Relative positions of points on the real line and balanced parentheses

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Abstract. Consider a finite set of positive real numbers $S$. For any real number $\lambda > 1$, a Dyck word denoted $\langle \langle S \rangle \rangle_\lambda \in \{a, b\}^*$, was defined in [1] in order to compute Hooley’s $\Delta$-function and its generalization. The aim of this paper is to prove that, given a real number $\lambda > 1$, any Dyck word can be expressed as $\langle \langle S \rangle \rangle_\lambda$ for some finite set $S$ of positive real numbers.

Keywords: Dyck word, positive real number, order relation.

1 Introduction

Consider the finite 2-letter alphabet $\Sigma = \{a, b\}$. Given a finite set $S$ of positive real numbers and a real parameter $\lambda > 1$, denote $\mu_0, \mu_1, ..., \mu_{k-1}$ the elements of $S \triangle \lambda S$ written in increasing order, i.e.

$$S \triangle \lambda S = \{\mu_0 < \mu_1 < ... < \mu_{k-1}\},$$

where $\triangle$ is the symmetric difference and $\alpha X := \{\alpha x : x \in X\}$ for $X \subseteq \mathbb{R}$ and $\alpha \in \mathbb{R}$. In [1] we defined the word

$$\langle \langle S \rangle \rangle_\lambda = w_0 w_1 ... w_{k-1} \in \Sigma^*,$$

where each letter $w_i \in \Sigma$, with $0 \leq i \leq k - 1$, is given by

$$w_i := \begin{cases} a & \text{if } \mu_i \in S, \\ b & \text{if } \mu_i \in \lambda S. \end{cases}$$

The original motivation to study $\langle \langle S \rangle \rangle_\lambda$ was that, for $\lambda = 2$ and $S$ being the set of divisors of a given integer $n \geq 1$, this word determines the non-zero coefficients of $(1 - q) P_n(q)$, where $P_n(q)$ are the polynomials introduced by Kassel and Reutenauer in order to enumerate the ideals of a given codimension in the group algebra $\mathbb{F}_q [\mathbb{Z} \oplus \mathbb{Z}]$ (see [2], [3] and [?]). If we take $\lambda = \exp(1)$, the word $\langle \langle S \rangle \rangle_\lambda$ can used to compute the Hooley’s $\Delta$-function. For details and generalizations of these results, see [1].

Consider the monoid $\mathcal{B}$, called the bicyclic semigroup, given by the presentation

$$\mathcal{B} := \langle a, b \mid a b = \varepsilon \rangle,$$
and the canonical projection $\pi : \Sigma^* \rightarrow B$. Recall that the Dyck language, denoted $\mathcal{D}$, is the kernel of $\pi$, i.e. $\mathcal{D} := \text{ker} \pi$. The elements of $\mathcal{D}$ are called Dyck words. If we identify the letters $a$ and $b$ with the parentheses (“” and “”) respectively, then Dyck words are nothing but balanced parentheses and conversely.

Given a real number $\lambda > 1$, define the language $L_\lambda := \{ \langle \langle S \rangle \rangle_\lambda : S \text{ is a finite subset of } [0, +\infty[ \} \subseteq \Sigma^*$.

The inclusion $L_\lambda \subseteq \mathcal{D}$ follows by Proposition 12 (i) in [1]. The aim of this paper is to prove following theorem.

**Theorem 1.** The equality $L_\lambda = \mathcal{D}$ holds for each real number $\lambda > 1$.

## 2 Irreducible Dyck words

The language of reducible Dyck words is the submonoid

$$\tilde{\mathcal{D}} := \{ \varepsilon \} \cup \{ u v : u, v \in \mathcal{D} \setminus \{ \varepsilon \} \}$$

of $\mathcal{D}$. The elements of the complement of $\tilde{\mathcal{D}}$ in $\mathcal{D}$, denoted $\mathcal{P} := \mathcal{D} \setminus \tilde{\mathcal{D}}$

are called irreducible Dyck words.

**Lemma 2.** Consider an integer $n > 1$. Let $w_0, w_1, \ldots, w_{2^n - 1} \in \Sigma$ be the sequence of letters of

$$w := w_0 w_1 \ldots w_{2^n - 1} \in \Sigma^*.$$

Let $0 \leq i \leq 2^n - 1$ be the largest integer satisfying $w_i = a$. Define the word

$$w' := w_0 w_1 \ldots w_{i-2} w_{i-1} \widehat{w_i} w_{i+1} w_{i+2} \ldots w_{2^n-3} w_{2^n-2} \widehat{w_{2^n-1}},$$

obtained from $w$ deleting the letters $w_i$ and $w_{2^n-1}$, where the notation $\widehat{w_i}$ means that we exclude the letter $w_i$ from the word. If $w \in \mathcal{P}$ then $w' \in \mathcal{P}$.

