Chapter 4

Eigenvalues of Gram Matrices of a Class of Diagram Algebras
In this chapter, we introduce symmetric diagram matrices \( A^{s+r,s} \) of size \((s+r)C_s\). We compute the eigenvalues of the symmetric diagram matrices using elementary row and column operations by induction.

Using the eigenvalues of the symmetric diagram matrices, we compute the eigenvalues of Gram matrices of algebra of \( \mathbb{Z}_2 \)-relations, signed partition algebras and the partition algebras. In particular, the eigenvalues of the partition algebras are linear polynomials whose factors are integers.

### 4.1 Diagram Matrices

In this section, we introduce the symmetric diagram matrices \( A^{s+r,s} \) of size \((s+r)C_s\) based on the diagrams. By induction, we compute the eigenvalues of the symmetric diagram matrices \( A^{s+r,s} \) using elementary row and column operations.

**Definition 4.1.1.** Fix \( s \) and \( r \). Choose a \( d^+ \in R_k \) such that \( d^+ \) has \( s + r \)-connected components where \( R_k \) denote the set of all equivalence relations on \( \underline{k} = \{1, 2, \cdots, k\} \).

We shall draw a diagram \( d \) graphically using the graph \( d^+ \) with \( s \) through classes as follows:

1. **(i)** Draw \( d^+ \) in the top row and a copy of \( d^+ \) denoted by \( d^- \) in the bottom row.
2. **(ii)** Among the \( s + r \) connected components in the top row, choose \( s \) connected components and join each connected component with the respective connected component in the bottom row by vertical edges.
3. **(iii)** The resultant diagram is denoted by \( d \).

**Definition 4.1.2.** Let \( \Omega^{s+r,s} \) denote the collection of all diagrams with \( s \) through classes mentioned as above in Definition 4.1.1 and the number of such diagrams are denoted by \(|\Omega^{s+r,s}| = (s+r)C_s\).

**Example 4.1.3.** Let \( s + r = 9 \) and \( \rho = (\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{8\}, \{9\}) \). The diagrams with three through classes corresponding to the set partition \( \rho \) are
Definition 4.1.4. Fix \( s \) and \( r \). Consider the following variables \( \{x_0, x_1, \cdots, x_{\min\{s,r\}}\} \). Define a matrix \( A^{s+r,s} \) of size \((s+r)C_s\) with the entries \( \{x_0, x_1, \cdots, x_{\min\{s,r\}}\} \) as follows:

\[
A^{s+r,s} = (a_{ij})_{(s+r)C_s \times (s+r)C_s} \quad \text{with} \quad a_{ij} = x_{\min\{s,r\}} - f
\]

where \( d_i, d_j \in \Omega^{s+r,s} \) and \( f \) denotes the number of horizontal edges in \( d_i \) which are replaced by through classes in \( d_j \) and vice versa.

Remark 4.1.5. For the sake of convenience, we shall replace every through class in the above mentioned diagram by a through class \((\mid\mid)\) obtained by joining a single vertex in the top row and the corresponding vertex in the bottom row. Similarly, every horizontal edge in the top and bottom row is replaced by a vertex \((\star)\) respectively.

Lemma 4.1.6. The matrix \( A^{s+r,s} \) is symmetric.

Proof.

Since the top and bottom row of the diagrams in \( \Omega^{s+r,s} \) are the same, the matrix \( A^{s+r,s} \) is symmetric.

Definition 4.1.7. The matrix \( A^{s+r,s} \) is called as symmetric diagram matrix.

Remark 4.1.8. The number of times \( x_{\min\{s,r\} - t} \) occurs in any row or column of a symmetric diagram matrix \( A^{s+r,s} \) of size \((s+r)C_s\) is given by \( sC_t, C_t \).
The eigenvalues of the symmetric diagram matrix $A^{s+r,s}$ can be obtained using induction. The main theorem of this paper gives all the distinct eigenvalues of the symmetric diagram matrix $A^{s+r,s}$.

4.1.1 Main Theorem

**Theorem 4.1.9.** The set of all distinct eigenvalues of the symmetric diagram matrix $A^{s+r,s}$ of size $(s+r)\times s$ with entries $\{x_0, x_1, \cdots, x_{\min\{s,r\}}\}$ are given by

$$\sum_{t=0}^{\min\{s,r\}} \left[ \sum_{j=0}^{l} (-1)^j lC_j (s-l)\hat{C}(t-j) (r-t)\hat{C}(t-j) \right] x_{\min\{s,r\}-t}$$

for all $0 \leq l \leq \min\{s, r\}$.

**Lemma 4.1.10.** If $a_0, a_1, \cdots, a_r$ are $(r+1)$-variables and if $a_l = \sum_{j=0}^{l} (-1)^j lC_j a_{t-j}$ then

$$a_{l+1}^t = a_l^t - a_{l-1}^t.$$  

**Proof.** Consider

$$a_l^t - a_{l-1}^t = \sum_{j=0}^{l} (-1)^j lC_j a_{t-j} - \sum_{j=0}^{l-1} (-1)^j lC_j a_{t-1-j}$$

$$= \sum_{j=0}^{l} (-1)^j lC_j a_{t-j} - \sum_{j=1}^{l+1} (-1)^{j-1} lC_{j-1} a_{t-j'}$$

$$= a_l^t + \sum_{j=1}^{l+1} (-1)^{j} \left[ lC_j + lC_{j-1} \right] a_{t-j} + (-1)^{l+1} a_{t-(l+1)}$$

$$= \sum_{j=0}^{l+1} (-1)^j (l+1)C_j a_{t-j}$$

$$= a_{l+1}^t$$

**Corollary 4.1.11.** In particular, if $a_i = (s-l)C_i (r-l)\hat{C}_i$ for $0 \leq i \leq \min\{s-l, r-l\}$ then $a_i^{l+1} = \sum_{j=0}^{l+1} (-1)^j l+1C_j (s-(l+1))\hat{C}(t-j) (r-(l+1))\hat{C}(t-j)$.

**Proof.** The proof follows from induction and Lemma 4.1.10.
4.1.2 Arrangement of Diagrams in $\Omega^{s+r,s}$

**Definition 4.1.12.** Let $I, J \subset \{1, 2, \cdots, s + r\}$ with $I \cap J = \phi$. Define,

$$\Omega_{I,J} = \{ d \in \Omega^{s+r,s} \mid \text{the } i^{th} \text{ connected component of } d \text{ is a through class } (\mid) \text{ for all } i \in I \text{ and the } j^{th} \text{ connected component of } d \text{ consists of two dots for all } j \in J \}.$$

*In particular, if $I = \{1\}$ and $J = \{s + r\}$ then*

$$\Omega_{\{1\},\{s+r\}} = \{ d \in \Omega^{s+r,s} \mid \text{the first connected component of } d \text{ is a through class } (\mid) \text{ and the } (s + r)^{th} \text{ connected component consists of two dots } \}.$$ 

**Lemma 4.1.13.** Let $\Omega_{I,J}$ be as in Definition 4.1.12. Then

$$|\Omega_{I,J}| = (s + r - |I| - |J|)C_{s-|I|}.$$ 

*In particular,*

$$|\Omega_{\{1\},\{s+r\}}| = (s + r - 2)C_{s-1}.$$ 

**Proof.** The proof follows from the Definition of $\Omega_{I,J}$. □

We shall arrange the diagrams in $\Omega^{s+r,s}$ as follows:

(i) The collection of diagrams whose first connected component is a through class and the last connected component is a horizontal edge is denoted by $\Omega_{\{1\},\{s+r\}}$.

Also, $|\Omega_{\{1\},\{s+r\}}| = (s + r - 2)C_{s-1}$ and the diagrams in $\Omega_{\{1\},\{s+r\}}$ look like

$$\begin{array}{c}
\text{d}_i' \\
\vdots
\end{array}$$

The diagrams in $\Omega_{\{1\},\{s+r\}}$ are indexed inductively as follows:

Suppose $d_i', 1 \leq i \leq (s + r - 2)C_{s-1}$ is the $i^{th}$ diagram in $\Omega^{s+r-2,s-1}$, and which is the the subdiagram of $d_i$ lying inbetween the first and last connected component will be the $i^{th}$ diagram in $\Omega_{\{1\},\{s+r\}}$ inductively.
(ii) For $1 \leq i \leq (s+r-2)C_{(s-1)}$, let $d_i \in \Omega_{\{1\},\{s+r\}}$ then $d_i^*$ is the diagram same as $d_i$ except the first and last connected component where the first connected component of $d_i^*$ is a horizontal edge and the last connected component is a through class. Collection of such diagrams is denoted by $\Omega_{\{s+r\},\{1\}}$. The diagram $d_i^*$ will look like

```
  . . d_i^* .
```

where $d_i^*$ is same as $d_i$ except the first and last connected components.

(iii) For $1 \leq i \leq (s+r-2)C_{(s-2)}$, the collection of diagrams whose first and last connected components are through classes is denoted by $\Omega_{\{1,s+r\},\{\} \}$. The diagrams in $\Omega_{\{1,s+r\},\{\} \}$ will look like

```
  . . .
```

(iv) For $1 \leq i \leq (s+r-2)C_s$, the collection of all diagrams whose first and last connected components are horizontal edges is denoted by $\Omega_{\{\} \},\{1,s+r\}}$. The diagrams in $\Omega_{\{\} \},\{1,s+r\}}$ will look like

```
  . . .
```

Lemma 4.1.14. Let $d_i, d_j \in \Omega_{\{m\},\{f\}}$, $d_i^*, d_j^* \in \Omega_{\{f\},\{m\}}$ and $a_{ij} = x_{\min(r,s)}-t$ then

(i) $a_{ij} = a_{i^*j^*}$.

