The First Eigenvalue of the Kohn-Laplace Operator in the Heisenberg Group

Najoua Gamara* Akram Makni

Abstract

In this paper, by extending the notions of harmonic transplantation and harmonic radius in the Heisenberg group, we give an upper bound for the first eigenvalue for the following Dirichlet problem:

$$(P_\Omega) \begin{cases} -\Delta_{H^1} u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

where $\Omega$ is a regular bounded domain of $\mathbb{H}^1$ with smooth boundary and $\Delta_{H^1}$ is the Kohn-Laplace operator. Using the results of P.Pansu given in [6, 7], which give the relation between the volume of $\Omega$ and the perimeter of its boundary, we prove the following result

$$\lambda_1(\Omega) \leq C_{\Omega} \frac{l_{11}^2}{\max_{\xi \in \Omega} r_{\Omega}(\xi)}$$

where $l_{11}$ is the first strictly positive zero of the Bessel function of first kind and order 1, $C_{\Omega}$ is a constant depending of $\Omega$ and $r_{\Omega}(\xi)$ is the harmonic radius of $\Omega$ at a point $\xi$ of $\Omega$.

Key words: Kohn-Laplace operator, First eigenvalue, Harmonic Radius, Harmonic Transplantation, Green’s Function, Heisenberg group.

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*Corresponding authors; ngamara7@gmail.com
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1 Introduction

The method of harmonic transplantation as proposed by Hersch in [1] is a technique to construct test functions for variational problems of the form

$$J[\Omega] = \inf_{u \in H^1_0} \int_{\Omega} |\nabla u|^2 \, dy,$$

where $\Omega$ is a domain of $\mathbb{R}^n$, $n \geq 3$ with smooth boundary. The harmonic transplantation is a generalization of the conformal transplantation for simply connected planar domain. This method has numerous applications in mathematical physics and function theory [1, 2, 3]. In particular it has been used to give estimates from above of the functional $J[\Omega]$, in contrast to the method based on symmetrization techniques which gives estimates from below. The harmonic transplantation is connected to the Green’s function $G_{\{\Omega;x\}}$ of $\Omega$ with Dirichlet boundary condition, which is the solution of

$$\begin{cases}
\Delta G_{\{\Omega;x\}} = -\delta_x & \text{in } \Omega \\
G_{\{\Omega;x\}} = 0 & \text{on } \partial \Omega.
\end{cases}$$

The Green’s function can be decomposed as:

$$G_{\{\Omega;x\}}(y) = \gamma \left( F_x(y) - H_{\{\Omega;x\}}(y) \right)$$

with

$$\gamma = \begin{cases}
\frac{1}{2\pi} & n = 2 \\
\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} & n \geq 3
\end{cases}$$

$$F_x(y) = F(\|y - x\|) = \begin{cases}
-\ln(\|y - x\|) & n = 2 \\
\|y - x\|^{-n} & n \geq 3
\end{cases}$$

and $H_{\{\Omega;x\}}$ is the solution of

$$\begin{cases}
\Delta H_{\{\Omega;x\}} = 0 & \text{in } \Omega \\
H_{\{\Omega;x\}} = F_x & \text{on } \partial \Omega,
\end{cases}$$

The functions $F_x$ and $H_{\{\Omega;x\}}$ are called respectively the singular part and the regular part of $G_{\{\Omega;x\}}$. The leading term of the regular part of the Green’s function

$$t(x) := H_{\{\Omega;x\}}(x)$$

is called Robin’s function of $\Omega$ at $x$. For $n \geq 3$, the harmonic radius $r(x)$ of $\Omega$ at $x$ defined in [3] by

$$F(r(x)) = t(x)$$
which gives
\[ r(x) = H_{\Omega,x}^{-1}(x). \]

The harmonic radius vanishes on the boundary \( \partial \Omega \) and takes its maximum at the so called harmonic centers of \( \Omega \).

Let \( \Omega \) be a domain of \( \mathbb{R}^n \) with smooth boundary, \( n \geq 3 \) and fix \( x \in \Omega \). If \( u : B(0,r(x)) \to \mathbb{R}_+ \) is a radial function and \( \mu \) is defined by \( u(y) = \mu \left( G_{B(0,r(x);0)} \right)(y) \), then
\[ U(y) = \mu \left( G_{\Omega,x} \right)(y) \]
defines the harmonic transplantation of \( u \) into \( \Omega \).

The harmonic transplantation has the following properties:

1. It preserves the Dirichlet integral
\[ \int_{B(0,r(x))} |\nabla u|^2 \, dy = \int_{\Omega} |\nabla U|^2 \, dy \quad (1.1) \]

2. For every positive function \( \varphi \)
\[ \int_{B(0,r(x))} \varphi(u) \, dy \leq \int_{\Omega} \varphi(U) \, dy. \quad (1.2) \]

The first eigenfunction of the Dirichlet problem on \( B(0,r(x)) \) is radial and positive, then as proved by Hersch in [4], by using Rayleigh’s principle, (1.1) and (1.2), one obtain \( \forall x \in \Omega \)
\[ \lambda_1(\Omega) \leq \frac{\int_{B(0,r(x))} |\nabla U|^2 \, dy}{\int_{\Omega} |U|^2 \, dy} \leq \frac{\int_{B(0,r(x))} |\nabla u|^2 \, dy}{\int_{B(0,r(x))} |u|^2 \, dy} = \lambda_1(B(0,r(x))), \]
hence
\[ \lambda_1(\Omega) \leq \lambda_1(B(0,\max_{\Omega} r(y))). \quad (1.3) \]

Recall that the harmonic transplantation was also extended to spaces of constant curvature by C. Bandle, A. Brillard, and M. Fluscher, one can see [5].

In this work, we will extend the notions of harmonic transplantation and harmonic radius to the Heisenberg group \( \mathbb{H}^1 \). More precisely, we will define counter parts of the harmonic transplantation and the harmonic radius in the subriemannian settings. Our aim is to give an upper bound for the first eigenvalue for the following Dirichlet problem:

\[ (P_{\Omega}) \left\{ \begin{array}{ll} -\Delta_{\mathbb{H}^1} u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{array} \right. \]

where \( \Omega \) is a regular bounded domain of the Heisenberg group \( \mathbb{H}^1 \) with smooth boundary.

The existence of eigenvalues for the Dirichlet problem \((P_{\Omega})\) has been proved by X.Luo and P.Niu in [10] [11] [12] using the Kohn inequality (one can see [13]) for the vector fields generating the horizontal tangent bundle of the Heisenberg group together with the spectral properties of compact operators.

The method, we will use in the present work is based on technics related to the harmonic transplantation method and a result of P.Pansu given in [6] [7] which give the relation between the volume of \( \Omega \) and the perimeter of its boundary. But, we know that we won’t lead to a similar formula to (1.3).
since Heisenberg balls or Koranyi balls are not isoperimetric regions in the Heisenberg group $\mathbb{H}^1$.

The isoperimetric problem as introduced by P. Pansu in $[6, 7]$ consists to find subsets $I$ of $\mathbb{H}^1$ among all subsets $I'$ with the same Riemannian volume as $I$ associated to the left invariant Riemannian metric such that

\[ P_{\mathbb{H}^1}(I) \leq P_{\mathbb{H}^1}(I') \]  

(1.4)

where $P_{\mathbb{H}^1}(I)$ denotes the horizontal perimeter of $I$. The set $I$ is called an isoperimetric region in $\mathbb{H}^1$.

The existence of isoperimetric regions was proved by G. P. Leonardi and S. Rigot in $[9]$ for the more general context of Carnot groups.

For any bounded open set $\Omega$ in $\mathbb{H}^1$ with $C^1$ boundary, P. Pansu proved in $[6, 7]$ the existence of a constant $C > 0$, so that

\[ |\Omega|^\frac{4}{3} \leq CP_{\mathbb{H}^1}(\Omega). \]  

(1.5)

and conjectured that any isoperimetric region of $\mathbb{H}^1$ bounded by a smooth surface is congruent to a bubble set:

\[ \mathbb{B}(0, R) = \left\{ \left( z, t \right) \in \mathbb{H}^1, |t| < R^2 \arccos \left( \frac{|z|}{R} \right) + |z| \sqrt{R^2 - |z|^2}, |z| < R \right\}. \]

for some $R > 0$. Recall that $\mathbb{B}(0, R)$ is obtained by rotating around the $t$-axis the geodesic joining the two poles $-\frac{\pi}{2}R^2$ and $\frac{\pi}{2}R^2$. The boundary of $\mathbb{B}(0, R)$ is

\[ S_R = \left\{ \left( z, t \right) \in \mathbb{H}^1, |t| = R^2 \arccos \left( \frac{|z|}{R} \right) + |z| \sqrt{R^2 - |z|^2}, |z| < R \right\}. \]

In $[8]$, M. Ritore and C. Rosales proved that an isoperimetric region in the class of Heisenberg group sub domains with $C^2$ surfaces as boundaries is congruent to a subset of $\mathbb{H}^1$ having $S_R$ as boundary. A similar result was obtained by R. Monti and M. Rickly in $[21]$ for convex euclidian domains $\mathbb{H}^1$.

