Effros, Baire, Steinhaus and Non-Separability

By A. J. Ostaszewski

Abstract. We give a short proof of an improved version of the Effros Open Mapping Principle via a shift-compactness theorem (also with a short proof), involving ‘sequential analysis’ rather than separability, deducing it from the Baire property in a general Baire-space setting (rather than under topological completeness). It is applicable to absolutely-analytic normed groups (which include complete metrizable topological groups), and via a Steinhaus-type Sum-set Theorem (also a consequence of the shift-compactness theorem) includes the classical Open Mapping Theorem (separable or otherwise).

Keywords: Open Mapping Theorem, absolutely analytic sets, base-σ-discrete maps, demi-open maps, Baire spaces, Baire property, group-action shift-compactness.

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1 Introduction

We generalize a classic theorem of Effros [Eff] beyond its usual separable context. Viewed, despite the separability, as a group-action counterpart of the Open Mapping Theorem OMT (that a surjective continuous linear map between Fréchet spaces is open – cf. [Rud]), it has come to be called the Open Mapping Principle – see [Anc §1]. Our ‘non-separable’ approach is motivated by a sequential property related to the Steinhaus-type Sum-set Theorem (that 0 is an interior point of $A - A$, for non-meagre $A$ with BP, the Baire property – [Oxt], [Pic]), because of the following argument (which goes back to Pettis [Pet]).

Consider $L : E \to F$, a linear, continuous surjection between Fréchet spaces, and $U$ a neighbourhood (nhd) of the origin. Choose $A$ an open nhd of the origin with $A - A \subseteq U$; as $L(A)$ is non-meagre (since $\{nL(A) : n \in \mathbb{N}\}$ covers $F$) and has BP (see Proposition 2 in §2.3), $L(A) - L(A)$ is a nhd of the origin by the Sum-set Theorem. But of course

$$L(U) \supseteq L(A) - L(A),$$
so $L(U)$ is a nhd of the origin. So $L$ is an open mapping.

Throughout this paper, without further comment, all spaces considered will be metrizable, but not necessarily separable. We recall the Birkhoff-Kakutani theorem (cf. [HewR, §II.8.3]), that a metrizable group $G$ with neutral element $e_G$ has a right-invariant metric $d^G_R$. Passage to $||g|| := d^G_R(g, e_G)$ yields a (group) norm (invariant under inversion, satisfying the triangle inequality, cf. [ArhT], [BinO1] – see §4.1), which justifies calling these normed groups; any Fréchet space qua additive group, equipped with an F-norm ([KalPR, Ch. 1 §2]), is a natural example (cf. Auth in §2.2). Recall that a Baire space is one in which Baire’s theorem holds – see [AaL]. Below we need the following.

**Definitions 1** (cf. [Pet]). For $G$ a metrizable group, say that $\varphi : G \times X \to X$ is a Nikodym group action (or that it has the Nikodym property – cf. [BinO4]) if for every non-empty open neighbourhood $U$ of $e_G$ and every $x \in X$ the set $Ux = \varphi_x(U) := \varphi(x, U)$ contains a non-meagre Baire set. (Here Baire set, as opposed to Baire space as above, means ‘set with the Baire property’.)

2. $A^q$ denotes the quasi-interior of $A$ – the largest open set $U$ with $U \setminus A$ meagre (cf. [Ost1, §4]); other terms (‘analytic’, ‘base-\(\sigma\)-discrete’, ‘group action’) are recalled later.

Concerning when the above property holds see §2.3. Our main results are Theorems S and E below, with Corollaries in §2.3 including OMT; see below for commentary.

**Theorem S (Shift-compactness Theorem).** For $T$ a Baire non-meagre subset of a metric space $X$ and $G$ a group, Baire under a right-invariant metric, and with separately continuous and transitive Nikodym action on $X$:

for every convergent sequence $x_n$ with limit $x$ and any Baire non-meagre $A \subseteq G$ with $e_G \in A^q$ and $A^q x \cap T^q \neq \emptyset$, there are $\alpha \in A$ and an integer $N$ such that $\alpha x \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

In particular, this is so if $G$ is analytic and all point-evaluation maps $\varphi_x$ are base-\(\sigma\)-discrete.

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1 This proof is presumably well-known – so simple and similar to that for the automatic continuity of homomorphisms – but we have no textbook reference; cf. [KalPR, Cor. 1.5].
This theorem has wide-ranging consequences, including Steinhaus’ Sumset Theorem – see the survey article [Ost4], and the recent [BinO3]; cf. §4.2-4.

**Theorem E (Effros Theorem – Baire version).** If
(i) the normed group $G$ has separately continuous and transitive Nikodym action on $X$;
(ii) $G$ is Baire under the norm topology and $X$ is non-meagre
then for any open neighbourhood $U$ of $e_G$ and any $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of $x$, so that in particular the point-evaluation maps $g \to g(x)$ are open for each $x$. That is, the action of $G$ is micro-transitive.

In particular, this holds if $G$ is analytic and Baire, and all point-evaluation maps $\varphi_x$ are base-$\sigma$-discrete.

By Proposition B2 (§2.3) $X$, being non-meagre here, is also a Baire space.

The classical counterpart of Theorem E has $G$ a Polish group; van Mill’s version [vMil1] requires the group $G$ to be analytic (i.e. the continuous image of some Polish space, cf. [JayR], [Kec2]). The Baire version above improves the version given in [Ost3], where the group is almost complete. (The two cited sources taken together cover the literature.)

A result due to Loy [Loy] and to Hoffmann-Jørgensen [HofJ] Th. 2.3.6 p. 355 asserts that a Baire, separable, analytic topological group is Polish (as a consequence of an analytic group being metrizable – for which see again [HofJ] Th. 2.3.6)), so in the analytic separable case Theorem E reduces to its classical version.

Unlike the proof of the Effros Theorem attributed to Becker in [Kec1] Th. 3.1, the one offered here does not employ the Kuratowski-Ulam Theorem (the Category version of the Fubini Theorem), a result known to fail beyond the separable context (as shown in [Pol], cf. [vMilP], but see [FreNR]).

For recent work on the circumstances when Theorems E and S are equivalent, see [BinO4]. For further commentary (connections between convexity and the Baire property, relation to van Mill’s separation property in [vMil2], certain specializations) see §4.

## 2 Analyticity, micro-action, shift-compactness

We recall some definitions from general topology, before turning to ones that are group-related. We refer to [Eng] for general topological usage (but prefer
‘meagre’ to ‘of first category’); see also §4.5.6.

