Qualitative analysis on logarithmic Schrödinger equation with general potential

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Abstract. In this paper, we study the existence, uniqueness, nondegeneracy, and some qualitative properties of positive solutions for the logarithmic Schrödinger equations:

$$-\Delta u + V(|x|)u = u \log u^2, \quad u \in H^1(\mathbb{R}^N).$$

Here \( N \geq 2 \), and \( V \in C^2((0, +\infty)) \) is allowed to be singular at 0 and repulsive at infinity (i.e., \( V(r) \to -\infty \) as \( r \to \infty \)). Under some general assumptions, we show the existence, uniqueness, and nondegeneracy of this equation in the radial setting. Specifically, these results apply to singular potentials such as \( V(r) = \alpha_1 \log r + \alpha_2 r^{\alpha_3} + \alpha_4 \) with \( \alpha_1 > 1 - N, \alpha_2, \alpha_3 \geq 0, \) and \( \alpha_4 \in \mathbb{R} \), which is repulsive for \( \alpha_1 < 0 \) and \( \alpha_2 = 0 \). We also investigate the connection between some power-law nonlinear Schrödinger equation with a critical frequency potential and the logarithmic-law Schrödinger equation with \( V(r) = \alpha \log r, \alpha > 1 - N \), proving convergence of the unique positive radial solution from the power-type problem to the logarithmic-type problem. Under a further assumption, we also derive the uniqueness and nondegeneracy results in \( H^1(\mathbb{R}^N) \) by showing the radial symmetry of solutions.

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1. Introduction

We investigate the existence, uniqueness, and nondegeneracy of positive solutions to

$$-\Delta u + V(|x|)u = u \log u^2, \quad u \in H^1(\mathbb{R}^N),$$

where \( N \geq 2 \), \( V \in C^2((0, +\infty)) \) is allowed to be singular at 0 and unbounded from below at infinity. Problem (1.1) comes from the study of standing waves.
to the nondispersive logarithmic Schrödinger equation

\[ i\hbar \frac{\partial \Phi}{\partial t} = -\Delta \Phi + V(x)\Phi - \Phi \log |\Phi|^2 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}, \tag{1.2} \]

where \( \hbar \) is the Planck constant and \( i \) is the imaginary unit. For a constant potential \( V \), this equation was introduced in \([4,5]\) to satisfy a specific tensorization property. See also \([11]\) for a brief introduction on this property. The equation (1.2) has been applied in several fields such as the Bose–Einstein condensation, nuclear physics, and the optics. See \([2,4,5,11,14,44,45]\) and the references therein for more discussions on (1.2). We also refer to \([13]\) for the dispersive case.

For the standing waves problem (1.2), mathematical analysis has been carried out in recent years on the existence and multiplicity of solutions (\([18,22,24,34,35,37,42,43]\)). When \( V \) is constant, up to a shift (see e.g., \([18,22,37]\)), (1.1) is equivalent to the following equation

\[ -\Delta u = u \log u^2, \quad u \in H^1(\mathbb{R}^N). \tag{1.3} \]

It has been pointed out in \([4,5,18,38]\) that (1.3) has a positive solution, i.e., the Gaussian. In \([18]\), d’Avenia et al. also proved the existence of infinitely many radially symmetric sign-changing solutions to (1.3) when \( N \geq 3 \). Existence results are also derived for various nonconstant potentials. The authors in \([24,34]\) proved the existence of solutions for different types of potentials that are bounded from below. In \([22,43]\), the authors consider the semiclassical bound states of (1.1) under some general assumptions that allow the potential to be unbounded from below at a finite number of singular points or at infinity.

Another interest in the study of nonlinear Schrödinger-type equations is the uniqueness of the positive solution. We note that the study of the uniqueness for nonlinear Schrödinger equation with power-type nonlinearity was started by Coffman \([17]\), and had been widely extended to general nonlinearities and general potentials in \([8,15,27–33,41]\). Some of these results can be partially applied to the logarithmic equation (1.3). The uniqueness result of Serrin-Tang \([33]\) and the symmetry result of d’Avenia et al. \([18, Proof of Theorem 1.2]\) imply that the Gaussian is the unique positive radial solution when \( N \geq 3 \). The special case \( N = 2 \) was treated by Troy \([38]\). When \( 1 \leq N \leq 9 \), Troy \([38]\) proved that the Gaussian is the only solution to (1.3), among positive \( u \) satisfying \((u(r), u'(r)) \to (0, 0)\) as \( r \to \infty \), by energy estimates and Ricatti equation estimates. However, it is not clear whether there is a unified way to derive the existence and uniqueness results for problem (1.1) with a general potential and for \( N \geq 2 \).

In this paper, we will deal with (1.1) under some general assumptions on the potential \( V \) that is allowed to be singular at 0 and unbounded from below at infinity. A typical example of this is the following problem that appeared in \([22]\) as a limit equation of a semiclassical problem with singularities,

\[ -\Delta u + (\alpha \log |x|)u = u \log u^2, \quad u \in H^1(\mathbb{R}^N). \tag{1.4} \]

It has been proved in \([22]\) that this equation has a positive ground state when \( \alpha > 0 \). Here we assume \( \alpha > 1 - N \), and note that the potential \( \alpha \log |x| \) is
repulsive when $\alpha \in (1 - N, 0)$. We will obtain general existence, uniqueness, and nondegeneracy results for (1.1) which covers (1.4). Moreover, we also discuss the connection between (1.4) and a power nonlinear equation with a critical frequency ([9, 10]), which extends the result of [40].

To state the results, following the ideas of Byeon–Oshita [8] and Kabeya–Tanaka [27], for $\delta \geq 0$, we consider the following perturbed problem which is slightly more general than (1.1),

$$-\Delta u + (V + \delta a)u = (1 + \delta b)u \log u^2, \quad u \in H^1(\mathbb{R}^N),$$  \hspace{0.5cm} (1.5)

where $a, b$ and $V$ satisfy

(Vab) $a \in C^2_0(\mathbb{R}^N), b \in C^3_0(\mathbb{R}^N)$ are radially symmetric functions, $b \geq 0$ and $b = 1$ in a neighborhood of 0.

(V1) $V \in C^2((0, +\infty)), \liminf_{r \to +\infty} \frac{V(r)}{\log r} \in (1 - N, +\infty)$, $V(|x|) \in L^q_{loc}(\mathbb{R}^N)$ for some $q > N$.

We have the following result on existence.

**Theorem 1.1.** Assume (Vab), (V1) and $\delta \geq 0$. Then (1.5) has a positive radial solution $u \in C^2_0(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \{0\})$, where $\gamma := Nq$. Moreover, for each $\tau \in (0, \frac{1}{2})$, there holds

$$\lim_{r \to \infty} u(r) e^{\tau r^2} = 0.$$ \hspace{0.5cm} (1.6)

Assumption (V1) implies that repulsion would happen in the potential. In fact, as $r \to +\infty$, $V(r)$ can approach the minus infinity with a logarithmic strength. To overcome the noncompactness caused by this repulsion, we will make use of the singular nature of logarithms and the radial lemma of Strauss [36] that for $u \in H^1_r(\mathbb{R}^N)$,

$$u(r) = O(r^{\frac{1-N}{2}}) \quad \text{as} \quad r \to \infty.$$ \hspace{0.5cm} (1.7)

Here, $H^1_r(\mathbb{R}^N)$ denotes the space of radial functions in $H^1(\mathbb{R}^N)$. Then a positive solution to (1.5) can be obtained as a minimizer on the Nehari manifold in a suitable subspace of $H^1_r(\mathbb{R}^N)$.

To achieve the uniqueness of positive radial solutions in $H^1_r(\mathbb{R}^N)$, it is natural to investigate every solution satisfying the decay property (1.7). In fact, we will show the uniqueness under a weaker decay condition. To state the problem, we define

$$\Theta := \left\{ \theta \in (1 - N, +\infty) \left| \liminf_{r \to \infty} (V(r) - \theta \log r) > -\infty \right. \right\}.$$  \hspace{0.5cm}

Note that under the assumption (V1), $\Theta$ is a nonempty subset of

$$\left(1 - N, \liminf_{r \to +\infty} \frac{V(r)}{\log r} \right].$$

For each $\theta \in \Theta$, we consider the uniqueness of solutions to the following problem:

$$\begin{align*}
\begin{cases}
 u'' + \frac{N-1}{r} u' - (V + \delta a)u + (1 + \delta b)u \log u^2 = 0, \quad u > 0 \text{ in } (0, +\infty), \\
 u'(0) = 0, \lim_{r \to \infty} r^{-\frac{N}{2}} u(r) = 0.
\end{cases}
\end{align*}$$  \hspace{0.5cm} (1.8)
By the singularity of the potential and nondecay assumption of $u$ in (1.8) when $\theta > 0$, we have to determine firstly the positivity of $u(0)$ and a fast decay estimate for each solution $u$ of (1.8). These will be done in Proposition 2.3. In particular, every solution to (1.8) should be in $H^1_r(\mathbb{R}^N)$ and should satisfy (1.5) by Proposition 2.3. To achieve uniqueness, we need a further assumption. Set

$$G(r) = V(r) + \frac{(N-1)(N-3)}{4r^2} + (N-1)\log r$$

and assume

(V2) When $N = 2$ or $3$, $G' > 0$ in $(0, +\infty)$ and $\liminf_{r \to 0^+} G'(r) > 0$. When $N \geq 4$, $\limsup_{r \to 0^+} r^3 G'(r) < 0$ and $r^3 G'(r)$ has a unique simple zero in $(0, +\infty)$.

We have

**Theorem 1.2.** Assume (Vab), (V1), and (V2). Then there is $\delta_0 > 0$ such that for every $\delta \in [0, \delta_0]$, problem (1.8) admits a unique solution. Especially, the positive radial solution to problem (1.5) is unique in $H^1_r(\mathbb{R}^N)$.

Note that if $\liminf_{r \to +\infty} V(r) > -\infty$, we can take $\theta = 0 \in \Theta$ in problem (1.8). Then we obtain a direct corollary of Theorem 1.2.

**Corollary 1.3.** Assume (Vab), (V1), (V2), and $\liminf_{r \to +\infty} V(r) > -\infty$. Then there is $\delta_0 > 0$ such that for $\delta \in [0, \delta_0]$, the following problem has a unique solution:

$$\begin{cases}
  u'' + \frac{N-1}{r} u' - (V + \delta a) u + (1 + \delta b) u \log u^2 = 0, & u > 0 \text{ in } (0, +\infty), \\
  u'(0) = 0, \lim_{r \to +\infty} u(r) = 0.
\end{cases}$$

**Remark 1.4.** We give some more comments on Theorem 1.2 and Corollary 1.3.

(i) For $\delta = 0$, the uniqueness of radial solutions to (1.8) under (V1) and the following weaker condition than (V2):

(V2') when $N = 2$ or $3$, $G' \geq 0$ in $(0, +\infty)$; and when $N \geq 4$,

$$\limsup_{r \to 0^+} r^3 G'(r) < 0$$

and the zero set of $r^3 G'(r)$ in $(0, +\infty)$ is a connected nonempty set.

(ii) A sufficient condition for (V2') is the following: $V \in C^2((0, +\infty))$ satisfies $V' > \frac{1-N}{r}$ for $N \geq 2$, and additionally $rV'$ increases when $N \geq 4$. In fact, it is clear when $N = 2, 3$. When $N \geq 4$, we have $\lim_{r \to 0^+} (rV' + N - 1) \geq 0$, and hence, $\lim_{r \to +\infty} (rV' + N - 1) > 0$. Then $r^3 G'(r) = r^2 (rV' + N - 1)(N - 1)(N - 3)/2$ is increasing, which satisfies $\lim_{r \to 0^+} r^3 G'(r) < 0$ and $\lim_{r \to +\infty} r^3 G'(r) = +\infty$. Hence, the assumption (V2') in (i) is verified.
(iii) We give typical examples of $V$ for the uniqueness of radial solutions to (1.8) with $\delta = 0$: $\alpha_1 \log r + \alpha_2 r^{\alpha_3} + \alpha_4$ with $\alpha_1 > 1 - N$, $\alpha_2, \alpha_3 \geq 0$ and $\alpha_4 \in \mathbb{R}$.

(iv) Corollary 1.3 generalizes the uniqueness results of [18] and [38] for constant potentials and $\delta = 0$. Especially, a problem raised by Troy in [38, Problem 1] is resolved by combining Corollary 1.3 and the symmetry result of [18, Proof of Theorem 1.2] that is true for $N \geq 2$.

(v) There is an example that multiple positive radial solutions exist for repulsive potential: $V(r) = -\mu r^2$ for $\mu \in (0, \frac{1}{4})$. See [12] for the explicit expressions of these solutions.

Now, we study the uniqueness of positive solutions to problem (1.1). To this end, we need to show any positive solution of (1.1) is radially symmetric by assuming

(V3) $V \in C((0, +\infty))$ is increasing and nonconstant.

