Connecting SLE and minisuperspace Liouville gravity

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Abstract

We show that Fokker-Planck equation for chordal SLE process under a simple rescaling of the probability density can be traced to the minisuperspace Wheeler-de Witt equation for boundary operator in 2d Liouville gravity. Insertion of an operator, calculating SLE critical exponent, corresponds to adding matter contribution to WdW equation. This observation may be useful for understanding of why SLE critical exponents are given by KPZ gravitational scaling dimensions. Possible applications of the obtained relation are discussed.
1 Introduction

Stochastic Loewner evolution (SLE), introduced in [1], proved to be a very useful tool in investigation of random processes on two dimensional plane. It has been shown that SLE process provides mathematically rigorous framework for describing the probability measure on critical curves in lattice models with boundary. In particular, it allows for rigorous proof of various intersection formulas and determination of critical exponents for $c \leq 1$ conformal theories [2].

The chordal SLE process is defined on the conformal upper-half plane in the following way. Consider a non-self-intersecting curve $\gamma_t$ starting at the origin, where $t$ is some parameter along the curve. Let $g_t(z)$ be a continuous set of conformal maps transforming the upper-half plane with a cut along $\gamma_t$ back onto upper-half plane. Then one can show that it satisfies a simple differential equation

$$\frac{\partial_t g_t(z)}{g_t(z) - a_t} = \frac{2}{g_t(z) - a_t},$$

first derived by Löwner in 1923 [3]. To fix the redundant $SL(2, \mathbb{R})$ invariance the condition $g_t(z) \approx z + \frac{2t}{z}$ at $z = \infty$ is implied. This choice of boundary condition also encodes a special parameterization of the curve. The real function $a_t$ corresponds to image of the end of the curve. Schramm [1] suggested to look at the equation (1) as inducing probability measure on the set of conformal maps $g_t$ and thus on curves $\gamma_t$ from some chosen measure on $a_t$. It can be shown that the only choice of the measure for $a_t$, which respects conformal invariance, reflection symmetry and produces non-self-intersecting curves, is the one dimensional Brownian motion $\xi_t$ with diffusion coefficient $\kappa$. Defined in this way, SLE process provides a natural probability measure on random curves with many useful properties, see e.g. [4] for review from a physical perspective.

In the works by Bauer and Bernard [5,6] as well as in the subsequent papers [7,8,9] a nontrivial connection between SLE and conformal field theories with central charge $c \leq 1$ was revealed. In particular, in [5] a natural lift of SLE$_\kappa$ to a formal group formed by exponentiation of lower part of Virasoro algebra was considered. It was shown that the martingales (conserved quantities) for this lifted SLE correspond to null vectors in the Verma module of Virasoro algebra, with the CFT central charge related to $\kappa$ by

$$c = 1 - \frac{3(\kappa - 4)^2}{2\kappa}.$$

However, it seems that the SLE/CFT duality cannot explain [5] one of the most enigmatic properties of critical conformally-invariant curves, namely the appearance of KPZ gravitational scaling dimensions [10] as critical scaling exponents. The KPZ formula relates the exponents of random paths in plane geometry to corresponding exponents on fluctuating plane. This relation has been extensively studied and used to derive exponents of random
paths in various lattice models by Duplantier and Kostov \cite{11}, see \cite{12} for recent review in the context of \textit{SLE}. In the framework of \textit{SLE}/CFT duality KPZ dimension appears as a scaling exponent of the one point function near the tip of the curve \cite{5}. It is fair to say, that fundamental reason of why critical exponents of random curves on the plane are given by gravitational dimensions remains to be understood.

