Schauder’s estimates for nonlocal equations with singular Lévy measures

Mingyan Wu

Based on a joint work with Zimo Hao and Guohuan Zhao

School of Mathematics and Statistics, Wuhan University

The 15th Workshop on Markov Processes and Related Topics

Jilin University, July 11-15, 2019
Plan of this talk

1. Singular Lévy measures
2. Aims and Assumptions
3. The Littlewood-Paley characteristic of Hölder spaces
4. Schauder’s estimates
5. Sketch of proofs
Part 1: Introduction
Lévy measures

Definition 1 (Lévy measures)

ν is a Lévy measure on $\mathbb{R}^d$, if it is a $\sigma$-finite (positive) measure such that

$$
\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) \nu(dx) < +\infty.
$$

Definition 2 ($\alpha$-stable Lévy measures)

For $\alpha \in (0, 2)$, Lévy measure $\nu^{(\alpha)}$ is an $\alpha$-stable Lévy measure, if it has the polar coordinates form

$$
\nu^{(\alpha)}(A) = \int_0^\infty \left(\int_{S^{d-1}} \frac{1_A(r\theta)\Sigma(d\theta)}{r^{1+\alpha}}\right) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),
$$

where $\Sigma$ is a finite measure over the unit sphere $S^{d-1}$ (called spherical measure of $\nu^{(\alpha)}$).
Scaling property $\nu^{(\alpha)}(d(\lambda z)) = \lambda^{-\alpha} \nu^{(\alpha)}(dz)$ for any $\lambda > 0$.

Moments property For any $\gamma_1 > \alpha > \gamma_2 \geq 0$,

$$\int_{\mathbb{R}^d} (|z|^{\gamma_1} \wedge |z|^{\gamma_2}) \nu^{(\alpha)}(dz) < \infty.$$ 

**Definition 3 (Non-degenerate Lévy measures)**

One says that an $\alpha$-stable Lévy measure $\nu^{(\alpha)}$ is **non-degenerate** if

$$\int_{S^{d-1}} |\theta_0 \cdot \theta|^\alpha \Sigma(\theta) > 0 \quad \text{for every } \theta_0 \in S^{d-1}.$$
Non-degenerate $\alpha$-stable Lévy measures

Example 4 (Standard $\alpha$-stable Lévy measures)

If $\Sigma$ is rotationally invariant with $\Sigma(S^{d-1}) = |S^{d-1}|$, then $\nu^{(\alpha)}$ is the standard or strict $\alpha$-stable Lévy measure and

$$\nu^{(\alpha)}(dy) = \frac{dy}{|y|^{d+\alpha}}.$$  

The $d$-dimensional Lévy process associated with this Lévy measure is called d-dimensional $\alpha$-stable process.
Non-degenerate $\alpha$-stable Lévy measures

Example 4 (Standard $\alpha$-stable Lévy measures)

If $\Sigma$ is rotationally invariant with $\Sigma(\mathbb{S}^{d-1}) = |\mathbb{S}^{d-1}|$, then $\nu^{(\alpha)}$ is the standard or strict $\alpha$-stable Lévy measure and

$$
\nu^{(\alpha)}(dy) = \frac{dy}{|y|^{d+\alpha}}.
$$

The $d$-dimensional Lévy process associated with this Lévy measure is called $d$-dimensional $\alpha$-stable process.

- If $W_t = (W_t^1, \cdots, W_t^d)$ is a $d$-dimensional Brownian Motion, then $W_t^i$ are i.i.d $1$-dimensional Brownian Motions.
Example 4 (Standard $\alpha$-stable Lévy measures)

If $\Sigma$ is rotationally invariant with $\Sigma(S^{d-1}) = |S^{d-1}|$, then $\nu^{(\alpha)}$ is the standard or strict $\alpha$-stable Lévy measure and

$$\nu^{(\alpha)}(dy) = \frac{dy}{|y|^{d+\alpha}}.$$ 

The $d$-dimensional Lévy process associated with this Lévy measure is called $d$-dimensional $\alpha$-stable process.

