Average cost optimal control under weak ergodicity hypotheses: Relative value iterations

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Abstract. We study Markov decision processes with Polish state and action spaces. The action space is state dependent and is not necessarily compact. We first establish the existence of an optimal ergodic occupation measure using only a near-monotone hypothesis on the running cost. Then we study the well-posedness of Bellman equation, or what is commonly known as the average cost optimality equation, under the additional hypothesis of the existence of a small set. We deviate from the usual approach which is based on the vanishing discount method and instead map the problem to an equivalent one for a controlled split chain. We employ a stochastic representation of the Poisson equation to derive the Bellman equation. Next, under suitable assumptions, we establish convergence results for the ‘relative value iteration’ algorithm which computes the solution of the Bellman equation recursively. In addition, we present some results concerning the stability and asymptotic optimality of the associated rolling horizon policies.

1. Introduction

The long run average or ‘ergodic’ cost is popular in applications when transients are fast and/or unimportant and one is optimizing over possible asymptotic behaviors. The dynamic programming equation for this criterion, in the finite state-action case, goes back to Howard [29]. A recursive algorithm to solve it in the aforementioned case is the so called relative value iteration scheme [42], dubbed so because it is a modification of the value iteration scheme for the (simpler) discounted cost criterion. This modification consists of subtracting at each step a suitable offset and track only the ‘relative’ values. Suitable counterparts of this algorithm for a general state space are available, if at all, only under rather strong conditions (see, e.g., Section 5.6 of [28]). Our aim here is to consider a special case of immense practical importance, viz., that of a near-monotone or inf-compact cost which penalizes instability [7], [8], and to establish both the dynamic programming equation and the relative value iteration scheme for it. Perforce the latter involves iteration in a function space and as far as implementation is concerned, would have to be replaced by suitable finite approximations through either state aggregation or parametrized approximation of the value function. But the validity of such an approximate scheme depends on provable convergence properties of the algorithm. Our aim is to provide this.

The results on convergence of the relative value iteration presented here may be viewed as discrete time counterparts of the results of [3]. It is not, however, the case that they can be derived simply from the results of [3], which relies heavily on the analytic machinery of the partial differential equations arising therein. This, in particular, leads to convenient regularity results which are not available here.

For studies on the average cost optimality equation (ACOE) of Markov decision processes (MDP) on Borel state space, we refer the reader to [14, 19–21, 27, 28, 30, 37, 40, 41]. All these papers assume only the (weak) Feller property on the transition kernel, whereas in this paper the kernel is assumed

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2000 Mathematics Subject Classification. Primary: 90C40, Secondary: 93E20.
Key words and phrases. ergodic control, Bellman equation, inf-compact cost, relative value iteration.
to be strong Feller (with the exception of Lemma 2.1 and Theorem 2.1). Classical approaches based on the vanishing discount argument such as [14, 21, 27, 37] need to ensure some variant of pointwise boundedness of the relative discounted value function. This typically requires additional hypotheses, or it is directly imposed as an assumption. The weakest condition appears in [21] where only the limit infimum of relative discounted value functions is required to be pointwise bounded in the vanishing discount limit. It follows from [23, Theorem 4.1] that if the solution of the ACOE is bounded then the relative discounted value functions are also bounded uniformly in the discount factor. This is a very specific case though, and for the more general case studied in this paper it is unclear how our assumptions compare with those of [21].

Pointwise boundedness of discounted relative value functions was verified from scratch for a specific application in [1]. The techniques therein, which leverage near-monotonicity in a manner different from here, may be more generally applicable.

Some of the aforementioned works derive an average cost optimality inequality (ACOI) as opposed to an equation. The ACOE is derived in [14, 19, 20, 28, 30, 40, 41]. Also worth noting is [20] which derives the ACOE for a classical inventory problem under a weak condition known as $K$-inf-compactness. The works in [30, 41] derive the ACOE under additional uniform stability conditions which we avoid. The works [40, 41] also use a minorization condition like us, but the purpose there is to facilitate a fixed point argument which is possible due to the stronger stability assumptions.

Our focus is on the ACOE rather than the ACOI because our eventual aim is to establish convergence of relative value iteration for which this is explicitly used. Moreover, for this convergence result we require uniqueness of the solution to the ACOE within a suitable class of functions.

Studies such as [21, 27, 37] work with standard Borel state spaces whereas we work with Polish spaces. We assume that the running cost is near-monotone (see (C) in Subsection 2.2), a notion more general than the more commonly used ‘inf-compactness’. The latter requires the level sets of the running cost functions to be compact, necessitating in particular that for non-$\sigma$-compact spaces, they be extended real-valued. On the contrary, a $K$-inf-compact cost (see (A1) in Subsection 3.1) together with (C) allows for more flexibility.

Furthermore, the above works do not address the relative value iteration which is our main focus here. This algorithm, after the seminal work of [42] for the finite state case, has been extended to denumerable state spaces in [10, 11, 13]. An analogous treatment for a general metric state space appears in [28, Section 5.6]. This directly assumes equicontinuity of the iterates, for which problems with convex value functions [24] have been cited as an example. We do not make any such assumption. The related though distinct algorithm of policy iteration has been analyzed in [33]. This work also uses the ‘pseudo-atom’ construction as we do, in order to obtain a solution to the fixed policy Poisson equation. We use it to derive the Bellman equation itself using a representation of the value function.

Another important part of this work concerns the stability and asymptotic optimality of rolling horizon policies. Analogous results in the literature have been reported only under very strong blanket ergodicity assumptions [11, 26]. For a review of this topic, see [15]. In this paper, we avoid any blanket ergodicity assumptions and impose a stabilizability hypothesis, namely, that under some Markov control the process is geometrically ergodic with a Lyapunov function that has the same growth as the running cost (see (H2) and Remark 6.1 in Section 6). This property is natural for ‘linear-like’ problems, and is also manifested in queueing problems with abandonment, or problems with the structure in Example 6.1. Under this hypothesis, we assert in Theorems 6.1 and 6.3, global convergence for the relative value iteration, and show in Theorem 6.2 that the rolling horizon procedure is stabilizing after a finite number of iterations. Then, under a ‘uniform’ $\psi$-irreducibility condition, Theorem 6.4 shows that the rolling horizon procedure is asymptotically optimal. The latter is an important problem of current interest (see, e.g., [12, 28]). Our results also contain computable error bounds.
The article is organized as follows. Section 2 has three main parts. We first review the formalism and basic notation of Markov decision processes in Subsection 2.1 and then, in Subsection 2.2, we establish the existence of an optimal ergodic occupation measure, thus extending the results of the convex analytic framework in [8] to MDPs on a Polish state space. Subsection 2.4 introduces an equivalent controlled split chain and the associated pseudo-atom. Section 3 then derives the dynamic programming equation to characterize optimality, extending the approach of [7] for countable state space - compact action space case. Section 4 establishes the convergence of the ‘value iteration’, which is the name we give to the analog of value iteration for discounted cost with no discounting, but with the cost-per-stage function modified by subtracting from it the optimal cost. The latter is in principle unknown, so this is not a legitimate algorithm. It does, however, pave the way to prove convergence of the true relative value iteration scheme, which we do in Section 5. Section 6 is devoted to the analysis of the rolling horizon procedure.

1.1. Notation. We summarize some notation used throughout the paper. We use $\mathbb{R}^d$ (and $\mathbb{R}_+^d$), $d \geq 1$, to denote the space of real-valued $d$-dimensional (nonnegative) vectors, and write $\mathbb{R}$ for $d = 1$. Also, $\mathbb{N}$ denotes the natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $x, y \in \mathbb{R}$, we let

$$x \vee y := \max\{x, y\} \quad \text{and} \quad x \wedge y := \min\{x, y\}.$$  

The Euclidean norm on $\mathbb{R}^d$ is denoted by $| \cdot |$. We use $A^c$, $\tilde{A}$, and $\mathbb{I}_A$ to denote the complement, the closure, and the indicator function of a set $A$, respectively.

For a Polish space $X$ we let $\mathcal{B}(X)$ stand for its Borel $\sigma$-algebra, and $\mathfrak{P}(X)$ for the space of probability measures on $\mathcal{B}(X)$ with the Prokhorov topology. We let $\mathcal{M}(X)$, $\mathcal{L}(X)$, and $C(X)$, denote the spaces of real-valued Borel measurable functions, lower semi-continuous functions bounded from below, and continuous functions on $X$, respectively. Also, $\mathcal{M}_b(X)$, $\mathcal{L}_b(X)$, and $C_b(X)$, denote the corresponding subspaces consisting of bounded functions.

For a Borel probability measure $\mu$ on $\mathcal{B}(X)$ and a measurable function $f : X \to \mathbb{R}$, which is integrable under $\mu$, we often use the convenient notation $\mu(f) := \int_X f(x) \mu(dx)$.

For $f \in \mathcal{M}(X)$, we define

$$\|g\|_f := \sup_{x \in X} \frac{|g(x)|}{1 + |f(x)|}, \quad g \in \mathcal{M}(X),$$

and $O(f) := \{g \in \mathcal{M}(X) : \|g\|_f < \infty\}$.

1.2. Assumptions. In this subsection, we outline the various assumptions used in this article. These have been introduced closer to their first use and after the relevant notation is in place. Not all of them are required for everything.

Assumption 2.1 introduced in Subsection 2.1 is the basic assumption regarding the minimal regularity hypothesis about the transition kernel, the set-valued map specifying available controls at each state, and the cost function. This is assumed throughout this work.

Assumption (C) in Subsection 2.2 refines further the assumption on the cost function. This too holds throughout and is first used in Theorem 2.1.

Assumption (A0) in Subsection 2.4 is an adaptation of the standard ‘minorization’ condition for the construction of the Athreya-Ney-Nummelin pseudo-atom for our purposes and facilitates the derivation of the Poisson equation for the split chain in this section. (A1) and (A2) in Subsection 3.1 are additional assumptions for the passage from the Poisson equation to the Bellman equation in Theorem 3.1.

Assumption A3.1 in Subsection 3.3 strengthens our regularity requirement on the controlled transition kernel, from weak feller to strong Feller. This is required in the derivation of the Bellman equation and is therefore operative throughout the rest of the article. It also plays a role in the subsequent analysis of the relative value iteration algorithm.
One of our main results is the convergence of the relative value iteration to solve the Bellman equation. This requires the additional assumption (H1) in Subsection 4.2, which proves the convergence of value iteration (Lemma 4.1, Theorem 4.1) assuming the optimal cost to be known. Convergence of relative value iteration in Theorem 5.1 of Section 5 follows from this. (Part (c) of the theorem invokes (H2) from the subsequent section, but parts (a), (b) do not require it.)

Assumption (H2) and its equivalent statement (H2’) are used to justify the rolling horizon procedure in Theorem 6.1, Theorem 6.2, which requires stronger conditions. The corresponding convergence result for optimal policies in Theorem 6.3 needs additional conditions that are embedded in the statement of the theorem itself.

2. Preliminaries

In this paper, we consider a controlled Markov chain otherwise referred to as a Markov decision process (MDP), taking values in a Polish space \( \mathbb{X} \).

2.1. The MDP model. Recall the notation introduced in Subsection 1.1. According to the most prevalent definition in the literature (see [17, 28]), an MDP is represented as a tuple \((\mathbb{X}, \mathbb{U}, \mathcal{U}, \mathbb{P}, c)\), whose elements can be described as follows.

(a) The state space \( \mathbb{X} \) is a Polish space (complete, separable, metric). Its elements are called states.

(b) \( \mathbb{U} \) is a Polish space, referred to as the action or control space.

(c) The map \( \mathcal{U} : \mathbb{X} \to \mathcal{B}(\mathbb{U}) \) is a strict, measurable multifunction. The set of admissible state/action pairs is defined as

\[
\mathbb{K} := \{(x, u) : x \in \mathbb{X}, u \in \mathcal{U}(x)\},
\]

endowed with the relative topology corresponding to \( \mathbb{X} \times \mathbb{U} \).

(d) The transition probability \( \mathbb{P}(\cdot | x, u) \) is a stochastic kernel on \( \mathbb{K} \times \mathcal{B}(\mathbb{X}) \), that is, \( \mathbb{P}(\cdot | x, u) \) is a probability measure on \( \mathcal{B}(\mathbb{X}) \) for each \((x, u) \in \mathbb{K}\), and \((x, u) \mapsto \mathbb{P}(A | x, u)\) is in \( \mathcal{M}(\mathbb{K}) \) for each \( A \in \mathcal{B}(\mathbb{X}) \).

(e) The map \( c : \mathbb{K} \to \mathbb{R} \) is measurable, and is called the running cost or one stage cost. We assume that it is bounded from below in \( \mathbb{K} \), so without loss of generality, it takes values in \([1, \infty]\).

The (admissible) history spaces are defined as

\[
\mathbb{H}_0 := \mathbb{X}, \quad \mathbb{H}_t := \mathbb{K}^{t-1} \times \mathbb{X}, \quad t \in \mathbb{N},
\]

and the canonical sample space is defined as \( \Omega := (\mathbb{X} \times \mathbb{U})^\infty \). These spaces are endowed with their respective product topologies and are therefore Polish spaces. The state, action (or control), and information processes, denoted by \( \{X_t\}_{t \in \mathbb{N}_0}, \{U_t\}_{t \in \mathbb{N}_0} \) and \( \{H_t\}_{t \in \mathbb{N}_0} \), respectively, are defined by the projections

\[
X_t(\omega) := x_t, \quad U_t(\omega) := u_t, \quad H_t(\omega) := (x_0, \ldots, x_{t-1}, u_t, \ldots) \in \Omega.
\]

An admissible control strategy, or policy, is a sequence \( \xi = \{\xi_t\}_{t \in \mathbb{N}_0} \) of stochastic kernels on \( \mathbb{H}_t \times \mathcal{B}(\mathbb{U}) \) satisfying the constraint

\[
\xi_t(U(x_t) | h_t) = 1, \quad x_t \in \mathbb{X}, \ h_t \in \mathbb{H}_t.
\]

The set of all admissible strategies is denoted by \( \mathcal{U} \). It is well known (see [34, Prop. V.1.1, pp. 162–164]) that for any given \( \mu \in \mathfrak{P}(\mathbb{X}) \) and \( \xi \in \mathcal{U} \) there exists a unique probability measure \( \mathbb{P}_\xi^\mu \) on \((\Omega, \mathcal{B}(\Omega))\) satisfying

\[
\mathbb{P}_\xi^\mu(X_0 \in D) = \mu(D), \quad \forall \ D \in \mathcal{B}(\mathbb{X}),
\]

\[
\mathbb{P}_\xi^\mu(U_t \in C \mid H_t) = \xi_t(C \mid H_t) \quad \mathbb{P}_\xi^\mu \text{-a.s.}, \quad \forall C \in \mathcal{B}(\mathbb{U}),
\]

\[
\mathbb{P}_\xi^\mu(U_t \in C \mid H_t) = \xi_t(C \mid H_t) \quad \mathbb{P}_\xi^\mu \text{-a.s.}, \quad \forall C \in \mathcal{B}(\mathbb{U}),
\]
\[ P^\xi_\mu (X_{t+1} \in D \mid H_t, U_t) = P(D \mid X_t, U_t) \quad \mathbb{P}^\xi_\mu \text{-a.s.}, \quad \forall D \in \mathcal{B}(X). \]

The expectation operator corresponding to \( \mathbb{P}^\xi_\mu \) is denoted by \( \mathbb{E}^\xi_\mu \). If \( \mu \) is a Dirac mass at \( x \in X \), we simply write these as \( P^\xi_\mu \) and \( \mathbb{E}^\xi_\mu \).

A strategy \( \xi \) is called randomized Markov if there exists a sequence of measurable maps \( \{v_t\}_{t \in \mathbb{N}_0} \), where \( v_t : X \to \mathcal{P}(U) \) for each \( t \in \mathbb{N}_0 \), such that
\[
\xi_t(\cdot \mid H_t) = v_t(X_t)(\cdot) \quad \mathbb{P}^\xi_\mu \text{-a.s.}
\]

With some abuse of notation, such a strategy is identified with the sequence \( v = \{v_t\}_{t \in \mathbb{N}_0} \). Note then that \( v_t \) may be written as a stochastic kernel \( v_t(x) \) on \( X \times \mathcal{B}(U) \) which satisfies \( v_t(U(x) \mid x) = 1 \).

We say that a Markov randomized strategy \( \xi \) is simple, or precise, if \( \xi_t \) is a Dirac mass, in which case \( v_t \) is identified with a Borel measurable function \( v_t : X \to U \). In other words, \( v_t \) is a measurable selector from the set-valued map \( U(x) \) [22].

We add the adjective stationary to indicate that the strategy does not depend on \( t \in \mathbb{N}_0 \), that is, \( v_t = v \) for all \( t \in \mathbb{N}_0 \). We let \( \mathfrak{U}_{sm} \) denote the class of stationary Markov randomized strategies, henceforth referred to simply as stationary strategies.

The basic structural hypotheses on the model, which are assumed throughout the paper, are as follows.

**Assumption 2.1.** The following hold:

(i) The transition probability \( P(dy \mid x, u) \) is weakly continuous, that is, the map
\[
(x, u) \mapsto H_f(x, u) := \int_X f(y) P(dy \mid x, u)
\]

is continuous for every \( f \in C_b(X) \).

(ii) The set-valued map \( U : X \to \mathcal{B}(U) \) is upper semi-continuous and closed-valued.

(iii) The running cost \( c : K \to [1, \infty] \) is lower semi-continuous.

**Assumption 2.1** is assumed throughout the paper, and repeated only for emphasis. More specific assumptions are imposed later in Subsection 2.4 and Section 3.

