A Semiclassical Formulation of the Chiral Magnetic Effect and Chiral Anomaly in Even $d + 1$ Dimensions

Ömer F. Dayı* and Mahmut Elbistan*†

*Physics Engineering Department, Faculty of Science and Letters, Istanbul Technical University, TR-34469, Maslak-Istanbul, Turkey
†Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, China

In terms of the matrix valued Berry gauge field strength for the Weyl Hamiltonian in any even spacetime dimensions a symplectic form whose elements are matrices in spin indices is introduced. Definition of the volume form is modified appropriately. A simple method of finding the path integral measure and the chiral current in the presence of external electromagnetic fields is presented. It is shown that within this new approach the chiral magnetic effect as well as the chiral anomaly in even $d + 1$ dimensions are accomplished straightforwardly.

1E-mail addresses: dayi@itu.edu.tr, elbistan@impcas.ac.cn
1 Introduction

Semiclassical analysis of dynamical systems naturally possesses ambiguities: It is defined within a classical phase space, though it claims to embrace quantum mechanical effects. Nevertheless, it is useful to get a better understanding of some quantum mechanical phenomena especially in many body systems. In this respect incorporating quantum mechanical effects in classical kinetic theory is of concern. An outstanding achievement for this purpose was to embody chiral anomaly in classical kinetic theory [1, 2]. In [1] 3 + 1 spacetime dimensional Fermi liquid theory was modified introducing the Berry curvature which is a monopole field deforming the initial phase space [3]. They successfully obtained the chiral magnetic effect and chiral anomaly in external electromagnetic fields. The same results were accomplished in [2] by showing that the monopole field is the field strength of the Berry gauge field emerging from the Weyl Hamiltonian of a massless Dirac particle in 3 + 1 dimensions. Recently, several aspects of the chiral kinetic theory were studied extensively [4–10]. However, in higher dimensions calculation of the chiral magnetic effect and chiral anomaly within the same formulation is missing. In fact the chiral magnetic effect and the chiral vortical effect for even \( d + 1 \) dimensions were conjectured [11]. These conjectures were supported with the results of [12]. Our main goal is to calculate the chiral magnetic effect and chiral anomaly in even \( d + 1 \) dimensions within the same theory. We will show that the source of both effects is the Dirac monopole field defined through the Berry field strength.

Incorporating spin in classical description of particles is one of the intriguing questions of semiclassical dynamics. For instance in spacetime dimensions higher than 3 + 1, Weyl Hamiltonians lead to the non-Abelian Berry gauge fields which are matrices in “spin indices”. A formulation of anomalies in higher dimensional classical chiral kinetic theory was carried out in [13]. They surmount the difficulty of treating the matrix valued Berry gauge fields by introducing group valued degrees of freedom and “dequantizing spin”. However this makes unclear the explicit form of the equations of motion which would yield a generalization of the chiral magnetic effect in higher dimensions. On the other hand, it is possible to deal with matrix valued Hamiltonians in the presence of the Berry gauge fields systematically [14]. Inspired with this approach, though following mainly the differential form methods employed in [13], we present a formalism of the classical chiral kinetic theory in even \( d + 1 \) dimensions yielding both the chiral anomaly and chiral magnetic effect in external electromagnetic fields. We also show that the chiral vortical effect can be formulated similarly.

The main ingredient of our approach is to extend the Hamiltonian methods to cover the symplectic forms which are matrix valued in spin space. Considering \( d + 1 \) spacetime dimensional systems, elements of the symplectic matrix are labeled as usual with the phase space indices \( a, b = 1, \cdots, 2d \). But, for \( d > 3 \) we let them to be matrices in spin space. Therefore, the symplectic matrix elements carry the phase space indices \( a, b \), as well as the spin indices \( \alpha, \beta = 1, \cdots, 2\left(\frac{d-3}{2}\right) \), but we distinguish between these indices. The former ones label the dynamical variables of the semiclassical kinetic theory where probability functions may depend only on classical phase space variables. Hence, we define the equations of motion as matrix valued in spin space. This is in accord with the fact that in a Fermi liquid theory the quasi-particles are described in general with matrices in the spin indices [15]. However, it is usually sufficient to consider distributions yielding scalar particle density. Thus a procedure of reducing our matrix valued quantities to scalars should be given. First of all, we need an appropriate definition of measure for the phase space path integral which should be a scalar. The most orthodox choice would be to define it as the determinant of the related \( (2d \times 2\left(\frac{d-3}{2}\right)) \) dimensional matrix, considering phase space and spin indices on the same footing. However, we distinguish between the phase space and spin indices. Thus, we first ignore the spin indices and obtain the Pfaffian of symplectic matrix considering only the phase space indices \( a, b \). This Pfaffian will be a matrix in the spin indices \( \alpha, \beta \). We define the measure as its trace. Similarly, to define the currents within the kinetic theory we take the trace of the related matrix valued currents accompanied by the reduction of the matrix valued velocities to scalars. We will demonstrate that our method permits us to calculate the
\(d+1\) dimensional chiral magnetic effect. It coincides with the conjectured chiral magnetic effect in \([11]\). We will also comment that the conjectured chiral vortical effect can be derived similarly. Moreover, within the same formalism the chiral anomaly in even \(d+1\) dimensions is engendered correctly.

In Section 2 we recall the definition of the Berry gauge fields for Weyl Hamiltonians in general. In Section 3 the \(3+1\) dimensional semiclassical chiral kinetic theory is considered using the differential form formalism of dynamical systems. We basically review the approach of \([13]\) to the chiral (non-Abelian) anomaly though we deal only with the external electromagnetic fields. However, we also show how this formalism can be used to achieve the phase space measure as well as the solutions of the equations of motion for the first time derivatives of the phase space variables straightforwardly.

The main results are presented in Section 4. We present a method of extending the formulation of Section 3 to cover the symplectic forms which are matrices in spin indices. We demonstrate that it permits us to find not only the chiral anomaly but also the \(d+1\) dimensional chiral magnetic effect as conjectured in \([11]\). We show how our method can be used to obtain the conjectured chiral vortical effect.

In Section 5 we outlined our results and discussed its applicability to some other physical system.

