Transpositional sequences and multigraphs

Alissa Ellis Yazinski  Raymond R. Fletcher*  Donald Silberger†

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Abstract

If \( s := \langle s_0, s_1, \ldots, s_k \rangle \) is a sequence of length \(|s| = k\) of permutations on the set \( n := \{0, 1, \ldots, n - 1\} \) then \( \bigcirc s := s_0 \circ s_1 \circ \cdots \circ s_k \in \text{Sym}(n) \), and \( \text{Seq}(s) := \{ \psi := \langle \psi(0), \ldots, \psi(k-1) \rangle : \psi \in \text{Sym}(k) \} \) denotes the set of rearrangements of \( s \). Our overall interest is the set \( \text{Prod}(s) := \{ \bigcirc r : r \in \text{Seq}(s) \} \subseteq \text{Sym}(n) \). We focus on transpositional sequences; that is, on those \( s \), each of whose terms is a transposition \((xy)\). For \( u \) a transpositional sequence in \text{Sym}(n), there is a natural correspondence between \( \text{Seq}(u) \) and its transpositional multigraph \( T(u) := \langle n; E(u) \rangle \) on the vertex set \( n \), where the \( k \) simple edges \((xy)\) in the collection \( E(u) \) of multiedges in \( T(u) \) are the \( k \) (not obligatorily distinct) terms in the sequence \( u \).

This paper considers antipodal sorts of sequences \( s \) in \text{Sym}(n):

We call \( s \) permutationally complete, aka perm-complete, if \( \text{Prod}(s) \subseteq \{ \text{Alt}(n), \text{Sym}(n) \setminus \text{Alt}(n) \} \), where \( \text{Alt}(n) \subseteq \text{Sym}(n) \) is the alternating subgroup. So, \( \text{Prod}(s) \) is as large as possible if \( s \) is perm-complete.

We provide sufficient criteria for a transpositional sequence \( u \) in \text{Sym}(n) to be perm-complete and also sufficient criteria for \( u \) not to be perm-complete.

We call \( s \) conjugacy invariant, aka CI, if the elements in \( \text{Prod}(s) \) are mutually conjugate. \( \text{Prod}(s) \) is small if \( s \) is CI. We specify the CI transpositional \( u \) in \text{Sym}(n).

1 Introduction

Unless specified otherwise, ‘sequence in \( X \)’ means finite sequence whose terms are elements in the set \( X \). For \( f := \langle x_0, x_1, \ldots, x_k \rangle \) a sequence, \( x_i < x_j \) iff \( 0 \leq i < j \leq k \), and \( x_i \leq x_j \) iff \( 0 \leq i < j \leq k \).

For \( n \) a positive integer, \( \text{Sym}(n) \) and \( \text{Alt}(n) \) denote respectively the symmetric group and the alternating group on the set \( n := \{0, 1, \ldots, n - 1\} \). When \( s := \langle s_0, s_1, \ldots, s_k \rangle \) is a permutational sequence, i.e., a sequence in \text{Sym}(n), then \(|s| = k\) is its length, and \( \bigcirc s := s_0 \circ s_1 \circ \cdots \circ s_k \) is its compositional product.

\( \text{Seq}(s) \) denotes the set of sequences that are arrangements of the terms of \( s \). That is to say, \( \text{Seq}(s) \) denotes the set \( \{ \psi := \langle \psi(0), \ldots, \psi(k-1) \rangle : \psi \in \text{Sym}(k) \} \) of \( k \) permutations \( \psi \). Obviously \( \psi \in \text{Seq}(s) \Rightarrow |\psi| = |s| \).

Our general subject is the family of sets \( \text{Prod}(s) := \{ \bigcirc r : r \in \text{Seq}(s) \} \) for the sequences \( s \) in \text{Sym}(n). However, fully to characterize the family of all such \( \text{Prod}(s) \) seems daunting. So we confine ourselves to the subclass, of that class of \( s \), which is treated in the papers [2, 3, 5, 6], whose results we extend.

Plainly, either \( \text{Prod}(s) \subseteq \text{Alt}(n) \) or \( \text{Prod}(s) \subseteq \text{Sym}(n) \setminus \text{Alt}(n) \). Also, \( |\text{Prod}(s)| \leq |\text{Seq}(s)| \leq |s|! \).

When \( f \in \text{Sym}(n) \), the expression \( \text{supp}(f) \) denotes the set of \( x \in n \) for which \( xf \neq f \). If \( s \) is a permutational sequence then \( \text{Supp}(s) \) denotes the family of all \( \text{supp}(g) \) for which \( g \) is a term in \( s \).

By a transposition we mean a permutation \( f \in \text{Sym}(n) \) for which there exist elements \( a \neq b \) in \( n \) with \( af = b \), with \( bf = a \) and with \( xf = x \) for all \( x \in n \setminus \{a, b\} \). For \( n \geq 2 \), the set of transpositions in \text{Sym}(n) is written \( 1^{n-2}2^1 \). By a transpositional sequence we mean a sequence in the subset \( 1^{n-2}2^1 \) of \text{Sym}(n).

By the transpositional multigraph \( T(u) \) of a sequence \( u \) in \( 1^{n-2}2^1 \), we mean the labeled multigraph on the vertex set \( n \) that has an \((xy)\) as a multiedge of multiplicity \( \mu(g) \geq 0 \) if and only if the transposition \( g := (xy) \) occurs exactly \( \mu(g) \) times as a term in \( u \). For convenience, we will usually take it that \( T(u) \) is
connected, in which event of course $\bigcup\text{Supp}(u) = \text{supp}(u) = n$. It is obvious that $\mathcal{T}(r) = \mathcal{T}(u)$, which is to say that $\mathcal{T}(r)$ is the same labeled multigraph as $\mathcal{T}(u)$, if and only if $r \in \text{Seq}(u)$.

The multigraph $\mathcal{T}(u)$ is simple, i.e., is a graph, if and only if $u$ is injective. Where we omit the prefix “multi” from “multentity”, we are tacitly indicating that the entity is simple; but our writing that $X$ is a multithing does not prohibit $X$ from its being a simple thing. (E.g., a multiedge can be of multiplicity 1.)

For $\mathcal{T}(u)$ a simple tree, this graph has been used in [3] to specify the set of all $r \in \text{Seq}(u)$ for which $\bigcirc r = \bigcirc u$, thus inducing a natural partial of $\text{Seq}(u)$. Also, both [2] and [6] show that, if the multigraph $\mathcal{T}(u)$ is simple, then every element in $\text{Prod}(u)$ is a cyclic permutation of the set $n$ if and only if $\mathcal{T}(u)$ is a tree. See also [5].

**Definition 1.** A sequence $s$ in $\text{Sym}(n)$ is **permutationally complete** iff $\text{Prod}(s) \in \{\text{Alt}(n), \text{Blt}(n)\}$ where $\text{Blt}(n) := \text{Sym}(n) \setminus \text{Alt}(n)$; that is to say, $s$ is permutationally complete iff $\text{Prod}(s) \in \text{Sym}(n)/\text{Alt}(n)$.

‘Permutationally complete’ is abbreviated perm-complete.

A sequence $s$ in $\text{Sym}(n)$ is perm-complete if and only if $\text{Prod}(s)$ is of largest possible size, $|\text{Prod}(s)| = n!/2$.

In §2 we elaborate criteria that imply the perm-completeness of a sequence $u$ in $1^{n-2}2^1$, and we provide other criteria which entail that such a $u$ cannot be perm-complete.

If the product function $\bigcirc$ maps $\text{Seq}(s)$ onto an element in the family $\text{Sym}(n)/\text{Alt}(n)$, and if $r$ is a sequence produced by inserting into $s$ an additional term $f \in \text{Sym}(n)$, then plainly $\bigcirc$ maps $\text{Seq}(r)$ onto an element in $\text{Sym}(n)/\text{Alt}(n)$; viz. Theorem [2,1]. So we can confine our attention in §2 to those transpositional $u$ which are injective, and whose transpositional multigraphs are consequently simple; i.e., they are “graphs”.

These graphs facilitate the identification of infinite classes of $u$ which are perm-complete and also of infinite classes of $u$ which fail to be perm-complete. For instance, if $\mathcal{T}(u)$ is the complete graph $K_n$, then $u$ is perm-complete, but if $\mathcal{T}(u)$ is a tree with $n \geq 3$ then $u$ is not perm-complete. Therefore, every injective perm-complete transpositional sequence $u$ has a minimal perm-complete subsequence.

In §2 we will specify, for each $n \geq 2$, a family of minimal perm-complete injective sequences in $1^{n-2}2^1$.

**Definition 2.** We call a permutational sequence $s$ **conjugacy invariant**, aka CI, iff every element in $\text{Prod}(s)$ is conjugate to $\bigcirc s$.

We lose no generality if we ignore the fact that $\mathcal{T}(u)$ is labeled. Indeed, we call an unlabeled multigraph $\mathcal{G}$ perm-complete if $\mathcal{G}$ is isomorphic to $\mathcal{T}(u)$ for some perm-complete $u$. Likewise, $\mathcal{G}$ is CI if $u$ is CI.

### 2 Permutational completeness

**Theorem 2.1.** Every supersequence in $\text{Sym}(n)$ of a perm-complete sequence $s$ in $\text{Sym}(n)$ is perm-complete.

**Proof.** Without loss of generality, let $\text{Prod}(s) = \text{Alt}(n)$. Pick $g \in \text{Sym}(n)$. The mapping, $\text{Alt}(n) \to \text{Sym}(n)$

\[ f \mapsto f \circ g \] defined by $f \mapsto f \circ g$, takes $\text{Prod}(s)$ either into $\text{Alt}(n)$ or into $\text{Blt}(n)$, and it is bijective. Since $|\text{Alt}(n)| = n!/2 = |\text{Blt}(n)|$, we conclude that $\text{Prod}(w) \in \text{Sym}(n)/\text{Alt}(n)$ where $w := (s, g) = (s_0, s_1, \ldots, s_{k-1}, g)$. 

Call a family $A$ connected if $\bigcup A$ cannot be expressed as the disjoint union, $\bigcup A = \bigcup B \bigcup \bigcup C$, of nonempty subfamilies $B$ and $C$ of $A$. Observe that if $s$ is perm-complete then $\text{Supp}(s)$ is connected.

In §2, we restrict our concern to those $u := \langle u_0, u_1, \ldots, u_{k-1} \rangle$ in $1^{n-2}2^1$ for which $\bigcup \text{Supp}(u) = n$, and for which the family $\text{Supp}(u)$ is connected. It is easy to see that if $g \in \text{Prod}(u)$ then $g^{-1} \in \text{Prod}(u)$ too.

### 2.1 Criteria ensuring that $u$ is not perm-complete

**Theorem 2.2.** Let $\mathcal{G}$ be a connected graph with vertex set $n \geq 3$, and which has a vertex $a$ of degree 1. Then $\mathcal{G}$ is not perm-complete. Consequently, no tree having three or more vertices is perm-complete.

**Proof.** Pretend that $\mathcal{G}$ is perm-complete, and let $u$ be a transpositional sequence for which $\mathcal{G} = \mathcal{T}(u)$. Then there exists $r \in \text{Seq}(u)$ with $a = a \bigcirc r$. There are subsequences $f$ and $g$ of $r$ such that $r = (f, (a) b, g)$ for some $b \in n \setminus \{a\}$. Since by hypothesis $\text{deg}_1(a) = 1$, the transposition $(a) b$ is the only term in the injective sequence $r$ with $a$ in its support, we get that $a = a \bigcirc r = a \bigcirc f \circ (a) b \circ \bigcirc g = (a b) \circ \bigcirc g = b \bigcirc g \neq a$. 

\(^2\)We call $s$ injective iff $s_i = s_j \iff i = j$ for $s_i$ and $s_j$ terms in $s$; i.e., iff the function $s : j \mapsto s_j \in \text{Sym}(n)$ is injective.

\(^3\)We write $g^{-1}$ to designate the inverse of $g$, where other people may prefer instead to write $g^{-1}$. 

\[ \text{Supp}(u) = \text{supp}(u) = n. \]
A transpositional sequence \( u \) in \( \text{Sym}(n) \) with \( 3 \leq |u| < n \) fails to be perm-complete, since \( |\text{Prod}(u)| \leq |\text{Seq}(u)| \leq |u|! < n!/2 \). Thus Theorem 2.2 implies that there exist non-perm-complete injective \( u \) of length \( \left( \frac{n - 1}{2} \right) + 1 \).

Although \( |\text{Seq}(r)| = |r|! \) when \( r \) is an injective sequence in \( \text{Sym}(n) \), it is rare that \( |\text{Prod}(r)| = |r|! \).

**Theorem 2.3.** Let \( G \) be a connected graph on the vertex set \( n \geq 4 \), and let \( G \) have adjacent vertices \( x \) and \( y \) each of which is of degree 2. Then \( G \) is not perm-complete.

**Proof.** Pretend that \( G \) is perm-complete, and assume that \( u \) is an injective sequence in \( 1^{n-2} \) with \( T(u) \) isomorphic to \( G \). There are \( \begin{aligned} \text{elements } a \text{ and } b \text{ in } n \setminus \{x,y\} \text{ such that } (a,x), (x,y), \text{ and } (y,b) \text{ are edges of } G. \text{ So let } r \in \text{Seq}(u) \text{ satisfy both } x = x \circ r \text{ and } y = y \circ r. \text{ Let } r' \text{ be the sequence of length } |u| - 3 \text{ obtained by removing the terms } (a,x) \text{ and } (y,b) \text{ from } r. \text{ Let } r' = fghk \text{ be the factorization of } \overrightarrow{r} \text{ into the four } \overrightarrow{r} \text{ consecutive segments engendered by the removal from } r \text{ of those three terms. Of course } \{x, y\} \cap (\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(h) \cup \text{supp}(k)) = \emptyset. \text{ There are essentially three cases.}

**Case:** \((a,x) <_r (x,y) <_r (y,b)\). So \( r = (f, (a,x), g, (y,b), h, (y,b), k)\). Then \( x \circ r = x[\text{f}\circ(a,x)\circ g\circ(x,y) \circ h\circ(y,b)\circ k] = x[\text{f}\circ(a,x)\circ g\circ(x,y) \circ h\circ(y,b)\circ k] = a[\text{g}\circ(x,y) \circ h\circ(y,b)\circ k] = \text{a[\text{g}\circ(x,y) \circ h\circ(y,b)\circ k]}\).

**Subcase:**\( b = a \circ h\). Then \( x \circ r = y \circ k \neq y \times y \neq y \times y \neq x \times y \times y \neq x \times y \neq x \times y \neq x \).

**Subcase:**\( b = c \neq : a \circ h\). Then \( x \circ r = c \circ h \neq x \).

**Case:** \((a,x) <_r (y,b) <_r (x,y)\). Here, \( r = (f, (a,x), g, (y,b), h, (x,y), k)\). Now \( y \circ r = b[\text{g}\circ(x,y) \circ k] = b \circ h \neq y \).

In each of these three cases we see that either \( x \circ r \neq x \) or \( y \circ r \neq y \), contrary to our requirement on \( r \). □

**Remarks.** Surely both of the complete graphs \( K_2 \) and \( K_3 \) are perm-complete. In fact, the triangle \( K_3 \) is minimally so, in the sense that the removal of one edge produces a graph which is not perm-complete.

Next, we prepare the way for two more-general theorems, each of which provides sufficient conditions for non-perm-completeness.

Since, in the present context, the transformational multigraph of each minimal perm-complete sequence is simple, we let \( G \) be a simple connected graph whose vertex set is \( n \),

**Fix a sequence**\((u_0, u_1, \ldots, u_k)\) for which \( G = T(u) \), where \( u_i := (x_i, y_i) \) for each \( i \leq k \).

\( G \) denotes the digraph obtained by replacing each edge \((x,y)\) of \( G \) with the two arcs \( x \rightarrow y \) and \( x \leftarrow y \).