**Definition 2.1.** For \( v \in \mathfrak{U}_{sm} \) we use the abbreviated notation
\[
P_v(A \mid x) := \int_{U(x)} P(A \mid x, u) v(du \mid x), \quad \text{and} \quad c_v(x) := \int_{U(x)} c(x, u) v(du \mid x).
\]

Also, \( P_v f := \int_X f(y) P_v(dy \mid x) \) for a function \( f \in \mathcal{M}(X) \), assuming that the integral is well defined. Similarly, we write \( P_u(A \mid x) := P(A \mid x, u) \) for \( u \in U(x) \), and define \( P_u f \) analogously. When needed to avoid ambiguity, we denote the chain controlled under \( v \) as \( \{X_n\}_{n \in \mathbb{N}_0} \).

2.1.1. **Control objective.** The control objective is to minimize over all admissible \( \xi = \{\xi_n\}_{n \in \mathbb{N}_0} \) the average (or ‘ergodic’) cost
\[
\mathcal{E}(\mu, \xi) := \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}^\xi_\mu \left[ \sum_{n=0}^{N-1} c(X_n, U_n) \right], \quad \mu \in \mathfrak{P}(X), \; \xi \in \mathfrak{U}.
\]

We let
\[
J(\mu) := \inf_{\xi \in \mathfrak{U}} \mathcal{E}(\mu, \xi), \quad \text{and} \quad \beta := \inf_{\mu \in \mathfrak{P}(X)} J(\mu).
\]

We say that an admissible strategy \( \xi \) is optimal if \( \mathcal{E}(\mu, \xi) = J(\mu) \) for all \( \mu \in \mathfrak{P}(X) \). The class of Markov stationary strategies that are optimal is denoted by \( \mathfrak{U}_{sm}^* \).

In the next section we introduce the concept of an optimal ergodic occupation measure, and assume that, under a near-monotone type hypothesis on the running cost, such a measure exists. We use this result in Section 3 to derive a solution to the Bellman equation.
2.2. Existence of an optimal ergodic occupation measure. Recall that \( \zeta \in \Psi(K) \) is called an ergodic occupation measure if it satisfies

\[
\int_K \left( f(x) - \int_X f(y) P(dy \mid x, u) \right) \zeta(dx, du) = 0 \quad \forall f \in C_b(X).
\]

We let \( \mathcal{M}_{\text{erg}} \) stand for the class of ergodic occupation measures. Any \( \zeta \in \mathcal{M}_{\text{erg}} \) can be disintegrated as

\[
\zeta(dx, du) = \pi_\zeta(dx) v_\zeta(du \mid x) \quad \text{\( \pi \)-a.s.,}
\]

where \( \pi_\zeta \in \Psi(X) \) and \( v_\zeta \) is a stochastic kernel on \( X \times \mathcal{B}(U) \) which satisfies \( v_\zeta(U(x) \mid x) = 1 \). We denote this disintegration as \( \zeta = \pi_\zeta \ast v_\zeta \).

**Remark 2.1.** Note that (2.2) does not define \( v_\zeta \) on the entire space, and thus \( v_\zeta \) cannot be viewed as an element of \( \mathcal{U}_{\text{sm}} \). However, if \( v \in \mathcal{U}_{\text{sm}} \) is any strategy that agrees \( \pi_\zeta \)-a.e. with \( v_\zeta \), then \( \pi_\zeta(\cdot) = \int_X P_v(\cdot \mid x) \pi_\zeta(dx) \), or, in other words, \( \pi_\zeta \) is an invariant probability measure for the chain controlled under \( v \). Note that such a strategy can be easily constructed. For example, for arbitrary \( v_0 \in \mathcal{U}_{\text{sm}} \), we can define \( v = v_\zeta \) on the support of \( \pi_\zeta \) and \( v = v_0 \) on its complement.

**Definition 2.2.** We say that \( \zeta^* \in \mathcal{M}_{\text{erg}} \) is optimal if

\[
\int_K c \, d\zeta^* = \beta
\]

and denote the set of optimal ergodic occupation measures by \( \mathcal{M}_{\text{erg}}^* \).

The convex analytic method introduced in [6] (see also [8]) is a powerful tool for the analysis of ergodic occupation measures. Two main models have been considered: MDPs with a blanket stability property, and MDPs with a *near-monotone* running cost. Near-monotonicity is a structural assumption, which, stated in simple terms, postulates that the running cost is strictly larger than the optimal average value in (2.1) on the complement of some compact set. More precisely, this assumption is stated as follows:

(C) Consider a continuous one-one embedding \( \Psi : K \to K^* \) of \( K \) into a Polish space \( K^* \) such that \( \Psi(K) \) is compact in \( K^* \). (Existence and examples of such embeddings follow.) By abuse of notation, we identify \( K \) with its image \( \Psi(K) \) under this map and \( K^* \) with \( \Psi(K) \).

Furthermore, we assume that there exists a compact set \( \tilde{K} \subset K \) and an \( \varepsilon_0 > 0 \) such that

\[
c(x, u) \geq \beta + \varepsilon_0 \quad \forall (x, u) \in \tilde{K} \setminus \tilde{K}.
\]

This implies in particular that \( \{(x_n, u_n) \} \subset K, (x_n, u_n) \to \partial K := K^* \setminus K, \) then

\[
\liminf_{n \uparrow \infty} c(x_n, u_n) \geq \beta + \varepsilon_0.
\]

It will be convenient for us to take \( K^* \) to be the closure of \( K \) (\( \approx \Psi(K) \)) embedded in \( X^* \times U^* \) where \( X^*, U^* \) are resp., compact dense embeddings of \( X, U \) into suitable Polish spaces, assumed to exist. We shall assume that this is so.

For MDPs on a countable state space a natural counterpart of assumption (C) is enough to guarantee the existence of an optimal ergodic occupation measure as shown in [8]. In Theorem 2.1, we extend this result to MDPs on a Polish space under Assumption 2.1 and the above assumption.

As an example, consider the case where the state space \( X \) is locally compact and \( K = X \times U \) for a compact action space \( U \). Suppose that a sequence \( \{\tilde{\zeta}_n\}_{n \in \mathbb{N}} \) of mean empirical measures converges vaguely to a positive measure \( \mu \in \Psi(K) \), meaning that \( \int_K f \, d\tilde{\zeta}_n \to \int_K f \, d\mu \) as \( n \to \infty \) for all \( f \in C_c(K) \), where \( C_c(K) \) denotes the subspace of \( C_b(X) \) consisting of functions with compact support. A key lemma then asserts that \( \mu(K) > 0 \), the normalized measure \( \frac{\mu}{\mu(K)} \) on \( K \) is an ergodic occupation measure. This is established in [8, Lemma 2.6] for models with a countable state space, and the proof can be adapted to MDPs with a locally compact state space. An important
ingredient in this proof is employing the Alexandroff extension, commonly known as the one-point compactification, and then applying Prokhorov’s theorem to the compactified space \( X \cup \{ \infty \} \).

The Alexandroff extension has a simple and geometrically meaningful structure, but it does not result in a Hausdorff compactification unless the original space is locally compact. For models with general Polish state and action spaces, a general scheme that is always available is to employ Urysohn’s theorem to embed \( K \) in the Hilbert cube, and use the closure of its image as a compactification. This is done as follows.

**Definition 2.3** (Embedding in the Hilbert cube). As is well known, \( K \), being a Polish space, can be homeomorphically embedded as a \( G_δ \) subset of the Hilbert cube \([0,1]^{\infty}\) by a homeomorphism \( \Psi: K \leftrightarrow \Psi(K) \subset [0,1]^{\infty} \) [5, Propositions 7.2 and 7.3]. Thus we can identify \( K \) with \( \Psi(K) \). Let \( K^* := \overline{\Psi(K)} \), and view \( K \) as being densely homeomorphically embedded in \( K^* \) with \( \partial K := K^* \setminus K \). We may view \( \Psi(K) \) as being isometrically embedded in \( \Psi(K^*) \) in the obvious manner. The latter is compact by Prokhorov’s theorem.

This may not always be convenient and one may use better problem-specific choices. As an example, consider \( K := \) a closed bounded subset of \( C_1[0,1] \), the space of continuous functions on \([0,1]\) which are continuously differentiable on \((0,1)\) with left, resp. right limits at 0, 1, equipped with the norm \( \|f\| := \sup_{x \in [0,1]} |f(x)| + \sup_{x \in (0,1)} |f'(x)| \). Its natural embedding into \( C[0,1] := \) the space of continuous real-valued functions on \([0,1]\) with the sup-norm, is compact and dense. Thus taking any bounded subset of \( C_1[0,1] \) as state space with extended real valued cost \( c(x,u) := \|f\| := \|f\|_1 + \sup_{x \in (0,1)} |f''(x)| \), satisfies the above conditions. Further examples can be constructed using compact embedding theorems for Sobolev and Hölder spaces such as the ones provided by the Rellich-Kondrachov theorem. Another example is a bounded subset of the space of probability measures on a euclidean space with finite \( p \)-th moment, \( p \geq 1 \), with the Wasserstein-\( p \) distance, embedded in the space of all probability measures on the underlying space with Prohorov topology. A set of laws with uniformly bounded \( p \)-th moment is necessarily tight, hence the embedding is compact. Density follows easily. Yet another simple example is the natural embedding of the open unit ball \( \{ f : \|f\| < 1 \} \) in \( L_2[0,1] \) with norm topology with its natural embedding into \( L_2[0,1] \) with the weak* topology.

Let \( \tilde{F} \) to be the class of functions in \( C_b(\mathbb{X}^*) \). The functions in \( \tilde{F} \) can also be viewed as functions on \( K^* \) by letting \( \tilde{f}(x,u) \equiv f(x) \) for \( u \in \mathcal{U}(x) \). Abusing the notation, we use the same symbol \( \tilde{F} \) to denote the pullback of the family \( F \) by the map \( \Psi^{-1} \). These are functions on \( \Psi(\mathbb{X}) \), that is, \( \tilde{f}(z) \) for \( z \in \Psi(\mathbb{X}) \) is identified with \( f(\Psi^{-1}(z)) \). Since \( f(x,u) \) in the family \( \tilde{F} \subset C_b(K^*) \) does not depend on \( u \), abusing the notation, we denote it simply as \( \tilde{f}(x) \) whenever this is convenient.

In the study of the average cost problem, empirical occupation measures play an important role. These are defined as follows.

**Definition 2.4.** For any given \( \mu_0 \in \mathfrak{P}(\mathbb{X}) \) and \( \xi \in \mathcal{U} \), we define the family of mean empirical measures \( \{ \tilde{\zeta}_t \in \mathfrak{P}(K), \ t > 0 \} \) by:

\[
\int_{K} h(x,u) \tilde{\zeta}_t(dx,du) := \frac{1}{t} \sum_{m=0}^{t-1} \mathbb{E}_{\mu_0}^\xi [h(X_m,\xi_m)] \quad \forall h \in C_b(K) .
\]

Naturally, \( \tilde{\zeta}_t \) depends on \( \mu_0 \) and \( \xi \), but we suppress this dependence in the notation.

We state and prove a key lemma which is analogous to the one mentioned earlier for the locally compact case. Consider a sequence \( \{ \tilde{\zeta}_n \}_{n \in \mathbb{N}} \) of mean empirical measures viewed as a sequence in \( \mathfrak{P}(K^*) \) using the embedding in Definition 2.3. By Prokhorov’s theorem, moving to a subsequence if necessary, also denoted as \( \{ \tilde{\zeta}_n \}_{n \in \mathbb{N}} \), we have \( \tilde{\zeta}_n \Rightarrow \tilde{\zeta} \) for some \( \tilde{\zeta} \in \mathfrak{P}(K^*) \). Since \( K^* \) is the disjoint union of \( K \) and \( \partial K \), it is clear that \( \tilde{\zeta} \) must be of the form

\[
\tilde{\zeta} = a \tilde{\zeta}_0 + (1-a) \tilde{\zeta}_1
\]

(2.6)
for some $a \in [0, 1]$, $\zeta_0 \in \Psi(\partial K)$, and $\zeta_1 \in \Psi(K)$.

**Lemma 2.1.** If $a < 1$, then $\zeta_1 \in \mathcal{M}_{\text{erg}}$. The same conclusion applies for a sequence $\{\zeta_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\text{erg}}$.

**Proof.** Using the strong law of large numbers for martingales given by

$$
\frac{1}{t} \sum_{m=1}^{t} \left( f(X_m) - \mathbb{E}_{\mu_0}^f [f(X_m) \mid X_{m-1}, U_{m-1}] \right) \xrightarrow{t \to \infty} 0 \ a.s.
$$

for $f \in C_b(\mathcal{X})$, we obtain, upon taking expectations, that

$$
\int_K \left( f(x) - \int_X f(y) P(dy \mid x, u) \right) \zeta_t(dx, du) \xrightarrow{t \to \infty} 0. \tag{2.7}
$$

Then

$$
\lim_{n \to \infty} \int_K f \, d\zeta_n = \lim_{n \to \infty} \int_{K^*} f \, d\zeta_n = \int_{K^*} f \, d\zeta = a \int_{\partial K} f \, d\zeta_0 + (1 - a) \int_K f \, d\zeta_1 \forall f \in C_b(K^*) \tag{2.8}
$$

by the hypothesis that $\zeta_n \Rightarrow \zeta$. As shown in [18, Theorem 4.5], if $\mathcal{X}$ is Polish, then any subset $F \subset C_b(\mathcal{X})$ which separates points in $\mathcal{X}$ and is also an algebra is a separating class for Borel probability measures, meaning that if $\mu', \mu'' \in \Psi(\mathcal{X})$ satisfy $\int f \, d\mu' = \int f \, d\mu''$ for all $f \in F$ then $\mu' = \mu''$. The method that we use in this proof reduces the problem of proving that $\zeta_1 \in \mathcal{M}_{\text{erg}}$ to establishing equality of two given measures in $\Psi(\mathcal{X})$. Therefore, it suffices to continue with a class $F$ that only separates probability measures. By adding a constant to any $f \in F$, we may suppose that each $f \in F$ is bounded away from zero from below. We begin with a special subclass of such $f$. Recall that given a compatible metric $d : K \times K \to [0, 1]$, and a countable dense set $\{s_n\}$ in $K$, we can homeomorphically embed $K$ into $[0, 1]^\infty$ via the map $\Phi : s \in K \mapsto [(d(s, s_1), d(s, s_2), \ldots)] \in \Phi(K) \subset [0, 1]^\infty$ (See Definition 2.3). Then for any $n \geq 1$, the set $K_n := \{[(d(s, s_1), d(s, s_2), \ldots, d(s, s_n))]\}$ is locally compact in the relative topology of $[0, 1]^n$. Let $K_n^* = K \cup \{\infty\}$ denote its one point compactification. Consider $f$ above of the form $f(s) = g(d(s, s_1), \ldots, d(s, s_n))$ for some $g : K_n^* \to \mathbb{R}$ vanishing at the point $K_n^* \setminus K_n$, restricted to $K_n$. Then in the right hand side of (2.8), the first term is zero. Let $C_n \subset C_b(K)$ denote the collections of such $f$, indexed by $n \geq 1$.

Next, extend $H_f$ to $K^*$ by defining it to be $\liminf_{y' \in K, y' \to y} H_f(y')$ for $y \in K^* \setminus K$. By abuse of notation, we denote this extension by $H_f$ again. Note that $H_f$ is lower semicontinuous on $K^*$ by construction. Using Skorokhod’s theorem, construct on some probability space $K^*$-valued random variables $\chi_n, n \geq 0$, and $\hat{\chi}$ such that the laws of $\chi_n$ (resp., $\hat{\chi}$) are $\zeta_n$ (resp., $\zeta$) and $\chi_n \to \hat{\chi}$ a.s.
Then we have
\[
\liminf_{n \to \infty} \int_{K} \left( \int_{X} f(y) P(dy \mid x, u) \right) \zeta_n(dx, du) \\
= \liminf_{n \to \infty} \int_{K^*} H_f(x, u) \zeta_n(dx, du) \\
= \liminf_{n \to \infty} E[H_f(\zeta_n)] \\
\geq (1 - a)E[H_f(\hat{\chi})I_K] + aE[H_f(\hat{\chi})I_{K^* \setminus K}] \tag{2.9}
\]
where ‘(a)’ follows from the lower semicontinuity of $H_f$ and ‘(b)’ follows from the fact that $H_f \geq 0$ on $K^* \setminus K$. Combining (2.7) with the above and using Fubini’s theorem, we get
\[
\int_{K} f(x) \zeta_1(dx, du) \geq \int_{X} f(y) \left( \int_{K} P(dy \mid x, u) \zeta_1(dx, du) \right) \quad \forall f \in F. \tag{2.10}
\]
For $A \in \mathcal{B}(X)$, define
\[
\eta_1(A) := \int_{(A \times U) \cap K} \zeta_1(dx, du), \\
\eta_2(A) := \int_{K} P(A \mid x, u) \zeta_1(dx, du). \tag{2.11}
\]
Then, (2.10) can be written as
\[
\int_{X} f(x) \eta_1(dx) \geq \int_{X} f(y) \eta_2(dx) \quad \forall f \in F. \tag{2.12}
\]
Such $f$ separate points of $K_n$ and therefore form a separating class for $\mathcal{B}(K_n)$. It follows that the set $C := \cup_{n \geq 1} C_n$ is a separating class for finite positive measures on $K$. Hence
\[
\eta_1(B) \geq \eta_2(B) = \int_{K} P(B \mid x, u) \zeta_1(dx, du) \quad \forall B \in \mathcal{B}(X). \tag{2.13}
\]
However, $\eta_1(X) = \eta_2(X) = 1$ by (2.11). Thus equality must hold in (2.13) for all $B \in \mathcal{B}(X)$, which means that $\zeta_1 \in \mathcal{M}_{\text{erg}}$ by the definition of $\mathcal{M}_{\text{erg}}$.

