A (2+1)-dimensional integrable spin model (the M-XXII equation) and Differential geometry of curves/surfaces

R. Myrzakulov

Centre for Nonlinear Problems, PO Box 30, 480035, Alma-Ata-35, Kazakhstan

Abstract

Using the differential geometry of curves and surfaces the Lakshmanan equivalent counterpart of the M-XXII equation is found. It is shown that these equations are too gauge equivalent to each other. Also the gauge equivalence between the Strachan and M-XXIIq equations is established. Some integrals of motion are presented. It is well known that the spectral parameter of the some (2+1)-dimensional soliton equations satisfies the following equations: $\lambda_t = \kappa \lambda^n \lambda_y$ (a nonisospectral problems). We present the simplest exact solutions of this equation.
1. Introduction

Dynamics of numerous nonlinear phenomena can be modelled by the nonlinear partial differential equations (NPDE), which describe an evolution of curves and surfaces [1-6]. A remarkable subclass of NPDE are soliton equations. The soliton theory as a new branch of mathematical physics, has been developed during the last three decades. Solitons has found an enormous variety of applications in various branches of science: biology, chemistry, physics, geometry and so on [7]. Recently many efforts have been made to generalize the soliton theory to the (2+1)-dimensional case [8-11]. In contrast with the (1+1)-dimensional case, the (2+1)-dimensional soliton equations display a richer phenomenology.

Integrable spin systems, besides being mathematically beautiful, have the important physical applications. It is well known that in the study of NPDE, the gauge [12] and Lakshmanan [3,13] equivalences take place between the some NPDE. Recently a new class integrable and nonintegrable spin systems including a multidimensional systems and the (2+1)-dimensional version of Lakshmanan equivalence were presented [14,18,19]. In this letter we will find the Lakshmanan and gauge equivalent counterpart of one of these systems - the Myrzakulov XXII (M-XXII) equation.

2. The Myrzakulov XXII equation

Consider the M-XXII equation [14]

\[ -iS_t = \frac{1}{2} (\{S, S_y\} + 2iuS)_x + \frac{i}{2} V_1 S_x - 2ib^2 S_y \]  

\[ u_x = -\tilde{S} (\tilde{S}_x \wedge \tilde{S}_y), \quad V_{1x} = \frac{1}{4b^2} (\tilde{S}_x)_y. \]

(1a) (1b)

where \( \tilde{S} = (S_1, S_2, S_3) \) is a spin vector, \( \tilde{S}^2 = E = \pm 1 \). These equations are integrable. The corresponding Lax representation is given by [14]

\[ \Phi_x = \{-i(\lambda^2 - b^2)S + \frac{\lambda - b}{2b} SS_x\} \Phi \]  

\[ \Phi_t = 2\lambda^2 \Phi_y + \{(\lambda^2 - b^2)(2A + B) + (\lambda - b)C\} \Phi \]

(2a) (2b)

with

\[ A = \frac{1}{4} ([S, S_y] + 2iuS) + \frac{i}{4} V_1 S, \quad B = \frac{i}{2} V_1 S, \quad C = -\frac{V_1}{4b^2} SS_x + \frac{i}{2b} \{S_{xy} - [S_x, A]\}. \]

Here spin matrix has the form

\[ S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^2 = I, \quad r^2 = \pm 1, \quad S^\pm = S_1 \pm iS_2. \]

3. The Lakshmanan equivalent counterpart of the Myrzakulov XXII equation
Now find the Lakshmanan equivalent counterpart of the M-XXII equation (1) for the case $E = +1$ (for the case $E = -1$ see, e.g., [14]). To this end we use the two geometrical approaches (D- and C-approaches).

a) The D-approach. Consider a space $\mathbb{R}^3$ with an orthonormal basis $\vec{e}_i, i = 1, 2, 3$. In this space consider the motion of curves. Model of this curves are given by[14]

$$
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_x = C
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix},
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_y = D
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix},
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_t = G
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}
$$

