WELL-POSEDNESS FOR MAGNETOVISCOELASTIC FLUIDS IN 3D

HENGRONG DU, YUANZHEN SHAO, AND GIERI SIMONETT

Abstract. We show that the system of equations describing a magnetoviscoelastic fluid in three dimensions can be cast as a quasilinear parabolic system. Using the theory of maximal $L^p$-regularity, we establish existence and uniqueness of local strong solutions and we show that each solution is smooth (in fact analytic) in space and time. Moreover, we give a complete characterization of the set of equilibria and show that solutions that start out close to a constant equilibrium exist globally and converge to a (possibly different) constant equilibrium. Finally, we show that every solution that is eventually bounded in the topology of the state space exists globally and converges to the set of equilibria.

1. Introduction

We will study the following system of equations that models the evolution of a magnetoviscoelastic fluid

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu_s \Delta u + \nabla \pi = -\nabla \cdot (\nabla m \otimes \nabla m) + \nabla \cdot (F F^T) & \text{in } \mathbb{R}_+ \times \Omega, \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
\partial_t F + u \cdot \nabla F - (\nabla u)^T F = \kappa \Delta F & \text{in } \mathbb{R}_+ \times \Omega, \\
F = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
\partial_t m + u \cdot \nabla m = -\alpha m \times (m \times \Delta m) - \beta m \times \Delta m & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_\nu m = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
|m| = 1 & \text{in } \mathbb{R}_+ \times \Omega, \\
(u(0), F(0), m(0)) = (u_0, F_0, m_0) & \text{in } \Omega.
\end{cases}
$$

(1.1)

Here, $\Omega \subset \mathbb{R}^3$ is a bounded connected $C^3$-domain with outward unit normal field $\nu$. The unknowns $(u, F, m) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^3 \times M^3 \times \mathbb{R}^3$ denote the fluid velocity, the deformation tensor field and the magnetization field, respectively, while $\pi$ is the pressure. Moreover, $M^3$ stands for the set of all $(3 \times 3)$-real matrices. The parameters $\alpha, \beta > 0$ are the so-called Gilbert damping and the exchange constant, while $\mu_s$ and $\kappa$ are the dynamic viscosity and dissipative coefficient, respectively.
The notation $\nabla m \circ \nabla m$ means $\nabla m(\nabla m)^T$. Hence, $\nabla m \circ \nabla m$ is a symmetric tensor with coefficients $[
abla m \circ \nabla m]_{ij} = \partial_i m \cdot \partial_j m$.

(1.1) is a coupled system of equations containing

- the incompressible Navier–Stokes equations for the velocity field $u$ and including in addition magnetic and elastic terms in the stress tensor,
- a transport-stretch-dissipative system for the deformation tensor $F$,
- a convected Landau–Lifshitz–Gilbert system for the magnetization field $m$.

As a multi-physical hydrodynamics model, (1.1) enjoys the following energy dissipation property:

$$\frac{d}{dt} \int_\Omega \frac{1}{2} (|u|^2 + |F|^2 + |\nabla m|^2) \, dx = - \int_\Omega \left( \mu_s |\nabla u|^2 + \kappa |\nabla F|^2 + \alpha |\Delta m + |\nabla m|^2 m|^2 \right) \, dx,$$

see Proposition 4.1.

The system (1.1) was first introduced in [3, 7]. This model can describe the motion of fluids with micromagnetic and elastic particles such as ferrofluids [2, 4] and magnetorheological fluids [24]. Existence of weak solutions in 2D was established in [3] under a smallness condition on the initial data by using a Galerkin approximation. In [12], the authors extended these results under more general assumptions on the elastic energy density. Moreover, they proved local-in-time existence of strong solutions and they established a weak-strong uniqueness property. Recently, the authors in [5] obtained global weak solutions to (1.1) with partial regularity in a 2D periodic domain by a careful blow-up analysis near singularities.

The main difficulty in constructing global weak solutions to (1.1) is caused by the lack of sufficient integrability in the a priori energy estimates for the stress tensor term $\nabla m \circ \nabla m$.

In 3D, the authors in [7, 8, 21] consider the simplified system

$$\begin{align*}
\partial_t u + u \cdot \nabla u - \mu_s \Delta u + \nabla \pi &= - \nabla \cdot (\nabla m \circ \nabla m) + \nabla \cdot (FF^T), \\
\nabla \cdot u &= 0, \\
\partial_t F + u \cdot \nabla F - (\nabla u)^T F &= \kappa \Delta F, \\
\partial_t m + u \cdot \nabla m &= \Delta m - \frac{1}{\varepsilon^2}(|m|^2 - 1)m,
\end{align*}$$

(1.2)

where the constraint $|m| \equiv 1$ is replaced by the Ginzburg–Landau penalization term $\frac{1}{2}(|m|^2 - 1)^2$.

In [7], the author adapted the approach in [14] to show the existence of weak solutions to (1.2) with the combination of a Galerkin approximation scheme and a fixed point argument. The weak-strong uniqueness of solutions to (1.2) was established in [21] under the Prodi–Serrin condition. In case the initial values have higher regularity, the authors in [8] obtained the well-posedness of strong solutions to (1.2) via a priori estimates that are uniform in the approximate solutions.

We would like to point out that the regularization term $\kappa \Delta F$ in (1.1) and (1.2) with $0 < \kappa \ll 1$ plays an important role in the mathematical analysis. If $\kappa = 0$, the evolution of the deformation tensor field becomes hyperbolic, and in this case, even in 2D, the existence of weak solutions to incompressible viscoelastic fluids ($m = 0$) with large initial data remains an open problem. Local well-posedness of
strong solutions to (1.2) without the regularization term was established in [25] in a periodic domain in two or three dimensions.

From the viewpoint of modeling, \( F = 0 \) represents a liquid phase that contains no elastic solid particles. We refer the reader to [15] for more details.

To the best of our knowledge, there are so far no existence results for system (1.1) in 3D. In our approach, we consider (1.1) as a quasilinear system and prove that the system is parabolic. We can then apply the theory of maximal \( L^p \)-regularity to establish short time existence and uniqueness of strong solutions, see Theorem 2.5.

In Sections 3 and 4, we show that the set of equilibria of (1.1) is given by
\[
E = \{(0, 0, m_*, \pi_*)\},
\]
where \( m_* \in H^2_q(\Omega, \mathbb{R}^3) \) solves the nonlinear constrained elliptic problem
\[
\begin{cases}
\Delta m_* + |\nabla m_*|^2 m_* = 0 & \text{in } \Omega, \\
|m_*| = 1 & \text{in } \Omega, \\
\partial_\nu m_* = 0 & \text{on } \partial \Omega
\end{cases}
\]
and \( \pi_* = -\frac{1}{2} |\nabla m_*|^2 + C \) for some constant \( C \).

In particular, we have that
\[
E_c := \{(0, 0, m_*, \pi_*) \in \mathbb{R}^3 \times \mathbb{M}^3 \times \mathbb{S}^2 \times \mathbb{R} \} \subset E.
\]
We call \( E_c \) the set of constant equilibria. We can prove that all constant equilibria are normally stable, and that each solution that starts out close to a constant equilibrium exists globally and converges to a (possibly different) constant equilibrium. Moreover, we show that any solution that is bounded in an appropriate topology exists globally and converges to the set \( E \) of equilibria.

