Between equilibrium fluctuations and Eulerian scaling: Perturbation of equilibrium for a class of deposition models

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Abstract

We investigate propagation of perturbations of equilibrium states for a wide class of 1D interacting particle systems. The class of systems considered incorporates zero range, $K$-exclusion, mysanthropic, ‘bricklayers’ models, and much more. We do not assume attractivity of the interactions. We apply Yau’s relative entropy method rather than coupling arguments.

The result is partial extension of T. Seppäläinen’s recent paper [11]. For $0 < \beta < 1/5$ fixed, we prove that, rescaling microscopic space and time by $N$, respectively $N^{1+\beta}$, the macroscopic evolution of perturbations of microscopic order $N^{-\beta}$ of the equilibrium states is governed by Burgers’ equation. The same statement should hold for $0 < \beta < 1/2$ as in Seppäläinen’s cited paper, but our method does not seem to work for $\beta \geq 1/5$.

1 Introduction

In the recent paper [11] T. Seppäläinen proves that in the so-called totally asymmetric stick process (equivalent to Hammersley’s process as seen from a traveling second class particle), small perturbations of microscopic order $N^{-\beta}$ of equilibrium states, macroscopically propagate according to Burgers’ equation, if hydrodynamic limit is taken where space and time are rescaled by $N$, respectively $N^{1+\beta}$. This result is valid for any $0 < \beta < 1/2$ fixed and goes even beyond the appearance of shocks in the solution of Burgers’ equation. Seppäläinen’s proof relies on the combinatorial peculiarities of Hammersley’s model and on coupling arguments. It is conjectured in [11] that the result should be valid in much wider context, actually Burgers’ equation should govern propagation of disturbances of equilibria (in this scaling
regime) for essentially all interacting particle systems with one conserved observable, which under Eulerian scaling lead to a nonlinear 1-conservation law. Seppälaïnen’s cited result and also our present paper conceptually is closely linked to the work of R. Esposito, R. Marra and H-T. Yau, [4], where this kind of intermediate scaling was first applied for the simple exclusion model in $d = 3$.

In the present paper we partially extend Seppälaïnen’s result. We prove a very similar result universally holding for a wide class of interacting particle systems. Our proof is structurally robust, it does not rely on any combinatorial properties of the models considered. We apply Yau’s relative entropy method rather than coupling arguments. We pay, of course, a price for this generality: (1) applying the relative entropy method, our results stay valid only up to the emergence of shocks in the Burgers’ solution and (2) we can prove our theorem only for $\beta \in (0, 1/5)$ instead of the ideal $\beta \in (0, 1/2)$.

Technically speaking, the proof is a careful application of the relative entropy method. However, we should emphasize that there is some new idea in the ‘one-block replacement’ step, where the standard large deviation argument is replaced by a central limit estimate — and a stronger result is gotten. See Lemma 2 and its proof. Also: since in our scaling regime we have to consider mesoscopic blocks of size $N^{2\beta}$ rather than large microscopic blocks, in the one block estimate so-called non-gradient arguments (e.g. spectral gap estimates) are involved.

The paper is organized as follows. In section 2 we present the models considered and some preliminary computations (infinitesimal generators, equilibria, reversed processes, eulerian hydrodynamic limits, formal perturbations). In section 3 the main result is precisely formulated in terms of relative entropies. Section 4 contains the proof. This is broken up in several subsections, according to what we consider a logical structure.

2 Preliminaries

2.1 The models

2.1.1 Notation, state space

Throughout this paper we denote by $\mathbb{T}^N$ the discrete tori $\mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. 
Let $z_{\text{min}}, z_{\text{max}} \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $z_{\text{min}} < z_{\text{max}}$, and $S := [z_{\text{min}}, z_{\text{max}}] \cap \mathbb{Z}$.

The state space of the interacting particles system considered is

$$\Omega^N := S^{T^N}.$$

Configurations will be denoted

$$\underline{z} := (z_j)_{j \in T^N} \in \Omega^N,$$

2.1.2 Rate functions, infinitesimal generator and examples

Following [3], [10] and [2] we require that the rate function $c : S \times S \to [0, \infty)$ satisfy the following conditions:

(A) For any $x, y \in S$

$$c(z_{\text{min}}, y) = 0 = c(x, z_{\text{max}}),$$

Note, that this condition is restrictive only if either $-\infty < z_{\text{min}}$ or $z_{\text{max}} < +\infty$. It guarantees that, with probability 1, the local ‘spins’ $z_j$ stay confined within the bounds $[z_{\text{min}}, z_{\text{max}}]$. In order to avoid degeneracies we also assume that for $x \in (z_{\text{min}}, z_{\text{max}}]$ and $y \in [z_{\text{min}}, z_{\text{max}})$

$$c(x, y) > 0. \quad (1)$$

(B) For any $x, y, z \in S$

$$c(x, y) + c(y, z) + c(z, x) = c(y, x) + c(z, y) + c(x, z).$$

(C) For any $x, y, z \in S \setminus \{z_{\text{min}}\}$

$$c(x, y - 1)c(y, z - 1)c(z, x - 1) = c(y, x - 1)c(z, y - 1)c(x, z - 1).$$

This condition is equivalent to requiring that there exist a function $r : S \to (0, \infty)$, with $r(z_{\text{min}}) = 0$, such that for any $x, y \in S \setminus \{z_{\text{min}}\}$

$$\frac{c(x, y - 1)}{c(y, x - 1)} = \frac{r(x)}{r(y)}.$$

If $-\infty < z_{\text{min}}$ or $z_{\text{max}} < +\infty$, we formally extend $r$ to $\mathbb{Z}$ as $r(x) = 0$ for $x < z_{\text{min}}$, and $r(x) = \infty$ for $x > z_{\text{max}}$. 

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Remarks: (1) The monotonicity condition $c(x,y+1) \leq c(x,y) \leq c(x+1,y)$ would imply *attractivity* of the processes defined below. We do not require this property of the rate functions. Our arguments do not rely on coupling ideas.

(2) In the case of unbounded $z$-variable, $\max\{|z_{\text{min}}|, |z_{\text{max}}|\} = \infty$, we shall also impose some growth condition on the rate function $c(x,y)$. See condition (D) below.

The elementary movements of our Markov process are: $(z_j, z_{j+1}) \rightarrow (z_j - 1, z_{j+1} + 1)$ with rate $c(z_j, z_{j+1})$. More formally, we define $\Theta_j : \Omega^N \rightarrow \Omega^N$:

$$\left(\Theta_j z\right)_i = z_i - \delta_{i,j} + \delta_{i,j+1}.$$ 

The infinitesimal generator of the process defined on the torus $T^N$ is

$$L^N f(z) = \sum_{j \in T^N} c(z_j, z_{j+1}) \left(f(\Theta_j z) - f(z)\right).$$

Clearly, due to the nondegeneracy condition [1], the only conserved quantity of the process is $\sum_j z_j$.

**Remark on notation:** Consequently, we shall denote by $\mathbf{z} = (z_j)_{j \in T^N}$ an element of the state space $\Omega^N$ and by $\zeta(s)$ the Markov process on $\Omega^N$ with infinitesimal generator $L^N$.

There are three essentially different classes of examples.

(1) *Bounded occupation number.* The only example with $z_{\text{min}} = 0$ and $z_{\text{max}} = 1$ is the *completely asymmetric simple exclusion model*. For any $K > 0$ one can easily check that there exists a finite-parameter family of models with $z_{\text{min}} = 0$ and $z_{\text{max}} = K$ satisfying conditions A to C. These are usually called *generalized $K$-exclusion models*.

(2) *Occupation number bounded from below.* There exists an infinite-parameter family of models with $z_{\text{min}} = 0$ and $z_{\text{max}} = +\infty$. In particular, with

$$c(x,y) = r(x) = \mathbb{1}_{\{x>0\}} r(x),$$

we get the *zero range models.*
(3) **Unbounded signed occupation number.** From the infinite-parameter family of possible models with $z_{\text{min}} = -\infty$ and $z_{\text{max}} = +\infty$ we point out the following: let $r : \mathbb{Z} \to (0, \infty)$ satisfy
\[ r(z)r(-z + 1) = 1. \]

Define
\[ c(x, y) = r(x) + r(-y) \]

Following [1], [2] we call these models *bricklayers models.*

If the occupation number is not bounded (i.e. the state space is not compact) we need some additional conditions on the growth of the rates. In order to avoid lengthy technical computations we only consider two special cases: the zero range model and the bricklayers model, defined in examples (2) and (3). For these models we need the following extra conditions:

(D) (Growth condition for zero range and bricklayers models)

(i) $\sup_{x \in \mathbb{N}} |r(x + 1) - r(x)| \leq a_1 < \infty$.