2.1 Analyticity

We say that a subspace $S$ of a metric space $X$ has a *Souslin-$\mathcal{H}$ representation* if there is a *determining system* $\langle H(i|n) \rangle := \langle H(i|n) : i \in \mathbb{N}^n \rangle$ of sets in $\mathcal{H}$ with ([Rog], [Han1], [Han2])

$$S = \bigcup_{i \in I} \bigcap_{n \in \mathbb{N}} H(i|n), \quad (I := \mathbb{N}^n, \  i|n := (i_1, ..., i_n)).$$

A topological space is an (absolutely) analytic space if it is embeddable as a Souslin-$\mathcal{F}$ set in its own metric completion (with $\mathcal{F}$ the closed sets); in particular, in a complete metric space $G_\delta$-subsets (being $F_\sigma\delta$) are analytic. For more recent generalizations see e.g. [NamP]. According to Nikodym’s theorem, if $\mathcal{H}$ above comprises Baire sets, then also $S$ is Baire (the Baire property is preserved by the Souslin operation): so analytic subspaces are Baire sets. For background – see [Kec2] Th. 21.6 (the Lusin-Sierpiński Theorem) and the closely related Cor. 29.14 (Nikodym Theorem), cf. the treatment in [Kur] Cor. 1 p. 482, or [JayR] pp. 42-43. For the extended Souslin operation of non-separable descriptive theory see also [Ost2]. This motivates our interest in analyticity as a carrier of the Baire property, especially as continuous images of separable analytic sets are separable, hence Baire.

However, the continuous image of an analytic space is not in general analytic – for an example of failure see [Han3] Ex. 3.12. But this does happen when, additionally, the continuous map is base-$\sigma$-discrete, as defined below (*Hansell’s Theorem*, [Han3] Cor. 4.2). This technical condition is the standard assumption for preservation of analyticity and holds automatically in the separable realm. Special cases include closed surjective maps and open-to-analytic injective maps (taking open sets to analytic sets). To define the key concept just mentioned, recall that for an (indexed) family $\mathcal{B} := \{B_t : t \in T\}$:

(i) $\mathcal{B}$ is *index-discrete* in the space $X$ (or just *discrete* when the index set $T$ is understood) if every point in $X$ has a nhd meeting the sets $B_t$ for at most one $t \in T$,

(ii) $\mathcal{B}$ is *$\sigma$-discrete* if $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each set $\mathcal{B}_n$ is discrete as in (i), and

(iii) $\mathcal{B}$ is a *base for $\mathcal{A}$* if every member of $\mathcal{A}$ is the union of a subfamily of $\mathcal{B}$. For $\mathcal{T}$ a topology (the family of all open sets) with $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$, this reduces to $\mathcal{B}$ being simply a (topological) *base.*
Definitions. 1. ([Mic1], Def. 2.1) Call \( f : X \to Y \) base-\( \sigma \)-discrete (or co-\( \sigma \)-discrete, [Han3, §3]) if the image under \( f \) of any discrete family in \( X \) has a \( \sigma \)-discrete base in \( Y \).

2. ([Han3, §2], cf. [KanP]). An indexed family \( A := \{ A_t : t \in T \} \) is \( \sigma \)-discretely decomposable if there are discrete families \( A_n := \{ A_{tn} : t \in T \} \) such that \( A_t = \bigcup_n A_{tn} \) for each \( t \).

3. ([Mic1], Def. 3.3). Call \( f : X \to Y \) index-\( \sigma \)-discrete if the image under \( f \) of any discrete family \( E \) in \( X \) is \( \sigma \)-discretely decomposable in \( Y \). (Note \( f(E) \) is regarded as indexed by \( E \), so could be discrete without being index-discrete.)

2.2 Action, micro-action, shift-compactness

Recall that a normed group \( G \) acts continuously on \( X \) if there is a continuous mapping \( \varphi : G \times X \to X \) such that \( \varphi(e_G, x) = x \) and \( \varphi(gh, x) = \varphi(g, \varphi(h, x)) \) \((x \in X, g, h \in G)\). The action \( \varphi \) is separately continuous if \( g : x \mapsto \varphi(g, x) \) is continuous for each \( g \), and \( \varphi_x : g \mapsto \varphi(g, x) \) is continuous for each \( x \); in such circumstances:

(i) the elements \( g \in G \) yield autohomeomorphisms of \( X \) via \( g : x \mapsto \varphi(g, x) := \varphi(g, x) \) (as \( g^{-1} \) is continuous), and

(ii) point-evaluation of these homeomorphisms, \( \varphi_x(g) = g(x) \), is continuous. In certain situations joint continuity of action is implied by separate continuity (see [Bon] and literature cited in [Ost2]).

The action is transitive if for any \( x, y \) in \( X \) there is \( g \in G \) such that \( g(x) = y \). For later purposes (§2.3 and 3), say that the action of \( G \) on \( X \) is weakly micro-transitive if for \( x \in X \) and each nhd \( A \) of \( e_G \) the set

\[ \text{cl}(Ax) = \text{cl}\{ax : a \in A\} \]

has \( x \) as an interior point (in \( X \)). The action is micro-transitive (‘transitive in the small’ – for details see [vMilII]) if for \( x \in X \) and each nhd \( A \) of \( e_G \) the set

\[ Ax = \{ax : a \in A\} \]

is a nhd of \( x \). This (norm) property implies that \( Ux \) is open for \( U \) open in \( G \) (i.e. that here each \( \varphi_x \) is an open mapping). We refer to \( Ax \) as an \( x \) orbit (the \( A \)-orbit of \( x \)). The following group action connects the Open Mapping Theorem to the present context.

Example (Induced homomorphic action). A surjective, continuous homomorphism \( \lambda : G \to H \) between normed groups induces a transitive
action of $G$ on $H$ via $\varphi^\lambda(g, h) := \lambda(g)h$ (cf. [Ost2] Th. 5.1), specializing for $G, H$ Fréchet spaces (regarded as normed, additive groups) and $\lambda = L : G \to H$ linear (Ancel [Anc] and van Mill [vMil]) to

$$\varphi^L(a, b) := L(a) + b.$$ 

Of course for Fréchet spaces, by the Open Mapping Theorem itself, $\varphi^L$ has the Nikodym property.

