$\psi : SH^*(M, d\theta + \beta) \to SH^*(M, d\theta; \Delta_{\beta}).$$

Proof. By Lemma 31 the map $\psi^m = \psi^m_\mu$ on homology is independent of $\mu$ for large $\mu$, and by Lemma 32 the maps $\psi^m$ commute with continuation maps. The direct limit is an exact functor, so $\psi = \lim \psi^m$ is an isomorphism. \[\square\]

Remark 34. The theorem can sometimes be applied to deformations $\omega_s$ which are not compactly supported by using Gray’s stability theorem e.g. see Lemma 51.

Remark 35. Let $\beta \in H^2(M; \mathbb{R})$ come from $H^2(\partial M; \mathbb{R})$ by the Thom construction. Then $SH^*(M, d\theta + \beta) \cong SH^*(M, d\theta; \Lambda)$, the ordinary symplectic cohomology with underlying ring $\Lambda$. Indeed, suppose $\beta$ vanishes on

$$\text{Fix}(\varphi_\mu) = \lim_{\mu \to -\infty} \varphi_\mu(M).$$

Let $H = h(R)$ be a convex Hamiltonian defined in a neighbourhood $R < R_0$ of $\text{Fix}(\varphi_\mu)$ where $\beta$ vanishes, such that $h'(R) \to \infty$ as $R \to R_0$. Let $H^m = h$ if $h' \leq m$ and let $H^m$ be linear with slope $m$ elsewhere. Then the Floer solutions concerned in the symplectic chain groups will all lie in the subset of $M$ where $\beta = 0$.

6.10. Technical remark. We assumed in 6.9 that all critical points of $H$ in the interior of $M$ lie in a neighbourhood $V \subset M \setminus \text{supp} \beta$. We can do this as follows.

Pick a small neighbourhood $V$ around $\text{Crit}(H)$ so that $\beta|_V = d\alpha$ is exact. We may assume that $\alpha$ is supported in $V$. To construct the isomorphism of 6.9 we need to homotope $d\theta + \mu^{-1}\beta$ to $d\theta + \mu^{-1}(\beta - d\alpha)$ for all large $\mu$. This can be done by a Moser isotopy compactly supported in $V$ via the exact deformation $\omega_s = d\theta + \mu^{-1}(\beta - s d\alpha)$. Since $\partial_s \omega_s = -\mu^{-1} d\alpha$, for large $\mu$ the Moser isotopy $\phi_s$ is close to the identity. Therefore during the isotopy the critical points of $\phi_s^* H$ stay within $V$. This guarantees that the Palais-Smale Lemma 22 can be applied for $V$ independently of large $\mu$, and the construction 6.9 can be carried out with minor modifications.
7. ALE spaces

7.1. Hyperkähler manifolds. We suggest [5] for a detailed account of Hyperkähler manifolds and ALE spaces.

Recall that a symplectic manifold \((M, \omega)\) is Kähler if there is an integrable \(\omega\)–compatible almost complex structure \(I\). Equivalently, a Riemannian manifold \((M, g)\) is Kähler if there is an orthogonal almost complex structure \(I\) which is covariant constant with respect to the Levi-Civita connection. \((M, g)\) is called hyperkähler if there are three orthogonal covariant constant almost complex structures \(I, J, K\) satisfying the quaternion relation \(IJK = -1\).

The hyperkähler manifold \((M, g)\) is therefore Kähler with respect to each of the (integrable) complex structures \(I, J, K\), with corresponding Kähler forms

\[
\omega_I = g(I \cdot, \cdot), \quad \omega_J = g(J \cdot, \cdot), \quad \omega_K = g(K \cdot, \cdot).
\]

Indeed, there is an \(S^2\) worth of Kähler forms: any vector \((u_I, u_J, u_K) \in S^2 \subset \mathbb{R}^3\) gives rise to a complex structure \(I_u = u_I I + u_J J + u_K K\) and a Kähler form

\[
\omega_u = u_I \omega_I + u_J \omega_J + u_K \omega_K.
\]

We will always think of \(M\) as a complex manifold with respect to \(I\), and we recall from [6] that \(\omega_J + i \omega_K\) is a holomorphic symplectic structure on \(M\) (a non-degenerate closed holomorphic \((2, 0)\) form). The form \(\omega_J + i \omega_K\) determines a trivialization of the canonical bundle \(\Lambda^{2,0} T^* M\), so \(c_1(M) = 0\) and the Conley-Zehnder indices give a \(\mathbb{Z}\)–grading on symplectic cohomology (see [5,1]).