In the present work: we won’t use isoperimetric regions and the isoperimetric inequality in the Heisenberg group $\mathbb{H}^1$ with the method of harmonic transplantation to determine an upper bound of the first eigenvalue $\lambda_1(\Omega)$ of $(P_{\Omega})$ as it was the case for the Euclidian settings. Instead, we will use the result of P. Pansu $[6, 7]$ given above by (1.5). Our main results are

**Theorem 6.2**

Let $\Omega$ be a bounded domain of $\mathbb{H}^1$ with smooth boundary, such that $\Omega$ and $\mathbb{H}^1 \setminus \Omega$ satisfy the uniform exterior ball property (3.1) and let $\xi$ be any point in $\Omega$, we have

\[ \lambda_1(\Omega) \leq C_{\Omega}(\xi) \frac{l_{11}^2}{r_{\Omega}^2(\xi)} \]

where $l_{11}$ is the first strictly positive zero of the Bessel function $J_1(l)$ and $r_{\Omega}(\xi)$ is the harmonic radius of $\Omega$ at $\xi$.

□

**Theorem 6.3**

Let $\Omega$ be a bounded domain of $\mathbb{H}^1$ with smooth boundary, such that $\Omega$ and $\mathbb{H}^1 \setminus \Omega$ satisfy the uniform exterior ball property (3.1) and let $\xi$ be any point in $\Omega$, we have

\[ \lambda_1(\Omega) \leq C_{\Omega} \max_{\xi \in \Omega} r_{\Omega}^2(\xi). \]
with \(C_\Omega = \min_{\xi \in A_\Omega} C_\Omega(\xi)\), where \(A_\Omega\) is the set of harmonic centers of \(\Omega\).

\[\square\]

The second theorem is a refined version of the first one.

The paper is organized as follows: in Section 2, we will introduce the general framework, we begin by recalling the local structure of the Heisenberg group, the Carnot-Carathéodory distance and the Koranyi balls. The section 3 is devoted to the study of some properties of Green’s functions of the Kohn-Laplace operator and their level sets. These properties lead to extend the notion of harmonic radius to the Heisenberg group \(\mathbb{H}^1\), which is the purpose of section 4. In section 5, we will use the Green’s functions of Koranyi balls to extend the harmonic transplantation to the Heisenberg group \(\mathbb{H}^1\). Finally, in section 6, we prove our main result, Theorem 6.2, then we give a refined version of this result which is Theorem 6.3. For the proof, we proceed as follows, we perturb the Dirichlet problem \((P_\Omega)\) on a Koranyi’s ball, then we compare the Rayleigh ratios of its solutions with those of their corresponding harmonics transplantations.

\[\square\]

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## 2 The Heisenberg group \(\mathbb{H}^1\)

The Heisenberg group \(\mathbb{H}^1\) is the Lie group whose underlying manifold is \(\mathbb{C} \times \mathbb{R} = \mathbb{R}^3\), with coordinates \((x, y, t)\), the group law \(o\) is defined as follows: for all \(\eta = (x, y, t)\), \(\xi = (x_0, y_0, t_0) \in \mathbb{R}^3\), \(\eta o \xi = (x + x_0, y + y_0, t + t_0 + 2(x_0 y - x y_0))\). For \(\eta \in \mathbb{H}^1\), the left translation by \(\xi\) is the diffeomorphism \(L_\eta(\xi) = \eta o \xi\). The Heisenberg group dilations are the \(\mathbb{H}^1\) transformations:

\[
\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 t), \lambda > 0.
\]

The homogeneous norm of the space is given for all \(\eta \in \mathbb{H}^1\) by

\[
\rho(\eta) = \left( (x^2 + y^2 + t^2) \frac{1}{4} \right),
\]

and the natural distance is accordingly defined for all \(\eta, \xi \in \mathbb{H}^1\) by:

\[
d_{\mathbb{H}^1}(\xi, \eta) = \rho(\xi^{-1} o \eta).
\]

The open balls \(B(\xi, r)\) associated to this distance are called the Koranyi balls

\[
B(\xi, r) = \{ \eta \in \mathbb{H}^1, d_{\mathbb{H}^1}(\xi, \eta) < r \}.
\]

A function \(u : \Omega \to \mathbb{R}\) defined on a domain \(\Omega\) of \(\mathbb{H}^1\) is called radial or \(\rho\)-radial if there exists a function \(\varphi : \mathbb{R}^+ \to \mathbb{R}\) such that \(u(\xi) = \varphi(\rho(\xi))\), for all \(\xi \in \Omega\).

The following left-invariant vector fields

\[
X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}
\]
give a base for the tangent space $T_\eta \mathbb{H}^1$. Whereas The horizontal distribution $H$ of $\mathbb{H}^1$ is generated by the vector fields $X$ and $Y$. The horizontal projection of a vector $V$ on $H$ will be denoted by $V_H$, so a vector $V$ is called horizontal if $V = V_H$. For any $C^1$ function $\varphi$ defined in an open set of $\mathbb{H}^1$, the horizontal gradient of $\varphi$ is given by

$$\nabla_{\mathbb{H}^1} \varphi = (X \varphi)X + (Y \varphi)Y.$$  
(2.1)

We denote the Lie bracket of two vectors fields $V_1$ and $V_2$ defined on $\mathbb{H}^1$ by $[V_1, V_2]$. We have, the following identities for the base $(X, Y, T)$

$$[X, T] = [Y, T] = 0 \text{ and } [X, Y] = -4T.$$ 

We consider on $\mathbb{H}^1$, the left-invariant Riemannian metric $g_{\mathbb{H}^1} = \langle ., . \rangle$ given in $\mathbb{H}^1$ for which the triplet $(X, Y, T)$ is an orthonormal basis for all point in $\mathbb{H}^1$. The corresponding Levi-Civita connexion denoted by $D$ satisfies

$$D_X X = 0, \quad D_Y Y = 0, \quad D_T T = 0$$

$$D_X Y = -2T, \quad D_Y X = 2Y, \quad D_Y T = -2X$$

$$D_Y Y = 2T, \quad D_T X = 2Y, \quad D_T Y = -2X.$$  

The volume of a domain $\Omega \subset \mathbb{H}^1$ denoted by $|\Omega|$, is the Riemannian volume associated to the the left invariant Riemannian metric $g_{\mathbb{H}^1}$, which coincides with the Lebesgue measure in $\mathbb{R}^3$. For a $C^1$ surface $\Sigma$ in $\mathbb{H}^1$ with $N$ as unit vector normal to $\Sigma$, we define its area by

$$A(\Sigma) = \int_{\Sigma} |N_{\mathbb{H}^1}| \, d\Sigma,$$  
(2.2)

where $N_H = N - <N, T > T$ denotes the orthogonal projection of $N$ onto the horizontal distribution, and $d\Sigma$ is the Riemannian area element induced by the metric $g_{\mathbb{H}^1}$ on $\Sigma$. If $\Sigma$ is a $C^1$ surface of a bounded domain $\Omega$, so $A(\Sigma)$ coincides with $P_{\mathbb{H}^1}(\Omega)$ the horizontal perimeter of $\Omega$ is defined by

$$P_{\mathbb{H}^1}(\Omega) = \sup \left\{ \int_{\Omega} \text{div} V \, dv; \ V \text{ horizontal of class} C^1, \, |V| \leq 1 \right\},$$  
(2.3)

where $\text{div} V$ is the Riemannian divergence of the vector field $V$, and $dv$ is the element of volume associated to $g_{\mathbb{H}^1}$. For more details one can see [14, 15, 20].