In the case of a sequence $\{\zeta_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\text{erg}}$ such that $\zeta_n \Rightarrow \zeta = a\zeta_0 + (1 - a)\zeta_1$ as above, observe that the left-hand side of (2.7) over this sequence is identically equal to 0 by the definition of an ergodic occupation measure. Thus, the proof of the statement is identical to the above. \hfill \square

We continue by showing that (C) implies the existence of an optimal ergodic occupation measure in the sense of Definition 2.2.

**Theorem 2.1.** Under (C), we have $\mathcal{M}_{\text{erg}}^* \neq \emptyset$. In addition if $\zeta^* \in \mathcal{M}_{\text{erg}}^*$, $\pi_{\zeta^*} \in \mathcal{P}(X)$ and $\nu_{\zeta^*}$ satisfy (2.2), and $\hat{v} \in \mathcal{A}_{\max}$ agrees $\pi_{\zeta^*}$-a.e. with $\nu_{\zeta^*}$, then
\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E}_0^x \left[ \sum_{n=0}^{N-1} c_{\hat{v}}(X_n) \right] = J(x) = \beta \, \pi_{\zeta^*}$-a.e. \tag{2.14}
\]
Proof. Let \( \{\zeta_k\}_{k \in \mathbb{N}} \) be such that \( \int c \, d\zeta_k \downarrow \beta \) as \( k \to \infty \). We select a subsequence such that \( \zeta_k \Rightarrow \tilde{\zeta} \in \mathcal{P}(\mathcal{K}^*) \), and write \( \tilde{\zeta} = a \zeta' + (1-a) \zeta'' \), with \( a \in [0,1] \), \( \zeta' \in \mathcal{P}(\partial \mathcal{K}^*) \) and \( \zeta'' \in \mathcal{P}(\mathcal{K}) \).

Since \( c \) is lower semi-continuous on \( \mathcal{K} \), there exists a sequence \( c_n \in C_b(\mathcal{K}) \) such that \( c_n \uparrow c \) pointwise. Choose a \( n_0 \geq 1 \) such that for \( n \geq n_0 \), \( \{x_m\} \subset \mathcal{K}, x_m \to \partial \mathcal{K} := \mathcal{K} \setminus \mathcal{K} \), then

\[
\lim_{k \to \infty} \inf_\pi c_n(x_k, u) > \beta + 2\varepsilon
\]

for some \( \varepsilon > 0 \). Then we have

\[
\beta \geq \liminf_{k \to \infty} \int_{\mathcal{K}^*} c \, d\zeta_k \geq \liminf_{k \to \infty} \int_{\mathcal{K}^*} c_n \, d\zeta_k
\]

\[
\geq a \int c_n \, d\zeta' + (1-a) \int c_n \, d\zeta'' \geq a(\beta + \varepsilon) + (1-a) \int_{\mathcal{K}} c_n \, d\zeta''.
\]

(2.15)

for all \( n \geq n_0 \). By the above lemma, \( \zeta'' \in \mathcal{M}_{\text{erg}} \), implying \( \int_{\mathcal{K}} c \, d\zeta'' \geq \beta \). Letting \( n \to \infty \) in (2.15), we obtain

\[
\beta \geq a(\beta + \varepsilon) + (1-a) \int_{\mathcal{K}} c \, d\zeta''
\]

\[
\geq a(\beta + \varepsilon) + (1-a) \beta.
\]

This shows that \( a = 0 \) and \( \int_{\mathcal{K}} c \, d\zeta'' = \beta \). Therefore \( \{\zeta_k\} \) are tight and \( \zeta'' \in \mathcal{M}_{\text{erg}}^* \), hence \( \mathcal{M}_{\text{erg}}^* \neq \phi \).

It remains to establish (2.14). If \( \zeta^* = \pi_{\zeta^*} \otimes v_{\zeta^*} \in \mathcal{M}_{\text{erg}}^* \) and \( \hat{v} \in \mathcal{U}_{\text{sm}} \) agrees \( \pi_{\zeta^*} \)-a.e. with \( v_{\zeta^*} \), then an application of Birkhoff’s ergodic theorem shows that

\[
\beta = \int_{\mathcal{K}} c \, d\zeta^* = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_x \left[ \sum_{n=0}^{N-1} c_{\hat{v}}(X_n) \right] \pi_{\zeta^*} \text{-a.e.}
\]

(2.16)

This completes the proof. \( \square \)

Remark 2.2. The pair \( (\hat{v}, \pi_{\zeta^*}) \) in Theorem 2.1 is a stationary minimum pair in the sense of [44, Definition 2.2] (see also [45]). It is worthwhile comparing the assumptions in [44] to the ones in this paper. In [44] the state space \( \mathcal{X} \) is Borel, \( \mathcal{U} \) is countable, and \( \mathcal{K} \) is a Borel subset of \( \mathcal{X} \times \mathcal{U} \). Existence of a stationary minimum pair is established under the assumption that \( c \) is strictly unbounded and the transition kernel satisfies a majorization condition. The latter involves weak continuity of the kernel \( P \) and lower semi-continuity of \( c \) when these are restricted to \( D \times \mathcal{U} \), where \( D \subset \mathcal{X} \) is a closed set that appears in the majorization condition [44, Assumption 3.1].

By comparison, we allow \( \mathcal{U} \) to be Polish, the running cost satisfies (C) and is not necessarily strictly unbounded, and we don’t need a majorization condition. However, we assume that \( \mathcal{X} \) is Polish, that \( \mathcal{U} \) is upper semicontinuous, weak continuity of \( P \) and lower semi-continuity of \( c \) on \( \mathcal{K} \), which are more restrictive than [44, Assumption 3.1].

2.3. Discussion. To guide the reader in the approach we follow to establish the Bellman equation and the existence of an optimal stationary Markov policy, we review the case of an MDP on a countable state space with compact action space under the near monotone hypothesis [7]. Let the state space be \( \mathbb{N}_0 := \{0,1,2,\ldots\} \), and \( \mathcal{U}(x) = \mathcal{U} \) for all \( x \in \mathbb{N}_0 \). Suppose the state 0 is reachable with positive probability from every other state under some control. Under the near-monotone hypothesis in (C), we obtain an optimal ergodic occupation measure \( \zeta^* = \pi_{\zeta^*} \otimes v_{\zeta^*} \). Let \( K \subset \mathbb{N}_0 \) denote the support of \( \pi_{\zeta^*} \). Then necessarily, \( 0 \in K \). Then \( v_{\zeta^*} \) is defined on \( K \) via the disintegration of \( \zeta^* \), and thus the Markov chain ‘controlled’ by \( v_{\zeta^*} \) is well defined when restricted to \( K \). We would like to extend \( v_{\zeta^*} \) to some policy \( v_* \in \mathcal{U}_{\text{sm}} \) which is optimal in the sense of the definition in Subsection 2.1.1. Let \( \tau_A \) denote the first return time to a set \( A \), defined by

\[
\tau_A := \min\{n \geq 1 : X_n \in A\}.
\]
Let $\tau_0 := \tau_{\{0\}}$. A key observation is that $v_{\zeta^*}$ satisfies
\[
\mathbb{E}^v_{\tau_0}\left[\sum_{n=0}^{t_0-1} (c_{v_{\zeta^*}}(X_n) - \beta)\right] = \inf_{v \in \mathcal{U}_{\text{sm}}} \mathbb{E}^v_{\tau_0}\left[\sum_{n=0}^{t_0-1} (c_v(X_n) - \beta)\right] \quad \forall x \in K. \quad (2.17)
\]
This can be shown by following the proof of Lemma 3.1 which establishes an analogous result for the model in this paper. Therefore, any Markov control that arises from the disintegration of an optimal ergodic occupation measure attains the infimum on the right-hand side of (2.17) for $x \in K$. Let
\[
V(x) := \inf_{v \in \mathcal{U}_{\text{sm}}} \mathbb{E}^v_x\left[\sum_{n=0}^{t_0-1} (c_v(X_n) - \beta)\right], \quad x \in \mathbb{N}_0. \quad (2.18)
\]
with $\beta$ as in (2.3), and suppose that the right-hand side of (2.18) is finite for all $x \in \mathbb{N}$. Then it is straightforward to show, using a one step analysis, that $V$ satisfies
\[
V(x) = \min_{u \in \mathcal{U}} \left[ c(x,u) - \beta + \sum_{y \in \mathbb{N}_0 \setminus \{0\}} V(y) P(y|x,u) \right] \quad \forall x \in \mathbb{N},
\]
in other words, we have the Bellman equation on the entire state space except possibly at $x = 0$. Now, since $\beta$ is the ergodic value, we have
\[
\mathbb{E}^v_0\left[\sum_{n=0}^{t_0-1} (c_v(X_n) - \beta)\right] \geq 0,
\]
with equality when $v = v_{\zeta^*}$. In particular, $V(0) = 0$. But this shows that the Bellman equation also holds for $x = 0$. One crucial step in this derivation is the finiteness of the right-hand side of (2.18). Since $c - \beta \geq 0$ on the complement of a finite set by the near-monotone hypothesis, it is easy to show that it suffices to assume that there exists some $v \in \mathcal{U}_{\text{sm}}$ which satisfies
\[
\mathbb{E}^v_x\left[\sum_{n=0}^{t_0-1} c_v(X_n)\right] < \infty \quad \forall x \in \mathbb{N}. \quad (2.19)
\]
The fact that $0$ is an atom plays of course an important role in showing that the Bellman equation is satisfied at $x = 0$. For the model in this paper, we circumvent this difficulty by imposing a suitable hypothesis and adopting the splitting method introduced by Athreya–Ney and Nummelin. This is discussed in the next subsection.

2.4. The split-chain and the pseudo-atom. We introduce here the notions of the split chain and pseudo-atom, originally due to Athreya and Ney [4], and Nummelin [35] for uncontrolled Markov chains. We follow the treatment of [2, Section 8.4]. See [32] for an extended treatment, albeit in the uncontrolled framework.

The basic assumption concerns the existence of a 1-small set which is compatible with the near-monotonicity condition (C). More precisely, the transition probability $P$ is assumed to satisfy the following minorization hypothesis.

(A0) There exists a bounded set $\mathcal{B} \subset \mathcal{X}$ which satisfies
\[
\inf_{(x,u) \in \mathcal{B} \times \mathcal{U}} c(x,u) > \beta,
\]
such that for some measure $\nu \in \mathfrak{M}(\mathcal{X})$ with $\nu(\mathcal{B}) = 1$, and a constant $\delta > 0$, we have $P(A|x,\cdot) \geq \delta \nu(A) 1_\mathcal{B}(x)$ for all $A \in \mathcal{B}(\mathcal{X})$ and $\sum_{n \geq 1} P(X_n \in \mathcal{B}) > 0$ under all $v \in \mathcal{U}_{\text{sm}}$. Here, $\beta$ is as defined in (2.3).
Remark 2.3. If $X = \mathbb{R}^d$ and the transition kernel has a continuous density $\varphi$, then a necessary and sufficient condition for the minimization condition in (A0) is that the function $\Gamma: \mathcal{B} \to \mathbb{R}_+$ defined by

$$\Gamma(y) := \inf_{(x,u) \in (\mathcal{B} \times \mathcal{U}) \cap K} \varphi(y | x, u)$$

is not equal to 0 $\nu$-a.e. In particular, if the density $\varphi$ is strictly positive, then (A0) is automatically satisfied.

Definition 2.5 (Pseudo-atom). Let

$$\mathcal{X} := (X \times \{0\}) \cup (B \times \{1\})$$

and $\mathcal{B}(\mathcal{X})$ denote its Borel $\sigma$-algebra. For a probability measure $\mu \in \mathfrak{P}(X)$ we define the corresponding probability measure $\hat{\mu}$ on $\mathcal{B}(\mathcal{X})$ by

$$\hat{\mu}(A \times \{0\}) := (1 - \delta)\mu(A \cap B) + \mu(A \cap B^c), \quad A \in \mathcal{B}(X),$$

$$\hat{\mu}(A \times \{1\}) := \delta \mu(A), \quad A \in \mathcal{B}(B). \quad (2.20)$$

Let $\hat{\mathcal{B}} := \mathcal{B} \times \{1\}$, and refer to it as the pseudo-atom.

Definition 2.6 (Split chain). Given the controlled Markov chain $(X, U, \mathcal{U}, P, c)$ as described in Section 2, we define the corresponding split chain $(\mathcal{X}, \mathcal{U}, \mathcal{U}, Q, \hat{c})$, with state space $\mathcal{X}$, and transition kernel given by

$$Q(dx \mid (x, i), u) := \begin{cases} \hat{P}(dy \mid x, u) & \text{if } (x, i) \in (X \times \{0\}) \setminus (\mathcal{B} \times \{0\}), \\ \frac{1}{1 - \delta} (\hat{P}(dy \mid x, u) - \delta \nu(dy)) & \text{if } (x, i) \in \mathcal{B} \times \{0\}, \\ \delta \nu(dy) & \text{if } (x, i) \in \mathcal{B} \times \{1\}. \end{cases} \quad (2.21)$$

The running cost $\hat{c}$ is defined in ??.

Using Definition 2.5 and (2.21), the kernel $Q$ of the split chain can be expressed as follows:

$$Q(A \times \{0\} \mid (x, 0), u) := \left[ P(A \cap \mathcal{B} \mid x, u) - \delta \nu(A \cap \mathcal{B}) + \frac{1}{1 - \delta} P(A \cap \mathcal{B}^c \mid x, u) \right] 1_B(x) \quad (2.22)$$

$$Q(A \times \{1\} \mid (x, 0), u) := \frac{\delta}{1 - \delta} (P(A \mid x, u) - \delta \nu(A)) 1_B(x) + \delta P(A \mid x, u) 1_{B^c}(x) \quad (2.23)$$

for $A \in \mathcal{B}(X)$,

$$Q(dx \times \{0\} \mid (x, 1), u) := (1 - \delta) \nu(dy), \quad (2.24)$$

$$Q(dx \times \{1\} \mid (x, 1), u) := \delta \nu(dy).$$

Note that $B^c \times \{1\}$ is not visited.

Given an initial distribution $\mu_0$ of $X_0$, the corresponding initial distribution $\hat{\mu}_0$ of the split chain is determined according to (2.20). We let $\hat{X}_n = (X_n, i_n) \in \mathcal{X}$ denote the state process of the split chain.

Next we define an equivalent running cost for the split chain. Consider a function $\hat{c}: \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ satisfying

$$\hat{c}(x, 0), u = c(x, u), \quad (x, u) \in (B^c \times \mathcal{U}) \cap K,$n

$$\delta \hat{c}(x, 1), u + (1 - \delta) \hat{c}(x, 0), u = c(x, u), \quad (x, u) \in (\mathcal{B} \times \mathcal{U}) \cap K,$n

with $\hat{c}(x, 1), u$ not depending on $u$. 
Let \( \mathcal{P}(\mathcal{X}) \) denote the class of probability measures \( \hat{\mu} \) on \( \mathcal{B}(\mathcal{X}) \) which satisfy (1 - \( \delta \))\( \hat{\mu}(A \times \{1\}) = \delta \hat{\mu}(A \times \{0\}) \) for all \( A \in \mathcal{B}(\mathcal{B}) \). It follows by (2.28), that for any initial \( \mu_0 \in \mathcal{P}(\mathcal{X}) \), we have \( \mu_0 Q_u \in \mathcal{P}(\mathcal{X}) \). In other words, \( \mathcal{P}(\mathcal{X}) \) is invariant under the action of \( Q \). This property implies that

\[
\mathbb{E}_\mu^\xi \left[ \sum_{n=0}^{N-1} \hat{c}(\hat{X}_n, U_n) \right] = \mathbb{E}_\mu^\xi \left[ \sum_{n=0}^{N-1} c(X_n, U_n) \right] \quad \forall \xi \in \mathcal{U}.
\]

In particular, the ergodic control problem of the split chain under the cost-per-stage function \( \hat{c} \) is equivalent to the original ergodic control problem.

With the above property in mind, we introduce the following definition.

**Definition 2.7.** We define the cost-per-stage function \( \hat{c}: \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) for the split chain by

\[
\hat{c}(x,0,u) := \begin{cases} c(x,u) & \text{for all } x \in (\mathcal{B}^c \times \mathcal{U}) \cap K, \\ \frac{c(x,u)}{1-\delta} & \text{for all } x \in (\mathcal{B} \times \mathcal{U}) \cap K, \end{cases}
\]

(2.25)

\[
\hat{c}(x,1,u) := 0 \quad \forall x \in \mathcal{B}.
\]

For \( v \in \mathcal{U}_{\text{adm}} \), we let \( \hat{c}_v \) be as in Definition 2.1 with \( c(\cdot) \) replaced by \( \hat{c}(\cdot) \).

An equivalent description of the split chain is as follows. Let \( \{\xi_n\} \) denote the control process.

1. If \( X_n = x \in \mathcal{B}, \xi_n = u \) and \( i_n = 0 \), then \( X_{n+1} = y \) according to the transition probability

\[
\frac{1}{1-\delta} \left( P(dy \mid x,u) - \delta \nu(dy) \right).
\]

Furthermore, if \( y \in \mathcal{B} \), then \( i_{n+1} = 1 \) with probability \( \delta \) and \( i_{n+1} = 0 \) with probability \( 1 - \delta \).

