Punctured Haag duality in locally covariant quantum field theories

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Abstract

We investigate a new property of nets of local algebras over 4-dimensional globally hyperbolic spacetimes, called punctured Haag duality. This property consists in the usual Haag duality for the restriction of the net to the causal complement of a point \( p \) of the spacetime. Punctured Haag duality implies Haag duality and local definiteness. Our main result is that, if we deal with a locally covariant quantum field theory in the sense of Brunetti, Fredenhagen and Verch, then also the converse holds. The free Klein-Gordon field provides an example in which this property is verified.

1 Introduction

The charged sectors of a net of local observables in a 4-dimensional globally hyperbolic spacetime have been investigated in [4]. The sectors define a \( C^* \)-category in which, except when there are geometrical obstructions, the charge structure arises from a tensor product, a symmetry and a conjugation. Geometrical obstructions occur when the spacetime has compact Cauchy surfaces: in this situation neither the classification of the statistics of sectors, nor the existence of a conjugation, have been established. Important progress in this direction has been achieved in [9]. In that paper the key assumption is that, given the net of local observables over \( \mathcal{M} \), its restriction to the causal complement of a point \( p \) of \( \mathcal{M} \) fulfils Haag duality. This allows the author to classify the statistics of sectors, and to provide several results towards the proof of the existence of a conjugation. The property assumed in [9], which we call punctured Haag duality, is the subject of the present paper. We will demonstrate that a net satisfying punctured Haag
duality is locally definite and fulfills Haag duality. Furthermore, if we deal with a locally covariant quantum field theory \[1\], then we are able to show that punctured Haag duality is equivalent to Haag duality plus local definiteness. This will allow us to provide an example, the free Klein-Gordon field, satisfying punctured Haag duality.

According to \[1\], once a locally covariant quantum field theory is given, to any 4-dimensional globally hyperbolic spacetime \( \mathcal{M} \), there corresponds a net of local algebras \( \mathcal{A}_K(M) \) satisfying the Haag-Kastler axioms \[5\]. Namely, \( \mathcal{A}_K(M) \) is an inclusion preserving map \( \mathcal{K}(M) \ni \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(M) \) assigning to each element \( \mathcal{O} \) of a collection \( \mathcal{K}(M) \) of subregions of \( M \) a C*-subalgebra \( \mathcal{A}(\mathcal{O}) \) of a C*-algebra \( \mathcal{A}(M) \), and satisfying

\[
\mathcal{O}_1 \perp \mathcal{O} \quad \Rightarrow \quad [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O})] = 0 \quad \text{locality.}
\]

In words locality means that if \( \mathcal{O}_1 \) is causally disjoint from \( \mathcal{O} \), then the algebras \( \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}) \) commute elementwise. \( \mathcal{A}(\mathcal{O}) \) is the algebra generated by all the observables measurable within the region \( \mathcal{O} \). The main new feature with respect to the Haag-Kastler framework is that if \( \psi \) is an isometric embedding from \( M \) into another 4-dimensional globally hyperbolic spacetime \( M_1 \), then there is a C*-morphism \( \alpha_\psi \) from \( \mathcal{A}(M) \) into \( \mathcal{A}(M_1) \) which maps isomorphically \( \mathcal{A}(\mathcal{O}) \) onto \( \mathcal{A}(\psi(\mathcal{O})) \) for each \( \mathcal{O} \in \mathcal{K}(M) \) (see \[1\] or Section 4.1 for details).

Now, turning to the subject of the present paper, given a state \( \omega \) of \( \mathcal{A}(M) \) and denoting by \( \pi \) the GNS representation of \( \omega \), we say that \( (\mathcal{A}_K(M), \omega) \) satisfies punctured Haag duality if for any point \( p \) of \( M \) the following identity

\[
\pi(\mathcal{A}(\mathcal{D}_1))'' = \cap \{ \pi(\mathcal{A}(\mathcal{D}))' \mid \mathcal{D} \in \mathcal{K}_s(M), \mathcal{D} \perp (\mathcal{D}_1 \cup \{p\}) \}
\]

is verified for any \( \mathcal{D}_1 \in \mathcal{K}_s(M) \) such that \( \mathcal{D}_1 \perp \{p\} \), where \( \mathcal{K}_s(M) \) is the sub-collection of \( \mathcal{K}(M) \) formed by the regular diamonds of \( M \) (see Section 3.1).

In the Haag-Kastler framework, punctured Haag duality implies Haag duality and local definiteness (Proposition 4.1). The main result of the present paper is that, in the setting of a locally covariant quantum field theory, punctured Haag duality is equivalent to Haag duality plus local definiteness (Theorem 4.4 and Corollary 4.5).

The basic idea of the proof of Theorem 4.4 is the following. The condition \( \mathcal{D} \perp \{p\} \) means that the closure \( \overline{\mathcal{D}} \) of \( \mathcal{D} \) is contained in the open set \( M \setminus J(p) \). The spacetime \( M_p \equiv M \setminus J(p) \) is globally hyperbolic, hence, there is a net of local algebras \( \mathcal{A}_K(M_p) \) associated with \( M_p \). Now, as the injection \( \iota_p \) of \( M_p \) into \( M \) is an isometric embedding, local covariance allows
us to identify the net $\mathcal{A}_K(M_p)$ with the restriction of $\mathcal{A}(M)$ to $M \setminus J(p)$, and the algebra $\mathcal{A}(M_p)$ with a subalgebra $\mathcal{A}(M)$. Then, if $\omega$ is a state of $\mathcal{A}(M)$, punctured Haag duality for $(\mathcal{A}_K(M), \omega)$ seems to be equivalent to Haag duality for $(\mathcal{A}_K(M_p), \omega|_{\mathcal{A}(M_p)})$. This is actually true, but, there is a subtle point that has to be carefully considered: the regular diamonds of $M_p$ might not be regular diamonds of $M$. However, we will be able to circumvent this problem by means of the following result: the set $K_\diamondsuit(M_p)$ and the set $\{D \in K_\diamondsuit(M_p) \mid D \subset M \setminus J(p)\}$ have a common “dense” subset (Proposition 3.3).

As an easy consequence of this result, we will show that the free Klein-Gordon field over a 4-dimensional globally hyperbolic spacetime, in the representation associated with a pure quasi-free state satisfying the microlocal spectrum condition, fulfills punctured Haag duality (Proposition 4.6). The same holds for pure adiabatic vacuum states of order $N > \frac{5}{2}$ (Remark 4.7).

2 Preliminaries on spacetime geometry

We recall some basics on the causal structure of spacetimes and establish our notation. Standard references for this topic are [8, 11, 12].

**Spacetimes** A spacetime $M$ consists in a Hausdorff, paracompact, smooth, oriented 4-dimensional manifold $M$ endowed with a smooth metric $g$ with signature $(-, +, +, +)$, and with a time-orientation, that is a smooth vector field $v$ satisfying $g(v_p, v_p) < 0$ for each $p \in M$. (Throughout this paper smooth means $C^\infty$).

A curve $\gamma$ in $M$ is a continuous, piecewise smooth, regular function $\gamma : I \rightarrow M$, where $I$ is a connected subset of $\mathbb{R}$ with nonempty interior. It is called timelike, lightlike, spacelike if respectively $g(\dot{\gamma}, \dot{\gamma}) < 0$, $= 0$, $> 0$ all along $\gamma$, where $\dot{\gamma} = \frac{d\gamma}{dt}$. Assume now that $\gamma$ is causal, i.e. a nonspacelike curve; we can classify it according to the time-orientation $v$ as future-directed (f-d) or past-directed (p-d) if respectively $g(\dot{\gamma}, v) < 0$, $> 0$ all along $\gamma$. When $\gamma$ is f-d and there exists $\lim_{t \to \sup I} \gamma(t)$ ($\lim_{t \to \inf I} \gamma(t)$), then it is said to have a future (past) endpoint. In the negative case, it is said to be future (past) endless; $\gamma$ is said to be endless if none of them exist. Analogous definitions are assumed for p-d causal curves.

The *chronological future* $I^+(S)$, the *causal future* $J^+(S)$ and the future
domain of dependence. $D^{+}(S)$ of a subset $S \subset M$ are defined as:

$I^{+}(S) \equiv \{ x \in M \mid \text{there is a f-d timelike curve from } S \text{ to } x \}$;

$J^{+}(S) \equiv S \cup \{ x \in M \mid \text{there is a f-d causal curve from } S \text{ to } x \}$;

$D^{+}(S) \equiv \{ x \in M \mid \text{any p-d endless causal curve through } x \text{ meets } S \}$.

