LIMIT THEOREMS FOR THE FIBONACCI QUANTUM WALK

CLEMENT AMPADU

31 Carrolton Road
Boston, Massachusetts, 02132
USA
e-mail: drampadu@hotmail.com

Abstract

We study the discrete-time quantum walk in one-dimension governed by the Fibonacci transformation. We show localization does not occur for the Fibonacci quantum walk by investigating the stationary distribution of the walk, in addition, we obtain the weak limit theorem.

KEY WORDS: quantum walk, limit theorems, Fibonacci transformation

PACS: 03.67.-a

I. Introduction

Limit theorems for quantum walks has been well studied by many authors [1-19], for example. Related to quantum physics localization of the quantum has been investigated [20-23], for example. Let \( P(N_t = n) \) be the probability that the quantum walker is at position \( n \in \mathbb{Z} \) at time \( t \), a criterion for localization is the following \( \limsup_{t \to \infty} P(N_t = 0) > 0 \).

In this paper we study the Fibonacci quantum walk as defined by A. Romanelli [24]. The Fibonacci sequence gives rise to rich behaviors in quantum systems. For example in [25] the Fibonacci quantum was shown to produce an unexpected sub-ballistic wave function spreading by the power-law tail of the standard deviation \( \sigma(t) \sim t^c \) with \( 0.5 < c < 1 \). This result was also confirmed in [26,27] which studied the quantum walk subjected to noise with a Levy waiting-time distribution [28]. In [29] the Fibonacci quantum walk is shown to produce sub-ballistic behavior for the quantum kicked rotor in resonance, and in [30] tight-binding in electrons.
This paper is organized as follows: in Section II we define the Fibonacci quantum walk. In Section III we present the main results with proof. We show non-existence of localization in the Fibonacci quantum walk via Theorem 1, and in Theorem 2 we give the weak limit theorem. Section IV is devoted to an open problem.

II. Definition of the Fibonacci quantum walk

The Fibonacci quantum walk is usually generated from a large sequence of two time-step unitary operators $U_1$ and $U_2$ for each time $t$. Given the two initial values of the succession $U_1$ and $U_2$, the sequence is obtained using the rule $U_{k+1} = U_k U_{k-1}$. To obtain the operators $U_1$ and $U_2$, consider the standard quantum walk corresponding to a one-dimensional evolution of a quantum system, in a direction which depends on an additional degree of freedom, the chirality, with two possible states: “left” $|L\rangle$ or “right” $|R\rangle$. Assume that the walker can move freely over a series of interconnected sites labeled by an index $n$. In the quantum walk the motion of the particle is selected by the chirality. At each time step a unitary transformation of the chirality takes place and the walker moves according to its final chirality state. The global Hilbert space of the system is the tensor product $H_s \otimes H_c$ where $H_s$ is the Hilbert space associated to the motion on the line and $H_c$ is the chirality Hilbert space.

Let us call $T_-$ ($T_+$) the operators that move the walker one site to the left (right) on the line in $H_s$ and let $|L\rangle\langle L|$ and $|R\rangle\langle R|$ be the chirality projector operators in $H_c$ and consider the unitary operator $U_i(\theta_i) = (T_- \otimes |L\rangle\langle L| + T_+ \otimes |R\rangle\langle R|) \circ (I \otimes K(\theta_i))$, where $K(\theta_i) = \sigma_z e^{-i\theta_i\sigma_y}$ is a unitary operator acting on $H_c$, and $\sigma_x$ and $\sigma_z$ being the standard Pauli matrices, and $I$ is the identity operator in $H_s$. Then one step of the quantum walk is given by $|\Psi(t+1)\rangle = U_i(\theta_i) |\Psi(t)\rangle$. 

