Local Privacy, Data Processing Inequalities, and Minimax Rates

John C. Duchi†
jduchi@eecs.berkeley.edu

Michael I. Jordan‡,∗
jordan@stat.berkeley.edu

Martin J. Wainwright†,∗
wainwrig@stat.berkeley.edu

Department of Statistics∗
University of California, Berkeley,
Berkeley, CA, 94720

Abstract

Working under a model of privacy in which data remains private even from the statistician, we study the tradeoff between privacy guarantees and the utility of the resulting statistical estimators. We prove bounds on information-theoretic quantities, including mutual information and Kullback-Leibler divergence, that depend on the privacy guarantees. When combined with standard minimax techniques, including the Le Cam, Fano, and Assouad methods, these inequalities allow for a precise characterization of statistical rates under local privacy constraints. We provide a treatment of several canonical families of problems: mean estimation, parameter estimation in fixed-design regression, multinomial probability estimation, and nonparametric density estimation. For all of these families, we provide lower and upper bounds that match up to constant factors, and exhibit new (optimal) privacy-preserving mechanisms and computationally efficient estimators that achieve the bounds.

1 Introduction

A major challenge in statistical inference is that of characterizing and balancing statistical utility with the privacy of individuals from whom data is obtained [20, 21, 28]. Such a characterization requires a formal definition of privacy, and differential privacy has been put forth as one such formalization [e.g., 24, 9, 25, 34, 35]. In the database and cryptography literatures from which differential privacy arose, early research was mainly algorithmic in focus, and researchers have used differential privacy to evaluate privacy-retaining mechanisms for transporting, indexing, and querying data. More recent work aims to link differential privacy to statistical concerns [22, 51, 33, 48, 15, 46]; in particular, researchers have developed algorithms for private robust statistical estimators, point and histogram estimation, and principal components analysis, among other procedures. Guarantees of optimality in this line of work have typically been with respect to a fixed class of estimators, where the goal is to approximate the estimators themselves under privacy-respecting transformations of the data. There has also been recent work in a more inferential vein that treats the data as random, and aims to recover aspects of the underlying population. Such analyses have been carried out within the context of classification problems as well as the “probably approximately correct” framework of statistical learning theory [e.g. 37, 7].

In this paper, we take a fully inferential point of view on privacy by bringing differential privacy into contact with statistical decision theory. Our focus is on the fundamental limits of differentially-private estimation. We obtain independence from specific estimation procedures by
studying the minimax risks associated with various estimation problems, and we obtain independence from specific privacy-preserving mechanisms by treating differential privacy as an abstract constraint on estimators. Within this framework, we derive both lower bounds and matching upper bounds on minimax risk. Our lower bounds are obtained by integrating differential privacy into the classical paradigms for bounding minimax risk via the inequalities of Le Cam, Fano, and Assouad, whereas the matching upper bounds are obtained by proposing and analyzing specific differentially-private procedures.

We study the strong setting of local privacy, in which providers trust no one, not even the statistician collecting the data. Although local privacy is a relatively stringent requirement, we view this setting as a natural step in identifying minimax risk bounds under privacy constraints. Indeed, local privacy is one of the oldest forms of privacy: its essential form dates back to Warner [50], who proposed it as a remedy for what he termed “evasive answer bias” in survey sampling. We hope that we can leverage deeper understanding of this classical setting to treat other privacy-preserving approaches to data analysis.

More formally, let \( X_1, \ldots, X_n \in \mathcal{X} \) be observations drawn according to some distribution \( P \), and let \( \theta = \theta(P) \) be some parameter of this unknown distribution. Our goal is to estimate \( \theta \) based on access to obscured views, \( Z_1, \ldots, Z_n \in \mathcal{Z} \), of the original data. The original random variables \( \{X_i\}_{i=1}^n \) and the privatized observations \( \{Z_i\}_{i=1}^n \) are linked via a family of conditional distributions \( Q_i(Z_i \mid X_i = x, Z_1, \ldots, Z_{i-1} = z_{i-1}) \). To simplify notation, we typically omit the subscript in \( Q_i \), as it is clear from the context. We assume the channel \( Q \) is sequentially interactive, meaning that it has the conditional independence structure

\[
\{X_i, Z_1, \ldots, Z_{i-1}\} \rightarrow Z_i \quad \text{and} \quad Z_i \perp X_j \mid \{X_i, Z_1, \ldots, Z_{i-1}\} \quad \text{for} \ j \neq i,
\]

as illustrated on the left of Figure 1. Since it acts as a conduit from the original to the privatized data, we refer to \( Q \) as a channel distribution. A special case of a sequentially interactive channel is the non-interactive case, in which each \( Z_i \) depends only on \( X_i \), and so that \( Z_i \perp \{X_j, Z_j, j \neq i\} \mid X_i \).

Our work is based on the following definition of local differential privacy. For a given privacy parameter \( \alpha \geq 0 \), we say that \( Z_i \) is an \( \alpha \)-differentially locally private view of \( X_i \) if for all \( z_1, \ldots, z_{i-1} \) and \( x, x' \in \mathcal{X} \) we have

\[
\sup_{S \in \sigma(\mathcal{Z})} Q_i(Z_i \in S \mid X_i = x, Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}) \leq \exp(\alpha),
\]

where \( \sigma(\mathcal{Z}) \) denotes an appropriate \( \sigma \)-field on \( \mathcal{Z} \). Definition (1) does not constrain \( Z_i \) to be a release of data based on exclusively on \( X_i \); the channel \( Q_i \) may change based on prior private observations.
that is, the channel may be interactive \cite{24}. We also consider the simplification \cite{50,27} where \( Z_i \) is generated based only on \( X_i \); in this non-interactive case (see the right side of Figure 1) the bound \((1)\) reduces to

\[
\sup_{S \in \sigma(Z)} \sup_{x,x' \in \mathcal{X}} Q(Z_i \in S \mid X_i = x) Q(Z_i \in S \mid X_i = x') \leq \exp(\alpha). \tag{2}
\]

These definitions capture a type of plausible-deniability: no matter what data \( Z \) is released, it is nearly equally as likely to have come from one point \( x \in \mathcal{X} \) as any other. It is also possible to interpret differential privacy within a hypothesis testing framework, where \( \alpha \) controls the error rate in tests for the presence or absence of individual data points in a dataset \cite{51}. Such guarantees against discovery, together with the treatment of issues of side information or adversarial strength that are problematic for other formalisms, have been used to make the case for differential privacy within the computer science literature; see, for example, the papers \cite{27,24,5,30}.

Although differential privacy provides an elegant formalism for limiting disclosure and protecting against many forms of privacy breach, it is a stringent measure of privacy, and it is conceivably overly stringent for statistical practice. Indeed, Fienberg et al. \cite{29} criticize the use of differential privacy in releasing contingency tables, arguing that known mechanisms for differentially private data release can give unacceptably poor performance. As a consequence, they advocate—in some cases—recourse to weaker privacy guarantees to maintain the utility and usability of released data. There are, however, results that are more favorable for differential privacy; for example, Smith \cite{48} shows that in some parametric problems, the non-local form of differential privacy \cite{24} can be satisfied while yielding asymptotically optimal parametric rates of convergence for some point estimators. Dwork and Lei \cite{22} suggest that it is possible to perform private inference with robust estimators (again, in the non-local form of differential privacy). Resolving such differing perspectives requires investigation into whether particular methods have optimality properties that would allow a general criticism of the framework, and characterizing the trade-offs between privacy and statistical efficiency. Such are the goals of the current paper.

1.1 Our contributions

The main contribution of this work is to provide general techniques for deriving minimax bounds under local privacy constraints and to illustrate the use of these techniques to compute the minimax rates for several canonical problems: (a) mean estimation; (b) parameter estimation in fixed design regression; (c) multinomial probability estimation; and (d) density estimation. We now outline our main contributions, giving some pointers to related work. Because a deeper comparison of the current work with prior research requires formally defining our minimax framework and presentation of our main results, we defer more expansive discussion of this related work to Section 6. We do wish to emphasize, however, that our minimax rates are for estimation of population quantities, in accordance with our connections to statistical decision theory; most prior work in the differential privacy literature focuses on accurate approximation of estimators in a conditional analysis in which the data are treated as fixed.

Many standard methods for obtaining minimax bounds involve information-theoretic quantities relating distributions that may have generated the data \cite{53,52,49}. In particular, let \( P_1 \) and \( P_2 \) denote two possible distributions that might have generated the data \( X_i \), and for \( \nu \in \{1,2\} \), define
the marginal distribution $M_\nu^n$ to be the distribution on $Z^n$ given by
\begin{equation}
M_\nu^n(S) := \int Q^n(S \mid x_1, \ldots, x_n) dP_\nu(x_1, \ldots, x_n) \quad \text{for } S \in \sigma(Z^n).
\end{equation}

Here $Q^n(\cdot \mid x_1, \ldots, x_n)$ denotes the joint distribution on $Z^n$ of the private sample $Z_{1:n}$, conditioned on the initial data $X_{1:n} = x_{1:n}$, based on the protocol for communication the inference algorithm and data providers use. The mutual information of samples drawn according to distributions of the form (3) and the KL divergence between such distributions are key objects in statistical discriminability and minimax rates [e.g., 34, 33, 52, 49], where they are often applied in one of three techniques for proving lower bounds: Le Cam’s, Fano’s, and Assouad’s methods [53, 49].

Keeping in mind the centrality of these information-theoretic quantities, our main results can be summarized at a high-level as follows. Theorem 1 provides a general result that bounds the KL divergence between distributions $M_1^n$ and $M_2^n$, as defined by the marginal (3), by a quantity dependent on the differential privacy parameter $\alpha$ and the total variation distance between $P_1$ and $P_2$. The essence of Theorem 1 is that
\begin{equation}
D_{\text{kl}}(M_1^n \parallel M_2^n) \lesssim \alpha^2 n \|P_1 - P_2\|_{TV}^2,
\end{equation}
where $\lesssim$ denotes inequality up to constant factors. When $\alpha^2 < 1$, which is the usual region of interest, this result shows that for statistical procedures whose minimax rate of convergence can be determined by classical information-theoretic methods, the additional requirement of $\alpha$-local differential privacy causes the effective sample size of any statistical procedure to be reduced from $n$ to at most $\alpha^2 n$. Section 3.1 contains the formal statement of this theorem, while Section 3.2 provides corollaries that show its use in application to minimax risk bounds. We follow this in Section 3.3 with applications of these results to estimation of one-dimensional means and fixed-design regression problems, providing corresponding upper bounds on the minimax risk. In accord with our general analysis, we see the reduction of effective sample size from $n$ to $\alpha^2 n$, but we also exhibit some striking difficulties of locally differentially private estimation in non-compact spaces. Indeed, if we wish to estimate the mean of a random variable $X$ satisfying $\text{Var}(X) \leq 1$, the minimax rate of estimation of $E[X]$ decreases from the parametric $1/n$ rate to $1/\sqrt{n\alpha^2}$.

Theorem 1 is appropriate for many one dimensional problems, but it does not address difficulties inherent in higher dimensional problems. With this motivation, our next two main results (Theorems 2 and 3) generalize Theorem 1 and incorporate dimensionality in an essential way: each provides bounds on information-theoretic quantities by dimension-dependent analogues of total-variation. Theorem 2 begins with multiple distributions $M_\nu^n$ of the form (3), where $\nu$ ranges over a set $\mathcal{V}$ indexing a set of possible distributions on the data $X$, and we define the mean distribution $\overline{M}^n = |\mathcal{V}| \sum_{\nu \in \mathcal{V}} M_\nu^n$. Controlling the average deviation $D_{\text{kl}}(M_\nu^n \parallel \overline{M}^n)$ over $\nu$ is essential in information theoretic techniques such as Fano’s method [53, 52] for proving minimax lower bounds. Theorem 2 allows us to relate the covariance structure of the elements $\nu \in \mathcal{V}$ to this average KL divergence. Theorem 3 by contrast, begins with pairs of distributions $M_{n+j}^n, M_{n+j}^n$ for $j \in \{1, \ldots, d\}$ for some integer $d$, and provides a variational upper bound on the summed pairs of divergences $\sum_j D_{\text{kl}}(M_{n+j}^n \parallel M_{n+j}^n)$.

As a consequence of Theorems 2 and 3, we obtain that for many $d$-dimensional estimation problems the effective sample size is reduced from $n$ to $n\alpha^2/d$; as we illustrate with various examples, this dimension-dependent reduction in sample size can have dramatic consequences. We provide the main statement and consequences of Theorem 2 in Section 4 showing its application to obtaining
minimax rates for mean estimation in both classical and high-dimensional settings. In Section 3 we present Theorem 3 showing how it provides (sharp) minimax lower bounds for multinomial and probability density estimation. Our results enable us to derive (often new) optimal mechanisms for these problems. One interesting consequence of our results is that Warner’s randomized response procedure [50] from the 1960s is an optimal mechanism for multinomial estimation.

**Notation:** For distributions \( P \) and \( Q \) defined on a space \( \mathcal{X} \), each absolutely continuous with respect to a distribution \( \mu \) (with corresponding densities \( p \) and \( q \)), the KL divergence between \( P \) and \( Q \) is

\[
D_{\text{kl}} (P||Q) := \int_{\mathcal{X}} dP \log \frac{dP}{dQ} = \int_{\mathcal{X}} p \log \frac{p}{q} d\mu.
\]

Letting \( \sigma(\mathcal{X}) \) denote the (an appropriate) \( \sigma \)-field on \( \mathcal{X} \), the total variation distance between two distributions \( P \) and \( Q \) is

\[
||P - Q||_{TV} := \sup_{S \in \sigma(\mathcal{X})} |P(S) - Q(S)| = \frac{1}{2} \int_{\mathcal{X}} |p(x) - q(x)| d\mu(x).
\]

Let \( P_X \) and \( P_Y \) denote marginal distributions of random vectors \( X \) and \( Y \) and \( P_Y(\cdot \mid X) \) denote the distribution of \( Y \) conditional on \( X \). The mutual information between \( X \) and \( Y \) is

\[
I(X;Y) := \mathbb{E}_P [D_{\text{kl}} (P_Y(\cdot \mid X)||P_Y(\cdot))] = \int D_{\text{kl}} (P_Y(\cdot \mid X = x)||P_Y(\cdot)) dP_X(x).
\]

A random variable \( Y \) has Laplace(\( \alpha \)) distribution if the density \( p_Y \) of \( Y \) is \( p_Y(y) = \frac{\alpha}{2} \exp (-\alpha|y|) \), where \( \alpha > 0 \). For matrices \( A, B \in \mathbb{R}^{d \times d} \), we use the notation \( A \preceq B \) to mean that \( B - A \) is positive semidefinite. For real sequences \( \{a_n\} \) and \( \{b_n\} \), we use \( a_n \lesssim b_n \) to mean that there is a universal constant \( C < \infty \) such that \( a_n \leq Cb_n \) for all \( n \), and \( a_n \asymp b_n \) to denote that \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \).

## 2 Background and problem formulation

We first establish the minimax framework we use throughout this paper; see references [52, 53, 49] for further background. Let \( \mathcal{P} \) denote a class of distributions on the sample space \( \mathcal{X} \), and let \( \theta(P) \in \Theta \) denote a function defined on \( \mathcal{P} \). The space \( \Theta \) in which the parameter \( \theta(P) \) takes values depends on the underlying statistical model (e.g., for univariate mean estimation, it is a subset of the real line). Let \( \rho \) denote a semi-metric on the space \( \Theta \), which we use to measure the error of an estimator for the parameter \( \theta \), and let \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a non-decreasing function with \( \Phi(0) = 0 \) (for example, \( \Phi(t) = t^2 \)).

In the classical setting, the statistician is given direct access to i.i.d. observations \( X_i \) drawn according to some \( P \in \mathcal{P} \). The local privacy setting involves an additional ingredient, namely, a conditional distribution \( Q \) that transforms the sample \( \{X_i\}_{i=1}^n \) into the private sample \( \{Z_i\}_{i=1}^n \) taking values in \( \mathcal{Z} \). Based on the observations \( Z_1, \ldots, Z_n \), our goal is to estimate the unknown parameter \( \theta(P) \in \Theta \). An estimator \( \hat{\theta} \) is a measurable function \( \hat{\theta} : \mathcal{Z}^n \to \Theta \), and we assess the quality of the estimate \( \hat{\theta}(Z_1, \ldots, Z_n) \) in terms of the risk

\[
\mathbb{E}_{P,Q} \left[ \Phi(\rho(\hat{\theta}(Z_1, \ldots, Z_n), \theta(P))) \right].
\]
For instance, for a univariate mean problem with $\rho(\theta, \theta') = |\theta - \theta'|$ and $\Phi(t) = t^2$, this error metric is the mean-squared error. For any fixed conditional distribution $Q$, the minimax rate is

$$M_n(\theta(P), \Phi \circ \rho, Q) := \inf_{Q \in Q_{\alpha}} \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_{P,Q} \left[ \Phi(\rho(\hat{\theta}(Z_1, \ldots, Z_n), \theta(P))) \right],$$

where we take the supremum (worst-case) over distributions $P \in \mathcal{P}$, and the infimum is taken over all estimators $\hat{\theta}$.

For each $\alpha > 0$, let $Q_{\alpha}$ denote the set of all conditional distributions guaranteeing $\alpha$-local privacy (1). By minimizing over all $Q \in Q_{\alpha}$, we obtain the central object of study for this paper, which characterizes the optimal rate of estimation in terms of the privacy parameter $\alpha$.

**Definition 1.** Given a family of distributions $\theta(P)$ and a privacy parameter $\alpha > 0$, the $\alpha$-minimax rate in the metric $\rho$ is

$$M_n(\theta(P), \Phi \circ \rho, \alpha) := \inf_{Q \in Q_{\alpha}} \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_{P,Q} \left[ \Phi(\rho(\hat{\theta}(Z_1, \ldots, Z_n), \theta(P))) \right].$$

**From estimation to testing:** A standard first step in proving minimax bounds is to reduce the estimation problem to a testing problem [53, 52, 49]. We use two types of testing problems: one a standard multiple hypothesis test, the second based on multiple binary hypothesis testing problems.

We begin with the simpler of the two. Given an index set $\mathcal{V}$ of finite cardinality, consider a family of distributions $\{P_\nu, \nu \in \mathcal{V}\}$ contained within $\mathcal{P}$. This family induces a collection of parameters $\{\theta(P_\nu), \nu \in \mathcal{V}\}$; it is said to be a $2\delta$-packing in the $\rho$-semimetric if

$$\rho(\theta(P_\nu), \theta(P_{\nu'})) \geq 2\delta \quad \text{for all } \nu \neq \nu'.$$

We use this family to define the **canonical hypothesis testing problem**:

- first, nature chooses a random variable $V$ from the uniform distribution over $\mathcal{V}$;
- second, conditioned on the choice $V = \nu$, the random sample $X = (X_1, \ldots, X_n)$ is drawn from the $n$-fold product distribution $P^n_\nu$.

In the classical setting, the sample $X$ is observed directly by the statistician, while the local privacy constraint means that a new random sample $Z = (Z_1, \ldots, Z_n)$ is generated by sampling $Z_i$ from the distribution $Q(\cdot | X_1, \ldots, X_n)$. By construction, conditioned on the choice $V = \nu$, the private sample $Z$ is distributed according to the marginal measure $M^n_\nu$ defined in equation (3).

Given the observed vector $Z$, the goal is to determine the value of the underlying index $\nu$. We refer to any measurable mapping $\psi : \mathbb{Z}^n \rightarrow \mathcal{V}$ as a test function. Its associated error probability is $\mathbb{P}(\psi(Z_1, \ldots, Z_n) \neq V)$, where $\mathbb{P}$ denotes the joint distribution over the random index $V$ and $Z$. The classical reduction from estimation to testing [e.g., 49, Section 2.2] guarantees that the minimax error (4) has lower bound

$$M_n(\Theta, \Phi \circ \rho, Q) \geq \Phi(\delta) \inf_{\psi} \mathbb{P}(\psi(Z_1, \ldots, Z_n) \neq V),$$

where the infimum ranges over all testing functions.

Following this reduction, the remaining challenge is to lower bound the probability of error in the underlying multi-way hypothesis testing problem. There are a variety of techniques for
this, and we in canonical hypothesis testing setting, we focus on two bounds on the probability of error (7) due to Le Cam and Fano. The simplest form of Le Cam’s inequality [e.g., 53, Lemma 1] is applicable when there are only two values \( \nu, \nu' \) in \( \mathcal{V} \). In this case, one has the equality

\[
\inf_{\psi} \mathbb{P}(\psi(Z_1, \ldots, Z_n) \neq V) = \frac{1}{2} - \frac{1}{2} \| M^n_\nu - M^n_{\nu'} \|_{TV},
\]

where the marginal \( M \) is defined as in expression (3). More generally, Fano’s inequality [52, 32, Lemma 4.2.1] holds when nature chooses uniformly at random from a set \( \mathcal{V} \) of cardinality larger than two, and takes the form

\[
\inf_{\psi} \mathbb{P}(\psi(Z_1, \ldots, Z_n) \neq V) \geq \left[ 1 - \frac{I(Z_1, \ldots, Z_n; V) + \log 2}{\log |\mathcal{V}|} \right].
\]

The second reduction we consider—which transforms estimation problems into multiple binary hypothesis testing problems—uses the structure of the hypercube in an essential way. For some \( d \in \mathbb{N} \), we set \( \mathcal{V} = \{-1, 1\}^d \), the \( d \)-dimensional hypercube. We say that the family \( P_\nu \) induces a \( 2\delta \)-Hamming separation for \( \Phi \circ \rho \) if there exists a function \( v: \theta(\mathcal{P}) \rightarrow \{-1, 1\}^d \) satisfying

\[
\Phi(\rho(\theta, \theta(P_\nu))) \geq 2\delta \sum_{j=1}^d 1\{v(\theta)_j \neq \nu_j\}.
\]

By letting \( \mathbb{P}_{+j} \) denote the joint distribution over the random index \( V \) and \( Z \) conditional on the \( j \)-th coordinate \( V_j = 1 \) (and similarly for \( \mathbb{P}_{-j} \)), we obtain the following sharper version of Assouad’s lemma [4] (see also the papers [3, 19]; we provide a proof in Appendix F.1).

**Lemma 1** (Sharp Assouad). Under the conditions of the previous paragraph, we have

\[
\mathcal{M}_n(\theta(\mathcal{P}), \Phi \circ \rho, Q) \geq \delta \sum_{j=1}^d \inf_{\psi} \mathbb{P}_{+j}(\psi(Z_1, \ldots, Z_n) \neq +1) + \mathbb{P}_{-j}(\psi(Z_1, \ldots, Z_n) \neq -1).
\]

With the definition of the marginals \( M^n_{\pm j} = 2^{-d+1} \sum_{\nu: \nu_j = \pm 1} M^n_\nu \), expression (8) shows that Lemma 1 is equivalent to the lower bound

\[
\mathcal{M}_n(\theta(\mathcal{P}), \Phi \circ \rho, Q) \geq \delta \sum_{j=1}^d \left[ 1 - \| M^n_{+j} - M^n_{-j} \|_{TV} \right].
\]

As a consequence of the preceding reductions to testing and the error bounds (8), (9), and (11), we obtain bounds on the \( \alpha \)-private minimax rate (5) by controlling total variation distances of the form \( \| M^n_{1} - M^n_{2} \|_{TV} \) or the mutual information between the random parameter index \( V \) and the sequence of random variables \( Z_1, \ldots, Z_n \). We devote the following sections to these tasks.

### 3 Pairwise bounds under privacy: Le Cam and local Fano methods

We begin with results upper bounding the symmetrized Kullback-Leibler divergence under a privacy constraint. We then develop some consequences of this result for both Le Cam’s method and a local form of Fano’s method. Using these methods, we derive sharp minimax rates under local privacy for estimating 1-dimensional means and for fixed design regression in \( d \)-dimensions.
3.1 Pairwise upper bounds on Kullback-Leibler divergences

Many statistical problems depend on comparisons between a pair of distributions $P_1$ and $P_2$ defined on a common space $X$. Any conditional distribution $Q$ transforms such a pair of distributions into a new pair $(M_1, M_2)$ via marginalization

$$M_j(S) := \int_X Q(S \mid x) dP_j(x) \quad \text{for } j = 1, 2.$$  

Our first main result bounds the symmetrized Kullback-Leibler (KL) divergence between these induced marginals as a function of the privacy parameter $\alpha > 0$ associated with the conditional distribution $Q$ and the total variation distance between $P_1$ and $P_2$.

**Theorem 1.** For any $\alpha \geq 0$, let $Q$ be any conditional distribution that guarantees $\alpha$-differential privacy. Then for any pair of distributions $P_1$ and $P_2$, the induced marginals $M_1$ and $M_2$ satisfy the bound

$$D_{\text{kl}}(M_1 \| M_2) + D_{\text{kl}}(M_2 \| M_1) \leq \min\{4, e^{2\alpha}\}(e^\alpha - 1)^2 \|P_1 - P_2\|_{\text{TV}}^2. \tag{12}$$

**Remarks:** For $\alpha \leq \frac{23}{35}$ we have the inequality $e^\alpha - 1 \leq \sqrt{2}\alpha$. Consequently, by applying Pinsker’s inequality to the total variation distance between $P_1$ and $P_2$, Theorem 1 implies that

$$D_{\text{kl}}(M_1 \| M_2) + D_{\text{kl}}(M_2 \| M_1) \leq 8\alpha^2 \|P_1 - P_2\|_{\text{TV}}^2 \leq 4\alpha^2 \min\{D_{\text{kl}}(P_1 \| P_2), D_{\text{kl}}(P_2 \| P_1)\} \tag{13}$$

for $\alpha \in [0, \frac{23}{35}]$. This inequality allows us to relate the symmetrized KL divergence between $M_1$ and $M_2$ directly to the KL divergences between $P_1$ and $P_2$. We may also relate the total variation distances directly using Pinsker’s inequality.

