Sum Uncertainty Relations: Uncertainty Regions for Qubits and Qutrits

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Received: 19 August 2020 / Accepted: 7 March 2021 / Published online: 20 April 2021 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
We investigate the notion of uncertainty region using the variance based sum uncertainty relation for qubits and qutrits. We compare uncertainty region of the qubit (a 2-level system) with that of the qutrit (3-level system) by considering sum uncertainty relation for two non-commuting Pauli-like observables, acting on the two dimensional qubit Hilbert space. We identify that physically valid uncertainty region of a qubit is smaller than that of a qutrit. This implies that an enhanced precision can be achieved in the measurement of incompatible Pauli-like observables acting on the 2-dimensional subspace of a qutrit Hilbert space. We discuss the implication of the reduced uncertainties in the steady states of \( \Lambda, V, \Xi \) types of 3-level atomic systems. Furthermore, we construct a two-qubit permutation symmetric state, corresponding to a 3-level system and show that the reduction in the sum uncertainty value - or equivalently, increased uncertainty region of a qutrit system – is a consequence of quantum entanglement in the two-qubit system. Our results suggest that uncertainty region can be used as a dimensional witness.

Keywords Sum uncertainty relation · Uncertainty region · Entanglement · 3-Level system · Steady-state population

1 Introduction
Quantum theory prevents assignment of precise values for two or more incompatible observables simultaneously. Heisenberg’s heuristic argument [1] highlighted this uncertainty associated with the non-commuting position (\( Q \)) and momentum (\( P \)) observables, in terms of the constraints placed on the product of their standard deviations. If position of a particle is measured, prediction of its momentum gets inaccurate and vice versa. A mathematically
formal version of the position-momentum uncertainty relation \((\Delta Q)(\Delta P) \geq \frac{\hbar}{2}\) was subsequently formulated by Kennard [2]. Furthermore, Robertson [3] (motivated by Weyl’s arguments [4]) extended the uncertainty relations to any arbitrary pairs of non-commuting observables \(A, B\).

Different forms of uncertainty relations have been formulated over the years [5–19], capturing the trade-off between two or more non-commuting observables. It has been shown that uncertainty relations play a crucial role in quantum information processing tasks like quantum key distribution [15, 16, 20–22]. Non-trivial state-dependent uncertainty relations which are experimentally verifiable are found to be of importance in device-independent cryptography [10].

A broader perspective on uncertainty relations, based on the concept of uncertainty regions, is recently being explored [17–19, 23] and provides a geometric visualization of the uncertainty relation. For any uncertainty relation, the corresponding uncertainty region is the legitimate domain of standard deviation (or any other measure of uncertainties) of a pair (or triple) of observables, in the entire range of their possible values [23]. Points \((\Delta A_1, \Delta A_2)\) inside the uncertainty region specify the uncertainty in the simultaneous measurement of a pair of observables \(A_1, A_2\). Different types of uncertainty relations can be chosen for analysing the uncertainty regions [17–19, 23]. In this work we have chosen variance based sum uncertainty relation, a state-independent uncertainty relation, proposed by Hofmann and Takeuchi [9], to analyze the uncertainty regions/minimum of the variance based sum uncertainty relation for two non-commuting Pauli-like observables.

The structure of the paper is as follows: In Section 2, we outline the geometry of uncertainty region corresponding to variance based sum-uncertainty relation for a pair of incompatible Pauli-like observables, when they are measured in quantum states of qubits (2-level systems) and qutrits (3-level systems). We show that uncertainty region of qutrits is larger, containing points with enhanced measurement precision for incompatible Pauli-like observables, in comparison with that of qubits. In Section 3, we express the minimum of the sum of variances of two noncommuting Pauli-like observables \(A(ij) = \sigma^{(ij)} \cdot \hat{a}, A(ij) = \sigma^{(ij)} \cdot \hat{b}, \hat{a} \cdot \hat{b} = 0\) of a 3-level system, given that both the observables are restricted to the 2-dimensional (qubit) subspace (labelled by the pair of indices \((ij), i < j = 1, 2, 3\)) of a 3-level (qutrit) system, in terms of the populations \(\rho_{ii}, \rho_{jj}\) in the \(i^{th}\) and \(j^{th}\) levels. We discuss implication of the reduced uncertainties in the steady states of \(\Lambda, V\) and \(\Xi\) types of 3-level atomic systems. Section 4 details the construction of permutation symmetric two-qubit system corresponding to a qutrit state and explicit evaluation of equivalent sum uncertainty relation for Pauli-like observables. In Section 5, we establish that separable two-qubit states can never achieve utmost precision in the measurement of incompatible Pauli-like observables. We also show that maximum precision in measurement of the non-commuting Pauli observables is possible using entangled symmetric two-qubit states. Section 6 provides concluding remarks.

2 Uncertainty Regions for Qubits and Qutrits

The well-known generalized uncertainty relation [3] for observables \(A_1, A_2\) is given by

\[
(\Delta A_1)(\Delta A_2) \geq \frac{1}{2} |\{[A_1, A_2]\}|
\]  

(1)
where \([A_1, A_2] = A_1A_2 - A_2A_1\) is the commutator and \(\Delta A_1, \Delta A_2\) defined by
\[
\Delta A_1 = \sqrt{\Delta^2 A_1}, \quad \Delta^2 A_1 = \langle A_1^2 \rangle - \langle A_1 \rangle^2 \\
\Delta A_2 = \sqrt{\Delta^2 A_2}, \quad \Delta^2 A_2 = \langle A_2^2 \rangle - \langle A_2 \rangle^2.
\]
are the standard deviations of \(A_1, A_2\) in any quantum state \(\rho\). Here, \(\langle \cdots \rangle = \text{Tr}(\rho \cdots)\) is the expectation value of any observable in the state \(\rho\).

