A Sampler in Analysis

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Preface

The mathematical area of analysis is often described as the study of limits, continuity, and convergence, as in calculus. It is also very much concerned with estimates, whether or not a limit is involved. Here we look at several topics involving norms on vector spaces and linear mappings, with prerequisites along the lines of advanced calculus and basic linear algebra. The main idea is to explore intermediate ranges of abstraction and sophistication, without getting bogged down with too many technicalities. Lebesgue integrals are not required, but could easily be incorporated by readers familiar with that theory.

We begin with some inequalities related to convexity in the first chapter, which can be applied to sums or integrals. The next three chapters focus on finite-dimensional vector spaces and linear transformations between them. Some properties of infinite sums are described in Chapter 5, as well as a class of infinite-dimensional spaces known as $\ell^p$ spaces. The latter give examples of Banach and Hilbert spaces, which are considered more abstractly in Chapter 6. An important tool for dealing with bounded linear operators on $\ell^p$ or $L^p$ spaces is Marcel Riesz’ convexity theorem, presented in Chapter 8.9. As a further introduction to real-variable methods in harmonic analysis, estimates for dyadic maximal and square functions are discussed in Chapter 8. A brief review of some basic notions about metric spaces is included in Appendix A.

Of course, there are numerous excellent texts on these and related subjects, a selection of which can be found in the bibliography. Indeed, it is hoped that readers might pursue specific topics more fully, according to their interests. Here one might find a few tricks of the trade, or simplified special cases, which illustrate broader concepts. I would like to dedicate this book to my fellow students from Washington University in St Louis, and to the faculty there, from whom we learned a great deal.
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Chapter 1

Preliminaries

1.1 Real and complex numbers

As usual, the real line is denoted $\mathbb{R}$, the complex plane is denoted $\mathbb{C}$, and the set of integers is denoted $\mathbb{Z}$. If $A$ is a subset of $\mathbb{R}$ and $b$ is a real number such that $a \leq b$ for all $a \in A$, then $b$ is said to be an upper bound for $A$. A real number $c$ is said to be the least upper bound or supremum of $A$ if $c$ is an upper bound for $A$ and $c \leq b$ for every real number $b$ which is an upper bound for $A$. One version of the completeness of the real numbers asserts that a nonempty subset $A$ of $\mathbb{R}$ with an upper bound has a least upper bound. It is easy to see from the definition that the supremum $\sup A$ of $A$ is unique when it exists.

Similarly, if $A \subseteq \mathbb{R}$ and $y \in \mathbb{R}$ satisfy $y \leq x$ for every $x \in A$, then $y$ is said to be a lower bound of $A$. If $z$ is a real number such that $z$ is a lower bound for $A$ and $y \leq z$ for every real number $y$ which is a lower bound for $A$, then $z$ is said to be a greatest lower bound or infimum of $A$. It follows from the completeness of the real numbers that every nonempty subset $A$ of $\mathbb{R}$ with a lower bound has a greatest lower bound. This can be obtained as the supremum of the set of lower bounds for $A$, or as the negative of the supremum of

\begin{equation}
-A = \{-a : a \in A\}.
\end{equation}

Again, it is easy to see directly from the definition that the infimum $\inf A$ of $A$ is unique when it exists.

It is sometimes convenient to use extended real numbers, which are real numbers together with $+\infty$, $-\infty$, with standard conventions concerning arithmetic operations and ordering. More precisely,

\begin{equation}
-\infty < x < +\infty
\end{equation}

and

\begin{equation}
x + (+\infty) = (+\infty) + x = +\infty, \quad x + (-\infty) = (-\infty) + x = -\infty
\end{equation}
for every \( x \in \mathbb{R} \). If \( x \) is a positive real number, then

\[
(1.4) \quad x \cdot (+\infty) = (+\infty) \cdot x = +\infty, \quad x \cdot (-\infty) = (-\infty) \cdot x = -\infty
\]

while the product of \( \pm \infty \) with a negative real number changes the sign. The product of \( \pm \infty \) with \( \pm \infty \) is defined to be \( \pm \infty \), where the signs are multiplied in the usual way. One can also define \( x/\pm \infty \) to be 0 for every \( x \in \mathbb{R} \), but expressions such as \( \infty - \infty \), \( \infty/\infty \), and \( 0/0 \) are not defined. If one allows extended real numbers, then every nonempty set \( A \subseteq \mathbb{R} \) has a supremum and an infimum, where \( \sup A = +\infty \) if \( A \) does not have a finite upper bound, and \( \inf A = -\infty \) if \( A \) does not have a finite lower bound. In situations where all of the quantities of interest are nonnegative, it may be appropriate to interpret \( 1/0 \) as being equal to \( +\infty \).

If \( a \) and \( b \) are real numbers with \( a < b \), then there are four types of intervals in the real line with endpoints \( a \) and \( b \), i.e., the open interval \((a, b)\), the half-open, half-closed intervals \((a, b]\), \([a, b)\), and the closed interval \([a, b]\). These four types of intervals are defined as follows:

\[
(a, b) = \{ x \in \mathbb{R} : a < x < b \};
\]
\[
(a, b] = \{ x \in \mathbb{R} : a < x \leq b \};
\]
\[
[a, b) = \{ x \in \mathbb{R} : a \leq x < b \};
\]
\[
[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}.
\]

The length of each of these intervals is defined to be \( b - a \), and the length of an interval \( I \) may be denoted \( |I| \).

We also consider \([a, b]\) to be defined when \( a = b \), in which event the interval consists of a single point and has length equal to 0. For an interval which is open at the left endpoint \( a \), we may allow \( a = -\infty \), and for an interval which is open at the right endpoint \( b \), we may allow \( b = +\infty \). Hence the real line may be expressed as \((-\infty, +\infty)\). In these cases, we say that the interval is unbounded, while an interval with finite endpoints is said to be bounded.

If \( x \) is a real number, then the absolute value of \( x \) is denoted \( |x| \) and defined to be \( x \) when \( x \geq 0 \) and to be \(-x\) when \( x \leq 0 \). Thus \( |x| \) is always a nonnegative real number, \( |x| = 0 \) if and only if \( x = 0 \), and

\[
(1.5) \quad |x + y| \leq |x| + |y|
\]

and

\[
(1.6) \quad |x \cdot y| = |x| \cdot |y|
\]

for every \( x, y \in \mathbb{R} \). These properties are not difficult to verify.

Suppose that \( z = x + iy \) is a complex number, where \( x, y \in \mathbb{R} \). One may refer to \( x, y \) as the real and imaginary parts of \( z \), denoted \( \text{Re} z \), \( \text{Im} z \). The complex conjugate of \( z \) is denoted \( \overline{z} \) and defined by

\[
(1.7) \quad \overline{z} = x - iy.
\]
It is easy to see that
\[(1.8) \quad z + w = \overline{z + w}\]
and
\[(1.9) \quad z \cdot w = \overline{z} \cdot \overline{w}\]
for every $z, w \in \mathbb{C}$. Note that
\[(1.10) \quad z + \overline{z} = 2 \text{Re} \ z \quad \text{and} \quad z - \overline{z} = 2i \text{Im} \ z\]
for every $z \in \mathbb{C}$, and that the complex conjugate of $\overline{z}$ is equal to $z$.

The *modulus* of $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, is denoted $|z|$ and defined to be the nonnegative real number given by
\[(1.11) \quad |z| = \sqrt{x^2 + y^2}.\]
Thus the modulus of $z$ is the same as the absolute value of $z$ when $z \in \mathbb{R}$, and
\[(1.12) \quad |\text{Re} \ z|, |\text{Im} \ z| \leq |z|\]
for every $z \in \mathbb{C}$. Of course, the modulus of $z$ is the same as the modulus of the complex conjugate of $z$, and it is easy to see that
\[(1.13) \quad |z|^2 = z \cdot \overline{z}\]
for every $z \in \mathbb{C}$. This implies that
\[(1.14) \quad |z \cdot w| = |z| \cdot |w|\]
for every $z, w \in \mathbb{C}$, because of (1.9).

Similarly, we would like to check that
\[(1.15) \quad |z + w| \leq |z| + |w|\]
for every $z, w \in \mathbb{C}$. Using (1.13) applied to $z + w$ and then (1.8), we get that
\[(1.16) \quad |z + w|^2 = (z + w)(\overline{z + w}) = |z|^2 + z \overline{w} + w \overline{z} + |w|^2.\]
We also have that
\[(1.17) \quad z \overline{w} + w \overline{z} = z \overline{w} + (\overline{z} \overline{w}) = 2 \text{Re} \ z \overline{w} \leq 2 |z| |w| = 2 |z| |w|,\]
and hence that
\[(1.18) \quad |z + w|^2 \leq |z|^2 + 2 |z| |w| + |w|^2 = (|z| + |w|)^2.\]
This implies (1.15), as desired.

A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to *converge* to another complex number $z$ if for every $\epsilon > 0$ there is a positive integer $N$ such that
\[(1.19) \quad |z_n - z| < \epsilon\]
One can check that the limit $z$ of the sequence $\{z_n\}_{n=1}^{\infty}$ is unique when it exists, in which case we put
\[(1.20) \quad \lim_{n \to \infty} z_n = z.\]

If $\{w_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ are two sequences of complex numbers which converge to the complex numbers $w$, $z$, respectively, then the sequences $\{w_n + z_n\}_{n=1}^{\infty}$, $\{w_n \cdot z_n\}_{n=1}^{\infty}$ of sums and products converge to the sum $w + z$ and product $w \cdot z$ of the limits, respectively. A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers converges to a complex number $z$ if and only if the sequences of real and imaginary parts of the $z_n$’s converge to the real and imaginary parts of $z$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers which is monotone increasing, which is to say that $x_n \leq x_{n+1}$ for each $n$. One can check that $\{x_n\}_{n=1}^{\infty}$ converges if and only if the set of $x_n$’s has an upper bound, in which case the limit of the sequence is equal to the supremum of this set. For any sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers,
\[(1.21) \quad x_j \to +\infty \text{ as } j \to \infty\]
if for each $L \geq 0$ there is a positive integer $N$ such that
\[(1.22) \quad x_n \geq L \quad \text{for every } n \geq N.\]

If $\{x_n\}_{n=1}^{\infty}$ is an unbounded monotone increasing sequence of real numbers, then $x_n \to +\infty$ as $n \to \infty$. Similar remarks apply to monotone decreasing sequences of real numbers.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. For each positive integer $k$, put
\[(1.23) \quad A_k = \sup\{a_n : n \geq k\},\]
which may be $+\infty$. Thus $A_{k+1} \leq A_k$ for every $k$. The upper limit of $\{a_n\}_{n=1}^{\infty}$ is denoted $\limsup_{n \to \infty} a_n$ and defined to be the infimum of the $A_k$’s, which may be $+\infty$. Similarly, if
\[(1.24) \quad B_l = \inf\{a_n : n \geq l\},\]
then $B_l \leq B_{l+1}$ for every $l$, and the lower limit $\liminf_{n \to \infty} a_n$ of $\{a_n\}_{n=1}^{\infty}$ is defined to be the supremum of the $B_l$’s. By construction, $B_l \leq A_k$ for every $k$ and $l$, and hence
\[(1.25) \quad \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n.\]

One can check that $a_n \to a$ as $n \to \infty$ if and only if
\[(1.26) \quad \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = a.\]

A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to be a Cauchy sequence if for every $\epsilon > 0$ there is a positive integer $N$ such that
\[(1.27) \quad |z_l - z_n| < \epsilon\]
for each \( l, n \geq N \). It is easy to see that \( \{z_n\}_{n=1}^{\infty} \) is a Cauchy sequence if and only if the corresponding sequences of real and imaginary parts of the \( z_n \)'s are Cauchy sequences, and that convergent sequences are automatically Cauchy sequences. It is not difficult to show that the upper and lower limits of a Cauchy sequence of real numbers are finite and equal, and hence that every Cauchy sequence of real numbers converges. It follows that every Cauchy sequence of complex numbers converges too.

An infinite series of complex numbers \( \sum_{j=0}^{\infty} a_j \) is said to converge if the corresponding sequence of partial sums \( \sum_{j=0}^{n} a_j \) converges, in which case the sum of the series is defined to be the limit of the sequence of partial sums. If \( \sum_{j=0}^{\infty} a_j \) converges, then
\[
\lim_{j \to \infty} a_j = 0. \tag{1.28}
\]
The partial sums of an infinite series whose terms are nonnegative real numbers are monotone increasing, and therefore the series converges if and only if the partial sums are bounded. An infinite series \( \sum_{j=0}^{\infty} a_j \) of complex numbers is said to converge absolutely if
\[
\sum_{j=0}^{\infty} |a_j| \tag{1.29}
\]
converges. One can check that the partial sums of an absolutely convergent series form a Cauchy sequence, and therefore converge.

If \( A \) is a subset of a set \( X \), then \( 1_A(x) \) denotes the indicator function of \( A \) on \( X \). This is the function equal to 1 when \( x \in A \) and to 0 when \( x \in X \setminus A \), and it is sometimes called the characteristic function associated to \( A \). A function on the real line, or on an interval in the real line, is called a step function if it is a finite linear combination of indicator functions of intervals. Equivalently, this means that there is a finite partition of the domain into intervals on which the function is constant. In this book, one is normally welcome to restrict one's attention to functions on the real line that are step functions, at least in the context of integrating functions on \( \mathbb{R} \). Step functions are convenient because their integrals can be reduced immediately to finite sums. Results about other functions can often be derived from those for step functions by approximation.

### 1.2 Convex functions

Let \( I \) be an open interval in the real line, which may be unbounded. A real-valued function \( \phi(x) \) on \( I \) is said to be convex if
\[
\phi(\lambda x + (1 - \lambda) y) \leq \lambda \phi(x) + (1 - \lambda) \phi(y) \tag{1.30}
\]
for every \( x, y \in I \) and \( \lambda \in [0, 1] \).

If \( \phi(x) \) is an affine function, which is to say that \( \phi(x) = ax + b \) for some real numbers \( a \) and \( b \), then \( \phi(x) \) is a convex function on the whole real line, with equality in (1.30) for all \( x, y, \) and \( \lambda \). Equivalently, both \( \phi(x) \) and \( -\phi(x) \) are convex, which characterizes affine functions. It is easy to see that \( \phi(x) = |x| \)
is a convex function on the whole real line too. If \( \phi(x) \) is an arbitrary convex function on \( I \), and if \( c \) is a real number, then the translation \( \phi(x - c) \) of \( \phi(x) \) is a convex function on

\[
I + c = \{ x + c : x \in I \}.
\]

In particular, for each real number \( c \), \(|x - c|\) defines a convex function on \( \mathbb{R} \).

**Lemma 1.32** A real-valued function \( \phi(x) \) on \( I \) is convex if and only if

\[
\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(s)}{u - s} \leq \frac{\phi(u) - \phi(t)}{u - t}
\]

for every \( s, t, u \in I \) with \( s < t < u \).

If \( s, t, u \) are as in the lemma, then

\[
t = \frac{t - s}{u - s} u + \frac{u - t}{u - s} s,
\]

where

\[
0 < \frac{t - s}{u - s} < 1
\]

and

\[
\frac{u - t}{u - s} = 1 - \frac{t - s}{u - s}.
\]

If \( \phi(x) \) is convex, then

\[
\phi(t) \leq \frac{t - s}{u - s} \phi(u) + \frac{u - t}{u - s} \phi(s).
\]

One can rewrite this in two different ways to get (1.33). Conversely, one can work backwards, and rewrite either of the inequalities in (1.33) to get (1.37), which gives (1.30) when \( s, t, \) and \( u \) correspond to \( x, y, \) and \( \lambda \) as in (1.34).

**Lemma 1.38** A function \( \phi(x) \) on \( I \) is convex if and only if for each \( t \in I \) there is a real-valued affine function \( A(x) \) on \( \mathbb{R} \) such that \( A(t) = \phi(t) \) and

\[ A(x) \leq \phi(x) \text{ for every } x \in I. \]

To see that this condition is sufficient for \( \phi \) to be convex, let \( x, y, \) and \( \lambda \) be given in the usual way. If \( A \) is an affine function associated to

\[
t = \lambda x + (1 - \lambda) y
\]

as in the statement of the lemma, then

\[
\phi(\lambda x + (1 - \lambda) y) = A(\lambda x + (1 - \lambda) y) = \lambda A(x) + (1 - \lambda) A(y) \leq \lambda \phi(x) + (1 - \lambda) \phi(y).
\]
Conversely, suppose that \( \phi(x) \) is convex, and let \( t \in I \) be given. We would like to choose a real number \( a \) so that
\[
A(x) = \phi(t) + a(x - t)
\]
satisfies \( A(x) \leq \phi(x) \) for all \( x \in I \), which is equivalent to
\[
a(x - t) \leq \phi(x) - \phi(t)
\]
for \( x \in I \). This is trivial when \( x = t \), and otherwise we can rewrite (1.42) as
\[
a \leq \frac{\phi(x) - \phi(t)}{x - t}
\]
when \( x > t \), and as
\[
\frac{\phi(t) - \phi(s)}{t - s} \leq a
\]
when \( x < t \). It follows from Lemma 1.32 that
\[
\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t}
\]
for every \( s, u \in I \) such that \( s < t < u \). Hence
\[
D_l = \sup \left\{ \frac{\phi(t) - \phi(s)}{t - s} : s \in I, s < t \right\}
\]
and
\[
D_r = \inf \left\{ \frac{\phi(u) - \phi(t)}{u - t} : u \in I, u > t \right\}
\]
are well-defined and satisfy
\[
D_l \leq D_r.
\]
To get (1.43) and (1.44), it suffices to choose \( a \in \mathbb{R} \) such that
\[
D_l \leq a \leq D_r.
\]
This completes the proof of Lemma 1.38.

A real-valued function \( \phi(x) \) on \( I \) is said to be **strictly convex** if
\[
\phi(\lambda x + (1 - \lambda) y) < \lambda \phi(x) + (1 - \lambda) \phi(y)
\]
for every \( x, y \in I \) such that \( x \neq y \) and each \( \lambda \in (0, 1) \).

**Lemma 1.51** A real-valued function \( \phi \) on \( I \) is strictly convex if and only if for every point \( t \in I \) there is a real-valued affine function \( A(x) \) on \( \mathbb{R} \) such that \( A(t) = \phi(t) \) and \( A(x) < \phi(x) \) for all \( x \in I \setminus \{t\} \).
This is the analogue of Lemma 1.38 for strictly convex functions, which can be obtained in practically the same manner as before. For the existence of $A$ when $\phi$ is strictly convex, one can start with $A$ as in the previous lemma, and use strict convexity to show that $A(x) \neq \phi(x)$ when $x \neq t$.

The convexity of a real-valued function $\phi$ on $I$ can also be characterized by the property that for each $x, y \in I$ with $x < y$,

$$
(1.52) \quad \phi(t) \leq B(t) \text{ for every } t \in [x, y],
$$

where $B$ is the affine function on the real line which is equal to $\phi$ at $x$ and $y$. Strict convexity corresponds to

$$
(1.53) \quad \phi(t) < B(t) \text{ when } t \in (x, y).
$$

This is easy to check, just using the definitions.

Note that convex functions are automatically continuous. This follows by trapping a convex function on both sides of a point between affine functions with the same value at that point.

**Lemma 1.54** If $\phi$ is a continuous real-valued function on $I$, and if for each $x, y$ in $I$ there is a $\lambda_{x,y} \in (0, 1)$ such that (1.30) holds with $\lambda = \lambda_{x,y}$, then $\phi$ is convex.

This is often stated in the special case where $\lambda_{x,y} = 1/2$ for every $x, y \in I$, in which event one can iterate the condition and pass to a limit to get the desired inequality for arbitrary $\lambda$. Alternatively, for each $x, y \in I$ with $x < y$, let $L(x, y)$ be the set of $\lambda \in [0, 1]$ such that (1.30) holds, which is a closed set when $\phi$ is continuous, and which automatically contains 0 and 1. If $L(x, y) \neq [0, 1]$, then one can get a contradiction under the conditions of the lemma, by considering a maximal open interval in $[0, 1] \setminus L(x, y)$, and showing that it has to contain an element of $L(x, y)$.

### 1.3 Some related inequalities

Suppose that $\phi$ is a convex function on an open interval $I \subseteq \mathbb{R}$, as in the previous section. If $K$ is an interval in $\mathbb{R}$ of positive length and $f$ is an integrable function on $K$ such that $f(x) \in I$ for all $x \in K$, then

$$
(1.55) \quad |K|^{-1} \int_K f(x) \, dx \in I,
$$

and

$$
(1.56) \quad \phi\left(|K|^{-1} \int_K f(x) \, dx\right) \leq |K|^{-1} \int_J \phi(f(x)) \, dx.
$$

This is called **Jensen’s inequality**.
Let us first consider the analogous statement for finite sums. If \( x_1, x_2, \ldots, x_n \) are elements of \( I \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are nonnegative real numbers such that \( \sum_{i=1}^{n} \lambda_i = 1 \), then

\[
\sum_{i=1}^{n} \lambda_i x_i \in I, 
\]

and

\[
\phi\left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i \phi(x_i). 
\]

This is the same as (1.30) when \( n = 2 \), and one can apply (1.30) repeatedly to get the general case. One can also use the characterization of convexity in Lemma 1.38, as in (1.40). If \( f \) is a step function, then (1.56) follows directly from (1.58). In general, one can reduce to the case of finite sums through suitable approximations, or employ Lemma 1.38 in the same way as for sums.

It is well known that \( \phi(t) = |t|^p \) is a convex function on the real line when \( p \) is a real number such that \( p \geq 1 \), and moreover that \( |t|^p \) is strictly convex when \( p > 1 \). In particular,

\[
\left| K \right|^{-1} \int_{K} \left| f(x) \right|^p dx \leq \left| K \right|^{-1} \int_{K} |f(x)|^p dx 
\]

for real-valued functions \( f \) on an interval \( K \) of positive length.

Let \( p, q \) be real numbers such that \( p, q \geq 1 \) and

\[
\frac{1}{p} + \frac{1}{q} = 1. 
\]

In this event we say that \( p \) and \( q \) are conjugate exponents. If \( f, g \) are nonnegative real-valued functions on an interval \( K \), then Hölder’s inequality states that

\[
\int_{K} f(x) g(x) dx \leq \left( \int_{K} f(y)^p dy \right)^{1/p} \left( \int_{K} g(z)^q dz \right)^{1/q}. 
\]

We can also allow \( p \) or \( q \) to be 1 and the other to be \( +\infty \), which is consistent with (1.60). If \( p = 1 \) and \( q = +\infty \), then the substitute for (1.61) is

\[
\int_{K} f(x) g(x) dx \leq \left( \int_{K} f(y) dy \right) \left( \sup_{z \in K} g(z) \right). 
\]

Let us now prove (1.61) when \( p, q > 1 \), beginning with some initial reductions. The inequality is trivial if \( f \) or \( g \) is identically 0, or zero “almost everywhere”, since the left side of (1.61) is then equal to 0. Thus we may suppose that

\[
\left( \int_{K} f(y)^p dy \right)^{1/p} \text{ and } \left( \int_{K} g(z)^q dz \right)^{1/q} 
\]

are nonzero. We may suppose further that these expressions are both equal to 1, because the general case would follow by multiplying \( f \) and \( g \) by positive constants. For any nonnegative real numbers \( s, t \),

\[
st \leq \frac{s^p}{p} + \frac{t^q}{q}. 
\]
This is a version of the geometric-arithmetic mean inequalities, which can be treated as an exercise in calculus, or derived from the convexity of the exponential function. Note that the inequality is strict when \( s^p \neq t^q \). Applying (1.64) to \( s = f(x) \) and \( t = g(x) \) and then integrating in \( x \), we get that

\[
(1.65) \quad \int_K f(x) g(x) \, dx \leq \frac{1}{p} \int_K f(x)^p \, dx + \frac{1}{q} \int_K g(x)^q \, dx.
\]

This implies (1.61) when the integrals of \( f^p \) and \( g^q \) are equal to 1, as desired.

Similarly,

\[
(1.66) \quad \sum_{j=1}^n a_j b_j \leq \left( \sum_{k=1}^n a_k^p \right)^{1/p} \left( \sum_{l=1}^n b_l^q \right)^{1/q}
\]

when \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are nonnegative real numbers and \( p, q \geq 1 \) are conjugate exponents. If \( p = 1 \) and \( q = \infty \), then this should be interpreted as

\[
(1.67) \quad \sum_{j=1}^n a_j b_j \leq \left( \sum_{k=1}^n a_k \right) \left( \max\{b_l : 1 \leq l \leq n\} \right).
\]

Let \( f \) and \( g \) be nonnegative functions on an interval \( K \) again, and let \( p \) be a real number, \( p \geq 1 \). Minkowski’s inequality states that

\[
(1.68) \quad \left( \int_K (f(x) + g(x))^p \, dx \right)^{1/p} \leq \left( \int_K f(x)^p \, dx \right)^{1/p} + \left( \int_K g(x)^p \, dx \right)^{1/p}.
\]

The analogue of (1.68) for \( p = +\infty \) is the elementary inequality

\[
(1.69) \quad \sup_{x \in K} (f(x) + g(x)) \leq \sup_{x \in K} f(x) + \sup_{x \in K} g(x).
\]

Let us suppose that \( 1 < p < +\infty \), since (1.68) is trivial when \( p = 1 \). We begin with

\[
(1.70) \quad \int_K (f(x) + g(x))^p \, dx
\]

\[
= \int_K f(x) (f(x) + g(x))^{p-1} \, dx + \int_K g(x) (f(x) + g(x))^{p-1} \, dx.
\]

If \( q > 1 \) is the conjugate exponent of \( p \), then Hölder’s inequality implies that

\[
(1.71) \quad \int_K f(x) (f(x) + g(x))^{p-1} \, dx
\]

\[
\leq \left( \int_K f(y)^p \, dy \right)^{1/p} \left( \int_K (f(z) + g(z))^{q(p-1)} \, dz \right)^{1/q}
\]

\[
= \left( \int_K f(y)^p \, dy \right)^{1/p} \left( \int_K (f(z) + g(z))^p \, dz \right)^{1-1/p}.
\]
1.3. SOME RELATED INEQUALITIES

There is an analogous estimate for \( \int_K g(x) (f(x) + g(x))^{p-1} \, dx \), which leads to

\[
(1.72) \int_K (f(x) + g(x))^p \, dx 
\leq \left\{ \left( \int_K f(y)^p \, dy \right)^{1/p} + \left( \int_J g(y)^p \, dy \right)^{1/p} \right\} \left( \int_K (f(z) + g(z))^p \, dz \right)^{1-1/p}.
\]

It is easy to derive (1.68) from this.

Minkowski’s inequality for finite sums can be expressed as

\[
(1.73) \left( \sum_{j=1}^n (a_j + b_j)^p \right)^{1/p} \leq \left( \sum_{j=1}^n a_j^p \right)^{1/p} + \left( \sum_{j=1}^n b_j^p \right)^{1/p}
\]
when \( 1 \leq p < \infty \), and

\[
(1.74) \max\{a_j + b_j : 1 \leq j \leq n\} \leq \max\{a_j : 1 \leq j \leq n\} + \max\{b_j : 1 \leq j \leq n\}
\]
when \( p = \infty \), where \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are nonnegative real numbers. These inequalities can be shown in the same way as for integrals. As an alternate approach, fix \( p, 1 < p < \infty \), since the \( p = 1 \) and \( p = \infty \) cases are easy, and suppose for the moment that

\[
(1.75) \left( \sum_{j=1}^n a_j^p \right)^{1/p} = \left( \sum_{j=1}^n b_j^p \right)^{1/p} = 1.
\]
If \( t \) is a real number such that \( 0 \leq t \leq 1 \), then

\[
(1.76) \left( \sum_{j=1}^n (t a_j + (1-t) b_j)^p \right)^{1/p} \leq 1.
\]
To see this, rewrite (1.75) and (1.76) as

\[
(1.77) \sum_{j=1}^n a_j^p = \sum_{j=1}^n b_j^p = 1
\]
and

\[
(1.78) \sum_{j=1}^n (t a_j + (1-t) b_j)^p \leq 1,
\]
respectively. To go from (1.77) to (1.78), it suffices to know that

\[
(1.79) (t a_j + (1-t) b_j)^p \leq t a_j^p + (1-t) b_j^p
\]
for each \( j \), which follows from the convexity of the function \( \phi(x) = x^p, x \geq 0 \). Once one has (1.76) under the assumption (1.75), it is not difficult to derive (1.73) in the general case. Basically, the parameter \( t \) compensates for \( (\sum_{j=1}^n a_j^p)^{1/p} \) and \( (\sum_{j=1}^n b_j^p)^{1/p} \) not being equal.
Fix a positive integer $n$, and suppose that $\{a_j\}_{j=1}^n$ is a finite sequence of nonnegative real numbers. Let $p$ and $q$ be positive real numbers, with $p < q$. Clearly

\begin{equation}
\max_{1 \leq j \leq n} a_j \leq \left( \sum_{j=1}^n a_j^p \right)^{1/p}.
\end{equation}

(1.80)

Moreover,

\begin{equation}
\left( \sum_{j=1}^n a_j^q \right)^{1/q} \leq \left( \sum_{j=1}^n a_j^p \right)^{1/p},
\end{equation}

(1.81)

because

\begin{equation}
\sum_{j=1}^n a_j^q \leq \left( \max_{1 \leq k \leq n} a_k \right)^{q-p} \left( \sum_{l=1}^n a_l^p \right) \leq \left( \sum_{r=1}^n a_r^p \right)^{1+(q-p)/p}
\end{equation}

(1.82)

and $1 + (q-p)/p = q/p$.

In the other direction,

\begin{equation}
\left( \sum_{j=1}^n a_j^p \right)^{1/p} \leq n^{1/p} \max_{1 \leq j \leq n} a_j,
\end{equation}

(1.83)

and

\begin{equation}
\left( \sum_{j=1}^n a_j^p \right)^{1/p} \leq n^{(1/p)-(1/q)} \left( \sum_{j=1}^n a_j^q \right)^{1/q}.
\end{equation}

(1.84)

The first inequality is trivial, and the second can be rewritten as

\begin{equation}
\left( \frac{1}{n} \sum_{j=1}^n a_j^q \right)^{q/p} \leq \frac{1}{n} \sum_{j=1}^n a_j^q,
\end{equation}

(1.85)

which is an instance of (1.58) applied to $\phi(x) = x^{q/p}$.

If $0 < p < 1$ and $u$, $v$ are nonnegative real numbers, then

\begin{equation}
(u + v)^p \leq u^p + v^p.
\end{equation}

(1.86)

This is a special case of (1.81), with $q = 1$ and $n = 2$. This leads to

\begin{equation}
\sum_{j=1}^n (b_j + c_j)^p \leq \sum_{j=1}^n b_j^p + \sum_{j=1}^n c_j^p,
\end{equation}

(1.87)

for nonnegative real numbers $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$, and

\begin{equation}
\int_K (f(x) + g(x))^p \, dx \leq \int_K f(x)^p \, dx + \int_K g(x)^p \, dx
\end{equation}

(1.88)

for nonnegative functions $f$, $g$ on an interval $K$.
1.3. SOME RELATED INEQUALITIES

Suppose that $0 < p, q, r, < \infty$ and

\[(1.89) \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.\]

If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are nonnegative real numbers, then

\[(1.90) \quad \left( \sum_{j=1}^{n} (a_j b_j)^r \right)^{1/r} \leq \left( \sum_{j=1}^{n} a_j^p \right)^{1/p} \left( \sum_{j=1}^{n} b_j^q \right)^{1/q}.
\]

This follows from Hölder’s inequality. Similarly, for nonnegative functions $f, g$ on an interval $K$,

\[(1.91) \quad \left( \int_{K} (f(x) g(x))^r \, dx \right)^{1/r} \leq \left( \int_{K} f(x)^p \, dx \right)^{1/p} \left( \int_{K} g(x)^q \, dx \right)^{1/q}.
\]

One can also allow for infinite exponents in the usual way.
Chapter 2
Norms on vector spaces

In this book, all vector spaces use the real or complex numbers as their underlying scalar field. We may sometimes wish to restrict ourselves to one or the other, but frequently both are fine. Let us make the standing assumption that all vector spaces are finite-dimensional in this and the next two chapters.

2.1 Definitions and examples

Let \( V \) be a real or complex vector space. By a norm on \( V \) we mean a nonnegative real-valued function \( \| \cdot \| \) on \( V \) such that \( \| v \| = 0 \) if and only if \( v \) is the zero vector in \( V \),

\[
(2.1) \quad \| t v \| = |t| \| v \|
\]

for every \( v \in V \) and \( t \in \mathbb{R} \) or \( \mathbb{C} \), as appropriate, and

\[
(2.2) \quad \| v + w \| \leq \| v \| + \| w \|
\]

for every \( v, w \in V \). As a basic class of examples, let \( V \) be \( \mathbb{R}^n \) or \( 
\mathbb{C}^n \), and consider

\[
(2.3) \quad \| v \|_p = \left( \sum_{j=1}^{n} |v_j|^p \right)^{1/p}
\]

when \( 1 \leq p < \infty \), and

\[
(2.4) \quad \| v \|_\infty = \max_{1 \leq j \leq n} |v_j|.
\]

The triangle inequality for these norms follows from (1.73) and (1.74).

If

\[
(2.5) \quad B_1 = \{ v \in V : \| v \| \leq 1 \}
\]

is the closed unit ball corresponding to a norm \( \| v \| \) on \( V \), then it is easy to see that \( B_1 \) is a convex set in \( V \). This means that

\[
(2.6) \quad tv + (1 - t)w \in B_1
\]
whenever \( v, w \in B_1 \) and \( t \) is a real number such that \( 0 \leq t \leq 1 \), which follows from (2.1) and (2.2). Conversely, if \( \|v\| \) is a nonnegative real-valued function on \( V \) such that \( \|v\| = 0 \) if and only if \( v = 0 \), \( \|v\| \) satisfies (2.1), and the unit ball \( B_1 \) is convex, then one can show \( \|v\| \) also satisfies (2.2), and hence that \( \|v\| \) is a norm on \( V \). In effect, this was mentioned already in Section 1.3, as an alternate approach to Minkowski’s inequality for finite sums.

If \( V \) is a vector space, and \( \| \cdot \| \) is a norm on \( V \), then
\[
\|v\| - \|w\| \leq \|v - w\| \tag{2.7}
\]
for every \( v, w \in V \). This follows from
\[
\|v\| \leq \|w\| + \|v - w\|, \tag{2.8}
\]
and the analogous inequality with the roles of \( v \) and \( w \) interchanged. Suppose that \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), which is not a real restriction, since every real or complex vector space of positive finite dimension is isomorphic to one of these. Let \( |x| \) denote the standard Euclidean norm on \( \mathbb{R}^n \) or \( \mathbb{C}^n \), which is the same as the norm \( \|x\|_2 \) in (2.3). One can check that there is a positive constant \( C \) such that
\[
\|v\| \leq C \|v\| \tag{2.9}
\]
for every \( v \in V \), by expanding \( v \) in the standard basis for \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \), and using the triangle inequality and homogeneity of \( \|v\| \). This and (2.7) imply that \( \|v\| \) is a continuous real-valued function on \( V \), with respect to the standard Euclidean metric and topology. Thus the minimum \( b > 0 \) of \( \|v\| \) among the vectors \( v \in V \) with \( |v| = 1 \) is attained, by well-known results about continuity and compactness, and \( b > 0 \). It follows that
\[
b \|v\| \leq \|v\| \tag{2.10}
\]
for every \( v \in V \), because of the homogeneity property of the norms \( \|v\| \) and \( |v| \).

### 2.2 Dual spaces and norms

Let \( V \) be a vector space, real or complex. By a linear functional on \( V \) we mean a linear mapping from \( V \) into the field of scalars, i.e., the real or complex numbers, as appropriate. The dual of \( V \) is the space of linear functionals on \( V \), which is a vector space over the same field of scalars as \( V \), with respect to pointwise addition and scalar multiplication. The dual of \( V \) is denoted \( V^* \), and it is well known that \( V^* \) is also finite-dimensional when \( V \) is, with the same dimension as \( V \).

If \( \| \cdot \| \) is a norm on \( V \), then the corresponding dual norm \( \| \cdot \|^* \) on \( V^* \) is defined as follows. If \( \lambda \) is a linear functional on \( V \), then
\[
\|\lambda\|^* = \sup\{\|\lambda(v)\| : v \in V, \|v\| \leq 1\}. \tag{2.11}
\]
Equivalently,
\begin{equation}
|\lambda(v)| \leq \|\lambda\|^* \|v\|
\end{equation}
for every \( v \in V \), and \( \|\lambda\|^* \) is the smallest nonnegative real number with this property. It is not difficult to verify that \( \| \cdot \|^* \) defines a norm on \( V^* \). In particular, the finiteness of \( \|\lambda\|^* \) can be derived from the remarks at the end of the previous section.

For example, suppose that \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \). We can identify \( V^* \) with \( \mathbb{R}^n \) or \( \mathbb{C}^n \), respectively, by associating to each \( w \) in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) the linear functional \( \lambda_w \) on \( V \) given by
\begin{equation}
\lambda_w(v) = \sum_{j=1}^{n} w_j v_j.
\end{equation}

Let \( 1 \leq p, q \leq \infty \) be conjugate exponents, which is to say that \( \frac{1}{p} + \frac{1}{q} = 1 \), and let us check that \( \|\lambda_w\|^* = \|w\|_q \) is the dual norm for \( \|v\| = \|v\|_p \).

First, we have that
\begin{equation}
|\lambda_w(v)| \leq \|w\|_q \|v\|_p
\end{equation}
for all \( w \) and \( v \), by Hölder’s inequality. To show that \( \|\lambda_w\|^* = \|w\|_q \), we would like to check that for each \( w \) there is a nonzero \( v \) such that
\begin{equation}
|\lambda_w(v)| = \|w\|_q \|v\|_p.
\end{equation}

Let \( w \) be given. We may as well assume that \( w \neq 0 \), since otherwise any \( v \) would do. Let us also assume for the moment that \( p > 1 \), so that \( q < \infty \). Under these conditions, we can define \( v \) by
\begin{equation}
v_j = \overline{w_j} |w_j|^{q-2}
\end{equation}
when \( w_j \neq 0 \), and by \( v_j = 0 \) when \( w_j = 0 \). Here \( \overline{w_j} \) is the complex conjugate of \( w_j \), which is not needed when we are working with real numbers instead of complex numbers. With this choice of \( v \), we have that
\begin{equation}
\lambda_w(v) = \sum_{j=1}^{n} |w_j|^q = \|w\|_q^q.
\end{equation}

It remains to check that
\begin{equation}
\|v\|_p = \|w\|_q^{q-1}.
\end{equation}

If \( q = 1 \), then \( p = \infty \), and (2.18) reduces to
\begin{equation}
\max_{1 \leq j \leq n} |v_j| = 1.
\end{equation}

In this case \( |v_j| = 1 \) for each \( j \) such that \( v_j \neq 0 \), which holds for at least one \( j \) because \( w \neq 0 \). Thus we get (2.19). If \( q > 1 \), then \( |v_j| = |w_j|^{q-1} \) for each \( j \), and one can verify (2.18) using the identity \( p(q - 1) = q \).
Finally, if \( p = 1 \), and hence \( q = \infty \), then choose \( l, 1 \leq l \leq n \), such that
\[
|w_l| = \max_{1 \leq j \leq n} |w_j| = \|w\|_\infty.
\]

Define \( v \) by
\[
v_l = \frac{w_l}{|w_l|} - 1
\]
and \( v_j = 0 \) when \( j \neq l \). This leads to
\[
\lambda_{w}(v) = |w_l| = \|w\|_\infty
\]
and \( \|v\|_1 = 1 \), as desired.

### 2.3 Second duals

Let \( V \) be a vector space, and \( V^* \) its dual space. The dual of \( V^* \) is denoted \( V^{**} \).

There is a canonical mapping from \( V \) into \( V^{**} \), defined as follows. Let \( v \in V \) be given. For each \( \lambda \in V^* \), we get a scalar by taking \( \lambda(v) \). The mapping \( \lambda \mapsto \lambda(v) \) is a linear functional on \( V^* \), and hence an element of \( V^{**} \). Since we can do this for every \( v \in V \), we get a mapping from \( V \) into \( V^{**} \). One can check that this mapping is linear and an isomorphism from \( V \) onto \( V^{**} \). For instance, everything can be expressed in terms of a basis for \( V \).

Now suppose that we have a norm \( \| \cdot \| \) on \( V \). This leads to a dual norm \( \| \cdot \|^* \) on \( V^* \), as in the previous section, and a double dual norm \( \| \cdot \|^{**} \) on \( V^{**} \). Using the canonical isomorphism between \( V \) and \( V^{**} \) just described, we can think of \( \| \cdot \|^{**} \) as defining a norm on \( V \). We would like to show that
\[
\|v\|^{**} = \|v\|
\]
for every \( v \in V \).

Note that this holds for the \( p \)-norms \( \| \cdot \|_p \) on \( \mathbb{R}^n \) and \( \mathbb{C}^n \), by the analysis of their duals in the preceding section.

Let \( v \in V \) be given. By definition of the dual norm \( \| \lambda \|^* \), we have that
\[
|\lambda(v)| \leq \|\lambda\|^* \|v\|
\]
for every \( \lambda \in V^* \), and hence that
\[
\|v\|^{**} \leq \|v\|.
\]

It remains to show that the opposite inequality holds, which is trivial when \( v = 0 \). Thus it suffices to show that there is a nonzero \( \lambda_0 \in V^* \) such that
\[
\lambda_0(v) = \|\lambda_0\|^* \|v\|
\]
when \( v \neq 0 \).

**Theorem 2.26** Let \( V \) be a real or complex vector space, and let \( \| \cdot \| \) be a norm on \( V \). If \( W \) is a linear subspace of \( V \) and \( \mu \) is a linear functional on \( W \) such that
\[
|\mu(w)| \leq \|w\| \quad \text{for every } w \in W,
\]
then there is a linear functional \( \tilde{\mu} \) on \( V \) such that \( \tilde{\mu} = \mu \) on \( W \) and \( \|\tilde{\mu}\|^* \leq 1 \).
The existence of a nonzero $\lambda_0 \in V^*$ satisfying (2.25) follows easily from this, by first defining $\lambda_0$ on the span of $v$ so that $\lambda_0(v) = \|v\|$, and then extending to a linear functional on $V$ with norm 1.

To prove the theorem, let us begin by assuming that $V$ is a real vector space. Afterwards, we shall discuss the complex case.

