GENTLY KILLING S–SPACES

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ABSTRACT. We produce a model of ZFC in which there are no locally compact first countable S–spaces, and in which $2^\aleph_0 < 2^\aleph_1$. A consequence of this is that in this model there are no locally compact, separable, hereditarily normal spaces of size $\aleph_1$, answering a question of the second author [7].

1. Introduction and Notation

In Problem 9 of [7], Nyikos asks if there is a ZFC example of a separable, hereditarily normal, locally compact space of cardinality $\aleph_1$. He notes there that for a negative answer, it suffices to produce a model of set theory in which there are neither Q–sets nor locally compact, locally countable, hereditarily normal S–spaces.

We provide such a model in this paper. In fact, in our model $2^\aleph_0 < 2^\aleph_1$ (so in particular there are no Q–sets) and there are no locally compact, first countable S–spaces at all (hence no locally compact, locally countable, hereditarily normal S–spaces).

In fact, we obtain something even more general. Recall that an S–space is a regular, hereditarily separable space which is not hereditarily Lindelöf. By switching the “separable” and “Lindelöf” we get the definition of an L–space. A simultaneous generalization of hereditarily separable and hereditarily Lindelöf spaces is the class of spaces of countable spread—those spaces in which every discrete subspace is countable. One of the basic facts in this little corner of set-theoretic topology is that if a regular space of countable spread is not hereditarily separable, it contains an L–space, and if it is not hereditarily Lindelöf it contains an S–space [8]. In our model, every locally compact 1st countable space of countable spread is hereditarily Lindelof; consequently, there are no S–spaces in locally compact 1st countable spaces of countable spread. This result, reminiscent of one half of a celebrated 1978 result of Szentmiklossy, will be discussed further at the end of the paper in connection with a fifty-year-old problem of M. Katětov.

These concepts and results have elegant translations in terms of Boolean algebras via Stone duality. The Stone space $\mathcal{S}(A)$ of a Boolean algebra $A$

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is hereditarily Lindelöf iff every ideal of \( A \) is countably generated, and first countable iff every maximal ideal is countably generated. Also, \( S(A) \) is of countable spread iff every minimal set of generators for an ideal is countable. (An ideal is said to be minimally generated if it has a generating set \( D \) such that no member of \( D \) is in the ideal generated by the remaining members.) Hence we now know that \( 2^{\aleph_0} < 2^{\aleph_1} \) is consistent with the following statement: if a Boolean algebra \( A \) has the property that every minimal set of generators for an ideal is countable, and every maximal ideal of \( A \) is countably generated, then every ideal of \( A \) is countably generated. On the other hand, this statement has long been known to be incompatible with CH.

Note that there are restrictions on such models. In [6] it is shown that CH implies the existence of a locally compact first countable \( S \)-space, and in Chapter 2 of [10] this is shown to follow from the weaker axiom \( b = \aleph_1 \). Thus the fact that our model satisfies \( b = \aleph_2 \) is no accident of the proof — something along these lines is required.

As far as background goes, we will assume a reasonable familiarity with topological notions such as filters of closed sets and free sequences. We also use a lot of set theory — we will assume that the reader is used to working with proper notions of forcing.

Our main tool is the use of totally proper notions of forcing that satisfy the \( \aleph_2 \)-p.i.c. (properness isomorphism condition). We will take a moment to recall the needed definitions.

**Definition 1.1.**

1. Let \( P \) be a notion of forcing, and \( N \) a countable elementary submodel of \( H(\lambda) \) for some large regular lambda with \( P \in N \). An \((N, P)\)-generic sequence is a decreasing sequence of conditions \( \{p_n : n \in \omega\} \subseteq N \cap P \) such that for every dense open \( D \subseteq P \) in \( N \), there is an \( n \) with \( p_n \in D \).
2. A notion of forcing \( P \) is said to be totally proper (also known as NNR proper) if for every \( N \) as above and \( p \in N \cap P \), there is an \((N, P)\)-generic sequence \( \{p_n : n \in \omega\} \) with \( p_0 = p \) that has a lower bound.

The following claim summarizes the properties of totally proper notions of forcing that we will need. The proofs are not difficult, and they are explicitly worked out in [3] and [4].

**Claim 1.2.** Let \( P \) be a totally proper notion of forcing.

1. \( P \) adds no new reals; in fact, forcing with \( P \) adds no new countable sequences of elements from the ground model.
2. If \( G \subseteq P \) is generic, then \( G \) is countably closed. In fact, every countable subset of \( G \) has a lower bound in \( G \).

The following definition is from Chapter VII of [9].

**Definition 1.3.** \( P \) satisfies the \( \aleph_2 \)-p.i.c. provided the following holds (for \( \lambda \) a large enough regular cardinal):

If
1. \( i < j < \aleph_2 \)
2. \( N_i \) and \( N_j \) are countable elementary submodels of \( H(\lambda) \)
3. \( i \in N_i, \ j \in N_j \)
4. \( N_i \cap \aleph_2 \subseteq j \)
5. \( N_i \cap i = N_j \cap j \)
6. \( h \) is an isomorphism from \( N_i \) onto \( N_j \)
7. \( h(i) = j \)
8. \( h \) is the identity map on \( N_i \cap N_j \)
9. \( P \in N_i \cap N_j \)
10. \( p \in N_i \cap P \)

then (letting \( \dot{G} \) be the \( P \)-name for the generic set) there is a \( q \in P \) such that:

11. \( q \models "(\forall r \in N_i \cap P)\{r \in \dot{G} \iff h(r) \in \dot{G}\}" \)
12. \( q \models "p \in \dot{G}" \)
13. \( q \) is \((N_i, P)\)-generic.

Notice that if \( N_i \) and \( N_j \) are as in the above definition, then \( N_i \) and \( N_j \) contain the same hereditarily countable sets. This follows because \( h \) is an isomorphism. In particular, \( N_i \cap \omega_1 \) and \( N_j \cap \omega_1 \) are the same ordinal. We also note that in both of the previous two definitions, it does not matter if we require that the models under consideration contain a fixed parameter \( x \in H(\lambda) \).

The properties of \( \aleph_2 \)-p.i.c. forcings that we utilize will be spelled out when we build our model in the last section of the paper.

2. Handling \( P \)-ideals

**Definition 2.1.** A \( P \)-ideal in \([\omega_1]^{\aleph_0}\) (the set of all countable subsets of \( \omega_1 \)) is a set \( \mathcal{I} \subseteq [\omega_1]^{\aleph_0} \) such that

- if \( A \) and \( B \) are in \( \mathcal{I} \), then so is \( A \cup B \)
- if \( A \in \mathcal{I} \) and \( B \subseteq A \), then \( B \in \mathcal{I} \)
- if \( A \in \mathcal{I} \) and \( B =^* A \), then \( B \in \mathcal{I} \)
- if \( A_n \in \mathcal{I} \) for each \( n \in \omega \), then there is an \( A \in \mathcal{I} \) such that \( A_n \subseteq^* A \) for each \( n \).

**Definition 2.2.** Let \( \mathcal{I} \) be a \( P \)-ideal in \([\omega_1]^{\aleph_0}\) generated by a set of size \( \aleph_1 \). A generating sequence for \( \mathcal{I} \) is a sequence \( \{A_\alpha : \alpha < \omega_1\} \) such that

- \( A_\alpha \subseteq \alpha \)
- if \( \alpha < \beta \) then \( A_\alpha \subseteq^* A_\beta \)
- if \( A \in \mathcal{I} \), then there is an \( \alpha \) with \( A \subseteq A_\alpha \).

Clearly every such \( \mathcal{I} \) has a generating sequence.
Our goal in this section is (assuming CH holds) to define a notion of forcing (which we call \(P(I)\)) with the property that any \(P\)-ideal \(I \subseteq [\omega_1]^{\aleph_0}\) in the ground model admits an uncountable set \(A\) in the extension satisfying \([A]^{\aleph_0} \subseteq I\) or \([A]^{\aleph_0} \cap I = \emptyset\). The partial order we use is a modification of one of the posets from [2], itself a modification of the notion of forcing used in [1].

Assume CH, and let \(I = \langle I_\xi : \xi < \kappa \rangle\) be a sequence of \(P\)-ideals in \([\omega_1]^{\aleph_0}\).

Let \(\{A_\xi^\alpha : \alpha < \omega_1\}\) be a generating sequence for \(I_\xi\) (such a sequence exists because CH holds). The notion of forcing we define depends on our choice of generating sequences, but we abuse notation and call the notion of forcing \(P(I)\).

**Definition 2.3.** A promise is a function \(f\) such that

- \(\text{dom } f\) is an uncountable subset of \(\omega_1\)
- \(f(\alpha)\) is a finite subset of \(\alpha\)

**Definition 2.4.** A condition \(p \in P(I)\) is a pair \((a_p, \Phi_p)\) such that

1. \(a_p\) is a function
2. \(\text{dom } a_p\) is a countable subset of \(\kappa \times \omega_1\)
3. \(\text{ran } a_p \subseteq 2\)
4. for \(\xi < \kappa, [p]_\xi := \{\zeta < \omega_1 : a_p(\xi, \zeta) = 1\}\) is in \(I_\xi\) (so \([p]_\xi = \emptyset\) for all but countably many \(\xi\))
5. \(\Phi_p\) is a countable collection of pairs \((v, f)\), where \(v \subseteq \kappa\) is finite and \(f\) is a promise.

A condition \(q\) extends \(p\) if

6. \(a_q \supseteq a_p, \Phi_q \supseteq \Phi_p\)
7. for \((v, f) \in \Phi_p, \)

\[
Y(v, f, q, p) = \{\alpha \in \text{dom } f : (\forall \xi \in v)([q]_\xi \subseteq [p]_\xi \subseteq A_\xi^\alpha \setminus f(\alpha))\}
\]

is uncountable, and

\[
(v, f \upharpoonright Y(v, f, q, p)) \in \Phi_q.
\]

The intent of \(P(I)\) is to attempt to adjoin for each \(\xi < \kappa\) an uncountable set \(A_\xi\) with \([A_\xi]^{\aleph_0}\) contained in \(I_\xi\). A condition gives us an approximation to \(A_\xi\) for countably many \(\xi\), as well as some constraints on future growth of these approximations. A pair \((v, f) \in \Phi_p\) puts limits on how our approximation to \(A_\xi\) can grow for the finitely many \(\xi \in v\). It may be that the forcing fails to produce an uncountable \(A_\xi\) for some \(\xi\), but we show that we can do so in every situation where we need it.