2. If \( X_n = x \in \mathcal{B} \) and \( i_n = 1 \), then \( X_{n+1} = y \in \mathcal{B} \) with probability \( \nu(dy) \) and \( i_{n+1} = 1 \) with probability \( \delta \) and \( = 0 \) with probability \( 1 - \delta \).

3. If \( X_n = x \notin \mathcal{B} \) and \( i_n = 0 \), then \( X_{n+1} = y \) according to \( P(dy \mid x,u) \) and \( i_{n+1} \) is as in (1) above.

4. The set \( \mathcal{B}^c \times \{1\} \) is never visited.

This gives a causal description of the split chain. We dub the control \( \xi = \{\xi_n\} \) as an admissible control. Intuitively, it can depend at time \( n \) on the past history till \( n \), i.e., on \( (X_m, i_m), m \leq n, \xi_k, k < n \), and in addition, on any extraneous randomization conditionally independent of the ‘future’ \((X_m, i_m), \xi, m > n\), given the history till \( n \).

It is clear that an admissible strategy \( \xi \in \mathcal{U} \), or a Markov randomized strategy \( v = \{v_t\}_{t \in \mathbb{N}_0} \), maps in a natural manner to a corresponding control for the split chain, which is also denoted as \( \xi \) or \( v \), respectively. We use the symbols, \( \mathbb{E}_{(x,i)}^\xi, \mathbb{E}_{(x,i)}^v \) to denote the expectation operator on the path space of the split chain controlled under \( \xi \in \mathcal{U}, v \in \mathcal{U}_{\text{adm}} \) resp., and adopt the analogous notation as in Definition 2.1, e.g., \( Q_v \) and \( \{X_n^v\}_{n \in \mathbb{N}_0} \). In addition, we let

\[
\bar{\tau} := \min\{n \geq 1 : \hat{X}_n \in \bar{\mathcal{B}}\},
\]

(2.26)

that is, the first return time to \( \bar{\mathcal{B}} := \mathcal{B} \times \{1\} \).

Let

\[
\delta_o := \frac{1 - \delta}{\delta} \left( \inf_{(x,u) \in (\mathcal{B} \times \mathcal{U}) \cap K} P(\mathcal{B} \mid x,u) - \delta \right)^{-1}.
\]

(2.27)

Since \( \delta > 0 \) in (A0) can always be chosen so that \( (x,u) \mapsto P(\mathcal{B} \mid x,u) - \delta \) is strictly positive on \( (\mathcal{B} \times \mathcal{U}) \cap K \), we may assume that \( \delta_o \) is a (finite) positive constant. We have the following simple lemma.
Lemma 2.2. For any $v \in \mathcal{U}_{\text{sm}}$ it holds that

$$\mathbb{E}^v_{(x,0)} \left[ \sum_{k=0}^{\tau-1} 1_{B \times \{0\}}(\tilde{X}_k) \right] \leq \delta_0.$$  

Proof. This follows directly from the fact that $Q(B \times \{1\} \mid (x,0),u) \geq \delta_0^{-1}$ for $x \in B$ by (2.23) and (2.27).

Let $\mu_0$ be an initial distribution of $\{X_n\}_{n \in \mathbb{N}}$. Adopting the notation $Q_u(\cdot \mid z) = Q(\cdot \mid z,u)$ for $z \in \mathcal{X}$, an easy calculation using Definition 2.6 shows that $\hat{\mu}_0 Q_u(\cdot) := \int_{\mathcal{X}} Q_u(\cdot \mid z) \mu_0(dz)$ is given by

$$\hat{\mu}_0 Q_u(\emptyset) = \int_{\mathcal{X}} [(1 - \delta) P(A \cap B \mid x,u) + P(A \cap B^c \mid x,u)] \mu_0(dx), \quad A \in \mathcal{B}(\mathcal{X}),$$

$$\hat{\mu}_0 Q_u(\{1\}) = \int_{\mathcal{X}} \delta P(A \cap B \mid x,u) \mu_0(dx), \quad A \in \mathcal{B}(\mathcal{B}). \quad (2.28)$$

It is important to note, as seen by (2.28), that the marginal of the law of $(\tilde{X}_n, U_n)$, $n \geq 0$, on $(K)^{\infty}$ coincides with the law of $(X_n, U_n)$, $n \geq 0$, but the split chain has a pseudo-atom $B \times \{1\}$ with many desirable properties that will become apparent in the next section (see [2, Theorem 8.4.1, p. 289] and [4,35]).

2.5. Some basic notions. We now recall some standard background from the theory of Markov chains on a general state space, see, e.g., [32] for a more detailed treatment. For $v \in \mathcal{U}_{\text{sm}}$ we define the resolvent $\mathcal{R}_v$ by

$$\mathcal{R}_v(x,A) := \sum_{n=1}^{\infty} 2^{-n} P^n_v(x,A).$$

Consider the chain $\{X_n\}_{n \geq 0}$ controlled by $v \in \mathcal{U}_{\text{sm}}$. Recall that a measure $\psi$ on $\mathcal{B}(\mathcal{X})$ is called a (maximal) irreducibility measure for the chain if $\psi$ is absolutely continuous with respect to $\mathcal{R}_v(x,\cdot)$ for all $x \in \mathcal{X}$ (and $\psi$ is maximal among such measures). In turn, the chain itself is said to be $\psi$-irreducible. Let $\mathcal{B}^+(\mathcal{X})$ denote the class of Borel sets $A$ satisfying $\psi(A) > 0$. Let $\tau_A$ denote the first return time to a set $A$, defined by

$$\tau_A := \min \{n \geq 1 : X_n \in A\}.$$  

For a $\psi$-irreducible chain, a set $C$ is petite if there exists a positive constant $c$ such that $\mathcal{R}_v(x,A) \geq c \hat{\psi}(A)$ every $A \in \mathcal{B}(\mathcal{X})$ and $x \in C$, and some finite positive measure $\hat{\psi}$ equivalent to $\psi$. Recall also that a $\psi$-irreducible chain is called Harris if $P_x(\tau_A < \infty) = 1$ for every $A \in \mathcal{B}^+(\mathcal{X})$ and $x \in \mathcal{X}$, and it is called positive Harris if it admits an invariant probability measure.

Let $f : \mathcal{X} \to [1,\infty)$ be a measurable map. For a $\psi$-irreducible chain, a set $D \in \mathcal{B}(\mathcal{X})$ is called $f$-regular [32] if

$$\sup_{x \in D} \mathbb{E}_x \left[ \sum_{n=0}^{\tau_A-1} f(X_n) \right] < \infty \quad \forall A \in \mathcal{B}^+(\mathcal{X}).$$

If there is countable cover of $\mathcal{X}$ by $f$-regular sets, then the chain is called $f$-regular. An $f$-regular chain is always positive Harris with a unique invariant probability measure $\pi$ and satisfies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}_x [f(X_n)] = \pi(f) := \int_{\mathcal{X}} f(x) \pi(dx) \quad \forall x \in \mathcal{X}.$$
3. The Bellman equation

In view of the definitions of the preceding section, we lift the control problem in Subsection 2.1.1 to an an equivalent problem on the controlled split chain \( (\mathcal{X}, \mathcal{U}, U(x), Q, c) \) described in Definitions 2.6 and 2.7. In other words, we seek to minimize over all admissible \( \xi \in \mathcal{U} \) the cost

\[
\limsup_{N \to \infty} \frac{1}{N} E_{(x,i)}^{\xi} \left[ \sum_{n=0}^{N-1} c(\bar{X}_n, U_n) \right].
\]

3.1. Two assumptions. We need two additional assumptions. To state the first, we borrow the notion of \( \mathcal{K} \)-inf-compactness from [22]. Recall that a function \( f: S \to \mathbb{R} \), where \( S \) is a topological space is called inf-compact (on \( S \)), if the set \( \{ x \in S : f(x) \leq \kappa \} \) (possibly empty) is compact in \( S \) for all \( \kappa \in \mathbb{R} \). A function \( f: \mathcal{K} \to \mathbb{R} \) is called \( \mathcal{K} \)-inf-compact if for every compact set \( K \subset X \) the function is inf-compact on \( (K \times \mathcal{U}) \cap \mathcal{K} \).

The first assumption is a structural hypothesis on the running cost and is stated as follows:

(A1) One of the following holds.

(i) For some \( x \in X \), we have \( J(x) < \infty \), and the running cost \( c \) is inf-compact on \( \mathcal{K} \).
(ii) The running cost \( c \) is \( \mathcal{K} \)-inf-compact and (C) holds.

It is clear that part (i) of (A1) implies (C). Therefore, as shown in Theorem 2.1, under (A1), there exists an optimal ergodic occupation measure.

Remark 3.1. Hypothesis (A1) (i) cannot be satisfied unless \( \mathcal{K} \) is \( \sigma \)-compact. A non-trivial example of such a Polish space is \( \prod_{i \in \mathbb{N}} \{ \lambda e_i : \lambda \geq 0 \} \) where \( \{ e_i \}_{i \in \mathbb{N}} \) is a complete orthonormal basis for a Hilbert space with relative topology inherited from the ambient Hilbert space. This space is not locally compact. Note also that an inf-compact \( c \) is automatically \( \mathcal{K} \)-inf-compact [22, Lemma 2.1 (ii)].

The second assumption is analogous to (2.19) for denumerable MDPs. We start with the following definition.

Definition 3.1. Let \( \mathcal{T} \) be as defined in (2.26). We say that \( v \in \mathcal{U}_{sm} \) is \( c \)-stable if for the chain controlled by \( v \) the map

\[
x \mapsto \bar{E}_{(x,0)}^{\mathcal{T} v} \left[ \sum_{k=0}^{\mathcal{T} - 1} c_v(\bar{X}_k) \right]
\]

is locally bounded on \( X \), and by that we mean that it is bounded on every bounded set of \( X \).

We impose the following assumption.

(A2) There exists a \( c \)-stable \( v \in \mathcal{U}_{sm} \).

Assumptions (A0)–(A2) are in effect throughout the rest of the paper, unless mentioned otherwise. To see how they are used, consider the following. Let \( \hat{v} \in \mathcal{U}_{sm} \) be such that it agrees \( \pi_{\hat{v}} \)-a.e. with the control \( v_{\hat{v}} \) obtained via the disintegration of an optimal ergodic occupation measure \( \zeta^* = \pi_{\zeta^*} \otimes v_{\zeta^*} \), whose existence is guaranteed by (A1). It is then clear by (A0) and Proposition 5.1.1, p. 97, [32] that the chain controlled by \( \hat{v} \) is \( \nu \)-irreducible and aperiodic. Thus, the invariant probability measure \( \pi_{\hat{v}} \) is unique for the chain controlled by \( \hat{v} \) and is (trivially) mutually absolutely continuous with respect to \( \nu \) on its support. This implies that \( J(x, \hat{v}) = \beta \), a constant that does not depend on \( x \in X \). Compare this with the counterexample in [17, Example 1, p. 178]. Also, (A2) should be compared with part (b) of [44, Theorem 3.5]. It is clear that (A0) implies that the split chain controlled by a \( c \)-stable \( v \in \mathcal{U}_{sm} \) is positive Harris.

In the rest of the paper we let

\[
\zeta^* = \pi_{\zeta^*} \otimes v_{\zeta^*} \in \mathcal{M}_{\text{erg}}
\]

be a generic optimal ergodic measure. It is clear that \( \bar{E}_{(x,0)}^{\mathcal{T} \pi_{\zeta^*}} \left[ \sum_{k=0}^{\mathcal{T} - 1} c_{\pi_{\zeta^*}}(X_k) \right] \) is well defined \( \pi_{\zeta^*} \)-a.e.

We continue with the following lemma.
Lemma 3.1. Any c-stable \( v \in \mathcal{U}_{sm} \) (and therefore every \( v \in \mathcal{U}_{sm} \)) satisfies

\[
\mathbb{E}_v^{\pi_{r-1}} \left[ \sum_{k=0}^{\tau-1} (c_v(X_k) - \beta) \right] \geq \mathbb{E}_v^{\pi_{r-1}} \left[ \sum_{k=0}^{\tau-1} (\hat{c}_{v_{r-1}}(X_k) - \beta) \right] \quad \pi_{r-1} \text{-a.e.}
\]

In particular, \((A2)\) implies that \( x \mapsto \mathbb{E}_v^{\pi_{r-1}} \left[ \sum_{k=0}^{\tau-1} (\hat{c}_{v_{r-1}}(X_k) - \beta) \right] \) is locally bounded \( \pi_{r-1} \text{-a.e.} \).

Proof. If not, then we have the reverse inequality on some set \( A \in \mathcal{B}(X) \) with \( \pi_{r-1}(A) > 0 \), that is,

\[
\mathbb{E}_v^{\pi_{r-1}} \left[ \sum_{k=0}^{\tau-1} (c_v(X_k) - \beta) \right] < \mathbb{E}_v^{\pi_{r-1}} \left[ \sum_{k=0}^{\tau-1} (\hat{c}_{v_{r-1}}(X_k) - \beta) \right] \quad \forall x \in A, \quad (3.2)
\]

with \( +\infty \) a possible value for the right hand side. To simplify the expressions let

\[
\mathcal{J}(v) := \sum_{k=0}^{\tau-1} (c_v(X_k) - \beta), \quad v \in \mathcal{U}_{sm}.
\]

Since \( \pi_{r-1}(A) > 0, \nu(A) > 0 \) and \( A \) is in the support of the resolvent \( R_{\pi_{r-1}}(x, \cdot) \) for \( \pi_{r-1} \text{-a.e.} \). Hence \((3.2)\) implies that

\[
\mathbb{E}_v^{\pi_{r-1}} \left[ \mathbf{1}_{(\tau > \tau_A)} \mathbb{E}_{\mathcal{X}_A}^{\pi_{r-1}} \left[ \mathcal{J}(v_{r-1}) \right] \right] > \mathbb{E}_v^{\pi_{r-1}} \left[ \mathbf{1}_{(\tau > \tau_A)} \mathbb{E}_{\mathcal{X}_A}^{\pi_{r-1}} \left[ \mathcal{J}(v) \right] \right]. \quad (3.3)
\]

Consider \( \check{v} = (\check{v}_n, \ n \in \mathbb{N}_0) \) defined by Let

\[
\check{v}_n := \begin{cases} v & \text{if } \tau_A \leq n < \check{\tau}, \\ v_{r-1} & \text{otherwise.} \end{cases}
\]

It is clear that this can be extended to a (nonstationary) strategy \( \check{v} \in \mathcal{U} \) over the infinite horizon, by using the \( k \)th return time to \( \mathcal{B} \), denoted as \( \check{\tau}_k \), and the number of cycles \( \tau(n) \) completed at time \( n \in \mathbb{N} \), which is defined by

\[
\tau(n) := \max \{ k : n \geq \check{\tau}_k \}.
\]

Using the strong Markov property and \((3.3)\), we obtain

\[
0 = \mathbb{E}_v^{\pi_{r-1}} \left[ \mathcal{J}(v_{r-1}) \right] \\
= \mathbb{E}_v^{\pi_{r-1}} \left[ \mathbf{1}_{(\tau \leq \tau_A)} \mathcal{J}(v_{r-1}) \right] + \mathbb{E}_v^{\pi_{r-1}} \left[ \mathbf{1}_{(\tau > \tau_A)} \mathbb{E}_{\mathcal{X}_A}^{\pi_{r-1}} \left[ \mathcal{J}(v_{r-1}) \right] \right] \\
> \mathbb{E}_v^{\check{v}} \left[ \mathbf{1}_{(\tau \leq \tau_A)} \mathcal{J}(v_{r-1}) \right] + \mathbb{E}_v^{\check{v}} \left[ \mathbf{1}_{(\tau > \tau_A)} \mathbb{E}_{\mathcal{X}_A}^{\check{v}} \left[ \mathcal{J}(v) \right] \right] \\
= \mathbb{E}_v^{\check{v}} \left[ \sum_{k=0}^{\tau-1} (\check{c}_v(X_k) - \beta) \right].
\]

For \( m > 0 \), let \( \hat{e}_m^m := \min\{m, \check{c}_v\} \). The preceding inequality shows that, for some \( \varepsilon > 0 \), we have

\[
\mathbb{E}_v^{\check{v}} \left[ \sum_{k=0}^{\tau-1} (\check{e}_m(X_k) - \beta) \right] < -\varepsilon \quad \forall m \in \mathbb{N}. \quad (3.4)
\]

We claim that \((3.4)\) contradicts the fact that \( \beta \) is the optimal ergodic value. Indeed, it is rather standard to show (see the proof of Theorem 5.1 of \([25]\)) that

\[
\frac{1}{T} \sum_{t=0}^{T-1} (\check{e}_m(X_t) - \beta) \xrightarrow{T \to \infty} \frac{1}{\mathbb{E}_{\check{v}}^{\pi_{r-1}}[\check{\tau}]} \mathbb{E}_{\check{v}}^{\check{v}} \left[ \sum_{k=0}^{\tau-1} (\check{e}_m(X_k) - \beta) \right] \quad \check{v}^{\check{v}} \text{-a.s.,}
\]
which together with (3.4) implies (since $c^m_\circ$ is bounded) that

$$
\lim_{T \to \infty} \frac{1}{T} \bar{\mathbb{E}}(x,1) \left[ \sum_{t=0}^{T-1} (c^m_\circ(\bar{X}_t) - \beta) \right] < -\varepsilon_1 \quad \forall m \in \mathbb{N},
$$

(3.5)

for some $\varepsilon_1 > 0$. Let $c^m := \min\{m,c\}$, and $\beta_m$ denote the optimal ergodic value for $c^m$ in place of $c$, defined as in assumption (C). We first show that $\beta_m \to \beta$ as $m \to \infty$. By Theorem 2.1, there exists $\zeta_m \in \mathcal{M}_{\text{erg}}$ such that

$$
\beta_m = \int_K c^m \, d\zeta_m \quad \forall m > \beta + 2\bar{\varepsilon}.
$$

As argued in the proof of Theorem 2.1, $\zeta_m$ converges along some subsequence to a measure $a\xi + (1-a)\zeta'' \in \mathcal{P}(K^*)$, with $\xi \in \mathcal{P}(\partial K^*)$ and $\zeta'' \in \mathcal{P}(K)$. We employ a family $\{c^m_n, m, n \in \mathbb{N}\}$ of lower semi-continuous functions on $K^*$ defined as in (C) with $c$ replaced by $c^m$. Then, analogously to Subsection 2.2, for any fixed $m > \beta + 2\bar{\varepsilon}$, we have

$$
\lim_{k \to \infty} \beta_k \geq \lim_{k \to \infty} \int_K c^m \, d\zeta_k \geq a(\beta + \bar{\varepsilon}) + (1-a) \int_K c^m \, d\zeta''.
$$

Taking limits as $m \to \infty$ and using monotone convergence, we obtain

$$
\beta \geq \lim_{k \to \infty} \beta_k \geq a(\beta + \bar{\varepsilon}) + (1-a) \int_K c \, d\zeta''.
$$

(3.6)

This implies that $a < 1$, and therefore $\zeta'' \in \mathcal{M}_{\text{erg}}$ by Lemma 2.1. But then $\int_K c \, d\zeta'' \geq \beta$ by (2.3), and the equality $\lim_{k \to \infty} \beta_k = \beta$ follows from (3.6). Parenthetically, we mention that the above argument also shows that the sequence $\{\zeta_m\}_{m \in \mathbb{N}}$ is tight. Continuing, (3.5) implies that

$$
\beta_m - \beta \leq \lim_{T \to \infty} \frac{1}{T} \bar{\mathbb{E}}(x,1) \left[ \sum_{t=0}^{T-1} (c^m_\circ(\bar{X}_t) - \beta) \right] = -\varepsilon_1 \quad \forall m \in \mathbb{N},
$$

which is a contradiction. Note that for a stationary policy that is not $c$-stable, the claim is vacuously true because the left hand side of the inequality is $+\infty$. This completes the proof. \qed

Let $v \in \mathcal{U}_{\text{sm}}$ be $c$-stable. It follows from Lemma 3.1 that the strategy which agrees with $v_{\xi^*}$ on the support of $\pi_{\xi^*}$ and with $v$ on its complement is also $c$-stable.