(ii) $a_{i^*j} = x_{\min(r,s)-(t+1)}$. 

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Proof.

Proof of (i): Let $d_i, d_j \in \Omega_{\{m\},\{f\}}$ and $a_{ij} = x_{\min\{r,s\}} - t$ where $t$ denotes the number of horizontal edges in $d_i$ which are replaced by through classes in $d_j$ and vice versa. From the definition of $\Omega_{\{m\},\{f\}}$, we know that the first and last connected components of $d_i$ and $d_j$ are the same.

By the definition of $\Omega_{\{f\},\{m\}}$, we know that for every $d_i \in \Omega_{\{1\},\{s+r\}}$ there exists a unique $d_i^* \in \Omega_{\{m\},\{f\}}$ which differs only in the first and last connected component. Also, the first and last connected components of $d_i$ and $d_j$ are the same.

Thus, $a_{ij} = a_{i^*j^*}$.

Proof of (ii): By the definition of $\Omega_{\{m\},\{f\}}$, the $t$ number of horizontal edges which are replaced by the through classes and vice versa lies in between the first and last connected components of $d_i$ and $d_j$. Also, the diagram $d_i^* \in \Omega_{\{f\},\{m\}}$ is same as the diagram $d_i \in \Omega_{\{m\},\{f\}}$ except the first and last connected component.

Therefore, including the first and last connected component there will be $t + 1$ number of horizontal edges in $d_i^*$ which are replaced by through classes in $d_j$ and vice versa.

Thus, $a_{i^*j} = x_{\min\{s,r\}} - (t+1)$.

Lemma 4.1.15. Let $d_i \in \Omega_{\{m\},\{f\}}$ and $d_j \in \Omega_{\{m\},\{f\}}$ or $\Omega_{\{f\},\{m\}}$ then

$$a_{ij} = a_{i^*j^*}, \quad 1 \leq i \leq (s+r-2)C_{(s-1)}.$$

Proof. Let $a_{ij} = x_{\min\{r,s\}} - t$ where $t$ denotes the number horizontal edges in $d_i$ which are replaced by through classes and vice versa. From the definition of $\Omega_{\{f\},\{m\}}$, we know that the diagram $d_i^*$ is same as the diagram $d_i \in \Omega_{\{m\},\{f\}}$ except at the first and last connected components where the first(last) connected component of $d_i$ is a through class(horizontal edge) but the first(last) connected component of $d_i^*$ is a horizontal edge(through class).

$a_{i^*j}$ denotes the entry corresponding to the product of diagrams $d_i^*$ and $d_j$ then by the definition of $\Omega_{\{m\},\{f\}}$ and $\Omega_{\{f\},\{m\}}$ we have,

$$a_{ij} = a_{i^*j^*}, \quad 1 \leq i \leq (s+r-2)C_{(s-1)}.$$

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We shall now prove the main theorem of this paper using the arrangement of diagrams given above and induction.

Proof of Main theorem: The proof of the main theorem is using induction on the size of the matrix.

Case (i): Let $s = 1, r = 1$ and

$$\Omega^{2,1} = \{ \begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \end{array} \}$$

By Definition 4.1.4, the entries of the symmetric diagram matrix $A^{2,1}$ are $\{x_0, x_1\}$ where

$$A^{2,1} = \begin{pmatrix} x_1 & x_0 \\ x_0 & x_1 \end{pmatrix}$$

Apply the following row operation and column operation on $A^{2,1}$:

$$R_d \leftrightarrow R_d - R_{d^*}, C_{d^*} \leftrightarrow C_{d^*} + C_d$$

where the diagrams $d$ and $d^*$ differ only at the first and last connected component. The first connected component of $d(d^*)$ is through class(horizontal edge) and the last connected component of $d(d^*)$ is horizontal edge(through class).

The reduced matrix is

$$\begin{pmatrix} x_1 - x_0 & 0 \\ 0 & x_1 + x_0 \end{pmatrix}.$$  

Thus, the eigenvalues of the symmetric diagram matrix $A^{2,1}$ are $x_1 - x_0$ and $x_1 + x_0$.

Case (ii): Let $s = 2, r = 2$ and

$$\Omega^{4,2} = \{ \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \}$$

By Definition 4.1.4, the entries of the symmetric diagram matrix $A^{4,2}$ are $\{x_0, x_1, x_2\}$ where
$A^{4,2} = \begin{pmatrix}
  x_2 & x_1 & x_1 & x_0 & x_1 & x_1 \\
  x_1 & x_2 & x_0 & x_1 & x_1 & x_1 \\
  x_1 & x_0 & x_2 & x_1 & x_1 & x_1 \\
  x_0 & x_1 & x_1 & x_2 & x_1 & x_1 \\
  x_1 & x_1 & x_1 & x_1 & x_1 & x_0 \\
  x_1 & x_1 & x_1 & x_1 & x_0 & x_2
\end{pmatrix}$

Apply the following row and column operation on $A^{4,2}$

$$R_d \leftrightarrow R_d - R_{d^*}, C_{d^*} \leftrightarrow C_{d^*} + C_d$$

for all $d \in \Omega_{(1),\{4\}}$ and $d^* \in \Omega_{\{1\},\{4\}}$ which differs only at the first and last connected component as in Definition 4.1.12. Then the reduced matrix is as follows:

$$\begin{pmatrix}
  A_{1,1} & 0 \\
  * & A_{1,2}
\end{pmatrix}$$

where $A_{1,1} = \begin{pmatrix} y_1 & y_0 \\ y_0 & y_1 \end{pmatrix}$ is a symmetric diagram matrix of size 2 with $y_1 = x_2 - x_1$ and $y_0 = x_1 - x_0$ and $A_{1,2} = \begin{pmatrix} x_2 + x_1 & x_1 + x_0 & x_1 & x_1 \\
  x_1 + x_0 & x_2 + x_1 & x_1 & x_1 \\
  x_1 + x_0 & x_2 + x_1 & x_1 & x_1 \\
  2x_1 & 2x_1 & x_2 & x_0 \\
  2x_1 & 2x_1 & x_0 & x_2
\end{pmatrix}$

The diagrams corresponding to the entries of the symmetric diagram matrix $A_{1,1}$ belong to $\Omega_{(1),\{4\}}$ and the diagrams corresponding to the entries of the matrix $A_{1,2}$ belong to $\Omega_{\{4\},\{1\}}, \Omega_{(1,4),\{\}}$ and $\Omega_{\{1\},\{1,4\}}$.

Using induction, the eigenvalues of the symmetric diagram matrix $A_{1,1}^{4,2}$ are $y_1 - y_0$ and $y_1 + y_0$, i.e., $x_2 + x_0 - 2x_1$ and $x_2 - x_0$.

Apply the following row and column operation on the matrix $A_{1,2}$:

$$R_d \leftrightarrow R_d - R_{d^*}, C_{d^*} \leftrightarrow C_{d^*} + C_d$$

for the diagrams $d \in \Omega_{\{2,4\},\{1,3\}}$ and $d^* \in \Omega_{\{3,4\},\{1,2\}}$.

Thus, the reduced matrix is as follows:

$$\begin{pmatrix}
  A_{2,1} & 0 \\
  * & A_{2,2}
\end{pmatrix}$$
where $A_{2,1} = (x_2 - x_0)$ is a matrix of size 1 and $A_{2,2} = \begin{pmatrix} x_2 + 2x_1 + x_0 & x_1 & x_1 \\ 4x_1 & x_2 & x_0 \\ 4x_1 & x_0 & x_2 \end{pmatrix}$.

The diagram corresponding to the entry of the matrix $A_{2,1}$ belong to $\Omega_{\{4,2\},\{1,3\}}$ and the diagrams corresponding to the entries of the matrix $A_{2,2}$ belong to $\Omega_{\{4,3\},\{1,2\}}, \Omega_{\{1,4\},\{2,3\}}$ and $\Omega_{\{2,3\},\{1,4\}}$.

The eigenvalue of the matrix $A_{2,1}$ is $x_2 - x_0$.

Apply the following row and column operations on the matrix $A_{2,2}$:

$$C_1 \leftrightarrow C_1 + C_2 + C_3, \quad R_2 \leftrightarrow R_2 - R_1, \quad R_3 \leftrightarrow R_3 - R_1$$

Thus, the reduced matrix is as follows:

$$\begin{pmatrix} A_{3,1} & * \\ 0 & A_{3,2} \end{pmatrix}$$

where $A_{3,1} = (x_2 + 4x_1 + x_0)$ and $A_{3,2} = \begin{pmatrix} x_2 - x_1 & x_0 - x_1 \\ x_0 - x_1 & x_2 - x_1 \end{pmatrix}$ which is same as $A_{1,1}$.

Therefore, the eigenvalues of the symmetric diagram matrix $A^{s+r,s}$ are $x_2 - x_0, x_2 + 4x_1 + x_0$ and $x_2 - 2x_1 + x_0$.

In general, we shall do this for the diagrams in $\Omega^{s+r,s}$ and shall find the eigenvalues of the symmetric diagram matrix $A^{s+r,s}$ using induction.