## 2 Weyl Hamiltonian and the Berry Gauge Field

The massless Dirac Hamiltonian in \(d+1\) dimensions is

\[
\mathcal{H} = \alpha \cdot p, \tag{2.1}
\]

in \(\hbar = c = 1\), units. \(p\) is the \(d\) dimensional momentum vector and \(\alpha_A; A = 1, \ldots, d\), are the \(2[(d+1)/2] \times 2[(d+1)/2]\) dimensional matrices which satisfy the anticommutation relations

\[
\{\alpha_A, \alpha_B\} = 2\delta_{AB}. \tag{2.2}
\]

They can be expressed as \(\alpha = \gamma^0\gamma\), in terms of the gamma matrices which obey the Clifford algebra, \(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\), where the metric tensor is \(g^{\mu\nu} = \text{diag}(1, -1, \ldots, -1)\); \(\mu, \nu = 0, 1, \ldots, d\).

In the chiral representation \(\alpha\) is block diagonal, so that (2.1) yields the Weyl Hamiltonian

\[
\mathcal{H}_w = \Sigma \cdot p, \tag{2.3}
\]

where \(\Sigma_A\) are \(2[(d-1)/2] \times 2[(d-1)/2]\) matrices. Quantum mechanical description of the Weyl particle is furnished with the eigenvalue equation

\[
\mathcal{H}_w \psi_E(p) = E\psi_E(p).
\]

The energy eigenvalues are \(E = (p, -p)\) where \(p = |p|\). Focusing on the positive energy solutions \(|\psi^\alpha\rangle\); \(\alpha = 1, \ldots, 2[(d-1)/2]\), one can define the Berry gauge field as

\[
A_\alpha^{\alpha\beta} = i\langle\psi^\alpha| \frac{\partial}{\partial p_\alpha} |\psi^\beta\rangle. \tag{2.4}
\]

Although \(A\) is Abelian for \(d = 3\), it becomes to be non-Abelian for higher dimensions when there is degeneracy. Thus, in general the Berry field strength is given by

\[
\mathcal{G}_A^{\alpha\beta} = \frac{\partial A^{\alpha\beta}_A}{\partial p_\alpha} - \frac{\partial A^{\alpha\beta}_A}{\partial p_\beta} - i[A_A, A_B]^{\alpha\beta}. \tag{2.5}
\]

We will present the \(d = 3\) and \(d = 5\) Berry gauge fields explicitly, respectively, in the next section and Appendix A.
3 The Chiral Anomaly and Chiral Magnetic Effect in 3 + 1 Dimensions

In [2] it was observed that for \( d = 3 \), the Berry field strength extracted from the diagonalization of Weyl Hamiltonian describes a monopole located at \( \mathbf{p} = 0 \). They clarified the role of this monopole field in obtaining the CME and the chiral anomaly. In [13] it was argued that in all even dimensions chiral (non-Abelian) anomaly arises in the presence of the Berry gauge fields within the differential form formulation of classical Hamiltonian dynamics. Here we basically review the approach of [13] considering external electromagnetic fields in \( d = 3 \). However, we will also show how the differential form formalism can be employed to acquire the first time derivatives of phase space variables which are needed to formulate the CME as in [2].

In the chiral representation of gamma matrices

\[
\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli spin matrices, the Weyl Hamiltonian is acquired:

\[
\mathcal{H}_W^{(3)} = \sigma \cdot \mathbf{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}.
\]

To formulate the Hamiltonian dynamics one introduces the one-form

\[
\eta_H = p_a dx_a + A_a(x,t) dx_a - A_a(p) dp_a - H dt,
\]

where \( a = 1, 2, 3 \). We work in \( e = c = 1 \), units. \( A \) is the vector potential of the external magnetic field \( B \):

\[
F_{ab} = \frac{\partial A_b}{\partial x_a} - \frac{\partial A_a}{\partial x_b} = \epsilon_{abc} B_c.
\]

\( H = p + A_0 \) where \( p \) is the Weyl Hamiltonian (3.2) diagonalized and projected onto the positive energy eigenstate. \( A_0(x,t) \) is the scalar potential of the external electric field \( E = \partial A/\partial t - \nabla A_0 \). \( A_a(p) \) denotes the Abelian Berry gauge field. Inspecting the positive energy solution of (3.2) one can show that its field strength is given in terms of \( b = \hat{\mathbf{p}}/2p^2 \), as

\[
G_{ab} = \frac{\partial A_b}{\partial p_a} - \frac{\partial A_a}{\partial p_b} = \epsilon_{abc} b_c.
\]

It is the field of a monopole located at \( \mathbf{p} = 0 \): \( \nabla \cdot \mathbf{b} = 2\pi \delta^3(p) \).

The exterior derivative of \( \eta_H \) provides the sympletic two-form

\[
w_H = d\eta_H = dp_a \wedge dx_a + F - G - \hat{p}_a dp_a \wedge dt + E_a dx_a \wedge dt,
\]

where \( F = \frac{1}{2} F_{ab} dx_a \wedge dx_b \) and \( G = \frac{1}{2} G_{ab} dp_a \wedge dp_b \). In terms of \( w_H \) the volume form in \( 6 + 1 \) dimensional phase space is defined by

\[
\Omega = \frac{1}{3!} w_H^3 \wedge dt.
\]

It can be written in terms of the canonical volume element of the phase space \( dV^{(3)} \) as

\[
\Omega = \sqrt{w} dV^{(3)} \wedge dt.
\]

Here, \( \sqrt{w} \equiv \sqrt{\det(w)} \), is the Pfaffian of the matrix

\[
\begin{pmatrix} F_{ab} & -\delta_{ab} \\ \delta_{ab} & -G_{ab} \end{pmatrix}.
\]
It was calculated explicitly in [3] as $\sqrt{w} = (1 + B \cdot b)$.

The equations of motion can be derived by demanding that $w_H$ satisfies

$$i_v w_H = 0,$$  \hspace{1cm} (3.6)

which is the interior product of the vector field

$$v = \frac{\partial}{\partial t} + \dot{x}_a \frac{\partial}{\partial x_a} + \dot{p}_a \frac{\partial}{\partial p_a},$$

with $w_H$. Indeed, (3.6) gives rise to the equations of motion obtained in [2]:

$$\dot{p}_a = \dot{x}_b F_{ab} + E_a,$$ \hspace{1cm} (3.7a)

$$\dot{x}_a = \dot{p}_b G_{ab} + \dot{\rho}_a.$$ \hspace{1cm} (3.7b)

Liouville equation will be provided by the time evolution of the volume form $\Omega$ which can be found by calculating its Lie derivative associated with $v$. It can be accomplished in two different ways. First one can employ (3.5) to observe that

$$L_v \Omega = (i_v d + d_i_v) \left( \sqrt{w} dV^{(3)} \wedge dt \right) = \left( \frac{\partial}{\partial t} \sqrt{w} + \frac{\partial}{\partial x_a} (\sqrt{w} \dot{x}_a) + \frac{\partial}{\partial p_a} (\sqrt{w} \dot{p}_a) \right) dV^{(3)} \wedge dt. \hspace{1cm} (3.8)$$