The result of Theorem 1 is similar to a result due to Dwork et al. [25, Lemma III.2], who show that $D_{\text{kl}}(Q(\cdot \mid x)\|Q(\cdot \mid x')) \leq \alpha(e^\alpha - 1)$ for any $x, x' \in X$, which implies $D_{\text{kl}}(M_1 \| M_2) \leq \alpha(e^\alpha - 1)$ by convexity. This upper bound is weaker than Theorem 1 since it lacks the term $\|P_1 - P_2\|_{\text{TV}}^2$. This total variation term is essential to our minimax lower bounds: more than providing a bound on KL divergence, Theorem 1 shows that differential privacy acts as a contraction on the space of probability measures. This contractivity holds in a strong sense: indeed, the bound (12) shows that even if we start with a pair of distributions $P_1$ and $P_2$ whose KL divergence is infinite, the induced marginals $M_1$ and $M_2$ always have finite KL divergence for any fixed privacy level $\alpha$.

We provide the proof of Theorem 1 in Section 7. Here we develop a corollary that has useful consequences for minimax theory under local privacy constraints. Suppose that conditionally on $V = \nu$, we draw a random vector $X = (X_1, \ldots, X_n)$ from the product measure $\prod_{i=1}^n P_{\nu,i}$, and that we draw the $\alpha$-locally private sample $Z = (Z_1, \ldots, Z_n)$ according to the channel $Q(\cdot \mid X_{1:n})$. Conditioned on $V = \nu$, the sample $Z$ is distributed according to the measure $M^\nu$ defined previously (4). Note that because we allow interactive protocols, the distribution $M^\nu$ need not be a product distribution in general. Given this set-up, we have the following:

**Corollary 1.** For any conditional distribution $Q$ that guarantees $\alpha$-local differential privacy (11) and any sequence of pairs of distributions $\{P_{\nu,i}\}$ and $\{P'_{\nu,i}\}$,

$$D_{\text{kl}}(M^\nu \| M'_{\nu}) + D_{\text{kl}}(M'_{\nu} \| M^\nu) \leq 4(e^\alpha - 1)^2 \sum_{i=1}^n \|P_{\nu,i} - P'_{\nu,i}\|_{\text{TV}}^2. \tag{14}$$

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See Section 7.2 for the proof, which requires a few intermediate steps to obtain the additive inequality. Inequality (14) also immediately implies a mutual information bound, which may be useful in applications of Fano’s inequality. In particular, if we define the mean distribution \( M_n = \frac{1}{|V|} \sum_{\nu \in V} M_{\nu} \), then by the definition of mutual information, we have

\[
I(Z_1, \ldots, Z_n; V) = \frac{1}{|V|} \sum_{\nu \in V} D_{\text{KL}}(M_{\nu} \parallel M_n) \leq 4(\epsilon^\alpha - 1)^2 \sum_{i=1}^{n} \frac{1}{|V|^2} \sum_{\nu, \nu' \in V} \| P_{\nu,i} - P_{\nu',i} \|_{\text{TV}},
\]

the first inequality following from the joint convexity of the KL divergence and the final inequality from Corollary 1.

Remarks: Mutual information bounds under local privacy have appeared previously. McGregor et al. \[43\] study relationships between communication complexity and differential privacy, showing that differentially private schemes allow low communication. They provide a result \[43, \text{Prop. 7}\] that guarantees

\[
I(X_1; Z_1) \leq 3 \alpha n;
\]

they strengthen this bound to

\[
I(X_1; Z_1) \leq (3/2) \alpha^2 n
\]

when the \( X_i \) are i.i.d. uniform Bernoulli \( \{0, 1\} \) variables. Since the total variation distance is at most 1, our result also implies this scaling (for arbitrary \( X_i \)), but it is stronger since it involves the total variation terms \( \| P_{\nu,i} - P_{\nu',i} \|_{\text{TV}} \), which are essential in our minimax results. In addition, Corollary 1 allows for any (sequentially) interactive mechanism generating the observed variables \( Z_1; Z_n \); each \( Z_i \) may depend on the private answers \( Z_{1:i-1} \) of other data providers.

3.2 Consequences for minimax theory under local privacy constraints

We now turn to some consequences of Theorem 1 for minimax theory under local privacy constraints. For ease of presentation, we analyze the case of independent and identically distributed (i.i.d.) samples, meaning that \( P_{\nu,i} \equiv P_\nu \) for \( i = 1, \ldots, n \). We show that in both Le Cam’s inequality and the local version of Fano’s method, the constraint of \( \alpha \)-local differential privacy reduces the effective sample size (at least) from \( n \) to \( 4\alpha^2 n \).

Consequence for Le Cam’s method: The classical non-private version of Le Cam’s method bounds the usual minimax risk

\[
\mathcal{M}_n(\theta(P), \Phi \circ \rho) := \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_P \left[ \Phi(\rho(\hat{\theta}(X_1, \ldots, X_n), \theta(P))) \right],
\]

for estimators \( \hat{\theta} : \mathcal{X}^n \rightarrow \Theta \) by a binary hypothesis test. One version of Le Cam’s lemma \[8\] asserts that, for any pair of distributions \( \{P_1, P_2\} \) such that \( \rho(\theta(P_1), \theta(P_2)) \geq 2\delta \), we have

\[
\mathcal{M}_n(\theta(P), \Phi \circ \rho) \geq \Phi(\delta) \left\{ \frac{1}{2} - \frac{1}{2\sqrt{2}} \sqrt{nD_{\text{KL}}(P_1 \parallel P_2)} \right\}.
\]

Returning to the \( \alpha \)-locally private setting, in which the estimator \( \hat{\theta} \) depends only on the private variables \( (Z_1, \ldots, Z_n) \), we measure the \( \alpha \)-private minimax risk \( \mathcal{M}_n(\theta(P), \Phi \circ \rho) \). By applying Le Cam’s method to the pair \( (M_1, M_2) \) along with Corollary 1 in the form of inequality (14), we find:
Corollary 2 (Private form of Le Cam bound). Given observations from an α-locally differential private channel for some $\alpha \in [0, \frac{22}{35}]$, the α-private minimax risk is lower bounded as

$$M_n(\Theta, \Phi \circ \rho, \alpha) \geq \Phi(\delta) \left\{ 1 - \frac{1}{2} \frac{\sqrt{8n\alpha^2 \|P_1 - P_2\|_{TV}^2}}{2\sqrt{2}} \right\}. \quad (17)$$

Using the fact that $\|P_1 - P_2\|_{TV} \leq \frac{1}{2} D_{KL}(P_1 || P_2)$, comparison with the original Le Cam bound (16) shows that for $\alpha \in [0, \frac{22}{35}]$, the effect of α-local differential privacy is to reduce the effective sample size from $n$ to $4\alpha^2 n$. We illustrate use of this private version of Le Cam’s bound in our analysis of the one-dimensional mean problem to follow.

Consequences for local Fano’s method: We now turn to consequences for the so-called local form of Fano’s method. This method based on constructing a family of distributions $\{P_\nu, \nu \in V\}$ that defines a $2\delta$-packing, meaning $\rho(\Theta(P_\nu), \Theta(P_{\nu'})) \geq 2\delta$ for all $\nu \neq \nu'$, satisfying

$$D_{KL}(P_\nu || P_{\nu'}) \leq \kappa^2 \delta^2 \quad \text{for some fixed } \kappa > 0. \quad (18)$$

We refer to any such construction as a $(\delta, \kappa)$ local packing. Recalling Fano’s inequality (9), the pairwise upper bounds (18) imply $I(Z_1, \ldots, Z_n; V) \leq n\kappa^2 \delta^2$ by a convexity argument. We thus obtain the local Fano lower bound [36, 8] on the classical minimax risk:

$$M_n(\Theta, \Phi \circ \rho) \geq \Phi(\delta) \left\{ 1 - \frac{n\kappa^2 \delta^2 + \log 2}{\log |V|} \right\}. \quad (19)$$

We now state the extension of this bound to the α-locally private setting.

Corollary 3 (Private form of local Fano inequality). Consider observations from an α-locally differential private channel for some $\alpha \in [0, \frac{22}{35}]$. Given any $(\delta, \kappa)$ local packing, the α-private minimax risk has lower bound

$$M_n(\Theta, \Phi \circ \rho, \alpha) \geq \Phi(\delta) \left\{ 1 - \frac{4n\alpha^2 \kappa^2 \delta^2 + \log 2}{\log |V|} \right\}. \quad (20)$$

Once again, by comparison to the classical version (19), we see that, for all $\alpha \in [0, \frac{22}{35}]$, the price for privacy is a reduction in the effective sample size from $n$ to $4\alpha^2 n$. The proof is again straightforward using Theorem 1. By Pinsker’s inequality, the pairwise bound (18) implies that

$$\|P_\nu - P_{\nu'}\|_{TV}^2 \leq \frac{1}{2} \kappa^2 \delta^2 \quad \text{for all } \nu \neq \nu'.$$

We find that $I(Z_1, \ldots, Z_n; V) \leq 4n\alpha^2 \kappa^2 \delta^2$ for all $\alpha \in [0, \frac{22}{35}]$ by combining this inequality with the upper bound (15) from Corollary 1. The claim (20) follows by combining this upper bound with the usual local Fano bound (19).

3.3 Some applications of Theorem 1

In this section, we illustrate the use of the α-private versions of Le Cam’s and Fano’s inequalities, established in the previous section as Corollaries 2 and 3 of Theorem 1. First, we study the problem of one-dimensional mean estimation. In addition to demonstrating how the minimax rate changes
as a function of $\alpha$, we also reveal some interesting (and perhaps disturbing) effects of enforcing $\alpha$-local differential privacy: the effective sample size may be even polynomially smaller than $\alpha^2 n$. Our second example studies fixed design linear regression, where we again see the reduction in effective sample size from $n$ to $\alpha^2 n$. We state each of our bounds assuming $\alpha \in [0, 1]$; the bounds hold (with different numerical constants) whenever $\alpha \in [0, C]$ for some universal constant $C$.

### 3.3.1 One-dimensional mean estimation

For some $k > 1$, consider the family

$$\mathcal{P}_k := \{\text{distributions } P \text{ such that } \mathbb{E}_P[X] \in [-1, 1] \text{ and } \mathbb{E}_P[|X|^k] \leq 1\},$$

and suppose that our goal is to estimate the mean $\theta(P) = \mathbb{E}_P[X]$. The next proposition characterizes the $\alpha$-private minimax risk in squared $\ell_2$-error

$$\mathcal{M}_n(\theta(\mathcal{P}_k), (\cdot)^2, \alpha) := \inf_{Q \in \mathbb{S}_\alpha} \inf_{\hat{\theta} \in \mathbb{P}_k} \mathbb{E}\left[ (\hat{\theta}(Z_1, \ldots, Z_n) - \theta(P))^2 \right].$$

**Proposition 1.** There exist universal constants $0 < c_\ell \leq c_u < \infty$ such that for all $k > 1$ and $\alpha \in [0, 1]$, the minimax error $\mathcal{M}_n(\theta(\mathcal{P}_k), (\cdot)^2, \alpha)$ is bounded as

$$c_\ell \min\left\{1, (n\alpha^2)^{-\frac{k-1}{k}}\right\} \leq \mathcal{M}_n(\theta(\mathcal{P}_k), (\cdot)^2, \alpha) \leq c_u \min\left\{1, \max\{1, u_k\} (n\alpha^2)^{-\frac{k-1}{k}}\right\},$$

where $u_k = \max\{1, (k - 1)^{-2}\}$.

We prove this result using the $\alpha$-private version (17) of Le Cam’s inequality, as stated in Corollary 2. See Section 7.3 for the details.

To understand the bounds (21), it is worthwhile considering some special cases, beginning with the usual setting of random variables with finite variance ($k = 2$). In the non-private setting in which the original sample $(X_1, \ldots, X_n)$ is observed, the sample mean $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n X_i$ has mean-squared error at most $1/n$. When we require $\alpha$-local differential privacy, Proposition 1 shows that the minimax rate worsens to $1/\sqrt{n\alpha^2}$. More generally, for any $k > 1$, the minimax rate scales as $\mathcal{M}_n(\theta(\mathcal{P}_k), (\cdot)^2, \alpha) \asymp (n\alpha^2)^{-\frac{k-1}{k}}$, ignoring $k$-dependent pre-factors. As $k \uparrow \infty$, the moment condition $\mathbb{E}[|X|^k] \leq 1$ becomes equivalent to the boundedness constraint $|X| \leq 1$ a.s., and we obtain the more standard parametric rate $(n\alpha^2)^{-1}$, where there is no reduction in the exponent.

More generally, the behavior of the $\alpha$-private minimax rates (21) helps demarcate situations in which local differential privacy may or may not be acceptable. In particular, for bounded domains—where we may take $k \uparrow \infty$—local differential privacy may be quite reasonable. However, in situations in which the sample takes values in an unbounded space, local differential privacy provides much stricter constraints. Indeed, in Appendix C, we discuss an example that illustrates the pathological consequences of providing (local) differential privacy for non-compact spaces.

### 3.3.2 Linear regression with fixed design

We turn now to the problem of linear regression. Concretely, for a given design matrix $X \in \mathbb{R}^{n \times d}$, consider the standard linear model

$$Y = X\theta^* + \varepsilon,$$  \hspace{1cm} (22)
where \( \varepsilon \in \mathbb{R}^n \) is a sequence of independent, zero-mean noise variables. By rescaling as needed, we may assume that \( \theta^* \in \Theta = \mathbb{B}_2(1) \), the Euclidean ball of radius one. Moreover, we assume that a scaling constant \( \sigma < \infty \) such that the noise sequence \( |\varepsilon_i| \leq \sigma \) for all \( i \). Given the challenges of non-compactness exhibited by the location family estimation problems (cf. Proposition 1), this type of assumption is required for non-trivial results. We also assume that \( X \) has rank \( d \); otherwise, the design matrix \( X \) has a non-trivial nullspace and \( \theta^* \) cannot be estimated even when \( \sigma = 0 \).

With the model (22) in place, let us consider estimation of \( \theta^* \) in the squared \( \ell_2 \)-error, where we provide \( \alpha \)-locally differentially private views of the response \( Y = \{Y_i\}_{i=1}^n \). By following the outline established in Section 3.2, we provide a sharp characterization of the \( \alpha \)-private minimax rate. In stating the result, we let \( \rho_j(A) \) denote the \( j \)th singular value of a matrix \( A \).

**Proposition 2.** Consider estimation in the fixed design regression model (22), where the variables \( Y_i \) are \( \alpha \)-locally differentially private for some \( \alpha \in [0,1] \). There exist universal constants \( 0 < c_\ell \leq c_u < \infty \) such that

\[
 c_\ell \min \left\{ 1, \frac{\sigma^2 d}{n \alpha^2 \rho_{\max}(X/\sqrt{n})} \right\} \leq \mathcal{M}_n \left( \Theta, \|\cdot\|_2^2, \alpha \right) \leq c_u \min \left\{ 1, \frac{\sigma^2 d}{\alpha^2 n \rho_{\min}(X/\sqrt{n})} \right\}.
\]

(23)

See Section 7.4 for the proof.

To interpret the bounds (23), it is helpful to consider some special cases. First consider the case of an orthonormal design, meaning that \( \frac{1}{n}X^\top X = I_{d \times d} \). The bounds (23) imply that \( \mathcal{M}_n(\Theta, \|\cdot\|_2^2, \alpha) \propto \sigma^2 d/(\alpha n^2) \), so that the \( \alpha \)-private minimax rate is fully determined (up to constant pre-factors). Standard minimax rates for linear regression problems scale as \( \sigma^2 d/n \); thus, by comparison, we see that requiring differential privacy indeed causes an effective sample size reduction from \( n \) to \( n \alpha^2 \). More generally, up to the difference between the maximum and minimum singular values of the design \( X \), Proposition 2 provides a sharp characterization of the \( \alpha \)-private rate for fixed-design linear regression. As the proof makes clear, the upper bounds are attained by adding Laplacian noise to the response variables \( Y_i \) and solving the resulting normal equations as in standard linear regression. In this case, the standard Laplacian mechanism [24] is optimal.

### 4 Mutual information under local privacy: Fano’s method

As we have previously noted, Theorem 1 provides indirect upper bounds on the mutual information. However, since the resulting bounds involve pairwise distances only, as in Corollary 1, they must be used with local packings. Exploiting Fano’s inequality in its full generality requires a more sophisticated upper bound on the mutual information under local privacy, which is the main topic of this section. We illustrate this more powerful technique by deriving lower bounds for mean estimation problems in both classical as well as high-dimensional settings under the non-interactive privacy model (2).

#### 4.1 Variational bounds on mutual information

We begin by introducing some definitions needed to state the result. Let \( V \) be a discrete random variable uniformly distributed over some finite set \( \mathcal{V} \). Given a family of distributions \( \{P_\nu, \nu \in \mathcal{V}\} \),
we define the mixture distribution
\[ \mathcal{P} := \frac{1}{|V|} \sum_{\nu \in V} P_{\nu}. \]
A sample \( X \sim \mathcal{P} \) can be obtained by first drawing \( V \) from the uniform distribution over \( V \), and then conditionally on \( V = \nu \), drawing \( X \) from the distribution \( P_{\nu} \). By definition, the mutual information between the random index \( V \) and the sample \( X \) is
\[ I(X; V) = \frac{1}{|V|} \sum_{\nu \in V} D_{\text{KL}}(P_{\nu} \| \mathcal{P}), \]
a representation that plays an important role in our theory. As in the definition (3), any conditional distribution \( Q \) induces the family of marginal distributions \( \{M_{\nu}, \nu \in V\} \) and the associated mixture \( M := \frac{1}{|V|} \sum_{\nu \in V} M_{\nu} \). Our goal is to upper bound the mutual information \( I(Z_1, \ldots, Z_n; V) \), where conditioned on \( V = \nu \), the random variables \( Z_i \) are drawn according to \( M_{\nu} \).

Our upper bound is variational in nature: it involves optimization over a subset of the space \( L^\infty(X) := \{f : X \to \mathbb{R} \mid \|f\|_\infty < \infty\} \) of uniformly bounded functions, equipped with the usual norm \( \|f\|_\infty = \sup_{x \in X} |f(x)| \). We define the 1-ball of the supremum norm \( B_\infty(X) := \{\gamma \in L^\infty(X) \mid \|\gamma\|_\infty \leq 1\} \).

We show that this set describes the maximal amount of perturbation allowed in the conditional \( Q \). Since the set \( X \) is generally clear from context, we typically omit this dependence. For each \( \nu \in V \), we define the linear functional \( \varphi_{\nu} : L^\infty(X) \to \mathbb{R} \) by
\[ \varphi_{\nu}(\gamma) = \int_X \gamma(x)(dP_{\nu}(x) - d\mathcal{P}(x)). \]
With these definitions, we have the following result:

**Theorem 2.** Let \( \{P_{\nu}\}_{\nu \in V} \) be an arbitrary collection of probability measures on \( X \), and let \( \{M_{\nu}\}_{\nu \in V} \) be the set of marginal distributions induced by an \( \alpha \)-differentially private distribution \( Q \). Then
\[ \frac{1}{|V|} \sum_{\nu \in V} [D_{\text{KL}}(M_{\nu} \| M) + D_{\text{KL}}(M \| M_{\nu})] \leq \frac{(e^\alpha - 1)^2}{|V|} \sup_{\gamma \in B_\infty(X)} \sum_{\nu \in V} (\varphi_{\nu}(\gamma))^2. \]

It is important to note that, at least up to constant factors, Theorem 2 is never weaker than the results provided by Theorem 1 including the bounds of Corollary 1. By definition of the linear functional \( \varphi_{\nu} \), we have
\[ \sup_{\gamma \in B_\infty(X)} \sum_{\nu \in V} (\varphi_{\nu}(\gamma))^2 \leq 4 \sum_{\nu \in V} \|P_{\nu} - \mathcal{P}\|_{TV}^2, \]
where inequality (i) follows by interchanging the summation and supremum. Overall, we have
\[ I(Z; V) \leq 4(e^\alpha - 1)^2 \frac{1}{|V|} \sum_{\nu \in V} \|P_{\nu} - \mathcal{P}\|_{TV}^2 \leq 4(e^\alpha - 1)^2 \frac{1}{|V|^2} \sum_{\nu, \nu' \in V} \|P_{\nu} - P_{\nu'}\|_{TV}^2. \]
The strength of Theorem 2 arises from the fact that inequality (i)—interchanging the order of the supremum and summation—may be quite loose.

We now present a corollary that extends Theorem 2 to the setting of repeated sampling, providing a tensorization inequality analogous to Corollary 1. Let \( V \) be distributed uniformly at random in \( V \), and assume that given \( V = \nu \), the observations \( X_i \) are sampled independently according to the distribution \( P_\nu \) for \( i = 1, \ldots, n \). For this corollary, we require the non-interactive setting (2) of local privacy, where each private variable \( Z_i \) depends only on \( X_i \).

Corollary 4. Suppose that the distributions \( \{Q_i\}_{i=1}^n \) are \( \alpha \)-locally differentially private in the non-interactive setting (2). Then

\[
I(Z_1, \ldots, Z_n; V) \leq n(e^\alpha - 1)^2 \left\| \sup_{\gamma \in B_\infty} \sum_{\nu \in V} (\varphi_\nu(\gamma)) \right\|^2.
\]

We provide the proof of Corollary 4 in Section 8.2. We conjecture that the bound (26) also holds in the fully interactive setting, but given well-known difficulties of characterizing multiple channel capacities with feedback [16, Chapter 15], it may be challenging to show.

Theorem 2 and Corollary 4 relate the amount of mutual information between the random perturbed views \( Z \) of the data to geometric or variational properties of the underlying packing \( V \) of the parameter space \( \Theta \). In particular, Theorem 2 and Corollary 4 show that if we can find a packing set \( V \) that yields linear functionals \( \varphi_\nu \) whose sum has good “spectral” properties—meaning a small operator norm when taking suprema over \( L_\infty \)-type spaces—we can provide sharper results.

4.2 Applications of Theorem 2 to mean estimation

In this section, we show how Theorem 2, coupled with Corollary 4, leads to sharp characterizations of the \( \alpha \)-private minimax rates for classical and high-dimensional mean estimation problems. Our results show that for in \( d \)-dimensional mean-estimation problems, the requirement of \( \alpha \)-local differential privacy causes a reduction in effective sample size from \( n \) to \( n\alpha^2/d \). Throughout this section, we assume that the channel \( Q \) is non-interactive, meaning that the random variable \( Z_i \) depends only on \( X_i \), and so that local privacy takes the simpler form (2). We also state each of our results for privacy parameter \( \alpha \in [0, 1] \), but note that all of our bounds hold for any constant \( \alpha \), with appropriate changes in the numerical pre-factors.

Before proceeding, we describe two sampling mechanisms for enforcing \( \alpha \)-local differential privacy. Our methods for achieving the upper bounds in our minimax rates are based on unbiased estimators of a data vector, often the observation \( X \). Let us assume we wish to construct an \( \alpha \)-private unbiased estimate \( Z \) for the vector \( v \in \mathbb{R}^d \). The following sampling strategies are based on a radius \( r > 0 \) and a bound \( B > 0 \) specified for each problem, and they require the Bernoulli random variable

\[
T \sim \text{Bernoulli}(\pi_\alpha), \quad \text{where} \quad \pi_\alpha := e^\alpha/(e^\alpha + 1).
\]

**Strategy A:** Given a vector \( v \) with \( \|v\|_2 \leq r \), set \( \bar{v} = rv/\|v\|_2 \) with probability \( \frac{1}{2} + \|v\|_2/2r \) and \( \bar{v} = -rv/\|v\|_2 \) with probability \( \frac{1}{2} - \|v\|_2/2r \). Then sample \( T \sim \text{Bernoulli}(\pi_\alpha) \) and set

\[
Z \sim \begin{cases} 
\text{Uniform}(z \in \mathbb{R}^d : \langle z, \bar{v} \rangle > 0, \|z\|_2 = B) & \text{if} \ T = 1 \\
\text{Uniform}(z \in \mathbb{R}^d : \langle z, \bar{v} \rangle \leq 0, \|z\|_2 = B) & \text{if} \ T = 0.
\end{cases}
\]

(27a)
Figure 2. Private sampling strategies. (a) Strategy (27a) for the \( \ell_2 \)-ball. Outer boundary of highlighted region sampled uniformly with probability \( e^{\alpha}/(e^{\alpha}+1) \). (b) Strategy (27b) for the \( \ell_\infty \)-ball. Circled point set sampled uniformly with probability \( e^{\alpha}/(e^{\alpha}+1) \).

**Strategy B:** Given a vector \( v \) with \( \|v\|_\infty \leq r \), construct \( \tilde{v} \in \mathbb{R}^d \) with coordinates \( \tilde{v}_j \) sampled independently from \( \{-r, r\} \) with probabilities \( 1/2 - v_j/(2r) \) and \( 1/2 + v_j/(2r) \). Then sample \( T \sim \text{Bernoulli}(\pi_\alpha) \) and set

\[
Z \sim \begin{cases} 
\text{Uniform}(z \in \{-B, B\}^d : \langle z, \tilde{v} \rangle > 0) & \text{if } T = 1 \\
\text{Uniform}(z \in \{-B, B\}^d : \langle z, \tilde{v} \rangle \leq 0) & \text{if } T = 0.
\end{cases}
\]

See Figure 2 for visualizations of these sampling strategies. By inspection, each is \( \alpha \)-differentially private for any vector satisfying \( \|v\|_2 \leq r \) or \( \|v\|_\infty \leq r \) for Strategy A or B, respectively. Moreover, each strategy is efficiently implementable: Strategy A by normalizing a sample from the \( \mathcal{N}(0, I_d \times d) \) distribution, and Strategy B by rejection sampling over the scaled hypercube \( \{-B, B\}^d \).

Our sampling strategies specified, we study the \( d \)-dimensional problem of estimating the mean \( \theta(P) := \mathbb{E}_P[X] \) of a random vector. We consider a few different metrics for the error of a mean estimator to flesh out the testing reduction in Section 2. Due to the difficulties associated with differential privacy on non-compact spaces (recall Section 3.3.1), we focus on distributions with compact support. We defer all proofs to Appendix A they use a combination of Theorem 2 with Fano’s method.