**Step 1:**

We shall split $\Omega^{s+r,s}$ into four subsets as follows:

$$\Omega^{s+r,s} = \Omega_{\{1\},\{s+r\}} \cup \Omega_{\{s+r\},\{1\}} \cup \Omega_{\{1,s+r\},\{\}} \cup \Omega_{\{\},\{1,s+r\}}$$

such that for all $d \in \Omega_{\{1\},\{s+r\}}$ there exist a unique $d^* \in \Omega_{\{s+r\},\{1\}}$ which differs only at the first and $(s+r)^{th}$ connected components where the first connected component of $d(d^*)$ is through class (horizontal edge) and the $(s+r)^{th}$ connected component of $d(d^*)$ is horizontal edge (through class).

Clearly,
\[ |\Omega_{\{1, s+r\}}| = (s+r-2)C_{s-1} = |\Omega_{\{s+r\}, \{1\}}|, |\Omega_{\{1, s+r\}, \{\} \}} = (s+r-2)C_{s-2} \text{ and } |\Omega_{\{\}, \{1, s+r\}}| = (s+r-2)C_{s}. \]

Apply the following row and column operations on the symmetric diagram matrix \( A^{s+r, s} \) of size \( (s+r)C_s \):

\[ R_{d_i} \leftrightarrow R_{d_i} - R_{d_i^*}, \quad \forall d_i \in \Omega_{\{1\}, \{s+r\}} \]

and

\[ C_{d_i} \leftrightarrow C_{d_i} + C_{d_i^*}, \quad \forall d_i^* \in \Omega_{\{s+r\}, \{1\}}. \]

Using lemmas 4.1.14 and 4.1.15, the reduced matrix is as follows:

\[
\begin{pmatrix}
  A_{1,1} & 0 \\
  * & A_{1,2}
\end{pmatrix}
\]

where \( A_{1,1} \) is a symmetric diagram matrix of size \( (s+r-2)C_{(s-1)} \) with entries

\( \{y_0, y_1, \ldots, y_{\min\{s-1, r-1\}}\} \) and \( y_i = x_{i+1} - x_i \).

The diagrams corresponding to the entries of the symmetric diagram matrix \( A_{1,1} \) belong to \( \Omega_{\{1\}, \{s+r\}} \).

Using induction the eigenvalues of the symmetric diagram matrix \( A_{1,1} \) of size \( (s+r-2)C_{(s-1)} \) are given as

\[
\min\{s-1, r-1\} \sum_{t=0}^{l-1} \left\{ \sum_{j=0}^{l-1} (-1)^j (l-1)C_j ([s-1]-(l-1))C([t-j]) |([r-1]-(l-1))C([t-j]) \right\} y_{\min\{s-1, r-1\}-t}
\]

for all \( 0 \leq l-1 \leq \min\{s-1, r-1\} \).

Substitute \( y_{\min\{s-1, r-1\}-t} = x_{\min\{s-1, r-1\}-t+1} - x_{\min\{s-1, r-1\}-t} \) in expression (4.2) we
get,

\[
\begin{align*}
\min \{s-1, r-1\} & = \sum_{t=0}^{l-1} \left\{ \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) \right\} \\
& \quad \left\{ x_{\min(s-1, r-1)-t+1} - x_{\min(s-1, r-1)-t} \right\} \\
& = \sum_{t=0}^{l-1} \left\{ \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) \right\} x_{\min(s-1, r-1)-t+1} \\
& \quad - \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) \} x_{\min(s-1, r-1)-t+1} \\
& = \sum_{t=0}^{l-1} \left\{ \sum_{j=0}^{l-1} \frac{\min(s-1, r-1)}{t} \left( (s-l) C_t (r-l) C_t + \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) (r-l) C(t-j) \right) \right\} x_{\min(s-1, r-1)-t+1} \\
& \quad + \frac{\min(s-1, r-1)}{t} \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) (r-l) C(t-j) \} x_{\min(s, r)-t} \forall 1 \leq l \leq \min \{s, r\} \\
\end{align*}
\]

Thus, the eigenvalues of the symmetric diagram matrix \(A_{1,1}\) of size \((s+r-4)C(s-2)\) are given by

\[
\begin{align*}
\sum_{t=0}^{l-1} \left\{ \sum_{j=0}^{l-1} (-1)^j (l-1) C_j (s-l) C(t-j) (r-l) C(t-j) \right\} x_{\min(s, r)-t} \forall 1 \leq l \leq \min \{s, r\} \}
\end{align*}
\]

(4.3)

Therefore, the eigenvalues of the symmetric diagram matrix \(A^{s+r,s}\) of size \((s+r-2)C(s-1)\) are
\[
\sum_{t=0}^{\min\{s,r\}} \left\{ \sum_{j=0}^{l} (-1)^j \binom{l-1}{j} C_j \binom{s-l}{j} \binom{t-j}{r-l} \right\} x_{\min\{s,r\}-t} \quad \forall \ 1 \leq l \leq \min\{s,r\}.
\]  

(4.4)

The number of distinct eigenvalues of the symmetric diagram matrix \(A^{s+r,s}\) so far computed which is given in (4.4) are \(\min\{s,r\}\).

We shall now prove that the eigenvalues of the submatrix \(A_{1,2}\) are also as given in expression (4.3).

**Step 2:** The diagrams corresponding to the entries of the matrix \(A_{1,2}\) belong to \(\Omega_{I_i,J_i}\) for \(1 \leq i \leq 3\) where \(I_1 = \{s + r\}, I_2 = \{1, s + r\}, I_3 = \{\}\), \(J_1 = \{1\}, J_2 = \{\}\) and \(J_3 = \{1, s + r\}\).

We shall apply the following row and column operations on \(A_{1,2}\):

\[R_d \leftrightarrow R_d - R_d^*, \quad C_d^* \leftrightarrow C_d^* + C_d \quad \forall \ d \in \Omega_{I_i,J_i\cup\{s+r-1\}} \text{ and } d^* \in \Omega_{I_i\cup\{s+r-1\},J_i\cup\{2\}}\]

for all \(1 \leq i \leq 3\).

Using Lemmas 4.1.14 and 4.1.15 and applying suitable row and column operations to the reduced matrix looks like,

\[
\begin{pmatrix}
A_{2,1} & 0 \\
* & A_{2,2}
\end{pmatrix}
\]

The diagrams corresponding to the entries of the matrix \(A_{2,1}\) belong to \(\Omega_{I_i\cup\{2\},J_i\cup\{s+r-1\}}\) for all \(1 \leq i \leq 3\).

The size of the matrix \(A_{2,1}\) is \((s+r-4)C_{s-2} + (s+r-4)C_{s-3} + (s+r-4)C_{s-1} = \sum_{j=1}^{3} (s+r-4)C_{s-j}\).

Now, we shall show that the eigenvalues of the matrix \(A_{2,1}\) belong to the collection of all eigenvalues of the matrix \(A_{1,1}\) obtained in Step 1.

We know that the diagrams corresponding to the entries of the symmetric diagram matrix \(A_{1,1}\) obtained in Step 1 belong to \(\Omega_{\{1\},\{s+r\}}\).

Apply the following row and column operations on the symmetric diagram matrix \(A_{1,1}\):

\[R_d \leftrightarrow R_d - R_d^*, \quad C_d^* \leftrightarrow C_d^* + C_d \]

for all \(d \in \Omega_{\{1\}\cup\{2\},\{s+r-1\}\cup\{s+r\}} \text{ and } d^* \in \Omega_{\{1\}\cup\{s+r-1\},\{2\}\cup\{s+r\}}\).

Using induction on the number of through classes the reduced matrix is as follows:
\[
\begin{pmatrix}
B_{1,1} & 0 \\
* & B_{1,2}
\end{pmatrix}
\]

where the diagrams corresponding to the entries of the matrix \(B_{1,1}\) belong to \(\Omega_{\{1,2\},\{s+r-1,s+r\}}\) and the diagrams corresponding to the entries of the matrix \(B_{1,2}\) belong to \(\Omega_{\{1,s+r-1\},\{2,s+r\}}\), \(\Omega_{\{1,2,s+r-1\},\{s+r\}}\) and \(\Omega_{\{1\},\{2,s+r-1,s+r\}}\).

The size of the matrix \(B_{1,1}\) is \((s+r-4)C(s-2)\) and the size of the matrix \(B_{1,2}\) is \(\sum_{j=1}^{3}(s+r-4)C(s-j)\).

Since, \(B_{1,1}\) and \(B_{1,2}\) are the submatrices of the symmetric diagram matrix \(A_{1,1}\), by induction the eigenvalues of the matrices \(B_{1,1}\) and \(B_{1,2}\) belong to the collection of all eigenvalues given in expression 4.3.

It is clear from the definition of \(\Omega_{I,J}\) defined in Definition 4.1.12 that the submatrices \(B_{1,2}\) and \(A_{2,1}\) are the same. Hence the eigenvalues of the matrix \(B_{1,2}\) and \(A_{2,1}\) are the same.

The diagrams corresponding to the entries of the matrix \(A_{2,2}\) belong to \(\Omega_{I_i\cup\{s+r-1\},J_i\cup\{2\}}, \Omega_{I_i\cup\{2,s+r-1\},J_i}\) and \(\Omega_{I_i\cup\{2,s+r-1\},J_i}\) for \(1 \leq i \leq 3\).