**Definitions.** 1. $\text{Auth}(X)$ denotes the autohomeomorphisms of a metric space $(X, d^X)$; this is a group under composition. $\mathcal{H}(X)$ comprises those $h \in \text{Auth}(X)$ of bounded norm:

$$||h|| := \sup_{x \in X} d^X(h(x), x) < \infty.$$ 

2. For a normed group $G$ acting on $X$, say that $X$ has the crimping property (property C for short) w.r.t. $G$ if, for each $x \in X$ and each sequence $\{x_n\} \to x$, there exists in $G$ a sequence $\{g_n\} \to e_G$ with $g_n(x) = x_n$. (This and a variant occurs in [Ban, Ch. III; Th.4] and [ChCh]; for the term see [BinO2].)

For a subgroup $G \subseteq \mathcal{H}(X)$, say that $X$ has the crimping property w.r.t. $G$ if $X$ has the crimping property w.r.t. to the natural action $(g, x) \to g(x)$ from $G \times X \to X$. (This action is continuous relative to the left or right norm topology on $G$ – cf. [Dug] XII.8.3, p. 271.)

3. As a matter of convenience, say that the Effros property (or property E) holds for the group $G$ acting on $X$ if the action is micro-transitive, as above.

4. For a subgroup $G \subseteq \text{Auth}(X)$ say that $X$ is $G$-shift-compact (or, shift-compact under $G$) if for any convergent sequence $x_n \to x_0$, any open subset $U$ in $X$ and any Baire set $T$ co-meagre in $U$, there is $g \in G$ with $g(x_n) \in T \cap U$ along a subsequence. Call the space shift-compact if it is $\mathcal{H}(X)$-shift-compact (cf. [MilO], [Ost5]).

In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of Prop. B2 below.

We shall prove in §3.1 equivalence between the Effros and Crimping properties:

**Theorem EC.** The Effros property holds for a group $G$ acting on $X$ iff $X$ has the Crimping property w.r.t. $G$.

We now clarify the role of shift-compactness.
Proposition B1. For any subgroup $G \subseteq H(X)$, if $X$ is $G$-shift-compact, then $X$ is a Baire space.

Proof. We argue as in [vMil2] Prop 3.1 (1). Suppose otherwise; then $X$ contains a non-empty meagre open set. By Banach’s Category Theorem (or localization principle, for which see [JayR] p. 42, or [Kel] Th. 6.35), the union of all such sets is a largest open meagre set $M$, and is non-empty. Thus $X \setminus M$ is a co-meagre Baire set. For any $x \in M$ the constant sequence $x_n \equiv x$ is convergent and, since $X \setminus M$ is co-meagre in $X$, there is $g \in G$ with $g(x) \in X \setminus M$. But, as $g$ is a homeomorphism, $g(M)$ is a non-empty open meagre set, so is contained in $M$, implying $g(x) \in M$, a contradiction. □

A similar argument gives the following and clarifies an assumption in Theorem E.

Proposition B2 (cf. [vMil2]; [HofJ, Prop. 2.2.3]). If $X$ is non-meagre and $G$ acts transitively on $X$, then $X$ is a Baire space.

Proof. As above, refer again to $M$, the union of all meagre open sets, which, being meagre, has non-empty complement. For $x_0$ in this complement and any non-empty open $U$ pick $u \in U$ and $g \in G$ such that $g(x_0) = u$. Now, as $g$ is continuous, $g^{-1}(U)$ is a nhd of $x_0$, so is non-meagre, since every nhd of $x_0$ is non-meagre. But $g$ is a homeomorphism, so $U = g(g^{-1}(U))$ is non-meagre. So $X$ is Baire, as every non-empty open set is non-meagre. □

2.3 Nikodym actions

The following result generalizes one that, for separable groups $G$, is usually a first step in proving the weakly micro-transitive variant of the classical Effros Theorem (cf. Ancel [Anc] Lemma 3, [Ost3] Th. 2). Indeed, one may think of it as giving a form of ‘very weak micro-transitivity’.

Proposition 1. If $G$ is a normed group, acting transitively on a non-meagre space $X$ with each point evaluation map $\varphi_x : g \mapsto g(x)$ base-\(\sigma\)-discrete relative – then for each non-empty open $U$ in $G$ and each $x \in X$ the set $U x$ is non-meagre in $X$.

In particular, if $G$ is analytic, then $G$ is a Nikodym action.

Proof. We first work in the right norm topology, i.e. derived from the assumed right-invariant metric $d^G_H(s, t) = ||st^{-1}||$. Suppose that $u \in U$, and
so without loss of generality assume that \( U = B_\varepsilon(u) = B_\varepsilon(e_G)u \) (open balls of radius some \( \varepsilon > 0 \)); then put \( y := ux \) and \( W = B_\varepsilon(e_G) \). Then \( Ux = Wy \). Next work in the left norm topology, derived from \( d^G_L(s, t) = ||s^{-1}t|| = d^G_L(s^{-1}, t^{-1}) \) (for which \( W = B_\varepsilon(e_G) \) is still a nhd of \( e_G \)). As each set \( hW \) for \( h \in G \) is now open (since now the left shift \( g \to hg \) is a homeomorphism), the open family \( \mathcal{W} = \{gW : g \in G\} \) covers \( G \). As \( G \) is metrizable (and so has a \( \sigma \)-discrete base), the cover \( \mathcal{W} \) has a \( \sigma \)-discrete refinement, say \( \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \), with each \( \mathcal{V}_n \) discrete. Put \( X_n := \bigcup \{Vy : V \in \mathcal{V}_n\} \); then \( X = \bigcup_{n \in \mathbb{N}} X_n \), as \( X = Gy \), and so \( X_n \) is non-meagre for some \( n \), for \( n = N \) say. Since \( \varphi_y \) is base-\( \sigma \)-discrete, \( \{Vy : V \in \mathcal{V}_N\} \) has a \( \sigma \)-discrete base, say \( \mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m \), with each \( \mathcal{B}_m \) discrete. Then, as \( \mathcal{B} \) is a base for \( \{Vy : V \in \mathcal{V}_N\} \),

\[
X_N = \bigcup_{m \in \mathbb{N}} \left( \bigcup \{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N)B \subseteq Vy\} \right).
\]

So for some \( m \), say for \( m = M \),

\[
\bigcup \{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N)B \subseteq Vy\}
\]

is non-meagre. But as \( \mathcal{B}_M \) is discrete, by Banach’s Category Theorem (cf. Prop. B1), there are \( \hat{B} \in \mathcal{B}_M \) and \( \hat{V} \in \mathcal{V}_N \) with \( \hat{B} \subseteq \hat{V}y \) such that \( \hat{B} \) is non-meagre. As \( \mathcal{V} \) refines \( \mathcal{W} \), there is some \( \hat{g} \in G \) with \( \hat{V} \subseteq \hat{g}W \), so \( \hat{B} \subseteq \hat{V}y \subseteq \hat{g}Wy \), and so \( \hat{g}Wy \) is non-meagre. As \( \hat{g}^{-1} \) is a homeomorphism of \( X \), \( Wy = Ux \) is also non-meagre in \( X \).

