Optimal Universal Controllers for Roll Damping

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Abstract

Roll damping is an important problem of ship motion control since excessive roll motion may cause motion sickness of human occupants and damage fragile cargo. Actuators used for roll damping (fins, rudders and thrusters) inevitably create a rotating yaw moment, interfering thus with the vessel’s autopilot (heading control system). To reach and maintain the “trade-off” between the concurrent goals of accurate vessel steering and roll damping, an optimization procedure in general needs to take place where the cost functional penalizes the roll angle, the steering error and the control effort. Since the vessel’s motion is influenced by the uncertain wave disturbance, the optimal value of this functional and the resulting optimal process are also uncertain. Standard approaches, prevailing in the literature, approximate the wave disturbance by the “colored noise” with a known spectral density, reducing the optimization problem to conventional loop-shaping, LQG or $H_\infty$ control. In this paper, we propose a novel approach to optimal roll damping, approximating the disturbance by a polyharmonic signal with known frequencies yet uncertain amplitudes and phase shifts. For this class of external disturbances, an optimal universal controller (OUC) can be found, delivering the optimal solution for any uncertain parameters of the signal. Using numerical simulations, we compare our design method with classical approaches to optimal roll damping and show that OUC controllers exhibit better performance.

Keywords: Roll damping, ship motion control, ship maneuvering

1. Introduction

Roll damping is a classical problem in ship motion control (Fossen, 1994; Perez, 2006; Perez and Blanke, 2012). Passive roll damping can be provided by special equipment such as bilge keels, water-tanks and moving weights (Perez, 2006; Perez and Blanke, 2012; Marzouk and Nayfeh, 2009); however, these devices cannot be easily adapted to the unsteady environment and the changing wave’s spectrum. This limitation can be overcome by active (controlled) roll damping, which can be provided by gyroscopic stabilizers, stabilizing fins and/or actuators (rudders and thrusters) used for the vessel’s steering. This is illustrated by the rudder roll stabilization (RRS), proposed originally for a vessel equipped with a single rudder (Cowley and Lambert, 1972; Carley, 1975; Lloyd, 1975). Since fins, rudders and thrusters affect both yaw and roll motion of the vessel, the roll damping controller should be integrated with the heading controller (autopilot). These control systems can share some actuators and pursue concurrent goals of roll damping control and course steering.

A vessel’s coupled yaw-roll motion can be modeled by a dynamical system, whose inputs are the rudder’s and fins’ angles and whose outputs stand for the ship’s heading and roll. After linearizing this model, classical methods of linear control, e.g. loop shaping and Quantitative Feedback Theory (Cowley and Lambert, 1972; Carley, 1975; Horowitz and Sidi, 1978; Blanke and Christensen, 1993; Hearns and Blanke, 1998) can be applied to stabilize yaw and roll motion. To cope with nonlinearities, methods of feedback linearization and sliding mode control can be used (Lauvdal and Fossen, 1997; Liu...
et al., 2016). For vessels equipped with fin stabilizers, classical methods usually decouple the roll motion from the yaw motion (Surendran et al., 2007; Hinostroza et al., 2015); however, as it was observed in (Carley and Duberley, 1972) ignoring internal cross-couplings often reduces the overall performance.

The roll dynamics of a vessel appear to be non-minimum phase, leading thus to the fundamental limitation (Carley, 1975; Goodwin et al., 2000): a controller stabilizing the vessel’s heading cannot fully attenuate the wave-induced roll oscillations. A natural question arises, namely which level of the roll oscillation damping can be provided without deteriorating the yaw control. Mathematically, the latter goal is usually formulated as optimality of a special performance index, which penalizes the time-averaged steering error, roll angle and the control effort. Besides the control input, such a functional implicitly depends on the uncertain wave disturbance that affects the ship’s motion. Unlike the aforementioned stabilization techniques, optimization-based algorithms assume that some model of the disturbance is known. Most typically, the wave-induced motion is approximated by either a “colored noise” or a random polyharmonic signal (Perez and Blanke, 2012; Fossen, 1994).

The wave model of the first type approximates the wave disturbance by the output of some low-pass shaping filter, fed by a white noise. This approach, prevailing in the literature, reduces roll damping control design to standard methods of optimal controller synthesis, such as the linear-quadratic Gaussian (LQG) control (van der Klugt, 1987; van Amerongen et al., 1990), $H_\infty$ control (Sharif et al., 1995; Blanke et al., 2000; Crossland, 2003; Stoustrup et al., 1994) and model-predictive control (MPC) (Perez, 2006). As usual in stochastic and minimax control, optimal controllers do not deliver optimal solutions for any specific realization of the stochastic disturbance, providing optimality either “on average” (in the sense of expectation) or in the “worst-case” scenario. Another downside of the mentioned methods is the necessity to estimate the spectral density of the wave motion.

An alternative “discrete” model of the wave motion, often used in marine engineering (Perez, 2006; Nicolau et al., 2005; Longuet-Higgins, 1963), approximates the wave motion by the sum of sinusoids with known frequencies, where the constant amplitudes are obtained via sampling of the spectral density and random phase shifts are uniformly distributed in $[0, 2\pi]$ in order to get different realizations. For this model of the wave disturbance and linearized vessel’s yaw-roll dynamics, the optimal roll damping may be considered as a linear-quadratic optimization problem, where the control system is affected by a partially uncertain polyharmonic signal. A relevant extension of the classical LQR control to cope with such problems has been developed in (Yakubovich, 1995; Lindquist and Yakubovich, 1997, 1999; Proskurnikov and Yakubovich, 2006, 2012; Proskurnikov, 2015). It appears that (under natural assumptions) an optimal universal controller (OUC) exists, which is independent of the uncertain signal’s parameters, delivers the optimal process for arbitrary values of these parameters. Furthermore, the OUC can be found in the class of linear stabilizing controllers; a convenient parametrization of such OUCs has been found (Yakubovich, 1995).

