On compact operators between lattice normed spaces

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Abstract

In this paper we continue the study of compact-like operators in lattice normed spaces started recently by Aydin, Emelyanov, Erkurse\'sun Özcan and Marabeh. We show among others, that every $p$-compact operator between lattice normed spaces is $p$-bounded. The paper contains answers of almost all questions asked by these authors.

1 Introduction

In [3], the authors introduced a new notion of compact operators in Lattice-normed spaces and studied some of their properties. These operators act on spaces equipped with vector valued norms taking their values in some vector lattices. Recall that an operator from a normed space $X$ to a normed space $Y$ is said to be compact if the image of every norm bounded sequence $(x_n)$ in $X$ has a norm convergent subsequence. This notion has been generalized in the setting of lattice normed spaces giving rise to two new notions: sequentially $p$-compactness and $p$-compactness ($p$ referred to the vector valued norm). Notice that these notions coincide in the classical case of Banach spaces. In general setting with vector lattice valued norms boundedness and convergence are considered with respect to these ‘norms’. Also as notions of relatively uniform convergence and almost order boundedness have been
generalized, new properties for the operator are considered like semicompactness. Recall that if \((E, p, V)\) and \((F, q, W)\) are Lattice-normed spaces and \(T\) is a linear operator from \(E\) to \(F\), then \(T\) is said to be \(p\)-compact (respectively, \(rp\)-compact) if for every \(p\)-bounded net \((x_\alpha)\) in \(E\), there is a subnet \((Tx_\varphi(\beta))\) that \(p\)-converges (respectively, \(rp\)-converges) to some \(y \in F\). The operator is said to be sequentially \(p\)-compact if nets and subnets are replaced by sequences and subsequences above. In this paper we prove some new results in this direction. Namely we show that every \(p\)-compact operator is \(p\)-bounded. As a consequence we get that every \(rp\)-compact is \(p\)-bounded. Also we give an example of sequentially \(p\)-compact operator which fails to be \(p\)-bounded. As a consequence we deduce that a sequentially \(p\)-compact need not be \(p\)-compact. In fact these two notions are totally independent. Example of \(p\)-compact operators that fail to be sequentially \(p\)-compact is given. As mentioned above the study of \(p\)-compact operators between lattice normed spaces was started in [3]. That paper contains several new results but also some open questions. Almost all these questions will be answered in our paper.

2 Preliminaries

The goal of this section is to introduce some basic definitions and facts. For general informations on vector lattices, Banach spaces and lattice-normed spaces, the reader is referred to the classical monographs [1] and [3].

Consider a vector space \(E\) and a real Archimedean vector lattice \(V\). A map \(p : E \rightarrow V\) is called a vector norm if it satisfies the following axioms:

1) \(p(x) \geq 0;\) \(p(x) = 0 \iff x = 0; (x \in E)\).

2) \(p(x_1 + x_2) \leq p(x_1) + p(x_2); (x_1, x_2 \in E)\).

3) \(p(\lambda x) = |\lambda|p(x); (\lambda \in \mathbb{R}, x \in E)\).

A triple \((E, p, V)\) is a lattice-normed space if \(p(.)\) is a \(V\)-valued vector norm in the vector space \(E\). When the space \(E\) is itself a vector lattice the triple \((E, p, V)\) is called a lattice-normed vector lattice. A set \(M \subset E\) is called \(p\)-bounded if \(p(M) \subset [-e, e]\) for some \(e \in V_+\). A subset \(M\) of a lattice-normed vector lattice \((E, p, V)\) is called \(p\)-almost order bounded if, for any \(w \in V_+\), there is \(x_w \in E_+\) such that \(p((|x| - x_w^+) = p(|x| - x_w \land |x|) \leq w\) for any \(x \in M\).
Let \((x_\alpha)_{\alpha \in \Delta}\) be a net in a lattice-normed space \((E, p, V)\). We say that \((x_\alpha)_{\alpha \in \Delta}\) is \(p\)-convergent to an element \(x \in E\) and write \(x_\alpha \xrightarrow{p} x\), if there exists a decreasing net \((e_\gamma)_{\gamma \in \Gamma}\) in \(V\) such that \(\inf_{\gamma \in \Gamma} (e_\gamma) = 0\) and for every \(\gamma \in \Gamma\) there is an index \(\alpha(\gamma) \in \Delta\) such that \(p(v - v_\alpha) \leq e_\gamma\) for all \(\alpha \geq \alpha(\gamma)\). Notice that if \(V\) is Dedekind complete, the dominating net \((e_\gamma)\) may be chosen over the same index set as the original net. We say that \((x_\alpha)\) is \(p\)-unbounded convergent to \(x\) (or for short, \(up\)-convergent to \(x\)) if \(|x_\alpha - x| \wedge u \xrightarrow{p} 0\) for all \(u \in V_+\). It is said to be relatively uniformly \(p\)-convergent to \(x \in X\) (written as, \(x_\alpha \xrightarrow{rp} x\)) if there is \(e \in E_+\) such that for any \(\varepsilon > 0\), there is \(\alpha_\varepsilon\) satisfying \(p(x_\alpha - x) \leq \varepsilon e\) for all \(\alpha \geq \alpha_\varepsilon\).