**Definition 2.5.** Let \(p\) be a condition in \(P(I)\), let \(D\) be a dense open subset of \(P(I)\), and let \(v\) be a finite subset of \(\kappa\). An ordinal \(\alpha\) is bad for \((v, p, D)\) if there is an \(F_\alpha \in [\alpha]^{<\aleph_0}\) such that there is no \(q \leq p\) in \(D\) with

\[
[q]_\xi \setminus [p]_\xi \subseteq A_\xi^\alpha \setminus F_\alpha
\]
Proof. Let \( a \) be the function with domain \( \text{Bad}(v, p, D) \) that sends \( \alpha \) to \( F_\alpha \), so \( f \) is a promise. Let \( r \) be the condition in \( \mathcal{P}(\mathbb{I}) \) with \( a_r = a_\alpha \), and \( \Phi_r = \Phi_p \cup \{(v, f)\} \). Clearly \( r \) extends \( p \). Now let \( q \leq r \) be in \( D \). By definition, there are uncountably many \( \alpha \in \text{dom}(f) \) such that if \( \xi \in v \) then \([q]_\xi \setminus [r]_\xi \) is a subset of \( A_\alpha^\xi \setminus f(\alpha) \). This is a contradiction, as any \( \alpha \) is bad for \( (v, p, D) \), yet \( q \in D \) and

\[
[q]_\xi \setminus [p]_\xi \subseteq A_\alpha^\xi \setminus f(\alpha)
\]

for all \( \xi \in v \).

\( \Box \)

**Theorem 1.** \( \mathcal{P}(\mathbb{I}) \) satisfies the \( \aleph_2 \)-p.i.c.

**Proof.** Let \( i, j, N_i, N_j, h, \) and \( p \) be as in Definition 1.3. For \( r \in N_i \cap \mathcal{P}(\mathbb{I}) \), we define

\[
(2.3) \quad r \cup h(r) := (a_r \cup h(a_r), \Phi_r \cup h(\Phi_r)).
\]

**Lemma 2.7.** Assume that \( r \in N_i \cap \mathcal{P}(\mathbb{I}) \).

1. \( r \cup h(r) \in \mathcal{P}(\mathbb{I}) \).
2. \( r \cup h(r) \) extends both \( r \) and \( h(r) \).
3. If \( s \in N_i \cap \mathcal{P}(\mathbb{I}) \) and \( r \leq s \), then \( r \cup h(r) \leq s \cup h(s) \).

**Proof.** Left to reader.

\( \Box \)

Now let \( \delta = N_i \cap \omega_1 = N_j \cap \omega_1 \), and let \( \{D_n : n \in \omega\} \) enumerate the dense open subsets of \( \mathcal{P}(\mathbb{I}) \) that are members of \( N_i \). Our goal is to build a decreasing sequence of conditions \( \{p_n : n \in \omega\} \) in \( N_i \cap \mathcal{P}(\mathbb{I}) \) such that \( p_0 = p \), \( p_{n+1} \in N_i \cap D_n \), and such that the sequence \( \{p_n \cup h(p_n) : n \in \omega\} \) has a lower bound \( q \). The next lemma shows that this will be sufficient.

**Lemma 2.8.** Let \( \{p_n : n \in \omega\} \) be an \( (N_i, \mathcal{P}(\mathbb{I})) \)-generic sequence.

1. \( \{h(p_n) : n \in \omega\} \) is an \( (N_j, \mathcal{P}(\mathbb{I})) \)-generic sequence.
2. If \( \{p_n \cup h(p_n) : n \in \omega\} \) has a lower bound \( q \), then \( q \) satisfies conditions 11 and 13 of Definition 1.3.

**Proof.** The first clause follows immediately from the fact that \( h \) is an isomorphism mapping \( N_i \) onto \( N_j \). For the second clause, note

\[
(2.4) \quad q \models \text{“}r \in N_i \cap \hat{G} \text{”} \iff r \in N_i \text{ and } \exists n(p_n \leq r).
\]

This is because for each \( r \in N_i \cap \mathcal{P}(\mathbb{I}) \), the set of conditions that extend \( r \) or that are incompatible with \( r \) is a dense open subset of \( \mathcal{P}(\mathbb{I}) \) that is in \( N_i \), and hence for some \( n \) either \( p_n \) extends \( r \) or \( p_n \) incompatible with \( r \). Similarly, we have

\[
(2.5) \quad q \models \text{“}r \in N_j \cap \hat{G} \text{”} \iff r \in N_j \text{ and } \exists n(h(p_n) \leq r).
\]
Now clause 11 of Definition 1.3 follows easily. Clause 13 holds because the $p_n$’s are an $(N_i, P(I))$–generic sequence.

Recall that $\delta = N_i \cap \omega_1 = N_j \cap \omega_1$, and let $\{\gamma_n : n \in \omega\}$ enumerate $N_i \cap \kappa$. We construct by induction on $n \in \omega$ objects $p_n, F_n, q_n$ and $u_n$ such that

(i) $p_0 = p, F_0 = \emptyset, u_0 = \emptyset$
(ii) $q_n = p_n \cup h(p_n)$
(iii) $p_{n+1} \in N_i \cap D_n$
(iv) $F_n$ is a finite subset of $\delta$
(v) $u_n$ is a finite subset of $N_i \cap \kappa$
(vi) $p_{n+1} \leq p_n$
(vii) $F_{n+1} \supseteq F_n$
(viii) $u_{n+1} \supseteq u_n$
(ix) $\{\gamma_m : m < n\} \subseteq u_n$
(x) for $\gamma \in u_{n+1} \cup h(u_{n+1}), [q_{n+1}]_{\gamma} \setminus [q_n]_{\gamma} \subseteq A^\gamma_\delta \setminus F_{n+1}$
(xi) if $(v, f) \in \Phi_{q_k}$ for some $k$, then there is a stage $n \geq k$ for which

\begin{equation}
 v \subseteq u_{n+1} \cup h(u_{n+1})
\end{equation}

and

\begin{equation}
 \{\alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v)(A^\xi_\delta \setminus F_{n+1} \subseteq A^\xi_\alpha \setminus f(\alpha))\}
\end{equation}

is uncountable.

We assume that we have fixed a bookkeeping system so that at each stage of the induction we are handed a pair $(v, f)$ from some earlier $\Phi_{q_k}$ for which we must ensure (xi), and such that every such $(v, f)$ appearing along the way is treated in this manner.

There is nothing to be done at stage 0, so assume we have carried out the induction through stage $n$. At stage $n + 1$, we will be handed $p_n, F_n, q_n,$ and $u_n$, and our bookkeeping hands us $(v, f) \in \Phi_{q_k}$ for some $k \leq n$.

To start, we choose $u_{n+1} \supseteq u_n \cup \{\gamma_n\}$ satisfying (v), but large enough so that $v \subseteq u_{n+1} \cup h(u_{n+1})$. This means that (v), (viii), and (ix) hold.

**Claim 2.9.** If $f$ is a promise, $B \subseteq \text{dom } f$ uncountable, $v \subseteq \kappa$ finite, and $\beta < \omega_1$, then there is a finite $F \subseteq \beta$ such that

\[ \{\alpha \in B : (\forall \xi \in v)(A^\xi_\beta \setminus F \subseteq A^\xi_\alpha \setminus f(\alpha))\} \]

is uncountable.

**Proof.** Straightforward, by induction on $|v|$.  

(Although the preceding claim has a trivial proof, it does not generalize to the context of the next section and in some sense this fact is the reason why the next section is so complicated.)
Now apply the preceding claim to \( v, f, Y(v, f, q_n, q_k), u_{n+1} \cup h(u_{n+1}), \) and \( \delta \) to get a finite \( \bar{F} \subseteq \delta \) such that

\[
(2.8) \quad \{ \alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in u_{n+1} \cup h(u_{n+1})) (A^\xi_\delta \setminus \bar{F} \subseteq A^\xi_\delta \setminus f(\alpha)) \}
\]
is uncountable. In particular, our choice of \( u_{n+1} \) implies

\[
(2.9) \quad \{ \alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v) (A^\xi_\delta \setminus \bar{F} \subseteq A^\xi_\delta \setminus f(\alpha)) \}
\]
is uncountable. Now let \( F_{n+1} = F_n \cup \bar{F} \). Clearly we have satisfied (iv) and (vii).

Next, we choose \( \beta < \omega_1 \) such that

\[
(2.10) \quad N_i \models \beta \notin \text{Bad}(p_n, D_n).
\]

For each \( \gamma \in u_{n+1} \cup h(u_{n+1}) \) there is a finite \( G_\gamma \subseteq \beta \) such that \( A^\gamma_\beta \setminus G_\gamma \subseteq A^\gamma_\delta \setminus F_{n+1} \), so there is a finite \( G \subseteq \beta \) such that

\[
(2.11) \quad (\forall \gamma \in u_{n+1} \cup h(u_{n+1})) (A^\gamma_\beta \setminus G \subseteq A^\gamma_\delta \setminus F_{n+1}).
\]

Note that both \( \beta \) and \( G \) are in \( N_i \cap N_j \), and hence are fixed by \( h \). By (2.10), we can find \( p_{n+1} \in N_i \) such that \( p_{n+1} \leq p_n, p_{n+1} \in D_n \), and

\[
(2.12) \quad N_i \models (\forall \gamma \in u_{n+1}) ([p_{n+1}]_\gamma \setminus [p_n]_\gamma \subseteq A^\gamma_\beta \setminus G).
\]

Applying \( h \), we see that

\[
(2.13) \quad N_j \models (\forall \gamma \in h(u_{n+1})) ([h(p_{n+1})]_\gamma \setminus [h(p_n)]_\gamma \subseteq A^\gamma_\beta \setminus G).
\]

Thus

\[
(2.14) \quad (\forall \gamma \in u_{n+1} \cup h(u_{n+1})) ([q_{n+1}]_\gamma \setminus [q_n]_\gamma \subseteq A^\gamma_\beta \setminus G \subseteq A^\gamma_\delta \setminus F_{n+1}).
\]

Our choice of \( p_{n+1} \) (and \( q_{n+1} \)) satisfies (ii), (iii), (vi), and (x). Since \( \bar{F} \subseteq F_{n+1} \), we have that (xi) is satisfied for this particular \((v, f)\).