Note that (V3) implies $\liminf_{r \to +\infty} V(r) > -\infty$. We have

**Theorem 1.5.** Assume (V3) and let $u \in C(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ be a solution to

\[
\begin{cases}
-\Delta u + V(|x|)u = u \log u^2, \\
u > 0, u \to 0 \text{ as } |x| \to \infty.
\end{cases}
\]

Then $u$ is radially symmetric about 0. Moreover, $u'(r) < 0$ for each $r > 0$.

A direct corollary of Theorem 1.5 and Corollary 1.3 is the uniqueness of positive solutions to problem (1.1).

**Corollary 1.6.** Assume (V1), (V2'), and (V3). Then problem (1.1) (or (1.9)) has a unique positive solution.

**Remark 1.7.** A typical example of $V$ satisfying (V1),(V2), and (V3) is that $\alpha_1 \log r + \alpha_2 r^{\alpha_3} + \alpha_4$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_3 > 0$, $\alpha_4 \in \mathbb{R}$ and $\alpha_1 + \alpha_2 \neq 0$. Especially, the assumptions cover the confining harmonic potential $\alpha r^2$ with $\alpha > 0$ that has been widely studied in physics. See for example [6,7].

We also study the nondegeneracy of the unique radial solution. We say a positive radial solution $w$ to (1.1) is nondegenerate in $H^1_r(\mathbb{R}^N)$ (resp. $H^1(\mathbb{R}^N)$) if the problem

\[-\Delta \psi + V(|x|)\psi = (\log w^2 + 2)\psi\]

has no nontrivial solution in $H^1_r(\mathbb{R}^N)$ (resp. $H^1(\mathbb{R}^N)$). We have the following:

**Theorem 1.8.** Under the assumptions (V1) and (V2), the unique positive radial solution to (1.1) is nondegenerate in $H^1_r(\mathbb{R}^N)$. Under the assumptions (V1), (V2), and (V3), the unique positive solution to (1.1) is nondegenerate in $H^1(\mathbb{R}^N)$. 
For an equation with a $C^2$-smooth variational structure, the nondegeneracy of a solution can be characterized as the nondegenerate Hessian at the corresponding critical point of the variational functional. This requires a $C^2$-smooth variational structure associated with the problem. However, by the non-lipschitzian property of the logarithmic nonlinearity, there is no $C^2$-smooth variational structure for problem (1.1). We find a method in Section 4.1 to recover smoothness partially so that we can apply the idea of [8,27] to show Theorem 1.8.

As we have discussed, problem (1.4) was first studied in [22] as a limit equation for some semiclassical logarithmic Schrödinger equation. In [22], it was also pointed out that the semiclassical state behaves similarly to that studied in [9,10] on the semiclassical power nonlinear Schrödinger equations with a critical frequency. So it seems that there should be a connection between these two types of limit equations. We will reveal this connection by studying the limit profile of the following equation:

$$-\Delta u + |x|^\alpha u = |u|^{2\sigma} u, \quad u \in H^1(\mathbb{R}^N),$$

(1.10)

where $\alpha > 1 - N$ and $\sigma \in (0, 2/(N - 2)^+)$. Here we denote that, throughout this paper, $d^+ = \max\{d, 0\}$ and $d^- = -\min\{d, 0\}$ for $d \in \mathbb{R}$. Moreover, we take $1/0^+ = +\infty$. By [8], we see that (1.10) has exactly one positive solution which is radially symmetric if $\alpha \geq 0$ and $\sigma \in (0, 2/(N - 2)^+)$. We will see that (1.10) has a unique positive radial solution even when $1 - N < \alpha < 0$ under some additional conditions. Above all, for every $\alpha > 1 - N$, we can give the convergence result of the unique positive radial solution to (1.10) as $\sigma \to 0^+$. In fact, we have

Theorem 1.9. Assume that $\alpha > 1 - N$ and $\sigma \in (0, 2/(N - 2)^+)$. The following statements hold.

(i) If

$$-\sigma \alpha < 2 \min\{1, N - 1 + \alpha\},$$

(1.11)

then (1.10) has a nonnegative nontrivial solution $u_{\sigma} \in H^1_r(\mathbb{R}^N)$ satisfying $u_{\sigma} > 0$ in $\mathbb{R}^N \setminus \{0\}$ and

$$u_{\sigma}(x) \leq C_{\sigma} \exp\left(-c_{\sigma}|x|^{\alpha+2}\right), \quad x \in \mathbb{R}^N,$$

where $C_{\sigma}, c_{\sigma}$ are positive constants independent of $x$.

(ii) Assume $-\sigma \alpha < 1$ in addition to (1.11). Then, $u_{\sigma} \in C^{2-\gamma'}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ for each $\gamma' \in (-\alpha, 1)$ and $u_{\sigma} > 0$ solves (1.10) uniquely in $H^1_r(\mathbb{R}^N)$.

(iii) For any $\gamma'' \in (0, 1)$, as $\sigma \to 0^+$, $\sigma^{\frac{1}{2\gamma''}} u_{\sigma}(\sigma^{-\frac{1}{2\gamma''}} \cdot)$ converges to the unique positive radial solution of (1.4), in $H^1_r(\mathbb{R}^N)$ and $C^{2-\gamma''}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus B_{\gamma''}(0))$.

Note that assumptions (1.11) and $-\sigma \alpha < 1$ hold automatically for either $\alpha \geq 0$ or $\sigma$ small enough.

The solution to (1.10) will be found in a suitable space and proved to belong to $H^1_r(\mathbb{R}^N)$. The proof of existence depends on a generalization of the radial lemma of Strauss [36]. We remark that when $\alpha = 0$, Theorem 1.9(iii) is
in fact the convergence result considered in [40, Theorem 1.1]. We also refer to [23,26] for related works on convergence behaviors of sublinear elliptic equations and eigenvalue problems, and [19,25] on monotone properties of the ground states to the scalar field equations.

The remainder of the paper is organized as follows. In Sect. 2, we introduce a suitable work space and show the existence of a radial solution to (1.5) and prove its positivity and the Gaussian decay estimate (1.6). Section 3 will be devoted to proving the uniqueness and radial symmetry of positive solutions to (1.1). In addition, in Sect. 4, we also deal with the nondegeneracy of the unique radial solution to (1.1), respectively, in $H^1_r(\mathbb{R}^N)$ and in $H^1(\mathbb{R}^N)$. In Sect. 5, by a revised radial lemma in a suitable function space and a decay estimate, we first obtain a positive radial solution to (5.1) and show the uniqueness of this solution in the radial setting. Then we show that the solution converges to the unique positive radial solution of (1.4) as $\sigma \to 0^+$, which extends the result of [40]. Finally, in Appendix 5, some technical results that are useful for the proof of uniqueness will be given.

2. Existence and decay estimate

Throughout the section, we assume (Vab) and (V1). First, we show that the problem

$$-\Delta u + V_\delta u = B_\delta u \log u^2, \quad u \in H^1(\mathbb{R}^N),$$

(2.1)

admits a radial ground state solution, where $V_\delta = V + \delta a$ and $B_\delta = 1 + \delta b$. Let

$$\alpha_0 := \begin{cases} \frac{1}{2} \left( \liminf_{r \to +\infty} \frac{V(r)}{\log r} + 1 - N \right) & \text{if } \liminf_{r \to +\infty} \frac{V(r)}{\log r} \leq 0, \\ 0 & \text{if } \liminf_{r \to +\infty} \frac{V(r)}{\log r} > 0. \end{cases}$$

(2.2)

Note that $\alpha_0 \leq 0$ and $1 - N < \alpha_0 < \liminf_{r \to +\infty} \frac{V(r)}{\log r}$. Set

$$E := \left\{ u \in H^1(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} (V - \alpha_0 \log |x|)^+ u^2 < +\infty \right. \right\}$$

with the norm

$$\|u\| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \int_{\mathbb{R}^N} (V - \alpha_0 \log |x|)^+ u^2 \right)^{\frac{1}{2}}.$$

Note that

$$\liminf_{r \to +\infty} \frac{V(r) - \alpha_0 \log r}{\log r} = \liminf_{r \to +\infty} \frac{V(r)}{\log r} - \alpha_0 > 0.$$  

(2.3)

Then the embedding $E \subset L^p(\mathbb{R}^N)$ is compact for $p \in [2, 2^*)$, where $2^* = 2N/(N - 2)^+$. As in [22], the Gagliardo–Nirenberg interpolation inequality is frequently used in this section. For every $p \in (2, 2^*)$, we have

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C \|u\|_{L^2(\mathbb{R}^N)}^{1 - \nu_p} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\nu_p} \quad \text{for all } u \in H^1(\mathbb{R}^N),$$

(2.4)
where $\nu_p = \frac{N(p-2)}{2p} \in (0, 1)$ satisfies $\nu_p \to 0$ as $p \to 2$. Moreover, by combining Young’s inequality and (2.4), for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that
\[
\|u\|^2_{L^p(R^N)} \leq C_\varepsilon \|u\|^2_{L^2(R^N)} + \varepsilon \|\nabla u\|^2_{L^2(R^N)}.
\] (2.5)

We consider another equivalent norm on $E$ for convenience.

**Lemma 2.1.** For each $\delta \geq 0$, there is $\mu_\delta > 0$ such that the norm $\| \cdot \|_\delta$ defined by
\[
\|u\|^2_\delta = \int_{R^N} (|\nabla u|^2 + (V_\delta - \alpha_0 \log |x|)^2 u^2 + \mu_\delta u^2), \quad u \in E
\]
is equivalent to $\| \cdot \|$ on $E$.

**Proof.** It suffices to find $\mu_\delta > 0$ such that for every $u \in E$,
\[
\int_{R^N} (|\nabla u|^2 + (V_\delta - \alpha_0 \log |x|)^2 u^2 + \mu_\delta u^2) \geq \frac{1}{2} \|u\|^2.
\] (2.6)

In fact, by (2.3) and (Vab), there is $R_1 > 0$ such that $\text{supp}(V_\delta - \alpha_0 \log |x|)^- \subset B_{R_1}(0)$. Then by the Hölder’s inequality and (2.5), for some $C_1 > 0$, there holds
\[
\int_{R^N} (V_\delta - \alpha_0 \log |x|^-)^2 u^2 \leq \|V_\delta - \alpha_0 \log |x|\|_{L^q(B_{R_1}(0))}\|u\|^2_{L^{q'}(B_{R_1}(0))}
\]
\[
\leq \frac{1}{2} \|\nabla u\|^2_{L^2(R^N)} + C_1 \|u\|^2_{L^2(R^N)}
\]
for all $u \in E$. So, setting $\mu_\delta = C_1 + \frac{1}{2} + \delta \max_{x \in R^N} |a(x)|$, we have (2.6). \(\square\)

By [22, Lemma 2.1], there hold $uv \log u^2 \in L^1(R^N)$ for $u, v \in E$. Especially, by [22, Remark 2.1], we can define the $C^1$-functional in $E$,
\[
I_\delta(u) = \frac{1}{2} \int_{R^N} |\nabla u|^2 + (V_\delta + B_\delta)u^2 - \frac{1}{2} \int_{R^N} B_\delta u^2 \log u^2.
\]

Denote $E_\tau := E \cap H^1_{\tau}(R^N)$. Note that by the radial lemma of Strauss [36] (see also [3]), there is $C_N > 0$ independent of $u$, such that
\[
|u(x)| \leq C_N |x|^{\frac{N-2}{2}} \|u\|_{H^1(R^N)}.
\] (2.7)

We consider the minimization problem
\[
c_\delta = \inf_{u \in N^\delta_\tau} I_\delta(u),
\] (2.8)
where
\[
N^\delta_\tau = \left\{ u \in E_\tau \setminus \{0\}; 0 = J_\delta(u) := I_\delta(u)u = \int_{R^N} |\nabla u|^2 + V_\delta u^2 - \int_{R^N} B_\delta u^2 \log u^2 \right\}.
\]

**Proposition 2.2.** The following statements hold for each $\delta \geq 0$.

(i) For each $u \in E_\tau \setminus \{0\}$, there is a unique $t_u \in R$ such that $e^{t_u/2} u \in N^\delta_\tau \setminus \{0\}$. Moreover, $t_u = (\int_{R^N} B_\delta u^2)^{-1} J_\delta(u)$, and it is the unique maximum point of the function $t(\in R) \mapsto I_\delta(e^{t/2} u)$.

