Local invariants of divergence-free webs

Wojciech Domitrz · Marcin Zubilewicz

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Abstract
The objects of our study are webs in the geometry of volume-preserving diffeomorphisms. We introduce two local invariants of divergence-free webs: a differential one, directly related to the curvature of the natural connection of a divergence-free 2-web introduced by Tabachnikov (Diff Geom Appl 3:265-284, 1993), and a geometric one, inspired by the classical notion of planar 3-web holonomy defined by Blaschke and Bol (Geometrie der Gewebe. Grundlehren der mathematischen Wissenschaften, vol. 49. Springer, Berlin, 1938). We show that triviality of either of these invariants characterizes trivial divergence-free web-germs up to equivalence. We also establish some preliminary results regarding the full classification problem, which jointly generalize the theorem of Tabachnikov on normal forms of divergence-free 2-webs. They are used to provide a canonical form and a complete set of invariants of a generic divergence-free web in the planar case. Lastly, the relevance of local triviality conditions and their potential applications in numerical relativity are discussed.

1 Introduction

Many lines of current research in differential geometry will inevitably lead to at least a passing consideration of almost-product structures. These objects, which take their name from a splitting of the tangent bundle into a direct sum of two complementary distributions they induce, lie at the foundation of several geometries of considerable, continuing interest. Examples include the geometry of symplectic pairs, in which one subbundle is the skew-orthogonal complement of the other with respect to the ambient symplectic form [5]; bi-Lagrangian manifolds, which come equipped with two complementary foliations with Lagrangian leaves [9, 10, 19] acting as double
polarizations in the quantum-mechanical setting [12]; para-Kähler structures [1, 8], i.e. integrable almost-product structures with compatible neutral metrics, which turn out to be equivalent to bi-Lagrangian structures [9]; and foliated (pseudo-)Riemannian manifolds [18], widely used in numerical relativity to fix a privileged “time” axis by means of families of simultaneity hypersurfaces [2, 4, 11, 14].

Over the years, much effort has been made in order to understand each of the aforementioned structures individually. To complement these studies, it might be fruitful to adopt a broader point of view and set out to document their shared traits. For instance: each of these objects allows us to define a canonical volume form on the underlying manifold. This volume form, taken together with two complementary foliations generated by the integrable subbundles induced by the almost-product structures, constitute what Tabachnikov named a divergence-free 2-web in his work on local properties of bi-Lagrangian and bi-Legendrian manifolds [17]. This paper aims to expand upon his work by investigating known geometrical invariants of divergence-free webs, introducing novel ones and providing potential applications in computational physics.

In physics, divergence-free 2-webs accompanying the metric structure appear naturally in the context of numerical relativity. A choice of a spacelike foliation by hypersurfaces of simultaneity, its timelike 1-dimensional complement and the associated Lorenzian volume element, lead naturally to an invariant formulation of differential equations governing the stress-energy tensor with respect to “time”. Among them are the energy-momentum conservation laws expressed using the divergence operator $\text{div} X$, which is defined in terms of the volume element $dV$ by the equality $\mathcal{L}_X dV = (\text{div} X) \cdot dV$ for the Lie derivative $\mathcal{L}_X$ along the vector field $X$. Insight into structures combining the two foliations and the volume element might simplify handling divergences in actual computations and pave the way for further improvements in numerical methods designed with these conservation laws in mind.

In order to motivate further discussion, we will now introduce a generalization of the notion of divergence-free web which accommodates structures with larger number of foliations. Let us state the relevant definitions, together with some elementary properties and examples of such objects.

An $n$-web is a smooth manifold $M$ of dimension $m$ equipped with a collection of $n$ foliations $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of $M$. In this paper we consider only the webs for which the following general position condition holds: for every $p \in M$

$$\text{codim} \bigcap_{i=1}^n T_p \mathcal{F}_i = \sum_{i=1}^n \text{codim} T_p \mathcal{F}_i. \quad (1.1)$$

In this paper, webs will not be distinguished according to the ordering of the constituent foliations, although an explicit ordering will prove useful in actual computations. Hence we notate them as either $\mathcal{W} = (M, \mathfrak{F})$ with $\mathfrak{F} = \text{Fol}(\mathcal{W}) = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ or $\mathcal{W} = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ according to circumstances.

The codimension of an $n$-web $\mathcal{W}$ is defined as the nondecreasing sequence of the numbers codim $\mathcal{F}$ for $\mathcal{F} \in \text{Fol}(\mathcal{W})$. If codim $\mathcal{W} = (k, k, \ldots, k)$, the web $\mathcal{W}$ is said to have codimension $k$. An $n$-web together with a nowhere-vanishing volume form
on $M$ will be called a divergence-free n-web, following [17]. In most circumstances, an n-web $\mathcal{W}$ equipped with a volume form $\Omega$ will be notated by the symbol $\mathcal{W}_\Omega$.

Given two webs $\mathcal{W}_M = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ and $\mathcal{W}_N = (N, G_1, \ldots, G_n)$, a diffeomorphism $\varphi : M \rightarrow N$ carrying the leaves of $\mathcal{F}_j$ onto the leaves of $G_{\sigma(j)}$ for some permutation $\sigma \in S_n$ will be called an equivalence between $\mathcal{W}_M$ and $\mathcal{W}_N$. If, in addition, $\mathcal{W}_M$, $\mathcal{W}_N$ come equipped with volume forms $\Omega_M$, $\Omega_N$ and $\varphi^*(\Omega_N) = \Omega_M$, then $\varphi$ is called an equivalence of divergence-free webs. We will mostly focus on local equivalences, which correspond to germs of the above maps.

Our variant of the general position condition (1.1) guarantees that every n-web $\mathcal{W} = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ is locally trivial in the sense that for each point $p \in M$ one can find a $\mathcal{W}$-adapted local coordinate chart $(U, \xi)$ with $\xi(x) = (x_1, x_2, \ldots, x_m)$. It is defined as a chart which is

1. centered at $p$, meaning that $\xi(p) = 0$;
2. cubical, i.e. $\xi(U) = \bigcap_{i=1}^m (a_i, b_i)$ for some $a, b \in \mathbb{R}^m$;
3. such that each plaque $L \in \mathcal{F}_i$ has the form

$$L = \{ x \in \mathbb{R}^m : \forall k=1, \ldots, \text{codim} \mathcal{F}_i \ x_{m_i+k} = \gamma_k \} \quad (1.2)$$

for some real constants $\gamma_k \in \mathbb{R}$, where $m_j = \sum_{k=1}^{j-1} \text{codim} \mathcal{F}_k$ for $j = 1, 2, \ldots, n + 1$.

This diffeomorphism $\xi : U \rightarrow C = \bigcap_{i=1}^m (a_i, b_i)$ can be interpreted as a local equivalence between $\mathcal{W}$ and a certain n-web $\mathcal{W}_0$ of codimension $\text{codim} \mathcal{W}$ on $\mathbb{R}^m$, the foliations of which are families of parallel affine subspaces of $\mathbb{R}^m$. One defines a trivial divergence-free n-web $\mathcal{W}_{0,\Lambda}$ as the web $\mathcal{W}_0$ equipped with the Lebesgue volume form $\Lambda = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$; a divergence-free web $\mathcal{W}_\Omega$ is locally trivial if it is locally equivalent to $\mathcal{W}_{0,\Lambda}$.

**Example 1** A symplectic pair [5] is a smooth manifold $M$ of dimension $2n$ equipped with a pair of closed 2-forms $\omega$, $\eta$ which satisfy $TM = \ker \omega \oplus \ker \eta$. Forming their sum $\Omega_+ = \omega + \eta$ is one way to give $M$ a structure of a symplectic manifold. The two kernel distributions integrate to a pair of complementary foliations $\mathcal{F}$, $\mathcal{G}$ which, taken together with $\Omega^0_+$, form a divergence-free 2-web $\mathcal{W}_\Omega = (M, \mathcal{F}, \mathcal{G}, \Omega^0_+)$). In $\mathcal{W}_\Omega$-adapted coordinates $(x_1, \ldots, x_{2k}, y_1, \ldots, y_{2(n-k)})$ one can express $\omega$ and $\eta$ as

$$\omega = \sum_{1 \leq i < j \leq 2k} f_{ij}(x, y) \ dx_i \wedge dx_j, \quad \eta = \sum_{1 \leq i < j \leq 2(n-k)} g_{ij}(x, y) \ dy_i \wedge dy_j. \quad (1.3)$$

The equalities $d\omega = d\eta = 0$ yield $f_{ij} = f_{ij}(x)$ and $g_{ij} = g_{ij}(y)$, hence by applying the classical Darboux theorem independently to $\omega$ and $\eta$ we bring $\Omega_+$ into the standard form $\sum_{i=1}^k dx_{2i-1} \wedge dx_{2i} + \sum_{j=1}^{n-k} dy_{2j-1} \wedge dy_{2j}$. Since symplectic pairs are locally trivial, their associated divergence-free 2-webs are also locally trivial.

As in the case of n-webs without extra structure, every divergence-free n-web $\mathcal{W}$ with $\sum_{\mathcal{F} \in \text{Fol}(\mathcal{W})} \text{codim} \mathcal{F} < \dim M$ is locally equivalent to a trivial one: by picking a $\mathcal{W}$-adapted coordinate system $(x_1, x_2, \ldots, x_m)$ on $U \subseteq M$ and stretching the space along the $x_m$-axis by an appropriately chosen transformation of the
form \( x \mapsto y = (x_1, \ldots, x_{m-1}, y_m(x)) \) one exhibits a local equivalence between \( \mathcal{W} \) on \( U \) and the trivial divergence-free web \( \mathcal{W}_{0;\Lambda} \) of codimension \( \text{codim } \mathcal{V} \). When the number \( \sum_{F \in \text{Fol}(\mathcal{W})} \text{codim } F \) reaches \( \dim M \), the situation becomes more involved, as indicated by the following elementary theorem.

**Theorem 1** Let \( \mathcal{W}_0 = (\mathbb{R}^m, \mathcal{F}_1, \ldots, \mathcal{F}_n) \) be an \( n \)-web with \( T \mathcal{F}_i = \bigcap_{k=m+1}^{m_i+1} \ker dx_k \) for some integers \( 0 = m_0 < m_1 < \cdots < m_n < m_{n+1} \leq m \). Assume that \( m_{n+1} = \sum_{k=1}^n \text{codim } F_k = m \) and fix \( \Lambda = dx_1 \wedge \cdots \wedge dx_m \in \Omega^m(\mathbb{R}^m) \). Given a volume form \( \Omega = h(x) dx_1 \wedge \cdots \wedge dx_m \) for some smooth function-germ \( h \) at 0, the divergence-free web-germs at 0 given by \( \mathcal{W}_{0;\Omega} = (\mathbb{R}^m, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega) \) and \( \mathcal{W}_{0;\Lambda} = (\mathbb{R}^m, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Lambda) \) are equivalent if and only if

\[
\frac{\partial \log h}{\partial x_k \partial x_l} = 0 \quad (1.4)
\]

for each pair of indices \( 1 \leq i < j \leq n \) and for each \( k = m_j + 1, \ldots, m_{j+1} \).

**Proof** Let \( \varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0) \) be a diffeomorphism-germ witnessing the local equivalence between \( \mathcal{W}_{0;\Omega} \) and \( \mathcal{W}_{0;\Lambda} \). By permuting the variables of the codomain chart we can assume that \( \varphi(x_1, \ldots, x_m) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)) \), where \( x_i = (x_{m_i+1}, \ldots, x_{m_i+1}) \). Now, by expanding the equality \( \varphi^* \Lambda = \Omega \) we arrive at an equivalent condition

\[
h(x_1, \ldots, x_m) = \det \frac{d\varphi_1}{dx_1}(x_1) \cdots \det \frac{d\varphi_n}{dx_n}(x_n), \quad (1.5)
\]

where \( \frac{d\varphi_k}{dx_k} \) denotes the Jacobian matrix of \( \varphi_k \) with respect to variables \( x_k \). From this the necessity of condition (1.4) follows immediately.

For sufficiency, we note that condition (1.4) is equivalent to the existence of some smooth functions \( g_k \in C^\infty(\mathbb{R}^d, 0) \) with \( c_k = \text{codim } F_k \) such that \( h(x_1, \ldots, x_m) = g_1(x_1) \cdots g_n(x_n) \). In this case, the diffeomorphism-germ \( \varphi(x_1, \ldots, x_m) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)) \) given by \( \varphi_i(x_i) = (x_{m_i+1}, \ldots, x_{m_i+1} - 1, G_i(x_i)) \), where

\[
G_i(x_i) = \int_0^{x_{m_i+1}} g_i(x_{m_i+1}, \ldots, x_{m_i+1} - 1, t) \, dt, \quad (1.6)
\]

is a valid local equivalence between \( \mathcal{W}_{0;\Omega} \) and \( \mathcal{W}_{0;\Lambda} \). \( \square \)

The main goal of this paper is to pinpoint and examine several kinds of local invariants of general divergence-free webs \( \mathcal{W}_\Omega \), the existence of which is hinted at by this observation. One of these invariants can in fact be defined by naively collecting the quantities from (1.4) into a single geometric object.
Definition 2  The symmetric covariant 2-tensor field $\mathcal{K}(\mathcal{W}_\Omega) \in \Gamma(S^2(TM))$ defined in any local $\mathcal{W}_\Omega$-adapted coordinates $(x_1, \ldots, x_m)$ by

$$\mathcal{K}(\mathcal{W}_\Omega) = \sum_{i \neq j} \sum_{k=m_i+1}^{m_i+1} \sum_{l=m_j+1}^{m_j+1} \frac{\partial \log h}{\partial x_k \partial x_l}(x) \, dx_k dx_l,$$

(1.7)

where $\Omega = h(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$ and $m_i = \sum_{k=1}^{i-1} \text{codim} \mathcal{F}_k$ for $i = 1, \ldots, n+1$, is called the nonuniformity tensor of the divergence-free $n$-web $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$.

A direct calculation proves that this tensor field is well defined. Moreover, it can be proven in a similar way that if $\varphi : \mathcal{W}_\Omega \to \mathcal{W}_\eta$ is an equivalence of divergence-free webs, then $\varphi^* \mathcal{K}(\mathcal{W}_\eta) = \mathcal{K}(\mathcal{W}_\Omega)$.

This diffeo-geometric object holds minimal amount of information which ensures the equivalence between its vanishing and the local triviality of the web $\mathcal{W}_\Omega$ by means of Theorem 1. While its construction may seem ad hoc, we will show in Theorems 12 and 15 that it can be derived from another, more natural invariant, which was introduced by Tabachnikov in [17] in the special case of divergence-free 2-webs in order to determine the realizable local normal forms of bi-Legendrian manifolds.

The invariant in question is the curvature of a certain natural connection associated with $\mathcal{W}_\Omega$ defined on the determinant bundle $\text{det} T\mathcal{F}$ for a fixed $\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)$. It was proved in [17] that its vanishing characterizes locally trivial divergence-free 2-webs; it also played a role in the solution to the problem of finding a normal form of $\mathcal{W}_\Omega$ in $\mathcal{W}_\Omega$-adapted coordinates provided in the same paper. One of our goals is to define the natural connection for divergence-free $n$-webs with $n > 2$ and use its curvature to extend the results of [17] to this more general setting.

We will express the connection and its curvature using Cartan’s method of moving frames [7]. With its help, the connection itself can be succinctly characterized as the unique linear connection $\omega$ on the Whitney sum of determinant bundles $\text{det} T\mathcal{F}$ for certain foliations $\mathcal{F}^c$ complementary to $\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)$ induced from any principal $G$-connection $\Theta$ (hereafter referred to as $\mathcal{W}_\Omega$-connection) equivariant with respect to the Lie group $G$ of differentials of $\mathcal{W}_0$-equivalence germs, where $\mathcal{W}_0$ is the trivial divergence-free web. This representation of $\omega$ yields a clear relationship between the curvature invariants of an $H$-structure for a given subgroup $H \leq G$ (e.g. a bi-Lagrangian manifold) and those of its induced divergence-free web, since a $H$-connection associated with a reduction of the structure group from $G$ to $H$ is itself a $G$-connection [13]. For example, we will show that the nonuniformity tensor defined above is a certain invariant part of the Ricci tensor of a $G$-connection (Theorem 15); since the natural connection of a bi-Lagrangian manifold is in particular a $G$-connection, we obtain the result of I. Vaisman [19] characterizing Ricci-free bi-Lagrangian manifolds in terms of the induced volume form, which was also noted in [17].

Also, of note is a side remark found in [17, Fig. 2], where an interesting geometric interpretation of the curvature form associated to a natural connection of a divergence-free 2-web in the planar case was mentioned in passing. It was pointed out that its
Fig. 1 Left: Tabachnikov’s interpretation of divergence-free 2-web’s curvature in the planar case [17]. Right: the action of map-germs \( r_p;\mathcal{F}, r_p;\mathcal{G} \). The regions bounded by the leaves of \( \mathcal{W}_\Omega \) have \( \Omega \)-volumes equal to \( \varepsilon \).

only nonzero coefficient measures the degree to which the equality \( ac = bd \) between the volumes of adjacent quadrilateral regions \( A, B, C, D \) enclosed by the leaves of the web fails (Fig. 1). Since the notions of divergence-free 2-web and Lagrangian 2-web coincide, a question arises whether a similar interpretation can be found for webs (both divergence-free and Lagrangian) of higher (co-)dimensions. In the case of divergence-free webs, we provide a positive answer by introducing a local invariant inspired by the classical works of Thomsen, Blaschke and Bol on the holonomy of planar webs [6, 16].

