Homotopy properties and lower-order vertices in higher-spin equations

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Abstract

New homotopy approach to the analysis of nonlinear higher-spin equations is developed. It is shown to directly reproduce the previously obtained local vertices. Simplest cubic (quartic in Lagrangian nomenclature) higher-spin interaction vertices in four-dimensional theory are examined from locality perspective by the new approach and shown to be local. The results are obtained in a background independent fashion.

Keywords: higher-spin theory, locality, interaction vertices

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1. Introduction

The full nonlinear theory of interacting higher-spin (HS) gauge fields is currently known at the level of equations of motion [1]. At the action level only some lower-order results are available (see e.g. [2–17]). At cubic level, however, the coupling constants are not fixed by the Noether procedure pretty much as the Yang–Mills coupling constants are only get fixed by Jacoby identities at the quartic order. Nevertheless in [4] the coupling constants were fixed for the 4d vertices with spins $s_1, s_2, s_3$ obeying the triangle inequality while in [5] these results were extended to any spins. Extension to any space-time dimension was obtained in [16] via holographic computation. While lacking conventional action principle prevents one from treating the HS theory at the full fledged quantum level, the Klebanov–Polyakov HS AdS/CFT conjecture [19] (see also [20–22]) points out to its quantum consistency. The first nontrivial evidence of HS holographic duality at tree level for certain three-point correlation functions was presented by Giombi and Yin in [23, 24]. At quantum level a remarkable cancellation of one loop determinant divergencies was found in [25, 26]. It followed that the spectrum of HS fields itself essentially fine tunes this cancellation.

A web of HS dualities relates in particular the simplest free $O(N)$ model to a highly non-trivial 4d bulk HS theory. For this particular case all boundary correlation functions in a singlet sector are explicitly known [23, 27, 28] and the HS bulk action can be reconstructed order by order using the boundary data. This program has been initiated in [13] and then further put forward in [14] and [16]. The resulting local HS cubic action was shown to be consistent with the HS algebra structure constants [29]. How this method should be applied at higher orders is not quite clear since already at quartic order the HS action is expected to be non-local [14, 15] while a part of holographic reconstruction routine is a systematic discarding of boundary terms which operation requires precise definition of the non-local class of functions. Roughly speaking, the bulk terms can be represented as boundary ones and vice versa whenever one is free to use non-local operators of a kind $\frac{1}{2} \partial$. This makes the problem of locality in HS theory of great importance.

One of the approaches on the way to understand (non-)locality in HS theory is based on the analysis of holographic Mellin amplitudes along the lines of [14, 30, 31]. Despite some difficulties in defining Mellin amplitudes for the free theory correlators an encouraging result was obtained recently in [32] that seemingly singles out a specific form of divergencies in quartic scalar interactions.

In this paper we reconsider the problem within the bulk HS gauge theory with no reference to holography. Specifically, by examining 4d HS equations perturbatively we find restrictions imposed by locality on some simplest HS vertices. Namely, we reconstruct interaction vertices which look schematically

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from HS equations of [1], where $\omega(Y; K|x)$ and $C(Y; K|x)$ are HS fields that apart from their dependence on space-time coordinates $x$ depend also on auxiliary spinor variables $Y$ and Klein operators $K$. In doing so one faces cohomological freedom of representatives in the spinor space which may affect the form of $\Upsilon$-vertices and their (non-)local behavior. Ideally, one would like to have a perturbative expansion procedure allowing to discard this cohomological freedom yet having the resulting $\Upsilon$-vertices within the proper locality class. Constructing such perturbation theory is not trivial. Particularly, one of most natural choices based on the conventional homotopy (see e.g. [33]) leads to non-local obstructions starting from the second order [23] (see also [34]). At this order it was shown in [35, 36] that up to local freedom there exists a unique field redefinition that respects the holomorphic factorization of the equations and results in local cubic vertices on AdS background. That these vertices do agree with HS AdS/CFT expectation was then confirmed in [37–39].

To extend the results of [35, 36] to higher perturbation orders it is necessary to elaborate a systematic perturbative approach based on the homotopy techniques different from the conventional one. In [40] it was suggested that the proper approach is based on the certain shifted homotopy allowing to decrease the level of nonlocality as a consequence of Pfaffian locality theorem (PLT) proven in that reference. In this paper we further elaborate properties of shifted homotopy operators applying them to the computation of perturbative corrections and, in particular, reproducing this way the previously obtained results of [35].

The shifted homotopy technique is modified in two respects. First, the shifted homotopy operators involve the shifts of arguments of the dynamical HS fields. The number of such free parameters grows with the order of perturbative expansion. The freedom in the field variable choice due to these parameters is very limited confined by a finite amount of free such parameters at a given perturbation order. Compared to the generic field redefinitions that have functional ambiguity the freedom in homotopy parameters represents a very specific finite-dimensional subset. Second, we no longer use the AdS space as vacuum solution. The new technique allows us to make the computation for general HS potentials $\omega$ while reconstructing the $C$-dependence order by order. From this point of view vertex $\Upsilon(\omega, \omega, C)$ is the simplest cubic vertex that shows up. We confine ourselves to $\Upsilon(\omega, \omega, C)$ and $\Upsilon(\omega, C, C)$ and show how the free homotopy parameters allow us to arrive at their local form. Let us stress that the proposed approach is free from any kind of divergencies and regularizations and needs no field redefinitions: the freedom in the choice of field variables is encoded in a few free homotopy parameters.

We would also like to point out that the locality notion considered in our paper might differ from the conventional space-time locality. What we take as (non-)local is attributed to spinor space rather than space-time, i.e. to derivatives in auxiliary spinor rather than space-time variables. The two are related via the unfolding routine that generally assumes non-linear and infinite derivative one-to-one map. Therefore we are dealing with spin locality rather than the space-time one (see also discussion in [40]). At the level considered in our paper both notions are equivalent. Our main findings are the following.

- One of the main results of the paper is the development of the useful properties of shifted homotopies of [40]. Remarkable feature of these homotopies is that they manifest a striking interplay with HS star product relying on the specific form of the latter. No relations of this kind exist in the conventional case being simply outside the box.
• While $\Upsilon(\omega, \omega, C)$ vertex is manifestly local in any perturbative scheme we show that the homotopy parameters can be chosen in such a way that it exhibits spin ultra-local form. Namely, it reveals no dependence on (anti)holomorphic auxiliary spinor variables in the $C$ field at all. This property is a natural generalization of the central on-mass-shell theorem [33, 41] obtained for AdS background. The conventional homotopy belongs to this class.

• Using the result of [40] which says that the conventional homotopy is not respected by PLT in the zero-form sector of HS equations we extract vertex $\Upsilon(\omega, C, C)$ using shifted homotopies. We show that those (and only those) homotopy parameters that are prescribed by the PLT of [40] generate local $\Upsilon(\omega, C, C)$ generalizing that of [35] obtained for AdS background.

• Finally, we show that though the admissible homotopies result in equivalent local HS vertices up to some local field redefinitions there is a one-parameter family of generalized homotopies that reproduces identically equivalent vertices $\Upsilon(\omega, C, C)$ and $\Upsilon(\omega, \omega, C)$.

Higher order vertices will be considered elsewhere. It will be interesting to see if the vertex $\Upsilon(\omega, \omega, C, C)$ shares spin ultra-local properties in the (anti)holomorphic sector or not. The result of [36] showing that (anti)holomorphic sector vanishes on AdS background points out that possibility.

The paper is organized as follows. In section 2 we sketch the locality problem and recall the HS equations as well as their perturbative treatment. In section 3 we construct the generalized homotopies and explore their properties. In section 4 we further develop the perturbation theory based on the shifted homotopies allowing us to compute HS vertices. Then in section 4.2 we briefly review PLT that constrains HS vertices to their local form. In section 4.3 HS vertex $\Upsilon(\omega, C, C)$ is explicitly calculated. In section 5 we discuss homotopy shifts that produce local effects at given order and in section 5.3 it is demonstrated that such shifts may result in local field redefinitions. We conclude in section 6. In appendix it is demonstrated how homotopy parameters that violate PLT bound result in non-local $\Upsilon(\omega, C, C)$ vertex.

2. Higher-spin equations and locality

Let us briefly recall the structure of HS equations. For more detail we refer the reader to [33]. Within the frame-like approach, HS dynamics is governed by one-form $\omega(y, \bar{y}; K|x)$ and zero-form $C(y, \bar{y}; K|x)$ generated by all possible polynomials of $sp(4)$ spinors $Y_A = (y_\alpha, \bar{y}_\dot{\alpha})$. \$\alpha, \dot{\alpha} = 1, 2\$. HS algebra is conveniently generated with the use of star product

\[ [y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}, \quad [\bar{y}_\dot{\alpha}, \bar{y}_\dot{\beta}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [y_\alpha, \bar{y}_\dot{\beta}] = 0 \]

\hspace{1cm} (2.1)

defined as

\[ f(y) * g(y) = f(y)e^{i\epsilon_{\alpha\beta}\bar{y}_\dot{\alpha}\bar{y}_\dot{\beta}}g(y), \]

\hspace{1cm} (2.2)

where $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are two invariant $sp(2)$ forms. Fields $\omega$ and $C$ belong to different representations of HS algebra distinguished by the extra dependence on the Klein operators $K = (k, \bar{k})$. Particularly, a spin $s$-field is encoded by fields $\omega_i$ and $C_s$ where $\omega_i$ spans a finite dimensional module of HS algebra as opposed to $C_s$ being an infinite dimensional one. In other words, a given spin $s$-field is stored in polynomial $\omega$ and unbounded $C$. Components of $C(y, \bar{y}; K|x)$ are designed to contain HS Weyl tensors along with all their on-shell nontrivial space-time derivatives.
The precise field dependence on $K$ will be specified later on and for now a schematic form of dynamical equations is given by (1.1) and (1.2). Their r.h.s.s say that HS interactions are driven by HS algebra (first terms in (1.1) and (1.2)) and its deformation which leads to HS vertices $\Upsilon(\omega, C \ldots C)$ for the zero-form sector and $\Upsilon(\omega, \omega, C \ldots C)$ for one-forms. The form of these vertices can in principle be determined from integrability requirement $d^2 = 0$ up to field redefinition. In practice however this kind of analysis gets increasingly complicated as the order of $C$ grows [41, 42].