Proceeding in the same manner we get,

**Step j**: In general, the diagrams corresponding to the entries of the matrix \(A_{j-1,2}\) obtained in step \(j - 1\) belong to \(\Omega_{I_i\cup\{s+r-(j-2)\},J_i\cup\{j-1\}}, \Omega_{I_i\cup\{j-1,s+r-(j-2)\},J_i}\) and \(\Omega_{I_i\cup\{j-1,s+r-(j-2)\},J_i}\) with \(I_i \cap J_i = \emptyset\) for all \(1 \leq i \leq 3^{j-2}\).

Apply the following row and column operations on \(A_{j-1,2}\):

\[
R_d \leftrightarrow R_d - R_{d^*}, \quad C_d \leftrightarrow C_{d^*} + C_d
\]

for all \(d \in \Omega_{I_i\cup\{j\},J_i\cup\{s+r-(j-1)\}}\) and \(d^* \in \Omega_{I_i\cup\{s+r-(j-1)\},J_i\cup\{j\}}\).

Using Lemmas 4.1.14 and 4.1.15 and interchanging the rows and columns suitably the reduced matrix is as follows:

\[
\begin{pmatrix}
A_{j,1} & 0 \\
* & A_{j,2}
\end{pmatrix}
\]

The diagrams corresponding to the entries of the matrix \(A_{j,1}\) belong to \(\Omega_{I_i\cup\{j\},J_i\cup\{s+r-(j-1)\}}\) for all \(1 \leq i \leq 3^{j-1}\). The size of the matrix \(A_{j,1}\) is \(\sum_{i=1}^{3^{j-1}}(s+r-((I_i|+|J_i))C(s-|I_i|))\).
Now, we shall show that the eigenvalues of the matrix $A_{j,1}$ and the eigenvalues of the matrix $B_{j-1,2}$ obtained from $B_{1,2}$ inductively as in Step 2 are the same. Using induction, we know that the diagrams corresponding to the entries of the symmetric diagram matrix $B_{j-2,2}$ obtained in Step $j - 2$ belong to $\Omega_{I_i \cup \{s+r-(j-3)\}, J_i' \cup \{j-2\}}, \Omega_{I_i' \cup \{j-2, s+r-(j-3)\}, J_i''}$ and $\Omega_{I_i' \cup \{j-2, s+r-(j-3)\}, J_i''}$ for all $1 \leq i \leq 3^{j-3}$. Apply the following row and column operations on the symmetric diagram matrix $B_{j-2,2}$:

\[
R_d \leftrightarrow R_d - R_{d'}, \quad C_{d'} \leftrightarrow C_{d'} - C_d
\]

for all $d \in \Omega_{I_i \cup \{j-1\}, J_i' \cup \{s+r-(j-2)\}}$ and $d^* \in \Omega_{I_i' \cup \{s+r-(j-2)\}, J_i' \cup \{j-1\}}$, $1 \leq i \leq 3^{j-2}$.

Using induction on the number of through classes, Lemma 4.1.14, 4.1.15 and interchanging rows and column suitably the reduced matrix is as follows:

\[
\begin{pmatrix}
B_{j-1,1} & 0 \\
* & B_{j-1,2}
\end{pmatrix}
\]

where the diagrams corresponding to the entries of the matrix $B_{j-1,1}$ belong to $\Omega_{I_i \cup \{j-1\}, J_i' \cup \{s+r-(j-2)\}}$ and the diagrams corresponding to the entries of the matrix $B_{j-1,2}$ belong to $\Omega_{I_i' \cup \{s+r-(j-2)\}, J_i' \cup \{j-1\}}, \Omega_{I_i' \cup \{j-1, s+r-(j-2)\}, J_i''}$ and $\Omega_{I_i' \cup \{j-1, s+r-(j-2)\}}$ for all $1 \leq i \leq 3^{j-2}.$

The size of the matrix $B_{j-1,2}$ is $\sum_{m=1}^{3^{j-1}} (s+r-|I_i|-|J_i|)C(\cdot, |I_i|)$. Since, $B_{j-1,1}$ and $B_{j-1,2}$ are the submatrices of the symmetric diagram matrix $A_{1,1}$, the eigenvalues of the matrices $B_{j-1,1}$ and $B_{j-1,2}$ belong to the collection of all eigenvalues given in expression 4.3.

It is clear from the definition of $\Omega_{I, J}$ defined in Definition 4.1.12 that the submatrices $B_{j-1,2}$ and $A_{j,2}$ are the same. Hence, eigenvalues of the matrix $B_{j-1,2}$ and $A_{j,1}$ are the same.

The diagrams corresponding to the entries of the matrix $A_{j,2}$ belong to $\Omega_{I_i \cup \{s+r-(j-1)\}, J_i \cup \{j\}}, \Omega_{I_i \cup \{j, s+r-(j-1)\}, J_i}$ and $\Omega_{I_i \cup \{j, s+r-(j-1)\}, J_i}$ for $1 \leq i \leq 3^{j-1}$.

**Note 4.1.16.** Some of $I_i$ and $J_i$ may be empty. When the set is empty we leave that set and continue the process. Also, we always consider the matrix with coefficient 1 for the highest order in the determinant.
This process is continued till step $p$ if $s + r = 2p$ and $p - 1$ if $s + r = 2p + 1$.

If $s + r = 2p + 1$ then we apply the following row and column operations on $A_{p-1,2}$:

$$R_d \leftrightarrow R_d - R_{d^*} \text{ and } C_{d^*} \leftrightarrow C_{d^*} + C_d$$

for all $d \in \Omega_{I_i \cup \{p\}, J_i \cup \{p+1\}}$ and $d \in \Omega_{I_i \cup \{p+1\}, J_i \cup \{p+2\}}$, $d^* \in \Omega_{I_i \cup \{p+1\}, J_i \cup \{p\}}$ and $d^* \in \Omega_{I_i \cup \{p+2\}, J_i \cup \{p+1\}}$, $1 \leq i \leq 3p - 1$.

In both the cases (i.e., $s + r = 2p$ and $s + r = 2p + 1$), using Lemmas 4.1.14, 4.1.15 and by interchanging rows and columns suitably the reduced matrix is as follows:

$$\left( \begin{array}{cc} A_{p,1} & 0 \\ * & A_{p,2} \end{array} \right).$$

The diagrams corresponding to the entries of the matrix $A_{p,2}$ belong to $\Omega_{I_i, J_i}$ for all $1 \leq i \leq 3p$.

Finally, we shall apply the following row and column operations on the matrix $A_{p,2}$:

(i) Fix a $d \in \Omega_{\{s+r\}, \{1\}} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$C_d \leftrightarrow C_d + \sum_{d'} C_{d'}$$

where $d' \in \Omega_{\{s+r\}, \{1\}} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{s+r\}, \{1\}} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}.$$

(ii) Fix a $d \in \Omega_{\{1,s+r\}, \{\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$C_d \leftrightarrow C_d + \sum_{d'} C_{d'}$$

where $d' \in \Omega_{\{1,s+r\}, \{\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{1,s+r\}, \{\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}.$$

(iii) Fix a $d \in \Omega_{\{\}, \{1,s+r\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$C_d \leftrightarrow C_d + \sum_{d'} C_{d'}$$

where $d' \in \Omega_{\{\}, \{1,s+r\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}$.

$$R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{\}, \{1,s+r\} \setminus \bigcup_{i=1}^{3p} \Omega_{I_i, J_i}.$$

Using induction, Lemma 4.1.14, Lemma 4.1.15 and after applying suitable row and column operations on the reduced matrix, it becomes,
where the size of the matrix $A_{p+1,1}$ is 3 and the matrix $A_{p+1,2}$ is one among the submatrices obtained after applying row and column operations on the symmetric diagram matrix $A_{1,1}$ inductively.

Thus, the eigenvalues of the matrix $A_{p+1,2}$ belong to the collection of all eigenvalues given in expression (4.3.)

Now, we are left out to find the eigenvalues of the matrix $A_{p+1,1}$. For that we add all the entries to one column and subtract the corresponding row with other rows. The reduced matrix is as follows:

\[
\begin{pmatrix}
A' & * \\
0 & A''
\end{pmatrix}
\]

where $A'$ is a $1 \times 1$ matrix and $A''$ is a $2 \times 2$ matrix.

Since, we have only performed addition on the columns the entry of the matrix $A'$ is the sum of the entries of the symmetric diagram matrix $A^{s+r,s}$ of size $(s+r)C_s$ which is given by

\[
\sum_{t=0}^{\min\{s,r\}} sC_t rC_t x_{\min\{s,r\}-t}.
\]

we perform the following row and column operations on the matrix $A''$:

\[
R_1 \leftrightarrow R_1 + R_2 \text{ and } C_2 \leftrightarrow C_2 - C_1
\]

Thus, the eigenvalues of the matrix $A''$ also belong to the collection of all eigenvalues given in expression (4.3).

Thus, we have computed all the $\min\{s,r\} + 1$ number of distinct eigenvalues of the symmetric diagram matrix $A^{s+r,s}$.

Thus, the eigenvalues of the symmetric diagram matrix $A^{s+r,s}$ of size $(s+r)C_s$ are given by

\[
\sum_{t=0}^{\min\{s,r\}} \left\{ \sum_{j=0}^{t} (-1)^j lC_j (s-l)C_{(t-j)} (r-l)C_{(t-j)} \right\} x_{\min\{s,r\}-t}
\]

for all $0 \leq l \leq \min\{s,r\}$. 

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4.2 Eigenvalues of Gram matrices for a class of diagram algebras

4.2.1 Eigenvalues of the Gram matrices of the partition algebras

In this section, we compute the eigenvalues of Gram matrices of partition algebras where the block submatrices of the Gram matrix are realized as the direct product of the symmetric diagram matrices.

