CHEBYSHEV POLYNOMIALS AND INEQUALITIES FOR KLEINIAN GROUPS

HALA ALAQAD, JIANHUA GONG, AND GAVEN MARTIN

ABSTRACT. The principal character of a representation of the free group of rank two into $\text{PSL}(2, \mathbb{C})$ is a triple of complex numbers that determines an irreducible representation uniquely up to conjugacy. It is a central problem in the geometry of discrete groups and low dimensional topology to determine when such a triple represents a discrete group which is not virtually abelian, that is a Kleinian group. A classical necessary condition is Jørgensen’s inequality. Here we use certainly shifted Chebyshev polynomials and trace identities to determine new families of such inequalities, some of which are best possible. The use of these polynomials also shows how we can identify the principal character of some important subgroups from that of the group itself.

1. Introduction

A connection between the geometry of discrete groups and certain shifted Chebyshev polynomials is given in [5, Theorem 2.3]. It relates the principal character (defined below) of a representation of the free group $\langle a, b \rangle$ of rank two to the principal characters of $\langle a^n, b \rangle$. Using this result, the authors extended Jørgensen’s inequality for discrete groups as well as other inequalities and gave a quantitative measure of the fact that certain groups are isolated in the topology of algebraic convergence. Asymptotically sharp estimates concerning the way loxodromic elements can degenerate into parabolic elements in sequences of discrete groups were also obtained. Here we extend these results in different directions using a family of polynomial trace identities first identified in [7]. The theory of these polynomials is further developed in [13] where they arise from a two variable polynomial Pell equation. We recall that the Chebychev polynomials arise as the solution of the usual polynomial Pell equation.

1.1. Notation and definitions. Consider the Poincaré upper half-space model $\mathbb{H}^3$ of 3-dimensional hyperbolic space $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ equipped with the hyperbolic metric

$$d^2 s = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2},$$

the boundary of 3-dimensional hyperbolic space is the Riemann sphere $\partial \mathbb{H}^3 = \mathbb{C}$, and we denote $\mathbb{H}^3 = \mathbb{H}^3 \cup \mathbb{C}$.

2000 Mathematics Subject Classification. Primary 30C60, 30F40; Secondary 20H10, 53A35.

Key words and phrases. Principal character, Chebyshev polynomials, Kleinian group, Jørgensen’s inequality.

Research supported by UAEU UPAR Grant (31S315). This paper is partly contained in the first author’s PhD thesis.
Let $\text{Isom}^+(\mathbb{H}^3)$ be the topological group of orientation preserving hyperbolic isometries of $\mathbb{H}^3$, and let $\text{M"ob}^+(\mathbb{C})$ be the topological group of the normalized orientation preserving M"obius transformations

$$\text{M"ob}^+(\mathbb{C})=\left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}\text{ and } ad-bc=1 \right\}. $$

Then these two topological groups are isomorphic as each isometry of $\text{Isom}^+(\mathbb{H}^3)$ is an extension of the element of $\text{M"ob}^+(\mathbb{C})$ (via the Poincaré extension [11]), and also each M"obius transformation of $\text{M"ob}^+(\mathbb{C})$ is the restriction of the extended isometry of $\text{Isom}^+(\mathbb{H}^3)$ on $\mathbb{H}^3$. Next the topological group projective special linear group $\text{PSL}(2,\mathbb{C})=\text{SL}(2,\mathbb{C})/\{\pm \text{Id}\}$, is also isomorphic as a topological group to each of the previous groups as each element gives the same M"obius transformation on the Riemann sphere $\mathbb{C}$.

A subgroup of $\text{Isom}^+(\mathbb{H}^3)$ is called a Kleinian group if it is non-elementary and discrete, where a subgroup of $\text{Isom}^+(\mathbb{H}^3)$ is called elementary if its accumulation set in $\mathbb{H}^3$ of an orbit of an arbitrary point in $\mathbb{H}^3$ is finite. Equivalently a group is elementary if it is a finite extension of an abelian group. The elementary groups are classified, see for instance [11]. In [8] Gehring and Martin introduced the following three complex parameters defined by the traces of generators for a group $\langle f, g \rangle$ generated by two elements $f$ and $g$ in $\text{PSL}(2,\mathbb{C})$:

$$\gamma(f, g) = \text{tr}(f, g) - 2, \quad \beta(f) = \text{tr}^2(f) - 4, \quad \beta(g) = \text{tr}^2(g) - 4$$

where $[f, g] = fgf^{-1}g^{-1}$ is the multiplicative commutator. The triple

$$(\gamma(f, g), \beta(f), \beta(g)) \in \mathbb{C}^3$$

is called the principal character of the representation of the group $\langle a, b \rangle \to \text{PSL}(2,\mathbb{C})$ defined by $a \mapsto f$ and $b \mapsto g$.

Furthermore, [8, Lemma 2.2] confirmed that if the parameter $\gamma(f, g) \neq 0$, then $\langle f, g \rangle$ is determined uniquely up to conjugacy by the principal character. While $\gamma(f, g) = 0$ implies that $f$ and $g$ share a common fixed point in $\mathbb{C}$, and this is not possible in a Kleinian group. Notice the normalizations of the trace have made the principal character of the identity group equal to $(0,0,0)$. We record this discussion in the following lemma.

**Lemma 1.** Every Kleinian group $\langle f, g \rangle$ generated by two elements $f$ and $g$ of $\text{Isom}^+(\mathbb{H}^3)$ is determined uniquely up to conjugacy by its principal character, the triple of complex parameters $$(\gamma(f, g), \beta(f), \beta(g)).$$

Thus, the complex parameters are an important tool in studying Kleinian groups. The complex parameters for a two-generator group conveniently encode various important geometric quantities such as translation length, holonomy, and complex hyperbolic distance introduced in the following definitions (also see [10]).

**Definition 2.** Suppose that $f$ is a non-identity element of $\text{Isom}^+(\mathbb{H}^3)$, $f$ is said to be

(a) parabolic if it has a single fixed point in $\mathbb{C}$,
(b) elliptic if it has two fixed points in $\mathbb{C}$ and a fixed point in $\mathbb{H}^3$,
(c) loxodromic if it has two fixed points in $\mathbb{C}$ and no fixed points in $\mathbb{H}^3$.

In particular, if $f$ is an elliptic or a loxodromic element of $\text{Isom}^+(\mathbb{H}^3)$, then the hyperbolic line in $\mathbb{H}^3$ whose end points are the fixed points of $f$ on $\overline{\mathbb{C}}$ is called the
The axis of $f$, denoted by $\text{axis}(f)$. The axis of an elliptic element is its fixed point set, and the axis of a loxodromic element its setwise fixed hyperbolic line.

**Definition 3.** Let $f$ and $g$ be elliptic or loxodromic elements of $\text{Isom}^+(\mathbb{H}^3)$, and suppose that $p$ is a hyperbolic line perpendicular to $\text{axis}(f)$.

1. The hyperbolic distance between two hyperbolic lines $p$ and $f(p)$ is called the translation length of $f$, denoted by $\tau_f$.
2. The dihedral angle between the plane containing $\text{axis}(f)$ and $p$ and the plane containing $\text{axis}(f)$ and $f(p)$ is called the holonomy of $f$, denoted by $\theta_f$.
3. The complex number $\delta + i\theta$ is called the complex hyperbolic distance between the axes of $f$ and $g$ if $\delta$ is the hyperbolic distance between $\text{axis}(f)$ and $\text{axis}(g)$ and $\theta$ is the holonomy of the element of $\text{Möb}^+(\mathbb{C})$ whose natural extension moves $\text{axis}(f)$ to $\text{axis}(g)$, and whose axis contains the common perpendicular between $\text{axis}(f)$ and $\text{axis}(g)$.