For \( x \in n \), the \( u \)-path from \( x \) is the subdigraph, \( \overrightarrow{u}_x := x \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_m \rightarrow y \), of \( \overrightarrow{G} \), where the vertices of this path are chosen (and given new names) in the following fashion:

- Let \( j(1) \) be the least integer \( \ell \) such that \( x \in \text{supp}(u_{\ell}) \). So \( u_{j(1)} = (x \bar{z}) \) for some \( z \in n \).
- Suppose the integers \( j(1) < j(2) < \cdots < j(i) \) to have been chosen with \( (z_{d-1} z_d) = u_{j(d)} \) for each \( d \in \{2, 3, \ldots, i\} \), let \( j(i + 1) \) be the smallest integer \( v > j(i) \) with \( z_i \in \text{supp}(u_v) \) if any such \( v \) exists, and in this event, define \( (z_{i} \ z_{i+1}) := u_{j(i+1)} \); but if there is no such \( v \) then define \( (z_m, y) := (z_{i-1}, z_i) \).

- Let \( u_x \) be the subsequence of \( u \) whose terms contribute the respective arcs that comprise \( \overrightarrow{u}_x \).

**Lemma 2.4.** \( \{\overrightarrow{u}_x : x \in n\} \) is a partition of the set of arcs comprising the digraph \( \overrightarrow{T}(u) \).

**Proof.** Since \( n = \bigcup \text{Supp}(u) \), we have that \( u_x \neq \emptyset \) for every \( x \in n \). Let \( a \rightarrow b \) be an arc in \( \overrightarrow{T}(u) \). Then \( (a,b) \) is a term \( u_i \) in the sequence \( u \). If \( i = 0 \) then let \( g \) be the identity permutation \( i/n \); but, if \( i > 0 \), let \( g := u_0 \circ u_1 \circ \cdots \circ u_{i-1} \). Then \( v_g = a \). Then \( v_g = a \), and so \( a \rightarrow b \) is an arc of \( u_x \). Furthermore, if \( v \neq q \in n \), then \( qg \neq a \) and thus \( a \rightarrow b \) is not an arc of \( u_q \). □

**Definition 3.** A perm-complete sequence \( s \in \text{Sym}(n) \) is **minimally perm-complete** iff the removal of any term of \( s \) results in a sequence which is not perm-complete.

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4For the notion of a connected graph, one may consult [3] or almost any other textbook on graph theory.

5not necessarily distinct

6some of which may be empty
Definition 4. A set $C$ of edges of a connected graph $\mathcal{G}$ is a cut set of $\mathcal{G}$ iff the removal of $C$ from the edge set of $\mathcal{G}$ results in a graph that is the union $\mathcal{G}_0 \cup \mathcal{G}_1$ of two disjoint subgraphs of $\mathcal{G}$, with each edge in $C$ having one vertex in $\mathcal{G}_0$ and the other in $\mathcal{G}_1$.

The next two theorems facilitate the identification of non-perm-complete transpositional sequences.

Theorem 2.5. Let $\mathcal{G} := (V; E)$ be a simple connected graph whose vertex set is $n$ and whose edge set is $E$, and which has a cut set $C \subseteq E$. Let $\mathcal{G}_0 := (V_0; E_0)$ and $\mathcal{G}_1 := (V_1; E_1)$ be the disjoint subgraphs of $\mathcal{G}$ gained by the removal of $C$ from $E$, where $V_i$ and $E_i$ are respectively the vertex sets and the edge sets of the two $\mathcal{G}_i$. Let $\mathcal{G}_0$ be a forest, let $2|C| < |V_0|$ and let $|V_1| \geq 2$. Then $\mathcal{G}$ fails to be perm-complete.

Proof. Assume that $\mathcal{G}$ is perm-complete. Then $\mathcal{G} = T(s)$ for some injective sequence $s := (s_0, s_1, \ldots, s_k)$ of transpositions in Sym$(n)$, where of course $|E| = k$. If $|E|$ is even then $\text{Alt}(s) = \text{Alt}(n)$, so $\text{Alt}(n) \subset \text{Prod}(s)$. But if $|E|$ is odd then $(u w) \in \text{Prod}(s)$ for some elements $u \neq w$ in $V_i$. In both cases there exists $f \in \text{Prod}(s)$ with $x f = x$ for every $x \in V_0$. The perm-completeness of $s$ implies that $f = \bigcirc t$ for some $t \in \text{Seq}(s)$.

Let $x$ be any arbitrary element in $V_0$. Since $x = x \bigcirc t$, the $t$-path $t_x \rightarrow$ that starts and ends at $x$. So, since the subgraph $\mathcal{G}_0$ is a forest, and since the sequence $t$ is injective, we can show that the path $t_x \rightarrow$ uses up two edges $e_x \notin t_x \rightarrow C$ in $C$ that contribute, to $t_x \rightarrow$, an arc $t_x \rightarrow$ from $V_0$ to $V_1$ and another arc $t_x \rightarrow$ back from $V_1$ to $V_0$.

Pretend that the arc $t_x \rightarrow$ occurs not only in the path $t_x \rightarrow$, but also in the path $t_x$, for some $x' \in V_0 \setminus \{x\}$. Then Lemma 2.4 implies that the set of arcs comprising $t_x$, is the same collection of arcs that comprise $t_x$. Viewed as a subsequence of $t$, the word $t_{x'} \rightarrow$, is a cyclic conjugate of the word $t_{x'}$.

Without loss of generality, take it that $h' \leq h$, where the transpositions $h'$ and $h$ are the first terms in $t$, under the orderings $\leq_t$ and $\leq$ in their respective supports.

Assume that $h' = h = (x x')$. Then $x \in \text{supp}(h_{1})$ where $h_{1}$ is the term immediately following $h'$ in the subsequence $t_{x'}$. Similarly, $x' \in \text{supp}(h_{1})$, where $h_{1}$ is the immediate successor of $h$ in the subsequence $t_{x}$. But obviously then $h_{1} = h_{1} = (x x') = h$ in violation of the injectivity of the sequence $t$. It follows that $h' > h$. So there is a prefix $p := \langle h', h_{2}, h_{3}, \ldots, h_{k}, h \rangle$ of the word $t_{x'}$, which induces a digraph $p^{t}$ whose vertices are the integer endpoints of the arcs in $p^{t}$, and which extends from $x' \in \text{supp}(h_{k})$ to $x \in \text{supp}(h)$.

Of course $p$ is a subsequence of $t$. Notice that $\text{supp}(h_{k}) \cap \text{supp}(h) = \{x\}$, and that $t_{0} \leq h' \leq h_{k} = h < h$, where $t_{0}$ is the first term in the sequence $t$. That $x \in \text{supp}(h_{k})$ violates a manufacturing criterion for the sequence $t_{x}$: to wit: Under the ordering $\leq$, the first term in $t_{x}$ was specified to be the first term in the sequence $t$, having $x$ in its support. That first term of $t_{x}$ is $h_{1} > h_{k}$. So $f \notin \text{Prod}(s)$. Having verified that each vertex in $V_0$ uses up at least two edges in $C$ if indeed $f = \bigcirc t$, we infer that $|V_0| \leq 2|C|$ if $s$ is perm-complete. So, since $2|C| < |V_0|$ by hypothesis, we conclude that $s$ is not perm-complete.

A modification of the proof of Theorem 2.5 will establish

Corollary 2.6. Let the hypothesis $2|C| < |V_0|$ in Theorem 2.5 be replaced by the hypothesis $|C| \leq |V_0|$, but let the other hypotheses of the theorem hold. Then $\mathcal{G}$ fails to be perm-complete.

Theorem 2.7. Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be connected graphs on the disjoint vertex sets $V_0$ and $V_1$, with $V_0 \cup V_1 = n$ and $\min(|V_0|, |V_1|) \geq 2$. Let $C$ be a nonempty set of edges, each of which has one of its vertices in $V_0$ and the other in $V_1$. Let $\mathcal{G} := (n; E) = \mathcal{G}_0 \cup C \cup \mathcal{G}_1$. Let $|C| < \min(|V_0|, |V_1|)$. Then $\mathcal{G}$ is not perm-complete.

Proof. Let $c := |C| < \min\{m, p\}$ where $V_0 = \{x_0, x_1, \ldots, x_{m-1}\}$ and where $V_1 = \{y_0, y_1, \ldots, y_{p-1}\}$. Assume that $\mathcal{G}$ is perm-complete. Then $\mathcal{G} = T(s)$ for some sequence $s$ of transpositions in Sym$(n)$.

Case: $|E|$ is odd. Let $f := (x_0 y_0 x_1 y_1 \ldots x_c y_c) = f \in \text{Bit}(n)$. Choose $r \in \text{Seq}(s)$ such that $f = \bigcirc r$. By Lemma 2.4, each of the $2c + 2$ distinct paths $r_z$ in $\mathcal{G}$, one for each $z \in \{x_0, y_0, \ldots, x_c, y_c\}$, contains an arc in $\mathcal{G}$ that is contained in no $r_{z'}$ with $z' \in \{x_0, y_0, \ldots, x_c, y_c\} \setminus \{z\}$. But $\mathcal{G}$ has only $2c$ arcs in all. Hence, $f \notin \text{Prod}(s)$. Thus we see that $\mathcal{G}$ fails to be perm-complete in the case that $|E|$ is odd.

Case: $|E|$ is even.

Subcase: $c$ is odd. Let $g := (x_0 y_0 x_1 y_1 \ldots x_c y_c) \in \text{Alt}(n)$. Choose $w \in \text{Seq}(s)$ for which $g = \bigcirc w$. As in the odd $|E|$ case, each of the $2c + 2$ paths $w_z$ in $\mathcal{G}$ for the $z \in \{x_0, y_0, x_1, y_1, \ldots, x_c, y_c\}$ uses an arc in
that is contained in no path $w_z$ with $z' \in \{x_0, y_0, x_1, y_1, \ldots, x_c, y_c\} \setminus \{z\}$ – an impossibility since $\overrightarrow{C}$ has only $2c$ arcs. So $g \notin \text{Prod}(s)$. We infer that here too $\mathcal{G}$ is not perm-complete.

Subcase: $c$ is even. We amalgamate two 2-cycles of $g$ to create a 4-cycle, thus producing the even permutation $h:=(x_0 y_0 x_1 y_1)(x_2 y_2)\ldots(x_c y_c)$. Having assumed $h \in \text{Prod}(s)$, we can choose $u \in \text{Seq}(s)$ for which $h = \bigcirc u$. Once again we have that the set of $2c + 2$ paths $u_z$ is obliged to use $2c + 2$ arcs in $\overrightarrow{C}$, but cannot do so since $\overrightarrow{C}$ has only $2c$ arcs. Again we get that $\mathcal{G}$ is not perm-complete. □

2.2 Criteria ensuring permutational completeness

When $\mathcal{G} := \langle n; E \rangle$ is a graph with vertex set $n$ and edge set $E$, and when $W \subseteq n$, then $\langle W \rangle$ denotes the subgraph $\langle W; D \rangle$ of $\mathcal{G}$ whose vertex set is $W$, and whose edge set $D$ consists of every edge $(x, y) \in E$ for which $(x, y) \in W$. This subgraph $\langle W \rangle$ of $\mathcal{G}$ is said to be induced by $W$ in $\mathcal{G}$.

We say that a subgraph $S$ of a graph $\mathcal{H}$ spans $\mathcal{H}$ if the vertex set of $S$ is that of $\mathcal{H}$. If a subgraph $S$ of $\mathcal{H}$ spans $\mathcal{H}$, and if no two distinct edges of $S$ share a vertex, then we call $S$ a perfect matching for $\mathcal{H}$.

Theorem 2.8. For $t$ an injective perm-complete transpositional sequence in $\text{Sym}(n)$, let $\mathcal{G} := \langle n; E \rangle = T(t)$. Let $\emptyset \neq W \subseteq n$, and let $x \notin n$ be a new vertex. Let $\mathcal{H} := \langle V_0; E_0 \rangle$ be the supergraph of $\mathcal{G}$ for which $V_0 := n \cup \{x\}$ is the vertex set of $\mathcal{H}$, and where $E_0 := E \cup \{(x, w) : w \in W\}$ is the edge set of $\mathcal{H}$. Let $s$ be an injective transpositional sequence in $\text{Sym}(V_0)$ such that $(p_q)g$ is a term of $s$ if and only if $(p_q) \in E_0$. Let the integer $|E_0|$ be even(odd). Given a permutation $f \in \text{Sym}(V_0)$ that is, correspondingly, even(odd):

2.8.1 If $xf \in W$ then $f \in \text{Prod}(s)$.

2.8.2 If $w_0 f = w_1 \neq w_0$ for some $\{w_0, w_1\} \subseteq W$, then $f \in \text{Prod}(s)$.

2.8.3 If $\langle W \rangle$ contains a perfect matching, and if $xf = x$ as well, then $f \in \text{Prod}(s)$.

Proof. We establish the theorem for the case where $|E_0|$ is even, and omit the (identical) proof for the case where $|E_0|$ is odd. So now let $|E_0|$ be even. Since $\mathcal{G}$ is perm-complete, we have that $|\text{Prod}(t)| = n!/2$.

Let $W := \{w_0, w_1, \ldots, w_{k-1}\} \subseteq n$ with $|W| = k$. We write $f^+ := f \cup \{(x, x)\} \in \text{Sym}(V_0)$; i.e., $f^+$ is just $f$ augmented by the 1-cycle $(x)$.

To prove 2.8.1, let

$$Q := \{h^+ \circ (w_0 x) \circ (w_1 x) \circ \cdots \circ (w_{k-1} x) : h \in \text{Prod}(t)\}.$$ 

Define $\varphi : \text{Prod}(t) \to Q$ by $\varphi(h) := h^+ \circ (w_0 x) \circ (w_1 x) \circ \cdots \circ (w_{k-1} x)$. Plainly $\varphi$ is a bijection from $\text{Prod}(t)$ onto $Q$. It follows that $|Q| = n!/2$. Now let $M := \{g : xy = w_0 \text{ and } g \in \text{Alt}(V_0)\}$. Observe that $Q \subseteq M$.

Given $g \in M$, we have $\{(x, w_0), (z_g, x)\} \subseteq g$ for some $z_g \in n$. Let $g^* := (g \setminus \{(x, w_0), (z_g, x)\}) \cup \{(x, w_0)\}$. The function $^*: g \mapsto g^*$ obviously maps $M$ bijectively onto $\text{Blt}(n)$. Hence $|M| = n!/2$. Therefore $Q = M$. But $Q \subseteq \text{Prod}(s)$.

The assertion 2.8.1 follows.

To prove 2.8.2, let $P := \{(w_0 x) \circ h^+ \circ (w_1 x) \circ (w_2 x) \circ \cdots \circ (w_{k-1} x) : h \in \text{Prod}(t)\}$. Define the function $\psi : \text{Prod}(t) \to P$ by $\psi(h) := (w_0 x) \circ h^+ \circ (w_1 x) \circ (w_2 x) \circ \cdots \circ (w_{k-1} x)$. Notice that $\psi$ is a bijection from $\text{Prod}(t)$ onto $P$. So $|P| = n!/2$. Let $L := \{g : w_0 g = w_1 \text{ and } g \in \text{Alt}(V_0)\}$. Then $P \subseteq L$.

For $g \in L$, let $y_g := w_{1g}$. Let $g^\dagger := (g \setminus \{(w_0, w_1), (w_1, y_g)\}) \cup \{(w_0, y_g)\}$. The function $^\dagger : g \mapsto g^\dagger$ obviously maps $L$ bijectively onto $\text{Blt}(V_0 \setminus \{w_1\})$. However, $|V_0 \setminus \{w_1\}| = n$. So $|L| = n!/2$. Thus $P = L$. But $P \subseteq \text{Prod}(s)$.

The assertion 2.8.2 follows.

To prove 2.8.3, take $|W| = k = 2m \geq 2$ to be even, and let $A := \{(x_0 y_0), (x_1 y_1), \ldots, (x_{m-1} y_{m-1})\}$ be a perfect matching of $\langle W \rangle$. Since $\mathcal{H}$ has an even number of edges, $\mathcal{G}$ also has an even number of edges. Thus $\text{Prod}(t) = \text{Alt}(n)$. So it suffices to show for each $h \in \text{Alt}(n)$ that $h^+ = \bigcirc s \in \text{Prod}(V_0)$ for some sequence $s$ such that $\mathcal{H} = T(s)$.