In case we choose \((u_0, F_0) = (0, 0)\), system (1.1) reduces to the well-known Landau-Lifshitz-Gilbert equation
\[
\begin{cases}
\partial_t m = -\alpha m \times (m \times \Delta m) - \beta m \times \Delta m & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_\nu m = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
|m| = 1 & \text{in } \mathbb{R}_+ \times \Omega, \\
m(0) = m_0 & \text{in } \Omega.
\end{cases}
\]
In this case, we obtain the energy dissipation relation
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla m|^2 \, dx = - \int_{\Omega} \alpha |\Delta m + |\nabla m|^2 m|^2 \, dx.
\]
By the same arguments as in Section 4, we can conclude that the set of equilibria of (1.4) is given by the solutions of (1.3). Hence, all the results established for system (1.1) remain true for the Landau-Lifshitz-Gilbert equation. A similar result was obtained in [17] in case \( \Omega = \mathbb{R}^n \) with \( n \geq 3 \).

Finally, we mention that all of our results remain valid in 2D, that is, in case \( \Omega \subset \mathbb{R}^2 \) and \((u, F, m) : \Omega \to \mathbb{R}^2 \times \mathbb{M}^2 \times \mathbb{R}^3\).

**Notation:** For the readers’ convenience, we list here some notation and conventions used throughout the manuscript.

In the following, all vectors \( a = (a_1, \cdots, a_n) \in \mathbb{R}^n \) are viewed as column vectors. For two vectors \( a, b \in \mathbb{R}^n \), the Euclidean inner product is denoted by \( a \cdot b \). Given
two matrices \( A, B \in \mathbb{M}^n \), the Frobenius matrix inner product \( A : B \) is given by
\[
A : B = \text{Tr}(AB^T),
\]
where \( ^T \) is the transpose. Suppose \( \Omega \) is an open subset of \( \mathbb{R}^n \). If \( u \in C^1(\Omega; \mathbb{R}^n) \), we set \( \nabla u(x) = e_j \otimes \partial_j u(x) \) for \( x \in \Omega \). Hence, for \( u = (u_1, \ldots, u_n) \in C^1(\Omega; \mathbb{R}^n) \), we have
\[
[\nabla u(x)]_{ij} = \partial_{ij} u(x), \quad 1 \leq i, j \leq n, \quad x \in \Omega. \tag{1.5}
\]
We note that \( [\nabla u(x)]^T \) corresponds to the Fréchet derivative of \( u \) at \( x \in \Omega \).

If \( A = [a_{ij}] \in C^1(\Omega; \mathbb{M}^n) \), its divergence \( \nabla \cdot A \) is the vector function defined by
\[
(\nabla \cdot A)(x) = (\partial_j A(x))^T e_j, \quad x \in \Omega. \tag{1.6}
\]
Hence, if \( A = [a_{ij}] \in C^1(\Omega; \mathbb{M}^n) \), its divergence is given by
\[
[(\nabla \cdot A)(x)]_i = \partial_j a_{ji}(x), \quad i = 1, \ldots, n, \quad x \in \Omega.
\]
Here and in the sequel, we use the summation convention, indicating that terms with repeated indices are added. We note that (1.5) and (1.6) imply
\[
\nabla \cdot (\nabla u) = \Delta u, \quad u \in C^2(\Omega; \mathbb{R}^n),
\]
and
\[
(\nabla \cdot A) \cdot u = \nabla \cdot (Au) - A \cdot \nabla u, \quad A \in C^1(\Omega; \mathbb{M}^n), \quad u \in C^1(\Omega; \mathbb{R}^n). \tag{1.7}
\]
For a matrix \( A = [a_{ij}] \in C^1(\Omega; \mathbb{M}^n) \), we set \( |\nabla A|^2 = \partial_j A : \partial_j A \).

For functions \( f, g \in L_2(\Omega; \mathbb{R}^m) \),
\[
(f|g)_{\Omega} = \int_{\Omega} f \cdot g \, dx
\]
denotes the \( L_2 \)-inner product. For any Banach space \( X \), \( s \geq 0, \ p, q \in (1, \infty) \), \( B^s_{pq}(\Omega; X) \) denote the \( X \)-valued Besov spaces, whereas \( H^s_q(\Omega; X) \) are the Bessel-potential spaces. When the choice of \( X \) is clear from the context, we will just write \( B^s_{pq}(\Omega) \) or \( H^s_q(\Omega) \).

For \( p \in (1, \infty) \) and \( \mu \in (0, 1) \), the \( X \)-valued \( L_p \)-spaces with temporal weight are defined by
\[
L_{p,\mu}((0, T); X) := \{ f : (0, T) \to X : t^{1-\mu} f(t) \in L_p((0, T); X) \}.
\]
Similarly,
\[
H^1_{p,\mu}((0, T); X) := \{ f \in L_{p,\mu}((0, T); X) \cap H^1((0, T); X) : f'(t) \in L_{p,\mu}((0, T); X) \}.
\]
For any two Banach spaces \( X \) and \( Y \), the notation \( \mathcal{L}(X, Y) \) stands for the set of all bounded linear maps from \( X \) to \( Y \) and \( \mathcal{L}(X) := \mathcal{L}(X, X) \).

2. Existence and uniqueness of solutions

In this section, we show how to formulate system (1.1) as a quasilinear equation. Using the theory of maximal \( L_p \)-regularity, we establish existence and uniqueness of local in time solutions, and we show that solutions have additional time regularity. We start by expressing the term
\[
\alpha m \times (m \times \Delta m) + \beta m \times \Delta m
\]
in a form that is more convenient for our analysis. By the well-known identity \( a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \), we have
\[
m \times (m \times \Delta m) = (m \cdot \Delta m)m - |m|^2 \Delta m.
\]
By using the facts that \(|m| = 1\) and \(0 = \Delta|m|^2 = 2|\nabla m|^2 + 2m \cdot \Delta m\), we obtain
\[
m \times (m \times \Delta m) = -(\Delta m + |\nabla m|^2 m), \tag{2.1}
\]
provided \(m\) is sufficiently smooth. Setting
\[
M(m) = \begin{bmatrix}
0 & -m_3 & m_2 \\
m_3 & 0 & -m_1 \\
-m_2 & m_1 & 0
\end{bmatrix}, \quad m = (m_1, m_2, m_3),
\]
we can write \(m \times \Delta m = M(m) \Delta m\). Hence, under the constraint \(|m| \equiv 1\), (1.1) is equivalent to the following system
\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu_s \Delta u + \nabla \pi = -\nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (FF^T) & \text{in } \mathbb{R}_+ \times \Omega, \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\n\partial_t F - \kappa \Delta F = (\nabla u)^T F - u \cdot \nabla F & \text{in } \mathbb{R}_+ \times \Omega, \\
F = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\n\partial_t m + u \cdot \nabla m = (\alpha I_3 - \beta M(m)) \Delta m + \alpha |\nabla m|^2 m & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_{\nu} m = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
|m| = 1 & \text{in } \mathbb{R}_+ \times \Omega, \\
(u(0), F(0), m(0)) = (u_0, F_0, m_0) & \text{in } \Omega,
\end{cases}
\tag{2.2}
\]
where \(I_3\) is the \(3 \times 3\) identity matrix. Neglecting the constraint \(|m| \equiv 1\), we have
\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu_s \Delta u + \nabla \pi = -\nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (FF^T) & \text{in } \mathbb{R}_+ \times \Omega, \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\n\partial_t F - \kappa \Delta F = (\nabla u)^T F - u \cdot \nabla F & \text{in } \mathbb{R}_+ \times \Omega, \\
F = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\n\partial_t m + u \cdot \nabla m = (\alpha I_3 - \beta M(m)) \Delta m + \alpha |\nabla m|^2 m & \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_{\nu} m = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
(u(0), F(0), m(0)) = (u_0, F_0, m_0) & \text{in } \Omega.
\end{cases}
\tag{2.3}
\]
We will first study the unconstrained system (2.3), and then show in a second step that the constraint \(|m| \equiv 1\) is preserved in case \(|m_0| \equiv 1\).

