Exact solutions in metric $f(R)$-gravity for static axisymmetric spacetime

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Axially symmetric static vacuum exact solution (ASSVES) in Weyl coordinates are studied in $f(R)$ theories of gravity. In particular, we obtain a general explicit expression for the dependence between $df(R)/dR$ and the $r$ and $z$ coordinates and then the corresponding explicit form of $f(R)$ is obtained. Next, we present in detail the explicit solution of the modified field equations corresponding to the Newtonian potential due to a punctual mass placed at the origin of coordinates and also the Schwarzschild solution to the modified field equations.

I. INTRODUCTION

In the last years, $f(R)$-gravities as modified theories have much attention as one of promising candidates for to overcome the so-called cosmological constant or dark energy problem (see [1] for recent reviews). Thus, there has been a wide physical and mathematical stimulus for their study, leading to a strong number of interesting results in the context of the exact solutions.

However, to find exact solutions in $f(R)$-gravities is a very difficult problem due to the highly nonlinearity of the equations involved. One have to impose some symmetry in order to simplify the problem. Spherically symmetric solution are the most widely studied exact solutions in the context of $f(R)$-gravity [2] mainly due to the solvability of the equations and also more importantly due to the astrophysical interest present in this kind of symmetry. Also in the recent years some exact solutions for static cylindrically symmetric in $f(R)$-gravity are presented [3]. Now, although there has been a lot of work in the last years we think it would be fair to say that, except for maybe [4], we do not have a fully integrated explicit exact axially symmetric solution of $f(R)$-gravity.

Our purpose here is to consider axially symmetric static vacuum solution in Weyl coordinates in $f(R)$ theories of gravity. In particular, we obtain the general explicit expression for the dependence between $df(R)/dR$ and the $r$ and $z$ coordinates and then the corresponding explicit form of $f(R)$ is obtained. Next, we present in detail the explicit solution of the modified field equations corresponding to the Newtonian potential due to a punctual mass placed at the origin of coordinates and also the Schwarzschild solution to the modified field equations.

The paper is organized as follows. In section II the action for $f(R)$ gravity is given by

$$S = \int \left( \frac{1}{16\pi G} f(R) + \mathcal{L}_m \right) \sqrt{-g} d^4 x,$$

where $G$ is the gravitational constant, $R$ is the Ricci (curvature) scalar and $\mathcal{L}_m$ is the matter Lagrangian. The field equation resulting from this action are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (\tilde{T}^g_{\mu\nu} + \tilde{T}^m_{\mu\nu}),$$

where the gravitational stress-energy tensor is

$$8\pi G \tilde{T}_{\mu\nu}^g = T_{\mu\nu}^g = \frac{1}{F_R} \left[ \frac{1}{2} g_{\mu\nu} \left( f(R) - RF_R \right) \right. \left. + F^{\alpha\beta}_R (g_{\alpha\nu} g_{\beta\mu} - g_{\mu\nu} g_{\alpha\beta}) \right],$$

with $F_R \equiv df(R)/dR$ and $\tilde{T}_{\mu\nu}^m \equiv T_{\mu\nu}^m/F_R$, being $T_{\mu\nu}^m$ the energy-stress tensor derived from the matter Lagrangian $\mathcal{L}_m$ in the action (1). We can write (2), equivalently, in the form

$$F_{R} R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F_R + g_{\mu\nu} \Box F_R = 8\pi G T_{\mu\nu}^m.$$  

The contraction of above field equations gives the following relation between $f(R)$ and its derivative $F_R$:

$$F_{R} R - 2f(R) + 3\Box F_R = 8\pi G T^m,$$

where, $T^m = T^m_{\mu\nu}$ is the trace of the energy-stress.

Interested in the static axially symmetric solutions of (4), we start with the Weyl-Lewis-Papapetrou metric in the cylindrical coordinates $x^\alpha = (t, \varphi, r, z)$ given by

$$ds^2 = -e^{2\phi} dt^2 + e^{-2\phi}[r^2 d\varphi^2 + e^{2\lambda}(dr^2 + dz^2)],$$

section III we present in detail the explicit solution of the modified field equations corresponding to the Newtonian potential due to a punctual mass placed at the origin of coordinates and also the Schwarzschild solution to the modified field equations.
where $\phi$ and $\lambda$ are continuous functions of $r$ and $z$. Using [7], the modified Einstein equations [1] becomes

$$F_R R_{\mu\nu} - \nabla_\mu \nabla_\nu F_R - 8\pi G T^m_{\mu\nu} = g_{\mu\nu} B,$$