Let $W$ and $\mu$ be given as in the theorem, and let $\dim Z$ be the dimension of a linear subspace $Z$ of $V$. For each integer $j$ such that $\dim W \leq j \leq \dim V$, we would like to show that there is a linear subspace $W_j$ of $V$ and a linear functional $\mu_j$ on $W_j$ such that $W \subseteq W_j$, $\dim W_j = j$, $\mu_j = \mu$ on $W$, and

$$(2.28) \quad |\mu_j(w)| \leq \|w\| \quad \text{for every } w \in W_j.$$  

If we can do this with $j = \dim V$, then $W_j = V$, and this would give a linear functional on $V$ with the required properties.

Let us show that we can do this by induction. For the base case $j = \dim W$, we simply take $W_j = W$ and $\mu_j = \mu$. Suppose that $\dim W \leq j < \dim V$, and that $W_j$, $\mu_j$ are as above. We would like to choose $W_{j+1}$ and $\mu_{j+1}$ with the analogous properties for $j+1$ instead of $j$. To be more precise, we shall choose them in such a way that $W_j \subseteq W_{j+1}$ and $\mu_{j+1}$ is an extension of $\mu_j$ to $W_{j+1}$.

Under these conditions, $W_j$ is a proper subspace of $V$, and hence there is a $z \in V \setminus W_j$. Fix any such $z$, and take $W_{j+1}$ to be the span of $W_j$ and $z$. Thus

$$(2.29) \quad \dim W_{j+1} = \dim W_j + 1 = j + 1.$$  

Let $\alpha$ be a real number, to be chosen later in the argument. If we set $\mu_{j+1}(z)$ equal to $\alpha$, then $\mu_{j+1}$ is determined on all of $W_{j+1}$ by linearity and the condition that $\mu_{j+1}$ be an extension of $\mu_j$. Specifically, each $w \in W_{j+1}$ can be expressed in a unique way as $x + tz$ for some $x \in W_j$ and $t \in \mathbb{R}$, and

$$(2.30) \quad \mu_{j+1}(w) = \mu_j(x) + t \alpha.$$  

It remains to choose $\alpha$ so that $\mu_{j+1}$ satisfies the analogue of (2.28) for $j+1$, which is to say that

$$(2.31) \quad |\mu_{j+1}(w)| \leq \|w\| \quad \text{for every } w \in W_{j+1}.$$  

Equivalently, we would like to choose $\alpha$ so that

$$(2.32) \quad |\mu_j(x) + t \alpha| \leq \|x + tz\| \quad \text{for every } x \in W_j \text{ and } t \in \mathbb{R}.$$  

It suffices to show that

$$(2.33) \quad |\mu_j(x) + \alpha| \leq \|x + z\| \quad \text{for every } x \in W_j,$$

since the case where $t = 0$ in (2.32) corresponds exactly to our induction hypothesis (2.28), and one can eliminate $t \neq 0$ using homogeneity. Let us rewrite (2.33) as

$$(2.34) \quad -\mu_j(x) - \|x + z\| \leq \alpha \leq -\mu_j(x) + \|x + z\| \quad \text{for every } x \in W_j,$$
It follows from (2.28) that
\[ \mu_j(x - y) \leq \|x - y\| \quad \text{for every } x, y \in W_j. \]
Using the triangle inequality, we get that
\[ \mu_j(x - y) \leq \|x + z\| + \|y + z\| \quad \text{for every } x, y \in W_j, \]
and hence
\[ -\mu_j(y) - \|y + z\| \leq -\mu_j(x) + \|x + z\| \quad \text{for every } x, y \in W_j. \]
If \( A \) is the supremum of the left side of this inequality over \( y \in W_j \), and \( B \) is the infimum of the right side of this inequality over \( x \in W_j \), then \( A \leq B \), and any \( \alpha \in \mathbb{R} \) such that \( A \leq \alpha \leq B \) satisfies (2.34). This finishes the induction argument, and the proof of Theorem 2.26 when \( V \) is a real vector space.

Consider now the case of a complex vector space \( V \). The real part of a linear functional on \( V \) is also a linear functional on \( V \) as a real vector space, i.e., forgetting about multiplication by \( i \). Conversely, if \( \phi \) is a real-valued function on \( V \) which is linear with respect to vector addition and scalar multiplication by real numbers, then there is a unique complex linear functional \( \psi \) on \( V \) whose real part is \( \phi \), given by
\[ \psi(v) = \phi(v) - i \phi(iv). \]
For any complex number \( \zeta \),
\[ |\zeta| = \sup \{ \text{Re}(a \zeta) : a \in \mathbb{C}, |a| \leq 1 \}. \]
If \( V \) is equipped with a norm \( \|v\| \), then the norm of a complex linear functional \( \lambda \) on \( V \) can be expressed as
\[ \|\lambda\|^* = \sup \{ \text{Re}(a \lambda(v)) : v \in V, a \in \mathbb{C}, \|v\| \leq 1, |a| \leq 1 \}. \]
By linearity, \( \lambda(a v) = a \lambda(v) \), which implies that
\[ \|\lambda\|^* = \sup \{ \text{Re} \lambda(v) : v \in V, \|v\| \leq 1 \}. \]
Thus the norm of a complex linear functional on \( V \) is the same as the norm of its real part, as a linear functional on the real version of \( V \). To prove the extension theorem in the complex case, one can apply the extension theorem in the real case to the real part of the given complex linear functional on a complex linear subspace, and then complexify the real extension to get a complex linear extension with the same estimate for the norm.

### 2.4 Linear transformations

Let \( V_1 \) and \( V_2 \) be vector spaces, both real or both complex, equipped with norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), respectively. Here the subscripts are merely labels to distinguish
these norms, rather than referring to the $p$-norms described in Section 2.1. The corresponding operator norm $\|T\|_\text{op}$ of a linear transformation $T$ from $V_1$ into $V_2$ is defined by

\begin{equation}
\|T\|_\text{op} = \sup\{\|T(v)\|_2 : v \in V_1, \|v\|_1 \leq 1\}.
\end{equation}

Equivalently,

\begin{equation}
\|T(v)\|_2 \leq \|T\|_\text{op} \|v\|_1 \quad \text{for every } v \in V_1,
\end{equation}

and $\|T\|_\text{op}$ is the smallest nonnegative real number with this property. The finiteness of $\|T\|_\text{op}$ is easy to check using the remarks at the end of Section 2.1.

The space $\mathcal{L}(V_1, V_2)$ of all linear transformations from $V_1$ into $V_2$ is a vector space in a natural way, using pointwise addition and scalar multiplication of linear transformations, with the same scalar field as for $V_1$ and $V_2$. One can verify that the operator norm $\| \cdot \|_\text{op}$ on $\mathcal{L}(V_1, V_2)$ is a norm on this vector space. Note that the dual $V^*$ of a vector space $V$ is the same as $\mathcal{L}(V, \mathbb{R})$ or $\mathcal{L}(V, \mathbb{C})$, as appropriate, and the dual norm on $V^*$ associated to a norm on $V$ is the same as the operator norm with respect to the standard norm on $\mathbb{R}$ or $\mathbb{C}$.

Suppose that $V_3$ is another vector space, with the same field of scalars as $V_1$ and $V_2$, and equipped with a norm $\| \cdot \|_3$. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear mappings, then the composition $T_2 \circ T_1$ is the linear mapping from $V_1$ to $V_3$ given by

\begin{equation}
(T_2 \circ T_1)(v) = T_2(T_1(v)).
\end{equation}

It is easy to see that the operator norm of $T_2 \circ T_1$ is less than or equal to the product of the operator norms of $T_1$ and $T_2$ with respect to the given norms on $V_1$, $V_2$, and $V_3$.

Let $V_1$ and $V_2$ be vector spaces, both real or both complex, and let $T$ be a linear transformation from $V_1$ into $V_2$. There is a canonical dual linear transformation $T^* : V_2^* \to V_1^*$ corresponding to $T$, defined by

\begin{equation}
T^*(\mu) = \mu \circ T \quad \text{for every } \mu \in V_2^*.
\end{equation}

In other words, if $\mu$ is a linear functional on $V_2$, then $\mu \circ T$ is a linear functional on $V_1$, and $T^*(\mu)$ is this linear functional. If $R, T : V_1 \to V_2$ are linear mappings and $a, b$ are scalars, then

\begin{equation}
(a R + b T)^* = a R^* + b T^*.
\end{equation}

If $V_3$ is another vector space with the same field of scalars as $V_1$ and $V_2$, and if $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$ are linear mappings, then

\begin{equation}
(T_2 \circ T_1)^* = T_1^* \circ T_2^*.
\end{equation}

If $T$ is a linear mapping from $V_1$ to $V_2$, then we can pass to the second duals to get a linear transformation $T^{**} : V_1^{**} \to V_2^{**}$. As in Section 2.3, there are canonical isomorphisms between $V_1$ and $V_1^{**}$, and between $V_2$ and $V_2^{**}$, which allow one to identify $T^{**}$ with a linear mapping from $V_1$ to $V_2$. It is easy to see that this mapping is the same as $T$. 

CHAPTER 2. NORMS ON VECTOR SPACES
2.5. SOME SPECIAL CASES

The identity transformation \( I = I_V \) on a vector space \( V \) is the mapping that takes each \( v \in V \) to itself, and the dual of \( I_V \) is equal to the identity mapping \( I_{V^*} \) on \( V^* \). A one-to-one linear transformation \( T \) from \( V_1 \) onto \( V_2 \) is said to be invertible, which implies that there is a linear transformation \( T^{-1} : V_2 \to V_1 \) such that
\[
T^{-1} \circ T = I_{V_1} \quad \text{and} \quad T \circ T^{-1} = I_{V_2}.
\]

One can check that \( T : V_1 \to V_2 \) is invertible if and only if \( T^* : V_2^* \to V_1^* \) is invertible, in which event
\[
(T^{-1})^* = (T^*)^{-1}.
\]

If \( T_1 : V_1 \to V_2 \) and \( T_2 : V_2 \to V_3 \) are both invertible, then their composition \( T_2 \circ T_1 : V_1 \to V_3 \) is invertible too, with
\[
(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}.
\]

Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be norms on \( V_1 \) and \( V_2 \) again, and let \( \| \cdot \|_1^* \) and \( \| \cdot \|_2^* \) be the corresponding dual norms on \( V_1^* \) and \( V_2^* \). We also have the associated operator norm \( \| \cdot \|_{op} \) on \( L(V_1, V_2) \), and the operator norm \( \| \cdot \|_{op^*} \) on \( L(V_2^*, V_1^*) \) determined by the dual norms on \( V_1^* \) and \( V_2^* \). It is easy to see that
\[
\|T^*\|_{op^*} \leq \|T\|_{op}
\]
for each linear mapping \( T : V_1 \to V_2 \), directly from the definitions. Using linear functionals as in (2.25), one can show that the opposite inequality holds, so that
\[
\|T\|_{op} = \|T^*\|_{op^*}.
\]

Alternatively, one can get the opposite inequality by applying (2.51) to \( T^* \) instead of \( T \) and identifying \( T^{**} \) with \( T \).

2.5 Some special cases

Let \( V \) be \( \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), and let \( \| \cdot \|_p \) be the norm on \( V \) described in Section 2.1 for some \( p, 1 \leq p \leq \infty \). If \( T \) is a linear mapping from \( V \) into \( V \), then we can express \( T \) in terms of an \( n \times n \) matrix \( (a_{j,k}) \) of real or complex numbers, as appropriate, through the formula
\[
(T(v))_j = \sum_{k=1}^n a_{j,k} v_k.
\]

Here \( v_k \) is the \( k \)th component of \( v \in V \), \( (T(v))_j \) is the \( j \)th component of \( T(v) \), and conversely any \( n \times n \) matrix \( (a_{j,k}) \) of real or complex numbers determines such a linear transformation \( T \). Let us write \( \|T\|_{op,pp} \) for the operator norm of \( T \) with respect to the norm \( \| \cdot \|_p \), used on \( V \) both as the domain and range of \( T \). These operator norms can be given explicitly when \( p = 1, \infty \), by
\[
\|T\|_{op,11} = \max_{1 \leq k \leq n} \sum_{j=1}^n |a_{j,k}|
\]
\[ (2.55) \quad \|T\|_{op,\infty} = \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{j,k}|. \]

To see this, let \( e_1, \ldots, e_n \) denote the standard basis vectors of \( V \), so that the \( k \)th coordinate of \( e_k \) is equal to 1 and the other coordinates are equal to 0. The right side of (2.54) is the same as

\[ (2.56) \quad \max_{1 \leq k \leq n} \|T(e_k)\|_1. \]

This is obviously less than or equal to \( \|T\|_{op,11} \), by definition of the operator norm. The opposite inequality can be derived by expressing any \( v \in V \) as a linear combination of the \( e_l \)'s and estimating \( \|T(v)\|_1 \) in terms of the \( \|T(e_l)\|_1 \)'s.

Similarly, for \( p = \infty \), we use the fact that

\[ (2.57) \quad \|T(w)\|_\infty = \max_{1 \leq j \leq n} |(T(w))_j| \]

for every \( w \in V \), by the definition of the \( \| \cdot \|_\infty \) norm. Clearly

\[ (2.58) \quad |(T(w))_j| \leq \sum_{k=1}^{n} |a_{j,k}| \]

when \( w \in V \) and \( \|w\|_\infty \leq 1 \), so that

\[ (2.59) \quad \|T\|_{op,\infty} \leq \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{j,k}|. \]

To get the opposite inequality, one can observe that for each \( j \) there is a \( w \in V \) such that \( \|w\|_\infty = 1 \) and equality holds in (2.58).

If \((a_{j,k})\) happens to be a diagonal matrix, so that \( a_{j,k} = 0 \) when \( j \neq k \), then \( \|T\|_{op,pp} \) is equal to the maximum of \( |a_{j,j}|, 1 \leq j \leq n \), for every \( p \). Otherwise, it may not be so easy to compute \( \|T\|_{op,pp} \) when \( 1 < p < \infty \). A famous theorem of Schur states that

\[ (2.60) \quad \|T\|_{op,pp} \leq \|T\|_{op,11}^{1/p} \|T\|_{op,\infty}^{1-1/p}. \]

To show this, fix \( p \in (1, \infty) \), and observe that

\[ (2.61) \quad |(T(v))_j|^p \leq \left( \sum_{k=1}^{n} |a_{j,k}| \right)^{p-1} \sum_{l=1}^{n} |a_{j,l}| |v_l|^p \]

\[ \leq \|T\|_{op,\infty}^{p-1} \sum_{l=1}^{n} |a_{j,l}| |v_l|^p. \]

for each \( j = 1, \ldots, n \) and \( v \in V \). This uses Hölder’s inequality or simply the convexity of \( \phi(r) = |r|^p \) for the first inequality, and (2.55) for the second.
Therefore

\[
\sum_{j=1}^{n} |(T(v))_j|^p \leq \|T\|_{op, \infty}^{p-1} \sum_{j=1}^{n} \sum_{l=1}^{n} |a_{j,l}| |v_l|^p
\]

\[
\leq \|T\|_{op, \infty}^{p-1} \|T\|_{op, 1} \sum_{l=1}^{n} |v_l|^p.
\]

Schur’s theorem follows by taking the \(p\)th root of both sides of this inequality.

## 2.6 Inner product spaces

Let \(V\) be a real or complex vector space. An **inner product** on \(V\) is a scalar-valued function \(\langle \cdot, \cdot \rangle\) on \(V \times V\) with the following properties: (1) for each \(w \in V\), \(v \mapsto \langle v, w \rangle\) is a linear functional on \(V\); (2) if \(V\) is a real vector space, then

\[
\langle w, v \rangle = \langle v, w \rangle \quad \text{for every } v, w \in V,
\]

and if \(V\) is a complex vector space, then

\[
\langle w, v \rangle = \overline{\langle v, w \rangle} \quad \text{for every } v, w \in V;
\]

(3) the inner product is positive definite, in the sense that \(\langle v, v \rangle\) is a positive real number for every \(v \in V\) such that \(v \neq 0\). Note that \(\langle v, w \rangle = 0\) whenever \(v = 0\) or \(w = 0\), and that \(\langle v, v \rangle \in \mathbb{R}\) for every \(v \in V\) even when \(V\) is complex.

A vector space with an inner product is called an **inner product space**. If \((V, \langle \cdot, \cdot \rangle)\) is an inner product space, then we put

\[
\|v\| = \langle v, v \rangle^{1/2}
\]

for every \(v \in V\). The **Cauchy–Schwarz inequality** says that

\[
|\langle v, w \rangle| \leq \|v\| \|w\|
\]

for every \(v, w \in V\), and it can be proved using the fact that

\[
\langle v + a w, v + a w \rangle \geq 0
\]

for all scalars \(a\). One can show that \(\| \cdot \|\) satisfies the triangle inequality, and is therefore a norm on \(V\), by expanding \(\|v + w\|^2\) as a sum of inner products and applying the Cauchy–Schwarz inequality. For each positive integer \(n\), the standard inner products on \(\mathbb{R}^n\) and \(\mathbb{C}^n\) are given by

\[
\langle v, w \rangle = \sum_{j=1}^{n} v_j w_j
\]

on \(\mathbb{R}^n\) and

\[
\langle v, w \rangle = \sum_{j=1}^{n} v_j \overline{w_j}
\]

on \(\mathbb{C}^n\).
on \( \mathbb{C}^n \), and the associated norms are the standard Euclidean norms on \( \mathbb{R}^n, \mathbb{C}^n \).

Let \((V, \langle \cdot, \cdot \rangle)\) be a real or complex inner product space. A pair of vectors \(v, w \in V\) are said to be **orthogonal** if
\[
\langle v, w \rangle = 0, \tag{2.70}
\]
which may be expressed symbolically by \(v \perp w\). This condition is symmetric in \(v\) and \(w\), and implies that
\[
\|v + w\|^2 = \|v\|^2 + \|w\|^2. \tag{2.71}
\]

A collection \(v_1, \ldots, v_n\) of vectors in \(V\) is said to be **orthonormal** if \(v_j \perp v_l\) when \(j \neq l\) and \(\|v_j\| = 1\) for each \(j\). In this case, if \(c_1, \ldots, c_n\) are scalars and
\[
w = c_1 v_1 + \cdots + c_n v_n, \tag{2.72}
\]
then
\[
c_j = \langle w, v_j \rangle \tag{2.73}
\]
for each \(j\), and
\[
\|w\|^2 = \sum_{j=1}^{n} |c_j|^2. \tag{2.74}
\]

An **orthonormal basis** for \(V\) is an orthonormal collection of vectors in \(V\) whose linear span is equal to \(V\). For example, the standard bases in \(\mathbb{R}^n\) and \(\mathbb{C}^n\) are orthonormal with respect to the standard inner products.

Suppose that \(v_1, \ldots, v_n\) are orthonormal vectors in \(V\), and define a linear transformation \(P : V \to V\) by
\[
P(w) = \sum_{j=1}^{n} \langle w, v_j \rangle v_j. \tag{2.75}
\]

Observe that
\[
P(w) = w \tag{2.76}
\]
when \(w \in V\) is a linear combination of \(v_1, \ldots, v_n\). If \(w\) is any vector in \(V\), then
\[
\langle P(w), v_j \rangle = \langle w, v_j \rangle \tag{2.77}
\]
for each \(j\), and hence \((w - P(w)) \perp v_j\) for each \(j\). Thus \((w - P(w)) \perp w\), and
\[
\|w\|^2 = \|P(w)\|^2 + \|w - P(w)\|^2 = \sum_{j=1}^{n} |\langle w, v_j \rangle|^2 + \|w - P(w)\|^2. \tag{2.78}
\]

Suppose that \(w\) is an element of \(V\) which is not in the span of \(v_1, \ldots, v_n\). This implies that \(w - P(w) \neq 0\), and
\[
u = \frac{w - P(w)}{\|w - P(w)\|}. \tag{2.79}
\]
2.6. INNER PRODUCT SPACES

satisfies \( \|u\| = 1 \) and \( u \perp v_j \) for each \( j \). It follows that \( v_1, \ldots, v_n, u \) is an orthonormal collection of vectors in \( V \) whose linear span is the same as the span of \( v_1, \ldots, v_n, w \). By repeating the process, we can extend \( v_1, \ldots, v_n \) to an orthonormal basis of \( V \). In particular, every finite-dimensional inner product space has an orthonormal basis.

The orthogonal complement \( W^\perp \) of a linear subspace \( W \) of \( V \) is defined by

\[
W^\perp = \{ v \in V : \langle v, w \rangle = 0 \text{ for every } w \in W \},
\]

and is also a linear subspace of \( V \). Note that

\[
W \cap W^\perp = \{ 0 \},
\]

since \( v \perp v \) if and only if \( v = 0 \). Let \( v_1, \ldots, v_n \) be an orthonormal basis for \( W \), and let \( P : V \to V \) be defined as in (2.75). Thus

\[
P(v) \in W \quad \text{and} \quad v - P(v) \in W^\perp
\]

for every \( v \in V \). If \( v \in V \) and \( x, y \in W \) satisfy \( v - x, v - y \in W^\perp \), then \( x - y \in W \cap W^\perp \), and hence \( x - y = 0 \). Therefore \( P(v) \) is uniquely determined by (2.82), and does not depend on the choice of orthonormal basis \( v_1, \ldots, v_n \) for \( W \). This linear transformation is called the orthogonal projection of \( V \) onto \( W \), and may be denoted \( P_W \).

Note that

\[
\lambda_w(v) = \langle v, w \rangle
\]

defines a linear functional on \( V \) for each \( w \in V \), and that

\[
|\lambda_w(v)| \leq \|v\| \|w\|
\]

for every \( v \in V \), by the Cauchy–Schwarz inequality. Thus the dual norm of \( \lambda_w \) corresponding to the norm \( \| \cdot \| \) on \( V \) is less than or equal to \( \|w\| \). In fact, the dual norm of \( \lambda_w \) is equal to \( \|w\| \), because \( \lambda_w(w) = \|w\|^2 \).

Conversely, every linear functional \( \lambda \) on \( V \) can be represented as \( \lambda_w \) for some \( w \in V \). To see this, let \( v_1, \ldots, v_n \) be an orthonormal basis of \( V \). If

\[
w = \sum_{j=1}^n \lambda(v_j) v_j,
\]

then

\[
\langle v_j, w \rangle = \lambda(v_j)
\]

for each \( j \), and hence \( \lambda(v) = \lambda_w(v) \) for every \( v \in V \). It is easy to reverse this argument to show that \( w \) is uniquely determined by \( \lambda \).

**Remark 2.87** If \((V, \langle \cdot, \cdot \rangle)\) is a real or complex inner product space, and if \( \| \cdot \| \) is the norm associated to the inner product, then there is a simple formula for
the inner product in terms of the norm, through polarization. Specifically, the polarization identities are

\[(2.88) \quad 4 \langle v, w \rangle = \|v + w\|^2 - \|v - w\|^2\]

in the real case, and

\[(2.89) \quad 4 \langle v, w \rangle = \|v + w\|^2 - \|v - w\|^2 + i \|v + iw\|^2 - i \|v - iw\|^2\]

in the complex case. The norm also satisfies the parallelogram law

\[(2.90) \quad \|v + w\|^2 + \|v - w\|^2 = 2 (\|v\|^2 + \|w\|^2) \quad \text{for every } v, w \in V.\]

Conversely, if \(V\) is a vector space and \(\| \cdot \|\) is a norm on \(V\) which satisfies the parallelogram law, then there is an inner product on \(V\) for which \(\| \cdot \|\) is the associated norm. This is a well-known fact, which can be established using (2.88) or (2.89), as appropriate, to define \(\langle v, w \rangle\), and using the parallelogram law to show that this is an inner product.

## 2.7 Some more special cases

Let \(V_1, V_2\) be vector spaces, both real or both complex, equipped with norms \(\| \cdot \|_{V_1}, \| \cdot \|_{V_2}\), respectively. Suppose first that \(V_1\) is \(\mathbb{R}^n\) or \(\mathbb{C}^n\) for some positive integer \(n\), and that \(\| \cdot \|_{V_1}\) is the norm \(\| \cdot \|_1\) from Section 2.1. Let \(e_1, \ldots, e_n\) be the standard basis vectors in \(V_1\), so that the \(k\)th coordinate of \(e_k\) is equal to 1 for each \(k\), and the rest of the coordinates are equal to 0. If \(T\) is any linear mapping from \(V_1\) into \(V_2\), then

\[(2.91) \quad \|T\|_{op} = \max_{1 \leq k \leq n} \|T(e_k)\|_{V_2}.\]

This reduces to (2.54) when \(V_2 = V_1\) with the norm \(\| \cdot \|_1\) from Section 2.1, and essentially the same argument works for any norm on any \(V_2\). As before,

\[(2.92) \quad \|T(e_k)\|_{V_2} \leq \|T\|_{op}\]

for each \(k\) by definition of the operator norm, since \(e_k\) has norm 1 in \(V_1\) for each \(k\), which implies that \(\|T\|_{op}\) is less than or equal to the right side of (2.91). To get the opposite inequality, one can express any \(v \in V_1\) as \(\sum_{k=1}^n v_k e_k\), where \(v_1, \ldots, v_n\) are the coordinates of \(v\) in \(V_1 = \mathbb{R}^n\) or \(\mathbb{C}^n\), and observe that

\[(2.93) \quad \|T(v)\|_{V_2} \leq \sum_{k=1}^n |v_k| \|T(e_k)\|_{V_2} \leq \left( \max_{1 \leq k \leq n} \|T(e_k)\|_{V_2} \right) \|v\|_1.\]

Now let \(V_1\) be any vector space with any norm \(\| \cdot \|_{V_1}\), and let \(V_2\) be \(\mathbb{R}^n\) or \(\mathbb{C}^n\) with the norm \(\| \cdot \|_{\infty}\) from Section 2.1. Let \(T\) be a linear mapping from \(V_1\) into \(V_2\) again, and let \(\lambda_j(v)\) be the \(j\)th component of \(T(v)\) in \(\mathbb{R}^n\) or \(\mathbb{C}^n\), as appropriate, for \(j = 1, \ldots, n\). Thus \(\lambda_j\) is a linear functional on \(V_1\), with a dual
norm $\|\lambda_j\|_{V_1}$ with respect to the norm $\|\cdot\|_{V_1}$ on $V_1$ for each $j$. In this case, it is easy to see that

$$\|T\|_{op} = \max_{1 \leq j \leq n} \|\lambda_j\|_{V_1},$$

(2.94)

using the definitions of the dual norm, the operator norm, and the norm $\|\cdot\|_{\infty}$ on $V_2$. If $V_1 = V_2$ equipped with the norm $\|\cdot\|_{\infty}$ from Section 2.1, then (2.94) reduces to (2.55), because of the standard identification of the dual of the norm $\|\cdot\|_{\infty}$ on $\mathbb{R}^n$ or $\mathbb{C}^n$ with the norm $\|\cdot\|_1$ from Section 2.1, as in Section 2.2. Note that this case is dual to the previous one, in the sense that it can be applied to the dual of a linear mapping as in the previous paragraph. Similarly, the remarks in the previous paragraph can be applied to the dual of a linear mapping as in this paragraph.

### 2.8 Quotient spaces

Let $V$ be a real or complex vector space, and let $W$ be a linear subspace of $V$. The quotient $V/W$ of $V$ by $W$ is defined by identifying $v, v' \in V$ when $v - v' \in W$. More precisely, one can define an equivalence relation $\sim$ on $V$ by saying that $v \sim v'$ when $v - v' \in W$, and the elements of $V/W$ correspond to equivalence classes in $V$ determined by $\sim$. By standard arguments, $V/W$ is a vector space in a natural way, and there is a canonical quotient mapping $q$ from $V$ onto $V/W$ that sends each $v \in V$ to the equivalence class that contains it, and which is a linear mapping from $V$ onto $V/W$ whose kernel is $W$.

If $V$ is equipped with a norm $\|\cdot\|$, then there is a natural quotient norm $\|\cdot\|_Q$ on $V/W$ defined by

$$\|q(v)\|_Q = \inf\{\|v + w\| : w \in W\}.$$  

(2.95)

It is not too difficult to show that this does determine a norm on $V/W$. More precisely, to check that $\|q(v)\|_Q > 0$ when $v \in V\setminus W$ and hence $q(v) \neq 0$ in $V/W$, one can use the remarks at the end of Section 2.1, and the fact that linear subspaces of $\mathbb{R}^n$ and $\mathbb{C}^n$ are closed with respect to the standard topology on those spaces. Note that the operator norm of $q$ is less than or equal to 1 with respect to the given norm on $V$ and the corresponding quotient norm on $V/W$, at that it is equal to 1 when $W = V$.

The dual $(V/W)^*$ of $V/W$ can be identified with a subspace of $V^*$ in a natural way. Of course, every linear functional on $V/W$ determines a linear functional on $V$, by composition with the quotient mapping $q$. The linear functionals on $V$ that occur in this way are exactly those that are equal to 0 on $W$. If $\lambda$ is a linear functional on $V$ that is equal to 0 on $W$ and $k$ is a nonnegative real number, then the statements

$$|\lambda(v)| \leq k \|v\| \quad \text{for every } v \in V$$

(2.96)

and

$$|\lambda(v)| \leq k \inf\{\|v + w\| : w \in W\} \quad \text{for every } v \in V$$

(2.97)
are equivalent to each other. This means that the dual norm of $\lambda$ as a linear functional on $V$ with respect to $\| \cdot \|$ is the same as the dual norm of the linear functional on $V/W$ that corresponds to $\lambda$ under the quotient mapping with respect to the quotient norm on $V/W$.

If $v$ is any element of $V$, then there is a $w_0 \in W$ such that

$$\|v + w_0\| \leq \|v + w\|$$

for every $w \in W$, so that the infimum in the definition of the quotient norm is attained. To see this, we may as well suppose that $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, since every real or complex vector space of positive finite dimension is isomorphic to one of these. As in Section 2.1, the norm $\| \cdot \|$ defines a continuous function on $\mathbb{R}^n$ or $\mathbb{C}^n$, as appropriate, with respect to the standard Euclidean metric and topology. It is also well known that linear subspaces of $\mathbb{R}^n$ and $\mathbb{C}^n$ are closed sets with respect to the standard Euclidean metric and topology. Remember too that $\| \cdot \|$ is bounded from below by a positive constant multiple of the standard Euclidean metric on $\mathbb{R}^n$ or $\mathbb{C}^n$, as appropriate, as in (2.10). Using this, it suffices to consider a bounded subset of $W$ when minimizing $\|v + w\|$ over $w \in W$. This permits the existence of the minimum to be derived from well-known results about minimizing continuous functions on compact sets, because closed and bounded subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$ are compact.

### 2.9 Projections

Let $V$ be a real or complex vector space, and let $U$ and $W$ be linear subspaces of $V$. Suppose that $U \cap W = \{0\}$, and that every $v \in V$ can be expressed as

$$v = u + w$$

for some $u \in U$ and $w \in W$. If $u' \in U$ and $w' \in W$ also satisfy $v = u' + w'$, then

$$u - u' = w' - w,$$

and this implies that $u = u'$ and $w = w'$, because $u - u' \in U$, $w - w' \in W$, and $U \cap W = \{0\}$. In this case, $U$ and $W$ are said to be complementary in $V$.

Consider the mapping $P : V \to V$ defined by

$$P(v) = u$$

for each $v \in V$, where $u \in U$ is as in (2.99). It is easy to see that $P$ is a linear mapping of $V$ onto $U$ with kernel equal to $W$, and that

$$P(u) = u$$

for every $u \in U$. Conversely, suppose that $P$ is a linear mapping from $V$ onto a linear subspace $U$ of $V$ such that the restriction of $P$ to $U$ is equal to the identity mapping on $U$. If $W$ is the kernel of $P$, then

$$v - P(v) \in W$$
2.9. PROJECTIONS

for every \( v \in V \), and it follows that \( U \) and \( W \) are complementary in \( V \).

A linear mapping \( P : V \to V \) is said to be a projection if

\[
(2.104) \quad P \circ P = P.
\]

This is equivalent to saying that the restriction of \( P \) to \( U = P(V) \) is the identity mapping on \( U \), as in the previous paragraph, so that \( U \) is complementary to the kernel \( W \) of \( P \). If \( P \) is a projection on \( V \), then it is easy to see that \( I - P \) is also a projection on \( V \), where \( I \) denotes the identity mapping on \( V \), because

\[
(2.105) \quad (I - P) \circ (I - P) = I - P - P + P \circ P = I - P.
\]

More precisely, \( I - P \) maps \( V \) onto the kernel \( W \) of \( P \), and the kernel of \( I - P \) is \( U = P(V) \). Of course, orthogonal projections onto linear subspaces of inner product spaces are projections in this sense.

Let \( U \) and \( W \) be linear subspaces of \( V \) again, and let \( q \) be the canonical quotient mapping from \( V \) onto \( V/W \), as in the previous section. It is easy to see that \( U \) and \( W \) are complementary in \( V \) if and only if the restriction of \( q \) to \( U \) is a one-to-one mapping from \( U \) onto \( V/W \). Suppose that this is the case, and let \( P \) be the corresponding projection of \( V \) onto \( U \) with kernel \( W \).

Let \( \| \cdot \| \) be a norm on \( V \), let \( \| \cdot \|_{op} \) be the corresponding operator norm for linear mappings on \( V \), and let \( \| \cdot \|_Q \) be the corresponding quotient norm on \( V/W \). If \( u \in U \) and \( w \in W \), then

\[
(2.106) \quad \|u\| = \|P(u + w)\| \leq \|P\|_{op}\|u + w\|.
\]

This implies that

\[
(2.107) \quad \|P\|\|u\| \leq \|q(u)\|_Q \leq \|u\|
\]

for every \( u \in U \).

Observe that

\[
(2.108) \quad \|P\|_{op} = \|P \circ \|_{op} \leq \|P\|^2_{op}.
\]

Thus

\[
(2.109) \quad \|P\|_{op} \geq 1
\]

when \( P \neq 0 \). If \( \|P\|_{op} = 1 \), then we get that

\[
(2.110) \quad \|q(u)\|_Q = \|u\|
\]

for every \( u \in U \). Orthogonal projections onto nontrival subspaces of inner product spaces have operator norm equal to 1, for instance.

Suppose that \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), and let \( I \) be a subset of the set \( \{1, \ldots, n\} \) of positive integers less than or equal to \( n \). Let \( U_I \) be the linear subspace of \( V \) consisting of vectors \( u \) such that \( u_j = 0 \) when \( j \notin I \), and let \( W_I \) be the complementary subspace consisting of vectors \( w \) such that \( w_j = 0 \) when \( j \in I \). The associated projection \( P_I \) of \( V \) onto \( U_I \) with kernel \( W_I \) sends \( v \in V \) to the vector whose \( j \)th coordinate is equal to \( v_j \) when \( j \in I \), and to 0 when \( j \notin I \). If \( V \) is equipped with a norm \( \| \cdot \|_p \) as in Section 2.1 for some
CHAPTER 2. NORMS ON VECTOR SPACES

Let \( V \) be any real or complex vector space with a norm \( \| \cdot \| \) again, and let \( u_1 \) be an element of \( V \) such that \( \| u_1 \| = 1 \). As in Section 2.3, there is a linear functional \( \lambda \) on \( V \) such that \( \lambda(u_1) = 1 \) and the dual norm of \( \lambda \) with respect to the given norm \( \| \cdot \| \) on \( V \) is equal to 1.

Under these conditions, one can check that
\[
P(v) = \lambda(v) \  u_1
\]
(2.112)
is a projection of \( V \) onto the 1-dimensional linear subspace \( U \) of \( V \) spanned by \( u_1 \) with operator norm equal to 1.

2.10 Extensions and liftings

Let \( V_1 \) and \( V_2 \) be vector spaces, both real or both complex, and equipped with norms. If \( U_1 \) is a linear subspace of \( V_1 \), and \( T \) is a linear mapping from \( U_1 \) into \( V_2 \), then it is easy to see that there is an extension \( \hat{T} \) of \( T \) to a linear mapping from \( V_1 \) into \( V_2 \). The operator norm of \( \hat{T} \) is automatically greater than or equal to the operator norm of \( T \), and one would like to choose \( \hat{T} \) so that its operator norm is as small as possible. If \( P_1 \) is a projection from \( V_1 \) onto \( U_1 \), then the composition \( T \circ P_1 \) is an extension of \( T \) to \( V_1 \) whose operator norm is less than or equal to the product of the operator norm of \( T \) on \( U_1 \) and the operator norm of \( P_1 \) on \( V_1 \). Conversely, if \( V_2 = U_1 \) and \( T \) is the identity mapping on \( U_1 \), then an extension of \( T \) to a linear mapping from \( V_1 \) into \( V_2 \) is the same as a projection from \( V_1 \) onto \( U_1 \).

If \( V_2 \) is 1-dimensional, then this extension problem is equivalent to the one for linear functionals discussed in Section 2.3. Similarly, suppose that \( V_2 \) is \( \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), equipped with the norm \( \| \cdot \|_\infty \) defined in Section 2.1. In this case, a linear mapping \( T \) from another vector space into \( V_2 \) is equivalent to \( n \) linear functionals on the vector space, and the operator norm of a \( T \) is equal to the maximum of the dual norms of the corresponding \( n \) linear functionals, as in the second part of Section 2.7. This permits the extension problem for \( T \) to be reduced to its counterpart for linear functionals again.

Now let \( W_2 \) be a linear subspace of \( V_2 \), and let \( L \) be a linear mapping from \( V_1 \) into \( V_2/W_2 \). It is easy to see that there is a linear mapping \( \hat{L} \) from \( V_1 \) into \( V_2 \) whose composition with the canonical quotient mapping \( q_2 \) from \( V_2 \) onto \( V_2/W_2 \) is equal to \( L \), and one would like to choose \( \hat{L} \) so that its operator norm is as small as possible. This problem is dual to the extension problem discussed in the previous paragraphs. Of course, the operator norm of \( \hat{L} \) is greater than or equal to the operator norm of \( L \), with respect to the quotient norm on \( V_2/W_2 \) that corresponds to the given norm on \( V_2 \). One way to approach this problem is to use a linear subspace \( U_2 \) of \( V_2 \) which is complementary to \( W_2 \), so that the restriction of \( q_2 \) to \( U_2 \) is a one-to-one linear mapping of \( U_2 \) onto \( V_2/W_2 \). In this
2.11. MINIMIZING DISTANCES

In the case considered, one can get a lifting \( \tilde{L} \) of \( L \) to \( V_2 \) by composing \( L \) with the inverse of the restriction of \( q_2 \) to \( U_2 \). Conversely, if \( V_1 = V_2/W_2 \) and \( L \) is the identity mapping on \( V_2/W_2 \), then a lifting of \( L \) to a linear mapping \( \tilde{L} \) from \( V_2/W_2 \) into \( V_2 \) whose composition with \( q_2 \) is the identity mapping on \( V_2/W_2 \) would map \( V_2/W_2 \) onto a linear subspace \( U_2 \) of \( V_2 \) which is complementary to \( W_2 \).

Suppose that \( V_1 \) has dimension 1, and let \( v_1 \) be an element of \( V_1 \) with norm 1. Let \( L \) be a linear mapping from \( V_1 \) into \( V_2/W_2 \), and let \( v_2 \) be an element of \( V_2 \) such that \( q_2(v_2) = L(v_1) \) and the norm of \( v_2 \) in \( V_2 \) is equal to the quotient norm of \( L(v_1) \) in \( V_2/W_2 \). The existence of \( v_2 \) follows from the discussion of minimization at the end of Section 2.8. If \( \tilde{L} \) is the linear mapping from \( V_1 \) into \( V_2 \) that sends \( v_1 \) to \( v_2 \), then \( \tilde{L} \) is a lifting of \( L \) with the same operator norm as \( L \). Similarly, if \( V_1 \) is \( \mathbb{R}^n \) or \( \mathbb{C}^n \) with the norm \( \| \cdot \|_1 \) from Section 2.1, then one can get a lifting \( \tilde{L} \) of \( L \) to a linear mapping from \( V_1 \) into \( V_2 \) with the same operator norm as \( L \) by lifting the \( n \) vectors \( L(e_j) \) in \( V_2/W_2 \) to \( V_2 \) for each of the standard basis vectors \( e_j \) in \( \mathbb{R}^n \) or \( \mathbb{C}^n \), since the operator norm of a linear mapping on \( V_1 \) may be computed as in the first part of Section 2.7.

2.11 Minimizing distances

Let \( V \) be a real or complex vector space with a norm \( \| \cdot \| \), and let \( W \) be a linear subspace of \( V \). If \( v \) is any element of \( V \), then there is a \( w_1 \in W \) such that

\[
\| v - w_1 \| \leq \| v - w \|
\]

for every \( w \in W \). This is equivalent to the minimization problem discussed at the end of Section 2.8, with \( w \) replaced by \( -w \) and \( w_1 = -w_0 \).

Suppose for the moment that there is an inner product \( \langle \cdot, \cdot \rangle \) on \( V \) for which \( \| \cdot \| \) is the corresponding norm, and let \( P_W(v) \) be the orthogonal projection of \( v \) onto \( W \), as in Section 2.6. Thus \( P_W(v) \in W \) and \( v - P_W(v) \in W^\perp \), as in (2.82). If \( w \) is any element of \( W \), then it follows that \( v - w \in W \), and hence

\[
\| v - w \|^2 = \| v - P_W(v) \|^2 + \| P_W(v) - w \|^2,
\]

because \( (v - P_W(v)) \perp (P_W(v) - w) \). This implies that \( P_W(v) \) minimizes the distance to \( v \) among elements of \( W \) in this case, and that \( P_W(v) \) is the only element of \( W \) with this property.

Let \( \| \cdot \| \) be any norm on \( V \) again, and let \( \| \cdot \|_{op} \) be the corresponding operator norm for linear mappings on \( V \). If \( P_1 \) is a projection of \( V \) onto \( W \), then the kernel of \( I - P_1 \) is equal to \( W \), and hence

\[
\| v - P_1(v) \| = \|(I - P_1)(v)\| = \| (I - P_1)(v - w) \| \\ \leq \| I - P_1 \|_{op} \| v - w \|
\]

for every \( w \in W \). If \( \| I - P_1 \|_{op} = 1 \), then it follows that

\[
\| v - P_1(v) \| \leq \| v - w \|
\]
for every \( w \in W \), so that \( w_1 = P_1(v) \in W \) satisfies (2.113).

Suppose for the sake of convenience now that \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), which is not a real restriction, since every real or complex vector space of positive finite dimension is isomorphic to one of these. Let \( E \) be a nonempty subset of \( V \) which is closed with respect to the standard Euclidean metric and topology on \( V \). If \( v \in V \), then there is a \( w_1 \in E \) which minimizes the distance to \( v \) with respect to the norm \( \| \cdot \| \) on \( V \), in the sense that (2.113) holds for every \( w \in E \). This follows from the same type of argument using continuity and compactness as before. More precisely, although \( E \) may not be bounded, and hence not compact, it suffices to consider a bounded subset of \( E \) for this minimization problem.