If \( G \) is analytic, then as \( U \) is open, it is also analytic (since open sets are \( \mathcal{F}_\varepsilon \) and Souslin-\( \mathcal{F} \) subsets of analytic sets are analytic, cf. [JayR]), and hence so is \( \varphi_x(U) \). Indeed, since \( \varphi_x \) is continuous and base-\( \sigma \)-discrete, \( Ax \) is analytic (Hansell’s Theorem, §2.1), so Souslin-\( \mathcal{F} \), and so Baire by Nikodym’s Theorem (§2.1). □

**Definition.** (Ancel [Anc].) Call the map \( \varphi_x \) **countably-covered** if there exist self-homeomorphisms \( h^n_x \) of \( X \) for \( n \in \mathbb{N} \) such that for any open nhd \( U \) in \( G \) the sets \( \{h^n_x(\varphi_x(U)) : n \in \mathbb{N}\} \) cover \( X \).

**Proposition 1’** (cf. Ancell [Anc]) *For the action \( \varphi : G \times X \to X \) with \( X \) non-meagre, if each map \( \varphi_x \) is countably-covered and takes open sets to sets with the Baire property, then the action has the Nikodym property.*

**Proof.** If \( \varphi_x \) is countably-covered, then there exist self-homeomorphisms \( h^n_x \) of \( X \) for \( n \in \mathbb{N} \) such that for any open nhd \( U \) in \( G \) the sets \( \{h^n_x(\varphi_x(U)) : n \in \mathbb{N}\} \):
For $E$ separable, an immediate consequence of continuous maps taking open sets to analytic sets (which are Baire sets) and of Prop. 1’ is that $\varphi^L$ is a Nikodym action.

For the general context, one needs demi-open continuous maps, which preserve almost completeness (absolute $G_\delta$ sets modulo meagre sets – see [Mic2] and its antecedent [Nol]), as it is not known which linear maps are base-$\sigma$-discrete – a delicate matter to determine, since the former include continuous linear surjections (by Lemma 1 below) and preserve almost analyticity as opposed to analyticity.

For present purposes, however, the monotonicity property below suffices. We omit the proof of the following observation (for which see the opening step in [Rud, 2.11], or [Con, Ch. 3 §12.3], or the Appendix). For the underlying translation-invariant metric of a Fréchet space denote below by $B(a, r)$ the open $r$-ball with centre $a$.

**Lemma 1.** For a continuous linear map $L : X \to Y$ from a Fréchet space $X$ to a normed space $Y$, for $s < t < r$

$$\text{int} (\text{cl} L(B(0, s))) \subseteq L(B(0, t)) \subseteq L(B(0, r)).$$

Hence for $L(a, r)$ convex, either $L(B(a, r))$ is meagre or differs from $\text{int} L(B(a, r))$ by a meagre set.

**Proposition 2.** For $L$ a continuous linear surjection from a Fréchet space $E$ to a non-meagre normed space $F$, the action $\varphi^L$ has the Nikodym property.

**Proof.** As in Prop 1’ for $L : E \to F$ a continuous linear surjection, $\{\varphi^L_x : x \in F\}$ are countably-covered. Indeed, fixing $x \in F$

$$h^x_n(z) := n(z - x) \quad (n \in \mathbb{N} \text{ and } z \in F)$$

is on the one hand a self-homeomorphism satisfying $h^x_n(\varphi_x(L(V))) = L(nV)$, since $n[(L(v) + x) - x] = nL(v) = L(nv)$; on the other hand the family

$$\{h^x_n(L(V) + x) : n \geq 1\}$$
covers $F$, as \( \{ nV : n \in \mathbb{N} \} \) covers $E$ for $V$ any open nhd of the origin in $E$ (by the ‘absorbing’ property, cf. [Con 4.1.13], [Rud 1.33]). In particular, $nL(B(0,1))$ is non-meagre for some $n$, and so $L(B(0,s))$ is non-meagre for any $s$. By Lemma 1, $L(B(0,t))$ for any $t > s$ contains the non-meagre Baire set $\text{cl}L(B(0,s))$. \qed

Corollary 1 below is now immediate; it is used in [Ost2, Th. 5.1] to prove the ‘Semi-Completeness Theorem’, an Ellis-type theorem [Ell] Cor. 2] (cf. [Ost6]) giving a one-sided continuity condition which implies that a right-topological group generated by a right-invariant metric is a topological group.

Corollary 1 (cf. [Ost2] Th. 5.1, ‘Open Homomorphism Theorem’). If the continuous surjective homomorphism $\lambda$ between normed groups $G$ and $H$, with $G$ analytic and $H$ a Baire space, is base-$\sigma$-discrete, then $\lambda$ is open; in particular, for $\lambda$ bijective, $\lambda^{-1}$ is continuous.

Corollary 2. For $L : E \to F$ a continuous surjective linear map between Fréchet spaces, the point evaluations $\varphi^L_b$ for $b \in F$ are open, and so $L$ is an open mapping.

Proof. By surjectivity of $L$, the action is transitive, and by Prop 2 the action $\varphi^L$ has the Nikodym property. So by Theorem E above the point-evaluations maps $\varphi^L_b$ are open. Hence so also is $L$. \qed

3 Proofs

3.1 Proof that $E \iff C$

In [BinO1] Th. 3.15 we showed that if the Effros property holds for the action of a group $G$ on $X$, then $X$ has the crimping property w.r.t. $G$. We recall the argument, as it is short. Suppose that $x = \lim x_n$. For each $n$, take $U = B^{G}_{1/n}(e_G)$; then $Ux := \{ u(x) : u \in U \}$ is an open nhd of $x$, and so there exists $h_{n,m} \in U$ with $h_{n,m}(x) = x_m$ for all $m$ large enough, say for all $m > m(n)$. Without loss of generality we may assume that $m(1) < m(2) < \ldots$. Put $h_m := e_G$ for $m < m(1)$, and for $m(k) \leq m < m(k+1)$ take $h_m := h_{k,m}$. Then $h_m \in B^{G}_{1/k}(e_G)$, so $h_m$ converges to $e_G$ and $h_m(e_G) = x_m$. 

