The regularity properties and blow-up for convolution wave equations and applications

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Abstract

In this paper, the Cauchy problem for linear and nonlinear convolution wave equations are studied. The equation involves convolution terms with a general kernel functions whose Fourier transform are operator functions defined in a Banach space $E$ together with some growth conditions. Here, assuming enough smoothness on the initial data and the operator functions, the local, global existence, uniqueness and regularity properties of solutions are established in terms of fractional powers of given sectorial operator function. Furthermore, conditions for finite time blow-up are provided.

By choosing the space $E$ and the operators, the regularity properties the wide class of nonlocal wave equations in the field of physics are obtained.

Key Word: nonlocal wave equations, Boussinesq equations, wave equations, abstract differential equations, blow-up of solutions, Fourier multipliers

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1. Introduction

The aim here, is to study the existence, uniqueness, regularity properties and blow-up on finite point of solutions to the initial value problem (IVP) for convolution abstract wave equation (WE)

$$u_{tt} - a \ast \Delta u + A \ast u = \Delta \left[ g \ast f (u) \right], \quad t \in \mathbb{R}^n_+ = \mathbb{R}^n \times (0, T),$$

$$u (x, 0) = \varphi (x), \quad u_t (x, 0) = \psi (x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where $A = A (x)$ is a linear and $g = g (x), f (u)$ are nonlinear operator functions defined in a Banach space $E$; $a$ is a complex valued functon on $\mathbb{R}^n$, $T \in (0, \infty]$, $\Delta$ denotes the Laplace operator in $\mathbb{R}^n$, $\varphi (x)$ and $\psi (x)$ are the given $E$–valued initial functions.

Remark 1.1. Let $u \in Y^{2, s, p} = W^{2, s, p} (\mathbb{R}^n_+; E (A), E)$, then by J. Lions-J. Peetre result (see e.g. [27, §1.8.2] the trace operator $u \rightarrow \frac{\partial u}{\partial n} (x, t)$ is bounded from $Y^{2, s, p}$ to $C \left( \mathbb{R}^n; (Y^{s, p}, X_p)_{\theta_j, p} \right)$, where

$$X_p = L^p (\mathbb{R}^n; E), \quad Y^{s, p} = W^{s, p} (\mathbb{R}^n; E (A), E), \quad \theta_j = \frac{1 + j p}{2p}, \quad j = 0, 1,$$
Moreover, if \( u(x,\cdot) \in (Y^{s,p},X_p)_{\theta,p} \), then under some assumptions that will be stated in the Section 3, \( f(u) \in E \) for all \( x, t \in \mathbb{R}_p \) and the map \( u \rightarrow f(u) \) is bounded from \((Y^{s,p},X_p)_{\theta,p}\) into \( E \). Hence, the nonlinear equation (1.1) is satisfied in the Banach space \( E \). Here, \( E(A) \) denotes a domain of \( A \) equipped with graphical norm, \((Y^{s,p},X_p)_{\theta,p}\) is a real interpolation space between \( X_p, Y^{s,p} \) for \( \theta \in (0,1), p \in [1,\infty] \) (see e.g. [27, §1.3]). The spaces \( X_p, Y^{s,p}, Y^{2,s,p} \) will be defined in Section 1.

The predictions of classical (local) elasticity theory become inaccurate when the characteristic length of an elasticity problem is comparable to the atomic length scale. To solve this situation, a nonlocal theory of elasticity was introduced (see [1-3] and the references cited therein) and the main feature of the new theory is the fact that its predictions were more down to earth than those of the classical theory. For other generalizations of elasticity we refer the reader to [4-6]. The global existence of the Cauchy problem for Boussinesq type nonlocal equations has been studied by many authors (see [11, 14, 21]). Note that, the existence and uniqueness of solutions and regularity properties for different type wave equations were considered e.g. in [4-6], [8], [10], [17,18] and [31, 32]. Wave type equations occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describe spinodal decomposition and in the absence of mechanical stresses (see [19, 20, 29, 33]).

The \( L^p \) well-posedness of the Cauchy problem (1.1) – (1.2) depends crucially on the presence of a suitable kernel. Then the question that naturally arises is which of the possible forms of the operator functions and kernel functions are relevant for the global well-posedness of the Cauchy problem (1.1) – (1.2). In this study, as a partial answer to this question, we consider the problem (1.1) – (1.2) with a general class of kernel functions with operator coefficients provide local and global existence and regularity properties of (1.1) – (1.2) in terms of fractional powers of operator \( A \) in frame of \( E \)-valued \( L^p \) spaces. The kernel functions most frequently used in the literature are particular cases of this general class of kernel functions in the scalar case, i.e. when \( E = \mathbb{C} \) (here, \( \mathbb{C} \) denote the set of complex numbers). In contrast to the above works, we consider the IVP for nonlocal wave equation with operator coefficients in \( E \)-valued function spaces. By choosing the space \( E \), operators \( A \) and \( g \) in (1.1) – (1.2), we obtain different classes of nonlocal wave equations which occur in application. Let we put \( E = l_q \) and choose \( A, g \) as infinite matrices \([a_{mj}], [g_{mj}]\), respectively for \( m,j = 1, 2, \ldots, \infty \). Consider IVP for infinity many system of nonlocal WEs

\[
\begin{align*}
\partial_t^2 u_m - a \Delta u_m + \sum_{j=1}^{\infty} a_{mj} * u_m = \\
\sum_{j=1}^{\infty} \Delta g_{mj} u_m * f_m (u_1, u_2, \ldots, u_m), t \in [0,T], x \in \mathbb{R}^n, \\
u_m (x,0) = \varphi_m (x), \partial_t u_m (x,0) = \psi_m (x), m = 1, 2, \ldots, \infty,
\end{align*}
\]
where $a_{mj} = a_{mj}(x)$, $g_{mj} = (x)$ are complex valued functions, $f_m$ are nonlinear functions and $u_j = u_j(x, t)$.

Then from our results we obtain the existence, uniqueness and regularity properties of the problem (1.3) in terms of fractional powers of matrix operator $A$ in frame of $l_q$-valued $L^p$ spaces.

Moreover, let we choose $E = L^{p_1}(0, 1)$ and $A$ to be degenerated differential operator in $L^{p_1}(0, 1)$ defined by

$$D(A) = \{ u \in W^{[2], p_1}_\gamma (0, 1), \alpha_k u^{[\nu_k]}(0) + \beta_k u^{[\nu_k]}(1) = 0, \ k = 1, 2 \},$$

$$A(x) u = b_1(x, y) u^{[2]} + b_2(x, y) u^{[1]}, \ x \in \mathbb{R}^n, \ y \in (0, 1), \ \nu_k \in \{0, 1\},$$

where $u^{[i]} = \left( y^{\gamma} \frac{d}{dy} \right)^\gamma u$ for $0 \leq \gamma < \frac{1}{p_1}$, $b_1 = b_1(x, y)$ is a continous, $b_2 = b_2(x, y)$ is a bounded functon in $y \in [0, 1]$ for a.e. $x \in \mathbb{R}^n$, $\alpha_k$, $\beta_k$ are complex numbers and $W^{[2], p_1}_\gamma (0, 1)$ is a weighted Sobolev space defined by

$$W^{[2], p_1}_\gamma (0, 1) = \{ u : u \in L^{p_1}(0, 1), \ u^{[2]} \in L^{p_1}(0, 1),$n$$ \| u \|_{W^{[2], p_1}_\gamma} = \| u \|_{L^{p_1}} + \| u^{[2]} \|_{L^{p_1}} < \infty.$$n

Then from general results we also obtain the existence, uniquenes s and regularity properties for the nonlocal mixed problem for nonlocal degenerate PDE

$$u_{tt} - a \Delta u + \left( b_1 \frac{\partial^{[2]} u}{\partial y^2} + b_2 \frac{\partial^{[1]} u}{\partial y} \right) * u = \Delta g * f(u),$$

$$x \in \mathbb{R}^n, \ y \in (0, 1), \ t \in (0, T), \ u = u(x, y, t),$$

$$\alpha_{ki} u^{[\nu_k]}(x, 0, t) + \beta_{ki} u^{[\nu_k]}(x, 1, t) = 0, \ k = 1, 2, (1.6)$$

$$u(x, y, 0) = \varphi(x, y), \ u_t(x, y, 0) = \psi(x, y). \ (1.7)$$

Then from our general results we deduced the existence, uniquenes s and regularity properties of the problem (1.5) – (1.7) in terms of fractional powers of operator $A$ defined by (1.4) in frame of $L^{p_1}(0, 1)$-valued $L^p$ spaces.

It should be noted that, the regularity properties of nonlinear wave equations in terms of interpolation of spaces are very hard to obtain by the usual classical methods.

The IVP for abstract hyperbolic equations were studied e.g. in [2], [12] and [22, 23].

The strategy is to express the equation (1.1) as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on $L^p$ norms
of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate for solutions of the linearized convolution wave equation. The methods of harmonic analysis, operator theory, interpolation of Banach spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

In order to state our results precisely, we introduce some notations and some function spaces.

1. Definitions and Background

Let $E$ be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$
\|f\|_p = \|f\|_{L^p(\Omega; E)} = \left( \int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

and

$$
\|f\|_{L^\infty(\Omega; E)} = \text{ess sup}_{x \in \Omega} \|f(x)\|_E.
$$

Let $E_1$ and $E_2$ be two Banach spaces. $(E_1, E_2)_{\theta,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ denotes the real interpolation spaces defined by $K$-method [27, §1.3.2]. Let $E_1$ and $E_2$ be two Banach spaces. $B(E_1, E_2)$ will denote the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

$\mathbb{N}$—denote the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. Here,

$$
S_\phi = \{\lambda \in \mathbb{C}, \ |\arg \lambda| \leq \phi, \ 0 \leq \phi < \pi \}.
$$

A closed linear operator $A$ is said to be sectorial in a Banach space $E$ with bound $M > 0$ if $D(A)$ and $R(A)$ are dense on $E$, $N(A) = \{0\}$ and

$$
\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M |\lambda|^{-1}
$$

for any $\lambda \in S_\phi$, $0 \leq \phi < \pi$, where $I$ is the identity operator in $E$, $B(E)$ is the space of bounded linear operators in $E$; $D(A)$ and $R(A)$ denote domain and range of the operator $A$. It is known that (see e.g. [27, §1.15.1]) there exist the fractional powers $A^\theta$ of a sectorial operator $A$. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$
\|u\|_{E(A^\theta)} = \left( \|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \theta < \infty.
$$

A sectorial operator $A(\xi)$ for $\xi \in \mathbb{R}^n$ is said to be uniformly sectorial in a Banach space $E$, if $D(A(\xi))$ is independent of $\xi$ and the uniform estimate

$$
\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M |\lambda|^{-1}
$$
A linear operator $A = A(\xi)$ belongs to $\sigma(M_0, \omega, E)$ (see [23] §11.2) if $D(A)$ is dense on $E, D(A(\xi))$ is independent of $\xi \in \mathbb{R}^n$ and for $\text{Re}\lambda > \omega$ the uniform estimate holds
\[ \left\| \left( A(\xi) - \lambda^2 I \right)^{-1} \right\|_{B(E)} \leq M_0 |\text{Re}\lambda - \omega|^{-1}. \]

**Remark 1.1.** It is known (see e.g. [22, §1.6], Theorem 6.3) that if $A \in \sigma(M_0, \omega, E)$ and $0 \leq \alpha < 1$ then it is generates a bounded group operator $U_A(t)$ satisfying
\[ \|U_A(t)\|_{B(E)} \leq M e^{\omega|t|}, \quad \|A^\alpha U_A(t)\|_{B(E)} \leq M |t|^{-\alpha}, \quad t \in [0, T]. \quad (2.1) \]

Let $E$ be a Banach space. $S = S(\mathbb{R}^n; E)$ denotes $E$-valued Schwartz class, i.e. the space of all $E$-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n; \mathbb{C})$ denoted by $S$. Let $S'(\mathbb{R}^n; E)$ denote the space of all continuous linear functions from $S$ into $E$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n; E)$ is norm dense in $L^p(\mathbb{R}^n; E)$ when $1 \leq p < \infty$. Let $m$ be a positive integer. $W^{m,p}(\Omega; E)$ denotes an $E$-valued Sobolev space of all functions $u \in L^p(\Omega; E)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^n} \in L^p(\Omega; E)$ with the norm
\[ \|u\|_{W^{m,p}(\Omega; E)} = \|u\|_{L^p(\Omega; E)} + \| \sum_{k=1}^n \frac{\partial^m u}{\partial x_k^n} \|_{L^p(\Omega; E)} < \infty. \]

Let $W^{s,p}(\mathbb{R}^n; E)$ denotes the fractional Sobolev space of order $s \in \mathbb{R}$, that is defined as:
\[ W^{s,p}(E) = W^{s,p}(\mathbb{R}^n; E) = \{ u \in S'(\mathbb{R}^n; E), \|u\|_{W^{s,p}(\mathbb{R}^n; E)} < \infty \}. \]

It clear that $W^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E)$. Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embedded into $E$. Here, $W^{s,p}(\mathbb{R}^n; E_0, E)$ denote the Sobolev-Lions type space i.e.,
\[ W^{s,p}(\mathbb{R}^n; E_0, E) = \{ u \in W^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0), \|u\|_{L^p(\mathbb{R}^n; E)} + \|u\|_{W^{s,p}(\mathbb{R}^n; E)} < \infty \}. \]

In a similar way, we define the following Sobolev-Lions type space:
\[ W^{2,s,p}(\mathbb{R}^n_T; E_0, E) = \{ u \in S'(\mathbb{R}^n_T; E_0), \partial_t^2 u \in L^p(\mathbb{R}^n_T; E), \|u\|_{W^{2,s,p}(\mathbb{R}^n_T; E_0, E)} = \| F_x^{-1} \left( I + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \|_{L^p(\mathbb{R}^n_T; E)} < \infty \}. \]
\[ \| \partial_t^2 u \|_{L^p(\mathbb{R}^n; E)} + \left\| \mathbb{F}^{-1} \left( I + |\xi|^2 \right) \mathbb{F} u \right\|_{L^p(\mathbb{R}^n; E)} < \infty \].