The main tool to study (2.3) is the theory of maximal \(L_p\)-regularity. For \(\theta \in (0, \pi]\), the open sector with angle \(2\theta\) is denoted by
\[
\Sigma_\theta := \{\omega \in \mathbb{C} \setminus \{0\} : |\arg \omega| < \theta\}.
\]

**Definition 2.1.** Let \(X\) be a complex Banach space, and \(A\) be a densely defined closed linear operator in \(X\) with dense range. \(A\) is called sectorial if \(\Sigma_\theta \subset \rho(-A)\) for some \(\theta > 0\) and
\[
\sup \{\|\lambda(\lambda + A)^{-1}\|_{L(X)} : \lambda \in \Sigma_\theta\} < \infty.
\]

The class of sectorial operators in \(X\) is denoted by \(\mathcal{S}(X)\).
To introduce the notion of maximal $L_p$-regularity, let us consider the following abstract Cauchy problem on $[0,T]$
\[
\begin{cases}
\partial_t u(t) + Au(t) = f(t), & t \in (0,T), \\
u(0) = 0.
\end{cases}
\] (2.4)

**Definition 2.2.** Assume that $X_1 \xrightarrow{d} X_0$ is some densely embedded Banach couple. Suppose that $\mathcal{A} \in \mathcal{S}(X_0)$ with $D(\mathcal{A}) = X_1$. The operator $\mathcal{A}$ is said to have the property of maximal $L_p$-regularity if for any fixed $T > 0$ and
\[
f \in L_p((0,T); X_0),
\]
(2.4) has a unique solution
\[
u \in L_p((0,T); X_1) \cap H^1_p((0,T); X_0).
\]
We denote the set of all operators $\mathcal{A} \in \mathcal{S}(X)$ which enjoy the property of maximal $L_p$-regularity by
\[
\mathcal{A} \in \mathcal{MR}_p(X_1, X_0).
\]
We refer to [19] for additional background information.

Let $P_H : L_q(\Omega; \mathbb{R}^3) \rightarrow L_{q,\sigma}(\Omega; \mathbb{R}^3)$ be the Helmholtz projection, where
\[
L_{q,\sigma}(\Omega; \mathbb{R}^3) := P_H(L_q(\Omega; \mathbb{R}^3))
\]
is the space of all solenoidal vector fields in $L_q(\Omega; \mathbb{R}^3)$. Setting
\[
H^2_{q,\sigma}(\Omega; \mathbb{R}^3) := H^2_q(\Omega; \mathbb{R}^3) \cap L_{q,\sigma}(\Omega; \mathbb{R}^3),
\]
we let $\mathcal{A}_q : D(\mathcal{A}_q) \rightarrow L_{q,\sigma}(\Omega; \mathbb{R}^3)$ be the Stokes operator, defined by
\[
\mathcal{A}_q u := -\mu_s P_H \Delta u, \quad D(\mathcal{A}_q) := \{u \in H^2_{q,\sigma}(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \partial \Omega\}.
\]

Similarly, we can define $\mathcal{G}_q : D(\mathcal{G}_q) \rightarrow L_q(\Omega; \mathbb{M}^3)$ by
\[
\mathcal{G}_q F := -\kappa \Delta F, \quad D(\mathcal{G}_q) := \{F \in H^2_q(\Omega; \mathbb{M}^3) : F = 0 \text{ on } \partial \Omega\}.
\]

Further, given any $m \in C(\overline{\Omega}; \mathbb{R}^3)$, the operator $\mathcal{D}_q(m) : D(\mathcal{D}_q(m)) \rightarrow L_q(\Omega; \mathbb{R}^3)$ is defined by
\[
\mathcal{D}_q(m) h := -(\alpha I_3 - \beta \mathcal{M}(m)) \Delta h, \quad D(\mathcal{D}_q(m)) := \{h \in H^2_q(\Omega; \mathbb{R}^3) : \partial_n h = 0 \text{ on } \partial \Omega\}.
\]

Next, we set
\[
[B_q(m)h]_i = \partial_i m \cdot \Delta h + \nabla m : \partial_i \nabla h, \quad i = 1, 2, 3.
\] (2.5)

It follows that
\[
B_q(m)m = \nabla \cdot (\nabla m \circ \nabla m) \quad \text{for each } m \in H^2_q(\Omega; \mathbb{R}^3). \tag{2.6}
\]

Note that $\mathcal{B}_q(m) \in L(D(\mathcal{D}_q(m)), L_q(\Omega; \mathbb{R}^3))$ for any $m \in C^1(\overline{\Omega}; \mathbb{R}^3)$. Finally, we define the spaces
\[
X_0 = L_{q,\sigma}(\Omega; \mathbb{R}^3) \times L_q(\Omega; \mathbb{M}^3) \times L_q(\Omega; \mathbb{R}^3)
\]
and
\[
X_1 = D(\mathcal{A}_q) \times D(\mathcal{G}_q) \times D(\mathcal{D}_q(m)).
\]
It is well known that $A_q$ and $G_q$ enjoy the property of maximal $L_p$-regularity, cf. [9,10,22] and [19, Section 6.3, Chapter 7]. To deal with $D_q(m)$ for $m \in C(\overline{\Omega}; \mathbb{R}^3)$, we set
\[
D_q(m(x), \xi) := (\alpha I_3 - \beta M(m(x)))|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^3,
\]
for the symbol of the differential operator $D_q(m)$. An easy computation shows that
\[
\sigma(D_q(m(x), \xi)) = \{\alpha, \alpha \pm i\beta|m(x)|\}, \quad x \in \Omega, \quad \xi \in \mathbb{S}^2,
\]
where $\sigma$ denotes the spectrum. Since $m \in C(\overline{\Omega}; \mathbb{R}^3)$, $D_q(m(x), \xi)$ is normally elliptic for every $x \in \overline{\Omega}$, see for instance [19, Definition 6.1.1]. By [19, Theorem 6.3.2], $D_q(m)$ has the property of maximal $L_p$-regularity. Then, the operator $A_q : X_1 \to X_0$ defined by
\[
A_q(m) = \begin{bmatrix} A_q & 0 & P_H B_q(m) \\ 0 & G_q & 0 \\ 0 & 0 & D_q(m) \end{bmatrix}
\]
(2.7)
enjoys the property of maximal $L_p$-regularity for every $m \in C^1(\overline{\Omega}; \mathbb{R}^3)$ as well, due to its upper triangular structure.

