Invariance and hierarchy-equivalence

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Abstract

Two type structures are hierarchy-equivalent if they induce the same set of hierarchies of beliefs. This note shows that the behavioral implications of “cautious rationality and common cautious belief in cautious rationality” (Catonini and De Vito 2021) do not vary across hierarchy-equivalent type structures.

1 Introduction

The aim of this note is to state and prove an invariance result for the behavioral implications of some epistemic assumptions. Precisely, say that two (epistemic) type structures are hierarchy-equivalent if they represent the same set of hierarchies of beliefs. Consider now the following epistemic assumptions, studied in Catonini and De Vito (2021):

1. cautious rationality and common cautious belief in cautious rationality (RcCBcRc);
2. rationality, transparency of cautiousness, and common cautious belief in both.

Such epistemic assumptions are represented by events (Borel sets) in a type structure. The main result of this note (Theorem 1) says that the behavioral implications of the above epistemic assumptions do not vary across hierarchy-equivalent type structures.

This note is organized as follows. Section 2 introduces an important, technical result that will be used in the proofs that follow. Section 3 briefly reviews the formalism of hierarchies of (lexicographic) beliefs and type structures, and it introduces the concept of hierarchy-equivalence. Section 4 states and proves the main result. Section 5 concludes.

2 Preliminaries

All the spaces considered in this paper are assumed to be topological spaces. A Souslin space is a topological space that is the image of a complete, separable metric space under a continuous surjection. In particular, a Polish space (i.e., a topological space which is homeomorphic to a complete, separable metric space) is Souslin. For any Borel probability measure \( \mu \) on a topological space \( X \), we let \( \mu^* \) denote the outer measure induced by \( \mu \). Moreover, if \( f : X \to Y \) is a Borel map between topological spaces \( X \) and \( Y \), we let \( \mu \circ f^{-1} \) denote the image measure of \( \mu \) under \( f \). With this, we state a technical result that will be used in the proof of Lemma 2 in Section 4.

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Lemma 1 Fix Souslin spaces $X$ and $Y$, and a Borel map $f : X \to Y$. Then, for every Borel probability measure $\mu$ on $X$, and for every $E \subseteq Y$,

$$\left(\mu \circ f^{-1}\right)^*(E) = \mu^* \left(f^{-1}(E)\right).$$

Proof. Every Borel probability measure on a Souslin space is Radon, hence perfect (see Bogachev 2007, Theorem 7.4.3 and Theorem 7.5.10). It follows from Theorem 3.6 in Peskir (1991) (see also Hoffmann-Jørgensen 2003, and Dudley 2014, Section 3.4) that the Borel map $f : X \to Y$ is $\mu$-perfect (that is, it satisfies $\left(\mu \circ f^{-1}\right)^*(E) = \mu^* \left(f^{-1}(E)\right)$ for all $E \subseteq Y$) for every measure $\mu$ on $X$. 

Remark 1 Lemma 1 can be equivalently stated in terms of inner measure $\mu_*$; that is, under the stated assumptions, $\left(\mu \circ f^{-1}\right)_*(E) = \mu_* \left(f^{-1}(E)\right)$ for every $E \subseteq Y$.

Given a sequence $(X_n)_{n \in \mathbb{N}}$ of pairwise disjoint Souslin spaces, the set $X := \bigcup_{n \in \mathbb{N}} X_n$ is endowed with the direct sum topology\(^1\) so that $X$ is a Souslin space. Moreover, we endow each finite or countable product of Souslin spaces with the product topology, hence the product space is Souslin as well.

We let $\mathcal{M}(X)$ denote the set of Borel probability measures on a topological space $X$. The set $\mathcal{M}(X)$ is endowed with the weak*-topology. So, if $X$ is Souslin, then $\mathcal{M}(X)$ is also Souslin. We let $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) denote the set of all finite (resp. length-$n$) sequences of Borel probability measures on $X$, that is,

$$\mathcal{N}(X) := \bigcup_{n \in \mathbb{N}} \mathcal{N}_n(X) \quad \text{and} \quad \mathcal{N}_n(X) := \bigcup_{n \in \mathbb{N}} (\mathcal{M}(X))^n.$$

Each $\overline{\pi} := (\mu^1, \ldots, \mu^n) \in \mathcal{N}(X)$ is called lexicographic probability system (LPS). In view of our assumptions, the topological space $\mathcal{N}(X)$ is Souslin.

