Gyrokinetic Field Theory as a Gauge Transform

or: gyrokinetic theory without Lie transforms

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Abstract

Gyrokinetic theory is a basis for treating magnetised plasma dynamics slower than particle gyrofrequencies where the scale of the background is larger than relevant gyroradii. The energy of field perturbations can be comparable to the thermal energy but smaller than the energy of the background magnetic field. Properly applied, it is a low-frequency gauge transform rather than a treatment of particle orbits, and more a representation in terms of gyrocenters rather than particles than an approximation. By making all transformations and approximations in the field/particle Lagrangian one preserves exact energetic consistency so that time symmetry ensures energy conservation and spatial axisymmetry ensures toroidal angular momentum conservation. This method draws on earlier experience with drift-kinetic models while showing the independence of gyrokinetic representation from particularities of Lie transforms or specific ordering limits, and that the essentials of low-frequency magnetohydrodynamics, including the equilibrium, are recovered. It gives a useful basis for total-f electromagnetic gyrokinetic or gyrofluid computation. Various versions of the representation based upon choice of parallel velocity space coordinate are illustrated.

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I. BACKGROUND AND INTRODUCTION

The most modern form of gyrokinetic theory appeared about two decades ago in two papers by Sugama and Brizard [1, 2]. The gyrokinetic approach to low frequency motion had emerged before 1980 first as an ordering scheme [3, 4], and then as a method to average fast time scales out of the collisionless Boltzmann equation describing evolution of a distribution function accounting for motion of charged particles in the presence of prescribed or self consistent electromagnetic fields governed by Maxwell’s equations [5–8]. The key to self consistency was a method to re-cast charge density in the form of a gyrocenter charge density and a polarisation density, which allowed solving for a low frequency electrostatic potential in the absence of finite explicit charge density. The low frequency approximation in this context is equivalent to quasineutrality: the divergences of the current and magnetic potential should vanish and the actual charge density should vanish even in the presence of finite ExB vorticity, whose existence implies a finite divergence of the perpendicular electric field. This was a matter of using the existing approximations to form a gyrocenter representation and derive a gyrokinetic Poisson equation [9]. About the same time the theory given a stronger mathematical foundation by demonstrating that the original results could be recovered by applying a Lie transform to the Lagrangian of the charge particles, so that all ordering assumptions could be collected into the starting point of the theory. The equations then descend rigorously from the Euler-Lagrange equations, following earliest variational treatments of drift-kinetic motion [10], first for drift centers and then for finite-gyroradius gyrocenters [11–15]. The Lie transform contains an opposite pull-back transformation which allows systematic derivation of the gyrocenter representation and thereby the gyrokinetic Poisson equation [14, 16]. The strategy of maintaining “canonical representation” in the Lagrangian by systematic application of the transform’s gauge freedom within a generally covariant version of the theory was explicitly established, and then several forms were presented under various levels of approximation [17–20]. This includes a version which was explicitly electromagnetic [21]. In an important concurrent line the representation of magnetohydrodynamics (MHD) by gyrokinetic theory was explicitly established [22–24]. The low-frequency form of MHD, called Reduced MHD, restricts the dynamics to eliminate the “fast wave” [25, 26]. The form of MHD most relevant to tokamak dynamics adds “low-beta” restrictions to this $\beta = 2\mu_0 p/B^2 \ll 1$, and is captured by gyrokinetic theory by allowing
fluctuations in the magnetic potential only parallel to the background magnetic field (i.e., \( \vec{A} = \vec{A}_1 B / B \)). Demonstrations of such “shear-Alfvén” dynamics were given by Lee et al. [27, 28]. These methods can also be described by a geometric viewpoint [14, 29]. A fully relativistic, electromagnetic treatment considering the representation of the complete Maxwell equations and exact conservation laws was given by Brizard [30]. This paper was the central precursor to the field theory papers referenced above [1, 2]. These have different emphases, both with regard to relativistic or mostly low-frequency forms, and continuum or particle representations. Fully equivalent, they allow choice in the approach to the theory. Both explicitly use the Noether theorem to obtain the conservation laws which therefore follow rigorously once the appropriate choice of Lagrangian has been made, and it is a particle/field system Lagrangian, not just a particle one — this is the step which turns the gyrokinetic representation into a field theory. The use of quasineutrality itself is no longer arbitrary; it follows directly from the assumption in the system Lagrangian that the electric field energy is small compared to the ExB kinetic energy of the particles, and then the Euler-Lagrange equations for both particles and fields maintain exact consistency. The above has been comprehensively reviewed by Brizard and Hahm [31]. Important demonstrations contained there are that the pullback transform and variational method to obtain the fields are mathematically identical, and that the Lie transform for the Lagrangian and Poisson bracket transform for the kinetic equation yield identical results, even if the field theory methods make the connection to the rest of physics that much clearer. However, the field theory methods have the advantage of restricting ordering assumptions to the starting point with no loss of consistency, whereas ordering applied directly to the equations without regard to the consistency of the starting point most often leads to a breach of some or all of the conservation laws. In a physical situation where ingredients with small energy content can have large effect (e.g., flows [32–34], or parallel currents [35–38], in pressure driven dynamics), it is imperative to maintain exact consistency in the equations in any computational model.

Energetic consistency refers not only to the existence of an exact conservation law for some quantity definable as energy within the model, but more generally to the principles of physical symmetry by which fields and particles interact with each other in the dynamics within the model. Conserved energy and momenta are specific quantities derived within the model along with its equations for the evolution of the fields and particles. The archetype for this is a Lagrangian system, with free pieces for each constituent and interac-
tion pieces describing their exchanges, such as described in the text by Landau and Lifshitz [39]. The Euler-Lagrange equations for the particle positions give their evolution, and the Euler-Lagrange equations for the field potentials give the field equations. Application of Liouville’s theorem to the particle equations gives the evolution of the particle distribution function (see, e.g., pp 48–52 of the text by Tolman [40]). In each application, these equations are specific to the particular model (Lagrangian); that is, there is no external field equation one can appropriate, but each model has its own field equation arising from its own Lagrangian; otherwise, energetic consistency will be broken. Any version of the gyrokinetic model is consistent as long as all approximations are made in constructing the Lagrangian while the derivation of the resulting equations for the particles and fields remains exact. In this context, gyrokinetic theory is a low-frequency gauge transformation of the Maxwell-Boltzmann system Lagrangian, not an approximate “orbit averaging” done on the equations themselves (this equivalence is present only in the first-order linearised version of the gyrokinetic representation).

Symmetry of interaction (Newton’s Third Law) is an automatic feature of any such system, and conservation laws are described by application of Noether’s theorem. In gyrokinetic theory applied to tokamaks or other axisymmetric equilibria, time symmetry leads to energy conservation and axisymmetry leads to toroidal momentum conservation, not just for particles moving in prescribed fields but generally for the field/particle system. The gyrokinetic field theory papers introduced these applications to our context [1, 2], and a detailed exposition of why and how it works, including discussion of the importance of canonical representation in the Lagrangian, is given in a previous work on energetic consistency and momentum conservation in gyrokinetic field theory [41]. In this context, canonical representation refers to the strategy of the Lie transform as discussed by Hahm [17] to arrange all dependence upon space-time dependent field quantities into the time component of the Lagrangian, so that the canonical momenta, dependent upon phase space coordinates, are independent of both time and toroidal angle. The derivations described below are arranged from the start to follow this structure, so as to automatically guarantee the existence of energy and toroidal momentum conservation laws, that is, general energetic consistency.
II. OUTLINE OF GYROKINETIC THEORY AS A GAUGE TRANSFORM

Gyrokinetic theory is not an orbit average over equations, but a set of operations on a Lagrangian which involves a change of representation from particle to gyrocenter variables. What those gyrocenter variables actually are is the result of choices made during the gauge transformation. A gauge transformation is a combination of operations involving coordinate changes and the addition of total differentials to the original Lagrangian to produce another one which reflects the same dynamics in different language as the original one. The low-frequency approximations enter through a chosen ordering scheme, in which the only really essential element is the smallness of the ExB vorticity or other dynamical frequencies compared to gyrofrequencies. Since it is the slowest, the ion gyrofrequency sets this limit. Since they are the fastest, the electron and/or shear Alfvén parallel transit frequencies are considered. The ExB vorticity is considered since it underlies any turbulent dynamics involving ExB motion. The approximations involved in these are well satisfied in tokamaks, usually by at least two orders of magnitude, even in steep gradient regimes. The only significant exception is the borderline case of the outboard midplane in present-day spherical tokamaks where the magnetic field strength drops to relatively small values and the gradient scale length drops to below a centimeter (e.g., [42]). For L-H transition databases on conventional tokamaks, however, this frequency ordering is well satisfied [43, 44].

The procedure we will use closely follows Littlejohn’s variational method from 1983 [13], in combination with the field theoretical methods from Landau and Lifshitz [39]. The dynamical role of the field potentials is taken on equal footing to the gyrocenter motion, treating the field/particle system as a whole. No separation between equilibrium or dynamics except for the background magnetic field is prescribed, expecting the electromagnetic version of the theory to recover not only equilibrium flows but also the MHD (Grad-Shafranov) equilibrium self-consistently.

We strictly maintain the original gyrokinetic strategy to preserve canonical representation by transforming field variable quantities strictly into the time component of the system Lagrangian (whether as part of the Hamiltonian or of the field Lagrangian density). In Landau-Lifshitz terms we have the free-particle Lagrangian (in our terms the part not dependent upon dynamical fields), the interaction Lagrangian (the part involving both fields and particles), and the free-field Lagrangian. Canonical representation refers to the interac-
tion Lagrangian appearing only in the Hamiltonian such that the canonical momenta involve only the phase space coordinates. All terms due to the field potentials appear only in $H$, so that the phase space Jacobian is a background quantity, symmetry of the background is not broken. This allows easily realisable versions of the relevant conservation law proofs as well as facilitating proof of correspondence to conventional models [41]. It also facilitates computations.