Indeed, given any $f = (f_1, f_2, f_3) \in L_p((0,T); X_0)$, we consider the system
\[
\begin{cases}
\partial_t z + A_q(m)z = f(t), & t \in (0,T), \\
z(0) = 0,
\end{cases}
\]
(2.8)
where $z = (v,G,h)$. By the maximal $L_p$-regularity property of $G_q$ and $D_q(m)$, one can find for each $m \in C(\overline{\Omega}; \mathbb{R}^3)$ a (unique) solution
\[
(g, h) \in L_p((0,T); H_q^2(\Omega; \mathbb{R}^3)) \cap H_q^1((0,T); L_q(\Omega; \mathbb{R}^3))
\]
for the system
\[
\begin{cases}
\partial_t G + G_q G = f_2 & \text{in } \Omega, \\
G = 0 & \text{on } \partial \Omega, \\
\partial_t h + D_q(m)h = f_3 & \text{in } \Omega, \\
\partial_t h = 0 & \text{on } \partial \Omega, \\
(G(0), h(0)) = (0,0).
\end{cases}
\]
Easy computations show that $B_q(m)h \in L_p((0,T); L_q(\Omega; \mathbb{R}^3))$ for $m \in C^1(\overline{\Omega}; \mathbb{R}^3)$. From the maximal $L_p$-regularity property of $A_q$, we thus infer that there exists a (unique) vector $v \in L_p((0,T); H_q^1(\Omega; \mathbb{R}^3)) \cap H_q^1((0,T); L_q(\Omega; \mathbb{R}^3))$ that solves
\[
\begin{cases}
\partial_t v + A_q v = -P_H B_q(m)h + f_1 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega, \\
v(0) = 0.
\end{cases}
\]
Hence $(v,G,h)$ is the unique solution of (2.8). This shows that
\[
A_q(m) \in \mathcal{MR}_p(X_1, X_0) \quad \text{for each } m \in C^1(\overline{\Omega}; \mathbb{R}^3).
\]
In addition, we define for $z = (u,F,m)$
\[
G(z) := \left( P_H \left[ \nabla \cdot (FF^T) - u \cdot \nabla u \right], (\nabla u)^T F - u \cdot \nabla F, \alpha |\nabla m|^2 m - u \cdot \nabla m \right).
\]
(2.10)
Given any $1 < p, q < \infty$, $T > 0$ and $\mu \in (1/p, 1]$, we set
\[
E_{0,\mu}(T) := L_{p,\mu}((0,T); X_0) \quad \text{and} \quad E_{1,\mu}(T) := L_{p,\mu}((0,T); X_1) \cap H_{p,\mu}^1((0,T); X_0).
\]
It is well known that
\[ E_{1,\mu}(T) \hookrightarrow C([0,T];X_{\gamma,\mu}) \quad \text{where} \quad X_{\gamma,\mu} := (X_0, X_1)_{\mu^{-1}/p,p}. \]
See [18], or [19, Theorem 3.4.8]. Observe that by [1, Theorem 3.4] and [23, Theorem 4.3.3], the triple \((u, F, m) \in B_{\mu^{-2}/p}^2(\Omega; \mathbb{R}^{15})\) belongs to \(X_{\gamma,\mu}\) iff
\[
\begin{align*}
&u \in B_{\mu^{-2}/p}^2(\Omega; \mathbb{R}^3) \quad \text{and} \quad u = 0 \text{ on } \partial\Omega, \\
&F \in B_{\mu^{-2}/p}^2(\Omega; \mathbb{M}^3) \quad \text{and} \quad F = 0 \text{ on } \partial\Omega, \\
&m \in B_{\mu^{-2}/p}^2(\Omega; \mathbb{R}^3) \quad \text{and} \quad \partial_{\nu} m = 0 \text{ on } \partial\Omega,
\end{align*}
\] (2.11)
where \(B_{\mu^{-2}/p}^2(\Omega; \mathbb{R}^3) := B_{\mu^{-2}/p}^2(\Omega; \mathbb{R}^3) \cap L_{q,\sigma}(\Omega; \mathbb{R}^3)\). In order for \(\partial_{\nu} m\) to be defined, we assume that \(2\mu - 2/p - 1/q > 1\).

One readily verifies that
\[ A_q \in C^\omega(X_{\gamma,\mu}, \mathcal{L}(X_1, X_0)), \quad G \in C^\omega(X_{\gamma,\mu}, X_0), \] (2.12)
with \(\omega\) being the notation for real analyticity, as long as
\[ X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega}; \mathbb{R}^{15}). \]
The above embedding holds whenever \(\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]\).

By the definitions (2.5), (2.7) and (2.10), and the relation (2.6), one sees that system (2.3) can be recast as the abstract evolutionary system
\[ \partial_t z + A_q(m) z = G(z), \quad z(0) = z_0 = (u_0, F_0, m_0). \] (2.13)
We have the following result on existence and uniqueness of solutions of (2.13).

**Proposition 2.3.** Suppose \(\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]\) and let \(z_0 \in X_{\gamma,\mu}\). Then there exists \(T = T(z_0)\) such that the evolution equation (2.13) admits a unique solution \(z \in E_{1,\mu}(T)\). Each solution can be extended to a maximal existence interval \([0, T_+(z_0))\) in the sense that
\[\begin{itemize}
\item[(i)] either \(T_+(z_0) = \infty\) or
\item[(ii)] \(\lim_{t \to T_+(z_0)} z(t)\) does not exist in \(X_{\gamma,\mu}\).
\end{itemize}\]
Moreover, \(z\) enjoys the additional regularity properties
\[ z \in C([0, T_+); X_{\gamma,\mu}) \cap C^\omega((0, T_+); X_1) \cap C^\omega((0, T_+) \times \Omega; \mathbb{R}^{15}). \] (2.14)

**Proof.** The existence, uniqueness and time regularity follow from (2.9), (2.12) and [19, Theorems 5.1.1 and 5.2.1 and Corollary 5.1.2], see also [13, Theorem 2.1]. The joint space-time regularity (2.14) can be proved by means of the parameter trick in [6], see also [19, Section 9.4.1]. \(\Box\)

Next, we show that solutions of (2.13) give rise to solutions of (2.3), and vice versa.

**Proposition 2.4.** Let \(T > 0\) be given. The following statements are equivalent:
\[\begin{itemize}
\item[(a)] (2.13) has a solution \((u, F, m) \in E_{1,\mu}(T)\).
\item[(b)] (2.3) has a solution \((u, F, m, \pi) \in E_{1,\mu}(T) \times L_{p,\mu}((0,T); \dot{H}^1_q(\Omega))\).
\end{itemize} \]
Proof. (a)$\Rightarrow$(b): Suppose $z = (u, F, m) \in E_{1,\mu}(T)$ solves (2.13) on $[0, T]$. Let
\[ v = \mu \Delta u - u \cdot \nabla u - \nabla \cdot (\nabla m \circ \nabla m) + \nabla \cdot (FF^T). \]

Then $v \in L_{p,\mu}((0, T); L_q(\Omega; \mathbb{R}^3))$. For $t \in (0, T)$, let $\nabla \psi(t) \in L_q(\Omega; \mathbb{R}^3)$ be the unique solution of
\[ (\nabla \psi(t)|\nabla \phi)_\Omega = (v(t)|\nabla \phi)_\Omega, \quad \forall \phi \in \dot{H}^1_q(\Omega), \]
where $(\cdot | \cdot)_\Omega$ is the inner product of $L_2(\Omega; \mathbb{R}^3)$ and $q'$ is the Hölder dual of $q$. Then $P_H v(t) = v(t) - \nabla \psi(t)$ by the definition of the Helmholtz projection. Let $\pi = \psi_v$. Then $\pi \in L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$, and noting that $P_H \partial \pi = \partial \pi$, we conclude that $(u, F, m, \pi)$ is a solution of (2.3) in the regularity class $E_{1,\mu}(T) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$.