**Theorem 4.2.1.**

(i) The set of all distinct eigenvalues of Gram matrix $G_s$ of the partition algebra with entries $\{X_0, X_1, \cdots, X_{\min\{s,r\}}\}$ are given by

$$\sum_{t=0}^{\min\{s,r\}} \left( \sum_{j=0}^{t} (-1)^j l C_j (s-t) C (t-j) (r-t-l) C (t-j) \right) X_{\min\{s,r\}-t}$$

for all $0 \leq l \leq \min\{r,s\}$ and $0 \leq r \leq k - s$

where $X_{\min\{s,r\}-t} = (-1)^t t! \prod_{m=1}^{r-1} [x - (s + l)]$, $k$ and $s$ are fixed integers.

(ii) The eigenvalues of the Gram matrix $G_s$ are integers which are given by

$$\sum_{t=0}^{\min\{s,r\}} \left( \sum_{j=0}^{t} (-1)^j l C_j (s-t) C (t-j) (r-t-l) C (t-j) \right) X_{\min\{s,r\}-t} = \prod_{i=0}^{l-1} [x - (s - 1 + i)]$$

$$\prod_{j=0}^{\min\{s,r\}-l-1} [x - (2s + j)]$$

for all $0 \leq l \leq \min\{s,r\}$.

**Proof.**

**Proof of (i):** Let $U_{R^d}^{R^d} \in \mathcal{J}_s^r$ such that $\phi'(R^d) = \lambda'$ where $\mathcal{J}_s^r$ and $\Omega_s^r$ are as in Notation 3.1.6 and Definition 3.1.1 respectively.

We shall draw a diagram using the diagram $R^d$ with $s$ through classes as follows:

(i) Draw $R^d$ denoted by $R^d_+$ in the top row and a copy of $R^d$ denoted by $R^d_-$ in the bottom row.
(ii) Among the $s+r$ connected components in the top row choose $s$ connected components and join each connected component in the top row with the respective connected component in the bottom row by vertical edges.

(iii) Denote the collection of such diagrams as $\mathcal{J}_s^{r,s}$. In particular $U_{Rd}^{rd} \in \Omega_{Rd}^{s+r,r}$.

It is clear that the number of such diagrams with $s$ through classes is $(s+r)C_s$ and $\mathcal{J}_s^{r} = \bigcup_{Rd} \Omega_{Rd}^{s+r,r}$ where $\mathcal{J}_s^{r}$ is as in Notation 3.1.6.

Suppose $U_{Rd}^{rd} \in \mathcal{J}_s^{r}$ then the entry in the Gram matrix corresponding to the product $U_{Rd}^{rd} \cdot U_{Rd}^{rd}$ is either $x^{r''}$ with $r'' < r$ or 0 mod $\lambda$.

**Case (i):** If the entry corresponding to the product $U_{Rd}^{rd} \cdot U_{Rd}^{rd}$ in the Gram matrix $G_s$ is $x^{r''}$ with $r'' < r$ and while applying the column operations while eliminate the entries corresponding to the diagrams coarser than $U_{Rd}^{rd}$ the entry $x^{r''}$ becomes zero by Lemma 3.1.22 in [15].

**Case (ii):** If the entry corresponding the product $U_{Rd}^{rd} \cdot U_{Rd}^{rd}$ in the Gram matrix $G_s$ is 0 mod $\lambda$ and it remains zero even after applying column operations while eliminate the entries corresponding to the diagrams coarser than $U_{Rd}^{rd}$ by Lemma 3.1.21 and Lemma 3.1.25 in [15]. Rearranging the diagrams $U_{Rd}^{rd} \in \mathcal{J}_s^{r}$ in such a way that

$$A'_{r,r} = \prod_{Rd} A^{s+r,s}_{Rd}$$

where $A^{s+r,s}_{Rd}$ is a symmetric diagram matrix of size $(s+r)C_s$ with entries

$$\{X_0, X_1, \cdots, X_{\min\{s,r\}}\}$$

and the diagrams corresponding to the entries of the symmetric diagram matrix $A^{s+r,s}_{Rd}$ belong to $\Omega_{Rd}^{s+r,r}$.

Also, the by Theorem 3.1.31(c) $X_{\min\{s,r\} - t} = (-1)^t \frac{t!}{l!} \prod_{m=t}^{r-1} [x - (s + m)]$.

By theorem 4.1.9, the eigenvalues of the symmetric diagram matrix $A^{s+r,s}_{Rd}$ of size $(s+r)C_s$ are given by

$$\sum_{t=0}^{\min\{s,r\}} \left[ \sum_{j=0}^{l} (-1)^j lC_j (s-l)C_{(t-j)} (r-l)C_{(t-j)} \right] X_{\min\{s,r\} - t}$$

for all $0 \leq l \leq \min\{s,r\}$.

Thus, the eigenvalues of the block submatrix $A'_{r,r}$ which is a direct product of symmetric diagram matrices $A^{s+r,s}_{Rd}$ are given by
The set of all distinct eigenvalues of Gram matrices

for all \(0 \leq l \leq \min\{s, r\}\).

Therefore, the eigenvalues of the Gram matrix \(G_s\) are given by

for all \(0 \leq l \leq \min\{s, r\}\) and \(0 \leq r \leq k - s\).

**proof of (ii):** The proof of (ii) follows by comparing the coefficients on both the sides.

\[\square\]

### 4.2.2 Eigenvalues of Gram Matrices for Signed Partition Algebras and the algebra of \(\mathbb{Z}_2\)-relations:

**Theorem 4.2.2.**

(i) The set of all distinct eigenvalues of Gram matrices \(G_{2s_1+s_2}\) of the algebra of \(\mathbb{Z}_2\)-relations with entries \(\{X_0, X_1, \ldots, X_{\min\{s_1, r_1\}}\}\) and \(\{X'_0, X'_1, \ldots X'_{\min\{s_2, r_2\}}\}\)

are given by

\[
(a) \quad \sum_{l=0}^{\min\{s_1, r_1\}} \left[ \sum_{j=0}^{l} (-1)^j \ t C_j (s_1-l) C(t-j) (r_1-l) C(t-j) \right] X_{\min\{s_1, r_1\}-t}
\]

for all \(0 \leq l \leq \min\{s_1, r_1\}\) and \(0 \leq r_1 \leq k - s_1 - s_2\)

where \(X_{\min\{s_1, r_1\}-t} = (-1)^t t! 2^t \prod_{i=t}^{r_1-1} [x^2 - x - 2(s_1 + i)], k, s_1\) and \(s_2\) are fixed integers

(b) \quad \sum_{l=0}^{\min\{s_2, r_2\}} \left[ \sum_{j=0}^{m} (-1)^j \ t C_j (s_2-l) C(t-j) (r_2-l) C(t-j) \right] X'_{\min\{s_2, r_2\}-t}

for all \(0 \leq l \leq \min\{s_2, r_2\}\) and \(0 \leq r_2 \leq k - s_1 - s_2\)

where \(X'_{\min\{s_2, r_2\}-t} = (-1)^t t! \prod_{i=t}^{r_1-1} [x-(s+l)], k, s_1\) and \(s_2\) are fixed integers.

(ii) The set of all distinct eigenvalues of block submatrices

\[
\left( \tilde{A}_{2r_1+r_2, 2r_1+r_2} \right)_{0 \leq 2r_1+r_2 \leq 2k-2s_1-2s_2-1} \quad \text{of the Gram matrix} \ G_{2s_1+s_2} \quad \text{of signed partition algebra with entries} \ \{X_0, X_1, \ldots, X_{\min\{s_1, r_1\}}\}\) and \(\{X'_0, X'_1, \ldots X'_{\min\{s_2, r_2\}}\}\)

are given by...
(a) \[ \sum_{t=0}^{\min\{s_1, r_1\}} \left[ \sum_{j=0}^{l} (-1)^j t C_j (s_1 - l) C(t-j) (r_1 - l) C(t-j) \right] X_{\min\{s_1, r_1\} - t} \]

for all \( 0 \leq l \leq \min\{s_1, r_1\} \) and \( 0 \leq r_1 \leq k - s_1 - s_2 \)

where \( X_{\min\{s_1, r_1\} - t} = (-1)^t t! 2^t \prod_{i=0}^{r_1-1} \left[ x^2 - x - 2(s_1 + i) \right] k, s_1 \) and \( s_2 \) are fixed integers.

(b) \[ \sum_{t=0}^{\min\{s_2, r_2\}} \left[ \sum_{j=0}^{m} (-1)^j t C_j (s_2 - l) C(t-j) (r_2 - l) C(t-j) \right] X_{\min\{s_2, r_2\} - t} \]

for all \( 0 \leq l \leq \min\{s_2, r_2\} \) and \( 0 \leq r_2 \leq k - s_1 - s_2 - 1 \)

where \( X_{\min\{s_2, r_2\} - t} = (-1)^t t! \prod_{m=0}^{r_1-1} \left[ x - (s_2 + l) \right] k, s_1 \) and \( s_2 \) are fixed integers.

**Proof.**

Since the \{e\}-connected components cannot be replaced by \( \mathbb{Z}_2 \)-connected components, we can rearrange the diagrams such that the block matrices \( A'_{2r_1,2r_1} \otimes A'_{2r_2,2r_2} (\tilde{A}'_{2r_1,2r_1} \otimes \tilde{A}'_{2r_2,2r_2}) \) becomes the tensor product of matrices say \( A'_{2r_1,2r_1} \otimes A'_{2r_2,2r_2} (\tilde{A}'_{2r_1,2r_1} \otimes \tilde{A}'_{2r_2,2r_2}) \) respectively.

