Constructing cospectral hypergraphs

Aida Abiad *†‡ Antonina P. Khramova §

Abstract

Spectral hypergraph theory mainly concerns using hypergraph spectra to obtain structural information about the given hypergraphs. The study of cospectral hypergraphs is important since it reveals which hypergraph properties cannot be deduced from their spectra. In this paper, we show a new method for constructing cospectral uniform hypergraphs using two well-known hypergraph representations: adjacency tensors and adjacency matrices.

1 Introduction

Spectral hypergraph theory seeks to deduce structural properties about the hypergraph using its spectra. This field has received a lot of attention in the last two decades, see for example [4, 5, 6, 12, 16, 17, 20, 21, 22, 24, 25]. The study of cospectral hypergraphs is important since it reveals which hypergraph properties cannot be deduced from their spectra. While the construction of cospectral graphs has been investigated extensively in the literature (see e.g. [1, 7, 8, 15, 19]), much less is known about the construction of cospectral hypergraphs. In this regard, Bu, Zhou, and Wei [3] presented a switching method for constructing $E$-cospectral hypergraphs which is based on the widely used Godsil-McKay switching (GM-switching) for graphs [7]. Another extension of GM-switching was shown by Banerjee and Sarkar in [18] for a matrix representation of a hypergraph using a natural generalization of the adjacency matrix for simple graphs.

In this paper, we show a new method for constructing $E$-cospectral hypergraphs (with respect to their $E$-characteristic polynomials). We also propose a new method for constructing uniform hypergraphs which are cospectral with respect to their adjacency matrices. Both of these methods are based on the recently introduced graph switching by Wang, Qiu and Hu (WQH-switching) [23, 15].

* a.abiad.monge@tue.nl, Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands
† Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
‡ Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium
§ a.khramova@tue.nl, Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands
2 Cospectral and $E$-cospectral hypergraphs using adjacency tensors

For a positive integer $n$, let $[n] = \{1, \ldots, n\}$. An order $k$ dimension $n$ tensor $A = (a_{i_1 \ldots i_k}) \in \mathbb{C}^{n \times \cdots \times n}$ is a multidimensional array with $n^k$ entries, where $i_j \in [n]$, $j = 1, \ldots, k$. For example, in case $k = 1$, $A$ is a column vector of dimension $n$, and in case $k = 2$, $A$ is an $n \times n$ matrix.

The following tensor multiplication was introduced by Shao [20] as a generalization of the matrix multiplication.

**Definition 2.1.** [20] Let $A$ and $B$ be order $m \geq 2$ and order $k \geq 1$, dimension $n$ tensors, respectively. The product $AB$ is the following tensor $C$ of order $(m-1)(k-1) + 1$ and dimension $n$ with entries:

$$c_{i \alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n]} a_{i i_2 \ldots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}},$$

where $i \in [n]$, $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$.

In particular, according to [20, Example 1.1], for an order $k \geq 2$ dimension $n$ tensor $A$ and a vector $x = (x_1, \ldots, x_n)^\top$ we can derive that the product $Ax$ is a vector with $i$-th component calculated by

$$(Ax)_i = \sum_{i_2, \ldots, i_k \in [n]} a_{i i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}.$$

In 2005, Qi [13] and Lim [11] independently introduced the concept of tensor eigenvalues with two different definitions. As we will see below, both of them generalize the notion of matrix eigenvalue in their own way. Since then, a vast number of authors have used such definitions to study spectral properties of hypergraphs [4, 5, 20, 21, 22, 24, 25].

Next, we introduce the definitions of characteristic and $E$-characteristic polynomials of a tensor.

Let $A$ be an order $k$ dimension $n$ tensor. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $A$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x_{[k-1]}$, where $x_{[k-1]} = (x_1^{k-1}, \ldots, x_n^{k-1})^\top$. The determinant of $A$, denoted by $\det(A)$, is the resultant of the system of polynomials $f_i(x_1, \ldots, x_n) = (Ax)_i$ for all $i \in [n]$. The characteristic polynomial of $A$ is defined as $\Phi_A(\lambda) = \det(\lambda I_n - A)$, where $I_n$ is the unit tensor of order $k$ and dimension $n$, i.e. a tensor of elements $\delta_{i_1 \ldots i_k}$ such that

$$\delta_{i_1 \ldots i_k} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_k, \\ 0, & \text{otherwise}. \end{cases}$$

It is known that the eigenvalues of $A$ are exactly the roots of $\Phi_A(\lambda)$ [20].

On the other hand, for an order $k \geq 2$ dimension $n$ tensor $A$, a number $\lambda \in \mathbb{C}$ is called an $E$-eigenvalue of $A$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$ and $x^\top x = 1$. In [14], the $E$-characteristic polynomial of $A$ is defined as
\[ \phi_A(\lambda) = \begin{cases} 

\text{Res}_x(Ax - \lambda(x^T x)^{\frac{k-2}{2}}), & \text{if } k \text{ is even}, \\

\text{Res}_{x,\beta} \left( A x - \beta x^T x - \beta^2 x \right), & \text{if } k \text{ is odd}, 
\end{cases} \]

where ‘Res’ is the resultant of the system of polynomials. It is known that \( E \)-eigenvalues of \( A \) are roots of \( \phi_A(\lambda) \) \[14\]. If \( k = 2 \), then \( \phi_A(\lambda) = \Phi_A(\lambda) \) is just the characteristic polynomial of the square matrix \( A \).

