Parameter estimation for SDEs related to stationary Gaussian processes

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Abstract: In this paper, we study central and non-central limit theorems for partial sum of functionals of general stationary Gaussian fields. We apply our result to study drift parameter estimation problems for some stochastic differential equations related to stationary Gaussian processes.

Key words: Central and non-central limit theorems; stationary Gaussian processes; Stein’s method; drift parameter estimation; fractional Gaussian processes.

1 Introduction

While the statistical inference of Itô type diffusions has a long history, the statistical analysis for equations driven by fractional Brownian motion (fBm) is obviously more recent. The development of stochastic calculus with respect to the fBm allowed to study such models. We will recall several approaches to estimate the parameters in fractional models but we mention that the below list is not exhaustive:

- The MLE approach in \cite{17}, \cite{26}. In general the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In general, the MLE is not easily computable.

- A least squares approach has been proposed in \cite{14}. The study of the asymptotic properties of the estimator is based on certain criteria formulated in terms of the Malliavin calculus (see \cite{23}). In the ergodic case, the statistical inference for several fractional Ornstein-Uhlenbeck (fOU) models has been recently developed in the papers \cite{14}, \cite{2}, \cite{3}, \cite{13}, \cite{15}, \cite{7}. The case of non-ergodic fOU process of the first kind and of the second kind can be found in \cite{7} and \cite{12} respectively.

Our aim is to bring a new idea to develop the statistical inference for stochastic differential equations related to stationary Gaussian processes by proposing a suitable criteria. Our approach is based on Malliavin calculus and it makes in principle our estimator easier to be simulated. Moreover, the models studied in \cite{14}, \cite{2}, \cite{3}, \cite{13} become particular cases in our approach (see section 4).

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2 Elements of Gaussian analysis and Malliavin calculus

For the essential elements of Gaussian analysis and Malliavin calculus that are used in this paper see the references ([24], [23]).

Now recall that, if $X, Y$ are two real-valued random variables, then the total variation distance between the law of $X$ and the law of $Y$ is given by

$$d_{TV}(X, Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P[X \in A] - P[Y \in A]|.$$  

If $X, Y$ are two real-valued integrable random variables, then the Wasserstein distance between the law of $X$ and the law of $Y$ is given by

$$d_W(X, Y) = \sup_{f \in \text{Lip}(1)} |Ef(X) - Ef(Y)|$$

where $\text{Lip}(1)$ indicates the collection of all Lipschitz functions with Lipschitz constant $\leq 1$.

The following well-known direct consequence of the Borel-Cantelli Lemma (see e.g. [18]), will allows us to turn convergence rates in the $p$-th mean into pathwise convergence rates.

**Lemma 1** Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables. If for every $p \geq p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,

$$(E|Z_n|^p)^{1/p} \leq c_p \cdot n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable $\eta_\varepsilon$ such that

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\gamma+\varepsilon} \quad \text{almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $E|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.

3 General context

3.1 Central and non-central limit central theorems

Consider a centered stationary Gaussian process $Z = (Z_k)_{k \in \mathbb{Z}}$ with covariance

$$r_Z(k) := E(Z_0Z_k)$$

and $|r_Z(k)| \leq 1$ for every $k \in \mathbb{Z}$.

Let us define the following normalized process

$$Y_k := \frac{Z_k}{\sqrt{r_Z(0)}}, \quad k \in \mathbb{Z}. \quad (1)$$

We also define

$$V_{q,n}(Z) = \sum_{k=1}^{q/2} I_{2k}(f_{2k,n}) \quad (2)$$
where

\[ f_{2k,n} := \frac{d_{q,2k}(Z)}{\sqrt{n}} \sum_{i=0}^{n-1} \varepsilon_i^{\otimes 2k} \]

with \( Y_i = Y(\varepsilon_i) \) and \( d_{q,2k}(Z) \) are constants.

If \( \sum_{k \in \mathbb{Z}} |r_Y(k)|^2 < \infty \), we define

\[ v_q(Z) := \lim_{n \to \infty} E[V_{q,n}^2(Z)]. \]

We have the following central and non-central limit theorems.

**Theorem 2** Let \((V_{2,n}(Z))_{n \geq 0}\) be the sequence defined in (2), we let \( Y_k = \frac{Z_k}{\sqrt{r_Z(0)}} \), \( k \in \mathbb{Z} \) with covariance \( r_Y(k) = E[Y_0Y_k] \). Denote \( N \sim N(0,1) \).

1) Then there exists a constant \( C > 0 \) depending on \( q \) and \( r_Z(0) \) such that,

\[
\frac{d_{TV} \left( \frac{V_{q,n}(Z)}{\sqrt{E[V_{q,n}^2(Z)]}}, N \right)}{C} \leq \frac{C}{E[V_{2,n}^2(Y)]} \sqrt{E[V_{2,n}^2(Y)]} \sqrt{\kappa_4(V_{2,n}(Y))) + \kappa_4(V_{2,n}(Y))}. \tag{3}
\]

We also have in the case when, \( \sum_{k \in \mathbb{Z}} |r_Y(k)|^2 < \infty \),

\[
\frac{d_{TV} \left( \frac{V_{q,n}(Z)}{v_q(Z)}, N \right)}{C} \leq \frac{C}{v_q(Z)} \left[ \sqrt{E[V_{2,n}^2(Y)]} \sqrt{\kappa_4(V_{2,n}(Y))) + \kappa_4(V_{2,n}(Y)) + |v_q(Z) - E[V_{2,n}^2(Y)]|} \right]. \tag{4}
\]

In addition:

- If \( \sum_{k \in \mathbb{Z}} (r_Y(k))^2 \leq \infty \),

\[
\kappa_4(V_{2,n}(Y)) \leq Cn^{-1} \left( \sum_{|k|<n} (r_Y(k))^4 \right)^{3/4}. \tag{5}
\]

- If \( r_Y(k) = k^{-\frac{1}{2}} \),

\[
\frac{\kappa_4(V_{2,n}(Y))}{\left( E[V_{2,n}^2(Y)] \right)^2} \leq \frac{C}{\log(n)}. \tag{6}
\]

where \( E[V_{2,n}^2(Z)] \sim 4d_{q,2}(Z) \log(n) \).

2) Assume that \( r_Y(k) = |k|^{-\alpha} \) with \( 0 < \alpha < \frac{1}{2} \). Then

\[
\frac{V_{q,n}(Z)}{n^{2-\alpha}} \xrightarrow{law} \frac{d_{q,2}(Z)}{\sqrt{D}} \mathbb{F}_\infty. \tag{7}
\]
where \( D = \int_{-\infty}^{\infty} e^{iy} |y|^{\alpha-1} \, dy = 2\Gamma(\alpha) \cos(\alpha\pi/2) \), and
\[
F_{\infty} = \int \int_{\mathbb{R}^2} e^{i(x+y)} e^{i(x+y) - \frac{1}{i(x+y)}} |xy|^{\alpha-1} W(dx)W(dy)
\]
with \( W \) is a random spectral measure of the white noise on \( \mathbb{R} \).