Hofmann and Takeuchi [9] reformulated the uncertainty relation (1) in the form of sum of variances. Given a set of non-commuting operators \(A_i\), \(i = 1, 2, \ldots, n\), they have shown that
\[
\sum_{i=1}^{n} \Delta^2 A_i \geq k_A, \quad k_A \text{ being a non-negative real number.} \tag{3}
\]
It is a \textit{state-independent} uncertainty relation [9] with non-trivial bound for incompatible observables.

We now set up the sum-uncertainty relation in (3) for a pair of observables \(A_1, A_2\) acting on the most general state of a qubit:
\[
\rho_{\text{qubit}} = \frac{1}{2} \left[ I_2 + r_1 \sigma_1 + r_2 \sigma_2 + r_3 \sigma_3 \right], \quad r_1^2 + r_2^2 + r_3^2 \leq 1 \tag{4}
\]
where \(\sigma_i, i = 1, 2, 3\) are Pauli spin operators, \(I_2\) is the two-dimensional identity operator and \(r = (r_1, r_2, r_3), (|r| \leq 1)\) is a real three dimensional vector, the mean spin vector of the state \(\rho_{\text{qubit}}\). On choosing \(A_1 = \sigma_1\) and \(A_2 = \sigma_2\), we get \(\langle A_1^2 \rangle = \langle A_2^2 \rangle = 1, \langle A_1 \rangle = r_1, \langle A_2 \rangle = r_2\). With \(\Delta^2 A_1 = 1 - r_1^2, \Delta^2 A_2 = 1 - r_2^2\), the sum-uncertainty relation becomes
\[
\Delta^2 A_1 + \Delta^2 A_2 = 2 - (r_1^2 + r_2^2) \geq 1, \quad 0 \leq \Delta A_1 \leq 1, \quad 0 \leq \Delta A_2 \leq 1. \tag{5}
\]
In general, we consider the \textit{orthogonal} Pauli-observables
\[
A_1 = \sigma \cdot \hat{a}, \quad A_2 = \sigma \cdot \hat{b}, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3), \quad \hat{a} \cdot \hat{b} = 0 \tag{6}
\]
and we readily have \(\langle A_1 \rangle = \hat{a} \cdot r, \langle A_2 \rangle = \hat{b} \cdot r, \langle A_1^2 \rangle = \hat{a} \cdot \hat{a} = 1, \langle A_2^2 \rangle = \hat{b} \cdot \hat{b} = 1\) leading to
\[
\Delta^2 A_1 = 1 - (\hat{a} \cdot r)^2, \quad \Delta^2 A_2 = 1 - (\hat{b} \cdot r)^2 \\
\Delta^2 A_1 + \Delta^2 A_2 = 2 - (\hat{a} \cdot r)^2 - (\hat{b} \cdot r)^2. \tag{7}
\]
As \(|r| \leq 1, |\hat{a}| = |\hat{b}| = 1\), we have \(\hat{a} \cdot r \leq 1, \hat{b} \cdot r \leq 1\) and from (7) we obtain the following sum-uncertainty relation for orthogonal Pauli-observables on a qubit:
\[
\Delta^2 A_1 + \Delta^2 A_2 \geq 1 \quad \text{with} \quad 0 \leq \Delta A_1 \leq 1, \quad 0 \leq \Delta A_2 \leq 1. \tag{8}
\]
The sum uncertainty relation (8) implies that the points \((\Delta A_1, \Delta A_2)\) lying outside the circular quadrant form the uncertainty region for Pauli-observables \(A_1, A_2\) measured on a qubit, as can be seen in Fig. 1.

Notice that the points \((\Delta A_1, \Delta A_2)\) close to the origin correspond to measurements that result in better accuracy. But as the uncertainty region does not contain points below the circular arc (See Fig. 1), precise joint measurements of Pauli-observables on a qubit are impossible and Fig. 1 provides clear visualization of this fact.

We now consider a 3-level system,
\[
\rho_{\text{qudit}} = \omega |\psi \rangle \langle \psi | \oplus (1 - \omega), \quad 0 \leq \omega \leq 1, \tag{9}
\]
obtained by appending an ancillary level to a single-qubit pure state $|\psi\rangle$. Here (9) corresponds to the state of a qutrit whose explicit form is given by

$$\rho_{\text{qutrit}} = \begin{pmatrix} \frac{\omega(1+r_1^2)}{2} & \frac{\omega(r_1 - ir_2^2)}{2} & 0 \\ \frac{\omega(r_1 + ir_2^2)}{2} & \frac{\omega(1-r_1^2)}{2} & 0 \\ 0 & 0 & 1 - \omega \end{pmatrix}, \quad r_1^2 + r_2^2 + r_3^2 = 1 \quad (10)$$

Here, $r_1, r_2, r_3$ are the components of the unit mean spin vector $\hat{r}$, corresponding to the pure state $|\psi\rangle$ (See (9)) and $\omega$ is a real parameter.

