Sharp Uncertainty Principles on Finsler Manifolds: 
The effect of curvature

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Abstract

In this paper we investigate the validity of uncertainty principles (Heisenberg-Pauli-Weyl inequality, Hardy inequality and their interpolation inequality) on non-compact, complete Finsler manifolds with vanishing mean covariation. The present study is highly influenced by the flag/Ricci curvature of the manifold:

• When the manifold has non-positive flag curvature, all three uncertainty principles hold with the same sharp constants as in their Euclidean counterparts. However, the existence of positive extremals associated to the sharp constants in the Heisenberg-Pauli-Weyl and interpolation inequalities imply the flatness of the manifold. Improved Hardy inequalities are also provided where the remainder terms are expressed by the flag curvature; in fact, we show that more powerful curvature implies more improved Hardy inequalities.

• When the manifold has non-negative Ricci curvature, the validity of Heisenberg-Pauli-Weyl and interpolation inequalities with their sharp Euclidean constants imply themselves the flatness of the manifold.

Although we are dealing with Finsler manifolds, our rigidity results are genuinely new also in the Riemannian context. Our approach exploits various elements from comparison geometry and fine, quantitative estimates on the Finsler-Laplace operator.

Keywords: uncertainty principles; Heisenberg-Pauli-Weyl inequality; Hardy inequality; interpolation inequality; Finsler manifold; sharp constant; curvature; rigidity.

MSC: 53C20, 53C60, 58J60.

1 Introduction and main results

1.1 Motivation

It is well known that the Heisenberg-Pauli-Weyl and Hardy inequalities in \( \mathbb{R}^n \) are endpoints of the interpolation inequality

\[
\left( \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^n} \frac{|u(x)|^{2p-2}}{|x|^{2q-2}} \, dx \right) \geq \frac{(n-q)^2}{p^2} \left( \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^q} \, dx \right)^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \quad (I)
\]

where \( n > 2 \) and \( p > 2 > q > 0 \). Indeed, on the one hand, if \( p \to 2 \) and \( q \to 0 \), then \( (I) \) becomes the Heisenberg-Pauli-Weyl inequality

\[
\left( \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^n} |x|^2 u(x)^2 \, dx \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} u(x)^2 \, dx \right)^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \quad (\text{HPW})
\]

where the constant \( \frac{n^2}{4} \) is sharp and the extremals are given (up to a constant) by the family of Gaussian functions \( u_\lambda(x) = e^{-\lambda|x|^2} \), \( \lambda > 0 \). As usual, by extremals we understand those functions for which the studied inequality (written with its optimal constant) becomes equality. On the other hand, when \( p \to 2 \) and \( q \to 2 \), then \( (I) \) reduces to the Hardy inequality

\[
\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u(x)^2}{|x|^2} \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \quad (H)
\]
where \((n-2)^2\) is also sharp, but there is no extremal function. Since \((\text{HPW})\) and \((\text{H})\) are sometimes called *uncertainty principles*, we shall use this notion for the aforementioned three inequalities.

Uncertainty principles have been widely investigated for the last decades in various contexts. In the classical Euclidean case, one may consult important contributions in Adimurthi, Chaudhuri and Ramaswamy [1], Barbatis, Filippas and Tertikas [3], Brezis and Vázquez [5], Caffarelli, Kohn and Nirenberg [7], Fefferman [13], Filippas and Tertikas [14], Ghoussoub and Moradifam [15, 16], Wang and Willem [27], and references therein.

Recently, certain uncertainty principles have been also studied in the context of *curved spaces*. As far as we know, the first study in this direction is due to Carro [8], where the Hardy inequality is studied on complete, non-compact Riemannian manifolds. As a source of inspiration, the latter work initiated a systematic study of the uncertainty principles on Riemannian manifolds by D’Ambrosio and Dipierro [9], do Carmo and Xia [10], Erb [11], Kombe and Özaydin [19, 20], Xia [29], Yang, Su and Kong [30], and on Finsler manifolds by Kristály and Ohta [18].

The purpose of the present paper is to point out the *influence of curvature* concerning the validity of sharp uncertainty principles on reversible Finsler manifolds with vanishing mean covariation. Although we are dealing with Finsler manifolds, we emphasize that our results are novel also on Riemannian manifolds. Moreover, readers familiar with Riemannian geometry rather than Finsler geometry may interpret the results within the Riemannian context by simply replacing a few notions as it will be shown later (see e.g. Remark 1.1). Our achievements can be roughly summarized as follows:

- When the Finsler manifold has *non-positive flag curvature*, we prove the validity of the uncertainty principles with the same sharp constants as in the Euclidean case. However, if one expects the *existence of extremals* with the same sharp Euclidean constants in the interpolation inequality or Heisenberg-Pauli-Weyl inequality, it turns out that the manifold is *flat*, see Theorems 1.1 & 1.3. Moreover, since no extremal is expected in the Hardy inequality, some improved forms are presented where the reminder terms involve the *curvature*. In fact, we show that more powerful curvature implies more improved Hardy inequalities, see Theorems 1.5 & 1.6. The sharpness of the constant in the Hardy inequality is also established.

- When the Finsler manifold has *non-negative Ricci curvature*, the validity of the interpolation inequality or Heisenberg-Pauli-Weyl inequality with their sharp Euclidean constants imply themselves the *flatness* of the manifold, see Theorems 1.2 & 1.4.

We emphasize that the nature/proof of the above results genuinely depends on the sign/size of the curvature where fine volume and Laplacian comparison arguments will be exploited.

### 1.2 Statement of main results

In the rest of this section, let \((M, F)\) be a complete, \(n\)–dimensional, reversible Finsler manifold without boundary. In order to present our results, we need some notations/notions; see Section 2 for the precise definitions. Namely, \(d_F : M \times M \to \mathbb{R}\) is the natural distance function generated by the Finsler metric \(F\), the function \(F^* : T^* M \to [0, \infty)\) is the polar transform of \(F\), \(Du(x) \in T^*_x M\) is the derivative of \(u\) at \(x \in M\), \(dV_F(x)\) is the Busemann-Hausdorff measure on \((M, F)\), and \(\text{Vol}_F(S)\) is the Finsler volume of a set \(S \in M\). Finally, \(K \leq c\) (resp. \(K = c\)) means that the flag curvature on \((M, F)\) is bounded from above by \(c \in \mathbb{R}\) (resp., is equal to \(c\)) for any choice of parameters, \(\text{Ric}_M \geq 0\) means that the Ricci curvature on \((M, F)\) is non-negative for every vector, and \(S = 0\) means that \((M, F)\) has vanishing mean covariation.

In the next subsections we state our main results together with several comments.
1.2.1 Interpolation inequality

Let $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that

$$0 < q < 2 < p \quad \text{and} \quad 2 < n < \frac{2(p - q)}{p - 2}. \quad (1.1)$$

For a given $x_0 \in M$, we consider the interpolation inequality

$$\left( \int_M F^*(x, Du(x))^2 dV_F(x) \right) \left( \int_M \frac{|u(x)|^{2p-2}}{d_F(x_0, x)^{2q-2}} dV_F(x) \right) \geq \frac{(n-q)^2}{p^2} \left( \int_M \frac{|u(x)|^p}{d_F(x_0, x)^q} dV_F(x) \right)^2, \quad \forall u \in C^\infty_0(M). \quad (\tilde{I})_{x_0}$$

Our first main result reads as follows.

**Theorem 1.1** [Interpolation inequality, non-positively curved case] Let $(M, F)$ be a complete, $n$–dimensional, reversible Finsler manifold with $S = 0$ and $K \leq 0$. Assume that the numbers $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ verify (1.1). Then the interpolation inequality $(\tilde{I})_{x_0}$ holds for every $x_0 \in M$. Moreover, we have:

(i) If $(M, F) = (\mathbb{R}^n, F)$ is a Minkowski space, then the constant $\frac{(n-q)^2}{p^2}$ is sharp and $w_\lambda(x) = \left( \lambda + F(x - x_0)^{2-q} \right)^{\frac{1}{2-p}} (\lambda > 0)$ is a class of extremal functions in $(\tilde{I})_{x_0}$ for every $x_0 \in M$.

(ii) If $(M, F)$ is a Berwald space of Hadamard-type, the following statements are equivalent:

(a) $\frac{(n-q)^2}{p^2}$ is sharp and there exists a positive extremal function in $(\tilde{I})_{x_0}$ for some $x_0 \in M$;

(b) $\frac{(n-q)^2}{p^2}$ is sharp and there exists a positive extremal function in $(\tilde{I})_{x_0}$ for every $x_0 \in M$;

(c) $(M, F)$ is isometric to an $n$–dimensional Minkowski space.

Note that Berwald spaces belong to the class of Finsler manifolds with $S = 0$. Furthermore, Riemannian manifolds and Minkowski spaces (i.e., smooth normed spaces) are Berwald spaces. As usual, a Finsler manifold is said to be of Hadamard-type if it is complete, simply connected with non-positive flag curvature.

The following result shows that the situation in the non-negatively curved case genuinely differs from its non-positively curved counterpart. Namely, we prove

**Theorem 1.2** [Interpolation inequality, non-negatively curved case] Let $(M, F)$ be a complete, $n$–dimensional, reversible Finsler manifold with $S = 0$ and $\text{Ric}_M \geq 0$. Assume that the numbers $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ verify (1.1). If the interpolation inequality $(\tilde{I})_{x_0}$ holds for some $x_0 \in M$, then $K = 0$.

In particular, if $(M, F)$ is a Berwald space, the following statements are equivalent:

(a) $(\tilde{I})_{x_0}$ holds for some $x_0 \in M$;

(b) $(\tilde{I})_{x_0}$ holds for every $x_0 \in M$;

(c) $(M, F)$ is isometric to an $n$–dimensional Minkowski space.

**Remark 1.1** The reader familiar with Riemannian geometry can read the above results by changing the expressions "Berwald space" to "Riemannian manifold", the "flag curvature" to "sectional curvature", and "Minkowski space" to "Euclidean space", respectively.

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1.2.2 Heisenberg-Pauli-Weyl inequality

In this subsection we are dealing with the Heisenberg-Pauli-Weyl uncertainty principle on Finsler manifolds which has the form

\[ \left( \int_M F^*(x, Du(x))^2 dV_F(x) \right) \left( \int_M d_F(x_0, x)^2 u(x)^2 dV_F(x) \right) \geq \frac{n^2}{4} \left( \int_M u(x)^2 dV_F(x) \right)^2, \forall u \in C_0^\infty(M). \]  

(\text{HPW})_{x_0}

**Theorem 1.3** [Heisenberg-Pauli-Weyl inequality, non-positively curved case] Let \((M, F)\) be a complete, \(n\)-dimensional, reversible Finsler manifold with \(S = 0\) and \(K \leq 0\). Then the Heisenberg-Pauli-Weyl inequality \((\text{HPW})_{x_0}\) holds for every \(x_0 \in M\). Moreover, we have:

(i) If \((M, F) = (\mathbb{R}^n, F)\) is a Minkowski space, then \(\frac{n^2}{4}\) is sharp and \(u_\lambda(x) = e^{-\lambda F(x-x_0)^2} (\lambda > 0)\) is a class of extremal functions in \((\text{HPW})_{x_0}\) for every \(x_0 \in M\).