- If $W_t = (W_t^1, \cdots, W_t^d)$ is a $d$-dimensional Brownian Motion, then $W_t^i$ are i.i.d 1-dimensional Brownian Motions.
- Let $L_t = (L_t^1, \cdots, L_t^d)$ be a $d$-dimensional $\alpha$-stable process. $L_t^i$ may not be independent or 1-dimensional standard $\alpha$-stable processes.
Non-degenerate $\alpha$-stable Lévy measures

Example 4 (Standard $\alpha$-stable Lévy measures)

If $\Sigma$ is rotationally invariant with $\Sigma(\mathbb{S}^{d-1}) = |\mathbb{S}^{d-1}|$, then $\nu^{(\alpha)}$ is the standard or strict $\alpha$-stable Lévy measure and

$$\nu^{(\alpha)}(dy) = \frac{dy}{|y|^{d+\alpha}}.$$ 

The $d$-dimensional Lévy process associated with this Lévy measure is called $d$-dimensional $\alpha$-stable process.

- If $W_t = (W_t^1, \cdots, W_t^d)$ is a $d$-dimensional Brownian Motion, then $W_t^i$ are i.i.d 1-dimensional Brownian Motions.
- Let $L_t = (L_t^1, \cdots, L_t^d)$ be a $d$-dimensional $\alpha$-stable process. $L_t^i$ may not be independent or 1-dimensional standard $\alpha$-stable processes.

Question

If $L_t^i, i = 1, \cdots, d$ are i.i.d 1-dimensional standard $\alpha$-stable processes, then what is $L_t = (L_t^1, \cdots, L_t^d)$?
Non-degenerate $\alpha$-stable Lévy measures

Example 5 (Cylindrical $\alpha$-stable Lévy measures)

If $\Sigma = \sum_{k=1}^{d} \delta_{e_k}$, where $e_k = (0, \cdots, 1_{k^{th}}, \cdots, 0)$, then

$$\nu^{(\alpha)}(dx) = \sum_{k=1}^{d} \delta_{e_k}(dx) \frac{dx_k}{|x_k|^{\alpha+1}},$$

called cylindrical $\alpha$-stable Lévy measures. Moreover, this Lévy measure is associated with a $d$-dimensional Lévy process $(L_t^1, L_t^2, \cdots, L_t^d)$, where $L_t^1, L_t^2, \cdots, L_t^d$ are i.d.d $1$-dimensional standard $\alpha$-stable processes.
Infinitesimal generators

- Standard $\alpha$-stable Lévy process with $\alpha \in (0, 1)$:
  \[
  \mathcal{L} f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x + z) - f(x)}{|z|^{d+\alpha}} \, dz = \Delta^{\alpha/2} f(x).
  \]

- Cylindrical $\alpha$-stable Lévy process with $\alpha \in (0, 1)$:
  \[
  \mathcal{L} f(x) = \sum_{i=1}^d \text{p.v.} \int_{\mathbb{R}} \frac{f(x + z e_i) - f(x)}{|z|^{1+\alpha}} \, dz,
  \]
  where $e_i = (0, \ldots, 1_{i_{th}}, \ldots, 0)$.
Fourier’s multipliers

- **Fourier’s multipliers**

  - Standard $\alpha$-stable Lévy process:
    \[
    \mathcal{F}(\mathcal{L} f)(\xi) = |\xi|^\alpha \mathcal{F} f(\xi) = \mathcal{F}(\Delta^{\frac{\alpha}{2}} f)(\xi),
    \]
    where $\phi(\xi) := |\xi|^\alpha \in C^\infty(\mathbb{R}^d \setminus \{0\})$.

  - Cylindrical $\alpha$-stable Lévy process:
    \[
    \mathcal{F}(\mathcal{L} f)(\xi) = \sum_{i=1}^{d} |\xi_i|^\alpha \mathcal{F} f(\xi),
    \]
    where $\phi(\xi) := \sum_{i=1}^{d} |\xi_i|^\alpha \in C^\infty(\mathbb{R}^d \setminus \bigcup_{i=1}^{d} \{x_i = 0\})$.