At its core are certain smooth map-germs \( r_p;\mathcal{F} \) associated with each pair \( (\mathcal{W}_\Omega, \mathcal{F}) \) consisting of a codimension-1 divergence-free web-germ \( \mathcal{W}_\Omega \) at \( p \in M \) and one of its foliations \( \mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega) \). The effect of applying \( r_p;\mathcal{F} \) to a point \( q \in M \) lying on one side of the hypersurface \( F_p \in \mathcal{F} \) crossing \( p \) is to transport it to \( q' = r_p;\mathcal{F}(q) \) lying on the other side of \( F_p \) along the curve formed by intersecting the leaves of each \( \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega) \setminus \{\mathcal{F}\} \) crossing \( q \) in such a way, that the regions \( R_{p, q} \) and \( R_{p, q'} \) enclosed by the leaves of \( \mathcal{W}_\Omega \) crossing one of the points \( p, q \) and \( p, q' \) respectively have equal \( \Omega \)-volumes (Fig. 1). If we choose two different foliations \( \mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega) \), then the composition \( \ell_p;\mathcal{F},\mathcal{G} = r_p;\mathcal{G} \circ r_p;\mathcal{F} \circ r_p;\mathcal{G} \circ r_p;\mathcal{F} \) transports each point through the concatenation of the corresponding curves. This piecewise-smooth curve is closed whenever \( \mathcal{W}_\Omega \) is trivial. We will prove that nontriviality of \( \ell_p;\mathcal{F},\mathcal{G} \) is measured by the coefficient of the nonuniformity tensor \( K(\mathcal{W}_\Omega) \) corresponding to the directions complementary to \( T\mathcal{F}, T\mathcal{G} \) (Theorem 32).

Aggregating these maps into a single mathematical entity defines the aforementioned local geometric invariant, called the volume-preserving reflection holonomy of \( \mathcal{W}_\Omega \). In Theorem 35, we relate the vanishing of this invariant to a generalization of S. Tabachnikov’s interpretation of the curvature of the natural connection in terms of volumes of adjacent quadrilaterals as stated above.

Lastly, we note that assessing local triviality of the divergence 2-web induced by the space-like foliation by hyperplanes of simultaneity in Lorentzian geometry may prove beneficial to numerical approaches to relativistic fluid dynamics [2, 3]. By working, if possible, in a coordinate system which not only has privileged space and time
directions, as in the classical 3+1 formalism [4], but also makes the metric volume density constant, we can simplify the calculation involving divergences of the fluid velocity field occurring in e.g. the continuity equation, where the volume density plays an important role [11]. We will elaborate on this remark in the final part of this paper.

The structure of the paper is as follows. In Sect. 2, we introduce the connections associated with a given divergence-free \( n \)-web, derive their representation in coordinates, provide some examples and relate the curvature of the natural connection of a divergence-free \( n \)-web to the Ricci tensor of the corresponding principal connection. In Sect. 3, we show that the nonuniformity tensor determines a divergence-free \( n \)-web uniquely given fixed initial data, and use the results surrounding this theorem to give a canonical form and the moduli space of the webs’ volume forms in the planar case. In Sect. 4 we define the geometric invariants mentioned above and establish several geometric conditions for local triviality of divergence-free \( n \)-webs. In the last section we give potential applications in numerical relativity.

Also, bearing in mind the remark preceding Theorem 1, from Sect. 2 onward we shall assume that divergence-free webs \( \mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega) \) satisfy

\[
\sum_{\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)} \text{codim} \mathcal{F} = \dim M.
\]

### 2 Connections associated with divergence-free webs

First, we construct a set of affine connections compatible with the structure of a given divergence-free \( n \)-web \( \mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega) \). While in general there are many such connections, they share a number of properties that allow us to extract well-defined differential invariants that do not depend on a choice of the connection. We proceed via Cartan’s method of moving frames (see e.g. [7]).

#### 2.1 Web-adapted coframes and principal connections

Our preferred coframes \((\xi^1, \xi^2, \ldots, \xi^m)\) are those which satisfy

(a) \( T \mathcal{F}_i = \bigcap_{j=m_i+1}^{m_i+1} \ker \xi^j \) for \( i = 1, 2, \ldots, n \), where \( m_i = \sum_{j=1}^{i-1} \text{codim} \mathcal{F}_j \),

(b) \( \Omega = \xi^1 \wedge \xi^2 \wedge \cdots \wedge \xi^m. \)

Locally, each choice of such 1-forms for which the ideals \( I_i = \langle \xi^{m_i+1}, \ldots, \xi^{m_i+1} \rangle \) with \( m_i = \sum_{j=1}^{i-1} \text{codim} \mathcal{F}_j \) are all closed under the exterior derivative fully defines a divergence-free \( n \)-web structure by taking (a) and (b) as definitions of \( \mathcal{F}_i \) and \( \Omega \). The integrability of \( T \mathcal{F}_i \) follows from Frobenius’s theorem. Such coframes will be called \( \mathcal{W}_\Omega \)-adapted.

Each choice of a \( \mathcal{W}_\Omega \)-adapted coframe partitions the index set \( [m] = \{1, 2, \ldots, m\} \) into \( n \) subsets

\[
\pi_i = \{ k \in [m] : T \mathcal{F}_i \subseteq \ker \xi^k \} = \{ m_i + 1, \ldots, m_{i+1} \},
\]

\[
m_i = \sum_{j=1}^{i-1} \text{codim} \mathcal{F}_j \quad (2.1)
\]
for \( i = 1, 2, \ldots, n \). This partition \( \pi \) is a recurring theme of nearly every coordinate computation throughout this paper. For notational convenience, let us introduce the symbol \( \sim \) for the equivalence relation induced by \( \pi \). More explicitly:

\[
i \sim j \iff i, j \in \pi_k \text{ for some } k = 1, 2, \ldots, n, \tag{2.2}\]

and \( i \not\sim j \) otherwise.

**Proposition 3** Let \( M \) be a \( m \)-dimensional smooth manifold and let \( \mathcal{W}_\Omega = (M, F_1, \ldots, F_n, \Omega) \) be a divergence-free \( n \)-web. There exists a torsionless affine connection \( \Theta \) whose connection 1-forms \( \theta_{ij} \) with respect to each \( \mathcal{W}_\Omega \)-adapted coframe satisfy

1. \( \forall j \sim k \theta_{ij}^k = 0 \), \quad (\mathcal{F}_i \text{ are } \Theta\text{-parallel})
2. \( \sum_{k=1}^m \theta_{ik}^k = 0 \). \quad (the volume form \( \Omega \) is \( \Theta \)-parallel)

**Proof** Fix a \( \mathcal{W}_\Omega \)-adapted coframe \((\xi_1, \xi_2, \ldots, \xi^m)\) on a neighbourhood \( U \subseteq M \) of a given point \( p \in M \). Our goal is to find a \( m \times m \) matrix of 1-forms \( \Theta = [\theta_{ij}^k] \) which for each \( j = 1, 2, \ldots, m \) satisfy the structure equation

\[
d\xi^j + \sum_k \theta_{ij}^k \wedge \xi^k = 0 \tag{2.3}\]

of a torsionless connection and have properties (1) and (2) given in the statement. The construction of \( \Theta \) is as follows. Since \( T.F_i \) are involutive, one can find a collection of 1-forms \( \alpha_{ij}^k \in \Omega^1(U) \) with \( \alpha_{ij}^k = 0 \) for \( j \not\sim k \) such that \( d\xi^j = \sum_k \alpha_{ij}^k \wedge \xi^k \) for each \( j = 1, 2, \ldots, m \). Let \( \beta = \sum_k \alpha_{ij}^k \). Expand this 1-form into components with respect to the coframe: \( \beta = \sum_k f_k \xi^k \) for some \( f_k \in C^\infty(M) \). For each \( j, k = 1, 2, \ldots, m \) put

\[
\theta_{ij}^k = -\alpha_{ij}^k + \delta_{jk} f_k \xi^k \tag{2.4}\]

where \( \delta_{jk} \) denotes the Kronecker’s delta.

This defines the desired connection locally. To define it globally, use a partition of unity \( \{g_U\}_{U \in \mathcal{U}} \) corresponding to the covering of \( M \) by the coframe-domains \( U \in \mathcal{U} \) to define \( \Theta \) on each \( U \in \mathcal{U} \) with respect to the \( \mathcal{W}_\Omega \)-adapted coframe \((\xi_U)^1, \ldots, (\xi_U)^m\)

as

\[
\Theta = \sum_{V \in \mathcal{U} : U \cap V \neq \emptyset} g_V(x) (Q_U^V \Theta_V Q_U^{1V}) - \left( (dQ_U^V)Q_U^{1V} \right), \tag{2.5}\]

where \( \Theta_U \) are the matrices of connection 1-forms on \( U \) obtained above, while \( Q_{UV} \) are the transition matrices satisfying \( (\xi_U)^k = \sum_l (Q_{UV})_{lj}^k (\xi_V)^l \) on \( U \cap V \) [13]. Note that \( Q_{UV} \) is a product of two matrix-valued functions \( RP \), where \( R_{ij}^k = 0 \) for \( k \not\sim l \) and \( P \) is a constant permutation matrix corresponding to \( \sigma \in S_m \) for which \( \sigma(k) \sim \sigma(l) \) if and only if \( k \sim l \). Moreover, \( \det Q_{UV} = 1 \), which leads to \( \tr((dQ_{UV})Q_{UV}^{-1}) = 0 \). Using these facts, it can be checked directly by means of the structure equation that this indeed yields a torsionless connection satisfying (1) and (2).
**Definition 4** An affine torsionless connection $\Theta$ satisfying (1) and (2) of Proposition 3 will be called a $W_\Omega$-connection.

**Proposition 5** Let $\varphi$ be a local equivalence of divergence-free n-webs between $W_\Omega$ and $W_\hat{\Omega}$. If $\Theta$ is a $W_\Omega$-connection, then $\varphi^*\Theta$ is a $W_\Omega$-connection.

**Proof** Choose a coframe $(\xi^1, \ldots, \xi^m)$ on the codomain of $\varphi$ and express $\Theta$ as $[\theta^i_k]$ in terms of this coframe. Then the connection matrix of $\varphi^*\Theta$ in the coframe $(\varphi^*\xi^1, \ldots, \varphi^*\xi^m)$ is exactly $[\varphi^*\theta^i_k]$ by the structure equation (2.3). The coframe $(\varphi^*\xi^k)_k$ is $W_\Omega$-adapted, since $\varphi^*\tilde{\Omega} = \Omega$ and for each $\mathcal{F} \in \text{Fol}(W_\Omega)$ the equality $d\varphi(T\mathcal{F}) = T\mathcal{G}$ holds for some $\mathcal{G} \in W_\hat{\Omega}$. Moreover, the matrix $[\varphi^*\theta^i_k]$ satisfies (1) and (2) of Proposition 3 by linearity of pullback, hence it represents a $W_\Omega$-connection.

**Lemma 6** Let $W_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a divergence-free n-web. Given a fixed $W_\Omega$-connection $\hat{\Theta}$, the mapping $\Theta \mapsto \hat{\Theta} - \hat{\Theta}$ defines a bijection between the space of $W_\Omega$-connections and the space of symmetric $TM$-valued covariant 2-tensor fields $D$ satisfying

1. $\forall_i \forall_{v,w\in T\mathcal{F}_i} D(v, w) \in T\mathcal{F}_i$,
2. $\forall_i \forall_{v\in T\mathcal{F}_i} \forall_{w\in \cap_{k\neq i} T\mathcal{F}_k} D(v, w) = 0$,
3. $\forall_{v\in TM} \text{tr} D(v, \cdot) = 0$.

**Proof** Pick a $W_\Omega$-adapted coframe $(\xi^1, \ldots, \xi^m)$. Let $(e_1, \ldots, e_m)$ be the dual frame. The quantity $D = \sum_{i,j,k} D^k_{ij} e_k \otimes \xi^i \otimes \xi^j$ can be thought of as a matrix of 1-forms $[\Delta^k_j]$, where $\Delta^k_j = \sum_{i} D^k_{ij} \xi^i$. The matrix $\Theta = \hat{\Theta} + [\Delta^k_j]$ is a matrix of a torsionless connection if and only if $D$ is a symmetric $TM$-valued 2-tensor. Necessity follows from Cartan’s lemma on division of 2-forms [7, Theorem 3.4] applied to the difference of structure equations (2.3) of $\Theta$, $\hat{\Theta}$, while sufficiency is a consequence of the way both entities transform under a change of frame. Recall the relation $j \sim k$ defined in (2.2). Properties (1)–(3) of $D$ can be expressed in terms of the coframe as:

1. $D^k_{ij} = 0$ for $i \sim k$ and $j \sim k$;
2. $D^k_{ij} = 0$ for $i \sim j$;
3. $\sum_j D^j_{ij} = 0$; or in terms of $\Delta^k_j$ as: (1)–(2) $\Delta^k_j = 0$ for $k \sim k$ (assuming the symmetry of $D$); (3) $\sum_k \Delta^k_j = 0$. A quick comparison with the statement of Proposition 3 makes it clear that $D$ has these properties if and only if $\Theta$ is a $W_\Omega$-connection.

In the course of the proof above we have established that for each fixed $k = 1, 2, \ldots, m$ the matrices $D^k_{ij}$ formed by the coefficients of a difference tensor in a fixed $W_\Omega$-adapted coframe are exactly the symmetric matrices which are zero outside of a square diagonal block spanned by entries with $i,j = m_k + 1, \ldots, m_{k+1}$, where $m_k = \sum_{l=1}^{k-1} \text{codim} \mathcal{F}_l$. The dimension of the space of these symmetric matrices for fixed $k$ is equal to $\frac{1}{2}c_k(c_k + 1)$, where $c_k = \text{codim} \mathcal{F}_k$. These matrices are further bound by equations $\sum_j D^j_{ij} = 0$ for $k = 1, \ldots, m$, which allow us to express each $D^k_{ij}$ in terms of the their off-diagonal entries. This makes the differentials $(dD^k_{ij})$ for $i \leq j$ and $i \sim j \sim k$ without $i = j = k$ a valid coframe of the bundle of difference
tensors $\mathcal{D}$. Hence, if we put $c_i = \text{codim} \mathcal{F}_i$, the dimension of $\mathcal{D}$ is
\begin{equation}
\dim \mathcal{D} = \sum_{i=1}^{n} c_i \left( \frac{1}{2} c_i (c_i + 1) - 1 \right) = \sum_{i=1}^{n} \frac{1}{2} c_i (c_i - 1) (c_i + 2).
\end{equation}

In particular, if all $c_i$ are equal to 1, the bundle $\mathcal{D}$ has dimension 0.

**Corollary 7** Let $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a codimension-1 divergence-free web. A $\mathcal{W}_\Omega$-connection exists and is unique. $\square$

### 2.2 The natural connection of a divergence-free web

When the codimension of any of the foliations $\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)$ of a divergence-free $n$-web $\mathcal{W}_\Omega$ exceeds 1, the uniqueness claim regarding the $\mathcal{W}_\Omega$-connections clearly breaks down. Note, however, that there are some invariant quantities independent of the choice of $\mathcal{W}_\Omega$-connection.

Define $\pi_i$ as in (2.1) and $j \sim k$ as in (2.2). Given a $\mathcal{W}_\Omega$-adapted coframe $(\xi^1, \ldots, \xi^m)$ with the dual frame $(e_1, \ldots, e_m)$, for each $\mathcal{W}_\Omega$-connection $\Theta = [\theta^i_k]$ and each fixed $i = 1, 2, \ldots, n$ the sums $\sum_{k \in \pi_i} \theta^i_k$ remain the same irrespective of the choice of $\Theta$, since $\sum_{k \in \pi_i} D^i_j k = 0$ for each difference tensor $D$ of Lemma 6 by properties (2) and (3).

The partial traces $\sum_{k \in \pi_i} \theta^i_k$ can be interpreted in terms of the covariant derivative $\nabla$ of $\Theta$, which acts on vector fields $X = \sum_k X^k e_k$ by $\nabla X = \sum_k (dX^k + \sum_j X^j \theta^i_k) \otimes e_k$. The corresponding action on multivectors of the form $e_i = e_{m_i+1} \wedge \cdots \wedge e_{m_i+1}$ with $m_i = \sum_{k=1}^{i-1} \text{codim} \mathcal{F}_k$ for $i = 1, 2, \ldots, n$ is
\begin{equation}
\nabla (e_{m_i+1} \wedge \cdots \wedge e_{m_i+1}) = \sum_{k \in \pi_i} \sum_{j=1}^{m} \sum_{j \neq k} (-1)^{k-1} \theta^i_k \otimes e_j \wedge e_{m_i+1} \wedge \cdots \hat{e}_k \wedge \cdots \wedge e_{m_i+1} \\
= \sum_{k \in \pi_i} \sum_{j \neq k} (-1)^{k-1} \theta^i_k \otimes e_j \wedge e_{m_i+1} \wedge \cdots \hat{e}_k \wedge \cdots \wedge e_{m_i+1} + \sum_{k \in \pi_i} \theta^i_k \otimes e_{m_i+1} \wedge \cdots \wedge e_k \wedge \cdots \wedge e_{m_i+1},
\end{equation}

where a hat denotes omission. Note that the first summand vanishes as a consequence of property (1) of Proposition 3, yielding $\nabla e_i = (\sum_{k \in \pi_i} \theta^i_k) \otimes e_i$ for $i = 1, 2, \ldots, n$.

The above two paragraphs allow us to conclude that the Whitney sum of line bundles generated by the $e_i$ carries a unique linear connection arising naturally from the structure of the web via associated connections. We will now define these objects invariantly in terms of components of the web $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$.

First, let $\mathcal{F}^i_c$ denote the foliation complementary to $\mathcal{F}_i$ with respect to $\mathcal{W}_\Omega$, which we define as the result of integrating the (involutive) tangent distribution $\bigcap_{j \neq i} T \mathcal{F}_j$. It is worthwhile to note that, since $\bigcap_{i=1}^{n} T \mathcal{F}_i = \{0\}$, the tangent bundle decomposes
both into $TM = \bigoplus_{i=1}^{n} T \mathcal{F}_i$ and $TM = T \mathcal{F}_i \oplus T \mathcal{F}_i$ for each $i = 1, \ldots, n$. The values of the corresponding projections at the tangent vector $X \in TM$, which we denote by $X_{\mathcal{F}_i} \in T \mathcal{F}_i$ and $X_{\mathcal{F}_i} \in T \mathcal{F}_i$, are the unique vectors which satisfy $X = \sum_{k=1}^{n} X_{\mathcal{F}_i} = X_{\mathcal{F}_i} + X_{\mathcal{F}_i}$ for $i = 1, 2, \ldots, n$.

Now, the line bundles mentioned above are exactly the determinant bundles $\det T \mathcal{F}_i$, i.e. the bundles of top-degree multivectors in $T \mathcal{F}_i$. The preceding discussion can be summarized in the following proposition.