(ii) If \((M, F)\) is a Berwald space of Hadamard-type, the following statements are equivalent:

(a) \(\frac{n^2}{4}\) is sharp and there exists a positive extremal function in \((\text{HPW})_{x_0}\) for some \(x_0 \in M\);

(b) \(\frac{n^2}{4}\) is sharp and there exists a positive extremal function in \((\text{HPW})_{x_0}\) for every \(x_0 \in M\);

(c) \((M, F)\) is isometric to an \(n\)-dimensional Minkowski space.

**Remark 1.2** Theorem 1.3(ii) shows that the constant \(\frac{n^2}{4}\) is not sharp in the Heisenberg-Pauli-Weyl inequality \((\text{HPW})_0\) on the hyperbolic space \(\mathbb{H}^n\), pointing out that the statement of Theorem 4.2 in Kombe and Özaydin \[20\] is not correct. In fact, we provide further arguments to support our point of view; we first point out the computational mistake in the proof of \[20\, Theorem 4.2\] and then we state the following modified sharp Heisenberg-Pauli-Weyl inequality on \(\mathbb{H}^n\) (in the spirit of Erb \[11\]):

\[ \left( \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n} \right) \left( \int_{\mathbb{H}^n} d^2 u^2 dV_{\mathbb{H}^n} \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{H}^n} \left( 1 + \frac{n-1}{n} (d \coth d - 1) \right) u^2 dV_{\mathbb{H}^n} \right)^2, \forall u \in C_0^\infty(\mathbb{H}^n), \]

where \(\frac{n^2}{4}\) is sharp and equality holds for the hyperbolic Gaussian family of functions \(u(x) = e^{-\alpha d^2}\), \((\alpha > 0)\). Hereafter, \(\nabla_{\mathbb{H}^n}\) and \(d = d_{\mathbb{H}^n}(0, x)\) denote the hyperbolic gradient and the hyperbolic distance between \(0\) and \(x\) in the Poincaré ball model, respectively. Details are presented in \[13\].

**Theorem 1.4** [Heisenberg-Pauli-Weyl inequality, non-negatively curved case] Let \((M, F)\) be a complete, \(n\)-dimensional, reversible Finsler manifold with \(S = 0\) and \(\text{Ric}_M \geq 0\). If the Heisenberg-Pauli-Weyl inequality \((\text{HPW})_{x_0}\) holds for some \(x_0 \in M\), then \(K = 0\).

In particular, if \((M, F)\) is a Berwald space, the following statements are equivalent:

(a) \((\text{HPW})_{x_0}\) holds for some \(x_0 \in M\);

(b) \((\text{HPW})_{x_0}\) holds for every \(x_0 \in M\);

(c) \((M, F)\) is isometric to an \(n\)-dimensional Minkowski space.

**Remark 1.3** Although the statements of the latter two results are similar to Theorems 1.1 and 1.2 there are some differences in their proofs. These facts arise basically from the shape of extremals in the Minkowski case which require different comparison approaches in the generic Finsler case.
1.2.3 Hardy inequality: improvements by curvature

For every $c \leq 0$ we consider the function $c t_c : (0, \infty) \to \mathbb{R}$ defined by

$$c t_c(\rho) = \begin{cases} \frac{1}{\rho} & \text{if } c = 0, \\ \frac{1}{\sqrt{|c|}} \coth(\sqrt{|c|} \rho) & \text{if } c < 0. \end{cases}$$

Let also $D_c : [0, \infty) \to \mathbb{R}$ defined by $D_c(0) = 0$ and $D_c(\rho) = \rho c t_c(\rho) - 1$, $\rho > 0$. It is clear that $D_c \geq 0$.

**Theorem 1.5** [Quantitative Hardy inequality via curvature] Let $(M, F)$ be a complete, $n$–dimensional ($n \geq 3$), reversible Finsler manifold with $S = 0$ and $K \leq c \leq 0$. Then, for every $x_0 \in M$ and $u \in C_0^\infty(M)$ we have

$$\int_M F^*(x, Du(x))^2 dV_F(x) \geq \frac{(n-2)^2}{4} \int_M \left( 1 + \frac{2(n-1)}{n-2} D_c(d_F(x_0, x)) \right) \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x).$$

In addition, if $(M, F)$ is simply connected, the constant $\frac{(n-2)^2}{4}$ is sharp.

**Remark 1.4** (i) When $(M, F)$ is a Minkowski space (in particular, the Euclidean space), Theorem 1.5 reduces to the result of Van Schaftingen [26, Proposition 7.5]. Note that in this case $K = 0$.

(ii) The proof of Theorem 1.5 shows in fact that

$$\frac{(n-2)^2}{4} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\int_M F^*(x, Du(x))^2 dV_F(x)}{\int_M \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x)}.$$

The latter relation has been deduced recently for the hyperbolic spaces by Kombe and Özaydin [19, 20] and more generally, for Riemannian manifolds with negative sectional curvature by Yang, Su and Kong [30]. None of these results give a quantitative version of the Hardy inequality.

Theorem 1.5 shows that more curvature implies more improved Hardy inequality:

**Corollary 1.1** [Improved Hardy inequality via curvature] Under the assumptions of Theorem 1.5, for every $x_0 \in M$ we have

$$\int_M F^*(x, Du(x))^2 dV_F(x) \geq \frac{(n-2)^2}{4} \int_M \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x) + \frac{3|c|(n-1)(n-2)}{2} \int_M \frac{u(x)^2}{\pi^2 + |c|d_F(x_0, x)^2} dV_F(x), \ \forall u \in C_0^\infty(M).$$

**Theorem 1.6** [Double improved Hardy inequality via curvature] Let $\Omega$ be a bounded open domain with smooth boundary in a complete, $n$–dimensional ($n \geq 3$), reversible Finsler manifold $(M, F)$ with $S = 0$ and $K \leq c \leq 0$. If $x_0 \in \Omega$ and $R > \sup_{x \in \Omega} d_F(x, x_0)$, then for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega F^*(x, Du(x))^2 dV_F(x) \geq \frac{(n-2)^2}{4} \int_\Omega \left( 1 + \frac{2(n-1)}{n-2} D_c(d_F(x_0, x)) \right) \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x) + \frac{l_{F^*}}{4} R_\Omega,$$

where $l_{F^*}$ denotes the uniformity constant of $F^*$ and

$$R_\Omega = \int_\Omega \left( 1 + 2(n-1) \ln \left( \frac{eR}{d_F(x_0, x)} \right) D_c(d_F(x_0, x)) \right) \frac{u(x)^2}{d_F(x_0, x)^2 \ln^2 \left( \frac{eR}{d_F(x_0, x)} \right)} dV_F(x).$$
Remark 1.5  (i) The uniformity constant $l_{F^*}$ of $F^*$ on $M$ measures the deviation of $F^*$ (and $F$) from Riemannian structures, see Egloff \[12\] and Ohta \[21\]. In fact, $l_{F^*} \leq 1$ and $l_{F^*} = 1$ if and only if $(M, F)$ is Riemannian; see Section 2 for details.

(ii) In the limiting case when $c = 0$ (thus $D_c(\rho) = D_0(\rho) = 0$ for every $\rho \geq 0$), the inequality in Theorem 1.6 takes the more familiar form

$$
\int_\Omega F^*(x, Du(x))^2 dV_F(x) \geq \frac{(n - 2)^2}{4} \int_\Omega \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x) + \frac{l_{F^*}}{4} \int_\Omega d_F(x_0, x)^2 \ln^2 \left( \frac{c_R}{d_F(x_0, x)} \right) dV_F(x),
$$

see [1] [14] in the Euclidean case.

Remark 1.6  It seems similar rigidity results for the Hardy inequalities as in Theorems 1.2 and 1.4 cannot be established on non-negatively curved spaces. In the proof of Theorems 1.2 and 1.4 the existence of extremals in the flat (Minkowski) case is crucial which fails in the case of Hardy inequalities.

In order to prove the above results, we first recall in Section 2 some elements from Finsler geometry as curvature notions, volume comparison and differentials on Finsler manifolds, and basic results about the Finsler-Laplace operator.

2 Preliminaries

2.1. Finsler manifolds. Let $M$ be a connected $n$-dimensional $C^\infty$ manifold and $TM = \bigcup_{x \in M} T_x M$ be its tangent bundle. The pair $(M, F)$ is a reversible Finsler manifold if the continuous function $F : TM \to [0, \infty)$ satisfies the conditions

(a) $F \in C^\infty(TM \setminus \{0\})$;

(b) $F(x, ty) = t F(x, y)$ for all $t \in \mathbb{R}$ and $(x, y) \in TM$;

(c) $g_{ij}(x, y) := \frac{1}{2} F^2(x, y) y^i y^j$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

If $g_{ij}(x) = g_{ij}(x, y)$ is independent of $y$ then $(M, F)$ is called Riemannian manifold. A Minkowski space consists of a finite dimensional vector space $V$ and a Minkowski norm which induces a Finsler metric on $V$ by translation, i.e., $F(x, y)$ is independent on the base point $x$; in such cases we often write $F(y)$ instead of $F(x, y)$. While there is a unique Euclidean space (up to isometry), there are infinitely many (isometrically different) Minkowski spaces. A Finsler manifold $(M, F)$ is a locally Minkowski space if there exists a local coordinate system $(x^i)$ on $M$ with induced tangent space coordinates $(y^i)$ such that $F$ depends only on $y = y^i \partial / \partial x^i$ and not on $x$.

We consider the polar transform of $F$, defined for every $(x, \alpha) \in T^* M$ by

$$
F^*(x, \alpha) = \sup_{y \in T_x M \setminus \{0\}} \frac{\alpha(y)}{F(x, y)}. \tag{2.1}
$$

Note that for every $x \in M$, the function $F^*(x, \cdot)$ is a Minkowski norm on $T^*_x M$. Since $F^*(x, \cdot)^2$ is twice differentiable on $T^* M \setminus \{0\}$, we consider the matrix

$$
g_{ij}^*(x, \alpha) := \frac{1}{2} F^*(x, \alpha)^2 |_{\alpha^i \alpha^j}
$$

for every $\alpha = \sum_{i=1}^n \alpha^i dx^i \in T^* M \setminus \{0\}$ in a local coordinate system $(x^i)$. The quantity

$$
l_{F^*} = \inf_{x \in M} \inf_{\alpha, \beta \in T^*_x M \setminus \{0\}} \frac{g_{ij}^*(x, \alpha)(\beta, \beta)}{F^*(x, \beta)^2}
$$

is called the uniformity constant of $F^*$ on $M$.
is the \textit{uniformity constant of }$F^*$ \textit{which measures how far }$F^*$ \textit{and }$F$ \textit{are from Riemannian structures. Indeed, one can see that }$l_{F^*} \leq 1,$ \textit{and }$l_{F^*} = 1$ \textit{if and only if }$(M, F)$ \textit{is a Riemannian manifold, see Eglöf [12] p. 330 and Ohta [21] Proposition 4.1].