One of the goals of this paper is to provide some tentative explanation of this phenomenon in the framework \textit{SLE}. The well known fact, that gravitational dimensions naturally appear in the Liouville 2d gravity \cite{10}, suggests that there should be some underlying relation between the two subjects. Here observe that \textit{SLE} process indeed corresponds to the minisuperspace approach to Liouville theory. Consider Loewner equation \cite{11} at some given point on the boundary as an ordinary stochastic Langevin equation. We show that the corresponding Fokker-Planck equation after a simple rescaling of probability density takes the form of minisuperspace Wheeler-de Witt (WdW) equation of 2d gravity \cite{13} for the purely gravitational boundary wave function. Comparison of two equations leads to following identification of parameters

$$\kappa = 2\gamma^2,$$

where $\gamma$ enters in the Liouville factor of two dimensional metric. This relation turns out to be in a precise agreement with the relation between $\kappa$ and CFT central charge \cite{2}. We also find that mathematical expectation of the operator computing the scaling exponent for the chordal \textit{SLE} process \cite{2} satisfies the WdW equation with additional CFT matter. The aforementioned rescaling of the probability density has a nice interpretation on the 2d gravity side, where it is nothing but the relation between vertex operator and wave function. Thus we establish that \textit{SLE} probability density corresponds to the boundary vertex operator. Since the minisuperspace approach encodes the KPZ relation in 2d gravity, our observations may shed some light on the appearance of the gravitational dimensions as \textit{SLE} scaling exponents.

In the minisuperspace approach the equation for gravitational wave function is approximated by only the zero mode part. Nevertheless this approach is proved to be quite powerful, e.g. it gives correct answer for the exact bulk one-point function in Liouville theory \cite{14}. The WdW zero mode wave function appears also in the matrix model approach to 2d gravity \cite{13}, similar objects has been considered in the $O(n)$ model on random surface \cite{15} \cite{16}. The equation we derive here is satisfied rather by the boundary quantum gravity operator \cite{17}, which wave function corresponds to an operator "creating a boundary" in the language of \cite{16}, or the trace of a random path in the language of \textit{SLE}. The bulk gravitational operator then corresponds to an operator "creating a loop" on a random surface. Taking into account the observed relationship between boundary wave function and \textit{SLE} trace it might be conjectured, that the bulk wave function corresponds to an \textit{SLE}-type probability measure for random loops. For $c=0$ such a measure was constructed by Lawler and Werner (see, e.g., \cite{18} and references therein), for
generic values of central charge $c \leq 1$ the existence of such measure is conjectured in the Malliavin-Kontsevich-Suhov theory [19].

This paper is organized as follows. In sec. 2 we review the euclidean quantum mechanics language for the Itô calculus and apply it to derivation of Fokker-Plank equation for $SLE$. In sec. 3 we show how WdW equation appears for pure $SLE$ process. In sec. 4 we review the calculation of one-sided scaling exponent and show that it corresponds to WdW equation with addition of matter. Discussion and comments are presented in the last section.

2 Quantum mechanics and Itô calculus

We start with a brief reminder on the relation between Itô formulas for stochastic processes [20] and euclidean quantum mechanics [21].

To define a measure for the stochastic process $y_t$ one connects it to the Brownian motion $\xi_t$ via the Langevin equation

$$dy_t = u(y_t, t)dt + v(y_t, t)d\xi_t,$$

where $u$ and $v$ are some given functions. The probability distribution for Brownian motion can be represented as path integral in euclidean quantum mechanics

$$P_\xi(q', 0; q, t) = \langle q | e^{-\frac{1}{2} \int_0^t \dot{\xi}^2 dt} | q' \rangle = \int_{(q', 0)}^{(q, t)} e^{-\frac{1}{2} \int_0^t \dot{\xi}^2 dt} D\xi_t.$$

Therefore we can use the equation (4) to find the induced measure on $y_t$ from the brownian measure on $\xi_t$. To illustrate this consider the example of $SLE$, i.e. set $u = y$ and $v = -1$

$$dy_t = \frac{2}{y_t} dt - d\xi_t.$$

This is nothing but the $SLE$ equation (11) written in Langevin form for the shifted variable $y_t(z) = g_t(z) - \xi_t$ and for some point $z$ on the boundary so that $y_t(z)$ is a real variable. Now we make the change of variables $\xi \rightarrow y_t(\xi)$ in the path integral (5)

