TOPOLOGY OF SCRAMBLED SIMPLICLES

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ABSTRACT. In this paper we define a family of topological spaces, which contains and vastly generalizes the higher-dimensional Dunce hats. Our definition is purely combinatorial, and is phrased in terms of identifications of boundary simplices of a standard d-simplex. By virtue of the construction, the obtained spaces may be indexed by words, and they automatically carry the structure of a \(\Delta\)-complex.

As our main result, we completely determine the homotopy type of these spaces. In fact, somewhat surprisingly, we are able to prove that each of them is either contractible or homotopy equivalent to an odd-dimensional sphere. We develop the language to determine the homotopy type directly from the combinatorics of the indexing word.

As added benefit of our investigation, we are able to emulate the Dunce hat phenomenon, and to obtain a large family of both \(\Delta\)-complexes, as well as simplicial complexes, which are contractible, but not collapsible.

1. A COMBINATORIAL FAMILY OF \(\Delta\)-COMPLEXES

1.1. Introduction.

Imagine we are given a \(d\)-simplex \(\Delta^d\), viewed as a \(\Delta\)-complex in a standard way: the vertices are indexed with numbers 1, \ldots, \(d+1\), and the order of the vertices in each boundary simplex is induced by that global order. There is, in a certain sense, a unique way to identify the boundary simplices with each other, if we want to identify as many simplices as possible, while at the same time preserving the property that the quotient has the induced \(\Delta\)-complex structure. This is achieved by taking any two boundary simplices of the same dimension and gluing them by the unique linear isomorphism which preserves the order of the vertices. When \(d=1\), we obtain a circle. When \(d=2\), we obtain the so-called Dunce hat. This is a classical \(\Delta\)-complex, which is contractible but not collapsible. Its second barycentric subdivision is a simplicial complex that is of course also contractible but not collapsible.

Graphically, we think about this boundary simplex identification as scrambling. Viewed as a \(\Delta\)-complex, this maximally scrambled \(d\)-simplex will have a single simplex in each dimension. This paper grew out of the attempts by the author to better understand and to generalize this well-known construction by relaxing the scrambling condition. Our suggestion is that, guided by a certain combinatorial principle, we identify only some of the boundary simplices of the same dimension, instead of gluing together all of them.

Specifically, we start by putting labels on the vertices of the original \(d\)-simplex. The label of vertex \(i\) is denoted by \(a_i\), for all \(i=1,\ldots,d+1\). We think about these labels as letters, so the ordered sequence of labels \((a_1,\ldots,a_{d+1})\) gives a word \(w\). Given any subsimplex \((i_1,\ldots,i_t)\), where \(i_1<\cdots<i_t\), of the original \(d\)-simplex, the corresponding ordered label sequence \((a_{i_1},\ldots,a_{i_t})\) gives a subword of \(w\). Using the same order-preserving

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1 We use the terminology of Hatcher, see [Ha02]; alternatively these spaces were called semisimplicial sets in [EZ50], triangulated spaces in [GM96], or, simply trisps in [Ko08].
linear isomorphism as above, we now identify any two boundary simplices, which yield the same subword. These simplices are necessarily of the same dimension, as this is just the length of the subword minus one. It is easy to see that the obtained space has the induced structure of a $\Delta$-complex, which we denote $\Delta_w$.

The maximally scrambled case above corresponds to putting the same label on all the vertices of $\Delta^d$, or, in other words, the word $w$ consists of a single letter repeated $d + 1$ times. At another extreme, if all the labels are different, then no identifications take place at all, and our space is just the original $d$-simplex itself. The whole family $\{\Delta_w\}$ can then be seen as approximating between simplices and higher-dimensional Dunce hats.

There is a rich and well-developed theory of combinatorially defined simplicial complexes, see, e.g., [Ko08]. However, to our knowledge very little work has been done in the category of $\Delta$-complexes, see, e.g., [Ko00, Ko02], where combinatorially defined $\Delta$-complexes, called there triangulated spaces, have been applied to analyze spaces of polynomials with multiple roots. Accordingly, we view the family $\{\Delta_w\}$ as a source of interesting combinatorially defined $\Delta$-complexes.

In this paper, we are able to completely determine the homotopy type of the $\Delta$-complexes $\Delta_w$, see our main Theorem 5.14. Somewhat surprisingly, all these spaces are either contractible or homotopy equivalent to odd-dimensional spheres. The combinatorial rule which reads off from the word $w$ the homotopy type of the corresponding space $\Delta_w$ is as follows. If $w = a_1v_1a_2v_2a_3\ldots a_qv_qa_q$, where $a_1, \ldots, a_q$ are (not necessarily distinct) letters, and $v_1, \ldots, v_q$ are (possibly empty) words, such that $a_i$ does not occur in $v_i$, for all $1 \leq i \leq q$, then $\Delta_w$ is homotopy equivalent to $S^{2q-1}$, else $\Delta_w$ is contractible.

As the added benefit, we are able to emulate the Dunce hat phenomenon, and to obtain a large family of both $\Delta$-complexes, as well as simplicial complexes, which are contractible, but not collapsible.

1.2. Preliminaries.

We use the notation $[n] = \{1, \ldots, n\}$, for all natural numbers $n$. Let us recall the definition of the $\Delta$-complexes.

**Definition 1.1.** The gluing data which defines a $\Delta$-complex $X$ consists of two parts:

- a family of sets $S_0, S_1, S_2, \ldots$;
- for each $0 \leq m \leq n$ and each order-preserving injection $f : [m + 1] \leftrightarrow [n + 1]$, we have a set map $B_f : S_n \to S_m$.

This data is subject to the following conditions:

1. for any pair of composable order-preserving injections $g : [k + 1] \leftrightarrow [m + 1]$ and $f : [m + 1] \leftrightarrow [n + 1]$, we have $B_{fg} = B_g \circ B_f$;
2. for any identity map $id_n : [n + 1] \leftrightarrow [n + 1]$, we have $B_{id_n} = id_{S_n}$.

The sets $S_n$ are the sets of $n$-simplices of $X$ and the maps $B_f$ are the boundary maps. Assume now that for some $n \geq 1$, we have picked $\sigma \in S_{n-1}$ and $\tau \in S_n$. For an order-preserving injection $f : [n] \leftrightarrow [n + 1]$, set $\text{sgn} f := (-1)^k$, where $k$ is the unique element in $[n + 1] \setminus \text{Im} f$. We now set

$$[\sigma : \tau] := \sum_f \text{sgn} f,$$

where the sum is taken over all order-preserving injections $f : [n] \leftrightarrow [n + 1]$, such that $B_f(\tau) = \sigma$. We refer to [Hatcher 2.1] and [Ko08 2.3] for further general details on $\Delta$-complexes.
For the sake of being self-contained, we define the notions which we need in this paper. In the next two definitions, assume we are given two $\Delta$-complexes $X$ and $\tilde{X}$ with respective gluing data $(\{S_n\}, \{B_f\})$ and $(\{\tilde{S}_n\}, \{\tilde{B}_f\})$.

**Definition 1.2.** We say that $X$ is isomorphic to $\tilde{X}$, if there exists a family of bijections $\alpha_n : S_n \to \tilde{S}_n$, for all $n \geq 0$, satisfying a commuting relation
\[
\tilde{B}_f \circ \alpha_n = \alpha_m \circ B_f : S_n \to \tilde{S}_m,
\]
for all order-preserving injections $f : [m + 1] \to [n + 1]$.

Such a family of bijections $\{\alpha_n\}$ is also called a $\Delta$-complex isomorphism between the complexes $X$ and $\tilde{X}$.