Whatever these vertices are they stem from HS symmetry represented by the star product. This is a source of possible non-localities. Indeed, star product (2.2) is a non-local operation in that it mixes any number of derivatives of two functions. Still, vertices containing star product can be local. As an example take the vertex $\omega \ast \omega$. Recall, that for a given spin $s$ the corresponding $\omega_s$ is a (degree $2(s - 1)$ [33]) polynomial in $Y_s$. Product $\omega_s \ast \omega_s$ is also a polynomial and therefore is local. Another cubic vertex entering (1.2) is $[\omega, C]_\ast$ which is local by similar argument. Despite $C_s(y, \bar{y}|x)$ is not a polynomial, product $C_s \ast \omega_s$ contain only finite amount of derivatives as follows from (2.2). Analogous reasoning brings one to a conclusion that the simplest cubic vertex $\Upsilon(\omega, \omega, C)$ is local as well. These exhaust the list of manifestly local vertices. Those containing more than one $C$ are potentially non-local raising a question of admissible class of functions that HS vertices should belong to. At this stage two different potential situations are not excluded. Either all $\Upsilon$-vertices are spin local for a given set of spins entering the vertex or some include infinite number of derivatives. To understand the structure of HS vertices we need to proceed to generating HS equations that eventually lead to (1.1) and (1.2).

2.1. Generating equations

Dynamical HS equations (1.1) and (1.2) are reproduced order by order from the following HS generating system [1]

\begin{align*}
&d_s W + W \ast W = 0, \\
&d_s S + W \ast S + S \ast W = 0, \\
&d_s B + [W, B]_\ast = 0, \\
&S \ast S = i(\theta^A \theta_A + \eta B \ast \gamma + \bar{\eta} B \ast \bar{\gamma}), \\
&[S, B]_\ast = 0.
\end{align*}

(2.3)\hspace{1cm}(2.4)\hspace{1cm}(2.5)\hspace{1cm}(2.6)\hspace{1cm}(2.7)

Here $W(Z, Y; K|x)$ is a one-form that eventually encodes HS one-form $\omega(Y; K|x)$ in (1.1) and (1.2). Apart from its dependence on generating variables $Y_A$, $W$ also depends on new auxiliary variables $Z_A = (z_\alpha, \bar{z}_\dot{\alpha})$. Together with $Y$’s these enhance spinor space along with its star product

\begin{equation}
(f \ast g)(Z, Y) = \frac{1}{(2\pi)^4} \int dUdVf(Z + U, Y + V)g(Z - V, Y + V)e^{U\eta V}. \tag{2.8}
\end{equation}

For $Z$-independent functions (2.8) reduces to (2.2). The following commutation relations can be easily derived using (2.8)

\begin{align*}
[Y_A, Y_B]_\ast &= -[Z_A, Z_B]_\ast = 2i\epsilon_{AB}, \\
[Y_A, Z_B]_\ast &= 0.
\end{align*}

(2.9)
The fields \( W, S \) and \( B \) depend on a pair of Klein operators \( K = (k, \bar{k}) \) that obey
\[
\{k, y_\alpha\} = \{k, \bar{z}_\alpha\} = 0, \quad [k, y_\alpha] = [k, \bar{z}_\alpha] = 0, \quad k^2 = 1. \tag{2.10}
\]
Similarly for \( \bar{k} \). In other words \( (k, \bar{k}) \) anticommutes with (anti)holomorphic variables. Property \( k^2 = \bar{k}^2 = 1 \) implies that field dependence on Klein operators is at most bilinear, e.g.
\[
W(Z, Y; K|x) = \sum_{m,n=0} W^{m,n}(Z, Y|x) k^m \bar{k}^n. \tag{2.11}
\]
Such dependence splits HS spectrum into propagating sector and topological one [33]. The physical sector is singled out by
\[
B(Z, Y; K|x) = W(Z, Y; -K|x). \tag{2.12}
\]

Another master field entering system (2.3)–(2.7) is \( S(Z, Y; K|x) \). This field is purely auxiliary being expressed via \( C \)-field on-shell. As opposed to \( W, S \) is a one-form in additional direction spanned by anti-commuting differentials \( \theta_A = (\theta_\alpha, \bar{\theta}_\dot{\alpha}) \) being a space-time zero-form
\[
S = \theta^\alpha S_\alpha + \bar{\theta}^\dot{\alpha} \bar{S}_\dot{\alpha}, \quad [\theta_A, \theta_B] = \{\theta_A, d_1\} = 0. \tag{2.13}
\]
The commutation rules for Klein operators are in accord with prescription (2.10)
\[
\{\theta_\alpha, k\} = \{\theta_\alpha, \bar{k}\} = 0, \quad [\theta_\alpha, \bar{k}] = [\bar{\theta}_\dot{\alpha}, k] = 0. \tag{2.14}
\]
Lastly, in the propagating sector of the theory, the dependence of \( S \) on Klein operators is similar to (2.11)
\[
S(Z, Y; K|x) = S(Z, Y; -K|x). \tag{2.15}
\]
Equation (2.6) contains the central elements
\[
\gamma = e^{i\zeta_\alpha y^\alpha} k \theta^\alpha \theta_\alpha, \quad \bar{\gamma} = e^{i\bar{\zeta}_{\dot{\alpha}} \bar{y}^\dot{\alpha}} \bar{k} \bar{\theta}^\dot{\alpha} \bar{\theta}_{\dot{\alpha}}, \tag{2.16}
\]
that commute with any element \( f(Z, Y; K; \theta; dx) \). This can be checked by noting using (2.8) that
\[
e^{i\zeta_\alpha y^\alpha} * f(z, \bar{z}, y, \bar{y}) = f(-z, \bar{z}, -y, \bar{y}) * e^{i\zeta_\alpha y^\alpha}, \tag{2.17}
\]
similarly for \( e^{i\bar{\zeta}_{\dot{\alpha}} \bar{y}^\dot{\alpha}} \) and that \( \theta^3 = \bar{\theta}^3 = 0 \) due to two-component indices. Finally, \( \eta \) and \( \bar{\eta} \) are the only free phase parameters \( (\eta \bar{\eta} = 1) \) of equations (2.3)–(2.7). Generally, they break parity of HS interaction unless \( \eta = 1 \) or \( \bar{\eta} = i \) in which cases the theory is called \( A \)- and \( B \)-models correspondingly [22].

2.2. Perturbative expansion: homotopy trick and field redefinitions

To understand the way system (2.3)–(2.7) reproduces (1.1) and (1.2) one starts with perturbative expansion. The proper vacuum for HS theory is given by
\[
B_0 = 0, \tag{2.18}
\]
\[
S_0 = \theta^A Z_A, \tag{2.19}
\]
\[
W_0 = \Omega = \frac{i}{4} (\omega_{\alpha \beta} y^\alpha y^\beta + \bar{\omega}_{\dot{\alpha} \dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} + 2\epsilon_{\alpha \dot{\alpha}} y^\alpha \bar{y}^{\dot{\alpha}}). \tag{2.20}
\]
It is easy to check that (2.3)–(2.7) is fulfilled provided $\omega_{\alpha\beta}, \bar{\omega}_{\dot{\alpha}\dot{\beta}}$ and $e_{\alpha\dot{\alpha}}$ are AdS$_4$ connection one-forms satisfying Cartan structure equations. Nonzero vacuum value for $S$-field creates a pattern for finding $Z$-dependence of fields. For example, at first order one has

$$[S_0, B_1]_* + [S_1, B_0]_* = 0.$$  \hfill (2.21)

Using (2.8) we have

$$[S_0, f(Z, Y; K)]_* = -2i\theta^A \frac{\partial}{\partial Z^A} f = -2i d_Z f$$  \hfill (2.22)

and therefore $B_1$ is $Z$-independent

$$B_1 = C(Y; K).$$  \hfill (2.23)

This is the zero-form $C$ that appears in (1.1) and (1.2). Let us stress that it can be identified with the $d_Z$ cohomological part of $B$.

$S_1$ should be expressed in terms of $C$. To find it one takes (2.6) from which we have

$$-2i d_Z S_1 = i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma}.$$  \hfill (2.24)

This is a typical partial differential equation that determines HS field $Z$-dependence. Terms on the r.h.s. are originated from already given lower order contributions. Freedom in solutions to such equations corresponds to the gauge and field redefinition freedom.

A natural way of solving equation

$$d_Z f(Z; Y; \theta) = J(Z; Y; \theta)$$  \hfill (2.25)

with $d_Z J = 0$ is by the homotopy trick. The simplest choice of the homotopy operator for the exterior differential $d_Z$ is

$$\partial = (Z^A + Q^A) \frac{\partial}{\partial \theta^A}$$  \hfill (2.26)

provided that

$$\frac{\partial Q^\theta}{\partial Z^A} = 0.$$  \hfill (2.27)

Clearly, it obeys $\partial^2 = 0$ (let us note that $Q^A$ can be an operator). One has

$$N := d_Z \partial + \partial d_Z = \theta^A \frac{\partial}{\partial \theta^A} + (Z^A + Q^A) \frac{\partial}{\partial Z^A}.$$  \hfill (2.28)

Introducing the almost inverse operator

$$N^* g(Z; Y; \theta) := \int_0^1 dr \frac{1}{r} g(rZ - (1 - r)Q; Y; t\theta), \quad g(-Q; Y; 0) = 0$$  \hfill (2.29)

and the resolution operator

$$\Delta_Q := \partial N^*, \quad \Delta_Q g(Z; Y; \theta) = (Z^A + Q^A) \frac{\partial}{\partial \theta^A} \int_0^1 dr \frac{1}{r} g((rZ - (1 - r)Q; Y; t\theta)$$  \hfill (2.30)

gives the resolution of identity

$$\{d_Z, \Delta_Q\} = 1 - h_Q,$$  \hfill (2.31)

where $h_Q$ is the following projector to the cohomology space.
\[ h_Q f(Z, \theta) = f(-Q, 0). \]  
\[ (2.32) \]

Resolution of identity allows one to write a particular solution to (2.25) in the form
\[ f_0(J) = \Delta Q J \]
provided that \( h_Q J = 0 \) which condition is always true since \( J(Z; Y; \theta) \) in (2.25) is at least linear in \( \theta^A \). General solution to (2.25) therefore has the form
\[ f(Z; Y; \theta) = f_0(J)(Z; Y; \theta) + h(Y) + d_2 \epsilon(Z; Y; \theta), \]
where \( h(Y) \) is in \( d_2 \)-cohomology while the last term is \( d_2 \)-exact. Clearly, solutions (2.33) with different homotopy parameters \( Q_1 \) and \( Q_2 \) can differ by a solution to the homogeneous equation, i.e.
\[ f_0(J) - f_0(J) = h_{1,2}(Y) + d_2 \epsilon_{1,2}(Z; Y; \theta) \]
with some \( h_{1,2}(Y) \) and \( \epsilon_{1,2}(Z; Y; \theta) \). (Explicit expressions for \( h_{1,2}(Y) \) and \( \epsilon_{1,2}(Z; Y; \theta) \) follow from equation (3.11).) In practical computations, the addition of the cohomological terms \( h_{1,2}(Y) \), that depend on the dynamical fields, implies a (nonlinear) field redefinition. This is because the dynamical fields \( C(Y; K) \) and \( \omega(Y; K) \) belong to the \( d_2 \)-cohomology. Hence, the homotopy choice in (2.33) affects the choice of variables. The \( d_2 \epsilon_{1,2}(Z; Y; \theta) \) term affects the gauge choice.