2.2. Chern connection and Berwald spaces. Let $\pi^*TM$ be the pull-back bundle of the tangent bundle $TM$ generated by the natural projection $\pi : TM \setminus \{0\} \rightarrow M$, see Bao, Chern and Shen [2] p. 28. The vectors of the pull-back bundle $\pi^*TM$ are denoted by $(v; w)$ with $(x, y) = v \in TM \setminus \{0\}$ and $w \in T_xM$. For simplicity, let $\partial_t|_v = (v; \partial/\partial x^i)|_x$ be the natural local basis for $\pi^*TM$, where $v \in T_xM$. One can introduce on $\pi^*TM$ the fundamental tensor $g$ and Cartan tensor $A$ by

$$g^v := g(\partial_t|_v, \partial_t|_v) = g_{ij}(x, y) \quad \text{and} \quad A^v := A(\partial_t|_v, \partial_t|_v, \partial_t|_v) = \frac{F(x, y)}{2} \left(\frac{F(x, y)^2}{2}\right)_{y, y, y_k},$$

respectively, where $v = \pi^*(\partial/\partial x^i)|_x$. Unlike the Levi-Civita connection in the Riemannian case, there is no unique natural connection in the Finsler geometry. Among these connections on the pull-back bundle $\pi^*TM$, we choose a torsion free and almost metric-compatible linear connection on $\pi^*TM$, the so-called Chern connection, see Bao, Chern and Shen [2] Theorem 2.4.1. The coefficients of the Chern connection are denoted by $\Gamma^i_{jk}$, which are instead of the well known Christoffel symbols from Riemannian geometry. A Finsler manifold is of \textit{Berwald type} if the coefficients $\Gamma^i_{jk}(x, y)$ in natural coordinates are independent of $y$. It is clear that Riemannian manifolds and (locally) Minkowski spaces are Berwald spaces. The Chern connection induces in a natural manner on $\pi^*TM$ the \textit{curvature tensor} $R$, see Bao, Chern and Shen [2] Chapter 3]. By means of the connection, we also have the \textit{covariant derivative} $D_vu$ of a vector field $u$ in the direction $v \in T_xM$. Note that $v \mapsto D_vu$ is not linear. A vector field $u = u(t)$ along a curve $\sigma$ is parallel if $D_\sigma u = 0$. A $C^\infty$ curve $\sigma : [0, a] \rightarrow M$ is a \textit{geodesic} if $D_\sigma \dot{\sigma} = 0$. Geodesics are considered to be parametrized proportionally to arc-length. The Finsler manifold is \textit{complete} if every geodesic segment can be extended to $\mathbb{R}$.

2.3. Curvatures. Let $u, v \in T_xM$ be two non-collinear vectors and $S = \text{span}\{u, v\} \subset T_xM$. By means of the curvature tensor $R$, the \textit{flag curvature} of the flag $\{S, v\}$ is defined by

$$K(S; v) = \frac{g^v(R(U, V)V, U)}{g^v(V, V)g^v(U, U) - g^v(U, V)^2},$$

where $U = (v; u), V = (v; v) \in \pi^*TM$. If for some $c \in \mathbb{R}$ we have $K(S; v) \leq c$ (resp., $K(S; v) = c$) for every choice of $U$ and $V$, we shall write $K \leq c$ (resp., $K = c$). In particular, if $K \leq 0$ then we say that $(M, F)$ \textit{has non-positive flag curvature}. If $(M, F)$ is Riemannian, the flag curvature reduces to the well known sectional curvature.

Let $v \in T_xM$ be such that $F(x, v) = 1$ and let $\{e_i\}_{i=1,...,n}$ with $e_n = v$ be a basis for $T_xM$ such that $\{\langle v; e_i \rangle\}_{i=1,...,n}$ is an orthonormal basis for $\pi^*TM$. Let $S_i = \text{span}\{e_i, v\}, i = 1, ..., n-1$. The \textit{Ricci curvature} $\text{Ric} : TM \rightarrow \mathbb{R}$ is defined by

$$\text{Ric}(v) = \sum_{i=1}^{n-1} K(S_i; v)F(v)^2.$$ 

We say that the \textit{Ricci curvature is non-negative} on $(M, F)$, if $\text{Ric}(v) \geq 0$ for every $v \in TM$, and we denote as $\text{Ric}_M \geq 0$.

2.4. Metric function and volume form. Let $\sigma : [0, r] \rightarrow M$ be a piecewise $C^\infty$ curve. The value $L_F(\sigma) = \int_0^r F(\sigma(t), \dot{\sigma}(t)) \, dt$ denotes the \textit{integral length} of $\sigma$. For $x_1, x_2 \in M$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise $C^\infty$ curves $\sigma : [0, r] \rightarrow M$ such that $\sigma(0) = x_1$ and $\sigma(r) = x_2$. Define the \textit{distance function} $d_F : M \times M \rightarrow [0, \infty)$ by

$$d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1, x_2)} L_F(\sigma).$$

(2.4)
Clearly, $d_F$ verifies the properties of the metric (i.e., $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, $d_F$ is symmetric, and it verifies the triangle inequality). The open metric ball with center $x_0 \in M$ and radius $\rho > 0$ is defined by $B(x_0, \rho) = \{x \in M : d_F(x_0, x) < \rho\}$. In particular, when $(M, F) = (\mathbb{R}^n, F)$ is a Minkowski space, one has $d_F(x_1, x_2) = F(x_2 - x_1).

Let $\{\partial/\partial x^i\}_{i=1,\ldots,n}$ be a local basis for the tangent bundle $TM$, and $\{dx^i\}_{i=1,\ldots,n}$ be its dual basis for $T^*M$. Let $B_x(1) = \{y = (y^i) : F(x, y^i \partial/\partial x^i) < 1\}$ be the unit tangent ball at $T_xM$. The Busemann-Hausdorff volume form $dV_F$ on $(M, F)$ is defined by

$$dV_F(x) = \sigma_F(x)dx^1 \wedge \ldots \wedge dx^n,$$

(2.5)

where $\sigma_F(x) = \omega_n \omega_{B_x(1)}$. Hereafter, $\text{Vol}(S)$ denotes the Euclidean volume of the set $S \subset \mathbb{R}^N$. The Finslerian-volume of a bounded open set $S \subset M$ is $\text{Vol}_F(S) = \int_S dV_F(x) = \text{Haus}_{d_F}(S)$, where $\text{Haus}_{d_F}(S)$ is the Hausdorff measure of $S$ with respect to the metric $d_F$. In general, one has for every $x \in M$ that

$$\lim_{\rho \to 0^+} \frac{\text{Vol}_F(B(x, \rho))}{\omega_n \rho^n} = 1.$$

(2.6)

When $(\mathbb{R}^n, F)$ is a Minkowski space, then on account of (2.5), $\text{Vol}_F(B(x, \rho)) = \omega_n \rho^n$ for every $\rho > 0$ and $x \in \mathbb{R}^n$.

2.5. Finsler-Laplacian operator. The Legendre transform $J^*: T^*M \to TM$ associates to each element $\alpha \in T^*_xM$ the unique maximizer on $T_xM$ of the map $y \mapsto \alpha(y) - \frac{1}{2} F^2(x, y)$. This element can also be interpreted as the unique vector $y \in T_xM$ with the properties

$$F(x, y) = F^*(x, \alpha) \quad \text{and} \quad \alpha(y) = F(x, y)F^*(x, \alpha).$$

(2.7)

In particular, if $\alpha = \sum_{i=1}^n \alpha^i dx^i \in T^*_xM$ and $\xi = \sum_{i=1}^n \xi^i (\partial/\partial x^i) \in T_xM$, one has

$$J^*(x, \alpha) = \sum_{i=1}^n \frac{\partial}{\partial \alpha_i} \left( \frac{1}{2} F^*(x, \alpha)^2 \right) \frac{\partial}{\partial x^i}.$$  

(2.8)

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The gradient of $u$ is defined by $\nabla u(x) = J^*(x, Du(x))$, where $Du(x) \in T^*_xM$ denotes the (distributional) derivative of $u$ at $x \in M$. In general, $u \mapsto \nabla u$ is not linear. If $x_0 \in M$ is fixed, then due to Ohta and Sturm [22] and relation (2.7), one has

$$F(x, \nabla d_F(x_0, x)) = F^*(x, Dd_F(x_0, x)) = Dd_F(x_0, x)(\nabla d_F(x_0, x)) = 1 \quad \text{for a.e. } x \in M.$$

(2.9)

In fact, relations from (2.9) are valid for every $x \in M \setminus \{x_0 \cup \text{Cut}(x_0)\}$, where $\text{Cut}(x_0)$ denotes the cut locus of $x_0$, see Bao, Chern and Shen [2, Chapter 8]. Note that $\text{Cut}(x_0)$ has null Lebesgue (thus Hausdorff) measure for every $x_0 \in M$.

Let $X$ be a vector field on $M$. In a local coordinate system $(x^i)$, on account of (2.5), the divergence is defined by $\text{div}(X) = \frac{1}{\sigma_F} \frac{\partial}{\partial x^i} (\sigma_F X^i)$. The Finsler-Laplace operator $\Delta u = \text{div}(\nabla u)$ acts on $W^{1,2}_{\text{loc}}(M)$ and for every $v \in C_0^\infty(M)$,

$$\int_M v \Delta u dV_F(x) = - \int_M Dv(\nabla u) dV_F(x),$$

(2.10)

see Belloni, Kawohl and Juutinen [4], Ohta and Sturm [22] and Shen [25].
2.6. Comparison principles. Let \( \{e_i\}_{i=1,...,n} \) be a basis for \( T_xM \) and \( g^{ij}_v = g^v(e_i,e_j) \). The mean distortion \( \mu : TM \setminus \{0\} \to (0,\infty) \) is defined by \( \mu(v) = \sqrt{\det(g^{ij}_v)} \). The mean covariation \( S : TM \setminus \{0\} \to \mathbb{R} \) is defined by

\[
S(x,v) = \frac{d}{dt}(\ln(\mu(\tilde{\sigma}_v(t))))|_{t=0},
\]

where \( \tilde{\sigma}_v \) is the geodesic such that \( \tilde{\sigma}_v(0) = x \) and \( \tilde{\sigma}_v(0) = v \). We say that \( (M,F) \) has vanishing mean covariation if \( S(x,v) = 0 \) for every \( (x,v) \in TM \), and we denote by \( S = 0 \). We notice that any Berwald space has vanishing mean covariation, see Shen [23].

**Theorem 2.1** (see [28] Theorem 5.1) Let \( (M,F) \) be an \( n \)-dimensional Finsler manifold with \( S = 0 \) and \( K \leq c \leq 0 \), and let \( x_0 \in M \) be fixed. Then, we have

\[
\Delta d_F(x_0,x) \geq (n-1)c \cdot (d_F(x_0,x)) \ 	ext{for a.e.} \ x \in M.
\]

In the proof of our results the volume comparison will play a crucial role. On account of Shen [23], Wu and Xin [28] Theorems 6.1 & 6.3 and Zhao and Shen [31], we adapt the following version:

**Theorem 2.2** [Volume comparison] Let \( (M,F) \) be a complete, \( n \)-dimensional, reversible Finsler manifold with \( S = 0 \). Then, the following statements hold.

