Strichartz inequalities for the Schrödinger equation with the full Laplacian on H-type groups

Heping Liu and Manli Song

Abstract

We prove the dispersive estimates and Strichartz inequalities for the solution of the Schrödinger equation related to the full Laplacian on H-type groups, by means of Besov spaces defined by a Littlewood-Paley decomposition related to the spectral of the full Laplacian. Let $p$ be the dimension of the center on those groups and we assume that $p > 1$. This requires a careful analysis due also to the non-homogeneous nature of the full Laplacian.

1 Introduction

The aim of this paper is to study Strichartz inequalities for the solution for the following Cauchy problem of the Schrödinger equation related to the full Laplacian on H-type groups $G$ with topological dimension $n$ and homogeneous dimension $N$:

\[
\begin{aligned}
\partial_t u - i\mathcal{L}u &= f \in L^1((0,T),L^2) \\
\big|_{t=0} u &= u_0 \in \dot{B}^{1/2}_{1,2}
\end{aligned}
\tag{1.1}
\]

where $\mathcal{L}$ is the full Laplacian on $G$ and the Besov spaces $\dot{B}^{\rho}_{q,r}(\mathcal{L})$ (written by $\dot{B}^{\rho}_{q,r}$ for short) are defined by a Littlewood-Paley decomposition related to the full Laplacian.

Strichartz estimates play an important role on the study of nonlinear partial differential equations. A key point consists in estimating the decay in time of the $L^\infty$ norm of the free solution. Together with a standard argument ([9], [14]), we can immediately get the Strichartz estimates.

In [1], H. Bahouri, P. Gérard et C.-J. Xu found sharp dispersive estimates and Strichartz inequalities for the Cauchy problem for the wave equation related to the Kohn-Laplacian $\Delta$ on the Heisenberg group, using the Besov spaces $\dot{B}^{\rho}_{q,r}(\Delta)$. Such estimates do not exist for the Schrödinger equation. To avoid the particular behaviour of the Schrödinger operator on the Heisenberg group. The sublaplacian have been replaced by the full Laplacian. The Cauchy problems for the wave and Schrödinger equation with the full Laplacian on the Heisenberg group were studied in [6] and [7] respectively. In the end of [6], the authors also gave a remark that
the behavior of the Schrödinger operator $e^{it\mathcal{L}}$ by analysing it with the space $\dot{B}^0_{q,r}(\Delta)$. Later, in [10], Martin Del Hierro generalized the dispersive and Strichartz estimates for the wave equation on H-type groups, using the Besov spaces $\dot{B}^0_{q,r}(\Delta)$. In a recent paper [16], H. Liu and M. Song have proved the Strichartz inequalities for the wave equation with the full Laplacian on H-type groups.

Our purpose is to show that the Schrödinger equation related to the full Laplacian on H-type groups is also dispersive, using the Besov space $\dot{B}^0_{q,r}$. In comparison with [10], the full Laplacian does not have the homogeneous properties, which involves some technical difficulties. Furthermore, in H-type groups, only the Heisenberg groups have a one dimensional center. In this article we only consider the H-type groups whose center is bigger than one, which makes the issue becomes very complicated.

It is well-known that the solution of the non-homogeneous equation (1.1) is given by the sum

$$ u = v + w$$

where

$$v(t) = e^{it\mathcal{L}}u_0$$

is the solution of (1.1) with $f = 0$ and

$$w(t) = \int_0^t e^{i(t-\tau)\mathcal{L}}f(\tau)\,d\tau$$

is the solution of (1.1) with $u_0 = 0$.

We can now state the main results of the paper. We first give the dispersive estimate on the free solution.

**Theorem 1.1** If $v$ is the free solution of the Schrödinger equation (1.1), then

$$||v(t)||_{L^\infty(G)} \leq C|t|^{-p/2}||u_0||_{\dot{B}^{n-1}_1},$$

and the result is sharp in time.

In comparison with the results for the Schrödinger equation solution in [7] and [10], we have obtained an improvement on the time decay, respectively due to the bigger size of the H-type group center and the replacement of the full Laplacian.

We also get a very useful estimate on the solution of the Schrödinger equation.

**Theorem 1.2** For $i = 1, 2$, let $q_i, r_i \in [2, \infty]$ and $\rho_i \in \mathbb{R}$ such that

a) $\frac{2}{q_i} = p(\frac{1}{2} - \frac{1}{r_i});$

b) $\rho_i = -(n-1)(\frac{1}{2} - \frac{1}{r_i}),$

except for $(q_i, r_i, p) = (2, \infty, 2)$. Let $q'_i, r'_i$ denote the conjugate exponent of $q_i$ and $r_i$. The solution of the Cauchy problem (1.1) $u$ satisfies the estimate

$$||u||_{L^{q_i}(0,T),\dot{B}^{r_i}_{q_i,2}} \leq C(||u_0||_{L^2(G)} + ||f||_{L^{q'_2}(0,T),\dot{B}^{-r'_2}_{q'_2,2}})$$

where the constant $C > 0$ does not depend on $u_0, f$ or $T$. 
2 H-type groups and spherical Fourier transform

2.1. H-type groups. Let $\mathfrak{g}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$. Its center is denoted by $\mathfrak{z}$. $\mathfrak{g}$ is said to be of H-type if $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and for every $s \in \mathfrak{z}$, the map $J_s : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_s u, w \rangle := \langle s, [u, w] \rangle, \forall u, w \in \mathfrak{z}^\perp$$

is an orthogonal map whenever $|s| = 1$.

An H-type group is a connected and simply connected Lie group $G$ whose Lie algebra is of H-type.

For a given $0 \neq a \in \mathfrak{z}^*$, the dual of $\mathfrak{z}$, we can define a skew-symmetric mapping $B(a)$ on $\mathfrak{z}^\perp$ by

$$\langle B(a) u, w \rangle = a([u, w]), \forall u, w \in \mathfrak{z}^\perp$$

We denote by $z_a$ the element of $\mathfrak{z}$ determined by

$$\langle B(a) u, w \rangle = a([u, w]) = \langle J_{za} u, w \rangle$$

Since $B(a)$ is skew symmetric and non-degenerate, the dimension of $\mathfrak{z}^\perp$ is even, i.e. $\dim \mathfrak{z}^\perp = 2d$.