**Proposition/Definition 8** The action of a $\mathcal{W}_\Omega$-adapated connection $\Theta$ on $\mathcal{X}(M)$ induces a linear connection $\omega$ on the bundle $\bigoplus_{i=1}^{n} \det T \mathcal{F}_i$ which is independent of the choice of $\Theta$. This connection will be called the natural $\mathcal{W}_\Omega$-connection. □

**Remark** Assume that $\mathcal{W}_\Omega$ is a codimension-1 divergence-free $n$-web. The determinant bundles $\det T \mathcal{F}_i$ reduce to $T \mathcal{F}_i$, allowing us to identify the (unique) $\mathcal{W}_\Omega$-connection $\Theta$ with the natural connection $\omega$ in this case.

By uniqueness of natural connections and Proposition 5, the pullback of the natural connection of $\mathcal{W}_\Omega$, via a local equivalence of divergence-free $n$-webs between $\mathcal{W}_\Omega$ and $\mathcal{W}_\Omega$, coincides with the natural connection of $\mathcal{W}_\Omega$. To further justify the use of the adjective natural in the above definition, we will show that we can characterize $\mathcal{W}_\Omega$-connections in terms of the natural connection.

**Proposition 9** Let $(M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a divergence-free $n$-web. An affine torsionless connection $\Theta$ on $M$ is a $\mathcal{W}_\Omega$-connection if and only its action on $\bigoplus_{i=1}^{n} \det T \mathcal{F}_i$ is well-defined and coincident with the action of the natural $\mathcal{W}_\Omega$-connection $\omega$.

**Proof** If $\Theta$ is a $\mathcal{W}_\Omega$-connection, then the action of $\Theta$ on $\bigoplus_{i=1}^{n} \det T \mathcal{F}_i$ is equal to that of $\omega$ by the definition of $\omega$. For the converse, suppose that $(\xi^1, \xi^2, \ldots, \xi^m)$ is a $\mathcal{W}_\Omega$-adapted coframe with dual frame $(e_1, e_2, \ldots, e_m)$. The connection $\Theta = [\theta^j_k]_i$ gives rise to a covariant derivative operator $\nabla$ which acts on multivectors $e_i = e_{m_i+1} \wedge \cdots \wedge e_{m_i+1} \in \Gamma(\det T \mathcal{F}_i)$ for $i = 1, 2, \ldots, n$ by means of (2.7), where $m_i = \sum_{k=1}^{l} \text{codim} \mathcal{F}_k$. This action coincides with the action of the natural connection $\omega$ by our assumption, which implies that the right-hand side must be equal to $(\sum_{k=m_i+1}^{m} \theta^k_k) \otimes e_i$, for some $\mathcal{W}_\Omega$-connection $\tilde{\Theta} = [\tilde{\theta}^j_k]$. From this we deduce: (1) $\theta^k_k = 0$ for $j \sim k$, where we have used notation from (2.2), and (2) $\sum_{k=1}^{m} \theta^k_k = \sum_{k=1}^{m} \tilde{\theta}^k_k = 0$. Since $(\xi^1, \ldots, \xi^m)$ is $\mathcal{W}_\Omega$-adapted, these two conditions correspond to (1) and (2) of Proposition 3, which together with the lack of torsion make $\Theta$ a $\mathcal{W}_\Omega$-adapted connection. □

**Remark** The equivalence above cannot be inferred for all (not necessarily torsionless) affine connections, since there are 1-forms $\theta^j_k = \sum_i \Gamma^{ij}_k \xi^i$ with coefficients $\Gamma^{ij}_k$, which are non-symmetric in $i, j$, but nevertheless satisfy $\theta^k_j = 0$ for $j \sim k$ and $\sum_k \theta^k_k = 0$.

### 2.3 Coordinate expressions

Fix a $\mathcal{W}_\Omega$-adapted system of coordinates $(x_1, x_2, \ldots, x_m)$, the standard frame $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m})$ and its dual coframe $(dx_1, dx_2, \ldots, dx_m)$. Express the volume
form in terms of the coframe as \( \Omega = h(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m \). By picking a \( \mathcal{W}_\Omega \)-adapted coframe, for example, \( \xi^i = dx_i \) for \( i = 1, 2, \ldots, m-1 \) and \( \xi^m = h(x) \, dx_m \), and carrying out the computations outlined in the proof of Proposition 3, one finds the matrix \( \Theta = [\theta^j_k] \) of a certain \( \mathcal{W}_\Omega \)-connection

\[
\theta^j_k = \begin{cases} 
\frac{\partial \log h}{\partial x_j} \cdot \xi^j & \text{if } j = k \neq m, \\
-d \log h + \frac{1}{h} \frac{\partial \log h}{\partial x_m} \cdot \xi^m & \text{if } j = k = m, \\
0 & \text{otherwise,} 
\end{cases}
\]

which can be used to compute the action of the natural connection on the determinant bundles \( \bigoplus_{i=1}^n \det \mathcal{T}_F^c_i \). It is given by \( \nabla e_i = (\sum_{k=m+1}^i \theta^k_j) \otimes e_i = \sum_j \omega^j_i \otimes e_j \) with \( m_i = \sum_{k=1}^{i-1} \text{codim } \mathcal{F}_k \) as described in the remarks surrounding equality (2.7). After expressing the result in the coordinate coframe \( (dx_1, dx_2, \ldots, dx_n) \) on \( \bigoplus_{i=1}^n \det \mathcal{T}_F^c_i \) by means of the gauge transformation \( \omega = Q^{-1} \tilde{\omega} Q + Q^{-1} dQ \) with the transition matrix \( Q \) defined by the relation \( \xi^{m_i+1} \wedge \cdots \wedge \xi^{m_{i+1}} = \sum_k Q^j_k \cdot dx^k \), where \( dx' = dx_{m_i+1} \wedge \cdots \wedge dx_{m_{i+1}} \in \det \mathcal{T}_F^c_i \) for \( i = 1, \ldots, n \), we get the final expression for the connection 1-form \( \omega \) of the natural connection in the coordinate coframe. It takes the form

\[
\omega = \begin{pmatrix} 
\omega_1 & 0 & \cdots & 0 \\
0 & \omega_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n 
\end{pmatrix},
\]

where

\[
\omega^i_j = \sum_{k=m_i+1}^{m_{i+1}} \frac{\partial \log h}{\partial x_k} (x) \, dx_k.
\]

With this in hand, it is straightforward to compute the curvature 2-form (denoted by \( \Xi \)) via Cartan’s formula, which yields

\[
\Xi = \begin{pmatrix} 
\Xi_1 & 0 & \cdots & 0 \\
0 & \Xi_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Xi_n 
\end{pmatrix},
\]

where

\[
\Xi^i_j = d\omega^i_j = \sum_{j \neq i} \left( \sum_{k=m_{j+1}}^{m_j} \sum_{l=m_{j+1}}^{m_{j+1}} \frac{\partial \log h}{\partial x_k} (x) \frac{\partial \log h}{\partial x_l} \right),
\]

As a corollary we obtain the following result.
A divergence-free n-web $\mathcal{W}_\Omega$ is locally trivial if and only if its natural connection $\omega$ is flat.

**Proof** Note that the curvature 2-forms $\Xi_i^j$ (2.12) vanish everywhere if and only if the assumptions of Theorem 1 are met. \hfill \Box

**Example 2** Let $M = \mathbb{R}^2 \setminus \{0\}$, $\Omega = dx \wedge dy$ and suppose that $\mathcal{F}, \mathcal{G}$ are foliations of $M$ by open half-lines emanating from 0 and by concentric circles centered at 0 respectively. Then the divergence-free 2-web $\mathcal{W}_\Omega = (M, \mathcal{F}, \mathcal{G}, \Omega)$ is locally trivial. A local web-equivalence $\varphi$ witnessing its triviality at a point $p \in M$ is given by a restriction of the covering map $\pi : \mathbb{R}_+ \times \mathbb{R} \to M; (\rho, \phi) \mapsto (\sqrt{2\rho} \cos \phi, \sqrt{2\rho} \sin \phi)$ to a neighbourhood of $p$.

The map $\pi$ carries lines into geodesics of the natural connection $\omega$ of $\mathcal{W}_\Omega$ by Proposition 5. In polar coordinates $(\rho, \phi)$ they are given by equations $ar^2 + b\phi = c$ for some $a, b, c \in \mathbb{R}$; in other words, they are Fermat spirals (Fig. 2). As a parametrized geodesic $\gamma(t)$, a generic Fermat spiral can be characterized using $\omega$ as the unique curve with given initial velocity $\dot{\gamma}(0)$ whose angular velocity $\dot{\gamma}_G$ with respect to the origin is constant, and such that at every instant $t \in \mathbb{R}$ the area $\Omega(\dot{\gamma}_F, \dot{\gamma}_G)$ of an infinitesimal rectangle spanned by the projections $\dot{\gamma}_F \in T\mathcal{F}, \dot{\gamma}_G \in T\mathcal{G}$ with $\dot{\gamma} = \dot{\gamma}_F + \dot{\gamma}_G$ remains the same.

**Example 3** Let $M$ be a unit 2-sphere $S^2 \subseteq \mathbb{R}^3$ without the north and south poles $(x, y, z) = (0, 0, \pm 1) \in S^2$. Let $\mathcal{F}, \mathcal{G}$ be foliations of $M$ by circles of constant latitude $z = c \in \mathbb{R}$ and semicircles of constant longitude $ax + by = 0$ for $a, b \in \mathbb{R}, (a, b) \neq 0$ respectively. Denote by $\Omega$ the Riemannian volume form $x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$ induced from the ambient space. The divergence-free 2-web $\mathcal{W}_\Omega = (M, \mathcal{F}, \mathcal{G}, \Omega)$ is locally trivial. A global coordinate system witnessing the triviality of $\mathcal{W}_\Omega$ is the cylindrical coordinate system $(\phi, z)$, where $z$ is the linear
coordinate along the north-south axis and \( \phi \) is the longitude of a given point. The geodesics of the natural connection \( \omega \) can be computed as in Example 2; they are the spherical spirals satisfying \( az + b \phi = c \) for some \( a, b, c \in \mathbb{R} \) (Fig. 2).

These curves possess a property similar to the one exhibited by Fermat spirals. Note that a single transverse geodesic together with a semicircle of constant longitude subdivide the sphere into several quasi-rectangular strips. The property in question is: the strips which are not adjacent to a pole have equal areas. This can easily be seen in cylindrical coordinates.

### 2.4 Construction in terms of Bott’s connection

That the coordinate expression for the curvature 2-form essentially agrees with the one derived in the article of Tabachnikov [17] for divergence-free 2-webs is not a coincidence; the natural connection \( \omega \) (Definition 8) and Tabachnikov’s connection \( \nabla \), while differing in the way they arise, represent the same connection.

To recall the construction of \( \nabla \), let us first introduce a partial connection acting on the quotient bundle \( \nu F = TM/T F \) for some foliation \( F \) of \( M \). Given a vector field \( X \in \Gamma(T F) \) and a section \( \bar{Y} \in \Gamma(\nu F) \) with a representative \( Y \in \mathcal{X}(M) \), define the \( D^F \)-covariant derivative of \( \bar{Y} \) along \( X \) by

\[
(D^F)_X \bar{Y} = [X, Y] \mod T F.
\]  

(2.13)

The Bott’s connection is the corresponding partial connection \( D^F : \Gamma(T F) \times \Gamma(\nu F) \to \Gamma(\nu F) \). It is always flat. This can be verified by picking a local basis of \( D^F \)-parallel sections in coordinates \( (x_1, \ldots, x_m) \) adapted to \( F \) in which \( T F = \bigcap_{i=k+1}^m \ker dx_i \), for example \( (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}) \mod T F \).

Now, in the 2-web case, the presence of a complementary foliation \( G \) allows us to decompose the tangent bundle into \( TM = T F \oplus T G \), leading to an identification of \( \nu F \) with \( T G \) via a projection onto the second summand. Thus, \( D^F \) acts naturally on \( T G \) along \( T F \), while \( D^G \) acts on \( T F \) along \( T G \); both of these extend naturally to the action on the corresponding determinant bundles \( \det T F, \det T G \) and their duals. A volume form \( \Omega \) on \( M \) defines a bundle isomorphism \( \mu : \det T F \to \det T^* G \) by the formula \( \eta \mapsto \eta \cup \Omega \). The aforementioned Tabachnikov’s connection of a divergence-free 2-web \( (M, F, G, \Omega) \) on \( \det T F \) is the linear connection \( \nabla : \mathfrak{X}(M) \times \Gamma(\det T F) \to \Gamma(\det T F) \) whose action is given by

\[
\nabla_X \eta = \mu^{-1} D^F_{X,F} (\mu \eta) + D^G_{X,G} \eta,
\]  

(2.14)

where \( X = X_F + X_G \) for \( X_F \in \Gamma(T F), X_G \in \Gamma(T G) \).

**Proposition 11** Let \( \mathcal{W}_\Omega = (M, F_1, \ldots, F_n, \Omega) \) be a divergence-free n-web and, for each \( i = 1, 2, \ldots, n \), let \( F_i^c \) be a foliation with tangent distribution \( T F_i^c = \bigcap_{j \neq i} T F_j \). The covariant derivative \( D^i : \mathfrak{X}(M) \times \Gamma(\det T F_i^c) \to \Gamma(\det T F_i^c) \) associated with the restriction of its natural connection \( \omega \) to \( \det T F_i^c \) is exactly the Tabachnikov’s connection \( \nabla^i \) on \( \det T F_i^c \) of the divergence-free 2-web \( (M, F_i^c, F_i, \Omega) \).
Proof Let \((\xi^1, \ldots, \xi^m)\) be a \(\mathcal{W}_\Omega\)-adapted coframe on \(M\) with the dual coframe \((e_1, \ldots, e_m)\) and define the relation \(j \sim k\) as in (2.2). For a \(\mathcal{W}_\Omega\)-connection \(\Theta = [\theta^i_j]\) and the associated covariant derivative \(D\), property (1) of Proposition 3 means that for each \(i = 1, 2, \ldots, n\) and \(X = \sum_{k=1}^n X^k e_k \in \Gamma(T_\mathcal{F}_i)\), the vector field \(D_Y X = \sum_{j=1}^n (V X^j) e_j + \sum_{k=1}^n \sum_{j} \theta^j_k (V) X^k e_j\) is in \(\Gamma(T_\mathcal{F}_i)\). Since \(T_\mathcal{F}_i^c = \bigcap_{j \neq i} T_\mathcal{F}_j\), the same is true with \(T_\mathcal{F}_i\) replaced by its complementary foliation \(T_\mathcal{F}_i^c\). By the lack of torsion of \(\Theta\), for \(X \in \Gamma(T_\mathcal{F}_i)\) and \(Y \in \Gamma(T_\mathcal{F}_i^c)\) we obtain

\[D_X Y = (D_X Y)_{\mathcal{F}_i} = (D_X Y)_{\mathcal{F}_i^c} - (D_Y X)_{\mathcal{F}_i^c} = [X, Y]_{\mathcal{F}_i} = D_{X|_{\mathcal{F}_i}} Y, \tag{2.15}\]

where the mapping \(v \mapsto v_{\mathcal{F}_i}\) is the projection from \(TM = T_\mathcal{F}_i \oplus T_{\mathcal{F}_i}^c\) to \(T_{\mathcal{F}_i}^c\) along \(T_\mathcal{F}_i\).

Let \(\mu_i : \det T_\mathcal{F}_i^c \to \det T^* \mathcal{F}_i\) be the map \(\mu_i(\eta) = \eta \circ \Omega\). Property (2) is equivalent to \(D \Omega = 0\), which together with \(D \Gamma(T_\mathcal{F}_i^c) \subseteq \Omega^1(M) \otimes \Gamma(T_\mathcal{F}_i^c)\) for \(i = 1, 2, \ldots, n\) gives \(D_Y (\eta \circ \Omega) = (D_Y \eta) \circ \Omega\) for each \(Y \in \mathfrak{X}(M)\) and \(\eta \in \det T_\mathcal{F}_i^c\), or, in other words, \(D_Y \circ \mu_i = \mu_i \circ D_Y\) for each \(Y \in \mathfrak{X}(M)\). Let \(G = \mathcal{F}_i^c\) and \(\mathcal{H} = \mathcal{F}_i\). Using the fact that \(D\) commutes with \(\mu_i\) and the equality (2.15) applied to a fixed \(\eta \in \Gamma(\det T_\mathcal{F}_i^c)\) via the Leibniz rule we arrive at

\[D_X \eta = D_X G \eta + D_X H \eta\]
\[= \mu_i^{-1} D_X G (\mu \eta) + D_X H \eta\]
\[= \mu_i^{-1} D_{X G} (\mu \eta) + D_{X H} \eta = \nabla^i \eta, \tag{2.16}\]

where \(X = X_G + X_H\) for \(X_G \in \Gamma(T \mathcal{G})\), \(X_H \in \Gamma(T \mathcal{H})\). Since the action of \(\omega\) on \(\bigoplus_{i=1}^n \det T_\mathcal{F}_i^c\) comes from the action of \(D\) regardless of the choice of \(\Theta\) by Proposition 9, this concludes the proof. \(\square\)

2.5 Nonuniformity tensor

The notion of \(\mathcal{W}_\Omega\)-connection of a divergence-free \(n\)-web \(\mathcal{W}_\Omega\) allows us to give an interpretation of the nonuniformity tensor \(K(\mathcal{W}_\Omega)\) (Definition 2) in terms of more familiar differential-geometric objects.

Theorem 12 The Ricci tensor \(R^c\) of the \(\mathcal{W}_\Omega\)-connection \(\Theta\) of a divergence-free \(n\)-web \(\mathcal{W}_\Omega\) of codimension 1 is equal to the nonuniformity tensor \(K(\mathcal{W}_\Omega)\).

