GRADIENT ESTIMATE FOR HARMONIC FUNCTIONS ON KÄHLER MANIFOLDS

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Abstract. We prove a sharp integral gradient estimate for harmonic functions on noncompact Kähler manifolds. As application, we obtain a sharp estimate for the bottom of spectrum of the $p$-Laplacian and prove a splitting theorem for manifolds achieving this estimate.

1. Introduction

This paper studies harmonic functions and spectral information of complete non-compact manifolds. On a Riemannian manifold $(M^n, g)$ the Laplace operator $\Delta$ acting on functions is essentially self adjoint and has its $L^2$ spectrum contained in $[0, \infty)$. Properties of harmonic functions are well understood for manifolds with Ricci curvature bounded from below. If the Ricci curvature is non-negative, Yau’s Liouville theorem [15] proves that there are no positive harmonic functions on $M$. Furthermore, there are important works concerning the space of polynomially growing harmonic functions, for example [2, 4].

On the other hand, when the Ricci curvature has lower bound $\text{Ric} \geq -(n-1) K$, for some $K > 0$, then there may exist positive harmonic functions. In this case, Yau’s gradient estimate asserts that

\begin{equation}
|\nabla \ln u|^2 \leq (n-1)^2 K,
\end{equation}

for any positive harmonic function $u$ on $M$. This estimate is sharp, as it can be seen for example on the hyperbolic space $\mathbb{H}^n$.

Assume now that $(M^n, g)$ is Kähler, where $m$ is the complex dimension. On $M$ we consider the Riemannian metric

\[ds^2 := \text{Re} \left( g_{\alpha \overline{\beta}} dz^\alpha d\overline{z}^\beta \right).\]

If $\{e_k\}_{k=1,2m}$ is an orthonormal frame in this metric, so that $e_{2k} = Je_{2k-1}$, then

\[\nu_\alpha = \frac{1}{2} (e_{2\alpha-1} - Je_{2\alpha})\]

is a unitary frame, where $\alpha = 1, 2, \ldots, m$. Assume the Ricci curvature of this Riemannian metric is bounded below by $\text{Ric} \geq -2 (m+1)$, or equivalently that $R_{\alpha \beta} \geq - (m+1) \delta_{\alpha \beta}$ in the unitary frame $\{\nu_\alpha\}_{\alpha=1,m}$. Then Yau’s gradient estimate (1.1) implies

\begin{equation}
|\nabla \ln u|^2 \leq 4m^2 + 2m - 2,
\end{equation}

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for any positive harmonic function \( u \) on \( M \). For \( m \geq 2 \) the estimate (1.2) is no longer sharp in the class of complete Kähler manifolds with \( \text{Ric} \geq -2(m+1) \). In fact, G. Liu proved \([10]\) that there exists a constant \( \varepsilon(m) > 0 \) so that
\[
|\nabla \ln u|^2 \leq 4m^2 + 2m - 2 - \varepsilon(m),
\]
for any \( u > 0 \) harmonic. In view of known examples, it is an interesting question whether the improved gradient estimate
(1.3)
\[
|\nabla \ln u|^2 \leq 4m^2 + 2m - 2 - \varepsilon(m),
\]
holds for any positive harmonic function \( u \) on a Kähler manifold with \( \text{Ric} \geq -2(m+1) \). There exist positive harmonic functions on the complex hyperbolic space \( \mathbb{CH}^m \) for which equality holds.

In this paper we have established some sharp integral gradient estimates which present supporting evidence for (1.3).

**Theorem 1.1.** Let \((M,g)\) be a complete Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2(m+1) \). Then any positive harmonic function \( u \) satisfies the integral gradient estimate
(1.4)
\[
\int_M u |\nabla \ln u|^p \, \phi^2 \leq ((2m)^p + \epsilon) \int_M u \phi^2 + \frac{\epsilon(m)}{\epsilon} \int_M u |\nabla \phi|^2,
\]
for any \( p \leq 2(m+2) \), any \( \epsilon > 0 \), and any \( \phi \geq 0 \) with compact support in \( M \).

For \( p = 2 \) the estimate was first established by the first author in \([12]\), so our contribution here is to prove it for higher exponents \( p \leq 2m + 4 \). While in doing this we are inspired by the ideas in \([12]\), Theorem 1.1 will require some delicate new estimates. Let us briefly describe the idea of proof for (1.4). Recall that Yau’s gradient estimate for Riemannian manifolds uses the maximum principle, the Bochner formula applied to the function \( f = \ln u \), and a clever manipulation of the hessian term \( |f_{ij}|^2 \). To get a sharp estimate for Kähler manifolds, the hessian term needs to be dealt with differently. To prove (1.4) we will use integration by parts and we will estimate the complex hessian \( |f_{\alpha\bar{\beta}}|^2 \) and the reminder \( |f_{\alpha\beta}|^2 \) in different ways. This strategy seems to break down when \( p > 2(m+2) \), because some additional terms appear that are difficult to control.

However, we have obtained an integral estimate valid for all exponents \( p \geq 2 \) provided the manifold satisfies an additional assumption. Recall that the bottom of spectrum of the Laplace operator \( \Delta \) is characterized by
(1.5)
\[
\lambda_1(M) = \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2}.
\]
According to \([12]\), \( \lambda_1(M) \leq m^2 \) holds on any complete Kähler manifold with \( \text{Ric} \geq -2(m+1) \). This estimate is sharp, being achieved on \( \mathbb{CH}^m \) and on other examples \([3, 9]\). We have the following result.

**Theorem 1.2.** Let \((M,g)\) be a complete Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2(m+1) \). Assume in addition that \( M \) has maximal bottom of spectrum for the Laplacian, \( \lambda_1(M) = m^2 \). Then any positive harmonic function \( u \) satisfies the integral gradient estimate
\[
\int_M u |\nabla \ln u|^p \, \phi^2 \leq ((2m)^p + \epsilon) \int_M u \phi^2 + \frac{\epsilon(p,m)}{\epsilon} \int_M u |\nabla \phi|^2,
\]
for any \( p \geq 2 \), any \( \epsilon > 0 \), and any \( \phi \geq 0 \) with compact support in \( M \).

Theorems 1.1 and 1.2 have applications to spectral estimates. Using judicious test functions in (1.5) it is possible to obtain upper bound estimates for the bottom spectrum of the Laplacian. The most natural test functions \( \phi \) in (1.5) are those depending only on distance function; this eventually needs application of the Laplacian comparison theorem. Using this approach, Cheng proved the sharp upper bound

\[
\lambda_1 (M) \leq \frac{(n - 1)^2}{4}
\]

(1.6)
on any Riemannian manifold satisfying \( \text{Ric} \geq -(n + 1) \). This result was generalized in [11] to the bottom spectrum \( \lambda_{1,p} (M) \) of the \( p \)-Laplacian

\[
\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right),
\]

which is characterized by

\[
\lambda_{1,p} (M) = \inf_{\phi \in C_0^\infty (M)} \frac{\int_M |\nabla \phi|^p}{\int_M |\phi|^p}.
\]

(1.7)

It was proved in [11] that

\[
\lambda_{1,p} (M) \leq \left( \frac{n - 1}{p} \right)^p
\]

(1.8)
on any Riemannian manifold with \( \text{Ric} \geq - (n + 1) \). This can be seen as a generalization of Cheng’s estimate, because by Hölder inequality (1.8) implies (1.6).

Both (1.6) and (1.8) are no longer sharp if \((M^m, g)\) is Kähler with Ricci curvature bounded by \( \text{Ric} \geq -2 (m + 1) \). Furthermore, the Laplace comparison theorem with comparison space \( \mathbb{C}^m \) fails for only Ricci curvature bounds [10], so different ideas are now required. In [12] the first author proved the sharp spectral estimate

\[
\lambda_1 (M) \leq m^2,
\]

(1.9)

by using a positive harmonic function (for example, the Green’s function) as a test function in (1.5) and applying the integral gradient estimate in Theorem 1.1 for \( p = 2 \).

As application of Theorems 1.1 and 1.2 we are able to extend (1.9) for the \( p \) Laplacian.

**Theorem 1.3.** Let \((M, g)\) be a Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2 (m + 1) \). Then the bottom spectrum \( \lambda_{1,p} (M) \) of the \( p \)-Laplacian is bounded by

\[
\lambda_{1,p} (M) \leq \left( \frac{2m}{p} \right)^p,
\]

(1.10)

for any \( p \leq 2m + 4 \). If, moreover, \((M, g)\) has maximal bottom of spectrum of the Laplacian,

\[
\lambda_1 (M) = m^2,
\]

then

\[
\lambda_{1,p} (M) = \left( \frac{2m}{p} \right)^p,
\]

(1.11)

for any \( p \geq 2 \).
Let us note that by Hölder inequality, the assumption that \( \lambda_1(M) = m^2 \) implies 
\[ \lambda_{1,p}(M) \geq \left( \frac{2m}{p} \right)^p \] for any \( p \geq 2 \). So to prove (1.11) we showed the converse inequality that \( \lambda_{1,p}(M) \leq \left( \frac{2m}{p} \right)^p \) for any \( p \geq 2 \). Hence, (1.11) can be rephrased that if \( \lambda_1(M) \) is maximal relative to the Ricci curvature bound, then \( \lambda_{1,p}(M) \) is maximal as well. The converse of this statement is not known.

Finally, as Theorem 1.3 is sharp, we address the equality case. A remarkable theory developed by P. Li and J. Wang [6, 7, 8, 9] proves rigidity of complete manifolds with more than one end and achieving maximal bottom of spectrum. As this theory uses harmonic functions associated to the number of ends of a manifold, it can be applied here to study rigidity in Theorem 1.3.

**Theorem 1.4.** Let \( (M, g) \) be a Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2(m + 1) \). Assume that \( p \leq 2m \) and 
\[ \lambda_{1,p}(M) = \left( \frac{2m}{p} \right)^p. \]
Then either \( M \) has one end or it is diffeomorphic to \( \mathbb{R} \times N \), for a compact \( 2m - 1 \) dimensional manifold \( N \), and the metric on \( M \) is given by 
\[ ds_M^2 = dt^2 + e^{-4t} \omega_2^2 + e^{-2t} (\omega_3^2 + ... + \omega_{2m}^2), \]
where \( \{\omega_2, ..., \omega_{2m}\} \) is an orthonormal coframe for \( N \).

The proof of this theorem uses the new estimates obtained in Theorem 1.1 applied to a harmonic function constructed under the assumption that the manifold has more than one end. The rigidity is obtained by reading the equality from the estimates in Theorem 1.1. The restriction \( p \leq 2m \) is assumed in order to use a result in [14] that rules out the existence of two infinite volume ends.

The structure of the paper is as follows. In Section 2 we prove the gradient estimate Theorems 1.1 and 1.2. This is applied in Section 3 to obtain the spectral estimates Theorem 1.3. In Section 4 we study the rigidity result in Theorem 1.4.
**Theorem 2.1.** Let \((M, g)\) be a complete Kähler manifold of complex dimension \(m\), with \(\text{Ric} \geq -2(m+1)\). Then any positive harmonic function \(u\) satisfies the integral gradient estimate

\[
\int_M u |\nabla \ln u|^p \phi^2 \leq ((2m)^p + \epsilon) \int_M u \phi^2 + \frac{c(m)}{\epsilon} \int_M u |\nabla \phi|^2,
\]

for any \(p \leq 2(m+2)\), any \(\epsilon > 0\), and any \(\phi \geq 0\) with compact support in \(M\). Here \(c(m) > 0\) is a constant depending only on \(m\).