Let \( 1 \leq p \leq q < \infty \). A function \( \Psi \in L^\infty(\mathbb{R}^n) \) is called a Fourier multiplier from \( L^p(\mathbb{R}^n; E) \) to \( L^q(\mathbb{R}^n; E) \) if the map \( P: u \to \mathbb{F}^{-1}\Psi(\xi)\mathbb{F}u \) for \( u \in S(\mathbb{R}^n; E) \) is well defined and extends to a bounded linear operator
\[
P: L^p(\mathbb{R}^n; E) \to L^q(\mathbb{R}^n; E).
\]

A Banach space \( E \) is called a UMD space if the Hilbert operator
\[
(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy
\]
is initially defined on \( S(\mathbb{R}; E) \) and is bounded in \( L^p(\mathbb{R}; E) \), \( p \in (1, \infty) \). UMD spaces include e.g. \( L_p, l_p \) spaces and Lorentz spaces \( L_{pq}, p, q \in (1, \infty) \) (see e.g. [13]).

A set \( K \subset B(E_1, E_2) \) is called \( R \)-bounded (see e.g. [13]) if there is a constant \( C > 0 \) such that for all \( T_1, T_2, ..., T_m \in K \) and \( u_1, u_2, ..., u_m \in E_1, m \in \mathbb{N} \)
\[
\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_j u_j \right\|_{E_2} \, dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} \, dy,
\]
where \( \{r_j\} \) is a sequence of independent symmetric \( \{-1;1\} \)-valued random variables on \([0,1]\). The smallest \( C \) for which the above estimate holds is called the \( R \)-bound of \( K \) and denoted by \( R(K) \).

Note that, in Hilbert spaces every norm bounded set is \( R \)-bounded. Therefore, all sectorial operators are \( R \)-sectorial in Hilbert spaces.

Sometimes we use one and the same symbol \( C \) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say \( \alpha \), we write \( C_\alpha \). Moreover, for \( u, v > 0 \) the relations \( u \lesssim v, u \approx v \) means that there exist positive constants \( C, C_1, C_2 \) independent on \( u \) and \( v \) such that, respectively
\[
u \leq C_1 v \quad C_1 v \leq u \leq C_2 v.
\]

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priori estimates for solution of the linearized problem (1.1) – (1.2). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1) – (1.2). In the Section 4, we show the same applications of the problem (1.1) – (1.2).

Sometimes we use one and the same symbol \( C \) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say \( h \), we write \( C_h \).
2. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of the Cauchy problem for the convolution linear WE

\[ u_{tt} - a \ast \Delta u + A \ast u = g(x, t), \ x \in \mathbb{R}^n, \ t \in (0, T), \ T \in (0, \ \infty], \]  

(2.1)

\[ u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x) \text{ for a.e. } x \in \mathbb{R}^n, \]  

(2.2)

where \( A = A(x) \) is a linear operator function in a Banach space \( E \) and \( a \geq 0 \).

Let \( A \) be a sectorial operator in \( E \). Here,

\[ X_p = L^p(\mathbb{R}^n; E), \ X_p(A^\gamma) = L^p(\mathbb{R}^n; E(A^\gamma)), \ 1 \leq p, \ q \leq \infty, \]

\[ Y_{s,p} = Y_{s,p}^* (E) = W_{s,p}^* (\mathbb{R}^n; E), \ Y_{s,p}^* (E) = Y_{s,p}^* (E) \cap X_q, \]

\[ \|u\|_{Y_{s,p}^*} = \|u\|_{W_{s,p}^* (\mathbb{R}^n; E)} + \|u\|_{X_q} < \infty, \]

\[ W_{s,p}^* (A^\gamma) = W_{s,p}^* (\mathbb{R}^n; E(A^\gamma)), \ 0 < \gamma \leq 1, \]

\[ Y_{s,p} = Y_{s,p} (A, E) = W_{s,p}^* (\mathbb{R}^n; E(A), E), \ Y_{2,s,p} = Y_{2,s,p} (A, E) = W_{2,s,p}^* (\mathbb{R}^n; E(A), E), \ Y_{q,p}^* (A; E) = Y_{s,p}^* (E) \cap X_q (A), \]

\[ \|u\|_{Y_{q,p}^* (A, E)} = \|u\|_{Y_{s,p}^* (E)} + \|u\|_{X_q (A)} < \infty, \]

\[ E_{0p} = (Y_{s,p} (A, E), X_p)_{\frac{1}{1-p}} \to, \ E_{1p} = (Y_{s,p} (A, E), X_p)_{\frac{1}{1-p}}, \]

Let \( \hat{X}_p (A^\alpha) \) denotes the \( D(A^\alpha) \)-value function space with norm

\[ \|u\|_{\hat{X}_p (A^\alpha)} = \|A^\alpha \ast u\|_{X_p}. \]

Let

\[ Y_0 (A^\alpha) = E_{0p} \cap \hat{X}_1 (A^\alpha), \ Y_1 (A^\alpha) = E_{1p} \cap \hat{X}_1 (A^\alpha). \]

Remark 2.1. By properties of real interpolation of Banach spaces and interpolation of the intersection of the spaces (see e.g. [27, §1.3]) we obtain

\[ E_{0p} = (Y_{s,p} (A, E) \cap X_p, X_p)_{\frac{1}{1-p}} = (Y_{s,p} (E), X_p)_{\frac{1}{1-p}} \cap (X_p (A), X_p)_{\frac{1}{1-p}} = \]

\[ W_{s,1 - \frac{1}{p}, p}^* (\mathbb{R}^n; E) \cap L^p (\mathbb{R}^n; (E(A), E)_{\frac{1}{1-p}}) = \]

\[ W_{s,1 - \frac{1}{p}, p}^* (\mathbb{R}^n; (E(A), E)_{\frac{1}{1-p}}, E). \]

In a similar way, we have

\[ E_{1p} = (Y_{s,p} (A, E) \cap X_p, X_p)_{\frac{1}{1-p}} = W_{s,1 - \frac{1}{p}, p}^* (\mathbb{R}^n; (E(A), E)_{\frac{1}{1-p}}, E). \]
Remark 2.2. Let $A$ be a sectorial operator in a Banach space $E$. In view of interpolation by domain of sectorial operators (see e.g. [27, §1.8.2]) we have the following relation

$$E (A^{1-\theta+\varepsilon}) \subset (E (A), E)_{\theta,p} \subset E (A^{1-\theta-\varepsilon})$$

for $0 < \theta < 1$ and $0 < \varepsilon < 1 - \theta$.

Note that from J. Lions-J. Peetre result (see e.g. [27, §1.8.2] we obtain the following result.

**Lemma A.** The trace operator $u \to \frac{\partial u}{\partial t} (x,t)$ is bounded from $Y^{2,s,p} (A,E)$ into

$$C \left( \mathbb{R}^n; (Y^{s,p} (A,E), X_p)_{\theta_j,p} \right), \quad \theta_j = \frac{1+jp}{2p}, \quad j = 0, 1.$$

Let $\hat{A} (\xi)$ be the Fourier transformation of $A (x)$, i.e. $\hat{A} (\xi) = \mathbb{F} (A (x))$. We assume that $\hat{A} (\xi)$ is uniformly sectorial operator in a Banach space $E$. Let

$$\eta = \eta (\xi) = \left[ \hat{a} (\xi) |\xi|^2 + \hat{A} (\xi) \right]^\frac{1}{2}.$$

Let $A$ be a generator of a strongly continuous cosine operator function in a Banach space $E$ defined by formula

$$C (t) = \frac{1}{2} \left( e^{it\hat{A}^\frac{1}{2}} + e^{-it\hat{A}^\frac{1}{2}} \right)$$

(see e.g. [12, §11]). Then, from the definition of sine operator-function $S (t)$ we have

$$S (t) u = \int_0^t C (\sigma) u d\sigma, \text{ i.e. } S (t) u = \frac{1}{2i} A^{-\frac{1}{2}} \left( e^{it\hat{A}^\frac{1}{2}} - e^{-it\hat{A}^\frac{1}{2}} \right).$$

Let

$$\eta_{\pm} (\xi) = e^{it\eta (\xi)} \pm e^{-it\eta (\xi)}, \quad C (t) = C (\xi, t) = \frac{\eta_+ (\xi)}{2}, \quad (2.3)$$

$$S (t) = S (\xi, t) = \eta^{-1} (\xi) \frac{\eta_- (\xi)}{2i}.$$

**Condition 2.1.** Assume: (1) $\eta (\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$; (2) $\hat{a} \in C^{(m)} (\mathbb{R}^n)$ such that

$$\hat{a} (\xi) |\xi|^2 \in S_{\varphi_1}, \quad \left( 1 + |\xi|^2 \right)^{-\frac{n}{2} - 2} |D^\beta \hat{a} (\xi)| \leq C_0, \quad (2.0)$$

$$m = |\beta| > 1 + \frac{n}{p}, \quad p \in (1, \infty) \text{ for all } \xi \in \mathbb{R}^n;$$

(3) $\hat{A} (\xi)$ is an uniformly $R$–sectorial operator in UMD space $E$ such that $\hat{A} (\xi) \in \sigma (M_0, \omega, E)$; (4) $\hat{A} (\xi)$ is a differentiable operator function with independent of
Theorem 2.1. Assume the Condition 2.1 holds and where \( C \) i.e. problem (2.1).
Moreover, the following estimate holds
\[
\| D^\beta \hat{A}(\xi) \eta^{-\gamma}(\xi) \|_{B(E)} \leq M \text{ for } 0 < \gamma < 1 - \frac{1}{2p};
\]
(5) \( \varphi \in E_{0p} \) and \( \psi \in E_{1p} \).
First we need the following lemmas:
Lemma 2.1. Let the assumption (1) of Condition 2.1 holds. Then, problem (2.1) – (2.2) has a unique solution.
Proof. By using of the Fourier transform, we get from (2.1) – (2.2):
\[
\hat{u}_{tt}(\xi,t) + \eta^2(\xi) \hat{u}(\xi,t) = \hat{g}(\xi,t), \tag{2.4}
\]
\[
\hat{u}(\xi,0) = \hat{\varphi}(\xi), \hat{u}_t(\xi,0) = \hat{\psi}(\xi),
\]
where \( \hat{u}(\xi,t) \) is a Fourier transform of \( u(x,t) \) in \( x \) and \( \hat{\varphi}(\xi), \hat{\psi}(\xi) \) are Fourier transform of \( \varphi \) and \( \psi \), respectively. By virtue of \( [12, \S 11.2.4] \) we obtain that \( \eta(\xi) \) is a generator of a strongly continuous cosine operator function and problem (2.4) has a unique solution for all \( \xi \in \mathbb{R}^n \) expresssing as
\[
\hat{u}(\xi,t) = C(\xi,t) \hat{\varphi}(\xi) + S(\xi,t) \hat{\psi}(\xi) + \int_0^t S(\xi,t - \tau) \hat{g}(\xi,\tau) \, d\tau, \tag{2.5}
\]
i.e. problem (2.1) – (2.2) has a unique solution
\[
u(x,t) = C_1(t) \varphi + S_1(t) \psi + Qg, \tag{2.6}
\]
where \( C_1(t), S_1(t), Q \) are linear operator functions defined by
\[
C_1(t) \varphi = \mathcal{F}^{-1} [C(\xi,t) \hat{\varphi}(\xi)] , 
S_1(t) \psi = \mathcal{F}^{-1} [S(\xi,t) \hat{\psi}(\xi)],
\]
\[
Qg = \mathcal{F}^{-1} \hat{Q}(\xi,t), \hat{Q}(\xi,t) = \int_0^t \mathcal{F}^{-1} [S(\xi,t - \tau) \hat{g}(\xi,\tau)] \, d\tau.
\]

Theorem 2.1. Assume the Condition 2.1 holds and \( s > 1 + \frac{n}{p} \) with \( p \in (1, \infty) \). Let \( 0 < \alpha < 1 - \frac{n}{2p} \). Then for \( \varphi \in Y_0(A^s), \psi \in Y_1(A^s) \) and \( g(x,t) \in Y_1^{s,p} \) problem (2.1) – (2.2) has a unique generalized solution \( u(x,t) \in C^2([0,T] ; X_{\infty}) \). Moreover, the following estimate holds
\[
\| A^\alpha * u \|_{X_{\infty}} + \| A^\alpha * u_t \|_{X_{\infty}} \leq C_0 \left[ \| \varphi \|_{Y_0^{\alpha}(A)} + \right. \tag{2.7}
\]
\[
\| \psi \|_{Y_1^{\alpha}(A)} + \int_0^t \left( \| g(\cdot,\tau) \|_{Y_1^{s,p}} + \| g(\cdot,\tau) \|_{X_1} \right) \, d\tau \right],
\]
uniformly in \( t \in [0, T] \), where the constant \( C_0 > 0 \) depends only on \( A \), the space \( E \) and initial data.