The easiest way to see the holonomy $\theta_f$ is to use a conjugacy to arrange things so that $\text{axis}(f)$ lies on $x_3$-axis, then it is simply the angle between the vertical projections to $\mathbb{C}$ of $p$ and $f(p)$ at the origin. An elementary calculation (see e.g. [11]) gives a way to find the parameters of a two-generator group $\langle f, g \rangle$ in terms of geometric quantities $\tau_f, \theta_f$, and $\delta + i\theta$ as following:

\begin{align}
\beta(f) &= 4 \sinh^2 \left( \frac{\tau_f + i\theta_f}{2} \right), \\
\beta(g) &= 4 \sinh^2 \left( \frac{\tau_g + i\theta_g}{2} \right), \\
\gamma(f, g) &= \frac{\beta(f)\beta(g)}{4} \sinh^2(\delta + i\theta).
\end{align}

The identification of precise inequalities for discrete groups of Möbius transformations started with Jørgensen’s famous inequality [12] from 1976, after earlier results of Shimizu from 1963 [19] and Leutbecher from 1967 [13] which gave estimates in the important special case when a generator is parabolic. Jørgensen’s inequality is the first important universal constraint in studying the geometry of Kleinian groups. We state it here to exhibit the relationship between the principal characters and discreteness criteria.

**Lemma 4.** (Jørgensen’s inequality). Let $(f, g)$ be a Kleinian group. Then

\begin{equation}
|\gamma(f, g)| + |\beta(f)| \geq 1.
\end{equation}

The inequality is sharp. It is achieved for representations of the $(2, 3, p)$-triangle groups, all $p \geq 7$, and the figure eight knot complement group $(\beta(f) = 0, \gamma(f, g) = \frac{1+i\sqrt{3}}{2})$.

Other inequalities can be found in [5, 6, 7, 8, 9, 10] and also [20]. One can also find many applications of these inequalities, the first due to Jørgensen concerns the local compactness of the space of principal characters under algebraic convergence, and the “thick and thin” decomposition of hyperbolic 3-manifolds and so forth. If we write (1.3) as

\[|\gamma(f, g) - \gamma_0| + |\beta(f) - \beta_0| \geq 1, \quad \gamma_0 = \beta_0 = 0,\]

then we see that (1.3) measures the isolation of the discrete elementary groups with principal character $(0, 0, z)$ from the non-elementary Kleinian groups. Thus we are particularly interested in inequalities of the form

\[|\gamma(f, g) - \gamma_0| + |\beta(f) - \beta_0| \geq \delta\]
when \((\gamma_0, \beta_0, z)\) is the principal character of an elementary discrete group, in particular the spherical triangle groups \(A_4, S_4\) and \(A_5\). An elementary compactness argument shows that \(\delta > 0\) is each case, but sharp bounds on \(\delta\) in turn imply sharp bounds on the distance between vertices of the trivalent singular graph of a hyperbolic orbifold and this has implicates for the hyperbolic volume of such spaces.

2. Two-generator Kleinian Groups

It has been long known that the class of two generator groups holds special importance. This follows from another of Jørgensen’s results which shows that a group is discrete if and only if all its two generator subgroups are discrete. Similar results in fact hold in all dimensions and generally negatively curved metrics, [16, 17]. This explains the following emphasis on two-generator groups in what follows.

We begin with an initial observation.

**Theorem 5.** Let \(\langle f, g \rangle\) be a Kleinian group generated by two elements \(f\) and \(g\) in \(\text{Isom}^+(\mathbb{H}^3)\). If \(f^n\) is not the identity for some \(n \in \mathbb{N}\), then \(\langle f^n, g \rangle\) is a Kleinian subgroup of \(\langle f, g \rangle\).

**Proof.** Certainly \(\langle f^n, g \rangle\) is discrete as a subgroup of a discrete group. If \(f\) is parabolic, loxodromic, that this group is non-elementary and therefore Kleinian follows from the classification of the elementary discrete groups ([11 Section 5.1]) as long as \(f^n\) is not the identity. Suppose \(f\) is elliptic of order \(p \geq 7\) and \(f^n\) is not the identity. Then the group is again Kleinian from the classification if \(g\) is not elliptic order \(q \leq 6\). If \(g\) is elliptic of lower order and \(f\) is also elliptic, then the group is Kleinian unless \(f^n\) (which is elliptic) and \(g\) share a finite fixed point or a point at \(\infty\) - so the axes of \(f^n\) and \(g\) meet and hence so also do those of \(f\) and \(g\) and this is not possible. \(\square\)

**Corollary 6.** Let \(\langle f, g \rangle\) be a Kleinian group generated by two elements \(f\) and \(g\). Then \(\langle f^n, g^m \rangle\) is a Kleinian subgroup of \(\langle f, g \rangle\), unless either \(f^n, g^m\) is the identity.

**Proof.** We apply Lemma 5 then \(\langle g, f^n \rangle = \langle f^n, g \rangle\) is Kleinian unless \(f^n\) is the identity and hence \(\langle f^n, g^m \rangle = \langle g^m, f^n \rangle\) is Kleinian unless either \(f^n, g^m\) is the identity. \(\square\)

**Corollary 7.** Let \(\langle f, g \rangle\) be a Kleinian group generated by two elements \(f\) and \(g\). If \((gf)^n\) is not the identity, then \(\langle (gf)^n, f \rangle\) is a Kleinian subgroup of \(\langle f, g \rangle\).

**Proof.** We need only note that \(\langle fg, f \rangle = \langle f, g \rangle\) from which the claim clearly follows. \(\square\)

**Corollary 8.** Let \(\langle f, g \rangle\) be a Kleinian group generated by two elements \(f\) and \(g\). If \(f\) is not elliptic of order \(p \leq 6\) and \([g, f]^n\) is not the identity, then \(\langle [g, f]^n, f \rangle\) is a Kleinian subgroup of \(\langle f, g \rangle\).

**Proof.** Obviously, \(\langle [g, f]^n, f \rangle\) is a subgroup of \(\langle f, g \rangle\) and so discrete. Since \(f\) is loxodromic or parabolic or elliptic of order \(p \geq 7\), then

\[
\langle f, gf^{-1} \rangle = \langle f, gf^{-1}f^{-1} \rangle
\]

is a Kleinian group. We may then apply Theorem 5 to deduce the result. \(\square\)
Actually the result is true for $f$ elliptic of order $2 \leq p \leq 6$ apart from some special cases. Maskit examined the case $p = 6$ in [18]. The key issue is whether $\langle f, gfg^{-1} \rangle$ the group generated by two elliptics is Kleinian. Again the classification tells us that it is unless $f$ has order two, or $\langle f, gfg^{-1} \rangle$ is a finite spherical triangle group. In this latter case there are only finitely many possibilities that can occur.

3. Trace Polynomials of Two Parameters

We will use Chebyshev polynomials of the first kind and their use in calculating geometric quantities such as translation length and holonomy to find formulae for calculating trace polynomials for the two complex parameters $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$ associated with specific words in a Kleinian group $\langle f, g \rangle$.