The choice of \( Q = 0 \) corresponds to the conventional homotopy operator used in [1]. As explained below homotopies with non-zero shifts \( Q \) play important role in higher orders.

Proceeding further with the conventional resolution we solve (2.24) and proceed to (2.4) that determines \( Z \)-dependence of \( W_1 \)
\[ -2id_2 W_1 + D_{12} S_1 = 0, \quad W_1 = \omega(Y; K|x) + \frac{1}{2i} \Delta_0 D_{12} S_1, \]
\[ (2.36) \]
where \( D_{12} S_1 := d_2 S_1 + [\Omega, S_1]_\omega \), with the AdS\(_4\) flat connection (2.20). Here \( \omega(Y; K|x) \) is the HS one-form potential that appears in (1.1) and (1.2). To obtain first-order form of (1.2) one plugs \( B_1 \) into (2.5) to have
\[ d_2 C = -[\Omega, C]_\omega. \]
\[ (2.37) \]
First-order form of (1.1) results from the substitution of (2.36) into (2.3)
\[ d_2 \omega = -\{\Omega, \omega\} + \frac{i}{4} D_{12} \Delta_0 D_{12} \Delta_0 (\eta C \ast \gamma + \eta^\ast C \ast \gamma). \]
\[ (2.38) \]
This way one finds linear contribution to \( \Upsilon(\omega, \omega, C) \) at \( \omega = \Omega \) having the form [33]
\[ \Upsilon(\Omega, \Omega, C) = \frac{i}{4} D_{12} \Delta_0 D_{12} \Delta_0 (\eta C \ast \gamma + \eta^\ast C \ast \gamma) \]
\[ (2.39) \]
\[ = \frac{i}{4} \left( \eta H^\alpha \partial^2_\alpha C(0; y, k; \tilde{k}) \tilde{k} + \eta H^\alpha \partial^2_\alpha C(0; y, k; \tilde{k}) \tilde{k} \right). \]
\[ (2.40) \]
Higher order interactions can be extracted analogously. The resulting vertices in (1.1) and (1.2) are manifestly HS gauge covariant. A systematic analysis of perturbation series based on conventional homotopy was elaborated in [43] (see also [44]).

It turns out however that starting from \( \Upsilon(\Omega, C, C) \) application of conventional homotopy results in infinite derivative tail and is not consistent with locality. It was first observed in [23] that boundary correlation functions resulting from such a vertex diverge. Later it was shown [35] that there is a field redefinition that respects the holomorphic factorization properties of the solution, making this vertex local and consistent with holographic limit [37, 38].
This fact suggests that by considering a wider class of homotopy operators for perturbative series one may hope to construct spin local or minimally non-local HS interactions. In what follows we elaborate some details of this programme.

3. Homotopies and star product

3.1. Shifted homotopies

In this section we analyze properties of shifted resolution operators $\Delta_Q$ (2.30) starting with those insensitive to the range of indices of the variables $Z^A, Y^A, \theta^A$.

3.1.1. General relations. Operators $\Delta_P$ and $\Delta_Q$ anticommute,

$$\Delta_P \Delta_Q = - \Delta_Q \Delta_P .$$

Indeed, by virtue of (2.30),

$$\Delta_P \Delta_Q f(Z; Y; \theta) = \int_0^1 \mathrm{d} \tau \int_0^1 \mathrm{d} t (Z^B + P^B)(\tau (Z^A + P^A) - P^A + Q^A) \times f_{AB} (\tau Z - t(1 - \tau)P - (1 - t)Q; Y; \tau \theta) ,$$

where

$$f_{AB} (Z; Y; \theta) := \frac{\partial^2}{\partial \theta^A \partial \theta^B} f(Z; Y; \theta) .$$

Using that

$$(Z^B + P^B)(\tau (Z^A + P^A) - P^A + Q^A) f_{AB} = (Z^B + P^B)(Z^A + Q^A) f_{AB}$$

since $f_{AB} = -f_{BA}$ and changing the integration variables

$$\tau_3 = \tau t, \quad \tau_2 = 1 - t, \quad \tau_1 = 1 - \tau_2 - \tau_3 .$$

Equation (3.2) can be rewritten in the form

$$\Delta_P \Delta_Q f(Z; Y; \theta) = \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3)(Z^B + P^B)(Z^A + Q^A) f_{AB} (\tau_1 Z - \tau_1 P - \tau_3 Q; Y; \tau_1)$$

making (3.1) obvious.

Formula (3.5) admits a neat generalization for successive resolution operators

$$\Delta_{Q_n} \ldots \Delta_{Q_1} f(Z; Y; \theta) = \int_{[0,1]^{n+1}} d^{n+1} \tau \delta(1 - \sum_{j=1}^{n+1} \tau_j) \prod_{j=1}^{n} (Z^A + Q^A) \times f_{\lambda_0 \ldots \lambda_n} (\tau_{n+1} Z - \sum_{j=1}^{n} \tau_j Q_j; Y; \tau_{n+1} \theta) ,$$

where

$$f_{\lambda_0 \ldots \lambda_n} = \frac{\partial}{\partial \theta^{\lambda_0}} \ldots \frac{\partial}{\partial \theta^{\lambda_n}} f(Z; Y; \theta) .$$

Also note that due to (2.32)
$h_p \triangle Q_n \ldots \triangle Q_1 f(Z; Y; \theta) = \int_{[0,1]^{n+1}} d^{n+1} \tau \delta \left(1 - \sum_{j=1}^{n+1} \tau_j \right) \prod_{j=1}^{n} \left( - P^j + Q^j \right) \times f_{A_n \ldots A_1} \left( - \tau_{n+1}^P - \sum_{j=1}^{n} \tau_j Q_j; Y; 0 \right), \tag{3.7}

meaning that $h_p \triangle Q_n \ldots \triangle Q_1$ is totally antisymmetric in indices $P, Q_j$. In particular,

$$h_p \triangle Q = - h_Q \triangle P, \tag{3.8}$$

which has a consequence

$$h_p \triangle P = 0. \tag{3.9}$$

Other useful relations are

$$h_p h_Q = h_Q, \quad \triangle_p h_Q = 0 \tag{3.10}$$

and the consequence of resolution of identity (2.31)

$$\triangle_B - \triangle_A = [d\zeta, \triangle A \triangle B] + h_A \triangle B. \tag{3.11}$$

In general in solving (2.25) one is not confined to a particular shift $Q$ in resolution operator $\triangle_Q$. Properly normalized linear combination also gives a solution to (2.25). One can take integrals over shift parameters

$$\triangle (\rho) := \int d\rho (Q) \triangle Q \tag{3.12}$$

with the normalization condition

$$\int d\rho (Q) = 1. \tag{3.13}$$

3.1.2. Two-component relations. Now we restrict indices of $Z_A, Y_A, \theta^A$ variables to take only two values

$$\left( Z_A, Y_A, \theta^A \right) \longrightarrow (z_\alpha, y_\alpha, \theta^\alpha). \tag{3.14}$$

The respective shifts will be denoted by lower case Latin letters. In this case formulas (3.6) and (3.7) give

$$\triangle_{b\alpha} \triangle_{a\beta} f(z, y) \theta^\beta \theta_\beta = 2 \int_{[0,1]^3} d^3 \tau \delta \left(1 - \tau_1 - \tau_2 - \tau_3 \right) (z + b)_\gamma (z + a)_\gamma f(\tau_1 z - \tau_3 b - \tau_2 a, y), \tag{3.15}$$

$$h_c \triangle_{b\alpha} \triangle_{a\beta} f(z, y) \theta^\beta \theta_\beta = 2 \int_{[0,1]^3} d^3 \tau \delta \left(1 - \tau_1 - \tau_2 - \tau_3 \right) (b - c)_\gamma (a - c)_\gamma f(- \tau_1 c - \tau_3 b - \tau_2 a, y). \tag{3.16}$$

From (3.16) it follows in particular that

$$h_{(\mu+1)_{\mu-\rho}, \triangle_{q_1} \triangle_{q_2}} = 0, \quad \forall \mu \in \mathbb{C}. \tag{3.17}$$

Applying (3.15) and (3.16) to $\gamma$ one finds

$$\triangle_{b\alpha} \gamma = 2 \int_{[0,1]^3} d^3 \tau \delta \left(1 - \tau_1 - \tau_2 - \tau_3 \right) (z + b)_\gamma (z + a)_\gamma e^{i(\tau_1 z - \tau_3 b - \tau_2 a)} \gamma, \tag{3.18}$$
\[ h_c \Delta_b \Delta_a \gamma = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3)(b - c)(a - c)^2 \gamma e^{-i(\gamma_1 c + \gamma_2 a + \gamma_3 b)\cdot \alpha} k. \tag{3.19} \]

Note that in accordance with (3.1) the prefactor in (3.18) is antisymmetric in \( a, b \) while the exponential is symmetric. Analogously, the r.h.s. of (3.19) is totally antisymmetric in \( a, b, c \). Also, from (3.19) it follows that
\[ h_{a+\alpha y} \Delta_b \Delta_c \gamma = h_a \Delta_b \Delta_c \gamma, \quad \forall \alpha \in \mathbb{C}. \tag{3.20} \]

Let us also note that any homotopy containing \( y \)-shift leaves no effect on the exponential when applied to \( \gamma \). Indeed, from (2.16) and (2.30) it follows that
\[ \Delta_q \alpha y \gamma = 2(z^\beta + q^\beta + \alpha y^\beta) \int_0^1 dt e^{i(\gamma_0 - (1-t)q)\alpha} \gamma k \tag{3.21} \]
as parameter \( \alpha \) drops out from the exponential being present in prefactor. This property implies that the \( y \)-shifted homotopies do not affect locality at the order they are first applied. Note however that local field redefinition at a given order may affect the structure of higher-order non-local vertices.