For a given $0 \neq a \in \mathfrak{z}^*$, we can choose an orthonormal basis

$$\{E_1(a), E_2(a), \cdots, E_d(a), \overline{E}_1(a), \overline{E}_2(a), \cdots, \overline{E}_d(a)\}$$

of $\mathfrak{z}^\perp$ such that

$$B(a)E_i(a) = |z_a|J_{\frac{z_a}{|z_a|}}E_i(a) = |a|\overline{E}_i(a)$$

and

$$B(a)\overline{E}_i(a) = -|a|E_i(a)$$

We set $p = \dim \mathfrak{z}$. Throughout this paper we assume that $p > 1$. We can choose an orthonormal basis $\{e_1, e_2, \cdots, e_p\}$ of $\mathfrak{z}$ such that $a(e_1) = |a|, a(e_j) = 0, j = 2, 3, \cdots, p$. Then we can denote the element of $\mathfrak{g}$ by

$$(z, t) = (x, y, t) = \sum_{i=1}^{d} (x_i E_i + y_i \overline{E}_i) + \sum_{j=1}^{p} s_j e_j$$

We identify $G$ with its Lie algebra $\mathfrak{g}$ by exponential map. The group law on H-type group $G$ has the form

$$(z, s)(z', s') = (z + z', s + s' + \frac{1}{2} [z, z'])$$

(2.1)

where $[z, z']_j = \{z, U^j z'\}$ for a suitable skew symmetric matrix $U^j, j = 1, 2, \cdots, p$.

Theorem 2.1 $G$ is an H-type group with underlying manifold $\mathbb{R}^{2d+p}$, with the group law (2.1) and the matrix $U^j, j = 1, 2, \cdots, p$ satisfies the following conditions:

(i) $U^j$ is a $2d \times 2d$ skew symmetric and orthogonal matrix, $j = 1, 2, \cdots, p$.

(ii) $U^jU^i + U^iU^j = 0, i, j = 1, 2, \cdots, p$ with $i \neq j$. 
Proof. See [2].

Remark 2.1 It is well known that $H$-type algebras are closely related to Clifford modules (see [17]). $H$-type algebras can be classified by the standard theory of Clifford algebras. Specially, on $H$-type group $G$, there is a relation between the dimension of the center and its orthogonal complement space. That is $p + 1 \leq 2d$ (see [13]).

Remark 2.2 We identify $G$ with $\mathbb{R}^{2d} \times \mathbb{R}^p$. We shall denote the topological dimension of $G$ by $n = 2d + p$. Following Folland and Stein (see [4]), we will exploit the canonical homogeneous structure, given by the family of dilations $\{\delta_r\}_{r>0}$,

$$\delta_r(z, s) = (rz, r^2s).$$

We then define the homogeneous dimension of $G$ by $N = 2d + 2p$.

The left invariant vector fields which agree respectively with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{p} \left( \sum_{l=1}^{2d} z_l U^k_{l,j} \right) \frac{\partial}{\partial s_k},$$

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} \sum_{k=1}^{p} \left( \sum_{l=1}^{2d} z_l U^k_{l,j+d} \right) \frac{\partial}{\partial s_k},$$

where $z_l = x_l, z_{l+d} = y_l, l = 1, 2, \ldots, d$.

The vector fields $S_k = \frac{\partial}{\partial s_k}, k = 1, 2, \ldots, p$ correspond to the center of $G$. In terms of these vector fields we introduce the sublaplacian $\Delta$ and full Laplacian $L$ respectively

$$\Delta = -\sum_{j=1}^{n} (X_j^2 + Y_j^2) = -\Delta_z + \frac{1}{4}|z|^2 S - \sum_{k=1}^{p} \langle z, U^k \nabla_z \rangle S_k$$

$$L = \Delta + S.$$

2.2. Spherical Fourier transform. Korányi, Damek and Ricci (see [3] and [15]) have computed the spherical functions associated to the Gelfand pair $(G, O(22d))$ (we identify $O(d)$ with $O(2d) \otimes I_{dp}$). They involve, as on the Heisenberg group, the Laguerre functions

$$L_m^{(\alpha)}(\tau) = L_m^{(\alpha)}(\tau) e^{-\tau/2}, \tau \in \mathbb{R}, m, \alpha \in \mathbb{N}$$

where $L_m^{(\alpha)}$ is the Laguerre polynomial of type $\alpha$ and degree $m$.

We say a function $f$ on $G$ is radial if the value of $f(z, s)$ depends only on $|z|$ and $s$. We denote by $S_{rad}(G)$ and $L_{rad}^q(G), 1 \leq q \leq \infty$, the spaces of radial functions in $S(G)$ and in $L^p(G)$, respectively. In particular, the set of $L_{rad}^1(G)$ endowed with the convolution product

$$f_1 * f_2(g) = \int_G f_1(gg')^{-1} f_2(g') dg', g \in G.$$
is a commutative algebra.

Let \( f \in L^1_{\text{rad}}(G) \), we define the spherical Fourier transform

\[
\mathfrak{F}(f)(\lambda, m) = \hat{f}(\lambda, m) = \left( \frac{m + d - 1}{m} \right)^{-1} \int_{\mathbb{R}^{2d+p}} e^{i\lambda s} f(z, s) \mathcal{L}_m^{(d-1)} \left( \frac{|\lambda|}{2} |z|^2 \right) dz ds.
\]

By a direct computation, we have \( \mathfrak{F}(f_1 * f_2) = \mathfrak{F}(f_1) \cdot \mathfrak{F}(f_2) \). Thanks to a partial integration on the sphere \( S^{p-1} \) we deduce from the Plancherel theorem on the Heisenberg group its analogue for the H-type groups.

