A fractional numerical study on advection-dispersion equation with fractional order

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Abstract. This paper proposes a fractional numerical study on Advection-Dispersion Equation (ADE) with Fractional Order (FO) via the Caputo Fractional Reduced Differential Transform Method (CFRDTM). CFRDTM is the combination of the Caputo Fractional Derivative (CFD) and the well known Transform Method (RDTM). A convergent series solution for ADE with FO is obtained via CFRDTM. The performance of CFRDTM is tested on two illustrative examples. Hence, CFRDTM is found to be accurate and efficient.

Keywords: Fractional order, Fractional advection-dispersion equation, Caputo fractional derivative, Reduced differential transform method

2020 MSC: 26A33, 35R11

1 Introduction
Most of the real life problems emanated from sciences and engineering are modelled by fractional differential equations, see [1-3]. In many cases these equations can be solved analytically and some cases where the equations have no analytical solutions, one way is to use numerical methods. According to [4], ADEs with FO have been widely used in groundwater hydrological research. Many researchers have studied the solutions of ADE with different FO such as [5-10], just to mention a few. In this paper, CFRDTM is proposed to obtain an efficient and accurate solution to ADE with FO. The emphasis is given to the most popular Caputo fractional operator which is more suitable for the study of fractional derivatives. The organization of the rest of the paper is as follows; Preliminaries are captured in Section 2. In Section 3, CFRDTM and its fundamental properties are presented. Section 4 consists of the illustrative examples and discussion of results. The conclusion of the paper is presented in Section 5.


2 Preliminaries

Definition 1: The Riemann-Liouville Fractional Integral Operator (RLFIO) of $f(t)$ is given by

$$J^\alpha_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tau > 0$$

(1)

where $\alpha \geq 0$ is the order, $f(t) \in C_\mu$ is a function, $\mu \geq 1, C_\mu$ is a space and $\Gamma(\alpha)$ is the gamma function of $\alpha$.

Definition 2: The Riemann-Liouville Fractional Derivative Operator (RLFDO) of $f(t)$ is given by

$$D^\alpha_f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, t > 0, n-1 < \alpha < n, n \in N$$

(2)

Definition 3: The Caputo Fractional Derivative (CFD) of $f(t)$ is given by

$$C^\alpha_D f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$$

(3)

for $\alpha \in (n-1, n], t > 0$.

Definition 4: The Caputo Time-Fractional Derivative Operator (CTFDO) of order $\alpha > 0$ is as follows

$$\frac{\partial^n u(x,t)}{\partial t^n}, \alpha = n$$

(4)

where $n$ is the smallest integer that exceeds $\alpha$, $u = u(x,t)$ and $u^{(n)}(x,t) = \frac{\partial^n u(x,t)}{\partial t^n}$.

In the Caputo fractional differential equation, initial conditions have clear physical interpretation which is the main advantage of CFDO over RLFO [11].

Definition 5. The series representation of the form

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{(n\alpha)!}, z \in C$$

(5)

is called the Mittag-Leffler Function (MLF) [12].

For more details about the properties and applications of fractional calculus; see [12, 13].

3 CFRDTM and its Fundamental Properties

The Caputo Fractional Reduced Differential Transform (CFRDT) of the function $a(x,t)$ is as follows

$$A^k_a(x) = \left[ \frac{\partial^k a(x,t)}{\partial t^k} \right]_{t=0}, \alpha \in (0,1], \quad k = 0,1,\ldots,n$$

(6)

where $\partial^k a(x,t) = \frac{\partial^k a(x,t)}{\partial t^k}$. Conversely the inverse CFRDT of $A^k_a(x)$ is defined as

$$a(x,t) = \sum_{k=0}^{n} A^k_a(x)(t-t_0)^{k/\alpha}, 0 < \alpha \leq 1$$

(7)
3.1 Fundamental Properties of CFRDTM

By means of (6) and (7), the fundamental properties of the method were given below.