**Proof.** Since \( \frac{V_{q,n}(Z)}{\sqrt{E[V_{q,n}^2(Z)]}} \in \mathbb{D}^{1,2} \), by [21, Proposition 2.4] we have
\[
\left| d_{TV} \left( \frac{V_{q,n}(Z)}{\sqrt{E[V_{q,n}^2(Z)]}}, N \right) \right| \leq 2E \left| 1 - \left\langle \frac{D}{\sqrt{E[V_{q,n}^2(Z)]}}, -DL^{-1}\frac{V_{q,n}(Z)}{\sqrt{E[V_{q,n}^2(Z)]}} \right\rangle_{\mathcal{H}} \right|
\]
On the other hand, exploiting the fact that
\[
E \left[ E \left[ (I_{2k}(f_{2k,n}))^2 \right] - \langle DI_{2k}(f_{2k,n}), -DL^{-1}I_{2k}(f_{2k,n}) \rangle_{\mathcal{H}} \right] = 0
\]
we obtain
\[
E \left| 1 - \left\langle \frac{D}{\sqrt{E[V_{q,n}^2(Z)]}}, -DL^{-1}\frac{V_{q,n}(Z)}{\sqrt{E[V_{q,n}^2(Z)]}} \right\rangle_{\mathcal{H}} \right|
\]
\[
\leq \frac{1}{E[V_{q,n}^2(Z)]} \left( \sum_{k=1}^{\alpha/2} \sqrt{Var \left( (2k)^{-1} \| DI_{2k}(f_{2k,n}) \|_{\mathcal{H}}^2 \right) } + \sum_{1 \leq k < t \leq q/2} (2l)^{-1}E \left| \langle DI_{2k}(f_{2k,n}), DI_{2l}(f_{2l,n}) \rangle_{\mathcal{H}} \right| \right)
\]
Moreover, by [22, Lemma 3.1] we have
\[
\sqrt{Var \left( (2k)^{-1} \| DI_{2k}(f_{2k,n}) \|_{\mathcal{H}}^2 \right) } = (2k)^{-2} \sum_{j=1}^{2k-1} j^2 \left( \frac{2k}{j} \right)^4 (4k - 2j)! \| f_{2k,n} \otimes f_{2k,n} \|_{\mathcal{H}^{4k-2j}}^2,
\]
and for \( k < l \)
\[
E \left[ \left( (2l)^{-1} \langle DI_{2k}(f_{2k,n}), DI_{2l}(f_{2l,n}) \rangle_{\mathcal{H}} \right)^2 \right] \leq (2k)! \left( \frac{2l}{2k-1} \right)^2 (2l - 2k)!E \left[ (I_{2k}(f_{2k,n}))^2 \right] \| f_{2k,n} \otimes f_{2k,n} \|_{\mathcal{H}^{4k}}^2
\]
\[+ 2k^2 \sum_{j=1}^{2k-1} (l - 1)^2 \left( \frac{2k-1}{j-1} \right)^2 \left( \frac{2l-1}{j-1} \right)^2 (2k + 2l - 2j)! \| f_{2k,n} \otimes f_{2k,n} \|_{\mathcal{H}^{2j}}^2 + \| f_{2k,n} \otimes f_{2k,n} \|_{\mathcal{H}^{2j}}^2 \right).
\]
Combining this together with the fact that for every \( 1 \leq s \leq 2k - 1 \) with \( k \in \{1, \ldots, q/2\} \)
\[
\| f_{2k,n} \otimes f_{2k,n} \|_{\mathcal{H}^{4k-2s}}^2 \leq d_{q,2k}(Z)n^{-2} \sum_{k_1,k_2,k_3,k_4=1} r_{Y}(k_1-k_2)r_{Y}(k_3-k_4)r_{Y}^{2k-s}(k_1-k_3)r_{Y}^{2k-s}(k_2-k_4)
\]
\[
\leq d_{q,2k}(Z)n^{-2} \sum_{k_1,k_2,k_3,k_4=1} r_{Y}(k_1-k_2)r_{Y}(k_3-k_4)r_{Y}(k_1-k_3)r_{Y}(k_2-k_4)
\]
\[
= d_{q,2k}(Z)\kappa_4(V_{2,n}(Y))
\]
we deduce that there exists a constant $C$ depending on $q$ and $r_Z(0)$ such that

$$\sqrt{\text{Var} \left( (2k)^{-1} \| DI_{2k}(f_{2k,n}) \|_{H}^2 \right)} \leq C \sqrt{\kappa_4(V_{2,n}(Y))}$$

and

$$(2l)^{-1} E \left| \langle DI_{2k}(f_{2k,n}), DI_{2l}(f_{2l,n}) \rangle_{H} \right| \leq \left( E \left[ (2l)^{-1} \| DI_{2k}(f_{2k,n}), DI_{2l}(f_{2l,n}) \|_{H}^2 \right]^2 \right)^{1/2} \leq C \sqrt{E \left[ (I_{2k}(f_{2k,n}))^2 \right] \sqrt{\kappa_4(V_{2,n}(Y)) + \kappa_4(V_{2,n}(Y))}} \leq C \sqrt{\sum_{i,j=0}^{n-1} r_Y^2(i-j) \sqrt{\kappa_4(V_{2,n}(Y)) + \kappa_4(V_{2,n}(Y))}}.$$

Furthermore,

$$d_{TV} \left( \frac{V_{q,n}(Z)}{\sqrt{E \left[ V_{q,n}^2(Z) \right]}}, N \right) \leq C \sqrt{\sum_{i,j=0}^{n-1} r_Y^2(i-j) \sqrt{\kappa_4(V_{2,n}(Y)) + \kappa_4(V_{2,n}(Y))}} \leq C \sqrt{E \left[ V_{2,n}^2(Y) \right] \sqrt{\kappa_4(V_{2,n}(Y)) + \kappa_4(V_{2,n}(Y))}}.$$

Thus the estimate (3) is obtained. By a similar argument we obtain (4).

Moreover, if $\sum_{k \in \mathbb{Z}} (r_Y(k))^2 < \infty$, it follows from (6) that

$$\kappa_4(V_{2,n}(Y)) \leq C n^{-1} \left( \sum_{|k|<n} (r_Y(k))^{4/3} \right)^3 \leq C n^{-1} \left( \sum_{|k|<n} (r_Y(k))^{4/3} \right)^3.$$
and if \( r_Y(k) = k^{-\frac{1}{2}} \) for \( k \in \mathbb{Z}^* \), it follows from (23) that

\[
E\left[V_{q,n}^2(Z)\right] = d_{q,2}(Z) \frac{2}{n} \sum_{i,j=0}^{n-1} |r_Y(i-j)|^2 + \sum_{k=2}^{q/2} d_{q,2k}(Z) \frac{(2k)!}{n} \sum_{i,j=0}^{n-1} |r_Y(i-j)|^{2k}
\]

\[
= 2d_{q,2}(Z) \left( 1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-j)^{-1} \right) + \sum_{k=2}^{q/2} d_{q,2k}(Z) \frac{(2k)!}{n} \sum_{i,j=0}^{n-1} |r_Y(i-j)|^{2k}
\]

\[
= 2d_{q,2}(Z) \left( 1 + 2 \sum_{j=1}^{n-1} j^{-1} - \frac{2}{n} \sum_{j=1}^{n-1} j^{-2} \right) + \sum_{k=2}^{q/2} d_{q,2k}(Z) \frac{(2k)!}{n} \sum_{i,j=0}^{n-1} |r_Y(i-j)|^{2k}
\]

\[
\sim 4d_{q,2}(Z) \log(n). \tag{8}
\]

Thus, as in [7] we conclude that

\[
\frac{\kappa_4(V_{2,n}(Y))}{\left(E\left[V_{2,n}^2(Y)\right]\right)^2} = \frac{\kappa_4(V_{2,n}(Y))}{\left(r_Z^{-1}(0)E\left[V_{2,n}^2(Z)\right]\right)^2}
\]

\[
= \frac{1}{n^2 \left(r_Z^{-1}(0)E\left[V_{2,n}^2(Z)\right]\right)^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} r_Y(k_1-k_2)r_Y(k_3-k_4)r_Y(k_1-k_3)r_Y(k_2-k_4)
\]

\[
\leq \frac{C}{n^2 \log^2(n)} \sum_{k_1,k_2,k_3,k_4=1}^{n} r_Y(k_1-k_2)r_Y(k_3-k_4)r_Y(k_1-k_3)r_Y(k_2-k_4)
\]

\[
\leq \frac{C}{\log(n)}. \tag{9}
\]

Hence the inequalities (5) and (6) are satisfied.

The convergence (7) is a direct consequence of [11, Theorem 1].

3.2 Hermite variation

Let \( q \in \mathbb{N}^* \) be even, and let \( H_q \) be the \( p \)th Hermite polynomial. So, \( H_q \) has the following decomposition

\[
H_q(x) = \sum_{k=0}^{q} \frac{q!(-1)^k}{k!(q-2k)!} 2^k x^{q-2k}.
\]

Define for every \( q \geq 2 \) even,

\[
Q_{q,n}(Z) := \frac{1}{n} \sum_{k=0}^{n-1} H_q(Z_k) \tag{10}
\]

and

\[
\gamma_Z(q) := E[H_q(Z_0)].
\]
Then, we can write
\[ \gamma_Z(q) := \sum_{k=0}^{q-2} \frac{q! (-1)^k}{k!(q-2k)!2^k} E(Z_0^{q-2k}) \]
\[ = \frac{q!}{2^{q/2}} \sum_{k=0}^{q-2} \frac{(-1)^k}{k!(\frac{q}{2}-k)!} \left[ E(Z_0^2) \right]^{\frac{q}{2}-k} \]
\[ = \frac{q!}{(\frac{q}{2})!2^{q/2}} \left( E(Z_0^2) - 1 \right)^{q/2}, \]  \hspace{1cm} (11)

We have the following almost sure convergence.

**Theorem 3** Suppose that $Z$ is ergodic. Then, as $n \to \infty$

\[ Q_{q,n}(Z) \longrightarrow \gamma_Z(q) \]  \hspace{1cm} (12)

almost surely.

