Multiphysics mixed finite element method with Nitsche’s technique for Stokes-poroelasticity problem

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Abstract

In this paper, we propose a multiphysics mixed finite element method with Nitsche’s technique for Stokes-poroelasticity problem. Firstly, we present a multiphysics reformulation of poroelasticity part of the original problem by introducing two pseudo-pressures to reveal the underlying deformation and diffusion multi physical processes in the Stokes-poroelasticity problem. Then, we prove the existence and uniqueness of weak solution of the reformulated and original problem. And we use Nitsche’s technique to approximate the coupling condition at the interface to propose a loosely-coupled time-stepping method– multiphysics mixed finite element method for space variables, and we decouple the reformulated problem into three sub-problems at each time step–a Stokes problem, a generalized Stokes problem and a mixed diffusion problem. Also, we give the stability analysis and error estimates of the loosely-coupled time-stepping method. Finally, we show the numerical tests to verify the theoretical results, which has a good stability and no locking” phenomenon.

Keywords: Stokes-poroelasticity problem, Nitsche’s technique, multiphysics mixed finite element method, time-stepping method, “locking” phenomenon.

1. Introduction

Stoke-poroelasticity model is an interaction of a free fluid and poroelastic, which is widely applied in reservoir engineering [1–3], the groundwater flow [4–6], the poroelastic structure and crack [7–9], in biomechanics [10–12], and so on. For the the Stokes-poroelasticity problem, there are few theoretical results. As for the numerical results, a domain decomposition (DD) method is proposed in [13, 14]. The monolithic multigrid

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method have been successfully applied for the system of Stokes equations in [15] and poroelasticity equations in [16], the Stokes-poroelasticity problem in [17, 18]. The authors of [19] adopt an extended domain decomposition method for the fluid-poroelastic structure interaction (FPSI) problem. In [20], a loosely coupled scheme using a Nitsche type weak coupling together with an interface pressure stabilization in time is proposed. However, the accuracy is relatively low and the computational efficiency of the scheme is reduced by using iterative correction. Recently, the multiphysics finite element method is proposed by [21] for the poroelasticity problem to reveals the underlying deformation and diffusion multi physical processes and overcome the locking phenomenon.

In this paper, we consider the flow of a viscous fluid in a channel bounded by a poroelastic medium, which is described as follows:

\[-2\mu_f \nabla \cdot D(v) + \nabla p_f = f, \quad (x, t) \in \Omega_f \times (0, T], \quad (1.1)\]
\[\nabla \cdot v = g, \quad (x, t) \in \Omega_f \times (0, T], \quad (1.2)\]
\[-2\mu_p \nabla \cdot D(U) - \lambda_p \nabla (\nabla \cdot U) + \alpha \nabla p_p = h, \quad (x, t) \in \Omega_p \times (0, T], \quad (1.3)\]
\[-k \nabla p_p = q, \quad (x, t) \in \Omega_p \times (0, T], \quad (1.4)\]
\[(s_0 p_p + \alpha \nabla \cdot U)_t + \nabla \cdot q = s, \quad (x, t) \in \Omega_p \times (0, T], \quad (1.5)\]

where $\Omega_f$ is a Stokes flow domain, and $\Omega_p$ is a poroelastic material. $v$ is the fluid velocity, $\sigma_f(v, p_f) = -p_f I + 2\mu_f D(v)$ is the fluid stress tensor, $p_f$ is the fluid pressure, $\mu_f$ is the fluid viscosity and $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the rate-of-strain tensor. $h$, $s$, $f$ and $g$ are the body force. The stress tensor of the poroelastic medium is $\sigma_p = \sigma^E - \alpha p_p I$, where $\sigma^E = \lambda_p (\nabla \cdot U) I + 2\mu_p D(U)$. $\lambda_p$ and $\mu_p$ denote the Lamé coefficients for the skeleton, and $D(U) = \frac{1}{2}(\nabla U + (\nabla U)^T)$. The equation of (1.4) is called by Darcy’s law, where the flux $q$ is the relative velocity of the fluid within the porous structure and $p_p$ is the fluid pressure. The hydraulic conductivity is denoted by $k$, which is in general a symmetric positive definite tensor. The coefficient $s_0 \in (0, 1)$ is the storage coefficient, and the Biot-CWillis constant $\alpha$ is the pressure-storage coupling coefficient. The fluid domain is bounded by a deformable porous matrix consisting of a skeleton and connecting pores filled with fluid, the following two configurations are considered: (i) the channel extends to the external boundary (Figure [1]), and (ii) the channel is surrounded by the poroelastic media (Figure [2]).
Configuration (i) is suitable FPSI in arteries, and the configuration (ii) is often applied to the fractured reservoirs. As for configuration (i), we prescribe the following boundary and initial conditions:

\[
\sigma_f n_f = -p_{in}(t)n_f, \quad (x, t) \in \Gamma_f^{in} \times (0, T],
\]
\[
\sigma_f n_f = 0, \quad (x, t) \in \Gamma_f^{out} \times (0, T],
\]
\[
U = 0, \quad (x, t) \in \Gamma_p^{in} \cup \Gamma_p^{out} \times (0, T],
\]
\[
n_p \cdot \sigma^E n_p = 0, \quad (x, t) \in \Gamma_p^{ext} \times (0, T],
\]
\[
U \cdot \tau_p = 0, \quad (x, t) \in \Gamma_p^{ext} \times (0, T],
\]
\[
p_p = 0, \quad (x, t) \in \Gamma_p^{ext} \times (0, T], \quad q \cdot n_p = 0, \quad (x, t) \in \Gamma_p^{in} \cup \Gamma_p^{out} \times (0, T],
\]
\[
v^0 = 0, \quad U^0 = 0, \quad p_p^0 = 0.
\]

As for the configuration (ii), we can suppose that \(U = 0\) and \(q \cdot n_p = 0\) on \(\partial \Omega_p := \Gamma_p^{ext}\).

Next, we give the interface conditions on \(\Gamma\). To do that, we denote the outward normal to the fluid domain by \(n\) and the tangential unit vector on the interface \(\Gamma\) by \(\tau\). The interface conditions are given by

\[
(v - \frac{\partial U}{\partial t}) \cdot n = q \cdot n, \quad (x, t) \in \Gamma \times (0, T],
\]
\[ \mathbf{v} \cdot \tau = \frac{\partial \mathbf{U}}{\partial t} \cdot \tau, \quad (x,t) \in \Gamma \times (0,T], \quad (1.14) \]
\[ -\mathbf{n} \cdot \sigma_{fn} = p_p, \quad (x,t) \in \Gamma \times (0,T], \quad (1.15) \]
\[ \mathbf{n} \cdot \sigma_{fn} - \mathbf{n} \cdot \sigma_p = 0, \quad (x,t) \in \Gamma \times (0,T], \quad (1.16) \]
\[ \tau \cdot \sigma_{fn} - \tau \cdot \sigma_p = 0, \quad (x,t) \in \Gamma \times (0,T]. \quad (1.17) \]

The condition (1.14) can be replaced by Beavers-CJoseph-Saffman condition (cf. [22]):
\[ \beta (\mathbf{v} - \frac{\partial \mathbf{U}}{\partial t}) \cdot \tau = -\tau \cdot \sigma_{fn}, \quad (x,t) \in \Gamma \times (0,T], \quad (1.18) \]
where the parameter \( \beta \) quantifies the resistance that the porous matrix opposes to fluid flow in the tangential direction.

As for the problem (1.1)-(1.17), we use the multiphysics mixed finite element method with Nitsche’s technique to solve the Stokes-poroelasticity problem. Firstly, we define the weak solution of the reformulated problem and prove the existence and uniqueness of weak solution. Then, we use Nitsche’s technique to deal with the interface condition to propose a fully discrete multiphysics finite element method, which decouples the problem into three sub-problems at each time step: a Stokes problem, a generalized Stokes problem and a mixed diffusion problems. And we give the stability analysis and error estimates of the decoupled finite element method. Finally, we give the numerical tests to verify the effectiveness and feasibility of our method. To the best of our knowledge, it is first time to use the multiphysics finite element method with Nitsche’s technique to solve the Stokes-poroelasticity problem.

The remainder of this paper is organized as follows. In Section 2 we reformulate the original problem based on multiphysics approach and prove the existence and uniqueness of weak solution of the reformulated problem. In Section 3 we use Nitsche’s technique to deal with the interface condition and propose a fully discrete multiphysics mixed finite element method to decouple the problem into three sub-problems at each time step. And we give the stability analysis and optimal error estimates of the proposed numerical method. In Section 4 we show the numerical tests to verify the theoretical results, which has a good stability and no locking” phenomenon. Finally, we draw a conclusion to summary the main results in this paper.

2. Multiphysics approach and PDE analysis

To reveal the multi physics processes underlying in the original problem, we introduce a new variable \( \phi := \nabla \cdot \mathbf{U} \), and define the following two pseudo-presures
\[ \xi := \alpha p_p - \lambda_p \phi, \quad \eta := s_0 p_p + \alpha \phi. \]
Remark 2.1. The coupled problem (2.3)-(2.8) reveals the underlying deformation and diffusion multiphysics process, which occurs in the Stokes-poroelasticity problem (1.1)-(1.5) if there hold

\[ p_p = k_1 \xi + k_2 \eta, \quad \phi = k_1 \eta - k_3 \xi, \] (2.1)

where

\[ k_1 = \frac{\alpha}{\alpha^2 + \lambda_p s_0}, \quad k_2 = \frac{\lambda_p}{\alpha^2 + \lambda_p s_0}, \quad k_3 = \frac{s_0}{\alpha^2 + \lambda_p s_0}. \] (2.2)

Then the problem (1.1)-(1.5) can be reformulate into

\[ -2\mu_f \nabla \cdot D(v) + \nabla p_f = f, \quad (x, t) \in \Omega_f \times (0, T], \] (2.3)
\[ \nabla \cdot v = g, \quad (x, t) \in \Omega_f \times (0, T], \] (2.4)
\[ -2\mu_p \nabla \cdot D(U) + \nabla \xi = h, \quad (x, t) \in \Omega_p \times (0, T], \] (2.5)
\[ k_3 \xi + \nabla \cdot U = k_1 \eta, \quad (x, t) \in \Omega_p \times (0, T], \] (2.6)
\[ -k \nabla (k_1 \xi + k_2 \eta) = q, \quad (x, t) \in \Omega_p \times (0, T], \] (2.7)
\[ \eta_t + \nabla \cdot q = s, \quad (x, t) \in \Omega_p \times (0, T]. \] (2.8)

**Remark 2.1.** The coupled problem (2.3)-(2.8) reveals the underlying deformation and diffusion multiphysics process, which occurs in the Stokes-poroelasticity problem (1.1)-(1.5). It should be noted that the original pressure is eliminated in the reformulation, which will be helpful to overcome the "locking" phenomenon.

To define a weak solution of the problem (2.3)-(2.8), we introduce the following function spaces

\[ V^f = \{ \varphi_f \in H^1(\Omega_f)^d : \varphi_f = 0, x \in \Gamma^{out}_f \}, \quad Q^f = \{ \psi_f \in L^2(\Omega_f) \}, \]
\[ X^p = \{ \varphi_p \in H^1(\Omega_p)^d : \varphi_p = 0, x \in \Gamma^{in}_p \cup \Gamma^{out}_p, \varphi_p \cdot n_p = 0, x \in \Gamma^{ext}_p \}, \]
\[ V^p = \{ r \in H(div; \Omega_p) : r \cdot n_p = 0, x \in \Gamma^{in}_p \cup \Gamma^{out}_p \}, \]
\[ M^p = \{ w \in L^2(\Omega_p) \}, \quad Q^p = \{ z \in H^1(\Omega_p) \}, \]

with the norm \( \|v\|^2_{H(div;\Omega_p)} = \|v\|^2_{L^2(\Omega_p)} + \|\nabla \cdot v\|^2_{L^2(\Omega_p)}. \)

For convenience, we assume that the functions \( f, g, h \) and \( s \) all are independent of \( t \) in the remaining of the paper. We note that all the results of this paper can be easily extended to the case of time-dependent source functions.

**Definition 2.1.** Let \( v^0 \in H^1(\Omega_f) \), \( U^0 \in H^1(\Omega_p) \), \( p^0_p \in L^2(\Omega_p) \), \( f, g \in L^2(\Omega_f) \), \( h, s \in L^2(\Omega_p) \), for any \( t \in (0, T] \), we say that \( v \in L^\infty(0, T; V^f), p_f \in L^\infty(0, T; Q^f), U \in L^\infty(0, T; X^p), q \in L^\infty(0, T; V^p), p_p \in L^\infty(0, T; Q^p) \) are a weak solution of the problem (1.1)-(1.5) if there hold

\[-\langle \sigma_f n_f, \varphi_f \rangle_T + (2\mu_f D(v), D(\varphi_f))_{\Omega_f} - (p_f, \nabla \cdot \varphi_f)_{\Omega_f} + \langle p_{in}(t)n_f, \varphi_f \rangle_{\Gamma^{in}_p} = (f, \varphi_f)_{\Omega_f} \]
(\nabla \cdot v, \psi_f)_{\Omega_f} = (g, \psi(2.4)0)

\begin{align}
-\langle \sigma_p n_p, \varphi_p \rangle_{\Gamma} + 2\mu_p (D(U), D(\varphi_p))_{\Omega_p} + \lambda_p (\nabla \cdot U, \nabla \cdot \varphi_p)_{\Omega_p} - \alpha(p_p, \nabla \cdot \varphi_p)_{\Omega_p} &= (h, \varphi(2.4)1) \\
2^{-1}(q, r)_{\Omega_p} - (p_p, \nabla \cdot r)_{\Omega_p} + (p_p \cdot n_p, r)_{\Gamma} &= 0, \quad (2.12) \\
((s_0 p_p + \alpha \nabla \cdot U)_t, z)_{\Omega_p} + (\nabla \cdot q, z)_{\Omega_p} &= (s, z)_{\Omega_p}, \quad (2.13)
\end{align}

for any \((\varphi_f, \psi_f, \varphi_p, r, z) \in V^f \times Q^f \times X^p \times V^p \times Q^p\).

Similarly, we can define the weak solution to the problem \[(2.3)-(2.8)\] if there hold

\begin{align}
-\langle \sigma_f n_f, \varphi_f \rangle_{\Gamma} + (2\mu_f D(v), D(\varphi_f))_{\Omega_f} + \langle p_{in}(t) n_f, \varphi_f \rangle_{\Gamma_{in}} &= (f, \varphi_f)_{\Omega_f}, \quad (2.14) \\
(\nabla \cdot v, \psi_f)_{\Omega_f} &= (g, \psi_f)_{\Omega_f}, \quad (2.15) \\
-\langle \sigma_p n_p, \varphi_p \rangle_{\Gamma} + 2\mu_p (D(U), D(\varphi_p))_{\Omega_p} - (\xi, \nabla \cdot \varphi_p)_{\Omega_p} &= (h, \varphi_p)_{\Omega_p}, \quad (2.16) \\
k_3(\xi, w)_{\Omega_p} + (\nabla \cdot U, w)_{\Omega_p} &= k_1(\eta, w)_{\Omega_p}, \quad (2.17) \\
k^{-1}(q, r)_{\Omega_p} - (k_1 \xi + k_2 \eta, \nabla \cdot r)_{\Omega_p} + (k_1 \xi + k_2 \eta) \cdot n_p, r_{\Gamma} &= 0, \quad (2.18) \\
(\partial \eta, z)_{\Omega_p} + (\nabla \cdot q, z)_{\Omega_p} &= (s, z)_{\Omega_p}, \quad (2.19)
\end{align}

for any \((\varphi_f, \psi_f, \varphi_p, r, w, z) \in V^f \times Q^f \times X^p \times V^p \times M^p \times Q^p\).