Finally, we need an identity that makes vertices (1.1) and (1.2) resulting from generating equations (2.3)–(2.7) manifestly \( z \)-independent. That this should be so is granted by consistency of (2.3)–(2.7). The precise mechanism responsible for \( z \)-cancellation is stored in identity resolution (2.31) which makes some combinations of shifted homotopies manifestly \( z \)-independent. A particularly useful relation of this type is
\[ (\Delta_d - \Delta_c)(\Delta_a - \Delta_b) \gamma = (h_d - h_c) \Delta_a \Delta_b \gamma, \tag{3.22} \]
which is true for any homotopy parameters \( a, b, c, d \). It can be proven as follows. First, using (3.11) along with \( d_c \gamma = 0, h_a \gamma = 0, h_b \gamma = 0 \) one finds \( (\Delta_a - \Delta_b) \gamma = d_c \Delta_b \Delta_c \gamma \). Then, moving \( d_c \) through \( (\Delta_d - \Delta_c) \) with the aid of identity resolution (2.31) and using that \( \Delta_a \Delta_b \Delta_c = 0 \ \forall a, b, c \) one obtains (3.22).

Equation (3.22) says in particular that its l.h.s. is \( z \)-independent. Let us note that identity resolution (2.31) involves partial integration which makes direct check of (3.22) quite non-trivial. As a simple consequence of (3.22) at \( d = a \) one finds
\[ (\Delta_c \Delta_b - \Delta_c \Delta_a + \Delta_a \Delta_b) \gamma = h_c \Delta_b \Delta_a \gamma. \tag{3.23} \]

Also applying \( h_d \) to the both sides of this relation and using (3.10) one obtains
\[ h_c \Delta_b \Delta_a \gamma = h_d \Delta_c \Delta_a \gamma + h_d \Delta_a \Delta_b \gamma = h_c \Delta_b \Delta_a \gamma. \tag{3.24} \]

In fact, equation (3.24) expresses triangle identity of [45] that played crucial role in the early analysis of nonlinear corrections to HS equations in [41, 42, 46] performed with the help of the triangle function
\[ \Delta(a_1, a_2, a_3) := \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3)(a_{1a} a_{2a}^* + a_{2a} a_{3a}^* - a_{1a} a_{3a}^*) \delta^2 \left( \sum_{i=1}^3 \tau_i a_i \right) \tag{3.25} \]

obeying triangle identity
\[ \Delta(a, b, c) + \Delta(c, d, a) = \Delta(a, b, d) + \Delta(b, c, d). \tag{3.26} \]

Comparing (3.25) with (3.19) we observe that \( h_c \Delta_b \Delta_a \gamma \) is a Fourier transform of \( \Delta(a, b, c) \)
\[ (h_c \Delta_b \Delta_a \gamma)(y) = 2 \int d^2 \alpha \Delta(c + u, b + u, a + u) \delta^{(2)}(y - \alpha) \tag{3.27} \]
from where it is obvious that (3.24) is a consequence of (3.26). Note also that identity (3.20) follows immediately from (3.27) upon the integration variable change $u_\beta \to u_\beta + \alpha y_\beta$.

### 3.2. Star-exchange homotopy relations

One of the technical challenges in HS perturbation theory based on conventional homotopy is a permanent interplay between such seemingly unrelated operations as star product and homotopy integration. This interplay renders algebraic structures obscure and hinders one from constructing functional space that respects both operations. Remarkably shifted homotopies obey relations that link the two operations as we now show. These properties do not rely on particular range of indices of variables $z,y,\theta$ and we take them as being generic in this section. The same time we discard the right sector of dotted spinors that can be treated analogously.

Shifted homotopy operators obey remarkable star-exchange relations with $z$-independent star-product elements. Namely, let us consider a homotopy action on a star product $C(y;k) \ast \phi(z,y;k;\theta)$. Using (2.8) and (2.30) one can check that

$$\Delta_{q+\alpha y}\left(C(y;k) \ast \phi(z,y;k;\theta)\right) = C(y;k) \ast \Delta_{q+(1-\alpha)p+\alpha y} \phi(z,y;k;\theta),$$

(3.28)

where $q$ is a $y$-independent parameter and $\alpha$ is a number. We have also introduced the notation

$$p_\alpha C(y;k) \equiv C(y;k) p_\alpha := -i\frac{\partial}{\partial y}\left(C_1(y) + C_2(y)k\right),$$

(3.29)

where $C(y;k) = C_1(y) + C_2(y)k$. Note that $p$ acts on the argument of the $z$-independent function only which in our case is $C$ no matter if it appears on the left or on the right. The $z$-independence of $C$ is crucial for (3.28) to take place. Let us stress also, that though $C = C(y;k)$ depends on $k$, operator $p_\alpha$ is defined to commute with $k$, $[p_\alpha,k] = 0$, and therefore is insensitive to such dependence. Its action on $C$ by our definition can be either written from left or right (3.29).

While it is not difficult to check (3.28) directly using (2.8) and (2.30), in doing so one encounters certain cancellations in star product computation that come along simultaneously both in the exponential and pre-exponential which may seem coincidental. Still, equation (3.28) is not that surprising after all as can be seen from the following consideration.

Suppose one is to solve the equation

$$d_z f = C(y;k) \ast \phi(z,y;k;\theta)$$

(3.30)

with $d_z$-closed $\phi(z,y;k;\theta)$. One way to do it is by using (2.30) which gives $f = \Delta_\alpha \left(C(y;k) \ast \phi(z,y;k;\theta)\right)$ with some homotopy parameter $a$. On the other hand, since $C(y;k)$ is $z$-independent and hence commutes with $d_z$, one can solve (3.30) in the form $f = C(y;k) \ast \Delta_\alpha \phi(z,y;k;\theta)$ with some other parameter $b$ thus suggesting that $a$ and $b$ should be related. The precise relation is given by (3.28). This reasoning makes it clear why $C(y;k)$ should be $z$-independent to yield (3.28).

Analogously, it can be shown that

$$\Delta_{q+\alpha y} \left(\phi(z,y;\theta) \ast k^\nu \ast C(y;k)\right) \equiv \Delta_{q+(-1)^\nu(1+\alpha)p+\alpha y} \left(\phi(z,y;\theta) \ast k^\nu\right) \ast C(y;k)$$

(3.31)

Note that the additional sign factor $(-1)^\nu$ is due to the definition (3.29) of the shift $p$ as derivative over the total argument of $C(y;k)$.

There are two special points $\alpha = \pm 1$ related to the normal ordering of star-product variables $y \pm z$. Right product (3.31) remains invariant for $\alpha = -1$, whereas left one (3.28) for $\alpha = 1$. 


Since any perturbative correction depends on star products of the Z-independent fields \( \omega(Y;K) \) and \( C(Y;K) \) using star-exchange formulas any perturbative result can be reduced to the form where shifted homotopy operators act only on the central two-form elements \( \gamma \) and \( \bar{\gamma} \) (2.16). Since \( \gamma \) is central, one is able to calculate \( C^* \Delta_{q} \gamma \rightarrow \Delta_{q'} \gamma^* C \) using for example (3.28). Taking into account that \( \gamma \) is linear in \( k \) (analogously for \( \bar{\gamma} \)) this results in the following rule

\[
\Delta_{q} \gamma * C(y;k) = C(y;k) * \Delta_{\bar{q}+2p} \gamma, \quad \bar{q} = q + \alpha y.
\]

Formula (3.32) will be used steadily in our analysis of HS vertices.

Analogously it is straightforward to check the following useful identities with \( y \)-independent \( q' \):

\[
h_{q + \alpha y}(C(y;k) * \phi(z,y;k;\theta)) = C(y;k) * h_{q+(1-\alpha)p+\alpha y}\phi(z,y;k;\theta),
\]

\[
h_{q + \alpha y}(\phi(z,y;\theta) * k'' * C(y;k)) = h_{q+(1-1)p+\alpha y}(\phi(z,y;\theta) * k'') * C(y;k).
\]

It should be stressed that formulae presented in this section heavily rely on the specific form of star product (2.8). An important consequence of our analysis is that the class of homotopies with shift parameters \( Q(2.26) \) linear in derivatives \( P_{A} \) of the Z-independent fields \( C(Y;K) \) and \( \omega(Y;K) \) and/or \( Y \)-variables, that obviously obey the Z-independence condition (2.27), remains closed under the star-product exchange formulas. It is this class of linear shifted homotopies that turns out to be most appropriate for the perturbative analysis of HS equations.

4. Lower-order vertices

The transition from one homotopy to another not only affects the gauge choice in the Z-space (i.e. \( d_{Z} \)-exact terms) but also the cohomology representative as is most obvious from (2.35). The latter freedom encodes different choices of field variables. Hence, the proper choice of the homotopy operators can lead directly to the local result with no need of further field redefinitions. In the rest of this paper we demonstrate how this works in practice.

Using generalized resolutions (2.30) we can step away from conventional homotopy approach and reconsider perturbative analysis of (2.3)–(2.7). Recall that within the standard approach one starts with the AdS background as vacuum solution (2.18)–(2.20). Identities (3.28) and (3.31) actually allow us to develop perturbative expansion for any HS one-form \( \omega(Y;K) \) reproducing directly vertices (1.1) and (1.2) as \( C \)-expansion in spirit of [41].

