Surfaces of finite \textit{III}-type

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Abstract— In this paper, we consider surfaces of revolution in the 3-dimensional Euclidean space \(E^3\) with nonvanishing Gauss curvature. We introduce the finite Chen type surfaces with respect to the third fundamental form of the surface. We present a special case of this family of surfaces of revolution in \(E^3\), namely, surfaces of revolution with \(R\) is constant, where \(R\) denotes the sum of the radii of the principal curvature of a surface.

Keywords— Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Beltrami operator, Surfaces of revolution.

I. INTRODUCTION

One of the most interesting and profound aspects of differential geometry is the idea of surfaces of finite type which was born by B-Y. Chen in the early 1970s and since then, it has become a source of interest for many researchers in this field. The reader can refer to [10] for more details. In the framework of this kind of study the first-named author with S. Stamatakis have given in [18] a new generalization to this area of study by giving a similar definition of surfaces of finite type.

Let \(r\) be an isometric immersion of a surface \(S\) in the 3-dimensional Euclidean space \(E^3\). We represent by \(\Delta'\) the Laplacian operator of \(S\) acting on the space of smooth functions \(C^\infty(S)\). Then \(S\) is said to be of finite \(J\)-type, \(J = I, II, III\), if the position vector \(r\) of \(S\) can be decomposed as a finite sum of eigenvectors of \(\Delta'\) of \(S\), that is

\[
    r = c + \sum_{i=1}^{k} \lambda_i x_i, \quad (1)
\]

where

\[
    \Delta' x_i = \lambda_i x_i, \quad i = 1, \ldots, k,
\]

\(c\) is a fixed vector and \(\lambda_1, \lambda_2, \ldots, \lambda_k\) are eigenvalues of the operator \(\Delta'\).

Surfaces of finite type according to the second or third fundamental forms became a topic of active research in the last years. Surfaces of finite type in \(E^3\) regarding to the second fundamental form were investigated for some important classes of surfaces. More precisely, the class of ruled surfaces where studied in [7], while in [3], H. AL-Zoubi studied tubes in \(E^3\). Other classes such as translation surfaces, Quadric surfaces, surfaces of revolution, helicoidal surfaces, cyclides of Dupin and spiral surfaces, the classification of its finite II-type surfaces still unknown. According to the third fundamental form, ruled surfaces in [4], tubes in [5] and quadric surfaces are the only classes were investigated in \(E^3\).

This type of study can be also extended to any smooth map, not necessary for the position vector of the surface, for example, the Gauss map of a surface. Regarding this see [8].

Another generalization to the above, one can study surfaces in \(E^3\) whose position vector \(r\) satisfies the following condition

\[
    \Delta' r = Ar, \quad J = II, III, \quad (2)
\]

where \(A \in \mathbb{R}^{3\times 3}\).

Related to this, in [19] it was proved that the spheres and the catenoids are the only surfaces of revolution satisfying the above equation. Similarly, in [1] it was shown that, helicoids and spheres are the only quadric surfaces in \(E^3\) that satisfy (2). Next, in [2] condition (2) was studied for the class of translation surfaces. In fact authored ascertained that Scherk's surface is the only translation surface in the Euclidean 3-space that satisfies (2), finally, in [17] the authors studied the class of translation surfaces in \(\mathbb{S}^3\) satisfying (2). Surfaces satisfying condition (2) are said to be of coordinate finite \(J\)-type [13].

In order to achieve our goal, we briefly introduce a formula for \(\Delta'' r\) and \(\Delta'' N\) by using tensors calculations, where \(N\) denotes the normal vector field of \(S\). Further, in the last section, we contribute to the solution of our main result.

II. BASIC CONCEPTS

Let \(S\) be a smooth surface in \(E^3\) given by a patch \(r = r(u^1, u^2)\) on a region \(U = (a, b) \times \mathbb{R}\) of \(\mathbb{R}^2\) in which does not contain parabolic points. We denote by

\[
    I = g_{u^1u^1} du^1 du^1, \quad II = b_{u^1u^1} du^1, \quad III = c_{u^1u^1} du^1
\]

the first, second and third fundamental forms of \(S\) respectively. For two sufficiently differentiable functions \(f(u^1, u^2)\) and \(g(u^1, u^2)\) on \(S\), the first differential parameter of Beltrami with
respect to the third fundamental form $J$ is defined by [15]

$$\nabla^i (f,g) := d^i f \cdot g_i,$$

where $f_i = \frac{\partial f}{\partial u^i}$ and $(d^i)$ denotes the inverse tensor of $(g_i)$, $(b_i)$ and $(e_i)$ for $J = I$, $II$ and $III$ respectively.

