Two-dimensional turbulence of dilute polymer solutions

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We investigate theoretically and numerically the effect of polymer additives on two-dimensional turbulence by means of a viscoelastic model. We provide compelling evidence that at vanishingly small concentrations, such that the polymers are passively transported, the probability distribution of polymer elongation has a power law tail: its slope is related to the statistics of finite-time Lyapunov exponents of the flow, in quantitative agreement with theoretical predictions. We show that at finite concentrations and sufficiently large elasticity the polymers react on the flow with manifold consequences: velocity fluctuations are drastically depleted, as observed in soap film experiments; the velocity statistics becomes strongly intermittent; the distribution of finite-time Lyapunov exponents shifts to lower values, signalling the reduction of Lagrangian chaos.

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Since the discovery of the conspicuous drag reduction obtained by dissolving minute amounts of long chain molecules in a liquid, turbulence of dilute polymer solutions has attracted a lot of attention in view of its industrial applications (see, e.g. Refs. [1, 2, 3]). The fluid mechanics of polymer solutions is appropriately described by viscoelastic models that are able to reproduce the rheological behavior and many other experimental observations [4]. For example, it has been shown by Sureshkumar et al. that the drag reduction effect can be captured by numerical simulations of the channel flow of viscoelastic fluids [5]. Although the parameters used in those simulations do not match the experimental ones, the qualitative agreement is remarkable, and all the hallmarks of the turbulent flow of polymer solutions are recovered in numerical experiments.

Following this premise, it is natural to ask whether a two-dimensional viscoelastic model can reproduce the recent results by Amarouchene and Kelley [6] showing that the turbulent flow of soap films is spectacularly affected by polymer additives (see also Refs. [7, 8]). Here we show that this is indeed the case, and that the suppression of large-scale velocity fluctuations observed experimentally has a simple theoretical explanation. However, the influence of polymers is not limited to the depletion of mean square velocity, which is a genuinely two-dimensional effect. In the viscoelastic case we observe a strong intermittency, with exponential tails of the velocity probability density. As for the Lagrangian statistics, we show that the values of finite-time Lyapunov exponents lower significantly upon polymer addition, which therefore reduces the chaoticity of the flow. These effects are expected to be independent of the space dimensionality, and thus relevant to three-dimensional turbulence as well.

We also investigate the limit of vanishingly small polymer concentrations, in which the polymer molecules have no influence on the advecting flow. In this case the velocity field evolves according the two-dimensional Navier-Stokes equation with friction, and is therefore smooth at scales smaller than the injection lengthscale [9, 10]. For passive polymers, space dimensionality plays only a minor role, and our system is an instance of a generic random smooth flow to which the theory of passive polymers developed by Chertkov [11] and Balkovsky et al. [12, 13] applies. We check this theory against our numerical results, and find an excellent quantitative agreement.

To describe the dynamics of a dilute polymer solution we adopt the linear viscoelastic model (Oldroyd-B)

\[ \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + \frac{2\eta}{\tau} \nabla \cdot \sigma - \alpha u + f \]  

(1)

\[ \partial_t \sigma + (u \cdot \nabla) \sigma = (\nabla u)^T \cdot \sigma + \sigma : (\nabla u) - \frac{2(\sigma - I)}{\tau} \]  

(2)

