The breakup of invariant tori is one of the key mechanisms of the transition to chaos in Hamiltonian dynamics. For two dimensional systems and for frequencies like the golden mean, it has been observed that, at the transition, a sequence of periodic orbits approaches geometrically a torus of the given frequency, with a nontrivial scaling behavior. This self-similarity has been described in terms of a nontrivial fixed point of a renormalization-group transformation. The sequence of periodic orbits responsible for the breakup is generated by the continued fraction expansion of the frequency. For the extension to systems with three degrees of freedom (d.f.) involving three incommensurate frequencies, we lack a theory that generalizes the continued fraction. Numerically, three d.f. Hamiltonian systems (or equivalently four dimensional volume-preserving maps) have been studied with an extension of Greene’s criterion. The conclusion of these analysis was that there is no geometrical accumulation of periodic orbits around the critical torus, and thus absence of universality.

The aim of this Letter is to show that there is another fixed set Λ which should have roughly the following properties: Λ has an attractive integrable fixed point H₀ (trivial fixed point) that has a smooth invariant torus of a given frequency Ω₀. Every Hamiltonian in its domain of attraction D has a smooth invariant torus with frequency vector Ω. The aim is to show that there is another fixed set Λ which lies on the boundary ∂D (the critical surface) and that is attractive for every Hamiltonian on ∂D. The numerical implementation for two d.f. gives support to this picture. This is by no means trivial since the construction of the renormalization iteration is based on properties that are valid close to H₀. The numerical results indicate that the domain of convergence of the iteration R indeed extends up to the critical surface. A mathematical justification of this observation, and the formulation of conditions for its validity, are completely open problems.

In this paper, we study the extension of these ideas to systems with three frequencies, by analyzing an approximate renormalization transformation based on the work of Escande and Doveil. Its properties are found in agreement with the general picture described above. We find universal exponents associated with the fixed set Λ. They are universal in the sense that they only depend on the frequency vector considered, and not on the chosen one-parameter family. These results give new insights for the set-up of an exact renormalization scheme in the spirit of Refs. 1–8.

The transformation we define acts on the following class of Hamiltonians with three d.f., quadratic in the actions A = (A₁, A₂, A₃), and described by three even scalar functions of the angles φ = (φ₁, φ₂, φ₃):

\[
H(A, \phi) = \frac{1}{2} (1 + m(\phi)) (\Omega \cdot A)^2 + [\omega_0 + g(\phi)\Omega] \cdot A + f(\phi),
\]

where m, g, and f are of zero average. The vector ω₀ is the frequency vector of the considered torus and Ω is a vector not parallel to ω₀, with norm one: ||Ω|| =
\(|\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2\)^{1/2} = 1.\) This model is the simplest one involving invariant tori with three frequencies. It can be thought as an intermediate case between two and three d.f. Its behavior is more complex than in two d.f.: For instance, the geometric Aubry-Mather theory is, to our knowledge, not available [23]. The transformation \(R\) is defined for a fixed frequency vector \(\omega_0\) with three incommensurate components. We choose \(\omega_0 = (\sigma^2, \sigma, 1)\), where \(\sigma \approx 1.3247\) is the spiral mean: it satisfies \(\sigma^2 = \sigma + 1\) [24]. Since \(\omega_0\) is Diophantine, the KAM theorem applies to Hamiltonians [1] (although they are isoenergetically degenerate [24]) and shows the existence of a torus \(T\) with frequency vector \(\omega_0\), for a sufficiently small and smooth perturbation consisting of \(m, g\) and \(f\). The domain of existence of \(T\) corresponds to a neighborhood of the trivial fixed point

\[
H_0(A) = (\Omega \cdot A)^2/2 + \omega_0 \cdot A,
\]

for which \(T\) is located at \(A = 0\). From some of its properties, \(\sigma\) plays a similar role as the golden mean in the two d.f. case [24]. The analogy comes from the fact that one can generate rational approximants by iterating a single unimodular matrix \(N\). In what follows, we denote resonance an element of the sequence \(\nu_k = N^{k-1} \nu_1, k \geq 1\) where \(\nu_1 = (1, 0, 0)\) and

\[
N = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & -1
\end{pmatrix}.
\]

