Residual Power Series Technique for Simulating Fractional Bagley–Torvik Problems Emerging in Applied Physics

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Featured Application: Fractional differential equations play a significant role in modeling certain dynamical systems arising in many fields of applied sciences and engineering. In this paper, the authors develop an attractive analytic-numeric technique, residual power series (RPS) method, for solving fractional Bagley–Torvik equations with a source term involving Caputo fractional derivative. In regard to its simplicity, the method can be applicable to a wide class of fractional partial differential equations, fractional fuzzy differential equations, fractional oscillator equations, and so on.

Abstract: Numerical simulation of physical issues is often performed by nonlinear modeling, which typically involves solving a set of concurrent fractional differential equations through effective approximate methods. In this paper, an analytic-numeric simulation technique, called residual power series (RPS), is proposed in obtaining the numerical solution a class of fractional Bagley–Torvik problems (FBTP) arising in a Newtonian fluid. This approach optimizes the solutions by minimizing the residual error functions that can be directly applied to generate fractional PS with a rapidly convergent rate. The RPS description is presented in detail to approximate the solution of FBTPs by highlighting all the steps necessary to implement the algorithm in addressing some test problems. The results indicate that the RPS algorithm is reliable and suitable in solving a wide range of fractional differential equations applying in physics and engineering.

Keywords: numerical simulation; residual power series algorithm; fractional Bagley–Torvik model; Newtonian fluid; Caputo fractional derivative

1. Introduction

Fluid is a fixed-volume state when determining a temperature and pressure, which flows and changes constantly when exposed to external shear forces or stresses without separating the mass. Theoretically, it can be ideally described through well-known differential equations, but, in fact, there is no ideal model except in the laboratory. Therefore, the best way to deal with many unpredictable situations is to study them statistically or numerically using the fractional meaning. However, mathematical modeling of fractional differential equations (FDEs) is a very useful and practical subject in applied physics, computer science, and engineering, which facilitates a better understanding of dynamic physical processes in terms of spatial and temporal parameters and illustrates their structures, which depends not only on the current time, but also in previous history, memory, mass
movement, and material transfer mechanisms [1–4]. Unlike the classical calculus, which has unique definitions and clear geometrical and physical interpretations, there are numerous definitions of the fractional operations. Riemann–Liouville, Riesz, Grünwald–Letnikov, and Caputo are some examples of these definitions [5–8]. Recently, FDEs have attracted the attention of numerous researchers for its considerable importance in many scientific applications, including fluid dynamics, signal processing, viscoelasticity, bioengineering, finance, Hamiltonian chaos, and vibrations [1–6]. In this light, there exists no classic, precise method that yields an analytical solution in a closed-form for these models; therefore, approximate and numerical methods have been developed to handle such FDEs. Among these methods, the spectral collocation methods [6], homotopy analyses transform method [7], reproducing kernel method [8], multistep method [9], and operational matrix approaches [10].

The current work chiefly aims for using the residual power series method to investigate and construct the approximate solution for a class of fractional Bagley–Torvik problem, in which the governing FBTP is given by the fractional differential operator

\[ Aω''(t) + BD^α_0ω(t) + Cω(t) = ϕ(t), t \in [0, T], \]

along with the initial condition

\[ ω(0) = μ₀ \text{ and } ω'(0) = μ₁, \]

where \( μ₀, μ₁ ∈ R, A, B, \) and \( C \) are parameters such that \( A, B, C ≠ 0 \), \( A \) is the plate of mass, \( B \) is the surface area in term of viscosity \( μ \) and fluid density \( ρ \) with \( B = 2A \sqrt{μρ} \), and \( C \) is the stiffness of the spring to which \( A \) is attached. The continuous function \( ϕ(t) \) can be used to represent a loading force term or sinks, and \( ω(t) \) represents the displacement of \( A \) and \( B \) to be determined. Furthermore, \( D^α_0 \) is the Caputo fractional derivative of order \( α \). Here, it is assumed that FBTPs (1) and (2) had a unique and sufficiently smooth solution in the domain of interest.

The FBTP (1) represents a suitable mathematical model for describing the motion of a rigid plate immersed in a Newtonian fluid that was proposed by Bagley and Torvik during the application of fractional operator on viscoelasticity theory [11–13]. Anyhow, advanced numerical methods are found in the literature for approximating the FBTP solutions, including the collocation method [14], spectral Tau method [15], differential transform method [16], pseudo-spectral method [17], and fractional-order Legendre collocation method [18]. On the other hand, there is a modern, distinctive, and nonclassical curriculum based on computational and logical thinking, innovation and motivating learners to better understand the real applications of various issues arising in the fields of sciences, which is science, technology, engineering and mathematics (STEM) education. What distinguishes it from traditional education is the mixed learning environment, which is based on applying the scientific method to the real issues of daily life. Interest in this type of education began in the United States in 2009 and many educational programs and methods have been developed and improved to deal with this curriculum. For more details about STEM education, see [19–22] and references therein.