**Proposition 2.1** For all \( f \in S_{\text{rad}}(G) \), we have

\[
\sum_{m \in \mathbb{N}} \left( \frac{m + d - 1}{m} \right)^{-1} \int_{\mathbb{R}^p} \left| \hat{f}(\lambda, m) \right| |\lambda|^d d\lambda < \infty
\]

we have

\[
f(z, s) = \left( \frac{1}{2\pi} \right)^{d+p} \sum_{m \in \mathbb{N}} \int_{\mathbb{R}^p} e^{-i\lambda s} \hat{f}(\lambda, m) \mathcal{L}_m^{(d-1)} \left( \frac{|\lambda|}{2} |z|^2 \right) |\lambda|^d d\lambda
\]

(2.3)

the sum being convergent in \( L^\infty \) norm.

Moreover, if \( f \in S_{\text{rad}}(G) \), the functions \( Lf \) is also in \( S_{\text{rad}}(G) \) and its spherical Fourier transform is given by

\[
\mathfrak{F}(Lf)(\lambda, m) = ((2m + d)|\lambda| + |\lambda|^2) \hat{f}(\lambda, m).
\]

The full Laplacian \( L \) is a positive self-adjoint operator densely defined on \( L^2(G) \). So by the spectral theorem, for any bounded Borel function \( h \) on \( \mathbb{R} \), we have

\[
\mathfrak{F}(hf)(\lambda, m) = h((2m + d)|\lambda| + |\lambda|^2) \hat{f}(\lambda, m).
\]

### 3 Littlewood-Paley decomposition

In this paper we use the Besov spaces defined by a Littlewood-Paley decomposition related to the spectral of the full Laplacian \( L \). Let \( R \) be a non-negative, even function in \( C_0^\infty(\mathbb{R}) \) such that \( \text{supp} R \subseteq \{ \tau \in \mathbb{R} : \frac{1}{2} \leq |\tau| \leq 4 \} \) and

\[
\sum_{j \in \mathbb{Z}} R(2^{-2j}\tau) = 1, \forall \tau \neq 0.
\]

For \( j \in \mathbb{Z} \), we denote by \( \psi_j \) the kernel of the operator \( R(2^{-2j}L) \) and we set \( \Delta_j f = f * \psi_j \). As \( R \in C_0^\infty(\mathbb{R}) \), Hulanicki proved that \( \psi_j \in S_{\text{rad}}(G) \) (see [11]) and

\[
\mathfrak{F}(\psi_j)(\lambda, m) = R(2^{-2j}((2m + d)|\lambda| + |\lambda|^2)).
\]

By [5] (see Proposition 6), there exists \( C > 0 \) such that

\[
\|\psi_j\|_{L^1(G)} \leq C, \forall j \in \mathbb{Z}.
\]
By standard arguments (see [5], Proposition 9), we can deduce from (3.1) that
\[
\|L^{\sigma/2} \Delta_j f\|_{L^q(G)} \leq C 2^{j\sigma} \|\Delta_j f\|_{L^q(G)}, \sigma \in \mathbb{R}, j \in \mathbb{Z}, 1 \leq q \leq \infty, f \in \mathcal{S}'(G)
\] (3.2)
where both sides of (3.2) are allowed to be infinite.

By the spectral theorem, for any \(f \in L^2(G)\), the following homogeneous Littlewood-Paley decomposition holds:
\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } L^2(G)
\]
So
\[
\|f\|_{L^\infty(G)} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^\infty(G)}, f \in L^2(G),
\] (3.3)
where both sides of (3.3) are allowed to be infinite.

Let \(1 \leq q, r \leq \infty, \rho < N/q\), we define the homogeneous Besov space \(\dot{B}^\rho_{q,r}\) as the set of distributions \(f \in \mathcal{S}'(G)\) such that
\[
\|f\|_{\dot{B}^\rho_{q,r}} = \left(\sum_{j \in \mathbb{Z}} 2^{j\rho r} \|\Delta_j f\|_q^r\right)^{1/r} < \infty
\]
and \(f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'(G)\).

We collect in the following proposition all the properties we need about the spaces \(\dot{B}^\rho_{q,r}\).

**Proposition 3.1** Let \(q, r \in [1, \infty] \) and \(\rho < \frac{N}{q}\).
(i) The space \(\dot{B}^\rho_{q,r}\) is a Banach space with the norm \(\| \cdot \|_{\dot{B}^\rho_{q,r}}\);
(ii) the definition of \(\dot{B}^\rho_{q,r}\) does not depend on the choice of the function \(R\) in the Littlewood-Paley decomposition;
(iii) for \(-\frac{N}{q} < \rho < \frac{N}{q}\) the dual space of \(\dot{B}^\rho_{q,r}\) is \(\dot{B}^{-\rho}_{q',r'}\);
(iv) for \(\alpha \in [n, N]\) we have the continuous inclusion
\[
\dot{B}^{\rho_1}_{q_1,r} \subset \dot{B}^{\rho_2}_{q_2,r}, \quad \frac{1}{q_1} - \frac{\rho_1}{\alpha} = \frac{1}{q_2} - \frac{\rho_2}{\alpha}, \rho_1 \geq \rho_2;
\]
(v) for all \(q \in [2, \infty]\) we have the continuous inclusion \(\dot{B}^{0}_{q,2} \subset L^q\);
(vi) \(\dot{B}^{0}_{2,2} = L^2\);
(vii) for \(\theta \in [0, 1]\) we have
\[
[\dot{B}^{\rho_1}_{q_1,r_1}, \dot{B}^{\rho_2}_{q_2,r_2}]^\theta = \dot{B}^\rho_{q,r}
\]
with \(\rho = (1 - \theta)\rho_1 + \theta \rho_2, \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \text{ and } \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}\).

We omit the proof of the proposition which is analogous to Proposition 3.3 in [6].
4 Dispersive estimates

It is a very classical way to get a dispersive estimate if we want to reach Strichartz inequalities. Hence, first we want to do is to get a dispersive estimate $\|e^{it\Delta}\psi_j\|_{L^\infty(G)}$.