The word resonance refers to the fact that the small denominators \(\omega_0 \cdot \nu_k\) that appear in the perturbation series or in the KAM iteration, tend to zero geometrically as \(k\) increases (\(\omega_0 \cdot \nu_k = \sigma^{2-k} \to 0\) as \(k \to \infty\)). Our hypothesis (which is also the starting point of a generalization of Greene’s criterion in Ref. [10]) is that this sequence plays a leading role in the breakup of the invariant torus with frequency vector \(\omega_0\). We build an approximate scheme by considering the three main resonances \(\nu_1, \nu_2, \nu_3\). The renormalization focuses on the next smaller scale represented by the resonances \(\nu_2, \nu_3\), together with \(\nu_4 = N\nu_3 = \nu_1 - \nu_3\). It includes a partial elimination of the perturbation (the part which can be considered nonresonant on the smaller scale, namely the mode \(\nu_1\)), a shift of the resonances, a rescaling of the actions and of the energy, and a translation in the action variables. It is, in spirit, close to the type of transformation considered in Refs. [7,8].

The approximations involved in this scheme are the two main ones used by Escande and Doveil:

a) A quadratic approximation in the actions: the rescaled Hamiltonian \(R(H)\) is in general, higher than quadratic in the actions; in order to remain in the same family of Hamiltonians [2], we neglect these higher order terms.

b) A three-resonance approximation: we only keep the three main resonances at each iteration of the transformation. This is the simplest generalization to three d.f. of the Escande-Doveil approach.

Renormalization transformation. — The approximate transformation we define acts on a reduced family of Hamiltonians [1]. We consider the three most relevant Fourier modes \(\{\nu_i, i = 1, 2, 3\}\) at each step of the transformation. Hamiltonian [1] can be written as

\[
H(A, \varphi) = H_0(A) + \sum_{i=1}^3 h_i(A) \cos(\nu_i \cdot \varphi),
\]

where \(H_0\) is given by Eq. (2), and \(h_i(A) = m_{\nu_i}(\Omega \cdot A)^2/2 + g_{\nu_i} \Omega \cdot A + f_{\nu_i}\).

Our transformation combines thus five steps:

1) A canonical transformation that eliminates the first main resonance \(\nu_1\) to the order \(O(\varepsilon)\). This is performed by a Lie transformation \(U_0 : (\varphi, A) \mapsto (\varphi', A')\), generated by \(S(A, \varphi) = S_1(\varphi) \sin(\nu_1 \cdot \varphi)\). The Hamiltonian expressed in the new coordinates is given by \(H' = \exp(S)H \equiv H + \{S, H\} + \{S, \{S, H\}\}/2! + \cdots\), where \(\{,\}\) is the Poisson bracket between two scalar functions of the actions and angles. The generating function \(S\) is determined by the requirement that the order \(O(\varepsilon)\) of the mode \(\nu_1\) vanishes: \(\{S, H_0\} + h_1(A) \cos(\nu_1 \cdot \varphi) = 0\).

This equation has the solution \(S(A, \varphi) = -h_1(A) \sin(\nu_1 \cdot \varphi)/\omega(A) \cdot \nu_1\), where \(\omega(A) = \omega_0 + (\Omega \cdot A)\Omega\). This step generates arbitrary orders in the action variables. In order to map the family of Hamiltonians [1] into itself, we expand \(H'\) to quadratic order in the actions, and we neglect higher orders. The justification for this approximation is that as the torus is located at \(A = 0\) for \(H_0\), one can expect that for small \(\varepsilon\), it is close to \(A = 0\). We notice that \(h_2(A)\) and \(h_3(A)\) are not changed to the order \(O(\varepsilon^3)\). Furthermore, we neglect all the Fourier modes except \(\nu_2, \nu_3, \nu_4\). We expand the Hamiltonian to the order \(O(\varepsilon^3)\). This leads to the expression of \(H'\):

\[
H' = H_0 + h_2 \cos(\nu_2 \cdot \varphi) + h_3 \cos(\nu_3 \cdot \varphi) + \{S, h_1\}/2 + \{S, h_3\},
\]

where \(\langle \rangle\) denotes the mean value defined as \(\langle h_i(A) \rangle = \int h_i(A, \varphi) d^3\varphi/(2\pi)^3\). The last term of Eq. (3) contains the Fourier mode \(\nu_4\) of amplitude

\[
h_4(A) = \langle S_1 \nu_1 \cdot \partial h_3 + h_3 \nu_3 \cdot \partial S_1 \rangle/2,
\]

where \(\partial\) denotes the derivative with respect to \(A\). We expand \(h_4\) to quadratic order in the actions.

2) A shift of the resonances \(\nu_k \mapsto \nu_{k-1}\): a linear transformation \((A, \varphi) \mapsto (N^{-1}A, N\varphi)\), where \(N\) is \(N\) transposed. This step changes the frequency \(\omega_0\) into \(N\omega_0 = \sigma^{-1}\omega_0\) (since \(\omega_0\) is an eigenvector of \(N\) by construction).