It is clear from the definition of $Q$ that the first exit distribution of the split-chain from $\mathcal{B} \times \{1\}$ does not depend on $x \in \mathcal{B}$. Thus $x \mapsto \bar{\mathbb{E}}(x,1)[\tau]$ is constant on $\mathcal{B}$. This implies that, for all $f \in C_b(\mathcal{X})$, with $\bar{f}$ defined analogously to (2.25) so that $\bar{f}((x,1)) = 0$ for all $x \in \mathcal{B}$, we have

$$
\pi_{\xi^*}(f) := \int_{\mathcal{X}} f(y) \pi_{\xi^*}(dy) = \frac{\bar{\mathbb{E}}(x,1) \left[ \sum_{k=0}^{\tau-1} \bar{f}(\bar{X}_k) \right]}{\bar{\mathbb{E}}(x,1)[\tau]} \quad \forall x \in \mathcal{B}.
$$

(3.7)

In fact, (3.7) holds for any $f \in L^1(\mathcal{X}; \pi_0)$ by [36, Proposition 5.9]. Therefore, we have

$$
\bar{\mathbb{E}}(x,1) \left[ \sum_{k=0}^{\tau-1} (\bar{c}_\circ(\bar{X}_k) - \beta) \right] = 0 \quad \forall x \in \mathcal{B}.
$$

(3.8)

It then follows by Lemma 3.1 that the function

$$
\tilde{G}_0^{(i)}(x) = \tilde{G}_0(x,i) := \bar{\mathbb{E}}(x,i) \left[ \sum_{k=0}^{\tau-1} (\bar{c}_\circ(\bar{X}_k) - \beta) \right], \quad (x,i) \in \mathcal{X},
$$

(3.9)

is locally bounded from above. On the other hand, by Lemma 2.2 and the fact that $\bar{c}_\circ \geq \beta$ on $\mathcal{B}^c \times \{0\}$ we have $\inf_{x} \tilde{G}_0^{(i)} \geq -\beta \delta_0$.

In Subsection 3.2 we show that $\tilde{G}_0^{(i)}(x)$ solves the Poisson equation.
3.2. Solution to the Poisson equation. Let \( \hat{v} \in U_{\text{sm}} \) be \( c \)-stable, and such that it agrees \( \pi_{\zeta^*} \)-a.e. with the control \( v_{\zeta^*} \), obtained via the disintegration of an optimal ergodic occupation measure \( \zeta^* = \pi_{\zeta^*} \oplus v_{\zeta^*} \). By one step analysis, using (2.22)–(2.25) and (3.9), adopting the notation in Definition 2.1, we obtain

\[
\begin{align*}
\hat{G}_{\hat{v}}^{(1)}(x) &= -\beta + (1 - \delta) \int_{B} \hat{G}_{\hat{v}}^{(1)}(y) \nu(dy) + \delta \int_{B} \hat{G}_{\hat{v}}^{(1)}(y) \nu(dy), \quad x \in B, \\
\hat{G}_{\hat{v}}^{(0)}(x) &= c_0(x) - \beta + \frac{c_0(x)}{1 - \delta} + \frac{1}{1 - \delta} \int_{B} \hat{G}_{\hat{v}}^{(1)}(y) P_{\hat{v}}(dy | x) \\
&\quad + \frac{\delta}{1 - \delta} \int_{B} \hat{G}_{\hat{v}}^{(1)}(y) [P_{\hat{v}}(dy | x) - \delta \nu(dy)], \quad x \in B, \\
\end{align*}
\]

and

\[
\begin{align*}
\hat{G}_{\hat{v}}^{(1)}(x) &= c_0(x) - \beta + (1 - \delta) \int_{B} \hat{G}_{\hat{v}}^{(0)}(y) P_{\hat{v}}(dy | x) + \int_{B} \hat{G}_{\hat{v}}^{(0)}(y) P_{\hat{v}}(dy | x) \\
&\quad + \delta \int_{B} \hat{G}_{\hat{v}}^{(1)}(y) P_{\hat{v}}(dy | x), \quad x \in B^c.
\end{align*}
\]

Let

\[
\begin{align*}
\bar{v}(x, u) := c(x, u) - \beta, \quad \text{and} \quad \bar{v}_{\hat{v}}(x) := c_{\hat{v}}(x) - \beta.
\end{align*}
\]

Multiplying (3.10) and (3.11) by \( \delta \) and \( (1 - \delta) \), respectively, and adding them together, we obtain

\[
(1 - \delta) \hat{G}_{\hat{v}}^{(0)}(x) + \delta \hat{G}_{\hat{v}}^{(1)}(x) = \bar{v}_{\hat{v}}(x) + \int_{B} [(1 - \delta) \hat{G}_{\hat{v}}^{(0)}(y) + \delta \hat{G}_{\hat{v}}^{(1)}(y)] P_{\hat{v}}(dy | x) + \int_{B} \hat{G}_{\hat{v}}^{(0)}(y) P_{\hat{v}}(dy | x), \quad x \in B.
\]

We define

\[
\begin{align*}
G_{\hat{v}}(x) := \begin{cases} 
(1 - \delta) \hat{G}_{\hat{v}}^{(0)}(x) + \delta \hat{G}_{\hat{v}}^{(1)}(x), & \text{for } x \in B, \\
\hat{G}_{\hat{v}}^{(0)}(x), & \text{otherwise}.
\end{cases}
\end{align*}
\]

It follows by (3.12), (3.14), and (3.15) that

\[
G_{\hat{v}}(x) = \bar{v}_{\hat{v}}(x) + \int_{B} G_{\hat{v}}(y) P_{\hat{v}}(dy | x) = \bar{v}_{\hat{v}}(x) + P_{\hat{v}} G_{\hat{v}}(x), \quad x \in \mathbb{X}.
\]

It is clear that (3.8) implies that \( \hat{G}_{\hat{v}}^{(1)} \equiv 0 \) on \( B \). Thus

\[
\int_{B} G_{\hat{v}}(y) \nu(dy) = \int_{B} (1 - \delta) \hat{G}_{\hat{v}}^{(0)}(y) \nu(dy) = \beta
\]

by (3.10) and (3.15).

Note that \( \hat{G}_{\hat{v}}^{(i)} \) and \( G_{\hat{v}}^{(i)} \) are well defined \( \pi_{\zeta^*} \)-a.e. via (3.9) and (3.15).

3.3. Derivation of the Bellman equation (ACOE). Starting in this section, and throughout the rest of the paper, we enforce the following structural hypothesis on the controlled chain. This assumption is implicit in all the results of the paper which follow, unless otherwise mentioned.

**Assumption 3.1.** \( P(dy | x, u) \) is strongly continuous (or strong Feller), that is, the map \( K \ni (x, u) \mapsto \int_{\mathbb{X}} f(y) P(dy | x, u) \) is continuous for every \( f \in M_b(\mathbb{X}) \).

**Remark 3.2.** Assumption 3.1 implies that the family \( \{P(\cdot | x, u) : (x, u) \in K\} \) is tight for any compact set \( K \subset \mathbb{K} \). Indeed, for any sequence \( (x_n, u_n) \in K \) converging to some \( (x, u) \) in this set, we have \( P(\cdot | x_n, u_n) \Rightarrow P(\cdot | x, u) \). Then the above set, being the continuous image of a compact set, is compact. By Prokhorov’s theorem, it is tight.
Remark 3.3. If $X = \mathbb{R}^d$, a sufficient condition for Assumption 3.1 is that
\[ P(dy \mid x, u) = \varphi(y \mid x, u)\lambda(dy) \]
for a density function $\varphi$ with respect to $\lambda$, the Lebesgue measure on $\mathbb{R}^d$, and that the map
\[ (y, x, u) \in \mathbb{R}^d \times K \mapsto \varphi(y \mid x, u) \in [0, \infty) \]
is continuous. This implies the continuity of the measure-valued map $(x, u) \mapsto \varphi(y \mid x, u)dy$ in total variation norm by Scheffe’s theorem, which in turn implies Assumption 3.1.

Definition 3.2. Define
\[ \tilde{V}^{(0)}_*(x) := \inf_{u \in \mathcal{U}(x, 0)} \mathbb{E}_{(x, 0)}^u \left[ \sum_{k=0}^{\tau-1} (\dot{c}_k(X_k) - \beta) \right] , \quad x \in X, \]
Also let \( \tilde{V}^{(1)}_*(x) = 0 \) for $x \in \mathcal{B}$, and
\[ V_*(x) := \begin{cases} (1 - \delta)\tilde{V}^{(0)}_*(x), & \text{for } x \in \mathcal{B}, \\ \tilde{V}^{(0)}_*(x), & \text{otherwise}. \end{cases} \]

Recall the definition of $O(f)$ in Subsection 1.1, and that $\mathcal{L}(X)$ denotes the class of real-valued lower semi-continuous functions which are bounded from below in $X$.

Theorem 3.1. The function $V_*$ in Definition 3.2 is in the class $\mathcal{L}(X)$ and satisfies
\[ V_*(x) = \min_{u \in \mathcal{U}(x)} \left[ \bar{c}(x, u) + \int_X V_*(y) P(dy \mid x, u) \right] \quad \forall x \in X, \quad (3.17) \]
with $\bar{c}$ as in (3.13). Moreover, every $v_* \in \mathcal{U}_{sm}$ which satisfies
\[ v_*(x) \in \text{Arg} \min_{u \in \mathcal{U}(x)} \left[ \bar{c}(x, u) + P_u V_*(x) \right] \quad (3.18) \]
is an optimal stationary Markov strategy. In addition, (3.17) has, up to an additive constant, a unique solution in $\mathcal{L}(X) \cap O(V_*)$.

Proof. As in (3.11), using standard dynamic programming arguments in place of one step analysis, we obtain
\[ \tilde{V}^{(0)}_*(x) = \min_{u \in \mathcal{U}(x)} \left[ \frac{c(x, u)}{1 - \delta} - \beta + \int_{\mathcal{B}} \tilde{V}^{(0)}_*(y) \left[ P(dy \mid x, u) - \delta \nu(dy) \right] \right] \]
\[ + \frac{1}{1 - \delta} \int_{\mathcal{B}^c} \tilde{V}^{(0)}_*(y) P(dy \mid x, u) , \quad x \in \mathcal{B}, \quad (3.19) \]
and
\[ \tilde{V}^{(0)}_*(x) = \min_{u \in \mathcal{U}(x)} \left[ c(x, u) - \beta + (1 - \delta) \int_{\mathcal{B}} \tilde{V}^{(0)}_*(y) P(dy \mid x, u) + \int_{\mathcal{B}^c} \tilde{V}^{(0)}_*(y) P(dy \mid x, u) \right] \quad (3.20) \]
for $x \in \mathcal{B}^c$.

On the other hand, since $\tilde{V}^{(0)}_* = \tilde{g}^{(0)}_{V_*} \nu$-a.e. by Lemma 3.1, then (3.10) shows that
\[ 0 = \tilde{V}^{(1)}_*(x) = -\beta + (1 - \delta) \int_{\mathcal{B}} \tilde{V}^{(0)}_*(y) \nu(dy) \quad \forall x \in \mathcal{B}. \quad (3.21) \]
It then follows by (3.19)–(3.21) and Definition 3.2, that $V_*$ satisfies
\[ V_*(x) = \min_{u \in \mathcal{U}(x)} \left[ \bar{c}(x, u) + P_u V_*(x) \right]. \quad (3.22) \]
Since the kernel $P$ is strongly continuous and $V_*$ is bounded from below in $X$ by Lemma 2.2, the map $(x, u) \mapsto P_u V_*(x)$ is lower semi-continuous on $K$. Therefore, since $\mathcal{U}$ is upper semi-continuous,
the map \((x, u) \mapsto \tau(x, u) + P_v V_\ast (x)\) is \(K\)-inf-compact by [22, Lemma 2.1 (i)]. Hence, applying Theorem 2.1 of [22] to (3.22), we deduce that \(V_\ast \in \mathcal{L}(\mathcal{X})\). Since \(V_\ast\) is bounded from below in \(\mathcal{X}\) by Lemma 2.2, existence and optimality of \(v_\ast\) in (3.18) follows by a standard argument using Birkhoff’s ergodic theorem.

We continue with the proof of uniqueness. Since \(\tilde{V}_\ast(0)\) is bounded from below in \(\mathcal{X}\), it is standard to show, using (3.20) and Fatou’s lemma, that

\[ \tilde{V}_\ast(0)(x) \geq \tilde{E}^{v_\ast}_{x,0} \left[ \sum_{k=0}^{\tau - 1} (\tilde{e}_{v_\ast}(\tilde{X}_k) - \beta) \right], \quad x \in \mathcal{X}, \tag{3.23} \]

with \(v_\ast\) as in (3.18). Definition 3.2 shows that we must have equality in (3.23). In turn, applying Dynkin’s formula to (3.20) we obtain

\[ \tilde{V}_\ast(0)(x) = \lim_{n \to \infty} \tilde{E}^{v_\ast}_{x,0} \left[ \sum_{k=0}^{\tau \wedge n - 1} (\tilde{e}_{v_\ast}(\tilde{X}_k) - \beta) \right], \quad x \in \mathcal{X}, \]

and

\[ \lim_{n \to \infty} \sup \tilde{E}^{v_\ast}_{x,0} \left[ \tilde{V}_\ast(0)(\tilde{X}_\tau) \mathbb{1}_{\{\tau > n\}} \right] = 0. \tag{3.24} \]

Let \(V \in \mathcal{L}(\mathcal{X}) \cap \mathcal{O}(V_\ast)\) be a solution of (3.17) and \(\hat{v} \in \mathcal{U}_{sm}\) a selector from its minimizer. Going to the split chain and scaling with an additive constant, we obtain functions \(\tilde{V}^{(i)}(x), i = 0, 1\), which satisfy (3.19)–(3.21) (with \(\tilde{V}^{(i)}\) replaced by \(\tilde{V}(i)\)), and

\[ V(x) := \begin{cases} (1 - \delta)\tilde{V}^{(0)}(x), & \text{for } x \in \mathcal{B}, \\ \tilde{V}^{(0)}(x), & \text{otherwise}. \end{cases} \]

In analogy to (3.23), we also have

\[ \tilde{V}^{(0)}(x) \geq \tilde{E}^{\hat{v}}_{x,0} \left[ \sum_{k=0}^{\tau - 1} (\tilde{e}_{\hat{v}}(\tilde{X}_k) - \beta) \right] \geq \tilde{V}_\ast^{(0)}(x), \quad x \in \mathcal{X}, \tag{3.25} \]

where for the second inequality we use Definition 3.2. Thus, if \(v_\ast\) is as in (3.18), then using the kernel \(Q_{v_\ast}\) of the split chain in (2.21), we deduce that \(\tilde{V}^{(i)} - \tilde{V}_\ast^{(i)}\) is a nonnegative local supermartingale under \(Q_{v_\ast}\). Since \(\tilde{V}^{(1)} - \tilde{V}_\ast^{(1)} = 0\), and \(V \in \mathcal{O}(V_\ast)\), using Dynkin’s formula, we obtain from (3.24) and the supermartingale inequality that

\[ \tilde{V}^{(0)}(x) - \tilde{V}_\ast^{(0)}(x) \leq 0 \quad \forall x \in \mathcal{X}. \tag{3.26} \]

Therefore, \(\tilde{V}^{(0)} = \tilde{V}_\ast^{(0)}\) on \(\mathcal{X}\) by (3.25) and (3.26). This completes the proof. \(\square\)

Remark 3.4. If we relax the strong Feller hypothesis in Assumption 3.1, and assume instead that the transition kernel is weak Feller, we can obtain an ACOI with a lower semi-continuous potential function. Indeed, if we let

\[ \tilde{V}_\ast(x) := \sup_{r > 0} \inf_{y \in B_r(x)} V_\ast(y), \]

with \(B_r(x)\) denoting the open ball of radius \(r\) centered at \(x\), then \(\tilde{V}_\ast \in \mathcal{L}(\mathcal{X})\). Therefore, by (3.16) we have

\[ V_\ast(x) \geq \inf_{u \in \mathcal{U}(x)} \left[ \tilde{\tau}(x, u) + P_v V_\ast(x) \right] \geq \inf_{u \in \mathcal{U}(x)} \left[ \tilde{\tau}(x, u) + P_u \tilde{V}_\ast(x) \right], \tag{3.27} \]

and the term on the right-hand side of (3.27) is in \(\mathcal{L}(\mathcal{X})\). Since \(\tilde{V}_\ast\) is the largest lower semi-continuous function dominated by \(V_\ast\) [41], we obtain

\[ \tilde{V}_\ast(x) \geq \inf_{u \in \mathcal{U}(x)} \left[ \tilde{\tau}(x, u) + P_u \tilde{V}_\ast(x) \right]. \]
It is standard to show that any measurable selector from this equation is optimal. We refer the reader to [30, 41] on how to improve this to an ACOE under additional hypotheses.