**Proof.** Suppose that $w' \not\in \mathcal{P}$. We have that $w' \neq \varepsilon$, because $|w| \geq 4$ and $|w'| = |w| - 2$. So, $w' = u v$ for some $u, v \in \mathcal{D} \setminus \{ \varepsilon \}$. If $u$ is not a prefix of $w$, then $v$ only contains the letter $b$, because $w_j = b$ for all $j$ satisfying $i < j \leq 2^n - 1$. The letters $a$ and $b$ appear the same number of times in $v$. So, $u$ is a proper prefix of $w$, i.e. $w \not\in \mathcal{P}$. Therefore $w' \in \mathcal{P}$ provided that $w \in \mathcal{P}$. \qed

**Lemma 3.** For any real number $\lambda > 1$, we have that $\mathcal{P} \subseteq L_\lambda$. 


Proof. Given \( w \in \mathcal{P} \). It easily follows from the definition of \( \mathcal{P} \) that \( |w| = 2n \) for some integer \( n \geq 1 \). We proceed by induction on \( n \geq 1 \) in order to prove that \( w = \langle S \rangle_\lambda \in L_\lambda \) for some finite set \( S \) of positive real numbers satisfying that \( S \) and \( \lambda S \) are disjoint.

For \( n = 1 \), the only possibility is that \( w = ab \). So, \( S = \{1\} \) satisfies \( w = \langle S \rangle_\lambda \in L_\lambda \). Furthermore, \( S \) and \( \lambda S \) are disjoint, because \( \lambda > 1 \).

Given \( n > 1 \), suppose that any \( w \in \mathcal{P} \) satisfies \( w \in L_\lambda \) provided that \( |w| = 2(n-1) \). Consider a word \( w \in \mathcal{P} \) having length \( |w| = 2n \).

Let \( w_0, w_1, \ldots, w_{2n-1} \in \Sigma \) be the sequence of letters of

\[
w = w_0 w_1 \ldots w_{2n-1}.
\]

Define the word

\[
w' := w_0 w_1 \ldots w_{i-2} w_{i-1} \hat{w}_i w_{i+1} w_{i+2} \ldots w_{2n-3} w_{2n-2} \hat{w}_{2n-1},
\]

as in Lemma 2 where \( i \) is the largest integer on the interval \( 0 \leq i < 2n \), satisfying \( w_i = a \). Notice that \( |w'| = 2(n-1) \). Furthermore, \( w' \) is an irreducible Dyck word by Lemma 2.

Using the induction hypothesis, \( w' = \langle S' \rangle_\lambda \) for some finite set of real numbers \( S' \) such that \( S' \cap \lambda S' = \emptyset \). Let \( \mu'_0, \mu'_1, \mu'_2, \ldots, \mu'_{2(n-1)-1} \) be the elements of \( S' \cap \lambda S' \) written in increasing order, i.e.

\[
S' \triangle \lambda S' = \left\{ \mu'_0 < \mu'_1 < \mu'_2 < \ldots < \mu'_{2(n-1)-1} \right\}.
\]

The inequality \( 0 \leq i \leq 2(n-1) - 1 \) follows from the hypothesis that \( w \) is an irreducible Dyck word, because \( ab \) cannot be a proper suffix of an irreducible Dyck word having length at least \( 4 \). So, the geometric mean between \( \mu'_{i-1} \) and \( \mu'_i \), denoted \( m := (\mu'_{i-1} \mu'_i)^{1/2} \), is well-defined. It follows that the set \( S := S' \cup \{m\} \) satisfies \( S \cap \lambda S = \emptyset \). Furthermore, \( S \triangle \lambda S \) can be expressed as

\[
\left\{ \mu'_0 < \mu'_1 < \ldots < \mu'_{i-2} < \mu'_{i-1} < m < \mu'_i < \mu'_{i+1} < \ldots < \mu'_{2n-2} < \lambda m \right\}.
\]

Combining \( w' = \langle S' \rangle_\lambda \) with the facts \( m \in S \) and \( \lambda m \in \lambda S \), the equality \( w = \langle S \rangle_\lambda \in L_\lambda \) follows from the expression above.

Therefore, \( \mathcal{P} \subseteq L_\lambda \). \( \square \)

3 Proof of the main result

Lemma 4. Let \( S \) be a finite set of positive real numbers. Consider a real number \( \lambda > 1 \). For any real number \( \alpha > 0 \), we have that \( \langle S \rangle_\lambda = \langle \alpha S \rangle_\lambda \).