  - **Note** It is more difficult to deal with cylindrical Lévy measures than standard Lévy measures.
Our work

- We want to show **Schauder’s estimates** for the following nonlocal equations:

\[
\partial_t u = \mathcal{L}_{\kappa,\sigma}^{(\alpha)} u + b \cdot \nabla u + f, \quad u(0) = 0,
\]  

(2.1)

where \( \mathcal{L}_{\kappa,\sigma}^{(\alpha)} \) is an \( \alpha \)-stable-like operator with the form:

\[
\mathcal{L}_{\kappa,\sigma}^{(\alpha)} u(t, x) := \int_{\mathbb{R}^d} (u(t, x + \sigma(t, x) z) - u(t, x))
\]

\[
- \sigma(t, x) z^{(\alpha)} \cdot \nabla u \kappa(t, x, z) \nu^{(\alpha)}(dz),
\]

where \( \nu^{(\alpha)} \) is a non-degenerate \( \alpha \)-stabe Lévy measure and \( z^{(\alpha)} := z 1_{\{z \leq 1\}} 1_{\alpha = 1} + z 1_{\alpha \in (1, 2)} \).

- **Schauder’s estimates:**

\[
\|u\|_{C^{\alpha+\beta}} \leq c \|f\|_{C^\beta}.
\]

**PDE** Schauder’s estimates play a basic role in constructing classical solutions for quasilinear PDEs.

**SDE** The Schauder estimate can be used to solve the existence and uniqueness of the solution for SDE. (The Zvonkin transform)
Supercritical Case: $\alpha \in (0, 1)$

- **Supercritical case:** If $\alpha \in (0, 1)$, then

$$\partial_t u = \mathcal{L}^{(\alpha)}_{\kappa, \sigma} u + b \cdot \nabla u + f, \quad u(0) = 0,$$

with

$$\mathcal{L}^{(\alpha)}_{\kappa, \sigma} u(t, x) := \int_{\mathbb{R}^d} \left( u(t, x + \sigma(t, x)z) - u(t, x) \right) \kappa(t, x, z) \nu^{(\alpha)}(dz).$$

When $\alpha \in (0, 1)$, the drift term will play the important role instead of the diffusion term.
Previous results

2009  (Bass) Consider the elliptic equation $L_\alpha u = f$, where $\alpha \in (0, 2)$ and

$$L_\alpha u = \int_{\mathbb{R}^d} (u(x + z) - u(x) - z1_{|z| \leq 1}1_{\alpha \in [1, 2)} \cdot \nabla u(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} \, dz.$$ 

2012  (Silvestrei) Consider the following parabolic equation:

$$\partial_t u + b \cdot \nabla u + (-\Delta)^{\alpha/2} = f,$$

where $\alpha \in (0, 1)$ and $b$ is bounded but not necessarily divergence free.

2018  (Zhang, Zhao) Consider

$$\partial_t u = L_\alpha u + b \cdot \nabla u + f,$$

where $b$ is bounded globally Hölder function and

$$L_\alpha u = \int_{\mathbb{R}^d} (u(x + z) - u(x) - z^{(\alpha)} \cdot \nabla u(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} \, dz$$

with $\alpha \in (0, 2)$ and $z^{(\alpha)} = z1_{|z| \leq 1}1_{\alpha = 1} + z1_{\alpha \in (1, 2)}$.

2019  (Chaudru de Raynal, Menozzi, Priola) Consider

$$\partial_t u + L_\alpha u + b \cdot \nabla u = -f, u(T) = g,$$

where $b$ is unbounded local Hölder function and

$$L_\alpha u = \int_{\mathbb{R}^d} (u(x + z) - u(x)) \nu(dz)$$

with singular Lévy measure $\nu$ and $\alpha \in (1/2, 1)$. 

References:
Assumptions on diffusion coefficients

Recall

\[ \partial_t u = \mathcal{L}^{(\alpha)}_{\kappa,\sigma} u + b \cdot \nabla u + f, \quad u(0) = 0, \tag{2.2} \]

where

\[ \mathcal{L}^{(\alpha)}_{\kappa,\sigma}(t, x) := \int_{\mathbb{R}^d} (u(t, x + \sigma(t, x)z) - u(t, x) - \sigma(t, x)z^{(\alpha)} \cdot \nabla u) \kappa(t, x, z) \nu^{(\alpha)}(dz). \]

