ORLICZ-LORENTZ GAUGE NORM INEQUALITIES FOR NONNEGATIVE INTEGRAL OPERATORS

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1. Introduction

Let $k \in M_+(\mathbb{R}^n)$ and $K \in M_+(\mathbb{R}^{2n}_+)$; here $\mathbb{R}_+ = (0, \infty)$ and, for example, $M_+(\mathbb{R}^{2n}_+)$ denotes the class of nonnegative, Lebesgue-measurable functions on the product space $\mathbb{R}^{2n}_+, n \in \mathbb{Z}_+$.

We consider two kinds of operators, namely, convolution operators $T_k$ defined at $f \in M_+(\mathbb{R}^n_+)$ by

$$(T_k f)(x) = (k * f)(x) = \int_{\mathbb{R}^n} k(x - y) f(y) \, dy, \quad x \in \mathbb{R}^n,$$

and general nonnegative integral operators $T_K$ defined at $f \in M_+(\mathbb{R}_+)$ by

$$(T_K f)(x) = \int_{\mathbb{R}_+} K(x, y) f(y) \, dy, \quad x \in \mathbb{R}_+.$$

We are interested in Orlicz gauge norms $\rho_1$ and $\rho_2$ on $M_+(\mathbb{R}_+)$ for which

$$\rho_1((T_f)^*) \leq C \rho_2(f^*),$$

where $C > 0$ is independent of $f$. In (1), $T = T_k$ or $T = T_K$ and, accordingly, $f \in M_+(\mathbb{R}^n_+)$ or $f \in M_+(\mathbb{R}_+)$. The function $f^*$ in (1) is the nonincreasing rearrangement of $f$ on $\mathbb{R}_+$, with

$$f^*(t) = \mu_f^{-1}(t),$$

in which

$$\mu_f(\lambda) = |\{x : f(x) > \lambda\}|.$$

The gauge norm $\rho$ is given in terms of an $N$-function

$$\Phi(x) = \int_0^x \phi(y) \, dy, \quad x \in \mathbb{R}_+,$$

$\phi$ a nondecreasing function mapping $\mathbb{R}_+$ onto itself, and a locally-integrable (weight) function $u$ on $\mathbb{R}_+$. Specifically, the gauge norm $\rho = \rho_{\Phi, u}$ is defined at $f \in M_+(\mathbb{R}_+)$ by

$$\rho_{\Phi, u}(f) = \inf \{ \lambda > 0 : \int_{\mathbb{R}_+} \Phi \left( \frac{f(x)}{\lambda} \right) u(x) \, dx \leq 1 \}.$$

Thus, in (1), $\rho_1 = \rho_{\Phi_1, u_1}$ and $\rho_2 = \rho_{\Phi_2, u_2}$. The gauge norms in (1) involving rearrangements are referred to as Orlicz-Lorentz norms.
2. The Convolution Operators $T_k$ on $M_+(\mathbb{R}^n)$

The fundamental inequality used in the analysis of $T_k$ is a rewritten form of the O’Neil rearranged convolution inequality.

\begin{equation}
(2) \quad \int_0^t (f * g)^*(s) \, ds \leq \int_0^t f^*(s) \, ds \int_0^t g^*(s) \, ds + t \int_t^\infty f^*(s)g^*(s) \, ds,
\end{equation}

$f, g \in M_+(\mathbb{R}^n)$, $t \in \mathbb{R}_+$.

The inequality \( \rho_1((T_kf)^*) \leq C \rho_2(f^*) \)

is shown in following form

\begin{equation}
(3) \quad \rho_1(T_Lf^*) \leq C \rho_2(f^*), \quad f \in M_+(\mathbb{R}_+),
\end{equation}
in which $L = L(x, y)$, $x, y \in \mathbb{R}_+$ is the so-called iterated rearrangement of $K(x, y)$, which rearrangement is nonincreasing in each of $x$ and $y$.

Using the concept of the down dual of a gauge norm the inequalities (2) and (3) for nonincreasing functions are, in each case, reduced to gauge norm inequalities for general functions in Theorem 2.1 and Theorem 3.4, respectively. Sufficient conditions for stronger integral inequalities like

\begin{equation}
(4) \quad \Phi_1^{-1} \left( \int_{\mathbb{R}_+} \Phi_1(\omega(x))((Tf)(x))t(x) \, dx \right) \leq \Phi_2^{-1} \left( \int_{\mathbb{R}_+} \Phi_2(u(y)f(y)v(y)) \, dy \right),
\end{equation}

with $c > 0$ independent of $f \in M_+(\mathbb{R}_+)$.

The integral inequality (4) is the same as the corresponding gauge norm inequality \( \rho_{\Phi_1, \omega}(\omega Tf) \leq C \rho_{\Phi_2, \omega}(\omega f) \),

when $\Phi_1(s) = s^q$ and $\Phi_2(s) = s^p$, $1 < p < \infty$. The necessary and sufficient conditions are spelled out in this case.

Fundamental to our approach is the O’Neil’s rearrangement convolution inequality which we write in the form

\begin{equation}
(5) \quad \int_0^t (f * g)^*(s) \, ds \leq \int_0^t f^*(s) \, ds \int_0^t g^*(s) \, ds + t \int_t^\infty f^*(s)g^*(s) \, ds,
\end{equation}

$f, g \in M_+(\mathbb{R}^n)$, $t \in \mathbb{R}_+$.