(a) If \( K \leq 0 \) and \( (M,F) \) is simply connected, the function \( \rho \mapsto \frac{\Vol_F(B(x,\rho))}{\rho^n} \) is non-decreasing, \( \rho > 0 \). In particular, from (2.6) we have

\[
\Vol_F(B(x,\rho)) \geq \omega_n \rho^n \ 	ext{for all} \ x \in M \text{ and} \ \rho > 0.
\]

If equality holds in (2.11), then \( K = 0 \).

(b) If \( \Ric_M \geq 0 \), the function \( \rho \mapsto \frac{\Vol_F(B(x,\rho))}{\rho^n} \) is non-increasing, \( \rho > 0 \). In particular, from (2.6) we have

\[
\Vol_F(B(x,\rho)) \leq \omega_n \rho^n \ 	ext{for all} \ x \in M \text{ and} \ \rho > 0.
\]

If equality holds in (2.12), then \( K = 0 \).

3 Interpolation inequalities on Finsler manifolds

3.1 Non-positively curved case: proof of Theorem 1.1

**Proof of Theorem 1.1** (first part). Way may apply Theorem 2.1 (with \( c = 0 \), since \( K \leq 0 \)), obtaining

\[
d_F(x_0,x) \Delta d_F(x_0,x) \geq n-1 \ 	ext{for a.e.} \ x \in M.
\]

Let us fix \( u \in C^\infty_0(M) \). Thus, on account of (2.10), one has

\[
\int_M \frac{|u(x)|^p}{d_F(x_0,x)^q} dV_F(x) \leq \frac{1}{n-1} \int_M \frac{|u(x)|^p}{d_F(x_0,x)^{q-1}} \Delta d_F(x_0,x) dV_F(x) \quad (3.1)
= -\frac{1}{n-1} \int_M D \left( \frac{|u(x)|^p}{d_F(x_0,x)^{q-1}} \right) (\nabla d_F(x_0,x)) dV_F(x)
= -\frac{p}{n-1} \int_M \frac{|u(x)|^{p-2}u(x)}{d_F(x_0,x)^{q-1}} D(|u(x)|) (\nabla d_F(x_0,x)) dV_F(x)
+ \frac{q-1}{n-1} \int_M \frac{|u(x)|^p}{d_F(x_0,x)^q} Dd_F(x_0,x) (\nabla d_F(x_0,x)) dV_F(x).
\]
Since $Dd_F(x_0, x)(\nabla d_F(x_0, x)) = 1$ (see (2.3)), a reorganization of the above estimate implies that

$$\frac{n - q}{p} \int_M \frac{|u(x)|^p}{d_F(x_0, x)^q} dV_F(x) \leq - \int_M \frac{|u(x)|^{p-2}u(x)}{d_F(x_0, x)^{q-1}} D(|u(x)|) \left( \nabla d_F(x_0, x) \right) dV_F(x).$$  \hspace{1cm} (3.2)

Now, since $F^*$ is reversible, from (2.1) and (2.5), we obtain for a.e. $x \in M$ that

$$|D(|u(x)|) \left( \nabla d_F(x_0, x) \right)| \leq F^*(x, D(|u(x)|))F(x, \nabla d_F(x_0, x)) = F^*(x, Du(x)).$$

Consequently, by (3.2) we have

$$\frac{n - q}{p} \int_M \frac{|u(x)|^p}{d_F(x_0, x)^q} dV_F(x) \leq \int_M F^*(x, Du(x)) \left( \frac{|u(x)|^{p-1}}{d_F(x_0, x)^{q-1}} \right) dV_F(x).$$  \hspace{1cm} (3.3)

By applying the Schwartz inequality in (3.3), it yields

$$\frac{(n - q)^2}{p^2} \left( \int_M \frac{|u(x)|^p}{d_F(x_0, x)^q} dV_F(x) \right)^2 \leq \left( \int_M F^*(x, Du(x))^2 dV_F(x) \right) \left( \int_M \frac{|u(x)|^{2p-2}}{d_F(x_0, x)^{2q-2}} dV_F(x) \right),$$

which is precisely the inequality $(\mathbf{I})_{x_0}$, concluding the proof of the first part.

**Proof of Theorem 1.1 (i).** Let $x_0 \in M$ be fixed. Since $(M, F) = (\mathbb{R}^n, F)$ is a Minkowski space (thus $S = 0$ and $K = 0$), on account of the first part, inequality $(\mathbf{I})_{x_0}$ holds and it takes the form

$$\left( \int_{\mathbb{R}^n} F^*(Dw(x))^2 dx \right) \left( \int_{\mathbb{R}^n} \frac{|w(x)|^{2p-2}}{F(x - x_0)^{2q-2}} dx \right) \geq \frac{(n - q)^2}{p^2} \left( \int_{\mathbb{R}^n} \frac{|w(x)|^p}{F(x - x_0)^q} dx \right)^2, \forall w \in C_0^\infty(\mathbb{R}^n).$$

We show that the constant $\frac{(n - q)^2}{p^2}$ is sharp in the above inequality and a whole class of minimizers is provided by

$$w_\lambda(x) = (\lambda + F(x - x_0)^{2-q})^{\frac{1}{2-q}}, \lambda > 0. \hspace{1cm} (3.4)$$

Note that the function $w_\lambda$ can be approximated by smooth functions with compact support, since $p > 2 > q$. In order to prove the claim, we may assume that $x_0 = 0$. We define three functions $P, Q, R : (0, \infty) \rightarrow \mathbb{R}$ by

$$P(\lambda) = \int_{\mathbb{R}^n} \frac{|w_\lambda(x)|^p}{F(x)^q} dx, \; Q(\lambda) = \int_{\mathbb{R}^n} F^*(Dw_\lambda(x))^2 dx, \; R(\lambda) = \int_{\mathbb{R}^n} \frac{|w_\lambda(x)|^{2p-2}}{F(x)^{2q-2}} dx. \hspace{1cm} (3.5)$$

We first state that the functions $P, Q$ and $R$ are well defined. By applying the layer cake representation and a change of variable, it yields that

$$P(\lambda) = \int_{\mathbb{R}^n} \frac{|w_\lambda(x)|^p}{F(x)^q} dx = \int_{\mathbb{R}^n} \frac{(\lambda + F(x)^{2-q})^{\frac{p}{2-q}}}{F(x)^q} dx$$

$$= \int_0^\infty \text{Vol} \left\{ x \in \mathbb{R}^n : \frac{(\lambda + F(x)^{2-q})^{\frac{p}{2-q}}}{F(x)^q} > t \right\} dt \left( t = \frac{(\lambda + \rho^{2-q})^{\frac{p}{2-q}}}{\rho^q} \right)$$

$$= \omega_n \int_0^\infty \rho^n h(\lambda, \rho) d\rho, \hspace{1cm} (3.6)$$
where \( h : (0, \infty)^2 \to \mathbb{R} \) is given by

\[
h(\lambda, \rho) = \frac{(\lambda + \rho^2 - q)^{2p-2}}{\rho^{q+1}} \left( 2\rho^{q+2} - q - q\lambda \right).
\]  
(3.7)

On account of (1.1), the improper integral in (3.6) is convergent:
- at 0, since \( n + (-q - 1) + 1 = n - q > 0 \), and
- at \( \infty \), since

\[
n + (2 - q)\frac{2p - 2}{2 - p} - (q + 1) + (2 - q) + 1 = n - \frac{2p - q}{p - 2} < 0.
\]

Thus, the function \( P \) is well defined. Furthermore, (2.9) yields that

\[
Q(\lambda) = \frac{(2 - q)^2}{(p - 2)^2} R(\lambda) = \frac{(2 - q)^2}{(p - 2)^2} \omega_n \int_0^\infty \rho^n g(\lambda, \rho) d\rho,
\]  
(3.8)

where

\[
g(\lambda, \rho) = \frac{(\lambda + \rho^2 - q)^{2p-4}}{\rho^{2q-1}} \left( \rho^{2q-2} - \left(2p - 2\right)\left(2 - q\right) + 2(q - 1)\lambda \right).
\]

The relations in (1.1) imply again the convergence of the integral in \( Q \). Moreover, integrating by parts, it yields that

\[
P(\lambda) = \omega_n \int_0^\infty \rho^n h(\lambda, \rho) d\rho = \frac{p(2 - q)}{(n - q)(p - 2)} \int_0^\infty \rho^n g(\lambda, \rho) d\rho = \frac{p(2 - q)}{(n - q)(p - 2)} R(\lambda).
\]

Consequently, for every \( \lambda > 0 \) we have

\[
\frac{Q(\lambda) R(\lambda)}{P(\lambda)^2} = \frac{(n - q)^2}{p^2},
\]  
(3.9)

which proves (i).

**Proof of Theorem (ii).** Since \((M, F)\) is a Berwald space of Hadamard type, then \( S = 0 \); thus, the first part of the theorem implies that (1.1) holds for every \( x_0 \in M \). Clearly, one has (b) \( \Rightarrow \) (a).

**Proof of (a) \( \Rightarrow \) (c).** According to the hypothesis, \( \frac{(n-q)^2}{p^2} \) is sharp and there exists a positive extremal function \( w_0 \) in (1.1) for some \( x_0 \in M \). In particular, in relation (3.11) we should have the equality

\[
\int_M \frac{w_0(x)^p}{d_F(x_0, x)^q} \omega_F(x)^q = \frac{1}{n - 1} \int_M \frac{w_0(x)^p}{d_F(x_0, x)^q} \Delta d_F(x_0, x) \omega_F(x).
\]  
(3.10)

Since \( w_0 > 0 \) and \( d_F(x_0, x) \Delta d_F(x_0, x) \geq n - 1 \) for every \( x \in M \setminus \{x_0\} \) (note that \( \text{Cut}(x_0) = \emptyset \)), relation (3.10) implies that

\[
d_F(x_0, x) \Delta d_F(x_0, x) = n - 1 \quad \text{for} \quad x \in M \setminus \{x_0\}.
\]