### 4.2.1 Minimax rates

We begin by bounding the minimax rate in the squared \( \ell_2 \)-metric. For a parameter \( p \in [1, 2] \) and radius \( r < \infty \), consider the family

\[
P_{p,r} := \{ \text{distributions } P \text{ supported on } \mathbb{B}_p(r) \subset \mathbb{R}^d \}.
\]

where \( \mathbb{B}_p(r) = \{ x \in \mathbb{R}^d \mid \|x\|_p \leq r \} \) is the \( \ell_p \)-ball of radius \( r \).
**Proposition 3.** For the mean estimation problem, there exist universal constants $0 < c_\ell \leq c_u < \infty$ such that for all $p \in [1, 2]$ and privacy levels $\alpha \in [0, 1]$,

$$c_\ell r^2 \min \left\{ 1, \frac{1}{\sqrt{n\alpha^2}}, \frac{d}{n\alpha^2} \right\} \leq \mathcal{M}_n(\theta(P_{p,r}), \|\cdot\|_2^2, \alpha) \leq c_u r^2 \min \left\{ \frac{d}{n\alpha^2}, 1 \right\}.$$  

This bound does not depend on the norm $p$ bounding $X$ so long as $p \in [1, 2]$, which is consistent with the classical mean estimation problem. Proposition 3 demonstrates the substantial difference between $d$-dimensional mean estimation in private and non-private settings: more precisely, the privacy constraint leads to a multiplicative penalty of $d/\alpha^2$ in terms of mean-squared error. Indeed, in the non-private setting, the standard mean estimator $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n X_i$ has mean-squared error at most $r^2/n$, since $\|X\|_2 \leq \|X\|_p \leq r$ by assumption. Thus, Proposition 3 exhibits an effective sample size reduction of $n \mapsto n\alpha^2/d$.

To show the applicability of the general metric construction in Section 2, we now consider estimation in $\ell_\infty$-norm; estimation in this metric is natural in scenarios where one wishes only to guarantee that the maximum error of any particular component in the vector $\theta$ is small. We focus in this scenario on the family $P_{\infty,r}$ of distributions $P$ supported on $B_\infty(r) \subset \mathbb{R}^d$.

**Proposition 4.** For the mean estimation problem, there exist universal constants $0 < c_\ell \leq c_u < \infty$ such that for all $\alpha \in [0, 1]$,

$$c_\ell \min \left\{ r, \frac{r \sqrt{d \log(2d)}}{\sqrt{n\alpha^2}} \right\} \leq \mathcal{M}_n(\theta(P_{\infty,r}), \|\cdot\|_\infty, \alpha) \leq c_u \min \left\{ r, \frac{r \sqrt{d \log(2d)}}{\sqrt{n\alpha^2}} \right\}.$$  

Proposition 4 provides a similar message to Proposition 3 on the loss of statistical efficiency. This is clearest from an example: let $X_i$ be random vectors bounded by 1 in $\ell_\infty$-norm. Then classical results on sub-Gaussian random variables (e.g., [11]) immediately imply that the standard non-private mean $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n X_i$ satisfies $\mathbb{E}[\|\hat{\theta} - \mathbb{E}[X]\|_\infty] \leq \sqrt{\log(2d)/n}$. Comparing this result to the rate $\sqrt{d \log(2d)/n}$ of Proposition 3, we again see the effective sample size reduction $n \mapsto n\alpha^2/d$.

Recently, there has been substantial interest in high-dimensional problems, in which the dimension $d$ is larger than the sample size $n$, but there is a low-dimensional latent structure that makes inference possible. (See the paper by Negahban et al. [44] for a general overview.) Accordingly, let us consider an idealized version of the high-dimensional mean estimation problem, in which we assume that $\theta(P) = \mathbb{E}[X] \in \mathbb{R}^d$ has (at most) one non-zero entry, so $\|\mathbb{E}[X]\|_0 \leq 1$. In the non-private case, estimation of such a $s$-sparse predictor in the squared $\ell_2$-norm is possible at rate $\mathbb{E}[\|\hat{\theta} - \mathbb{E}[X]\|_2^2] \leq s \log(d/s)/n$, so that the dimension $d$ can be exponentially larger than the sample size $n$. With this context, the next result exhibits that privacy can have a dramatic impact in the high-dimensional setting. Consider the family

$$P_{\infty,r}^s := \left\{ \text{distributions } P \text{ supported on } B_\infty(r) \subset \mathbb{R}^d \text{ with } \|\mathbb{E}[X]\|_0 \leq s \right\}.$$  

**Proposition 5.** For the 1-sparse mean estimation problem, there exist universal constants $0 < c_\ell \leq c_u < \infty$ such that for all $\alpha \in [0, 1]$,

$$c_\ell \min \left\{ r^2, \frac{r^2 d \log(2d)}{n\alpha^2} \right\} \leq \mathcal{M}_n(\theta(P_{\infty,r}^1), \|\cdot\|_2^2, \alpha) \leq c_u \min \left\{ r^2, \frac{r^2 d \log(2d)}{n\alpha^2} \right\}.$$  

See Section A.3 for a proof. From Proposition 5, it becomes clear that in locally private but non-interactive (2) settings, high dimensional estimation is effectively impossible.
4.2.2 Optimal mechanisms: attainability for mean estimation

In this section, we describe how to achieve matching upper bounds in Propositions 3 and 4 using simple and practical algorithms—namely, the “right” type of stochastic perturbation of the observations $X_i$ coupled with a standard mean estimator. We show the optimality of privatizing via the sampling strategies (27a) and (27b); interestingly, we also show that privatizing via Laplace perturbation is strictly sub-optimal. To give a private mechanism, we must specify the conditional distribution $Q$ satisfying $\alpha$-local differential privacy used to construct $Z$. In this case, given an observation $X_i$, we construct $Z_i$ by perturbing $X_i$ in such a way that $\mathbb{E}[Z_i \mid X_i = x] = x$. Each of the strategies (27a) and (27b) also requires a constant $B$, and we show how to choose $B$ for each strategy to satisfy the unbiasedness condition $\mathbb{E}[Z \mid X = x] = x$.

We begin with the mean estimation problem for distributions $P_{p,r}$ in Proposition 3 for which we use the sampling scheme (27a). That is, let $X = x \in \mathbb{R}^d$ satisfy $\|x\|_2 \leq \|x\|_p \leq r$. Then we construct the random vector $Z$ according to strategy (27a), where we set the initial vector $v = x$ in the sampling scheme. To achieve the unbiasedness condition $\mathbb{E}[Z \mid x] = x$, we set the bound

$$B = \frac{e^\alpha + 1}{e^\alpha - 1} \frac{d \sqrt{d} \Gamma\left(\frac{d-1}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)}$$

(see Appendix F.2 for a proof that $\mathbb{E}[Z \mid x] = x$ with this choice of $B$). Notably, the choice (29) implies $B \leq cr\sqrt{d}/\alpha$ for a universal constant $c < \infty$, since $d \sqrt{d} / \Gamma\left(\frac{d}{2} + 1\right) \lesssim \sqrt{d}$ and $e^\alpha - 1 = O(\alpha^2)$. As a consequence, generating each $Z_i$ by this perturbation strategy and using the mean estimator $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n Z_i$, the estimator $\hat{\theta}$ is unbiased for $\mathbb{E}[X]$ and satisfies

$$\mathbb{E} \left[ \|\hat{\theta} - \mathbb{E}[X]\|^2_2 \right] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(Z_i) \leq \frac{B^2}{n} \leq \frac{c r^2 d}{n \alpha^2}$$

for a universal constant $c$.

In Proposition 4, we consider the family $P_{\infty,r}$ of distributions supported on the $\ell_\infty$-ball of radius $r$. In our mechanism for attaining the upper bound, we use the sampling scheme (27b) to generate the private $Z_i$, so that for an observation $X = x \in \mathbb{R}^d$ with $\|x\|_\infty \leq r$, we resample $Z$ (from the initial vector $v = x$) according to strategy (27b). Again, we would like to guarantee the unbiasedness condition $\mathbb{E}[Z \mid X = x] = x$, for which we use a result of Duchi et al. [18]. In that paper, we show that taking

$$B = \frac{r \sqrt{d}}{\alpha}$$

(30)

for a (particular) universal constant $c$, we obtain the desired unbiasedness [18, Corollary 3]. Since the random variable $Z$ satisfies $Z \in \mathcal{B}_{\infty}(r)$ with probability 1, each coordinate $[Z]_j$ of $Z$ is sub-Gaussian. As a consequence, we obtain via standard bounds [11] that

$$\mathbb{E}[\|\hat{\theta} - \theta\|_\infty^2] \leq \frac{B^2 \log(2d)}{n} = \frac{c^2 r^2 d \log(2d)}{n \alpha^2}$$

for a universal constant $c$, proving the upper bound in Proposition 4.

To conclude this section, we note that the strategy of adding Laplacian noise to the vectors $X$ is sub-optimal. Indeed, consider the the family $P_{2,1}$ of distributions supported on $\mathcal{B}_{2}(1) \subset \mathbb{R}^d$ as in Proposition 3. To guarantee $\alpha$-differential privacy using independent Laplace noise vectors
for $x \in \mathbb{B}_2(1)$, we take $Z = x + W$ where $W \in \mathbb{R}^d$ has components $W_j$ that are independent and distributed as Laplace($\alpha/\sqrt{d}$). We have the following information-theoretic result: if the $Z_i$ are constructed via the Laplace noise mechanism,

$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \left\| \hat{\theta}(Z_1, \ldots, Z_n) - \mathbb{E}_P[X] \right\|_2^2 \right] \gtrsim \min \left\{ \frac{d^2}{n\alpha^2}, 1 \right\}. \tag{31}$$

See Appendix A.4 for the proof of this claim. The poorer dimension dependence exhibited by the Laplace mechanism (31) in comparison to Proposition 3 demonstrates that sampling mechanisms must be chosen carefully, as in the strategies (27a)–(27b), in order to obtain statistically optimal rates.

5 Bounds on multiple pairwise divergences: Assouad’s method

Thus far, we have seen how Le Cam’s method and Fano’s method, in the form of Theorem 2 and Corollary 4, can give sharp minimax rates for various problems. However, their application appears to be limited to the non-interactive channels, satisfying the simpler definition (2) of local privacy. In this section, we show that a privatized form of Assouad’s method (in the form of Lemma 1) can be used to sharp minimax rates in interactive settings. In particular, it can be applied when the loss is sufficiently “decomposable”, so that the coordinate-wise nature of the Assouad construction can be brought to bear. Concretely, we show an upper bound on a sum of paired KL-divergences, which combined with Assouad’s method provides sharp lower bounds for several problems, including multinomial probability estimation and nonparametric density estimation. Each of these problems can be characterized in terms of an effective dimension $d$, and our results (paralleling those of Section 4) show that the requirement of $\alpha$-local differential privacy causes a reduction in effective sample size from $n$ to $n\alpha^2/d$.

5.1 Variational bounds on paired divergences

For a fixed $d \in \mathbb{N}$, we consider collections of distributions that are indexed using the Boolean hypercube $\mathcal{V} = \{-1, 1\}^d$. For each $i \in [n]$ and $\nu \in \mathcal{V}$, we let the distribution $P_{\nu,i}$ be supported on the fixed set $\mathcal{X}$. For each $\nu \in \mathcal{V}$, we define the product distribution $P^n_\nu = \prod_{i=1}^n P_{\nu,i}$, and for each $j \in [d]$, we define the paired mixtures

$$P^n_{+j} = \frac{1}{2^{d-1}} \sum_{\nu: \nu_j = 1} P^n_\nu, \quad P^n_{-j} = \frac{1}{2^{d-1}} \sum_{\nu: \nu_j = -1} P^n_\nu, \quad \text{and} \quad P^n_{\pm j, i} = \frac{1}{2^{d-1}} \sum_{\nu: \nu_j = \pm 1} P_{\nu,i}. \tag{32}$$

(Note that $P^n_{+j}$ is not necessarily a product distribution.) Recalling the marginal channel (3), we may then define the marginal mixture distributions

$$M^n_{+j}(S) := \frac{1}{2^{d-1}} \sum_{\nu: \nu_j = 1} M^n_\nu(S) = \int Q^n(S \mid x_{1:n}) dP^n_{+j}(x_{1:n}) \quad \text{for} \ j = 1, \ldots, d,$$

with the distributions $M^n_{-j}$ defined in analogous manner. For a given pair of distributions $(M, M')$, we let $D^n_{\text{kl}}(M \parallel M') = D_{\text{kl}}(M || M') + D_{\text{kl}}(M' || M)$ denote the symmetrized KL-divergence.

Recalling the 1-ball of the supremum norm (24), we have the following theorem:
Theorem 3. Under the conditions of the previous paragraph, for any \( \alpha \)-locally differentially private channel \( Q \), we have

\[
\sum_{j=1}^{d} D_{\text{kl}}^Q (M_{n,j}^{+} \| M_{n,j}^{-}) \leq 2(e^\alpha - 1)^2 \sum_{i=1}^{n} \sup_{\gamma \in B_{\infty}(X)} \left( \int_X \gamma(x) dP_{+,i}(x) - dP_{-,i}(x) \right)^2.
\]

Theorem 3 generalizes Theorem 1, which corresponds to the special case \( d = 1 \). It has parallels with Theorem 2, in that taking the supremum outside the summation is essential to obtain sharp results. We provide the proof of Theorem 2 in Section 9.

Theorem 3 allows us to prove sharper lower bounds on the minimax risk. In particular, a combination of Pinsker’s and the Cauchy-Schwarz inequalities implies that

\[
\sum_{j=1}^{d} \| M_{n,j}^{+} - M_{n,j}^{-} \|_{\text{TV}} \leq \frac{1}{2} \sqrt{d \left( \sum_{j=1}^{d} D_{\text{kl}} (M_{n,j}^{+} \| M_{n,j}^{-}) + D_{\text{kl}} (M_{n,j}^{-} \| M_{n,j}^{+}) \right)}.
\]

Thus, in combination with the sharper Assouad inequality (11), whenever \( P_{\nu} \) induces a 2\( \delta \)-Hamming separation for \( \Phi \circ \rho \) we have

\[
\mathcal{M}_n(\theta(P), \Phi \circ \rho) \geq d\delta \left[ 1 - \left( \frac{1}{4d} \sum_{j=1}^{d} (D_{\text{kl}} (M_{n,j}^{+} \| M_{n,j}^{-}) + D_{\text{kl}} (M_{n,j}^{-} \| M_{n,j}^{+})) \right)^{\frac{1}{2}} \right].
\]

The combination of inequality (34) with Theorem 3 is the foundation for the remainder of this section: multinomial estimation in Section 5.2 and density estimation in Section 5.3.

5.2 Multinomial estimation under local privacy

For our first illustrative application of Theorem 3 we return to the original motivation for studying local privacy [50]: avoiding answer bias in surveys. Consider the probability simplex

\[
\Delta_d := \left\{ \theta \in \mathbb{R}^d \mid \theta \geq 0 \text{ and } \sum_{j=1}^{d} \theta_j = 1 \right\}.
\]

Any vector \( \theta \in \Delta_d \) specifies a multinomial random variable taking \( d \) states, in particular with probabilities \( P_{\theta}(X = j) = \theta_j \) for \( j \in \{1, \ldots, d\} \). Given a sample from this distribution, our goal is to estimate the probability vector \( \theta \). Warner [50] studied the Bernoulli variant of this problem (corresponding to the special case \( d = 2 \)), and proposed a simple privacy-preserving mechanism known as randomized response: for a given survey question, respondents provide a truthful answer with probability \( p > 1/2 \) and a lie with probability \( 1 - p \). Here we study the minimax rates of multinomial estimation under \( \alpha \)-local differential privacy (1).

5.2.1 Minimax rates of convergence for multinomial estimation

Our first result provides bounds on the minimax error measured in either the squared \( \ell_2 \)-norm or the \( \ell_1 \)-norm for (sequentially) interactive channels. The \( \ell_1 \)-norm norm is sometimes more appropriate for probability estimation due to its connections with total variation distance and testing.
Proposition 6. For the multinomial estimation problem, for any $\alpha$-locally differentially private channel $\mathcal{C}$, there exist universal constants $0 < c_{\ell} \leq c_u < 5$ such that for all $\alpha \in [0, 1]$,  
\begin{equation}
    c_{\ell} \min \left\{ 1, \frac{1}{\sqrt{n} \alpha^2}, \frac{d}{n \alpha^2} \right\} \leq \mathcal{M}_n (\Delta_d, \|\cdot\|_2^2, \alpha) \leq c_u \min \left\{ 1, \frac{d}{n \alpha^2} \right\},
\end{equation}
and  
\begin{equation}
    c_{\ell} \min \left\{ 1, \frac{d}{\sqrt{n} \alpha^2} \right\} \leq \mathcal{M}_n (\Delta_d, \|\cdot\|_1, \alpha) \leq c_u \min \left\{ 1, \frac{d}{\sqrt{n} \alpha^2} \right\}.
\end{equation}

See Appendix A for the proofs of the lower bounds in these results. We provide simple estimation strategies achieving the upper bounds in the next section.

As in the previous section, let us compare the given private rates to the classical rate in which there is no privacy. The standard maximum likelihood estimate (MLE) $\hat{\theta}$ sets $\hat{\theta}_j$ to the proportion of samples taking the value $j$; it has mean-squared error  
\begin{equation}
    E \left[ \| \hat{\theta} - \theta \|_2^2 \right] = \sum_{j=1}^{d} E \left[ (\hat{\theta}_j - \theta_j)^2 \right] = \frac{1}{n} \sum_{j=1}^{d} \theta_j (1 - \theta_j) \leq \frac{1}{n} \left( 1 - \frac{1}{d} \right) < \frac{1}{n}.
\end{equation}

An analogous calculation for the $\ell_1$-norm yields  
\begin{equation}
    E[\| \hat{\theta} - \theta \|_1] \leq \sum_{j=1}^{d} E[|\hat{\theta}_j - \theta_j|] \leq \sum_{j=1}^{d} \sqrt{\text{Var}(\hat{\theta}_j)} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{d} \sqrt{\theta_j (1 - \theta_j)} < \frac{\sqrt{d}}{\sqrt{n}}.
\end{equation}

Consequently, for estimation either in $\ell_1$ or $\ell_2$-norm, the effect of providing differential privacy at a level $\alpha$ causes a reduction in the effective sample size of $n \rightarrow n \alpha^2 / d$.

5.2.2 Optimal mechanisms: attainability for multinomial estimation

An interesting consequence of the lower bound (35) is the following: a minor variant of Warner’s randomized response strategy is an optimal mechanism. There are also other relatively simple estimation strategies that achieve convergence rate $d/n\alpha^2$; for instance, the Laplace perturbation approach [24] is another. Nonetheless, its ease of use, coupled with our optimality results, provide support for randomized response as a desirable method for private estimation of population probabilities.

Let us now demonstrate that these strategies attain the optimal rate of convergence. Since there is a bijection between multinomial observations $x \in \{1, \ldots, d\}$ and the $d$ standard basis vectors $e_1, \ldots, e_d \in \mathbb{R}^d$, we abuse notation and represent observations $x$ as either when designing estimation strategies. In randomized response, we construct the private vector $Z \in \{0, 1\}^d$ from a multinomial observation $x \in \{e_1, \ldots, e_d\}$ by sampling $d$ coordinates independently via the procedure  
\begin{equation}
    [Z]_j = \begin{cases} 
        x_j & \text{with probability } \frac{\exp(\alpha/2)}{1 + \exp(\alpha/2)}, \\
        1 - x_j & \text{with probability } \frac{1}{1 + \exp(\alpha/2)}.
    \end{cases}
\end{equation}

The distribution (37) is $\alpha$-differentially private: indeed, for $x, x' \in \Delta_d$ and any $z \in \{0, 1\}^d$, we have  
\begin{equation}
    \frac{Q(Z = z \mid x)}{Q(Z = z \mid x')} = \exp \left( \frac{\alpha}{2} \left( \|z - x\|_1 - \|z - x'\|_1 \right) \right) \in [\exp(-\alpha), \exp(\alpha)],
\end{equation}

20
where the triangle inequality guarantees that $||z - x||_1 - ||z - x'||_1 \leq ||x - x'||_1 \leq 2$. We now compute the expected value and variance of the random variables $Z$. Using the definition (37), we have

$$E[Z | x] = \frac{e^{\alpha/2}}{1 + e^{\alpha/2}}x + \frac{1}{1 + e^{\alpha/2}}(1 - x) = \frac{e^{\alpha/2} - 1}{e^{\alpha/2} + 1}x + \frac{1}{1 + e^{\alpha/2}}1.$$

Since the random variables $Z$ are Bernoulli, we have the variance bound

$$E[\|Z - E[Z]\|^2] < d/4 + 1 < d.$$

Letting $\Pi_\Delta_d$ denote the projection operator onto the simplex, we arrive at the natural estimator

$$\hat{\theta}_{part} := \frac{1}{n} \sum_{i=1}^n (Z_i - 1/(1 + e^{\alpha/2})) \frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1} \quad \text{and} \quad \hat{\theta} := \Pi_\Delta_d(\hat{\theta}_{part}). \quad (38)$$

The projection of $\hat{\theta}_{part}$ onto the probability simplex can be done in time linear in the dimension $d$ of the problem (10), so the estimator (35) is efficiently computable. Since projections onto convex sets are non-expansive, any pair of vectors in the simplex are at most $\ell_2$-distance $\sqrt{2}$ apart, and $E_\theta[\hat{\theta}_{part}] = \theta$ by construction, we have

$$E[\|\hat{\theta} - \theta\|^2] \leq \min \left\{2, E \left[\|\hat{\theta}_{part} - \theta\|^2\right]\right\} \leq \min \left\{2, \frac{d}{n} \left(\frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1}\right)^2 \right\} \lesssim \min \left\{1, \frac{d}{na^2}\right\}.$$ 

Similar results hold for the $\ell_1$-norm: using the same estimator, since Euclidean projections to the simplex are non-expansive for the $\ell_1$ distance, we have

$$E[\|\hat{\theta} - \theta\|_1] \leq \min \left\{1, \sum_{j=1}^d E \left[|\hat{\theta}_{part,j} - \theta_j|\right]\right\} \approx \min \left\{1, \sum_{j=1}^d \frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1} \frac{1}{n-1} \right\} \approx \min \left\{1, \frac{d}{\sqrt{na^2}}\right\}.$$ 

An analogous argument shows that the strategy of Dwork et al. [24], where we add Laplacian noise to the data, achieves the same convergence rates. In detail, the mechanism is based on forming a random vector $W \in \mathbb{R}^d$ has independent coordinates, each distributed as Laplace($\alpha/2$). The sum $x + W$ is $\alpha$-differentially private for $x \in \Delta_d$, and setting $\hat{\theta}_{part} = (1/n) \sum_{i=1}^n (X_i + W_i)$ and $\hat{\theta} = \Pi_\Delta_d(\hat{\theta}_{part})$ yields the rate $\min\{1, d/na^2\}$ as before.

### 5.3 Density estimation under local privacy

In this section, we show that the effects of local differential privacy are more severe for nonparametric density estimation: instead of just a multiplicative loss in the effective sample size as in previous sections, imposing local differential privacy leads to a different convergence rate. This result holds even though we solve a problem in which the function estimated and the samples themselves belong to compact spaces.

A probability density with respect to Lebesgue measure on the interval $[0, 1]$ is a non-negative integrable function $f : [0, 1] \rightarrow \mathbb{R}_+$ that is normalized ($\int_0^1 f(x)dx = 1$). The Sobolev classes [e.g., 49, 26] are subsets of densities that satisfy certain generalized smoothness conditions. More precisely, let $\{\varphi_j\}_{j=1}^\infty$ be any orthonormal basis for $L^2([0,1])$. Then any function $f \in L^2([0,1])$ can be expanded as a sum $\sum_{j=1}^\infty \theta_j \varphi_j$ in terms of the basis coefficients $\theta_j := \int f(x)\varphi_j(x)dx$. By Parseval’s theorem, we are guaranteed that $\{\theta_j\}_{j=1}^\infty \in \ell^2(\mathbb{N})$. The Sobolev space $\mathcal{F}_\theta[C]$ is obtained by enforcing a particular decay rate on the basis coefficients, as formalized in the following:
Definition 2 (Elliptical Sobolev space). For a given orthonormal basis \( \{ \varphi_j \} \) of \( L^2([0,1]) \), smoothness parameter \( \beta > 1/2 \) and radius \( C \), the Sobolev class of order \( \beta \) is given by
\[
F_\beta[C] := \left\{ f \in L^2([0,1]) \mid f = \sum_{j=1}^{\infty} \theta_j \varphi_j \text{ such that } \sum_{j=1}^{\infty} j^{2\beta} \theta_j^2 \leq C^2 \right\}.
\]

If we choose the trigonometric basis as our orthonormal basis, membership in the class \( F_\beta[C] \) corresponds to smoothness constraints on the derivatives of \( f \). More precisely, for \( j \in \mathbb{N} \), consider the orthonormal basis for \( L^2([0,1]) \) of trigonometric functions:
\[
\varphi_0(t) = 1, \quad \varphi_{2j}(t) = \sqrt{2} \cos(2\pi j t), \quad \varphi_{2j+1}(t) = \sqrt{2} \sin(2\pi j t).
\]
(39)

Let \( f \) be a \( \beta \)-times almost everywhere differentiable function for which \( |f^{(\beta)}(x)| \leq C \) for almost every \( x \in [0,1] \) satisfying \( f^{(k)}(0) = f^{(k)}(1) \) for \( k \leq \beta - 1 \). Then, uniformly over all such \( f \), there is a universal constant \( c \leq 2 \) such that that \( f \in F_\beta[cC] \) (see, for instance, [49, Lemma A.3]).

Now suppose our goal is to estimate a density function \( f \in F_\beta[C] \) and that quality is measured in terms of the integrated squared error (squared \( L^2[0,1] \)-norm)
\[
\| \hat{f} - f \|_2^2 := \int_0^1 (\hat{f}(x) - f(x))^2 \, dx.
\]
In the non-private setting, it is well-known [53, 52, 49] that the minimax risk in this squared norm scales as
\[
\mathcal{M}_n \left( F_\beta, \| \cdot \|_2^2, \infty \right) \asymp n^{-\frac{2\beta}{2\beta+1}}.
\]
(40)
The goal of this section is to understand how this minimax rate changes when we add an \( \alpha \)-privacy constraint to the problem. Our main result is to demonstrate that the classical rate (40) is no longer attainable when we require \( \alpha \)-local differential privacy. In particular, we prove a lower bound that is substantially larger. In Sections 5.3.2 and 5.3.3, we show how to achieve this lower bound using histogram and orthogonal series estimators.

