GLOBAL TRANSVERSAL STABILITY OF EUCLIDEAN PLANES UNDER SKEW MEAN CURVATURE FLOW EVOLUTIONS

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ABSTRACT. In this paper, we prove that 2 dimensional transversal small perturbations of d dimensional Euclidean planes under the skew mean curvature flow lead to global solutions which converge to the unperturbed planes in suitable norms. And we clarify the long time behaviors of the solutions in Sobolev spaces.

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1. Introduction

Let \( \Sigma \) be a \( d \)-dimensional oriented manifold and \((N, h)\) be a \((d + 2)\)-dimensional oriented Riemannian manifold. Assume that \( I \) is an interval containing \( t = 0 \), and \( F : I \times \Sigma \to N \) is a family of immersions. For each given \( t \in I \), denote \( \Sigma_t = F(t, \Sigma) \) the submanifold and \( \mathbf{H}(F) \) its mean curvature. Denote tangent bundle of the submanifold by \( T\Sigma_t \) and the normal bundle by \( N\Sigma_t \) respectively. There exists a natural induced complex structure \( J(\mathbf{F}) \) for \( N\Sigma_t \) (rank two) via simply rotating a vector in the normal space by \( \frac{\pi}{2} \) positively. To be precise, for any point \( z = F(t, x) \in \Sigma_t \) and normal vector \( \nu \in N_z\Sigma_t \), define \( J(F) \) by letting \( J(F)\nu \perp \nu = 0 \) and \( \omega(F_*, (e_1), \ldots, F_*(e_d), \nu, J(F)\nu) > 0 \), where \( \omega \) is the volume form of \( N \) and \( \{e_1, \ldots, e_d\} \) is an oriented basis of \( T\Sigma_t \). We shall call the binormal vector \( J(F)\mathbf{H}(F) \in N\Sigma_t \) the skew mean curvature vector. And the skew mean curvature

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flow (SMCF) for immersion \( F : I \times \Sigma \to \mathcal{N} \) is defined by

\[
\begin{cases}
\partial_t F = J(F)H(F), \\
F(0, x) = F_0(x), \ x \in \Sigma
\end{cases}
\]

In view of geometric flows, the SMCF evolves a codimension 2 submanifold along its binormal direction with a speed given by its mean curvature. It is remarkable that SMCF was historically derived from both physics and mathematics. The motivations from physics are vortex filament equations (VFE), the localized induction approximation (LIA) of the hydrodynamical Euler equations, and asymptotic dynamics of vortices in superfluidity and superconductivity.

In fact, the 1-dimensional SMCF in the Euclidean space \( \mathbb{R}^3 \) is the vortex filament equation

\[
\partial_t u = \partial_s u \times u, \ u : (s, t) \in \mathbb{R} \times \mathbb{R} \mapsto u(x, t) \in \mathbb{R}^3
\]

where \( t \) denotes time, \( s \) denotes the arc-length parameter of the curve \( u(t, \cdot) \), and \( \times \) denotes the cross product in \( \mathbb{R}^3 \). The VFE describes the free motion of a vortex filament, see [3, 7].

The SMCF also appears in the study of asymptotic dynamics of vortices in the context of superfluidity and superconductivity. For the Gross-Pitaevskii equation, which models Bose-Einstein condensates, physicists conjecture that the vortices would evolve along the SMCF. This was first verified by Lin [12] for the vortex filaments in \( \mathbb{R}^3 \). Similar phenomena were also observed for other PDEs, for example, it was shown for the Ginzburg-Landau heat flow that the energy asymptotically concentrates on the codimension 2 vortices whose motion is governed by the mean curvature flow.

The other motivation is the LIA of hydrodynamical Euler equations, which describes the limit of a generalized Biot-Savart formula. In fact, let \( M \subset \mathbb{R}^d \) with \( d \geq 3 \) be a closed oriented submanifold of codimension 2. Consider the vorticity 2-form \( \eta_M \) supported on this submanifold: \( \eta_M = C \cdot \delta_M \). We shall call \( M \) a higher dimensional vortex filament or membrane. Then for any dimension \( d \geq 3 \) the divergence-free vector field \( \mathbf{v} \) in \( \mathbb{R}^d \) satisfying \( \operatorname{curl} \mathbf{v} = \eta_M \) is given by a generalized Biot-Savart formula, which only holds for points away from \( M \). In order to derive the formula of \( \mathbf{v} \) for points within \( M \), one defines truncation vector field \( \mathbf{v}_\epsilon \) by the above generalized Biot-Savart formula and some truncation. It was shown that the limit of \( (\ln \epsilon)^{-1} \mathbf{v}_\epsilon(x) \) for \( x \in M \) as \( \epsilon \to 0 \) is the mean curvature of \( M \). Hence, the LIA approximation for a vortex membrane (or higher filament) \( M \) in \( \mathbb{R}^d \) up to a suitable scaling coincides with the SMCF. These facts were discovered by Shashikanth [17] and generalized by Khesin [9].