(b)$\Rightarrow$(a): Suppose $(u, F, m, \pi) \in E_{1,\mu}(T) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$ solves (2.3). Applying $P_H$ to the equation governing $u$ in (2.3), it is an easy task to check that $(u, F, m) \in E_{1,\mu}(T)$ solves (2.13). \qed

We are now ready for our main result on existence and uniqueness of solutions for system (2.2), or equivalently, system (1.1).

**Theorem 2.5.** Let $p, q \in (1, \infty)$ and $\mu \in \left(\frac{1}{p} + \frac{1}{p'} + \frac{1}{q}, 1\right]$. Suppose that
\[ z_0 = (u_0, F_0, m_0) \in B^2_{2p,\sigma}(\Omega; \mathbb{R}^3) \times B^2_{2q,\sigma}(\Omega; \mathbb{M}^3) \times B^2_{2p,\sigma}(\Omega; \mathbb{R}^3) \]
satisfies the compatibility conditions $(u_0, F_0, \partial_t m_0) = 0$ on $\partial \Omega$. Then there exists a unique solution
\[ (u, F, m, \pi) \in [H^1_{p,\mu}(0, T); X_0] \times L_{p,\mu}((0, T); X_1) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega)) \]
of (2.3) for some $T = T(z_0) > 0$. Each solution can be extended to a maximal existence interval $[0, T_+(z_0))$. Moreover, $(z, \pi) = (u, F, m, \pi)$ enjoys the additional regularity
\[ z \in C([0, T_+); X_{\mu,\sigma}) \cap C^{\sigma}((0, T_+); X_1) \] and $(z, \pi) \in C^{\sigma}((0, T_+) \times \Omega; \mathbb{R}^{16})$. \quad (2.15)
If $|m_0| \equiv 1$, then the solution also satisfies
\[ |m(t)| \equiv 1, \quad t \in [0, T_+(z_0)). \quad (2.16) \]

**Proof.** The assertions in the first part of the statement follow readily from Propositions 2.3 and 2.4. It then only remains to show that the condition $|m(t)| \equiv 1$ holds for every $t \in [0, T_+(z_0))$, provided $|m_0| \equiv 1$.

Suppose then that $|m_0| \equiv 1$. Let $T \in (0, T_+(z_0))$ be fixed and set $\varphi = |m|^2 - 1$. We note that
\[ m \in C^1((0, T); H^2_q(\Omega; \mathbb{R}^3)) \]
\[ \varphi \in C((0, T]; B^2_{2q,\sigma}(\Omega)) \cap C^1((0, T); H^2_q(\Omega)) \]
Indeed, (2.15) implies that $m \in C([0, T]; B^2_{2q,\sigma}(\Omega; \mathbb{R}^3)) \cap C^1((0, T); H^2_q(\Omega))$. The condition $\mu \in \left(\frac{1}{p} + \frac{1}{p'} + \frac{1}{q}, 1\right]$ guarantees that $B^2_{2q,\sigma}(\Omega)$ and $H^2_q(\Omega)$ are Banach algebras. The asserted regularity of $\varphi$ thus holds. Taking the dot product of the equation
\[ \partial_t m + u \cdot \nabla m = \alpha (\Delta m + |\nabla m|^2 m) - \beta m \times \Delta m \]
with \( m \) and using the relations \( \partial_t |m|^2 = 2 \partial_t m \cdot m, \Delta |m|^2 = 2 \Delta m \cdot m + 2 |\nabla m|^2 \) results in
\[
\begin{cases}
\partial_t \varphi + u \cdot \nabla \varphi - \alpha \Delta \varphi - 2 \alpha |\nabla m|^2 \varphi = 0 \quad \text{in } \Omega, \\
\partial_t \varphi = 0 \quad \text{on } \partial \Omega, \\
\varphi(0) = 0.
\end{cases}
\] (2.17)

Multiplying both sides of (2.17) with \( \phi \) and integrating over \( \Omega \) yields
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \varphi^2 \, dx + \alpha \int_{\Omega} |\nabla \varphi|^2 \, dx = 2 \alpha \int_{\Omega} |\nabla m|^2 \varphi^2 \, dx, \quad t \in (0, T).
\]

As \( m \in C([0, T]; B_{ap}^{2-2p}((\Omega; \mathbb{R}^3)) \rightarrow C([0, T]; C^1(\Omega; \mathbb{R}^3)) \), we obtain the following integral inequality
\[
\frac{d}{dt} \int_{\Omega} \varphi^2(t) \, dx \leq 4 \alpha \| \nabla m(t) \|_{L_\infty}^2 \int_{\Omega} \varphi^2(t) \, dx, \quad t \in (0, T).
\]

Applying the Gronwall inequality we get
\[
\max_{0 \leq t \leq T} \int_{\Omega} \varphi^2(t) \, dx \leq \exp \left( 4 \alpha \int_0^T \| \nabla m(t) \|_{L_\infty}^2 \, dx \right) \int_{\Omega} \varphi^2(0) \, dx = 0.
\]

This implies \( \varphi \equiv 0 \) in \( Q_T = [0, T] \times \Omega \). In other words, \( |m| \equiv 1 \) in \( Q_T \). As this is true for every \( T \in (0, T_+(z_0)) \) we obtain that \( |m(t)| \equiv 1 \) for any \( t \in (0, T_+(z_0)) \). As (2.2) and (1.1) are equivalent, we have proved the assertions of the theorem. \( \square \)

**Remark 2.6.** Let \( p, q, \mu \) be as in Theorem 2.5. The assertions of Theorem 2.5, with exception of the higher time regularity stated in (2.15), still hold if we pose the nonhomogeneous boundary conditions
\[
(u, F, m) = (u_D, F_D, m_N) \quad \text{on } \partial \Omega,
\] (2.18)

where
\[
\begin{align*}
u_D &\in F_{p \mu}^{3-1/2q}((0, T); L_q(\partial \Omega; \mathbb{R}^3)) \cap L_p(\partial \Omega; \mathbb{R}^3)), \\
F_D &\in F_{p \mu}^{3-1/2q}((0, T); L_q(\partial \Omega; \mathbb{M}^3)) \cap L_p(\partial \Omega; \mathbb{M}^3)), \\
m_N &\in F_{p \mu}^{3-1/2q}((0, T); L_q(\partial \Omega; \mathbb{R}^3)) \cap L_p(\partial \Omega; \mathbb{R}^3))
\end{align*}
\]

and the initial data satisfy the compatibility conditions
\[
(u_D(0), F_D(0), m_N(0)) = (u_0, F_0, \partial_\mu m_0) \quad \text{on } \partial \Omega.
\]

See [19, Theorem 6.3.2]. Here \( F_{p \mu}^s \) are the Triebel-Lizorkin spaces with temporal weight.

### 3. Stability and Asymptotic Behavior

The last two sections are devoted to a discussion of the asymptotic behavior of solutions \( (u, F, m, \pi) \) to (1.1). In view of Proposition 2.4, the pressure \( \pi \) can be obtained from \( z = (u, F, m) \). For this reason, it suffices to restrict our attention to a solution \( z = (u, F, m) \) of (2.13).