We claim (5) amounts to the domination of $(f * g)(t)$ by the expression

\begin{equation}
(6) \quad f^*(t) \int_0^t g^*(s) \, ds + g^*(t) \int_0^t f^*(s) \, ds + \int_t^\infty f^*g^* \,
\end{equation}
in the HLP sense. Indeed, the first two terms in (6) add up to \( \frac{d}{dt} \left( \int_0^t f^* \, ds \int_0^t g^* \, ds \right) \), so the integral between 0 and $t$ of these terms is the first term on the right side of (5). Again,

\begin{align*}
\int_0^t \int_s^\infty f^*g^* &\leq \int_0^t \int_s^t f^*g^* + t \int_t^\infty f^*g^* \\
&\leq \int_0^t f^*(s) \int_s^t g^* + t \int_t^\infty f^*g^* \\
&\leq \int_0^t f^* \int_0^t g^* + t \int_t^\infty f^*g^*.
\end{align*}
Consider norm inequality. We combine Theorems 1.7 and 4.1 from [1] concerning such integral inequalities.

\[ \int_0^t (f * g)^* \leq 2 \int_0^t \left[ \int_0^s f^* + \int_s^\infty f^* g^* \right] ds \]

ensures that

\[ \rho((f * g)^*) \leq 2 \left[ \rho_d \left( \int_0^t f^* \right) + \rho_d \left( \int_0^t g^* \right) + \rho_d \left( \int_0^\infty f^* g^* \right) \right], \]

where

\[ \rho_d(h) = \sup_{\rho_k \leq 1} \int_{R_+} h k^*, \quad h, k \in M_+(R_+), \]

with \( \Psi(t) = \int_0^t \phi^{-1}(s) ds \) being the \( N \)-function complementary to \( \Phi \).

According to [3] Theorem, \( \rho_d(h) = \rho \left( \int_0^t h/t \right), \quad h \in M_+(R_+) \).

In Theorem 1 to follow, which summarizes the foregoing discussions, \( (Ik^*)(t) = \int_0^t k^* \) and \( (T_k k^*)(t) = I(1k^*)(t) = \int_0^t (t - s)k^*(s) ds \).

**Theorem 2.1.** Fix \( k \in M_+(R^n) \) and let \( \Phi_1 \) and \( \Phi_2 \) be \( N \)-functions, with \( \Phi_2(2t) \approx \Phi_2(t) \), \( t \gg 1 \). Settings \( \rho_i = \rho_{\Phi_i}, \quad i = 1, 2 \), we have

\[ \rho_1(T_k f) \leq C \rho_2(f), \quad f \in M_+(R^n), \]

provided

\[ \begin{align*}
(i) \quad & \rho_1 \left( \frac{1}{t} \int_0^t f(s) \int_s^t k^* ds \right) \leq C \rho_2(f), \quad f \in M_+(R_+); \\
(ii) \quad & \rho_2 \left( \frac{1}{t} \int_0^t g(s) \int_s^t k^* ds \right) \leq C \rho_1(g), \quad g \in M_+(R_+); \\
(iii) \quad & \tilde{\rho}_2 \left( \frac{(I k^*)(t)}{t} \int_t^\infty g(y) dy \right) \leq C \tilde{\rho}_1(g), \quad g \in M_+(R+); \\
(iv) \quad & \tilde{\rho}_2 \left( \frac{1}{t} \int_0^t (I k^*)(s) s g(s) ds \right) \leq C \tilde{\rho}_1(g), \quad g \in M_+(R),
\end{align*} \]

here \( \tilde{\rho}_i = \rho_{\Phi_i}, \quad i = 1, 2. \)

There are no known conditions that are necessary and sufficient for any of the norm inequalities (i) - (iv), unless \( \Phi_1(t) = t^p, \Phi_2(t) = t^q, \quad 1 < p \leq q < \infty. \)

There are, however, such conditions for an integral inequality stronger that the corresponding norm inequality. We combine Theorems 1.7 and 4.1 from [3] concerning such integral inequalities.

**Theorem 2.2.** Consider \( K(x, y) \in M_+(R_+^2) \), which, for fixed \( y \in R_+ \), increases in \( x \) and, for fixed \( x \in R_+ \), decreases in \( y \), and which satisfies the growth condition

\[ K(x, y) \leq K(x, z) + K(z, y), \quad 0 < y < z < x. \]

Let \( t, u, v \) be nonnegative, measurable (weight) functions on \( R_+ \) and suppose \( \Phi_1 \) and \( \Phi_2 \) are \( N \)-functions having complimentary functions \( \Psi_1 \) and \( \Psi_2 \), respectively, with \( \Phi_1 \circ \Phi_2^{-1} \) convex. Then there exists \( C > 0 \) such that

\[ \Phi_1^{-1} \left( \int_{R_+} \Phi_1(w(x)(T_K f)(x))t(x) dx \right) \leq \Phi_2^{-1} \left( \int_{R_+} \Phi_2(Cu(x)f(x))v(x) dx \right) \]
for all \( f \in M_+(\mathbb{R}^+), \) if and only if there is a \( c > 0, \) independent of \( \lambda, x > 0, \)

\[
\int_0^x \Psi_2 \left( c \frac{\alpha(\lambda, x) K(x, y)}{\lambda u(y)v(y)} \right) v(y) dy \leq \alpha(\lambda, x) < \infty
\]

(10)

and

\[
\int_0^x \Psi_2 \left( c \frac{\beta(\lambda, x)}{\lambda u(y)v(y)} \right) v(y) dy \leq \beta(\lambda, x) < \infty,
\]

where

\[
\alpha(\lambda, x) = \Phi_2 \circ \Phi_1^{-1} \left( \int_x^\infty \Phi_1(\lambda u(y)) t(y) dy \right)
\]

and

\[
\beta(\lambda, x) = \Phi_2 \circ \Phi_1^{-1} (\Phi_1(\lambda u(y)K(y, x)) t(y) dy).
\]

In the case \( K(x, y) = \chi_{(0,x)}(y) \) only the first of the conditions in (9) is required.