(ii) There exists $x_0 \in \mathbb{N}$ and $a_2 > 0$ such that $r(x) - r(y) \geq a_2$ for all $x \geq y + x_0$. (That means that for $x \in \mathbb{N}$ $r(x)$ is essentially linear.)

These conditions will guarantee the existence of dynamics and cf. [8] the uniform spectral gap estimate stated in Lemma 5.

2.2 Equilibrium states and reversed process

2.2.1 Stationary measures

From the growth condition D it follows that
\[ Z := \sum_{n=1}^{\infty} \prod_{k=1}^{n} r(-k + 1) + 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} r(k)^{-1} < \infty. \]

We define the following probability measure on $S$
\[
\pi(x) := \begin{cases} 
Z^{-1} \prod_{k=1}^{x} r(k)^{-1} & \text{if } x \geq 0, \\
Z^{-1} \prod_{k=1}^{-x} r(-k + 1) & \text{if } x \leq 0.
\end{cases}
\]
For $\theta \in \mathbb{R}$ let

$$F(\theta) := \log \sum_{z \in S} e^{\theta z} \pi(z)$$

and

$$\theta_{\text{min}} := \inf \{ \theta : F(\theta) < \infty \} \quad \theta_{\text{max}} := \sup \{ \theta : F(\theta) < \infty \}$$

For $\theta \in (\theta_{\text{min}}, \theta_{\text{max}})$ we define the probability measures

$$\pi_\theta(z) := \pi(z) \exp \{ \theta z - F(\theta) \}$$

on $S$. Expectation, variance and covariance with respect to the measure $\pi_\theta$ will be denoted by $E_\theta(\cdots)$, $\text{Var}_\theta(\cdots)$ and $\text{Cov}_\theta(\cdots)$, respectively.

According to [3], [10], [2], conditions A to C guarantee that for any $\theta \in (\theta_{\text{min}}, \theta_{\text{max}})$ the product measure

$$\pi^N_\theta := \prod_{j \in \mathbb{T}^N} \pi_\theta$$

is stationary for the Markov process. However, due to the conservation of $\sum_j z_j$, on the finite tori $T^N$ these measures are not ergodic. It is a standard matter to check that the measures conditioned on the value of $\sum_j z_j$, $\pi^N_k(\underline{z}) := \pi^N(\underline{z}) \sum_{\sum_j z_j = k}$, $k \in \mathbb{Z} \cap [Nz_{\text{min}}, Nz_{\text{max}}]$, are ergodic. We shall refer to $\pi^N_\theta$, respectively, $\pi^N_k$ as grand canonical, respectively, canonical measures for our model. (The different uses of the subscript should not cause any confusion.)

2.2.2 The reversed process

The elementary movements of the reversed process are $(z_{j-1}, z_j) \rightarrow (z_{j-1} + 1, z_j - 1)$ with rate $c(z_j, z_{j-1})$.

Define $\Theta_j^* : \Omega^N \rightarrow \Omega^N$,

$$(\Theta_j^* \underline{z})_i = z_i - \delta_{i,j} + \delta_{i,j-1}.$$  

The reversed generator on the torus $\mathbb{T}^N$:

$$L^N \ast f(\underline{z}) = \sum_{j \in \mathbb{T}^N} c(z_j, z_{j-1}) (f(\Theta_j^* \underline{z}) - f(\underline{z})).$$

Note, that the reversed process is the same for any $\pi^N_\theta$, $\theta \in (\theta_{\text{min}}, \theta_{\text{max}})$, or $\pi^N_k$, $k \in \mathbb{Z} \cap [Nz_{\text{min}}, Nz_{\text{max}}].$
2.2.3 Some expectations

We denote

\[ v(\theta) := E_\theta(z) = \sum_{z \in S} z \pi_\theta(z) = F'(\theta). \]

Elementary computations show

\[ v'(\theta) = F''(\theta) = \text{Var}_\theta(z) > 0, \]

thus \( (\theta_{\text{min}}, \theta_{\text{max}}) \ni \theta \mapsto v(\theta) \in (z_{\text{min}}, z_{\text{max}}) \) is invertible. With some abuse of notation denote the inverse function by \( \theta(v) \).

Further notation: we shall denote

\[ \Phi_j := c(z_{j+1}, z_j), \]

\[ \hat{\Phi}(v) := E_{\theta(v)}(\Phi_j) = \sum_{x,y \in S} \pi_{\theta(v)}(x) \pi_{\theta(v)}(y) c(x, y). \]

Clearly, if \(-\infty < z_{\text{min}} < z_{\text{max}} < \infty\) then \( \hat{\Phi}(v) \) is bounded. On the other hand, for the zero range models and bricklayers’ models with rate function \( r \) satisfying condition (D), straightforward estimates show that

\[ \hat{\Phi}(v) \leq C|v| \]

and also that \( \Phi_j \) has finite exponential moment with respect to any grand canonical measure.

**Remark on notation of finite-base cylinder functions:** If \( \Psi : S^m \to \mathbb{R} \), then we shall denote \( \Psi_j := \Psi(z_j, \ldots, z_{j+m-1}) \). The indices \( j \in \mathbb{T}^N \) are always meant periodically, mod \( N \). Expectation of \( \Psi_j \) with respect to the grand canonical measure \( \pi_{\theta(v)}^N \) is denoted

\[ \hat{\Psi}(v) := E_{\theta(v)}(\Psi_j) = \sum_{z_1, \ldots, z_m \in S} \pi_{\theta(v)}(z_1) \ldots \pi_{\theta(v)}(z_m) \Psi(z_1, \ldots, z_m). \]

2.3 Hydrodynamic limits

2.3.1 Eulerian scaling and its formal perturbation

For the local density \( v(t, x) \) of the conserved quantity \( \sum_j z_j \), under Eulerian scaling, by applying Yau’s relative entropy method (see [12], chapter 6 of [6], or section 8 of [5]), one gets the pde:

\[ \partial_t v + \partial_x \hat{\Phi}(v) = 0. \quad (2) \]
2.3.2 Perturbation of the Euler equation

Throughout the rest of this paper \( v_0 \in (z_{\text{min}}, z_{\text{max}}) \) will be fixed and the shorthand notation
\[
a_0 := \Phi(v_0), \quad b_0 := \Phi'(v_0), \quad c_0 := \Phi''(v_0)
\]
will be used. Note that \( b_0 \) is the characteristic speed for the hyperbolic pde (2), corresponding to \( v_0 \). Furthermore, it is assumed that \( c_0 \neq 0 \).

We now consider a small perturbation of the trivial constant solution \( v(t, x) \equiv v_0 \) of (2). We fix \( \beta > 0 \) and insert in (2)
\[
u(\varepsilon)(t, x) := v_0 + \varepsilon\beta u(\varepsilon^{1+\beta}t, \varepsilon(x - b_0t)).
\]
Letting \( \varepsilon \to 0 \), formally the inviscid Burgers’ equation is gotten for \( u \):
\[
\partial_t u + \frac{c_0}{2} \partial_x (u^2) = 0.
\]

3 The main result

3.1 Further notation and terminology

Let \( v_0 \in (z_{\text{min}}, z_{\text{max}}) \) be fixed and \( a_0, b_0 \) and \( c_0 \) as defined in (3), \( c_0 \neq 0 \) is assumed. We also denote \( \theta_0 := \theta(v_0) \).

Furthermore, let \( u(t, x), t \in [0, T], x \in \mathbb{T}, \) be smooth solution of Burgers’ equation (4). We shall use as absolute reference measure the stationary measure
\[
\pi^N := \prod_{j \in \mathbb{T}^N} \pi_{\theta_0}.
\]

We define
\[
\theta^N(t, x) := N^\beta \left( \theta(v_0 + N^{-\beta}u(t, x - N^\beta b_0t)) - \theta_0 \right)
\]
i.e. \( \theta(v_0 + N^{-\beta}u(t, x - N^\beta b_0t)) = \theta_0 + N^{-\beta}\theta^N(t, x) \).

The partial derivatives of \( \theta^N(t, x) \) are easily computed:
\[
\theta^N_t(t, x) := \partial_t \theta^N(t, x) = \theta'(v_0 + N^{-\beta}u(t, x - N^\beta b_0t)) \partial_t u(t, x - N^\beta b_0t)
\]
\[
\theta^N_x(t, x) := \partial_x \theta^N(t, x) = -\theta^N_x(t, x) \times (c_0u(t, x - N^\beta b_0t) + N^\beta b_0)
\]
In the computation of \( \partial_t \theta^N \) we use the fact that \( u \) is smooth solution of (4).
The **time dependent reference measure** (not to be confused with the absolute reference measure!) is

\[ \nu^N_t := \prod_{j \in \mathbb{T}^N} \pi_{\theta_0 + N^{-\beta} \theta^N(t, j/N)} = \prod_{j \in \mathbb{T}^N} \pi_{\theta_0 + N^{-\beta} u(t, j/N + N^{-\beta} b_0 t)} \].