A hypergraph \( G = (V(G), E(G)) \) is called \( k \)-uniform if each edge of \( G \) contains exactly \( k \) distinct vertices. All hypergraphs in this paper are uniform and simple. The adjacency tensor of \( G \), denoted by \( A_G \), is an order \( k \) dimension \( |V(G)| \) tensor with entries \[4\]:

\[
\left[ a_{i_1, i_2 \ldots i_k} \right] = \begin{cases} 

\frac{1}{(k-1)!}, & \{i_1, i_2, \ldots, i_k\} \in E(G), \\

0, & \text{otherwise}.
\end{cases}
\]

We say that two \( k \)-uniform hypergraphs are cospectral (\( E \)-cospectral) if their adjacency tensors have the same characteristic polynomial (\( E \)-characteristic polynomial).

In order to state our main result in this section, we need some preliminary work. We shall follow the same notation as in \[3\].

Let \( G = (V(G), E(G)) \) be a \( k \)-uniform hypergraph. The degree of a vertex \( u \in V(G) \) is the number of edges that contain \( u \). For any edge \( \{u_1, \ldots, u_k\} \in E(G) \), we say that \( u_1 \) is a neighbour of \( \{u_2, \ldots, u_k\} \). The neighbourhood of \( \{u_2, \ldots, u_k\} \) is denoted by \( \Gamma(u_2, \ldots, u_k) \).

We say that \( A \) is a symmetric tensor if \( a_{i_1, i_2 \ldots i_k} = a_{\sigma(1), \sigma(2) \ldots \sigma(k)} \) for any permutation \( \sigma \) on \( [k] \). From the definition of the adjacency tensor, it is easy to observe that for a hypergraph \( G \), the tensor \( A_G \) is symmetric.

The following lemma can be obtained from \[20\ Eq. (2.1)\].

**Lemma 2.2.** Let \( A = (a_{i_1 \ldots i_k}) \) be an order \( k \geq 2 \) dimension \( n \) tensor, and let \( Q = (q_{i,j}) \) be an \( n \times n \) matrix. Then

\[
(QAQ^T)_{i_1 \ldots i_k} = \sum_{j_1 \ldots j_k \in [n]} a_{j_1 \ldots j_k} q_{i_1,j_1} q_{i_2,j_2} \cdots q_{i_k,j_k}.
\]

From Lemma 2.2 we obtain the following result.

**Lemma 2.3.** Let \( A' = QAQ^T \), where \( A \) is a tensor of dimension \( n \) and \( Q \) is an \( n \times n \) matrix. If \( A \) is symmetric, then \( A' \) is symmetric.

Additionally, let \( Q \) be a real orthogonal matrix. In \[20\], Shao pointed out that \( A \) and \( A' = QAQ^T \) are orthogonally similar tensors as defined by Qi \[13\], which implies that they have the same set of \( E \)-eigenvalues. In this case, the \( E \)-characteristic polynomials are also the same (see also \[10\]):

**Lemma 2.4.** Let \( A' = QAQ^T \), where \( A \) is a tensor of dimension \( n \) and \( Q \) is an \( n \times n \) real orthogonal matrix. Then \( A \) and \( A' \) have the same \( E \)-characteristic polynomial.
Let $I_n$ be the unit tensor of order $k$ and dimension $n$, i.e. a tensor of elements $\delta_{i_1 \cdots i_k}$ such that
\[
\delta_{i_1 \cdots i_k} = \begin{cases} 
1, & i_1 = i_2 = \cdots = i_k, \\
0, & \text{otherwise.} 
\end{cases}
\]

A claim analogous to Lemma 2.4 can be made for characteristic polynomials of tensors, as it is a straightforward consequence of [20, Theorem 2.1]:

**Lemma 2.5.** Let $A' = QAQ^\top$, where $A$ is a tensor of dimension $n$ and $Q$ is an $n \times n$ real orthogonal matrix such that $QI_nQ^\top = I_n$. Then $A$ and $A'$ have the same characteristic polynomial.

Note that the additional identity $QI_nQ^\top = I_n$ does not hold in general for a real orthogonal matrix $Q$, making cospectrality of tensors a stronger property than $E$-cospectrality.

Lemma 2.4 will be the key ingredient for proving the $E$-cospectrality of the hypergraphs constructed using the method described in Section 2.1.

### 2.1 Constructing $E$-cospectral hypergraphs using adjacency tensors

Inspired by the WQH-switching [23] for graphs, we propose a method to construct $E$-cospectral hypergraphs.