**Lemma 2.1.** Every weak solution \((v, p_f, U, q, \xi, \eta)\) of the problem \[(2.14)-(2.19)\] satisfies the following energy law:

\begin{align}
J(t) + \int_0^t [2\mu_f \|D(v)\|_{L^2(\Omega_f)}^2 + k^{-1}\|q\|_{L^2(\Omega_p)}^2 + \beta \|v - U_t\|_{L^2(\Gamma)}^2] dt - \int_0^t F(t) dt = J(0), \quad (2.20)
\end{align}

for all \(t \in (0, T]\), where

\begin{align}
J(t) &= \frac{1}{2}[2\mu_f \|D(U(t))\|_{L^2(\Omega_p)}^2 + k_2 \|\eta(t)\|_{L^2(\Omega_p)}^2 + k_3 \|\xi(t)\|_{L^2(\Omega_p)}^2] - 2(h, U(t))_{\Omega_p}, \quad (2.21) \\
F(t) &= (f, v)_{\Omega_f} + \langle p_{in}(t) n_f, v \rangle_{\Gamma_{in}} + (g, p_f)_{\Omega_f} + (s, p_p)_{\Omega_p}. \quad (2.22)
\end{align}

**Proof.** Differentiating \[(2.17)\] with respect to \(t\), taking \(\varphi_f = v, r = q, \psi_f = p_f, \varphi_p = U_t, w = \xi, z = p_p = k_1 \xi + k_2 \eta\), we have

\begin{align}
-\langle \sigma_f n_f, v \rangle_{\Gamma} + 2\mu_f \|D(v)\|_{L^2(\Omega_f)}^2 &= (f, v)_{\Omega_f} + \langle -p_{in}(t) n_f, v \rangle_{\Gamma_{in}} + (g, p_f)_{\Omega_f}, \quad (2.23) \\
\mu_p \frac{d}{dt} \|D(U)\|_{L^2(\Omega_p)}^2 + \frac{k_3}{2} \frac{d}{dt} \|\eta\|_{L^2(\Omega_p)}^2 + \frac{k_3}{2} \frac{d}{dt} \|\xi\|_{L^2(\Omega_p)}^2 - \langle \sigma_p n_p, U_t \rangle_{\Gamma}
\end{align}
\[ + \langle p_p \mathbf{n}_p, \mathbf{q} \rangle_{\Gamma} + k^{-1} \| \mathbf{q} \|^2_{L^2(\Omega_p)} = (s, p_p)_{\Omega_p} + \frac{d}{dt} (\mathbf{h}, \mathbf{U})_{\Omega_p}. \] (2.24)

Adding (2.23) and (2.24), we get
\[ \beta \| \mathbf{v} - \mathbf{U}_t \cdot \tau \|^2_{L^2(\Gamma)} + 2 \mu_f \| \mathbf{D} (\mathbf{v}) \|^2_{L^2(\Omega_f)} + k^{-1} \| \mathbf{q} \|^2_{L^2(\Omega_p)} + \mu_p \frac{d}{dt} \| \mathbf{D}(\mathbf{U}(t)) \|^2_{L^2(\Omega_p)} + \frac{k_2}{2} \frac{d}{dt} \| \mathbf{\eta}(t) \|^2_{L^2(\Omega_p)} + \frac{k_3}{2} \frac{d}{dt} \| \mathbf{\xi}(t) \|^2_{L^2(\Omega_p)} \]
\[ \quad - \mathbf{h}, \mathbf{U}(t) \|_{\Omega_p} \]
\[ = (f, \mathbf{v})_{\Omega_f} + \langle -p_m(t) \mathbf{n}_f, \mathbf{v} \rangle_{\Gamma_f} + (g, p_f)_{\Omega_f} + (s, p_p)_{\Omega_p} + \frac{d}{dt} (\mathbf{h}, \mathbf{U})_{\Omega_p}. \] (2.25)

Integrating (2.25) from \([0, t] \), we obtain
\[ \int_0^t [\beta \| \mathbf{v} - \mathbf{U}_t \cdot \tau \|^2_{L^2(\Gamma)} + 2 \mu_f \| \mathbf{D} (\mathbf{v}) \|^2_{L^2(\Omega_f)} + k^{-1} \| \mathbf{q} \|^2_{L^2(\Omega_p)}] \]
\[ + \mu_p \| \mathbf{D}(\mathbf{U}(t)) \|^2_{L^2(\Omega_p)} + \frac{k_2}{2} \| \mathbf{\eta}(t) \|^2_{L^2(\Omega_p)} + \frac{k_3}{2} \| \mathbf{\xi}(t) \|^2_{L^2(\Omega_p)} - (\mathbf{h}, \mathbf{U}(t)) \|_{\Omega_p} \]
\[ = (f, \mathbf{v})_{\Omega_f} + \langle -p_m(t) \mathbf{n}_f, \mathbf{v} \rangle_{\Gamma_f} + (g, p_f)_{\Omega_f} + (s, p_p)_{\Omega_p} \]
\[ + \int_0^t [\beta \| \mathbf{v} - \mathbf{U}_t \cdot \tau \|^2_{L^2(\Gamma)} + 2 \mu_f \| \mathbf{D} (\mathbf{v}) \|^2_{L^2(\Omega_f)} + k^{-1} \| \mathbf{q} \|^2_{L^2(\Omega_p)}] \]
\[ \int_0^t F(t) dt = E(0), \] (2.26)

which implies that (2.20) holds. The proof is complete.

Taking the similar argument of Lemma 2.1 we can get the following result, and here we omit the detail of the proof.

**Lemma 2.2.** Every weak solution \((\mathbf{v}, p_f, \mathbf{U}, \mathbf{q}, p_p)\) of the problem (2.9)-(2.13) satisfies the following energy law
\[ E(t) + \int_0^t [2 \mu_f \| \mathbf{D} (\mathbf{v}) \|^2_{L^2(\Omega_f)} + k^{-1} \| \mathbf{q} \|^2_{L^2(\Omega_p)} + \beta \| \mathbf{v} - \mathbf{U}_t \cdot \tau \|^2_{L^2(\Gamma)}] dt - \int_0^t F(t) dt = E(0), \] (2.26)

for all \(t \in (0, T)\), where
\[ E(t) = \frac{1}{2} [2 \mu_p \| \mathbf{D}(\mathbf{U}(t)) \|^2_{L^2(\Omega_p)} + \lambda_p \| \nabla \cdot \mathbf{U}(t) \|^2_{L^2(\Omega_p)} + s_0 \| p_p(t) \|^2_{L^2(\Omega_p)} - 2(\mathbf{h}, \mathbf{U}(t))_{\Omega_p}], \]
\[ F(t) = (f, \mathbf{v})_{\Omega_f} + \langle -p_m(t) \mathbf{n}_f, \mathbf{v} \rangle_{\Gamma_f} + (g, p_f)_{\Omega_f} + (s, p_p)_{\Omega_p}. \] (2.28)

From Lemma 2.1, we obtain the following estimates:

**Lemma 2.3.** There exists a positive constant \(C_1\) such that
\[ \sqrt{2 \mu_f} \| \mathbf{D} (\mathbf{v}) \|_{L^2(0, T, L^2(\Omega_f))} + \sqrt{k^{-1}} \| \mathbf{q} \|_{L^2(0, T, L^2(\Omega_p))} \]
\[ + \sqrt{\mu_p} \| \mathbf{D}(\mathbf{U}) \|_{L^\infty(0, T, L^2(\Omega_p))} + \sqrt{3} \| \mathbf{v} - \mathbf{U}_t \cdot \tau \|_{L^2(0, T, L^2(\Gamma))} \]
\[ + \sqrt{\frac{k_2}{2}} \| \mathbf{\eta} \|_{L^\infty(0, T, L^2(\Omega_p))} + \sqrt{\frac{k_3}{2}} \| \mathbf{\xi} \|_{L^\infty(0, T, L^2(\Omega_p))} \leq C_1, \] (2.29)

where \(C_1\) is a constant dependent on the initial data and source functions of \(\| \mathbf{v}^0 \|_{H^1(\Omega_f)}\), \(\| p_f \|_{L^2(\Omega_f)}\), \(\| p_p \|_{L^2(\Omega_p)}\), \(\| \mathbf{U}^0 \|_{H^1(\Omega_f)}\), \(\| f \|_{L^2(\Omega_f)}\), \(\| g \|_{L^2(\Omega_f)}\), \(\| \mathbf{h} \|_{L^2(\Omega_p)}\), \(\| s \|_{L^2(\Omega_p)}\).
Lemma 2.4. Suppose that $v^0$, $U^0$ and $p^0_t$ are sufficiently smooth, then there exist positive constants $C_2 = C_2(C_1, \|q^0\|_{L^2(\Omega_p)})$ and $C_3 = C_3(C_1, C_2, \|U^0\|_{H^2(\Omega_p)}, \|p^0_t\|_{H^2(\Omega_p)})$ such that

\[
\sqrt{H_f} \|D(v)\|_{L^\infty(0,T;L^2(\Omega_f))} + \sqrt{(2k)^{-1}} \|q\|_{L^\infty(0,T;L^2(\Omega_p))} + \sqrt{2\mu_f} \|D(U_t)\|_{L^2(0,T;L^2(\Omega_p))} \\
+ \sqrt{\frac{\beta}{2}} \|v - U_t\|_\tau_{L^2(0,T;L^2(\Omega_f))} + \sqrt{k_2} \|\eta\|_{L^2(0,T;L^2(\Omega_p))} + \sqrt{k_3} \|\xi_t\|_{L^2(0,T;L^2(\Omega_p))} \leq C(2.30) \\
\sqrt{2\mu_f} \|D(v_t)\|_{L^2(0,T;L^2(\Omega_f))} + \sqrt{k_2} \|\eta\|_{L^2(0,T;L^2(\Omega_p))} + \sqrt{k_3} \|\xi_t\|_{L^2(0,T;L^2(\Omega_p))} \leq C(2.31)
\]

Proof. Differentiating $(2.14)$, $(2.16)$, $(2.17)$ and $(2.18)$ with respect to $t$, taking $\varphi_f = v$, $r = q$, $\psi_f = p_{f,t}$, $\varphi_p = U_t$, $w = \xi_t$, $z = k_1 \xi_t + k_2 \eta_t$, we have

\[
-\langle \sigma_f, n_f, v \rangle_\Gamma + \mu_f \frac{d}{dt} \|D(v)\|^2_{L^2(\Omega_f)} = (g, p_{f,t})_\Omega_f, \quad (2.32) \\
-\langle \sigma_p, n_p, U_t \rangle_\Gamma + \langle p_{f,t}, n_p, q \rangle_\Gamma + 2\mu_p \|D(U_t)\|^2_{L^2(\Omega_p)} + (2k)^{-1} \frac{d}{dt} \|q\|^2_{L^2(\Omega_p)} \\
+ k_2 \|\eta\|^2_{L^2(\Omega_p)} + k_3 \|\xi_t\|^2_{L^2(\Omega_p)} = (s, p_{f,t} + p_{p,t})_\Omega_p. \quad (2.33)
\]

Adding $(2.32)$ and $(2.33)$, we have

\[
\beta \frac{d}{dt} \| (v - U_t) \cdot \tau \|^2_{L^2(\Gamma)} + \mu_f \frac{d}{dt} \|D(v)\|^2_{L^2(\Omega_f)} + (2k)^{-1} \frac{d}{dt} \|q\|^2_{L^2(\Omega_p)} \\
+ 2\mu_p \|D(U_t)\|^2_{L^2(\Omega_p)} + k_2 \|\eta\|^2_{L^2(\Omega_p)} + k_3 \|\xi_t\|^2_{L^2(\Omega_p)} = \frac{d}{dt} [(g, p_f)_\Omega_f + (s, p_{f,t} + p_{p,t})_\Omega_p]. \quad (2.34)
\]

Integrating in $t$, we get

\[
\beta \| (v(t) - U_t(t)) \cdot \tau \|^2_{L^2(\Gamma)} + \mu_f \|D(v(t))\|^2_{L^2(\Omega_f)} + (2k)^{-1} \|q(t)\|^2_{L^2(\Omega_p)} \\
+ \int_0^t [2\mu_p \|D(U_t)\|^2_{L^2(\Omega_p)} + k_2 \|\eta\|^2_{L^2(\Omega_p)} + k_3 \|\xi_t\|^2_{L^2(\Omega_p)}] dt \\
= \beta \| (v(0) - U_t(0)) \cdot \tau\|_{L^2(\Gamma)} + \mu_f \|D(v(0))\|^2_{L^2(\Omega_f)} + (2k)^{-1} \|q(0)\|^2_{L^2(\Omega_p)} \\
[(g, p_f(0) - p^0_f)_\Omega_f + (s, p(t) - p^0_p)_\Omega_p],
\]

which implies that $(2.30)$ holds.

Differentiating $(2.14)$, $(2.15)$, $(2.16)$, $(2.18)$ and $(2.19)$ with respect to $t$, differentiating $(2.17)$ with respect to $t$ two times, taking $\varphi_f = v_t$, $r = q_t$, $\psi_f = p_{f,t}$, $\varphi_p = U_{tt}$, $w = \xi_t$, $z = p_{p,t} = k_1 \xi_t + k_2 \eta_t$, we have

\[
\beta \| (v_t - U_{tt}) \cdot \tau \|^2_{L^2(\Gamma)} + 2\mu_f \|D(v_t)\|^2_{L^2(\Omega_t)} + k_3 \|\xi_t\|^2_{L^2(\Omega_p)} \\
+ \mu_p \frac{d}{dt} \|D(U_t)\|^2_{L^2(\Omega_p)} + \frac{k_2}{2} \frac{d}{dt} \|\eta\|^2_{L^2(\Omega_p)} + \frac{k_3}{2} \frac{d}{dt} \|\xi_t\|^2_{L^2(\Omega_p)} = 0. \quad (2.35)
\]

Integrating $(2.35)$ in $t$, we get

\[
\int_0^t [ \beta \| (v_t - U_{tt}) \cdot \tau \|^2_{L^2(\Gamma)} + 2\mu_f \|D(v_t)\|^2_{L^2(\Omega_t)} + k_3 \|\xi_t\|^2_{L^2(\Omega_p)} ] dt
\]
Suppose that main result.

and taking Lemma 2.5.

Every weak solution (\(v, p_f, U, \xi, \eta, q, p_p\)) of the problem (2.14)-(2.19) satisfies the following relations:

\[
C_v(t) := \langle v(t) \cdot n_f, 1 \rangle_{\partial \Omega_f} = (g, 1)_{\Omega_f}, \quad (2.36)
\]

\[
C_{p_f}(t) := (p_f(t), 1)_{\Omega_f} = -\langle \sigma_f n_f, x \rangle_{\Gamma} + 2\mu_f C_v(t) + \langle p_{in}(t)n_f, x \rangle_{\Gamma_p} - (f, x)_{\Omega_f}, \quad (2.37)
\]

\[
C_\eta(t) := (\eta(t), 1)_{\Omega_p} = t[(s, 1)_{\Omega_p} - C_q] + (\eta(0), 1)_{\Omega_p}, \quad t \geq 0, \quad (2.38)
\]

\[
C_\xi(t) := (\xi(t), 1)_{\Omega_p} = \frac{1}{2\mu_pk_3 + 1}[2\mu_pk_1C_\eta - (h, x)_{\Omega_p} - \langle \sigma_p n_p, x \rangle_{\Gamma_r}], \quad (2.39)
\]

\[
C_{U}(t) := \langle U(t) \cdot n_p, 1 \rangle_{\partial \Omega_p} = k_1 C_\eta(t) - k_3 C_\xi(t), \quad (2.40)
\]

\[
C_{p_p}(t) := (p_p(t), 1)_{\Omega_p} = k_1 C_\eta(t) + k_2 C_\xi(t), \quad (2.41)
\]

\[
C_q(t) := \langle q(t) \cdot n_p, 1 \rangle_{\partial \Omega_p} = (k \nabla C_{p_p} \cdot n_p, 1)_{\partial \Omega_p}, \quad (2.42)
\]

Proof. Letting

\[
C_v(t) := \langle v(t) \cdot n_f, 1 \rangle_{\partial \Omega_f}, \quad C_U(t) := \langle U(t) \cdot n_p, 1 \rangle_{\partial \Omega_p}, \quad C_q(t) := \langle q(t) \cdot n_p, 1 \rangle_{\partial \Omega_p},
\]

\[
C_{p_f}(t) := (p_f(t), 1)_{\Omega_f}, \quad C_\eta(t) := (\eta(t), 1)_{\Omega_p}, \quad C_\xi(t) := (\xi(t), 1)_{\Omega_p}, \quad C_{p_p}(t) := (p_p(t), 1)_{\Omega_p},
\]

and taking \(\varphi_f = x, \psi_f = 1, \varphi_p = x, z = 1, w = 1\) in (2.14)-(2.19), we have

\[
-\langle \sigma_f n_f, x \rangle_{\Gamma} + 2\mu_f C_v(t) - C_{p_f}(t) + \langle p_{in}(t)n_f, x \rangle_{\Gamma_p} = (f, x)_{\Omega_f}, \quad (2.43)
\]

\[
C_v(t) = (g, 1)_{\Omega_f}, \quad (2.44)
\]

\[
-\langle \sigma_p n_p, x \rangle_{\Gamma} + 2\mu_p C_U(t) - C_\xi(t) = (h, x)_{\Omega_p}, \quad (2.45)
\]

\[
k_3 C_\xi(t) + C_U(t) = k_1 C_\eta(t), \quad (2.46)
\]

\[
\frac{1}{t}[C_\eta(t) - (\eta(0), 1)] + C_q(t) = (s, 1)_{\Omega_p}, \quad (2.47)
\]

which imply that (2.36)-(2.42) hold. The proof is complete. \(\square\)

With the help of Lemma 2.1, Lemma 2.2 and Lemma 2.5, we obtain the following main result.

**Theorem 2.1.** Suppose that \(v^0 \in H^1(\Omega_f), U^0 \in H^1(\Omega_p), p_f^0 \in L^2(\Omega_p), f, g \in L^2(\Omega_f), h, s \in L^2(\Omega_p),\) for any \(t \in (0, T],\) then there exists a unique weak solution to the problem (1.1)-(1.5), likewise, there exists a unique weak solution to the problem (2.3)-(2.8).
Proof. Note that the energy laws established in Lemma 2.1 and Lemma 2.2 guarantee the required uniform estimates, so it is standard to prove the existence of weak solution by the standard Galerkin method and compactness argument (cf. [23]), here we omit the details.