These definitions have a dual in which “future” is replaced by “past” and the + by −. So, we define $I(S) \equiv I^{+}(S) \cup I^{-}(S)$, $J(S) \equiv J^{+}(S) \cup J^{-}(S)$ and $D(S) \equiv D^{+}(S) \cup D^{-}(S)$. We recall that $I^{+}(S)$ is an open set, and that $\overline{J^{+}(S)} = \overline{I^{+}(S)}$ and $(J^{+}(S))^{o} = I^{+}(S)$.

Two sets $S, V \subset M$ are causally disjoint, $S \perp V$, if $S \subseteq M \setminus J(V)$ or, equivalently, if $V \subseteq M \setminus J(S)$. A set $S$ is achronal if $S \cap I(S) = \emptyset$; it is acausal if $\{ p \} \perp \{ q \}$ for each pair $p, q \in S$. A (acausal) Cauchy surface $\Sigma$ of $M$ is an achronal (acausal) set verifying $D(\Sigma) = M$. Any Cauchy surface is a closed, connected, Lipschitz hypersurface of $M$. A spacelike Cauchy surface is a smooth Cauchy surface whose tangent space is everywhere spacelike. Any spacelike Cauchy surface is acausal.

Global hyperbolicity A spacetime $M$ is strongly causal if for each point $p$ the following condition holds: any neighbourhood $U$ of $p$ contains a neighbourhood $V$ of $p$ such that for each $q_1, q_2 \in V$ the set $J^{+}(q_1) \cap J^{-}(q_2)$ is either empty or contained in $V$. $M$ is globally hyperbolic if it admits a Cauchy surface or, equivalently, if it is strongly causal and for each pair of points $p_1, p_2$ the set $J^{+}(p_1) \cap J^{-}(p_2)$ is empty or compact. Assume that $M$ is globally hyperbolic. Then $M$ can be foliated by spacelike Cauchy surfaces [2]. Namely, there is 3-dimensional smooth manifold $\Sigma$ and a diffeomorphism $F : \mathbb{R} \times \Sigma \rightarrow M$ such that: for each $t \in \mathbb{R}$ the set $\Sigma_t \equiv \{ F(t, y) \mid y \in \Sigma \}$ is a spacelike Cauchy surface of $M$; for each $y \in \Sigma$, the curve $t \in \mathbb{R} \rightarrow F(t, y) \in M$ is a f-d (by convention) endless timelike curve. For any relatively compact $S \subset M$ the following properties hold:

1. $J^{+}(S) = J^{+}(\Sigma)$;
2. $D^{+}(S)$ is compact;
3. for each Cauchy surface $\Sigma$ the set $J^{+}(\Sigma) \cap \Sigma$ is either empty or compact;
4. $J^{+}(S \cup \{ p \}) = J^{+}(\overline{S \cup \{ p \}})$ for any $p \in M$ such that $\overline{S \cap \{ p \}} = \emptyset$.

The category $\mathcal{MAn}$ [1] Let $M$ and $M_1$ be globally hyperbolic spacetimes with metric $g$ and $g_1$ respectively. A smooth function $\psi$ from $M_1$ into $M$ is called an isometric embedding if $\psi : M_1 \rightarrow \psi(M_1)$ is a diffeomorphism and $\psi_* g_1 = g|_{\psi(M_1)}$. The category $\mathcal{MAn}$ is the category whose objects are the 4-dimensional globally hyperbolic spacetimes; the arrows $\text{Hom}(M_1, M)$ are the isometric embeddings $\psi : M_1 \rightarrow M$ preserving the orientation and the time-orientation of the embedded spacetime, and that satisfy the property

$\forall p, q \in \psi(M_1), \ J^{+}(p) \cap J^{-}(q)$ is either empty or contained in $\psi(M_1)$. 

4
The composition law between two arrows $\psi$ and $\phi$, denoted by $\psi \circ \phi$, is given by the usual composition between smooth functions; the identity arrow $id_M$ is the identity function of $M$.

## 3 Causal excisions and regular diamonds

We present two index sets for nets of local algebras over a globally hyperbolic spacetime $M$: the set $\mathcal{K}(M)$ and the set of the regular diamonds $\mathcal{K}_o(M)$. Furthermore, we introduce the spacetime $M_p$, the causal excision of $p \in M$, and study how $\mathcal{K}(M_p)$ and $\mathcal{K}_o(M_p)$ are embedded in $M$.

### 3.1 The sets $\mathcal{K}(M)$ and $\mathcal{K}_o(M)$

Let us consider a globally hyperbolic spacetime $M$ with metric $g$. The set $\mathcal{K}(M)$ is defined as the collection of the open sets $O$ of $M$ which are relatively compact, open, connected and enjoy the following property: for each pair $p, q \in O$ the set $J^+(p) \cap J^-(q)$ is either empty or contained in $O$. The importance of $\mathcal{K}(M)$ for the locally covariant quantum field theories derives from the following properties that can be easily checked:

\[ O \in \mathcal{K}(M) \Rightarrow O \in \mathcal{Wan} \quad (2) \]
\[ \psi \in \text{Hom}(M_1, M) \Rightarrow \psi(\mathcal{K}(M_1)) \subseteq \mathcal{K}(M), \quad (3) \]

where for $O \in \mathcal{Wan}$ we mean that $O$ with the metric $g|_O$ and with induced orientation and time orientation is globally hyperbolic. Property (2) allows one to associate a net of local algebras with $M$; property (3) makes the algebras associated with isometric regions of different spacetimes isomorphic. $\mathcal{K}(M)$ however is a too big index set for studying properties of the net like Haag duality and punctured Haag duality (see Section 4.2). For this purpose the set of the regular diamonds of $M$ is well suited. Given a smooth manifold $N$, let

\[ \mathcal{G}(N) \equiv \{ \ G \subset N \mid G \text{ and } N \setminus \overline{G} \text{ are nonempty and open, } \overline{G} \text{ is compact and contractible to a point in } G; \partial G \text{ is a two-sided, locally flat embedded } C^0\text{-submanifold of } N \text{ having finitely many connected components, and in each connected component there are points near to which } \partial G \text{ is } C^\infty\text{-embedded } \} \]

Then, a regular diamond $\mathcal{D}$ is an open subset of $M$ of the form $\mathcal{D} = (\mathcal{D}(G))^0$ where $G \in \mathcal{G}(\mathcal{C})$ for some spacelike Cauchy surface $\mathcal{C}$; $\mathcal{D}$ is said to be based on $\mathcal{C}$, while $G$ is called the base of $\mathcal{D}$. We denote by $\mathcal{K}_o(M)$ the set of the
regular diamonds of $\mathcal{M}$, and by $\mathcal{K}_o(\mathcal{M}, \mathcal{C})$ those elements of $\mathcal{K}_o(\mathcal{M})$ which are based on the spacelike Cauchy surface $\mathcal{C}$.

The set $\mathcal{K}_o(\mathcal{M})$ is a base of the topology of $\mathcal{M}$ and $\mathcal{K}_o(\mathcal{M}) \subset \mathcal{K}(\mathcal{M})$. Moreover, let us consider $\mathcal{D} \in \mathcal{K}_o(\mathcal{M}, \mathcal{C})$ and an open set $U$ such that $\overline{\mathcal{D}} \subset U$. Then there exists ([11, Lemma 3]) $\mathcal{D}_1 \in \mathcal{K}_o(\mathcal{M}, \mathcal{C})$ such that $\overline{\mathcal{D}} \subset \mathcal{D}_1 \subset (\mathcal{D}(U \cap \mathcal{C}))^o$. (4)

Now, notice that $\psi \in \text{Hom}(\mathcal{M}_1, \mathcal{M}) \Rightarrow \psi(\mathcal{K}_o(\mathcal{M}_1)) \subset \mathcal{K}(\mathcal{M})$, but $\psi(\mathcal{K}_o(\mathcal{M}_1))$ might not be contained in $\mathcal{K}_o(\mathcal{M})$. To see this, consider a spacelike Cauchy surface $\mathcal{C}_1$ of $\mathcal{M}_1$ and notice that $\psi(\mathcal{C}_1)$ is a spacelike hypersurface of $\mathcal{M}$. If $\mathcal{D}_1$ is a regular diamond of $\mathcal{M}_1$ of the form $(\mathcal{D}(G_1))^o$, then $\psi((\mathcal{D}(G_1))^o) = (\mathcal{D}(\psi(G_1)))^o$ and $\psi(G_1) \in \mathcal{G}(\psi(\mathcal{C}_1))$

However, in general a spacelike Cauchy surface $\mathcal{C}$ of $\mathcal{M}$ such that $\psi(\mathcal{C}_1) \subset \mathcal{C}$ does not exist (see for instance Remark [42]). More in general, we do not know whether $\psi(G_1) \in \mathcal{G}(\mathcal{C})$ for some spacelike Cauchy surface $\mathcal{C}$ of $\mathcal{M}$. So, we have no way to conclude that $\psi(\mathcal{D}) \in \mathcal{K}_o(\mathcal{M})$.