For every Borel probability measure $\mu$ on $X$, the support of $\mu$, denoted by $\text{Supp}\mu$, is the smallest closed subset $C \subseteq X$ such that $\mu(C) = 1$. The support of an LPS $\overline{\pi} := (\mu^1, \ldots, \mu^n) \in \mathcal{N}(X)$ is defined as $\text{Supp}\overline{\pi} := \bigcup_{k \leq n} \text{Supp}\mu^k$. So, an LPS $\overline{\pi} := (\mu^1, \ldots, \mu^n) \in \mathcal{N}(X)$ is of full-support if $\text{Supp}\overline{\pi} = X$. We write $\mathcal{N}^+(X)$ for the set of full-support LPS’s.

Fix Souslin spaces $X$ and $Y$, and a Borel map $f : X \to Y$. For each $n \in \mathbb{N}$, the map $\hat{f}(n) : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is defined by

$$\left(\mu^1, \ldots, \mu^n\right) \mapsto \hat{f}(n) \left(\left(\mu^1, \ldots, \mu^n\right)\right) := \left(\mu^k \circ f^{-1}\right)_{k \leq n}.$$

With this, the map $\hat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ defined by

$$\hat{f}(\overline{\pi}) := \hat{f}(n)(\overline{\pi}), \overline{\pi} \in \mathcal{N}_n(X),$$

is called the image LPS map of $f$. Alternatively put, the map $\hat{f}$ is the union of the maps $\left(\hat{f}(n)\right)_{n \in \mathbb{N}}$, and it is Borel measurable\(^2\).

Given Souslin spaces $X$ and $Y$, we let $\text{Proj}_X$ denote the canonical projection from $X \times Y$ onto $X$; in view of our assumption, the map $\text{Proj}_X$ is continuous. The marginal measure of

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\(^1\)In this topology, a set $O \subseteq X$ is open if and only if $O \cap X_n$ is open in $X_n$ for all $n \in \mathbb{N}$. The assumption that the spaces $X_n$ are pairwise disjoint is without any loss of generality, since they can be replaced by a homeomorphic copy, if needed (see Engelking 1989, p. 75).

\(^2\)For details and proofs related to Borel measurability and continuity of the involved maps, the reader can consult Catonini and De Vito (2018).
\[ \mu \in \mathcal{M}(X \times Y) \text{ on } X \text{ is defined by } \text{marg}_X \mu := \mu \circ \text{Proj}_X^{-1}. \] Consequently, the marginal of \( \mu \in \mathcal{N}(X \times Y) \) on \( X \) is defined by \( \text{marg}_X \mu := \text{Proj}_X(\mu) \), and the function \( \text{Proj}_X : \mathcal{N}(X \times Y) \to \mathcal{N}(X) \) is continuous and surjective.

Finally, for any set \( X \), we let \( \text{Id}_X \) denote the identity map on \( X \), that is, \( \text{Id}_X(x) := x \) for all \( x \in X \).

### 3 Hierarchies of lexicographic beliefs and type structures

Throughout, we consider finite games. A finite game is a structure \( G := \langle I, (S_i, \pi_i)_{i \in I} \rangle \), where (a) \( I \) is a finite set of players with cardinality \( |I| \geq 2 \); (b) for each player \( i \in I \), \( S_i \) is a finite, non-empty set of strategies; and (c) \( \pi_i : S \to \mathbb{R} \) is the payoff function. Each strategy set \( S_i \) is given the obvious topology, i.e., the discrete topology.

Fix a finite game \( G := \langle I, (S_i, \pi_i)_{i \in I} \rangle \). A type structure (associated with \( G \)) formalizes Harsanyi’s (1967-68) implicit approach to model hierarchies of beliefs.

**Definition 1** An \( (S_i)_{i \in I} \)-based lexicographic type structure is a structure \( T := \langle S_i, T_i, \beta_i \rangle_{i \in I} \) where

1. for each \( i \in I \), \( T_i \) is a Souslin space;
2. for each \( i \in I \), the function \( \beta_i : T_i \to \mathcal{N}(S_{\sim i} \times T_{\sim i}) \) is Borel measurable.

We call each space \( T_i \) type space and we call each \( \beta_i \) belief map. Members of type spaces, viz. \( t_i \in T_i \), are called types. Each element \( (s_i, t_i)_{i \in I} \in \prod_{i \in I} (S_i \times T_i) \) is called state (of the world).

In what follows, we will omit the qualifier “lexicographic,” and simply speak of type structures when the underlying strategy sets \( (S_i)_{i \in I} \) are clear from the context.