**Lemma 36.** Let \(L \subset \mathbb{H}\) be an \(I\)–complex vector subspace of the space of quaternions with \(\dim_{\mathbb{R}} L = 2\). Then \(L\) is a real Lagrangian subspace with respect to \(\omega_J\) and \(\omega_K\), and a symplectic subspace with respect to \(\omega_I\). After an automorphism of \(\mathbb{H}\), \(L\) is identified with \(\mathbb{C} \oplus 0 \subset \mathbb{H}\).

**Proof.** \(L\) is a complex 1-dimensional subspace of \((\mathbb{H}, I)\), so \(L\) is complex Lagrangian with respect to the \(I\)-holomorphic symplectic form \(\omega_c = \omega_J + i \omega_K\). Thus \(L\) is a real Lagrangian vector subspace of \(\mathbb{H}\) with respect to \(\omega_J\) and \(\omega_K\).

Moreover, given any vector \(e_1 \in L\), let \(e_2 = I e_1, e_3 = J e_1\) and \(e_4 = K e_1\). Then \(L = \text{span}\{e_1, e_2\}\) and \(\omega_I (e_1, e_2) = g(e_2, e_2) > 0\), so \(L\) is symplectic with respect to \(\omega_I\) and corresponds to \(\mathbb{C} \oplus 0\) in the hyperkähler basis \(e_1, \ldots, e_4\). \(\square\)

7.2. Hyperkähler quotients. Let \(M\) be a simply connected hyperkähler manifold. Let \(G\) be a compact Lie group \(G\) acting on \(M\) and preserving \(g, I, J, K\). Then corresponding to the forms \(\omega_I, \omega_J, \omega_K\) there exist moment maps \(\mu_I, \mu_J, \mu_K\). Recall that if \(\zeta\) is in the Lie algebra \(\mathfrak{g}\) of \(G\), then it generates a vector field \(X_\zeta\) on \(M\). A moment map \(\mu : M \to \mathfrak{g}^\vee\) is a \(G\)–equivariant map such that

\[
\mu_m(\zeta) = \omega(X_\zeta(m), \cdot) \quad \text{at} \quad m \in M.
\]

For simply connected \(M\) such a \(\mu\) exists and is determined up to the addition of an element in \(Z = (\mathfrak{g}^\vee)^G\), the invariant elements of the dual Lie algebra \(\mathfrak{g}^\vee\).

Putting these moment maps together yields \(\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathbb{R}^3 \otimes \mathfrak{g}^\vee\), and for \(\zeta \in \mathbb{R}^3 \otimes Z\) we may define the hyperkähler quotient space

\[
X_\zeta = \mu^{-1}(\zeta)/F.
\]

If \(F\) acts freely on \(\mu^{-1}(\zeta)\) then \(X_\zeta\) is a smooth manifold of dimension \(\dim M - 4 \dim F\) and the structures \(g, I, J, K\) descend to \(X_\zeta\) making it hyperkähler (see [6]).
7.3. ALE spaces and ADE singularities.

**Definition 37.** Let $\Gamma$ be any finite subgroup of $SU(2)$ (or, equivalently, $SL_2(\mathbb{C})$). An ALE space (asymptotically locally Euclidean) is a hyperkähler 4--manifold with precisely one end which at infinity is isometric to the quotient $\mathbb{C}^2/\Gamma$, where $\mathbb{C}^2/\Gamma$ is endowed with a metric that differs from the Euclidean metric by order $O(r^{-4})$ terms and which has the appropriate decay in the derivatives.

Kronheimer proved in [9] that ALE spaces are (particularly nice) models for the minimal resolution of the quotient singularities $\mathbb{C}^2/\Gamma$. More precisely, any ALE space is diffeomorphic to such a minimal resolution, and vice-versa.

We now recall Kronheimer’s construction [8] of ALE spaces as hyperkähler quotients. Let $R$ be the left regular representation of $\Gamma \subset SU(2)$ endowed with the natural Euclidean metric, $R = \bigoplus_{\gamma \in \Gamma} C e_\gamma \cong \mathbb{C}^{|\Gamma|}$.

Denoting by $\mathbb{C}^2$ the natural left $SU(2)$--module, let $M = (\mathbb{C}^2 \otimes C \text{Hom}_C(R,R))^\Gamma$ be the pairs of endomorphisms $(\alpha,\beta)$ of $R$, which are invariant under the induced left action of $\Gamma$. We make $M$ into a hyperkähler vector space by letting $I$ act by $i$ and $J$ by $J(\alpha,\beta) = (-\beta^*,\alpha^*)$.