The wave vector $|\Psi(t)\rangle$ is expressed by $|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} \left( a_n(t) |n\rangle + b_n(t) |n\rangle \right)$, where we have associated the upper (lower) component with the left (right) chirality, the states $|n\rangle$ are eigenstates of the position operator corresponding to the site $n$ on the line. The unitary evolution for $|\Psi(t)\rangle$ corresponding to 

$$|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} \left( a_n(t) |n\rangle + b_n(t) |n\rangle \right)$$

can be written as

$$a_n(t+1) = a_{n+1}(t) \cos \theta + b_{n+1}(t) \sin \theta$$

$$b_n(t+1) = a_{n-1}(t) \sin \theta - b_{n-1}(t) \cos \theta$$

To build the operators $U_1$ and $U_2$ we replace $\theta$ immediately above with $\theta_1$ and $\theta_2$ respectively.

If we create the spatial Fourier transform of the amplitude $(a_n(t), b_n(t))^T$ by multiplying both sides of $a_n(t+1) = a_{n+1}(t) \cos \theta + b_{n+1}(t) \sin \theta$, and $b_n(t+1) = a_{n-1}(t) \sin \theta - b_{n-1}(t) \cos \theta$ by $e^{i \frac{\phi - \pi}{2} n}$, with $\phi \in [-\pi, \pi]$, and summing over the integer index $n$, we get

$$\left( \begin{array}{c} F(\phi, t+1) \\ G(\phi, t+1) \end{array} \right) = \left( \begin{array}{c} F(\phi, t) \\ G(\phi, t) \end{array} \right) M(\phi, \theta)$$

where $F(\phi, t) = \sum_n e^{i \frac{\phi - \pi}{2} n} a_n(t)$, $G(\phi, t) = \sum_n e^{i \frac{\phi - \pi}{2} n} b_n(t)$,

and $M(\phi, \theta) = \left( \begin{array}{cc} e^{-i\phi} \cos \theta & i e^{-i\phi} \sin \theta \\ i e^{i\phi} \sin \theta & e^{i\phi} \cos \theta \end{array} \right)$. Thus, in the Fourier space the dynamics of the Fibonacci quantum walk is determined by the unitary matrix $M(\phi, \theta)$. If we call $M_1$ and $M_2$ the matrix $M(\phi, \theta)$ evaluated at $(\phi_1, \theta_1)$ and $(\phi_2, \theta_2)$ respectively, then in the Fourier space the dynamics of the Fibonacci quantum walk is determined as follows: Given the two initial values of the succession $M_1$ and $M_2$ the sequence is obtained using the rule $M_{k+1} = M_k M_{k-1}$.

Notice we can write $M(\phi, \theta) = \left( \begin{array}{cc} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{array} \right) \left( \begin{array}{cc} \cos \theta & i \sin \theta \\ i \sin \theta & -i \cos \theta \end{array} \right)$. Let $U = \left( \begin{array}{cc} \cos \theta & i \sin \theta \\ i \sin \theta & -i \cos \theta \end{array} \right)$,
then we can write $U = P + Q$, where

$$P = \begin{pmatrix} \cos \theta & i \sin \theta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ i \sin \theta & -i \cos \theta \end{pmatrix}.$$  So in the Fourier space we can write the evolution of the Fibonacci quantum walk as

$$|\Psi_{t+1}\rangle = \sum_{n \in \mathbb{Z}} |n\rangle \otimes \left(P|\psi_i(n+1)\rangle + Q|\psi_i(n-1)\rangle\right).$$  Let $\|y\|^2 = \langle y | y \rangle$, then the probability that the quantum walker $N_i$ is at position $n$ at time $t$ is defined by $P(N_i = n) = \|\psi_i(n)\|^2$. The Fourier transform $|\hat{\Psi}_i(k)\rangle$ of $|\psi_i(n)\rangle$ is defined as $|\hat{\Psi}_i(k)\rangle = \sum_{n \in \mathbb{Z}} e^{-i kn} |\psi_i(n)\rangle$. By the inverse Fourier transform we have $|\psi_i(n)\rangle = \int_{-\pi}^{\pi} e^{i kn} |\hat{\Psi}_i(k)\rangle \frac{dk}{2\pi}$. The time evolution of $|\hat{\Psi}_i(k)\rangle$ is