In this paper, we apply Yakubovich’s theory of OUC to the problem of optimal roll damping. This paper extends our previous work (Kapitanyuk et al., 2016), which considered a simplified model of the vessel with a single rudder and no stabilizing fins. We illustrate the efficiency of OUCs in the optimal roll damping problem and compare it with classical controllers by using numerical simulations that utilize the “benchmark” vessel’s model from (Perez, 2006). The OUC theory provides a method for combined fin-rudder stabilization control design, avoiding the undesired counteraction between different actuators and improving the resulting efficiency of the control system. Unlike the usual LQR (Perez and Blanke, 2012), the OUC does not need to measure the full state vector and provides optimality for any polyharmonic signal from the specified class; to find OUC, one does not need to solve the Riccati equation. Unlike LQG and $H_\infty$ approaches, the OUC design does not require one to know the spectral density of the wave motion (or, equivalently, the structure of the shaping filter). The OUC depends only on the fixed wave’s frequencies and ensures optimality of the cost functional for any realization of the random disturbance.

The paper is organized as follows. In Section 2, mathematical models of the vessel’s motion and wave disturbances are considered. In Section 3 the theory of OUC in general problems of linear-quadratic optimization with uncertain disturbances is introduced. In Section 4, we apply this theory to design an optimal roll damping controller, whose performance is studied numerically in Section 5.
2. Mathematical models

We first introduce mathematical models of the ship’s yaw-roll motion and the wave disturbances.

2.1. The vessel’s motion

The movements of a marine vessel (as a rigid body) have six degrees of freedom. The standard 6-DoF mathematical model can be found in (Perez and Blanke, 2012; Fossen, 1994). However, it is more convenient to use a simplified reduced-order model (van Amerongen et al., 1990; Fossen, 1994; Perez, 2006), which is derived under two simplifying assumptions: 1) the effects of the pitch and heave motion of the vessel on its surge, sway, roll and yaw dynamics are negligible; 2) the vessel’s speed is changing slowly relative to the remaining coordinates. Under these assumptions, the yaw and the roll controllers can be designed for a simplified linearized model.

In the original papers on rudder roll damping (Cowley and Lambert, 1972; Lloyd, 1975), the simplest configuration of the vessel with one rudder has been considered, whose angle is the single control input of the system. In general, the vessel can be equipped with multiple actuators (rudders, azimuth and tunnel thrusters, waterjets etc.); however, for the sake of autopilot and roll damping control design they are usually replaced by an equivalent “virtual rudder”, whose “angle” stands for the scaled rotating yaw moment, distributed among the actuators by a separate control allocation system (Johansen et al., 2008). In addition to this, we allow the vessel to have two synchronized stabilizing fins, whose angle serves as the second control input.

Denoting the rudder, the fin, the roll and the yaw (or heading) angles by, respectively, $\delta_{\text{rud}}(t)$, $\delta_{\text{fin}}(t)$, $\varphi(t)$ and $\psi(t)$ (Fig. 1), the reduced-order vessel’s model has the structure illustrated in Fig. 2. The system is affected by the environmental disturbance, represented by its roll and yaw components\footnote{For clarity, in this paper we consider the “motion superposition” model (Perez, 2006), where the disturbance is modeled as an uncertain displacement from the original trajectory of the vessel. An alternative approach, referred to as the “force superposition” (Perez, 2006), treats the disturbance as an additional force, acting on the ship’s hull.} $d_{\varphi}(t)$, $d_{\psi}(t)$. The transfer functions from $\delta_{\text{rud}}$ and $\delta_{\text{fin}}$ to $\varphi$ and $\psi$, denoted by $W_{\varphi}(s)$, $W_{\psi}(s)$, and $W_{\varphi r}(s)$, $W_{\psi r}(s)$ respectively, can be approximated as follows (Perez, 2006, Sect. 8.2)

\[
\begin{align*}
W_{\varphi}(s) &\approx \frac{K_{\varphi}(q_1 - s)(q_2 + s)}{(p_1 + s)(p_2 + s)(s^2 + 2\zeta_\varphi \omega_\varphi s + \omega_\varphi^2)} \\
W_{\psi}(s) &\approx \frac{K_{\psi}(q_3 - s)(q_4 + s)}{s(p_1 + s)(p_2 + s)(s^2 + 2\zeta_\psi \omega_\psi s + \omega_\psi^2)} \\
W_{\varphi r}(s) &\approx \frac{K_{\varphi r}(q_5 - s)(q_6 + s)}{(p_1 + s)(p_2 + s)(s^2 + 2\zeta_r \omega_r s + \omega_r^2)} \\
W_{\psi r}(s) &\approx \frac{K_{\psi r}(q_7 - s)(q_8 + s)}{s(p_1 + s)(p_2 + s)(s^2 + 2\zeta_q \omega_q s + \omega_q^2)}
\end{align*}
\]

where $q_i > 0$, $p_j > 0$, $\omega_\varphi$, $\omega_\psi$, $\omega_r$, $\omega_q > 0$ and $\zeta_\varphi$, $\zeta_\psi$, $\zeta_r$, $\zeta_q \in (0; 1)$ are constants.