In 2013, the RPS method was proposed by Abu Arqub [23] as a powerful and effective approximate algorithm to solve a class of uncertain initial value problems. Later, RPSM has been used in generating a fractional power series (FPS) solutions for strongly nonlinear FDEs in the form of a rapidly convergent with a minimum size of calculations without any restrictive hypotheses. Thus, this adaptive can be used as an alternative technique in solving several nonlinear problems arising in engineering and physics [24–28].

The purpose of this paper is to present an RPS method to construct the approximate solution for FBTP involving the Caputo fractional derivative using the concept of residual error function. The remaining part of this paper is structured as follows. In Section 2, some popular definitions and results of fractional calculus are recalled briefly. In Section 3, the RPS technique is described. In Section 4, the suggested method is implemented for solving the FBTE. Lastly, concluding remarks are provided in Section 5.
2. Mathematical Preliminaries

In this section, we recall some definitions and results concerning the fractional Caputo concept and FPS representations.

**Definition 1.** The Riemann–Liouville fractional integral operator of order $\gamma$, over the interval $[a, b]$ for a function $\omega \in L_1[a, b]$ is defined by

$$I_{\varepsilon}^{\gamma} \omega(t) = \frac{1}{\Gamma(\gamma)} \int_{\varepsilon}^{t} \omega(\delta) (t - \delta)^{-\gamma} d\delta, \quad 0 < \gamma < 1, \quad a < t < b,$$

where $\varepsilon > 0$. The following are some interesting properties of the operator $D_{\varepsilon}^{\alpha}$:

- For any constant $c \in \mathbb{R}$, then $D_{\varepsilon}^{\alpha} c = 0$,
- $D_{\varepsilon}^{\alpha} (t - \varepsilon)^{\gamma} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - r)} (t - \varepsilon)^{\alpha - r}, \quad m - 1 < \alpha \leq m, \quad m \in \mathbb{N}, \quad r \in \mathbb{R}, \quad \alpha > 0, \quad \alpha < 1,$
- $D_{\varepsilon}^{\alpha} \Gamma(t) = \omega(t), \quad \forall t > \varepsilon$.
- $D_{\varepsilon}^{\alpha} \omega(t) = \omega(t) - \sum_{j=0}^{m-1} \frac{\omega(j)}{j!} (t - \varepsilon)^{j}$.

In addition, it should be noted that, for an arbitrary function $\omega(t), 0 \leq \tau_0 < t$, the Caputo fractional derivative can be given as follows:

$$D_{\varepsilon}^{\alpha} \omega(t) = \sum_{n=0}^{\infty} \frac{\omega(n)}{\Gamma(n + 1 - \alpha)} (t - \tau_0)^{n \alpha}, \quad \tau_0 < t < \tau_0 + R^{1/\alpha}, \quad R > 0,$$

where $\omega(n)(\tau_0)$ means the application of the fractional derivative $n$ times.

**Definition 3.** A general fractional power series of the form

$$\sum_{j=0}^{\infty} a_j (t - \tau_0)^{j \alpha} = a_0 + a_1 (t - \tau_0)^{1 \alpha} + a_2 (t - \tau_0)^{2 \alpha} + \ldots,$$

where $t \geq \tau_0$, is called generalized fractional power series (FPS) about $t = \tau_0, 0 < \alpha \leq 1$, and $a_j, \quad j \in \mathbb{N}$, denote the coefficients of the series.

As the classical power series, it clear that all terms of the FPS (3) vanish as soon as $t = \tau_0$ except the first term, which means the FPS is convergent when $t = \tau_0$. Anyhow, for $t \geq \tau_0$, this series is definitely convergent for $|t - \tau_0| < R^{1/\alpha}, (R > 0)$, where $R^{1/\alpha}$ is the radius of convergence of the FPS. On the other hand, a function $\omega(t)$ is analytical at $t = \tau_0$ if $\omega(t)$ can be written as a form of FPS (3).

**Theorem 1.** If $\omega(t)$ has the FPS about $t = \tau_0$ as follows:

$$\omega(t) = \sum_{j=0}^{\infty} a_j (t - \tau_0)^{j \alpha}, \quad \tau_0 < t < \tau_0 + R^{1/\alpha}, \quad R > 0,$$

Theorem 1. If $\omega(t)$ has the FPS about $t = \tau_0$ as follows:
where \(-1 < \alpha \leq m\), \(\omega(t) \in C\left[t_0, \tau_0 + R^{1/\alpha}\right]\), \(D^{\alpha}_{t_0^+} \omega(t) \in C\left(t_0, \tau_0 + R^{1/\alpha}\right)\), and \(D^{k}_{t_0^+} \omega(t)\) is well defined on \((\tau_0, \tau_0 + R^{1/\alpha})\) for \(k = 1, 2, \ldots, m - 1\), and \(j = 0, 1, 2, \ldots\), where \(D^{k\omega}_{t_0^+} = D^{\alpha}_{t_0^+} D^{\alpha}_{t_0^+} \cdots D^{\alpha}_{t_0^+} (j\text{-times})\). Then, the coefficients, \(a_j\), of the FPS representation are given by \(a_j = \frac{1}{\Gamma(j+1)} D^{\alpha}_{t_0^+} \omega(t_0)\).