Let $h \in \text{Alt}(n)$. Choose $r \in \text{Seq}(t)$ such that $h = \bigcirc r$. We expand the length-$|t|$ sequence $r$ to a sequence $s$ of $|t| + 2m$ distinct transpositions in $\text{Sym}(V_0)$, by replacing each of the $m$ special terms, $(x_i y_i)$, in $r$ with the corresponding three-term sequence $\langle (x_i x_i), (x_i y_i), (y_i x_i)\rangle$. Plainly $h^+ = \bigcirc s$. Therefore $h^+ \in \text{Prod}(s)$.

The assertion 2.8.3 follows.

Corollary 2.9. A rectangle with one of its two diagonals is a minimal perm-complete transpositional graph.
Proof. Let $t := ((0 1), (0 2), (0 3), (1 3), (2 3))$. It is obvious from Theorems 2.2 and 2.3 that the removal of any of the five terms of $t$ results in a transpositional sequence in $\text{Sym}(4)$ which is not perm-complete. Therefore it suffices to show that $t$ itself is perm-complete.

Since the triangle graph is perm-complete, Theorem 2.3 implies that $\text{Prod}(t)$ contains every $h \in \text{Blt}(4)$ except possibly for the missing diagonal, $(1 2)$. But $(1 2) = (0 1) \circ (2 3) \circ (0 2) \circ (1 3) \circ (0 3) \in \text{Prod}(t)$. □

By a bike on $n + 2$ vertices we mean any graph isomorphic to the labeled graph $\mathcal{B}_n$, whose edge set has these $2n + 1$ edges: the “axle” (0 1) and the $2n$ “spokes” (0 $i$) and (1 $i$) for the $i \in \{2, 3, \ldots, n + 1\}$.

We already observed that the tree with one edge, $\mathcal{B}_0 = \mathcal{K}_2$, and the triangle, $\mathcal{B}_1 = \mathcal{K}_3$, are minimal perm-complete. By Corollary 2.3 we have that the proper subgraph $\mathcal{B}_2$ of $\mathcal{K}_4$ is minimal perm-complete.

As usual, $\omega := \{0, 1, 2, \ldots\}$. Let $\langle x_1, x_2, \ldots \rangle$ be an injective sequence in $\omega \setminus 2 := \{2, 3, 4, \ldots\}$. We recursively define an infinite sequence $\langle c_{2t} \rangle_{t=1}^{\infty}$ of finite sequences of transpositions in $\text{Sym}(\omega)$ thus:

$c_{(2)} := ((1 2), (0 2))$
$c_{(2t+2)} := c_{(2t)}(0, x_{2t+1})$, $(1, x_{2t+1}), (1, x_{2t+2}), (0, x_{2t+2})$

Lemma 2.10. Let $r_{(2t)} := ((0 x_1), c_{(2t)}, (1 x_1))$ for $t > 0$ an integer. Then $\bigcirc r_{(2t)} = (0 1)(x_1 x_2 \cdots x_{2t})$.

Proof. Since $\bigcirc r_{(2)} = (0 x_1) \circ (1 x_2) \circ (0 x_2) \circ (1 x_1) = (0 1)(x_1 x_2)$, the basis holds for an induction on $t$.

Now pick $t \geq 1$, and suppose that $\bigcirc r_{(2t)} = (0 1)(x_1 x_2 \ldots x_{2t})$. Then

$\bigcirc r_{(2t+2)} = \bigcirc r_{(2t)} \circ (0 1)(x_1 x_2 \ldots x_{2t}) \circ (1 x_{2t+1}) \circ (0 x_{2t+1}) \circ (1 x_{2t+2}) \circ (0 x_{2t+2}) \circ (1 x_1) = (0 1)(x_1 x_2 \ldots x_{2t+2}) \circ (1 x_1) \circ (0 x_{2t+1}) \circ (1 x_{2t+1}) \circ (0 x_{2t+2}) \circ (1 x_1) = (0 1)(x_1 x_2 \ldots x_{2t+1} x_{2t+2})$.

So $\bigcirc r_{(2t+2)} = (0 1)(x_1 x_2 \ldots x_{2t+2})$. □

Theorem 2.11. $\mathcal{B}_n$ is a minimal perm-complete graph for every nonnegative integer $n$.

Proof. Recall that the theorem holds for $0 \leq n \leq 2$. So we will establish it for $n \geq 3$. We show that the removal of an edge from $\mathcal{B}_n$ results in a subgraph which fails to be perm-complete. So, if $\mathcal{B}_n$ is perm-complete then it is minimal as such.

The removal of a spoke from $\mathcal{B}_n$ results in a subgraph that has a vertex of degree 1. By Corollary 2.2 such a subgraph is not perm-complete. So consider the subgraph $\mathcal{G}_n := \mathcal{B}_n \setminus (0 1)$ obtained by removing the axle from $\mathcal{B}_n$. Now $\mathcal{G}_n = \mathcal{G}_{n,0} \cup \mathcal{E} \cup \mathcal{G}_{n,1}$ is a disjoint union, where $\mathcal{G}_{n,0}$ is the one-edge subgraph (0 2), where $\mathcal{G}_{n,1}$ is the triangle on the $n$ vertices $- 1, 3, 4, \ldots n, n + 1$ and whose edge set is $\{(1 j) : 3 \leq j \leq n + 1\}$, and where $\mathcal{E}$ is the subgraph whose vertex set is all of $n + 2$ and whose edge set is $C := \{(1 2) \cup \{(0 j) : 3 \leq j \leq n + 1\})$. But $C$ is the cut set connecting $\mathcal{G}_{n,0}$ to $\mathcal{G}_{n,1}$ to form $\mathcal{G}_n$. So Corollary 2.3 implies $\mathcal{G}_n$ is not perm-complete.

It remains only to show that $\mathcal{B}_n$ perm-complete. The basis of an induction is already established. So pick an integer $n \geq 3$, and suppose for any nonnegative $i < n$ that any graph isomorphic to $\mathcal{B}_i$ is perm-complete. Let $s$ be a transpositional sequence in $\text{Sym}(n + 2)$ such that $\mathcal{B}_n = T(s)$.

Of course $\text{Prod}(s) \subseteq \text{Blt}(n + 2)$. But we do need to show that $\text{Blt}(n + 2) \subseteq \text{Prod}(s)$.

Claim: For every even positive integer $2t \leq n$, the set $\text{Prod}(s)$ contains every $f \in \text{Blt}(n + 2)$ which has a cyclic component of length $2t$.

To prove this Claim, pick $2t \in \{2, 3, \ldots, n\}$. Let $\langle x_1, x_2, \ldots, x_{2t-1}, x_{2t} \rangle$ be any injective sequence in the set $\{2, 3, \ldots, n, n + 1\}$, and let $X$ be the $(2t)$-membered set $\{x_1, x_2, \ldots, x_{2t}\}$. Pick a sequence $v$ of transpositions such that $\mathcal{B}_n \setminus X = T(v)$, where $\mathcal{B}_n \setminus X$ is the graph obtained by removing the $2t$ vertices in $X$ from $\mathcal{B}_n$. Since $B_n \setminus X$ is isomorphic to $\mathcal{B}_{n-2}$, we have by the inductive hypothesis that $\mathcal{B}_n \setminus X$ is perm-complete. It follows that $\text{Prod}(v) = \text{Blt}(n + 2) \setminus X$.

Let $r$ be the transpositional sequence $r_{(2t)}$ of Lemma 2.10. Define $Q := \{0 \circ g : g \in \text{Blt}(n \setminus X)\}$. Then, by Lemma 2.10 we get that $Q = \{(x_1 x_2 \ldots x_{2t})(0 1) \circ g : g \in \text{Blt}(n \setminus X)\}$. Furthermore, $Q \subseteq \text{Blt}(n + 2)$. For each $g \in \text{Blt}(n \setminus X)$, the concatenation $r v_g$ is an element in $\text{Seq}(s)$, where $g = \bigcirc v_f$ for some $v_f \in \text{Seq}(v)$. Therefore $Q \subseteq \text{Prod}(s)$. Thus, when both $f \in \text{Blt}(n + 2)$, and $f$ has an even length cycle whose support is a subset of $\{2, 3, \ldots, n + 1\}$, then $f \in \text{Prod}(s)$.

For every $x \in \{2, 3, \ldots, n + 1\}$, the graph $T(a_x) := \mathcal{B}_n \setminus \{x\}$ is perm-complete by the inductive hypothesis, and hence by Theorem 2.8.1 we have that $\text{Prod}(a_x)$ contains every $f_x \in \text{Blt}((n + 2) \setminus \{x\})$ such that $x f_x = x$; those $f_x$ include every one with an even-length cyclic component in $(n + 2) \setminus \{x\}$. The claim is established.

The theorem follows from the Claim, since every $f \in \text{Blt}(n + 2)$ has at least one even-length cycle. □
We call a vertex \( v \) of a graph \( \mathcal{G} \) **central** iff \( v \) is adjacent to every other vertex of \( \mathcal{G} \).

**Corollary 2.12.** If a connected graph \( \mathcal{G} \) has at least two central vertices then \( \mathcal{G} \) is perm-complete.

**Proof.** The corollary is immediate by Theorems 2.11 and 2.1. \( \square \)

**Corollary 2.13.** Every finite complete graph is perm-complete.

The following examples provide instances where the converse of Corollary 2.12 fails.

**Proposition 2.14.** Each of the following five transpositional sequences is minimally perm-complete:

- \( a := \langle (0,1), (0,2), (0,3), (0,4), (1,2), (2,3), (3,4) \rangle \)
- \( b := \langle a, (25), (35) \rangle \)
- \( c := \langle (0,1), (0,2), (0,3), (1,2), (14), (2,3), (3,4) \rangle \)
- \( d := \langle b, (46), (56) \rangle \)

**Partial Proof.** We shall establish our claim about \( a \), and leave the other four sequences for our reader.

It is easy to see by Corollaries 2.2 and 2.3 that the removal of an edge from the graph \( T(a) \) produces a graph which is not perm-complete. So it remains only show that \( a \) is perm-complete.

\( B_2 \) is perm-complete. Referring to Theorem 2.8 identify \( \mathcal{G} \) to be the copy of \( B_2 \) whose vertex set is \( \{0, 1, 2, 3\} \), whose \( W \) is \( \{0, 3\} \), and whose \( x \) is the vertex 4. By Theorem 2.8 and symmetry considerations, it is easy to see that \( \text{Prod}(a) \) contains every element in \( \text{Bit}(5) \) except maybe \((14)\). But, since \((14) = (02) \circ (34) \circ (01) \circ (23) \circ (04) \circ (12) \circ (03)\), we have that \((14) \in \text{Prod}(a)\). So \( a \) is minimally perm-complete.

Lest it be surmised that every graph which is an amalgamation of triangles is perm-complete, we offer

**Proposition 2.15.** Let \( e := \langle (0,1), (0,3), (1,2), (13), (2,3), (24), (34), (37), (45), (47), (56), (57), (67) \rangle \). The transpositional sequence \( e \) is not perm-complete.

**Proof.** \( T(e) \) consists of two copies of \( B_2 \) conjoined by a three-element cut set. So Theorem 2.7 implies that \( T(e) \) is not perm-complete. \( \square \)

By an \( n \)-wheel we mean any graph isomorphic to \( W_n := \langle n + 1; E \rangle \), where \( E \) contains the following 2\( n \) edges: \((0,i)\) for every \( i \in \{1, 2, \ldots, n\} \) and \((i, i + 1)\) for every \( i \in \{1, 2, \ldots, n - 1\} \) and finally also \((1,n)\).

By Corollary 2.6 and Theorem 2.7 if \( W_n \) is perm-complete then \( W_n \) is minimally perm-complete.

**Conjecture.** \( W_n \) is perm-complete for every \( n \geq 3 \).

### 3 Conjugacy invariance

The present section will lay the ground work for, and thereafter establish, the following characterization of the conjugacy invariant transpositional sequences having multigraphs on the vertex set \( n \) that are connected.

**Theorem 3.1.** Let \( u \) be a transpositional sequence in \( \text{Sym}(n) \) with \( 2 \leq n \in \mathbb{N} \) whose multigraph \( T(u) \) is connected on the vertex set \( n \). If \( n = 2 \) then \( u \) is both perm-complete and CI. If \( n = 3 \) then \( u \) is CI if and only if either \( |u| \) is odd or \( T(u) \) is a multitree with at least one simple multiedge.

For \( n \geq 4 \), the sequence \( u \) is CI if and only if \( T(u) \) is a multitree in which no vertex is an endpoint of more than one non-simple multiedge, and in which each even-multiplicity multiedge is a multitree whose non-leaf vertex has only one non-leaf neighbor.

#### 3.1 Constant-product sequences

We say that a permutational sequence \( s \) is **constant-product** iff \( |\text{Prod}(s)| = 1 \). The class of constant-product \( s \) is antipodal to the class of perm-complete \( s \).

It is clear that \( s := \langle s_0, s_1, \ldots, s_k \rangle \) is constant-product if \( s_i \circ s_j = s_j \circ s_i \) whenever \( 0 \leq i < j \leq k \). Moreover, \( s_i \circ s_j = s_j \circ s_i \) if either \( \text{supp}(s_i) \cap \text{supp}(s_j) = \emptyset \) or \( s_i \) and \( s_j \) are powers \( s_i = f^p \) and \( s_j = f^q \) of a common permutation \( f \); that is to say, \( s \) is constant-product if \( s \) is boring.

Do there exist non-boring constant-product permutational sequences?
We paraphrase a theorem of Eden and Schützenberger (Page 144 of [3]), which remarks upon certain injective transpositional sequences \( u \), and which touches on this question.

For each \( v \in n \), let \( u_{(v)} \) be the subsequence of \( u \) of which \( u_{(v),j} \) is a term if and only if \( v \in \text{supp}(u_{(v),j}) \).

**Eden-Schützenberger Theorem.** When the transpositional multigraph \( T(u) \) is a simple tree and also \( s \in \text{Seq}(u) \), then \( \bigcirc s = \bigcirc u \) if and only if \( s_{(v)} = u_{(v)} \) for every \( v \in n \).

The paucity of non-boring constant-product permutational sequences, raises our interest to its superclass \( O(n) \) of permutational sequences \( s \) for which the order of the permutation \( \bigcirc x \) is constant over all \( x \in \text{Seq}(s) \). The class of conjugacy invariant sequences is a natural proper subclass of \( O(n) \).

### 3.2 Preliminaries

We call a binary relation \( a \subseteq X \times X \) conjugate to \( b \subseteq X \times X \) and write \( a \cong b \), if \( b = \{(xf, yf) : (x, y) \in a\} \) for some permutation \( f \in \text{Sym}(X) \). Equivalently, \( a \cong b \) iff \( g^{-1} \circ a \circ g = b \) for some \( g \in \text{Sym}(X) \). Plainly \( \cong \) is an equivalence relation on the family \( \mathcal{P}(X \times X) := \{r : r \subseteq X \times X\} \) of all binary relations on the set \( X \).

We define the world of \( c \subseteq X \times X \) to be \( \mathcal{S}(c) := \text{Dom}(c) \cup \text{Rng}(c) \). It is commonplace that \( a \cong b \) if and only if \( b = g^{-1} \circ a \circ g \) for some \( g \in \text{Sym}(\mathcal{S}(a) \cup \mathcal{S}(b)) \). Of course \( b = g^{-1} \circ a \circ g \) if and only if \( g \circ b = a \circ g \).

In this paper we restrict our attention to those binary relations which are permutations on the set \( n \). Whenever \( (a, b) \subseteq \text{Sym}(n) \), we have not only that \( a \circ b \cong b \circ a \) but also that \( a \cong a^{-1} \).

For \( n > 0 \) an integer, \([n]\) denotes the set \( \{1, 2, \ldots, n\} \). (But remember that \( n \) denotes \( \{0, 1, \ldots, n-1\} \).)