The 3-dimensional subspace
\[
\mathcal{E}_0 := \{0\} \times \{0\} \times \mathbb{R}^3 \quad \text{of } X_1
\]
is clearly contained in the set \( \mathcal{E}_1 \) of equilibria of (2.13), where \( 0_3 \) is the \( 3 \times 3 \) matrix with zero entries. We refer to Remark 4.3(a) for more information on \( \mathcal{E}_1 \).
At each \( z_* = (0, 0_3, m_*) \in \mathcal{E}_0 \), the linearization of (2.13) is given by
\[
\partial_t z + A_* z = 0, \quad z(0) = z_0,
\] (3.1)
where \( z = (u, F, m) \) and
\[
A_* z = (-\mu_* P_h \Delta u, -\kappa \Delta F, (\beta M(m_*) - \alpha I_3) \Delta m).
\]
Since \( \Omega \) is bounded, the spectrum of \( A_* \) consists only of eigenvalues. Suppose that \( A_* z = \lambda z \) for some \( \lambda \in \mathbb{C} \). By elliptic regularity theory, we can assume that \( z \in H_0^2(\Omega; \mathbb{C}^{15}) \) for \( q \geq 2 \). Taking the inner product of \( A_* z = \lambda z \) with \( \overline{z} \), where \( \overline{z} \) denotes the complex conjugate of \( z \), direct computations lead to
\[
\text{Re} \lambda (\|u\|^2 + \|F\|^2 + \|m\|^2) = \mu_* \|\nabla u\|^2 + \kappa \|\nabla F\|^2 + \alpha \|\nabla m\|^2,
\]
which implies that \( \text{Re} \lambda \geq 0 \). Here we have used the anti-symmetry of \( M(m_*) \) to conclude that
\[
\text{Re} (M(m_*) \Delta m_{\overline{\eta}}) = 0.
\]
Indeed, as the entries of \( M(m_*) \) are constant, we obtain
\[
(M(m_*) \Delta m_{\overline{\eta}}) = (\Delta(M(m_*) m))_{\overline{\eta}} = (M(m_*) m_{\overline{\eta}})_{\overline{\eta}},
\]
where we set \( z \cdot \overline{w} = z_j \overline{w}_j \) for \( z, w \in \mathbb{C}^3 \). The anti-symmetry of \( M(m_*) \) implies
\[
(M(m_*) \Delta m_{\overline{\eta}})_{\overline{\eta}} = -(\Delta m_{\overline{\eta}} M(m_*) m)_{\overline{\eta}} = -(M(m_*) m_{\overline{\eta}})_{\overline{\eta}}.
\]
This readily yields \( \text{Re} (M(m_*) m_{\overline{\eta}})_{\overline{\eta}} = 0 \). When \( \text{Re} \lambda = 0 \), one concludes from the above that
\[
\|\nabla u\|^2 = \|\nabla F\|^2 = \|\nabla m\|^2 = 0.
\]
Combined with the boundary conditions, this shows that \( z = (u, F, m) \in \mathcal{E}_0 \). Further, we infer that \( A_* z = 0 \). Thus \( \sigma(A_*) \cap i\mathbb{R} = \{0\} \) and \( N(A_*) = \mathcal{E}_0 \).

To show \( \{0\} \) is a semi-simple eigenvalue, we will prove that \( N(A_*) = N(A_*)^2 \). Assume that \( w = (v, f, h) \in N(A_*) \). Then there exists \( z = (0, 0_3, m) \in N(A_*) \) such that \( A_* w = z \). Then by the divergence theorem, the boundary condition \( \partial_t h = 0 \) and the fact that \( m \) as well as \( m_* \) are constant,
\[
\|z\|^2 = (A_* w, z) = (\beta M(m_*) - \alpha I_3) \Delta h|m| = 0.
\]
We conclude that \( z = 0 \) and thus \( w \in N(A_*) \). This shows that \( \{0\} \) is semi-simple. As \( \mathcal{E}_0 \) is a linear space, we clearly have \( \mathcal{T}_0 \mathcal{E}_0 = N(A_*) \).

If follows from [20, Remark 2.2], see also [19, Remarks 5.3.2], that all equilibria close to \( z_* \) are contained in a manifold \( \mathcal{M} \) of dimension \( 3 = \dim(N(A_*)) \), where we used the fact that the center space \( X^c \) coincides with \( N(A_*) \) as \( \{0\} \) is semi-simple. Since the dimension of \( \mathcal{E}_0 \) is also \( 3 \), we conclude that there exists an open neighborhood \( V_* \subset X_1 \) of \( z_* \) such that \( \mathcal{M} \cap V_* = \mathcal{E}_0 \cap V_* \). Hence, the neighborhood \( V_* \) contains no other equilibria than the elements of \( \mathcal{E}_0 \), that is, \( V_* \cap \mathcal{E}_0 = V_* \cap \mathcal{E}_1 \).

We have, thus, shown that \( z_* \) is normally stable, see [19, Theorem 5.3.1] for a definition.

**Theorem 3.1.** Let \( p, q \in (1, \infty) \) and \( \mu \in \left[ \frac{1}{2} + \frac{1}{p} + \frac{1}{2q}, 1 \right] \).

Then each equilibrium \( z_* = (0, 0_3, m_*) \) with \( m_* \in \mathbb{S}^2 \) is stable in the topology of \( X_{\gamma, \mu} \). There exists \( \varepsilon > 0 \) such that any solution \((u, F, m, \pi)\) of (1.1) with initial value \( z_0 = (u_0, F_0, m_0) \in X_{\gamma, \mu} \) satisfying \( \|z_0 - z_*\|_{X_{\gamma, \mu}} \leq \varepsilon \) exists globally and converges to some \( z_\infty = (0, 0_3, m_\infty) \) with \( m_\infty \in \mathbb{S}^2 \) in the topology of \( X_{\gamma, 1} \) at an exponential rate as \( t \to \infty \).
Proof. Given an equilibrium $z_* = (0, 0_1, m_*) \in \mathcal{E}_0$, we infer from [19, Theorem 5.3.1] and [16, Proposition 5.1] that there exists $\varepsilon > 0$ such that any solution $\varepsilon = (u, F, m)$ of (2.13) with initial data $z_0 = (u_0, F_0, m_0) \in X_{\gamma, \mu}$ satisfying the conditions $|m_0| = 1$ and $\|z_0 - z_*\|_{X_{\gamma, \mu}} \leq \varepsilon$ exists globally and converges at an exponential rate to some $z_* = (0, 0_3, m_*)$ with $m_* = \text{constant}$, in the topology of $X_{\gamma, 1}$ as $t \to \infty$. By Proposition 2.4, we can determine a pressure $\pi$ such that $(z, \pi)$ solves (2.3) on $\mathbb{R}_+$. Furthermore, since $|m_0| = 1$, we infer that $|m(t)| \equiv 1$ for all $t \geq 0$, which implies that $(z, \pi)$ solves (1.1) on $\mathbb{R}_+$. Finally, we conclude that $m_* \in S^2$.

4. Lyapunov functional and global solutions

Let

$$E := E(u, F, m) := \frac{1}{2} \int_{\Omega} (|u|^2 + |F|^2 + |\nabla m|^2) \, dx. \quad (4.1)$$

We show that the energy $E$ is dissipated.

Proposition 4.1. Let $(u, F, m, \pi)$ be a solution of (1.1) with initial value $z_0$ satisfying the assumptions of Theorem 2.5. Then

$$\frac{d}{dt} E(t) = - \int_{\Omega} (\mu_s |\nabla u(t,x)|^2 + \kappa |\nabla F(t,x)|^2 + \alpha |\Delta m(t,x) + |\nabla m(t,x)|^2 m(t,x)|^2) \, dx$$

for $t \in (0, T_+(z_0))$. Moreover, $E$ is a strict Lyapunov functional for (1.1).