Suppose that there are two group of weak solutions: $v_1$, $p_{f,1}$, $U_1$, $p_{p,1}$, $q_1$ and $v_2$, $p_{f,2}$, $U_2$, $p_{p,2}$, $q_2$. Let $v = v_1 - v_2$, $p_f = p_{f,1} - p_{f,2}$, $U = U_1 - U_2$, $p_p = p_{p,1} - p_{p,2}$, $q = q_1 - q_2$. It suffices to show that $v \equiv 0$, $p_f \equiv 0$, $U \equiv 0$, $p_p \equiv 0$, $q \equiv 0$. By the linearity of equations, we immediately imply that $v$, $p_f$, $U$, $p_p$, $q$ satisfy (1.5) with $f = h = g = s = p_{in}(t) = 0$ and $v^0 = 0$, $U^0 = 0$, $p^0_p = 0$, thus $F(t) = 0$, $E(0) = 0$.

Using (2.26), we have

$$
\mu_p \|D(U(t))\|_{L^2(\Omega_p)}^2 + \lambda_p \|\nabla \cdot U(t)\|_{L^2(\Omega_p)}^2 + \frac{s_0}{2} \|p(t)\|_{L^2(\Omega_p)}^2 + \int_0^t \left[ 2\mu_f \|D(v)\|_{L^2(\Omega_f)}^2 + k^{-1} \|q\|_{L^2(\Omega_p)}^2 + \beta \|(v - U_t)\cdot \tau\|_{L^2(\Gamma)}^2 \right] dt = 0,
$$

which implies that $D(U) = 0$, $q = 0$, $p_p = 0$, $D(v) = 0$.

Therefore, we see that $U = 0$, $v = 0$, $p_f = 0$ by using Lemma 2.5 and $U \in H^1(\Omega_p)$, $v \in H^1(\Omega_f)$. The proof is complete. \hfill \Box

3. Fully discrete multiphysics mixed finite element method with Nitsche’s technique

3.1. Multiphysics mixed finite element method

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be fixed, quasi-uniform meshes defined on the domains $\Omega_f$ and $\Omega_p$ with the maximum mesh size $h$. We require that $\Omega_f$ and $\Omega_p$ are polygonal or polyhedral domains and that they conform at the interface $\Gamma$. We denote $V_h^f \subset V^f$, $Q_h^f \subset Q^f$ by the finite element spaces for the velocity and pressure approximation on the fluid domain $\Omega_f$, $V_h^p \subset V^p$, $M_h^p \subset M^p$, $Q_h^p \subset Q^p$ by the spaces for velocity and pressure approximation on the porous matrix $\Omega_p$ and $X_h^p \subset X^p$ by the approximation spaces for the structure displacement, respectively.

Using the Nitsche’s technique to deal with the interface conditions (cf. [24]), we get

$$
-I_f^\tau = -\int_{\Gamma} (n \cdot \sigma_{f,h}(v_h, p_{f,h}) n (\varphi_{f,h} - \varphi_{p,h} - r_h) \cdot n + \tau \cdot \sigma_{f,h}(v_h, p_{f,h}) n (\varphi_{f,h} - \varphi_{p,h}) \cdot \tau ) dx \\
+ \int_{\Gamma} \gamma_f \mu_f h^{-1} [ (v_h - \partial_t U_h - q_h) \cdot n (\varphi_{f,h} - \varphi_{p,h} - r_h) \cdot n + (v_h - \partial_t U_h) \cdot \tau (\varphi_{f,h} - \varphi_{p,h}) \cdot \tau ] dx,
$$

where $\gamma_f > 0$ is a penalty parameter.

Furthermore, we introduce

$$
-S_{\Gamma}^{\psi} = -\int_{\Gamma} (n \cdot \sigma_{f,h}(\psi_{f,h}, -\psi_{f,h}) n (v_h - \partial_t U_h - q_h) \cdot n + \tau \cdot \sigma_{f,h}(\psi_{f,h}, -\psi_{f,h}) n (v_h - \partial_t U_h) \cdot \tau ) dx,
$$

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where $\varsigma$ can be chosen 1, 0 or $\pm 1$.

Also, we can accommodate the weak enforcement of the BeaversJosephCSaffman condition \((1.18)\) by

$$-I_1^+ - S_{1^+\varsigma} = - \int_{\Gamma} n \cdot \sigma_{f,h}(v_h, p_{f,h}) n (\varphi_{f,h} - \varphi_{p,h} - r_h) \cdot n \, dx$$

$$- \int_{\Gamma} n \cdot \sigma_{f,h}(\varsigma \varphi_{f,h}, -\psi_{f,h}) n (v_h - \partial_t U_h - q_h) \cdot n \, dx$$

$$+ \int_{\Gamma} \gamma_f \mu_f h^{-1}(v_h - \partial_t U_h - q_h) \cdot n (\varphi_{f,h} - \varphi_{p,h} - r_h) \cdot n \, dx$$

$$+ \int_{\Gamma} \beta(v_h - \partial_t U_h) \cdot \tau(\varphi_{f,h} - \varphi_{p,h}) \cdot \tau \, dx.$$  

**Remark 3.1.** Comparing $I_1^+ + S_{1^+\varsigma}$ with $I_1^+ + S_{1^{-}\varsigma}$, we find that the operators corresponding to \((1.18)\) can be seen as a particular form of the more general case that is obtained when no-slip conditions \((1.14)\) are enforced weakly, namely $I_1^+ + S_{1^+\varsigma}$. For this reason, we perform the analysis of the numerical scheme in the latter form.

The semi-discrete scheme of the problem \((2.3)-(2.8)\) is: find $v_h \in L^\infty(0, T; V_h^f)$, $p_{f,h} \in L^\infty(0, T; Q_h^p)$, $U_h \in L^\infty(0, T; X_h^p)$, $q_h \in L^\infty(0, T; V_h^p)$, $\xi_h \in L^\infty(0, T; M_h^p)$, $\eta_h \in L^\infty(0, T; Q_h^p)$ such that

$$(2\mu_f D(v_h), D(\varphi_{f,h}))_{\Omega_f} - (p_{f,h}, \nabla \cdot \varphi_{f,h})_{\Omega_f} + (\nabla \cdot v_h, \psi_{f,h})_{\Omega_f}$$

$$+ 2\mu_p (D(U_h), D(\varphi_{p,h}))_{\Omega_p} - (\xi_h, \nabla \cdot \varphi_{p,h})_{\Omega_p} + (\nabla \cdot U_h, w_h)_{\Omega_p}$$

$$+ k_3(\xi_h, w_h)_{\Omega_p} - k_1(\eta_h, w_h)_{\Omega_p} + k^{-1}(q_h, r_h)_{\Omega_p} = (f, \varphi_{f,h})_{\Omega_f} + (-p_m(t) n_f, \varphi_{f,h})_{\Gamma_f^n} + (g, \psi_{f,h})_{\Omega_f} + (h, \varphi_{p,h})_{\Omega_p} + (s, z_h)_{\Omega_p}.$$  

(3.1)

for any $(\varphi_{f,h}, \psi_{f,h}, r_h, w_h, z_h, \varphi_{p,h}) \in V^f_h \times Q^p_h \times M^p_h \times Q^p_h \times X^p_h$, where $F(t; \varphi_{f,h}, \psi_{f,h}, \varphi_{p,h}, z_h) = (f, \varphi_{f,h})_{\Omega_f} + (-p_m(t) n_f, \varphi_{f,h})_{\Gamma_f^n} + (g, \psi_{f,h})_{\Omega_f} + (h, \varphi_{p,h})_{\Omega_p} + (s, z_h)_{\Omega_p}$.

Let $\Delta t$ denote the time step, $t_n = n\Delta t$, $0 \leq n \leq N$, and define the (backward) discrete time derivative by $d_t u^n := \frac{u^n - u^{n-1}}{\Delta t}$. Then, the fully multiphysics mixed finite element method for the problem \((2.3)-(2.8)\) is: find $(v^n_h, p^n_{f,h}, q^n_h, \xi^n_h, \eta^n_h, U^n_h) \in V_h^f \times Q_h^p \times V^p_h \times M^p_h \times Q^p_h \times X^p_h$ such that

$$(2\mu_f D(v^n_h), D(\varphi_{f,h}))_{\Omega_f} - (p^n_{f,h}, \nabla \cdot \varphi_{f,h})_{\Omega_f} + (\nabla \cdot v^n_h, \psi_{f,h})_{\Omega_f}$$

$$+ 2\mu_p (D(U^n_h), D(\varphi_{p,h}))_{\Omega_p} - (\xi^n_h, \nabla \cdot \varphi_{p,h})_{\Omega_p} + (\nabla \cdot U^n_h, w_h)_{\Omega_p}$$

$$+ k_3(\xi^n_h, w_h)_{\Omega_p} - k_1(\eta^n_h, w_h)_{\Omega_p} + k^{-1}(q^n_h, r_h)_{\Omega_p}$$

$$= F(t_n; \varphi_{f,h}, \psi_{f,h}, \varphi_{p,h}, z_h),$$

for any $(\varphi_{f,h}, \psi_{f,h}, r_h, w_h, z_h, \varphi_{p,h}) \in V^f_h \times Q^p_h \times V^p_h \times M^p_h \times Q^p_h \times X^p_h$. 

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We also recall the following inverse inequality
\[ h\|D(u_h)\|_L^2(\Gamma) \leq C_{TI} \|D(u_h)\|_L^2(\Omega), \quad (3.3) \]
where \( C_{TI} \) is a positive constant uniformly upper bounded with respect to the mesh size \( h \).

Define
\[ E^{n}_{p,h} := \frac{1}{2}(2\mu_p\|D(U^n_h)\|_L^2(\Omega_p) + k_3\|\xi^n_h\|_L^2(\Omega_p) + k_2\|\eta^n_h\|_L^2(\Omega_p)). \]

**Theorem 3.1.** For any \( \tau^1_f, \xi^1_f \) satisfying
\[ 1 - (\varsigma + 1)\tau^1_f C_{TI} - \frac{\xi^1_f}{4} > 0 \]
provided that \( \gamma_f > (\varsigma + 1)(\tau^1_f)^{-1} \), then there exist constants \( 0 < c < 1 \) and \( C > 1 \) (uniformly independent of the mesh characteristic size \( h \)) such that
\[
E^{N}_{p,h} + c\Delta t \sum_{n=1}^{N} \left[ 2\mu_f\|D(v^n_h)\|_L^2(\Gamma) + k^{-1}\|q^n_h\|_L^2(\Gamma) \right. \\
+ \frac{\Delta t}{2}(2\mu_p\|d_tD(U^n_h)\|_L^2(\Omega_p) + k_3\|d_t\xi^n_h\|_L^2(\Omega_p) + k_2\|d_t\eta^n_h\|_L^2(\Omega_p)) \\
+ \mu_f h^{-1}\|((v^n_h - q^n_h - d_tU^n_h) \cdot n)\|_L^2(\Gamma) + \|(v^n_h - d_tU^n_h) \cdot \tau\|_L^2(\Gamma)] \\
\left. \leq E^{0}_{p,h} + \Delta t \sum_{n=1}^{N} \frac{C}{\mu_f}\|F(t_n)\|^2 \right. \\
\]
with \( c < \min\{(1 - (\varsigma + 1)\tau^1_f C_{TI} - \frac{\xi^1_f}{4}), (\gamma_f - (\varsigma + 1)(\tau^1_f)^{-1})\} \) and \( C > (2\xi^1_f)^{-1} \).

**Proof.** Taking \( \varphi_{f,h} = v^n_h, r_h = q^n_h, \psi_{f,h} = p^n_{f,h}, \varphi_{p,h} = d_tU^n_h, w_h = d_t\xi^n_h, z_h = k_1\xi^n_h + k_2\eta^n_h \) in (3.2), we have
\[
(2\mu_f D(v^n_h), D(v^n_h))_{\Omega_f} - (p^n_{f,h}, \nabla \cdot v^n_h))_{\Omega_f} + (\nabla \cdot v^n_h, p^n_{f,h})_{\Omega_f} \\
+ k^{-1}(q^n_h, q^n_h)_{\Omega_p} - ((k_1\xi^n_h + k_2\eta^n_h), \nabla \cdot q^n_h)_{\Omega_p} + (\nabla \cdot q^n_h, (k_1\xi^n_h + k_2\eta^n_h))_{\Omega_p} \\
= 2\mu_f\|D(v^n_h)\|_L^2(\Omega_f) + k^{-1}\|q^n_h\|_L^2(\Omega_p). \\
\]
The interface terms for the coupling between the fluid and the structure can be bounded as follows
\[
- \int_{\Gamma} n \cdot \sigma_{f,h}(v^n_h, p^n_{f,h}) n(v^n_h - q^n_h - d_tU^n_h) \cdot n \, dx - \int_{\Gamma} n \cdot \sigma_{f,h}(\varsigma v^n_h, -p^n_{f,h}) n(v^n_h - d_tU^n_h - q^n_h) \cdot n \, dx \\
= -(1 + \varsigma) \int_{\Gamma} n \cdot (2\mu_f D(v^n_h)) n(v^n_h - d_tU^n_h - q^n_h) \cdot n \, dx \\
\leq 2\mu_f(1 + \varsigma)\|D(v^n_h)\|_L^2(\Gamma)\|D(v^n_h)\|_L^2(\Gamma) \\
\leq \mu_f(1 + \varsigma)(\tau^1_f)^{-1}\|D(v^n_h)\|_L^2(\Gamma) + \mu_f(1 + \varsigma)(\tau^1_f)^{-1}\|D(v^n_h)\|_L^2(\Gamma). \\
\]

(3.5)
Also, for the structure problem, we have
\begin{align*}
    2\mu_p (D(U_h^n), d_t D(U_h^n))_{\Omega_p} - (\xi_h^n, d_t \nabla \cdot U_h^n)_{\Omega_p} + (d_t \eta_h^n, k_1 \xi_h^n + k_2 \eta_h^n)_{\Omega_p} \\
    + (\nabla \cdot U_h^n, d_t \xi_h^n)_{\Omega_p} + k_3 (\xi_h^n, d_t \xi_h^n)_{\Omega_p} - (\eta_h^n, d_t \xi_h^n)_{\Omega_p} \leq 0 \quad \text{for } n = 1, \ldots , N
\end{align*}

which implies that (3.4) holds. The proof is complete.