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For the converse, suppose that the Effros property fails for $G$ acting on $X$. Then for some open nhd $U$ of $e_G$ and some $x \in X$, $U x := \{u(x) : u \in U\}$ is not an open nhd of $x$. So for each $n$ there is a point $x_n \in B_{1/n}(x) \setminus U x$. As $x_n$ converges to $x$ there are homeomorphisms $h_n$ converging to the identity $e_G$ with $h_n(x) = x_n$. As $U$ is an open nhd of $e_G$ and since $h_n$ converges to $e_G$, there is $N$ such that $h_n \in U$ for $n > N$. In particular, for any $n > N$, $h_n(x) = x_n \in U x$, a contradiction.

### 3.2 Weak S

We view Th. S as having ‘two tasks’: to find a ‘translator of the sequence’ $\tau$, and to locate it in a given Baire non-meagre subset of the group – provided that subset satisfies a consistency condition (a necessary condition).

For clarity we break the tasks the into two steps – the first delivering a weaker version of S in Proposition 3 below. The arguments are based on the following lemma. We note a corollary, observed earlier by van Mill in the case of metric topological groups ([vMil2, Prop. 3.4]), which concerns a co-meagre set, but we need its refinement to a localized version for a non-meagre set.

**Separation Lemma.** Let $G$ be a normed group, with separately continuous and transitive Nikodym action on a non-meagre space $X$. Then for any point $x$ and any $F$ closed nowhere dense, $W_{x,F} := \{\alpha \in G : \alpha(x) \notin F\}$ is dense open in $G$. In particular, $G$ separates points from nowhere dense closed sets.

**Proof.** The set $W_{x,F}$ is open, being of the form $\varphi_x^{-1}(X \setminus F)$ with $\varphi_x$ continuous (by assumption). By the Nikodym property, for $U$ any non-empty open set in $G$, the set $U x$ is non-meagre, and so $U x \setminus F$ is non-empty, as $F$ is meagre. But then for some $u \in U$ we have $u(x) \notin F$. □

**Corollary 2.** If $G$ is a normed group, Baire in the norm topology with transitive and separately continuously Nikodym action on a non-meagre space $X$ space, and $T$ is co-meagre in $X$– then for countable $D \subseteq X$, the set $\{g : g(D) \subseteq T\}$ is a dense $G_\delta$.

In particular, this holds if $G$ is analytic and each point-evaluation map $\varphi_x : g \to g(x)$ is base-$\sigma$-discrete.

**Proof.** Without loss of generality, the co-meagre set is of the form $T = U \setminus \bigcup_{n \in \omega} F_n$ with each $F_n$ closed and nowhere dense, and $U$ open. Then, by
the Separation Lemma and as $G$ is Baire,
\[ \{ g \in G : g(D) \subseteq T \} = \bigcap_{n \in \omega} \{ g : g(D) \cap F_n = \emptyset \} = \bigcap_{d \in D, n \in \omega} \{ g : g(d) \notin F_n \} \]
is a dense $G_\delta$. □

**Proposition 3.** If $T$ is a Baire non-meagre subset of a metric space $X$ and $G$ a normed group, Baire in its norm topology, acting separately continuously and transitively on $X$, with the Nikodym property – then, for every convergent sequence $x_n$ with limit $x_0$ there is $\tau \in G$ and an integer $N$ with $\tau x_0 \in T$ and
\[ \{ \tau(x_n) : n > N \} \subseteq T. \]

**Proof.** Write $T := M \cup (U \setminus \bigcup_{n \in \omega} F_n)$ with $U$ open, $M$ meagre and each $F_n$ closed and nowhere dense in $X$. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x_0 = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$. Put
\[ C := \bigcap_{m,n \in \omega} \{ \alpha \in G : \alpha(u_m) \notin F_n \}, \]
a dense $G_\delta$ in $G$; then, by the Separation Lemma above, as $G$ is Baire,
\[ \{ \alpha \in G : \alpha(u_0) \in U \} \cap C \]
is non-empty. For $\alpha$ in this set we have $\alpha(u_0) \in U \setminus \bigcup_{n \in \omega} F_n$. Now $\alpha(u_n) \to \alpha(u_0)$, by continuity of $\alpha$, and $U$ is open. So for some $N$ we have for $n > N$ that $\alpha(u_n) \in U$. Since $\{ \alpha(u_m) : m = 1,2,.. \} \in X \setminus \bigcup_{n \in \omega} F_n$, we have for $n > N$ that $\alpha(u_n) \in U \setminus \bigcup_{n \in \omega} F_n \subseteq T$.

Finally put $\tau := \alpha \sigma$; then $\tau(x_0) = \alpha \sigma(x_0) \in T$ and $\{ \tau(x_n) : n > N \} \subseteq T$. □

### 3.3 Proof of S

We work in the right norm topology and use the notation of the preceding proof (of Proposition 3), so that $U$ here is the quasi-interior of $T$ and $\sigma x_0 = u_0$. As $e_G \in A^q$ and $A$ is a non-meagre Baire set, we may without loss of generality write $A = B_{e_G}(e_G) \setminus \bigcup_n G_n$, where each $G_n$ is closed nowhere dense with $e_G \notin G_n$ and $B_{e_G}(e_G)$ is the quasi-interior of $A$.

As $A^q x_0 \cap T^q$ is non-empty, there is $\alpha_0 \in B_{e_G}(e_G)$ with $\alpha_0 x_0 \in U$ (but, we want a better $\alpha$ so that $\alpha x_0 \in T$ and $\alpha \in A$). Put $\beta_0 = \alpha_0 \sigma^{-1}$; then
\[ \beta_0 = \alpha_0 \sigma^{-1} \in B_{e_G}(e_G) \sigma^{-1} \cap \{ \alpha : \alpha(x_0) \in U \} \sigma^{-1} = B_{e_G}(e_G) \sigma^{-1} \cap \{ \beta : \beta(\sigma x_0) \in U \} = B_{e_G}(e_G) \sigma^{-1} \cap \{ \beta : \beta(u_0) \in U \}, \]
i.e. the open set \( \{ \beta : \beta(u_0) \in U \} \cap B_\varepsilon(e_G) \sigma^{-1} \) is non-empty. So
\[
(C \setminus \bigcup_n G_n \sigma^{-1}) \cap \{ \beta : \beta(u_0) \in U \} \cap B_\varepsilon(e_G) \sigma^{-1} \neq \emptyset,
\]
since \( G \) is a Baire space and each \( G_n \sigma^{-1} \) is closed and nowhere dense in \( G \) (as the right shift \( g \rightarrow g \sigma^{-1} \) is a homeomorphism).