Chebyshev polynomials of the first kind are defined by the recursion formula

$$T_0(z) = 1,$$
$$T_1(z) = z,$$
$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad n \in \mathbb{N}.$$  

or by the explicit formula $T_n(z) = \frac{1}{2} \left( (z - \sqrt{z^2 - 1})^n + (z + \sqrt{z^2 - 1})^n \right)$. For example, the first nine Chebyshev polynomials are $T_0(z) = 1$, and

$$T_1(z) = z,$$
$$T_2(z) = 2z^2 - 1,$$
$$T_3(z) = 4z^3 - 3z,$$
$$T_4(z) = 8z^4 - 8z^2 + 1,$$
$$T_5(z) = 16z^5 - 20z^3 + 5z,$$
$$T_6(z) = 32z^6 - 48z^4 + 18z^2 - 1,$$
$$T_7(z) = 64z^7 - 112z^5 + 56z^3 - 7z,$$
$$T_8(z) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1.$$  

We also recall that the Chebyshev polynomials $T_n$ have the defining property

$$T_n(\cosh(z)) = \cosh(nz), \quad n \in \mathbb{N}.$$  

Now we start to find the formulae for the trace polynomials $\gamma(f^n, g)$ in terms of $\gamma$ and $\beta$ in the following theorem which is known, but which we prove for the convenience of the reader.

**Theorem 9.** Let $\langle f, g \rangle$ be a Kleinian group generated by two elements $f$ and $g$ in $\text{Isom}^+(\mathbb{H}^3)$ with two complex parameters $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$, where $f$ is elliptic or loxodromic. Then,

$$\beta(f^n) = 2T_n \left( 1 + \frac{\beta}{2} \right) - 2, \quad n \in \mathbb{N},$$

$$\gamma(f^n, g) = \frac{\beta(f^n)}{\beta} \gamma, \quad n \in \mathbb{N}.$$  

**Proof.** (1) Since $\cosh^2 \left( \frac{\tau}{2} \right) - \sinh^2 \left( \frac{\tau}{2} \right) = 1$ and $\cosh(z) = \cosh^2 \left( \frac{\tau}{2} \right) + \sinh^2 \left( \frac{\tau}{2} \right)$ give $\cosh(z) = 2 \sinh^2 \left( \frac{\tau}{2} \right) + 1$, therefore,

$$\cosh(nz) = 1 + 2 \sinh^2 \left( \frac{nz}{2} \right), \quad n \in \mathbb{N}.$$  

Since $f$ is elliptic or loxodromic, we may assume that $f$ has two fixed points $0$ and $\infty$, then, up to conjugacy, $f$ can be represented by $f = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{array} \right) \in \text{PSL}(2, \mathbb{C})$ and hence $f^n = \left( \begin{array}{cc} \lambda^n & 0 \\ 0 & \frac{1}{\lambda^n} \end{array} \right)$, where $\lambda$ can be expressed as $e^{2\pi i \tau}$ for a suitable $\tau = \tau_f + i\theta_f$. Thus, $\beta(f^n) = (\lambda^n - \frac{1}{\lambda^n})^2 = 4 \left( \frac{e^{2\pi i \tau} - e^{-2\pi i \tau}}{2} \right)^2 = 4 \sinh^2 \left( \frac{nz}{2} \right)$. That
is,
\begin{equation}
\beta(f^n) = 4 \sinh^2 \left( \frac{n\tau}{2} \right), \text{ for } n \in \mathbb{N}, \tau = \tau_f + i\theta_f.
\end{equation}

where \(\tau_f\) and \(\theta_f\) are the translation length and the holonomy of \(f\), respectively.

It follows from the identities (3.5) and (3.6) that
\begin{equation}
\cosh(nz) = 1 + \frac{\beta(f^n)}{2}, \text{ for } n \in \mathbb{N}.
\end{equation}

Applying for the defining property (3.2) and the previous identity (3.7), that give
\begin{equation}
T_n \left( 1 + \frac{\beta}{2} \right) = 1 + \frac{\beta(f^n)}{2} \text{ and hence }
\end{equation}
\begin{equation}
\beta(f^n) = 2T_n \left( 1 + \frac{\beta}{2} \right) - 2, \text{ for } n \in \mathbb{N}.
\end{equation}

(2) Notice that one can represent \(f^n\) and \(g\) as the following:
\begin{equation}
f^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \frac{1}{\lambda^n} \end{pmatrix} \text{ and } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}),
\end{equation}
where \(bc \neq 0\) because that \(\langle f, g \rangle\) is Kleinian. Thus, the commutator is
\begin{equation}
[f^n, g] = \begin{pmatrix} ad - bc\lambda^{2n} & ab\lambda^{2n} - ab \\ \frac{ad}{\lambda^n} - cd & ad - \frac{bc}{\lambda^n} \end{pmatrix}.
\end{equation}

Using \(ad = 1 + bc\),
\begin{equation}
\text{tr} [f^n, g] = -bc(\lambda^{2n} + \frac{1}{\lambda^{2n}}) + 2 + 2bc
\end{equation}
\begin{equation}
= -bc(\lambda^n - \frac{1}{\lambda^n})^2 + 2.
\end{equation}

Since \(f\) is non-parabolic, \(\beta(f) = (\lambda - \frac{1}{\lambda})^2 \neq 0\) and hence
\begin{equation}
\gamma(f^n, g) = -bc(\lambda^n - \frac{1}{\lambda^n})^2
\end{equation}
\begin{equation}
= -\frac{(\lambda^n - \frac{1}{\lambda^n})^2}{(\lambda - \frac{1}{\lambda})^2} bc(\lambda - \frac{1}{\lambda})^2
\end{equation}
\begin{equation}
= \frac{\beta(f^n)}{\beta(f)} \gamma, \text{ for } n \in \mathbb{N}.
\end{equation}

The result follows. \(\square\)

For example, we have the following particular formulas.
\begin{align}
\gamma(f, g) &= \gamma, \\
\gamma(f^2, g) &= \gamma(\beta + 4), \\
\gamma(f^3, g) &= \gamma(\beta + 3)^2, \\
\gamma(f^4, g) &= \gamma(\beta + 4)(\beta + 2)^2, \\
\gamma(f^5, g) &= \gamma(\beta^2 + 5\beta + 5)^2, \\
\gamma(f^6, g) &= \gamma(\beta + 4)(\beta + 3)^2(\beta + 1)^2.
\end{align}

Next, we notice that if \(\langle f, g \rangle\) is a Kleinian group and if \((gf)^n+1\) is not the identity, then
\begin{equation}
\gamma((gf)^n g, f) = \gamma((gf)^n g, f) = \gamma((gf)^{n+1}, f).
\end{equation}
This has the implication that \( \gamma((gf)^n g, f) \neq 0 \). Otherwise, if \( \gamma((gf)^n g, f) = 0 \), then \( \gamma((gf)^{n+1}, f) = 0 \) which means that \((gf)^{n+1}\) and \(f\) share a fixed point. Then \(gf\) and \(f\) share a fixed point so that \(\ell (gf, f) = \ell (g, f)\) is elementary, a contradiction.

We now develop a recursion formula for the commutators \(\gamma_n = \gamma(f^n, g)\). Remarkably (3.11) and (3.12) show both \(\beta(f^n)\) and \(\gamma(f^n, g)\) satisfy the same recursion relation.