The FPS (4) about \(t = \tau_0\) can be rewritten as \(\omega(t) = \omega_n(t) + R_n(t)\), where \(\omega_n(t)\) represent the \(n\)th approximate series of \(\omega(t)\) and \(R_n(t)\) the tail of FPS, which can be given, respectively, by

\[
\omega_n(t) = \sum_{j=0}^{n} a_j (t - \tau_0)^{j\alpha},
\]

and \(R_n(t) = I^{(n-1)\alpha}_{t_0^+} D^{(n-1)\alpha}_{t_0^+} \omega(t) = \omega(t) - \sum_{j=0}^{n} \frac{D^{\alpha}_{t_0^+} \omega(t_0)}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{j\alpha}\). Anyhow, the FPS is convergent to the exact solution whenever \(\lim_{n \to \infty} R_n(t) = 0\).

**Corollary 1.** Let \(\alpha \in (m - 1, m]\), \(D^{\alpha}_{t_0^+} \omega(t)\) exist for \(j = 0, 1, 2, \ldots, n + 1\), and \(\omega(t)\) has the FPS representation (4) such that \(|D^{(n+1)\alpha}_{t_0^+} \omega(t)| \leq M, M > 0\) for some \(n \in \mathbb{N}\). Then, for all \(\left(t_0, \tau_0 + R^{1/\alpha}\right)\), the reminder \(R_n(t)\) of the FPS \(\omega(t)\) satisfies

\[
|R_n(t)| = \frac{M}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{(n+1)\alpha}.
\]

**Proof.** From the assumption \(|D^{(n+1)\alpha}_{t_0^+} \omega(t)| \leq M\), we have

\[
-M \leq D^{(n+1)\alpha}_{t_0^+} \omega(t) \leq M.
\]

Thus, by applying the operator \(I^{(n+1)\alpha}_{t_0^+}\) on both sides of (7), one can get that

\[
\frac{-M}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{(n+1)\alpha} \leq \frac{a_j}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{(n+1)\alpha} \leq \frac{M}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{(n+1)\alpha}.
\]

Hence, \(\left|R_n(t)\right| = \frac{M}{\Gamma((n+1)\alpha+1)} (t - \tau_0)^{(n+1)\alpha}\).

### 3. Mathematical Model Formulation

In this section, the fractional Bagley–Torvik equation (FBTE) is formulated subject to suitable initial conditions to construct the RPS solution utilizing the RPS algorithm based on the truncated residual error function.

Consider the FBTEs described in Equations (1) and (2) at \(\alpha = 3/2\); thus, to achieve our goal in applying the FRPS method, let us first convert the FBTE, \(\omega''(t) + bD^{3/2}_0 \omega(t) + c\omega(t) = \varphi(t)\), into an equivalent system of fractional-order \(\alpha = 1/2\) by setting \(\omega(t) = x_1(t), D^{1/2}_0 \omega(t) = x_2(t), D^{1/2}_0 \omega(t) = x_3(t), D^{1/2}_0 \omega(t) = x_4(t)\), and \(\omega''(t) = D^{3/2}_0 x_1(t) = D^{3/2}_0 x_2(t) = D^{3/2}_0 x_3(t) = D^{3/2}_0 x_4(t)\); thus, we have

\[
\begin{align*}
D^{\alpha}_0 x_1(t) &= x_2(t), \\
D^{\alpha}_0 x_2(t) &= x_3(t), \\
D^{\alpha}_0 x_3(t) &= x_4(t), \\
D^{\alpha}_0 x_4(t) &= \varphi(t) - bx_4(t) - cx_1(t),
\end{align*}
\]

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subject to the nonhomogeneous initial conditions
\[ x_1(0) = \mu_0, \ x_2(0) = 0, \ x_3(0) = \mu_1, \ x_4(0) = 0. \] (9)

According the FRPS method [24–27], let us assume that the solutions of IVPs (8) and (9) can be written by
\[ x_i(t) = \sum_{k=0}^{\infty} a_{i,k} \frac{t^\alpha}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq \frac{1}{2}, \quad i = 1, 2, 3, 4, \] (10)
where the truncated series solution of \( x_i(t) \) is
\[ x_i^n(t) = \sum_{k=0}^{n} a_{i,k} \frac{t^\alpha}{\Gamma(k\alpha + 1)}. \] (11)

Now, the residual functions can be defined by
\[ \text{Res}_x(t) = D_{0^+}^\alpha x_i(t) - x_{i+1}(t), \quad i = 1, 2, 3, \]
\[ \text{Res}_{x_4}(t) = D_{0^+}^\alpha x_4(t) + bx_4(t) + cx_1(t) - \varphi(t), \]
and the \( n \)th-truncated residual functions by
\[ \text{Res}_{x_i}^n(t) = D_{0^+}^\alpha x_i^n(t) - x_{i+1}^n(t), \quad i = 1, 2, 3, \]
\[ \text{Res}_{x_4}^n(t) = D_{0^+}^\alpha x_4^n(t) + bx_4^n(t) + cx_1^n(t) - \varphi(t), \] (12)
whereas \( \text{Res}_{x_i}(t) = \lim_{n \to \infty} \text{Res}_{x_i}^n(t) = 0, \quad i = 1, 2, 3, 4, \) and \( D_{0^+}^{(n-1)\alpha} \text{Res}_{x_i}(0) = D_{0^+}^{(n-1)\alpha} \text{Res}_{x_i}^n(0) = 0 \) for each \( k = 1, 2, \ldots, n \). However, \( D_{0^+}^{(n-1)\alpha} \text{Res}_{x_i}(t) = 0, \quad n = 1, 2, \ldots, i = 1, 2, 3, 4. \)