Our main tool is to apply oscillating integral estimates to the Schrödinger equation. First of all, we recall the stationary phase lemma (see [19], Chapter VIII):

Lemma 4.1 (Stationary phase estimate) Let $g \in C^\infty([a,b])$ be real-valued such that $|g''(x)| \geq \delta$ for any $x \in [a,b]$ with $\delta > 0$. Then for any function $h \in C^\infty([a,b])$, there exists a constant $C$ which does not depend on $\delta, a, b, g$ or $h$, such that

$$\left| \int_a^b e^{ig(x)}h(x) \, dx \right| \leq C\delta^{-1/2} \left[ \|h\|_\infty + \int_a^b |h'(x)| \, dx \right].$$

Next, we will need some estimates of the Laguerre functions:

Lemma 4.2

$$\left| (\tau \frac{d}{d\tau})^k \mathcal{L}_m^{(d-1)}(\tau) \right| \leq C_{k,d}(2m + d)^{d-1/4} \quad (4.1)$$

for all $0 \leq k \leq d$.

Proof. We refer the reader to the proof of Lemma 3.2 in [10].

Remark 4.1 In fact, for $0 \leq k \leq d - 1$, we have a better estimate

$$\left| (\tau \frac{d}{d\tau})^k \mathcal{L}_m^{(d-1)}(\tau) \right| \leq C_{k,d}(2m + d)^{d-1}. \quad (4.2)$$

Furthermore, we will exploit the following estimates, which can be easily proved by comparing the sums with the corresponding integrals:

Lemma 4.3 Fix $\beta \in \mathbb{R}$. There exists $C_\beta > 0$ such that for $A > 0$ and $d \in \mathbb{Z}_+$, we have

$$\sum_{m \in \mathbb{N}} \frac{1}{2m+d\geq A} (2m + d)^{\beta} \leq C_\beta A^\beta + 1, \quad \beta < -1; \quad (4.2)$$

$$\sum_{m \in \mathbb{N}} \frac{1}{2m+d\leq A} (2m + d)^{\beta} \leq C_\beta A^\beta + 1, \quad \beta > -1. \quad (4.3)$$

Finally, we introduce the following properties of the Fourier transform of surface-carried measures (see [18], Theorem 1.2.1):
Theorem 4.1 Let $S$ be a smooth hypersurface in $\mathbb{R}^p$ with non-vanishing Gaussian curvature and $d\mu$ a $C_0^\infty$ measure on $S$. Suppose that $\Gamma \subset \mathbb{R}^p \setminus \{0\}$ is the cone consisting of all $\xi$ which are normal of some point $x \in S$ belonging to a fixed relatively compact neighborhood $N$ of supp$\mu$. Then,

$$
\left( \frac{\partial}{\partial \xi} \right)^\alpha \hat{d}\mu(\xi) = O \left( (1 + |\xi|)^{-M} \right), \forall M \in \mathbb{N}, \text{ if } \xi \not\in \Gamma,
$$

and our assertion simply reads

$$
\hat{d}\mu(\xi) = \sum e^{-i(x,\xi)}a_j(\xi), \text{ if } \xi \in \Gamma,
$$

where the (finite) sum is taken over all points $x \in N$ having $\xi$ as a normal and

$$
\left| \left( \frac{\partial}{\partial \xi} \right)^\alpha a_j(\xi) \right| \leq C_\alpha (1 + |\xi|)^{-(p-1)/2 - |\alpha|}.
$$

Here, we need the following properties of the Fourier transform of the measure $d\sigma$ on the sphere $S^{p-1}$. Obviously, $d\sigma$ is radial. So we write $\hat{d}\sigma(\xi) = \hat{d}\sigma(\xi)$. By Theorem 4.1, we have the radical decay properties of the Fourier transform of the spherical measure.

Lemma 4.4 For any $r > 0$, the estimate holds

$$
\hat{d}\sigma(r) = e^{ir\phi_+}(r) + e^{-ir\phi_-}(r),
$$

where

$$
|\phi^{(k)}_\pm(r)| \leq c_k (1 + r)^{- (p-1)/2 - k}, \text{ for all } k \in \mathbb{N}.
$$

We can now prove the following:

Lemma 4.5 There exists a $C > 0$, which depends only on $d$ and $p$, such that for any $\rho \in [n - 1, N - 2]$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we have:

$$
||e^{it\mathcal{L}}\psi_j||_{L^\infty(G)} \leq C|t|^{-1/2}2^{j\rho}.
$$

Proof. Fixing $t \in \mathbb{R}^*, j \in \mathbb{Z}$ and $(z, s) \in G$, by the inversion Fourier formula, we have

$$
e^{it\mathcal{L}}\psi_j(z, s) = \left( \frac{1}{2\pi} \right)^{d+p} \int_{\mathbb{R}^p} e^{-i\lambda s} e^{it((2m+d)|\lambda|+|\lambda|^2)}
$$

$$
\times R \left( 2^{-2j} \left( (2m + d)|\lambda| + |\lambda|^2 \right) \right) \mathcal{L}_m^{(d-1)} \left( \frac{|\lambda|}{2}|z|^2 \right) |\lambda|^d d\lambda
$$

$$
= \left( \frac{1}{2\pi} \right)^{d+p} \sum_{m \in \mathbb{N}} I_m
$$

where

$$
I_m = \int_{\mathbb{R}^p} e^{-i\lambda s} e^{it((2m+d)|\lambda|+|\lambda|^2)} R \left( 2^{-2j} \left( (2m + d)|\lambda| + |\lambda|^2 \right) \right) \mathcal{L}_m^{(d-1)} \left( \frac{|\lambda|}{2}|z|^2 \right) |\lambda|^d d\lambda
$$

and our assertion simply reads

$$
\sum_{m \in \mathbb{N}} |I_m| \lesssim \begin{cases} 
|t|^{-1/2}2^{j(2d+p-1)}, & j > 0 \\
|t|^{-1/2}2^{j(2d+2p-2)}, & j \leq 0.
\end{cases}
$$
Putting $\sigma = \frac{2}{j}$ and $M = 2m + d$, we first integrate on $\mathbb{R}^+$, then

$$ I_m = \int_{\mathbb{R}^p} e^{it(M|\lambda| + |\lambda|^2 - \lambda \cdot \sigma)} \mathcal{R}(2^{-2j}(M|\lambda| + |\lambda|^2)) \mathcal{S}_m^{(d-1)}(\frac{|\lambda|}{2}|z|^2)|\lambda|^d \ d\lambda $$

$$ = \int_{S^{p-1}} I_{\epsilon,m} \ d\sigma(\epsilon) $$

where

$$ I_{\epsilon,m} = \int_0^{+\infty} e^{it(M\lambda + \lambda^2 - \lambda \cdot \epsilon \cdot \sigma)} \mathcal{R}(2^{-2j}(M\lambda + \lambda^2)) \mathcal{S}_m^{(d-1)}(\frac{\lambda}{2}|z|^2)\lambda^{d+p-1} \ d\lambda. $$