Let \( V \) be any real or complex vector space again, and let \( B_1 \) be the closed unit ball associated to the norm \( \| \cdot \| \) on \( V \), as in Section 2.1. Let us say that \( B_1 \) is strictly convex if

\[
\| t v + (1 - t) w \| < 1.
\]

for every \( v, w \in V \) with \( \| v \| = \| w \| = 1 \) and \( v \neq w \) and every real number \( t \) with \( 0 < t < 1 \). The unit ball in any inner product space is strictly convex, as one can show by analyzing the case of equality in the proof of the triangle inequality. If \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) with the norm \( \| \cdot \|_p \) as in Section 2.1 for some \( p, 1 < p < \infty \), then one can check that the unit ball is strictly convex, using the strict convexity of the function \( |r|^p \). If \( n \geq 2 \) and \( p = 1 \) or \( \infty \), then it is easy to see that the unit ball is not strictly convex.

Let \( E \) be a nonempty convex set in \( V \), and let \( v \) be an element of \( V \). Suppose that \( w_1, w_2 \in E \) both minimize the distance to \( v \) with respect to \( \| \cdot \| \) in \( V \), in the sense that

\[
\| v - w_1 \| = \| v - w_2 \| \leq \| v - w \|
\]

for every \( w \in E \). If \( 0 < t < 1 \), then \( w = t w_1 + (1 - t) w_2 \in E \), because \( E \) is convex. However, if \( B_1 \) is strictly convex and \( w_1 \neq w_2 \), then the norm of

\[
v - w = t (v - w_1) + (1 - t) (v - w_2),
\]

is strictly less than the common value of the norms of \( v - w_1 \) and \( v - w_2 \), contradicting (2.118). This shows that \( w_1 = w_2 \) under these conditions when \( B_1 \) is strictly convex.

Note that we could simply take \( t = 1/2 \) in the preceding argument. If the norm on \( V \) is associated to an inner product, then the strict convexity property of the unit ball with \( t = 1/2 \) follows from the parallelogram law (2.90).
Chapter 3

Structure of linear operators

In this chapter, we continue to restrict our attention to finite-dimensional vector spaces.

3.1 The spectrum and spectral radius

Let $V$ be a complex vector space with positive dimension, and let $T$ be a linear operator from $V$ into $V$. The spectrum of $T$ is the set of complex numbers $\alpha$ such that $\alpha$ is an eigenvalue of $T$, which is to say that there is a $v \in V$ such that $v \neq 0$ and

\[ T(v) = \alpha v. \]

(3.1)

In this case, $v$ is said to be an eigenvector of $T$ with eigenvalue $\alpha$. If $\alpha$ is an eigenvalue of $T$, then

\[ \{ v \in V : T(v) = \alpha v \} \]

(3.2)

is a nontrivial linear subspace of $V$, called the eigenspace of $T$ associated to $\alpha$.

It is well known that a one-to-one linear mapping $R : V \to V$ automatically maps $V$ onto itself, and hence is invertible, because $R(V)$ is a linear subspace of $V$ with the same dimension as $V$. If $R$ is not invertible on $V$, then it follows that $R$ is not one-to-one, so that the kernel of $R$ is nontrivial. By definition, $\alpha \in \mathbb{C}$ is an eigenvalue of $T$ when the kernel of $T - \alpha I$ is nontrivial, where $I$ denotes the identity transformation on $V$. Equivalently, $\alpha \in \mathbb{C}$ is not in the spectrum of $T$ when $T - \alpha I$ is an invertible linear operator on $V$.

A famous theorem states that every linear operator $T$ on $V$ has at least one eigenvalue. To see this, note that $\alpha \in \mathbb{C}$ lies in the spectrum of $T$ exactly when the determinant of $T - \alpha I$ is 0. The determinant of $T - \alpha I$ is a polynomial in $\alpha$, whose degree is equal to the dimension of $V$. By the “Fundamental Theorem of Algebra”, det$(T - \alpha I)$ has at least one root, as desired.
CHAPTER 3. STRUCTURE OF LINEAR OPERATORS

This argument also shows that the number of distinct eigenvalues of $T$ is less than or equal to the dimension of $V$, since a polynomial of degree $n$ has at most $n$ roots. The spectral radius $\text{rad}(T)$ of $T$ is defined to be the maximum of $|\alpha|$, where $\alpha \in \mathbb{C}$ is an eigenvalue of $T$. Thus $T - \alpha I$ is invertible when $|\alpha| > \text{rad}(T)$, and $\text{rad}(T)$ is the largest nonnegative real number with this property.

Let $\| \cdot \|$ be a norm on $V$, and let $\| \cdot \|_{\text{op}}$ be the corresponding operator norm for linear transformations on $V$, as in Section 2.4. If $\alpha \in \mathbb{C}$ is an eigenvalue of $T$, and $v \in V$ is a nonzero eigenvector corresponding to $\alpha$, then

$$|\alpha| \|v\| \leq \|T\|_{\text{op}} \|v\|,$$

and hence $|\alpha| \leq \|T\|_{\text{op}}$. It follows that $\text{rad}(T) \leq \|T\|_{\text{op}}$.

Now let $n$ be a positive integer, and let us check that

$$\text{rad}(T^n) = \text{rad}(T)^n.$$

If $\alpha$ is an eigenvalue of $T$, then $\alpha^n$ is obviously an eigenvalue of $T^n$ for each $n$, and hence

$$\text{rad}(T^n) \leq \text{rad}(T^n).$$

To get the opposite inequality, suppose that $\beta$ is an eigenvalue of $T^n$, and let us show that $\alpha$ is an eigenvalue of $T$ for some complex number $\alpha$ such that $\alpha^n = \beta$. Let $\alpha_1, \ldots, \alpha_n$ be the $n$th roots of $\beta$, so that

$$z^n - \beta = (z - \alpha_1) \cdots (z - \alpha_n).$$

This implies that

$$T^n - \beta I = (T - \alpha_1 I) \cdots (T - \alpha_n I),$$

where the product of linear operators on $V$ is defined by their composition. If $T - \alpha_j I$ is invertible on $V$ for each $j = 1, \ldots, n$, then it follows that $T^n - \beta I$ is also invertible on $V$, because the composition of invertible operators is invertible. If $\beta$ is an eigenvalue of $T$, then $T^n - \beta I$ is not invertible, and hence $T - \alpha_j I$ is not invertible for some $j$. This says exactly that $\alpha_j$ is an eigenvalue of $T$ for some $j$, as desired.

### 3.2 Adjoint

In this section, both real and complex vector spaces are allowed. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. If $T$ is a linear operator on $V$, then there is a unique linear operator $T^*$ on $V$ such that

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle$$

for every $v, w \in V$, called the adjoint of $T$. More precisely, for each $w \in V$,

$$\mu_w(v) = \langle T(v), w \rangle.$$
defines a linear functional on $V$. This implies that there is a unique element $T^*(w)$ of $V$ such that

$$\mu_w(v) = \langle v, T^*(w) \rangle$$

for every $v \in V$, as in Section 2.6. One can check that $T^*$ is linear on $V$, using the fact that $T^*(w)$ is uniquely determined by (3.10). Alternatively, if we fix an orthonormal basis for $V$, then we can express $T$ in terms of a matrix relative to this basis. Remember that the transpose of a matrix $(a_{j,l})$ is the matrix $(b_{j,l})$ given by $b_{j,l} = a_{l,j}$. If $V$ is a real vector space, then $T^*$ is the linear transformation on $V$ that corresponds to the transpose of the matrix for $T$ with respect to the same basis. In the complex case, the entries of the matrix for $T^*$ are the complex conjugates of the entries of the transpose of the matrix for $T$. It is easy to see that $T^*$ is uniquely determined by (3.8), so that different orthonormal bases for $V$ lead to the same linear transformation $T^*$ when one computes $T^*$ in terms of matrices.

Although we are using the same notation here for the adjoint as we did in Section 2.4 for dual linear mappings, we should be careful about some of the differences. In the real case, we can identify $V$ with its dual space $V^*$, since every linear functional on $V$ can be represented as

$$\lambda_w(v) = \langle v, w \rangle$$

for some $w \in V$. In this case, it is easy to see that the adjoint of $T$ corresponds exactly to the dual linear transformation defined previously. However, in the complex case, the mapping from $w \in V$ to the linear functional $\lambda_w \in V^*$ is not quite linear, but rather conjugate-linear, in the sense that multiplication of $w$ by a complex number $a$ corresponds to multiplying $\lambda_w$ by the complex conjugate $\overline{a}$ of $a$. Thus the adjoint of $T$ is not quite the same as the dual linear transformation defined earlier in the complex case, which is also reflected in the linearity properties of the mapping from $T$ to $T^*$ discussed next.

Note that $I^* = I$, where $I$ is the identity transformation on $V$. If $S$ and $T$ are linear transformations on $V$ and $a$, $b$ are scalars, then

$$\text{(3.12)} \quad (aS + bT)^* = aS^* + bT^*$$

when $V$ is a real vector space, and

$$\text{(3.13)} \quad (aS + bT)^* = \overline{a}S^* + \overline{b}T^*$$

when $V$ is a complex vector space. Also, $(T^*)^* = T$. and

$$\text{(3.14)} \quad (ST)^* = T^*S^*.$$ 

If $T$ is invertible, then $T^*$ is invertible, and

$$\text{(3.15)} \quad (T^*)^{-1} = (T^{-1})^*.$$ 

It follows that $T - \lambda I$ is invertible if and only if $T^* - \lambda I$ is invertible for each $\lambda \in \mathbb{R}$ in the real case, and similarly that $T - \lambda I$ is invertible if and only if $T^* - \overline{\lambda}I$ is invertible for each $\lambda \in \mathbb{C}$ in the complex case.
Let $\| \cdot \|$ be the norm on $V$ associated to the inner product $\langle \cdot, \cdot \rangle$, and let $\| \cdot \|_{op}$ be the corresponding operator norm for linear transformations on $V$ with respect to $\| \cdot \|$. Let us check that
\begin{equation}
\| T \|_{op} = \sup \{|\langle T(v), w \rangle| : v, w \in V, \|v\|, \|w\| \leq 1\}\}
\end{equation}
for any linear transformation $T$ on $V$. The right side of (3.16) is clearly less than or equal to the operator norm of $T$, because of the Cauchy–Schwarz inequality. To get the opposite inequality, one can take $w = T(v)/\|T(v)\|$ in the right side of (3.16) when $T(v) \neq 0$. It follows that
\begin{equation}
\| T^* \|_{op} = \| T \|_{op},
\end{equation}
because the right side of (3.16) is equal to the analogous quantity for $T^*$.

### 3.3 Self-adjoint linear operators

Let $(V, \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, as in the previous section. A linear transformation $A$ on $V$ is said to be self-adjoint if
\begin{equation}
A^* = A.
\end{equation}
This is equivalent to the condition that
\begin{equation}
\langle A(v), w \rangle = \langle v, A(w) \rangle
\end{equation}
for every $v, w \in V$. Thus the identity operator $I$ on $V$ is self-adjoint.

Let $W$ be a linear subspace of $V$, and let $P_W$ be the orthogonal projection of $V$ onto $W$, as in Section 2.6. Remember that $P_W(v)$ is characterized by the conditions $P_W(v) \in W$ and $v - P_W(v) \in W^\perp$, for each $v \in V$. This implies that
\begin{equation}
\langle P_W(v), w \rangle = \langle P_W(v), P_W(w) \rangle = \langle v, P_W(w) \rangle
\end{equation}
for every $v, w \in V$, and hence that $P_W$ is self-adjoint on $V$.

If $A$ and $B$ are self-adjoint linear operators on $V$, then their sum $A + B$ is self-adjoint. Similarly, if $A$ is a self-adjoint linear operator on $V$ and $t$ is a real number, then $tA$ is self-adjoint as well. Note that it is important to take $t \in \mathbb{R}$ here, even when $V$ is a complex vector space.

If $A$ is a self-adjoint linear operator on $V$ and $V$ is complex, then it is easy to see that
\begin{equation}
\langle A(v), v \rangle \in \mathbb{R}
\end{equation}
for every $v \in V$. Using this, one can check that the eigenvalues of $A$ are also real numbers.

Suppose that $A$ is a self-adjoint linear operator on a real or complex inner product space $(V, \langle \cdot, \cdot \rangle)$, and that $v \in V$ is an eigenvector of $A$ with eigenvalue $\lambda$. If $y \in V$ and $y \perp v$, then
\begin{equation}
\langle v, A(y) \rangle = \langle A(v), y \rangle = \lambda \langle v, y \rangle = 0,
\end{equation}
and hence $y \in \ker A$. Thus $A$ is a normal operator.

Note that if $A$ is a self-adjoint linear operator on $V$, then $A$ is also normal. This is because $\langle Au, v \rangle = \langle u, A^*v \rangle = \langle A^*u, v \rangle = \langle A^*A^*v, v \rangle$, and hence $A$ is self-adjoint.

### 3.3.1 Spectral theorem

The spectral theorem states that if $A$ is a self-adjoint linear operator on $V$, then there exists an orthonormal basis of $V$ consisting of eigenvectors of $A$. This is a powerful result that allows us to diagonalize self-adjoint operators.

### 3.3.2 Orthogonal projections

An orthogonal projection $P_W$ on $V$ is a self-adjoint operator. The range $W$ of $P_W$ is the orthogonal projection of $V$ onto $W$, and the null space $W^\perp$ is the orthogonal projection of $V$ onto $W^\perp$. The orthogonal projection $P_W$ is characterized by the conditions $P_W(v) \in W$ and $v - P_W(v) \in W^\perp$, for each $v \in V$.

The orthogonal projection $P_W$ is self-adjoint on $V$, because
\begin{equation}
\langle P_W(v), w \rangle = \langle v, P_W(w) \rangle
\end{equation}
for every $v, w \in V$, and hence that $P_W$ is self-adjoint on $V$.
so that $A(y) \perp v$. If $w \in V$ is an eigenvector of $A$ with eigenvalue $\mu \neq \lambda$, then

\begin{equation}
\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle A(v), w \rangle = \langle v, A(w) \rangle = \langle v, \mu w \rangle = \mu \langle v, w \rangle,
\end{equation}

which implies that $\langle v, w \rangle = 0$.

If the dimension $n$ of $V$ is positive, then one can use an orthonormal basis of $V$ to show that $V$ is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ with its standard inner product, and we may as well take $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for the moment. By the usual considerations of continuity and compactness, $\langle A(v), v \rangle$ attains its maximum and minimum on the unit sphere

\begin{equation}
\{ v \in V : \|v\| = 1 \}.
\end{equation}

It is well known that the critical points of $\langle A(v), v \rangle$ on the unit sphere are exactly the eigenvectors of $A$ with norm 1, and hence that the maximum and minimum of $\langle A(v), v \rangle$ on the unit sphere are attained at eigenvectors of $A$. In particular, $A$ has a nonzero eigenvector $v$, and $A$ maps $W = \{ w \in V : \langle w, v \rangle = 0 \}$ to itself, as in the previous paragraph. By repeating the process, one can show that there is an orthonormal basis of $V$ consisting of eigenvectors of $A$.

A linear operator $T$ on a complex inner product space $V$ is said to be normal if $T$ and $T^*$ commute, which is to say that

\begin{equation}
T \circ T^* = T^* \circ T.
\end{equation}

If $T$ can be diagonalized in an orthonormal basis, then $T^*$ is diagonalized by the same basis, and $T$ is normal. Conversely, one can show that a normal operator $T$ on $V$ can be diagonalized in an orthonormal basis, as follows. Any linear operator $T$ on $V$ can be expressed as $A + iB$, where

\begin{equation}
A = \frac{T + T^*}{2} \quad \text{and} \quad B = \frac{T - T^*}{2i}
\end{equation}

are self-adjoint. Thus $A$ and $B$ can each be diagonalized in an orthonormal basis of $V$, and one would like to show that they can both be diagonalized by the same orthonormal basis when $T$ is normal, which implies that $A$ and $B$ commute. If $A$ and $B$ are commuting linear transformations on any vector space, then it is easy to see that the eigenspaces of $A$ are invariant under $B$. To diagonalize $T$ in an orthonormal basis, one can first use a diagonalization of $A$ to decompose $V$ into an orthogonal sum of eigenspaces of $A$, and then diagonalize the restriction of $B$ to each of the eigenspaces of $A$.

A linear operator $T$ on $V$ is said to be an orthogonal transformation when $V$ is real, or a unitary transformation when $V$ is complex, if

\begin{equation}
\langle T(v), T(w) \rangle = \langle v, w \rangle
\end{equation}
for every $v, w \in V$. This implies that

$$\|T(v)\| = \|v\|$$

for every $v \in V$, and the converse holds because of polarization, as in Remark 2.87. This condition obviously implies that the kernel of $T$ is trivial, and hence that $T$ is invertible, because $V$ is supposed to be finite-dimensional. More precisely, it is easy to see that $T$ is orthogonal or unitary, as appropriate, if and only if $T$ is invertible and

$$T^{-1} = T^*.$$ (3.30)

In particular, unitary operators are normal, because $T$ automatically commutes with $T^{-1}$.

### 3.4 Anti-self-adjoint operators

Let $(V, \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, as before. A linear operator $R$ on $V$ is said to be anti-self-adjoint if

$$R^* = -R,$$ (3.31)

which is equivalent to asking that

$$\langle R(v), w \rangle = -\langle v, R(w) \rangle$$ (3.32)

for every $v, w \in V$. If $V$ is a complex vector space, then $R$ is anti-self-adjoint if and only if $R = iB$ for some self-adjoint linear operator $B$ on $V$. Note that the sum of two anti-self-adjoint linear operators on $V$ is also anti-self-adjoint, as is the product of an anti-self-adjoint linear operator and a real number.

If $V$ is a real inner product space and $R$ is an anti-self-adjoint linear operator on $V$, then

$$\langle R(v), v \rangle = -\langle v, R(v) \rangle = -\langle R(v), v \rangle$$ (3.33)

for every $v \in V$, using the symmetry of the inner product on $V$ in the second step. This implies that

$$\langle R(v), v \rangle = 0$$ (3.34)

for every $v \in V$, and hence that any eigenvalue of $R$ must be equal to 0, if there is one. The analogous argument in the complex case would only give that

$$\langle R(v), v \rangle$$ (3.35)

is purely imaginary for each $v \in V$ when $R$ is anti-self-adjoint, and hence that the eigenvalues of $R$ are purely imaginary, which also follow from the representation of $R$ as $iB$ for some self-adjoint linear operator $B$ on $V$. As in the case of self-adjoint operators, if $v, w \in V$ satisfy $R(v) = 0$ and $v \perp w$, then it is easy to see that $v \perp R(w)$ too.

If $R$ is an anti-self-adjoint linear operator on $V$, then

$$\langle R^2(v), w \rangle = -\langle R(v), R(w) \rangle = \langle v, R^2(w) \rangle$$ (3.36)
for every $v, w \in W$. This shows that $R^2 = R \circ R$ is a self-adjoint linear operator on $V$, and in particular that $R^2$ can be diagonalized in an orthonormal basis in $V$, as in the previous section. If we take $v = w$ in (3.36), then we get that

$$\langle R^2(v), v \rangle = -\langle R(v), R(v) \rangle = -\|R(v)\|^2$$

for every $v \in V$. Of course, $R(v) = 0$ implies that $R^2(v) = R(R(v)) = 0$ trivially, and (3.37) shows that $R^2(v) = 0$ implies that $R(v) = 0$ in this case.

If $T$ is any linear operator on $V$, then $T$ can be expressed as

$$T = A + R,$$

where $A = (T + T^*)/2$ is self-adjoint, and $R = (T - T^*)/2$ is anti-self-adjoint. If $T$ commutes with $T^*$, then $A$ commutes with $R$, and hence $A$ commutes with $R^2$. As in the previous section, one can show that there is an orthonormal basis of $V$ in which $A$ and $R^2$ are simultaneously diagonalized under these conditions. One also gets that the eigenspaces of $A$ are invariant under $R$, because $A$ commutes with $R$. In particular, these remarks can be applied to the case of an orthogonal linear transformation $T$ on a real inner product space $V$.

### 3.5 The $C^*$-identity

Any linear operator $T$ on a real or complex inner product space $V$ satisfies

$$\|T^* T\|_{op} \leq \|T^*\|_{op} \|T\|_{op} = \|T\|^2_{op},$$

using (3.17) in the second step. Conversely,

$$\|T(v)\|^2 = \langle T(v), T(v) \rangle = \langle T^*(T(v)), v \rangle$$

for every $v \in V$, and hence

$$\|T(v)\|^2 \leq \|(T^* T)(v)\| \|v\| \leq \|T^* T\|_{op} \|v\|^2.$$

Thus $\|T\|^2_{op} \leq \|T^* T\|_{op}$, which implies that

$$\|T^* T\|_{op} = \|T\|^2_{op}.$$

This is known as the $C^*$-identity.

Let $(V_1, \langle \cdot, \cdot \rangle_1), (V_2, \langle \cdot, \cdot \rangle_2)$ be inner product spaces which are both real or both complex, and with norms $\| \cdot \|_1, \| \cdot \|_2$ associated to their inner products, respectively. Let $\|T\|_{op,ab}$ be the operator norm of a linear mapping $T : V_a \to V_b$ using $\| \cdot \|_a$ on the domain and $\| \cdot \|_b$ on the range, where $a, b = 1, 2$. This can be characterized in terms of inner products by

$$\|T\|_{op,ab} = \sup\{ \|\langle T(v), w \rangle_b \| : v \in V_a, w \in V_b, \|v\|_a, \|w\|_b \leq 1 \},$$

as in (3.16).
If \( T \) is a linear mapping from \( V_1 \) into \( V_2 \), then there is a unique linear mapping \( T^* : V_2 \rightarrow V_1 \) such that
\[
\langle T(v), w \rangle_2 = \langle v, T^*(w) \rangle_1
\]
for every \( v \in V_1 \) and \( w \in V_2 \), again called the adjoint of \( T \). As before,
\[
\mu_w(v) = \langle T(v), w \rangle_2
\]
defines a linear functional on \( V_1 \) for each \( w \in V_2 \), which can be represented as
\[
\mu_w(v) = \langle v, T^*(w) \rangle_1
\]
for a unique element \( T^*(w) \) of \( V_1 \), and one can check that \( T^* : V_2 \rightarrow V_1 \) is linear. Otherwise, one can get \( T^* \) using orthonormal bases for \( V_1 \) and \( V_2 \), with respect to which the matrix for \( T^* \) is equal to the transpose or the complex conjugate of the transpose of the corresponding matrix for \( T \), depending on whether \( V_1 \), \( V_2 \) are real or complex vector spaces. As in Section 3.2, the adjoint of \( T : V_1 \rightarrow V_2 \) is very similar to the dual linear mapping discussed in Section 2.4, but there are some differences, especially when \( V_1 \) and \( V_2 \) are complex.

If \( T : V_1 \rightarrow V_2 \) is a linear mapping and if \( a \) is a real or complex number, as appropriate, then
\[
(aT)^* = aT^*
\]
in the real case, and
\[
(aT)^* = \overline{aT^*}
\]
in the complex case. If \( S, T : V_1 \rightarrow V_2 \) are linear mappings, then
\[
(S + T)^* = S^* + T^*.
\]
It is easy to see that \((T^*)^* = T\) for every \( T : V_1 \rightarrow V_2 \). If \( V_1, V_2, \) and \( V_3 \) are inner product spaces, all real or all complex, and if \( T_1 : V_1 \rightarrow V_2 \) and \( T_2 : V_2 \rightarrow V_3 \) are linear mappings, then
\[
(T_2 \circ T_1)^* = T_1^* \circ T_2^*
\]
as linear mappings from \( V_3 \) into \( V_1 \). A linear mapping \( T : V_1 \rightarrow V_2 \) is invertible if and only if \( T^* : V_2 \rightarrow V_1 \) is invertible, in which case
\[
(T^{-1})^* = (T^*)^{-1}.
\]
Using (3.43), we get that
\[
\|T^*\|_{op,21} = \|T\|_{op,12},
\]
for any linear mapping \( T : V_1 \rightarrow V_2 \), as in (3.17). The \( C^* \)-identities
\[
\|T^* T\|_{op,11} = \|TT^*\|_{op,22} = \|T\|_{op,12}^2
\]
can be verified in this setting as well.
3.6 The trace norm

Let \( V_1 \) and \( V_2 \) be vector spaces, both real or both complex, equipped with norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), respectively. Let \( V_1^* \), \( V_2^* \) and \( \| \cdot \|_1^* \), \( \| \cdot \|_2^* \) be the corresponding dual spaces and norms, as in Section 2.2.

Any linear mapping \( T : V_1 \rightarrow V_2 \) can be expressed as

\[
T(v) = \sum_{j=1}^{N} \lambda_j(v) w_j,
\]

(3.54)

where \( N \) is a positive integer, \( \lambda_1, \ldots, \lambda_N \in V_1^* \), and \( w_1, \ldots, w_N \in V_2 \). The trace norm of \( T \) relative to \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) is defined to be the infimum of

\[
\sum_{j=1}^{N} \| \lambda_j \|_1^* \| w_j \|_2
\]

(3.55)

over all such representations of \( T \), and is denoted \( \|T\|_{tr,12} \) to indicate the role of the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \).

If \( \lambda \in V_1 \), \( w \in V_2 \), and \( A(v) = \lambda(v) w \), then

\[
\| A \|_{op,12} = \| \lambda \|_1^* \| w \|_2,
\]

(3.56)

where \( \| A \|_{op,12} \) is the operator norm of \( A \) with respect to the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \). Thus

\[
\| T \|_{op,12} \leq \sum_{j=1}^{N} \| \lambda_j \|_1^* \| w_j \|_2
\]

(3.57)

for each representation (3.54) of \( T \), and therefore

\[
\| T \|_{op,12} \leq \| T \|_{tr,12}.
\]

(3.58)

Using this, one can check that the trace norm is a norm on the vector space of linear mappings from \( V_1 \) to \( V_2 \). If \( A(v) = \lambda(v) w \), where \( \lambda \in V_1^* \) and \( w \in V_2 \), then

\[
\| \lambda \|_1^* \| w \|_2 = \| A \|_{op,12} \leq \| A \|_{tr,12} \leq \| \lambda \|_1^* \| w \|_2,
\]

(3.59)

and hence the operator and trace norms of \( A \) are the same.

Suppose that \( V_3 \) is another vector space which is real or complex depending on whether \( V_1 \), \( V_2 \) are real or complex, and that \( \| \cdot \|_3 \) is a norm on \( V_3 \). If \( T_1 : V_1 \rightarrow V_2 \), \( T_2 : V_2 \rightarrow V_3 \) are linear mappings, then

\[
\| T_2 \circ T_1 \|_{tr,13} \leq \| T_1 \|_{op,12} \| T_2 \|_{tr,23}
\]

(3.60)

and

\[
\| T_2 \circ T_1 \|_{tr,13} \leq \| T_1 \|_{tr,12} \| T_2 \|_{op,23}.
\]

(3.61)

Here \( \| \cdot \|_{op,ab} \) and \( \| \cdot \|_{tr,ab} \) are the operator and trace norms for linear mappings from \( V_a \) to \( V_b \), \( a,b = 1,2,3 \). This follows by converting representations of the form (3.54) for \( T_1 \) or \( T_2 \) into similar representations for \( T_1 \circ T_2 \).
Let us briefly review the notion of the trace of a linear mapping. Fix a vector space $V$, and suppose that $A$ is a linear mapping from $V$ to itself. Let $v_1, \ldots, v_n$ be a basis for $V$, so that every element of $V$ can be expressed as a linear combination of the $v_j$’s in exactly one way. With respect to this basis, $A$ can be described by an $n \times n$ matrix $(a_{j,k})$ of real or complex numbers, as appropriate, through the formula

$$ A(v_k) = \sum_{j=1}^{n} a_{j,k} v_j. \quad (3.62) $$

The trace of $A$ is denoted $\text{tr} A$ and defined by

$$ \text{tr} A = \sum_{j=1}^{n} a_{j,j}. \quad (3.63) $$

Clearly $\text{tr} A$ is linear in $A$, and one can check that

$$ \text{tr} (A \circ B) = \text{tr} (B \circ A) \quad (3.64) $$

for any linear transformations $A$ and $B$ on $V$. In particular,

$$ \text{tr} (T \circ A \circ T^{-1}) = \text{tr} A \quad (3.65) $$

for every invertible linear transformation $T$ on $V$. This implies that the trace does not depend on the choice of basis for $V$.

Suppose further that $V$ is equipped with a norm $\| \cdot \|$, and let $\| \cdot \|_{tr}$ be the corresponding trace norm for operators on $V$. If $A$ is any linear transformation on $V$, then

$$ | \text{tr} A | \leq \| A \|_{tr}. \quad (3.66) $$

To prove this, it suffices to show that

$$ | \text{tr} A | \leq \sum_{l=1}^{N} \| \lambda_l \|^{*} \| w_l \| \quad (3.67) $$

whenever $\lambda_1, \ldots, \lambda_N \in V^*$, $w_1, \ldots, w_N \in V$, and $A(v) = \sum_{l=1}^{N} \lambda_l(v) w_l$. By linearity, it is enough to check that

$$ | \text{tr} A | \leq \| \lambda \|^{*} \| w \| \quad (3.68) $$

when $\lambda \in V^*$, $w \in V$, and $A(v) = \lambda(v) w$. In this case, $\text{tr} A = \lambda(w)$, and $|\lambda(w)| \leq \| \lambda \|^{*} \| w \|$ by definition of the dual norm.

Let us return to the setting of two vector spaces $V_1, V_2$, with norms $\| \cdot \|_1$, $\| \cdot \|_2$. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_1$ are linear mappings, then

$$ \text{tr}_{V_1} (T_2 \circ T_1) = \text{tr}_{V_2} (T_1 \circ T_2). \quad (3.69) $$
3.6. The Trace Norm

Here the trace on the left applies to linear operators on $V_1$, and the trace on the right applies to linear operators on $V_2$, as indicated by the notation. By (3.66),

$$ (3.70) \quad |\text{tr}_{V_1}(T_2 \circ T_1)| = |\text{tr}_{V_2}(T_1 \circ T_2)| $$

is less than or equal to the trace norm of the composition of $T_1$ and $T_2$ in either order. Hence it is less than or equal to the product of the trace norm of $T_1$ and the operator norm of $T_2$, or the operator norm of $T_1$ times the trace norm of $T_2$.

Remember that $L(V_1, V_2)$ denotes the vector space of linear mappings from $V_1$ into $V_2$. If $R$ is a linear mapping from $V_2$ into $V_1$, then

$$ (3.71) \quad T \mapsto \text{tr}_{V_1}(R \circ T) $$

is a linear functional $L(V_1, V_2)$. This defines a linear isomorphism from $L(V_2, V_1)$ onto $L(V_1, V_2)^*$. Fix a linear mapping $R : V_2 \to V_1$, and let us check that the dual norm of (3.71) with respect to the trace norm on $L(V_1, V_2)$ is equal to $\|R\|_{op,21}$. We have already seen that

$$ (3.72) \quad |\text{tr}_{V_1}(R \circ T)| \leq \|R\|_{op,21} \|T\|_{tr,12}, $$

which says exactly that the aforementioned dual norm of (3.71) is less than or equal to $\|R\|_{op,21}$. To establish the opposite inequality, let $\lambda \in V_1^*$ and $w \in V_2$ be given, and put $T_0(v) = \lambda(v)w$. Thus $T_0$ is a linear mapping from $V_1$ to $V_2$,

$$ (3.73) \quad (R \circ T_0)(v) = \lambda(v) R(w), $$

and

$$ (3.74) \quad \text{tr}_{V_1}(R \circ T_0) = \lambda(R(w)). $$

By definition, $|\text{tr}_{V_1}(R \circ T_0)|$ is less than or equal to the dual norm of (3.71) times the trace norm of $T_0$. The trace norm of $T_0$ is equal to $\|\lambda\|_1 \|w\|_2$, and hence $|\lambda(R(w))|$ is less than or equal to the dual norm of (3.71) times $\|\lambda\|_1 \|w\|_2$. Since $\lambda \in V_1^*$ and $w \in V_2$ are arbitrary, this implies that $\|R\|_{op,21}$ is less than or equal to the dual norm of (3.71), so that the two are the same.

Similarly, if $T$ is a linear mapping from $V_1$ into $V_2$, then

$$ (3.75) \quad R \mapsto \text{tr}_{V_1}(R \circ T) $$

is a linear functional on $L(V_2, V_1)$, and this defines a linear isomorphism from $L(V_1, V_2)$ onto $L(V_2, V_1)^*$. It follows from the previous discussion that the dual norm of (3.75) with respect to the operator norm on $L(V_2, V_1)$ is equal to $\|T\|_{tr,12}$, by the results in Section 2.3. In other words, we just saw that the operator norm on $L(V_2, V_1)$ corresponds to the dual of the trace norm on $L(V_1, V_2)$, and this implies that the dual of the operator norm on $L(V_2, V_1)$ corresponds to the trace norm on $L(V_1, V_2)$, as in Section 2.3.
3.7 The Hilbert–Schmidt norm

Let \( (V_1, \langle \cdot, \cdot \rangle_1) \), \( (V_2, \langle \cdot, \cdot \rangle_2) \) be inner product spaces, both real or both complex, with norms \( \| \cdot \|_1, \| \cdot \|_2 \) associated to their inner products, as usual. If \( \{a_j\}_{j=1}^p \), \( \{b_k\}_{k=1}^q \) are orthonormal bases for \( V_1 \), \( V_2 \), respectively, then

\[
v = \sum_{j=1}^p \langle v, a_j \rangle_1 a_j, \quad w = \sum_{k=1}^q \langle w, b_k \rangle_2 b_k
\]

for every \( v \in V_1 \), \( w \in V_2 \), and we can express a linear mapping \( T : V_1 \to V_2 \) as

\[
T(v) = \sum_{j=1}^p \sum_{k=1}^q \langle v, a_j \rangle_1 \langle T(a_j), b_k \rangle_2 b_k.
\]

Because \( \{a_j\}_{j=1}^p \) and \( \{b_k\}_{k=1}^q \) are orthonormal bases for \( V_1 \) and \( V_2 \), we get that

\[
\sum_{j=1}^p \|T(a_j)\|_2^2 = \sum_{j=1}^p \sum_{k=1}^q |\langle T(a_j), b_k \rangle_2|^2 = \sum_{j=1}^p \sum_{k=1}^q |\langle a_j, T^*(b_k) \rangle_1|^2 = \|T^*(b_k)\|_1^2.
\]

The Hilbert–Schmidt norm of \( T \) is denoted \( \|T\|_{HS} \) and defined to be the square root of the common value of these sums. It follows that the Hilbert–Schmidt norms of \( T \) and \( T^* \) are the same, and do not depend on the particular choices of orthonormal bases for \( V_1 \) and \( V_2 \).

If we express \( T(v) \) as

\[
T(v) = \sum_{j=1}^p \langle v, a_j \rangle_1 T(a_j),
\]

and \( (T^* \circ T)(v) \) as

\[
(T^* \circ T)(v) = \sum_{j=1}^p \sum_{l=1}^p \langle v, a_j \rangle_1 \langle T(a_j), T(a_l) \rangle_1 a_l,
\]

then we see that

\[
\text{tr}_{V_1} (T^* \circ T) = \sum_{j=1}^p \|T(a_j)\|_1^2 = \|T\|_{HS}^2.
\]

Here the left side is the trace of \( T^* \circ T \) as an operator on \( V_1 \), and similarly the trace of \( T \circ T^* \) on \( V_2 \) is equal to \( \|T\|_{HS}^2 \). One can check that

\[
\langle A, B \rangle_{\mathcal{L}(V_1, V_2)} = \text{tr}_{V_1} (B^* \circ A) = \text{tr}_{V_2} (A \circ B^*)
\]
defines an inner product on the vector space \( \mathcal{L}(V_1, V_2) \) of linear transformations from \( V_1 \) into \( V_2 \), so that the Hilbert–Schmidt norm is exactly the norm on \( \mathcal{L}(V_1, V_2) \) associated to this inner product. More precisely, if \( R \) is a linear transformation on a real or complex inner product space \( V \), then

\[
\text{tr}_V R^* = \text{tr}_V R
\]

in the real case, and

\[
\text{tr}_V R^* = \overline{\text{tr}_V R}
\]

in the complex case, as one can verify using the description of the matrix of \( R^* \) with respect to an orthonormal basis for \( V \) mentioned in Section 3.2. This implies that (3.82) satisfies the symmetry property required to be an inner product.

### 3.8 Schmidt decompositions

Let \((V_1, \langle \cdot, \cdot \rangle_1), (V_2, \langle \cdot, \cdot \rangle_2)\) be inner product spaces again, both real or both complex, with norms \( \| \cdot \|_1, \| \cdot \|_2 \) associated to their inner products. A Schmidt decomposition for a linear mapping \( T : V_1 \to V_2 \) is a representation of \( T \) as

\[
T(v) = \sum_{j=1}^{r} \lambda_j \langle v, u_j \rangle_1 w_j,
\]

where \( r \) is a positive integer, \( u_1, \ldots, u_r \) and \( w_1, \ldots, w_r \) are orthonormal vectors in \( V_1 \) and \( V_2 \), respectively, and \( \lambda_1, \ldots, \lambda_r \) are scalars. The existence of a Schmidt decomposition uses the fact that \( T^* \circ T \) is self-adjoint on \( V_1 \), and hence can be diagonalized in an orthonormal basis. It also uses the observation that

\[
T(y) \perp T(z)
\]

in \( V_2 \) when \( y, z \in V_1 \), \( y \perp z \), and \( y \) is an eigenvector for \( T^* \circ T \).

If \( T \) has a Schmidt decomposition (3.85), then it is easy to see that

\[
\|T\|_{op} = \max(|\lambda_1|, \ldots, |\lambda_r|),
\]

and

\[
\|T\|_{HS} = \left( \sum_{j=1}^{r} |\lambda_j|^2 \right)^{1/2}.
\]

Let us check that

\[
\|T\|_{tr} = \sum_{j=1}^{r} |\lambda_j|.
\]

Clearly

\[
\|T\|_{tr} \leq \sum_{j=1}^{r} |\lambda_j|,
\]
by the definition of the trace norm, and

\[ (3.91) \quad \sum_{j=1}^{r} |\lambda_j| = \sum_{j=1}^{r} |\langle T(u_j), w_j \rangle|_2. \]

Let \( y_1, \ldots, y_k \in V_1 \) and \( z_1, \ldots, z_k \in V_2 \) be arbitrary orthonormal collections of vectors, and let us check that

\[ (3.92) \quad \sum_{l=1}^{k} |\langle R(y_l), z_l \rangle|_2 \leq \| R \| _{tr} \]

for every linear mapping \( R : V_1 \to V_2 \). If \( R \) is of the form \( R(v) = \langle v, a \rangle b \) for some \( a \in V_1 \) and \( b \in V_2 \), then

\[ (3.93) \quad \sum_{l=1}^{k} |\langle R(y_l), z_l \rangle|_2 = \sum_{l=1}^{k} |\langle y_l, a \rangle|_1 |\langle b, z_l \rangle|_2 \leq \| a \| _1 \| b \| _2, \]

using the Cauchy–Schwarz inequality in the second step. Thus the left side of (3.92) is less than or equal to the operator norm of \( R \) when \( R \) has rank 1, which implies (3.92) in general. If \( T \) has Schmidt decomposition (3.85), then we can apply (3.92) with \( R = T, y_l = u_l, \) and \( z_l = w_l \), to get that the right side of (3.91) is less than or equal to the trace norm of \( T \), as desired.

### 3.9 \( S_p \) norms

Let \((V_1, \langle \cdot, \cdot \rangle_1), (V_2, \langle \cdot, \cdot \rangle_2)\) be inner product spaces again, both real or both complex, with norms \( \| \cdot \|_1, \| \cdot \|_2 \) associated to their inner products, and let \( p \) be a real number, \( 1 < p < \infty \). If \( T \) is a linear mapping from \( V_1 \) to \( V_2 \) with Schmidt decomposition (3.85), and \( y_1, \ldots, y_k \in V_1, z_1, \ldots, z_k \in V_2 \) are orthonormal collections of vectors, then

\[ (3.94) \quad \left( \sum_{h=1}^{k} |\langle T(y_h), z_h \rangle|_2^p \right)^{1/p} \leq \left( \sum_{j=1}^{r} |\lambda_j|^p \right)^{1/p}. \]

Equivalently,

\[ (3.95) \quad \left( \sum_{h=1}^{k} \sum_{l=1}^{r} \lambda_l \langle y_h, u_l \rangle_1 \langle w_l, z_h \rangle_2 \right)^{1/p} \leq \left( \sum_{j=1}^{r} |\lambda_j|^p \right)^{1/p}. \]

To see this, observe that

\[ (3.96) \quad \left( \sum_{l=1}^{r} |\langle y_h, u_l \rangle_1|^2 \right)^{1/2} \leq \| y_h \|_1 = 1 \]
3.9 \( S_p \) Norms

and

\[
\left( \sum_{l=1}^{r} |\langle w_l, z_h \rangle_2|^2 \right)^{1/2} \leq \|z_h\|_2 = 1
\]

for each \( h \), and that

\[
\left( \sum_{h=1}^{k} |\langle y_h, u_l \rangle_1|^2 \right)^{1/2} \leq \|u_l\|_1 = 1,
\]

and

\[
\left( \sum_{h=1}^{k} |\langle w_l, z_h \rangle_2|^2 \right)^{1/2} \leq \|w_l\|_2 = 1
\]

for each \( l \). Hence

\[
\sum_{l=1}^{r} |\langle y_h, u_l \rangle_1 \langle w_l, z_h \rangle_2| \leq 1
\]

for each \( h \), and

\[
\sum_{h=1}^{k} |\langle y_h, u_l \rangle_1 \langle w_l, z_h \rangle_2| \leq 1
\]

for each \( l \), by the Cauchy–Schwarz inequality. The desired estimate (3.95) can now be derived from Schur’s theorem in Section 2.5.

