Dynamical Wormholes and Energy Conditions

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Abstract

A class of exact solutions of the Einstein field equations representing non-static wormholes that obey the weak and dominant energy conditions is presented. Hence, in principle, these wormholes can be built with less exotic matter than the static ones.

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The idea of wormholes can be traced back as early as to 1916, when Flamm [1] studied the just published Schwarzschild solution and found that its embedding indicates a nontrivial topology. This concept was further developed by Weyl [2], Einstein and Rosen [3], and Wheeler [4]. However, it did not receive much attention until recently when Morris and Thorne [5] pointed out that advanced civilizations might be able to build a wormhole and to use it in interstellar travels, and even more to convert it into a time machine and travel back in time. One of the main obstacles to build such a wormhole is that the matter needed necessarily violates the weak energy condition (WEC) [5 - 7].

In this Letter, we present a class of solutions of the Einstein field equations, which represents non-static wormholes. These solutions are obtained by the so-called “cut and paste” approach commonly used in the recent studies of bubbles [8] as well as wormholes [6, 9]. However, our solutions are differentiated from the already known ones due to the striking feature that they represent wormholes not necessarily built with “exotic” matter. In fact, they satisfy both of the weak and dominant energy conditions. In this sense they are sharing the same properties as those studied by the present authors in the context of the Brans-Dicke theory of gravity [10]. Therefore, to construct a wormhole for interstellar and time travels now seems less forbidden than before.
Before proceeding, we would like to note that the results presented here do not invalidate the early conclusions of Morris and Thorne [5] about the static wormholes but rather show the fact that the properties of dynamic wormholes are quite different from the static ones. Dynamic wormholes that satisfy the WEC have been also studied recently in [11, 12]. However, the ones presented here are different from those given in the above citations and were briefly reported in [13].

The rest of this Letter is organized as follows: Using the “cut and paste” approach, we first construct a class of solutions of the Einstein field equations, which represents a spherically symmetric non-static bubble. Then, we analyze these solutions and show that the bubble acts as the throat of a wormhole which connects two asymptotically flat universes without horizons and singularities. Finally, we show that such constructed bubbles satisfy the weak and dominant (but not strong) energy conditions [14].

To begin with, let us consider the Schwarzschild solution of the Einstein vacuum equations

\[ ds^2 = f dT^2 - f^{-1} dR^2 - R^2 d^2 \Omega, \]  

(1)

where the function \( f \) is defined as usual \( f \equiv 1 - 2m/R \), and \( d^2 \Omega \equiv d^2 \theta + \sin^2 \theta d^2 \phi \). The coordinates have the ranges \(-\infty < T < +\infty, 0 \leq R < +\infty, 0 \leq \theta \leq \pi, \) and \( 0 \leq \phi \leq 2\pi \). Following Ref. [10], we perform the
following coordinate transformations

\[ T = M(t - \psi) + N(t + \psi), \quad R = U(t - \psi) + V(t + \psi), \quad \text{for } \psi \geq 0 \]  

(2a)

\[ T = M(t + \psi) + N(t - \psi), \quad R = U(t + \psi) + V(t - \psi), \quad \text{for } \psi \leq 0 \]  

(2b)

where \( M, N, U \) and \( V \) are functions of the class \( C^4 \) in their indicated arguments in the sense defined in [14], and \( t \) and \( \psi \) are new coordinates with \(-\infty < t, \psi < +\infty\). In terms of \( t \) and \( \psi \), the metric (1) becomes

\[ ds^2 = F dt^2 + 2G dt d\psi - H d\psi^2 - R^2 d\Omega, \]  