**Lemma 10.** Let \(\langle f, g \rangle\) be a Kleinian group generated by two elements \(f\) and \(g\) in Isom\(^+\)(\(\mathbb{H}^3\)) with two complex parameters \(\gamma = \gamma(f, g)\) and \(\beta = \beta(f, g)\), where \(f\) is elliptic or loxodromic. Let \(\beta_0 = \beta(f)\) and \(\gamma_0 = \gamma(f^n, g)\). Then

\[
\beta_{n+1} = (2 + \beta)\beta_n - \beta_{n-1} + 2\beta_1,
\]

and

\[
\gamma_{n+1} = (2 + \beta)\gamma_n - \gamma_{n-1} + 2\gamma_1.
\]

**Proof.** Since \(\beta_0 = \beta(f)\), so \(\beta_0 = 0\) and \(\beta_1 = \beta\). Then using (3.8) and the recursion formula (3.1) we may calculate that

\[
\beta_{n+1} = 2T_{n+1} \left(1 + \frac{\beta}{2}\right) - 2 = 4 \left(1 + \frac{\beta}{2}\right) T_n \left(1 + \frac{\beta}{2}\right) - 2T_{n-1} \left(1 + \frac{\beta}{2}\right) - 2
\]

\[
= (2 + \beta)2T_n \left(1 + \frac{\beta}{2}\right) - 2T_{n-1} \left(1 + \frac{\beta}{2}\right) - 2
\]

\[
= (2 + \beta) \left[2T_n \left(1 + \frac{\beta}{2}\right) - 2\right] - \left[2T_{n-1} \left(1 + \frac{\beta}{2}\right) - 2\right] - 4 + 2(2 + \beta)
\]

\[
= (2 + \beta)\beta_n - \beta_{n-1} + 2\beta.
\]

Next we put \(\gamma_0 = 0\) and \(\gamma_1 = \gamma(f, g)\) and use the above result (3.4) to calculate

\[
\gamma_{n+1} = \frac{\beta(f^{n+1})}{\beta} \gamma(f, g) = \frac{(2 + \beta)\beta_n - \beta_{n-1} + 2\beta}{\beta} \gamma(f, g)
\]

\[
= (2 + \beta)\frac{\beta_n}{\beta} \gamma(f, g) - \frac{\beta_{n-1}}{\beta} \gamma(f, g) + 2\gamma(f, g)
\]

\[
= (2 + \beta)\gamma_n - \gamma_{n-1} + 2\gamma_1.
\]

This completes the proof. \(\square\)

In particular, we note the following for future use. Note the interesting factor \(\gamma(\gamma - \beta)\) in the even terms. This arises as it identifies dihedral subgroups of a
Kleinian group which are elementary and have \( \gamma = \beta \).

\[
\begin{align*}
\gamma_0 &= 0, \quad \gamma_1 = \gamma \\
\gamma_2 &= \gamma (\gamma - \beta) \\
\gamma_3 &= \gamma (\gamma - \beta - 1)^2 \\
\gamma_4 &= \gamma (\gamma - \beta)(\gamma - \beta - 2)^2 \\
\gamma_5 &= \gamma (1 + 3\beta + \beta^2 - 3\gamma - 2\beta\gamma + \gamma^2)^2 \\
\gamma_6 &= \gamma (\gamma - \beta)(\gamma - \beta - 1)^2 (\gamma - \beta - 3)^2 \\
\gamma_7 &= \gamma (-1 - 6\beta - 5\beta^2 - \beta^3 + 10\beta\gamma + 3\beta^2\gamma - 5\gamma^2 - 3\beta\gamma^2 + \gamma^3)^2 \\
\gamma_8 &= \gamma (\gamma - \beta)(\gamma - \beta - 2)^2 (2 + 4\beta + \beta^2 - 4\gamma - 2\beta\gamma + \gamma^2)^2 \\
\gamma_9 &= \gamma (\gamma - \beta - 1)^2 (-1 - 9\beta - 6\beta^2 - \beta^3 + 9\gamma + 12\beta\gamma + 3\beta^2\gamma - 6\gamma^2 - 3\beta\gamma^2 + \gamma^3)^2 \\
\gamma_{10} &= \gamma (\gamma - \beta)(5 + 5\beta + \beta^2 - 5\gamma - 2\beta\gamma + \gamma^2)^2 (1 + 3\beta + \beta^2 - 3\gamma - 2\beta\gamma + \gamma^2)^2.
\end{align*}
\]

As a consequence of \( \beta_n \) and \( \gamma_n \) satisfying the same recursion relation we also have the following easy consequence.

**Lemma 11.** Let \( \beta_n = \beta(f^n) \) and \( \gamma_n = \gamma(f^n, g) \). Then \( \alpha_n = \gamma(f^n, g) - \beta(f^n) \) satisfies the recursion relation

\[
\alpha_{n+1} = (2 + \beta)\alpha_n - \alpha_{n-1} + 2\alpha_1.
\]

The recursion relation \( \alpha_n \) has the explicit solution

\[
\alpha_n = -\frac{2^{-n} (2^{1+n} - (2 + \beta - \sqrt{3\sqrt{4 + \beta}})^n - (2 + \beta + \sqrt{3\sqrt{4 + \beta}})^n)}{\beta} \gamma
\]

while \( \gamma_n \) has the solution

\[
\gamma_n = -\frac{2^{-n} \alpha (2^{1+n} - (2 + \beta - \sqrt{3\sqrt{4 + \beta}})^n - (2 + \beta + \sqrt{3\sqrt{4 + \beta}})^n)}{\beta}.
\]

We deduce that \( \lambda_n = \gamma_n \alpha_n = \gamma_n (\gamma_n - \beta_n) \) is given by the formula

\[
\lambda_n = \frac{\lambda_1}{4^n \beta^2} \left[ -2^{1+n} + (2 + \beta - \sqrt{3\sqrt{4 + \beta}})^n + (2 + \beta + \sqrt{3\sqrt{4 + \beta}})^n \right]^2.
\]

These calculation enable us to calculate various principal characters for subgroups.

**Theorem 12.** Let \((f, g)\) be a group with principal character

\[
(\gamma, \beta, \tilde{\beta}) = (\gamma(f, g), \beta(f), \beta(g)).
\]

Then the subgroups below have the associated principal characters

\[
\begin{align*}
(f^n, g) &\quad \rightarrow \quad (\gamma_n, \beta_n, \tilde{\beta}), \\
\langle f^n, g f^n g^{-1} \rangle &\quad \rightarrow \quad (\lambda_n, \beta_n, \beta_n).
\end{align*}
\]

We remark here that a Kleinian group generated by two elements of equal trace admits an absolute lower bound on the commutator parameter. Thus if \((f, g)\) is Kleinian we must have

\[
|\gamma_n| \geq 0.198 \ldots.
\]

There is an important special case where \( g \) has order two which motivates the above calculation. This is because of the following lemma [7].
Lemma 13. Let \( \langle f, g \rangle \) be a Kleinian group with principal character
\[
(\gamma, \beta, \tilde{\beta}) = (\gamma(f, g), \beta(f), \beta(g)).
\]
Then there is a discrete group \( \langle f, \phi \rangle \) with principal character \( (\gamma, \beta, -4) \), so that in particular \( \phi \) has order two. This group is also non-elementary if \( f \) does not have order \( p \in \{2, 3, 4, 5, 6\} \). If \( f \) does have finite order \( p \leq 6 \), then the following must also be true.

- \( f \) has order 2 and \( \langle f, \phi \rangle \) is the Klein 4-group.
- \( f \) has order 3 and \( \langle f, \phi \rangle \) is one of the groups \( A_4 \) or \( S_4 \).
- \( f \) has order 4 and \( \langle f, \phi \rangle \) is the group \( S_4 \).
- \( f \) has order 5 and \( \langle f, \phi \rangle \) is the group \( A_5 \).
- \( f \) has order 6 and \( \langle f, \phi \rangle \) is the \((2, 3, 6)\) Euclidean triangle group.

Further, there is also a group with principal character \( (\beta - \gamma, \beta, -4) \).