The velocity field \( u \) is incompressible, the symmetric matrix \( \sigma \) is the conformation tensor of polymer molecules, and its trace \( \text{tr} \sigma \) is a measure of their elongation [14]. The parameter \( \tau \) is the (slowest) polymer relaxation time. The energy source \( f \) is a large-scale random, zero-mean, statistically homogeneous and isotropic, solenoidal vector field. The pressure term \( -\nabla p \) ensures incompressibility of the velocity field. The matrix of velocity gradients is defined as \( (\nabla u)_{ij} = \partial_i u_j \) and \( I \) is the unit tensor. The solvent viscosity is denoted by \( \nu \) and \( \eta \) is the zero-shear contribution of polymers to the total solution viscosity \( \nu_t = \nu(1+\eta) \). The dissipative term \( -\alpha u \) models the mechanical friction between the soap film and the surrounding air [15], and plays a prominent role in the energy budget of Newtonian two-dimensional turbulence [16]. It should be remarked that a model that describes more accurately the polymer dynamics is the FENE-P model, which accounts for the nonlinear character of polymer elasticity, culminating in a finite molecular extensibility [4]. Although here we limit ourselves to the linear case because it allows a simpler theoretical treatment, our conclusions apply to the nonlinear case as well, provided that the maximal polymer elongation is very large compared to the equilibrium length.
Passive polymers. The effect of polymer concentration \( n \) is included in Eq. (1) through the parameter \( \eta \propto n \). In the limit \( \eta = 0 \) polymers are passively transported and stretched by Newtonian two-dimensional turbulence. The flow is driven at the largest scales and develops an enstrophy cascade towards the small scales, while the inverse energy flux is immediately halted by friction. The ensuing velocity field is therefore everywhere smooth. We briefly recall that according to Refs.\[11, 12, 13\] the stationary statistics and to develop a consistent theory of linear passive theory. Accounting for the nonlinear elastic modulus of polymer molecules allows to recover a stationary statistics and to develop a consistent theory of passive polymers at all Weissenberg numbers \[20\]. In the following we do not pursue that approach, but we rather focus on a different mechanism that limits polymer elongation: the feedback of polymers on the advecting flow.

Active polymers. When \( \eta > 0 \), polymers can affect significantly the velocity dynamics, provided that they are sufficiently elongated — a condition that is met at \( Wi > 1 \). This strong feedback regime is characterized in two dimensions by a suppression of large-scale velocity fluctuations (see Fig. 2), an effect first observed in soap film experiments \[6\]. In Fig. 3 we present the time evolution of the total kinetic energy of the system, showing that after polymer injection a drastic depletion of kinetic energy occurs. This should be contrasted with...
The three-dimensional case where, on the opposite, velocity fluctuations are larger in the viscoelastic case than in the Newtonian one [22].

\[ F = \epsilon + \frac{2\eta}{\tau^2} \left( \langle \text{tr}\sigma \rangle - \text{tr}1 \right) + \alpha \langle |u|^2 \rangle \]

where \( \epsilon = \nu \langle |\nabla u|^2 \rangle \) is the viscous dissipation and \( F \) is the average energy input, which is flow-independent for a Gaussian, \( \delta \)-correlated random forcing \( f \). To obtain eq. (4) we multiply eq. (1) by \( u \), add to it the trace of eq. (2) times \( \eta \nu / \tau \), and average over space and time. Since in two dimensions kinetic energy flows towards large scales, it is mainly drained by friction, and viscous dissipation is vanishingly small in the limit of very large Reynolds numbers [16]. Neglecting \( \epsilon \) and observing that in the Newtonian case \( (\eta = 0) \) the balance (4) yields

\[ F = \alpha \langle |u|^2 \rangle_{N}, \]

we obtain

\[ \langle |u|^2 \rangle = \langle |u|^2 \rangle_{N} - \frac{2\eta}{\alpha \tau^2} \left( \langle \text{tr}\sigma \rangle - \text{tr}1 \right). \]

Since, as a consequence of incompressibility and chaoticity of the flow, it can be shown from eq. (2) that \( \text{tr}\sigma \geq \text{tr}1 \), we finally have \( \langle |u|^2 \rangle \leq \langle |u|^2 \rangle_{N} \), in agreement with numerical results. This simple energy balance argument can be generalized to nonlinear elastic models. As viscosity tends to zero, the average polymer elongation increases so as to compensate for the factor \( \nu \) in eq. (5), resulting in a finite effect also in the infinite \( Re \) limit. Since energy is essentially dissipated by linear friction, the depletion of \( \langle |u|^2 \rangle \) entails immediately the reduction of energy dissipation. The main difference between two-dimensional “friction reduction” and three-dimensional drag reduction resides in the lengthscales involved in the energy drain – large scales in 2D vs small scales in 3D.