(3)

where \( F \equiv f T^2_t - f^{-1} R^2_t, \ G \equiv f T_t T_\psi - f^{-1} R_t R_\psi, \) and \( H \equiv f^{-1} R_\psi^2 - f T_\psi^2. \) From Eqs. (2) one can see that in each of the two regions, \( \psi \geq 0 \) and \( \psi \leq 0 \), the metric (3) is locally isometric to the one given by Eq. (1). In other words, they are related to each other by the well-defined coordinate transformations (2a) and (2b). However, across the hypersurface \( \psi = 0 \), the functions \( T \) and \( R \) given by Eqs. (2) are only continuous functions of \( \psi \) (class \( C^0 \)). Consequently, the metric coefficients \( F, G, \) and \( H \) are discontinuous functions of \( \psi \) across \( \psi = 0 \). In order that the Einstein equations hold in the sense of distributions [15] the metric coefficients need to be at least \( C^0 \) across the hypersurface. It is easy to show that the condition

\[ \dot{M}^2 - \dot{N}^2 = \frac{(U + V)^2}{(U + V - 2m)^2}(\dot{U}^2 - \dot{V}^2), \quad (\psi = 0), \]  

(4)
where an overdot denotes the ordinary differentiation with respect to $t$, guarantees the continuity of the metric coefficients across $\psi = 0$.

For our purpose, we find sufficient to consider only the special solutions of Eq. (4) with $\dot{M}^2 - \dot{N}^2 = \mu$, $M = AN + B$, and $U = aV + b$, where $\mu, A, B, a$ and $b$ are arbitrary constants. When $\mu = 0$, the solutions reduce to the ones considered by Visser [9], which represent static wormholes and therefore belong to the class studied by Morris and Thorne [5]. The weak energy condition is violated in those solutions. In the rest of the Letter we shall focus our attention on the solutions with $\mu \neq 0$. The integration of Eq. (4) yields

$$
\begin{align*}
N(t) &= \varepsilon_1 \mu_1 t + N_0, \\
\varepsilon_2 \mu_2 t &= V(t) + \frac{2m}{1 + a} \ln(\hat{R}(t) - 2m) + V_0,
\end{align*}
$$

(5)

where $N_0$ and $V_0$ are integration constants, $\mu_1 \equiv \sqrt{\mu/(A^2 - 1)}$, $\mu_2 \equiv \sqrt{\mu/(a^2 - 1)}$, $\varepsilon_1, \varepsilon_2 = \pm 1$, and $\hat{R}(t) \equiv R(t, \psi = 0) = (1 + a)V(t) + b$. Once Eq. (4) is solved, the metric coefficients of (3) are in turn fixed in terms of $t$ and $\psi$. As mentioned previously, such solutions are isometric to the Schwarzschild metric (1) in the regions $\psi \geq 0$ and $\psi \leq 0$, respectively. Therefore, in these two regions the energy-stress tensor vanishes. For metrics of the class $C^0$ the energy-stress tensor is distribution valued [15] across the
hypersurface $\psi = 0$. In the present case we find

$$T_{\mu\nu} = \tau_{\mu\nu} \delta(\psi) = \{\sigma u_\mu u_\nu - \tau (\theta_\mu \theta_\nu + \phi_\mu \phi_\nu)\} \delta(\psi),$$  \hspace{1cm} (6)$$

where $\delta(\psi)$ denotes the usual Dirac delta function, $u_\mu = \sqrt{F} \delta_{\mu t}$, $\theta_\mu = \hat{R}(t) \delta_{\mu \theta}$, and $\phi_\mu = \hat{R}(t) \sin \theta \delta_{\mu \phi}$. The function $\sigma$ denotes the surface energy density of the bubble and $\tau$ the tensions in the tangential directions measured by an observer with the four-velocity $u_\mu$, and are given respectively by

$$\sigma = \frac{\sigma_0}{\hat{R}(t)}, \quad \tau = \frac{\sigma_0 (\hat{R}(t) - m)}{2 \hat{R}(t) (\hat{R}(t) - 2m)},$$  \hspace{1cm} (7)$$

where $\sigma_0 = 2 \varepsilon_2 (1 + A)/[\mu_2 (a - A)]$ is a constant [16]. We conclude that our solutions represent a spherical bubble connecting two regions, each of which is locally isometric to part of the Schwarzschild space-time. Since the radius of the bubble $\hat{R}(t)$ is time-dependent, the bubble in the present case is not static. From Eq. (5) we see that $\hat{R}(t) \geq 2m$. That is, the bubble is always greater than or equal to the Schwarzschild sphere, where the equality holds only at $t = +\infty$. In order that the hypersurface $\psi = 0$ be non-spacelike, we need to impose the condition $\mu(a - A)(1 + a)(1 + A) > 0$.