The \( S_p \) norm of \( T \) is denoted \( \|T\|_{S_p} \) and defined by

\[
\|T\|_{S_p} = \left( \sum_{j=1}^{r} |\lambda_j|^p \right)^{1/p}
\]

when \( 1 \leq p < \infty \), and

\[
\|T\|_{S_{\infty}} = \max(|\lambda_1|, \ldots, |\lambda_r|).
\]

This is equal to the trace norm of \( T \) when \( p = 1 \), the Hilbert–Schmidt norm of \( T \) when \( p = 2 \), and the operator norm of \( T \) when \( p = \infty \). Equivalently,

\[
\|T\|_{S_p} = \sup \left( \sum_{h=1}^{k} |\langle T(y_h), z_h \rangle_2|^p \right)^{1/p}
\]

when \( 1 \leq p < \infty \), where the supremum is taken over all orthonormal collections of vectors \( y_1, \ldots, y_k \) and \( z_1, \ldots, z_k \) in \( V_1 \) and \( V_2 \), since (3.94) shows that the supremum is attained by the Schmidt decomposition. Using (3.104), one can check that the \( S_p \) norm satisfies the triangle inequality, and hence is a norm on the vector space \( \mathcal{L}(V_1, V_2) \) of linear mappings from \( V_1 \) into \( V_2 \).

Similarly, if \( p \geq 2 \) and \( y_1, \ldots, y_k \in V_1 \) are orthonormal, then

\[
\left( \sum_{h=1}^{k} \|T(y_h)\|_2^p \right)^{1/p} \leq \|T\|_{S_p}
\]
for every linear mapping $T : V_1 \rightarrow V_2$. To see this, let (3.85) be a Schmidt decomposition for $T$, and observe that

$$
\|T(y_h)\|_2 = \left( \sum_{l=1}^{r} |\lambda_l|^2 |\langle y_h, u_l \rangle_1|^2 \right)^{1/2}
$$

(3.106)

for each $h$. If $q = p/2$ and $\mu_j = |\lambda_j|^2$, then (3.105) can be re-expressed as

$$
\left( \sum_{h=1}^{k} \left( \sum_{l=1}^{r} \mu_l |\langle y_h, u_l \rangle_1|^2 \right)^{q} \right)^{1/q} \leq \left( \sum_{j=1}^{r} \mu_j^q \right)^{1/q}.
$$

(3.107)

The orthonormality of $y_1, \ldots, y_k$ and $u_1, \ldots, u_r$ in $V_1$ imply that (3.96) holds for each $h$ and that (3.98) holds for each $l$, as before. The desired estimate again follows from Schur’s theorem in Section 2.5.

### 3.10 Duality

Let $(V_1, \langle \cdot, \cdot \rangle_1)$, $(V_2, \langle \cdot, \cdot \rangle_2)$ be inner product spaces again, both real or both complex, and with norms $\| \cdot \|_1$, $\| \cdot \|_2$ associated to their inner products. Let $T$ be a linear mapping from $V_1$ into $V_2$, and let $R$ be a linear mapping from $V_2$ into $V_1$. Suppose that $T$ has Schmidt decomposition (3.85), so that

$$
(R \circ T)(v) = \sum_{j=1}^{r} \lambda_j \langle v, u_j \rangle_1 R(w_j)
$$

(3.108)

for each $v \in V_1$. Thus

$$
\text{tr}_{V_1}(R \circ T) = \sum_{j=1}^{r} \lambda_j \langle R(w_j), u_j \rangle_1.
$$

(3.109)

If $1 < p, q < \infty$ are conjugate exponents, then we get that

$$
|\text{tr}_{V_1}(R \circ T)| \leq \left( \sum_{j=1}^{j} |\lambda_j|^p \right)^{1/p} \left( \sum_{j=1}^{r} |\langle R(w_j), u_j \rangle_1|^q \right)^{1/q},
$$

(3.110)

by Hölder’s inequality. This implies that

$$
|\text{tr}_{V_1}(R \circ T)| \leq \|T\|_{S_p} \|R\|_{S_q},
$$

(3.111)

using the definition of the $S_p$ norm of $T$ in terms the Schmidt decomposition, and the analogue of (3.104) for $R$ and $q$. This also works when $p = \infty$ or $q = \infty$, by the same argument.

The preceding inequality implies that

$$
T \mapsto \text{tr}_{V_1}(R \circ T)
$$

(3.112)
has dual norm less than or equal to $\|R\|_{S_q}$ with respect to the norm $\|T\|_{S_p}$ on $\mathcal{L}(V_1, V_2)$ when $1 \leq p, q \leq \infty$ are conjugate exponents. To show that the dual norm is equal to $\|R\|_{S_q}$, it suffices to check that

$$\text{tr}_{V_1}(R \circ T) = \|T\|_{S_p} \|R\|_{S_q} \tag{3.113}$$

for some $T \neq 0$. Let us begin this time with a Schmidt decomposition

$$R(w) = \sum_{j=1}^{r} \mu_j \langle w, w_j \rangle_2 u_j \tag{3.114}$$

for $R$, where $u_1, \ldots, u_r$ and $w_1, \ldots, w_r$ be orthonormal vectors in $V_1$ and $V_2$, respectively, and $\mu_1, \ldots, \mu_k$ are real or complex numbers, as appropriate. Let us also restrict our attention now to linear mappings $T$ from $V_1$ into $V_2$ with Schmidt decomposition (3.85), using the same orthonormal vectors $u_1, \ldots, u_r$ and $w_1, \ldots, w_r$ as for $R$. In this case, the trace of $R \circ T$ reduces to

$$\text{tr}_{V_1}(R \circ T) = \sum_{h=1}^{k} \mu_h \lambda_h, \tag{3.115}$$

and for any $\mu_1, \ldots, \mu_k$ one can choose $\lambda_1, \ldots, \lambda_k$, not all equal to 0, such that (3.113) holds, as in Section 2.2.
Chapter 4

Seminorms and sublinear functions

As in the previous two chapters, we continue to restrict our attention to finite-dimensional vector spaces in this chapter.

4.1 Seminorms

A seminorm on a real or complex vector space \( V \) is a nonnegative real-valued function \( N \) on \( V \) such that
\[
N( tv) = |t| \, N(v)
\]
for every \( v \in V \) and \( t \in \mathbb{R} \) or \( \mathbb{C} \), as appropriate, and
\[
N(v + w) \leq N(v) + N(w)
\]
for every \( v, w \in V \). Note that \( N(0) = 0 \), as one can see by applying (4.1) with \( t = 0 \). Thus a seminorm \( N \) on \( V \) is a norm when \( N(v) > 0 \) for every \( v \in V \) with \( v \neq 0 \). If \( \lambda \) is a linear functional on \( V \), then
\[
N_\lambda(v) = |\lambda(v)|
\]
is a seminorm on \( V \), and the sum and maximum of finitely many seminorms on \( V \) are also seminorms on \( V \).

If \( N \) is a seminorm on \( V \) and \( v, w \in V \), then
\[
N(v) - N(w) \leq N(v - w)
\]
and
\[
N(w) - N(v) \leq N(w - v) = N(v - w),
\]
by the triangle inequality. Hence
\[
|N(v) - N(w)| \leq N(v - w).
\]
Suppose for the moment that $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, and let $|v|$ be the standard Euclidean norm on $V$. It is easy to see that there is a nonnegative real number $C$ such that

\begin{equation}
N(v) \leq C |v|
\end{equation}

for every $v \in V$, by expressing $v$ as a linear combination of the standard basis vectors for $V$ and using (4.1) and (4.2). This together with (4.6) implies that $N$ is a continuous function on $V$ with respect to the standard Euclidean metric and topology.

Suppose that $V$ is a real vector space, $N$ is a seminorm on $V$, and $\lambda$ is a linear functional on $V$ such that

\begin{equation}
\lambda(v) \leq N(v)
\end{equation}

for every $v \in V$. This implies that

\begin{equation}
-\lambda(v) = \lambda(-v) \leq N(-v) = N(v)
\end{equation}

for every $v \in V$, and hence that

\begin{equation}
|\lambda(v)| \leq N(v).
\end{equation}

Similarly, if $V$ is a complex vector space, $N$ is a seminorm on $V$, and $\lambda$ is a linear functional on $V$ such that

\begin{equation}
\text{Re} \lambda(v) \leq N(v)
\end{equation}

for every $v \in V$, then we get that

\begin{equation}
\text{Re} t \lambda(v) = \text{Re} \lambda(t v) \leq N(t v) = N(v)
\end{equation}

for every $v \in V$ and $t \in \mathbb{C}$ with $|t| = 1$. This implies again that (4.10) holds for every $v \in V$.

### 4.2 Sublinear functions

A **sublinear function** on a real or complex vector space $V$ is a real-valued function $p(v)$ on $V$ such that

\begin{equation}
p(t v) = t p(v)
\end{equation}

for every $v \in V$ and nonnegative real number $t$, and

\begin{equation}
p(v + w) \leq p(v) + p(w)
\end{equation}

for every $v, w \in V$. In particular, $p(0) = 0$, as one can see by applying (4.13) with $t = 0$. Note that seminorms are sublinear functions, but sublinear functions are not required to be nonnegative. Linear functionals on real vector spaces are sublinear functions, as are the real parts of linear functionals on complex vector spaces.
vector spaces. The sum and maximum of finitely many sublinear functions are sublinear functions, which includes the maximum of a sublinear function and 0, to get a nonnegative sublinear function.

Let $p$ be a sublinear function on $V$, and observe that
\begin{equation}
0 = p(0) \leq p(v) + p(-v)
\end{equation}
for every $v \in V$. If
\begin{equation}
p(-v) = p(v)
\end{equation}
for every $v \in V$, then it follows that
\begin{equation}
p(v) \geq 0
\end{equation}
for every $v \in V$, and that $p$ is a seminorm on $V$ when $V$ is a real vector space.

Similarly, if $V$ is a complex vector space, and
\begin{equation}
p(tv) = p(v)
\end{equation}
for every $t \in \mathbb{C}$ with $|t| = 1$, then $p$ is a seminorm on $V$.

If $p$ is a sublinear function on $V$, then
\begin{equation}
p(v) - p(w) \leq p(v-w)
\end{equation}
and
\begin{equation}
p(w) - p(v) \leq p(w-v)
\end{equation}
for every $v, w \in V$, and hence
\begin{equation}
|p(v) - p(w)| \leq \max(p(v-w), p(w-v))
\end{equation}
for every $v, w \in V$. Note that
\begin{equation}
N(v) = \max(p(v), p(-v))
\end{equation}
is a seminorm on $V$ when $V$ is a real vector space, by the remarks in the previous paragraphs. Of course, a complex vector space may also be considered to be a real vector space, by forgetting about multiplication by $i$. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, then $N(v)$ is bounded by a constant multiple of the standard Euclidean norm of $v$, as in the previous section. This implies that $p(v)$ is continuous with respect to the standard Euclidean metric and topology on $V$, as before.

4.3 Another extension theorem

**Theorem 4.23** Let $V$ be a real vector space, and let $p$ be a sublinear function on $V$. If $W$ is a linear subspace of $V$ and $\mu$ is a linear functional on $W$ such that
\begin{equation}
\mu(w) \leq p(w) \quad \text{for every } w \in W,
\end{equation}
then there is a linear functional $\hat{\mu}$ on $V$ which is equal to $\mu$ on $W$ and satisfies
\begin{equation}
\hat{\mu}(v) \leq p(v) \quad \text{for every } v \in V.
\end{equation}
This is the analogue of Theorem 2.26 in Section 2.3 for sublinear functions instead of norms. Note that the statement and proof of Theorem 2.26 already work in exactly the same way for seminorms instead of norms. The case of sublinear functions is essentially the same, except for a few simple changes following the differences in the statements of the theorems.

As in (2.32), we would like to show that there is an \( \alpha \in \mathbb{R} \) such that
\[
\mu_j(x) + t \alpha \leq p(x + tz) \quad \text{for every } x \in W_j \text{ and } t \in \mathbb{R}.
\]

This is equivalent to
\[
\mu_j(x) + \alpha \leq p(x + z), \quad \mu_j(x) - \alpha \leq p(x - z) \quad \text{for every } x \in W_j,
\]
because one can convert (4.26) into (4.27) when \( t \neq 0 \) using homogeneity, and the \( t = 0 \) case of (4.26) follows from the induction hypothesis for \( \mu_j \) and \( W_j \).

Let us rewrite (4.27) as
\[
\mu_j(x) - p(x - z) \leq \alpha \leq p(x + z) - \mu_j(x) \quad \text{for every } x \in W_j.
\]

To show that there is an \( \alpha \in \mathbb{R} \) that satisfies (4.28), it suffices to verify
\[
\mu_j(x) - p(x - z) \leq p(y + z) - \mu_j(y) \quad \text{for every } x, y \in W_j,
\]
which reduces to
\[
\mu_j(x + y) \leq p(x - z) + p(y + z) \quad \text{for every } x, y \in W_j.
\]
The subadditivity property of \( p(v) \) implies that this condition holds if
\[
\mu_j(x + y) \leq p(x + y) \quad \text{for every } x, y \in W_j,
\]
which is the same as
\[
\mu_j(w) \leq p(w) \quad \text{for every } w \in W_j.
\]
This holds by induction hypothesis, which completes the proof of Theorem 4.23.

### 4.4 Minkowski functionals

Let \( V \) be a real or complex vector space, and let \( A \) be a subset of \( V \) such that \( 0 \in A \). Suppose also that \( A \) has the absorbing property that for each \( v \in V \) there is a positive real number \( t \) such that
\[
t v \in A.
\]
If \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) for some positive integer \( n \), and if \( 0 \) is an element of the interior of \( A \) with respect to the standard Euclidean metric and topology on \( V \), then \( A \) obviously has this property.
Under these conditions, the Minkowski functional on $V$ associated to $A$ is defined by

$$N_A(v) = \inf\{r > 0 : r^{-1}v \in A\}$$

for each $v \in V$. Note that $N(v) \geq 0$ for every $v \in V$, $N(0) = 0$, and

$$N_A(tv) = tN(v)$$

for every nonnegative real number $t$. By construction,

$$N_A(v) \leq 1$$

for every $v \in A$. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, and if $A$ is an open set in $V$ with respect to the standard Euclidean metric and topology, then

$$N_A(v) < 1$$

for every $v \in A$. This is because $r^{-1}v \in A$ for all $r$ sufficiently close to 1 when $v \in A$ and $A$ is an open set.

Let $-A$ be the set of vectors in $V$ of the form $-v$ with $v \in A$, and let $tA$ be the set of vectors in $V$ of the form $tv$ with $v \in A$ for each real or complex number $t$, as appropriate. Thus

$$N_A(v) = \inf\{r > 0 : v \in rA\}$$

for each $v \in V$. If $-A = A$, then

$$N_A(-v) = N_A(v)$$

for every $v \in V$. Similarly, if $V$ is a complex vector space, and if $tA = A$ for every complex number $t$ with $|t| = 1$, then

$$N_A(tv) = N_A(v)$$

for every $v \in V$ and $t \in \mathbb{C}$ with $|t| = 1$.

Let us suppose from now on in this section that $A$ is star-like about 0, which means that $A$ contains every line segment in $V$ between 0 and any other element of $A$. Equivalently, $A$ is star-like about 0 if

$$tA \subseteq A$$

for every nonnegative real number $t$ with $t \leq 1$. If $v \in V$ satisfies $N_A(v) < 1$, then there is a positive real number $r < 1$ such that $v \in rA$, and it follows that $v \in A$. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some $n$ and $A$ is an open set in $V$ with respect to the standard metric and topology, then we get that

$$A = \{v \in V : N_A(v) < 1\}.$$  

Similarly, if $A$ is a closed set in $V = \mathbb{R}^n$ or $\mathbb{C}^n$, then

$$A = \{v \in V : N_A(v) \leq 1\}.$$
To see this, it remains to check that \( v \in A \) when \( v \in V \) satisfies \( N_A(v) = 1 \). In this case, \( r^{-1}v \in A \) for some positive real numbers \( r \) that are arbitrarily close to 1, which implies that \( v \in A \) when \( A \) is closed.

Suppose now that \( A \) is a convex set in \( V \). Note that this implies that \( A \) is star-like about 0, because 0 \( \in A \). We would like to show that

\[
N_A(v + w) \leq N_A(v) + N_A(w)
\]

for every \( v, w \in V \).

Let \( v, w \in V \) be given, and let \( r_v, r_w \) be positive real numbers such that

\[
r_v > N_A(v), \quad r_w > N_A(w).
\]

This implies that \( r_v^{-1}v, r_w^{-1}w \) are elements of \( A \), because of the definition of \( N_A \) and the fact that \( A \) is star-like about 0. If \( A \) is convex, then it follows that

\[
(r_v + r_w)^{-1}(v + w) = \frac{r_v}{r_v + r_w} (r_v^{-1}v) + \frac{r_w}{r_v + r_w} (r_w^{-1}w)
\]

is an element of \( A \) too, so that

\[
N_A(v + w) < r_v + r_w.
\]

This implies (4.44), since \( r_v, r_w \) can be arbitrarily close to \( N_A(v), N_A(w) \).

### 4.5 Convex cones

Let \( V \) be a vector space over the real numbers. A nonempty set \( E \subseteq V \) is said to be a cone if

\[
tv \in E
\]

for every \( v \in E \) and nonnegative real number \( t \), or equivalently,

\[
tE \subseteq E
\]

for every \( t \geq 0 \). In particular, \( 0 \in E \), since we can take \( t = 0 \), and \( E \neq \emptyset \). Note that every linear subspace of \( V \) is a cone.

We shall be especially interested in convex cones, which are cones that are also convex sets. Equivalently, a nonempty set \( E \subseteq V \) is a convex cone if

\[
rv + tw \in E
\]

for every \( v, w \in E \) and nonnegative real numbers \( r, t \). Thus linear subspaces of \( V \) are also convex cones. If \( V = \mathbb{R}^n \), then it is easy to see that

\[
\{v \in \mathbb{R}^n : v_j \geq 0 \text{ for } j = 1, \ldots, n\}
\]

is a convex cone in \( V \).
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If $E$ is any nonempty subset of $V$, then we can get a cone $C(E)$ in $V$ from $E$, by taking
\begin{equation}
C(E) = \bigcup_{t \geq 0} tE = \{ tv : v \in E, t \geq 0 \}.
\end{equation}

Of course, $C(E) = E$ when $E$ is already a cone in $V$. One can check that $C(E)$ is also convex in $V$ when $E$ is convex.

If $p$ is a sublinear function on $V$, then it is easy to see that
\begin{equation}
C_p = \{ v \in V : p(v) \leq 0 \}
\end{equation}
is a convex cone in $V$. This cone has another important property, which is that it is a closed set in a suitable sense. To make this precise, it is convenient to suppose that $V = \mathbb{R}^n$ for some positive integer $n$. As in Section 4.2, sublinear functions on $\mathbb{R}^n$ are continuous with respect to the standard Euclidean metric and topology, which implies that $C_p$ is a closed set.

Let $V$ be any real vector space again, and let $\| \cdot \|$ be a norm on $V$. If $E$ is a nonempty subset of $V$, then put
\begin{equation}
p_E(v) = \inf \{ \| v - w \| : w \in E \}
\end{equation}
for each $v \in V$. Thus $p_E(v) = 0$ for every $v \in E$, and conversely $p_E(v) = 0$ implies that $v \in E$ when $E$ is a closed set in a suitable sense. In particular, this works when $V = \mathbb{R}^n$ and $E$ is a closed set with respect to the standard Euclidean metric and topology, because of the remarks at the end of Section 2.1, about the relationship between an arbitrary norm on $\mathbb{R}^n$ and the standard Euclidean norm. Of course, if $V = \mathbb{R}^n$, then one can simply take $\| \cdot \|$ to be the standard Euclidean norm on $\mathbb{R}^n$.

If $E$ is a convex cone in $V$, then one can check that $p_E$ is a sublinear function on $V$. It follows that every closed convex cone in $\mathbb{R}^n$ can be expressed as in (4.53) for some sublinear function $p$ on $\mathbb{R}^n$.

4.6 Dual cones

Let $V$ be a real vector space again, and let $E$ be a nonempty subset of $V$. Consider the set $E' \subseteq V^*$ consisting of all linear functionals $\lambda$ on $V$ such that
\begin{equation}
\lambda(v) \geq 0
\end{equation}
for every $v \in E$. It is easy to see that this is a convex cone in $V^*$, known as the dual cone associated to $E$. One can also check that
\begin{equation}
C(E)' = E',
\end{equation}
so that we may as well restrict our attention to convex cones $E$ in $V$.

Suppose for the moment that $V = \mathbb{R}^n$ for some positive integer $n$, and let us identify $V^*$ with $\mathbb{R}^n$ as in Section 2.2. Observe that $E'$ is the same as
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the dual cone associated to the closure $\overline{E}$ of $E$ with respect to the standard Euclidean metric and topology. Thus we may as well restrict our attention to closed subsets of $\mathbb{R}^n$, and hence to closed convex cones. Similarly, the dual cone $E'$ of any set $E \subseteq \mathbb{R}^n$ automatically corresponds to a closed subset of $\mathbb{R}^n$, and therefore to a closed convex cone.

If $V$ is any real vector space and $E$ is a nonempty subset of $V$, then let $E''$ be the dual cone associated to $E' \subseteq V^*$, which is a convex cone in the second dual $V^{**}$ of $V$. As in Section 2.3, $V^{**}$ can be identified with $V$ in a natural way. Using this identification, one can check that

$$E \subseteq E''.$$ \hspace{1cm} (4.57)

Suppose for convenience that $V = \mathbb{R}^n$ for some $n$ again. If $E$ is a closed convex cone in $V$, then

$$E = E''.$$ \hspace{1cm} (4.58)

To see this, it suffices to show that $E'' \subseteq E$, since we already know (4.57). If $v$ is any element of $\mathbb{R}^n$ that is not in $E$, then we would like to show that $v$ is also not in $E''$. Equivalently, we would like to show that there is a linear functional $\lambda$ on $V$ such that $\lambda \geq 0$ on $E$ and $\lambda (v) < 0$.

It is easy to find such a linear functional $\lambda$ initially on the linear span $W$ of $v$ in $V$, and we would like to extend $\lambda$ to all of $V$ using Theorem 4.23 in Section 4.3. More precisely, let $p$ be a sublinear function on $V$ such that $E$ is the set where $p \leq 0$, which exists when $E$ is a closed convex cone in $V = \mathbb{R}^n$, as in the previous section. It is convenient to ask also that $p \geq 0$ everywhere on $V$, so that $E$ is the set where $p = 0$. This holds by construction when $p = p_E$ as before, and otherwise one can simply replace $p$ with $\max(p, 0)$.

Let $\mu$ be the linear functional defined on the linear span $W$ of $V$ by

$$\mu(t v) = t p(v)$$ \hspace{1cm} (4.59)

for each $t \in \mathbb{R}$. Thus

$$\mu(t v) = p(t v)$$ \hspace{1cm} (4.60)

when $t \geq 0$, and

$$\mu(t v) = t p(v) \leq 0 \leq p(t v)$$ \hspace{1cm} (4.61)

when $t \leq 0$, so that $\mu \leq p$ on $W$. Theorem 4.23 implies that there is an extension $\tilde{\mu}$ of $\mu$ to a linear functional on $V$ such that $\tilde{\mu} \leq p$ on all of $V$. If $\lambda = -\tilde{\mu}$, then $\lambda (v) = -p(v) < 0$ and $\lambda \geq -p$ on all of $V$, which implies that $\lambda \geq 0$ on $E$, as desired.

As an example, consider the case where $E$ is the set of $v \in \mathbb{R}^n$ such that $v_j \geq 0$ for $j = 1, \ldots, n$, as in (4.51). In this case, one can check that $E' = E$, using the standard identification of $\mathbb{R}^n$ with its own dual space. In particular, $E'' = E$. 

4.7 Nonnegative self-adjoint operators

Let \((V, \langle \cdot, \cdot \rangle)\) be a real or complex inner product space. A self-adjoint linear transformation \(A\) on \(V\) is said to be nonnegative if

\[
\langle A(v), v \rangle \geq 0
\]

for every \(v \in V\). The identity transformation obviously has this property, for instance. If \(W\) is a linear subspace of \(V\), then we have seen that the orthogonal projection \(P_W\) of \(V\) onto \(W\) is self-adjoint, as in Section 3.3. In this case, we also have that

\[
\langle P_W(v), v \rangle = \langle P_W(v), P_W(v) \rangle = \|P_W(v)\|^2
\]

for every \(v \in V\), as in (3.20), and hence that \(P_W\) is nonnegative.

If \(a \in V\), then it is easy to see that

\[
A_a(v) = \langle v, a \rangle a
\]

defines a nonnegative self-adjoint linear operator on \(V\), which is the same as the orthogonal projection onto the span of \(a\) when \(\|a\| = 1\). Of course, every self-adjoint linear operator \(A\) on \(V\) is a linear combination of rank-one operators like this, as a consequence of diagonalization. If the eigenvalues of \(A\) are nonnegative, then it follows that \(A\) is nonnegative, because \(A\) can be expressed as a linear combination of rank-one operators like this with nonnegative coefficients, by diagonalization. Conversely, one can check that the eigenvalues of a nonnegative self-adjoint linear operator \(A\) are nonnegative, by applying the nonnegativity condition to the eigenvectors of \(A\).

If \(T\) is any linear operator on \(V\), then \(T^* T\) is self-adjoint and nonnegative, because

\[
(T^* T)^* = T^* (T^*)^* = T^* T
\]

and

\[
\langle (T^* T)(v), v \rangle = \langle T(v), T(v) \rangle = \|T(v)\|^2 \geq 0
\]

for each \(v \in V\). In particular, if \(B\) is a self-adjoint linear operator on \(V\), then \(B^*\) is a nonnegative self-adjoint linear operator on \(V\). Conversely, one can use diagonalizations to show that every nonnegative self-adjoint linear operator \(A\) on \(V\) can be expressed as \(B^2\) for some nonnegative self-adjoint linear operator \(B\) on \(V\). Note that \(-B^2\) is a nonnegative self-adjoint linear operator on \(V\) when \(B\) is anti-self-adjoint.

If \(A\) is a nonnegative self-adjoint linear transformation on \(V\), then the trace of \(A\) is equal to the sum of the eigenvalues of \(A\), with their appropriate multiplicity, because of diagonalization. More precisely,

\[
\text{tr } A = \|A\|_{S_1} = \|A\|_{tr},
\]

as in Sections 3.8 and 3.9. Similarly, let us check that

\[
\text{tr } AB \geq 0
\]
for every pair of nonnegative self-adjoint linear operators $A, B$ on $V$. If $A = A_a$ is as in (4.64), then
\begin{equation}
\text{tr} AB = \langle B(a), a \rangle \geq 0.
\end{equation}
(4.69)
Otherwise, $A$ can be expressed as a sum of rank 1 operators of this type, as before, and (4.68) follows.

Conversely, if $B$ is a self-adjoint linear operator on $V$ such that (4.68) holds for every nonnegative self-adjoint linear operator $A$ on $V$, then $B$ is also non-negative. This follows by applying this condition to $A = A_a$ as in (4.64), as in the previous paragraph.

Let $\mathcal{L}(V)$ be the vector space of all linear operators on $V$, and let $\mathcal{L}_{sa}(V)$ be the collection of self-adjoint linear operators on $V$. Thus $\mathcal{L}_{sa}(V)$ is a linear subspace of $\mathcal{L}(V)$ when $V$ is a real vector space. If $V$ is a complex vector space, then $\mathcal{L}(V)$ is a complex vector space, but $\mathcal{L}_{sa}(V)$ is a real vector space, which may be considered as a real-linear subspace of $\mathcal{L}(V)$.

As in Section 3.7,
\begin{equation}
\langle A, B \rangle_{\mathcal{L}(V)} = \text{tr} AB^*
\end{equation}
(4.70)
defines an inner product on $\mathcal{L}(V)$, for which the corresponding norm is the Hilbert–Schmidt norm. This reduces to
\begin{equation}
\langle A, B \rangle_{\mathcal{L}_{sa}(V)} = \text{tr} AB
\end{equation}
(4.71)when $A$ and $B$ are self-adjoint. Using this inner product, we can identify $\mathcal{L}_{sa}(V)$ with its own dual space in the usual way.

The set of nonnegative self-adjoint linear operators on $V$ forms a convex cone in $\mathcal{L}_{sa}(V)$, and it is also a closed set in $\mathcal{L}_{sa}(V)$ in a suitable sense. This cone is equal to its own dual cone, when we identify the dual of $\mathcal{L}_{sa}(V)$ with itself using the inner product in the previous paragraph. This follows from the earlier remarks about traces of products of self-adjoint linear operators on $V$. 
Chapter 5

Sums and $\ell^p$ spaces

5.1 Nonnegative real numbers

Let $E$ be a nonempty set, and let $f$ be a nonnegative real-valued function on $E$. If $A$ is a nonempty subset of $E$ with only finitely many elements, then the sum

$$\sum_{x \in A} f(x) \quad (5.1)$$

can be defined in the usual way. The sum

$$\sum_{x \in E} f(x) \quad (5.2)$$

is then defined as the supremum of the finite subsums (5.1), over all finite nonempty subsets $A$ of $E$. More precisely, the sum (5.2) is considered to be $+\infty$ as an extended real number when there is no finite upper bound for the finite subsums (5.1).

Of course, if $E$ is itself a finite set, then this definition of the sum (5.2) reduces to the usual one. If $E$ is the set $\mathbb{Z}_+$ of positive integers, then the usual definition of the sum of an infinite series is equivalent to

$$\sum_{j=1}^{\infty} f(j) = \sup_{n \geq 1} \sum_{j=1}^{n} f(j), \quad (5.3)$$

which is again interpreted as being $+\infty$ when there is no finite upper bound for the partial sums. In this case, this definition of the infinite sum is equivalent to the previous one. More precisely, this definition of the sum is less than or equal to the previous one, because the partial sums $\sum_{j=1}^{n} f(j)$ are subsums of the form (5.1) for each $n$. Similarly, the previous definition of the sum is less than or equal to this one, because every finite set $A \subseteq E = \mathbb{Z}_+$ is contained in a set of the form $\{1, \ldots, n\}$ for some $n$, so that the corresponding subsum (5.1) is less than or equal to the partial sum $\sum_{j=1}^{n} f(j)$.
5.2. SUMMABLE FUNCTIONS

Suppose that the finite subsums (5.1) are bounded by a nonnegative real number $C$, so that the sum (5.2) is also less than or equal to $C$. In this case, it is easy to see that

\[ E(f, \epsilon) = \{ x \in E : f(x) \geq \epsilon \} \]  

(5.4)

has at most $C/\epsilon$ elements for each $\epsilon > 0$. Otherwise, if $f(x) \geq \epsilon$ for more than $C/\epsilon$ elements $x$ of $E$, then there would be a finite set $A \subseteq E$ such that (5.1) is larger than $C$. In particular, $E(f, \epsilon)$ has only finitely many elements for each $\epsilon > 0$. This implies that

\[ \{ x \in E : f(x) > 0 \} = \bigcup_{n=1}^{\infty} E(f, 1/n) \]  

(5.5)

has only finitely or countably many elements, so that (5.2) can be reduced to a finite sum or an ordinary infinite series.

If $f$, $g$ are nonnegative real-valued functions on $E$, then one can check that

\[ \sum_{x \in E} (f(x) + g(x)) = \sum_{x \in E} f(x) + \sum_{x \in E} g(x), \]  

(5.6)

by reducing to the case of finite sums. This includes the case where some of the sums are infinite, with the usual conventions for sums of extended real numbers. Similarly, if $a$ is a nonnegative real number, then

\[ \sum_{x \in E} a f(x) = a \sum_{x \in E} f(x), \]  

(5.7)

with the convention that $a \cdot (+\infty) = +\infty$ when $a > 0$. In this context, it is also appropriate to make the convention that $0 \cdot (+\infty) = 0$, since the left side of (5.7) is automatically equal to 0 when $a = 0$.

5.2 Summable functions

A real or complex-valued function $f$ on a nonempty set $E$ is said to be summable on $E$ if

\[ \sum_{x \in E} |f(x)| \]  

(5.8)

is finite. If $f$ and $g$ are summable functions on $E$, then it is easy to see that $f + g$ is summable, since

\[ |f(x) + g(x)| \leq |f(x)| + |g(x)| \]  

(5.9)

for each $x \in E$. Similarly, if $f$ is a summable function on $E$ and $a$ is a real or complex number, as appropriate, then $a f$ is a summable function on $E$ too. Thus the real or complex-valued summable functions on $E$ form a vector space over the real or complex numbers, as appropriate.
Let $f$ be a real or complex-valued summable function on $E$, and let $\epsilon > 0$ be given. By definition of (5.8), there is a finite set $A_{\epsilon} \subseteq E$ such that

\[ \sum_{x \in E} |f(x)| \leq \sum_{x \in A_{\epsilon}} |f(x)| + \epsilon. \]  

Equivalently,

\[ \sum_{x \in A} |f(x)| \leq \sum_{x \in A_{\epsilon}} |f(x)| + \epsilon \]

for every finite set $A \subseteq E$. If $B$ is a finite subset of $E$ which is disjoint from $A_{\epsilon}$, then we get that

\[ \sum_{x \in B} |f(x)| \leq \epsilon, \]  

by applying the previous inequality to $A = A_{\epsilon} \cup B$. This will be used frequently to approximate various sums by finite sums.

We would like to define the sum $\sum_{x \in E} f(x)$ of a real or complex-valued summable function $f$ on $E$. One way to do this is to express $f$ as a linear combination of nonnegative real-valued summable functions on $E$, and then use the previous definition of the sum for nonnegative real-valued functions on $E$. Another way to do this is to use the fact that the set of $x \in E$ such that $f(x) \neq 0$ has only finitely or countably many elements, as in the previous section, and then treat the sum as a finite sum or an infinite series, by enumerating the elements of this set. The summability of $f$ on $E$ implies that such an infinite series would be absolutely convergent, and hence convergent. The value of the sum would also not depend on the choice of the enumeration of the set where $f \neq 0$, because absolutely-convergent infinite series are invariant under rearrangements.

Whichever way one uses to define the sum, a key point is that it should be approximable by finite sums in a natural way. More precisely, for each $\epsilon > 0$, there should be a finite set $A_{\epsilon} \subseteq E$ such that

\[ \left| \sum_{x \in E} f(x) - \sum_{x \in A} f(x) \right| \leq \epsilon \]

for every finite set $A \subseteq E$ such that $A_{\epsilon} \subseteq A$. One can check that both of the approaches to defining the sum described in the previous paragraph have this approximation property. Practically any reasonable way of defining the sum as a limit of finite subsums should also have this property, because of the approximation property (5.12) mentioned earlier.

At any rate, it is easy to see that any definition of the sum that satisfies (5.13) is uniquely determined by this property. In particular, this can be helpful for showing that such a definition does not depend on any auxiliary choices used to define the sum. In effect, (5.13) characterizes the sum as a somewhat fancy limit of the finite subsums $\sum_{x \in A} f(x)$. This can be made precise by treating these subsums as a net, which is indexed by the collection of all finite subsets $A$ of $E$. This also uses the natural partial ordering on the collection of all finite subsets of $E$ by inclusion.
If $f$ is any real or complex-valued summable function on $E$, then

$$
\left| \sum_{x \in E} f(x) \right| \leq \sum_{x \in E} |f(x)|.
$$

(5.14)

This follows easily by approximating the sum by finite subsums, and using the triangle inequality. If $f$ and $g$ are summable functions on $E$, and $a$ and $b$ are real or complex numbers, as appropriate, then one can also check that

$$
\sum_{x \in E} (a f(x) + b g(x)) = a \sum_{x \in E} f(x) + b \sum_{x \in E} g(x).
$$

(5.15)

As usual, this can be shown by approximating the sums by finite subsums. Using these two properties of the sum, we get that

$$
\left| \sum_{x \in E} f(x) - \sum_{x \in E} g(x) \right| = \left| \sum_{x \in E} (f(x) - g(x)) \right| \leq \sum_{x \in E} |f(x) - g(x)|
$$

(5.16)

for any pair of summable functions $f$, $g$ on $E$.

If $f$ is any real or complex-valued function on $E$, then the support of $f$ is the set $\text{supp } f$ of $x \in E$ such that $f(x) \neq 0$. If the support of $f$ has only finitely many elements, then the sum $\sum_{x \in E} f(x)$ reduces to an ordinary finite sum. The sum $\sum_{x \in E} f(x)$ of a summable function $f$ on $E$ can be characterized as the unique linear functional on the vector space of all summable functions on $E$ that satisfies (5.14) and reduces to the ordinary finite sum when $f$ has finite support. This follows by approximating an arbitrary summable function on $E$ by functions with finite support using (5.12), and then using (5.16) to analyze the corresponding sums.

### 5.3 Convergence theorems

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of real or complex-valued summable functions on a nonempty set $E$, and suppose that

$$
\lim_{j \to \infty} f_j(x) = f(x)
$$

(5.17)

for every $x \in E$. Of course,

$$
\lim_{j \to \infty} \sum_{x \in E} f_j(x) = \sum_{x \in E} f(x)
$$

(5.18)

when $E$ is finite, but the situation is more complicated when $E$ is infinite. For example, if $E$ is the set of positive integers, $f_j(j) = 1$ for each $j$, and $f_j(x) = 0$ when $x \neq j$, then

$$
\sum_{x \in E} f_j(x) = 1
$$

(5.19)
for each $j$, but $\{f_j(x)\}_{j=1}^\infty$ converges to 0 for each $x \in E$. Alternatively, if we take $f_j(x) = 1/j$ when $x \leq j$ and $f_j(x) = 0$ when $x > j$, then we get the same conclusions, but with $\{f_j\}_{j=1}^\infty$ converging to 0 uniformly on $E$.

However, there are some positive results, as follows. Suppose that $\{f_j(x)\}_{j=1}^\infty$ is a monotone increasing sequence of nonnegative real valued functions on $E$, so that
\begin{equation}
(5.20) \quad f_j(x) \leq f_{j+1}(x)
\end{equation}
for every $x \in E$ and $j \geq 1$. Put
\begin{equation}
(5.21) \quad f(x) = \sup_{j \geq 1} f_j(x)
\end{equation}
for each $x \in E$, which may be $+\infty$. Thus
\begin{equation}
(5.22) \quad f_j(x) \to \sup_{l \geq 1} f_l(x) \quad \text{as} \quad j \to \infty
\end{equation}
for every $x \in E$, with the usual interpretation when $f(x) = +\infty$. Under these conditions, the monotone convergence theorem implies that
\begin{equation}
(5.23) \quad \sum_{x \in E} f_j(x) \to \sum_{x \in E} f(x) \quad \text{as} \quad j \to \infty,
\end{equation}
with suitable interpretations when any of the quantities involved are infinite.

Of course,
\begin{equation}
(5.24) \quad \sum_{x \in E} f_j(x) \leq \sum_{x \in E} f_{j+1}(x) \leq \sum_{x \in E} f(x)
\end{equation}
for each $j$, by monotonicity. If $\sum_{x \in E} f_l(x) = +\infty$ for some $l$, then
\begin{equation}
(5.25) \quad \sum_{x \in E} f_j(x) = \sum_{x \in E} f(x) = +\infty
\end{equation}
for each $j \geq l$, and (5.23) is trivial. Similarly, if $f(x_0) = +\infty$ for some $x_0 \in E$, then $\sum_{x \in E} f(x)$ is interpreted as being equal to $+\infty$, and
\begin{equation}
(5.26) \quad \sum_{x \in E} f_j(x) \geq f_j(x_0)
\end{equation}
also tends to $+\infty$ as $j \to \infty$. Suppose then that each $f_j$ is summable, and that $f(x)$ is finite for every $x \in E$. In this case, one can get (5.23) using the fact that
\begin{equation}
(5.27) \quad \sum_{x \in A} f(x) = \lim_{j \to \infty} \sum_{x \in A} f_j(x) \leq \lim_{j \to \infty} \sum_{x \in E} f_j(x)
\end{equation}
for every finite set $A \subseteq E$, since $\sum_{x \in E} f(x)$ is equal to the supremum of $\sum_{x \in A} f(x)$ over all finite subsets $A$ of $E$.

Now let $\{f_j\}_{j=1}^\infty$ be any sequence of nonnegative real-valued functions on $E$, and put
\begin{equation}
(5.28) \quad f(x) = \liminf_{j \to \infty} f_j(x).
\end{equation}
5.3. CONVERGENCE THEOREMS

Under these conditions, Fatou’s lemma states that

\[(5.29) \quad \sum_{x \in E} f(x) \leq \liminf_{j \to \infty} \sum_{x \in E} f_j(x),\]

again with suitable interpretations when any of the quantities are infinite. If \(A\) is a finite subset of \(E\), then

\[(5.30) \quad \sum_{x \in A} f(x) \leq \liminf_{j \to \infty} \sum_{x \in A} f_j(x)\]

is a well known property of the lower limit. This implies that

\[(5.31) \quad \sum_{x \in A} f(x) \leq \liminf_{j \to \infty} \sum_{x \in E} f_j(x)\]

because the sum of \(f_j(x)\) over \(x \in A\) is less than or equal to the sum of \(f_j(x)\) over \(x \in E\) when \(A \subseteq E\). In order to get (5.29), it suffices to take the supremum of the left side of (5.31) over all finite subsets \(A\) of \(E\).

Suppose now that \(\{f_j\}_{j=1}^{\infty}\) is a sequence of real or complex valued functions on \(E\) that converges pointwise to another function \(f\) on \(E\). Suppose also that \(h\) is a nonnegative real-valued function on \(E\) which is summable, and that

\[(5.32) \quad |f_j(x)| \leq h(x)\]

for every \(x \in E\) and \(j \geq 1\). This implies that

\[(5.33) \quad |f(x)| \leq h(x)\]

for every \(x \in E\), because \(\{f_j(x)\}_{j=1}^{\infty}\) converges pointwise to \(f(x)\) for every \(x \in E\) by hypothesis. Note that \(f_j\) is a summable function on \(E\) for each \(j\), and that \(f\) is a summable function on \(E\) too, since \(h\) is summable.

Under these conditions, the dominated convergence theorem implies that

\[(5.34) \quad \lim_{j \to \infty} \sum_{x \in E} f_j(x) = \sum_{x \in E} f(x).\]

To prove this, it suffices to show that

\[(5.35) \quad \lim_{j \to \infty} \sum_{x \in E} |f_j(x) - f(x)| = 0.\]

Let \(\epsilon > 0\) be given, and let us show that there is a \(L \geq 1\) such that

\[(5.36) \quad \sum_{x \in E} |f(x) - f_j(x)| < \epsilon\]

for every \(j \geq L\).