**Definition 1.3.** We define a new $\Delta$-complex, with gluing data $(\{T_n\}, \{C_f\})$, which we call the join of $X$ and $\tilde{X}$, and denote by $X \ast \tilde{X}$. To that end, we set
\[
T_n := \bigcup_{i+j+\alpha = n} \{(\sigma, \tilde{\sigma}) \mid \sigma \in S_i, \tilde{\sigma} \in \tilde{S}_j\}.
\]

Let now $f : [m + 1] \to [n + 1]$ be an order-preserving injection, and pick $\sigma \in S_i$, $\tilde{\sigma} \in \tilde{S}_j$, such that $i + j + 1 = n$. The map $f$ can be represented by order-preserving bijections $f : [\alpha + 1] \to [\tilde{\alpha} + 1]$, and $\tilde{f} : [\tilde{\alpha} + 1] \to [j + 1]$, where $\alpha := |\text{Im} f \cap [i + 1]| - 1$, and $\tilde{\alpha} := |\text{Im} \tilde{f} \cap [i + 1]| - 1$. We then set $C_f((\sigma, \tilde{\sigma})) := (B_f(\sigma), \tilde{B}_f(\tilde{\sigma}))$.

When $X$ and $Y$ are $\Delta$-complexes, we shall abuse notations and denote by the same letters the corresponding CW complexes and corresponding topological spaces, i.e., their geometric realizations. There will be different ways by which we shall relate our objects to each other. In connection with that we would like to remind the reader about the following sequence of implications:

\[
\begin{align*}
X \text{ and } Y \text{ are isomorphic as } \Delta\text{-complexes} & \quad \Downarrow \\
X \text{ and } Y \text{ are isomorphic as } \text{CW complexes} & \quad \Downarrow \\
X \text{ and } Y \text{ are homeomorphic} & \quad \Downarrow \\
X \text{ and } Y \text{ are homotopy equivalent} & \quad \Downarrow
\end{align*}
\]

There is a notion of elementary collapses for CW complexes, see [Co73, §4]. We do not need the full generality for $\Delta$-complexes. The following operation is very close to elementary collapses of simplicial complexes and is sufficient for our purposes.

**Definition 1.4.** Consider a $\Delta$-complex $X$ given by the gluing data $(\{S_n\}, \{B_f\})$. Assume we have $\tau \in S_n$, and $\sigma \in S_{n-1}$, such that
\begin{enumerate}
  \item there exists a unique order-preserving injection $f : [n] \to [n + 1]$, such that $B_f(\tau) = \sigma$;
  \item if we have $\tilde{\tau} \in S_n$, and an order-preserving injection $\tilde{f} : [n] \to [n + 1]$, such that $\tilde{B}_f(\tilde{\tau}) = \sigma$, then $\tilde{f} = f$ and $\tau = \tilde{\tau}$;
  \item the simplex $\tau$ is maximal in the following sense: there does not exist another simplex $\delta \in S_{n+1}$, such that $B_\delta(\delta) = \tau$, for some order-preserving injection $g : [n + 1] \to [n + 2]$.
\end{enumerate}

Removing $\tau$ from $S_n$, $\sigma$ from $S_{n-1}$, and restricting the maps $B_f$ accordingly yields a new $\Delta$-complex, which we shall call $X \setminus \{\sigma, \tau\}$. We say that it is obtained from $X$ by an elementary collapse.
Sometimes, we call the pair \((\sigma, \tau)\) itself an elementary collapse. When \(X\) and \(Y\) are \(\Delta\)-complexes, we have the following sequence of strict implications:

\[
\begin{align*}
\text{there exists a sequence of elementary collapses reducing } X & \text{ to } Y \\
\downarrow & \\
\text{there exists a strong deformation retraction from } X & \text{ onto } Y \\
\downarrow & \\
X \text{ and } Y & \text{ are homotopy equivalent.}
\end{align*}
\]

Finally, we note that all homology groups which we consider in this paper are taken with integer coefficients.

1.3. The scrambled simplices.

Let us now describe the language in which we want to talk about the scrambled simplices.

**Definition 1.5.** Given any set \(S\), we define a word \(w\) in alphabet \(S\) to be any finite ordered tuple \((a_1, \ldots, a_n)\) of elements of \(S\); we allow repetitions in that tuple. The elements \(a_1, \ldots, a_n\) are referred to as letters of \(w\). The number \(n\) is called the length of \(w\), which we denote by \(l(w)\). We set \(\text{supp}(w) := \{a_1, \ldots, a_n\}\), and call it the support set of \(w\).

Note, that \(|\text{supp}(w)| \leq l(w)\), and in general it is possible to have the strict inequality.

We shall write \(w = a_1, \ldots, a_n\), where for all \(1 \leq i \leq n\), we have \(a_i \in S\). Oftentimes we shall skip the commas and simply write \(w = a_1 \ldots a_n\). We shall use the power notation to denote repetitions of letters, so \(w = a^3\) means \(w = aaa\), and \(w = (a^2b)^3b = a^2ba^2b^2 = aabaabb\). For any \(0 \leq k \leq n\), we shall say that the word \(a_1 \ldots a_k\) is a prefix of \(w\); when additionally \(k \leq n - 1\), we shall say that it is a proper prefix of \(w\).

We say that \(w = a_1^{i_1} \ldots a_t^{i_t}\) is the reduced form\(^2\) of the word \(w\), if \(a_i \neq a_{i+1}\), for all \(1 \leq i \leq t - 1\), and \(a_i \geq 1\), for all \(1 \leq i \leq t\). Obviously, every word has a unique reduced form, and, when not stated otherwise, we shall assume that our words are written in a reduced form.

**Definition 1.6.** Assume we are given a word \(w = a_1 \ldots a_n\), and a subset \(I \subseteq [n]\), say \(I = \{i_1, \ldots, i_k\}\), where \(i_1 < \cdots < i_k\). We set \(w_I := a_{i_1} \ldots a_{i_k}\), and call it the \(I\)-subword of \(w\).

It is convenient to identify a subset \(I \subseteq [n]\), such that \(|I| = k\), with an order-preserving injection \(I : [k] \hookrightarrow [n]\).

**Definition 1.7.** Given a word \(w = a_1 \ldots a_n\), \(a_i \in S\), \(n \geq 1\), the scrambled simplex \(\Delta_w\) is the \(\Delta\)-complex defined as follows.

- For all \(l \geq 0\), we set \(S_I\) to be the set of all \(I\)-subwords of \(w\), such that \(|I| = l + 1\).
- Assume we are given an order-preserving injection \(f : [m + 1] \hookrightarrow [l + 1]\), where \(0 \leq m \leq l\). Take \(w_I \in S_{f(I)}\), where \(|I| = l + 1\). We have an order-preserving injection \(I : [l + 1] \hookrightarrow [n]\), and can consider the composition \(I \circ f : [m + 1] \hookrightarrow [n]\). We now set
  \[
  B_I(w_I) := w_{\text{Im}(I \circ f)}. 
  \]

Clearly, the \(\Delta\)-complex \(\Delta_w\) only depends on the underlying partition of \([t + 1]\) corresponding to the word \(w\) in the following sense: given words \(w\) in alphabet \(S\) and \(w'\) in alphabet \(S'\), and a renaming function \(f : S \to S'\), such that \(f(w) = w'\), then \(f\) induces a \(\Delta\)-complex isomorphism, see Definition\(^2\) between \(\Delta_w\) and \(\Delta_{w'}\).

Intuitively, the gluing data of \(\Delta_w\) simply records what happens when we delete letters. As mentioned above, there is an alternative description of \(\Delta_w\) as a quotient complex of the

\(^2\)We find convenient to slightly abuse notations here and use the same letters \(a_i\).
$d$-simplex $\Delta^d$, where $d = l(w) - 1$. In this description, we start with a $d$-simplex $\Delta$. Its boundary simplices are indexed by the subsets of $[d + 1]$, so let $\Delta^I$ denote the boundary simplex corresponding to $I \subseteq [d + 1]$. Now, if $w_I = w_J$, then we identify $\Delta^I$ with $\Delta^J$ using a linear isomorphism which preserves the order of the vertices. In particular, if $w = a_1 \ldots a_{d+1}$, for $a_i \neq a_j$, for all $i, j$, then $\Delta_w$ is just a $d$-simplex. Accordingly, we think of $\Delta_w$ as a $d$-simplex whose boundary has been scrambled in a certain pattern, given by the word $w$.