3.2 Causal excisions

Consider a globally hyperbolic spacetime $\mathcal{M}$, with metric $g$, and a point $p$ of $\mathcal{M}$. As the set $\mathcal{M} \setminus J(p)$ is open and connected, the manifold

$$\mathcal{M}_p \equiv \mathcal{M} \setminus J(p)$$

endowed with the metric $g_p \equiv g|_{\mathcal{M} \setminus J(p)}$, (5) and with the induced orientation and time-orientation, is a spacetime. $\mathcal{M}_p$ inherits from $\mathcal{M}$ the strong causality condition because this property is stable under restriction to open subsets. Moreover, for each pair $p_1, p_2 \in \mathcal{M} \setminus J(p)$ the set $J^+(p_1) \cap J^-(p_2)$ is either empty or compact and contained in $\mathcal{M} \setminus J(p)$ (the compactness follows from the global hyperbolicity of $\mathcal{M}$). Hence $\mathcal{M}_p$ is globally hyperbolic, thus $\mathcal{M}_p \in \text{Man}$. We call $\mathcal{M}_p$ the causal excision of $p$. Let us now denote by $\iota_p$ the injection of $\mathcal{M}_p$ into $\mathcal{M}$, that is

$$\mathcal{M} \setminus J(p) \ni q \mapsto \iota_p(q) = q \in \mathcal{M}.$$ 

Clearly, $\iota_p \in \text{Hom}(\mathcal{M}_p, \mathcal{M})$. 

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Proposition 3.1. Given a globally hyperbolic spacetime $\mathcal{M}$, let $\mathcal{M}_p$ be the causal excision of $p \in \mathcal{M}$. If $\mathcal{C}$ is an acausal (spacelike) Cauchy surface of $\mathcal{M}$ that meets $p$, then $\mathcal{C} \setminus \{p\}$ is an acausal (spacelike) Cauchy surface of $\mathcal{M}_p$. Conversely, if $\mathcal{C}_p$ is an acausal Cauchy surface of $\mathcal{M}_p$, then $\mathcal{C}_p \cup \{p\}$ is an acausal Cauchy surface of $\mathcal{M}$.

Proof. ($\Rightarrow$) $\mathcal{C} \setminus \{p\}$ is an acausal set of $\mathcal{M}_p$ and $D^+(\mathcal{C} \setminus \{p\}) \subseteq D^+(\mathcal{C}) \setminus J(p)$. Conversely, if $q \in D^+(\mathcal{C}) \setminus J(p)$ each p-d endless causal curve through $q$ meets $\mathcal{C}$ but not $p$. Hence $D^+(\mathcal{C} \setminus \{p\}) \supseteq D^+(\mathcal{C}) \setminus J(p)$. This implies $D^+(\mathcal{C} \setminus \{p\}) = D^+(\mathcal{C}) \setminus J(p)$ and the dual equality $D^-(\mathcal{C} \setminus \{p\}) = D^-(\mathcal{C}) \setminus J(p)$. Therefore $D(\mathcal{C} \setminus \{p\}) = D^+(\mathcal{C}) \setminus J(p) \cup D^-(\mathcal{C}) \setminus J(p) = (D^+(\mathcal{C}) \cup D^-(\mathcal{C})) \setminus J(p) = \mathcal{M} \setminus J(p)$. Clearly if $\mathcal{C}$ is spacelike, then $\mathcal{C} \setminus \{p\}$ is spacelike. This completes the proof. ($\Leftarrow$) Set $\mathcal{C} \equiv \mathcal{C}_p \cup \{p\}$ and notice that $\mathcal{M} \setminus J(p) = D(\mathcal{C}_p) \subseteq D(\mathcal{C})$. Now, given $q \in J^+(p)$ let us consider a p-d endless causal curve $\gamma$ through $q$ that does not meet $p$. Because of global hyperbolicity $\gamma$ leaves $J^+(p)$. Each connected component of $\gamma \cap (\mathcal{M} \setminus J(p))$ is a f-d endless causal curve of $\mathcal{M}_p$, therefore it meets $\mathcal{C}_p$. This entails $J(p) \subset D(\mathcal{C})$ and that $D(\mathcal{C}) = \mathcal{M}$. In order to prove that $\mathcal{C}$ is acausal, assume that there exists a f-d causal curve $\gamma : [0, 1] \rightarrow \mathcal{M}$ that joins two points $q_1$ and $q_2$ lying on $\mathcal{C}$. If one of the two points is $p$, then $\gamma(t) \in J(p)$ for each $t \in [0, 1]$, and this leads to a contradiction. If $q_1, q_2 \neq p$, and $\gamma \cap J(p) = \emptyset$, then $\gamma$ would be a f-d causal curve of $\mathcal{M}_p$ joining two points of $\mathcal{C}_p$, and this leads to a contradiction. The same happens in the case where $q_1, q_2 \neq p$ and $\gamma \cap J(p) \neq \emptyset$. In fact, if $\gamma(t_1) \in J^+(p)$, then $\gamma(t) \in J^+(p)$ for each $t \geq t_1$. Analogously, if $\gamma(t_1) \in J^-(p)$, then $\gamma(t) \in J^-(p)$ for each $t \leq t_1$.  

Remark 3.2. Let $\mathcal{C}_p$ be a spacelike Cauchy surface of $\mathcal{M}_p$. Because of Proposition 3.1, $\mathcal{C}_p \cup \{p\}$ is an acausal Cauchy surface of $\mathcal{M}$. However, it is not smooth in general. Consider for instance the Minkowski space $\mathbb{M}^4$. Let

$$
\mathbb{M}_2^4 \equiv \{(t, \vec{x}) \in \mathbb{M}^4 \mid -t^2 + \langle \vec{x}, \vec{x} \rangle > 0\},
$$

$$
\mathbb{C}_2 \equiv \{(t, \vec{x}) \in \mathbb{M}^4 \mid -4 \cdot t^2 + \langle \vec{x}, \vec{x} \rangle = 0, \ t > 0\},
$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product of $\mathbb{R}^3$. $\mathbb{M}_2^2$ is the causal excision of $\varnothing = (0, 0, 0, 0)$, and $\mathbb{C}_2$ is a spacelike Cauchy surface of $\mathbb{M}_2^4$. Clearly, $\mathbb{C}_2 \cup \{\varnothing\}$ is a nonsmooth hypersurface of $\mathbb{M}^4$.

We now turn to study how the injection $\iota_p$ embeds $\mathcal{K}(\mathcal{M}_p)$ and $\mathcal{K}_c(\mathcal{M}_p)$ into $\mathcal{M}$. Concerning $\mathcal{K}(\mathcal{M}_p)$ one can easily prove that

$$
\mathcal{K}(\mathcal{M}_p) = \{\emptyset \in \mathcal{K}(\mathcal{M}) \mid \emptyset \perp \{p\}\}.
$$

(6)
As for regular diamonds, in general we do not know whether \( K_0(M_p) \subset K_0(M) \) (see the observation made in the previous section). However, the set
\[
K_0(M_p \wedge M) \equiv K_0(M_p) \cap K_0(M) \tag{7}
\]
of the regular diamonds shared by \( M_p \) and \( M \) is not empty. In fact, notice that if \( \mathcal{C} \) is a spacelike Cauchy surface of \( M \) that meets \( p \), then by Proposition 3.1 \( \mathcal{C}_p \equiv \mathcal{C} \setminus \{p\} \) is a spacelike Cauchy surface of \( M_p \). Since \( \mathcal{G}(\mathcal{C}_p) \subset \mathcal{G}(\mathcal{C}) \), we conclude that
\[
K_0(M_p, \mathcal{C}_p) = \{ \mathcal{D} \in K_0(M, \mathcal{C}) \mid \mathcal{D} \perp \{p\} \} \subset K_0(M_p \wedge M). \tag{8}
\]
Furthermore, \( K_0(M_p \wedge M) \) is a “dense” subset of both \( K_0(M_p) \) and \( \{ \mathcal{D} \in K_0(M) \mid \mathcal{D} \perp \{p\} \} \), as the following proposition shows.