So there is \( \beta \) with \( \beta(u_0) \in U \) such that \( \alpha := \beta \sigma \in B_\varepsilon(e_G) \setminus \bigcup_n G_n = A \). That is, \( \alpha x_0 = \beta u_0 \in U \); so \( \beta(u_n) \in U \) for large \( n \), for \( n > N \) say, as \( \alpha x_0 = \lim \alpha x_n = \lim \beta \sigma x_n = \lim \beta u_n \). But \( \{ \beta(u_m) : m = 1, 2, .. \} \in X \setminus \bigcup_n F_n \), as \( \beta \in C \); so \( \beta(u_n) \in U \setminus \bigcup_n F_n \subseteq T \) for \( n > N \).

Finally, \( \alpha(x_0) = \beta \sigma(x_0) \in T \) and \( \{ \alpha(x_n) : n > N \} \subseteq T \). □

### 3.4 Proof that \( S \implies E \)

Assume \( G \) acts transitively on \( X \) and that \( X \) is non-meagre. Let \( B := B_\varepsilon(e_G) \) and suppose that for some \( x \) the set \( Bx \) is not a nhd of \( x \). Then there is \( x_n \rightarrow x \) with \( x_n \notin Bx \) for each \( n \). Take \( A := B_{\varepsilon/2}(e_G) \) and note first that \( A \) is a symmetric open set \( (A^{-1} = A \) since \( ||g|| = ||g^{-1}|| \)\), and secondly that by the Nikodym property \( Ax \) contains a non-meagre, Baire subset \( T \). So by Theorem S, as \( Ax \) meets \( T^q \), there are \( a \in A \) (which being open has the Baire property) and a co-finite \( \mathbb{M}_n \) such that \( ax_m \in Ax \) for \( m \in \mathbb{M}_n \). For any such \( m \), choose \( b_m \in A \) with \( ax_m = b_m x \). Then \( x_m = a^{-1} b_m x \in A^2 x \subseteq Bx \), a contradiction (note that \( a^{-1} \in A \), by symmetry).

As earlier, in the special case that \( G \) is (metrizable and) analytic, \( A \) is analytic, since open sets are \( \mathcal{F}_\sigma \) and Souslin-\( \mathcal{F} \) subsets of analytic sets are analytic, cf. [JayR] Th. 2.5.3, by Prop. 3 \( Ax \) is Baire non-meagre, as \( \varphi_x \) is base-\( \sigma \)-discrete.

### 4 Complements

1. **Normed groups.** For \( T \) an algebraic group with neutral element \( e = e_G \), say that \( \| \cdot \| : T \to \mathbb{R}_+ \) is a **group-norm** ([ArhT], [BinOl]) if the following properties hold:
   
   - **Subadditivity** (Triangle inequality): \( \|st\| \leq \|s\| + \|t\| \);
   - **Positivity**: \( \|t\| > 0 \) for \( t \neq e \) and \( \|e\| = 0 \);
   - **Inversion** (Symmetry): \( \|t^{-1}\| = \|t\| \).

Then \((T, \|\cdot\|)\) is called a **normed-group.**
The group-norm generates a right and a left norm topology via the right-invariant and left-invariant metrics $d^R_T(s, t) := ||st^{-1}||$ and $d^L_T(s, t) := ||s^{-1}t|| = d^R_T(s^{-1}, t^{-1})$. In the right norm-topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s, t)$, so in particular the group is a right topological group; likewise in the left norm-topology the left shift $\lambda_s(t) = st$ is a uniformly continuous homeomorphism. Since $d^L_T(t, e) = d^L_T(e, t^{-1}) = d^R_T(e, t)$, convergence at $e$ is identical under either topology. One may refer to whichever topology is appropriate, since despite their differences, they are homeomorphic. Thus one may say, for instance, that separability is a norm property, as separability of either norm topology implies that of the other.

See [Ost5] for a characterization of almost-complete normed groups as non-meagre almost-analytic (i.e. analytic modulo a meagre set).

2. Genericity Our focus on various shift-compactness theorems derives from their close affinity with the literature of ‘generic’ automorphisms (for which see [Mac], also described in the introduction to [Ost5]) and from a multitude of its applications, for which see e.g. [BinO1], and these offering a unifying sequential compactness-like, or combinatorial, perspective on category-measure duality and on other, apparently unrelated, problems.

3. Assumptions in Theorem S. With regard to the assumption of separate continuity, note that a theorem of Bouziad ([Bou Th. 3]) implies that a separately continuous action by a metrizable left-topological Baire group acting on a metric space is in fact jointly continuous; we retain the only apparently weaker hypothesis of separate continuity, because there are variants of Theorem S, where joint continuity is absent (and the group is not Baire): see [MilO]. The result is connected with van Mill’s Separation Property: say that $X$ has SP ([vMil2]) with respect to a group of homeomorphisms $\mathcal{G}$ if for any countable set $D$ in $X$ and any meagre set $M$ in $X$, there is a homeomorphism $g \in \mathcal{G}$ such that $g(D) \cap M = \emptyset$; it is interesting to recall here the Galvin-Mycielski-Solovay characterization of subsets of $\mathbb{R}$ of strong measure zero as sets $A$ such that for any meagre $M$ there is $g$ with $(g + A) \cap M = \emptyset$ (see [Mil §3]). Stated equivalently, SP asserts that for $D$ countable and $T$ co-meagre, there is $g \in \mathcal{G}$ with $g(D) \subseteq T$ (see also the next remark). Compared to this restatement, Theorem S refers on the one hand to a smaller class of countable sets (convergent sequences, or their co-finite parts), but on the other hand asserts embeddability into a larger class of sets – sets $T$ that
are ‘locally’ rather than globally co-meagre; for further information see also [MilO].

4. Shift-compactness and the SP property. In view of the strength of Th. S and to place our results in context, we briefly summarize some of the relevant results of [vMil2, Th. 1.1]. A (separable metric) space with SP is Baire, by the proof which we imported for Propositions B1 and B2 in §2; likewise, an almost complete non-meagre space with SP is completely Baire. Since an absolutely co-analytic space is Polish if it is ‘completely Baire’, i.e. closed-hereditarily Baire (closed subspaces are Baire), for which see Kechris [Kec2, Cor. 21.21], it follows that an absolutely Borel space with the SP is Polish ([vMil2, Th. 1.1]). More generally, if an analytic group $G$ acts on a space $X$ and SP holds w.r.t. $G$, then $X$ is Polish. Van Mill also shows from his Prop 3.4 (cf. our Prop. 2) that a locally compact homogeneous space has the SP. It seems likely that, just as with Proposition B, more of these arguments can be copied across in the language of Th. S.