$$P_y(q', 0; q, t) = \int_{(q', 0)}^{(q, t)} e^{-\frac{1}{2} \int_0^t (\dot{y} - \frac{2}{y})^2 dt} J[y] Dy,$$

where we omitted the subscript $t$ to simplify the notations. Jacobean $J[y]$ comes from the transformation of path integral measure. It can be computed directly [21] and has the form

$$J[y] = e^{\int_0^t \frac{1}{y^2} dt}$$
One can also read off its form from the following considerations. The path integral \( P(y, t) \) satisfies the Shrödinger equation

\[
\frac{\partial}{\partial t} P(y, t) = -\frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \kappa \frac{\partial}{\partial y} P(y, t) - \frac{4}{y} P(y, t) \right),
\]

where the total derivative on the rhs leads to conservation of the overall probability. Therefore the role of the Jacobean \( \frac{\partial}{\partial y} \) is to adjust the measure in the appropriate way.

The Shrödinger equation \( (9) \) can be interpreted as the Fokker-Planck equation (backward Kolmogorov equation) in stochastic calculus. To treat the expectation values of operators

\[
\langle O(y, t) \rangle = \int dq \langle q|\hat{O}(\hat{q}, t)e^{-\frac{\hat{H}}{\kappa}}|y\rangle = \int dq O(q, t) \langle q|e^{-\frac{\hat{H}}{\kappa}}|y\rangle
\]

one makes use of Heisenberg representation. The Heisenberg equation on \( (10) \) differs from \( (9) \) due to presence of linear momentum term in the quantum Hamiltonian

\[
\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{y} \hat{p} + \frac{\kappa}{y^2},
\]

where

\[
[\hat{p}, \hat{y}] = \kappa, \quad \hat{p} = \kappa \frac{\partial}{\partial y},
\]

and has the form

\[
\left[ \frac{\partial}{\partial t} + \frac{\kappa}{y^2} \frac{\partial^2}{\partial y^2} + \frac{2}{y} \frac{\partial}{\partial y} \right] \langle O(y, t) \rangle = 0.
\]

Stochastic counterpart of the Heisenberg equation is the Itô formula \( [20] \). Given a random process \( (11) \) any function \( h(y_t, t) \) of random variable \( y_t \) and time \( t \) satisfies the Itô equation

\[
dh(y_t, t) = dt \frac{\partial}{\partial t} h(y_t, t) + dy_t \frac{\partial}{\partial y} h(y_t, t) + (dy_t)^2 \frac{1}{2} \frac{\partial^2}{\partial y^2} h(y_t, t),
\]

where we assume the following formal multiplication rules for differentials

\[
dt \cdot dt = dt \cdot d\xi_t = d\xi_t \cdot dt = 0, \quad d\xi_t \cdot d\xi_t = \kappa dt
\]

Using \( [6, 14, 15] \) one can easily check that Itô formula is equivalent to \( (13) \) after taking the expectation values of operators.
3 Minisuperspace Wheeler-de Witt equation and $SLE_\kappa$

Consider the following ansatz for the probability density
\[ P(y,t) = e^{-\frac{E\kappa}{2} t} P(y) = e^{-\frac{E\kappa}{2} t} y^{\frac{1}{2} + \frac{2}{\kappa}} \Psi(y). \] (16)

Here $\Psi(y)$ is the (rescaled) stationary wave function in the euclidean quantum mechanics. Equation (9) now reads
\[
\left[- \left( y \frac{\partial}{\partial y} \right)^2 + \frac{2E\kappa^2 y^2}{\kappa^2} + \left( \frac{1}{2} - \frac{2}{\kappa} \right)^2 \right] \Psi(y) = 0.
\] (17)

In this equation one can recognize the WdW equation in Liouville theory. Let us briefly remind how it appears in the minisuperspace approach to 2d gravity [13].