Type structures generate a collection of hierarchies of beliefs for each player. More precisely, to illustrate how a type induce a hierarchy of beliefs, we first review how the set of hierarchies is defined. For each \( i \in I \), set \( X_i^1 := S_{\sim i} \) and recursively,

\[ X_i^{m+1} := X_i^m \times \prod_{j \neq i} \mathcal{N}(X_j^m). \]

The set of all possible hierarchies of beliefs (LPS’s) for player \( i \) is

\[ H_i^0 := \prod_{m=1}^{\infty} \mathcal{N}(X_i^m). \]

Since each strategy set \( S_i \) is finite, it follows form standard arguments (see Catonini and De Vito 2018) that \( X_i^m \ (i \in I, m \geq 1) \) and \( H_i^0 \ (i \in I) \) are Polish (hence Souslin) spaces.

Next, fix a type structure \( T := \langle S_i, T_i, \beta_i \rangle_{i \in I} \) associated with the game \( G := \langle I, (S_i, \pi_i)_{i \in I} \rangle \). We define, for each player \( i \in I \), a hierarchy map \( d_i : T_i \to H_i^0 \) which associates each \( t_i \in T_i \) with a hierarchy of beliefs. Such map is defined recursively.

- \( (m = 1) \) For each \( i \in I \) and each \( t_i \in T_i \), define the first-order hierarchy map \( d_i^1 : T_i \to \mathcal{N}(X_i^1) \) as

\[ d_i^1(t_i) := \text{marg}_{S_{\sim i}}(\beta_i(t_i)). \]

For each \( i \in I \), let \( d_{i-} := (d_i^1)_{j \neq i} : T_{\sim i} \to \prod_{j \neq i} \mathcal{N}(X_j^1). \)

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Then, for each $i \in I$, define $\rho_{-i}^1 : S_{-i} \times T_{-i} \to X_i^1$ as

$$\rho_{-i}^1 := (\text{Id}_{S_{-i}}, d_{-i}^1).$$

Since each belief map $\beta_i$ is Borel measurable, standard arguments (Catonini and De Vito 2018) show that all the maps defined above are Borel measurable.

- $(m + 1, m \geq 1)$ Suppose we have already defined, for each $i \in I$, Borel measurable maps $d_i^m : T_i \to \mathcal{N}(X_i^m)$ and $\rho_i^m : S_{-i} \times T_{-i} \to X_i^m$. For each $i \in I$ and each $t_i \in T_i$, define $d_{i}^{m+1} : T_i \to \mathcal{N}(X_i^{m+1})$ as

$$d_{i}^{m+1}(t_i) := \tilde{\rho}_{-i}^m(\beta_i(t_i)).$$

For each $i \in I$, let $d_{-i}^{m+1} := \left( d_{j}^{m+1} \right)_{j \neq i} : T_{-i} \to \prod_{j \neq i} \mathcal{N}(X_j^{m+1}).$

Then, for each $i \in I$, the map $\rho_{-i}^{m+1} : S_{-i} \times T_{-i} \to X_i^{m+1}$ is defined as

$$\rho_{-i}^{m+1} := (\rho_{-i}^m, d_{-i}^{m+1}).$$

All the maps defined above are Borel measurable.

With this, for each $i \in I$ the map $d_i : T_i \to H_i^0$ is defined by

$$d_i(t_i) := (d_i^1(t_i), d_i^2(t_i), ...).$$

For future reference, we point out the following fact, whose proof can be found in Catonini and De Vito (2018).

**Remark 2** Fix a type $t_i \in T_i$ in a type structure. Then, for each $m \geq 1$,

$$\text{marg}_{X_i^m}(d_i^{m+1}(t_i)) = d_i^m(t_i).$$

That is, type $t_i$ induces a coherent hierarchy of beliefs.

Next step is to formally define the notion of hierarchy-equivalence for type structures. To this end, we first provide the definition of hierarchy morphism.

**Definition 2** Fix type structures $\mathcal{T} := \langle S, T, \beta \rangle_{i \in I}$ and $\mathcal{T}^o := \langle S, T^o, \beta^o \rangle_{i \in I}$ associated with a finite game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$. A map $(\varphi_i)_{i \in I} : T \to T^o$ is a hierarchy morphism (from $\mathcal{T}$ to $\mathcal{T}^o$) if, for all $i \in I$, $t_i \in T_i$ and $n \geq 1$,

$$d_i^n(t_i) = d_{i}^{n} \left( \varphi_i(t_i) \right).$$

In words, a hierarchy morphism from $\mathcal{T}$ to $\mathcal{T}^o$ is a (not necessarily measurable) map which preserves the hierarchies of beliefs.$^3$

**Definition 3** Fix type structures $\mathcal{T} := \langle S, T, \beta \rangle_{i \in I}$ and $\mathcal{T}^o := \langle S, T^o, \beta^o \rangle_{i \in I}$ associated with a finite game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$. We say that $\mathcal{T}$ and $\mathcal{T}^o$ are hierarchy-equivalent if there exist hierarchy morphisms $(\varphi_i)_{i \in I} : T \to T^o$ and $(\varphi^o_i)_{i \in I} : T^o \to T$.