The usual assumptions of low-frequency theory are quasineutrality (the neglect of space-charge effects while allowing a finite Laplacian of the field potential upon which finite vorticity depends), and shear-Alfvén magnetic responses. These are effected within the theory by neglecting the $\epsilon_0 E^2/2$ electric field energy in the field Lagrangian and by allowing only a parallel component of the magnetic potential $A_\parallel$ among magnetic disturbances. Specifically, neglecting $\epsilon_0 E^2/2$ against ExB kinetic energy $\rho_m (E/B)^2/2$ is a statement that the (mostly ion) plasma polarisability $\rho_m/B^2$ overshadows the permittivity of free space $\epsilon_0$, so that the overall charge density is neglected despite the nonzero divergence of the electric field. Since $\epsilon_0 \mu_0 = 1/c^2$ this also means that the Alfvén velocity $v_A$ is small compared to $c$, the speed of light: $v_A^2$ is $B^2/\mu_0 \rho_m$ so that $\epsilon_0 B^2/\rho_m$ is also $v_A^2/c^2$, the more familiar measure. The second statement is the neglect of $A_\perp$ disturbances due to $\beta \ll 1$ and $k_\perp v_A \gg \omega$ being well satisfied, with $\omega$ tracking $\partial/\partial t$ in any time-dependent response. Hence in force balance $\nabla^2_\perp (2\mu_0 p + B^2) \approx 0$ the changes to the field strength $B^2$ due to the pressure $p$ are neglected, and in the dynamics only the parallel electric field should involve inductive responses. Dynamical magnetic compressibility is therefore disallowed. However, the magnetic compressibility implied by the existence of diamagnetic flows and heat fluxes and polarisation currents is explicitly kept in the theory, such that any low-beta compressibility effects are retained.

Ultimately, however, gyrokinetic theory is about the representation, not the ordering. We have gyrokinetic polarisation (Poisson) and induction (Ampère) equations in which a polarisation density appears, while the gyrokinetic equation itself has no $\partial/\partial t$ terms associated with polarisation drifts. The polarisation current is recovered by taking the time derivative of the polarisation equation and the MHD Ohm’s law is recovered by taking the time derivative of the induction equation. These two statements recover nonlinear low-frequency MHD, including the Grad-Shafranov equilibrium. The representation is the result of having preserved canonical representation. With this maintained the result of conventional gyrokinetic
approaches is easily recovered in the appropriate limit.

The Lie transform which corresponds to this gauge transform version was given previously \[45\], including the correspondence to low-frequency MHD and to previously derived forms of the gyrokinetic Lagrangian and equations. Generally, energetic consistency and momentum conservation, including the route back to MHD, follows easily in this formulation \[41\].

III. GAUGE TRANSFORM TO GET THE LAGRANGIAN

Herein we derive the gyrokinetic Lagrangian as a field theory. Finite ExB Mach number (flow amplitude) is allowed by taking a maximal ordering on the ratio of kinetic to thermal energy (i.e., they appear at the same level in the expansion). The theory is cast as a gauge transform which does not require Lie transform techniques. The method closely follows Littlejohn’s differential gauge transform method \[13\], just that we generalise the role of the field potentials to become dependent variables, enforce canonical representation on \(L\) including them, and treat the result as a field theory rather than a Lagrangian for individual gyrocenters.

We consider generally a particle Lagrangian \(L_p\) which gets transformed to a gyrocenter one. The structure is

\[ L_p dt = p \cdot dZ - H dt \]

(1)

cast as a fundamental one-form, where the components of \(p\) are canonical momenta, the components of \(Z\) are the phase space coordinates, and \(H\) is the Hamiltonian (the time component). In general this is six-dimensional (6D) dynamics, but in its gyrokinetic representation the gyromotion involving perpendicular velocity space components is separated away so that the actual dynamics covers the 4D space of \((R, z_\parallel)\) consisting of gyrocenter positions and the parallel canonical momentum. Collisions bring in the 5th coordinate, usually the magnetic moment \(\mu\) conserved by the drift motion. The sixth coordinate is the gyrophase angle \(\vartheta\) which only appears in the gyromotion since \(H\) and \(p\) and the rest of \(L_p\) is gyrophase independent.

Starting with the Landau-Lifshitz treatment as a background \[39\], we restrict to non-relativistic situations with the time not being varied. We have the free particle and interaction Lagrangians as

\[ L_p = \frac{m}{2} \ddot{x} \cdot \dot{x} + eA \cdot \dot{x} - e\phi \]

(2)
for the particles, and the free field piece $L_f$ which we treat later. The Legendre transform is applied as

$$ p \equiv \partial L_p / \partial \dot{x} \quad H \equiv p \cdot \dot{x} - L_p \quad L_p = p \cdot \dot{x} - H $$

(3)

after which $H$ and therefore $L_p$ are functions of $(x, p)$ and not $(x, v)$. Then, $L_p$ is turned into a fundamental one-form by considering the differential action

$$ L_p dt = p \cdot dx - H dt \quad \text{where} \quad H = m \frac{U^2}{2} + e\phi \quad mU = p - eA $$

(4)

Since we will use $A$ as an anchor for low-frequency drift motion we split the canonical momentum into an equilibrium part and a dynamical part,

$$ L_p dt = p \cdot dx - H dt \quad \text{where} \quad p = eA + z \quad \text{and} \quad H = m \frac{v^2}{2} + e\phi $$

(5)

This gets us to the structure referred to in the beginning, but now with $p$ as total canonical momentum and $z$, the dynamical part, as velocity coordinate. Note that $x$ is used in place of $Z$ at this point because all the canonical momenta are spatial when we start.

To get a low-frequency low-beta kinetic Lagrangian we assume $\phi$ is a dynamical field but that $A$ evolves through small shear-Alfvén disturbances parallel to $B$. We re-cast $A$ generally in terms of an equilibrium piece $A$ assumed to be static, and add to it the dynamical piece $A_\parallel b$ assuming $A_\parallel$ to be the other dynamical field. Now, $A$ and $B = \nabla \times A$, with $B = |B|$ and $b = B/B$, are assumed to be static quantities describing the background magnetic field, while the particle coordinates and $\phi$ and $A_\parallel$ constitute the set of dependent variables which represent the dynamical system. We assume the gyromotion is a fast frequency to be eliminated, retaining dynamics on the time scale of the ExB vorticity and the parallel transit frequencies. This leads to $m/e$ as the formal small parameter for expansion (which tracks the ratio $\Omega_{exb}/\Omega_i$ between the ExB vorticity and ion gyrofrequency).

We are using $A$ to anchor drifts, so we cannot use canonical variables directly. The gyrokinetic Lagrangian therefore represents a non-canonical transformation. However, we do want canonical representation, which means that the whole Lagrangian except for $H$ is static (dependent on geometry, coordinates, and constants only) and, in a tokamak, axisymmetric. It also means that the resulting phase-space Jacobian retains the symmetry of the background, and that there are no extra $\partial / \partial t$ terms on fields in the kinetic equation. We get canonical representation using the gauge freedom of the transformation: generating functions of the coordinate changes, and the gauge terms (pure differentials) added to the
Lagrangian. All time- (and in a tokamak, toroidal angle-) dependence involving fields is moved into \( H \) and out of the part of \( L_p \) containing the canonical momenta.

We expand the spatial canonical momentum in terms of parallel and perpendicular motion, reckoned against the background magnetic field,

\[
\mathbf{z} = z_\parallel \mathbf{b} + z_\perp
\]

At this point, \( z_\parallel \) contains \( A_\parallel \) and \( z_\perp \) can contain a mean flow, denoted \( u_0 \), as well as the gyromotion velocity. These appear in the Lagrangian, which is

\[
L_p \, dt = \left( eA + z_\parallel \mathbf{b} + z_\perp \right) \cdot \mathbf{dx} - H \, dt
\]

\[
H = mU^2 + \frac{1}{2m} |z_\perp|^2 + e\phi \quad mU = z_\parallel - eA_\parallel
\]

where \( U \), now just the parallel velocity, is not a coordinate but a function of dependent variables. Since there is no dynamical \( A_\perp \) kept in the low-beta shear-Alfvén case, \( z_\perp \) is the same as \( mv_\perp \), which may or may not involve \( u_0 \). We will demand any \( u_0 \) not be specified but emerge naturally from the derivation process.

The next step most closely follows Littlejohn’s drift-kinetic gauge treatment [13]. The small parameter is formally any factor of \( m/e \) which tracks drifts. Flows (gradients of \( \phi \)) will enter naturally, later. A drift-kinetic representation will treat the gyromotion \( \mathbf{w} \) but leave a mean flow \( u_0 \) as part of \( \mathbf{p} \) whose \( \partial/\partial t \) represents the polarisation drift. In a gyrokinetic representation we treat \( \mathbf{w} \) and \( u_0 \) together in \( z_\perp \), with any dependence upon \( \phi \) absent from \( \mathbf{p} \) but present in \( H \). Polarisation no longer enters as a drift, but as a density in the field equation for \( \phi \) which we will see later. As we will show, the same expression for \( \nabla \cdot \mathbf{J} = 0 \) is recovered after \( \partial/\partial t \) is taken on this field equation. We emphasise that “gyrokinetic” refers to the representation, not the finite-gyroradius (FLR) effects, and that a model with zero-FLR and polarisation density is still a gyrokinetic one. This will become obvious only when the self consistent field equations are at hand.