**Proof.** Let \(u : M \to \mathbb{R}\) be a positive harmonic function on a Kähler manifold \((M, g)\) with \(\text{Ric} \geq -2(m+1)\). In complex coordinates, this means that

\[
R_{\alpha\beta} \geq -(m+1)g_{\alpha\beta}.
\]

Here and throughout, \(\{\nu_\alpha\}_{\alpha=1,m}\) is a local unitary frame. Denote with

\[
Q = |\nabla \ln u|^2.
\]

Let \(\phi\) be a cut-off function with compact support in \(M\) and fix any \(k \geq 0\). To prove this theorem we use a strategy inspired from [12]. Let us note that the two end example from Theorem 1.4 admits a positive harmonic function \(w\) so that \(|\nabla w| = 2mw\) and the hessian of \(w\) satisfies

\[
w_{\alpha\bar{\beta}} = -mg_{\alpha\bar{\beta}}w + w^{-1}w_\alpha w_{\bar{\beta}}
\]

\[
w_{\alpha\beta} = \frac{m+1}{m}w^{-1}w_\alpha w_{\bar{\beta}},
\]

Because for this example \(|w_{\alpha\bar{\beta}}|^2 \neq |w_{\alpha\beta}|^2\), we will compute each expressions separately.

Using integration by parts. We have

\[
\int_M u^{-1} |u_{\alpha\bar{\beta}}|^2 Q^k \phi^2 = -\int_M u_\alpha \left(u^{-1} u_{\alpha\bar{\beta}} Q^k \phi^2\right)_{\bar{\beta}}
\]

\[
= -\int_M u^{-1} u_{\alpha\bar{\beta}} u_\alpha Q^k \phi^2 + \int_M u^{-2} u_{\alpha\bar{\beta}} u_\alpha u_{\beta\bar{\gamma}} Q^k \phi^2
\]

\[
- k \int_M u^{-1} u_{\alpha\bar{\beta}} u_\alpha Q^k \phi_{\bar{\beta}}^2 - \int_M u^{-1} u_{\alpha\bar{\beta}} u_\alpha \left(\phi^2\right)_{\bar{\beta}} Q^k.
\]

Since \(u\) is harmonic, we have that \(u_{\alpha\bar{\beta}} = 0\). This implies

\[
\int_M u^{-1} |u_{\alpha\bar{\beta}}|^2 Q^k \phi^2 = \int_M u^{-2} u_{\alpha\bar{\beta}} u_\alpha u_{\beta\bar{\gamma}} Q^k \phi^2
\]

\[
- k \int_M u^{-1} \text{Re} \left(u_{\alpha\bar{\beta}} u_\alpha Q_{\bar{\beta}}\right) Q^{k-1} \phi^2
\]

\[
- \int_M u^{-1} \text{Re} \left(u_{\alpha\bar{\beta}} u_\alpha \left(\phi^2\right)_{\bar{\beta}}\right) Q^k,
\]

where \(\text{Re}(z)\) denotes the real part of \(z\).

Furthermore, since

\[
Q = u^{-2} |\nabla u|^2
\]

\[
= 4u^{-2} u_\gamma u_{\bar{\gamma}},
\]

it follows that

\[
Q_{\bar{\beta}} = 4u^{-2} u_{\gamma\bar{\beta}} u_\gamma + 4u^{-2} u_{\gamma\bar{\beta}} u_{\bar{\gamma}} - 2Q u_{\beta\bar{\gamma}} u^{-1}.
\]
We use this to compute
\begin{equation}
(2.4) \quad \text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha Q_{\tilde{\beta}} \right) = 4u^{-2} |u_{\tilde{\alpha} \tilde{\beta}} u_\alpha|^2 + 4u^{-2} \text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_{\tilde{\gamma}} u_\gamma \right) - 2u^{-1} Q u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_\beta
\end{equation}
Notice that
\begin{equation}
2\text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_{\tilde{\gamma}} u_\gamma \right) = |u_{\tilde{\alpha} \tilde{\beta}} u_\alpha + u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2 - |u_{\tilde{\alpha} \tilde{\beta}} u_\alpha|^2 - |u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2.
\end{equation}
We write
\begin{equation}
|u_{\tilde{\alpha} \tilde{\beta}} u_\alpha + u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2 = \frac{1}{4} \left( |\nabla u|^2 \right)_\alpha \beta = \frac{1}{4} Q_{\alpha \beta} u^2 + \frac{1}{2} Qu_\alpha u_\beta,
\end{equation}
and hence get that
\begin{equation}
|u_{\tilde{\alpha} \tilde{\beta}} u_\alpha + u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2 = \frac{1}{16} |Q_{\alpha \beta} u + 2Qu_\beta|^2 u^2 = \frac{1}{64} |\nabla Q|^2 u^4 + \frac{1}{16} Q^3 u^4 + \frac{1}{16} Q \left( \nabla Q, \nabla u \right) u^3.
\end{equation}
In conclusion,
\begin{equation}
2\text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_{\tilde{\gamma}} u_\gamma \right) = \frac{1}{64} |\nabla Q|^2 u^4 + \frac{1}{16} Q^3 u^4 + \frac{1}{16} Q \left( \nabla Q, \nabla u \right) u^3 - |u_{\tilde{\alpha} \tilde{\beta}} u_\alpha|^2 - |u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2.
\end{equation}
Plugging this in (2.4) it follows that
\begin{equation}
(2.5) \quad \text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha Q_{\tilde{\beta}} \right) = 2u^{-2} |u_{\tilde{\alpha} \tilde{\beta}} u_\alpha|^2 - 2u^{-2} |u_{\alpha \tilde{\beta}} u_\tilde{\alpha}|^2 + \frac{1}{32} |\nabla Q|^2 u^2 + \frac{1}{8} Q^3 u^2 + \frac{1}{8} Q \left( \nabla Q, \nabla u \right) u - 2u^{-1} Q u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_\beta.
\end{equation}
Similarly, one can also prove the following identity that will be used later
\begin{equation}
(2.6) \quad \text{Re} \left( u_{\alpha \tilde{\beta}} u_\alpha u_{\tilde{\gamma}} u_\gamma \right) = 2u^{-2} |u_{\alpha \tilde{\beta}} u_\alpha|^2 - 2u^{-2} |u_{\tilde{\alpha} \tilde{\beta}} u_\tilde{\alpha}|^2 + \frac{1}{32} |\nabla Q|^2 u^2 + \frac{1}{8} Q^3 u^2 + \frac{1}{8} Q \left( \nabla Q, \nabla u \right) u - 2u^{-1} Q \text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_\beta \right).
\end{equation}
Plugging (2.5) into (2.3) we obtain
\begin{equation}
(2.7) \quad \int_M u^{-1} |u_{\alpha \beta}|^2 Q^k \phi^2 = (2k + 1) \int_M u^{-2} u_{\tilde{\alpha} \tilde{\beta}} u_\alpha u_{\tilde{\gamma}} u_\alpha Q^k \phi^2 + 2k \int_M u^{-3} |u_{\alpha \tilde{\beta}} u_\alpha|^2 Q^{k-1} \phi^2 - 2k \int_M u^{-3} |u_{\tilde{\alpha} \tilde{\beta}} u_\tilde{\alpha}|^2 Q^{k-1} \phi^2 - \frac{k}{32} \int_M |\nabla Q|^2 Q^{k-1} \phi^2 - \frac{k}{8} \int_M u Q^{k+2} \phi^2 - \frac{k}{8} \int_M \left( \nabla Q, \nabla u \right) Q^k \phi^2 - \int_M u^{-1} \text{Re} \left( u_{\tilde{\alpha} \tilde{\beta}} u_\alpha \left( \phi^2 \right)_\beta \right) Q^k.
\end{equation}
Integrating by parts, we get that
\[- \frac{k}{8} \int_M \langle \nabla Q, \nabla u \rangle Q^k \phi^2 = - \frac{k}{8(k+1)} \int_M \langle \nabla Q^{k+1}, \nabla u \rangle \phi^2 \]
\[= \frac{k}{8(k+1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{k+1}.\]

Using this in (2.7), we conclude that
\[\int_M u^{-1} |u\alpha\beta|^2 Q^k \phi^2 + \frac{k}{32} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 \]
\[= (2k+1) \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 + 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 \]
\[-2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 - \frac{k}{8} \int_M u Q^{k+2} \phi^2 + F_1(k),\]
where
\[F_1(k) = \frac{k}{8(k+1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{k+1} - \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} (\phi^2)_{\beta} \right) Q^k.\]

We now proceed similarly and compute
\[\int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 = - \int_M u_{\alpha} \left( u^{-1} u_{\alpha\beta} Q^k \phi^2 \right)_{\beta} \]
\[= - \int_M u^{-1} u_{\alpha\beta} u_{\alpha} Q^k \phi^2 + \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 \]
\[-k \int_M u^{-1} u_{\alpha\beta} u_{\alpha} Q^k \phi^2 - \int_M u^{-1} u_{\alpha\beta} u_{\alpha} (\phi^2)_{\beta} Q^k.\]

Note that by Ricci identities,
\[-u_{\alpha\beta\gamma} u_{\gamma} = -u_{\beta\alpha\gamma} u_{\gamma} - R_{\alpha\beta\gamma} u_{\gamma} \]
\[\leq \frac{m+1}{4} |\nabla u|^2.\]

Hence, we get
\[\int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 \leq \frac{m+1}{4} \int_M u Q^{k+1} \phi^2 \]
\[+ \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u_{\alpha} u_{\beta} \right) Q^k \phi^2 \]
\[-k \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} Q^k \phi^2 \right) Q^{k-1} \phi^2 \]
\[- \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} (\phi^2)_{\beta} \right) Q^k.\]
By (2.6) it follows that

\[
\int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 + \frac{k}{32} \int_M u |\nabla Q|^2 Q^{-1} \phi^2 
\leq \frac{m+1}{4} \int_M u Q^{k+1} \phi^2 + (2k+1) \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u_{\alpha} u_{\beta} \right) Q^k \phi^2 
+ 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 - 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 
- \frac{k}{8} \int_M u Q^{k+2} \phi^2 + \frac{k}{8(2k+1)} \int_M \langle \nabla u, \nabla \phi \rangle Q^{k+1} 
- \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} \left( \phi^2 \right)_{\beta} \right) Q^k.
\]

Finally, from

\[
u_{\alpha\beta} u_{\alpha} + u_{\alpha} u_{\beta} = \frac{1}{4} u^2 Q_{\beta} + \frac{1}{2} u Q u_{\beta}
\]

we obtain that

\[
\text{Re} \left( u_{\alpha\beta} u_{\alpha} u_{\beta} \right) = -u_{\alpha\beta} u_{\alpha} u_{\beta} 
+ \frac{1}{16} u^2 \langle \nabla Q, \nabla u \rangle + \frac{1}{8} u^3 Q^2.
\]

Hence, we get

\[
(2k+1) \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u_{\alpha} u_{\beta} \right) Q^k \phi^2 
= - (2k+1) \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 
- \frac{2k+1}{16(k+1)} \int_M Q^{k+1} \langle \nabla u, \nabla \phi \rangle 
+ \frac{2k+1}{8} \int_M u Q^{k+2} \phi^2.
\]

By (2.9) we conclude that

\[
\int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 + \frac{k}{32} \int_M u |\nabla Q|^2 Q^{-1} \phi^2 
\leq \frac{m+1}{4} \int_M u Q^{k+1} \phi^2 - (2k+1) \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 
+ 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 - 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 
+ \frac{k+1}{8} \int_M u Q^{k+2} \phi^2 + \mathcal{F}_2(k),
\]

where

\[
\mathcal{F}_2(k) = -\frac{1}{16(k+1)} \int_M \langle \nabla u, \nabla \phi \rangle Q^{k+1} - \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} \left( \phi^2 \right)_{\beta} \right) Q^k.
\]
Recall that in (2.8) we proved the following:

\[
\int_M u^{-1} |u_{\alpha \bar{\beta}}|^2 Q^k \phi^2 + \frac{k}{32} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 \\
= (2k + 1) \int_M u^{-2} u_{\bar{\alpha} \beta} u_{\alpha} u_{\bar{\beta}} Q^k \phi^2 + 2k \int_M u^{-3} |u_{\alpha \bar{\beta}} u_{\bar{\alpha}}|^2 Q^{k-1} \phi^2 \\
- 2k \int_M u^{-3} |u_{\bar{\alpha} \beta} u_{\bar{\alpha}}|^2 Q^{k-1} \phi^2 - \frac{1}{8} \int_M u Q^{k+2} \phi^2 + \mathcal{F}_1(k),
\]

where

\[
\mathcal{F}_1(k) = \frac{k}{8(k+1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{k+1} - \int_M u^{-1} \text{Re} \left( u_{\bar{\alpha} \beta} u_{\bar{\alpha}} \left( \phi^2 \right) _{\bar{\beta}} \right) Q^k.
\]

Adding (2.11) and (2.12) implies that

\[
\int_M u^{-1} |u_{\alpha \bar{\beta}}|^2 Q^k \phi^2 + \int_M u^{-1} |u_{\alpha \bar{\beta}}|^2 Q^k \phi^2 \\
+ \frac{k}{16} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 \\
\leq \frac{m+1}{4} \int_M u Q^{k+1} \phi^2 + \frac{1}{8} \int_M u Q^{k+2} \phi^2 \\
+ \mathcal{F}_1(k) + \mathcal{F}_2(k).
\]

Note that (2.13) is exactly the identity that one gets by multiplying the (Riemannian) Bochner formula

\[
\frac{1}{2} \Delta |\nabla u|^2 = |u_{ij}|^2 + \text{Ric} \left( \nabla u, \nabla u \right)
\]

by \(u^{-1} Q^k \phi^2\) and integrating it on \(M\). However, inspired by (2.2), we will use different estimates for (2.11) and (2.12).