Along with the transfer function, one can introduce the state-space model of the system

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + G d(t)
\end{align*}
\]

Here the vessel’s reduced state vector $x(t) = (\varphi, p, \psi, r, v)^\top$ consist of the roll angle $\varphi$, the roll rate $p = \dot{\varphi}$, the yaw angle $\psi$, the yaw rate $r = \dot{\psi}$ and the sway velocity $v$. The disturbance $d(t) = (d_\varphi, d_\psi)^\top$ stands for the wave-induced motion of the vessel. The vector $y(t) = (\varphi, \psi)^\top \in \mathbb{R}^2$ stands for the system’s output, whose components $\varphi$ and $\psi$ are measured, respectively, by a vertical reference unit (VRU) sensor (Baloch, 1998) and a gyro or GPS compass and the control input is presented by the vector $\delta(t) = (\delta_{\text{rud}}, \delta_{\text{fin}})^\top$. We omit the exact formulas for $A$, $B$, $C$, $G$, $G$, since they are not explicitly used in the controller design.

2.2. The disturbance model

The environmental disturbances, influencing a marine craft’s motion, are due to the waves, the wind and the current. The fast oscillations in the roll and the heading angles are mainly caused by

Figure 1: The rudder ($\delta_{\text{rud}}$), the fin ($\delta_{\text{fin}}$), roll ($\varphi$) and yaw ($\psi$) angles.
the waves, whereas the current and the wind are changing much more slowly and their effect is usually modeled as a constant roll angle and stationary heading deviation. Henceforth, the disturbance $d(t)$ stands for the wave-induced motion only. In this paper, we use a polyharmonic approximation of this motion (Perez and Blanke, 2012; Fossen, 1994)

$$
\begin{align*}
    d_\phi(t) &= \sum_{i=1}^{p} a_i^\phi \sin(\omega_i t + \phi_i^\phi), \\
    d_\psi(t) &= \sum_{i=1}^{p} a_i^\psi \sin(\omega_i t + \phi_i^\psi).
\end{align*}
$$

Here the spectrum $\omega_1, \ldots, \omega_p \geq 0$ is known. The special case $p = 1$ corresponds to the model of regular waves; however, a real state of the sea is best described by a random or irregular wave model. This stochastic process can be approximated by the model (3) with $p$ being sufficiently large. The constant amplitudes $a_i^\phi$ and $a_i^\psi$ are obtained via sampling the spectral density with a small enough step $\Delta \omega$ to ensure that the fundamental period of the finite sum of sinusoidal components is longer than the desired duration of the simulation. The random phase shifts $\phi_i^\phi$ and $\phi_i^\psi$ used to generate different realizations of the stochastic process are uniformly distributed in $[0, 2\pi]$. Although the model (3) of irregular waves can describe a sea state quite accurately, the direct use of it in the control design is difficult due to the high dimension. The better strategy is to consider a few “dominating” frequencies corresponding to the peaks of the spectral density. In general, the localization and the shape of the spectral density highly depend on many parameters of motion such as the average speed of the vessel, sailing conditions and a frequency response of the vessel’s hull; however, these “dominating” frequencies can be efficiently estimated in real time, see e.g. (Bellet et al., 2015; Bobtsov et al., 2012; Fedele and Ferrise, 2012; Hou, 2012) and references therein. For simplicity and clarity of presentation, we proceed to assume that the number and the values of such frequencies are known.

It should be noted that in the existing control literature the wave motion is usually approximated by the “colored noise”, that is, the output from a low-pass shaping filter fed by the white noise signal. The simplest approximation for the shaping filter’s transfer function (that is, the wave spectrum), is

$$
H(s) = \frac{K_w s}{s^2 + 2\zeta_0 \omega_0 s + \omega_0^2}.
$$

Here the constant $K_w > 0$ determines the wave strength, $\omega_0$ is the encounter frequency and $\zeta_0$ is the damping ratio (Perez and Blanke, 2012). Unlike our approach, using only the information about the frequencies, the existing approaches, as discussed in Introduction, typically use all parameters of the transfer function $H(s)$, whose identification is a self-standing non-trivial problem.

3. Linear-quadratic optimization in presence of uncertain polyharmonic signals

In this section, the basic ideas of the theory of OUC are given for the reader’s convenience, following the survey paper (Proskurnikov, 2015).

We start with introducing some notation. The set of complex $m \times n$ matrices is denoted by $\mathbb{C}^{m \times n}$. The Hermitian complex-conjugate transpose of a matrix $M \in \mathbb{C}^{m \times n}$ is denoted by $M^* \in \mathbb{C}^{n \times m}$. We use $i \triangleq \sqrt{-1}$ to denote the imaginary unit. The real part of a number $z \in \mathbb{C}$ is denoted by $\text{Re} \, z$.

3.1. A family of uncertain optimization problems

Consider a linear time-invariant MIMO system, influenced by an exogenous signal

$$
\dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\
y(t) = Cx(t) + Du(t) + Gd(t).
$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^k$ stand for, respectively, the state vector, the control and the observed output. The signal $d(t) \in \mathbb{R}^l$ is a polyharmonic process with known spectrum $\omega_1, \ldots, \omega_N$

$$
d(t) = \text{Re} \sum_{j=1}^{N} d_j e^{j\omega_j t},
$$

where
whose complex amplitudes $d_i \in \mathbb{C}^i$ (absorbing also the phase shifts) are uncertain. The components of this exogenous signal may include disturbances, measurement noises and reference signals.