Proof. Let $z_0$ be an initial value satisfying the assumptions of Theorem 2.5. Then (1.1) admits a unique solution $(u, F, m, \pi)$ in the regularity class stated in the Theorem. In particular, $z = (u, F, m)$ enjoys the regularity property

$$z \in C([0, T_+); X_{\gamma, \mu}) \cap C^1((0, T_+); X_1),$$

with $T_+ = T_+(z_0)$. In the following, we suppress the time variable $t \in (0, T_+)$. A straightforward computation, using the boundary condition $\partial_m m = 0$, yields

$$\frac{d}{dt} E = \int_{\Omega} (\partial_t u \cdot u + \partial_t F \cdot F - \partial_t \Delta m \cdot \Delta m) \, dx.$$

We have

$$\int_{\Omega} \partial_t u \cdot u \, dx$$

$$= \int_{\Omega} \left[ \mu_s \Delta u - u \cdot \nabla u - \nabla \pi - \nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (FF^T) \right] \cdot u \, dx \quad (4.2)$$

$$= \int_{\Omega} \left( -\mu_s |\nabla u|^2 - (u \cdot \nabla m) \cdot \Delta m - F F^T : \nabla u \right) \, dx,$$

where we used $\nabla \cdot u = 0$, the boundary condition $u = 0$ on $\partial \Omega$, (1.7), and the relations

$$\nabla \cdot (\nabla m \odot \nabla m) = \nabla m \cdot \Delta m + \frac{1}{2} \nabla (|\nabla m|^2), \quad (\nabla m \cdot \Delta m) \cdot u = (u \cdot \nabla m) \cdot \Delta m.$$

Moreover,

$$\int_{\Omega} \partial_t F : F \, dx = \int_{\Omega} \left[ (\nabla u)^T F - u \cdot \nabla F + \kappa \Delta F \right] : F \, dx$$

$$= \int_{\Omega} (F F^T : \nabla u - \kappa |\nabla F|^2) \, dx, \quad (4.3)$$

$$\int_{\Omega} \partial_t m \cdot \Delta m \, dx$$

$$= \int_{\Omega} \left( -\mu_s |\nabla m|^2 \cdot \Delta m - F F^T : \nabla m \right) \, dx,$$
where we employed the condition $\nabla \cdot u = 0$, the boundary condition $F = 0$ on $\partial \Omega$, and the relations
\[ (\nabla u)^T F : F = FF^T : \nabla u, \quad 2(u \cdot \nabla) F = u \cdot \nabla |F|^2. \]
Observing that $(\partial_t m + u \cdot \nabla m) \cdot m = 0$, we obtain
\[
\int_{\Omega} (\partial_t m + u \cdot \nabla m) \cdot \Delta m \, dx \\
= \int_{\Omega} (\partial_t m + u \cdot \nabla m) \cdot (\Delta m + |\nabla m|^2 m) \, dx \\
= \int_{\Omega} (\alpha (\Delta m + |\nabla m|^2 m) - \beta m \times \Delta m) \cdot (\Delta m + |\nabla m|^2 m) \, dx \\
= \alpha \int_{\Omega} |\Delta m + |\nabla m|^2 m|^2 \, dx
\]
as $m \times \Delta m$ is perpendicular to both $m$ and $\Delta m$. Combining the results in (4.2)–(4.4) readily yields the assertion.

Hence, $E$ is non-increasing along solutions and, thus, is a Lyapunov functional. If, for any time $t \in I := (t_1, t_2) \subset (0, T_+(z_0))$ with some $0 \leq t_1 < t_2$, $\frac{d}{dt} E(t) = 0$, then
\[ \|\nabla u(t)\|_2 = \|\nabla F(t)\|_2 = \|\Delta m(t) + |\nabla m(t)|^2 m(t)\|_2 = 0. \]
Combining with the boundary conditions, we infer that
\[ u(t) = 0, \quad F(t) = 0, \quad t \in I. \]
This readily yields $(\partial_t u(t), \partial_t F(t)) = (0, 0)$ for all $t \in I$. Moreover, the condition $\|\Delta m(t) + |\nabla m(t)|^2 m(t)\|_2 = 0$ implies that
\[ \Delta m(t) + |\nabla m(t)|^2 m(t) = 0 \quad \text{in } \Omega, \quad t \in I. \tag{4.5} \]
Taking the cross product of both sides of (4.5) by $m(t)$ yields
\[ m(t) \times \Delta m(t) = -|\nabla m(t)|^2 m(t) \times m(t) = 0 \quad \text{in } \Omega. \]
Therefore, $\partial_t m(t) = 0$ for all $t \in I$. Hence, $(\partial_t u(t), \partial_t F(t), \partial_t m(t)) = (0, 0, 0)$ for all $t \in I$, and this means that the system is at equilibrium for $t \in I$. To sum up, we have proved that $E : X_{\gamma, \mu} \to \mathbb{R}$ is a strict Lyapunov functional for (1.1).

The arguments above additionally yield a characterization of the set of equilibria of (1.1).

**Corollary 4.2.** The set of equilibria of (1.1) is given by
\[ \mathcal{E} = \{ (0, 0, m_*, \pi_*) \}, \tag{4.6} \]
where $m_* \in H^2_0(\Omega)$ solves the constrained nonlinear elliptic problem
\[
\begin{cases}
\Delta m_* + |\nabla m_*|^2 m_* = 0 & \text{in } \Omega, \\
|m_*| \equiv 1 & \text{in } \Omega, \\
\partial_{\nu} m_* = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{4.7}
\]
and $\pi_* = -\frac{1}{2} |\nabla m_*|^2 + C$ for some constant $C$. 

(Note: The provided text is a continuation of the previous discussion without the page number and author details for clarity.)
Proof. We have already shown in the proof of Proposition (4.1) that any equilibrium of (1.1) is given by \((0, q, m_*)\), where \(m_*\) solves (4.7). Hence, at equilibrium, we are left with the relation \(\nabla \pi_* = -\nabla \cdot (\nabla m_* \otimes \nabla m_*)\). We have
\[
[-\nabla \cdot (\nabla m_* \otimes \nabla m_*)]_i = -\partial_i m_* \cdot \Delta m_* - (\partial_i \partial_j m_*) \cdot \partial_j m_* \\
= \partial_i m_* \cdot m_* |\nabla m_*|^2 - \frac{1}{2} \partial_i |\nabla m_*|^2 = -\frac{1}{2} \partial_i |\nabla m_*|^2,
\]
where we used the relations \(\Delta m_* = -|\nabla m_*|^2 m_*\) and \(\partial_i m_* \cdot m_* = \frac{1}{2} \partial_i |m_*|^2 = 0\). Hence, \(\nabla \pi_* = -\frac{1}{2} \nabla |\nabla m_*|^2\) and the assertion for \(\pi_*\) follows. \(\square\)

Remark 4.3. (a) We note that for solutions of system (2.3), that is, in case the condition \(|m| \equiv 1\) is dropped in (2.2), we can only conclude that
\[
\frac{d}{dt} E = -\int_\Omega (\mu |\nabla u|^2 + \kappa |\nabla F|^2 + \alpha (|\Delta m|^2 + |\nabla m|^2 (m \cdot \Delta m))) \, dx.
\]
As the term \(m \cdot \Delta m\) does not have a sign, we can no longer derive the characterization (4.6) for the set of equilibria, \(E_1\), of (2.3), respectively (2.13). However, as shown in Section 3, we can conclude that for every \(z_* \in E_0\) there exists a neighborhood \(V_*\) in \(X_1\) such that \(E_1 \cap V_* = E_0 \cap V_*\).