The second differential parameter of Beltrami with respect to the fundamental form $J = I$, $II$, $III$ of $S$ is defined by [10]

$$\Delta^I f_i = -d^i \nabla^j f_j,$$  \hspace{1cm} \text{(3)}

where $f$ is sufficiently differentiable function, $\nabla^j$ is the covariant derivative in the $u^i$ direction with respect to the fundamental form $J$ [7].

If we apply (3) for the position vector $r$ with respect to the fundamental form $III$, it can be easily verified that [18]

$$\Delta^I r = \text{grad}^III R - R N.$$  \hspace{1cm} \text{(4)}

By applying (3) for the normal vector $N$ we get [18]

$$\Delta^I N = 2 N.$$  \hspace{1cm} \text{(5)}

From (5) we find that the Gauss map of every surface $S$ in $E^3$ is of finite $III$-type 1 with eigenvalue $\lambda = 2$ [18].

Let $S'$ be a parallel surface of $S$ in (directed) distance $\mu = \text{const} \neq 0$, so that $1 - 2 \mu H + \mu^2 K \neq 0$, where $K$ is the Gauss curvature and $H$ is the mean curvature of $S$ respectively. Then $S'$ possesses the position vector $r^* = r + \mu N$.

Denoting by $K'$ and $H'$ the Gauss and mean curvature of $S'$ respectively, we mention the following relations [15]

$$K' = \frac{K}{1 - 2 \mu H + \mu^2 K},$$

$$H' = \frac{H - \mu K}{1 - 2 \mu H + \mu^2 K}.$$

Hence we get

$$R' = \frac{2 H'}{K'} = R - 2 \mu,$$ \hspace{1cm} \text{(6)}

On the other hand, the surfaces $S$, $S'$ have common unit normal vector and spherical image. Thus $III = III'$ and $\Delta^I = \Delta'^I$. We prove now the following theorem for later use.

**Theorem 1.** Let $S$ be a minimal surface in $E^3$. Then $S'$ is parallel surface of $S$ if and only if the sum of the principal radii of curvature $R$ of $S'$ is constant.

**Proof.** Suppose that $S$ is minimal surface in $E^3$, which is defined on a simply connected domain $D$ in the $(u^1, u^2)$-plane. Let

$$S': r^* = r + \mu N, \mu \neq 0$$

be parallel surface of $S$.

From (6) and taking into account $H = 0$, we find $R^* = -2 \mu = \text{const}$.. Hence the first part of the theorem is proved.

Conversely, let $R^* = \text{const} \neq 0$. Then from Theorem (4.4) (see [18]), $S'$ is of null $III$- type 2. Therefore from (1) there exist nonconstant eigenvectors $x_1(u^1, u^2)$ and $x_2(u^1, u^2)$ defined on the same domain $D$ such that

$$r^* = x_1 + \lambda x_2,$$  \hspace{1cm} \text{(7)}

where $\Delta^I x_1 = \lambda_1 x_1$, $\Delta^I x_2 = \lambda_2 x_2$, and here we have $\lambda_1 = 0$ because $S'$ is of null $III$- type 2.

Once we have $\Delta^I r^* = \Delta^I x_1 + \Delta^I x_2$, it then follows that

$$\Delta^I r^* = \lambda_2 x_2.$$  \hspace{1cm} \text{(8)}

On the other hand, since $R^* = \text{const} \neq 0$, we get

$$\Delta^I r^* = -R^* N.$$  \hspace{1cm} \text{(9)}

Thus from (8) and (9), one finds

$$\lambda_2 x_2 = -R^* N$$

or $x_2 = c N$, where $c = -\frac{R^*}{\lambda_2}$, and then (7) becomes

$$r^* = x_1 + c N.$$  \hspace{1cm} \text{(10)}

The differential of the above equation is

$$dr^* = dx_1 + cdN.$$  \hspace{1cm} \text{(11)}

Taking the inner product of both sides of (11) with $N$ yields

$$\langle dx_1, N \rangle > 0.$$  \hspace{1cm} \text{(12)}

Now we want to show that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$. It is enough to prove that

$$x_{1u^1} \times x_{1u^2} \neq 0, \forall (u^1, u^2) \in D,$$

where $\times$ is the Euclidean cross product. We have

$$x_{1} = r^* - \mu N^*.$$  \hspace{1cm} \text{(13)}

Using the Wangerin equations [15]