The following example illustrates the above concepts.

$^3$A type morphism is a hierarchy morphism; the reverse implication does not hold (see Friedenberg and Meier 2011).
Example 1 Consider a finite game between two players, Ann (a) and Bob (b), such that
\[
S_a := \{s_a\},
\]
\[
S_b := \{s_b, \bar{s}_b\},
\]
and \( \bar{s}_b \) is strictly dominant for Bob. Append to this game a finite type structure \( \mathcal{T} = \langle S_i, T_i, \beta_i \rangle \in \{a,b\} \) where each type set is a singleton, that is, \( T_a := \{t_a\} \) and \( T_b := \{\bar{t}_b, \bar{\bar{t}}_b\} \).

For Bob, \( \beta_b(\bar{t}_b) \) is the trivial probability measure on \( \{(s_a, t_a)\} \). Ann’s type is associated with the LPS \( \beta_a(t_a) := (\mu_a^1, \mu_a^2, \mu_a^3) \) described in the following table:
\[
\begin{array}{c|ccc}
\mu_a^1 & (\bar{s}_b, \bar{t}_b) & (\bar{s}_b, \bar{\bar{t}}_b) & (s_b, \bar{t}_b) \\
\mu_a^2 & 1 & 0 & 1 \\
\mu_a^3 & 0 & 1 & 0 \\
\end{array}
\]
Clearly, each type in \( \mathcal{T} \) induces a unique hierarchy of beliefs.

Consider now a different type structure \( \mathcal{T}^0 = \langle S_i, T_i^0, \beta_i^0 \rangle \in \{a,b\} \) where \( T_a^0 := \{t_a\} \) and \( T_b^0 := \{\bar{t}_b, \bar{\bar{t}}_b\} \), and the belief maps are as follows. Bob’s belief map satisfies \( \beta_b^0(\bar{t}_b) = \beta_b(\bar{t}_b) = \beta_b(\bar{\bar{t}}_b) \).

Ann’s type is associated with the LPS \( \beta_a^0(t_a) := (\nu_a^1, \nu_a^2, \nu_a^3) \) described in the following table:
\[
\begin{array}{c|ccc}
\nu_a^1 & (\bar{s}_b, \bar{t}_b) & (\bar{s}_b, \bar{\bar{t}}_b) & (s_b, \bar{t}_b) \\
\nu_a^2 & 0 & 1/2 & 1/2 \\
\nu_a^3 & 0 & 1/2 & 1/2 \\
\end{array}
\]
Note that \( \mathcal{T} \) and \( \mathcal{T}^0 \) are hierarchy-equivalent. To see this, define the map \( \varphi^0 := (\varphi_a^0, \varphi_b^0) : T_a^0 \times T_b^0 \rightarrow T_a \times T_b \) as follows:
\[
\varphi_a^0(t_a) := t_a,
\]
\[
\varphi_b^0(\bar{t}_b) = \varphi_b^0(\bar{\bar{t}}_b) := \bar{t}_b.
\]
It can be checked that \( \varphi^0 \) is a hierarchy morphism from \( \mathcal{T}^0 \) to \( \mathcal{T} \). So, the set of belief hierarchies induced by types in \( \mathcal{T}^0 \) is included in the set of belief hierarchies induced by types in \( \mathcal{T} \). Since each type in \( \mathcal{T} \) induces a unique hierarchy of beliefs, the conclusion follows.

4 The main result

In this section we state and prove the result of this note. We first review the epistemic condition of interest (subsection 4.1), then we show that such conditions are invariant (in terms of behavioral implications) across hierarchy-equivalent type structures (subsection 4.2).

4.1 Epistemic events

For this subsection, we fix a finite game \( G := \langle I, (S_i, \pi_i)_{i \in I} \rangle \), and we append to \( G \) a type structure \( \mathcal{T} := \langle S_i, T_i, \beta_i \rangle \in I \).

For any two vectors \( x := (x_l)_{l=1}^n, y := (y_l)_{l=1}^n \in \mathbb{R}^n \), we write \( x \geq_L y \) if either (a) \( x_l = y_l \) for every \( l \leq n \), or (b) there exists \( m \leq n \) such that \( x_m > y_m \) and \( x_l = y_l \) for every \( l < m \); we write \( x >_L y \) if condition (b) holds.