**Proposition 3.3.** For each pair \( \mathcal{D}, \mathcal{D}_1 \in K_0(M_p) \) such that \( \overline{\mathcal{D}} \subset \mathcal{D}_1 \), there exists \( \mathcal{D}_o \in K_0(M_p \wedge M) \) such that \( \overline{\mathcal{D}} \subset \mathcal{D}_o \), \( \overline{\mathcal{D}_o} \subset \mathcal{D}_1 \). The same result holds true for each \( \mathcal{D}, \mathcal{D}_1 \in \{ \mathcal{D} \in K_0(M) \mid \mathcal{D} \perp \{p\} \} \) such that \( \overline{\mathcal{D}} \subset \mathcal{D}_1 \).

**Proof.** The proof follows from Propositions A.3, A.4. \( \square \)

## 4 Punctured Haag duality

This section is devoted to the investigation of punctured Haag duality. We will start by recalling the axioms of a locally covariant quantum field theory. Afterwards, we will show necessary and sufficient conditions for punctured Haag duality, both in the Haag-Kastler framework and in the setting of the locally covariant quantum field theories. Finally, we will apply these results to the theory of the free Klein-Gordon field.

### 4.1 Locally covariant quantum field theories

The locally covariant quantum field theory is a categorical approach to the theory of quantum fields incorporating the covariance principle of general relativity [1]. In order to introduce the axioms of the theory, we give a preliminary definition. Let us denote by \( \mathfrak{Alg} \) the category whose objects are unital C*-algebras and whose arrows \( \text{Hom}(A_1, A_2) \) are the unit-preserving injective C*-morphisms from \( A_1 \) into \( A_2 \). The composition law between the arrows \( \alpha_1 \) and \( \alpha_2 \), denoted by \( \alpha_1 \circ \alpha_2 \), is given by the usual composition between C*-morphisms; the unit arrow \( \text{id}_A \) of \( \text{Hom}(A, A) \) is the identity morphism of \( A \).
A **locally covariant quantum field theory** is a covariant functor $\mathcal{A}$ from the category $\mathbf{Man}$ (see Section 2) into the category $\mathbf{Alg}$, that is, a diagram

$$
\begin{array}{c}
M_1 \xrightarrow{\psi} M_2 \\
\downarrow \mathcal{A} \downarrow \\
\mathcal{A}(M_1) \xrightarrow{\alpha_\psi} \mathcal{A}(M_2),
\end{array}
$$

where $\alpha_\psi \equiv \mathcal{A}(\psi)$, such that $\alpha_{id_M} = id_{\mathcal{A}(M)}$, and $\alpha_\phi \circ \alpha_\psi = \alpha_{\phi \circ \psi}$ for each $\psi \in \text{Hom}(M_1, M)$ and $\phi \in \text{Hom}(M, M_2)$. The functor $\mathcal{A}$ is said to be **causal** if, given $\psi_i \in \text{Hom}(M_i, M)$ for $i = 1, 2$,

$$
\psi_1(M_1) \perp \psi_2(M_2) \Rightarrow [\alpha_{\psi_1}(\mathcal{A}(M_1)), \alpha_{\psi_2}(\mathcal{A}(M_2))] = 0,
$$

where $\psi_1(M_1) \perp \psi_2(M_2)$ means that $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally disjoint in $M$. From now on $\mathcal{A}$ will denote a causal locally covariant quantum field theory.

We now turn to the notion of a state space of $\mathcal{A}$. To this aim, let $\mathbf{Sts}$ be the category whose objects are the state spaces $S(A)$ of unital C*-algebras $A$, namely $S(A)$ is a subset of the states of $A$ closed under finite convex combinations and operations $\omega(\cdot) \rightarrow \omega(A^* \cdot A)/\omega(A^* A)$ for $A \in \mathcal{A}$. The arrows between two objects $S(A)$ and $S'(A')$ are the positive maps $\gamma^*: S(A) \rightarrow S'(A')$, arising as dual maps of injective morphisms of C*-algebras $\gamma: A' \rightarrow A$, by $\gamma^* \omega(A) \equiv \omega(\gamma(A))$ for each $A \in \mathcal{A}$. The composition law between two arrows, as the definition of the identity arrow of an object, are obvious. A **state space** for $\mathcal{A}$ is a contravariant functor $S$ between $\mathbf{Man}$ and $\mathbf{Sts}$, that is, a diagram

$$
\begin{array}{c}
M_1 \xrightarrow{\psi} M_2 \\
\downarrow S \downarrow \\
S(M_1) \xleftarrow{\alpha^*_\psi} S(M_2),
\end{array}
$$

where $S(M_1)$ is a state space of the algebra $\mathcal{A}(M_1)$, such that $\alpha^*_{id_M} = id_{S(M)}$, and $\alpha^*_\phi \circ \alpha^*_\psi = \alpha^*_{\phi \circ \psi}$ for each $\psi \in \text{Hom}(M_1, M)$ and $\phi \in \text{Hom}(M, M_2)$.

In conclusion let us see how a net of local algebras over $M \in \mathbf{Man}$ can be recovered by a locally covariant quantum field theory $\mathcal{A}$. For this purpose, recall that by (2) any $O \in \mathcal{K}(M)$, considered as a spacetime with the metric $g|_O$, belongs to $\mathbf{Man}$. The injection $\iota_{M,O}$ of $O$ into $M$ is an element of $\text{Hom}(O, M)$ because of the definition of $\mathcal{K}(M)$. Then, by using
\[\alpha_{t_{M,O}} \in \text{Hom}(\mathcal{A}(\emptyset), \mathcal{A}(M))\] to define \(A(\emptyset) \equiv \alpha_{t_{M,O}}(\mathcal{A}(\emptyset))\), it turns out that the correspondence \(\mathcal{A}(M) \ni \emptyset \mapsto A(\emptyset) \subset \mathcal{A}(M)\), defined as

\[\mathcal{A}(M) \ni \emptyset \mapsto A(\emptyset) \subset \mathcal{A}(M), \quad (9)\]
is a net of local algebras satisfying the Haag-Kastler axioms. As for the local covariance of the theory, let \(M_1 \in \text{Man}\) with the metric \(g_1\). Notice that if \(\psi \in \text{Hom}(M, M_1)\), then \(\psi(\emptyset) \in \mathcal{K}(M_1)\) for each \(\emptyset \in \mathcal{K}(M)\). Since \(t_{M_1, \psi(\emptyset)} \circ t_{M, \emptyset}\) is an isometry from the spacetime \(\emptyset\) into the spacetime \(\psi(\emptyset)\) — the latter equipped with the metric \(g_1|_{\psi(\emptyset)}\) — one has that

\[\alpha_\psi : A(\emptyset) \subset \mathcal{A}(M) \rightarrow A(\psi(\emptyset)) \subset \mathcal{A}(M_1) \quad (10)\]
is an \(\mathbb{C}^*\)-isomorphism.

### 4.2 Punctured Haag duality in the Haag-Kastler framework

We investigate punctured Haag duality \([4]\) in the Haag-Kastler framework. This means that we will study punctured Haag duality on the net of local algebras \(\mathcal{A}(M)\), associated with a spacetime \(M \in \text{Man}\) by \([3]\), without making use of the local covariance \([10]\). In this framework, we will obtain two necessary conditions for punctured Haag duality.

To begin with, let \(\omega\) be a state of the algebra \(\mathcal{A}(M)\) and let \(\pi\) be the GNS representation associated with \(\omega\). We recall that \((\mathcal{A}(M), \omega)\) is said to be locally definite if for each \(p \in M\),

\[\mathbb{C} \cdot 1 = \cap \{\pi(A(D))^\prime \prime | p \in D \in \mathcal{K}_\emptyset(M)\};\]
it said to satisfy Haag duality if for each \(D_1 \in \mathcal{K}_\emptyset(M)\),

\[\pi(A(D_1))^\prime \prime = \cap \{\pi(A(D))'| \pi \in \mathcal{A}(M), D \perp D_1\}.\]

These properties (and also punctured Haag duality) are defined in terms of the local algebras associated with regular diamonds. The reason is that Haag duality has been proved, in models of quantum fields, only when the local algebras are defined in regions of the spacetime like the regular diamonds \([7, 10, 6]\).

Punctured Haag duality is strongly related to Haag duality and local definiteness, as the following proposition shows.