(7)

where $B = \frac{1}{2}(F_R R - \Box F_R - 8\pi G T^m)$. For the metric [3], the non-zero components of the Ricci curvature tensor are simply given by

$$R_{00} = e^{4\phi-2\lambda} \nabla^2 \phi,$$

(8a)

$$R_{11} = r^2 e^{-2\lambda} \nabla^2 \phi,$$

(8b)

$$R_{22} = -\nabla^2 \lambda + \nabla^2 \phi + 2 \frac{\lambda}{r} \lambda, r - 2 \phi_z^2,$$

(8c)

$$R_{43} = -\nabla^2 \lambda + \nabla^2 \phi - 2 \phi_z^2,$$

(8d)

$$R_{23} = \frac{1}{r} \lambda, z - 2 \phi, r \phi, z,$$

(8e)

while a straightforward computation of the curvature scalar of this metric leads to following result:

$$R = 2 e^{2\phi - 2\lambda} (-\nabla^2 \lambda + \nabla^2 \phi + \frac{1}{r} \lambda, r - \phi_z^2),$$

(9)

where $\nabla^2$ is the usual Laplace operator in cylindrical coordinates. Thus, from (7) and (8) we have the following equations system:

$$\frac{F_R R_{00} - 8\pi G T^m_{00}}{g_{00}} = B,$$

(10a)

$$\frac{F_R R_{11} - 8\pi G T^m_{11}}{g_{11}} = B,$$

(10b)

$$\frac{F_R R_{22} - 8\pi G T^m_{22} - F_R, 22}{g_{22}} = B,$$

(10c)

$$\frac{F_R R_{33} - 8\pi G T^m_{33} - F_R, 33}{g_{33}} = B,$$

(10d)

$$\frac{F_R R_{23} - 8\pi G T^m_{23} - F_R, 23}{g_{23}} = 0.$$

(10e)

The last equations system allow us to write the following independent field equations:

$$\nabla^2 \phi = -\frac{4\pi G}{F_R} e^{2(\phi - \lambda)} [T^m_{00} - T^m_{11}],$$

(11a)

$$\lambda, r = r (\phi_r^2 - \phi_z^2) + \frac{4\pi G r}{F_R} e^{2(\phi - \lambda)} [T^m_{22} - T^m_{33}] + \frac{r}{2F_R} [F_R, 22 - F_R, 33],$$

(11b)

$$\lambda, z = 2r \phi, r \phi, z + \frac{8\pi G r}{F_R} T^m_{23} + \frac{r F_R, 23}{F_R}.$$  

(11c)

Obviously it is not an easy task to find a general solution to the above equations, so in the following sections we will discuss some particular solutions.

## III. VACUUM SOLUTIONS IN $f(R)$ GRAVITY FOR STATIC AXIALLY SYMMETRIC SPACE-TIME

To simplify the problem we suppose the vacuum case. Correspondingly, we put:

$$\nabla^2 \phi = 0,$$

(12a)

$$\lambda, r = r (\phi_r^2 - \phi_z^2) + \frac{r}{2F_R} (F_R, rr - F_R, zz),$$

(12b)

$$\lambda, z = 2r \phi, r \phi, z + \frac{r F_R, zz}{F_R},$$

(12c)

$$r F_R, rr (F_R, rr - F_R, zz) + r F_R \nabla^2 (F_R, z) + F_R, rz (F_R, rz - 2r F_R, r, r) = 0,$$

(12d)

$$R = -2 e^{2\phi - 2\lambda} (\lambda, rr + \lambda, zz + \phi_r^2 + \phi_z^2),$$

(12e)

$$f(R) = \frac{1}{2} F_R R + \frac{3}{2} \Box F_R,$$

(12f)

being [12d] the condition of integrability of $\lambda$. Otherwise, we need obtain a explicit for of $f(R)$. Its very easy to prove that for the metric [4] we have:

$$\Box F_R = e^{2(\phi - \lambda)} (F_R, rr + F_R, zz).$$

(13)

So, by inserting (13) in (12f) and using (12e), we have

$$f(R) = \frac{1}{2} F_R R \left[ 1 - \frac{3W(r)}{2F_R} \right],$$

(14)

where

$$W(r) = \frac{F_R, rr + F_R, zz}{\lambda, rr + \lambda, zz + \phi_r^2 + \phi_z^2}.$$  

(15)

With the aim to obtain some solutions to the [12] equation systems, we assume some simplifications. We first suppose that is possible to write

$$F_R(r, z) = U(r) V(z).$$

(16)

So, by substituting back into (12d) we have:

$$2U, r, r (U, r, r - U) - 2r U U, rr = \frac{V, zz, V - V, z, z, V}{V, z, V}.$$  

(17)