Remark 3.5. Our approach differs from the standard approach of deriving the Bellman equation using a vanishing discount argument. We briefly indicate here how near-monotonicity or inf-compactness of the cost function can help us with the standard methodology. One important consequence of near-monotonicity is that we can prove that the discounted value function attains its minimum on a fixed compact set as the discount parameter varies. Thus if we can establish equicontinuity of the relative discounted value functions as the discount factor varies (e.g., using convexity when available as in [24], or in Example 6.1 in Section 6), one can argue that as the discount parameter tends to 1, the relative discounted value functions either remain bounded on compacts or tend to infinity uniformly on compacts along a subsequence. Eliminating the latter possibility by a suitable choice of the offset in the definition of the relative discounted value function shows uniform boundedness over compacts. This idea is used in [1] for deriving the Bellman equation for the average cost for a specific class of problems, and can potentially be generalized. See also [21, Theorem 6].

Remark 3.6. It is worth noting that the derivation of the Bellman equation (3.22) does not require the strong continuity of Assumption 3.1; weak continuity will suffice. We do, however, require strong continuity in order to obtain a solution in the class $\mathcal{L}(X)$.

4. The value iteration

Throughout this section as well as Section 6, $v_\ast \in \mathcal{U}_{\text{stat}}^\ast$ is some optimal stationary Markov strategy which is kept fixed.

4.1. The value iteration algorithm. We start with the following definition.

Definition 4.1. [Value Iteration] Given $\Phi_0 \in \mathcal{L}(X)$ which serves as an initial condition, we define the value iteration (VI) by

$$\Phi_{n+1}(x) = \overline{T}\Phi_n(x) := \min_{u \in \mathcal{U}(x)} \left[ \tau(x,u) + P_u \Phi_n(x) \right], \quad n \in \mathbb{N}_0. \quad (4.1)$$

Since $\overline{T}: \mathcal{L}(X) \to \mathcal{L}(X)$, it is clear that the algorithm lives in the space of lower semi-continuous functions which are bounded from below in $X$. It is also clear that $\overline{T}$ is a monotone operator on $\mathcal{L}(X)$, that is, for any $f, f' \in \mathcal{L}(X)$ with $f \leq f'$, we have $\overline{T}f \leq \overline{T}f'$.

4.1.1. The value iteration for the split chain. Using (2.22)–(2.24) we can also express the algorithm via the split chain as follows. The value iteration functions $\{\Phi_n(i), n \in \mathbb{N}_0, i = 0, 1\}$, are defined as follows. Let $V_0: X \to \mathbb{R}$ be a nonnegative continuous function. The initial condition is $\Phi_0^{(1)} = 0$, and $\Phi_0^{(0)}(x) = V_0(x)\left[(1 - \delta)^{-1}1_B(x) + 1_{B^c}(x)\right]$, and for each $n \in \mathbb{N}$, define

$$\Phi_n(x) = \begin{cases} (1 - \delta)\Phi_n^{(0)}(x) + \delta \Phi_n^{(1)}(x), & \text{for } x \in B, \\ \Phi_n^{(0)}(x), & \text{otherwise}. \end{cases} \quad (4.2)$$

Thus, the algorithm takes the form

$$\Phi_{n+1}^{(0)}(x) = \frac{1}{1 - \delta} \min_{u \in \mathcal{U}(x)} \left[ \tau(x,u) + \int_X \Phi_n(y) P(dy \mid x,u) \right]$$

$$- \frac{\delta}{1 - \delta} \int_B \Phi_n(y) \nu(dy), \quad x \in B, \quad (4.3)$$

$$\Phi_{n+1}^{(0)}(x) = \min_{u \in \mathcal{U}(x)} \left[ \tau(x,u) + \int_X \Phi_n(y) P(dy \mid x,u) \right], \quad x \in B^c, \quad (4.4)$$
(4.5)

\[
\tilde{\Phi}^{(1)}_{n+1}(x) = -\beta + \int_{\mathcal{B}} \Phi_n(y) \nu(dy), \quad x \in \mathcal{B}.
\]

**Notation 4.1.** We adopt the following simplified notation. We let \( \hat{\nu}_n \in \mathcal{U}_{\text{sm}} \) be a measurable selector from the minimizer of (4.1), and define

\[
\hat{P}_n(\cdot | x) := P(\cdot | x, \hat{\nu}_n(x)), \quad \text{and} \quad \hat{\tau}_n(x) := c(x, \hat{\nu}_n(x)) - \beta.
\]

Note that these depend on the initial value \( \Phi_0 \).

We fix an optimal strategy \( v_* \in \mathcal{U}_{\text{sm}}^* \), and let \( P_*(dy | x) \) denote the transition kernel under \( v_* \). In addition, we let \( c_*(x) = c(x, v_*(x)) \) and \( \tau_* = c_* - \beta \).

With this notation, for \( n \in \mathbb{N}_0 \), we have

\[
\Phi_{n+1}(x) = \min_{u \in \mathcal{U}(x)} [\tau(x, u) + P_u \Phi_n(x)]
\]

\[
= \hat{\tau}_n(x) + \hat{P}_n \Phi_n(x) \quad \forall x \in \mathcal{X},
\]

and

\[
V_*(x) = \tau_*(x) + P_* V_n(x) \quad \forall x \in \mathcal{X}.
\]

It follows from optimality of \( v_* \) and \( \hat{\nu}_n \) that

\[
\Phi_{n+1} \leq \tau_* + P_* \Phi_n,
\]

and

\[
V_* \leq \hat{\tau}_n + \hat{P}_n V_*.
\]

**4.2. General results on convergence of the VI.** Recall the function \( V_* \) from Theorem 3.1, and let \( \tau_* \) denote the associated invariant probability measure. Consider the following hypothesis.

(H1) \( \tau_*(V_*) < \infty \).

For \( c \) bounded, finiteness of the second moments of \( \bar{\tau} \) implies (H1) (see, for example, [7, p. 66]).

In general, (H1) is equivalent to the finiteness of the second moments of the modulated first hitting times to \( \mathcal{B} \times \{1\} \) on a full and absorbing set.

For a constant \( \kappa \in \mathbb{R} \) we define the set

\[
\mathcal{V}(\kappa) := \{ f \in \mathcal{L}(\mathcal{X}) \cap \mathcal{O}(V_*): f \geq V_* - \kappa, \ \tau_*(f) \leq \kappa + 1 \}.
\]

Under (H1), we show that the VI converges pointwise for any \( \Phi_0 \in \mathcal{V}(\kappa) \). In order to prove this result, we need the following lemma.

**Lemma 4.1.** Under (H1), if \( \Phi_0 \in \mathcal{V}(\kappa) \) for some \( \kappa \in \mathbb{R} \), then \( \Phi_n \in \mathcal{V}(\kappa) \) for all \( n \in \mathbb{N} \), or in other words, the set \( \mathcal{V}(\kappa) \) is invariant under the action of \( \mathcal{T} \). In addition, \( \tau_*(\Phi_{n+1}) \leq \tau_*(\Phi_n) \) for all \( n \in \mathbb{N}_0 \).

**Proof.** Subtracting (4.8) from (4.9) we obtain

\[
\Phi_{n+1} - V_* \leq P_*(\Phi_n - V_*),
\]

while by subtracting (4.10) from (4.7) we have

\[
\Phi_{n+1} - V_* \geq \hat{P}_n(\Phi_n - V_*).
\]

Applying (14.4) of [32, Theorem 14.0.1] to \( g \) for any \( g \in \mathcal{O}(V_*) \), we have: there exists a constant \( \bar{m}(x) \) depending on \( x \) such that

\[
\|P_\nu^ng(x)\|_{V_*} \leq \bar{m}(x)\|g\|_{V_*} + \frac{\tau_*(g)}{V_*(x) + 1} \quad \forall x \in \mathcal{X}, \quad \forall n \in \mathbb{N}.
\]
Therefore \( \pi_\ast(\Phi_{n+1}) \leq \pi_\ast(\Phi_n) \) by (4.12). From this it follows by induction that \( \pi_\ast(\Phi_n) \leq \kappa + 1 \forall n \geq 0 \), if it is so for \( n = 0 \). Likewise, \( \Phi_{n+1} - V_* \geq \inf_X (\Phi_n - V_*) \) by (4.13). From this it follows by induction that \( \phi_n \geq V_* - \kappa \forall n \geq 0 \), if it is so for \( n = 0 \).

The result then follows from these. \( \square \)

**Theorem 4.1.** Assume (H1), and suppose \( \Phi_0 \in \mathcal{V}(\kappa) \) for some \( \kappa \in \mathbb{R} \). Then the following hold

\[
\Phi_n \xrightarrow{n \to \infty} V_* + \lim_{n \to \infty} \pi_\ast(\Phi_n - V_*) \quad \text{in } L^1(X;\pi_\ast) \quad \text{and } \pi_\ast\text{-a.s.}.
\]

(4.14)

Also,

\[
\lim_{n \to \infty} |\nu(\Phi_n) - \pi_\ast(\Phi_n - V_*)| = \beta.
\]

(4.15)

**Proof.** Let \( \{X^*_n\}_{n \in \mathbb{Z}} \) denote the stationary optimal process controlled by \( v_* \). If \( \Phi_0 \in \mathcal{V}(\kappa) \), we have

\[
\sup_{n \in \mathbb{N}} \int |\Phi_n(x) - V_*(x)| \pi_\ast(dx) < \infty
\]

by Lemma 4.1. Then (4.12) implies that the process

\[
M_k := \{\Phi_{-k}(X^*_k) - V_*(X^*_k)\}_{k \leq 0}
\]

is a backward submartingale with respect to the filtration \( \{\mathcal{F}_k\}_{k \leq 0} := \{\sigma(X^*_\ell, \ell \leq k)\}_{k \leq 0} \). By Corollary V-3.13, p. 119, [34], \( M_k \) converges a.s. and in the mean to some random variable \( M^* \). The latter implies the convergence in \( L^1(X;\pi_\ast) \) claimed in (4.14). By the ergodicity of \( \{X^*_n\}_{n \in \mathbb{Z}} \) the \( M^* \) is a constant \( \pi_\ast\text{-a.s.} \). This is because \( M^* \) is measurable with respect to the tail \( \sigma\)-field \( \cap_{k \leq 0} \sigma(X^*_m, m \leq k) \) which is a.s. trivial by the ergodicity of \( \{X^*_n\} \).

Convergence of \( \Phi_n \) in \( L^1(X;\pi_\ast) \), and hence also in \( L^1(B;\nu) \) by (A0), implies that \( \nu(\Phi_n) \) converges. Since \( \nu(V_*) = \beta \) by (3.21), (4.15) then follows from (4.14). \( \square \)

**Corollary 4.1.** Assume (H1), and suppose that \( V_* \) is bounded. Then \( \Phi_n(x) - V_*(x) \) converges to a constant \( \pi_\ast\text{-a.e.} \) as \( n \to \infty \) for any initial condition \( \Phi_0 \in \mathcal{L}_b(X) \).

**Proof.** This clearly follows from Theorem 4.1, since if \( \Phi_0 \in \mathcal{L}_b(X) \), then \( \Phi_0 \in \mathcal{V}(\kappa) \) for some \( \kappa \in \mathbb{R} \). \( \square \)

5. Relative value iteration

We consider three variations of the relative value iteration algorithm (RVI). All these start with initial condition \( V_0 \in \mathcal{L}(X) \).

Let

\[
\mathcal{S}f(x) := \inf_{u \in \mathcal{U}(x)} \left[ c(x,u) + P_u f(x) \right], \quad f \in \mathcal{L}(X).
\]

The iterates \( \{V_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X) \) are defined by

\[
V_n(x) = \mathcal{T} V_{n-1}(x) := \mathcal{S} V_{n-1}(x) - \nu(V_{n-1}), \quad x \in X. \tag{5.1}
\]

An important variation of this is

\[
\bar{V}_n(x) = \bar{T} \bar{V}_{n-1}(x) := \mathcal{S} \bar{V}_{n-1}(x) - \min \bar{V}_{n-1}, \quad x \in X. \tag{5.2}
\]

Also, we can modify (5.1) to

\[
\tilde{V}_n(x) = \tilde{T} \tilde{V}_{n-1}(x) := \mathcal{S} \tilde{V}_{n-1}(x) - \tilde{V}_{n-1}(\hat{x}), \quad x \in X, \tag{5.3}
\]

where \( \hat{x} \in \mathcal{B} \) is some point that is kept fixed.

We let \( \hat{v}_n \) be a measurable selector from the minimizer of (5.1)–(5.3) (note that all three minimizers agree if the algorithms start with the same initial condition). We refer to \( \{\hat{v}_n\} \) as the receding horizon control sequence.
Lemma 5.1. Provided that $\Phi_0 = V_0$, then we have
\[ \Phi_n(x) - \Phi_n(y) = V_n(x) - V_n(y) \quad \forall x, y \in X, \forall n \in \mathbb{N}, \] (5.4)
and the same applies if $V_n$ is replaced by $\tilde{V}_n$ or $\hat{V}_n$. In addition, the convergence of $\{\Phi_n\}$ implies the convergence of $\{V_n\}$, and also that of $\{\tilde{V}_n\}, \{\hat{V}_n\}$ in the same space.

Proof. A straightforward calculation shows that
\[ V_n(x) - \Phi_n(x) = n\beta - \sum_{k=0}^{n-1} \nu(V_k), \]
from which (5.4) follows. The proofs for $\tilde{V}_n$ and $\hat{V}_n$ are completely analogous.

We have
\[ V_{n+1}(x) - \Phi_{n+1}(x) = V_n(x) - \Phi_n(x) + \beta - \nu(V_n), \]
which implies that
\[ \nu(V_{n+1}) = \nu(\Phi_{n+1}) - \nu(\Phi_n) + \beta. \]
Therefore, convergence of $\{\Phi_n\}$ implies that $\nu(V_n) \to \beta$ as $n \to \infty$. In turn, this implies the convergence of $\{V_n\}$ by (5.4). In the case of $\{\tilde{V}_n\}$ we obtain $\min_x \tilde{V}_n \to \beta$ as $n \to \infty$, and analogously for $\{\hat{V}_n\}$.

The following theorem, under hypotheses (a)–(b), is a direct consequence of Theorem 4.1, Corollary 4.1, and Lemma 5.1. Hypothesis (H2) is given in the beginning of the next section.

Theorem 5.1. Let one of the following assumptions be satisfied.

(a) (H1) holds and $V_0 \in \mathcal{V}(\kappa)$ for some $\kappa \in \mathbb{R}$. Here, $\mathcal{V}(\kappa)$ is as defined in (4.11).
(b) (H1) holds, $V_\ast$ is bounded, and $V_0 \in \mathcal{L}_b(X)$.
(c) (H2) holds and $V_0 \in \mathcal{L}(X) \cap \mathcal{O}(V_\ast)$.

Then the value iteration functions in (5.1)–(5.3) converge $\pi_\ast$-a.e. to $V_\ast$ as $n \to \infty$. In addition, if (c) holds, then convergence is pointwise for all $x \in X$.

The assertions concerning (H2) are proved in the next section. They are included in Theorem 5.1 in order to give a unified statement.

6. Stability of the Rolling Horizon Procedure

Consider the following hypothesis.

(H2) There exist constants $\theta_1 > 0$ and $\theta_2$ such that
\[ \min_{u \in \mathcal{U}(x)} c(x, u) \geq \theta_1 V_\ast(x) - \theta_2 \quad \forall x \in X. \]

Without loss of generality, we assume that $\theta_1 \in (0, 1)$.

Remark 6.1. Hypothesis (H2) can be written in the following equivalent, but seemingly more general form.