Let $Y_i$ be the process defined in (1). We have,

\[ H_q(Z_i) - EH_q(Z_i) = \sum_{k=1}^{q/2} b_{q,2k} H_2k(Y_i) \]

where for any $k \in \{1, \ldots, \frac{q}{2} - 1\}$

\[ b_{q,q-2k}(Z) = (-1)^k \left( r_{\frac{q}{2}} Z(0) - r_{\frac{q}{2}-1} Z(0) \right) a^q_{q-2k}a^q_{q-4} \cdots a^q_{q-2k+2} \]
\[ + (-1)^{k-1} \left( r_{\frac{q}{2}} Z(0) - r_{\frac{q}{2}-2} Z(0) \right) a^q_{q-4}a^q_{q-6} \cdots a^q_{q-2k+2} \]
\[ + \ldots \]
\[ + (-1)^1 \left( r_{\frac{q}{2}} Z(0) - r_{\frac{q}{2}-k} Z(0) \right) a^q_{q-2k} \]

and $b_{q,q}(Z) = r_{\frac{q}{2}} Z(0)$, with for every $p$ even

\[ a^p_{p-2k} = \frac{p!(-1)^k}{k!(p-2k)!2^k} \quad k = 0, \ldots, p/2 \]

which verify

\[ H_p(x) = \sum_{k=0}^{p/2} a^p_{p-2k}x^{p-2k}. \]

Define

\[ V_{Q_{q,n}}(Z) := \sqrt{n} \left( Q_{q,n}(Z) - \gamma_Z(q) \right). \]
As a consequence, we can write
\[ V_{Q,q,n}(Z) = \sqrt{n} (Q_{q,n}(Z) - \gamma_Z(q)) \]
\[ = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} [H_q(Z_i) - EH_q(Z_i)] \]
\[ = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \frac{q/2}{k=1} b_{q,2k}(Z) H_{2k}(Y_i) \]
\[ = \sum_{k=1}^{q/2} b_{q,2k}(Z) H_{2k}(Z) \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \xi_i^{\otimes 2k} \]
\[ = \sum_{k=1}^{q/2} I_{2k} \left( b_{q,2k}(Z) \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \xi_i^{\otimes 2k} \right) \]

where \( Y_i = Y(\varepsilon_i) \).
Furthermore,
\[ E \left[ V_{Q,q,n}^2(Z) \right] = \sum_{k=1}^{q/2} b_{q,2k}(Z)^2 \frac{(2k)!}{n} \sum_{i,j=0}^{n-1} |r_Y(i - j)|^{2k} \]
\[ = \sum_{k=1}^{q/2} b_{q,2k}(Z)^2 (2k)! \left( 1 + \frac{2}{n} \sum_{j=1}^{n-1} (n - 1 - j |r_Y(j)|^{2k} \right) \]
\[ = \sum_{k=1}^{q/2} b_{q,2k}(Z)^2 (2k)! \left( 1 + 2 \sum_{j=1}^{n-1} |r_Y(j)|^{2k} - \frac{2}{n} \sum_{j=1}^{n-1} j |r_Y(j)|^{2k} \right) . \] (13)

We will need the following technical lemma.

**Lemma 4** Let \((Z_k)_{k \geq 0}\) be a stationary Gaussian sequence with \(E(Z_0^2) < \infty\), and let \(\lambda > 0\) and \(q \in \mathbb{N}^*\) even. Consider the sequence
\[ R_{Q,q}(\lambda, Z_k) := H_q(Z_k - e^{-\lambda k} Z_0) - H_q(Z_k) . \]

Then for every \(p \geq 1\) there exits a constant \(c(\lambda, q)\) depending on \(\lambda, q\) and \(E(Z_0^2)\) such that
\[ \left\| \frac{1}{n} \sum_{k=0}^{n-1} R_{Q,q}(\lambda, Z_k) \right\|_{L^p(\Omega)} \leq c(\lambda, q) . \] (14)

Moreover for every \(\varepsilon > 0\)
\[ \frac{1}{n \varepsilon} \sum_{k=0}^{n-1} R_{Q,q}(\lambda, Z_k) \rightarrow 0 \] (15)
almost surely as \(n \rightarrow \infty\).
For the convergence (15), it is a direct consequence of (14) and Lemma 1. Thus (14) is obtained.

Theorem 5
Let
\[ R_{Q,q}(\lambda, Z_k) = \sum_{i=0}^{q} \frac{q!(-1)^i}{k!(q-2i)!2^i} \sum_{j=1}^{q-2i} (-1)^j \binom{q-2i}{j} e^{-\lambda jk} Z_0^j Z_k^{q-2i-j}. \]

Combining this with the fact that \( Z \) is stationary and Gaussian, we obtain
\[ \|R_{Q,q}(\lambda, Z_k)\|_{L^p(\Omega)} \leq c_0(\lambda, q)e^{-\lambda k} \]
Thus (14) is obtained.

For the convergence (15), it is a direct consequence of (14) and Lemma 1.

Applying Theorem 2 and Lemma 4 we conclude the following result.

Theorem 5
Let \( (V_{Q,n}(Z))_{n \geq 0} \) and \( (R_{Q,q}(\lambda, Z_k))_{n \geq 0} \) be the sequences defined in the above.
1) Then there exists \( C \) depending on \( \lambda, q \) and \( r_Z(0) \) such that
\[
d_W \left( \sqrt{\frac{n}{E[V_{Q,n}^2(Z)]}} (Q_{q,n}(Z_k - e^{-\lambda k} Z_0) - \gamma_Z(q)), N \right) \leq \frac{C}{E[V_{Q,n}^2(Z)]} \left( \sqrt{\frac{E[V_{Q,n}^2(Y)]}{n}} + \sqrt{E[V_{Q,n}^2(Y)]} \sqrt{\kappa_4(V_{Q,n}^2(Y)) + \kappa_4(V_{Q,n}^2(Y))} \right) \]
On the other hand if \( \sum_{k \in \mathbb{Z}} |r_Y(k)|^2 < \infty \), we can write
\[
d_W \left( \sqrt{\frac{n}{V_{Q}(Z)}} (Q_{q,n}(Z_k - e^{-\lambda k} Z_0) - \gamma_Z(q)), N \right) \leq \frac{C}{V_{Q}(Z)} \left( \sqrt{\frac{V_{Q}(Z)}{n}} + \sqrt{E[V_{Q,n}^2(Y)]} \sqrt{\kappa_4(V_{Q,n}^2(Y)) + \kappa_4(V_{Q,n}^2(Y))} + |V_{Q}(Z) - E[V_{Q,n}^2(Y)]| \right) \]
where \( V_{Q}(Z) := \lim_{n \to \infty} E[V_{Q,n}^2(Z)] \). Moreover, \( \kappa_4(V_{Q,n}^2(Y)) \) verifies (9) and (10).
2) Assume that \( r_Y(k) = |k|^{-\alpha} \) with \( 0 < \alpha < \frac{1}{2} \). Then
\[
\frac{n^\alpha}{\sqrt{V_{Q}(Z)}} (Q_{q,n}(Z_k - e^{-\lambda k} Z_0) - \gamma_Z(q)) \xrightarrow{law} \frac{b_{q,2}}{\sqrt{D}} F_\infty
\]
where \( D \) and \( F_\infty \) are defined in Theorem 2.

Proof. We have
\[
\sqrt{\frac{n}{E[V_{Q,n}^2(Z)]}} (Q_{q,n}(Z_k - e^{-\lambda k} Z_0) - \gamma_Z(q)) = \sqrt{\frac{n}{E[V_{Q,n}^2(Z)]}} (Q_{q,n}(Z) - \gamma_Z(q)) + \frac{1}{\sqrt{nE[V_{Q,n}^2(Z)]}} \sum_{k=0}^{n-1} R_{Q,q}(\lambda, Z_k) = \frac{V_{Q,n}(Z)}{\sqrt{E[V_{Q,n}^2(Z)]}} + \frac{1}{\sqrt{nE[V_{Q,n}^2(Z)]}} \sum_{k=0}^{n-1} R_{Q,q}(\lambda, Z_k). \]
Hence,

\[ d_W \left( \sqrt{\frac{n}{E[V_{Q,q,n}^2(Z)]}} \left( Q_{q,n} \left( Z_k - e^{-\lambda_k} Z_0 \right) - \gamma Z(q) \right), N \right) \]

\[ \leq d_W \left( \frac{V_{Q,q,n}(Z)}{\sqrt{E[V_{Q,q,n}^2(Z)]}}, N \right) + \left\| \frac{1}{\sqrt{nE[V_{Q,q,n}^2(Z)]}} \sum_{k=0}^{n-1} R_{Q,q}(\lambda, Z_k) \right\|_{L^1(\Omega)}. \]

Combining this with (3) and (14) we obtain (16). Similar argument leads to (17). Moreover, from (19), (7) and (14) we deduce (18).

3.3 Power variation

Consider, for every \( q \in \mathbb{N}^* \) even, the following power variation

\[ P_{q,n}(Z) := \frac{1}{n} \sum_{i=0}^{n-1} (Z_i)^q. \] (20)

Define

\[ \delta Z(q) := E[(Z_0)^q] = \frac{q!}{(\frac{q}{2})!2^{q/2}} \left[ E(Z_0^2) \right]^{q/2}. \] (21)

We have the following almost sure convergence.