5.3.1 Lower bounds on density estimation

We begin by giving our main lower bound on the minimax rate of estimation of densities when observations from the density are differentially private. We provide the proof of the following proposition in Section C.1.

Proposition 7. Consider the class of densities \( F_\beta \) defined using the trigonometric basis (39). There exists a constant \( c_\beta > 0 \) such that for any \( \alpha \)-locally differentially private channel [1] with \( \alpha \in [0,1] \), the private minimax risk has lower bound
\[
\mathcal{M}_n \left( F_\beta[1], \| \cdot \|_2^2, \alpha \right) \geq c_\beta (na^2)^{-\frac{2\beta}{2\beta+2}}.
\]
(41)
The most important feature of the lower bound (41) is that it involves a different polynomial exponent than the classical minimax rate (40). Whereas the exponent in classical case (40) is \( 2\beta/(2\beta+1) \), it reduces to \( 2\beta/(2\beta+2) \) in the locally private setting. For example, when we estimate Lipschitz densities (\( \beta = 1 \)), the rate degrades from \( n^{-2/3} \) to \( n^{-1/2} \). Moreover, this degradation occurs even though the sample is drawn from a compact space and the set \( F_\beta \) is also compact.
Interestingly, no estimator based on Laplace (or exponential) perturbation of the observations \( X_i \) themselves can attain the rate of convergence (11). This fact follows from results of Carroll and Hall [12] on nonparametric deconvolution. They show that if observations \( X_i \) are perturbed by additive noise \( W \), where the characteristic function \( \phi_W \) of the additive noise has tails behaving as \( |\phi_W(t)| = O(|t|^{-a}) \) for some \( a > 0 \), then no estimator can deconvolve \( X + W \) and attain a rate of convergence better than \( n^{-2/3+2a+1} \). Since the Laplace distribution’s characteristic function has tails decaying as \( t^{-2} \), no estimator based on the Laplace mechanism (applied directly to the observations) can attain rate of convergence better than \( n^{-2/3+5} \). In order to attain the lower bound (11), we must thus study alternative privacy mechanisms.

### 5.3.2 Achievability by histogram estimators

We now turn to the mean-squared errors achieved by specific practical schemes, beginning with the special case of Lipschitz density functions (\( \beta = 1 \)). In this special case, it suffices to consider a private version of a classical histogram estimate. For a fixed positive integer \( k \in \mathbb{N} \), let \( \{X_j\}_{j=1}^k \) denote the partition of \( \mathcal{X} = [0, 1] \) into the intervals

\[
X_j = [(j-1)/k, j/k) \quad \text{for } j = 1, 2, \ldots, k-1, \quad \text{and } X_k = [(k-1)/k, 1].
\]

Any histogram estimate of the density based on these \( k \) bins can be specified by a vector \( \theta \in k\Delta_k \), where we recall \( \Delta_k \subset \mathbb{R}_k^k \) is the probability simplex. Letting \( 1_E \) denote the characteristic (indicator) function of the set \( E \), any such vector \( \theta \in \mathbb{R}_k \) defines a density estimate via the sum

\[
f_\theta := \sum_{j=1}^k \theta_j 1_{X_j}.
\]

Let us now describe a mechanism that guarantees \( \alpha \)-local differential privacy. Given a sample \( \{X_1, \ldots, X_n\} \) from the distribution \( f \), consider the vectors

\[
Z_i := e_k(X_i) + W_i, \quad \text{for } i = 1, 2, \ldots, n,
\]

where \( e_k(X_i) \in \Delta_k \) is a \( k \)-vector with \( j^{th} \) entry equal to one if \( X_i \in X_j \) and zeroes in all other entries, and \( W_i \) is a random vector with i.i.d. Laplace(\( \alpha/2 \)) entries. The variables \( \{Z_i\}_{i=1}^n \) so-defined are \( \alpha \)-locally differentially private for \( \{X_i\}_{i=1}^n \). Using these private variables, we form the density estimate \( \hat{f} := f_\theta = \sum_{j=1}^k \theta_j 1_{X_j} \) based on the vector \( \theta := \Pi_k \left( \frac{1}{n} \sum_{i=1}^n Z_i \right) \), where \( \Pi_k \) denotes the Euclidean projection operator onto the set \( k\Delta_k \). By construction, we have \( \hat{f} \geq 0 \) and \( \int_0^1 \hat{f}(x)dx = 1 \), so \( \hat{f} \) is a valid density estimate. The following result characterizes its mean-squared estimation error.

**Proposition 8.** Consider the estimate \( \hat{f} \) based on \( k = (n\alpha^2)^{1/4} \) bins in the histogram. For any 1-Lipschitz density \( f : [0, 1] \to \mathbb{R}_+ \), the MSE is upper bounded as

\[
\mathbb{E}_f \left[ \left\| \hat{f} - f \right\|_2^2 \right] \leq 5(\alpha^2 n)^{-\frac{1}{4}} + \sqrt{\alpha n}^{-3/4}.
\]

For any fixed \( \alpha > 0 \), the first term in the bound (43) dominates, and the \( O((\alpha^2 n)^{-\frac{1}{4}}) \) rate matches the minimax lower bound (11) in the case \( \beta = 1 \). Consequently, the privatized histogram estimator is minimax-optimal for Lipschitz densities. This result provides the private analog of the classical (non-private) result that histogram estimators are minimax-optimal for Lipschitz densities. See Section C.2 for a proof of Proposition 8. We remark in passing that a randomized response scheme parallel to that of Section 5.2.2 achieves the same rate of convergence, showing that this classical mechanism is again an optimal scheme.
5.3.3 Achievability by orthogonal projection estimators

For higher degrees of smoothness ($\beta > 1$), standard histogram estimators no longer achieve optimal rates in the classical setting \[47\]. Accordingly, we now turn to developing estimators based on orthogonal series expansion, and show that even in the setting of local privacy, they can achieve the lower bound \[41\] for all orders of smoothness $\beta \geq 1$.

Recall the elliptical Sobolev space (Definition \[2\]), in which a function $f$ is represented in terms of its basis expansion $f = \sum_{j=1}^{\infty} \theta_j \varphi_j$. This representation underlies the orthonormal series estimator, which operates as follows. Given a sample $X_{1:n}$ drawn i.i.d. according to a density $f \in L^2([0,1])$, first compute the empirical basis coefficients

$$\hat{\theta}_j = \frac{1}{n} \sum_{i=1}^{n} \varphi_j(X_i) \quad \text{for } j \in \{1, \ldots, k\},$$

(44)

where the value $k \in \mathbb{N}$ is chosen either a priori based on known properties of the estimation problem or adaptively, for example, using cross-validation \[26, 49\]. Using these empirical coefficients, the density estimate is $\hat{f} = \sum_{j=1}^{k} \hat{\theta}_j \varphi_j$.

In our local privacy setting, we consider a mechanism that, instead of releasing the vector of coefficients $(\varphi_1(X_i), \ldots, \varphi_k(X_i))$ for each data point, employs a random vector $Z_i = (Z_{i,1}, \ldots, Z_{i,k})$ with the property that $\mathbb{E}[Z_{i,j} \mid X_i] = \varphi_j(X_i)$ for each $j = 1, 2, \ldots, k$. We assume the basis functions are $B_0$-uniformly bounded, that is, $\sup_j \sup_x |\varphi_j(x)| \leq B_0 < \infty$. This boundedness condition holds for many standard bases, including the trigonometric basis \[39\] that underlies the classical Sobolev classes and the Walsh basis. We generate the random variables from the vector $v \in \mathbb{R}^k$ defined by $v_j = \varphi_j(X)$ in the hypercube-based sampling scheme \[27b\], where we assume that the outer bound $B > B_0$. With this sampling strategy, iteration of expectation yields

$$\mathbb{E}[Z_j \mid X = x] = c_k \frac{B}{B_0 \sqrt{k}} \left( \frac{e^\alpha}{e^\alpha + 1} - \frac{1}{e^\alpha + 1} \right) \varphi_j(x),$$

(45)

where $c_k > 0$ is a constant (which is bounded independently of $k$). Consequently, it suffices to take $B = O(B_0 \sqrt{k}/\alpha)$ to guarantee the unbiasedness condition $\mathbb{E}[Z_{i,j} \mid X_i] = \varphi_j(X_i)$.

Overall, the privacy mechanism and estimator perform the following steps:

- given a data point $X_i$, set the vector $v = [\varphi_j(X_i)]_{j=1}^{k}$
- sample $Z_i$ according to the strategy \[27b\], starting from the vector $v$ and using the bound $B = B_0 \sqrt{k}/c_k (e^\alpha - 1)$.
- compute the density estimate

$$\hat{f} := \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} Z_{i,j} \varphi_j,$$

(46)

The resulting estimate enjoys the following guarantee, whose proof we provide in Section C.3.

**Proposition 9.** Let $\{\varphi_j\}$ be a $B_0$-uniformly bounded orthonormal basis for $L^2([0,1])$. There exists a constant $c$ (depending only on $C$ and $B_0$) such that, for any $f$ in the Sobolev space $\mathcal{F}_\beta[C]$, the estimator \[40\] with $k = (n\alpha^2)^{1/(2\beta+2)}$ has MSE upper bounded as

$$\mathbb{E}_f \left[ \|f - \hat{f}\|_2^2 \right] \leq c \left( n\alpha^2 \right)^{\frac{2\beta}{2\beta+2}}.$$

(47)

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Propositions 8 and 9 make clear that the minimax lower bound (41) is sharp, as claimed. We have thus attained a variant of the known minimax density estimation rates (40) but with a polynomially worse sample complexity as soon as we require local differential privacy.

Before concluding our exposition, we make a few remarks on other potential density estimators. Our orthogonal series estimator (46) and sampling scheme (45), while similar in spirit to that proposed by Wasserman and Zhou [51, Sec. 6], is different in that it is locally private and requires a different noise strategy to obtain both $\alpha$-local privacy and the optimal convergence rate. Lastly, similarly to our remarks on the insufficiency of standard Laplace noise addition for mean estimation, it is worth noting that density estimators that are based on orthogonal series and Laplace perturbation are sub-optimal: they can achieve (at best) rates of $(na^2)^{-\frac{2\beta}{2\beta+3}}$. This rate is polynomially worse than the sharp result provided by Proposition 9. Again, we see that appropriately chosen noise mechanisms are crucial for obtaining optimal results.

6 Comparison to related work

There has been a substantial amount of work in developing differentially private mechanisms, both in local and non-local settings, and a number of authors have attempted to characterize optimal mechanisms. For example, Kasiviswanathan et al. [37], working within a local differential privacy setting, study Probably-Approximately-Correct (PAC) learning problems and show that the statistical query model [38] and local learning are equivalent up to polynomial changes in the sample size. In our work, we are concerned with a finer-grained assessment of inferential procedures—that of rates of convergence of procedures and their optimality. In the remainder of this section, we discuss further connections of our work to previous research on optimality, global (non-local) differential privacy, as well as error-in-variables models.

6.1 Sample versus population estimation

The standard definition of differential privacy, due to Dwork et al. [24], is somewhat less restrictive than the local privacy considered here. In particular, a conditional distribution $Q$ with output space $Z$ is $\alpha$-differentially private if

$$\sup \left\{ \frac{Q(S \mid x_{1:n})}{Q(S' \mid x'_{1:n})} \mid x_i, x'_i \in X, S \in \sigma(Z), d_{\text{ham}}(x_{1:n}, x'_{1:n}) \leq 1 \right\} \leq \exp(\alpha),$$

(48)

where $d_{\text{ham}}$ denotes the Hamming distance between sets. Local privacy as previously defined (1), and studied by Warner [50], Evfimievski et al. [27] and others, is more stringent.

Several researchers have considered quantities similar to our minimax criteria under local (2) or non-local (48) differential privacy [6, 33, 35, 17]. However, the objective has been substantially different from ours: instead of bounding errors based on population-based quantities, they provide bounds in which the data are assumed to be held fixed. More precisely, let $\theta : X^n \to \Theta$ denote an estimator, and let $\theta(x_{1:n})$ be a sample quantity based on $x_1, \ldots, x_n$. Prior work is based on conditional minimax risks of the form

$$\mathcal{M}_n^{\text{cond}}(\theta(X), \Phi \circ \rho, \alpha) := \inf_{\tilde{\theta}} \sup_{x_{1:n} \in X^n} \mathbb{E}_Q \left[ \Phi(\rho(\theta(x_{1:n}), \tilde{\theta})) \mid X_{1:n} = x_{1:n} \right],$$

(49)

where $\tilde{\theta}$ is drawn according to $Q(\cdot \mid x_{1:n})$, the infimum is taken over all $\alpha$-differentially private channels $Q$, and the supremum is taken over all possible samples of size $n$. Note that the only
randomness in this conditional minimax risk is provided by the channel; the data are held fixed, so there is no randomness from an underlying data-generating distribution. A partial list of papers that use definitions of this type include Beimel et al. [3, Section 2.4], Hardt and Talwar [35, Definition 2.4 in the full paper], Hall et al. [33, Section 3], and De [17].

In general, the conditional (49) and population minimax risk (5) can differ substantially, and such differences are precisely those addressed by the theory of statistical inference. The goal of inference is to draw conclusions about the population-based quantity \( \theta(P) \) based on the sample. Moreover, lower bounds on the conditional minimax risk (49) do not imply bounds on the rate of estimation for the population \( \theta(P) \). Since the conditional minimax risk (49) involves a supremum over all possible samples \( x \in \mathcal{X} \), the opposite is usually true: population risks provide lower bounds on the conditional minimax risk, as we show below.

An illustrative example is useful to understand the differences. Consider estimation of the mean \( \theta \) of a normal distribution with known standard deviation \( \sigma \). The conditional minimax risk of the mean statistic is

\[
Q(\hat{\theta} - \theta \mid x_{1:n}) > \delta \mid x_{1:n}) \leq \frac{1}{2} \quad \text{for all samples } x_{1:n} \in \mathbb{R}^n.
\]

Fix \( N(\delta) \in \mathbb{N} \) and choose points \( \theta_{\nu}, \nu \in [N(\delta)] \), such that all are at least \( 2\delta \)-separated, that is, \( |\theta_{\nu} - \theta_{\nu'}| \geq 2\delta \) for all \( \nu \neq \nu' \). Then the sets \( \{\theta \in \mathbb{R} \mid |\theta - \theta_{\nu}| \leq \delta \} \) are all disjoint. As a consequence, we have for any samples \( x_{1:n} \) and \( x_{1:n}^{\nu} \), indexed by \( \nu \), with \( d_{\text{ham}}(x_{1:n}, x_{1:n}^{\nu}) \leq 1 \) that

\[
Q(\exists \nu \in V \text{ s.t. } |\hat{\theta} - \theta_{\nu}| \leq \delta \mid x_{1:n}) = \sum_{\nu=1}^{N(\delta)} Q(\hat{\theta} - \theta_{\nu} \mid x_{1:n}) \geq e^{-\alpha} \sum_{\nu=1}^{N(\delta)} Q(\hat{\theta} - \theta_{\nu} \mid x_{1:n}^{\nu}).
\]

We may take each sample \( x_{1:n}^{\nu} \) such that \( \theta(x_{1:n}^{\nu}) = \frac{1}{n} \sum_{i=1}^{n} x_{i}^{\nu} = \theta_{\nu} \) (for example, for each \( \nu \in [N(\delta)] \) set \( x_{1}^{\nu} = n\theta_{\nu} - \sum_{i=1}^{n} x_{i} \)) and by assumption, we have

\[
1 \geq Q(\exists \nu \in V \text{ s.t. } |\hat{\theta} - \theta_{\nu}| \leq \delta \mid x_{1:n}) \geq e^{-\alpha} N(\delta) \frac{1}{2}.
\]

Taking \( N(\delta) > 2e^\alpha \) yields a contradiction. Since our argument applies to an arbitrary \( \delta > 0 \), the claim follows.

There are variations on this result. For instance, even if the output of the mean estimator is restricted to \([-1, 1]\), the conditional minimax risk remains constant. Similar arguments apply to weakenings of differential privacy (e.g., \( \delta \)-approximate \( \alpha \)-differential privacy [23]). Thus, we see that conditional and population risks are very different quantities.

More generally, the population minimax risk usually lower bounds the conditional minimax risk. Suppose we measure minimax risks in some given metric \( \rho \) (so the loss \( \Phi(t) = t \)). Let \( \hat{\theta} \) be any
estimator based on the original sample \(X_{1:n}\), and let \(\hat{\theta}\) be any estimator based on the privatized sample. We then have the following series of inequalities:

\[
\mathbb{E}_{Q,P}[\rho(\theta(P), \hat{\theta})] \leq \mathbb{E}_{Q,P}[\rho(\theta(P), \tilde{\theta})] + \mathbb{E}_{Q,P}[\rho(\tilde{\theta}, \hat{\theta})] \\
\leq \mathbb{E}[\rho(\theta(P), \tilde{\theta})] + \sup_{x_{1:n} \in \mathcal{X}^n} \mathbb{E}_{Q,P}[\rho(\tilde{\theta}(x_{1:n}), \tilde{\theta})] \mid X_{1:n} = x_{1:n}].
\]

(50)

The population minimax risk \([5]\) thus lower bounds the conditional minimax risk \([49]\) via

\[
\mathcal{M}_n^{\text{cond}}(\tilde{\theta}(\mathcal{X}), \rho, \alpha) \geq \mathcal{M}_n(\theta(P), \rho, \alpha) - \mathbb{E}[\rho(\theta(P), \tilde{\theta})].
\]

In particular, if there exists an estimator \(\tilde{\theta}\) based on the original (non-private data) such that \(\mathbb{E}[\rho(\theta(P), \tilde{\theta})] \leq \frac{1}{2} \mathcal{M}_n(\theta(P), \rho, \alpha)\) we are guaranteed that

\[
\mathcal{M}_n^{\text{cond}}(\tilde{\theta}(\mathcal{X}), \rho, \alpha) \geq \frac{1}{2} \mathcal{M}_n(\theta(P), \rho, \alpha),
\]

showing that the conditional minimax risk is lower bounded by a constant multiple of the population minimax risk. This lower bound holds for each of the examples in Sections \([3, 5]\) lower bounds on the \(\alpha\)-private population minimax \([5]\) risk are strictly stronger than lower bounds on the conditional minimax risk.

To illustrate one application of the lower bound \([50]\), consider the estimation of the sample mean of a data set \(x_{1:n} \in \{0,1\}^n\) under \(\alpha\)-local privacy. This problem has been considered before; for instance, Beimel et al. \([6]\) study distributed computational protocols for this problem. In Theorem 2 of their work, they show that if a protocol has \(\ell\) interactive rounds of communication, the squared error in estimating the sample mean \((1/n) \sum_{i=1}^n x_i\) is \(\Omega(1/(n\alpha^2 \ell^2))\). The standard mean estimator \(\tilde{\theta}(x_{1:n}) = (1/n) \sum_{i=1}^n x_i\) has error \(\mathbb{E}[|\theta(x_{1:n}) - \theta|] \leq n^{-\frac{1}{2}}\). Consequently, if we combine the lower bound \([50]\) with Proposition \([1]\) we find that

\[
c \frac{1}{\sqrt{n\alpha^2}} - \frac{1}{\sqrt{n}} \leq \mathcal{M}_n(\theta(P), |\cdot|, \alpha) - \sup_{\theta \in [\theta_0, \theta_1]} \mathbb{E}[|\tilde{\theta}(x_{1:n}) - \theta|] \leq \mathcal{M}_n^{\text{cond}}(\theta(\{-1,1\}), |\cdot|, \alpha).
\]

for some universal constant \(c > 0\). Thus, a corollary of our results is such an \(\Omega(1/(n\alpha^2))\) lower bound on the conditional minimax risk for mean estimation, allowing for sequential interactivity but not multiple “rounds.” An inspection of Beimel et al.’s proof technique \([1]\) Section 4.2] shows that their lower bound also implies a lower bound of \(1/n\alpha^2\) for estimation of the population mean \(\mathbb{E}[X]\) in one dimension in non-interactive \([2]\) settings; it is, however, unclear how to extend their technique to more general problems or interactive settings.

### 6.2 Local versus non-local privacy

It is also worthwhile to make some comparisons to work on non-local forms of differential privacy, mainly to understand the differences between local and global forms of privacy. Chaudhuri and Hsu \([14]\) provide lower bounds for estimation of certain one-dimensional statistics based on a two-point family of problems. Their techniques differ from those of the current paper, and do not appear to provide bounds on the statistic being estimated, but rather one that is near to it. Beimel et al. \([7]\) provide some bounds on sample complexity in the “probably approximate correct” (PAC)
framework of learning theory, though extensions to other inferential tasks are unclear. Other work on non-local privacy [e.g., 33, 15, 48] shows that for various types of estimation problems, adding Laplacian noise leads to degraded convergence rates in at most lower-order terms. In contrast, our work shows that in the setting of local privacy, the Laplacian mechanism can be highly sub-optimal, meaning that it actually degrades the convergence rate.

To understand convergence rates for non-local privacy, let us return to estimation of a multinomial distribution in $\Delta_d$, based on observations $X_i \in \{e_j\}_{j=1}^d$. In this case, adding a noise vector $W \in \mathbb{R}^d$ with i.i.d. entries distributed as Laplace($\alpha n$) provides differential privacy [23]; the associated mean-squared error is at most

$$E_\theta\left[\frac{1}{n} \sum_{i=1}^n X_i + W - \theta\right]^2 = E\left[\left\|\frac{1}{n} \sum_{i=1}^n X_i - \theta\right\|^2_2\right] + E[\|W\|_2^2] \leq \frac{1}{n} + \frac{d}{n^2 \alpha^2}.$$

In particular, in the asymptotic regime $n \gg d$, there is no penalty from providing differential privacy except in higher-order terms. Similar results hold for histogram estimation [33], classification problems [15], and classical point estimation problems [48]; in this sense, local and global forms of differential privacy can be rather different.

6.3 Error-in-variables models

As a final remark on related work, we touch briefly on errors-in-variables models [13, 31], which have been the subject of extensive study. In such problems, one observes a corrupted version $Z_i$ of the true covariate $X_i$. Privacy analysis is one of the few settings in which it is possible to precisely know the conditional distribution $Q(\cdot | X_i)$. However, the mechanisms that are optimal from our analysis—in particular, those in strategies (27a) and (27b)—are more complicated than adding noise directly to the covariates, which leads to complications. Known (statistically) efficient error-in-variables estimation procedures often require either solving certain integrals or estimating equations, or solving non-convex optimization problems [e.g., 39, 41]. Some recent work [40] shows that certain types of non-convex programs arising from errors-in-variables can be solved efficiently; extending these types of results to private settings could be interesting. In density estimation, as noted, corrupted observations lead to nonparametric deconvolution problems that appear harder than estimation under privacy constraints. (See Section 5.3.1 for our earlier discussion.) Further investigation of computationally efficient procedures for nonlinear error-in-variables models for privacy-preservation is an interesting direction for future research.

7 Proof of Theorem 1 and related results

We now turn to the proofs of our results, beginning with Theorem 1 and related results. In all cases, we defer the proofs of more technical lemmas to the appendices.

7.1 Proof of Theorem 1

Observe that $M_1$ and $M_2$ are absolutely continuous with respect to one another, and there is a measure $\mu$ with respect to which they have densities $m_1$ and $m_2$, respectively. The channel probabilities $Q(\cdot | x)$ and $Q(\cdot | x')$ are likewise absolutely continuous, so that we may assume they
have densities \( q(\cdot \mid x) \) and write \( m_i(z) = \int q(z \mid x) dP_i(x) \). In terms of these densities, we have

\[
D_{kl}(M_1 \| M_2) + D_{kl}(M_2 \| M_1) = \int m_1(z) \log \frac{m_1(z)}{m_2(z)} d\mu(z) + \int m_2(z) \log \frac{m_2(z)}{m_1(z)} d\mu(z)
= \int (m_1(z) - m_2(z)) \log \frac{m_1(z)}{m_2(z)}
\]

Consequently, we need to bound both the difference \( m_1 - m_2 \) and the log difference of the marginal densities. The following two auxiliary lemmas are useful:

**Lemma 3.** For any \( \alpha \)-locally differentially private conditional, we have

\[
|m_1(z) - m_2(z)| \leq c_\alpha \inf_x q(z \mid x) (e^\alpha - 1) \|P_1 - P_2\|_{TV},
\]

where \( c_\alpha = \min\{2, e^\alpha\} \).

**Lemma 4.** Let \( a, b \in \mathbb{R}_+ \). Then

\[
|\log a| \leq \frac{|a - b|}{\min(a, b)}.
\]

We prove these two results at the end of this section and in Appendix [F.3] respectively.

With the lemmas in hand, let us now complete the proof of the theorem. From Lemma 4, the log ratio is bounded as

\[
\left| \log \frac{m_1(z)}{m_2(z)} \right| \leq \frac{|m_1(z) - m_2(z)|}{\min\{m_1(z), m_2(z)\}}.
\]

Applying Lemma 3 to the numerator yields

\[
\left| \log \frac{m_1(z)}{m_2(z)} \right| \leq c_\alpha (e^\alpha - 1) \|P_1 - P_2\|_{TV} \inf_x q(z \mid x) \leq c_\alpha (e^\alpha - 1) \frac{\|P_1 - P_2\|_{TV} \inf_x q(z \mid x)}{\min\{m_1(z), m_2(z)\}},
\]

where the final step uses the inequality \( \min\{m_1(z), m_2(z)\} \geq \inf_x q(z \mid x) \). Putting together the pieces leads to the bound

\[
\left| \log \frac{m_1(z)}{m_2(z)} \right| \leq c_\alpha (e^\alpha - 1) \|P_1 - P_2\|_{TV}.
\]

Combining with inequality (51) yields

\[
D_{kl}(M_1 \| M_2) + D_{kl}(M_2 \| M_1) \leq c_\alpha^2 (e^\alpha - 1)^2 \|P_1 - P_2\|_{TV}^2 \int \inf_x q(z \mid x) d\mu(z).
\]

The final integral is at most one, which completes the proof of the main theorem.