**Proposition 3.2.** Let \( b := \langle b_0, b_1, \ldots, b_{k-1} \rangle \) be a sequence in \( \text{Sym}(n) \). For \( b^{(i)} := \langle b_i, b_{i+1}, \ldots, b_{k-2}, b_{k-1}, b_0, b_1, \ldots, b_{i-1} \rangle \), known as a “cyclic conjugate” of \( b \), satisfies \( b^{(i)} \cong b \).

**Proof.** Let \( p := \langle b_0, b_1, \ldots, b_{k-1} \rangle \) and \( s := \langle b_i, b_{i+1}, \ldots, b_{k-1} \rangle \). Then \( \bigcirc b^{(i)} = \bigcirc ps = \bigcirc s \circ \bigcirc p \cong \bigcirc p \circ \bigcirc s = \bigcirc ps = \bigcirc b \).

**Definition.** For \( s := \langle s_0, s_1, \ldots, s_{k-1} \rangle \) a sequence, \( s^R := \langle s_{k-1}, \ldots, s_0 \rangle \) is called the reverse of \( s \).

**Proposition 3.3.** Let \( t := \langle t_0, t_1, \ldots, t_{k-1} \rangle \) be a transpositional sequence in \( \text{Sym}(n) \). Then \( \bigcirc (t^R) \cong \bigcirc t \).

**Proof.** \( \bigcirc (t^R) = t_{k-1} \circ t_{k-2} \circ \cdots \circ t_1 \circ t_0 = t_{k-1}^{-1} \circ t_{k-2}^{-1} \circ \cdots \circ t_1^{-1} \circ t_0^{-1} = (t_0 \circ t_1 \circ \cdots \circ t_{k-1}^{-1})^{-1} = (\bigcirc t)^{-1} \cong \bigcirc t \) since \( t_i^{-1} = t_i \) when \( t_i \) is a transposition.

Dan Franklin: By Proposition 3.3 if \( s \) is a transpositional sequence and \( f \in \text{Prod}(s) \), then \( f^- \in \text{Prod}(s) \).

**Terminology.** When \( w := \langle x \rangle \) is a length-one sequence, then \( x \) may serve as a nickname for \( w \). If a sequence \( w \) is of length \( |w| = k \) in \( X \), and if \( w \) occurs exactly \( m \) times as a term in \( r = \langle w, w, \ldots, w \rangle \), then we write \( r := w^b(m) \). That is, \( w^b(m) \) is the “block” consisting of exactly \( m \) adjacent occurrences of \( w \). Thus \( r \) has length \( m \) when seen as a sequence in the set \( \{w\} \), but \( |r| = mk \) when \( r \) is viewed as a sequence in \( X \).

Whereas \( w^b(m) \) denotes a sequence comprised of \( m \) adjacent occurrences of the subsequence \( w \), the expression \( \bigcirc w^m \) denotes the compositional product \( \bigcirc w \circ \bigcirc w \circ \cdots \circ \bigcirc w \) of \( m \) adjacent occurrences of the permutation \( \bigcirc w \). That is to say, if \( r := w^b(m) \) is a sequence in \( \text{Sym}(n) \) then \( \bigcirc r = \bigcirc (\bigcirc w^m) = (\bigcirc w)^m \).

Each sequence in \( \text{Sym}(2) \) is both perm-complete and CI. If \( |s| < 3 \) for \( s \) a sequence in \( \text{Sym}(n) \) then \( s \) is CI. However, for \( s \) in \( \text{Sym}(n) \) with \( n \geq 3 \) and with \( |s| \geq 3 \), the plot thickens.

When \( a \neq b \) are vertices in a multigraph \( \mathcal{G} \), the multiplicity in \( \mathcal{G} \) of its multiedge \( (a, b) \) is the number \( \mu_{\mathcal{G}}(a, b) \) of simple edges in the bundle comprising that multiedge. Thus, when \( \mu_{\mathcal{G}}(a, b) = 0 \), there is no
simple edge in $G$ connecting $a$ with $b$. But, when $\mu_G(a,b) = 1$, then the multiedge $(a,b)$ is itself simple in $G$. For $u \in 1^{n-2}2^1$, the multiplicity $\mu_u(a,b)$ in $u$ of the transposition $(a,b)$ as a term in $u$ equals $\mu_T(u)(a,b)$.

**Reminder:** $f \in 1^{n-2}2^1$ says merely that $f$ is a transposition in $\text{Sym}(n)$. A multigraph $G$ we call CI iff $G$ is isomorphic to $T(t)$ for a CI sequence $t$ in $1^{n-2}2^1$. Without ado we will apply obviously corresponding terminology interchangeably to transpositional sequences and to isomorphisms of multigraphs.

### 3.3 Conjugacy invariant transpositional sequences

We proceed to identify the CI transpositional sequences $u$ in $\text{Sym}(n)$. It suffices to treat those such $u$ for which $T(u)$ is a connected multigraph on the vertex set $n$; this narrow focus is embodied in Theorem 3.1.

**Theorem 3.4.** Let $T(u)$ be a multitree with no even-multiplicity multiedges, and none of whose vertices lie on more than one non-simple multiedge. Then $\text{Prod}(u) \subseteq n^1$.

**Proof.** We induce on $|u| \geq n - 1$.

- **Basis Step:** The theorem is easily seen to hold when $T(u)$ is simple. Proofs occur in [2] and in [6].
- **Inductive Step:** Pick $k > n$. Suppose the theorem holds for all $u$ for which $|u| \in \{n - 1, n, \ldots, k - 1\}$. Let $u \in \{k - 1, k\}$, and let $u$ satisfy the hypotheses of the theorem.

Let $(x,y)$ be a multiedge of $T(u)$ such that neither $x$ nor $y$ is an endpoint of any non-simple multiedge $(x',y') \neq (x,y)$. Let $v$ be a sequence created by inserting into $u$ two additional occurrences, $(x,y)_1$ and $(x,y)_2$, of the transposition $(x,y)$. Thus $v = (a, (x,y)_1, b, (x,y)_2, c)$ for some subsequences $a$, $b$, and $c$ of $u$ for which $u = abc$. If $|b| = 0$ then obviously $\overline{v} = \overline{u} \in n^1$. So suppose that $|b| > 0$.

Let the first term of $b$ be $(t,z)$. If $(x,y) = (t,z)$ then $(x,y)_1 \circ (t,z) = \iota$, and so again $\overline{v} = \overline{u}$. But, if $(x,y) \cap (t,z) = \emptyset$, then $(x,y)_1 \circ (t,z) = (z,y) \circ (x,y)_1$, and $(x,y)_1$ will have migrated one space to the right in $v$ towards $(x,y)_2$. So take it that $y = t$ and that $\{x,y\} \cap \{y,z\} = 1$.

Now, $(x,y)_1 \circ (y,z) = (x,z) \circ (x,y)_1$. The tree $T(v)$ does not have the triangle $(xy)_1, (yz), (xz)$ as a subgraph. So the transposition $(x,z)$ does not occur as a term in $u$. Indeed, if $v$ satisfies the hypotheses of the lemma, then the multiplicity in $v$ of $(yz)$ is 1, since the multiplicity of $(xy)$ in $v$ is greater than one. Thus the tree $T(w)$ is just the modification of $T(v)$ obtained by the replacement of the simple multiedge $(yz)$ of $T(v)$ by the simple multiedge $(xz)$.

That is, $w$ has a single occurrence of the transposition $(x,z)$ as a term but has no $(yz)$ terms, whereas $v$ has a single occurrence of $(yz)$ but has no occurrences of $(xz)$. Clearly $w$ also satisfies the hypotheses of the lemma, and $|w| = |v| \in \{k + 1, k + 2\}$, since $w = (a', (x,y)_1, b', (x,y)_2, c)$ where $a' = (a, x,z)$ and where $b'$ is the sequence created by removing the leftmost term $(y,z)$ of $b$. So in this fashion too, $(x,y)_1$ migrates one space rightward towards $(x,y)_2$. The rightward migrations of $(x,y)_1$ continue until $(x,y)_1$ either abuts on $(x,y)_2$ or on some occurrence of $(xy)$ to the left of $(x,y)_2$. Thus the rightward migrations of $(x,y)_1$ ultimately result in a sequence $w'$ with $|w'| \leq k$ and for which $\overline{w'} = \overline{u}$. Thus the inductive step is successful, and the theorem follows.

**Theorem 3.5.** Let $u$ be a sequence in $1^{12}2^1$ with $\bigcup \text{Supp}(u) = 3$. Then $u$ is CI if and only if either $|u|$ is odd or $T(u)$ is a multitree with at least one simple multiedge.

**Proof.** If $|u|$ is odd then $\text{Prod}(u) \subseteq 1^{12}2^1$, and so $u$ is CI. For the rest of the proof we take $|u|$ to be even.

Let $T(u)$ be a tree with a simple multiedge $(0,1)$. If the multiedge $(1,2)$ is simple too, then $u$ is CI. So take it that $u := ((0,1), (1,2)^{(2i+1)})$ for some $i \geq 1$. Let $r \in \text{Seq}(u)$. Then $r = ((12)^{(2i)}, (01), (12)^{(2i+1)-j})$ for some $j \in 2i + 2$. So $\overline{r} = (12)^j \circ (01) \circ (12)^{(2i+1-j)}$. If $j$ is even then $2i+1-j$ is odd, whence $\overline{r} = (01) \circ (12) = (021) \in 3^1$, and if $j$ is odd then $2i+1-j$ is even, and so $\overline{r} = (12) \circ (01) = (012) \in 3^1$. Therefore $u$ is CI in the event that $T(u)$ is a multitree, one of whose multiedges has multiplicity one.

To establish the converse, we first consider the case where $T(u)$ is a multitree, and assume it has no simple multiedge. We can take it that $u := ((01)^{(2i)}, (12)^{(2i)})$, where $i \geq 2$ and $j \geq 2$ and $i+j$ is even. The argument about this multitree obviously reduces to only two cases.

- **Case** $i = j = 2$. Then $\overline{u} = i3 \not\equiv (012) = (01) \circ (12)^2$.
- **Case** $i = j = 3$. Then $\overline{u} = (021) \not\equiv i3 = (021)^3 = (01) \circ (12)^3$.

Now suppose that $T(u)$ is a multitriangle with $u := ((01)^{(2a)}, (12)^{(2b)}, (20)^{(2c)})$, where $1 \leq \min\{a,b,c\}$ and where $a + b + c$ is even. The argument again reduces to two cases.

- **Case** $a = b = 1$ and $c = 2$. Then $\overline{u} = (021) \not\equiv i3 = (01) \circ (20) \circ (12) \circ (20)$.
Case. $a = b = c = 2$. Then $\bigcirc u = i3 \not\in \{0 1 2\} = (0 1) \circ (1 2) \circ (2 0)$.

Henceforth $u$ is a sequence in $1^{n-2}2^1$ for which $T(u)$ a connected multigraph whose vertex set is $n$. We have characterized the CI sequences for $n = 4$. From now on, $n \geq 4$. The $u$ we will be treating are of two sorts: One: $T(u)$ is a multitree. Two: $T(u)$ has a circuit subgraph. First we treat Sort One.

By an $m$-twig of a multigraph $G$ we mean any multiplicity-$m$ multiedge $(v, w)$, one of whose vertices has exactly one neighbor in $G$. If $w$ is the only neighbor of the vertex $v$, then $v$ is the leaf of the multitwig.

**Theorem 3.6.** Let the transpositional multitree $T(u)$ have exactly $b$ multiedges of even multiplicity, where $u := \langle u_0, u_1, \ldots, u_{k-1} \rangle$ is of length $|u| := k \geq 3$ in $1^{n-2}2^1$ with $n \geq 4$. Let the following two conditions hold:

3.6.1 No vertex lies on more than one non-simple multiedge.

3.6.2 Each even-multiplicity multiedge is a multitwig whose non-leaf vertex has exactly two neighbors. Then $\text{Prod}(u) \subseteq 1^b(n-b)^1$, and therefore $u$ is CI.

**Proof.** Given $n \geq 4$, we induce on $b \in \{0, 1, \ldots, n-1\}$.

**Basis Step: $b = 0$.** This is just Theorem 3.4

**Inductive Step: Suppose, for each $m \in \{4, 5, \ldots, n-1\}$ and each $X \subseteq n$ with $|X| = m$, that the theorem holds for every transpositional sequence $t$ in $\text{Sym}(X)$ for which $T(t)$ is a multitree with vertex set $X$. By hypothesis, $u$ is a sequence in $1^{n-2}2^1$ that satisfies 3.6.1 and 3.6.2, where $T(u)$ has exactly $b$ even-multiplicity multitwigs, and where all of the non-multitwig multiedges of $T(u)$ are of odd multiplicity. Suppose $b \geq 1$.

Let $(0 1)$ be an even-multiplicity multitwig of $T(u)$ with leaf 0. Let $v$ be the subsequence of $u$ obtained by removing all occurrences of $(0 1)$ as terms in $u$. Then $T(v)$ is a multitree on the set $X := n \setminus \{0\}$. Obviously $T(v)$ is a multitree that satisfies 3.6.1 and 3.6.2 and that has exactly $b-1$ even-multiplicity multitwigs. Since $|X| = n - 1$, the inductive hypothesis implies that $\text{Prod}(v) \subseteq 1^{b-1}((n-1)-(b-1))^1 = 1^{b-1}(n-b)^1$ and that $v$ is CI. By 3.6.2, the only multiedge of $T(v)$, other than $(0 1)$, to share the vertex 1 is a simple multiedge $(1 x)$ of $T(v)$, and $(1 x)$ is the only term of $u$ that fails to commute with $(0 1)$. So $f \leftrightarrow f \cup (0 1)$ is a one-to-one matching $\text{Prod}(v) \leftrightarrow \text{Prod}(u)$. It follows that $\text{Prod}(u) \subseteq 1^b(n-b)^1$ since $\text{Prod}(v) \subseteq 1^{b-1}(n-b)^1$ by the inductive hypothesis.

Lemma 3.5 gives necessary and sufficient conditions for $u$ to be CI when $n \leq 3$. Theorems 3.4 and 3.6 give sufficient conditions for $u$ to be CI when $n \geq 4$. We will show that those conditions are also necessary for $n \geq 4$. The crux is to establish that, if the connected multigraph $T(u)$ on the vertex set $n \geq 4$, fails to be a multitree satisfying both 3.5.1 and 3.5.2, then $u$ is not CI. This project involves two subprojects:

The first such subproject will show that, if $T(u)$ is a “pathological” multitree – which is to say, one for which either 3.6.1 or 3.6.2 fails, then $u$ cannot be CI.

The last will show that, if $n \geq 4$ and $T(u)$ has a circuit submultigraph, then again $u$ cannot be CI.

For the balance of §3, the expression $u$ will denote a sequence in $1^{n-2}2^1$ with $|u| \geq n - 1 \geq 3$, and such that the transpositional multigraph $T(u)$ is connected on the vertex set $n$.

**Subproject One: To prove that, if $T(u)$ is a pathological multitree, then $u$ fails to be CI**

We call a sequence $s$ **reduced** iff no entity occurs more than three times as a term in $s$.

When at least one entity occurs as a term in a sequence $a$ more than three times, we may produce a reduced subsequence $c$ of $a$ by means of a string of “elementary reductions”:

If $x$ occurs as a term more than three times in $a$, then a subsequence $b$ of $a$ is an **elementary reduction** of $a$ if $b$ is obtained by removing from $a$ two occurrences of $x$. The resulting such $b$ is of length $|a| - 2$.

A **reduction** of $a$ is any reduced subsequence of $a$ that results from a sequence of elementary reductions.

Clearly each sequence $u$ in $1^{n-2}2^1$ has a unique reduced subsequence. If $r$ is a reduced subsequence of a transpositional sequence $s$ then of course $\text{Seq}(r)$ is the set of all reduced subsequences of elements in $\text{Seq}(s)$.