The Lie group $F = \text{Aut}_C(R,R)^\Gamma/\{\text{scalar maps}\}$ of unitary automorphisms of $R$ which are $\Gamma$--invariant act by conjugation on $M$, $f \cdot (\alpha,\beta) = (f\alpha f^{-1}, f\beta f^{-1})$, where we quotiented by the scalar matrices since they act trivially. The corresponding Lie algebra $\mathfrak{f}$ corresponds to the traceless elements of $\text{Hom}_C(R,R)$, and the moment maps are:

$$\mu_I(\alpha,\beta) = \frac{1}{2} i(\{\alpha,\alpha^*\} + \{\beta,\beta^*\}), \quad (\mu_J + i\mu_K)(\alpha,\beta) = [\alpha,\beta].$$

By McKay’s correspondence, this description can be made explicit. Recall that $R = \bigoplus n_i R_i$, where the $R_i$ are the complex irreducible representations of $\Gamma$ of complex dimension $n_i$. Then $\mathbb{C}^2 \otimes R_i \cong \bigoplus_j A_{ij} R_j$ where $A$ is the adjacency matrix describing an extended Dynkin diagram of ADE type (the correspondence between $\Gamma$ and the type of diagram is described in the Introduction). It follows that

$$M = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

where each edge $i \rightarrow j$ of the extended Dynkin diagram appears twice, once for each choice of orientation. Moreover,

$$F = (\oplus_i U(n_i))/\{\text{scalar maps}\}$$

where the unitary group $U(n_i)$ acts naturally on $\mathbb{C}^{n_i}$.

The hyperkähler quotient for $\zeta \in Z = \text{centre}(\mathfrak{f'})$ is therefore

$$X_\zeta = \mu^{-1}(\zeta)/F.$$

**Definition 38.** Let $\mathfrak{h}_\mathbb{R}$ denote the real Cartan algebra associated to the Dynkin diagram for $\Gamma$. Let the hyperplanes $D_\theta = \ker \theta$ denote the walls of the Weyl chambers, where the $\theta$ are the roots. We identify the centre $Z$ with $\mathfrak{h}_\mathbb{R}$ by dualizing the map

$$\text{centre}(\mathfrak{f}) \rightarrow \mathfrak{h}_\mathbb{R}', \quad i \pi_k \mapsto n_k \theta_k,$$
where \( \pi_k : R \to \mathbb{C}^n \otimes R_k \) are the projections to the summands.

We call \( \zeta \in \mathbb{R}^3 \otimes \mathbb{Z} \) generic if it does not lie in \( \mathbb{R}^3 \otimes D_\theta \) for any root \( \theta \), i.e. \( \theta(\zeta_1), \theta(\zeta_2), \theta(\zeta_3) \) are not all zero for any root \( \theta \).

**Theorem 39** (Kronheimer, [8]). Let \( \zeta \in \mathbb{R}^3 \otimes \mathbb{Z} \) be generic. Then \( X_\zeta \) is a smooth hyperkähler four-manifold with the following properties.

1. \( X_\zeta \) is a continuous family of hyperkähler manifolds in the parameter \( \zeta \);
2. \( X_0 \) is isometric to \( \mathbb{C}^2/\Gamma \);
3. there is a map \( \pi : X_\zeta \to X_0 \) which is an \( I \)-holomorphic minimal resolution of \( \mathbb{C}^2/\Gamma \), and \( \pi \) varies continuously with \( \zeta \);
4. in particular, \( \pi \) is a biholomorphism away from \( \pi^{-1}(0) \) and \( \pi^{-1}(0) \) consists of a collection of \( I \)-holomorphic spheres with self-intersection \( -2 \) which intersect transversely according to the Dynkin diagram from \( \Gamma \);
5. \( H^2(X_\zeta; \mathbb{R}) \cong \mathbb{Z} \) such that \( [\omega_1], [\omega_2], [\omega_3] \) map to \( \zeta_1, \zeta_2, \zeta_3 \).
6. \( H^2(X_\zeta; \mathbb{Z}) \cong \{ \text{root lattice for } \Gamma \} \), such that the classes \( \Sigma \) with self-intersection \( -2 \) correspond to the roots;
7. \( X_\zeta \) and \( X_{\zeta'} \) are isometric hyperkähler manifolds if \( \zeta, \zeta' \) lie in the same orbit of the Weyl group;
8. Every ALE space asymptotic to \( \mathbb{C}^2/\Gamma \) is isomorphic to \( X_\zeta \) for some generic \( \zeta \).

### 7.4. Plumbing construction of ALE spaces

Our goal is to prove that for any ALE space \( X \), \( SH^*(X; \omega) = 0 \) for a generic choice of (non-exact) symplectic form \( \omega \). By Theorem 39(b) the cohomology class \( [\omega_1] \) ranges linearly in \( \zeta_1 \) over all of \( H^2(X; \mathbb{R}) \). Therefore it suffices to consider the hyperkähler quotient \( X = X_\zeta \) for all generic \( \zeta = (\zeta_1, 0, 0) \in \mathbb{Z} \otimes \mathbb{R}^3 \).