\textbf{(H}^{\beta}_{\kappa}) \text{ For some } c_0 \geq 1 \text{ and } \beta \in (0, 1), \text{ it holds that for all } x, z \in \mathbb{R}^d,

\[ c_0^{-1} \leq \kappa(t, x, z) \leq c_0, \quad \left[ \kappa(t, \cdot) \right]_{C^\beta} := \sup_{h} \frac{\| \kappa(t, \cdot + h, z) - \kappa(t, \cdot, z) \|_{\infty}}{|h|^{\beta}} \leq c_0, \]

and in the case of \( \alpha = 1, \)

\[ \int_{r \leq |z| \leq R} z \kappa(t, x, z) \nu^{(\alpha)}(dz) = 0 \text{ for every } 0 < r < R < \infty. \]

\textbf{(H}^{\gamma}_{\sigma}) \text{ For some } c_0 \geq 1 \text{ and } \gamma \in (0, 1], \text{ it holds that for all } x, \xi \in \mathbb{R}^d,

\[ c_0^{-1} |\xi|^2 \leq \xi^T \sigma(t, x) \xi \leq c_0 |\xi|^2, \quad |\sigma(t, x) - \sigma(t, y)| \leq c_0 |x - y|^\gamma. \]
Assumptions on drift coefficients

\((H_{b}^{\beta})\) For some \(c_0 \geq 1\) and \(\beta \in (0, 1)\), it holds that for all \(t \in \mathbb{R}\),

\[
|b(t, 0)| \leq c_0, \quad [b(t, \cdot)]_{C^{\beta}} := \sup_{0 < |h| \leq 1} \frac{\|b(t, \cdot + h) - b(t, \cdot)\|_{\infty}}{|h|^\beta} \leq c_0.
\]

That is the Local H"older regularity.

- Here, \(b\) is not necessarily bounded in \(x\). For example, \(b(x) = x\) satisfies

\[
[b]_{C^{s}} < \infty, \quad \forall s \in (0, 1).
\]

It is related to the nonlocal Ornstein-Uhlenbeck operator:

\[
\Delta^{\alpha/2} - x \cdot \nabla.
\]

- For any fixed \(x\), function \(t \to b(t, x)\) is bounded because

\[
|b(t, x) - b(t, y)| \leq [b(t, \cdot)]_{C^{\beta}} |x - y| \mathbf{1}_{\{|x-y| > 1\}}
+ [b(t, \cdot)]_{C^{\beta}} |x - y|^\beta \mathbf{1}_{\{|x-y| \leq 1\}}. \tag{2.3}
\]
Littlewood-Paley Decomposition

- Let $\phi_0$ be a radial $C^\infty$-function on $\mathbb{R}^d$ with
  \[ \phi_0(\xi) = 1 \text{ for } \xi \in B_1 \text{ and } \phi_0(\xi) = 0 \text{ for } \xi \notin B_2. \]
  
  For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define
  \[ \phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi). \]

- It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{-(j-1)}\xi) \geq 0$ and
  \[ \text{supp} \phi_j \subset B_{2j+1} \setminus B_{2j-1}, \quad \sum_{j=0}^{k} \phi_j(\xi) = \phi_0(2^{-k}\xi) \to 1, \quad k \to \infty. \]
Littlewood-Paley Decomposition

- If $|j - j'| \geq 2$, then $\text{supp}(\phi_1(2^{-j} \cdot)) \cap \text{supp}(\phi_1(2^{-j'} \cdot)) = \emptyset$. 
Block operators

- For $j \in \mathbb{N}_0$, the block operator $\Delta_j$ is defined on $\mathcal{S}'(\mathbb{R}^d)$ by
  \[
  \Delta_j f(x) := (\phi_j \hat{f})(x) = \tilde{\phi}_j * f(x) = 2^{j-1} \int_{\mathbb{R}^d} \tilde{\phi}_1(2^{j-1}y)f(x - y)dy.
  \]

- For $j \in \mathbb{N}_0$, by definition it is easy to see that
  \[
  \Delta_j = \Delta_j \tilde{\Delta}_j, \quad \text{where} \quad \tilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1} \quad \text{with} \quad \Delta_{-1} \equiv 0,
  \]
  and $\Delta_j$ is symmetric in the sense that
  \[
  \langle \Delta_j f, g \rangle = \langle f, \Delta_j g \rangle.
  \]

- Noticing that
  \[
  \sum_{j=0}^{k} \Delta_j f = 2^{dk} \int_{\mathbb{R}^d} \tilde{\phi}_0(2^k (x - y)) f(y)dy \to f,
  \]
  we have the Littlewood-Paley decomposition of $f$:
  \[
  f = \sum_{j=0}^{\infty} \Delta_j f.
  \]
Then, we have the following definition.