We introduce an arbitrary spatial coordinate change in which

\[
\mathbf{x} = \mathbf{R} + \mathbf{r}
\]

where we can choose \( \mathbf{r} \) according to how we want to arrange the representation. We expect its magnitude to satisfy \( r \ll L_B \) where \( L_B \) is the scale of variation of the magnetic field.
(assumed to be of order the toroidal major radius in a tokamak). We then Taylor-expand $A$ and $\phi$ in powers of $A$ and arrange them by order. This ultimately leads to a long-wavelength version of the model since we will find a posteriori that $\rho_s^2 \nabla^2$ should be small (its terms arise at second order). The one-form $L_\nu \, dt$ splits according to orders as

$$L_0 \, dt = eA \cdot dR - e\phi \, dt$$ (10)

$$L_1 \, dt = e \mathbf{r} \cdot \nabla A \cdot dR + eA \cdot dr + z_\perp \cdot dR - \left( e \mathbf{r} \cdot \nabla \phi + m \frac{U^2}{2} + \frac{1}{2m} |z_\perp|^2 \right) \, dt$$ (11)

$$L_2 \, dt = e \mathbf{r} \cdot \nabla A \cdot dr + \left( z_\parallel b + z_\perp \right) \cdot dr - \frac{1}{2} e \left( \mathbf{r} \mathbf{r} : \nabla \nabla \right) \phi \, dt$$ (12)

where the dependence of $A$ and $\phi$ and $\nabla$ is now upon $R$ after the Taylor expansion. Due to the factors of $m/e$ the spatial variation of $b$ enters one order lower than that of $A$ so we do not expand it.

The lowest-order one-form is $L_0 \, dt$. Varying $R$ we find

$$e \left( \nabla A \cdot dR - dR \cdot \nabla A \right) = e \nabla \phi \, dt$$ (13)

The solution of this is

$$\dot{\mathbf{R}}_0 = \frac{1}{B^2} \left( \nabla \phi \cdot (\nabla A) - (\nabla A) \cdot \nabla \phi \right) + \mathbf{b} (\mathbf{b} \cdot \dot{\mathbf{R}})$$ (14)

with order zero denoted by the subscript. This solution describes the lowest-order ExB drift. We note that at this level the parallel component $\mathbf{b} \cdot \dot{\mathbf{R}}$ is indeterminate. Since the parallel dynamics enters at the next order however we leave this component at zero and specify

$$\mathbf{u}_0 \equiv \dot{\mathbf{R}}_0 = \mathbf{u}_E \quad \mathbf{u}_E \equiv \frac{1}{B^2} \nabla \phi \cdot \mathbf{F} \quad \mathbf{F} = \nabla A - (\nabla A)^T$$ (15)

with superscript $T$ denoting the transpose. This operation is how the drift tensor $\mathbf{F}$ enters the problem. Establishing $\mathbf{u}_0$ as $\mathbf{u}_E$ is the main result of this step.

The next-order one-form is $L_1 \, dt$. We first subtract the total differential

$$dS_1 = d (\mathbf{r} \cdot A) = dr \cdot A + dR \cdot \nabla A \cdot r$$ (16)

where we note that $A$ depends on $R$ but is static. The subtraction yields

$$L_1 \, dt - dS_1 = e \mathbf{r} \cdot \mathbf{F} \cdot dR + \left( z_\parallel b + z_\perp \right) \cdot dR + \cdots$$ (17)
where we’ve written only the terms appearing as part of p. We now choose r. Customary operations in the theory set r to cancel the gyromotion w out of p, leaving u₀ there as a background term. In our case we choose r to cancel the entire z⊥, leaving only the parallel piece in the canonical momentum, which is now just eA + z∥b. This sets

\[ er \cdot F + z_\perp = 0 \]  

and the solution is

\[ r = \frac{1}{eB^2} z_\perp \cdot F \]  

where we arbitrarily choose b · r = 0. This r is the same gyro-drift radius as in the Lie-transform version of this model [45]. We also find

\[ er \cdot \nabla \phi = -u_E \cdot z_\perp \]  

Discarding the total differential term, the first-order one-form correction is

\[ L_1 dt - dS_1 = z_\parallel b \cdot dR - \left( \frac{mU^2}{2} + \frac{1}{2m} |z_{\perp} - m u_E|^2 - \frac{v_E^2}{2} \right) dt \]  

where using the evaluation of r · ∇φ we have completed the squares on the last two terms. This step is how u₀, now u_E, explicitly enters the expression of H.

The result up to this point is almost good enough to build the model, since we have obtained the quadratic field term in φ necessary to build its field equation. However, we haven’t found any constraints on z⊥ yet. To do that we have to proceed to the next order and consider the details of gyromotion as Littlejohn did.

At second order the Lagrangian correction is

\[ L_2 dt - dS_2 = \left( er \cdot \nabla A + z_\parallel b + z_\perp \right) \cdot dr - \frac{1}{2} e (rr \cdot \nabla \nabla) \phi dt \]  

The first step is to use b · dr = 0 to strip the z∥ term and then subtract

\[ dS_2 = d \left( \frac{1}{2} er \cdot \nabla A \cdot r \right) = \frac{1}{2} er \cdot \nabla A \cdot dr + \frac{1}{2} e dr \cdot \nabla A \cdot r \]  

with dA contributions an order smaller than those from the gyromotion. This is to symmetrise the form with ∇A, obtaining

\[ L_2 dt - dS_2 = \left( \frac{1}{2} er \cdot F + z_\perp \right) \cdot dr - \frac{1}{2} e (rr \cdot \nabla \nabla) \phi dt \]
Using Eq. (18), this can be re-cast as

$$L_2 dt - dS_2 = \frac{1}{2} z_\perp \cdot dr - \frac{1}{2} e (rr : \nabla\nabla) \phi dt$$  \hspace{1cm} (25)

To evaluate the first term we examine the details of the gyromotion.

We set up an auxiliary basis $e_1$ and $e_2$ for the plane perpendicular to $b$, with restrictions and coordinate sense

$$e_1 \cdot e_2 = 0 \quad e_{1,2} \cdot b_2 = 0 \quad e_1 \times e_2 \cdot b = 1$$  \hspace{1cm} (26)

giving the signs. In the Hamiltonian through first order, $H_0 + H_1$, we identify the quantity

$$z_\perp - m u_E \equiv m w$$  \hspace{1cm} (27)

as the gyromotion velocity in the rest frame of the lowest order velocity $u_0$, which is $u_E$. We introduce the gyrophase angle $\vartheta$ hence covering the plane with $w$ expressed in terms of $w$ and $\vartheta$. Due to the large $\Omega$ the fast part of $dr$ is solely due to $d(w \cdot F)$ with contributions due to $u_0$ down an order. Contributions due to $\nabla e_{1,2}$ will give formal gyrophase invariance, but all the others due to the spatial variation of $B$ are neglected. The motion $(\Omega dt)$ is described locally as a geometric circle with angle variation $d\vartheta$, with $w$ the directed gyration velocity, of magnitude $w$, and $\vartheta$ the gyrophase angle. Using Eqs. (18,19), we are left with $w \cdot b \times dw$ which is then worked through as $dw \cdot w \times b$ according to

$$w = -w (e_1 \sin \vartheta + e_2 \cos \vartheta)$$  \hspace{1cm} (28)

$$dw = -w (e_1 \cos \vartheta - e_2 \sin \vartheta) d\vartheta - w dR \cdot (\nabla e_1 \sin \vartheta + \nabla e_2 \cos \vartheta) + \frac{dw}{w} w$$  \hspace{1cm} (29)

$$w \times b = w (e_2 \sin \vartheta - e_1 \cos \vartheta)$$  \hspace{1cm} (30)

The sense of the motion for ions is clockwise, for $b$ out of the plane toward the observer, hence the signs. We then use $e_1 \cdot e_2 = 0$ hence $\nabla (e_1 \cdot e_2) = 0$ to find

$$w \cdot dr = dw \cdot w \times b = \frac{w^2}{\Omega} (d\vartheta - dR \cdot \nabla e_1 \cdot e_2)$$  \hspace{1cm} (31)

This is the minimal description of gyromotion which preserves gyrophase invariance through rotations $\vartheta = \vartheta' + \alpha(R)$. The gyromotion appears in the second order Lagrangian as

$$L_2 dt - dS_2 = \frac{mw^2}{2\Omega} (d\vartheta - dR \cdot \nabla e_1 \cdot e_2) - \frac{1}{2} e (rr : \nabla\nabla) \phi dt$$  \hspace{1cm} (32)
Averaging the gradient components in \((rr:\nabla\nabla)\) over \(\vartheta\), by setting \(rr \to (r^2/2)g_\perp\), with \(g_\perp\) the perpendicular metric, produces the main FLR correction, and we obtain

\[
L_2 dt - dS_2 = M (d\vartheta - dR \cdot \nabla e_1 \cdot e_2) - \frac{e}{4} \frac{r^2}{\nabla_\perp^2 \phi} dt
\]

where we have identified

\[
M = \frac{mw^2}{2\Omega}
\]

as the conserved quantity multiplying \(d\vartheta\). Since \(\vartheta\)-dependence has been eliminated everywhere else in \(L_p\), we now note that \(M\) is a constant of the motion and suitable for use as a coordinate.

The piece due to \(W = \nabla e_1 \cdot e_2\) is small but formally important since the \(d\vartheta\) piece by itself is not gyrophase invariant. If \(\vartheta = \vartheta' + \alpha(R)\) then the combination \(d\vartheta - W \cdot dR\) is invariant (use the dependence of \(e_{1,2}\) on \(\vartheta\) and \(d\alpha = dR \cdot \nabla \alpha\) to show it). In practice the gyromotion drops out of the kinetic equation anyway, since \(\partial/\partial \vartheta\) and \(dM/dt\) vanish. The \(MW\) piece is a small \(O(r/L_B)^2\) correction to the \((r/L_B)\) drifts and does not introduce any new charge-separation effects. In practice no numerical simulation to date keeps it. For profile scales \(L_\perp \ll L_B\) the higher-order terms from the contribution of \(u_E^2\) to \(r^2\) are larger and these receive the attention.