**Lemma 40.** The exceptional divisors in \( X \) are exact Lagrangian spheres with respect to \( \omega_1 \) and \( \omega_3 \) and they are symplectic spheres with respect to \( \omega_2 \). Moreover, the areas \( \langle \omega_1, \Sigma_m \rangle \) of the exceptional spheres \( \Sigma_m \) range linearly in \( \zeta_1 \) over all possible positive values.

**Proof.** The first statement is an immediate consequence of Lemma 39(a) using the fact that the exceptional divisors in \( X \) are holomorphic spheres by Theorem 39(a). Note that if a sphere is Lagrangian then it is exact since \( H^1(S^2; \mathbb{R}) = 0 \). The second statement is immediate since \( [\omega_1] \) ranges linearly in \( \zeta_1 \) over \( H^2(X; \mathbb{R}) \) and the \( \Sigma_m \) generate \( H_2(X; \mathbb{Z}) \), by Theorem 39(b). \( \Box \)

The space \( (X, \omega_1) \) is the plumbing of copies of \( T^*\mathbb{C}P^1 \), plumbed according to the Dynkin diagram for \( \Gamma \). Indeed, by mimicking the proof of Weinstein’s Lagrangian neighbourhood theorem, one observes that a neighbourhood of the collection of exceptional Lagrangian spheres is symplectomorphic to a plumbing of copies of small disc cotangent bundles \( DT^*\mathbb{C}P^1 \). That neighbourhood can be chosen so that its complement is a symplectic collar diffeomorphic to \( (S^3/\Gamma) \times [1, \infty) \), since \( X \) is biholomorphic to \( \mathbb{C}^2/\Gamma \) away from 0.

**Remark 41.** We will show that the exact Lagrangians inside an ALE space \( X \) must be spheres. To prove this holds also for the above plumbing, we don’t actually need to know that the plumbing \( Y \) is all of \( X \), the embedding \( Y \hookrightarrow X \) provided by Weinstein’s theorem is enough. Indeed, the argument relies on contradicting Corollary 17 by showing that \( c_1 \) maps to 0 via \( SH^*(Y, \omega_1; \Delta_{\omega_1}) \to H_{n-*}(\mathcal{L}L) \otimes \Lambda \).
This is true since $c_1 \mathcal{L}$ is in the image of $SH^*(X, \omega_J; \Delta_{\omega_J}) \to SH^*(Y, \omega_J; \Delta_{\omega_J})$ by Theorem 12 and we will prove that $SH^*(X, \omega_J; \Delta_{\omega_J}) = 0$.

7.5. Contact hypersurfaces inside ALE spaces.

Lemma 42. Recall that any $(u_I, u_J, u_K) \in S^2 \subset \mathbb{R}^3$ gives rise to a Kähler form

$\omega_u = u_I \omega_I + u_J \omega_J + u_K \omega_K$.

Then $(X, \omega_u)$ is a symplectic manifold such that $\pi^{-1}(S^3/\Gamma)$ is a contact hypersurface in $X$ for all sufficiently large $r$, so that $X$ can be thought of as a symplectic manifold with contact type boundary with an infinite collar attached. Moreover, $X$ is exact symplectic precisely when $u_I = 0$.

Proof. Recall $\pi : X \to \mathbb{C}^2/\Gamma$ denotes the resolution. Let $\omega'_u$ denote the corresponding combination of forms for $\mathbb{C}^2/\Gamma$ can be described explicitly as follows (following [5]). The moment map equations are $[\alpha, \beta] = 0$ and $[\alpha, \alpha^*] + [\beta, \beta^*] = -2i \zeta_1$. Since $\alpha, \beta$ commute by the first equation, they have a common eigenvector $e$, say $(\alpha, \beta)e = (a, b)e$. By $\Gamma$-invariance, $e^\gamma = R(\gamma) \cdot e$ is also a common eigenvector such that $(\alpha, \beta)e^\gamma = (\gamma \cdot (a, b))e^\gamma$.

The map $X \to \mathbb{C}^2/\Gamma$, $(\alpha, \beta) \mapsto \Gamma \cdot (a, b)$ is then an $I$-holomorphic minimal resolution. In fact $\pi$ is also compatible with $J$ and $K$ if we identify $\mathbb{C}^2/\Gamma = \mathbb{H}/\Gamma$.

Theorem 43. The $S^1$-action $\lambda \cdot (a, b) = (\lambda a, \lambda b)$ on $\mathbb{C}^2/\Gamma$ lifts to a unique $I$-holomorphic $S^1$-action on $(X, \omega_I)$. Moreover the $S^1$-action preserves the contact hypersurface $\pi^{-1}(S^3/\Gamma)$ inside $(X, \omega_I)$ described in Lemma 42 and the contact form $\theta_I$ can be chosen to be $S^1$-equivariant.