Note that we have the following inequalities

\[
|u_{\alpha \beta} u_{\bar{\alpha}}|^2 \leq \frac{1}{4} |u_{\alpha \bar{\beta}}|^2 |\nabla u|^2
\]

and

\[
0 \leq u^{-1} \left| u_{\bar{\alpha} \beta} - u^{-1} u_{\bar{\alpha} \beta} + \frac{1}{4m} Q g_{\bar{\alpha} \beta} u \right|^2 \\
= u^{-1} |u_{\bar{\alpha} \beta}|^2 - 2u^{-2} u_{\bar{\alpha} \beta} u_{\bar{\alpha} \bar{\beta}} + m - \frac{1}{16m} u Q^2.
\]

Using (2.13) we have

\[
\int_M u^{-1} |u_{\alpha \beta}|^2 Q^k \phi^2 \geq 4 \int_M u^{-3} |u_{\alpha \bar{\beta}} u_{\bar{\alpha}}|^2 Q^{k-1} \phi^2
\]

and from (2.15) we get

\[
\int_M u^{-1} |u_{\alpha \bar{\beta}}|^2 Q^k \phi^2 \geq 2 \int_M u^{-2} u_{\bar{\alpha} \beta} u_{\alpha} u_{\bar{\beta}} Q^k \phi^2 - \frac{m-1}{16m} \int_M u Q^{k+2} \phi^2.
\]
Plugging these two inequalities into (2.13) implies

\[
4 \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 + \frac{k}{16} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2
\leq -2 \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 + \frac{m+1}{4} \int_M u Q^{k+1} \phi^2
+ \frac{3m-1}{16m} \int_M u Q^{k+2} \phi^2 + F_1(k) + F_2(k).
\]

We have the following inequality

\[
0 \leq \left| u_{\alpha\beta} u_{\alpha} - \frac{m-1}{4m} u Q u_{\beta} \right|^2
= \left| u_{\alpha\beta} u_{\alpha} \right|^2 - \frac{m-1}{2m} u Q u_{\alpha\beta} u_{\alpha} u_{\beta} + \frac{1}{64} \left( \frac{m-1}{m} \right)^2 Q^3 u^4,
\]

from which we deduce that

\[
(2.18) \quad -2 \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2
\leq - \frac{m-1}{m} \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2
+ \frac{1}{32} \left( \frac{m-1}{m} \right)^2 \int_M u Q^{k+2} \phi^2.
\]

By (2.17) and (2.18) we get

\[
2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2 - 2k \int_M u^{-3} |u_{\alpha\beta} u_{\alpha}|^2 Q^{k-1} \phi^2
\leq - \frac{k^2}{32} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 - \frac{(2m-1)k}{m} \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2
+ \frac{(4m^2 - 3m + 1)k}{32m^2} \int_M u Q^{k+2} \phi^2 + \frac{(m+1)k}{8} \int_M u Q^{k+1} \phi^2
+ \frac{k}{2} \left( F_1(k) + F_2(k) \right).
\]

Using this into (2.12) we obtain

\[
(2.19) \quad \int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 + \frac{k(k+1)}{32} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2
\leq \left( 1 + \frac{k}{m} \right) \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2
- \frac{(3m-1)k}{32m^2} \int_M u Q^{k+2} \phi^2 + \frac{(m+1)k}{8} \int_M u Q^{k+1} \phi^2
+ \left( \frac{k}{2} + 1 \right) F_1(k) + \frac{k}{2} F_2(k).
\]
Plugging (2.16) into (2.19) it follows that
\begin{equation}
(2.20) \quad \left(1 - \frac{k}{m}\right) \int_M u^{-2} u_{\alpha\beta} u^\alpha u^\beta F^2 + \frac{k(k+1)}{32} \int_M u |\nabla Q|^2 F^{k-1} \phi^2 \\
\leq \left(\frac{m-1}{16m} - \frac{(3m-1)k}{32m^2}\right) \int_M u F^{k+2} \phi^2 + \frac{(m+1)k}{8} \int_M u F^{k+1} \phi^2 \\
+ \left(\frac{k}{2} + 1\right) F_1(k) + \frac{k}{2} F_2(k).
\end{equation}

This holds for any \( k \geq 0 \). Choosing \( k = m \) into (2.20) implies
\begin{equation}
(2.21) \quad \int_M u F^{m+2} \phi^2 \leq 4m^2 \int_M u F^{m+1} \phi^2 \\
+ \frac{16m}{m+1} ((m+2) F_1(m) + m F_2(m)).
\end{equation}

By Young’s inequality we have
\[ 4m^2 Q^{m+1} \leq \frac{m+1}{m+2} Q^{m+2} + \frac{1}{m+2} (4m^2)^{m+2}. \]

In conclusion, (2.21) implies that
\begin{equation}
(2.22) \quad \int_M u F^{m+2} \phi^2 \leq \left(4m^2\right)^{m+2} \int_M u \phi^2 + \frac{16m}{m+1} \frac{m+2}{m+1} F,
\end{equation}

where
\[ F = \frac{m(2m+3)}{16(m+1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{m+1} - (m+2) \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u^\alpha (\phi^2)_\beta \right) F^m \\
- m \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u^\alpha (\phi^2)_\beta \right) F^m. \]

We now estimate \( F \) as follows. Recall that (2.13) proved
\begin{equation}
(2.23) \quad \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 + \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 \\
+ \frac{m+1}{16} \int_M u |\nabla Q|^2 Q^{m-1} \phi^2 \\
\leq \frac{m}{4} \int_M u F^{m+1} \phi^2 + \frac{1}{8} \int_M u F^{m+2} \phi^2 + \frac{2m-1}{16(m+1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{m+1} \\
- \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u^\alpha (\phi^2)_\beta \right) F^m - \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u^\alpha (\phi^2)_\beta \right) F^m.
\end{equation}

By the Cauchy-Schwarz inequality we have
\[ \int_M u^{-1} |u_{\alpha\beta} u^\alpha (\phi^2)_\beta| Q^m \leq \frac{1}{2} \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 \\
+ \frac{1}{2} \int_M u |\nabla \phi|^2 Q^{m+1}, \]

and
\[ \int_M u^{-1} |u_{\alpha\beta} u^\alpha (\phi^2)_\beta| Q^m \leq \frac{1}{2} \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 \\
+ \frac{1}{2} \int_M u |\nabla \phi|^2 Q^{m+1}. \]
By (2.23) this implies that

\begin{equation}
\frac{1}{2} \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 + \frac{1}{2} \int_M u^{-1} |u_{\alpha\bar{\beta}}|^2 Q^m \phi^2
\end{equation}

\leq \frac{m+1}{4} \int_M u Q^{m+1} \phi^2 + \frac{1}{8} \int_M u Q^{m+2} \phi^2

+ \frac{2m-1}{8 (m+1)} \int_M |\nabla u| |\nabla \phi| Q^{m+1} + \int_M u |\nabla \phi|^2 Q^{m+1}.

Using Yau’s gradient estimate \( Q \leq c(m) = 4m^2 + 2m - 2 \) it results that

\begin{equation}
\int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 + \int_M u^{-1} |u_{\alpha\bar{\beta}}|^2 Q^m \phi^2
\end{equation}

\leq c(m) \int_M u \left( \phi^2 + |\nabla \phi|^2 \right),

for some constant \( c(m) \) depending only on dimension \( m \). Hence, for any \( \varepsilon > 0 \) small enough, we get

\begin{equation}
\int_M u^{-1} |u_{\alpha\beta} u_{\bar{\alpha}}(\phi^2)_{\bar{\beta}}| Q^m + \int_M u^{-1} |u_{\alpha\bar{\beta}} u_{\bar{\alpha}}(\phi^2)_{\bar{\beta}}| Q^m
\end{equation}

\leq \frac{\varepsilon}{c(m)} \int_M u^{-1} |u_{\alpha\beta}|^2 Q^m \phi^2 + \frac{\varepsilon}{c(m)} \int_M u^{-1} |u_{\alpha\bar{\beta}}|^2 Q^m \phi^2

+ \frac{c(m)}{\varepsilon} \int_M u Q^{m+1} |\nabla \phi|^2

\leq \varepsilon \int_M u \phi^2 + \frac{c(m)}{\varepsilon} \int_M u |\nabla \phi|^2,

where in the last line we used (2.23).

Using (2.20) we estimate \( \mathcal{F} \) from (2.22) by

\begin{equation}
|\mathcal{F}| \leq c(m) \varepsilon \int_M u \phi^2 + \frac{c(m)}{\varepsilon} \int_M u |\nabla \phi|^2.
\end{equation}

Hence, (2.22) implies

\begin{equation}
\int_M u Q^{n+2} \phi^2 \leq \left( (4m^2)^{m+2} + \varepsilon \right) \int_M u \phi^2 + \frac{c(m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\end{equation}

where \( c(m) \) depends only on dimension. This proves the theorem for \( p = 2 (m + 2) \).

For \( p < 2 (m + 2) \) this follows immediately from Young’s inequality. \( \square \)

We now prove that Theorem 2.1 can be in fact extended to all values of \( p \geq 2 \), provided in addition that \( \lambda_1(M) \) is maximal.