The following algorithm shows us the FRPS strategy to determine the coefficients \( a_{i,k}, i = 1, 2, 3, 4, k = 1, 2, \ldots, n \) of Equation (11) in order to predict and obtain the RPS solution of FBTEs (1) and (2).

In particular, the 5th RPS approximate solution of FBTEs (1) and (2) by using the Algorithm 1 can be given by
\[ \omega_5(t) = \mu_0 + \mu_1 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + (\varphi(0) - \mu_0) \frac{t^{\alpha}}{\Gamma(4\alpha + 1)} + \left( D_{0^+}^\alpha \varphi(0) - b(\varphi(0) - \mu_0) \right) \frac{t^{5\alpha}}{\Gamma(5\alpha + 1)}. \]

On the other aspect, the analytic solution of FBTE (1) along with homogeneous initial conditions \( \omega(0) = \omega'(0) = 0 \) has been given in [29] by
\[ \omega(t) = \int_0^t G(t - \xi) \varphi(\xi) d\xi, \]
where \( G(t) \) is the Green function, which is defined by
\[ G(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^{j^2 + 1} E_{1/2,2,3/3}^{(j)}(-b\sqrt{t}), \]
and \( E_{\zeta,\eta}^{(j)} \) denotes the \( j \)th derivative of the Mittag–Leffler function in two parameters \( \zeta, \eta > 0 \), which is given by
\[ E_{\zeta,\eta}^{(j)}(t) = \sum_{k=0}^{\infty} \frac{(k + j)!^k}{k! \Gamma(k\zeta + j\zeta + \eta)}. \]
Algorithm 1 To find out the coefficients, $a_{i,j}, k = 1, 2, \ldots, n$, for $i = 1, 2, 3, 4$, in the series expansion (11), do the following steps:

- **Step 1:** Assume that the solutions of the fractional IVPs (8) and (9) have the following FPS about $t_0 = 0$:
  \[ x_i(t) = \sum_{k=0}^{\infty} a_{i,k} \frac{t^k}{\Gamma((k + 1)\alpha)}, \]
  where $0 < \alpha \leq \frac{1}{2}, t \in [0, \rho), \rho = R^{1/\alpha}$ is the radius of convergence of the FPS for $i = 1, 2, 3, 4$.

- **Step 2:** Define the $n$th-truncated series $x^n_i(t)$ of $x_i(t)$ such that
  \[ x^n_i(t) = \sum_{k=0}^{n} a_{i,k} \frac{t^k}{\Gamma((k + 1)\alpha)}.
  \]

- **Step 3:** Consider the initial conditions $x_1(0) = \mu_0$, $x_2(0) = \mu_1$ and $x_3(0) = x_4(0) = 0$, then the 0th-RPS approximate solutions are
  \[ x^n_1(t) = \mu_0 + \sum_{k=1}^{n} a_{1,k} \frac{t^k}{\Gamma((k + 1)\alpha)}, \]
  \[ x^n_2(t) = \mu_1 + \sum_{k=1}^{n} a_{2,k} \frac{t^k}{\Gamma((k + 1)\alpha)}, \]
  and
  \[ x^n_3(t) = \sum_{k=1}^{n} a_{3,k} \frac{t^k}{\Gamma((k + 1)\alpha)}, \]
  and
  \[ x^n_4(t) = \sum_{k=1}^{n} a_{4,k} \frac{t^k}{\Gamma((k + 1)\alpha)}. \]

- **Step 4:** Define the $n$th-residual functions $\text{Res}_{x^n_i}(t)$ such that
  \[ \text{Res}^n_i(t) = D^n_{0+} x^n_i(t) - x^n_{i+1}(t), \]
  \[ \text{Res}^n_i(t) = D^n_{0+} x^n_i(t) + bx^n_i(t) + cx^n_i(t) - q(t). \]

- **Step 5:** Substitute the $n$th-truncated series in Step 3 into $\text{Res}^n_i(t)$ in Step 4 such that
  \[ \text{Res}^n_i(t) = \left( \frac{a_{i,1} + a_{i,2} \frac{t^\alpha}{\Gamma((\alpha + 1)\alpha)} + \ldots + a_{i,(n-1)} \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha + 1)}}{\Gamma(n(\alpha + 1))} \right) \]
  \[ + \left( \frac{a_{i,0} + a_{i,1} \frac{t^\alpha}{(\alpha + 1)} + \ldots + a_{i,n} \frac{t^{n\alpha}}{\Gamma(n(\alpha + 1))}}{\Gamma(n(\alpha + 1))} \right) \]
  \[ + \left( \frac{a_{4,1} + a_{4,2} \frac{t^\alpha}{(\alpha + 1)} + \ldots + a_{4,(n-1)} \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha + 1)}}{\Gamma(n(\alpha + 1))} \right) \]
  \[ + \left( \frac{a_{4,0} + a_{4,1} \frac{t^\alpha}{(\alpha + 1)} + \ldots + a_{4,n} \frac{t^{n\alpha}}{\Gamma(n(\alpha + 1))}}{\Gamma(n(\alpha + 1))} \right) - q(t). \]