SMCF also emerges in different mathematical problems. In study of nonlinear Grassmannians, Haller-Vizman [6] noted that SMCF is the Hamiltonian flow of the volume
functional on the space consisting of all co-dimensional 2 immersions of a given Riemannian manifold. Notice that this space admits a generalized Marsden-Weinstein symplectic structure [16] which provides the volume form. The SMCF of surfaces in $\mathbb{R}^4$ is included by a vast energy conserved motion project raised by Lin and his collaborators [13, 14]. Terng [21] also proposed SMCF under the name of star MCF.

Let us describe the non-exhaustive list of works on SMCF. The case $d = 1$, i.e. the VFE, has been intensively studied by many authors from many views such as integrable systems, low regularity, ill-posedness for irregular data, self-similar solutions etc. We recommend the reader to read the survey Vega [23] and Gomez’s thesis [5] for VFE. The works on the case $d \geq 2$ are much less. Song-Sun [19] proved local existence of SMCF for $F : \Sigma \to \mathbb{R}^4$ with compact oriented surface $\Sigma$. This was generalized by Song [20] to $F : \Sigma \to \mathbb{R}^{d+2}$ with compact oriented surface $\Sigma$ for all $d \geq 2$. Khesin-Yang [10] constructed an example showing that the SMCF can blow-up in finite time. Jerrard proposed a notion of weak solutions to the SMCF in [8]. Song [18] proved that the Gauss map of a $d$ dimensional SMCF in $\mathbb{R}^{d+2}$ satisfies a Schrödinger map flow equation (see e.g. [1, 2, 4, 22]).

For $d \geq 2$, the global existence theory of SMCF is largely open even for small data. This is what we aim to solve in this paper. Let us consider the case $\Sigma = \mathbb{R}^d$, $\mathcal{N} = \mathbb{R}^{d+2}$ in SMCF. It is easy to see $F(t, x_1, ..., x_d) = (x_1, ..., x_d, 0, 0)$ is a solution for SMCF, i.e. the $d$-dimensional planes. We consider the transversal perturbations of this plane, i.e. we seek for a graph like solution of the form

$$F(t, x) = (x_1, ..., x_d, u_1(t, x), u_2(t, x)).$$

(1.3)

Our following main theorem states that if the initial perturbation $u_1(0, \cdot), u_2(0, \cdot)$ is sufficiently small, then SMCF has a global graph like solution and the asymptotic behaviors can be clearly determined.

In fact, we have

**Theorem 1.1.** Let $d \geq 3$, $k$ be the smallest integer larger than $\frac{1}{2}(d + 7)$. Assume that $u_1^0, u_2^0 : \mathbb{R}^d \to \mathbb{R}$ are functions belonging to $H^k$. There exists a sufficiently small constant $\epsilon > 0$ such that if

$$\|u_1^0\|_{H^k} + \|u_2^0\|_{H^k} \leq \epsilon,$$

then there exists a global solution to SMCF of the form (1.3) such that $u_1(t) = u_1^0, u_2(t) = u_2^0$ when $t = 0$. Moreover, one has

$$\langle t \rangle^4 (\|u_1(t)\|_{W^{2,\infty}_x} + \|u_2(t)\|_{W^{2,\infty}_x}) + \|u_1(t)\|_{H^k} + \|u_2(t)\|_{H^k} \leq \epsilon$$

(1.4)
for any \( t \in \mathbb{R} \). And there exist time independent complex valued functions \( \phi_{\pm} \in H^2 \) such that

\[
\lim_{t \to \pm \infty} \| u_1(t) + i u_2(t) - e^{it\Delta} \phi_{\pm} \|_{H^1_t} = 0.
\]

(1.5)

- Let \( d = 2 \), \( q \in (1, 2) \). Assume that \( u^0_1, u^0_2 : \mathbb{R}^2 \to \mathbb{R} \) are functions belonging to \( H^6 \). There exists a sufficiently small constant \( \epsilon > 0 \) such that if

\[
\| u^0_1 \|_{H^6 \cap W^{3,q}_x} + \| u^0_2 \|_{H^6 \cap W^{3,q}_x} \leq \epsilon,
\]

then for some \( p > 2 \) there exists a global solution to SMCF of the form (1.3) such that \( u_1(t) = u^0_1 \), \( u_2(t) = u^0_2 \) when \( t = 0 \). Moreover, one has

\[
(1.6)
\]

\[
\langle t \rangle^{1-p} (\| u_1(t) \|_{W^{3,p}_x} + \| u_2(t) \|_{W^{3,p}_x}) + \| u_1(t) \|_{H^6_t} + \| u_2(t) \|_{H^6_t} \leq \epsilon
\]

for any \( t \in \mathbb{R} \). And there exist time independent complex valued functions \( \phi_{\pm} \in H^1 \) such that

\[
\lim_{t \to \pm \infty} \| u_1(t) + i u_2(t) - e^{it\Delta} \phi_{\pm} \|_{H^1_t} = 0.
\]