(ii) $c_\delta > 0$ and the minimization problem (2.8) is attained by a radial solution, which is positive in $R^N \setminus \{0\}$, to (2.1).
Proof. 1. The first conclusion follows from a direct calculation,
\[
\frac{d}{dt} I_\delta(e^{\frac{t}{2}} u) = \frac{1}{2} e^t \left( J_\delta(u) - t \int_{\mathbb{R}^N} B_\delta u^2 \right).
\]

2. We write
\[
I_\delta(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (V_\delta - \alpha_0 \log |x| + \mu_\delta + B_\delta) u^2 - \frac{1}{2} \int_{\mathbb{R}^N} (u^2 \log(e^{\alpha_\delta|x|} - \alpha_0 u^2) + \delta b u^2 \log u^2).
\]

Fix \( p \in (2, 2^*) \) such that
\[
p - 2 + \nu_p p < 2. \tag{2.9}
\]

By (2.7), for some \( C_3 > 0 \)
\[
u_1 u^2 \log(e^{\alpha_\delta|x|} - \alpha_0 u^2) + \delta b u^2 \log u^2
\leq u^2 (\log(e^{\alpha_\delta u^2})^+ + \delta b u^2 (\log u^2)^+
\leq u^2 (\log(e^{\alpha_\delta u^2})^+ + u^2 \left( \log(e^{\alpha_\delta C_N \|u\|_{H^1(\mathbb{R}^N)}^2 - \frac{2\alpha_0}{N}})^+ + \delta b u^2 (\log u^2)^+
\leq C_3 \left( \|u|^p + \|u|^p_{H^1(\mathbb{R}^N)} \right) \bigg| u \bigg|^{p_1} \bigg) \quad \text{in} \quad \mathbb{R}^N,
\]

where \( p_1 := p - \frac{(p - 2)\alpha_0}{1 - N} \in (2, p] \). By (2.6) and (2.10), we have \( I_\delta(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^p \) for some \( C > 0 \). Hence, there is \( m_\delta > 0 \) such that
\[
I_\delta(u) \geq \frac{1}{8} m_\delta^2 \quad \text{for} \quad \|u\| = m_\delta \quad \text{and} \quad I_\delta(u) \geq 0 \quad \text{for} \quad \|u\| \leq m_\delta.
\]

Then
\[
c_\delta = \inf_{u \in \mathcal{N}_p} \sup_{t \in \mathbb{R}} I_\delta(e^{\frac{t}{2}} u) \geq \frac{1}{8} m_\delta^2 > 0.
\]

To show that \( c_\delta \) is achieved, we assume \( u_n \in \mathcal{N}_p^\delta \) is such that \( I_\delta(u_n) \rightarrow c_\delta > 0 \). Since
\[
\int_{\mathbb{R}^N} B_\delta u_n^2 = 2I_\delta(u_n) - J_\delta(u_n) = 2I_\delta(u_n), \tag{2.11}
\]

we see that \( \|u_n\|_{L^2(\mathbb{R}^N)} \) is bounded. By \( J_\delta(u_n) = 0 \) and (2.10), we have
\[
\|u_n\|^2_{I_\delta} = \int_{\mathbb{R}^N} (u_n^2 \log(e^{\alpha_\delta|x|} - \alpha_0 u_n^2) + \delta b u_n^2 \log u_n^2)
\leq C_p \left( \|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u_n\|^p_{H^1(\mathbb{R}^N)} \right) \|u_n\|^p_{L^p(\mathbb{R}^N)}
\leq C' \left( \|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u_n\|^p_{H^1(\mathbb{R}^N)} \right) \|\nabla u_n\|^p_{L^p(\mathbb{R}^N)}
\leq C'' \left( \|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u_n\|^p_{H^1(\mathbb{R}^N)} \right)
\leq C'' \left( \|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u_n\|^p_{L^p(\mathbb{R}^N)} \right).
\]

By (2.9), we have \( p - p_1 + \nu_{p_1} p_1 < p - 2 + \nu_p p < 2 \). Then \( \|u_n\|_{I_\delta} \) is bounded.
Up to a subsequence, we assume $u_n \to u$ in $E$ and $u_n \to u$ in $L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. By (2.11), we obtain $\int_{\mathbb{R}^N} B_\delta u^2 = 2c_\delta > 0$ and $u \neq 0$. By (i), there is $t_u$ such that $e^{t_u/2}u \in N^\delta_r$. To prove the attainability of $c_\delta$, it suffices to show that $t_u = 0$.

Note that (2.10) and the boundedness of $\|u_n\|_\delta$ imply that

$$u_n^2 \left( \log(e^{\mu\delta} |x|^{-\alpha_0} u_n^2) \right)^+ + \delta b u_n^2 \left( \log u_n^2 \right)^+ \leq C(|u_n|^p + |u_n|^p) \quad \text{in} \quad \mathbb{R}^N.$$ 

Then by the dominated convergence theorem, up to a subsequence, we can conclude that

$$\int_{\mathbb{R}^N} u_n^2 \left( \log(e^{\mu\delta} |x|^{-\alpha_0} u_n^2) \right)^+ + \delta b u_n^2 \left( \log u_n^2 \right)^+ \to (2.12) \quad \int_{\mathbb{R}^N} u^2 \left( \log(e^{\mu\delta} |x|^{-\alpha_0} u^2) \right)^+ + \delta b u^2 \left( \log u^2 \right)^+.$$ 

Therefore, by (i), (2.12), the weakly lower semicontinuity of norm and the Fatou’s Lemma, we obtain

$$t_u \int_{\mathbb{R}^N} B_\delta u^2 = J_\delta(u) \leq \liminf_{n \to +\infty} J_\delta(u_n) = 0.$$ 

On the other hand, by

$$2c_\delta \leq 2I_\delta(e^{t_u/2}u) = e^{t_u} \int_{\mathbb{R}^N} B_\delta u^2,$$ 

we have $t_u \geq 0$. Then $t_u = 0$ and $u \in N^\delta_r$.

Since $c_\delta$ is also achieved by $|u|$, we may assume that $u \geq 0$ and solves (2.1). By the regularity theory, $u \in C^{2-\gamma}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$. Noting that $u$ satisfies

$$-\Delta u + V^+_\delta u \geq B_\delta u \log u^2 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$ 

by the maximum principle in [39], we have $u > 0$ in $\mathbb{R}^N \setminus \{0\}$. \hfill \Box

The positivity and the decay estimate of the solution is obtained in the following.

**Proposition 2.3.** Let $u(r)$ solves (1.8). Then

(i) $u(0) > 0$,

(ii) $\lim_{r \to +\infty} u(r)e^{\tau r^2} = 0$ for any $\tau \in (0, 1/2)$.

**Proof.** (i) Assume on the contrary that $u(0) = u'(0) = 0$. We have

$$(r^{N-1}u')' - r^{N-1}V_\delta(r)u + r^{N-1}B_\delta(r)u^2 \log u^2 = 0, \quad r \in (0, +\infty),$$

or

$$u(r) = \int_0^r s^{1-N} ds \int_0^s \left( t^{N-1}V_\delta(t)u(t) - t^{N-1}B_\delta(t)u^2(t) \log u^2(t) \right) dt.$$ 

Set $v(r) = \max_{0 \leq s \leq r} u(s)$. Then $0 < u(r) \leq v(r)$ for $r > 0$ and $v(r)$ is increasing with respect to $r$. Fix $r_0 > 0$ small enough such that $v(r_0) < e^{-1}$. 

Let $r_1 \in (0, r_0]$ be such that $v(r_0) = u(r_1)$; by monotonicity, we have

$$v(r_0) = u(r_1) = \int_0^{r_1} s^{1-N} ds \int_0^{s} (t^{N-1}V(t)u(t) - t^{N-1}B_\delta(t)u^2(t) \log u^2(t)) dt$$

$$\leq \int_0^{r_0} s^{1-N} ds \int_0^{s} (t^{N-1}|V_\delta(t)|v(s) - t^{N-1}B_\delta(t)v^2(s) \log v^2(s)) dt$$

$$\leq C \int_0^{r_0} \left[ s^{1-N}v(s) \left( \int_0^{s} t^{(N-1)^2} \cdot \frac{N-N}{N-\tau} dt \right) \frac{N-1}{N} \left( \int_0^{s} t^{N-1}|V_\delta(t)|N dt \right)^{\frac{1}{N}}$$

$$- v(s) \log v^2(s) \right] ds$$

$$\leq C \int_0^{r_0} \left( v(s) - v(s) \log v^2(s) \right) ds.$$

Then by [1, Lemma 1.4.1], we have $v \equiv 0$ in $(0, r_0)$. This is a contradiction.

(ii) Set $v(r) = r^{-\frac{2}{2}}u(r)$ and we have

$$v'' + \frac{N-1+\theta}{r}v' + \frac{\theta}{2r^2} \left( \frac{\theta}{2} + N - 2 \right) v - V_\delta(v) + \theta B_\delta v \log r + B_\delta v \log v^2 = 0.$$

Let $R_1 > 0$ be such that $V_\delta = V$ and $B_\delta = 1$ for $r \geq R_1$. Set

$$\mu = - \inf_{r \geq R_1} \left( V(r) - \theta \log r - \frac{\theta}{2r^2} \left( \frac{\theta}{2} + N - 2 \right) \right).$$

Then in $[R_1, +\infty)$, $v$ satisfies

$$v'' + \frac{N-1+\theta}{r}v' + v \log(e^\mu v^2) \geq 0. \quad (2.13)$$

Fix $\tau_1 \in (r, \frac{1}{2})$. Since $\lim_{r \to +\infty} v = 0$, we can find $R > R_1$ to be such that for $r \geq R$,

$$(4\tau_1^2 - 2\tau_1)r^2 - 2\tau_1(N - 1 + \theta) - 2\tau_1 < 0 \quad \text{and} \quad 4v^2(r)e^\mu < e^{-2}.$$

Now set

$$v_0(r) = 2v(R)e^{\tau_1 R^2}e^{-\tau_1 r^2}, \quad r \geq R.$$

We have

$$v_0'' + \frac{N-1+\theta}{r} v_0' + v_0 \log(e^\mu v_0^2) \leq 0. \quad (2.14)$$

We claim that

$$v(r) \leq v_0(r) \quad \text{for} \quad r \geq R.$$

Assume by contradiction that

$$\sup_{r \geq R} (v(r) - v_0(r)) > 0.$$
By \( v(R) < v_0(R) \) and \( \lim_{r \to +\infty} (v(r) - v_0(r)) = 0 \), there is a local maximum point \( R_2 \in (R, +\infty) \) of \( v - v_0 \) such that \( v(R_2) > v_0(R_2) \) and \( v'(R_2) - v_0'(R_2) = 0 \). By the strict decreasing property of \( s \log s \) for \( s \in (0, e^{-2}) \), we have

\[
v_0(R_2) \log(e^h v_0^2(R_2)) > v(R_2) \log(e^h v^2(R_2)).
\]

Then by (2.13) and (2.14), \( v''(R_2) - v_0''(R_2) > 0 \). This is a contradiction.

\( \square \)

**Remark 2.4.** Modifying the proof of Proposition 2.3 (i) slightly, we can get the uniqueness of the following initial value problem under assumptions (Vab) and (V1):

\[
\begin{aligned}
u'' + \frac{N - 1}{r} u' - V_\delta(r) u + B_\delta(r) u \log u^2 &= 0, \\
u(0) &= \beta, \quad u'(0) = 0,
\end{aligned}
\]

where \( \beta \in \mathbb{R} \).

In fact, the assumption that \( V(|x|) \in L^q(B_1(0)) \) ensures the uniqueness of the solution near 0. On the other hand, the uniqueness in \((0, +\infty)\) follows from the uniqueness criteria for initial value problems (see [1, Theorem 3.5.1]).

**Completion of the proof of Theorem 1.1.** Proposition 2.2(ii) gives the existence of a radial state solution and Proposition 2.3 implies this solution is in fact positive and has the desired decay property.

\( \square \)

### 3. Uniqueness and symmetry

In this section, we show Theorem 1.2 and Theorem 1.5.

#### 3.1. Uniqueness in the radial setting

Assuming (Vab), (V1) and (V2), to show the uniqueness of

\[
\begin{aligned}
u'' + \frac{N - 1}{r} u' - V_\delta(r) u + B_\delta(r) u \log u^2 &= 0, \\
u(0) &= \beta, \quad u'(0) = 0, \quad \beta \in \mathbb{R},
\end{aligned}
\]

we set \( V_\delta = V + \delta a, \ K_\delta = B_\delta^{-1} = (1 + \delta b)^{-1} \) and \( v = K_\delta^{-\frac{1}{2}} r^{\frac{N - 1}{2}} u \) in (3.1). Then \( v \) satisfies

\[
K_\delta v'' + \frac{K_\delta'}{2} v' - G_\delta v + v \log v^2 = 0,
\]

where

\[
G_\delta = K_\delta V_\delta - \frac{K_\delta''}{4} + \frac{3(K_\delta')^2}{16 K_\delta} + \frac{(N - 1)(N - 3) K_\delta}{4 r^2} - \frac{\log K_\delta}{2} + (N - 1) \log r.
\]

Let

\[
E_\delta(v; u) = \frac{1}{2} K_\delta (v')^2 - \frac{1}{2} G_\delta(r) v^2 + \frac{1}{2} (v^2 \log v^2 - v^2).
\]
Then
\[ E_\delta'(r) = -\frac{1}{2} G_\delta'(r) v^2. \] (3.4)

We have

**Lemma 3.1.** Assume (Vab), (V1), (V2) and let \( u \) be a solution to (3.1). Then there is \( \delta_0 > 0 \) independent of \( u \), such that for \( \delta \in [0, \delta_0] \), \( E_\delta(r; u) > 0 \) if \( r > 0 \) and \( \lim_{r \to +\infty} E_\delta(r; u) = 0 \).