$$|\hat{\Psi}_{t+1}(k)\rangle = M(k)|\hat{\Psi}_i(k)\rangle,$$  where $M(k) = \begin{pmatrix} e^{-ik} \cos \theta & e^{-ik} \sin \theta \\ ie^{ik} \sin \theta & -ie^{ik} \cos \theta \end{pmatrix}$. By induction on $t$, we get

$$|\hat{\Psi}_t(k)\rangle = M(k)^t |\hat{\Psi}_0(k)\rangle.$$  In particular, the probability distribution can be written as

$$P(N_i = n) = \left| \int_{-\pi}^{\pi} M(k)^t |\hat{\Psi}_0(k)\rangle e^{i kn} \frac{dk}{2\pi} \right|^2.$$  To end this section we should remark that the definition of the Fibonacci quantum walk here can partly be found in [24].

III. Main Results

From $M(k) = \begin{pmatrix} e^{-ik} \cos \theta & e^{-ik} \sin \theta \\ ie^{ik} \sin \theta & -ie^{ik} \cos \theta \end{pmatrix}$, it is easily seen that the eigenvalues of $M(k)$ are given by $\lambda_1(k) = e^{iw(k)}$ and $\lambda_2(k) = e^{-iw(k)}$ where $w(k)$ is determined by

$$\sin w(k) = \sqrt{1 - \sin k \cos \theta}.$$  The normalized eigenvectors corresponding to the eigenvalues $\lambda_j(k)$, $1 \leq j \leq 2$ are given by,

$$V_j(k) = N_j \begin{pmatrix} i \sin(2k) \cot \theta - \lambda_j e^{-ik} \csc \theta \rangle + \cot \theta \\ 1 \end{pmatrix},$$  where $N_j$ is an appropriate normalization factor. Recall that the degeneracy of the eigenvalues is a necessary condition for localization. However for the Fibonacci quantum walk in this paper, none of the
eigenvalues are independent of $k$, so we can start by saying that localization does not occur in the Fibonacci quantum walk. In particular, we expect that our limit theorem does not have a $\delta$–measure corresponding to localization.

To show localization does not occur, rigorously, let us take the initial state as

$$\left| \psi_0(n) \right\rangle = \begin{cases} T \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, & \text{if } n = 0 \\ T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } n \neq 0 \end{cases},$$

where $|\alpha|^2 + |\beta|^2 = 1$, and $T$ is the transposed operator. We should note that $|\tilde{\psi}_0(k)\rangle = |\psi_0(0)\rangle$. Now we evaluate the following limits,

$$\lim_{t \to \infty} P(N_{2t} = 0), \quad \lim_{t \to \infty} P(N_{2t} = n), \quad \lim_{t \to \infty} P(N_{2t+1} = 0), \quad \text{and} \quad \lim_{t \to \infty} P(N_{2t+1} = n).$$

We should note that the Fourier transform $|\tilde{\psi}_0(k)\rangle$ can be expressed by the normalized eigenvectors as

$$|\tilde{\psi}_0(k)\rangle = \sum_{j=1}^{2} \left( \lambda_j(k) \right)^* \left( \left| v_j(k) \right\rangle \langle v_j(k) \right| v_j(k) \left\rangle \right)$$

which implies that

$$\left| \tilde{\psi}_0(k) \right\rangle = M(k)^T \left| \tilde{\psi}_0(0) \right\rangle = \sum_{j=1}^{2} \lambda_j(k)^* \left( \left| v_j(k) \right\rangle \langle v_j(k) \right| v_j(k) \left\rangle \right).$$

So by the inverse Fourier transform we get

$$|\psi_t(n)\rangle = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \lambda_j(k)^* \left( \left| v_j(k) \right\rangle \langle v_j(k) \right| v_j(k) \left\rangle \right) e^{iun} \frac{dk}{2\pi}.$$ If $n \neq 0$, then we recall that

$$|\tilde{\psi}_0(k)\rangle = |\psi_0(0)\rangle = T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right],$$

and it is easy to check that

$$|\psi_t(n)\rangle = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \lambda_j(k)^* \left( \left| v_j(k) \right\rangle \langle v_j(k) \right| v_j(k) \left\rangle \right) e^{iun} \frac{dk}{2\pi} = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right].$$