4.1. \( \Upsilon(\omega,\omega,C) \)–vertex

At first order on AdS we know that conventional homotopy works fine reproducing canonical form (2.40) that identifies HS Weyl tensors with derivatives of HS potentials. We may now redo this analysis for generic \( \omega \) thus reproducing the entire vertex \( \Upsilon(\omega,\omega,C) \). Let us start with (2.24) from which we find

\[
S_{1} = -\frac{\eta}{2} \Delta_{0} (C * \gamma) + \text{c.c.} = -\frac{\eta}{2} C * \Delta_{p} \gamma + \text{c.c.}
\]

Here we made use of (3.28). (Recall that index \( p \) denotes differentiation (3.29).) From (2.4) we then have

\[
W_{1} = \frac{1}{2i} \Delta_{0} (d_{s}S_{1} + \omega * S_{1} + S_{1} * \omega) + \text{c.c.}
\]
Substituting here (4.1) and using
\[ d_\gamma C + [\omega, C]_\gamma = 0 \] (4.3)
and (3.31) one gets
\[ W_1 = -\frac{\eta}{4}\left( C * \omega * \Delta_{p+1}\Delta_{p+2}\gamma - \omega * C * \Delta_{p+1}\Delta_p\gamma \right) + \text{c.c.}, \] (4.4)
where \( t_\alpha \) acts on \( \omega \)
\[ t_\alpha \omega(Y; K) := -i\frac{\partial}{\partial \alpha^\gamma}\omega(Y; K). \] (4.5)

To obtain \( \Upsilon(\omega, \omega, C) \) it remains to plug (4.4) into (2.3) which after some elementary algebra making use of (3.23) and (3.9) gives
\[ d_\omega + \omega * \omega = \frac{\eta}{4}\left( C * \omega * \Delta_{p+1}\Delta_{p+2}\gamma - \omega * C * \Delta_{p+1}\Delta_p\gamma \right) + \text{c.c.}, \] (4.6)
where
\[ X_{\omega\omega C} = h_{p+n+2\gamma}, \] (4.7)
\[ X_{C\omega\omega} = h_{p+n+1,2\gamma}, \] (4.8)
\[ X_{\omega C\omega} = -h_{p+n+1,2\gamma}. \] (4.9)

Carrying out star-product integration on the r.h.s. of (4.6) and using (3.19) we find
\[ \Upsilon(\omega, \omega, C) = \Upsilon_{\omega\omega C} + \Upsilon_{C\omega\omega} + \Upsilon_{\omega C\omega}, \] (4.10)
where
\[ \Upsilon_{\omega\omega C} = \frac{\eta}{24} \int_{[0,1]^3} d^3\delta(1 - \tau_1 - \tau_2 - \tau_3)e^{i(1 - \tau_3)\partial_\gamma^\alpha}\partial_\alpha^\gamma, \] (4.11)
\[ \Upsilon_{C\omega\omega} = \frac{\eta}{24} \int_{[0,1]^3} d^3\delta(1 - \tau_1 - \tau_2 - \tau_3)e^{i(1 - \tau_3)\partial_\gamma^\alpha}\partial_\alpha^\gamma, \] (4.12)
\[ \Upsilon_{\omega C\omega} = \frac{\eta}{24} \int_{[0,1]^3} d^3\delta(1 - \tau_1 - \tau_2 - \tau_3)e^{i(1 - \tau_3)\partial_\gamma^\alpha}\partial_\alpha^\gamma. \] (4.13)

Here \( \bar{\star} \) denotes the star product with respect to the leftover \( \bar{y} \) variables and \( \partial_1, \partial_2, \text{ and } \partial_3 \) are differentiations of first and second \( \omega \)'s correspondingly counted from left to right. Remarkable property of the obtained vertex is that all its structures contain only derivatives of \( C(0, y; K) \) having no dependence on \( y \) in \( C(y, \bar{y}; K) \) at all. This fact is not accidental being a consequence of PLT as explained in the next section. Such a form of deformation effectively makes spin—s contribution from \( C \) finite component for any given \( \omega \). On AdS background, where \( \omega = \Omega \)
(2.20), the vertex is given by (2.40) and contains $\partial \partial C(0, \bar{y}, K)$. Equations (4.11)–(4.13) generalize this property to cubic (quartic order in the Lagrangian nomenclature). A local vertex that contains at most a finite number of derivatives of $C$'s at $y = 0$ we call **ultra-local**. While the locality of $\Upsilon(\omega, \omega, C)$ is granted in any perturbative scheme, the fact that it appears in a ultra-local form can be crucial for higher-order locality.

4.2. Classes of functions and Pfaffian locality theorem

As already mentioned, at the level $O(C^2)$ conventional homotopy leads to a non-local result for, say, $\Upsilon(\Omega, C, C)$. The non-locality results from the application of the conventional homotopy for the reconstruction of the $z$-dependence of $B$ field from (2.7) and upon substitution it to the physical sector equation (2.5) yields $\Upsilon(\omega, C, C)$ which appears non-local even on AdS, i.e. at $\omega = \Omega$. The origin of this phenomenon is as follows. Suppose one solves for $\omega = z^i \partial_i \bar{y}$, $y \partial \bar{z}$ orders gives the locality of ultra-local form can be crucial for higher-order locality.

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where
\begin{align}
T \circ T' &= T + T' - 2TT', \\
A'' &= (1-T')A' + (1-T)A' + TBB' - T'B', \\
B'' &= (1-T')B' + (1-T)B' + TAB' - T'A', \\
p''ij &= p^{ij} + p^{'ij} + (A' + B')(A' - B') - (A' - B')(A' + B').
\end{align}

A striking property of star product (2.8) which follows from (4.18)–(4.21) is as follows. Suppose one solves for field $z$-dependence using conventional homotopy all the way in perturbation then, as shown in [40], coefficients $T$, $A$, $B$ and $P$ satisfy
\begin{align}
\sum (-)^i A^j &= -T, \\
\sum (-)^i B^j &= 0, \\
\sum (-)^i P^ij &= B^j.
\end{align}

(Sign alternation $(-)^i$ appears due to the presence of the outer Klein operator $k$ in $\gamma$ (2.16).) This property is also invariant under star product (2.8). For instance, as $S$ field at some orders $S'$ and $S''$ satisfy (4.22)–(4.24) then so does $S''' = S S'$. Looking at (2.6) we see that it has $S S$ term which belongs to (4.22)–(4.24) as soon as one uses conventional homotopy. Another contribution to (2.6) is $B S$. If $B$ is solved by using the conventional homotopy then $B S$ does not belong to the class (4.22)–(4.24) simply because the star product of $B$ with $\gamma$ swaps $z$ and $y$. However, it turns out that a proper shifted homotopy exists that brings $B S$ to the class (4.22)–(4.24). Indeed, in order $B S$ to respect (4.22)–(4.24)
\begin{equation}
e^{i(Tz\alpha' + A'\partial^{*}\alpha' + B'\partial^{*}\alpha' + 1/2\theta^{*}\partial^{*}\theta)} + e^{i(z\alpha')} = e^{i(1-T)z\alpha' - A'\partial^{*}\alpha' - B'\partial^{*}\alpha' + 1/2\theta^{*}\partial^{*}\theta}
\end{equation}
the following conditions
\begin{align}
\sum (-)^i A^j &= 0, \\
\sum (-)^i B^j &= 1 - T, \\
\sum (-)^i P^ij &= -A^j
\end{align}
should be imposed. Equations (4.26)–(4.28) place strong constraints on $B$-field. Indeed, as $B''''$ originates from homotopy acting on $S S$ where the two fields belong to different classes (4.22)–(4.24) and (4.26)–(4.28) the fact that $B''''$ remains in (4.26)–(4.28) is non-trivial and requires specific homotopy. Let us take most general homotopy up to $y$-shifts (local effects) to see if it is capable to reconcile the result with (4.22)–(4.24)
\begin{equation}
z_{\alpha} \rightarrow z_{\alpha} - i \sum \theta^{ij} \partial_{\alpha} v^{ji},
\end{equation}
where $v^{ij}$ are some numbers. Performing star product of $S$ and $B''$ and applying homotopy (4.29) to the obtained result entails the following coefficients in $B''$
\footnote{From (4.23) it follows that $B_i = 0$ if $i$ takes only one value. Hence, in accordance with the analysis of section 4.1, from here it follows that the correction in the sector linear in the zero-forms $C$ is $y$-independent being ultra-local provided that one stays in the proper class.}
\[ A'' = T''((1 - T')A' + (1 - T)A'' + TB' - T'B'), \]  
\[ B'' = (1 - T')B' + (1 - T)B'' + TA' - T'A'' + (1 - T'')(T \circ T')v''', \]  
\[ P''^{ij} = P''^{ij} + (A' + B')(A' - B') - (A' - B') (A' + B') - \frac{1}{T''} (A'' v'' - A'' v'''), \]  
where \( T, T' \) and \( T'' \) are the homotopy integration variables entering in \( S \), \( B' \) and \( B'' \) correspondingly. Assuming now that dashed and double-dashed fields are from (4.26)–(4.28) and non-dashed are from (4.22)–(4.24) we find following \[40\]
\[ \sum (-)^{j''} v_{j''} = 1. \]  
(4.33)

Thus the homotopy prescription used for one-forms (\( S \) and \( W \)) requires modified homotopy (4.29) and (4.33) for zero-form \( B \) in order to preserve functional classes (4.22)–(4.24) for one-forms and (4.26)–(4.28) for zero-forms. A short explanation to this phenomenon is the presence of inner Klein operator in \( \gamma \) which comes with \( B \)-field only, affecting the structure of the perturbative expansion. The effect it brings is compensated by modified homotopy. That such a compensation mechanism works at all at the condition (4.33) is highly non-trivial and is related to a very special form of star product (2.8). Clearly the conventional homotopy \( v'' = 0 \) violates (4.33).

Another important fact shown in \[40\] is that structure lemma and, hence, PLT is respected by the \( d \) differential which may have a nontrivial effect for shifted homotopies. The fact that shifted homotopies obeying conditions of PLT decrease the level of nonlocality follows from the \( Z \)-dominance Lemma proven in \[40\].