**Proof.** We write \( E_\delta(r) = E_\delta(r; u) \) for brevity. By (V1), there is \( r_n \in [1/n, 2/n] \) for each \( n \), such that
\[
r_n^{N-1} V(r_n) = n \left( \int_{\frac{1}{n}}^{\frac{2}{n}} r^{N-1} V(r) dr \right) \leq \frac{n}{N} \left( \int_{\frac{1}{n}}^{\frac{2}{n}} r^{N-1} |V(r)|^N dr \right) \leq \frac{n}{N} \left( \int_{\frac{1}{n}}^{\frac{2}{n}} \frac{1}{r^{N-1}} |V(r)|^N dr \right) \to 0 \quad \text{as} \quad n \to +\infty.
\]

Hence \( \liminf_{r \to 0^+} r^{N-1} V(r) \leq 0 \). Then by (3.2), (Vab), Proposition 2.3 (ii) and Lemma A.1, we have
\[
\liminf_{r \to 0^+} G_\delta(r) v^2 \leq \begin{cases} +\infty, & \text{if } N = 2, \\ 0, & \text{if } N \geq 3, \end{cases} \quad \text{and} \quad \lim inf_{r \to +\infty} G_\delta(r) v^2 = 0 \quad \text{if } N \geq 2.
\] (3.5)

On the other hand, by
\[
v'(r) = -\frac{1}{4} K_\delta^{-\frac{N}{2}} K_\delta r^{\frac{N-1}{2}} u + \frac{N-1}{2} K_\delta^{-\frac{1}{2}} r^{\frac{N-3}{2}} u + K_\delta^{-\frac{1}{2}} r^{\frac{N-1}{2}} u',
\] (3.2), (Vab), Proposition 2.3 (ii) and Lemma A.1, we can get
\[
\lim_{r \to 0^+} v'(r) = \begin{cases} +\infty, & \text{if } N = 2, \\ (1 + \delta)^{\frac{1}{2}} u(0), & \text{if } N = 3, \\ 0, & \text{if } N \geq 4, \end{cases} \quad \text{and} \quad \lim_{r \to +\infty} v'(r) = 0. \] (3.6)

By (3.3), (3.5) and (3.6), we get
\[
\limsup_{r \to 0^+} E_\delta(r) \geq \begin{cases} +\infty, & \text{if } N = 2, \\ \frac{1}{2} (1 + \delta)^{-\frac{1}{2}} (u(0))^2, & \text{if } N = 3, \\ 0, & \text{if } N \geq 4. \end{cases} \] (3.7)

and
\[
\limsup_{r \to +\infty} E_\delta(r) = 0. \] (3.8)

**Case** \( N = 2 \) or 3

In this case, we show that there is \( \delta > 0 \) such that for \( \delta \in [0, \delta_0] \), \( E_\delta \) is strictly decreasing in \((0, +\infty)\). By (3.7) and (3.8), it suffices to show that \( E_\delta'(r) \neq 0 \) for each \( r > 0 \). Arguing indirectly, assume there are \( \delta_n \to 0 \) and \( r_n > 0 \) such that \( E_{\delta_n}'(r_n) = 0 \). By (3.4), we have \( G_{\delta_n}'(r_n) = 0 \). By (Vab) and
(V2), \( r_n \) is bounded. Otherwise, along a subsequence, \( G'(r_n) = G'_{\delta_n}(r_n) = 0 \).

This is a contradiction. On the other hand, by (Vab) and (V2), there is \( \rho_1 > 0 \) independent of \( n \) such that \( G'_{\delta_n}(r) > 0 \) for \( r \in (0, \rho_1) \). Then we may assume that \( r_n \to r_0 \geq \rho_1 > 0 \). This implies \( G'(r_0) = 0 \) and it is a contradiction.

**Case \( N \geq 4 \)**

Let \( r_0 > 0 \) be the unique simple zero of \( r^3G'(r) \). By \( \limsup_{r \to 0^+} r^3G'(r) < 0 \), we have \( r^3G'(r) < 0 \) in \((0, r_0)\), \( r^3G'(r) > 0 \) in \((r_0, \infty)\) and \( (r^3G'(r))'|_{r=r_0} > 0 \). Note also that \( r^3G'_{\delta}(r) = r^3G'(r) \) for large \( r \). Then, from (Vab) and (V2), there is \( \delta_0 > 0 \) such that for each \( \delta \in [0, \delta_0] \), \( r^3G'_{\delta}(r) \) has a unique simple zero \( r_{\delta} \) satisfying \( \lim_{\delta \to 0} r_{\delta} = r_0 \), \( G'_{\delta}(r) < 0 \) for \( 0 < r < r_{\delta} \) and \( G'_{\delta}(r) > 0 \) for \( r > r_{\delta} \). Hence \( E_{\delta} \) is strictly increasing in \((0, r_{\delta})\) and strictly decreasing in \((r_{\delta}, \infty)\). This proves the lemma for \( N \geq 4 \).

We remark that in either case (\( N = 2, 3 \) or \( N = 4 \)), the fact that \( E_{\delta}(r) \) is strictly decreasing for large \( r \) implies that \( \lim_{r \to +\infty} E_{\delta}(r) = 0 \). \( \Box \)

Now we are ready to prove the uniqueness of (3.1).

**Proof of Theorem 1.2.** Assume by contradiction that for some \( \delta \in [0, \delta_0] \), problem (3.1) has two solutions \( u_1, u_2 \) such that \( 0 < u_1(0) < u_2(0) \). By Lemma A.6, we may assume that \( u_2 \) intersects \( u_1 \) only once. Then by Lemma A.2, we can get, in \((0, +\infty)\),

\[
\frac{d}{dr}\left(\frac{u_1}{u_2}\right) > 0, \quad \text{or equivalently,} \quad \frac{d}{dr}\left(\frac{v_2}{v_1}\right)^2 = \frac{d}{dr}\left(\frac{u_2}{u_1}\right)^2 < 0,
\]

where \( v_i = K^{-\frac{1}{2}}_{\delta} r^{\frac{N-1}{2}} u_i, i = 1, 2 \). We can also see that

\[
0 < \left(\frac{v_2(r)}{v_1(r)}\right)^2 = \left(\frac{u_2(r)}{u_1(r)}\right)^2 < \left(\frac{u_2(0)}{u_1(0)}\right)^2 \quad \text{for} \quad r > 0. \quad (3.9)
\]

Setting \( E_i(r) = E_{\delta}(r; u_i), i = 1, 2 \), it follows from (3.3) and (3.4) that

\[
\frac{d}{dr}\left(\frac{v_2}{v_1}\right)^2 E_1 - E_2 = \left(\frac{v_2}{v_1}\right)^2 \frac{dE_1}{dr} - \frac{dE_2}{dr} + E_1 \frac{d}{dr}\left(\frac{v_2}{v_1}\right)^2 = E_1 \frac{d}{dr}\left(\frac{u_2}{u_1}\right)^2 < 0,
\]

and

\[
\left(\frac{v_2}{v_1}\right)^2 E_1 - E_2 = \left(\frac{v_2}{v_1}\right)^2 \left(\frac{1}{2} K_{\delta}(v_1')^2 - \frac{1}{2} G_{\delta}(r)v_1^2 + \frac{1}{2} v_1^2 \left(\log v_1^2 - 1\right)\right)
\]

\[
- \left(\frac{1}{2} K_{\delta}(v_2')^2 - \frac{1}{2} G_{\delta}(r)v_2^2 + \frac{1}{2} v_2^2 \left(\log v_2^2 - 1\right)\right)
\]

\[
= \frac{1}{2} \left( K_{\delta} \left(\frac{v_2}{v_1}\right)^2 (v_1')^2 - K_{\delta}(v_2')^2 + v_2^2 \log v_1^2 - v_2^2 \log v_2^2 \right).
\]
Let \((0, r_1)\) be an nonempty interval where \(K_\delta = (1 + \delta)^{-1}\). By (3.6), we can check,
\[
\left(\frac{v_2}{v_1}\right)^2 (v'_1)^2 - (v'_2)^2 = K_\delta^{-\frac{1}{2}} r^{N-2} \left(\frac{u_2 u'_1}{u_1} - u'_2\right) \left((N - 1)u_2 + \frac{ru_2 u'_1}{u_1} + ru'_2\right) \to 0, \quad \text{as } r \to 0.
\]
As a result,
\[
\lim_{r \to 0} \left(\frac{v_2}{v_1}\right)^2 \left(E_1 - E_2\right) = \frac{1}{2} v_2(0)^2 \log \left(\frac{u_1(0)}{u_2(0)}\right)^2 = 0. \quad (3.11)
\]
By Lemma 3.1 and (3.9), we obtain that
\[
\lim_{r \to +\infty} \left(\frac{v_2(r)}{v_1(r)}\right)^2 \left(E_1(r) - E_2(r)\right) = 0. \quad (3.12)
\]
Then we arrive at a contradiction by (3.10), (3.11) and (3.12). \qed

3.2. Radial symmetry

Assume (V3) and let \(u \in C(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})\) solves (1.9). By means of the moving plane method (see [20]), we show that \(u\) is radially symmetric about 0 and \(u'(r) < 0\) for each \(r > 0\).

Proof of Theorem 1.5. Denote \(x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}\), and for \(\lambda \in \mathbb{R}\), set \(\Sigma_\lambda = \{x \in \mathbb{R}^N|x_1 < \lambda\}\), \(x_\lambda = (2\lambda - x_1, x')\), \(u_\lambda(x) = u(x_\lambda)\) and \(U_\lambda = u_\lambda - u\).

Then in \(\Sigma_\lambda\), we have
\[
-\Delta U_\lambda + V(|x|)U_\lambda = (V(|x|) - V(|x_\lambda|))u_\lambda + u_\lambda \log u_\lambda^2 - u \log u^2.
\] \quad (3.13)

Step 1. Take \(R > 1\) such that \(u(x) < \min\{e^{-1}, u(0)\}\) if \(|x| \geq R\). We show that \(U_\lambda \geq 0\) in \(\Sigma_\lambda \setminus B_R(0)\) for each \(\lambda \leq 0\).

Otherwise, since \(U_\lambda(x) \to 0\) as \(|x| \to +\infty\) and \(U_\lambda|_{\partial \Sigma_\lambda} = 0\), we assume \(U_\lambda\) reaches its negative minimum at some \(\hat{x} \in \Sigma_\lambda \setminus B_R(0)\). We note that \(\hat{x}, \hat{x}_\lambda \neq 0\) by the choice of \(R\). So \(U_\lambda\) is \(C^2\) near \(\hat{x}\). Noting that \(s \log s^2\) is strictly decreasing in \((0, e^{-1})\), we have \(u_\lambda \log u_\lambda^2 - u \log u^2 > 0\) at \(\hat{x}\). Since we can assume without loss of generality that \(V(r) \geq 1\) for \(r \geq 1\), we have
\[
\Delta U_\lambda(\hat{x}) < V(|\hat{x}|)U_\lambda(\hat{x}) + (V(|\hat{x}_\lambda|) - V(|\hat{x}|))u_\lambda(\hat{x}) \leq 0.
\]
This is a contradiction since \(\hat{x}\) is the minimum point of \(U_\lambda\).

Step 2. Set \(\lambda_0 = \sup \{\lambda < 0|U_{\lambda'} \geq 0\} \in \Sigma_{\lambda'}\) for any \(\lambda' \in (-\infty, \lambda]\). Step 1 implies that \(U_\lambda \geq 0\) in \(\Sigma_{\lambda}\) for each \(\lambda \leq -R\) and hence \(\lambda_0 \geq -R\). We claim that \(U_{\lambda} > 0\) in \(\Sigma_{\lambda}\) for \(\lambda \leq \lambda_0\). In fact, by (3.13), there holds
\[
-\Delta U_\lambda + U_{\lambda}(V(|x|) - \log U_{\lambda}^2)
= (V(|x|) - V(|x_\lambda|))u_\lambda + \int_{u}^u (\log s^2 - \log(s - u)^2)ds \geq 0,
\] \quad (3.14)
where \( V(|x|) - \log U_{\lambda}^2 \) is bounded from below in \( \Sigma_\lambda \). By maximum principle ([16,21]), either \( U_\lambda \equiv 0 \) or \( U_\lambda > 0 \) in \( \Sigma_\lambda \). Since \( V'(r) \neq 0 \), we have \( U_\lambda > 0 \).

**Step 3.** We prove \( \lambda_0 = 0 \). Assume by contradiction that \( \lambda_0 < 0 \), we prove that there exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0] \)

\[
U_{\lambda_0 + \delta} \geq 0 \text{ in } \Sigma_{\lambda_0 + \delta}.
\]

Arguing by contradiction, for \( \varepsilon_n \to 0^+ \), assume that \( x^n \in \Sigma_{\lambda_0 + \varepsilon_n} \) attains the negative minimum of \( U_{\lambda_0 + \varepsilon_n} \). We note that by Step 1, \( |x^n| \leq R \) for all \( i \). We assume along a subsequence, \( x^n \to x_0 \). Then

\[
U_{\lambda_0}(x_0) \leq 0, \quad \nabla U_{\lambda_0}(x_0) = 0,
\]

which implies \( x_0 \in \partial \Sigma_{\lambda_0} \). By (3.14) and Hopf Lemma ([16,21]), we get a contradiction

\[
\frac{\partial U_{\lambda_0}(x_0)}{\partial x_1} < 0.
\]

Now we have shown that \( u_\lambda \leq u \) and \( \frac{\partial u}{\partial x_1} > 0 \) in \( \Sigma_0 \) by Step 3. Then we can complete the proof since similar arguments hold for any direction in \( \mathbb{R}^N \).