Proof. The identity

\[
(\alpha S) \triangle (\lambda \alpha S) = \alpha (S \triangle \lambda S)
\]

holds for all \( \alpha \in \mathbb{R} \). The function \( \left[0, +\infty\right) \rightarrow [0, +\infty] \), given by \( x \mapsto \alpha x \), is strictly increasing provided that \( \alpha > 0 \). Hence, \( \langle S \rangle_\lambda = \langle \alpha S \rangle_\lambda \) for all \( \alpha \in [0, +\infty] \). \( \square \)
Lemma 5. Let $S_1$ and $S_2$ be two nonempty finite sets of positive real numbers. Consider a real number $\lambda > 1$. If $\frac{\min S_2}{\max S_1} > \lambda$ then $\langle (S_1 \cup S_2) \rangle_\lambda = \langle (S_1) \rangle_\lambda \langle (S_2) \rangle_\lambda$.

Proof. On the one hand, the set-theoretical identity

$$(S_1 \cup S_2) \triangle [\lambda (S_1 \cup S_2)] = [S_1 \triangle (\lambda S_1)] \cup [S_2 \triangle (\lambda S_2)]$$

holds, provided that $S_1 \cap S_2 = \emptyset$. On the other hand, $S_1 \cap S_2 = \emptyset$ provided that $\frac{\min S_2}{\max S_1} > \lambda$. Hence, $\langle (S_1 \cup S_2) \rangle_\lambda = \langle (S_1) \rangle_\lambda \langle (S_2) \rangle_\lambda$ if $\frac{\min S_2}{\max S_1} > \lambda$. □

Lemma 6. Given a real number $\lambda > 1$, for all $u, v \in \Sigma^*$, if $u \in L_\lambda$ and $v \in L_\lambda$ then $u v \in L_\lambda$.

Proof. Suppose that $u = \langle (S_1) \rangle_\lambda$ and $v = \langle (S_2) \rangle_\lambda$ for two finite sets of positive real numbers $S_1$ and $S_2$. If either $S_1 = \emptyset$ or $S_2 = \emptyset$ then $u v \in L_\lambda$ trivially follows.

Suppose that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. By Lemma 4, $\langle (S_2) \rangle_\lambda = \langle (\alpha S_2) \rangle_\lambda$, for any real number $\alpha > 0$. Take an arbitrary $\alpha > 0$ large enough in order to guarantee that $\frac{\min \alpha S_2}{\max S_1} > \lambda$.

It follows by Lemma 5 that $u v = \langle (S_1 \cup \alpha S_2) \rangle_\lambda$. Therefore, $u v \in L_\lambda$. □

We now proceed to prove our main result.

Proof (of Theorem). As it was mentioned in the introduction, the inclusion $L_\lambda \subseteq D$ was already proved in [1].

Notice that $L_\lambda$ contains the empty word, because $\langle \emptyset \rangle_\lambda = \varepsilon$. By Lemma 3, $\mathcal{P} \subseteq L_\lambda$. By Lemma 6, $\mathcal{P}^* \subseteq L_\lambda$. It is well-known that $D$ is freely generated by $\mathcal{P}$, i.e. every word in $D$ may be formed in a unique way by concatenating a sequence of words from $\mathcal{P}$. So, using the equality $D = \mathcal{P}^*$, we conclude that $D \subseteq L_\lambda$.

Therefore, $L_\lambda = D$. □

4 Some computations

Let $\lambda = e = \exp(1)$ be the Euler number. Applying the method used in the proof of Lemma 3, we can compute the sets for the corresponding irreducible Dyck words given in the following table.

| $S$ | $\langle (S) \rangle_e$ |
|-----|---------------------|
| 1   | ab                  |
| $1, e^{1/2}$ | aabb               |
| $1, e^{1/4}$ | aaabbb             |
| $1, e^{1/2}, e^{3/4}$ | aaabbb             |
| $1, e^{1/4}, e^{3/4}, e^2$ | aaabbb             |
Applying Lemma 5 to the equalities

\[ u := aaababbb = \langle \langle 1, e^{1/2}, e^{3/4}, e^{5/4} \rangle \rangle e, \]
\[ v := aaababbabb = \langle \langle 1, e^{1/2}, e^{3/4}, e^{5/4}, e^2 \rangle \rangle e, \]

we obtain the set corresponding to the following reducible Dyck word

\[ u v = aaababbbbaababb = \langle \langle 1, e^{1/2}, e^{3/4}, e^{5/4}, e^3, e^{7/2}, e^{15/4}, e^{37/4}, e^5 \rangle \rangle e. \]

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