The uncertainty region of the qutrit in (9) has been examined in Ref. [23], for orthogonal Pauli observables

$$A_1 = \sigma \cdot \hat{a} \oplus 0, \quad A_2 = \sigma \cdot \hat{b} \oplus 0, \quad \hat{a} \cdot \hat{b} = 0. \quad (11)$$

It can be seen that [23]

$$\langle A_1 \rangle = \omega(\hat{a} \cdot \hat{r}), \quad \langle A_2 \rangle = \omega(\hat{b} \cdot \hat{r}), \quad \langle A_1^2 \rangle = \langle A_2^2 \rangle = \omega, \quad (12)$$

leading to

$$\Delta^2 A_1 = \omega - \omega^2 (\hat{a} \cdot \hat{r})^2, \quad \Delta^2 A_2 = \omega - \omega^2 (\hat{b} \cdot \hat{r})^2. \quad (13)$$
On fixing $\Delta^2 A_1$ and minimizing $\Delta^2 A_2$, one obtains [23]

$$\text{(14)} 
(\Delta A_2)_{\text{min}} = \Delta A_1 \sqrt{(1 - \Delta^2 A_1)}. $$

Similarly, fixing $\Delta^2 A_2$ and minimizing $\Delta^2 A_1$ results in [23]

$$\text{(15)}
(\Delta A_1)_{\text{min}} = \Delta A_2 \sqrt{(1 - \Delta^2 A_2)}. $$

From (14), (15), we see that when $\Delta A_1 = 0$, $\Delta A_2$ can also become zero and vice versa. This means, origin $(0, 0)$ of the co-ordinate system, a point corresponding to utmost precision in simultaneous measurement, is physically realizable for measurement of orthogonal Pauli observables on the qutrit state $\rho_{\text{qutrit}}$ [23]. The uncertainty region of the qutrit is larger in comparison with that of a qubit, containing points near the origin and origin itself, as can be readily seen in Fig. 2.

It is evident from the above observations that simultaneous measurements with utmost precision are impossible when non-commuting Pauli measurements are performed on a 2-dimensional Hilbert space (qubit) whereas a 3-dimensional Hilbert space (qutrit) admits enhanced precision. In the following, we show that the uncertainty sum for two Pauli-like observables in an arbitrary 3-level system reduces below that for a 2-level system.
3 Sum Uncertainty Relation for 3-level Atomic Systems

In this section we explore the sum uncertainty relation for two Pauli-like observables (i.e., atomic operators acting on any 2-level subspace of a 3-level atomic system)

\[
A^{(ij)}_1 = \sigma^{(ij)} \cdot \hat{a}, \quad \sigma^{(ij)} = \left( \sigma^{(ij)}_1, \sigma^{(ij)}_2, \sigma^{(ij)}_3 \right)
\]

\[
A^{(ij)}_2 = \sigma^{(ij)} \cdot \hat{b}, \quad i < j = 1, 2, 3,
\]

where \( \hat{a} \cdot \hat{b} = 0, \hat{a} \cdot \hat{a} = 1 = \hat{b} \cdot \hat{b} \) and

\[
\sigma^{(12)}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma^{(13)}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma^{(23)}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
\sigma^{(12)}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma^{(13)}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \sigma^{(23)}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}
\]

\[
\sigma^{(12)}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma^{(13)}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma^{(23)}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

In an arbitrary 3-level atomic system, characterized by the density matrix,

\[
\varrho_{\text{qutrit}} = \begin{pmatrix} \varrho_{11} & \varrho_{12} & \varrho_{13} \\ \varrho^*_{12} & \varrho_{22} & \varrho_{23} \\ \varrho^*_{13} & \varrho^*_{23} & \varrho_{33} \end{pmatrix},
\]

we obtain

\[
\left( A^{(ij)}_1 \right)^2 = \text{Tr} \left[ \varrho_{\text{qutrit}} \left( \sigma^{(ij)} \cdot \hat{a} \right)^2 \right] = \varrho_{ii} + \varrho_{jj},
\]

\[
\left( A^{(ij)}_2 \right)^2 = \text{Tr} \left[ \varrho_{\text{qutrit}} \left( \sigma^{(ij)} \cdot \hat{b} \right)^2 \right] = \varrho_{ii} + \varrho_{jj},
\]

and

\[
\left\langle A^{(ij)}_1 \right\rangle = \text{Tr} \left[ \varrho_{\text{qutrit}} \left( \sigma^{(ij)} \cdot \hat{a} \right) \right] = n^{(ij)} \cdot \hat{a},
\]

\[
\left\langle A^{(ij)}_2 \right\rangle = \text{Tr} \left[ \varrho_{\text{qutrit}} \left( \sigma^{(ij)} \cdot \hat{b} \right) \right] = n^{(ij)} \cdot \hat{b},
\]

where

\[
n^{(ij)} = \text{Tr} \left[ \varrho_{\text{qutrit}} \sigma^{(ij)} \right] = \left( 2 \text{ Re} \varrho_{ij}, \ 2 \text{ Im} \varrho_{ij}, \ \varrho_{ii} - \varrho_{jj} \right).
\]