Performing the change of variable $x = 2^{-2j}M\lambda$, we obtain

$$ I_{\epsilon,m} = 2^{j(2d+2p)}K_{\epsilon,m} $$

where

$$ K_{\epsilon,m} = \int_0^{+\infty} e^{i2jG_{j,\sigma,\epsilon,m}(x)} h_{j,z,m}(x) \ dx $$

(4.4)

Here,

$$ G_{j,\sigma,\epsilon,m}(x) = x + \frac{2^{2j}}{M^2}x^2 - \frac{\epsilon \cdot \sigma}{M}x $$

$$ h_{j,z,m}(x) = R(x + \frac{2^{2j}}{M^2}x^2) \mathcal{S}_m^{(d-1)}(\frac{2^{2j-1}x|z|^2}{M})x^{d+p-1} \ M^{d-p}. $$

So

$$ \text{supp} \ h_{j,z,m} \subseteq \{ x \in \mathbb{R}^+ : \frac{1}{2} \leq x + \frac{g_{2j}}{M^2}x^2 \leq 4 \} = [a_{j,m}, b_{j,m}] $$

where

$$ a_{j,m} = \frac{1}{1 + \sqrt{1 + 2^{2j+1}M^{-2}}} \quad b_{j,m} = \frac{8}{1 + \sqrt{1 + 2^{2j+4}M^{-2}}} $$

Note that

$$ a_{j,m}, b_{j,m} \sim \min(1, 2^{-j}M). $$

(4.5)

For $x \in [a_{j,m}, b_{j,m}]$, we have

$$ G''_{j,\sigma,\epsilon,m}(x) = \frac{2^{2j+1}}{M^2} $$

(4.6)

Moreover, by Lemma 4.2 and (4.5), one can easily verify that

$$ ||h_{j,z,m}||_{L^\infty[a_{j,m}, b_{j,m}]} + ||h'_{j,z,m}||_{L^1[a_{j,m}, b_{j,m}]} \lesssim \begin{cases} M^{-\frac{(p+1)}{2}}, & M \geq 2^j, \\ 2^{-j(d+p-1)}M^{d-2}, & M < 2^j. \end{cases} $$

Applying the stationary phase lemma 4.1, we obtain a consistent estimate

$$ |K_{\epsilon,m}| \lesssim \begin{cases} |t|^{-1/2}2^{-2j}M^{-p}, & M \geq 2^j, \\ |t|^{-1/2}2^{-j(d+p+1)}M^{d-1}, & M < 2^j. \end{cases} $$

Hence, we have

$$ |I_m| \lesssim \begin{cases} |t|^{-1/2}2^{(2d+2p-2)}M^{-p}, & M \geq 2^j, \\ |t|^{-1/2}2^{j(d+p+1)}M^{d-1}, & M < 2^j. \end{cases} $$

(4.7)
Noting that \( p > 1 \), for \( j \leq 0 \), we have \( \sum_{m \in \mathbb{N}} |I_m| \lesssim |t|^{-1/2} 2^j (2d+2p-2) \). For \( j > 0 \), \( \sum_{m \in \mathbb{N}} |I_m| \lesssim |t|^{-1/2} 2^j (2d+p-1) \) follows from (4.7) by applying Lemma 4.3 separately to the sums \( \sum_{M \geq 2^j} |I_m| \) and \( \sum_{M < 2^j} |I_m| \).

Next, we integrate first over \( S^{p-1} \) to estimate \( I_m \),

\[
I_m = \int_0^{+\infty} \tilde{d}\sigma(\lambda|s|) e^{it(M\lambda + \lambda^2)} R(2^{-2j}(M\lambda + \lambda^2)) \xi^{(d-1)}(\frac{\lambda}{2}|s|^2) \lambda^{d+p-1} d\lambda.
\]

Performing the change of variable \( x = 2^{-2j} M\lambda \), we obtain

\[
I_m = 2^{j(2d+2p)} \int_0^{+\infty} e^{it2^{2j}(x + \frac{2j}{M} x^2 \pm |x|^{M/2})} \tilde{d}\sigma(\frac{2^{2j}|s|x}{M}) h_{j,z,m}(x) dx \quad (4.8)
\]

It follows from Lemma 4.4 that

\[
I_m = (2\pi)^{p/2} \sum_{\pm} I^\pm_m, \quad (4.9)
\]

where

\[
I^\pm_m = 2^{j(2d+2p)} \int_0^{+\infty} e^{it2^{2j}(x + \frac{2j}{M} x^2 \pm |x|^{M/2})} \phi_{\pm}(\frac{2^{2j}|s|x}{M}) h_{j,z,m}(x) dx
\]

From Lemma 4.2, Lemma 4.4 and (4.9) one can verify that

\[
\| \phi_{\pm}(\frac{2^{2j}|s|x}{M}) h_{j,z,m} \|_{L^\infty[a_j,m,b_j,m]} + \| \frac{\partial}{\partial x}(\phi_{\pm}(\frac{2^{2j}|s|x}{M}) h_{j,z,m}) \|_{L^1[a_j,m,b_j,m]} \lesssim \left\{ \begin{array}{ll}
|s|^{-p-1/2} & M \geq 2j, \\
|s|^{-p-1/2-3/2} & M < 2j \end{array} \right.
\]

Exploiting the stationary phase lemma 4.4 we obtain

\[
|I_m^\pm| \lesssim \left\{ \begin{array}{ll}
|t|^{-1/2} |s|^{-p-1/2} 2^{j(2d+p-1)} M^{-p+1/2} & M \geq 2j, \\
|t|^{-1/2} |s|^{-p-1/2} 2^{j(d+p-1/2)} M^{-d-1} & M < 2j \end{array} \right.
\]