Now we need to verify that the sequence \( \{q_n : n \in \omega\} \) has a lower bound \( q \). To start, we define

\[
a_q = \bigcup_{n \in \omega} a_{q_n}
\]

\[
[q]_\xi = \bigcup_{n \in \omega} [q_n]_\xi
\]

**Claim 2.10.**

1. \( a_q : (N_i \cup N_j) \cap \kappa \to 2 \)
2. If \( \xi \in N_i \cap \kappa \), then \( [q]_\xi = \bigcup \{[p_n]_\xi : n \in \omega\} \). If \( \xi \in N_j \cap \kappa \), then \( [q]_\xi = \bigcup \{[h(p_n)]_\xi : n \in \omega\} \).
3. \( [q]_\xi \in \mathcal{I}_\xi \) for \( \xi < \kappa \).
Proof of Claim. Part 1 of the claim follows because the sequence \( \{p_n : n \in \omega\} \) (resp. \( \{h(p_n) : n \in \omega\} \)) meets every dense set in \( P(\mathbb{I}) \) that is a member of \( N_i \) (resp. \( N_j \)). Part 2 follows as in the proof of Lemma 2.7. For the last part, if \( \xi \notin (N_i \cup N_j) \cap \kappa \) there is nothing to check, so assume \( \xi \in (N_i \cup N_j) \cap \kappa \), and fix \( n \) such that \( \xi \in \{\gamma_n, h(\gamma_n)\} \). Our construction guarantees that \( [q]_{\xi} \subseteq [q_n]_{\xi} \cup A^\xi \), and this latter set is in \( \mathcal{I}_\xi \). 

\( \square \)

Claim 2.11. If \( k \in \omega \) and \( (v, f) \in \Phi_{q_k} \), then

\[
(2.16) \quad K(v, f, k) := \{\alpha \in \text{dom} f : (\forall \xi \in v)(([q]_{\xi} \setminus [q_k]_{\xi} \subseteq A^\xi \setminus f(\alpha)))\}
\]

is uncountable.

Proof. Let \( n \geq k \) be such that our bookkeeping handed us the promise \( (v, f) \) at stage \( n + 1 \) of the construction. The actions we took at stage \( n + 1 \) ensure that

\[
(2.17) \quad A := \{\alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v)(A^\xi \setminus F_{n+1} \subseteq A^\xi \setminus f(\alpha))\}
\]

is uncountable. We claim that \( A \subseteq K(v, f, k) \); to see this fix \( \alpha \in A \), and let \( \xi \in v \) be arbitrary. We must verify that \( [q]_{\xi} \setminus [q_k]_{\xi} \) is a subset of \( A^\xi \setminus f(\alpha) \).

\[
[q]_{\xi} \setminus [q_k]_{\xi} = ([q]_{\xi} \setminus [q_n]_{\xi}) \cup ([q_n]_{\xi} \setminus [q_k]_{\xi}) \\
\subseteq \bigcup_{m \geq n} ([q_m]_{\xi} \setminus [q_n]_{\xi}) \cup A^\xi \setminus f(\alpha) \\
\subseteq A^\xi \setminus F_{n+1} \cup A^\xi \setminus f(\alpha) \\
\subseteq A^\xi \setminus f(\alpha)
\]

Notice that in obtaining the second line, we used that \( \alpha \in Y(v, f, q_n, q_k) \), and to obtain the third line we used requirement (x) of our construction and the fact that \( v \subseteq u_{n+1} \cup h(u_{n+1}) \).

\( \square \)

Now we define

\[
(2.18) \quad \Phi_q = \bigcup_{n \in \omega} \Phi_{q_n} \cup \bigcup_{n \in \omega} \{ (v, f \upharpoonright K(v, f, n)) : (v, f) \in \Phi_{q_n} \}
\]

and \( q = (u_q, x_q, \Phi_q) \) is a lower bound for the sequence \( \{q_n : n \in \omega\} \) as desired.

\( \square \)

Notice that in our proof, the only relevant properties of \( h \) were that it is an isomorphism from \( N_i \) onto \( N_j \) that is the identity on \( N_i \cap N_j \) — the other requirements from Definition 1.3 were not used. In particular, our proof goes through in the case that \( h \) is actually the identity map (so \( N_i = N_j \)). Thus we obtain the following.

Theorem 2. \( P(\mathbb{I}) \) is totally proper.

We are still not through, however, as we have not yet verified that \( P(\mathbb{I}) \) lives up to its billing.
Definition 2.12. Let $f$ be a promise and $v \subseteq \kappa$ finite. For $\xi \in v$, we define a set $\text{Ban}_\xi(v, f)$ by
\begin{equation}
\beta \in \text{Ban}_\xi(v, f) \iff \{\alpha \in \text{dom } f : \beta \in A^\xi_\alpha \setminus f(\alpha)\}
\end{equation}
is countable. If $\xi \notin v$ then let $\text{Ban}_\xi(v, f) = \emptyset$.

Proposition 2.13. If $\xi < \kappa$ and there is no uncountable $A \subseteq \omega_1$ with $[A]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$, then $\text{Ban}_\xi(v, f)$ is countable.

Proof. We can assume that $\xi \in v$ as otherwise there is nothing to prove. Our assumption on $\mathcal{I}_\xi$ means that there is an infinite $B \subseteq A$ with $B \in \mathcal{I}_\xi$. For each $\alpha \in \text{dom } f$, there is a finite set $F_\alpha$ for which $B \setminus F_\alpha \subseteq A^\xi_\alpha \setminus f(\alpha)$. Thus there is a single finite $F$ for which
\begin{equation}
\{\alpha \in \text{dom } f : B \setminus F \subseteq A^\xi_\alpha \setminus f(\alpha)\}
\end{equation}
is uncountable. Therefore any member of $B \setminus F$ is not in $\text{Ban}_\xi(v, f)$, a contradiction. \hfill \Box

Proposition 2.14. If $\xi < \kappa$ and there is no uncountable $A \subseteq \omega_1$ with $[A]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$, then for each $\gamma < \omega_1$, the set of conditions $p$ for which $[p]_\xi \gamma$ is non-empty is dense in $P(\mathcal{I})$.

Proof. Let $\xi$ and $\gamma$ be as in the assumption, and let $p \in P(\mathcal{I})$ be arbitrary. By the previous proposition,
\begin{equation}
\bigcup \{\text{Ban}_\xi(v, f) : (v, f) \in \Phi_p\}
\end{equation}
is countable (as $\Phi_p$ is countable), hence there is an $\alpha > \gamma$ not in $\text{Ban}_\xi(v, f)$ for any $(v, f) \in \Phi_p$. It is straightforward to see that there is a $q \leq p$ with $\alpha \in [q]_\xi$. \hfill \Box

Conclusion 1. Assume CH, and let $\mathcal{I} = \langle I_\xi : \xi < \kappa \rangle$ be a list of P–ideals in $[\omega_1]^{\aleph_0}$. Then there is a totally proper notion of forcing $P(\mathcal{I})$, satisfying the $\aleph_2$–p.i.c., so that in the generic extension, for each $\xi < \kappa$ there is an uncountable $A_\xi \subseteq \omega_1$ for which either $[A_\xi]^{\aleph_0} \subseteq I_\xi$ or $[A_\xi]^{\aleph_0} \cap I_\xi = \emptyset$.

Proof. We have all the ingredients of the proof already. By Theorems 1 and 2, we know $P(\mathcal{I})$ is totally proper and satisfies the $\aleph_2$–p.i.c. Fix $\xi < \kappa$, assume that $G \subseteq P(\mathcal{I})$ is generic over $V$, and work for a moment in $V[G]$.

If in $V$ there is an uncountable $A_\xi$ with $[A_\xi]^{\aleph_0} \cap I_\xi = \emptyset$, then $A_\xi$ still has this property in $V[G]$. (Note that since $P(\mathcal{I})$ is totally proper, no new countable subsets of $\omega_1$ are added, so $I_\xi$ is unchanged by passing to $V[G]$.) If no such set exists in $V$, then the set
\begin{equation}
A_\xi := \bigcup_{p \in G} [p]_\xi
\end{equation}
is uncountable by the previous proposition, and $[A_\xi]^{\aleph_0} \subseteq I_\xi$ by definition of our forcing notion. \hfill \Box
3. Handling Relevant Spaces

Our goal in this section is to build, assuming that CH holds, a totally proper notion of forcing having the $\aleph_2$–p.i.c. that destroys all first countable, countably compact, non–compact S–spaces in the ground model. In fact, we do a little better than this — if $X$ is a first countable, countably compact, non–compact regular space with no uncountable free sequences, then after we force with our poset, $X$ acquires an uncountable free sequence. The partial order we use is a modification of that used in [4], although things do not work as smoothly as they did in the last section.

**Proposition 3.1.** Suppose $U$ is a countably complete (not necessarily maximal) filter of closed subsets of the space $X$, and suppose $Z \subseteq X$ meets every set in $U$. If $\text{cl}_{X} Z_0 \notin U$ for every countable subset $Z_0$ of $Z$, then $X$ has an uncountable free sequence.

The proof of the proposition is straightforward. As a corollary, we note that $U$ is generated by separable sets if $X$ has no uncountable free sequences, and so under CH the filter $U$ is generated by a family of size at most $\aleph_1$.

**Definition 3.2.** If $U$ if a filter of closed subsets of $X$, we say that $Y \subseteq X$ is $U$–large if $Y \cap A \neq \emptyset$ for every $A \in U$. We say that $Y \subseteq X$ diagonalizes $U$ if $Y$ is $U$–large and $Y \setminus A$ is countable for every set $A \in U$.

Notice that if $U$ is countably complete and $U$ is generated by a set of size at most $\aleph_1$, then every $U$–large set $Y$ has a subset $Z$ that diagonalizes $U$. If in addition $U$ is not fixed, then every uncountable subset of $Z$ will diagonalize $U$ as well.

Let us call a space $X$ *relevant* if $X$ is first countable, countably compact, non–compact, regular, $|X| = \aleph_1$, and $X$ has no uncountable free sequences. For each relevant $X$, we fix a maximal filter of closed sets $U_X$ that is not fixed. Since we have assumed CH holds and $X$ is relevant, we can fix a set $Y_X \subseteq X$ that diagonalizes $U_X$. By passing to a subset if necessary, we may assume that $Y_X$ is right–separated in type $\omega_1$.

Since $U_X$ is a maximal filter of closed sets, this means that $Y_X$ is a sub–Ostaszewski subspace of $X$, i.e., every closed subset of $Y_X$ is either countable or co–countable. The filter $U_X$ is reconstructible from $Y_X$ as the set of all closed subsets of $X$ that meet $Y_X$ uncountably often.

We assume that each $Y_X$ has $\omega_1$ as an underlying set, and that this correspondence is set up so that initial segments are open. Thus given a collection of relevant spaces, a countable ordinal $\alpha$ is viewed as a point in each of the spaces.

We also fix a function $B$ so that for each relevant space $X$ and ordinal $\alpha < \omega_1$, $\{ B(X, \alpha, n) : n \in \omega \}$ is a decreasing neighborhood base for $\alpha$ as a point in $X$. We will need one more definition before defining our notion of forcing.