We will employ the contrapositive version of the following obvious fact:

**Lemma 3.7.** A **reduced subsequence** of a CI transpositional sequence is CI.

We henceforth take it that all of our transpositional sequences are reduced, unless specified otherwise.
Lemma 3.8. If $\mu_u(01) = 2$, but if (01) is not a multitwig of the multtree $T(u)$, then $u$ fails to be CI.

Proof. In the spirit developed earlier, “$\mu_u$” is an abbreviation for “$\mu_{T(u)}$”.

Let $\mu_u(01) = 2$ and the multiedge (01) of $T(u)$ not be a multitwig. Let $v$ be the subsequence of $u$ resulting from the removal of $u$ of its two occurrences of (01) as terms. $T(v)$ is the disjoint union $G_0 \cup G_1$ of two multitrees, each of which has a vertex set containing more than one vertex since the excised multiedge (01) of $T(u)$ was not a multitwig. So $v$ consists of two nonempty complementary subsequences $v_0$ and $v_1$, with $|v_0| + |v_1| = |v| = |u| - 2 \geq n - 3 \geq 1$, and for which $G_0 = T(v_0)$ and $G_1 = T(v_1)$. That is to say, the terms of $v_i$ are the simple edges of $G_i$ for each $i \in \{0, 1\}$.

Let $f_0$ be the component of $\bigcirc v_0$ such that $0 \in supp(f_0)$, and let $f_1$ be the component of $\bigcirc v_1$ such that $1 \in supp(f_1)$, observing that neither $f_0$ nor $f_1$ is a 1-cycle. Since our real concern is Seq($u$), we can take it that $u = (v_0, (01))^{\beta(2), v_1}$ and that $v = v_0 v_1$. Of course, 0 is a vertex in $G_0$ and 1 is a vertex in $G_1$. Then $f_0$ and $f_1$ are disjoint nontrivial cyclic components of the permutation $\mu u = \bigcirc v = \bigcirc v_0 \bigcirc v_1$.

Define $u' := (v_0, (01), v_1) \in$ Seq($u$). All of the components of $\bigcirc u$ other than $f_0$ and $f_1$ are components also of $\bigcirc u'$. So the only change made to $\bigcirc u$ that creates $\bigcirc u'$ is the replacement of the two components $f_0$ and $f_1$, with a new pair (0) and $h$, where $h$ is a cycle with $1 \in supp(h)$, and with $|h| = |f_0| + |f_1| - 1$. So $\mu u' \not\simeq \mu u$, and hence $u$ is not CI.

Lemma 3.8 shows without loss of generality for $n \geq 4$ that, if the transpositional multtree $T(u)$ has an even-multiplicity multiedge which is not a multitwig, then $u$ cannot be CI.

Corollary 3.9. Let $\mu_u(01) = \mu_u(12) = 2$. Then $u$ is not CI.

Proof. Pretend that $u$ is CI. It follows by Lemma 3.8 that both of the multiedges (01) and (12) of the multtree $T(u)$ are multitwigs. Therefore, since $n \geq 4$, there exists $x \in n \setminus 3$ for which (1x) is a multiedge of $T(u)$. Let $v$ be the subsequence of $u$ that is produced by the removal from $u$ of both of the terms that are occurrences of the transposition (01) and both of the terms that are occurrences of (12). Then $|v| = |u| - 4 \geq 5 - 4 = 1$. Let $f$ be the component of $\bigcirc v$ with either $f = (1)$ or $1 \in supp(f)$. Observe that $\{0, 2\} \not\subseteq supp(f) = \emptyset$. Since our interest lies in the sets Seq($u$) and Seq($v$), we can suppose that $u = (01)^{\beta(2), v_0} (12)^{\beta(2), v_1}$. Of course $f$ is a component of $\bigcirc v = \bigcirc u$. Defining $u' := \{(01), (12)^{\beta(2)}, v\} \in$ Seq($u$), we see that $\bigcirc u' = (012) \circ \bigcirc v$, a permutation which is identical to the permutation $\bigcirc u$ in all component cycles that are disjoint from $3 \cup supp(f)$. Where (0), (2), and $f$ are components of $\bigcirc u$, the permutation $\bigcirc u'$ instead has the cycle $(012) \circ f$ of length $|f| + 2$. Thus $\bigcirc u' \not\simeq \bigcirc u$, and so $u$ is not CI.

Corollary 3.10. Let $u := (01)^{\beta(2), v_0} (12)^{\beta(3), v_1}$ and $v$ be sequences in $1^{n-2}2^2$, where neither (01) nor (12) is a term in $v$. Then $u$ is not CI.

Proof. Assume that $u$ is CI. By Lemma 3.8, the multiedge (01) of the multtree $T(u)$ is a multitwig of $T(u)$. So there is a component $f$ of the permutation $\bigcirc v$ to which exactly one of the following two cases applies.

Case: Either $2 \in supp(f)$ or $f = (2)$, and $\bigcirc u = (01)^2 \circ (12)^{\beta(3)} \circ \bigcirc v = (12) \circ \bigcirc v$. So $f_2 := (12) \circ f$ is a cyclic component of $\bigcirc u$. Note: $|f_2| = |f| + 1$, since the point 1 is incorporated into the cycle in order to create $f_2$. [Paradigm example: When $f := (234)$ then $f_2 = (12) \circ f = (12) \circ (234) = (1342)].$ Define $u_2 := (12, (01), (12), (01), (12), v) \in$ Seq($u$). Then $\bigcirc u_2 = (01) \circ \bigcirc v = (012) \circ \bigcirc v$, and $f$ is a cyclic component of $\bigcirc u_2$.

Case: Either $1 \in supp(f)$ or $f = (1)$, and $\bigcirc u = (12) \circ \bigcirc v$. So $f_1 := (12) \circ f$ is a cyclic component of $\bigcirc u$. Let $u_1 := (01), (12)^{\beta(3)}, (01), v)$. Then $\bigcirc u_1 = (02) \circ \bigcirc v = (02) \circ \bigcirc v$, and $f$ is a component of $\bigcirc u_1$. But $|f_1| = |f| + 1$.

We showed, for each $i \in \{1\}$, that $|f_i| = |f| + 1$. Moreover, $\bigcirc u$ has one more 1-cycle and one fewer 2-cycles than $\bigcirc u_i$, as well as other cyclic components of $\bigcirc u_i$ are the same as those of $\bigcirc u$. Hence $\bigcirc u_i \not\simeq \bigcirc u$ in both cases. Thus our assumption fails. Therefore $u$ is not CI.

Lemma 3.11. Let $u := (01)^{\beta(3), v_0} (12)^{\beta(3), v_1} v_2$ where neither (01) nor (12) is a term in the sequence $v_0 v_1 v_2$, and where no vertex of $T(v_i)$ is a vertex in $T(v_j)$ if $i \neq j$. Then $u$ is not CI.

Proof. We can suppose at least one of the three subsequences $v_i$, to be nonvacuous since $n \geq 4$. Each $T(v_i)$ is a (possibly one-vertex) submultigraph of the transpositional multtree $T(u)$, where for each $i \in \{1\}$ we are given that $i$ is a vertex in $T(v_i)$. Now, $\bigcirc u = (021) \circ \bigcirc v_0 \bigcirc v_1 \bigcirc v_2$. 

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For each $i \in 3$, let $f_i$ be the component of $\bigcirc v_i$ for which either $i \in \text{supp}(f_i)$ or $f_i = (i)$. Then $\bigcirc u$ has a cyclic component $f$ of length $|f| = |f_0| + |f_1| + |f_2|$ with $3 \subseteq \text{supp}(f)$.

Let $u := ((1 2), (0 1), v_0, (1 2), (0 1), v_1, (1 2), (0 1) v_2)$. So $\bigcirc u' = (0 1 2) \circ \bigcirc v_0 \circ (0 1 2) \circ \bigcirc v_1 \circ (0 1 2) \circ \bigcirc v_2$ lacks the cyclic component $f$ of $\bigcirc u$, but in place of $f$ it has the three cycles $f_0, f_1$ and $f_2$, and otherwise the cycles of $\bigcirc u'$ are identical to those of $\bigcirc u$. So $\bigcirc u' \neq \bigcirc u$ although $u' \in \text{Seq}(u)$. Therefore $u$ is not CL. \hfill $\Box$

**Corollary 3.12.** Let $\mu_u(01) = 2$, let $\mu_u(12) = 1 = \mu_u(13)$, and let $u := ((0 1)\beta(2), (1 2), (1 3), v_2v_3)$, where the two submultigraphs $T(v_2)$ and $T(v_3)$ of $T(u)$ are disjoint. Then $u$ is not CL.

**Proof.** Assume that $u$ is CL. By Lemma 3.3, the multiedge $(0 1)$ of $T(u)$ is a multitwist with leaf 0. We take it that $f_2$ is a cyclic component of $\bigcirc v_2$ for which either $2 \in \text{supp}(f_2)$ or $f_2 = (2)$, and likewise that $f_3$ is a cyclic component of $\bigcirc v_3$ for which either $3 \in \text{supp}(f_3)$ or $f_3 = (3)$. Now $\bigcirc u = (0)fg$, where $f$ is a cycle incorporating the point 1 together with the points in $f_2$ and $f_3$ into a single cycle of consequent length $|f| = 1 + |f_2| + |f_3|$, where $g$ is a permutation that involves the points in $\{4, 5, \ldots, n-1\}$ which occur neither in $f_2$ nor in $f_3$. On the other hand, defining $u' := ((0 1), (1 2), (0 1), (1 3), v_2v_3) \in \text{Seq}(u)$, we find that $\bigcirc u' = f_2'f_3'g$, where $f_2'$ is a cycle of length $|f_2'| = 1 + |f_2|$ that incorporates the points 0 and the points in the cycle $f_2$, and where $f_3'$ is a cycle of length $|f_3'| = 1 + |f_3|$ that incorporates the point 1 and the points in the cycle $f_2$. So $\bigcirc u' \neq \bigcirc u$, violating our assumption that $u$ is CL. \hfill $\Box$

Subproject One is completed. We summarize it in the following immediate conjunction of Lemma 3.3 Corollaries 3.9 and 3.10, Lemma 3.11 and Corollary 3.12.

**Theorem 3.13.** Let $T(u)$ be a multitree with $n \geq 4$. Then $u$ is CL if and only if it satisfies 3.6.1 and 3.6.2.

**Subproject Two:** Proving for $n \geq 4$ that, if $u$ is CL, then $T(u)$ has no circuits.

For $n \geq 4$, our focus now is upon those sequences $u$ in $1^n-2^1$ for which the transpositional multigraph $T(u)$ is connected on the vertex set $n$, but is not a multitree; instead, $T(u)$ has at least one circuit subgraph. We will now provide, some convenient additional terminological background.

Although we write a sequence usually between pointy brackets – e.g., $(x_0, x_1, \ldots, x_{k-1})$ – with its terms separated by commas, when ambiguity is not at issue, we may write it with (some or all of) its terms concatenated (i.e., without commas.) However, when $f$ and $g$ are permutations whose supports are distinct, we have been writing $f \circ g$ as $fg$ in order to indicate this disjointness. Context will make it clear whether an expression denotes disjoint permutations instead of concatenated sequences.

When a sequence is of length one, we call its single term primitive.

A few specific sequences, to which we frequently refer, will we honor with the adjective basic.

Thus far, all of the sequences we have treated in detail are permutational sequences; their terms either are permutations or are characters denoting sequences of permutations. Indeed, almost all of our permutational sequences are transpositional: Their terms are either transpositions or characters denoting sequences of transpositions. Non-basic permutational sequences get lower-case bold-face Latin-letter names.

For the present subproject, when $n \geq 4$, we shall have recourse to two basic transpositional sequences, $\sigma(n)$ and $\tau(n)$. But we shall use number (integer) sequences as well; our basic number sequence is written $\nu(n)$. Number sequences other than $\nu(n)$ will usually receive lower-case Latin letter designations.

**Definition 5.** $\sigma(n) := ((0 1), (1 2), \ldots, (n - 2 \ n - 1))$ and $\tau(n) := (\tau(n), (n - 1 \ 0))$. Also, $\nu(n) := (1, \ldots, n - 1)$.

Of course $T(\sigma(n))$ is a simple circuit multigraph on $n$ vertices, with $n \geq 4$ understood, and $T(\tau(n))$ is the simple branchless multitree resulting from the removal of the simple multiedge $(n - 1 \ 0)$ from $T(\sigma(n))$.

Before we treat circuit-containing connected multigraphs with $n \geq 4$, we recall that Theorem 3.5 settles the case for $n \leq 3$. Now, for $n \geq 4$, we show that, if the transpositional multigraph $T(u)$ contains a 4-vertex simple subgraph which is a triangle sprouting a twig, then $u$ is not CL. Remember: $4 := \{0, 1, 2, 3\}$.

**Theorem 3.14.** Let $n \geq 4$, and let $u$ be a sequence in $1^n-2^1$ which has $h := ((0 1), (1 2), (0 2), (0 3))$ as a subsequence. Then $u$ is not CL.

\footnote{We write $u \setminus h$ to designate the subsequence of $u$ obtained by removing from $u$ its subsequence $h$.}
Proof. Let $W := \{c : c$ is a cyclic component of $\bigcirc(u \setminus h)$ with $4 \cap \supp(c) \neq \emptyset\}$. Let $w \in \Sym(n)$ be the permutation having $W$ as its set of cyclic components. It suffices to show that $\bigcirc p \circ w \neq \bigcirc h \circ w$ for some $p \in \Seq(h)$. There are five cases to treat.

**Case 1:** $|4 \cap \supp(c)| = 1$ for every $c \in W$. Then $W = \{(0s_0), (1s_1), (2s_2), (3s_3)\}$ for some sequences $s_i$ in $\{0, 1, \ldots, n-1\}$. Consider the following three rearrangements $p_i \in \Seq(h)$:

- $p_1 := ((02), (03), (12), (01))$ and $p_2 := ((02), (01), (03), (12))$ and $p_3 := ((02), (03), (01), (12))$

Then $\bigcirc p_1 = (0)(123)$ and $\bigcirc p_2 = (013)(2)$, and $\bigcirc p_3 = (01)(23)$. Consequently $\bigcirc p_1 \circ w = (123) \circ w = (123) \circ (0s_0)(1s_1)(2s_2)(3s_3) = (0s_0)(1s_1)(2s_2)(3s_3)$. Similarly, $\bigcirc p_2 \circ w = (013) \circ w = (0s_1)(1s_3)(3s_0)(2s_2)$ and $\bigcirc p_3 \circ w = (01)(23) \circ w = (0s_1)(1s_3)(3s_0)(2s_3)$. Summarizing, we have that $\bigcirc p_1 \circ w = (0s_0)(1s_2)(2s_3)(3s_1)$ and $\bigcirc p_2 \circ w = (0s_1)(1s_3)(3s_0)(2s_2)$ and $\bigcirc p_3 \circ w = (0s_1)(1s_3)(3s_0)(2s_3)(3s_2)$.

In order to establish that $u$ is not CI, it suffices to show that these three permutations $\bigcirc p_i \circ w$ are not members of the same conjugacy class. Observe that, for each $i \in \{1, 2, 3\}$, the permutation $\bigcirc p_i \circ w$ has exactly two cyclic components, $a_i$ and $b_i$. To argue by contradiction, we assume the multiset equalities $\{|a_1|, |b_1|\} = \{|a_2|, |b_2|\} = \{|a_3|, |b_3|\}$. Spelled out, these multiset equalities are

$\{|(0s_0)|, |(1s_2)(2s_3)(3s_1)|\} = \{|(0s_1)(1s_3)(3s_0)|, |(2s_2)|\} = \{|(0s_1)(1s_3)(3s_0)|, |(2s_3)(3s_2)|\}$, whence

$\{|1 + |s_0|, 3 + |s_2| + |s_3| + |s_1|\} = \{|1 + |s_1| + |s_3| + |s_0|, 1 + |s_2|\} = \{|2 + |s_1| + |s_0|, 2 + |s_3| + |s_2|\}$.