By setting each side equal to $l^2$, an arbitrary constant of separation, we obtain

$$2U, r, r (U, r, r - U) - 2r U U, rr = l^2$$

(18a)

$$\frac{V, zz, V - V, z, z, V}{V, z, V} = l^2.$$  

(18b)

We can write (18a) as

$$\frac{dM(r)}{dr} + \frac{M(r)}{r} = \frac{l^2}{2},$$

(19)

where $M(r) = U^{-1} U, r$. We will now obtain a solution of (19), to do it, we suppose that is possible to write

$$M = M_h + M_p,$$

(20)
where $M_h$ is the general solution of the homogeneous differential equation

$$\frac{dM_h(r)}{dr} + \frac{M_h(r)}{r} = 0,$$  \hfill (21)

which is

$$M_h = \frac{n}{r},$$  \hfill (22)

being $n$ an arbitrary constant. Whereas we can see that $M_p = -l^2 r/4$ is a particular solution of (19). Consequently we have the differential equation

$$M(r) = U^{-1} U_r = \frac{n}{r} - \frac{l^2 r}{4},$$  \hfill (23)

which solution is

$$U(r) = cr^n e^{-l^2 r^2/4},$$  \hfill (24)

with $c$ an arbitrary constant. On the other hand, its very easy to prove that

$$V(z) = e^{b z},$$  \hfill (25)

being $b$ and arbitrary constant, is solution of (18) if $b$ and $l$ both satisfy the condition

$$bl^2 = 0.$$  \hfill (26)

So, we have the following possible solutions for $F_R$:

(i) $b = 0$ and $l = 0$. In this case

$$F_R = cr^n.$$  \hfill (27)

(ii) $b \neq 0$ and $l = 0$. In this case

$$F_R = cr^n e^{b z}.$$  \hfill (28)

(iii) $b = 0$ and $l \neq 0$. In this case

$$F_R = cr^n e^{-l^2 r^2/4}.$$  \hfill (29)

Consequently, by substituting (27) (or (28)) in (14) we receive

$$f(R) = \frac{1}{2} R \frac{df}{dR},$$  \hfill (30)

which solution can be written as

$$f(R) = kr^{1/2},$$  \hfill (31)

being $k$ an arbitrary constant.

On the other hand, by inserting (28) in (12b) and (12c), we have

$$\lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{1}{2r} [n(n-1) - b^2 r^2],$$  \hfill (32a)

$$\lambda_z = 2r \phi_r \phi_z + bn,$$  \hfill (32b)

respectively. Whereas, by taking $F_R = cr^n e^{-l^2 r^2/4}$ we obtain from (14)

$$f(R) = \frac{1}{2} F_R R [1 - 3L(r)],$$  \hfill (33)

being

$$L(r) = \frac{(l^2 r^2 - 4n^2 - 4(l^2 r^2 + 4n)}{(3l^2 r^2 + 4n - 4)(l^2 r^2 - 4n)}.$$  \hfill (34)

And, by inserting (29) in (12b) and (12c), we have

$$\lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{1}{32r} [l^2 r^2 - 4n] \times [2(n-1) \ln r - \frac{1}{2} (br^2 - 4n)]$$

$$\lambda_z = 2r \phi_r \phi_z,$$  \hfill (35a)

respectively.

\section*{IV. PARTICULAR SOLUTIONS IN f(R) GRAVITY FOR THE VACUUM STATIC AXIALLY SYMMETRIC SPACE-TIME}

In this section we present two applications of the result obtained in the last section. We first suppose a given metric potential and then we obtain the another metric potential in “presence” of $f(R)$, by using $F_R = cr^n e^{b z}$, and $F_R = cr^n e^{-l^2 r^2/4}$, the previous results.

\subsection*{A. The punctual mass $m$ at the origin}

The metric potential corresponding to a punctual mass $m$ at the origin of a coordinate system is given by

$$\phi = -\frac{m}{\sqrt{r^2 + z^2}},$$  \hfill (36)

so, by using the result $F_R = cr^n e^{b z}$, we obtain for $\lambda$:

$$\lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{1}{2r} [n(n-1) - b^2 r^2],$$  \hfill (37)

$$\lambda_z = 2r \phi_r \phi_z + bn,$$  \hfill (38)

consequently, $\lambda$ is given by

$$\lambda = \frac{\sqrt{r^2 + z^2}}{4},$$  \hfill (39)

$$\lambda = -2m r^2 + (r^2 + z^2)^2 \times [2(n-1) \ln r - \frac{1}{2} (br^2 - 4n)]$$