The dynamics of the bubble can be studied using the kinematical quantities

$$\hat{R}' = \beta \left( \frac{\hat{R}(t) - 2m}{\hat{R}(t)} \right)^{1/2}, \quad \hat{R}'' = m \beta^2 \frac{1}{\hat{R}^2(t)};$$  \hspace{1cm} (8)$$

where $\beta \equiv \varepsilon_2 \mu_2 (1 + a) \sqrt{(1 - a)(1 - A)/[2 \mu(a - A)]}$, and a prime denotes differentiation with respect to the proper time measured by the observer
who is at rest relative to the bubble. The above equations show that in the present case the bubble either expands ($\beta > 0$) or collapses ($\beta < 0$), depending on the choice of the free parameters. Now we shall consider the following representative cases:

(a) $\varepsilon_2 = -1, \ 0 < a < 1, \ 0 < A < 1, \ A > a, \ b = 2m, \text{ and } V_0 = 0$. Then, we find

\[
\begin{align*}
\text{Exp}(-\mu_2 t) &= [(1 + a)V(t)]^{\frac{2m}{\mu_2}}\text{Exp}[V(t)], \\
R(t, |\psi|) &= aV(t - |\psi|) + V(t + |\psi|) + 2m, \\
\dot{R}(t) &= (1 + a)V(t) + 2m. \quad (9)
\end{align*}
\]

From the above equation, it is easy to show that at any moment, say, $t = t_1$, we always have $R(t, |\psi|) \to +\infty$ as $\psi \to \pm\infty$, and that $R(t, |\psi|) \geq 2m$ for any $t$ and $\psi$, where the equality holds only when $t = +\infty$. Thus, in the present case the bubble acts as the throat of a wormhole that connects two asymptotically flat Schwarzschild universes. Note that in [9] non-static wormholes were also studied. However, our solutions are different from those. This can be seen clearly by the following considerations. From Eqs.(4), (5) and (9) we find that $[\partial R(t, |\psi|)/\partial |\psi|]|_{\psi=0} = -(1 - a)\mu_2 f < 0$. Therefore, in the present case the radius of the wall is initially decreasing as away from it. Then, according to the results obtained in Ref. 8, the surface energy density of the bubble is positive [cf. Eq.(7)]. When the radial coordinate reaches its
minimum, say, $R_{\text{min}}$, which is always greater than or equal to $2m$ (Again, the equality holds only when $t = +\infty$), it starts to increase. As $|\psi| \to +\infty$, we have $R(t, |\psi|) \to +\infty$. In [9], the case with $[\partial R(t, |\psi|)/\partial |\psi||_{\psi=0} > 0$ was studied. As a result, the surface energy density is always negative and violates all the energy conditions [8, 9]. It should be noted that usually [17] the signs of the angular component of the extrinsic curvature tensor of the wall were taken as the criterion to classify the spatial topology of the spacetimes. This is correct for the static case. However, when the spacetime is time-dependent, it is true only in the neighborhood of the bubble. The global topology of the spacetime could be completely different. This fact has been noticed in [18] quite recently and the present solutions provide another example.