Next, we design a loosely-coupled time-stepping method to decouple the problem (3.2) into three sub-problems at each time step. The loosely-coupled time-stepping algorithm is:

Step 1: Given \( v_h^{n-1}, p_f^{n-1}, \gamma_{h}^{n-1}, q_h^{n-1} \in V_h^f \times Q_h^f \times \Omega_h^p \times V_h^p \), solve \( U_h^n, \xi_h^n \in X_h^p \times M_h^p \) such that

\begin{align*}
2\mu_p (D(U_h^n), D(\varphi_{p,h}))_{\Omega_p} - (\xi_h^n, \nabla \cdot \varphi_{p,h})_{\Omega_p} + (\nabla \cdot U_h^n, w_h)_{\Omega_p} + k_3 (\xi_h^n, w_h)_{\Omega_p} \\
+ \int_{\Gamma} \gamma_f \mu_f h^{-1} d_t U_h^n \cdot \tau_p \varphi_{p,h} \cdot \tau_p \, dx + \int_{\Gamma} \gamma_f \mu_f h^{-1} d_t U_h^n \cdot n_p \varphi_{p,h} \cdot n_p \, dx
\end{align*}

such that

\begin{align*}
    0 &= - \int_{\Gamma} \sigma_{f,h}^{n-1} n_p (-\varphi_{p,h}) \cdot n_p \, dx - \int_{\Gamma} \tau_p \cdot \sigma_{f,h}^{n-1} n_p (-\varphi_{p,h}) \cdot \tau_p \, dx \\
    &+ \int_{\Gamma} \gamma_f \mu_f h^{-1} v_h^{n-1} \cdot \tau_p \varphi_{p,h} \cdot \tau_p \, dx + \int_{\Gamma} \gamma_f \mu_f h^{-1} (v_h^{n-1} - q_h^{n-1}) \cdot n_p \varphi_{p,h} \cdot n_p \, dx \\
    &+ (h, \varphi_{p,h})_{\Omega_p} + k_1 (\gamma_{h}^{n-1}, w_h)_{\Omega_p}.
\end{align*}

Step 2: Given \( v_h^{n-1}, p_f^{n-1}, s_h, U_h^n \in V_h^f \times Q_h^f \times M_h^p \times X_h^p \), solve \( q_h^n, \eta_h^n \in V_h^p \times Q_h^p \) such that

\begin{align*}
    - (k_2 \eta_h^n, \nabla \cdot r_h)_{\Omega_p} + k^{-1} (q_h^n, r_h)_{\Omega_p} + (d_t \eta_h^n, z_h)_{\Omega_p} + (\nabla \cdot q_h^n, z_h)_{\Omega_p} \\
    + \int_{\Gamma} \gamma_f \mu_f h^{-1} q_h^n \cdot n_p r_h \cdot n_p \, dx + s_f q (d_t q_h^n \cdot n_p, r_h \cdot n_p)
\end{align*}
\begin{align}
    & (k_1 \xi_h^n, \nabla \cdot r_h)_{\Omega_p} + \int_{\Gamma} \gamma_{f} \mu_f h^{-1} (v_h^{n-1} - d_t U_h^{n-1}) \cdot n_p r_h \cdot n_p \, dx \\
    & + \int_{\Gamma} n_p \cdot \sigma_{f} h^{-1} n_p r_h \cdot n_p \, dx + (s, z_h)_{\Omega_p}. \quad (3.8)
\end{align}

Step 3: Solve $v_h^n, p_{f,h}^n \in V_f \times Q_f^f$ such that

\begin{align}
    & (2 \mu_f D(v_h^n), D(\varphi_{f,h}))_{\Omega_f} - (p_{f,h}^n, \nabla \cdot \varphi_{f,h})_{\Omega_f} + (\nabla \cdot v_h^n, \psi_{f,h})_{\Omega_f} \\
    & + s_{f,p}(d_t p_{f,h}, \psi_{f,h}) + s_{f,v}(d_t v_h^n, \varphi_{f,h} \cdot n_f) \\
    & - \int_{\Gamma} \sigma_{f,h}(\varphi_{f,h}, -\psi_{f,h}) n_f \cdot v_h^n \, dx + \int_{\Gamma} \gamma_{f} \mu_f h^{-1} v_h^n \cdot \varphi_{f,h} \, dx \\
    & = \int_{\Gamma} \sigma_{f,h} h^{-1} n_f \cdot \varphi_{f,h} \, dx - \int_{\Gamma} \sigma_{f,h}(\varphi_{f,h}, -\psi_{p,h}) n_f d_t U_h^n \, dx \\
    & - \int_{\Gamma} n_f \cdot \sigma_{f,h}(\varphi_{f,h}, -\psi_{f,h}) n_f q_h^n \cdot n_f \, dx \\
    & + \int_{\Gamma} \gamma_{f} \mu_f h^{-1} d_t U_h^n \cdot \varphi_{f,h} \, dx + \int_{\Gamma} \gamma_{f} \mu_f h^{-1} q_h^n \cdot n_f \varphi_{f,h} \cdot n_f \, dx \\
    & + (f, \varphi_{f,h})_{\Omega_f} + \langle -p_{in}(t) n_f, \varphi_{f,h} \rangle_{\Gamma_f} + (g, \psi_{f,h})_{\Omega_f}. \quad (3.9)
\end{align}

**Remark 3.2.** The stabilization terms $s_{f,v}, s_{f,q}$ (cf. [25]) can be chosen by

\begin{align}
    s_{f,q}(d_t q_h \cdot n, r_h \cdot n) & := \gamma_{stab}' \gamma_f \mu_f \frac{\Delta t}{h} \int_{\Gamma} d_t q_h \cdot n r_h \cdot n \, dx, \\
    s_{f,v}(d_t v_h^n \cdot n, \varphi_{f,h} \cdot n) & := \gamma_{stab}' \gamma_f h \mu_f \frac{\Delta t}{h} \int_{\Gamma} d_t v_h^n \cdot n \varphi_{f,h} \cdot n \, dx,
\end{align}

and $s_{f,p}(d_t p_{f,h}, \psi_{f,h})$ is a stabilization term (cf [20]) with

\begin{align}
    s_{f,p}(d_t p_{f,h}, \psi_{f,h}) & := \gamma_{stab} \frac{h \Delta t}{\gamma_f h \mu_f} \int_{\Gamma} d_t p_{f,h}^n \psi_{f,h} \, dx.
\end{align}

3.2. Stability analysis

Following [20] and using Theorem 3.1, we can easily give the stability analysis of the explicit scheme.

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Lemma 3.1. For any \(\varepsilon_f^1, \tau_f^1 > 0\), we have

\[
E_{p,h}^N + \Delta t \frac{\gamma_{stab}}{2} \frac{h}{\gamma_{f^u_f}} \|p_{f,h}^N\|^2_{L^2(\Gamma)} + \Delta t \frac{\gamma_{stab}^f}{2} \frac{\mu_f f}{h} \left( \|v_h^N \cdot n_f\|^2_{L^2(\Gamma)} + \|q_h^N \cdot n_p\|^2_{L^2(\Gamma)} \right) \\
+ \Delta t \sum_{n=1}^N \left[ 2\mu_f (1 - (\varepsilon + 1)^3 C_{TT} - \frac{\varepsilon_f^1}{4}) \|D(v_h^n)\|^2_{L^2(\Omega_f)} + k^{-1} \|q_h^n\|^2_{L^2(\Omega_p)} \right] \\
+ \frac{\Delta t}{2} \left( 2\mu_p d_t d_t D(U_h^n) \right) \|L_2(\Omega_p) + k_3 \|d_t q_h^n\|^2_{L^2(\Omega_p)} + k_2 \|d_t \eta_h^n\|^2_{L^2(\Omega_p)} \right) \\
+ (\gamma_f - (\varepsilon + 1)^3 C_{TT} - (\varepsilon_f^1)^{-1}) \mu_f h^{-1} ((v_h^n - q_h^n - d_t U_h^n) \cdot n)_{L^2(\Gamma)} + \|v_h^n - d_t U_h^n\cdot \tau\|_{L^2(\Gamma)} \right) \\
+ \frac{\gamma_{stab}^f}{2} \frac{h}{\gamma_{f^u_f}} \|p_{f,h}^N - p_{f,h}^{N-1}\|^2_{L^2(\Gamma)} \\
+ \frac{\gamma_{stab}^f}{2} \frac{h}{\gamma_{f^u_f}} (\|v_h^n - v_h^{n-1}\|_{L^2(\Gamma)} + \|q_h^n - q_h^{n-1}\|_{L^2(\Gamma)}) \right]
\]

\[
\leq E_{p,h}^0 + \Delta t \sum_{n=1}^N \left[ T_{1,a} + T_{1,b} + T_{1,c} + T_{2} + T_{3} + T_{4} \right] (v_h^n, q_h^n, d_t U_h^n, \xi_h^n) \\
+ \Delta t \frac{\gamma_{stab}^f}{2} \frac{h}{\gamma_{f^u_f}} \left( \|v_h^0 \cdot n_f\|^2_{L^2(\Gamma)} + \|q_h^0 \cdot n_p\|^2_{L^2(\Gamma)} \right) \\
+ \Delta t \sum_{n=1}^N \left[ 2\varepsilon_f^1 \mu_f \right] \|F(t_n)\|^2.
\]

Proof. Denoting \(\tilde{A}_h(\cdot, \cdot)\) by the collection of terms on the left hand side of (3.2), and \(y_h = \{U_h, v_h, p_{f,h}, q_h, \xi_h, \eta_h\}\) by the vector of all the solution components and with \(z_h\).

Thus, the loosely-coupled time-stepping method is equivalent to the following formulation

\[
\tilde{A}_h(y_h^n, z_h) + s_{f,j}(d_t p_{f,h}, \psi_{f,h}) + s_{f,q}(d_t q_h \cdot n, r_{h} \cdot n) + s_{f,v}(d_t v_h^n \cdot n, \varphi_{f,h} \cdot n) \\
= \mathcal{F}_n(t_n, z_h) + \int_{\Gamma} \gamma_{f} \mu_f h^{-1} (v_h^n - v_h^{n-1}) \cdot \tau (-\varphi_{f,h}) \cdot \tau \, dx
\]

\[
+ \int_{\Gamma} \gamma_{f} \mu_f h^{-1} ((v_h^n - v_h^{n-1}) - d_t (U_h^n - U_h^{n-1})) \cdot n(-r_{h}) \cdot n \, dx
\]

\[
+ \int_{\Gamma} \gamma_{f} \mu_f h^{-1} ((v_h^n - v_h^{n-1}) - (q_h^n - q_h^{n-1})) \cdot n(-\varphi_{f,h}) \cdot n \, dx
\]

\[
- \int_{\Omega} 2\mu_f (D(v_h^n) - D(v_h^{n-1})) \cdot n \cdot (\varphi_{f,h} - r_{h} - \varphi_{p,h}) \cdot n + \tau (\varphi_{f,h} - \varphi_{p,h}) \cdot \tau \, dx
\]

\[
+ \int_{\Omega} n \cdot (p_{f,h}^{n-1} - p_{f,h}^{n-1}) \cdot n(\varphi_{f,h} - r_{h} - \varphi_{p,h}) \cdot n \, dx - \int_{\Omega} k_1 (\eta_{h} - \eta_{h}^{n-1}) w_1 \, dx.
\]
Using \((3.11)\), following the proof of Theorem 3.1, summing up with respect to the index \(n\), multiplying by \(\Delta t\), we see that \((3.10)\) holds. The proof is finished.

The stability of the loosely-coupled time-stepping method follows from \((3.10)\) provided that the terms \(T_{1,a}^{n}, T_{1,b}^{n}, T_{1,c}^{n}, T_{2}^{n}, T_{3}^{n}, T_{4}^{n}\) defined in \((3.11)\) can be bounded. To do that, we give the following lemma.

**Lemma 3.2.** For any \(\epsilon_{j}^{2}, \xi_{j}^{3}, \tau_{j}^{3} > 0\), there holds

\[
\Delta t \sum_{n=1}^{N}[T_{1,a}^{n} + T_{1,b}^{n} + T_{1,c}^{n} + T_{2}^{n} + T_{3}^{n} + T_{4}^{n}](v_{h}^{n}, q_{h}^{n}, d_{t}U_{h}^{n}, d_{t}\xi_{h}^{n})
\leq \frac{\gamma_{f}h}{2}\Delta t(2\|v_{h}^{0}\cdot n\|^{2}_{L^{2}(\Gamma)} + \|v_{h}^{0}\cdot \tau\|^{2}_{L^{2}(\Gamma)} + \|q_{h}^{0}\cdot n\|^{2}_{L^{2}(\Gamma)} + \|d_{t}U_{h}^{0}\cdot n\|^{2}_{L^{2}(\Gamma)})
\]  
\[+ \frac{c_{2}C_{TI}}{2}\mu_{f}\|D(v_{h}^{0})\|^{2}_{L^{2}(\Omega_{f})} + \Delta t \sum_{n=1}^{N}\frac{\mu_{f}}{2h}(\gamma_{f}(\xi_{j}^{3} + \tau_{j}^{3} + 1) + 2(\epsilon_{j}^{2})^{-1})
\times (\|v_{h}^{n} - q_{h}^{n} - d_{t}U_{h}^{n}\cdot n\|^{2}_{L^{2}(\Gamma)} + \|\mu_{f}(\gamma_{f}/2 + (\epsilon_{j}^{2})^{-1})\|v_{h}^{n} - d_{t}U_{h}^{n}\cdot \tau\|^{2}_{L^{2}(\Gamma)})
\]  
\[+ 2c_{2}C_{TI}\mu_{f}\|D(v_{h}^{n})\|^{2}_{L^{2}(\Omega_{f})} + k_{1}\Delta t h(\|d_{t}n_{h}^{n}\|^{2}_{L^{2}(\Omega_{h})} + \|d_{t}\xi_{h}^{n}\|^{2}_{L^{2}(\Omega_{h})})
\]  
\[+ (\bar{\tau}_{j}^{3})^{-1} \frac{h}{2}\|\gamma_{f}\mu_{f}\|^{2}_{L^{2}(\Gamma)} + \frac{(\xi_{j}^{3} - 1)}{2}\frac{\gamma_{f}\mu_{f}}{h}b
\times (\|v_{h}^{n} - v_{h}^{n-1}\|^{2}_{L^{2}(\Gamma)} + \|q_{h}^{n} - q_{h}^{n-1}\|^{2}_{L^{2}(\Gamma)})].
\]

**Proof.** As for \(T_{1,a}^{n}\), it is easy to check that

\[
T_{1,a}^{n}(v_{h}^{n}, q_{h}^{n}, d_{t}U_{h}^{n}, d_{t}\xi_{h}^{n})
= \int_{\Gamma} \gamma_{f}\mu_{f}h^{-1}(v_{h}^{n} - v_{h}^{n-1})\cdot \tau(v_{h}^{n} - d_{t}U_{h}^{n})\cdot \tau dx - \int_{\Gamma} \gamma_{f}\mu_{f}h^{-1}(v_{h}^{n} - v_{h}^{n-1})\cdot \tau v_{h}^{n}\cdot \tau dx
\]
\[= \frac{\gamma_{f}\mu_{f}}{2h}\|v_{h}^{n} - d_{t}U_{h}^{n}\cdot \tau\|^{2}_{L^{2}(\Gamma)} + \Delta t \frac{\gamma_{f}\mu_{f}}{2h}(\|d_{t}v_{h}^{n}\cdot \tau\|^{2}_{L^{2}(\Gamma)}
\]  
\[- d_{t}\|v_{h}^{n}\cdot \tau\|^{2}_{L^{2}(\Gamma)} - \Delta t\|d_{t}v_{h}^{n}\cdot \tau\|^{2}_{L^{2}(\Gamma)}).
\]

As for \(T_{1,b}^{n}\), we introduce \(u_{h}^{n} = v_{h}^{n} - d_{t}U_{h}^{n}\) and obtain

\[
T_{1,b}^{n}(v_{h}^{n}, q_{h}^{n}, d_{t}U_{h}^{n}, d_{t}\xi_{h}^{n})
= \int_{\Gamma} \gamma_{f}\mu_{f}h^{-1}(u_{h}^{n} - u_{h}^{n-1})\cdot n(u_{h}^{n} - q_{h}^{n})\cdot n dx - \int_{\Gamma} \gamma_{f}\mu_{f}h^{-1}(u_{h}^{n} - u_{h}^{n-1})\cdot n u_{h}^{n}\cdot n dx
\]
\[= \frac{\gamma_{f}\mu_{f}}{2h}\|u_{h}^{n} - q_{h}^{n}\|^{2}_{L^{2}(\Gamma)} + \Delta t \frac{\gamma_{f}\mu_{f}}{h}(\|d_{t}u_{h}^{n}\cdot n\|^{2}_{L^{2}(\Gamma)}
\]  
\[- d_{t}\|u_{h}^{n}\cdot n\|^{2}_{L^{2}(\Gamma)} - \Delta t\|d_{t}u_{h}^{n}\cdot n\|^{2}_{L^{2}(\Gamma)}).
\]

\[
= \frac{\gamma_{f}\mu_{f}}{2h}\|v_{h}^{n} - q_{h}^{n} - d_{t}U_{h}^{n}\)\cdot n\|^{2}_{L^{2}(\Gamma)} - \Delta t \frac{\gamma_{f}\mu_{f}}{h}d_{t}\|v_{h}^{n} - d_{t}U_{h}^{n}\)\cdot n\|^{2}_{L^{2}(\Gamma)}.
\]
To estimate $T_{1,c}^n$, we introduce the auxiliary variable $w_h = v_h - q_h$ at the interface $\Gamma$ and get

$$\begin{align*}
T_{1,c}^n(v_h^n, q_h^n, d_t U_h^n, d_t \xi_h^n) &= \int_\Gamma \gamma_f \mu h^{-1}(w_h^n - w_h^{n-1}) \cdot n(w_h^n - d_t U_h^n) \cdot n \, dx - \int_\Gamma \gamma_f \mu f h^{-1}(w_h^n - w_h^{n-1}) \cdot n w_h^n \cdot n \, dx \\
&\leq \epsilon_f h^{-1} \langle w_h^n - w_h^{n-1} \rangle \cdot n \langle w_h^n - d_t U_h^n \rangle \cdot n \, dx + \frac{\Delta t}{h} \gamma_f \mu f h^{-1} \|d_t w_h^n \cdot n\|_{L^2(\Gamma)} \\
&\quad - d_t \|w_h^n \cdot n\|_{L^2(\Gamma)} - d_t \|d_t w_h^n \cdot n\|_{L^2(\Gamma)} \\
&\leq \frac{\gamma_f \mu f}{h^2} \langle \epsilon_f^3 \rangle \|w_h^n - w_h^{n-1} \rangle \cdot n \langle w_h^n - d_t U_h^n \rangle \cdot n \|_{L^2(\Gamma)} + \frac{\Delta t}{h} \gamma_f \mu f h^{-1} \|d_t w_h^n \cdot n\|_{L^2(\Gamma)} \\
&\quad + \langle \epsilon_f^3 \rangle - 1 \|w_h^n - w_h^{n-1} \rangle \cdot n \|_{L^2(\Gamma)} - \frac{\Delta t}{h} \gamma_f \mu f \|d_t w_h^n \cdot n\|_{L^2(\Gamma)}.
\end{align*}$$