| S/N | Functional Form | Transformed Form |
|-----|----------------|-----------------|
| 1   | \( p(x,t) = \varphi(x,t) \pm \xi(x,t) \) | \( P_k(x) = \Phi_k(x) \pm \Xi_k(x) \) |
| 2   | \( p(x,t) = c\varphi(x,t) \) | \( P_k(x) = c\Phi_k(x) \), where \( c \) is a constant. |
| 3   | \( p(x,t) = \varphi(x,t)\xi(x,t) \) | \( P_k(x) = \sum_{i=0}^{N} \Phi_i(x)\Xi_{k-i}(x) \) |
| 4   | \[ p(x,t) = \frac{x^{\alpha} - t^{\alpha}}{\Gamma(1+\alpha)} \] | \[ P_k(x) = \frac{x^{\alpha} - \delta_n(k-n)}{\Gamma(1+\alpha)} \], \( m,n \in \mathbb{N} \) |
| 5   | \[ p(x,t) = \frac{\partial_x^{\alpha} \varphi(x,t)}{\partial t^{\alpha}} \] | \[ P_k(x) = \frac{\partial_x^{\alpha} \Phi_k(x)}{\partial t^{\alpha}} \], where \( m \in \mathbb{N} \) |
| 6   | \[ p(x,t) = x^{\alpha} \] | \[ P_k(x) = x^{\alpha} \delta_{(k-1)}, i = 1,\ldots,m \] |
| 7   | \[ p(x,t) = e^{\lambda t} \] | \[ P_k(x) = \frac{\lambda^k}{k!} \] |

4 Illustrative Examples and Discussion of Results

4.1 Illustrative Examples

Example 1

Consider the ADE with FO of the form \[ \frac{\partial}{\partial t} D_t^\alpha u(x,t) = \mu u_{xx} - u_x, t > 0, \alpha \in (0,1) \] (8)

with initial condition (IC)

\[ u(x,0) = \exp(-x) \] (9)

Taking CFRDT of (8) and (9), yields

\[ U_{k+1}(x) = \frac{\Gamma(\alpha(k+1))}{\Gamma(\alpha(k+1)+1)} \left[ \frac{\mu}{\alpha^2} U_k(x) - \frac{\partial}{\partial x} U_k(x) \right] \] (10)

and

\[ U_0(x) = \exp(-x) \] (11)

respectively. Using (10) and (11), the following \( \{U_k(x)\}_{k=0}^{\infty} \) values are obtained

\[ U_1(x) = \frac{\exp(-x)B}{\alpha}, U_2(x) = \frac{\exp(-x)B^2}{(2\alpha)!}, U_3(x) = \frac{\exp(-x)B^3}{(3\alpha)!}, \]

\[ U_4(x) = \frac{\exp(-x)B^4}{(4\alpha)!}, \ldots, U_n(x) = \frac{\exp(-x)B^n}{(n\alpha)!}, \ldots \] (12)

with

\[ B = (1 + \mu) \] (13)
By means of (7), one obtains

\[ u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{\alpha k} \]  \hspace{1cm} (14)

Substituting (11) and (12) into (14) and simplifying further, one obtains

\[ u(x,t) = \lim_{n \to \infty} \sum_{m=0}^{n} \frac{\exp(-x) (Bt^{\alpha})^m}{(m\alpha)!} \]  \hspace{1cm} (15)

where \( B \) is given by (13). Equation (15) is the same solution as that given in [10] using fractional variational iteration method. The results generated via CFRDTM using (15) are displayed in the Figures 1-9.