- **Step 6:** Set $n = 1$ in Step 5, then by using $\text{Res}^n_i(t)|_{t=0} = 0$, the 1st unknown coefficients $a_{i,1}$ for $i = 1, 2, 3, 4$ will be obtained. Therefore, the 1st approximate PS solutions $x^n_i(t), i = 1, 2, 3, 4$, are also obtained.

- **Step 7:** For $n = 2, 3, \ldots, k$, do the following subroutine:
  
  (A) Apply the operator $D^n_{0+}$, $(n-1)$-times, on both sides of the $n$th-residual functions $\text{Res}^n_i(t), i = 1, 2, 3, 4$, in Step 4 such that $D^n_{0+}^{(n-1)\alpha} \text{Res}^n_i(t)$.

  (B) Compute the resulting equation at $t = 0$ with equality to 0 such that $D^n_{0+}^{(n-1)\alpha} \text{Res}^n_i(t)|_{t=0} = 0$, with the help of $D^n_{0+} \rho^q = 0$ for $q > \alpha$ at $t = 0$.

  (C) Find the $n$th unknown coefficients $a_{i,n}$ and do Step 7 for $n = n + 1$ until the arbitrary $k$.

- **Step 8:** Substitute the values of $a_{i,n}$ back into $n$th-truncated series in Step 2. Then, collect the obtained approximation solutions $x^n_i(t)$ and try to find a general pattern with the term of infinite series so that the exact solution $\omega(t)$ of FBTEs (1) and (2) is obtained, then STOP.

4. Numerical Experiments

In this section, some illustrative examples are performed to demonstrate the efficiency and superiority of the RPS algorithm. All computations are done using Wolfram Mathematica 10.0 software package (Wolfram Research, Inc.: Champaign, IL, USA) [30].
Example 1. We consider the following FBTE [31]:

\[ \alpha^{\nu}(t) + D_{0+}^{3/2} \omega(t) + \omega(t) = \varphi(t), \quad t > 0, \]

with the initial conditions

\[ \omega(0) = 0, \quad \omega'(0) = 0 \]

where the forcing term is

\[ \varphi(t) = \varphi(v-1)t^{n-2} + b \frac{\Gamma(v+1)}{\Gamma(v-1/2)} t^{n-3/2} + ct^v, \]

and the exact solution is \( \omega(t) = t^2 \).

This model is a special case of FBTE that arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [32]. To apply the proposed algorithm, we have to solve the equivalent system by letting \( \omega(t) = x_1(t), D_{0+}^{\alpha} x_1(t) = x_2(t), D_{0+}^{\alpha} x_2(t) = x_3(t), D_{0+}^{\alpha} x_3(t) = x_4(t), \) and \( D_{0+}^{\alpha} x_4(t) + x_4(t) + x_1(t) = \varphi(t) \) subject to the initial conditions \( x_i(0) = 0, \quad i = 1, 2, 3, 4, \) where \( 0 < \alpha \leq 1/2 \), and the \( n \)th-truncated series is

\[ x_i^n(t) = \sum_{k=1}^{n} a_{ik} \frac{t^k}{\Gamma(k\alpha + 1)}, i = 1, 2, 3, 4. \]

For numerical considerations, choose \( v = 2 \) such that the corresponding forcing term of Equation (13) is \( \varphi(t) = 2 + 4 \sqrt{\frac{t}{\pi}} + t^2 \). In this sense, the equivalent system can be given by

\[
\begin{align*}
D_{0+}^{\alpha} x_1(t) & = x_2(t), \\
D_{0+}^{\alpha} x_2(t) & = x_3(t), \\
D_{0+}^{\alpha} x_3(t) & = x_4(t), \\
D_{0+}^{\alpha} x_4(t) + x_4(t) + x_1(t) & = 2 + 4 \sqrt{\frac{t}{\pi}} + t^2,
\end{align*}
\]

whereas the \( n \)th-residual functions are

\[ \text{Res}_n x_1^n(t) = D_{0+}^{\alpha} x_1^n(t) - x_2^n(t), \quad \text{Res}_n x_2^n(t) = D_{0+}^{\alpha} x_2^n(t) - x_3^n(t), \quad \text{Res}_n x_3^n(t) = D_{0+}^{\alpha} x_3^n(t) - x_4^n(t), \quad \text{Res}_n x_4^n(t) = D_{0+}^{\alpha} x_4^n(t) + x_4^n(t) + x_1^n(t) - \left(2 + 4 \sqrt{\frac{t}{\pi}} + t^2\right). \]

Thus, using the procedures of the RPS algorithm [25–28], the 4th RPS approximate solution of FBTEs (13) and (14) can be given by

\[ \omega_4(t) = \frac{2^k}{\Gamma(k\alpha + 1)}. \]

Consequently, the RPS solution at \( \alpha = 1/2 \) will be \( \omega(t) = t^2 \), which is fully compatible with the exact solution investigated earlier in [32].