Proof Pick a \(\mathcal{W}_\Omega\)-adapted coordinate system \((x_1, \ldots, x_n)\). Coefficients of the curvature tensor \(R^l_{ijk}\) in the coordinate basis \(\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)\) can be computed from the curvature 2-form \(\Xi\) (2.12) and are equal to

\[R^l_{ijk} = \Xi_k \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{kl} \sum_{m \neq k} \frac{\partial^2 \log h}{\partial x_k \partial x_m} (\delta_{m j} \delta_{k i} - \delta_{m i} \delta_{k j})
= \delta_{kl} \left( (1 - \delta_{k i}) \delta_{k j} \frac{\partial^2 \log h}{\partial x_k \partial x_i} - (1 - \delta_{k j}) \delta_{k i} \frac{\partial^2 \log h}{\partial x_k \partial x_j} \right), \tag{2.17}\]
where \( \delta_{ij} \) is the Kronecker’s delta. To compute the Ricci tensor, we contract the second lower index with the upper one. As a result we get

\[
Rc_{ik} = \sum_j R^j_{ijk} = (1 - \delta_{ki}) \frac{\partial^2 \log h}{\partial x_k \partial x_i}.
\]

(2.18)

The expression above is symmetric in its indices, and is 0 if the indices are equal. We can rewrite this tensor using the symmetric product to obtain

\[
Rc = \sum_{i \neq j} \frac{\partial^2 \log h}{\partial x_j \partial x_i} dx_i dx_j = K(W_\Omega).
\]

(2.19)

\[\square\]

**Corollary 13** For a codimension-1 divergence-free n-web \( W_\Omega \) with the \( W_\Omega \)-connection \( \Theta \), the following properties are equivalent.

(1) \( W_\Omega \) is locally trivial.

(2) \( \Theta \) is flat.

(3) \( \Theta \) is Ricci-flat.

\[\square\]

The results above concern only the codimension-1 case. We can use them, however, to recover the nonuniformity tensor \( K(W_\Omega) \) from a \( W_\Omega \)-connection \( \Theta \) for a divergence-free n-web \( W_\Omega = (M, F_1, \ldots, F_n, \Omega) \) of arbitrary, possibly non-constant codimension.

The method is based on the following observation, which is more easily seen in \( W_\Omega \)-adapted coordinates: the Ricci tensor \( Rc \) contains all the data already present in the curvature \( K(W_\Omega) \) of not only the codimension-1 web \( W_\Omega \) itself, but also of all webs \( W_{\pi;\Omega} = (M, F_{\pi_1}, \ldots, F_{\pi_n}, \Omega) \) obtained by forming a partition \( \pi = \{\pi_1, \ldots, \pi_n\} \) of the index set \([m]\) and letting the foliations \( F_{\pi_j} \) be generated by the integrable distributions \( \bigcap_{k \in \pi_j} T F_k \). Moreover, the coordinate expressions suggest that the Ricci tensor \( Rc \) of the \( W_\Omega \)-connection can be written as a sum of \( K(W_{\pi;\Omega}) \) and a certain block-diagonal term. It turns out that this decomposition is invariant with respect to \( W_{\pi;\Omega} \)-equivalences; hence, one can use an auxiliary codimension-1 web to obtain a characterization of the nonuniformity tensor of an arbitrary divergence-free n-web \( W_\Omega \) as one of the invariant components of the Ricci tensor \( Rc \) of an \( W_\Omega \)-connection \( \Theta \). The process will be detailed below.

With the help of Lemma 6 it is possible to compute the set of Ricci tensors \( Rc \) of all \( W_\Omega \)-connections \( \Theta \) given a single one. In the course of the calculations one will eventually notice that certain coefficients of \( Rc \) inside a fixed \( W_\Omega \)-adapted coordinate chart do not change when replacing \( \Theta \) with its perturbed variant.

**Lemma 14** Let \( W_\Omega = (M, F_1, \ldots, F_n, \Omega) \) be a divergence-free n-web. In any \( W_\Omega \)-adapted coordinate system \((x_1, \ldots, x_m)\) the Ricci tensor \( Rc \) of any \( W_\Omega \)-connection \( \Theta \) satisfies

\[
Rc_{kl}(x) = \frac{\partial^2 \log h}{\partial x_k \partial x_l}(x),
\]

(2.20)
for all \( i \neq j, m_i < k \leq m_{i+1} \) and \( m_j < l \leq m_{j+1} \), where \( m_s = \sum_{k=1}^{s-1} \text{codim } \mathcal{F}_k \) for \( s = 1, \ldots, n + 1 \).

**Proof** Let \( \Theta \) be a fixed \( \mathcal{W}_\Omega \)-connection and let \( D \) be a difference tensor described by Lemma 6. These two objects correspond to matrices of 1-forms \( \Theta = [\theta^i_k] \) and \( \Delta = [\Delta^i_k] \) with \( \Delta^i_k = \sum_l D^i_{kj} \, dx_l \) in the standard coframe \( (dx_1, dx_2, \ldots, dx_m) \). If we denote the Riemann curvature tensors corresponding to \( \Theta \) and \( \Theta + \Delta \) in the standard (co)frame by \( R \) and \( \tilde{R} \) respectively, and use the usual Christoffel symbols \( \Gamma^k_{ij} \) to denote the coefficients \( \theta^k_j (\frac{\partial}{\partial x^i}) \) of the connection \( \Theta \), the following identity, expressed using the Einstein summation convention for brevity, is obtained via direct calculation for each quadruple of indices \( i, j, k, l = 1, \ldots, m \).

\[
\tilde{R}_{ikj} = R_{ikj} + \frac{\partial}{\partial x_l} D^l_{kj} \Gamma^m_{lk} + D^m_{ik} \Gamma^l_{mj} + D^m_{ik} D^m_{kj} - \Gamma^m_{km} D^m_{ij}.
\] (2.21)

A contraction of the \( j \)-index with the \( l \)-index will yield an equality relating the corresponding Ricci tensors \( R_c \) and \( \tilde{R}_c \). By \( \text{tr}_v D = 0 \) for each \( v \in TM \), i.e. condition (3) of Lemma 6, and the symmetry of \( D \) in the lower indices, the contractions of the second, fifth and last summands vanish, leaving only

\[
\tilde{R}_c_{ij} = R_c_{ij} + \frac{\partial}{\partial x_k} D^k_{ij} + D^k_{im} \Gamma^m_{kj} + \Gamma^k_{im} D^m_{kj} + D^m_{im} D^m_{kj} - \Gamma^k_{km} D^m_{ij}.
\] (2.22)

To show that this expression reduces to \( \tilde{R}_c_{ij} = R_c_{ij} \), define \( j \sim k \) as in (2.2) and note that \( \Gamma^k_{ij} = 0 \) if: (1) \( j \sim k \) by condition (1) of Proposition 3; (2) \( i \sim k \) by the symmetry in lower indices, which follows from the lack of torsion; (3) \( i \sim j \) by the preceding two cases. The same holds for \( D^k_{ij} \), since any difference tensor \( D \) is a difference of two such Christoffel symbols \( \Gamma \) and \( \tilde{\Gamma} = \Gamma + D \). Thus, the only nonzero coefficients of both \( D^k_{ij} \) and \( \Gamma^k_{ij} \) are those with \( i \sim j \sim k \). Applying this to (2.22) yields the desired equality \( \tilde{R}_c_{ij} = R_c_{ij} \) for each \( i \sim j \).

The value of \( R_c_{ij} \) at any given point \( p \) inside a \( \mathcal{W}_\Omega \)-adapted chart can be determined by choosing \( \Theta \) to be any \( \mathcal{W}_\Omega \)-connection which coincides with the natural connection of a codimension-1 divergence-free \( m \)-web \((U, G_1, \ldots, G_m, \Omega)\) with \( T\mathcal{G}_i = \ker dx_i \) on a small neighbourhood \( U \) of \( p \). Such a connection exists by a partition of unity argument similar to the one employed in Proposition 3. Inside the set \( U \), the components \( R_c_{ij} \) with \( i \sim j \) are given in Theorem 12, which describes \( R_c \) in the codimension-1 case.

Let us provide an invariant description of the off-diagonal part of \( R_c \) computed above. A decomposition of the tangent bundle \( TM = \bigoplus_{i=1}^{m} T\mathcal{F}^c_i \) for \( T\mathcal{F}^c_i = \bigcap_{j \neq i} T\mathcal{F}_j \) allows us write the bundle of symmetric covariant 2-tensors \( S^2(M) \) as a Whitney sum of two subbundles invariant with respect to local equivalences of
The leading question of this part is: does the nonuniformity tensor vary while keeping the other at 0 one would obtain \( \hat{\Omega}_1 \) being 2 are not locally equivalent. Their local equivalence would assume the form \( \varphi(0) = \hat{\varphi}(0) \) (up to permutation of variables) and satisfy \( \varphi^* \Omega_1 = \Omega_2 \) or, more explicitly,

\[
e^{\frac{1}{2}x^2(1+y)^2} \dot{x}(1+x)(1+y) e^{\frac{1}{2}y^2} = (1+x)(1+y) e^{\frac{1}{2}x^2y^2}
\]

Setting \( x = y = 0 \) would lead to \( \hat{x}'(0)\hat{y}'(0) = 1 \). By letting one of the coordinates vary while keeping the other at 0 one would obtain \( \hat{x}(x) = \hat{x}'(0)(x + \frac{1}{2}x^2) \) and \( \hat{y}(y) = \hat{y}'(0)(y + \frac{1}{2}y^2) \), contradicting (3.2).
To spot the missing ingredient in the recipe for reconstruction of the web from its nonuniformity tensor, let us view the problem through the lens of $W_{\Omega}$-adapted coordinates $(x_1, \ldots, x_m)$. Bearing in mind the coordinate expression for $\mathcal{K}(\mathcal{V}_{\Omega})$ in terms of $\Omega = h(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$ found in the introduction, the reconstruction problem can be restated as follows: given several mixed partial derivatives $\frac{\partial^2 \log h}{\partial x_i \partial x_j}$ of a function-germ $\log h \in C^\infty(\mathbb{R}^d)$ at 0, find $h$. The solution is determined by $\mathcal{K}(\mathcal{V}_{\Omega})$ only up to a multiplication by smooth functions $f_i(x) = f_i(x_{m_i+1}, \ldots, x_{m_i+1})$ with $m_i = \sum_{k=1}^{i-1} \operatorname{codim} F_k$ for $i = 1, 2, \ldots, n$. There is a way to resolve this ambiguity using certain "initial conditions" for the function $h$, which determine uniquely the correction factors $f_i$. Moreover, any such initial condition can be brought into a trivial normal form via a change of $W_{\Omega}$-adapted coordinates. We will now give more details on these results.

In the following theorem we use the concept of the pullback $\iota^* E$ of a smooth vector bundle $E \to M$ along an embedded submanifold $S \hookrightarrow M$. Its sections, called sections of $E$ along $S$, are exactly the precompositions $\sigma \circ \iota$ of sections $\sigma \in \Gamma(E)$ with the submanifold inclusion $\iota$.

**Theorem 16** Let $\mathcal{W} = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ be the germ at $p \in M$ of some $n$-web on an $m$-dimensional manifold $M$ and, for each $i = 1, 2, \ldots, n$, let $\mathcal{F}^\circ_i$ be the foliations complementary to $\mathcal{F}_i$ generated by the tangent distributions $\bigcap_{j \neq i} T \mathcal{F}_j$. Let $F_i$ be the leaf-germ of $\mathcal{F}^\circ_i$ crossing $p$ relative to some $\mathcal{W}$-adapted chart with the corresponding inclusion $\iota_i : F_i \hookrightarrow M$. Given

(i) nonvanishing smooth section-germs $\Omega_{0,i} \in \Gamma(\iota_i^*(\bigwedge^m TM))$ at $p$ for $i = 1, 2, \ldots, n$ such that $(\Omega_{0,i})|_p = (\Omega_{0,j})|_p$ for each pair of different indices $i, j$,

and

(ii) a covariant 2-tensor field-germ $A$ at $p$ satisfying the identities of nonuniformity tensors

1. $A(X, Y) = A(Y, X)$ for every $X, Y \in \mathfrak{X}(M)$,
2. $A(X, Y) = 0$ for every $X, Y \in \Gamma(T \mathcal{F}^\circ_i)$, where $i = 1, 2, \ldots, n$, and
3. $X A(Y, Z) = Y A(X, Z)$ for $X \in \Gamma(T \mathcal{F}^\circ_i)$, $Y \in \Gamma(T \mathcal{F}^\circ_j)$ and $Z \in \Gamma(T \mathcal{F}^\circ_k)$, where $i, j, k = 1, 2, \ldots, n$ are such that $i \neq k$ and $j \neq k$,

there exists a unique volume form-germ $\Omega \in \Omega^m(M)$ at $p$ satisfying $\Omega \circ \iota_i = \Omega_{0,i}$ for $i = 1, \ldots, n$ such that $A = \mathcal{K}(\mathcal{W}_{\Omega})$, where $\mathcal{W}_{\Omega}$ is the induced divergence-free $n$-web-germ $(M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$.

**Proof** Fix a $\mathcal{W}$-adapted coordinate system $(x_1, \ldots, x_m)$ centered at $p$. In these coordinates $F_i = \{ x \in \mathbb{R}^m : \forall k \neq \pi_i, x_k = 0 \}$, where $\pi_i = \{ m_i + 1, \ldots, m_i+1 \}$ with $m_i = \sum_{k=1}^{i-1} \operatorname{codim} F_k$ for $i = 1, \ldots, n$. Write $\Omega_{0,i} = h_i(x_{m_i+1}, \ldots, x_{m_i+1}) \, dx_1 \wedge \cdots \wedge dx_m$ for some function-germs $h_i \in C^\infty(F_i)$ at 0, and let $\tilde{h} \in C(\bigcup_{i=1}^n F_i)$ be the function-germ defined by $\tilde{h}(x) = h_i(x_{m_i+1}, \ldots, x_{m_i+1})$ whenever $x \in F_i$. This function is well defined since $h_i(0) = h_j(0)$ for $i \neq j$ by assumption.

Our goal, expressed using the notation introduced in (2.2), is to produce a smooth extension $h \in C^\infty(U)$ of $\tilde{h}$ to a neighbourhood $U$ of 0 such that $\frac{\partial^2 \log \tilde{h}}{\partial x_j \partial x_k} = A_{jk}$ for each
pair of indices \( j \sim k \), and to show that it is unique; taking \( \Omega = h(x) \, dx_1 \wedge \cdots \wedge dx_m \) will end the proof. The above system of partial differential equations is equivalent to its integral counterpart

\[
h(x) = \left( \frac{h(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m) \, h(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_m)}{h(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_m)} \right) \exp \left( \int_0^{x_j} \int_0^{x_k} A_{jk}(x_1, \ldots, t_j, \ldots, t_k, \ldots, x_m) \, dt_k dt_j \right) \text{ for each } j \sim k.
\]

(3.3)

Using these equations we can express the value of \( h \) at \( x = (x_1, \ldots, x_m) \) in terms of components of \( A \) and values of \( h \) at points with strictly smaller number of nonzero coordinates. While there are several such expressions, one for each pair of indices \( j \sim k \), they are in fact equal to each other. To prove this, denote the respective right-hand sides of (3.3) by \( \rho_{jk} \) and rewrite the double integral of \( A_{jk} \) as

\[
\int_0^{x_j} \int_0^{x_k} \left( \int_0^{x_l} \frac{\partial}{\partial x_l} A_{jk}(x_1, \ldots, t_j, \ldots, t_k, \ldots, x_m) \, dt_l \right) \, dt_k dt_j + \int_0^{x_j} \int_0^{x_k} A_{jk}(x_1, \ldots, t_j, \ldots, t_k, \ldots, 0, \ldots, x_m) \, dt_k dt_j
\]

(3.4)

for some index \( l \sim k \). Then, expand the second summand using (3.3) for indices \( j, k \) at \( (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_m) \) and insert the outcome back into (3.3). By property (3) of the tensor field \( A \) and Fubini’s theorem, the expression for \( h(x) \) obtained in this way is symmetric with respect to indices \( j, l \), hence is equal to both \( \rho_{jk} \) and \( \rho_{lj} \). Since \( \rho_{jk} = \rho_{kj} \), the right-hand side of (3.3) does not depend on the choice of the indices \( j, k \).

This lets us prove by induction that the desired extension of \( \tilde{h} \) exists, is well-defined and unique. One way to set up the induction is to consider subsets of the set of indices \( I \subseteq [m] \) and linear subspaces \( F_I \) of points \( x = (x_1, \ldots, x_m) \) satisfying \( x_k = 0 \) for \( k \notin I \). Denote by \( \pi_i \) the partition of \([m]\) given by (2.1). The base case corresponds to the fact that the function \( h \) is defined on each \( F_I \) for \( I \subseteq \pi_i \) with \( i = 1, 2, \ldots, n \) and is equal to \( \tilde{h} \). Now, equality (3.3) allows us to extend \( h \) from \( F_{I'} \cup F_{I''} \) to \( F_I \) in a smooth and unique way if \( I = I' \cup \{k\} = I'' \cup \{l\} \) for some \( k \sim l \). If we consider any \( I \) not covered by the base case such that \( h \) extends smoothly and uniquely to each \( F_J \) for \( J \subset I \), then \( I \cap \pi_i \neq \emptyset \) and \( I \cap \pi_j \neq \emptyset \) for some \( i \neq j \); hence we can take \( I' = I \setminus \{k\} \) and \( I'' = I \setminus \{l\} \) for some \( k \in I \cap \pi_i \) and \( l \in I \cap \pi_j \) to extend \( h \) to \( F_I \) using (3.3), finishing the induction step and ending the proof.

\[ \square \]

**Corollary 17** A divergence-free \( n \)-web-germ \( W_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega) \) at \( p \in M \) is uniquely determined by its nonuniformity tensor-germ \( K(W_\Omega) \) at \( p \) and values of \( \Omega \) along the union \( \bigcup_{k=1}^n F_k \) of the leaf-germs \( F_k \in \mathcal{F}_k \) passing through \( p \).

**Proof** The nonuniformity tensor \( A := K(W_\Omega) \) satisfies the conditions listed in (ii) of Theorem 16. The volume form \( \Omega \) is a valid extension of the forms \( \Omega \circ \iota_i \) along the
inclusions $t_i : F_i \hookrightarrow \mathbb{R}^d$. Moreover, $K(\mathcal{W}_\Omega) = A$ holds, hence the claim follows from uniqueness in Theorem 16.

While the set of compatible tuples of volume form-germs along the leaves crossing $p$ is quite large, there is a way to put each such initial condition into a trivial normal form with an appropriate choice of a coordinate system.