**Theorem 6** Suppose that \( Z \) is ergodic. Then, as \( n \to \infty \)

\[ P_{q,n}(Z) \longrightarrow \delta Z(q) \] (22)

almost surely.

Let \( Y_i \) be the process defined in (1). Let \( c_{q,2k} = \frac{1}{(2k)!} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} x^q H_{2k}(x) \, dx \) be the coefficients of the monomial \( x^q \) expanded in the basis of Hermite polynomials:

\[ x^q = \sum_{k=0}^{q/2} c_{q,2k} H_{2k}(x). \]
Then we can write,

\[
V_{P,q,n}(Z) = \sqrt{n} (P_{q,n}(Z) - E[(Z_0)^q])
\]

\[
= \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} \left( E \left[ \left( \frac{Z_i}{\sqrt{r_Z(0)}} \right)^q \right] - E \left[ \left( \frac{Z_0}{\sqrt{r_Z(0)}} \right)^q \right] \right)
\]

\[
= \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} \sum_{k=1}^{q/2} c_{q,2k}(Z) H_{2k}(Y_i)
\]

\[
= \sum_{k=1}^{q/2} c_{q,2k}(Z) \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} I_{2k} \left( \varepsilon_i^{2k} \right)
\]

where \(Y_i = Y(\varepsilon_i) = \frac{Z_i}{\sqrt{r_Z(0)}}\).

Furthermore,

\[
E \left[ V_{Q,q,n}^2(Z) \right] = [r_Z(0)]^q \sum_{k=1}^{q/2} c_{q,2k}(Z)(2k)! \frac{n}{n} \sum_{i,j=0}^{n-1} |r_Y(i - j)|^{2k}
\]

\[
= [r_Z(0)]^q \sum_{k=1}^{q/2} c_{q,2k}(Z)(2k)! \left( 1 + \frac{2}{n} \sum_{j=1}^{n-1} (n - 1 - j)|r_Y(j)|^{2k} \right)
\]

\[
= [r_Z(0)]^q \sum_{k=1}^{q/2} c_{q,2k}(Z)(2k)! \left( 1 + 2 \sum_{j=1}^{n-1} |r_Y(j)|^{2k} - \frac{2}{n} \sum_{j=1}^{n-1} j|r_Y(j)|^{2k} \right). \quad (23)
\]

We will also need the following technical lemma.

**Lemma 7** Let \((Z_k)_{k \geq 0}\) be a stationary Gaussian sequence with \(E(Z_0^2) < \infty\), and let \(\lambda > 0\) and \(q \in \mathbb{N}^*\) even. Consider the sequence

\[
R_{P,q}(\lambda, Z_k) := \left( Z_k - e^{-\lambda k} Z_0 \right)^q - (Z_k)^q.
\]

Then for every \(p \geq 1\) there exists a constant \(c(\lambda, q)\) depending on \(\lambda, q\) and \(E(Z_0^2)\) such that

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} R_{P,q}(\lambda, Z_k) \right\|_{L^p(\Omega)} \leq \frac{c(\lambda, q)}{n}. \quad (24)
\]

Moreover for every \(\varepsilon > 0\)

\[
\frac{1}{n^\varepsilon} \sum_{k=0}^{n-1} R_{P,q}(\lambda, Z_k) \rightarrow 0 \quad (25)
\]

almost surely as \(n \rightarrow \infty\).
Proof. The proof is straightforward by using similar arguments as in the proof of Lemma 4. Applying Theorem 2 and Lemma 7 we conclude.

**Theorem 8** Let \( (V_{P,q,n}(Z))_{n \geq 0} \) and \( (R_{P,q}(\lambda,Z_k))_{n \geq 0} \) be the sequences defined in the above.

1) Then there exist \( C \) depending on \( \lambda, q \) and \( r_Z(0) \) such that

\[
d_W \left( \sqrt{\frac{1}{n E[V_{P,q,n}^2(Z)]}} \left( P_{q,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \delta_Z(q) \right), N \right) \leq \frac{C}{E[V_{P,q,n}^2(Z)]} \left( \sqrt{E[V_{P,q,n}^2(Y)]} + \sqrt{E[V_{P,q,n}^2(Y)] \sqrt{\kappa_4(V_{P,q,n}(Y)) + \kappa_4(V_{P,q,n}(Y))}} \right).
\]

(26)

On the other hand if \( \sum_{k \in \mathbb{Z}} |r_Y(k)|^2 < \infty \), we can write

\[
d_W \left( \sqrt{\frac{1}{v_{P,q}(Z)}} \left( P_{q,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \gamma_Z(q) \right), N \right) \leq \frac{C}{v_{P,q}(Z)} \left( \sqrt{\frac{v_{P,q}(Z)}{n}} + \sqrt{E[V_{P,q,n}^2(Y)]} \sqrt{\kappa_4(V_{P,q,n}(Y)) + \kappa_4(V_{P,q,n}(Y)) + \left| v_q(Z) - E[V_{P,q,n}^2(Y)] \right|} \right).
\]

(27)

where \( v_{P,q}(Z) := \lim_{n \to \infty} E[V_{P,q,n}^2(Z)] \). Moreover \( \kappa_4(V_{P,q,n}(Y)) \) verifies (2) and (6).

2) Assume that \( r_Y(k) = |k|^{-\alpha} \) with \( 0 < \alpha < \frac{1}{2} \). Then

\[
\frac{n^\alpha}{\sqrt{v_{P,q}(Z)}} \left( P_{q,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \gamma_Z(q) \right) \xrightarrow{law} \frac{c_q \sqrt{2}}{\sqrt{D}} F_\infty
\]

(28)

where \( D \) and \( F_\infty \) are defined in Theorem 2.

**Proof.** Similar proof as in Theorem 8.

### 3.4 Quadratic case

In this subsection we suppose that \( q = 2 \). Then, in this case, we have

\[
V_{Q,2,n}(Z) = V_{P,2,n}(Z) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (Z_k^2 - E[Z_k^2]) = \frac{E[Z_0^2]}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(Y_k) = E[Z_0^2] V_{2,n}(Y) = V_{2,n}(Z).
\]

Thus, we obtain the following theorem.

**Theorem 9** We have
• if $\sum_{k \in \mathbb{Z}} |r_Y(k)|^2 < \infty$, there exists $C > 0$, $n_0 \geq 1$ such that for every $n \geq n_0$

$$d_{TV} \left( \frac{V_{2,n}(Z)}{\sqrt{E[V_{2,n}^2(Z)]}}, N \right) \leq C \frac{\left( \sum_{|k| < n} |r_Y(k)|^{3/2} \right)^2}{\left( \sum_{|k| < n} |r_Y(k)|^2 \right)^{3/2} \sqrt{n}}, \quad (29)$$

and hence also

$$d_W \left( \frac{n}{\sqrt{E[V_{2,n}^2(Z)]}} \left( Q_{2,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \gamma_Z(2) \right), N \right) \leq C \frac{1 + \left( \sum_{|k| < n} |r_Y(k)|^{3/2} \right)^2}{\sqrt{n}}; \quad (30)$$

• if $|r_Y(k)| = |k|^{-\frac{1}{2}}$,

$$d_{TV} \left( \frac{V_{2,n}(Z)}{\sqrt{E[V_{2,n}^2(Z)]}}, N \right) \leq \frac{C}{(\log n)^{1/4}}; \quad (31)$$

and also

$$d_W \left( \frac{n}{\sqrt{E[V_{2,n}^2(Z)]}} \left( Q_{2,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \gamma_Z(2) \right), N \right) \leq \frac{C}{(\log n)^{1/4}}; \quad (32)$$

where in this case $E[V_{2,n}^2(Z)] \sim 4b_2^2(Z) \log(n) = 4r_Z^2(0) \log(n)$.

• if $r_Y(k) = |k|^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$, there exists $C > 0$ depending on $\alpha$

$$d_{TV} \left( \frac{V_{2,n}(Z)}{\sqrt{E[V_{2,n}^2(Z)]}}, \frac{1}{2} \sqrt{\frac{1 - 2\alpha}{D}} F_\infty \right) \leq \frac{C}{\sqrt{\log n}}. \quad (33)$$

and also

$$d_W \left( \frac{n}{\sqrt{E[V_{2,n}^2(Z)]}} \left( Q_{2,n} \left( Z_k - e^{-\lambda k} Z_0 \right) - \gamma_Z(2) \right), \frac{1}{2} \sqrt{\frac{1 - 2\alpha}{D}} F_\infty \right) \leq \frac{C}{\sqrt{\log n}}, \quad (34)$$

where $D$ and $F_\infty$ are defined in Theorem 2.