It remains to prove Lemma 3. For any \( z \in Z \), we have

\[
m_1(z) - m_2(z) = \int_{X} q(z \mid x) [dP_1(x) - dP_2(x)]
= \int_{X} q(z \mid x) [dP_1(x) - dP_2(x)]_+ + \int_{X} q(z \mid x) [dP_1(x) - dP_2(x)]_-
\leq \sup_{x \in X} q(z \mid x) \int_X [dP_1(x) - dP_2(x)]_+ + \inf_{x \in X} q(z \mid x) \int_X [dP_1(x) - dP_2(x)]_-
= \left( \sup_{x \in X} q(z \mid x) - \inf_{x \in X} q(z \mid x) \right) \int_X [dP_1(x) - dP_2(x)]_+.
\]
By definition of the total variation norm, we have \( \int [dP_1 - dP_2]_+ = \|P_1 - P_2\|_{TV} \), and hence
\[
|m_1(z) - m_2(z)| \leq \sup_{x, x'} |q(z | x) - q(z | x')| \|P_1 - P_2\|_{TV} .
\] (52)

For any \( \hat{x} \in X \), we may add and subtract \( q(z | \hat{x}) \) from the quantity inside the supremum, which implies that
\[
\sup_{x, x'} |q(z | x) - q(z | x')| = \inf_{\hat{x}} \sup_{x, x'} |q(z | x) - q(z | \hat{x})| + q(z | \hat{x}) - q(z | x')| \\
\leq 2 \inf_{\hat{x}} \sup_{x} |q(z | x) - q(z | \hat{x})| \\
= 2 \inf_{\hat{x}} q(z | \hat{x}) \sup_{x} \left| \frac{q(z | x)}{q(z | \hat{x})} - 1 \right|.
\]

Similarly, we have for any \( x, x' \)
\[
|q(z | x) - q(z | x')| = q(z | x') \left| \frac{q(z | x)}{q(z | x')} - 1 \right| \leq e^\alpha \inf_{\hat{x}} q(z | \hat{x}) \left| \frac{q(z | x)}{q(z | x')} - 1 \right|.
\]

Since for any choice of \( x, \hat{x} \), we have \( q(z | x)/q(z | \hat{x}) \in [e^{-\alpha}, e^\alpha] \), we find that (since \( e^\alpha - 1 \geq 1 - e^{-\alpha} \))
\[
\sup_{x, x'} |q(z | x) - q(z | x')| \leq \min\{2, e^\alpha\} \inf_{x} q(z | x) (e^\alpha - 1).
\]

Combining with the earlier inequality (52) yields the claim (51).

### 7.2 Proof of Corollary 1

Let us recall the definition of the induced marginal distribution (3), given by
\[
M_{\nu}(S) = \int X Q(S | x_{1:n}) dP^n_{\nu}(x_{1:n}) \quad \text{for } S \in \sigma(Z^n).
\]

For each \( i = 2, \ldots, n \), we let \( M_{\nu,i}(\cdot | z_1 = z_1, \ldots, z_{i-1} = z_{i-1}) = M_{\nu,i}(\cdot | z_{1:i-1}) \) denote the (marginal over \( X_i \)) distribution of the variable \( Z_i \) conditioned on \( Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1} \). In addition, use the shorthand notation
\[
D_{kl}(M_{\nu,i}||M'_{\nu,i}) := \int_{\mathbb{Z}^{i-1}} D_{kl}(M_{\nu,i}(\cdot | z_{1:i-1})||M'_{\nu,i}(\cdot | z_{1:i-1})) dM'_{\nu}\mu(z_{1}, \ldots, z_{i-1})
\]

To denote the integrated KL divergence of the conditional distributions on the \( Z_i \). By the chain-rule for KL divergences [32, Chapter 5.3], we obtain
\[
D_{kl}(M^n_{\nu}||M^n_{\nu}) = \sum_{i=1}^{n} D_{kl}(M_{\nu,i}||M'_{\nu,i}).
\]

By assumption (11), the distribution \( Q_{i}(\cdot | X_i, Z_{1:i-1}) \) on \( Z_i \) is \( \alpha \)-differentially private for the sample \( X_i \). As a consequence, if we let \( P_{\nu,i}(\cdot | Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}) \) denote the conditional
By construction, we have the chain rule and Theorem 1 we obtain

\[ D_{\text{kl}}(M_\nu^n||M_{\nu'}^n) = \sum_{i=1}^n \int_{z_{i-1}}^n D_{\text{kl}}(M_{\nu,i}(\cdot|z_{1:i-1})|M_{\nu',i}(\cdot|z_{1:i-1})) \ dM_{\nu,i}^{-1}(z_{1:i-1}) \]

\[ \leq \sum_{i=1}^n 4(e^\alpha - 1)^2 \int_{z_{i-1}}^n \|P_{\nu,i}(\cdot|z_{1:i-1}) - P_{\nu',i}(\cdot|z_{1:i-1})\|_{\text{TV}}^2 \ dM_{\nu,i}^{-1}(z_{1},...,z_{i-1}). \]

By the construction of our sampling scheme, the random variables \( X_i \) are conditionally independent given \( V = \nu \); thus the distribution \( P_{\nu,i}(\cdot|z_{1:i-1}) = P_{\nu,i} \), where \( P_{\nu,i} \) denotes the distribution of \( X_i \) conditioned on \( V = \nu \). Consequently, we have

\[ \|P_{\nu,i}(\cdot|z_{1:i-1}) - P_{\nu',i}(\cdot|z_{1:i-1})\|_{\text{TV}} = \|P_{\nu,i} - P_{\nu',i}\|_{\text{TV}}, \]

which gives the claimed result.

### 7.3 Proof of Proposition 1

The minimax rate characterized by equation (21) involves both a lower and an upper bound, and we divide our proof accordingly. We provide the proof for \( \alpha \in (0, 1] \), but note that a similar result (modulo different constants) holds for any finite value of \( \alpha \).

**Lower bound:** We use Le Cam’s method to prove the lower bound in equation (21). Fix a given constant \( \delta \in (0, 1] \), with a precise value to be specified later. For \( \nu \in \mathcal{V} \in \{-1, 1\} \), define the distribution \( P_\nu \) with support \( \{-\delta^{-1/k}, 0, \delta^{1/k}\} \) by

\[ P_\nu(X = \delta^{-1/k}) = \frac{\delta(1 + \nu)}{2}, \quad P_\nu(X = 0) = 1 - \delta, \quad \text{and} \quad P_\nu(X = -\delta^{-1/k}) = \frac{\delta(1 - \nu)}{2}. \]

By construction, we have \( E[|X|^k] = \delta(\delta^{-1/k})^k = 1 \) and \( \theta_\nu = E_\nu[X] = \delta^{k-1} + \nu \), whence the mean difference is given by \( \theta_1 - \theta_{-1} = 2\delta^{k-1} \). Applying Le Cam’s method (8) and the minimax bound (7) yields

\[ \mathfrak{m}_n(\Theta, (\cdot)^2, Q) \geq \left( \delta^{k-1} \right)^2 \left( \frac{1}{2} - \frac{1}{2} \|M_1^n - M_{-1}^n\|_{\text{TV}} \right), \]

where \( M_\nu^n \) denotes the marginal distribution of the samples \( Z_1, \ldots, Z_n \) conditioned on \( \theta = \theta_\nu \). Now Pinsker’s inequality implies that \( \|M_1^n - M_{-1}^n\|_{\text{TV}}^2 \leq \frac{1}{2} D_{\text{kl}}(M_\nu^n||M_{-1}^n) \), and Corollary 1 yields

\[ D_{\text{kl}}(M_1^n||M_{-1}^n) \leq 4(e^\alpha - 1)^2 n \|P_1 - P_{-1}\|_{\text{TV}}^2 = 4(e^\alpha - 1)^2 n \delta^2. \]

Putting together the pieces yields \( \|M_1^n - M_{-1}^n\|_{\text{TV}} \leq (e^\alpha - 1)\delta \sqrt{2n} \). For \( \alpha \in (0, 1] \), we have \( e^\alpha - 1 \leq 2\alpha \), and thus our earlier application of Le Cam’s method implies

\[ \mathfrak{m}_n(\Theta, (\cdot)^2, \alpha) \geq \left( \delta^{k-1} \right)^2 \left( \frac{1}{2} - \alpha \delta \sqrt{2n} \right). \]

Substituting \( \delta = \min\{1, 1/\sqrt{32n\alpha^2}\} \) yields the claim (21).
Upper bound: We must demonstrate an $\alpha$-locally private conditional distribution $Q$ and an estimator that achieves the upper bound in equation (21). We do so via a combination of truncation and addition of Laplacian noise. Define the truncation function $[.]_T : \mathbb{R} \to [-T, T]$ by

$$[x]_T := \max\{-T, \min\{x, T\}\},$$

where the truncation level $T$ is to be chosen. Let $W_i$ be independent Laplace($\alpha/(2T)$) random variables, and for each index $i = 1, \ldots, n$, define $Z_i := [X_i]_T + W_i$. By construction, the random variable $Z_i$ is $\alpha$-differentially private for $X_i$. For the mean estimator $\hat{\theta} := \frac{1}{n} \sum_{i=1}^{n} Z_i$, we have

$$E \left[ (\hat{\theta} - \theta)^2 \right] = \text{Var}(\hat{\theta}) + \left( E[\hat{\theta}] - \theta \right)^2 = \frac{4T^2}{n\alpha^2} + \frac{1}{n} \text{Var}([X_1]_T) + (E[Z_1] - \theta)^2. \quad (53)$$

We claim that

$$E[Z] = E[[X]_T] \in \left[ E[X] - \frac{1}{(k-1)T^{k-1}}, E[X] + \frac{1}{(k-1)T^{k-1}} \right]. \quad (54)$$

Indeed, by the assumption that $E[|X|^k] \leq 1$, we have by a change of variables that

$$\int_T^{\infty} x dP(x) = \int_T^{\infty} x^k dP(x) \leq \int_T^{\infty} \frac{1}{x^k} dx = \frac{1}{(k-1)T^{k-1}}.$$

Thus

$$E[[X]_T] \geq E[\min\{X, T\}] = E[\min\{X, T\} + [X - T]_+ - [X - T]_+] = E[X] - \int_T^{\infty} (x - T) dP(x) \geq E[X] - \frac{1}{(k-1)T^{k-1}}.$$

A similar argument yields the upper bound in equation (54).

From the bound (53) and the inequalities that since $[X]_T \in [-T, T]$ and $\alpha^2 \leq 1$, we have

$$E \left[ (\hat{\theta} - \theta)^2 \right] \leq \frac{5T^2}{n\alpha^2} + \frac{1}{(k-1)^2T^{2k-2}} \quad \text{valid for any } T > 0.$$

Choosing $T = (5(k-1))^{-\frac{1}{2k}}(n\alpha^2)^{1/(2k)}$ yields

$$E \left[ (\hat{\theta} - \theta)^2 \right] \leq \frac{5(5(k-1))^{-\frac{1}{2k}}(n\alpha^2)^{\frac{1}{k}}}{n\alpha^2} + \frac{1}{(k-1)^{2(5(k-1))^{-1}+1/k(n\alpha^2)^{1-1/k}}}$$

$$= 5^{1-1/k} \left( 1 + \frac{1}{k-1} \right) \frac{1}{(k-1)^{\frac{1}{k}(n\alpha^2)^{1-1/k}}}.$$

Since $(1+(k-1)^{-1})k^{-1} < (k-1)^{-1}+(k-1)^{-2}$ for $k \in (1, 2)$ and is bounded by $1+(k-1)^{-1} \leq 2$ for $k \in [2, \infty]$, the upper bound (21) follows.

7.4 Proof of Proposition 2

We now turn to the proof of minimax rates for fixed design linear regression.
Lower bound: We use a slight generalization of the $\alpha$-private form \[20\] of the local Fano inequality from Corollary 3. For concreteness, we assume throughout that $\alpha \in [0, \frac{23}{30}]$, but analogous arguments hold for any bounded $\alpha$ with changes only in the constant pre-factors. Consider an instance of the linear regression model \[22\] in which the noise variables $\{\varepsilon_i\}_{i=1}^n$ are drawn i.i.d. from the uniform distribution on $[-\sigma, +\sigma]$. Our first step is to construct a suitable packing of the unit sphere $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : \|u\|_2 = 1\}$ in $\ell_2$-norm:

Lemma 5. There exists a $1$-packing $\mathcal{V} = \{\nu^1, \ldots, \nu^N\}$ of the unit sphere $\mathbb{S}^{d-1}$ with $N \geq \exp(\delta d/8)$.

See Appendix D.1 for the proof of this claim.

For a fixed $\delta \in (0, 1]$ to be chosen shortly, define the family of vectors $\{\theta_\nu, \nu \in \mathcal{V}\}$ with $\theta_\nu := \delta \nu$. Since $\|\nu\|_2 \leq 1$, we have $\|\theta_\nu - \theta_{\nu'}\|_2 \leq 2\delta$. Let $P_{\nu,i}$ denote the distribution of $Y_i$ conditioned on $\theta^* = \theta_{\nu}$. By the form of the linear regression model \[22\] and our assumption on the noise variable $\varepsilon_i$, $P_{\nu,i}$ is uniform on the interval $[\langle \theta_\nu, x_i \rangle - \sigma, \langle \theta_\nu, x_i \rangle + \sigma]$. Consequently, for $\nu \neq \nu' \in \mathcal{V}$, we have

$$\|P_{\nu,i} - P_{\nu',i}\|_{TV} = \frac{1}{2} \int |p_{\nu,i}(y) - p_{\nu',i}(y)| dy \leq \frac{1}{2} \left[ \frac{1}{2\sigma} |\langle \theta_\nu, x_i \rangle - \langle \theta_{\nu'}, x_i \rangle| + \frac{1}{2\sigma} |\langle \theta_\nu, x_i \rangle - \langle \theta_{\nu'}, x_i \rangle| \right] = \frac{1}{2\sigma} |\langle \theta_\nu - \theta_{\nu'}, x_i \rangle|.$$

Letting $V$ denote a random sample from the uniform distribution on $\mathcal{V}$, Corollary 1 implies that the mutual information is upper bounded as

$$I(Z_1, \ldots, Z_n; V) \leq 4(e^\alpha - 1)^2 \sum_{i=1}^n \frac{1}{|\mathcal{V}|^2} \sum_{\nu, \nu' \in \mathcal{V}} \|P_{\nu,i} - P_{\nu',i}\|_{TV}^2 \leq \frac{(e^\alpha - 1)^2}{\sigma^2} \sum_{i=1}^n \frac{1}{|\mathcal{V}|^2} \sum_{\nu, \nu' \in \mathcal{V}} (\langle \theta_\nu - \theta_{\nu'}, x_i \rangle)^2 \leq \frac{(e^\alpha - 1)^2}{\sigma^2} \sum_{\nu, \nu' \in \mathcal{V}} (\theta_\nu - \theta_{\nu'})^\top X^\top X (\theta_\nu - \theta_{\nu'}).$$

Since $\theta_\nu = \delta \nu$, we have by definition of the maximum singular value that

$$(\theta_\nu - \theta_{\nu'})^\top X^\top X (\theta_\nu - \theta_{\nu'}) \leq \delta^2 \|\nu - \nu'\|_2^2 \rho_{\max}(X^\top X) \leq 4\delta^2 \rho_{\max}^2(X) = 4n\delta^2 \rho_{\max}^2(X/\sqrt{n}).$$

Putting together the pieces, we find that

$$I(Z_1, \ldots, Z_n; V) \leq \frac{4n\delta^2(e^\alpha - 1)^2}{\sigma^2} \rho_{\max}^2(X/\sqrt{n}) \leq \frac{8n\alpha^2 \delta^2}{\sigma^2} \rho_{\max}^2(X/\sqrt{n}),$$

where the second inequality is valid for $\alpha \in [0, \frac{23}{30}]$. Consequently, Fano’s inequality combined with the packing set $\mathcal{V}$ from Lemma 5 implies that

$$\mathcal{M}_n \left( \Theta, \|\cdot\|_2^2, \alpha \right) \geq \delta^2 \left( 4 \left( 1 - \frac{8n\alpha^2 \delta^2 \rho_{\max}^2(X/\sqrt{n})/\sigma^2 + \log 2}{d/8} \right) \right).$$

We split the remainder of the analysis into cases.
Case 1: First suppose that \( d \geq 16 \). Then setting \( \delta^2 = \min\left\{ 1, \frac{d\sigma^2}{128n\rho_{\max}(X/\sqrt{n})} \right\} \) implies that

\[
\frac{8n\delta^2\alpha^2\rho_{\max}^2(X/\sqrt{n})}{d/8} + \log 2 \leq 8 \left\lbrack \log \frac{2}{d} + \frac{64}{128} \right\rbrack < \frac{7}{8}
\]

As a consequence, we have the lower bound

\[
\mathcal{M}_n \left( \Theta, \| \cdot \|_2^2, \alpha \right) \geq \frac{1}{4} \min \left\{ 1, \frac{d\sigma^2}{128n\rho_{\max}(X/\sqrt{n})} \right\} \cdot \frac{1}{8},
\]

which yields the claim for \( d \geq 16 \).

Case 2: Otherwise, we may assume that \( d < 16 \). In this case, a lower bound for the case \( d = 1 \) is sufficient, since apart from constant factors, the same bound holds for all \( d < 16 \). We use the Le Cam method based on a two point comparison. Indeed, let \( \theta_1 = \delta \) and \( \theta_2 = -\delta \) so that the total variation distance is at upper bounded \( \| P_{1,i} - P_{2,i} \|_{TV} \leq \frac{1}{2} |x_i| \). By Corollary 2 we have

\[
\mathcal{M}_n \left( \Theta, (\cdot)^2, \alpha \right) \geq \delta^2 \left( 1 - \frac{\delta(e^\alpha - 1)}{\alpha} \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \right).
\]

Letting \( x = (x_1, \ldots, x_n) \) and setting \( \delta^2 = \min\{1, \sigma^2/(16(e^\alpha - 1)^2 \| x \|_2^2) \} \) gives the desired result.

Upper bound: We now turn to the upper bound, for which we need to specify a private conditional \( Q \) and an estimator \( \hat{\theta} \) that achieves the stated upper bound on the mean-squared error. Let \( W_i \) be independent Laplace(\( \alpha/(2\sigma) \)) random variables. Then the additively perturbed random variable \( Z_i = Y_i + W_i \) is \( \alpha \)-differentially private for \( Y_i \), since by assumption the response \( Y_i \in [\langle \theta, x_i \rangle - \sigma, \langle \theta, x_i \rangle + \sigma] \). We now claim that the standard least-squares estimator of \( \theta^* \) achieves the stated upper bound. Indeed, the least-squares estimate is given by

\[
\hat{\theta} = (X^\top X)^{-1}X^\top Y = (X^\top X)^{-1}X^\top (X\theta^* + \varepsilon + W).
\]

Moreover, from the independence of \( W \) and \( \varepsilon \), we have

\[
\mathbb{E} \left[ \| \hat{\theta} - \theta^* \|^2 \right] = \mathbb{E} \left[ \| (X^\top X)^{-1}X^\top (\varepsilon + W) \|^2 \right] = \mathbb{E} \left[ \| (X^\top X)^{-1}X^\top \varepsilon \|^2 \right] + \mathbb{E} \left[ \| (X^\top X)^{-1}X^\top W \|^2 \right].
\]

Since \( \varepsilon \in [-\sigma, \sigma]^n \), we know that \( \mathbb{E}[\varepsilon \varepsilon^\top] \preceq \sigma^2 I_{n \times n} \), and for the given choice of \( W \), we have \( \mathbb{E}[WW^\top] = (4\sigma^2/\alpha^2)I_{n \times n} \). Since \( \alpha \leq 1 \), we thus find

\[
\mathbb{E} \left[ \| \hat{\theta} - \theta^* \|^2 \right] \leq \frac{5\sigma^2}{\alpha^2} \text{tr} \left( X(X^\top X)^{-2}X^\top \right) = \frac{5\sigma^2}{\alpha^2} \text{tr} \left( (X^\top X)^{-1} \right).
\]

Noting that \( \text{tr}((X^\top X)^{-1}) \leq d/\rho_{\min}^2(X) = d/n\rho_{\min}^2(X/\sqrt{n}) \) gives the claimed upper bound.

8 Proof of Theorem 2 and related results

In this section, we collect together the proof of Theorem 2 and related corollaries.
8.1 Proof of Theorem 2

Let \( Z \) denote the domain of the random variable \( Z \). We begin by reducing the problem to the case when \( Z = \{1, 2, \ldots, k\} \) for an arbitrary positive integer \( k \). Indeed, in the general setting, we let \( K = \{K_i\}_{i=1}^k \) be any (measurable) finite partition of \( Z \), where for \( z \in Z \) we let \([z]_K = K_i\) for the \( K_i \) such that \( z \in K_i \). The KL divergence \( D_{kl} (M_\nu \| M) \) can be defined as the supremum of the (discrete) KL divergences between the random variables \([Z]_K\) sampled according to \( M_\nu \) and \( M \) over all partitions \( K \) of \( Z \); for instance, see Gray [32, Chapter 5]. Consequently, we can prove the claim for \( Z = \{1, 2, \ldots, k\} \), and then take the supremum over \( k \) to recover the general case. Accordingly, we can work with the probability mass functions \( m(z \mid \nu) = M_\nu(Z = z) \) and \( \overline{m}(z) = \overline{M}(Z = z) \), and we may write

\[
D_{kl} (M_\nu \| \overline{M}) + D_{kl} (\overline{M} \| M_\nu) = \sum_{z=1}^{k} (m(z \mid \nu) - \overline{m}(z)) \log \frac{m(z \mid \nu)}{\overline{m}(z)}. \tag{55}
\]

Throughout, we will also use (without loss of generality) the probability mass functions \( q(z \mid x) = Q(Z = z \mid X = x) \), where we note that \( m(z \mid \nu) = \int q(z \mid x) dP_\nu(x) \).

Now we use Lemma 4 from the proof of Theorem 1 to complete the proof of Theorem 2. Starting with equality (55), we have

\[
\frac{1}{|V|} \sum_{\nu \in V} [D_{kl} (M_\nu \| \overline{M}) + D_{kl} (\overline{M} \| M_\nu)] \leq \sum_{\nu \in V} \frac{1}{|V|} \sum_{z=1}^{k} |m(z \mid \nu) - \overline{m}(z)| \left| \log \frac{m(z \mid \nu)}{\overline{m}(z)} \right| \\
\leq \sum_{\nu \in V} \frac{1}{|V|} \sum_{z=1}^{k} |m(z \mid \nu) - \overline{m}(z)| \frac{|m(z \mid \nu) - \overline{m}(z)|}{\min \{\overline{m}(z), m(z \mid \nu)\}}.
\]

Now, we define the measure \( m^0 \) on \( Z = \{1, \ldots, k\} \) by \( m^0(z) := \inf_{x \in \mathcal{X}} q(z \mid x) \). It is clear that \( \min \{\overline{m}(z), m(z \mid \nu)\} \geq m^0(z) \), whence we find

\[
\frac{1}{|V|} \sum_{\nu \in V} [D_{kl} (M_\nu \| \overline{M}) + D_{kl} (\overline{M} \| M_\nu)] \leq \sum_{\nu \in V} \frac{1}{|V|} \sum_{z=1}^{k} \frac{(m(z \mid \nu) - \overline{m}(z))^2}{m^0(z)}.
\]

It remains to bound the final sum. For any constant \( c \in \mathbb{R} \), we have

\[
m(z \mid \nu) - \overline{m}(z) = \int_{\mathcal{X}} (q(z \mid x) - c) (dP_\nu(x) - dP(x)).
\]

We define a set of functions \( f : Z \times \mathcal{X} \to \mathbb{R} \) (depending implicitly on \( q \)) by

\[
\mathcal{F}_\alpha := \{ f \mid f(z, x) \in [1, e^\alpha] m^0(z) \text{ for all } z \in Z \text{ and } x \in \mathcal{X} \}.
\]

By the definition of differential privacy, when viewed as a joint mapping from \( Z \times \mathcal{X} \to \mathbb{R} \), the conditional p.m.f. \( q \) satisfies \( \{(z, x) \mapsto q(z \mid x)\} \in \mathcal{F}_\alpha \). Since constant (with respect to \( x \)) shifts do not change the above integral, we can modify the range of functions in \( \mathcal{F}_\alpha \) by subtracting \( m^0(z) \) from each, yielding the set

\[
\mathcal{F}_\alpha := \{ f \mid f(z, x) \in [0, e^\alpha - 1] m^0(z) \text{ for all } z \in Z \text{ and } x \in \mathcal{X} \}.
\]

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As a consequence, we find that
\[
\sum_{\nu \in \mathcal{V}} (m(z \mid \nu) - \overline{m}(z))^2 \leq \sup_{f \in \mathcal{F}_n} \left\{ \sum_{\nu \in \mathcal{V}} \left( \int_X f(z, x) \left( dP_{\nu}(x) - d\overline{P}(x) \right) \right)^2 \right\}
\]
\[
= \sup_{f \in \mathcal{F}_n} \left\{ \sum_{\nu \in \mathcal{V}} \left( \int_X (f(z, x) - m^0(z)) \left( dP_{\nu}(x) - d\overline{P}(x) \right) \right)^2 \right\}.
\]
By inspection, when we divide by \(m^0(z)\) and recall the definition of the set \(\mathcal{B}_\infty \subset L^\infty(\mathcal{X})\) in the statement of Theorem 2, we obtain
\[
\sum_{\nu \in \mathcal{V}} (m(z \mid \nu) - \overline{m}(z))^2 \leq (m^0(z))^2 \left( e^\alpha - 1 \right)^2 \sup_{\gamma \in \mathcal{B}_\infty} \sum_{\nu \in \mathcal{V}} \left( \int_X \gamma(x) \left( dP_{\nu}(x) - d\overline{P}(x) \right) \right)^2.
\]
Putting together our bounds, we have
\[
\frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \left[ D_{\text{kl}}(M_{\nu} || \overline{M}) + D_{\text{kl}}(\overline{M} || M_{\nu}) \right]
\]
\[
\leq (e^\alpha - 1)^2 \sum_{z=1}^k \frac{1}{|\mathcal{V}|} \frac{(m^0(z))^2}{m^0(z)} \sup_{\gamma \in \mathcal{B}_\infty} \sum_{\nu \in \mathcal{V}} \left( \int_X \gamma(x) \left( dP_{\nu}(x) - d\overline{P}(x) \right) \right)^2
\]
\[
\leq (e^\alpha - 1)^2 \frac{1}{|\mathcal{V}|} \sup_{\gamma \in \mathcal{B}_\infty} \sum_{\nu \in \mathcal{V}} \left( \int_X \gamma(x) \left( dP_{\nu}(x) - d\overline{P}(x) \right) \right)^2,
\]
since \(\sum_z m^0(z) \leq 1\), which is the statement of the theorem.