Then, one can show that the definition (3.4) implies

$$L_v \Omega = (i_v d + d_i_v) \left( \frac{1}{3!} w_H^3 \wedge dt \right) = d_i_v \left( \frac{1}{3!} w_H^3 \wedge dt \right) = \frac{1}{2} dw_H \wedge w_H^2,$$

where

$$dw_H = -\frac{1}{2} \frac{\partial G_{ab}}{\partial p_c} dp_c \wedge dp_a \wedge dp_b.$$

Making use of (3.3) one can express $dw_H$ as

$$dw_H = - (\nabla \cdot b) dp_1 \wedge dp_2 \wedge dp_3 = -2\pi \delta^3(p) dp_1 \wedge dp_2 \wedge dp_3.$$

The unique contribution from $w_H^2$ will be $E_a F_{bc} dx_a \wedge dx_b \wedge dx_c \wedge dt$, thus one finds that

$$\frac{1}{2} dw_H \wedge w_H^2 = 2\pi \delta^3(p) (E \cdot B) dV^{(3)} \wedge dt.$$

One reaches the conclusion that the Liouville equation is anomalous:

$$\left( \frac{\partial}{\partial t} \sqrt{w} + \frac{\partial}{\partial x_a} (\sqrt{w} \dot{x}_a) + \frac{\partial}{\partial p_a} (\sqrt{w} \dot{p}_a) \right) = 2\pi \delta^3(p) E \cdot B. \hspace{1cm} (3.9)$$

In quantum field theory the chiral anomaly is expressed as non-conservation of the classically conserved chiral current at the quantum level. In order to connect the anomaly contribution in (3.9) to the non-conservation of particle number introduce the probability function $f(x, p, t)$, which satisfies the collisionless Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_a} \dot{x}_a + \frac{\partial f}{\partial p_a} \dot{p}_a = 0. \hspace{1cm} (3.10)$$

It is appropriate to define the probability density function as $\rho(x, p, t) = \sqrt{w} f$. Hence, define the particle number density and the particle current density:

$$n(x, t) = \int \frac{d^3p}{(2\pi)^3} \rho(x, p, t), \quad j_a = \int \frac{d^3p}{(2\pi)^3} \rho(x, p, t) \dot{x}_a.$$
Utilizing (3.9) and (3.10) one can observe that

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \dot{x}_a)}{\partial x_a} + \frac{\partial (\rho \dot{p}_a)}{\partial p_a} = 2\pi f \delta^3(p) \mathbf{E} \cdot \mathbf{B}. \quad (3.11)$$

In order to derive non-conservation of the particle current integrate (3.11) over momentum degrees of freedom. There is no influx of particles from the negative energy sea as we only deal with the positive energy sector of the Weyl Hamiltonian. Thus, the momentum current density $\mathbf{j}_p = \rho \dot{p}$ vanishes at the boundary of the momentum space:

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial (\rho \dot{p}_a)}{\partial p_a} = 0.$$

Actually, the Berry monopole situated on the boundary $|p| = 0$ is responsible for the non-conservation of the chiral particle current. A detailed explanation was given in [2]. Therefore, the statement of non-conservation of the particle current follows,

$$\frac{\partial n(x,t)}{\partial t} + \nabla \cdot \mathbf{j} = \frac{1}{4\pi^2} f(x,p = 0,t) \mathbf{E} \cdot \mathbf{B}.$$

To derive the CME in line with [2] one needs to solve the equations of motion (3.7) for $\sqrt{w} \dot{x}_a$. Obviously, here this can be done directly, because $\sqrt{w}$ has already been calculated in [3]. However, it is possible to acquire the same result by inspecting the explicit form of $w^3_H$. As a byproduct the explicit form of $\sqrt{w}$ will also be provided. To present this method which will be extremely useful in higher dimensions, let us write $w^3_H$ explicitly as

$$w^3_H = dp_a \wedge dx_a \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6F \wedge \mathcal{G} \wedge dp_a \wedge dx_a$$

$$- 3\dot{p}_a dp_a \wedge dt \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6E_a F \wedge \mathcal{G} \wedge dx_a \wedge dt$$

$$+ 3E_a dx_a \wedge dt \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6\dot{p}_a F \wedge dp_a \wedge dt \wedge dp_b \wedge dx_b$$

$$- 6E_a \mathcal{G} \wedge dx_a \wedge dt \wedge dp_b \wedge dx_b + 6\dot{p}_a F \wedge \mathcal{G} \wedge dp_a \wedge dt.$$

Its Lie derivative associated with $\nu$ procures

$$L_\nu \Omega = \frac{1}{3!} dw^3_H = \left\{ \frac{\partial}{\partial t} (1 + \mathbf{B} \cdot \mathbf{b}) + \frac{\partial}{\partial \dot{x}} (\dot{\mathbf{p}} + \mathbf{E} \times \mathbf{b} + \mathbf{B} (\dot{\mathbf{p}} \cdot \mathbf{b})) \right\} dV^{(3)} \wedge dt. \quad (3.12)$$

Comparing (3.12) with (3.8) one can directly read $\sqrt{w}$, as well as the solutions of the equations of motion (3.7) as

$$\sqrt{w} = 1 + \mathbf{B} \cdot \mathbf{b},$$

$$\sqrt{w} \dot{x} = \dot{\mathbf{p}} + \mathbf{E} \times \mathbf{b} + \mathbf{B} (\dot{\mathbf{p}} \cdot \mathbf{b}),$$

$$\sqrt{w} \dot{\mathbf{p}} = \mathbf{E} + \dot{\mathbf{p}} \times \mathbf{B} + \mathbf{b} (\mathbf{E} \cdot \mathbf{B}).$$

Now, the particle current density $\mathbf{j}$ can be obtained as

$$\mathbf{j} = \int \frac{d^3p}{(2\pi)^3} \sqrt{w} \dot{x} f = \int \frac{d^3p}{(2\pi)^3} \dot{\mathbf{p}} f + \mathbf{E} \times \int \frac{d^3p}{(2\pi)^3} \mathbf{b} f + \mathbf{B} \int \frac{d^3p}{(2\pi)^3} \dot{\mathbf{p}} \cdot \mathbf{b} f.$$

The last term where the current is parallel to the magnetic field is the CME term that was mentioned in [2].
4 Chiral Kinetic Theory in $d+1$ Dimensions

We would like to consider kinetic theory of the $d+1 = 2n + 2$; $n = 1, 2, \ldots$, dimensional Weyl particles in the presence of the Berry gauge field $2.4$, the external magnetic field $F_{AB} = \partial A_B/\partial x_A - \partial A_A/\partial x_B$, and the electric field $E_A = -\partial A_0/\partial x_A + \partial A_A/\partial t$, pointing towards the $\hat{x}_A$ direction. $A_0(x, t)$ and $A(x, t)$, are the scalar and vector potentials. We will mainly follow the procedure of Section 3 with some modifications required to deal with spin degrees of freedom.