On the other hand, from Eq.(9) we find that $\hat{R}(t) \to +\infty$ as $t \to -\infty$, and $\hat{R}(t) \to 2m$ as $t \to +\infty$. That is, the corresponding solutions represent collapsing wormholes. The wormhole throat starts to collapse at $\hat{R}(t) = \infty$ and ends at $\hat{R}(t) = 2m$. From Eq. (8) one can show that to complete this process, the throat takes an infinite proper time. Consequently, a space adventurer will have enough time to pass through the throat of the wormhole from one asymptotically flat region to the other before the radius of the throat shrinks to $2m$, where the event horizon is developed. A distinguish feature of this wormhole is that it consists of matter that satisfies the weak and
dominant energy conditions [14] as long as $\dot{R}(t) \geq 3m$. The latter can be seen clearly from Eq. (7). Note that, although these two energy conditions are satisfied, the strong one is not, and the throat of the wormhole is still gravitational repulsive, since the “Newtonian mass” of the throat (that is proportional to $(\sigma - 2\tau)$) is less than or equal to zero. The equality holds only at the initial point $t = -\infty$. The repulsive character is needed in order to keep the throat open [5].

As mentioned before, the above results do not contradict to the ones presented in [5]. To show this, let us consider the embedding of our solutions in the three dimensional Euclidean space $ds^2 = dZ^2 + dR^2 + R^2 d\phi^2$. Following Ref. [5], we find $(dZ/d\psi)^2 = g(t, \psi)$, where $g(t, \psi)$ changes signs from point to point. That is, in the present case our solutions can not be embedded in a 3-dimensional Euclidean space and pictured as an ordinary Euclidean curved surface. From Ref. [5] we see that the violation of the weak energy condition is tightly related to such a three dimensional embedding. Recall that not any two dimensional metric can be embedded into a three dimensional Euclidean space. Classical examples are the Moebius strip and the Gauss-Bólyai-Lobachevski metric $ds^2 = (1 + r^2)^{-1} dr^2 + r^2 d\phi^2$ [19].

(b) $\varepsilon_2 = 1, \; a > 1, \; A > 1, \; a > A, \; b = 2m, \; \text{and} \; V_0 = 0$. We find that $Exp(\mu_2 t) = [(1 + a)V(t)]^{\frac{2m}{\mu_2}} Exp[V(t)]$, and that $R$ and $\dot{R}(t)$ are still given by Eq. (9). As in the previous case, one can show that these solutions represent
expanding bubbles, which connects two asymptotically flat universes. The bubble expands from $\hat{R}(t) = 2m$ to $\hat{R}(t) = +\infty$ by taking infinite proper time. Again, when $\hat{R}(t) \geq 3m$, the bubble satisfies the weak and dominant energy conditions.

From Eq. (7) we can see that when $\hat{R}(t)$ is approaching $2m$, the tensions in the tangent directions of the bubble tend to infinite. Thus, in the course of the collapse of the bubble, as described in the first case, it is not difficult to imagine that the bubble will explode due to the enormous tensions before its radius shrinks to $2m$. By properly arranging the parameters, the explosion could happen as early as wanted. After the explosion, the material may recompose and form another wormhole, the later evolution of which follows more or less the same process as described by the solutions of Case (b).

In this Letter we have shown that traversable wormholes can be built out of matter that satisfies the weak and dominant energy conditions. The violation of the strong energy condition nowadays does not seem to be a very serious drawback (Recall that cosmic bubbles and domain walls formed in the early Universe do not satisfy this condition either).

The recent studies of wormholes usually fall into two different directions. One is concerned with the energy conditions [7], and the other is concerned with the vacuum polarization due to the quantum effects [20, 21]. To the first, one can see that even it can be shown that the WEC is preserved at
the quantum level for the generic cases, the existence of wormholes can not be ruled out. As shown above, they can exist even in the classical level without violating the WEC. To the second, Hawking [20] and Visser [21] argued that, when the vacuum polarization effects are taken into account, one might finally show that such a building of a traversable wormhole is impossible, although Thorne and others [22] seem to defend the opposite opinion. The considerations of the latter are out of scope of this Letter and will be discussed somewhere else.