To show the accuracy of the method, some numerical results of the RPS solutions are given for inputs between 0 and 1 with a step of 0.1 in Table 1, which displays the comparison between the results obtained by RPS with those obtained by the application of reproducing kernel algorithm (RKA) [33], variational iteration method (VIM) [34], genetic algorithm method (GAM), pattern search technique (PST), and Podlubny matrix approach (PMA) that developed in [35]. From this table, it can be observed that the results obtained by the RPS approach correspond well to those obtained in [33–35]. Figure 1 shows the behavior of the exact and RPS solutions for different values of \( \alpha \), where \( \alpha \in [0.5, 0.45, 0.4, 0.35, 0.3] \) and \( t \in [0,1] \). The RPS solutions are in good agreement with each other and with the exact solution.
will be vanished. In particular, the RPS approximate solution at $t = 0$, which is fully compatible with the exact solution investigated earlier in [29].

Example 2. We consider the following special case of FBTE [29,31]:

$$\omega''(t) + D^{3/2}_{0} \omega(t) + \omega(t) = t + 1, \ 0 < t < 1, \ (15)$$

with the initial conditions

$$\omega(0) = 1, \ \omega'(0) = 1. \ (16)$$

The values of the assuming parameter in this example are $A = B = C = 1$, where the exact solution is given as $\omega(t) = t + 1$.

The RPS approximate solution of IVP (15) and (16) can be written as follows:

$$\omega(t) = \sum_{k=0}^{\infty} a_k \frac{t^{ka}}{\Gamma(ka + 1)},$$

with the assumptions $\omega(t) = x_1(t)$, $D^{\alpha}_{0} x_1(t) = x_2(t)$, $D^{\alpha}_{0} x_2(t) = x_3(t)$, $D^{\alpha}_{0} x_3(t) = x_4(t)$ and $D^{\alpha}_{0} x_4(t) + x_4(t) + x_1(t) = t + 1$ subject to $x_1(0) = 1, x_2(0) = 0, x_3(0) = 1$ and $x_4(0) = 0$, where $0 < \alpha \leq 1/2$. According to the RPS method, the 5th approximate solution of FBTEs (14) and (15) is $\omega_5(t) = 1 + \frac{t^\alpha}{\Gamma(2a+1)}$. If we keep the repeating of the RPS process, the unknown coefficients $a_k$ for $k \geq 6$ will be vanished. In particular, the RPS approximate solution at $\alpha = 1/2$ is $\omega(t) = 1 + t$, which is fully compatible with the exact solution investigated earlier in [29].

To show the accuracy of the RPS algorithm, numerical results of the 5th approximate solutions are given in Table 2 for inputs $t$ between 0 and 1 with a step size of 0.1. Table 2 displays the comparison between the results obtained by the RPS algorithm with those obtained by the application.

| $t$ | $\omega(t)$ | RPS | RKA | VIM | PMA | PSA | GAM |
|-----|-------------|-----|-----|-----|-----|-----|-----|
| 0.0 | 0.00        | 0.00| 0.00| 0.00| 0.00| 0.027300| 0.06730 |
| 0.1 | 0.01        | 0.01| 0.01| 0.01| 0.01005| 0.012392| 0.08925 |
| 0.2 | 0.04        | 0.04| 0.04| 0.04| 0.04063| 0.03892| 0.049646| 0.12802 |
| 0.3 | 0.09        | 0.09| 0.09| 0.09| 0.09266| 0.08873| 0.111143| 0.18375 |
| 0.4 | 0.16        | 0.16| 0.16| 0.16| 0.16748| 0.15867| 0.196074| 0.25676 |
| 0.5 | 0.25        | 0.25| 0.25| 0.25| 0.26679| 0.24871| 0.303374| 0.34756 |
| 0.6 | 0.36        | 0.36| 0.36| 0.36| 0.39277| 0.35883| 0.431620| 0.45684 |
| 0.7 | 0.49        | 0.49| 0.49| 0.49| 0.54806| 0.48903| 0.578880| 0.58548 |
| 0.8 | 0.64        | 0.64| 0.64| 0.64| 0.73588| 0.63928| 0.742531| 0.73455 |
| 0.9 | 0.81        | 0.81| 0.81| 0.81| 0.96007| 0.80958| 0.919015| 0.90530 |
| 1.0 | 1.00        | 1.00| 1.00| 1.00| 1.22519| 0.99993| 1.103535| 1.09922 |

Figure 1. Plots of exact and residual power series (RPS) approximate solutions at different values of $\alpha$: Pink-Dashed $\alpha = 0.3$; Green-Dashed $\alpha = 0.35$; Gray-Dashed $\alpha = 0.4$; Orange-Dashed $\alpha = 0.45$; Red-Dashed $\alpha = 0.5$; and Blue is exact.
of reproducing kernel algorithm (RKA) [33], genetic algorithm method (GAM), Pattern search technique (PST), and GAM hybrid with PST (GA-PS) [35]. The results of this table illustrate that the result obtained by our scheme is in good agreement with the state-of-the-art numerical solvers.