**Definition 6 (Besov spaces)**

For $s \in \mathbb{R}$, the Besov space $B_{\infty, \infty}^s$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$
\|f\|_{B_{\infty, \infty}^s} := \sup_{j \geq -1} 2^j s \|\Delta_j f\|_{\infty} < \infty.
$$

**Proposition 7 (H. Triebel)**

For any $0 < s / t \in \mathbb{N}_0$, it holds that

$$
\|f\|_{B_{\infty, \infty}^s} \asymp \|f\|_{C^s},
$$

where $C^s$ is the usual Hölder space. For $n \in \mathbb{N}$, we have $C^n \subset B_{\infty, \infty}^n$.

**Proposition 8 (Interpolation inequalities)**

For any $0 < s < t$, there is a constant $c > 0$ such that for any $\varepsilon \in (0, 1)$,

$$
\|f\|_{B_{\infty, \infty}^s} \leq \|f\|_{B_{\infty, \infty}^s}^{s/t} \|f\|_{B_{\infty, \infty}^{(t-s)/t}} \leq \varepsilon \|f\|_{B_{\infty, \infty}^t} + c \varepsilon^{-s/(t-s)} \|f\|_{\infty}.
$$
Part 2: Main results
**Definition 9 (Classical solutions)**

We call a bounded continuous function $u$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ a classical solution of PDE (2.2) if for some $\varepsilon \in (0, 1)$,

$$u \in \cap_{T > 0} L^\infty([0, T]; C^{1+\varepsilon})$$

with $\nabla u(\cdot, x) \in C([0, \infty))$ for any $x \in \mathbb{R}^d$, and for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$,

$$u(t, x) = \int_0^t \left(\mathcal{L}_{\kappa, \sigma}^{(\alpha)} u + b \cdot \nabla u + f\right)(s, x)ds.$$

**Lemma 10 (Maximum principles)**

Assume that $\sigma(t, x)$ and $\kappa(t, x, z) \geq 0$ are bounded measurable functions. Let $b(t, x)$ be measurable function and bounded in $t \in \mathbb{R}_+$ for any fixed $x \in \mathbb{R}^d$. For any $T > 0$ and classical solution $u$ of (2.2) in the sense of Definitions 9, it holds that

$$\|u\|_{L^\infty([0, T])} \leq T\|f\|_{L^\infty([0, T])}.$$
Theorem 11 (Schauder’s estimates)

Suppose that $\gamma \in (0,1]$, $\alpha \in (1/2,2)$ and $\beta \in ((1 - \alpha) \vee 0, (\alpha \wedge 1)\gamma)$. Under conditions $(H^{\beta}_{\kappa})$, $(H^{\gamma}_{\sigma})$, and $(H^{\beta}_{\nu})$, for any $T > 0$, there is a constant $c = c(T, c_0, d, \alpha, \beta, \gamma) > 0$ such that for any classical solution $u$ of PDE (2.2),

$$
\|u\|_{L^\infty([0,T],\mathcal{C}^{\alpha+\beta})} \leq c\|f\|_{L^\infty([0,T],\mathcal{C}^{\beta})}.
$$

- Since we consider classical solutions, $\alpha + \beta$ must be larger than 1 such that $\nabla u$ is meaningful. In addition, we must assume $\beta < \alpha$ for the moment problem. Thus, $1 - \alpha < \beta < \alpha$.
- The critical case $\alpha + \beta = 1$ is a technical problem. We have no ideas to fix it.
Existences