Choosing \( \hat{a} = \hat{n}^{(ij)} = n^{(ij)}/|n^{(ij)}|, \ \hat{b} = \hat{n}^{(ij)}_\perp \) and simplifying the sum of variances \( \Delta^2 A^{(ij)}_1 + \Delta^2 A^{(ij)}_2 \) in the 3-level system (17) (with the help of (18), (19), (20), (21), (22)), we obtain

\[
\left[ \Delta^2 A^{(ij)}_1 + \Delta^2 A^{(ij)}_2 \right] = 2 \left( \varrho_{ii} + \varrho_{jj} \right) - |n^{(ij)}|^2
\]

\[
= 2 \left( \varrho_{ii} + \varrho_{jj} \right) - \left[ 4 |\varrho_{ij}|^2 + (\varrho_{ii} - \varrho_{jj})^2 \right].
\]

Positive semidefiniteness of the \((ij)\)th \(2 \times 2\) block of \(\varrho_{\text{qutrit}}\) imposes the condition

\[
|n^{(ij)}| \leq \varrho_{ii} + \varrho_{jj}
\]
Fig. 3 Minimum value of the uncertainty sum \([\Delta^2 A_1^{(ij)} + \Delta^2 A_2^{(ij)}]_{\text{min}}\) as a function of the populations \(\rho_{ii}, \rho_{jj}\) (see (25)). Enhanced measurement precision of incompatible Pauli-like atomic observables is ensured whenever \([\Delta^2 A_1^{(ij)} + \Delta^2 A_2^{(ij)}]_{\text{min}} < 1\)

leading to the following minimum value for the uncertainty sum:

\[
[\Delta^2 A_1^{(ij)} + \Delta^2 A_2^{(ij)}]_{\text{min}} = 2 (\varrho_{ii} + \varrho_{jj}) - (\varrho_{ii} + \varrho_{jj})^2 .
\] (25)

It may be noted that if we restrict ourselves to the 2-level system i.e., \(i, j = 1, 2\), we obtain \([\Delta^2 A_1^{(12)} + \Delta^2 A_2^{(12)}]_{\text{min}} = 1\), as \((\varrho_{11} + \varrho_{22}) = \text{Tr}[\varrho] = 1\). In other words, the uncertainty sum is always greater than 1 in a 2-level system indicating that joint measurement of the non-commuting Pauli observables \(\sigma \cdot \hat{a}, \sigma \cdot \hat{b}\) is limited by the sum uncertainty relation (8).

On the other hand, when an additional level is included, the populations \(\varrho_{ii}, \varrho_{jj}\) of the \(i^{th}\) and \(j^{th}\) levels \((i > j = 1, 2, 3)\) play a crucial role in enhancing the measurement precision of the atomic observables \(\sigma^{(ij)} \cdot \hat{a}, \sigma^{(ij)} \cdot \hat{b}\).

In Fig. 3 we have plotted the minimum value of the uncertainty sum \([\Delta^2 A_1^{(ij)} + \Delta^2 A_2^{(ij)}]_{\text{min}}\) as a function of the populations \(\varrho_{ii}, \varrho_{jj}\) (see (25)). It is clearly seen that the minimum value of the uncertainty sum (25) can take values smaller than 1. This establishes the advantage of the additional level for improving precision in the measurement of atomic observables.

Considerable research interest has been evinced in exploring the response of \(\Lambda, \Xi, \nu\) types of 3-level atomic systems to lasing radiation [24]. In \(\Lambda\)-type system (see Fig. 4a) transition between the two lower levels \([3], [2]\) is forbidden and the upper level \([1]\) is commonly shared in atomic transitions with levels \([3]\) and \([2]\); in \(\nu\)-type system (Fig. 4b), transitions
from the lower level \( |3 \rangle \) with the two upper levels \( |1 \rangle \) and \( |2 \rangle \) are allowed, but the transition \( |1 \rangle \leftrightarrow |2 \rangle \) between the upper levels is forbidden. While atomic transitions \( |1 \rangle \leftrightarrow |2 \rangle \) and \( |2 \rangle \leftrightarrow |3 \rangle \) are allowed in \( \Xi \)-type system (Fig. 4c), the transition \( |1 \rangle \leftrightarrow |3 \rangle \) is forbidden. It is of interest to consider any two levels of the atomic 3-level system, between which transitions are allowed, as a qubit, and explore if there is any enhanced precision in the measurement of two non-commuting qubit operators. To this end, we consider the coherent population trapping state in a \( \Lambda \) type atomic system \([25–28]\): \[
\rho_{\Lambda}^{33} = 1/2 = \rho_{\Lambda}^{22}, \quad \rho_{\Lambda}^{11} = 0. \tag{26}
\]

It is clearly seen that the uncertainty sum (25) with \( i = 1, j = 2 \) and \( i = 1, j = 3 \) is given by \[
\left[ \Delta^2 A_{(12)}^{(1)} + \Delta^2 A_{(12)}^{(2)} \right]_{\min}^{\Lambda} = 2 (\rho_{11}^{\Lambda} + \rho_{22}^{\Lambda}) - (\rho_{11}^{\Lambda} + \rho_{22}^{\Lambda})^2 = 0.75
\]
\[
\left[ \Delta^2 A_{(13)}^{(1)} + \Delta^2 A_{(13)}^{(2)} \right]_{\min}^{\Lambda} = 2 (\rho_{11}^{\Lambda} + \rho_{33}^{\Lambda}) - (\rho_{11}^{\Lambda} + \rho_{33}^{\Lambda})^2 = 0.75
\]

revealing improved precision in the measurements of the qubit operator pairs \( \{A_{(12)}^{(1)}, A_{(12)}^{(2)}\} \) and \( \{A_{(13)}^{(1)}, A_{(13)}^{(2)}\} \).