Since $1 + |s_0| < 3 + |s_1| + |s_3| + |s_0|$, the equality $\{|1 + |s_0|, 3 + |s_2| + |s_3| + |s_1|\} = \{|1 + |s_1| + |s_3| + |s_0|, 1 + |s_2|\}$ implies that $1 + |s_0| = 1 + |s_2|$; so $|s_0| = |s_2|$. Therefore, $\{|1 + |s_1| + |s_3| + |s_0|, 2 + |s_3| + |s_2|\}$ implies that $1 + |s_0| = 2 + |s_3| + |s_2|$ since $1 + |s_0| < 2 + |s_1| + |s_0|$. Hence, $1 + |s_0| = 2 + |s_3| + |s_0|$, forcing us to the impossibility $|s_3| = -1$. So the assumed three multiset equalities cannot hold simultaneously. Therefore $\bigcirc p_i \circ w \neq \bigcirc h \circ w$ for at least one $i \in \{1, 2, 3\}$. We infer that $u$ is not CI in the Case 1 situation.

In the remaining four cases, $\psi$ denotes an arbitrary element in $\Sym(4)$.

**Case 2:** $W = \{w\}$, and $4 \subseteq \supp(w)$. That is, $w = (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})) \in \mathbb{N}^4$, where the four $s_{\psi(i)}$ are number sequences, the family of whose nonempty term sets is a partition of the set $n \setminus 4 := \{4, 5, \ldots, n - 1\}$. Consider the subset $\{p_4, p_5\} \subseteq \Seq(h)$ given by

- $p_4 := ((01), (03), (02), (12))$ and $p_5 := ((12), (01), (03), (02))$.

Then $\bigcirc p_4 = (02)(13)$ and $\bigcirc p_5 = (01)(23)$ and $\bigcirc h = (01) \circ (12) \circ (02) \circ (03) = (03)(12)$. Observe that $\{p_4, p_5, h\} = 2^2 \subset \Alt(4)$. Hence there exists $\{p, q\} \subseteq \Seq(h)$ for which $\bigcirc p = (\psi(0) \psi(1)) (\psi(2) \psi(3))$ and for which $\bigcirc q = (\psi(0) \psi(2)) (\psi(1) \psi(3))$. By straightforward computation we now obtain that

$\bigcirc p \circ w = (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)}))) \neq \bigcirc q \circ w$.

Thus we infer that $u$ fails to be CI in the Case 2 situation.

**Case 3:** $W = \{c_1, c_2\}$ where $|4 \cap \supp(c_i)| = 2$ for each $i \in \{1, 2\}$. So this time we can write $w = c_1c_2 = (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})))$. As in Case 2, here too we can provide $\{p, q\} \subseteq \Seq(h)$, for which $\bigcirc p = (\psi(0) \psi(1)) (\psi(2) \psi(3))$ and for which $\bigcirc q = (\psi(0) \psi(2)) (\psi(1) \psi(3))$. We compute that

$\bigcirc p \circ w = (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)}))) \neq (\psi(0) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})) = \bigcirc q \circ w$.

Thus in the situation of Case 3 we again find that $u$ is not CI.

**Case 4:** $W = \{c_1, c_2, c_3\}$ with $c_1 := (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})) \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})) \ (\psi(3) \ s_{\psi(3)}))$. Let $p$ and $q$ be as in Cases 2 and 3. Then

$\bigcirc p \circ w = (\psi(0) \ s_{\psi(0)} \ (\psi(1) \ s_{\psi(1)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)}))) \neq (\psi(0) \ s_{\psi(0)} \ (\psi(2) \ s_{\psi(2)} \ (\psi(3) \ s_{\psi(3)})) = \bigcirc q \circ w$.

Thus $u$ fails to be CI in the Case 4 situation as well.
Case 5: \( W = \{c_1, c_2\} \) with \( c_1 = (\psi(0) s_{\psi(0)}) \) and \( c_2 = (\psi(1) s_{\psi(1)} \psi(2) s_{\psi(2)} \psi(3) s_{\psi(3)}) \). That is to say, \( w = (\psi(0) s_{\psi(0)}) (\psi(1) s_{\psi(1)} \psi(2) s_{\psi(2)} \psi(3) s_{\psi(3)}) \). The \( \mathcal{O}_i \) of the following six \( r_i \in \text{Seq}(h) \) comprise the example class, \( 1^3 3 \subset A_5(4) \), the six possible 3-cycles:

\[
\begin{align*}
    r_1 &= ((0 1), (1 2), (0 3), (0 2)) & \text{for which we compute that } \circ r_1 &= (0)(1 3 2) \\
    r_2 &= ((0 2), (1 2), (0 3), (0 1)) & \text{for which we compute that } \circ r_2 &= (0)(1 2 3) \\
    r_3 &= ((1 2), (0 3), (0 2), (0 1)) & \text{for which we compute that } \circ r_3 &= (0 3 2)(1) \\
    r_4 &= ((0 1), (0 2), (1 2), (0 3)) & \text{for which we compute that } \circ r_4 &= (0 2 3)(1) \\
    r_5 &= ((1 2), (0 3), (0 1), (0 2)) & \text{for which we compute that } \circ r_5 &= (0 3 1)(2) \\
    r_6 &= ((1 2), (0 1), (0 2), (0 3)) & \text{for which we compute that } \circ r_6 &= (0 1 3)(2)
\end{align*}
\]

Subcase: \( \psi(0) \neq 3 \). Then \( \circ p = (\psi(0))(\psi(1) \psi(2) \psi(3)) \) and \( \circ q = (\psi(0))(\psi(1) \psi(3) \psi(2)) \) for some \( \{p, q\} \subseteq \{r_i : 1 \leq i \leq 6\} \).

We compute that \( \circ p \circ w = (\psi(0))(\psi(1) \psi(2) \psi(3)) \circ (\psi(0) s_{\psi(0)}) (\psi(1) s_{\psi(1)} \psi(2) s_{\psi(2)} \psi(3) s_{\psi(3)}) = (\psi(0) s_{\psi(0)}) (\psi(1) s_{\psi(1)} \psi(2) s_{\psi(2)} \psi(3) s_{\psi(3)}) \neq (\psi(0) s_{\psi(0)}) (\psi(1) s_{\psi(1)} \psi(2) s_{\psi(2)} \psi(3) s_{\psi(3)}) = \circ q \circ w \). Thus we see that \( u \neq \text{CI in the situation of Case 3 where also } \psi(0) \neq 3 \).

Subcase: \( \psi(0) = 3 \). There are two subsubcases, which are:

\[
\begin{align*}
    \text{I: } w &= (0 s_0 1 s_1 2 s_2)(3 s_3) \\
    \text{II: } w &= (0 s_0 2 s_2 1 s_1)(3 s_3)
\end{align*}
\]

We will show that the theorem holds for Subcase I but omit the similar proof for Subcase II.

We use rearrangements \( v_1 := h \) and \( v_2 := ((0 1), (0 3), (0 2), (1 2)) \) and \( v_3 := ((0 2), (0 3), (1 2), (0 1)) \) of \( h \), noting first that \( \circ v_1 = (0)(3)(12) \), that \( \circ v_2 = (0)(2)(13) \), and that \( \circ v_3 = (0)(1)(23) \), and hence that \( \circ v_1 \circ w = (0 s_3 3 s_0 1 s_2)(2 s_1) \) and \( \circ v_2 \circ w = (0 s_2)(1 s_3 3 s_1 2 s_0) \) and \( \circ v_3 \circ w = (0 s_0 1 s_2)(2 s_3 3 s_1) \).

Assume that \( \circ v_3 \circ w = \circ v_2 \circ w = \circ v_3 \circ w \). Then the following three multiset equalities must hold:

\[
\{3 + |s_3| + |s_0| + |s_2|, 1 + |s_1|\} = \{1 + |s_2|, 3 + |s_3| + |s_1| + |s_0|\} = \{2 + |s_0| + |s_2|, 2 + |s_3| + |s_1|\}
\]

Since 1 + |s_2| < 3 + |s_3| + |s_0| + |s_2|, the equality of the first two multisets implies that 1 + |s_1| = 1 + |s_2|, whence |s_1| = |s_2|. Since 1 + |s_1| < 2 + |s_3| + |s_1|, the equality of the first and third multisets therefore implies that 1 + |s_1| = 2 + |s_0| + |s_2| = 2 + |s_0| + |s_1|, whence 0 = 1 + |s_0|, which entails the impossibility |s_0| = -1.

So u fails to be CI in Case 5 as well. Since the five Cases are exhaustive, the theorem is proved. \( \square \)

Theorem 6.14 gives us that, if \( n \geq 4 \) and if \( \mathcal{T}(u) \) is a transpositional multigraph containing a triangular subgraph, then \( u \) is not CI. The remainder of Section 6.13 is devoted mainly to generalizing the proof of Theorem 6.14 in order to establish, for \( n \geq 4 \), that no connected transpositional multigraph on \( n \) vertices is CI if it contains a circuit subgraph on more than three vertices. To this purpose it is useful to describe those sequences \( s \in 1^n \times 2^{3} \) for which \( \mathcal{T}(s) \) is itself a circuit. The following three lemmas do so.

Recall the basic transpositional sequence \( s(n):= ((0 1), (1 2), \ldots, (n-3 n-2), (n-2 n-1)) \); that is, \( s(n) \) is the only sequence of the sequence \( s \), and that \( \mathcal{T}(s) \) is its own circuit. The following three lemmas do so.

Recall the basic transpositional sequence \( s(n):= ((0 1), (1 2), \ldots, (n-3 n-2), (n-2 n-1)) \); that is, \( s(n) \) is the only sequence of the sequence \( s \), and that \( \mathcal{T}(s) \) is its own circuit.

When \( s \) is a sequence, we write \( x \prec y \) to indicate that \( x \) precedes \( y \) as a term in \( s \).

**Lemma 3.15.** For \( n \geq 3 \), let \( g \) be any rearrangement of \( \tau(n) \). Then \( \circ g = (0 \ p \ n \ - 1 \ q) \in n^3 \) for some subsequence \( p \) of \( (1, 2, \ldots, n - 2) \) and with \( q := ((1, 2, \ldots, n - 2) \setminus p)^R \).

**Proof.** We induct on \( n \). Note that \( (0 1), (1 2) \) and \( (1 2), (0 1) \) are the only rearrangements of \( \tau(3) \), that \( (0 1) \circ (1 2) = (0 2 1) = (0 p 2 q) \) with \( p \) the empty sequence and reverse-complementary to \( q = (1) \) in the number sequence \( 1 \), and similarly that \( (0 1) \circ (0 1) = (0 1 2) = (0 p 2 q) \) where \( p = (1) \) and \( q = (1) \).

Choose an integer \( k \geq 3 \). Suppose the lemma holds for \( n = k \). Let \( g \) be a rearrangement of \( \tau(k+1) \). Let \( g' := g \setminus \{(k-1 k)\} \). Now, \( \text{supp}(t) \cap \text{supp}(k-1 k) = \emptyset \) for every term \( t \) in \( g' \) except for \( t = (k-2 k-1) \).

Hence, one of the following two equalities must hold:

\[
\begin{align*}
1. & \quad \circ g = (k-1 k) \circ g' \\
2. & \quad \circ g = (k-1 k) \circ g'
\end{align*}
\]
Lemma 3.16. These equalities are exactly what the lemma claims.

Lemma 3.17. Let $f \in \text{Seq}(\sigma(n))$ with $n \geq 3$. Then $\bigcirc f = (h)(\nu(n) \setminus h)^-$ for a subsequence $h \neq \emptyset$ of $\nu(n)$.

Proof. Case 1: $(01) < f (n - 1 0)$. Let $m$ be the smallest integer such that $(n - 0 0) < f (n - 1 n - 2) < f (n - 2 n - 3) < f \cdots < f (m + 1 m)$. We can decompose $f$ as follows:

$$f = \langle b_0, (n - 1 0), b_1, (n - 1 n - 2), b_2, (n - 2 n - 3), \ldots, b_{n - m - 1}, (m + 1 m), b_{n - m} \rangle.$$

Then $\bigcirc f = \bigcirc b_0 \circ (n - 1 0) \circ \bigcirc b_1 \circ (n - 1 n - 2) \circ \bigcirc b_2 \circ \cdots \circ \bigcirc b_{n - m - 1} \circ (m + 1 m) \circ \bigcirc b_{n - m}.$

Since $f \in \text{Seq}(\sigma(n))$, there are exactly two terms in $f$ whose supports contain $n - 1$: those two terms are $(n - 1 0)$ and $(n - 1 n - 2)$. Since those terms border the transpositional sequence $b_1$, they do not occur as terms in $b_1$. Consequently $n - 1 \not\in \text{supp}(\bigcirc b_1)$. By hypothesis $(01) < f (n - 1 0)$, and hence $(01) < f b_1$. Thus neither of the two terms of $f$ which have 0 in their supports are terms in $b_1$. Therefore supp$(\bigcirc b_1) \cap \text{supp}((n - 1 0)) = \emptyset$. So $(n - 1 0) \circ \bigcirc b_1 = \bigcirc b_1 \circ (n - 1 0)$. Thus we infer that

$$\bigcirc f = \bigcirc b_0 \circ \bigcirc b_1 \circ (n - 1 0) \circ (n - 2 n - 1) \circ \bigcirc b_2 \circ \cdots \circ \bigcirc b_{n - m - 1} \circ (m + 1 m) \circ \bigcirc b_{n - m}.$$

Similarly we see that supp$(\bigcirc b_2) \cap \text{supp}((n - 1 0) \circ (n - 1 n - 2)) = \emptyset$, and thus that

$$\bigcirc f = \bigcirc b_0 \circ \bigcirc b_1 \circ \bigcirc b_2 \circ (n - 1 0) \circ (n - 1 n - 2) \circ (n - 2 n - 3) \circ \cdots \circ \bigcirc b_{n - m - 1} \circ (m + 1 m) \circ \bigcirc b_{n - m}.$$

Continuing in this fashion, we eventually obtain that

$$\bigcirc f = \bigcirc b_0 \circ \bigcirc b_1 \circ \cdots \circ \bigcirc b_{n - m} \circ (n - 1 0) \circ (n - 1 n - 2) \circ \cdots \circ (m + 1 m).$$

Define $g := \langle b_0, b_1, \ldots, b_{n - m - 1}, b_{n - m} \rangle$. Note that $g = f \setminus ((n - 1 0), (n - 1 n - 2), \ldots, (m + 1 m))$. Since $f \in \text{Seq}(\sigma(n))$, we see that $g \in \text{Seq}(\tau(m + 1))$, recalling that $\tau(m + 1) = ((01), (12), \ldots, (m + 1 m))$. So by Lemma 3.15 we have that $\bigcirc g = (0 p m q)$, where $p$ is a subsequence of $(1, 2, \ldots, m - 1)$ and where $q = ((1, 2, \ldots, m - 1) \setminus \{p\})^R$. Thus $\bigcirc f = \bigcirc g \circ (n - 1 0) \circ (n - 1 n - 2) \circ (n - 2 n - 3) \circ \cdots \circ (m + 1 m) = (0 p m q) \circ (n - 1 0) \circ (n - 1 n - 2) \circ \cdots \circ (m + 1 m) = (0 p m + 1 m + 2 \ldots n - 2 n - 1(q m)).$ Setting $h := (0 p m + 1 m + 2 \ldots n - 2 n - 1)$, we see that $h$ is a nonempty subsequence of $\nu(n)$ and observe that $\langle q, m \rangle = (\nu(n) \setminus h)^R$. And therefore $(q m) = (\nu(n) \setminus h)^-$. So $\bigcirc f = (h)(\nu(n) \setminus h)^-$ as alleged.

Case 2: $(01) > f (n - 1 0)$. Since the argument parallels that for Case 1, we omit it.

Lemma 3.17. Let $n \geq 3$. Let $h$ be a proper nonempty subsequence of $\nu(n)$. Then $\bigcirc f = (h)(\nu(n) \setminus h)^-$ for some $f \in \text{Seq}(\sigma(n))$.