**Theorem 12 (Existences)**

Suppose that \( \gamma \in (0, 1] \), \( \alpha \in (1/2, 2) \) and \( \beta \in ((1 - \alpha) \vee 0, (\alpha \wedge 1) \gamma) \). Under conditions \((H_\kappa^b), (H_\sigma^\gamma), \text{ and } (H_b^\beta)\), for any \( f \in \cap_{T > 0} L^\infty([0, T], C^\beta) \), there is a unique classical solution \( u \) for (2.2) in the sense of Definition 9 such that for any \( T > 0 \) and some constant \( c > 0 \),

\[
\|u\|_{L^\infty([0,T], C^{\alpha+\beta})} \leq c \|f\|_{L^\infty([0,T], C^\beta)}, \quad \|u\|_{L^\infty(0,T)} \leq c \|f\|_{L^\infty(0,T)}.
\]
Part 3: Sketch of proofs
Main technics

**Step 1** Using perturbation argument to prove the Schauder estimate under $(H^\gamma, H^\gamma)$ and

$$[b(t, \cdot)]_{C^\beta} \leq c_0, \forall t \geq 0.$$

- Freeze the coefficients along the characterization curve.
- Use Duhamel’s formulas. (Heat kernel estimates of integral form, Littlewood-Paley’s decomposition, and interpolation inequalities.)

**Step 2** Using cut-off technics to obtain the desired Schauder estimate.

**Step 3** By Schauder’s estimates, using the continuity method and the vanishing viscosity approach to get existences of the classical solutions.
The characterization curve

- Fix \( x_0 \in \mathbb{R}^d \). Let \( \theta_t \) solve the following ODE in \( \mathbb{R}^d \):
  \[
  \dot{\theta}_t = -b(t, \theta_t), \quad \theta_0 = x_0.
  \]

Define
  \[
  \tilde{u}(t, x) := u(t, x + \theta_t), \quad \tilde{f}(t, x) := f(t, x + \theta_t), \quad \tilde{\sigma}(t, x) := \sigma(t, x + \theta_t),
  \]
  \[
  \tilde{\kappa}(t, x, z) := \kappa(t, x + \theta_t, z), \quad \tilde{\sigma}_0(t) := \tilde{\sigma}(t, 0), \quad \tilde{\kappa}_0(t, z) := \tilde{\kappa}(t, 0, z).
  \]

and
  \[
  \tilde{b}(t, x) := b(t, x + \theta_t) - b(t, \theta_t).
  \]

It is easy to see that \( \tilde{u} \) satisfies the following equation:
  \[
  \partial_t \tilde{u} = \mathcal{L}^{(\alpha)}_{\tilde{\kappa}_0, \tilde{\sigma}_0} u + \tilde{b} \cdot \nabla \tilde{u} + \left( \mathcal{L}^{(\alpha)}_{\tilde{\kappa}, \tilde{\sigma}} - \mathcal{L}^{(\alpha)}_{\tilde{\kappa}_0, \tilde{\sigma}_0} \right) \tilde{u} + \tilde{f}.
  \]
Heat kernels $p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)$

- For the case of $\alpha \in (0, 1)$. Let $N(dt, dz)$ be the Possion random measure with intensity measure
  
  $$\tilde{\kappa}_0(t, z)\nu^{(\alpha)}(dz)dt.$$  
  
  For $0 \leq s \leq t$, define
  
  $$X_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0} := \int_s^t \int_{\mathbb{R}^d} \tilde{\sigma}_0(r)zN(dr, dz),$$  
  
  whose generator is $\mathcal{L}_{\tilde{\kappa}_0,\tilde{\sigma}_0}^{(\alpha)}$.

- Under our conditions, the random variable $X_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}$ has a smooth density $p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)$. Moreover, for any $\beta \in [0, \alpha)$ and $m \in \mathbb{N}_0$, there exists a positive constant $c$ such that for all $0 \leq s < t$,
  
  $$\int_{\mathbb{R}^d} |x|^\beta |\nabla^m p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)|dx \leq c(t - s)^{(\beta-m)/\alpha}, \quad (5.1)$$  
  
  where the constant $c$ depends on $d, m, c_0, \nu^{(\alpha)}, \beta, \alpha$. 
Duhamel’s formulas