(6)

The **true distribution** of our process on \( \mathbb{T}^N \), at macroscopic time \( t \), i.e. at microscopic time \( N^{1+\beta} t \) is

\[ \mu^N_t := \mu^N_0 \exp \{ N^{1+\beta} t L^N \} \].

(7)

The Radon-Nikodym derivatives of these last two probability measures on \( \Omega^N \), with respect to the absolute reference measure \( \pi^N \), are

\[ f^N_t(z) := \frac{d\nu^N_t}{d\pi^N}(z) = \prod_{j \in \mathbb{T}^N} \exp \{ z_j N^{-\beta} \theta^N(t, j/N) - F(\theta_0 + N^{-\beta} \theta^N(t, j/N)) + F(\theta_0) \} \]

\[ h^N_t(z) := \frac{d\mu^N_t}{d\pi^N}(z) = \exp \{ N^{1+\beta} t L^N \} h^N_0(z) \]

### 3.2 What is to be proved?

We want to prove that if \( \mu^N_0 \) is close to \( \nu^N_0 \), in the sense of the relative entropy \( H(\mu^N_0 \mid \nu^N_0) \) being small, then \( \mu^N_t \) stays close to \( \nu^N_t \) in the same sense, uniformly for \( t \in [0, T] \).

How close? Given two smooth profiles \( u_i : \mathbb{T} \to \mathbb{R}, i = 1, 2 \), let

\[ \nu^N_i := \prod_{j \in \mathbb{T}^N} \pi_{\theta_0 + N^{-\beta} u_i(j/N)}, \quad i = 1, 2. \]

Then, an easy computation shows that the relative entropy \( H(\nu^N_2 \mid \nu^N_1) \) is

\[ H(\nu^N_2 \mid \nu^N_1) = \sum_{j \in \mathbb{T}^N} H(\pi_{\theta_0 + N^{-\beta} u_2(j/N)} \mid \pi_{\theta_0 + N^{-\beta} u_1(j/N)}) \]

\[ = N^{1-2\beta} \theta'_0 \int_{\mathbb{T}} (u_2 - u_1)(u_2 - \frac{F''_0}{2}(u_2 + u_1))dx + O(N^{1-3\beta}), \]

where \( \theta'_0 := \theta'(v_0) \) and \( F''_0 := F''(\theta_0) \). This suggests that one should prove

\[ H^N(t) := H(\mu^N_t \mid \nu^N_t) = o(N^{1-2\beta}), \]

(9)

uniformly for \( t \in [0, T] \).
3.3 Main result

Consider a generalized mysanthrope model with rate function satisfying conditions A-D. Let $v_0 \in (z_{\text{min}}, z_{\text{max}})$ be fixed so that $c_0$ defined in (3) is nonzero. Let $u : [0, T] \times T \to \mathbb{R}$ be a smooth solution of the inviscid Burgers’ equation (4). Further on, let $\nu_t^N$, respectively, $\mu_t^N$ be the time dependent reference measure, respectively, the true distribution of the mysanthrope process, defined in (5), respectively, (6).

Our main result is the following

**Theorem.** Let $\beta \in (0, 1/5)$ be fixed. Under the stated conditions, if

$$H(\mu_0^N \mid \pi^N) = O(N^{1-2\beta})$$

and (3) holds for $t = 0$, then (3) will hold uniformly for $t \in [0, T]$.

**Remark:** The statement should hold for $\beta < 1/2$, but, with our method, seemingly only $\beta < 1/5$ can be treated.

From this theorem, by applying the entropy inequality the next corollary follows:

**Corollary.** Under the conditions of the Theorem, for any smooth test function $\varphi : T \to \mathbb{R}$

$$N^{-1+\beta} \sum_{j \in T^N} \varphi((j - N^{1+\beta}b_0 t)/N) \left( \zeta_j(N^{1+\beta} t) - v_0 \right) \xrightarrow{P} \int_T \varphi(x) u(t, x) \, dx,$$

as $N \to \infty$.

4 Proof

Our strategy is to get a Gromwall type estimate. We shall prove

$$H^N(t) - H^N(0) \leq C \int_0^t H^N(s) \, ds + \text{Err}^N(t). \quad (10)$$

It is assumed that $H^N(0) = o(N^{1-2\beta})$ and the error estimate $\text{Err}^N(t) = o(N^{1-2\beta})$ is the main point.

**Important remark on further notation:** In the remaining part of the paper, without loss of generality, we assume

$$v_0 = 0, \quad \theta_0 = 0, \quad a_0 = 0.$$

This means that from now on $z, v, \theta, \Phi$ and $\hat{\Phi}$ stand for $z - v_0, v - v_0, \theta - \theta_0, \Phi - a_0$ and $\hat{\Phi} - a_0$. 
4.1 Estimating $\partial_t H^N(t)$

In order to prove an inequality like (10) we need to estimate $\partial_t H^N(t)$. Using the well known inequality

$$f L \log f \leq L f$$

which holds for every $f \geq 0$, straightforward computations lead to

$$\partial_t H^N(t) \leq N^{1+\beta} \int_{\Omega^N} \frac{L_{x}^N f^N_t}{f^N_t} \, d\mu^N_t - \int_{\Omega^N} \frac{\partial_t f^N_t}{f^N_t} \, d\mu^N_t. \quad (11)$$

(See chapter 6 of [6] or the paper [12] for details.)

**Further remarks on notation:** In subsections 4.1 and 4.2 $t \in [0,T]$ will be fixed. In order to avoid heavy notations, in these subsects ions we do not denote explicitly dependence on $t$. In particular we shall use the following shorthand notations

$$\theta^N(x) := \theta^N(t,x), \quad \theta^N_x(x) := \theta^N_x(t,x), \quad \theta^N_t(x) := \theta^N_t(t,x),$$

$$u^N(x) := u(t,x - N^\beta b_0 t)$$

Discrete gradient of functions $g: \mathbb{T} \to \mathbb{R}$ will be denoted

$$\nabla^N g(x) := N \left( g(x + 1/N) - g(x) \right).$$

4.1.1 Computation of $L^N f^N_t / f^N_t$

After straightforward calculations we have

$$\frac{L^N f^N_t}{f^N_t}(x) = \sum_{J \in \mathbb{T}^N} \left( e^{-N^{-1-\beta}(\nabla^N \theta^N)(j/N)} - 1 \right) \Phi_j$$

$$= -N^{-1-\beta} \sum_{J \in \mathbb{T}^N} \theta^N_x(j/N) \left( \Phi_j - \tilde{\Phi}(N^{-\beta} u^N (j/N)) \right)$$

$$-N^{-1-\beta} \sum_{J \in \mathbb{T}^N} \theta^N_t(j/N) \tilde{\Phi}(N^{-\beta} u^N (j/N))$$

$$+ \sum_{J \in \mathbb{T}^N} \left( e^{-N^{-1-\beta}(\nabla^N \theta^N)(j/N)} - e^{-N^{-1-\beta} \theta^N_x (j/N)} \right) \Phi_j$$

$$+ \sum_{J \in \mathbb{T}^N} A(N^{-1-\beta} \theta^N_x (j/N)) \Phi_j.$$
where in the last line the shorthand notation $A(x) := e^{-x} - 1 + x$ is used. The main term is the first sum on the right hand side. We introduce

$$\Psi_j := \Phi_j - b_0 z_j$$

$$\hat{\Psi}(v) := \mathbb{E}_{\theta(v)}(\Psi_j) = \hat{\Phi}(v) - b_0 v$$

and write in the main term

$$\Phi_j - \hat{\Phi}(N^{-\beta} u) = (\Psi_j - \hat{\Psi}(N^{-\beta} u)) + b_0 (z_j - N^{-\beta} u)$$

Thus, eventually we get

$$N^{1+\beta} \int_{\Omega} \frac{L^N f_t^N}{f_t^N} d\mu_t^N =$$

$$- \sum_{j \in \mathbb{T}^N} \theta_x^N(j/N) \int_{\Omega} (\Psi_j - \hat{\Psi}(N^{-\beta} u^N(j/N))) d\mu_t^N$$

$$- b_0 \sum_{j \in \mathbb{T}^N} \theta_x^N(j/N) \int_{\Omega} (z_j - N^{-\beta} u^N(j/N)) d\mu_t^N$$

$$+ \text{Err}_1^N(t) + \text{Err}_2^N(t) + \text{Err}_3^N(t),$$

where the error terms are

$$\text{Err}_1^N(t) = - \sum_{j \in \mathbb{T}^N} \theta_x^N(j/N) \hat{\Phi}(N^{-\beta} u^N(j/N)),$$

$$\text{Err}_2^N(t) = N^{1+\beta} \sum_{j \in \mathbb{T}^N} (e^{-N^{-1-\beta}(\nabla^N \theta^N)(j/N)} - e^{-N^{-1-\beta} \theta_x^N(j/N)}) \int_{\Omega} \Phi_j d\mu_t^N,$$

$$\text{Err}_3^N(t) = N^{1+\beta} \sum_{j \in \mathbb{T}^N} A(N^{-1-\beta} \theta_x^N(j/N)) \int_{\Omega} \Phi_j d\mu_t^N.$$ 