To improve the time decay, we will try to apply \( p \) times a non-critical phase estimate. We will exploit the following estimates for the derivatives of \( h_{j,z,m} \):

**Lemma 4.6** For any \( x \in [a_j,m,b_j,m] \), \( 0 \leq l \leq d \), we have

\[
|h_{j,z,m}^{(l)}(x)| \lesssim \left\{ \begin{array}{ll}
M^{-(p+\theta_l)}, & M \geq 2j, \\
2^{-j(d+p-1)} M^{d-l-\theta_l-1}, & M < 2j \end{array} \right.
\]

where \( \theta_l = \left\{ \begin{array}{ll}
1, & 0 \leq l \leq d-1 \\
1/4, & l = d \end{array} \right. \).
Proof. Recall that
\[ h_{j,z,m}(x) = R(x + \frac{2^j}{M^2} x^2) Q_m^{(d-1)}(\frac{2^{2j-1}|x|^2}{M^2}) x^{d+p-1} \]
By an induction we get
\[ h_{j,z,m}^{(l)}(x) = \sum_{\alpha \in F} A(l, \alpha) R^{(\alpha_1)}(x + \frac{2^j}{M^2} x^2) (1 + \frac{2^{2j+1}}{M^2} x)^{\alpha_2} (\frac{2^{2j+1}}{M^2})^{\alpha_3} \]
\[ \times \left[ (x \frac{d}{dx})^{\alpha_4} Q_m^{(d-1)}(\frac{2^{2j-1}|x|^2}{M^2}) x^{d+p-\alpha_5-1} \right] \]
where \( F = \{ \alpha = (\alpha_1, \ldots, \alpha_5) \in \mathbb{N}^5 : \alpha_1 = \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_5 = l, \alpha_4 \leq \alpha_5 \} \).
Applying Lemma 4.6 comes out easily.

We can now prove the following:

**Lemma 4.7** There exists a \( C > 0 \), which depends only on \( d \) and \( p \), such that for any \( j \in \mathbb{Z} \) and \( t \in \mathbb{R}^+ \) we have:
\[ ||e^{it\mathcal{L}} \psi_j||_{L^\infty(G)} \leq C |t|^{-p/2} 2^{j(n-1)}. \]

**Proof.** From Lemma 4.5 it suffices to prove the case \( |t| > 1 \). Without loss of generality, we can assume that \( t > 1 \).

For \( j > 0 \), recall from (4.4) that
\[ K_{\epsilon,m} = \int_0^{+\infty} e^{it2^j G_{j,\sigma,\epsilon,m}(x)} h_{j,z,m}(x) dx, \]
where
\[ G_{j,\sigma,\epsilon,m}(x) = 1 + \frac{2^{2j+1}}{M^2} x - \epsilon \cdot \sigma \]
and
\[ G_{j,\sigma,\epsilon,m}''(x) = \frac{2^{2j+1}}{M^2} \]
We divide \( \mathbb{N} \) into three (possible empty) disjoint subsets:
\[ A_1 = \{ m \in \mathbb{N} : M \geq 2^j, |\sigma| \leq M \}, \]
\[ A_2 = \{ m \in \mathbb{N} : M \geq 2^j, |\sigma| \geq M \}, \]
\[ A_3 = \{ m \in \mathbb{N} : M < 2^j \}. \]
Then our assertion reads:
\[ \sum_{m \in A_l} |I_m| \lesssim t^{-p/2} 2^{j(2d+p-1)}, l = 1, 2, 3. \]
For \( l = 1 \), by (4.11), we obtain
\[ |G_{j,\sigma,\epsilon,m}(x)| \gtrsim 1, \text{ for any } x \in [a_j,m, b_j,m]. \]
The phase function $G_{j,\sigma,\epsilon,\mu}(x)$ for $K_{\epsilon,\mu}$ has no critical points on $[a_{j,\mu}, b_{j,\mu}]$. By $Q$-fold integration by parts, we get

$$K_{\epsilon,\mu} = (it^{2j})^{-Q} \int_{0}^{+\infty} e^{it^{2j}G_{j,\sigma,\epsilon,\mu}(x)} D^Q h_{j,z,\mu}(x) \, dx,$$

(4.14)

where the differential operator $D$ is defined by

$$Dh_{j,z,\mu}(x) = \frac{d}{dx} \left( h_{j,z,\mu}(x) \frac{G'_{j,\sigma,\epsilon,\mu}(x)}{G_{j,\sigma,\epsilon,\mu}(x)} \right).$$

By a direct induction, we have

$$D^Q h_{j,z,\mu}(x) = \sum_{k=Qa_1+2a_2=k}^{2Q} \sum_{\alpha} C(\alpha, k, Q) \frac{h^{(\alpha_1)}_{j,z,\mu}(x)}{(G'_{j,\sigma,\epsilon,\mu})^k},$$

with $\alpha = (\alpha_1, \alpha_2) \in \{0, 1, \cdots, Q\} \times \mathbb{N}$.

The estimates (4.12) and (4.13) yield

$$||D^Q h_{j,z,\mu}||_{\infty} \lesssim \sup_{0 \leq \alpha_1 \leq Q} ||h^{(\alpha_1)}_{j,z,\mu}||_{\infty}.$$

Applying Lemma 4.6 for all $0 \leq Q \leq d$, we obtain

$$||D^Q h_{j,z,\mu}||_{\infty} \lesssim \sup_{0 \leq \alpha_1 \leq Q} ||h^{(\alpha_1)}_{j,z,\mu}||_{\infty} \lesssim M^{-(p+1/4)},$$

(4.15)

which yields the trivial estimate

$$|K_{\epsilon,\mu}| \lesssim M^{-(p+1/4)}.$$ 

(4.16)

Moreover, it follows from (4.5), (4.14) and (4.15) that, for any $0 \leq Q \leq d$, we get a uniform estimate (with respect to $\epsilon \in S^{p-1}$)

$$|K_{\epsilon,\mu}| \lesssim (t2j)^{-Q} M^{-(p+1/4)}.$$ 

(4.17)