It is immediate that $\dim \Delta_w = d$. Furthermore, we have a cellular isomorphism $\Delta^w \cong \Delta^\bar{w}$, where $\bar{w}$ is the word $w$ written backwards. This isomorphism does not have to be a $\Delta$-complex isomorphism, but it certainly induces a homeomorphism. For example, we have $\Delta_{abb} \cong \Delta_{bba} \cong \Delta_{aab}$.

2. Examples and first properties

2.1. The $\Delta$-complexes of subwords of a word of length at most 3.

We shall now describe, up to isomorphism, the cell complexes $\Delta_w$, when $l(w) \leq 3$. If $l(w) = 1$, then we only need to consider $w = a$, and clearly $\Delta_w$ is just a point.

When $l(w) = 2$, we have two cases: $w = ab$ and $w = aa$. We see that $\Delta_{ab}$ is a 1-simplex, and $\Delta_{aa}$ is the CW complex with one 0-cell and one 1-cell, which is homeomorphic to $S^1$.

When $l(w) = 3$, we have the cases: $w = abc$, $w = a^2b$, $w = aba$, and $w = a^3$. Accordingly, $\Delta_{abc}$ is a 2-simplex, $\Delta_{a^2b}$ is homeomorphic to a disc, so is contractible, and $\Delta_{aba}$ is homeomorphic to the topological space obtained from the disc by identifying two of its boundary points, which is homotopy equivalent to $S^1$. Finally, $\Delta_{a^3}$ is the classical Dunce hat, see, e.g., [Ze64] for further details. It is well-known that this space is contractible as well. All the three nontrivial cases are shown on Figure 2.1.

![Figure 2.1. The $\Delta$-complexes of subwords of a word of length at most three. The arrows on the edges indicate which edges are glued together and in which direction the gluing is done.](image)

2.2. Concatenation of words from disjoint alphabets.

Before we proceed with computing further examples, we make the following simple, but useful proposition, whose formal verification is left to the reader.

**Proposition 2.1.** Assume $w$ is a concatenation of two words $w = w_1 \circ w_2$, such that $\text{supp}(w_1)$ and $\text{supp}(w_2)$ are disjoint. Then we have a $\Delta$-complex isomorphism.

\[
\Delta_w \cong \Delta_{w_1} \ast \Delta_{w_2}.
\]
In a situation like this, we shall say that \( w \) is decomposable, and else we say that \( w \) is indecomposable. The Tables 2.1 and 2.2 list, up to renaming and up to reversing the order of the letters, all indecomposable words of length at most 5.

| \( l(w) \) | indecomposable words |
|---|---|
| 1 | \( a \) |
| 2 | \( a^2 \) |
| 3 | \( a^3, aba \) |
| 4 | \( a^4, aba^2, abab, ab^2a, abca \) |

Table 2.1. The 9 indecomposable words of length at most 4.

| \( |\text{supp}(w)| \) | indecomposable words |
|---|---|
| 1 | \( a^5 \) |
| 2 | \( aba^3, a^2ba^2, abab^2, ab^2ab, ab^3a, ab^2a^2, ababa \) |
| 3 | \( abca^2, abaca, abacb, ab^2ca \) |
| 4 | \( abcda \) |

Table 2.2. The 13 indecomposable words of length 5.

Clearly, when interested in determining the topology of the complexes \( \Delta_w \) it is fully sufficient to restrict ourselves to considering the indecomposable words only.

2.3. The higher-dimensional Dunce hats.
The indecomposable words \( w = a^n \) correspond to an interesting family of spaces. As mentioned above \( \Delta_{a^n} \) is a classical Dunce hat. It is then easy to understand \( \Delta_{a^t} \). Indeed, the 2-skeleton of this CW complex is homeomorphic to a Dunce hat, hence it is contractible. Therefore, \( \Delta_{a^n} \) is obtained by attaching a 3-cell to a contractible space. Contracting the 2-skeleton to a point we see that \( \Delta_{a^n} \) is homotopy equivalent to \( S^3 \).

In general, the spaces \( \Delta_{a^n} \) were defined by Andersen, Marjanović, and Schori, see [AMS93], using the symmetric products of intervals, see also Borsuk and Ulam, [BU31]. When \( n \) is odd, the spaces \( \Delta_{a^n} \) are called higher-dimensional Dunce hats. The topology of \( \Delta_{a^n} \) was completely determined in [AMS93].

Proposition 2.2. ([AMS93 Theorem 2.3]). When \( n \) is odd, the spaces \( \Delta_{a^n} \) are contractible, and when \( n \) is even, the space \( \Delta_{a^n} \) is homotopy equivalent to a sphere of dimension \( n - 1 \).

The case when \( n \) is odd is the interesting one and was proved using Homotopy Addition Theorem of Hu, [Hu53]. The case when \( n \) is even is a simple corollary. Since \( \Delta_{a^n} \) is always obtained from \( \Delta_{a^{n-1}} \) by attaching a single \( (n-1) \)-cell, our above argument for \( n = 4 \) works in general. The space \( \Delta_{a^{n-1}} \) is contractible, and its inclusion into \( \Delta_{a^n} \) is a cofibration, see, e.g., [Ko08], so we can just shrink it to a point without changing the homotopy type, and end up with an \( (n - 1) \)-sphere.

Alternatively, it is easy to prove Proposition 2.2 using a version of Whitehead’s theorem, as is done in this paper.
3. FORMULAS FOR THE REDUCED EULER CHARACTERISTICS

3.1. The arrow terminology and the recursive formula.
In this section, we compute the Euler characteristic of the complexes $\Delta_w$ for all words $w$. This simple derivation is the first step in the general understanding of the homotopy type of the spaces $\Delta_w$. We also design the combinatorial language, which we will need later to formulate our main result.

**Definition 3.1.** For any word $w$, let $\mathcal{E}(w)$ denote the reduced Euler characteristics of the $\Delta$-complex $\Delta_w$.

Note that $\mathcal{E}(w)$ counts the subwords of $w$ with weights, with each word giving the contribution $(-1)^{l(w)+1}$. For example, we have $\mathcal{E}(a^{2t}) = -1$, $\mathcal{E}(a^{2t+1}) = 0$, $\mathcal{E}(aba) = -1$, etc. Also, for the empty word we have $\mathcal{E}(\emptyset) = -1$. We shall prove the following, somewhat surprising result:

for all words $w$, we either have $\mathcal{E}(w) = 0$, or $\mathcal{E}(w) = -1$.

The more precise statement is given in Theorem 3.9.

Before we can prove the result on the Euler characteristics, we need to introduce some new terminology.

**Definition 3.2.** Assume $w$ is a word, and pick a letter $a \in \text{supp}(w)$. The new word, which we denote $w \downarrow a$, is obtained from $w$ by finding the leftmost occurrence of $a$ in $w$ and deleting everything to the left of it, including $a$ itself.

For example, $a^n \downarrow a = a^{n-1}$, $aba \downarrow a = ba$, $aba \downarrow b = a$, and if $w = va$, such that $a \notin \text{supp}(v)$, then $w \downarrow a = \emptyset$. Let us furthermore introduce the following short hand notation: for any word $v = a_1 \ldots a_t$, where $a_i$'s are letters, we set

$$w \downarrow v := (\ldots(w \downarrow a_1) \downarrow a_2) \downarrow \ldots \downarrow a_t.$$  

**Proposition 3.3.** Assume $w$ is a word, and $a$ is a letter in its support. There is a 1-1-correspondence between the subwords of $w \downarrow a$ and those subwords of $w$ which begin with the letter $a$. This correspondence is given by adding $a$ as the first letter to a subword of $w \downarrow a$.

**Proof.** Let $A$ denote the set of all subwords of $w \downarrow a$, and let $B$ denote the set of all subwords of $w$ which begin with the letter $a$. Let $\varphi : A \to B$ be the map which adds $a$ as the first letter, and let $\psi : B \to A$ be the map which deletes the first letter (which, by definition of $B$, must be $a$).