**Definition 3.3.** A promise $f$ is a function whose domain is an uncountable subset of $\omega_1$ and whose range is a subset of $\omega$. 
Until said otherwise, $\mathcal{X} = \{X_\xi : \xi < \kappa\}$ is a collection of relevant spaces, and CH holds.

**Definition 3.4.** A condition $p \in P(\mathcal{X})$ is a pair $(a_p, \Phi_p)$ such that
1. $a_p$ is a function
2. $\text{dom} \ a_p$ is a countable subset of $\{ (\xi, x) : \xi < \kappa \text{ and } x \in X_\xi \}$
3. $\text{ran} \ a_p \subseteq 2$
4. for each $\xi < \kappa$, $[p]_\xi := \{ x \in X_\xi : a_p(\xi, x) = 1 \}$ satisfies $\text{cl} X_\xi [p]_\xi \notin U_\xi$
5. $\Phi_p$ is a countable set of pairs $(v, f)$ where $v \subseteq \kappa$ is finite and $f$ is a promise.

A condition $q$ extends $p$ if
6. $a_q \supseteq a_p$, $\Phi_q \supseteq \Phi_p$
7. for $(v, f) \in \Phi_p$,

$$Y(v, f, q, p) := \{ \alpha \in \text{dom} \ f : (\forall \xi \in v)[ [q]_\xi \setminus [p]_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha)) ] \}$$

is uncountable, and

$$\text{(3.2) } (v, f \upharpoonright Y(v, f, q, p)) \in \Phi_q.$$

The notion of forcing we have described (seemingly) need not be proper. If, however, we put restrictions on the family $\mathcal{X}$ we get a proper notion of forcing. We will need some notation to express the necessary ideas.

**Definition 3.5.** Let $v \subseteq \kappa$ be finite. We define

$$X_v = \prod_{\xi \in v} X_\xi,$$

and we let $U_v$ be the filter of closed subsets of $X_v$ that is generated by sets of the form $\prod_{\xi \in v} A_\xi$, where $A_\xi \in U_\xi$.

Note that $U_v$ will be countably complete and generated by $\leq \aleph_1$ sets because each $U_\xi$ is.

**Definition 3.6.** Let $v \subseteq \kappa$ be finite, and let $f$ be a promise. A point $(x_\xi : \xi \in v) \in X_v$ is banned by $(v, f)$ if

$$\text{(3.4) } \{ \alpha \in \text{dom} \ f : (\forall \xi \in v)[x_\xi \in \mathcal{B}(X_\xi, \alpha, f(\alpha)) ] \}$$

is countable. We let $\text{Ban}(v, f)$ be the collection of all points in $X_v$ that are banned by $(v, f)$.

**Definition 3.7.** Let $v \subseteq \kappa$ be finite. We say $v$ is *dangerous* if there is a promise $f$ such that $\text{Ban}(v, f)$ is $U_v$–large. $\mathcal{X}$ is *safe* if no finite $v \subseteq \kappa$ is dangerous.

Our definition of “safe” was formulated so that the proof of the following theorem goes through — the proof of Claim 3.11 is the place where we really need it.
Theorem 3. If $\mathcal{X} = \{X_\xi : \xi < \kappa\}$ is safe, then $P(\mathcal{X})$ is totally proper.

Before we commence with the proof of this theorem, we need a definition and lemma.

Definition 3.8. Let $v \subseteq \kappa$ be finite, $p \in P(\mathcal{X})$, and let $D \subseteq P(\mathcal{X})$ be dense. An ordinal $\gamma < \omega_1$ is said to be bad for $(v, p, D)$ if there is an $n$ such that there is no $q \leq p$ in $D$ such that for all $\xi \in v$,

$$\left[ q_\xi \setminus [p]_\xi \subseteq B(X_\xi, \gamma, n) \right].$$

We let $\text{Bad}(v, p, D)$ be the collection of all $\gamma < \omega_1$ that are bad for $(v, p, D)$.

Lemma 3.9. $\text{Bad}(v, p, D)$ is countable.

Proof. Suppose not. The function $f$ with domain $\text{Bad}(v, p, D)$ that sends $\gamma$ to the $n$ that witnesses $\gamma \in \text{Bad}(v, p, D)$ is a promise. Now we define $r = (a_p, \Phi_p \cup \{(v, f)\})$. Clearly $r \leq p$ in $P(\mathcal{X})$, and since $D$ is dense there is a $q \leq r$ in $D$. Now $Y(v, f, q, r)$ is uncountable, and for $\gamma \in Y(v, f, q, r)$ and $\xi \in v$ we have

$$\left[ q_\xi \setminus [p]_\xi \subseteq B(X_\xi, \gamma, f(\gamma)) \right]$$

and this contradicts the definition of $f$. \qed

Lemma 3.10. Let $(v, f)$ be a promise, and suppose $(x_\xi : \xi \in v)$ is not in $\text{Ban}(v, f)$. Then there is $(U_\xi : \xi \in v)$ such that $U_\xi$ is a neighborhood of $x_\xi \in X_\xi$ and

$$\{\alpha \in \text{dom } f : (\forall \xi \in v)\left[ U_\xi \subseteq B(X_\xi, \alpha, f(\alpha)) \right] \}$$

is uncountable. In particular, $\text{Ban}(v, f)$ is a closed subset of $X_v$.

Proof. Let $\{V_n : n \in \omega\}$ be a neighborhood base for $(x_\xi : \xi \in v)$ in the (first countable) space $X_v$, and define

$$A = \{\alpha \in \text{dom } f : (\forall \xi \in v)\left[ x_\xi \in B(X_\xi, \alpha, f(\alpha)) \right] \}.$$

By assumption, $A$ is uncountable, and for each $\alpha \in A$ there is an $n$ for which

$$V_n \subseteq \prod_{\xi \in v} B(X_\xi, \alpha, f(\alpha)).$$

Thus there is a single $n$ for which

$$\{\alpha \in A : V_n \subseteq \prod_{\xi \in v} B(X_\xi, \alpha, f(\alpha))\}$$

is uncountable. The definition of the product topology then gives us the $U_\xi$’s that we need. \qed
Proof of Theorem 3. Let \( N < H(\lambda) \) be countable with \( P(\mathcal{X}) \in N \). Let \( p \in N \cap P(\mathcal{X}) \) be arbitrary, and let \( \{D_n : n \in \omega\} \) list the dense open subsets of \( P(\mathcal{X}) \) that are members of \( N \). Let \( \delta = N \cap \omega_1 \), and let \( \{\gamma_n : n < \omega\} \) enumerate \( N \cap \kappa \).

Since all the spaces in \( \mathcal{X} \) are countably compact and \( N \) is countable, there is a sequence \( \{\delta_n : n \in \omega\} \) increasing and cofinal in \( \delta \) such that for every \( \xi \in N \cap \kappa \), the sequence \( \{\delta_n : n \in \omega\} \) converges in \( X_\xi \) to a point \( z_\xi \).

**Claim 3.11.** If \( v = \{\xi_0, \ldots, \xi_{n-1}\} \subseteq N \cap \kappa \) and \( f \in N \) is a promise, then \((z_{\xi_0}, \ldots, z_{\xi_{n-1}})\) is not banned by \((v, f)\).

**Proof.** Since \( \mathcal{X} \) is safe and \((v, f) \in N \), there are sets \( A_i \in \mathcal{U}_\xi \cap N \) for \( i < n \) such that \( A_0 \times \cdots \times A_{n-1} \) is disjoint to \( \text{Ban}(v, f) \). Since \( A_i \cap \omega_1 \) is countable in \( X_\xi \), for all sufficiently large \( \ell \) we have \( \delta_\ell \in A_i \). Since this holds for each \( i \), for all sufficiently large \( \ell \) the \( n \)-tuple \((\delta_\ell, \ldots, \delta_\ell)\) is in \( A_0 \times \cdots \times A_{n-1} \). Since this latter set is closed, we have that \((z_{\xi_0}, \ldots, z_{\xi_{n-1}})\) is in \( A_0 \times \cdots \times A_{n-1} \), hence \((z_{\xi_0}, \ldots, z_{\xi_{n-1}})\) is not banned by \((v, f)\).

Let \( \{V(z_\xi, n) : n \in \omega\} \) be a decreasing neighborhood base for \( z_\xi \) in \( X_\xi \), with \( cl_{X_\xi} V(z_\xi, 0) \notin \mathcal{U}_\xi \); this uses the fact that each \( X_\xi \) is regular.

We define \( p_n \in P(\mathcal{X}) \), \( u_n \subseteq \kappa \) and \( g \in [\omega_1]^{\omega_0} \) such that
1. \( p_0 = p \), \( u_0 = \emptyset \), \( g(0) = 0 \)
2. \( p_{n+1} \leq p_n \)
3. \( p_{n+1} \in N \cap D_n \)
4. \( u_n \) is finite
5. \( u_{n+1} \supseteq u_n \)
6. \( h(n + 1) > h(n) \)
7. \( \{\gamma_m : m < n\} \subseteq u_n \)
8. for \( \gamma \in u_{n+1} \), \([p_{n+1}]_\gamma \setminus [p_n]_\gamma \subseteq V(z_\gamma, g(n + 1)) \)
9. if \((v, f)\) appears in \( \Phi_{p_k} \) for some \( k \), then there is an \( n \geq k \) for which \( v \subseteq u_{n+1} \) and

\[
(3.11) \quad \{\alpha \in Y(v, f, p_n, p_k) : (\forall \xi \in u_{n+1})[V(z_\xi, g(n + 1)) \subseteq B(X_\xi, \alpha, f(\alpha))]\}
\]

is uncountable.

Assume that a suitable bookkeeping procedure has been set up so that at each stage \( n + 1 \) we are handed a \((v, f)\) in \( \Phi_{p_k} \) for some earlier \( k \) for the purposes of ensuring condition 3, and in such a way that every such \((v, f)\) so appears.

There is nothing to be done at stage 0. At stage \( n + 1 \) we will be handed \( p_n, u_n, \) and \( g \downarrow n + 1 \), and our bookkeeping hands us a \((v, f)\) in \( \Phi_{p_k} \) for some \( k \leq n \).