Because \(h\) is summable on \(E\), there is a finite set \(E_{\epsilon} \subseteq E\) such that

\[(5.37) \quad \sum_{x \in E \setminus E_{\epsilon}} |h(x)| \leq \frac{\epsilon}{3}.\]
as in (5.12). Thus

\[(5.38) \quad \sum_{x \in E \setminus E_\epsilon} |f(x) - f_j(x)| \leq \sum_{x \in E \setminus E_\epsilon} 2|h(x)| \leq \frac{2\epsilon}{3}\]

for each \(j\). We also have that

\[(5.39) \quad \lim_{j \to \infty} \sum_{x \in E_\epsilon} |f(x) - f_j(x)| = 0,\]

because \(E_\epsilon\) is finite and \(\{f_j(x)\}_{j=1}^\infty\) converges to \(f(x)\) for each \(x \in E\). This implies that there is an \(L \geq 1\) such that

\[(5.40) \quad \sum_{x \in E_\epsilon} |f(x) - f_j(x)| < \frac{\epsilon}{3}\]

when \(j \geq L\). Combining (5.38) and (5.40), we get that (5.36) holds when \(j \geq L\), as desired.

### 5.4 Double sums

Let \(E_1\) and \(E_2\) be nonempty sets, and let \(E = E_1 \times E_2\) be their Cartesian product. If \(f(x, y)\) is a nonnegative real-valued function on \(E\), then we would like to check that

\[(5.41) \quad \sum_{(x, y) \in E} f(x, y) = \sum_{x \in E_1} \left( \sum_{y \in E_2} f(x, y) \right) = \sum_{y \in E_2} \left( \sum_{x \in E_1} f(x, y) \right).\]

More precisely, it may be that \(\sum_{y \in E_2} f(x', y) = +\infty\) for some \(x' \in E_1\), or that \(\sum_{x \in E_1} f(x, y') = +\infty\) for some \(y' \in E_2\), in which case the corresponding iterated sum in (5.41) is considered to be \(+\infty\) as well.

Let \(A\) be a finite subset of \(E\), and \(A_1, A_2\) be finite subsets of \(E_1, E_2\), respectively, such that

\[(5.42) \quad A \subseteq A_1 \times A_2.\]

Clearly

\[(5.43) \quad \sum_{(x, y) \in A} f(x, y) \leq \sum_{x \in A_1} \left( \sum_{y \in A_2} f(x, y) \right) = \sum_{y \in A_2} \left( \sum_{x \in A_1} f(x, y) \right).\]

This implies that \(\sum_{(x, y) \in A} f(x, y)\) is less than or equal to each of

\[(5.44) \quad \sum_{x \in E_1} \left( \sum_{y \in E_2} f(x, y) \right) \text{ and } \sum_{y \in E_2} \left( \sum_{x \in E_1} f(x, y) \right).\]

It follows that \(\sum_{(x, y) \in E} f(x, y)\) is less than or equal to the same iterated sums, because \(A\) is an arbitrary finite subset of \(E\).
5.4. DOUBLE SUMS

In the other direction, if \( A_1 \) and \( A_2 \) are arbitrary finite subsets of \( E_1 \) and \( E_2 \), respectively, then

\[
\sum_{x \in A_1} \left( \sum_{y \in A_2} f(x, y) \right) = \sum_{y \in A_2} \left( \sum_{x \in A_1} f(x, y) \right) = \sum_{(x,y) \in A_1 \times A_2} f(x, y),
\]

and therefore

\[
\sum_{x \in A_1} \left( \sum_{y \in A_2} f(x, y) \right) = \sum_{y \in A_2} \left( \sum_{x \in A_1} f(x, y) \right) \leq \sum_{(x,y) \in E} f(x, y).
\]

Using this, it is easy to see that

\[
\sum_{x \in A_1} \left( \sum_{y \in E_2} f(x, y) \right) \leq \sum_{(x,y) \in E} f(x, y)
\]

and

\[
\sum_{y \in A_2} \left( \sum_{x \in E_1} f(x, y) \right) \leq \sum_{(x,y) \in E} f(x, y).
\]

This implies that the iterated sums are less than or equal to the sum of \( f(x, y) \) over \( E \), as desired, by taking the suprema over \( A_1 \) and \( A_2 \).

Now let \( f(x, y) \) be a real or complex-valued function on \( E = E_1 \times E_2 \). If any of the sums

\[
\sum_{(x,y) \in E} |f(x, y)|, \quad \sum_{x \in E_1} \left( \sum_{y \in E_2} |f(x, y)| \right), \quad \sum_{y \in E_2} \left( \sum_{x \in E_1} |f(x, y)| \right)
\]

are finite, then they are all finite, and equal to each other, by the previous discussion. Suppose that this is the case, so that \( f(x, y) \) is a summable function on \( E \). We also get that

\[
\sum_{y \in D_2} |f(x, y)| < +\infty
\]

for every \( x \in D_1 \), and

\[
\sum_{x \in D_1} |f(x, y)| < +\infty
\]

for every \( y \in D_2 \), so that \( f(x, y) \) is a summable function on \( D_2 \) for each \( x \in D_1 \), and \( f(x, y) \) is a summable function on \( D_1 \) for each \( y \in D_2 \). Put

\[
f_1(x) = \sum_{y \in E_2} f(x, y)
\]

for each \( x \in E_1 \), and

\[
f_2(y) = \sum_{x \in E_1} f(x, y)
\]

for each \( y \in E_2 \). Thus

\[
|f_1(x)| \leq \sum_{y \in E_2} |f(x, y)|
\]
for every $x \in E_1$, and
\begin{equation}
|f_2(y)| \leq \sum_{x \in E_1} |f(x, y)|
\end{equation}
for every $y \in E_2$. This implies that $f_1(x)$ is a summable function on $E_1$, and that $f_2(y)$ is a summable function on $E_2$, because of the finiteness of the iterated sums in (5.49). Under these conditions, we also have that
\begin{equation}
\sum_{(x,y) \in E} f(x,y) = \sum_{x \in E_1} f_1(x) = \sum_{y \in E_2} f_2(y).
\end{equation}
One way to show this is to express $f(x, y)$ as a linear combination of nonnegative summable functions on $E$, and apply (5.41) in that case. Alternatively, one can approximate $f(x, y)$ by functions with finite support on $E$, as in (5.12). The equality of the iterated and double sums is clear for functions with finite support, and one can use the previous results for nonnegative functions to estimate the errors in the approximations.

### 5.5 $\ell^p$ Spaces

Let $E$ be a nonempty set, and let $p$ be a positive real number. A real or complex-valued function $f$ on $E$ is said to be $p$-summable if $|f(x)|^p$ is a summable function on $E$. The spaces of real and complex-valued $p$-summable functions on $E$ are denoted $\ell^p(E, \mathbb{R})$ and $\ell^p(E, \mathbb{C})$, respectively, although we may also use $\ell^p(E)$ to include both cases at the same time. If $f$ and $g$ are $p$-summable functions on $E$, then it is easy to see that $f + g$ is also $p$-summable, using the observation that
\begin{equation}
|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq 2 \max(|f(x)|, |g(x)|)
\end{equation}
for every $x \in E$, and hence
\begin{equation}
|f(x) + g(x)|^p \leq 2^p \max(|f(x)|^p, |g(x)|^p) \leq 2^p \left(|f(x)|^p + |g(x)|^p\right).
\end{equation}
Similarly, a function $f(x)$ is $p$-summable on $E$ when $f(x)$ is $p$-summable on $E$ and $a$ is a real or complex number, as appropriate, so that $\ell^p(E, \mathbb{R})$ and $\ell^p(E, \mathbb{C})$ are vector spaces with respect to pointwise addition and scalar multiplication of functions on $E$.

If $f$ is a real or complex-valued $p$-summable function on $E$, then we put
\begin{equation}
\|f\|_p = \left(\sum_{x \in E} |f(x)|^p\right)^{1/p}.
\end{equation}
Thus
\begin{equation}
\|a f\|_p = |a| \|f\|_p
\end{equation}
for every real or complex number $a$, as appropriate. In this context, Minkowski’s inequality states that
\begin{equation}
\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p
\end{equation}
when \( p \geq 1 \) and \( f_1, f_2 \) are \( p \)-summable functions on \( E \). This follows from the version (1.73) of Minkowski’s inequality for finite sums, although analogous arguments could be used more directly in this case. If \( 0 < p \leq 1 \), then

\[
\| f_1 + f_2 \|_p^p \leq \| f_1 \|_p^p + \| f_2 \|_p^p;
\]

as in (1.87).

Let \( \ell^\infty(E, \mathbb{R}) \) and \( \ell^\infty(E, \mathbb{C}) \) be the spaces of bounded real and complex-valued functions \( f \) on \( E \), respectively, which is to say that the values of \( f \) on \( E \) are contained in a bounded subset of \( \mathbb{R} \) or \( \mathbb{C} \). Of course, the sum and product of two bounded functions is bounded. If \( f \) is a bounded function on \( E \), then put

\[
\| f \|_\infty = \sup \{|f(x)| : x \in E\}.
\]

(5.63)

Clearly

\[
\| a f \|_\infty = |a| \| f \|_\infty
\]

for every real or complex number \( a \), as appropriate. One can also check that

\[
\| f_1 + f_2 \|_\infty \leq \| f_1 \|_\infty + \| f_2 \|_\infty
\]

and

\[
\| f_1 f_2 \|_\infty \leq \| f_1 \|_\infty \| f_2 \|_\infty
\]

(5.66)

for all bounded real or complex-valued functions \( f_1, f_2 \) on \( E \).

If \( f \) and \( g \) are real or complex-valued \( 2 \)-summable functions on \( E \), then their product \( fg \) is a summable function on \( E \), because

\[
|f(x)| |g(x)| \leq \max(|f(x)|^2, |g(x)|^2) \leq |f(x)|^2 + |g(x)|^2
\]

(5.67)

for every \( x \in E \). Alternatively, one can use the well-known fact that

\[
2ab \leq a^2 + b^2
\]

(5.68)

for all nonnegative real numbers \( a \) and \( b \) to get a better estimate. Put

\[
\langle f, g \rangle = \sum_{x \in E} f(x) g(x)
\]

(5.69)

in the case of real-valued functions on \( E \), and

\[
\langle f, g \rangle = \sum_{x \in E} f(x) \overline{g(x)}
\]

(5.70)

in the complex case. It is easy to see that (5.69) defines an inner product on \( \ell^2(E, \mathbb{R}) \), that (5.70) defines an inner product on \( \ell^2(E, \mathbb{C}) \), and that the corresponding norms are the same as the \( \ell^2 \) norm \( ||f||_2 \) defined earlier.
5.6 Additional properties

If \( f \) is a \( p \)-summable function on a nonempty set \( E \) for some \( p > 0 \), then it is easy to see that \( f \) is bounded on \( E \), and that

\[
\|f\|_\infty \leq \|f\|_p,
\]  

as in (1.80). Similarly, if \( p, q \) are positive real numbers with \( p \leq q \), and \( f \) is a \( p \)-summable function on \( E \), then \( f \) is \( q \)-summable, and

\[
\|f\|_q \leq \|f\|_p,
\]  

as in (1.81).

A real or complex-valued function \( f \) on \( E \) is said to “vanish at infinity” if for each \( \epsilon > 0 \) there is a finite set \( A_\epsilon \subseteq E \) such that

\[
|f(x)| < \epsilon
\]  

for every \( x \in E \setminus A_\epsilon \). Of course, any function on \( E \) with finite support satisfies this condition. Let \( c_0(E, \mathbb{R}) \), \( c_0(E, \mathbb{C}) \) denote the spaces of real and complex-valued functions on \( E \), respectively, with this property. As before, we may also use the notation \( c_0(E) \) to include both cases at the same time. Note that these are linear subspaces of the corresponding spaces of bounded functions on \( E \).

If \( f \) is \( p \)-summable on \( E \) for some \( p > 0 \), then \( f \) vanishes at infinity on \( E \). Equivalently, if there is an \( \epsilon > 0 \) such that \( |f(x)| \geq \epsilon \) for infinitely many \( x \in E \), then \( f \) is not \( p \)-summable for any \( p > 0 \). Note that a bounded function \( f \) on \( E \) vanishes at infinity if and only if for each \( \epsilon > 0 \) there is a function \( f_\epsilon \) with finite support on \( E \) such that

\[
\|f - f_\epsilon\|_\infty < \epsilon.
\]  

If \( f \) is a \( p \)-summable function on \( E \), then one can check that for each \( \epsilon > 0 \) there is a function \( f_\epsilon \) with finite support on \( E \) such that

\[
\|f - f_\epsilon\|_p < \epsilon.
\]  

In particular, this implies (5.74) in this case, because of (5.71).

Suppose now that \( \{f_j\}_{j=1}^\infty \) is a sequence of real or complex-valued functions on \( E \) that converges pointwise to a function \( f \) on \( E \). If there are positive real numbers \( p, C \) such that \( f_j \) is \( p \)-summable for each \( j \), with

\[
\|f_j\|_p \leq C
\]

for each \( j \), then \( f \) is \( p \)-summable too, and

\[
\|f\|_p \leq C.
\]  

This can be derived from Fatou’s lemma, as in Section 5.3.

Similarly, if \( f_j \) is a bounded function on \( E \) for each \( j \), with

\[
\|f_j\|_\infty \leq C
\]
for each \( j \), then \( f \) is bounded on \( E \) as well, and
\[
\|f\|_{\infty} \leq C. 
\]  
(5.79)

This is easy to verify, directly from the definitions. However, it is easy to give examples where \( f_j \) vanishes at infinity on \( E \) for each \( j \), but \( f \) does not vanish at infinity on \( E \). If \( f_j \) vanishes at infinity on \( E \) for each \( j \), and \( \{f_j\}_{j=1}^{\infty} \) converges to \( f \) uniformly on \( E \), then \( f \) vanishes at infinity on \( E \), by standard arguments. Of course, \( \{f_j\}_{j=1}^{\infty} \) converges to \( f \) uniformly on \( E \) if and only if \( \{f_j\}_{j=1}^{\infty} \) converges to \( f \) with respect to the \( \ell^{\infty} \) norm, in the sense that
\[
\lim_{j \to \infty} \|f_j - f\|_{\infty} = 0.
\]  
(5.80)

Let \( \{f_j\}_{j=1}^{\infty} \) be a sequence of functions on \( E \) in \( \ell^p(E, \mathbb{R}) \) or \( \ell^p(E, \mathbb{C}) \) for some \( p, 0 < p \leq \infty \). As usual, we say that \( \{f_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \ell^p(E, \mathbb{R}) \) or \( \ell^p(E, \mathbb{C}) \), as appropriate, if for each \( \epsilon > 0 \) there is an \( L(\epsilon) \geq 1 \) such that
\[
\|f_j - f_l\|_p < \epsilon
\]  
for every \( j, l \geq L(\epsilon) \). This is equivalent to saying that \( \{f_j\}_{j=1}^{\infty} \) is a Cauchy sequence with respect to the metric
\[
d_p(g, h) = \|g - h\|_p
\]  
(5.82)
on \( \ell^p(E, \mathbb{R}) \) or \( \ell^p(E, \mathbb{C}) \) when \( p \geq 1 \), and with respect to the metric
\[
d_p(g, h) = \|g - h\|_p^p
\]  
(5.83)when \( 0 < p < 1 \). We would like to show that \( \{f_j\}_{j=1}^{\infty} \) converges to some function \( f \) on \( E \) in \( \ell^p(E, \mathbb{R}) \) or \( \ell^p(E, \mathbb{C}) \), as appropriate, in the sense that
\[
\lim_{j \to \infty} \|f_j - f\|_p = 0,
\]  
(5.84)and thereby conclude that \( \ell^p(E, \mathbb{R}) \) and \( \ell^p(E, \mathbb{C}) \) are complete as metric spaces.

To do this, observe first that \( \{f_j(x)\}_{j=1}^{\infty} \) is a Cauchy sequence of real or complex numbers, as appropriate, for every \( x \in E \), because
\[
|f_j(x) - f_l(x)| \leq \|f_j - f_l\|_p
\]  
(5.85)for every \( x \in E, j, l \geq 1 \), and \( 0 < p \leq \infty \). Thus \( \{f_j(x)\}_{j=1}^{\infty} \) converges to a real or complex number \( f(x) \), as appropriate, for each \( x \in E \), since every Cauchy sequence of real and complex numbers converges. Note that
\[
\|f_j\|_p \leq \|f_{L(1)}\|_p + 1
\]  
(5.86)for every \( j \geq L(1) \) when \( p \geq 1 \), and that
\[
\|f_j\|_p^p \leq \|f_{L(1)}\|_p^p + 1
\]  
(5.87)
for every $j \geq L(1)$ when $0 < p \leq 1$, by applying the Cauchy condition (5.81) with $\epsilon = 1$ and $l = L(1)$, and using the corresponding form of the triangle inequality. This implies that $f \in \ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$, as appropriate, by the earlier remarks about pointwise convergent sequences of functions on $E$. We also get that
\[
\|f_j - f\|_p \leq \epsilon
\]
for every $j \geq L(\epsilon)$, by applying the earlier remarks to $f_j - f_l$ as a sequence in $l$ for each $j$ and using the Cauchy condition (5.81) again, so that (5.84) holds, as desired.

### 5.7 Bounded linear functionals

Let $p, q$ be real numbers such that $1 < p, q < \infty$ and
\[
\frac{1}{p} + \frac{1}{q} = 1,
\]
so that $p, q$ are conjugate exponents. If $f, g$ are real or complex-valued functions on a nonempty set $E$ which are $p, q$-summable, respectively, then their product $fg$ is a summable function on $E$, and
\[
\sum_{x \in E} |f(x)||g(x)| \leq \|f\|_p \|g\|_q.
\]
This is Hölder’s inequality in the present context, which follows easily from the version (1.66) for finite sums. We can also allow $p = 1, q = \infty$ or $p = \infty, q = 1$, using bounded functions on $E$ when the corresponding exponent is infinite.

A **bounded linear functional** on $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$ is a linear mapping $\lambda$ from this space into the real or complex numbers, as appropriate, for which there is a nonnegative real number $C$ such that
\[
|\lambda(f)| \leq C \|f\|_p
\]
for every $f \in \ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$. This definition makes sense for every $p$ in the range $0 < p \leq \infty$, but let us focus first on the case where $p \geq 1$, and consider $p < 1$ afterwards.

If $q$ is the conjugate exponent associated to $p \geq 1$, and $g$ is a real or complex-valued function on $E$ in $\ell^q(E, \mathbb{R})$ or $\ell^q(E, \mathbb{C})$, then
\[
\lambda_g(f) = \sum_{x \in E} f(x) g(x)
\]
defines a bounded linear functional on $\ell^p(E, \mathbb{R})$ of $\ell^p(E, \mathbb{C})$, as appropriate, that satisfies (5.91) with $C = \|g\|_q$, by Hölder’s inequality. One can also check that $\|g\|_q$ is the smallest value of $C$ for which (5.91) holds, which is analogous to the case of finite sums discussed in Section 2.2.
Conversely, let $\lambda$ be a bounded linear functional on $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$, $1 \leq p < \infty$, that satisfies (5.91). If $z \in E$, then let $\delta_z$ be the function on $E$ defined by $\delta_z(y) = 1$ when $y = z$, and $\delta_z(y) = 0$ otherwise. Put

$$g(z) = \lambda(\delta_z)$$

for every $z \in E$, so that

$$\lambda(f) = \sum_{x \in E} f(x) g(x)$$

when $f$ has finite support on $E$. If $A$ is a finite subset of $E$, then one can show that

$$\left( \sum_{x \in A} |g(x)|^q \right)^{1/q} \leq C$$

when $q < \infty$, and

$$\max_{x \in A} |g(x)| \leq C$$

when $q = \infty$, using suitable choices of functions $f$ supported on $A$. This is also very similar to the discussion in Section 2.2.

It follows that $g$ is $q$-summable when $q < \infty$, and that $g$ is bounded when $q = \infty$, with

$$\|g\|_q \leq C$$

in both cases. Thus we can define $\lambda_g$ as a bounded linear functional on $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$, as appropriate, and

$$\lambda(f) = \lambda_g(f)$$

when $f$ has finite support on $E$, as in (5.94). To show that this holds for every $p$-summable function $f$ on $E$, one can approximate $f$ by functions with finite support on $E$, as in (5.75) in the previous section. This also uses the fact that both $\lambda$ and $\lambda_g$ are bounded linear functionals on $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$, as appropriate.

If $p = \infty$, then it is better to consider bounded linear functionals on $c_0(E, \mathbb{R})$, $c_0(E, \mathbb{C})$ instead of $\ell^\infty(E, \mathbb{R})$, $\ell^\infty(E, \mathbb{C})$. As before, a bounded linear functional on $c_0(D, \mathbb{R})$ or $c_0(D, \mathbb{C})$ is a linear mapping from this space to the real or complex numbers, as appropriate, for which there is a nonnegative real number $C$ such that

$$|\lambda(f)| \leq C \|f\|_\infty$$

for every function $f$ on $E$ that vanishes at infinity. More precisely, this is a bounded linear functional on $c_0(E, \mathbb{R})$ or $c_0(E, \mathbb{C})$ with respect to the $\ell^\infty$ norm, which is the natural norm in this case.

If $g$ is a summable function on $D$, then we have seen that (5.92) defines a bounded linear functional $\lambda_g$ on $\ell^\infty(E, \mathbb{R})$ or $\ell^\infty(E, \mathbb{C})$, as appropriate, and that $\lambda_g$ satisfies (5.91) with $p = \infty$ and $C = \|g\|_1$. Hence the restriction of $\lambda_g$ to $c_0(E, \mathbb{R})$ or $c_0(E, \mathbb{C})$, as appropriate, is a bounded linear functional that satisfies (5.99) with $C = \|g\|_1$. One can check that $\|g\|_1$ is still the smallest value
of $C$ for which (5.99) holds, even when we restrict our attention to functions $f$ that vanish at infinity on $E$, instead of considering all bounded functions on $E$.

Conversely, suppose that $\lambda$ is a bounded linear functional on $c_0(D, \mathbb{R})$ or $c_0(D, \mathbb{C})$ that satisfies (5.99). Let $g$ be the function on $E$ defined by (5.93), as before, so that (5.94) holds when $f$ has finite support on $E$. If $A$ is a finite subset of $E$, then one can show that

$$\sum_{x \in A} |g(x)| \leq C,$$

(5.100)

using suitable choices of functions $f$ supported on $A$. This implies that $g$ is a summable function on $D$, with

$$\|g\|_1 \leq C.$$

(5.101)

To show that $\lambda(f) = \lambda_g(f)$ for every function $f$ that vanishes at infinity on $E$, one can approximate such a function $f$ by functions with finite support on $E$ with respect to the $\ell^\infty$ norm, as in (5.74) in the previous section.

If $g$ is a bounded real or complex-valued function on $E$, then we have seen that (5.92) defines a bounded linear functional $\lambda_g$ on $\ell^1(E, \mathbb{R})$ or $\ell^1(E, \mathbb{C})$, as appropriate, and that $\lambda_g$ satisfies (5.91) with $p = 1$ and $C = \|g\|_\infty$. If $0 < p < 1$, then we have also seen that every $p$-summable function $f$ on $E$ is summable and satisfies

$$\|f\|_1 \leq \|f\|_p,$$

(5.102)

as in (5.72) in the previous section, with $q = 1$. It follows that the restriction of $\lambda_g$ to $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$, as appropriate, is also a bounded linear functional that satisfies (5.91) with $C = \|g\|_\infty$. It is easy to see that $\|g\|_\infty$ is still the smallest value of $C$ for which (5.91) holds, because

$$\|\delta_z\|_p = 1$$

(5.103)

for every $z \in E$.

Conversely, suppose that $\lambda$ is a bounded linear functional on $\ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$ that satisfies (5.91), where $0 < p < 1$. As usual, we can define a function $g$ on $E$ by (5.93), so that (5.94) holds when $f$ has finite support on $E$. It is easy to see that $g$ is a bounded function on $E$, with

$$\|g\|_\infty \leq 1,$$

(5.104)

by applying these conditions to $f = \delta_z$ for each $z \in E$. One can then check that $\lambda(f) = \lambda_g(f)$ for every $p$-summable function $f$ on $E$, by approximating $f$ by functions with finite support on $E$, as in (5.75) in the previous section.

### 5.8 Another convergence theorem

Let $E$ be a nonempty set, and let $1 \leq p \leq \infty$ be given. Also let $\{f_j\}_{j=1}^\infty$ be a sequence of real or complex-valued functions on $E$ that converges pointwise
to another function $f$ on $E$. Suppose that $f_j$ is $p$-summable for each $j$ when $p < \infty$, and that $f_j$ is bounded for each $j$ when $p = \infty$, with

$$\|f_j\|_p \leq C$$

(5.105)

for some nonnegative real number $C$ and for every $j$ in both cases. As in Section 5.6, this implies that $f$ is $p$-summable when $p < \infty$, and that $f$ is bounded when $p = \infty$, with $\|f\|_p \leq C$ in both cases.

Let $1 \leq q \leq \infty$ be the exponent conjugate to $p$, so that $1/p + 1/q = 1$, and let $g$ be a real or complex-valued function on $E$ which is $q$-summable when $q < \infty$, and which vanishes at infinity on $E$ when $q = \infty$. In particular, $g$ is bounded when $q = \infty$. As in the previous section, $f_j g$ is summable on $E$ for each $j$, as is $f g$. Under these conditions, we would like to show that

$$\lim_{j \to \infty} \sum_{x \in E} f_j(x) g(x) = \sum_{x \in E} f(x) g(x).$$

(5.106)

The proof is similar to that of the dominated convergence theorem in Section 5.3, and in fact one can derive this from the dominated convergence theorem when $p = \infty$.

Equivalently, we would like to show that

$$\lim_{j \to \infty} \sum_{x \in E} (f_j(x) - f(x)) g(x) = 0.$$  

(5.107)

Let $\epsilon > 0$ be given, and let $A$ be a finite subset of $E$ such that

$$\left( \sum_{x \in E \setminus A} |g(x)|^q \right)^{1/q} < \epsilon$$

(5.108)

when $q < \infty$, and $|g(x)| < \epsilon$ for every $x \in E \setminus A$ when $q = \infty$. Using this, it is easy to see that

$$\left| \sum_{x \in E \setminus A} (f_j(x) - f(x)) g(x) \right| \leq \epsilon \|f_j - f\|_p \leq 2 C \epsilon,$$

(5.109)

for each $j$, by Hölder’s inequality. We also have that

$$\sum_{x \in A} (f_j(x) - f(x)) g(x) \leq \epsilon \|f_j - f\|_p \leq 2 C \epsilon,$$

for all sufficiently large $j$, because $\{f_j\}_{j=1}^\infty$ converges to $f$ pointwise on $E$, and because $A$ has only finitely many elements. The desired conclusion (5.107) follows by combining these two statements.

Note that (5.107) follows directly from Hölder’s inequality if we ask that

$$\lim_{j \to \infty} \|f_j - f\|_p = 0.$$  

(5.111)

In this case, it would also have been sufficient to ask that $g$ be bounded on $E$ when $q = \infty$. 


Chapter 6

Banach and Hilbert spaces

6.1 Basic concepts

Let \( V \) be a vector space over the real or complex numbers, and let \( \|v\| \) be a norm on \( V \). In this chapter, \( V \) is allowed to be infinite-dimensional, but the definition of a norm on \( V \) is the same as in the finite-dimensional case.

As usual, a sequence \( \{v_j\}_{j=1}^{\infty} \) of elements of \( V \) is said to converge to \( v \in V \) if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
\|v_j - v\| < \epsilon
\]
(6.1)
for every \( j \geq L \). In this case, we put
\[
\lim_{j \to \infty} v_j = v,
\]
(6.2)
and call \( v \) the limit of the sequence \( \{v_j\}_{j=1}^{\infty} \). It is easy to see that the limit of a convergent sequence is unique when it exists.

Similarly, a sequence \( \{v_j\}_{j=1}^{\infty} \) in \( V \) is a Cauchy sequence if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
\|v_j - v_l\| < \epsilon
\]
(6.3)
for every \( j, l \geq L \). Note that every convergent sequence is automatically a Cauchy sequence, by a simple argument.

If every Cauchy sequence of elements of \( V \) converges to an element of \( V \), then we say that \( V \) is complete. This is equivalent to the completeness of \( V \) as a metric space, with respect to the metric
\[
d(v, w) = \|v - w\|
\]
(6.4)
corresponding to the norm \( \|v\| \) on \( V \). If \( V \) is complete in this sense, then we say that \( V \) is a Banach space. If \( \langle v, w \rangle \) is an inner product on \( V \), and if \( V \) is complete with respect to the associated norm
\[
\|v\| = \langle v, v \rangle^{1/2},
\]
(6.5)
6.1. BASIC CONCEPTS

then we say that $V$ is a Hilbert space.

Of course, the real and complex numbers are complete with respect to their standard norms, as in Section 1.1. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, and $\|v\|$ is one of the norms $\|v\|_p$ defined in Section 2.1, $1 \leq p \leq \infty$, then it is easy to see that $V$ is complete, by reducing to the $n = 1$ case. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ equipped with any norm $\|v\|$, then one can also check that $V$ is complete with respect to $\|v\|$, by reducing to the case of the standard norm on $V$ using the remarks at the end of Section 2.1. This implies that any finite-dimensional vector space $V$ over the real or complex numbers is complete with respect to any norm $\|v\|$ on $V$, because $V$ is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ for some positive integer $n$, or $V = \{0\}$. If $E$ is a nonempty set and $V = \ell^p(E, \mathbb{R})$ or $\ell^p(E, \mathbb{C})$ for some $1 \leq p \leq \infty$, then $V$ is complete with respect to the corresponding norm $\|f\|_p$, as in Section 5.6.

Let $V$ be a real or complex vector space with a norm $\|v\|$. A subset $W$ of $V$ is said to be a closed set in $V$ if for every sequence \( \{w_j\}_{j=1}^{\infty} \) of elements of $W$ that converges to an element $w$ of $V$, we have that $w \in W$. This is equivalent to other standard definitions of closed sets in metric spaces, in terms of a set containing all of its limit points, or being a closed set when the complement is an open set. If $V$ is complete with respect to $\|v\|$, and $W$ is a closed linear subspace of $V$, then it follows that $W$ is complete with respect to the restriction of the norm $\|v\|$ to $v \in W$. This is because a Cauchy sequence in $W$ is also a Cauchy sequence in $V$ in this situation, which converges to an element of $V$ when $V$ is complete, and the limit is in $W$ when $W$ is a closed set in $V$.

In particular, if $E$ is a nonempty set, then $c_0(E, \mathbb{R})$ and $c_0(E, \mathbb{C})$ are closed linear subspaces of $\ell^\infty(E, \mathbb{R})$ and $\ell^\infty(E, \mathbb{C})$, as in Section 5.6. Hence $c_0(E, \mathbb{R})$ and $c_0(E, \mathbb{C})$ are complete with respect to the $\ell^\infty$ norm, as in the preceding paragraph, because $\ell^\infty(E, \mathbb{R})$ and $\ell^\infty(E, \mathbb{C})$ are complete.

Let $V$ be the vector space of continuous real or complex-valued functions on the closed unit interval $[0, 1]$, with respect to pointwise addition and scalar multiplication. Remember that continuous functions on $[0, 1]$ are automatically bounded, because $[0, 1]$ is compact. Thus

\begin{equation}
\|f\| = \sup_{0 \leq x \leq 1} |f(x)|
\end{equation}

defines a norm on $V$, known as the supremum norm. It is well known that $V$ is complete with respect to this norm, because of the fact that the limit of a uniformly-convergent sequence of continuous functions is also continuous. If $1 \leq p < \infty$, then

\begin{equation}
\|f\|_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p}
\end{equation}

also defines a norm on $V$, because of Minkowski’s inequality in Section 1.3. However, it is well known that $V$ is not complete with respect to this norm for any $p < \infty$. To get a complete space, one can use Lebesgue integrals. Similarly, the vector space of bounded continuous real or complex-valued functions on any topological space is complete with respect to the corresponding supremum norm.
norm. There are also $L^p$ spaces associated to any measure space, which are Banach spaces when $p \geq 1$, and Hilbert spaces when $p = 2$. Note that the $\ell^p$ spaces discussed in the previous chapter may be considered as $L^p$ spaces with respect to counting measure on a set $E$.

### 6.2 Sequences and series

Let $V$ be a real or complex vector space with a norm $\|v\|$. If $\{v_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ are sequences in $V$ that converge to $v, w \in V$, respectively, then

\[
\lim_{j \to \infty} (v_j + w_j) = v + w.
\]  

(6.8)

This can be shown in essentially the same way as for sequences of real or complex numbers. Similarly, if $\{t_j\}_{j=1}^\infty$ is a sequence of real or complex numbers, as appropriate, that converges to the real or complex number $t$, and if $\{v_j\}_{j=1}^\infty$ is a sequence of vectors in $V$ that converges to $v \in V$, then

\[
\lim_{j \to \infty} t_j v_j = tv.
\]  

(6.9)

Equivalently, this means that addition and scalar multiplication are continuous on $V$ with respect to the metric associated to the norm.

As in (2.7) in Section 2.1, one can check that

\[
\|v\| - \|w\| \leq \|v - w\|
\]  

(6.10)

for every $v, w \in V$, using the triangle inequality. If $\{v_j\}_{j=1}^\infty$ is a sequence in $V$ that converges to $v \in V$, then it follows that

\[
\lim_{j \to \infty} \|v_j\| = \|v\|,
\]  

(6.11)

as a sequence of real numbers. This is the same as saying that $\|v\|$ is a continuous real-valued function on $V$ with respect to the metric associated to the norm.

Let $\sum_{j=1}^\infty a_j$ be an infinite series whose terms $a_j$ are elements of $V$. As usual, we say that $\sum_{j=1}^\infty a_j$ converges in $V$ if the corresponding sequence of partial sums $\sum_{j=1}^n a_j$ converges in $V$, in which case we put

\[
\sum_{j=1}^\infty a_j = \lim_{n \to \infty} \sum_{j=1}^n a_j.
\]  

(6.12)

If $\sum_{j=1}^\infty a_j$ and $\sum_{j=1}^\infty b_j$ are convergent series with terms in $V$, then it is easy to see that $\sum_{j=1}^\infty (a_j + b_j)$ also converges, and that

\[
\sum_{j=1}^\infty (a_j + b_j) = \sum_{j=1}^\infty a_j + \sum_{j=1}^\infty b_j,
\]  

(6.13)
because of the corresponding fact about sums of convergent sequences mentioned earlier. Similarly, if \( \sum_{j=1}^{\infty} a_j \) is a convergent series with terms in \( V \), and if \( t \) is a real or complex number, as appropriate, then \( \sum_{j=1}^{\infty} t a_j \) also converges, and

\[
\sum_{j=1}^{\infty} t a_j = t \sum_{j=1}^{\infty} a_j.
\] (6.14)

An infinite series \( \sum_{j=1}^{\infty} a_j \) with terms in \( V \) is said to converge absolutely if

\[
\sum_{j=1}^{\infty} \|a_j\|
\] converges as an infinite series of nonnegative real numbers. In this case, one can show that the partial sums of \( \sum_{j=1}^{\infty} a_j \) form a Cauchy sequence, because

\[
\left\| \sum_{j=l}^{n} a_j \right\| \leq \sum_{j=l}^{n} \|a_j\|
\] for every \( n \geq l \geq 1 \). If \( V \) is complete, then it follows that \( \sum_{j=1}^{\infty} a_j \) converges in \( V \). We also get that

\[
\sum_{j=1}^{\infty} \|a_j\|
\] (6.15)

Conversely, suppose that \( \{v_j\}_{j=1}^{\infty} \) is a Cauchy sequence of elements of \( V \). It is easy to see that there is a subsequence \( \{v_{j_l}\}_{l=1}^{\infty} \) of \( \{v_j\}_{j=1}^{\infty} \) such that

\[
\|v_{j_l} - v_{j_{l+1}}\| < 2^{-l}
\] for each \( l \), so that

\[
\sum_{l=1}^{\infty} \|v_{j_l} - v_{j_{l+1}}\|
\] converges absolutely. Of course,

\[
\sum_{l=1}^{n} (v_{j_l} - v_{j_{l+1}})
\] (6.19)

for each \( n \), which implies that the series (6.19) converges in \( V \) if and only if \( \{v_{j_l}\}_{l=1}^{\infty} \) converges in \( V \). If \( \{v_{j_l}\}_{l=1}^{\infty} \) converges in \( V \), then one can check that \( \{v_j\}_{j=1}^{\infty} \) also converges to the same element of \( V \), because \( \{v_j\}_{j=1}^{\infty} \) is a Cauchy sequence. If every absolutely convergent series in \( V \) converges, then it follows that every Cauchy sequence in \( V \) converges, which is to say that \( V \) is complete.

Suppose now that the norm \( \|v\| \) on \( V \) is associated to an inner product \( \langle v, w \rangle \) on \( V \) in the usual way. Let \( \sum_{j=1}^{\infty} a_j \) be an infinite series whose terms are pairwise-orthogonal vectors in \( V \), in the sense that

\[
\langle a_j, a_k \rangle = 0
\] (6.21)
when \( j \neq k \). In this case,

\[
\left\| \sum_{j=1}^{n} a_j \right\|^2 = \sum_{j=1}^{n} \|a_j\|^2
\]

for each \( n \). If \( \sum_{j=1}^{\infty} a_j \) converges in \( V \), then \( \sum_{j=1}^{\infty} \|a_j\|^2 \) converges in \( \mathbb{R} \), and

\[
\left\| \sum_{j=1}^{\infty} a_j \right\|^2 = \sum_{j=1}^{\infty} \|a_j\|^2.
\]

Of course, the orthogonality condition (6.21) implies that

\[
\left\| \sum_{j=1}^{n} a_j \right\|^2 = \sum_{j=1}^{n} \|a_j\|^2
\]

for every \( n \geq l \geq 1 \). If \( \sum_{j=1}^{\infty} \|a_j\|^2 \) converges, then it follows that the partial sums of \( \sum_{j=1}^{\infty} a_j \) form a Cauchy sequence in \( V \). If \( V \) is complete, then \( \sum_{j=1}^{\infty} a_j \) converges in \( V \) under these conditions.

### 6.3 Minimizing distances

Let \( V \) be a real or complex vector space with an inner product \( \langle v, w \rangle \), and let \( \|v\| \) be the corresponding norm on \( V \). Also let \( E \) be a nonempty subset of \( V \), let \( v \) be an element of \( V \), and put

\[
\rho = \inf \{ \|v - w\| : w \in E \}.
\]

Thus for each positive integer \( j \) there is a \( w_j \in E \) such that

\[
\|v - w_j\| < \rho + \frac{1}{j}.
\]

Note that

\[
\left\| \frac{x + y}{2} \right\|^2 + \left\| \frac{x - y}{2} \right\|^2 = \frac{1}{2} (\|x\|^2 + \|y\|^2)
\]

for every \( x, y \in V \), by applying the parallelogram law (2.90) to \( x/2, y/2 \). If we take \( x = v - w_j \) and \( y = v - w_l \), then we get that

\[
\left\| v - \left( \frac{w_j + w_l}{2} \right) \right\|^2 + \frac{1}{4} \|w_j - w_l\|^2 = \frac{1}{2} (\|v - w_j\|^2 + \|v - w_l\|^2).
\]

Combining this with (6.26) gives

\[
\left\| v - \left( \frac{w_j + w_l}{2} \right) \right\|^2 + \frac{1}{4} \|w_j - w_l\|^2 < \rho^2 + \rho \left( \frac{1}{j} + \frac{1}{l} \right) + \frac{1}{2} \left( \frac{1}{j^2} + \frac{1}{l^2} \right).
\]
6.4. ORTHOGONAL PROJECTIONS

Suppose now that $E$ is a convex set in $V$. This implies that
\[(6.30) \quad \frac{w_j + w_l}{2} \in E\]
for each $j, l$, and hence that
\[(6.31) \quad \left\| v - \frac{w_j + w_l}{2} \right\| \geq \rho.\]
In this case, we get that
\[(6.32) \quad \frac{1}{4} \|w_j - w_l\|^2 < \rho \left( \frac{1}{j} + \frac{1}{l} \right) + \frac{1}{2} \left( \frac{1}{j^2} + \frac{1}{l^2} \right)\]
for each $j, l$. Thus $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in $V$ under these conditions.

If $V$ is complete, then it follows that $\{w_j\}_{j=1}^\infty$ converges to an element $w$ of $V$. If $E$ is a closed set in $V$, then $w \in E$. Moreover,
\[(6.33) \quad \|v - w\| = \rho,\]
so that $w$ minimizes the distance to $v$ among elements of $E$.

This argument also works in a large class of Banach spaces. More precisely, a norm $\|v\|$ on $V$ is said to be uniformly convex if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $\delta(\epsilon) < 1$ and
\[(6.34) \quad \left\| \frac{u + z}{2} \right\| \leq 1 - \delta(\epsilon)\]
for every $u, z \in V$ such that $\|u\| = \|z\| = 1$ and $\|u - z\| > \epsilon$. Equivalently, this means that
\[(6.35) \quad \|u - z\| \leq \epsilon\]
for every $u, z \in V$ such that $\|u\| = \|z\| = 1$ and $\|(u + z)/2\| > 1 - \delta(\epsilon)$. If $\|v\|$ is associated to an inner product on $V$, then it is easy to see that $\|v\|$ is uniformly convex, using the parallelogram law. It is well known that the $L^p$ norm is uniformly convex when $1 < p < \infty$, as a consequence of famous inequalities of Clarkson. If $\|v\|$ is uniformly convex, then one can modify the previous arguments to show that the minimum of the distance from a point $v \in V$ to a nonempty closed convex set $E \subseteq V$ is attained when $V$ is complete. As before, the main step is to show that a minimizing sequence $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence when $\|v\|$ is uniformly convex.

6.4 Orthogonal projections

Let $V$ be a real or complex vector space with an inner product $\langle v, w \rangle$, and let $\|v\|$ be the corresponding norm on $V$. Suppose that $V$ is complete, so that $V$ is a Hilbert space, and let $W$ be a closed linear subspace of $V$. If $v$ is any element
of \( V \), then there is a \( w \in W \) whose distance to \( v \) is minimal among elements of \( W \), as in the previous section. Equivalently,

\[
\|v - w\| \leq \|v - w + z\|
\]

for every \( z \in W \), because \( W \) is a linear subspace of \( V \). Using the inner product, we get that

\[
\|v - w\|^2 \leq \|v - w + z\|^2
\]

\[
= \|v - w\|^2 + \langle v - w, z \rangle + \|z\|^2
\]

\[
= \|v - w\|^2 + 2 \Re \langle v - w, z \rangle + \|z\|^2.
\]

More precisely, it is not necessary to take the real part of \( \langle v - w, z \rangle \) in the last step when \( V \) is a real vector space, but this is needed when \( V \) is complex. Of course, this inequality reduces to

\[
0 \leq 2 \Re \langle v - w, z \rangle + \|z\|^2,
\]

by subtracting \( \|v - w\|^2 \) from both sides.