In presence of the oscillatory disturbance (6), the solutions of (5) do not vanish at infinity. The goal of control is to guarantee boundedness of the solution $(x(t), u(t))$ and its optimality in the sense of the following quadratic performance index

$$J[x, u, d] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathcal{F}[x(t), u(t), d(t)] \, dt.$$  (7)

Here $\mathcal{F}$ is a quadratic form, which is assumed to be non-negative definite $\mathcal{F} \geq 0$. Considering the integrand in (7) as a measure of the solution’s “energy”, its average value $J$ can be thought of as the solution’s average “power”. Formally, the control goal can be formulated as follows

minimize $J(x(\cdot), u(\cdot), d(\cdot))$

subject to (5) and $\sup_{t \geq 0}|x(t)| + |u(t)| < \infty$.  (8)

In fact, (8) defines an infinite family of optimization problems, corresponding to different choices of the amplitudes $d_1, \ldots, d_N$. Obviously, the set of optimal processes also depends on the amplitudes and hence cannot be found explicitly. Nevertheless, it can be shown that an optimal universal controller (OUC) exists that provides an optimal process for any uncertain amplitudes $d_i$, solving thus the whole family of optimization problems (8).

**Definition 1.** A causal operator $U : y(\cdot) \mapsto u(\cdot)$ is an OUC for the family of optimization problems (8), if for any initial condition $x(0) \in \mathbb{R}^n$ and any amplitudes $d_1, \ldots, d_N$ in (6) there exists a unique solution of the closed-loop system

$$\dot{x} = Ax + Bu + Ed, \quad y = Cx + Du + Gd, \quad u(\cdot) = Uy(\cdot),$$

which is bounded and delivers an optimum to (8).

### 3.2. A class of linear OUC

Although the existence of OUCs may seem exceptional, such controllers exist under rather mild assumptions on the system and the cost functional.

We assume that the system (5) is stable, that is, $\det(sI_n - A) \neq 0$ whenever $\Re s \geq 0$. If the system is stabilizable and detectable, one may always augment it with an observer-based stabilizing controller, so the stability assumption can be adopted without loss of generality.

Let $F = F^T$ stand for the matrix of the quadratic form $\mathcal{F}(x, u, d)$ and $F_0 = F_0^T$ be the matrix of the quadratic form $\mathcal{F}_0(x, u) = \mathcal{F}(x, u, 0)$, that is,

$$\mathcal{F}(x, u, d) = \left[ \begin{array}{c} x \\ u \\ d \end{array} \right]^T F \left[ \begin{array}{c} x \\ u \\ d \end{array} \right] F_0 \left[ \begin{array}{c} x \\ u \end{array} \right] + 2d^T F_{dx} x + 2d^T F_{du} u + d^T F_{dd} d,$$

where $F_{dx}, F_{du}, F_{dd} = F_{dd}^T$ are matrices of appropriate dimensions. We introduce the rational complex-valued matrix $\Pi(\omega) = \Pi(\omega^*)$ as follows

$$\Pi(\omega) \geq \epsilon I_m, \quad \epsilon = \text{const} > 0.$$  (10)

The condition (10) is a standard solvability condition for classical LQR problems, providing the existence of the stabilizing solution to the Riccati equation (Anderson and Moore, 1990). It always holds when $F_0(x, u)$ is positively definite, which is a natural assumption in practice. The condition (10) cannot be discarded and, moreover, its “strong” violation in the sense that $\tilde{u}^* \Pi(\omega_0) \tilde{u} < 0$ for some $\omega_0 \in \mathbb{R}$ and $\tilde{u} \in \mathbb{C}^m$ implies the ill-posedness of the problem (8): $\inf J = -\infty$ for any signal (6).

Under non-restrictive assumptions, the OUC exists and can be found among linear controllers

$$N \left( \frac{d}{dt} \right) u(t) = M \left( \frac{d}{dt} \right) y(t),$$  (11)

where $N$ and $M$ stand for matrix polynomials; the matrix $N(s)$ is square and det $N \neq 0$. The relevant result is given by the following theorem.

**Theorem 1.** (Proskurnikov, 2015) Let the system (5) be stable and the inequality (10) hold. Then the linear controller (11) is an OUC for the family of problems (8) if the following two conditions hold

1. the closed-loop systems is stable, that is,

$$\det \begin{bmatrix} sI_n - A & -B \\ -M(s)C & N(s) - M(s)D \end{bmatrix} \neq 0,$$  (12)

2. $\forall s : \Re s \geq 0$;

Footnote 2: For a similar discrete-time optimization problem, the proof is available in (Lindquist and Yakubovich, 1999), and the continuous-time case is considered in the same way.
2. the closed-loop transfer function $W_{ud}$ from $d$ to $u$ satisfies the interpolation equations

$$W_{ud}(\omega_j) = R_j, \quad \forall j = 1, 2, \ldots, N, \quad (13)$$

where the constant matrices $R_j$ are as follows

$$R_j = -\Pi^{-1}(i\omega_j) \begin{bmatrix} A_{i\omega_j}^{-1}B & * \\ I_m & 0 \end{bmatrix} F \begin{bmatrix} A_{i\omega_j}^{-1}E \\ 0 & I_l \end{bmatrix}. \quad (14)$$