(3)

with

$$
C = \begin{pmatrix}
0 & k & 0 \\
-Ek & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix},
D = \begin{pmatrix}
0 & m_3 & -m_2 \\
E^{-1}m_3 & 0 & m_1 \\
Em_2 & -m_1 & 0
\end{pmatrix},
G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-E\omega_3 & 0 & \omega_1 \\
E\omega_2 & -\omega_1 & 0
\end{pmatrix}.
$$

Hence we have

$$
C_y - D_x + [C, D] = 0, \quad C_t - G_x + [C, G] = 0, \quad D_t - G_y + [D, G] = 0. \quad (4)
$$

From the first equation we obtain

$$
(m_1, m_2, m_3) = (\partial_x^{-1}(\tau_y + km_2), m_2, \partial_x^{-1}(k_y - \tau m_2)), \quad m_2 = \partial_x^{-1}(\tau m_3 - km_1)
$$

(5)

Now let $\vec{\mathcal{S}} = \vec{e}_1$. Then from equations (1) and (3) we get $\omega_j$. Let us now introduce the new function $q$ by

$$
q = \frac{k}{2b} \exp \left[ \frac{1}{8} \partial_x^{-1}(k^2b^{-2} - 4\tau) - 2b^2x \right] \quad (6)
$$

Then the function $q$ satisfies the following equations[14]

$$
iq_t + q_{yx} + \frac{i}{2} [(V_1q)_x - V_2q - qpq_y] = 0 \quad (7a)
$$

$$
ipt - pq_{yx} + \frac{i}{2} [(V_1p)_x + V_2p - qpp_y] = 0 \quad (7b)
$$

$$
V_{1x} = (pq)_y, \quad V_{2x} = pq_{yx} \quad (7c)
$$

where $p = Eq$. This set of equations is the L-equivalent counterpart of the M-XXII equation (1). As it seems to us, equations (7) are new. We will call (7) the M-XXII$\xi$ equation.

b) The C-approach. Note that the L-equivalent counterpart of equation (1) we can find also using the surface approach, e.g., the C-approach[14]. Let us show it. Consider the motion of surface in the 3-dimensional space which generated by a position vector $\vec{r}(x, y, t) = \vec{r}(x^1, x^2, t)$. According to the C-approach[14], $x$ and $y$ are local coordinates on the surface. The first and second fundamental forms in the usual notation are given by

$$
I = d\vec{r}d\vec{r} = Edx^2 + 2Fdx\,dy + Gdy^2, \quad II = -d\vec{\mathcal{N}}d\vec{\mathcal{N}} = Ldx^2 + 2Mdx\,dy + Ndy^2
$$

(8)
where
\[ E = r_x r_x = g_{11}, \quad F = r_x r_y = g_{12}, \quad G = r_y^2 = g_{22}, \]
\[ L = \bar{n} \bar{n} r_{xx} = b_{11}, \quad M = \bar{n} \bar{n} r_{xy} = b_{12}, \quad N = \bar{n} \bar{n} r_{yy} = b_{22}, \quad \bar{n} = \frac{(\bar{r}_x \wedge \bar{r}_y)}{||\bar{r}_x \wedge \bar{r}_y||}. \]