In the case of V-type 3-level atom, with the transition \( |3 \rangle \leftrightarrow |2 \rangle \) driven by a strong-coupling laser field and \( |3 \rangle \leftrightarrow |1 \rangle \) transition driven by an incoherent pump field the steady state populations are given by \([29]\) \[
\rho_{V}^{11} \approx 0.2, \quad \rho_{V}^{22} \approx \rho_{V}^{33} \approx 0.4. \tag{27}
\]
The uncertainty sum (25) of Pauli-like atomic observables associated with \( i = 1, j = 3 \) and \( i = 2, j = 3 \) are given by \[
\left[ \Delta^2 A_{(13)}^{(1)} + \Delta^2 A_{(13)}^{(2)} \right]_{\min}^{V} = 2 (\rho_{11}^{V} + \rho_{33}^{V}) - (\rho_{11}^{V} + \rho_{33}^{V})^2 = 0.84
\]
\[
\left[ \Delta^2 A_{(23)}^{(1)} + \Delta^2 A_{(23)}^{(2)} \right]_{\min}^{V} = 2 (\rho_{22}^{V} + \rho_{33}^{V}) - (\rho_{22}^{V} + \rho_{33}^{V})^2 = 0.96.
\]
Thus the steady state of V-type atomic qutrit (see (27)) offers advantage over 2-level atomic system in reducing the uncertainty sum of non-commuting Pauli-like atomic observables.

Populations in the steady state of a 3-level \( \Xi \) atomic system \([28, 30]\) satisfy the condition \[
\rho_{\Xi}^{11} = \rho_{\Xi}^{22} \leq \frac{1}{3}, \quad \frac{1}{3} \leq \rho_{\Xi}^{33} \leq \frac{1}{2}. \tag{28}
\]
We thus obtain the limiting value of the uncertainty sum as,

\[
\left[ \Delta^2 A^{(12)}_1 + \Delta^2 A^{(12)}_2 \right]_{\text{min}}^{\Xi} = 2 \left( 1 - \rho_{33}^{\Xi} \right) - \left( 1 - \rho_{33}^{\Xi} \right)^2 ; \quad \frac{1}{3} \leq \rho_{33}^{\Xi} \leq \frac{1}{2},
\]

\[
\implies \frac{3}{4} \leq \left[ \Delta^2 A^{(12)}_1 + \Delta^2 A^{(12)}_2 \right]_{\text{min}}^{\Xi} \leq \frac{8}{9},
\]

\[
\left[ \Delta^2 A^{(23)}_1 + \Delta^2 A^{(23)}_2 \right]_{\text{min}}^{\Xi} = 2 \left( 1 - \rho_{11}^{\Xi} \right) - \left( 1 - \rho_{11}^{\Xi} \right)^2 ; \quad 0 \leq \rho_{11}^{\Xi} \leq \frac{1}{3},
\]

\[
\implies \frac{8}{9} \leq \left[ \Delta^2 A^{23}_1 + \Delta^2 A^{23}_2 \right]_{\text{min}}^{\Xi} \leq 1,
\]

for the atomic Pauli-like observables associated with \( i = 1, j = 2 \) and \( i = 2, j = 3 \) atomic levels of the \( \Xi \)-type atomic system. More detailed investigations on improving measurement precision of non-commuting Pauli-like atomic observables of a driven 3-level system, beyond what can be achieved in a 2-level system, will be reported separately.

### 4 Transformation of Qutrit State into a Two-Qubit Symmetric State

We now analyze the enhanced accuracy of orthogonal Pauli measurements on qutrit states from a perspective based on the separability/non-separability of two-qubit symmetric states corresponding to qutrit states. In the following, we detail the construction of two-qubit symmetric states from qutrit states, in particular the state in (9). We also carry out an analysis of the role, if any, of two-qubit entanglement in the measurement precision possible in the qutrit state (9).

A two-qubit symmetric state belongs to the 3-dimensional maximum multiplicity space of the collective angular momentum \( j = j_1 + j_2, j_1 = j_2 = \frac{1}{2} \). With the dimensions of the qutrit space and that of the two-qubit symmetric state (expressed in angular momentum basis) being equal, they have a one-one correspondence. Here, we outline this correspondence and accomplish the construction of two-qubit symmetric states corresponding to qutrit states.

The qutrit state in (9), expressed as a \( 3 \times 3 \) matrix in (10), can be written equivalently as

\[
\rho_{\text{qutrit}} \equiv \begin{pmatrix}
\frac{\omega(1+r_1)}{2} & \frac{\omega(r_1-i r_2)}{2} & 0 & 0 \\
\frac{\omega(r_1+i r_2)}{2} & \frac{\omega(1-r_2)}{2} & 0 & 0 \\
0 & 0 & 1 - \omega & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (29)
\]