**Remark 2.3.** The integral inequality (9) with the kernel \( K \) of Theorem 2.2 replace by any \( K \in M_+(\mathbb{R}^+_{2m}) \) implies the norm inequality

(11)

Thus, in the generalization of (8) replace \( f \) by \( f \frac{wT K f}{C \rho \Phi_2, v(uf)} \) and suppose \( \Phi_1(1) = 1, i = 1, 2. \)

since \( \int_{\mathbb{R}^+} \Phi_2, v \left( \frac{uf}{C \rho \Phi_2, v(uf)} \right) \leq 1, \) we get

\[
\int_{\mathbb{R}^+} \Phi_1 \left( \frac{wT K f}{C \rho \Phi_2, v(uf)} \right) t \leq 1,
\]

whence (11) holds.

In Theorem 4 to follow, \( (Ik^*)(t) = \int_0^t k^* \) and \( (I_2 k^*)(t) = ((I \circ I)k^*)(t) = \int_0^1 k^*(t-s) ds. \)

**Theorem 2.4.** Let \( k, \rho_1 \) and \( \rho_2 \) be as in Theorem 2.2. Then, (7) holds if there exists \( c > 0, \)

independent of \( \lambda, x \in \mathbb{R}^+, \) such that

(5)

\[
\int_0^x \Psi_2 \left( c \frac{\alpha(\lambda, x)}{\lambda} \int_y^x k^* \right) dy \leq \alpha_1(\lambda, x) < \infty
\]

and

\[
\Psi_2 \left( c \frac{\beta(\lambda, x)}{\lambda} \right) \leq \beta_1(\lambda, x) < \infty,
\]

where

\[
\alpha_1(\lambda, x) = \Phi_2 \circ \Phi_1^{-1} \left( \int_x^\infty \Phi_1 \left( \frac{\lambda}{y} \right) dy \right)
\]

and

\[
\beta_1(\lambda, x) \Phi_2 \circ \Phi_1^{-1} \left( \int_x^\infty \Phi_1 \left( \frac{\lambda}{y} \int_x^y k^* \right) dy \right).
\]

(6)

\[
\int_0^x \Phi_1 \left( c \frac{\alpha_2(\lambda, x)}{\lambda} \int_y^x k^* \right) dy \leq \alpha_2(\lambda, x) < \infty
\]

and

\[
x \Phi_1 \left( c \frac{\beta_2(\lambda, x)}{\lambda} \right) \leq \beta_2(\lambda, x) < \infty,
\]
where
\[ \alpha_2(\lambda, x) = \Psi_1 \circ \Psi_2^{-1} \left( \int_{x}^{\infty} \Psi_2 \left( \frac{\lambda}{y} \right) \, dy \right) \]

and
\[ \beta_2(\lambda, x) = \Psi_1 \circ \Psi_2^{-1} \left( \int_{x}^{\infty} \Psi_2 \left( \frac{\lambda}{y} \int_{x}^{y} k^* \right) \, dy \right) \]

(vii)
\[ \int_{0}^{x} \Psi_2 \left( \frac{cy^{\alpha_3}(\lambda, x)}{\lambda(I_2 k^*)(y)} \right) \, dy \leq \alpha_3(\lambda, x) < \infty, \]

where
\[ \alpha_3(\lambda, x) = \Phi_2 \circ \Phi_1^{-1} \left( \int_{x}^{\infty} \Phi_1 \left( \frac{\lambda}{y} \right) \, dy \right) ; \]

(viii)
\[ \int_{0}^{x} \Phi_1 \left( \frac{cy^{\alpha_4}(\lambda, x)}{\lambda(I_2 k^*)(y)} \right) \, dy \leq \alpha_4(\lambda, x) < \infty, \]

where
\[ \alpha_4(\lambda, x) = \Psi_1 \circ \Psi_2^{-1} \left( \int_{x}^{\infty} \Psi_2 \left( \frac{\lambda}{y} \right) \, dy \right) . \]

Proof. The conditions (v) and (vi) result from a direct application of Theorem 2.2 to the operator with kernel
\[ K(x, y) = \int_{y}^{x} k^* , \]

which clearly increases in \( x \), decreases in \( y \) and satisfies the growth condition \( \mathfrak{S} \).

Writing the inequality (iv) in the equivalent form
\[ \tilde{\rho}_2 \left( \frac{1}{t} (I h)(t) \right) \leq C \tilde{\rho}_1 \left( s h(s)/(I k^*)(s) \right), \quad h \in M_+(\mathbb{R}_+) , \]

one sees the condition (viii) comes out of Theorem 4.2 in [1], which theorem accers that for the Hardy operator \( I \) only the first of the conditions in \( \mathfrak{S} \) is required.