**8.2 Proof of Corollary 4**

In the non-interactive setting (2), the marginal distribution \(M^\alpha_{\nu}\) is a product measure and \(Z_i\) is conditionally independent of \(Z_{1:i-1}\) given \(V\). Thus by the chain rule for mutual information [32, Chapter 5] and the fact (as in the proof of Theorem 2) that we may assume w.l.o.g. that \(Z\) has finite range \(I(Z_1, \ldots, Z_n; V) = \sum_{i=1}^n I(Z_i; V \mid Z_{1:i-1}) = \sum_{i=1}^n [H(Z_i \mid Z_{1:i-1}) - H(Z_i \mid V, Z_{1:i-1})].\)

Since conditioning reduces entropy and \(Z_{1:i-1}\) is conditionally independent of \(Z_i\) given \(V\), we have \(H(Z_i \mid Z_{1:i-1}) \leq H(Z_i)\) and \(H(Z_i \mid V, Z_{1:i-1}) = H(Z_i \mid V)\). In particular, we have
\[
I(Z_1, \ldots, Z_n; V) \leq \sum_{i=1}^n I(Z_i; V) = \sum_{i=1}^n \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} D_{\text{kl}}(M_{\nu,i} || \overline{M}_i) .
\]
Applying Theorem 2 completes the proof.
9 Proof of Theorem 3

The proof of this theorem combines the techniques we used in the proofs of Theorems [1] and [2], the first handles interactivity, while the techniques to derive the variational bounds are reminiscent of those used in Theorem [2]. Our first step is to note a consequence of the independence structure in Fig. [1] that is essential for tensorization steps. More precisely, we claim that for any set \( S \subseteq \sigma(\mathcal{Z}) \),

\[
M_{\pm j}(Z_i \in S \mid z_{1:i-1}) = \int Q(Z_i \in S \mid Z_{1:i-1} = z_{1:i-1}, X_i = x) dP_{\pm j,i}(x). \tag{56}
\]

We postpone the proof of this intermediate claim to the end of this section.

Now consider the summed KL-divergences. Let \( M_{\pm j,i}(\cdot \mid z_{1:i-1}) \) denote the conditional distribution of \( Z_i \) under \( P_{\pm j} \), conditional on \( Z_{1:j-1} = z_{1:i-1} \). As in the proof of Corollary [1], the chain-rule for KL-divergences [e.g. 32, Chapter 5] implies

\[
D_{kl} \left( M_{\pm j}^n \| M_{-j}^n \right) \leq \sum_{i=1}^{n} \left( E_{+j} \left[ D_{kl} \left( M_{+j,i}(\cdot \mid Z_{1:i-1}) \| M_{-j,i}(\cdot \mid Z_{1:i-1}) \right) \right] + E_{-j} \left[ D_{kl} \left( M_{-j,i}(\cdot \mid Z_{1:i-1}) \| M_{+j,i}(\cdot \mid Z_{1:i-1}) \right) \right] \right) \leq \sum_{i=1}^{n} \left( E_{+j} \left[ D_{sy}^{kl} \left( M_{+j,i}(\cdot \mid Z_{1:i-1}) \| M_{-j,i}(\cdot \mid Z_{1:i-1}) \right) \right] + E_{-j} \left[ D_{sy}^{kl} \left( M_{-j,i}(\cdot \mid Z_{1:i-1}) \| M_{+j,i}(\cdot \mid Z_{1:i-1}) \right) \right] \right) \leq 2 \sum_{i=1}^{n} \int_{Z_{i-1}} D_{sy}^{kl} \left( M_{+j,i}(\cdot \mid z_{1:i-1}) \| M_{-j,i}(\cdot \mid z_{1:i-1}) \right) d\mathcal{M}^{-1}(z_{1:i-1}),
\]

where we have used the definition of \( \mathcal{M} \) and that \( 2P = P_{+j} + P_{-j} \) for all \( j \). Summing over \( j \in [d] \) yields

\[
\sum_{j=1}^{d} D_{sy}^{kl} \left( M_{+j}^n \| M_{-j}^n \right) \leq 2 \sum_{i=1}^{n} \int_{Z_{i-1}} \sum_{j=1}^{d} D_{sy}^{kl} \left( M_{+j,i}(\cdot \mid z_{1:i-1}) \| M_{-j,i}(\cdot \mid z_{1:i-1}) \right) d\mathcal{M}^{-1}(z_{1:i-1}). \tag{57}
\]

We bound the underlined expression in inequality (57), whose elements we denote by \( \mathcal{T}_{j,i} \).

Without loss of generality (as in the proof of Theorem [2]), we may assume \( \mathcal{Z} \) is finite, and that \( \mathcal{Z} = \{1, 2, \ldots, k\} \) for some positive integer \( k \). Using the probability mass functions \( m_{\pm j,i} \) and
omitting the index \( i \) when it is clear from context, Lemma 4 implies

\[
\mathcal{T}_{j,i} = \sum_{i=1}^{k} (m_{+j}(z | z_{1;i-1}) - m_{+j}(z | z_{1;i-1})) \log \frac{m_{+j}(z | z_{1;i-1})}{m_{-j}(z | z_{1;i-1})} \\
\leq \sum_{i=1}^{k} (m_{+j}(z | z_{1;i-1}) - m_{+j}(z | z_{1;i-1}))^2 \frac{1}{\min\{m_{+j}(z | z_{1;i-1}), m_{-j}(z | z_{1;i-1})\}}.
\]

For each fixed \( z_{1;i-1} \), define the infimal measure \( m^0(z | z_{1;i-1}) := \inf_{x \in \mathcal{X}} q(z | X_i = x, z_{1;i-1}) \). By construction, we have \( \min\{m_{+j}(z | z_{1;i-1}), m_{-j}(z | z_{1;i-1})\} \geq m^0(z | z_{1;i-1}) \), and hence

\[
\mathcal{T}_{j,i} \leq \sum_{i=1}^{k} (m_{+j}(z | z_{1;i-1}) - m_{+j}(z | z_{1;i-1}))^2 \frac{1}{m^0(z | z_{1;i-1})}.
\]

Recalling equality (50), we have

\[
m_{+j}(z | z_{1;i-1}) - m_{+j}(z | z_{1;i-1}) = \int_{\mathcal{X}} q(z | x, z_{1;i-1}) (dP_{+j,i}(x) - dP_{-j,i}(x)) \\
= m^0(z | z_{1;i-1}) \int_{\mathcal{X}} \left( \frac{q(z | x, z_{1;i-1})}{m^0(z | z_{1;i-1})} - 1 \right) (dP_{+j,i}(x) - dP_{-j,i}(x)).
\]

From this point, the proof is similar to that of Theorem 2. Define the collection of functions

\[
\mathcal{F}_\alpha := \{ f : \mathcal{X} \times \mathbb{Z}^i \to [0, e^\alpha - 1] \}.
\]

Using the definition of differential privacy, we have \( \frac{q(z | x, z_{1;i-1})}{m^0(z | z_{1;i-1})} \in [1, e^\alpha] \), so there exists \( f \in \mathcal{F}_\alpha \) such that

\[
\sum_{j=1}^{d} \mathcal{T}_{j,i} \leq \sum_{j=1}^{d} \sum_{i=1}^{k} \left( m^0(z | z_{1;i-1}) \int_{\mathcal{X}} f(x, z, z_{1;i-1}) (dP_{+j,i}(x) - dP_{-j,i}(x)) \right)^2 \\
= \sum_{i=1}^{k} \left( d \int_{\mathcal{X}} f(x, z, z_{1;i-1}) (dP_{+j,i}(x) - dP_{-j,i}(x)) \right)^2.
\]

Taking a supremum over \( \mathcal{F}_\alpha \), we find the further upper bound

\[
\sum_{j=1}^{d} \mathcal{T}_{j,i} \leq \sum_{i=1}^{k} m^0(z | z_{1;i-1}) \sup_{f \in \mathcal{F}_\alpha} \left( d \int_{\mathcal{X}} f(x, z, z_{1;i-1}) (dP_{+j,i}(x) - dP_{-j,i}(x)) \right)^2.
\]

The inner supremum may be taken independently of \( z \) and \( z_{1;i-1} \), so we rescale by \( (e^\alpha - 1) \) to obtain our penultimate inequality

\[
\sum_{j=1}^{d} D^\text{KL}_{\text{sy}}(M_{+j,i}(\cdot | z_{1;i-1}), M_{-j,i}(\cdot | z_{1;i-1})) \\
\leq (e^\alpha - 1)^2 \sum_{i=1}^{k} m^0(z | z_{1;i-1}) \sup_{\gamma \in \mathcal{B}_\infty(\mathcal{X})} \left( \int_{\mathcal{X}} \gamma(x) (dP_{+j,i}(x) - dP_{-j,i}(x)) \right)^2.
\]
Noting that \( m^0 \) sums to a quantity \( \leq 1 \) and substituting the preceding expression in inequality (57) completes the proof.

Finally, we return to prove our intermediate marginalization claim (56). We have that

\[
M_{\pm j}(Z_i \in S \mid z_{1:i-1}) = \int Q(Z_i \in S \mid z_{1:i-1}, x_{1:n})dP_{\pm j}(x_{1:n} \mid z_{1:i-1})
\]

\[
= (i) \int Q(Z_i \in S \mid z_{1:i-1}, x_i)dP_{\pm j}(x_{1:n} \mid z_{1:i-1})
\]

\[
= (ii) \int Q(Z_i \in S \mid Z_{1:i-1} = z_{1:i-1}, X_i = x)dP_{\pm j,i}(x),
\]

where equality (i) follows by the assumed conditional independence structure of \( Q \) (recall Figure 1) and equality (ii) is a consequence of the independence of \( X_i \) and \( Z_{1:i-1} \) under \( P_{\pm j} \). That is, we have \( P_{\pm j}(X_i \in S \mid Z_{1:i-1} = z_{1:i-1}) = P_{\pm j,i}(S) \) by the definition of \( P_{\nu} \) as a product and that \( P_{\pm j} \) are a mixture of the products \( P_{\nu} \).

10 Conclusions

We have linked minimax analysis from statistical decision theory with differential privacy, bringing some of their respective foundational principles into close contact. Our main technique, in the form of the divergence inequalities in Theorems 1 and 2 and their Corollaries 1–4, shows that applying differentially private sampling schemes essentially acts as a contraction on distributions. These contractive inequalities allow us to give sharp minimax rates for estimation in locally private settings, and we think such results may be more generally applicable. With our examples in Sections 4.2, 5.2, and 5.3, we have developed a set of techniques that show that roughly, if one can construct a family of distributions \( \{P_{\nu}\} \) on the sample space \( \mathcal{X} \) that is not well “correlated” with any member of \( f \in L_{\infty}(\mathcal{X}) \) for which \( f(x) \in \{-1, 1\} \), then providing privacy is costly: the contraction Theorems 2 and 3 provide is strong.

By providing sharp convergence rates for many standard statistical estimation procedures under local differential privacy, we have developed and explored some tools that may be used to better understand privacy-preserving statistical inference. We have identified a fundamental continuum along which privacy may be traded for utility in the form of accurate statistical estimates, providing a way to adjust statistical procedures to meet the privacy or utility needs of the statistician and the population being sampled.

There are a number of open questions raised by our work. It is natural to wonder whether it is possible to obtain tensorized inequalities of the form of Corollary 4 even for interactive mechanisms. Another important question is whether the results we have provided can be extended to settings in which standard (non-local) differential privacy holds. Such extensions could yield insights into optimal mechanisms for differentially private procedures.

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A Proofs of multi-dimensional mean-estimation results

At a high level, our proofs of these results consist of three steps, the first of which is relatively standard, while the second two exploit specific aspects of the local privacy setting. We outline them here:

1. The first step is a standard reduction, based on inequalities (7)–(9) in Section 2, from an estimation problem to a multi-way testing problem that involves discriminating between indices \( \nu \) contained within some subset \( \mathcal{V} \) of \( \mathbb{R}^d \).

2. The second step is an appropriate construction of a maximal \( \delta \)-packing, meaning a set \( \mathcal{V} \subset \mathbb{R}^d \) such that each pair is \( \delta \)-separated and the resulting set is as large as possible. In addition, our arguments require that, for a random variable \( V \) uniformly distributed over \( \mathcal{V} \), the covariance \( \text{Cov}(V) \) has relatively small operator norm.

3. The final step is to apply Theorem 2 in order to control the mutual information associated with the testing problem. Doing so requires bounding the supremum in Corollary 4 via the operator norm of \( \text{Cov}(V) \).

The estimation to testing reduction of Step 1 was previously described in Section 2. Accordingly, the proofs to follow are devoted to the second and third steps in each case.

A.1 Proof of Proposition 3

We provide a proof of the lower bound, as we provided the argument for the upper bound in Section 4.2.2.

Constructing a good packing: Let \( k \) be an arbitrary integer in \( \{1,2,\ldots,d\} \). The following auxiliary result provides a building block for the packing set underlying our proof:

**Lemma 6.** For each integer \( k \), there exists a packing \( \mathcal{V}_k \) of the \( k \)-dimensional hypercube \( \{-1,1\}^k \) with \( \|\nu - \nu'\|_1 \geq k/2 \) for each \( \nu, \nu' \in \mathcal{V}_k \) with \( \nu \neq \nu' \) such that \( |\mathcal{V}_k| \geq \lceil \exp(k/16) \rceil \), and

\[
\frac{1}{|\mathcal{V}_k|} \sum_{\nu \in \mathcal{V}_k} \nu \nu^\top \preceq 25I_{k \times k}.
\]

See Appendix D.2 for the proof.

For a given \( k \leq d \), we extend the set \( \mathcal{V}_k \subset \mathbb{R}^k \) to a subset of \( \mathbb{R}^d \) by setting \( \mathcal{V} = \mathcal{V}_k \times \{0\}^{d-k} \). For a parameter \( \delta \in (0,1/2] \) to be chosen, we define a family of probability distributions \( \{P_\nu\}_{\nu \in \mathcal{V}} \) constructively. In particular, the random vector \( X \sim P_\nu \) (a single observation) is formed by the following procedure:

Choose index \( j \in \{1,\ldots,k\} \) uniformly at random and set

\[
X = \begin{cases} r e_j & \text{w.p. } \frac{1+\delta \nu_j}{2}, \\ -r e_j & \text{w.p. } \frac{1-\delta \nu_j}{2}. \end{cases}
\]

(58)
By construction, these distributions have mean vectors

\[ \theta_\nu := \mathbb{E}_{P_\nu}[X] = \frac{\delta r}{k} \nu. \]

Consequently, given the properties of the packing \( V \), we have \( X \in \mathbb{B}_1(r) \) with probability 1, and \( \| \theta_\nu - \theta_\nu' \|^2 \geq r^2 \delta^2 / k \). Thus we see that the mean vectors \( \{ \theta_\nu \}_{\nu \in V} \) provide us with an \( r \delta / \sqrt{k} \)-packing of the ball.

**Upper bounding the mutual information:** Our next step is to bound the mutual information \( I(Z_1, \ldots, Z_n; V) \) when the observations \( X \) come from the distribution (58) and \( V \) is uniform in the set \( V \). We have the following lemma, which applies so long as the channel \( Q \) is non-interactive and \( \alpha \)-locally private (2). See Appendix E.1 for the proof.

**Lemma 7.** Fix \( k \in \{1, \ldots, d\} \). Let \( Z_i \) be \( \alpha \)-locally differentially private for \( X_i \), and let \( X \) be sampled according to the distribution (58) conditional on \( V = \nu \). Then

\[ I(Z_1, \ldots, Z_n; V) \leq n \frac{25 e^\alpha \delta^2}{16 k} (e^\alpha - e^{-\alpha})^2. \]

**Applying testing inequalities:** We now show how a combination of the hypercube packing specified by Lemma 6 and the sampling scheme (58) give us our desired lower bound. Fix \( k \leq d \) and let \( V = V_k \times \{0\}^{d-k} \) be the packing of \( \{-1, 1\}^k \times \{0\}^{d-k} \) defined following Lemma 6. Combining Lemma 7 and the fact that the vectors \( \theta_\nu \) provide an \( r \delta / \sqrt{k} \) packing of the ball of cardinality at least \( \exp(k/16) \), Fano’s inequality implies that for any \( k \in \{1, \ldots, d\} \),

\[ M_n(\theta(P), \|\cdot\|_2^2, \alpha) \geq \frac{r^2 \delta^2}{4k} \left( 1 - \frac{25 ne^\alpha \delta^2 (e^\alpha - e^{-\alpha})^2}{16 k} + \log 2 \right) \]

Because of the 1-dimensional mean-estimation lower bounds provided in Section 3.3.1, we may assume w.l.o.g. that \( k \geq 32 \). Setting \( \delta^2_{n,\alpha,k} = \min\{1, k^2/(50 ne^\alpha (e^\alpha - e^{-\alpha})^2)\} \), we obtain

\[ M_n(\theta(P), \|\cdot\|_2^2, \alpha) \geq \frac{r^2 \delta^2_{n,\alpha,k}}{4k} \left( 1 - \frac{1}{2} - \frac{\log 2}{2} \right) \geq cr^2 \min\left\{ \frac{1}{k}, \frac{k}{ne^\alpha (e^\alpha - e^{-\alpha})^2} \right\} \]

for a universal (numerical) constant \( c \). Since \( e^\alpha (e^\alpha - e^{-\alpha})^2 < 16 \alpha^2 \) for \( \alpha \in [0, 1] \), we obtain the lower bound

\[ M_n(\theta(P), \|\cdot\|_2^2, \alpha) \geq cr^2 \max_{k \in [d]} \left\{ \min\left\{ \frac{1}{k}, \frac{k}{n \alpha^2} \right\} \right\} \]

for \( \alpha \in [0, 1] \) and a universal constant \( c > 0 \). Setting \( k \) in the preceding display to be the integer in \( \{1, \ldots, d\} \) nearest \( \sqrt{n \alpha^2} \) gives the result of the proposition.

### A.2 Proof of Proposition 4

Since the upper bound was established in Section 4.2.2, we focus on the lower bound.
**Constructing a good packing:** In this case, the packing set is very simple: set \( V = \{ \pm e_j \}_{j=1}^d \) so that \(|V| = 2d\). Fix some \( \delta \in [0,1] \), and for \( \nu \in V \), define a distribution \( P_{\nu} \) supported on \( X = \{-r,r\}^d \) via

\[
P_{\nu}(X = x) = (1 + \delta \nu^\top x/r) / 2^d.
\]

In words, for \( \nu = e_j \), the coordinates of \( X \) are independent uniform on \( \{-r,r\} \) except for the coordinate \( j \), for which \( X_j = r \) with probability \( 1/2 + \delta \nu_j \) and \( X_j = -r \) with probability \( 1/2 - \delta \nu_j \). With this scheme, we have \( \theta(P_{\nu}) = r \delta \nu \), and since \( \|\delta r \nu - \delta r \nu'\|_\infty \geq \delta r \), we have constructed a \( \delta r \) packing in \( \ell_\infty \)-norm.

**Upper bounding the mutual information:** Let \( V \) be drawn uniformly from the packing set \( V = \{ \pm e_j \}_{j=1}^d \). With the sampling scheme in the previous paragraph, we may provide the following upper bound on the mutual information \( I(Z_1, \ldots, Z_n; V) \) for any non-interactive private distribution \( Q \):

**Lemma 8.** For any non-interactive \( \alpha \)-differentially private distribution \( Q \), we have

\[
I(Z_1, \ldots, Z_n; V) \leq n \frac{e^\alpha}{4d} (e^\alpha - e^{-\alpha})^2 \delta^2.
\]

See Appendix \[E.2\] for a proof.

**Applying testing inequalities:** Finally, we turn to application of the testing inequalities. Lemma \[A.3\] in conjunction with the standard testing reduction and Fano’s inequality \[9\], implies that

\[
\mathcal{M}_n(\theta(P), \|\cdot\|_\infty, \alpha) \geq \frac{r \delta}{2} \left( 1 - \frac{e^\alpha \delta^2 n(e^\alpha - e^{-\alpha})^2/(4d) + \log 2}{\log(2d)} \right).
\]

There is no loss of generality in assuming that \( d \geq 2 \), in which case the choice

\[
\delta^2 = \min \left\{ 1, \frac{d \log(2d)}{e^\alpha(e^\alpha - e^{-\alpha})^2 n} \right\}
\]

yields the proposition.

### A.3 Proof of Proposition \[5\]

For this proposition, the construction of the packing and lower bound used in the proof of Proposition \[4\] also apply. Under these packing and sampling procedures, note that the separation of points \( \theta(P_{\nu}) = r \delta \nu \) in \( \ell_2 \)-norm is \( r \delta \). It thus remains to provide the upper bound. In this case, we use the sampling strategy \([27b]\), as in Proposition \[4\] and Section \[4.2.2\] noting that we may take the bound \( B \) on \( \|Z\|_\infty \) to be \( B = c \sqrt{d} r / \alpha \) for a constant \( c \). Let \( \theta^* \) denote the true mean, assumed to be \( s \)-sparse. Now consider estimating \( \theta^* \) by the \( \ell_1 \)-regularized optimization problem

\[
\hat{\theta} := \arg\min_{\theta \in \mathbb{R}^d} \left\{ \frac{1}{2n} \left\| \sum_{i=1}^n (Z_i - \theta) \right\|_2^2 + \lambda \|\theta\|_1 \right\}.
\]

Defining the error vector \( W = \theta^* - \frac{1}{n} \sum_{i=1}^n Z_i \), we claim that

\[
\lambda \geq 2 \|W\|_\infty \quad \text{implies that} \quad \|\hat{\theta} - \theta\|_2 \leq 3\lambda \sqrt{s}.
\]
This result is a consequence of standard results on sparse estimation (e.g., Negahban et al. [44, Theorem 1 and Corollary 1]).

Now we note if \( W_i = \theta^* - Z_i \), then \( W_i = \frac{1}{n} \sum_{i=1}^{n} W_i \), and by construction of the sampling mechanism (27b) we have \( \|W\|_\infty \leq c \sqrt{dr/\alpha} \) for a constant \( c \). By Hoeffding’s inequality and a union bound, we thus have for some (different) universal constant \( c \) that

\[
P(\|W\|_\infty \geq t) \leq 2d \exp \left( -\frac{n\alpha^2 t^2}{r^2 d} \right)
\]

for \( t \geq 0 \).

By taking \( t^2 = r^2 d (\log(2d) + \epsilon^2) / (cn\alpha^2) \), we find that \( \|W\|_\infty \leq r^2 d (\log(2d) + \epsilon^2) / (cn\alpha^2) \) with probability at least 1 - \( \exp(-\epsilon^2) \), which gives the claimed minimax upper bound by appropriate choice of \( \lambda = c \sqrt{d \log d / n\alpha^2} \) in inequality (59).

### A.4 Proof of inequality (31)

We prove the bound by an argument using the private form of Fano’s inequality from Corollary 3. The proof makes use of the classical Varshamov-Gilbert bound (e.g. [53, Lemma 4]):

**Lemma 9** (Varshamov-Gilbert). There is a packing \( V \) of the \( d \)-dimensional hypercube \( \{-1,1\}^d \) of size \( |V| \geq \exp(d/8) \) such that

\[
\|\nu - \nu'\|_1 \geq d/2 \quad \text{for all distinct pairs } \nu, \nu' \in V.
\]

Now, let \( \delta \in [0,1] \) and the distribution \( P_\nu \) be a point mass at \( \delta \nu / \sqrt{d} \). Then \( \theta(P_\nu) = \delta \nu / \sqrt{d} \) and \( \|\theta(P_\nu) - \theta(P_{\nu'})\|_2^2 \geq \delta^2 \). In addition, a calculation implies that if \( M_1 \) and \( M_2 \) are \( d \)-dimensional Laplace(\( \kappa \)) distributions with means \( \theta_1 \) and \( \theta_2 \), respectively, then

\[
D_{\text{kl}}(M_1 \| M_2) = \sum_{j=1}^{d} (\exp(-\kappa |\theta_{1,j} - \theta_{2,j}|) + \kappa |\theta_{1,j} - \theta_{2,j}| - 1) \leq \frac{\kappa^2}{2} \|\theta_1 - \theta_2\|_2^2.
\]

As a consequence, we have that under our Laplacian sampling scheme for the \( Z \) and with \( V \) chosen uniformly from \( V \),

\[
I(Z_1, \ldots, Z_n; V) \leq \frac{1}{|V|^2} \sum_{\nu, \nu' \in V} \sum_{\nu, \nu' \in V} D_{\text{kl}}(M_\nu \| M_{\nu'}) \leq \frac{n\alpha^2}{2d|V|^2} \sum_{\nu, \nu' \in V} \left( \frac{\delta}{\sqrt{d}} \right)^2 \left( |\nu - \nu'| \right)^2 \leq \frac{2n\alpha^2 \delta^2}{d}.
\]

Now, applying Fano’s inequality (31) in the context of the testing inequality (7), we find that

\[
\inf_{\theta} \sup_{\nu \in V} \mathbb{E}_{P_\nu} \left[ \|\hat{\theta}(Z_1, \ldots, Z_n) - \theta(P_\nu)\|_2^2 \right] \geq \frac{\delta^2}{4} \left( 1 - \frac{2n\alpha^2 \delta^2 / d + \log 2}{d/8} \right).
\]

We may assume (based on our one-dimensional results in Proposition 11) w.l.o.g. that \( d \geq 10 \). Taking \( \delta^2 = d^2 / (48n\alpha^2) \) then implies the result (31).
Proof of Proposition 6

Before proving the bounds, however, we outline our technique, which borrows from that in Section A and which we also use to prove the lower bounds on density estimation. The outline is as follows:

1. As in step (1) of Section A, our first step is a standard reduction using the sharper version of Assouad’s method (Lemma 1) from estimation to a multiple binary hypothesis testing problem. Specifically, we perform a (essentially standard) reduction of the form (10).