Definition 4 A strategy \( s_i \in S_i \) is optimal under \( \beta_i(t_i) := (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_i \times T_i) \) if, for every \( s'_i \in S_i \),
\[
\left( \pi_i(s_i, \text{marg}_{S_i} \mu_i) \right)_{l=1}^n \geq_L \left( \pi_i(s'_i, \text{marg}_{S_i} \mu_i) \right)_{l=1}^n.
\]
We say that \( s_i \) is a lexicographic best reply to \( \text{marg}_{S_i} \beta_i(t_i) \) if it is optimal under \( \beta_i(t_i) \).
This is the usual definition of optimality for a strategy, but this time optimality is taken lexicographically.

**Definition 5** A type $t_i \in T_i$ is **cautious** (in $\mathcal{T}$) if $\text{marg}_{S_{-i}} \beta_i(t_i) \in \mathcal{N}^+ (S_{-i})$.

In words, this notion of cautiousness requires that the first-order belief of a type be a full-support LPS.

For strategy-type pairs we define the following notions.

**Definition 6** Fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$.

1. Say $(s_i, t_i)$ is **rational** (in $\mathcal{T}$) if $s_i$ is optimal under $\beta_i(t_i)$.
2. Say $(s_i, t_i)$ is **cautiously rational** (in $\mathcal{T}$) if it is rational and $t_i$ is cautious.

The following definition is essential.

**Definition 7** Fix a non-empty event $E \subseteq S_{-i} \times T_{-i}$ and a type $t_i \in T_i$ with $\beta_i(t_i) := (\mu^1_i, \ldots, \mu^n_i)$. We say that $E$ is **cautiously believed under $\beta_i(t_i)$ at level $m \leq n$** if the following conditions hold:

(i) $\mu^l_i(E) = 1$ for all $l \leq m$;
(ii) for every elementary cylinder $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$, if $E \cap \hat{C}_{s_{-i}} \neq \emptyset$ then $\mu^l_i(E \cap \hat{C}_{s_{-i}}) > 0$ for some $l \leq m$.

We say that $E$ is **cautiously believed under $\beta_i(t_i)$ at some level** $m \leq n$.

We say that $t_i \in T_i$ **cautiously believes** $E$ if $E$ is cautiously believed under $\beta_i(t_i)$.

We are now ready to formally state the epistemic conditions of interest.

For each player $i \in I$, we let $R^1_i$ denote the set of cautiously rational strategy-type pairs. Let $B^\beta_i : \Sigma_{S_{-i} \times T_{-i}} \to \Sigma_{S_i \times T_i}$ be the operator defined by

$$B^\beta_i(E_{-i}) := \{(s_i, t_i) \in S_i \times T_i : t_i \text{ cautiously believes } E_{-i}, E_{-i} \in \Sigma_{S_{-i} \times T_{-i}}$$

As shown in Catonini and De Vito (2021), the set $B^\beta_i(E_{-i})$ is Borel in $S_i \times T_i$ if $E_{-i} \subseteq S_{-i} \times T_{-i}$ is an event; so the operator $B^\beta_i$ is well-defined.

For each $m \geq 1$, define $R^{m+1}_i$ recursively by

$$R^{m+1}_i := R^m_i \cap B^\beta_i(R^{m}_i)$$

where $R^{0}_i := \prod_{j \neq i} R^1_j$. Note that

$$R^{m+1}_i = R^1_i \cap \left( \bigcap_{l \leq m} B^\beta_i \left( R^l_i \right) \right)$$

and each $R^m_i$ is Borel in $S_i \times T_i$ (see Catonini and De Vito 2021).

We write $R^\infty_i := \cap_{m \in \mathbb{N}} R^m_i$ for each $i \in I$. If $(s_i, t_i)_{i \in I} \in \prod_{i \in I} R^{m+1}_i$, we say that there is **cautious rationality and $m$th-order cautious belief in cautious rationality ($R^mB^\betaR^c$)** at this state. If $(s_i, t_i)_{i \in I} \in \prod_{i \in I} R^\infty_i$, we say that there is **cautious rationality and common cautious belief in cautious rationality ($R^cCB^\betaR^c$)** at this state.
4.2 The invariance result

We consider only the epistemic assumption of $R^c CB^c R^c$, since the proof for the different epistemic assumption mentioned in the introduction is identical.

**Theorem 1** Fix type structures $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^\circ = \langle S_i, T_i^\circ, \beta_i^\circ \rangle_{i \in I}$ associated with a finite game $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$. If $\mathcal{T}$ and $\mathcal{T}^\circ$ are hierarchy-equivalent, then, for each $i \in I$ and for each $m \geq 0$,

$$\text{Proj}_{S_i}(R_i^m) = \text{Proj}_{S_i}(R_i^{\circ,m}).$$

The following lemma plays a crucial role for the proof of Theorem 1.