Finally, the inequality (iii) is equivalent to the dual inequality
\[ \rho_1 \left( \frac{1}{t} \int_{0}^{t} (I_2 k^*)(s) h(s) s \, ds \right) \leq C \rho_2(h), \quad h \in M_+(\mathbb{R}_+) , \]

which can be written in the form
\[ \rho_1 \left( \frac{1}{t} (I f)(t) \right) \leq C \rho_2 \left( s f(s)/(I_2 k^*)(s) \right), \quad f \in M_+(\mathbb{R}_+) . \]

The condition for this is (vii). \( \square \)

The integral inequality (vi) is equivalent to the norm inequality (viii) when \( \Phi_1 \) and \( \Phi_2 \) are power functions, say \( \Phi_1(t) = t^p, \Phi_2(t) = t^p, \quad 1 < p \leq q < \infty. \) Moreover, in this case the conditions (v) to (viii) simplify. They become
\[ (v') \quad \int_{0}^{x} \left( \int_{y}^{x} k^* \right)^{p'/q'} \, dy \leq C x^{p'/q'} \]

and
\[ \left( \int_{x}^{\infty} \left( \frac{1}{y} \int_{x}^{y} k^* \right)^q \, dy \right)^{p'/q'} \leq C x^{-1} ; \]

(vii')
\[ \int_{0}^{x} \left( \int_{y}^{x} k^* \right)^q \, dy \leq C x^{d'/r} \]
and
\[
\left( \int_0^\infty \left( \int_x^y f(x) \right)^{q'} \, dy \right)^{q/r'} \leq C x^{1-1};
\]
\[
(viii') \quad \int_{\mathbb{R}^n} -\theta^r \left( \frac{y}{1/2} \right)^{q'} \, dy \leq C x^{1/q'};
\]
\[
(viii'') \quad \int_0^x \left( \frac{y}{(1/2)^k(y)} \right)^q \, dy \leq C x^{3/r}.
\]

It is shown in \cite{6} that O'Neil condition inequality is sharp when \( f \) and \( g \) are radially decreasing on \( \mathbb{R}^n \). Altogether then, we have

**Theorem 2.5.** Fix the indices \( p \) and \( q, 1 < p \leq q < \infty \), and suppose \( k \) is radially decreasing on \( \mathbb{R}^n \). Then, one has the inequality
\[
\left[ \int_{\mathbb{R}^n} (T_k f)^q \right]^{1/q} \leq C \left[ \int_{\mathbb{R}^n} f^p \right]^{1/p},
\]
with \( C > 0 \) independent of \( f \in M_+(\mathbb{R}^n) \), if and only if the conditions \((v') - (viii'')\) hold.

We remark that a convolution operator \( T_k \) whose kernel \( k(x) = k(|x|) \) decreases in \( |x| \) on \( \mathbb{R}^n \) is known as a potential operator; see \cite{6} and the reference therein. In the formulas \((v') - (viii'')\),
\[
k^*(t) = k \left( \frac{\Gamma(n/2 + 1)^{1/n}}{\sqrt{\pi}} t^{1/n} \right), \quad t \in \mathbb{R}_+.
\]

3. **General nonnegative integral operators on \( M_+(\mathbb{R}^n) \)**

As a first step in our study of \( (1) \) for \( T = T_K \) we focus on the related inequality
\[
\rho_i(T_K f^*) \leq C \rho_2(f^*), \quad f \in M_+(\mathbb{R}^n).
\]

**Theorem 3.1.** Fix \( K \in M_+(\mathbb{R}^n) \) and let \( \Phi_1 \) and \( \Phi_2 \) be \( N \)-function, with \( \Phi_2(2t) \approx \Phi_2(t), \quad t \gg 1 \). Given weight functions \( u_1, u_2 \in M_+(\mathbb{R}^n), \int_{\mathbb{R}^n} u_i = \infty \), one has \((12)\) for \( \rho_i = \rho_{\Phi_i, u_i}, i = 1, 2, \) if
\[
\rho_{\Phi_2, u_2}(Sg/u_2) \leq C \rho_{\Phi_1, u_1}(g/u_1), \quad g \in M_+(\mathbb{R}^n),
\]
\[
u_2(x) = \int_0^x u_2(z) \, dz, \quad \text{and} \quad \Psi_i(t) = \int_0^t \phi_i^{-1}, \quad i = 1, 2.
\]

**Proof.** The identity
\[
\int_{\mathbb{R}^n} g T_K f^* = \int_{\mathbb{R}^n} f^* T_K g
\]
readily yields that \((12)\) holds if and only if
\[
(\rho_1')^d(T_K g) \leq C \rho_1'(g),
\]
where
\[
(14) \quad \rho_1'(g) = \rho_{\Phi_1, u_1}(g/u_1)
\]
and
\[
(15) \quad (\rho_2^d)'(h) = \rho_{\Phi_2, u_2} \left( \int_0^x h/u_2(x) \right), \quad h \in M_+(\mathbb{R}^n).
\]

For \((14)\), see \cite[Theorem 6.2]{1}; \((15)\) is straightforward. The proof is complete on ranking \( h = T_K'(g) \). \( \square \)
To replace $T_K f^*$ in (12) by $(T_K f)^*$ we will require

$$\rho \left( t^{-1} \int_0^t f^* \right) \leq C \rho_1(f^*), \quad f \in M_+(\mathbb{R}_+) \text{.}$$

Conditions sufficient for such an inequality to hold are given in Theorem 3.2.