Interpolating (4.10) and (4.17), we get that for all $0 \leq \theta \leq d$

$$|K_{\epsilon,\mu}| \lesssim (t2j)^{-\theta} M^{-(p+1/4)}.$$ 

Since $p \leq 2d - 1$, we have $(p + 1)/2 \leq d$. Hence, let $\theta = (p + 1)/2$,

$$|K_{\epsilon,\mu}| \lesssim t^{-p/2} 2^{-j(p+1)} M^{-(p+1/4)}.$$ 

Finally, the desired estimate holds

$$\sum_{m \in A_1} |I_m| \lesssim t^{-p/2} 2^{j(2d+p-1)} \sum_{m \in \mathbb{N}} M^{-(p+1/4)} \lesssim t^{-p/2} 2^{j(2d+p-1)}.$$

For $l = 2$, the estimate (4.10) yields

$$|I_m^\pm| \lesssim t^{-p/2} 2^{j(2d+p-1)} M^{-p}.$$
Then it follows from (4.9) that
\[ \sum_{m \in A_2} |I_m| \lesssim t^{-p/2} 2^{j((2d+p-1)} \sum_{m \in \mathbb{N}} M^{-p} \lesssim t^{-p/2} 2^{j(2(d+p-1)}. \]

For \( l = 3 \), when \( |\sigma| \gtrsim 2^j \), the estimate (4.10) yields
\[ |I_m^\pm| \lesssim t^{-p/2} 2^{j} M^{d-1}. \]

Thanks to (4.3), we have
\[ \sum_{m \in A_3} |I_m| \lesssim t^{-p/2} 2^{jd} \sum_{M < 2^j} M^{d-1} \lesssim t^{-p/2} 2^{jd} \lesssim t^{-p/2} 2^{j((2d+p-1)}. \]

When \( |\sigma| \lesssim 2^j \), the estimate
\[ |G'_{j,\sigma,\epsilon,m}(x)| \gtrsim \frac{2^j}{M} \] (4.18)
holds for any \( x \in [a_{j,m}, b_{j,m}] \).

According to Lemma 4.6, for any \( 0 \leq \alpha_1 \leq d \)
\[ ||h_{j,z,m}^{\alpha_1}||_\infty \lesssim 2^{-j(d+p-1)} \left( \frac{2^j}{M} \right)^{\alpha_1} M^{d-5/4} \] (4.19)
It implies the following trivial estimate
\[ |K_{\epsilon,m}| \lesssim 2^{-j(d+p-1)} M^{d-5/4}. \] (4.20)

Furthermore, analogous to the case \( r = 1 \), (4.12), (4.18) and (4.19) yield, for any \( 0 \leq Q \leq d \)
\[ |D^Q h_{j,z,m}| \lesssim \sum_{k=Q,2 \leq k} \sum_{\alpha_1+\alpha_2=k} |h_{j,z,m}^{\alpha_1}| |G'_{j,\sigma,\epsilon,m}^{\alpha_2}| k \lesssim 2^{-j(d+p-1)} M^{d-5/4} \]
We can also obtain a uniform estimate
\[ |K_{\epsilon,m}| \lesssim (t 2^{j})^{-Q} 2^{-j(d+p-1)} M^{d-5/4}. \] (4.21)

Interpolating (4.20) and (4.21), for any \( 0 \leq \alpha_1 \leq d \)
\[ |K_{\epsilon,m}| \lesssim (t 2^{j})^{-\theta} 2^{-j(d+p-1)} M^{d-5/4}. \]

Let \( \theta = p/2 \)
\[ |K_{\epsilon,m}| \lesssim t^{-p/2} 2^{-j(d+p-1)} M^{d-5/4}. \]

Finally, because of (4.3) and \( p \geq 2 \), the desired estimate holds
\[ \sum_{m \in A_3} |I_m| \lesssim t^{-p/2} 2^{j(d+1)} \sum_{M < 2^j} M^{d-5/4} \lesssim t^{-p/2} 2^{j(2d+3/4)} \lesssim t^{-p/2} 2^{j(2d+p-1)}. \]
For \( j \leq 0 \), recall from (4.8) that
\[
I_m = 2^{j(2d+2p)} \int_0^{+\infty} e^{it2^j \varphi_{j,m}(x)} H_{j,z,s,m}(x) \, dx
\]
where
\[
\varphi_{j,m}(x) = x + \frac{2^{2j}}{M^2} x^2
\]
\[
H_{j,z,s,m}(x) = \hat{d}\sigma \left( \frac{2^j |s| x}{M} \right) h_{j,z,m}(x)
\]
First we obtain a trivial estimate
\[
|I_m| \lesssim 2^{j(2d+2p)} M^{-(p+1)}
\] (4.22)
We will discuss it in the following cases.

**Case 1.** \( \frac{2^j |s|}{M} \leq 1 \). In this case, we will exploit the vanishing property of the Fourier transform of the spherical measure at the origin. One can easily get from Lemma 4.4 that for any \( k \in \mathbb{N} \),
\[
\left| \frac{\partial^k}{\partial x^k} \left( \hat{d}\sigma \left( \frac{2^j |s| x}{M} \right) \right) \right| \leq c_k.
\] (4.23)
Analogous to \( r = 1 \) when dealing with \( j > 0 \), using integration by parts, for any \( 0 \leq Q \leq d \),
\[
I_m = 2^{j(2d+2p)} (it2^j)^{-Q} \sum_{k=Q\alpha_1+2\alpha_2=k}^{2Q} \sum_{\alpha_1=0}^{2Q} C(\alpha, k, Q) \int_0^{+\infty} e^{it2^j \varphi_{j,m}(x)} H_{j,z,s,m}^{(\alpha_1)}(x) (\varphi'_{j,m}(x))^{\alpha_2} \left( \frac{2^j |s| x}{M} \right)^{\alpha_2} \, dx
\]
with \( \alpha = (\alpha_1, \alpha_2) \in \{0, 1, \ldots, Q\} \times \mathbb{N} \).