**Theorem 2.2.** Let \((M, g)\) be a complete Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2(m+1) \). Assume in addition that \( M \) has maximal bottom of spectrum for the Laplacian,

\( \lambda_1(M) = m^2. \)

Then any positive harmonic function \( u \) satisfies the integral gradient estimate

\begin{equation}
\int_M u |\nabla \ln u|^p \phi^2 \leq ((2m)^p + \varepsilon) \int_M u \phi^2 + \frac{c(p,m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\end{equation}

for any \( p \geq 2, \) any \( \varepsilon > 0 \), and any \( \phi \geq 0 \) with compact support in \( M \). Here \( c(p,m) \) depends only on \( p \) and \( m \).
Proof. Start with the inequality

\[
0 \leq u^{-1} |u^{-1} u_{\alpha \beta} u_{\bar{\beta}} - m (m + 1) u_\alpha|^2 \\
= u^{-3} |u_{\alpha \beta} u_{\bar{\beta}}|^2 - 2m (m + 1) u^{-2} \text{Re} (u_{\alpha \beta} u_{\bar{\alpha}} u_{\bar{\beta}}) + \frac{1}{4} m^2 (m + 1)^2 u Q \\
\leq \frac{1}{4} u^{-1} |u_{\alpha \beta}|^2 Q - 2m (m + 1) u^{-2} \text{Re} (u_{\alpha \beta} u_{\bar{\alpha}} u_{\bar{\beta}}) + \frac{1}{4} m^2 (m + 1)^2 u Q.
\]

This implies that for any \( k \geq 1 \),

\[
\int_M u^{-1} |u_{\alpha \beta}|^2 Q^k \phi^2 \geq 8m (m + 1) \int_M u^{-2} \text{Re} (u_{\alpha \beta} u_{\bar{\alpha}} u_{\bar{\beta}}) Q^{k-1} \phi^2 - m^2 (m + 1)^2 \int_M u Q^k \phi^2.
\]

Combining this with (2.13), we get

\[\tag{2.29}
\int_M u^{-1} |u_{\alpha \beta}|^2 Q^k \phi^2 + \frac{k}{16} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 \\
\leq -8m (m + 1) \int_M u^{-2} \text{Re} (u_{\alpha \beta} u_{\bar{\alpha}} u_{\bar{\beta}}) Q^{k-1} \phi^2 \\
+ \frac{1}{8} \int_M u Q^{k+2} \phi^2 + \frac{m + 1}{4} \int_M u Q^{k+1} \phi^2 + m^2 (m + 1)^2 \int_M u Q^k \phi^2 \\
+ F_1(k) + F_2(k).
\]

Recall that by (2.10) we have

\[\tag{2.30}
\int_M u^{-2} \text{Re} (u_{\alpha \beta} u_{\bar{\alpha}} u_{\bar{\beta}}) Q^{k-1} \phi^2 = - \int_M u^{-2} u_{\bar{\alpha}} u_{\alpha} u_{\bar{\beta}} Q^{k-1} \phi^2 \\
+ \frac{1}{8} \int_M u Q^{k+1} \phi^2 \\
- \frac{1}{16k} \int_M Q^k \left\langle \nabla u, \nabla \phi^2 \right\rangle.
\]

Hence, by (2.29) and (2.30) it follows that

\[
\int_M u^{-1} |u_{\alpha \beta}|^2 Q^k \phi^2 + \frac{k}{16} \int_M u |\nabla Q|^2 Q^{k-1} \phi^2 \\
\leq 8m (m + 1) \int_M u^{-2} u_{\bar{\alpha}} u_{\alpha} u_{\bar{\beta}} Q^{k-1} \phi^2 \\
+ \frac{1}{8} \int_M u Q^{k+2} \phi^2 + \left( \frac{m + 1}{4} - m (m + 1) \right) \int_M u Q^{k+1} \phi^2 \\
+ m^2 (m + 1)^2 \int_M u Q^k \phi^2 \\
+ F_1(k) + F_2(k) + \frac{m (m + 1)}{2k} \int_M Q^k \left\langle \nabla u, \nabla \phi^2 \right\rangle.
\]
Finally, combining this with (2.16) implies

$$\int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2$$

$$\leq 4m (m + 1) \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^{k-1} \phi^2$$

$$+ \frac{3m - 1}{32m} \int_M u Q^{k+2} \phi^2 - \frac{m + 1}{2} \left( m - \frac{1}{4} \right) \int_M u Q^{k+1} \phi^2$$

$$+ \frac{1}{2} m^2 (m + 1)^2 \int_M u Q^k \phi^2$$

$$+ F_0(k),$$

where

$$F_0(k) = \frac{1}{2} F_1(k) + \frac{1}{2} F_2(k) + \frac{m (m + 1)}{4k} \int_M Q^k \langle \nabla u, \nabla \phi^2 \rangle,$$

and $F_1(k), F_2(k)$ are specified in (2.8) and (2.11), respectively. Denote with

(2.32) \[ A_k = \int_M u^{-2} u_{\alpha\beta} u_{\alpha} u_{\beta} Q^k \phi^2 - \frac{3m - 1}{32m} \int_M u Q^{k+2} \phi^2 \]

$$+ \frac{1}{8} m (m + 1) \int_M u Q^{k+1} \phi^2.$$ 

We now observe that (2.31) is equivalent to

(2.33) \[ A_k \leq 4m (m + 1) A_{k-1} + F_0(k), \]

for any $k \geq 1$. We estimate $F_0(k)$ as in the proof of Theorem 2.1. Note that

(2.34) \[ F_0(k) = \frac{2k - 1}{32 (k + 1)} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{k+1} \]

$$+ \frac{m (m + 1)}{4k} \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^k$$

$$- \frac{1}{2} \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} \phi^2 \right) Q^k$$

$$- \frac{1}{2} \int_M u^{-1} \text{Re} \left( u_{\alpha\beta} u_{\alpha} \phi^2 \right) Q^k.$$ 

As in (2.25) we get

$$\int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2 + \int_M u^{-1} |u_{\alpha\beta}|^2 Q^k \phi^2$$

$$\leq c(k, m) \int_M u \left( \phi^2 + |\nabla \phi|^2 \right),$$

which yields similarly to (2.27) that

$$|F_0(k)| \leq \varepsilon \int_M u \phi^2 + \frac{c(k, m)}{\varepsilon} \int_M u |\nabla \phi|^2$$

Using this in (2.33) implies that

$$A_k \leq 4m (m + 1) A_{k-1} + \varepsilon \int_M u \phi^2 + \frac{c(k, m)}{\varepsilon} \int_M u |\nabla \phi|^2,$$
for all \( k \geq 1 \). Iterating from \( k = 1, 2, 3 \ldots \) we obtain

\[
A_k \leq (4m (m + 1))^k A_0 + c_1 (k, m) \varepsilon \int_M u \phi^2 + \frac{c_1 (k, m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\]

where \( c_1 (k, m) \) depends only on \( k \) and dimension \( m \). By (2.32) we see that

\[
A_0 = \int_M u^{-2} a_{\alpha \beta} u_{\alpha} u_{\beta} \phi^2 - \frac{3m - 1}{32m} \int_M u Q^2 \phi^2 + \frac{1}{8} m (m + 1) \int_M u Q \phi^2.
\]

Using (2.20) for \( k = 0 \) it follows that

\[
A_0 \leq -\frac{m + 1}{32m} \int_M u Q^2 \phi^2 + \frac{1}{8} m (m + 1) \int_M u Q \phi^2 + \varepsilon \int_M u \phi^2 + \frac{c (m)}{\varepsilon} \int_M u |\nabla \phi|^2.
\]

We now use the assumption that \( \lambda_1 (M) = m^2 \) and obtain

\[
m^2 \int_M u \phi^2 \leq \int_M \left| \nabla \left( u \frac{4}{m} \phi \right) \right|^2
\]

\[
= \frac{1}{4} \int_M u^{-1} |\nabla u|^2 \phi^2 + \frac{1}{2} \int_M \langle \nabla u, \nabla \phi^2 \rangle + \int_M u |\nabla \phi|^2
\]

\[
= \frac{1}{4} \int_M u Q \phi^2 + \int_M u |\nabla \phi|^2.
\]

On the other hand, we have

\[
\int_M u Q \phi^2 \leq \frac{1}{8m^2} \int_M u Q^2 \phi^2 + 2m^2 \int_M u \phi^2,
\]

which combined with (2.37) implies

\[
\int_M u Q \phi^2 \leq \frac{1}{4m^2} \int_M u Q^2 \phi^2 + 4 \int_M u |\nabla \phi|^2.
\]

Using this in (2.36) yields

\[
A_0 \leq \varepsilon \int_M u \phi^2 + \frac{c (m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\]

for some constant \( c \) depending only on dimension. By (2.35) and (2.38) we conclude

\[
A_k \leq c_2 (k, m) \varepsilon \int_M u \phi^2 + \frac{c_2 (k, m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\]

where \( c_2 (k, m) \) depends only on \( k \) and dimension \( m \). Hence, by (2.32) we have proved that

\[
\int_M u^{-2} a_{\alpha \beta} u_{\alpha} u_{\beta} Q^k \phi^2
\]

\[
\leq \frac{3m - 1}{32m} \int_M u Q^{k+2} \phi^2 - \frac{1}{8} m (m + 1) \int_M u Q^{k+1} \phi^2
\]

\[
+ c_2 (k, m) \varepsilon \int_M u \phi^2 + \frac{c_2 (k, m)}{\varepsilon} \int_M u |\nabla \phi|^2.
\]
for any $k \geq 0$. Recall that by (2.20) we have
\[
\left( \frac{(3m - 1)k}{32m^2} - \frac{m - 1}{16m} \right) \int_M u Q^{k+2} \phi^2 \leq \left( \frac{k}{m} - 1 \right) \int_M u^{-2} u_{\kappa\beta} u_\kappa u_\beta Q^k \phi^2 \\
+ \frac{(m + 1)k}{8} \int_M u Q^{k+1} \phi^2 \\
+ c_3 (k, m) \int_M u \phi^2 + \frac{c_3 (k, m)}{\varepsilon} \int_M u |\nabla \phi|^2.
\]
Combining with (2.39) it follows that for $k \geq m$,
\[
\int_M u Q^{k+2} \phi^2 \leq 4m^2 \int_M u Q^{k+1} \phi^2 \\
+ c (k, m) \int_M u \phi^2 + \frac{c (k, m)}{\varepsilon} \int_M u |\nabla \phi|^2,
\]
where $c (k, m)$ depends only on $k$ and dimension $m$. This implies the desired result. \qed

3. Spectrum of $p$-Laplacian

As an application of the integral estimate, we prove a sharp upper bound for the bottom of the spectrum of
\[\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right).\]
It is known that this satisfies
\[(3.1) \quad \lambda_{1,p} (M) \leq \frac{\int_M |\nabla \psi|^p}{\int_M \psi^p},\]
for any $\psi \geq 0$ with compact support in $M$. Hence, to obtain an upper bound for $\lambda_{1,p} (M)$ we will apply (3.1) to a carefully chosen test function $\psi$. For this, let us recall some relation between $\lambda_{1,p} (M)$ for $p \geq 2$ and $\lambda_1 (M) = \lambda_{1,2} (M)$. First, observe that for any $\phi \geq 0$ with compact support in $M$,
\[
\lambda_1 (M) \int_M \phi^p = \lambda_1 (M) \left( \int_M \phi^p \right)^2 \\
\leq \int_M |\nabla \phi^p|^2 \\
= \frac{p^2}{4} \int_M |\nabla \phi|^2 \phi^{p-2} \\
\leq \frac{p^2}{4} \left( \int_M |\nabla \phi|^p \right)^\frac{2}{p} \left( \int_M \phi^p \right)^\frac{p-2}{p}.
\]
This proves that
\[
\left( \frac{4}{p^2} \lambda_1 (M) \right)^\frac{2}{p} \int_M \phi^p \leq \int_M |\nabla \phi|^p,
\]
for any $\phi \geq 0$ with compact support in $M$. Hence
\[(3.2) \quad \lambda_{1,p} (M) \geq \left( \frac{4}{p^2} \lambda_1 (M) \right)^\frac{2}{p}.\]
According to a result of Sung-Wang, it is possible to obtain a reversed inequality, but which is not sharp anymore. By (3.8) in [14] we know that

\[
\frac{p}{2} \lambda_{1,p}(M) \int_M \phi^2 \leq \int_M |\nabla \ln v|^{p-2} |\nabla \phi|^2,
\]

for any \( \phi \) with compact support, where \( v > 0 \) is a positive eigenfunction of the \( p \) Laplacian, \( \text{div} \left( |\nabla v|^{p-2} \nabla v \right) = -\lambda_{1,p}(M) v^{p-1} \). According to Theorem 2.2 in [14], on a complete manifold with \( \text{Ric} \geq -2 (m+1) \), we have a gradient estimate

\[
|\nabla \ln v| \leq \sigma,
\]

where \( \sigma \) is the first positive root of the equation

\[
F(\sigma) := (p-1) \sigma^p - \sqrt{2 (m+1) (2m-1)} \sigma^{p-1} + \lambda_{1,p}(M) = 0.
\]

It is easy to see that

\[
F \left( \frac{1}{p} \sqrt{2 (m+1) (2m-1)} \right) = \lambda_{1,p}(M) - \left( \frac{1}{p} \sqrt{2 (m+1) (2m-1)} \right)^p \leq 0,
\]

where the last line follows applying (1.8) for the Ricci curvature bound \( \text{Ric} \geq -2 (m+1) \). This implies

\[
\sigma \leq \sqrt{2 (m+1) (2m-1)}
\]

By (3.3) and (3.5) we conclude that

\[
\lambda_{1,p}(M) \geq \frac{p}{2} \lambda_{1,p}(M) \left( \frac{p}{\sqrt{2 (m+1) (2m-1)}} \right)^{p-2}.
\]

From here we infer in particular that \( \lambda_{1,p}(M) > 0 \) implies \( \lambda_1(M) > 0 \). It is known that a manifold with positive bottom of spectrum is non-parabolic, so it admits a positive minimal Green’s function \( G \) for the Laplacian. The Green’s function \( G(x_0,x) \) with a pole at \( x_0 \) is harmonic on \( M \setminus \{x_0\} \) and will be used as a test function in (3.1). We will prove the following result.