It is obvious that $\varphi$ is well-defined, since $a$ can always be added on the left to any subword of $w \downarrow a$. The $\psi$ is well-defined is equally obvious, since when $av$ is a subword of $w$, $v$ will be the subword of $w \downarrow a$, essentially by the definition of $w \downarrow a$. The two maps are also inverses of each other, so we are done with the proof.

The next proposition allows us to use recursion to calculate the function $\mathcal{E}$.

**Proposition 3.4.** Assume $w$ is an arbitrary word, then we have the following recursive formula

$$(3.1) \quad \mathcal{E}(w) = - \sum_{x \in \text{supp}(w)} \mathcal{E}(w \downarrow x) - 1.$$  

**Proof.** Recall that the left hand side of (3.1) counts the subwords of $w$, with each word giving the contribution $(-1)^{l(w)+1}$. The empty word gives the term $-1$, and all other words
we can rewrite as
\[ E(w) = -1 + \sum_{x \in \text{supp}(w)} E_x, \]
where for each \( x \in \text{supp}(w) \), the term \( E_x \) denotes the total contribution of the subwords of \( w \) which start with \( x \). By Proposition 3.3 there is a bijection between the set of such words and the subwords of \( w \). This bijection changes the length of the word by 1, hence it changes the sign of the contribution. We can therefore conclude that for all \( x \in \text{supp}(w) \), we have the equality \( E_x = -E(w \downarrow x) \). Substituting this into (3.2) we obtain the identity (3.1).

In particular, if \( \text{supp}(w) = \{a\} \), then for all \( n \geq 1 \), we get \( E(a^n) = -E(a^{n-1}) - 1 \), which we can rewrite as \( E(a^{n-1}) + E(a^n) = -1 \). This is consistent with the direct observation that
\[ E(a^n) = \begin{cases} -1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \]

If \( \text{supp}(w) = \{a, b\} \), then (3.1) says
\[ E(w) = -E(w \downarrow a) - E(w \downarrow b) - 1. \]
Let \( \text{alt}(t) \) denote the word of length \( t \) consisting of two alternating letters, say \( \text{alt}(4) = abab \).
As another example, we calculate \( f_t := E(\text{alt}(t)) \). Clearly, we have \( f_0 = -1 \) and \( f_1 = f_2 = 0 \). Equation (3.4) tells us that \( f_t = -f_{t-1} - f_t - 2 - 1 \), for all \( t \geq 2 \). Reindexing and moving terms yields the following identity:
\[ f_n + f_{n+1} + f_{n+2} = -1, \text{ for all } n \geq 0. \]
Comparing (3.5) for two consecutive values of \( n \), we conclude that \( f_n \) is periodic with period 3, in other words that \( f_n = f_{n+3} \), for all \( n \geq 0 \). We conclude that
\[ E(\text{alt}(n)) = \begin{cases} -1, & \text{if } n \text{ is divisible by } 3; \\ 0, & \text{otherwise.} \end{cases} \]

Later we shall see that \( \Delta_{\text{alt}(n)} \) is actually contractible unless \( n \) is divisible by 3. If \( n = 3k \), then we will show that \( \Delta_{\text{alt}(3k)} \cong S^{3k-1} \).

In fact, the equations (3.3) and (3.6) can easily be generalized to words, where a certain set of \( t \) letters repeats in a circular manner: \( w = a_1 \ldots a_t a_1 \ldots a_t \ldots \). For such a word we have \( E(w) = -1 \) if the length of \( w \) is divisible by \( t + 1 \), and \( E(w) = 0 \) otherwise.

3.2. Elimination of circular words and the main Euler characteristics theorem.

The following proposition provides the crucial step in our computation.

Proposition 3.5. Assume \( w = auav \), where \( a \) is a letter, and \( u \) and \( v \) are words, such that \( a \notin \text{supp}(u) \). Then we have the equality \( E(w) = E(v) \).

Remark 3.6. We would like to point out that it is allowed for words \( u \) and \( v \) in Proposition 3.5 to be empty. When \( v \) is empty, we recover a simple corollary of Proposition 4.1. When \( u \) is empty, we obtain the equality \( E(a^2v) = E(v) \), for an arbitrary word \( v \).

Proof of Proposition 3.5.

By the recursive formula (3.1), we have
\[ E(w) = -\sum_{x \in \text{supp}(w), x \neq a} E(w \downarrow x) - E(w \downarrow a) - 1. \]
Note that $w \downarrow a = uav$. Applying (3.1) to that word, we obtain

$$\mathcal{E}(uav) = - \sum_{x \in \text{supp}(uav), x \neq a} \mathcal{E}(uav \downarrow x) - \mathcal{E}(uav \downarrow a) - 1. \tag{3.8}$$

Let us now substitute (3.8) into (3.7). We get

$$\mathcal{E}(w) = - \sum_{x \in \text{supp}(w), x \neq a} \mathcal{E}(w \downarrow x) + \left( \sum_{x \in \text{supp}(uav), x \neq a} \mathcal{E}(uav \downarrow x) + \mathcal{E}(uav \downarrow a) + 1 \right) - 1 = \mathcal{E}(uav \downarrow a) + \sum_{x \in \text{supp}(w), x \neq a} (\mathcal{E}(uav \downarrow x) - \mathcal{E}(w \downarrow x)) = \mathcal{E}(v), \tag{3.9}$$

where the penultimate equality is obtained by using the fact that $\text{supp}(w) = \text{supp}(uav)$, and the last equality follows from the equalities $uav \downarrow a = v$, and $uav \downarrow x = w \downarrow x$, whenever $x \in \text{supp}(w), x \neq a$. □

**Definition 3.7.** Let $w$ be an arbitrary word. We introduce the following terminology.

- We call $w$ **circular**, if it is of the form $w = ava$, where $a$ is a letter, and $v$ is a word, which is possibly empty, such that $a \notin \text{supp}(v)$.
- We call $w$ **spherical**, if it can be represented as a concatenation of circular words.
- We call $w$ **conical**, if it is of the form $w = av$, where $a$ is a letter, and $v$ is a word, which is possibly empty, such that $a \notin \text{supp}(v)$.

Note, that we consider the empty word to be spherical, and we view it as a concatenation of the empty set of circular words. We do not consider the empty word to be either circular, or conical.

Clearly, when $w$ is conical, the $\Delta$-complex $\Delta_w$ is a cone with apex $a$; in particular, $\mathcal{E}(w) = 0$. Furthermore, if $w = ava$ is circular, then Proposition 4.1 tells us that $\Delta_w$ is homotopy equivalent to a circle, and so $\mathcal{E}(w) = -1$. This explains our terminology.

**Proposition 3.8.** The following is true for all words.

1. A decomposition of a spherical word into circular one is unique.
2. If a word $w$ is not spherical, then there is a unique decomposition $w = uv$, where $u$ is a spherical word, which is possibly empty, and $v$ is a conical word.

**Proof.** To see (1), let $a$ be the first letter of a spherical word $w$. The first circular word in the decomposition of $w$ must be the subword between the two leftmost occurrences of $a$, including the letter $a$ on both ends. Proceeding left to right we see that the entire decomposition is unique.

To see (2) proceed as in the argument above. Clearly, unless the word is spherical, we will end up with the decomposition $w = w_1 \ldots w_k a\overline{v}$, where the words $w_1, \ldots, w_k$ are circular, and $a$ is a letter, such that $a \notin \text{supp}(\overline{v})$. Setting $u := w_1 \ldots w_k$, and $v := a\overline{v}$, we obtain the desired decomposition. □

As the first example, consider the words $w = a^n$. When $n = 2$, such a word is circular. Hence, when $n$ is even, such a word is spherical. When $n$ is odd, we can decompose $w = a^{n-1} \cdot a$, where $a^{n-1}$ is spherical and $a$ is conical.