(H2') There exists $v \in \mathcal{U}_{sm}$, and a function $V_v : X \to [1, \infty)$ satisfying
\[ \min_{u \in \mathcal{U}(x)} c(x, u) \geq \theta_1 V_v(x) - \theta_2 \quad \forall x \in X, \]
for some constants $\theta_1 > 0$ and $\theta_2$, and
\[ P_v V_v(x) - V_v(x) \leq C \mathbb{1}_B(x) - c_v(x) \]
for some constant $C$. 

It is clear that (H2) implies (H2′) (take \( v \) an optimal stationary policy), while the converse follows by the stochastic representation of \( V_\ast \) and Definition 3.2, whereby \( V_\ast(\cdot) \geq V_\ast(\cdot) \). From the definition of \( V_\ast \), it is clear by ‘one step analysis’ that \( V_\ast(x) \geq v_1 \min_u c(x,u) \) for suitable constants \( v_1, v_2 > 0 \). Hence the above hypothesis says that \( V_\ast \) and \( c_{\min}(\cdot):=\min_u c(\cdot,u) \) have comparable growth. From the definition of \( V_\ast \), it also follows that
\[
\int V_\ast(y) P(dy| x,v_\ast(x)) - V_\ast(x) \leq -c_{\min}(x) + C
\]
for some \( C > 0 \). Then by Theorem 15.0.1, pp. 362-3, of [32], a necessary condition for the above is that the process be geometrically ergodic under \( v^\ast \).

Concerning the value iteration, we have the following:

**Theorem 6.1.** Assume (H2), and suppose that the initial condition \( \Phi_0 \) lies in \( \mathcal{L}(X) \cap \mathcal{O}(V_\ast) \). Then, there exists a constant \( \tilde{C}_0 \) depending on \( \Phi_0 \) such that
\[
|\Phi_n(x) - V_\ast(x)| \leq \tilde{C}_0 (1 + (1 - \theta_1)^n V_\ast(x)) \quad \forall x \in X, \forall n \in \mathbb{N}.
\]
In addition, \( \Phi_n(x) - V_\ast(x) \) converges to a constant \( \pi_\ast \)-a.e. as \( n \to \infty \).

**Proof.** Under (H2) we obtain
\[
P_\ast V_\ast(x) = \beta - c_\ast(x) + V_\ast(x) \leq \beta + \theta_2 + (1 - \theta_1)V_\ast(x).
\]
Let \( \rho := 1 - \theta_1 \), and define
\[
f_n(x) := \Phi_n(x) - (1 - \rho^n)(V_\ast(x) - \frac{\theta_1 + \theta_2}{\theta_1}).
\]
Recall (4.6) and (4.8). We have
\[
f_{n+1}(x) - \tilde{P}_n f_n(x) = \tilde{c}(n)(x) - \theta_1 \rho^n V_\ast(x) - \frac{\theta_1 + \theta_2}{\theta_1} + (1 - \rho^n)(\tilde{P}_n - I)V_\ast(x)
\]
\[
\geq \tilde{c}(n)(x) - \theta_1 \rho^n V_\ast(x) - \frac{\theta_1 + \theta_2}{\theta_1} - (1 - \rho^n)\tilde{c}(n)(x)
\]
\[
= \rho^n (-\theta_1 V_\ast(x) + \theta_2 + c_\ast(x)) \geq 0 \quad \forall (x,n) \in X \times \mathbb{N},
\]
where we also used (4.10). Iterating the above inequality, we get \( f_n \geq \inf_X \Phi_0 \) for all \( n \in \mathbb{N} \). Assuming without loss of generality that \( \Phi_0 \) is nonnegative, this implies that
\[
(1 - \rho^n)(V_\ast(x) - \frac{\theta_1 + \theta_2}{\theta_1}) \leq \Phi_n(x).
\]
On the other hand, by (4.12) and (6.1), we obtain
\[
\Phi_n(x) \leq V_\ast(x) + \tilde{C}_0(\tilde{C}_1 + \rho^n V_\ast(x))
\]
for some constants \( \tilde{C}_0 \) and \( \tilde{C}_1 \) which depend on \( \Phi_0 \). Since \( \pi_\ast(\Phi_n - V_\ast) \) is bounded from above by (6.3), and bounded from below by (6.2), the result follows by the same argument as was used in the proof of Theorem 4.1. \( \square \)

**Definition 6.1.** We say that \( v \in \mathcal{U}_{\text{min}} \) is stabilizing if
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E}^x \left[ \sum_{k=0}^{N-1} c_v(X_k) \right] < \infty \quad \forall x \in X,
\]
and denote the class of stabilizing controls by \( \mathcal{U}_{\text{stab}} \).

Recall the definition of \( \hat{v}_n \) in Notation 4.1. We have the following theorem.

**Theorem 6.2.** Under (H2), for every \( \Phi_0 \in \mathcal{L}(X) \cap \mathcal{O}(V_\ast) \) there exists \( N_0 \in \mathbb{N} \) such that the stationary Markov control \( \hat{v}_n \) is stabilizing for any \( n \geq N_0 \).
Proof. Combining (4.7), (6.2), and (6.3), we obtain
\[
\tilde{P}_n \left( (1 - \rho^n)(V_*(x) - \frac{\beta + \theta_2}{\theta_1}) \right) \leq \tilde{P}_n \Phi_n(x)
\]
\[
= -\tilde{c}_n(x) + \Phi_{n+1}(x)
\]
\[
\leq -\tilde{c}_n(x) + V_*(x) + C_0(\tilde{C}_1 + \rho^{n+1}V_*(x))
\]
Rearranging, this gives
\[
(1 - \rho^n)\tilde{P}_n V_*(x) \leq -\tilde{c}_n(x) + (1 - \rho^n)\frac{\beta + \theta_2}{\theta_1} + C_0\tilde{C}_1 + (1 + C_0\rho)\rho^nV_*(x) + (1 - \rho^n)V_*(x).
\]
From (H2) we have \( V_*(x) \leq \frac{-\tilde{c}_n + \beta + \theta_2}{\theta_1} \), and using this in (6.4) gives
\[
(1 - \rho^n)\tilde{P}_n V_*(x) \leq -\left(1 - \frac{1+C_0\rho}{\theta_1}\right)\tilde{c}_n(x) + (1 + C_0\rho^{n+1})\frac{\beta + \theta_2}{\theta_1} + C_0\tilde{C}_1 + (1 - \rho^n)V_*(x).
\]
Let \( N_0 \in \mathbb{N} \) be large enough such that \( \frac{1+C_0\rho}{\theta_1} < 1 \) and let \( n \geq N_0 \). Since \( 0 < \rho < 1 \), adding and subtracting \( (1 - \rho^n) V_*(x) \) on the right hand side of (6.5) and using it to form a telescoping sum, the fact that \( V_*(x) \) is bounded from below in \( X \) coupled with Fatou’s lemma leads to
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_x^{\Phi_n} \left[ \sum_{k=0}^{N-1} c_n(X_k) \right] \leq \beta + \frac{(1 + C_0\rho^{n+1})(\beta + \theta_2) + \theta_1\tilde{C}_0\tilde{C}_1}{\theta_1 - (1 + C_0\rho)\rho^n} \forall n \geq N_0,
\]
with \( c_n(x) \) := \( c(x, \tilde{v}_n(x)) \). This shows that \( \tilde{v}_n \) is stabilizing for all \( n \geq N_0 \).

We improve the convergence result in Theorem 6.1.

**Theorem 6.3.** Assume (H2). Then, for every initial condition \( \Phi_0 \in \mathcal{L}(X) \cap \mathcal{O}(V_*) \), the sequence \( \Phi_n(x) - V_*(x) \) converges pointwise to a constant.

Proof. Without loss of generality, we may translate the initial condition \( \Phi_0 \) by a constant so that \( \Phi_n \to V_* \) as \( n \to \infty \) \( \tau_\sigma \)-a.s. Then of course
\[
\Phi_n^{(1)} \to \tilde{V}_*^{(1)} = 0 \text{ on } B.
\]
Recall also, that these functions are constant on \( B \). Let \( \Psi_n := \Phi_n - V_* \). Then \( |\Psi_n| = \epsilon_n \) on \( B \), for some sequence \( \epsilon_n \to 0 \). Also
\[
|\Psi_n(x)| \leq \tilde{C}_0(1 + (1 - \theta_1)^nV_*(x)) \forall x \in X
\]
by Theorem 6.1, and
\[
P^n_\tau V_*(x) \leq \frac{\beta + \theta_2}{\theta_1} + (1 - \theta_1)^nV_*(x)
\]
by (6.1). Let \( \tau_m \) denote the first exit time from the ball of radius \( m \) centered at some fixed point \( x_0 \). Applying the optional sampling theorem to (4.12) relative to the stopping time \( \check{\tau} \land n \land \tau_m \), we obtain
\[
\check{\Psi}_n^{(0)}(x) \leq \check{\mathbb{E}}_{n,0}^{v_*} \left[ \check{\Psi}_n^{(1)}(\check{X}_\check{\tau})I_{\{\check{\tau} \leq m \land \tau_m \}} + \check{\Psi}_n^{(0)}(\check{X}_\check{\tau})I_{\{n < \check{\tau} < \tau_m \}} + \check{\Psi}_n^{(0)}(\check{X}_\tau)I_{\{\tau \leq \check{\tau} \land n \}} \right]
\]
\[
\leq \sup_{k \geq n} \epsilon_k + \check{\mathbb{E}}_{n,0}^{v_*} \left[ \check{\Psi}_n^{(1)}(\check{X}_n)I_{\{\check{\tau} > n \}} \right] + \check{\mathbb{E}}_{n,0}^{v_*} \left[ \check{\Psi}_n^{(0)}(\check{X}_\tau)I_{\{\tau \leq n \}} \right].
\]
By (4.2), we have
\[
|\check{\Psi}_n^{(0)}| \leq \frac{1}{1 - \delta} |\Psi_n| + \frac{\delta \epsilon_n}{1 - \delta}. \tag{6.11}
\]
Therefore
\[
\limsup_{n \to \infty} \check{\mathbb{E}}_{n,0}^{v_*} \left[ \check{\Psi}_n^{(0)}(\check{X}_n)I_{\{\check{\tau} > n \}} \right] \leq 0
\]
by (6.8), (6.9), and (6.11) and the fact that \( \check{\mathbb{E}}_{n,0}^{v_*}[\check{\tau}] < \infty \).
Lemma 2.2 also shows that \( \limsup \) reduces to \( \lambda \). We claim that \( \ell \) tends to 0 as \( m \to \infty \) for any fixed \( n \in \mathbb{N} \).

A slight modification of (6.10) also shows that \( \limsup \) tends to 0 as \( m \to \infty \) for any fixed \( n \in \mathbb{N} \).

Notation 4.1

By a standard application of the optional sampling theorem to the Poisson equation \( P V_n + c \_

Taking expectations, using (6.15) and (6.17), we see that the third term on the right-hand side of (6.10) tends to 0 as \( m \to \infty \) for any fixed \( n \in \mathbb{N} \).

For all \( m,k \in \mathbb{N} \), we see that there exists some constant \( \kappa_2 \) such that

\[
\mathbb{E}_{x}^{(m)} \left[ V_n^*(X_{t_m}) 1_{\{ t_m \leq n \}} \right] \leq \kappa_2 \left( \mathbb{E}_{x}^{(m)} \left[ (t_n - m) \right] + (\gamma - \theta) \right) \]

for all \( m,k \in \mathbb{N} \). This shows that the third term on the right-hand side of (6.10) tends to 0 as \( m \to \infty \) for any fixed \( n \in \mathbb{N} \).

A slight modification of (6.10) also shows that \( \limsup_{n \to \infty} \frac{\Phi_n(0)}{n} \) tends to 0 as \( m \to \infty \) for any fixed \( n \in \mathbb{N} \).

In the second part of the proof, we establish that \( \lim\inf_{n \to \infty} \Phi_n(0) \) agrees with the value of superior limit in (6.13). Let \( \hat{u}_n \) denote the nonstationary Markov policy \((\hat{u}_n, \hat{u}_{n-1}, \ldots, \hat{u}_1)\) with \( \hat{u}_n \) as defined in Notation 4.1. We claim that

\[
\mathbb{E}_{x}^{(m)} \left[ \hat{u}_n(0, x, y) \right] \to \mathbb{E}_{x}^{(m)} \left[ \hat{u}_n(0, x, y) \right] \quad \text{as} \quad m \to \infty.
\]

Summing \((1 - \delta) \times \text{equation } (4.3)\) and \( \delta \times \text{equation } (4.5)\) for \( x \in \mathcal{B} \) and using (4.4) for \( x \in \mathcal{B}^c \), we recover (4.1) in view of (4.2). To prove the claim, we apply Dynkin’s formula for the stopping time \( \tau \land \tau_m \land n \) and Fatou’s Lemma as \( m \uparrow \infty \) to the value iteration till \( \tau \land \tau_m \land n \) over the split chain, in order to obtain

\[
\Phi_n(x) \geq \mathbb{E}_{x}^{(m)} \left[ \tau \land n \right] \sum_{k=0}^{\tau \land n} (\bar{c}_n - \beta) + \mathbb{E}_{x}^{(m)} \left[ \Phi_n(0) \right] \quad \text{as} \quad m \to \infty.
\]

leading to

\[
\Phi_n(x) \geq \mathbb{E}_{x}^{(m)} \left[ \tau \land n \right] \sum_{k=0}^{\tau \land n} (\bar{c}_n - \beta) + \mathbb{E}_{x}^{(m)} \left[ \Phi_n(0) \right] \quad \text{as} \quad m \to \infty.
\]

for \( x \in \mathcal{X} \). By \((A0)\), there exists some positive constant \( \varepsilon_0 \) such that \( \tilde{c}(x, u) \geq \beta + \varepsilon_0 \) for \( (x, u) \in (\mathcal{B}^c \times \mathcal{U}) \). Therefore,

\[
\sum_{k=0}^{\tau \land n} (\bar{c}_n - \beta) \geq \varepsilon_0(\tau \land n) - (\beta + \varepsilon_0) \sum_{k=0}^{\tau \land n} \mathbb{1}_{\mathcal{B} \times \{0\}}(X_k).
\]

Taking expectations, using Lemma 2.2, we see that (6.16) reduces to

\[
\Phi_n(0) \geq \varepsilon_0 \mathbb{E}_{x}^{(m)} \left[ \tau \land n \right] - (\beta + \varepsilon_0) \delta_0 + \mathbb{E}_{x}^{(m)} \left[ \Phi_n(0) \right] \quad \text{as} \quad m \to \infty.
\]

We use (6.13) and the hypothesis that \( \Phi_n(0) \) is bounded from below, to obtain from (6.17) that

\[
\limsup_{n \to \infty} \frac{\Phi_n(0)}{n} \leq \kappa_3 + \frac{1}{\varepsilon_0} \mathbb{E}_{x}^{(m)} \left[ \Phi_n(0) \right] \quad \text{as} \quad m \to \infty.
\]

for some constant \( \kappa_3 \) which depends on \( \Phi_0 \). This establishes (6.14).

Continuing, we assume without loss of generality (as in the first part of the proof), that the initial condition \( \Phi_0 \) is translated by a constant so that \( \kappa_0 = 0 \) in (6.13). Recall that \( \Psi_n = \Phi_n - V_n \).
Applying Dynkin’s formula together with Fatou’s lemma to (4.13) relative to the stopping time $\bar{\tau} \land n$, we obtain

$$
\tilde{\Psi}^{(0)}_{2n}(x) \geq \tilde{\Psi}^{(1)}(\bar{X}_{\bar{\tau}}, 1_{\{\tilde{\tau} \leq n\}} + \tilde{\Psi}^{(0)}(\bar{X}_n)1_{\{\tilde{\tau} > n\}})
$$

$$
\geq - \sup_{k \geq n} \tilde{\epsilon}_k + \tilde{\Psi}^{(2n)}_{(x, 0)}\left[\tilde{\Psi}^{(0)}(\bar{X}_n)1_{\{\tilde{\tau} > n\}}\right].
$$

Let $\tilde{N}_0 \in \mathbb{N}$ be such that

$$
2\left(1 + C_0\rho\right)\rho^{\tilde{N}_0}/(1 - \rho^{\tilde{N}_0}) < \theta_1.
$$

From (H2), we have, $\tilde{c}_n \geq \theta_1 V_\star - \theta_2 - \beta$, which, if we use in (6.4), we obtain

$$
P_n V_\star(x) \leq \tilde{C}_0 + (1 - \frac{\theta_1}{2}) V_\star(x) \quad \forall n \geq \tilde{N}_0,
$$

with

$$
\tilde{C}_0 := \frac{\beta + \theta_2}{\theta_1} + \frac{1}{1 - \rho^{\tilde{N}_0}}(\tilde{C}_1 + \beta + \theta_2).
$$

In turn, (6.19) shows that

$$
\mathbb{E}^{\tilde{c}_n}_{(x, 0)}[V_\star(X_n)] \leq \frac{2\tilde{C}_0}{\theta_1} + (1 - \frac{\theta_1}{2})^n V_\star(x) \quad \forall n \geq \tilde{N}_0.
$$

Shifting our attention to the split-chain, it is clear from (6.8) and (6.20) that for some constant $\kappa_4$, we have

$$
\mathbb{E}^{\tilde{c}_n}_{(x, 0)}[\tilde{\Psi}^{(0)}(\bar{X}_n)] \leq \kappa_4\left(1 + (1 - \frac{\theta_1}{2})^n V_\star(x)\right) \quad \forall n \geq \tilde{N}_0.
$$

By (6.14) and (6.21), we obtain

$$
\liminf_{n \to \infty} \mathbb{E}^{\tilde{c}_n}_{(x, 0)}[\tilde{\Psi}^{(0)}(\bar{X}_n)1_{\{\tilde{\tau} > n\}}] \geq 0,
$$

which together with (6.18) shows that $\liminf_{n \to \infty} \tilde{\Psi}^{(0)}_{2n}(x) \geq 0$. Using Dynkin’s formula for $\tilde{\Psi}^{(0)}_{2n+1}$ in an analogous manner to (6.18), we obtain the same conclusion for this function. Thus we have shown that

$$
\lim_{n \to \infty} \tilde{\Psi}^{(0)}_{2n}(x) = 0,
$$

which completes the proof.