In this case, the set of equations of the C-approach[14], becomes
\[ \bar{r}_t = W_1 \bar{r}_x + W_2 \bar{r}_y + W_3 \bar{n} \]
\[ (9a) \]
\[ \bar{r}_{xx} = \Gamma_{11}^1 \bar{r}_x + \Gamma_{11}^2 \bar{r}_y + L \bar{n} \]
\[ (9b) \]
\[ \bar{r}_{xy} = \Gamma_{12}^1 \bar{r}_x + \Gamma_{12}^2 \bar{r}_y + M \bar{n} \]
\[ (9c) \]
\[ \bar{r}_{yy} = \Gamma_{12}^1 \bar{r}_x + \Gamma_{12}^2 \bar{r}_y + N \bar{n} \]
\[ (9d) \]
\[ \bar{n}_x = p_1 \bar{r}_x + p_2 \bar{r}_y \]
\[ (9e) \]
\[ \bar{n}_y = q_1 \bar{r}_x + q_2 \bar{r}_y \]
\[ (9f) \]
where \( W_j \) are some functions, \( \Gamma^k_{ij} \) are the Christoffel symbols of the second kind defined by the metric \( g_{ij} \) and \( g^{ij} = (g_{ij})^{-1} \) as
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right) \]  
\[ \text{(10)} \]
The coefficients \( p_i, q_i \) are given by
\[ p_i = -b_{1j} g^{ji}, \quad q_i = -b_{2j} g^{ji} \]  
\[ \text{(11)} \]
The compatibility conditions \( \bar{r}_{xx} = \bar{r}_{xy} \) and \( \bar{r}_{yy} = \bar{r}_{yx} \) yield the following Mainardi-Peterson-Codazzi equations (MPCE)
\[ R^l_{ijk} = b_{ijb_k^l} - b_{ikb_j^l} \quad \frac{\partial b_{ij}}{\partial x^k} - \frac{\partial b_{ik}}{\partial x^j} = \Gamma^s_{ik} b_{is} - \Gamma^s_{ij} b_{ks} \]  
\[ \text{(12)} \]
where \( b^l_j = g^{jl} b_{il} \) and the curvature tensor has the form
\[ R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ij} \Gamma^l_{ks} - \Gamma^s_{ik} \Gamma^l_{js} \]  
\[ \text{(13)} \]
Let \( Z = (r_x, r_y, n)^t \). Then
\[ Z_x = AZ, \quad Z_y = BZ \]  
\[ \text{(14)} \]
where
\[ A = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ p_1 & p_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \\ q_1 & q_2 & 0 \end{pmatrix} \]
\[ \text{(15)} \]
Hence we get the new form of the MPCE(12)
\[ A_y - B_x + [A, B] = 0 \]  
\[ \text{(16)} \]
Let us introduce the orthogonal trihedral[14]

\[ \vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \vec{n}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 \] (17)

Let \( \vec{r}_x^2 = E = \pm 1 \) and \( F = 0 \). Then the vectors \( \vec{e}_j \) satisfy equations(3), where

\[ k = \frac{L}{2}, \quad \tau = MG^{-1/2}. \] (18)

So this approach is equivalent to the previous approach.

4. The gauge equivalent counterpart of the Myrzakulov XXII equation

Note that the M-XXII\(_q\) equation(7) is integrable as the L-equivalent counterpart of the integrable M-XXII\(_s\) equation(1). It is means that it must allows the Lax representation. Let us find it. To this end consider the gauge transformation

\[ \Phi = g^{-1}\Psi_1 \] (19)

If we take \( g \) as \( g = \Psi_1 \) as \( \lambda = b \) then after the some calculations we came to the following Lax representation for the M-XXII\(_q\) equation(7)

\[ \Psi_{1x} = \{-i(\lambda^2 - \frac{pq}{4})\sigma_3 + \lambda Q\}\Psi_1, \quad \Psi_{1t} = 2\lambda^2\Psi_{1y} + (\lambda^2 B_2 + \lambda B_1 + B_0)\Psi_1 \] (20)

with

\[ Q = \begin{pmatrix} 0 & q \\ -p & 0 \end{pmatrix}, \quad B_2 = \frac{i}{2}V_1\sigma_3, \quad B_1 = i\sigma_3Q_y - \frac{1}{2}V_1Q, \quad B_0 = \frac{1}{4}V_2 - \frac{i}{8}pqV_1. \]

So between the M-XXII\(_s\)(1) and M-XXII\(_q\)(7) equations take places the gauge equivalence.