Let \( U^{(d,P)}_{(d,P)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} (U^{(d,P)}_{(d,P)} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}) \) such that \( \phi((d, P)) = \lambda \left( \tilde{\phi}(\tilde{d}, \tilde{P}) = \lambda \right) \)

where \( \lambda \in \Omega_{s_1,s_2}^{r_1,r_2} \) as in Definition 3.1.1, \( \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} \) and \( \mathbb{J}_{2s_1+s_2}^{2r_1+r_2} \) are as in Notation 3.1.6.

We shall draw a diagram using the diagram \( d\left(\tilde{d}\right)\) with \( 2s_1 + s_2 \) through classes respectively as follows.

(i) Draw \( d\left(\tilde{d}\right)\) denoted by \( d^+ \) in the top row and a copy of \( d^+ \left(\tilde{d}^+\right) \) denoted by \( d^- \left(\tilde{d}^-\right) \) in the bottom row.

(ii) Fix the positions of \( \mathbb{Z}_2 \)-connected component and among the \( s_1 + r_1 \) number of pairs \{e\}-connected components in the top row choose \( s_1 \) pairs of \{e\} connected components and join each pair of \{e\}-connected component in the top row with the respective pair of \{e\}-connected component in the bottom row by vertical edges.

Similarly, Fix the positions of \{e\}-connected components and among the \( s_2 + r_2 \) number of pairs \( \mathbb{Z}_2 \)-connected components in the top row choose \( s_2 \) number of
Z₂-connected components and join each Z₂-connected component in the top row with the respective Z₂-connected component in the bottom row by vertical edges.

(iii) Denote the collection of diagrams obtained by fixing Z₂-connected components as

\[ \Omega_{d}^{s_{1}+r_{1},s_{1}} \left( \Omega_{d}^{s_{1}+r_{1},s_{1}} \right) \]  

and denote the collection of diagrams obtained by fixing \{e\} - connected components as Ω_{d}^{s_{2}+r_{2},s_{2}}

\[ \left( \Omega_{d}^{s_{1}+r_{1},s_{1}} \right) . \]

It is clear that the number of such diagrams with \( 2s_{1} + s_{2} \) through classes is

\( (s_{1} + r_{1})C_{s_{1}}(s_{2} + r_{2})C_{s_{2}} \) and \( \mathbb{J}^{2s_{1}+r_{1}+s_{2}} = \bigcup_{d} \Omega_{d}^{s_{1}+r_{1},s_{1}} \times \Omega_{d}^{s_{2}+r_{2},s_{2}} \left( \mathbb{J}_{2s_{1}+s_{2}} = \bigcup_{d} \Omega_{d}^{s_{1}+r_{1},s_{1}} \times \Omega_{d}^{s_{2}+r_{2},s_{2}} \right) \)

where \( \mathbb{J}_{2s_{1}+s_{2}} \) and \( \mathbb{J}_{2s_{1}+s_{2}} \) are as in Notation 3.1.6.

Suppose \( U^{(d',p')} \in \mathbb{J}^{2s_{1}+r_{1}+r_{2}} \) then the entry corresponding to the product \( U^{(d,P)} \cdot U^{(d',p')} \) in the Gram matrix is either \( x^{2s_{1}+r_{2}} \) with \( r_{1}'' < r_{1} \) and \( r_{2}'' < r_{2} \) or 0 mod \( \lambda \).

**Case (i):** If the entry corresponding to the product \( U^{(d,P)} \cdot U^{(d',p')} \) is \( x^{2s_{1}+r_{2}} \) with \( r_{1}'' < r_{1} \) and \( r_{2}'' < r_{2} \) then it becomes zero after applying column operations while eliminate the entries corresponding to the diagrams coarser than \( U^{(d,P)} \) by Lemma 3.1.22 in [15].

**Case (ii):** If the entry corresponding to the product \( U^{(d,P)} \cdot U^{(d',p')} \) is 0 mod \( \lambda \) then it remains zero even after applying the column operations while eliminate the entries corresponding to the diagrams coarser than \( U^{(d,P)} \) by Lemma 3.1.21 and Lemma 3.1.25 in [15].

Rearranging the diagrams \( U^{(d,P)} \in \mathbb{J}^{2s_{1}+r_{2}} \) in such a way that

\[ A'_{2r_{1},2r_{1}} = \prod_{d} A_{d}^{s_{1}+r_{1},s_{1}} \]  

and \( A'_{2r_{2},2r_{2}} = \prod_{d} A_{d}^{s_{2}+r_{2},s_{2}} \)

where \( A_{d}^{s_{1}+r_{1},s_{1}} \) and \( A_{d}^{s_{2}+r_{2},s_{2}} \) are symmetric diagram matrices of size \( (s_{1}+r_{1})C_{s_{1}} \) and \( (s_{2}+r_{2})C_{s_{2}} \) respectively. The entries of the symmetric diagram matrices \( A_{d}^{s_{1}+r_{1},s_{1}} \) and \( A_{d}^{s_{2}+r_{2},s_{2}} \) are \{ \( X_{0}, X_{1}, \ldots, X_{\min\{s_{1},r_{1}\}} \) \} and \{ \( X_{0}', X_{1}', \ldots, X_{\min\{s_{2},r_{2}\}} \) \} respectively and the diagrams corresponding to the entries of the symmetric diagram matrices \( A_{d}^{s_{1}+r_{1},s_{1}} \) and \( A_{d}^{s_{2}+r_{2},s_{2}} \) belong to \( \Omega_{d}^{s_{1}+r_{1},s_{1}} \) and \( \Omega_{d}^{s_{2}+r_{2},s_{2}} \) respectively.
Similarly, we can rearrange the diagrams in signed partition algebra in such a way that
\[ \tilde{A}_{2r_1,2r_1} = \prod_d \tilde{A}_{d}^{s_1+r_1,s_1} \] and \[ \tilde{A}_{r_2,r_2} = \prod_d \tilde{A}_{d}^{s_2+r_2,s_2} \]
where \( \tilde{A}_{d}^{s_1+r_1,s_1} \) and \( \tilde{A}_{d}^{s_2+r_2,s_2} \) are symmetric diagram matrices of size \((s_1+r_1)C_{s_1}\) and \((s_2+r_2)C_{s_2}\) respectively. The entries of the symmetric diagram matrices \( \tilde{A}_{d}^{s_1+r_1,s_1} \) and \( \tilde{A}_{d}^{s_2+r_2,s_2} \) are \( \{X_0, X_1, \cdots, X_{\min\{s_1,r_1\}}\} \) and \( \{X'_0, X'_1, \cdots, X'_{\min\{s_2,r_2\}}\} \) respectively and the diagrams corresponding to the entries of the symmetric diagram matrices \( \tilde{A}_{d}^{s_1+r_1,s_1} \) and \( \tilde{A}_{d}^{s_2+r_2,s_2} \) belong to \( \Omega_{d}^{s_1+r_1,s_1} \) and \( \Omega_{d}^{s_2+r_2,s_2} \) respectively.

Also, the by Theorem 3.1.31(a) \( X_{\min\{s_1,r_1\}-t} = (-1)^t \ t! \ 2^{t} \prod_{m=t}^{r_1-1} \lfloor x^2 - x - 2(s_1 + m) \rfloor \)
and \( X'_{\min\{s_2,r_2\}-t} = (-1)^t \ t! \ \prod_{m=t}^{r_2-1} \lfloor x - (s_2 + m) \rfloor \).

By theorem 4.1.9, the eigenvalues of the symmetric diagram matrix \( A_{d}^{s_1+r_1,s_1} (\tilde{A}_{d}^{s_1+r_1,s_1}) \)
of size \((s_1+r_1)C_{s_1}\) are given by
\[
\sum_{t=0}^{\min\{s_1,r_1\}} \left( \sum_{j=0}^{l} (-1)^j \ tC_j \ (s_1-t)C(t-j) \ (r_1-t)C(t-j) \right) \ X_{\min\{s_1,r_1\}-t}
\]
for all \( 0 \leq l \leq \min\{s_1,r_1\} \).

By theorem 4.1.9, the eigenvalues of the symmetric diagram matrix \( A_{d}^{s_2+r_2,s_2} (\tilde{A}_{d}^{s_2+r_2,s_2}) \)
of size \((s_2+r_2)C_{s_2}\) are given by
\[
\sum_{t=0}^{\min\{s_2,r_2\}} \left( \sum_{j=0}^{m} (-1)^j \ tC_j \ (s_2-t)C(t-j) \ (r_2-t)C(t-j) \right) \ X'_{\min\{s_2,r_2\}-t}
\]
for all \( 0 \leq l \leq \min\{s_2,r_2\} \).

Thus, the eigenvalues of the block submatrix \( A'_{2r_1,2r_1} (\tilde{A}'_{2r_1,2r_1}) \) which is a direct product of symmetric diagram matrices \( A_{d}^{s_1+r_1,s_1} (\tilde{A}_{d}^{s_1+r_1,s_1}) \) are given by
\[
\sum_{t=0}^{\min\{s_1,r_1\}} \left( \sum_{j=0}^{l} (-1)^j \ tC_j \ (s_1-t)C(t-j) \ (r_1-t)C(t-j) \right) \ X_{\min\{s_1,r_1\}-t}
\]
for all \( 0 \leq l \leq \min\{s_1,r_1\} \).