Note that, unlike the classical LQR problem, where the optimal controller is uniquely defined from the Riccati equation, the OUC in the problem (8) is not unique; to find it, one need not solve Riccati equations. We will use Theorem 1 in a special situation, where $F$ depends only on the output and the control, i.e. $F$ admits the decomposition

$$F = \begin{bmatrix} C & D & G^* \\ 0 & I_m & 0 \end{bmatrix}, \quad (15)$$

where $\tilde{F} = F^* \in \mathbb{C}^{n+m}$. In this situation, one has

$$\Pi(i\omega) = \begin{bmatrix} W_0^y(i\omega) \\ I_m \end{bmatrix}^* \tilde{F} \begin{bmatrix} W_0^y(i\omega) \\ I_m \end{bmatrix},$$

$$R_j = -\Pi^{-1}(i\omega_j) \begin{bmatrix} W_0^y(i\omega_j) \\ I_m \end{bmatrix}^* \tilde{F} \begin{bmatrix} W_0^y(i\omega_j) \\ I_m \end{bmatrix}.$$  

Here $W_0^y(s)$ and $W_0^y(s)$ stand for the open-loop transfer functions from respectively $u$ and $d$ to $y$

$$W_0^y(s) := CA_s^{-1}B + D, \quad W_0^y(s) := CA_s^{-1}E + G.$$

Recalling that $A$ is a Hurwitz matrix, it can be shown that the closed-loop system is stabilized by the controller (11), whose coefficients are as follows

$$M(s) = \Delta(s)r(s),$$

$$N(s) = M(s)[CA_s^{-1}B + D] + \rho(s)I_m,$$

$$\Delta(s) := \det(A_s - 1) - \det(sI_n - A). \quad (16)$$

Here $r(s)$ is a matrix polynomial and $\rho(s)$ is a scalar Hurwitz polynomial with $\deg \rho \geq \deg M$. Such a controller is “feasible” in the sense that its transfer matrix $N^{-1}M$, as well as the closed-loop system’s transfer matrices from $d$ to $x, u$, are proper. For the controller (11), (16), one obtains

$$W_{ud}(s) = \frac{M(s)}{\rho(s)}W_0^y(s), \quad (17)$$

and the interpolation constraints (13) boil down to

$$\Delta(i\omega_j)r(i\omega_j)W_0^y(i\omega_j) = \rho(i\omega_j)R_j. \quad (18)$$

The constraints (18) can be satisfied when

$$\det[W_0^y(i\omega_j)W_0^y(i\omega_j)^*] \neq 0 \quad \forall j = 1, \ldots, N. \quad (19)$$

Here $W_0^y(s)$ is the open-loop transfer matrix from $d$ to $y$. The conditions (19) typically hold when $\dim y \geq \dim d$. Furthermore, if (19) holds, the coefficients of $r$ and $\rho$ can be chosen as continuous functions of $\omega_j$, so that the controller is robust to small deviations in the spectrum $\omega_j \approx \omega_j$. Choosing an arbitrary Hurwitz polynomial $\rho$ of degree $\deg \rho \geq 2N + \deg \delta - 1$, one needs to find the matrix polynomial $r$ with $\deg r \leq 2N - 1$, satisfying the conditions

$$r(i\omega_j) = r^0(i\omega_j),$$

$$r^0(s) := \frac{\rho(s)R_j}{\Delta(s)}W_0^y(s)[W_0^y(s)W_0^y(s)]^{-1}. \quad (20)$$

Separating the real and imaginary parts, one obtains $2N$ equations for $2N$ real coefficients of $r$

It appears that any OUC (11) is equivalent, in some sense (Yakubovich, 1995; Proskurnikov, 2015), to the controller (16) with some polynomials $r, \rho$, satisfying the interpolation constraints (18).

**Remark 1.** Note that the controller (16) in fact does not depend on the state-space model (5), involving only the system’s characteristic polynomial $\Delta(s)$ and the open-loop transfer function $W_0^y(s) := D + C(sI - A)^{-1}B$ from $u$ to $y$ (Fig. 3). In the case where $F = F(u, y)$ depends only on $y$ and $u$, the interpolation conditions (18) also involve only the values of $W_0^y(\omega_j)$ and $W_0^y(\omega_j)$ rather than the whole state model (5). Hence, in this special situation, the design of OUC requires only the knowledge of $\Delta(s), W_0^y(s)$ and $W_0^y(s)$, which are independent of the minimal state-space realization.

**Remark 2.** As discussed in (Lindquist and Yakubovich, 1999), the important property of the OUC (11) is its robustness against small changes in the frequencies $\omega_j$, whereas the straightforward LQR-based design leads to a controller that is formally optimal yet non-robust to deviations in spectrum. The results from (Lindquist and Yakubovich, 1999) deal with discrete-time systems, but this robustness property is retained by the continuous-time OUC (11).
neglected in the roll damping system design. The slower than the ship’s roll motion, and hence are linear path. However, these dynamics are much e.g. when autopilot steers the vessel along a curvi-
point¯ψ. In practice, ¯ψ(t) can be a function of time, e.g. when autopilot steers the vessel along a curvi-

4. Optimal Universal Roll Damping Controllers

In this section, we reduce the optimal roll damping problem to a special case of the problem (8). The cost functional will depend only on the control effort and output. In view of Remark 1, in this situation one does not need to know a special state-space representation of the open-loop system, requiring only its characteristic polynomial and transfer matrices $W_{yu}^0, W_{yd}^0$. In this sense, an optimal controller can be designed in the frequency domain.