2. Having constructed appropriately separated binary hypothesis tests, we use apply Theorem 3 via inequality (34) to control the testing error in the binary testing problem. Applying the theorem requires bounding certain suprema related to the covariance structure of randomly selected elements of $V = \{-1, 1\}^d$, as in the arguments in Section A. In this case, though, the symmetry of the binary hypothesis testing problems eliminates the need for carefully constructed packings of step A(2).

With this outline in mind, we turn to the proofs of inequalities (35) and (36). As we proved the upper bounds in Section 5.2.2, this section focuses on the argument for the lower bound. We provide the full proof for the mean-squared Euclidean error, after which we show how the result for the $\ell_1$-error follows.

Our first step is to provide a lower bound of the form (10), giving a Hamming separation for the squared error. To that end, fix $\delta \in [0, 1]$, and for simplicity, let us assume that $d$ is even. In this case, we set $V = \{-1, 1\}^{d/2}$, and for $\nu \in V$ let $P_\nu$ be the multinomial distribution with parameter $\theta_\nu := 1/d + \frac{1}{d} \left[ \nu \right] \in \Delta_d$.

For any estimator $\hat{\theta}$, by defining $\hat{\nu}_j = \text{sign}(\hat{\theta}_j - 1/d)$ for $j \in [d/2]$ we have the lower bound

$$\|\hat{\theta} - \theta_\nu\|^2 \geq \delta^2 \frac{d}{d/2} \sum_{j=1}^{d/2} 1\{\hat{\nu}_j \neq \nu_j\},$$

so that by the sharper variant (31) of Assouad’s Lemma, we obtain

$$\max_{\nu \in V} \mathbb{E}_{P_\nu}[\|\hat{\theta} - \theta_\nu\|^2] \geq \frac{\delta^2}{4d} \left[ 1 - \left( \frac{1}{2d} \sum_{j=1}^{d/2} D_{\text{kl}}(M^n_{\nu,j} \| M^n_{-\nu,j}) + D_{\text{kl}}(M^n_{\nu,j} \| M^n_{-\nu,j}) \right)^{\frac{1}{2}} \right]. \quad (60)$$

Now we apply Theorem 3, which requires bounding sums of integrals $\int \gamma(P_{+j} - P_{-j})$, where $P_{+j}$ is defined in expression (32). We claim the following inequality:

$$\sup_{\gamma \in \mathcal{B}_\infty(X)} \sum_{j=1}^{d/2} \left( \int_X \gamma(x) dP_{+j}(x) - dP_{-j}(x) \right)^2 \leq \frac{8\delta^2}{d}. \quad (61)$$

Indeed, by construction $P_{+j}$ is the multinomial with parameter $(1/d)1 + \frac{\delta}{d}[e_j^\top - e_j^\top] \in \Delta_d$ and similarly for $P_{-j}$, where $e_j \in \{0, 1\}^{d/2}$ denotes the jth standard basis vector. Abusing notation and identifying $\gamma$ with vectors $\gamma \in [-1, 1]^d$, we have

$$\int_X \gamma(x) dP_{+j}(x) - dP_{-j}(x) = \frac{2\delta}{d} \gamma^\top \left[ e_j \right],$$

where $e_j = (e_j^1, \ldots, e_j^{d/2})^\top$ denotes the $j$th standard basis vector.
whence we find
\[ \sum_{j=1}^{d/2} \left( \int_X \gamma(x) dP_{+j}(x) - dP_{-j}(x) \right)^2 = \frac{4\delta^2}{d^2} \sum_{j=1}^{d/2} \begin{bmatrix} e_j \\ -e_j \end{bmatrix} \begin{bmatrix} e_j \\ -e_j \end{bmatrix}^\top \gamma = \frac{4\delta^2}{d^2} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \gamma \leq \frac{8\delta^2}{d}, \]

because the operator norm of the matrix is bounded by 2. This gives the claim (61).

Substituting the bound (61) into the bound (60) via Theorem 3, we obtain
\[ \max_{\nu \in \mathcal{V}} \mathbb{E}_{P_\nu} \left[ \|\hat{\theta} - \theta_{\nu}\|_2^2 \right] \geq \frac{\delta^2}{4d} \left[ 1 - (4n(e^\alpha - 1)^2 \delta^2/d^2)^{1/2} \right]. \]

Choosing \( \delta^2 = \min\{1, d^2/(16n(e^\alpha - 1)^2)\} \) gives the lower bound
\[ \mathcal{M}_n(\Delta_d, \|\cdot\|_2^2, \alpha) \geq \min \left\{ \frac{1}{4d}, \frac{d}{64n(e^\alpha - 1)^2} \right\}. \]

To complete the proof, we note that we can prove the preceding upper bound for any even \( d_0 \in \{2, \ldots, d\} \); this requires choosing \( \nu \in \mathcal{V} = \{-1, 1\}^{d_0/2} \) and constructing the multinomial vectors
\[ \theta_{\nu} = \frac{1}{d_0} \begin{bmatrix} 1_{d_0} \\ 0_{d-d_0} \end{bmatrix} + \frac{\delta}{d_0} \begin{bmatrix} \nu \\ -\nu \end{bmatrix} \in \Delta_d, \quad \text{where} \quad 1_{d_0} = [1 1 \ldots 1]^\top \in \mathbb{R}^{d_0}. \]

Repeating the proof *mutatis mutandis* gives the bound
\[ \mathcal{M}_n(\Delta_d, \|\cdot\|_2^2, \alpha) \geq \max_{d_0 \in \{2,4,\ldots,2\lfloor d/2 \rfloor\}} \min \left\{ \frac{1}{4d_0}, \frac{d_0}{64n(e^\alpha - 1)^2} \right\}. \]

Choosing \( d_0 \) to be the even integer closest to \( \sqrt{n\alpha^2} \) in \( \{1, \ldots, d\} \) and noting that \((e^\alpha - 1)^2 \leq 3\alpha^2\) for \( \alpha \in [0,1] \) gives the claimed result (35).

In the case of measuring error in the \( \ell_1 \)-norm, we provide a completely identical proof, except that we have the separation \( \|\hat{\theta} - \theta_{\nu}\|_1 \geq (\delta/d) \sum_{j=1}^{d/2} 1\{\nu_j \neq \nu_j\} \), and thus inequality (60) holds with the initial multiplier \( \delta^2/(4d) \) replaced by \( \delta/(4d) \). Parallel reasoning to the \( \ell_2 \) case then gives the minimax lower bound
\[ \mathcal{M}_n(\Delta_d, \|\cdot\|_1, \alpha) \geq \frac{\delta}{4d_0} \left[ 1 - (4n(e^\alpha - 1)^2 \delta^2/d_0^2)^{1/2} \right] \]

for any even \( d_0 \in \{2, \ldots, d\} \). Choosing \( \delta = \min\{1, d_0^2/(16n(e^\alpha - 1)^2)\} \) gives the claim (36).

**C Proofs of density estimation results**

In this section, we provide the proofs of the results stated in Section 5.3 on density estimation. We defer the proofs of more technical results to later appendices. Throughout all proofs, we use \( c \) to denote a universal constant whose value may change from line to line.
Figure 3. Panel (a): illustration of 1-Lipschitz continuous bump function $g_1$ used to pack $F_\beta$ when $\beta = 1$. Panel (b): bump function $g_2$ with $|g_2''(x)| \leq 1$ used to pack $F_\beta$ when $\beta = 2$.

C.1 Proof of Proposition 7

As with our proof for multinomial estimation, the argument follows the general outline described at the beginning of Section B. We remark that our proof is based on an explicit construction of densities identified with corners of the hypercube, a more classical approach than the global metric entropy approach of Yang and Barron [52] (cf. [53]). We use the local packing approach since it is better suited to the privacy constraints and information contractions that we have developed. In comparison with our proofs of previous propositions, the construction of a suitable packing of $F_\beta$ is somewhat more challenging: the identification of densities with finite-dimensional vectors, which we require for our application of Theorem 3 is not immediately obvious. In all cases, we guarantee that our density functions $f$ belong to the trigonometric Sobolev space, so we may work directly with smooth density functions $f$.

Constructing well-separated densities: We begin by describing a standard framework for defining local packings of density functions. Let $g_\beta : [0, 1] \rightarrow \mathbb{R}$ be a function satisfying the following properties:

(a) The function $g_\beta$ is $\beta$-times differentiable with

$$0 = g_\beta^{(i)}(0) = g_\beta^{(i)}(1/2) = g_\beta^{(i)}(1) \text{ for all } i < \beta.$$ 

(b) The function $g_\beta$ is centered with $\int_0^1 g_\beta(x) dx = 0$, and there exist constants $c, c_{1/2} > 0$ such that

$$\int_0^{1/2} g_\beta(x) dx = -\int_{1/2}^1 g_\beta(x) dx = c_{1/2} \text{ and } \int_0^1 \left(g_\beta^{(i)}(x)\right)^2 dx \geq c \text{ for all } i < \beta.$$ 

(c) The function $g_\beta$ is non-negative on $[0, 1/2]$ and non-positive on $[1/2, 1]$, and Lebesgue measure is absolutely continuous with respect to the measures $G_j, j = 1, 2$, given by

$$G_1(A) = \int_{A \cap [0, 1/2]} g_\beta(x) dx \text{ and } G_2(A) = -\int_{A \cap [1/2, 1]} g_\beta(x) dx. \quad (62)$$
Lastly, for almost every $x \in [0,1]$, we have $|g^{(\beta)}_\beta(x)| \leq 1$ and $|g_\beta(x)| \leq 1$.

As illustrated in Figure 3, the functions $g_\beta$ are smooth “bump” functions.

Fix a positive integer $k$ (to be specified in the sequel). Our first step is to construct a family of “well-separated” densities for which we can reduce the density estimation problem to one of identifying corners of a hypercube, which allows application of Lemma 1. Specifically, we must exhibit a condition similar to the separation condition (10). For each $j \in \{1,\ldots,k\}$ define the function

$$g_{\beta,j}(x) := \frac{1}{k \beta} g_\beta \left( k \left( x - \frac{j-1}{k} \right) \right) 1 \left\{ x \in \left[ \frac{j-1}{k}, \frac{j}{k} \right] \right\}.$$ 

Based on this definition, we define the family of densities

$$\left\{ f_\nu := 1 + \sum_{j=1}^{k} \nu_j g_{\beta,j} \text{ for } \nu \in V \right\} \subseteq \mathcal{F}_\beta.$$ 

(63)

It is a standard fact [53, 49] that for any $\nu \in V$, the function $f_\nu$ is $\beta$-times differentiable, satisfies $|f^{(\beta)}_\nu(x)| \leq 1$ for all $x$. Now, based on some density $f_\nu \in \mathcal{F}_\beta$, let us define the sign vector $v(f) \in \{-1,1\}^k$ to have entries

$$v_j(f) := \text{argmin}_{s \in \{-1,1\}} \int_{\left[ \frac{j-1}{k}, \frac{j}{k} \right]} (f(x) - sg_{\beta,j}(x))^2 \, dx.$$ 

Then by construction of the $g_\beta$ and $v$, we have for a numerical constant $c$ (whose value may depend on $\beta$) that

$$\|f - f_\nu\|_2^2 \geq c \sum_{j=1}^{k} 1 \{ v_j(f) \neq \nu_j \} \int_{\left[ \frac{j-1}{k}, \frac{j}{k} \right]} (g_{\beta,j}(x))^2 \, dx = \frac{c}{k^{2\beta+1}} \sum_{j=1}^{k} 1 \{ v_j(f) \neq \nu_j \}.$$ 

By inspection, this is the Hamming separation required in inequality (10), whence the sharper version (34) of Assouad’s Lemma 1 gives the result

$$\mathcal{M}_n\left( \mathcal{F}_\beta[1], \|\cdot\|_2^2, \alpha \right) \geq \frac{c}{k^{2\beta}} \left[ 1 - \left( \frac{1}{4k} \sum_{j=1}^{k} (D_{\text{kl}}(M_{\nu_j}^n \| M_{\nu_j}^n) + D_{\text{kl}}(M_{\nu_j}^n \| M_{\nu_j}^n)) \right)^{\frac{1}{2}} \right],$$ 

(64)

where we have defined $P_{\pm j}$ to be the probability distribution associated with the averaged densities $f_{\pm j} = 2^{1-k} \sum_{\nu: v_j = \pm 1} f_\nu$.

Applying divergence inequalities: Now we must control the summed KL-divergences. To do so, we note that by the construction (63), symmetry implies that

$$f_{+j} = 1 + g_{\beta,j} \quad \text{and} \quad f_{-j} = 1 - g_{\beta,j} \quad \text{for each } j \in [k].$$ 

(65)

We then obtain the following result, which bounds the averaged KL-divergences.
Lemma 10. For any $\alpha$-locally private conditional distribution $Q$, the summed KL-divergences are bounded as

$$\sum_{j=1}^{k} \left( D_{\text{kl}} \left( M_{j}^{\alpha} || M_{j}^{\beta} \right) + D_{\text{kl}} \left( M_{j}^{\beta} || M_{j}^{\alpha} \right) \right) \leq 4c_{1/2}^{2} n \left( e^{\alpha} - 1 \right)^{2} \frac{k^{2/3+1}}{k^{2/3+1}}.$$

The proof of this lemma is fairly involved, so we defer it to Appendix E.3. We note that, for $\alpha \leq 1$, we have $(e^\alpha - 1)^2 \leq 3\alpha^2$, so we may replace the bound in Lemma 10 with the quantity $cn\alpha^2/k^{2\beta+1}$ for a constant $c$. We remark that standard divergence bounds using Assouad’s lemma [53, 49] provide a bound of roughly $n/k^{2\beta}$; our bound is thus essentially a factor of the “dimension” $k$ tighter.

The remainder of the proof is an application of inequality (63). In particular, by applying Lemma 10, we find that for any $\alpha$-locally private channel $Q$, there are constants $c_0,c_1$ (whose values may depend on $\beta$) such that

$$\mathcal{M}_n \left( \mathcal{F}_\beta, \| \cdot \|_2^2, Q \right) \geq \frac{c_0}{k^{2\beta}} \left[ 1 - \left( \frac{c_1n\alpha^2}{k^{2\beta+2}} \right)^{1/2} \right].$$

Choosing $k_{n,\alpha,\beta} = \left( 4c_1n\alpha^2 \right)^{-2/3+2}$ ensures that the quantity inside the parentheses is at least $1/2$. Substituting for $k$ in the preceding display proves the proposition.

C.2 Proof of Proposition 8

Note that the operator $\Pi_k$ performs a Euclidean projection of the vector $(k/n) \sum_{i=1}^{n} Z_i$ onto the scaled probability simplex, thus projecting $\hat{f}$ onto the set of probability densities. Given the non-expansivity of Euclidean projection, this operation can only decrease the error $\|\hat{f} - f\|_2^2$. Consequently, it suffices to bound the error of the unprojected estimator; to reduce notational overhead we retain our previous notation of $\hat{\theta}$ for the unprojected version. Using this notation, we have

$$\mathbb{E} \left[ \|\hat{f} - f\|_2^2 \right] \leq \sum_{j=1}^{k} \mathbb{E}_f \left[ \int_{1/k}^{1} \left( f(x) - \frac{k}{n} \sum_{i=1}^{n} [e_k(X_i)]_j \right)^2 dx \right] + \sum_{j=1}^{k} \int_{1/k}^{1} \mathbb{E} \left[ \left( \frac{k}{n} \sum_{i=1}^{n} W_{ij} \right)^2 \right]$$

$$= \sum_{j=1}^{k} \int_{1/k}^{1} \mathbb{E}_f \left[ \left( f(x) - \frac{k}{n} \sum_{i=1}^{n} [e_k(X_i)]_j \right)^2 \right] dx + k \frac{4k^2}{k \alpha^2}.$$ (66)

Next we bound the error term inside the expectation (66). Defining $p_j := \mathbb{P}_f(X \in \mathcal{X}_j) = \int_{\mathcal{X}_j} f(x) dx$, we have

$$k \mathbb{E}_f \left[ [e_k(X)]_j \right] = kp_j = k \int_{\mathcal{X}_j} f(x) dx \in \left[ f(x) - \frac{1}{k}, f(x) + \frac{1}{k} \right]$$

for any $x \in \mathcal{X}_j$, for any $x \in \mathcal{X}_j$. 

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by the Lipschitz continuity of \( f \). Thus, expanding the bias and variance of the integrated expectation above, we find that

\[
E_f \left[ \left( f(x) - \frac{k}{n} \sum_{i=1}^{n} [e_k(X_i)]_j \right)^2 \right] \leq \frac{1}{k^2} + \text{Var} \left( \frac{k}{n} \sum_{i=1}^{n} [e_k(X_i)]_j \right)
\]

\[
= \frac{1}{k^2} + \frac{k^2}{n} \text{Var}(e_k(X)_j) = \frac{1}{k^2} + \frac{k^2}{n} p_j(1 - p_j).
\]

Recalling the inequality (66), we obtain

\[
E_f \left[ \| \hat{f} - f \|_2^2 \right] \leq \sum_{j=1}^{k} \int_{-1}^{1} \left( \frac{1}{k^2} + \frac{k^2}{n} p_j(1 - p_j) \right) dx + \frac{4k^2}{n\alpha^2} = \frac{1}{k^2} + \frac{4k^2}{n\alpha^2} + \frac{k^2}{n} \sum_{j=1}^{k} p_j(1 - p_j).
\]

Since \( \sum_{j=1}^{k} p_j = 1 \), we find that

\[
E_f \left[ \| \hat{f} - f \|_2^2 \right] \leq \frac{1}{k^2} + \frac{4k^2}{n\alpha^2} + \frac{k^2}{n},
\]

and choosing \( k = (n\alpha^2)^{\frac{1}{4}} \) yields the claim.

### C.3 Proof of Proposition 9

We begin by fixing \( k \in \mathbb{N} \); we will optimize the choice of \( k \) shortly. Recall that, since \( f \in F_\beta[C] \), we have \( f = \sum_{j=1}^{\infty} \theta_j \varphi_j \) for \( \theta_j = \int f \varphi_j \). Thus we may define \( Z_j = \frac{1}{n} \sum_{i=1}^{n} Z_{i,j} \) for each \( j \in \{1, \ldots, k\} \), and we have

\[
\| \hat{f} - f \|_2^2 = \sum_{j=1}^{k} (\theta_j - Z_j)^2 + \sum_{j=k+1}^{\infty} \theta_j^2.
\]

Since \( f \in F_\beta[C] \), we are guaranteed that \( \sum_{j=1}^{\infty} j^{2\beta} \theta_j^2 \leq C^2 \), and hence

\[
\sum_{j>k} \theta_j^2 \leq \frac{1}{k^{2\beta}} \sum_{j>k} j^{2\beta} \theta_j^2 \leq \frac{1}{k^{2\beta}} C^2.
\]

For the indices \( j \leq k \), we note that by assumption, \( E[Z_{i,j}] = \int \varphi_j f = \theta_j \), and since \( |Z_{i,j}| \leq B \), we have

\[
E \left[ (\theta_j - Z_j)^2 \right] = \frac{1}{n} \text{Var}(Z_{1,j}) \leq B^2/n = \frac{B_0^2}{c_k} \frac{k}{n} \left( \frac{e^{\alpha} + 1}{e^{\alpha} - 1} \right)^2,
\]

where \( c_k = \Omega(1) \) is the constant in expression (45). Putting together the pieces, the mean-squared \( L^2 \)-error is upper bounded as

\[
E_f \left[ \| \hat{f} - f \|_2^2 \right] \leq c \left( \frac{k^2}{n\alpha^2} + \frac{1}{k^{2\beta}} \right),
\]

where \( c \) is a constant depending on \( B_0, c_k, \) and \( C \). Choose \( k = (n\alpha^2)^{1/(2\beta+2)} \) to complete the proof.
C.4 Insufficiency of Laplace noise for density estimation

Finally, we consider the insufficiency of standard Laplace noise addition for estimation in the setting of this section. Consider the vector \([\varphi_j(X_i)]_{j=1}^k \in [-B_0, B_0]^k\). To make this vector \(\alpha\)-differentially private by adding an independent Laplace noise vector \(W \in \mathbb{R}^k\), we must take \(W_j \sim \text{Laplace}(\alpha/(B_0k))\). The natural orthogonal series estimator [e.g., 51] is to take \(Z_i = [\varphi_j(X_i)]_{j=1}^k + W_i\), where \(W_i \in \mathbb{R}^k\) are independent Laplace noise vectors. We then use the density estimator (40), except that we use the Laplacian perturbed \(Z_i\). However, this estimator suffers the following drawback:

**Observation 1.** Let \(\hat{f} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^k Z_{i,j} \varphi_j\), where the \(Z_i\) are the Laplace-perturbed vectors of the previous paragraph. Assume the orthonormal basis \(\{\varphi_j\}\) of \(L^2([0,1])\) contains the constant function. There is a constant \(c\) such that for any \(k \in \mathbb{N}\), there is an \(f \in F_\beta[2]\) such that

\[
\mathbb{E}_f \left[ \| f - \hat{f} \|_2^2 \right] \geq c(n\alpha^2)^{-\frac{2\beta}{2\beta + 4}}.
\]

**Proof.** We begin by noting that for \(f = \sum_j \theta_j \varphi_j\), by definition of \(\hat{f} = \sum_j \hat{\theta}_j \varphi_j\) we have

\[
\mathbb{E}\left[ \| f - \hat{f} \|_2^2 \right] = \sum_{j=1}^k \mathbb{E}\left[ (\theta_j - \hat{\theta}_j)^2 \right] + \sum_{j \geq k+1} \theta_j^2 = \sum_{j=1}^k \frac{B_0^2k^2}{n\alpha^2} + \sum_{j \geq k+1} \theta_j^2.
\]

Without loss of generality, let us assume \(\varphi_1 = 1\) is the constant function. Then \(\int \varphi_j = 0\) for all \(j > 1\), and by defining the true function \(f = \varphi_1 + (k+1)^{-\beta} \varphi_{k+1}\), we have \(f \in F_\beta[2]\) and \(\int f = 1\), and moreover,

\[
\mathbb{E}\left[ \| f - \hat{f} \|_2^2 \right] \geq \frac{B_0^2k^3}{n\alpha^2} + \frac{1}{(k+1)^{-2\beta}} \geq C_{\beta,B_0}(n\alpha^2)^{-\frac{2\beta}{2\beta + 4}},
\]

where \(C_{\beta,B_0}\) is a constant depending on \(\beta\) and \(B_0\). This final lower bound comes by minimizing over all \(k\). (If \((k+1)^{-\beta}B_0 > 1\), we can rescale \(\varphi_{k+1}\) by \(B_0\) to achieve the same result and guarantee that \(f \geq 0\).)

This lower bound shows that standard estimators based on adding Laplace noise to appropriate basis expansions of the data fail: there is a degradation in rate from \(n^{-\frac{2\beta}{2\beta + 2}}\) to \(n^{-\frac{2\beta}{2\beta + 3}}\). While this is not a formal proof that no approach based on Laplace perturbation can provide optimal convergence rates in our setting, it does suggest that finding such an estimator is non-trivial.

D Packing set constructions

In this appendix, we collect proofs of the constructions of our packing sets.

D.1 Proof of Lemma 5

By the Varshamov-Gilbert bound [e.g., 53, Lemma 4], there is a packing \(\mathcal{H}_d\) of the \(d\)-dimensional hypercube \([-1,1]^d\) of size \(|\mathcal{H}_d| \geq \exp(d/8)\) satisfying \(\|u - v\|_1 \geq d/2\) for all \(u, v \in \mathcal{H}_d\) with \(u \neq v\). For each \(u \in \mathcal{H}_d\), set \(\nu_u = u/\sqrt{d}\), so that \(\|\nu_u\|_2 = 1\) and \(\|\nu_u - \nu_v\|_2^2 \geq d/d = 1\) for \(u \neq v \in \mathcal{H}_d\). Setting \(\mathcal{V} = \{\nu_u \mid u \in \mathcal{H}_d\}\) gives the desired result.
D.2 Proof of Lemma 6

We use the probabilistic method [2], showing that for random draws from the Boolean hypercube, a collection of vectors as claimed in the lemma exists with positive probability. Consider a set of $N$ vectors $\nu^i \in \{-1, 1\}^k$ sampled uniformly at random from the Boolean hypercube, and for a fixed $t > 0$, define the two “bad” events

$$B_1 := \{\exists i \neq j \mid \|\nu^i - \nu^j\|_1 < k/2\}, \quad \text{and} \quad B_2(t) := \left\{ \frac{1}{N} \sum_{i=1}^{N} \nu^i(\nu^i)^\top \not\leq (t + 1)I_{k \times k} \right\}.$$

We begin by analyzing $B_1$. Letting $\{W_t\}_{t=1}^k$ denote a sequence of i.i.d. Bernoulli $\{0, 1\}$ variables, for any $i \neq j$, the event $\{\|\nu^i - \nu^j\|_1 < k/2\}$ is equivalent to the event $\{\sum_{t=1}^{k} W_t < k/4\}$. Consequently, by combining the union bound with the the Hoeffding bound, we find

$$\mathbb{P}(B_1) \leq \binom{N}{2} \mathbb{P}\left( \|\nu_i - \nu_j\|_1 < k/2 \right) \leq \binom{N}{2} \exp(-k/8). \quad (67)$$

Turning to the event $B_2(t)$, we have $\frac{1}{N} \sum_{i=1}^{N} \nu_i(\nu^i)^\top \not\leq (t + 1)I_{k \times k}$ if and only if the maximum eigenvalue $\lambda_{\max}(\frac{1}{N} \sum_{i=1}^{N} \nu_i(\nu^i)^\top - I_{k \times k})$ is larger than $t$. Using sharp versions of the Ahlswede-Winter inequalities [1] (see Corollary 4.2 in the paper [42]), we obtain

$$\mathbb{P}(B_2(t)) \leq k \exp\left(-\frac{Nt^2}{k^2}\right). \quad (68)$$

Finally, combining the union bound with inequalities (67) and (68), we find that

$$\mathbb{P}(B_1 \cup B_2(t)) \leq \frac{N(N-1)}{2} \exp(-k/8) + d \exp\left(-\frac{Nt^2}{k^2}\right).$$

By inspection, if we choose $t = 24$ and $N = \lceil \exp(k/16) \rceil$, the above bound is strictly less than 1, so a packing satisfying the constraints must exist.