(1.7)

Theorem 1.1 also shows the \( d \) dimensional plane is transversally stable under the SMCF evolution. Let us describe the main idea in the proof of Theorem 1.1. The most challenging problem in dealing with SMCF is that it is highly quasilinear and the leading part is degenerate along the tangent bundle. In the graph like form (3.2), the degenerateness can be avoided using a suitable equivalent formulation of SMFC. The quasilinear nature seems to be unavoidable as far as we know. In fact, whether there exists a gauge transform to make SMCF semilinear is largely open, see Khesin-Yang [10] for some discussions. But there still exist some ways to deal with quasilinear PDEs, which may be hopeful in our case. In this paper, we adopt the strategy dating back to Klainerman [11], the idea is that dispersive estimates of linear part provide time decay of solutions in \( L^p \) norms with \( p > 2 \) and the high order energy estimates give a chance to overcome the derivative loss due to the quasilinear nature. The combination of dispersive estimates and energy estimates can close the bootstrap in the small data case, and thus finishing the proof.

Notations. The notation \( A \lesssim B \) means there exists some universal constant \( C > 0 \) such that \( A \leq CB \). Denote \( \langle x \rangle = \sqrt{1 + |x|^2} \). The Fourier transform \( f \mapsto \hat{f} \) is denoted by

\[
(1.8)
\]

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi \cdot x} \, dx.
\]

The usual Sobolev spaces \( H^s \) are defined by

\[
(1.9)
\]

\[
\| f \|_{H^s} = \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2_x}.
\]
2. Master equation

Let $F : \Sigma \to \mathbb{R}^{n+2}$, and $H$ be the mean curvature. Then

$$\Delta_g F = H,$$

where $\Delta_g$ denotes the Laplacian on $\Sigma$ of the induced metric $g$ on $T\Sigma_t$ given by

$$g_{ij} = \partial_{x_i} F \cdot \partial_{x_j} F = \delta_{ij} + \partial_i u \cdot \partial_j u. \tag{2.1}$$

Since one has

$$\Delta_g F^\alpha = g^{ij}(\frac{\partial^2}{\partial x_i \partial x_j} F^\alpha - \Gamma^\alpha_{ij} \frac{\partial}{\partial x_l} F^\alpha)$$

and the vector $g^{ij} \Gamma^l_{ij} \frac{\partial}{\partial x_l} F$ belongs to $T\Sigma_t$, we see

$$H = (\Delta_g F)^\perp = \sum_{i=1,2} (g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} F \cdot v_i) v_i,$$

where $\{v_i\}_{i=1}^2$ denotes the orthonormal basis of $N\Sigma_t$ such that $J(v_1) = v_2$, $J(v_2) = -v_1$. Then, the SMCF can be written as

$$\partial_t F^\alpha = (g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} F \cdot v_i) v_2 - (g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} F \cdot v_2) v_1.$$ 

Now, let us consider the case when $F$ is represented by a graph, i.e. $F(x,t) = (x_1,\ldots,x_d,u_1,u_2)$ where $u_1,u_2$ are functions of $x,t$. It is easy to see the $(F^1,\ldots,F^d)$ components of this graph like map $F$ satisfies the afore $d$-equations of SMCF. Set the orthonormal basis $\{v_1,v_2\}$ of $N\Sigma_t$ to be

$$v_1 = \frac{1}{\sqrt{1 + |\partial_s u_1|^2}}(\partial_s u_1, -1, 0)$$

$$v_2 = \frac{1}{|v_2|} \bar{v}_2$$

$$\bar{v}_2 = (\partial_s u_2, 0, -1) - [(\partial_s u_2, 0, -1) \cdot v_1])v_1.$$

Here in the above we denote $\partial_s u = (\partial_x u, \ldots, \partial_x u)$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n+2}$. Further calculations give

$$v_2 = \frac{1}{\Lambda} (\partial_s u_2 - \frac{\partial_s u_2 \cdot \partial_s u_1}{1 + |\partial_s u_1|^2} \partial_s u_1, \frac{\partial_s u_1 \cdot \partial_s u_2}{1 + |\partial_s u_1|^2}, -1)$$

$$\Lambda = \left( \frac{\partial_s u_2 - \frac{\partial_s u_2 \cdot \partial_s u_1}{1 + |\partial_s u_1|^2} \partial_s u_1}{1 + |\partial_s u_1|^2} + \frac{\partial_s u_1 \cdot \partial_s u_2}{1 + |\partial_s u_1|^2} + 1 \right)^\frac{1}{2}.$$