On account of (2.9), for \( x \in M \setminus \{x_0\} \) we have

\[
\Delta(d_F(x_0, x)^2) = 2 \text{div}(d_F(x_0, x) \nabla d_F(x_0, x)) = 2[Dd_F(x_0, x)(\nabla d_F(x_0, x)) + d_F(x_0, x) \Delta d_F(x_0, x)]
\]

\[
= 2[1 + d_F(x_0, x) \Delta d_F(x_0, x)],
\]

thus,

\[
\Delta(d_F(x_0, x)^2) = 2n \quad \text{for} \quad x \in M \setminus \{x_0\}.
\]  
(3.11)
Let us fix $\rho > 0$ arbitrarily. Note that the unit outward pointing normal vector to the sphere $S(x_0, \rho) = \partial B(x_0, \rho) = \{x \in M : d_F(x_0, x) = \rho \}$ is given by $n = \nabla d_F(x_0, x)$, $x \in S(x_0, \rho)$. Let us denote by $d\varsigma_F(x)$ the volume form on $S(x_0, \rho)$ induced from $dV_F(x)$. By applying Stokes' formula (see [23], [28, Lemma 3.2]) and the fact that $g_n(n) = F(x, n)^2 = F(x, \nabla d_F(x_0, x))^2 = 1$ (see (2.9)), on account of relation (3.11) we have that

$$2n \text{Vol}_F(B(x_0, \rho)) = \int_{B(x_0, \rho)} \Delta(d_F(x_0, x)^2)dV_F(x) = \int_{B(x_0, \rho)} \text{div}(\nabla(d_F(x_0, x)^2))dV_F(x)$$

$$= \int_{S(x_0, \rho)} g_n(n, \nabla d_F(x_0, x))d\varsigma_F(x)$$

$$= 2 \int_{S(x_0, \rho)} d_F(x_0, x)g_n(n, \nabla d_F(x_0, x))d\varsigma_F(x)$$

$$= 2\rho \int_{S(x_0, \rho)} g_n(n, n)d\varsigma_F(x) = 2\rho \int_{S(x_0, \rho)} d\varsigma_F(x)$$

$$= 2\rho A_F(S(x_0, \rho)), $$

where

$$A_F(S(x_0, \rho)) = \lim_{\varepsilon \to 0^+} \frac{\text{Vol}_F(B(x_0, \rho + \varepsilon)) - \text{Vol}_F(B(x_0, \rho))}{\varepsilon} := \frac{d}{d\rho} \text{Vol}_F(B(x_0, \rho))$$

is the $F$-surface area of $S(x_0, \rho)$. Thus, the above relations imply that

$$\frac{d}{d\rho} \text{Vol}_F(B(x_0, \rho)) \frac{\text{Vol}_F(B(x_0, \rho))}{\rho^n} = \frac{n}{\rho}. $$

By integrating this expression and taking into account relation (2.6), we conclude that

$$\text{Vol}_F(B(x_0, \rho)) = \omega_n \rho^n \text{ for all } \rho > 0. \quad (3.12)$$

Let $x \in M$ and $\rho > 0$ be arbitrarily fixed. Since $(M, F)$ is a Hadamard-type manifold, by the volume comparison (see Theorem 2.2(a)), the function $r \mapsto \frac{\text{Vol}_F(B(x, r))}{r^n}$ is non-decreasing on $(0, \infty)$. Thus, one has

$$\omega_n \leq \frac{\text{Vol}_F(B(x, \rho))}{\rho^n} \quad \text{(see (2.11))}$$

$$\leq \limsup_{r \to \infty} \frac{\text{Vol}_F(B(x, r))}{r^n} \quad \text{(monotonicity)}$$

$$\leq \limsup_{r \to \infty} \frac{\text{Vol}_F(B(x_0, r + d_F(x_0, x)))}{(r + d_F(x_0, x))^n} \quad \text{($B(x, r) \subset B(x_0, r + d_F(x_0, x))$)}$$

$$= \limsup_{r \to \infty} \left( \frac{\text{Vol}_F(B(x_0, r + d_F(x_0, x)))}{(r + d_F(x_0, x))^n} \cdot \frac{(r + d_F(x_0, x))^n}{r^n} \right)$$

$$= \omega_n. \quad \text{(see (3.12))}$$

Consequently,

$$\text{Vol}_F(B(x, \rho)) = \omega_n \rho^n \text{ for all } x \in M \text{ and } \rho > 0. \quad (3.13)$$

Now, the equality case in Theorem 2.2(a) implies that $K = 0$. Moreover, note that every Berwald space with $K = 0$ is necessarily a locally Minkowski space, see Bao, Chern and Shen [2] Section 10.5.
Therefore, the volume identity (3.13) actually implies that \((M, F)\) is isometric to a Minkowski space, concluding the proof of the implication \((a)\Rightarrow(c)\).

**Proof of \((c)\Rightarrow(b)\).** Let \(x_0 \in M\) be fixed arbitrarily, and assume that \(\Phi : (M, F) \to (\mathbb{R}^n, F_0)\) is an isometry, where \(F_0 : \mathbb{R}^n \to \mathbb{R}\) is a Minkowski norm. Since \(\Phi\) is an isometry, one has

\[
d_F(x_0, x) = F_0(\Phi(x) - \Phi(x_0)), \quad x \in M;
\]

\[
F(x, y) = F_0(d\Phi_x(y)), \quad x, y \in T_x M.
\]

[Since \((\mathbb{R}^n, F_0)\) is a Minkowski space, \(F_0(d\Phi_x(y))\) can be used instead of \(F_0(\Phi(x), d\Phi_x(y))\).] A simple computation based on (3.15) and the definition of the polar transform (see (2.1)) give

\[
F^*(x, \alpha) = F_0^*(\alpha d\Phi^{-1}_x), \quad x \in M, \alpha \in T^*_x M.
\]

Now, we consider the change of variable \(z = \Phi(x)\). First, relation (3.16) implies for every \(u \in C_0^\infty(M)\) that

\[
F^*(x, Du(x)) = F_0^*(D(u \circ \Phi^{-1})(z)).
\]

Then, since \(F_0\) is a Minkowski norm on \(\mathbb{R}^n\), by (2.5) we have that

\[
dV_F(x) = dV_{F_0}(z) = c_n dz,
\]

where

\[
c_n = \frac{\omega_n}{\text{Vol}(\{z \in \mathbb{R}^n : F_0(z) < 1\})}.
\]

Consequently, on account of (3.14), (3.18) and (3.17), inequality \((\hat{1})_{x_0}\) can be transformed into its equivalent form

\[
\left(\int_{\mathbb{R}^n} F_0^*(Dw(z))^2 dz\right) \left(\int_{\mathbb{R}^n} \frac{|w(z)|^{2p-2}}{F_0(z - z_0)^{2q-2}} dz\right) \geq \frac{(n-q)^2}{p^2} \left(\int_{\mathbb{R}^n} \frac{|w(z)|^p}{F_0(z - z_0)^q} dz\right)^2, \quad \forall w \in C_0^\infty(\mathbb{R}^n),
\]

where \(z_0 = \Phi(x_0) \in \mathbb{R}^n\). Now, it remains to apply (i).

**Remark 3.1** If \((M, F) = (M, g)\) is a Riemannian Hadamard-type manifold, the implication \((a)\Rightarrow(c)\) in Theorem 1.1(ii) has an elegant, geometric proof. Indeed, on account of Jost [17] Lemma 2.1.5 and relation (3.11), it follows that we have equality in the CAT(0)-inequality with reference point \(x_0 \in M\), i.e., for every geodesic segment \(\gamma : [0, 1] \to M\) we have

\[
d_g^2(x_0, \gamma(s)) = (1-s)d_g^2(x_0, \gamma(0)) + s d_g^2(x_0, \gamma(1)) - s(1-s)d_g^2(\gamma(0), \gamma(1)) \quad \text{for all } s \in [0, 1].
\]

Alexandrov’s rigidity result implies that the geodesic triangle formed by the points \(x_0, \gamma(0)\) and \(\gamma(1)\) is flat whenever the triangle is not degenerated to a segment, see e.g. Bridson and Haefliger [6]. Therefore, the conclusion that \((M, g)\) is isometric to the Euclidean space \(\mathbb{R}^n\) follows in a standard manner; the author thank J. Jost and A. Lytchak for pointing out this approach. We emphasize that this geometric approach does not apply for Finsler manifolds; indeed, a well known rigidity result says that if the CAT(0)-inequality holds on a Finsler manifold \((M, F)\), then \((M, F)\) is actually Riemannian, see [6].
Non-negatively curved case: proof of Theorem 1.2

Remark 3.2 We recall the functions $P, R$ and $h$ from (3.5) and (3.7). According to the identities (3.9) and (3.8) we have that
\[
P(2-q)R(\lambda) = P(\lambda), \; \lambda > 0.
\]

One the other hand, we also have
\[
P(\lambda) = \frac{2-p}{p}P^p(\lambda) + R(\lambda), \; \lambda > 0.
\]

Eliminating the term $R$ from the above two relations, we obtain the following first order ODE
\[
\frac{1}{2} \left( -n + \frac{2(p-q)}{p-2} \right) P(\lambda) = -\lambda P^p(\lambda), \; \lambda > 0,
\]
which will be crucial in the proof of Theorem 1.2. At the same time, due to (3.7), we also have
\[
P(\lambda) = \omega_n \lambda^{\frac{2-q}{p-2}} \int_0^\infty t^n h(1,t)dt.
\]

3.2 Non-negatively curved case: proof of Theorem 1.2

The proof is divided into three steps and one exploits some elements from the proof of Theorem 1.1.

Step 1. Let us fix $x_0 \in M$ arbitrarily. By assumption, $(\tilde{I})_{x_0}$ holds. In particular, one can see that $(M,F)$ is not compact; indeed, if we assume the compactness of $(M,F)$, then a test function $u(x) = c_0 \in C_0^\infty(M)$ with $c_0 \neq 0$ would imply a contradiction in $(\tilde{I})_{x_0}$.

We define the class of functions
\[
\tilde{w}_\lambda(x) = (\lambda + d_F(x_0,x)^{2-q})^{\frac{1}{2-p}}, \; \lambda > 0,
\]
which can be approximated by elements from $C_0^\infty(M)$. Thus, replacing the function $\tilde{w}_\lambda$ into the inequality $(\tilde{I})_{x_0}$, it yields
\[
\left( \int_M F^*(x,D\tilde{w}_\lambda(x))^2dV_F(x) \right)^p \left( \int_M \tilde{w}_\lambda(x)^{2-2q} dV_F(x) \right)^q \geq \frac{(n-q)^2}{p^2} \left( \int_M \tilde{w}_\lambda(x)^p dV_F(x) \right)^2.
\]

Since $F^*(x,Dd_F(x_0,x)) = 1$ for a.e. $x \in M$ (cf. (2.9)), one has
\[
\int_M F^*(x,D\tilde{w}_\lambda(x))^2dV_F(x) = \frac{(2-q)^2}{(p-2)^2} \int_M \frac{(\lambda + d_F(x_0,x)^{2-q})^{\frac{2q}{2-p}}}{d_F(x_0,x)^{2q-2}} dV_F(x).
\]

The latter expressions together with (3.22) imply that
\[
\frac{p(2-q)}{(p-2)(n-q)} \int_M \frac{(\lambda + d_F(x_0,x)^{2-q})^{\frac{2q}{2-p}}}{d_F(x_0,x)^{2q-2}} dV_F(x) \geq \int_M \frac{(\lambda + d_F(x_0,x)^{2-q})^{\frac{2q}{2-p}}}{d_F(x_0,x)^{2q}} dV_F(x).
\]