When \(E = V\) and \(p\) is the absolute value in \(E\), the \(p\)-convergence is the order convergence, the \(up\)-convergence is the unbounded order convergence, and the \(rp\)-convergence is the relatively uniformly convergence. We refer to [5] and [4] for the basic facts about nets in topological spaces and vector lattices respectively. We will use [6, 8] as unique source for unexplained terminology in Lattice-Normed Spaces. Since the most part of this paper is devoted to answer several open questions in [3], the reader must have that paper handy, from which we recall some definitions.

**Definition 1** Let \(X, Y\) be two lattice-normed spaces and \(T \in L(X, Y)\). Then

1. \(T\) is called \(p\)-compact if, for any \(p\)-bounded net \((x_\alpha)\) in \(X\), there is a subnet \(x_{\alpha(\beta)}\) such that \(Tx_{\alpha(\beta)} \xrightarrow{p} y\) in \(Y\) for some \(y \in Y\).

2. \(T\) is called sequentially \(p\)-compact if, for any \(p\)-bounded sequence \(x_\alpha\) in \(X\), there is a subsequence \((x_{\alpha_k})\) such that \(Tx_{\alpha_k} \xrightarrow{p} y\) in \(Y\) for some \(y \in Y\).

3. \(T\) is called \(p\)-semicompact if, for any \(p\)-bounded set \(A\) in \(X\), the set \(T(A)\) is \(p\)-almost order bounded in \(Y\).

**3 \(p\)-compact operators are \(p\)-bounded**

It is well known that compact operators between Banach spaces are bounded. This result remains valid for general situation of \(p\)-compact operators as it will be shown in our first result, which answers positively Question 2 in [3].
Theorem 2  Every $p$-compact operator between two Lattice-normed spaces is $p$-bounded.

Proof. Assume, by contradiction, that there exists a $p$-compact operator $T : (E, p, V) \rightarrow (F, q, W)$ which is not $p$-bounded. Then there exists a $p$-bounded subset $A$ of $E$ such that $T(A)$ is not $q$-bounded. So, for every $u \in W^+$ there exists some $x_u \in A$ satisfying $q(T(x_u)) \not\leq u$. Since the net $(x_u)_{u \in W^+}$ is $p$-bounded there is a subnet $(y_v = x_{\varphi(v)})_{v \in \Gamma}$ and an element $f \in F$ such that $(Ty_v) \xrightarrow{q} f$. It follows that the net $(Ty_v)$ has a $q$-bounded tail, which means that for some $v_0$ in $\Gamma$ and some $w \in W^+$ we have,

$$q(Tx_{\varphi(v)}) \leq w, \quad \text{for } v \geq v_0.$$  

(1)

Pick $v_1$ in $\Gamma$ such that $\varphi(v) \geq w$ for all $v \geq v_1$. It follows that for $v \geq v_0 \vee v_1$, we have $q(Tx_{\varphi(v)}) \not\leq \varphi(v)$ and so

$q(Tx_{\varphi(v)}) \not\leq w,$

which is a contradiction with (1) and the proof comes to its end. ■

The following lemma, which connects unbounded order convergence with pointwise convergence, is a known fact, although a quick proof is included for the sake of completeness.