□

**Remark 2.1.** In the rest of the paper, for a given domain $D \subset \mathbb{H}^1$ with boundary $\partial D$, $d\Sigma_D$ will denote the Riemannian element area on $\partial D$ induced by the Riemannian metric $g_{\mathbb{H}^1}$.

We denotes by $\Delta_{\mathbb{H}^1}$ the Kohn-Laplace operator on the Heisenberg group, which is defined by

$$\Delta_{\mathbb{H}^1} = X^2 + Y^2.$$ 

The Kohn-Laplace operator $\Delta_{\mathbb{H}^1}$ satisfies the Hörmander conditions, thus it is hypoelliptic on $\mathbb{H}^1$. Particularly, any harmonic function $u$ defined on an open set $\Omega$ of $\mathbb{H}^1$ ($\Delta_{\mathbb{H}^1} u = 0$, in $\Omega$) is $C^\infty$. 

6
A fundamental solution of $-\Delta_{\mathbb{H}^1}$ with pole at zero is given by $\Gamma(\xi) = \frac{1}{8\pi\rho(\xi)^2}$. Moreover, a fundamental solution with pole at $\xi$, is $\Gamma(\xi,\xi') = \frac{1}{8\pi d(\xi,\xi')^2}$.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$|\varphi|_4^2 \leq c|\nabla_{\mathbb{H}^1}\varphi|_2^2, \quad \forall \varphi \in C_0^\infty(\mathbb{H}^1)$$

This inequality ensures in particular that for every domain $\Omega$ of $\mathbb{H}^1$, $|\varphi| = |\nabla_{\mathbb{H}^1}\varphi|_2$ is a norm on $C_0^\infty(\Omega)$.

We shall denote by $S^{1,2}(\Omega)$ the Banach space of the functions $u \in L^2(\Omega)$ such that the distributional derivative $X_1u$, $X_2u \in L^2(\Omega)$, and by $S_0^{1,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm above. The space $S_0^{1,2}(\Omega)$ is a Hilbert space with the inner product $\langle u,v \rangle_{S_0^{1,2}(\Omega)} = \int_\Omega \langle \nabla_{\mathbb{H}^1}u, \nabla_{\mathbb{H}^1}v \rangle$.

In what follows, we let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots \to +\infty$ denote the successive eigenvalues for $(P_\Omega)$ with corresponding eigenfunctions: $u_1, u_2, \ldots, u_m, \ldots$ in $S^{1,2}(\Omega)$.

The Kohn-Laplace operator is related to the following result called the maximum principle

**Theorem 2.2.** Let $\Omega$ be an open bounded set of $\mathbb{H}^1$ and $u \in C^2 \cap C(\overline{\Omega})$. If $\Delta_{\mathbb{H}^1}u \geq 0$, then

$$\sup_{\overline{\Omega}} u = \sup_{\partial \Omega} u,$$

and if $\Delta_{\mathbb{H}^1}u \leq 0$,

$$\inf_{\overline{\Omega}} u = \inf_{\partial \Omega} u.$$

We will end this section by giving the volumes of Koranyi balls.

**Proposition 2.3.** For $R > 0$, the volume of the Koranyi’s ball $B(0,R)$ is given by

$$|B(0,R)| = \frac{\pi^2}{2}R^4.$$  \hspace{1cm} (2.4)

**Proof.** We consider the function

$$\phi : [0,1[ \times [0,\pi[ \times ]0,2\pi[ \to B(0,R) \setminus P, \quad (r,\theta,\varphi) \mapsto (r\sqrt{\sin \theta \cos \varphi}, r\sqrt{\sin \theta \sin \varphi}, r^2 \cos \theta),$$

where

$$P = \left\{(x,y,t) \in \mathbb{H}^1, y = 0, x \geq 0 \right\}.$$ 

So $\phi$ is a $C^1$-diffeomorphism and $\forall (r,\theta,\varphi) \in [0,1[ \times [0,\pi[ \times ]0,2\pi[,$ we have

$$\det (J\phi(r,\theta,\varphi)) = r^3.$$ 

We observe that the Lebesgue measure of the set $P$ is zero, so we conclude that

$$|B(0,R)| = \int_0^1 \int_0^\pi \int_0^{2\pi} |\det (J\phi(r,\theta,\varphi))| \, dr \, d\theta \, d\varphi = \frac{\pi^2}{2}R^4.$$ \hspace{1cm} \Box
3 Green’s function

Before giving the different properties of the Green’s function, we will introduce the notion of uniform exterior ball property.

Definition 3.1. We say that the domain \( \Omega \) of \( \mathbb{H}^1 \) satisfies the uniform exterior ball property, if the following condition holds

\[
\begin{align*}
\text{There exists } r_0 > 0 & \text{ such that } \\
\forall \xi \in \partial \Omega, \; \forall r \in [0, r_0] & \exists \eta \in \mathbb{H}^n \text{ such that } \\
B_d(\eta, r) \cap \Omega = \emptyset & \text{ and } \xi \in \partial B_d(\eta, r).
\end{align*}
\]

We consider an open set \( \Omega \) of the Heisenberg group \( \mathbb{H}^1 \) with smooth boundary, such that \( \Omega \) and \( \mathbb{H}^n \setminus \overline{\Omega} \) satisfy the uniform exterior ball property. This condition seems to be natural since the Koranyi balls of \( \mathbb{H}^1 \) satisfy such a property. In [24], Hansen and Hueber proved that this condition implies that the domain \( \Omega \) is then regular: for \( \varphi \) in \( C(\partial \Omega) \), the Dirichlet problem

\[
\begin{align*}
\Delta_{\mathbb{H}^1} u & = 0 \quad \text{in } \Omega \\
u & = \varphi \quad \text{on } \partial \Omega.
\end{align*}
\]

has a classical solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \). In particular, \( \Omega \) has a Green’s function for every \( \xi \in \Omega \), which we denote by \( G_{(\Omega; \xi)} \). More precisely, for every \( \xi \in \Omega \), there exists a classical solution \( H_{(\Omega; \xi)} \) of the Dirichlet problem

\[
\begin{align*}
\Delta_{\mathbb{H}^1} H_{(\Omega; \xi)} & = 0 \quad \text{in } \Omega \\
H_{(\Omega; \xi)} & = \Gamma_\xi \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Gamma_\xi \) is a fundamental solution of \( -\Delta_{\mathbb{H}^1} \) with pole at \( \xi \). Hence, the Green’s function is defined by

\[
G_{(\Omega; \xi)}(\eta) = \Gamma_\xi(\eta) - H_{(\Omega; \xi)}(\eta)
\]

with

\[
\Gamma_\xi(\eta) = \frac{1}{8\pi \rho^2(\xi^{-1} \eta)}.
\]

The functions \( \Gamma_\xi \) and \( H_{(\Omega; \xi)} \) are called respectively the singular part and the regular part of the Green’s function \( G_{(\Omega; \xi)} \).

\[
\square
\]

3.1 Examples

Finding Green’s functions for given domains is difficult, however for certain domains with nice geometries, it is possible. We will give two examples below.

Example 3.2. The Green’s function of the Koranyi ball \( B(\xi, R) \) with pole at \( \xi \) is given by

\[
G_{(B(\xi, R); \xi)}(\eta) = \frac{1}{8\pi \rho^2(\xi^{-1} \eta)} - \frac{1}{8\pi R^2},
\]

indeed, we have

\[
\begin{align*}
\Delta_{\mathbb{H}^1} \left( \frac{1}{8\pi R^2} \right) & = 0 \quad \text{in } B(\xi, R) \\
\frac{1}{8\pi R^2} & = \frac{1}{8\pi \rho^2(\xi^{-1} \eta)} \quad \text{on } \partial B(\xi, R).
\end{align*}
\]
Example 3.3. The Green’s function of the open set $D = \{ \eta(x,y,t) \in \mathbb{H}^1, t > 0 \}$, with pole at $\xi = (0,0,t_0)$, recall that the boundary of $D$ is $\partial D = \partial \{ \eta(x,y,t) \in \mathbb{H}^1, t = 0 \}$. We have

$$G_{D,\xi}(\eta) = \frac{1}{8\pi r^2(\xi^{-1} \eta)} - \frac{1}{8\pi r^2(\xi \eta)},$$

in fact

$$\begin{cases}
\Delta_{\mathbb{H}^1} \left( \frac{1}{8\pi r^2(\xi \eta)} \right) = 0 & \text{in } D \\
\frac{1}{8\pi r^2(\xi \eta)} = \frac{1}{8\pi r^2(\xi^{-1} \eta)} & \text{on } \partial D
\end{cases}$$

3.2 Properties of the Green’s function

Let $\Omega$ be a regular domain of $\mathbb{H}^1$ and $G_\xi$ the Green’s function associated to $\Omega$ with pole at the point $\xi$.