Therefore, the SMCF reduces to

$$\begin{cases}
\partial_t u_1 = g^{ij} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot v_2 \right) \frac{1}{\sqrt{1 + |\partial_s u_1|^2}} + g^{ij} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot v_1 \right) \frac{\partial_s u_1 \cdot \partial_s u_2}{(1 + |\partial_s u_1|^2) \Lambda} \\
\partial_t u_2 = -g^{ij} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot v_1 \right) \frac{1}{\Lambda}.
\end{cases} \tag{2.2}$$
To clarify the main linear part of the above equation, we calculate the expansions of $\nu_1, \nu_2, \Lambda$. We observe that when $|\nabla u_1| + |\nabla u_2|$ is sufficiently small, $\nu_1, \nu_2, \Lambda$ have the expansion

$$
\nu_1 = (\partial_s u_1, -1, 0)[1 - \frac{1}{2}|\partial_s u_1|^2 + O(|\partial_s u|^4)]
$$

$$
\Lambda = 1 + O(|\partial_s u|^2)
$$

$$
\nu_2 = (\partial_s u_2 - \frac{\partial_s u_2 \cdot \partial_s u_1}{1 + |\partial_s u_1|^2}\partial_s u_1, \frac{\partial_s u_1 \cdot \partial_s u_2}{1 + |\partial_s u_1|^2}, -1)[1 + O(|\partial_s u|^2)].
$$

Thus one has

$$
\frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \nu_2 = -\frac{\partial^2 u_2}{\partial x_i \partial x_j} + O(\partial^2 u |\partial_s u|^2)
$$

$$
\frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \nu_1 = -\frac{\partial^2 u_1}{\partial x_i \partial x_j} + O(\partial^2 u |\partial_s u|^2)
$$

Now, we see (2.2) reduces to

$$
\begin{cases}
\partial_s u_1 = -g^{ij}\frac{\partial^2 u_2}{\partial x_i \partial x_j} + O(g^{ij}\partial_s^2 u |\partial_s u|^2) \\
\partial_s u_2 = g^{ij}\frac{\partial^2 u_1}{\partial x_i \partial x_j} + O(g^{ij}\partial_s^2 u |\partial_s u|^2)
\end{cases}
$$

Let us calculate the expansion of $g^{ij}$ by (2.1). In fact, the determinant of $[g_{ij}]$ is $I + O(|\partial_s u|^2)$, then one has by (2.1) that

$$
g^{ij} = \delta_{ij} + O(|\partial_s u|^2).
$$

Therefore, (2.3) can be written as

$$
\begin{cases}
\partial_s u_1 = -\Delta u_2 + O(\partial_s^2 u |\partial_s u|^2) \\
\partial_s u_2 = \Delta u_1 + O(\partial_s^2 u |\partial_s u|^2)
\end{cases}
$$

where the $O(\partial_s^2 u |\partial_s u|^2)$ in fact contains many terms including the leading order cubic term like $\partial_s^2 u |\partial_s u|^2$ and remainder terms of higher powers of $|\partial_s u|$.

Let $\phi = u_1 + iu_2$. We see (2.4) is indeed a quasilinear Schrödinger equation with at least cubic interactions:

$$
i\partial_t \phi + \Delta \phi = O(\partial_s^2 \phi |\partial_s \phi|^2)\mathcal{E}
$$

where the RHS of (3.1) can be written as

$$
\sum c_{ijr} \partial_{x_i x_j}^2 \phi^+ \partial_{x_r} \phi^+ + \mathcal{R},
$$

where $c_{ijr}$ are constants, $\phi^+$ and $\phi^-$ denote $\phi$ and $\bar{\phi}$ respectively, and the remainder $\mathcal{R}$ point-wisely satisfies

$$
|\partial_s^l \mathcal{R}| \leq C_l \sum_{l_1 + \ldots + l_4 = l} |\partial_{x_i}^2 \phi^+| |\partial_{x_j}^2 \phi^+| |\partial_{x_k}^2 \phi^+| |\partial_{x_l}^2 \phi^+| |\partial_{x_m}^2 \phi^+| |\partial_{x_n}^2 \phi^+|,
$$

for any $l \geq 0$ provided that $\|\partial_s^l \phi^+\|_{L^\infty}$ is sufficiently small.
3. Local well-posedness and Energy estimates