We assume that the vessel’s heading is stabilized by a known autopilot (Fig. 4). Behind this statement, there are two practical considerations. First of all, it allows splitting of the adjustment procedure for a motion control system on the vessel in two sequential stages: the independent tuning of an autopilot and the following design of the roll damping controller. The second reason is the flexibility and the modularity; the roll damping system may be supplied by a manufacturer of the equipment such as high-performance rudders or active fins independent of the development of the autopilot, which is in itself a challenging task. The autopilot design problem has been thoroughly studied in the literature (Fossen, 1994; Perez, 2006; Nicolau et al., 2005; Veremey, 2014) and is beyond the scope of this paper. Furthermore, we assume that the roll damping system is aware of the measured heading of the vessel and the constant heading setpoint $\psi$. In practice, $\dot{\psi}(t)$ can be a function of time, e.g. when autopilot steers the vessel along a curvilinear path. However, these dynamics are much slower than the ship’s roll motion, and hence are neglected in the roll damping system design. The deviation among them (heading error) $e_\psi(t)$, along with the roll damping error $e_\phi(t)$ are the inputs to the roll damping system (Fig. 4). Mathematically,

$$e_\psi(t) := \dot{\psi}(t) + \ddot{\psi}, \quad e_\phi(t) := \phi(t) + \ddot{\phi}(t).$$

The rudder angle $\delta_{rud}(t)$ is the sum of the autopilot’s and the roll damping controller’s commands (Fig. 4), denoted respectively by $\delta_{AP}(t)$ and $u_1(t)$. The fin angle $\delta_{fin}(t)$ is used as the second control input $u_2(t)$. Denoting the autopilot’s transfer function by $W_{AP}(s)$, one has

$$\delta_{rud}(t) = \delta_{AP}(t) + u_1(t) = W_{AP}\left(\frac{d}{dt}\right)e_\psi(t) + u_1(t)$$

$$\delta_{fin}(t) = u_2(t).$$

The yaw-roll dynamics of the vessel, closed by the autopilot, are represented by the input-output model

$$y(t) = W_{yu}^0 \left(\frac{d}{dt}\right) u(t) + W_{yd}^0 \left(\frac{d}{dt}\right) d(t),$$

$$y(t) := \begin{bmatrix} e_\phi(t) \\ e_\psi(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad d(t) := \begin{bmatrix} \ddot{\phi}(t) \\ \ddot{\psi}(t) \end{bmatrix}.$$

Here $d_\phi(t), d_\psi(t)$ are the polyharmonic components of the wave-induced motion (3). Considering $\ddot{\psi}$ as a harmonic signal of zero frequency, $d(t)$ is a special case of (6) with $l = 3$ and $N = 1 + p$, where $\omega_j, k = 1, \ldots, p$ are the wave frequencies from (6) and $\omega_{1+p} = 0$. The transfer functions $W_{yu}^0, W_{yd}^0$ depend on the autopilot’s transfer function $W_{AP}$.
(from $v_\omega$ to $\delta_{AP}$) and the functions $W_{yaw}, W_{roll}$ from (1). The exact formulas for $W_{yaw}, W_{roll}$ are derived in Appendix A and it can be easily seen from these formulas that (19) always holds for any wave $\omega_1, \ldots, \omega_N \in \mathbb{R}$.

The cost functional penalizes the mean square values of the following three variables (i) the roll displacement ($v_\delta$), (ii) the heading deviation ($v_\phi$), and (iii) the control effort. Denoting the corresponding penalty weights by $\alpha, \beta, \gamma_{1,2} > 0$, we introduce the quadratic cost functional as follows

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(y(t), u(t)) dt,$$

$$F(y, u) := \alpha v_\delta^2 + \beta v_\phi^2 + \gamma_1 \omega_1^2 + \gamma_2 \omega_2^2.$$  

The Hermitian form $F$ can be represented in the form (14), where $F$ is defined by

$$\hat{F} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}.$$  

The matrix function $\Pi(\omega)$ and the matrices $R_j$ are defined by (15); $\Pi(\omega) > 0$ since $\gamma_1, \gamma_2 > 0$.

This formalization of the RRS problem makes it possible to apply the theory of optimal universal controllers, discussed in the previous section. To design OUC (11) with the coefficients (16), one has to choose $\rho(s)$ to be a scalar Hurwitz polynomial with $\deg \rho \geq \deg r + \deg \Delta$, whereas $r$ is a $2 \times 2$ matrix polynomial that satisfies (18). By fixing $\rho(\omega_j)$ and splitting the real and imaginary parts in the interpolation condition (18), one obtains a pair of real-valued matrix equations for the coefficients of $r(s)$. The only exception is $j = N = p + 1$: since $\omega_N = 0$, the equation (18) is real-valued. Hence we get $1 + 2p$ equations for the coefficients of the polynomial $r$. To satisfy them, the polynomial $r(s)$ should have $1 + 2p$ real-valued coefficients, i.e. it suffices to choose $\deg r = 2p$ and $\deg \rho \geq \deg \Delta + 2p$.

The just described algorithm to design an OUC for the roll damping problem can be summarized as follows:

1. choose a Hurwitz polynomial $\rho(s)$ with $\deg \rho(s) \geq 2p + \deg \Delta$;
2. compute the matrices $R_j$ from (15) (here $N = 1 + p$, $\omega_1, \ldots, \omega_p$ are the wave frequencies from (3) and $\omega_N = \omega_1 + p = 0$);
3. compute $W_{yaw}(\omega_j)$ (see Appendix A);
4. find the real coefficients of the matrix polynomial $r(s) = r_0 + \ldots + r_{2p}s^{2p}$ from (20);
5. the controller (11) with the coefficients (16) provides optimality of (22) for any uncertain amplitudes and phases.