Proof. Since $\Gamma$ is a complex group, it commutes with the diagonal $S^1$-action on $\mathbb{C}^2$, therefore the action is well-defined on $\mathbb{C}^2/\Gamma$. The lift of the action is

$\varphi_\lambda (\alpha, \beta) = (\lambda \alpha, \lambda \beta)$.

In particular, the $S^1$-action preserves $\omega_I$ because it preserves the metric $g$ and it commutes with the action of $I$.

Let $\theta_I$ denote the contact form constructed in Lemma 12 for the hypersurface $\pi^{-1}(S^3/\Gamma)$ and the symplectic form $\omega_I$. To make $\theta_I$ an $S^1$-equivariant contact form, we simply replace it by the $S^1$-averaged form $\overline{\theta}_I = \int_0^1 \varphi^*_u \omega_I dt$. Since $\varphi^*_u \omega_I = \omega_I$, it satisfies $d\overline{\theta}_I = \omega_I$ and the positivity condition

$\overline{\theta}_I \wedge d\overline{\theta}_I = \left(\int_0^1 \varphi^*_u \omega_I dt\right) \wedge \omega_I = \int_0^1 \varphi^*_u \omega_I \wedge \omega_I > 0$. □

Remark 44. The $S^1$-action does not preserve $\omega_J$ and $\omega_K$. That is why the symplectic cohomology for $\omega_I$ will be very different from the one for $\omega_J$ or $\omega_K$. 

Deformations of Symplectic Cohomology
7.7. **Changing the contact hypersurface to a standard** $S^3/\Gamma$. Our aim is to change the contact hypersurface in $(X,\omega_I)$ so that it becomes a standard $S^3/\Gamma$. We want to do this compatibly with the $S^1$-actions on $X$ and $\mathbb{C}^2/\Gamma$, so that the $S^1$-action on $(X,\omega_I)$ will coincide with the new Reeb flow. To do this, we need an $S^1$-equivariant version of Gray’s stability theorem.

**Lemma 45** ($S^1$-equivariant Gray stability). For $t \in [0,1]$, let $\xi_t = \ker \alpha_t$ be a smooth family of contact structures on some closed manifold $N^{2n-1}$. Then there is an isotopy $\psi_t$ of $N$ and a family of smooth functions $f_t$ such that

$$\psi_t^* \alpha_t = e^{f_t} \alpha_0.$$  

If there is an $S^1$-action on $N$ preserving each $\alpha_t$, then $f_t$ and $\psi_t$ are $S^1$-equivariant.

**Proof.** Let $X_t$ be a vector field inducing a flow $\psi_t$. By Cartan’s formula,

$$\partial_t \psi_t^* \alpha_t = \psi_t^* (\dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t) = \psi_t^* (\dot{\alpha}_t + di_{X_t} \alpha_t + i_{X_t} \rho_t).$$

Observe now that if $\psi_t$ satisfied the claim, then $\partial_t \psi_t^* \alpha_t = \dot{f}_t e^{f_t} \alpha_0 = \psi_t^* (f_t(\psi_t^{-1}) \alpha_t).$

We can reverse the argument to obtain the required $\psi_t$ if we can find a vector field $X_t$ in $\xi_t$ (so $i_{X_t} \alpha_t = 0$) satisfying the equation

$$i_{X_t} \rho_t = \dot{f}_t (\psi_t^{-1}) \alpha_t - \dot{\alpha}_t.$$  

Inserting the Reeb vector field $R_t$ we obtain $0 = \dot{f}_t (\psi_t^{-1}) - \dot{\alpha}_t (R_t)$. Solving the latter equation determines $f_t$ with $f_0 = 0$. Then the original equation determines $X_t \in \xi_t$ since $d\alpha_t$ is non-degenerate on $\xi_t$.

Suppose we had an $S^1$-action $\varphi_\lambda$ preserving $\alpha$, $\varphi_\lambda^* \alpha_t = \alpha_t$. Applying $\varphi_\lambda^*$ to the equation which determines $X_t$ at $x$ we obtain the equation

$$i_{\varphi_\lambda^* X_t} \rho_t = \dot{f}_t (\psi_t^{-1}) \alpha_t - \dot{\alpha}_t$$

at $y = \varphi_\lambda^{-1}(x)$. The solution $f_t$ does not change and so by uniqueness and $\varphi_\lambda^* X_t = X_t$, which proves that $f_t$ and $\psi_t$ are $S^1$-equivariant. □

**Lemma 46.** The contact hypersurface $\pi^{-1}(S^3/\Gamma)$ can be deformed inside $(X,\omega_I)$ into a copy of the standard $S^3/\Gamma$ via an $S^1$-equivariant contactomorphism.