**Theorem 3.1.** Let \((M,g)\) be a Kähler manifold of complex dimension \( m \), with \( \text{Ric} \geq -2 (m+1) \). Then the bottom spectrum \( \lambda_{1,p}(M) \) of the \( p \)-Laplacian is bounded by

\[
\lambda_{1,p}(M) \leq \left( \frac{2m}{p} \right)^p,
\]

for any \( 2 \leq p \leq 2m + 4 \).

**Proof.** It suffices to prove the theorem for \( p = 2m + 4 \), as the estimate for smaller \( p \) follows from Hölder inequality. Hence, throughout this proof, \( p = 2m + 4 \).

Let us assume by contradiction that \( \lambda_{1,p}(M) > \left( \frac{2m}{p} \right)^p \). Then there exists \( \varepsilon > 0 \) so that

\[
\lambda_{1,p}(M) \geq \left( \frac{2m}{p} \right)^p + 2\varepsilon.
\]

Consider \( G(x_0,x) \) the positive minimal Green’s function, which exists by (3.6). Define

\[
\psi(x) := G^+(x_0,x) \phi(x),
\]
for a cut-off function $\phi(x)$ with support in $B(x_0,2R) \setminus B(x_0,1)$ given by

$$
\phi(x) = \begin{cases} 
  r(x) - 1 & \text{on } B(x_0,2) \setminus B(x_0,1) \\
  1 & \text{on } B(x_0,R) \setminus B(x_0,2) \\
  R^{-1}(2R - r(x)) & \text{on } B(x_0,2R) \setminus B(x_0,R)
\end{cases}
$$

Note that

$$
|\nabla \psi| \leq G^{1 \over p} |\nabla \phi| + \frac{1}{p} |\nabla \ln G| G^{1 \over p} \phi.
$$

Then it follows that

$$
\int_M |\nabla \psi|^p \leq p^{-p} \int_M G |\nabla \ln G|^p \phi^p + \sum_{k=0}^{p-1} \binom{p}{k} \left( \int_M \frac{1}{p} |\nabla G| G^{1 \over p - 1} \right)^k \left( |\nabla \phi| G^{1 \over p} \right)^{p-k}.
$$

Since $|\nabla \phi| \leq c$ and by Yau’s estimate $|\nabla \ln G| \leq c(m)$ on the support of $\phi$, we get that

$$
\int_M |\nabla \psi|^p \leq p^{-p} \int_M G |\nabla \ln G|^p \phi^p + c(m) \sum_{k=0}^{p-1} \int_M G |\nabla \ln G|^k |\nabla \phi|^{p-k}.
$$

By (3.1), (3.7) and (3.8) we conclude that

$$
\varepsilon^2 \int_M \partial_0 \phi^p \leq c \int_M G |\nabla \phi|.
$$

In particular, this proves that there exists a constant $\varepsilon_0 > 0$ so that

$$
\varepsilon_0 \int_{B(x_0,R) \setminus B(x_0,2)} G(x_0,x) \, dx \leq \frac{1}{R} \int_{B(x_0,2R) \setminus B(x_0,R)} G(x_0,x) \, dx
$$

$$
+ c \int_{B(x_0,2R) \setminus B(x_0,1)} G(x_0,x) \, dx,
$$

for any $R > 2$. Recall from (13) that

$$
\frac{1}{C} R \leq \int_{B(x_0,R)} G(x_0,x) \, dx \leq CR,
$$

for some constant $C > 0$ dependent only on $m$ and $\lambda_1(M)$. This contradicts (3.9). The theorem is proved.

We now obtain (1.11) by applying Theorem 2.2.
Theorem 3.2. Let \((M, g)\) be a Kähler manifold of complex dimension \(m\), with \(\text{Ric} \geq -2(m+1)\). Assume that \((M, g)\) has maximal bottom of spectrum of the Laplacian, \(\lambda_1 (M) = m^2\).

Then \(\lambda_{1,p} (M) = \left(\frac{2m}{p}\right)^p\), for any \(p \geq 2\).

Proof. The proof of \(\lambda_{1,p} (M) \leq \left(\frac{2m}{p}\right)^p\) follows as in Theorem 3.1. The converse inequality results from (3.2). \(\square\)

4. Rigidity for maximal bottom spectrum

In this section, we follow a theory developed by P. Li and J. Wang [6, 7, 8, 9] and study the rigidity of manifolds that achieve the estimate for the bottom spectrum of the \(p\)-Laplacian in Theorem 3.1, and have more than one end. The strategy is to use harmonic functions constructed in [5] for manifolds with more than one end, whose behavior depends on whether the end has finite or infinite volume.

Assuming the manifold has at least two ends, we will first prove that one of these ends must have finite volume. For a harmonic function \(u\) associated to any two ends of the manifold, where one of them has finite volume, Theorem 2.1 proves a gradient estimate that implies Theorem 3.1. When \(\lambda_{1,p} (M)\) is maximal, one can infer from the proof of Theorem 2.1 that all inequalities used there must turn into equalities. This will imply the splitting of the manifold topologically into product of the real line with a compact manifold, and will determine the metric as well. For this approach to work, it is crucial that the boundary terms expressed in \(F\) in (2.22) converge to zero for a carefully chosen cut-off function. This turns out to be the case eventually. Although each term in \(F\) does not converge to zero on a given end, it can be computed explicitly and it yields the same absolute constant but with different signs on the two ends. Hence, after cancellation we are able to conclude the rigidity question. It should be noted that this complication arises only in the Kähler case (cf. [12]).

First, note that if \(\lambda_{1,p} (M)\) is maximal, then \(\lambda_{1,q} (M)\) is also maximal, for any \(q \geq p\). Similarly to (3.2), this follows from Hölder inequality. Hence, throughout this section we will assume that \(p = 2m\) and \(\lambda_{1,2m} (M)\) is maximal, which is to say \(\lambda_{1,2m} (M) = 1\). The following result therefore implies Theorem 1.4.

Theorem 4.1. Let \((M, g)\) be a Kähler manifold of complex dimension \(m \geq 2\) and with \(\text{Ric} \geq -2(m+1)\). Suppose \(M\) has maximal bottom of spectrum of the \(2m\) Laplacian,

\[
\lambda_{1,2m} (M) = 1.
\]

Then either \(M\) has one end or it is diffeomorphic to \(\mathbb{R} \times N\), for a compact \(2m-1\) dimensional manifold \(N\), and the metric on \(M\) is given by

\[
ds^2_M = dt^2 + e^{-4t} \omega_2^2 + e^{-2t} \left( \omega_3^2 + ... + \omega_{2m}^2 \right),
\]

where \(\{\omega_2, ..., \omega_{2m}\}\) is an orthonormal coframe for \(N\).

The proof will be done in several steps. From now on we will assume that \(M\) satisfies the hypothesis of Theorem 4.1 and that \(M\) has at least two ends. We first record the following result.
Proposition 4.2. Let \((M, g)\) be a Kähler manifold of complex dimension \(m \geq 2\) and with \(\text{Ric} \geq -2(m+1)\). Assume that \(M\) has maximal bottom of spectrum of the \(2m\) Laplacian,

\[ \lambda_{1,2m}(M) = 1. \]

Then

\[ \lambda_1(M) > \frac{m + 1}{2} \]

and \(M\) has only one infinite volume end.

Proof. From (3.6) we have

\[ \lambda_1(M) \geq m \left( \frac{2m^2}{2m^2 + m - 1} \right)^{m-1}. \]

It follows through elementary calculations that

\[ \lambda_1(M) > \frac{m + 1}{2} \]

for any \(m \geq 2\). Indeed, it can be checked that the function

\[ f(m) = (m-1) \ln \left( \frac{2m^2}{2m^2 + m - 1} \right) - \ln \left( \frac{m + 1}{2m} \right) \]

is decreasing on \([6, \infty)\) and has positive limit at infinity. This implies (4.1).

By Theorem B in [9], this proves that there exists only one infinite volume end. \(\square\)

Proposition 4.2 implies that there exists an infinite volume end \(E\) of \(M\), and \(F = M \setminus E\) is a finite volume end. According to a result of Li and Tam [5], there exists a positive harmonic function \(u : M \to (0, \infty)\) with the following behavior at infinity.

On the infinite volume end \(E\) the function \(u\) is bounded, \(\liminf_E u = 0\), and \(u\) has finite Dirichlet energy, \(\int_E |\nabla u|^2 < \infty\). Moreover, it was proved in Lemma 1.1 of [5] that there exists a constant \(C > 0\) so that

\[ (4.2) \int_{E \setminus B(x_0, R)} u^2 \leq Ce^{-2\sqrt{\lambda_1(M)}R}. \]

On the finite volume end \(F\) the function is unbounded, \(\limsup_F u = \infty\). Moreover, by Theorem 1.4 in [6] we have

\[ (4.3) \text{Vol}(F \setminus B(x_0, R)) \leq ce^{-2\sqrt{\lambda_1(M)}R}. \]

The next result follows from Theorem 2.1 for \(p = 2m\) by carefully keeping track of all boundary terms.

Proposition 4.3. Let \((M, g)\) be a Kähler manifold of complex dimension \(m \geq 2\) and with \(\text{Ric} \geq -2(m+1)\). Assume that \(M\) has maximal bottom of spectrum of the \(2m\) Laplacian, \(\lambda_{1,2m}(M) = 1\). Let \(u > 0\) be the harmonic function defined above and \(\phi\) a non-negative cut-off function satisfying \(\phi + |\nabla \phi| \leq c(m)\). Denoting \(Q = |\nabla \ln u|^2\), we have

\[ (4.4) \left(4m^2\right)^{m-1} \int_M uQ^2 + R_2 \leq \int_M uQ^m \phi^{2m} \leq \left(4m^2\right)^{m-1} \int_M uQ^m \phi^{2m} + R_1 \]
and

\[
\int_M u^{-2} u_{\bar{a}\beta} u_{\alpha} u_{\beta} Q^{m-2} \phi^{2m} \leq (4m^2)^{m-2} \frac{m(m-1)}{4} \int_M u Q \phi^{2m} + R_A
\]

\[
\int_M u^{-2} u_{\bar{a}\beta} u_{\alpha} u_{\beta} Q^{m-2} \phi^{2m} \geq (4m^2)^{m-2} \frac{m(m-1)}{4} \int_M u Q \phi^{2m} + R_A
\]

where

\[
R_i = a_i(m) \int_M \left\langle \nabla u, \nabla \phi^{2m} \right\rangle Q^{m-1} + b_i(m) \int_M u^{-1} \Re \left( u_{\bar{a}\beta} u_{\alpha} \phi^{2m} \right) Q^{m-2}
\]

\[
+ c_i(m) \int_M u^{-1} \Re \left( u_{\alpha\beta} u_{\bar{a}} \phi^{2m} \right) Q^{m-2} + d_i(m) \int_M u |\nabla \phi|^2,
\]

for some constants \( a_i, b_i, c_i, d_i \) depending only on \( m \).