Dealing with the non-Abelian fields $A_A$, the appropriate definition of the matrix valued symplectic two-form is

$$\tilde{W}_H \equiv dp_A \wedge dx_A + F - G - \hat{p}_A dp_A \wedge dt + E_A dx_A \wedge dt,$$

where, $F = \frac{1}{2} F_{AB} dx_A \wedge dx_B$, and $G = \frac{1}{2} G_{AB} dp_A \wedge dp_B$. The symplectic two-form $\tilde{W}_H$ is a $2^{n-1} \times 2^{n-1}$ matrix in spin indices. We suppress the matrix indices and do not write the unit matrix $I$ explicitly, unless it is necessary.

Let the matrix valued $(\dot{X}_A, \dot{P}_A)$ denote time evolution of the phase space variables $(x_A, p_A)$ which are acquired through the interior product of the matrix valued vector field

$$\tilde{V} = \frac{\partial}{\partial t} + \dot{X}_A \frac{\partial}{\partial x_A} + \dot{P}_A \frac{\partial}{\partial p_A},$$

with the symplectic two-form $\tilde{W}_H$ as

$$i_L \tilde{W}_H = 0.$$

This gives rise to

$$\dot{P}_A = \dot{X}_B F_{AB} + E_A, \quad (4.1a)$$

$$\dot{X}_A = G_{AB} \dot{P}_B + \hat{p}_A, \quad (4.1b)$$

as the equations of motion. We would like to stress that $(\dot{X}_A, \dot{P}_A)$ are matrices.

4.1 Liouville Equation and the Chiral Anomaly

The novelty in this formulation is the fact that we do not treat spin degrees of freedom as dynamical variables. Classical dynamics is asserted through the phase space variables $(x, p)$, so that we define the phase space volume form as

$$\tilde{\Omega} \equiv \frac{(-1)^{n+1}}{(2n + 1)!} \tilde{W}_H^{2n+1} \wedge dt.$$

We express it in terms of the 2d dimensional Liouville measure $dV$, as

$$\tilde{\Omega} = \tilde{W}_{1/2} dV \wedge dt. \quad (4.3)$$

$\tilde{W}_{1/2}$ is a two-form in the phase space variables $(x_A, p_A)$, so that $\tilde{W}_{1/2}$ is the Pfaffian of the $(4n + 2) \times (4n + 2)$ matrix

$$\left( \begin{array}{cc} F_{AB} & -\delta_{AB} \\ \delta_{AB} & -G_{AB} \end{array} \right).$$

In order to derive the Liouville equation we need the Lie derivative of the volume form $\tilde{\Omega}$. By making use of (4.3) it can be written formally as

$$L_\tilde{V} \tilde{\Omega} = (i_L d + dt_\tilde{V}) \tilde{W}_{1/2} dV \wedge dt = \left( \frac{\partial}{\partial t} \tilde{W}_{1/2} + \frac{\partial}{\partial x_A} (\tilde{X}_A \tilde{W}_{1/2}) + \frac{\partial}{\partial p_A} (\tilde{W}_{1/2} \dot{P}_A) \right) dV \wedge dt. \quad (4.4)$$

However, in order to acquire it explicitly the definition (4.2) should be employed,

$$L_\tilde{V} \tilde{\Omega} = \frac{(-1)^{n+1}}{(2n + 1)!} d\tilde{W}_H^{2n+1}. \quad (4.5)$$
Among the several terms in $\tilde{W}_{\mu}^{2n+1}$, the chiral anomaly stems from the one which includes the singularity in momentum space:

$$(-1)^n(2n+1)! \frac{n \times n \times n}{(n!)^2} G_{\ldots}G \mathcal{E}_A \mathcal{F} \ldots \mathcal{F} \ dx_A \wedge dt.$$  \hfill (4.6)

We suppress wedge products of the curvature forms.

To describe quantum mechanical particles possessing spin within classical phase space we introduced matrix valued quantities representing the spin degrees of freedom. On the other hand, measure of the related path integral should be a scalar. Thus an appropriate definition of the classical limit is needed. For accomplishing the semiclassical approximation we take the trace over spin indices and adopt $\text{Tr} \tilde{W}_{1/2}$ as the definition of the related path integral measure. As it is demonstrated in Appendix B after taking the trace the unique term which survives in (4.5) is originated from (4.6) and it leads to the following anomalous Liouville equation,

$$\text{Tr} [L \tilde{V} \tilde{\Omega}] = -\frac{1}{(n!)^2} \text{Tr} [d(\tilde{G}_{\ldots}G)]\mathcal{E}_A \mathcal{F} \ldots \mathcal{F} \ dx_A \wedge dt.$$  

In Appendix B we also briefly reported, following [16], how the singularity is calculated to be

$$\text{Tr} [d(\tilde{G}_{\ldots}G)] = \frac{1}{2^n} \frac{2^n}{\partial p_A} \text{Tr} [\epsilon_{ABC\ldots DE} \tilde{G}_{BC\ldots} \tilde{G}_{DE}] \delta^{2n+1} p$$

$$= \frac{(-1)^{n+1}(2n)!}{2^n} (\nabla \cdot b) d^{2n+1} p,$$

where $b$ is the $2n + 1$ dimensional Dirac monopole field

$$b = \frac{p}{2p^{2n+1}}.$$  \hfill (4.7)

Obviously it satisfies

$$\nabla \cdot b = \frac{\text{Vol}(S^{2n})}{2} \delta^{2n+1}(p),$$

where $\text{Vol}(S^{2n})$ denotes volume of the $2n$-sphere. Hence, we attain

$$\text{Tr} [L \tilde{V} \tilde{\Omega}] = \frac{(-1)^{n+1}(2n)!}{(n!)^2} \frac{2^n}{2^{2n+1}} \text{Vol}(S^{2n}) \delta^{2n+1}(p) \epsilon_{ABC\ldots DE} \mathcal{E}_A \mathcal{F} \ldots \mathcal{F} \ dV \wedge dt.$$  \hfill (4.8)