The density matrix \( \rho_{AB} \) of a two-qubit symmetric state is given by [31]

\[
\rho_{AB} = \frac{1}{4} \left[ I_2 \otimes I_2 + \sum_{i=1}^{3} s_i (\sigma_i \otimes I_2 + I_2 \otimes \sigma_i) + \sum_{i,j=1}^{3} t_{ij} (\sigma_i \otimes \sigma_j) \right], \quad (30)
\]
where $t_{ij} = t_{ji}$. The elements $\rho_{ij}, i, j = 1, 2, 3, 4$ of the $4 \times 4$ matrix $\rho_{AB}$ are explicitly given by

$$
\rho_{11} = \frac{1}{4} (1 + 2s_3 + t_{33}), \quad \rho_{22} = \frac{1}{4} (1 - t_{33}) = \rho_{33}
$$

$$
\rho_{12} = \rho_{13} = \frac{1}{4} (s_1 - is_2 + t_{13} - it_{23}) = \rho_{21}^* = \rho_{31}^*
$$

$$
\rho_{14} = \frac{1}{4} (t_{11} - t_{22} - 2it_{12}) = \rho_{41}^*
$$

$$
\rho_{24} = \rho_{34} = \frac{1}{4} (s_1 - is_2 - t_{13} + it_{23}) = \rho_{42}^* = \rho_{43}^*
$$

$$
\rho_{23} = \frac{1}{4} (t_{11} + t_{22}) = \rho_{32}, \quad \rho_{44} = \frac{1}{4} (1 - 2s_3 + t_{33})
$$

(31)

The standard basis for a two-qubit state is given by the direct product basis (uncoupled basis) consisting of orthonormal vectors

$$
|1/2; 1/2\rangle, \; |1/2; -1/2\rangle, \; | -1/2; 1/2\rangle, \; | -1/2; -1/2\rangle
$$

(32)

where $|m_1; m_2\rangle = |m_1\rangle \otimes |m_2\rangle$, $m_1, m_2 = 1/2, -1/2$.

One can readily express the direct product basis $\{|m_1; m_2\rangle\}$ in terms of the collective angular momentum basis (coupled basis) $\{|jm\rangle\}$, ($j = 1, 0, -j \leq m \leq j$ for each $j$) and vice versa, through

$$
|11\rangle = |1/2; 1/2\rangle, \quad |10\rangle = \frac{1}{\sqrt{2}} (|1/2; -1/2\rangle + | -1/2; 1/2\rangle)
$$

$$
|1 - 1\rangle = | -1/2; -1/2\rangle, \quad |00\rangle = \frac{1}{\sqrt{2}} (|1/2; -1/2\rangle - | -1/2; 1/2\rangle).
$$

(33)

From (33), it follows that the unitary matrix

$$
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}, \quad U^\dagger U = U U^\dagger = I_4
$$

(34)

where $^\dagger$ denotes hermitian conjugate, corresponds to the transformation from coupled basis to uncoupled basis. The similarity transformation $U^\dagger \rho_{\text{qutrit}} U$ effects the transformation of the state $\rho_{\text{qutrit}}$ (See (9), (29)) to its equivalent two-qubit symmetric state $\rho_{AB}$. Explicitly, we have

$$
\rho_{AB} = U^\dagger \rho_{\text{qutrit}} U = \frac{1}{2} \begin{pmatrix}
(1 + r_3)\omega & 0 & 0 & (r_1 - ir_2)\omega \\
0 & 1 - \omega & 1 - \omega & 0 \\
0 & 1 - \omega & 1 - \omega & 0 \\
(r_1 + ir_2)\omega & 0 & 0 & (1 - r_3)\omega
\end{pmatrix}
$$

(35)

On comparing the elements (See (31)) of $\rho_{AB}$ with the corresponding elements in (35), we obtain the following relation between the parameters $\omega, r_1, r_2, r_3, (r_1^2 + r_2^2 + r_3^2 = 1)$ of the qutrit state in (9) and the parameters $s_i, t_{ij}, i, j = 1, 2, 3$ of the symmetric two-qubit state.
\( \rho_{AB} \) in (30). That is, the non-zero parameters \( s_i, t_{ij} \), \( i, j = 1, 2, 3 \) of \( \rho_{AB} \) (See (30)) are seen to be
\[
\begin{align*}
s_3 &= \omega_3, \quad t_{12} = t_{21} = \omega_r, \\
t_{11} &= (1 - \omega) + \omega_1, \quad t_{22} = (1 - \omega) - \omega_1, \quad t_{33} = 2\omega - 1.
\end{align*}
\]
Thus, a two-qubit symmetric state \( \rho_{AB} \) in (30)) with its elements given in (36) corresponds to the qutrit state \( \rho_{\text{qutrit}} \) in (9).