It is noted in [BinO2] that a Polish space which is strongly locally homogeneous has SP.

5. Index-$\sigma$-discrete maps. In many circumstances it is easier to work with index-$\sigma$-discrete maps: see [Ost1] for a brief discussion of this point and for relations with the automatic continuity of homomorphisms (noted earlier in [Nol]). Furthermore, if $f : X \to Y$ is injective and closed-analytic, i.e. carries closed sets to analytic sets (or, alternatively open-analytic, mutatis mutandis), then $f$ is base-$\sigma$-discrete – in fact index-$\sigma$-discrete ([Han3] Prop. 3.14). On the other hand, if $f$ is surjective and closed with $Y$ metrizable, then $f$ is base-$\sigma$-discrete ([Han3] Prop. 3.10). So if the group action is such that each $\varphi_x$ is open-analytic and injective (as when a vector space acts on itself), then each $\varphi_x$ is index- and so base-$\sigma$-discrete.

6. Generalizations of analyticity. A Hausdorff space is almost analytic if it is analytic modulo a meagre set. Similarly, a space $X'$ is absolutely $G_\delta$, or an absolute-$G_\delta$, if $X'$ is a $G_\delta$ in all spaces $X$ containing $X'$ as a subspace. This latter property is equivalent to complete metrizability in the realm of metrizable spaces [Eng Th. 4.3.24] (and to topological/Čech completeness in the realm of completely regular spaces – [Eng, §3.9]). So a metrizable absolute-$G_\delta$ is analytic. A metric space is almost complete if it contains a dense absolute $G_\delta$. The notion of ‘almost completeness’ is due to Frolík in [Frol] (but its name to Michael [Mic2] – see also [AaL] and [BinO1]).
Consequently, it is natural to assume that when a group $G$ acts on a space $X$ each point evaluation map $\varphi_x : g \to g(x)$ is not only continuous but also base-$\sigma$-discrete; the Nikodym property above is implied by the former property when $G$ is analytic. Note that [Han3] Ex. 3.12 shows that, for $D = (\mathbb{R}, d_{\text{discrete}})$ the discrete space of cardinality the continuum, the projection from $D \times [0,1] \to [0,1]$ is an open mapping that is not base-$\sigma$-discrete (by reference to the closed discrete graph of a bijection between $D$ and $[0,1]$).

Index-$\sigma$-discrete maps (easier to work with) are base-$\sigma$-discrete ([Han3] Prop. 3.7(i)); the latter combined with continuity preserves analyticity, as mentioned earlier. Evidently, if all evaluation maps $\varphi_x$ are continuous and base-$\sigma$-discrete and $A$ is an open subset of an analytic group $G$, acting on $X$, then $A$ is analytic, and so $\varphi_x(A) = Ax$ is analytic; thus the point evaluations are then open-analytic. So our Nikodym hypothesis on the action is not far from demanding that point evaluations be open-analytic.

7. Convexity and Baire Property. We have seen above that analyticity confers the Baire property. Convexity may also confer it: for $A$ convex, if $\text{int}A$, its interior, is non-empty, then $\text{int}A$ is dense in $A$, and so $A$ has the Baire property: $A\setminus\text{int}(A) \subseteq \overline{\text{cl}(A)}\setminus\text{int}(\overline{\text{cl}(A)})$, a nowhere dense set; so $A$ differs from its interior by a nowhere dense subset of $\overline{\text{cl}(A)}\setminus\text{int}(\overline{\text{cl}(A)})$. If $A$ is non-meagre its closure has non-empty interior, raising the question of whether $A$ itself has non-empty interior.

The Banach context is particularly transparent; Banach’s lemma (see Appendix) yields: for a continuous linear map $T : X \to Y$ from a Banach space $X$ to a normed space $Y$ and $x \in X$, if $B(0,\rho) \subseteq \text{cl}(T(B(x,r)))$ with $r, \rho > 0$, then $B(0,\rho) \subseteq T(B(x,r))$.

So, as above, by translation, interpret this lemma as asserting that for $r > 0$ and $A := L(B(0,r))$ one has $\text{int}(\overline{\text{cl}(A)}) \subseteq \text{int}(A)$. So $A\setminus\text{int}(A) \subseteq \overline{\text{cl}(A)}\setminus\text{int}(\overline{\text{cl}(A)})$, a nowhere dense set, so $A$ differs from its interior by a nowhere dense subset of $\overline{\text{cl}(A)}\setminus\text{int}(\overline{\text{cl}(A)})$, and so has the Baire property.

In the Fréchet case, with the balls referring to the underlying translation-invariant metric, one may apply Lemma 1 of §2.3. Taking the convex set $B(0,r)$ with $r > 0$, suppose $T(B(0,r))$ is non-meagre; then $T(B(0,s))$ is non-meagre for some $0 < s < r$. By the claim, for $s < t < r$,

$$\emptyset \neq \text{int}(\overline{\text{cl}(T(B(0,s)))}) \subseteq T(B(0,t)).$$

As $T(B(0,t))$ is convex, $\text{int}T(B(0,t))$ is convex, and being non-empty is dense in $\overline{\text{cl}(T(B(0,t)))}$, i.e. $\overline{\text{cl}(T(B(0,t)))} = \overline{\text{cl}(\text{int}(T(B(0,t))))}$. So $T(B(0,t))$
differs from \( \text{int} T(B(0, t)) \) by a meagre set, as

\[
T(B(0, t)) \setminus \text{int} T(B(0, t)) \subseteq \text{cl} T(B(0, t)) \setminus \text{int} T(B(0, t)) = \text{cl}(\text{int} T(B(0, t))) \setminus \text{int} T(B(0, t)).
\]

Finally, taking \( s < t_n \not\nearrow r \), one has

\[
T(B(0, r)) \setminus \text{int} T(B(0, r)) \subseteq \bigcup_{n \in \mathbb{N}} \text{cl}(\text{int} T(B(0, t_n))) \setminus \text{int} T(B(0, t_n)).
\]

So for \( B(0, r) \) convex, either \( T(B(0, r)) \) is meagre or differs from \( \text{int} T(B(0, r)) \) by a meagre set.