In particular, if $n \neq 0$, then

$$P(N_{2t} = n) = P(N_{2t+1} = n) = 0,$$

so we have $\lim_{t \to \infty} P(N_{2t} = n) = \lim_{t \to \infty} P(N_{2t+1} = n) = 0$. If $n = 0$,

$$P(N_{2t} = n) = P(N_{2t+1} = n) = 0,$$

then we note that

$$|\psi_{2t(2t+1)}(0)\rangle = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \lambda_j(k)^{2(2t+1)} \left( \left| v_j(k) \right\rangle \langle v_j(k) \right| v_j(k) \left\rangle \right) e^{iun} \frac{dk}{2\pi}.$$
have to evaluate the following integrals, 
\[ \int_{-\pi}^{\pi} \lambda_1(k) e^{2i \langle v_1(k) | \psi_0(0) \rangle v_1(k)} \frac{dk}{2\pi}, \]
\[ \int_{-\pi}^{\pi} \lambda_2(k) e^{2i \langle v_2(k) | \psi_0(0) \rangle v_2(k)} \frac{dk}{2\pi}, \]
\[ \int_{-\pi}^{\pi} \lambda_1(k) e^{2i \langle \psi_0(0) | v_1(k) \rangle v_1(k)} \frac{dk}{2\pi}, \]
and
\[ \int_{-\pi}^{\pi} \lambda_2(k) e^{2i \langle \psi_0(0) | v_2(k) \rangle v_2(k)} \frac{dk}{2\pi}. \]

To make the calculation manageable, let us write the eigenvalues of \( M(k) \) for \( j = 1, 2 \) as \( \lambda_j(k) = e^{i \nu(k)(-1)^{j+1}} \), then we see that we can write
\[ \lambda_j(k) \langle v_j(k) | \psi_0(0) \rangle \langle \psi_0(0) | v_j(k) \rangle = e^{i \nu(k)(-1)^{j+1}} \left[ i N_j \sin(2k) \cot \theta - i N_j e^{-\alpha} \csc \theta + N_j \cot \theta \left( -i a N_j \sin(2k) \cot \theta + i a N_j e^\alpha \csc \theta + a N_j \cot \theta + \overline{N}_j \beta \right) \right] \]
\[ - i a |N_j|^2 \sin(2k) \cot \theta + i a |N_j|^2 e^\alpha \csc \theta + a |N_j|^2 \cot \theta + |N_j|^2 \beta \]

So,
\[ |\phi_{2(2r+1)}(0)\rangle = \sum_{j=1}^2 \int_{-\pi}^{\pi} e^{i \nu(k)(-1)^{j+1}} \left[ i N_j \sin(2k) \cot \theta - i N_j e^{-\alpha} \csc \theta + N_j \cot \theta \left( -i a N_j \sin(2k) \cot \theta + i a N_j e^\alpha \csc \theta + a N_j \cot \theta + \overline{N}_j \beta \right) \right] \frac{dk}{2\pi} \]
which implies that
\[ |\phi_{2(2r+1)}(0)\rangle = \int_{-\pi}^{\pi} e^{i \nu(k)(-1)^{j+1}} \left[ i N_j \sin(2k) \cot \theta - i N_j e^{-\alpha} \csc \theta + N_j \cot \theta \left( -i a N_j \sin(2k) \cot \theta + i a N_j e^\alpha \csc \theta + a N_j \cot \theta + \overline{N}_j \beta \right) \right] \frac{dk}{2\pi} \]
\[ + \int_{-\pi}^{\pi} e^{i \nu(k)(-1)^{j+1}} \left[ i N_j \sin(2k) \cot \theta - i N_j e^{-\alpha} \csc \theta + N_j \cot \theta \left( -i a N_j \sin(2k) \cot \theta + i a N_j e^\alpha \csc \theta + a N_j \cot \theta + \overline{N}_j \beta \right) \right] \frac{dk}{2\pi} \]