The effect of polymer additives cannot be merely represented by a rescaling of velocity fluctuations by a given factor. In Fig. 4 we show the probability distribution of a velocity component, \( u_x \). The choice of the \( x \) direction is immaterial by virtue of statistical isotropy. In the Newtonian case the distribution is remarkably close to the subgaussian density \( F \exp(-c|u|^3) \) stemming from the balance between forcing and nonlinear terms in the Navier-Stokes equation, in agreement with the prediction by Falkovich and Lebedev [23]. On the contrary, the distribution in the viscoelastic case is markedly supergaussian, with approximately exponential tails. This strong intermittency in the velocity dynamics is due to the alternation of quiescent low-velocity phases ruled by polymer feedback and bursting events where inertial nonlinearities take over.

![FIG. 3: Dilute polymers reduce the level of velocity fluctuations \( \int |u(x,t)|^2 \, dx \). Polymers are introduced in the flow at \( t = 0 \). In the inset, the mean square elongation \( \int \text{tr}\sigma(x,t) \, dx \) as a function of time.](image)

![FIG. 4: Intermittency of velocity fluctuations induced by polymer additives. The probability density function \( P(u_x) \) of the velocity component \( u_x \) for the Newtonian (continuous line) and for the viscoelastic case with strong feedback (dashed line). Same parameters as in Fig. 2. Also shown the distribution \( F(2/3)3^{3/2} \exp(-c|u|^3)/(4\pi c) \) with \( c = 2.1 \cdot 10^{-3} \) (dotted line).](image)
for all $n$, a result which guarantees the stationarity of the statistics of $\text{tr} \boldsymbol{\sigma}$ in presence of feedback, while imposing a less restrictive condition on $S(\gamma)$ than the one proposed in Ref. [13].

FIG. 5: Finite-time Lyapunov exponents decrease in presence of polymers. The Cramér rate function $S(\gamma)$ for the Newtonian (continuous line) and for the viscoelastic case with strong feedback ($Wi = \lambda_N \tau = 1.6$, dashed line). Viscosity $\nu = 6 \cdot 10^{-3}$, relaxation time $\tau = 2$, $\eta = 0.2$ (dashed), $\eta = 2$ (dotted). For sake of completeness, we also show $S(\gamma)$ for a mild feedback case ($Wi = 0.4$, $\eta = 0.2$, dash-dotted line). In the latter case, the Lyapunov exponent is practically identical to the Newtonian value, and polymers affect only the right tail of $S(\gamma)$ reducing appreciably the probability of large stretching events $\gamma \gg \lambda_N$.

Finally, we discuss the influence of polymer concentration on the properties of the flow. As shown in Fig. 5, the Cramér functions at two very different values of $\eta$ are practically indistinguishable. The level of velocity fluctuations (not shown) appears to be independent of concentration as well. This property follows from the viscoelastic equations assuming that the term $\propto 1/\tau$ in eq. (2) can be neglected if polymers are substantially stretched. In that case the dynamics of the conformation tensor is invariant under rescaling by a constant factor, allowing to absorb $\eta$ in the definition of $\boldsymbol{\sigma}$, and making the velocity dynamics independent of concentration. This observation poses an interesting question: can there be a concentration-dependent onset of friction or drag reduction within a linear viscoelastic model? Since polymers are increasingly stretched as $\eta \to 0$ there are two alternatives: either this is a singular limit, and the passive case is not recovered but for $\eta$ strictly equal to zero, or the feedback is not uniquely ruled by polymer elongation, and there are other relevant mechanisms for polymer activity, e.g. via the creation of strong gradients of the conformation tensor. The limit of vanishingly small yet finite $\eta$ is very demanding at the computational level and its investigation will require further numerical work.

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