Proof. If the lemma holds for $h$ then it holds also for its complement $\nu(n) \setminus h$ in $\nu(n)$. For, if $\bigcirc f = (h)(\nu(n) \setminus h)^-$, then $\bigcirc f^R = (\bigcirc f)^-= (\nu(n) \setminus h)^-(\nu(n) \setminus h)^- = (\nu(n) \setminus h)(\nu(n) \setminus h)^-$. We induce on $n \geq 3$.

Basis Step. There are the six nonempty proper subsequences of $\nu(3)$; they are

$$u := \langle 0 \rangle, \quad v := \langle 1 \rangle, \quad w := \langle 2 \rangle, \quad x := \langle 0, 1 \rangle, \quad y := \langle 0, 2 \rangle, \quad z := \langle 1, 2 \rangle.$$
We use the fact noted in the preceding paragraph. If \( f_n \coloneqq (0(1), (1,2), (2,0)) \) then \( \bigcirc f_n = (0)(21) = (u)(v(3)\setminus u)^{-} = (z)(v(3)\setminus z)^{-} \). If \( f_n = (0(1), (2,0)) \) then \( \bigcirc f_n = (0)(21) = (v)(v(3)\setminus v)^{-} = (y)(v(3)\setminus y)^{-} \). If \( f_n = (1,2), (0,1), (2,0) \) then \( \bigcirc f_n = (0)(12) = (w)(v(3)\setminus w)^{-} = (x)(v(3)\setminus x)^{-} \). The arbitrary length-one sequence \( \langle x \rangle \) in \( \nu(k + 1) \) is a special case. Choose the transpositional sequence \( f \in \mathrm{Seq}(\sigma(k + 1)) \) to be

\[
\bigcirc f = \langle (x \ x \ x \ x), (x \ x \ x \ x + 1 \ x \ x \ x \ x \ x - 2 \ x \ x \ x \ x \ x - 2 \ 2 \ 1 \ 0) \rangle = (x)(\nu(k + 1) \setminus \langle x \rangle)^{-},
\]

as desired.

Let \( h \coloneqq \{x_1, x_2, \ldots, x_n\} \) be a subsequence of \( \nu(k + 1) \) and let \( h' \coloneqq \nu(k + 1) \setminus h = \{y_1, y_2, \ldots, y_n\} \) be the complement in \( \nu(k + 1) \) of \( h \). Of course \( s + t = k + 1 \).

By the first paragraph in this proof, we can set it both that \( x_k = k \). and also that there exist \( s \) disjoint subsequence \( a_i \) of \( \nu(k + 1) = \langle a_1, a_2, a_3, \ldots, a_s, x_s, x_{s+1}, \ldots, x_n \rangle \). Indeed, \( h' = a_1a_2\ldots a_s \), where \( h' \) is expressed here as the concatenation of the subsequence \( a_i \). The \( f \in \mathrm{Seq}(\sigma(k + 1)) \), whose existence this lemma alleges, must satisfy \( \bigcirc f = \langle x_1, x_2, \ldots, x_s \rangle (\nu(3)\setminus x_1) \setminus \langle x_1, x_2, \ldots, x_s \rangle \) be the permutation whose family of cyclic components of the permutation \( \sigma \). Let \( T(\bigcirc f) \) be the permutation whose family of components is \( \{x_1, x_2, \ldots, x_s\} \). Now pretend that \( \bigcirc f \) is not CI.

To prove when we exhibit rearrangements \( \{p, q\} \subseteq \mathrm{Seq}(\sigma) \) such that \( \bigcirc (p, q) \setminus \sigma = \bigcirc p \circ w \neq \bigcirc q \circ w = \bigcirc (q, f) \setminus \sigma \). Moreover, since \( \supp(p) \cap \supp(\bigcirc p \circ w) = \emptyset = \supp(q) \cap \supp(\bigcirc q \circ w) \), it will suffice to insist only that \( \bigcirc p \circ w \neq \bigcirc q \circ w \). There are three cases.

**Case One:** \( |n \cap \supp(C)| = 1 \) for each cycle \( C \) in \( \mathcal{W} \).

We write \( w = (0, s_0) (1, s_1) \ldots (n - 1, s_{n-1}) \), where the \( s_i \) are finite sequences \( s_i \in \mathbb{N} \) in \( k \setminus n \). By Lemma 3.17 for each \( i \in n \) there exists \( p^{(i)} \in \mathrm{Seq}(\sigma) \) such that \( \bigcirc p^{(i)} = (i)(0 \ 1 \ 2 \ldots i - 2 \ i - 1 \ i + 1 \ i + 2 \ldots n - 2 \ n - 1) \). Hence \( \bigcirc p^{(i)} \circ w = (i \ s_i) (0 \ s_0 \ 1 \ s_1 \ 2 \ s_2 \ldots i - 2 \ s_{i-2} \ i - 1 \ s_{i-1} \ i + 1 \ s_{i+1} \ i + 2 \ldots n - 2 \ s_{n-1} \ n - 1 \ s_n) \) for each \( i \) in \( n \). Now pretend that \( \bigcirc p^{(i)} \circ w \backsim \bigcirc p^{(i)} \circ w \) for all \( i \in n \). Then all \( n \) of the cycle-length multisets \( K[p^{(i)}] \) of these permutations \( \bigcirc p^{(i)} \circ w \) must be identical. Specifying the \( K[p^{(i)}] \) for each \( i \in n \), we see that

\[
K[p^{(i)}] := \{1 + |s_i|, n - 1 + \sum_i |s_j| : i \neq j \in n\}.
\]

Obviously \( 1 + |s_i| < n - 1 + \sum_i |s_j| : t \neq j \in n \) whenever \( i \neq t \). So our assumption that all of the \( K[p^{(i)}] \) are identical implies that \( |s_i| = |s_j| \) for all \( i, j \in n \).

Again invoking Lemma 3.17 we can find \( q \in \mathrm{Seq}(\sigma) \) for which \( \bigcirc q = (1)(0) (23 \ldots n - 2, n - 1) \). Then \( \bigcirc q \circ w = (0 \ s_0 \ 1 \ s_1) (2 \ s_2 \ldots 4 \ s_4 \ldots n - 2 \ s_{n-1} \ n - 1 \ s_2) \), and so \( K[q] = \{2 + |s_1| + |s_0|, n - 2 + \sum_{j=2}^{n+1}(n - 1) |s_j| \} \). Under the assumption that \( f \) is CI, we must have that \( \bigcirc p^{(i)} \circ w = \bigcirc q \circ w \) for all \( i \in n \), whereupon \( K[p^{(i)}] = K[q] \). Since all of the integers \( |s_i| \) were found to be equal, for \( i = 0 \) we must infer that \( \{1 + |s_0|, n - 1 + (n - 1) |s_0| \} = \{2 + 2 \cdot |s_0|, n - 2 + (n - 2) |s_0| \} \), an impossibility since \( \{1, n - 1\} \cap \{2, n - 2\} = \emptyset \) when \( n \geq 4 \). So the Theorem holds in the Case-One situation.

---

8 modulo \( k + 1 \) of course
9 some of which may be vacuous
10 as well as remaining the complement in \( \nu(k + 1) \) of \( h \)
11 which are not required to be nonempty
The next Case requires an ancillary fact.

**Claim 1.** If $n \geq 4$ and if $W$ contains a cycle $C$ with $|n \cap \text{supp}(C)| \geq 2$ then there exists $p \in \text{Seq}(\sigma(n))$ such that the permutation $(\bigcirc p \circ w)(n \cup \text{supp}(w))$ has at least three cycles.

**Proof of Claim.** We first suppose that there exists $C := (x \ s_x \ y \ s_y) \in W$ with $n \cap \text{supp}(C) = \{x, y\}$. Without loss of generality, we take it that $(x, y)$ is a subsequence of $\nu(n)$, and we invoke Lemma 3.17 to find some $p \in \text{Seq}(\sigma(n))$ for which $\bigcirc p = (x y)(\nu(n) \setminus \{x, y\})$.

Let $w'$ be the permutation whose family of cyclic components is $W \setminus \{C\}$. Since two permutations commute if their supports are disjoint, we compute: $\bigcirc p \circ w = \bigcirc p \circ w' C = (x y)(\nu(n) \setminus \{x, y\}) \circ \bigcirc w' C = (\nu(n) \setminus \{x, y\}) \circ w(x y) \circ C = (\nu(n) \setminus \{x, y\}) \circ w'(x y) \circ (x s_x \ y s_y) = (\nu(n) \setminus \{x, y\}) \circ w'(x s_y)(y s_x)$. This is thus clear that the permutation $(\bigcirc p \circ w')(n \cup \text{supp}(w))$ has at least three cyclic components.

More generally, now suppose there exists $C := (x s_x y s_y z s_z) \in W$ where $s_x s_y s_z$ is an injective sequence in the set $k \setminus n$. Let $q = (m_1, m_2, m_3)$ be a rearrangement of the number sequence $(x y z)$ for which $m_1 < m_2 < m_3$. Surely either $(q) = (z y x)$ or $(q)^{-1} = (z y x)$.

For $(q) := (z y x)$, by Lemma 3.17, there exists $p \in \text{Seq}(\sigma(n))$ with $\bigcirc p = (q)(\nu(n) \setminus q)$. Again, let $w'$ be the permutation whose family of components is $W \setminus \{C\}$. Then $\bigcirc p \circ w = \bigcirc p \circ w' C = (q)(q') \circ w' C$, where $q' := \nu(n) \setminus q$. Thus $\bigcirc p \circ w = [(q') \circ w'][(z y x) \circ C] = [(q') \circ w'][z y x (x s_x y s_y z s_z)] = [(q') \circ w'(x s_y)(y s_x)(z s_z)]$ for three number sequences $s_i$. So $\bigcirc p \circ w$ has at least three cycles. Thus the claim holds for $(q) := (z y x)$. On the other hand, if $(q)^{-1} := (z y x)$, then Lemma 3.17 provides a $p_1 \in \text{Seq}(\sigma(n))$ for which $\bigcirc p_1 = (\nu(n) \setminus q)(q'^{-1})$, and we omit the repetitive rest of the argument. Claim 1 follows.

**Case Two:** The family $W$ of cycles contains exactly one element $C$, and $n \subseteq \text{supp}(C)$.

Pick an integer $i$ with $0 \leq i, i + 1, i + 2 < k$. The cycle $C$ is expressible in one of these two ways:

- **Order 1:** $C = (i \ s_i \ i + 1 \ s_{i+1} \ i + 2 \ s_{i+2})$
- **Order 2:** $C = (i \ s_i \ i + 2 \ s_{i+2} \ i + 1 \ s_{i+1})$

For the subsequence $a_i := ((i + 1 + 2), (i + 1))$ of $\sigma(n) \in \text{Seq}(\sigma(n))$, if Order 1 prevails then

$$\bigcirc a_i \circ C = (i + 1 + 2) \circ (i \ s_i \ i + 1 \ s_{i+1} \ i + 2 \ s_{i+2}) = (i \ s_{i+1} \ i + 2 \ s_i \ i + 1 \ s_{i+2}).$$

Thus, if $C$ is of the form in Order 1, then $\bigcirc a_i \circ C$ is a single cycle of the same length as that of $C$. But if, instead, Order 2 prevails, then

$$\bigcirc a_i \circ C = (i + 2 + 1) \circ (i \ s_i \ i + 2 \ s_{i+2} \ i + 1 \ s_{i+1}) = (i \ s_{i+2} \ i + 1 \ s_i \ i + 2 \ s_{i+1}).$$

So here too, when $C$ is of the form Order 2, then $\bigcirc a_i \circ C$ is a single cycle whose length is $|C|$.  

**Subcase:** $n$ is even. Then the transpositional sequence $\sigma(n)$ has an even number of terms. So we may write $\sigma(n)$ as a sequence $v_1, v_2, \ldots, v_t$ of $t := n/2$ pairs $v_i := ((2t - 2 \ 2t - 1), (2t - 1 \ 2t))$ of transpositions that are adjacent and consecutive in $\sigma(n)$. Define $v'_i := v_i^R$ if $(2t - 2, 2t - 1, 0) = (n - 2, n - 1, 0)$ occurs in Order 1 in $C$. For each $i \in [t]$, we write $v'_i$ as $v_i^R$ if $(2t - 2, 2t - 1, 2t)$ occurs in Order 1 in the cycle $\bigcirc v'_{i+1} \circ \bigcirc v'_{i+2} \circ \cdots \circ \bigcirc v'_{i-1} \circ \bigcirc v'_i \circ C$. In the corresponding Order 2 situation we make the opposite definitions for the $v'_i$; that is, $v'_i := v_i$ for each $i \in [t]$. As a consequence of our observations prior to the present Subcase, the permutation $\bigcirc v'_1 \circ \bigcirc v'_2 \circ \cdots \circ \bigcirc v'_t \circ C$ is a single cycle whose length is $|C|$.

Of course $p := v'_1 v'_2 \ldots v'_t \in \text{Seq}(\sigma(n))$. Claim 1 tells us that there exists $q \in \text{Seq}(\sigma(n))$ for which $\bigcirc q \circ C$ has at least three component cycles. So $\bigcirc p \circ C \not\cong \bigcirc q \circ C$.

**Subcase:** $n$ is odd. This time $t := (n - 1)/2$ and, for $1 \leq i \leq t$, we define the $v_i$ and the $v'_i$ as above. Here, $\sigma(n) = \{v_1, v_2, \ldots, v_t, (n - 1, 0)\}$. Now let $p := (p', (n - 1, 0)) := (v'_1, v'_2, \ldots, v'_t, (n - 1, 0))$. Then $p \in \text{Seq}(\sigma(n))$. So $\bigcirc p \circ C = [\bigcirc p' \circ C]((n - 1, 0)$ is the product of the cycles $\bigcirc p' \circ C$ and $(n - 1, 0)$. Moreover, $(n - 1, 0) \subseteq \text{supp}(\bigcirc p \circ C)$. Therefore $\bigcirc p \circ C$ has exactly two cyclic components. Claim 1 promises us a $q \in \text{Seq}(\sigma(n))$ for which $\bigcirc q \circ C$ has more than two cyclic components, whence $\bigcirc q \circ C \not\cong \bigcirc p \circ C$. So the theorem holds under Case Two circumstances.

**Case Three:** $W$ contains a cycle $C$ for which $1 < |n \cap \text{supp}(C)| < n$. As before, let $w \in \text{Sym}(k)$ be the permutation whose family of cyclic components is $W$, and let $w'$ be the permutation whose family of (nontrivial) cyclic components is $W \setminus \{C\}$. Since $|n \cap \text{supp}(C)| < n$, there exists $m \in n \setminus \text{supp}(C)$. If

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12We may write $a \circ b$ as $ab$ in order to emphasize that supp(a) ∩ supp(b) = ∅.
m \not\in \text{supp}(w')$, then let $Q$ be the trivial cycle. But if $m \in \text{supp}(w')$ then let $Q$ be the unique cycle in $W \setminus \{C\}$ such that $m \in \text{supp}(Q)$, and let $w''$ be the permutation whose family of nontrivial cyclic components is $W \setminus \{C, Q\}$. Let $h$ be the subsequence of $\nu(n)$ with $\text{supp}(h) = n \cap \text{supp}(Q)$. If $Q = \{m\}$, we let $h := \{m\}$.

Since $h$ is a nonempty proper subsequence of $\nu(n)$, by Lemma 3 there exists $p \in \text{Seq}(\sigma(n))$ such that $\bigcirc p = (h)(\nu(n) \setminus h)^-$. Thus $\bigcirc p \circ w = \bigcirc p \circ CQw'' = (h)(\nu(n) \setminus h)^- \circ CQw'' = ((\nu(n) \setminus h)^- \circ Cw')(h \circ Q)$, which is the product of two permutations, $(\nu(n) \setminus h)^- \circ Cw''$ and $(h) \circ Q$, whose supports are disjoint.

Now, if $\bigcirc p \circ w((n \cup \text{supp}(w)))$ has fewer than three cyclic components, then we employ Claim 1 to obtain some $q \in \text{Seq}(\sigma(n))$ such that $\bigcirc q \circ w((n \cup \text{supp}(w)))$ has at least three cycles, whence $\bigcirc q \circ w \not\equiv \bigcirc p \circ w$. So it remains only to deal with the situation where $\bigcirc p \circ w((n \cup \text{supp}(w)))$ has at least three cycles.