For the detailed derivation of the OUC controller one may represent the transfer functions (1) as follows

$$W_{\phi r}(s) = \frac{sb_{\phi r}(s)}{a(s)}, \quad W_{\phi f}(s) = \frac{b_{\phi f}(s)}{a(s)},$$

$$W_{\psi f}(s) = \frac{sb_{\psi f}(s)}{a(s)}, \quad W_{\psi r}(s) = \frac{b_{\psi r}(s)}{a(s)},$$

In order to stabilize the vessel’s heading, the autopilot controller is chosen to be

$$W_{ap}(s) = \frac{b_{ap}(s)}{a_{ap}(s)}.$$  

A straightforward computation of $W_{yaw}(s), W_{roll}(s)$ (see Appendix A) shows that

$$W_{yaw}(s) = \frac{1}{\Delta(s)} \begin{pmatrix} s a_{ap}(s) b_{\phi r}(s) & b_{\phi u_2}(s) \\ a_{ap}(s) b_{\phi r}(s) & a_{ap}(s) b_{\phi f}(s) \end{pmatrix},$$

$$W_{roll}(s) = \begin{pmatrix} s b_{\phi r}(s) b_{\phi f}(s) & 1 & s b_{\phi r}(s) b_{\phi f}(s) \\ a(s) a_{ap}(s) & a(s) a_{ap}(s) & a(s) a_{ap}(s) \end{pmatrix},$$

$$\Delta(s) = a(s) a_{ap}(s) - b_{\phi r}(s) b_{ap}(s),$$

$$b_{\phi u_2}^0 = a_{ap}(s) b_{\phi f}(s) + b_{ap}(s) b_{\phi r}(s) b_{\phi f}(s) - b_{\phi f}(s) b_{\phi r}(s).$$

Obviously, $\deg \Delta(s) = \deg a(s) + \deg a_{ap}(s)$.

The application of this procedure to a specific vessel’s model is illustrated in the next section.

5. Numerical simulation

In this section we consider a numerical example to illustrate the proposed approach. The exact expressions for the transfer functions (23) of a vessel from (Perez, 2006, Appendix B) obtained via linearization of the nonlinear 4-DoF model at the constant speed 8 m/s are represented below

$$a(s) = s(s + 0.4375)(s + 0.040401)(s^2 + 0.2164s + 1.31),$$

$$b_{\phi r}(s) = -0.159(s - 0.4919)(s + 0.3005),$$

$$b_{\phi f}(s) = -0.078(s + 0.1785)(s^2 + 0.2586s + 1.324),$$

$$b_{\psi f}(s) = 0.402(s + 0.4501)(s + 0.3036),$$

$$b_{\psi r}(s) = 0.006(s - 0.9642)(s^2 + 0.1974s + 0.2361),$$

8
For this simulation we assume that the stabilizing autopilot (24) has the following form
\[ a_{ap}(s) = (s + 10), \quad b_{ap}(s) = 57(s + 0.5263). \]

To obtain the proper time series of the polyharmonic approximation of the irregular wave (3) we use the methodology presented in (Perez and Blanke, 2012, Sect. 4.2.5). Note that the resulting realization is obtained for the long-crested irregular sea at 15 kts in beam seas for significant wave height of 3 m and the peak frequency 1.15 rad/s; the response amplitude operator has been taken from (Perez and Blanke, 2012, Table B.9.).

For this simulation we assume that the stabilizing autopilot (24) has the following form
\[ a_{ap}(s) = (s + 10), \quad b_{ap}(s) = 57(s + 0.5263). \]

The conventional loop shaping controller has been chosen as the third approach for the comparison. To use this method, we completely ignore the yaw dynamics, working directly with the transfer function \( W^0_{\varphi, 2} \) (see Appendix A). Since we know the dominating frequency of the disturbance, we may select the structure of the notch filter centered at that point to minimize the amplitudes of the frequency response around it. The controller takes the following form
\[ W_c(s) = \frac{u_1(s)}{e_\varphi(s)} = -10s^2 + 0.2(1.15)s + 1.15^2. \]

The results of simulation are presented in Fig. 6–9, showing the dynamics of, respectively, the roll angle, the heading deviation and the rudder angle implementing aforementioned controllers.

As one can see, the proposed OUC controller demonstrates the best performance among all comparing controllers, although it utilizes the control
more actively. All controllers show low influence on the heading deviation; however, the performance of the roll damping is different. The loop shaping controller shows the weakest damping ability, despite the fact that it utilizes the fins more than optimal regulators. This simulation demonstrates the benefits of the combined rudder-fin control strategy avoiding undesired interaction between these actuators. The LQR shows the least control efforts providing the decent roll damping performance. The main disadvantage of the LQR is the lack of knowledge about the structure of the disturbance signal. For this simulation we have considered the worst case scenario when the peak of the spectral density is located close to the natural roll frequency of the vessel; however, if it is not the case the roll damping performance may significantly degrade. A key advantage of the OUC is the simplicity of the design procedure, which does not require to know the exact shape of the wave’s spectral density and, furthermore, allows to find the explicit dependence of the controller’s parameters from the coefficients of the cost functional. One can adjust the coefficients of the RRS controller “on the fly”, using e.g. some adaptive estimator of the disturbance’s spectrum (Bohtsov et al., 2012; Belleter et al., 2015).