Another example is provided by the alternating words $w = \text{alt}(n)$. The decomposition $(ab)^3 = aba \cdot bab$ shows that $(ab)^3$ is spherical, and hence $(ab)^3t$ is spherical for any $t$. When $n = 3t + 1$, we get the decomposition $w = (ab)^3t \cdot ab$, where $ab$ is conical. When
Theorem 3.9. For an arbitrary word \( w \), we have

\[
\mathcal{E}(w) = \begin{cases} 
-1, & \text{if } w \text{ is spherical;} \\
0, & \text{otherwise.}
\end{cases}
\]

Proof. If \( w \) is spherical, then it follows from repeated application of Proposition 3.5 that \( \mathcal{E}(w) = \mathcal{E}(\emptyset) = -1 \). If \( w \) is not spherical, by Proposition 3.8(2), we can write \( w = uv \), where \( u \) is spherical and \( v \) is conical. Repeatedly applying Proposition 3.5 we obtain \( \mathcal{E}(w) = \mathcal{E}(v) = 0 \). \( \square \)

4. Fundamental group of \( \Delta_w \)

We start with a simple observation.

Proposition 4.1. When the word \( w \) is circular, the \( \Delta \)-complex \( \Delta_w \) is homotopy equivalent to \( S^1 \).

Proof. Assume \( w = ava \), where \( a \) is a letter, and \( v \) is a word, such that \( a \notin \text{supp}(v) \). Let \( \tilde{a} \) be a letter, such that \( \tilde{a} \notin \text{supp}(w) \), and set \( \tilde{w} := av\tilde{a} \). By the concatenation formula (2.1), the cell complex \( \Delta_{\tilde{w}} \) is obtained from \( \Delta_v \) by coning twice, with apexes \( a \) and \( \tilde{a} \). In particular, the space \( \Delta_{\tilde{w}} \) is contractible. On the other hand, the \( \Delta \)-complex \( \Delta_{\tilde{w}} \) is obtained from \( \Delta_{\tilde{a}} \) by identifying two of its vertices, namely the ones labeled by \( a \) and by \( \tilde{a} \). The proposition follows now from an easy general fact, that gluing together two vertices in a connected cell complex \( K \) results in a space which is homotopy equivalent to a wedge of \( K \) with \( S^1 \). \( \square \)

Proposition 4.1 covers many cases of the words from Tables 2.1 and 2.2. When \( l(w) = 4 \), it covers words \( ab^2a \) and \( abca \), and when \( l(w) = 5 \), we get the words \( ab^3a \), \( ab^2ca \), and \( abeda \).

It turns out, that not only the circular words are the only ones for which \( \Delta_w \) is homotopy equivalent to \( S^1 \), but, in fact, these are the only words, for which \( \Delta_w \) is not simply connected.

Theorem 4.2. Let \( w \) be an arbitrary word. The fundamental group of the \( \Delta \)-complex \( \Delta_w \) is given by

\[
\pi_1(\Delta_w) = \begin{cases} 
\mathbb{Z}, & \text{if } w \text{ is circular;} \\
0, & \text{otherwise.}
\end{cases}
\]

Proof. By Proposition 4.1 we know that when \( w \) is circular, the \( \Delta \)-complex \( \Delta_w \) is actually homotopy equivalent to a circle, so obviously, \( \pi_1(\Delta_w) \cong \mathbb{Z} \).

Assume now that the word \( w = a_1 \ldots a_t \) is not circular. Take \( x := a_1 \) to be the base point, and consider \( \pi_1(\Delta_w, x) \). To start with, if \( x \notin \{a_2, \ldots, a_t\} \), then \( \Delta_w \) is a cone with apex \( x \), so the fundamental group is trivial. Assume therefore, that \( x \in \{a_2, \ldots, a_t\} \). Let \( m \) denote the minimal index such that \( m \geq 2 \) and \( a_m = x \). Since the word \( w \) is not circular, we also have \( m \leq t - 1 \).

The subword \( a_1a_m = xx \) indexes an edge of \( \Delta_w \), which is a loop based at \( x \). If we fix a choice of orientation on that loop, we obtain a representation of an element \( h_x \in \pi_1(\Delta_w, x) \). That element is easy to understand. Namely, consider the subword \( a_1a_2a_m = xx \). Independently of the fact, whether \( a_1 = x \), the boundary of the corresponding 2-simplex, viewed as a loop in \( \Delta_w \) tells us that \( h_x = 0 \).
It is now a general fact about $\Delta$-complexes, that any element of the fundamental group $g \in \pi_1(\Delta_0, x)$ can be represented as a sequence of edges $x b_1, \ldots, b_k x$, such that $k \geq 0$, and $b_i \in \text{supp}(w)$, for all $1 \leq i \leq k$. Call this an edge representation.

Assume that the fundamental group $\pi_1(\Delta_0, x)$ is non-trivial. Consider all nontrivial elements of $\pi_1(\Delta_0, x)$, and all their edge representations. Pick among all these representations one which minimizes $k$, say it consists of $l + 1$ edges, and let $g$ denote the represented element. Since the loop $h_2$ represents a trivial element, we must have $l \geq 1$. Furthermore, since $l$ is minimal possible, we have $x \notin \{b_1, \ldots, b_l\}$.

Assume first $l = 1$. We have $b_1 \neq x$. Either $a_1 b_1 a_m = x b_1 x$ or $a_1 a_m b_1 = x b_1$ is a subword of $w$. The corresponding 2-simplex provides a path homotopy between the concatenation of $x b_1$ and $b_1 x$ and one of the orientations of the loop $x x$. Since the latter represents a trivial element of the fundamental group, we conclude that $g = 0$, yielding a contradiction.

Assume now, that $l \geq 2$. We have $x \notin \{b_1, \ldots, b_l\}$. Either $x b_1 b_2$ or $x b_2 b_1$ is a subword of $w$, and therefore indexes a 2-simplex of $\Delta_0$. This means that the concatenation of edges $x b_1$ and $b_1 b_2$ is path-homotopic to the edge $x b_2$, contradicting the assumption that $l$ is smallest possible.

We conclude that the group $\pi_1(\Delta_0, x)$ is trivial. \hfill $\square$

5. The main theorem

5.1. Orders on exponential presentations of subwords.
Assume $a_1^{m_1} \cdots a_t^{m_t}$ is the reduced form of a word $w$, see Subsection 1.3. All of the subwords of $w$ can be written as $v = a_1^{\beta_1} \cdots a_t^{\beta_t}$, where $0 \leq \beta_i \leq a_i$, for all $i = 1, \ldots, t$; note that we are forced to allow $\beta_i = 0$. This is not necessarily a reduced form of the word $v$ and it is clearly not unique. For example, when $w = aba = a b^1 a^1$, we have two presentations for the subword $v = a$; namely, $v = a^0 b^0 a^0$ and $v = a^0 b^0 a^1$.

Once the word $w$ fixed, it is sufficient to simply write the tuples of the exponents to (non-uniquely) record the subwords. So, in the previous example, we could use the tuple $(1, 1, 1)$ to encode $w$ itself, while the two presentations of $v$ would be denoted by the tuples $(1, 0, 0)$ and $(0, 0, 1)$. We shall call such a tuple the exponential presentation of the word $v$ as a subword of $w$, and we let $F_w(v)$ denote the set of all these exponential presentations. So, in the above example, we have $F_{aba}(a) = \{(1, 0, 0), (0, 0, 1)\}$.

The set of $t$-tuples of numbers may be equipped with various standard orders. Here we will need the domination order, which is a partial order and is denoted by $\succ$, and two total orders: the lexicographic order, denoted by $\geq_{\text{lex}}$ and the colexicographic order, denoted by $\geq_{\text{colex}}$. Let us recall what these orders are.

- In the domination (partial) order, we say that $(x_1, \ldots, x_t) \geq (y_1, \ldots, y_t)$ if and only if $x_i \geq y_i$, for all $i = 1, \ldots, t$.
- In the lexicographic order, we order the tuples as words in a dictionary. In other words, $(x_1, \ldots, x_t) \geq_{\text{lex}} (y_1, \ldots, y_t)$ if and only if there exists $1 \leq k \leq t$, such that $x_1 = y_1, \ldots, x_{k-1} = y_{k-1}$, and $x_k > y_k$.
- The colexicographic order is essentially the same as the lexicographic one, except we read the $t$-tuples from right to left instead. That is, $(x_1, \ldots, x_t) \geq_{\text{colex}} (y_1, \ldots, y_t)$ if and only if there exists $1 \leq k \leq t$, such that $x_t = y_t, \ldots, x_{k+1} = y_{k+1}$, and $x_k > y_k$.