Consider the system of Liouville ($\phi$) and conformal matter ($\varphi_i$) fields with the action
\[
S(\phi, \varphi_i, \hat{g}) = S_{\text{liouv}}(\phi, \hat{g}) + S_{\text{matter}}(\varphi_i, \hat{g}) = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left( (\hat{\nabla} \phi)^2 + \frac{\mu}{\gamma^2} e^{\gamma \phi} + \left( \gamma + \frac{2}{\gamma} \right) \phi R(\hat{g}) \right) + S_{\text{matter}}(\varphi_i, \hat{g}),
\] (18)

where the Liouville field $\phi$ is defined as a scale factor of the two-dimensional metric $g = e^{\gamma \phi} \hat{g}$ relative to some reference metric $\hat{g}$, $\mu$ is the bulk cosmological constant and the background charge $Q = \gamma + 2/\gamma$. Liouville part of the action is a conformal field theory with central charge $26 - c$ if the parameter $\gamma$ is related to the matter central charge $c$ as
\[
\gamma = \frac{1}{\sqrt{12}} \left( \sqrt{25 - c} - \sqrt{1 - c} \right).
\] (19)

The wave functions of the coupled system factorize onto matter and gravitational part $\Psi = \Psi_{\text{matter}} \otimes \Psi_{\text{liouv}}$. Any spinless primary operator of conformal dimension $\delta$ in the bulk (or in the boundary) can be dressed by the exponent of Liouville field, such that the resulting operator can be integrated over bulk (resp. boundary)
\[
\int d^2z e^{\alpha \phi} \Phi_0(z, \bar{z}), \quad \oint e^{\alpha \phi} \Phi_0(x),
\] (20)

where $\alpha$ is determined by the condition that the sum of CFT and Liouville conformal dimensions equals one
\[
\delta - \frac{1}{2} \alpha (\alpha - Q) = 1.
\] (21)
The corresponding wave function should therefore satisfy the operator equation

$$(L_0^{\text{liouv}} + \delta - 1)\Psi = 0. \tag{22}$$

If matter boundary conditions are diffeomorphism invariant, then $\Psi$ should depend on a diffeomorphism-invariant variable. The natural variable to use is the length of the boundary $l = \oint e^{\gamma \phi}$. Then in the minisuperspace approximation we restrict to the field configurations, independent of $\sigma$ coordinate on the worldsheet, e.g. to the zero mode part $\phi_0$ of Liouville field. In this approximation (22) reduces just to $T_{00} \Psi = 0$, which is nothing but the Wheeler-de Witt equation, stating that gravitational wave function is invariant under time diffeomorphisms. In this form it is also equivalent to Shrödinger equation in Liouville quantum mechanics and $\phi_0$ - to the quantum mechanics coordinate. In terms of the parameter $l = e^{2\gamma \phi_0}$ the Wheeler-de Witt equation becomes

$$\left[-\left(\frac{1}{l} \frac{\partial}{\partial l}\right)^2 + \frac{\mu}{\gamma^4} l^2 + \nu^2\right] \Psi(l) = 0, \tag{23}$$

where for the boundary operator

$$\nu^2 = \nu_{\text{boundary}}^2 = 2 \delta \gamma^2 + \left(\frac{1}{2} - \frac{1}{\gamma^2}\right)^2. \tag{24}$$

Originally this equation was written for bulk operators [13] for which the index $\nu$ doubles: $\nu_{\text{bulk}} = 2 \nu_{\text{boundary}}$. Although the boundary WdW equation has not been considered in the literature, there is some indirect indication that it should have twice smaller index compared to bulk equation, i.e. as in (24).