It is clearly seen that the integrals in \( |\phi_{2r+1}(0)\rangle \) do not exist, so it certain that \( \lim_{t \to \infty} P(N_{2r} = 0) \)
and \( \lim_{t \to \infty} P(N_{2r} = 0) \) do not exist. Since we have shown that
\[ \lim_{t \to \infty} P(N_{2r} = 0) = \lim_{t \to \infty} P(N_{2r+1} = 0) = \text{Does not exist} \]
and \( \lim_{t \to \infty} P(N_{2r} = n) = \lim_{t \to \infty} P(N_{2r+1} = n) = 0 \)
for any initial state, it follows rigorously that localization does not occur in the Fibonacci quantum walk. In particular, the non-existence of localization is given by the following.
Theorem 1: \( \lim_{t \to \infty} P(N_{2t} = 0) = \lim_{t \to \infty} P(N_{2t+1} = 0) = \text{Does not exist} \), and 
\( \lim_{t \to \infty} P(N_{2t} = n) = \lim_{t \to \infty} P(N_{2t+1} = n) = 0 \)

Now we obtain the limit theorem for the Fibonacci quantum walk. Using the method of Grimmett et.al [31], we see that the \( r - \)th moment of \( N_t \) is given by

\[
E\left( N_t^r \right) = \sum_{n \in \mathbb{Z}} n^r P(N_t = n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left| \frac{\partial}{\partial k} \varphi(k) \right| D\varphi(k) = \int_{-\pi}^{\pi} \sum_{j=1}^{2\pi} \lambda_j^{-r}(k) \left| \langle \psi_j(k) | \varphi(0) \rangle \right|^2 + O(t^{-r-1})
\]

where \( D = i \left( \frac{d}{dk} \right) \) and \( (t)_r = t(t-1) \cdots (t-r+1). \) Let \( h_j(k) = \frac{D\lambda_j(k)}{\lambda_j(k)} \), then

\[
E\left( \left( \frac{N_t}{t} \right)^r \right) = \frac{1}{\omega} \sum_{j=1}^{2\pi} h_j(k)^r \left| \langle \psi_j(k) | \varphi(0) \rangle \right|^2 \text{ as } t \to \infty. \]

Substituting \( h_j(k) = x \), we have

\[
\lim_{t \to \infty} E\left( \left( \frac{N_t}{t} \right)^r \right) = \int_{\mathbb{R}} x^r f(x) \, dx , \text{ where } f(x) = f_K(x; \varphi, \gamma) and |\gamma| = |\cos \theta| . \]

Since \( f(x) \) is a density function, the proof is complete. In particular, we the following.

**Theorem 2:** \( \frac{N_t}{t} \Rightarrow Z \), where \( \Rightarrow \) means weak convergence and \( Z \) has the density function

\[
f(x) = (c_0; x)f_K(x; \cos \theta) , \text{ where } f_K(x; a) = \frac{\sqrt{1-|a|^2}}{\pi(1-x^2)^{1/2}} I_{(-|a|,|a|)}(x) , \]

\[
I_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \text{, and } c_0 \text{ is determined by the initial state of the particle undergoing the Fibonacci quantum walk in one dimension.}
\]

**IV. Open Problem**

Consider the quantum walk on the \( k - \)dimensional lattice governed by the unitary matrix
$U = A^\otimes k$, where $A = \begin{bmatrix} 0 & 0 & i\cos \theta & i\sin \theta \\ i\sin \theta & -i\cos \theta & 0 & 0 \\ i\cos \theta & i\sin \theta & 0 & 0 \\ 0 & 0 & i\sin \theta & -i\cos \theta \end{bmatrix}$. It is an open problem to consider the Fibonacci quantum walk as an $m$–state quantum walk without memory on the $k$–dimensional lattice and obtain the associated limit theorems.