**Theorem 2.6.** Let $G$ be a $k$-uniform hypergraph on $n$ vertices that satisfies the following conditions:

1. The vertex set $V(G)$ is partitioned into three sets $C_1 \cup C_2 \cup D$ with $|C_1| = |C_2| = t$.
2. For any edge $\{u_1, \ldots, u_k\} \in E(G)$, there is at most one vertex in $C_1 \cup C_2$, i.e. $|\{u_1, \ldots, u_k\} \cap (C_1 \cup C_2)| \leq 1$.
3. For any $k-1$ distinct vertices $u_2, \ldots, u_k$ from $D$, we have $\Gamma(u_2, \ldots, u_k) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_1| = |\Gamma(u_2, \ldots, u_k) \cap C_2|$.

To construct a hypergraph $H$, for any $\{u_2, \ldots, u_k\} \subseteq D$ that has all its neighbours in $C_1$ (or $C_2$), switch the adjacency of $\{u_1, \ldots, u_k\}$ for all $u_1 \in C_1 \cup C_2$. Then $H$ is a $k$-uniform $E$-cospectral hypergraph with $G$.

**Proof.** To prove this result, we will show that
\[
A_H = QA_GQ^\top,
\]
where $A_G$ and $A_H$ are the adjacency tensors of $G$ and $H$, and
\[
Q = \begin{pmatrix}
I_t - \frac{1}{t}J_t & \frac{1}{t}J_t & 0 \\
\frac{1}{t}J_t & I_t - \frac{1}{t}J_t & 0 \\
0 & 0 & I_{n-2t}
\end{pmatrix}.
\]
Since $Q$ is a real orthogonal matrix, the $E$-cospectrality of $G$ and $H$ follows from Lemma 2.4.

Let $A' = QA_GQ^T$. According to Lemma 2.3, $A'$ is symmetric. By Lemma 2.2,

$$(A')_{i_1 \ldots i_k} = \sum_{j_1, \ldots, j_k \in V(G)} a_{j_1 \ldots j_k} q_{i_1 j_1} \cdots q_{i_k j_k}.$$

We need to show that $A' = A_H$.

First, consider the case $\{|i_1, \ldots, i_k\} \cap (C_1 \cup C_2)| = 0$, or $\{i_1, \ldots, i_k\} \subseteq D$. Then, for any $s \in [k]$, $q_{i_s j_s} = 1$ if and only if $i_s = j_s$ and is equal to 0 otherwise. Hence

$$(A')_{i_1 \ldots i_k} = a_{i_1 \ldots i_k} \text{ if } \{i_1, \ldots, i_k\} \subseteq D.$$

Next, if $\{|i_1, \ldots, i_k\} \cap (C_1 \cup C_2)| \geq 2$, then $a_{i_1 \ldots i_k} = 0$ since every edge of $H$ has no more than one vertex in $C_1 \cup C_2$. We consider three cases for $\{j_1, \ldots, j_k\}$:

- If $\{|j_1, \ldots, j_k\} \cap (C_1 \cup C_2)| = 0$ then without loss of generality we may assume that $i_1 \in (C_1 \cup C_2)$. Since $j_1 \in D$, we have $q_{i_1 j_1} = 0$.
- If $\{|j_1, \ldots, j_k\} \cap (C_1 \cup C_2)| \geq 2$ then $\{j_1, \ldots, j_k\}$ is not an edge of $G$ and $a_{j_1 \ldots j_k} = 0$.
- If $\{|j_1, \ldots, j_k\} \cap (C_1 \cup C_2)| = 1$ then without loss of generality we may assume that $i_1, i_2 \in (C_1 \cup C_2)$. Then, either $j_1 \in D$ or $j_2 \in D$, hence we have $q_{i_1 j_1} = 0$ or $q_{i_2 j_2} = 0$.

This argument implies that every term in the sum that defines $(A')_{i_1 \ldots i_k}$ is equal to zero, and so

$$(A')_{i_1 \ldots i_k} = 0 = a_{i_1 \ldots i_k} \text{ if } \{|i_1, \ldots, i_k\} \cap (C_1 \cup C_2)| \geq 2.$$

The final case is $\{|i_1, \ldots, i_k\} \cap (C_1 \cup C_2)| = 1$. Without loss of generality we may assume $i_1 \in (C_1 \cup C_2)$ and $i_2, \ldots, i_k \in D$. Since $q_{s, j_s} = 0$ for any $s \in \{2, \ldots, k\}$ unless $j_s = i_s$ when $q_{s, j_s} = 1$, and also $q_{i_1 j_1} = 0$ if $j_1 \in D$, we have

$$(A')_{i_1 \ldots i_k} = \sum_{j_1 \in (C_1 \cup C_2)} a_{j_1 i_2 \ldots i_k} q_{i_1 j_1}.$$

We assume $i_1 \in C_1$, and the case $i_1 \in C_2$ can be considered analogously.