Comparing (23) to (17) we can identify $\text{SLE}$ and WdW parameters. First, the length of boundary $l$ corresponds to stochastic variable $y$. This is not surprising since one can look at the $\text{SLE}$ process, as inducing random measure on the conformal factor of the upper-half-plane metric $ds^2 = |\partial_z g_t(z)|^2 dz d\bar{z}$. Second, the "energy" $E$ has to be identified with the cosmological constant $\mu$. This is also rather natural identification since both parameters can be rescaled arbitrarily: in $\text{SLE}$ due to the scaling invariance $y \rightarrow \lambda y$, $t \rightarrow \lambda^2 t$, and in Liouville theory by shifting the field $\phi$ by a constant. Third, the conformal dimension of matter is set to zero $\delta = 0$, i.e we consider pure gravitational wavefunctions in the Liouville-matter system. Finally, we get

$$\kappa = 2 \gamma^2. \tag{25}$$

This last formula is in precise agreement with the relation between the CFT central charge $c$ and $\text{SLE}$ parameter $\kappa$ [2]. Under these identifications Fokker-Plank equation (17) takes exactly the form (23). There is also a nice interpretation of the rescaling of probability
density $16$ by $y^{\frac{1}{2} + \frac{2}{\kappa}}$. This is equivalent to the relation between boundary Liouville vertex operator and WdW wave function

$$V(\phi) = e^{\frac{Q}{4} \phi} \Psi(l).$$

Therefore the $SLE$ probability measure defined by the Fokker-Planck equation $17$ corresponds to purely gravitational boundary vertex operator.

In the next section we extend this correspondence to the operators with nonzero matter conformal dimension $\delta \neq 0$.

4 $SLE_\kappa$ scaling dimensions and WdW

Various critical exponents associated with random paths on two dimensional plane have been studied for a long time $11$, see also review $12$ for complete list of references. In the context of $SLE$ the critical exponents (crossing probabilities) have been derived in $2$.

The simplest critical $SLE$ exponent is the so-called one-sided chordal intersection exponent. We refer to the first paper in $2$ for rigorous definition. For our purposes it suffices to say that one has to compute the scaling of expectation value of the derivative of the conformal transformation $y_t'(z)^\delta$ at the point $z$ on the boundary and close to the origin. Using $1$ one can easily show that

$$y_t'(z)^\delta = \exp \left( -\delta \int_0^t \frac{2}{y_s^2} ds \right).$$

The scaling law of this operator for small $z$ determines the critical exponent $\Delta$

$$h(y, t) = E \left[ \exp \left( -\delta \int_0^t \frac{2}{y_s^2} ds \right) \right] \sim \left( \frac{y}{\sqrt{t}} \right)^{\Delta(\delta, \kappa)}.$$

which is found to be

$$\Delta(\delta, \kappa) = \frac{\kappa - 4 + \sqrt{(\kappa - 4)^2 + 16\delta\kappa}}{2\kappa} = \frac{1}{2} - \frac{1}{\gamma^2} + \sqrt{\left( \frac{1}{2} - \frac{1}{\gamma^2} \right)^2 + \frac{2\delta}{\gamma^2}}.$$

In this expression one can recognize the KPZ gravitational scaling dimension $10$. The latter appears in the one-point function of the dressed operator $\Phi_0$ of bare dimension $\delta$ at fixed length $l$ in Liouville theory

$$\frac{1}{Z(A)} \int \mathcal{D}\phi \mathcal{D}\varphi_i s^{-S(\phi, \varphi_i)} \delta \left( \oint ds e^{\frac{2}{\gamma} \phi} - l \right) \oint s e^{\alpha \phi} \Phi_0 \sim l^{1-\Delta},$$

and therefore related to Liouville dressing exponent $\alpha$ $21$ as $\alpha = 1 - \Delta/\gamma$. 

8
It is easy to derive the Itô equation (14) for the expectation value (28). Similar to (14) we arrive at

\[
\left[ \frac{\partial}{\partial t} + \kappa \frac{\partial^2}{\partial y^2} + \frac{2}{y} \frac{\partial}{\partial y} - \frac{2\delta}{y^2} \right] h(y,t) = 0.
\] (31)

Following the discussion in sec. 2 one can write down the corresponding Fokker-Planck equation, which is just the Shrödinger picture (10) of the previous equation. Using then the same ansatz as for the wave function in (16) the equation becomes

\[
\left[ - \left( \frac{y}{\partial y} \right)^2 + \frac{2E}{\kappa^2} y^2 + \frac{4\delta}{\kappa} + \left( \frac{1}{2} - \frac{2}{\kappa} \right)^2 \right] \Psi(y) = 0,
\] (32)

The same identification of parameters as in the previous section leads us to the minisuperspace WdW equation for the wave function of the gravitationally dressed boundary operator with a nonzero matter conformal dimension \(\delta\) (23).