Using (3.13)-(3.15), we have

$$\Delta t \sum_{n=1}^N \left[ T_{1,a}^n + T_{1,b}^n + T_{1,c}^n \right](v_h^n, q_h^n, d_t U_h^n, d_t \xi_h^n) \leq \frac{\gamma_f \mu f}{h^2} \left[ \langle \epsilon_f^3 \rangle \|w_h^n - w_h^{n-1} \rangle \cdot n \langle w_h^n - d_t U_h^n \rangle \cdot n \|_{L^2(\Gamma)} + \frac{\Delta t}{h} \gamma_f \mu f \|d_t w_h^n \cdot n\|_{L^2(\Gamma)} \\
+ \langle \epsilon_f^3 \rangle - 1 \|w_h^n - w_h^{n-1} \rangle \cdot n \|_{L^2(\Gamma)} - \frac{\Delta t}{h} \gamma_f \mu f \|d_t w_h^n \cdot n\|_{L^2(\Gamma)} \right].$$

As for $T_2^n$ and $T_3^n$, we have

$$\begin{align*}
T_2^n(v_h^n, q_h^n, d_t U_h^n, d_t \xi_h^n) &\leq \epsilon_f^2 \gamma_f CT 1 h \mu \|D(v_h^n)\|_{L^2(\Omega_f)} + \|D(v_h^{n-1})\|_{L^2(\Omega_f)} \\
&\quad + \langle \epsilon_f^3 \rangle h^{-1} \mu f \langle w_h^n - d_t U_h^n \rangle \cdot n \|d_t w_h^n \cdot n\|_{L^2(\Gamma)}, \\
T_3^n(v_h^n, q_h^n, d_t U_h^n, d_t \xi_h^n) &\leq \epsilon_f^3 \frac{h}{2\gamma_f \mu f} \|p_h^n - p_h^{n-1}\|_{L^2(\Gamma)} + \epsilon_f^3 \frac{\gamma_f \mu f}{2h} \|w_h^n - q_h^n - d_t U_h^n \cdot n\|_{L^2(\Gamma)}.
\end{align*}$$

To determine an appropriate upper bound for $T_4^n$, we get

$$\begin{align*}
T_4^n(v_h^n, q_h^n, d_t U_h^n, d_t \xi_h^n) &= k_1 \Delta t \int_{\Omega_p} (d_t \eta_h^n)(d_t \xi_h^n) \, dx \\
&\leq \frac{k_1 \Delta t}{2} \|d_t \eta_h^n\|_{L^2(\Omega_p)}^2 + \frac{k_1 \Delta t}{2} \|d_t \xi_h^n\|_{L^2(\Omega_p)}^2.
\end{align*}$$

The proof is complete. □

Using Lemma 3.1 and Lemma 3.2 we can obtain the stability of the loosely-coupled time-stepping method.
Theorem 3.2. For any $\epsilon_1^3, \epsilon_1^3, \epsilon_3^3, \epsilon_2^2 > 0$ with $(2 - 2(\epsilon + 1)\epsilon_1^3)C_{TI} - \epsilon_1^3 - 2\epsilon_2^2C_{TI} > 0$ and $(2 - (\epsilon_3^3 + \epsilon_3^3 + 1)) = \delta > 0$ provided that the penalty and stabilization parameters are large enough, more precisely

$$\gamma_{stab} \geq (2\epsilon_1^3)^{-1}, \quad \gamma'_{stab} \geq (\epsilon_3^3)^{-1} - 1$$

$$\gamma_f \geq \frac{2(\epsilon + 1)(\epsilon_1^3)^{-1} + 2(\epsilon_2^2)^{-1}}{\delta},$$

and if $k_2 > k_1, k_3 > k_1$ and $\Delta t < Ch$, then there exist positive constants $c_f^1$ and $C_f^1$ such that

$$E_{p,h}^N + \Delta t\gamma_{stab} \frac{h}{\gamma_f \mu_f} \|p_f^N\|_{L^2(\Gamma)}^2 + \Delta t\gamma'_{stab} \frac{\gamma_f \mu_f}{h} (\|v_h^N \cdot n_f\|_{L^2(\Gamma)}^2 + \|q_h^N \cdot n_p\|_{L^2(\Gamma)})$$

$$+ \gamma_f \mu_f h^{-1}(\|v_h^N - q_h^n - d_t U_h^n\|_{L^2(\Omega_p)}) + \gamma_f h^{-1}(\|v_h^N - d_t U_h^n\|_{L^2(\Omega_p)}) + \psi_{stab} h\Delta t^2 C_f^1\sum_{n=1}^N \|F(t_n)\|^2$$

with

$$c_f^1 \leq \min\left\{ \frac{(2 - 2(\epsilon + 1)\epsilon_1^3)C_{TI} - \epsilon_1^3 - 2\epsilon_2^2C_{TI}}{2}, (k_3 - k_1)(k_2 - k_1), \right.$$

$$\left. (\gamma_f(2 - (\epsilon_3^3 + \epsilon_3^3 + 1)) - 2(\epsilon + 1)(\epsilon_1^3)^{-1} - 2(\epsilon_2^2)^{-1}), (\gamma_f - 2(\epsilon + 1)(\epsilon_1^3)^{-1} - 2(\epsilon_2^2)^{-1}) \right\},$$

$$C_f^1 \geq \max\left\{ \frac{(2\epsilon_2^2)C_{TI}}{2}, \frac{\gamma_{stab} h\Delta t}{\gamma_f}, \frac{\gamma'_{stab}}{2} \right\}.$$

Proof. Using (3.10) and (3.12), we obtain

$$E_{p,h}^N + \Delta t\gamma_{stab} \frac{h}{\gamma_f \mu_f} \|p_f^N\|_{L^2(\Gamma)}^2 + \Delta t\gamma'_{stab} \frac{\gamma_f \mu_f}{h} (\|v_h^N \cdot n_f\|_{L^2(\Gamma)}^2 + \|q_h^N \cdot n_p\|_{L^2(\Gamma)})$$

$$+ \gamma_f \mu_f h^{-1}(\|v_h^N - q_h^n - d_t U_h^n\|_{L^2(\Omega_p)}) + \gamma_f h^{-1}(\|v_h^N - d_t U_h^n\|_{L^2(\Omega_p)})$$

$$+ \Delta t(2\epsilon_2^2)\|d_t D(U_h^n)\|_{L^2(\Omega_p)}^2 + (k_3 - k_1)\|d_t \xi_h^n\|_{L^2(\Omega_p)}^2 + (k_2 - k_1)\|d_t \eta_h^n\|_{L^2(\Omega_p)}^2)$$

$$+ \frac{1}{2}(\gamma_f(2 - (\epsilon_3^3 + \epsilon_3^3 + 1)) - 2(\epsilon + 1)(\epsilon_1^3)^{-1} - 2(\epsilon_2^2)^{-1})\mu_f h^{-1}(\|v_h^n - q_h^n - d_t U_h^n\|_{L^2(\Gamma)}^2)$$

$$+ \frac{1}{2}(\gamma_f - 2(\epsilon + 1)(\epsilon_1^3)^{-1} - 2(\epsilon_2^2)^{-1})\mu_f h^{-1}(\|v_h^n - d_t U_h^n\|_{L^2(\Gamma)}^2) + \psi_{stab} h\Delta t^2 C_f^1\sum_{n=1}^N \|F(t_n)\|^2$$

(3.17)
where \( k \) for all \( s \) where \( (3.16) \) holds. The proof is complete.

Taking

\[
2 - 2(\xi + 1)C_{TI} - \frac{\epsilon_f}{2} - 2C_{TI} > 0, \quad \gamma_f - 2(\xi + 1)(\xi_f)^{-1} - 2(\xi_f)^{-1} > 0,
\]

\( k_3 - k_1 > 0, \quad k_2 - k_1 > 0, \quad \gamma_{stab} - (\xi_f)^{-1} > 0, \quad \gamma_{stab} - (\xi_f)^{-1} + 1 > 0,
\]

\[
\gamma_f(2 - (\xi_f^3 + \xi_f^3 + 1)) - 2(\xi + 1)(\xi_f^3)^{-1} - 2(\xi_f)^{-1} > 0,
\]

we imply that \( 3.16 \) holds. The proof is complete.

### 3.3. Error analysis

Let \( W_{f,h}, \ W_{p,h} \) be the \( L^2 \)-projection operators onto \( Q_h^f, Q_h^p \) satisfying

\[
(p_f - W_{f,h}p_f, \psi_{f,h})_{\Omega_f} = 0, \quad \forall \psi_{f,h} \in Q_h^f, \quad (3.18)
\]

\[
(p_p - W_{p,h}p_p, \pi_h)_{\Omega_p} = 0, \quad \forall \pi_h \in Q_p^h. \quad (3.19)
\]

It is well-known that the approximation properties holds (cf. \[26\]):

\[
\|p_f - W_{f,h}p_f\|_L^2(\Omega_f) \leq Ch^{r_1}\|p_f\|_{H^{r_1}(\Omega_f)}, \quad 0 \leq r_1 \leq s_f + 1, \quad (3.20)
\]

\[
\|p_p - W_{p,h}p_p\|_L^2(\Omega_p) \leq Ch^{r_2}\|p_p\|_{H^{r_2}(\Omega_p)}, \quad 0 \leq r_2 \leq s_p + 1, \quad (3.21)
\]

where \( s_f \) and \( s_p \) the degrees of piecewise polynomials of the spaces \( Q_h^f \) and \( Q_h^p \).

Next, we define a Stokes-like projection operator \( I_h : V_h^f \to V_h^f \times Q_h^f, \ X_h^p \to X_h^p \times M_h^p \) for all \( \varphi_f \in V_h^f, \ \varphi_p \in X_h^p \) by

\[
(\nabla \cdot \varphi_f, \psi_{f,h})_{\Omega_f} = (\nabla \cdot I_h \varphi_f, \psi_{f,h})_{\Omega_f}, \quad \forall \psi_{f,h} \in Q_h^f, \quad (3.22)
\]

\[
(\nabla \cdot \varphi_p, w_h)_{\Omega_p} = (\nabla \cdot I_h \varphi_p, w_h)_{\Omega_p}, \quad \forall w_h \in M_h^p. \quad (3.23)
\]

From \[27\], we know that there hold the following estimates

\[
\|\varphi_f - I_h \varphi_f\|_{H^{r_1}(\Omega_f)} \leq Ch^{r_3}\|\varphi_f\|_{H^{r_3+1}(\Omega_f)}, \quad 0 \leq r_3 \leq k_f, \quad (3.24)
\]

\[
\|\varphi_p - I_h \varphi_p\|_{H^{r_1}(\Omega_p)} \leq Ch^{r_4}\|\varphi_p\|_{H^{r_4+1}(\Omega_p)}, \quad 0 \leq r_4 \leq k_p, \quad (3.25)
\]

where \( k_f \) and \( k_p \) the degrees of polynomials in the spaces \( V_h^f \) and \( X_h^p \).
Let \( \Pi_h \) be an interpolation onto \( \mathbb{V}^p_h \) satisfying for any \( \theta > 0 \) and \( \mathbf{r} \in \mathbb{V}^p \cap H^\theta(\Omega_p) \)

\[
(\nabla \cdot \Pi_h \mathbf{r}, z_h)_{\Omega_p} = (\nabla \cdot \mathbf{r}, z_h)_{\Omega_p}, \quad \forall z_h \in Q^p_h, \quad (3.26)
\]

\[
\langle \Pi_h \mathbf{r} \cdot \mathbf{n}_p, \mathbf{r}_h \cdot \mathbf{n}_p \rangle_\Gamma = \langle \mathbf{r} \cdot \mathbf{n}_p, \mathbf{r}_h \cdot \mathbf{n}_p \rangle_\Gamma, \quad \forall \mathbf{r}_h \in \mathbb{V}_h^p, \quad (3.27)
\]

From [26], it is well-known that there hold

\[
\| \mathbf{r} - \Pi_h \mathbf{r} \|_{L^2(\Omega_p)} \leq C h^{r_s} \| \mathbf{r} \|_{H^{s+1}(\Omega_p)}, \quad 1 \leq r_s \leq k_s + 1, \quad (3.28)
\]

\[
\| \Pi_h \mathbf{r} \|_{H^1(\Omega_p)} \leq C (\| \mathbf{r} \|_{H^p(\Omega_p)} + \| \nabla \cdot \mathbf{r} \|_{L^2(\Omega_p)}), \quad (3.29)
\]

where \( k_s \) the degree of piecewise polynomial of the spaces \( \mathbb{V}_h^p \).

For any \( z \in H^1(\Omega_p) \), we define its elliptic projection \( S_h : H^1(\Omega_p) \rightarrow Q^p_h \) by

\[
(K \nabla S_h z, \nabla z_h)_{\Omega_p} = (K \nabla z, \nabla z_h)_{\Omega_p}, \quad \forall z \in Q^p_h, \quad (3.30)
\]

\[
(S_h z, 1)_{\Omega_p} = (z, 1)_{\Omega_p}, \quad (3.31)
\]

and the projection operator satisfies the approximation property(cf. [26]):

\[
\| S_h z - z \|_{L^2(\Omega_p)} \leq C h^{r_6} \| z \|_{H^{r_6+1}(\Omega_p)}, \quad 1 \leq r_6 \leq s_p + 1. \quad (3.32)
\]

Next, we introduce the following notations

\[
E^n_v := v(t_n) - v^n_h, \quad E^n_{p_f} := p_f(t_n) - p_{f,h}^n, \quad E^n_\mathbf{U} := U(t_n) - U^n_h,
\]

\[
E^n_\xi := \xi(t_n) - \xi^n_h, \quad E^n_\eta := \eta(t_n) - \eta^n_h, \quad E^n_{p_p} := p_p(t_n) - p_{p,h}^n, \quad E^n_\mathbf{q} := q(t_n) - q^n_h.
\]

It is easy to check that

\[
E^n_{p_p} = k_1 E^n_\xi + k_2 E^n_\eta, \quad \nabla \cdot E^n_\mathbf{U} = k_1 E^n_\eta - k_3 E^n_\xi,
\]

\[
E^n_\mathbf{q} = -k_\nabla E^n_{p_p}, \quad E^n_{\sigma_f} = 2\mu_f D(E^n_\mathbf{q}) - E^n_{p_p} I.
\]

Also, we split the errors into two parts and denote as follows:

\[
E^n_v := v(t_n) - \mathcal{I}_h(v(t_n)) + \mathcal{I}_h(v(t_n)) - v^n_h := \Lambda^n_v + \Theta^n_v,
\]

\[
E^n_{p_f} := p_f(t_n) - \mathcal{W}_{f,h}(p_f(t_n)) + \mathcal{W}_{f,h}(p_f(t_n)) - p_{f,h}^n := \Lambda^n_{p_f} + \Theta^n_{p_f},
\]

\[
E^n_\mathbf{U} := U(t_n) - \mathcal{I}_h(U(t_n)) + \mathcal{I}_h(U(t_n)) - U^n_h := \Lambda^n_\mathbf{U} + \Theta^n_\mathbf{U},
\]

\[
E^n_\xi := \xi(t_n) - \mathcal{S}_h(\xi(t_n)) + \mathcal{S}_h(\xi(t_n)) - \xi^n_h := \Lambda^n_\xi + \Theta^n_\xi,
\]

\[
E^n_\eta := \eta(t_n) - \mathcal{S}_h(\eta(t_n)) + \mathcal{S}_h(\eta(t_n)) - \eta^n_h := \Lambda^n_\eta + \Theta^n_\eta,
\]

\[
E^n_{p_p} := p_p(t_n) - \mathcal{W}_{p,h}(p_p(t_n)) + \mathcal{W}_{p,h}(p_p(t_n)) - p_{p,h}^n := \Lambda^n_{p_p} + \Theta^n_{p_p},
\]

\[
E^n_\mathbf{q} := q(t_n) - \Pi_h(q(t_n)) + \Pi_h(q(t_n)) - q^n_h := \Lambda^n_\mathbf{q} + \Theta^n_\mathbf{q}.
\]
Theorem 3.3. The solution of the problem [3.7]-(3.9) satisfies the following error estimates:

\[
\begin{align*}
\frac{\sqrt{2}}{2} & \left[ \sqrt{c} \left\| \Theta^n_{U} \right\|_{L^\infty(0,T;H^1(\Omega_p))} + \sqrt{k_3} \left\| \Phi^n_{\xi} \right\|_{L^\infty(0,T;L^2(\Omega_p))} + \sqrt{k_2} \left\| \Phi^n_{\eta} \right\|_{L^\infty(0,T;L^2(\Omega_p))} \right] \\
& + \sqrt{c}^p \left\| \Theta^n_{q} \right\|_{L^2(0,T;L^2(\Omega_p))} + \sqrt{c} \left\| \Theta^n_{p} \right\|_{L^2(0,T;H^1(\Omega_f))} \\
& + \left\| \Theta^n_{p_f} \right\|_{L^2(0,T;L^2(\Omega_f))} + \left\| \Theta^n_{p_p} \right\|_{L^2(0,T;L^2(\Omega_p))} + \left\| \Theta^n_{\varphi} \right\|_{L^2(0,T;L^2(\Omega_p))} \\
& + \sqrt{\gamma_{M_f} k_{f-1}} \left( \Theta^n_{\varphi} - d_i \Theta^n_{U} - \Theta^n_{q} \right) \cdot \mathbf{n} \left\|_{L^2(0,T;L^2(\Gamma))} \right) \\
& + \sqrt{\gamma_{M_f} k_{f-1}} \left( \Theta^n_{\varphi} - d_i \Theta^n_{U} \right) \cdot \mathbf{\tau} \left\|_{L^2(0,T;L^2(\Gamma))} \right) \\
\leq C \left( \exp(T) \left( h_T \| \mathbf{v} \|_{L^2(0,T;H^s+1(\Omega_f))} + h_T \| \mathbf{p} \|_{L^2(0,T;H^s(\Omega_f))} + h_T^2 \| \mathbf{p} \|_{L^2(0,T;H^s+2(\Omega_f))} \right) + h_T^3 \| \mathbf{q} \|_{L^2(0,T;H^5(\Omega_f))} + h_T^4 \| \mathbf{U} \|_{L^2(0,T;H^s+1(\Omega_f))} + h_T^4 \| \mathbf{d} \|_{L^2(0,T;H^s+1(\Omega_f))} \right) \\
& + \frac{1}{2} \left( 1 + \delta \right) \Delta t^2 \left( \left\| d_i \mathbf{v}_h \cdot \mathbf{\tau} \right\|_{L^2(0,T;L^2(\Gamma))} \right) + \sqrt{\frac{2 + \gamma_{stab}^{'}}{1 + \Delta t}} \left\| d_i \mathbf{v}_h \right\|_{L^2(0,T;L^2(\Gamma))} \\
& + \sqrt{\frac{1 + \gamma_{stab}^{'}}{1 + \Delta t}} \left\| d_i \mathbf{q}_h \cdot \mathbf{n} \right\|_{L^2(0,T;L^2(\Gamma))} + \left\| d_i \mathbf{U}_h \cdot \mathbf{n} \right\|_{L^2(0,T;L^2(\Gamma))} \\
& + \frac{\Delta t^2}{2 \epsilon_1} \left( \left\| \mu_f \nabla \cdot \mathbf{p}_f \right\|_{L^2(0,T;L^2(\Gamma))} \right) + \frac{\Delta t^2}{2 \epsilon_1} \left( \left\| \mu_f \nabla \cdot \mathbf{p}_h \right\|_{L^2(0,T;L^2(\Gamma))} \right) \\
& + \frac{\Delta t^2}{3 \epsilon_1} \left( \left\| d_i \mathbf{n}_h \right\|_{L^2(0,T;H^1(\Omega_p))} \right) + \frac{\Delta t^2 C}{\epsilon_1} \left( \left\| d_i \mathbf{v}_h \right\|_{L^2(0,T;H^1(\Omega_f))} \right),
\end{align*}
\]

where

\[
0 \leq r_1 \leq s_f + 1, \quad 0 \leq r_2 \leq s_p + 1, \quad 0 \leq r_3 \leq k_f, \\
0 \leq r_4 \leq k_p, \quad 1 \leq r_5 \leq k_s + 1, \quad 1 \leq r_6 \leq s_p + 1.
\]

Proof. Subtracting [3.7]-(3.9) from [2.14]-[2.19], summing the equations, using the definition of the projection operators \( \mathcal{W}_{f,h}, \mathcal{W}_{p,h}, \mathcal{T}_h, \Pi_h, \mathcal{S}_h, \mathcal{I}_h \) and setting \( \varphi_{f,h} = \Theta^n_{\varphi}, \varphi_{p,h} = d_i \Theta^n_{U}, \mathbf{w}_h = \Phi^n_{\xi}, \mathbf{r}_h = \Theta^n_{q}, \mathbf{z}_h = \Theta^n_{p_p} = \mathbf{k}_1 \Phi^n_{\xi} + k_2 \Phi^n_{\eta}, \) we have

\[
\begin{align*}
2 \mu_p & \mathbf{D}(\Lambda^n_{U}), \mathbf{D}(\varphi_{p,h}))_p + 2 \mu_p (\mathbf{D}(\Theta^n_{U}), \mathbf{D}(\varphi_{p,h}))_p - (\mathbf{\Psi}_\xi, \mathbf{\nabla} \cdot \varphi_{p,h})_p \\
& - (\mathbf{\Phi}_\xi, \mathbf{\nabla} \cdot \varphi_{p,h})_p + (\mathbf{\nabla} \cdot d_i \Theta^n_{U}, \mathbf{w}_h)_p + k_3 (d_i \Phi^n_{\xi}, \mathbf{w}_h)_p - k_1 (d_i \Phi^n_{\eta}, \mathbf{w}_h)_p \\
& - (\mathbf{\Theta}_p^\prime, \mathbf{\nabla} \cdot \mathbf{r}_h)_p + k_3 (\Lambda^n_{q}, \mathbf{r}_h)_p + k_3 (\Lambda^n_{q}, \mathbf{r}_h)_p \\
& +(d_i \Phi^n_{\xi}, \mathbf{w}_h)_p + (\mathbf{\nabla} \cdot \Theta^n_{q}, \mathbf{w}_h)_p + (2 \mu_f \mathbf{D}(\Lambda^n_{x}), \mathbf{D}(\varphi_{f,h}))_f + (2 \mu_f \mathbf{D}(\Theta^n_{\varphi}), \mathbf{D}(\varphi_{f,h}))_f \\
& - (\Lambda^n_{p_f}, \mathbf{\nabla} \cdot \varphi_{f,h})_f - (\Theta^n_{p_f}, \mathbf{\nabla} \cdot \varphi_{f,h})_f + (\mathbf{\nabla} \cdot \Phi^n_{\xi}, \varphi_{f,h})_f \\
& - s_{f,h} (d_i \Phi^n_{\xi}, \varphi_{f,h}) - s_{f,h} (d_i \Phi^n_{\eta}, \mathbf{r}_h \cdot \mathbf{n}) - s_{f,r} (d_i \Phi^n_{\xi}, \varphi_{f,h} \cdot \mathbf{n}) - s_{f,n} d_i \Phi^n_{\xi} \cdot \mathbf{n} dx
\end{align*}
\]
where $R \leq \int \tau \cdot \Theta^a_{\Omega_f} \cdot \tau dx - \int \tau \cdot \sigma_{f,h}(\psi_{f,h} + \psi_{p,h}) \cdot \tau dx$.

Integrating (3.34) over $[0, t]$ yields

$$\int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{U}) \cdot n dx + \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{q}) \cdot n dx \quad (3.34)$$

$$\left. \begin{array}{l}
\int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{U}) \cdot n dx + \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{q}) \cdot n dx \\
+ \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{U}) \cdot \tau dx + \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{q}) \cdot \tau dx \\
+ \int \tau \cdot \Theta^a_{\sigma_f} \cdot n dx + \int \tau \cdot \sigma_{f,h}(\psi_{f,h} + \psi_{p,h}) \cdot n dx \\
+ \int \tau \cdot \Theta^a_{\sigma_f} \cdot \tau dx + \int \tau \cdot \sigma_{f,h}(\psi_{f,h} + \psi_{p,h}) \cdot \tau dx
\end{array} \right\} \leq \frac{1}{4}(s_{f,p}(d_t p_{f,h}, \Theta^a_{\sigma_f} - \Theta^a_{p,h}) + s_{f,q}(d_t q_{f,h} \cdot n, \Theta^a_{\sigma_f} - \Theta^a_{p,h}) + s_{f,v}(d_t v_{f,h} \cdot n, \Theta^a_{\sigma_f} - \Theta^a_{p,h} - \Theta^a_{\psi} \cdot n))$$

Integrating (3.34) over $[0, t]$, we get

$$\begin{align*}
&\int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{U}) \cdot n dx + \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{q}) \cdot n dx \\
&+ \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{U}) \cdot \tau dx + \int \gamma_f \mu_f h^{-1}(\Theta^a_{\Omega_f} - d_t \Theta^a_{q}) \cdot \tau dx \\
&+ \int \tau \cdot \Theta^a_{\sigma_f} \cdot n dx + \int \tau \cdot \sigma_{f,h}(\psi_{f,h} + \psi_{p,h}) \cdot n dx \\
&+ \int \tau \cdot \Theta^a_{\sigma_f} \cdot \tau dx + \int \tau \cdot \sigma_{f,h}(\psi_{f,h} + \psi_{p,h}) \cdot \tau dx
\end{align*}$$

$$\begin{align*}
&\leq \frac{1}{4}(s_{f,p}(d_t p_{f,h}, \Theta^a_{\sigma_f} - \Theta^a_{p,h}) + s_{f,q}(d_t q_{f,h} \cdot n, \Theta^a_{\sigma_f} - \Theta^a_{p,h}) + s_{f,v}(d_t v_{f,h} \cdot n, \Theta^a_{\sigma_f} - \Theta^a_{p,h} - \Theta^a_{\psi} \cdot n)) \quad (3.35)
\end{align*}$$
Using integration by parts in time, we obtain

\[-\int_0^t \int_{\Gamma} \gamma_f \mu_f h^{-1}(n_h - \mathbf{v}^{n-1}_h) \cdot \mathbf{v}^{n-1}_h \cdot \tau dx dt\]
\[-\int_0^t \int_{\Gamma} \gamma_f \mu_f h^{-1}((\mathbf{v}^n_h - \mathbf{v}^{n-1}_h) - d_t(U^n_h - U^{n-1}_h)) \cdot \mathbf{n}(-\Theta^n_q) \cdot \mathbf{n} dx dt\]
\[-\int_0^t \int_{\Gamma} \gamma_f \mu_f h^{-1}((\mathbf{v}^n_h - \mathbf{v}^{n-1}_h)) \cdot (-d_t\Theta^n_q) \cdot \mathbf{n} dx dt\]
\[+ \int_0^t \int_{\Gamma} k_1(\eta^n_h - \eta^{n-1}_h)d_t\Theta^n_x \cdot \mathbf{n} dx dt + \int_0^t \int_{\Omega_p} R^n_h \cdot \Theta^n q \cdot \mathbf{n} dx dt\]
\[+ \int_0^t \int_{\Gamma} 2\mu_f(\mathbf{D}(\mathbf{v}^n_h) - \mathbf{D}(\mathbf{v}^{n-1}_h)) \cdot \mathbf{n} [\mathbf{n}(\Theta^n_q - d_t\Theta^n_U - \Theta^n_q) \cdot \mathbf{n} + \tau(\Theta^n_q - d_t\Theta^n_U) \cdot \tau] dx dt\]
\[-\int_0^t \int_{\Gamma} \mathbf{n} \cdot (p^n_h - p^{n-1}_h) \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U - \Theta^n_q) \cdot \mathbf{n} dx dt,\]

where \(\Theta^n_{p} := k_1\Phi^n_x + k_2\Phi^n_q\) and \(\varepsilon^n_T := \frac{1}{2}[e(t(\Theta^n_U)^2_H(\Omega_p) + k_3\|\Phi^n_q\|^2_{L^2(\Omega_p)} + k_2\|\Phi^n_q\|^2_{L^2(\Omega_p)}].\)

Using Cauchy-Schwarz inequality and Young inequality, we have

\[
(2\mu_f \mathbf{D}(\Lambda^n_q), \mathbf{D}(\Theta^n_q)_{|_{\Omega_f}} + (\Lambda^n_{p}, \nabla \cdot \Theta^n_q)_{|_{\Omega_f}} + k^{-1}(\Lambda^n_q, \Theta^n_q){}_{|_{\Omega_p}} \leq C\varepsilon^{-1}(2\mu_f \|\mathbf{D}(\Lambda^n_q)\|^2_{H^1(\Omega_f)} + \|\Lambda^n_{p}\|^2_{L^2(\Omega_f)} + \|\Lambda^n_q\|^2_{L^2(\Omega_p)}) + \varepsilon^{-1}(\|\nabla \cdot \Theta^n_q\|^2_{H^1(\Omega_f)} + \|\Theta^n_q\|^2_{L^2(\Omega_f)} + \|\Theta^n_q\|^2_{L^2(\Omega_p)})
\]

Using integration by parts in time, we obtain

\[
\int_0^t 2\mu_f(\mathbf{D}(\Lambda^n_q), di\mathbf{D}(\Theta^n_q))_{|_{\Omega_p}} dt + \int_0^t (\Lambda^n_q, di\nabla \cdot \Theta^n_q)_{|_{\Omega_p}} dt \leq C(\varepsilon^{-1}\|\Lambda^n_q\|^2_{H^1(\Omega_p)} + \|di\Lambda^n_q\|^2_{L^2(0,t;H^1(\Omega_p))} + \varepsilon^{-1}\|\Lambda^n_q\|^2_{L^2(\Omega_p)}) + \|di\Lambda^n_q\|^2_{L^2(0,t;L^2(\Omega_p))})(\varepsilon^{-1}\|\Theta^n_q\|^2_{H^1(\Omega_p)} + \|\Theta^n_q\|^2_{L^2(0,t;H^1(\Omega_p))}).
\]

the interface terms can be bounded as follows,

\[
\int_{\Gamma} \mathbf{n} \cdot \Theta^n_{\sigma,f} \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U - \Theta^n_q) \cdot \mathbf{n} dx
\]
\[+ \int_{\Gamma} \mathbf{n} \cdot \sigma_{f,h}(\Theta^n_q - \Theta^n_q) \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U - \Theta^n_q) \cdot \mathbf{n} dx
\]
\[= (1 + \zeta)\int_{\Gamma} \mathbf{n} \cdot (2\mu_f \mathbf{D}(\Theta^n_q)) \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U - \Theta^n_q) \cdot \mathbf{n} dx
\]
\[\leq 2\mu_f(1 + \zeta)\|\mathbf{D}(\Theta^n_q)\|_{L^2(\Gamma)}\|\Theta^n_q - d_t\Theta^n_U - \Theta^n_q\|_{L^2(\Gamma)}\]
\[\leq \mu_f(1 + \zeta)\|\mathbf{D}(\Theta^n_q)\|^2_{H^1(\Omega_f)} + \mu_f(1 + \zeta)(\varepsilon^{-1})^{-1}\|\Theta^n_q - d_t\Theta^n_U - \Theta^n_q\|^2_{L^2(\Gamma)}.
\]

As for the interface terms, we have

\[
\int_{\Gamma} \tau \cdot \Theta^n_{\tau,f} \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U) \cdot \tau dx + \int_{\Gamma} \tau \cdot \sigma_{f,h}(\Theta^n_q - \Theta^n_q) \cdot \mathbf{n}(\Theta^n_q - d_t\Theta^n_U) \cdot \tau dx
\]
\[\leq \mu_f(1 + \zeta)\|\mathbf{D}(\Theta^n_q)\|^2_{H^1(\Omega_f)} + \mu_f(1 + \zeta)(\varepsilon^{-1})^{-1}\|\Theta^n_q - d_t\Theta^n_U\|_{L^2(\Gamma)}.
\]
\[
\int_{\Gamma} \mathbf{n} \cdot \Lambda^n_{\sigma_f} \mathbf{n}(\Theta^a_{\nu} - d_t \Theta^a_{U} - \Theta^a_{q}) \cdot \mathbf{n} \, dx
\]

\[
\leq \mu_f \epsilon_1 \epsilon_l C \| \Lambda^a_{\nu} \|_{H^1(\Omega_f)}^2 + \mu_f (\epsilon_1 h)^{-1} \| (\Theta^a_{\nu} - d_t \Theta^a_{U} - \Theta^a_{q}) \|_{L^2(\Gamma)}^2
\]

\[+ C \epsilon_1 \| \Lambda^a_{\nu} \|_{L^2(\Omega_f)}^2 + \epsilon_l^{-1} \| (\Theta^a_{\nu} - d_t \Theta^a_{U} - \Theta^a_{q}) \|_{L^2(\Gamma)}^2, \tag{3.39}\]

\[
\int_{\Gamma} \tau \cdot \Lambda^a_{\sigma_f} \mathbf{n}(\Theta^a_{\nu} - d_t \Theta^a_{U}) \cdot \tau \, dx
\]

\[
\leq \mu_f \epsilon_1 \epsilon_l C \| \Lambda^a_{\nu} \|_{H^1(\Omega_f)}^2 + \mu_f (\epsilon_1 h)^{-1} \| (\Theta^a_{\nu} - d_t \Theta^a_{U}) \|_{L^2(\Gamma)}^2 \tag{3.40}\]

\[
+ C \epsilon_1 \| \Lambda^a_{\nu} \|_{L^2(\Omega_f)} \epsilon_l^{-1} \| (\Theta^a_{\nu} - d_t \Theta^a_{U}) \|_{L^2(\Gamma)}^2, \]