We have now accounted for \(w^2\) and \(r^2\) appearing only through their magnitudes so we combine the result \(L_{0,1,2} dt\) as

\[
L_p dt = (eA + z_\parallel b - MW) \cdot dR + M d\vartheta - H dt
\]

with Hamiltonian

\[
H = \frac{mU^2}{2} + M\Omega + \left(1 + \frac{r^2}{4} \nabla_\perp^2\right) e\phi - M\frac{u_E^2}{2}
\]

where

\[
mU = z_\parallel - eA_\parallel \quad u_E^2 = \frac{1}{B^2} |\nabla_\perp \phi|^2 \quad r^2 = \frac{2M\Omega + mu_E^2}{m\Omega^2}
\]

where the extra differentials \(dS_{1,2}\) are dropped (formally changing the representation). This can be shown (cf. Ref. [47]) to be a low-\(k_\perp\) and low-\(\beta\) version of the result of Ref. [21].

The importance of the second-order step is to establish \(M\) as a coordinate (the notation is from Hahm [11]). We may define a magnetic moment \(\mu\) as

\[
M\Omega \equiv \mu B \quad \text{hence} \quad \mu = \frac{e}{m} M
\]

13
Since $M$ is invariant, so is $\mu$. It is interpreted as the magnetic moment of the gyrocenter in the reference frame co-moving with velocity $u_E$. Note that $M$ is first defined after noting in Eq. (32) that the combination $mw^2/2\Omega$ multiplies $d\vartheta$ and is therefore invariant, hence re-writing as in Eq. (33), we can replace the perpendicular energy term in $H$ with simply $M\Omega$. This is the step in which the fast gyromotion is decoupled from the rest of the dynamics, since $z_\perp$ no longer appears explicitly.

A. The Field Lagrangian

The field theory embeds this into a phase space

$$L = \sum_{sp} \int d\Lambda f L_p + \int dV L_f$$

where for shear-Alfvén conditions the field Lagrangian density is

$$L_f = \frac{1}{2} \left( \epsilon_0 E^2 - \mu_0^{-1} B^2 \right)$$

with $B = \nabla \times A$ in entirety. Under quasineutrality $\epsilon_0$ is taken to be small, and then under shear-Alfvén conditions the field Lagrangian reduces to

$$L_f = -\frac{1}{2\mu_0} B^2_\perp \quad \text{where} \quad B_\perp = \nabla \times (A_\parallel b)$$

Only the perturbation due to $B_\perp$ appears in the field term, as otherwise the full equilibrium current would have to be carried by the gyrocenters. We now have $B = \nabla \times A$ as the background magnetic field serving as an anchor, while $B_\perp$ carried by $A_\parallel$ accounts for the dynamics.

With these approximations we now re-write the system Lagrangian as

$$L = \sum_{sp} \int d\Lambda f L_p + \int dV L_f$$

with gyrocenter and field Lagrangian pieces

$$L_p = \left( eA + z_\parallel b \right) \cdot \dot{R} + M \left( \dot{\vartheta} - W \cdot \dot{R} \right) - H \quad L_f = -\frac{1}{2\mu_0} B^2_\perp$$

with Hamiltonian

$$H = m\frac{U^2}{2} + M\Omega + \left( 1 + \frac{r^2}{4} \nabla_\perp^2 \right) e\dot{\phi} - m\frac{u_E^2}{2}$$
and as auxiliaries the parallel velocity and the squares of the ExB velocity and the gyro-drift radius

$$mU = z\parallel - eA\parallel \quad u_E^2 = \frac{1}{B^2} \left| \nabla_\perp \phi \right|^2 \quad r^2 = \frac{2M\Omega + mu_E^2}{m\Omega^2}$$  \hspace{1cm} (45)

This is now a complete description of the dynamical system. In following subsections, we derive the Euler-Lagrange equations for the gyrocenters, the gyrokinetic equation for their distribution function, and then the Euler-Lagrangian equations for the field variables giving their self-consistent equations in the model.

B. The Euler-Lagrange Equations for Gyrocenters

Gyrocenter motion itself arises from $L_p$ only. We note that derivatives arise from variations with respect to the phase space coordinates holding each other fixed. Hence we note again that the spatial gradient operator is taken with respect to gyrocenter positions $\mathbf{R}$ holding $z\parallel$ fixed. We define geometric quantities

$$eA^* = eA + z\parallel b - M\mathbf{W} \quad \mathbf{B}^* = \nabla \times A^*$$  \hspace{1cm} (46)

The derivatives of $H$ are

$$\nabla H = e\nabla \phi_E + M\nabla \Omega_E - eU\nabla A\parallel$$  \hspace{1cm} (47)

spatially, and in the velocity space coordinates

$$\frac{\partial H}{\partial z\parallel} = mU \quad \frac{\partial H}{\partial M} = \Omega_E \quad \frac{\partial H}{\partial \vartheta} = 0$$  \hspace{1cm} (48)

Auxiliary quantities are

$$e\phi_E = e\phi - m\frac{u_E^2}{2} \left( 1 - \frac{\Omega_{exb}}{2\Omega} \right) \quad \Omega_E = \Omega + \frac{1}{2}\Omega_{exb} \quad \Omega_{exb} = \frac{1}{B} \nabla_\perp^2 \phi$$  \hspace{1cm} (49)

Varying $\mathbf{R}$ and $z\parallel$ together in $L_p$ dt and setting the coefficients of the variation components to zero yields the drift motion

$$B^*\dot{R} = \frac{1}{e} b \times \nabla H + \frac{\partial H}{\partial z\parallel} \mathbf{B}^* \quad B^*\dot{z}\parallel = -\mathbf{B}^* \cdot \nabla H$$  \hspace{1cm} (50)

and varying $\vartheta$ and $M$ yields the gyromotion

$$\dot{M} = 0 \quad \dot{\vartheta} = \frac{\partial H}{\partial M} + \mathbf{W} \cdot \dot{\mathbf{R}}$$  \hspace{1cm} (51)
In the drift motion the general form of the phase space volume element is

\[ B^\parallel = e \frac{\partial A^*}{\partial z} \cdot B^* \] (52)

where in our case the \( z \)-coordinate is \( z^\parallel \).

Since \( z^\parallel \) enters \( A^* \) only through \( b \), we have

\[ e \frac{\partial A^*}{\partial z} = b \quad \text{hence} \quad B^\parallel = b \cdot B^* \] (53)

recovering the more usual expressions, also involving \( b \times \nabla H \) in the drift motion. In fact these forms are not general, since other choices for the \( z \)-coordinate produce their own representations for \( \frac{\partial A^*}{\partial z} \) and hence the other expressions. But the forms in Eqs. (50,52) are general to any choice determining the \( z \)-coordinate and \( A^* \).

C. The Gyrokinetic Equation

The operations to get to this are familiar. We first observe that not only \( \nabla \cdot B^* = 0 \) but also

\[ \frac{1}{e} \nabla \times \left( e \frac{\partial A^*}{\partial z} \right) - \frac{\partial B^*}{\partial z} = 0 \] (54)

for any representation. It follows that the requirement of phase space incompressibility

\[ \nabla \cdot B^\parallel \dot{R} + \frac{\partial}{\partial z} B^\parallel \dot{z} = 0 \] (55)

is satisfied. The distribution function \( f \) is just the density of gyrocenters in phase space, so we have its continuity equation

\[ \frac{\partial f}{\partial t} + \dot{R} \cdot \nabla f + \dot{z} \frac{\partial f}{\partial z} = 0 \] (56)

in advection form using the phase-space incompressibility. There is no \( M \) term since \( \dot{M} = 0 \) and no \( \vartheta \) term since \( \partial f / \partial \vartheta = 0 \). This is our application of Liouville’s theorem. In our case we started with

\[ z \rightarrow z^\parallel \quad e \frac{\partial A^*}{\partial z^\parallel} = b \] (57)

so that the results for the gyrocenter motion lead to

\[ B^\parallel \frac{\partial f}{\partial t} + \frac{1}{e} b \cdot \nabla H \times \nabla f + B^* \cdot \left( \frac{\partial H}{\partial z^\parallel} \nabla f - \frac{\partial f}{\partial z^\parallel} \nabla H \right) = 0 \] (58)

This is our gyrokinetic equation. One thing to note is that the only appearance of \( MW \) is in its contribution to \( B^* \) and \( B^\parallel \) and it is small there and introduces no new effects. This is why it is usually dropped in computations.
D. The Field Equations

Given the system Lagrangian in Eq. (42) we find the equations for $\phi$ and $A_\parallel$ by varying them in $L_p\,dt$ and setting the coefficients of the variations (i.e., the functional derivatives) to zero.

The induction equation for $A_\parallel$ is found by varying $L$ with respect to $A_\parallel$,

$$b \cdot \nabla \times \left[ \nabla \times \left( A_\parallel b \right) \right] = \mu_0 J_\parallel$$ (59)

where

$$J_\parallel = \sum_{sp} \int dW eU f$$ (60)

is the gyrocenter current and arises from the appearance of $A_\parallel$ in $H$. Eq. (59) is the gyrokinetic Ampère’s law.

The polarisation equation for $\phi$ is found by varying $L$ with respect to $\phi$,

$$\nabla \cdot \left( N_E \nabla_\perp \phi + \nabla_\perp^2 P_E \right) + \rho_G = 0$$ (61)

where

$$\rho_G = \sum_{sp} ne \quad N_E = \sum_{sp} \frac{nm}{B^2} \left( 1 - \frac{\Omega_{e\text{eb}}}{2\Omega} \right) \quad P_E = \sum_{sp} \frac{m}{2eB^2} \left( p_\perp + nm \frac{u_E^2}{2} \right)$$ (62)

are the gyrocenter charge density, the polarisability, and the generalised FLR correction to the gyrocenter charge density, with moment quantities given by

$$n = \int dW f \quad p_\perp = \int dW M\Omega f$$ (63)

These represent the gyrocenter density and perpendicular pressure. Eq. (61) is the gyrokinetic Poisson equation. In this case all the contributions arise from $H$ solely.

Together, Eqs. (58,59,61) describe the complete self consistent dynamical system which arises from the description due to the Lagrangian in Eq. (42).