There are three possibilities:

**Case 1.** The set $\{i_2, \ldots, i_k\}$ has every vertex of $C_1$ as a neighbour and no neighbours in $C_2$. This implies $i_1$ is one of these neighbours, so $a_{i_1 i_2 \ldots i_k} = \frac{1}{(k-1)!}$. Then

$$(A')_{i_1 \ldots i_k} = \sum_{j_1 \in (C_1 \cup C_2)} a_{j_1 i_2 \ldots i_k} q_{i_1 j_1} \frac{1}{(k-1)!} \sum_{j_1 \in C_1} q_{i_1 j_1}$$

$$= \frac{1}{(k-1)!} \sum_{j_1 \in C_1, j_1 \neq i_1} q_{i_1 j_1} + \frac{1}{(k-1)!} q_{i_1 i_1}$$

$$= \frac{1}{(k-1)!} \left(-\left(t-1\right)\frac{1}{t} + \frac{t-1}{t}\right) = 0 \neq a_{i_1 i_2 \ldots i_k}.$$
Case 2. The set \( \{i_2, \ldots, i_k\} \) has every vertex of \( C_2 \) as a neighbour and no neighbours in \( C_1 \), implying \( a_{i_1 i_2 \ldots i_k} = 0 \). Then
\[
(A')_{i_1 \ldots i_k} = \sum_{j_1 \in (C_1 \cup C_2)} a_{j_1 i_1 \ldots i_k} q_{i_1 j_1} = \sum_{j_1 \in C_2} a_{j_1 i_1 \ldots i_k} q_{i_1 j_1} = t \cdot \frac{1}{(k-1)!} \cdot \frac{1}{t} = \frac{1}{(k-1)!} \neq a_{i_1 i_2 \ldots i_k}.
\]

Case 3. The set \( \{i_2, \ldots, i_k\} \) has \( r \leq t \) neighbours in \( C_1 \) as well as in \( C_2 \). Then
\[
(A')_{i_1 \ldots i_k} = \sum_{j_1 \in C_1} a_{j_1 i_1 \ldots i_k} q_{i_1 j_1} + \sum_{j_1 \in C_2} a_{j_1 i_1 \ldots i_k} q_{i_1 j_1} = \sum_{j_1 \in C_1} a_{j_1 i_1 \ldots i_k} + r \cdot \frac{1}{(k-1)!} \cdot \frac{1}{t}.
\]

Depending on whether \( i_1 \) is one of the \( r \) neighbours in \( C_1 \) or not, we have either
\[
(A')_{i_1 \ldots i_k} = -\frac{1}{t} \cdot r \cdot \frac{1}{(k-1)!} + r \cdot \frac{1}{(k-1)!} \cdot \frac{1}{t} = 0 \text{ when } a_{i_1 i_2 \ldots i_k} = 0,
\]

or
\[
(A')_{i_1 \ldots i_k} = -\frac{1}{t} \cdot (r - 1) \cdot \frac{1}{(k-1)!} + \frac{t - 1}{t} \cdot \frac{1}{(k-1)!} + r \cdot \frac{1}{(k-1)!} \cdot \frac{1}{t} = \frac{1}{(k-1)!} \text{ when } a_{i_1 i_2 \ldots i_k} \neq 0.
\]

Either way we derive that in Case 3,
\[
(A')_{i_1 \ldots i_k} = a_{i_1 i_2 \ldots i_k}.
\]

By considering every possible case we see that \( A' \) is the adjacency tensor of a hypergraph that can be obtained from \( G \) by switching adjacency of every edge \( \{i_1, i_2, \ldots, i_k\} \) such that \( \{i_2, \ldots, i_k\} \subseteq D \) and \( i_1 \in (C_1 \cup C_2) \) in case \( \Gamma(i_2, \ldots, i_k) \cap (C_1 \cup C_2) \in \{C_1, C_2\} \), which is exactly \( H \) by construction. Hence \( A' = A_H \) as required.

**Remark 2.7.** Note that the matrix \( Q \) defined in the proof of Theorem 2.6 in general does not satisfy \( Q \mathcal{L}_n Q^\top = \mathcal{I}_n \). Hence the hypergraph \( H \) obtained from \( G \) as the result of switching described in Theorem 2.6 is \( E \)-cospectral but not necessarily cospectral with \( G \).

We will refer to the switching of Theorem 2.6 as \( E \)-WQH switching. The paper [3] describes a switching to construct \( E \)-cospectral hypergraphs by a method which is similar to GM-switching [7]. This will be referred to as \( E \)-GM switching.