The local well-posedness follows by (3.1) and Theorem 1.1 of [15], where local well-posedness was established for small data in $H^s$ with $s > \frac{d+5}{2}$ for general quasilinear Schrödinger equations with cubic interactions. Thus we have

**Lemma 3.1.** Let $s > \frac{d+5}{2}$. Then there exists a small constant $\varepsilon > 0$ such that if

$$
\|u^0_1\|_{H^s} + \|u^0_2\|_{H^s} \leq \varepsilon,
$$

then there exists a local solution $F$ to SMCF of the form

$$
F(t, x) = (x_1, \ldots, x_d, u_1(t, x), u_2(t, x)), \quad (u_1, u_2) \big|_{t=0} = (u^0_1, u^0_2),
$$

and $u_1, u_2 \in C([-1, 1]; H^s)$.

The energy estimates are due to Song [19, 20]. We recall the results in the following lemma.

Let $F : \mathbb{R}^n \to \mathbb{R}^{n+m}$ be a map given by

$$
F(x_1, \ldots, x_n) = (x_1, \ldots, x_n, u_1(x_1, \ldots, x_n), \ldots, u_m(x_1, \ldots, x_n)).
$$

Let $A$ denote the corresponding second fundamental of $\Sigma_t := \text{Graph}(u)$. Define the Sobolev norm of $A$ by

$$
\|A\|_{H^l,2} = \left( \sum_{k=0}^l \int_{\Sigma_t} |\nabla^l A|^p \, d\mu \right)^{\frac{1}{p}},
$$

where $d\mu$ denotes the volume form of $\Sigma_t$ induced by the metric $g$.

And define the usual Sobolev norm of the Hessian of $D^2u$ by

$$
\|D^2u\|_{W^{l,2}} = \left( \sum_{k=0}^l \int_{\mathbb{R}^d} |D^l D^2u|^p \, dx \right)^{\frac{1}{p}}.
$$

**Lemma 3.2** ([19, 20]). With above notions and notations, there holds point-wisely that

$$
|A|^2_g \leq |D^2u|^2 \leq (1 + |Du|^2)^3 |A|^2_g.
$$

Let $|Du| \leq \alpha$, then for any $l \geq 0$,

$$
\|A\|_{H^{l+2}} \leq C_\alpha \sum_{k=1}^{l+1} \|D^2u\|^k_{W^{l+2}}.
$$

Let $|Du| \leq \alpha, |D^2u| \leq \beta$, then for any $l \geq 0$,

$$
\|D^2u\|_{W^{l,2}} \leq C_{\alpha, \beta} \left( \sum_{k=1}^l \|A\|_{H^{l+2}}^k + \sum_{k=2}^l \|D^2u\|_{W^{l,2}}^k \right).
$$
The smooth solution $F(t,x)$ of SMCF satisfies

$$\frac{d}{dt} \int_{\Sigma_t} |\nabla^i A|^2_g d\mu \leq C \max_{\Sigma_t} |A|^2_g \int_{\Sigma_t} |\nabla^i A|^2_g d\mu.$$  

We remark that (3.8) was not directly stated in [19, 20]. In $d = 2$, [19] proved a stronger result:

$$\|D^2 u\|_{W^{l,2}} \leq C_{\alpha, \beta} \sum_{k=1}^{l} \|A\|_{H^{k,2}}.$$  

for any $l \geq 0$, if $|Du| \leq \alpha$, $|D^2 u| \leq \beta$. But in $d \geq 3$ one only has (3.8) while (3.10) fails.

But (3.8) suffices to work in small data case by noticing that the RHS of (3.8) is at least quadratic in $\|D^2 u\|_{W^{l,2}}$.

We also remark that although (3.6), (3.7), (3.9) were proved in [19, 20] for compact $\Sigma$, its proof works for general manifolds where Gagliardo-Nirenberg inequalities hold.

4. Proof of main theorem in $d \geq 3$

By Duhamel principle, the solution of (3.1) can be expressed by

$$\phi(t) = e^{i\Delta t} \phi_0 + \int_0^t e^{i(t-s)\Delta} O(\partial_x^2 \phi |\partial_x \phi|^2)(s) ds.$$  

Let $k$ be the smallest integer such that $k > \frac{d}{2} + 3.5$. Let $T$ be the maximal time such that

$$\sup_{t \in [-T, T]} \left( (t)^{\frac{d}{2}} \|\phi(t)\|_{W^{2,\infty}} + \|\phi(t)\|_{H^k} \right) \leq C_0 \epsilon.$$  

Let’s prove Theorem 1.1 by bootstrap.