Moreover in this case $E[V_{2,n}^2(Z)] \sim 42^{1-2\alpha} |r_Z(0)|^2$.

**Proof.** Since $\frac{V_{2,n}(Z)}{\sqrt{E[V_{2,n}^2(Z)]}} = \frac{V_{2,n}(Y)}{\sqrt{E[V_{2,n}^2(Y)]}}$, then (29) is a direct consequence of [19] Theorem 3 (see also [6], [21]). Hence also, from (19), (29) and (14) we deduce (30).
Combining (3) and (6) we obtain (31), and also the estimates (19), (31) and (14) lead to (32).

Now, suppose that \( r_Y(k) = |k|^{-\alpha} \) with \( 0 < \alpha < \frac{1}{2} \). It is easy to see that (23) leads to

\[
E \left[ V_{2,n}^2(Z) \right] = 2|r_Z(0)|^2 \left( 1 + 2 \sum_{j=1}^{n-1} |r_Y(j)|^2 - \frac{2}{n} \sum_{j=1}^{n-1} j |r_Y(j)|^{2k} \right)
\]

\[
\sim 4n^{1-2\alpha} |r_Z(0)|^2.
\]

Thus, from [19, Theorem 5] and [11, Theorem 1], the estimate (33) is obtained. Hence also, from (19), (33) and (14) we deduce (34).

### 3.5 Improve the rate convergence

Consider a centered stationary Gaussian process \( Z = (Z_k)_{k \in \mathbb{Z}} \) with covariance \( r_Z(k) = E(Z_0 Z_k) < \infty \) and \( |r_Z(k)| \leq 1 \) for \( k \in \mathbb{Z} \).

Define the centered stationary Gaussian process \( Z^{(p)} \) as follows:

\[ Z^{(1)}_k = Z_{k+1} - Z_k, \quad k \in \mathbb{Z} \]

and for every \( p \geq 2 \)

\[ Z^{(p)}_k = Z^{(p-1)}_{k+1} - Z^{(p-1)}_k, \quad k \in \mathbb{Z}. \]

Now, applying Theorem 2 we conclude the following result.

**Theorem 10** Assume that \( k^{-\alpha} r_Z(k) \) converges to a constant for some \( \alpha < \frac{1}{2} \). Then for every \( p \geq 2 \) there exists \( C \) depending on \( p, q \) and \( r_Z(0) \) such that

\[
d_{TV} \left( \frac{V_{q,n}(Z^{(p)})}{\sqrt{E \left[ V_{q,n}^2(Z^{(p)}) \right]}}, N \right) \leq C n^{\alpha-2p+\frac{3}{2}}. \tag{35}
\]

In particular, if \( \alpha = 2H - 2 \) then for every \( p \geq 2 \) there exists \( C \) depending on \( p, q \) and \( r_Z(0) \) such that

\[
d_{TV} \left( \frac{V_{q,n}(Z^{(p)})}{\sqrt{E \left[ V_{q,n}^2(Z^{(p)}) \right]}}, N \right) \leq C n^{2H-2p-\frac{1}{2}}. \tag{36}
\]

This leads that if \( p \geq 2 \), then \( \frac{V_{q,n}(Z^{(p)})}{\sqrt{E \left[ V_{q,n}^2(Z^{(p)}) \right]}}, N \) is asymptotically normal for every \( H \in (0, 1) \).
4 Applications to Ornstein-Uhlenbeck processes

4.1 Fractional Ornstein-Uhlenbeck process

In this section suppose that \( X = \{X_t, t \geq 0\} \) is an Ornstein-Uhlenbeck process driven by a fractional Brownian motion \( B^H = \{B^H_t, t \geq 0\} \) of Hurst index \( H \in (0, 1) \). That is, \( X \) is the solution of the following linear stochastic differential equation

\[
X_0 = 0; \quad dX_t = -\theta X_t dt + dB^H_t, \quad t \geq 0, \tag{37}
\]

where whereas \( \theta > 0 \) is considered as unknown parameter.

The solution \( X \) has the following explicit expression:

\[
X_t = \int_0^t e^{-\theta(t-s)} dB^H_s.
\]

We can also write

\[
X_t = Z^\theta_t - e^{-\theta t} Z^\theta_0
\]

where

\[
Z^\theta_t = \int_{-\infty}^t e^{-\theta(t-s)} dB^H_s.
\]

Moreover, \( Y^\theta \) is an ergodic stationary Gaussian process.

We will need the following result.

**Lemma 11** Let \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \), \( m, m' > 0 \) and \(-\infty \leq a \leq b \leq c < d < \infty\). Then

\[
E \left( \int_a^b e^{ms} dB^H(s) \int_c^d e^{ms} dB^H(t) \right) = H(2H - 1) \int_a^b dse^{ms} \int_c^d dt e^{m't}(t-s)2H-2.
\]

**Proof.** We use the same argument as in the proof of [10, Lemma 2.1]. ■

**Lemma 12** Let \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), \( m, m' > 0 \) and let \( Z^\theta \) be the process defined in (39). Then,

\[
E \left( Z^\theta_0 \right)^2 = H\Gamma(2H)\theta^{-2H} \tag{39}
\]

and for large \( |t| \)

\[
E \left[ Z^\theta_0 Z^\theta_t \right] \sim \frac{H(2H - 1)}{\theta^2} |t|^{2H-2}. \tag{40}
\]

**Proof.** see [10, Theorem 2.3] or Lemma 17. ■
4.2 Construction and asymptotic behavior of the estimators

Fix \( q \geq 2 \) and assume that \( q \) is even. From (38) we can write

\[
Q_{q,n}(X) = Q_{q,n}(Z^\theta) + \frac{1}{n} \sum_{k=0}^{n-1} R_{Q,q}(\theta, Z_k^\theta) \quad (41)
\]

Combining (41), Lemma 4 and the fact that \( Z^\theta \) is ergodic we conclude that, almost surely,

\[
\lim_{n \to \infty} Q_{q,n}(X) = \lim_{n \to \infty} Q_{q,n}(Z^\theta) = \gamma_{Z^\theta}(q) = \frac{q!}{(\frac{q}{2})!2^{q/2}} \left( H \Gamma(2H)\theta^{-2H} - 1 \right)^{q/2} =: \mu_q(\theta).
\]

Hence we obtain the following estimator for \( \theta \)

\[
\hat{\theta}_{q,n} = \mu_q^{-1} [Q_{q,n}(X)].
\]

As consequence, we have the following strong consistency of \( \hat{\theta}_{q,n} \).

**Theorem 13** Let \( H \in (0,1) \). Then, as \( n \to \infty \)

\[
\hat{\theta}_{q,n} \to \theta
\]

almost surely.

Combining (41) and Theorem 8 we conclude the following result.

**Theorem 14** Denote \( N \sim \mathcal{N}(0,1) \). If \( H \in (0, \frac{3}{4}] \), then there exists \( C \) depending on \( q, H \) and \( \theta \) such that

\[
dW \left( \sqrt{\frac{n}{E[V_{q,n}^2(Z^\theta)]}} \left( \mu_q \left( \hat{\theta}_{q,n} \right) - \mu_q(\theta) \right), N \right) \leq C \begin{cases} 
  n^{-\frac{1}{4}}, & \text{if } 0 < H < \frac{5}{8} \\
  n^{-\frac{1}{4}} \log^2(n), & \text{if } H = \frac{5}{8} \\
  n^{2H - \frac{3}{2}}, & \text{if } \frac{5}{8} < H < \frac{3}{4} \\
  \log^{-\frac{3}{4}}(n), & \text{if } H = \frac{3}{4}.
\end{cases} \quad (43)
\]

In particular,

- If \( H \in (0, \frac{3}{4}) \)

\[
\sqrt{n} \left( \mu_q \left( \hat{\theta}_{q,n} \right) - \mu_q(\theta) \right) \xrightarrow{law} \mathcal{N} \left( 0, \sigma_q^2(Z^\theta) \right)
\]

where \( \sigma_q^2(Z^\theta) = \sum_{k=1}^{q/2} b_{q,2k}(Z^\theta)(2k)! \left( 1 + 2 \sum_{j=1}^{\infty} \frac{|r_{q,\theta}(j)2k|}{r_{q,\theta}^2(0)} \right) = \lim_{n \to \infty} E \left[ V_{q,n}^2(Z^\theta) \right] \),
• if $H = \frac{3}{4}$

$$\sqrt{\frac{n}{\log(n)}} \left( \mu_q(\hat{\theta}_{q,n}) - \mu_q(\theta) \right) \xrightarrow{law} \mathcal{N} \left( 0, 4b^2_{q,2}(Z^\theta) \right)$$

(45)

where in this case $E \left[ V^2_{q,n}(Z^\theta) \right] \sim 4b^2_{q,2}(Z^\theta) \log(n)$.