Clearly, all of these orders are inherited by the set $F_w(v)$, for arbitrary $w$ and $v$.

5.2. Shifted presentations.
The next definition shall help us to standardize the ways we deal with subwords of a word.
Definition 5.1. Assume, we are given a word $w$, whose reduced form is $a_1^i \ldots a_t^i$, and we are given a subword $v$. We say that $(\beta_1, \ldots, \beta_t) \in F_w(v)$ is left-shifted if it is maximal in $F_w(v)$ with respect to the lexicographic order. We say that it is right-shifted if it is maximal in $F_w(v)$ with respect to the colexicographic order.

Obviously, for an arbitrary word $w$ and a subword $v$, both the left-shifted and the right-shifted presentations of $v$ exist and are unique.

Definition 5.2. Assume again that a word $w$ is given by its reduced form $a_1^i \ldots a_t^i$, and let $p$ be an arbitrary index, $1 \leq p \leq t$. Set $w' := a_1^i \ldots a_p^i$, and $w'' := a_p^i \ldots a_t^i$. We say that the tuple $(\beta_1, \ldots, \beta_t) \in F_w(v)$ is $p$-shifted if $(\beta_1, \ldots, \beta_p) \in F_w(v')$ is left-shifted and $(\beta_p, \ldots, \beta_t) \in F_w(v'')$ is right-shifted, where $v' = a_1^i \ldots a_p^i$, and $v'' = a_p^i \ldots a_t^i$.

In the special cases $p = 1$ and $p = t$, we recover the notions of being right-shifted and being left-shifted, respectively. In general, in contrast to the left- and right-shifted cases, we can only conclude that a $p$-shifted presentation exists. For example, a $p$-shifted presentation can easily be constructed by starting with any tuple $(\beta_1, \ldots, \beta_t)$ in $F_w(v)$, then first left-shifting the tuple $(\beta_1, \ldots, \beta_p)$, and then right-shifting the tuple $(\beta_p, \ldots, \beta_t)$. However, the $p$-shifted tuple is by no means unique. In the example above, where $w = aba$ and $v = a$, both $(1, 0, 0)$ and $(0, 0, 1)$ are $2$-shifted. We do however have the following proposition.

Proposition 5.3. Assume $w$ is a word, given by its reduced form $a_1^i \ldots a_t^i$, and let $v$ be a subword of $w$. Assume we have an index $1 \leq p \leq t$, and a $p$-shifted exponential presentation of $v$, $\beta = (\beta_1, \ldots, \beta_i)$, such that $\beta_p \geq 1$. Then the $p$-shifted exponential presentation of $v$ is unique.

Proof. Assume that we have another $p$-shifted exponential presentation of $v$, say $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_i)$. Let $(\beta_i, \ldots, \beta_i)$ be the tuple of all $\beta_i, \ldots, \beta_i \neq 0$, such that $1 \leq i < \cdots < i_k \leq p - 1$, and $a_1, \ldots, a_{i_k} \neq a_p$. Symmetrically on the right, let $(\beta_j, \ldots, \beta_j)$ be the tuple of all $\beta_j, \ldots, \beta_j \neq 0$, such that $p + 1 \leq j_1 < \cdots < j_m \leq t$, and $a_{j_1}, \ldots, a_{j_m} \neq a_p$. Let the tuples $(\tilde{\beta}_i, \ldots, \tilde{\beta}_i)$ and $(\tilde{\beta}_j, \ldots, \tilde{\beta}_j)$ be defined the same way for $\tilde{\beta}$.

Assume $a_1^{i_1} \ldots a_{i_k}^{i_k} \neq a_1^{i_1} \ldots a_{i_k}^{i_k} \ast a_{i_k}^{i_k} \ldots a_t^{i_k}$. Flipping the word, if necessary, we can assume, without loss of generality, that $a_1^{i_1} \ldots a_{i_k}^{i_k}$ is a proper prefix of $a_1^{i_1} \ldots a_{i_k}^{i_k}$. But that would mean that the whole subword $a_1^{i_1} \ldots a_p^{i_p}$ would have an exponential presentation within $(a_1, \ldots, a_0)$. This contradicts to our assumption that the exponential presentation $(\beta_1, \ldots, \beta_p)$ is left-shifted.

We conclude that $(\beta_1, \ldots, \beta_i) = (\tilde{\beta}_1, \ldots, \tilde{\beta}_i)$ and $(\beta_j, \ldots, \beta_j) = (\tilde{\beta}_j, \ldots, \tilde{\beta}_j)$. Therefore, the presentations $\beta$ and $\tilde{\beta}$ may only differ on the exponents of those $a_i$, for $i_k < i < j_1$, for which $a_i = a_p$. However, since $\beta_p > 0$, we must have $\beta_i = \tilde{\beta}_i$, whenever $i_k < i < j_1$, $i \neq p$, and $a_i = a_p$. This obviously determines the rest of the exponential presentation uniquely, and we conclude that $\beta = \tilde{\beta}$.

5.3. Whitehead’s and Hurewicz’ Theorems.

In this short subsection we list two classical results which we use for our computations. Both are obtained by combining versions of Whitehead’s and Hurewicz’ theorems.

Theorem 5.4. A simply connected CW complex $X$ whose homology groups $\tilde{H}_i(X; \mathbb{Z})$ are trivial is contractible.

Theorem 5.5. Let $\varphi : X \to Y$ be a map between simply connected CW complexes that induces isomorphism maps $\varphi_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$, for all $n$. Then, the map $\varphi$ is
a homotopy equivalence. If, furthermore, $\phi$ is an inclusion map, then there exists a strong deformation retraction from $Y$ to $X$.

We refer to various sources, such as [Wh78], [Ko08] Corollary 6.32 and Proposition 6.34, [Ha02] Theorem 4.4.5 and Corollary 4.4.33, and [Sp] Theorem 7.6.25.

### 5.4. Algebraic Morse Theory.

In this subsection we present a short extract from the algebraic Morse theory, which is sufficient for our purposes. We include sketches of proofs to stay self-contained. The reader is advised to consult [Ko08] Section 11.3, [Ko05], and the references therein for a more complete picture. For what follows, we recall that when $P$ is a partially ordered set, and $x \in P$, we set $P_{\leq x} := \{ y \in P \mid y \geq x \}$, and $P_{> x} := \{ y \in P \mid y > x \}$.

**Definition 5.6.** Assume we are given a finite $\Delta$-complex $X$. Let $P(X)$ denote its face poset. A sequence of pairs of simplices $((\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n))$ of $X$ is called a **collapsing order** if the following conditions are satisfied:

1. for all $1 \leq i \leq n$, we have $\dim \sigma_i = \dim \tau_i - 1$;
2. $[\sigma_i : \tau_i] = \pm 1$;
3. $P(X)_{\geq \sigma_i} \subseteq \{ \sigma_1, \ldots, \sigma_i, \tau_1, \ldots, \tau_i \}$.

**Lemma 5.7.** Assume we have a $\Delta$-complex $X$ of dimension $d + 1$, and a pair of simplices $(\sigma, \tau)$, such that $\dim \tau = d + 1$, $\dim \sigma = d$, $[\sigma : \tau] = \pm 1$, and $P(X)_{> \sigma} = \{ \tau \}$. Let $\tilde{X}$ be the $\Delta$-complex obtained from $X$ by removing $\sigma$ and $\tau$.

Then, the inclusion map $\iota : \tilde{X} \rightarrow X$ induces isomorphism on homology groups with integer coefficients. If, in addition, the spaces $X$ and $\tilde{X}$ are simply connected, then there is a strong deformation retraction from $X$ to $\tilde{X}$.