1. Leading cubic terms. Among all the terms in the leading cubic part of

$$\|\phi(t)\|_{W^{2,\infty}} \leq t^{-\frac{d}{2}} \|\phi_0\|_{H^{2,2}} + t^{-\frac{d}{2}} \sum_{l=0}^{2} \|\partial^l O(\partial_x^2 \phi |\partial_x \phi|^2)(s)\|_{L^2} ds.$$  

For $1 < |t| < T$, then (4.1) and linear dispersive estimates of $e^{i\Delta t}$ give

$$\|\partial^l O(\partial_x^2 \phi |\partial_x \phi|^2)(s)\|_{L^2} ds.$$  

1. Leading cubic terms. Among all the terms in the leading cubic part of

$$\sum_{l=0}^{2} \|\partial^l O(\partial_x^2 \phi |\partial_x \phi|^2)(s)\|_{L^2} ds,$$

the most hard term on the RHS of (4.3) is

$$\int_0^t \|\partial_x^3 \phi|^2 |\partial_x \phi| ds.$$  

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$$\sum_{l=0}^{2} \|\partial^l O(\partial_x^2 \phi |\partial_x \phi|^2)(s)\|_{L^2} ds,$$

the most hard term on the RHS of (4.3) is

$$\int_0^t \|\partial_x^3 \phi|^2 |\partial_x \phi| ds,$$
whose integrand at first glance has least time decay due to the fact we only assume \( W^{2,\infty} \) decays in (4.2). However, this is also admissible by interpolation. In fact, Gagliardo-Nirenberg inequality shows
\[
\|\partial_x^3 \phi\|_{L^\infty_x} \lesssim \|\partial_x^2 \phi\|_{L^2_x} \|\partial_x^k \phi\|_{L^2_x}^{1-\theta} \quad \theta = \frac{2k - 6 - d}{2k - 4 - d}.
\]

When \( k > \frac{1}{2}(d + 7) \), one has
\[
-\frac{d}{4} - \frac{d}{4} \theta < -1.
\]

Then (4.2) shows
\[
\sup_{t \in [-T, T]} \int_0^t \|\partial_x^3 \phi\|_{L^2_x}^2 |\partial_x \phi|_{L^2_x} ds \\
\lesssim \sup_{t \in [-T, T]} \int_0^t \|\partial_x^3 \phi\|_{L^\infty_x} \|\partial_x^2 \phi\|_{L^\infty_x} \|\partial_x^3 \phi\|_{L^2_x} ds + \int_0^1 \|\partial_x^3 \phi\|_{L^2_x} \|\partial_x^2 \phi\|_{L^\infty_x} ds \\
\lesssim \epsilon^3 \int_1^\infty s^{-\frac{d}{4}(1+\theta)} ds + \epsilon \int_0^1 \|\phi\|_{H^2}^2 ds \\
\lesssim \epsilon^3.
\]

The other term in the leading cubic part of RHS of (4.3) requiring attention is the top derivative term
\[
\int_0^\infty \|\partial_x^4 \phi\|_{L^2_x}^2 |\partial_x \phi|_{L^2_x} ds.
\]
This is admissible by controlling \( \|\partial_x^4 \phi\|_{L^2_x} \) via \( \|\phi\|_{H^2} \) due to the assumption \( k > \frac{d}{2} + 3.5 \).

\section{Remained higher order terms.}

The higher order remainder are easy to dominate since we always have
\[
\int_0^\infty \|R\|_{L^2} ds \lesssim \int_0^\infty \|\phi\|_{H^2}^2 |\partial_x u|_{L^2}^2 ds,
\]
which is integrable since \( t^{-\frac{2}{3}} \in L^1([1, \infty)) \).

In a summary we have proved in Step 2 that
\[
(4.4) \quad \|\phi(t)\|_{W^{2,\infty}} \leq C_1(t)^{-\frac{d}{2}}(\|\phi_0\|_{H^2} + C_0^3 \epsilon^3)
\]
for some \( C_1 > 0 \).

\section{Step 3.}

Let us deal with the \( H^k \) norm in (4.2). By (3.6) and (4.2), we have
\[
|A|^2_k \lesssim \epsilon^2(t)^{-\frac{d}{4}}.
\]
Then (3.9) and Gronwall inequality show for any \( t \in [0, T] \)
\[
\int_\Sigma |\nabla^i A|^2_k(t) d\mu \lesssim 2 \int_\Sigma |\nabla^i A|^2_k(0) d\mu.
\]
if $0 < \epsilon \ll 1$. Then (3.7) shows for any $t \in [0, T]$

\begin{equation}
\int_{\Sigma_t} |\nabla l A|_g^2(t) d\mu \lesssim \|\phi_0\|_{H^k} \ll 1,
\end{equation}

provided that $0 < \epsilon \ll 1$. Since the RHS of (3.8) with $l = k - 2$ is quadratic in $\|D^2 u\|_{W^{k-2,2}}$ and $\|D^2 u\|_{W^{k-2,2}}$ is small by bootstrap assumption, (4.6) further implies for any $t \in [0, T]$, (4.7)

\begin{equation}
\|D^2 u(t)\|_{W^{k-2,2}} \leq C_2 \|\phi_0\|_{H^k}
\end{equation}

for some $C_2 > 0$ if $0 < \epsilon \ll 1$. (4.9) provides admissible bounds of $\|\phi\|_{H^l}$ with $2 \leq l \leq k$.