**Proof.** From Lemma 2.1 we obtain that problem (2.1) – (2.2) has a unique generalized solution \( u(x, t) \in C^2 ([0, T]; \mathcal{Y}^{s,p} (A; E)) \) for \( \varphi \in \mathbb{E}_{0p}, \psi \in \mathbb{E}_{1p} \) and \( g (., t) \in \mathcal{Y}^{s,p}_1 \). Let \( N \in \mathbb{N} \) and

\[
\Pi_N = \{ \xi : \xi \in \mathbb{R}^n, |\xi| \leq N \}, \quad \Pi'_N = \{ \xi : \xi \in \mathbb{R}^n, |\xi| \geq N \}.
\]

From (2.6) we deduced that

\[
|A^\alpha u|_{X_\infty} \lesssim \left| \|F^{-1} C (\xi, t) \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} S (\xi, t) \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} C (\xi, t) \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty(\Pi'_N)} + \right| \|F^{-1} S (\xi, t) \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty(\Pi'_N)} + \right| \frac{1}{2} \left| \|F^{-1} \hat{A}^\alpha \hat{Q} (\xi, t) \hat{\varphi} (\xi, \tau)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} \hat{A}^\alpha \hat{Q} (\xi, t) \hat{\psi} (\xi, \tau)\|_{L^\infty(\Pi'_N)}.
\]

By virtue of Remarks 2.1, 2.2 and properties of sectorial operators we have the following uniform estimate

\[
\left| \|F^{-1} \hat{A}^\alpha \hat{Q} (\xi, t) \hat{\varphi} (\xi, \tau)\|_{L^\infty(\Pi_N)} \leq C \|g\|_{X_1}.
\]

Hence, due to uniform boundedness of operator functions \( C (\xi, t), S (\xi, t) \), in view of (2.3) and by Minkowski’s inequality for integrals we get the uniform estimate

\[
\left| \|F^{-1} C (\xi, t) \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} S (\xi, t) \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty(\Pi_N)} \lesssim \left[ |A^\alpha \varphi|_{X_1} + |A^\alpha \psi|_{X_1} + \|g\|_{X_1} \right].
\]

Moreover, from (2.6) we deduced that

\[
\left| \|F^{-1} C (\xi, t) \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} S (\xi, t) \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty(\Pi'_N)} \lesssim \left| \|F^{-1} C (\xi, t) \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty(\Pi_N)} + \right| \|F^{-1} S (\xi, t) \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty(\Pi'_N)} \lesssim \left| \|F^{-1} \hat{A}^\alpha \hat{Q} (\xi, t) \hat{\varphi} (\xi, \tau)\|_{L^\infty} \right| \left| \|F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} C (\xi, t) \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{A}^\alpha \hat{\varphi} (\xi)\|_{L^\infty} \right| \left| \|F^{-1} \hat{A}^\alpha \hat{Q} (\xi, t) \hat{\psi} (\xi, \tau)\|_{L^\infty} \right| \left| \|F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{s}{2}} S (\xi, t) \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{A}^\alpha \hat{\psi} (\xi)\|_{L^\infty} \right|
\]

(2.10)
where the space $L^\infty (\Omega; E)$ is denoted by $L^\infty$. It is clear to see that

$$
\frac{\partial}{\partial \xi_k} \left[ (1 + |\xi|^2)^{-\frac{s}{2}} \hat{A}^\alpha (\xi) C (\xi, t) \Phi_0 (\xi) \right] =
-s \xi_k \left( 1 + |\xi|^2 \right)^{-\frac{s}{2} - 1} \hat{A}^\alpha (\xi) C (\xi, t) \Phi_0 (\xi) +
(1 + |\xi|^2)^{-\frac{s}{2}} \left\{ \frac{t}{4} \hat{A}^\alpha (\xi) \eta_- (\xi) \left( 2 \xi_k \hat{a} (\xi) + |\xi|^2 \frac{\partial}{\partial \xi_k} \hat{a} (\xi) + \frac{\partial}{\partial \xi_k} \hat{A} (\xi) \right) +
\alpha C (\xi, t) \hat{A}^{\alpha - 1} (\xi) \frac{\partial}{\partial \xi_k} \hat{A} (\xi) \Phi_0 (\xi) + \hat{A}^\alpha (\xi) C (\xi, t) \frac{\partial}{\partial \xi_k} \Phi_0 (\xi) \right\},
$$

$$
\frac{\partial}{\partial \xi_k} \left[ (1 + |\xi|^2)^{-\frac{s}{2}} \hat{A}^\alpha (\xi) S (\xi, t) \Phi_1 (\xi) \right] =
-s \xi_k \left( 1 + |\xi|^2 \right)^{-\frac{s}{2} - 1} \hat{A}^\alpha (\xi) S (\xi, t) \Phi_1 (\xi) +
(1 + |\xi|^2)^{-\frac{s}{2}} \left\{ \frac{t}{4} \hat{A}^\alpha (\xi) \eta_+ (\xi) \left( 2 \xi_k \hat{a} (\xi) + |\xi|^2 \frac{\partial}{\partial \xi_k} \hat{a} (\xi) + \frac{\partial}{\partial \xi_k} \hat{A} (\xi) \right) +
\alpha S (\xi, t) \hat{A}^{\alpha - 1} (\xi) \frac{\partial}{\partial \xi_k} \hat{A} (\xi) \Phi_1 (\xi) + \hat{A}^\alpha (\xi) C (\xi, t) \frac{\partial}{\partial \xi_k} \Phi_1 (\xi) \right\},
$$

where

$$
\Phi_0 (\xi) = \left[ \hat{A}^1 - \frac{1}{4p} - \varepsilon_0 + \left( 1 + |\xi|^2 \right)^{s(1 - \frac{1}{4p})} \right]^{-1}, \quad 0 < \varepsilon_0 < 1 - \frac{1}{2p},
$$

$$
\Phi_1 (\xi) = \left[ \hat{A}^\frac{3}{2} - \frac{1}{4p} - \varepsilon + \left( 1 + |\xi|^2 \right)^{s(\frac{3}{2} - \frac{1}{4p})} \right]^{-1}, \quad 0 < \varepsilon < \frac{1}{2} - \frac{1}{2p}.
$$

By assumption on $\hat{A}^\alpha (\xi)$, we have the uniform estimates

$$
\left\| \hat{A}^\alpha (\xi) C (\xi, t) \Phi_0 (\xi) \right\|_{B(E)} \leq C \left\| \hat{A}^\alpha (\xi) \hat{A}^{-(1 - \frac{1}{4p} - \varepsilon_0)} (\xi) \right\|_{B(E)} \leq C_0,
$$

$$
\left\| \hat{A}^\frac{3}{2} (\xi) \eta^{-1} (\xi) \right\|_{B(E)} \leq \left\| \hat{A}^\alpha (\xi) \hat{A}^{\frac{3}{2}} (\xi) \Phi_1 (\xi) \right\|_{B(E)} \leq C_0.
$$
\[ \left\| \hat{A}^{\alpha} (\xi) S (\xi, t) \Phi_1 (\xi) \right\|_{B(E)} \leq C \left\| \hat{A}^{\alpha} (\xi) \hat{A}^{-(1 - \frac{2}{p})} (\xi) \right\|_{B(E)} \leq C_1. \]

Then by calculating \( \frac{\partial}{\partial \xi} \Phi_0 (\xi) \) and \( \frac{\partial}{\partial \xi} \Phi_1 (\xi) \) and in view of the assumptions on \( \frac{\partial}{\partial \xi} \hat{A} (\xi) \) we obtain
\[
\hat{A}^{\alpha} (\xi) \frac{\partial}{\partial \xi} \Phi_0 (\xi) \in B (E), \hat{A}^{\alpha} (\xi) \frac{\partial}{\partial \xi} \Phi_1 (\xi) \in B (E).
\]

By assumption (4), in view of \( s > 1 + \frac{2}{p} \) from (2.3), (2.11) for \( \beta = (\beta_1, \beta_2, ..., \beta_n) \) and \( \beta_k \in \{0, 1\} \) we have the following uniform estimates
\[
\sup_{\xi \in \mathbb{R}^n, t \in [0, T]} |\xi|^{\beta + \frac{2}{p}} \left\| D^\beta \left[ (1 + |\xi|^2)^{\frac{\beta}{2}} \hat{A}^{\alpha} (\xi) C (\xi, t) \Phi_0 (\xi) \right] \right\|_{B(E)} \leq C_1,
\]
\[
\sup_{\xi \in \mathbb{R}^n, t \in [0, T]} |\xi|^{\beta + \frac{2}{p}} \left\| D^\beta \left[ (1 + |\xi|^2)^{\frac{\beta}{2}} \hat{A}^{\alpha} (\xi) S (\xi, t) \Phi_1 (\xi) \right] \right\|_{B(E)} \leq C_2 \quad (2.12)
\]

Moreover, in view of (2.12) we show that the operator functions
\[
|\xi|^{\beta + \frac{2}{p}} \left\| D^\beta \left[ (1 + |\xi|^2)^{\frac{\beta}{2}} \hat{A}^{\alpha} (\xi) C (\xi, t) \Phi_0 (\xi) \right] \right\|,
\]
\[
|\xi|^{\beta + \frac{2}{p}} \left\| D^\beta \left[ (1 + |\xi|^2)^{\frac{\beta}{2}} \hat{A}^{\alpha} (\xi) S (\xi, t) \Phi_1 (\xi) \right] \right\|
\]
are uniformly \( R \)-bounded in \( E \). Hence, by Fourier multiplier theorems (see e.g. [13, Theorem 4.3]) we get that the functions \( (1 + |\xi|^2)^{\frac{\beta}{2}} \hat{A}^{\alpha} (\xi) C (\xi, t) \Phi_i (\xi) \) are \( L^p (\mathbb{R}^n; E) \rightarrow L^\infty (\mathbb{R}^n; E) \) Fourier multipliers. Then by Minkowski’s inequality for integrals, from (2.3), (2.10) and (2.11) – (2.12) we have
\[
\left\| F^{-1} C (\xi, t) \hat{A}^{\alpha} \hat{\varphi} (\xi) \right\|_{L^\infty} + \left\| F^{-1} S (\xi, t) \hat{A}^{\alpha} \hat{\psi} (\xi) \right\|_{L^\infty} \lesssim
\]
\[
\left\| F^{-1} C (\xi, t) \hat{\varphi} \right\|_{L^\infty} + \left\| F^{-1} S (\xi, t) \eta^{-2} \hat{\varphi} \right\|_{L^\infty} \lesssim \left[ \| \varphi \|_{E_{0,p}} + \| \psi \|_{E_{1,p}} + \| g \|_{W^{r,p}} \right]. \quad (2.13)
\]

Moreover, by virtue of Remarks 2.1, 2.2 and by reasoning as the above, we have the following estimate
\[
\left\| F^{-1} \hat{A}^{\alpha} \hat{Q} (\xi, t) \right\|_{X_\infty} \leq C \int_0^T \left( \| g (\cdot, \tau) \|_{W^{r,p}} + \| g (\cdot, \tau) \|_{X_1} \right) d\tau \quad (2.14)
\]
uniformly in \( t \in [0, T] \). Thus, from (2.6), (2.13) and (2.14) we obtain
\[
\| A^{\alpha} \ast u \|_{X_\infty} \leq C \left[ \| \varphi \|_{E_{0,p}} + \| A^{\alpha} \varphi \|_{X_1} + \right] \quad (2.15)
\]
\[
\|\psi\|_{E_1} + \|A^\alpha \psi\|_{X_1} + \int_0^t (\|g(\cdot, \tau)\|_{Y^{s,p}} + \|g(\cdot, \tau)\|_{X_1}) d\tau.
\]

By differentiating (2.6), in a similar way we get
\[
\|A^\alpha * u_t\|_{X_\infty} \leq C \left[ \|\varphi\|_{E_0} + \|A^\alpha * \varphi\|_{X_1} + \int_0^t \left( \|g(\cdot, \tau)\|_{Y^{s,p}} + \|g(\cdot, \tau)\|_{X_1} \right) d\tau \right].
\]

Then from (2.15) and (2.16) in view of Remarks 2.1, 2.2 we obtain the assertion.

**Theorem 2.2.** Let the Condition 2.1 holds, \( s > 1 + \frac{n}{p} \) and let \( 0 < \alpha < 1 - \frac{1}{2p} \). Then for \( \varphi \in E_0, \psi \in E_1 \) and \( g \in Y^{s,p} \) the problem (2.1) - (2.2) has a unique generalized solution \( u \in C^2([0, T]; Y^{s,p}) \) and the following uniform estimate holds
\[
\left( \|A^\alpha * u\|_{Y^{s,p}} + \|A^\alpha * u_t\|_{Y^{s,p}} \right) \leq \left( \|\varphi\|_{E_0} + \|\psi\|_{E_1} + \int_0^t \|g(\cdot, \tau)\|_{Y^{s,p}} d\tau \right).
\]

**Proof.** From (2.5) and (2.11) we get the following uniform estimate
\[
\left( \left\| \mathbb{F}^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} A^\alpha \hat{u} \right\|_{X_p} + \left\| \mathbb{F}^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} A^\alpha \hat{u}_t \right\|_{X_p} \right) \leq \left( \|\varphi\|_{E_0} + \|\psi\|_{E_1} + \int_0^t \|g(\cdot, \tau)\|_{Y^{s,p}} d\tau \right). \tag{2.18}
\]

By using the Fourier multiplier theorem [13, Theorem 4.3] and by reasoning as in Theorem 2.1 we get that \( C(\xi, t), S(\xi, t) \) and \( \hat{A}^\alpha S(\xi, t) \) are Fourier multipliers in \( L^p(\mathbb{R}^n; E) \) uniformly with respect to \( t \in [0, T] \). So, the estimate (2.18) by using the Minkowski’s inequality for integrals implies (2.17).

3. Local well posedness of IVP for nonlinear nonlocal WE
In this section, we will show the local existence and uniqueness of solution
for the nonlinear problem (1.1) – (1.2).

For this aim we need the following lemmas. Here, we will denote $L^p(\mathbb{R}^n; E)$,
$W^{s,p}(\mathbb{R}^n; E)$ by $X_p$ and $Y^{s,p}$, respectively. Here, we assume that $E$ is a
Banach algebra. By reasoning as in [8, 13, 26], we show the following lemmas
concerning the behaviour of the nonlinear term in $E$ valued space $Y^{s,p}$.