We shall rewrite relation (3.23) in terms of the function $\mathcal{P} : (0, \infty) \to \mathbb{R}$, defined by
\[
\mathcal{P}(\lambda) = \int_M \frac{(\lambda + d_F(x_0,x)^{2-q})^{\frac{2q}{2-p}}}{d_F(x_0,x)^q} dV_F(x).
\]
First, we check that $\mathcal{P}$ is well defined and differentiable. To do this, by using the layer cake representation, a similar argument as in (3.6) and Theorem 2.2(b) (see (2.12)) imply that for every $\lambda > 0$,

$$0 < \mathcal{P}(\lambda) = \lim_{\lambda \to 0^+} \int_0^\infty \operatorname{Vol}_F \left\{ x \in M : \frac{(\lambda + d_F(x_0, x)^2 - q)^{\frac{p}{p-q}}}{d_F(x_0, x)^q} > t \right\} dt$$

$$= \lim_{\lambda \to 0^+} \int_0^\infty \operatorname{Vol}_F \{ x \in M : d_F(x_0, x) < \rho \} h(\lambda, \rho)d\rho$$

$$= \lim_{\lambda \to 0^+} \int_0^\infty \operatorname{Vol}_F(B(x_0, \rho))h(\lambda, \rho)d\rho$$

$$\leq \omega_n \int_0^\infty \rho^n h(\lambda, \rho)d\rho$$

$$= P(\lambda) < \infty,$$

where $P$ is from (3.5). Now, inequality (3.23) can be rewritten equivalently to

$$\frac{1}{2 - q} \left( -n + \frac{2(p - q)}{p - 2} \right) \mathcal{P}(\lambda) \geq -\lambda \mathcal{P}'(\lambda), \ \lambda > 0. \quad (3.24)$$

**Step 2.** We claim that

$$\mathcal{P}(\lambda) \geq P(\lambda) \text{ for all } \lambda > 0. \quad (3.25)$$

First, we shall prove that

$$\liminf_{\lambda \to 0^+} \frac{\mathcal{P}(\lambda)}{P(\lambda)} \geq 1. \quad (3.26)$$

On account of (2.6), for every $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that

$$\operatorname{Vol}_F(B(x_0, \rho)) \geq (1 - \varepsilon)\omega_n \rho^n \text{ for all } \rho \in [0, \rho_\varepsilon].$$

By using a change of variable of the form $\rho = \lambda^\frac{1}{2-q}t$, one has

$$\mathcal{P}(\lambda) = \lim_{\lambda \to 0^+} \int_0^\infty \operatorname{Vol}_F(B(x_0, \rho))h(\lambda, \rho)d\rho$$

$$\geq (1 - \varepsilon)\omega_n \int_0^{\rho_\varepsilon} \rho^n h(\lambda, \rho)d\rho$$

$$= (1 - \varepsilon)\omega_n \lambda^{\frac{n}{2-q}} \int_0^{\rho_\varepsilon} \rho^p \lambda^{-\frac{p}{p-2}} \rho^{\frac{p}{p-2}} h(1, t)dt.$$ 

Combining this estimate with (3.21) and using the fact that $2 - q > 0$, it turns out that

$$\liminf_{\lambda \to 0^+} \frac{\mathcal{P}(\lambda)}{P(\lambda)} \geq 1 - \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, relation (3.26) holds.

On the other hand, by relations (3.20) and (3.24) it follows that

$$\frac{\mathcal{P}'(\lambda)}{\mathcal{P}(\lambda)} \geq \frac{P'(\lambda)}{P(\lambda)}, \ \lambda > 0.$$
After an integration, one can see that \( \lambda \mapsto \frac{\mathcal{P}(\lambda)}{P(\lambda)} \) is non-decreasing. Therefore, by (3.26), for every \( \lambda > 0 \) we have
\[
\frac{\mathcal{P}(\lambda)}{P(\lambda)} \geq \liminf_{\lambda \to 0^+} \frac{\mathcal{P}(\lambda)}{P(\lambda)} \geq 1,
\]
which concludes the proof of (3.25).

**Step 3.** Relation (3.25) is equivalent to
\[
\int_0^\infty (\text{Vol}_F(B(x_0, \rho)) - \omega_n \rho^n) h(\lambda, \rho) d\rho \geq 0 \quad \text{for all } \lambda > 0.
\]

Note that \( h(\lambda, \rho) > 0 \) for all \( \lambda > 0 \) and \( \rho > 0 \), and due to (2.12), one has \( \text{Vol}_F(B(x_0, \rho)) \leq \omega_n \rho^n \) for all \( \rho > 0 \); therefore, we necessarily have that \( \text{Vol}_F(B(x_0, \rho)) = \omega_n \rho^n \) for a.e. \( \rho > 0 \). By continuity, we actually have
\[
\text{Vol}_F(B(x_0, \rho)) = \omega_n \rho^n \quad \text{for all } \rho \geq 0.
\]

(3.27)

Now, let \( x \in M \) and \( \rho > 0 \) be arbitrarily fixed. Note that by Theorem 2.2(b) the function \( r \mapsto \frac{\text{Vol}_F(B(x, r))}{r^n} \) is non-increasing on \((0, \infty)\). Therefore, we have
\[
\omega_n \geq \frac{\text{Vol}_F(B(x, \rho))}{\rho^n} \quad \text{(see (2.12))}
\]
\[
\geq \limsup_{r \to \infty} \frac{\text{Vol}_F(B(x, r))}{r^n} \quad \text{(monotonicity)}
\]
\[
\geq \limsup_{r \to \infty} \frac{\text{Vol}_F(B(x_0, r - d_F(x_0, x)))}{r^n} \quad \text{\( (B(x, r) \supset B(x_0, r - d_F(x_0, x)) \))}
\]
\[
= \limsup_{r \to \infty} \frac{(\text{Vol}_F(B(x, r - d_F(x, x)))}{(r - d_F(x, x))^n} \cdot \frac{(r - d_F(x, x))^n}{r^n}
\]
\[
= \omega_n. \quad \text{(see (3.27))}
\]

Consequently, one has
\[
\text{Vol}_F(B(x, \rho)) = \omega_n \rho^n \quad \text{for all } x \in M, \ \rho \geq 0.
\]

(3.28)

Thus, the equality case in Theorem 2.2(b) implies that \( K = 0 \), which ends the first part of the proof.

In order to prove the equivalence between (a), (b) and (c) when \((M, F)\) is a Berwald space, we proceed similarly as above. Namely, (a) \(\Rightarrow\) (c) follows from the fact that every Berwald space with \( K = 0 \) is a locally Minkowski space, thus by (3.28), \((M, F)\) is isometric to a Minkowski space. (c) \(\Rightarrow\) (b) follows from Theorem 1.1(1), while (b) \(\Rightarrow\) (a) is trivial. \(\square\)

4 Heisenberg-Pauli-Weyl inequalities on Finsler manifolds

4.1 Non-positively curved case: proof of Theorem 1.3

Let \( x_0 \in M \) and \( u \in C_0^\infty(M) \) be fixed arbitrarily. Since \( K \leq 0 \), Theorem 2.1 implies that
\[
\int_M \Delta (d_F(x_0, x)^2) u(x)^2 dV_F(x) = 2 \int_M [1 + d_F(x_0, x) \Delta d_F(x_0, x)] u(x)^2 dV_F(x)
\]
\[
\geq 2n \int_M u(x)^2 dV_F(x).
\]

(4.1)
By (2.10) we have that
\[
\int_M \Delta(d_F(x_0, x)^2) u(x)^2 dV_F(x) = - \int_M D(u(x)^2)(\nabla(d_F(x_0, x)^2)) dV_F(x)
\]
\[
= -4 \int_M u(x) d_F(x_0, x) D u(x)(\nabla d_F(x_0, x)) dV_F(x).
\]

On account of (2.1) and (2.9) we have
\[
|D u(x)(\nabla d_F(x_0, x))| \leq F^*(x, D u(x)) F(x, \nabla d_F(x_0, x)) = F^*(x, D u(x)),
\]
and the Schwartz inequality implies
\[
\left( \int_M u(x) d_F(x_0, x) D u(x)(\nabla d_F(x_0, x)) dV_F(x) \right)^2 \leq \left( \int_M D u(x)^2 dV_F(x) \right) \left( \int_M F^*(x, D u(x))^2 dV_F(x) \right).
\]
The latter relation with (4.1) yields the Heisenberg-Pauli-Weyl inequality (HPW)_{x_0} concluding the first part of the proof.

**Proof of Theorem 1.3 (i).** Let us fix \( x_0 \in M \) arbitrarily. Taking into account that \((M, F) = (\mathbb{R}^n, F)\) is a Minkowski space, a similar argument as in the proof of Theorem 1.1 (i) shows that (HPW)_{x_0} is equivalent to
\[
\left( \int_{\mathbb{R}^n} F^*(D u(x))^2 dx \right) \left( \int_{\mathbb{R}^n} F(x-x_0)^2 u(x)^2 dx \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} u(x)^2 dx \right)^2, \forall u \in C^\infty_0(\mathbb{R}^n).
\]
\[(\mathbb{R}^n, F)\] being a Minkowski space (thus a simply connected, complete Berwald space with \( K = 0 \)), inequality (4.2) holds true due to the first part of the proof.

It remains to prove the sharpness of the constant \( \frac{n^2}{4} \). Indeed, if we replace the Gaussian function 
\[u_\lambda(x) = e^{-\lambda F^2(x-x_0)} (\lambda > 0)\] into (4.2), it suffices to verify the equality
\[2\lambda \int_{\mathbb{R}^n} F(x-x_0)^2 e^{-2\lambda F(x-x_0)^2} dx = \frac{n}{2} \int_{\mathbb{R}^n} e^{-2\lambda F(x-x_0)^2} dx, \lambda > 0.\]

Note that (4.3) follows by standard computations based on Wulff-type polar coordinates, i.e., \((\rho, \theta) = \left(F(x-x_0), \frac{x-x_0}{F(x-x_0)}\right)\), by using the identity
\[\int_0^\infty \rho^{n+1} e^{-\rho^2} d\rho = \frac{n}{2} \int_0^\infty \rho^{n-1} e^{-\rho^2} d\rho.
\]

**Proof of Theorem 1.3 (ii).** The arguments are similar as in the proof of Theorem 1.1 (ii); thus, we provide only the key points.

**Proof of (a)⇒(c).** If \( \frac{n^2}{4} \) is sharp and \( u_0 > 0 \) is an extremal in (HPW)_{x_0} for some \( x_0 \in M \), by (4.1) it follows that \( \Delta(d_F(x_0, x)^2) = 2n \) for every \( x \in M \setminus \{x_0\} \). Now we can repeat the same argument as in the proof of the implication (a)⇒(c) in Theorem 1.1 (ii), obtaining that \((M, F)\) is isometric to an \( n \)-dimensional Minkowski space.

The proof of the implications (c)⇒(b)⇒(a) are trivial in view of the above arguments.
Remark 4.1 If we consider the function $T : (0, \infty) \to \mathbb{R}$ defined by

$$T(\lambda) = \int_{\mathbb{R}^n} e^{-2\lambda F(x-x_0)^2} \, dx, \quad \lambda > 0,$$

the identity (4.3) becomes

$$-\lambda T'(\lambda) = \frac{n}{2} T(\lambda), \quad \lambda > 0. \quad (4.4)$$

Moreover, by the layer cake representation and changing a variable, one has

$$T(\lambda) = 4\lambda \omega_n \int_0^\infty \rho^{n+1} e^{-2\lambda \rho^2} \, d\rho = \frac{2}{(2\lambda)^{\frac{n}{2}}} \omega_n \int_0^\infty t^{n+1} e^{-t^2} \, dt. \quad (4.5)$$

These facts will be used in the proof of Theorem 1.4.

4.2 Non-negatively curved case: proof of Theorem 1.4

Again, the proof is divided into three steps.