Choose \( u_{n+1} \subseteq N \cap \kappa \) finite with \( u_n \cup v \cup \{\gamma_n\} \subseteq u_{n+1} \). Clearly \( u_{n+1} \) satisfies 4, 5, and 7.
Let $f'$ be the promise $f \upharpoonright Y(v, f, p_n, p_k)$. Clearly $f'$ is in $N$. By Claim \ref{claim:3.11}, we know that $(z_\xi : \xi \in u_{n+1})$ is not banned by $(u_{n+1}, f')$. Thus by an application of Lemma \ref{lemma:3.10} we can choose a value for $g(n+1) > g(n)$ large enough so that

\[
\{ \alpha \in \text{dom } f' : (\forall \xi \in u_{n+1}) [V(z_\xi, g(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))] \}
\]

is uncountable. Now we choose $\ell < \omega$ large enough so that $\delta_\ell \not\in \text{Bad}(u_{n+1}, p_n, D_n)$ and

\[
(\forall \xi \in u_{n+1}) [\delta_\ell \in V(z_\xi, h(n+1))]
\]

Next choose $m$ large enough so that

\[
(\forall \xi \in u_{n+1}) [\mathcal{B}(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, h(n+1))].
\]

Since $\mathcal{B} \in N$, we can apply the definition of $\delta_\ell \not\in \text{Bad}(u_{n+1}, p_n, D_n)$ to get $p_{n+1} \leq p_n$ in $N \cap D_n$ such that

\[
(\forall \xi \in u_{n+1}) [p_{n+1}]_\xi \in \mathcal{B}(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, h(n+1))].
\]

Now why does the sequence $\{p_n : n \in \omega\}$ have a lower bound?

Define $a_q = \bigcup_{n \in \omega} a_{p_n}$. Note that $a_q$ is a function satisfying requirements 1–3 of Definition \ref{def:3.4}, and $[a_q]_\xi \neq \emptyset$ only if $\xi \in N \cap \kappa$. If $\xi \in N \cap \kappa$, then $\xi = \gamma_m$ for some $m \in \omega$, and our construction guarantees that

\[
[a_q]_\xi \subseteq [p_m]_\xi \cup V(z_\xi, 0)
\]

and so $\text{cl}_{X_\xi}[a_q]_\xi \not\in \mathcal{U}_\xi$.

Now suppose $(v, f, k) \in \Phi_{p_k}$ for some $k \in \omega$. Define

\[
K(v, f, k) = \{ \alpha \in \text{dom } f : (\forall \xi \in v) [\text{[}x_q]_\xi \setminus [p_k]_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))] \}.
\]

Claim 3.12. $K(v, f, k)$ is uncountable.

Proof. Let $n \geq k$ be as in condition \ref{condition:3.3} for $(v, f)$, so

\[
A := \{ \alpha \in Y(v, f, p_n, p_k) : (\forall \xi \in v) [V(z_\xi, h(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))] \}
\]

is uncountable. For $\alpha \in A$ and $\xi \in v$, we have

\[
[a_q]_\xi \setminus [p_k]_\xi = \bigcup_{m \geq n} [p_m]_\xi \setminus [p_n]_\xi \cup [p_n]_\xi \setminus [p_k]_\xi
\]

\[
\subseteq \bigcup_{m \geq n} [p_m]_\xi \setminus [p_n]_\xi \cup \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad \text{(as } A \subseteq Y(v, f, p_n, p_k))
\]

\[
\subseteq V(z_\xi, h(n+1)) \cup \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad \text{(by 8 of our construction)}
\]

\[
\subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad \text{(as } \alpha \in A)
\]

Thus $A \subseteq K(v, f, k)$. \hfill \Box
We want to show that for (3.23) hence is uncountable. Note that this reduces to showing (3.24) is uncountable. If (3.20) is uncountable.

Proof. For each \( p \in \mathcal{P} \) there is a set \( n \in \omega \) such that \( q = (a_q, \Phi_q) \) is a lower bound for \( \{p_n : n \in \omega \} \).

**Proposition 3.13.** A singleton is safe, so if \( \mathcal{X} = \{X\} \) then \( P(\mathcal{X}) \) is totally proper.

**Proof.** Suppose \( (\{X\}, f) \) form a counterexample. Then \( \text{Ban}(\{X\}, f) \) is a \( \mathcal{U}_X \)-large subset of \( X \). Since \( X \) has no uncountable free sequences, there is a countable \( A = \{x_n : n \in \omega\} \subseteq \text{Ban}(\{X\}, f) \) such that \( \text{cl}_X A \in \mathcal{U}_X \) and hence

\[
B := \text{dom} f \cap \text{cl}_X A
\]

is uncountable. If \( \alpha \in B \), then there is an \( n \in \omega \) with \( x_n \in B(X, \alpha, f(\alpha)) \). Thus there is a single \( n \) for which the set of \( \alpha \in B \) with \( x_n \in B(X, \alpha, f(\alpha)) \) is uncountable, and this contradicts the fact that \( x_n \in \text{Ban}(v, f) \).

Since the union of an increasing chain of safe collections is itself safe, we know that maximal safe collections of relevant spaces exist.

**Proposition 3.14.** Assume \( \mathcal{X} = \{X_\xi : \xi < \kappa\} \) is safe, \( u \subseteq \kappa \) is finite, and \( p \in P(\mathcal{X}) \). There is a set \( A \in \mathcal{U}_u \) such that for any \( (x_\xi : \xi \in u) \in A \), there is a \( q \leq p \) such that \( x_\xi \in [q]_\xi \) for all \( \xi \in u \).

**Proof.** For each \( \xi \in u \) we define a set \( A_\xi \in \mathcal{U}_\xi \) as follows:

Let \( \{(v_n, f_n) : n \in \omega\} \) list all members of \( \Phi_p \) with \( \xi \in v_n \) (the assumption that this set is infinite is purely for notational convenience). For each \( n \in \omega \) there is a set

\[
B_n := \prod_{\xi \in v_n} B^n_\xi \in \mathcal{U}_{v_n}
\]

that is disjoint to \( \text{Ban}(v_n, f_n) \). Note that this means that for every \( w \subseteq v_n \) and \( (x_\xi : \xi \in w) \in \prod_{\xi \in w} B^n_\xi \), the set

\[
\{\alpha \in \text{dom} f_n : (\forall \xi \in w)[x_\xi \in B(X_\xi, \alpha, f(\alpha))]\}
\]

is uncountable.

We let \( A_\xi = \bigcup_{n \in \omega} B^n_\xi \), and we check that \( A = \prod_{\xi \in u} A_\xi \) is as required.

So suppose \( x_\xi \in A_\xi \) for \( \xi \in u \), and define

\[
a_q = a_p \cup \{\langle \xi, x_\xi, 1 \rangle : \xi \in u\}.
\]

We want to show that for \( (v, f) \in \Phi_p \) the set

\[
K(v, f, p) = \{\alpha \in \text{dom} f : (\forall \xi \in v)[[a_q]_\xi \setminus [p]_\xi \subseteq B(X_\xi, \alpha, f(\alpha))\} \}
\]

is uncountable. Note that this reduces to showing

\[
\{\alpha \in \text{dom} f : (\forall \xi \in u \cap v)[x_\xi \in B(X_\xi, \alpha, f(\alpha))]\}
\]
is uncountable, and this follows easily from the fact that the set in (3.22) is uncountable.

Thus if we define

$$\Phi_q = \Phi_p \cup \{(v, f \upharpoonright K(v, f, p)) : (v, f) \in \Phi_p\},$$

then \(q = (a_q, \Phi_q)\) is the desired extension of \(p\). \(\Box\)

**Corollary 3.15.** If \(v \subseteq \kappa\) is finite, \(Z \subseteq X_v\) is \(U_v\)-large, and \(p \in P(\mathcal{X})\), then there is a \(q \leq p\) and \((x_\xi : \xi \in v) \in Z\) such that \(x_\xi \in [q]_\xi\) for all \(\xi \in v\).

**Theorem 4.** Suppose \(\mathcal{X}\) is a maximal safe family, and let \(X\) be an arbitrary relevant space. If \(G \subseteq P(\mathcal{X})\) is generic, then

$$V[G] \models \text{“}X\text{ has an uncountable free sequence”}.$$ 

**Proof.** CASE 1: \(X \in \mathcal{X}\)

In this case \(X = X_\xi\) for some \(\xi < \kappa\). Let

$$A = \bigcup_{p \in G} [p]_\xi.$$ 

The filter \(U_\xi\) generates a countably complete filter of closed subsets of \(X_\xi\) in the extension; we will abuse notation a little bit and call this filter \(U_\xi\) as well. Note that a set is \(U_\xi\)-large in \(V[G]\) if and only if it meets every set \(A \in U_\xi \cap V\).

Now let \(A = \bigcup_{p \in G} [p]_\xi\). Clearly \(A\) is a subset of \(X_\xi\) in the extension, and since \(G\) is countably closed, if we are given a countable \(A_0 \subseteq A\) there is a \(p \in G\) with \(A_0 \subseteq [p]_\xi\). This means (in \(V[G]\)) that the closure of every countable subset of \(A\) is not in \(U_\xi\). Given a set \(Z \subseteq U_\xi\), we can apply Corollary 3.13 with \(v = \{\xi\}\) to conclude that \(A \cap Z\) is non-empty. Thus in \(V[G]\) the set \(A\) is \(U_\xi\)-large. By Proposition 3.1, \(X_\xi\) has an uncountable free sequence.

CASE 2: \(X \notin \mathcal{X}\)

In this case, by the maximality of \(\mathcal{X}\) there is a finite \(v \subseteq \kappa\) such that \(\{X_\xi : \xi \in v\} \cup \{X\}\) is dangerous. To save ourselves from notational headaches, we assume that \(v = n\), and we will refer to \(X\) as \(X_n\). We will also let \(w\) stand for \(n + 1\) so the notation \(U_w\) and \(X_w\) will have the obvious meaning.

Let \(f\) be a promise witnessing that \(\{X_i : i \leq n\}\) is dangerous. In \(V[G]\), for \(i < n\) we let \(A_i = \bigcup_{r \in G} [r]_i\) be the subset of \(X_i\) obtained from the generic filter.

By a density argument, there is a \(p \in G\) such that \((v, f) \in \Phi_p\). Thus if \(q \leq p\) in \(P(\mathcal{X})\) the set

$$Y(v, f, q, p) = \{\alpha \in \text{dom} f : (\forall i < n)\left[[q]_i \setminus [p]_i \subseteq B(X_i, \alpha, f(\alpha))\right]\}$$

is uncountable.
Claim 3.16. In $V[G]$, if $A'_i$ is a countable subset of $A_i \setminus [p]_i$ for each $i < n$, then
\begin{equation}
\{ \alpha \in \text{dom } f : (\forall i < n) [A'_i \subseteq B(X_i, \alpha, f(\alpha))] \}
\end{equation}
is uncountable.