On the other hand, within the semiclassical approximation we define

$$\sqrt{W} \equiv \text{Tr} [\tilde{W}_{1/2}], \text{ Tr} [\tilde{X}_A \tilde{W}_{1/2}] \equiv \sqrt{W} \tilde{x}_A, \text{ Tr} [\tilde{W}_{1/2} \tilde{P}_A] \equiv \sqrt{W} \tilde{p}_A,$$

where $(\tilde{x}_A, \tilde{p}_A)$ denote the classical velocities. Hence, equating the trace of (4.4) with (4.8) we obtain the semiclassical anomalous Liouville equation as

$$\left( \frac{\partial}{\partial t} \sqrt{W} + \frac{\partial}{\partial x_A} (\sqrt{W} \tilde{x}_A) + \frac{\partial}{\partial p_A} (\sqrt{W} \tilde{p}_A) \right) = \frac{(-1)^{n+1}(2n)!}{(n!)^2} \frac{2^n}{2^{2n+1}} \text{Vol}(S^{2n}) \delta^{2n+1}(p) \epsilon_{ABC\ldots DE} \mathcal{E}_A \mathcal{F} \ldots \mathcal{F}. $$

In order to connect it to the non-conservation of the chiral particle number we employ the phase space distribution $f(x, p, t)$ satisfying the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_A} \tilde{x}_A + \frac{\partial f}{\partial p_A} \tilde{p}_A = 0,$$
We deal with the CME which is generated by the terms depending on the external magnetic field. Our formalism of kinetic theory directly provides the solutions of the equations of motion (4.1) for 4.2 The Chiral Magnetic Effect
manifestation of the chiral anomaly in any even dimensions.

\[ \frac{\partial n}{\partial t} + \nabla \cdot j = \frac{(-1)^{n+1}}{n!} \frac{1}{(2\pi)^{n+1}} \int f(x,p=0,t) \epsilon_{ABC} \ldots \epsilon_{DE} \frac{n \text{ times}}{2^n} \left( \hat{\rho} \cdot \hat{b} \right) f(x,p,t), \]

which is the non-conservation of the phase space probability. Now, introduce the chiral particle density \( n(x,t) = \int \frac{d^3p}{(2\pi)^3} \rho p \) and the chiral current density \( j_\lambda = \int \frac{d^3p}{(2\pi)^3} \rho \hat{x}_\lambda \), for establishing non-conservation of the chiral current as

\[ \frac{\partial n}{\partial t} + \nabla \cdot j = \frac{(-1)^{n+1}}{n!} \frac{1}{(2\pi)^{n+1}} \int f(x,p=0,t) \epsilon_{ABC} \ldots \epsilon_{DE} \frac{n \text{ times}}{2^n} \left( \hat{\rho} \cdot \hat{b} \right) f(x,p,t). \]

It is derived by integrating (4.9) over the momentum space and setting \( \text{Vol}(S^{2n}) = \frac{2^{n+1} \pi^n n!}{(2n)!} \). Obviously, we also employed \( \int \frac{d^3p}{(2\pi)^3} \rho \hat{x}_\lambda = 0 \), as it was discussed in Section 3. (4.10) is the semiclassical manifestation of the chiral anomaly in any even dimensions.

### 4.2 The Chiral Magnetic Effect

Our formalism of kinetic theory directly provides the solutions of the equations of motion (4.1) for \( \hat{X}_\lambda \hat{W}_{1/2}, \hat{W}_{1/2} \hat{P}_\lambda \) as well as \( \hat{W}_{1/2} \) in terms of the phase space variables \( (x_\lambda, p_\lambda) \), by equating the right hand sides of (4.4) and (4.5):

\[ \frac{(-1)^{n+1}}{(2n+1)!} d\hat{W}_{2n+1}^\mu = \frac{\partial}{\partial t} \hat{W}_{1/2} + \frac{\partial}{\partial x_\lambda} (\hat{X}_\lambda \hat{W}_{1/2}) + \frac{\partial}{\partial p_\lambda} (\hat{W}_{1/2} \hat{P}_\lambda). \]

To derive the chiral current it is sufficient to present \( \hat{X}_\lambda \hat{W}_{1/2} \) which includes several terms like

\[ \hat{X}_\lambda \hat{W}_{1/2} = \frac{(-1)^n}{(2n)!} \frac{1}{2^n} \epsilon_{ABC} \ldots \epsilon_{DE} \hat{F}_{BC} \ldots \hat{F}_{DE} \epsilon_{LM} \hat{F}_{LM}, \]

In the semiclassical approximation we define the current as

\[ J_\lambda = \int \frac{d^3p}{(2\pi)^3} \epsilon_{ABC} \ldots \epsilon_{DE} \hat{F}_{BC} \ldots \hat{F}_{DE} \epsilon_{LM} \hat{F}_{LM} f(x,p,t). \]

We deal with the CME which is generated by the terms depending on the external magnetic field \( \hat{F}_{AB} \). As it is shown in Appendix B, once we take the trace over the spin indices there remains only one term depending on the external magnetic field \( \hat{F}_{AB} \), which is generated by the last term given in (4.11).

Therefore the chiral magnetic current is calculated to be

\[ J_\lambda^{\text{CME}} = \frac{1}{2^n (n!)^2} \int \frac{d^3p}{(2\pi)^3} \epsilon_{ABC} \ldots \epsilon_{DE} \hat{F}_{BC} \ldots \hat{F}_{DE} \epsilon_{LM} \hat{F}_{LM} f(x,p,t) \]

\[ = \frac{(-1)^{n+1}}{2^n (n!)^2} \int \frac{d^3p}{(2\pi)^3} \epsilon_{ABC} \ldots \epsilon_{DE} \hat{F}_{BC} \ldots \hat{F}_{DE} (\hat{p} \cdot \hat{b}) f(x,p,t), \]
where $b$ is the Dirac monopole field given in (4.7). When we deal with an isotropic momentum distribution $f = f(E)$, the angular part of (4.12) can be computed, so that we establish the chiral magnetic current as

$$j_{\text{CME}}^A = \frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2} \frac{\text{Vol}(S^{2n})}{2(2\pi)^{2n+1}} \epsilon_{\text{BC...DE}} F_{BC...DE} \int dE f(E),$$

$$= \frac{(-1)^{n+1}}{2^n(2\pi)^{n+1}n!} \epsilon_{\text{ABC...DE}} F_{BC...DE} \int dE f(E).$$

This is the chiral magnetic current conjectured in [11].