**Proposition 4.1.** Assume that \((\mathcal{A}(M), \omega)\) satisfies punctured Haag duality. Then, \((\mathcal{A}(M), \omega)\) satisfies Haag duality. Furthermore, if \(\omega\) is pure and

\[\pi(\mathcal{A}(M)) \subseteq \left( \cup \{\pi(A(\emptyset))^\prime \prime | \emptyset \in \mathcal{K}(M), M \setminus \overline{\emptyset} \neq \emptyset \} \right)^\prime \prime, \quad (11)\]

where \(\pi\) is the GNS representation of \(\omega\), then \((\mathcal{A}(M), \omega)\) is locally definite.
Proof. Obviously $(\mathcal{A}_{\mathcal{K}(\mathcal{M})}, \omega)$ fulfills Haag duality. As for the proof of local definiteness, let us define $\mathcal{R}_p \equiv \cap \{ \pi(A(D))'' | p \in D \in \mathcal{K}(\mathcal{M}) \}$, for $p \in \mathcal{M}$. Notice that, as $\mathcal{K}(\mathcal{M})$ contains a neighbourhoods basis for each point $p$ of $\mathcal{M}$, locality entails that $\mathcal{R}_p \subset \pi(A(D))' \cap \mathcal{D} \perp \{ p \}$. Combining this with punctured Haag duality we have that $\mathcal{R}_p \subset \pi(A(D))''$, therefore
\[
\mathcal{R}_p \subset \pi(A(D))'' \cap \pi(A(D))' \quad \forall \mathcal{D} \in \mathcal{K}(\mathcal{M}), \mathcal{D} \perp \{ p \}. 
\] (11)

Now, let $p_1 \in \mathcal{M}$ be such that $\{ p \} \perp \{ p_1 \}$. For each $\mathcal{D} \in \mathcal{K}(\mathcal{M})$ containing $p_1$, we can find $\mathcal{D}_1 \in \mathcal{K}(\mathcal{M})$ such that $\mathcal{D}_1 \subset \mathcal{D}$ and $\mathcal{D}_1 \perp \{ p \}$. This and (11) imply that $\mathcal{R}_p \subset \mathcal{R}_{p_1}$ and, by the symmetry of $\perp$, that $\mathcal{R}_p = \mathcal{R}_{p_1}$. For a generic $q \in \mathcal{M}$ we can find a finite sequence $p_1, \ldots, p_n$, such that $\{ p \} \perp \{ p_1 \}, \{ p_1 \} \perp \{ p_2 \}, \ldots, \{ p_n \} \perp \{ q \}$. Consequently, $\mathcal{R}_p = \mathcal{R}_q$ and $\mathcal{R}_p \subset \pi(A(\mathcal{D}))'' \cap \pi(A(\mathcal{D}))'$ for each $\mathcal{D} \in \mathcal{K}(\mathcal{M})$. We also have that $\mathcal{R}_p \subset \pi(\mathcal{A}(\emptyset))'' \cap \pi(\mathcal{A}(\emptyset))'$ for any $\emptyset \in \mathcal{K}(\mathcal{M})$ such that $\mathcal{M} \setminus \mathcal{D}$ is nonempty. The irreducibility of $\pi$ and (11) complete the proof. \hfill \qed

For later purposes, it is useful to note that if $(\mathcal{A}_{\mathcal{K}(\mathcal{M})}, \omega)$ satisfies Haag duality, then $(\mathcal{A}_{\mathcal{K}(\mathcal{M})}, \omega)$ is outer regular, that is for any $\mathcal{D}_1 \in \mathcal{K}(\mathcal{M})$ we have

\[
\pi(A(\mathcal{D}_1))'' = \cap \{ \pi(A(D))'' | \overline{D_1} \subset \mathcal{D} \in \mathcal{K}(\mathcal{M}) \}. 
\] (12)

In fact, consider $\mathcal{D}_2 \in \mathcal{K}(\mathcal{M})$ such that $\mathcal{D}_2 \perp \mathcal{D}_1$. This means that $\overline{D_1}$ is contained in the open set $\mathcal{M} \setminus J(\overline{D}_2) = \mathcal{M} \setminus J(D_2)$. By (11) there is a regular diamond $\mathcal{D}$ such that $\overline{D_1} \subset \mathcal{D}$ and $\mathcal{D} \perp \mathcal{D}_2$. This with the observation that $\pi(A(D))'' \subset \pi(A(D_2))'$, imply that
\[
\pi(A(\mathcal{D}_1))'' \subset \cap \{ \pi(A(D))'' | D \in \mathcal{K}(\mathcal{M}), \overline{D_1} \subset D \} 
\subset \cap \{ \pi(A(D_2))' | D_2 \in \mathcal{K}(\mathcal{M}), \mathcal{D}_2 \perp \mathcal{D}_1 \} = \pi(A(\mathcal{D}_1))''
\]
completing the proof of (12).

### 4.3 Punctured Haag duality in a locally covariant quantum field theory

In the Haag-Kastler framework we have seen that punctured Haag duality implies Haag duality and local definiteness. On the other hand, in the setting of the locally covariant quantum field theories, the relation between these properties is stronger. To make a precise claim, let us consider a locally covariant quantum field theory $\mathcal{A}$, and let $\mathcal{G}_{\text{HL}}(\mathcal{A})$, $\mathcal{G}_{\text{Ph}}(\mathcal{A})$ be two sets of state spaces of $\mathcal{A}$ defined as follows:
• \( \mathcal{S}_{HL}(\mathcal{A}) \) is the family of state spaces \( S \) of \( \mathcal{A} \) such that, for any \( M \in \mathcal{M} \) and for any pure state \( \omega \in S(M) \) the pair \((\mathcal{A}_{\mathcal{K}(M)}, \omega)\) is locally definite and satisfies Haag duality;

• \( \mathcal{S}_{pH}(\mathcal{A}) \) is the family of state spaces \( S \) of \( \mathcal{A} \) such that, for any \( M \in \mathcal{M} \) and for any pure state \( \omega \in S(M) \) the pair \((\mathcal{A}_{\mathcal{K}(M)}, \omega)\) satisfies punctured Haag duality and (III).

Then, we will prove that \( \mathcal{S}_{HL}(\mathcal{A}) = \mathcal{S}_{pH}(\mathcal{A}) \). Notice that \( \mathcal{S}_{pH}(\mathcal{A}) \subseteq \mathcal{S}_{HL}(\mathcal{A}) \) because of Proposition 4.1. Moreover, one can easily see that, if \( S \in \mathcal{S}_{HL}(\mathcal{A}) \) and \( \omega \in S(M) \), then \( \omega \) is a pure state of \( \mathcal{A}(M) \) and \((\mathcal{A}_{\mathcal{K}(M)}, \omega)\) fulfills (III). So, what remains to prove is that \((\mathcal{A}_{\mathcal{K}(M)}, \omega)\) satisfies punctured Haag duality.

To begin with, let us take \( S \in \mathcal{S}_{HL}(\mathcal{A}) \), \( M \in \mathcal{M} \), \( p \in M \), and a state \( \omega \in S(M) \). Let \( M_p \) be the causal excision of \( p \), and let

\[
\mathcal{A}_{\mathcal{K}(M_p)} : \mathcal{K}(M_p) \ni \emptyset \rightarrow A_p(\emptyset) \subseteq \mathcal{A}(M_p)
\]

be the net associated with \( M_p \), where the subscript \( p \) is added in order to avoid confusion between the elements of \( \mathcal{A}_{\mathcal{K}(M)} \) and those of \( \mathcal{A}_{\mathcal{K}(M_p)} \).

Observing that \( \iota_p \in \text{Hom}(M_p, M) \), we can define

\[
\omega_p(A) \equiv \alpha^*_{\iota_p} \omega(A) = \omega(\alpha_{\iota_p}(A)), \quad A \in \mathcal{A}(M_p).
\]

Since \( \omega \in S(M) \) and \( \omega_p = \alpha^*_{\iota_p} \omega \), by the definition of state space \( \omega_p \) belongs to \( S(M_p) \). The first step of our proof consists in showing that \((\mathcal{A}_{\mathcal{K}(M_p)}, \omega_p)\) satisfies Haag duality that, according to the definition of \( \mathcal{S}_{HL}(\mathcal{A}) \) it is equivalent to prove that \( \omega_p \) is pure. To this aim let us define

\[
V_{\iota_p} \pi_p(A) \Omega_p \equiv \pi(\alpha_{\iota_p}(A)) \Omega, \quad A \in \mathcal{A}(M_p),
\]

where \((\mathcal{H}, \pi, \Omega)\) and \((\mathcal{H}_p, \pi_p, \Omega_p)\) are respectively the GNS constructions associated with \( \omega \) and \( \omega_p \).