**Theorem 3.2.** Let $\Phi$ be an $N$-function satisfying $\Phi(2t) \approx \Phi(t)$, $t \gg 1$, and suppose $u$ is weight on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} u = \infty$. Then,

$$\rho_{\Phi,u} \left( t^{-1} \int_0^t f^* \right) \leq C \rho_{\Phi,u}(f^*), \quad f \in M_+(\mathbb{R}_+) \text{,}$$

provided

$$\int_0^x \Phi(c \alpha(\lambda, x) / \lambda) u(y) \, dy \leq \alpha(\lambda, x) < \infty,$$

with $c > 0$ independent of $\lambda, x \in \mathbb{R}_+$, where

$$\alpha(\lambda, x) = \int_0^\infty 2 \Psi(\lambda u(y) / u(y)) \, dy$$

and

$$\int_0^x \Phi(c \beta(\lambda, x) / \lambda U(y)) u(y) \, dy \leq \beta(\lambda, x) < \infty,$$

with $c > 0$ independent of $\lambda, x > 0$, where

$$\beta(\lambda, x) = \int_x^\infty \Phi(\lambda / y) u(y) \, dy.$$

**Proof.** In Theorem 3.1 take $K(x, y) = \chi_{0,x}(y) / x$, $\Psi_1 = \Psi_2 = \Psi$ and $u_1 = u_2 = u$ to get

$$\langle Sg \rangle(x) = \int_0^x \int_y^\infty g(z) \frac{dz}{z} = \int_0^x g + x \int_x^\infty g(y) \frac{dy}{y},$$

whence (13) reduces to

$$\rho_{\Phi,u} \left( \int_0^x g(U(x)) \right) \leq C \rho_{\Phi,u}(g/u) \tag{18}$$

and

$$\rho_{\Phi,u} \left( x \int_0^\infty g(y) \frac{dy}{y} / U(x) \right) \leq C \rho_{\Phi,u}(g/u) \text{.}$$

The first inequality in (18) is a consequence of the modular inequality

$$\int_{\mathbb{R}_+} \Psi \left( c \int_0^x g/u(x) \right) \leq \int_{\mathbb{R}_+} \Psi(g/u)u,$$

which, according to [BK, Theorem 4.1] hold, if and only if the first inequality in (17) does.

Again, by duality, the second inequality in (18) holds when the modular inequality

$$\int_{\mathbb{R}_+} \Phi \left( c \int_0^x g \right) u \leq \int_{\mathbb{R}_+} \Phi(g/u)u$$

does, which inequality holds if and only if one has the second condition in (17). □
In Theorem 3.3 below we show the boundedness of $TKf$ depends on that of $TKf^*$, where the kernel $L$ is the iterated rearrangement of $K$ considered in 2. Thus, for each $x \in \mathbb{R}_+$, we rearrange the function $k_x(y) = K(x,y)$ with respect to $y$ to get $(k_x^*)(s) = K^*(x,s) = k_s(x)$ and then rearrange the function of $x$ so obtained to arrive at $(K^*)^*(t,s) = L(t,s)$. It is clear from its construction that $K(t,s)$ is nonincreasing in each of $s$ and $t$.

**Theorem 3.3.** Consider $K \in M_+(\mathbb{R}_+^2)$ and set $L(t,s) = (K^*)^*(t,s)$, $s,t \in \mathbb{R}_+$. suppose $\Phi_1$ and $\Phi_2$ are N-functions, with $\Phi_1(2t) \approx \Phi_1(t)$, $t \gg 1$, and let $u_1$ and $u_2$ be weight functions, with $\int_{\mathbb{R}_+} u_1 = \infty$. Then, given the conditions (15) for $\Psi = \Psi_1$ and $u = u_1$ one has

$$
\rho_1((TKf)^*) \leq C\rho_2(f^*),
$$

provided

$$
\rho_1((TLf)^*) \leq C\rho_2(f^*), \quad f \in M_+(\mathbb{R}_+).
$$

**Proof.** We claim

$$(19) \quad (TKf)^*(t) \leq (TLf)^*(t), \quad t \in \mathbb{R}_+,
$$

in which, say, $(TKf)^*(t) = t^{-1}\int_t^1 (TKf)^*$.

Indeed, given $E \subset \mathbb{R}_+$, $|E| = t$,

$$
\int_E TKf \leq \int_E \int_{\mathbb{R}_+} K^*(x,s)f^*(s) \, ds
= \int_{\mathbb{R}_+} f^*(s) \int_{\mathbb{R}_+} \chi_E(x)K^*(x,s) \, dx
\leq \int_{\mathbb{R}_+} f^*(s) \int_0^t L(u,s) \, du
= \int_0^t (TLf^*)(u) \, du.
$$

Taking the supremum over all such $E$, then dividing by $t$ yields (19).

Next, the inequality

$$
\rho_{\Phi_1,u_1}((TKf)^*) \leq C\rho_{\Phi_1,u_1}((TLf)^*)
$$

is equivalent to

$$
\rho_{\Phi_1,u_1}((TKf)^*) \leq C\rho_{\Phi_1,u_1}(TLf^*),
$$

given (17) for $\Phi = \Phi_1$ and $u = u_1$. For, in that case,

$$
\rho_{\Phi_1,u_1}((TKf)^*) \leq \rho_{\Phi_1,u_1}(TKf)^*
\leq \rho_{\Phi_1,u_1}(TKf^*)
\leq C\rho_{\Phi_1,u_1}(TLf^*),
$$

The assertion of the theorem now follows. \[\square\]

**Theorem 3.4.** Let $K, L, \Phi_1, \Phi_2, u_1$ and $u_2$ be as in theorem 3.3. Assume, in addition, that $\Phi_2(2t) \approx \Phi_2(t)$, $t \gg 1$, $\int_{\mathbb{R}_+} u_2 = \infty$ and that conditions (18) hold for $\Phi = \Phi_1$, $u = u_1$, then

$$
\rho_{\Phi_1,u_1}((TKf)^*) \leq C\rho_{\Phi_2,u_2}(f^*)
$$

provided

$$
\rho_{\Phi_2,u_2}(H_1f/u_2) \leq C\rho_{\Phi_1,u_1}(f/u_1)
$$

and

$$
\rho_{\Phi_1,u_1}(H_2g) \leq C\rho_{\Phi_2,u_2}(g(i^2)), \quad f, i \in M_+(\mathbb{R}_+),
$$