5. On the gauge equivalence between the M-XXII\(_q\) equation and the Strachan equation

Now let us consider the following transformation

\[ q' = q \exp(-\frac{i}{2}\partial_x^{-1}|q|^2) \] (21)

Then the new variable \( q' \) satisfies the Strachan equation[15]

\[ iq'_t + q'_{xy} + i(Vq')_x = 0, \quad V_x = E(|q'|^2)_y. \] (22)

We see that the M-XXII\(_q\) equation(7) and the Strachan equation(22) is gauge equivalent to each other. The tranformation (21) induces the following tranformation of the Jost function\(\Psi_1\)

\[ \Psi_1 = f^{-1}\Psi_2 \] (23)
where
\[ f = \exp \left( -\frac{i}{4} \partial_x^{-1} |q|^2 \sigma_3 \right) = \Psi_1^{-1} |_{\lambda=0}. \] (24)

Then the new Jost function \( \Psi_2 \) satisfies the following set of equations \cite{15}
\[ \Psi_{2x} = \{-i\lambda^2 \sigma_3 + \lambda Q'\} \Psi_2, \quad \Psi_{2t} = 2\lambda^2 \Psi_{2y} + \{\lambda^2 B'_2 + \lambda B'_1 + B'_0\} \Psi_2 \] (25)
with
\[ Q = \begin{pmatrix} 0 & q' \\ -p' & 0 \end{pmatrix}, \]
and \( B'_j \) are given in \cite{15,14,13}.

6. On some integrals of motion

As integrable the above presented equations allow an infinite number of integrals of motion. It is interesting that the some important integrals of motion follow from the geometrical formalism that was presented in section II. So in 2+1 dimensions we have the following

**Theorema-1:**

The 2+1 dimensional nonlinear evolution equations admit the following integrals of motion
\[ K_1 = \int \kappa m_2 dx dy, \quad K_2 = \int \tau m_2 dx dy \] (26)
In particular for the 2+1 dimensional spin systems this theorem we can reformulate in the following way

**Theorema-2:**

The 2+1 dimensional spin systems admit the following integrals of motion
\[ K_1 = \int \vec{S} \cdot (\vec{S}_x \wedge \vec{S}_y) dx dy, \quad K_2 = \int \frac{[\vec{S} \cdot (\vec{S}_x \wedge \vec{S}_y)] [\vec{S} \cdot (\vec{S}_x \wedge \vec{S}_{xx})]}{|\vec{S}_x|^2} dx dy. \] (27)

Note that in the last case \( G = \frac{1}{4\pi} K_1 \) is the well known topological charge. The proves of these theorems are given in \cite{14}.

7. A nonisospectral problems

In contrast with the 1+1 dimensional case, where \( \lambda_t = 0 \), in our case the spectral parameter \( \lambda_t \neq const \) and satisfies the following equation
\[ \lambda_t = 2\lambda^2 \lambda_y. \] (28)

This equation we can solve using the following Lax representation
\[ h_x = -i\lambda^2 \sigma_3 h, \quad h_t = 2\lambda^2 h_y. \] (29)
The trivial solution is \( \lambda = \lambda_1 = const. \) To find the other solution let us consider the following general equation
\[ \lambda_t = \kappa \lambda^n \lambda_y \] (30)
where $\kappa = \text{const.}$ Let
\begin{equation}
\lambda_y = \sum_j b_j \lambda^j, \quad \lambda_t = \sum_j d_j \lambda^j. \tag{31}
\end{equation}
where $b_j, d_j$ are some functions in general of $y, t$. In particular we can take
\begin{equation}
\lambda_t = \frac{\lambda}{a - \kappa t}, \quad \lambda_y = \frac{\lambda}{y + c}. \tag{32}
\end{equation}
where $a(c)$ is real (complex) constant. Hence follows that solution of equation (30) has the form
\begin{equation}
\lambda = \lambda_2 = \left(\frac{y + c}{a - \kappa t}\right)^{\frac{1}{2}}. \tag{33}
\end{equation}
So if $n = 1$ we have
\begin{equation}
\lambda_2 = \frac{y + c}{a - \kappa t}. \tag{34}
\end{equation}
If $n = 2$ we get
\begin{equation}
\lambda_2 = \left(\frac{y + c}{a - \kappa t}\right)^{\frac{1}{2}} \tag{35}
\end{equation}
and so on. In our case the solution of (28) has the form (35) with $\kappa = 2$. We note the corresponding solutions of the soliton equations is called the overlapping or breaking solutions[17]. In this case soliton equations must be solve with help the non-isospectral version of the inverse scattering transform (IST) method. Note that unlike with the 1+1 dimensions, where $\lambda_t = 0$, in 2+1 dimensions we have the following integral of motion for the spectral parameter
\begin{equation}
K = \int \lambda dy, \quad K_t = 0. \tag{36}
\end{equation}