\( \square \)

4. Nondegeneracy

Assume (V1), (V2). Let \( w > 0 \) be the unique radial positive solution to (1.1). If \( \psi \in H^1(\mathbb{R}^N) \) weakly solves

\[
- \Delta \psi + V(|x|)\psi = (\log w^2 + 2)\psi,
\]

then we have

\[
\int_{\mathbb{R}^N} |\nabla \psi|^2 + \int_{\mathbb{R}^N} ((V - \alpha_0 \log |x| + \mu_0) + (\log |x|^{-\alpha_0} w^2)) \psi^2 = \int_{\mathbb{R}^N} ((\log |x|^{-\alpha_0} w^2)^+ + 2 + \mu_0) \psi^2,
\]

where \( \mu_0 > 0 \) is the constant determined in Lemma 2.1 for \( \delta = 0 \). Noting that \( |x|^{-\alpha_0} w \to 0 \) as \( |x| \to +\infty \) by Proposition 2.3 (ii), we have

\[
\int_{\mathbb{R}^N} (\log |x|^{-\alpha_0} w^2)^+ < +\infty.
\]

Then by Lemma 2.1, we can conclude that \( \psi \in E \). Choosing \( R > 0 \) such that \( |x|^{-\alpha_0} w^2 < w < 1 \) when \( |x| \geq R \), then

\[
\int_{\mathbb{R}^N} (\log w)^- \psi^2 = -\int_{\mathbb{R}^N \setminus B_R(0)} (\log w)\psi^2 < -\int_{\mathbb{R}^N \setminus B_R(0)} (\log |x|^{-\alpha_0} w^2)\psi^2 = \int_{\mathbb{R}^N} (\log |x|^{-\alpha_0} w^2)^- \psi^2.
\]

Therefore, \( \int_{\mathbb{R}^N} (\log w)^- \psi^2 < +\infty \). By \( \int_{\mathbb{R}^N} (\log |x|^{-\alpha_0} w^2)^+ < +\infty \) and (4.1)

\[
\int_{\mathbb{R}^N} |\log w|\psi^2 < +\infty \quad \text{and} \quad \int_{\mathbb{R}^N} |V|\psi^2 < +\infty.
\]

To achieve the nondegeneracy, we first show that \( \psi = 0 \) if \( \psi \in E_r \) and next show \( \psi \in E_r \) if we further assume (V3).
4.1. Nondegeneracy in the radial setting

Proof. Proof of \( \psi = 0 \) if \( \psi \in E_\rho \) First, note that

\[
0 = \int_{\mathbb{R}^N} (\nabla \psi \nabla w + V \psi w - \psi w \log w^2 - 2\psi w) = -2 \int_{\mathbb{R}^N} \psi w. \tag{4.3}
\]

Assume \( \psi \in E_\rho \setminus \{0\} \). We can choose a nonnegative radial function \( b \in C^\infty_0(\mathbb{R}^N) \) such that \( b = 1 \) in \( B_{r_1}(0) \) in a neighborhood of 0 such that

\[
\int_{\mathbb{R}^N} b\psi^2 > 0.
\]

Set \( a = b \log w^2 \). By Theorem 1.1, there is \( \delta > 0 \) such that \( w \) is the unique solution to

\[
-\Delta u + V_\delta(|x|)u = B_\delta(|x|)u \log w^2,
\]

where \( V_\delta = V + \delta a = V + \delta b \log w^2 \) and \( B_\delta = 1 + \delta b \). We may also assume that \( \delta \) is chosen small enough such that

\[
\int_{\mathbb{R}^N} b\psi^2 - \delta(\int_{\mathbb{R}^N} bw\psi)^2 > 0. \tag{4.5}
\]

By Proposition 2.2,

\[
2I_\delta(w) = 2 \inf_{u \in N_\rho^\delta} I_\delta(u) = \inf_{u \in N_\rho^\delta} (2I_\delta(u) - J_\delta(u)) = \inf_{u \in N_\rho^\delta} \left( \int_{\mathbb{R}^N} B_\delta u^2 = \int_{\mathbb{R}^N} B_\delta w^2. \right) \tag{4.6}
\]

Take a radial function \( \eta \in C^\infty_0(B_1(0); [0, 1]) \) such that \( \eta = 1 \) in \( B_{1/2}(0) \) and \( |\nabla \eta| \leq 4 \). For each \( n \geq 1 \), we denote \( \psi_n = \eta(n^{-1} \cdot \psi) \). Noting that \( \psi \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\}) \) by regularity, we can find \( s_n > 0 \) such that \( w + s \psi_n > 0 \) for each \( s \in (-s_n, s_n) \). Then the map \( s \mapsto \int_{\mathbb{R}^N} (w + s \psi_n)^2 \log(w + s \psi_n)^2 \) is \( C^2 \)-continuous for \( s \in (-s_n, s_n) \). For each \( s \in (-s_n, s_n) \), denote

\[
P_n(s) := J_\delta(w + s \psi_n), \quad Q_n(s) := \int_{\mathbb{R}^N} B_\delta(w + s \psi_n)^2 \quad \text{and} \quad t_n(s) := \frac{P_n(s)}{Q_n(s)}.
\]

Then by Proposition 2.2 (i), we have

\[
e^{rac{t_n(s)}{2}} (w + s \psi_n) \in N_\rho^\delta \quad \text{for} \quad s \in (-s_n, s_n).
\]

Let us check by (4.1), (4.2), (4.3) and (4.4),

\[
P'_n(0) = 2 \int_{\mathbb{R}^N} (\nabla w \nabla \psi_n + V_\delta w \psi_n - B_\delta w \psi_n \log w^2 - B_\delta w \psi_n) \]

\[
-2\delta \int_{\mathbb{R}^N} bw\psi, \quad \left\{ \begin{array}{l}
P''_n(0) = 2 \int_{\mathbb{R}^N} (|\nabla \psi_n|^2 + V_\delta \psi_n^2 - B_\delta \psi_n^2 \log w^2 - 3B_\delta \psi_n^2) \]

\[
-4\delta \int_{\mathbb{R}^N} b\psi^2 - 2 \int_{\mathbb{R}^N} B_\delta \psi^2, \quad \left\{ \begin{array}{l}
Q'_n(0) = 2 \int_{\mathbb{R}^N} B_\delta w \psi_n \to 2 \int_{\mathbb{R}^N} \delta bw\psi, \quad Q''_n(0) \to 2 \int_{\mathbb{R}^N} B_\delta \psi^2 \end{array} \right. \tag{4.7}
\]
\[ t_n(0) = 0, \quad t'_n(0)Q_n(0) = P'_n(0), \quad t''_n(0)Q_n(0) + 2t'_n(0)Q'_n(0) = P''_n(0). \]  
(4.8)

Then we have by (4.7), (4.8) and (4.5),
\[
\frac{d^2}{ds^2} \bigg|_{s=0} \left( e^{t_n(s)} Q_n(s) \right) = t''_n(0)Q_n(0) + 2t'_n(0)Q'_n(0) = P''_n(0).
\]

This is a contradiction because the function \( s \mapsto e^{t_n(s)} Q_n(s), s \in (-s_n, s_n) \) attains its minimum at \( s = 0 \) by (4.6).

\[ \Box \]

4.2. Nondegeneracy in \( H^1(\mathbb{R}^N) \)

In this subsection, we assume (V1), (V2) and (V3). It suffices to show the following result to obtain \( \psi = 0 \) in \( H^1(\mathbb{R}^N) \).

**Lemma 4.1.** Let \( w \) be the unique solution to (1.1), and \( \psi \in H^1(\mathbb{R}^N) \) satisfies
\[ -\Delta \psi + V(|x|) \psi = (\log w^2 + 2) \psi. \]

Then \( \psi(x) = \psi(|x|) \).

**Proof.** The proof is similar to [8, Lemma A.4], but more subtle due to lack of regularity. If \( \psi \) is not radially symmetric, we assume without loss of generality that
\[
\phi(x) := \psi(x_1, x_2, \cdots, x_N) - \psi(-x_1, x_2, \cdots, x_N) \neq 0,
\]
where \( x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N \). We note that
\[ -\Delta \phi + V(|x|) \phi = (\log w^2 + 2) \phi. \]  
(4.9)

So by the regularity theory, \( \phi \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\}) \). Let \( \Omega \) be a connected component of \( \{ x \in \mathbb{R}^N | \phi(x) > 0 \} \). Since \( \phi(x) = 0 \) when \( x_1 = 0 \), we may assume
\[ \Omega \subset \{ x = (x_1, x_2, \cdots, x_N) | x_1 > 0 \}. \]

For \( \varepsilon > 0 \), set \( \Omega_\varepsilon := \{ x \in \Omega | \phi(x) > \varepsilon \} \), which is a bounded subset of \( \Omega \). By Sard’s theorem, there exists \( \varepsilon_m > 0 \) with \( \lim_{m \to \infty} \varepsilon_m = 0 \) such that \( \{ \varepsilon_m \}_{m=1}^\infty \) are regular values of \( \phi \). Note that
\[
-\Delta \frac{\partial w}{\partial x_1} + V(|x|) \frac{\partial w}{\partial x_1} + V'(|x|) \frac{x_1}{|x|} w = (\log w^2 + 2) \frac{\partial w}{\partial x_1}. \]  
(4.10)

By (4.9) and (4.10), we have
\[
\phi \Delta \frac{\partial w}{\partial x_1} - \frac{\partial w}{\partial x_1} \Delta \phi - V'(|x|) \frac{x_1}{|x|} w \phi = 0.
\]
Integrating by parts on $\Omega_{\varepsilon_m}$, we obtain
\[
\int_{\partial\Omega_{\varepsilon_m}} \frac{\partial^2 w}{\partial x_1 \partial \nu} \phi - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} w \phi = \int_{\partial\Omega_{\varepsilon_m}} \frac{\partial \phi}{\partial \nu} \frac{\partial w}{\partial x_1},
\]
where $\nu$ denotes the outward unit vector normal to $\partial\Omega_{\varepsilon_m}$. By
\[
\text{supp} \ (V(|x|) - \log w^2 - 2) \subset B_R(0)
\]
for some $R > 0$ and $\frac{\partial w}{\partial x_1} < 0$ in $\{x|x_1 > 0\}$, we can check that
\[
\varepsilon_m \int_{\Omega_{\varepsilon_m}} \Delta \frac{\partial w}{\partial x_1} - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} w \phi
= \varepsilon_m \int_{\Omega_{\varepsilon_m}} (V(|x|) - \log w^2 - 2) \frac{\partial w}{\partial x_1} - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} (\phi - \varepsilon_m) w
\leq \varepsilon_m \int_{B_R(0)} (V(|x|) - \log w^2 - 2) \frac{\partial w}{\partial x_1} - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} (\phi - \varepsilon_m) w
= C \varepsilon_m - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} (\phi - \varepsilon_m) w,
\]
where $C$ is a constant independent of $m$. Therefore, by $V'(r) \geq 0$, we have
\[
\limsup_{m \to \infty} \left( \int_{\Omega_{\varepsilon_m}} \frac{\partial^2 w}{\partial x_1 \partial \nu} \phi - \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} w \phi \right) \leq 0.
\]
We next claim that $\partial \Omega = \{x|x_1 = 0\}$. Otherwise, $\partial \Omega \cap \{x|x_1 > 0\} \neq \emptyset$. On $\partial \Omega \cap \{x|x_1 > 0\}$, we have $\frac{\partial w}{\partial x_1} < 0$ and, by the Hopf lemma, $\frac{\partial \phi}{\partial \nu} < 0$. We can check that
\[
\liminf_{m \to \infty} \int_{\Omega_{\varepsilon_m}} \frac{\partial \phi}{\partial \nu} \frac{\partial w}{\partial x_1} \geq \int_{\partial \Omega \cap \{x|x_1 > 0\}} \frac{\partial \phi}{\partial \nu} \frac{\partial w}{\partial x_1} > 0,
\]
which is a contradiction. Then $\partial \Omega = \{x|x_1 = 0\}$, and thus $\Omega = \{x|x_1 > 0\}$. By (4.11) and (4.12), we have
\[
\limsup_{m \to \infty} \int_{\Omega_{\varepsilon_m}} V'(|x|) \frac{x_1}{|x|} (\phi - \varepsilon_m) w \leq - \liminf_{m \to \infty} \int_{\Omega_{\varepsilon_m}} \frac{\partial \phi}{\partial \nu} \frac{\partial w}{\partial x_1} \leq 0.
\]
This is impossible since $V' \neq 0$ by (V3).