In the case when $H \in \left( \frac{3}{4}, 1 \right)$, we have

$$\frac{1}{n^{2H - \frac{3}{4}}} \left( \mu_q(\hat{\theta}_{q,n}) - \mu_q(\theta) \right) \xrightarrow{law} \frac{b^2_{q,2}(Z^\theta)}{\sqrt{D}} F_\infty$$

(46)

where $F_\infty$ is defined in Theorem 2.

Thus we deduce the asymptotic distribution of $\hat{\theta}_{q,n}$.

**Theorem 15** If $H \in (0, \frac{3}{4}]$, then

$$\sqrt{\frac{n}{E \left[ V^2_{q,n}(Z^\theta) \right]}} \left( \hat{\theta}_{q,n} - \theta \right) \xrightarrow{law} \mathcal{N} \left( 0, \left( \mu'_q(\theta) \right)^2 \right).$$

(47)

If $H \in \left( \frac{3}{4}, 1 \right)$, then

$$\frac{1}{n^{2H - \frac{3}{2}}} \left( \hat{\theta}_{q,n} - \theta \right) \xrightarrow{law} \frac{b_{q,2}(Z^\theta)}{\mu'_q(\theta)\sqrt{D}} F_\infty$$

(48)

**Proof.** We can write

$$\sqrt{n} \left( \mu_q(\hat{\theta}_{q,n}) - \mu_q(\theta) \right) = \mu'_q(\xi_{q,n})\sqrt{n} \left( \hat{\theta}_{q,n} - \theta \right)$$

where $\xi_{q,n}$ is a random variable between $\theta$ and $\hat{\theta}_{q,n}$.

Combining this with Theorem 14 we obtain the desired conclusion. $lacksquare$

**Quadratic case:** Now we will discuss the particular case when $q = 2$.

Define

$$Y^\theta_t := \frac{Z^\theta_t}{\sqrt{\gamma_Z(2)}} = \frac{Z^\theta_t}{\sqrt{E[[Z^\theta_0]^2]}} = \frac{Z^\theta_t}{\sqrt{H\Gamma(2H)\theta^{-2H}}}.$$  

(49)

Combining (41) and Theorem 9 we obtain the desired result.

**Theorem 16** If $H \in (0, \frac{3}{4}]$, then

$$d_W \left( \frac{H\Gamma(2H)}{E \left[ V^2_{2,n}(Z^\theta) \right]} \left( \hat{\theta}_{2,n} - 2H, N \right) \right) \leq C \times \left\{ \begin{array}{ll}
\frac{1}{\sqrt{n}}, & \text{if } 0 < H < \frac{2}{3} \\
\frac{\log^2(n)}{\sqrt{n}}, & \text{if } H = \frac{2}{3} \\
n^{6H - \frac{3}{2}}, & \text{if } \frac{2}{3} < H < \frac{3}{4} \\
\frac{1}{(\log(n))^{1/4}}, & \text{if } H = \frac{3}{4}.
\end{array} \right.$$  

(50)
As consequence, for every $H \in \left(0, \frac{3}{4}\right)$

$$
\sqrt{n} \left( \hat{\theta}_{2,n} - \theta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \frac{\theta^{4H+2} \sigma^2(Z^0)}{(H \Gamma(2H + 1))^2} \right)
$$

(51)

and if $H = \frac{3}{4}$

$$
\sqrt{\frac{n}{\log(n)}} \left( \hat{\theta}_{2,n} - \theta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \frac{H}{\theta} \right)
$$

(52)

where in this case $E[V_{2,n}^2(Z^0)] \sim (\Gamma(2H + 1)\theta^{-2H})^2 \log(n)$.

If $\frac{3}{4} < H < 1$,

$$
d_W \left( \frac{HT \Gamma(2H)}{\sqrt{E[V_{2,n}^2(Z^0)]}} \sqrt{n} \left( \hat{\theta}_{2,n}^{-2H} - \theta^{-2H} \right) \right)^\frac{1}{2} \leq C \sqrt{\frac{4H - 3}{D \log(n)}}
$$

(53)

where $E[V_{2,n}^2(Z^0)] \sim \frac{n^{4H-3}}{4H-3} (\Gamma(2H + 1)\theta^{-2H})^2$.

As consequence

$$
\sqrt{n} \left( \theta - \hat{\theta}_{2,n} \right) \xrightarrow{\text{law}} \frac{\theta}{2H} F_\infty.
$$

(54)

### 4.3 OU driven by fractional Ornstein-Uhlenbeck process

In this section suppose that $X = \{X_t, t \geq 0\}$ is an Ornstein-Uhlenbeck process driven by fractional Ornstein-Uhlenbeck process $V = \{V_t, t \geq 0\}$ given by the following linear stochastic differential equations

$$
\begin{align*}
X_0 &= 0; & dX_t &= -\theta X_t dt + dV_t, & t \geq 0 \\
V_0 &= 0; & dV_t &= -\rho V_t dt + \sigma dB_t^H, & t \geq 0,
\end{align*}
$$

(55)

where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion of Hurst index $H \in (0, 1)$, whereas $\theta > 0$ and $\rho > 0$ are considered as unknown parameters such that $\theta \neq \rho$.

For [13] we have the following results related to $X_t$:

$$
X_t = \frac{\rho}{\rho - \theta} X_t^\theta + \frac{\theta}{\theta - \rho} X_t^\theta
$$

(56)

where for $m > 0$

$$
X_t^m = \int_0^t e^{-m(t-s)} dB^H_s.
$$

(57)

On the other hand, we can also write the system (55) as follows

$$
dX_t = - (\theta + \rho) X_t dt - \rho \sigma dt + dB_t^H.
$$
where for $0 \leq t \leq T$

$$\Sigma_t = \int_0^t X_s ds = V_t - X_t = \frac{X_t - X_t^\rho}{\rho - \theta}$$  \hspace{1cm} (58)

We can also write

$$X^m_t = Z^m_t - e^{-mt} Z^m_0$$  \hspace{1cm} (59)

where

$$Z^m_t = \int_{-\infty}^t e^{-mt} dB^H_s.$$  \hspace{1cm} (60)

As consequence

$$X_t = \frac{\rho}{\rho - \theta} Z^\rho_t + \frac{\theta}{\theta - \rho} Z^\theta_t - \left( \frac{\rho e^{-\rho t}}{\rho - \theta} Z^\rho_0 + \frac{\theta e^{-\theta t}}{\theta - \rho} Z^\theta_0 \right)$$

$$:= Z^\rho_t \frac{\rho}{\rho - \theta} + \frac{\theta}{\theta - \rho} R(\rho, Z^\rho) + \frac{\theta}{\theta - \rho} R(\theta, Z^\theta)$$  \hspace{1cm} (61)

and

$$\Sigma_t = \frac{Z^\theta_t - Z^\rho_t}{\rho - \theta} - \frac{e^{-\theta t} Z^\theta_0 - e^{-\rho t} Z^\rho_0}{\rho - \theta}$$

$$:= \Sigma^\rho_t \frac{\rho}{\theta - \rho} + \frac{\theta}{\theta - \rho} \left( R(\theta, Z^\rho) - R(\rho, Z^\theta) \right).$$  \hspace{1cm} (62)

On the other hand, the process

$$\left( Z^m_t, Z^m_t' \right)$$  \hspace{1cm} (63)

is an ergodic stationary Gaussian process.

**Lemma 17** Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $m, m' > 0$ and let $Z^m$ be the process defined in (60). Then,

$$\lambda(m, m') := E \left[ Z^m_0 Z^m_0' \right] = \frac{H \Gamma(2H)}{m + m'} \left( m^{1-2H} + (m')^{1-2H} \right)$$  \hspace{1cm} (64)

and for large $|t|$,

$$E \left[ Z^m_0 Z^m_t' \right] \sim \frac{H(2H - 1)}{mm'} |t|^{2H-2}.$$  \hspace{1cm} (65)

This implies that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$

$$\eta_X(\theta, \rho) := E \left[ (Z^\theta_0 Z^\rho_0)^2 \right] = \frac{H \Gamma(2H)}{\rho^2 - \theta^2} \left[ \rho^{-2-2H} - \theta^{-2-2H} \right],$$

$$\eta_\Sigma(\theta, \rho) := E \left[ (\Sigma^\theta_0 Z^\rho_0)^2 \right] = \frac{H \Gamma(2H)}{\rho^2 - \theta^2} \left[ \theta^{-2H} - \rho^{-2H} \right],$$
and
\[ E \left[ \left( Z_0^0, \Sigma_0^0 \right) \right] = 0. \]

**Proof.** By using [10, Proposition A.1], we can write
\[
E \left[ Z_0^m Z_0^{m'} \right] = \frac{mm'}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{mu} e^{m'v} E \left( B_u^{H} B_v^{H} \right) \, du \, dv
= \frac{mm'}{2} \int_{0}^{\infty} \int_{0}^{\infty} e^{mu} e^{m'v} (-u)^{2H} + (-v)^{2H} - |v - u|^{2H} \, du \, dv
= \frac{mm'}{2} \int_{0}^{\infty} \int_{0}^{\infty} e^{mu} e^{m'v} (u^{2H} + v^{2H} - |v - u|^{2H}) \, du \, dv
= \frac{\Gamma(2H + 1)}{2(m + m')} \left( m^{1-2H} + (m')^{1-2H} \right).
\]

Thus the estimate (64) is proved.