**Lemma 2** Fix type structures $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^\circ = \langle S_i, T_i^\circ, \beta_i^\circ \rangle_{i \in I}$ associated with a finite game $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$. If, for each $i \in I$ and for each $m \geq 1$,

(a) there are $t_i \in T_i$ and $t_i^\circ \in T_i^\circ$ such that $d_i^m(t_i) = d_i^{\circ,m}(t_i^\circ)$,

(b) $\text{Proj}_{S_i}(R_i^{m-1}) = \text{Proj}_{S_i}(R_i^{\circ,m-1}),$

then, for each $s_i \in S_i$, $(s_i, t_i) \in R_i^m$ only if $(s_i, t_i^\circ) \in R_i^{\circ,m}$.

**Proof.** We prove by induction on $m \geq 1$ that, for each $i \in I$, the following statements hold:

(i) for each $t_i \in T_i$ and $t_i^\circ \in T_i^\circ$, if $d_i^m(t_i) = d_i^{\circ,m}(t_i^\circ)$ and $\text{Proj}_{S_i}(R_i^{m-1}) = \text{Proj}_{S_i}(R_i^{\circ,m-1})$,

then, for each $s_i \in S_i$, $(s_i, t_i) \in R_i^m$ only if $(s_i, t_i^\circ) \in R_i^{\circ,m};$

(ii) $(\rho_i^{m+1})^{-1}(\rho_i^{\circ,m+1}(R_i^{m-1})) \subseteq R_i^{\circ,m}.$

This will yield the result.

**Basis step.** The conclusion of part (i) is immediate: indeed, $d_i^1(t_i) = \text{marg}_{S_i} \beta_i(t_i) = \text{marg}_{S_i} \beta_i^\circ(t_i) = d_i^{\circ,1}(t_i^\circ)$, and, by definition, $R_i^0 := S_i \times T_i$ and $R_i^{\circ,0} := S_i \times T_i^\circ$. Hence, $(s_i, t_i) \in R_i^1$ implies $(s_i, t_i^\circ) \in R_i^{\circ,1}$. To show part (ii), fix any $(s_{-i}, t_{-i}^\circ) \in (\rho_i^{\circ,2})^{-1}(\rho_i^{\circ,1}(R_i^{1-1}))$. Then there exists $t_{-i} \in T_{-i}$ such that $(s_{-i}, t_{-i}) \in R_{-i}^1$ and $\rho_i^{\circ,2}((s_{-i}, t_{-i}^\circ)) = \rho_i^{\circ,1}((s_{-i}, t_{-i}^\circ))$. Hence, $d_{-i}^1(t_i) = d_{-i}^{\circ,1}(t_{-i}^\circ)$ and so, by part (i) (established for $m = 1$), we obtain $(s_i, t_i^\circ) \in R_i^{\circ,1}.$

**Inductive step.** Suppose that the result is true for $m \geq 1$. We show that it is also true for $m + 1$. As for part (i), consider $t_i \in T_i$ and $t_i^\circ \in T_i^\circ$ and $s_i \in S_i$ so that

$$d_i^{m+1}(t_i) = d_i^{\circ,m+1}(t_i^\circ),$$

$$(s_i, t_i) \in R_i^{m+1}.$$

By coherence of belief hierarchies (Remark 2), we have $d_i^m(t_i) = d_i^{\circ,m}(t_i^\circ)$. Hence, by part (i) of the inductive hypothesis, $(s_i, t_i^\circ) \in R_i^{\circ,m}$. Thus, it suffices to show that $t_i^\circ$ cautiously believes $R_i^{\circ,m}$. To this end, set

$$\beta_i(t_i) := \nu_i = (\nu_1, ..., \nu_n),$$

$$\beta_i^\circ(t_i) := \nu_i^\circ = (\mu_1, ..., \mu_n).$$

(Since $d_i^{m+1}(t_i) = d_i^{\circ,m+1}(t_i^\circ)$, LPS’s $\nu_i$ and $\nu_i^\circ$ have the same length.) For each $p \geq 1$, we let

$$\nu_i^p := (\nu_1^p, ..., \nu_n^p)$$

and

$$\nu_i^p := (\mu_1^p, ..., \mu_n^p)$$

denote the $p$-th order belief induce by $t_i$ and $t_i^\circ$, respectively.