By Duhamel’s formula and operating the block operator $\Delta_j$ on both sides, we have

$$\Delta_j \tilde{u}(t, x) = \int_0^t \Delta_j P_{s,t} \left( \mathcal{L}_{\tilde{\kappa}, \tilde{\sigma}}^{(\alpha)} - \mathcal{L}_{\tilde{\kappa}_0, \tilde{\sigma}_0}^{(\alpha)} \right) \tilde{u}(s, x) \, ds$$

$$+ \int_0^t \Delta_j P_{s,t} (\tilde{b} \cdot \nabla \tilde{u})(s, x) \, ds + \int_0^t \Delta_j P_{s,t} \tilde{f}(s, x) \, ds,$$

where

$$\int_0^t P_{s,t} f(s, x) \, ds = \int_0^t \int_{\mathbb{R}^d} P_{\tilde{\kappa}_0, \tilde{\sigma}_0}^{(\alpha)}(y) f(s, x + y) \, dy \, ds.$$
Convolution to Product

- Recall $\|u(t)\|_{B^\alpha_\infty, \infty} = \sup_{j \geq 0} 2^{\alpha j} \|\Delta_j u(t)\|_{\infty}$.

- Noticing

$$|\Delta_j u(t, \theta_t)\| = |\Delta_j \tilde{u}(t, 0)|,$$

we get $\|\Delta_j u(t)\|_{\infty}$ by taking the supremum of the $\theta_t$’s initial data $x_0$ for $|\Delta_j \tilde{u}(t, 0)|$.

- Therefore, we only need to consider values at the origin point:

$$\Delta_j \tilde{u}(t, 0) = \int_0^t \Delta_j P_{s,t} \left( \mathcal{L}^{(\alpha)}_{\tilde{\kappa}, \tilde{\sigma}} - \mathcal{L}^{(\alpha)}_{\tilde{\kappa}_0, \tilde{\sigma}_0} \right) \tilde{u}(s, 0) ds$$

$$+ \int_0^t \Delta_j P_{s,t} (\tilde{b} \cdot \nabla \tilde{u})(s, 0) ds + \int_0^t \Delta_j P_{s,t} \tilde{f}(s, 0) ds.$$  \hspace{1cm} (5.4)

Here,

$$\int_0^t \Delta_j P_{s,t} f(s, 0) ds = \int_0^t \int_{\mathbb{R}^d} p_{\tilde{\kappa}_0, \tilde{\sigma}_0}(y) \cdot \Delta_j f(s, y) dy ds.$$  \hspace{1cm} (5.5)

**Convolution $\implies$ Product**
A crucial Lemma for heat kernels

**Lemma 13 (Heat kernel estimates)**

For any $T > 0$, $\beta \in [0, \alpha)$, and $n \in \mathbb{N}_0$, there is a constant $c = (d, m, c_0, \nu(\alpha), \beta, \alpha)$ such that for any $s, t \in [0, T]$ and $j \in \mathbb{N},$

$$
\int_{\mathbb{R}^d} |x|^{\beta} |\Delta_j p_{s,t}^{\tilde{\kappa}_0, \tilde{\sigma}_0}(x)| \, dx \leq c 2^{(m-n)j} (t - s)^{-\frac{n}{\alpha}} ((t - s)^{\frac{1}{\alpha}} + 2^{-j})^\beta,
$$

(5.6)

and

$$
\int_0^t \int_{\mathbb{R}^d} |x|^{\beta} |\Delta_j p_{s,t}^{\tilde{\kappa}_0, \tilde{\sigma}_0}(x)| \, dx \, ds \leq c 2^{-(\alpha+\beta)j}.
$$

(5.7)

- **Note** We use the regularity of time to get the regularity of space.

- Under $(H_{\kappa}^{\beta})$, $(H_{\sigma}^{\beta})$ and $(H_{b}^{\beta})$, we have

$$
\begin{align*}
|\tilde{\kappa}(t, x, z) - \tilde{\kappa}_0(t, z)| &\leq [\kappa(t, \cdot, z)]_{C^\beta} |x|^{\beta} \leq c_0 |x|^{\beta}; \\
|\tilde{\sigma}(t, x) - \tilde{\sigma}_0(t)| &\leq [\sigma(t, \cdot)]_{C^\gamma} |x|^{\gamma} \leq c_0 |x|^{\gamma}; \\
|\tilde{b}(t, x)| &\leq [b(t, \cdot)]_{C^\beta} |x|^{\beta} \leq c_0 |x|^{\beta}.
\end{align*}
$$

(5.8)
Thank you for your attention!

Thank Zimo Hao for his advices to this presentation.