Thus, the eigenvalues of the block submatrix \( A'_{r_2,r_2} (\tilde{A}'_{r_2,r_2}) \) which is a direct product of symmetric diagram matrices \( A_{d}^{s_2+r_2,s_2} (\tilde{A}_{d}^{s_2+r_2,s_2}) \) are given by
\[
\sum_{t=0}^{\min\{s_2,r_2\}} \left( \sum_{j=0}^{l} (-1)^j \ tC_j \ (s_2-t)C(t-j) \ (r_2-t)C(t-j) \right) \ X'_{\min\{s_2,r_2\}-t}
\]
for all \( 0 \leq l \leq \min\{s_2,r_2\} \).

Therefore, the eigenvalues of the Gram matrix \( G_{2s_1+s_2} \) of the algebra of \( \mathbb{Z}_2 \)-relations

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are given by

\[(a) \quad \sum_{t=0}^{\min\{s_1, r_1\}} \left[ \sum_{j=0}^{l} (-1)^j t C_j (s_1-t) C(t-j) (r_1-t) C(t-j) \right] X_{\min\{s_1, r_1\}-t}\]

for all \(0 \leq l \leq \min\{s_1, r_1\}\) and \(0 \leq r_1 \leq k - s_1 - s_2\)

where \(X_{\min\{s_1, r_1\}-t} = (-1)^t t! \prod_{i=t}^{r_1-1} [x^2 - x - 2(s_1 + i)], k, s_1\) and \(s_2\) are fixed integers.

\[(b) \quad \sum_{t=0}^{\min\{s_2, r_2\}} \left[ \sum_{j=0}^{m} (-1)^j t C_j (s_2-t) C(t-j) (r_2-t) C(t-j) \right] X'_{\min\{s_2, r_2\}-t}\]

for all \(0 \leq l \leq \min\{s_2, r_2\}\) and \(0 \leq r_2 \leq k - s_1 - s_2\)

where \(X'_{\min\{s_2, r_2\}-t} = (-1)^t t! \prod_{m=t}^{r_1-1} [x - (s + l)], k, s_1\) and \(s_2\) are fixed integers.

\[\square\]

4.3 Illustrations

To compute the eigenvalues of the symmetric diagram matrix \(A^{7,4}\) of size 35, we
need the eigenvalues of the symmetric diagram matrices \(A^{2,1}\) and \(A^{5,3}\) of sizes 3 and 10 respectively. First, we shall compute the eigenvalues of the symmetric diagram matrix \(A^{2,1}\) of size 3.

\[A^{2,1} = \begin{pmatrix} x_1 & x_0 & x_0 \\ x_0 & x_1 & x_0 \\ x_0 & x_0 & x_1 \end{pmatrix}\]

**Step 1:** Apply the following row and column operations on \(A^{2,1}\):

\[R_1 \leftrightarrow R_1 - R_2, C_2 \leftrightarrow C_2 + C_1 \text{ and } R_2 \leftrightarrow R_2 - R_3, C_3 \leftrightarrow C_2 + C_3\]

then the reduced matrix is as follows:

\[
\begin{pmatrix}
 x_1 - x_0 & 0 & 0 \\
 0 & x_1 - x_0 & 0 \\
 x_0 & 2x_0 & x_1 + 2x_0
\end{pmatrix}
\]

Thus, the eigenvalues of the symmetric diagram matrix \(A^{2,1}\) of size 3 are

\[x_1 - x_0, x_1 + 2x_0.\]

Secondly, we shall compute the eigenvalues of the symmetric diagram matrix \(A^{5,3}\)
of size 10.

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Step 1: apply the following row and column operations on the symmetric diagram matrix $A^{5,3}$.

$$R_i \leftrightarrow R_i - R_{3+i} \quad \text{and} \quad C_{3+i} \leftrightarrow C_{3+i} + C_i \quad \text{for} \quad 1 \leq i \leq 3$$

then the reduced matrix is as follows:

$$
\begin{bmatrix}
A_{1,1} & 0 \\
* & A_{1,2}
\end{bmatrix}
$$

where $A_{1,1} = \begin{pmatrix} y_1 & y_0 & y_0 \\ y_0 & y_1 & y_0 \\ y_0 & y_0 & y_1 \end{pmatrix}$ is a symmetric diagram matrix of size 3 with $y_1 = x_2 - x_1$ and $y_0 = x_1 - x_0$. Using induction the eigenvalues of the symmetric diagram matrix $A_{1,1}$ are $y_1 - y_0$ and $y_1 + 2y_0$, i.e., $x_2 - 2x_1 + x_0$ and $x_2 + x_1 - 2x_0$ respectively.

Step 2: Apply the following row and column operations on $A_{1,2}$.

$$R_4 \leftrightarrow R_4 - R_6, R_7 \leftrightarrow R_7 - R_9 \quad \text{and} \quad C_6 \leftrightarrow C_6 + C_4, C_9 \leftrightarrow C_9 + C_7$$

By interchanging the rows and columns of the reduced matrix suitably, we get

$$
\begin{bmatrix}
A_{2,1} & 0 \\
* & A_{2,2}
\end{bmatrix}
$$
where \( A_{2,1} = \begin{pmatrix} x_2 - x_0 & x_1 - x_0 \\ 2x_1 - 2x_0 & x_2 - x_1 \end{pmatrix} \) is a submatrix of the symmetric diagram matrix \( A_{1,1} \) obtained in Step 1. Therefore, the eigenvalues of \( A_{2,1} \) are same as the eigenvalues of \( A_{1,1} \).

**Step 3:** Apply the following row and column operations on \( A_{2,2} \).

\[
R_5 \leftrightarrow R_5 - R_6, R_8 \leftrightarrow R_8 - R_9 \text{ and } C_6 \leftrightarrow C_6 + C_5, C_9 \leftrightarrow C_9 + C_8
\]

By interchanging the rows and columns of the reduced matrix suitably, we get

\[
\begin{pmatrix} A_{3,1} & 0 \\ * & A_{3,2} \end{pmatrix}
\]

where \( A_{3,2} = \begin{pmatrix} x_2 + 3x_1 + 2x_0 & 2x_1 + x_0 & x_1 \\ 4x_1 + 2x_0 & x_2 + 2x_1 & x_0 \\ 6x_1 & 3x_0 & x_2 \end{pmatrix} \), \( A_{3,1} \) is same as \( A_{2,1} \) which is a submatrix of the symmetric diagram matrix \( A_{1,1} \) obtained in Step 1. Therefore, the eigenvalues of \( A_{3,1} \) are same as the eigenvalues of \( A_{1,1} \).

**Step 4:** Apply the following row and column operation on \( A_{3,2} \).

\[
C_1 \leftrightarrow C_1 + C_2 + C_3, R_2 \leftrightarrow R_2 - R_1 \text{ and } R_3 \leftrightarrow R_3 - R_1
\]

The reduced matrix is as follows:

\[
\begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}
\]

where \( A' \) is a \( 1 \times 1 \) matrix whose entry is sum of the entries of the symmetric diagram matrix \( A^{5,3} \) i.e., \( x_2 + 6x_1 + 3x_0 \) and \( A'' = \begin{pmatrix} x_2 - x_0 & x_0 - x_1 \\ 2x_0 - 2x_1 & x_2 - x_1 \end{pmatrix} \) which is again a submatrix of the symmetric diagram matrix \( A_{1,1} \) obtained in Step 1. Therefore, the eigenvalues of \( A'' \) are same as the eigenvalues of \( A_{1,1} \).

Therefore, the eigenvalues of the symmetric diagram matrix \( A^{5,3} \) are

\[
x_2 + 6x_1 + 3x_0, x_2 - 2x_1 + x_0 \text{ and } x_2 + x_1 - 2x_0
\]

Now, we shall compute the eigenvalues of the symmetric diagram matrix \( A^{7,4} \) using induction.
The following are the diagrams in $\Omega^{s+r,s}$ when $s = 4$ and $r = 3$:

$$d_1 = \begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
where $A_{1,1}$ is a symmetric diagram matrix of size 10 and

$$A_{1,1} = \begin{pmatrix}
  x_3 - x_2 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 \\
  x_2 - x_1 & x_3 - x_2 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 \\
  x_2 - x_1 & x_2 - x_1 & x_3 - x_2 & x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 \\
  x_2 - x_1 & x_1 - x_0 & x_3 - x_2 & x_2 - x_1 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 \\
  x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_3 - x_2 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 \\
  x_1 - x_0 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 & x_3 - x_2 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 \\
  x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_3 - x_2 & x_2 - x_1 & x_2 - x_1 \\
  x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_1 - x_0 & x_3 - x_2 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 \\
  x_1 - x_0 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 & x_3 - x_2 & x_1 - x_0 \\
  x_2 - x_1 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_1 - x_0 & x_1 - x_0 & x_3 - x_2 & x_1 - x_0
\end{pmatrix}.$$
Using induction, the eigenvalues of the symmetric diagram matrix \( A_{1,1} \) are \( x_3 + 5x_2 - 3x_1 - 3x_0, x_3 - 3x_2 + 3x_1 - x_0 \) and \( x_3 - 3x_1 + 2x_0 \).