Figure 1: The results generated via CFRDTM with \( \alpha = 1, \mu = 1 \)

Figure 2: The results generated via CFRDTM with \( \alpha = 1, \mu = 2 \)
Figure 3: The results generated via CFRDTM with $\alpha = 1, \mu = 3$

Figure 4: The results generated via CFRDTM with $\alpha = 0.9, \mu = 1$
Figure 5: The results generated via CFRDTM with $\alpha = 0.9, \mu = 2$

Figure 6: The results generated via CFRDTM with $\alpha = 0.9, \mu = 3$
Figure 7: The results generated via CFRDTM with $\alpha = 0.8, \mu = 1$

Figure 8: The results generated via CFRDTM with $\alpha = 0.8, \mu = 2$

Figure 9: The results generated via CFRDTM with $\alpha = 0.8, \mu = 3$

Example 2
Consider the ADE with FO of the form [10]

$$\partial_t^\alpha D_t^\alpha u = \kappa u_{xx} - \nu u_x, \ t > 0, \ 0 < \alpha \leq 1$$

with IC

$$u(x,0) = \sin(x)$$

(16)

(17)

By means of CFRDT, (16) and (17) become

$$U_{k+1} = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} \left( \kappa \frac{\partial^2}{\partial x^2} U_k(x) - \nu \frac{\partial}{\partial x} U_k(x) \right)$$

and

$$U_0(x) = \sin(x)$$

(18)

(19)

respectively. Using (18) and (19), the values of $\{U_k(x)\}_{k=0}^\infty$ are obtained as follows:
$$U_1(x) = \frac{-1}{\alpha!}(\kappa \sin(x) + \nu \cos(x))$$

$$U_2(x) = \frac{1}{(2\alpha)!}\left((\kappa^2 - \nu^2)\sin(x) + 2\kappa\nu\cos(x)\right)$$

$$U_3(x) = \frac{1}{(3\alpha)!}\left((\nu^3 - 3\kappa^2\nu)\cos(x) + (3\nu^2\kappa - \kappa^3)\sin(x)\right)$$

and so on. By means of the inversion formula of CFRDT, one gets

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{\alpha k}$$

Substituting (19), (20), (21) and (22) into (23), yields

$$u(x,t) = \sin(x) - \frac{t^\alpha}{\alpha!}\left((\kappa \sin(x) + \nu \cos(x)\right)$$

$$+ \frac{t^{2\alpha}}{(2\alpha)!}\left((\kappa^2 - \nu^2)\sin(x) + 2\kappa\nu\cos(x)\right)$$

$$+ \frac{t^{3\alpha}}{(3\alpha)!}\left((\nu^3 - 3\kappa^2\nu)\cos(x) + (3\nu^2\kappa - \kappa^3)\sin(x)\right)+...$$

Equation (24) is the same solution as that given in [10] using fractional variational iteration method.
The results generated via CFRDTM using (24) are displayed in the Figures 10-13.

Figure 10: The results generated via CFRDTM with $\alpha = 1, \kappa = 0.6, \nu = 0.2$
Figure 11: The results generated via CFRDTM with $\alpha = 0.75, \kappa = 0.6, \nu = 0.2$

Figure 12: The results generated via CFRDTM with $\alpha = 0.50, \kappa = 0.6, \nu = 0.2$

Figure 13: The results generated via CFRDTM with $\alpha = 0.25, \kappa = 0.6, \nu = 0.2$
4.2 Discussion of Results

It is observed from Figures 1-9 that increase in the values of $\mu$ leads to increase in the values of CFRDTM. It is also observed that the fractional order $\alpha$ has influence on the results generated via CFRDTM. As $\alpha$ decreases, the values of CFRDTM increase. It is observed from Figures 10-13 that the results generated via CFRDTM decrease as the fractional order $\alpha$ decreases with different values of $\kappa$ and $\nu$.

5 Conclusion

In this paper, CFRDTM for the solution of ADE with FO is proposed. The solutions of ADE with FO are constructed in the form of a convergent series by means of CFRDTM. Furthermore, it is evident that CFRDTM with $\alpha = 1$ reduces to the exact solution of the classical ADE. Moreover, the physical behaviour of the solution obtained via CFRDTM has been shown in terms of plots for different FO. Hence, it is concluded that CFRDTM is accurate, efficient, easy to implement and a good alternative approach to other existing tools or approaches for the solution of ADE with FO.

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