(20)
where
\[(H_1 f)(x) = \int_0^x M_1(x, y) f(y) \, dy\]
and
\[(H_2 g)(y) = \int_0^y M_2(y, x) g(x) \, dx,\]
with \(i(x) = x^{-1},\)
\[M_1(x, y) = \int_y^x L(y, z) \, dz \quad \text{and} \quad M_2(y, x) = \int_0^{x^{-1}} L(y^{-1}, z) \, dz.\]

Proof. In view of theorem 3.3, we need only verify conditions (20) imply (21)
\[\rho_{\Phi_{1, u_1}}(T_L f^*) \leq C \rho_{\Phi_{2, u_2}}(f^*), \quad f \in M_+(\mathbb{R}_+).\]
Now, according to Theorem 3.1, (21) will hold if one has (13) with \(K = L.\) Again, (13) is equivalent to the dual inequality
\[\rho_{\Phi_{1, u_1}}(S'g) \leq C \rho_{\Phi_{2, u_2}}(gU_2/u_2),\]
where
\[(S'g)(y) = \int_0^\infty \left[ \int_0^x L(y, z) \, dz \right] g(x) \, dx = \left( \int_y^\infty + \int_y^\infty \right) \left[ \int_0^x L(y, z) \, dz \right] g(x) \, dx = \int_y^\infty \left[ \int_0^x L(y, z) \, dz \right] g(x) \, dx\]
has associate operator
\[\int_0^x \left[ \int_0^x L(y, z) \, dz \right] f(y) \, dy = (H_1 f)(x)\]
and this operator is to satisfy
\[\rho_{\Phi_{2, u_2}}(H_1, U_2) \leq C \rho_{\Phi_{1, u_1}}(f/u_1).\]

Again, if there is to exist \(C > 0\) so that
\[(22) \quad \rho_{\Phi_{2, u_2}} \left( \int_0^y \left[ \int_0^x L(y, z) \, dz \right] g(x) \right) \leq C \rho_{\Phi_{2, u_2}}(gU_2/u_2),\]
then, for such \(C,\)
\[\int_0^\infty \Phi_1 \left( \int_0^y \left[ \int_0^x L(y, z) \, dz \right] g(x) \, dx/C \rho_{\Phi_{2, u_2}}(gU_2/u_2) \right) u_1(y) \, dy \leq 1\]
for all \(g \in M_+(\mathbb{R}_+), g \neq 0 a.e.\) That is, on making the changes of variables \(y \to y^{-1}\) then \(x \to x^{-1},\) one will have
\[\int_0^\infty \Phi_1 \left( \int_0^y \left[ \int_0^{x^{-1}} L(y^{-1}, z) \, dz \right] g(x^{-1}) \, dx \, x^2/C \rho_{\Phi_{2, u_2}}(gU_2/u_2) \right) u_1(y^{-1}) \, dy \leq 1.\]

\[\int_0^\infty \Phi_2(gU_2/xu_2)u_2 = \int_0^\infty \Phi_2(g \circ iu_2 \circ i/\lambda u_2 \circ i)i^2u_2 \circ i,\]
so
\[\rho_{\Phi_{2, u_2}}(gU_2/u_2) = \rho_{\Phi_{2, i^2u_2/0}}(g \circ iu_2 \circ i/u_2 \circ i),\]
whence (22) amounts to
\[\rho_{\Phi_{1, i^2u_1/0}}(H_2(i^2g \circ iu_2 \circ i/u_2 \circ i)) \leq C \rho_{\Phi_{2, i^2u_2}}(g \circ u_2 \circ i/u_2 \circ i)\]
or

\[ \rho_{\Phi_1, i^2 u_{101}}(H_2 g) \leq C \rho_{\Phi_2, i^2 u_{201}}(g/i^2), \]

since \( i^2 g \circ i u_2 \circ i/u_2 \circ i \) is arbitrary.

We have to this point shown that the inequality (11) holds for \( \rho_i = \rho_{\Phi_i, u_{i1}}, \ i = 1, 2 \), whenever the inequalities (20) holds for \( H_1 \) and \( H_2 \). In Theorem 3.5 below we give four conditions which, together with (17) for \( \Phi_1 \) and \( \Phi_2 \), guarantee (20).

The kernel \( M_1(x, y) \) of the operator \( H_1 \) is increasing in \( x \) and decreasing in \( y \). Similarly, the kernel \( M_2(y, x) \) of \( H_2 \) is increasing in \( y \) and decreasing in \( x \). The operators \( H_1 \) and \( H_2 \) will be so-called generalized Hardy operator (GHOs) if their kernels satisfy the growth conditions

\[ M_1(x, y) \leq M_1(x, z) + M_1(z, y), \quad y < z < x, \]

(23)

and

\[ M_2(y, x) \leq M_2(y, z) + M_2(z, x), \quad x < z < y. \]

Neither of the conditions in (23) are guarantees to hold. They have to be assumed in theorem 3.5 below so that we may apply Theorem 1.7 in [1] concerning GHOs. Theorem 4.1 in the next section gives a class of kernels for which (23) is satisfied.