Property 3.4. The function $G_\xi$ is positive on $\Omega$.

Proof. We have $\lim_{\eta \to \xi} G_\xi(\eta) = +\infty$, so $\forall M > 0$, there exists $r > 0$ such that

$$G_\xi(\eta) > M > 0, \forall \eta \in \overline{B(\xi,r)} \quad (3.5).$$

Let $D = \Omega \setminus \overline{B(\xi,r)}$, the function $G_\xi$ is harmonic on $D$. As $G_\xi$ vanishes on $\partial \Omega$ and is strictly positive on $\partial B(\xi,r)$, thus by using the maximum principle given in Theorem 2.2 we obtain

$$G_\xi \geq 0 \text{ on } D. \quad (3.6)$$

Through (3.5) and (3.6), we conclude that $G_\xi \geq 0$ on $\Omega$.

Property 3.5. The Green’s function is symmetric, it is a consequence of the fact that the Kohn-Laplace operator is a self adjoint operator.

Property 3.6. If $\Omega_1 \subset \Omega_2$, we have for all $\xi \in \Omega_1$

$$G_{\{\Omega_1;\xi\}} \leq G_{\{\Omega_2;\xi\}} \text{ on } \Omega_1$$

Proof. We consider for $\xi \in \Omega_1$, the function defined on $\Omega_1$

$$\psi_\xi = G_{\{\Omega_2;\xi\}} - G_{\{\Omega_1;\xi\}},$$

so

$$\psi_\xi = H_{\{\Omega_1;\xi\}} - H_{\{\Omega_2;\xi\}},$$
thus, \( \psi_\xi \) is harmonic on \( \Omega_1 \). On the other hand, as \( G_{\{\Omega_2, \xi\}} \) is positive on \( \Omega_2 \), then \( \forall \eta \in \partial \Omega_1 \),

\[
\psi_\xi = G_{\{\Omega_2, \xi\}} \geq 0,
\]

by using the maximum principle, we have \( \forall \eta \in \Omega_1 \)

\[
\psi_\xi \geq 0.
\]

Therefore

\[
G_{\{\Omega_2, \xi\}} \geq G_{\{\Omega_1, \xi\}}.
\] (3.7)

For any real \( c \), we denote by \( \Omega(c) = \{ \eta \in \Omega, G_\xi(\eta) > c \} \), the level set of the Green’s function on \( \Omega \) and by \( \partial \Omega(c) = \{ \eta \in \Omega, G_\xi(\eta) = c \} \) its boundary.

**Theorem 3.7.** Let \( G_\xi \) be the Green’s function of \( \Omega \) with pole at \( \xi \). For almost every \( \tau \), it holds

\[
\int_{\partial \Omega(\tau)} \frac{\langle \nabla \mathcal{H}^1 G_\xi, N_1 \rangle}{|\nabla G_\xi|} d\Sigma_{\Omega(\tau)} = 1.
\]

\[
\square
\]

**Proof.** Let \( D = \Omega(\tau) \setminus \overline{B(\xi, \epsilon)} \). So \( G_\xi = \Gamma_\xi - H_\xi \) is harmonic on \( D \) and we have

\[
\int_D \Delta_{\mathcal{H}^1} G_\xi dv = 0.
\]

An integration by parts gives

\[
\int_D \Delta_{\mathcal{H}^1} G_\xi dv = \int_{\partial \Omega(\tau)} \langle \nabla_{\mathcal{H}^1} G_\xi, N_1 \rangle d\Sigma_{\Omega(\tau)} - \int_{\partial B(\xi, \epsilon)} \langle \nabla_{\mathcal{H}^1} G_\xi, N_2 \rangle d\Sigma_{B(\xi, \epsilon)}
\]

where

\[
N_1 = -\frac{\nabla G_\xi}{|\nabla G_\xi|}
\]

and

\[
N_2 = -\frac{\nabla \rho(\xi^{-1} o \eta)}{|\nabla \rho(\xi^{-1} o \eta)|}.
\]

Thus

\[
\int_{\partial \Omega(\tau)} \langle \nabla_{\mathcal{H}^1} G_\xi, N_1 \rangle d\Sigma_{\Omega(\tau)} = \int_{\partial B(\xi, \epsilon)} \langle \nabla_{\mathcal{H}^1} G_\xi, N_2 \rangle d\Sigma_{B(\xi, \epsilon)}.
\]

On one hand

\[
\int_{\partial B(\xi, \epsilon)} \langle \nabla_{\mathcal{H}^1} G_\xi, N_2 \rangle d\Sigma_{B(\xi, \epsilon)} = \int_{\partial B(\xi, \epsilon)} \langle \nabla_{\mathcal{H}^1} \Gamma_\xi, N_2 \rangle d\Sigma_{B(\xi, \epsilon)} - \int_{\partial B(\xi, \epsilon)} \langle \nabla_{\mathcal{H}^1} H_\xi, N_2 \rangle d\Sigma_{B(\xi, \epsilon)},
\]

then, by proceeding with an integration by parts in the second integral of the equality above, we obtain
\[ \int_{\partial B(\xi,\varepsilon)} \langle \nabla_{\mathbb{H}^1} H_\xi, N_2 \rangle d\Sigma_{B(\xi,\varepsilon)} = \int_{B(\xi,\varepsilon)} \Delta_{\mathbb{H}^1} H_\xi dv = 0. \]

On the other hand
\[ \int_{\partial B(\xi,\varepsilon)} \langle \nabla_{\mathbb{H}^1} \Gamma_\xi, N_2 \rangle d\Sigma_{B(\xi,\varepsilon)} = -1, \]
therefore
\[ \int_{\partial \Omega(\tau)} \langle \nabla_{\mathbb{H}^1} G_\xi, N_1 \rangle d\Sigma_\Omega(\tau) = -1. \]

The result follows. \( \square \)

4 The harmonic radius in the Heisenberg group

4.1 Definition and examples

The harmonic radius of a regular domain \( \Omega \subset \mathbb{H}^1 \) at a fixed point \( \xi \), denoted by \( r_\Omega(\xi) \), is defined by
\[ \Gamma(r_\Omega(\xi)) = H_{\{\Omega;\xi\}}(\xi) \]
where
\[ \Gamma(r) = \frac{1}{8\pi r^2} \]
and \( H_{\{\Omega;\xi\}} \) is the regular part of the Green’s function \( G_{\{\Omega;\xi\}} \).

Using the expression of the function \( \Gamma \), we obtain
\[ r_\Omega(\xi) = \frac{1}{\sqrt{8\pi H_{\{\Omega;\xi\}}(\xi)}} \]

\( \square \)

Remark 4.1. As \( H_{\{\Omega;\xi\}} \) is strictly positive, the harmonic radius is well defined.

The difficulty to explicit the harmonic radius at all points of any domain \( \Omega \) is due to the difficulty to explicit Green’s functions for such domain as we noticed it earlier. So as examples, we will come back to the ones given in the last section.

Example 4.2. Let \( \Omega \) be the Koranyi’s ball \( B(\xi,R) \). The associated Green’s function with pole at the center \( \xi \) of the ball, is
\[ G_{\{\Omega;\xi\}}(\eta) = \frac{1}{8\pi \rho^2 (\xi^{-1} \circ \eta)} - \frac{1}{8\pi R^2}, \]
so for all \( \eta \in B(\xi,R) \), \( H_{\{\Omega;\xi\}}(\eta) = \frac{1}{8\pi R^2} \).

Hence, we deduce that the harmonic radius at the center of the ball is equal to its radius
\[ r_\Omega(\xi) = R. \]
Example 4.3. Let $D = \{ \eta(x,y,t) \in \mathbb{H}^1, \ t > 0 \}$. For $\xi = (0,0,t_0)$, we have

$$G_{\{\Omega;\xi\}}(\eta) = \frac{1}{8\pi \rho^2(\xi^{-1}o\eta)} - \frac{1}{8\pi \rho^2(\xi o\eta)},$$

so

$$H_{\{\Omega;\xi\}}(\eta) = \frac{1}{8\pi \rho^2(\xi o\eta)},$$

and

$$H_{\{\Omega;\xi\}}(\xi) = \frac{1}{8\pi \rho^2(\xi o\xi)} = \frac{1}{16\pi t_0},$$

then

$$r_\Omega(\xi) = \sqrt{2t_0}. \quad \square$$

4.2 Properties of The harmonic radius

In the sequel and for the sake of simplicity, the functions $G_{\{\Omega;\xi\}}$ and $H_{\{\Omega;\xi\}}$ will be respectively denoted by $G_\xi$ and $H_\xi$.