Whereas, if $F_R = cr^n e^{-l^2 r^2/4}$, we have for the derivatives of $\lambda$:

$$\lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{1}{32r} [16n(n-1) - 4(2n+1)l^2 r^2 + r^4 l^4]$$

$$\lambda_z = 2r \phi_r \phi_z,$$  \hfill (40b)
which solution is
\[ \lambda = \frac{\tilde{\lambda}}{128(r^2 + z^2)^2}, \]
being
\[ \tilde{\lambda} = [(l^2r^3 + l^2rz^2)^2 - 64m^2r^2]x^2 + [64n(n - 1) \ln(r) - 8(2n + 1)l^2r^2](r^2 + z^2)^2. \]

**B. The Schwarzschild solution**

In prolate spheroidal coordinates \((x, y)\) related to the cylindrical coordinates \((r, z)\) through the relations
\[ r^2 = m(x^2 - 1)(1 - y^2), \quad z = mx. \]
The Einstein vacuum equations in \(f(R)\) gravity, for static axially symmetric space-time can be cast:

\begin{align*}
\nabla^2 \phi &= 0 \quad (43a) \\
\lambda_{,x} &= \tilde{\lambda}_{,x} + \beta r_{,x} + \Omega z_{,x}, \quad (43b) \\
\lambda_{,y} &= \tilde{\lambda}_{,y} + \beta r_{,y} + \Omega z_{,y}, \quad (43c)
\end{align*}

where
\[ \phi = \frac{1}{2} \ln \left[ \frac{x - 1}{x + 1} \right], \quad \tilde{\lambda} = \frac{1}{2} \ln \left[ \frac{x^2 - 1}{x^2 - y^2} \right] \]
are the metric potentials corresponding to the usual Schwarzschild solution in the usual Einstein’s gravity, and
\[ \beta = \frac{r}{2F_R} (F_{R,rr} - F_{R,zz}), \quad \Omega = \frac{rF_{R,yz}}{F_R}. \]
so, in the same way as the last case, we will obtain an expthelricit form of \(\lambda\) by take the different values of \(F_R\). So, in the case \(F_R = c r^a e^{b z}\), we have
\begin{align*}
\lambda_{,x} &= \frac{x(1 - y^2)}{2(x^2 - 1)(x^2 - y^2)} + \frac{n(n - 1)x}{2(x^2 - 1)} \\
&\quad - \frac{b^2m^2}{2} x(1 - y^2) + b m n y \\
\lambda_{,y} &= \frac{y}{x^2 - y^2} - \frac{n(n - 1)y}{2(1 - y^2)} \\
&\quad + \frac{b^2m^2y}{2}(x^2 - 1) + b m n x,
\end{align*}
which solution is
\[ \lambda = \tilde{\lambda} + \frac{n(n - 1)}{4} \ln \left[ \frac{(x^2 - 1)(1 - y^2)}{4} \right] + \frac{b^2m^2Q}{4} + b m n x y, \]
\[ Q = x^2y^2 - x^2 - y^2. \]
whereas in the case \(F_R = c r^n e^{-\ell^2 r^2/8}\), we have for the derivatives of \(\lambda\):
\begin{align*}
\lambda_{,x} &= \tilde{\lambda}_{,x} + \frac{x}{32(x^2 - 1)} P, \quad (49a) \\
\lambda_{,y} &= \tilde{\lambda}_{,y} - \frac{y}{32(1 - y^2)} P, \quad (49b) \\
P &= [16n(n - 1) + \ell^4 r^4 - 4l^2(2n + 1)r^2],
\end{align*}
which solution is
\[ \lambda = \tilde{\lambda} + \frac{n(n - 1)}{4} \ln \left[ \frac{(x^2 - 1)(1 - y^2)}{Q + 2} + 8(2n + 1)Q. \right. \]

The issue of static and axially symmetric solutions in \(f(R)-\)gravity is an important theme in the context of the exact solutions. In this paper, we have presented axially symmetric static vacuum solution in Weyl coordinates in \(f(R)\) theories of gravity. In particular, by the introduction of the integrability condition of one of the metric potentials of the Weyl-Lewis-Papapetrou line element and using the method of separation of variables we have obtained a general explicit expression for the dependence between \(df(R)/dR\) and the \(r\) and \(z\) coordinates and then the corresponding general explicit form of \(f(R)\). We have also presented in detail the explicit solution of the modified field equations corresponding to the Newtonian potential due to a punctual mass placed at the origin of coordinates and also the Schwarzschild solution to the modified field equations.

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