**Theorem 3.5.** Let \( K, L, \Phi_1, \Phi_2, u_1, u_2, M_1, M_2, H_1 \) and \( H_2 \) be as in Theorem 3.4. Assume, in addition, that \( \Phi_1 \circ \Phi_2^{-1} \) is convex and that \( M_1 \) and \( M_2 \) satisfy the growth conditions (23). Then, one has

\[ \int_{R_+} \Phi_1(T_K f)^* u_1 \leq C \int_{R_+} \Phi_2(f^*) u_2, \]

(24)

provided

\[ \int_0^y \Phi_1 \left( \frac{\alpha_1(\lambda, x) M_1(x, y)}{\lambda u_1(y)} \right) u_1(y) \, dy \leq \alpha_1(\lambda, x) < \infty, \]

\[ \int_0^y \Phi_1 \left( \frac{\beta_1(\lambda, x)}{\lambda u_1(y)} \right) u_1(y) \, dy \leq \beta_1(\lambda, x) < \infty, \]

(25)

and

\[ \int_0^y \frac{\alpha_2(\lambda, y) M_2(y, x)}{u_2(x^{-1})} \, x^{-2} u_2(x^{-1}) \, dx \leq \alpha_2(\lambda, y) < \infty \]

and

\[ \int_0^y \frac{\beta_2(\lambda, y)}{u_2(x^{-1})} \, x^{-2} u_2(x^{-1}) \, dx \leq \beta_2(\lambda, y). \]

Here,

\[ \alpha_1(\lambda, x) = \Psi_1 \circ \Psi_2^{-1} \left( \int_x^\infty \Psi_2 \left( \frac{\lambda}{u_2(y)} \right) u_2(y) \, dy \right), \]

\[ \beta_1(\lambda, x) = \Psi_1 \circ \Psi_2^{-1} \left( \int_x^\infty \Psi_2 \left( \frac{\lambda M_1(x, y)}{u_2(y)} \right) u_2(y) \, dy \right) \]

\[ \alpha_2(\lambda, y) = \Phi_2 \circ \Phi_1^{-1} \left( \int_y^\infty \Phi_1(\lambda) x^{-2} u_1(x^{-1}) \, dx \right) \]

and

\[ \beta_2(\lambda, y) = \Phi_2 \circ \Phi_1^{-1} \left( \int_y^\infty \Phi_1(\lambda M_2(x, y)) x^{-2} u_1(x^{-1}) \, dx \right). \]
**Proof.** We prove the result involving $H_2$; the proof for $H_1$ is similar. Now, the norm inequality for $H_2$ for holds if one has the integral inequality
\[
\Phi^{-1}_1 \left( \int_{\mathbb{R}^+} \Phi_1 \left( \frac{H_2 g}{C} \right) y^{-2} u_1(y^{-1}) \, dy \right) \leq \Phi^{-1}_2 \left( \int_{\mathbb{R}^+} \Phi_2 \left( \frac{y^2 g(y)}{\lambda} \right) y^{-2} u_2(y^{-1}) \, dy \right).
\]

Indeed, in the latter replace $g(y)$ by $g(y)/\rho_{\Phi_2, y^{-2}u_2(y^{-1})} = g(y)/\lambda$, to get
\[
\int_{\mathbb{R}^+} \Phi_1 \left( \frac{H_2 g}{C \lambda} \right) y^{-2} u_1(y^{-1}) \, dy \leq \Phi_1 \circ \Phi^{-1}_2 \left( \int_{\mathbb{R}^+} \Phi_2 \left( \frac{y^2 g(y)}{\lambda} \right) y^{-2} u_2(y^{-1}) \, dy \right) = \Phi_1 \circ \Phi^{-1}_2(1) = 1,
\]
where we have assumed, without loss of generality, that $\Phi(1) = \Phi_2(1) = 1$. Hence,
\[
\rho_{\Phi_1, y^{-2}u_1(y^{-1})}(H_2 g) \leq C \lambda = C \rho_{\Phi_2, y^{-2}u_2(y^{-1})}(g/\lambda^2).
\]
But, (27) is valid if and only if the third and fourth conditions in (27) hold. \hfill \Box

4. **Examples**

**Theorem 4.1.** Let $k$ be nonnegative, nonincreasing function on $\mathbb{R}^+$. Then, the growth conditions (25) are satisfied for $K(x, y) = k(x + y)$, $x, y \in \mathbb{R}^+$.

**Proof.** We observe that $K(x, y) = L(x, y)$ since $K$ decreases in each of $x$ and $y$. So,
\[
M_1(x, y) = \int_0^x k(y + s) \, ds \quad \text{and} \quad M_2(x, y) = \int_0^{x-1} k(y + s) \, ds.
\]

Now, given $y < z < x$,
\[
M_1(x, y) = \int_0^x k(y + s) \, ds = \int_0^z k(y + s) \, ds + \int_z^x k(y + s) \, ds
= M_1(y, z) + \int_0^{x-z} k(y + z + s) \, ds
\leq M_1(y, z) + \int_0^x k(z + s) \, ds
= M_1(y, z) + M_1(x, z).
\]

Again, given $x < z < y$,
\[
M_2(x, y) = \int_0^{x-1} k(y^{-1} + s) \, ds = \int_0^{z-1} k(y^{-1} + s) \, ds + \int_{z^{-1}}^{x-1} k(y^{-1} + s) \, ds
= M_2(z, y) + \int_0^{x^{-1}-z^{-1}} k(y^{-1} + z^{-1} + s) \, ds
\leq M_2(z, y) + \int_0^{x^{-1}} k(z^{-1} + s) \, ds
= M_2(z, y) + M_2(x, z).
\]

\hfill \Box

**Theorem 4.2.** Fix the indices $p$ and $q$, $1 < p < q < \infty$, and suppose $K(x, y) = k(x + y)$, $x, y \in \mathbb{R}^+$, where $k$ is nonnegative and nonincreasing on $\mathbb{R}^+$.

\[
\left( \int_{\mathbb{R}^+} (T_K f)^q \, dy \right)^{1/q} \leq C \left( \int_{\mathbb{R}^+} f^p \, dy \right)^{1/p},
\]