Property 4.4. For any regular domain $\Omega$ of $\mathbb{H}^1$ and any $\xi \in \Omega$, we have

$$\inf_{\eta \in \partial \Omega} \rho(\xi^{-1}o\eta) \leq r_\Omega(\xi) \leq \sup_{\eta \in \partial \Omega} \rho(\xi^{-1}o\eta) \quad (4.1)$$

\[ \square \]

Proof. Since $H_\xi$ is harmonic on $\Omega$, so by using the maximum principle for the Kohn-Laplace operator, we obtain

$$\inf_{\eta \in \Omega} H_\xi(\eta) = \inf_{\eta \in \partial \Omega} H_\xi(\eta), \quad (4.2)$$

and

$$\sup_{\eta \in \Omega} H_\xi(\eta) = \sup_{\eta \in \partial \Omega} H_\xi(\eta). \quad (4.3)$$

On the other hand, $\forall \eta \in \partial \Omega$

$$H_\xi(\eta) = \frac{1}{8\pi \rho^2(\xi^{-1}o\eta)}, \quad (4.4)$$

thus, by using (4.2), (4.3) and (4.4), we obtain

$$\inf_{\eta \in \partial \Omega} \frac{1}{8\pi \rho^2(\xi^{-1}o\eta)} \leq H_\xi(\eta) \leq \sup_{\eta \in \partial \Omega} \frac{1}{8\pi \rho^2(\xi^{-1}o\eta)}. \quad (4.5)$$

The result follows using the definition of the harmonic radius. \[ \square \]
Property 4.5. The harmonic radius is monotone, it means that, if $\Omega_1 \subset \Omega_2$, then for all $\xi$ in $\Omega_1$

$$r_{\Omega_1}(\xi) \leq r_{\Omega_2}(\xi)$$

Proof. We consider the function

$$\psi_\xi = G_{\{\Omega_2, \xi\}} - G_{\{\Omega_1, \xi\}}$$

we have

$$\psi_\xi = H_{\{\Omega_1, \xi\}} - H_{\{\Omega_2, \xi\}}.$$  

As $\psi_\xi$ is positive, we obtain

$$H_{\{\Omega_1, \xi\}}(\xi) \geq H_{\{\Omega_2, \xi\}}(\xi).$$

The result follows.

Property 4.6. Boundary Behavior

Since the regular part of the Green’s function coincides with its singular part on the boundary of the domain, the harmonic radius vanishes at every point of the boundary. More precisely, Let $\xi, \eta \in \partial \Omega$, then

$$\lim_{\eta \to \xi} H_\xi(\eta) = \lim_{\eta \to \xi} \Gamma_\xi(\eta) = \infty.$$  

Hence $r_\Omega(\xi) = 0$.

Furthermore, we have the following expansion of the Harmonic radius near the boundary of domains $\Omega$ such that $\Omega$ and $\mathbb{H}^1 \setminus \overline{\Omega}$ satisfy [3.1]. For a point $\xi \in \Omega$, let us denote $d_\xi = d(\xi, \partial(\Omega))$ its distance from the boundary of $\Omega$, then

$$H_\xi(\xi) = \frac{1}{(2d_\xi)^2} + o(d_\xi^{-2}).$$

For a proof one can see [16]. Hence, the harmonic radius has the following expansion near the boundary

$$r_\Omega(\xi) = 2d_\xi + o(d_\xi)$$

In the following, we will give the relation between the harmonic radius of $\Omega$ and those of its level sets.

Proposition 4.7. Let $\Omega(\tau) = \{\eta \in \Omega, G_\xi(\eta) > \tau\}$ with $\tau$ a positive real and $\xi \in \Omega(\tau)$. We have the following relation between the harmonic radius of $\Omega(\tau)$ at $\xi$ denoted by $r_{\Omega(\tau)}(\xi)$ and the one of $\Omega$ at the same point

$$r_{\Omega(\tau)}(\xi) = \frac{r_\Omega(\xi)}{\sqrt{\tau 8\pi r_\Omega^2(\xi) + 1}}$$

(4.6)
Proof. We define on $\Omega(\tau)$, the function

$$G^\tau_\xi(\eta) = G_\xi - \tau = \frac{1}{8\pi \rho^2(\xi^{-1} \eta)} - H_\xi(\eta) - \tau.$$ 

As $G^\tau_\xi$ vanishes on $\partial \Omega(\tau)$ and the function $H^\tau_\xi = H_\xi + \tau$ is harmonic on $\Omega(\tau)$, we conclude that $G^\tau_\xi$ is the Green’s function associated to the set $\Omega(\tau)$ and $H^\tau_\xi = H_\xi + \tau$ is its regular part, thus the harmonic radius of $\Omega(\tau)$ at $\xi$, is given by

$$r_{\Omega(\tau)}(\xi) = \frac{1}{\sqrt{8\pi H^\tau_\xi(\xi)}} = \frac{1}{\sqrt{8\pi (H_\xi(\xi) + \tau)}} = \frac{1}{\sqrt{8\pi \left(\frac{1}{8\pi \rho^2(\xi)} + \tau\right)}} r_{\Omega}(\xi) = \frac{1}{\sqrt{\tau 8\pi \rho^2(\xi) + 1}}.$$ 

Next, we will give the relation between the harmonic radius of $\Omega$ at a given point $\xi \in \Omega$ and the volume of $\Omega$.

**Theorem 4.8.** Let $\Omega$ be a domain of $\mathbb{H}^1$ such that $\Omega$ and $\mathbb{H}^1 \setminus \Omega$ satisfy the uniform exterior property (3.1) and let $\xi$ be a point of $\Omega$. We have the following inequality

$$r_{\Omega}(\xi) \leq \alpha_\Omega(\xi) \left(\frac{2|\Omega|}{\pi^2}\right)^{\frac{1}{4}},$$

with

$$\alpha_\Omega(\xi) = \sup_{\eta \in \partial \Omega} \frac{d_{\mathbb{H}^1}(\xi, \eta)}{\inf_{\eta \in \partial \Omega} d_{\mathbb{H}^1}(\xi, \eta)}$$

Proof. Since $\Omega$ and $\mathbb{H}^1 \setminus \Omega$ satisfy (3.1), the ball $B(\xi, \inf_{\eta \in \partial \Omega} d_{\mathbb{H}^1}(\xi, \eta)) \subset \Omega$, hence

$$|B(\xi, \inf_{\eta \in \partial \Omega} d_{\mathbb{H}^1}(\xi, \eta))| \leq |\Omega|,$$

thus

$$\frac{\pi^2}{2} \left(\inf_{\eta \in \partial \Omega} d_{\mathbb{H}^1}(\xi, \eta)\right)^4 \leq |\Omega|,$$

which implies

$$\inf_{\eta \in \partial \Omega} d_{\mathbb{H}^1}(\xi, \eta) \leq \left(\frac{2|\Omega|}{\pi^2}\right)^{\frac{1}{4}}.$$ 

The result follows using (4.1) and (4.8). We derive the following result for the Koranyi’s ball of center 0 and radius $r_{\Omega}(\xi)$.
Corollary 4.9. For any \( \xi \in \Omega \), we have
\[
|B(\xi, r_\Omega(\xi))| \leq \alpha_\Omega^4(\xi)|\Omega|
\] (4.9)
\[
\square
\]

**Proof.** The result is a direct consequence of (2.4) and (4.7).
\[
\square
\]

**Example 4.10.** We come back to Example 4.2, where, we showed that the harmonic radius of the Koranyi’s ball \( B(\xi, r) \) at \( \xi \) is equal to the radius \( r \). Since
\[
\sup_{\eta \in \partial B(\xi, r)} d_{H^1}(\xi, \eta) = \inf_{\eta \in \partial B(\xi, r)} d_{H^1}(\xi, \eta) = r,
\]
we derive that
\[
\alpha_{B(\xi, r)}(\xi) = 1.
\]
Then, using (2.4), we obtain the equality case in (4.7) and (4.9).
\[
\square
\]