Note that the switching described in Theorem 2.6 is based on the simplified version of WQH-switching for ordinary graphs. However, the cospectrality of the hypergraphs constructed using the more general version of the switching can also be argued in a similar
Let $G$ be a $k$-uniform hypergraph such that $V(G)$ admits partition $C_1 \cup C_2 \cup \cdots \cup C_m \cup D$ for some integer $m \geq 1$ with $|C_i| = |C_{i+1}|$ for any odd $i \in [2m]$, any edge has at most one vertex in $V(G) \setminus D$, and for each odd $i \in [2m]$ and a subset $\{u_2, \ldots, u_k\} \subseteq D$ we have either $\Gamma(u_2, \ldots, u_k) \cap (C_i \cup C_{i+1}) \in \{C_i, C_{i+1}\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_i| = |\Gamma(u_2, \ldots, u_k) \cap C_{i+1}|$. Then a hypergraph $H$ is constructed by taking all subsets $\{u_2, \ldots, u_k\} \subseteq D$ such that $\Gamma(u_2, \ldots, u_k) \cap (C_i \cup C_{i+1}) \in \{C_i, C_{i+1}\}$ for some odd $i \in [2m]$ and switching adjacency between the subset and all the vertices in $C_i \cup C_{i+1}$. A similar observation can be made regarding to $E$-GM switching, which is based on the simplified version of GM-switching but can be extended to its general version [7].

**Remark 2.8.** If $G$ is a hypergraph that admits the conditions of Theorem 2.6 and $|C_1 \cup C_2| = 2$, then the hypergraph constructed as a result of $E$-WQH switching is isomorphic to $G$. It is easily observed that the isomorphism is the permutation of the two vertices of $C_1 \cup C_2$. The same observation is true for $E$-GM switching.

In $E$-GM switching, it is required to partition the vertices of a hypergraph $G$ into two sets $C$ and $D$, where no two vertices of $C$ are in the same edge, and any $(k-1)$-subset of $D$ has either 0, $\frac{|C|}{2}$, or $|C|$ neighbours in $C$. We will call $C$ the switching set of $G$.

**Remark 2.9.** If $G$ is a $k$-uniform hypergraph satisfying the conditions of Theorem 2.6 and $|C_1 \cup C_2| = 4$, then the hypergraph constructed as a result of $E$-WQH switching is isomorphic to the one constructed using $E$-GM switching. Indeed, any subset of $k-1$ vertices in $D$ must have 0, 2, or 4 neighbours in $C_1 \cup C_2$. This implies that the conditions of $E$-GM switching are satisfied for $G$ with the switching set $C := C_1 \cup C_2$. Applying $E$-GM switching is equivalent to applying $E$-WQH switching and a permutation of the two vertices in $C_1$ and the two vertices in $C_2$.

Finally we provide an example of two $E$-cospectral non-isomorphic hypergraphs that can be obtained through the new $E$-WQH switching from Theorem 2.6 but not via the $E$-GM switching shown in [3].

**Example 2.10.** Let $G$ be a 3-uniform hypergraph on 9 vertices $u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3$ with edges

$$
\begin{align*}
\{v_1, v_2, u_1\}, \{v_1, v_2, u_2\}, \{v_1, v_2, u_3\}, \{v_2, v_3, u_1\}, \{v_2, v_3, u_4\}, \\
\{v_1, v_3, u_2\}, \{v_1, v_3, u_3\}, \{v_1, v_3, u_4\}, \{v_1, v_3, u_5\}.
\end{align*}
$$

If we take $C_1 := \{u_1, u_2, u_3\}$ and $C_2 := \{u_4, u_5, u_6\}$, then every edge has exactly one vertex in $C_1 \cup C_2$, and every 2-subset of $\{v_1, v_2, v_3\}$ has either three neighbours in $C_1$ and none in $C_2$, or the same number of neighbours in both $C_1$ and $C_2$. Hence this hypergraph admits the switching described in Theorem 2.6, and the result is a hypergraph $H$ with edge set

$$
\begin{align*}
\{v_1, v_2, u_4\}, \{v_1, v_2, u_5\}, \{v_1, v_2, u_6\}, \{v_2, v_3, u_1\}, \{v_2, v_3, u_4\}, \\
\{v_1, v_3, u_2\}, \{v_1, v_3, u_3\}, \{v_1, v_3, u_4\}, \{v_1, v_3, u_5\},
\end{align*}
$$

7
i.e. only the first three edges are switched. \( G \) and \( H \) are clearly \( E \)-cospectral but not isomorphic, since only one of them has an isolated vertex. Moreover, the hypergraph \( H \) cannot be constructed from \( G \) using the \( E \)-GM switching, since the switching set would have to include \( C_1 \cup C_2 \) and simultaneously cannot include any of the vertices from \( \{v_1, v_2, v_3\} \). The set \( C_1 \cup C_2 \) by itself does not satisfy the conditions of \( E \)-GM switching.

3 Constructing cospectral hypergraphs using matrix representation

One of the disadvantages of studying the hypergraph spectrum using tensors, as in the previous section, is the computational complexity: computing eigenvalues of the adjacency tensor is known to be an NP-hard problem \([9]\). On the other hand, a different way of representing a hypergraph on \( n \) vertices can be found in the literature (see, for example, \([6, 12, 17, 16]\)). Here we have an \( n \times n \) matrix where its \((i, j)\)-entry is the number of edges that vertices labelled \( i \) and \( j \) share. We use the slightly altered definition from \([2]\) for the adjacency matrix \( A = (A_{ij}) \) of a \( k \)-uniform hypergraph \( G \):

\[
A_{ij} = \begin{cases} 
0, & i = j, \\
\frac{1}{k-1} |\{e \in E(G) \mid i, j \in e\}|, & i \neq j.
\end{cases}
\]

Two hypergraphs are said to be cospectral with respect to their matrix representation if their adjacency matrices have the same spectrum.