5. Proof of Theorem 1.9
Throughout this section, we assume $N \geq 2$, $\alpha > 1-N$ and $\sigma \in (0, 2/(N - 2)^+)$. We first consider the existence and uniqueness of the positive radial solution to
\[
- \Delta u + |x|^{\alpha \sigma} u = |u|^{2\sigma} u, \quad u \in H^1(\mathbb{R}^N).
\]
To this end, we need a suitable function space. For any \( \phi \in C^\infty_0(\mathbb{R}^N) \), define
\[
\|\phi\|_\sigma = \left( \int_{\mathbb{R}^N} |\nabla \phi|^2 + |x|^{\alpha \sigma} \phi^2 \right)^{\frac{1}{2}},
\]
and let \( X^\sigma_r \) be the completion of \( \{ \phi \in C^\infty_0(\mathbb{R}^N) \mid \phi(x) = \phi(|x|) \} \) with respect to \( \| \cdot \|_\sigma \). Note that \( X^\sigma_r \subset H^1_r(\mathbb{R}^N) \) when \( \alpha \geq 0 \) and \( X^\sigma_r \supset H^1_r(\mathbb{R}^N) \) when \( \alpha < 0 \). We have

**Lemma 5.1.** Assume (1.11). There is \( C_N > 0 \) independent of \( \sigma \) such that for any \( u \in X^\sigma_r \),
\[
|u(x)| \leq C_N \| u \|_\sigma |x|^{\frac{1}{2}N - \frac{\alpha \sigma}{2}}, \quad |x| \neq 0.
\]
Moreover, for each \( p \in [2\sigma + 2, 2^*), X^\sigma_r \) embeds to \( L^p(\mathbb{R}^N) \) compactly.

**Proof.** By density, for \( u \in X^\sigma_r \cap C^\infty_0(\mathbb{R}^N) \),
\[
\|u\|_\sigma^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + |x|^{\alpha \sigma} u^2 \, dx = \omega_N \int_0^{+\infty} r^{N-1} ((u')^2 + r^{\alpha \sigma} u^2) \, dr,
\]
where \( \omega_N \) denotes the surface area of the unit sphere in \( \mathbb{R}^N \). We have
\[
\frac{d}{dr} \left(r^{N-1 + \frac{\alpha \sigma}{2}} u^2 \right) = 2ur^{N-1 + \frac{\alpha \sigma}{2}} \frac{du}{dr} + (N - 1 + \frac{\alpha \sigma}{2})r^{N-2 + \frac{\alpha \sigma}{2}} u^2
\geq 2ur^{N-1 + \frac{\alpha \sigma}{2}} \frac{du}{dr}.
\]
Integrating over \([r, +\infty)\), we obtain
\[
r^{N-1 + \frac{\alpha \sigma}{2}} u^2(r) \leq 2 \int_r^{+\infty} s^{\frac{\alpha \sigma}{2}} u \left| \frac{du}{ds} \right| s^{N-1} \, ds
\leq \int_r^{+\infty} s^{N-1} ((u')^2 + s^{\alpha \sigma} u^2) \, ds \leq C_N^2 \| u \|_\sigma^2.
\]
We have verified the first part of Lemma 5.1. To show the compactness, assume \( u_n \rightharpoonup 0 \) in \( X^\sigma_r \). Fixing \( p \in [2\sigma + 2, 2^*), \) we have for \( \kappa \in (0, 1) \) and \( R > 0 \):
\[
\int_{|x|>R} |u_n|^p = \int_{|x|>R} \left( |x|^{\frac{\alpha \sigma}{2}} |u_n| \right)^{2-2\kappa} |x|^{-\alpha \sigma(1-\kappa)} |u_n|^{p-2+2\kappa} \\
\leq \left( \int_{\mathbb{R}^N} |x|^{\alpha \sigma} u_n^2 \right)^{1-\kappa} \left( \int_{|x|>R} |x|^{-\alpha \sigma(\kappa-1)} |u_n|^{(p-2)\kappa-1+2} \right)^{\kappa} \\
\leq \| u_n \|_\sigma^{2(1-\kappa)} \left( \int_{|x|>R} |x|^{-\alpha \sigma(\kappa-1)} |u_n|^{(p-2)\kappa-1+2} \right)^{\kappa} \\
\leq C_N^{p-2+2\kappa} \| u_n \|_\sigma^p \left( \int_{|x|>R} |x|^{\frac{1-N}{2} - \frac{\alpha \sigma}{4} (p-2)\kappa-1+2+1-N+\frac{\alpha \sigma}{2}} \right)^{\kappa}.
\]
Note that by (1.11), \( \frac{1-N}{2} - \frac{\alpha \sigma}{4} < 0 \) and
\[
\left( \frac{1-N}{2} - \frac{\alpha \sigma}{4} \right) (p-2) - \alpha \sigma \leq \left( 1 - N - \frac{\alpha \sigma}{2} - \alpha \right) \sigma < 0.
\]
Thus, we can fix \( \kappa > 0 \) sufficiently small such that
\[
\left( \frac{1 - N}{2} - \frac{\alpha \sigma}{4} \right) (p - 2) \kappa^{-1} - \alpha \sigma \kappa^{-1} + 1 - N + \frac{\alpha \sigma}{2}
\leq \left( 1 - N - \frac{\alpha \sigma}{2} - \alpha \right) \kappa^{-1} + 1 - N + \frac{\alpha \sigma}{2} < -N.
\]
Then we have
\[
\lim_{R \to \infty} \sup_n \int_{|x| > R} |u_n|^p = 0.
\]
We can finally verify \( u_n \to 0 \) in \( L^p(\mathbb{R}^N) \) by the compact embedding from \( X_\sigma \) to \( L^p_{\text{loc}}(\mathbb{R}^N) \).

The existence is a corollary of Lemma 5.1 and the uniqueness is a result of a modified decay estimate and the arguments in [8].

**Proposition 5.2.** Assume (1.11). There is a positive solution \( u_\sigma \in X_\sigma \) to (5.1) satisfying
\[
u_\sigma(x) \leq C_\sigma \exp \left( -c_\sigma |x|^{\frac{\alpha + \alpha_2}{2}} \right),
\]
where \( C_\sigma, c_\sigma \) are positive constants independent of \( x \). Assuming further that \( -\sigma \alpha < 1 \), then \( u_\sigma \) is the unique positive radial solution to (5.1).

**Proof.** (i) By the compact embedding \( X_\sigma \subset L^{2\sigma + 2}(\mathbb{R}^N) \) and the mountain pass theorem, the following equation has a nonnegative nontrivial radial solution \( u \in X_\sigma \),
\[
-\Delta u + |x|^{\alpha \sigma} u = |u|^{2\sigma} u, \quad \text{lim sup} |x|^{\frac{N+1}{2} + \frac{\alpha \sigma}{4}} u(|x|) < +\infty.
\]
By (1.11), \( u \in C(\mathbb{R}^N) \) since there holds \( |x|^{\alpha \sigma} \in L^q(B_1(0)) \) for some \( q > N/2 \). By the maximum principle, we can conclude that \( u \) is positive in \( \mathbb{R}^N \setminus \{0\} \). To show that \( u \) is a solution of (5.1), we claim that (5.2), which ensures the square integrability, holds for any solution to (5.3). By (1.11), we know \( \frac{1-N}{2} - \frac{\alpha \sigma}{4} < \frac{\alpha}{2} \). Then by Lemma 5.1, there is \( R > 0 \) such that if \( |x| \geq R \), then
\[
-\Delta u + \frac{1}{2} |x|^{\alpha \sigma} u \leq -\Delta u + (|x|^{\alpha \sigma} - |u|^{2\sigma}) u = 0.
\]
On the other hand, fix \( c_\sigma \in (0, \frac{\sqrt{2}}{2 + \alpha \sigma}) \) and denote \( v(x) = \exp \left( -c_\sigma |x|^{\frac{\alpha \sigma + 2}{2}} \right) \). Then there is \( R' \geq R \) such that if \( |x| \geq R' \), then
\[
-\Delta v + \frac{1}{2} |x|^{\alpha \sigma} v \geq 0.
\]
Let \( C_\sigma > 0 \) be such that \( u(x) \leq C_\sigma v(x) \) for \( |x| = R' \). We have
\[
0 \leq \int_{|x| > R'} |\nabla (u - C_\sigma v) + \frac{1}{2} |x|^{\alpha \sigma}((u - C_\sigma v)^+)\|^2
\leq \int_{|x| > R'} \left( -\Delta (u - C_\sigma v) + \frac{1}{2} |x|^{\alpha \sigma} (u - C_\sigma v) \right) (u - C_\sigma v)^+ \leq 0.
\]
This implies \( u \leq C_\sigma v \).
(ii) If we assume further that $-\sigma \alpha < 1$, then by the regularity theory, $u_\sigma \in C^{2-\gamma'}(R^N) \cap C^2(R^N \setminus \{0\})$ for each $\gamma' \in (0, \alpha \sigma, 1)$. Then $u'_0(0) = 0$ and similar to the proof of Proposition 2.3 (i), we can conclude that $u_\sigma > 0$. To see the uniqueness, we check the arguments of [8, Theorem 1.1] to
\begin{equation}
\begin{aligned}
u'' + \frac{N-1}{r} u' - r^{\alpha \sigma} u + |u|^{2\sigma} u = 0, & \quad u > 0 \quad \text{in} \quad (0, +\infty), \\
u'(0) = 0, & \quad \limsup_{r \to 0^+} r^{N-1+\frac{\alpha \sigma}{\sigma}} u(r) < +\infty.
\end{aligned}
\end{equation}

We first check that under condition (1.11), $V(r) = r^{\alpha \sigma}$ satisfies the second assumption of [8, (V2)], i.e., the function
\[ H(r) = \left(\frac{2(N-1)}{\sigma + 2} + \alpha\right) r^{\alpha \sigma + 2} - D(N, \sigma), \]
where $D(N, \sigma)$ is a negative constant if $N = 2$ and a positive constant if $N \geq 3$, satisfies that $\inf_{r > 0} H(r) > 0$ for $N = 2$, and $H$ has the unique simple zero in $(0, \infty)$ and $\limsup_{r \to 0^+} H(r) < 0$ for $N \geq 3$. In fact, the condition (1.11) implies that $H(r)$ is strictly increasing, $\lim_{r \to \infty} H(r) = +\infty$, and $H(0) > 0$ if $N = 2$ and $H(0) < 0$ if $N \geq 3$.

Now the only point to apply [8, Theorem 1.1] for (5.4) is that $V(r) = r^{\alpha \sigma}$ does not meet the assumption [8, (V1)]. However checking the proof [8, Theorem 1.1], we can see that the fact $u(r) \leq C \exp \{-cr^{\frac{\alpha \sigma + 2}{2}}\}$ is sufficient to continue the arguments. \hfill \Box

Now we consider the asymptotic behavior of the unique positive radial solution $u_\sigma$ as $\sigma \to 0^+$. Set
\[ v_\sigma = \sigma^{\frac{\alpha \sigma}{2+\alpha \sigma}} u_\sigma(\sigma^{-\frac{1}{2+\alpha \sigma}}). \]
Then $v_\sigma$ satisfies
\[ -\Delta v + \sigma^{-1} |x|^{\alpha \sigma} v = \sigma^{-1} |v|^{2\sigma} v, \]

or equivalently,
\[ -\Delta v + \sigma^{-1} (|x|^{\alpha \sigma} - 1) v = \sigma^{-1} (|v|^{2\sigma} - 1) v. \]

Consider the functional corresponding to (5.5)
\[ I_{1,\sigma}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + \sigma^{-1} |x|^{\alpha \sigma} v^2) - \frac{1}{\sigma(2\sigma + 2)} \int_{\mathbb{R}^N} |v|^{2\sigma + 2}, \quad v \in X^\sigma. \]

By the standard argument, we can see that
\[ I_{1,\sigma}(v_\sigma) = \inf_{v \in X^\sigma} \sup_{t \geq 0} I_{1,\sigma}(tv). \]

We define
\[ I_1(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + \alpha v^2 \log |x|) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \, dx \]
\[ - \frac{1}{2} \int_{\mathbb{R}^N} v^2 \log v^2 \, dx, \]

where $v$ belongs to
\[ Y_r := \left\{ v \in H^1_r(\mathbb{R}^N) \mid \int_{|x| \geq 1} v^2 \log |x| < +\infty \right\}, \]
equipped with the norm
\[ \|v\|_Y := \left( \int_{\mathbb{R}^N} |\nabla v|^2 + (1 + (\log |x|)^+)^2 \right)^{1/2}. \]

Note that
\[ \lim_{\sigma \to 0^+} I_{1,\sigma}(v) = I_1(v) \quad \text{for each} \quad v \in C^1_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N). \quad (5.7) \]

**Proposition 5.3.** Let \( \alpha > 1 - N. \) Then
\[ 0 < \liminf_{\sigma \to 0^+} I_{1,\sigma}(v_\sigma) \leq \limsup_{\sigma \to 0^+} I_{1,\sigma}(v_\sigma) < +\infty, \quad (5.8) \]
\[ 0 < \limsup_{\sigma \to 0^+} \|v_\sigma\|_Y \leq \limsup_{\sigma \to 0^+} \|v_\sigma\|_Y < +\infty. \quad (5.9) \]