**Lemma 3.1.** Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; E)$ with $f(0) = 0$. Then for any
$u \in Y^{s,p} \cap L^\infty$, we have $f(u) \in Y^{s,p} \cap X_{\infty}$. Moreover, there is some constant
$A(M)$ depending on $M$ such that for all $u \in Y^{s,p} \cap L^\infty$ with $\|u\|_{X_{\infty}} \leq M$,
\[
\|f(u)\|_{Y^{s,p}} \leq C(M) \|u\|_{Y^{s,p}}. \tag{3.1}
\]

**Proof.** For $s = 0$ in view of $f(0) = 0$, we get
\[
f(u) = u \int_0^1 f(\sigma u) d\sigma.
\]
It follows that
\[
\|f(u)\|_{X_p} \leq C(M) \|u\|_{X_p}.
\]
If $s > 0$ is a positive integer, we have
\[
\|f(u)\|_{Y^{s,p}} \leq C \left[ \|f(u)\|_{X_p} + \sum_{k=1}^n \left\| \frac{\partial^s u}{\partial x_k} f(u) \right\|_{X_p} \right]. \tag{3.2}
\]
By calculation of derivativie and applying Holder inequality we get
\[
\left\| \frac{\partial^s u}{\partial x_i} f(u) \right\|_{X_p} \leq \sum_{l=1}^s \sum_{\alpha} \left\| f^{(l)}(u) \frac{\partial^{\beta_1} u}{\partial x_i} \frac{\partial^{\beta_2} u}{\partial x_i} \cdots \frac{\partial^{\beta_l} u}{\partial x_i} \right\|_{X_p} \leq \sum_{l=1}^s \sum_{\alpha} \left\| f^{(l)}(u) \right\|_{X_\infty} \prod_{k=1}^l \left\| \frac{\partial^{\beta_k} u}{\partial x_i} \right\|_{X_{p_k}}, \quad i = 1, 2, \ldots, n, \tag{3.3}
\]
where
\[
\beta = (\beta_1, \beta_2, \ldots, \beta_l), \quad \beta_k \geq 1, \quad \beta_1 + \beta_2 + \cdots + \beta_l = l, \quad p_k = \frac{pl}{\beta_k}.
\]
Applying Gagliardo-Nirenberg’s inequality in $E$-valued $X_p$ spaces, we have
\[
\left\| \frac{\partial^{\beta_k} u}{\partial x_i} \right\|_{X_{p_k}} \leq C \|u\|_{X_\infty}^{1-\frac{\beta_k}{l}} \left\| \frac{\partial^s u}{\partial x_i} \right\|_{X_p}^{\frac{\beta_k}{l}}. \tag{3.4}
\]
Hence, from (3.3) and (3.4) we deduced
\[
\left\| \frac{\partial^s u}{\partial x_i} f(u) \right\|_{X_p} \leq C(M) \left\| \frac{\partial^s u}{\partial x_i} \right\|_{X_p}. \tag{3.5}
\]
Then combining (3.2), (3.3) and (3.5) we obtain (3.1).

If \( s \) is not integer number, let \( m = \lfloor s \rfloor \). From the above proof, we have

\[
\| f(u) \|_{\gamma_{m,p}} \leq C(M) \| u \|_{\gamma_{m,p}}, \quad \| f(u) \|_{\gamma_{m+1,p}} \leq C(M) \| u \|_{\gamma_{m+1,p}}.
\]

Then using interpolation between \( W^{m+1,p} \) and \( W^{m,p} \) yields (3.1) for all \( s \geq 0 \).

By using Lemma 3.1 and properties of convolution operators we obtain

**Corollary 3.1.** Let \( s \geq 0, f \in C^{[s]+1}(\mathbb{R}; E) \) with \( f(0) = 0 \). Moreover, assume \( \Phi \in L^\infty(\mathbb{R}^n; B(E)) \). Then for any \( u \in Y^{s,p} \cap L^\infty \), we have \( f(u) \in Y^{s,p} \cap X_\infty \). Moreover, there is some constant \( A(M) \) depending on \( M \) such that for all \( u \in Y^{s,p} \cap L^\infty \) with \( \| u \|_{X_\infty} \leq M \),

\[
\| \Phi \ast f(u) \|_{Y^{s,p}} \leq C(M) \| u \|_{Y^{s,p}}.
\]

**Lemma 3.2.** Let \( s \geq 0, f \in C^{[s]+1}(\mathbb{R}; E) \). Then for any \( m \) there is some constant \( K(M) \) depending on \( M \) such that for all \( u, v \in Y^{s,p} \cap X_\infty \) with \( \| u \|_{X_\infty} \leq M, \| v \|_{X_\infty} \leq M, \| u \|_{Y^{s,p}} \leq M, \| v \|_{Y^{s,p}} \leq M \),

\[
\| f(u) - f(v) \|_{Y^{s,p}} \leq K(M) \| u - v \|_{Y^{s,p}}.
\]

By reasoning as in [13, Lemma 3.4] and [26, Lemma X 4] we have, respectively

**Corollary 3.2.** Let \( s > \frac{n}{2}, f \in C^{[s]+1}(\mathbb{R}; E) \). Then for any \( M \) there is a constant \( K(M) \) depending on \( M \) such that for all \( u, v \in Y^{s,p} \) with \( \| u \|_{Y^{s,p}} \leq M \),

\[
\| f(u) - f(v) \|_{Y^{s,p}} \leq K(M) \| u - v \|_{Y^{s,p}}.
\]

**Lemma 3.3.** If \( s > 0 \), then \( Y_{\infty}^{s,p} \) is an algebra. Moreover, for \( f, g \in Y_{\infty}^{s,p} \),

\[
\| fg \|_{Y^{s,p}} \leq C \left[ \| f \|_{X_\infty} \| g \|_{Y^{s,p}} + \| f \|_{Y^{s,p}} \| g \|_{X_\infty} \right].
\]

By using, The Corollary 3.1 and Lemma 3.3 we obtain

**Lemma 3.4.** Let \( s \geq 0, f \in C^{[s]+1}(\mathbb{R}; E) \) and \( f(u) = O\left(|u|^\gamma \right) \) for \( u \to 0, \gamma \geq 1 \) a positive integer. If \( u \in Y_{\infty}^{s,p} \) and \( \| u \|_{X_\infty} \leq M \), then

\[
\| f(u) \|_{Y^{s,p}} \leq C(M) \| u \|_{Y^{s,p}} \| u \|_{X_\infty}^{\gamma-1}.
\]

**Corollary 3.3.** Let \( s \geq 0, f \in C^{[s]+1}(\mathbb{R}; E) \) and \( f(u) = O\left(|u|^\gamma \right) \) for \( u \to 0, \gamma \geq 1 \) a positive integer. Moreover, assume \( \Phi \in L^\infty(\mathbb{R}^n; B(E)) \). If \( u \in Y_{\infty}^{s,p} \) and \( \| u \|_{X_\infty} \leq M \), then

\[
\| \Phi \ast f(u) \|_{Y^{s,p}} \leq C(M) \| u \|_{Y^{s,p}} \| u \|_{X_\infty}^{\gamma-1}.
\]
Lemma 3.5. Let \( s \geq 0, f \in C^{[s]+1}(\mathbb{R}; E) \) and \( f(u) = O\left(|u|^{\gamma+1}\right) \) for \( u \to 0, \gamma \geq 0 \) be a positive integer. If \( u, v \in Y^{s,p}_\infty, \|u\|_{Y^{s,p}_\infty} \leq M, \|v\|_{Y^{s,p}_\infty} \leq M \) and \( \|u\|_{X^*_\infty} \leq M, \|v\|_{X^*_\infty} \leq M \), then

\[
\|f(u) - f(v)\|_{Y^{s,p}_\infty} \leq C(M) \left[\left(\|u\|_{X^*_\infty} - \|v\|_{X^*_\infty}\right)(\|u\|_{Y^{s,p}} + \|v\|_{Y^{s,p}_\infty}) + \|u\|_{X^*_\infty} + \|v\|_{X^*_\infty}\right]^\gamma^{-1},
\]

\[
\|f(u) - f(v)\|_{X^*_i} \leq C(M) \left(\|u\|_{X^*_\infty} + \|v\|_{X^*_\infty}\right)^\gamma^{-1} \left(\|u\|_{X^*_p} + \|v\|_{X^*_p}\right) \|u - v\|_{X^*_p}.
\]

Let \( E_0 \) denotes the real interpolation space between \( Y^{s,p}_\infty(A, E) \) and \( X_p \) with \( \theta = \frac{1}{s+p} \), i.e.

\[
E_0 = (Y^{s,p}_\infty(A, E), X_p)^{\frac{s}{s+p}},
\]

Here, \( Y_0(A^\alpha) \) and \( Y_1(A^\alpha) \) are the spaces defined in Section 2.

Remark 3.1. By using J.Lions-I. Petree result (see e.g. [27, § 1.8]) we obtain that the map \( u \to u(t_0), t_0 \in [0, T] \) is continuous and surjective from \( Y^{s,p}_\infty(A, E) \) onto \( E_0 \), and there is a constant \( C_1 \) such that

\[
\|u(t_0)\|_{E_0} \leq C_1 \|u\|_{Y^{s,p}_\infty(A, E)}, \ 1 \leq p \leq \infty.
\] (3.6)

Let

\[
C^2(Y_1^{s,p}(A)) = C(2)([0, T]; Y_1^{s,p}(A, E)), \ C^{2,s}(A, E) = C(2)([0, T]; Y^{s,p}_\infty(A, E)).
\]

Definition 3.1. Let \( T > 0 \) and \( \varphi \in Y_0(A^\alpha), \psi \in Y_1(A^\alpha) \). The function \( u \in C^2(Y_1^{s,p}(A)) \) satisfies the problem (1.1) – (1.2) is called the continuous solution or the strong solution of (1.1) – (1.2). If \( T < \infty \), then \( u(x, t) \) is called the local strong solution of the problem (1.1) – (1.2). If \( T = \infty \), then \( u(x, t) \) is called the global strong solution of (1.1) – (1.2).

Condition 3.1. Assume:

1. the Condition 2.1 holds for \( s > \frac{\alpha}{2} \) and \( 0 < \alpha < 1 - \frac{1}{2p} \);
2. the kernel \( g = g(x) \) is a bounded integrable operator function in \( E \) such that \( \Delta g \in L^\infty(\mathbb{R}^n; B(E)) \);
3. the function \( u \to f(u) \): continuous from \( u \in E_0 \) into \( E, f \in C^k(\mathbb{R}; E) \) with \( k \) an integer, \( k \geq s > \frac{\alpha}{2} \) and \( f(u) = O\left(|u|^{\gamma+1}\right) \) for \( u \to 0, \gamma \geq 1 \) be a positive integer.

Remark 3.2. We will use Lemmas 3.1-3.5 and Corollary 3.3 in the following results. Note that, inspite of in these Lemmas and Corollary were assumed \( E \) to be Banach algebra, here it is sufficient to take \( E \) UMD space. Really, since the solution \( u \) of (1.1) – (1.2) is assumed to be from the space \( Y^{s,p}_\infty(A, E) \).

Then by assumption (3) of the Condition 3.1 and by Remark 1.1 the function \( u \to f(u) \) continuous from \( u \in Y^{s,p}_\infty \) into \( E \). Hence, Lemmas 3.1-3.5 and Corollary 3.3 are yield for \( u \in Y^{s,p}_\infty(A, E) \), when \( E \) is only UMD spaces.

Let

\[
\hat{Y}_1^{s,p}(A^\alpha; E) = \hat{Y}^{s,p}(A^\alpha; E) \cap X_1(A^\alpha), \hat{Y}^{s,p}(A^\alpha; E) = \{ u \in Y^{s,p}(A^\alpha; E),
\]

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we got that ∥ψ∥_{E_0} + ∥A^α * ψ∥_{X_1} + ∥ψ∥_{E_1} + ∥A^α * ψ∥_{X_1} ≤ δ, \tag{3.7} \end{equation}

problem (1.1) – (1.2) has a unique local strange solution u ∈ C^2 (Y^{s,p}_1 (A)). Moreover,

\begin{equation}
\sup_{t \in [0,T]} \left( ∥u (., t)∥_{Y^{s,p}_t (A, E)} + ∥u_t (., t)∥_{Y^{s,p}_t (A, E)} \right) \leq Cδ, \tag{3.8} \end{equation}

where the constant C depends only on A, E, g, f and initial values.