E Information bounds

In this appendix, we collect the proofs of lemmas providing mutual information and KL-divergence bounds.

E.1 Proof of Lemma 7

Our strategy is to apply Theorem 2 to bound the mutual information. Without loss of generality, we may assume that $r = 1$ so the set $\mathcal{X} = \{\pm e_j\}_{j=1}^k$, where $e_j \in \mathbb{R}^d$. Thus, under the notation of Theorem 2 we may identify vectors $\gamma \in L^\infty(\mathcal{X})$ by vectors $\gamma \in \mathbb{R}^{2k}$. If we define $\bar{\gamma} = \frac{1}{\sqrt{2}} \sum_{\nu \in \mathcal{V}} \nu$ to be the mean element of the packing set, the linear functional $\varphi_\nu$ defined in Theorem 2 is

$$\varphi_\nu(\gamma) = \frac{1}{2k} \left[ \sum_{j=1}^{k} \gamma(e_j) \frac{1 + \nu_j \delta}{2} + \sum_{j=1}^{k} \gamma(-e_j) \frac{1 - \nu_j \delta}{2} \right] - \frac{1}{2k} \left[ \sum_{j=1}^{k} \gamma(e_j) \frac{1 + \bar{\gamma}_j \delta}{2} + \sum_{j=1}^{k} \gamma(-e_j) \frac{1 - \bar{\gamma}_j \delta}{2} \right]$$

$$= \frac{1}{2k} \sum_{j=1}^{k} \left[ \frac{\delta}{2} \gamma(e_j)(\nu_j - \bar{\gamma}_j) - \frac{\delta}{2} \gamma(-e_j)(\nu_j - \bar{\gamma}_j) \right] = \frac{\delta}{4k} \bar{\gamma}^\top \begin{bmatrix} I_{k \times k} & 0_{k \times d-k} \\ 0_{k \times d-k} & I_{k \times k} \end{bmatrix} (\nu - \bar{\gamma}).$$

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Define the matrix
\[ A := \begin{bmatrix} I_{k \times k} & 0_{k \times d-k} \\ -I_{k \times k} & 0_{k \times d-k} \end{bmatrix} \in \{-1, 0, 1\}^{2k \times d}. \]

Then we have that
\[
\frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \varphi_\nu(\gamma)^2 = \frac{\delta^2}{(4k)^2} \gamma^\top A \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} (\nu - \mathcal{V})(\nu - \mathcal{V})^\top A^\top \gamma \\
= \frac{\delta^2}{(4k)^2} \gamma^\top A \left( \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \nu \nu^\top - \mathcal{V} \mathcal{V}^\top \right) A^\top \gamma \leq \frac{\delta^2}{(4k)^2} \gamma^\top A \left( \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \nu \nu^\top \right) A^\top \gamma \\
\leq \frac{25}{16} \frac{\delta^2}{k^2} \gamma^\top A I_{2d \times 2d} A^\top \gamma = \left( \frac{5\delta}{4k} \right)^2 \gamma^\top \begin{bmatrix} I_{k \times k} & -I_{k \times k} \\ -I_{k \times k} & I_{k \times k} \end{bmatrix} \gamma.
\]

Here the final inequality used our assumption on the sum of outer products in \( \mathcal{V} \).

We complete our proof using the bound (69). The operator norm of the matrix specified in (69) is 2. As a consequence, since we have the containment
\[
\mathbb{B}_\infty = \left\{ \gamma \in \mathbb{R}^{2k} : \|\gamma\|_\infty \leq 1 \right\} \subset \left\{ \gamma \in \mathbb{R}^{2k} : \|\gamma\|_2 \leq 2k \right\}
\]
we have the inequality
\[
\sup_{\gamma \in \mathbb{B}_\infty} \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \varphi_\nu(\gamma)^2 \leq \frac{25\delta^2}{16k^2} \cdot 2 \cdot 2k = \frac{25}{4} \frac{\delta^2}{k}
\]
Applying Theorem 2 completes the proof.

### E.2 Proof of Lemma 8

It is no loss of generality to assume the radius \( r = 1 \). We use the notation of Theorem 2, recalling the linear functionals \( \varphi_\nu : L^\infty(\mathcal{X}) \to \mathbb{R} \). Because the set \( \mathcal{X} = \{-1, 1\}^d \), we can identify vectors \( \gamma \in L^\infty(\mathcal{X}) \) with vectors \( \gamma \in \mathbb{R}^{2d} \). Moreover, we have (by construction) that
\[
\varphi_\nu(\gamma) = \sum_{x \in \{-1,1\}^d} \gamma(x) p_\nu(x) - \sum_{x \in \{-1,1\}^d} \gamma(x) \mathcal{V}(x) \\
= \frac{1}{2d} \sum_{x \in \mathcal{X}} \gamma(x) (1 + \delta \nu^\top x - 1) = \frac{\delta}{2d} \sum_{x \in \mathcal{X}} \gamma(x) \nu^\top x.
\]

For each \( \nu \in \mathcal{V} \), we may construct a vector \( u_\nu \in \{-1, 1\}^{2d} \), indexed by \( x \in \{-1,1\}^d \), with
\[
u = e_j, \text{ we see that } u_{e_1}, \ldots, u_{e_d} \text{ are the first } d \text{ columns of the standard Hadamard transform matrix (and } u_{-e_j} \text{ are their negatives). Then we have that } \sum_{x \in \mathcal{X}} \gamma(x) \nu^\top x = \gamma^\top u_\nu, \text{ and } \varphi_\nu(\gamma) = \gamma^\top u_\nu u_\nu^\top \gamma.\]
Note also that \( u_\nu u_\nu^\top = u_{-\nu} u_{-\nu}^\top \), and as a consequence we have

\[
\sum_{\nu \in \mathcal{V}} \varphi_\nu(\gamma)^2 = \frac{\delta^2}{4d} \gamma^\top \sum_{\nu \in \mathcal{V}} u_\nu u_\nu^\top \gamma = \frac{2\delta^2}{4d} \gamma^\top \sum_{j=1}^d u_{e_j} u_{e_j}^\top \gamma. \tag{70}
\]

But now, studying the quadratic form (70), we note that the vectors \( u_{e_j} \) are orthogonal. As a consequence, the vectors (up to scaling) \( u_{e_j} \) are the only eigenvectors corresponding to positive eigenvalues of the positive semidefinite matrix \( \sum_{j=1}^d u_{e_j} u_{e_j}^\top \). Thus, since the set

\[
\mathcal{B}_\infty = \left\{ \gamma \in \mathbb{R}^{2d} : \|\gamma\|_\infty \leq 1 \right\} \subset \left\{ \gamma \in \mathbb{R}^{2d} : \|\gamma\|_2^2 \leq 4^d \right\},
\]

we have via an eigenvalue calculation that

\[
\sup_{\gamma \in \mathcal{B}_\infty} \sum_{\nu \in \mathcal{V}} \varphi_\nu(\gamma)^2 \leq \frac{2\delta^2}{4d} \sup_{\gamma : \|\gamma\|_2 \leq 2^d} \gamma^\top \sum_{j=1}^d u_{e_j} u_{e_j}^\top \gamma = \frac{2\delta^2}{4d} \|u_{e_1}\|_2^4 = 2\delta^2
\]

since \( \|u_{e_j}\|_2^2 = 2^d \) for each \( j \). Applying Theorem 2 and Corollary 4 completes the proof.

E.3 Proof of Lemma 10

This result relies on inequality (33) from Theorem 3, along with a careful argument to understand the extreme points of \( \gamma \in L^\infty([0,1]) \) that we use when applying the result. First, we take the packing \( \mathcal{V} = \{-1,1\}^\beta \) and densities \( f_\nu \) for \( \nu \in \mathcal{V} \) as in the construction (63). Overall, our first step is to show for the purposes of applying inequality (33), it is no loss of generality to identify \( \gamma \in L^\infty([0,1]) \) with vectors \( \gamma \in \mathbb{R}^{2k} \), where \( \gamma \) is constant on intervals of the form \([i/2k, (i+1)/2k]\). With this identification complete, we can then provide a bound on the correlation of any \( \gamma \in \mathcal{B}_\infty \) with the densities \( f_{\pm j} \) defined in (65), which completes the proof.

With this outline in mind, let the sets \( D_i, i \in \{1, 2, \ldots, 2k\} \), be defined as \( D_i = [(i-1)/2k, i/2k] \) except that \( D_{2k} = [(2k-1)/2k, 1] \), so the collection \( \{D_i\}_{i=1}^{2k} \) forms a partition of the unit interval \([0,1]\). By construction of the densities \( f_\nu \), the sign of \( f_\nu - 1 \) remains constant on each \( D_i \). Let us define (for shorthand) the linear functionals \( \varphi_j : L^\infty([0,1]) \to \mathbb{R} \) for each \( j \in \{1, \ldots, k\} \) via

\[
\varphi_j(\gamma) := \int \gamma(dP_{+j} - dP_{-j}) = \sum_{i=1}^{2k} \int_{D_i} \gamma(x)(f_{+j}(x) - f_{-j}(x))dx = 2 \int_{D_{j-1} \cup D_{j}} \gamma(x)g_{\beta,j}(x)dx,
\]

where we recall the definitions (65) of the mixture densities \( f_{\pm j} = 1 \pm g_{\beta,j} \). Since the set \( \mathcal{B}_\infty \) from Theorem 3 is compact, convex, and Hausdorff, the Krein-Milman theorem [15, Proposition 1.2] guarantees that it is equal to the convex hull of its extreme points; moreover, since the functionals \( \gamma \mapsto \varphi_j^2(\gamma) \) are convex, the supremum in Theorem 3 must be attained at the extreme points of \( \mathcal{B}_\infty([0,1]) \). As a consequence, when applying the divergence bound

\[
\sum_{j=1}^k \left( D_{kl}(M_{++j}^n M_{--j}^n) + D_{kl}(M_{--j}^n M_{++j}^n) \right) \leq 2n(e^n - 1)^2 \sup_{\gamma \in \mathcal{B}_\infty} \sum_{j=1}^k \varphi_j^2(\gamma), \tag{71}
\]
we can restrict our attention to \( \gamma \in \mathbb{B}_\infty \) for which \( \gamma(x) \in \{-1, 1\} \).

Now we argue that it is no loss of generality to assume that \( \gamma \), when restricted to \( D_i \), is a constant (apart from a measure zero set). Fix \( i \in [2k] \), and assume for the sake of contradiction that there exist sets \( B_i, C_i \subset D_i \) such that \( \gamma(B_i) = \{1\} \) and \( \gamma(C_i) = \{-1\} \), while \( \mu(B_i) > 0 \) and \( \mu(C_i) > 0 \) where \( \mu \) denotes Lebesgue measure. We will construct vectors \( \gamma_1 \) and \( \gamma_2 \in \mathbb{B}_\infty \) and a value \( \lambda \in (0, 1) \) such that

\[
\int_{D_i} \gamma(x)g_{\beta,j}(x)dx = \lambda \int_{D_i} \gamma_1(x)g_{\beta,j}(x)dx + (1 - \lambda) \int_{D_i} \gamma_2(x)g_{\beta,j}(x)dx
\]

simultaneously for all \( j \in [k] \), while on \( D_i^c = [0, 1] \setminus D_i \), we will have the equivalence

\[
\gamma_1|_{D_i^c} \equiv \gamma_2|_{D_i^c} \equiv \gamma|_{D_i^c}.
\]

Indeed, set \( \gamma_1(D_i) = \{1\} \) and \( \gamma_2(D_i) = \{-1\} \), otherwise setting \( \gamma_1(x) = \gamma_2(x) = \gamma(x) \) for \( x \not\in D_i \). For the unique index \( j \in [k] \) such that \([ (j - 1)/k, j/k ) \supset D_i \), we define

\[
\lambda := \frac{\int_{B_i} g_{\beta,j}(x)dx}{\int_{D_i} g_{\beta,j}(x)dx} \quad \text{so} \quad 1 - \lambda = \frac{\int_{C_i} g_{\beta,j}(x)dx}{\int_{D_i} g_{\beta,j}(x)dx}.
\]

By the construction of the function \( g_\beta \), the functions \( g_{\beta,j} \) do not change signs on \( D_i \), and the absolute continuity conditions on \( g_\beta \) specified in equation (62) guarantee \( 1 > \lambda > 0 \), since \( \mu(B_i) > 0 \) and \( \mu(C_i) > 0 \). We thus find that for any \( j \in [k] \),

\[
\int_{D_i} \gamma(x)g_{\beta,j}(x)dx = \int_{B_i} \gamma_1(x)g_{\beta,j}(x)dx + \int_{C_i} \gamma_2(x)g_{\beta,j}(x)dx
\]

\[
= \int_{B_i} g_{\beta,j}(x)dx - \int_{C_i} g_{\beta,j}(x)dx = \lambda \int_{D_i} g_{\beta,j}(x)dx - (1 - \lambda) \int_{D_i} g_{\beta,j}(x)dx
\]

\[
= \lambda \int_{D_i} \gamma_1(x)g_{\beta,j}(x)dx + (1 - \lambda) \int_{D_i} \gamma_2(x)g_{\beta,j}(x)dx.
\]

(Notably, for \( j \) such that \( g_{\beta,j} \) is identically 0 on \( D_i \), this equality is trivial.) By linearity and the strong convexity of the function \( x \mapsto x^2 \), then, we find that for sets \( E_j := D_{2j - 1} \cup D_{2j} \),

\[
\sum_{j=1}^{k} \varphi_j^2(\gamma) = \sum_{j=1}^{k} \left( \int_{E_j} \gamma(x)g_{\beta,j}(x)dx \right)^2
\]

\[
< \lambda \sum_{j=1}^{k} \left( \int_{E_j} \gamma_1(x)g_{\beta,j}(x)dx \right)^2 + (1 - \lambda) \sum_{j=1}^{k} \left( \int_{E_j} \gamma_2(x)g_{\beta,j}(x)dx \right)^2.
\]

Thus one of the densities \( \gamma_1 \) or \( \gamma_2 \) must have a larger objective value than \( \gamma \). This is our desired contradiction, which shows that (up to measure zero sets) any \( \gamma \) attaining the supremum in the information bound (71) must be constant on each of the \( D_i \).

Having shown that \( \gamma \) is constant on each of the intervals \( D_i \), we conclude that the supremum (71) can be reduced to a finite-dimensional problem over the subset

\[
\mathcal{B}_{1, 2k} := \left\{ u \in \mathbb{R}^{2k} \mid \|u\|_\infty \leq 1 \right\}
\]

\(^1\)For a function \( f \) and set \( A \), the notation \( f(A) \) denotes the image \( f(A) = \{ f(x) \mid x \in A \} \).
of \( \mathbb{R}^{2k} \). In terms of this subset, the supremum \((71)\) can be rewritten as the upper bound

\[
\sup_{\gamma \in \mathbb{B}_\infty} \sum_{j=1}^{k} \varphi_j(\gamma)^2 \leq \sup_{\gamma \in \mathbb{B}_{1,2k}} \sum_{j=1}^{k} \left( \gamma_{2j-1} \int_{D_{2j-1}} g_{\beta,j}(x)dx + \gamma_{2j} \int_{D_{2j}} g_{\beta,j}(x)dx \right)^2
\]

By construction of the function \( g_\beta \), we have the equality

\[
\int_{D_{2j-1}} g_{\beta,j}(x)dx = -\int_{D_{2j}} g_{\beta,j}(x)dx = \int_0^{\frac{1}{k}} g_{\beta,1}(x)dx = \int_0^{\frac{1}{k}} \frac{1}{k\beta} g_\beta(kx)dx = \frac{c_{1/2}}{k_{\beta+1}}.
\]

This implies that

\[
\frac{1}{2e^n(e^n - 1)^2n} \sum_{j=1}^{k} \left( D_{kl}(M^n_{+j}||M^n_{-j}) + D_{kl}(M^n_{+j}||M^n_{-j}) \right) \leq \sup_{\gamma \in \mathbb{B}_\infty} \sum_{j=1}^{k} \varphi_j(\gamma)^2
\]

\[
\leq \sup_{\gamma \in \mathbb{B}_{1,2k}} \sum_{j=1}^{k} \left( \frac{c_{1/2}}{k_{\beta+1}} \gamma^\top (e_{2j-1} - e_{2j}) \right)^2 = \frac{c_{1/2}}{k_{\beta+1}} \sup_{\gamma \in \mathbb{B}_{1,2k}} \gamma^\top \sum_{j=1}^{k} (e_{2j-1} - e_{2j})(e_{2j-1} - e_{2j})^\top \gamma, \quad (72)
\]

where \( e_j \in \mathbb{R}^{2k} \) denotes the \( j \)th standard basis vector. Rewriting this using the Kronecker product \( \otimes \), we have

\[
\sum_{j=1}^{k} (e_{2j-1} - e_{2j})(e_{2j-1} - e_{2j})^\top = I_{k \times k} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leq 2I_{2k \times 2k}.
\]

Combining this bound with our inequality \((72)\), we obtain

\[
\sum_{j=1}^{k} \left( D_{kl}(M^n_{+j}||M^n_{-j}) + D_{kl}(M^n_{+j}||M^n_{-j}) \right) \leq 4n(e^n - 1)^2 \frac{c_{1/2}}{k_{2\beta+2}} \sup_{\gamma \in \mathbb{B}_{1,2k}} \|\gamma\|_2^2 = 4c_{1/2} \frac{n(e^n - 1)^2}{k_{2\beta+1}}.
\]

F Technical arguments

In this appendix, we collect proofs of technical lemmas and results needed for completeness.

F.1 Proof of Lemma 11

Fix an (arbitrary) estimator \( \hat{\theta} \). By assumption \((10)\), we have

\[
\Phi(\rho(\theta, \theta(P_\nu))) \geq 2\delta \sum_{j=1}^{d} \mathbb{1}\{[\nu(\theta)]_j \neq \nu_j\}.
\]

Taking expectations, we see that

\[
\sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \Phi(\rho(\tilde{\theta}(Z_1, \ldots, Z_n), \theta(P))) \right] \geq \max_{\nu \in \mathcal{V}} \mathbb{E}_P \left[ \Phi(\rho(\tilde{\theta}(Z_1, \ldots, Z_n), \theta(\nu))) \right]
\]

\[
\geq \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} \mathbb{E}_P \left[ \Phi(\rho(\tilde{\theta}(Z_1, \ldots, Z_n), \theta(\nu))) \right]
\]

\[
\geq \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} 2\delta \sum_{j=1}^{d} \mathbb{E}_P \left[ \mathbb{1}\{[\nu(\tilde{\theta})]_j \neq \nu_j\} \right]
\]

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as the average is smaller than the maximum of a set and using the separation assumption (10). Recalling the definition (32) of the mixtures $P_{\pm j}$, we swap the summation orders to see that

$$
\frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} P_\nu \left( [\nu(\hat{\theta})]_j \neq \nu_j \right) = \frac{1}{|\mathcal{V}|} \sum_{\nu, \nu_j = 1} P_\nu \left( [\nu(\hat{\theta})]_j \neq \nu_j \right) + \frac{1}{|\mathcal{V}|} \sum_{\nu, \nu_j = -1} P_\nu \left( [\nu(\hat{\theta})]_j \neq \nu_j \right) = \frac{1}{2} P_{+j} \left( [\nu(\hat{\theta})]_j \neq \nu_j \right) + \frac{1}{2} P_{-j} \left( [\nu(\hat{\theta})]_j \neq \nu_j \right).
$$

This gives the statement claimed in the lemma, while taking an infimum over all testing procedures $\psi : \mathcal{Z}^n \to \{-1, +1\}$ gives the claim (11).

### F.2 Proof of unbiasedness for sampling strategy (27a)

We compute the expectation of a random variable $Z$ sampled according to the strategy (27a), i.e. we compute $\mathbb{E}[Z \mid v]$ for a vector $v \in \mathbb{R}^d$. By scaling, it is no loss of generality to assume that $\|v\|_2 = 1$, and using the rotational symmetry of the $\ell_2$-ball, we see it is no loss of generality to assume that $v = e_1$, the first standard basis vector.

Let the function $s_d$ denote the surface area of the sphere in $\mathbb{R}^d$, so that

$$
s_d(r) = \frac{d\pi^{d/2}}{\Gamma(d/2 + 1)} r^{d-1}
$$

is the surface area of the sphere of radius $r$. (We use $s_d$ as a shorthand for $s_d(1)$ when convenient.) Then for a random variable $W$ sampled uniformly from the half of the $\ell_2$-ball with first coordinate $W_1 \geq 0$, symmetry implies that by integrating over the radii of the ball,

$$
\mathbb{E}[W] = e_1 \frac{2}{s_d} \int_0^1 s_{d-1} \left( \sqrt{1 - r^2} \right) r dr.
$$

Making the change of variables to spherical coordinates (we use $\phi$ as the angle), we have

$$
\frac{2}{s_d} \int_0^1 s_{d-1} \left( \sqrt{1 - r^2} \right) r dr = \frac{2}{s_d} \int_0^{\pi/2} s_{d-1} (\cos \phi) \sin \phi d\phi = \frac{2s_{d-1}}{s_d} \int_0^{\pi/2} \cos^{d-2}(\phi) \sin(\phi) d\phi.
$$

Noting that $\frac{d}{d\phi} \cos^{d-1}(\phi) = -(d - 1) \cos^{d-2}(\phi) \sin(\phi)$, we obtain

$$
\frac{2s_{d-1}}{s_d} \int_0^{\pi/2} \cos^{d-2}(\phi) \sin(\phi) d\phi = -\left. \frac{\cos^{d-1}(\phi)}{d-1} \right|_0^{\pi/2} = \frac{1}{d-1},
$$

or that

$$
\mathbb{E}[W] = e_1 \frac{(d-1)\pi^{d-1}}{d\pi^{\frac{d-1}{2}} \Gamma(d/2 + 1)} \frac{1}{d-1} = e_1 \frac{\Gamma(d/2 + 1)}{\sqrt{\pi} \Gamma(d/2 + 1) c_d}.
$$

where we define the constant $c_d$ to be the final ratio.

Allowing again $\|v\|_2 \leq r$, with the expression (73), we see that for our sampling strategy for $Z$, we have

$$
\mathbb{E}[Z \mid v] = v B \frac{e^\alpha}{r c_d \left( e^\alpha + 1 \right)} \left( e^\alpha - \frac{1}{e^\alpha + 1} \right) = B \frac{c_d e^\alpha - 1}{r c_d e^\alpha + 1}.
$$
Consequently, the choice
\[ B = \frac{e^\alpha + 1}{e^\alpha - 1} c_d = \frac{e^\alpha + 1}{e^\alpha - 1} r \sqrt{\pi d} \Gamma\left(\frac{d-1}{2} + 1\right) \]
yields \( \mathbb{E}[Z \mid v] = v \). Moreover, we have
\[ \|Z\|_2 = B \leq \frac{e^\alpha + 1}{e^\alpha - 1} 3 \sqrt{\pi d} \]
by Stirling’s approximation to the \( \Gamma \)-function. By noting that \( (e^\alpha + 1)/(e^\alpha - 1) \leq 3/\alpha \) for \( \alpha \leq 1 \), we see that \( \|Z\|_2 \leq 8r \sqrt{d}/\alpha \).

F.3 Proof of Lemma 4

For any \( x > 0 \), the concavity of the logarithm implies that
\[ \log(x) \leq x - 1. \]
Setting alternatively \( x = a/b \) and \( x = b/a \), we obtain the inequalities
\[ \log \frac{a}{b} \leq \frac{a}{b} - 1 = \frac{a-b}{b} \quad \text{and} \quad \log \frac{b}{a} \leq \frac{b}{a} - 1 = \frac{b-a}{a}. \]
Using the first inequality for \( a \geq b \) and the second for \( a < b \) completes the proof.

G Effects of differential privacy in non-compact spaces

In this appendix, we present a somewhat pathological example that demonstrates the effects of differential privacy in non-compact spaces. Let us assume only that \( \theta \in \mathbb{R} \) and \( \alpha < \infty \), and we denote \( P_\theta \) to be the collection of probability measures with variance 1 having \( \theta \) as a mean. In contrast to the non-private case, where the risk of the sample mean scales as \( 1/n \), we obtain
\[ M_n(\mathbb{R}, (\cdot)^2, \alpha) = \infty \]
for all \( n \in \mathbb{N} \). To see this, consider the Fano inequality version \( 9 \). Fix \( \delta > 0 \) and choose \( \{\theta_1 = 0, \theta_2 = 2\delta, \ldots, \theta_N = 2N\delta\} \) where \( N = N(\delta, n) = \max\{\left\lceil \exp(64(\alpha - 1)^2n) \right\rceil, 2^4\} \). Then by applying Corollary 1, we have for \( \mathcal{V} = [N] \) that
\[ \mathcal{M}_n(\mathbb{R}, (\cdot)^2, \alpha) \geq \delta^2 \left( 1 - \frac{4(e^\alpha - 1)^2n}{\log N(\delta, n)} \right). \]
We have \( \|P_\nu - P_{\nu'}\|_{TV} \leq 1 \) for any two distributions \( P_\nu \) and \( P_{\nu'} \), which implies
\[ \mathcal{M}_n(\mathbb{R}, (\cdot)^2, \alpha) \geq \delta^2 \left( 1 - \frac{16(e^\alpha - 1)^2n + \log 2}{\log N(\delta, n)} \right) \geq \delta^2 \left( 1 - \frac{1}{2} \right) = \frac{1}{2} \delta^2. \]
Since \( \delta > 0 \) was arbitrary, this proves the infinite minimax risk bound \( 74 \). The construction to achieve \( 74 \) is somewhat contrived, but it suggests that care is needed when designing differentially private inference procedures, and shows that even in cases when it is possible to attain a parametric rate of convergence, there may be no (locally) differentially private inference procedure.
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