Table 2. Numerical comparison of the approximate solution of Example 2.

| t   | ω(t) | RPS | RKA | PSA | GAM | GA-PS |
|-----|------|-----|-----|-----|-----|-------|
| 0.0 | 1.00 | 1.00| 1.00| 0.691604 | 1.024862 | 1.016007 |
| 0.1 | 1.01 | 1.01| 1.01| 0.623749 | 1.121206 | 1.109804 |
| 0.2 | 1.04 | 1.04| 1.04| 0.859697 | 1.220821 | 1.199804 |
| 0.3 | 1.09 | 1.09| 1.09| 1.122266 | 1.323041 | 1.299333 |
| 0.4 | 1.16 | 1.16| 1.16| 1.337736 | 1.426952 | 1.404972 |
| 0.5 | 1.25 | 1.25| 1.25| 1.501839 | 1.531330 | 1.507972 |
| 0.6 | 1.36 | 1.36| 1.36| 1.628907 | 1.634569 | 1.607429 |
| 0.7 | 1.49 | 1.49| 1.49| 1.734458 | 1.744591 | 1.706705 |
| 0.8 | 1.64 | 1.64| 1.64| 1.830520 | 1.828738 | 1.799987 |
| 0.9 | 1.81 | 1.81| 1.81| 1.925308 | 1.913640 | 1.883785 |
| 1.0 | 2.00 | 2.00| 2.00| 2.024099 | 1.985057 | 1.953762 |

Example 3. We consider the following homogeneous FBTE [36]:

\[
\omega''(t) + \beta \sqrt{\pi} D_{0+}^{3/2} \omega(t) + \omega(t) = 0, \quad t \geq 0, \tag{17}
\]

with the initial conditions

\[
\omega(0) = 1, \quad \omega'(0) = 0, \tag{18}
\]

where \( \beta \) is a parameter.

This fractional model was developed to design a highly accurate microelectromechanical instrument for measuring the viscosity of liquids encountered during oil drilling. The exact solution reduced to \( \omega(t) = \cos(t) \) as soon as \( \beta \rightarrow 0 \).

The RPS solution of IVP (17) and (18) can be written as follows:

\[
\omega(t) = \sum_{k=0}^{\infty} a_k \frac{t^k t}{(k\alpha + 1)}, \quad 0 < \alpha \leq 1/2,
\]

with the assumptions \( \omega(t) = x_1(t), \ D_{0+}^\alpha x_1(t) = x_2(t), \ D_{0+}^\alpha x_2(t) = x_3(t), \ D_{0+}^\alpha x_3(t) = x_4(t) \) and \( D_{0+}^\alpha x_4(t) + \beta \sqrt{\pi} x_4(t) + x_1(t) = 0 \) subject to \( x_1(0) = 1, \ x_2(0) = x_3(0) = x_4(0) = 0 \), where \( 0 < \alpha \leq 1/2 \). Thus, according to the RPS algorithm, the 5th RPS approximate solution of FBTEs (17) and (18) is \( \omega_5(t) = 1 - \frac{t^{\alpha}}{\Gamma(2\alpha + 1)} + \beta \sqrt{\pi} \frac{t^{3\alpha}}{\Gamma(5\alpha + 1)} \). In particular, for \( \alpha = 1/2 \) and \( \beta = 0 \), we have \( \omega_5(t) = 1 - \frac{1}{2} t^2 \).

Absolute errors of the 5th approximate solution for FPTE (17) and (18) are computed for \( \alpha = 1/2 \), with selected nods of \( t \) with step size 0.16 and summarized in Table 3, while Table 4 shows the numerical results of the RPS algorithm and exponential integrators method (EIM) [36] for the parameter value \( \beta = 0.2 \) and different values of \( t \) in \( [0, 5] \).
Table 3. Numerical results at \( \alpha = 1/2 \) for Example 3.

| \( t \) | Exact \( RPS \) | Absolute Error |
|---|---|---|
| 0.16 | 0.9872272834 | 0.9872 | \( 2.72834 \times 10^{-5} \) |
| 0.32 | 0.9492354181 | 0.9488 | \( 4.35418 \times 10^{-4} \) |
| 0.48 | 0.8869949228 | 0.8848 | \( 2.19492 \times 10^{-3} \) |
| 0.64 | 0.8020957579 | 0.7952 | \( 6.89576 \times 10^{-3} \) |
| 0.80 | 0.6967067094 | 0.6800 | \( 1.67067 \times 10^{-2} \) |
| 0.96 | 0.5735199861 | 0.5392 | \( 3.4320 \times 10^{-2} \) |

Table 4. Comparison of the results with \( \beta = 0.2 \) of Example 3.