**Proof.** By (2.5), (2.6)) the problem of finding a solution u of (1.1) – (1.2) is equivalent to finding a fixed point of the mapping

\begin{equation}
G (u) = C_1 (t) \varphi (x) + S_1 (t) \psi (x) + Q (u), \tag{3.9} \end{equation}

where C_1 (t), S_1 (t) are defined by (2.6) and Q (u) is a map defined by

\begin{equation}
Q (u) = - \int_0^t F^{-1} \left[ U (\xi, t - \tau) |ξ|^2 \hat{g} (ξ) \hat{f} (u) (ξ, \tau) \right] d\tau. \tag{3.10} \end{equation}

We define the metric space

\begin{equation}
C (T, A) = C^{2}_δ (Y^{s,p}_1 (A)) = \left\{ u ∈ C^{2,s} (A, E), ∥u∥_{C^{2,s} (T, A)} ≤ 5C_0δ \right\} \tag{3.11} \end{equation}

equipped with the norm defined by

\begin{equation}
∥u∥_{C(T,A)} = \sup_{t \in [0,T]} \left[ ∥A^α \ast u (., t)∥_{X_∞} + ∥u (., t)∥_{Y^{s,p}} + ∥A^α \ast u_t (., t)∥_{X_∞} + ∥u_t (., t)∥_{Y^{s,p}} \right], \tag{3.12} \end{equation}

where δ > 0 satisfies (3.7) and C_0 is a constant in Theorem 2.1 and 2.2. It is easy to prove that C(T, A) is a complete metric space. From imbedding in Sobolev-Lions space Y^{s,p}_1 (A, E) (see e.g. [30], Theorem 1) and trace result (3.6) we got that ∥u∥_{X_∞} ≤ 1 if we take that δ is enough small. For φ ∈ Y^{0}_0 (A^α) and ψ ∈ Y^{1}_1 (A^α), let

\begin{equation}
∥φ∥_{E_0} + ∥A^α \ast φ∥_{X_1} + ∥ψ∥_{E_1} + ∥A^α \ast ψ∥_{X_1} = δ. \tag{3.13} \end{equation}

So, we will find T and M so that G is a contraction on C^{2,s,p}_1 (T, A). By Theorems 2.1, 2.2 and Corollary 3.3 ∆g \ast f (u) ∈ Y^{s,p}_1. So, problem (1.1) – (1.2) has a solution satisfies the following

\begin{equation}
G (u) (x, t) = C_1 (t) \varphi + S_1 (t) \psi + Qu, \tag{3.14} \end{equation}

\begin{equation}
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where \( C_1 (t), S_1 (t) \) are defined by (2.5) and (2.6). By assumptions, it is easy to see that the map \( G \) is well defined for \( f \in C^{[\alpha]+1} (\mathbb{E}_{0p}; E) \). First, let us prove that the map \( G \) has a unique fixed point in \( C (T, A) \). For this aim, it is sufficient to show that the operator \( G \) maps \( C (T, A) \) into \( C (T, A) \) and \( G \) is strictly contractive if \( \delta \) is suitable small. In fact, by (2.7) in Theorem 2.1, Corollary 3.3 and in view of (3.7), we have

\[
\| A^\alpha * G (u) \|_{X_{\infty}} + \| A^\alpha * G (u)_t \|_{X_{\infty}} \leq 2C_0 \left( \| \varphi \|_{Y_0^\alpha (A^\alpha)} + \right)
\]

(3.11)

\[
\| \psi_{Y_0^\alpha (A^\alpha)} + \int_0^t (\| \Delta g * f ((u)) \|_{Y_{s,p}} + \| \Delta g * f ((u)) \|_{X_1}) d\tau \right) \leq
\]

(3.12)

\[
2C_0 \int_0^t \left( \| u (\tau) \|_{Y_{s,p}} + \| u (\tau) \|^2_{Y_{s,p}} + \| u (\tau) \|^2_{X_1} \right) d\tau \leq
\]

(3.13)

On the other hand, by (2.17) in Theorem 2.2, Corollary 3.3 and (3.7), we get

(3.14)

Hence, combining (3.11) with (3.12) we obtain

(3.15)

Therefore, taking that \( \delta \) is enough small such that \( C (5C_0 \delta)^{\gamma} < \frac{1}{5} \), then by Theorems 2.1, 2.2 and (3.13), \( G \) maps \( C (T, A) \) into \( C (T, A) \).

Now, we are going to prove that the map \( G \) is strictly contractive. Let \( u_1, u_2 \in C (T, A) \) given. From (3.10) we get

\[
G (u_1) - G (u_2) =
\]

(3.16)
By (2.7) in Theorem 2.1 and Corollary 3.3, we have
\[
\|A^p [G(u_1) - G(u_2)]\|_{X_\infty} + \|A^p [G(u_1) - G(u_2)]_t\|_{X_\infty} \leq (3.14)
\]
\[
\int_0^t \left(\|\Delta g \ast [f(u_1) - f(u_2)]\|_{Y_{s,p}} + \|\Delta g \ast [f(u_1) - f(u_2)]\|_{X_1}\right) d\tau \leq
\]
\[
\int_0^t \{\|u_1 - u_2\|_{X_\infty} (\|u_1\|_{Y_{s,p}} + \|u_2\|_{Y_{s,p}}) (\|u_1\|_{X_\infty} + \|u_2\|_{X_\infty})^{\gamma - 1} +
\]
\[
\|u_1 - u_2\|_{Y_{s,p}} (\|u_1\|_{X_\infty} + \|u_2\|_{X_\infty})^{\gamma} +
\]
\[
(\|u_1\|_{X_\infty} + \|u_2\|_{X_\infty})^{\gamma - 1} \|u_1 + u_2\|_{X_\infty} \|u_1 - u_2\|_{X_\infty}\}
\leq
\]
\[
C \left(\|u_1\|_{C(T,A)} + \|u_2\|_{C(T,A)}\right)^\gamma \|u_1 - u_2\|_{C(T,A)}.
\]
On the other hand, by (2.17) in Theorem 2.2, Corollary 3.3 and (3.7), we get
\[
(\|A^p [G(u_1) - G(u_2)]\|_{Y_{s,p}} + \|A^p [G(u_1) - G(u_2)]_t\|_{Y_{s,p}}) \leq
\]
\[
C \int_0^t \|\Delta g \ast [f(u_1)(\tau) - f(u_2)(\tau)]\|_{Y_{s,p}} d\tau \leq (3.15)
\]
\[
\int_0^t \{\|u_1 - u_2\|_{X_\infty} (\|u_1\|_{Y_{s,p}} + \|u_2\|_{Y_{s,p}}) (\|u_1\|_{X_\infty} + \|u_2\|_{X_\infty})^{\gamma - 1} +
\]
\[
\|u_1 - u_2\|_{Y_{s,p}} (\|u_1\|_{X_\infty} + \|u_2\|_{X_\infty})^{\gamma}\} d\tau \leq
\]
\[
C \left(\|u_1\|_{C(T,A)} + \|u_2\|_{C(T,A)}\right)^\gamma \|u_1 - u_2\|_{C(T,A)}.
\]
Combining (3.14) with (3.15) yields
\[
\|G(u_1) - G(u_2)\|_{C(T,A)} \leq (3.16)
\]
\[
C \left(\|u_1\|_{C(T,A)} + \|u_2\|_{C(T,A)}\right)^\gamma \|u_1 - u_2\|_{C(T,A)}.
\]
Taking \(\delta\) is enough small, from (3.16) we obtain that \(G\) is strictly contractive in \(C(T,A)\). Using the contraction mapping principle we get that \(G(u)\) has a unique fixed point \(u(x,t) \in C(T,A)\) and \(u(x,t)\) is the solution \((1.1) - (1.2)\).
Let us show that this solution is a unique in $C^{2,s}(A,E)$. Let $u_1, u_2 \in C^{2,s}(A,E)$ are two solutions of (1.1) − (1.2). Then for $u = u_1 - u_2$, we have

$$u_{tt} - a \ast \Delta u + A \ast u = \Delta g \ast [f(u_1) - f(u_2)]$$  \hspace{1cm} (3.17)

Hence, by Minkowski’s inequality for integrals and by Theorem 2.2 from (3.17) we obtain

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{s,p}} \, d\tau.$$  \hspace{1cm} (3.18)

From (3.18) and Gronwall’s inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e. problem (1.1) − (1.2) has a unique solution in $C^{2,s}(A,E)$.

Consider the problem (1.1) − (1.2), when $\varphi \in E_{0p}$ and $\psi \in E_{1p}$. Let

$$C^{(i)}(Y^{s,2}) = C^{(i)}([0, \infty); Y^{s,2}(A,E)), i = 0, 1, 2.$$  \hspace{1cm} (3.19)

Theorem 3.2. Let the Condition 3.2 holds. Then there exists a constant $\delta > 0$, such that for any $\varphi \in E_{0p}$, $\psi \in E_{1p}$ satisfying

$$\|\varphi\|_{E_{0p}} + \|\psi\|_{E_{1p}} \leq \delta,$$  \hspace{1cm} (3.20)

problem (1.1) − (1.2) has a unique local strange solution $u \in C^{(2)}(Y^{s,p})$. Moreover,

$$\sup_{t \in [0,T]} \left(\|u(.,t)\|_{Y^{s,p}(A^{\alpha};E)} + \|u_t(.,t)\|_{Y^{s,p}(A^{\alpha};E)}\right) \leq C\delta,$$  \hspace{1cm} (3.21)

where the constant $C$ only depends on $f$ and initial data.

Proof. Consider a metric space defined by

$$W_0^{s,p} = \left\{ u \in C^{(2)}(Y^{s,p}) : \|u\|_{Y^{s,p}} \leq 3C_0\delta \right\},$$

equipped with the norm

$$\|u\|_{W_0^{s,p}} = \sup_{t \in [0,T]} \left(\|u\|_{Y^{s,p}(A^{\alpha};E)} + \|u_t\|_{Y^{s,p}(A^{\alpha};E)}\right),$$

where $\delta > 0$ satisfies (3.20) and $C_0$ is a constant in Theorem 2.1. It is easy to prove that $W_0^{s,p}$ is a complete metric space. From Sobolev imbedding theorem we know that $\|u\|_{\infty} \leq 1$ if we take that $\delta$ is enough small. By Theorem 2.2 and
Theorem 2.1, we get that

\[ G \]

Theorem 3.1, we obtain that

\[ G \]

C

Therefore, from (3

contraction mapping principle, we know that

\[ G \]

Taking that

\[ \delta \]

is enough small such that

\[ C \]

is suitable small. In fact, by (2

solution which can be written as (3

Theorem 2.2 and Lemma 3.1 we get

Corollary 3.1, ∆g * f (u) ∈ Y^{s,p}. Thus the problem (1.1) – (1.2) has a unique solution which can be written as (3.9). We should prove that the operator \( G(u) \) defined by (3.9) is strictly contractive if \( \delta \) is suitable small. In fact, by (2.17) in Theorem 2.2 and Lemma 3.1 we get

\[
\|A^α * G(u)\|_{Y^{s,p}} + \|A^α * G_1(u)\|_{Y^{s,p}} \leq C_0 \left( \|\varphi\|_{E_{0p}} + \|\psi\|_{E_{1p}} + \int_0^t \|K(u)\|_{Y^{s,p}} \, dt \right) \leq C_0 \delta + C_0 \int_0^t \|K(u)\|_{Y^{s,p}} \, dt \leq C_0 \delta + C_\|u\|_{Y^{s,p}}, \tag{3.22}
\]

where

\[ K(u)(\cdot, \tau) = S(x, t - \tau) \Delta g * f(u)(x, \tau). \]

Therefore, from (3.22) we have

\[
\|G(u)\|_{Y^{s,p}} \leq 2C_0 \delta + C_\|u\|_{Y^{s,p}}. \tag{3.23}
\]

Taking that \( \delta \) is enough small such that \( C (3C_0 \delta)^α < 1/3 \), from (3.23) and from Theorems 2.1, 2.2 we get that \( G \) maps \( W_0^{s,p} \) into \( W_0^{s,p} \). Then, by reasoning as in Theorem 3.1 we obtain that \( G : W_0^{s,p} \rightarrow W_0^{s,p} \) is strictly contractive. Using the contraction mapping principle, we know that \( G(u) \) has a unique fixed point \( u \in C^{(2)}(Y^{s,2}) \) and \( u(x, t) \) is the solution of the problem (1.1) – (1.2). Moreover, by virtue of Theorem 2.1 from (3.20) we obtain (3.21).

We claim that the solution of (1.1) – (1.2) is also unique in \( C^{(1)}(Y^{s,2}) \). In fact, let \( u_1 \) and \( u_2 \) be two solutions of the problem (1.1) – (1.2) and \( u_1, u_2 \in C^{(2)}(Y^{s,2}) \). Using the contraction mapping principle, we know that \( G(u) \) has a unique fixed point \( u \in C^{(2)}(Y^{s,2}) \). Using the contraction mapping principle, we know that \( G(u) \) has a unique fixed point \( u \in C^{(2)}(Y^{s,2}) \). Let \( u = u_1 - u_2 \), then

\[
u_{tt} - a \Delta u + A \ast u = \Delta [g * (f(u_1) - f(u_2))].
\]

This fact is derived in a similar way as in Theorem 3.1, by using Theorems 2.1, 2.2 and Gronwall’s inequality.

Let

\[ C^{(2,s)}(Y^{s,p}) = C^{(2)}([0, T]; Y^{s,p}(A; E)). \]

**Theorem 3.3.** Let the Condition 3.2 hold. Then there is some \( T > 0 \) such that the problem (1.1) – (1.2) for initial data \( \varphi \in E_{0p} \) and \( \psi \in E_{1p} \) is well posed with solution in \( C^{(1)}([0, T]; Y^{s,p}(A, E)) \).

**Proof.** Consider the convolution operator

\[ u \rightarrow \Delta [g * f(u)]. \]
In view of assumptions and Fourier multiplier results in $X_p$ spaces (see e.g. [12, Theorem 4.3]) we have

$$||\Delta g * v||_{Y_{s',p}} \lesssim \rho \left\| (1 + \xi^2)^{\frac{n}{2} - 1} \xi^2 \hat{g} (\xi) \hat{v} (\xi) \right\| \lesssim \| v \|_{Y_{s,p}},$$

i.e. $\Delta g * v$ is a bounded linear operator on $Y^{s,p}$. Then by Corollary 3.1, $K(u)$ is locally Lipschitz on $Y^{s,p}$. Then by reasoning as in Theorem 3.2 and [13, Theorem 1.1] we obtain that $G: W_0^{1,p} \rightarrow W_0^{1,p}$ is strictly contractive. Using the contraction mapping principle, we get that the operator $G(u)$ defined by (3.5) has a unique fixed point $u(x,t) \in C(2) (Y^{s,p})$ and $u(x,t)$ is the solution of the problem (1.1) – (1.2). Moreover, we show that the solution $u(x,t)$ of (1.1) – (1.2) is also unique in $C(2) (Y^{s,p})$. In fact, let $u_1$ and $u_2$ be two solutions of the problem (1.1) – (1.2) and $u_1, u_2 \in C(2) (Y^{s,p})$. Let $u = u_1 - u_2$, then

$$u_{tt} - a \cdot \Delta u + A \cdot u = \Delta [g * (f(u_1) - f(u_2))].$$

This fact is derived in a similar way as in Theorem 3.2, by using Theorems 2.1, 2.2 and Gronwall’s inequality.