### 4.1.2 Computation of $\partial_t f_t^N / f_t^N$

Now we turn our attention to the second term on the right side of (11). From (8) and (5) we get:

$$\frac{\partial_t f_t^N}{f_t^N}(z) = \sum_{j \in \mathbb{T}^N} N^{-\beta} \theta_t^N(j/N)(z_j - N^{-\beta} u^N(j/N))$$

$$= - \sum_{j \in \mathbb{T}^N} \theta_x^N(j/N)(c_0 N^{-\beta} u^N(j/N) + b_0)(z_j - N^{-\beta} u^N(j/N))$$
In the last sum we write
\[ c_0 N^{-\beta} u = \bar{\Psi}'(N^{-\beta} u) - (\bar{\Psi}'(N^{-\beta} u) - c_0 N^{-\beta} u) \]
and note that the second term is a small error.

Eventually we get:
\[- \int_{\Omega N} \frac{\partial_t f_i^N}{f_i^N} \, d\mu^N_t(z) = \]
\[ \sum_{j \in T^N} \theta_x^N(j/N) \bar{\Psi}'(N^{-\beta} u^N (j/N)) \int_{\Omega N} (z_j - N^{-\beta} u^N (j/N)) \, d\mu^N_t(z) \]
\[- b_0 \sum_{j \in T^N} \theta_x^N(j/N) \int_{\Omega N} (z_j - N^{-\beta} u^N (j/N)) \, d\mu^N_t + \text{Err}^N_4(t) \]
where
\[ \text{Err}^N_4(t) = - \sum_{j \in T^N} \theta_x^N(j/N) (\bar{\Psi}'(N^{-\beta} u^N (j/N)) - c_0 N^{-\beta} u^N (j/N)) \]
\[ \times \int_{\Omega N} (z_j - N^{-\beta} u^N (j/N)) \, d\mu^N_t(z). \]

Note that, when inserting in (11), the second sums on the right hand side of (12) and (16) cancel out.

4.1.3 Blocks
Throughout the paper the one-block size \( l \) will be chosen, depending on the system size \( N \), so that asymptotically
\[ l \gg N^{2\beta}. \]

We introduce the block averages
\[ \Psi_j^l := l^{-1} \sum_{i=0}^{l-1} \Psi_{j+i}, \quad z_j^l := l^{-1} \sum_{i=0}^{l-1} z_{j+i}. \]

The main terms (i.e. the first sums on the right hand side) in (12), respectively, in (16) become
\[- \sum_{j \in T^N} \theta_x^N(j/N) \int_{\Omega N} (\Psi_j^l - \bar{\Psi}(N^{-\beta} u^N (j/N))) \, d\mu^N_t + \text{Err}^N_5(t), \quad (18) \]
respectively,
\[
\sum_{j \in \mathcal{T}^{N}} \theta_x^N(j/N) \hat{\Psi}'(N^{-\beta}u^N(j/N)) \int_{\Omega^N} (z_j^l - N^{-\beta}u^N(j/N)) \, d\mu^N_t(z) \tag{19}
\]
\[+ \text{Err}^{N,t}_6(t). \]

After rearrangement of sums the error terms \(\text{Err}^{N,t}_5(t)\), respectively, \(\text{Err}^{N,t}_6(t)\) are written as
\[
\text{Err}^{N,t}_5(t) = - \sum_{j \in \mathcal{T}^{N}} \left( t^{-1} \sum_{i=0}^{t-1} \theta_x^N((j-i)/N) - \theta_x^N(j/N) \right) \int_{\Omega^N} \hat{\Psi}_j \, d\mu^N_t(z) \tag{20}
\]
\[
\text{Err}^{N,t}_6(t) = \sum_{j \in \mathcal{T}^{N}} \left( t^{-1} \sum_{i=0}^{t-1} \theta_x^N((j-i)/N) \hat{\Psi}'(N^{-\beta}u^N((j-i)/N)) - \theta_x^N(j/N) \hat{\Psi}'(N^{-\beta}u^N(j/N)) \right) \int_{\Omega^N} z_j \, d\mu^N_t(z). \tag{21}
\]

### 4.1.4 Sumup and estimate of the error terms (so far)

Summing up, from (11), (12), (16), (18) and (19), so far we have got:
\[
\partial_t H^N(t) \leq - \sum_{j \in \mathcal{T}^{N}} \theta_x^N(j/N) \int_{\Omega^N} \left( \hat{\Psi}_j - \hat{\Psi}(N^{-\beta}u^N(j/N)) \right) \int_{\Omega^N} (z_j^l - N^{-\beta}u^N(j/N)) \, d\mu^N_t(z) 
\]
\[+ \text{Err}^N_1(t) + \text{Err}^N_2(t) + \text{Err}^N_3(t) + \text{Err}^N_4(t) + \text{Err}^{N,t}_5(t) + \text{Err}^{N,t}_6(t) \tag{22}
\]

with the error terms given in (13), (14), (15), (17), (20), (21), respectively.

For the estimate of the these terms we use the following lemma:

**Lemma 1.** Let \( \Psi : \mathbb{Z}^m \to \mathbb{R} \) be a finite cylinder function and denote \( \Psi_j := \Psi(z_j, \ldots, z_{j+m-1}) \). Assume that, for \( |\gamma| < \gamma_0 \), \( E_\pi(\exp\{\gamma\Psi\}) < \infty \). Then there exists a constant \( C < \infty \) depending only on \( m \) and \( \gamma_0 \), such that for any \( \psi_N : \mathbb{T}^N \to \mathbb{R} \),
\[
\sum_{j \in \mathcal{T}^{N}} \psi_N(j) \int_{\Omega^N} \hat{\Psi}_j \, d\mu^N_t \leq C \max_{j \in \mathcal{T}^N} |\psi_N(j)| \left( N^{1-\beta} + N E_\pi(\Psi) \right),
\]

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uniformly for $t \in [0, T]$.

Proof. We may assume that $\max_{j \in \mathbb{T}^N} |\psi_N(j)| = 1$ and $E_\pi \Psi(\zeta) = 0$. We set $\gamma_1 := \gamma_0 N^{-\beta} < \gamma_0$ then with the entropy inequality:

$$\left| \sum_{j \in \mathbb{T}^N} \psi_N(j) \int_{\Omega_N} \Psi_j d\mu_t^N \right| \leq \frac{1}{\gamma_1} H(\mu_t^N | \pi_t^N) + \frac{1}{\gamma_1} \log E_\pi \exp \left\{ \gamma_1 \sum_{j \in \mathbb{T}^N} \psi_N(j) \Psi_j \right\}.$$ 

Applying the Hölder inequality to the second term, and using that $\Psi_j$ and $\Psi_k$ are independent if $|j - k| > m$ we have

$$\left| \sum_{j \in \mathbb{T}^N} \psi_N(j) \int_{\Omega_N} \Psi_j d\mu_t^N \right| \leq \frac{1}{\gamma_1} H(\mu_t^N | \pi_t^N) + \frac{1}{\gamma_1 m} \sum_{j \in \mathbb{T}^N} \Lambda(\gamma_1 m \psi_N(j)),$n

where we use the notation $\Lambda(\gamma) := \log E_\pi \exp \{ \gamma \Psi(\zeta) \}$.

Now, $\Lambda(0) = \Lambda'(0) = 0$, thus we have the asymptotics $\Lambda(\gamma) = O(\gamma^2)$ for $\|\gamma\| \ll 1$. Since $\max_{j \in \mathbb{T}^N} |\psi_N(j)| = 1$ and $\gamma_1 = O(N^{-\beta})$ there exists a positive constant $C_1$ such that $\Lambda(\gamma_1 m \psi_N(j)) \leq C_1 \gamma_1^2$ for every $j \in \mathbb{T}^N$. There also exists a constant $C_2$ with $H(\mu_t^N | \pi_t^N) \leq C_2 N^{1-2\beta}$. From these the lemma follows with $C = C_2 / \gamma_0 + C_1 \gamma_0 m$.