Let \( t \) be a real number, and put

\[
f(t) = 2 \Re \langle v - w, t z \rangle + \|t z\|^2 = 2 t \Re \langle v - w, z \rangle + t^2 \|z\|^2.
\]

If \( z \in W \), then \( t z \in W \), and hence \( f(t) \geq 0 \), by (6.38). Thus the minimum of \( f(t) \) is attained at \( t = 0 \), which implies that the derivative of \( f(t) \) at \( t = 0 \) is also equal to 0. This shows that

\[
\Re \langle v - w, z \rangle = 0
\]

for every \( z \in W \), which is the same as saying that

\[
\langle v - w, z \rangle = 0
\]

for every \( z \in W \) in the real case. In the complex case, one can get (6.41) by applying (6.40) to \( z \) and to \( i z \).

Conversely, (6.41) implies that

\[
\|v - w + z\|^2 = \|v - w\|^2 + \|z\|^2
\]

for every \( z \in W \), and hence that \( w \) minimizes the distance to \( v \) among elements of \( W \). As in Section 2.6, \( w \in W \) is uniquely determined by the condition that (6.41) holds for every \( z \in W \). Put \( P_W(v) = w \), which is the orthogonal projection of \( v \) onto \( W \). Note that

\[
\|v\|^2 = \|v - P_W(v)\|^2 + \|P_W(v)\|^2,
\]

which is the same as (6.42) with \( z = w \). In particular,

\[
\|P_W(v)\| \leq \|v\|
\]
for every \( v \in V \).

If \( v_1, v_2 \in V \), then \( P_W(v_1) + P_W(v_2) \in W \) and

\[
(v_1 + v_2) - (P_W(v_1) + P_W(v_2)) = (v_1 - P_W(v_1)) + (v_2 - P_W(v_2))
\]

is orthogonal to every element of \( W \), by the corresponding properties of \( P_W(v_1) \) and \( P_W(v_2) \). This implies that

\[
P_W(v_1 + v_2) = P_W(v_1) + P_W(v_2),
\]

because \( P_W(v_1 + v_2) \) is characterized by these conditions, as in the preceding paragraph. Similarly,

\[
P_W(t v) = t P_W(v)
\]

for every \( v \in V \) and \( t \in \mathbb{R} \) or \( \mathbb{C} \), as appropriate, because \( t P_W(v) \in W \) and

\[
t v - t P_W(v) = t (v - P_W(v))
\]

is orthogonal to every element of \( W \), by the corresponding properties of \( P_W(v) \). Thus \( P_W(v) \) is a linear mapping from \( V \) into \( W \). Of course, \( P_W(v) = v \) when \( v \in W \).

### 6.5 Orthonormal sequences

Let \( V \) be a real or complex vector space with an inner product \( \langle v, w \rangle \) again, and let \( \|v\| \) be the corresponding norm on \( V \). Suppose that \( e_1, e_2, e_3, \ldots \) is an infinite sequence of orthonormal vectors in \( V \), so that

\[
\langle e_j, e_k \rangle = 0 \quad \text{when} \quad j \neq k, \quad \text{and} \quad \|e_j\| = 1 \quad \text{for each} \quad j.
\]

Note that any sequence of vectors in \( V \) can be modified to get an orthonormal sequence with the same linear span, using the Gram–Schmidt process. This was already used in Section 2.6 to show that every finite-dimensional inner product space has an orthonormal basis.

Put

\[
P_n(v) = \sum_{j=1}^{n} \langle v, e_j \rangle e_j
\]

for each \( v \in V \) and positive integer \( n \). This is the same as the orthogonal projection of \( V \) onto the linear subspace \( W_n \) spanned by \( e_1, \ldots, e_n \) in \( V \), as in Section 2.6. Remember that

\[
\|v\|^2 = \|P_n(v)\|^2 + \|v - P_n(v)\|^2 = \sum_{j=1}^{n} |\langle v, e_j \rangle|^2 + \|v - P_n(v)\|^2,
\]

as in (2.78). In particular,

\[
\sum_{j=1}^{n} |\langle v, e_j \rangle|^2 \leq \|v\|^2
\]
for every $v \in V$ and $n \geq 1$. Remember also that $P_n(v) \in W_n$ minimizes the distance to $v$ among elements of $W_n$, as in Section 2.11.

Note that $\bigcup_{n=1}^{\infty} W_n$ is a linear subspace of $V$, because $W_n$ is a linear subspace of $V$ for each $n$, and because $W_n \subseteq W_{n+1}$ for each $n$, by construction. Let $W$ be the closure of $\bigcup_{n=1}^{\infty} W_n$ in $V$, which is the set of $v \in V$ with the property that for each $\epsilon > 0$ there is a $w \in \bigcup_{n=1}^{\infty} W_n$ such that

$$\|v - w\| < \epsilon.$$  

(6.53)

Thus $\bigcup_{n=1}^{\infty} W_n \subseteq W$ automatically, and one can check that $W$ is a closed linear subspace of $V$.

Equivalently, $v \in V$ is an element of $W$ if and only if

$$\lim_{n \to \infty} \|v - P_n(v)\| = 0.$$  

More precisely, if $v$ satisfies (6.54), then it is easy to see that $v \in W$, because $P_n(v) \in W_n$ for each $n$. Conversely, suppose that $v \in W$, and let $\epsilon > 0$ be given. By definition of $W$, there is a positive integer $k$ and a $w \in W_k$ such that (6.53) holds. This implies that

$$\|v - P_n(v)\| \leq \|v - w\| < \epsilon$$  

(6.55)

for every $n \geq k$, as desired, because $w_k \in W_n$ for each $n \geq k$, and because $P_n(v)$ minimizes the distance to $v$ among elements of $W_n$, as before.

We would like to put

$$P(v) = \sum_{j=1}^{\infty} \langle v, e_j \rangle e_j = \lim_{n \to \infty} P_n(v)$$  

(6.56)

for every $v \in V$, but we need to be careful about the existence of the limit. If $v \in W$, then (6.54) implies that $\{P_n(v)\}_{n=1}^{\infty}$ converges to $v$, so that the definition of $P(v)$ makes sense and $P(v) = v$. Otherwise, if $v$ is any element of $V$, then

$$\sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2$$  

(6.57)

converges and is less than or equal to $\|v\|^2$, because of (6.52). If $V$ is complete, then it follows that the series in (6.56) converges, as in Section 6.2. In this case, it is easy to see that $P(v) \in W$ for every $v \in V$, because $P_n(v) \in W_n$ for each $n$. One can also check that $P$ is a linear mapping from $V$ into $W$ under these conditions. By construction, we also have that

$$\|P(v)\|^2 = \sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2 \leq \|v\|^2$$  

(6.58)

for every $v \in V$. 


Let us suppose from now on in this section that $V$ is complete, and thus a Hilbert space. Observe that

$$
\langle P(v), e_l \rangle = \lim_{n \to \infty} \langle P_n(v), e_l \rangle = \langle v, e_l \rangle
$$

for every $v \in V$ and $l \geq 1$. This uses the fact that

$$
\langle P_n(v), e_l \rangle = \langle v, e_l \rangle
$$

for each $n \geq l$, because of the orthonormality of the $e_j$'s. This also implicitly uses the Cauchy–Schwarz inequality, in order to take the limit outside of the inner product, which is basically the same as the continuity of the inner product with respect to the associated norm. It follows that $v - P(v)$ is orthogonal to $e_l$ for each $l$, which implies that $v - P(v)$ is orthogonal to every element of $\bigcup_{n=1}^{\infty} W_n$, because of the linearity properties of the inner product. Using continuity of the inner product again, we get that $v - P(v)$ is orthogonal to every element of the closure $W$ of $\bigcup_{n=1}^{\infty} W_n$. As in Section 2.6, $P(v)$ is uniquely determined by the conditions that $P(v) \in W$ and $v - P(v)$ is orthogonal to every element of $W$, and hence $P$ is the same as the orthogonal projection $P_W$ of $V$ onto $W$, as in the previous section.

If $\{a_j\}_{j=1}^{\infty}$ is any sequence of real or complex numbers, as appropriate, such that $\sum_{j=1}^{\infty} |a_j|^2$ converges in $\mathbb{R}$, then the same arguments show that

$$
\sum_{j=1}^{\infty} a_j e_j
$$

converges in $V$ to an element of $W$, and that

$$
\left\| \sum_{j=1}^{\infty} a_j e_j \right\|^2 = \sum_{j=1}^{\infty} |a_j|^2.
$$

If $\{b_j\}_{j=1}^{\infty}$ is another sequence of real or complex numbers, as appropriate, such that $\sum_{j=1}^{\infty} |b_j|^2$ converges, then one can check that

$$
\left\langle \sum_{j=1}^{\infty} a_j e_j, \sum_{k=1}^{\infty} b_k e_k \right\rangle = \sum_{j=1}^{\infty} a_j b_j
$$

in the real case, and

$$
\left\langle \sum_{j=1}^{\infty} a_j e_j, \sum_{k=1}^{\infty} \overline{b_k} e_k \right\rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}
$$

in the complex case, using the orthonormality of the $e_j$'s and the continuity properties of the inner product, as before. Note that the infinite series on the right sides of (6.63) and (6.64) are absolutely convergent under these conditions, as in Section 5.5. If $V = W$, then the $e_j$'s are said to form an orthonormal basis of $V$. In this case, we get a natural isomorphism between $V$ and $\ell^2(\mathbb{Z}_{+}, \mathbb{R})$ or $\ell^2(\mathbb{Z}_{+}, \mathbb{C})$, as appropriate, associated to this orthonormal basis for $V$.  

6.6 Bounded linear functionals

Let $V$ be a real or complex vector space with a norm $\|v\|$. A linear functional $\lambda$ on $V$ is said to be bounded if there is a nonnegative real number $C$ such that

$$|\lambda(v)| \leq C \|v\| \quad (6.65)$$

for every $v \in V$. If $V$ has finite dimension, then every linear functional on $V$ is bounded, as in Section 2.2. If $\lambda$ is a bounded linear functional on $V$, then it is easy to see that $\lambda$ is continuous on $V$ with respect to the metric associated to the norm. In particular, if $\{v_j\}_{j=1}^{\infty}$ is a sequence of vectors in $V$ that converges to another vector $v \in V$, as in Section 6.1, then it is easy to see that

$$\lim_{j \to \infty} \lambda(v_j) = \lambda(v) \quad (6.66)$$

in $\mathbb{R}$ or $\mathbb{C}$, as appropriate. Conversely, one can check that a linear functional $\lambda$ on $V$ is bounded when it is continuous at 0. Note that continuity of a linear functional on $V$ at 0 implies continuity at every point in $V$, by linearity.

Suppose for the moment that $V$ is equipped with an inner product $\langle v, w \rangle$, and that $\|v\|$ is the norm associated to this inner product. If $w \in V$, then

$$\lambda_w(v) = \langle v, w \rangle \quad (6.67)$$

defines a bounded linear functional on $V$, since

$$|\lambda_w(v)| \leq \|v\| \|w\| \quad (6.68)$$

for every $v \in V$, by the Cauchy–Schwarz inequality. Conversely, if $V$ is complete, and if $\lambda$ is a bounded linear functional on $V$, then there is a unique $w \in V$ such that $\lambda(v) = \lambda_w(v)$ for every $v \in V$. The uniqueness of $w$ is a simple exercise that does not use the completeness of $V$, and so we proceed now to the proof of the existence of $w$. This is trivial when $\lambda(v) = 0$ for every $v \in V$, and hence we suppose that $\lambda(v_0) \neq 0$ for some $v_0 \in V$.

Let $Z$ be the kernel of $\lambda$, which is to say that

$$Z = \{v \in V : \lambda(v) = 0\} \quad (6.69)$$

It is easy to see that $Z$ is a closed linear subspace of $V$, because of the continuity of $\lambda$ that follows from boundedness. Thus the orthogonal projection $P_Z$ of $V$ onto $Z$ may be defined as in Section 6.4. Consider

$$w_0 = v_0 - P_Z(v_0) \quad (6.70)$$

Note that $w_0 \neq 0$, since $v_0 \notin Z$ by hypothesis, and that $w_0$ is orthogonal to every element of $Z$, as in Section 6.4. In particular,

$$\langle v_0, w_0 \rangle = \langle v_0 - P_Z(v_0), w_0 \rangle = \langle w_0, w_0 \rangle = \|w_0\|^2 > 0. \quad (6.71)$$

Put $w = \lambda(v_0)\|w_0\|^{-2}w_0$ in the real case, and $w = \overline{\lambda(v_0)}\|w_0\|^{-2}w_0$ in the complex case. By construction, $\lambda_w(v_0) = \lambda(v_0)$, and $\lambda_w(z) = 0$ for every $z \in Z$. 

This implies that \( \lambda_w(v) = \lambda(v) \) for every \( v \in V \), as desired, because \( V \) is spanned by \( v_0 \) and \( Z \) in this situation.

Let \( V \) be a real or complex vector space with an arbitrary norm \( \|v\| \) again. Suppose that \( W \) is a linear subspace of \( V \), and that \( \lambda \) is a linear functional on \( W \) that satisfies (6.65) for some \( C \geq 0 \) and every \( v \in V \). Under these conditions, the Hahn–Banach theorem states that there is an extension of \( \lambda \) to a linear functional on \( V \) that satisfies (6.65) for every \( v \in V \), with the same constant \( C \). If \( V \) is finite-dimensional, then this is the same in essence as Theorem 2.26 in Section 2.3. Otherwise, there is an argument using the axiom of choice, with the previous construction as an important part of the proof. In some situations, one can use a sequence of extensions as before to extend \( \lambda \) to a dense linear subspace of \( V \), and then extend \( \lambda \) to all of \( V \) using continuity. At any rate, an important consequence of the Hahn–Banach theorem is that for each \( v \in V \) with \( v \neq 0 \) there is a bounded linear functional \( \lambda \) on \( V \) such that \( \lambda(v) \neq 0 \).

More precisely, one can first define \( \lambda \) on the 1-dimensional linear subspace of \( V \) spanned by \( v \), and then use the Hahn–Banach theorem to extend \( \lambda \) to a bounded linear functional on all of \( V \).

### 6.7 Dual spaces

Let \( V \) be a real or complex vector space with a norm \( \|v\| \) again, and let \( V^* \) be the space of all bounded linear functionals on \( V \). This is also a vector space over the real or complex numbers in a natural way, because the sum of two bounded linear functionals on \( V \) is also bounded, as is the product of a bounded linear functional on \( V \) by a scalar. If \( \lambda \) is a bounded linear functional on \( V \), then the dual norm \( \|\lambda\|^* \) of \( \lambda \) is defined by

\[
\|\lambda\|^* = \sup\{ |\lambda(v)| : v \in V, \|v\| \leq 1 \}.
\]

(6.72)

This is the same as (2.11) in Section 2.2, except that now we need to ask that \( \lambda \) be a bounded linear functional on \( V \) to ensure that the supremum is finite. As before, \( \lambda \) satisfies (6.65) with \( C = \|\lambda\|^* \), and this is the smallest value of \( C \) for which (6.65) holds.

It is easy to see that \( \|\lambda\|^* \) is a norm on \( V^* \), as in the finite-dimensional case. Let us check that \( V^* \) is automatically complete with respect to the dual norm. Let \( \{\lambda_j\}_{j=1}^\infty \) be a sequence of bounded linear functionals on \( V \) which is a Cauchy sequence with respect to the dual norm. This means that for each \( \epsilon > 0 \) there is an \( L(\epsilon) \geq 1 \) such that

\[
\|\lambda_j - \lambda_l\|^* < \epsilon
\]

(6.73)

for every \( j, l \geq L(\epsilon) \), and hence

\[
|\lambda_j(v) - \lambda_l(v)| \leq \epsilon \|v\|
\]

(6.74)

for every \( v \in V \) and \( j, l \geq L(\epsilon) \). In particular, \( \{\lambda_j(v)\}_{j=1}^\infty \) is a Cauchy sequence of real or complex numbers, as appropriate, for each \( v \in V \). Thus \( \{\lambda_j(v)\}_{j=1}^\infty \) converges to a real or complex number \( \lambda(v) \) for each \( v \in V \), by the completeness
of $\mathbf{R}$, $\mathbf{C}$. One can check that $\lambda$ defines a linear functional on $V$, because $\lambda_j$ is linear on $V$ for each $j$. We also have that

$$\tag{6.75} |\lambda_j(v) - \lambda(v)| \leq \epsilon \|v\|$$

for every $v \in V$ and $j \geq L(\epsilon)$, by taking the limit as $l \to \infty$ in (6.74). Applying this with $\epsilon = 1$ and $j = L(1)$, we get that

$$\tag{6.76} |\lambda(v)| \leq |\lambda_{L(1)}(v)| + \|v\| \leq (\|\lambda_{L(1)}\| + 1) \|v\|$$

for every $v \in V$, so that $\lambda$ is a bounded linear functional on $V$. Using (6.75) again, we get that $\{\lambda_j\}_{j=1}^\infty$ converges to $\lambda$ with respect to the dual norm, as desired.

Let $V^{**}$ be the space of bounded linear functionals on $V^*$, with respect to the dual norm $\|\lambda\|^{**}$ on $V^*$. If $v \in V$, then

$$\tag{6.77} L_v(\lambda) = \lambda(v)$$

defines a linear functional on $V^*$, which satisfies

$$\tag{6.78} |L_v(\lambda)| = |\lambda(v)| \leq \|\lambda\|^{v} \|v\|$$

for every $\lambda \in V^*$, by the definition of $\|\lambda\|^{**}$. This implies that $L_v$ is a bounded linear functional on $V^*$. More precisely, if $\|L\|^{**}$ is the dual norm of a bounded linear functional $L$ on $V^*$ with respect to the dual norm $\|\lambda\|^{**}$ on $V^*$, then (6.78) implies that

$$\tag{6.79} \|L_v\|^{**} \leq \|v\|$$

for every $v \in V$. Using the Hahn–Banach theorem, one can check that

$$\tag{6.80} \|L_v\|^{**} = \|v\|$$

for every $v \in V$. The main point is to show that if $v \neq 0$, then there is a $\lambda \in V^*$ such that $\|\lambda\|^{**} = 1$ and $\lambda(v) = \|v\|$, so that equality holds in (6.78). As usual, one can start by defining $\lambda$ on the 1-dimensional subspace of $V$ spanned by $v$, and then extend $\lambda$ to all of $V$ using the Hahn–Banach theorem.

A Banach space $V$ is said to be reflexive if every bounded linear functional on $V^{**}$ is of the form $L_v$ for some $v \in V$. Note that $V$ has to be complete for this to hold, since we already know that $V^{**}$ is complete, because it is a dual space. It is easy to see that Hilbert spaces are reflexive, using the characterization of their dual spaces in the previous section. It is also well known that $L^p$ spaces are reflexive when $1 < p < \infty$, because the dual of $L^p$ can be identified with the corresponding $L^q$ space, where $1 < q < \infty$ is conjugate to $p$ in the usual sense that $1/p + 1/q = 1$. In particular, $\ell^p$ spaces are reflexive when $1 < p < \infty$, by the characterization of their dual spaces in Section 5.7. We also saw in Section 5.7 that the dual of $c_0(E)$ can be identified with $\ell^1(E)$ for any nonempty set $E$, and that the dual of $\ell^1(E)$ can be identified with $\ell^{\infty}(E)$. If $E$ is an infinite set, then $c_0(E)$ is a proper linear subspace of $\ell^{\infty}(E)$, and it follows that $c_0(E)$ is not reflexive.
6.8. BOUNDED LINEAR MAPPINGS

Let $V_1$ and $V_2$ be vector spaces, both real or both complex, and equipped with norms $\| \cdot \|_1$, $\| \cdot \|_2$, respectively. A linear mapping $T$ from $V_1$ into $V_2$ is said to be bounded if

\[
\|T(v)\|_2 \leq C \|v\|_1
\]

for some $C \geq 0$ and every $v \in V_1$. If $V_1$ has finite dimension, then one can check that every linear mapping from $V_1$ into $V_2$ is bounded, using a basis for $V_1$ and the remarks at the end of Section 2.1 to reduce to the case where $V_1$ is $\mathbb{R}^n$ or $\mathbb{C}^n$ equipped with the standard norm. If $V_2 = \mathbb{R}$ or $\mathbb{C}$, as appropriate, then a bounded linear mapping from $V_1$ into $V_2$ is the same as a bounded linear functional on $V_1$. The boundedness of any linear mapping is equivalent to suitable continuity conditions, as in the context of linear functionals.

Let $\mathcal{B}(V_1, V_2)$ be the space of bounded linear mappings from $V_1$ into $V_2$. It is easy to see that this is a vector space with respect to pointwise addition and scalar multiplication. If $T$ is a bounded linear mapping from $V_1$ into $V_2$, then the operator norm of $T$ is defined by

\[
\|T\|_{op} = \sup\{\|T(v)\|_2 : v \in V_1, \|v\|_1 \leq 1\},
\]

as in (2.42) in Section 2.4. Equivalently, (6.81) holds with $C = \|T\|_{op}$, and this is the smallest value of $C$ for which (6.81) holds. One can check that (6.82) defines a norm on $\mathcal{B}(V_1, V_2)$. If $V_2 = \mathbb{R}$ or $\mathbb{C}$, as appropriate, then the operator norm reduces to the dual norm on $(V_1)^*$ defined in the previous section. If $V_2$ is any vector space which is complete with respect to the norm $\| \cdot \|_2$, then one can show that $\mathcal{B}(V_1, V_2)$ is complete with respect to the operator norm, in the same way as for dual spaces.

Let $V_3$ be another vector space, which is real or complex depending on whether $V_1$ and $V_2$ are real or complex, and let $\| \cdot \|_3$ be a norm on $V_3$. If $T_1$ is a bounded linear mapping from $V_1$ into $V_2$, and $T_2$ is a bounded linear mapping from $V_2$ into $V_3$, then it is easy to see that their composition $T_2 \circ T_1$ is a bounded linear mapping from $V_1$ into $V_3$. Moreover,

\[
\|T_2 \circ T_1\|_{op,13} \leq \|T_1\|_{op,12} \|T_2\|_{op,23},
\]

where $\| \cdot \|_{op,ab}$ is the operator norm for a linear mapping from $V_a$ into $V_b$, with $a, b = 1, 2, 3$.

A bounded linear mapping $T : V_1 \rightarrow V_2$ is said to be invertible if it is a one-to-one linear mapping from $V_1$ onto $V_2$ whose inverse $T^{-1}$ is bounded as a linear mapping from $V_2$ into $V_1$. Note that the composition of invertible mappings is also invertible. If $T$ is invertible, then

\[
\|T(v)\|_2 \geq c \|v\|_1
\]

for some $c > 0$ and every $v \in V_1$. More precisely, this holds with $c$ equal to the reciprocal of the operator norm of $T^{-1}$, by applying the boundedness of $T^{-1}$ to $T^{-1}(T(v)) = v$. Conversely, suppose that $T$ is a bounded linear mapping from
V_1 into V_2 that satisfies (6.84). In particular, v = 0 when T(v) = 0, so that T is one-to-one. If T maps V_1 onto V_2, then (6.84) implies that T^{-1} is bounded, with operator norm less than or equal to 1/c.

If V_1 is complete and T : V_1 → V_2 is a bounded linear mapping that satisfies (6.84), then it is easy to see that T(V_1) is also complete. This is because a sequence \{v_j\}_{j=1}^{\infty} of elements of V_1 is a Cauchy sequence in V_1 if and only if \{T(v_j)\}_{j=1}^{\infty} is a Cauchy sequence in V_2, and \{v_j\}_{j=1}^{\infty} converges to v ∈ V_1 if and only if \{T(v_j)\}_{j=1}^{\infty} converges to T(v) in V_2. In this case, it follows that T(V_1) is a closed linear subspace in V_2. To see this, let \{v_j\}_{j=1}^{\infty} be a sequence of elements of V_1 such that \{T(v_j)\}_{j=1}^{\infty} converges to some z ∈ V_2, and let us check that z = T(v) for some v ∈ V_1. Note that \{T(v_j)\}_{j=1}^{\infty} is a Cauchy sequence in V_2, since it converges in V_2. As before, this implies that \{v_j\}_{j=1}^{\infty} is a Cauchy sequence in V_1, so that \{v_j\}_{j=1}^{\infty} converges to some v ∈ V, because V is complete. Thus \{T(v_j)\}_{j=1}^{\infty} converges to T(v) in V_2, because T is bounded, and hence z = T(v), as desired.

Let V be a real or complex vector space with a norm ||v||, and let T be a bounded linear operator on V. If j is a positive integer, then let T^j be the composition of j T’s, so that T^1 = T, T^2 = T o T, and so on. It will be convenient to interpret T^j as being the identity operator I on V when j = 0. Observe that

\[(I - T) \left( \sum_{j=0}^{n} T^j \right) = \left( \sum_{j=0}^{n} T^j \right) (I - T) = I - T^{n+1} \tag{6.85} \]

for each nonnegative integer n, as in the case of ordinary geometric series of real and complex numbers. Of course,

\[||T^j||_{op} \leq ||T||_{op}^j \tag{6.86} \]

for each j, by (6.83). If ||T||_{op} < 1, then we get that

\[\sum_{j=0}^{\infty} ||T^j||_{op} \leq \sum_{j=0}^{\infty} ||T||_{op}^j = \frac{1}{1 - ||T||_{op}}, \tag{6.87} \]

by the usual formula for the sum of an geometric series.

This shows that the infinite series

\[\sum_{j=0}^{\infty} T^j \tag{6.88} \]

converges absolutely in the vector space BL(V) = BL(V,V) of bounded linear operators on V when ||T||_{op} < 1. If V is complete, then we have seen that BL(V) is complete with respect to the operator norm, and hence that (6.88) converges in BL(V). We also get that

\[(I - T) \left( \sum_{j=0}^{\infty} T^j \right) = \left( \sum_{j=0}^{\infty} T^j \right) (I - T) = I, \tag{6.89} \]
by taking the limit as \( n \to \infty \) in (6.85), and using the fact that \( T^{n+1} \to 0 \) as \( n \to \infty \) when \( \|T\|_{op} < 1 \). Thus \( I - T \) is invertible on \( V \) when \( \|T\|_{op} < 1 \) and \( V \) is complete.

Let us continue to ask that \( V \) be complete. If \( T \) is any bounded linear operator on \( V \), and \( \lambda \) is a real or complex number, as appropriate, such that \( |\lambda| > \|T\|_{op} \), then \( \lambda I - T \) is invertible on \( V \). This follows from the preceding argument applied to \( \lambda^{-1} T \). Similarly, if \( R \) is a bounded linear operator on \( V \) which is also invertible, and if \( T \) is a bounded linear operator on \( V \) that satisfies

\[
\|R^{-1}\|_{op} \|T\|_{op} < 1,
\]

then

\[
R - T = R (I - R^{-1} T)
\]

is invertible on \( V \).

Suppose that \( V \) is a complex Banach space, and let \( T \) be a bounded linear operator on \( V \). The spectrum of \( T \) is the set of complex numbers \( \lambda \) such that \( \lambda I - T \) is not invertible on \( V \). As usual, eigenvalues of \( T \) are elements of the spectrum, but the converse does not hold in infinite dimensions. Note that

\[
|\lambda| \leq \|T\|_{op},
\]

for every \( \lambda \in \mathbb{C} \) in the spectrum of \( T \), as in the preceding paragraph. If \( \lambda \in \mathbb{C} \) is not in the spectrum of \( T \), so that \( \lambda I - T \) is invertible on \( V \), then \( \mu I - T \) is also invertible for every complex number \( \mu \) sufficiently close to \( \lambda \), by the remarks in the previous paragraph. This implies that the spectrum of \( T \) is a closed set in the complex plane. A famous theorem states that the spectrum of \( T \) is always nonempty. The main idea in the proof is that otherwise \( (\lambda I - T)^{-1} \) would be a holomorphic function of \( \lambda \) on the complex plane that tends to 0 as \( |\lambda| \to \infty \).

Let \( V_1 \) and \( V_2 \) be Banach spaces, both real or both complex, and with norms \( \| \cdot \|_1, \| \cdot \|_2 \), respectively. Also let

\[
B_1 = \{v \in V_1 : \|v\|_1 \leq 1\}
\]

be the closed unit ball in \( V_1 \). A linear mapping \( T \) from \( V_1 \) into \( V_2 \) is said to be compact if the closure of \( T(B_1) \) in \( V_2 \) is a compact set. This is equivalent to asking that \( T(B_1) \) be totally bounded in \( V_2 \), which means that for each \( \epsilon > 0 \), \( T(B_1) \) can be covered by finitely many balls of radius \( \epsilon \) in \( B_2 \). In particular, totally bounded sets are bounded, and hence compact linear mappings are bounded. It is easy to see that bounded subsets of finite-dimensional spaces are totally bounded, so that bounded linear mappings of finite rank are compact. One can also check that the composition of a bounded linear mapping with a compact linear mapping is compact, where the compact operator is either first or second in the composition.

Let \( cL(V_1, V_2) \) be the space of compact linear mappings from \( V_1 \) into \( V_2 \). This is a linear subspace of the vector space \( B\mathcal{L}(V_1, V_2) \) of bounded linear mappings from \( V_1 \) into \( V_2 \), which is closed with respect to the operator norm on \( B\mathcal{L}(V_1, V_2) \). This means that if \( \{T_j\}_{j=1}^\infty \) is a sequence of compact linear mappings from \( V_1 \)
into $V_2$ that converges to a bounded linear mapping $T : V_1 \to V_2$ with respect to the operator norm, then $T$ is compact too. In particular, $T$ is compact if it is the limit of a sequence of bounded linear mappings of finite rank with respect to the operator norm. In some situations, including mappings between Hilbert spaces, one can show that every compact linear mapping is the limit of a sequence of bounded linear mappings of finite rank with respect to the operator norm.

Let $T$ be a compact linear mapping from a Banach space $V$ into itself. If $\lambda$ is a real or complex number, as appropriate, such that $\lambda \neq 0$ and $\lambda I - T$ is not invertible, then it can be shown that $\lambda$ is an eigenvalue of $T$, and that the corresponding eigenspace is finite-dimensional. It can also be shown that for each $r > 0$, there are only finitely many eigenvalues $\lambda$ with $|\lambda| \geq r$.

### 6.9 Self-adjoint linear operators

Let $V$ be a real or complex vector space with an inner product $\langle v, w \rangle$, and suppose that $V$ is complete with respect to the corresponding norm $\|v\|$, so that $V$ is a Hilbert space. As in Section 3.3, a bounded linear operator $A$ on $V$ is said to be self-adjoint if

$$\langle A(v), w \rangle = \langle v, A(w) \rangle$$

(6.94)

for every $v, w \in V$. As before, the identity operator $I$ on $V$ is self-adjoint, as is the orthogonal projection $P_W$ of $V$ onto a closed linear subspace $W$ of $V$. The sum of two bounded self-adjoint linear operators on $V$ is also self-adjoint, and the product of a bounded self-adjoint linear operator on $V$ and a real number is self-adjoint too.

Suppose for the moment that $V$ is a complex Hilbert space. If $A$ is a bounded self-adjoint linear operator on $V$, then

$$\langle A(v), v \rangle = \langle v, A(v) \rangle = \overline{\langle A(v), v \rangle}$$

(6.95)

for every $v \in V$, and hence

$$\langle A(v), v \rangle \in \mathbb{R}$$

(6.96)

for every $v \in V$. Using this, it is easy to see that the eigenvalues of $A$ are real numbers, as before. Let us check that the spectrum of $A$, as defined in the previous section, is also contained in the real line under these conditions. Equivalently, this means that $\lambda I - A$ is invertible on $V$ for every complex number $\lambda$ with nonzero imaginary part.

Observe that

$$\text{Im} \langle (\lambda I - A)(v), v \rangle = (\text{Im} \lambda) \|v\|^2$$

(6.97)

for every $v \in V$, by (6.96), so that

$$|\langle (\lambda I - A)(v), v \rangle| \geq |\text{Im} \lambda| \|v\|^2$$

(6.98)

for every $v \in V$. The Cauchy–Schwarz inequality implies that

$$|\langle (\lambda I - A)(v), v \rangle| \leq \| (\lambda I - A)(v) \| \|v\|,$$
from which we get that

\[(\lambda I - A)(v) \geq |\text{Im } \lambda| \|v\|\]  \hspace{1cm} (6.100)

for every \(v \in V\). This is the same type of condition as (6.84) in the previous section, since \(\text{Im } \lambda \neq 0\), by hypothesis. In order to show that \(\lambda I - A\) is invertible on \(V\), it suffices to check that \(\lambda I - A\) maps \(V\) onto itself.

Suppose for the sake of a contradiction that \(W = (\lambda I - A)(V)\) is a proper linear subspace of \(V\). Note that \(W\) is a closed linear subspace of \(V\), because of (6.100) and the completeness of \(V\), as in the previous section. Let \(v\) be any element of \(V \setminus W\), and put

\[y = v - P_W(v),\]  \hspace{1cm} (6.101)

where \(P_W(v)\) is the orthogonal projection of \(v\) onto \(W\), as in Section 6.4. Thus \(y \neq 0\), because \(v \notin W\) and \(P_W(v) \in W\), and \(y\) is orthogonal to every element of \(W\). The latter condition is the same as saying that

\[\langle (\lambda I - A)(v), y \rangle = 0\]  \hspace{1cm} (6.102)

for every \(v \in V\). In particular, we can apply this to \(v = y\), to get that

\[\langle (\lambda I - A)(y), y \rangle = 0.\]  \hspace{1cm} (6.103)

This implies that \(y = 0\), by (6.98), contradicting the hypothesis that \(y \neq 0\). It follows that \(W = V\), so that \(\lambda I - A\) is invertible on \(V\), as desired.

Let \(V\) be a real or complex Hilbert space again. A bounded self-adjoint linear operator \(A\) on \(V\) is said to be nonnegative if

\[\langle A(v), v \rangle \geq 0\]  \hspace{1cm} (6.104)

for every \(v \in V\). Suppose that \(A\) satisfies the strict positivity condition that

\[\langle A(v), v \rangle \geq c \|v\|^2\]  \hspace{1cm} (6.105)

for some \(c > 0\) and every \(v \in V\), and let us check that \(A\) is invertible on \(V\). By the Cauchy-Schwarz inequality,

\[\langle A(v), v \rangle \leq \|A(v)\| \|v\|\]  \hspace{1cm} (6.106)

for every \(v \in V\), and hence

\[\|A(v)\| \geq c \|v\|\]  \hspace{1cm} (6.107)

for every \(v \in V\). This is the same as (6.84) in this context, and it suffices to show that \(A\) maps \(V\) onto itself.

As before, \(W = A(V)\) is a closed linear subspace of \(V\) under these conditions. If \(W \neq V\), then there is a \(y \in V\) such that \(y \neq 0\) and \(y\) is orthogonal to every element of \(W\). Equivalently, this means that

\[\langle A(v), y \rangle = 0\]  \hspace{1cm} (6.108)
for every $v \in V$, and for $v = y$ in particular, so that $\langle A(y), y \rangle = 0$. This implies that $y = 0$, by the strict positivity of $A$, contradicting the hypothesis that $y \neq 0$. Thus $A(V) = V$, and hence $A$ is invertible on $V$, as desired.

Similarly, if $A$ is a bounded self-adjoint linear operator on $V$ that satisfies (6.107) for some $c > 0$ and every $v \in V$, then $A$ is invertible on $V$. As before, $W = A(V)$ is a closed linear subspace of $V$ under these conditions, and we want to show that $W = V$. Otherwise, there is a $y \in V$ such that $y \neq 0$ and $y$ is orthogonal to every element of $W$, so that

$$\langle v, A(y) \rangle = \langle A(v), y \rangle = 0$$

(6.109)

for every $v \in V$. This implies that $A(y) = 0$, and hence that $y = 0$, because of (6.107). Thus $A(V) = V$, so that $A$ is invertible on $V$, as desired.

In analogy with the finite-dimensional case, it can be shown that a compact self-adjoint linear operator $T$ on $V$ can be diagonalized in an orthonormal basis for $V$. Using this, one can show that any compact linear mapping between Hilbert spaces has a Schmidt decomposition as in Section 3.8, but perhaps with infinite sequences of orthonormal vectors, and coefficients $\lambda_j$ converging to 0 as $j \to \infty$. If the $\lambda_j$’s are $p$-summable for some $p > 0$, then the operator is said to be in the $S_p$ class.
Chapter 7

Marcel Riesz’ convexity theorem

Let \((a_{j,k})\) be an \(n \times n\) matrix of complex numbers, and let \(A(x, y)\) be the bilinear form defined for \(x, y \in \mathbb{C}^n\) by

\[
A(x, y) = \sum_{j=1}^{n} \sum_{k=1}^{n} y_j a_{j,k} x_k.
\]  (7.1)

For \(1 < p < \infty\), let \(M_p\) be the quantity

\[
M_p = \sup \left\{ |A(x, y)| : x, y \in \mathbb{C}^n, \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq 1, \left( \sum_{j=1}^{n} |y_j|^{p'} \right)^{1/p'} \leq 1 \right\},
\]  (7.2)

where \(p'\) denotes the conjugate exponent of \(p\), \(1/p + 1/p' = 1\). When \(p = 1\), \(p' = \infty\), put

\[
M_1 = \sup \left\{ |A(x, y)| : x, y \in \mathbb{C}^n, \sum_{k=1}^{n} |x_k| \leq 1, \max_{1 \leq j \leq n} |y_j| \leq 1 \right\},
\]  (7.3)

and when \(p = \infty\), \(p' = 1\), set

\[
M_\infty = \sup \left\{ |A(x, y)| : x, y \in \mathbb{C}^n, \max_{1 \leq k \leq n} |x_k| \leq 1, \sum_{j=1}^{n} |y_j| \leq 1 \right\}.
\]  (7.4)

**Theorem 7.5** As a function of \(1/p \in [0, 1]\), \(\log M_p\) is convex.

More precisely, if \(1 \leq p < q \leq \infty\), \(0 < t < 1\), \(1 < r < \infty\), and

\[
\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q},
\]  (7.6)
then
\[(7.7) \quad M_r \leq M_p^t M_q^{1-t}.\]

If \(M_p = 0\) for some \(p\), then \(A \equiv 0\) and \(M_p = 0\) for every \(p\), and hence we may as well assume that \(A \neq 0\) in the arguments that follow. The special case of \(p = 1\), \(q = \infty\) corresponds exactly to the theorem of Schur discussed in Section 2.5. Note that the analogous inequality holds when the \(a_{j,k}\)'s are real numbers, and we use \(x, y \in \mathbb{R}^n\) in the definition of \(M_p\), by the same proof.

We can also describe \(M_p\) as
\[(7.8) \quad M_p = \sup \left\{ \left( \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{j,k} x_k|^p \right)^{1/p} : x \in \mathbb{C}^n, \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq 1 \right\}\]
when \(1 \leq p < \infty\), and
\[(7.9) \quad M_\infty = \sup \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^{n} a_{j,k} x_k \right| : x \in \mathbb{C}^n, \max_{1 \leq k \leq n} |x_k| \leq 1 \right\}.

This definition of \(M_p\) is greater than or equal to the previous one by Hölder’s inequality, and for each \(x \in \mathbb{C}^n\), there is a \(y \in \mathbb{C}^n\) for which equality holds and \(\left( \sum_{j=1}^{n} |y_j|^p' \right)^{1/p'} \) or \(\max_{1 \leq j \leq n} |y_j|\) is equal to 1, according to whether \(p' < \infty\) or \(p' = \infty\). Equivalently, \(M_p\) is the operator norm of the linear transformation on \(\mathbb{C}^n\) associated to the matrix \((a_{j,k})\) with respect to the \(p\)-norm \(\| \cdot \|_p\) defined in Section 2.1. Similarly,
\[(7.10) \quad M_p = \sup \left\{ \left( \sum_{k=1}^{n} \sum_{j=1}^{n} |y_j a_{j,k}|^{p'/p'} \right)^{1/p'} : y \in \mathbb{C}^n, \left( \sum_{j=1}^{n} |y_j|^p \right)^{1/p'} \leq 1 \right\}\]
when \(1 < p \leq \infty\), and
\[(7.11) \quad M_1 = \sup \left\{ \max_{1 \leq k \leq n} \left| \sum_{j=1}^{n} y_j a_{j,k} \right| : y \in \mathbb{C}^n, \max_{1 \leq j \leq n} |y_j| \leq 1 \right\},\]
which says that \(M_p\) is equal to the operator norm of the dual linear transformation on \(\mathbb{C}^n\), associated to the transpose matrix, and with respect to the dual norm \(\| \cdot \|_{p'}\). One can check that \(M_s\) is a continuous function of \(1/s\), \(1/s \in [0, 1]\), using the inequalities (1.80), (1.81), (1.83), and (1.84).

As in Lemma 1.54, we would like to show that for each \(p, q \in [1, \infty]\) there is a \(t \in (0, 1)\) such that \((7.7)\) holds. Fix a real number \(r, 1 < r < \infty\), and let \(r'\) be its conjugate exponent. There exist \(x^0, y^0 \in \mathbb{C}^n\) at which the supremum in the definition (7.2) of \(M_r\) is attained, i.e., which satisfy
\[(7.12) \quad |A(x^0, y^0)| = M_r\]
and the normalizations
\[(7.13) \quad \left( \sum_{k=1}^{n} |x_0^k|^r \right)^{1/r} = 1\]
and

\[ (\sum_{j=1}^{n} |y_{j}^{0}| r')^{1/r'} = 1. \]  

(7.14)

This follows from standard considerations of continuity and compactness.