Since $(s_i, t_i) \in R_i^{m+1} \subseteq R_i^m \cap B_i^\circ(R_i^{\circ,m})$, event $R_i^{\circ,m}$ is cautiously believed under $\beta_i(t_i) := (\nu_1, ..., \nu_n)$ at some level $k \leq m$. So, by Proposition 2 in Catonini and De Vito (2021),

$$\text{Proj}_{S_{-i}}(R_i^m) = \bigcup_{l=1}^k \text{Supp}\text{marg}_{S_{-i}} \nu_i^l.$$
By the inductive hypothesis, \( \text{Proj}_{S_i} (R^m_i) = \text{Proj}_{S_i} (R^{o,m}_{i-1}) \); moreover,

\[
\text{marg}_{S_i} \nu_i^l = \text{marg}_{S_i} \mu_i^l
\]

for all \( l = 1, \ldots, m \) (i.e., the first-order beliefs induced by \( \nu_i \) and \( \mu_i \) coincide—see again Remark 2). Hence, it is enough to show that \( \mu_i^l (R^{o,m}_{i-1}) = 1 \) for all \( l = 1, \ldots, k \).

Note that \( \rho_{i-1}^{m+1} (R^m_i) \) is not necessarily a Borel set. Yet, for all \( l = 1, \ldots, k \),

\[
\left( \nu_i^{m+1,l} \right)^* (\rho_{i-1}^{m+1} (R^m_i)) = \left( \nu_i^l \circ (\rho_{i-1}^{m+1})^{-1} \right)^* (\rho_{i-1}^{m+1} (R^m_i)) = \left( \nu_i^l \right)^* \left( (\rho_{i-1}^{m+1})^{-1} (\rho_{i-1}^{m+1} (R^m_i)) \right) \geq \left( \nu_i^l \right)^* (R^m_i) = \nu_i^l (R^m_i) = 1,
\]

where the first equality holds by definition, the second equality follows from Lemma 1, the inequality holds by a trivial fact about inverse images of functions and by monotonicity of outer measures, while the third equality holds because \( R^m_i \) is Borel.

Recall that \( \nu_i^{m+1} = \mu_i^{m+1} \), since \( d_i^{m+1} (t_i) = d_i^{o,m+1} (t_i^o) \). Hence, for all \( l = 1, \ldots, k \),

\[
\left( \nu_i^{m+1,l} \right)^* (\rho_{i-1}^{m+1} (R^m_i)) = \left( \nu_i^{m+1,l} \right)^* (\rho_{i-1}^{m+1} (R^m_i)) = 1.
\]

With this, using the fact that \( \mu_i^{m+1,l} = \mu_i^l \circ (\rho_{i-1}^{o,m+1})^{-1} \) and Lemma 1, we obtain, for all \( l = 1, \ldots, k \),

\[
1 = \left( \mu_i^l \circ (\rho_{i-1}^{o,m+1})^{-1} \right)^* (\rho_{i-1}^{m+1} (R^m_i)) = \left( \mu_i^l \right)^* \left( (\rho_{i-1}^{o,m+1})^{-1} (\rho_{i-1}^{m+1} (R^m_i)) \right).
\]

By part (ii) of the inductive hypothesis,

\[
(\rho_{i-1}^{o,m+1})^{-1} (\rho_{i-1}^{m+1} (R^m_i)) \subseteq R^{o,m}_{i-1}.
\]

Hence, for all \( l = 1, \ldots, k \),

\[
\mu_i^l (R^{o,m}_{i-1}) = \left( \mu_i^l \right)^* (R^{o,m}_{i-1}) \geq \left( \mu_i^l \right)^* \left( (\rho_{i-1}^{o,m+1})^{-1} (\rho_{i-1}^{m+1} (R^m_i)) \right) = 1,
\]

where the first equality holds because \( R^{o,m}_{i-1} \) is a Borel set, and the inequality follows from monotonicity of outer measures. This concludes the proof of the inductive step of part (i).

To prove part (ii), fix any \( (s_{i-1}, t_{i-1}^s) \in \left( (\rho_{i-1}^{o,m+2})^{-1} (\rho_{i-1}^{m+2} (R^{m+1}_{i-1})) \right) \). Then there exists \( t_{i-1} \in T_{i-1} \) such that \( (s_{i-1}, t_{i-1}) \in R^{m+1}_{i-1} \) and \( \rho_{i-1}^{o,m+2} ((s_{i-1}, t_{i-1}^s)) = \rho_{i-1}^{m+2} ((s_{i-1}, t_{i-1})) \). Hence, \( d_i^{m+1} (t_{i-1}) = d_i^{o,m+1} (t_{i-1}^s) \) and so, by part (i) (established for \( m + 1 \)), we obtain \( (s_{i-1}, t_{i-1}^s) \in R^{o,m+1}_{i-1} \). \( \blacksquare \)