**Proof.** A direct analysis of the chain complex $C_*(X, \tilde{X})$ shows that the homology groups $H_*(X, \tilde{X})$ are trivial for all $n$. The long exact sequence of the pair $(X, \tilde{X})$ then implies that $\iota$ induces isomorphism on homology groups. Furthermore, the statement about the strong deformation retraction is a direct corollary of Theorem 5.5. □

**Theorem 5.8.** Assume we have a finite $\Delta$-complex $X$, and a collapsing sequence $(\sigma_1, \tau_1), \ldots, (\sigma_q, \tau_q)$. Let $\tilde{X}$ be the $\Delta$-complex obtained from $X$ by removing the set of simplices $\{ \sigma_1, \ldots, \sigma_q, \tau_1, \ldots, \tau_q \}$.

Then, the inclusion map $\iota : \tilde{X} \rightarrow X$ induces isomorphisms on homology groups with integer coefficients. If, in addition, the spaces $X$ and $X \setminus \{ \sigma_1, \ldots, \sigma_q, \tau_1, \ldots, \tau_q \}$ are simply connected, for all $1 \leq k \leq q$, then there is a strong deformation retraction from $X$ to $\tilde{X}$.

**Proof.** Apply Lemma 5.7 first to $X$ and pair $(\sigma_1, \tau_1)$, then to $X \setminus \{ \sigma_1, \tau_1 \}$ and pair $(\sigma_2, \tau_2)$, etc., until we reach $\tilde{X}$. Take the concatenation of all the isomorphisms obtained at each step. □

### 5.5. The proof of the main theorem.

Recall, that by Proposition 5.8, every spherical word $w$ has a unique decomposition $w = a_1 v_{\alpha_1} a_1 \ldots a_t v_{\alpha_t} a_t$, where, for all $1 \leq i \leq t$, the word $a_i v_{\alpha_i} a_i$ is circular.

**Definition 5.9.** Let $w = a_1 v_{\alpha_1} a_1 \ldots a_t v_{\alpha_t} a_t$ be a representation of a spherical word, as concatenation of circular ones. We call $v = a_1^* \ldots a_t^*$ the **fundamental subword** of $w$.

The simplex indexed by the fundamental subword will encode the topology of $\Delta_w$. 

Definition 5.10. We define a function $\xi : \mathbb{Z}_+ \to \mathbb{Z}_+$ as follows. For an arbitrary nonnegative integer $n$ we set

$$\xi(n) := \begin{cases} n + 1, & \text{if } n \text{ is even;} \\ n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, the function $\xi$ is a bijection, and $\xi^2$ is the identity map. The function $\xi$ simply negates the last bit in the binary representation of a number, and can also be defined by a closed formula $\xi(n) = 4\lfloor n/2 \rfloor + 1 - n$.

Definition 5.11. Assume we are given an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, and an index $t$, such that $1 \leq t \leq n$, and the numbers $\alpha_1, \ldots, \alpha_{t-1}$ are even.

Let $\beta = (\beta_1, \ldots, \beta_t)$ be an $n$-tuple, such that $\beta \leq \alpha$. We let $h_t(\beta)$ denote the minimal index $k$ between 1 and $t$, such that $\beta_k \leq \alpha_k - 1$. If no such $k$ exists, i.e., if $\beta_i = \alpha_i$, for all $i = 1, \ldots, t$, then we set $h_t(\beta) := t$. We call $h_t(\beta)$ the $t$-height of $\beta$ (w.r.t. $\alpha$).

When $v$ is a subword of $w$, we set $h_t(v) := h_t(w)$, where $v = (\beta_1, \ldots, \beta_t)$, is the left-shifted exponential presentation of $v$ (and the $t$-height is taken w.r.t. the exponential presentation of $w$).

Proposition 5.12. Assume a word $w$ is given by its reduced form $\alpha_1^{a_1} \cdots \alpha_t^{a_t}$, such that the numbers $\alpha_1, \ldots, \alpha_t$ are even.

1. If $\alpha_t$ is odd, then the topological space $\Delta_w$ is contractible.
2. If $\alpha_t$ is even, then the $(l(w) - 2)$-skeleton of $\Delta_w$ is contractible. In particular, the topological space $\Delta_w$ is homotopy equivalent to a $(l(w) - 1)$-sphere.

Proof. We will first show the statement (1). Clearly, in this case, the word $w$ is not circular. So, by Theorem 5.12 the space $\Delta_w$ is simply connected. In order to use the Whitehead’s Theorem 5.13 we need to see that all integral homology groups of $\Delta_w$ vanish.

Given a subword $v$, for brevity we shall set $h := h_t(v)$. Let $\Sigma_0$ denote the set of all left-shifted presentations such that $\beta_h$ is even, and let $\Sigma_1$ denote the set of all left-shifted presentations such that $\beta_h$ is odd. Clearly, $\Sigma_0$ and $\Sigma_1$ are disjoint, and simplices of $\Delta_w$ are indexed by $\Sigma_0 \cup \Sigma_1$.

We now define a matching $\mu_t$ between the sets $\Sigma_0$ and $\Sigma_1$. Namely, for $v = (\beta_1, \ldots, \beta_t)$, we set

$$(5.1) \quad \mu_t(v) := (\alpha_1, \ldots, \alpha_{h-1}, \xi(\beta_h), \beta_{h+1}, \ldots, \beta_t).$$

We start by verifying that $\mu_t(v)$ is well-defined. All we need to check is that $\xi(\beta_h) \leq \alpha_h$. If $\beta_h \leq \alpha_h - 1$ this is obvious. Otherwise, we have $(\beta_1, \ldots, \beta_t) = (\alpha_1, \ldots, \alpha_t)$. In this case $h = t$, and $\xi(\alpha_t) = \alpha_t - 1$, since $\alpha_t$ is odd.

Next, let us show that $\mu_t(v)$ is a left-shifted exponential presentation of a subword of $w$. For example, if $w = a^2 b^2 a$, and $v = (2, 1, 1)$, then $\mu_t(v) = (2, 0, 1)$, which is a left-shifted exponential presentation of the subword $a^3$. If $\beta_h$ is even, then $\xi(\beta_h) = \beta_h + 1$, and we obviously get a left-shifted exponential presentation. Assume therefore that $\beta_h$ is odd, so $\xi(\beta_h) = \beta_h - 1$. Let $m$ be the minimal index, such that $m \geq h + 1$, and $\beta_m > 0$. If that index does not exist, we must have $\beta_t = 0$, for all $h \leq i \leq t$, and so $\mu_t(v)$ is obviously left-shifted. Else, the index $m$ is well-defined, and we have

$$(5.2) \quad \mu_t(v) = (\alpha_1, \ldots, \alpha_{h-1}, \beta_h - 1, \ldots, 0, \beta_m, \ldots).$$

If $a_m \neq a_h$, then this $t$-tuple is for sure left-shifted. If, on the other hand, $a_m = a_h$, then the $t$-tuples
\[ \beta = (\alpha_1, \ldots, \alpha_{h-1}, \beta_h, 0, \ldots, 0, \beta_m, \ldots) \quad \text{and} \quad \beta' = (\alpha_1, \ldots, \alpha_{h-1}, \beta_h + 1, 0, \ldots, 0, \beta_m - 1, \ldots) \]

are both exponential presentations of \( v \), where we recall that here \( \beta_h + 1 \leq \alpha_h \). Since clearly \( \beta' >_{\text{lex}} \beta \), we can conclude that the \( t \)-tuple \( \beta \) was not left-shifted to start with, yielding a contradiction with our initial assumptions. We therefore conclude that \( \mu_{i}(v) \) is a left-shifted exponential presentation of a subword of \( w \).

We can next see that

\[ (5.2) \quad h = h_{i}(\mu_{i}(v)). \]

Clearly, the only way this could fail to be true would be if \( \bar{\xi}(\beta_h) = \alpha_h \). Of course, this is impossible if \( \alpha_h \) is even. If \( \alpha_h \) is odd, then \( h = t \), and we still obtain the identity \((5.2)\).

Furthermore, it follows immediately from \( \bar{\xi}^2 = \text{id} \) and \((5.2)\) that \( \mu_{i}^2 = \text{id} \). In fact, the map \( \mu_{i} \) provides a bijection between \( \Sigma_0 \) and \( \Sigma_1 \).