**Proof.** Consider $X_t = X_{t,h_1,0,0}$ and denote by $\omega_t$ its form $\omega_I$, $(0 \leq t \leq 1)$, and let $\pi_t: X_t \to X_0 = \mathbb{H}/\Gamma$ denote the minimal resolution. By Lemma 42 each $X_t$ comes with an $S^1$-equivariant contact form $\theta_t$ with $d\theta_t = \omega_t$ and such that $\theta_0$ is the standard contact form on $S^3/\Gamma \subset X_0$. This defines a family of $S^1$-equivariant contact forms $\alpha_t = (\pi_t)_* \theta_t$ on $S^3/\Gamma$. By Lemma 45 there is an $S^1$-equivariant isomorphism $(S^3_t/\Gamma,e^{f_t} \alpha_t) \to (S^3_t/\Gamma,\alpha_0)$. In particular, this proves that $X$ arises by attaching an infinite collar to the manifold

$$\{(R,x) : R \leq e^{f_t(x)}, x \in \pi^{-1}(S^3/\Gamma)\} \subset X$$

along the boundary $\{(e^{f_t(x)},x)\} \subset X$ which is a standard contact $S^3_t/\Gamma$. □

7.8. **Non-vanishing of the exact symplectic cohomology.**

**Theorem 47.** $SH^*(X,\omega_u) \neq 0$ for $u = (0,u_J,u_K) \in S^2$, indeed $c_* : H^*(X) \otimes \Lambda \to SH^*(X,\omega_u)$ is an injection.
Proof. The exceptional spheres in $X$ are exact by Lemma [10]. For each such $S^2$ we have a commuting diagram by Theorem [16] using the bundles described in [12]:

$$H_{4-*}LS^2 \otimes \Lambda \cong \text{SH}^*(T^*S^2, d\theta) \xrightarrow{i_*} \text{SH}^*(X, \omega_u)$$

$$H_{4-*}(S^2) \otimes \Lambda \cong H^*(S^2) \otimes \Lambda \xrightarrow{i^*} H^*(X) \otimes \Lambda$$

The left vertical map is induced by the inclusion of constant loops and it is injective on homology because it has a left inverse by evaluation at 0. Since $H^*(X)$ is generated by the exceptional spheres by Theorem [39] and $i^*$ is the projection to the summands of $H^*(X)$, the claim follows.

7.9. Vanishing of the non-exact symplectic cohomology.

**Theorem 48.** $\text{SH}^*(X, \omega_f) = 0$.

**Proof.** By Theorem [3] the symplectic cohomology changes by an isomorphism if we choose a different contact hypersurface in the collar. By Lemma [16] we changed the hypersurface by an $S^1$-equivariant contactomorphism so that the collar of $X$ (after metric rescaling) can be assumed to be the standard $S^3/T \times [1, \infty)$ with $S^1$-action $(a, b) \mapsto (\lambda a, \lambda b)$. The symplectic $S^1$-action $\varphi_\lambda$ on $(X, \omega_f)$ defines a vector field $X_{\varphi_\lambda}(x) = \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} \varphi_{\lambda(2\pi \lambda)}(x)$.

By Cartan’s formula $0 = \partial_\lambda \varphi_\lambda^* \omega = \varphi_\lambda^* \mathcal{L}_{X_\lambda} \omega = d i_{X_\lambda} \omega$. Thus, since $H^1(X; \mathbb{R}) = 0$, we obtain a Hamiltonian via $i_{X_\lambda} \omega = -dH_\lambda$. Moreover, accelerating the flow by a factor $k$, we obtain an $S^1$-action $\varphi_{k\lambda}$ with Hamiltonian $H_k = kH_{\lambda}$. On the collar, $H_k(a, b) = k\pi(|a|^2 + |b|^2)$ and since $R = |a|^2 + |b|^2$, the Hamiltonian is linear at infinity: $h_k(R) = k\pi R$.

The 1-periodic orbits of $H_k$ either lie in $\pi^{-1}(0)$ or come from lifts of nonconstant 1-periodic orbits on $S^2/T$ for the flow $(a, b) \mapsto (\lambda^k a, \lambda^k b)$. But for generic $k$, there are no 1-periodic orbits of $H_k$ on $S^2/T$ except for 0. So we reduce to calculating the Maslov indices of 1-periodic orbits in $\pi^{-1}(0)$.