In fact the precise values of \( \gamma \) and \( \beta \) in each of these cases can be found in [15]. The two groups with principal characters \( (\gamma, \beta, -4) \) and \( (\beta - \gamma, \beta, -4) \) are the two \( \mathbb{Z}_2 \) extensions of the group \( \langle f, gfg^{-1} \rangle \) generated by elements of the same order and, in the non-parabolic case, the elements of order two are suitable involutions in the bisector of the common perpendicular to the lines \( \text{axis}(f) \) and \( \text{axis}(g) = \text{axis}(gfg^{-1}) \).

The virtue of Lemma 13 is that any inequality which holds for the triple of parameters \( (\gamma, \beta, \tilde{\beta}) \) of a Kleinian group must hold (with a small finite list of exceptions in the spherical and Euclidean triangle groups) for the parameters \( (\gamma, \beta, -4) \). We now exploit this.

We recall the Fricke identity (see [3])
\[
\text{tr}[f, g] = \text{tr}^2(f) + \text{tr}^2(g) + \text{tr}(fg)\text{tr}(g) - 2,
\]
which gives the following version of the Fricke identity in terms of our complex parameters
\[
\gamma (f, g) = \beta (f) + \beta (g) + \beta (fg) - \text{tr}(f)\text{tr}(g)\text{tr}(fg) + 8.
\]
Now if \( g \) is elliptic of order 2, then \( \text{tr}(g) = 0 \) and \( \beta (g) = -4 \), so
\[
\beta (fg) = \gamma - \beta - 4.
\]
Notice that since \( \langle f, fg \rangle = (f, g) \) in general, we have the following lemma, again using Lemma 13.

Lemma 14. The triple of parameters \( (\gamma, \beta, -4) \) is the principal character of a representation of a two-generator Kleinian group if and only if \( (\gamma, \gamma - \beta - 4, -4) \) and \( (\beta - \gamma, -\gamma - 4, -4) \) are also.

By way of example for what will follow, this implies the following partly known generalization of Jørgensen’s inequality.

Lemma 15. Let \( \langle f, g \rangle \) be a Kleinian group with principal character \( (\gamma, \beta, \tilde{\beta}) \) and \( f \) not of order 2. Then both
\[
|\gamma - \beta - 4| + |\gamma| \geq 1 \quad \text{and} \quad |\gamma + 4| + |\beta - \gamma| \geq 1.
\]
Both inequalities are sharp and realized in the \((2, 3, 7)\) hyperbolic triangle group.
The same is true if \( f \) is primitive elliptic of order 5, for then \( \beta(f) + 4 = \frac{1}{2} (3 + \sqrt{5}) = 2.61 \cdots \). If \( f \) is not a primitive elliptic of order 5, then by Lemma 13 \( \langle f, gfg^{-1} \rangle \) must be the group \( A_5 \). Then \( gfg^{-1}f^{-1} = [f, g] \) must be elliptic of order 2, 3 or 5. If \( [f, g] \) has order 2 or 3, then \( \gamma \in -1, 2, -3 \) and the inequality is trivial. We are left with the case \( \beta(f) = -4\sin^2 \frac{\pi}{5} = \frac{1}{2} (-5 - \sqrt{5}) = -3.618 \cdots \), and

\[
\gamma(f, g) \in \{ 2 \cos \frac{\pi}{5} - 2, 2 \cos \frac{2\pi}{5} - 2 \} = \{-0.381966, -1.38197\}.
\]

Thus \( \beta(f) = \frac{1}{2} (-5 - \sqrt{5}) \) and \( \gamma(f, g) = \frac{1}{2} (-3 + \sqrt{5}) \). Then

\[
\gamma - \beta = \frac{1}{2} (-3 + \sqrt{5}) + \frac{1}{2} (5 + \sqrt{5}) = 1 + \sqrt{5}
\]

and

\[
|\gamma - \beta - 4| + |\gamma| = 3 - \sqrt{5} + \frac{1}{2} (3 - \sqrt{5}) = \frac{3}{2} (3 - \sqrt{5}) = 1.1459 \cdots \geq 1.
\]

A similar analysis deals with the second inequality.

The previous argument suggests where we might find equality. If \( \beta = -3 \), then the first inequality, with equality, reads as \( |\gamma| + |\gamma - 1| = 1 \) and so \( \gamma \in [0,1] \). Such an example can be found in [4, Theorem 4.17] in the \((2,3,7)\) hyperbolic triangle group where

\[
\gamma = 4 \left( \cos^2 \frac{2\pi}{7} - \sin^2 \frac{\pi}{7} \right) = 0.80193 \cdots \in [0,1].
\]

If \( \beta = -3 \), then the equality corresponding the second inequality is \( |\gamma + 4| + |\gamma + 3| = 1 \) and hence \( \gamma \in [-4, -3] \). Now we take \( \gamma = -4 \) from the second case in [4, Theorem 4.17] in the \((2,3,7)\) hyperbolic triangle group and hence the triangle group with principal character \((-4, -3, -4)\) achieves the sharpness of the second inequality. \(\square\)

### 4. Further inequalities

We define a sequence \( a_{n+1}^{u,v} \) recursively by the relation

\[
a_{n+1}^{u,v} = (2 + u)a_n^{u,v} - a_{n-1}^{u,v} + 2v, \quad a_0^{u,v} = 0, \quad a_1^{u,v} = v.
\]

Then Lemma 13 together with (3.11) and (3.12) give use the following easy consequence.

**Theorem 16.** Let \((\gamma, \beta, -4)\) be the principal character of a representation of a two-generator Kleinian group for some \(\gamma, \beta \in \mathbb{C}\). Then for every \(n \geq 1\)

\[
|a_n^{2+\beta,\gamma}| + |a_n^{2+\beta,\beta}| \geq 1
\]

unless \(a_{n}^{2+\beta,\beta} = 0\) and \(\beta = -4\sin^2 \left(\frac{\pi}{n}\right)\), for some integer \(p\).

**Proof.** Because of (3.11) and (3.12) this is simply Jørgensen’s inequality applied to the group \(\langle f^n, g \rangle\) which is Kleinian if \(f^n\) is not the identity (Theorem 5). But \(f^n\) equal to the identity implies \(a_{n}^{2+\beta,\beta} = 0\) and that \(f\) is elliptic of order \(n\). \(\square\)
Then from Lemma 15 we have the following corollary.

**Corollary 17.** Let \((\gamma, \beta, -4)\) be the principal character of a representation of a two-generator Kleinian group for some \(\gamma, \beta \in \mathbb{C}\). Then for every \(n \geq 1\)

\[
|a_n^{2+\beta,\gamma} - a_n^{2+\beta,\beta} - 4| + |a_n^{2+\beta,\gamma} + 4| + |a_n^{2+\beta,\gamma} - a_n^{2+\beta,\beta}| \geq 1
\]

unless \(a_n^{2+\beta,\beta} = 0\) and \(\beta = -4 \sin^2 \left(\frac{p\pi}{n}\right)\), for some integer \(p\).

Now the recursion identity for the Chebychev polynomials also gives

\[
\beta((fg)^{n+1}) = \beta((gf)^{n+1}) = 2T_n \left(1 + \frac{\beta(fg)}{2}\right) - 2
\]

Thus we also have, with

\[
\tilde{\gamma}_n = \gamma((fg)^n, g), \quad \tilde{\beta}_n = \beta((fg)^n)
\]

\[
\tilde{\gamma}_{n+1} = (\gamma_1 - \beta - 2)\tilde{\gamma}_n - \tilde{\gamma}_{n-1} + 2\tilde{\gamma}_1, \quad \text{for } n \in \mathbb{N}
\]

\[
\tilde{\beta}_{n+1} = (\gamma_1 - \beta - 2)\tilde{\beta}_n - \tilde{\beta}_{n-1} + 2\tilde{\beta}_1, \quad \text{for } n \in \mathbb{N}
\]

and this calculation applied to the group \(((fg)^n, g)\) gives us the following corollary.