We next show that the sequence $\{\hat{v}_n\}_{n \in \mathbb{N}}$ is asymptotically optimal.

**Theorem 6.4.** In addition to (H2), we assume the following:

(a) The running cost $c$ is inf-compact on $K$.

(b) There exists $\psi \in \mathbb{P}(X)$ such that under the stabilizing policies $\hat{v}_n$ in Theorem 6.2, the controlled chain is positive Harris recurrent and the corresponding invariant probability measures are absolutely continuous with respect to $\psi$.

Then for every $\Phi_0 \in \mathcal{L}(X) \cap O(V_\star)$ the sequence $\{\hat{v}_n\}_{n \in \mathbb{N}}$ is asymptotically optimal in the sense that

$$
\lim_{n \to \infty} \hat{n}_n(c_n) = \beta
$$

where $\hat{n}_n$ denotes the invariant probability measure of the chain under the control $\hat{v}_n$.

**Proof.** First, by (6.6), we have

$$
\hat{n}_n(c_n) \leq \beta + \frac{(1 + C_0\rho^{n+1})(\beta + \theta_2)}{\theta_1 - (1 + C_0\rho)\rho^n} \quad \forall n \geq N_0,
$$

with $N_0$ as in the proof of Theorem 6.2.
Combining (6.2), (6.3), and (H2), we obtain
\[
|\Phi_{n+1} - \Phi_n| \leq C_0 C_1 + (C_0 + 1)\rho^n V_\star(x) \\
\leq C_0 C_1 + \theta_1^{-1} \theta_2 (C_0 + 1)\rho^n + \theta_1^{-1} (C_0 + 1)\rho^n c_n.
\]  
(6.23)

Writing (4.7) as
\[
\hat{P}_n \Phi_n = - (\hat{c}_n - \Phi_{n+1} + \Phi_n) + \Phi_n,
\]  
(6.24)
and combining this with (6.23), we obtain
\[
\hat{P}_n \Phi_n \leq \beta + C_0 C_1 + \theta_1^{-1} \theta_2 (C_0 + 1)\rho^n - (1 - \theta_1^{-1} (C_0 + 1)\rho^n) c_n + \Phi_n.
\]  
(6.25)

On the other hand, by (H2) and (6.3), we have
\[
c_n \geq \theta_1 (1 + C_0\rho^n)^{-1} (\Phi_n - C_0 C_1) - \theta_2.
\]  
(6.26)

Select \(N_1\) such that \(2(C_0 + 1)\rho^{N_1} < \theta_1\). Then, (6.25) and (6.26) imply that there exists a constant \(C_2\) such that
\[
\hat{P}_n \Phi_n \leq C_2 + (1 - \frac{\theta_2}{\theta_1}) \Phi_n \quad \forall n \geq N_1.
\]  
(6.27)

Note that \(\Phi_{n+1} - \Phi_n \in \mathcal{O}(c_n)\) by (6.23). Thus, (6.22), (6.24), and (6.27) imply that
\[
\hat{\pi}_n (\hat{c}_n - \Phi_{n+1} + \Phi_n) = 0 \quad \forall n \geq N_0 \vee N_1.
\]  
(6.28)

From the definition of ergodic occupation measures, it follows that they form a closed set and therefore so do the invariant probability measures under stationary strategies, which are marginals thereof. Since the latter are absolutely continuous with respect to \(\psi\), it follows that their Radon-Nikodym derivatives with respect to \(\psi\) are uniformly integrable and therefore weakly compact in \(L^1\) by the Dunford-Pettis compactness criterion [16, p. 27-II], otherwise there would be a limit point of the invariant probability measures that is not absolutely continuous with respect to \(\psi\). By the Eberlein-Smulian theorem [16, p. 27-II], this is equivalent to weak sequential compactness in \(L^1\). Therefore every sequence of \(\Lambda_n := \frac{\text{d}\hat{\pi}_n}{\text{d}\psi}\) contains a subsequence which converges weakly in \(L^1\).

Consider such a subsequence, which we denote as \(\{\Lambda_n\}_{n \in \mathbb{N}}\) for simplicity, and let \(\Lambda\) be its limit. Define \(\hat{\pi}(A) := \psi(1_A \Lambda)\) for \(A \in \mathcal{B}(X)\). For every \(f \in C_b(X)\) we have
\[
\hat{\pi}_n (f) = \psi (f \Lambda_n) \xrightarrow{n \to \infty} \psi (f \Lambda) = \hat{\pi}(f).
\]

On the other hand we have
\[
\hat{\pi}_n (A) = \psi (1_A \Lambda_n) \xrightarrow{n \to \infty} \psi (1_A \Lambda),
\]  
(6.29)
where \(\hat{\pi}_n (A) = \hat{\pi}_n (1_A)\) by a liberal use of the notation. Since, \(\psi (f \Lambda) = \hat{\pi}(f)\) for all \(f \in C_b(X)\), it follows of course that \(\psi (1_A \Lambda) = \hat{\pi}(A)\). Thus \(\lim_{n \to \infty} \hat{\pi}_n (A) = \hat{\pi}(A)\) by (6.29). Let \(f_n := \Phi_{n+1} - \Phi_n\). Then
\[
\hat{\pi}_n (\{|f_n| > \epsilon\}) = \int_{\{|f_n| > \epsilon\}} \text{d}\Lambda_n \psi \leq \sup_{m \in \mathbb{N}} \int_{\{|f_n| > \epsilon\}} \text{d}\Lambda_m \psi \xrightarrow{n \to \infty} 0
\]  
(6.30)
by uniform integrability, since \(\psi (\{|f_n| > \epsilon\}) \to 0\) as \(n \to \infty\) by Theorem 6.1. Equation (6.30) implies that \(f_n \to 0\) in \(\hat{\pi}_n\)-measure in the sense of [38, p. 385]. It is also straightforward to verify using (6.22) and (6.23) and the inf-compactness of \(c\), that \(f_n\) is tightly and uniformly \(\{\hat{\pi}_n\}\)-integrable in the sense of definitions [38, (2.4)-(2.5)]. Hence,
\[
\hat{\pi}_n (f_n) \xrightarrow{n \to \infty} 0
\]  
(6.31)
by [38, Theorem 2.8]. Since (6.31) holds over any sequence over which \(\Lambda_n\) converges in \(\sigma(L^1, L^\infty)\), it is clear that it must hold over the original sequence \(\{n\}\). The result then follows by (6.28) and (6.31).
\[\square\]
Remark 6.2. Concerning the positive Harris assumption in Theorem 6.4, it is clear that the Lyapunov equation (6.27) implies that the controlled chain is bounded in probability. If in addition the chain is a \( \psi \)-irreducible \( T \)-model (see [39, p. 177]) then it is positive Harris recurrent [39, Theorem 3.4].

The result in Theorems 6.2 and 6.4 justify in particular the use of \( \hat{v}_n \) for large \( n \) as a ‘rolling horizon’ approximation of optimal long run average policy, as is often done in Model Predictive Control, a popular approach in control engineering practice (see, e.g., [31]), wherein one works with time horizons of duration \( T \gg 1 \) and at each time instant \( t \), the Markov control strategy optimal for the finite horizon control problem on the horizon \([t, t+1, \ldots, t+T]\) is used.

We present an important class of problems for which (H2) is satisfied.

Example 6.1. Consider a linear quadratic Gaussian (LQG) system

\[
X_{t+1} = AX_t + BU_t + DW_t, \quad t \geq 0
\]

\[
X_0 \sim \mathcal{N}(x_0, \Sigma_0),
\]

where \( X_t \in \mathbb{R}^d \) is the system state, \( U_t \in \mathbb{R}^{d_u} \) is the control, \( W_t \in \mathbb{R}^{d_w} \) is a white noise process, and \( \mathcal{N}(x, \Sigma) \) denotes the normal distribution in \( \mathbb{R}^d \) with mean \( x \) and covariance matrix \( \Sigma \). We assume that each \( W_t \sim \mathcal{N}(0, I_{d_w}) \) is i.i.d. and independent of \( X_0 \), and that \((A, B)\) is stabilizable. The system is observed via a finite number of sensors scheduled or queried by the controller at each time step. Let \( \{\gamma_t\} \) be a Bernoulli process indicating if the data is lost in the network: each observation is either received (\( \gamma_t = 1 \)) or lost (\( \gamma_t = 0 \)). A scheduled sensor attempts to send information to the controller through the network; depending on the state of the network, the information may be received or lost. The query process \( \{Q_t\} \) takes values in the finite set of allowable sensor queries denoted by \( Q \). The observation process \( \{Y_t\} \) is given by

\[
Y_t = \gamma_t (C_{Q_{t-1}}X_t + F_{Q_{t-1}}W_t), \quad t \geq 1,
\]

if \( \gamma_t = 1 \), otherwise no observation is received. The value of \( \gamma_t \) is assumed to be known to the controller at every time step. In (6.33), \( C_q \) and \( F_q \) are matrices which depend on the query \( q \in Q \). Their dimension is not fixed but depends on the number of sensors queried by \( q \).

For each query \( q \in Q \), we assume that \( \det(F_qF_q^\top) \neq 0 \) and (primarily to simplify the analysis) that \( DF_q^\top = 0 \). Also without loss of generality, we assume that \( B \) is full rank; if not, we restrict control actions to the row space of \( B \).

The observed information is lost with a probability that depends on the query, that is,

\[
P(\gamma_{t+1} = 0) = \lambda(Q_t),
\]

where the loss rate \( \lambda: Q \rightarrow [0, 1) \).

The running cost is the sum of a positive querying cost \( c: Q \rightarrow \mathbb{R} \) and a quadratic plant cost \( c_p: \mathbb{R}^d \times \mathbb{R}^{d_u} \rightarrow \mathbb{R} \) given by

\[
c_p(x, u) = x^\top Rx + u^\top Mu,
\]

where \( R, M \in M^+ \). Here, \( M^+ (M^+_0) \) denotes the cone of real symmetric, positive definite (positive semi-definite) \( d \times d \) matrices.

The system evolves as follows. At each time \( t \), the controller takes an action \( v_t = (U_t, Q_t) \), and the system state evolves as in (6.32). Then the observation at \( t+1 \) is either lost or received, determined by (6.33) and (6.34). The decision \( v_t \) is non-anticipative, that is, it depend only on the history \( F_t \) of observations up to time \( t \) defined by

\[
F_t := \sigma(x_0, \Sigma_0, Y_1, \gamma_1, \ldots, Y_t, \gamma_t).
\]

This model is an extension of the one studied in [43]. More details can be found in [9] which considers an even broader class of problems where the loss rate depends on the ‘network congestion’.
We convert the partially observed controlled Markov chain in (6.32)–(6.34) to an equivalent completely observed one. Standard linear estimation theory tells us that the expected value of the state \( \hat{X}_t := \mathbb{E}[X_t | \mathcal{F}_t] \) is a sufficient statistic. Let \( \hat{\Pi}_t \) denote the error covariance matrix given by

\[
\hat{\Pi}_t = \text{cov}(X_t - \hat{X}_t) = \mathbb{E}[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T].
\]

The state estimate \( \hat{X}_t \) can be recursively calculated via the Kalman filter

\[
\hat{X}_{t+1} = A\hat{X}_t + BU_t + \hat{K}_{Q_t}\gamma_{t+1}(\hat{\Pi}_t)(Y_{t+1} - CQ_t(A\hat{X}_t + BU_t)), \tag{6.35}
\]

with \( \hat{X}_0 = x_0 \). The Kalman gain \( \hat{K}_{q,\gamma} \) is given by

\[
\hat{K}_{q,\gamma}(\hat{\Pi}) := \Xi(\hat{\Pi})\gamma \bar{C}_q \bar{C}_q^T \Xi(\hat{\Pi})C_q^T + F_qF_q^T)^{-1},
\]

\[
\Xi(\hat{\Pi}) := DD^T + A\hat{\Pi}A^T,
\]

and the error covariance evolves on \( M^+_0 \) as

\[
\hat{\Pi}_{t+1} = \Xi(\hat{\Pi}_t) - \hat{K}_{Q_t}\gamma_{t+1}(\hat{\Pi}_t)CQ_t\Xi(\hat{\Pi}_t), \quad \hat{\Pi}_0 = \Sigma_0.
\]

When an observation is lost (\( \gamma_t = 0 \)), the gain \( \hat{K}_{q,\gamma} = 0 \) and the observer (6.35) simply evolves without any correction factor.

Define \( \mathcal{T}_q : M^+_0 \to M^+_0 \) by

\[
\mathcal{T}_q(\hat{\Pi}) := \Xi(\hat{\Pi}) - \hat{K}_{q,1}(\hat{\Pi})C_q\Xi(\hat{\Pi}), \quad q \in \mathbb{Q},
\]

and an operator \( \hat{\mathcal{T}}_q \) on functions \( f : M^+_0 \to \mathbb{R} \),

\[
\hat{\mathcal{T}}_qf(\hat{\Pi}) = ((1 - \lambda(q))f(\mathcal{T}_q(\hat{\Pi})) + \lambda(q)f(\Xi(\hat{\Pi}))).
\]

It is clear then that \( \hat{\Pi}_t \) forms a completely observed controlled Markov chain on \( M^+_0 \), with action space \( \mathbb{Q} \), and kernel \( \hat{\mathcal{T}}_q \). Admissible and Markov policies are defined as usual but with \( v_t = Q_t \), since the evolution of \( \hat{\Pi}_t \) does not depend on the state control \( U_t \).

As shown in [43], there is a partial separation of control and observation for the ergodic control problem which seeks to minimize the long-term average cost,

\[
J^v := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^v \left[ \sum_{t=0}^{T-1} (c(Q_t) + c_p(X_t, U_t)) \right].
\]

The dynamic programming equation is given by

\[
V_\ast(\hat{\Pi}) + \varphi' = \min_{q \in \mathbb{Q}} \left\{ c(s, q) + \text{trace}(\Pi^\ast\hat{\Pi}) + \hat{\mathcal{T}}_qV_\ast(\hat{\Pi}) \right\}, \tag{6.36}
\]

with \( \Pi^\ast := R - \Pi^\ast + A^T\Pi^\ast A \), and \( \Pi^\ast \in M^+ \) the unique solution of the algebraic Riccati equation

\[
\Pi^\ast = R + A^T\Pi^\ast A - A^T\Pi^\ast B(M + B^T\Pi^\ast B)^{-1}B^T\Pi^\ast A.
\]

If \( q^* : M^+_0 \to \mathbb{Q} \) is a selector of the minimizer in (6.36), then the policy given by \( v^* = \{U_t^*, q^*(\hat{\Pi}_t)\}_{t \geq 0} \), with

\[
U_t^* := -K^*\hat{X}_t,
\]

\[
K^* := (M + B^T\Pi^\ast B)^{-1}B^T\Pi^\ast A,
\]

and \( \{\hat{X}_t\} \) as in (6.35), is optimal, and satisfies

\[
J^{v^*} = \inf_v J^v = \varphi^* + \text{trace}(\Pi^\ast DD^T).
\]

In addition, the querying component of any optimal stationary Markov policy is an a.e. selector of the minimizer in (6.36).
The analysis of the problem also shows that $V_\ast$ is concave and non-decreasing in $M^+_0$, and thus
\[ V_\ast(\Sigma) \leq m^+_1 \text{trace}(\Sigma) + m^+_0, \] (6.37)
for some positive constants $m^+_1$ and $m^+_0$. Note that the running cost corresponding to the equivalent completely observed problem is
\[ r(q, \Sigma) := c(q) + \text{trace}(\tilde{\Pi}^\ast \Sigma). \] (6.38)
It thus follows by (6.37) and (6.38) and the fact that $\tilde{\Pi}^\ast \in M^+$, that (H2) is satisfied for this problem. Note also that the RVI, VI are given by
\[ \varphi_{n+1}(\Sigma) = \min_{q \in Q} \left\{ r(q, \Sigma) + \tilde{T}_q \varphi_n(\Sigma) \right\} - \varphi_n(0), \]
\[ \bar{\varphi}_{n+1}(\Sigma) = \min_{q \in Q} \left\{ r(q, \Sigma) + \tilde{T}_q \bar{\varphi}_n(\Sigma) \right\} - \bar{\varphi}_n, \quad \bar{\varphi}_0 = \varphi_0, \]
respectively, where both algorithms are initialized with the same function $\varphi_0 : M^+_0 \to \mathbb{R}_+.$

**Acknowledgements**

Most of this work was done during the visits of AA to the Department of Electrical Engineering of the Indian Institute Technology Bombay and of VB to the Department of Electrical and Computer Engineering at the University of Texas at Austin, and the finishing touches were given when both authors were at the Institute of Mathematics of the Polish Academy of Sciences (IMPAN) in Warsaw, for a workshop during the 2019 Simons Semester on Stochastic Modeling and Control.

The work of AA was supported in part by the National Science Foundation through grant DMS-1715210, and in part the Army Research Office through grant W911NF-17-1-001, while the work of VB was supported by a J. C. Bose Fellowship. VB acknowledges some early discussions with Prof. Debasish Chatterjee which spurred some of this work.

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