Let us now consider the set of pairs \( \{(\sigma, \mu_{i}(\sigma)) \mid \sigma \in \Sigma_0 \} \). By what we have shown, this is a complete decomposition of the set of simplices of \( \Delta_{w} \). Let us now order these pairs in any order which does not increase the dimension of \( \sigma \). A crucial property which we have here is the following: if \( \sigma \in \Sigma_0 \) and \( \gamma \) covers \( \sigma \) in \( \mathcal{P}(\Delta_{w}) \), then either \( \gamma = \mu_{i}(\sigma) \), or \( \gamma \in \Sigma_0 \). To see that the suggested order is actually a collapsing order, simply check the three conditions of Definition 5.6. The first two conditions follow from the construction of \( \mu_{i} \), and the last one follows from the above mentioned property. We can thus apply Theorem 5.8 to conclude that the space \( \Delta_{w} \) is contractible.

We shall now show the statement (2), so assume that \( \alpha_{t} \) is even. We can clearly assume that \( \alpha_{1} + \cdots + \alpha_{t} \geq 4 \), since the claim is trivially true for \( w = a_{1}^2 \). It is now easily seen that the matching \( \mu_{i} \) is still well-defined and that all the simplices of \( \Delta_{w} \) are matched, except for the single top-dimensional simplex.

Let \( X \) be the complex obtained from \( \Delta_{w} \) by removing the single top-dimensional cell. Note, that this cell has dimension \( \geq 3 \). The matching \( \mu_{i} \) implies that all integral homology groups of \( X \) vanish. Furthermore, \( X \) is simply connected, since it is obtained from a simply connected space \( \Delta_{w} \) by removing a cell of dimension at least \( 3 \). Whitehead’s theorem now implies that \( X \) is contractible. Since the subcomplex inclusion is a cofibration, this subcomplex can be shrunk to a point, yielding a homotopy equivalence. Attaching the top simplex onto this point yields a sphere.

\[ \square \]

**Proposition 5.13.** Assume a word \( w \) is given by its reduced form \( a_{1}^{\alpha_{1}} \cdots a_{t}^{\alpha_{t}} \), such that not all \( \alpha_{i} \) are even. Let \( k \) be the minimal index, such that \( \alpha_{k} \) is odd, and assume that \( k \leq t - 1 \). Let \( \tilde{w} \) be obtained from \( w \) by removing one letter \( a_{k+1} \) from the power \( a_{k+1}^{a_{k+1}} \), i.e., we set

\[ \tilde{w} := a_{1}^{\alpha_{1}} \cdots a_{k}^{\alpha_{k}} a_{k+1}^{\alpha_{k+1} - 1} a_{k+2}^{\alpha_{k+2}} \cdots a_{t}^{\alpha_{t}}; \]

note, that when \( \alpha_{k+1} = 1 \) and \( a_{k} = a_{k+2} \) this is not a reduced form of the word \( \tilde{w} \).

Then there exists a strong deformation retraction from \( \Delta_{w} \) to \( \Delta_{\tilde{w}} \).

**Proof.** The simplices of \( \Delta_{w} \) which do not belong to \( \Delta_{\tilde{w}} \), can be indexed by \((k + 1)-\)shifted exponential presentations \( (\beta_{1}, \ldots, \beta_{t}) \), such that \( \beta_{k+1} = \alpha_{k+1} \). By Proposition 5.3 such a presentation is unique. The map \( \mu_{k} \) from Proposition 5.12 provides a complete matching on this set of simplices.

\[ \square \]

We are now ready to state and to prove our main theorem.

**Theorem 5.14.** Let \( w \) be an arbitrary word.

1. If \( w \) is not spherical, then the \( \Delta \)-complex \( \Delta_{w} \) is contractible.
Figure 5.1. The face posets of $\Delta_{aba^2}$ and $\Delta_{a^2ba}$ and the induced matchings of simplices.

(2) Assume $w$ is spherical, and let $v = a_1^2 \ldots a_t^2$ be its fundamental subword. Then there exists a strong deformation retraction from $\Delta_w$ to the subcomplex $\Delta_v$. In particular, as a topological space $\Delta_w$ is homotopy equivalent to a $(2t - 1)$-sphere.

Proof. Using Proposition 5.13 we can reduce every spherical word to the fundamental subword. If the original word is not spherical, we can reduce it to the word of the form $a_1^2 \ldots a_t a_i$. Flipping the word and applying the same proposition, we arrive at the word $a$, giving just a point. This means, that the original space was contractible.

Two examples of matchings produced by Theorem 5.14 are shown on Figure 5.1. Note, that even though $\Delta_{aba^2}$ and $\Delta_{a^2ba}$ are isomorphic as cell complexes (the indexed word is flipped), the final matchings yielded by the theorem are quite different.

5.6. Example of an application: the alternating words.
Recall, that $\text{alt}(n)$ denotes the word $w = abababab\ldots$, such that $l(w) = n$. Depending on the parity of $n$ we either have $w = (ab)^n$ or $w = (ab)^n a$.

The specific matching given by Theorem 5.14 for $w = \text{alt}(n)$ will be

\begin{align}
(a^2b^2)^k ab \leftrightarrow (a^2b^2)^k ba, \\
(a^2b^2)^k ab^2 \leftrightarrow (a^2b^2)^k ba^3, \\
(a^2b^2)^k a^2b \leftrightarrow (a^2b^2)^k a^3, \\
(a^2b^2)^k a^3b \leftrightarrow (a^2b^2)^k a^2,
\end{align}

for all $k \geq 0$, and all words $\sigma$. In particular, $a$ is matched with the empty simplex. The matching rules will of course only be applied if both simplices are in $\Delta_v$. In particular, it is easy to see that when 3 divides $l(w)$, there will be one unmatched simplex, namely the one indexed by the fundamental subword of $w$.

Theorem 5.15. When 3 does not divide $n$, the $\Delta$-complex $\Delta_{\text{alt}(n)}$ is collapsible. If 3 divides $n$, then one can collapse $\Delta_{\text{alt}(n)}$ onto $\Delta_v$, where $v$ is the fundamental subword of $\text{alt}(n)$. In
particular, we have

\[ \Delta_{ab(n)} \cong \begin{cases} S^{2n/3-1}, & \text{if } 3 \text{ divides } n; \\ \text{point}, & \text{otherwise}. \end{cases} \]

**Proof.** The only strengthening of the general theorem here is that the strong deformation retraction is replaced by collapses in \( \Delta \)-complexes. This can be done, since all pairs \((\sigma, \tau)\) in the collapsing order prescribed by Theorem [5.14] satisfy \([\sigma : \tau] = \pm 1\), as can be seen by direct examination of the rules (5.3).

So the alternating words for which we get non-trivial topology are \( w = aba \), with \( \Delta_{aba} \cong S^1 \), \( w = (ab)^3 \), with \( \Delta_{(ab)^3} \cong S^3 \), \( w = (ab)^4a \), with \( \Delta_{(ab)^4a} \cong S^5 \), etc.

5.7. **Last remarks.**

The following three facts are easy verifications which are left to the reader.

- All words for which \( \Delta_w \), is a pseudomanifold have the reduced form \( a_1^2 \ldots a_i^2 \).
- All words for which \( \Delta_w \), is a manifold have the reduced form \( a_1^2 \ldots a_i^2 \), with an additional condition \( a_i \neq a_j \), for \( i \neq j \).
- When \( w = a_1^{\alpha_1} \ldots a_i^{\alpha_i} \), such that \( \alpha_i \geq 2 \) for all \( i \), then both the \( \Delta \)-complex \( \Delta_w \), as well as the simplicial complex \( bd \Delta_w \), are not collapsible.

In particular, the non-spherical words of the type \( a_1^{\alpha_1} \ldots a_i^{\alpha_i} \), with \( \alpha_i \geq 2 \) for all \( i \), provide a rich source of contractible, but not collapsible simplicial complexes. The classical Dunce hat is the special case of that given by the word \( w = a^3 \).

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