Suppose $\bigcirc p \circ w((n \cup \text{supp}(w)))$ has at least three cycles. Then there are two possibilities to treat; to wit:

1. $(\nu(n) \setminus h)^- \circ Cw''\mid Y$ has more than one cycle, where $Y := \text{supp}(Cw'') \cup \text{supp}((\nu(n) \setminus h)^-)$. \\
2. $(h) \circ Q \mid X$ has more than one cycle, where $X := \text{supp}(Q) \cup (n \setminus \text{supp}((\nu(n) \setminus h)^-)$. \\

**FIRST POSSIBILITY:** The permutation $(\nu(n) \setminus h)^- \circ Cw''\mid Y$ has more than one cycle. Here we need

**Claim 2.** Each orbit of $(\nu(n) \setminus h)^- \circ Cw''\mid Y$ contains at least one element in $\text{supp}((\nu(n) \setminus h)^-)$. \\

**Proof of Claim.** Let $E$ be an orbit of $(\nu(n) \setminus h)^- \circ Cw''\mid Y$. Let $e \in E$. Then $E = \{e((\nu(n) \setminus h)^- \circ Cw'')^i : i \in \mathbb{Z}\}$. We are done if $e \in \text{supp}((\nu(n) \setminus h)^-)$. So suppose $e \not\in \text{supp}((\nu(n) \setminus h)^-)$. Then, since $e \in Y$, it follows that $e \in \text{supp}(Cw'')$, and hence that either $e \in \text{supp}(C)$ or $e \in \text{supp}(w'')$.

First, suppose that $e \in \text{supp}(C)$. Then $eC^i \subseteq \text{supp}(C) \subseteq \text{supp}(Cw'')$ for all $i \in \mathbb{Z}$. Also, there is a least positive integer $l$ with $eC^l \in n \cap \text{supp}(C)$. Now, $n \cap \text{supp}(C) \subseteq \text{supp}((\nu(n) \setminus h)^-)$. Hence $eC^l \subseteq \text{supp}((\nu(n) \setminus h)^-)$. Since $eC^l \not\sim \text{supp}((\nu(n) \setminus h)^-) \forall i \in \{0, 1, \ldots, l-1\}$, we have that $eC^l = (\nu(n) \setminus h)^- \circ Cw'' \mid E$. So $eC^l \subseteq E \cap \text{supp}((\nu(n) \setminus h)^-)$. Thus $E \cap \text{supp}((\nu(n) \setminus h)^-) \neq \emptyset$, as claimed.

Next, suppose instead that $e \in \text{supp}(w'')$. Then $e \in \text{supp}(F)$ for some cycle $F \subseteq W \setminus \{C, Q\}$. Since $F \subseteq W$, we have that $n \cap \text{supp}(F) = (\text{supp}(h) \cup \text{supp}((\nu(n) \setminus h)^-)) \cap \text{supp}(F) \neq \emptyset$. Also, since $F \subseteq W \setminus \{Q\}$, we have that $\text{supp}(F) \cap \text{supp}(Q) = \emptyset$, and hence that $\text{supp}(F) \cap \text{supp}(h) = \emptyset$ since $\text{supp}(h) \subseteq \text{supp}(Q)$. So $\text{supp}(F) \cap \text{supp}((\nu(n) \setminus h)^-) = \emptyset$. This time let $l$ denote the least positive integer such that $eF^l \subseteq \text{supp}(h) \cup \text{supp}((\nu(n) \setminus h)^-)$. Let $w'''$ be the permutation whose family of component cycles is $W \setminus \{C, Q, F\}$. Since $eF^l \not\sim \text{supp}((\nu(n) \setminus h)^- \circ Cw''') \forall i \in \{0, 1, \ldots, l-1\}$, it follows that $eF^l = (\nu(n) \setminus h)^- \circ Cw''' = (\nu(n) \setminus h)^- \circ Cw''$, whence $eF^l \subseteq E$. But then $eF^l \subseteq E \cap \text{supp}((\nu(n) \setminus h)^-)$. Therefore, we have that $E \cap \text{supp}((\nu(n) \setminus h)^-) \neq \emptyset$. The proof of Claim 2 is complete.

**Claim 3.** There exist orbits $A \neq B$ of $(\nu(n) \setminus h)^- \circ Cw''\mid Y$ and elements $x$ and $y$ in $\text{supp}((\nu(n) \setminus h)^-) \forall x \in A$ and $y \in B$, and such that $y(\nu(n) \setminus h)^- = x$. \\

**Proof of Claim.** Let $B$ be an orbit of $G \circ Cw''\mid Y$, where $G := (\nu(n) \setminus h)^-$. By Claim 2, there exists $b \in B \cap \text{supp}(G)$. If $bG^j \subseteq B$ for every $i \in \mathbb{N}$, then $\text{supp}(G) \subseteq B$, contrary to Claim 2, since by hypothesis $G \circ Cw''\mid Y$ has at least two orbits. So $bG^j \subseteq B$ while $bG^{j+1} \not\subseteq B$ for some $j \in \mathbb{N}$. Let $y := bG^j$, let $x := yG$, and let $A$ be the orbit of $G \circ Cw''\mid Y$ for which $x \in A$. Claim 3 follows.

Let $x, y, A, B$, and $G := (\nu(n) \setminus h)^- \in \text{Claim 3}$, and let $d$ be the subsequence of $\nu(n)$ such that the term set of $d$ is $\{x\} \cup \text{supp}(h)$. If $h = \{m\}$, let the term set of $d$ be $\{m, x\}$. Since $n \cap \text{supp}(C) \subseteq \text{supp}(G)$, and since $n \cap \text{supp}(C) \neq \text{supp}(G)$, it follows that there is no element in $\text{supp}(G)$, and hence in $n$, which is not a term in $d$. So $d$ is a proper subsequence of $\nu(n)$. The number sequence $d$ was produced by inserting $x$ as a term into the sequence $h$, and so the sequence $\nu(n) \setminus d$ is obtained by deleting the term $x$ from the sequence $\nu(n) \setminus h$. Since $|d| \geq 2$, there exists $z \in \text{supp}(d)$ such that $z(d) = x$. But $z \not\in x$, and so $z \in \text{supp}(h)$; we can write $h = (z s_z)$, and so $(z) \circ (z x) \circ (z s_z) = (z x s_z) = (z x s_z)$. Similarly, since $yG = x$, we may write $G = (y x s_z)$. Delete the term $x$ from $G$, and obtain that $(\nu(n) \setminus d)^- = (y s_z)$. Thus, $(y x) \circ G = (y x) \circ (\nu(n) \setminus h)^- = (y x) \circ (y x s_z) = (x y x s_z) = (x y x s_z)$. These equalities enable us to expand the product: $\bigcirc q \circ w = \bigcirc q \circ CQw'' = (d)(\nu(n) \setminus d)^- \circ CQw'' = (z x) \circ (h)(\nu(n) \setminus h)^- \circ CQw'' = (z x) \circ ((y x) \circ (h)(\nu(n) \setminus h)^- \circ CQw'' = (z x) \circ ((y x) \circ \bigcirc p \circ w$.

Recall that $x$ and $y$ are elements in distinct orbits $A$ and $B$ of $G \circ Cw''\mid Y$. Recall also that $\bigcirc p \circ w = [(h) \circ Q][G \circ Cw''].$ But $\text{supp}((h) \circ Q) \cap \text{supp}(G \circ Cw') = \emptyset$, and $A$ and $B$ are distinct orbits of $\bigcirc p \circ w$.
as well. Also, since \( z \in \supp((h)) \subseteq \supp((h) \circ Q) \), we have that \( z \) must belong to a third orbit \( D \) of \( \circ p \circ w(n \cup \supp(w)) \). Consequently the sets \( A, B, C \) will amalgamate to form a single orbit \( A \cup B \cup C \) of \( \circ p \circ w(n \cup \supp(w)) \). Thus the permutation \( \circ q \circ w(n \cup \supp(w)) \) will possess exactly two fewer orbits than the permutation \( \circ p \circ w(n \cup \supp(w)) \). Hence \( \circ q \circ w \not\supseteq \circ p \circ w \).

SECOND POSSIBILITY: \((h) \circ Q\) has more than one orbit.

If \( |n \cap \supp(Q)| = 1 \), then \((h) \circ Q\) is a single cycle, contrary to the present hypothesis. Thus \( |\supp((h))| = |n \cap \supp(Q)| > 1 \). So, arguing as in the First Possibility, we can find \( \{x, y\} \subseteq \supp((h)) \) and distinct orbits \( A \) and \( B \) of \((h) \circ Q\) with \((x, y) \in A \times B\) and such that \((x(h)) = y\). Let \( d \) be the sequence obtained by deleting the term \( x \) from the sequence \( h \). By Lemma 3.17 there exists \( q \in \text{Seq}(\sigma(n)) \) such that \( \circ q = (d)(\nu(n) \setminus d)^\circ \).

Observe that \((d) = (h)(x y)\), and that if an element \( z \in \supp((\nu(n) \setminus d)^\circ) \) satisfies \( z(\nu(n) \setminus d)^\circ = x^\circ \) then \((\nu(n) \setminus d)^\circ = (z x) \circ (\nu(n) \setminus h)^\circ \circ (x t)\) where \( t := x(\nu(n) \setminus d)^\circ \). So \( \circ q \circ w = \circ q \circ CQw'' = (d)(\nu(n) \setminus d)^\circ CQw'' = (h)(x y)(\nu(n) \setminus h)^\circ \circ (x t) \circ CQw'' = (h)(\nu(n) \setminus h)^\circ CQw''(x y) \circ (x t) = \circ q \circ w \circ (x y)(t x)\).

As in the First Possibility, we encounter \( x, y \), and \( t \) as elements in distinct orbits \( A, B \), and \( C \) of the permutation \( \circ p \circ w(n \cup \supp(w)) \). By an argument similar to that in the First Possibility, we infer that \( \circ q \circ w \not\supseteq \circ p \circ w \). So \( f \) is not CI in Case Three too, and thus Theorem 3.18 is proved. \( \Box \)

We have completed the proof of Theorem 3.19 which tells us exactly which connected transpositional multigraphs are CI. This renders it easy to specify the class of all CI transpositional multigraphs on the vertex set \( n \). Recall that, where \( n \in \{1, 2\} \), every transpositional sequence is both permutationally complete and conjugacy invariant. The following summarizes the main results in §3.

**Theorem 3.19.** For \( n \geq 3 \), let \( u \) be a sequence in \( 1^{n-2}2^1 \). Let \( \{T(u_i) : i \in m\} \) be the set of components of the transpositional multigraph \( T(u) \), where each \( u_i \) is the subsequence of \( u \) for which the vertex set of \( T(u_i) \) is \( V_i = \bigcup \text{Supp}(u_i) \), and where of course \( \{V_i : i \in m\} \) is a partition of the set \( n \). Then:

- \( u \) is conjugacy invariant if and only if \( u_i \) is conjugacy invariant for every \( i \in m \).
- \( u_i \) is CI for \( |V_i| = 3 \) if and only if either \( |u_i| \) is odd or \( T(u_i) \) is a multitree with a single multiedge.
- \( u_i \) is CI for \( |V_i| \geq 4 \) if and only if \( T(u_i) \) is a multitree, no vertex of which is on more than one nonsimple multiedge, and each even-multiplicity multiedge of which is a multiedge whose non-leaf vertex has exactly two neighbors.

**Proof.** The theorem’s first claim is obvious. Its second claim is immediate from Theorem 5.5. Its third claim merely combines Theorems 3.6, 3.13, 3.14 and 3.18. \( \Box \)

### 3.4 Unfinished work

If a sequence \( s \) in \( \text{Sym}(n) \) is perm-complete then of course Prod\((s)\) is a coset of the subgroup Alt\((n)\) of \( \text{Sym}(n) \). Ross Willard asks for what other \( s \) are there subgroups \( H_s < \text{Sym}(n) \) for which Prod\((s)\) \( \in \text{Sym}(n)/H_s \).

An obvious task ahead pertaining to conjugacy invariance is the formidable one of providing necessary and sufficient criteria for deciding conjugacy invariance of every permutational sequence \( s \) in \( \text{Sym}(n) \). The ultimate goal is criteria enabling one to recognize the family \( C_s \) of conjugacy classes \( C \) of \( \text{Sym}(n) \) for which \( C \cap \text{Prod}(s) \neq \emptyset \).

Every \( f \in \text{Sym}(n) \) has an infinite number of factorizations into products of transpositions. But if the lengths of the nontrivial cyclic components of \( f \) are \( \ell_1, \ell_2, \ldots, \ell_d \) then the length of every minimal \( t \) is \( (\sum_{i=1}^{d} \ell_i) - d \), and if \( f \) has, by our definition given now, exactly \( \Phi(\text{Type}(f)) > \sum_{i=1}^{d} \ell_i^2 \) distinct minimal length transpositional factorizations if \( |\supp(f)| \geq 5 \).

**Problem.** Specify exact values for \( \Phi(\text{Type}(f)) \). The enumeration gets nontrivial when \( f \) is not single-cycled.

Clearly, if every term \( s_i \) of the sequence \( s := (s_0, s_1, \ldots, s_m) \) in \( \text{Sym}(n) \) has a factorization, \( s_i = \bigcirc t_i = t_{i,0} \circ t_{i,1} \circ \cdots \circ t_{i,\ell} \) into a product of transpositions such that the conglomerate transpositional sequence \( t := t_0 t_1 \cdots t_m \) is conjugacy invariant, then the permutational sequence \( s \) itself is conjugacy invariant. Thus we quickly get a sufficient condition for \( s \) to be conjugacy invariant. However, that condition is not necessary to assure the conjugacy invariance of a permutational sequence.

**Counterexample.** Let \( s := ((0 \ 1 \ 2), (0 \ 2 \ 1)^{\oplus(2)}) \). We omit the easy verification that Prod\((s)\) \( \subseteq 3^1 \), whence \( s \) is conjugacy invariant. However, \((0 \ 1 \ 2) \) has exactly three distinct factorizations as a product of two
transpositions; these are:

\[(0 1 2) = (0 1) \circ (0 2) \quad (0 1 2) = (0 2) \circ (1 2) \quad (0 1 2) = (1 2) \circ (0 1)\]

Of course (0 2 1) likewise has exactly three such factorizations, and since the permutation (0 2 1) occurs exactly twice as a term in \(s\), we infer each sequence \(t\) in \(S_1^2\) that results from factorizations of each term of \(s\) into products of two transpositions per term is six terms long.

The reader can check that there are exactly three distinct \(\text{Seq}(t_i)\) that result from the possible length-6 conglomerate transpositional sequences. As usual, each such \(\text{Seq}(t_i)\), for \(i \in 3\), determines a transpositional multigraph \(T(t_i)\) on the vertex set \(3\). We list the multiedge sets of these three multigraphs; they are:

\[E_0 := \{(0 1)^{(2)}, (1 2)^{(2)}, (0 2)^{(2)}\} \quad E_1 = \{(0 1)^{(2)}, (1 2), (0 2)^{(3)}\} \quad E_2 = \{(0 1)^{(3)}, (1 2)^{(3)}\}\]

By Theorem 3.19 none of these three transpositional multigraphs \(T(t_i)\) is conjugacy invariant.

This counterexample exhibits a conjugacy invariant permutational sequence \(s\) which lacks a conjugacy invariant conglomerate transpositional sequence that results from transpositional factorizations of the terms in \(s\). We leave it to the reader to corroborate that the permutational sequence \(\langle (0 1 2), (0 3 2), (0 3 1) \rangle\) in \(\text{Sym}(4)\) is a second, perhaps more interesting, such counterexample.

A transposition is a special sort of “single-cycled” permutation; i.e., an \(f \in \bigcup\{1^{n-c}c^1 : 2 \leq c \leq n\}\). Arthur Tuminaro [7] kicked off the study of conjugacy invariance of sequences of single-cycled permutations.

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