**Corollary 18.** Let \(\gamma, \beta \in \mathbb{C}\). Then there is a Kleinian group with principal character \((\gamma, \beta, -4)\) if and only if for every \(n \geq 1\)

\[
|a_n^{2-\beta,\gamma}| + |a_n^{2-\beta,\gamma-\beta-2}| \geq 1
\]

5. **Trace Polynomials Linear in \(\beta\)**

In [7, Lemma 2.1], the following trace polynomial of two complex variables \(\gamma\) and \(\beta\) that is linear in \(\beta\) is identified.

\[
\gamma(f, gfg^{-1}) = \gamma(\gamma - \beta).
\]

One may observe that \(\gamma(f, [g, f]) = \gamma(f, gfg^{-1})\) and also that

\[
\beta([g, f]) = \text{tr}^2 ([g, f]) - 4 = (\gamma(g, f) + 2)^2 - 4 = \gamma^2 + 4\gamma.
\]

This reads as

\[
\beta([g, f]) = \gamma(\gamma + 4).
\]

The following theorem shows that the trace polynomials associated the word \([g, f]^{n+1}\) are linear in \(\beta\) for each \(n \in \mathbb{N}\). These calculations quickly lead to the following theorem showing there are infinitely many trace polynomials of two complex variables \(\gamma\) and \(\beta\) which are linear in \(\beta\).

**Theorem 19.** Suppose that \((f, g)\) is a Kleinian group generated by two elements \(f\) and \(g\) with two complex parameters \(\gamma = \gamma(f, g)\) and \(\beta = \beta(f)\), and suppose that \([g, f]\) is elliptic or loxodromic. Then the following recursion formulas for the trace polynomials \(\gamma_{n+1} = \gamma([g, f]^{n+1})\) holds.

\[
\gamma_0 = 0, \quad \gamma_1 = \gamma(\gamma - \beta), \quad \gamma_{n+1} = (\gamma^2 + 4\gamma + 2)\gamma_n - \gamma_{n-1} + 2\gamma(\gamma - \beta), \quad \text{for } n \in \mathbb{N}.
\]
In particular,

\[ \gamma_2 = \gamma(\gamma - \beta)(\gamma + 2)^2, \]
\[ \gamma_3 = \gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2, \]
\[ \gamma_4 = \gamma(\gamma - \beta)(\gamma + 2)^2(\gamma^2 + 4\gamma + 2)^2, \]
\[ \gamma_5 = \gamma(\gamma - \beta)(\gamma^2 + 3\gamma + 1)^2(\gamma^2 + 5\gamma + 5)^2. \]

6. Inequalities for Kleinian Groups

In this section, we apply the ideas above to establish new inequalities for two-generator Kleinian groups \( \langle f, g \rangle \) with two complex parameters \( \gamma = \gamma(f, g) \) and \( \beta = \beta(f) : \)

\[ |\gamma(f, g)| + |\beta(f) - \beta_0| \geq r, \]

where \( \beta_0 \) is typically real.

We first recall that the space of principal characters of Kleinian groups is locally compact. This is basically a consequence of Jørgensen’s algebraic convergence theorem [12] after some normalizations see [13, Theorem 6.17].

Lemma 20. Let \( M \geq 0 \). The set of triples \( (\gamma, \beta, \beta_0) \) with \( |\beta| \leq M \) and \( \beta \neq -4 \), and associated to a Kleinian group via

\[ \gamma = \gamma(f, g), \quad \beta = \beta(f), \quad \beta_0 = \beta(g) \]

is compact.

In fact as \( \beta \to \infty \) or \( \beta = -4 \) it is possible that \( \gamma \to 0 \). However this is tightly controlled. We use our earlier results to quantify this in the following theorem.

Theorem 21. Let \( \langle f, g \rangle \) be a Kleinian group and \( f \) not order two. Then

\[ |\gamma(f^2, g)| \geq 2 - 2 \cos\left(\frac{\pi}{7}\right) = 0.198 \cdots. \]

This estimate is sharp and achieved in the \((2, 3, 7)\) hyperbolic triangle group.

Proof. We recall [2] Theorem 4.1] which states that if \( \langle u, v \rangle \) is a discrete group generated by two elements of equal trace, then

\[ |\gamma(u, v)| \geq 2 - 2 \cos\left(\frac{\pi}{7}\right) = 0.198 \cdots. \]

unless \( \gamma(u, v) \in \{0, \beta\} \) or \( \beta(u) = -4 \), and that this inequality is sharp. Under our hypotheses we have \( \langle fg, g \rangle \) discrete and so \( \langle fg, g(fg)g^{-1} \rangle = \langle fg, gf \rangle \) is discrete and generate by elements of equal trace. Let \( \gamma = \gamma(f, g) \) and \( \beta = \beta(f) \), then \( \gamma(fg, g) = \gamma \). Using (5.1), Lemma 13, (3.19), and (3.9), we calculate the parameters to be

\[ \gamma(fg, gf) = \gamma(fg, g) \gamma(fg, g) - \beta(fg) = \gamma(\gamma - (\gamma - \beta - 4)) \]
\[ = \gamma(\beta + 4) = \gamma(f^2, g). \]

That \( \langle fg, g \rangle \) is Kleinian implies both \( \gamma \neq 0 \) and \( \gamma \neq \beta \). The result now follows. \( \square \)
Lemma 22. Suppose that \(<f, g>\) is a Kleinian group generated by two elements \(f\) and \(g\), with \(g\) of order 2, principal character \((\gamma, \beta, -4)\), and \(fg\) is not parabolic. Suppose that \(<f, g>\) achieves the minimum value of

\[
|\gamma| + |\beta - \beta_0|
\]

Then

\[
|\gamma - \beta - 4| \leq 2 \left| T_{n+1} \left( \frac{1}{2} (\gamma - \beta - 2) \right) - 1 \right|, \quad \text{for } n \in \mathbb{N}.
\]

Proof. Since \(<f, g>\) is a Kleinian group, by Corollary 7 and (3.19), \(<gf^n, f>\) is a Kleinian group and \(\beta(fg) = \gamma - \beta - 4\). By minimality we must have

\[
|\gamma| + |\beta - \beta_0| \leq |\gamma(fg)^{n+1}, f| + |\beta - \beta_0|,
\]

which gives

\[
|\gamma| \leq |\gamma(fg)^{n+1}, f|.
\]

Applying Theorem 10,

\[
|\gamma| \leq \frac{|\beta((fg)^n)|}{|\beta(fg)|} |\gamma|.
\]

Since \(\gamma \neq 0\), dividing by \(|\gamma|\) gives

\[
|\beta(fg)| \leq |\beta((fg)^n)|.
\]

Assuming that \(fg\) is not parabolic, so \(\beta(fg) \neq 0\). Then Theorem 10 shows us that this is equivalent to the inequality,

\[
|\gamma - \beta - 4| \leq 2 \left| T_n \left( \frac{1}{2} (\gamma - \beta - 2) \right) - 1 \right|.
\]

This is what we wanted to prove. \(\square\)

We make a brief remark about the parabolic case.

Theorem 23. Let \(<f, g>\) be Kleinian and \(fg\) parabolic. Then

\[
|\gamma| + |\beta - \beta_0| \geq 1
\]

Proof. The Shimizu-Leutbecher inequality (or Jørgensen’s inequality in the parabolic case) imply that \(|\gamma(fg, g)| = |\gamma(f, g)| \geq 1\). \(\square\)

In what follows we will typically ignore the parabolic case and leave that to the reader.