**Proposition 4.11.** For almost every positive real \( \tau \), we have
\[
\alpha_{\Omega(\tau)}(\xi) \leq \alpha_{\Omega}(\xi),
\] (4.10)
\[
\square
\]

**Proof.** The Green’s function of \( \Omega(\tau) \) is given by
\[
G_{\Omega(\tau), \xi}(\eta) = \frac{1}{8\pi \rho^2(\xi^{-1}0\eta)} - H_\xi(\eta) - \tau,
\]
so, for all \( \eta_\tau \in \partial \Omega(\tau) \)
\[
\frac{1}{8\pi \rho^2(\xi^{-1}0\eta_\tau)} = H_\xi(\eta_\tau) + \tau.
\]
Next, we use (4.5), to obtain
\[
8\pi \left( \sup_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2 + \tau \leq \frac{1}{8\pi \rho^2(\xi^{-1}0\eta_\tau)} \leq 8\pi \left( \inf_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2 + \tau,
\]

hence
\[
\frac{\left( \sup_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2}{1 + \tau 8\pi \left( \sup_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2} \geq \rho^2(\xi^{-1}0\eta_\tau) \geq \frac{\left( \inf_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2}{1 + \tau 8\pi \left( \inf_{\eta \in \partial \Omega} d_{H^1}(\xi, \eta) \right)^2}.
\]

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Therefore, we obtain
\[
\left( \sup_{\eta \in \partial\Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2 \geq \frac{\sup_{\eta \in \partial\Omega} \rho^2(\xi^{-1}\eta_{\tau})}{1 + \tau 8\pi \left( \sup_{\eta \in \partial\Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2}
\]
(4.11)
and
\[
\left( \inf_{\eta_{\tau} \in \partial\Omega(\tau)} d_{\mathbb{H}^1}(\xi, \eta_{\tau}) \right)^2 = \inf_{\eta_{\tau} \in \partial\Omega(\tau)} \rho^2(\xi^{-1}\eta_{\tau}) \geq \frac{\left( \inf_{\eta \in \partial\Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2}{1 + \tau 8\pi \left( \inf_{\eta \in \partial\Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2}.
\]
(4.12)
By using (4.11) and (4.12), we obtain
\[
\alpha^2_{\Omega(\tau)}(\xi) \leq \frac{1 + \tau 8\pi \left( \inf_{\eta \in \partial\Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2}{1 + \tau 8\pi \left( \sup_{\eta \in \Omega} d_{\mathbb{H}^1}(\xi, \eta) \right)^2} \leq \alpha^2_{\Omega}(\xi).
\]
As a consequence of the proposition above, we obtain

**Corollary 4.12.** For almost every positive real \( \tau \),
\[
|B(0, r_{\Omega(\tau)}(\xi))| \leq \alpha^4_{\Omega}(\xi) |\Omega(\tau)|
\]
(4.13)

**Proof.** We apply (4.9) to the domain \( \Omega(\tau) \), then we use (4.10), the result follows.

\( \square \)

## 5 Harmonic Transplantation

### 5.1 A result of P. Pansu in \( \mathbb{H}^1 \)

We have the following result

If \( \Omega \subset \mathbb{H}^1 \) is a \( C^1 \) bounded domain, then \( P_{\mathbb{H}^1}(\Omega) \), the horizontal perimeter of \( \Omega \) coincides with the area of \( \partial\Omega \).

\[
A(\partial\Omega) = \int_{\partial\Omega} |N_{\mathbb{H}^1}| d\Sigma, \quad \text{(5.1)}
\]

where \( N_{\mathbb{H}^1} = N - <N, T > T \) denotes the orthogonal projection of \( N \) onto the horizontal distribution, and \( d\Sigma \) is the Riemannian area element induced by the metric \( g_{\mathbb{H}^1} \) on \( \partial\Omega \) (one can see [14] [15] [20]).

\( \square \)
Recall that $P_{\mathbb{H}^1}(\Omega)$ is defined by

$$P_{\mathbb{H}^1}(\Omega) = \sup \left\{ \int_\Omega \text{div} V \, dv ; \text{V horizontal of classe } C^1, \, |V| \leq 1 \right\},$$

where $\text{div} V$ is the Riemannian divergence of the vector field $V$, and $dv$ is the volume element associated to the left-invariant Riemannian metric $g_{\mathbb{H}^1}$.

As a consequence, one obtain

**Corollary 5.1.** (\cite{[14], [15]})
For a set $D$ of $\mathbb{H}^1$ bounded by a surface $\{u = 0\}$ of class $C^1$

$$P_{\mathbb{H}^1}(D) = \int_{\{u = 0\}} \frac{\left| \nabla_{\mathbb{H}^1} u \right|}{|\nabla u|} d\Sigma_D.$$

\[\square\]

We have the following result of P. Pansu in \cite{6, 7}

**Theorem 5.2.** Let $\Omega \subset \mathbb{H}^1$ be a $C^1$ bounded domain, then

$$|\Omega|^\frac{4}{3} \leq C P_{\mathbb{H}^1}(\Omega)$$

with $C = \left(\frac{3}{2\pi}\right)^{\frac{1}{4}}$

\[\square\]

### 5.2 Harmonic Transplantation

**Definition 5.3.** Let $\Omega$ be a bounded regular subset of $\mathbb{H}^1$ satisfying \cite{[6], [11]} and $\xi$ a point in $\Omega$. If $u : B(0, r_{\Omega}(\xi)) \to \mathbb{R}_+$ is a $\rho$-radial function on the Koranyi’s ball of center $0$ and radius $r_{\Omega}(\xi)$ and $\varphi$ is a function defined by $\varphi(G_{\{B(0, r_{\Omega}(\xi)), 0\}}) = u$. Then the harmonic transplantation $U_\xi$ of $u$ into $\Omega$ is the function

$$U_\xi = \varphi(G_\xi).$$

\[\square\]

**Proposition 5.4.** Let $\Omega$ be a regular bounded domain of $\mathbb{H}^1$ and $\xi$ any point in $\Omega$, then

$$\int_{\Omega} |\nabla_{\mathbb{H}^1} U_\xi|^2 \, dv = \int_{B(0, r_{\Omega}(\xi))} |\nabla_{\mathbb{H}^1} u|^2 \, dv.$$

\[\square\]

**Proof.** Using the co-area formula given in \cite{[17], [18]}, we obtain

$$\int_{\Omega} |\nabla_{\mathbb{H}^1} U_\xi|^2 \, dv = \int_0^\infty \left( \int_{\{G_\xi = \tau\}} \frac{\langle \nabla_{\mathbb{H}^1} U_\xi, \nabla_{\mathbb{H}^1} U_\xi \rangle}{|\nabla G_\xi|} \, d\Sigma_{\mathbb{H}^1}(\tau) \right) \, d\tau$$

On the other hand

$$\nabla_{\mathbb{H}^1} U_\xi = \mu'(G_\xi)\nabla_{\mathbb{H}^1} G_\xi.$$

Since
\[
\int_{\Omega} |\nabla_{H^1} U_\xi|^2 \, dv = \int_{0}^{\infty} \left( \int_{\{G_\xi = \tau\}} \frac{\mu'(G_\xi)}{\nabla G_\xi} \left( \frac{\nabla_{H^1} G_\xi}{\nabla G_\xi} \right)^2 \, d\Sigma_{\Omega(\tau)} \right) \, d\tau
\]

\[
= \int_{0}^{\infty} |\mu'(\tau)|^2 \left( \int_{\Omega(\tau)} \frac{\nabla_{H^1} G_\xi}{\nabla G_\xi} \, d\Sigma_{\Omega(\tau)} \right) \, d\tau.
\]

Therefore, using Theorem 3.7, we obtain

\[
\int_{\Omega} |\nabla_{H^1} U_\xi|^2 \, dv = \int_{0}^{\infty} |\mu'(\tau)|^2 \, d\tau.
\]

We remark that the integral above is independent of \(\Omega\), thus by the same way, we prove that

\[
\int_{B(0,r\alpha(\xi))} |\nabla_{H^1} u|^2 \, dv = \int_{0}^{\infty} |\mu'(\tau)|^2 \, d\tau,
\]

which completes the proof of the proposition.

\[\square\]