$$N_{\theta} = -b_0 g^0_k r^* \theta,$$

and the equation (13), it follows that

$$x_{1u^1} \times x_{1u^2} = (r^*_1 - \mu N_{\theta}) \times (r^*_2 - \mu N_{\theta})$$

$$= (r^*_1 \times r^*_2) - \mu (r^*_{1u^1} \times N_{\theta}) + \mu (r^*_{2u^2} \times N_{\theta}) + \mu^2 (N_{\theta1} \times N_{\theta2})$$

$$= (1 - 2 \mu H + \mu^2 K) (r^*_1 \times r^*_2) \neq 0, \forall (u^1, u^2) \in D.$$  \hspace{1cm} \text{(14)}

Hence, on account of (12) and (14), we conclude that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$ with $N$ it's Gauss map.
Since $\Delta^\text{III} x_1 = 0$. Consequently, from Theorem (3.1) (see [18]), $x_1(u', u^2)$ is a minimal surface. Thus from (10) we obtain that $S'$ is parallel surface of a minimal. Now we mention and prove our main theorem.

**Theorem 2.** The only surfaces of revolution in $E^3$ of which the sum of the radii of the principal curvature $R$ is constant are

- parts of spheres which are of finite III-type 1,
- catenoids which are of finite null III-type 1, and
- the parallel surfaces to the catenoids, which are of finite null III-type 2.

**III. PROOF OF THEOREM 2**

Let $C$ be a smooth curve lies on the $xz$-plane parametrized by

$$x(u) = (f(u), 0, g(u)), u \in J, (J \subset \mathbb{R}),$$

where $f$, $g$ are smooth functions and $f$ is a positive function. When $C$ is revolved about the $z$-axis, the resulting point set $S$ is called the surface of revolution generated by the curve $C$. In this case, the $z$-axis is called the axis of revolution of $S$ and $C$ is called the profile curve of $S$. On the other hand, a subgroup of the rotation group which fixes the vector $(0, 0, 1)$ is called the profile curve of $S$.

The partial derivatives of (15) are

$$r_u = (f'(u) \cos v, f'(u) \sin v, g'(u)), \quad r_v = (-f(u) \sin v, f(u) \cos v, 0).$$

The components $g_{ij}$ of the first fundamental form in (local) coordinates are the following

$$g_{11} = \langle r_u, r_u \rangle = 1, \quad g_{12} = \langle r_u, r_v \rangle = 0, \quad g_{22} = \langle r_v, r_v \rangle = f''.$$

Denoting by $\kappa$ the curvature of the curve $C$ and $R_1, R_2$ the principal radii of curvature of $S$, we have

$$R_1 = \frac{1}{\kappa}, \quad R_2 = \frac{f'}{g'}.$$

The Gauss curvature and the mean curvature of $S$ are respectively

$$K = \frac{1}{R_1 R_2} = \frac{kg'}{f} = -f'', \quad 2H = \frac{1}{R_1} + \frac{1}{R_2} = \kappa + \frac{g'}{f}.$$

The Gauss map $N$ of $S$ is computed as follows

$$N(u,v) = (-g'\cos v, -g'\sin v, -f').$$

Now, by using the natural frame $\{N_u, N_v\}$ of $S$ defined by

$$N_u = (g'' \cos v, g'' \sin v, f''), \quad N_v = (g' \sin v, g' \cos v, 0)$$

the components $e_{ij}$ of the third fundamental form in (local) coordinates are the following

$$e_{11} = \langle N_u, N_u \rangle = (g'')^2 + (f'')^2, \quad e_{12} = \langle N_u, N_v \rangle = 0, \quad e_{22} = \langle N_v, N_v \rangle = (g')^2.$$

The Beltrami operator $\Delta^\text{III}$ in terms of local coordinates $(u, v)$ of $S$ can be expressed as follows

$$\Delta^\text{III} = -\frac{1}{\kappa^2} \frac{\partial^2}{\partial u^2} + \frac{g'k'' - kg''}{g'k^3} \frac{\partial}{\partial u} - \frac{1}{g''^2} \frac{\partial^2}{\partial v^2}. \quad (18)$$

On account of (16) we put

$$f' = \cos \varphi, \quad g' = \sin \varphi,$$

where $\varphi = \varphi(u)$. Then $\kappa = \varphi'$ and the parametric representation (17) of the unit vector $N$ of $S$ becomes

$$N(u, v) = \{ -\sin \varphi \cos v, -\sin \varphi \sin v, \cos \varphi \}. \quad (19)$$