At second order \( O(CC) \) the number of free parameters in (4.29) when solving for \( B \) is two. Condition (4.33) reduces it further to a one-parameter family
\[ \nu_2 - \nu_1 = 1. \]  
(4.34)

The resulting contribution based on homotopy (4.34) to the cubic vertex \( \Upsilon(\omega, C, C) \) turns out to be local generalizing the known cubic vertex \( \Upsilon(\Omega, C, C) \) in AdS4 \[35\]. Moreover the one-parameter freedom (4.34) appears to be spurious dropping off from the final result as explained in section 4.3. We now analyze \( O(C^3) \)-type vertices in more detail.

### 4.3. \( \Upsilon(\omega, C, C) \)—vertex

Let us start by sketching how (4.33) leads to correct local vertex \( \Upsilon(\Omega, C, C) \). To solve for \( B \) to the second order we take \( S_1 \) (4.1) giving
\[ S_1 = \theta^\alpha S_{1\alpha} + c.c., \]  
\[ S_{1\alpha} = \eta \int_0^1 dt z_{\alpha} C(-tz, \bar{y}; K)e^{it\alpha\gamma_\alpha} k \]  
(4.35)

and from (2.7) have
\[ [S_0, B_2] + [S_1, B_1] = 0 \implies \quad 2i\theta^\alpha \frac{\partial}{\partial z^\alpha} B_2 = S_1 * C - C * S_1, \]  
(4.36)

which results in
\[ 2i \frac{\partial}{\partial z^\alpha} B_2 = A_\alpha - B_\alpha. \]  
(4.37)
where
\begin{align}\nonumber
A_\alpha &= \eta \int_0^1 t(z_\alpha - i \partial z_\alpha) e^{i t(z_\alpha - i \partial z_\alpha)(y^\alpha - i \partial y^\alpha) - i t(z_\alpha - i \partial z_\alpha) y^\alpha} C(0, y; K) \equiv C(0, y; K) k, \\
B_\alpha &= \eta \int_0^1 t(z_\alpha + i \partial z_\alpha) e^{i t(z_\alpha + i \partial z_\alpha)(y^\alpha + i \partial y^\alpha) + i t(z_\alpha + i \partial z_\alpha) y^\alpha - i t(z_\alpha + i \partial z_\alpha) y^\alpha} C(0, y; K) \equiv C(0, y; K) k.
\end{align}

We are ignoring $\bar{\eta}$ terms for brevity and $*$ denotes the star product with respect to the right variables $\bar{y}^\alpha$. Now we can solve for $B_2$ using shifted homotopy as follows
\begin{align}
2i B_2 &= \Delta_q \left( \theta^\alpha (A_\alpha - B_\alpha) \right), \quad q = v_1 p_1 + v_2 p_2,
\end{align}
where $v_1$ and $v_2$ are some numbers. Imposing homotopy condition (4.34), after some algebra involving partial integration one can obtain
\begin{align}
B_2 &= B^{\text{loc}}_{2\eta} - \frac{\eta}{2} \int_0^1 \text{d}t \ C (\bar{y}^\alpha; K) \equiv C ((t - 1) y; K) k, \\
\end{align}
where
\begin{align}
B^{\text{loc}}_{2\eta} &= \frac{\eta}{2} \int_{[0,1]} \text{d}^3 \tau (\delta'(X) - i z_\alpha y^\alpha \delta(X)) e^{r z_\alpha (\partial y^\alpha + i \partial y^\alpha + i r z_\alpha \partial y^\alpha + i r z_\alpha y^\alpha - i r z_\alpha \partial y^\alpha - i r \partial y^\alpha) - i r z_\alpha y^\alpha} \times C(0, y; K) \equiv C(0, y; K) k,
\end{align}
which turns out independent of the remaining parameter $v_1 + v_2$. Here $B^{\text{loc}}_{2\eta}$ coincides with that found in [48] which is known to reproduce correct local $Y(\Omega, C; C)$. Let us stress that our approach is free of any field redefinitions thanks to the proper homotopy choice leading directly to the local result. The resulting $B_2$ differs from $B^{\text{loc}}_{2\eta}$ by the second term on the r.h.s. of (4.41) which, containing no contractions between first arguments of the two factors of $C$, is local.

After this old fashion-style exercise we proceed to a more systematic analysis of the vertex $Y(\Omega, C; C)$. We want to obtain (1.2) up to terms quadratic in $C$. To do that one solves (2.7) up to this order. Namely, from (4.36) and (4.1) we have (ignoring $\bar{\eta}$-terms)
\begin{align}
-2i d_1 B_2 &= -\frac{\eta}{2} C * C * \Delta_{p_1} \gamma + \frac{\eta}{2} C * \Delta_{p_1} \gamma * C = -\frac{\eta}{2} C * C * (\Delta_{p_1} - \Delta_{p_1 + 2p_2}) \gamma.
\end{align}
Here $p_1$ and $p_2$ differentiate first and second $C$‘s correspondingly as seen from left. Its solution within some $q$-homotopy (2.30) reads
\begin{align}
B_2 := B_2^q = \frac{\eta}{4i} C * C * \Delta_q (\Delta_{p_2} - \Delta_{p_2 + 2p_2}) \gamma, \quad q = v_1 p_1 + v_2 p_2.
\end{align}
Postponing the analysis of the freedom associated with $y$-shifts in $q$ till section 5 let us take $q$ (4.40) (note that the difference in the position of $\Delta_q$ in (4.40) and (4.45) is equivalent to the shift $v_1 \rightarrow v_1 + 1$ that, as shown below, does not affect the final result).

Using (3.11) and (2.31) along with the fact that $\Delta_a \Delta_b \Delta_c = 0 \forall a, b, c$ (4.45) gives
\begin{align}
B_2^q &= \frac{\eta}{4i} C * C * (1 - h_q) \Delta_{p_2 + 2p_2} \Delta_{p_2} \gamma.
\end{align}
Thanks to (3.17), for $q$ (4.40),

\[18\]
provided that \( v_2 - v_1 = 1 \) which is just the PLT condition. Thus, in this case the final result is independent of the remaining freedom in \( v_{1,2} \) giving

\[
B_2 = \frac{\eta}{4i} C \ast C \ast \Delta_{p_1+2p_2, \Delta_{p_2} \gamma}.
\]

As will be shown in the next section, these parameters however contribute in presence of \( y \)-dependent shifts affecting local terms. In accordance with PLT of [40] \( B_2 \) (4.48) must result in the local vertex.

From (2.5) we have at given order

\[
d_4 C + [\omega, C]_\ast + d_4 B_2 + [\omega, B_2]_\ast + [W_1, C]_\ast = 0.
\]

Substituting here \( B_2 \) (4.48) and \( W_1 \) from (4.4), after simple algebra using (3.32) and (3.22) one eventually finds

\[
d_4 C + [\omega, C]_\ast = \Upsilon_{\omega CC} + \Upsilon_{CC\omega} + \Upsilon_{C\omega C} + \text{c.c.},
\]

where

\[
\Upsilon_{\omega CC} = \frac{\eta}{4i} C \ast C \ast X_{\omega CC}, \quad \Upsilon_{CC\omega} = \frac{\eta}{4i} C \ast C \ast \omega \ast X_{CC\omega}, \quad \Upsilon_{C\omega C} = \frac{\eta}{4i} C \ast \omega \ast C \ast X_{\omega C}, \quad (4.51)
\]

\[
X_{\omega CC} = h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+\tau} \gamma,
\]

\[
X_{CC\omega} = h_{p_2+2\tau} \Delta_{p_2+\tau} \Delta_{p_1+2p_2+\tau} \gamma,
\]

\[
X_{\omega C} = (h_{p_1+2p_2+2\tau} - h_{p_2}) \Delta_{p_2+\tau} \Delta_{p_1+2p_2+\tau} \gamma.
\]

Local structures (4.52)–(4.54) are independent of the leftover parameter (4.34). Note also that vertices \( \Upsilon(\omega, C, C) \) are built from \( h_4 \Delta_{\delta_c} \gamma \) structures related to the triangle function (3.25) via (3.27).

It is straightforward to compute \( \Upsilon(\omega, C, C) = \Upsilon_{\omega CC} + \Upsilon_{CC\omega} + \Upsilon_{C\omega C} \) (4.51) that enters (1.2) using (3.19). The final result is

\[
\Upsilon_{\omega CC} = \frac{\eta}{2i} \int_{[0,1]^3} d^3 \tau d(1 - \tau_1 - \tau_2 - \tau_3)(\partial_1^{\omega} + \partial_2^{\omega}) \partial_3^\omega
\]

\[
\omega((-1 - \tau_3) y, \bar{y}, \bar{y}, K) \bar{\omega}(-1 - \tau_3) y + i\tau_3 \partial_{\bar{y}} \bar{y}, \bar{y}, K) \tag{4.55}
\]

\[
\Upsilon_{CC\omega} = \frac{\eta}{2i} \int_{[0,1]^3} d^3 \tau d(1 - \tau_1 - \tau_2 - \tau_3)(\partial_1^{\omega} + \partial_2^{\omega}) \partial_3^\omega
\]

\[
C((-1 - \tau_3) y + i(1 - \tau_1) \partial_{\bar{y}} \bar{y}, \bar{y}, K) \bar{\omega}(-1 - \tau_3) y - i\tau_3 \partial_{\bar{y}} \bar{y}, \bar{y}, K \tag{4.56}
\]

\[
\Upsilon_{C\omega C} = \frac{\eta}{2i} \int_{[0,1]^3} d^3 \tau d(1 - \tau_1 - \tau_2 - \tau_3)(\partial_1^{\omega} + \partial_2^{\omega}) \partial_3^\omega
\]

\[
\left\{ C((-1 - \tau_3) y + i(1 - \tau_2) \partial_{\bar{y}} \bar{y}, \bar{y}, K) \bar{\omega}(-1 - \tau_3) y - i(1 - \tau_3) \partial_{\bar{y}} \bar{y}, \bar{y}, K \right. \tag{4.57}
\]

where \( \partial_{1,2} \) differentiates \( C \)'s. The result generalizes local vertex \( \Upsilon(\Omega, C, C) \) obtained in [35] and remains local at cubic level. Let us stress that the remaining star product \( C \ast C \sim e^{\Theta_{\Delta}^\gamma} \) does not imply non-locality. Indeed, since the vertex \( O(CC) \) is local in its holomorphic part the
contribution from the anti-holomorphic one automatically terminates when restricted to given three spins $s_1, s_2, s_3$. Relaxing (4.33) makes the corresponding vertex non-local. In appendix it is demonstrated that no parameters $v_1$ and $v_2$ in (4.45) lead to a local vertex $Υ(ω, C, C)$ other than those specified by (4.34).