**Proposition 5.5.** Let \(\Omega\) be a regular bounded domain of \(H^1\) satisfying (3.1) and \(\xi\) any point in \(\Omega\). For any positive continuous function \(f\), we have

\[
\int_{B(0,r\alpha(\xi))} f(u) \, dv \leq C_{\Omega}(\xi) \int_{\Omega} f(U_\xi) \, dv
\]

where \(C_{\Omega}(\xi) = 16\sqrt{2} \alpha^6_{\Omega}(\xi) C^2\)

\[\square\]

**Proof.** We apply the co-area formula, we obtain

\[
\int_{\Omega} f(U_\xi(\eta)) \, dv = \int_{0}^{\infty} \left( \int_{\{G_\xi = \tau\}} \frac{f(\varphi(G_\xi(\eta)))}{\nabla G_\xi(\eta)} \, d\Sigma_{\Omega(\tau)} \right) \, d\tau
\]

\[
= \int_{0}^{\infty} f(\varphi(\tau)) \left( \int_{\{G_\xi = \tau\}} \frac{1}{\nabla G_\xi(\eta)} \, d\Sigma_{\Omega(\tau)} \right) \, d\tau.
\]

On one hand,

\[
\int_{\{G_\xi = \tau\}} \frac{1}{\nabla G_\xi(\eta)} \, d\Sigma_{\Omega(\tau)} = \int_{\{G_\xi = \tau\}} \frac{|\nabla_{H^1} G_\xi(\eta)|^2}{|\nabla G_\xi(\eta)|} \, d\Sigma_{\Omega(\tau)} \cdot \int_{\{G_\xi = \tau\}} \frac{1}{|\nabla G_\xi(\eta)|} \, d\Sigma_{\Omega(\tau)}
\]

\[
\geq \left( \int_{\{G_\xi = \tau\}} \frac{|\nabla_{H^1} G_\xi(\eta)|}{|\nabla G_\xi(\eta)|} \, d\Sigma_{\Omega(\tau)} \right)^2
\]

\[
= (P_{H^1}(\{G_\xi > \tau\}))^2
\]

We apply Theorem 5.2 to obtain

\[
\int_{\{G_\xi = \tau\}} \frac{1}{|\nabla G_\xi(\eta)|} \, d\Sigma_{\Omega(\tau)} \geq \frac{1}{C^2} \{G_\xi > \tau\}\frac{3}{2}
\]
On the other hand, $G_{B(0,r_{\Omega}(\xi)),0} = \frac{1}{8\pi \rho^2(\eta)} - \frac{1}{8\pi r^2(\xi)}$, so by denoting $G_{B(0,r_{\Omega}(\xi)),0} = \tilde{G}_0$, we have

\[
\int_{B(0,r_{\Omega}(\xi))} f(u(\eta)) \, dv = \int_0^{r_{\Omega}(\xi)} \left( \int_{\{\tilde{G}_0 = \tau\}} \frac{f(\varphi(\tilde{G}_0(\eta)))}{|\nabla \tilde{G}_0(\eta)|} d\Sigma_{\{G_0 > \tau\}} \right) d\tau 
\]

\[
= \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) \left( \int_{\{\tilde{G}_0 = \tau\}} \frac{1}{|\nabla \tilde{G}_0(\eta)|} d\Sigma_{\{G_0 > \tau\}} \right) d\tau 
\]

\[
= \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) \left( 4\pi (r_{\Omega(\tau)}(\xi))^{3} \int_{\{\rho(\eta) = r_{\Omega(\tau)}(\xi)\}} \frac{1}{|\nabla \rho(\eta)|} d\Sigma_{B(0,r_{\Omega(\tau)}(\xi))} \right) d\tau. 
\]

Recall the volume of the Koranyi’s ball $B(0,r_{\Omega(\tau)}(\xi))$

\[
\int_{B(0,r_{\Omega(\tau)}(\xi))} dv = |B(0,r_{\Omega(\tau)}(\xi))| = \frac{\pi^2}{2} (r_{\Omega(\tau)}(\xi))^4. 
\]

Applying again the co-area formula, we find

\[
\int_{B(0,r_{\Omega(\tau)}(\xi))} dv = \int_0^{r_{\Omega(\tau)}(\xi)} \left( \int_{\{\rho(\eta) = r\}} \frac{1}{|\nabla \rho(\eta)|} d\Sigma_{B(0,r)} \right) dr 
\]

thus

\[
\int_{\{\rho(\eta) = r_{\Omega(\tau)}(\xi)\}} \frac{1}{|\nabla \rho(\eta)|} d\Sigma_{B(0,r_{\Omega(\tau)}(\xi))} = \frac{\partial |(B(0,r_{\Omega(\tau)}(\xi))|}{\partial r_{\Omega(\tau)}(\xi)} = 2\pi^2 (r_{\Omega(\tau)}(\xi))^3 
\]

which gives using (4.13),

\[
\int_{B(0,r_{\Omega(\xi)})} f(u(\eta)) \, dv = \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) 8\pi^3 (r_{\Omega(\tau)}(\xi))^6 d\tau 
\]

\[
= \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) 16\sqrt{2} |B(0,r_{\Omega(\tau)}(\xi))|^\frac{3}{2} d\tau 
\]

\[
\leq \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) \alpha_0^6(\xi) 16\sqrt{2} |\{G_\xi > \tau\}|^\frac{3}{2} d\tau 
\]

\[
\leq \alpha_0^6(\xi) 16\sqrt{2} C^2 \int_0^{r_{\Omega}(\xi)} f(\varphi(\tau)) \left( \int_{\{G_\xi = \tau\}} \frac{1}{|\nabla G_\xi(\eta)|} d\Sigma_{\{G_\xi = \tau\}} \right) d\tau 
\]

\[
\leq C_\Omega(\xi) \int_{\Omega} f(U_\xi(\eta)) \, dv. 
\]
6 The first eigenvalue of the Kohn-Laplace operator

In this section, we will use Proposition 5 to find an upper bound for the first eigenvalue \( \lambda_1 \) of a regular bounded domain in the Heisenberg group \( \mathbb{H}^1 \) associated to the following problem:

\[
\begin{cases}
-\Delta_{\mathbb{H}^1} u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

called The Kohn-Laplace problem. We recall that \( \Delta_{\mathbb{H}^1} = X^2 + Y^2 \) is the Kohn-Laplace operator in the Heisenberg group \( \mathbb{H}^1 \), where \( X = \partial_x + 2y \partial_t \) and \( Y = \partial_y - 2x \partial_t \).

The existence of the eigenvalues of \( (P_{\Omega}) \) was proved by X. Luo and P. Niu in [10, 11, 12] by using the Kohn inequality ([13]) for the two vector fields \( X, Y \) with the spectral properties of compact operators.

The problem \( (P_{\Omega}) \) does not admit a radial solution in the ball case, which imposes a constraint for the use of the method of Harmonic transplantation in the order to find an upper bound for the first eigenvalue as in the Euclidean case. Indeed, if \( u \) is \( \rho \)-radial, so there exists a real function \( f \) such that \( u(\eta) = f(\rho(\eta)) \), thus

\[
\Delta_{\mathbb{H}^1} u(\eta) = \frac{x^2 + y^2}{\rho^2(\eta)} \left( f''(\rho(\eta)) + \frac{3}{\rho(\eta)} f'(\rho(\eta)) \right)
\]

and by using the polar coordinates, we obtain

\[
\Delta_{\mathbb{H}^1} u(\eta) = \sin(\theta) \left( f''(r) + \frac{3}{r} f'(r) \right).
\]

So, to obtain a radial solution of the Kohn-Laplace problem for the Koranyi’s ball \( B(0, R) \), we have to solve the following problem:

\[
\begin{cases}
-\sin(\theta) \left( f''(r) + \frac{3}{r} f'(r) \right) = \lambda f(r) & \text{in } ]0, R[ \times ]0, \Pi[ \\
f(R) = 0 & \text{on } \partial B(0, R)
\end{cases}
\]

which obviously does not admit a radial solution.