E. The Conventional Tokamak Case

In a tokamak, the background magnetic field is axisymmetric. The general form is

$$\mathbf{B} = \nabla \times \mathbf{A} = I \nabla \varphi + \nabla \psi \times \nabla \varphi$$ (64)
where \( \varphi \) is the geometric toroidal angle about the symmetry axis \((R = 0)\). In the conventional tokamak limit both the plasma beta and the effective field line pitch angle away from purely toroidal are small, so that we can re-cast the geometry as

\[
I = B_0 R_0 = \text{constant} \quad b = R \nabla \varphi \quad B = \frac{I}{R}
\]  

(65)

where \( R \) is the toroidal major radius and \( R_0 \) and \( B_0 \) are constants giving the reference values of \( R \) and \( B \). This results in

\[
\mathbf{B}_\perp = \nabla \times \left( A_{\parallel} R \nabla \varphi \right) \quad B_\perp = \frac{1}{R} \left| \nabla_\perp (A_{\parallel} R) \right|
\]  

(66)

in the field Lagrangian density. If the full equilibrium current is considered to be carried by the gyrocenters, then the field Lagrangian density is generalised to

\[
\mathcal{L}_f = -\frac{1}{2 \mu_0 R^2} \left| \nabla_\perp \left( \psi + A_{\parallel} R \right) \right|^2
\]

(67)

Hence \( A_{\parallel} R \) acts as a perturbation to \( \psi \) in a conventional tokamak model. Moreover, in this case \( \nabla \varphi \cdot \nabla \times (z_{\parallel} R \nabla \varphi) \) vanishes when setting up the coordinate Jacobian, so that \( B_{\parallel}^* \) reduces to \( B \).

In the conventional tokamak case the gyrokinetic Ampère’s law in Eq. (59) becomes

\[
-R^2 \nabla \cdot \frac{1}{R^2} \nabla_\perp \left( \psi + A_{\parallel} R \right) = \mu_0 R J_{\parallel}
\]  

(68)

if \( \psi \) is included in \( \mathcal{L}_f \). Using this form with \( \psi \) requires the equilibrium current to be contained in \( J_{\parallel} \). If the latter is not, then \( \psi \) is not included in \( \mathcal{L}_f \) and then as a result does not appear in the gyrokinetic Ampère’s law.

**IV. GENERAL PHASE SPACE JACOBIAN AND THE FOUR-BRACKET FORM**

We always assume the use of \( M \) as one of the velocity-space coordinates since it is a conserved quantity in the motion and the gyromotion has little consequence for the rest of the dynamics. But the \( z \)-coordinate can be chosen differently (e.g., unperturbed energy or angular momentum). To show the usefulness of the derivation method we leave the \( z \)-coordinate general and allow \( \mathbf{A}^* \) to be a general function of all the coordinates. The result is a generalised bracket form which is helpful for building consistent computations. We do assume \( \mathbf{A}^* \) remains independent of time so that canonical representation is maintained.
The gyrocenter Lagrangian is in general

\[ L_p = eA^* \cdot \dot{R} + M\dot{\vartheta} - H \]  

(69)

The gyromotion separates out in the same way as above. We note that \(L_p\) can be re-written in terms of phase space four-vectors

\[ L_p = eA^*_a \dot{Z}^a + M\dot{\vartheta} - H \]  

(70)

with indices \(a\) one of \(\{ijk, z\}\) with \(z\) the fourth coordinate. The indices \(\{ijk\}\) range over the spatial dimensions, and we will use \(e^{ijk}\) as the Levi-Civita pseudotensor of rank three with units \(g^{-1/2}\), the inverse spatial volume element given by

\[ \frac{1}{\sqrt{g}} = \nabla x^1 \times \nabla x^2 \cdot \nabla x^3 \]  

(71)

with the three spatial coordinates in order in a right handed system. In general the component \(A^*_z\) is zero but the \(A^*_a\) can have derivatives in any coordinate except \(\vartheta\). Varying the differential action \(L_p\, dt\) we have

\[ \delta L_p \, dt = e\delta Z^a A^*_{b,a} dZ^b + eA^*_b d(\delta Z^b) - \delta Z^a H_a \, dt + \delta M(\cdots) + \delta \vartheta(\cdots) \]  

(72)

concentrating only on the four coordinates spanned by \(\{a\}\) or \(\{b\}\). Subtracting the relevant differential we obtain

\[ \delta L_p \, dt - ed(A^*_b \delta Z^b) = \delta Z^a \left[ e \left( A^*_{b,a} - A^*_{a,b} \right) dZ^b - H_a \, dt \right] + \delta M(\cdots) + \delta \vartheta(\cdots) \]  

(73)

This yields as Euler-Lagrange equations for the four coordinates \((R, z)\)

\[ e \left( A^*_{p,a} - A^*_{a,p} \right) dZ^p = H_a \, dt \]  

(74)

and re-labelling the existing summed index \(b\) as \(p\) for convenience with what happens next.

We anticipate solving this by operating from the left with the 4D Levi-Civita pseudotensor without units, denoted \(\mathcal{E}^{abcd}\), which takes values ±1 or 0 according to positive/negative perturbation of indices from \(\{1234\}\) or repeated indices. Then, doubly contracting with \(A^*_{c,d}\) produces

\[ eA^*_{c,d} \mathcal{E}^{abcd} \left( A^*_{p,a} - A^*_{a,p} \right) dZ^p = \mathcal{E}^{abcd} H_a A^*_{c,d} \, dt \]  

(75)

In each step we will use the fact that there is no \(A^*_z\) so that index \(z\) must be among the derivatives.
First considering index $b$ is $z$ so that \{\text{acd}\} are the spatial \{\text{ijk}\} indices (the sign remains since it is two exchanges between $ijkz$ and $izjk$), we have

$$eA_{j,k}^* \delta^{ijk} \left( A_{p,i}^* - A_{i,p}^* \right) dZ^p = \epsilon^{ijk} H_i A_{j,k}^* dt$$  \hspace{1cm} (76)$$

where the lower case $\epsilon^{ijk}$ is the 3D Levi-Civita pseudotensor without units. We note that

$$\epsilon^{ijk} = \sqrt{g} \epsilon^{ijk} \epsilon^{ijk} A_{j,k}^* = - (B^*)^i \quad A_{i,l}^* - A_{i,l}^* = \epsilon_{ilm} (B^*)^m$$  \hspace{1cm} (77)$$

where \{\text{lm}\} are also spatial indices. We expand in terms of index $p$ being spatial or $z$, so that

$$eA_{j,k}^* \delta^{ijk} \left( A_{i,l}^* - A_{i,l}^* \right) dZ^l - A_{i,z}^* dz = \epsilon^{ijk} H_i A_{j,k}^* dt$$  \hspace{1cm} (78)$$

The spatial terms cancel mutually since

$$\epsilon^{ijk} A_{j,k}^* = - \sqrt{g} (B^*)^i \quad A_{i,l}^* - A_{i,l}^* = \epsilon_{ilm} (B^*)^m$$  \hspace{1cm} (79)$$

so that the combination is proportional to $(B^*)^i \epsilon_{ilm} (B^*)^m = 0$. The last piece is

$$\sqrt{g} (B^*)^i eA_{i,z}^* \dot{Z}^z = \epsilon^{ijk} H_i A_{j,k}^*$$  \hspace{1cm} (80)$$

and here we define

$$B^* = eA_{i,z}^* (B^*)^i$$  \hspace{1cm} (81)$$

as written in Eq. \([52]\), so that

$$\dot{Z}^z = \mathcal{E}^{izjk} H_i A_{j,k}^*$$  \hspace{1cm} (82)$$

now establishing the units of $\mathcal{E}^{abcd}$ to be $(\sqrt{g} B^*)^{-1}$.

Next we choose index $b$ to be spatial so that one of the others is $z$, but $c$ cannot be $z$, which leaves only $a$ and $d$ as choices. We expand according to whether index $a$ is spatial or $z$ and the same for index $p$, so that

$$eA_{j,k}^* \delta^{ijk} \left( A_{l,z}^* \right) dZ^l + eA_{k,z}^* \delta^{ijkz} \left[ (A_{i,l}^* - A_{i,l}^*) dZ^l - A_{i,z}^* dz \right] = \mathcal{E}^{ijkz} H_i A_{k,z}^* dt$$  \hspace{1cm} (83)$$

noting that any occurrence of $A_{z}^*$ drops out. Noting that \{\text{zijk}\} is 3 exchanges away from \{\text{ijkz}\} we have

$$-eA_{j,k}^* \delta^{ijk} \left( A_{l,z}^* \right) dZ^l + eA_{k,z}^* \delta^{ijkz} \left[ (A_{i,l}^* - A_{i,l}^*) dZ^l - A_{i,z}^* dz \right] = \mathcal{E}^{ijkz} H_i A_{k,z}^* dt$$  \hspace{1cm} (84)$$
The $A^*_{l,z}$ term drops due to antisymmetry of $\epsilon^{ijk}$ leaving

$$ e(B^*)^i \left( A^*_{l,z} \right) dZ^l + eA^*_{k,z} \epsilon^{ijk} \left[ \epsilon_{l,m} (B^*)^m dZ^l \right] = \mathcal{S}^{ijkz} H_i A^*_{k,z} dt \quad (85) $$

Finally, contracting the 3D Levi-Civita tensors, we find

$$ eA^*_{i,z} (B^*)^i dZ^j = \frac{1}{\sqrt{g}} \mathcal{S}^{ijkz} H_i A^*_{k,z} dt \quad (86) $$

Collectively we have proven

$$ eA^*_{c,d} \mathcal{S}^{abcd} (A^*_{p,a} - A^*_{a,p}) = \sqrt{g} B^*_{||} \delta^b_p \quad (87) $$

with $B^*_{||}$ defined in Eq. (81) which is equivalent to Eq. (52). The solution to the Euler-Lagrange equations is

$$ \dot{Z}^b = \mathcal{E}^{abcd} H_{a} A^*_{c,d} \quad (88) $$

The gyrokinetic equation is then

$$ \frac{\partial f}{\partial t} + \mathcal{E}^{abcd} H_{a} f_{,b} A^*_{c,d} = 0 \quad (89) $$

for any choice of coordinates under canonical representation (this is Eq. 24 of Ref. [41]). In general, on the right hand side will be placed a collision operator and perhaps external source and sink terms. This form greatly facilitates the construction of computational models since with a good discretisation of a bracket structure (e.g., the Arakawa [48] or Morinishi [49] ones), the conservation properties of the bracket are preserved and therefore the numerical scheme will be closely conservative (this will depend on the timestep scheme).