### 4.3 The Chiral Vortical Effect

When one considers the vortical flow of the fluid in $3 + 1$ dimensions, there exists an induced current linear in the vorticity which is known as the chiral vortical effect [17–19]. It is similar to the CME although the vorticity is an intrinsic property of the fluid in contrast to the external magnetic field: The vorticity $\omega_i \equiv (1/2)\epsilon_{ijk} \omega^{jk}$, gives rise to the effective magnetic field $2|p|\omega$, where $|p|$ is the energy of the co-moving Weyl particle [20]. Hence, we can incorporate $\omega_{AB}$, denoting the vorticity in $d + 1$ dimensions, into the present formalism by switching off the external fields and dealing with the symplectic two-form

$$\tilde{W}_H \equiv dp_A \wedge dx_A + \Omega - \hat{p}_A dp_A \wedge dt,$$

where $\Omega = |p|\omega_{AB} dx_A \wedge dx_B$. By performing the same computations which yielded the CME (4.12), we obtain the chiral vortical effect in any even dimensions for an isotropic momentum distribution as

$$j_{\text{CVE}}^A = \frac{(-1)^{n+1}}{2^n(2\pi)^{n+1}n!} \epsilon_{\text{ABC...DE}} \omega_{BC...DE} \int dE E^n f(E).$$

This is in accord with the chiral vortical effect conjectured in [11].

### 5 Discussions

We first calculated the Abelian Berry gauge field arising from the $3 + 1$ dimensional Weyl Hamiltonian and demonstrated that the Berry field strength yields a monopole situated at the origin of phase space. In the Appendix, we explicitly showed that similar results hold in $5 + 1$ dimensions where the Berry gauge field is non-Abelian. In fact in any even $d + 1$ dimensions the related Berry field strength engenders a Dirac monopole field. This monopole field is responsible for the chiral anomaly manifested itself in the kinetic theory of electrons as non-conservation of particle current. We showed that this monopole is also the source of the CME. We presented an efficient method of finding the path integral measure and solutions of the equations of motion for the first derivatives of the phase space variables weighted by this measure. This furnished the possibility of obtaining the chiral current directly inspecting Liouville equation. Hence we calculated the CME and chiral anomaly in any even dimensions within the same formulation. The accomplished CME and chiral vortical effect are in accord with the ones conjectured in [11]. The main novelty is to keep the spin dependence explicit without attributing them some dynamical variables.

It seems that there is no obstacle of incorporating external non-Abelian gauge fields to our method in the line with [13, 21].

A semiclassical study of the massive Dirac particle engaging the non-Abelian Berry gauge fields was presented in [22]. There some dynamical variables have been associated with the spin degrees of freedom. In principle the approach of dealing with the non-Abelian Berry gauge fields presented here
can be adopted to perform a similar study in the ordinary classical phase space without enlarging it with some new dynamical variables.

The formalism based on the matrix valued symplectic form is not restricted to even dimensions. It can be employed to establish solution of the equations of motion in terms of phase space variables in any dimensions. These solutions which exhibit the spin dependence explicitly can be useful to formulate some interesting physical phenomena like the spin Hall effect.

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**Appendices**

### A Chiral Kinetic Theory in 5 + 1 Dimensions

To illustrate the whole machinery explicitly we would like to discuss the case which the lowest dimensions where the Berry gauge fields are non-Abelian. A representation of \( \alpha_i; i = 1, \ldots, 5 \), satisfying (2.2) can be given as the direct product of the 3 + 1 dimensional gamma matrices (3.1) with \( \sigma_0 = \text{diag}(1,1) \) and \( \sigma_3 = \text{diag}(1, -1) \):

\[
\begin{align*}
\alpha_1 &= \sigma_0 \otimes \gamma_0 \gamma_1, \\
\alpha_2 &= \sigma_0 \otimes \gamma_0 \gamma_2, \\
\alpha_3 &= \sigma_0 \otimes \gamma_0 \gamma_3, \\
\alpha_4 &= i \sigma_0 \otimes \gamma_0 \gamma_5, \\
\alpha_5 &= \sigma_3 \otimes \gamma_0.
\end{align*}
\]

In this representation the \( \Sigma \) matrices are

\[
\Sigma_a = \begin{pmatrix}
\sigma_a & 0 \\
0 & -\sigma_a
\end{pmatrix}; \quad a = 1, 2, 3, 4, 5.
\]

Therefore, the Weyl Hamiltonian is expressed as

\[
\mathcal{H}_W = \begin{pmatrix}
\sigma_a p_a & i(p_4 + ip_5) \\
-i(p_4 - ip_5) & -\sigma_a p_a
\end{pmatrix}.
\]

We construct the normalized, positive energy eigenstates \( \mathcal{H}_W |\psi^\alpha\rangle = p|\psi^\alpha\rangle; \alpha = 1, 2, \) as

\[
|\psi^1\rangle = \frac{1}{\sqrt{2p(p - p_3)}} \begin{pmatrix}
p_4 - ip_5 \\
p_4 + ip_5 \\
0 \\
1
\end{pmatrix},
|\psi^2\rangle = \frac{1}{\sqrt{2p(p - p_3)}} \begin{pmatrix}
i(p_4 + ip_5) \\
0 \\
p - p_3 \\
-(p_1 + ip_2)
\end{pmatrix}.
\]

Plug these degenerate eigenvectors into the definition (2.4) to build the non-Abelian Berry gauge field components \( A^\alpha_\beta \) as

\[
\begin{align*}
A_1 &= \frac{1}{2p(p - p_3)} \begin{pmatrix}
-p_2 \\
\sqrt{p_4^2 + p_5^2} \\
-i \sqrt{p_4^2 + p_5^2} \\
\frac{p_4 - p_5}{p_2}
\end{pmatrix}, \\
A_2 &= \frac{1}{2p(p - p_3)} \begin{pmatrix}
p_1 \\
\sqrt{p_4^2 + p_5^2} \\
\sqrt{p_4^2 + p_5^2} \\
-p_1
\end{pmatrix}, \\
A_3 &= 0, \\
A_4 &= \frac{1}{2p(p - p_3)} \begin{pmatrix}
p_5 \left[\frac{2p(p_1 - p_3)}{p_1^2 + p_2^2} - 1\right] \\
-i \sqrt{p_4^2 + p_5^2} (p_1 - ip_2) \\
\frac{p_4 + ip_5}{p_1 - ip_2} \\
-p_5
\end{pmatrix}, \\
A_5 &= \frac{1}{2p(p - p_3)} \begin{pmatrix}
-p_4 \left[\frac{2p(p_1 - p_3)}{p_1^2 + p_2^2} - 1\right] \\
-\sqrt{p_4^2 + p_5^2} (p_1 - ip_2) \\
\frac{p_4 - ip_5}{p_1 - ip_2} \\
-p_4
\end{pmatrix}.
\]