Using (4.1) we get

\begin{equation}
\|\phi(t)\|_{L^2_x} \lesssim \|\phi_0\|_{L^2_x} + \int_0^t \sum_{l=0}^2 \|D^l O D^2 \phi D_x \phi\|_{L^2_x} ds.
\end{equation}

In Step 2 we have seen the RHS of (4.8) is dominated by $C_2 \|\phi_0\|_{L^2_x} + C_2^2 \epsilon^3$. As a summary, we have obtained admissible bounds of $\|\phi\|_{H^k}$:

\begin{equation}
\|u(t)\|_{H^k} \leq C_2 \|\phi_0\|_{H^k} + C_3 \epsilon^3
\end{equation}

for some $C_2 > 0$ and any $t \in [0, T]$. The inverse direction $t \in [-T, 0]$ follows by a time reflection and defining $J(H)$ as the opposite direction rotation. Hence, (4.9) holds for all $t \in [-T, T]$.

**Step 4.** Set $C_0$ to satisfy

$$C_0 > 4C_1 + 4C_2,$$

then choose $\epsilon > 0$ to be sufficiently small such that

$$C_0^2 \epsilon^2 \leq \frac{1}{4}.$$

By Step 1, (4.4) in Step 2, (4.9) in Step 3 and bootstrap, we have proved (4.2) holds with $C_0 \epsilon$ replaced by $\frac{1}{2} \epsilon$. Thus $T = \infty$. And hence (1.4) holds for all $t \in \mathbb{R}$.

5. **Proof of main theorem in $d = 2$**

Let’s prove Theorem 1.1 for $d = 2$ by bootstrap as well. Set $2 \leq p < \infty$, $1 < q < 2$ to satisfy

\begin{equation}
\frac{1}{q} - \frac{1}{p} > \frac{1}{2}, \quad \frac{1}{q} - \frac{2}{p} \leq \frac{1}{2}.
\end{equation}

Let $T$ be the maximal time such that

\begin{equation}
\sup_{t \in [-T, T]} \left( (t)^{\frac{1}{p} - \frac{1}{q}} \|\phi(t)\|_{W^{k-3,p}} + \|\phi(t)\|_{H^{k-2}} \right) \leq C_0 \epsilon.
\end{equation}

The following bootstrap is similar to $d \geq 3$ with several modifications.

**Step 1’.** By local theorem in Lemma 3.1, we see $T \geq 1$. **Proof of main theorem in $d = 2$**

Let’s prove Theorem 1.1 for $d = 2$ by bootstrap as well. Set $2 \leq p < \infty$, $1 < q < 2$ to satisfy

\begin{equation}
\frac{1}{q} - \frac{1}{p} > \frac{1}{2}, \quad \frac{1}{q} - \frac{2}{p} \leq \frac{1}{2}.
\end{equation}

Let $T$ be the maximal time such that

\begin{equation}
\sup_{t \in [-T, T]} \left( (t)^{\frac{1}{p} - \frac{1}{q}} \|\phi(t)\|_{W^{k-3,p}} + \|\phi(t)\|_{H^{k-2}} \right) \leq C_0 \epsilon.
\end{equation}

The following bootstrap is similar to $d \geq 3$ with several modifications.
Step 2’. Assume that $1 < |t| < T$, then (4.1), linear dispersive estimates of $e^{i\Delta t}$ yield

$$\|\phi(t)\|_{W^{3,\infty}} \leq t^{-\left(\frac{1}{q} - \frac{1}{p}\right)}\|\phi_0\|_{H^3} + \int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \sum_{l=0}^3 \|\partial^l t O(\partial^3 s^2 \partial^l_x \phi^2)(s)\|_{L^2_t} ds.$$  