**Step 2:** The diagrams corresponding to the entries of the matrix \( A_{1,2} \) belong to \( \Omega_{I_i,J_i} \) where \( I_1 = \{7\}, I_2 = \{1,7\}, I_3 = \{\} \) and \( J_1 = \{1\}, \{\}, \{1,7\} \) respectively. Apply the following row and column operations on \( A_{1,2} \):

\[
R_d \leftrightarrow R_d - R_{d'} \quad \forall d \in \Omega_{I_i \cup \{2\}, J_i \cup \{6\}} \text{ and } C_{d'} \leftrightarrow C_{d'} + C_d \quad \forall d' \in \Omega_{I_i \cup \{6\}, J_i \cup \{2\}} \quad \forall 1 \leq i \leq 3.
\]

By interchanging the rows and columns suitably the reduced matrix is as follows:

\[
\begin{pmatrix}
A_{2,1} & 0 \\
* & A_{2,2}
\end{pmatrix}
\]

where \( A_{2,1} = \begin{pmatrix}
x_3 - x_1 & x_2 - x_0 & x_2 - x_0 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 & x_2 - x_1 \\
x_2 - x_0 & x_3 - x_1 & x_2 - x_0 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 & x_2 - x_1 \\
x_2 - x_0 & x_2 - x_0 & x_3 - x_1 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 & x_2 - x_1 \\
2x_1 - 2x_0 & 2x_2 - 2x_1 & 2x_2 - 2x_1 & x_3 - x_2 & x_2 - x_1 & x_2 - x_1 & x_1 - x_0 \\
2x_2 - 2x_1 & 2x_1 - 2x_0 & 2x_2 - 2x_1 & x_2 - x_1 & x_3 - x_2 & x_2 - x_1 & x_1 - x_0 \\
2x_2 - 2x_1 & 2x_2 - 2x_1 & 2x_1 - 2x_0 & x_2 - x_1 & x_2 - x_1 & x_3 - x_2 & x_1 - x_0 \\
2x_2 - 2x_1 & 2x_2 - 2x_1 & 2x_2 - 2x_1 & x_1 - x_0 & x_1 - x_0 & x_3 - x_2
\end{pmatrix}
\]

is same as \( B_{1,2} \) a submatrix of \( A_{1,1} \) obtained after applying the row and column operations. Thus, the eigenvalues of the matrix \( A_{2,1} \) are same as the eigenvalues of matrix \( A_{1,1} \).

**Step 3:** The diagrams corresponding to the entries of the matrix \( A_{2,2} \) belong to \( \Omega_{I_i,J_i} \) where \( I_1 = \{6,7\}, I_2 = \{1,6,7\}, I_3 = \{6\}, I_4 = \{2,6,7\}, I_5 = \{1,2,6,7\}, I_6 = \{2,6\}, I_7 = \{7\}, I_8 = \{1,7\}, I_9 = \{\} \) and \( J_1 = \{1,2\}, J_2 = \{2\}, J_3 = \{1,2,7\}, J_4 = \{1\}, J_5 = \{\}, J_6 = \{1,7\}, J_7 = \{1,2,6\}, J_8 = \{2,6\}, J_9 = \{1,2,6,7\} \) respectively. Apply the following row and column operations on \( A_{2,1} \):

\[
R_d \leftrightarrow R_d - R_{d'} \quad \forall d \in \Omega_{I_i \cup \{3\}, J_i \cup \{5\}} \text{ and } C_{d'} \leftrightarrow C_{d'} + C_d \quad \forall d' \in \Omega_{I_i \cup \{5\}, J_i \cup \{3\}} \quad \forall 1 \leq i \leq 3^2.
\]

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By interchanging the rows and columns suitably the reduced matrix is as follows:

\[
\begin{pmatrix}
A_{3,1} & 0 \\
* & A_{3,2}
\end{pmatrix}
\]

where \(A_{3,1} = \begin{pmatrix}
x_3 + x_2 - x_1 - x_0 & x_2 - x_0 & x_2 - x_1 & x_2 - x_0 & x_2 - x_1 \\
2x_2 - 2x_0 & x_3 - x_1 & x_1 - x_0 & 2x_2 - 2x_1 & x_2 - x_1 \\
4x_2 - 4x_1 & 2x_1 - 2x_0 & x_3 - x_2 & 2x_2 - 2x_1 & x_1 - x_0 \\
2x_2 - 2x_0 & 2x_2 - 2x_1 & x_2 - x_1 & x_3 - x_1 & x_1 - x_0 \\
4x_2 - 4x_1 & 2x_2 - 2x_1 & x_1 - x_0 & 2x_1 - 2x_0 & x_3 - x_2
\end{pmatrix}
\]
is the same as \(B_{2,2}\) obtained from \(B_{1,2}\) after applying row and column operations. Thus, the eigenvalues of the matrix \(A_{3,2}\) are same as the eigenvalues of matrix \(A_{1,1}\).

**Step 4:** The diagrams corresponding to the entries of the matrix \(A_{3,2}\) belong to \(\Omega_{I_i,J_i}\) where \(1 \leq i \leq 3^3\). Apply the following row and column operations on \(A_{3,1}\):

\[
R_d \leftrightarrow R_d - R_{d'} \quad \forall d \in \Omega_{I_i,J_i} \cup \{3,4\} \text{ and } \forall d \in \Omega_{I_i,J_i} \cup \{5\}
\]

\[
C_{d'} \leftrightarrow C_{d'} + C_d \quad \forall d \in \Omega_{I_i,J_i} \cup \{4\} \text{ and } \forall d \in \Omega_{I_i,J_i} \cup \{3\} \forall 1 \leq i \leq 3^3.
\]

By interchanging the rows and columns suitably the reduced matrix is as follows:

\[
\begin{pmatrix}
A_{4,1} & 0 \\
* & A_{4,2}
\end{pmatrix}
\]

where \(A_{4,1}\) is same as \(A_{3,1}\). Thus, the eigenvalues of the matrix \(A_{4,1}\) are same as the eigenvalues of matrix \(A_{1,1}\).

**Step 5:** Apply the following row and column operations on \(A_{4,2}\):

(i) Fix a \(d \in \Omega_{\{7,\{1\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i,J_i}}\).

\[
C_d \leftrightarrow C_d + \sum_{d'} C_{d'}
\]

where \(d' \in \Omega_{\{7,\{1\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i,J_i}}\).

\[
R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{7,\{1\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i,J_i}}\).
\]

(ii) Fix a \(d \in \Omega_{\{1,7,\{\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i,J_i}}\).
\[ C_d \leftrightarrow C_d + \sum_{d'} C_{d'} \]

where \( d' \in \Omega_{\{1,7\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i, J_i}} \).

\[ R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{1,7\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i, J_i}}. \]

(iii) Fix a \( d \in \Omega_{\{1,7\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i, J_i}} \).

\[ C_d \leftrightarrow C_d + \sum_{d'} C_{d'} \]

where \( d' \in \Omega_{\{1,7\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i, J_i}} \).

\[ R_{d'} \leftrightarrow R_{d'} - R_d \text{ for all } d' \in \Omega_{\{1,7\} \setminus \bigcup_{i=1}^{3^4} \Omega_{I_i, J_i}}. \]

interchanging the rows and columns suitably, the reduced matrix is as follows:

\[
\begin{pmatrix}
A_{5,1} & 0 \\
* & A_{5,2}
\end{pmatrix}
\]

where

\[
A_{5,1} = \begin{pmatrix}
x_3 + 7x_2 + 9x_1 + 3x_0 & 3x_2 + 6x_1 + x_0 & 2x_2 + 3x_1 \\
6x_2 + 12x_1 + 2x_0 & x_3 + 6x_2 + 3x_1 & 3x_1 + 2x_0 \\
8x_2 + 12x_1 & 6x_1 + 4x_0 & x_3 + 4x_2
\end{pmatrix}
\]

and \( A_{5,2} \) is same as \( A_{3,1} \). Thus, the eigenvalues of the matrix \( A_{5,2} \) are same as the eigenvalues of matrix \( A_{4,1} \).

**Step 6:** apply the following row and column operations on \( A_{5,1} \):

\[ C_1 \leftrightarrow C_1 + C_2 + C_3, R_2 \leftrightarrow R_2 - R_1 \text{ and } R_3 \leftrightarrow R_3 - R_1 \]

then the reduced matrix is as follows:

\[
\begin{pmatrix}
A' & * \\
0 & A''
\end{pmatrix}
\]

where \( A' \) is a \( 1 \times 1 \) matrix whose entry is sum of the entries of the symmetric diagram matrix \( A^{7,4} \) i.e., \( x_3 + 12x_2 + 18x_1 + 4x_0 \) and \( A'' = \begin{pmatrix}
x_3 + 3x_2 - 3x_1 - x_0 & 2x_0 - 2x_2 \\
3x_0 - 3x_2 & x_3 + 2x_2 - 3x_1
\end{pmatrix} \).

We perform the following row and column operation on \( A'' \):

\[ R_1 \leftrightarrow R_1 + R_2 \text{ and } C_2 \leftrightarrow C_2 - C_1 \]

then the reduced matrix is as follows:
Thus, the eigenvalues of the matrix $A''$ are $x_3 - 3x_1 + 2x_0$ and $x_3 + 5x_2 - 3x_1 - 3x_0$.

Therefore, the eigenvalues of the symmetric diagram matrix $A_{7,4}$ are $x_3 + 12x_2 + 18x_1 + 4x_0, x_3 + 5x_2 - 3x_1 - 3x_0, x_3 - 3x_2 + 3x_1 - x_0$ and $x_3 - 3x_1 + 2x_0$.

\* \* \* \* \*