3) We rescale the energy (or equivalently the time) by a factor \(\sigma\), in order to keep the frequency fixed at \(\omega_0\). The vector \(\Omega\) is changed into \(N\Omega\). We define the image \(\Omega'\) of \(\Omega\) by \(\Omega' = N\Omega/||N\Omega||\), in order to have \(\Omega'\) of
unit norm. We rewrite the mean-value term of Eq. (4) as \( \langle \{ S, h_1 \} \rangle = \mu (\Omega \cdot A)^2 + o (\Omega \cdot A + \text{const}) \).

4) In order to keep the image of \( g \) with zero mean-value, we eliminate \( a \) by a translation in the action variables \( A \rightarrow A + a \) (of order \( O(\varepsilon^2) \) in the \( \Omega \) direction). The constant part of the quadratic term of the resulting Hamiltonian is \( \sigma ||N\Omega||^2 (1 + \mu) (\Omega \cdot A)^2 / 2 \).

5) In order to map this Hamiltonian back into the form \( H \), we rescale the actions: \( \hat{H}(\mathbf{A}, \varphi) = \lambda \hat{H}(A/\lambda, \varphi) \), with \( \lambda = \sigma ||N\Omega||^2 (1 + \mu) \).

In summary, the transformation is equivalent to a mapping acting on a 11-dimensional space

\[
(m_{\nu_1}, g_{\nu_1}, f_{\nu_1}, m_{\nu_2}, g_{\nu_2}, f_{\nu_2}, m_{\nu_3}, g_{\nu_3}, f_{\nu_3}, \Omega) \rightarrow (m'_{\nu_1}, g'_{\nu_1}, f'_{\nu_1}, m'_{\nu_2}, g'_{\nu_2}, f'_{\nu_2}, m'_{\nu_3}, g'_{\nu_3}, f'_{\nu_3}, \Omega'),
\]

defined by the following relations

\[
m'_{\nu_1} = m_{\nu_1+1}/(1 + \mu),
\]

\[
g'_{\nu_1} = \sigma ||N\Omega|| g_{\nu_1+1},
\]

\[
f'_{\nu_1} = \lambda f_{\nu_1+1}, \quad \text{for} \ i = 1, 2
\]

\[
m'_{\nu_2} = 2h_4^{(2)}/(1 + \mu),
\]

\[
g'_{\nu_2} = \sigma ||N\Omega|| h^{(1)}_4,
\]

\[
f'_{\nu_2} = \lambda h^{(0)}_4,
\]

\[
\Omega' = \tilde{N} \Omega / ||\tilde{N}\Omega||,
\]

(5)

where \( h_4^{(i)} \) is the coefficient in \((\Omega \cdot A)^i\) of \( h_4 \) given by Eq. (4).

Critical surface of the transformation.— The numerical implementation of this scheme shows that there are two main domains separated by a critical surface: one where the iteration converges to \( H_0 \) and the other where it diverges to infinity. The renormalization-group picture for two-dimensional systems with golden mean frequency showed that this surface is the stable manifold of a nontrivial fixed point (or nontrivial fixed set related to this nontrivial fixed point by symmetries \( \Omega \)). Here, we cannot expect any relatively stable nontrivial fixed point: this has been highlighted in Ref. \( [10,22] \) and can be explained by considering the map \( \hat{H} \). This map has no stable point (there is only one hyperbolic fixed point which corresponds to \( \omega_0 \)). The eigenvalues of \( \hat{N} \) are \( \sigma^{-1} \) and \( -\sqrt{\sigma \varepsilon^{\pm \alpha}} \) where \( \alpha \approx 2 \pi \times 0.1120 \). The map \( \hat{H} \) leads asymptotically to a rotation of angle \( \alpha \). As \( \alpha \) is close to \( 2\pi/9 \), we expect the results to oscillate approximately with period 9 as it has been observed in Ref. \( [10] \). Figure 1 shows the scaling factor \( \lambda_{k+9} \) as a function of \( \lambda_k \) after \( k \) iterations on the critical surface. The points near the diagonal correspond to approximate period 9 behavior. There are however strong deviations from this behavior. Figure 2 shows the statistical distribution of the scaling factors.