5. Y-dependent shifts and local freedom

Perturbation theory based on generalized resolution (2.30) is shown to properly reproduce simplest local vertices (4.11)–(4.13) and (4.55)–(4.56). In this analysis the shift parameters $q_i$ of $δ_p$ were linear combinations of derivatives $\partial_i$ acting on the fields $C$ and $ω$. The freedom in relative coefficients turned out to be constrained by locality implemented via PLT that keeps track of derivative shifts in homotopies (4.29). Generally, however, one can add $y$-terms in $q_i$ as in, e.g. (3.21). The effect they produce is local at given order and, naively, can be discarded. Indeed, thanks to the star-product exchange formulas (3.28) and (3.31) the resolution operators (2.30) can be arranged to act on $γ$ only. As a result, the $y$-contribution resides solely in the pre-exponential as in (3.21). It should be stressed however that the freedom in local field redefinition may affect the structure of non-localities at higher orders. Therefore, we would like to control local $y$-shifts in (2.30) aiming at future higher-order application.

5.1. Uniform $y$-shift freedom of $Υ(ω, ω, C)$

In our analysis we have shown that standard approach based on the conventional homotopy used in (4.1) and (4.4) not only reproduces canonical form of the central on-mass-shell theorem but also leads to its ultra-local completion (4.11)–(4.13). Remarkably, there exists a one-parameter extension of the conventional homotopy that leaves vertex (4.11)–(4.13) unaffected. To obtain it similarly to (4.1) and (4.2) taking into account (3.28) one can solve for $S_1$ and $W_1$ in the following form

$$S_1 = -\frac{\eta}{2} δ_{\alpha(p+y)} (C * γ) + \text{c.c.} = -\frac{\eta}{2} C* δ_{p+αy} γ + \text{c.c.}, \quad (5.1)$$

$$W_1 = \frac{1}{2i} δ_{\alpha(p+y)} (d_i S_1 + ω * S_1 + S_1 * ω) + \text{c.c.} = -\frac{\eta}{4i} (C * ω* δ_{p+i+αy} δ_{p+2i+αy} γ - ω * C* δ_{p+i+αy} δ_{p+2i+αy} γ) + \text{c.c.}. \quad (5.2)$$

Homotopies, that contain one and the same coefficient in $y$-shifts ($α$ in (5.1) and (5.2)), will be referred to as uniformly shifted. That (5.1) and (5.2) still reproduce (4.11)–(4.13) is a consequence of identity (3.20), which guarantees that (4.7)–(4.9) while receiving uniform $y$-shifts in homotopy parameters remain unchanged. $α = 0$ corresponds to the conventional homotopy in (5.1) and (5.2) reproducing (4.1) and (4.4). Let us stress that should one took non-uniform shifts by taking different parameters, say, $α_1$ in $S_1$ and $α_2$ in $W_1$ the resulting vertex $Υ(ω, ω, C)$ would have been different from that in (4.11)–(4.13) affecting the form of the free equation (2.40) which is not allowed. Correspondingly, expressions (5.1) and (5.2) provide a one-parameter family of homotopies that give the same form of central on-mass-shell theorem (2.40) as well as of its completion $Υ(ω, ω, C)$ (4.11)–(4.13). Note that $Y$-shifts, uniform or not, do not affect the class of homotopies (4.33) prescribed by PLT [40].
5.2. Uniform y-shift freedom of \( \Upsilon(\omega, C, C) \)

Consider now the second-order contribution to the zero-form sector of \( B \) using the \( \alpha \)-modified uniform homotopies. Analogously to the one-form sector we can directly generalize (4.48) by the uniform y-shift. Choosing appropriately shifted homotopies from (5.1) and (4.36) we find

\[
B_2 = \frac{\eta}{4i} C^* C \Delta \left( \Delta_{p_1 + 2p_2, \alpha y} - \Delta_{p_2} \right) \gamma. \tag{5.3}
\]

While the obtained \( B_2 \) differs from (4.48) at \( \alpha \neq 0 \) the physical sector equation (4.50) remains unchanged thanks to identity (3.20), reproducing same vertices (4.55) as at \( \alpha = 0 \).

Despite physical vertices \( \Upsilon(\omega, \omega, C) \) and \( \Upsilon(\omega, C, C) \) go through the uniform homotopy deformation this is no longer so for higher-order vertices where such a homotopy freedom may affect locality. However the non-uniform shifts produce non-zero local effects in the lowest orders which is interesting to inspect. Let us examine the zero-form sector.

5.3. Non-uniform y-shift in \( B_2 \)

In the previous section it was shown how shifted homotopy leads to the expression for \( B_2 \) that differs from the one from [48] by the \( Z \)-independent local term in (4.41). Adding a \( y \) shift to the homotopy index it is possible to obtain the additional local term. Namely, starting from the general expression for \( B_2 \)

\[
B_2 = \frac{\eta}{4} C^* C \Delta q \left( \Delta_{p_1 + 2p_2} - \Delta_{p_2} \right) \gamma \tag{5.4}
\]

we specify index \( q \) in the form

\[
q = q + \alpha y, \quad q = v_1 p_1 + v_2 p_2. \tag{5.5}
\]

Coefficients in front of \( p_1 \) and \( p_2 \) (recall the definition of \( p \) (3.29)) are chosen to obey PLT condition (4.34) while \( \alpha \) is free.

To proceed we use (3.18) whence it follows

\[
\Delta_{p_1 + \alpha y} \Delta \gamma = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (z + b + \alpha y) (a - b - \alpha y) e^{(\tau_2 - \gamma) \alpha y \gamma} k
\]

\[
= \Delta b \Delta_a \gamma + 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (z + A) (\gamma - \alpha y) e^{(\tau_2 - \gamma) \alpha y \gamma} k
\]

\[
= \Delta b \Delta_q \gamma + 2 \int_{[0,1]^3} \frac{d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (\partial_{\tau_1} - \partial_{\tau_2}) e^{(\tau_2 - \gamma) \alpha y \gamma} k. \tag{5.6}
\]

By partial integration this yields

\[
\Delta_{p_1 + \alpha y} \Delta \gamma = \Delta b \Delta_a \gamma - 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (\partial(\tau_1) - \partial(\tau_2)) e^{(\tau_2 - \gamma) \alpha y \gamma} k. \tag{5.7}
\]

Therefore by virtue of (5.7) from (4.45) it follows for \( q \) (5.5)

\[
B_2^2 = \frac{\eta}{4} C^* C \Delta q + \alpha y \left( \Delta_{p_1 + 2p_2} - \Delta_{p_2} \right) \gamma
\]

\[
= B_2^2 + \frac{\eta \alpha}{2} C^* C \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_2 - \tau_3) \left( e^{(\gamma - \gamma 2p_2) + \gamma 2p_2} - e^{(\gamma - \gamma 2p_2 - \gamma 2p_2)} \right) k. \tag{5.8}
\]
As anticipated, the $y$-shift in $\Delta_{q+\alpha y}$ does not affect the exponential and hence locality. As mentioned in section 4.3 the $\alpha$-independent part $B^q_2$ gives $B_2$ of [48] with the local extra term.

Choosing appropriately $q$ and $\alpha$ one can further simplify the extra term on the r.h.s. of (4.41). The two most interesting options are

$$q = p_1 + 2p_2, \quad \alpha = -1, \quad (5.9)$$

$$q = p_2, \quad \alpha = 1. \quad (5.10)$$

In the case (5.9) the $\alpha$-dependent terms from (5.8) can be rewritten in the following way

$$-\frac{\eta}{2} C * C * \int_{[0,1]^2} d^2 \tau \delta(1 - \tau_2 - \tau_3) \left( e^{-\gamma^\nu (\partial_1 + 2\partial_2) \alpha} - e^{-\gamma^\nu \partial_2 \alpha - \tau_1 \gamma^\nu (\partial_1 + \partial_2) \alpha} \right) k. \quad (5.11)$$

Straightforward computation of this expression gives

$$-\frac{\eta}{2} C (0, \bar{y}, K) \star C (-y, \bar{y}; K) k + \frac{\eta}{2} \int_0^1 dt C (ty, \bar{y}; K) \bar{C} ((t-1)y, \bar{y}; K) k, \quad (5.12)$$

which yields by virtue of (4.41)

$$\frac{in}{4} C * C * \Delta_{p_1+2p_2-y} (\Delta_{p_1+2p_2} - \Delta_{p_2}) \gamma = B^\text{loc}_2 - \frac{\eta}{2} C (0, \bar{y}; K) \star C (-y, \bar{y}; K) k. \quad (5.13)$$

Analogously, in the case (5.10) the expression for $B_2$ can be written in the form

$$\frac{in}{4} C * C * \Delta_{p_1+y} (\Delta_{p_1+2p_2} - \Delta_{p_2}) \gamma = B^\text{loc}_2 - \frac{\eta}{2} C (y, \bar{y}; K) \star C (0, \bar{y}; K) k. \quad (5.14)$$

A natural choice for $B_2$ respecting the symmetry reversing the order of product factors is

$$B_2 = \frac{in}{8} C * C * \left( \Delta_{p_1+2p_2-y} + \Delta_{p_1+2p_2+y} \right) (\Delta_{p_1+2p_2} - \Delta_{p_2}) \gamma$$

$$= B^\text{loc}_2 - \frac{\eta}{4} C (y, \bar{y}; K) \star k C (y, \bar{y}; K) - \frac{\eta}{4} C (y, \bar{y}; K) \star k C (0, \bar{y}; K). \quad (5.15)$$

Using shifted homotopy Ansatz we were not able to reproduce $B^\text{loc}_2$ with no additional local terms. However the terms on the r.h.s. of (5.13) and (5.14) contributing to dynamical equations feature quasi ultra-local form, i.e. such that the argument of one of the factors of $C$ is zero and there is at most a finite number of contractions between them.

Also note that in presence of the $y$-shift the final result does depend on $v_1 + v_2$ which freedom only affects local terms.