First, we apply Lemma 22 for the Chebychev polynomial \(T_2 = 2z^2 - 1\):

\[
|\gamma - \beta - 4| \leq 2 \left| T_2 \left( \frac{1}{2} (\gamma - \beta - 2) \right) - 1 \right| = \left| \frac{1}{2} (\gamma - \beta - 2)^2 - 4 \right| \leq |\gamma - \beta - 4| |\gamma - \beta|.
\]

Thus at a minimum of \(|\gamma| + |\beta|\) we have \(|\gamma - \beta| \geq 1\). This provides a new approach of Jørgensen’s inequality since

\[
|\gamma| + |\beta| \geq |\gamma - \beta| \geq 1.
\]

Next,
Theorem 24. Suppose that \( \langle f, g \rangle \) is a Kleinian group generated by two elements \( f \) and \( g \) with principal character \( (\gamma, \beta, -4) \). Then

\[
|\gamma| + |\beta + 1|^2 \geq 1.
\]

Hence

\[
|\gamma| + |\beta + 1| \geq 1.
\]

Both these inequalities are sharp. They hold with equality in the generalized triangle group \( \Gamma(6, 2, 3) \) with parameters \( (-1, -1, -4) \).

Proof. Again the result follows from Shimitzu-Neutbecher if \( fg \) is parabolic and so we assume otherwise. Consider the Chebychev polynomial \( T_3(z) = 4z^3 - 3z \).

Lemma 22 gives

\[
|\gamma - \beta - 4| \leq \left| T_3\left(\frac{1}{2}(\gamma - \beta - 2)\right) - 1 \right|
\]

\[
= \left| (\gamma - \beta - 4)(\gamma - \beta - 1)^2 \right|
\]

\[
\leq |\gamma - \beta - 4||\gamma - (\beta + 1)|^2.
\]

Dividing by \( |\gamma - \beta - 4| \), we find that at the minimum,

\[
1 \leq |\gamma - (\beta + 1)|^2.
\]

This establishes the inequalities. The example of the generalized triangle group can be found in Hagelberg, Maclachlan and Rosenberger [11]. □

Theorem 25. Suppose that \( \langle f, g \rangle \) is a Kleinian group generated by two elements \( f \) and \( g \) with principal character \( (\gamma, \beta, -4) \). Then

\[
|\gamma| + |\beta + 2| \geq \frac{\sqrt{5} - 1}{2}.
\]

This inequality is sharp for the \((2, 4, 5)\) hyperbolic triangle group with the parameters \( \left(\frac{\sqrt{5} - 1}{2}, -2, -4\right) \).

Proof. Apply Lemma 22 with the Chebychev polynomial \( T_4 = 8z^4 - 8z^2 + 1, \)

\[
|\gamma - \beta - 4| \leq 2|T_4\left(\frac{1}{2}(\gamma - \beta - 2)\right)| - 1
\]

\[
= \left| (\gamma - \beta - 2)^2(\gamma - \beta)(\gamma - \beta - 4) \right|
\]

\[
\leq |\gamma - \beta - 2|^2 |\gamma - \beta| |\gamma - \beta - 4|.
\]

Dividing by \( |\gamma - \beta - 4| \),

\[
1 \leq |\gamma - \beta - 2|^2 |\gamma - \beta|
\]

\[
\leq |\gamma - \beta - 2|^2 (|\gamma - \beta - 2| + 2).
\]

Let \( x = |\gamma - \beta - 2| \), then \( x \leq |\gamma| + |\beta + 2| \) and hence

\[
1 \leq x^2(x + 2).
\]

Solving the inequality gives \( x \geq \frac{\sqrt{5} - 1}{2} \), i.e., \( \frac{\sqrt{5} - 1}{2} \leq x \leq |\gamma| + |\beta + 2| \). So we conclude that

\[
|\gamma| + |\beta + 2| \geq \frac{\sqrt{5} - 1}{2}.
\]
The \((2, 4, 5)\) hyperbolic triangle group has the triple of parameters \(\left(\frac{\sqrt{5}-1}{2}, -2, -4\right)\) which verifies the sharpness.

For the next theorem we recall \(2 \cos \frac{\pi}{5} - 2 = \frac{5-3}{2}\).

**Theorem 26.** Suppose that \((f, g)\) is a Kleinian group generated by two elements \(f\) and \(g\) with principal character \((\gamma, \beta, -4)\). Then

\[
|\gamma| + \left|\beta + \frac{3 \pm \sqrt{5}}{2}\right| \geq \frac{3 - \sqrt{5}}{2}.
\]

This inequality is sharp for the \(\mathbb{Z}_2\)-extension of \((10, 10, 5)\) hyperbolic triangle group with the triple of parameters \(\left(\frac{\sqrt{5}-3}{2}, -\frac{3\pm\sqrt{5}}{2}, -4\right)\).

**Proof.** We consider the Chebychev polynomial \(T_5 = 16z^5 - 20z^3 + 5z\) and use the inequality in Lemma 22 to obtain

\[
|\gamma - \beta - 4| \leq 2|T_5(\frac{1}{2}(\gamma - \beta - 2)) - 1| = |(\gamma - \beta - 2)^5 - 5(\gamma - \beta - 2)^3 + 5(\gamma - \beta - 2) - 2|.
\]

Let \(x = \gamma - \beta - 2\), then \(\gamma - \beta - 4 = x - 2 \neq 0\) and hence

\[
|x - 2| \leq |x^5 - 5x^3 + 5x - 2| = \left| (x - 2) (x^2 + x - 1) \right|^2 \leq |x - 2| |x^2 + x - 1|^2
\]

Dividing \(|x - 2|\) gives \(1 \leq |x^2 + x - 1|^2\). Thus,

\[
1 \leq |x^2 + x - 1| = \left| \left( x - \frac{-1 + \sqrt{5}}{2} \right) \left( x - \frac{-1 - \sqrt{5}}{2} \right) \right| = \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right|,
\]

which gives the inequality

\[
(6.6) \quad \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right| \geq 1.
\]

Rearranging the first factor of the previous inequality,

\[
1 \leq \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right| \leq \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right|^2 + \sqrt{5} \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right| = \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right|^2 + \sqrt{5} \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right|.
\]
Let \( s = \left| \gamma - \beta - \frac{3-\sqrt{5}}{2} \right| \), then

\[
s^2 + \sqrt{5}s \geq 1,
\]

which gives the solution \( s \geq \frac{3-\sqrt{5}}{2} \) and hence we obtain one inequality

\[
\frac{3 - \sqrt{5}}{2} \leq \left| \gamma - \beta - \frac{3 - \sqrt{5}}{2} \right| \leq |\gamma| + \left| \beta + \frac{3 - \sqrt{5}}{2} \right|.
\]

Next, we can finish the other inequality by rearranging the second factor of the inequality (6.6),

\[
1 \leq \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \leq |\gamma| + \left| \beta + \frac{3 + \sqrt{5}}{2} \right|.
\]

Next, we consider \( y = \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \), then the above inequality becomes

\[
y^2 + \sqrt{5}y \geq 1,
\]

solving it gives \( y \geq \frac{3-\sqrt{5}}{2} \) and hence

\[
\frac{3 - \sqrt{5}}{2} \leq \left| \gamma - \beta - \frac{3 + \sqrt{5}}{2} \right| \leq |\gamma| + \left| \beta + \frac{3 + \sqrt{5}}{2} \right|.
\]