We also want to point out that in this work we have not considered the influence of the saturation in the actuators on the performance of a motion control system focusing on the design procedure and comparison with similar linear controllers. Further consideration of these effects is an essential part of our ongoing research; however, it is possible to easily augment the OUC with either a heuristic algorithm such as an automatic gain control (AGC) (van Amerongen et al., 1990; van der Klugt, 1987) or more sophisticated methods based on the ideas of the control allocation with nonlinear servomechanism (Zaccarian, 2009; Johansen and Fossen, 2013).

6. Conclusion

In this paper, we offer a novel approach to the design of the roll damping system for marine vessels, based on the idea of optimal universal controllers (OUC). Unlike the existing approaches, such a controller does not require the full information about the wave’s spectral density, but only the knowledge of its dominant frequencies. A topic of ongoing research is to employ adaptive control methods to enable the controller’s functioning in the fully uncertain environment, in particular, combining the roll damping controller with an estimator of the dominating encounter wave frequencies.

Appendix A. Transfer matrices of the ship-autopilot system

In this section we are going to present the transformation procedure on how to obtain the models
in equations (21) based on the general dynamics of
the vessel described by the transfer function
\[
\varphi(t) = W_{\varphi r} \left( \frac{d}{dt} \delta_{r u d}(t) + W_{\varphi f} \left( \frac{d}{dt} \delta_{f i n}(t) \right) \right),
\]
\[
\psi(t) = W_{\varphi r} \left( \frac{d}{dt} \delta_{r u d}(t) + W_{\varphi f} \left( \frac{d}{dt} \delta_{f i n}(t) \right) \right).
\]
The observed outputs of the system are
\[
e_{\varphi}(t) = \psi(t) + d_{\varphi}(t) - \bar{\psi}, \quad e_{\psi} = \varphi(t) + d_{\varphi}(t),
\]
where \(\bar{\psi}\) is the heading setpoint. We introduce the
two control inputs as follows
\[
u_1 = \delta_{r u d}(t) - W_{AP} \left( \frac{d}{dt} e_{\varphi}(t) \right),
\]
\[
u_2 = \delta_{f i n}(t),
\]
where \(W_{AP}\) is the autopilot’s transfer function, sta-
bilizing the vessel’s yaw motion. Putting the equa-
tions together, one arrives at the following
\[
\begin{bmatrix}
1 & -W_{\varphi r} W_{ap}
0 & 1 - W_{\varphi r} W_{ap}
\end{bmatrix}
\begin{bmatrix}
e_{\varphi} \\
e_{\psi}
\end{bmatrix} = \begin{bmatrix}
W_{\varphi r} & W_{\varphi f} \\
W_{\varphi r} & W_{\varphi f}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} +
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\bar{\psi}} \\
d_{\varphi} \\
d_{\psi}
\end{bmatrix}.
\]
Assuming that the autopilot stabilizes the yaw
loop i.e. \(1 - W_{\varphi r} W_{ap} \neq 0\) this yields
\[
\begin{bmatrix}
e_{\varphi} \\
e_{\psi}
\end{bmatrix} = \begin{bmatrix}
W_{\varphi r u_1} & W_{\varphi r u_2} \\
W_{\varphi r u_1} & W_{\varphi r u_2}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} +
\begin{bmatrix}
W_{\varphi r u_1} & 1 & 0 \\
W_{\varphi r u_2} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\bar{\psi} \\
d_{\varphi} \\
d_{\psi}
\end{bmatrix},
\]
where
\[
W_{\varphi r u_1} = (1 - W_{\varphi r} W_{ap})^{-1} W_{\varphi r},
\]
\[
W_{\varphi r u_2} = W_{\varphi f} + (1 - W_{\varphi r} W_{ap})^{-1} W_{\varphi r} W_{ap} W_{\varphi f},
\]
\[
W_{\psi r u_1} = (1 - W_{\psi r} W_{ap})^{-1} W_{\psi r},
\]
\[
W_{\psi r u_2} = (1 - W_{\psi r} W_{ap})^{-1} W_{\psi r} W_{ap} W_{\psi f},
\]
\[
-W_{\varphi r u_1} = W_{\varphi d \psi} = (1 - W_{\varphi r} W_{ap})^{-1} W_{\varphi r} W_{ap},
\]
\[
-W_{\psi r u_1} = W_{\psi d \psi} = (1 - W_{\psi r} W_{ap})^{-1}.
\]
Recalling that
\[
y = \begin{bmatrix}
e_{\varphi} \\
e_{\psi}
\end{bmatrix}, \quad u = \begin{bmatrix}
\nu_1 \\
\nu_2
\end{bmatrix}, \quad d = \begin{bmatrix}
\bar{\psi} \\
d_{\varphi} \\
d_{\psi}
\end{bmatrix},
\]
the transfer matrices from \(u\) and \(d\) respectively to \(y\) are given by
\[
W_{\varphi r u} = \begin{bmatrix}
W_{\varphi r u_1} & W_{\varphi r u_2}
W_{\psi r u_1} & W_{\psi r u_2}
\end{bmatrix}, W_{\psi d} = \begin{bmatrix}
W_{\varphi d \psi} & 1 \\
W_{\psi d \psi} & 0
\end{bmatrix}.
\]

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