It follows from (4.3), Lemma 4.6 and (4.4) that
\[
||H_{j,z,s,m}^{(\alpha_1)}||_{\infty} \lesssim M^{-(p+1/4)}, \text{ for any } 0 \leq \alpha_1 \leq d.
\] (4.24)
Note that
\[
\varphi'_{j,m}(x) \gtrsim 1
\] (4.25)
\[
\varphi''_{j,m}(x) = \frac{2^{2j}}{M^2}.
\] (4.26)
So (4.3), (4.24), (4.25) and (4.26) imply that, for any \( 0 \leq Q \leq d \),
\[
|I_m| \lesssim 2^{j(2d+2p)} (it2^j)^{-Q} M^{-(p+1/4)}
\] (4.27)
Interpolating (4.22) and (4.27), we get that for any \( 0 \leq \theta \leq d \),
\[
|I_m| \lesssim 2^{j(2d+2p)} (it2^j)^{-\theta} M^{-(p+1/4)}
\]
Let \( \theta = p/2 \),
\[
|I_m| \lesssim t^{-p/2} 2^{j(2d+2p)} M^{-(p+1/4)}
\]
There exists a \( C > 0 \), which depends only on \( d \) and \( p \), such that for any \( t \in \mathbb{R}^* \) and \( u_0 \in \mathcal{S}(G) \) we have:

\[
\|e^{it\Delta} u_0\|_{L^\infty(G)} \leq C|t|^{-p/2}\|u_0\|_{B^s_{1,1}}^{n-1}
\] (4.29)
We can obtain Corollary 4.1 by the same proof as in [7] (Corollary 10).

In the end of the section, let us show as in [10] the sharpness of the time decay in Corollary 4.1. First we recall the asymptotic expansion of oscillating integrals:

**Proposition 4.1** Suppose \( \phi \) is a smooth function on \( \mathbb{R}^p \) and has a nondegenerate critical point at \( x_0 \). If \( \psi \) is supported in a sufficiently small neighborhood of \( x_0 \), then

\[
\left| \int_{\mathbb{R}^p} e^{it\phi(x)} \psi(x) \, dx \right| \sim |t|^{-p/2}, \quad \text{as } t \to \infty.
\]

A proof can be found in [19] (Proposition 6, p.344).

Let \( Q \in C_0^\infty(D_0) \) with \( Q(\cdot|\lambda|) \delta_{m,0} \) determines a solution of the Cauchy problem (4.1) with \( f = 0 \),

\[
u((z,s),t) = C \int_{\mathbb{R}^p} e^{it(d|\lambda| + |\lambda|^2 - |\lambda|^2)} Q(|\lambda|)|\lambda|^d \, d\lambda.
\]

Consider \( u((0,ts_0),t) \) for a fixed \( s_0 \) such that \( |s_0| = 3d \).

\[
u((0,ts_0),t) = C \int_{\mathbb{R}^p} e^{it(d|\lambda| + |\lambda|^2 - \lambda \cdot s_0)} Q(|\lambda|)|\lambda|^d \, d\lambda.
\]

This oscillating integral has a phase \( \phi(\lambda) := d|\lambda| + |\lambda|^2 - \lambda \cdot s_0 \) with a unique critical point \( \lambda_0 = \frac{s_0}{3} \) which is not degenerate. Indeed, the Hessian is equal to

\[
H(\lambda) = \frac{1}{|\lambda|} \left( (2|\lambda| + d) \delta_{k,l} - \frac{\lambda_k \lambda_l}{|\lambda|^2} \right)_{1 \leq k,l \leq p}
\]

Let \( s_0 = (0, \ldots, 0, 3d) \), so \( \lambda_0 = \frac{s_0}{3} = (0, \ldots, 0, d) \). The Hessian at \( \lambda_0 \) is

\[
H(\lambda_0) = \left[ \begin{array}{cccc}
3 & & \\
& \ddots & \\
& & 3
\end{array} \right]
\]

Applying asymptotic expansion of oscillating integrals, we get

\[
u((0,ts_0),t) \sim |t|^{-p/2}.
\]

**5 Strichartz estimates**

We are now to prove our Strichartz estimates.
Theorem 5.1 For \( i = 1, 2 \), let \( q_i, r_i \in [2, \infty) \) and \( \rho_i \in \mathbb{R} \) such that
\[
a) \quad \frac{2}{q_i} = p \left( \frac{1}{2} - \frac{1}{r_i} \right);
\]
\[
b) \quad \rho_i = -(n-1) \left( \frac{1}{2} - \frac{1}{r_i} \right),
\]
except for \((q_i, r_i, p) = (2, \infty, 2)\). Then the following estimates are satisfied:
\[
\| e^{it\mathcal{L}} u_0 \|_{L^{q_1}(\mathbb{R}, \dot{B}^{\rho_1}_{r_1,2})} \leq C \| u_0 \|_{L^2(G)},
\]
\[
\| \int_0^t e^{i(t-\tau)\mathcal{L}} f(\tau) \, d\tau \|_{L^{q_1}((0,T), \dot{B}^{\rho_1}_{r_1,2})} \leq C \| f \|_{L^{q_2'}((0,T), \dot{B}^{-\rho_2'}_{r_2,2})},
\]
where the constant \( C > 0 \) does not depend on \( u_0 \), \( f \) or \( T \).

Once we have obtained the estimate in Lemma 4.7, the proof is classical and a good reference is, for example, the papers by Ginibre and Velo [9] or by Keel and Tao [14]. A detailed presentation in this framework is also given by [7] in the proof of Theorem 11.

Going back to the Schrödinger equation (1.1), by theorem 5.1, Theorem 1.2 is straightforward.

Remark 5.1 We compare the results by G. Furioli and A. Veneruso [7]. Theorem 1.1 and Theorem 1.2, which are general results on H-type groups with the center dimension \( p \geq 2 \), are also compatible with those on the Heisenberg group. Hence, the results in our paper apply to all the H-type groups.

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Heping Liu  
School of Mathematical Sciences  
Peking University  
Beijing 100871  
People’s Republic of China  
E-mail address: hpliu@pku.edu.cn

Manli Song  
School of Mathematical Sciences  
Peking University  
Beijing 100871  
People’s Republic of China  
E-mail address: songmanli@pku.edu.cn