Next, we consider \( f \) of order 10, then

\[
\beta = -4\sin^2\left(\frac{\pi}{10}\right) = -\frac{3 - \sqrt{5}}{2} \quad \text{or} \quad \beta = -4\sin^2\left(\frac{3\pi}{10}\right) = -\frac{3 + \sqrt{5}}{2}
\]

and hence the term \( |\beta + \frac{3 + \sqrt{5}}{2}| \) vanishes. It follows that \( \gamma = \frac{\sqrt{5}-3}{2} \) or \( \frac{3-\sqrt{5}}{2} \). We choose the first case of \( \gamma \) to compute \( \beta([f, g]) \):

\[
\beta([f, g]) = \text{tr}^2 [f, g] - 4 = (\gamma + 2)^2 - 4
\]

\[
= \left( \frac{\sqrt{5} - 3}{2} + 2 \right)^2 - 4
\]

\[
= \left( 2 \cos\left(\frac{\pi}{5}\right) \right)^2 - 4 = -4\sin^2\left(\frac{\pi}{5}\right).
\]

Thus, \([f, g]\) is elliptic of order 5. Let \( h = g f^{-1} g^{-1} \), then it is elliptic of order 10, and so the product \( fh = [f, g] \) is elliptic of order 5.

Now choose \( g \) of order 2, this gives a \( \mathbb{Z}_2 \)-extension \( \Gamma \) of the group \( \langle f, g f g^{-1} \rangle \).

Since \( \langle f, g f^{-1} \rangle = \langle f, g f^{-1} g^{-1} \rangle \), \( \langle f, g f g^{-1} \rangle \) is the \((10, 10, 5)\) hyperbolic triangle group and hence \( \Gamma \) is a Kleinian group with principal character

\[
\left( \frac{\sqrt{5} - 3}{2}, \frac{3 \pm \sqrt{5}}{2}, -4 \right),
\]
which gives the sharpness,

\[ |\gamma| + \left| \beta + \frac{3 \pm \sqrt{5}}{2} \right| = \frac{3 - \sqrt{5}}{2}. \]

\[ \square \]

**Theorem 27.** Suppose that \( \langle f, g \rangle \) is a Kleinian group generated by two elements \( f \) and \( g \) with principal character \( (\gamma, \beta, -4) \). Then

\[ |\gamma| + \left| \beta + 2 + \sqrt{2} \right| \geq 0.117875. \]

**Proof.** We apply the Chebychev polynomial \( T_8(z) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 \), then Lemma 22 gives

\[ \frac{|\gamma - \beta - 2|}{2|T_8\left(\frac{1}{2}(\gamma - \beta - 2)\right) - 1|} = \frac{|(\gamma - \beta - 2)^8 - 8(\gamma - \beta - 2)^6 + 20(\gamma - \beta - 2)^4 - 16(\gamma - \beta - 2)^2|}{2}. \]

Let \( x = \gamma - \beta - 2 \), then \( \gamma - \beta - 4 = x - 2 \neq 0 \). It follows that

\[ |x - 2| \leq |x|^2 |x - 2| |x + 2| |(x^2 - 2)|^2. \]

Dividing \( |x - 2| \),

\[ 1 \leq |x|^2 |x + 2| |(x^2 - 2)|^2 \]

\[ = |\gamma - \beta - 2|^2 |\gamma - \beta| |(\gamma - \beta - 2)^2 - 2|^2. \]

Let \( y = |\gamma - \beta - 2 - \sqrt{2}| \), then the previous inequality becomes

\[ 1 \leq \left( y + \sqrt{2} \right)^2 \left( y + 2 + \sqrt{2} \right) y^2 \left( y + 2\sqrt{2} \right)^2. \]

Solving the inequality gives \( 0.117875 \leq y \) and hence

\[ 0.117875 \leq |\gamma - \beta - 2 - \sqrt{2}| \leq |\gamma| + |\beta + 2 + \sqrt{2}|. \]

\[ \square \]

**References**

[1] Beardon, Alan. F. The geometry of discrete groups, Graduate texts in mathematics 91, Springer-Verlag, 1983.

[2] Cao, Chun. Some Trace Inequalities for Discrete Groups of Möbius Transformations. *Proceedings of the American Mathematical Society* **123** (1995), 3807–3815.

[3] Fricke, R.; Klein, F. Vorlesungen über die theorie der automorphen funktionen, Chapter 2. Teubner, Leipzig, 1897.

[4] Gehring, F. W.; Gilman, J.P.; Martin G. J. Kleinian groups with real parameters. *Communications in Contemporary Mathematics* **3** (2001), 163–186.

[5] Gehring, F. W.; Martin, G. J. Chebyshev polynomials and discrete groups. *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), 114–125. Conf. Proc. Lecture Notes Anal., I. Int. Press, Cambridge, MA, 1994.

[6] Gehring, F. W.; Martin, G. J. Inequalities for Möbius transformations and discrete groups. *J. Reine Angew. Math.* **418** (1991), 31–76.

[7] Gehring, F. W.; Martin, G. J. Commutators, collars and the geometry of Möbius groups. *Journal D’Analyse Mathematique* **63** (1994), 175-219.

[8] Gehring, F. W.; Martin, G. J. Stability and extremality in Jørgensen’s inequality. *Complex variables* **12** (1989), 277-282.

[9] Gehring, F. W.; Martin, G. J. 6-torsion and hyperbolic volume. *Proc. Amer. Math. Soc.* **117** (1999), 727-735.
[10] Gehring, F. W.; Maclachlan, C.; Martin, G. J. On the Discreteness of the Free Product of Finite Cyclic groups. *Mitteilungen aus dem Mathematischen Seminar Giessen* **228** (1996), 9-15.
[11] Hagelberg, M.; Maclachlan, C.; Rosenberger, G. On discrete generalised triangle groups. *Proc. Edinburgh Math. Soc.* **38** (1995), 397–412.
[12] Jörgensen, T. On discrete groups of Möbius transformations. *Amer. J Math* **98** (1976), 739-749.
[13] Leutbecher, A. Über spitzen diskontinuierlicher lineargebrchertransformationen. *Math. Zeit.* **100** (1967), 183-200.
[14] Marshall, T.H.; Martin, G. J. Polynomial Trace Identities in $SL(2, \mathbb{C})$, Quaternion Algebras, and Two-generator Kleinian Groups. *Math. Arxiv* 111643. *Handbook of complex analysis*, 2020 (to appear).
[15] Martin, G. J. The Geometry and Arithmetic of Kleinian Groups. Handbook of Group Actions, Volume I (Advanced lectures in Mathematics Volume 31). *International Press of Boston, Inc.*, 411-494, 2015.
[16] Martin, G. J. On discrete Möbius groups in all dimensions: A generalization of Jørgensen’s inequality. *Acta Math.* **163** (1989), 253–289.
[17] Martin, G. J. On discrete isometry groups of negative curvature. *Pacific journal of mathematics* **160** (1992), 109–127.
[18] Maskit, B. Some special 2-generator Kleinian groups. *Proc. Amer. Math. Soc.* **106** (1989), 175–186.
[19] Shimizu, H. On discontinuous groups operating on the product of half-spaces. *Ann. of Math.* **77** (1963), 33-71.
[20] Tan, D. On two-generator discrete groups of Möbius transformations. *Proc. Amer. Math. Soc.* **106** (1989), 763-770.

**Department of Mathematical Sciences, United Arab Emirates University**

*Email address:* hala.a@uae.ac.ae

**Department of Mathematical Sciences, United Arab Emirates University**

*Email address:* j.gong@uae.ac.ae

**Institute for Advanced Study, Massey University,**

*Email address:* g.j.martin@massey.ac.nz