V. OTHER CHOICES FOR PHASE SPACE COORDINATES

The usefulness of this version of the Euler-Lagrange derivation is to show the generality of this definition of $B^*_{||}$, which reduces to $\mathbf{b} \cdot \mathbf{B}^*$ only if $e \partial A^*/\partial z = \mathbf{b}$. We first recover the standard forms using $v_{||}$ for the parallel phase-space coordinate, for use as a guide. Then, we give two examples: an electrostatic model with use of unperturbed energy $z = mv_{||}^2/2 + M \Omega$ together with $M$ as velocity space coordinates (“energy representation”), and an electromagnetic model with use of total canonical angular momentum $z = e(\psi + A_{||} R) + mv_{||} R$ together with $M$ as velocity space coordinates (“momentum representation”). The latter seems to be promising for studies of gyrokinetic MHD.
A. conventional representation

The conventional representation uses $v_\parallel$ as the $z$-coordinate. In the electrostatic case, this is the same as using $z_\parallel$ as above since the only difference is the factor of $m$ which is normalised away. Starting with

$$A^* = A + \frac{m}{e} v_\parallel b$$

(90)

we obtain

$$\frac{e}{m} \frac{\partial A^*}{\partial v_\parallel} = b \quad B^* = B + \frac{m}{e} v_\parallel \nabla \times b$$

(91)

and then

$$B^*_\parallel = b \cdot B^* = B + \frac{m}{e} v_\parallel b \cdot \nabla \times b$$

(92)

which are what is usually given [17], also following the conventional language of the drift-kinetic predecessor of gyrokinetic theory [10]. Since the Hamiltonian $H$ has spatial gradients only through $\phi$ plus FLR corrections and $M\Omega$, the ExB and grad-B drifts arise from the spatial drifts in $b \times \nabla H$, while the curvature drift arises from the part due to $\nabla \times b$ (whose perpendicular component is proportional to $b \cdot \nabla b$, the actual curvature) in $B^*$ where one factor of $mv_\parallel$ arises from $\partial H/\partial z$ and the other $v_\parallel$ from $B^*$ to produce the multiplier in $mv_\parallel^2 b \cdot \nabla b$ with which one is familiar. Only if we keep $A_\parallel$ or ExB velocity corrections in the canonical momentum [19, 20], especially if time dependent, does any of this significantly change. But then canonical representation is broken and we do not pursue that here. We turn to the energy representation to highlight the effect on the form of $B^*$ and $B^*_\parallel$ and on the resulting equations.

B. energy representation

In an electrostatic model with an energy representation the unperturbed energy $mv^2/2$ is used as the $z$-coordinate, and the parallel velocity function, Hamiltonian, and spatial canonical momentum are

$$m \frac{U^2}{2} = z - M\Omega \quad H = z + e\phi_E \quad eA^* = eA + mUb - MW$$

(93)

with $\phi_E$ containing the FLR effects. Then through the dependence of $U$ upon $z$, we obtain

$$B^*_\parallel = e \frac{\partial A^*}{\partial z} \cdot B^* = \frac{1}{U} b \cdot B^*$$

(94)
For the Euler-Lagrange equations we still have $\dot{M} = 0$ but

$$\dot{\vartheta} = \frac{eB}{mu}b \cdot \dot{R} + W \cdot \dot{R}$$

(95)

awaits solution of the other dimensions. Solving Eq. (88) for the rest we have

$$B_{\parallel} \dot{R} = b \times \nabla \phi_E + UB^* \quad B^*_\parallel \dot{z} = -UB^* \cdot \nabla (e\phi_E)$$

(96)

(don’t forget to apply $\dot{M} = 0$ in evaluating the $\delta R$ terms) and now the gyromotion agrees with Eq. (51) since $b \cdot \dot{R} = U$ still holds. We see that $B^* = \nabla \times A^*$ contains not only the curvature drift but also the $\nabla B$-drift. This can be useful in an electrostatic model and indeed such a model has been constructed with trapped-electron dynamics in mind since under delta-f conditions both energy and $M$ are constants of the motion [50]. However, the singularity at $U = 0$ and the complications with $A_\parallel$ as part of $A^*$ make questionable the usefulness of the energy representation in a total-f electromagnetic model.

C. momentum representation

Another choice is to combine the $\psi$ and $A_\parallel R$ in a large-scale electromagnetic model neglecting $W$ so that we re-define

$$z = e(\psi + A_\parallel R) + mv_\parallel R$$

(97)

with the parallel velocity function becoming

$$mUR = z - e(\psi + A_\parallel R)$$

(98)

with $H$ remaining as in Eq. (44) but with this version of $U$. In this case

$$A^* = A_{pol} + z\nabla \varphi$$

(99)

where $A_{pol}$ is the magnetic potential for $B_{tor} = I\nabla \varphi$, the toroidal magnetic field. Now, the modified magnetic field $B^*$ is simply $B_{tor}$, since the curl of $z\nabla \varphi$ with $z$ held fixed is zero, and $\psi$ is part of $z$. We then have

$$e \frac{\partial A^*}{\partial z} = \nabla \varphi \quad \text{hence} \quad B^*_\parallel = B_{tor} \cdot \nabla \varphi = \frac{I}{R^2}$$

(100)
Since \( z \) is the full toroidal canonical angular momentum it is conserved except for \( \partial H / \partial \phi \). In the equilibrium magnetic field both \( M \) and \( z \) are conserved and the motion is purely spatial. Now, both the \( \nabla B \) and curvature drifts arise from \( \nabla H \) since
\[
\nabla H = e \nabla \phi_E + M \nabla \Omega_E - m U^2 \nabla \log R - e \frac{U}{R} \nabla \left( \psi + A_{\parallel} R \right)
\] (101)
in this representation. In a non field-aligned global model we can choose spatial coordinates \( \{xy\phi\} \) such that
\[
x = \log \frac{R}{R_0}, \quad y = \frac{Z}{R_0}, \quad \sqrt{g} = R_0 R^2
\] (102)
The Euler-Lagrange equations for the gyrocenters are
\[
\frac{dx}{dt} = -\frac{1}{e} \frac{\partial H}{IR_0 \partial y}, \quad \frac{dy}{dt} = \frac{1}{e} \frac{\partial H}{IR_0 \partial x}, \quad \frac{d\phi}{dt} = \frac{\partial H}{\partial z}, \quad \frac{dz}{dt} = -\frac{\partial H}{\partial \phi}
\] (103)
with \( \sqrt{g}B^*_{\parallel} = IR_0 \) a constant. In a 2D equilibrium relaxation model it is even simpler since \( \partial / \partial \phi = 0 \) and hence \( dz/dt = 0 \), leaving
\[
\frac{\partial f}{\partial t} + \frac{1}{IR_0} [H, f]_{xy} = C(f)
\] (104)
with a collision operator. Together with the field equations (Eqs. [59][61]) this would describe the 2D electromagnetic gyrokinetic model. It could be useful to computational studies of equilibrium relaxation with an X-point (arising from contributions due to external coils).

D. section summary

The main point of these examples is that where the various drifts arise (from \( H \) or from \( B^* \)) depends on the representation and no one of them is more valid than another. This pertains especially to the forms of \( B^* \) and \( B^*_{\parallel} \) so one has to examine the representation used to be able to check whether the choices written down are consistent. Ultimately this is a matter of knowing the starting point (choice of coordinates and of the forms in \( L\,dt \)) so that the resulting equations and geometric quantities are properly checked.

VI. MHD AND TOKAMAK EQUILIBRIUM

In general magnetohydrodynamics (MHD) the relevant low frequency approximation is the combination \( \beta \ll 1 \) and \( \omega \ll k_\perp v_A \) in the ordering [23][26]. In the context of global
tokamak MHD this was first cast as an aspect ratio expansion: the ratio of $a$, the reference minor radius, to $R_0$, the reference major radius (usually the geometric axis), should satisfy $a/R_0 \ll 1$. The reasoning is to keep shear-Alfvén dynamics, which have $\omega \sim v_A/qR_0$, but not the compressional Alfvén dynamics, with $\omega \sim v_A/a$, which requires $a/qR_0$ to be a small parameter. Since in the Grad-Shafranov equation $F_{\text{dia}} = RB_{\text{tor}}$ enters squared, the departures of $F_{\text{dia}}$ from $I = \text{constant}$ enter at $O(a/qR_0)^2$ for the current along with $O(\beta)$ for the pressure. These are at most one to four percent in conventional tokamaks, which is why this “Reduced MHD” treatment is in use. The near-constancy of $F_{\text{dia}}$ is easily verified in standard equilibrium computations of conventional tokamaks, which can be defined as $a/R_0 \sim 1/3$ and a $q$-profile giving values in the range 1 to 2 at the magnetic axis and 3 to 10 at the edge. This means that the square of $\rho/q$, with $\rho$ the minor radius of a given flux surface with $q = q(\rho)$, is small everywhere. This is in line with keeping $A_\parallel$, but not $A_\perp$, together with $\phi$ in the field variables. Then, for tokamaks we further take $F_{\text{dia}} = I$ and $B = I/R$ and $b \to R \nabla \phi$ in the Lagrangian.