(b) It is claimed in [11, Lemma 5.2], see also [19, Lemma 12.2.4], that the nonlinear problem (4.7) admits only constant solutions \(m_* \in \mathbb{S}^2\). However, this assertion is not correct in the form stated, as the following example shows: Let \(\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < |x| < r_2\}\) and \(m_* : \Omega \to \mathbb{S}^2\) be defined by \(m_*(x) = x/|x|\). Then \(m_*\) is a (non-constant) solution of (4.7).

Theorem 4.4. Let \(p, q, \mu, z_0\) and \(T_+(z_0)\) be as in Theorem 2.5. Suppose that the solution \((u, F, m, \pi)\) of (1.1) satisfies
\[
z = (u, F, m) \in BC([\delta, T_+(z_0)]; X_{\gamma, T})
\]
for some \(\delta \in (0, T_+(z_0))\) and \(\bar{\mu} \in (\mu, 1]\). Then \(z\) exists globally and \(\text{dist}(u(t), E) \to 0\) in \(X_{\gamma, 1}\) as \(t \to \infty\), where \(E\) is the set of equilibria of (1.1).

Proof. Given any initial value \(z_0\), we define the \(\omega\)-limit set of (2.13) as
\[
\omega(z_0) := \{w \in X_{\gamma, \mu} : \exists t_n \to \infty \text{ s.t. } \|z(t_n) - w\|_{X_{\gamma, 1}} \to 0 \text{ as } n \to \infty\}.
\]
[19, Theorem 5.7.1] implies that \(z(\cdot)\) exists globally and the orbit \(\{z(t)\}_{t \geq \delta}\) is relatively compact in \(X_{\gamma, 1}\). By [19, Theorem 5.7.2], \(\omega(z_0)\) is nonempty, compact and \(\omega(z_0) \subset E\). Further, we can infer that \(\text{dist}(z(t), E) \to 0\) in \(X_{\gamma, 1}\) as \(t \to \infty\). \(\square\)

Acknowledgements

The first author would like to thank Professor Changyou Wang for suggesting this problem and many helpful discussions.

References

[1] H. Amann, On the strong solvability of the Navier-Stokes equations. J. Math. Fluid Mech. 2, no. 1, 16-98 (2000).
[2] D. Andelman and R.E. Rosensweig, The Phenomenology of Modulated Phases: From Magnetic Solids and Fluids to Organic Films and Polymers. Polymers, Liquids and Colloids in Electric Fields: Interfacial instabilities, orientation, and phase-transitions (Y. Tsori and U. Steiner, eds.), Series in Soft Condensed Matter, vol. 2, World Scientific, 2009, pp. 1-56.
[3] B. Benesova, J. Forster, C. Liu, A. Schlömerkemper, Existence of weak solutions to an evolutionary model for magnetoelasticity. SIAM J. Math. Anal. 50, no. 1, 1200–1236 (2018).

[4] P. Berger, N.B. Adelman, K.J. Beckman, D.J. Campbell, A.B. Ellis, G.C. Lisensky, Preparation and Properties of an Aqueous Ferrofluid. Journal of Chemical Education 76, no. 7, 943 (1999).

[5] F. De Anna, J. Kortum, A. Schlömerkemper, Struwe-like solutions for an evolutionary model of magnetoelastic fluids. J. Differential Equations 309, 455–507 (2022).

[6] J. Escher, J. Prüss, G. Simonett, A new approach to the regularity of solutions for parabolic equations. Evolution equations, 167-190, Lecture Notes in Pure and Appl. Math., 234, Dekker, New York, 2003.

[7] J. Forster, Variational approach to the modeling and analysis of magnetoelastic materials. Ph.D. thesis, Universität Würzburg, 2016.

[8] H. Garcke, P. Knopf, S. Mitra, A. Schlömerkemper, Strong well-posedness, stability and optimal control theory for a mathematical model for magneto-viscoelastic fluids. arXiv preprint arXiv:2108.03094 (2021).

[9] M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis, Maximal $L^p – L^q$-estimates for the Stokes equation: a short proof of Solonnikov’s theorem. J. Math. Fluid Mech. 12, no. 1, 47-60 (2010).

[10] Y. Giga, Domains of fractional powers of the Stokes operator in $L^r$ spaces. Arch. Rational Mech. Anal. 89, no. 3, 251-265 (1985).

[11] M. Hieber, M. Nesensohn, J. Prüss, K. Schade, Dynamics of nematic liquid crystal flows: the quasilinear approach. Ann. Inst. H. Poincaré Anal. Non Linéaire 33, no. 2, 397-408 (2014).

[12] M. Kaouešek, J. Kortum, A. Schlömerkemper, Mathematical analysis of weak and strong solutions to an evolutionary model for magnetoviscoelasticity. Discrete Contin. Dyn. Syst. Ser. S 14, no. 1, 17-39 (2021).

[13] M. Köhne, J. Prüss, M. Wilke, On quasilinear parabolic evolution equations in weighted $L^p$-spaces. J. Evol. Equ. 10, no. 2, 443-463 (2010).

[14] F.-H. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals. Commun. Pure Appl. Math. 48, no. 5, 501-537 (1995).

[15] C. Liu and N. J. Walkington, An Eulerian description of fluids containing visco-elastic particles. Arch. Rational Mech. Anal. 159, no. 3, 229-252 (2001).

[16] G. Mazzongè, J. Prüss, G. Simonett, A maximal regularity approach to the study of motion of a rigid body with a fluid-filled cavity. J. Math. Fluid Mech. 21, no. 3, Paper No. 44, 20 pp. (2019).

[17] C. Melcher, Global solvability of the Cauchy problem for the Landau–Lifshitz–Gilbert equation in higher dimensions. Indiana Univ. Math. J. 61, no. 3, 1175-1200 (2012).

[18] J. Prüss, G. Simonett, Maximal regularity for evolution equations in weighted $L^p$-spaces. Arch. Math. (Basel) 82, no. 5, 415-431 (2004).

[19] J. Prüss, G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations. Monographs in Mathematics. Birkhäuser Verlag. 2016.

[20] J. Prüss, G. Simonett, R. Zacher, On convergence of solutions to equilibria for quasilinear parabolic problems. J. Differential Equations 246 , no. 10, 3902-3931 (2009).

[21] A. Schlömerkemper, J. Žabenský, Uniqueness of solutions for a mathematical model for magneto-viscoelastic flows. Nonlinearity 31, no. 6, 2989-3012 (2018).

[22] V. A. Solonnikov, Estimates for solutions of nonstationary Navier-Stokes equations. J. Math. Sci. 8, no. 4, 467-529 (1977).

[23] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators. North-Holland Publishing Co., Amsterdam-New York, 1978.

[24] N.M. Wereley (ed.), Magnetorheology. RSC Smart Materials, The Royal Society of Chemistry, 2014.

[25] W. Zhao, Local well-posedness and blow-up criteria of magneto-viscoelastic flows. Discrete Contin. Dyn. Syst. 38, no. 9, 4637-4655 (2018).
Department of Mathematics, Vanderbilt University, Nashville, Tennessee, USA
Email address: hengrong.du@vanderbilt.edu

The University of Alabama, Tuscaloosa, Alabama, USA
Email address: yshao8@ua.edu

Department of Mathematics, Vanderbilt University, Nashville, Tennessee, USA
Email address: gieri.simonett@vanderbilt.edu