Observe that

\[ |A(x_{0}, y_{0})| = \left| \sum_{j=1}^{n} \sum_{k=1}^{n} y_{j}^{0} a_{j,k} x_{k}^{0} \right| \]

\[ = \left( \sum_{i=1}^{n} |y_{i}^{0}|^{r'} \right)^{1/r'} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} x_{k}^{0} \right)^{r'} \]  

(7.15)

The first step uses only the definition of \( A \). If the second equality were replaced with \( \leq \), then it would be a consequence of Hölder’s inequality. If equality did not hold, then we could replace \( y_{0} \) with an element of \( C^{n} \) which satisfies (7.14) and for which equality does hold, increasing the value of \( |A(x_{0}, y_{0})| \). Similarly,

\[ |A(x_{0}, y_{0})| = \left| \sum_{j=1}^{n} \sum_{k=1}^{n} y_{j}^{0} a_{j,k} x_{k}^{0} \right| \]

\[ = \left( \sum_{k=1}^{n} \sum_{j=1}^{n} y_{j}^{0} a_{j,k} \right)^{1/r'} \left( \sum_{l=1}^{n} |x_{l}^{0}|^{r} \right)^{1/r} \]  

(7.16)

Because of the second equality in (7.15), there is a \( \mu \geq 0 \) such that

\[ \left| \sum_{k=1}^{n} a_{j,k} x_{k}^{0} \right| = \mu |y_{j}^{0}|^{r' - 1} \]  

(7.17)

for \( j = 1, 2, \ldots, n \). This can be derived from the proof of Hölder’s inequality, by analyzing the conditions in which equality holds. Similarly, there is a \( \nu \geq 0 \) such that

\[ \left| \sum_{j=1}^{n} y_{j}^{0} a_{j,k} \right| = \nu |x_{k}^{0}|^{r - 1} \]  

(7.18)

for \( k = 1, 2, \ldots, n \). From (7.15) and (7.16), we have that

\[ M_{r} = \left( \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} x_{k}^{0} \right)^{1/r'} = \left( \sum_{k=1}^{n} \sum_{j=1}^{n} y_{j}^{0} a_{j,k} \right)^{1/r'}, \]  

(7.19)

using also (7.12), (7.13), and (7.14). Substituting (7.17) in the first equality, we get that

\[ M_{r} = \mu \left( \sum_{j=1}^{n} |y_{j}^{0}|^{r(r' - 1)} \right)^{1/r}. \]  

(7.20)

Because \( r(r' - 1) = r' \), since \( 1/r + 1/r' = 1 \), we can apply (7.14) to get that \( M_{r} = \mu \). For the same reasons, \( M_{r} = \nu \).
Now suppose that \( p, q, \) and \( t \) are real numbers such that \( 1 \leq p < q \leq \infty, \) \( 0 < t < 1, \) and \( 1/r = t/p + (1-t)/q. \) Thus \( p' < \infty, \) where \( q' \) is the conjugate exponent of \( q. \) Observe that

\[
\left( \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{j,k} a_{k}^0|^{p} \right)^{1/p} \leq M_p \left( \sum_{k=1}^{n} |x_{k}^0|^p \right)^{1/p}
\]

and

\[
\left( \sum_{k=1}^{n} \sum_{j=1}^{n} |y_{j}^0 a_{j,k}|^{q'} \right)^{1/q'} \leq M_q \left( \sum_{j=1}^{n} |y_{j}^0|^{q'} \right)^{1/q'}.
\]

Applying the previous computations, we get that

\[
M_r \left( \sum_{j=1}^{n} |y_{j}^0|^{p(r'-1)} \right)^{1/p} \leq M_p \left( \sum_{k=1}^{n} |x_{k}^0|^p \right)^{1/p}
\]

and

\[
M_r \left( \sum_{k=1}^{n} |x_{k}^0|^q(r-1) \right)^{1/q} \leq M_q \left( \sum_{j=1}^{n} |y_{j}^0|^{q'} \right)^{1/q'}.
\]

We are going to need some identities with indices. Let us first check that

\[
t \left( \frac{1}{r} - \frac{1}{p} \right) = (1-t) \left( \frac{1}{r'} - \frac{1}{q'} \right).
\]

Because \( 1/r = t/p + (1-t)/q, \) we have that

\[
t \left( \frac{1}{r} - \frac{1}{p} \right) = t \left( 1-t \right) \left( \frac{1}{p'} + \frac{1}{q} \right).
\]

Similarly, \( 1/r' = t/p' + (1-t)/q', \) and

\[
(1-t) \left( \frac{1}{r'} - \frac{1}{q'} \right) = (1-t) t \left( \frac{1}{p'} - \frac{1}{q} \right) = (1-t) t \left( \frac{1}{p} + \frac{1}{q} \right).
\]

This proves (7.25).

Suppose that

\[
t \frac{1}{r} = \frac{1-t}{r'}.
\]

This implies that

\[
t \frac{1}{p} = \frac{1-t}{q'},
\]

by (7.25). Hence

\[
r' - 1 = \frac{1-t}{t} \quad \text{and} \quad r - 1 = \frac{t}{1-t},
\]

because \( r (r' - 1) = r' \) and \( r' (r - 1) = r, \) since \( 1/r + 1/r' = 1. \) Therefore

\[
p (r' - 1) = q' \quad \text{and} \quad q' (r - 1) = p.
\]
We can take the $t$th and $(1 - t)$th powers of (7.23) and (7.24), respectively, and then multiply to get

\begin{equation}
M_r \left( \sum_{j=1}^{n} |y_j^0|^{p(r'-1)} \right)^{t/p} \left( \sum_{k=1}^{n} |x_k^0|^{q'(r-1)} \right)^{(1-t)/q'} 
\leq M_p^t M_q^{1-t} \left( \sum_{k=1}^{n} |x_k^0|^p \right)^{t/p} \left( \sum_{j=1}^{n} |y_j^0|^{q'} \right)^{(1-t)/q'}.
\end{equation}

Assuming (7.28), this reduces to

\begin{equation}
M_r \leq M_p^t M_q^{1-t}.
\end{equation}

because the factors involving $x^0$ and $y^0$ on the left and right sides of (7.32) exactly match up under these conditions, by the computations in the preceding paragraph. To summarize, for each $p, q$ with $1 \leq p < q \leq \infty$, there is a $t \in (0, 1)$ such that (7.28) holds when $r$ is given by $1/r = t/p + (1-t)/q$. For this choice of $t$, we get the inequality (7.33). Theorem 7.5 now follows from Lemma 1.54, with the small adaptation to functions on closed intervals.
Chapter 8

Some dyadic analysis

8.1 Dyadic intervals

Normally, a reference to “the unit interval” in the real line might suggest the closed interval $[0, 1]$, but here it will be convenient to use $(0, 1)$ instead, for minor technical reasons.

**Definition 8.1** The dyadic subintervals of $[0, 1)$ are the intervals of the form $[j2^{-k}, (j+1)2^{-k})$, where $j$ and $k$ are nonnegative integers, and $j + 1 \leq 2^k$. In particular, the length of a dyadic interval in $[0, 1)$ is of the form $2^{-k}$, where $k$ is a nonnegative integer.

The dyadic intervals in $\mathbb{R}$ can be defined in the same way, with arbitrary integers $j$ and $k$. The half-open, half-closed condition leads to nice properties in terms of disjointness, as in the next two lemmas, whose simple proofs are left as exercises.

**Lemma 8.2** For each nonnegative integer $k$, $[0, 1)$ is the union of the dyadic subintervals of length $2^{-k}$, and these subintervals are pairwise disjoint.

**Lemma 8.3** If $J_1$ and $J_2$ are two dyadic subintervals of $[0, 1)$, then either $J_1 \subseteq J_2$, or $J_2 \subseteq J_1$, or $J_1 \cap J_2 = \emptyset$.

More precisely, if $J_1$, $J_2$ are dyadic subintervals of $[0, 1)$ such that the length of $J_2$ is less than or equal to the length of $J_1$, then either $J_2 \subseteq J_1$ or $J_1 \cap J_2 = \emptyset$.

**Lemma 8.4** If $J$ is a dyadic subinterval of $[0, 1)$ of length $2^{-k}$, and if $n$ is an integer greater than $k$, then $J$ is the union of the dyadic subintervals of $J$ of length $2^{-n}$, and these subintervals are pairwise disjoint. Every dyadic subinterval of $[0, 1)$ of length $2^{-n}$ is contained in a unique dyadic subinterval of $[0, 1)$ of length $2^{-k}$ when $n \geq k$.

This is easy to see.
Lemma 8.5 If $\mathcal{F}$ is an arbitrary collection of dyadic subintervals of $[0, 1)$, then there is a subcollection $\mathcal{F}_0$ of $\mathcal{F}$ such that

$$(8.6) \quad \bigcup_{J \in \mathcal{F}_0} J = \bigcup_{J \in \mathcal{F}} J$$

and the elements of $\mathcal{F}_0$ are pairwise disjoint.

To prove this, we take $\mathcal{F}_0$ to be the set of maximal elements of $\mathcal{F}$, i.e., the set of $J \in \mathcal{F}$ such that $J \subseteq J'$ for some $J' \in \mathcal{F}$ only when $J' = J$. Every interval in $\mathcal{F}$ is contained in a maximal interval in $\mathcal{F}$, since every dyadic subinterval of $[0, 1)$ is contained in only finitely many dyadic subintervals of $[0, 1)$. Thus every element of $\mathcal{F}$ is contained in an element of $\mathcal{F}_0$, which implies (8.6). Any two maximal elements of $\mathcal{F}$ which are distinct are disjoint, by Lemma 8.3, which implies the second property of $\mathcal{F}_0$ in the lemma.

Let $f$ be a real or complex-valued function on the unit interval $[0, 1)$ which is sufficiently well-behaved for integrals of $f$ over subintervals of $[0, 1)$ to be defined. One is welcome to restrict one’s attention to step functions here, and we shall simplify this a bit further in a moment. For each nonnegative integer $k$, let $E_k(f)$ be the function on $[0, 1)$ defined by

$$(8.7) \quad E_k(f)(x) = 2^{-k} \int_J f(y) \, dy,$$

where $J$ is the dyadic subinterval of $[0, 1)$ with length $2^{-k}$ that contains $x$. Of course, $E_k(f)$ is linear in $f$.

Lemma 8.8 (a) For each $f$, $E_k(f)$ is constant on the dyadic subintervals of $[0, 1)$ of length $2^{-k}$.

(b) If $f$ is constant on the dyadic subintervals of $[0, 1)$ of length $2^{-k}$, then $E_k(f) = f$.

(c) For any $f$, $E_j(E_k(f)) = E_k(f)$ and $E_k(E_j(f)) = E_k(f)$ when $j \geq k$.

(d) If $g$ is a function on $[0, 1)$ which is constant on the dyadic subintervals of $[0, 1)$ of length $2^{-k}$, then $E_k(gf) = g E_k(f)$ for each $f$.

This is easy to verify, directly from the definitions. Note that the first part of (c) holds simply because $E_k(f)$ is constant on dyadic subintervals of $[0, 1)$ of length $2^{-j}$ when $j \geq k$. In the second part of (c), one is first averaging $f$ on the smaller dyadic intervals of length $2^{-j}$ to get $E_j(f)$, and then averaging the result on the larger dyadic intervals of length $2^{-k}$ to get $E_k(E_j(f))$, and the conclusion is that this is the same as averaging over the dyadic intervals of length $2^{-k}$ directly.

Definition 8.9 A function $f$ on $[0, 1)$ is a dyadic step function if it is a finite linear combination of indicator functions of dyadic subintervals of $[0, 1)$.

Lemma 8.10 Let $f$ be a function on $[0, 1)$. The following are equivalent:

(a) $f$ is a dyadic step function;
(b) There is a nonnegative integer \( k \) such that \( f \) is constant on every dyadic subinterval of \([0, 1)\) of length \( 2^{-k} \):

(c) \( E_k(f) = f \) for some nonnegative integer \( k \), and hence for all sufficiently large integers \( k \).

One can check this using the previous lemma. From now on, one is welcome to restrict one’s attention to dyadic step functions in this chapter.

Lemma 8.11 For any functions \( f, g \) on \([0, 1)\) and nonnegative integer \( j \),

\[
(8.12) \quad \int_{[0,1)} E_j(f)(x) g(x) \, dx = \int_{[0,1)} f(x) E_j(g)(x) \, dx = \int_{[0,1)} E_j(f)(x) E_j(g)(x) \, dx.
\]

Lemma 8.13 For any functions \( f, g \) on \([0, 1)\) and positive integers \( j, k \) with \( j \neq k \),

\[
(8.14) \quad \int_{[0,1)} E_0(f)(x) (E_j(g)(x) - E_{j-1}(g)(x)) \, dx = 0
\]

and

\[
(8.15) \quad \int_{[0,1)} (E_j(f)(x) - E_{j-1}(f)(x)) (E_k(g)(x) - E_{k-1}(g)(x)) \, dx = 0.
\]

The computations for these two lemmas are straightforward and left to the reader.

Let \( I \) be a dyadic subinterval of \([0, 1)\), and let \( I_l \) and \( I_r \) be the two dyadic subintervals of \( I \) of half the size of \( I \). The Haar function \( h_I(x) \) on \([0, 1)\) associated to the interval \( I \) is defined by

\[
(8.16) \quad h_I(x) = -|I|^{1/2} \quad \text{when } x \in I_l \quad = |I|^{1/2} \quad \text{when } x \in I_r \quad = 0 \quad \text{when } x \in [0, 1) \setminus I.
\]

Observe that

\[
(8.17) \quad \int_{[0,1)} h_I(x) \, dx = 0
\]

and

\[
(8.18) \quad \int_{[0,1)} h_I(x)^2 \, dx = 1.
\]

In addition, there is a special Haar function \( h_0(x) \) on \([0, 1)\) defined by \( h_0(x) = 1 \) for every \( x \in [0, 1) \), for which we also have

\[
(8.19) \quad \int_{[0,1)} h_0(x)^2 \, dx = 1.
\]
If $I$ and $J$ are distinct dyadic subintervals of $[0, 1)$, then $h_I$ and $h_J$ satisfy the orthogonality property
\begin{equation}
\int_{[0,1)} h_I(x) h_J(x) \, dx = 0. \tag{8.20}
\end{equation}
For if $I$ and $J$ are disjoint, then $h_I(x) h_J(x) = 0$ for every $x \in [0, 1)$, and the integral vanishes trivially. Otherwise, one of the intervals $I$ and $J$ is contained in the other, and we may as well assume that $J \subseteq I$, since the two cases are completely symmetric. Because $J \neq I$, $J \subseteq I_l$ or $J \subseteq I_r$, $h_I$ is constant on $J$, and (8.20) follows from (8.17). If $I$ is any dyadic subinterval of $[0, 1)$, then
\begin{equation}
\int_{[0,1)} h_0(x) h_I(x) \, dx = 0, \tag{8.21}
\end{equation}
by (8.17).

For each function $f$ on $[0, 1)$ and nonnegative integer $k$,
\begin{equation}
E_k(f) = \langle f, h_0 \rangle h_0 + \sum_{|I| \geq 2^{-k+1}} \langle f, h_I \rangle h_I. \tag{8.22}
\end{equation}
Here the sum is taken over all dyadic subintervals $I$ of $[0, 1)$ with $|I| \geq 2^{-k+1}$, and is interpreted as being 0 when $k = 0$. Also, $\langle f, h_0 \rangle$, $\langle f, h_I \rangle$ are the integrals of $f$ times $h_0$, $h_I$, respectively. In particular, dyadic step functions are finite linear combinations of Haar functions. If $f$ is a dyadic step function on $[0, 1)$, then
\begin{equation}
f = \langle f, h_0 \rangle h_0 + \sum_I \langle f, h_I \rangle h_I, \tag{8.23}
\end{equation}
where the sum is taken over all dyadic subintervals $I$ of $[0, 1)$. The sum is actually a finite sum, since $\langle f, h_I \rangle = 0$ for all but finitely many $I$. This expression for $f$ follows from the orthonormality conditions for the Haar functions described earlier.

### 8.2 Maximal functions

As mentioned in the previous section, one is welcome to restrict one’s attention to real or complex-valued functions on $[0, 1)$ that are dyadic step functions in this chapter. The **dyadic maximal function** $M(f)$ associated to a function $f$ on $[0, 1)$ is defined by
\begin{equation}
M(f)(x) = \sup_{k \geq 0} |E_k(f)(x)|. \tag{8.24}
\end{equation}
Equivalently, $M(f)(x)$ is equal to
\begin{equation}
\sup \left\{ \left| \frac{1}{|J|} \int_J f(y) \, dy \right| : J \text{ is a dyadic subinterval of } [0, 1) \text{ and } x \in J \right\}. \tag{8.25}
\end{equation}
For each nonnegative integer \( l \), put
\[
M_l(f)(x) = \max_{0 \leq k \leq l} |E_k(f)(x)|,
\]
which is the same as
\[
M_l(f)(x) = \max \left\{ \left| \frac{1}{|J|} \int_J f(y) \, dy \right| : J \subseteq [0, 1), x \in J, \text{ and } |J| \geq 2^{-l} \right\},
\]
where the maximum is again taken over dyadic subintervals \( J \) of \([0, 1)\). Thus
\[
M_l(f) \leq M_r(f) \quad \text{when } r \geq l
\]
and
\[
M(f)(x) = \sup_{l \geq 0} M_l(f)(x) \quad \text{for every } x \in [0, 1).
\]

For any pair of functions \( f_1, f_2 \) on \([0, 1)\),
\[
M(f_1 + f_2) \leq M(f_1) + M(f_2)
\]
and
\[
M_l(f_1 + f_2) \leq M_l(f_1) + M_l(f_2)
\]
for each \( l \geq 0 \). Also,
\[
M(cf) = |c| M(f)
\]
and
\[
M_l(cf) = |c| M_l(f)
\]
for any function \( f \) and constant \( c \). Thus \( M(f), M_l(f) \) are sublinear in \( f \).

**Lemma 8.34** If \( f \) is constant on the dyadic subintervals of \([0, 1)\) of length \( 2^{-l} \), then \( M(f) \) is constant on the dyadic subintervals of \([0, 1)\) of length \( 2^{-l} \), and \( M(f) = M_l(f) \). For any function \( f \),
\[
M_l(f) = M(E_l(f)),
\]
and \( M_l(f) \) is constant on dyadic intervals of length \( 2^{-l} \).

**Exercise.**

**Corollary 8.36** If \( f \) is a dyadic step function, then \( M(f) \) is too, and \( M(f) = M_j(f) \) for sufficiently large \( j \).

**Lemma 8.37** (Supremum bound for \( M(f) \)) If \(|f(x)| \leq A \) for some \( A \geq 0 \) and every \( x \in [0, 1) \), then \( M(f)(x) \leq A \) for every \( x \in [0, 1) \).

This is an easy consequence of the definitions. Lemma 8.37 also works if \(|f(x)| \leq A \) for every \( x \in [0, 1) \) except for a small set that does not affect the integrals.
Proposition 8.38 (Weak-type estimate for $M(f)$) For every $\lambda > 0$,  
\begin{equation}
\label{eq:8.39}
|\{x \in [0,1) : M(f)(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{[0,1)} |f(w)| \, dw.
\end{equation}

The left-hand side of (8.39) refers to the measure of the set in question, the meaning of which is clarified by the proof.

Let $\lambda > 0$ be given, let $F$ be the collection of dyadic intervals subintervals $L$ of $[0,1)$ such that
\begin{equation}
\label{eq:8.40}
\frac{1}{|L|} \int_L f(y) \, dy > \lambda,
\end{equation}
and let us check that
\begin{equation}
\label{eq:8.41}
\{x \in [0,1) : M(f)(x) > \lambda\} = \bigcup_{L \in F} L.
\end{equation}

If $x \in [0,1)$ and $M(f)(x) > \lambda$, then there is a dyadic interval $L$ in $[0,1)$ such that $x \in L$ and $L$ satisfies (8.40), because of (8.25), and hence the left side of (8.41) is contained in the right side of (8.41). Conversely, if $L \in F$, then
\begin{equation}
\label{eq:8.42}
M(f)(x) \geq \frac{1}{|L|} \int_L f(y) \, dy > \lambda
\end{equation}
for every $x \in L$, and therefore $L$ is contained in the left side of (8.41). Thus the right side of (8.41) is contained in the left side, and (8.41) follows.

As in Lemma 8.5, if $F_0$ consists of the maximal elements of $F$, then
\begin{equation}
\label{eq:8.43}
\bigcup_{L \in F_0} L = \bigcup_{L \in F} L,
\end{equation}
and the intervals in $F_0$ are pairwise disjoint. Thus
\begin{equation}
\label{eq:8.44}
|\{x \in [0,1) : M(f)(x) > \lambda\}| = \sum_{L \in F_0} |L|.
\end{equation}
If $f$ is a dyadic step function which is constant on the dyadic subintervals of $[0,1)$ of length $2^{-l}$, then $M(f)$ is constant on the dyadic intervals of length $2^{-l}$, and the elements of $F_0$ have length $\geq 2^{-l}$.

Each interval $L \in F_0$ satisfies (8.40), which gives
\begin{equation}
\label{eq:8.45}
|L| < \frac{1}{\lambda} \int_L f(y) \, dy \leq \frac{1}{\lambda} \int_{[0,1)} |f(y)| \, dy,
\end{equation}
and hence
\begin{equation}
\label{eq:8.46}
\sum_{L \in F_0} |L| < \sum_{L \in F_0} \frac{1}{\lambda} \int_L |f(y)| \, dy = \frac{1}{\lambda} \int_{[0,1)} |f(y)| \, dy,
\end{equation}
using the disjointness of the intervals $L \in F_0$. Therefore
\begin{equation}
\label{eq:8.47}
|\{x \in [0,1) : M(f)(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{y \in [0,1) : M(f)(y) > \lambda\}} |f(y)| \, dy,
\end{equation}
which implies (8.39).
Lemma 8.48 For each $\lambda > 0$,

\begin{equation}
\{|x \in [0, 1) : M(f)(x) > 2\lambda| \leq \frac{1}{\lambda} \int_{\{u \in [0, 1) : |f(u)| > \lambda\}} |f(u)| \, du. \tag{8.49}\end{equation}

Let $\lambda > 0$ be given, and put

\begin{equation}
f_1(x) = f(x) \text{ when } |f(x)| \leq \lambda, \quad f_1(x) = 0 \text{ when } |f(x)| > \lambda, \tag{8.50}\end{equation}

and

\begin{equation}
f_2(x) = f(x) \text{ when } |f(x)| > \lambda, \quad f_2(x) = 0 \text{ when } |f(x)| \leq \lambda. \tag{8.51}\end{equation}

Thus $f(x) = f_1(x) + f_2(x)$ and $M(f_1)(x) \leq \lambda$ for every $x \in [0, 1)$. This implies that

\begin{equation}
M(f)(x) \leq \lambda + M(f_2)(x) \tag{8.52}\end{equation}

for every $x$, and hence

\begin{equation}
\{|x \in [0, 1) : M(f)(x) > 2\lambda| \leq \{|x \in [0, 1) : M(f_2)(x) > \lambda|\}. \tag{8.53}\end{equation}

We can apply Proposition 8.38 with $f$ replaced by $f_2$ to get that

\begin{equation}
\{|x \in [0, 1) : M(f_2)(x) > \lambda| \leq \frac{1}{\lambda} \int_{[0,1)} |f_2(u)| \, du, \tag{8.54}\end{equation}

and the lemma follows.

Lemma 8.55 If $g(x)$ is a nonnegative real-valued function on $[0, 1)$, and $p$ is a positive real number, then

\begin{equation}
\int_{[0,1]} g(x)^p \, dx = \int_{0}^{\infty} p \lambda^{p-1} |\{x \in [0, 1) : g(x) > \lambda\}| \, d\lambda. \tag{8.56}\end{equation}

One can see this by integrating $p \lambda^{p-1}$ on the set

\begin{equation}
\{(x, \lambda) \in \mathbb{R}^2 : x \in [0, 1), 0 < \lambda < g(x)\} \tag{8.57}\end{equation}

first in $\lambda$, and then in $x$, and first in $x$, and then in $\lambda$.

Proposition 8.58 For each real number $p > 1$,

\begin{equation}
\int_{[0,1]} M(f)(x)^p \, dx \leq \frac{2^p p}{p-1} \int_{[0,1]} |f(y)|^p \, dy. \tag{8.59}\end{equation}

To prove this, we apply Lemma 8.55 with $g = M(f)$ to get that

\begin{equation}
\int_{[0,1]} M(f)(x)^p \, dx = \int_{0}^{\infty} p \lambda^{p-1} |\{x \in [0, 1) : M(f)(x) > \lambda\}| \, d\lambda. \tag{8.60}\end{equation}
8.3. SQUARE FUNCTIONS

By (8.49) with $\lambda$ replaced by $\lambda/2$,

\[
(8.61) \int_{[0,1]} M(f)(x)^p \, dx \leq \int_0^\infty p \lambda^{p-1} \left( \frac{2}{\lambda} \int_{\{u \in [0,1] : |f(u)| > \lambda/2\}} |f(u)| \, du \right) \, d\lambda \\
= \int_0^\infty \int_{\{u \in [0,1] : |f(u)| > \lambda/2\}} 2p \lambda^{p-2} |f(u)| \, du \, d\lambda.
\]

Interchanging the order of integration leads to

\[
(8.62) \int_{[0,1]} M(f)(x)^p \, dx \leq \int_{[0,1]} \int_0^{2|f(u)|} 2p \lambda^{p-2} |f(u)| \, d\lambda \, du.
\]

Because $p > 1$,

\[
(8.63) \int_{[0,1]} M(f)(x)^p \, dx \leq \int_{[0,1]} 2p (p-1)^{-1} (2 |f(u)|)^{p-1} |f(u)| \, du \\
= \frac{2^p p}{p-1} \int_{[0,1]} |f(u)|^p \, du,
\]

as desired.

8.3 Square functions

The dyadic square function $S(f)$ associated to a function $f$ on $[0,1)$ is defined by

\[
(8.64) S(f)(x) = \left( |E_0(f)(x)|^2 + \sum_{j=1}^\infty |E_j(f)(x) - E_{j-1}(f)(x)|^2 \right)^{1/2}.
\]

For each nonnegative integer $l$, put

\[
(8.65) S_l(f)(x) = \left( |E_0(f)(x)|^2 + \sum_{j=1}^l |E_j(f)(x) - E_{j-1}(f)(x)|^2 \right)^{1/2},
\]

where the sum on the right side is interpreted as being 0 when $l = 0$. Thus

\[
(8.66) S_l(f)(x) \leq S_p(f)(x) \quad \text{when} \quad l \leq p
\]

and $S(f)(x) = \sup_{l \geq 0} S_l(f)(x)$. It is easy to see that $S(f)$, $S_l(f)$ are sublinear in $f$, in the sense that

\[
(8.67) S(f_1 + f_2) \leq S(f_1) + S(f_2)
\]

and $S(cf) = |c| S(f)$ for all functions $f_1$, $f_2$, and $f$ on $[0,1)$ and all constants $c$, and similarly for $S_l(f)$. 
Lemma 8.68  If \( f \) is constant on the dyadic subintervals of \([0, 1)\) of length \(2^{-l}\), then \( S(f) \) is constant on the dyadic subintervals of \([0, 1)\) of length \(2^{-l}\), and \( S(f) = S_i(f) \). For any \( f \),

\[
S_i(f) = S(E_i(f)),
\]

and \( S_i(f) \) is constant on dyadic intervals of length \(2^{-l}\).

Exercise.

Corollary 8.70  If \( f \) is a dyadic step function on \([0, 1)\), then \( S(f) \) is too, and \( S(f) = S_j(f) \) for sufficiently large \( j \).

Lemma 8.71  For any function \( f \) on \([0, 1)\),

\[
\int_{[0,1]} S_l(f)(x)^2 \, dx = \int_{[0,1]} |E_l(f)(x)|^2 \, dx
\]

for every \( l \geq 0 \), and

\[
\int_{[0,1]} S(f)(x)^2 \, dx = \int_{[0,1]} |f(x)|^2 \, dx.
\]

Of course

\[
E_l(f) = E_0(f) + \sum_{j=1}^{l} (E_j(f) - E_{j-1}(f)),
\]

and to prove the lemma one uses the orthogonality conditions in Lemma 8.13.

8.4 Estimates, 1

Proposition 8.75  If \( 0 < p < 2 \), then there is a positive real number \( C_1(p) \) such that

\[
\int_{[0,1]} S(f)(x)^p \, dx \leq C_1(p) \int_{[0,1]} M(f)(x)^p \, dx
\]

for any function \( f \) on \([0,1)\).

Let \( p < 2 \), a function \( f \) on \([0,1)\), and \( \lambda > 0 \) be given, and consider

\[
|\{x \in [0,1) : S(f)(x) > \lambda\}|.
\]

Let \( \mathcal{F} \) denote the set of dyadic subintervals \( J \) of \([0,1)\) such that

\[
\frac{1}{|J|} \left| \int_J f(y) \, dy \right| > \lambda.
\]

If \( \mathcal{F}_0 \) is the set of maximal intervals in \( \mathcal{F} \), then

\[
\bigcup_{J \in \mathcal{F}_0} J = \bigcup_{J \in \mathcal{F}} J
\]
and

\[(8.80) \quad J_1 \cap J_2 = \emptyset \quad \text{when} \quad J_1, J_2 \in \mathcal{F}_0, \quad J_1 \neq J_2,\]

as in Lemma 8.5.

Suppose that \([0, 1)\) is not an element of \(\mathcal{F}_0\), and let \(\mathcal{F}_1\) be the set of dyadic subintervals \(L\) of \([0, 1)\) for which there is a \(J \in \mathcal{F}_0\) such that

\[(8.81) \quad J \subseteq L \quad \text{and} \quad |J| = |L|/2.\]

Because \(\mathcal{F}_0\) consists of maximal intervals in \(\mathcal{F}\), each \(L \in \mathcal{F}_1\) does not lie in \(\mathcal{F}\), and hence

\[(8.82) \quad \frac{1}{|L|} \left| \int_L f(y) \, dy \right| \leq \lambda
\]

for every \(L \in \mathcal{F}_1\).

The elements of \(\mathcal{F}_1\) need not be disjoint, and so we let \(\mathcal{F}_{10}\) be the set of maximal elements of \(\mathcal{F}_1\). As usual,

\[(8.83) \quad \bigcup_{L \in \mathcal{F}_{10}} L = \bigcup_{L \in \mathcal{F}_1} L
\]

and

\[(8.84) \quad L_1 \cap L_2 = \emptyset \quad \text{when} \quad L_1, L_2 \in \mathcal{F}_{10}, \quad L_1 \neq L_2.\]

Let \(f_\lambda(x)\) be the function on \([0, 1)\) defined by

\[(8.85) \quad f_\lambda(x) = \begin{cases} \frac{1}{|L|} \int_L f(y) \, dy & \text{when} \quad x \in L, \quad L \in \mathcal{F}_{10}, \\ f(x) & \text{when} \quad x \in [0, 1) \setminus \left( \bigcup_{L \in \mathcal{F}_{10}} L \right). \end{cases}\]

**Lemma 8.86** If \(K\) is a dyadic subinterval of \([0, 1)\) such that

\[(8.87) \quad K \setminus \left( \bigcup_{L \in \mathcal{F}_{10}} L \right) \neq \emptyset
\]

or \(L \subseteq K\) for some \(L \in \mathcal{F}_{10}\), then

\[(8.88) \quad \frac{1}{|K|} \int_K f(u) \, du = \frac{1}{|K|} \int_K f_\lambda(u) \, du.\]

Under these conditions, \(K\) is the disjoint union of the intervals \(L \in \mathcal{F}_{10}\) such that \(L \subseteq K\) and \(K \setminus \left( \bigcup_{L \in \mathcal{F}_{10}} L \right)\). The integral of \(f\) over \(K\) is equal to the sum of the integrals of \(f\) over these sets, which is the same as the integral of \(f_\lambda\) over \(K\).

**Corollary 8.89** If \(x \in [0, 1) \setminus \left( \bigcup_{L \in \mathcal{F}_{10}} L \right)\), then \(S(f)(x) = S(f_\lambda)(x)\).
For these $x$’s, Lemma 8.86 implies that $E_j(f)(x) = E_j(f^\lambda)(x)$ for every nonnegative integer $j$, and hence $S(f)(x) = S(f^\lambda)(x)$.

Using the corollary, it is easy to see that

\[(8.90) \quad \{|x \in [0,1): S(f)(x) > \lambda|\} \leq \sum_{L \in \mathcal{F}_{10}} |L| + \{|x \in [0,1): S(f^\lambda)(x) > \lambda|\}.
\]

For each $L \in \mathcal{F}_{10}$, there is a $J \in \mathcal{F}_0$ such that $J \subseteq L$ and $|J| = |L|/2$, and this leads to

\[(8.91) \quad \sum_{L \in \mathcal{F}_{10}} |L| \leq 2 \sum_{J \in \mathcal{F}_0} |J|.
\]

By (8.79) and (8.80),

\[(8.92) \quad \sum_{L \in \mathcal{F}_{10}} |L| \leq 2 \left| \bigcup_{J \in \mathcal{F}_0} J \right| = 2 \left| \bigcup_{J \in \mathcal{F}} J \right|.
\]

As in (8.41),

\[(8.93) \quad \bigcup_{J \in \mathcal{F}} J = \{x \in [0,1): M(f)(x) > \lambda\}.
\]

Therefore

\[(8.94) \quad \sum_{L \in \mathcal{F}_{10}} |L| \leq 2 \{|x \in [0,1): M(f)(x) > \lambda|\},
\]

and hence

\[(8.95) \quad \{|x \in [0,1): S(f)(x) > \lambda|\} \leq 2 \{|x \in [0,1): M(f)(x) > \lambda|\} + |\{x \in [0,1): S(f^\lambda)(x) > \lambda\}|.
\]

By Lemma 8.71,

\[(8.96) \quad \lambda^2 \{|x \in [0,1): S(f^\lambda)(x) > \lambda|\} \leq \int_{[0,1)} S(f^\lambda)(x)^2 \, dx = \int_{[0,1)} |f^\lambda(x)|^2 \, dx.
\]

Thus

\[(8.97) \quad \{|x \in [0,1): S(f)(x) > \lambda|\} \leq 2 \{|x \in [0,1): M(f)(x) > \lambda|\} + \lambda^2 \int_{[0,1)} |f^\lambda(x)|^2 \, dx.
\]

**Lemma 8.98** $|f^\lambda| \leq \min(\lambda, M(f))$.

Indeed, for each dyadic subinterval $I$ of $[0,1)$, we have that

\[(8.99) \quad \frac{1}{|I|} \int_I f(y) \, dy \leq M(f)(x)
\]
automatically when \( x \in I \), and

\[
\frac{1}{I} \int_I f(y) \, dy \leq \lambda
\]

(8.100)

when \( I \in \mathcal{F}_1 \) or \( x \in I \setminus \bigcup_{L \in \mathcal{F}_0} L \), since \( I \notin \mathcal{F} \) in these two cases. With the help of Lemma 8.86, one can actually get the stronger estimate

\[
M(f_\lambda) \leq \min(\lambda, M(f)).
\]

(8.101)

Because of the lemma, we may replace (8.97) with

\[
|\{ x \in [0, 1) : S(f)(x) > \lambda \}| \leq 2 \min(\lambda, M(f))(x) + \lambda^{-2} \int_{[0,1]} \min(\lambda, M(f))(x)^2 \, dx.
\]

(8.102)

At the beginning of this argument, just before (8.81), we assumed that \([0, 1)\) is not an element of \( \mathcal{F}_0 \). If \([0, 1)\) is an element of \( \mathcal{F}_0 \subseteq \mathcal{F} \), then

\[
M(f)(x) > \lambda \text{ for every } x \in [0, 1), \text{ and hence}
\]

\[
\{ x \in [0, 1) : S(f)(x) > \lambda \} \leq \{ x \in [0, 1) : M(f)(x) > \lambda \}.
\]

(8.103)

Thus (8.102) holds in general. By Lemma 8.55,

\[
\int_{[0,1]} S(f)(x)^p \, dx = \int_0^\infty p \lambda^{p-1} |\{ x \in [0, 1) : S(f)(x) > \lambda \}| \, d\lambda
\]

(8.105)

and

\[
\int_{[0,1]} M(f)(x)^p \, dx = \int_0^\infty p \lambda^{p-1} |\{ x \in [0, 1) : M(f)(x) > \lambda \}| \, d\lambda.
\]

(8.106)

Therefore

\[
\int_{[0,1]} S(f)(x)^p \, dx
\leq 2 \int_{[0,1]} M(f)(x)^p \, dx + \int_0^\infty p \lambda^{p-3} \int_{[0,1]} \min(\lambda, M(f)(x))^2 \, dx \, d\lambda.
\]

We can interchange the order of integration and replace the second term on the right side of (8.107) with

\[
\int_{[0,1]} \int_0^\infty p \lambda^{p-3} \min(\lambda, M(f)(x))^2 \, d\lambda \, dx.
\]

(8.108)
The integral in $\lambda$ can be computed exactly, since
\[
\int_{M(f)(x)}^{\infty} p \lambda^{p-3} M(f)(x)^2 \, d\lambda = \frac{p}{2-p} M(f)(x)^p
\]
and
\[
\int_{0}^{M(f)(x)} p \lambda^{p-3} \lambda^2 \, d\lambda = M(f)(x)^p.
\]

Proposition 8.75 now follows by using these formulae in (8.107).

The coefficient $(2-p)^{-1}$ in the previous computations is not very nice, and one can get bounded constants for $p$ near 2 using interpolation arguments. One can also start with an estimate for $p = 4$ instead of $p = 2$, as in Section 8.10, and use the same method as here to get estimates for $0 < p < 4$ that remain bounded for $p$ near 2.

**Proposition 8.111 (Weak-type estimate for $S(f)$)** For any function $f$ on $[0, 1)$ and $\lambda > 0$,
\[
|\{x \in [0, 1) : S(f)(x) > \lambda\}| \leq \frac{3}{\lambda} \int_{[0,1)} |f(x)| \, dx.
\]

This follows from practically the same arguments as above. By (8.97) and Lemma 8.98,
\[
|\{x \in [0, 1) : S(f)(x) > \lambda\}|
\leq 2 \left|\{x \in [0, 1) : M(f)(x) > \lambda\}\right| + \lambda^{-1} \int_{[0,1)} |f_\lambda(x)| \, dx,
\]
where $f_\lambda(x)$ is as in (8.85). To get (8.112), one can use Proposition 8.38 and the observation that
\[
\int_{[0,1)} |f_\lambda(x)| \, dx \leq \int_{[0,1)} |f(x)| \, dx.
\]

**8.5 Estimates, 2**

**Proposition 8.115** If $0 < p < 2$, then there is a positive real number $C_2(p)$ such that
\[
\int_{[0,1)} M(f)(x)^p \, dx \leq C_2(p) \int_{[0,1)} S(f)(x)^p \, dx
\]
for any function $f$ on $[0, 1)$.

Let $p < 2$ and $f$ be given, and let $\lambda > 0$ be a positive real number.

**Lemma 8.117** The set
\[
\{x \in [0, 1) : S(f)(x) > \lambda\}
\]
is a union of dyadic subintervals of $[0, 1)$. 
If \( w \in [0, 1) \) and

\[
8.119 \quad S(f)(w) = \left( |E_0(f)(w)|^2 + \sum_{j=1}^{\infty} |E_j(f)(w) - E_{j-1}(f)(w)|^2 \right)^{1/2} > \lambda,
\]

then

\[
8.120 \quad \left( |E_0(f)(w)|^2 + \sum_{j=1}^{l} |E_j(f)(w) - E_{j-1}(f)(w)|^2 \right)^{1/2} > \lambda
\]

for some \( l \). Let \( I \) be the dyadic subinterval of \([0, 1)\) such that \( |I| = 2^{-l} \) and \( w \in I \). Because \( E_j(f) \) is constant on dyadic intervals of length \( 2^{-j} \), \( E_j(f)(y) = E_j(f)(w) \) when \( j \leq l \) and \( y \in I \), and hence

\[
8.121 \quad \left( |E_0(f)(y)|^2 + \sum_{j=1}^{l} |E_j(f)(y) - E_{j-1}(f)(y)|^2 \right)^{1/2} > \lambda
\]

for every \( y \in I \). Therefore \( S(f)(y) > \lambda \) for every \( y \in I \), and the lemma follows easily.

Let \( \mathcal{G}_0 \) be the collection of maximal dyadic subintervals of \([0, 1)\) contained in the set \((8.118)\). As usual,

\[
8.122 \quad \bigcup_{J \in \mathcal{G}_0} J = \{ x \in [0, 1) : S(f)(x) > \lambda \},
\]

and the intervals in \( \mathcal{G}_0 \) are pairwise disjoint. In particular,

\[
8.123 \quad \sum_{J \in \mathcal{G}_0} |J| = |\{ x \in [0, 1) : S(f)(x) > \lambda \}|.
\]

Suppose that \((8.118)\) is not equal to the whole unit interval \([0, 1)\). Let \( \mathcal{G}_1 \) be the collection of dyadic subintervals \( L \) of \([0, 1)\) for which there is a \( J \in \mathcal{G}_0 \) such that

\[
8.124 \quad J \subseteq L \quad \text{and} \quad |J| = |L|/2.
\]

Because the elements of \( \mathcal{G}_0 \) are maximal dyadic intervals contained in \((8.118)\), each \( L \in \mathcal{G}_1 \) is not a subset of \((8.118)\). Thus for each \( L \in \mathcal{G}_1 \) there is a point \( u \in L \) such that \( S(f)(u) \leq \lambda \). If \( \ell(L) \) denotes the nonnegative integer such that \( 2^{-\ell(L)} = |L| \), then

\[
8.125 \quad \left( |E_0(f)(u)|^2 + \sum_{j=1}^{\ell(L)} |E_j(f)(u) - E_{j-1}(f)(u)|^2 \right)^{1/2} \leq \lambda,
\]

where the sum on the left is interpreted as being 0 if \( \ell(L) = 0 \). More precisely, this inequality holds for at least one \( u \in L \), and hence at every \( u \in L \), because \( E_j(f) \) is constant on \( L \) when \( j \leq \ell(L) \).
CHAPTER 8. SOME DYADIC ANALYSIS

The intervals in \(G_1\) need not be pairwise disjoint, and we can pass to the subcollection \(G_{10}\) of maximal elements of \(G_1\) to get

\[
\bigcup_{L \in G_{10}} L = \bigcup_{L \in G_1} L
\]

(8.126)

and \(L_1 \cap L_2 = \emptyset\) when \(L_1, L_2 \in G_1\) and \(L_1 \neq L_2\). The definition of \(G_1\) implies that \(\bigcup_{J \in G_0} J \subseteq \bigcup_{L \in G_1} L\), and therefore

\[
\{x \in [0, 1) : S(f)(x) > \lambda\} \subseteq \bigcup_{L \in G_{10}} L.
\]

Also,

\[
\sum_{L \in G_{10}} |L| \leq \sum_{J \in G_0} 2 |J| = 2 \{x \in [0, 1) : S(f)(x) > \lambda\}.
\]

(8.127)

Let \(g_\lambda(x)\) be the function defined on \([0, 1)\) by

\[
g_\lambda(x) = \begin{cases} 
\frac{1}{|L|} \int_L f(y) \, dy & \text{when } x \in L, \ L \in G_{10} \\
f(x) & \text{when } x \in [0, 1) \setminus \left( \bigcup_{I \in G_{10}} I \right).
\end{cases}
\]

(8.129)

**Lemma 8.130** If \(K\) is a dyadic subinterval of \([0, 1)\) such that

\[
K \setminus \left( \bigcup_{I \in G_{10}} I \right) \neq \emptyset
\]

(8.131)

or \(L \subseteq K\) for some \(L \in G_{10}\), then

\[
\frac{1}{|K|} \int_K g_\lambda(u) \, du = \frac{1}{|K|} \int_K f(u) \, du.
\]

(8.132)

This uses the fact that \(K\) is the disjoint union of the \(L \in G_{10}\) with \(L \subseteq K\) and \(K \setminus \left( \bigcup_{I \in G_{10}} I \right)\), as in Lemma 8.86.