In this illustration we use a Lagrangian which neglects the FLR corrections and work in the $z = z_\parallel R$ representation. The equilibrium $\psi$ is still kept separate from $A_\parallel R$, so that $B$ still includes the poloidal field due to $\psi$, but the current carried by $f$ is assumed to include the equilibrium piece, so that $\psi$ is included in the field Lagrangian density. We have

$$L_p = eA^* \cdot \dot{R} + M \dot{\phi} - H \quad L_f = -\frac{B_\perp^2}{2\mu_0}$$

with

$$eA^* = eA + \frac{z}{e}\nabla \phi \quad H = m \frac{U^2}{2} + M\Omega + e\phi_E \quad mU = \frac{z}{R} - eA_\parallel$$

$$\phi_E = \phi - \frac{m}{e} \frac{u_E^2}{2} \quad u_E^2 = \frac{1}{B^2} |\nabla \phi|^2 \quad B_{\perp}^2 = \frac{1}{R^2} |\nabla \left(\psi + A_\parallel R\right)|^2$$

and

$$B^* = B \quad B_\parallel^* = \frac{B}{R} = \frac{I}{R^2}$$

The gyrokinetic equation then can be written

$$\frac{B}{R} \frac{\partial f}{\partial t} + \frac{1}{e} \nabla \phi \times \nabla H \cdot \nabla f + B \cdot \left(\frac{\partial H}{\partial z} \nabla f - \frac{\partial f}{\partial z} \nabla H\right) = 0$$

The field equations are then

$$\omega = \sum_{sp} \int dW e f \quad \text{where} \quad \omega = -\nabla \cdot \frac{\rho_m}{B^2} \nabla_\perp \phi$$
$$-\Delta^* \left( \psi + A_\parallel R \right) = \mu_0 J_\parallel R \quad \text{where} \quad \Delta^* = R^2 \nabla \cdot \frac{1}{R^2} \nabla_\perp. \quad (111)$$

The relevant moment variables are

$$\rho_m = \sum_{sp} \int dW \; m f \quad J_\parallel = \sum_{sp} \int dW \; e U f \quad (112)$$

giving the mass density and the parallel current.

The gist of this is that the two main Reduced MHD equations are found from the time derivatives of the two field equations, yielding the vorticity equation and the Ohm’s law, respectively. These are essentially gyrofluid moment equations, since the terms are evaluated using velocity-space moments of the terms in the gyrokinetic equation. This is facilitated by using the divergence form of the gyrokinetic equation,

$$\frac{B}{R} \frac{\partial f}{\partial t} + \nabla \cdot \left[ f \frac{1}{e} \nabla \phi \times \nabla H \right] - \frac{\partial}{\partial z} \left[ f B \cdot \nabla H \right] = 0 \quad (113)$$

and noting that \( \int dW/B^* \) commutes past \( \nabla \) and annihilates \( \partial/\partial z \). The derivatives of \( H \) are

$$\nabla H = e \nabla \phi_E - e \frac{U}{R} \nabla \left( A_\parallel R \right) - \left( mU^2 + M\Omega \right) \nabla \log R \quad \frac{\partial H}{\partial z} = \frac{U}{R} \quad (114)$$

We will be dealing with moments over unity and over \( z \), in the vorticity equation and Ohm’s law, respectively.

The first equation to consider is charge conservation. We take the time derivative of Eq. (110) and evaluate the \( \partial f/\partial t \) terms, to obtain

$$\frac{\partial \varphi}{\partial t} + \left[ \phi, \frac{\varphi}{B} \right] + \left[ \frac{mU^2}{2}, n_i \frac{R}{B} \right] = B \nabla_\parallel J_\parallel + \left[ \log R^2, p_\perp + p_\parallel \frac{B}{2} \right] \quad (115)$$

where the spatial bracket structure denotes

$$[f, g] = \nabla f \times \nabla g \cdot \nabla \varphi \quad (116)$$

and \( \nabla_\parallel \) combines the background magnetic field with the perturbation according to

$$B_T = I \nabla \varphi + \nabla \left( \psi + A_\parallel R \right) \times \nabla \varphi \quad B \nabla_\parallel = B_T \cdot \nabla \quad (117)$$

and the terms with \( \log R \) are the generalised curvature terms. Eq. (113) is our generalisation of the Reduced MHD vorticity equation. The conventional form is recovered by neglecting finite-Mach corrections and taking the pressure to be isotropic, leaving

$$\frac{\partial \varphi}{\partial t} + \left[ \phi, \frac{\varphi}{B} \right] = B \nabla_\parallel J_\parallel - \left[ \log R^2, p \frac{R}{B} \right] \quad (118)$$
which is the Reduced MHD vorticity equation.

The second equation to consider is the one for the parallel electric field. We take the time derivative of Eq. (59) and evaluate the $\partial f / \partial t$ terms, to obtain

$$-R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \right) = \mu_0 R \sum_{\text{sp}} \int dW e \left( U \frac{\partial f}{\partial t} + f \frac{\partial U}{\partial t} \right) \quad (119)$$

With $z = z_R$ we find as an auxiliary

$$\frac{\partial U}{\partial t} = -\frac{e}{m} \frac{\partial A}{\partial t} \quad (120)$$

since $R$ is not time-dependent. Pulling the $A_\parallel$ terms to the left side we arrive at

$$\left( d_e^2 - R^2 \nabla \cdot \frac{1}{R^2} \nabla \right) \left( R \frac{\partial A}{\partial t} \right) = \mu_0 R \sum_{\text{sp}} \int dW \frac{e}{m} \left( mU \frac{\partial f}{\partial t} \right) \quad (121)$$

where the skin depth $d_e$ is given by

$$d_e^2 = \sum_{\text{sp}} \frac{\mu_0 n e^2}{m} \quad n = \int dW f \quad (122)$$

Each species contributes to both sides of this through inverse mass, since the moments of $mU$ of the gyrokinetic equation scale like the pressure gradient or the parallel electric field. The latter comes from the $\partial / \partial z$ term in Eq. (113) since $\partial(mUR)/\partial z = 1$. In the MHD limit where $n_e e \nabla \phi \gg \nabla p$ for all species and the skin depth $d_e$ is taken to be small along with $m_e$, the only significant terms are the the electron contribution to $d_e$ and the terms involving $\mathbf{B} \cdot \nabla \phi$ and $[A_\parallel R, \phi]$ from the electron contribution to $\partial f / \partial t$. This leaves

$$\frac{\partial A}{\partial t} + \nabla_\parallel \phi = 0 \quad (123)$$

which is the Reduced MHD Ohm’s law. If collisional dissipation is added then the resistivity term adds to the right hand side.

The equilibrium condition for the currents is found by setting the right side of the Reduced MHD vorticity equation (Eq. 118) to zero. Assuming $p = p(\psi)$ and $B = I/R$ the curvature term yields

$$\left[ \log R^2, p \frac{R}{B} \right] = c I \frac{\partial p}{\partial \psi} \nabla R^2 \times \nabla \psi \cdot \nabla \phi \quad (124)$$

Assuming that $A_\parallel$ and $\partial / \partial \phi$ vanish, we also have

$$B \nabla_\parallel \rightarrow \nabla \psi \times \nabla \phi \cdot \nabla \quad \text{ and } \quad \Delta^* \psi \rightarrow -\mu_0 J_\parallel R \quad (125)$$
and therefore
\[ \nabla \psi \times \nabla \varphi \cdot \nabla \left[ \Delta^* \psi + \mu_0 \frac{\partial p}{\partial \psi} R^2 \right] = 0 \tag{126} \]
which is the parallel gradient of the Grad-Shafranov equation
\[ F_{dia} \frac{\partial F_{dia}}{\partial \psi} + \Delta^* \psi + \mu_0 \frac{\partial p}{\partial \psi} R^2 = 0 \tag{127} \]
since the first term is a flux function (i.e., of \( \psi \) only, like \( p \)). In other words, the full tokamak equilibrium is recovered, since the integral of Eq. (126) produces an arbitrary flux function which without loss of generality can be set such that Eq. (127) is recovered. This is the statement of tokamak equilibrium under Reduced MHD.

These operations summarise the capture of Reduced MHD and the Grad-Shafranov equilibrium by gyrokinetic theory \[45]\]. The time derivative of the gyrokinetic Poisson equation yields the Reduced MHD vorticity equation, and the time derivative of the gyrokinetic Ampère’s law yields the Reduced MHD Ohm’s law. The equilibrium of the Reduced MHD vorticity equation in the absence of strong flows yields the parallel gradient of the Grad-Shafranov equation, or the Reduced MHD equilibrium. This result is contained in the more general treatment \textit{On the gyrokinetic equilibrium} by Qin \textit{et al} in Ref. \[24]\.

VII. SUMMARY

This outline has shown how the gyrokinetic representation of low frequency plasma dynamics can be derived in a manner more transparent than the ones which use Lie transforms and differential form calculus. A small and straightforward re-casting of Littlejohn’s gauge transform method, originally used for the drift-kinetic case, is sufficient to derive the gyrokinetic Lagrangian allowing for shear-Alfvén electromagnetic electron responses and for finite-amplitude ExB flows. If the flow amplitude is small, such that the ExB contribution to gyroaveraging is negligible, and the plasma beta is low enough that electromagnetic induction in electron responses is negligible, then the model reduces to the conventional small-fluctuation, electrostatic one that is most familiar. The equations derived from the gyrokinetic Lagrangian are not only the kinetic equation of motion, one per species, but also the self consistent electric and shear-Alfvén magnetic field potentials, to which each species contributes through charge and current density. The symmetry principles of a Lagrangian field theory guarantee energetic consistency in what has just been derived. The equations
derived from the gyrokinetic Lagrangian can then serve as the basis for deriving the gyrofluid model within the same representation which then has the same level of consistency. These methods then form the basis for well-constructed numerical models in computations of turbulence, MHD, energetic particle dynamics, and equilibrium and transport in magnetised plasmas. The moment equations exhibiting the correspondences to MHD also indicate the usefulness of this approach to generally energetically consistent gyrofluid equations.