By Lemma [1] and the smoothness of $u(t, x)$ we readily get:

$$\text{Err}_1^N(t) = O(N^{1-3\beta}),$$
$$\text{Err}_2^N(t) = O(N^{-\beta}),$$
$$\text{Err}_3^N(t) = O(N^{1-4\beta}),$$
$$\text{Err}_4^N(t) = O(N^{1-4\beta}),$$
$$\text{Err}_5^{N,1}(t) = O(N^{-\beta l}),$$
$$\text{Err}_6^{N,1}(t) = O(N^{-2\beta l}).$$

4.2 One block replacement

On the right hand side of (22) we replace the block average $\Psi_j^i(z)$ by its ‘local equilibrium value’: $\hat{\Psi}(z_j^i)$. We denote

$$R(x, y) := \hat{\Psi}(x) - \hat{\Psi}(y) - \hat{\Psi}'(y)(x - y) \quad (23)$$
Then:
\[
\partial_t H^N(t) \leq - \sum_{j \in T^N} \theta^N_x(j/N) \int_{\Omega^N} R(z^j, N^{-\beta} u^N(j/N)) d\mu^N_t(z) \\
+ M^{N,l}(t) + O\left(N^{1-3\beta} \vee N^{-\beta l}\right),
\]
\[
\leq \sup_{0 < t < T} \left| \theta^N_x(j/N) \right| \sum_{j \in T^N} \int_{\Omega^N} |R(z^j, N^{-\beta} u^N(j/N))| d\mu^N_t(z) \\
+ M^{N,l}(t) + O\left(N^{1-4\beta} \vee N^{-\beta l}\right),
\]
(24)

where
\[
M^{N,l}(t) := - \sum_{j \in T^N} \theta^N_x(j/N) \int_{\Omega^N} (\Psi^l_j - \hat{\Psi}(z^j)) d\mu^N_t(z).
\]
(25)

The estimate of \( \int_0^t M^{N,l}(s) ds \) is done in the next subsection, by the so-called 'one block estimate'.

We estimate now the first term on the right hand side of (24). Assume \( N = ML \). By the entropy inequality
\[
\sum_{j \in T^N} \int_{\Omega^N} |R(z^j, N^{-\beta} u^N(j/N))| d\mu^N_t \leq \frac{1}{\gamma} H(\mu^N_t | \nu^N_t) \\
+ \frac{1}{\gamma} \log \left( \int_{\Omega^N} \exp \left\{ \gamma \sum_{j \in T^N} |R(z^j, N^{-\beta} u^N(j/N))| \right\} d\nu^N_t(z) \right)
\]
(26)

We estimate the integral in the second term on the right hand side of (26) using again the Hölder inequality:
\[
\int_{\Omega^N} \exp \left\{ \gamma \sum_{j \in T^N} |R(z^j, N^{-\beta} u^N(j/N))| \right\} d\nu^N_t(z)
\]
\[
= \prod_{i=1}^l \int_{\Omega^N} \exp \left\{ l\gamma \sum_{k=0}^{M-1} |R(z^j_{kl+i}, N^{-\beta} u^N((kl+i)/N))| \right\} d\nu^N_t(z)
\]
\[
\leq \left( \prod_{i=1}^l \int_{\Omega^N} \exp \left\{ l\gamma |R(z^j_{kl+i}, N^{-\beta} u^N((kl+i)/N))| \right\} d\nu^N_t(z) \right)^{1/l}
\]
\[
= \left( \prod_{j \in T^N} \int_{\Omega^N} \exp \left\{ l\gamma |R(z^j, N^{-\beta} u^N(j/N))| \right\} d\nu^N_t(z) \right)^{1/l}
\]
(27)

In the last setp we use the fact that for any fixed \( i \in [1, l] \) the block averages \( \zeta^l_{kl+i}, k = 0, 1, \ldots, M - 1 \), are independent under the measure \( \nu^N_t \). From
It is easy to see that the function
\[ x \mapsto R(x + N^{-\beta}u_N(j/N), N^{-\beta}u_N(j/N)) \] (28)
is asymptotically quadratic if \(|x| \ll 1\). If the variables \(z_i \in S\) are bounded than (28) is automatically bounded. If \(S\) is unbounded, but condition D holds, than (28) is asymptotically linearly bounded for \(|x| \gg 1\). Thus we may use Lemma 2 stated below, and eventually from (26), (27) we get for \(\gamma_0\) sufficiently small and \(l \geq 1/\gamma_0\):
\[
\sum_{j \in T} \int_{\Omega} |R(z_i^l, N^{-\beta}u_N(j/N))| d\mu_t^N(z) \leq \frac{1}{\gamma_0} H(\mu_t^N | \nu_t^N) + CNl^{-1}.
\]
Consequently, using this bound in (24) we find
\[
\partial_t H^N(t) \leq CH^N(t) + M^{N,l}(t) + O(1) \quad (29)
\]
holding uniformly for \(t \in [0,T]\).

Lemma 2. Let \(\zeta_1, \zeta_2, \ldots\) be i. i. d. random variables with zero mean. Assume
\[
\Lambda(\lambda) := \log \mathbb{E} \left( e^{\lambda \zeta_i} \right) < \infty. \quad (30)
\]
Let the smooth function \(G : \mathbb{R} \to \mathbb{R}_+\) be quadratically, respectively, linearly bounded for \(|x| \ll 1\), respectively, \(|x| \gg 1\), i.e., \(G(x) \leq C_1(|x| \wedge (x^2/2))\), with some finite constant \(C_1\). Then there exist constants \(\gamma_0 > 0\) and \(C < \infty\), such that for any \(0 < \gamma < \gamma_0\) and \(l \geq 1/\gamma_0\)
\[
\mathbb{E} \exp \left\{ \gamma l G((\zeta_1 + \cdots + \zeta_l)/l) \right\} < C. \quad (31)
\]
Remarks: (1) It is worth comparing the statement and proof of Lemma 2 with the corresponding places in previous works applying the one-block replacement, see, e.g., Proposition 1.6. in Part 6. of [6]. There usually a weaker statement \((o(l)\) instead of \(O(1)\) on the right hand side of (31)) is gotten by use of more sophisticated tools (large deviation principle instead of central limit estimate). Actually, we do need the sharper \(O(1)\) bound.
(2) The statement is easily extended: imposing more restrictive conditions on \(\Lambda(\lambda)\), the growth condition on \(G(x)\) can be relaxed. E.g., assuming \(\Lambda(\lambda) = O(\lambda^2)\) for \(|\lambda| \gg 1\), we may take \(G(x)\) quadratically (rather than
linearly) bounded at $|x| \gg 1$.

(3) Actually,

$$\lim_{l \to \infty} E \exp \{ \gamma l G((\zeta_1 + \cdots + \zeta_l)/l) \} = (1 - \gamma \Lambda''(0)G''(0))^{-1/2}.$$ 

But, since we need only the bound \[31\] and not the exact value of the limit, we leave the proof of this as a funny exercise for the reader.

**Proof.** First we prove the statement with the more restrictive assumption $\Lambda(\lambda) \leq C_2 \lambda^2/2$. Assume $\gamma < \left( C_1 C_2 \right)^{-1}$ and let $\xi$ be a standard Gaussian random variable, independent of the variables $\zeta_j$. We denote by $\langle \cdots \rangle$ expectation with respect to the random variables $\xi_j$. Then we have the following chain of (in)equalities:

$$E \exp \{ \gamma l G((\zeta_1 + \cdots + \zeta_l)/l) \} \leq E \exp \{ C_1 \gamma (\zeta_1 + \cdots + \zeta_l)^2/(2l) \}$$

$$= E \langle \exp \{ \sqrt{C_1 \gamma l} (\zeta_1 + \cdots + \zeta_l) \xi \} \rangle$$

$$= \langle E \exp \{ \sqrt{C_1 \gamma l} l (\zeta_1 + \cdots + \zeta_l) \xi \} \rangle$$

$$= \langle \exp \{ l \Lambda(\sqrt{C_1 \gamma l} \xi) \} \rangle$$

$$\leq \langle \exp \{ C_2 C_1 \xi^2 / 2 \} \rangle$$

$$= (1 - \gamma C_1 C_2)^{-1/2}.$$

Now we consider the general case. Choose $\alpha$ so large, that for any $x \in \mathbb{R}$

$$G(x) < \ln \cosh(\alpha x).$$

One can do this due to the bounds imposed on $G$. Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables which are also independent of the $\zeta_j$-s and have the common distribution $P(\xi_j = \pm \alpha) = 1/2$. We shall denote by $\langle \cdots \rangle$ expectation with respect to the random variables $\xi_j$. We choose $\lambda_0, C_3$ so that for $|\lambda| < \lambda_0$ the quadratic bound $\Lambda(\lambda) < C_3 \lambda^2 / 2$ holds and fix $\gamma < \lambda_0 / \alpha$. Then we have:

$$E \exp \{ \gamma l G((\zeta_1 + \cdots + \zeta_l)/l) \} \leq \cosh (\alpha (\zeta_1 + \cdots + \zeta_l)/l)^{[\gamma l]} \leq$$

$$\leq E\langle \exp \{ (\xi_1 + \cdots + \xi_{[\gamma l]}) (\zeta_1 + \cdots + \zeta_l)/l \} \rangle$$

$$= \langle E \exp \{ (\xi_1 + \cdots + \xi_{[\gamma l]}) (\zeta_1 + \cdots + \zeta_l)/l \} \rangle$$

$$= \langle \exp \{ l \Lambda(\xi_1 + \cdots + \xi_{[\gamma l]}/l) \} \rangle$$

$$\leq \langle \exp \{ C_3 (\xi_1 + \cdots + \xi_{[\gamma l]})^2/(2l) \} \rangle.$$
Now, since $\ln \cosh(\alpha x) \leq \alpha^2 x^2/2$, we can apply to the random variables $\xi_j$ the argument of the first part of this proof, with $C_2 = \alpha^2$ and $C_1 = C_3$, to get

$$E \exp \left\{ \gamma l G ((\zeta_1 + \cdots + \zeta_l)/l) \right\} \leq (1 - \gamma C_3 \alpha^2)^{-1/2}. $$

\[ \square \]

4.3 The one block estimate

The objective of this section is to provide an estimate for $\int_0^t M^{N,l}(s) \, ds$, where $M^{N,l}(s)$ is given in (25).

4.3.1 Cutoff

We cut off large values of the block averages. In case of compact state space, i.e. $-\infty < z_{\text{min}} < z_{\text{max}} < \infty$ this step is completely omitted. Clearly,

$$M^{N,l}(t) \leq A^{N,l}_K(t) + B^{N,l}_K(t), \quad (32)$$

where the terms on the right side are defined as

$$A^{N,l}_K(t) := \sum_{j \in \mathbb{T}^N} \theta^N_x(t, j/N) \int_{\Omega^N} (\Psi^l_j - \tilde{\Psi}(z^l_j)) 1_{\{|z^l_j| \leq K\}} \, d\mu^N_t(z),$$

$$B^{N,l}_K(t) := \sup_{0 < t < T} |\theta^N_x(t, j/N)| \sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi^l_j - \tilde{\Psi}(z^l_j)| 1_{\{|z^l_j| > K\}} \, d\mu^N_t(z),$$

where $\alpha > 0$ is a fixed constant which will only depend on the rate function.

For the estimate of $B^{N,l}_K(t)$ we need the following lemma (applied with $m = 1$ or $2$ only):

**Lemma 3.** Let $\Delta : \mathbb{Z}^m \to \mathbb{R}$ be a finite cylinder variable. Then there exists a map $K \mapsto \epsilon(K)$, such that $\lim_{K \to \infty} \epsilon(K) = 0$ and

$$\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi^l_j - \tilde{\Psi}(z^l_j)| 1_{\{|\Delta^l_j| > K\}} \, d\mu^N_t(z) \leq \epsilon(K) N^{1-2\beta}. $$

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Proof. The entropy inequality yields:
\[
\sum_{j \in \mathbb{N}} \int_{\Omega^N} |\Psi_j - \tilde{\Psi}(z_j)| \mathbb{1}_{\{|\Delta_j| > K\}} \, d\mu_t^N(z) \\
\leq \frac{1}{\gamma} \left( H(\mu_t^N | \pi^N) + \log E_{\pi^N} \exp \left\{ \gamma \sum_{j \in \mathbb{N}} |\Psi_j - \tilde{\Psi}(z_j)| \mathbb{1}_{\{|\Delta_j| > K\}} \right\} \right)
\]
We note that the \(j^{th}\) and \(k^{th}\) terms are independent in the last sum if \(|j - k| > l + m - 1\). By the H"older inequality, for \(l \geq m\), we have
\[
\log E_{\pi^N} \exp \left\{ \gamma \sum_{j \in \mathbb{T}^N} |\Psi_j - \tilde{\Psi}(z_j)| \mathbb{1}_{\{|\Delta_j| > K\}} \right\}
\leq nl^{-1} \log E_{\pi^N} \exp \left\{ 2l\gamma |\Psi - \tilde{\Psi}(z)| \mathbb{1}_{\{|\Delta| > K\}} \right\}.
\]
Next we use Cauchy-Schwarz inequality:
\[
E_{\pi^N} \exp \left\{ 2l\gamma |\Psi - \tilde{\Psi}(z)| \mathbb{1}_{\{|\Delta| > K\}} \right\}
\leq 1 + E_{\pi^N} \left( \mathbb{1}_{\{|\Delta| > K\}} \exp \left\{ 2l\gamma |\Psi - \tilde{\Psi}(z)| \right\} \right)
\leq 1 + \left\{ P_{\pi^N} \left( |\Delta| > K \right) \right\}^{1/2} \left\{ E_{\pi^N} \exp \left\{ 2l\gamma |\Psi - \tilde{\Psi}(z)| \right\} \right\}^{1/2}.
\]
From standard large deviation arguments it follows that there exists a function \([0, \infty) \ni \gamma \mapsto \Lambda(\gamma) \in [0, \infty)\) (finite for any finite \(\gamma\!\!\!), such that
\[
E_{\pi^N} \exp \left\{ 2l\gamma |\Psi - \tilde{\Psi}(z)| \right\} \leq \exp \left\{ l\Lambda(\gamma) \right\}.
\]
On the other hand, using again a H"older bound and a standard large deviation estimate, for large \(l\) we have
\[
P_{\pi^N} \left( |\Delta| > K \right) \leq m \exp \left\{ -H(K)/(2m) \right\},
\]
where \(x \mapsto I(x)\) is the rate function
\[
I(x) := \sup_{\lambda} \left( \lambda x - \log E_{\pi^N} \exp \left\{ \lambda \Delta \right\} \right).
\]
We define
\[
\gamma(K) := \sup \left\{ \gamma : \Lambda(\gamma) < I(K)/(2m) \right\} \land K.
\]
Since \(\lim_{x \to \infty} I(x) = \infty\), we also have \(\lim_{K \to \infty} \gamma(K) = \infty\). Now, putting together all our estimates, we get
\[
\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi_j - \tilde{\Psi}(z_j)| \mathbb{1}_{\{|\Delta_j| > K\}} \, d\mu_t^N(z)
\leq \frac{1}{\gamma(K)} \left( H(\mu_t^N | \pi^N) + Nl^{-1}(1 + \sqrt{m}) \right).
\]
Noting that $H(\mu^N_\pi | \pi^N) = O(N^{1-2\beta} )$ and $l \geq CN^{2\beta}$, the lemma follows with 
\[ \varepsilon(K) = C\gamma(K)^{-1}. \]
\[ \varepsilon(K) = C\gamma(K)^{-1}. \]

It is easy to see, that the functions $\Delta_j = z_j$ and $\Delta_j = \Psi_j$ satisfy the
conditions of the Lemma [3], thus it follows that there exists a map $K \to \varepsilon(K)$ with 
\[ \lim_{K \to \infty} \varepsilon(K) = 0 \]
and
\[ B^{N,j}_K(t) \leq \varepsilon(K)N^{1-2\beta}. \] (33)

4.3.2 General tools

We collect in this paragraph the general, model independent facts used in
the one-block estimate.

Let $\xi(s)$ be a Markov process on the countable state space $\Omega$, with ergodic
stationary measure $\pi$. Denote by $L$ and $L^*$ the infinitesimal generator and its
adjoint, acting on $L^2(\Omega, \pi)$. We denote by $D(f)$ the Dirichlet form associated
with the generator $L$ and stationary measure $\pi$:
\[ D(f) := -\int_\Omega Lf d\pi = -\int_\Omega L^*f d\pi \]
The spectral gap of the infinitesimal generator $L$ is $\rho^{-1}$ defined by
\[ \rho = \rho(L) := \sup_{f \in L^2(\Omega, \pi)} \frac{\text{Var}_\pi(f)}{D(f)} \in (0, \infty]. \]

Actually, this means that $(L + L^*)/2$, the symmetric part of $L$, has a gap of
size $\rho^{-1}$ in its spectrum, immediately to the left of the eigenvalue 0.

If $V : \Omega \to \mathbb{R}$ is a bounded measurable function we denote
\[ \overline{\sigma}(L + V(\cdot)) := \sup \{ \text{spectrum of } (L + L^*)/2 + V(\cdot) \}. \]

The following statement is the variational characterization of the ‘top
of the spectrum’ of a self-adjoint operator over a Hilbert space. It can be
found in any introductory textbook on functional analysis.