**Proposition 4.2.** \((\mathcal{A}_{\mathcal{K}(M_p)}, \omega_p)\) satisfies Haag duality. In particular:

a) the representation \( \pi \circ \alpha_{\iota_p} \) of \( \mathcal{A}(M_p) \) is irreducible; b) \( V_{\iota_p} \in (\pi_p, \pi \circ \alpha_{\iota_p}) \) is unitary, hence \( \omega_p \) is pure.

**Proof.** a) Let \( T \in (\pi \circ \alpha_{\iota_p}, \pi \circ \alpha_{\iota_p}) \). Given \( D_1 \in \mathcal{K}_o(M) \) such that \( p \in D_1 \), let us consider \( D \in \mathcal{K}_o(M), \ D \perp D_1 \). By (D) \( D \) belongs to \( \mathcal{K}(M_p) \) and by (III) \( A(D) = \alpha_{\iota_p}(A_p(D)) \). Thus, \( T \in (\pi \upharpoonright A(D), \pi \upharpoonright A(D)) \) for each regular diamond of \( M \) causally disjoint from \( D_1 \). By Haag duality
\[ T \in \pi(A(D_1))'' \text{ for each } D_1 \in \mathcal{K}_0(M) \text{ such that } p \in D_1; \text{ hence by local definiteness } T = c \cdot 1 \text{ completing the proof.} \]

b) Observe that for each \( A \in \mathcal{A}(M_p) \) we have \( \|V_{i_p} \pi_p(A) \Omega_p\|^2 = \|\pi(\alpha_{i_p}(A))\Omega\|^2 = \Omega, \pi(\alpha_{i_p}(A^* A))\Omega\) = \( \omega_p(A^* A) = \|\pi_p(A)\Omega_p\|^2. \) This entails that \( V_{i_p} \) is a unitary intertwiner between \( \pi_p \) and \( \pi \circ \alpha_{i_p} \) because \( \Omega_p \) is cyclic for \( \pi_p \) and \( \pi \circ \alpha_{i_p} \) is irreducible.

Therefore \( \pi_p \) is irreducible and, consequently, \( \omega_p \) is pure. Finally, as observed above, \((\mathcal{A}(M_p), \omega_p)\) satisfies Haag duality. \( \square \)

We do not know whether the sets \( \mathcal{K}_0(M_p) \) and \( \{ D \in \mathcal{K}_0(M) \mid D \perp \{ p \} \} \) are equal or not. If they were the same, punctured Haag duality for \((\mathcal{A}(M_0), \omega)\) would follow from Haag duality for \((\mathcal{A}(M_p), \omega_p)\). Nevertheless, by Proposition 4.3 the sets \( \mathcal{K}_0(M_p) \) and \( \{ D \in \mathcal{K}_0(M) \mid D \perp \{ p \} \} \) have a common “dense” subset: the set \( \mathcal{K}_0(M_p \wedge M) \). This is enough for our aim and will allow us to prove punctured Haag duality in two steps.

**Lemma 4.3.** For any \( D_1 \in \mathcal{K}_0(M_p \wedge M) \) the following identity holds

\[ \pi(A(D_1))'' = \cap \{ \pi(A(D))' \mid D \in \mathcal{K}_0(M), \ D \perp (D_1 \cup \{ p \}) \} \]

**Proof.** Let us consider \( D_2 \in \mathcal{K}_0(M_p), \ D_2 \perp D_1 \). The closure of \( D_2 \) is contained in the open set \( M \setminus J(D_1 \cup \{ p \}) \) (see Section 2). By Proposition 3.3 there is \( D_o \in \mathcal{K}_0(M_p \wedge M) \) satisfying the relations \( \bar{D}_2 \subset D_o, \ D_o \perp (D_1 \cup \{ p \}) \).

This leads to the following inclusions

\[ \pi(A(D_1))'' \subseteq \cap \{ \pi(A(D_o))' \mid D_o \in \mathcal{K}_0(M), \ D_o \perp (D_1 \cup \{ p \}) \} \]

\[ \subseteq \cap \{ \pi(A(D_2))' \mid D_2 \in \mathcal{K}_0(M_p), \ D_2 \perp D_1 \} \quad (\ast) \]

Recall now that by Proposition 1.2 b \( \pi(A(D_1))'' = V_{i_p} \pi_p(A_p(D_1))'' V_{i_p}^* \). As \((\mathcal{A}(M_p), \omega_p)\) satisfies Haag duality we have

\[ \pi(A(D_1))'' = V_{i_p} \left( \cap \{ \pi_p(A_p(D_2))' \mid D_2 \in \mathcal{K}_0(M_p), \ D_2 \perp D_1 \} \right) V_{i_p}^* \]

\[ = \cap \{ \pi(A(D_2))' \mid D_2 \in \mathcal{K}_0(M_p), \ D_2 \perp D_1 \} \]

Combining this with (\ast) we obtain the proof. \( \square \)

**Theorem 4.4.** Given \( S \in S_{\mathcal{H}}(\mathcal{A}) \), for any \( M \in \mathcal{M} \) and for any \( \omega \in S(M) \) the pair \((\mathcal{A}(M), \omega)\) satisfies punctured Haag duality.

**Proof.** Let \( \omega \in S(M) \) be a pure state of \( \mathcal{A}(M) \), and let \( \pi \) be the GNS representation associated with \( \omega \). Fix \( p \in M \) and \( D_1 \in \mathcal{K}_0(M) \) such that \( D_1 \perp \{ p \} \). Notice that if \( D_2 \in \mathcal{K}_0(M) \) such that \( \bar{D}_1 \subset D_2 \), then \( \bar{D}_1 \) is
contained in the open set \( D_2 \cap (M \setminus J(p)) \). By Proposition 3.3 there is \( D_o \in \mathcal{K}(M_p \land M) \) such that \( \overline{D}_1 \subseteq D_o, D_o \subseteq D_2 \) and \( D_o \perp \{p\} \). As \( D_o \) fulfils the hypotheses of Lemma 4.3, we have

\[
\pi(A(D_1))'' \subseteq \cap \{ \pi(A(D))' \mid D \in \mathcal{K}_o(M), \ D \perp (D_1 \cup \{p\}) \} \\
\subseteq \cap \{ \pi(A(D))' \mid D \in \mathcal{K}_o(M), \ D \perp (D_o \cup \{p\}) \} \\
= \pi(A(D_o))'' \subseteq \pi(A(D_2))''.
\]

These inclusions are verified for any \( D_2 \in \mathcal{K}_o(M) \) such that \( \overline{D}_1 \subseteq D_2 \). Hence, the outer regularity [12] implies that \((\mathcal{A}_{\mathcal{K}_o(M)}, \omega)\) satisfies punctured Haag duality.

This theorem and the observations at the beginning of this section lead to the following

**Corollary 4.5.** \( \mathcal{S}_{HL}(\mathcal{A}) = \mathcal{S}_{PH}(\mathcal{A}) \).

### 4.4 The case of the free Klein-Gordon field

The theory of the free Klein-Gordon field provides, as shown in [11], an example of a locally covariant quantum field theory \( \mathcal{W} \) with a state space \( S_\mu \). The functor \( \mathcal{W} \) is defined as the correspondence \( M \rightarrow \mathcal{W}(M) \) that associates to any \( M \in \mathfrak{Man} \) the Weyl algebra \( \mathcal{W}(M) \) of the free Klein-Gordon field over \( M \). \( S_\mu \) is defined as the correspondence \( M \rightarrow S_\mu(M) \) that to any \( M \) associates the collection of the states of \( \mathcal{W}(M) \) which are locally quasiequivalent to quasi-free states of \( \mathcal{W}(M) \) fulfilling the microlocal spectrum condition (or equivalently the Hadamard condition). We refer the reader to the cited paper and reference therein for a detailed description of this example. We now prove that for any \( \omega \in S_\mu(M) \) the pair \((\mathcal{W}_{\mathcal{K}(M)}, \omega)\) is locally definite and, if \( \omega \) is pure, satisfies Haag duality. Hence, \( S_\mu \in \mathcal{S}_{HL}(\mathcal{W}) \), and by Theorem 1.4 \((\mathcal{W}_{\mathcal{K}(M)}, \omega)\) satisfies punctured Haag duality for any pure state \( \omega \in S_\mu(M) \).