**Theorem 3.4.** Let the Condition 3.2 holds for $r > 2 + \frac{n}{p}$. Then there is some $T > 0$ such that problem (1.1) – (1.2) is well posed for $\varphi \in \mathbb{E}_0^{1,p}$ and $\psi \in \mathbb{E}_1^{1,p}$ with solution in $C(2) (Y^{s,p})$.

**Proof.** All we need here, is to show that $K * f(u)$ is Lipschitz on $Y^{s,p}$. Indeed, by reasoning as in Theorem 3.3 we have

$$||\Delta g * v||_{Y_{s+r-2,p}} \lesssim \rho \left\| (1 + \xi^2)^{\frac{n}{2} - 1} \xi^2 \hat{g} (\xi) \hat{v} (\xi) \right\| \lesssim \| v \|_{Y_{s,p}},$$

Then $\Delta g * v$ is a bounded linear map from $Y^{s,p}$ into $Y^{s+r-2,p}$. Since $s \geq 0$ and $r > 2 + \frac{n}{p}$ we get

$$s + r - 2 > \frac{n}{p}.$$  

The embedding theorem for $E$–valued Sobolev spaces (see e.g. [31]) implies that $\Delta g * v$ is a bounded linear map from $Y^{s,p}(A; E)$ into $Y^{s,p}(A; E)$. Lemma 3.2 implies the Lipschitz condition on $Y^{s,p}$. Then, by reasoning as in Theorem 3.3 we obtain the assertion.

The solution in theorems 3.2-3.4 can be extended to a maximal interval $[0, T_{\max})$, where finite $T_{\max}$ is characterized by the blow-up condition

$$\limsup_{T \to T_{\max}} \|u\|_{\tilde{Y}^{s,p}(A; \mathbb{R}; E)} = \infty.$$

**Lemma 3.8.** Let the Condition 3.2 hold and $u$ is a solution of (1.1) – (1.2). Then there is a global solution if for any $T < \infty$ we have

$$\sup_{t \in [0, T]} \left( \|u\|_{\tilde{Y}^{s,p}(A; \mathbb{R}; E)} + \|u_t\|_{\tilde{Y}^{s,p}(A; \mathbb{R}; E)} \right) < \infty.$$  

(3.24)
Proof. Indeed, by reasoning as in the second part of the proof of Theorem 3.1, by using a continuation of local solution of (1.1) – (1.2) and assuming contrary that, (3.24) holds and $T_0 < \infty$ we obtain contradiction, i.e. we get $T_0 = T_{\text{max}} = \infty$.

4. Conservation of energy and global existence.

In this section, we prove the existence and the uniqueness of the global strong solution for the problem (1.1) – (1.2). For this purpose, we are going to make a priori estimates of the local strong solution of (1.1) – (1.2).

Condition 4.1. Suppose the Condition 3.2 is satisfied. Assume $a \in L^2(\mathbb{R}^n)$ and the kernel $g$ is a bounded operator function in $E$, whose Fourier transform satisfies

$$0 < \|\hat{g}(\xi)\|_{B(E)} \lesssim (1 + |\xi|^2)^{-\frac{r}{2}}$$

for all $\xi \in \mathbb{R}^n$ and $r \leq 2(s + 1)$.

Moreover, let $\hat{g}(\xi)$ have fractional powers for all $\xi \in \mathbb{R}^n$. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Assume that the operator $\hat{g}(\xi)$ has a fractional power $\hat{g}^{\frac{1}{2}}(\xi)$ for all $\xi \in \mathbb{R}^n$. We consider the Fourier multiplier operator $B = B_g$ defined by

$$u \in D(B) = Y^{s,p}, \quad Bu = \mathcal{F}^{-1}\left[|\xi|^{-1} \hat{g}^{\frac{1}{2}}(\xi) \hat{u}(\xi)\right],$$

Then it is clear to see that

$$B^{-2}u = -\Delta g * u, \quad B^{-1}u = \mathcal{F}^{-1}\left[|\xi| \hat{g}^{\frac{1}{2}}(\xi) \hat{u}(\xi)\right]. \quad (4.1)$$

Let

$$C^{(1)}(L^p) = C^{(1)}([0, T); L^p(\mathbb{R}^n; E)), \quad C^{(2,s)}(A, E) = C^{(2)}([0, T]; Y^{s,p}(A; E)),$$

where $Y^{s,p}(A; E)$ was defined in Section 2.

First, we show the following

Lemma 4.1. Let the Condition 4.1 holds and $0 < \alpha < 1 - \frac{1}{2p}$. Assume there exist a solution $u \in C^{(2,s)}(A, E)$ of (1.1) – (1.2). Then

$$\hat{A}^\alpha Bu, \hat{A}^\alpha Bu \in C^{(1)}(L^p).$$

Proof. By Lemma 2.1, problem (1.1) – (1.2) is equivalent to the following integra equation,

$$u(x, t) = C_1(t) \varphi + S_1(t) \psi + Qg, \quad (4.2)$$

where $C_1(t)$, $S_1(t)$ are operator functions defined by (2.5) and (2.6), where $g$ replaced by $g * f(u)$ and

$$Qg = \int_0^t \mathcal{F}^{-1}\left[S(\xi, t - \tau) |\xi|^2 \hat{g}(\xi) \hat{f}(u)(\xi)\right] d\tau. \quad (4.3)$$
From (4.2) we get that
\[ u_t(x, t) = \frac{d}{dt}C_1(t) \varphi + \frac{d}{dt}S_1(t) \psi + \int_0^t F^{-1} \left[ C(\xi, t - \tau) |\xi|^2 \hat{g}(\xi) \hat{f}(G(u)(\xi)) \right] d\tau. \] (4.4)

Since \( C_1(t), S_1(t) \) and \( \frac{d}{dt}S(\xi, t) \) are uniformly bounded operators in \( E \) for fixet \( t \), by (4.1), (4.2), (4.4) and Fourier multiplier results in \( X_p \) spaces (see e.g. [12, Theorem 4.3]) we have
\[ \left\| \hat{A}^\alpha BC_1(t) \varphi \right\|_{L^p} = \left\| F^{-1} \left[ |\xi|^{-1} \hat{g}^{-\frac{1}{2}}(\xi) \hat{A}^\alpha C(\xi, t) \hat{\varphi} \right] \right\|_{L^p} \lesssim \left\| \varphi \right\|_{E_{\alpha p}} < \infty, \] (4.5)
\[ \left\| \hat{A}^\alpha BS_1(t) \varphi \right\|_{L^p} = \left\| F^{-1} \left[ |\xi|^{-1} \hat{g}^{-\frac{1}{2}}(\xi) \hat{A}^\alpha S(\xi, t) \hat{\psi} \right] \right\|_{L^p} \lesssim \left\| \psi \right\|_{E_{\alpha p}} < \infty. \] (4.6)

By differentiating (2.3), in a similar way we have
\[ \left\| \hat{A}^\alpha B \frac{d}{dt}C_1(t) \varphi \right\|_{L^p} = \left\| F^{-1} \left[ |\xi|^{-1} \hat{g}^{-\frac{1}{2}}(\xi) \hat{A}^\alpha \frac{d}{dt} C(\xi, t) \hat{\varphi} \right] \right\|_{L^p} \lesssim \left\| \varphi \right\|_{E_{\alpha p}} < \infty, \] (4.7)
\[ \left\| \hat{A}^\alpha B \frac{d}{dt}S_1(t) \varphi \right\|_{L^p} = \left\| F^{-1} \left[ |\xi|^{-1} \hat{g}^{-\frac{1}{2}}(\xi) \hat{A}^\alpha \frac{d}{dt} S(\xi, t) \hat{\psi} \right] \right\|_{L^p} \lesssim \left\| \psi \right\|_{E_{\alpha p}} < \infty. \]

For fixed \( t \), we have \( f(u) \in Y^{s,p} \). Moreover, by assumption on \( \hat{A}(\xi) \) we have the uniformly estimate
\[ \left\| \hat{A}^\alpha(\xi) \eta^{-1}(\xi) \right\|_{B(E)} \leq C_A. \]

Then by hypothesis on \( \hat{g}(\xi) \), due to \( s + r \geq 1 \) from (4.1) and (4.3) and Fourier multiplier results in \( X_p \) we get
\[ \left\| \hat{A}^\alpha BQg \right\|_{L^p} \leq \left\| F^{-1} \left[ |\xi|^{\frac{1}{2}} \hat{g}(\xi) \hat{A}^\alpha(\xi) \int_0^t S(\xi, t - \tau) \hat{f}(u)(\xi) d\tau \right] \right\|_{L^p} \lesssim C_A \left\| f(u) \right\|_{Y^{s,p}} < \infty. \] (4.7)

Then from (4.2) and (4.4) – (4.7) we obtain the assertion.
Lemma 4.2. Assume the Condition 4.1 holds with $a = 0$. Moreover, let
\[
\left\| (\hat{g}(\xi))^{-\frac{1}{2}} \right\|_{B(E)} = O \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}}.
\]
Suppose the solution of (1.1) – (1.2) exists in $C^{(2,s)}(A,E)$. If $B\psi \in L^p$ then $Bu_t \in C^{(1)}(L^p)$. Moreover, if $B\varphi \in L^p$, then $Bu \in C^{(1)}(L^p)$.

Proof. Integrating the equation (1.1) for $a = 0$ twice and calculating the resulting double integral as an iterated integral, we have
\[
\begin{align*}
    u(x,t) &= \varphi(x) + t\psi(x) - \\
    &\int_0^t (t - \tau) (A * u)(x,\tau) \, d\tau + \int_0^t (t - \tau) \Delta (g * f(u))(x,\tau) \, d\tau, \\
    \quad (4.8)
\end{align*}
\]
\[
\begin{align*}
    u_t(x,t) &= \psi(x) - \int_0^t (A * u)(x,\tau) \, d\tau + \int_0^t \Delta (g * f(u))(x,\tau) \, d\tau. \\
    \quad (4.9)
\end{align*}
\]
From (4.1) and (4.9) for fixed $t$ and $\tau$ we get $f(u) \in Y^{s,p}$ for all $t$. Also
\[
\|B\Delta (g * f(u))(x,\tau)\|_{L^p} \lesssim \left\| \mathcal{F}^{-1} \left[ |\xi|^{-\frac{1}{\sqrt{2}}} \hat{g} \frac{\hat{\varphi}}{\xi} (\xi) \right] \hat{f}(u)(\xi) \right\|_{L^p}. \quad (4.10)
\]
Then from (4.8) – (4.10) we obtain
\[
\begin{align*}
    \|Bu_t(x,t)\|_{L^2} &= \|B\psi(x)\|_{L^2} - \\
    &\int_0^t \|B (A * u)(x,\tau)\|_{L^p} \, d\tau - \int_0^t \|B\Delta (g * f(u))(x,\tau)\|_{L^p} \, d\tau.
\end{align*}
\]
By assumption on $A$, $g$ and by (4.1) for fixed $\tau$ we have $Bu_t \in C^{(1)}(L^p)$.
\[
\|B (A * u)(x,\tau)\|_{L^p} \lesssim \left\| \mathcal{F}^{-1} \left[ |\xi|^{-\frac{1}{\sqrt{2}}} \hat{A}(\xi) \left( \hat{g}^{-\frac{1}{2}} (\xi) \right) \hat{\varphi}(\xi,\tau) \right] \right\|_{L^p} \lesssim \|u(\cdot,\tau)\|_{Y^{s,p}(A)}.
\]
Moreover, by Lemma 3.3 we have $Bu_t \in C^{(1)}(L^p)$. The second statement follows similarly from (4.8).

From Lemma 4.2 we obtain the following result.

Result 4.1. Assume the Condition 4.1 are satisfied with $a = 0$ and
\[
\|\hat{g}(\xi)\|_{B(E)} = O \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}}.
\]
Suppose the solution of (1.1) – (1.2) exists in $C^{(2,s)}(A,H)$ for some $s \geq 0$. If $B\psi \in L^2$ then $Bu_t \in C^{(1)}(L^2)$. Moreover, if $B\varphi \in L^2$, then $Bu \in C^{(1)}(L^2)$.
Lemma 4.3. Assume the Condition 3.2 holds and $s + r \geq 1$. Let $u \in C^{(2,s)}(A, H)$ be a solution of (1.1) - (1.2) for any $t \in [0, T]$. Let $B\psi \in L^2$ and $(f(u), u) \in L^2$. Then the energy

$$E(t) = \|Bu_t\|^2_{L^2} + (B [A \ast u - a \ast \Delta u], Bu)_{L^2} + (f(u), u)_{L^2} \quad (4.11)$$

is constant.

Proof. By Theorem 4.1, $A^\alpha Bu, A^\alpha Bu_t \in L^2$ for $0 < \alpha < \frac{3}{2}$. By assumptions $(f(u), u) \in L^2$ and $A \ast u \in L^2$. By use of (1.1) and Parseval’s identity, it follows from straightforward calculation that

$$\frac{d}{dt} E(t) = 2(Bu_{tt}, Bu_t) + 2(Ba \ast \Delta u, Bu_t) +$$

$$2[B(A \ast u), Bu_t(t)] + 2(f(u), u_t) = 2B^2(u_{tt}, u_t) +$$

$$2B^2(a \ast \Delta u, u_t) + 2B^2(A \ast u, u_t) - 2B^2(\Delta [g \ast f(u)], u_t) =$$

$$2B^2(u_{tt} + a \ast \Delta u + A \ast u - \Delta [g \ast f(u)], u_t) =$$

$$B^2 \frac{d}{dt} [(u_{tt} - a \ast \Delta u + A \ast u - \Delta g \ast f(u), u)] = 0,$$

where $(u, v)$ denotes the inner product in $L^2(\mathbb{R}^n)$. Hence, we obtain the assertion.