**Fact 1.** For $\overline{\sigma}(L + V(\cdot))$ the following variational formula holds:
\[ \overline{\sigma}(L + V(\cdot)) = \sup_h \left( \int_\Omega V(\cdot)h d\pi - D(\sqrt{\mathbb{E}}_h) \right), \] (34)
where the supremum is taken over all probability densities with respect to
the stationary measure $\pi$. 

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The second fact is a perturbative estimate of $\sigma(L + \varepsilon V(\cdot))$. It can be found, e.g., as Theorem 1.1 in Appendix 3 of [6].

**Fact 2.** If $V : \Omega \to \mathbb{R}$ has zero mean, i.e., $\int_{\Omega} V d\pi = 0$, then, for every $\varepsilon < (2 \|V\|_\infty \rho(L))^{-1}$

$$\sigma(L + \varepsilon V(\cdot)) \leq \frac{\varepsilon^2 \rho(L)}{1 - 2 \|V\|_\infty \varepsilon \rho(L)} \text{Var}_\pi(V). \tag{35}$$

The third general fact to be used is a direct consequence of the Feynman-Kac formula and straightforward euclidean (inner product) manipulations. Its proof can be found, e.g., in [6] or as Lemma 7.2 in Appendix 1 of [6].

**Fact 3.** Assume now that $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a bounded function. The following bound holds

$$E_\pi \exp \left\{ \int_0^t V(s, \zeta(s)) \, ds \right\} \leq \exp \left\{ \int_0^t \sigma(L + V(s, \cdot)) \, ds \right\}, \tag{36}$$

where now $E_\pi$ denotes expectation over the Markov chain trajectories started from the stationary initial measure $\pi$.

### 4.3.3 Notations

We shall use the notation $\mu^N$, respectively, $\mu^l$ for a generic probability measure on $\Omega^N$, respectively, $\Omega^l$. We shall denote by $h^N(\underline{x})$, respectively, $h^l(\underline{x})$ their Radon-Nikodym derivatives with respect to the absolute reference measures $\pi^N$, respectively, $\pi^l$. Further on $\mu^{N,j\cdot}$ will denote the $[j, \ldots, j+l-1]$ marginal of $\mu^N$ and $\mu^{N,j \cdot} := N^{-1} \sum_{j \in \mathbb{Z}^N} \mu^{N,j \cdot}$ the average $l$-dimensional marginal of $\mu^N$. Correspondingly, $h^{N,j\cdot}(\underline{x})$, respectively, $h^{N,j \cdot}(\underline{x})$ will denote the Radon-Nikodym derivatives of $\mu^{N,j\cdot}$, respectively, $\mu^{N,j \cdot}$, with respect to the absolute reference measure $\pi^l$. 

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For \( k \in \mathbb{Z} \) fixed we denote:

\[
\Omega^l_k := \{ z \in \Omega^l : \sum_{i=1}^{l} z_i = k \},
\]

\[
m^l_k := \pi^l(\Omega^l_k),
\]

\[
w^l_k := \mu^l(\Omega^l_k),
\]

\[
\pi^l_k(z) := \pi^l(z \mid \sum_{i=1}^{l} z_i = k) = \mathbb{1}_{\{z \in \Omega^l_k\}} \frac{\pi^l(z)}{m^l_k},
\]

\[
\mu^l_k(z) := \mu^l(z \mid \sum_{i=1}^{l} z_i = k) = \mathbb{1}_{\{z \in \Omega^l_k\}} \frac{\mu^l(z)}{w^l_k},
\]

\[
h^l_k(z) := \mathbb{1}_{\{z \in \Omega^l_k\}} \frac{\mu^l_k(z)}{\pi^l_k(z)} = \mathbb{1}_{\{z \in \Omega^l_k\}} \frac{m^l_k h^l(z)}{w^l_k}.
\]

Denote by \( D^N \), \( D^l \) respectively \( D^l_k \) the following Dirichlet forms:

\[
D^N(f) := \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega^N} c(z_i, z_{i+1}) (f(\Theta_i z) - f(z))^2 \, d\pi^N(z)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega^N} c(z_i, z_{i-1}) (f(\Theta_i^* z) - f(z))^2 \, d\pi^N(z)
\]

\[
D^l(f) := \frac{1}{2} \sum_{i=1}^{l-1} \int_{\Omega^l} c(z_i, z_{i+1}) (f(\Theta_i z) - f(z))^2 \, d\pi^l(z)
\]

\[
= \frac{1}{2} \sum_{i=1}^{l} \int_{\Omega^l} c(z_i, z_{i-1}) (f(\Theta_i^* z) - f(z))^2 \, d\pi^l(z)
\]

\[
D^l_k(f) := \frac{1}{2} \sum_{i=1}^{l-1} \int_{\Omega^l_k} c(z_i, z_{i+1}) (f(\Theta_i^* z) - f(z))^2 \, d\pi^l_k(z)
\]

\[
= \frac{1}{2} \sum_{i=1}^{l} \int_{\Omega^l_k} c(z_i, z_{i-1}) (f(\Theta_i^* z) - f(z))^2 \, d\pi^l_k(z).
\]

In the definition of \( D^N \) periodic, in that of \( D^l \) and \( D^l_k \) free boundary conditions are understood.

It is easy to check that for any probability measure \( \mu^l \) on \( \Omega^l \)

\[
D^l(\sqrt{h^l}) = \sum_{k \in \mathbb{Z}} w^l_k D^l_k(\sqrt{h^l_k}). \tag{37}
\]

Further on, using convexity of the Dirichlet form one can readily prove that

\[
D^N(\sqrt{h^N}) \geq \frac{1}{l} \sum_{j \in \mathbb{T}^N} D^l(\sqrt{h^{N,l,j}}). \tag{38}
\]
4.3.4 Applying F-K formula

We return now to the concrete computations. Before the estimate of
\[ \int_0^t A_{K}^{N,l}(s) \, ds \]
we need some more notation (we do not denote explicitly dependence on the cutoff):

\[ V_{N,l}^{j}(z) := (\Psi_{l} - \hat{\Psi})(z_{j}) \mathbb{1}_{\{|z_{j}| \leq \alpha \} \leq K} \],
\[ V^{j}(z) := V_{1}^{N,l}(z) \],
\[ V_{N,l}^{j}(t,z) := \theta_{x}^{N} (N^{-1+\beta}t, j/N) V_{j}^{N,l}(z) \],
\[ V^{N,l}(t,z) := \sum_{j \in T^{N}} V_{j}^{N,l}(t,z) \].

We denote by \( \xi_{N}(t) \) the Markov process on \( \Omega^{N} \) with infinitesimal generator \( L^{N} \) and by \( E_{\mu_{0}^{N}} \), respectively, \( E_{\pi^{N}} \) the path measure of this process starting with initial distribution \( \mu_{0}^{N} \), respectively, \( \pi^{N} \).

By the definitions and the entropy inequality we have

\[ \int_0^t A_{K}^{N,l}(s) \, ds = \frac{1}{N^{1+\beta}} E_{\mu_{0}^{N}} \left( \int_0^{N^{1+\beta}t} V^{N,l}(s, \xi_{N}(s)) \, ds \right) \leq \frac{1}{\gamma N^{1+\beta}} \left( H(\mu_{0}^{N} | \pi^{N}) + \log E_{\pi^{N}} \exp \left\{ \int_0^{N^{1+\beta}t} \gamma V^{N,l}(s, \xi_{N}(s)) \, ds \right\} \right) \].