5 Discussion

In this paper we considered chordal \(SLE_\kappa\) process and showed that the Fokker-Planck equation for this process under a simple rescaling of the probability density takes the form of minisuperspace Wheeler-de Witt equation for 2d gravity. We also show that insertion of an operator calculating scaling exponent modifies FP equation in a very same manner as adding matter contribution to WdW equation. Since the latter encodes KPZ relation this result provides some insight to the appearance of gravitational scaling dimensions in SLE. Let us now comment on other possible applications of this result.

Despite being only an approximation, the minisuperspace approach nevertheless gives exact answers for some 2d gravity correlators, e.g. for the one-point function of the bulk Liouville exponent \(\alpha\) in the presence of boundary cosmological constant \(\mu_b\)

\[
\langle e^{\alpha \phi} \rangle = \frac{U(\alpha|\mu_b)}{|z - \bar{z}|^{2\Delta_\alpha}} = \frac{1}{|z - \bar{z}|^{2\Delta_\alpha}} \int_0^\infty \frac{dl}{l} e^{-\mu_b l} \Psi_\alpha(l).
\] (33)

Here \(\Psi_\alpha(l)\) satisfies eq. (23) with \(\nu = \nu_{\text{bulk}}\) and with the Liouville exponent \(\alpha\) related to \(\delta\) by eq. (21). Equation (23) appears also in the framework of random matrix formulation of 2d quantum gravity. In the Hermitean one-matrix model WdW wave function of \((2, 2m-1)\) minimal model coupled to gravity arises [13] at the \(m^{th}\) multicritical point in the double-scaling limit of the "macroscopic loop" operator, creating a hole of boundary length \(l\)

\[
w(l) = \frac{1}{l} \text{tr} e^{\lambda M}.
\] (34)
The correlation functions of multiple insertions of macroscopic operators have been computed in the matrix model approach \cite{22,13} and are in agreement \cite{23} with exact results, provided by Liouville theory \cite{14}. Recently the minisuperspace wave function was also interpreted as ZZ brane amplitude \cite{24}.

In the loop gas approach \cite{15,16} similar object appears as the solution to loop equations and has the same interpretation, as an operator creating closed boundary of length \(l\) on the worldsheet. Boundary operator, creating SLE trace, considered in the present paper, seemingly corresponds to the operator creating piece of open boundary at some point of the closed boundary \cite{16}. The loop gas approach is also in agreement \cite{25} with the boundary two-point function calculation of Fateev, Zamolodchikov and Zamolodchikov and Teschner, thus providing another indication of the validity of minisuperspace approach beyond the semiclassical limit.

Random matrix model interpretation of the boundary operator, as creating SLE curve, could provide some insight for the generalization of SLE-type measure to random loops. This problem was recently addressed by Lawler and Werner \cite{18}, and by Kontsevich and Suhov who conjectured the existence of such measure for \(c \leq 1\) \cite{19}. In the light of the relation observed here, one could conjecture that the insertion of bulk operator creating a loop should correspond to SLE-type process for random closed curves. Thus the WdW wave function, satisfying eq. (23) with index \(\nu = \nu_{\text{bulk}}\), may have some relevance to (some approximation of) the SLE-type probability measure on random loops.

We obtained the correspondence between SLE and minisuperspace Liouville theory for the simplest possible SLE scaling dimension. It would be very interesting to show whether there is a deeper connection between the two subjects. In view of this let us mention, that there exists a variety of different critical exponents for SLE process \cite{2}. It would be also interesting to extend this relation to the generalizations of SLE, such as SLE(\(\kappa, \rho\)) \cite{8}. Let us also note, that the connection between SLE and 2d Liouville gravity has been discussed from different perspectives in \cite{26,27}.

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