Also relation (18) takes the following form

$$\Delta^\text{III} = -\frac{1}{\varphi''^2} \frac{\partial^2}{\partial u^2} + \left( \frac{\varphi'' - \cos \varphi}{\varphi' \sin \varphi} \right) \frac{\partial}{\partial u} - \frac{1}{\sin^2 \varphi} \frac{\partial^2}{\partial v^2}. \quad (20)$$
For the sum of the principal radii of curvature \( R = R_1 + R_2 = \frac{2H}{K} \), one finds
\[
R = \frac{f}{\sin \varphi} + \frac{1}{\phi'}.
\] (21)

Taking the derivative of (21) we find
\[
R' = -\frac{\varphi''}{\varphi'^3} + \frac{2\sin \varphi}{\phi'\sin \varphi} + \frac{f}{\sin^2 \varphi}.
\] (22)

Let \((r_1, r_2, r_3)\) be the coordinate functions of (15). By virtue of (20) we obtain
\[
\Delta^{III} r_1 = \Delta^{III}(f \cos \nu) = \left( \varphi' \cos \varphi - \frac{1}{\phi'} + \frac{2\sin \varphi}{\phi' \sin \varphi} + \frac{f}{\sin^2 \varphi} \right) \cos \nu.
\] (23)

\[
\Delta^{III} r_2 = \Delta^{III}(f \sin \nu) = \left( \varphi' \cos \varphi - \frac{1}{\phi'} + \frac{2\sin \varphi}{\phi' \sin \varphi} + \frac{f}{\sin^2 \varphi} \right) \sin \nu.
\] (24)

\[
\Delta^{III} r_3 = \Delta^{III} g = -\frac{2\cos \varphi}{\phi'} + \varphi' \sin \varphi.
\] (25)

From (21) and (22), equations (23), (24) and (25) become respectively
\[
\Delta^{III} r_1 = \left( R \sin \varphi - \frac{R' \cos \varphi}{\phi'} \right) \cos \nu,
\] (26)

\[
\Delta^{III} r_2 = \left( R \sin \varphi - \frac{R' \cos \varphi}{\phi'} \right) \sin \nu,
\] (27)

\[
\Delta^{III} r_3 = -\frac{R' \sin \varphi}{\phi'} - R \cos \varphi.
\] (28)

We obtain the following two cases:

Case I. \( R = 0 \).

Thus \( H = 0 \). Consequently \( S \), being a minimal surface of revolution, is a catenoid [19].

Case II. \( R \) is const. \( \neq 0 \).

From (26), (27) and (28) we obtain
\[
\Delta^{III} r_1 = R \sin \varphi \cos \nu,
\]
\[
\Delta^{III} r_2 = R \sin \varphi \sin \nu
\]
\[
\Delta^{III} r_3 = -R \cos \varphi.
\] (29)

Let \((N_1, N_2, N_3)\) be the coordinate functions of \( N \). From (18), relations (29) can be written
\[
\Delta^{III} r_1 = -RN_1, \Delta^{III} r_2 = -RN_2, \Delta^{III} r_3 = -RN_3,
\]
and hence
\[
\Delta^{III} r = -RN.
\] (30)

In view of (5) and (30) we have
\[
(\Delta^{III})^m r = -(2^{m-1})RN.
\] (31)

Now, if \( S \) is of finite type \( k \), then there exist real numbers \( c_1, c_2, \ldots, c_k \) such that
\[
(\Delta^{III})^m r + c_1(\Delta^{III})^{k-1}r + \ldots + c_k r = 0.
\] (32)

From (30) and (31) relation (32) becomes
\[
-2^{k-1}RN - 2^{k-2}c_1RN - \ldots - c_k RN + c_k r = 0,
\]
\[
c N + c_k r = 0,
\] (33)

where \( c = -R(2^{k-1} + 2^{k-2}c_1 + \ldots + c_k) \) = const.,

Now, if \( c_k \neq 0 \), then from (33) we have \( r = \frac{-c}{c_k} N \), and hence we get \( |r| = \left| \frac{c}{c_k} \right| \) and so \( S \) is a sphere. On account of Theorem (3.3) (see [18]), \( S \) is of finite III-type 1. If \( c_k = 0 \), then \( S \) is of null type \( k \). Since \( R = \) const., thus according to Theorem (4.4) (see [18]) and Theorem (1), \( S \) is of null III-type 2 which is parallel surface of a minimal.

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