Lemma 3  Let $E = \mathbb{R}^X$ be the Riesz space of all real-valued functions defined on a nonempty set $X$. The following statements are equivalent:

(i) The net $(f_\alpha)_{\alpha \in A}$ is $uo$-convergent in $E$.

(ii) for every $x \in X$, the net $(f_\alpha(x))_{\alpha \in A}$ is convergent in $\mathbb{R}$.

Proof. (i) $\implies$ (ii) Assume that $f_\alpha \xrightarrow{uo} f$ in the Dedekind complete Riesz space $E$. Then there is a net $(g_\alpha)_{\alpha \in A}$ which decreases to 0 and for some $\alpha_0$ we have

$$|f_\alpha - f| \land 1 \leq g_\alpha \text{ for all } \alpha \geq \alpha_0. \quad (2)$$

Since $(g_\alpha(x))$ decreases to 0 for every $x \in X$, it follows easily from (2) that $f_\alpha(x) - f(x)$ converges to 0, as desired.

(ii) $\implies$ (i) Assume now that $f_\alpha$ converges simply to some $f \in E$ and let $h \in E^+$. Define a net $(g_\alpha)$ by putting

$$g_\alpha(x) = \sup_{\beta \geq \alpha} (|f_\beta - f| \land h)(x), \quad x \in X.$$
it is clear that \( g_\alpha \) decreases to 0 and \(|f_\alpha - f| \wedge h \leq g_\alpha\). This shows that \( f_\alpha \xrightarrow{\text{w}} f \) and we are done.  

Consider the Riesz space \( F \) of all bounded real valued functions defined on the real line with countable support and denote by \( E \) the direct sum \( \mathbb{R}1 \oplus F \), where \( 1 \) denotes the constant function taking the value 1. This example will be of great interest for us. The following lemma establishes some of its properties. Recall that a vector sublattice \( Y \) of a vector lattice \( X \) is said to be regular if every subset in \( Y \) having a supremum in \( Y \) has also a supremum in \( X \) and these suprema coincide. For more information about this notion and nice characterizations of it via unbounded order convergence the reader is referred to [4].

Lemma 4 The space \( E \) introduced above has the following properties.

(i) \( E \) is a regular vector sublattice of \( \mathbb{R}^\mathbb{R} \).

(ii) \( E \) is Dedekind \( \sigma \)-complete but not Dedekind complete.

Proof. (i) It is clear that \( E \) is a vector sublattice of \( \mathbb{R}^\mathbb{R} \). To show that it is regular assume that \((g_\alpha)_{\alpha \in A}\) is a net in \( E \) satisfying \( g_\alpha \downarrow 0 \) in \( E \). Let \( g = \inf_\alpha g_\alpha \) in \( \mathbb{R}^\mathbb{R} \) and \( x \in \mathbb{R} \). Then \( h = g(x)1_{\{x\}} \in E \) and \( 0 \leq h \leq g_\alpha \) for all \( \alpha \). This implies that \( h = 0 \) and then \( g(x) = 0 \). Hence \( g = 0 \) and the regularity is proved.

(ii) Let \((g_n)\) be an order bounded sequence in \( E \) and write \( g_n = \lambda_n + f_n \), with \( \lambda_n \in \mathbb{R} \) and \( f_n \in F \). Let \( \Omega \) be the union of the supports of \( f_n \), then \( \Omega \) is countable. Let \( g \) be the supremum of \((g_n)\) in \( \mathbb{R}^\mathbb{R} \), that is,

\[
g(a) = \sup g_n(a), \text{ for all } a \in \mathbb{R}.
\]

It will be sufficient to show that \( g \in E \). To this end observe that \( g(x) = \alpha := \sup_\alpha \alpha_n \) for all \( x \in \mathbb{R} \setminus \Omega \). Now put \( f = (g - \alpha)1_{\Omega} \). Then \( f \in F \) and \( g = \alpha + f \in E \) as required. Next we show that \( E \) is not Dedekind complete. Consider the net \((g_x)_{x \in [0,1]}\) in \( E \) defined by \( g_x = x1_{\{x\}} \). It is a bounded net in \( E \) and its supremum in \( \mathbb{R}^\mathbb{R} \) does not belong to \( E \). As \( E \) is regular in \( \mathbb{R}^\mathbb{R} \) this net can not have a supremum in \( E \).