\[
\int_{0}^{t} \int_{\Gamma} \gamma_f \mu_f h^{-1}(v^h - v^{h-1}) \cdot \tau(-d_t \Theta^a_{U}) \cdot \tau \, dx \, dt
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_l^{-1} (1 + \Delta t^{-1}) \| (v^h - v^{h-1}) \|_{L^2(0,t;L^2(\Gamma))}^2 \tag{3.41}\]

\[
+ \epsilon_1 \| (\Theta^a_{\nu} - d_t \Theta^a_{U}) \|_{L^2(0,t;L^2(\Gamma))}^2 + \Delta t \| \Theta^a_{\nu} \|_{L^2(0,t;L^2(\Gamma))}^2
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_l^{-1} (1 + \Delta t^{-1}) \Delta t^2 \| d_t v^h \|_{L^2(0,t;L^2(\Gamma))}^2 \tag{3.42}\]

\[
+ \epsilon_1 \| (\Theta^a_{\nu} - d_t \Theta^a_{U}) \|_{L^2(0,t;L^2(\Gamma))}^2 + \Delta t C \| \Theta^a_{\nu} \|_{L^2(0,t;H^1(\Omega_f))}^2
\]

\[
\int_{0}^{t} \int_{\Gamma} \gamma_f \mu_f h^{-1}((v^h - v^{h-1}) - d_t (U^h - U^{h-1})) \cdot \mathbf{n}(-\Theta^a_{q}) \cdot \mathbf{n} \, dx \, dt
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_l^{-1} \epsilon_1 (1 + \Delta t^{-1}) \int_{0}^{t} \| (v^h - v^{h-1}) \|_{L^2(\Gamma)}^2 \tag{3.43}\]

\[
+ \| d_t (U^h - U^{h-1}) \|_{L^2(\Gamma)}^2 + \epsilon_1 \int_{0}^{t} \| d_t \Theta^a_{U} \|_{L^2(\Gamma)}^2 \, dt
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_1 (1 + \Delta t^{-1}) \Delta t^2 \| d_t v^h \|_{L^2(0,t;L^2(\Gamma))}^2 + \| d_t U^h \|_{L^2(0,t;L^2(\Gamma))}^2 \]

\[
+ \frac{1}{2} \gamma_f \mu_f \epsilon_1 \Delta t \| d_t \Theta^a_{U} \|_{L^2(0,t;H^1(\Omega_f))}^2
\]

\[
\int_{0}^{t} \int_{\Gamma} \gamma_f \mu_f h^{-1}((v^h - v^{h-1}) - (q^h - q^{h-1})) \cdot \mathbf{n}(-d_t \Theta^a_{q}) \cdot \mathbf{n} \, dx \, dt
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_l^{-1} (1 + \Delta t^{-1}) \int_{0}^{t} \| (v^h - v^{h-1}) \|_{L^2(\Gamma)}^2 + \| (q^h - q^{h-1}) \|_{L^2(\Gamma)}^2 \, dt \tag{3.44}\]

\[
+ \epsilon_1 \int_{0}^{t} \| (\Theta^a_{\nu} - \Theta^a_{q} - d_t \Theta^a_{U}) \cdot \mathbf{n} \|_{L^2(\Gamma)}^2 + \Delta t \| \Theta^a_{\nu} \cdot \mathbf{n} \|_{L^2(\Gamma)}^2 + \Delta t \| \Theta^a_{q} \cdot \mathbf{n} \|_{L^2(\Gamma)}^2 \, dt
\]

\[
\leq \frac{1}{2} \gamma_f \mu_f \epsilon_l^{-1} (1 + \Delta t^{-1}) \Delta t^2 \| d_t v^h \|_{L^2(0,t;L^2(\Gamma))}^2 + \| d_t q^h \|_{L^2(0,t;L^2(\Gamma))}^2 \]

\[
+ C \Delta t \| \Theta^a_{\nu} \|_{L^2(0,t;H^1(\Omega_f))}^2 + \epsilon_1 (\| (\Theta^a_{\nu} - \Theta^a_{q} - d_t \Theta^a_{U}) \cdot \mathbf{n} \|_{L^2(0,t;L^2(\Gamma))}^2
\]

\[
+ \frac{C_{TI} \Delta t}{h} \| \Theta^a_{q} \|_{L^2(0,t;L^2(\Omega_f))}^2),\]

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\[
\int_0^t \int_\Gamma 2\mu_f (D(v_h^n) - D(v_h^{n-1})) \cdot n \left[ n(\Theta_v^n - d_t \Theta_U^n - \Theta_q^n) \cdot n + \tau(\Theta_v^n - d_t \Theta_U^n) \cdot \tau \right] \, dx \, dt \\
\leq \frac{\mu_f}{\epsilon_1} \int_0^t \| D(v_h^n - v_h^{n-1}) \|^2_{L^2(\Gamma)} \, dt \\
+ \epsilon_1 \mu_f \left( \| (\Theta_v^n - \Theta_q^n - d_t \Theta_U^n) \cdot n \|^2_{L^2((0,t;L^2(\Gamma)))} + \| (\Theta_v^n - d_t \Theta_U^n) \cdot \tau \|^2_{L^2((0,t;L^2(\Gamma)))} \right) \\
\leq \frac{\mu_f}{\epsilon_1} t \int_0^t \| d_t v_h^n \|^2_{L^2(0,t;H^1(\Omega_t))} \\
+ \epsilon_1 \mu_f \left( \| (\Theta_v^n - \Theta_q^n - d_t \Theta_U^n) \cdot n \|^2_{L^2((0,t;L^2(\Gamma)))} + \| (\Theta_v^n - d_t \Theta_U^n) \cdot \tau \|^2_{L^2((0,t;L^2(\Gamma)))} \right) \\
\leq \frac{\Delta t^2}{2\epsilon_1} \| d_t p_h^n \|^2_{L^2(0,t;L^2(\Gamma))} + \frac{\epsilon_1}{2} \left( \| (\Theta_v^n - \Theta_q^n - d_t \Theta_U^n) \cdot n \|^2_{L^2(0,t;L^2(\Gamma))} \right) \\
\leq \frac{\Delta t^2}{2\epsilon_1} \| d_t \eta_h^n \|^2_{L^2(0,t;L^2(\Omega_p))} + \frac{\epsilon_1}{2} \left( \| d_t \Theta_U^n \|^2_{L^2(0,t;L^2(\Omega_p))} \right) \\
\leq \frac{\Delta t^2}{3\epsilon_1} \| d_t \eta_h^n \|^2_{L^2(0,t;H^1(\Omega_p))} + \frac{\epsilon_1}{4} \left( \| \Theta_q^n \|^2_{L^2(0,t;L^2(\Omega_p))} \right) \]

For any \( \epsilon_1 > 0 \), the stabilization terms \( s_{f,p}(\cdot,\cdot) \), \( s_{f,v}(\cdot,\cdot) \), \( s_{f,q}(\cdot,\cdot) \) satisfy the following upper bounds, respectively,

\[
\int_0^t s_{f,p}(d_t p_h^n, \Theta_U^n) \, dt \\
\leq \frac{\gamma_{stab}}{\epsilon_1^2} \int_0^t \left( \frac{C_{TI} \Delta t}{h} \| \Theta_U^n \|^2_{L^2(0,t;L^2(\Gamma))} \right) \]

\[
\int_0^t s_{f,v}(d_t v_h^n \cdot n, \Theta_v^n \cdot n) \, dt \\
\leq \frac{\gamma_{stab}}{\epsilon_1 \mu_f} \left( \frac{\Delta t}{h} \| d_t v_h^n \|^2_{L^2(0,t;L^2(\Gamma))} + \epsilon_1 C \| \Theta_v^n \|^2_{L^2(0,t;H^1(\Omega_t))} \right) \\
\int_0^t s_{f,q}(d_t q_h^n \cdot n, \Theta_q^n \cdot n) \, dt \\
\leq \frac{\gamma_{stab}}{\epsilon_1 \mu_f} \left( \frac{\Delta t}{h} \| d_t q_h^n \|^2_{L^2(0,t;L^2(\Gamma))} + \epsilon_1 C \| \Theta_q^n \|^2_{L^2(0,t;L^2(\Omega_p))} \right)
\]
It is easy to check that

\[
\begin{align*}
\| (\Theta_{p_f}^n, \Theta_{p_h}^n, \Theta_{p}^n) \|_{Q \times M} & \leq C \sup_{\left( (\varphi_f, r_h), (\varphi_p, h) \right) \in V_h \times X_h} \left( \frac{1}{2} \left( \Theta_{p_f}^n \cdot \nabla \varphi_{f,h} \right)_{\Omega_f} + (\Theta_{p_h}^n \cdot \nabla r_h)_{\Omega_p} + (\Theta_{p}^n \cdot \nabla \varphi_{p,h})_{\Omega_p} \right) \\
& \quad + \frac{\left( n \Theta_{p_f}^n \cdot n \left( \varphi_{f,h} - \varphi_{p,h} - r_h \right) n \right)_{\Gamma}}{\| \left( (\varphi_f, r_h), (\varphi_p, h) \right) \| \times \times p} \\
& \leq C \sup_{\left( (\varphi_f, r_h), (\varphi_p, h) \right) \in V_h \times X_h} \left( \frac{-2\mu_f D(E_0^{\sigma}), D(\varphi_{f,h})_{\Omega_f} - 2\mu_p D(E_0^\nu), D(\varphi_{p,h})_{\Omega_p}}{\| \left( (\varphi_f, r_h), (\varphi_p, h) \right) \| \times \times p} \\
& \quad + \frac{\left( \Lambda_{p_f}^n, \nabla \varphi_{f,h} \right)_{\Omega_f} - \left( \Lambda_{p_h}^n, \nabla r_h \right)_{\Omega_p} - \left( \Lambda_{p}^n, \nabla \varphi_{p,h} \right)_{\Omega_p}}{\| \left( (\varphi_f, r_h), (\varphi_p, h) \right) \| \times \times p} \right) \\
& \quad + \frac{\left( \Lambda_{p_f}^n \cdot \nabla \varphi_{f,h} \right)_{\Omega_f} - \left( \Lambda_{p_h}^n \cdot \nabla r_h \right)_{\Omega_p} - \left( \Lambda_{p}^n, \nabla \varphi_{p,h} \right)_{\Omega_p}}{\| \left( (\varphi_f, r_h), (\varphi_p, h) \right) \| \times \times p}
\end{align*}
\]

and

\[
\epsilon_2 \left( \| \Theta_{p_f}^n \|_{L^2(0,t;L^2(\Omega_f))}^2 + \| \Theta_{p_h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + \| \Theta_{p}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \right)
\leq C \epsilon_2 \left( \| \Theta_{\xi}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Theta_{\nu}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Theta_{h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \right)
\]

\[
+ \| \Lambda_{p_f}^n \|_{L^2(0,t;L^1(\Omega_f))}^2 + \| d_t \Lambda_{\nu}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Lambda_{\xi}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + \| \Lambda_{p}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + \| \Lambda_{p}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Lambda_{h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \]
\]

Substituting (3.30)–(3.51) into (3.35) and taking \( \epsilon_1, \epsilon_2 \) small enough, we get

\[
\frac{1}{2} \left[ c \| \Theta_{\xi}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + k_3 \| \Phi_{\xi}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + k_2 \| \Phi_{\xi}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + c^2 \| \Theta_{h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \right]
\]

\[
+ \left[ c \| \Theta_{p_f}^n \|_{L^2(0,t;H^1(\Omega_f))}^2 + \| \Theta_{p_f}^n \|_{L^2(0,t;H^1(\Omega_f))}^2 + \| \Theta_{p_h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 + \| \Theta_{p}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \right)
\]

\[
+ \left[ c \| \Theta_{\nu}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Theta_{\nu}^n \|_{L^2(0,t;H^1(\Omega_p))}^2 + \| \Theta_{h}^n \|_{L^2(0,t;L^2(\Omega_p))}^2 \right]
\]

\[
\leq C \left( \left\| \Lambda_{p_f}^n \right\|_{L^2(0,t;L^2(\Omega_f))}^2 + \left\| \Lambda_{p_h}^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + \left\| \Lambda_{p}^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 \right)
\]

\[
+ \left\| \Lambda_{\nu}^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + \left\| \Lambda_{\xi}^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + \left\| \Lambda_{h}^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + \left\| \left( \Theta_{\xi}^n - d_t \Theta_{\xi}^n - \Theta_{\nu}^n \right) \cdot \tau \right\|_{L^2(0,t;L^2(\Gamma))}^2
\]

\[
+ \frac{(1 + \Delta t) \Delta t \gamma_{f,m}^2}{2 \epsilon_1} \left( \left\| d_t v_h^n \cdot \tau \right\|_{L^2(0,t;L^2(\Gamma))}^2 + (2 + \frac{\gamma_{stab}^2}{1 + \Delta t}) \left\| d_t q_h^n \cdot n \right\|_{L^2(0,t;L^2(\Gamma))}^2 \right)
\]

\[
+ \frac{\Delta t \gamma_{stab}^2}{2 \epsilon_1} \left( \left\| d_t \eta_h^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + \left\| d_t \eta_h^n \right\|_{L^2(0,t;H^1(\Omega_p))}^2 \right)
\]

\[
+ \frac{\mu_f}{\epsilon_1} \Delta t^2 \left( \left\| d_t \eta_h^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 \right) + \left\| \Delta t^2 \left( \left\| d_t \eta_h^n \right\|_{L^2(0,t;L^2(\Omega_p))}^2 + (\Delta t^2 \gamma_{stab}^2 + \frac{\gamma_{stab} h \Delta t}{2 \epsilon_1}) \left\| d_t \eta_h^n \right\|_{L^2(0,t;L^2(\Gamma))}^2 + \frac{\Delta t \gamma_{stab}^2}{3 \epsilon_1} \left\| d_t \eta_h^n \right\|_{L^2(0,t;H^1(\Omega_p))}^2 \right) \right)\]
Proof. The above estimates following immediately from an application of the triangle inequality on

\[ \mathbf{v}(t_n) - \mathbf{v}_h^n = \Lambda^n_{\mathbf{v}} + \Theta^n_{\mathbf{v}}, \quad p_f(t_n) - p_{f,h}^n = \Lambda^n_{p_f} + \Theta^n_{p_f}, \]
\[ \mathbf{U}(t_n) - \mathbf{U}_h^n = \Lambda^n_{\mathbf{U}} + \Theta^n_{\mathbf{U}}, \quad \xi(t_n) - \xi_h^n = \Lambda^n_{\xi} + \Theta^n_{\xi}, \]
\[ \eta(t_n) - \eta_h^n = \Lambda^n_{\eta} + \Theta^n_{\eta}, \quad p_p(t_n) - p_{p,h}^n = \Lambda^n_{p_p} + \Theta^n_{p_p}, \]

\[ \begin{align*}
\sqrt{2} & \left[ \sqrt{c} \left\| \mathbf{U}(t_n) - \mathbf{U}_h^n \right\|_{L^\infty(0,T;H^1(\Omega_p))} + \sqrt{h} \left\| \xi(t_n) - \xi_h^n \right\|_{L^\infty(0,T;L^2(\Omega_p))} \\
& + \sqrt{h} \left\| \eta(t_n) - \eta_h^n \right\|_{L^\infty(0,T;L^2(\Omega_p))} \right] \\
& + \sqrt{c} \left\| \mathbf{q}(t_n) - \mathbf{q}_h^n \right\|_{L^2(0,T;L^2(\Omega_f))} + \sqrt{c} \left\| \mathbf{v}(t_n) - \mathbf{v}_h^n \right\|_{L^2(0,T;H^1(\Omega_f))} \\
& + \left\| p_f(t_n) - p_{f,h}^n \right\|_{L^2(0,T;L^2(\Omega_f))} + \left\| \mathbf{p}(t_n) - \mathbf{p}_{p,h} \right\|_{L^2(0,T;L^2(\Omega_p))} + \left\| e(t_n) - e_h^n \right\|_{L^2(0,T;L^2(\Omega_p))} \\
& + \sqrt{\gamma_f \mu_f h^{-1}} \left\| \mathbf{v}(t_n) - \mathbf{d}_t \mathbf{U}(t_n) - \mathbf{q}(t_n) - \left( \mathbf{v}_h^n - \mathbf{d}_t \mathbf{U}_h^n - \mathbf{q}_h^n \right) \cdot \mathbf{n} \right\|_{L^2(0,T;L^2(\Gamma))} \\
& + \sqrt{\gamma_f \mu_f h^{-1}} \left\| \mathbf{V}(t_n) - \mathbf{d}_t \mathbf{U}(t_n) - \mathbf{v}_h^n - \mathbf{d}_t \mathbf{U}_h^n \right\| \cdot \mathbf{T} \left\| L^2(0,T;L^2(\Gamma)) \right\|
\end{align*} \]

\[ \leq C \sqrt{\exp(T)} (h^n \left\| \mathbf{v} \right\|_{L^2(0,T;H^{s+1}(\Omega_f)))} + h^n \left\| \mathbf{p} \right\|_{L^2(0,T;H^1(\Omega_f)))} + h^n \left\| \mathbf{q} \right\|_{L^2(0,T;H^s(\Omega_p))} + h^n \left\| \mathbf{U} \right\|_{L^\infty(0,T;H^{s+1}(\Omega_p))} + h^n \left\| \mathbf{d}_t \mathbf{U} \right\|_{L^2(0,T;H^{s+1}(\Omega_p))} + h^n \left\| \mathbf{d}_t \mathbf{v} \right\|_{L^2(0,T;L^2(\Gamma))} + h^n \left\| \mathbf{d}_t \mathbf{v} \right\|_{L^2(0,T;L^2(\Gamma))} + h^n \left\| \mathbf{d}_t \mathbf{v} \right\|_{L^2(0,T;L^2(\Gamma))} + h^n \left\| \mathbf{d}_t \mathbf{v} \right\|_{L^2(0,T;L^2(\Gamma))}, \]
\[ q(t_n) - q_h^n = A_q^n + \Theta_q^n. \]

Using (3.20), (3.24), (3.25), (3.28), (3.32) and Theorem 3.3 we see that (3.53) holds. The proof is complete. \qed

4. Numerical tests

**Test 1.** Taking the domain by \( \Omega = [0, 1] \times [-1, 1] \), and we associate the upper half with the Stokes flow, while the lower half represents the poroelastic structure. The appropriate interface conditions are enforced along the interface \( y = 0 \).