Proof. Since $G$ is countably closed, there is a $q \leq p$ in $G$ such that $A'_i \subseteq [q]_i \setminus [p]_i$ for all $i < n$. Now we apply the fact that $Y(v, f, q, p)$ is uncountable. \qed

Now back in $V$, our assumption is that $\text{Ban}(w, f)$ is $U_w$--large. Since $U_w$ is $\aleph_1$--complete and generated by $\aleph_1$ sets, we can choose $Z := \{ (x^\xi_i : i < w) : \xi < \omega_1 \} \subseteq \text{Ban}(w, f)$ diagonalizing $U_w$. By passing to a subsequence, we may assume that
\begin{equation}
\xi_0 \neq \xi_1 \Rightarrow x^\xi_0 \neq x^\xi_1
\end{equation}
for all $i \leq n$. Note also that
- $\{ (x^\xi_i : i < n) : \xi < \omega_1 \}$ diagonalizes $U_v$
- $\{ x^\xi_n : \xi < \omega_1 \}$ diagonalizes $U_X$

Claim 3.17. In $V[G]$, $I = \{ \xi < \omega_1 : (\forall i < n)x^\xi_i \in A_i \}$ is uncountable.

Proof. This will follow by an easy density argument in $V$. Given $\xi_0 < \omega_1$, the set $\{ (x^\xi_i : i < n) : \xi \geq \xi_0 \}$ still diagonalizes $U_v$, so in particular it is $U_v$--large. Now Corollary 3.15 tells us that the set of conditions forcing the existence of a $\xi > \xi_0$ such that $(\forall i < n)x^\xi_i \in [q]_i$ is dense in $P(X)$, hence $G$ contains such a condition. \qed

Since $I$ is uncountable, in $V[G]$ the set $\{ x^\xi_n : \xi \in I \}$ will diagonalize $U_X$.

Claim 3.18. In $V[G]$, if $I_0 \subseteq I$ is countable, then $\text{cl}_X \{ x^\xi_n : \xi \in I_0 \} \notin U_X$.

Proof. Suppose this fails, so there is a countable $I_0 \subseteq I$ witnessing it. In particular, all but countably many $\alpha < \omega_1$ are in $\text{cl}_X \{ x^\xi_n : \xi \in I_0 \}$. For $i < n$, we define
\begin{equation}
A'_i = \{ x^\xi_i : \xi \in I_0 \},
\end{equation}
and by Claim 3.16, the set
\begin{equation}
B = \{ \alpha \in \text{dom } f : (\forall i < n) [A'_i \subseteq B(X_i, \alpha, f(\alpha))] \}
\end{equation}
is uncountable. By throwing away a countable subset of $B$, we can assume that for all $\alpha \in B$, there is an $\xi \in I_0$ such that $x^\xi_n \in B(X_n, \alpha, f(\alpha))$. Thus there is a single $\xi \in I_0$ for which
\begin{equation}
\{ \alpha \in B : x^\xi_n \in B(X_n, \alpha, f(\alpha)) \}
\end{equation}
is uncountable. Now this contradicts the fact that $(x^\xi_i : i < n)$ is in $\text{Ban}(w, f)$ \qed
We have shown that in $V[G]$, there is a set that diagonalizes $\mathcal{U}_X$ with the property that the closure of every countable subset is not in $\mathcal{U}_X$. Now Proposition 3.1 tells is that $X$ has an uncountable free sequence. \qed

**Theorem 5.** If $\mathcal{X}$ is a safe collection of relevant spaces, then $P(\mathcal{X})$ satisfies the $\aleph_2$-p.i.c.

**Proof.** Let $i$, $j$, $N_i$, $N_j$, $h$, and $p$ be as in Definition 3.4. Just as in the previous section, if $r \in N_i \cap P(\mathcal{X})$, we define

\begin{equation}
(3.36) \quad r \cup h(r) := (a_r \cup h(a_r), \Phi_r \cup h(\Phi_r)).
\end{equation}

**Lemma 3.19.** Assume that $r \in N_i \cap P(\mathcal{X})$.

1. $r \cup h(r) \in P(\mathcal{X})$
2. $r \cup h(r)$ extends both $r$ and $h(r)$
3. if $s \in N_i \cap P(\mathcal{X})$ and $r \leq s$, then $r \cup h(r) \leq s \cup h(s)$

**Proof.** The proof is essentially the same as the one for Lemma 2.7. \qed

Just as in the proof of Theorem 4, it suffices to produce an $(N_i, P(\mathcal{X}))$-generic sequence $\{p_n : n \in \omega\}$ (with $p_0 = p$) such that $\{p_n \cup h(p_n) : n \in \omega\}$ has a lower bound.

Let $\{D_n : n \in \omega\}$ list the dense open subsets of $P(\mathcal{X})$ that are members of $N_i$. Let $\delta = N_i \cap N_1 = N_j \cap N_1$, and let $\{\gamma_n : n < \omega\}$ enumerate $N_i \cap \kappa$. Also fix a sequence $\{\delta_n : n \in \omega\}$ strictly increasing and cofinal in $\delta$ such that for each $\xi \in (N_i \cup N_j) \cap \kappa$, the sequence $\{\delta_n : n \in \omega\}$ converges in $X_\xi$ to a point $z_\xi$.

**Claim 3.20.** If $v \subseteq N_i \cap \kappa$ is finite and $f \in N_i$ is a promise, then $(z_\xi : \xi \in v)$ is not banned by $(v, f)$. The same holds with $N_i$ replaced by $N_j$.

For $\xi \in (N_i \cup N_j) \cap \kappa$, let $\{V(z_\xi, n) : n \in \omega\}$ be a decreasing neighborhood base for $z_\xi$ in $X_\xi$, with $\text{cl}_{X_\xi} V(z_\xi, 0) \not\subseteq \mathcal{U}_\xi$. We will define $p_n$, $q_n$, $u_n$, and $g \in \omega^\omega$ such that

1. $p_0 = p$, $q_0 = p_0 \cup h(p_0)$, $u_0 = \emptyset$, $g(0) = 0$
2. $p_{n+1} \leq p_n$
3. $p_{n+1} \in N_i \cap D_n$
4. $q_n = p_n \cup h(p_n)$
5. $u_n \subseteq N_i \cap \kappa$ is finite
6. $u_{n+1} \supseteq u_n$
7. $\{\gamma_m : m < n\} \subseteq u_n$
8. $g(n + 1) > g(n)$
9. for $\gamma \in u_{n+1} \cup h(u_{n+1})$, $[q_{n+1}]_{\gamma} \setminus [q_n]_{\gamma} \subseteq V(z_\gamma, g(n + 1))$
10. if $(v, f) \notin \Phi_{q_k}$ for some $k$, then there is a stage $n \geq k$ for which

\begin{equation}
(3.37) \quad v \subseteq u_{n+1} \cup h(u_{n+1})
\end{equation}

and

\begin{equation}
(3.38) \quad \{\alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v)[V(z_\xi, g(n + 1)) \subseteq B(X_\xi, \alpha, f(\alpha))]\}
\end{equation}
is uncountable.

Fix a bookkeeping procedure as in the proof of Theorem 3. At stage \( n + 1 \)
we will be handed \( p_n, q_n, u_n, g \upharpoonright n + 1 \), and \((v, f) \in \Phi_{q_k}\) for some \( k \leq n \).

Choose \( u_{n+1} \subseteq N_i \cap \kappa \) finite with \( u_n \cup \{\gamma_n\} \subseteq u_n \) and \( v \subseteq u_{n+1} \cup h(u_{n+1}) \)

To define \( g(n + 1) \), we need to split into cases depending on whether \((v, f)\)
comes from \( p_k \) or \( h(p_k) \).

Case 1: \((v, f) \in N_i \)

Note that \( Y(v, f, q_n, q_k) = Y(v, f, p_n, p_k) \), so \( f' = f \upharpoonright Y(v, f, p_n, p_k) \) is a
promise in \( N_i \). We know \((z_\ell : \ell \in v)\) is not banned by \((v, f')\), hence there is
a value \( g(n + 1) > g(n) \) large enough such that

\[ \{ \alpha \in \text{dom} f' : (\forall \xi \in v)[V(z_\ell, g(n + 1)) \subseteq B(X_\xi, \alpha, f(\alpha))]\} \]

is uncountable.

Case 2: \((v, f) \in N_j \setminus N_i \)

This case is analogous — we use the fact that \( Y(v, f, q_n, q_k) = Y(v, f, h(p_n), h(p_k)) \)
is in \( N_j \).

In either case, we have ensured that condition (10) of our construction is
satisfied for \((v, f)\).

Now choose \( \ell \leq \omega \) large enough so that

\[ \delta_\ell \notin \text{Bad}(u_{n+1}, p_n, D_n) \]

and

\[ (\forall \xi \in u_{n+1} \cup h(u_{n+1}))[\delta_\ell \in V(z_\xi, g(n + 1))]. \]

Choose \( m \) large enough so that

\[ (\forall u_{n+1} \cup h(u_{n+1}))[B(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, g(n + 1))]. \]

In \( N_i \), apply the definition of \( \delta_\ell \notin \text{Bad}(u_{n+1}, p_n, D_n) \) to get \( p_{n+1} \leq p_n \) in
\( N_i \cap D_n \) such that

\[ (\forall \xi \in u_{n+1})([p_{n+1}]_\xi \setminus [p_n]_\xi \subseteq B(X_\xi, \delta_\ell, m)). \]

Applying the isomorphism \( h \) tells us that

\[ (\forall \xi \in h(u_{n+1}))( [h(p_{n+1})]_\xi \setminus [h(p_n)]_\xi \subseteq B(X_\xi, \delta_\ell, m)). \]

The choice of \( m \), together with (3.39) and (3.40), tells us

\[ (\forall \xi \in u_{n+1} \cup h(u_{n+1}))([q_{n+1}]_\xi \setminus [q_n]_\xi \subseteq V(z_\xi, g(n + 1))). \]

Thus we have achieved everything required of us at stage \( n + 1 \). The
verification that \( \{q_n : n \in \omega\} \) has a lower bound proceeds just as in the
proof of Theorem 3.

Conclusion 2. Assume CH holds. There is a totally proper notion of forcing
\( P(X) \), satisfying the \( \aleph_2 \)-p.i.c., such that every relevant space in the
ground model acquires an uncountable free sequence in the generic extension.
4. THE ITERATION

We now construct a model of ZFC in which $2^\aleph_0 < 2^\aleph_1$ and there are no locally compact first countable $S$–spaces. Starting with a ground model $V$ satisfying $2^\aleph_0 = \aleph_1$ and $2^\aleph_1 = \aleph_{17}$, we will do a countable support iteration of length $\omega_2$.