5 Sum Uncertainty Relation for Two-Qubit State \( \rho_{AB} \)

Here, we set up the sum uncertainty relation for the two-qubit state \( \rho_{AB} \) (See (35)) in order to establish the equivalence of its uncertainty region with that of the qutrit state \( \rho_{\text{qutrit}} \) (See (9)). In order to do this, we need to recognize the two-qubit observables \( A_1, A_2 \) which are equivalent to \( A_1, A_2 \) (See (11)). For simplicity, and without loss of generality, we choose the orthonormal vectors in (11) to be \( \hat{a} = (1, 0, 0) \), \( \hat{b} = (0, 1, 0) \) so that \( A_1 = \sigma_1 \oplus 0 \), \( A_2 = \sigma_2 \oplus 0 \). It is not difficult to see that we can express \( A_1, A_2 \) as
\[
A_1 = \sigma_1 \oplus \theta_2, \quad A_2 = \sigma_2 \oplus \theta_2, \quad \theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
to facilitate their action on the qutrit state \( \rho_{\text{qutrit}} \) expressed as a 4\( \times \)4 matrix in (29). Corresponding to the basis transformation \( \rho_{\text{qutrit}} \rightarrow \rho_{AB} \) in (35), the observables \( A_1, A_2 \) (See (37)) undergo the similarity transformation
\[
\begin{align*}
A_1 &= U^\dagger A_1 U = \frac{1}{2} [\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2], \\
A_2 &= U^\dagger A_2 U = \frac{1}{2} [\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1].
\end{align*}
\]
Here \( U \) is the basis transformation matrix (See (34)) that takes \( \rho_{\text{qutrit}} \) to \( \rho_{AB} \) (See (35)). On explicit evaluation, we get
\[
\begin{align*}
\langle A_1 \rangle &= \text{Tr}(A_1 \rho_{AB}) = \omega r_1, \quad \langle A_1^2 \rangle = \text{Tr}(A_1^2 \rho_{AB}) = \omega, \\
\langle A_2 \rangle &= \text{Tr}(A_2 \rho_{AB}) = \omega r_2, \quad \langle A_2^2 \rangle = \text{Tr}(A_2^2 \rho_{AB}) = \omega, \\
\Delta^2 A_1 &= \omega - \omega^2 r_1^2, \quad \Delta^2 A_2 = \omega - \omega^2 r_2^2.
\end{align*}
\]
The expressions for \( \Delta^2 A_1, \Delta^2 A_2 \) in (39) are the same as that obtained in (13) for the qutrit state \( \rho_{\text{qutrit}} \) (See (9)). The uncertainty region of the two-qubit state \( \rho_{AB} \) is thus the same as that of the qutrit state \( \rho_{\text{qutrit}} \), with the origin \( (\Delta A_1, \Delta A_2) = (0, 0) \) being a physically realizable point (See Fig. 2).

As \( r_1^2 + r_2^2 + r_3^2 = 1 \), the sum uncertainty relation of the two-qubit state \( \rho_{AB} \) can be simplified to (See (39))
\[
\Delta^2 A_1 + \Delta^2 A_2 = 2\omega - \omega^2 k^2, \quad k = \sqrt{r_1^2 + r_2^2} = \sqrt{1 - r_3^2}.
\]
Figure 4 shows the variation of the uncertainty sum \( \Delta^2 A_1 + \Delta^2 A_2 \) as a function of the parameters \( 0 \leq \omega, k \leq 1 \).
5.1 Sum Uncertainty Relation for Symmetric Two-Qubit Separable States

Our intention in obtaining the two-qubit counterpart $\rho_{AB}$ of the qutrit state $\rho_{\text{qutrit}}$ lies in utilizing its composite nature and examine whether separability/non-separability of $\rho_{AB}$ has any role in the better precision observed in joint measurement of Pauli observables on the single party state $\rho_{\text{qutrit}}$. In order to carry out this task, we consider the most general bipartite, symmetric separable state

$$\rho_{\text{sep}} = \sum_{i} p_i (\rho_i \otimes \rho_i), \quad i = 1, 2, 3 \ldots$$ (41)

with $0 \leq p_i \leq 1, \sum_i p_i = 1$ being the probabilities. It has been shown in Ref. [32] that the single qubit density operators $\rho_i, i = 1, 2, 3 \ldots$ constituting any symmetric separable state $\rho_{\text{sep}}$ are necessarily pure. Thus,

$$\rho_i = \frac{1}{2} \left( I_2 + \sigma \cdot \hat{s}_i \right), \quad \hat{s}_i = (s_{1i}, s_{2i}, s_{3i}), \quad s_{1i}^2 + s_{2i}^2 + s_{3i}^2 = 1.$$ (42)

We evaluate the expectation values of $A_\alpha$ and $A_\alpha^2, \alpha = 1, 2$ (See (38)) in a product state $\rho_i \otimes \rho_i$ (where $\rho_i$ is given by (42)):

$$\langle A_1 \rangle_i = \text{Tr} \left[ (\rho_i \otimes \rho_i) A_1 \right] = \frac{1}{2} \left( s_{1i}^2 - s_{2i}^2 \right),$$

$$\langle A_1^2 \rangle_i = \text{Tr} \left[ (\rho_i \otimes \rho_i) A_1^2 \right] = \frac{1}{2} \left( 1 + s_{3i}^2 \right),$$

$$\langle A_2 \rangle_i = \text{Tr} \left[ (\rho_i \otimes \rho_i) A_2 \right] = s_{1i}s_{2i},$$

$$\langle A_2^2 \rangle_i = \text{Tr} \left[ (\rho_i \otimes \rho_i) A_2^2 \right] = \frac{1}{2} \left( 1 + s_{3i}^2 \right),$$

leading to

$$\left( \Delta^2 A_1 \right)_i = \frac{1}{2} \left[ 1 + s_{3i}^2 - \frac{1}{2} (s_{1i}^2 - s_{2i}^2)^2 \right]$$

$$\left( \Delta^2 A_2 \right)_i = \frac{1}{2} \left[ 1 + s_{3i}^2 - 2s_{1i}^2s_{3i}^2 \right]$$ (43)

Using $s_{1i}^2 + s_{2i}^2 + s_{3i}^2 = 1$ and on simplification, we get

$$\left( \Delta^2 A_1 \right)_i + \left( \Delta^2 A_2 \right)_i = \frac{3}{4} + \frac{3}{2} s_{3i}^2 - \frac{1}{4} s_{3i}^4$$ (44)