| \( t \) | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| RPS | 0.6100 | −0.3966 | −0.5372 |
| EIM | 0.6188 | −0.1398 | −0.2436 |

Example 4. We consider the following FBTE \([37,38]\):

\[
\omega''(t) + \frac{1}{2} D^{3/2}_0 \omega(t) + \frac{1}{2} \omega(t) = \varphi(t), \quad t \geq 0, \tag{19}
\]

with the initial conditions

\[
\omega(0) = 1, \quad \omega'(0) = 0, \tag{20}
\]

where

\[
\varphi(t) = \begin{cases} 
8, & 0 \leq t \leq 1, \\
0, & t > 1.
\end{cases}
\]

Here, the values of the assuming parameter are \( B = C = \frac{1}{2} \), and the exact solution is given by

\[
\omega(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} t^{k+1/2}}{2k!} E_{1/2,3k/2+2}^{(k)} \left( -\frac{\sqrt{t}}{2} \right),
\]

where \( E_{\nu,\delta}^{(k)}(t) \) is the Mittag–Leffler function of the two parameters.

This fractional model is the most popular case of FBTE, which was developed to design a highly accurate microelectromechanical instrument for measuring the viscosity of fluids encountered during oil well exploration.

Anyhow, the RPS solution of IVP (19) and (20) can be written as follows:

\[
\omega(t) = \sum_{k=0}^{\infty} a_k t^{\frac{k\alpha}{\Gamma(k\alpha+1)}},
\]

with the assumptions \( \omega(t) = x_1(t), \ D^\alpha_0 x_1(t) = x_2(t), \ D^\alpha_0 x_2(t) = x_3(t), \ D^\alpha_0 x_3(t) = x_4(t) \) and \( D^\alpha_0 x_4(t) + \frac{1}{2} x_4(t) + \frac{1}{2} x_1(t) = \varphi(t) \) subject to \( x_1(0) = 1 \) and \( x_2(0) = x_3(0) = x_4(0) = 0 \), where \( 0 < \alpha \leq 1/2 \). Thus, according to the RPS algorithm, the 5th approximate solution of FBTEs (19) and (20) is given as \( \omega_5(t) = 1 - \left( \varphi(0) - \frac{1}{2} \right) t^{\frac{\alpha^2}{\Gamma(4\alpha+1)}} + \left( \frac{1}{4} - \frac{1}{2} \varphi(0) \right) t^{\frac{\alpha^2}{\Gamma(5\alpha+1)}} \).

The resulting values of the RPS algorithm and some numerical methods, including the Fermat Tau method (FTM) \([38]\), the generalized Taylor method (GTM) \([36]\), and the fractional Taylor method (FrTM) \([37]\), for inputs \( t \) between 0 and 1 with a step of 0.1, are given in Table 5. From this table, it can be illustrated that the result obtained by our scheme is in good agreement with the state-of-the-art numerical solvers.
Table 5. Numerical comparison of the approximate solutions of Example 4.

| $\omega(t)$ | RPS | FTM [38] | GTM [36] | FrTM [37] |
|-------------|-----|----------|----------|-----------|
| 0.0         | 0.0 | 0.0      | 0.0      | 0.0       |
| 0.1         | 0.036487479 | 0.036487480 | 0.036487479 | 0.036485547 | 0.036487480 |
| 0.2         | 0.140639621 | 0.140639621 | 0.140639621 | 0.140634716 | 0.140639621 |
| 0.3         | 0.307484627 | 0.307484627 | 0.307484627 | 0.307476229 | 0.307484627 |
| 0.4         | 0.533284110 | 0.533284111 | 0.533284110 | 0.533271294 | 0.533284110 |
| 0.5         | 0.814756950 | 0.814756951 | 0.814756950 | 0.814735609 | 0.814756949 |
| 0.6         | 1.148837428 | 1.148837423 | 1.148837428 | 1.148805808 | 1.148837422 |
| 0.7         | 1.532565443 | 1.532565438 | 1.532565443 | 1.532521264 | 1.532565426 |
| 0.8         | 1.963029298 | 1.963029281 | 1.963029298 | 1.962974991 | 1.963029255 |
| 0.9         | 2.437334072 | 2.437334073 | 2.437334072 | 2.437455982 | 2.437333971 |
| 1.0         | 2.952584099 | 2.952584089 | 2.952584099 | 2.954070000 | 2.952583880 |

5. Conclusions

In this work, the RPS algorithm within a developed strategy has been successfully applied to provide the approximate solution of fractional Bagley-Torvik equation arising in fluid mechanics subjected to suitable initial conditions. The solution methodology depends on constructing a fractional power series form and deriving its residual error function under the meaning of Caputo. An efficacious experiment is implemented to verify the validity and reliability of the RPS algorithm. The numerical results indicated the simultaneous behavior between the exact solution and RPS approximate solution at different values of $\alpha$, which are also in good agreement with those obtained by other existing methods. Furthermore, in terms of accuracy and simplicity, it can be concluded that the RPS algorithm is straightforward, systematic, and can be applicable to a wide class of fractional models occurring in physics, engineering, and applied sciences.

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