**Lemma 18** Let $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a germ at $p \in M$ of a divergence-free $n$-web, and let $F_i$ be the leaf-germ of $\mathcal{F}_i$ crossing $p$, where $\mathcal{F}_i$ is given by $T\mathcal{F}_i^c = \bigcap_{j \neq i} T\mathcal{F}_j$, for $i = 1, 2, \ldots, n$.

1. There exists a $W \in q_y \iota$ inclusions Local invariants of divergence-free webs Page 21 of 41

(1) There exists a $\mathcal{W}_\Omega$-adapted coordinate system $(x_1, \ldots, x_m)$ in which $\Omega_{iq} = \Lambda_{iq}$ for all $q \in \bigcup_{i=1}^n F_i$, where $\Lambda = dx_1 \wedge \cdots \wedge dx_m$ is the unit volume form.

2. Let $m_i = \sum_{k=1}^{n-1} \text{codim } \mathcal{F}_k$ for $i = 1, 2, \ldots, n + 1$. Any two such coordinate systems $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ differ by a change of variables of the form

$$ (y_{m_i+1}, \ldots, y_{m_{i+1}}) = y_i \mapsto x_i(y_i) = (x_{m_i+1}(y_i), \ldots, x_{m_{i+1}}(y_i)) \quad (3.5) $$

with constant determinant $\det \frac{\partial x_i}{\partial y_i}(y_i)$ followed by a permutation of coordinates $x_i \mapsto x_{\sigma(i)}$ for some $\sigma \in S^n$ satisfying $\text{codim } \mathcal{F}_{\sigma(k)} = \text{codim } \mathcal{F}_k$, such that

$\text{sgn } \sigma_j \cdot \prod_{j=1}^n \det \frac{\partial x_i}{\partial y_i}(y_i) = 1$, where by $J_0$ we denote the set of indices $j = 1, \ldots, n$ for which codim $\mathcal{F}_j$ is odd.

**Proof** Pick a $\mathcal{W}_\Omega$-adapted coordinate system $(y_1, \ldots, y_m)$ centered at $p$, then express $\Omega$ as $h(y) \, dy_1 \wedge \cdots \wedge dy_m$ for some $h \in C^\infty(M)$ and make the substitution

$$ x_{m_i+1} = \frac{1}{h(0)} \int_0^{y_{m_i+1}} h(0, \ldots, 0, t, y_{m_i+2}, \ldots, y_{m_{i+1}}, 0, \ldots, 0) \, dt, \quad (3.6) $$

for each $i = 1, \ldots, n$, while keeping the other coordinates unaltered. If we name this transformation $\varphi$, then via direct calculation we obtain for each $i = 1, \ldots, n$ and $q \in F_i$ the equality $\varphi^*(h(0) \, dx_1 \wedge \cdots \wedge dx_m)|_q = \Omega_{iq}$. Further linear rescaling of coordinates yields the desired coordinate system.

Any transition map $y = \varphi(x)$ between two such systems $(y_1, \ldots, y_m)$ and $(x_1, \ldots, x_m)$ is an equivalence of the underlying $n$-web, and as such it takes the form $y_i = \varphi_i(x_{\sigma(i)})$ for $i = 1, \ldots, n$ and some permutation $\sigma \in S^n$ satisfying $\text{codim } \mathcal{F}_{\sigma(i)} = \text{codim } \mathcal{F}_i$ for each $i = 1, \ldots, n$. At each point $q \in F_i$ we have

$$ (\Omega_{0,i})_{iq} = dx_1 \wedge \cdots \wedge dx_m = dy_1 \wedge \cdots \wedge dy_m $$

$$ = \pm \left( \prod_{j \neq i} \det \frac{\partial \varphi_{\sigma^{-1}(j)}}{\partial x_j}(0) \right) \cdot \det \frac{\partial \varphi_{\sigma^{-1}(i)}}{\partial x_i}(q_{m_i+1}, \ldots, q_{m_{i+1}}) \, dx_1 \wedge \cdots \wedge dx_m. \quad (3.7) $$
This equality forces \( \det \frac{\partial \phi_i}{\partial x_{(i)}} \) to be constant for each \( i = 1, \ldots, n \). Evaluating it at 0 we obtain that

\[
\pm \prod_{i=1}^{n} \det \frac{\partial \phi_{\sigma^{-1}(j)}}{\partial x_j} = 1, \tag{3.8}
\]

where the correct sign is found by noting that if we write \( \sigma \) as a product of transpositions \( x_i \leftrightarrow x_j \) satisfying \( \text{codim } F_i = \text{codim } F_j = c \) for some \( c \in \mathbb{N} \), then each \( \tau \) corresponds to \( c \) transpositions of individual variables \( x_k \), contributing a factor \((-1)^c\) to \( \det d\phi \).

The above normalization of the coordinate system is a generalization of a theorem of Tabachnikov [17] on normal forms of the volume elements associated with divergence-free 2-webs. Pick an arbitrary divergence-free \( n \)-web \( W_\Omega \) and a coordinate system \( (x_1, \ldots, x_m) \) centered at \( p \in M \) normalized by means of Lemma 18 with respect to \( \Omega \). Let \( m = (x_1, x_2, \ldots, x_m) \) be the maximal ideal of the ring of smooth function-germs at the point \( p \). In these coordinates, the density \( h \in C^\infty(M, p) \) of the volume form-germ \( \Omega = h(x) \, dx_1 \wedge \cdots \wedge dx_m \) can be expanded into

\[
h(x) = 1 + \sum_{i<j, m_i<k \leq m_{i+1}, m_j<l \leq m_{j+1}} \kappa_{kl} x_k x_l + f(x), \tag{3.9}
\]

where \( m_i = \sum_{k=1}^{i-1} \text{codim } F_k \) and \( \kappa_{kl} \) are exactly the coefficients at \( p \) of the nonuniformity tensor \( K(W_\Omega)_p = \sum_{i,j} k_{ij} \, dx_i \, dx_j \), while the function-germ \( f \in \mathbb{R}^3 \) vanishes on the union of leaves passing through the origin \( F_i \in F^c \) of the complementary foliations \( F_i^c \) generated by the tangent distribution \( T \mathcal{F}_i = \bigcap_{j \neq i} T \mathcal{F}_j \). When the web consists of two foliations, the displayed equality reduces exactly to [17, Theorem 0.2, (ii)] when taken modulo \( m^3 \).

For a fixed \( n \)-web \( \mathcal{W} \), denote by \( \mathcal{G}_\mathcal{W} \) the group of all coordinate transformation-germs \( \varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0) ; (y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_m) \) preserving the web \( \mathcal{W} \) and satisfying \((\varphi^* \Lambda)_q = \Lambda_q \) for the unit volume form \( \Lambda = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m \) and each \( q \in \bigcup_{i=1}^{n} F_i \) with \( 0 \in F_i \in F_i^c \), as in Lemma 18. The reduction of the group of all \( \mathcal{W} \)-adapted coordinate transformations to \( \mathcal{G}_\mathcal{W} \), made possible by Lemma 18, allows us to reformulate the classification problem of divergence-free \( n \)-webs as follows.

**Theorem 19** Divergence-free \( n \)-web-germs with an underlying \( m \)-dimensional \( n \)-web \( \mathcal{W} \) are classified up to equivalence by orbits of the action of \( \mathcal{G}_\mathcal{W} \) by pullback on germs at 0 of covariant symmetric 2-tensor fields \( A \) on \( \mathbb{R}^m \) satisfying the identities of nonuniformity tensors listed in Theorem 16 via the correspondence \( \mathcal{W}_\Omega \mapsto \mathcal{G}_\mathcal{W}(\psi^{-1})^* K(\mathcal{W}_\Omega), \) where \( \psi(x) = (x_1, \ldots, x_m) \) is any \( \mathcal{W}_\Omega \)-adapted coordinate system normalized by means of Lemma 18.

**Proof** By the uniqueness claim of Theorem 16, an equivalence between two divergence-free web-germs \( W_\Omega \) and \( \mathcal{W}_\Omega \) is the same as a change of variables \( \varphi \in \mathcal{G}_\mathcal{W} \) between the corresponding normalized coordinate systems \( \tilde{\psi}, \psi \) which satisfies
\( \varphi^*(\psi^{-1})^*\mathcal{K}(\mathcal{W}_\Omega) = (\tilde{\psi}^{-1})^*\mathcal{K}(\mathcal{W}_\tilde{\Omega}) \). Since, again by Theorem 16, any tensor field-germ \( A \) of the form given above is a nonuniformity tensor of a divergence-free \( n \)-web-germ \( \mathcal{W}_\Omega \) in some normalized \( \mathcal{W}_\Omega \)-adapted coordinates, the two classification problems are equivalent. \( \square \)

Note that, in contrast to webs of higher codimension, the group \( \mathcal{G}_{\mathcal{W}} \) is finite-dimensional for codimension-1 webs \( \mathcal{W} \). In this case, each map \( \varphi \in \mathcal{G}_{\mathcal{W}} \) has to be linear, since the requirement \( \det \frac{\partial \varphi_i}{\partial x_{\sigma(i)}} = \text{const.} \) for each \( i = 1, \ldots, m \) enforced during a change of variables \( \varphi(x_1, \ldots, x_m) = (\phi_1(x_{\sigma(1)}), \ldots, \phi_m(x_{\sigma(m)})) \), \( \sigma \in S_m \), between any two coordinate systems normalized in the sense of Lemma 18 reduces to the constancy of \( d \varphi \). This opens up a possibility to classify some generic codimension-1 divergence-free \( n \)-webs using elementary tools.

**Example 5** Our goal is to determine a canonical form of a generic divergence-free 2-web’s volume element \( \Omega \). Consider a germ at 0 of a planar divergence-free 2-web \((\mathbb{R}^2, \mathcal{F}, \mathcal{G}, \Omega)\) with \( \Omega = h(x, y) \, dx \wedge dy \), \( T\mathcal{F} = \ker dx \) and \( T\mathcal{G} = \ker dy \). By Lemma 18, we can choose the coordinate system \((x, y)\) so that \( h(x, 0) = h(0, y) = 1 \). Moreover, the Lemma states that any other such coordinate system \((\tilde{x}, \tilde{y})\) differs from \((x, y)\) by a linear change of variables of the form

\[
(\tilde{x}, \tilde{y}) = (cx, c^{-1}y) \quad \text{or} \quad (\tilde{x}, \tilde{y}) = (cy, -c^{-1}x) \quad \text{for} \; c \in \mathbb{R} \setminus \{0\}.
\]

Since the nonuniformity tensor \( \mathcal{K}(\mathcal{W}_\Omega) = \kappa(x, y) \, dx \wedge dy = \frac{\partial \log h}{\partial x} dx \wedge dy \) is itself a divergence-free web-invariant, its covariant derivative with respect to the natural connection \( \Theta \) with covariant derivative \( \nabla \) is also an invariant. It is equal to \( \nabla \mathcal{K}(\mathcal{W}_\Omega) = (\kappa - \kappa \, d \log h) \otimes (dx \otimes dy) \); hence its value at the origin is \( \kappa \otimes (dx \otimes dy) \), since \( h \) is constant along the leaves \( \{x = 0\} \) and \( \{y = 0\} \).

Suppose now that the genericity condition \( \frac{\partial \kappa}{\partial x}(0) \neq 0, \frac{\partial \kappa}{\partial y}(0) \neq 0 \) is satisfied. Using \( \pi/2 \)-rotations and transformations of the form \((x, y) \mapsto (cx, \frac{1}{c} y)\) for \( c > 0 \) we put \( \nabla \mathcal{K}(\mathcal{W}_\Omega) \) into a form which satisfies \( \frac{\partial \kappa}{\partial x}(0) = \frac{\partial \kappa}{\partial y}(0) > 0 \). Now, the canonical form of the volume density function \( h \) can be computed using Hadamard’s lemma. It is equal to

\[
h(x, y) = 1 + xy(\kappa_0 + \frac{1}{2} a \cdot (x + y) + x^2 g_1(x) + xy g_{xy}(x, y) + y^2 g_2(y))
\]

for some function-germs \( g_1, g_2 \in C^\infty(\mathbb{R}, 0), g_{xy} \in C^\infty(\mathbb{R}^2, 0) \) and real constants \( \kappa_0 \in \mathbb{R}, a \in \mathbb{R}_+ \). The two scalar invariants correspond to

\[
\kappa_0 = \frac{\kappa(0)}{h(0)}, \quad a = \left| \frac{(h(0) \frac{\partial \kappa}{\partial x}(0) - \frac{\partial h}{\partial x}(0)\kappa(0)) \cdot (h(0) \frac{\partial \kappa}{\partial y}(0) - \frac{\partial h}{\partial y}(0)\kappa(0))}{h(0)^5} \right|^{1/2}
\]

in any \( \mathcal{W}_\Omega \)-adapted coordinate system \((x, y)\). Two divergence-free 2-web-germs satisfying the genericity condition are locally equivalent if and only if their canonical forms coincide.
4 Volume-preserving holonomy and geometric invariants

Let $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a codimension-1 divergence-free $n$-web on a smooth manifold $M$. Here, we give a certain geometrical interpretation of the nonuniformity tensor $K(\mathcal{W}_\Omega)$ inspired by the works of Blaschke, Thomsen and Bol on planar 3webs [6, 16] and based upon the interpretation of curvature of bi-Lagrangian structures given by Tabachnikov in dimension 2 [17]. In the latter work, the curvature of the bi-Lagrangian connection $\nabla$ at the point $p$ was approximated by certain ratios between adjacent volumes enclosed by the leaves of the web crossing $p$ (Fig. 1). Our approach goes along similar lines.

Let $M$ be a smooth manifold and let $\mathcal{W}$ be a codimension-1 $n$-web on $M$. For each $x, y \in \mathbb{R}$ let us treat $[x, y]$ as an ordinary closed interval $[\min(x, y), \max(x, y)]$ oriented positively if $x \leq y$ and negatively if $x > y$. Now, given any pair of vectors $a, b \in \mathbb{R}^m$, we let $[a, b] = \prod_{k=1}^m [a_k, b_k]$ in the sense given above with the product orientation.

**Definition 20** Let $p, q \in M$. A compact set $C \subseteq M$ which takes the form $[\varphi(p), \varphi(q)]$ for some $\mathcal{W}$-adapted chart $(U, \varphi)$ will be called a region bounded by the leaves of $\mathcal{W}$ crossing $p$ and $q$. When the chart $(U, \varphi)$ is implied, we will omit $\varphi$ and denote $C$ by $[p, q]$ for convenience.

Every region bounded by leaves $C$ has finite volume with respect to the volume form $\Omega$ (henceforth abbreviated by $\text{Vol}^\Omega(C)$). This $\Omega$-volume, defined as

$$\text{Vol}^\Omega(C) = \left| \int_C \Omega \right|,$$

is an essential ingredient in the construction of the reflection holonomy group of a divergence-free web. The reflection holonomy group shares many similarities with the holonomy group of a planar 3-web, which measures the degree of its non-hexagonality at a given point $p \in \mathbb{R}^2$ [6, 16] or, in other words, the extent to which planar, curvilinear figures formed by the leaves of $\mathcal{W}$ with vertices lying on the leaves crossing $p$ are not closed (see Fig. 3). The 3-web is hexagonal precisely when a certain natural connection associated with it is flat.

The property of non-hexagonality can be expressed more formally as nontriviality of the group generated by the smooth map-germ $\ell : (F, p) \to (F, p)$ taking one end of each such figure’s perimeter to the other. This transformation is itself a 6-fold composition of map-germs $s_{ij} : (F_i, p) \to (F_j, p)$, each transporting the points on $F_i \in \mathcal{F}_i$ to the points on $F_j \in \mathcal{F}_j$ along the leaves of $\mathcal{F}_k$ for $\mathcal{W} = (\mathbb{R}^2, \mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k)$ and some leaves $F_i, F_j$ crossing the center $p$.

We build the volume-preserving holonomy groups in a similar way, by first introducing a way to transport the points along the leaves of a codimension-1 divergence-free web $\mathcal{W}_\Omega$ with the help of the volume form $\Omega$. Afterwards, we measure the nonuniformity tensor $K(\mathcal{W}_\Omega)$ by means of compositions of such transport mappings.
4.1 Reflections, loops and reflection holonomy

Let us first clarify the construction of the point-transport map utilizing the volume form. The next lemma requires an auxiliary definition.

Definition 21 Let $\mathcal{W}$ be a codimension-1 $n$-web. Two regions $A, B$ bounded by leaves of $\mathcal{W}$ are said to be adjacent if $C = A \cup B$ is also a region bounded by leaves of $\mathcal{W}$ and $A \cap B$ is entirely contained in a leaf $F$ of some $\mathcal{F} \in \text{Fol}(\mathcal{W})$ (in which case we say that $A$ and $B$ are adjacent along $F$, or that $F$ subdivides the region $C$ into subregions $A$ and $B$).

Given a foliation $\mathcal{F}$ and an open subset $U$ of its domain, we denote by $\mathcal{F}|_U$ the foliation of $U$ whose leaves are exactly the connected components of $F \cap U$ for $F$ the leaves of $\mathcal{F}$.

Lemma 22 Let $\mathcal{W}_\Omega$ be a divergence-free codimension-1 $n$-web on $M$ and let $\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)$. There exists an open neighbourhood $U$ of $p$ such that to each point $q \in U$ not lying on the leaf $G$ of $\mathcal{G}|_U$ crossing $p$ for each $\mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ there corresponds a unique point $q' \in U$ different from $q$ such that the regions $A$ and $B$ bounded by leaves of $\mathcal{W}_\Omega$ crossing $p, q$ and $p, q'$ respectively are adjacent along $\mathcal{F}$ and have the same $\Omega$-volume. The correspondence $q \mapsto q'$ extends uniquely to a smooth map-germ $r_{p,\mathcal{F}} : (M, p) \to (M, p)$.