From the structure of $\rho_{\text{sep}}$ (See (41)), we readily have $\Delta^2 A_1 = \sum_i p_i \left( \Delta^2 A_1 \right)_i, \Delta^2 A_2 = \sum_i p_i \left( \Delta^2 A_2 \right)_i$ and hence we get (See (44))

$$\Delta^2 A_1 + \Delta^2 A_2 = \frac{3}{4} + \frac{3}{2} \sum_i p_i s_{3i}^2 - \frac{1}{4} \sum_i p_i s_{3i}^4$$ (45)

As $0 \leq s_{3i} \leq 1$, it readily follows that

$$\left( \Delta^2 A_1 + \Delta^2 A_2 \right)_{\text{min}} = \frac{3}{4}$$ (46)

which happens when $s_{3i} = 0$ for all $i = 1, 2, 3 \ldots$. In other words,

$$\Delta^2 A_1 + \Delta^2 A_2 \geq \frac{3}{4}$$ (47)
is the sum-uncertainty relation for symmetric separable two-qubit states, with its lowest bound being $3/4$. Thus the uncertainty sum $\rho_{\text{sep}}$ (See (41)), set up for the two-qubit observables $A_1, A_2$ in (38) cannot even go close to zero. This implies that symmetric separable states (See (41)) can never achieve maximum accuracy in joint measurements by $A_1, A_2$ in (38).

We now wish to check whether entanglement in the two-qubit state $\rho_{AB}$ contributes to enhanced precision in the measurements of the observables $A_1, A_2$. To this end, we evaluate the concurrence [33, 34], a measure of two-qubit entanglement, of the state $\rho_{AB}$. Concurrence of any arbitrary two-qubit state $\rho$ is defined as [33, 34]

$$C = \max \left( 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right) \quad (48)$$

where $\lambda_k, k = 1, 2, 3, 4$ are the eigenvalues of the matrix $\rho(\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y)$, arranged in the descending order (i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$).

The structure of $\rho_{AB}$ in (35) allows us to make use of the simplified expression for concurrence given in Ref. [35], and leads to

$$C_{AB} = \begin{cases} \omega (1 + \kappa) - 1 & \text{for } \omega (1 + \kappa) \geq 1 \\ 1 - \omega (1 + \kappa) & \text{for } \omega (1 + \kappa) \leq 1 \end{cases} \quad (49)$$

where $\kappa = \sqrt{1 - r_3^2}$ and $0 \leq r_3 \leq 1$. In other words, we have

$$C_{AB} = |\omega (1 + \kappa) - 1|, \quad \kappa = \sqrt{1 - r_3^2}, \quad 0 \leq r_3 \leq 1. \quad (50)$$

A contour plot of $C_{AB}$ as a function of the parameters $\omega$ and $\kappa$ is shown in Fig. 6.
Based on Figs. 5 and 6, we reach the following conclusions:

1. The uncertainty sum $\Delta^2 A_1 + \Delta^2 A_2$ can be reduced below the value $3/4$ only in an entangled state $\rho_{AB}$ (i.e., when $C_{AB} \neq 0$). In particular, $\Delta^2 A_1 + \Delta^2 A_2 \to 0$, in maximally entangled two-qubit symmetric states (i.e., when $C_{AB} \to 1$).

2. While no separable state can reduce the uncertainty sum $\Delta^2 A_1 + \Delta^2 A_2$ below the value $3/4$, there indeed exist entangled states with $\Delta^2 A_1 + \Delta^2 A_2 \geq 3/4$. This implies that, while qutrit states $\rho_{\text{qutrit}}$ that permit accurate simultaneous measurements of the observables $A_1$, $A_2$ are necessarily associated with entangled two-qubit states, the converse is not always true.

It can thus be concluded that entanglement in a two-qubit state constructed from a qutrit (a qubit appended with an additional level) plays a significant role in the precise joint measurements by a pair of orthogonal Pauli observables. It would be of interest to examine whether two-qubit states constructed from a qudit (a qubit with $d-1$ ancillary levels) exhibit a similar feature. A study of uncertainty region of such two-qubit states, dimensional dependence of uncertainty sum and accuracy of simultaneous measurements by any incompatible pair of observables in $d$-dimensional spaces form topics of further interest.

6 Conclusion

This work is a contribution to the ongoing study on uncertainty regions [17–19, 23] providing a different perspective in accounting for better measurement precision seen in qutrits.
For any arbitrary 3-level atomic systems we have obtained an expression for minimum value of uncertainty sum for Pauli-like observables in terms of atomic populations. This is useful to study if enhanced measurement precision (reduction in the uncertainty sum) can be realized in $\Lambda$, $V$ and $\Xi$ types of 3-level atomic systems, which are characterised by different schemes of allowed/forbidden atomic transitions between any two levels. We have also examined whether entanglement in a two-qubit state (which is constructed from a qutrit state of specific structure) is responsible for the inclusion of more precisely determinable points in the uncertainty region. We have shown that while simultaneous measurements of orthogonal Pauli-like observables with utmost accuracy is not possible for separable two-qubit states, entangled states allow for such precise measurements. The technique we have proposed for the construction of two-qubit symmetric states corresponding to qutrit states is helpful in such a construction from multi-level states obtained by appending several ancillary levels to a single qubit. It would be of interest to examine whether increase in the dimension of Hilbert space leads to better measurement precision of incompatible observables. Our work suggests that uncertainty region could be employed as a dimensional witness.

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