More specifically, let $\mathbb{P} = (P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2)$ be a countable support iteration defined by

- $P_0$ is the trivial poset
- if $\alpha = \beta + 1$, then $V_{P_\alpha} | = \dot{Q}_\alpha$ is Laver forcing
- if $\alpha$ is a limit ordinal, then $V_{P_\alpha} | = \dot{Q}_\alpha = \dot{P}(\mathbb{I}) * \dot{P}(\mathcal{X})$, where

  $V_{P_\alpha} | = \mathbb{I}$ is the collection of all $P$–ideals in $[\omega_1]^{\aleph_0}$,

  and

  $V_{P_\alpha * P(\mathbb{I})} | = \mathcal{X}$ is a maximal safe family of relevant spaces.

We don’t actually use much about Laver forcing; the relevant facts we need are that it is proper, assuming CH it satisfies the $\aleph_2$–p.i.c. (Lemma VIII.2.5 of [9]), and it adds a real $r \in {}^{<\omega}\omega$ that eventually majorizes every real in the ground model.

The point of using the partial orders from sections 2 and 3 is that they can handle all “candidates” from a given ground model, instead of just one at a time. This means that in $\omega_2$ stages we can catch our tail, even though there are $\aleph_{17}$ “candidates” to worry about at each stage of the iteration.

Having defined our iteration, we arrive at the main theorem of this paper.

Theorem 6. In the model $V[G_{\omega_2}]$, there are no locally compact first countable $S$–spaces, and $2^{\aleph_0} < 2^{\aleph_1}$. More generally, every locally compact first countable space of countable spread is hereditarily Lindelöf.

The rest of this section will comprise the proof of this theorem. We start by noting that for every $\alpha$,

$V_{P_\alpha} | = \dot{Q}_\alpha$ has the $\aleph_2$–p.i.c. .

This means

(4.1) $\alpha < \omega_2 \implies V_{P_\alpha} | = CH$

(so in particular the definition of $\dot{Q}_\alpha$ for limit $\alpha$ makes sense) and

(4.2) $P_{\omega_2}$ has the $\aleph_2$–c.c. .

The statement (4.1) is just Lemma VIII.2.4 of [3], while (4.2) is Claim VIII.2.9 from the same source. Note also that (4.2) together with the fact that we are adding many Laver reals in the iteration implies

(4.3) $V_{P_{\omega_2}} | = b = 2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_{17}$.

Thus the cardinal arithmetic in $V_{P_{\omega_2}}$ is as advertised, and we need only verify that every locally compact 1st countable space of countable spread is
hereditarily Lindelöf in $V[G_{\omega_2}]$. We first reduce our task by showing that it suffices to consider only $X$ with a certain form.

**Claim 4.1.** If $Z$ is a locally compact space of countable spread which is not hereditarily Lindelöf, then there are $X$, $Y$, and $\{U_\alpha : \alpha < \omega_1\}$ such that

- $X$ is a locally compact non-Lindelöf subspace of $Z$
- $Y \subseteq X$ is right separated in type $\omega_1$, witnessed by open sets $\{U_\alpha : \alpha < \omega_1\}$
- $X = \bigcup_{\alpha < \omega_1} U_\alpha$
- $X = \text{cl } Y$
- $\ell(X) = \aleph_1$

**Proof.** By a basic lemma [8], $Z$ has a right-separated subspace $Y$ of cardinality $\aleph_1$, $\{y_\alpha : \alpha < \omega_1\}$, and any such subspace is hereditarily separable because $Z$ is of countable spread. For each $y_\alpha$ pick an open neighborhood $W_\alpha$ whose closure is compact and misses all the later $y_\beta$. Every locally compact space is Tychonoff, so for each $\alpha$ there is a cozero-set neighborhood $V_\alpha$ of $y_\alpha$ inside $W_\alpha$. Let $V = \bigcup\{V_\alpha : \alpha \in \omega_1\}$. Then $V$ is locally compact, and it is not Lindelöf because each $V_\alpha$ contains only countably many $y_\alpha$. In fact, $\ell(V) = \aleph_1$ because we carefully took the union of the $V_\alpha$ instead of the union of the $W_\alpha$, and each $V_\alpha$ is sigma-compact. Now it is clear that $X = \text{cl}_V Y$ is as desired. \hfill \Box

We work now in the model $V[G_{\omega_2}]$ and assume for purposes of contradiction that $Z$ is a locally compact first countable space of countable spread which is not Lindelöf. Let $X$ and $Y$ be as in the previous claim. For each $y_\alpha \in Y$, we choose a neighborhood $V_\alpha$ such that $\text{cl } V_\alpha$ is a compact subset of $U_\alpha$. Let $A_\alpha = V_\alpha \cap Y \in [\omega_1]^{\omega_0}$.

**Claim 4.2.** $X$ satisfies Property $D$, i.e., every countable closed discrete subset of $X$ expands to a discrete collection of open sets.

**Proof.** This follows from the general result that every 1st countable regular space $X$ satisfying $\ell(X) < b$ satisfies Property $D$. The proof of this is only a minor modification of the proof of [11, 12.2] which was for $|X| < b$ because van Douwen could not find any use for the added generality given by $\ell(X) < b$. However, for the sake of self-containment we give the proof of this result here. Let $\ell(X) < b$ and let $D = \{x_n : n \in \omega\}$ be a countable closed discrete subspace of $X$. Using regularity, let $\{U_n : n \in \omega\}$ be a family of disjoint open sets such that $x_n \in U_m$ if and only if $x_n = x_m$. For each $n$ let $\{B^n_i : i \in \omega\}$ be a decreasing local base at $x_n$ such that $B^n_i \subseteq U_n$. Let $U = \bigcup\{U_n : n \in \omega\}$ and for each $y \in Y = X \setminus U$ let $V_y$ be an open neighborhood of $y$ whose closure misses $D$, and let $f_y : \omega \to \omega$ be such that $B^n_{f_y(n)}$ has closure missing $V_y$ for all $n$. Let $\{V_\alpha : \alpha < \kappa\}$ ($\kappa < b$) cover $Y$ and, using the definition of $b$, let $f : \omega \to \omega$ be such that $f_{f_n} <^* f$ for all $\alpha$. In other words, there exists $k \in \omega$ such that $f_{f_n}(n) < f(n)$ for all $n \geq k$. We then have all of $Y$...
covered by open sets each of which meets at most finitely many of the sets $B^\gamma_{f(n)}$, which is thus a locally finite collection of disjoint open sets. Hence it is a discrete open expansion of $D$, as desired. □

Our assumptions on $X$ imply that $|X| \leq \omega_2$ — every point in $X$ is the limit of a sequence from $Y$. We will assume that in fact $|X| = \aleph_2$ (this is the difficult case) and that the underlying set of $X$ is $\omega_2$, with $Y = \omega_1 \subseteq X$.

Since $X$ is first countable, we have that $w(X) \leq \aleph_2$, so let $B = \{ W_\xi : \xi < \omega_2 \}$ be a base for $X$. For technical reasons, we assume $W_\xi = U_\xi$ for $\xi < \omega_1$ with repetitions allowed in the case $w(X) = \aleph_1$. Let $\mathcal{B}$ be a $P_{\omega_2}^\omega$-name for $B$, and let $N$ be an elementary submodel of $H(\lambda)$ satisfying

- $|N| = \aleph_1$
- $X, P, B, \mathcal{B}, \{ U_\xi : \xi < \omega_1 \}, \{ V_\xi : \xi < \omega_1 \},$ and $G_{\omega_2}$ are in $N$
- $N \cap \omega_2 = \alpha$ for some $\alpha < \omega_2$

(The set of such $N$ is closed and unbounded in $[H(\lambda)]^{\aleph_1}$.)

For an ordinal $\beta < \omega_2$, define $B_\beta := \{ W_\xi \cap \beta : \xi < \beta \}$.

Claim 4.3. With $\alpha$ as above,

1. $B_\alpha$ is a base for the topology on $\alpha$ as a subspace of $X$
2. $B_\alpha \in V[G_\alpha]$

Proof.
1) Suppose $\beta < \alpha$ and $U \subseteq X$ is a neighborhood of $\beta$. Since $X$ is first countable and $\beta \in N$, there is a neighborhood $U'$ of $\beta$ such that $U' \in N$ and $U' \subseteq U$. Now

$$N \models (\exists \gamma \in \omega_2)[\beta \in W_\gamma \land W_\gamma \subseteq U'].$$

Thus there is such a $\gamma < \alpha$ and we are done.
2) For each pair $\bar{\beta} = (\beta_0, \beta_1) \in \alpha$, there is a condition $p_{\bar{\beta}} \in G_{\omega_2}$ that decides whether or not $\beta_1 \in W_{\beta_0}$, hence there is such a condition in $N$. Now the support of $p_{\bar{\beta}}$ is a countable subset of $\omega_2$ that is in $N$, hence there is a $\gamma < \alpha$ with the support of $p_{\bar{\beta}}$ a subset of $\gamma$. This means to decide whether or not $\beta_1$ is in $W_{\beta_0}$, we need only $\mathcal{B}$ and $G_{\omega_2} \upharpoonright P_\gamma = G_\gamma$. Thus $B_\alpha$ can be recovered from $\mathcal{B}$ and the sequence $(G_\gamma : \gamma < \alpha)$, both of which are in $V[G_\alpha]$. □

Now let $\mathfrak{R} = \{ N_\xi : \xi < \omega_2 \}$ be a continuous, increasing $\varepsilon$-chain of elementary submodels of $H(\lambda)$ such that

- each $N_\xi$ is as in the previous discussion
- $\langle N_\xi : \zeta < \xi \rangle \in N_{\zeta+1}$
- $[\omega_2]^{\aleph_0} \subseteq \bigcup_{\xi < \omega_2} N_\xi$

Now we define a function $F : \omega_2 \to \omega_2$ by letting $F(\xi)$ equal the least $\zeta$ such that

$$V[G_\xi] \cap [\xi]^{\aleph_0} \subseteq N_\zeta$$

and

$$N_\xi \cap [\xi]^{\aleph_0} \subseteq V[G_\xi].$$
Note that since both $V[G_\xi] \cap [\xi]^{\aleph_0}$ and $N_\xi \cap [\xi]^{\aleph_0}$ have cardinality at most $\aleph_1$, the function $F$ is defined for all $\xi < \omega_2$.

**Claim 4.4.** Suppose $\alpha < \omega_2$ has cofinality $\aleph_1$ and is closed under the function $F$. Then $N_\alpha \cap [\alpha]^{\aleph_0} = V[G_\alpha] \cap [\alpha]^{\aleph_0}$.

**Proof.** Suppose first that $A \in [\alpha]^{\aleph_0} \cap V[G_\alpha]$. Then there is a $\beta$ such that sup $A < \beta < \alpha$ and $A \in V[G_\beta]$. Now $F(\beta) < \alpha$ and $A \in N_{F(\beta)} \cap [\beta]^{\aleph_0} \subseteq N_\alpha \cap [\alpha]^{\aleph_0}$.