Proof Fix a $\mathcal{W}_\Omega$-adapted coordinate system $(x_1, \ldots, x_n)$ centered at $p$ such that $T \mathcal{F} = \ker dx_i$ and write $\Omega = h(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ for some $h \in C^\infty(M)$. Whenever two given points $q = (u_1, \ldots, u_n)$ and $q' = (u'_1, \ldots, u'_n)$ lie on the same leaf $F = \{\forall j \neq i \, x_j = u_j\}$ of $\mathcal{F}^c$, the difference between the integrals

$$g_i(q, u'_i) = g_i(u_1, u_2, \ldots, u_n, u'_i) = \int_0^{u_i} f_i(q, s) \, ds - \int_0^{u'_i} f_i(q, s) \, ds \quad (4.2)$$
of the function \( f_i \) defined by

\[
    f_i(q, s) = f_i(u_1, u_2, \ldots, \hat{u}_i, \ldots, u_n, s)
    = \int_0^1 \cdots \int_0^1 h(t_1u_1, \ldots, t_{i-1}u_{i-1}, s, t_{i+1}u_{i+1}, \ldots, t_nu_n) \, dt_1 \cdots \hat{dt}_i \cdots dt_n
\]  

(4.3)
is zero precisely when \( \int_{[p,q]} \Omega = -\int_{[p,q']} \Omega \), which, under the assumption \( u_j \neq 0 \) for \( j = 1, \ldots, n \), is true if and only if \( \text{Vol}^2(A) = \text{Vol}^2(B) \) and \( q \neq q' \). The uniqueness of the point \( q' \) for fixed \( q \) is guaranteed by the strict monotonicity of \( g \) with respect to \( u_i' \). Its local existence is deduced using the classical implicit function theorem applied to the function \( g \) to obtain a smooth function \( \rho_i : \tilde{U} \to \mathbb{R} \) satisfying

\[
    g_i(q, \rho_i(q)) = g_i(0) = \int_0^0 f_i(0, s) \, ds - \int_0^0 f_i(0, s) \, ds = 0,
\]  

(4.4)

where \( \tilde{U} \) is an open neighbourhood of \( p \) equal to \((-\varepsilon, \varepsilon)^n\) in coordinates for some \( \varepsilon > 0 \). Put

\[
    r_{p,\mathcal{F}}(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, \rho_i(u_1, \ldots, u_n), u_{i+1}, \ldots, u_n)
\]  

(4.5)
to obtain a smooth map \( r_{p,\mathcal{F}} : \tilde{U} \to M \) extending the correspondence \( q \mapsto q' \). Any other smooth extension of this correspondence coincides with \( r_{p,\mathcal{F}} \) on an open and dense subset of a sufficiently small open neighbourhood of \( p \) by uniqueness of \( q' \), hence is equal to \( r_{p,\mathcal{F}} \) as a map-germ by continuity. Thus, the smooth extension of \( q \mapsto q' \) to a neighbourhood of \( p \) is unique. Since \( r_{p,\mathcal{F}}(p) = p \), we can put \( U = \tilde{U} \cap r_{p,\mathcal{F}}^{-1}(\tilde{U}) \) to finish the proof.

\[\Box\]

**Definition 23** Let \( \mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega) \). The map-germ \( r_{p,\mathcal{F}} : (M, p) \to (M, p) \) defined in the above proposition is called the *volume-preserving reflection through \( \mathcal{F} \) anchored at \( p \).*

These reflections are defined in an invariant way, hence they behave well with respect to local equivalences of divergence-free webs.

**Proposition 24** If \( \varphi : (M, p) \to (N, s) \) is a germ of a local equivalence between codimension-1 webs \( \mathcal{W}_M \) and \( \mathcal{W}_N \) mapping the leaves of foliation \( \mathcal{F} \) onto the leaves of foliation \( \tau(\mathcal{F}) \) for some map \( \tau : \text{Fol}(\mathcal{W}_M) \to \text{Fol}(\mathcal{W}_N) \), then the corresponding reflections are conjugate:

\[
    r_{s,\tau(\mathcal{F})} = \varphi \circ r_{p,\mathcal{F}} \circ \varphi^{-1}
\]  

(4.6)
Proof Choose representatives of \( r_p^\Omega; F, r_s, \tau(F) \) with their respective domains \( U, V \) as in the statement of Lemma 22. Moreover, assume without loss of generality that \( \varphi = \varphi|_U \) is a divergence-free web-equivalence and \( \varphi(U) = V \). Now, if two different points \( q, q' \) span adjacent regions \( A, B \) bounded by the leaves of \( W_M \) of equal volume crossing \( p, q \) and \( p, q' \) respectively, their images \( \varphi(q), \varphi(q') \) also do so, since \( \varphi \) is a volume-preserving web-equivalence. This immediately leads to the implication

\[
q' = r_p^\Omega; F(q) \implies \varphi(q') = r_s, \tau(F)(\varphi(q))
\]

for \( q \) in a dense subset of \( U \) by uniqueness, which by continuity gives the sought relation (4.6).

In a similar vein one proves the following properties of volume-preserving reflection mappings.

Lemma 25 Let \( W_\Omega \) be a divergence-free \( n \)-web of codimension 1 on \( M \) with volume form \( \Omega \). Fix \( p \in M \) and \( F \in \text{Fol}(W_\Omega) \) and denote by \( r_p^\Omega; F \) the volume-preserving reflection of \( W_\Omega \) along \( F \). Then

1. \( r_p^\Omega; F \circ r_p^\Omega; F = \text{id} \),
2. if \( c \in \mathbb{R} \setminus \{0\} \) and \( r_p^\Omega; F \) is the volume-preserving reflection map-germ corresponding to the divergence-free \( n \)-web \( (M, \text{Fol}(W_\Omega), c\Omega) \), there is an equality \( r_p^\Omega; F = r_p^\Omega; F \).
3. in any \( W_\Omega \)-adapted coordinates chart \( (U, (x_1, \ldots, x_n)) \) around \( p \) in which \( T_F = \ker dx_i \), the condition \( x_i(q) > 0 \) implies \( x_i(r_p^\Omega; F(q)) < 0 \) for any \( q \in U \).

Proof Properties (1) and (2) both follow directly from Lemma 22 and its uniqueness statement. For the last property (3) use Lemma 22 together with the fact that the function \( x \mapsto \int_{[p, x]} \Omega \) is strictly monotone for generic \( x \in U \) with respect to \( x_i \).

The invariance statement of Proposition 24 carries over naturally to compositions of map-germs \( r_p^\Omega; F \) for \( F \in \text{Fol}(W_\Omega) \). The properties of \( r_p^\Omega; F \) listed above suggest that the shortest among possibly nontrivial such compositions which are actually comparable to the identity (hence can be used to gauge nontriviality of the web \( W_\Omega \)) are the commutators of two reflections.

Definition 26 Let \( p \in M \) and \( F, G \in \text{Fol}(W_\Omega) \). The germ of the composition

\[
\ell_p^\Omega; F, G = r_p^\Omega; G \circ r_p^\Omega; F \circ r_p^\Omega; G \circ r_p^\Omega; F
\]

will be called a (volume-preserving) loop through \( F \) and \( G \) anchored at \( p \).

Definition 27 Let \( p \in M \). The commutator \([R_p, R_p]\) of the group \( R_p \) generated by volume-preserving reflections \( r_p^\Omega; F \) anchored at \( p \) for \( F \in \text{Fol}(W_\Omega) \) will be called the (volume-preserving) reflection holonomy group of \( (M, F_1, \ldots, F_n, \Omega) \) at \( p \).

The reflection holonomy groups are generated by conjugates of the loops \( \ell_p^\Omega; F, G \) by compositions of volume-preserving reflections. Their triviality can be rephrased as commutativity of all pairs of reflections, i.e. all loops being equal to identities in the neighbourhood of a given point.
Example 6 Let $\mathcal{W}_{0; \Lambda} = (\mathbb{R}^n, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Lambda)$ be a trivial divergence-free $n$-web. In the standard coordinate system $(x_1, x_2, \ldots, x_n)$, in which $T \mathcal{F}_i = \ker dx_i$ and $\Lambda = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, the reflection maps anchored at $p \in \mathbb{R}^n$ become ordinary linear reflections

$$r_p; \mathcal{F}(q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_n) = (q_1, \ldots, q_{i-1}, -q_i + 2p_i, q_{i+1}, \ldots, q_n). \quad (4.9)$$

The reflection holonomy group $[\mathcal{R}_p, \mathcal{R}_p]$ of $\mathcal{W}_{0; \Lambda}$ is trivial.

The reflection holonomy groups $[\mathcal{R}_p, \mathcal{R}_p]$ can be treated as covariants of the web $\mathcal{W}_\Omega$ via the following construction. Every germ of a divergence-free web-equivalence $\varphi$ acts on the reflections at a point $p$ by conjugation: if $\varphi$ maps the leaves of $\mathcal{F} \in \text{Fol}(\mathcal{W}_\Omega)$ onto the leaves of $\tau(\mathcal{F})$, then $\varphi$ transforms $r_p; \mathcal{F}$ into $r_{\varphi(p)}; \mathcal{F} \circ \varphi^{-1}$. Hence, $\varphi$ defines a group isomorphism $\varphi_* : \mathcal{R}_p \rightarrow \mathcal{R}_{\varphi(p)}$, which restricts to an isomorphism of the corresponding commutators. If $p = \varphi(p)$, then $\varphi_*$ is simply the conjugation by $\varphi$ inside $\text{Diff}(M, p)$.

Definition 28 Let $(U, \varphi)$ be a $\mathcal{W}_\Omega$-adapted chart on some open neighbourhood $U$ centered at $p \in M$. Then the conjugacy class of $\varphi_*[\mathcal{R}_p, \mathcal{R}_p]$ inside the group $\text{Diff}(\mathbb{R}^n, 0)$ of diffeomorphism-germs will be called the (volume-preserving) reflection holonomy of $\mathcal{W}_\Omega$ at $p$.

The reflection holonomy is a local invariant of the web by construction. Its triviality is dependent upon the nonuniformity tensor of the web, as we will soon show. Below we give an elementary example of a divergence-free web with nontrivial holonomy.

Example 7 Let $(x, y)$ be the standard coordinates on $\mathbb{R}^2$ and let $\mathcal{W}_\Omega = (I^2, \mathcal{F}, \mathcal{G}, \Omega)$ be a divergence-free 2-web on $I = (-1, 1) \subseteq \mathbb{R}$ with $T \mathcal{F} = \ker dx, T \mathcal{G} = \ker dy$ and $\Omega = (1 + xy) \, dx \wedge dy$. Its nonuniformity tensor is $K(\mathcal{W}_\Omega) = (1 + xy)^{-2} \, dx \, dy \neq 0$. Elementary computations allow us to recover the closed form of the volume-preserving reflections anchored at 0:

$$r_{0, \mathcal{F}}(x, y) = (z(x, y), y),$$

$$r_{0, \mathcal{G}}(x, y) = (x, z(y, x)), \quad (4.10)$$

where the smooth function

$$z(x, y) = \begin{cases} \frac{\sqrt{4(1-xy)-x^2y^2}-2}{y}, & \text{if } y \neq 0, \\ -x, & \text{if } y = 0 \end{cases} \quad (4.11)$$

is defined on some open neighbourhood of 0. Using these we compute the closed form of the loop $\ell_{0; \mathcal{F}, \mathcal{G}}$, which turns out to be

$$\ell_{0; \mathcal{F}, \mathcal{G}}(x, y) = \left( \frac{\sqrt{4(1-xy)-x^2y^2}-2}{xy^2}, \frac{x^2y^3}{(\sqrt{4(1-xy)-x^2y^2}-2)^2} \right). \quad (4.12)$$

Since it generates the holonomy group of $\mathcal{W}_\Omega$ at 0, $[\mathcal{R}_0, \mathcal{R}_0]$ is nontrivial.
Fig. 4 The effect of the nonuniformity tensor $K(W_\Omega) = \sum_{i \neq j} \kappa_{ij} dx_i dx_j$ on reflections

**Lemma 29** Let $W_\Omega = (M, F_1, \ldots, F_n, \Omega)$ be a divergence-free n-web of codimension 1. Fix a point $p \in M$ and a $W_\Omega$-adapted coordinate system $(x_1, \ldots, x_n)$ centered at $p$. The volume-preserving reflection along the foliation $F_i \in \text{Fol}(W_\Omega)$ generated locally by $\ker dx_i$ satisfies

$$r_{p;F_i}(x) = (x_1, \ldots, x_{i-1}, z(x), x_{i+1}, \ldots, x_n)$$

(4.13)

where

$$z(x) = -x_i - \alpha_i x_i^2 - \alpha_i^2 x_i^3 - \sum_{j \neq i} \alpha_{ij} x_i^2 x_j + o(|x|^3).$$

(4.14)

with $\alpha_i = (\frac{\partial}{\partial x_i} \log h)(0)$ and $\alpha_{ij} = (\frac{\partial^2}{\partial x_i \partial x_j} \log h)(0)$.

**Lemma 30** Let $W_\Omega = (M, F_1, \ldots, F_n, \Omega)$ be a divergence-free n-web of codimension 1. Fix a point $p \in M$ and a $W_\Omega$-adapted coordinate system $(x_1, \ldots, x_n)$ centered at $p$. Express the volume form as $\Omega = h(x) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ and the nonuniformity tensor of $W_\Omega$ at $p$ as $K(W_\Omega)|_p = \sum_{i \neq j} \kappa_{ij} dx_i dx_j$, where $\kappa_{ij} = \frac{\partial \log h}{\partial x_i \partial x_j}(p)$. In this setting, the volume-preserving loop along the foliations $F_i, F_j$ with $T F_i = \ker dx_i$ and $T F_j = \ker dx_j$ satisfies

$$\ell_{p;F_i,F_j}(x) = (x_1, \ldots, x_{i-1}, u_i(x), x_{i+1}, \ldots, x_{j-1}, u_j(x), x_{j+1}, \ldots, x_n)$$

(4.15)

where the $i^{th}$ and $j^{th}$ coordinates of the image satisfy

$$u_i(x) = x_i + 2\kappa_{ij} x_i^2 x_j + o(|x|^3) \quad \text{and} \quad u_j(x) = x_j - 2\kappa_{ij} x_i x_j^2 + o(|x|^3).$$

(4.16)

respectively (cf. Fig. 4).

The proof of the two propositions above can be reduced to elementary (but quite tedious) calculations involving the implicit function $g_i$ defined by Eq. (4.4).
4.2 Geometric triviality conditions

For planar divergence-free 2-webs, the curvature of the natural connection was given a vivid and intuitive geometric interpretation in [17, Fig. 2]. We are now ready to give its extension to the case of an arbitrary codimension-1 divergence-free n-web, which will be used in the next section to characterize the curvature geometrically in full generality. Before proceeding, recall the definition of adjacency of regions bounded by leaves; it can be found in Definition 21.

Theorem 31 Let \( \mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega) \) be a codimension-1 divergence-free n-web. The following conditions are equivalent.

(1) The web \( \mathcal{W}_\Omega \) is locally trivial.

(2) For each pair \( \mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega) \) of two different foliations of \( M \), any region bounded by leaves \( K \), and any two open subsets of leaves \( F \in \mathcal{F}, G \in \mathcal{G} \) which subdivide \( K \) into four subregions \( A, B, C, D \) with \( (A \cup B) \cap (C \cup D) \subseteq F \) and \( (A \cup D) \cap (B \cup C) \subseteq G \), the respective \( \Omega \)-volumes \( a, b, c, d \) of \( A, B, C, D \) satisfy

\[
ac = bd. \quad (4.17)
\]

(3) For each pair \( \mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega) \) of two different foliations of \( M \), any region bounded by leaves \( K \), and any two open subsets of leaves \( F \in \mathcal{F}, G \in \mathcal{G} \) which subdivide \( K \) into four subregions \( A, B, C, D \) with \( (A \cup B) \cap (C \cup D) \subseteq F \) and \( (A \cup D) \cap (B \cup C) \subseteq G \) in such a way that the \( \Omega \)-volumes \( a, b, c, d \) of \( A, B, C, D \) satisfy \( a + b = c + d \), the equality \( a = b \) implies \( b = c = d \).

(4) For any region bounded by leaves \( K \) and each \( k = 1, 2, \ldots, n \) there exist open subsets of leaves \( F_i \in \mathcal{F}_i \) for \( i = 1, 2, \ldots, k \) which subdivide \( K \) into \( 2^k \) subregions with equal \( \Omega \)-volumes.

(5) The volume-preserving reflection holonomy of \( \mathcal{W}_\Omega \) at each point \( p \in M \) is trivial.

(6) The nonuniformity tensor \( K(\mathcal{W}_\Omega) \) vanishes identically.

Proof To prove (2) from (1), carry out direct computations of volumes of \( A, B, C, D \) inside a \( \mathcal{W}_\Omega \)-adapted coordinate system \( (x_1, \ldots, x_n) \) in which the volume form \( \Omega \) becomes \( dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \). Now, condition (3) follows from (2) by elementary algebra.

In order to deduce (4) from (3) we proceed by induction on \( k \). The base case \( k = 1 \) is covered by the intermediate value theorem; if \( K = [a, b] \) for \( a, b \in \mathbb{R}^n \) in some fixed \( \mathcal{W}_\Omega \)-adapted coordinate system \( (x_1, \ldots, x_n) \), then we apply the theorem to the continuous function \( t \mapsto \int_{[a,b]}^t \Omega \) with \( b_t = (b_1, \ldots, b_{t-1}, t, b_{t+1}, \ldots, b_n) \) to find a subregion with half the \( \Omega \)-volume of \( K \), where we treat \( [a, b_t] \) as a nonempty oriented chain according to Definition 20.

Assume the region \( K \) is subdivided into \( 2^k \) subregions \( K_j \) of equal \( \Omega \)-volumes by the open subsets of leaves \( F_i \in \mathcal{F}_i \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, 2^k \). Pick one of the subregions \( K_{j_0} \) and a leaf \( F_{k+1} \) of \( \mathcal{F}_{k+1} \) subdividing each region \( K_j \) into two subregions \( A_j, B_j \) with \( \Omega \)-volumes \( a_j, b_j \) in such a way that \( a_{j_0} = b_{j_0} \). Since for each region \( K_j \) the equality \( a_j + b_j = a_{j_0} + b_{j_0} \) holds, we obtain \( a_j = b_j = a_{j_0} = b_{j_0} \). for every subregion \( K_j \) adjacent to \( K_{j_0} \) as a consequence of (3). Apply the above
Fig. 5 If \( p \neq q \), then there is a strict containment relation between some subregions

reasoning inductively to arrive at \( a_i = b_j \) for \( i, j = 1, \ldots, 2^k \), thereby finishing the induction step and concluding the proof of (4).