Let us start by recalling that two states \( \omega, \omega_1 \) of \( \mathcal{W}(M) \) are said to be \textit{locally quasiequivalent} if for each \( O \in \mathcal{K}(M) \) there exists an isomorphism \( \rho_O : \pi(\mathcal{W}(O))'' \rightarrow \pi_1(\mathcal{W}(O))'' \) such that \( \rho_O \pi(A) = \pi_1(A) \) for each \( A \in \mathcal{W}(O) \), where \( \pi, \pi_1 \) are respectively the GNS representations of \( \omega \) and \( \omega_1 \). Furthermore, we need to recall the following fact ([10, Theorem 3.6]): for each \( M \in \mathfrak{Man} \), if \( \omega_\mu \) is a quasi-free state of \( \mathcal{W}(M) \) satisfying the microlocal spectrum condition, then \((\mathcal{W}_{\mathcal{K}(M)}, \omega_\mu)\) is locally definite and, if \( \omega_\mu \) is pure, it satisfies Haag duality.
Proposition 4.6. \( S_\mu \in \mathcal{S}_{HL}(\mathcal{W}) \). Therefore, \( (\mathcal{W}_{\mathcal{K}(M)}, \omega) \) satisfies punctured Haag duality for any pure state \( \omega \in S_\mu(M) \).

Proof. Fix \( M \in \mathfrak{Man} \) and consider a pure state \( \omega \) of \( \mathcal{W}(M) \) which is locally quasiequivalent to a quasi-free state \( \omega_\mu \) satisfying the microlocal spectrum condition. Let \( \pi \) and \( \pi_\mu \) be the GNS representations of \( \omega \) and \( \omega_\mu \) respectively. It has already been shown in [10] that \( (\mathcal{W}_{\mathcal{K}(M)}, \omega) \) satisfies Haag duality. Hence it remains to be proved that \( (\mathcal{W}_{\mathcal{K}(M)}, \omega) \) is locally definite. To this aim, fix \( p \in M \) and \( D_1 \in \mathcal{K}_o(M) \) such that \( p \in D_1 \). As observed above \( (\mathcal{W}_{\mathcal{K}(M)}, \omega_\mu) \) is locally definite. This entails that \( \cap\{\pi_\mu(W(D))'' \mid D \in \mathcal{K}_o(M), \ p \in D \subset D_1 \} = \mathbb{C} \cdot 1 \) because \( \mathcal{K}_o(M) \) contains a neighbourhoods basis of \( p \). Being \( \omega \) locally quasiequivalent to \( \omega_\mu \), there is an isomorphism \( \rho_{D_1} \) from \( \pi(W(D))'' \) onto \( \pi_\mu(W(D_1))'' \) such that \( \rho_{D_1}(A) = \pi_\mu(A) \) for each \( A \in W(D_1) \). Hence

\[
\rho_{D_1} : \cap\{\pi(W(D))'' \mid p \in D \subset D_1 \} \longrightarrow \cap\{\pi_\mu(W(D))'' \mid p \in D \subset D_1 \} = \mathbb{C} \cdot 1
\]

is an isomorphism and, consequently, \( (\mathcal{W}_{\mathcal{K}(M)}, \omega) \) is locally definite. \( \square \)

Remark 4.7. It is worth mentioning that the family of the adiabatic vacuum states of order \( N > \frac{5}{2} \), studied in [6], is contained in \( S_\mu \): any such state is locally quasiequivalent to a quasi-free state fulfilling the microlocal spectrum condition.

A Proof of the Proposition 3.3

The proof of the Proposition 3.3 comes by a slight modification of the Lemmas 5, 7 and 8 of [11]. So, we give a detailed description only of the modified parts of the proofs and refer the reader to the cited paper for the assertions that we will not prove.

We start by recalling some results of the cited paper. Let \( M \in \mathfrak{Man} \) and let \( F : \mathbb{R} \times \Sigma \longrightarrow M \) be a foliation of \( M \) by spacelike Cauchy surfaces. For each acausal (spacelike) Cauchy surface \( \mathcal{C} \), there is an associated pair \( (\tau_c, f_c) \) where \( \tau_c : \Sigma \longrightarrow \mathbb{R} \) is a continuous (smooth) function, while \( f_c \), defined as

\[
f_c(y) \equiv F(\tau_c(y), y), \quad y \in \Sigma,
\]

is an homeomorphism (diffeomorphism) \( f_c : \Sigma \longrightarrow \mathcal{C} \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]

is an homeomorphism (diffeomorphism) \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \). Given another acausal Cauchy surface \( \mathcal{C}_o \) and the corresponding pair \( (\tau_{c_o}, f_{c_o}) \), the map \( \Phi_{c_o,c} : \mathcal{C} \longrightarrow \mathcal{C}_o \) defined as

\[
\Phi_{c_o,c}(p) \equiv (f_{c_o} \circ f_c^{-1})(p), \quad \forall p \in \mathcal{C},
\]
is an homeomorphism (diffeomorphism if $C$ and $C_o$ are spacelike).

Now, consider the causal excision $M_p$ of $p \in M$, and a spacelike Cauchy surface $C_p$ of $M_p$. By Proposition 3.1, $C \equiv C_p \cup \{p\}$ is an acausal Cauchy surface of $M$.

**Lemma A.1.** For each continuous strictly positive function $\varepsilon : \Sigma \rightarrow \mathbb{R}$ there exists a spacelike Cauchy surface $C_o$ that meets $p$, such that $|\tau_e(y) - \tau_{e_o}(y)| < \varepsilon(y)$ for each $y \in \Sigma$.

**Proof.** Let us define $A^{\pm}_\lambda \equiv \{ F(\tau_e(y) \pm \lambda \cdot \varepsilon(y), y) \mid y \in \Sigma \}$, $0 < \lambda < 1$, and $\mathcal{N} \equiv (M \setminus (J^+(A^{\pm}_\lambda) \cup J^-(A^{\pm}_\lambda)))^c$. Notice that if $p = F(t_o, y_o)$ with $t_o > \tau_e(y_o) + \lambda \cdot \varepsilon(y_o)$, then $p \notin \mathcal{N}$. In fact, the f-d timelike curve

$$\gamma(t) \equiv F(\tau_e(y_o) + t \cdot \varepsilon(y_o), y_o), \quad t \in [\lambda, (t_o - \tau_e(y_o)) \cdot \varepsilon(y_o)^{-1}],$$

joins the point $F(\tau_e(y_o) + \lambda \cdot \varepsilon(y_o), y_o) \in A^{\pm}_\lambda$ with $p$. Analogously, if $p = F(t_o, y_o)$ with $t_o < \tau_e(y_o) - \lambda \cdot \varepsilon(y_o)$, then $p$ does not belong to $\mathcal{N}$. Hence

$$p \in \mathcal{N}, \quad p = F(t, y) \iff |t - \tau_e(y)| < \lambda \cdot \varepsilon(y) \Rightarrow |t - \tau_e(y)| < \varepsilon(y).$$

Now, as $\mathcal{N}$ is globally hyperbolic, there is a spacelike Cauchy surface $C_o$ of $\mathcal{N}$ that meets $p$. $C_o$ is also a spacelike Cauchy surface of $M$. Since $C_o \subset \mathcal{N}$, we have $|\tau_e(y) - \tau_{e_o}(y)| < \varepsilon(y)$ for each $y \in \Sigma$. □

**Lemma A.2.** Let $C_p$ and $C$ as above. Consider three connected, relatively compact, open subsets $G, U_1, U_2$ of $C_p$ that verify $C \subset U_1, \overline{U}_1 \subset U_2$. Then, there exists a smooth acausal Cauchy surface $C_o$ of $M$ that meets $p$, such that:

\begin{enumerate}
  \item $J(G) \cap C_o \subset \Phi_{e_o,c}(U_1)$
  \item $J(\Phi_{e_o,c}(U_1)) \cap C \subset U_2$
\end{enumerate}

**Proof.** The sets $\mathcal{N}_{1\pm}$, defined as $(D^+(U_1))^c$, are globally hyperbolic. Let us take two spacelike Cauchy surfaces $S_{1\pm}$ of $\mathcal{N}_{1\pm}$. Notice that $C_{1\pm} \equiv S_{1\pm} \cup (C \setminus U_1)$ are acausal Cauchy surfaces of $M$, and that, there are two strictly positive continuous functions $\varepsilon_{1\pm} : f_{c_1}(U_1) \rightarrow \mathbb{R}$ such that $S_{1\pm} = \{ F(\tau_e(y) \pm \varepsilon_{1\pm}(y), y) \mid y \in f_{c_1}^{-1}(U_1) \}$. Let us define

$$U_{1+} \equiv J^-(J^+(G) \cap S_{1+}) \cap C, \quad U_{1-} \equiv J^+(J^-(G) \cap S_{1-}) \cap C.$$