Remark 5 Consider the following operator:

\[
T : L_1[0,1] \longrightarrow c_0; \quad f \mapsto Tf = \left( \int_0^1 f(t) \sin nt \, dt \right)_{n \geq 1}.
\]
It is mentioned in [1], that \( T \) is not order bounded; it is perhaps more convenient to consider the same operator defined on \( L_1[0, 2\pi] \). In this case if we define \( u_n \) by \( u(t) = \sin nt \) for \( t \in [0, 2\pi] \), then \( |u_n| \leq 1 \), however \( (Tu_n) = (e_n) \) is not bounded in \( c_0 \), where \((e_n)\) denotes the standard basis of \( c_0 \). This statement implies also that \( T \) is not sequentially order compact. Because \((e_n)\) has no order bounded subsequence, it follows that \((Tu_n)\) can not admit an order convergent subsequence. So the statement made in [3] that \( T \) is \( p \)-bounded is not correct.

The above example is presented in [3] to show that sequentially \( p \)-compact operators need not be \( p \)-bounded. Although the operator given in that example fails to be sequentially \( p \)-compact, the assertion that sequentially \( p \)-compact operators need not be \( p \)-bounded is true. This will be shown in our next example.

**Example 6** Consider the Riesz spaces \( E \) and \( F \) defined just before Lemma 4 and let \( T \) be the projection defined on \( E \) with range \( F \) and kernel \( R_1 \). We claim that \( T \) is sequentially order compact, but not order bounded. Let \((f_n)\) be an order bounded sequence in \( E \). Then \(|f_n| \leq \lambda\) for some real \( \lambda > 0 \) and for all \( n \). Write \( f_n = g_n + \lambda_n \) with \( \lambda_n \) real and \( g_n \in F \) and observe that \(|g_n| \leq 2\lambda\) for all \( n \). We have also \(|g_n| \leq 2\lambda 1_A \in F \) where \( A \) is the union of the supports of \( g_n \), \( n = 1, 2, ... \). A standard diagonal process yields a subsequence \((g_{k_n})\) of \((f_n)\) which converges pointwise on \( A \) and then on \( \mathbb{R} \) since all functions \( g_{k_n} \) vanish on \( \mathbb{R} \setminus A \). Hence \((g_{k_n})\) is wo-convergent in \( \mathbb{R}^R \). As \((g_{k_n})\) is order bounded this implies that \((g_{k_n})\) is order convergent in \( \mathbb{R}^R \). Observe moreover that \( \sup_{p \geq n} g_{k_n} \) belongs to \( F \), which shows that \((g_{k_n})\) is order convergent in \( F \). The fact that \( T \) is not order bounded is more obvious: it is clear that the image of the net \((1_{x})_{x \in [0, 1]}\) by \( T \) is not order bounded in \( F \).

As an immediate consequence of Theorem 2 and Example 6 we deduce that sequentially \( p \)-compactness does not imply \( p \)-compactness. At this stage one might expect that the converse is true. Does \( p \)-compactness imply sequentially \( p \)-compactness? This is an open question left in [3]. Unfortunately the answer is again negative.

**Example 7** Let \( X \) be the set of all strictly increasing maps from \( \mathbb{N} \) to \( \mathbb{N} \) and \( E = \mathbb{R}^X \) be the space of all real-valued functions defined on \( X \), equipped with the product topology.
1. First we will prove that the identity map, \( I \), is a \( p \)-compact operator on the lattice-normed space \((E, |\cdot|, E)\). To this aim, pick a \( p \)-bounded net \((f_\alpha)_{\alpha \in A}\) in \(E\), that is, \(|f_\alpha| \leq f\) for some \(f \in E^+\) and for every \(\alpha \in A\). It follows that
\[
f_\alpha \in \prod_{x \in X} [-f(x), f(x)].
\]
Notice that the space \(\prod_{x \in X} [-f(x), f(x)]\), equipped with the product topology, is compact by Tychonoff’s Theorem. Thus \((f_\alpha)\) has a convergent subnet \((g_\beta)_{\beta \in B}\) in \(\prod_{x \in X} [-f(x), f(x)]\) to some \(g\). This means that
\[
g_\beta(x) \longrightarrow g(x) \text{ for all } x \in X.
\]
According to Lemma 3, \(g_\beta\) is \(uo\)-convergent to \(g\) in \(E\). Since bounded \(uo\)-convergent nets are order convergent, we have that \(g_\beta \xrightarrow{o} g\). This proves that \(I\) is a \(p\)-compact operator.