The source functions \( f, g, h \) and \( s \) are given by

\[
\begin{align*}
f &= \left( \begin{array}{c}
\pi t \cos(\frac{\pi y}{2}) \cos(\pi x) + \pi \mu f \cos(y) \cos(\pi t) \\
-\frac{\pi}{2} e^t \sin(\pi x) \sin(\frac{\pi y}{2})
\end{array} \right) , \\
h &= \left( \begin{array}{c}
\alpha \pi t \cos(\frac{\pi y}{2}) \cos(\pi x) + \mu_p \cos(y) \sin(\pi t) \\
-\frac{\pi}{2} \alpha e^t \sin(\pi x) \sin(\frac{\pi y}{2})
\end{array} \right) ,
\end{align*}
\]

\[
g = -2\pi \cos(\pi t) , \\
s = (s_0 - \frac{3}{4} \pi^2) e^t \sin(\pi x) \cos(\frac{\pi y}{2}) - 2\alpha \pi \cos(\pi t).
\]

The boundary conditions are

\[
\begin{align*}
U &= 0, \quad q = 0, \quad \xi = 0, \quad \eta = 0, \quad (x, t) \in \Gamma_p^\text{in} \cup \Gamma_p^\text{out} \cup \Gamma_p^\text{ext} \times (0, T] , \\
v &= 0, \quad (x, t) \in \Gamma_f^\text{in} \cup \Gamma_f^\text{out} \times (0, T] , \\
p_f &= 0, \quad (x, t) \in \Gamma_f^\text{out} \times (0, T] . 
\end{align*}
\]

The initial values are

\[
\begin{align*}
U^0 &= 0, \quad v^0 = 0, \quad \xi^0 = 0, \quad \eta^0 = 0.
\end{align*}
\]

And we take the total simulation time for this test case is \( T = 10^{-3} \) and the time step is \( \Delta t = 10^{-4} \), the mesh space discretization step with \( h = 0.014 \) as a exact solution. We use \( \mathbb{P}_2 - \mathbb{P}_1 \) element approximations for velocity and pressure in the fluid, combined with \( \mathbb{P}_2 - \mathbb{P}_1 \) element approximation of the relative velocity and pressure of the fluid within the porous structure and \( \mathbb{P}_2 \) element approximation of the structure displacement.
Table 1: Values of parameters

| Parameters                  | Notation | Values         |
|-----------------------------|----------|----------------|
| Viscosity                   | $\mu_f$  | 0.01           |
| Lemé constant               | $\mu_p$  | $1 \times 10^8$|
| Lemé constant               | $\lambda_p$ | $4.28 \times 10^6$|
| Mass storage coefficient    | $s_0$    | $5 \times 10^{-6}$|
| Biot-Willis constant        | $\alpha$ | 1              |
| Hydraulic conductivity      | $k$      | 1              |

Table 2: Error indicators and convergence order

| $h$   | $\varepsilon_f$ | Rate | $\varepsilon_p$ | Rate | $\varepsilon_{fp}$ | Rate | $\varepsilon_{pp}$ | Rate |
|-------|-----------------|------|-----------------|------|--------------------|------|--------------------|------|
| $h = 0.27$ | 1.8E-02 |      | 4.6E-07 |      | 6.7E-04 |      | 7.4E-04 |      |
| $h/2$  | 1.1E-02 | 0.71 | 2.5E-07 | 0.88 | 2.7E-04 | 1.31 | 3.1E-04 | 1.26 |
| $h/4$  | 6.8E-03 | 0.69 | 1.3E-07 | 0.94 | 1.1E-04 | 1.30 | 1.3E-04 | 1.25 |
| $h/8$  | 4.0E-03 | 0.77 | 4.1E-08 | 1.66 | 3.9E-05 | 1.50 | 4.5E-05 | 1.53 |

Table 2 gives the error and the corresponding order of convergence between numerical solution and exact solutions in different mesh $h$, which shows that the smaller error of the velocity $v$ and pressure $p_f$, the displacement $U$, pressure $p_p$ in different regions, it improved the calculation precision, where

\[
\varepsilon_f := \|v - v_h\|_{L^2(H^1)}, \quad \varepsilon_{fp} := \|p_f - p_{f,h}\|_{L^2(L^2)}, \\
\varepsilon_p := \|U - U_h\|_{L^\infty(H^1)}, \quad \varepsilon_{pp} := \|p_p - p_{p,h}\|_{L^2(L^2)}.
\]

The convergence order of velocity $v$ is greater than 0.5, and the convergence order of displacement $U$ and pressure $p_f, p_p$ can reach the first convergence gradually.
Figure 3: Numerical solution of $v$ in $t = 0.001$

Figure 4: Numerical solution of $p_f$ in $t = 0.001$
From Figure 3 to Figure 6, we find out that our loosely-coupled time-stepping method has good numerical stability and no "locking" phenomenon.

**Test 2.** In this example, we focus on fluid-structure interaction in the context of modeling the interaction between a stationary fracture filled with fluid and the surrounding poroelastic medium. We consider a general case where the hydraulic conductivity is a tensor given by \( k = \frac{K}{\mu_f} \), where \( K \) is the permeability tensor. The interface condition is the Beavers-Joseph-Saffman condition (1.18) with the coefficient \( \beta = \frac{\alpha_f \sqrt{3}}{\sqrt{\text{tr}(K)}} \). The reference domain is a square \([-100m, 100m]^2\). A fracture is positioned in the middle of the square, whose boundary is given by

\[
\hat{y}^2 = 0.8^2(\hat{x} - 35)(\hat{x} + 35).
\]

The fracture represents the reference fluid domain \( \hat{\Omega}^f \), while the reference poroelastic structure domain is given as \( \hat{\Omega}^p = \hat{\Omega} \setminus \hat{\Omega}^f \), one can see Figure 7. To obtain a more realistic domain, we transform the reference domain \( \hat{\Omega} \) onto the physical domain \( \Omega \) via the mapping

\[
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
= \begin{pmatrix}
    \hat{x} \\
    \hat{y}/5 - \hat{x}/10
\end{pmatrix} = \begin{pmatrix}
    \hat{x} \\
    5 \cos(\frac{\hat{x} + \hat{y}}{100}) \cos(\frac{\pi \hat{x} + \hat{y}}{100})^2 + \hat{y}/5 - \hat{x}/10
\end{pmatrix}.
\]
Figure 7: Physical computational domain $\Omega = \Omega^f \cup \Omega^p$

The flow is driven by the injection of the fluid into the fracture with the constant rate $g = 25\, kg/s$. On all external boundaries, we prescribe the no flow condition $\mathbf{q} \cdot \mathbf{n} = 0$, zero normal displacement $\mathbf{U} \cdot \mathbf{n} = 0$, and zero shear traction $\tau \cdot \sigma_p \mathbf{n} = 0$. The simulation time is $T = 10s$. The problem was solved using the time step $\Delta t = 0.1s$. The remaining parameters are given in Table 3. We adopt the $P_1 - P_1$ element approximation for the fluid velocity and pressure, complemented with the pressure stabilization $s_p(p_{f,h}, \psi_{f,h})$. For the relative velocity and pressure approximation of the fluid in the poroelastic medium, as for the structure displacement, we again use the $P_1 - P_1$ finite elements. However, due to the large time-step in this example, we are numerically closer to the divergence-free regime. Thus, we add two pseudo-pressure stabilization given by

$$
\begin{align*}
s_q(\xi_h, w_h) := \gamma_q h^2 k_1 \int_{\Omega_p} \nabla \xi_h \cdot \nabla w_h \, d\mathbf{x},
\end{align*}
$$

$$
\begin{align*}
s_q(\eta_h, z_h) := \gamma_q h^2 k_2 \int_{\Omega_p} \nabla \eta_h \cdot \nabla z_h \, d\mathbf{x},
\end{align*}
$$

where the stabilization parameter is selected as $\gamma_q = 10^{-3}$, and the choice of the penalty and stabilization parameters are $\gamma_f = 1500$, $\gamma_{stab} = 1$, $\gamma'_{stab} = 0$. 
Table 3: Values of parameters

| Parameters                          | Denotations | Values       |
|------------------------------------|-------------|--------------|
| Viscosity                          | $\mu_f$     | $10^{-3}$    |
| Lemé constant                      | $\mu_p$     | $2.92 \times 10^8$ |
| Lemé constant                      | $\lambda_p$ | $1.94 \times 10^{10}$ |
| Mass storage coefficient           | $s_0$       | $6.9 \times 10^{-5}$ |
| Biot-Willis constant               | $\alpha$    | 1            |
| Hydraulic conductivity             | $k$         | $1 \times 10^{-8}$ |
| Beavers-CJoseph-C-Saffman coefficient | $\beta$   | $3.47 \times 10^3$ |

Figure 8: Fluid velocity

Figure 9: Fluid pressure

Figure 8 and Figure 9 show the pressure and the velocity in the fracture at final time. At the beginning of the process the simulations capture the expected local pressure increase in the region of fluid injection, while at the final time, the fluid pressure is the largest at the tip of the crack. The pressure and displacement are shown in Figure 10 and Figure 11.
This test demonstrates the ability of our algorithm to handle complex two-dimensional simulations in different applications. Our model includes Darcy equations in the mixed formulation which are necessary to compute accurately the Darcy velocity in the surrounding rock.

5. Conclusions

In this paper, we propose a multiphysics finite element method based on Nitsche’s technique for Stokes-poroelasticity problem. To better describe the multiphysics process of deformation and diffusion for Stokes Stokes-poroelasticity problem, we first present a reformulation of the original problem by introducing two pseudo-pressures, which reveals the underlying deformation and diffusion multiphysics process. Then, we define the weak solution of the reformulated problem and prove the existence and uniqueness of weak solution of the original problem and the reformulated problem. We use Nitsche’s technique to approximate the coupling condition at the interface and analyze the stability of the reformulated full-coupling problem. For the reformulated full-coupling problem, we propose a loosely-coupled time-stepping method, which decouples the problem into three
sub-problems at each time step. And we prove the loosely-coupled time-stepping method with good stability and no "locking" phenomenon. Also, we give the error estimates of the loosely-coupled time-stepping method. A practical advantage of the time-stepping algorithm allows one to use any convergent Stokes solver together with any convergent diffusion equation solver to solve the Stokes-poroelasticity problem. From the point of calculation, this method is very effective and accurate. To the best of our knowledge, it is first time to use the multiphysics finite element method with Nitsche’s technique to solve the Stokes-poroelasticity problem and give the error estimates.

References

[1] M.B. Dusseault, M.S. Bruno, J. Barrera. Casing shear: causes, cases, cures. SPE Drilling and Completion, 2001, 16(2): 98-107.

[2] M.S. Bruno. Geomechanical and decision analyses for mitigating compaction-related casing damage. SPE Drilling and Completion, 2002, 17(3): 179-188.

[3] Y. Wang, M.B. Dusseault. A coupled conductive-convective thermoporoelastic solution and implications for wellbore stability. Journal of Petroleum Science and Engineering, 2003, 38(3): 187-198.

[4] J. Kim, H.A. Tchelepi, R. Juanes. Stability and convergence of sequential methods for coupled flow and geomechanics: drained and undrained splits. Computer Methods in Applied Mechanics and Engineering, 2011, 200(13-16): 1591-1606.

[5] A. Mikelić, M.F. Wheeler. Convergence of iterative coupling for coupled flow and geomechanics. Computational Geosciences, 2013, 17(3): 455-461.

[6] J.A. White, R.I. Borja. Block-preconditioned Newton-Krylov solvers for fully coupled flow and geomechanics. Computational Geosciences, 2011, 15(4): 647-659.

[7] A. Mikelić, M.F. Wheeler, E. Sanchez-Palencia. On the interface law between a deformable porous medium containing a viscous fluid and an elastic body. Mathematical Models and Methods in Applied Sciences, 2012, 22(11): 1250031.

[8] B. Ganis, M.E. Mear, A. Sakhaee-Pour, et al. Modeling fluid injection in fractures with a reservoir simulator coupled to a boundary element method. Computational Geosciences, 2014, 18(5): 613-624.

[9] M. Lesinigo, C. D’Angelo, A. Quarteroni. A multiscale Darcy-Brinkman model for fluid flow in fractured porous media. Numerische Mathematik, 2011, 117(4): 717-752.
[10] M. Prosi, P. Zunino, K. Perktold, et al. Mathematical and numerical models for transfer of low-density lipoproteins through the arterial walls: a new methodology for the model set up with applications to the study of disturbed luminal flow. Journal of Biomechanics, 2005, 38(4): 903-917.

[11] G.A. Holzapfel, G. Sommer, P. Regitnig. Anisotropic mechanical properties of tissue components in human atherosclerotic plaques. Journal of Biomechanical Engineering, 2004, 126(5): 657-665.

[12] S.M.K. Rausch, C. Martin, P.B. Bornemann, et al. Material model of lung parenchyma based on living precision-cut lung slice testing. Journal of the Mechanical Behavior of Biomedical Materials, 2011, 4(4): 583-592.

[13] J.M. Connors, J.S. Howell. A fluid-fluid interaction method using decoupled subproblems and differing time steps. Numerical Methods for Partial Differential Equations, 2012, 28(4): 1283-1308.

[14] J.M. Connors, J.S. Howell, W.J. Layton. Decoupled time stepping methods for fluid-fluid interaction. SIAM Journal on Numerical Analysis, 2012, 50(3): 1297-1319.

[15] S.P. Vanka. Block-implicit multigrid solution of Navier-Stokes equations in primitive variables. Journal of Computational Physics, 1986, 65(1): 138-158.

[16] F.J. Gaspar, F.J. Lisbona, C.W. Oosterlee, et al. A systematic comparison of coupled and distributive smoothing in multigrid for the poroelasticity system. Numerical Linear Algebra with Applications, 2004, 11(2-3): 93-113.

[17] I. Ambartsumyan, E. Khattatov, I. Yotov, et al. A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model. Numerische Mathematik, 2018, 140: 513-553.

[18] S. Muntz. Fluid structure interaction for fluid flow normal to deformable porous media. Kaiserslautern: Technische Universität Kaiserslautern, 2008.

[19] S. Badia, A. Quaini, A. Quarteroni. Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction. Journal of Computational Physics, 2009, 228(21): 7986-8014.

[20] E. Burman, M.A. Fernández. Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility. Computer Methods in Applied Mechanics and Engineering, 2009, 198(5-8): 766-784.
[21] X.B. Feng, Z.H. Ge, Y.K. Li. Analysis of a multiphysics finite element method for a poroelasticity model. IMA Journal of Numerical Analysis, 2018, 38(1): 330-359. arXiv:1411.7464 [math.NA], 2014.

[22] W.J. Layton, F. Schieweck, I. Yotov. Coupling fluid flow with porous media flow. SIAM Journal on Numerical Analysis, 2003, 40(6): 2195-2218.

[23] R. Temam. Navier-Stokes Equations. Studies in Mathematics and its Applications, Vol. 2, North-Holland, 1977.

[24] P. Hansbo. Nitsche’s method for interface problems in computational mechanics. Gamm-Mitteilungen, 2005, 28(2): 183-206.

[25] M. Bukač, I. Yotov, R. Zakerzadeh, et al. Partitioning strategies for the interaction of a fluid with a poroelastic material based on a Nitsche’s coupling approach. Computer Methods in Applied Mechanics and Engineering, 2015, 292: 138-170.

[26] P.G. Ciarlet. The Finite Element Method for Elliptic Problems. Amsterdam: North-Holland, 1978.

[27] M.A. Fernández. Incremental displacement-correction schemes for the explicit coupling of a thin structure with an incompressible fluid. Comptes Rendus Mathematique, 2011, 349(7-8): 473-477.