(A.1)
By employing (A.1) in the definition (2.5) we extract $G_{ij}^{a3}$ as follows,

\[ G_{12} = \frac{1}{2p^2(p-p_3)} \left( \frac{(p_3^2+p_4^2+p_5^2)-pp_3}{-(p_1+ip_2)\sqrt{(p_4^2+p_5^2)}} - (p_1+ip_2)\sqrt{(p_4^2+p_5^2)} \right), \]

\[ G_{13} = \frac{1}{2p^2} \left( \frac{p_2}{-i\sqrt{(p_4^2+p_5^2)}} + \sqrt{(p_4^2+p_5^2)} \right), \]

\[ G_{14} = \frac{1}{2p^2(p-p_3)} \left( \frac{-p_1p_5p_2p_4}{-p_1p_5} \right), \]

\[ G_{15} = \frac{1}{2p^2(p-p_3)} \left( \frac{(p_4+ip_5)(-i(p_1p_2-p_4p_5)+p_3p-p_3^2-p_1^2)}{p_4-p_2p_5} \right), \]

\[ G_{23} = \frac{1}{2p^2} \left( \frac{p_1p_4-p_2p_5}{p_1} \right), \]

\[ G_{24} = \frac{1}{2p^2(p-p_3)} \left( \frac{p_1p_4p_2p_5}{\sqrt{(p_4^2+p_5^2)}} \right), \]

\[ G_{25} = \frac{1}{2p^2(p-p_3)} \left( \frac{-(p_4+ip_5)(-i(p_1p_2-p_4p_5)+p_3p-p_3^2-p_1^2)}{p_4-p_2p_5} \right), \]

\[ G_{34} = \frac{1}{2p^2} \left( \frac{-(p_4+ip_5)(-i(p_1p_2)+p_3p-p_3^2-p_1^2)}{p_5} \right), \]

\[ G_{35} = \frac{1}{2p^2} \left( \frac{-p_1p_4}{-i\sqrt{(p_4^2+p_5^2)}} - (p_1+ip_2)\sqrt{(p_4^2+p_5^2)} \right), \]

\[ G_{45} = \frac{1}{2p^2(p-p_3)} \left( \frac{(p_1^2+p_2^2+p_3^2)-pp_3}{p_1-ip_2} \right). \]

They are all traceless.

The symplectic two-form, the Pfaffian and the matrix valued vector field are denoted, respectively, $\tilde{w}_H$, $\tilde{w}_H$, and $\tilde{v}$. We expose $\tilde{\Omega}$ explicitly as

\[
\tilde{w}_H^5 = dp_i \wedge dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \wedge dp_m \wedge dx_m
\]

\[-20\mathcal{F} \wedge \mathcal{G} \wedge dp_i \wedge dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k + 30\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{G} \wedge \mathcal{G} \wedge dp_i \wedge dx_i
\]

\[-5\hat{p}_i dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \wedge dp_m \wedge dx_m
\]

\[-20\mathcal{E}_i \mathcal{G} \wedge dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l
\]

\[+ 60\hat{p}_i \mathcal{G} \wedge \mathcal{F} \wedge dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k
\]

\[+ 60\hat{p}_i \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{F} \wedge dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k
\]

\[-20\hat{p}_i \mathcal{F} \wedge dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l
\]

\[+ 60\hat{p}_i \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{F} \wedge dx_i \wedge dt \wedge dp_j \wedge dx_j + 30\mathcal{E}_i \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{F} \wedge dx_i \wedge dt.\]
Although it is cumbersome, making use of $G_{ij}$ presented above we obtain

$$\text{Tr} [L_5 \tilde{\Omega}] = - \frac{\pi^2}{2} \delta^5(p) \epsilon^{ijklm} \epsilon_i \mathcal{F}_{jk} \mathcal{F}_{lm} \, dV^{(5)} \wedge dt.$$ (A.3)

Hence, the chiral current is anomalous:

$$\frac{\partial n}{\partial t} + \tilde{\nabla} \cdot \tilde{j} = - \frac{1}{(4\pi)^3} f(x, p = 0, t) \epsilon^{ijklm} \epsilon_i \mathcal{F}_{jk} \mathcal{F}_{lm}.$$ 

By inspecting (A.2) and comparing with the formal expression (4.4) for $d = 5$, we read directly:

$$\hat{w}_{1/2} = I + \frac{1}{2} \mathcal{F}_{ij} \tilde{G}_{ij} + \frac{1}{8} \mathcal{F}_{ij} \mathcal{F}_{kl} (\tilde{G}_{ij} \tilde{G}_{kl} + \tilde{G}_{ik} \tilde{G}_{lj} + \tilde{G}_{lj} \tilde{G}_{ik})$$

$$\hat{X}_r \hat{w}_{1/2} = \hat{p}_r + \mathcal{E}_i \tilde{G}_{ri} + \frac{1}{2} \mathcal{F}_{ij} (\hat{p}_r \tilde{G}_{ij} + 2 \hat{p}_j \tilde{G}_{ri}) +$$

$$\frac{1}{4} \mathcal{E}_i \mathcal{F}_{jk} (\tilde{G}_{ri} \tilde{G}_{jk} + \tilde{G}_{rj} \tilde{G}_{ki} + 2 \tilde{G}_{rk} \tilde{G}_{ij} + 2 \tilde{G}_{lj} \tilde{G}_{ri}) + \frac{1}{64} \mathcal{F}_{ij} \mathcal{F}_{kl} \mathcal{F}_{mn} \tilde{G}_{st} \hat{p}_p \epsilon_{rijkl} \epsilon_{mnst}p^{(A.4)}$$

$$\hat{w}_{1/2} \hat{P}_r = \mathcal{E}_r + \hat{p}_r \mathcal{F}_{ri} + \frac{1}{2} \mathcal{E}_i \tilde{G}_{ij} (\mathcal{F}_{ri} \mathcal{F}_{ij} + 2 \mathcal{F}_{rj} \mathcal{F}_{ri}) + \frac{1}{2} \hat{p}_r \tilde{G}_{jk} (\mathcal{F}_{ri} \mathcal{F}_{jk} + 2 \mathcal{F}_{rj} \mathcal{F}_{ki})$$

$$+ \frac{1}{64} \mathcal{F}_{mn} \mathcal{F}_{st} \mathcal{E}_p \tilde{G}_{ij} \tilde{G}_{kl} \epsilon_{rijkl} \epsilon_{mnst}p.$$ 