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Appendix A: On not gyroaveraging the magnetic potential

In conventional gyrokinetic theory all of the time-dependent fields are gyroaveraged in a form which can be represented as, e.g., $J_0\phi$ for the electrostatic potential. However, in most of my work the magnetic potential $A_\parallel$ is not gyroaveraged. The reason is that its dynamics is controlled by the electrons with ion contributions entering through the small mass ratio $m_e/M_i$. Here we note that $\rho_s^2 = (M_i/m_e)\rho_e^2$, so that any ion-based FLR effect combines with the mass ratio to produce a correction commensurate with an electron FLR effect. Unless the scale range $\rho^{-1}_s < k_\perp < \rho^{-1}_e$ is explicitly treated, any global computation of mesoscale MHD plus drift Alfvén turbulence will find this a negligible effect.

To see this we repeat the derivation of the Reduced MHD Ohm’s law in Eq. (123). The inverse skin depth defined in Eq. (122) is clearly dominated by the electron term. On the right side the moment with $mU$ is taken of the gyrokinetic equation. All the terms are of similar size, with the main ones being $ne\nabla_\parallel\phi$ and $\nabla_\parallel P_\parallel$ with $P_\parallel$ being the moment of $mU^2$ over $f$. Each species therefore contributes according to the multiplier $\mu_0 ne^2/m$ on the $\nabla_\parallel\phi$ term which gives the static parallel electric field, and according to the multiplier $\mu_0 e/m$ on the pressure terms. With all pressures of similar size, the electron contributions are dominant here too. Hence, gyroaveraging $A_\parallel$ does not bring any significant corrections and it is not necessary. The main exception to this might be global simulation of energetic particle MHD phenomena. We leave such things to future work.

Appendix B: Ordering and Cancellations

We have treated these before [41], but in the meantime there is a version of the polarisation cancellation in the momentum conservation law which is easier to see. It is given here.
The toroidal canonical momentum conservation law in phase space is
\[
\frac{\partial}{\partial t} P_\phi f + \frac{1}{\sqrt{g B}} \frac{\partial}{\partial Z_p} \sqrt{g B} P_\phi f \dot{Z}_p + f \frac{\partial H}{\partial \phi} = 0 \tag{B1}
\]
where the specific toroidal canonical momentum is
\[
P_\phi = z \parallel b_\phi + e \psi \tag{B2}
\]
and \(b_\phi\) is the toroidal covariant unit vector component (usually approximated as the toroidal major radius \(R\)), and we are using the \(z = z_\parallel\) representation in canonical structure (cf. Eqs. 34,50 of Ref. [41]).

The issue is the \(\psi\) terms which appear with the factor of \(ne\) and are of order unity. With a neoclassical flow, the parallel momentum itself is \(O(\delta)\) in the small parameter \(\delta = \rho_s/L_{\perp}\). The transport is given by a fluctuations in parallel momentum and ExB velocity, each another order down so that the momentum flux is \(O(\delta^3)\). Hence the belief that \(O(\delta^3)\) drifts in \(e\psi f \dot{R}\) might be commensurate with this.

However, the largest terms in this equation cancel exactly, leaving the toroidal ExB momentum term which in this context is \(O(\delta^2)\). Now the largest term is the parallel momentum content at \(O(\delta)\) and the transport at \(O(\delta^3)\) which is actually \(O(\delta^2)\) relative to the content. With the leading order eliminated each term is promoted by one order, and the result is a transport equation in which the ordering is the same as in any other transport equation: content at lowest \(O(1)\), and transport at \(O(\delta^2)\). It does not matter whether we call these \(O(\delta)\) and \(O(\delta^3)\), since the relative order is the same. This was already shown in Eqs. (73–80) of Ref. [41], but we sketch it again here.

The charge conservation law is found by varying \(\phi\) in the Lagrangian to obtain the polarisation equation (Eq. 61), and then taking the time derivative. We may write these generally as
\[
\nabla \cdot P = \rho_G = \sum_{sp} \int dW_e f \\
\nabla \cdot \frac{\partial P}{\partial t} = \sum_{sp} \int dW_e \frac{\partial f}{\partial t} = -\nabla \cdot J_G \tag{B3}
\]
with the gyrocenter charge density and current defined as
\[
\rho_G = \sum_{sp} \int dW_e f \\
J_G = \sum_{sp} \int dW_e f \dot{R} \tag{B4}
\]
We may always write \(\rho_G\) as the divergence of a polarisation vector \(\mathbf{P}\) because all dependence of \(L\) upon \(\phi\) except for the first \(e\phi\) term in \(H\) enters through gradients of \(\phi\) (formally, \(\mathbf{P}\)
is given by the functional derivative of $L$ with respect to $\mathbf{E}$ (cf. also Eqs. 15,16 of Ref. [51]). Note that the gyrocenter charge density is not zero and gyrocenter dynamics is not ambipolar – these concepts refer to the particle density and dynamics. Only under static conditions ($\partial/\partial t = 0$) is the gyrocenter dynamics ambipolar, and only in the absence of an electric field and FLR corrections does the gyrocenter charge density vanish. The charge conservation law is found by combining the divergences into a total one,

$$\nabla \cdot \mathbf{J} = 0 \quad \mathbf{J} = \mathbf{J}_p + \mathbf{J}_G \quad \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$  \hspace{1cm} (B5)

taking the time-dependent polarisation current $\mathbf{J}_p$ into account.

Now considering the $\psi$ terms in the toroidal canonical momentum conservation law, we have upon taking the velocity-space integral and species sum

$$\sum_{sp} \int d\mathcal{W} \frac{\partial}{\partial t} e f \psi + \nabla \cdot \sum_{sp} \int d\mathcal{W} e f \psi \dot{\mathbf{R}} + \cdots$$  \hspace{1cm} (B6)

where the other terms are left to be treated separately. Putting in for $\rho_G$ in the first term and $\mathbf{J}_G$ in the second, noting that $\psi$ pulls out of both the integral and the sum but not the divergence, we have

$$\psi \frac{\partial \rho_G}{\partial t} + \nabla \cdot \psi \mathbf{J}_G + \cdots$$  \hspace{1cm} (B7)

Putting the polarisation current in for $\rho_G$ we have

$$\psi \nabla \cdot \mathbf{J}_p + \nabla \cdot \psi \mathbf{J}_G + \cdots$$  \hspace{1cm} (B8)

In one term $\psi$ is inside the divergence, in the other not, so we do the first one by parts, and combine the total divergences

$$-\mathbf{J}_p \cdot \nabla \psi + \nabla \cdot (\psi \mathbf{J}) + \cdots$$  \hspace{1cm} (B9)

Since the divergence of the total current vanishes, this is re-cast as

$$-\mathbf{J}_p \cdot \nabla \psi + \mathbf{J} \cdot \nabla \psi + \cdots$$  \hspace{1cm} (B10)

Expressing the first term with the polarisation vector and noting that $\psi$ is static, we have the exact result that

$$\psi \frac{\partial \rho_G}{\partial t} + \nabla \cdot \psi \mathbf{J}_G = \frac{\partial}{\partial t} (-\mathbf{P} \cdot \nabla \psi) + \mathbf{J} \cdot \nabla \psi$$  \hspace{1cm} (B11)
Since $\psi$ is constant on a flux surface, it is proportional to $V$, the volume enclosed by the surface. Simply due to vector identities, the flux surface average of the contraction between any divergence-free vector and $\nabla V$ vanishes \[52\]. So we take the flux surface average (denoted by angle brackets) and bring the other terms back in, we arrive at

$$\frac{\partial}{\partial t} \left\langle M_\phi - \mathbf{P} \cdot \nabla \psi \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle \Pi_\phi \cdot \nabla \psi \right\rangle + \left\langle \sum_{sp} \int dW f \frac{\partial H}{\partial \phi} \right\rangle = 0 \quad (B12)$$

where

$$M_\phi = \sum_{sp} \int dW f z_\parallel b_\phi \quad \Pi_\phi = \sum_{sp} \int dW f z_\parallel b_\phi \dot{\mathbf{R}} \quad (B13)$$

are the gyrocenter parallel momentum and parallel momentum flux, the $f \partial H/\partial \phi$ term gives the wave flux transport, and $V' = dV/d\psi$ gives the dependence of the enclosed flux-surface volume $V$ upon $\psi$. This equation is the result of Section VII of Ref. \[41\], with subsections A and B showing that the $f \partial H/\partial \phi$ term is a total divergence. Note that the treatments of the $\psi$ terms and of the $f \partial H/\partial \phi$ term are separate and there are no exchanges between the two. In this equation, the two $f z_\parallel$ terms are the largest ones and their relation as a transport equation is the same as in those for particles or thermal energy. This is the reason that the lowest-order terms in $H$ due to the perturbed field variables are the main ones for turbulent transport. In rare circumstances the Reynolds stress terms from $f \partial H/\partial \phi$ can become concurrent but are never dominant. Lowest-order terms in $H$ always dominate in the $z_\parallel$ terms, and second order terms in $H$ or $L_f$ always dominate in the $\mathbf{P} \cdot \nabla \psi$ and and $f \partial H/\partial \phi$ terms.

Similar considerations apply also to the charge conservation equation itself, in which transport of $n$ by the part of $\dot{\mathbf{R}}$ due to $e\phi$ looks like a large term, but the same polarisation cancellation applies to this term, so that the ExB velocity transports $\rho_G$ which is replaced by the scalar quantity $\varpi = -\nabla \cdot \mathbf{P}$ functioning as a generalised vorticity. The equation for $\varpi$ is essentially a simple generalisation of the vorticity equation in any two-fluid model. This has been shown elsewhere \[27, 45\]. Again, all the dominant transport effects are given by the lowest order terms due to $\phi$ and $A_\parallel$ in $H$ and the drift tensor due to $A^*$, except for the dependence of $\varpi$ itself on the appearance of the ExB energy in $L$.

One can always check this by keeping some higher order terms in $H$ or $A^*$ in the theoretical model (for example, $mu_E^2$ compared to $2M\Omega$ in $r^2$ in Eq. \[37\] neglecting them a priori in computations, and then measuring them a posteriori in the results. A good example of how to do this is given in some recent papers \[47, 53\].