Now, let \( 0 < \varepsilon < 1 \)
\[
E \left( Z_t^m Z_t^{m'} \right) = e^{-m't} E \left( \int_{-\infty}^{0} e^{mu} dB_u^{H} \int_{-\infty}^{t} e^{m'v} dB_v^{H} \right)
= e^{-m't} E \left( \int_{-\infty}^{0} e^{mu} dB_u^{H} \int_{-\infty}^{t} e^{m'v} dB_v^{H} \right) + e^{-m't} E \left( \int_{-\infty}^{0} e^{mu} dB_u^{H} \int_{t}^{\infty} e^{m'v} dB_v^{H} \right)
:= A + B
\]
where, using [10, Proposition A.1] it is easy to see that
\[
|A| = O \left( e^{-m't} \right).
\]

On the other hand, by Lemma [11] and integrations by part
\[
B = H(2H - 1) e^{-m't} \int_{-\infty}^{0} du \, e^{mu} \int_{t}^{t} dv \, e^{m'v} (v - u)^{2H-2}
= H(2H - 1) e^{-m't} \int_{-\infty}^{0} du \, e^{mu} \int_{t}^{t} dv \, e^{m'v} (u + z)^{2H-2}
= H(2H - 1) e^{-m't} \int_{t}^{\infty} dz \, e^{m'z} z^{2H-2} \int_{t-z}^{0} e^{(m+m')u} \, du
= \frac{H(2H - 1)}{m + m'} \left( \int_{t}^{\infty} e^{-m(z-t)} \int_{t-z}^{2H-2} e^{2H-2} \, dz + e^{-m't(1-\varepsilon)} \int_{t}^{\infty} e^{-m(z-t)} \int_{t-z}^{2H-2} e^{2H-2} \, dz \right)
= \frac{H(2H - 1)}{m + m'} \left( \frac{t^{2H-2}}{m} + \frac{2H - 2}{m} \int_{t}^{\infty} e^{-m(z-t)} \int_{t-z}^{2H-3} e^{2H-2} \, dz + \frac{2H - 2}{m'} \int_{t}^{\infty} e^{-m(z-t)} \int_{t-z}^{2H-2} e^{2H-2} \, dz \right)
- \frac{(\varepsilon t)^{2H-2}}{m'} e^{-m'(1-\varepsilon)t} - \frac{2H - 2}{m'} \int_{t}^{t} e^{-m'(t-z)} \int_{t-z}^{2H-2} e^{2H-2} \, dz + e^{-m't(1-\varepsilon)} \int_{t}^{\infty} e^{-m(z-t)} \int_{t-z}^{2H-2} e^{2H-2} \, dz
\]
\[
= \frac{H(2H - 1)}{mm'} t^{2H-2} + O \left( t^{2H-2} \right),
\]

20
the last inequality comes from the fact that 
\[ \int_{t}^{\infty} e^{-m(z-t)}z^{2H-3}dz \leq t^{-1} \int_{0}^{\infty} e^{-my}dy \]
\[ \rightarrow 0, \quad \text{as } t \rightarrow \infty, \]
and
\[ t^{2-2H} \int_{et}^{t} e^{-m'(t-z)}z^{2H-3}dz \leq \varepsilon^{2H-3}t^{-1} \int_{et}^{t} e^{-m'(t-z)}dz \]
\[ = \varepsilon^{2H-3}t^{-1} \int_{0}^{(1-\varepsilon)t} e^{-m'y}dy \]
\[ \rightarrow 0, \quad \text{as } t \rightarrow \infty, \]

So, we conclude that the estimate (65) is obtained.

4.4 Construction and asymptotic behavior of the estimators

Case: \( X \) and \( \Sigma \) are observed

Combining (61), (62), (11), (63) and Lemma 4 we conclude that

\[ \lim_{n \to \infty} (Q_{q,n}(X), Q_{q,n}((\Sigma))) = \lim_{n \to \infty} (Q_{q,n}(Y^{\theta,\rho}), Q_{q,n}(\Sigma^{\theta,\rho})) \]
\[ = (\gamma_{Y^{\theta,\rho}}(q), \gamma_{\Sigma^{\theta,\rho}}(q)) \]
\[ = \frac{q!}{(q!)^{2^{q/2}}} \left( \left( \eta_X(\theta, \rho) - 1 \right)^{q/2}, \left( \eta_X(\theta, \rho) - 1 \right)^{q/2} \right) \]
\[ := F_q(\theta, \rho). \]

Hence, we obtain the following estimator for \((\theta, \rho)\).

\[ \left( \hat{\theta}_{q,n}, \hat{\rho}_{q,n} \right) = G_q (Q_{q,n}(X), Q_{q,n}(\Sigma)). \]

where \( G_q \) is the inverse function of \( F_q \).

By construction, we have the following strong consistence of \( \left( \hat{\theta}_{q,n}, \hat{\rho}_{q,n} \right) \).

**Theorem 18** Let \( H \in (0,1) \). Then, as \( n \to \infty \)

\[ \left( \hat{\theta}_{q,n}, \hat{\rho}_{q,n} \right) \rightarrow (\theta, \rho) \quad (66) \]

almost surely.

Combining (62), (61), Lemma 17 and Theorem 8 we conclude the following result
Theorem 19 Let $H \in (0, \frac{3}{4})$. Then
\[
\sqrt{n} \left[ (Q_{q,n}(X), Q_{q,n}(\Sigma)) - F_q(\theta, \rho) \right] \xrightarrow{\mathcal{L}} N(0, \Gamma(\theta, \rho))
\] (67)
where
\[
\Gamma(\theta, \rho) = \begin{pmatrix} \sigma_q^2(Y^{\theta,\rho}) & \sigma_q(Y^{\theta,\rho}, \Sigma^{\theta,\rho}) \\ \sigma_q(Y^{\theta,\rho}, \Sigma^{\theta,\rho}) & \sigma_q^2(\Sigma^{\theta,\rho}) \end{pmatrix}
\] (68)
where
\[
\sigma_q^2(Y^{\theta,\rho}) = \text{Var} \left[ H_q(Y_0^{\theta,\rho}) \right] + 2 \sum_{k=1}^{\infty} \text{Cov} \left[ H_q(Y_0^{\theta,\rho}), H_q(Y_k^{\theta,\rho}) \right]
\]
\[
\sigma_q^2(\Sigma^{\theta,\rho}) = \text{Var} \left[ H_q(\Sigma_0^{\theta,\rho}) \right] + 2 \sum_{k=1}^{\infty} \text{Cov} \left[ H_q(\Sigma_0^{\theta,\rho}), H_q(\Sigma_k^{\theta,\rho}) \right]
\]
\[
\sigma_q(Y^{\theta,\rho}, \Sigma^{\theta,\rho}) = \text{Cov} \left[ H_q(Y_0^{\theta,\rho}), H_q(\Sigma_0^{\theta,\rho}) \right] + \sum_{k \in \mathbb{Z}^*} \text{Cov} \left[ H_q(Y_0^{\theta,\rho}), H_q(\Sigma_k^{\theta,\rho}) \right].
\]

Theorem 20 Let $H \in (0, \frac{3}{4})$. Then
\[
\sqrt{n} \left( \bar{\theta}_{q,n} - \theta, \bar{\rho}_{q,n} - \rho \right) \xrightarrow{\mathcal{L}} N(0, J_{G_q}(\eta_X, \eta_{\Sigma}) \Gamma(\theta, \rho) \Gamma(\theta, \rho)^{t} J_{G_q}(\eta_X, \eta_{\Sigma}))
\] (69)
Proof. By Taylor’s formula we can write
\[
\sqrt{n} \left( \bar{\theta}_{q,n} - \theta, \bar{\rho}_{q,n} - \rho \right) = \sqrt{n} (Q_{q,n}(X) - \gamma_{Y^{\theta,\rho},\rho}(q), Q_{q,n}(\Sigma) - \gamma_{\Sigma^{\theta,\rho}}(q))^{t} J_{G_q}(\eta_X, \eta_{\Sigma}) + d_n
\]
where $d_n$ converges in distribution to zero, because
\[
\left\| d_n \right\| \leq c(\theta, \rho, H) \sqrt{T_n} \left\| Q_{q,n}(X) - \gamma_{Y^{\theta,\rho},\rho}(q), Q_{q,n}(\Sigma) - \gamma_{\Sigma^{\theta,\rho}}(q) \right\|^2.
\]
It is easy to see that if for any $w \in \Omega$ there exists $n_0(w) \in \mathbb{N}$ such that $X_n(w) = Y_n(w)$ for all $n \geq n_0(w)$ and $X_n \xrightarrow{\text{law}} 0$ as $n \to \infty$, then $Y_n \xrightarrow{\text{law}} 0$ as $n \to \infty$.
Combining this and above estimates we obtain the desired result. □