**Proof.** To prove the first inequality in (4.4) we use that \( \lambda_{1,2m}(M) = 1 \), so

\[
\int_M \left( u^{\frac{1}{2m}} \phi \right)^{2m} \leq \int_M \left| \nabla \left( u^{\frac{1}{2m}} \phi \right) \right|^{2m}.
\]

We have that

\[
\left| \nabla \left( u^{\frac{1}{2m}} \phi \right) \right|^2 = \left| u^{\frac{1}{2m}} \nabla \phi + \frac{1}{2m} u^{\frac{1}{2m}} \phi \nabla \ln u \right|^2
\]

\[
= \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q + u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right).
\]

We write the right hand side of (4.6) as

\[
\int_M \left| \nabla \left( u^{\frac{1}{2m}} \phi \right) \right|^{2m}
\]

\[
= \int_M \left( \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q + u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right) \right)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \int_M \left( \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q \right)^{m-j} \left( u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right) \right)^j
\]

\[
= \frac{1}{(4m^2)^m} \int_M u Q^m \phi^{2m}
\]

\[
+ m \int_M \left( \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q \right)^{m-1} \left( u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right) \right)
\]

\[
+ \sum_{j=2}^{m} \binom{m}{j} \int_M \left( \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q \right)^{m-j} \left( u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right) \right)^j.
\]

By Yau’s estimate \( Q \leq c(m) \), and by hypothesis \( \phi + |\nabla \phi| \leq c(m) \). So for \( j \geq 2 \) we have

\[
\left( \frac{1}{4m^2} \phi^2 u^{\frac{1}{m}} Q \right)^{m-j} \left( u^{\frac{1}{m}} |\nabla \phi|^2 + \frac{1}{m} u^{\frac{1}{m}} \phi \left( \nabla \ln u, \nabla \phi \right) \right)^j \leq c(m) |\nabla \phi|^2 u.
Hence, it follows that
\begin{equation}
\int_M \left| \nabla \left( u_1^m \phi \right) \right|^{2m} \leq \frac{1}{(4m^2)^m} \int_M u Q^m \phi^{2m} + \frac{1}{(4m^2)^{m-1}} \int_M \phi^{2m-1} \langle \nabla u, \nabla \phi \rangle Q^{m-1} + c(m) \int_M |\nabla \phi|^2 u.
\end{equation}

Consequently, (4.6) and (4.7) imply that
\begin{equation}
\int_M u \phi^{2m} \leq \frac{1}{(4m^2)^m} \int_M u Q^m \phi^{2m} + \frac{1}{(4m^2)^{m-1}} \int_M \phi^{2m-1} \langle \nabla u, \nabla \phi \rangle Q^{m-1} + c(m) \int_M |\nabla \phi|^2 u.
\end{equation}

Together with Young's inequality
\begin{equation}
(4m^2)^{m-1} Q \leq \frac{m-1}{m} (4m^2)^m + \frac{1}{m} Q^m,
\end{equation}
this proves
\begin{equation}
\int_M u Q \phi^{2m} \leq \frac{1}{(4m^2)^{m-1}} \int_M u Q^m \phi^{2m} + \frac{2(m-1)}{(4m^2)^{m-1}} \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} + c(m) \int_M |\nabla \phi|^2 u.
\end{equation}

This is the first inequality in (4.4).

We now prove the second inequality in (4.4). Recall that by (2.20) we get setting
\begin{equation}
\int_M u^{-2} u_{\alpha \beta} u_\alpha u_{\beta} Q^{m-2} \phi^{2m} \leq -\frac{m^2 - 5m + 2}{64m} \int_M u Q^m \phi^{2m} + \frac{m(m+1)(m-2)}{16} \int_M u Q^{m-1} \phi^{2m} + \frac{m^2}{4} F_1(m-2) + \frac{m(m-2)}{4} F_2(m-2),
\end{equation}
where
\begin{equation}
F_1(m-2) = \frac{m-2}{8(m-1)} \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} - \int_M u^{-1} \text{Re} \left( u_{\bar{\alpha} \bar{\beta}} u_\alpha \left( \phi^{2m} \right)_{\bar{\bar{\beta}}} \right) Q^{m-2}
\end{equation}
and
\begin{equation}
F_2(m-2) = -\frac{1}{16 (m-1)} \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} - \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha}} \left( \phi^{2m} \right)_{\bar{\bar{\beta}}} \right) Q^{m-2}.
\end{equation}
To simplify notation, we subsequently omit the dependency of $F_i$ on $(m-2)$. 
We estimate the left side of (4.9) by using (2.10) that

\[ \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u^*_{\bar{\alpha}\bar{\beta}} \right) Q^{m-2} \phi^{2m} \]

\[ = - \int_M u^{-2} u_{\bar{\alpha}\beta} u_{\alpha} u^*_{\bar{\beta}} Q^{m-2} \phi^{2m} \]

\[ - \frac{1}{16 (m - 1)} \int_M Q^{m-1} \langle \nabla u, \nabla \phi^{2m} \rangle \]

\[ + \frac{1}{8} \int_M u Q^{m} \phi^{2m}. \]

Moreover, we have the following estimate

\[ \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u^*_{\bar{\alpha}\bar{\beta}} \right) Q^{m-2} \phi^{2m} \]

\[ \leq \frac{1}{4} \int_M |u_{\alpha\beta}| Q^{m-1} \phi^{2m} \]

\[ \leq \frac{m}{2 (m + 1)} \int_M u^{-1} |u_{\alpha\beta}|^2 Q^{m-2} \phi^{2m} + \frac{m + 1}{32 m} \int_M u Q^{m} \phi^{2m}. \]

By (2.13) and (2.15) we get

\[ \int_M u^{-1} |u_{\alpha\beta}|^2 Q^{m-2} \phi^{2m} \leq - 2 \int_M u^{-2} u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta} Q^{m-2} \phi^{2m} + \frac{m + 1}{4} \int_M u Q^{m-1} \phi^{2m} \]

\[ + \frac{3 m - 1}{16 m} \int_M u Q^{m} \phi^{2m} + \mathcal{F}_1 + \mathcal{F}_2. \]

We use this in (4.11) to conclude

\[ \int_M u^{-2} \text{Re} \left( u_{\alpha\beta} u^*_{\bar{\alpha}\bar{\beta}} \right) Q^{m-2} \phi^{2m} \]

\[ \leq \frac{m}{m + 1} \int_M u^{-2} u_{\bar{\alpha}\beta} u_{\alpha} u^*_{\bar{\beta}} Q^{m-2} \phi^{2m} \]

\[ + \frac{m}{8} \int_M u Q^{m-1} \phi^{2m} + \frac{4 m^2 + m + 1}{32 m (m + 1)} \int_M u Q^{m} \phi^{2m} \]

\[ + \frac{m}{2 (m + 1)} (\mathcal{F}_1 + \mathcal{F}_2). \]

Now (4.10) and (4.12) imply that

\[ \int_M u^{-2} u_{\bar{\alpha}\beta} u_{\alpha} u^*_{\bar{\beta}} Q^{m-2} \phi^{2m} \]

\[ \geq \frac{3 m - 1}{32 m} \int_M u Q^{m} \phi^{2m} - \frac{m (m + 1)}{8} \int_M u Q^{m-1} \phi^{2m} \]

\[ - \frac{m + 1}{16 (m - 1)} \int_M Q^{m-1} \langle \nabla u, \nabla \phi^{2m} \rangle \]

\[ - \frac{m}{2} (\mathcal{F}_1 + \mathcal{F}_2). \]
Combining this with (4.9) yields
\[
\int_M u Q^m \phi^{2m} \leq 4m^2 \int_M u Q^{m-1} \phi^{2m} \\
+ \frac{4}{m-1} \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} \\
+ 16 \frac{m(m+2)}{m+1} F_1 + 16 \frac{m^2}{m+1} F_2.
\]

We use Young’s inequality
\[
4m^2 Q^{m-1} \leq \frac{m-2}{m-1} Q^m + \frac{1}{m-1} (4m^2)^{m-1} Q
\]
to conclude from (4.14) that
\[
\int_M u Q^m \phi^{2m} \leq \frac{4m^2}{m-1} \int_M u Q^m \phi^{2m} + 4 \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} \\
+ 16 \frac{m(m-1)(m+2)}{m+1} F_1 + 16 \frac{m^2(m-1)}{m+1} F_2.
\]

This completes the proof of (4.4).

To prove the first inequality in (4.5), we first use (4.15) into (4.9) to get
\[
\int_M u - \frac{2}{m} u \bar{\alpha} \bar{\beta} u \alpha u \bar{\beta} Q^{m-2} \phi^{2m} \\
\leq \frac{3m^2 - 7m + 6}{64m(m-1)} \int_M u Q^m \phi^{2m} + \frac{m(m+1)(m-2)}{16(m-1)} (4m^2)^{m-2} \int_M u Q^{m-1} \\
+ \frac{m^2}{4} F_1 + \frac{m(m-2)}{4} F_2.
\]

We plug (4.16) into this inequality and obtain
\[
\int_M u Q^m \phi^{2m} \leq (4m^2)^{m-2} \frac{m(m-1)}{4} \int_M u Q^m \phi^{2m} \\
+ R_3,
\]
where
\[
R_3 = a_3(m) \int_M \langle \nabla u, \nabla \phi^{2m} \rangle Q^{m-1} + b_3(m) \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\alpha} (\phi^{2m})_{\beta} \right) Q^{m-2} \\
+ c_3(m) \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha}} (\phi^{2m})_{\beta} \right) Q^{m-2} + d_3(m) \int_M |\nabla \phi|^2 u.
\]

This proves the first inequality in (4.5).

To prove the second inequality in (4.5), we plug (4.15) into (4.13) and get
\[
\int_M u^{-2} u_{\alpha \beta} u_{\alpha} u_{\beta} Q^{m-2} \phi^{2m} \geq \frac{2m^2 - 3m + 3}{32m(m-1)} \int_M u Q^m \phi^{2m} \\
- \frac{m(m+1)}{8(m-1)} (4m^2)^{m-2} \int_M u Q^m \phi^{2m} \\
- \frac{m+1}{16(m-1)} \int_M Q^{m-1} \langle \nabla u, \nabla \phi^{2m} \rangle \\
- \frac{m}{2} (F_1 + F_2).
\]
Using (4.18) into this inequality it follows that
\[
\int_M u^{-2} u_{\overline{\alpha} \beta} u_\alpha u_{\overline{\beta}} Q^{m-2} \phi^2 \geq (4m^2)^{m-2} \frac{m(m-1)}{4} \int_M u Q \phi^2 + R_4,
\]
for
\[
R_4 = a_4(m) \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{m-1} + b_4(m) \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} + c_4(m) \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} + d_4(m) \int_M |\nabla \phi|^2 u.
\]
This proves the result.

From the inequality (4.3) in Proposition 4.3 we see that
\[
R_2 \leq \int_M u Q^m \phi^{2m} - (4m^2)^{m-1} \int_M u Q \phi^2 \leq R_1,
\]
where for \(i \in \{1, 2\}\)
\[
R_i = a_i(m) \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{m-1} + b_i(m) \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} + c_i(m) \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} + d_i(m) \int_M u |\nabla \phi|^2,
\]
for some constants \(a_i, b_i, c_i, d_i\) depending only on \(m\). From (4.17) we conclude that there exist constants \(\alpha_i(m) > 0\) so that
\[
\left| \int_M u Q^m \phi^{2m} - (4m^2)^{m-1} \int_M u Q \phi^2 \right| \leq \alpha_1(m) \left| \int_M \langle \nabla u, \nabla \phi^2 \rangle Q^{m-1} \right| + \alpha_2(m) \left| \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} \right| + \alpha_3(m) \left| \int_M u^{-1} \text{Re} \left( u_{\overline{\alpha} \beta} u_\alpha (\phi^{2m})_{\overline{\beta}} \right) Q^{m-2} \right| + \alpha_4(m) \int_M u |\nabla \phi|^2.
\]

Our goal is to prove that the right side of (3.18) converges to zero. As in [8], we will use \(u\) to construct a cut-off function \(\phi\) on \(M\). Denote the level and sublevel sets of \(u\) with
\[
l(t) = \{ x \in M : u(x) = t \}, \quad L(a,b) = \{ x \in M : a < u(x) < b \}.
\]
Note that \(l(t)\) and \(L(a,b)\) may not be compact for all \(t\) and all \(a < b\).