Since the flow $\varphi_\lambda$ is holomorphic, the linearization over a periodic orbit will be a loop of unitary transformations. Its Maslov index can therefore be calculated as the winding number of the determinant of the linearization in the trivialization $\mathbb{C} \cdot (\omega_f + i\omega_K)$ of the canonical bundle. Since $\varphi_\lambda^* \omega_f(V, W) = g(J\varphi_\lambda^* V, \varphi_\lambda^* W) = g(J\lambda AV, \lambda W) = \lambda^2 \omega_f(V, W)$, and similarly for $K$, we deduce that $\varphi_\lambda$ acts on the canonical bundle by rotation by $\lambda^2$. Therefore the Maslov index increases by 2 for each full rotation of $\lambda$.

We deduce that the Maslov indices for $H_k$ grow to infinity as $k \to \infty$. Therefore the generators of $\text{SH}^*(H_{k+N}, \omega_f)$ have arbitrarily negative Conley-Zehnder indices as $N \to \infty$, and so the image of $\text{SH}^m(H_k, \omega_f)$ under the continuation map $\text{SH}^m(H_k, \omega_f) \to \text{SH}^m(H_{k+N}, \omega_f)$ vanishes for large $N$. Thus the direct limit $\text{SH}^m(X, \omega_f) = 0$ for all $m$. □

**Corollary 49.** Let $X$ be an ALE space. Given a generic class in $H^2(X; \mathbb{R})$, it is possible to choose a symplectic form $\omega$ on $X$ representing that class such that $\text{SH}^*(X, \omega) = 0$. 


Here, genericity refers to choosing \([\omega]\) in the complement of certain finitely many hyperplanes in \(H^2(X; \mathbb{R})\).

Proof. All the \(X_{\xi_1,0,0}\) for generic \(\xi_1\) are diffeomorphic (Theorem 50). We fix one such choice \(X = X_{a,0,0}\), and we consider the family of forms \(\omega_I\) induced on \(X\) by pull-back from \(X_{\xi_1,0,0}\) via the diffeomorphism \(X \cong X_{\xi_1,0,0}\). By Lemma 39 \([\omega_I]\) will range over all generic choices in \(H^2(X; \mathbb{R})\) (genericity of \(\omega_I\) corresponds to the genericity of \(\xi_1\)). The result now follows from Theorem 48. \(\square\)

7.10. Vanishing of the twisted symplectic cohomology.

Lemma 50. The non-compactly supported deformation from \(\omega_J\) to \(\omega_I\) can be made to satisfy Theorem 23, thus \(SH^*(X, \omega_J; \Delta_{\omega_J}) \cong SH^*(X, \omega_I)\).

Proof. Let \(\omega_\varepsilon = \omega_J + \varepsilon \omega_I\). By the proof of Lemma 42 we can find a family of contact forms \(\theta|_S\) on \(S = \pi^{-1}(S^2_p/\Gamma)\) with \(d\theta_\varepsilon = \omega_\varepsilon\). By Gray’s stability theorem, there is a family of contactomorphisms \(\psi_\varepsilon : S \to S\) such that \(\psi_\varepsilon^*\theta|_S = e^{f_\varepsilon} \theta|_S\). As we deform \(\omega_0\) to \(\omega_\varepsilon\) we simultaneously change the hypersurface in \(X\) by
\[
S \to X, (R, x) \mapsto (e^{-f_\varepsilon(R, x)} R, \psi_\varepsilon(R, x)),
\]
so that on the collar determined by this hypersurface the one-form is \(\theta_0\) instead of \(\theta_\varepsilon\). This change of hypersurface will change the symplectic cohomology by an isomorphism (Theorem 8). The “interior part” of \(X\) has changed by a diffeomorphism, and we have reduced the setup to the case where we deform \(\omega_0\) to a form \(\omega_\varepsilon\) which is cohomologous to \(\omega_\varepsilon\), but which equals \(d\theta_0\) on the collar.

Now it is possible to make a compactly supported deformation from \(\omega_J\) to \(\omega_\varepsilon\) and, for small \(\varepsilon\), Theorem 23 implies that \(SH^*(X, \omega_J; \Delta_{\omega_J}) \cong SH^*(X, \omega_\varepsilon)\). Rescale by \(1/\varepsilon\) via \(t \mapsto t^{1/\varepsilon}\) to deduce that
\[
SH^*(X, \omega_J; \Delta_{\omega_J}) \cong SH^*(X, \omega_\varepsilon/\varepsilon).
\]
Now \(\omega_\varepsilon/\varepsilon\) is cohomologous to \(\omega_I\). By applying Gray’s theorem as above, we can change \(\omega_\varepsilon/\varepsilon\) within its cohomology class so that on the collar it becomes equal to \(\omega_I\). Finally we apply a compactly-supported Moser symplectomorphism as in Lemma 9 to deform the form to \(\omega_I\) on all of \(X\). Hence
\[
SH^*(X, \omega_\varepsilon/\varepsilon) \cong SH^*(X, \omega_I). \quad \square
\]

Theorem 51. Let \(X\) be an ALE space and let \(d\theta\) denote any non-zero linear combination of \(\omega_J\) and \(\omega_K\). For any generic closed two-form \(\beta\) on \(X\),
\[
SH^*(X, d\theta; \Lambda_{\tau, \beta}) = 0.
\]
Again, generic is understood in the sense of Corollary 10.