Now, it is possible to build the classical current by means of the phase space probability function $f(x, p, t)$ and taking the trace over the spin indices of the solution (A.4) as

$$j_r = \int \frac{d^5p}{(2\pi)^5} \text{Tr} [\hat{X}_r \hat{w}_{1/2}] f$$

$$= 2 \int \frac{d^5p}{(2\pi)^5} \hat{p}_r f + \frac{1}{2} \int \frac{d^5p}{(2\pi)^5} \mathcal{E}_i \mathcal{F}_{jk} \text{Tr} [\tilde{G}_{ri} \tilde{G}_{jk} + 2 \tilde{G}_{rk} \tilde{G}_{ij}] f$$

$$+ \frac{1}{64} \int \frac{d^5p}{(2\pi)^5} \mathcal{F}_{ij} \mathcal{F}_{kl} \epsilon_{rijkl} \text{Tr} [\tilde{G}_{mn} \tilde{G}_{st} \hat{p}_p \epsilon_{mnst}] f.$$ (A.5)

The last term in (A.5) gives the 5 + 1 dimensional chiral magnetic effect

$$j_r^{CME} = -\frac{3}{8} \int \frac{d^5p}{(2\pi)^5} \mathcal{F}_{ij} \mathcal{F}_{kl} \epsilon_{rijkl} (\hat{p} \cdot \hat{b}) f.$$ 

When the Fermi-Dirac distribution is considered, it yields $j_r^{CME} = (-\mu/8(2\pi)^3) \epsilon_{rijkl} \mathcal{F}_{ij} \mathcal{F}_{kl}$, at finite chemical potential $\mu$.

## B Properties of the Berry Curvature and the $\Sigma$ Matrices

Interrelation between the Dirac monopole and the Berry curvature was established for an even $d + 1$ dimensional Weyl Hamiltonian in [16]. Here, we review the arguments of [16] and also demonstrate the trace properties of the wedge products of the Berry curvature $\tilde{G}$ used in Section 4.

In terms of the energy eigenstates one can introduce a unitary matrix $U$ which diagonalizes the Weyl Hamiltonian (2.3),

$$U \mathcal{H}_\text{w} U^\dagger = \rho (\mathcal{I}^+ - \mathcal{I}^-).$$

$\mathcal{I}^+$ and $\mathcal{I}^-$ project onto the positive and negative energy subspaces. The Berry gauge field (2.4) can be written by means of $U$ as

$$A = i \mathcal{I}^+ U \partial_p U^\dagger \mathcal{I}^+.$$
Thus, one can construct the Berry curvature (2.5) in terms of $U$.

In $d = 2n + 1$ dimensions the trace of the $m \leq n$ subsequent Berry field strengths can be expressed as

$$\epsilon_{A_1A_2...A_{2m+1}} \text{Tr}[G_{A_2A_3}...G_{A_{2m}A_{2m+1}}] = (2i)^m \epsilon_{A_1A_2...A_{2m+1}} \text{Tr}[P^+(\partial_{A_2}P^+)...(\partial_{A_{2m+1}}P^+)].$$

(B.1)

We introduced

$$P^+ = U^\dagger \mathcal{I}^+ U,$$

which can be written as

$$P^+ = \frac{1}{2}(\mathcal{H}_W p + 1).$$

Plugging it into (B.1) leads to

$$\epsilon_{A_1A_2...A_{2m+1}} \text{Tr}[P^+(\partial_{A_2}P^+)...(\partial_{A_{2m+1}}P^+)] = \frac{\epsilon_{A_1A_2...A_{2m+1}}}{(2p)^{2m+1}} \text{Tr}[\mathcal{H}_W(\partial_{A_2}\mathcal{H}_W)...(\partial_{A_{2m+1}}\mathcal{H}_W)]$$

$$= \frac{\epsilon_{A_1A_2...A_{2m+1}}}{(2p)^{2m+1}} \text{Tr}[\Sigma \cdot p \Sigma A_2...\Sigma A_{2m+1}].$$

(B.2)

Inspecting (B.1) and (B.2) one observes that the trace of the wedge products of the Berry curvature, $G$, can be expressed in terms of the trace of the antisymmetrized $\Sigma$ matrices:

$$\epsilon_{A_1A_2...A_{2m+1}} \text{Tr}[G_{A_2A_3}...G_{A_{2m}A_{2m+1}}] = (2i)^m \frac{p_A}{(2p)^{2m+1}} \epsilon_{A_1A_2...A_{2m+1}} \text{Tr}[\Sigma A_2...\Sigma A_{2m+1}].$$

$$\Sigma_A$$ obey the Clifford algebra,

$$\{\Sigma_A, \Sigma_B\} = 2\delta_{AB}.$$

Moreover, they are traceless,

$$\text{Tr}[\Sigma_A] = 0,$$

and in $2n + 2$ dimensional spacetime they satisfy the following identity,

$$\Sigma_1...\Sigma_{2n+1} = i^{n+2}1_{2^n \times 2^n}.$$

Thus the trace of $2n + 1$ antisymmetric product of the $\Sigma$ matrices yields

$$\frac{1}{(2n + 1)!} \epsilon_{A_1...A_{2n+1}} \text{Tr}[\Sigma A_1...\Sigma A_{2n+1}] = i^{n+2}2^n.$$

Actually one can observe that the trace of the product of even number of different $\Sigma$ matrices always vanishes because of satisfying the Clifford algebra:

$$\text{Tr}[\Sigma A_1...\Sigma A_{2m}] = 0.$$

Moreover, it can be easily shown that the trace of the product of $2m + 1$ different $\Sigma$ matrices is equal to the trace of the product of the remaining $2(n - m)$ $\Sigma$ matrices which is equal to zero. Therefore, the trace of the antisymmetrized product of the Berry field strength vanishes

$$\epsilon_{A_1A_2...A_{2m-1}A_{2m}...A_{2n+1}} \text{Tr}[G_{A_1A_2}...G_{A_{2m-1}A_{2m}}] = 0,$$

for the case $m < n$. When $m = n$ one finds

$$\epsilon_{A_1A_2A_3...A_{2n}A_{2n+1}} \text{Tr}[G_{A_2A_3}...G_{A_{2n}A_{2n+1}}] = (-1)^{n+1}(2n)\frac{p_A}{2p^{2n+1}},$$

which is the Dirac monopole field.
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