8. Topological and semitopological groups. When a group \( G \) equipped with a topology such that the (action) map \( (g_1, g_2) \to g_1 g_2 \) from \( G^2 \) to \( G \) is separately continuous (i.e. left and right shifts are continuous), then it is said to be a semitopological group. Thus a topological group is semitopological. If also the topology is metrizable and \( G \) is Baire, Theorem S applies and asserts here that if \( T \) is non-meagre in \( G \) and \( g_n \to g \), then for some \( t \in T \) the point \( tg \) is in \( T \) and almost all of the sequence \( tg_n \) is in \( T \). As in Theorem S, under these additional assumptions, group multiplication in \( G \) is jointly continuous; indeed, Bouziad [Bou] proves that a semitopological Baire \( p \)-space (and metric spaces are \( p \)-spaces) has jointly continuous multiplication (is ‘paratopological’).

9. Specializing the proof of Theorem S to semitopological groups. The argument proving Th. S is particularly transparent in the case of a semitopological group and follows a standard real-line argument, as follows. Suppose \( x_n \to x_0 \) and consider \( T := U \setminus \bigcup F_m \) with \( F_m \) closed and nowhere dense, and \( U \) open. For any \( u_0 \in T \cap U \), take \( \sigma(x) = xx_0^{-1} u_0 \), which is a continuous right-shift, so that \( u_n = \sigma(x_n) = x_n(x_0^{-1} u_0) \to u_0 = \sigma(x_0) \). Now

\[
u_0^{-1} U \cap \bigcap_{n, m} u_0^{-1}(X \setminus F_m)
\]

is a dense \( G_\delta \), since left-shifts are homeomorphisms. As \( G \) is Baire, there is \( g \) with

\[
g \in u_0^{-1} U \cap \bigcap_{n, m} u_n^{-1}(X \setminus F_m).
\]

Then \( u_0 g \in U \) and \( u_0 g \notin F_m \) for all \( m \), so \( u_0 g \in T \). By continuity of right-shifts, \( u_n g \to u_0 g \), so for large \( n \), say for \( n > N \), we have \( u_n g \in U \).
For each such \( n \) we have \( u_n g \notin F_n \) for all \( m \) and so \( u_n g \in T \). Thus, as \( u_n g = x_n(x_0^{-1}u_0)g \), we conclude the existence for some \( g \in G \) and \( u_0 \in T \) with \( t := u_0 g \in T \) that, as in Theorem S

\[
x_n x_0^{-1} t \in T \text{ for } n > N.
\]

10. From E back to S. In Proposition 3 density of the set \( W_{x,F} \) was deduced from a Lemma which, as noted, asserts a very weak microtransitivity. Unsurprisingly, Property E also implies density of that set (see below). However, Property E does not imply that \( G \) is Baire, since (see Introduction) there do exist meagre analytic groups acting on a non-meagre space which by van Mill’s result have Property E notwithstanding. So we can go no further with the density argument of Proposition 3. Of course, as in Proposition 3, if we know that \( G \) is Baire, then E implies S.

As for the density claim, consider \( \beta \in G \). Suppose first that \( \beta(x) \in F \). By the Effros property, the set \( B_\varepsilon(\beta)x = B_\varepsilon(e_G)\beta x \) is an open nhd of \( \beta x \). As \( F \) is nowhere dense and closed, there is \( y \in \text{int}(B_\varepsilon(e_G)\beta x) \setminus F \). So there is \( \gamma \in B_\varepsilon(e_G) \) such that \( y = \alpha x := \gamma \beta x \notin F \). So \( a = \gamma \beta \in B_\varepsilon(\beta) \cap \{ \alpha : \alpha(x) \notin F \} \).

If, on the other hand, \( \beta(x) \notin F \), then \( \beta \in B_\varepsilon(\beta) \cap \{ \alpha : \alpha(x) \notin F \} \). This proves the claim.

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Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE
a.j.ostaszewski@lse.ac.uk

22
Appendix

For convenience we include here the

**Lemma.** (see e.g. [TayL, Lemma 5.4]; [Ban, Ch. III; Proof of Th.3]) For a continuous linear map $T : X \to Y$ from a Banach space $X$ to a normed space $Y$, if $B(0, \rho) \subseteq \text{cl}(T(\bar{B}(a, r)))$, then $B(0, \rho) \subseteq T(\bar{B}(a, r))$.

**Proof.** Let $\varepsilon$ be arbitrary with $0 < \varepsilon < 1$ and suppose that $B(0, \rho) \subseteq \text{cl}(T(\bar{B}(a, r)))$. If $y \in B(0, \rho)$, then $||y - y_1|| < \varepsilon \rho$ for some $y_1 = Tx_1$ with $x_1 \in B(a, r)$. So $x_1 = a + z_1$ with $||z_1|| < r$. But $||y - Tx_1|| < \varepsilon \rho$ and, by homogeneity of the norm in $Y$, 

$$B(0, \varepsilon \rho) = \varepsilon B(0, \rho) \subseteq \text{cl}(T(\varepsilon a, \varepsilon r)),$$

so

$$||y - Tx_1 - Tx_2|| < \varepsilon^2 \rho.$$ 

for some $x_2 \in \bar{B}(\varepsilon a, \varepsilon r)$, i.e. $x_2 = \varepsilon a + z_2$ for some $z_2$ with $||z_2|| \leq \varepsilon r$. Continue by induction. Now

$$x_1 + ... + x_n = a(1 + \varepsilon + \varepsilon^2 + ... + \varepsilon^{n-1}) + z_1 + ... + z_n,$$

and

$$||z_1 + ... + z_n|| \leq ||z_1|| + ... + ||z_n|| \leq r(1 + ... + \varepsilon^{n-1}).$$

By completeness, $x_1 + ... + x_n \to x_* \in \bar{B}(a/(1 - \varepsilon), r/(1 - \varepsilon))$ and $Tx_* = y$. So

$$B(0, \rho) \subseteq T(\bar{B}(a/(1 - \varepsilon), r/(1 - \varepsilon)))$$

and so

$$B(0, (1 - \varepsilon)\rho) = (1 - \varepsilon)B(0, \rho) \subseteq T(\bar{B}(a, r)).$$

As $\varepsilon$ was arbitrary we conclude that

$$B(0, \rho) = \bigcup_{0 < \varepsilon < 1} B(0, (1 - \varepsilon)\rho) \subseteq T(\bar{B}(a, r)).$$

In particular $0 \in \text{int}(T(\bar{B}(a, r)))$. □