We apply the Feynman-Kac bound (36) and the variational formula (34) to the second term on the right hand side of the last inequality:

\[ \log E_{\pi^{N}} \exp \left\{ \int_0^{N^{1+\beta}t} \gamma V^{N,l}(s, \xi_{N}(s)) \, ds \right\} \leq \int_0^{N^{1+\beta}t} \sigma (L^{N} + \gamma V^{N,l}(s, \cdot)) \, ds \]
\[ = \int_0^{N^{1+\beta}t} \sup_{h^{N}} \left( \int_{\Omega^{N}} \gamma V^{N,l}(s, \cdot) h^{N} \, d\pi^{N} - D^{N} (\sqrt{h^{N}}) \right) \, ds. \]
Using (38) we bound the integrand in the last expression

\[
\sup_{h^N} \left( \int_{\Omega^N} \gamma V^{N,l}(s,\cdot)h^{N,l} d\pi^l - D^N \left( \sqrt{h^N} \right) \right)
\]

(40)

\[
= \sup_{h^N} \left( \sum_{j \in T^N} \int_{\Omega^l} \gamma V^{N,l}_j(s,\cdot)h^{N,l,j} d\pi^l - D^l \left( \sqrt{h^{N,l,j}} \right) \right)
\]

\[
\leq \sup_{h^N} \sum_{j \in T^N} \left( \int_{\Omega^l} \gamma V^{N,l}_j(s,\cdot)h^{N,l,j} d\pi^l - \frac{1}{l} D^l \left( \sqrt{h^{N,l,j}} \right) \right)
\]

\[
\leq \frac{1}{l} \sum_{j \in T^N} \sup_{h^l} \left( \int_{\Omega^l} \gamma V^{N,l}_j(s,\cdot)h^l d\pi^l - D^l \left( \sqrt{h^l} \right) \right).
\]

Next we use (37) and again the variational formula (34)

\[
\sup_{h^l} \left( \int_{\Omega^l} l\gamma V^{N,l}_j(s,\cdot)h^l d\pi^l - D^l \left( \sqrt{h^l} \right) \right)
\]

(41)

\[
= \sup_{h^l} \sum_{k} w_k^l \left( \int_{\Omega^l_k} l\gamma V^{N,l}_j(s,\cdot)h^l_k d\pi^l_k - D^l_k \left( \sqrt{h^l_k} \right) \right)
\]

\[
= \sup_{w^l} \sum_{k} \frac{1}{l} \sup_{h^l_k} \left( \int_{\Omega^l_k} l\gamma V^{N,l}_j(s,\cdot)h^l_k d\pi^l_k - D^l_k \left( \sqrt{h^l_k} \right) \right)
\]

\[
= \sup_{w^l} \sum_{k} \frac{1}{l} \sigma(L^l_k + l\gamma V^{N,l}_j(s,\cdot))
\]

\[
= \sup_{w^l} \sum_{k} \left( l\gamma \theta^N_x (s,j/N)E^l_k(V^l) + \sigma(L^l_k + l\gamma \theta^N_x (s,j/N)(V^l - E^l_k(V^l))) \right)
\]

In the first step we used (37). The second step is a straightforward identity. In the third step we have used again (34) and we introduced the notation \( L^l_k \) for the infinitesimal generator of the process restricted to \( \Omega^l_k \). Finally, in the last step we use the notation introduced at the beginning of the present paragraph.

### 4.3.5 Spectral estimates

The rest of the proof of the one block estimate relies on the following three steps: (1) a straightforward estimate of \( E^l_k(V^l) \) and \( \text{Var}^l_k(V^l) \); (2) a lower bound of order \( \sim l^{-2} \) on the spectral gap of \( L^l_k \), valid uniformly in \( k \in \mathbb{Z} \); (3) combining these two and the perturbational bound (35), an upper bound on \( \sigma(\ldots) \) appearing in the last expression.
Lemma 4. There exist constant $C(K) < \infty$ for every $K > K_0$, such that for any $l$ and $k$ the following bounds hold:

$$|E_k^l(V^l)| \leq C(K)l^{-1}, \quad \text{Var}_k^l(V^l) \leq C(K)l^{-1}. \quad (42)$$

Proof. For $|k| > Kl$ there is nothing to prove, so let $|k| \leq Kl$. Restricted on $\Omega_k^l$

$$V^l = \Psi^l - \hat{\Psi}(k/l) - (\Psi^l - \hat{\Psi}(k/l)) \mathbb{1}_{\{|\Psi^l| > K\}}.$$

Consequently,

$$|E_k^l(V^l)| \leq 2 \left|E_k^l(\Psi^l - \hat{\Psi}(k/l))\right| + E_k^l(|\Psi^l - E_k^l\Psi^l| \mathbb{1}_{\{|\Psi^l| > K\}}).$$

By the equivalence of ensembles (see e.g. Appendix 2. of [6] and also [8])

$$\left|E_k^l(\Psi^l - \hat{\Psi}(k/l))\right| \leq C(K)l^{-1}.$$ 

The second term can be estimated with the Cauchy-Schwarz inequality and with large deviation techniques (noting that because of the growth conditions on the rates we can choose such $\alpha > 0$ that $\alpha^{-1}K > |E_k^l\Psi^l|$ uniformly for $|k| < Kl$) and it can be easily shown to be smaller order then the first one. $\text{Var}_k^l(V^l)$ may be estimated with similar methods. \hfill \qed

Lemma 5. There exists a constant $C < \infty$, independent of $l$ and $k$, such that for any $f \in L^2(\Omega_k^l, \pi_k^l)$

$$\text{Var}_k^l(f) \leq Cl^2D_k^l(f). \quad (43)$$

Proof. For the details of the proof of this gap-estimate we refer to [3], [8], [4]. For models with bounded $z$-variable, $-\infty < z_{\text{min}} < z_{\text{max}} < \infty$, we note that

$$c(x, y) \geq \alpha r(x) \mathbb{1}_{\{x > z_{\text{min}}, y < z_{\text{max}}\}},$$

with some positive constant $\alpha$. Thus, it is sufficient to prove the gap estimate for the reversible process with rates $\tilde{c}(x, y) := r(x) \mathbb{1}_{\{x > z_{\text{min}}, y < z_{\text{max}}\}}$, which has the same ergodic stationary measures $\pi_k^l$ as our original process. For this latter process the induction steps of [3] or Appendix 3 of [4] apply without any essential modification.
In [8] the statement is proved for zero range model with rate function satisfying condition (D). Minor formal (but not essential) modifications of that argument yield the result for the bricklayers’ models with rate functions satisfying condition (D).

Remark: Actually we could consider a wider class of models with unbounded spin space, by imposing

$$\inf_y c(x, y) \geq \alpha r(x)$$

with some positive constant $\alpha$ and $r(x)$ obeying condition (D).

We remark that there exists a constant $C$ depending only on the solution $u(t,x)$ of the Burgers’ equation (4), and another constant $C(K)$ which depends also on the cutoff level $K$, such that

$$\sup_{0 < t < T} \left| \theta_x^N(t,j/N) \right| \leq C,$$  

$$\| V^l - E_k^l(V^l) \|_\infty \leq C(K)$$

Now, combining (35), (42), (43), (44) and (45), we get the following upper bound, which holds for every sufficiently small $\gamma$:

$$\sigma(L^l_k + l\gamma \theta_x^N(s,j/N)(V^l - E_k^l(V^l))) \leq \frac{C_1(K)l^3 \gamma^2}{1 - C_2(K) \gamma l^3}$$

Setting

$$\gamma := \gamma_0 t^{-3} \quad \text{with} \quad \gamma_0 < \min \{1, (2C_2(K))^{-1}\}$$

we have

$$\sigma(L^l_k + l\gamma \theta_x^N(s,j/N)(V^l - E_k^l(V^l))) \leq C(K)\gamma_0^2 l^{-3}.$$ 

Collecting all the estimates and going backwards through (41), (40), (39), we find eventually

$$\log E_{x,N} \exp \left\{ \int_0^{N^{1+\beta} l} \gamma V^{N,l} (s) N^{3} (s) \right\} \leq C(K)\gamma_0 N^{2+\beta} l^{-4}$$

and

$$\int_0^{l} A_{K}^{N,l}(s) \, ds \leq C(K)(N^{-3\beta} l^3 + N l^{-1})$$  

(46)
Consequently, from (46), (33) and (32), with any fixed $K < \infty$ we have

$$\int_0^t M^N_{N,l}(s) \, ds \leq \varepsilon(K) O\left(N^{1-2\beta}\right) + C(K)(N^{-3\beta}l^3 + N^{-1})$$

(47)

where $C(K)$ is a finite constant which may increase to infinity as $K \to \infty$, and $\varepsilon(K) \to 0$ as $K \to \infty$.

### 4.4 End of proof

We put together (29) and (47) to get, for any $K < \infty$ fixed (with a $C$ not depending on $K$)

$$H^N(t) \leq H^N(0) + C \int_0^t H^N(s) \, ds + \varepsilon(K) O\left(N^{1-2\beta}\right) + O\left(N^{1-3\beta} \vee N^{-\beta}l \vee Nl^{-1} \vee N^{-3\beta}l^3\right).$$

If

$$0 < \beta < \frac{1}{5}$$

then we can choose

$$N^{2\beta} \ll l \ll N^{(1+\beta)/3}$$

which ensures

$$O\left(N^{1-3\beta} \vee N^{-\beta}l \vee Nl^{-1} \vee N^{-3\beta}l^3\right) = o\left(N^{1-2\beta}\right).$$

Thus for every $K < \infty$

$$H^N(t) \leq H^N(0) + C \int_0^t H^N(s) \, ds + \varepsilon(K) N^{1-2\beta} + o(N^{1-2\beta}),$$

where $\lim_{K \to \infty} \varepsilon(K) = 0$, and from Gromwall indeed (9) follows, uniformly for $t \in [0, T]$.

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