A first switching method to construct cospectral hypergraphs regarding the previous adjacency matrix was shown by Sarkar and Banerjee in \([18]\). Their method is analogous to GM-switching for ordinary graphs in its most general form. In this section we show a different switching method which extends WQH-switching \([23, 15]\) to hypergraphs.

**Theorem 3.1.** Let \( G = (V(G), E(G)) \) be a \( k \)-uniform hypergraph whose vertex set admits a partition \( C_1 \cup C_2 \cup \cdots \cup C_{2m} \cup D \) for some integer \( m \geq 1 \), and such that the following conditions for \( G \) hold:

1. \(|C_i| = t\) for all \( i \in [2m] \) and some integer \( t \), while \(|D| = k - 1\).

2. Any edge of \( G \) has either 0 or \( k - 1 \) vertices in \( D \).

3. For any odd \( i \in [2m] \), we have either \( \Gamma(D) \cap (C_i \cup C_{i+1}) = C_i \) or \(|\Gamma(D) \cap C_i| = |\Gamma(D) \cap C_{i+1}|\), where \( \Gamma(D) \) denotes the set of neighbours of \( D \), or a subset of vertices \( v \) such that \( \{v, D\} \in E(G) \).

4. For the adjacency matrix \( A \) and each \( i, j \in [2m] \) there exists an integer \( \alpha_{ij} \) such that

\[
\sum_{u \in C_i} A_{uv} = \sum_{u \in C_i} A_{vu} = \alpha_{ij} \text{ for } v \in C_j \quad \text{and} \quad \sum_{u \in C_j} A_{uv} = \sum_{u \in C_j} A_{vu} = \alpha_{ij} \text{ for } v \in C_i.
\]
5. For any odd \( i, j \in [2m] \), \( \alpha_{ij} = \alpha_{i+1,j+1} \) and \( \alpha_{i+1,j} = \alpha_{i,j+1} \).

Let \( H \) be the hypergraph which is constructed from \( G \) as follows. For all odd \( i \in [2m] \), if \( \Gamma(D) \cap (C_i \cup C_{i+1}) = C_i \) then remove all edges of the form \( \{D,v\} \) with \( v \in C_i \) and add the edges of the form \( \{D,v\} \) for all \( v \in C_{i+1} \).

Then \( H \) and \( G \) are cospectral with respect to their matrix representation.

Observe that by the condition (4) of Theorem 3.1 we obtain that for all \( i, j \in [2m] \), we have \( \alpha_{ij} = \alpha_{ji} \).

Proof. Let \( X = I_t - \frac{1}{t} J_t \) and \( Y = \frac{1}{t} J_t \) be \( t \times t \) matrices. Without loss of generalization, we assume that the labelling of vertices of \( G \) is consistent with the partition \( C_1 \cup C_2 \cup \cdots \cup C_{2m-1} \cup C_{2m} \cup D \).

Consider the block matrix

\[
Q = \begin{pmatrix}
X & Y & O & O & \ldots & O \\
Y & X & O & O & \ldots & O \\
O & O & X & Y & \ldots & O \\
O & O & Y & X & \ldots & O \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
O & \ldots & \ldots & O & I_{k-1},
\end{pmatrix}
\]

It is clear that \( Q \) is an orthogonal matrix, which means that \( A' := QAQ^\top \) is a matrix with the same spectrum as \( A \). Hence all we need to prove is that \( A' \) is the adjacency matrix of \( H \).

We can write \( A \) as a block matrix

\[
A = \begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{12m} & B_1 \\
B_{21} & B_{22} & \cdots & B_{22m} & B_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{2m1} & B_{2m2} & \cdots & B_{2m2m} & B_{2m} \\
B_1^\top & B_2^\top & \cdots & B_{2m}^\top & t(J - I)
\end{pmatrix},
\]

where \( B_{ij} \) are \( t \times t \) blocks for all \( i, j \in [2m] \), while \( B_i \) are \( t \times (k - 1) \) blocks of all-one or all-zero rows. According to the conditions of the theorem satisfied by \( G \), the sum of rows and columns in \( B_{ij} \) is \( \alpha_{ij} \) for any \( i, j \in [2m] \). The matrix \( A' \) can also be represented as a block matrix in a similar way.