We will now prove the converse implication from (4) to (3). Fix a \( \mathcal{W}_\Omega \)-adapted coordinate system \((x_1, \ldots, x_n)\) on some open subset \( U \) containing \( K \) and assume that locally \( F, G \) are generated by \( \ker dx_i, \ker dx_j \) respectively, so that \( F \subseteq \{ x_i = p_i \} \) and \( G \subseteq \{ x_j = p_j \} \). By our assumption, there exist two other open subsets of leaves \( F' \subseteq \{ x_i = q_i \} \in \mathcal{F}, G' \subseteq \{ x_j = q_j \} \in \mathcal{G} \) subdividing \( K \) into subregions \( E_1, E_2, E_3, E_4 \) with \( \Omega \)-volumes all equal to \( e \in \mathbb{R}_+ \). Note that \( e = \frac{1}{4}(a + b + c + d) \). Since \( a + b = c + d = 2e \), the only possibility is that \( p_i = q_i \). Since \( a = b = e \), we similarly get \( p_j = q_j \). Hence \( F, F' \subseteq \{ x_i = p_i \} \) and \( G, G' \subseteq \{ x_j = p_j \} \), which imply \( A = E_1, B = E_2, C = E_3 \) and \( D = E_4 \) after a permutation of indices (Fig. 5). The subregions \( A, B, C, D \) have equal \( \Omega \)-volumes as intended.

To show that the ability to cut regions as in (3) trivializes the holonomy (5), pick a point \( p \in M \), a \( \mathcal{W}_\Omega \)-adapted coordinate system centered at \( p \) on an open set \( W \), an open neighbourhood \( V \subseteq W \) of \( p \) of the coordinate form \((-\varepsilon, \varepsilon)^n\) in which every volume-preserving reflection is well-defined and unique in the sense of Lemma 22, and a neighbourhood \( U \subseteq V \) of \( p \) such that every sufficiently long composition of reflection map-germs \( r_{p, F} \) for \( F \in \text{Fol}(\mathcal{W}_\Omega) \) maps \( U \) into \( V \). Our goal is to show that for fixed \( F, G \in \text{Fol}(\mathcal{W}_\Omega), F \neq G \) the equality \( \ell_{p; F, G}(q) = q \) holds for each \( q \in U \). It is sufficient to prove this for points \( q \) not lying on any plaque of \( \mathcal{W}_\Omega \) passing through \( p \); the result extends to the whole \( U \) by the continuity of \( \ell_{p; F, G} \).

Let \( q' = r_{p; F}(q) \). The regions bounded by leaves \( A = [p, q] \) and \( B = [p, q'] \) are well-defined and contained in \( V \) (since \( V \) is a coordinate cube), adjacent along the plaque \( F_p \in \mathcal{F} \) crossing \( p \) and having the same \( \Omega \)-volumes \( \varepsilon > 0 \). Denote by \( G_p \) the plaque of \( G \) crossing \( p \). We will now find a plaque \( \tilde{G} \in \mathcal{G} \) such that the region \( R \)
adjacent to $A \cup B$ along $G_p$ bounded in the $x_2$-direction by $\tilde{G}$ has $\Omega$-volume at least $2\varepsilon$.

One of the plaques $G_u, G_v \in \mathcal{G}$ crossing the points $u = r_p;\mathcal{G}(q)$ and $v = r_p;\mathcal{G}(q')$ must satisfy this criterion. To see this, denote the corresponding regions by $R_u, R_v$ and suppose that either of their $\Omega$-volumes $r_u, r_v$ is less than $2\varepsilon$ (we can assume that $r_u \leq r_v$ without loss of generality). The plaque $F_p$ subdivides $R_u$ into subregions $S_u = [p, u], T_u$ and $R_v$ into $S_v, T_v = [p, v]$. Note that $\Omega$-volumes $s_u, t_v$ of both $S_u$ and $T_v$ are $\varepsilon$ by Definition 23. Therefore, the inequality $r_u \leq 2\varepsilon$ implies that the $\Omega$-volume $t_u$ of $T_u$ is not greater than $\varepsilon$. Since the geometry of this setup forces either $T_u \subseteq T_v$ or $T_v \subseteq T_u$, by $t_u \leq t_v = \varepsilon$ we obtain that $T_u \subseteq T_v$, hence also $S_u \subseteq S_v$. This leads through $\varepsilon = s_u \leq s_v$ to $r_v = s_v + t_v = s_v + \varepsilon \geq 2\varepsilon$, proving that in this case we can put $\tilde{G} = G_v$, so that $R = R_v$.

Now, by picking a suitable plaque $G \in \mathcal{G}$ we can shrink $R$ to a region $E$ adjacent to $A \cup B$ along $F_p$ with $\Omega$-volume $\varepsilon$ equal to exactly $2\varepsilon$, which we subsequently subdivide by means of $G_p$ into subregions $C, D$ adjacent to $B, A$ respectively with the corresponding $\Omega$-volumes $c, d$. Since $a = b = \varepsilon$ and $c + d = 2\varepsilon$, by our assumption (3) we obtain $c = d = \varepsilon$. Since $A \cup B \cup C \cup D \subseteq V$ and reflections are well-defined and unique on $V$ in the sense of Lemma 22, we obtain that $C = [p, q']$ for $q'' = r_p;\mathcal{G}(q'), D = [p, q^{(3)}]$ for $q^{(3)} = r_p;\mathcal{F}(q'')$ and finally $A = [p, q] = [p, q^{(4)}]$ for $q^{(4)} = r_p;\mathcal{G}(q^{(3)})$, hence $q = q^{(4)} = \ell_{p;\mathcal{F},\mathcal{G}}(q)$ as claimed.

Let us assume (5) that the reflection holonomy at every point $p \in M$ is trivial. Since the loops $\ell_{p;\mathcal{F},\mathcal{G}}$ for each $p \in M$ and $\mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ are identities, all coefficients in their Taylor expansions of order higher than 1 vanish. The coefficients $\kappa_{ij}(p)$ of $K(\mathcal{W}_\Omega) = \sum_{i \neq j} \kappa_{ij} dx_i dx_j$ all occur as order 3 coefficients inside the expansion of some loop $\ell_{p;\mathcal{F},\mathcal{G}}$ by Lemma 30. Hence, the nonuniformity tensor vanishes identically, proving (6).

The last implication from (6) to (1) is exactly the statement of Theorem 1. □

Remark: In the statement of the above theorem, due to the fact that conditions (1), (5) and (6) are purely local, we are allowed to substitute conditions (2), (3), (4) with their localized versions (2'), (3'), (4') in which the phrase ”any region bounded by leaves $K$” is replaced by ”any sufficiently small region bounded by leaves $K$”. Validity of this assertion becomes more clear after a careful investigation of the proofs of the implications between (2), (3), (4) and (5); they all can be carried over almost verbatim to their localized form.

We will now give a quantitative version of condition (2) of Theorem 31, which gives information about the nonuniformity tensor directly from non-fulfillment of Tabachnikov’s local triviality condition [17, Fig. 2.].

Theorem 32. Let $\mathcal{W}_\Omega$ be a codimension-1 divergence-free n-web on a smooth manifold $M$ and let $p \in M$. Given sufficiently small $u, v \in \mathbb{R}^n$ defining a region bounded by leaves $K = [u, v]$ with $p \in \text{Int} K$ inside a fixed $\mathcal{W}_\Omega$-adapted coordinate system $(x_1, \ldots, x_n)$, and two plaques $F = \{x_i = p_i\} \in \mathcal{F}, G = \{x_j = p_j\} \in \mathcal{G}$ of $\mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ which subdivide $K$ into four subregions $A, B, C, D$ of the form...
which was to be proved in the planar case.

\[
A = K \cap \{ x_i \leq p_i \} \cap \{ x_j \geq p_j \}, \quad B = K \cap \{ x_i \geq p_i \} \cap \{ x_j \geq p_j \}, \\
D = K \cap \{ x_i \leq p_i \} \cap \{ x_j \leq p_j \}, \quad C = K \cap \{ x_i \geq p_i \} \cap \{ x_j \leq p_j \},
\]

(4.18)

de the \( \Omega \)-volumes \( a, b, c, d \) of the respective subregions \( A, B, C, D \) satisfy

\[
ac < bd \text{ if } \kappa_{ij} > 0, \quad ac > bd \text{ if } \kappa_{ij} < 0,
\]

(4.19)

where \( \kappa_{ij} \) are the \( dx_idx_j \)-coefficients of the nonuniformity tensor \( \mathcal{K}(\mathcal{W}_\Omega)|_p = \sum_{k \neq l} \kappa_{kl} dx_k dx_l \) at point \( p \).

**Proof** We first prove the theorem in the special case of planar divergence-free 2-webs \( \mathcal{W}_\Omega \) with a coordinate system \((x, y)\) centered at \( p \in M \) which is normalized in the sense of Lemma 18. In such coordinates the volume form \( \Omega = h(x, y) \, dx \wedge dy \) satisfies \( h(x, 0) = h(0, y) = 1 \). Denote by \( \kappa \) the only nonzero coefficient of the nonuniformity tensor \( \mathcal{K}(\mathcal{W}_\Omega)|_p = \kappa \, dx \, dy \) at point \( p \). In these coordinates we can express \( h(x, y) \) as

\[
h(x, y) = 1 + \kappa \, xy + g(x, y) \, xy
\]

(4.20)

for some continuous function \( g \) vanishing at 0. Hence, for fixed \( \varepsilon > 0 \) and small enough \( u = (u_1, u_2) \in \mathbb{R}^2 \) we have

\[
\left| \int_{[0,u]} \Omega - (u_1u_2 + \frac{1}{4} \kappa u_1^2 u_2^2) \right| < \frac{1}{4} \varepsilon u_1^2 u_2^2.
\]

(4.21)

Now, choose \( \varepsilon > 0 \) such that \( |\kappa| > \varepsilon \). Suppose that \( K = [u, v] \) for some \( u_1, u_2 > 0, v_1, v_2 < 0 \), and that \( u_1, u_2, v_1, v_2 \) are so close to 0 that each \((x, y) \in K\) satisfies (4.21) and \( 1 + (\kappa \pm \varepsilon)xy > 0 \). Then, since \( A = [v_1, 0] \times [0, u_2], B = [0, u_1] \times [0, u_2], \)
\( C = [0, u_1] \times [v_2, 0] \) and \( D = [v_1, 0] \times [v_2, 0] \), we obtain that the corresponding \( \Omega \)-volumes \( a, b, c, d \) satisfy

\[
bd - ac = \int_B \Omega \cdot \int_D \Omega - \int_A \Omega \cdot \int_C \Omega
\]

\[
> (u_1u_2 + \frac{1}{4}(\kappa - \varepsilon)u_1^2 u_2^2) \cdot (v_1v_2 + \frac{1}{4}(\kappa - \varepsilon)v_1^2 v_2^2)
\]

\[
- (v_1u_2 + \frac{1}{4}(\kappa - \varepsilon)v_1^2 u_2^2) \cdot (u_1v_2 + \frac{1}{4}(\kappa - \varepsilon)u_1^2 v_2^2)
\]

\[
= \frac{1}{4}u_1u_2v_1v_2(u_1 - v_1)(u_2 - v_2)(\kappa - \varepsilon),
\]

(4.22)

and similarly

\[
bd - ac < \frac{1}{4}u_1u_2v_1v_2(u_1 - v_1)(u_2 - v_2)(\kappa + \varepsilon).
\]

(4.23)

Since \( u_1u_2v_1v_2(u_1 - v_1)(u_2 - v_2) > 0 \), the sign of \( bd - ac \) coincides with that of \( \kappa \), which was to be proved in the planar case.

The same result holds irrespective of the choice of a \( \mathcal{W}_\Omega \)-coordinate system, since any such set of coordinates \((x, y)\) can be transformed into one of the above form, say
(\tilde{x}, \tilde{y})$, by letting \(\tilde{x} = \int_0^x h(t,0)dt\) and \(\tilde{y} = h(0,0)^{-1} \cdot \int_0^y h(0,s)ds\) as in Lemma 18. As a result of this transformation, the nonuniformity tensor coefficient changes from \(\kappa\) to \(\tilde{\kappa} = \kappa / h(0,0)\). It might change sign in the process, but its effect on the statement of the theorem is countered by the way the subregions \(A, B, C, D\) arrange themselves in the new set of coordinates; note that \(\{p : x(p) > 0\} = \{p : \tilde{x}(p) < 0\}\) and \(\{p : y(p) > 0\} = \{p : \tilde{y}(p) > 0\}\) if \(h(0,0) < 0\) and relabel the regions as necessary.

We now prove the theorem for an arbitrary codimension-1 divergence-free \(n\)-web \(\mathcal{W}_2\). First, express \(\Omega\) in the \(\mathcal{W}_2\) adapted coordinate chart \((U, \varphi)\) as \(\Omega = h dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\). Then, assuming without loss of generality that \(i = 1\) and \(j = 2\), define a family of planar divergence-free 2-webs \(\mathcal{W}_{\alpha,\beta}\) depending continuously on parameters \(\alpha, \beta \in \mathbb{R}^{n-2}\) in the following way.

Let \(U\) be the image of \(U\) by the projection \((x_1, \ldots, x_n) \mapsto (x_1, x_2) = (x, y)\) and let \(F, G\) be generated by \(\ker dx_1, \ker dy\) respectively. Let \(\Omega_{\alpha,\beta} = h_{\alpha,\beta}(x, y) \, dx \wedge dy\), where \(h_{\alpha,\beta}\) denotes the integral of \(h\) over fibers of the projection

\[
h_{\alpha,\beta}(x, y) = \frac{1}{\lambda_1 \cdots \lambda_n} \int_0^1 \cdots \int_0^1 h(x, y, \lambda_1(t_1), \ldots, \lambda_{n-2}(t_{n-2})) \, dt_{n-2} \cdots dt_1, \quad (4.24)
\]

with \(\lambda_k(t) = \alpha_k + t(\beta_k - \alpha_k)\). The webs \(\mathcal{W}_{\alpha,\beta}\) are defined as \(\mathcal{W}_{\alpha,\beta} = (U, F, G, \Omega_{\alpha,\beta})\).

Pick any two points \(u, v \in U\) and represent them as \(u = (q^{(1)}, \alpha)\) and \(v = (q^{(2)}, \beta)\) for \(q^{(1)}, q^{(2)} \in \mathbb{R}^2\) and \(\alpha, \beta \in \mathbb{R}^{n-2}\). When \(\alpha_k \neq \beta_k\) for each \(k = 1, \ldots, n-2\), then a change of variables in (4.24) yields

\[
\int_{[q^{(1)}, q^{(2)}]} \Omega_{\alpha,\beta} = \frac{1}{\prod_{k=1}^{n-2} (\alpha_k - \beta_k)} \int_{[u,v]} \Omega, \quad (4.25)
\]

where we treat \([a, b]\) as chains in the sense given in Definition 20. On the other hand, taking \(\alpha = \beta\) results in equality \(h_{\alpha,\beta}(x, y) = h(x, y, \alpha_1, \ldots, \alpha_{n-2})\). Straightforward calculation confirms that the nonuniformity tensor of \(\mathcal{W}_{0,0}\) at 0 is exactly \(\kappa_{ij} = \kappa_{12}\). Since \(h_{\alpha,\beta} \in C^\infty(U)\) depends continuously on \(\alpha, \beta \in \mathbb{R}^{n-2}\), so does the coefficient \(\tilde{\kappa}_{\alpha,\beta}\) of the nonuniformity tensor \(\mathcal{K}(\mathcal{W}_{\alpha,\beta}) = \tilde{\kappa}_{\alpha,\beta} \, dx \wedge dy\). By continuity, for all \(\alpha, \beta\) in some neighbourhood of 0 the sign of \(\kappa_{\alpha,\beta}\) is the same as the sign of \(\kappa_{ij}\). Moreover, regions \(A, B, C, D\) are all of the form \([p, u_k]\) for points \(u_k \in \mathbb{R}^n\), \(k = 1, 2, 3, 4\), differing only at the \(i\)th and \(j\)th coordinate. Hence, by the planar case and equality (4.25), the inequalities (4.19) hold for regions \(K = [u, v]\) with sufficiently small \(u, v \in \mathbb{R}^n\).

\(\square\)

### 4.3 Geometric conditions in higher codimensions

The following definition outlines one of the possible extensions of Definitions 20 and 21 to an arbitrary codimension (Fig. 6).
Definition 33 Let $\mathcal{W} = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ be an $n$-web on a smooth $m$-dimensional manifold $M$. A compact set $C \subseteq M$ which takes the form $C_1 \times C_2 \times \cdots \times C_n$ for connected closed subsets $C_i \subseteq \mathbb{R}^{e_i}$, $e_i = \text{codim} \mathcal{F}_i$, in some $\mathcal{W}$-adapted chart will be called a region bounded by leaves of $\mathcal{W}$. Two such regions $A, B$ are adjacent along $\mathcal{F} \in \text{Fol}(\mathcal{W})$ if $A \cup B$ is also a region bounded by the leaves of $\mathcal{W}$ and $A \cap B$ is a closed fragment of a smooth hypersurface $F \subseteq M$ satisfying $T \mathcal{F} \subseteq T F$. In this case we say that $F$ subdivides $A \cup B$ into subregions $A, B$.

The volume-preserving loops along $\mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ of a codimension-$1$ divergence-free $n$-web $\mathcal{W}_\Omega$ (Definition 26) allow us to probe the coefficients of the nonuniformity tensor $\mathcal{K}(\mathcal{W}_\Omega)$ in the directions transverse to $\mathcal{F}, \mathcal{G}$ by means of their Taylor expansions (Lemma 30). If $\mathcal{W}_\Omega$ has arbitrary codimension, we can still consider volume-preserving loops coming from local webs $\mathcal{W}_{\psi; \Omega}$ with foliations formed by level-sets of individual coordinate functions $(x_1, \ldots, x_m)$ inside a $\mathcal{W}_\Omega$-adapted chart $(U, \psi)$ centered at $p \in M$. As the natural connection $\Theta$ of $\mathcal{W}_{\psi; \Omega}$ with Ricci tensor $R_c$ is a $\mathcal{W}_\Omega$-connection on $U$, Theorem 15 implies that $\mathcal{K}(\mathcal{W}_\Omega) = \text{pr}_O(R_c) = \text{pr}_O(\mathcal{K}(\mathcal{W}_{\psi; \Omega}))$. With that in mind, we are able to recover $\mathcal{K}(\mathcal{W}_\Omega)$ from loops through the foliations generated by $\ker dx_i$, $\ker dx_j$ for $i \sim j$, where both $i \sim j$ and $i \sim j$ are defined in (2.2). This motivates the following definition of volume preserving holonomy.