Next, the following observation will be useful.
Remark 3 Fix type structures $T = (S_i, T_i, \beta_i)_{i \in I}$ and $T^0 = (S_i, T_i^0, \beta_i^0)_{i \in I}$. Suppose that there exists a hierarchy morphism $(\phi_i)_{i \in I} : T \to T^0$. Then, for each $i \in I$ and for each $E_i \subseteq S_i \times T_i,$

$$\text{Proj}_{S_i} (E_i) = \text{Proj}_{S_i} ((\text{Id}_{S_i}, \phi_i) (E_i)).$$

Proof of Theorem 1. Let $(\phi_i)_{i \in I} : T \to T^0$ (resp. $(\phi_i^0)_{i \in I} : T^0 \to T$) be a hierarchy morphism from $T$ to $T^0$ (resp. from $T^0$ to $T$). We will prove by induction on $m \geq 0$ that, for each $i \in I,$

$$(\text{Id}_{S_i}, \phi_i) (R_i^m) \subseteq R_i^0, \quad (\text{Id}_{S_i}, \phi_i^0) (R_i^0) \subseteq R_i^m.$$

With this, Remark 3 entails

$$\text{Proj}_{S_i} (R_i^m) = \text{Proj}_{S_i} ((\text{Id}_{S_i}, \phi_i) (R_i^m)) \subseteq \text{Proj}_{S_i} (R_i^0),$$

$$\text{Proj}_{S_i} (R_i^0) = \text{Proj}_{S_i} ((\text{Id}_{S_i}, \phi_i^0) (R_i^0)) \subseteq \text{Proj}_{S_i} (R_i^m),$$

hence $\text{Proj}_{S_i} (R_i^m) = \text{Proj}_{S_i} (R_i^0),$ as desired.

Basis step. The result is immediate because $R_i^0 := S_i \times T_i$ and $R_i^0 := S_i \times T_i^0$ for each $i \in I.$

Inductive step. Suppose that the result is true for $m \geq 0.$ By Remark 3, it follows that, for each $i \in I,$

$$\text{Proj}_{S_i} (R_i^m) = \text{Proj}_{S_i} (R_i^0). \quad (4.1)$$

We now show that the result is true for $m + 1.$ We will prove only that $(\text{Id}_{S_i}, \phi_i) (R_i^{m+1}) \subseteq R_i^{0,m+1},$ since the proof for $(\text{Id}_{S_i}, \phi_i^0) (R_i^{0,m+1}) \subseteq R_i^{0,m+1}$ is similar. Consider any $i \in I$ and $(s_i, t_i) \in R_i^{0,m+1}.$ By definition of hierarchy morphism, types $t_i \in T_i$ and $\phi_i (t_i) \in T_i^0$ induce the same hierarchy, in particular $d_i^{m+1} (t_i) = d_i^{0,m+1} (\phi_i (t_i)).$ Using the fact that (4.1) holds for each $i \in I,$ Lemma 2 yields $(s_i, \phi_i (t_i)) \in R_i^{0,m+1},$ as required.

5 Final remarks

Some remarks on Lemma 2 are in order. First, the proof of Lemma 2 follows the lines of the proof of Lemma D2 in Friedenberg and Keisler (2021). There are, however, some differences. Friedenberg and Keisler (2021) consider the epistemic assumptions of “rationality and common belief of rationality” within a standard (i.e., length-1 LPS’s) type structure formalism. Since belief is a monotone operator, an assumption like (b) in Lemma 2 is not needed for the proof of their result. Furthermore, Lemma D2 in Friedenberg and Keisler (2021) restricts attention to finite type structures—that is, each type set $T_i$ is a finite, discrete set. Hence, the issue of measurability for sets such as $\rho^{m+1}_{-i} (R_i^m)$ (see the inductive step in the proof of Lemma 2) does not arise in their proof. However, it can be shown (using the result in Lemma 1, as we did in the proof of Lemma 2) that their result can be extended to arbitrary, not necessarily finite type structures.

Finally, the proof of Theorem 1 can be easily adapted to show that the behavioral implications of “rationality and common belief in rationality” do not vary across hierarchy-equivalent type structures. We conjecture (but we have not proved) that an analogous result holds for the epistemic assumption of “rationality and common strong belief in rationality” (Battigalli and Siniscalchi 2002) within the framework of conditional type structures—i.e., type structures where types map to conditional probability systems. Such conjecture stems from the fact that, although formally distinct, the notions of cautious belief and strong belief have similar properties: they are both non-monotonic, and—more importantly—they satisfy a monotonicity property for events
with the same behavioral implications. Specifically, consider events $E$ and $F$ such that $E \subseteq F$ and their projections onto strategy sets are equal. It is known (see Catonini and De Vito 2021) that, in this case, cautious belief in $E$ implies cautious belief in $F$. An analogous conclusion also holds for strong belief.

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