The idea to find an upper bound for the first eigenvalue of \( (P_{\Omega}) \) is to perturb \( (P_{B(0, R)}) \) and consider the following problem

\[
\begin{cases}
-\Delta_{\mathbb{H}^1} u = \mu(R) \psi u & \text{in } B(0, R) \setminus \{0\} \\
u = 0 & \text{on } \partial B(0, R)
\end{cases}
\]

(6.2)

where \( \psi(\eta) = \frac{x^2 + y^2}{\rho^2(\eta)} = |\nabla_{\mathbb{H}^1} \rho|^2 \). We will prove that problem (6.2) admits a radial solution given by Bessel functions. To this aim, let us consider the function

\[
J_m(l) = \left( \frac{1}{2} \right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(m + k + 1)} \left( \frac{l}{2} \right)^{2k}
\]

which is the Bessel function of first kind with order \( m \), it is a solution of the following equation

\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( 1 - \frac{m^2}{x^2} \right) y = 0,
\]

called Bessel equation, we refer to [19]. The function \( J_m(l) \) admits an infinity of simple and positive zeros:
The function Gamma $\Gamma$ is defined on $]0, \infty[\,$, by $\Gamma(x) = \int_0^\infty \exp^{-t} t^{x-1} dt$ and verifies $\Gamma(n) = (n - 1)!$, $\forall n \in \mathbb{N}^*$.

To find a radial solution for problem (6.2), firstly, we have to solve the following problem

\[
\begin{aligned}
    f''(r) + 3 \frac{r}{r} f'(r) + \mu(R) f(r) &= 0 \quad \text{on} \quad ]0, R[ \\
    f(R) &= 0.
\end{aligned}
\]  

(6.3)

We consider $f(r) = \frac{1}{r} g(r)$ with $r > 0$, so

\[
\begin{aligned}
    f'(r) &= -\frac{1}{r^2} g(r) + \frac{1}{r} g'(r) \\
    f''(r) &= 2 \frac{1}{r^3} g(r) - \frac{2}{r^2} g'(r) + \frac{1}{r} g''(r)
\end{aligned}
\]

thus

\[
f''(r) + \frac{3}{r} f'(r) + \mu(R) f(r) = 0 \iff g''(r) + \frac{1}{r} g'(r) + (\mu(R) - \frac{1}{r^2}) g(r) = 0
\]

and performing the change of variables $l = \sqrt{\mu(R)} r$, we obtain

\[
g''(l) + \frac{1}{l} g'(l) + (\mu(R) - \frac{1}{l^2}) g(l) = 0 \iff g''(l) + \frac{1}{l} g'(l) + (1 - \frac{1}{l^2}) g(l) = 0.
\]

The equation

\[
g''(l) + \frac{1}{l} g'(l) + (1 - \frac{1}{l^2}) g(l) = 0.
\]

admits the Bessel function $J_1(l)$ as solution. Then $J_1(\sqrt{\mu} r)$ is a solution of

\[
g''(r) + \frac{1}{r} g'(r) + (\mu(R) - \frac{1}{r^2}) g(r) = 0
\]

which implies that $\frac{1}{r} J_1(\sqrt{\mu} r)$ is a solution of

\[
f''(r) + \frac{3}{r} f'(r) + \mu(R) f(r) = 0.
\]

So problem (6.3) admits as eigenvalues and associated eigenfunctions respectively

\[
\mu_{1j}(R) = \frac{l_{1j}^2}{R^2}, \quad f_{1j}(r) = \frac{1}{r} J_1 \left( \frac{l_{1j} r}{R} \right), \quad j = 1, 2, ...,\]

where the $l_{1j}$'s are the strictly positive zeros of the Bessel function $J_1(l)$.

Next, we remark that on $B(0, R) \setminus \{0\}$, we have

\[
\Delta_{\mathbb{S}^1} f_{1j}(\rho(\eta)) + \mu_{1j}(R) \psi f_{1j}(\rho(\eta)) = 0 \quad \forall j = 1, 2, ...
\]

Therefore, the radial eigenfunctions and the associated eigenvalues of problem (6.3)
\[ u_{1j}(\eta) = f_{1j}(\rho(\eta)), \quad \mu_{1j}(R) = \frac{l_{1j}^2}{R^2}, \quad \forall j = 1, 2, \ldots \]

are respectively solutions of problem (6.2). Hence, we derive the following result

**Corollary 6.1.** For \( j = 1, 2, \ldots \), we have

\[
\int_{B(0,R)} |\nabla_{H^1} u_{1j}|^2 \, dv = \mu_{1j}(R) \int_{B(0,R)} \psi u_{1j}^2 \, dv
\]

\[ \square \]

Now, we are ready to state our main result

**Theorem 6.2.** Let \( \Omega \) be a bounded domain of \( H^1 \) with smooth boundary, such that \( \Omega \) and \( H^1 \setminus \Omega \) satisfy the uniform exterior ball property (3.1) and let \( \xi \) be any point in \( \Omega \), we have

\[
\lambda_1(\Omega) \leq C_\Omega(\xi) \frac{l_{11}^2}{r_\Omega^2(\xi)}
\]

where \( l_{11} \) is the first strictly positive zero of the Bessel function \( J_1(l) \) and \( r_\Omega(\xi) \) is the harmonic radius of \( \Omega \) at \( \xi \).

\[ \square \]

**Proof.** Using Rayleigh’s principle, the first eigenvalue of problem \( (P_\Omega) \) is given by

\[
\lambda_1(\Omega) = \inf_{v \in S^1_0(\Omega)} \frac{\int_{\Omega} |\nabla_{H^1} v|^2 \, dv}{\int_{\Omega} v^2 \, dv}.
\]

Hence

\[
\lambda_1(\Omega) \leq \frac{\int_{\Omega} |\nabla_{H^1} U_{\xi,11}|^2 \, dv}{\int_{\Omega} U_{\xi,11}^2 \, dv}
\]

where \( U_{\xi,11} \) is the harmonic transplantation of \( u_{11} \) in \( \Omega \). Using Propositions 5.4 and Corollary 5.1 we obtain

\[
\lambda_1(\Omega) \leq C_\Omega(\xi) \frac{\int_{B(0,r_\Omega(\xi))} |\nabla_{H^1} u_{11}|^2 \, dv}{\int_{B(0,r_\Omega(\xi))} u_{11}^2 \, dv}
\]

\[
\leq C_\Omega(\xi) \frac{\int_{B(0,r_\Omega(\xi))} |\nabla_{H^1} u_{11}|^2 \, dv}{\int_{B(0,r_\Omega(\xi))} \psi u_{11}^2 \, dv}
\]

\[
\leq C_\Omega(\xi) \mu_{11}(r_\Omega(\xi)) = C_\Omega(\xi) \frac{l_{11}^2}{r_\Omega^2(\xi)}.
\]
We can refine the result of the theorem above, more precisely we have

**Theorem 6.3.** Let $\Omega$ be a bounded domain of $\mathbb{H}^1$ with smooth boundary, such that $\Omega$ and $\mathbb{H}^1 \setminus \Omega$ satisfy the uniform exterior ball property (3.1) and let $\xi$ be any point in $\Omega$, we have

$$\lambda_1(\Omega) \leq C_\Omega \frac{p_{\Omega}^2}{\max_{\xi \in \Omega} r_{\Omega}^2(\xi)}$$

with $C_\Omega = \min_{\xi \in A_\Omega} C_\Omega(\xi)$, where $A_\Omega$ is the set of harmonic centers of $\Omega$.

If $\Omega$ is a convex (Euclidean) domain of $\mathbb{H}^1$, then it has only one harmonic center. As an example, we will give the case of a Koranyi’s ball.

**Example 6.4.** In the case of the Koranyi’s ball $B(\xi, R)$, of center $\xi \in \mathbb{H}^1$ and radius $R$, using the result of Example 4.10 and Theorem 6.2, we obtain

$$\lambda_1(B(\xi, R)) \leq C_\Omega(\xi) \frac{p_{\Omega}^2}{R^2} = 16\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \frac{p_{\Omega}^2}{R^2}$$

As a continuation of this work, we can propose these open questions:

1. Compare the first eigenvalue of $(P_{\Omega})$, with the first eigenvalue of $(P_{\Omega'})$, where $\Omega'$ is an isoperimetric region having the same volume as $\Omega$.

2. Compare the first eigenvalue of $(P_{\Omega'})$, where $\Omega'$ is an isoperimetric region with the first eigenvalue of $(P_{B(0,R)})$, $B(0,R)$ is the bubble set with the same volume as $\Omega'$.

3. After completing questions 1. and 2., one can attempt to find an analogous formula for the one given by Hersch in the Euclidian settings by using perhaps different methods.

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**E-mails:** ngamara@taibahu.edu.sa, najoua.gamara@fst.rnu.tn
akremmakni@gmail.com

**Addresses:** College of Sciences, Taibah University, Saudi Arabia.
University Tunis El Manar, F.S.T, Mathematics Department, Tunisia.