For arbitrary numbers \(0 < \delta \varepsilon < \varepsilon < T < \beta T < \infty\) and \(R > 1\) we consider the cut-off function
\[
\phi = (\chi \psi)^2
\]
where $\chi$ is given by

\begin{align}
\chi = \begin{cases} 
(-\ln \delta)^{-1} (\ln u - \ln \delta \varepsilon) & \text{on } L(\delta \varepsilon, \varepsilon) \\
1 & \text{on } L(\varepsilon, T) \\
(\ln \beta)^{-1} (\ln (\beta T) - \ln u) & \text{on } L(T, \beta T) \\
0 & \text{otherwise}
\end{cases} 
\end{align}

and $\psi(r)$ is a function with support in $B(x_0, 2R)$ so that $\psi = 1$ on $B(x_0, R)$ and $|\nabla \psi| \leq \frac{c}{R}$. Eventually, we will let $R \to \infty$ and then $\delta \to 0$ and $\beta \to \infty$. The following observation will be important for this purpose.

Using the co-area formula and that $u$ is harmonic, it was proved in Lemma 5.1 of [8] that

$$\int_{l(t)} |\nabla u| = \int_{l(s)} |\nabla u| < \infty,$$

for any $t, s > 0$.

By the co-area formula, it follows that for any function $H: (0, \infty) \to \mathbb{R}$ we have

$$\int_{L(a,b)} |\nabla u|^2 H(u) = A \int_a^b H(s) \, ds,$$

where we have denoted with

$$A = \int_{l(t)} |\nabla u|.$$

We fix $0 < \delta \varepsilon < \varepsilon < T < \beta T$ and study (4.18) as $R \to \infty$. We first record two preliminary results.

**Lemma 4.4.** Under the assumptions of Theorem 4.1, for $\chi$ given in (4.20), we have

$$\left| (-\ln \delta)^{-1} \int_{L(\delta \varepsilon, \varepsilon)} u Q^m \chi^{4m-1} - \frac{A}{4m} (4m^2)^{m-1} \right| \leq c(m) A \left( -\ln \delta \right)^{-\frac{1}{2}},$$

and

$$\left| (\ln \beta)^{-1} \int_{L(T, \beta T)} u Q^m \chi^{4m-1} - \frac{A}{4m} (4m^2)^{m-1} \right| \leq c(m) A (\ln \beta)^{-\frac{1}{2}},$$

for a constant $c(m)$ depending only on $m$.

**Proof.** By Proposition 4.3 we have

$$\left(4m^2\right)^{m-1} \int_M u Q^2 \varphi^{2m} + R_2 \leq \int_M u Q^m \varphi^{2m} \leq \left(4m^2\right)^{m-1} \int_M u Q^2 \varphi^{2m} + R_1$$

for any non-negative cut-off $\varphi$ on $M$ so that $\varphi + |\nabla \varphi| \leq c(m)$.

We choose $\varphi = \chi^{4m-1} \psi^2$, where

$$\tilde{\chi}(u) = \begin{cases} 
(-\ln \delta)^{-1} (\ln u - \ln (\delta \varepsilon)) & \text{on } L(\delta \varepsilon, \varepsilon) \\
(\ln 2)^{-1} (\ln (2 \varepsilon) - \ln u) & \text{on } L(\varepsilon, 2 \varepsilon) \\
0 & \text{otherwise}
\end{cases}$$

and $\psi$ is as in (4.19).

Hence, we are applying Proposition 4.3 to a cut-off function $\varphi$ that is supported on the end $E$, and satisfies $\varphi^{2m} = \chi^{4m-1} \psi^{4m}$ on $L(\delta \varepsilon, \varepsilon)$. We now want to bound $R_1$ and $R_2$. 
We have

\begin{equation}
\int_M \left| \left\langle \nabla \varphi^{2m}, \nabla u \right\rangle \right| Q^{m-1} = \int_M \left| \left\langle \nabla \left( \bar{\chi}^{m-1} \psi^m \right), \nabla u \right\rangle \right| Q^{m-1} \leq (4m-1) \int_M \left| \left\langle \nabla \bar{\chi}, \nabla u \right\rangle \right| \bar{\chi}^{m-2} \psi^m Q^{m-1} + 4m \int_M \left| \left\langle \nabla \psi, \nabla u \right\rangle \right| \bar{\chi}^{m-1} \psi^{m-1} Q^{m-1}.
\end{equation}

On one hand, by (4.2) we have

\begin{align*}
\int_M \left| \left\langle \nabla \psi, \nabla u \right\rangle \right| \bar{\chi}^{m-1} \psi^{m-1} Q^{m-1} & \leq c(m) \int_{(E\setminus B(x_0, R)) \cap L(\delta \varepsilon, 2\varepsilon)} u^2 \\
& \leq \frac{c(m)}{\delta \varepsilon} \int_{E \setminus B(x_0, R)} u^2 \\
& \leq \frac{C}{\delta \varepsilon} e^{-2\sqrt{\lambda_1(M)} R}.
\end{align*}

On the other hand, by Yau’s gradient estimate and (4.21) we have

\begin{align*}
\int_M \left| \left\langle \nabla \bar{\chi}, \nabla u \right\rangle \right| \bar{\chi}^{m-1} \psi^{m-1} Q^{m-1} & \leq c(m) \int_M \left| \left\langle \nabla \bar{\chi}, \nabla u \right\rangle \right| \\
& = c(m) \frac{1}{(-\ln \delta)} \int_{L(\delta \varepsilon, \varepsilon)} u^{-1} |\nabla u|^2 \\
& + c(m) \frac{1}{\ln 2} \int_{L(\varepsilon, 2\varepsilon)} u^{-1} |\nabla u|^2 \\
& \leq c(m) A.
\end{align*}

Hence, by (4.24) we get as $R \to \infty$,

\begin{equation}
\int_M \left| \left\langle \nabla \varphi^{2m}, \nabla u \right\rangle \right| Q^{m-1} \leq c(m) A.
\end{equation}

A similar proof shows that

\begin{equation}
\int_M u |\nabla \varphi|^2 \leq c(m) A.
\end{equation}

Following the proof of (2.24) we obtain similarly from (2.13) for $k = m - 2$ that

\begin{equation}
\int_M u^{-1} |u_{\alpha \beta}|^2 Q^{m-2} \varphi^{2m} + \int_M u^{-1} |u_{\alpha \beta}|^2 Q^{m-2} \varphi^{2m} \leq c(m) \int_{L(\delta \varepsilon, 2\varepsilon)} u^{-1} |\nabla u|^2 + c(m) \int_M u |\nabla \varphi|^2 \\
\leq c(m) A (-\ln \delta).
\end{equation}
By the Cauchy-Schwarz inequality we get as $R \to \infty$
\[
\int_M u^{-1} \left| \text{Re} \left( u_{\bar{\alpha}} u_{\alpha} (\varphi^{2m})_\beta \right) \right| Q^{m-2} \\
\leq \frac{m}{2} \int_M u^{-1} |u_{\bar{\alpha}}_\beta||\nabla u||\nabla \varphi| \varphi^{2m-1} Q^{m-2} \\
\leq \frac{m}{2} \left( \int_M u^{-1} |u_{\bar{\alpha}}_\beta|^2 Q^{m-2} \varphi^{2m} \right)^{\frac{1}{2}} \left( \int_M u |\nabla \varphi|^2 Q^{m-1} \varphi^{2m-2} \right)^{\frac{1}{2}} \\
\leq c(m) (-\ln \delta)^{\frac{1}{2}} A,
\]
where in the last line we used (4.26) and (4.27). The estimate
\[
\int_M u^{-1} \left| \text{Re} \left( u_{\bar{\alpha}} u_{\alpha} (\varphi^{2m})_\beta \right) \right| Q^{m-2} \leq c(m) (-\ln \delta)^{\frac{1}{2}} A
\]
follows similarly. This proves that (4.28) $|\mathcal{R}_i| \leq c(m) A (-\ln \delta)^{\frac{1}{2}}$.

Making $R \to \infty$ in (4.23) and using (4.28) we get
\[
\left| \int_M u Q^m \chi^{4m-1} - (4m^2)^{m-1} \int_M u Q \chi^{4m-1} \right| \leq c(m) A (-\ln \delta)^{\frac{1}{2}}.
\]
Since
\[
\int_{L(\varepsilon, 2\varepsilon)} u Q^m \chi^{4m-1} \leq c(m) \int_{L(\varepsilon, 2\varepsilon)} u Q = c(m) A,
\]
we conclude from above that
\[
(4.29) \quad \left| \int_M u Q^m \chi^{4m-1} - (4m^2)^{m-1} \int_M u Q \chi^{4m-1} \right| \leq c(m) A (-\ln \delta)^{\frac{1}{2}}.
\]
Note that by (4.21)
\[
(4.30) \quad \int_M u Q \chi^{4m-1} = A \int_{\delta \varepsilon}^{\varepsilon} \frac{1}{t} \chi^{4m-1} (t) \, dt \\
= \frac{1}{4m} (-\ln \delta) A.
\]
Hence, we conclude from (4.29) that
\[
\left| \int_{L(\delta \varepsilon, \varepsilon)} u Q^m \chi^{4m-1} - \frac{A}{4m} (4m^2)^{m-1} (-\ln \delta) \right| \leq c(m) A (-\ln \delta)^{\frac{1}{2}}.
\]
This proves the first estimate of the lemma. The corresponding estimate on $L(T, \beta T)$ follows similarly, the only difference being that we use (4.3) to get
\[
\int_{L(T, \beta T)} |(\nabla \psi, \nabla u)| \leq \frac{c(m)}{R} \int_{L(T, \beta T) \cap (F \setminus B(x_0, R))} u \\
\leq c(m) \beta T \text{Vol} (F \setminus B(x_0, R)) \\
\leq c(m) \beta T e^{-2\sqrt{\lambda_1(M)} R}.
\]
Certainly, the right side converges to zero when $R \to \infty$ and $\beta, T$ are fixed. This proves the lemma. □
The next result is similar to Lemma 4.4.

**Lemma 4.5.** Under the assumptions of Theorem 4.1, for \( \chi \) given in (4.20) we have

\[
\left\| (-\ln \delta)^{-1} \int_{L(\delta \varepsilon, \varepsilon)} u_{\bar{\alpha}\beta} u_{\alpha} u_{\bar{\beta}} Q^{m-2} \chi^{4m-1} - \frac{(m-1) A}{16} (4m^2)^{m-2} \right\| \leq c(m) A (-\ln \delta)^{-\frac{1}{2}}
\]

and

\[
\left\| (\ln \beta)^{-1} \int_{L(T, \beta T)} u_{\bar{\alpha}\beta} u_{\alpha} u_{\bar{\beta}} Q^{m-2} \chi^{4m-1} - \frac{(m-1) A}{16} (4m^2)^{m-2} \right\| \leq c(m) A \ln \beta^{-\frac{1}{2}},
\]

where \( c(m) \) is a constant depending only on \( m \).