8. Conclusion

To conclude, we have found the L-equivalent counterpart of the M-XXII equation using the differential geometry of curves and surfaces. Also the gauge equivalence between this counterpart equation and the Strachan equation is established. Finally we present the (1+1)-dimensional reductions of the above considered equations. We have
the M-XXII$_s$ equation reduces to the following equation
\begin{equation}
-iS_t = \frac{1}{2}[S, S_{xx}] + \frac{i}{8b^2}S_x^2S_{xx} - 2ib^2S_x, \tag{37}
\end{equation}
the M-XXII$_q$ equation reduces to the equation
\begin{equation}
iq_t + q_{xx} + ipqq_x = 0, \quad ip_t - p_{xx} + iqpp_x = 0 \tag{38}
\end{equation}
the Strachan equation reduces to the equation
\begin{equation}
iq_t + q_{xx} + iq(pq)_x = 0, \quad ip_t - p_{xx} + i(qp)_x = 0 \tag{39}
\end{equation}
where $p = Eq$. Note that equations (38) and (39) are gauge equivalent each to other[16].

6
Finally we also would like to pose the following questions.

Questions
1. What is the Hirota (bilinear) form of the M-XX_s(1) and M-XXII_q (7) equations?
2. How construct the solutions of these equations?

If you have any results (or information) about these and other multidimensional spin equations please inform me.

References

[1] P. Pelce (ed): *Dynamics of Curved Fronts* (Academic Press, 1988).

[2] A. Sym, Springer Lecture Notes in Physics, edited by R. Martini (Springer, Berlin 1985) Vol 239, p 154

[3] M. Lakshmanan, Phys. Lett. A 64, 353 (1978); J. Math. Phys. 20, 1667 (1979)

[4] K. Nakayama, H. Segur and M. Wadati//Phys.Rev.Lett. v.69, N18,2603-2606(1992)

[5] A. Doliwa and P.M. Santini, Phys. Lett. A 185, 373 (1994)

[6] R. Balakrishnan, P.Guha // IMSc-95/24

[7] A.R. Bishop, L.J. Cambell and P.J. Channels, in *Fronts, Interfaces and Patterns* (North - Holland, New York, 1984)

[8] B.G.Konopelchenko//Solitons in multidimensions. World Scientific. Singapore. 1993.

[9] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear evolution equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991)

[10] V. E. Zakharov, *Solitons*, edited by R.K. Bullough and P.J. Caudrey (Springer, Berlin, 1980)

[11] M. Boiti, L. Martina and F. Pempinelli // Multidimensional localized solitons(25.08.1993).

[12] V. E. Zakharov, L. A. Takhtajan //Theor. Math. Phys. 1979. v.38. p.17.

[13] G. N. Nugmanova// The Myrzakulov equations: the gauge equivalent counterparts and soliton solutions. Alma-Ata (1992)

[14] R. Myrzakulov, On some integrable and nonintegrable soliton equations of magnets I-IV (HEPI, Alma-Ata) (1987)

[15] I. A. B. Strachan, J. Math. Phys. 34, 243 (1993)

[16] K. Sogo and M. Wadati// J.Phys.Soc.Jpn, v.52, 394(1983)
[17] O. I. Bogoyavlensky, Breaking Solitons (Moscow: Nauka) (1991)

[18] G.N.Nugmanova// On magnetoelastic solitons in ferromagnet. Preprint CNLP-1994-01. Alma-Ata. 1994.

[19] R. Myrzakulov // Soliton equations in 2+1 dimensions and Differential geometry of curves/surfaces. Preprint CNLP-1994-02. Alma-Ata. 1994.