Definition 34 Let $M$ be a $m$-dimensional smooth manifold and let $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a divergence-free $n$-web. Consider a $\mathcal{W}_\Omega$-adapted chart $(U, \psi)$ centered at $p \in M$ with coordinates $\psi = (x_1, \ldots, x_m)$ and the induced local codimension-$1$ divergence-free $m$-web $\mathcal{W}_{\psi; \Omega}$ whose defining foliations $\mathcal{G}_i$ satisfy $T \mathcal{G}_i = \ker dx_i$ for $i = 1, \ldots, m$.

1. The smallest normal subgroup $\mathcal{H}_{p; \psi}$ generated by all volume-preserving loops $\ell_{p; \mathcal{G}_i, \mathcal{G}_j}$ of $\mathcal{W}_{\psi; \Omega}$ with $i \sim j$ inside the group $\mathcal{R}_{p; \psi}$ of all volume-preserving reflections will be called the (volume-preserving) reflection holonomy group of $\mathcal{W}_\Omega$ at $p$ in the chart $(U, \psi)$.

2. Let $\mathcal{H}_p$ be the group generated by the union of all $\mathcal{H}_{p; \psi}$, where $\psi$ runs over all $\mathcal{W}_\Omega$-adapted coordinate charts $(U, \psi)$ centered at some fixed point $p \in M$, and...
let $(V, \psi)$ be a fixed $\mathcal{W}_\Omega$-adapted chart. Define $\psi^*_p : \mathcal{H}_p \to \mathcal{H}_{\psi(p)}$ by $\psi^*_p(f) = \psi \circ f \circ \psi^{-1}$. The conjugacy class of $\psi^*_p \mathcal{H}_p$ inside the group of diffeomorphism-germs $\text{Diff}(\mathbb{R}^m, 0)$ is called the (volume-preserving) reflection holonomy of $\mathcal{W}_\Omega$ at $p$.

These straightforward generalizations allow us to carry over the statement of Theorem 31 almost verbatim to the higher-codimensional case.

**Theorem 35** Let $M$ be a $m$-dimensional smooth manifold and let $\mathcal{W}_\Omega = (M, \mathcal{F}_1, \ldots, \mathcal{F}_n, \Omega)$ be a divergence-free $n$-web. The following conditions are equivalent.

1. The web $\mathcal{W}_\Omega$ is locally trivial.

2. For each pair $\mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ of two different foliations of $M$, any region bounded by leaves $K$, and any two hypersurfaces $F, G$ satisfying $T \mathcal{F} \subseteq T F, T \mathcal{G} \subseteq T G$ which subdivide $K$ into four subregions $A, B, C, D$ with $(A \cup B) \cap (C \cup D) \subseteq F$ and $(A \cup D) \cap (B \cup C) \subseteq G$, the respective $\Omega$-volumes $a, b, c, d$ of $A, B, C, D$ satisfy

\[ ac = bd. \] (4.26)

3. For each pair $\mathcal{F}, \mathcal{G} \in \text{Fol}(\mathcal{W}_\Omega)$ of two different foliations of $M$, any region bounded by leaves $K$, and any two hypersurfaces $F, G$ satisfying $T \mathcal{F} \subseteq T F, T \mathcal{G} \subseteq T G$ which subdivide $K$ into four subregions $A, B, C, D$ with $(A \cup B) \cap (C \cup D) \subseteq F$ and $(A \cup D) \cap (B \cup C) \subseteq G$ in such a way that the $\Omega$-volumes $a, b, c, d$ of $A, B, C, D$ satisfy $a + b = c + d$, the equality $a = b$ implies that $a = b = c = d$.

4. The volume-preserving reflection holonomy of $\mathcal{W}_\Omega$ at each point $p \in M$ is trivial.

5. The nonuniformity tensor $K(\mathcal{W}_\Omega)$ vanishes identically.

**Remark** The naive analogue of the splitting condition (4) of Theorem 31, namely “for any region bounded by leaves $K$ and each $k = 1, 2, \ldots, n$ there exist hypersurfaces $F_i$ satisfying $T F_i \subseteq T F_i$ for $i = 1, 2, \ldots, k$ which subdivide $K$ into $2^k$ subregions with equal $\Omega$ -volumes” is not equivalent to the other conditions, despite being satisfied by trivial divergence-free webs. For example, in codimension $(1, 2)$ the above property holds for every divergence-free 2-web $(M, \mathcal{F}, \mathcal{G}, \Omega)$ due to a variant of the Stone-Tukey ham sandwich theorem. More specifically, we subdivide a region bounded by leaves $K$ by a 2-dimensional plaque $F \in \mathcal{F}$ into two subregions with equal $\Omega$-volumes, and then we choose a coordinate plane $H$ with $T \mathcal{G} \subseteq T H$ subdividing both subregions into subregions with equal $\Omega$-volumes by means of the classical Borsuk-Ulam theorem.

**Proof of Theorem 35** The proof proceeds as in Theorem 31 with some minor modifications, which will be highlighted below.

The implications from (1) through (2) to (3) and from (5) to (1) are established in essentially the same way as their codimension-1 counterparts.

To deduce (4) from (3), note that the holonomy in arbitrary codimension is generated by conjugates of the holonomy group $\mathcal{H}_{p; \psi}$ inside some fixed coordinates; it is therefore sufficient to trivialize $\mathcal{H}_{p; \psi}$ for an arbitrary coordinate chart $(U, \varphi)$ by following closely the proof in the codimension 1-case. In order to use property (3), let the
coordinate function level-sets assume the role of the hypersurfaces $F$, $G$ subdividing the region $K$.

Finally, assuming (4), fix a $\mathcal{W}_\Omega$-adapted coordinate chart $(U, \varphi)$ with coordinates $(x_1, \ldots, x_m)$ and let $\mathcal{G}_i$ be the foliations with $T \mathcal{G}_i = \ker dx_i$ for $i = 1, \ldots, m$ comprising the local web $\mathcal{W}_{\varphi;\Omega}$ on $U$. Since the holonomy group $\mathcal{H}_{\rho;\varphi}$ is trivial by assumption, the volume-preserving loops $\ell_{\rho;\mathcal{G}_i}, \mathcal{G}_j$ are all identities for $i \nsim j$ (see (2.2)). The equality $K(\mathcal{W}_\Omega) = \text{pr}_O(K(\mathcal{W}_{\varphi;\Omega})) = \sum_{i \nsim j} \kappa_{ij} dx_i dx_j$, which holds by an application of Theorem 15 to the natural connection of $\mathcal{W}_{\varphi;\Omega}$, leads to $\kappa_{ij} = 0$ for each $i \nsim j$ and $p \in \mathcal{M}$ via Lemma 30. This is exactly condition (5). \hfill \square

5 Applications in general relativity

In this section we consider divergence-free 2-webs $\mathcal{W}_{dV}$ arising from a 4-dimensional spacetime $(\mathcal{M}, g)$ with metric $g$ of signature $(-+++)$, its natural volume element $dV$ and a single codimension-1 spacelike foliation $F$ of $\mathcal{M}$, as defined below.

The foliation $F$ is spacelike if the restriction of $g$ to each of its leaves is positive-definite. This means that the leaves of $F$ can be locally treated as hypersurfaces of simultaneity in the ambient spacetime. The assignment to each event $p \in \mathcal{M}$ a plaque $L \ni p$ of $F$ in a certain coordinate system adapted to the foliation $F$ can be treated as a measurement of local time $t$ by some observer; the plaque itself (roughly) represents the state of the world at time $t$. The orthogonal complement $T F^\perp$ of $T F$ with respect to $g$ is a 1-dimensional distribution, hence is integrable; the corresponding foliation $F^\perp$ is necessarily timelike and can be treated as trajectories of a family of Eulerian observers. Let $dV = \sqrt{-\det g}$ be the pseudo-Riemannian volume form of the metric $g$. The divergence-free 2-web $\mathcal{W}_{dV}$ in question is $(\mathcal{M}, F, F^\perp, dV)$.

The $\mathcal{W}_{dV}$-adapted coordinates are called normal (Eulerian) coordinates [11]. In these coordinates, the metric tensor splits into a sum

$$g = -\alpha^2 dt^2 + \gamma,$$

where the $t$-coordinate parametrizes the leaves of $F$, $\alpha$ is the lapse function and $\gamma$ is a positive-definite metric on $T F$. Of course, according to circumstances, one can find complementary foliations $\mathcal{G}$ which are better suited to the situation at hand; the above choice is motivated mainly by the clarity of exposition and can be altered without difficulty.

Example 8 Consider the event horizon exterior of the Schwarzschild spacetime $(\mathcal{M}, g)$ in standard radial coordinates $(t, r, \theta, \phi)$, where $\mathcal{M} = \mathbb{R}^4 \setminus \{r \leq 2m\}$ for some positive mass $m$ and

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2.$$

The hypersurfaces of constant time $\{t = c\}$ for $c \in \mathbb{R}$ define a foliation $F$ of $\mathcal{M}$. Its orthogonal complement $F^\perp$ is exactly the set of curves $\{r = c_r, \theta = c_{\theta}, \phi = c_{\phi}\}$ for
$$c_r, c_\theta, c_\phi \in \mathbb{R}$$ by virtue of the form of $g$ in these coordinates. We also have

$$dV = r^2 \sin \theta \cdot dt \wedge dr \wedge d\theta \wedge d\phi. \quad (5.3)$$

Since the coordinate system is $\mathcal{W}_{dV}$-adapted, the divergence-free 2-web $\mathcal{W}_{dV}$ is locally trivial by Theorem 1.

**Example 9** The constant-$T$ foliation $\mathcal{F}$ corresponding to Gullstrand-Painlevé coordinates $(T, r, \theta, \phi)$ on the Schwarzschild spacetime $(\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}), g)$, in which the metric $g$ takes the form

$$g = -\left(1 - \frac{2m}{r}\right) dT^2 + 2 \sqrt{\frac{2m}{r}} dT dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \quad (5.4)$$

is spacelike and generates a divergence-free 2-web $\mathcal{W}_{dV}$ which is not locally trivial. To see this, switch to a $\mathcal{W}_{dV}$-adapted coordinate system $(T, R, \theta, \phi)$ with $R = \frac{2}{3}r^{3/2} + \sqrt{2mT}$ (known as Lemaître coordinate system) to bring $g$ into the split form (5.1)

$$g = -dT^2 + r^{-1} dR^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \quad r = \left(\frac{3}{2}R - \frac{3}{2}\sqrt{2mT}\right)^{2/3}. \quad (5.5)$$

The claim follows from Theorem 1, since the corresponding volume form

$$dV = \frac{3}{2}(R - \sqrt{2mT}) \sin \theta \cdot dT \wedge dR \wedge d\theta \wedge d\phi \quad (5.6)$$

yields a nonvanishing nonuniformity tensor $K(\mathcal{W}_{dV}) = \sqrt{2m (R - \sqrt{2mT})}^{-2} dT dR$. Its form in the initial Gullstrand-Painlevé coordinate system is

$$K(\mathcal{W}_{dV}) = \frac{9}{2}mr^{-3} dT^2 + \frac{9}{4} \sqrt{2m} r^{-5/2} dT dr. \quad (5.7)$$

Since $g \left( \frac{\partial}{\partial T}, \frac{\partial}{\partial T} \right) = -1$, the variable $T$ can be interpreted as the proper time of Eulerian observers falling radially into the Schwarzschild black hole. Near the singularity itself, the nonuniformity tensor grows without bound.

The local triviality of the induced divergence-free 2-web has some relevance in relativistic fluid dynamics, where the laws of motion of the fluid expressed in $\mathcal{W}_{dV}$-adapted coordinates include the conservation of energy (5.9) and momentum (5.10), both of which depend on the spacetime volume element $dV$ in a nontrivial way. These laws, expressed using the notation introduced in [2, Part 11] and the Einstein summation convention, reduce to

$$\nabla_{\alpha} T^{\alpha\beta} = 0, \quad (5.8)$$

for the stress-energy tensor $T$ bound by certain constitutive relations, where $\nabla_{\alpha}$ denotes the Levi-Civita covariant differentiation. The above condition on the divergence of $T$
results in appearance of the Lorentzian volume density in the resulting equations. In normal coordinates, assuming that in the tensors below the 0th index corresponds to time-like components, while the indices \(i, j, k = 1, 2, 3\) refer to the spatial ones, the equations read [14]

\[
\frac{\partial}{\partial t} (\alpha^3 \gamma^{1/2} T^{00}) + \frac{\partial}{\partial x_j} (\alpha^3 \gamma^{1/2} T^{0j}) = \gamma^{1/2} \alpha^2 (\frac{\partial}{\partial t} \alpha) T^{00} - \frac{1}{2} \gamma^{1/2} \alpha \left( \frac{\partial}{\partial t} g_{jk} \right) T^{jk} \tag{5.9}
\]

\[
\frac{\partial}{\partial t} (\alpha^3 \gamma^{1/2} T^{10}) + \frac{\partial}{\partial x_j} (\alpha^3 \gamma^{1/2} T^{1j}) = \frac{1}{2} \gamma^{1/2} \alpha^2 \left( \frac{\partial}{\partial x_k} \alpha^2 \right) T^{00} - \gamma^{1/2} \alpha^3 g^{jk} \left( \frac{\partial}{\partial t} g_{jk} \right) T^{j0} - \gamma^{1/2} \alpha^3 \Gamma^j_{jk} T^{j0} - T^{0i} \frac{\partial}{\partial t} (\gamma^{1/2} \alpha^3) \\
+ 2 T^{0i} \gamma^{1/2} \alpha^2 \left( \frac{\partial}{\partial t} \alpha \right) + 2 \alpha^3 T^{ij} \left( \frac{\partial}{\partial x_j} \gamma^{1/2} \right), \tag{5.10}
\]

where \(\alpha\) is the lapse function, \(\Gamma^j_{jk}\) are the Christoffel symbols and \(\gamma^{1/2}\) is the coefficient of the spacelike volume element on \(T\mathcal{F}\) induced from \(dV\).

If the nonuniformity tensor \(K(\mathcal{W}_{dV})\) of the web \(\mathcal{W}_{dV}\) defined in the introduction vanishes identically, then by Theorem 1 one can pick a \(\mathcal{W}_{dV}\)-adapted coordinate system \((t, x_1, x_2, x_3)\) in which the density of the volume element \(dV = \alpha \gamma^{1/2} dt \wedge dx_1 \wedge dx_2 \wedge dx_3\) becomes constant. This simplifies the above equations, completely eliminating their dependence on \(\gamma^{1/2}\). If we assume that \(\alpha \gamma^{1/2} = 1\), then from (5.9) and (5.10) we obtain

\[
\frac{\partial}{\partial t} (\alpha^2 T^{00}) + \frac{\partial}{\partial x_j} (\alpha^2 T^{0j}) = \frac{1}{2} \left( \frac{\partial}{\partial t} \alpha^2 \right) T^{00} - \frac{1}{2} \left( \frac{\partial}{\partial t} g_{jk} \right) T^{jk} \tag{5.11}
\]

\[
\frac{\partial}{\partial t} (\alpha^2 T^{10}) + \frac{\partial}{\partial x_j} (\alpha^2 T^{1j}) = \frac{1}{2} \alpha^2 \left( \frac{\partial}{\partial x_k} \alpha^2 \right) T^{00} - \alpha^2 g^{jk} \left( \frac{\partial}{\partial t} g_{jk} \right) T^{j0} - \alpha^2 \Gamma^i_{jk} T^{j0} - \left( \frac{\partial}{\partial x_j} \alpha^2 \right) T^{ij}. \tag{5.12}
\]

This form of the equations is more convenient to work with, and due to its simplicity it might improve numerical accuracy if \(\alpha^2\) and \(g\) are well behaved in the new coordinates. Given \(\alpha^2\) and the other components of the metric, one can avoid taking any square roots and determinants in order to evolve the stress-energy tensor \(T\) in time. The equations can be simplified further if we are able to choose \(\mathcal{F}\) such that \(\mathcal{F}^\perp\) is totally geodesic. This amounts to the equality \(d\alpha|_{T\mathcal{F}} = 0\) by a direct verification of the identity \(g(\nabla_n n, v) = d \log \alpha(v)\) for the unit normal \(n \in T\mathcal{F}^\perp\) with \(||n||_g = -1\) and each \(v \in T\mathcal{F}\). In this setting, the leaves of the orthogonal foliation \(T\mathcal{F}^\perp\) are geodesics; a foliation \(\mathcal{F}\) of this kind is called a geodesic slicing of the spacetime [11, 10.2].

One last thing to note is that the Levi-Civita connection \(\nabla\) is not a \(\mathcal{W}_{dV}\)-connection in general (see Definition 4). It is so only if both \(\mathcal{F}\) and \(\mathcal{F}^\perp\) are totally geodesic, in the sense that for all \(X, Z \in \Gamma(T\mathcal{F})\) and \(Y, W \in \Gamma(T\mathcal{F}^\perp)\)

\[
\nabla_Z X \in \Gamma(T\mathcal{F}), \quad \nabla_W Y \in \Gamma(T\mathcal{F}^\perp). \tag{5.13}
\]

This condition is equivalent to \(d\alpha|_{T\mathcal{F}} = 0\) and \(K_{ij} = \frac{1}{2 \alpha} \frac{\partial}{\partial t} \nabla_{ij} = 0\) for \(i, j = 1, 2, 3\). As a consequence, it is usually not possible to determine local triviality of \(\mathcal{W}_{dV}\) using the Ricci tensor of the Levi-Civita connection alone, as the results of Sect. 2.5 seem to suggest, except in quite special geometries defined by the above two sets of equations.
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Declarations

Conflict of interest:  The authors declare that there is no conflict of interests.

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