**Corollary 8.133** If \(x \in [0, 1) \setminus \left( \bigcup_{I \in G_{10}} I \right)\), then \(M(f)(x) = M(g_\lambda)(x)\).

**Corollary 8.134** If \(x \in [0, 1) \setminus \left( \bigcup_{I \in G_{10}} I \right)\), then \(S(f)(x) = S(g_\lambda)(x)\). If \(L \in G_{10}\), \(v \in L\), and \(2^{-\ell(L)} = |L|\), then

\[
S(g_\lambda)(v) = \left( |E_0(f)(v)|^2 + \sum_{j=1}^{\ell(L)} |E_j(f)(v) - E_{j-1}(f)(v)|^2 \right)^{1/2}.
\]

(8.135)

These two corollaries follow from Lemma 8.130 and the relevant definitions.

**Corollary 8.136** \(S(g_\lambda) \leq \min(\lambda, S(f))\).
Corollary 8.134 implies that \( S(g_\lambda) \leq S(f) \), and we get \( S(g_\lambda) \leq \lambda \) using also (8.125) and (8.127).

Because of Corollary 8.133,

\[
\{ x \in [0, 1) : M(f)(x) > \lambda \} \subseteq \left( \bigcup_{I \in \mathcal{G}_{01}} I \right) \cup \{ x \in [0, 1) : M(g_\lambda)(x) > \lambda \},
\]

and hence

\[
\{ x \in [0, 1) : M(f)(x) > \lambda \} \leq \left( \sum_{I \in \mathcal{G}_{01}} |I| \right) + \{ x \in [0, 1) : M(g_\lambda)(x) > \lambda \}.
\]

Therefore

\[
\{ x \in [0, 1) : M(f)(x) > \lambda \} \leq \left( \sum_{I \in \mathcal{G}_{01}} |I| \right) + \{ x \in [0, 1) : M(g_\lambda)(x) > \lambda \},
\]

by (8.128). Of course,

\[
\{ x \in [0, 1) : M(g_\lambda)(x) > \lambda \} \leq \lambda^{-2} \int_{[0, 1)} M(g_\lambda)(u)^2 \, du,
\]

and

\[
\int_{[0, 1)} M(g_\lambda)(u)^2 \, du \leq C \int_{[0, 1)} |g_\lambda(y)|^2 \, dy
\]

for some \( C > 0 \), by Proposition 8.58. Moreover,

\[
\int_{[0, 1]} |g_\lambda(y)|^2 \, dy = \int_{[0, 1]} S(g_\lambda)(w)^2 \, dw
\]

\[
\leq \int_{[0, 1]} \min(\lambda, S(f)(w))^2 \, dw,
\]

by Lemma 8.71 and Corollary 8.136. It follows that

\[
\{ x \in [0, 1) : M(g_\lambda)(x) > \lambda \} \leq C \lambda^{-2} \int_{[0, 1]} \min(\lambda, S(f)(w))^2 \, dw,
\]

and consequently

\[
\{ x \in [0, 1) : M(f)(x) > \lambda \} \leq 2 \{ x \in [0, 1) : S(f)(x) > \lambda \} + C \lambda^{-2} \int_{[0, 1]} \min(\lambda, S(f)(w))^2 \, dw.
\]

We assumed near the beginning of the argument that the set (8.118) is not all of \([0, 1)\). If it is, then the preceding inequality holds trivially. The rest of the proof of Proposition 8.115 proceeds via computations like those in the previous section.
8.6 Duality, 1

If \( f_1, f_2 \) are functions on \([0, 1]\), and \( l \) is a nonnegative integer, then

\[
\int_{[0,1]} E_l(f_1) E_l(f_2) \, dx = \int_{[0,1]} \left( E_0(f_1) E_0(f_2) + \sum_{j=1}^{l} (E_j(f_1) - E_{j-1}(f_1)) (E_j(f_2) - E_{j-1}(f_2)) \right) \, dx,
\]

where the sum is interpreted as being 0 when \( l = 0 \). This is a “bilinear” version of (8.72) in Lemma 8.71, which can be verified in essentially the same way. By the Cauchy–Schwarz inequality for sums,

\[
\left| \int_{[0,1]} E_l(f_1) E_l(f_2) \, dx \right| \leq \int_{[0,1]} S_l(f_1) S_l(f_2) \, dx,
\]

and for suitable functions \( f_1 \) and \( f_2 \),

\[
\left| \int_{[0,1]} f_1(x) f_2(x) \, dx \right| \leq \int_{[0,1]} S(f_1)(x) S(f_2)(x) \, dx.
\]

**Proposition 8.148** For each \( q > 2 \), there is a \( C_3(q) > 0 \) such that

\[
\int_{[0,1]} |f(x)|^q \, dx \leq C_3(q) \int_{[0,1]} S(f)(x)^q \, dx.
\]

Let \( q > 2 \) be given, and let \( p, 1 < p < \infty \), be the exponent dual to \( q \), so that \( 1/p + 1/q = 1 \) and \( p < 2 \). By Hölder’s inequality,

\[
\int_{[0,1]} f_1(x) f_2(x) \, dx \leq \left( \int_{[0,1]} S(f_1)^q \, dy \right)^{1/q} \left( \int_{[0,1]} S(f_2)^p \, dw \right)^{1/p}.
\]

Propositions 8.58 and 8.75 yield

\[
\int_{[0,1]} f_1(x) f_2(x) \, dx \leq C \left( \int_{[0,1]} S(f_1)(y)^q \, dy \right)^{1/q} \left( \int_{[0,1]} |f_2(w)|^p \, dw \right)^{1/p}
\]

for some \( C > 0 \). In general, if

\[
\int_{[0,1]} f_1(x) f_2(x) \, dx \leq A \left( \int_{[0,1]} |f_2(w)|^p \, dw \right)^{1/p}
\]

for some \( A \geq 0 \) and arbitrary functions \( f_2 \) on \([0, 1]\), then

\[
\left( \int_{[0,1]} |f_1(x)|^q \, dx \right)^{1/q} \leq A,
\]

and the proposition follows.
8.7 Duality, 2

**Proposition 8.154** For each \( q > 2 \), there is a \( C_4(q) > 0 \) such that

\[
\int_{[0,1]} S(f)(x)^q \, dx \leq C_4(q) \int_{[0,1]} |f(x)|^q \, dx. \tag{8.155}
\]

Let \( q > 2 \) be given, and let \( p \) be the conjugate exponent to \( q \). It suffices to show that the proposition holds with \( S(f) \) replaced with \( S_l(f) \) for every \( l \), with a constant that does not depend on \( l \). To do this, it is enough to show that

\[
\left| \int_{[0,1]} \left( \alpha_0(x) E_0(f)(x) + \sum_{j=1}^{l} \alpha_j(x) (E_j(f)(x) - E_{j-1}(f)(x)) \right) \, dx \right| \tag{8.156}
\]

is less than or equal to a constant times the product of

\[
\left( \int_{[0,1]} |f(y)|^q \, dy \right)^{1/q} \tag{8.157}
\]

and

\[
\left( \int_{[0,1]} \left( \sum_{j=0}^{l} |\alpha_j(w)|^2 \right)^{p/2} \, dw \right)^{1/p} \tag{8.158}
\]

for arbitrary functions \( \alpha_0, \ldots, \alpha_l \) on \([0,1)\). By Lemma 8.11, (8.156) is equal to

\[
\left| \int_{[0,1]} \left( E_0(\alpha_0)(x) + \sum_{j=1}^{l} (E_j(\alpha_j)(x) - E_{j-1}(\alpha_j)(x)) \right) f(x) \, dx \right|. \tag{8.159}
\]

Hölder’s inequality implies that this is less than or equal to the product of (8.157) and

\[
\left( \int_{[0,1]} \left( \sum_{j=0}^{l} |\alpha_j(w)|^2 \right)^{p/2} \, dw \right)^{1/p} \tag{8.160}
\]

Thus we would like to show that (8.160) is less than or equal to a constant times (8.158). Proposition 8.115 implies that (8.160) is bounded by a constant times

\[
\left( \int_{[0,1]} S \left( E_0(\alpha_0) + \sum_{j=1}^{l} (E_j(\alpha_j) - E_{j-1}(\alpha_j)) \right) (x)^p \, dx \right)^{1/p}, \tag{8.161}
\]

and so we would like to show that (8.161) is bounded by a constant times (8.158). One can check that

\[
S \left( E_0(\alpha_0) + \sum_{j=1}^{l} (E_j(\alpha_j) - E_{j-1}(\alpha_j)) \right) (x) \tag{8.162}
\]
is equal to
\[(8.163) \quad \left( |E_0(\alpha_0)(x)|^2 + \sum_{j=1}^l |E_j(\alpha_j)(x) - E_{j-1}(\alpha_j)(x)|^2 \right)^{1/2}.
\]

It therefore remains to show that
\[(8.164) \quad \left( \int_{[0,1)} \left( |E_0(\alpha_0)(x)|^2 + \sum_{j=1}^l |E_j(\alpha_j)(x) - E_{j-1}(\alpha_j)(x)|^2 \right)^{p/2} dx \right)^{1/p}
\]
is bounded by a constant times (8.158). This can be done using the results discussed in the next section.

### 8.8 Auxiliary estimates

Let \( l \) be a nonnegative integer, and let \( \beta_0(x), \beta_1(x), \ldots, \beta_l(x) \) be nonnegative functions on \([0,1)\). Given \( p, r \geq 1 \), consider the problem of bounding

\[(8.165) \quad \left( \int_{[0,1)} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right)^{p/r} dx \right)^{1/p}
\]

by a constant times

\[(8.166) \quad \left( \int_{[0,1)} \left( \sum_{j=0}^l \beta_j(x)^r \right)^{p/r} dx \right)^{1/p},
\]

where the constant does not depend on \( l \) or \( \beta_0(x), \beta_1(x), \ldots, \beta_l(x) \). If \( r = \infty \), then
\[(8.167) \quad \left( \sum_{j=0}^l \beta_j(x)^r \right)^{1/r}, \quad \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right)^{1/r}
\]
should be replaced with
\[(8.168) \quad \max_{0 \leq j \leq l} \beta_j(x), \quad \max_{0 \leq j \leq l} E_j(\beta_j)(x),
\]
as usual.

**Lemma 8.169** For each \( p \geq 1 \), nonnegative integer \( j \), and nonnegative function \( \beta \) on \([0,1)\),
\[(8.170) \quad \int_{[0,1)} E_j(\beta)(x)^p \ dx \leq \int_{[0,1)} \beta(x)^p \ dx.
\]

If \( J \) is any interval in \([0,1)\), then
\[(8.171) \quad \left( \frac{1}{|J|} \int_{J} \beta(y) \ dy \right)^p \leq \frac{1}{|J|} \int_{J} \beta(y)^p \ dy,
\]
by Jensen’s inequality. Lemma 8.169 follows by summing this over the dyadic intervals $J$ of length $2^{-j}$.

Using Lemma 8.169, it is easy to see that (8.165) is less than or equal to (8.166) when $p = r$. When $r = \infty$, we might as well restrict our attention to the case where the $\beta_j$’s are all the same, and the question reduces to one about maximal functions. Lemma 8.37 and Proposition 8.58 yield suitable estimates for $1 < p \leq \infty$.

**Lemma 8.172** Suppose that $1 \leq r < p < \infty$, and let $s \in (1, \infty)$ be conjugate to $p/r$, so that $1/s + r/p = 1$. For each positive real number $A_0$, (8.165) is less than or equal to $A_0$ times (8.166) for arbitrary nonnegative functions $\beta_0, \beta_1, \ldots, \beta_l$ on $[0, 1)$ if and only if

\[
\int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right) h(x) \, dx 
\leq A_0 \left( \int_{[0,1]} \left( \sum_{j=0}^l \beta_j(y)^r \right)^{p/r} dy \right)^{r/p} \left( \int_{[0,1]} h(w)^s \, dw \right)^{1/s}
\]

for arbitrary nonnegative functions $\beta_0, \beta_1, \ldots, \beta_l$ and $h$ on $[0, 1)$.

This is basically the same observation as in (8.152) and (8.153), applied to this situation. Let us continue to assume that $1 \leq r < p < \infty$, and that $s$ is conjugate to $p/r$. We would like to show that (8.173) holds for a suitable choice of $A_0$. Because $r \geq 1$, $E_j(\beta_j)^r \leq E_j(\beta_j^r)$, by Jensen’s inequality. Hence

\[
\int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right) h(x) \, dx 
\leq \int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j^r)(x) \right) h(x) \, dx.
\]

This implies that

\[
\int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right) h(x) \, dx 
\leq \int_{[0,1]} \sum_{j=0}^l \beta_j(x)^r E_j(h)(x) \, dx,
\]

and thus

\[
\int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right) h(x) \, dx 
\leq \int_{[0,1]} \sum_{j=0}^l \beta_j(x)^r M(h)(x) \, dx.
\]

By Hölder’s inequality,

\[
\int_{[0,1]} \left( \sum_{j=0}^l E_j(\beta_j)(x)^r \right) h(x) \, dx 
\leq \left( \int_{[0,1]} \left( \sum_{j=0}^l \beta_j(y)^r \right)^{p/r} dy \right)^{r/p} \left( \int_{[0,1]} M(h)(w)^s \, dw \right)^{1/s}.
\]
It follows from Proposition 8.58 that (8.173) holds for some \( A_0 > 0 \). This shows that (8.165) is bounded by a constant times (8.166) when \( 1 \leq r < p < \infty \).

**Lemma 8.178** If \( 1 < p < \infty \), \( 1 < r < \infty \), and \( p', r' \) are the exponents conjugate to \( p \), \( r \), respectively, then for each positive real number \( A_0 \), (8.165) is less than or equal to \( A_0 \) times (8.166) for arbitrary nonnegative functions \( \beta_0, \beta_1, \ldots, \beta_l \) on \([0,1)\) if and only if the same is true with \( p, r \) replaced by \( p', r' \).

This also works for \( p, r = 1, \infty \), with minor adjustments of the usual type. To prove the lemma, the main step is to observe that (8.165) is bounded by \( A_0 \) times (8.166) for all nonnegative functions \( \beta_0(x), \beta_1(x), \ldots, \beta_l(x) \) on \([0,1)\) if and only if

\[
(8.179) \int_{(0,1)} \left( \sum_{j=0}^{l} E_j(\beta_j)(x) \gamma_j(x) \right) dx
\]

\[
\leq A_0 \left( \int_{(0,1)} \left( \sum_{j=0}^{l} \beta_j(x)^{p/r} dx \right)^{1/p} \left( \int_{(0,1)} \left( \sum_{j=0}^{l} \gamma_j(x)^{r'/r'} dx \right)^{1/r'} \right) \right.
\]

for all nonnegative functions \( \beta_0(x), \beta_1(x), \ldots, \beta_l(x) \) and \( \gamma_0(x), \gamma_1(x), \ldots, \gamma_l(x) \) on \([0,1)\). It follows from the lemma and the remarks preceding it that (8.165) is less than or equal to a constant times (8.166) when \( 1 < p < r < \infty \).

### 8.9 Interpolation

Let \( T \) be a linear operator acting on real or complex-valued dyadic step functions on \([0,1)\). Suppose that \( 1 \leq p < q \leq \infty \),

\[
(8.180) \left( \int_{(0,1)} |T(f)(x)|^p dx \right)^{1/p} \leq N_p \left( \int_{(0,1)} |f(x)|^p dx \right)^{1/p},
\]

and

\[
(8.181) \left( \int_{(0,1)} |T(f)(x)|^q dx \right)^{1/q} \leq N_q \left( \int_{(0,1)} |f(x)|^q dx \right)^{1/q}
\]

when \( q < \infty \) or

\[
(8.182) \sup_{x \in [0,1)} |T(f)(x)| \leq N_\infty \sup_{x \in [0,1)} |f(x)|
\]

when \( q = \infty \), for some \( N_p, N_q \geq 0 \) and each \( f \). If \( 0 < t < 1 \) and

\[
(8.183) \frac{1}{r} = \frac{t}{p} + \frac{1-t}{q},
\]

then

\[
(8.184) \left( \int_{(0,1)} |T(f)(x)|^r dx \right)^{1/r} \leq N_p^t N_q^{1-t} \left( \int_{(0,1)} |f(x)|^r dx \right)^{1/r}.
\]
8.10. ANOTHER ARGUMENT FOR $P = 4$

This can be derived from Theorem 7.5, as follows. For each positive integer $l$,

$$\left( \int_{[0,1]} |E_l(T(f))(x)|^p \, dx \right)^{1/p} \leq N_p \left( \int_{[0,1]} |f(x)|^p \, dx \right)^{1/p},$$

and analogously for $q$ instead of $p$. Theorem 7.5 can be applied to get that

$$\left( \int_{[0,1]} |E_l(T(f))(x)|^r \, dx \right)^{1/r} \leq N_p^t N_q^{1-t} \left( \int_{[0,1]} |f(x)|^r \, dx \right)^{1/r},$$

for all step functions $f$ on $[0,1)$ that are constant on dyadic intervals of length $2^{-l}$, by thinking of $E_l \circ T$ as a linear transformation on that space, which can be identified with $\mathbb{R}^n$ or $\mathbb{C}^n$, $n = 2^l$, as appropriate. Once one has (8.186) for step functions that are constant on dyadic intervals of length $2^{-l}$ for every $l$, it is easy to derive (8.184) for arbitrary dyadic step functions. Of course, one can extend this to other classes of functions too.

The maximal and square function operators discussed in this chapter are not linear, but the same interpolation inequalities can be applied to them. One can show this by approximating these operators by linear operators. Suppose that $T$ is a not-necessarily-linear operator acting on dyadic step functions on $[0,1)$ such that for each dyadic step function $f$ there is a linear operator $A$ on the same space of functions with the properties that

$$|A(h)(x)| \leq T(h)(x)$$

for every $h, x$ and

$$T(f)(x) = |A(f)(x)|.$$

If $T$ satisfies (8.180) and (8.181) or (8.182), then the analogous inequalities hold for these approximating linear operators $A$. By interpolation, the approximating linear operators $A$ satisfy (8.184), and therefore $T$ does too. One can approximate maximal functions in this way by linear operators of the form

$$E_{\alpha(x)}(f)(x),$$

where $\alpha(x)$ takes values in nonnegative integers. One can approximate square functions by linear operators of the form

$$\alpha_0(x) E_0(f)(x) + \sum_{i=1}^t \alpha_i(x)(E_i(f)(x) - E_{i-1}(f)(x)),$$

where $(\sum_{i=0}^t |\alpha_i(x)|^2)^{1/2} \leq 1$.

8.10 Another argument for $p = 4$

Let $f$ be a function on $[0,1)$, and consider

$$S(f)(x)^4 = \left( |E_0(f)(x)|^2 + \sum_{j=1}^{\infty} |E_j(f)(x) - E_{j-1}(f)(x)|^2 \right)^2.$$
Put
\[ R_j(f)(x) = \left( \sum_{k=j}^\infty |E_k(f)(x) - E_{k-1}(f)(x)|^2 \right)^{1/2} \]
for each positive integer \( j \). Thus
\[
S(f)(x)^4 = (|E_0(f)(x)|^2 + R_1(f)(x))^2 \\
= |E_0(f)(x)|^4 + 2|E_0(f)(x)|^2 R_1(f)(x)^2 + R_1(f)(x)^4,
\]
and
\[
R_1(f)(x)^4 = \sum_{j=1}^\infty |E_j(f)(x) - E_{j-1}(f)(x)|^4 \\
+ 2\sum_{j=1}^\infty |E_j(f)(x) - E_{j-1}(f)(x)|^2 R_{j+1}(f)(x)^2.
\]

If \( I \) is a dyadic interval of length \( 2^{-j} \), then
\[
\int_I (|E_j(f)(x)|^2 + R_{j+1}(f)(x)^2) \, dx = \int_I |f(x)|^2 \, dx.
\]
This is analogous to Lemma 8.71, using orthogonality properties on \( I \) analogous to those in Lemma 8.13 on \([0, 1]\). In particular,
\[
\int_I R_{j+1}(f)(x)^2 \, dx \leq \int_I |f(x)|^2 \, dx \leq \int_I M(|f|^2)(x) \, dx,
\]
where \( M(|f|^2) \) is the dyadic maximal function associated to \(|f|^2\). It follows that
\[
\int_{[0, 1]} |E_j(f)(x) - E_{j-1}(f)(x)|^2 R_{j+1}(f)(x)^2 \, dx \\
\leq \int_{[0, 1]} |E_j(f)(x) - E_{j-1}(f)(x)|^2 M(|f|^2)(x) \, dx
\]
for each \( j \geq 1 \), by expressing the integral over \([0, 1]\) as a sum of integrals over dyadic intervals of length \( 2^{-j} \), and using the fact that \(|E_j(f)(x) - E_{j-1}(f)(x)|^2\) is constant on dyadic intervals of length \( 2^{-j} \).

Using estimates like these, one can check that
\[
\int_{[0, 1]} S(f)(x)^4 \, dx \leq C \int_{[0, 1]} S(f)(x)^2 M(|f|^2)(x) \, dx
\]
for some constant \( C \geq 0 \) that does not depend on \( f \). This also uses the fact that \( M(|f|^2) \leq M(|f|^2) \) to deal with the diagonal terms. This gives another way to estimate the \( L^4 \) norm of \( S(f) \) in terms of the \( L^4 \) norm of \( f \). More precisely, one can first apply the Cauchy–Schwarz inequality to the right side of (8.198). This implies that the \( L^4 \) norm of \( S(f) \) is bounded by a constant times the product
8.11. RADEMACHER FUNCTIONS

of the square root of the $L^4$ norm of $S(f)$ and the fourth root of the $L^2$ norm of $M(|f|^2)$. Dividing both sides by the square root of the $L^4$ norm of $S(f)$ and then squaring, one gets that the $L^4$ norm of $S(f)$ is bounded by a constant times the square root of the $L^2$ norm of $M(|f|^2)$. The latter is bounded by a constant multiple of the $L^4$ norm of $f$, as desired, because of the $L^2$ estimates for the maximal function applied to $|f|^2$.

8.11 Rademacher functions

For each positive integer $j$, the $j$th Rademacher function $r_j$ is the dyadic step function on $[0, 1)$ which is constant on dyadic intervals of length $2^{-j}$ and whose values alternate between 1 and $-1$. Thus

\[ r_j(t) = 1 \]

when $k 2^{-j} \leq t < (k + 1) 2^{-j}$ and $k$ is an even integer, and

\[ r_j(t) = -1 \]

when $k$ is odd. In particular,

\[ |r_j(t)| = 1 \]

for each $j$ and $t$. If $I$ is a dyadic subinterval of $[0, 1)$ of length $|I| > 2^{-j}$, then

\[ \int_I r_j(t) \, dt = 0, \]

because the values of $r_j$ alternate between 1 and $-1$ on $I$. If $j$ and $l$ are distinct positive integers, then

\[ \int_0^1 r_j(t) r_l(t) \, dt = 0. \]

Thus the Rademacher functions are orthogonal with respect to the usual integral inner product

\[ \langle f_1, f_2 \rangle = \int_0^1 f_1(t) f_2(t) \, dt \]

for real-valued functions on the unit interval. Since

\[ \int_0^1 r_j(t)^2 \, dt = 1 \]

for each $j$, the Rademacher functions are orthonormal with respect to this inner product.

If $f$ is a real-valued dyadic step function on $[0, 1)$ and $p$ is a positive real number, then we put

\[ \|f\|_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p}. \]
This is a norm when $p \geq 1$, and a quasinorm when $0 < p < 1$. Jensen’s inequality implies that
\begin{equation}
\|f\|_p \leq \|f\|_q
\end{equation}
when $p \leq q$. If
\begin{equation}
f = \sum_{j=1}^{n} a_j r_j
\end{equation}
is a linear combination of Rademacher functions, then
\begin{equation}
\|f\|_2 = \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2},
\end{equation}
by orthonormality. Hence
\begin{equation}
\|f\|_p \leq \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} \leq \|f\|_q
\end{equation}
when $p \leq 2 \leq q$. It turns out that for each $q > 2$ there is a $B(q) > 0$ such that
\begin{equation}
\|f\|_q \leq B(q) \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2},
\end{equation}
and for each $p < 2$ there is a $B(p) > 0$ such that
\begin{equation}
\left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} \leq B(p) \|f\|_p.
\end{equation}
These constants do not depend on $n$ or the coefficients $a_1, \ldots, a_n$. By contrast,
\begin{equation}
\max_{0 \leq t < 1} |f(t)| = \sum_{j=1}^{n} |a_j|,
\end{equation}
and hence the analogous statement for $q = +\infty$ does not work.

If $q$ is an even integer, then we can expand $|f(t)|^q$ as a $q$-fold sum of products of Rademacher functions. The integral of a product of Rademacher functions is 1 when the corresponding indices are equal in pairs, and is 0 otherwise. This permits one to estimate $\|f\|_q^q$ by a multiple of
\begin{equation}
\left( \sum_{j=1}^{n} a_j^2 \right)^{q/2},
\end{equation}
as desired. Actually, it suffices to know that each index of a Rademacher function in a product is equal to at least one other index when the integral of the product is different from 0. If $q > 2$ is not an even integer, then we can apply the
8.12. WALSH FUNCTIONS

previous assertion to the smallest even integer \( Q > q \) and use the monotonicity of \( \| f \|_q \). One could use Hölder’s inequality instead, in the form

\[
\| f \|_q \leq \| f \|_Q^a \| f \|_2^{1-a}
\]

where

\[
\frac{1}{q} = \frac{a}{Q} + \frac{1-a}{2},
\]

to get a better constant. For \( p < 2 \), we can use Hölder’s inequality in the form

\[
\| f \|_2 \leq \| f \|_p^b \| f \|_4^{1-b}
\]

with

\[
\frac{1}{2} = \frac{b}{p} + \frac{1-b}{4}
\]

and replace \( \| f \|_4 \) by a multiple of \( \| f \|_2 \) to estimate \( \| f \|_2 \) in terms of \( \| f \|_p \).

One can also see this as a consequence of the analysis of the previous sections. If \( E_l \) is as defined in (8.7), then \( E_l(r_j) = 0 \) when \( j > l \) and \( E_l(r_j) = r_j \) when \( j \leq l \). Hence

\[
E_l(r_j) - E_{l-1}(r_j) = 0
\]

when \( j \neq l \), and

\[
E_j(r_j) - E_{j-1}(r_j) = r_j.
\]

Therefore

\[
E_j(f) - E_{j-1}(f) = a_j r_j
\]

for \( j = 1, \ldots, n \), and

\[
S(f) = \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2}.
\]

In particular, the square function \( S(f) \) is constant on \([0, 1)\).

8.12 Walsh functions

If \( A = \{j_1, \ldots, j_n\} \) is a finite set of positive integers, then the Walsh function \( w_A \) is the dyadic step function on the unit interval which is the product of the Rademacher functions with these indices, i.e.,

\[
w_A(t) = r_{j_1}(t) \cdots r_{j_n}(t).
\]

This should be interpreted as the constant function equal to 1 on \([0, 1)\) when \( A = \emptyset \). Thus

\[
|w_A(t)| = 1
\]

for each \( A \) and \( t \), and

\[
\int_0^1 w_A(t)^2 \, dt = 1.
\]
The Walsh functions are orthonormal with respect to the usual integral inner product, because the integral of a product of Rademacher functions on $[0, 1)$ is nonzero if and only if the indices of the Rademacher functions are equal in pairs, as in the previous section. One can check that the Walsh functions form an orthonormal basis for the space of all dyadic step functions on $[0, 1)$. More precisely, the Walsh functions associated to subsets $A$ of $\{1, \ldots, n\}$ form an orthonormal basis for the dyadic step functions that are constant on dyadic intervals of length $2^{-n}$. Remember that there are $2^n$ subsets of $\{1, \ldots, n\}$, which is the same as the number of dyadic subintervals of $[0, 1)$ of length $2^{-n}$.

If $A \subseteq \{1, \ldots, n\}$, then $w_A$ is constant on dyadic intervals of length $2^{-n}$, and

$$E_l(w_A) = w_A$$

when $l \geq n$. If also $n \in A$, then $E_{n-1}(w_A) = 0$, and therefore

$$E_l(w_A) = 0$$

for each $l < n$. It follows that

$$E_l(w_A) - E_{l-1}(w_A) = 0$$

when $l \neq n$, and

$$E_n(w_A) - E_{n-1}(w_A) = w_A.$$  

The *Walsh group* can be defined as the set of sequences of $\pm 1$'s, with respect to coordinatewise multiplication. Thus the Walsh group is the Cartesian product of a sequence of copies of the group with two elements, and is a compact Hausdorff topological space with respect to the product topology. The group structure is compatible with the topology, so that the Walsh group is a commutative topological group. Walsh functions correspond exactly to Fourier analysis on this group.
Appendix A

Metric spaces

A metric space is a set $M$ together with a nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y) = 0$ if and only if $x = y$,

$$d(x, y) = d(y, x)$$  \hspace{1cm} (A.1)

for every $x, y \in M$, and

$$d(x, z) \leq d(x, y) + d(y, z)$$  \hspace{1cm} (A.2)

for every $x, y, z \in M$. The function $d(x, y)$ is known as the metric on $M$, and represents the distance between $x$ and $y$ in the metric space. If $V$ is a real or complex vector space equipped with a norm $\|v\|$, then it is easy to see that

$$d(v, w) = \|v - w\|$$  \hspace{1cm} (A.3)

is a metric on $V$. In particular, the standard Euclidean metric on $\mathbb{R}^n$ is the metric that corresponds to the standard Euclidean norm on $\mathbb{R}^n$ in this way. If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the usual way, then the metric on $\mathbb{C}^n$ determined by the standard Euclidean norm corresponds exactly to the standard Euclidean metric on $\mathbb{R}^n$.

Let $(M, d(x, y))$ be a metric space. A sequence $\{x_j\}_{j=1}^\infty$ of elements of $M$ is said to converge to $x \in M$ if for each $\epsilon > 0$ there is an $L \geq 1$ such that

$$d(x_j, x) < \epsilon$$  \hspace{1cm} (A.4)

for each $j \geq L$. One can check that the limit $x$ of a convergent sequence $\{x_j\}_{j=1}^\infty$ is unique when it exists, in which case we put

$$\lim_{j \to \infty} x_j = x.$$  \hspace{1cm} (A.5)

A sequence of elements $\{x_j\}_{j=1}^\infty$ of $M$ is said to be a Cauchy sequence if for each $\epsilon > 0$ there is an $L \geq 1$ such that

$$d(x_j, x_l) < \epsilon$$  \hspace{1cm} (A.6)
for every \( j, l \geq L \). One can also check that every convergent sequence in \( M \) is a Cauchy sequence.

Conversely, if every Cauchy sequence in \( M \) converges to an element of \( M \), then \( M \) is said to be complete. As in Section 1.1, \( \mathbb{R} \) and \( \mathbb{C} \) are complete as metric spaces with respect to their standard metrics. This implies that \( \mathbb{R}^n \) and \( \mathbb{C}^n \) are complete with respect to their standard metrics for each positive integer \( n \), because a sequence of elements of \( \mathbb{R}^n \) or \( \mathbb{C}^n \) is a Cauchy sequence or a convergent sequence if and only if their corresponding \( n \) sequences of coordinates have the same property.

Let \( E \) be a subset of a metric space \( M \). A point \( p \in M \) is said to be in the closure \( \overline{E} \) of \( E \) in \( M \) if for each \( \epsilon > 0 \) there is a point \( q \in E \) such that
\[
d(p, q) < \epsilon.
\]
If \( p \in E \), then one can simply take \( q = p \), so that every element of \( E \) is automatically an element of \( \overline{E} \). If
\[
\overline{E} = E,
\]
then we say that \( E \) is a closed set in \( M \).

One can check that the closure of any set in \( M \) is closed. If \( \{x_j\}_{j=1}^{\infty} \) is a sequence of elements of a subset \( E \) of \( M \) that converges to an element \( x \) of \( M \), then it is easy to see that \( x \in \overline{E} \). Conversely, every element of \( \overline{E} \) is the limit of a sequence of elements of \( E \) that converges in \( M \).

If \( x \) is an element of a metric space \( M \) and \( r \geq 0 \), then the closed ball in \( M \) with center \( x \) and radius \( r \) is defined by
\[
B(x, r) = \{y \in M : d(x, y) \leq r\}.
\]
One can check that this is always a closed set in \( M \), using the triangle inequality.

A subset \( E \) of a metric space \( M \) is said to be dense in \( M \) if \( \overline{E} = M \). This is equivalent to saying that every element of \( M \) is the limit of a convergent sequence of elements of \( E \). The set \( \mathbb{Q} \) of rational numbers is dense in the real line with the standard metric, for instance.

A subset \( E \) of a metric space \( M \) is said to be bounded if it is contained in a ball, which is to say that
\[
E \subseteq B(p, r)
\]
for some \( p \in M \) and \( r \geq 0 \). In this case, we also have that
\[
E \subseteq B(q, r + d(p, q))
\]
for every \( q \in M \), by the triangle inequality. If \( E \subseteq M \) is bounded and nonempty, then the diameter is defined by
\[
diam E = \sup\{d(x, y) : x, y \in E\}.
\]
One can check that the closure \( \overline{E} \) of a bounded set \( E \subseteq M \) is bounded as well, and that the diameter of \( \overline{E} \) is the same as the diameter of \( E \).
If \((M, d(x, y))\) is a metric space and \(X\) is a subset of \(M\), then the restriction of the metric \(d(x, y)\) on \(M\) to \(x, y \in X\) satisfies the requirements of a metric on \(X\), so that \(X\) becomes a metric space too. If \(M\) is complete as a metric space and \(X\) is a closed subset of \(M\), then \(X\) is also complete as a metric space. To see this, observe that any Cauchy sequence \(\{x_j\}_{j=1}^\infty\) in \(X\) is a Cauchy sequence in \(M\) as well. If \(M\) is complete, then \(\{x_j\}_{j=1}^\infty\) converges to an element \(x\) of \(M\), and \(x \in X\) when \(X\) is a closed set in \(M\).

Suppose now that \((M, d(x, y))\) and \((N, \rho(u, v))\) are both metric spaces. Let \(f\) be a function on \(M\) with values in \(N\), which is the same as a mapping from \(M\) into \(N\), and which may be expressed symbolically by \(f : M \to N\). As usual, \(f\) is said to be continuous at a point \(x \in M\) if for every \(\epsilon > 0\) there is a \(\delta > 0\) such that

\[(A.13) \quad \rho(f(x), f(y)) < \epsilon\]

for every \(y \in M\) such that \(d(x, y) < \delta\). If \(f\) is continuous at \(x\), and if \(\{x_j\}_{j=1}^\infty\) is a sequence of elements of \(M\) that converges to \(x\), then it is easy to see that \(\{f(x_j)\}_{j=1}^\infty\) converges to \(f(x)\) in \(N\). Conversely, if \(f\) is not continuous at \(x\), then one can check that there is an \(\epsilon > 0\) and a sequence \(\{x_j\}_{j=1}^\infty\) of elements of \(M\) that converges to \(x\) such that

\[(A.14) \quad \rho(f(x), f(x_j)) \geq \epsilon\]

for each \(j\), so that \(\{f(x_j)\}_{j=1}^\infty\) does not converge to \(f(x)\) in \(N\).

A mapping \(f : M \to N\) is is said to be continuous if it is continuous at every point in \(M\). Suppose that \((M_1, d_1)\), \((M_2, d_2)\), and \((M_3, d_3)\) are metric spaces, and that \(f_1 : M_1 \to M_2\) and \(f_2 : M_2 \to M_3\) are continuous mappings between them. The composition \(f_2 \circ f_1\) is the mapping from \(M_1\) into \(M_3\) defined by

\[(A.15) \quad (f_2 \circ f_1)(x) = f_2(f_1(x))\]

for every \(x \in M_1\). One can check that \(f_2 \circ f_1\) is also a continuous mapping from \(M_1\) into \(M_3\) under these conditions, using either the definition of continuity in terms of \(\epsilon\)'s and \(\delta\)'s, or the characterization of continuity in terms of convergent sequences.

If \(f\) and \(g\) be continuous real or complex-valued functions on a metric space \(M\), then their sum \(f + g\) and product \(fg\) are also continuous functions on \(M\). More precisely, when we say that a real or complex-valued functions on \(M\) is continuous, we mean that it is continuous as a mapping into \(\mathbb{R}\) or \(\mathbb{C}\) with its standard metric. To show that \(f + g\) and \(fg\) are continuous, one can use the characterization of continuous functions in terms of convergent sequences to reduce to the analogous statements for sums and products of convergent sequences of real or complex numbers. Of course, one can also prove this more directly, using very similar arguments.

Let \(M\) be a set, and let \((N, \rho(u, v))\) be a metric space. A sequence \(\{f_j\}_{j=1}^\infty\) of mappings from \(M\) into \(N\) is said to converge pointwise to a mapping \(f : M \to N\) if \(\{f_j(x)\}_{j=1}^\infty\) converges as a sequence of elements of \(N\) to \(f(x)\) for every \(x \in M\).
Similarly, \( \{f_j\}_{j=1}^\infty \) is said to converge to \( f \) uniformly on \( M \) if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that

\[
A.16 \quad \rho(f_j(x), f(x)) < \epsilon
\]

for every \( j \geq L \) and \( x \in M \). The difference between uniform and pointwise convergence is that \( L \) depends only on \( \epsilon \) and not on \( x \) in the definition of uniform convergence, while \( L \) is allowed to depend on both \( \epsilon \) and \( x \) in the analogous formulation of pointwise convergence. If \( (M,d(x,y)) \) is also a metric space, and \( \{f_j\}_{j=1}^\infty \) is a sequence of continuous mappings from \( M \) into \( N \) that converges uniformly to a mapping \( f : M \to N \), then a well-known theorem states that \( f \) is also continuous.

A function \( f \) on a set \( M \) with values in a metric space \( N \) is said to be bounded if

\[
A.17 \quad f(M) = \{f(x) : x \in M\}
\]

is a bounded subset of \( N \). Note that the sum and product of bounded real or complex-valued functions on \( M \) are also bounded functions on \( M \). If a sequence \( \{f_j\}_{j=1}^\infty \) of bounded mappings from \( M \) into a metric space \( N \) converges uniformly to a mapping \( f : M \to N \), then it is easy to see that \( f \) is also bounded.

Let \( (M,d(x,y)) \), \( (N,\rho(u,v)) \) be metric spaces again, and let \( C_b(M,N) \) be the collection of bounded continuous mappings from \( M \) into \( N \). One can check that

\[
A.18 \quad \theta(f,g) = \sup \{\rho(f(x),g(x)) : x \in M\}
\]

defines a metric on \( C_b(M,N) \), known as the supremum metric. Note that a sequence \( \{f_j\}_{j=1}^\infty \) of bounded continuous mappings from \( M \) into \( N \) converges to a bounded continuous mapping \( f : M \to N \) with respect to the supremum metric if and only if \( \{f_j\}_{j=1}^\infty \) converges to \( f \) uniformly on \( M \).

If \( N \) is complete as a metric space, then \( C_b(M,N) \) is also complete with respect to the supremum metric. More precisely, if \( \{f_j\}_{j=1}^\infty \) is a sequence of bounded continuous mappings from \( M \) into \( N \) that is a Cauchy sequence with respect to \( A.18 \), then it is easy to see that \( \{f_j(x)\}_{j=1}^\infty \) is a Cauchy sequence in \( N \) for each \( x \in M \). If \( N \) is complete, then it follows that \( \{f_j(x)\}_{j=1}^\infty \) converges to an element \( f(x) \) of \( N \) for every \( x \in M \). Using the Cauchy condition with respect to \( A.18 \), one can check that \( \{f_j\}_{j=1}^\infty \) converges to \( f \) uniformly on \( M \), and hence that \( f \) is bounded and continuous on \( M \).

Let \( (M,d(x,y)) \) and \( (N,\rho(u,v)) \) be metric spaces. A mapping \( f : M \to N \) is said to be uniformly continuous if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
A.19 \quad \rho(f(x), f(y)) < \epsilon \quad \text{for every } x, y \in M \text{ with } d(x, y) < \delta.
\]

As before, the composition of two uniformly continuous mappings is uniformly continuous, as is the limit of a uniformly convergent sequence of uniformly continuous mappings. Similarly, the sum of two uniformly continuous real or complex-valued functions is also uniformly continuous, as is the product of a uniformly continuous function and a constant. The product of two bounded uniformly continuous real or complex-valued functions is uniformly continuous.
as well, but this does not always work without the additional hypothesis of boundedness.

Suppose that $E$ is a dense subset of $M$, and that $f$ is a uniformly continuous mapping from $E$ into $N$. If $N$ is complete, then there is a unique extension of $f$ to a uniformly continuous mapping from $M$ into $N$. To see this, let $x$ be any element of $M$, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of $E$ that converges to $x$, which exists because $E$ is dense in $M$. In particular, $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence, and one can use the uniform continuity of $f : E \to N$ to show that $\{f(x_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in $N$. If $N$ is complete, then it follows that $\{f(x_j)\}_{j=1}^{\infty}$ converges in $N$. One can also check that the limit of this sequence does not depend on the specific choice of the sequence $\{x_j\}_{j=1}^{\infty}$ of elements of $E$ converging to $x$, so that this defines a mapping from $M$ into $N$. This new mapping clearly agrees with the original one on $E$, and one can use the uniform continuity of the original mapping on $E$ to show that the new mapping is uniformly continuous on all of $M$. The uniqueness of the extension follows from the fact that two continuous mappings from $M$ into $N$ are the same when they agree on a dense set.

A subset $A$ of $M$ is said to be totally bounded if for each $\epsilon > 0$, $A$ can be covered by finitely many balls of radius $\epsilon$ in $M$. It is easy to see that totally bounded sets are bounded, and that bounded subsets of $\mathbb{R}^n$ with the standard metric are totally bounded. If $f : M \to N$ is uniformly continuous and $A \subseteq M$ is totally bounded, then $f(A)$ is totally bounded in $N$. 
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