2. We prove now that \(I\) is not sequentially \(p\)-compact. Let \((\varphi_n)\) be a sequence in \(\{-1, 1\}^X\) which has no convergent subsequence (see Example 3.3.22 in [7]). This sequence is order bounded in \(E\) and every subsequence \((\psi_n)\) of \((\varphi_n)\) does not converges in \(\{-1, 1\}^X\), that is, for some \(x \in X\), \(\psi_n(x)\) diverges. According to Lemma 3, \((\psi_n)\) is not \(uo\)-convergent in \(E\). Since \((\psi_n)\) is order bounded it does not converge in order. This finishes the proof.

In classical theory of Banach spaces the identity map is compact if and only if the space is finite-dimensional. In contrast of this the situation is not clear in general case. We already have seen an example of infinite-dimensional space on which the identity map is \(p\)-compact. This question has been investigated in [3] where the authors showed that \(I_{L_1[0,1]}\) fails to be compact however, \(I_{\ell_1}\) is \(p\)-compact. In the next example we show that \(I_{L_\infty[0,1]}\) is not \(p\)-compact, answering a question asked in [3].

**Example 8** The identity operator \(I\) on the lattice normed space \((L_\infty[0,1], |\cdot|, L_\infty[0,1])\) is neither \(p\)-compact nor sequentially \(p\)-compact. To this end, consider the sequence of Rademacher function given by :

\[
r_n : [0, 1] \longrightarrow \mathbb{R}
\]
\[
t \quad \mapsto \quad \text{sgn} \left( \sin \left( 2^n \pi t \right) \right)
\]
for all $n \in \mathbb{N}$, which is order bounded since $|r_n| = 1$. Suppose now that $(r_n)$ has an order convergent subnet $(r_{n_\alpha})_{\alpha \in \Gamma}$. Then $r_{n_\alpha} \xrightarrow{\sigma} r$ for some $r \in L_\infty[0,1]$. Let $\alpha \in A$. For every $\beta > \alpha$, $\int_0^1 r_{n_\alpha} r_{n_\beta} d\mu = 0$. On the other hand $(r_{n_\alpha} r_{n_\beta})_\beta$ converges in order to $r_{n_\alpha} r$ in $L_\infty[0,1]$ and then in $L_1[0,1]$. Since the integral is order continuous, we deduce that

$$\int_0^1 r_{n_\alpha} r d\mu = 0.$$  

This equality holds for every $\alpha \in A$, and a similar argument leads to

$$\int_0^1 r^2 d\mu = 0,$$

which is a contradiction since $|r| = 1$, and the claim is now proved.

## 4 Semicompact operators

The notion of semicompact operators has been introduced by Zaanen in [9] and extended in the framework of lattice normed spaces in [3].

Let $(X,p,E)$ be a lattice normed space and $(Y,q,F)$ be an lattice normed vector lattice. A linear operator $T : X \rightarrow Y$ is called $p$-semicompact if it maps $p$-bounded sets in $X$ to $p$-almost order bounded sets in $Y$. We recall that a subset $B$ of $Y$ is said to be $p$-almost order bounded if for any $w \in F_+$, there is $y_w \in Y$ such that

$$q((|y| - y_w)^+) = q(|y| - y_w \wedge |y|) \leq w \text{ for all } y \in B.$$  

Semicompact operators from Banach spaces to Banach lattices fail, in general, to be compact (see [1]). This yields trivially that $p$-semicompactness does not imply $p$-compactness. However, the converse is true in the classical case as has been shown in Theorem 5.71 in [1]. And one can expect to extend this result in general situation. This is already the subject of Question 4 in [3]. Unfortunately the answer is again negative. Before stating our counterexample let us recall that every order bounded operator from a vector lattice $E$ to a Dedekind complete vector lattice $F$ has a modulus ([1]).