**Step 1.** Let $x_0 \in M$ be fixed. By our hypothesis, the Heisenberg-Pauli-Weyl inequality $(\mathcal{HPW})_{x_0}$ holds; in particular, $(M, F)$ cannot be compact. We consider the class of functions

$$\tilde{u}_\lambda(x) = e^{-\lambda d F(x,x_0)^2}, \quad \lambda > 0.$$ 

Clearly, the function $\tilde{u}_\lambda$ can be approximated by elements from $C_0^\infty(M)$ for every $\lambda > 0$. By inserting $\tilde{u}_\lambda$ into $(\tilde{HPW})_{x_0}$, due to (2.9) we obtain that

$$2\lambda \int_M d_F(x_0,x)^2 e^{-2\lambda d F(x_0,x)^2} dV_F(x) \geq \frac{n}{2} \int_M e^{-2\lambda d F(x_0,x)^2} dV_F(x), \quad \lambda > 0. \quad (4.6)$$

We introduce the function $\mathcal{T} : (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{T}(\lambda) = \int_M e^{-2\lambda d F(x_0,x)^2} dV_F(x), \quad \lambda > 0.$$ 

By the layer case representation, $\mathcal{T}$ can be equivalently rewritten as

$$\mathcal{T}(\lambda) = 4\lambda \omega_n \int_0^\infty \text{Vol}_F(B(x_0, \rho)) \rho e^{-2\lambda \rho^2} \, d\rho. \quad (4.7)$$

Since $\text{Ric}_M \geq 0$, one account of (2.12) and (4.7), the function $\mathcal{T}$ is well defined and differentiable. Consequently, relation (4.6) is equivalent to

$$-\lambda \mathcal{T}'(\lambda) \geq \frac{n}{2} \mathcal{T}(\lambda), \quad \lambda > 0. \quad (4.8)$$

**Step 2.** We shall prove that

$$\mathcal{T}(\lambda) \geq T(\lambda) \text{ for all } \lambda > 0. \quad (4.9)$$

By (4.4) and (4.8) it turns out that

$$\frac{\mathcal{T}'(\lambda)}{\mathcal{T}(\lambda)} \leq \frac{T'(\lambda)}{T(\lambda)}, \quad \lambda > 0.$$
Integrating this inequality, it yields that the function \( \lambda \mapsto \frac{\mathcal{T}(\lambda)}{T(\lambda)} \) is non-increasing; in particular, for every \( \lambda > 0 \),

\[
\frac{\mathcal{T}(\lambda)}{T(\lambda)} \geq \liminf_{\lambda \to \infty} \frac{\mathcal{T}(\lambda)}{T(\lambda)}. \tag{4.10}
\]

Now, we shall prove that

\[
\liminf_{\lambda \to \infty} \frac{\mathcal{T}(\lambda)}{T(\lambda)} \geq 1. \tag{4.11}
\]

Due to relation (2.6), for every \( \varepsilon > 0 \) one can find \( \rho_\varepsilon > 0 \) such that

\[
\text{Vol}_F(B(x_0, \rho)) \geq (1 - \varepsilon)\omega_n\rho^n \quad \text{for all } \rho \in [0, \rho_\varepsilon].
\]

Consequently, one has

\[
\mathcal{T}(\lambda) = 4\lambda \int_0^\infty \text{Vol}_F(B(x_0, \rho))\rho e^{-2\lambda \rho^2} d\rho
\]

(see (4.7))

\[
\geq 4\lambda(1 - \varepsilon)\omega_n \int_0^{\sqrt{2}\rho_\varepsilon} \rho^{n+1} e^{-2\lambda \rho^2} d\rho
\]

\[
\geq \frac{2}{(2\lambda)^\frac{n}{2}}(1 - \varepsilon)\omega_n \int_0^{\sqrt{2}\rho_\varepsilon} t^{n+1} e^{-t^2} dt.
\]

Now, by (4.5), it yields that

\[
\liminf_{\lambda \to \infty} \frac{\mathcal{T}(\lambda)}{T(\lambda)} \geq 1 - \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrar, relation (4.11) holds, so (4.10). This ends the proof of the claim (4.9).

**Step 3.** Via (4.5) and (4.7), relation (4.9) is equivalent to

\[
\int_0^\infty (\text{Vol}_F(B(x_0, \rho)) - \omega_n\rho^n) \rho e^{-2\lambda \rho^2} d\rho \geq 0 \quad \text{for all } \lambda > 0.
\]

Due to (2.12), we shall have \( \text{Vol}_F(B(x_0, \rho)) = \omega_n\rho^n \) for all \( \rho \geq 0 \). Now, it remains to repeat the arguments after the relation (3.27), obtaining that \( K = 0 \). The rest of the proof is similar to that of Theorem 1.2.

**4.3 Heisenberg-Pauli-Weyl inequality on hyperbolic spaces**

For the hyperbolic space we use the Poincaré ball model \( \mathbb{H}^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \) endowed with the Riemannian metric \( g(x) = (g_{ij}(x))_{i,j=1,\ldots,n} = p(x)^2\delta_{ij} \), where \( p(x) = \frac{2}{1 - |x|^2} \). It is well known that \( (\mathbb{H}^n, g) \) is a Riemannian manifold of Hadamard-type with constant sectional curvature \(-1\). The Busemann-Hausdorff volume form is \( dV_{\mathbb{H}^n}(x) = p(x)^n dx \), while the hyperbolic gradient and Laplace-Beltrami operator are given by

\[
\nabla_{\mathbb{H}^n} u = \nabla u = \frac{\nabla u}{p^2} \quad \text{and} \quad \Delta_{\mathbb{H}^n} u = \Delta u = p^{-n} \text{div}(p^{n-2}\nabla u),
\]

where \( \nabla \) denotes the Euclidean gradient in \( \mathbb{R}^n \). The hyperbolic distance between the origin and \( x \in \mathbb{H}^n \) is given by

\[
d_{\mathbb{H}^n}(0, x) = \ln \left( \frac{1 + |x|}{1 - |x|} \right).
\]
Recently, Kombe and Özaydin [20] stated a sharp Heisenberg-Pauli-Weyl inequality on hyperbolic spaces. For completeness, we recall the statement of Theorem 4.2 from [20]:

\[ u \in C_0^\infty(\mathbb{H}^n), \quad d = d(x) = d_{\mathbb{H}^n}(0, x) \text{ and } n > 2. \text{ Then} \]

\[ \left( \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n} \right) \left( \int_{\mathbb{H}^n} d^2 u^2 dV_{\mathbb{H}^n} \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{H}^n} u^2 dV_{\mathbb{H}^n} \right)^2. \quad (4.12) \]

Moreover, equality holds in (4.12) if \( u(x) = A e^{-\alpha d^2} \), where \( A \in \mathbb{R}, \quad \alpha = \frac{n-1}{n-2} \left( n - 1 + 2\pi \frac{C_n-2}{C_n} \right) \) and \( C_n = \int_{\mathbb{H}^n} e^{-\alpha d^2} dV_{\mathbb{H}^n} \).

According to Theorem 1.3 relation (4.12) holds. However, the statement concerning the equality in (4.12) is not true, which has at least the following three independent arguments:

- If \( \frac{n^2}{4} \) is sharp with the hyperbolic Gaussian \( u(x) = e^{-\alpha d^2} \) as extremal function, Theorem 1.3(ii) immediately implies that \((\mathbb{H}^n, g)\) is isometric to an \( n \)–dimensional Minkowski space. But every Minkowski space which has a Riemannian structure is Euclidean; thus, if follows that \((\mathbb{H}^n, g)\) is isometric to the standard Euclidean space \( \mathbb{R}^n \), a contradiction.

- For \( n = 4 \), numeric calculus shows that the non-linear equation \( \alpha = \frac{n-1}{n-2} \left( n - 1 + 2\pi \frac{C_n-2}{C_n} \right) \), where \( C_n = \int_{\mathbb{H}^n} e^{-\alpha d^2} dV_{\mathbb{H}^n} \), cannot be solved in \( \alpha \).

- Since \( K = -1 \) in the hyperbolic space \( \mathbb{H}^n \), instead of (4.11) we have a sharper inequality, i.e.,

\[ \int_{\mathbb{H}^n} \Delta_{\mathbb{H}^n}(d^2) u^2 dV_{\mathbb{H}^n} = 2 \int_{\mathbb{H}^n} (1 + d\Delta_{\mathbb{H}^n} d) u^2 dV_{\mathbb{H}^n} \geq 2 \int_{\mathbb{H}^n} (1 + (n - 1)\text{coth}(d)) u^2 dV_{\mathbb{H}^n} \]

\[ = 2n \int_{\mathbb{H}^n} \left( 1 + \frac{n-1}{n} D_{-1}(d) \right) u^2 dV_{\mathbb{H}^n}. \]

Repeating the same argument as in the proof of Theorem 1.3 one has

\[ \left( \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n} \right) \left( \int_{\mathbb{H}^n} d^2 u^2 dV_{\mathbb{H}^n} \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{H}^n} \left( 1 + \frac{n-1}{n} D_{-1}(d) \right) u^2 dV_{\mathbb{H}^n} \right)^2. \quad (4.13) \]

Since \( D_{-1}(d) \geq 0, \) (4.13) implies (4.12). Moreover, \( \frac{n^2}{4} \) is sharp in (4.13) and an integration by part easily shows that the equality holds for the hyperbolic Gaussian family of functions \( u(x) = e^{-\alpha d^2} \), where \( \alpha > 0 \). A similar argument was carried out also in Erb [11]. Thus, if one would expect to have equality in (4.12) for \( u(x) = e^{-\alpha d^2} \), we necessarily would have \( D_{-1}(\rho) = 0 \) for every \( \rho \geq 0 \), a contradiction.