**Proof.** Fix a nonnegative nontrivial function \( v \in C^1_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N). \) We can find \( T_0 > 0 \) such that \( I_1(T_0v) < 0. \) By (5.7), for \( \sigma > 0 \) sufficiently small \( I_{1,\sigma}(T_0v) < 0. \) Note that the function \( t \mapsto I_1(tv) \) admits a unique extreme point in \((0, +\infty).\) Then we have
\[ \lim_{\sigma \to 0^+} I_{1,\sigma}(v_\sigma) \leq \lim_{\sigma \to 0^+} \max_{t \geq 0} I_{1,\sigma}(tv) = \lim_{\sigma \to 0^+} \max_{t \in [0, T_0]} I_{1,\sigma}(tv) = \max_{t \geq 0} I_1(tv) < +\infty. \]

Let us rewrite
\[
I_{1,\sigma}(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + \sigma^{-1}(|x|^\alpha - 1)^2v^2 - \sigma^{-1}(|v|^{2\sigma} - 1)v^2 \right) + \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |v|^{2\sigma + 2} \geq \frac{1}{2} I'_{1,\sigma}(v) = \int_{\mathbb{R}^N} \left( |\nabla v|^2 + \sigma^{-1}(|x|^\alpha - 1)^2v^2 - \frac{1}{2} \int_{\mathbb{R}^N} \sigma^{-1}(|v|^{2\sigma} - 1)v^2 \right) \geq \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + \alpha_0^2 \log |x| \right) - \frac{1}{2} \int_{\mathbb{R}^N} \sigma^{-1}(|v|^{2\sigma} - 1)v^2 := I_{2,\sigma}(v),
\]
where the last inequality holds because
\[ \sigma^{-1}(s^{t\sigma} - 1) \geq t \log s \quad \text{for each} \quad (t, s, \sigma) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty). \quad (5.10) \]

The functional \( I_{2,\sigma}(v) \) is well defined on \( Y_r. \) By the same arguments to Lemma 2.2, there is \( \mu_1 > 0 \) such that
\[ \int_{\mathbb{R}^N} \left( |\nabla v|^2 + (\alpha - \alpha_0)v^2 \log |x| + \mu_1 v^2 \right) dx \geq \frac{1}{2} \|v\|_Y^2, \quad v \in Y_r. \quad (5.12) \]

Here, we recall that \( \alpha_0 \) given by (2.2) satisfies \( \alpha_0 \leq 0 \) and \( 1 - N < \alpha_0 < \alpha. \) On the other hand, similar to (2.10), for fixed \( p \in (2, 2^*) \) satisfying (2.9) and \( p_1 = p - \frac{(p-2)\alpha_0}{1-N} \in (2, p], \) when \( \sigma \) is small enough, we have
\[ \sigma^{-1}(|v|^{2\gamma} - 1)v^2 - \alpha_0 v^2 \log |x| + \mu_1 v^2 \]
\[ = \int_0^1 (|v|^{2\gamma+2} - v^2) \log v^2 \mathrm{d}s + v^2 \log(v^2|x|^{-\alpha_0 \mu_1}) \leq \max\{0, (|v|^{2\gamma+2} - v^2) \log v^2\} + v^2 \log(v^2|x|^{-\alpha_0 \mu_1}) \leq C_3 \left( |v|^p + \|v\|_{H^1(\mathbb{R}^N)}^{p_1} \right)^{p-1} \quad \text{in} \quad \mathbb{R}^N, \quad (5.13) \]
where $C_3'$ is a constant independent of $\sigma$. Then by (5.12) and (5.13), for some $C > 0$ independent of $\sigma$, there holds

$$I_{2,\sigma}(v) \geq \frac{1}{4} \|v\|_{Y}^2 - C\|v\|^p_{Y}.$$ 

Hence there is $m > 0$ independent of $\sigma$ such that

$$I_{2,\sigma}(v) \geq \frac{1}{8} m^2 \text{ for } \|v\|_{Y} = m, \text{ and } I'_{1,\sigma}(v) v \geq \frac{1}{4} \|v\|_{Y}^2 \text{ for } \|v\|_{Y} \leq m. \quad (5.14)$$

By Proposition 5.2, $v_\sigma \in Y_r$. Then for small $\sigma > 0$

$$I_{1,\sigma}(v_\sigma) = \max_{t \geq 0} I_{1,\sigma}(tv_\sigma) \geq \max_{t \geq 0} I_{2,\sigma}(tv_\sigma) \geq I_{2,\sigma}(m\|v_\sigma\|_{Y}^{-1}v_\sigma) \geq \frac{1}{8} m^2.$$ 

This completes the proof of (5.8). Note that by (5.14), we have $\|v_\sigma\|_{Y} > m$. Otherwise, $0 = I'_{1,\sigma}(v_\sigma)v_\sigma \geq \frac{1}{4} \|v_\sigma\|_{Y}^2$. This is a contradiction. Then,

$$\liminf_{\sigma \to 0^+} \|v_\sigma\|_{Y} \geq m > 0.$$ 

To show the boundedness of $\|v_\sigma\|_{Y}$, we note that

$$\limsup_{\sigma \to 0^+} \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} v_\sigma^{2\sigma+2} = \limsup_{\sigma \to 0^+} \left( I_{1,\sigma}(v_\sigma) - \frac{1}{2} I'_{1,\sigma}(v_\sigma)v_\sigma \right) < +\infty.$$ 

Note that by the Gagliardo–Nirenberg interpolation inequality,

$$\|v_\sigma\|_{L^{p}(\mathbb{R}^N)} \leq C\|v_\sigma\|^{\frac{2p}{p-2}\sigma+2}_{L^2(\mathbb{R}^N)} \|\nabla v_\sigma\|_{L^2(\mathbb{R}^N)}^{\nu_{p,\sigma}} \leq C\|\nabla v_\sigma\|_{L^2(\mathbb{R}^N)}^{\nu_{p,\sigma}},$$

where

$$\nu_{p,\sigma} = \frac{N(p-2\sigma-2)}{p(2\sigma+2-N\sigma)} \to \nu_{p} \text{ as } \sigma \to 0^+,$$ 

$\nu_p$ is as in (2.4) and $C > 0$ is a constant independent of $\sigma$ small. Then by (5.10), (5.12) and (5.13), we have

$$I_{1,\sigma}(v_\sigma) \geq \frac{1}{4} \|v_\sigma\|_{Y}^2 - C(\|v_\sigma\|_{Y}^{\nu_{p,\sigma}} + \|v_\sigma\|_{Y}^{p-p_1+1+p_1\nu_{p_1,\sigma}}),$$

where $\nu_{p_1,\sigma} \to \nu_{p_1}$ as $\sigma \to 0^+$. Since $p-p_1+\nu_{p_1} < p-2+\nu_p < 2$ by (2.9), we have (5.9). 

Now we are ready to complete the proof of Theorem 1.9.

**Proof of Theorem 1.9.** (i) and (ii) follow from Proposition 5.2. We show (iii). By (5.9), up to a subsequence, we have $v_\sigma \to v$ in $Y_r$. Clearly $v$ is a solution to (1.4). We show $v_\sigma \to v$ in $Y_r$. To achieve this, we claim that there is $C > 0$ independent of $\sigma$ small such that

$$v_\sigma(x) \leq Ce^{-|x|}. \quad (5.15)$$

In fact, by (5.6) and (5.11), we have

$$-\Delta v_\sigma + (\alpha - \alpha_0)v_\sigma \log |x| \leq v_\sigma \left( \sigma^{-1}(v_\sigma^{2\sigma} - 1) - \alpha_0 \log |x| \right).$$

Similar to (5.13), we have by the boundedness of $\|v_\sigma\|_{H^1(\mathbb{R}^N)}$

$$-\Delta v_\sigma + (\alpha - \alpha_0)v_\sigma \log |x| \leq C_4(v_\sigma^{2\sigma} + v_\sigma^{p_1}).$$
Since \( v_\sigma(x) \to 0 \) uniformly by (2.7), we can get (5.15) by the comparison arguments. Now from (5.15) and the Sobolev inequality, we have in fact
\[
\int_{\mathbb{R}^N} \sigma^{-1}(v_\sigma^{2\sigma} - 1)v_\sigma^2 \to \int_{\mathbb{R}^N} v^2 \log v^2 \quad \text{and} \quad \int_{\mathbb{R}^N} \alpha v_\sigma^2 \log |x| \to \int_{\mathbb{R}^N} \alpha v^2 \log |x|.
\]
Then we have
\[
\limsup_{\sigma \to 0^+} \int_{\mathbb{R}^N} |\nabla v_\sigma|^2 = \limsup_{\sigma \to 0^+} \left( \int_{\mathbb{R}^N} \sigma^{-1}(v_\sigma^{2\sigma} - 1)v_\sigma^2 - \int_{\mathbb{R}^N} \sigma^{-1}(v_\sigma^{2\sigma}) |x|^{\alpha \sigma} - 1 \right)
\leq \limsup_{\sigma \to 0^+} \left( \int_{\mathbb{R}^N} \sigma^{-1}(v_\sigma^{2\sigma} - 1)v_\sigma^2 - \int_{\mathbb{R}^N} \alpha v_\sigma^2 \log |x| \right)
= \int_{\mathbb{R}^N} v^2 \log v^2 - \int_{\mathbb{R}^N} \alpha v^2 \log |x| = \int_{\mathbb{R}^N} |\nabla v|^2.
\]
Then \( v_\sigma \to v \) in \( Y_r \) up to a subsequence. By (5.9), \( v \) is nontrivial and hence positive. By Theorem 1.2, the convergence is independent of the choice of subsequences. Hence, \( v_\sigma \to v \) in \( Y_r \) as \( \sigma \to 0^+ \). By the regularity theory, for any \( \gamma'' \in (0, 1) \), \( v_\sigma \) is bounded in \( C^{2-\gamma''/2}(\mathbb{R}^N) \cap C^{2+\gamma''}(\mathbb{R}^N \setminus B_{\gamma''}(0)) \). Hence by Arzéla-Ascoli theorem, \( v_\sigma \to v \) in \( C^{2-\gamma''}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus B_{\gamma''}(0)) \). By uniqueness of \( v \) and \( v_\sigma \), the convergences are independent of subsequences. Finally, we remark that
\[
\sigma^{\frac{\alpha}{2}} u_\sigma(\sigma^{-\frac{3}{2}}) - v_\sigma = \sigma^{\frac{2\alpha}{1+2\sigma}} v_\sigma(\sigma^{-\frac{3}{1+2\sigma}}) - v_\sigma \to 0
\]
in \( Y_r \) and \( C^{2-\gamma''}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus B_{\gamma''}(0)) \)
to complete the proof. \( \square \)

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Appendix A

In this section, assuming (Vab), (V1) and \( \theta \in \Theta \), we give several technical results which are useful to show the uniqueness on the following equation:

\[
\begin{cases}
  u'' + \frac{N-1}{r}u' - V_\delta(r)u + B_\delta(r)u \log u^2 = 0, & u > 0 \text{ in } (0, +\infty) \\
u'(0) = 0, r^{-\frac{\theta}{2}}u(r) \to 0 & \text{as } r \to +\infty.
\end{cases}
\]  

(A.1)

Lemma A.1. Let \( u(r) \) solve (1.8). Then there is \( R > 0 \) such that \( u'(r) < 0 \) for \( r \geq R \). Moreover,

\[
\lim_{r \to \infty} r^{N-1}u'(r) = \liminf_{r \to \infty} r^{N-1}V_\delta(r)u(r) = 0.
\]

Proof. By Proposition 2.3, we can assume that \( R \) is sufficiently large such that for \( r \geq R \),

\[
(r^{N-1}u')' = r^{N-1}W(r) > 0,
\]

(A.2)

where \( W(r) := V_\delta(r)u(r) - B_\delta(r)u(r) \log u^2(r) \). Then \( r^{N-1}u' \) is strictly increasing in \([R, +\infty)\). If \( u'(r_1) \geq 0 \) for some \( r_1 > R \), then for \( r \geq r_1 \), we have \( u'(r) > r^{1-N}r_1^{N-1}u'(r_1) \geq 0 \). Hence for \( r \geq r_1 \), we obtain a contradiction that \( 0 < u(r_1) < u(r) \to 0 \) as \( r \to +\infty \). Thus, we see that \( u'(r) < 0 \) for \( r > R \).