(28)
with \( C > 0 \) independent of \( f \in M_+(\mathbb{R}_+) \), if and only if

\[
c \int_0^2 M_1(x, y)^q u_1(y) \frac{d y}{1-q} \leq \lambda(x)^{1-q} \\
c \int_0^2 u_1(y) \frac{d y}{1-q} \leq \beta_1(\lambda)^{1-q}
\]

(29)

\[
c \int_0^2 M_2(y, x)^{p'} x^{-2} u_2(\lambda^{-1}) \frac{d x}{1-p'} \leq \alpha_2(x)^{1-p'}
\]

and

(30)

\[
c \int_0^2 x^{-2} u_2(x^{-1}) \frac{d x}{1-p'} \leq \beta_2(x)^{1-p'}.
\]

Here,

\[
\begin{align*}
\alpha_1(x) &= \left( \int_x^\infty U_2(y)^{-p'} u_2(y) \frac{d y}{q/p'} \right)^{q/p'}, \\
\beta_1(x) &= \left( \int_x^\infty \left[ \frac{M_1(x, y)}{U_2(y)} \right]^{p'} u_2(y) \frac{d y}{q/p'} \right)^{q/p'}, \\
\alpha_2(y) &= \left( \int_y^\infty x^{-2} u_1(x^{-1}) \frac{d x}{p/q} \right)^{p/q}, \\
\text{and} \\
\beta_2(y) &= \left( \int_y^\infty M_2(x, y)^p x^{-2} u_1(x^{-1}) \frac{d x}{p/q} \right)^{p/q}.
\end{align*}
\]

**Proof.** The \( N \)-functions \( \Phi_1(t) = t^q \) and \( \Phi_2(t) = t^p \), as well as the weights \( u_1 \) and \( u_2 \) satisfy the conditions required in theorem 3.3. According to Theorem 4.1, so do the kernels \( M_1 \) and \( M_2 \) of \( H_1 \) and \( H_2 \), respectively. We conclude, then, that (24) holds for \( \lambda \) given (25), which in our case are the inequality (28), and the conditions (29). We observe that \( \lambda \) cancels out in the latter conditions and we are left with \( \alpha_1(1), x(= \alpha_1(x)), etc. \)

Consider a kernel of the form \( K(x, y) = k(\sqrt{x^2 + y^2}) \), where \( k(t) \) is nonincreasing in \( t \) on \( \mathbb{R}_+ \) and \( \int_0^\infty K(x, y) \frac{d y}{1} \) for all \( a, x > 0 \). In particular, \( K(x, y) \) is nonincreasing on \( \mathbb{R}_+ \) in each of \( x \) and \( y \). Again, \( M_1(x, y) = \int_0^y k(\sqrt{x^2 + z^2}) \frac{d z}{1} \) and \( M_2(x, y) = \int_0^y k(\sqrt{x^2 + z^2}) \frac{d z}{1} \) satisfy (23), so that (28) holds for \( T_K \), given (29).

In particular, the above is true for \( K(x, y) = (x^2 + y^2)^{-3/4} \). However, this kernel does not satisfy the classic Kantorovič condition usually involved to prove (28) for \( T_K \). Indeed,

\[
2^{-3/4}(x + y)^{-3/2} \leq K(x, y) \leq 2^{3/4}(x + y)^{-3/2},
\]

whence, for \( p > 1 \),

\[
\left[ \int_0^\infty K(x, y)^{p'} \frac{d y}{1} \right]^{1/p'} \approx x^{-1/p-1/2},
\]

and, therefore,

\[
\left[ \int_0^\infty \left[ \int_0^\infty K(x, y)^{p'} \frac{d y}{1} \right]^{q/p'} \frac{d x}{1} \right]^{1/q} = \left[ \int_0^\infty \frac{d x}{x^{q/p+q/2}} \right]^{1/q} = \infty.
\]

The kernel \( K(x, y) \) is not homogeneous of degree \(-1\), so Theorem in [5] does not apply to it.
We observe that in Theorem 3.3, the weights $v_i \equiv 1$, $i = 1, 2$, then the inequality $\rho_1((TKf)^2) \leq C\rho_2(f^2)$ is the same as $\rho_1(TKf) \leq C\rho_2(f)$.

Finally, R. O’Neil in [3] proved that, for $K \in M_+(\mathbb{R}^2_+)$, one has, for each $f \in M_+(\mathbb{R}^2_+),$

$$\frac{1}{x} \int_0^x (TKf^*)(y) \, dy \leq \int_0^\infty K^*(xy)f^*(y) \, dy.$$ 

Given $K(x, y) = k(\sqrt{x^2 + y^2})$ as above, $K^*(t) = k(t^{1/2})$, so the right side of the O’Neil inequality is

$$\int_0^\infty k(\sqrt{xy})f^*(y) \, dy.$$

On the other hand,

$$(TKf^*)(x) = \int_0^\infty k(\sqrt{x^2 + y^2})f^*(y) \, dy.$$ 

Thus, if $\Phi_1(2t) \approx \Phi_1(t)$, for $t \gg 1$, the O’Neil inequality yields

$$\int_0^\infty \Phi_1((TKf^*)(x))u_1(x) \, dx \leq \int_0^\infty \Phi_1 \left( \int_0^\infty k(\sqrt{xy})f^*(y) \, dy \right) \, dx.$$ 

Observing that $k(\sqrt{x^2 + y^2}) = k \left( \sqrt{\frac{x^2 + y^2}{xy}} \right) = k \left( \sqrt{\frac{x}{y} + \frac{y}{x}} \right)$, thus our bound is tighter.

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