**Proof.** By Proposition 4.3 we have

\[
\int_M u_{\bar{\alpha}\beta} u_{\alpha} u_{\bar{\beta}} Q^{m-2} \varphi^{2m} \leq \left( 4m^2 \right)^{m-2} \frac{m (m-1)}{4} \int_M u Q^{2m} + R_3
\]

and

\[
\int_M u_{\bar{\alpha}\beta} u_{\alpha} u_{\bar{\beta}} Q^{m-2} \varphi^{2m} \geq \left( 4m^2 \right)^{m-2} \frac{m (m-1)}{4} \int_M u Q^{2m} + R_4
\]

for any cut-off \( \varphi \) on \( M \). We choose the cut-off \( \varphi = \bar{\chi}^{4m-1} \psi^2 \) as in Lemma 4.4. From (4.28) we know that

\[
| R_i | \leq c(m) A (-\ln \delta)^{\frac{1}{2}}.
\]

By (4.30), this proves the estimate on \( L(\delta \varepsilon, \varepsilon) \). The proof of the estimate on \( L(T, \beta T) \) is similar.

We use Lemma 4.4 and Lemma 4.5 to estimate each term in (4.18) for the cut-off \( \phi \) specified in (4.19).

**Lemma 4.6.** Under the assumptions of Theorem 4.1, for \( \phi \) given in (4.19), we have as \( R \to \infty \),

\[
\int_M u |\nabla \phi|^2 \leq c(m) A (-\ln \delta)^{-1} + c(m) A \ln \beta^{-1},
\]

for a constant \( c(m) \) depending only on \( m \).

**Proof.** We have

\[
\int_M u |\nabla \phi|^2 \leq 8 \int_M u \chi^4 \psi^2 |\nabla \psi|^2 + 8 \int_M u \psi^4 \chi^2 |\nabla \chi|^2.
\]

Using (4.21) we have

\[
\int_M u \psi^4 \chi^2 |\nabla \chi|^2 \leq c(\ln \beta)^{-2} \int_{L(T, \beta T)} u^{-1} |\nabla u|^2 + c(-\ln \delta)^{-2} \int_{L(\delta \varepsilon, \varepsilon)} u^{-1} |\nabla u|^2 = cA (\ln \delta)^{-1} + cA (\ln \beta)^{-1}.
\]
By (4.3) and (4.2) we also have that

\[
\int_M u\chi^4 \psi^2 |\nabla \psi|^2 \leq c \int_{B(x_0, 2R) \setminus B(x_0, R)} u\chi \\
\leq c \int_{E \setminus B(x_0, R)} u\chi + c \int_{F \setminus B(x_0, R)} u\chi \\
\leq \frac{c}{\delta \varepsilon} \int_{E \setminus B(x_0, R)} u^2 \\
+ c\beta T \text{Vol}(F \setminus B(x_0, R)) \\
\leq C \left( \frac{1}{\delta \varepsilon} + \beta T \right) e^{-2\sqrt{\lambda_1}(M)R}.
\]

The result follows from (4.31) and (4.32).

We continue with the following.

**Lemma 4.7.** Under the assumptions of Theorem 4.1, for \( \phi \) given by (4.19), we have as \( R \to \infty \)

\[
\left| \int_M \phi^{2m-1} \langle \nabla u, \nabla \phi \rangle Q^{m-1} \right| \leq c(m) A (-\ln \delta)^{-\frac{1}{2}} + c(m) A (\ln \beta)^{-\frac{1}{2}},
\]

for a constant \( c(m) \) depending only on \( m \).

**Proof.** We have

\[
(4.33) \quad \int_M \phi^{2m-1} \langle \nabla u, \nabla \phi \rangle Q^{m-1} \\
= 2 \int_M \phi^{2m-1} \chi^2 \psi \langle \nabla u, \nabla \psi \rangle Q^{m-1} \\
+ 2 \int_M \phi^{2m-1} \chi \psi^2 \langle \nabla u, \nabla \chi \rangle Q^{m-1}.
\]

For the first term, we use Yau’s gradient estimate \( Q \leq c(m) \) and (4.32) to get

\[
(4.34) \quad \int_M \phi^{2m-1} \chi^2 \psi |\langle \nabla u, \nabla \psi \rangle| Q^{m-1} \\
\leq c(m) \int_{B(x_0, 2R) \setminus B(x_0, R)} u\chi \\
\leq C \left( \frac{1}{\delta \varepsilon} + \beta T \right) e^{-2\sqrt{\lambda_1}(M)R}.
\]

For the second term in (4.33) note that

\[
\int_M \phi^{2m-1} \chi \psi^2 \langle \nabla u, \nabla \chi \rangle Q^{m-1} \\
= (\ln \delta)^{-1} \int_{L(\delta \varepsilon, c)} uQ^m \chi^{4m-1} \psi^{4m} \\
- (\ln \beta)^{-1} \int_{L(T, \beta T)} uQ^m \chi^{4m-1} \psi^{4m}.
\]

By Lemma 4.4 we have as \( R \to \infty \),

\[
(4.35) \quad \left| \int_M \phi^{2m-1} \chi \psi^2 \langle \nabla u, \nabla \chi \rangle Q^{m-1} \right| \leq c(m) A (-\ln \delta)^{-\frac{1}{2}} + c(m) A (\ln \beta)^{-\frac{1}{2}}.
\]
By (4.33), (4.34) and (4.35) we conclude as $R \to \infty$,
\[
\left| \int_M \phi^{2m-1} (\nabla u, \nabla \phi) Q^{m-1} \right| \leq c(m) A (- \ln \delta)^{- \frac{1}{2}} + c(m) A (\ln \beta)^{- \frac{1}{2}}.
\]
This proves the result. \hfill \Box

We now prove the following.

**Lemma 4.8.** Under the assumptions of Theorem 4.1 for $\phi$ given in (4.13), we have as $R \to \infty$
\[
\left| \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_\alpha (\phi^{2m})_{\beta} \right) Q^{m-2} \right| \leq c(m) A (- \ln \delta)^{- \frac{1}{2}} + c(m) A (\ln \beta)^{- \frac{1}{2}}
\]
and
\[
\left| \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_\alpha (\phi^{2m})_{\beta} \right) Q^{m-2} \right| \leq c(m) A (- \ln \delta)^{- \frac{1}{2}} + c(m) A (\ln \beta)^{- \frac{1}{2}},
\]
for a constant $c(m)$ depending only on $m$.

**Proof.** We have
\[
(4.36) \quad \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_\alpha (\phi^{2m})_{\beta} \right) \phi^{2m-1} Q^{m-2}
= 2 \int_M u^{-1} \left( u_{\alpha \beta} u_\alpha \chi^2 \phi^{2m-1} Q^{m-2} \right)
+ 2 \int_M u^{-1} \left( u_{\alpha \beta} u_\alpha \psi \phi^{2m-1} Q^{m-2} \right).
\]
As in the proof of Lemma 4.7, we use (2.24), (4.2) and (4.3) to estimate
\[
(4.37) \quad \left| \int_M u^{-1} \left( u_{\alpha \beta} u_\alpha \psi \phi^{2m-1} Q^{m-2} \right) \right|
\leq \frac{1}{4} \int_M u^{-1} \left| u_{\alpha \beta} \right| \left| \nabla u \right| \left| \nabla \psi \right| \chi^2 \psi \phi^{2m-1} Q^{m-2}
\leq \frac{1}{4} \left( \int_M u^{-1} \left| u_{\alpha \beta} \right|^2 Q^{m-2} \phi^{2m} \right)^{\frac{1}{2}} \left( \int_M u Q^{m-1} Q^{m-1} \chi^2 \phi^{2m-2} \right)^{\frac{1}{2}}
\leq c \left( \frac{1}{\delta\epsilon} + \beta T \right) e^{-\sqrt{\lambda(M)R}}.
\]
Furthermore, we have
\[
\int_M u^{-1} \left( u_{\alpha \beta} u_\alpha \chi \right) \phi^{2m-1} Q^{m-2}
= (- \ln \delta)^{-1} \int_{L(4\xi, \epsilon)} u^{-2} u_{\alpha \beta} u_\alpha u_\beta Q^{m-2} \chi^4 m-1 \psi^4 m
- (\ln \beta)^{-1} \int_{L(T, \beta T)} u^{-2} u_{\alpha \beta} u_\alpha u_\beta Q^{m-2} \chi^4 m-1 \psi^4 m.
\]
Using Lemma 4.5 it follows that
\[
(4.38) \quad \left| \int_M u^{-1} \left( u_{\alpha \beta} u_\alpha \chi \right) Q^{m-2} \chi^4 m-1 \psi^4 m \right|
\leq c(m) A (- \ln \delta)^{- \frac{1}{2}} + c(m) A (\ln \beta)^{- \frac{1}{2}}.
\]
Using a cut-off function \( \varphi \) as in Lemma 4.4, and by (4.10), Lemma 4.4 and Lemma 4.5 it follows that as \( R \to \infty \),

\[
\int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \right) \varphi^{2m-1} \leq c(m) A (-\ln \delta)^{-\frac{1}{4}} + c(m) A (\ln \beta)^{-\frac{1}{4}}.
\]

This proves the first estimate of the lemma. For the other estimate, we proceed similarly and get

\[
\int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \right) Q^{m-2} \varphi^{2m-1} \leq 2 \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \chi \right) \varphi^{2m-1} Q^{m-2} \leq 2 \int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \right) \varphi^{2m-1} Q^{m-2} + C \left( \frac{1}{\delta \epsilon} + \beta T \right) e^{-\sqrt{\lambda_1(M)R}},
\]

where the last line follows similarly to (4.37). Furthermore, we have

\[
\int_M u^{-1} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \chi \right) \varphi^{2m-1} Q^{m-2} = \frac{1}{(-\ln \delta)} \int_{L(\delta \epsilon, \epsilon)} u^{-2} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \right) \chi^{4m-1} \varphi^{4} Q^{m-2} - \frac{1}{\ln \beta} \int_{L(T, \beta T)} u^{-2} \text{Re} \left( u_{\alpha \beta} u_{\bar{\alpha} \bar{\beta}} \right) \chi^{4m-1} \varphi^{4} Q^{m-2}.
\]

Using a cut-off function \( \varphi \) as in Lemma 4.3 and by (4.10), Lemma 4.3 and Lemma 4.5 it follows that as \( R \to \infty \),

\[
\int_M u^{-1} \text{Re} \left( u_{\alpha \beta} \chi \bar{\chi} \right) \varphi^{2m-1} Q^{m-2} = c(m) A (-\ln \delta)^{-\frac{1}{4}} + c(m) A (\ln \beta)^{-\frac{1}{4}}.
\]

We therefore obtain from above that as \( R \to \infty \),

\[
\int_M u^{-1} \text{Re} \left( u_{\alpha \beta} \varphi^{2m} \right) Q^{m-2} \leq c(m) A (-\ln \delta)^{-\frac{1}{4}} + c(m) A (\ln \beta)^{-\frac{1}{4}}.
\]

This proves the lemma. \( \square \)

We now finish the proof of Theorem 4.1. By (4.18) and Lemmas 4.6 4.7 and 4.8 we get as \( R \to \infty \)

\[
\int_M u \varphi^{2m} - (4m^2)^{m-1} \int_M u \varphi^{2m} \leq c(m) A (-\ln \delta)^{-\frac{1}{4}} + c(m) A (\ln \beta)^{-\frac{1}{4}},
\]

where \( A = \int_{\Omega} |\nabla u| < \infty \). Making \( \delta \to 0 \) and \( \beta \to \infty \) implies that all inequalities used in proving (4.41) in Proposition 4.3 must turn into equalities. Now (4.15) implies that

\[
(4.39) \quad |\nabla \ln u| = 2m,
\]

and by (2.15) we have that

\[
(4.40) \quad u_{\bar{\alpha} \beta} - u^{-1} u_{\bar{\alpha} \beta} + m g_{\alpha \beta} u = 0.
\]
Moreover, equality in (4.11) yields

\begin{equation}
|u_{\alpha\beta} u_{\bar{\alpha}} u_{\bar{\beta}}| = \frac{1}{4} |u_{\alpha\beta}| |\nabla u|^2
\end{equation}

(4.41)

\[ |u_{\alpha\beta}| = m (m + 1) u. \]

Note that (4.39), (4.40) and (4.41) are the same as the identities (12) in the proof of Theorem 4 of [12]. They imply the splitting as claimed in Theorem 4.1. This proves the result.

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