By using the above lemmas we obtain the following results

Theorem 4.1. Let the Condition 4.1 holds for $r > 2 + \frac{n}{2}$. Moreover, let $B\psi \in L^2$, $(f(u), u) \in L^2(\mathbb{R}^n; H)$ and there is some $k > 0$ so that $(f(u), u) \geq -k \|u(., t)\|^2$ for all $t \in [0, T]$. Then there is some $T > 0$ such that problem (1.1) - (1.2) has a global solution $u \in C^{(2,s)}(A, H)$.

Proof. Since $r > 2 + \frac{n}{2}$, by Theorem 3.4 we get local existence $u \in C^{(2,s)}(A, E)$ for some $T > 0$. Assume that $u$ exists on $[0, T)$. By Lemma 4.3, we obtain

$$\|Bu_t\|^2 + \|\hat{u}\|^2_{L^2} \|F^{-1}\mathring{\hat{g}} \ast \hat{u}\|^2 + (B(A \ast u), Bu) \leq E(0) + 2k \|u(., t)\|^2. \quad (4.12)$$

Let $Y^{s,2}$ denotes by $W^s$. By condition on $\mathring{\hat{g}}(\xi)$, we have

$$\|Bu_t\|^2_{L^2(A)} = \int_{\mathbb{R}^n} |\xi|^{-2} \|\hat{g}^{-1}(\xi)\|^2_{B(H)} \|A\hat{u}_t(\xi, t)\|^2_H \geq C_g^{-1} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{3}{2} - 1} \|A\hat{u}_t(\xi, t)\|^2_H \approx C_g^{-1} \|A\hat{u}_t(\xi, t)\|^2_{W^{\frac{3}{2} - 1}}, \quad (4.13)$$

where $C_g$ is the positive constant that appear in (3.19). By properties of norms in Hilbert spaces and by Cauchy-Schwarz inequality, from (4.12) and (4.13) we get

$$\frac{d}{dt} \|u(t)\|^2_{W^{\frac{3}{2} - 1}(A)} \leq 2 \|u_t(t)\|_{W^{\frac{3}{2} - 1}(A)} \|u(t)\|_{W^{\frac{3}{2} - 1}(A)} \leq$$
\[ \|u_t(t)\|_{W^{2-1,A}} + \|u(t)\|_{W^{2-1,A}} \leq C \|Bu_t(t)\|_{W^{2-1,A}}^2 + \|u(t)\|_{W^{2-1,A}}^2. \]

Gronwall’s lemma implies that \( \|u(t)\|_{W^{2-1,A}} \) is bounded in \([0,T]\). But, since \( \frac{2}{2} - 1 > \frac{n}{4} \), we conclude that \( \|u(t)\|_{L^\infty(A)} \) also is bounded in \([0,T]\). By Lemma 3.8 this implies a global solution.

5. Blow up in finite time

We will use the following lemma to prove blow up in finite time.

**Lemma 5.1** [16] Suppose \( H(t), t \geq 0 \) is a positive, twice differentiable function satisfying \( H''(t) - (1 + \nu) (H'(t))^2 \geq 0 \), where \( \nu > 0 \). If \( H(0) > 0 \) and \( H'(0) > 0 \), then \( H(t) \to \infty \) when \( t \to t_1 \) for some \( t_1 \leq H(0) \left[ \nu H'(0) \right]^{-1} \).

We rewrite the energy identity as
\[ E(t) = \|Bu_t\|^2 + \left( [B^2 (A*u - (a*\Delta)u)] , u \right) + (f(u), u) = E(0). \]

We prove here the following

**Theorem 5.1.** Assume the Condition 4.1 is satisfied and \( s + r \geq 1 \). Let \( B\varphi, B\psi \in L^2 \). If there are some positive numbers \( \nu, t_0 \) and \( b \) such that

\[ (1 + 2\nu) b + d \leq -E(0), \quad 4b (1 + \nu) (t + t_0) \leq 2b - 2E(0) \]

and
\[ 4\nu \|Bu\|^2 \|Bu_t\|^2 \leq \zeta_1 \|Bu\|^2 + \zeta_2 \|Bu_t\|^2 + \phi(t), \]

for the solution \( u \in C^{(2,s)}(A,E) \) of (1.1) - (1.2) and
\[ E(0) = \|B\psi\|^2 + \left( [B^2 [A*u - a*\Delta u]] , u \right) + (f(\varphi), \varphi) < 0, \]

for all \( t \geq 0 \), where
\[ (1 + 2\nu) b \leq -E(0), \]
\[ \zeta_1 = [2b - 2E(0) - 4b (1 + \nu) (t + t_0)], \]
\[ \zeta_2 = 4b (t + t_0) [(t + t_0) - (1 + \nu)], \]
\[ \phi(t) = 2b [-E(0) - (1 + 2\nu) b] (t + t_0)^2. \]

Then the solution \( u \) blows up in finite time.

**Proof.** Assume that there is a global solution. Then \( Bu(t), Bu_t(t) \in L^2 \) for all \( t > 0 \). Let
\[ H(t) = \|Bu\|^2 + b (t + t_0)^2. \]
for some positive \( b \) and \( t_0 \) that will be determined later. We have

\[
H^{(1)}(t) = 2 (Bu, Bu_t) + 2b (t + t_0), \quad (5.2)
\]

\[
H^{(2)}(t) = 2 \|Bu_t\|^2 + 2 (Bu, Bu_{tt}) + 2b.
\]

From (1.1) and (5.1) we get

\[
(Bu, Bu_{tt}) = (u, B^2 u_{tt}) = (u, B^2 [a \ast \Delta u - A \ast u + \Delta g \ast f (u)]) =
\]

\[
[(u, B^2 (a \ast \Delta u)) - (u, B^2 A \ast u) - (u, f (u))] = \|Bu_t\|^2 - E (0).
\]

From (5.2) and (5.3), we obtain

\[
H^{(2)}(t) \geq 4 \|Bu_t\|^2 - 2E (0) + 2b. \quad (5.4)
\]

On the other hand, in view of Cauchy-Schwartz inequality, we have

\[
\left( H^{(1)}(t) \right)^2 = \left[ 2 (Bu, Bu_t) + 2b (t + t_0) \right]^2 \leq
\]

\[
4 \left[ \|Bu\|^2 \|Bu_t\|^2 + b (t + t_0) \left( \|Bu\|^2 + \|Bu_t\|^2 \right) \right] +
\]

\[
4b^2 (t + t_0)^2. \quad (5.5)
\]

Hence, by (5.2), (5.4) and (5.5), we obtain

\[
H^{(2)} H - (1 + \nu) \left( H^{(1)} \right)^2 \geq
\]

\[
\left[ 4 \|Bu_t\|^2 + 2b - 2E (0) \right] \left[ \|Bu\|^2 + b (t + t_0)^2 \right] -
\]

\[
4 (1 + \nu) \left[ \|Bu\|^2 \|Bu_t\|^2 + b (t + t_0) \left( \|Bu\|^2 + \|Bu_t\|^2 \right) \right] -
\]

\[
4 (1 + \nu) b^2 (t + t_0)^2 \geq
\]

\[
2b [-E (0) - (1 + 2\nu) b] (t + t_0)^2 +
\]

\[
[2b - 2E (0) - 4b (1 + \nu) (t + t_0)] \|Bu\|^2 +
\]

\[
4b (t + t_0) [(t + t_0) - (1 + \nu)] \|Bu_t\|^2 - 4\nu \|Bu\|^2 \|Bu_t\|^2 \geq 0,
\]

when

\[
(1 + 2\nu) b \leq -E (0), \quad 4b (1 + \nu) (t + t_0) \leq 2b - 2E (0)
\]
and
\[ 4\nu \|Bu\|^2 \|Bu_t\|^2 \leq \varkappa_1 \|Bu\|^2 + \varkappa_2 \|Bu_t\|^2 + \phi(t), \]
for all \( t \geq 0 \), where
\[ (1 + 2\nu) b \leq -E(0), \]
\[ \varkappa_1 = [2b - 2E(0) - 4b (1 + \nu) (t + t_0)], \]
\[ \varkappa_2 = 4b (t + t_0) [(t + t_0) - (1 + \nu)], \]
\[ \phi(t) = 2b [-E(0) - (1 + 2\nu) b] (t + t_0)^2. \]
Then by Theorem 5.1 we obtain the assertion.

6. Applications

6.1. The Cauchy problem for the system of nonlocal WEs

Consider the problem (1.3). Let
\[ l_q = \left\{ u = \{u_j\}, \ j = 1, 2, ... N, \ \|u\|_{l_q} = \left( \sum_{j=1}^{\infty} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\}, \]
(see [23, § 1.18]. Let \( A_1 \) be the operator in \( l_p \) defined by
\[ A_1 = [a_{jm}(x)], \ a_{jm} = b_j(x) 2^{\sigma m}, \ m, j = 1, 2, ... \infty, \ D(A_1) = l_\sigma^* = \]
\[ \left\{ u = \{u_j\}, \ j = 1, 2, ... \infty, \ \|u\|_{l_\sigma^*} = \left( \sum_{j=1}^{\infty} 2^{\sigma j} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\}, \ \sigma > 0. \]
Let
\[ W^{s,p}(E) = W^{s,p}(\mathbb{R}^n; E), \ W^s(E) = W^{s,2}(\mathbb{R}^n; E), \]
\[ Y^{s,p,\sigma} = W^{s,p}(\mathbb{R}^n; l_q) \cap L^p(\mathbb{R}^n; l_\sigma^*), \ 1 \leq q \leq \infty, \]
\[ W_0(l_q) = W^{s(1 - \frac{d}{p})p}(\mathbb{R}^n; l_q) \cap L^p(\mathbb{R}^n; l_\sigma^{(1 - \frac{d}{p})}). \]
Let \( f = \{f_m\}, \ m = 1, 2, ... \infty \) and
\[ \eta_1 = \eta_1(\xi) = \left[ \hat{a}(\xi) \xi^2 + \hat{A}_1(\xi) \right]^{\frac{1}{2}}. \]
Here,
\[ E_{ip}(l_q) = W^{s(1 - \theta_i)p}(\mathbb{R}^n; l_q) \cap L^p(\mathbb{R}^n; l_\sigma^{(1 - \theta_i)}), \]
where
\[ \theta_j = \frac{1 + ip}{2p}, \ i = 0, 1. \]
From Theorem 3.1 we obtain the following result

**Theorem 6.1.** Assume: (1) $0 < \alpha < 1 - \frac{2}{p}$, $\varphi \in E_{0p} (l_q)$, $\psi \in E_{1p} (l_q)$ and $s > 1 + \frac{2}{p}$ for $p \in [1, \infty]$, $q \in (1, \infty)$; (2) the assumptions (1)-(2) of Condition 2.1 are satisfied; (3) $\hat{b}_j = b_j (\xi)$ are nonnegative bounded differentiable functions on $\mathbb{R}^n$ and $a + \hat{b}_j (\xi)$ are nonnegative bounded integrable functions, whose Fourier transform satisfies

$$
\int_{\mathbb{R}^n} |\hat{g}_{m, j}(\xi)|^2 \leq (1 + |\xi|^2)^{-\frac{s}{2}} \text{ for all } \xi \in \mathbb{R}^n \text{ and } r \geq 2;
$$

(4) the kernel $g_{m, j}$ are bounded integrable functions, whose Fourier transform satisfies

$$
0 \leq \sum_{j=m, j}^N |\hat{g}_{m, j}(\xi)|^2 \leq (1 + |\xi|^2)^{-\frac{s}{2}} \text{ for all } \xi \in \mathbb{R}^n \text{ and } r \geq 2;
$$

(5) the function

$$
u \to f (x, t, u) : \mathbb{R}^n \times [0, T] \times W_0 (l_q) \to l_q
$$

is a measurable in $(x, t) \in \mathbb{R}^n \times [0, T]$ for $u \in W_0 (l_q)$; Moreover, $f (x, t, u)$ is continuous in $u \in W_0 (l_q)$ and $f \in C^{s+1} (W_0 (l_q); l_q)$ uniformly in $x \in \mathbb{R}^n$, $t \in [0, T]$. Then problem (1.3) has a unique local strange solution

$$
u \in C^{(2)} ([0, T_0); Y_{\infty}^{s, p} (A_1, l_q)), \quad T_0
$$

where $T_0$ is a maximal time interval that is appropriately small relative to $M$.

Moreover, if

$$
\sup_{t \in [0, T_0]} \left( \|u\|_{Y_{\infty}^{s, p} (A_1; l_q)} + \|u_t\|_{Y_{\infty}^{s, p} (A_1; l_q)} \right) < \infty,
$$

then $T_0 = \infty$.

**Proof.** It is known that $L^p (\mathbb{R}^n; l_q)$ is a UMD space for $p, q \in (1, \infty)$ (see e.g [25]). By Remark 2.1, by definition of $W^{s, p} (A_1, l_q)$ and by real interpolation of Banach spaces (see e.g. [23, §1.3, 1.18]), we have

$$
E_{ip} = \left( W^{s, p} (\mathbb{R}^n; l_q), L_p (\mathbb{R}^n; l_q) \right) = W^{s(1-\theta_i), p} (\mathbb{R}^n; \eta^{(1-\theta_i)}, l_q) = W^{s(1-\theta_i), p} (\mathbb{R}^n; l_q) \cap L_p (\mathbb{R}^n; \eta^{(1-\theta_i)}) = E_{0i} (l_q), \quad i = 0, 1.
$$

By assumptions (1), (2) we obtain that $\hat{A}_1 (\xi)$ is uniformly sectorial in $l_q$, $\hat{A}_1 (\xi) \in \sigma (M_0, \omega, l_q)$, $\eta_1 (\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ and

$$
\|D^\alpha \hat{A}_1 (\xi) \eta_1^{-1} (\xi)\|_{B(l_q)} \leq M
$$

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for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( |\alpha| \leq n \). Hence, by (4), (5), all conditions of Theorem 3.2 are hold, i.e., we get the conclusion.