Since $J^+(G) \cap C_{1+}$ is a closed subset of $C_{1+}$ and $J^+(G) \cap C_{1+} = J^+(G) \cap S_{1+}$, we have that $U_{1+}$ is a closed subset of $U_1$ and $G \subset U_{1+}$. The same holds for $U_{1-}$. The set $W_1 \equiv U_{1+} \cup U_{1-}$ is closed, compact (because contained in a relatively compact set) and $G \subset W_1 \subset U_1$. We now apply the same
reasoning with respect to the inclusion \( U_1 \subset U_2 \). Namely, given two spacelike Cauchy surfaces \( S_{2\pm} \) of the spacetimes \((D^\pm(U_2))^\circ\), we consider the acausal Cauchy surfaces \( C_{2\pm} \) of \( M \) defined as \( C_{2\pm} \equiv S_{2\pm} \cup (C \setminus U_2) \). As above, there are two strictly positive continuous functions \( \varepsilon_{2\pm} \) such that \( S_{2\pm} = \{ F(\tau_c(y) \pm \varepsilon_{2\pm}(y), y) \mid y \in f_c^{-1}(U_2) \} \). Thus, we can find a compact set \( W_2 \) of \( C \) verifying \( U_1 \subset W_2 \subset U_2 \). Now let us define

\[
\varepsilon \equiv \min \left\{ \min_{y \in f_c^{-1}(W_1)} \{ \varepsilon_{1+}(y), \varepsilon_{1-}(y) \}, \min_{y \in f_c^{-1}(W_2)} \{ \varepsilon_{2+}(y), \varepsilon_{2-}(y) \} \right\}.
\]

By Lemma \ref{lem:causal_embedding} there is a spacelike Cauchy surface \( C_o \) of \( M \) that meets \( p \), such that \( |\tau_{c_o}(y) - \tau_c(y)| < \varepsilon \) for each \( y \in \Sigma \). Since \( \mathcal{G} \subseteq W_1 \), by the definition of \( \varepsilon \) the set \( J^+(\mathcal{G}) \cap C_o \) is in the past of \( J^+(\mathcal{G}) \cap S_{1+} \), while \( J^-(\mathcal{G}) \cap C_o \) is in the future of \( J^-(\mathcal{G}) \cap S_{1-} \). Hence, we have

\[
J(J(\mathcal{G}) \cap C_o) \cap C \subset W_1.
\]

This entails

\[
\Phi_{c,c_o}(J(\mathcal{G}) \cap C_o) = f_c \circ f_c^{-1}(J(\mathcal{G}) \cap C_o) \subset W_1
\]

\[\iff\]

\[J(\mathcal{G}) \cap C_o \subset \Phi_{c,c_o}(W_1) \Rightarrow J(\mathcal{G}) \cap C_o \subset \Phi_{c,c_o}(U_1),\]

completing the proof of the statement a). The same reasoning applied to the inclusion \( U_1 \subset U_2 \) leads to \( J(J(U_1) \cap C_o) \cap C \subset W_2 \). As \( \Phi_{c,c_o}(U_1) \) is contained in the closed set \( J(U_1) \cap C_o \), we have that \( J(\Phi_{c,c_o}(U_1)) \cap C \subset W_2 \subset U_2 \). □

**Proposition A.3.** Let \( D \in \mathcal{K}_o(M_p) \) and let \( V \) be an open set of \( M_p \) such that \( \overline{D} \subset V \). Then, there exist a spacelike Cauchy surface \( C_o \) of \( M \) that meets \( p \), and \( D_o \in \mathcal{K}_o(M, C_o) \) such that \( \overline{D} \subset D_o \), and \( \overline{D_o} \subset (D(V))^\circ \).

**Proof.** Assume that \( D \) is based on a spacelike Cauchy surface \( C_p \) of \( M_p \) and let \( G \subset C_p \) be the base of \( D \). The set \( U \equiv V \cap C_p \) is open in \( C_p \) and \( \overline{G} \subset U \).

According to \cite{Hawking Ellis} Lemma 3 we can find \( U_1, U_2 \subset \mathcal{G}(C_p) \) such that \( \overline{G} \subset U_1 \), \( \overline{U_1} \subset U_2 \) and \( \overline{U_2} \subset U \). Let \( C \) be the acausal Cauchy surface of \( M \) defined as \( C = C_p \cup \{p\} \). By Lemma \ref{lem:causal_embedding'}, there is a spacelike Cauchy surface \( C_o \) of \( M \) that meets \( p \), such that

\[
J(\overline{G}) \cap C_o \subset \Phi_{c_o,c}(U_1), \quad J(\overline{\Phi_{c_o,c}(U_1)}) \cap C \subset U_2.
\]

\( f_{c_o} \) is a diffeomorphism between \( \Sigma \) and \( C_o \) because \( C_o \) is spacelike. Notice now that \( f_c : \Sigma \to C_p \cup \{p\} \) is an homeomorphism because \( C = C_p \cup \{p\} \)
is an acausal, in general nonsmooth, Cauchy surface of \( M \). However, \( \mathcal{C}_p \) is spacelike, hence smooth. Then, it easily follows from the definition of \( f_e \) that \( f_e : \Sigma \setminus \{ f_e^{-1}(p) \} \longrightarrow \mathcal{C}_p \) is a diffeomorphism. Then, \( \mathcal{D}_o \equiv (D(\Phi_{e_o,e}(U_1)))^o \) is a regular diamond of \( M \) based on \( \mathcal{C}_o \). The previous inclusions entail that \( \overline{\mathcal{D}} \subset \mathcal{D}_o, \overline{\mathcal{D}_o} \subset (D(U_2))^o \subset (D(V))^o \) completing the proof.

Because \( \S \) the Proposition \( \S \) proves the first part of the Proposition 3.3. Concerning the second part, we have the following

**Proposition A.4.** Let \( \mathcal{D} \in \mathcal{K}_o(M) \) be such that \( \mathcal{D} \perp \{ p \} \) and let \( V \) be an open set of \( M \) such that \( \overline{\mathcal{D}} \subset V \). Then, there exist a spacelike Cauchy surface \( \mathcal{C}_o \) of \( M \), that meets \( p \), and \( \mathcal{D}_o \in \mathcal{K}_o(M,\mathcal{C}_o) \) such that \( \overline{\mathcal{D}} \subset \mathcal{D}_o, \overline{\mathcal{D}_o} \subset (D(V))^o \) and \( \mathcal{D}_o \perp \{ p \} \).

**Proof.** The proof is very similar to the proof of Proposition \( \S \). Assume that \( \mathcal{D} = (D(G))^o \) where \( G \in \mathfrak{G}(\mathcal{C}) \) for some spacelike Cauchy surface \( \mathcal{C} \) of \( M \). Notice \( \mathcal{G} \cap J(p) = \emptyset \). Let us define \( U = (\mathcal{C} \cap V) \setminus (\mathcal{C} \cap J(p)) \). \( U \) is an open set of \( \mathcal{C} \setminus (\mathcal{C} \cap J(p)) \) and \( \overline{\mathcal{G}} \subset U \). By \( \S \) Lemma 3\) we can find \( \overline{U}_1, \overline{U}_2 \subset \mathfrak{G}(\mathcal{C}) \) such that \( \overline{\mathcal{G}} \subset \overline{U}_1, \overline{U}_1 \subset \overline{U}_2 \) and \( \overline{U}_2 \subset U \). Now, notice that in general \( \mathcal{C} \) does not meet \( p \), hence the Lemmas \( \S \) and \( \S \) cannot be applied. In this case, however, we can use directly \( \S \) Lemma 5\) that asserts that for each \( \varepsilon > 0 \) there exists a spacelike Cauchy surface \( \mathcal{C}_o \) that meets \( p \) such that \( |\tau_{\mathcal{C}}(y) - \tau_{\mathcal{C}_o}(y)| < \varepsilon \) for any \( y \in f_{e_1}^{-1}(U) \). Proceeding as in Lemma \( \S \) one can choose \( \varepsilon \) in such a way that \( J(\overline{\mathcal{G}}) \cap \mathcal{C}_o \subset \Phi_{e_o,e}(U_1), J\left(\Phi_{e_o,e}(U_1)\right) \cap \mathcal{C} \subset U_2 \).

Proceeding now as in Proposition \( \S \) these inclusions lead to the proof of the statement.

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