The iterations on the critical surface converge to a non-periodic bounded set \( \Lambda \). Figure 3 shows the projection of \( \Lambda \) on the plane \((g_{\nu_1}, m_{\nu_1})\). This set has a codimension 1 stable manifold, i.e., one expansive direction transverse to the critical surface. This set plays, for the system we consider, the same role as the nontrivial fixed point of the renormalization-group transformation for quadratic irrational frequencies in two d.f. Hamiltonian systems. In particular, its existence implies universality for one-parameter families crossing the critical surface. We define exponents that characterize the universality class associated with the spiral mean. The mean-rescaling is defined by \( \lambda = \lim_{n \to \infty} (\prod_{k=1}^n \lambda_k)^{1/n} \), where \( \lambda_k \) is the value of the rescaling after \( k \) iterations on the critical surface. We also calculate the largest Lyapunov exponent \( \kappa \). The result we found is that these limits do not depend on the point on the critical surface where we start the iteration nor on the initial choice of \( \Omega \). The coefficients \( \kappa \) and \( \lambda \) depend only on \( \omega_0 \). Numerically, we find \( \kappa \approx 0.5870 \) and \( \lambda \approx 2.6640 \).

In conclusion, the numerical results indicate that for the spiral mean, the critical surface of the approximate renormalization transformation is the stable manifold of a critical set instead of a nontrivial fixed point. This feature depends strongly on the characteristics of the eigenvalues of \( N \). Consequently, we do not expect a priori that some specific choices of the frequency vector (or equivalently of \( N \)) could lead to a nontrivial fixed point of a renormalization-group transformation.

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FIG. 1. $\lambda_{k+9}$ as a function of $\lambda_k$.

FIG. 2. Distribution of the values of the rescalings $\lambda$ of the critical attractor $\Lambda$.

FIG. 3. Projection on the plane $(g_{\nu_1}, m_{\nu_1})$ of the critical attractor $\Lambda$.

[1] J.M. Greene, J. Math. Phys. (N.Y.) 20, 1183 (1979).
[2] L.P. Kadanoff, Phys. Rev. Lett. 47, 1641 (1981).
[3] S.J. Shenker and L.P. Kadanoff, J. Stat. Phys. 27, 631 (1982).
[4] R.S. MacKay, Physica (Amsterdam) 7D, 283 (1983).
[5] M. Govin, C. Chandre, and H.R. Jauslin, Phys. Rev. Lett. 79, 3881 (1997).
[6] C. Chandre, M. Govin, and H.R. Jauslin, Phys. Rev. E 57, 1536 (1998).
[7] C. Chandre, M. Govin, H.R. Jauslin, and H. Koch, Phys. Rev. E 57, 6612 (1998).
[8] J.J. Abad, H. Koch, and P. Wittwer, Nonlinearity 11, 1185 (1998).
[9] J.M. Mao and R.H.G. Helleman, Nuovo Cimento 104B, 177 (1989).
[10] R. Artuso, G. Casati, and D.L. Shepelyansky, Europhys. Lett. 15, 381 (1991).
[11] S. Tompaidis, Experimental Mathematics 5, 197 (1996).
[12] S. Kurosaki and Y. Aizawa, Prog. Theor. Phys. 98, 783 (1997).
[13] S.R. McKay, A.N. Berker, and S. Kirkpatrick, Phys. Rev. Lett. 48, 767 (1982).
[14] B. Derrida, J.P. Eckmann, and A. Erzan, J. Phys. A: Math. Gen. 16, 893 (1983).
[15] O.E. Lanford, in Statistical Mechanics and Field Theory: Mathematical Aspects, edited by T.C. Dorlas, N.M. Hugenholtz, and M. Winnink (Springer-Verlag, Berlin, 1986).
[16] D.A. Rand, Proc. R. Soc. Lond. A 413, 45 (1987).
[17] D.K. Umberger, J.D. Farmer, and I.I. Satija, Phys. Lett. A 114, 341 (1986).
[18] I.I. Satija, Phys. Rev. Lett. 58, 623 (1987).
[19] D.F. Escande and F. Doveil, J. Stat. Phys. 26, 257 (1981).
[20] D.F. Escande, Phys. Rep. 121, 165 (1985).
[21] C. Chandre, H.R. Jauslin, and G. Benfatto, J. Stat. Phys. 94, to appear (1999).
[22] R.S. MacKay, J.D. Meiss, and J. Stark, Phys. Lett. A 190, 417 (1994).
[23] J. Moser, Ergod. Th. & Dynam. Sys. 8, 251 (1988).
[24] S. Kim and S. Ostlund, Phys. Rev. A 34, 3426 (1986).
[25] E.M. Bollt and J.D. Meiss, Physica (Amsterdam) 66D, 282 (1993).
[26] C. Chandre and H.R. Jauslin, J. Math. Phys. 39, 5856 (1998).