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References

[1] L. Flamm, Physik Z. 17, 448 (1916).

[2] H. Weyl, Philosophy of Mathematics and Natural Science (Princeton University Press, Princeton, 1949).
[3] A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).

[4] J.A. Wheeler, *Geometrodynamics* (Academic, New York, 1962).

[5] M.S. Morris and Kip S. Thorne, Am. J. Phys. 56, 395 (1988).

[6] M.S. Morris, Kip S. Thorne, and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988); V.P. Flolov and I.D. Novikov, Phys. Rev. D42, 1057 (1990); J. Friedman, et al, ibid 42, 1915 (1990).

[7] U. Yurtsever, Class. Quantum Grav. 7, L251 (1990); J. Math. Phys. 31, 3064 (1990); Class. Quantum Grav. 8, 1127 (1991); G. Klinkrammer, Phys. Rev. D43, 2542 (1991); R.M. Wald and U. Yurtsever, Phys. Rev. D44, 403 (1991).

[8] H. Sato, Prog. Theor. Phys. 76, 1250 (1986); P. Laguna-Castillo and R.A. Matzner, Phys. Rev. D34, 2913 (1986); S.K. Blau, E.I. Guendelman, and A.H. Guth, 35, 1747 (1987); J. Ipser, 36, 1933 (1987); V.A. Berezin, V.A. Kuzmin, and I.I. Tkachev, 36, 2919 (1987); C. Barrabès and W. Israel, 43, 1129 (1991).

[9] M. Visser, Nucl. Phys. B328, 203 (1989); Phys. Rev. D39, 3182 (1989); ibid. 41, 1116 (1990).

[10] P.S. Letelier and A.Z. Wang, Phys. Rev. D48, 631 (1993).
[11] A. Ori, Phys. Rev. Lett. 71, 2517 (1993); A. Ori and Y. Soen, Phys. Rev. D49, 3990 (1994).

[12] S. Kar, Phys. Rev. D49, 862 (1994).

[13] P.S. Letelier and A.Z. Wang, in *Gravitation: the Spacetime Structure*, SILARG VIII, July 25 – 30, 1993, Aguas de Lindoia, Brazil, edited by P.S. Letelier and W.A. Rodrigues Jr. (World Scientific, Singapore, 1994), pp320–324.

[14] S.W. Hawking and G.T.R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973).

[15] W. Israel, Nuovo Cimento, 44B, 1 (1966); A.H. Taub, J. Math. Phys. 21, 1423 (1980); R. Geroch and J. Traschen, Phys. Rev. D36, 1017 (1987).

[16] The surface energy-momentum tensor of the wall is usually defined as

$$S_{\mu\nu} \equiv \int T_{\mu\nu} dn = \int \tau_{\mu\nu} \delta(\psi) H^{1/2} d\psi = H^{1/2} \tau_{\mu\nu},$$

where $n$ is the proper distance in the direction perpendicular to the wall. Thus, the surface energy density and tensions of the wall defined here are different from the ones in [8] by a factor $H^{1/2}|_{\psi=0} = \{2\mu(a-A)f(\hat{R})/[(A+1)(a+1)]\}^{1/2}$. Taking this fact into account, one can easily show that the conservation of the stress-energy of the wall $d(H^{1/2}\sigma \hat{R})/d\hat{R} = H^{1/2}(2\tau - \sigma)$ is satisfied, as it should be.
[17] For example, see J. Ipser and P. Sikivie, Phys. Rev. D30, 712 (1984); J. Ipser, Phys. Rev. D30, 2452 (1984); 36, 1933 (1987).

[18] N. Sakai and K. Maeda, Phys. Rev. D50, 5425 (1994).

[19] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).

[20] S.W. Hawking, Phys. Rev. D46, 603 (1992).

[21] M. Visser, Phys. Rev. D47, 554 (1993).

[22] S.W. Kim and Kip S. Thorne, Phys. Rev. D43, 3929 (1991).