In view of the aforementioned arguments, the following natural question arises: find the sharp constant \( C_{\text{HPW}} \geq \frac{n^2}{4} \) (and perhaps the extremal functions) in the 'pure' Heisenberg-Pauli-Weyl inequality

\[ \left( \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n} \right) \left( \int_{\mathbb{H}^n} d^2 u^2 dV_{\mathbb{H}^n} \right) \geq C_{\text{HPW}} \left( \int_{\mathbb{H}^n} u^2 dV_{\mathbb{H}^n} \right)^2, \forall u \in C_0^\infty(\mathbb{H}^n). \]
5 Hardy inequalities on non-positively curved Finsler manifolds

5.1 Proof of Theorem 1.5

First of all, by a convexity reason and (2.8), a Taylor expansion implies that

\[ F^*(x, \beta)^2 \geq F^*(x, \alpha)^2 + 2(\beta - \alpha)(J^*(x, \alpha)), \quad \forall \alpha, \beta \in T^*_x M. \]  

Let \( x_0 \in M \) and \( u \in C^\infty_0(M) \) be arbitrarily and fix \( \gamma = \frac{n-2}{2} > 0 \). We consider the function \( v(x) = d_F(x_0, x)^\gamma u(x) \). Thus, for \( u(x) = d_F(x_0, x)^{-\gamma} v(x) \) one has

\[ Du(x) = -\gamma d_F(x_0, x)^{-\gamma-1} v(x) Dd_F(x_0, x) + d_F(x_0, x)^{-\gamma} Dv(x). \]

Applying the inequality (5.1) with the choices \( \beta = -Du(x) \) and \( \alpha = \gamma d_F(x_0, x)^{-\gamma-1} v(x) Dd_F(x_0, x) \), and taking into account the symmetry of \( F^*(x, \cdot) \), it yields

\[
F^*(x, Du(x))^2 = F^*(x, -Du(x))^2 \\
\geq F^*(x, \gamma d_F(x_0, x)^{-\gamma-1} v(x) Dd_F(x_0, x))^2 \\
-2d_F(x_0, x)^{-\gamma} Dv(x) (J^*(x, \gamma d_F(x_0, x)^{-\gamma-1} v(x) Dd_F(x_0, x))).
\]

Since \( F^*(x, Dd_F(x_0, x)) = 1 \) (see (2.9)), \( J^*(x, Dd_F(x_0, x)) = \nabla d_F(x_0, x) \) and \( Dv(x) \in T^*_x M \), we obtain

\[
F^*(x, Du(x))^2 \geq \gamma^2 d_F(x_0, x)^{-2\gamma-2} v(x)^2 - 2\gamma d_F(x_0, x)^{-2\gamma-1} v(x) Dv(x) (\nabla d_F(x_0, x)).
\]

Integrating the latter inequality over \( M \), it yields

\[
\int_M F^*(x, Du(x))^2 dV_F(x) \geq \gamma^2 \int_M d_F(x_0, x)^{-2\gamma-2} v(x)^2 dV_F(x) + R_0,
\]

where

\[
R_0 = -2\gamma \int_M d_F(x_0, x)^{-2\gamma-1} v(x) Dv(x) (\nabla d_F(x_0, x)) dV_F(x) \\
= \frac{1}{2} \int_M D(v(x)^2) (\nabla (d_F(x_0, x)^{-2\gamma})) dV_F(x) \\
= -\frac{1}{2} \int_M v(x)^2 \Delta (d_F(x_0, x)^{-2\gamma}) dV_F(x) \\
= \gamma \int_M v(x)^2 d_F(x_0, x)^{-2\gamma-2} (-2\gamma - 1 + d_F(x_0, x) \Delta d_F(x_0, x)) dV_F(x) \\
\geq \frac{(n-1)(n-2)}{2} \int_M (d_F(x_0, x) \text{ct}_c(d_F(x_0, x)) - 1) \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x), \quad (\text{see Theorem 2.1})
\]

\[
= \frac{(n-1)(n-2)}{2} \int_M D_d (d_F(x_0, x)) \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x),
\]

which completes the first part of the proof.

We shall prove in the sequel that \( \gamma^2 = \frac{(n-2)^2}{4} \) is sharp in \((H)_{x_0}\). Fix the numbers \( R > r > 0 \) and a smooth cutoff function \( \psi : M \to [0,1] \) with \( \supp(\psi) = \overline{B(x_0, R)} \) and \( \psi(x) = 1 \) for \( x \in B(x_0, r) \), and for every \( \varepsilon > 0 \), let

\[
u_\varepsilon(x) = (\max \{ \varepsilon, d_F(x_0, x) \})^{-\gamma}, \quad x \in M.
\]
One the one hand,

\[ I_1(\varepsilon) := \int_M F^*(x, D(\psi u_\varepsilon)(x))^2 dV_F(x) \]
\[ = \int_{B(x_0,\varepsilon)} F^*(x, Du_\varepsilon(x))^2 dV_F(x) + \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} F^*(x, D(\psi u_\varepsilon)(x))^2 dV_F(x) \]
\[ = \gamma^2 \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} dF(x_0, x)^{-2\gamma-2} dV_F(x) + \tilde{I}_1(\varepsilon), \]

where the quantity

\[ \tilde{I}_1(\varepsilon) = \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} F^*(x, D(\psi u_\varepsilon)(x))^2 dV_F(x) \]

is finite and does not depend on \( \varepsilon > 0 \) whenever \( \varepsilon < r \). On the other hand,

\[ I_2(\varepsilon) := \int_M \left( 1 + \frac{2(n-1)}{n-2} D_c(d_F(x_0, x)) \right) \frac{(\psi u_\varepsilon)(x)^2}{d_F(x_0, x)^2} dV_F(x) \]
\[ \geq \int_M \frac{(\psi u_\varepsilon)(x)^2}{d_F(x_0, x)^2} dV_F(x) \]
\[ \geq \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} dF(x_0, x)^{-2\gamma-2} dV_F(x) =: \tilde{I}_2(\varepsilon). \]

By applying the layer cake representation, we deduce that for \( 0 < \varepsilon < r \), one has

\[ \tilde{I}_2(\varepsilon) = \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} dF(x_0, x)^{-2\gamma-2} dV_F(x) = \int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} dF(x_0, x)^{-n} dV_F(x) \]
\[ = \int_{r^{-n}}^{\varepsilon^{-n}} \text{Vol}_n(B(x_0, \rho^{-\frac{2}{\gamma}})) d\rho \]
\[ \geq \omega_n \int_{r^{-n}}^{\varepsilon^{-n}} \rho^{-1} d\rho \]
\[ = n \omega_n(\ln r - \ln \varepsilon). \]

In particular, \( \lim_{\varepsilon \to 0^+} \tilde{I}_2(\varepsilon) = +\infty \). Therefore, from the above relations it follows that

\[ (n-2)^2 \leq \frac{\int_{B(x_0,\varepsilon)\setminus B(x_0,\varepsilon)} F^*(x, Du(x))^2 dV_F(x)}{4} \leq \frac{\int_M F^*(x, D\psi u_\varepsilon(x))^2 dV_F(x)}{4} \]
\[ = \lim_{\varepsilon \to 0^+} \frac{I_1(\varepsilon)}{I_2(\varepsilon)} \leq \lim_{\varepsilon \to 0^+} \frac{\gamma^2 \tilde{I}_2(\varepsilon) + \tilde{I}_1(\varepsilon)}{I_2(\varepsilon)} \]
\[ = \gamma^2 = \frac{(n-2)^2}{4}, \]

which concludes the proof. \( \square \)

**Proof of Corollary 1.1** By the continued fraction representation of the function \( \rho \mapsto \coth(\rho) \), one has

\[ \rho \coth(\rho) - 1 \geq \frac{3\rho^2}{\pi^2 + \rho^2}, \forall \rho > 0. \]

Now, the inequality follows at once from this estimate and Theorem 1.5. \( \square \)
5.2 Proof of Theorem 1.6

Similarly to (5.1), the definition of the uniformity constant of $F^*$ implies that
\[ F^*(x, \beta) \geq F^*(x, \alpha)^2 + 2(\beta - \alpha)(J^*(x, \alpha)) + l_{F^*} F^*(x, \beta - \alpha)^2, \quad \forall \alpha, \beta \in T^*_x M. \tag{5.3} \]
Let $x_0 \in \Omega$, $u \in C_0^\infty(\Omega)$ and fix $\gamma = \frac{\alpha_0}{2} > 0$. We consider the function $v(x) = d_F(x_0, x)^\gamma u(x)$. Thus,
\[ Du(x) = -\gamma d_F(x_0, x)^{-\gamma - 1}v(x) Dd_F(x_0, x) + d_F(x_0, x)^{-\gamma}Dv(x). \]
Applying (5.3) with the choices $\beta = -Du(x)$ and $\alpha = \gamma d_F(x_0, x)^{-\gamma - 1}v(x)Dd_F(x_0, x)$, a similar argument as in the proof of Theorem 1.5 gives
\[ F^*(x, Du(x))^2 \geq \gamma^2 d_F(x_0, x)^{-2\gamma - 2}v(x)^2 - 2\gamma d_F(x_0, x)^{-2\gamma - 1}v(x)Dv(x)(\nabla d_F(x_0, x)) + l_{F^*} F^*(x, Du(x))^2. \]
After an integration over $\Omega$ of the above inequality, one can repeat the argument from the proof of Theorem 1.5 to the first two integrands, obtaining
\[ \int_{\Omega} F^*(x, Du(x))^2 dV_F(x) \geq \frac{(n - 2)^2}{4} \int_{\Omega} \left( 1 + \frac{2(n - 1)}{n - 2} Dc(d_F(x_0, x)) \right) \frac{u(x)^2}{d_F(x_0, x)^2} dV_F(x) + l_{F^*} \tilde{R}, \]
where
\[ \tilde{R} = \int_{\Omega} d_F(x_0, x)^{-2\gamma} F^*(x, Du(x))^2 dV_F(x). \]
Due to the fact that $R > \sup_{x \in \Omega} d_F(x, x_0)$, the function $h(x) = \ln \frac{R}{d_F(x_0, x)}$ is well defined on $\Omega \setminus \{x_0\}$ and $h \geq 1$. Let $z(x) = h(x)^{-1/2}v(x)$. Since
\[ Dv(x) = -\frac{z(x)}{2d_F(x_0, x)} h(x)^{-1/2} Dd_F(x_0, x) + h(x)^{1/2} Dz(x), \]
by (5.1) it turns out that
\[ F^*(x, Dv(x))^2 \geq \frac{z(x)^2}{4d_F(x_0, x)^2} h(x)^{-1} - \frac{z(x)}{d_F(x_0, x)} Dz(x)(\nabla d_F(x_0, x)). \]
Consequently,
\[ \tilde{R} = \int_{\Omega} d_F(x_0, x)^{-2\gamma} F^*(x, Du(x))^2 dV_F(x) \]
\[ \geq \frac{1}{4} \int_{\Omega} d_F(x_0, x)^{-2\gamma - 2} h(x)^{-1} z(x)^2 dV_F(x) - \frac{1}{2} \int_{\Omega} d_F(x_0, x)^{-2\gamma - 1} D(z(x)^2)(\nabla d_F(x_0, x)) dV_F(x) \]
\[ = \frac{1}{4} \int_{\Omega} d_F(x_0, x)^{-2} h(x)^{-2} u(x)^2 dV_F(x) - \frac{1}{4} \int_{\Omega} z(x)^2 \Delta(d_F(x_0, x)^{-2\gamma}) dV_F(x) \]
\[ = \frac{1}{4} \int_{\Omega} d_F(x_0, x)^{-2} h(x)^{-2} u(x)^2 dV_F(x) \]
\[ + \frac{1}{2} \int_{\Omega} z(x)^2 d_F(x_0, x)^{-2\gamma - 2} (-2\gamma - 1 + d_F(x_0, x) \Delta d_F(x_0, x)) dV_F(x) \]
\[ \geq \frac{1}{4} \int_{\Omega} d_F(x_0, x)^{-2} h(x)^{-2} u(x)^2 dV_F(x) + \frac{n - 1}{2} \int_{\Omega} Dc(d_F(x_0, x)) d_F(x_0, x)^{-2\gamma - 2} z(x)^2 dV_F(x) \]
\[ = \frac{1}{4} \int_{\Omega} (1 + 2(n - 1)h(x) Dc(d_F(x_0, x))) \frac{u(x)^2}{d_F(x_0, x)^2 h(x)^2} dV_F(x), \]
which concludes the proof.
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