Conversely, suppose $A \in [\alpha]^{\aleph_0} \cap N_\alpha$. Since sup $A < \alpha$ and $\alpha$ is a limit ordinal, there is a $\beta > \sup A$ below $\alpha$ with $A \in N_\beta$. Then $A \in V[G_{F(\beta)}] \cap [\beta]^{\aleph_0} \subseteq V[G_\alpha] \cap [\alpha]^{\aleph_0}$.

Let $\alpha_0 < \omega_2$ be large enough that $\{A_\xi : \xi < \omega_1\} \in V[G_{\alpha_0}]$ (the $A_\xi$‘s were defined right before Claim 4.2), and let $\alpha < \omega_2$ satisfy

1. $\alpha > \alpha_0$
2. $\text{cf}(\alpha) = \aleph_1$
3. $N_\alpha \cap \omega_2 = \alpha$
4. $N_\alpha \cap [\alpha]^{\aleph_0} = V[G_\alpha] \cap [\alpha]^{\aleph_0}$

Such an $\alpha$ can be found by using the preceding claim, as the set of ordinals satisfying (3) is closed unbounded in $\omega_2$.

**Claim 4.5.** $V[G_\alpha] \models \mathcal{I} := \{B \in [\omega_1]^{\aleph_0} : |A_\xi \cap B| < \aleph_0 \text{ for all } \xi < \omega_1\}$ is a P–ideal.

**Proof.** Clearly $\mathcal{I}$ is an ideal (and in $V[G_\alpha]$). Let $\{B_n : n \in \omega\} \subseteq \mathcal{I}$ be given; without loss of generality the $B_n$‘s are pairwise disjoint. Since $\text{cf}(\alpha) = \aleph_1$, there is a $\beta$ in the interval $(\alpha_0, \alpha)$ such that $\{B_n : n \in \omega\} \subseteq V[G_\beta]$. For each $\xi < \omega_1$, define a function $f_\xi \in \omega_\omega$ by

$$f_\xi(n) = 1 + \max(A_\xi \cap B_n).$$

Since $\alpha_0 < \beta$, each $f_\xi$ is in $V[G_\beta]$. Now in $V[G_\alpha]$ there is an $r \in \omega_\omega$ dominating $\{f_\xi : \xi < \omega_1\}$ — $r$ can be taken to be the Laver real added at stage $\beta + 2 < \alpha$. Now let

$$B := \bigcup_{n \in \omega} B_n \setminus r(n).$$

Clearly $B \in \mathcal{I}$ and $B_n \subseteq^* B$ for all $n \in \omega$.

Now let $X_\alpha$ be the topological space with underlying set $\alpha$ and base given by $B_\alpha$. Claim 4.3 tells us that $X_\alpha \in V[G_\alpha]$, and that in $V[G_{\omega_2}]$, $X_\alpha$ is a subspace of $X$. We will use this implicitly throughout the remainder of the section.

**Claim 4.6.**

1. If $A \in V[G_\alpha] \cap [X_\alpha]^{\aleph_0}$ has a limit point in $X$, then $A$ has a limit point in $X_\alpha$.
2. $V[G_\alpha] \models X_\alpha$ has Property D
Claim 4.4 we know that $D$ is a closed discrete subset of $X$, and by Claim 4.4 we know that $D \subseteq N_\alpha$. Thus $B$ is a discrete collection of open sets, without loss of generality members of $\mathbb{N}$. This gives us the required limit point for $A$ in $X$.

2) Suppose $D = \{x_n : n \in \omega\}$ is a closed discrete subset of $X_\alpha$ in $V[G_\alpha]$. By the first part of the Claim, $D$ is a closed discrete subset of $X$, and by Claim 4.4 we know that $D \subseteq N_\alpha$. Since $X$ satisfies Property D, $D$ expands to a discrete collection of open sets, without loss of generality members of our fixed base $B$. Since $D \subseteq N_\alpha$, there is such an expansion in $N_\alpha$. Now the countable subset of $\omega_2$ that indexes this cover is in $N_\alpha \cap [\alpha]^{\aleph_0}$, hence it is in $V[G_\alpha]$ as well. This gives us the required discrete family of open sets in $V[G_\alpha]$.

Proof. First note that any countable subset of $X$ in $V[G_\omega]$, if we attain our goal we will have a contradiction, proving that such a space $X$ does not exist in $V[G_\omega]$.

We work for a bit in $V[G_\alpha]$. The first thing we do is force with $P(\mathbb{I})$, where $\mathbb{I}$ lists all the $P$–ideals in $V[G_\alpha]$. If $H_0$ is a generic subset of $P(\mathbb{I})$, then in $V[G_\alpha][H_0]$, either there is an uncountable $B \subseteq \omega_1$ with $[B]^{\aleph_0} \subseteq \mathbb{I}$, or there is an uncountable $B \subseteq \omega_1$ with $[B]^{\aleph_0} \cap \mathbb{I} = \emptyset$.

Let us suppose the first possibility occurs. This means that every countable subset of $B$ has finite intersection with every $A_\xi$ (in $V[G_\alpha][H_0]$). This continues to hold in $V[G_\omega]$, so in $V[G_\omega]$ there is an uncountable $B \subseteq Y$ that meets each $V_\xi$ at most finitely often, i.e., $B$ has no limit points in $Y$. Thus $B$ is a discrete subspace of $Y \subseteq X_\alpha$, and we achieve our goal and reach a contradiction.

Now suppose the second possibility occurs. This means that in $V[G_\alpha][H_0]$, there is an uncountable $B$ such that every countably infinite subset of $B$ meets some $A_\xi$ in an infinite set.

Claim 4.7. $V[G_\alpha][H_0] \models Z := \text{cl}_{X_\alpha} B$ is countably compact and non–compact

Proof. First note that any countable subset of $Z$ from $V[G_\alpha][H_0]$ is in $V[G_\alpha]$, as $P(\mathbb{I})$ is totally proper. Given $B_0 \in [B]^{\aleph_0}$, there is a $\xi < \omega_1$ such that $B_1 = B_0 \cap A_\xi$ is infinite.

Now step into the model $V[G_\omega]$. Since $B_1 \subseteq A_\xi \subseteq V_\xi$ and cl$V_\xi$ is compact, $B_0$ has a limit point. Since $B_0$ is in the model $V[G_\alpha]$, our choice of $\alpha$ implies that $B_0$ has a limit point in $X_\alpha$.

Now $X_\alpha$ has Property D in $V[G_\alpha]$, and since no new countable subsets of $X_\alpha$ appear in $V[G_\alpha][H_0]$, $X_\alpha$ has Property D in this model as well.

This means that any alleged infinite closed discrete subset of cl$X_\alpha$ $B$ (in $V[G_\alpha][H_0]$) would expand to a discrete collection of open sets, thereby yielding an infinite subset of $B$ with no limit point in $X_\alpha$. We have already argued that this is impossible. Thus

$$V[G_\alpha][H_0] \models \text{cl}_{X_\alpha} B \text{ is countably compact.}$$
Now the open cover \( \{ X_\alpha \cap U_\xi : \xi < \omega_1 \} \) of \( X_\alpha \) is in \( V[G_\alpha] \) (here we use another assumption we made about \( B \)), and each of these sets meets \( B \) at most countably often, and so \( \text{cl}_{X_\alpha} B \) is not compact.

If it happens that \( Z \) contains an uncountable discrete subset, then we are done, so we may assume this does not happen. In particular, we may assume that \( Z \) contains no uncountable free sequence. By virtue of the preceding claim, this means that \( Z \) is a relevant space (terminology from the last section) in \( V[G_\alpha][H_0] \).

The next thing we do in our iteration is to force with \( P(\mathcal{X}) \), where

\[
V[G_\alpha][H_0] \models \mathcal{X} \text{ is a maximal safe collection of relevant spaces.}
\]

The results of the preceding section tell us that \( Z \) acquires an uncountable discrete subset after we do this forcing. Thus

\[
V[G_\alpha+1] \models X_\alpha \text{ has an uncountable discrete subset}
\]

and again we have achieved our goal, reaching a contradiction. Thus every first countable locally compact space of countable spread is hereditarily Lindelöf; in particular, there are no locally compact first countable S–spaces in \( V[G_\omega_2] \) and Theorem 6 is established.

Theorem 6 is reminiscent of the theorem of Szentmiklóssy recounted in [8] that MA\((\omega_1)\) implies that no compact space of countable tightness can contain an S–space or an L–space. Every compact space of countable spread is of countable tightness, and if a locally compact space is of countable spread, so is its one-point compactification. So our result may be looked upon as a mild version of one half of Szentmiklóssy’s theorem for models of \( 2^{\aleph_0} < 2^{\aleph_1} \). It would be very nice if we could get even a similarly mild version of the other half—it would settle a famous fifty year-old problem of Katětov:

**Problem.** If a compact space has hereditarily normal (“\( T_5 \)” ) square, must it be metrizable?

The second author showed that the answer is negative if there is a Q-set, so that in particular MA\((\omega_1)\) implies a negative answer. Gary Gruenhage showed that CH also implies a negative answer. Proofs appeared in [8] along with a theorem connecting Katětov’s problem with the theory of S and L spaces:

**Theorem 7.** If there does not exist a Q-set, and \( X \) is a compact nonmetrizable space with \( T_5 \) square, then at least one of the following is true:

1. \( X \) is an L-space
2. \( X^2 \) is an S-space
3. \( X^2 \) is of countable spread, and contains both an S-space and an L-space.
Parts (2) and (3) are ruled out in our model because of Katětov’s theorem that every compact space with $T_5$ square is perfectly normal, hence first countable. If it could be shown that there are no compact $L$–spaces (which are automatically first countable) in our model, then Katětov’s fifty-year old problem would be fully solved. It is not out of the question that first countable compact $L$–spaces can be gently killed, so that even if some of these spaces exist in this model, we can maybe throw in a few more notions of forcing to explicitly banish them.

There is a tantalizing sort of duality between our model and the model obtained by adding $\aleph_2$ random reals to a model of MA+$\mathfrak{c}=\aleph_2$. There, too, there are no $Q$-sets (even though $2^{\aleph_0}=2^{\aleph_1}$); but there, it is $L$–subspaces of compact spaces of countable spread that have been ruled out to date, so that (1) and (3) that are ruled out there, and it is the status of locally compact first countable $S$–spaces that is unknown.

If neither of these models works out, it is to be hoped that the techniques we have introduced in this paper will some day produce a model that does settle Katětov’s problem.

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