**Quadratic case:** Here we suppose that $q = 2$. In this case we have:
$F_2$ is a positive function of the variables $(x, y)$ in $(0, +\infty)^2$ defined by: for every $(x, y) \in (0, +\infty)^2$
\[
F_2(x, y) = H \Gamma(2H) \times \begin{cases} 
\frac{1}{y^2-x^2} (y^2-2H - x^2-2H, x^2-2H - y^2-2H) & \text{if } x \neq y \\
(1-H)x^2-2H, Hx^2-2H-2) & \text{if } y = x.
\end{cases}
\] (70)
Since for every $(x, y) \in (0, +\infty)^2$ with $x \neq y$
\[
J_{F_2} (x, y) = \Gamma(2H + 1) \begin{pmatrix} \frac{(1-H)x^2-2H-x^2-2H}{x^2-y^2} & \frac{(1-H)y^2-2H-y^2-2H}{x^2-y^2} \\ \frac{Hx^2-2H-1}{x^2-y^2} & \frac{Hy^2-2H-1}{x^2-y^2} \end{pmatrix}
\]
the determinant of $J_{F_2} (x, y)$ is non-zero on $(0, +\infty)^2$. So, $F_2$ is a diffeomorphism in $(0, +\infty)^2$ and its inverse $G_2$ has a Jacobian
\[
J_{G_2} (a, b) = \frac{\Gamma(2H + 1)}{\det J_{F_2} (x, y)} \begin{pmatrix} \frac{Hx^2-2H-1}{x^2-y^2} & \frac{(1-H)y^2-2H-y^2-2H}{x^2-y^2} \\ \frac{Hx^2-2H-1}{x^2-y^2} & \frac{Hy^2-2H-1}{x^2-y^2} \end{pmatrix}
\]
where $(x, y) = G_2 (a, b)$.
4.5 Fractional Ornstein-Uhlenbeck process with the second kind

Let \( U = \{ U_t, t \geq 0 \} \) be a fractional Ornstein-Uhlenbeck process with the second kind defined as

\[
U_0 = 0, \quad dU_t = -\alpha U_t dt + dY^{(1)}_t, \quad t \geq 0,
\]

where \( Y^{(1)}_t = \int_0^t e^{-s} dB_{as} \) with \( a_s = H e^{\frac{s}{2}} \) and \( B = \{ B_t, t \geq 0 \} \) is a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \), and where \( \alpha > 0 \) is a unknown real parameter.

The equation (71) admits an explicit solution

\[
U_t = e^{-\alpha t} \int_0^t e^{\alpha s} dY^{(1)}_s = e^{-\alpha t} \int_0^t e^{(\alpha - 1)s} dB_{as}.
\]

Hence we can also write

\[
U_t = U_t^\alpha + R(\alpha, U^\alpha)
\]

where

\[
U_t^\alpha = e^{-\alpha t} \int_{-\infty}^t e^{(\alpha - 1)s} dB_{as} = H^{(1-\alpha)} e^{-\alpha t} \int_0^t r^{(\alpha - 1)H} dB_r.
\]

Lemma 21 Let \( H \in (\frac{1}{2}, 1) \). Then,

\[
E[(U_0^\alpha)^2] = \frac{(2H - 1)H^{2H}}{\alpha} \beta(1 - H + \alpha H, 2H - 1).
\]

and for large \(|t|\)

\[
r_{U^\alpha}(t) = E[U_0^\alpha] = O \left( e^{-\min(\alpha, 1-H) t} \right).
\]

Proof. We prove the first point (73). We have

\[
E[(U_0^\alpha)^2] = H(2H - 1)H^{2(1-\alpha)H} \int_0^{\alpha^0} dy y^{(\alpha-1)H} \int_0^{\alpha^0} dx x^{(\alpha-1)H} |x - y|^{2H-2}
\]

\[
= 2H(2H - 1)H^{2(1-\alpha)H} \int_0^{\alpha^0} dy y^{(\alpha-1)H} \int_0^{y} dx x^{(\alpha-1)H} (y - x)^{2H-2}
\]

\[
= 2H(2H - 1)H^{2(1-\alpha)H} \int_0^{\alpha^0} dy y^{2\alpha H - 1} \int_0^1 dz z^{(\alpha-1)H} (1 - z)^{2H-2}
\]

\[
= \frac{(2H - 1)H^{2H}}{\alpha} \beta(1 - H + \alpha H, 2H - 1).
\]

Thus (73) is obtained.

For the point (74) see [16].
Combining (72), Lemma 4 and the fact that \( U_{\alpha} \) is ergodic we conclude that almost surely
\[
\lim_{n \to \infty} Q_{q,n}(U) = \lim_{n \to \infty} Q_{q,n}(U_{\alpha}) = \gamma_{U_{\alpha}}(q) = \frac{q!}{(\frac{q}{2})! 2^{q/2}} \left( \frac{2H - 1}{H^2} \frac{\beta(1 - H + \alpha H, 2H - 1)}{\alpha} \right)^{q/2} := \nu_q(\alpha).
\]
Hence we obtain the following estimator for \( \alpha \)
\[
\hat{\alpha}_{q,n} = \nu_q^{-1}[Q_{q,n}(U)].
\]
By construction, we have the following strong consistence of \( \hat{\alpha}_{q,n} \).

**Theorem 22** Let \( H \in \left( \frac{1}{2}, 1 \right) \). Then, as \( n \to \infty \)
\[
\hat{\alpha}_{q,n} \to \alpha
\]
aalmost surely.

Combining (72) and Theorem 8, Lemma 21 and Lemma 4 we conclude the following result.

**Theorem 23** Denote \( N \sim \mathcal{N}(0, 1) \). If \( H \in \left( \frac{1}{2}, 1 \right) \), then there exist \( C \) depending on \( q, H \) and \( \alpha \) such that
\[
\sqrt{n} \nu_q \left( \hat{\alpha}_{q,n} - \nu_q(\alpha) \right) \xrightarrow{\text{law}} \mathcal{N} \left( 0, \sigma^2(U_{\alpha}) \right)
\]
where \( \sigma^2(U_{\alpha}) = \sum_{k=1}^{q/2} b_{q,2k}(U_{\alpha})(2k)! \left( 1 + 2 \sum_{j=1}^{\infty} \frac{|r_{U_{\alpha}(j)}|^{2k}}{r_{U_{\alpha}(0)}^{2k}} \right) = \lim_{n \to \infty} E \left[ V_{q,n}^2(U_{\alpha}) \right] \).

**Theorem 24** Let \( H \in \left( \frac{1}{2}, 1 \right) \). Then
\[
\sqrt{n} \nu_q \left( \hat{\alpha}_{q,n} - \alpha \right) \xrightarrow{\text{law}} \mathcal{N} \left( 0, \nu_q'(\alpha)^{-2} \right).
\]

**Proof.** We can write
\[
\sqrt{n} \nu_q \left( \hat{\alpha}_{q,n} - \nu_q(\alpha) \right) = \nu_q'(\xi_{q,n}) \sqrt{n} \left( \hat{\alpha}_{q,n} - \alpha \right)
\]
where \( \xi_{q,n} \) is a random variable between \( \alpha \) and \( \hat{\alpha}_{q,n} \).
Combining this with Theorem 23 we obtain the desired conclusion. ■

**Quadratic case:** Here we suppose that \( q = 2 \). We will study the asymptotic distribution of \( \hat{\alpha}_{2,n} \).
Theorem 25 Let $H \in \left( \frac{1}{2}, 1 \right)$. Then there exists $C$ depending on $H$ and $\alpha$ such that

$$d_W \left( \frac{n}{\sqrt{E[V_{Z,n}^2(U^\alpha)]}} (\nu_2(\hat{\alpha}_{2,n}) - \nu_2(\alpha)), N \right) \leq \frac{C}{\sqrt{n}}. \quad (79)$$

Proof. From Lemma, we have for any $H \in (\frac{1}{2}, 1)$, $\sum_{k \in \mathbb{Z}} |r_{Z^\alpha}(k)|^{3/2} < \infty$ and $\sum_{k \in \mathbb{Z}} |r_{Z^\alpha}(k)|^{2} < \infty$. Hence by Theorem, the conclusion is obtained. ■

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