To show \( \lim_{r \to \infty} r^{N-1}u'(r) = 0 \), by contradiction and the monotonicity of \( r^{N-1}u' \), we assume that there are \( \delta > 0 \) and \( r_2 > R \) such that \( r^{N-1}u'(r) < -\delta \) for \( r \geq r_2 \). Then as \( r \to \infty \), we get a contradiction that

\[
1 \geq e^{r}u(r) = -e^{r}\int^{+\infty}_{r} u'(s)ds > \delta e^{r}\int^{2r}_{r} s^{1-N}ds \to +\infty.
\]

Since \( r^{N-1}u'(r) \to 0 \) as \( r \to \infty \), by (A.2), we can find \( r_n \in (R + n - 1, R + n) \), such that

\[
r_n^{N-1}W(r_n) = \int^{R+n}_{R+n-1} r^{N-1}W(r)dr \to 0.
\]

Together with (A.2) and Proposition 2.3, we have

\[
0 = \liminf_{r \to \infty} r^{N-1}W(r) = \liminf_{r \to \infty} r^{N-1}V(r)u(r).
\]

Lemma A.2. Suppose that \( u_1, u_2 \) are two solutions of (A.1) satisfying \( u_1(0) < u_2(0) \), then

\[
\rho := \inf \{ r \in (0, \infty) | u_1(r) > u_2(r) \}
\]

exists and satisfies \( u_1(\rho) = u_2(\rho) \), and

\[
u_1 < u_2, \quad \frac{d}{dr} \left( \frac{u_1}{u_2} \right) > 0 \quad \text{in } (0, \rho).
\]

Moreover if

\[
u_1 > u_2 > 0 \quad \text{in } (\rho, \infty),
\]
then
\[ \frac{d}{dr} \left( \frac{u_1}{u_2} \right) > 0 \text{ in } (0, \infty). \]

**Proof.** The proof is similar to [27, Lemma 1.2]. To show the existence of \( \rho \in (0, \infty) \), we only need to prove
\[ \{ r \in (0, \infty) | u_1(r) > u_2(r) \} \neq \emptyset. \] (A.3)

We note that
\[
\frac{d}{dr} \left( (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) \right) = \frac{d}{dr} \left( r^{N-1}(u'_1 u_2 - u'_2 u_1) \right) = (r^{N-1}u'_1)u_2 - (r^{N-1}u'_2)u_1 \] (A.4)

We know from \( u'_1(0) = u'_2(0) = 0 \), Proposition 2.3 (ii) and Lemma A.1 that
\[ \lim_{r \to 0} (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) = 0, \quad \lim_{r \to \infty} (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) = 0. \]

If (A.3) is not true, then by (A.4)
\[ \frac{d}{dr} \left( (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) \right) \geq 0. \]

So,
\[ (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) \equiv 0 \text{ in } (0, +\infty). \]

This contradicts
\[ \frac{d}{dr} \left[ (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) \right] = B_\delta r^{N-1}u_1u_2[(\log u_2^2 - \log u_1^2)] \neq 0. \]

If \( u_1 > u_2 \) in \((\rho, +\infty)\), then by (A.4), the function \( (r^{N-1}u_2^2) \frac{d}{dr} \left( \frac{u_1}{u_2} \right) \) is strictly increasing in \((0, \rho)\) and strictly decreasing in \((\rho, +\infty)\). Therefore, \( \frac{d}{dr} \left( \frac{u_1}{u_2} \right) > 0 \) in \((0, +\infty)\). \( \square \)

From the proof of Lemma A.2 we can also get the following:

**Lemma A.3.** Suppose \( \rho > 0 \) and \( u_1, u_2 \) are two solutions of
\[
u'' + \frac{N - 1}{r} u' - V_\delta u + B_\delta u \log u^2 = 0, \quad \text{in } (0, \rho), \]
\[ u'(0) = 0, \quad u > 0 \quad \text{in } (0, \rho) \]
such that
\[ u_1 < u_2 \quad \text{in } [0, \rho). \]

Then
\[ \frac{d}{dr} \left( \frac{u_1}{u_2} \right) > 0 \quad \text{in } (0, \rho). \]
Lemma A.4. For $b_0 > 0$, let $w$ be the unique solution to the initial value problem
\[
    w'' + \frac{N-1}{r}w' + b_0 w = 0, \quad w(0) = 1, \quad w'(0) = 0. \tag{A.5}
\]
Then there is $r_1 > 0$ such that $w(r_1) = 0$, $w > 0$ in $(0, r_1)$ and $w' < 0$ in $(0, r_1]$.

Proof. Let $\lambda_1$ be the first eigenvalue of
\[
    \left\{ \begin{array}{ll}
    -\Delta \phi = \lambda_1 \phi & \text{in } B_1(0), \\
    \phi = 0 & \text{on } \partial B_1(0),
    \end{array} \right.
\]
with the corresponding eigenfunction $\phi(x) = \phi(|x|)$ which satisfies $\phi(r) > 0$ in $(0, 1)$, $\phi(1) = 0$, $\phi'(0) = 0$ and $\phi'(r) < 0$, $0 < r \leq 1$. Then $\psi = \phi^{-1}(0)\phi(r_1^{-1})$ with $r_1 := b_0^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}$ satisfies (A.5) and coincides with $w$ in $(0, r_1]$.

For $\beta > 0$, we consider the following initial value problem:
\[
    \left\{ \begin{array}{ll}
    u'' + \frac{N-1}{r}u' - V_\delta(r)u + B_\delta(r)u \log u^2 = 0, \\
    u(0) = \beta, \quad u'(0) = 0.
    \end{array} \right. \tag{A.6}
\]
By Remark 2.4, we can denote the unique solution by $u(r; \beta)$. The next lemma sketches the graph of $u(r; \beta)$ for large $\beta$.

Lemma A.5. Denoting
\[
    v(r; \beta) := \beta^{-1}u((\log \beta)^{-\frac{1}{2}}r; \beta),
\]
we have $v(r; \beta) \to w(r)$ in $C_{loc}([0, +\infty)) \cap C^1_{loc}((0, \infty))$, as $\beta \to \infty$, where $w$ is the solution to the problem (A.5) with $b_0 = 2 B_\delta(0)$ in Lemma A.4.

Proof. Noting that $v(r; \beta)$ satisfies $v(0) = 1$, $v'(0) = 0$ and
\[
    (r^{N-1}v')' + 2B_\delta((\log \beta)^{-\frac{1}{2}}r)r^{N-1}v
    = (\log \beta)^{-1}r^{N-1} \left( V_\delta((\log \beta)^{-\frac{1}{2}}r)v - B_\delta((\log \beta)^{-\frac{3}{2}}r)\log v^2 \right).
\]
We claim that $v(\cdot; \beta)$ is bounded in $C_{loc}([0, +\infty))$ for $\beta \geq e$, i.e., for any $T > 0$, $\max_{r \in [0,T]} |v(r; \beta)|$ is bounded for $\beta \geq e$. It suffices to show that if $\max_{r \in [0,T]} |v(r; \beta)| \leq C$ for some $r_0 \geq 0$ and $C > 0$, then there is $\delta > 0$ independent of $\beta$ such that
\[
    \max_{r \in [0,r_0+\delta]} |v(r; \beta)| \leq C + 1. \tag{A.7}
\]
Without loss of generality, we assume that
\[
    r_1 := \inf \{ r > r_0 \mid |v(r; \beta) - v(r_0; \beta)| > 1 \} < +\infty.
\]
We estimate the lower bound for $r_1$ to show (A.7). Notice that $v(r; \beta)$ satisfies
\[
    v'(r) = r^{1-N} \int_0^r s^{N-1} \left( (\log \beta)^{-1}V_\delta((\log \beta)^{-\frac{1}{2}}s)v - (\log \beta)^{-1}B_\delta((\log \beta)^{-\frac{3}{2}}s)\log v^2 - 2B_\delta((\log \beta)^{-\frac{3}{2}}s)v \right) ds. \tag{A.8}
\]
We have by (A.8) and (V1) that
\[1 = |v(r_1) - v(r_0)| = \left| \int_{r_0}^{r_1} v'(r) dr \right| \]
\[\leq C \int_{r_0}^{r_1} r^{1-N} dr \int_{0}^{r} s^{N-1}(\log \beta)^{-1} |V_\delta((\log \beta)^{-\frac{1}{2}} s)| ds\]
\[+ 3(1 + |C \log C|) \int_{r_0}^{r_1} r dr\]
\[\leq C \int_{r_0}^{r_1} (\log \beta)^{-\frac{3}{2}} r^{1-N} dr \int_{0}^{r(\log \beta)^{-\frac{1}{2}}} s^{N-1} |V_\delta(s)| ds\]
\[+ 3(1 + |C \log C|) \int_{r_0}^{r_1} r dr\]
\[\leq C \int_{r_0}^{r_1} (\log \beta)^{-\frac{3}{2}} \|V_\delta(\cdot)\|_{L^N(|x| \leq r_1(\log \beta)^{-\frac{1}{2}})} dr + C'(r_1 - r_0)^2\]
\[\leq C' \left( \|V_\delta(\cdot)\|_{L^N(|x| \leq r_1(\log \beta)^{-\frac{1}{2}})} (r_1 - r_0) + (r_1 - r_0)^2 \right),\]
where \(C'\) is a constant independent of \(\beta\). Then we have proved (A.7) and verified the claim.

By (A.8) again, we can check that for any \(T > 0\),
\[\max_{r \in [0,T]} |v'(r)| \leq C \|V_\delta(\cdot)\|_{L^N(|x| \leq T(\log \beta)^{-\frac{1}{2}})} + 3(1 + |C \log C|) T.\]
So \(v(r; \beta)\) is bounded in \(C^1_{loc}([0, +\infty))\), and thus by the equation, bounded in \(C^2_{loc}((0, +\infty))\). Hence, by the Arzéla-Ascoli theorem, as \(\beta \to \infty\), \(v(r; \beta)\) converges in \(C_{loc}([0, +\infty)) \cap C^1_{loc}((0, +\infty))\) to the unique solution of (A.5) with \(b_0 = 2B_\delta(0)\).

\[\text{Lemma A.6. Suppose that problem (A.1) has two distinct solutions } u_1, u_2 \text{ such that } u_1(0) < u_2(0). \text{ Then there exists a solution } u_3 \text{ of (A.1) such that } u_3(0) \geq u_2(0) \text{ and } \# \{ r \in (0, \infty) \mid u_1(r) = u_3(r) \} = 1. \quad (A.9)\]

\[\text{Proof. The proof is similar to [27, Appendix A] and [8, Lemma A.3]. Let } u(r; \beta), \beta > 0 \text{ be the solution of the initial value problem (A.6). For } \beta \geq u_2(0), \text{ let }\]

\[n(\beta) = \# \{ r > 0 \mid u(r; \beta) = u_1(r) \}.\]

By Lemma A.2, we may assume that \(n(u_2(0)) \geq 2\). Then by the continuous dependence of solution to the initial value, \(n(\beta) \geq 2\) for \(\beta > u_2(0)\) sufficiently close to \(u_2(0)\). Denote

\[\beta^* := \sup \left\{ \beta > u_2(0) \mid n(\tilde{\beta}) \geq 2 \text{ for all } \tilde{\beta} \in (u_2(0), \beta) \right\}.\]

For \(\beta \in (u_2(0), \beta^*)\) let \(\rho_1(\beta)\) and \(\rho_2(\beta)\) be the first and the second intersection points of \(u(\cdot; \beta)\) and \(u_1\). Then we have \(u(\cdot; \beta) > 0\) in \((0, \rho_2(\beta))\) for \(\beta \in (u_2(0), \beta^*)\). Otherwise, we set

\[\beta_0 := \inf \left\{ \beta \in (u_2(0), \beta^*) \mid u(r; \beta) \leq 0 \text{ for some } r \right\}.\]
Then the solution curve of $u(r; \beta_0)$ is tangent to the $r$-axis, which is impossible by the uniqueness of the initial value problem. By Lemma A.5, we can find $\bar{\beta} > u_2(0)$, such that

$$\begin{array}{ll}
(i) & u(r; \bar{\beta}) \text{ hits zero at some } r_0 \in (0, \infty), \\
(ii) & \# \{ r \in (0, r_0)|u(r; \bar{\beta}) = u_1(r) \} = 1.
\end{array}$$

The existence of $\bar{\beta}$ implies $\beta^* < \infty$. Next we will prove that $u_3(r) := u(r; \beta^*)$ is the solution of (A.1) satisfying (A.9).

**Case 1.** $\rho_1(\beta^*)$ does not exist, i.e., $\rho_1(\beta) \to \infty$ as $\beta \to \beta^*$. This case will not happen. In fact, for $\beta \in (u_2(0), \beta^*)$, we have, by Lemma A.3,

$$\frac{d}{dr} \left( \frac{u_1}{u(\cdot; \beta)} \right) > 0 \quad \text{in } (0, \rho_1(\beta)).$$

Then,

$$u_1 < u(\cdot; \beta) < \frac{\beta}{u_1(0)} u_1 \quad \text{in } [0, \rho_1(\beta)).$$

Taking limit as $\beta \to \beta^*$, we get

$$u_1 \leq u_3 \leq \frac{\beta^*}{u_1(0)} u_1 \quad \text{in } (0, \infty),$$

which means $u_3$ solves (A.1). By the uniqueness of initial value problem, $u_3 > u_1$. This contradicts Lemma A.2.

**Case 2.** $\rho_1(\beta^*)$ exists, but $\rho_2(\beta^*)$ does not exist, i.e., $\rho_2(\beta) \to \infty$ as $\beta \to \beta^*$.

In this case, for $\beta \in (u_2(0), \beta^*)$, we have

$$0 < u_1(r) < u(r; \beta) \quad \text{in } (0, \rho_1(\beta)),
$$

$$0 < u(r; \beta) < u_1(r) \quad \text{in } (\rho_1(\beta), \rho_2(\beta)).$$

Taking the limits as $\beta \to \beta^*$, we get our desired $u_3$. $\square$

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