Let \( G \) be a function defined by (4.15).

**Theorem 6.2.** Assume: (a) (1)-(3) assumptions of Theorem 6.1 are satisfied for \( p = 2 \) and

\[
\| \hat{g} (\xi) \|_{B(l_2)} \lesssim \left( 1 + |\xi|^2 \right)^{-\frac{r}{2}} \text{ for } r \leq 2(s + 1),
\]

\[
\| \hat{g}^\frac{1}{2} (\xi) \|_{B(l_2)} \lesssim |\xi| \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \text{ for all } \xi \in \mathbb{R}^n;
\]

(c) \( f_m \in C^{[a]} (\mathbb{R}; l_2) \) with \( f(0) = 0 \) and

\[
\sum_{m=1}^{N} |\hat{f}_m (u) (\xi)|^2 < \infty \text{ for all } u = \{u_m\} \in C^{(2)} ([0, \infty); Y_{s,2} (A_1; l_2));
\]

(d) \( B\phi, B\psi \in L^2 (\mathbb{R}^n; l_2) \) and \( (\phi, f(\phi)) \in L^2 (\mathbb{R}^n; l_2); \) (e) there is some \( k > 0 \) so that

\[
(\phi, f(\phi))_{L^2(\mathbb{R}^n; l_2)} \geq -k \| \phi \|_{L^2(\mathbb{R}^n; l_2)}.
\]

Then there is some \( T > 0 \) such that problem (1.3) has a global solution

\[
u \in C^{(2)} ([0, \infty); Y_{s,2} (A_1; l_2)).
\]

**Proof.** From the assumptions (a), (b) it is clear to see that the Condition 4.1 holds for \( H = l_2 \) and \( r > 2 + \frac{4}{p} \). By (c), (d), (e) all other assumptions of Theorem 4.1 are satisfied. Hence, we obtain the assertion.

### 6.2. The mixed problem for degenerate nonlocal WE

Consider the problem (1.5) - (1.7). Let

\[
Y^{s,p,2} = W^{s,p} (\mathbb{R}^n; L^{p_1} (0, 1)) \cap L^p (\mathbb{R}^n; W^{[2(1-\theta_i)];p_1} (0, 1)), 1 \leq p \leq \infty,
\]

Let \( A_2 \) is the operator in \( L^{p_1} (0, 1) \) defined by (1.4) and let

\[
\eta_2 = \eta_2 (\xi) = \left[ a |\xi|^2 + \hat{A}_2 (\xi) \right]^\frac{1}{2}.
\]

Here,

\[
E_{ip} (L^{p_1}) = W^{[s(1-\theta_i)];p} (\mathbb{R}^n; L^{p_1} (0, 1)) \cap L^p (\mathbb{R}^n; W^{[2(1-\theta_i)];p_1} (0, 1)),
\]

where

\[
\theta_i = \frac{1 + ip}{2p}, i = 0, 1.
\]

Now, we present the following result:

**Condition 6.1** Assume;
(1) $0 \leq \gamma < \frac{1}{p_1}$ for $p_1 \in (1, \infty)$ and $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$;

(2) $0 < \alpha < 1 - \frac{1}{2p}$, $\varphi \in E_{0p} (L^{p_1})$, $\psi \in E_{1p} (L^{p_1})$ and $s > 1 + \frac{p}{p}$ for $p \in [1, \infty]$, $p_1 \in (1, \infty)$;

(2) $b_1$ and $b_2$ are complex valued functions on $(0, 1)$. Moreover, $b_1 \in C[0, 1]$, $b_1(0) = b_1(1)$, $b_2 \in L_\infty (0, 1)$ and $|b_2 (x)| \leq C \left| b_2 \frac{T^2 - \mu}{t^2} (x) \right|$ for $0 < \mu < \frac{1}{p}$ and for a.a. $x \in (0, 1)$;

(3) the assumptions (1)-(2) of Condition 2.1 are satisfied; $D^\alpha b_j$, $j = 1, 2$ are uniformly bounded on $R^n$ for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \leq n$ and $\eta_2 (\xi) \neq 0$ for all $\xi \in R^n$;

(4) for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| \leq n$ the uniform estimate holds

$$\left\| \left[ D^\alpha \hat{A}_2 (\xi) \right] \eta_2^{-1} (\xi) \right\|_{B(L^{p_1} (0, 1))} \leq M.$$  

(5) the function

$$u \rightarrow f(x, t, u) : R^n \times [0, T] \times W_0 (L^{p_1} (0, 1)) \rightarrow L^{p_1} (0, 1)$$

is a measurable in $(x, t) \in R^n \times [0, T]$ for $u \in W_0 (L^{p_1} (0, 1))$, $f(x, t, u)$. Moreover, $f(x, t, u)$ is continuous in $u \in W_0 (L^{p_1} (0, 1))$ and

$$f(x, t, u) \in C^{[s]+1} (W_0 (L^{p_1} (0, 1)); L^{p_1} (0, 1))$$

uniformly with respect to $x \in R^n$, $t \in [0, T]$.

**Theorem 6.3.** Assume that the Condition 6.1 is satisfied. Then problem (1.5) – (1.7) has a unique local strange solution

$$u \in C^{[2]} (0, T_0 ; Y_\infty^{s,p} (A_2, L^{p_1} (0, 1))) ,$$

where $T_0$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$\sup_{t \in [0, T_0]} \left( \| u \|_{Y_\infty^{s,p} (A_2; L^2 (0, 1))} + \| u_t \|_{Y_\infty^{s,p} (A_2; L^2 (0, 1))} \right) < \infty$$

then $T_0 = \infty$.

**Proof.** It is known (see e.g. [13]) that $L^{p_1} (0, 1)$ is a UMD space for $p_1 \in (1, \infty)$. By definition of $W^{s,p} (A_2, L^{p_1} (0, 1))$ and by real interpolation of Banach spaces (see e.g. [23, §1.3]) we have

$$E_{ip} = W^{s, -p} \left( R^n ; W^{[2]-p_1} (0, 1), L^{p_1} (0, 1), L^p R^n ; L^{p_1} (0, 1) \right)_{\theta, i, p} = W^{s(1-\theta), p} \left( R^n ; W^{[2(1-\theta)],[ p_1]} (0, 1), L^{p_1} (0, 1) \right) = E_{ip} (L^{p_1}).$$

In view of [26, Theorem 4.1] we obtain that $\hat{A}_2 (\xi)$ is uniformly sectorial in $L^{p_1} (0, 1)$ and

$$\hat{A}_2 (\xi) \in \sigma (M_0, \omega, L^{p_1} (0, 1)).$$
Moreover, by using the assumptions (1), (2) we deduced that $\eta_2(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ and

$$\left\| D^\alpha \hat{\varphi}_2(\xi) \eta_2^{-1}(\xi) \right\|_{B(L^{p_1}(0,1))} \leq M.$$ 

for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| \leq n$. Hence, by hypothesis (3), (4) of the Condition 5.1 we get that all hypothesis of Theorem 3.2 are hold, i.e., we obtain the conclusion. Let $G = (0,1) \times \mathbb{R}^n$

**Theorem 6.4.** Assume the Condition 6.1 is satisfied for $p_1 = 2$. Suppose $f \in C^{[s]}(\mathbb{R}; L^2((0,T)))$ with $f(0) = 0$. Let the kernel $g_{m,j}$ be bounded integrable functions and

$$\left\| \hat{g}(\xi) \right\|_{B(l^2)} \lesssim \left(1 + |\xi|^2\right)^{-\frac{s}{2}} \text{ for } r \leq 2(s + 1),$$

$$\left\| \hat{g}^{1/2}(\xi) \right\|_{B(l^2)} \lesssim |\xi|\left(1 + |\xi|^2\right)^{s/2} \text{ for all } \xi \in \mathbb{R}^n.$$ 

Moreover, let $B\varphi, B\psi \in L^2(G)$, and $(\varphi, f(\varphi)) \in L^2(G)$; (e) there is some $k > 0$ so that

$$(\varphi, f(\varphi))_{L^2(G)} \geq -k \| \varphi \|_{L^2(G)}.$$ 

Then there is some $T > 0$ such that the problem (1.5) – (1.7) has a global solution

$$u \in C^2([0, \infty); Y^{s,2}).$$

**Proof.** Indeed, by assumptions all conditions of Theorem 4.1. are satisfied for $H = L^2(0,1)$, i.e. we obtain the assertion.

1. **References**

1. M. Arndt and M. Griebel, Derivation of higher order gradient continuum models from atomistic models for crystalline solids Multiscale Modeling Simul. (2005)4, 531–62.

2. A. Ashyralyev, N. Aggez, Nonlocal boundary value hyperbolic problems involving Integral conditions, Bound.Value Probl., 2014 V (2014):214.

3. X. Blanc, C. LeBris, P. L. Lions, Atomistic to continuum limits for computational materials science, ESAIM- Math. Modelling Numer. Anal. (2007)41, 391–426.

4. J.L. Bona, R.L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm. Math. Phys. 118 (1988), 15–29.

5. A. Constantin and L. Molinet, The initial value problem for a generalized Boussinesq equation, Diff.Integral Eqns. (2002)15, 1061–72.
6. G. Chen and S. Wang, Existence and nonexistence of global solutions for
the generalized IMBq equation Nonlinear Anal.—Theory Methods Appl.
(1999)36, 961–80.

7. R. Coifman and Y. Meyer, Wavelets. Calderón-Zygmund and
multilinear operators, Cambridge University Press, 1997.

8. M. Dafermos, I. Rodnianski, Y. Shlapentokh-Rothman, Decay for
solutions of the wave equation on Kerr exterior spacetimes III: The full
subextremal case $|a|<M$, Anal. Math, 183 (2016), 787-913.

9. A. De Godefroy, Blow up of solutions of a generalized Boussinesq equation
IMA J. Appl. Math. (1998) 60 123–38.

10. N. Duruk, H.A. Erbay and A. Erkip, Global existence and blow-up for a
class of nonlocal nonlinear Cauchy problems arising in elasticity, Nonlin-
earity, (2010)23, 107–118.

11. A. C. Eringen, Nonlocal Continuum Field Theories, New York, Springer
(2002).

12. H. O. Fattorini, Second order linear differential equations in Banach spaces,
in North Holland Mathematics Studies, V. 108, North-Holland, Amster-
dam, 1985.

13. M. Girardi, L. Weis, Operator-valued Fourier multiplier theorems on
$L^p(\mathcal{X})$
and geometry of Banach spaces, J. Funct. Anal., 204(2) (2003), 320–354.

14. Z. Huang, Formulations of nonlocal continuum mechanics based on a new
definition of stress tensor Acta Mech. (2006)187, 11–27.

15. T. Kato, G. Ponce, Commutator estimates and the Euler and Navier–
Stokes equations, Comm. Pure Appl. Math. (1988)41, 891–907.

16. V. K. Kalantarov and O. A. Ladyzhenskaya, The occurrence of collapse for
quasilinear equation of parabolic and hyperbolic types Journal of Soviet
Mathematics (10) (1978) 53-70.

17. M. Lazar, G. A. Maugin and E. C. Aifantis, On a theory of nonlocal
elasticity of bi-Helmholtz type and some applications Int. J. Solids and
Struct. (2006)43, 1404–21.

18. F. Linares, Global existence of small solutions for a generalized Boussinesq
equation, J. Differential Equations 106 (1993), 257–293.

19. Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq
equation, SIAM J. Math. Anal. 26 (1995), 1527–1546.

20. V.G. Makhankov, Dynamics of classical solutions (in non-integrable sys-
tems), Phys. Lett. C 35(1978), 1–128.
21. C. Polizzotto, Nonlocal elasticity and related variational principles Int. J. Solids Struct. (2001) 38 7359–80.

22. A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer, Berlin, 1983.

23. L. S. Pulkina, A non local problem with integral conditions for hyperbolice quations, Electron. J. Differ. Equ., (1999)45, 1-6.

24. C. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces J. Mech. Phys. Solids (2000)48 175-209.

25. V. B. Shakhmurov, Embedding and separable differential operators in Sobolev-Lions type spaces, Math. Notes, 84(2008) (6), 906-926.

26. V. B. Shakhmurov, Linear and nonlinear abstract differential equations with small parameters, Banach J. Math. Anal. 10 (2016)(1), 147–168.

27. H. Triebel, Interpolation theory, Function spaces, Differential operators, North-Holland, Amsterdam, 1978.

28. H. Triebel, Fractals and spectra, Birkhauser Verlag, Related to Fourier analysis and function spaces, Basel, 1997.

29. G.B. Whitham, Linear and Nonlinear Waves, Wiley–Interscience, New York, 1975.

30. S. Wang, G. Chen, Small amplitude solutions of the generalized IMBq equation, J. Math. Anal. Appl. 274 (2002) 846–866.

31. S.Wang and G.Chen, Cauchy problem of the generalized double dispersion equation Nonlinear Anal. Theory Methods Appl. (2006)64 159–73.

32. Ta-Tsien Li, Y. Jinseni, Global $C^1$ solution to the mixed initial boundary value problems for quasilinear hyperbolic system, Chinese Ann. Math. (22)03 (2001), 325-336.

33. N.J. Zabusky, Nonlinear Partial Differential Equations, Academic Press, New York, 1967.