6. Conclusions

The important problem addressed in [40] and in this paper is the elaboration of the technical tools allowing to find a minimally nonlocal formulation of the HS theory and its further generalizations like those proposed recently in [49]. The main new tool consists of the application of the shifted homotopy techniques allowing to reduce the degree of nonlocality of HS vertices in accordance with the Pfaffian locality theorem (PLT) of [40] (see also section 4.2). In this paper we have applied this approach to the analysis of some vertices in HS theory demonstrating that the shifted homotopy from the proper class identified in [40] leads directly to the known local vertices of [35, 49] with no need of nonlocal field redefinitions used in those references.
In this paper we develop perturbative approach to HS equations (2.3)–(2.7) based on shifted resolutions (2.30) \( \Delta_q \) generalizing the conventional resolution \( \Delta_0 \) introduced in [1] that is known to render non-localities in cubic vertices [23] (see also [34]). Parameter \( q \) in its argument is any \( z \)-independent spinor variable (operator). The proposed homotopies drastically simplify the analysis of HS equations thanks to remarkable star-product exchange formulas (3.28) and (3.31) found in this paper, which link together such seemingly unrelated operations as star product and homotopy integration. They reduce the analysis of HS equations to the analysis of structures built from combinations of resolutions \( \Delta_a \) acting on \( \gamma \) and their further star products, where \( \gamma \) is the central element (2.16) from HS equations. Though at higher orders these structures get more and more involved like \( \Delta_a \Delta_b \gamma \ast \Delta_c \Delta_d \gamma \), \( \Delta_a (\Delta_b \gamma \ast \Delta_c \gamma) \), etc, they remain the only ones to be analyzed leading to enormous simplification of the perturbative analysis. The freedom in homotopy operator parameters reflects the freedom in the gauge choice and field redefinition of dynamical fields and therefore is tightly related to the locality problem. One of the key results of this paper is that PLT that constrains the homotopy freedom leaves one with the family of parameters resulting in \( \Upsilon(\omega, \omega, C) \) and \( \Upsilon(\omega, C, C) \) being ultra-local and local correspondingly for an arbitrary HS connection \( \omega \). Our results can be summarized as follows.

• One of the major and simple results of the paper are star-product exchange formulas (3.28) and (3.31) that relate shifted homotopies and the star product. The obtained expressions are largely rely on the form of the star product (2.8). By making it possible to interchange star product of HS fields with homotopy operation it shifts gears of the whole perturbation theory. The new setting reduces it eventually to the analysis of homotopies and their star products of the central elements \( \gamma \) and \( \bar{\gamma} \) only. In particular, it allows one analyzing HS vertices in generic backgrounds.

• Using the notion of shifted homotopies [40] we have studied simplest HS vertices \( \Upsilon(\omega, \omega, C) \) and \( \Upsilon(\omega, C, C) \). In doing so we have modified the standard perturbative approach that is based on the expansion around AdS background to the HS curvature \( C \)-expansion allowing one directly reproducing vertices in (1.1) and (1.2). From this point of view the found vertices represent cubic completion of Central on-shell theorem (2.40) and quadratic vertex \( \Upsilon(\Omega_{AdS}, C, C) \). By analyzing homotopy parameter space we check explicitly that a subspace that renders dynamical equations local is that governed by the PLT.

• We have also briefly addressed the problem of local field redefinitions. Within the proposed generalized set of homotopies \( \Delta_q \) there are two different kind of parameters encoded in \( q \). One is the shift in derivatives over the arguments of fields \( C \) and \( \omega \) and the other is just \( y \)’s. The former crucially affect local structure of the resulting dynamical equations. We show that in agreement with PLT there is a family of shift parameters that make equations local. On the other hand, shifts \( q \sim y \) at the lower level of perturbation theory are responsible for local effects remaining unconstrained by locality. It is important however to control local shifts as well as they can affect locality at higher orders.

In doing so we found a one-parameter family of homotopies consistent with the admissible class of [40] which while affecting HS master fields in their perturbative expansion provides the same dynamical equations. Particularly, this family respects Central on-shell

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5 Strictly speaking, this is true up to terms resulting from the application of \( d_\ell \) to dynamical fields \( \omega \) and \( C \) in the lower orders, which also contain the cohomology projectors \( h_\ell \) (2.32) like in (4.50)–(4.54), that demand a proper extension of the shifted homotopy technique to be presented elsewhere.
theorem. That freedom is not expected to leave higher order vertices unaffected and therefore provides a one parametric control over higher-order structures.

Let us note that the analysis carried out in this paper is mostly focused on the holomorphic sector of the HS equations (at this level the anti-holomorphic sector is reproduced analogously). As such it can be directly used for analysis of 3d HS equations.

There are plenty of problems left outside the scope of the paper. Particularly, we have considered the simplest HS vertices governed by deformation of HS algebra. These turned out to be (ultra) local. Among those bilinear in $C$ there is one, $\Upsilon(\omega, \omega, C, C)$, which we left aside. This vertex being sufficiently simple is crucial for HS locality as it contains structures typical for higher orders. Namely, unlike vertices considered in this paper it has star product of homotopy structures, e.g. $\triangle \triangle \gamma \ast \triangle \triangle \gamma$. It will be interesting to see whether it is local or not.

What is known at this stage is that its reduction to AdS, $\Upsilon(\Omega, \Omega, C, C)$, should be local being a part of HS cubic vertex found in [36]. This does not necessarily imply that $\Upsilon(\omega, \omega, C, C)$ is local for general $\omega$.

Another key test for (non-)locality is the $\Upsilon(\omega, C, C, C)$ vertex. At the level of vertices considered in this paper the notions of space-time locality and spin locality from our analysis are equivalent. At higher orders however the two may essentially differ. Vertex $\Upsilon(\omega, C, C, C)$ in this context is of particular interest given the result of [14], where its AdS reduction was holographically analyzed for scalar quartic self-interaction. Interestingly, its space-time representation presumably is non-local but have non-locality of a specific form [32].

Last but not the least, it will be interesting to understand the role of PLT at higher orders. At the level of vertices considered in our paper PLT prescribes unique (up to local redefinitions) form of HS vertices. Nonetheless this theorem looks so far as a crude estimate on exponential behavior in perturbation theory. What it says is that the class of homotopies in the one-form sector of HS equations is respected provided the zero-form homotopy sector is appropriately shifted (4.29) and (4.33). The key observation was that the inner Klein operator present in HS equations breaks down the class of functions generated by star products along with homotopy action unless that breaking is taken into account by properly shifted homotopies in zero-form sector. Strikingly, this is what is needed to come up with local lower order vertices. PLT however says nothing about concrete values of homotopy coefficients in (4.29) other than single constraint (4.33). It will be interesting to see if PLT admits a generalization strong enough to fully control over admissible structure of higher order vertices.

The important current problem in HS theory is to understand to which extent it is local. If non-local then one should elaborate a proper substitute for the notion of locality (minimal non-locality), i.e. the admissible functional class that respects physical predictions be these HS correlation functions, black hole solutions, charges, etc. The study of systematic means for addressing these kind of questions is initiated in [40] and in this paper.

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Appendix

Here we analyse in some more detail the structure of $\Upsilon(\omega, C, C)$ for (4.45) with unconstrained coefficients $v_1$ and $v_2$ to show that such vertex is non-local unless (4.34) imposed. Straightforward calculation yields

$$d_\epsilon C + [\omega, C]_* = \Upsilon_{CC} + \Upsilon_{C\omega C} + \Upsilon_{CC\omega} + c.c.,$$

(A.1)

where

$$\Upsilon_{CC} = \eta \frac{\omega}{4!} C \ast C \ast C \ast X_{CC}, \quad \Upsilon_{C\omega} = \eta \frac{C \ast \omega \ast X_{CC}}{4!}, \quad \Upsilon_{C\omega C} = \eta \frac{C \ast \omega \ast C \ast C_{CC}}{4!},$$

(A.2)

$$X_{CC} = h_{p_1 p_2 p_3} \delta_{p_1 + p_2 + p_3, t} \gamma - (h_{p_1 p_2 p_3} - h_{p_1 + p_3, 2} - h_{p_2 + p_3, 2}) \delta_{p_1 + p_2 + p_3, t} \gamma,$$

(A.3)

$$X_{CC\omega} = (h_{p_1 p_2 p_3} - h_{p_1 + p_3, 2} - h_{p_2 + p_3, 2}) \delta_{p_1 + p_2 + p_3, t} \gamma,$$

(A.4)

$$X_{CC\omega} = (h_{p_1 p_2 p_3} - h_{p_1 + p_3, 2} - h_{p_2 + p_3, 2}) \delta_{p_1 + p_2 + p_3, t} \gamma,$$

(A.5)

To see if the r.h.s. of (A.1) is local or not, consider typical expression that appears there

$$C \ast C \ast = e^{i p_1 p_2},$$

(A.6)

where $a_{1,2}$, $b_{1,2}$ and $c_{1,2}$ are some numbers. We discard the $\omega$-factor and corresponding homotopy differentiation parameter $t = -i\partial^\omega$ as it does not affect locality being at most polynomial in $y$'s for a given spin. Now, $C \ast C$ itself contains non-local contribution

$$C \ast C \sim e^{i p_1 p_2}$$

(A.7)

which should be cancelled out by homotopy action (A.6). Performing star product in (A.6) using (3.18) and keeping track of the non-local part we find it to be

$$(C \ast C) e^{i (a_2 - a_1)(b_2 - b_1)(c_2 - c_1)} p_1^a p_2^b.$$  

(A.8)

Due to integration over simplex (3.18) which cuts off $\tau_1 + \tau_2 + \tau_3 = 1$, the exponential containing $p_1^a p_2^b$ cancels out iff

$$a_2 - a_1 = b_2 - b_1 = c_2 - c_1 = 1.$$  

(A.9)

It is easy to see now that all structures in (A.3)–(A.5) meet locality requirement (A.9) provided that (4.34) is true, i.e. the homotopy for $B$ is taken within the class of functions (4.33) specified for zero-forms. This result is anticipated being in accordance with PLT of [40] stating that the Pfaffian matrix of derivatives must be degenerate for the proper class of homotopies. Being a $2 \times 2$ matrix for the terms bilinear in $C$ this implies that it is zero.

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