Let \( B'_{ij}, B'_i \) for \( i, j \in [2m] \) be the \( t \times t \) and \( t \times (k - 1) \) blocks of \( A' \), respectively. Let \( i \) and \( j \) be odd integers from \([2m]\). Then by block matrix multiplication, we can derive that

\[
\begin{pmatrix}
B'_{ij}
B'_{i+1,j+1}
\end{pmatrix} =
\begin{pmatrix}
(XB_{ij} + YB_{i+1,j})X + (XB_{i+1,j+1} + YB_{i+1,j+1})Y \\
(YB_{ij} + XB_{i+1,j})X + (YB_{i+1,j+1} + XB_{i+1,j+1})Y
\end{pmatrix}
= \begin{pmatrix}
(XB_{ij} + YB_{i+1,j})Y + (XB_{i+1,j+1} + YB_{i+1,j+1})X \\
(YB_{ij} + XB_{i+1,j})Y + (YB_{i+1,j+1} + XB_{i+1,j+1})X
\end{pmatrix}.
\]
Now, since the blocks $B_{i,j}$ all have constant sum of rows and columns, we have

$$J_t B_{i,j} = B_{i,j} J_t = \alpha_{i,j} J_t \implies Y B_{i,j} = B_{i,j} Y = \frac{\alpha_{i,j}}{t} J_t, \ X B_{i,j} = B_{i,j} X = B_{i,j} - \frac{\beta_{i,j}}{t} J_t.$$ 

Additionally, observe that

$$X^2 = X, \ Y^2 = Y, \ XY = YX = O.$$ 

From this, we can obtain through a straightforward calculation that, for any odd $i, j \in [2m]$, we have

$$B'_{i,j} = B_{i,j} - \frac{\alpha_{i,j}}{t} J_t + \frac{\alpha_{i+1,j+1}}{t} J_t = B_{i,j},$$

$$B'_{i+1,j} = B_{i+1,j} - \frac{\alpha_{i,j+1}}{t} J_t + \frac{\alpha_{i+1,j}}{t} J_t = B_{i+1,j},$$

$$B'_{i,j+1} = B_{i,j+1} - \frac{\alpha_{i+1,j}}{t} J_t + \frac{\alpha_{i,j+1}}{t} J_t = B_{i+1,j+1},$$

$$B'_{i+1,j+1} = B_{i+1,j+1} - \frac{\alpha_{i+1,j+1}}{t} J_t + \frac{\alpha_{i,j}}{t} J_t = B_{i+1,j+1}.$$ 

Next, for an odd $i \in [2m]$ we have $B'_i = XB_i + YB_{i+1}$ and $B'_{i+1} = YB_i + XB_{i+1}$. There are two possible cases.

The first case is when $B_i$ is an all-ones block while $B_{i+1} = O$, so that $\Gamma(D) \cap (C_i \cup C_{i+1}) = C_i$. Then, since $J_t B_i = t B_i$, we have $B'_i = O$ and $B'_{i+1} = B_i$, meaning that the adjacency of all $k$-sets of the form $\{v, D\}$ for $v \in C_i \cup C_{i+1}$ was switched. This is consistent with the switching used to construct $\mathcal{H}$.

The second case is when $B_i$ and $B_{i+1}$ both have exactly $r$ all-ones rows, meaning that $|\Gamma(D) \cap C_i| = |\Gamma(D) \cap C_{i+1}| = r$. Then $J_t B_i = J_t B_{i+1} = rB$ where $B$ is an all-ones block of size $t \times (k - 1)$, and from this we can derive $B'_i = B_i$ and $B'_{i+1} = B_{i+1}$.

Combining the observations above we can conclude that $\mathcal{A}'$ is the adjacency matrix of $\mathcal{H}$. 

Note that the $\frac{1}{k-1}$ factor in the definition of the adjacency matrix $\mathcal{A}$ has no bearing on the argument of the proof, which implies that the above switching is also applicable when using the adjacency matrix definition from [6].

**Remark 3.2.** Let $\mathcal{G}$ be a hypergraph that satisfies the conditions of Theorem 3.1 with a partition of vertices $C_1 \cup C_2 \cup D$ and $\Gamma(D) \cap (C_1 \cup C_2) = C_1$. Then the application of the switching described in Theorem 3.1 is equivalent to applying the switching described by Sarkar and Banerjee in [18]. The partition of the vertices in that case would be into two subsets $\mathcal{C} := C_1 \cup C_2$ and $\mathcal{D}$.

We end up with an example that illustrates the use of Theorem 3.1. Note that the following cospectral pair of uniform hypergraphs cannot be obtained using the switching for hypergraphs using matrix representation from [18].
Example 3.3. Let $G$ be a 3-uniform hypergraph on 14 vertices $u_1, u_2, \ldots, u_{12}, v_1, v_2$ and having edge set

$\{u_1, u_2, u_3\}, \{u_1, u_4, u_5\}, \{u_2, u_5, u_6\}, \{u_3, u_4, u_6\}, \{u_7, u_{10}, u_{12}\}, \{u_8, u_{10}, u_{11}\},$

$\{u_9, u_{11}, u_{12}\}, \{u_1, v_1, v_2\}, \{u_2, v_1, v_2\}, \{u_3, v_1, v_2\}, \{u_7, v_1, v_2\}, \{u_{10}, v_1, v_2\}$.