References

[1] N. Konno, and T. Machida, Limit theorems for quantum walks with memory, Quantum Information and Computation, Vol.10, No.11&12, (2010)

[2] C. Di Franco, et.al, Alternate two-dimensional quantum walk with a single-qubit coin, arXiv: 1107.4400 (2011)

[3] C. Liu, and N. Petulante, Asymptotic evolution of quantum walks on the $N$ – cycle subject to decoherence on both the coin and position degree of freedom, Phys. Rev. A 84, 012317 (2011)

[4] K. Chisaki, et.al, Crossovers induced by discrete-time quantum walks, Quantum Information and Computation 11 (2011)

[5] K. Chisaki, et.al, Limit theorems for the discrete-time quantum walk on a graph with joined half lines, arXiv: 1009.1306 (2010)

[6] T. Machida, Limit theorems for a localization model of 2-state quantum walks, International Journal of Quantum Information, Vol.9, No.3, (2011)

[7] T. Machida, and N. Konno, Limit theorem for a time-dependent coined quantum walk on the line, F. Peper et al. (Eds.): IWNC 2009, Proceedings in Information and Communications Technology, Vol.2, (2010)

[8] F. Sato, and M. Katori, Dirac equation with ultraviolet cutoff and quantum walk, Phys. Rev. A81 012314, (2010)

[9] S. Salimi, Continuous-time quantum walks on star graphs, Annals of Physics 324 (2009)

[10] N. Konno, One-dimensional discrete-time quantum walks on random environments, Quantum Information Processing, Vol.8, No.5, (2009)

[11] K. Chisaki, et.al, Limit theorems for discrete-time quantum walks on trees, Interdisciplinary Information Sciences 15, (2009)
[12] A. Belton, Quantum random walks and thermalisation, arXiv: 0810.2927 (2009)

[13] E. Segawa, and N. Konno, Limit theorems for quantum walks driven by many coins, International Journal of Quantum Information, Vol.6, Issue 6, (2008)

[14] S. Salimi, Quantum central limit theorem for continuous-time quantum walks on odd graphs in quantum probability theory, Int J Theor Phys 47 (2008)

[15] M. A. Jafarizadeh, et al., Investigation of continuous-time quantum walk by using Krylov subspace-Lanczos algorithm, European Journal B 59 (2007)

[16] N. Konno, Continuous-time quantum walks on trees in quantum probability theory, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol.9, No.2, (2006)

[17] N. Konno, Limit theorem for continuous-time quantum walk on the line, Physical Review E, Vol.72, 026113 (2005)

[18] C. Ampadu, Limit theorems for quantum walks associated with Hadamard matrices, Physical Review A 84, 012324 (2011)

[19] N. Konno, Limit theorems and absorption problems for one-dimensional correlated random walks, Stochastic Models, Vol.25, Issue 1, (2009)

[20] M.C. Banuls, et al., Quantum walks with a time-dependent coin, Phys. Rev. A 73, 062304 (2006)

[21] N. Inui, and N. Konno, Localization of multi-state quantum walk in one dimension, Physica A (2005)

[22] N. Konno, Localization of an inhomogeneous discrete-time quantum walk on the line, Quantum Information Processing, 9 (2010)

[23] A. Wojcik, et al., Quasiperiodic dynamics of a quantum walk on the line, Phys.Rev.Lett, 93 (2004)

[24] A. Romanelli, The Fibonacci quantum walk and its classical trace map, Physica A, Volume 388, Issue 18 (2009)

[25] P. Riberio, et al., Phys.Rev.Lett. 93, 190503 (2004)

[26] A. Romanelli, et al., Phys.Rev.E 76, 037202 (2007)

[27] A. Romanelli, Phys.Rev.A 76, 054306 (2007)

[28] P. Levy, Theorie de l’Addition de Variables Aleatoires, Gauthier-Villiers, Paris (1937)

[29] A. Romanelli, et al., Phys.Lett.A 365, 200 (2007)

[30] S. Abe and H. Hiramoto, Phys.Rev.A 36, 5349 (1987)

[31] G. Grimmett, et al., Phys.Rev.E. 69, 026119 (2004)