1’. Leading cubic term. Among all the terms in the leading cubic part

$$\sum_{l=0}^3 \|\partial^l t O(\partial^3 s^2 \partial^l_x \phi^2)(s)\|_{L^2_t},$$

are dominated by

$$\int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \sum_{l=0}^3 \|\partial^l x_1 \phi_{l}\|_{L^1_t} \|\partial^l x_1 \phi\|_{L^2_t} ds$$

where $r$ is defined by $0 < \frac{1}{r} = \frac{1}{q} - \frac{2}{p} < \frac{1}{2}$. Then by embedding $H^1 \hookrightarrow L^r$ in $d = 2$, (5.2) and (5.1) show

$$\sup_{t \in [-T, T]} \int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \sum_{l=0}^3 \|\partial^l x_1 \phi_{l}\|_{L^1_t} \|\partial^l x_1 \phi\|_{L^2_t} ds \lesssim \sup_{t \in [-T, T]} \epsilon^3 \int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} s^{-2\left(\frac{1}{p} - \frac{1}{2}\right)} ds + \int_0^1 ... ds \lesssim t^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \epsilon^3.$$  

2’. Remained higher order terms. The higher order remainder terms are easy to dominate since we always have

$$\int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \|R\|_{W^{3,\infty}} ds \lesssim \int_0^t (t - s)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \|\phi\|_{H^3_t} \|\partial t u\|_{L^2_t} ds.$$  

Step 3’. Let us deal with the $H^6$ norm in (5.2). By bootstrap assumption and interpolation, we have

$$\|D^2 u\|_{L^6_t} \lesssim \|\phi\|_{L^6_t} \|D^3 \phi\|_{L^6_t} \lesssim \epsilon^2 \langle t \rangle^{-\left(\frac{1}{q} - \frac{1}{p}\right)},$$

where $\theta = \frac{1}{2} - \frac{2}{3p}$. Thus (3.6) yields

$$\|A_g\|_{H^6} \lesssim \epsilon^2 \langle t \rangle^{-2\left(\frac{1}{p} - \frac{1}{2}\right)},$$

whose RHS is integrable due to (5.1). Then (3.9) and Gronwall inequality show

$$\int_{\Sigma} \|\nabla^2 A_g^2(t)\|_{H^6} \leq 2 \int_{\Sigma} \|\nabla^2 A_g^2(0)\|_{H^6}$$

if $0 < \epsilon \ll 1$. Moreover, (3.8), (3.7) and a time reflection argument as before further imply for an $t \in [-T, T]$,

$$\|D^2 u(t)\|_{H^6} \lesssim \|D^2 u(0)\|_{H^4}.$$
if $0 < \epsilon \ll 1$. (5.6) provides admissible bounds of $\|\phi\|_{H^l}$ with $2 \leq l \leq 6$. Using (4.1) we get

\begin{equation}
\|\phi(t)\|_{L^2_x} \lesssim \|\phi_0\|_{L^2_x} + \int_0^t \sum_{l=0}^1 \|\partial^l O(\partial_x^2 \phi | \partial_x \phi|^2)(s)\|_{L^2_x} ds.
\end{equation}

The RHS of (5.7) is dominated by $C\epsilon^3 + C\|\phi_0\|_{L^2_x}$, since

\begin{align*}
\int_0^t \sum_{l=0}^1 \|\partial^l O(\partial_x^2 \phi | \partial_x \phi|^2)(s)\|_{L^2_x} ds &
\leq \int_0^t \|\phi\|_{H^l_x}\|\partial_x \phi\|_{L^\infty_x}^2 ds + \int_0^t \|\partial_x^2 \phi\|_{L^\infty_x}\|\partial_x \phi\|_{L^\infty_x}\|\partial_x^2 \phi\|_{L^2_x} ds + \int_0^t \|\phi\|_{H^l_x}\|\partial_x \phi\|_{L^2_x}^2 ds
\lesssim \epsilon^3 \int_0^t (s)^{-\frac{1}{2} - \frac{1}{p}} ds
\lesssim \epsilon^3
\end{align*}

where in the third line we applied (5.4), $t^{-\left(\frac{1}{q} - \frac{1}{p}\right)}$ decay of $\|\partial_x \phi\|_{L^\infty_x}$ (interpolation $L^p$ decay proved in Step 2') and $\|\phi\|_{H^p}$ bounds by bootstrap. As a summary, we have obtained admissible bounds of $\|\phi\|_{L^\infty_t H^6_x}$.

**Step 4'.** By Step 1', Step 2', Step 3' and bootstrap, we have proved $T = \infty$. And thus (1.6) holds for all $t \in \mathbb{R}$.

6. **Proof of scattering**

(1.6) and (1.5) in fact are scattering. This follows if one has shown

\begin{equation}
\int_0^\infty \|O(\partial_x^2 \phi | \partial_x \phi|^2)\|_{H^l_x} ds \leq 1,
\end{equation}

where $l_d = 2$ for $d \geq 3$ and $l_d = 1$ when $d = 2$. In $d \geq 3$, (6.1) has been proved. For $d = 2$, it follows from (5.7).

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