Proof. By the proof of Corollary 10 we may assume that \(\omega_I\) represents \([\beta]\). In particular, \(SH^*(X, \omega_I) = 0\). Note that a non-zero linear combination of \(\omega_J\) and \(\omega_K\) is of the form \(c \omega_u\) for some \(c > 0\) and \(u = (0, u_J, u_K) \in S^2\), and is therefore exact. The proof of Lemma 50 can easily be adapted to \(d\theta = c \omega_u\), so \(SH^*(X, d\theta; \Delta_{\omega_J}) = 0\). By Lemma 14 \(\Lambda_{\tau, \omega_J}\) only depends on the cohomology class of \(\omega_I\) up to isomorphism, therefore \(SH^*(X, d\theta; \Lambda_{\tau, \beta}) = 0\). \(\square\)
7.11. Exact Lagrangians in ALE spaces.

**Theorem 52.** Let $X$ be an ALE space. Then any exact Lagrangian submanifold $j : L \hookrightarrow (X, \omega_J)$ must be a sphere, in particular $L$ cannot be unorientable. This result also holds if we replace $\omega_J$ by any non-zero combination of $\omega_J$ and $\omega_K$.

**Proof.** Since $SH^*(X, \omega_J; \Lambda_{\omega_I}) = 0$, Corollary 17 implies that the transgression $\tau(j^*\omega_I)$ cannot vanish. But for orientable $L$ which are not spheres all transgressions must vanish since $\pi_2(L) = 0$. Therefore the only allowable orientable exact Lagrangians are spheres. The unorientable case follows by Remark 18.

**Corollary 53.** Let $(Y, d\theta)$ be the plumbing of copies of $T^*S^2$ as prescribed by any ADE Dynkin diagram. Then any exact Lagrangian $L \subset Y$ must be a sphere, in particular $L$ cannot be unorientable.

**Proof.** This follows immediately by Section 7.4 (or, as mentioned in Remark 41, by embedding $Y$ into an ALE space $X$ with the same Dynkin diagram).

**References**

[1] V. I. Arnol'd, *Dynamical Systems VI*, Singularity theory I, Springer-Verlag, 1993.

[2] Y. Eliashberg, L. Polterovich, *Unknottedness of Lagrangian surfaces in symplectic 4-manifolds*, Internat. Math. Res. Notices 1993, no. 11, 295–301.

[3] L. C. Evans, *Partial Differential Equations*, AMS, GSM 19, 1998.

[4] R. Hind, *Lagrangian isotopies in Stein manifolds*, preprint [arXiv:math/0311093] 2003.

[5] N. Hitchin, *Hyper-Kähler manifolds*. Séminaire Bourbaki, Vol. 1991/92. Astérisque No. 206 (1992), Exp. No. 748, 3, 137–166.

[6] N. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. 108 (1987), no. 4, 535–589.

[7] H. Hofer, D. Salamon, *Floer homology and Novikov rings*, The Floer memorial volume, 483–524, Progr. Math., 133, Birkhäuser, Basel, 1995.

[8] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989), no. 3, 665–683.

[9] P. B. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989), no. 3, 685–697.

[10] A. Ritter, *Novikov-symplectic cohomology and exact Lagrangian embeddings*, Geometry & Topology 13 (2009), 943–978.

[11] D. Salamon, *Lectures on Floer homology*, Symplectic geometry and topology (Park City, UT, 1997), 143–229, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.

[12] D. McDuff, D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, 6. American Mathematical Society, Providence, RI, 1994.

[13] P. Seidel, *A biased view of symplectic cohomology*, Current Developments in Mathematics, Volume 2006 (2008), 211-253.

[14] P. Slodowy, *Four lectures on simple groups and singularities*, Communications of the Mathematical Institute (Utrecht), 11, 1980.

[15] C. Viterbo, *Functors and computations in Floer homology with applications. I.*, Geom. Funct. Anal. 9 (1999), no. 5, 985–1033.

[16] C. Viterbo, *Exact Lagrange submanifolds, periodic orbits and the cohomology of free loop spaces*, J. Differential Geom. 47 (1997), no. 3, 420–468.

[17] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, NY 1978.