**Example 9** Let $E$ be a Dedekind complete Banach lattice with order continuous norm and $T$ be a norm-compact operator in $\mathcal{L}(E)$ such that $T$ has no
modulus, and therefore $T$ can not be order bounded. For the existence of such operator we refer the reader to the Krengel’s example in [1, p 277]. Consider now the following lattice-normed vector spaces $(E, \| \cdot \|, \mathbb{R})$ and $(E, p, \mathbb{R}^2)$, where $p(x) = \begin{pmatrix} \|x\| \\ 0 \end{pmatrix}$ for all $x \in E$. It is straightforward to prove that $T$ is again $p$-compact operator and we claim that $T$ is not $p$-semicompact. To this end we will argue by contradiction and we assume that $T$ is $p$-semicompact. Fix an element $u \in E^+$ and let $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then there exists $z_w$ such that $p((T(x) - z_w)^+) \leq w$ for all $x \in [-u, u]$, which means that $(T(x) - z_w)^+ = 0$. Noting that this occurs for $x$ and $-x$ we see that $|T(x)| \leq z_w$ for all $x \in [-u, u]$.

This shows that $T$ is order bounded, a contradiction, and our proof comes to an end.

A slight modification of the proof of Example 9 leads to a more general result. The proof of it will be left for the reader.

**Proposition 10** Let $(E, p, V)$ be a lattice normed space and $(F, q, W)$ a lattice normed vector lattice. We assume that $q(F)^d$ is not trivial. Then every semicompact operator $T : (E, p, V) \to (F, q, W)$ is $p$-bounded as an operator from $(E, p, V)$ to $(F, | \cdot |, W)$.

## 5 $rp$-compact operators

As every $rp$-compact operator between lattice-normed spaces is $p$-compact, the following result is an immediate consequence of Theorem 2.

**Theorem 11** Let $(E, p, V)$ and $(F, q, W)$ be lattice-normed spaces and $T$ be in $\mathcal{L}(E, F)$. If $T$ is $rp$-compact then $T$ is $p$-bounded.

In the following example we will prove that sequentially $p$-compact operators need not be $rp$-compact.

**Example 12** Let $E$ be the Riesz space defined above. We claim that the identity operator $I : E \to E$ is sequentially $p$-compact but fails to be $rp$-compact. Let $(x_n)$ be a bounded sequence in $E$, that is, $|x_n| \leq x$ for some.
$x \in E^+$. Write $x = \alpha + f$, and $x_n = \alpha_n + f_n$ where $\alpha \in \mathbb{R}^+$ and $f \in F$ and $\alpha_n \in \mathbb{R}$, $f_n \in F$ for $n = 1, 2, \ldots$. It is easily seen that $|\alpha_n| \leq \alpha$, $|f_n| \leq x + \alpha$. By a standard diagonal argument there exists a subsequence such that $f_{\varphi(n)}(a)$ converges for every $a \in \mathbb{R}$ and $\alpha_{\varphi(n)}$ converges in $\mathbb{R}$. This shows that $x_n$ converges pointwise on $\mathbb{R}$ and its limit is clearly in $E$. By Lemma 3, $x_n \xrightarrow{u_0} x$ in $\mathbb{R}^R$ and then $x_n \xrightarrow{\alpha} x$ in $\mathbb{R}^R$ as it is an order bounded sequence. Now using Lemma 27 in [2] and Lemma 4, we deduce that $x \in E$. On the other hand, let $\mathcal{F}$ be the collection of finite subsets of $\mathbb{R}_+$ ordered by inclusion and consider the net $(g_A)_{A \in \mathcal{F}}$ where $g_{\alpha} = 1_\alpha$. Then $(g_{\alpha})$ is order bounded in $E$ but has no convergent subnet. Since $g_{\alpha} \uparrow 1_{\mathbb{R}_+}$ in $\mathbb{R}^R$ and $E$ is regular in $\mathbb{R}^R$, it follows that $(g_{\alpha})$ is not order convergent in $E$ and so are all its subnets.

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