Consider the vertex partition $C_1 := \{u_1, u_2, u_3\}$, $C_2 := \{u_4, u_5, u_6\}$, $C_3 := \{u_7, u_8, u_9\}$, $C_4 := \{u_{10}, u_{11}, u_{12}\}$, and $D := \{v_1, v_2\}$. Then the switching of Theorem 3.1 can be applied to obtain a hypergraph $H$ with edge set

$\{u_1, u_2, u_3\}, \{u_1, u_4, u_5\}, \{u_2, u_5, u_6\}, \{u_3, u_4, u_6\}, \{u_7, u_{10}, u_{12}\}, \{u_8, u_{10}, u_{11}\},$

$\{u_9, u_{11}, u_{12}\}, \{u_4, v_1, v_2\}, \{u_5, v_1, v_2\}, \{u_6, v_1, v_2\}, \{u_7, v_1, v_2\}, \{u_{10}, v_1, v_2\}$.

The constructed hypergraph $H$ has a subset of vertices $\{u_1, u_2, u_3\}$ of degree 2 which themselves form an edge. There is no such subset in $G$, hence $G$ and $H$ are not isomorphic. Furthermore, there is no partition of the vertices of $G$ that satisfies the conditions of the GM-switching based method described in [18], meaning that this example can only be obtained using the new method from Theorem 3.1.

Acknowledgements

We thank Utku Okur and Joshua Cooper for carefully reading the manuscript and pointing out the distinction between cospectrality and $E$-cospectrality of tensors. We also thank the anonymous referees for their useful feedback. Aida Abiad is partially supported by FWO (Research Foundation Flanders) via the grant 1285921N. This research is supported by NWO (Dutch Research Council) via an ENW-KLEIN-1 project (OCENW.KLEIN.475).

References

[1] Abiad A, Haemers WH. Cospectral graphs and regular orthogonal matrices of level 2. Electron. J. Comb. 2012; 19(3):#P13.

[2] Banerjee A. On the spectrum of hypergraphs. Linear Algebra Appl. 2021;614:82–110.

[3] Bu C, Zhou J, Wei Y. $E$-cospectral hypergraphs and some hypergraphs determined by their spectra. Linear Algebra Appl. 2014;459:397–403.

[4] Cooper J, Dutle A. Spectra of uniform hypergraphs. Linear Algebra Appl. 2012;436(9):3268–3292.

[5] Cooper J, Dutle A. Computing hypermatrix spectra with the Poisson product formula. Linear Multilinear Algebra. 2015;63(5):956–970.

[6] Feng K, Li WW. Spectra of hypergraphs and applications. J. Number Theory. 1996;60(1):1–22.
[7] Godsil CD, McKay BD. Constructing cospectral graphs. Aequationes Math. 1982;25:257–268.

[8] Halbeisen L, Hungerbühler N. Generation of isospectral graphs. J. Graph Theory. 1999;31.3:255–265.

[9] Hillar CJ, Lim LH. Most tensor problems are NP-hard. J. ACM. 2013;60(6):1–39.

[10] Li AM, Qi L, Zhang B. E-characteristic polynomials of tensors. Commun. Math. Sci. 2013;11(1):33–53.

[11] Lim LH. Singular values and eigenvalues of tensors: a variational approach. 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 2005. 2005;1:129–132.

[12] Lin H, Zhou B. Spectral radius of uniform hypergraphs. Linear Algebra Appl. 2017;527:32–52.

[13] Qi L. Eigenvalues of a real supersymmetric tensor. J. Symbolic Comput. 2005;40(6):1302–1324.

[14] Qi L. Eigenvalues and invariants of tensors. J. Math. Anal. Appl. 2007;325(2):1363–1377.

[15] Qiu L, Ji Y, Wang W. On a theorem of Godsil and McKay concerning the construction of cospectral graphs. Linear Algebra Appl. 2020;603:265–274.

[16] Rodriguez JA. On the Laplacian Eigenvalues and Metric Parameters of Hypergraphs. Linear Multilinear Algebra. 2002;50(1):1–14.

[17] Saha SS, Sharma K, Panda SK. On the Laplacian spectrum of k-uniform hypergraphs. Linear Algebra Appl. 2022;655:1–27.

[18] Sarkar A, Banerjee A. Joins of hypergraphs and their spectra. Linear Algebra Appl. 2020;603:101–129.

[19] Schwenk AJ. Almost all trees are cospectral. In: Harary F, editor. New directions in the theory of graphs. New York: Academic Press; 1973. pp. 275–307.

[20] Shao JY. A general product of tensors with applications. Linear Algebra Appl. 2013;439(8):2350–2366.

[21] Shao JY, Qi L, Hu S. Some new trace formulas of tensors with applications in spectral hypergraph theory. Linear Multilinear Algebra. 2015;63(5):971–992.

[22] Wang WH. The minimum spectral radius of the r-uniform supertree having two vertices of maximum degree. Linear Multilinear Algebra. 2022;70(15):2898–2918.
[23] Wang W, Qiu L, Hu Y. Cospectral graphs, GM-switching and regular rational orthogonal matrices of level \( p \). Linear Algebra Appl. 2019;563:154–177.

[24] Xiao P, Wang L. The maximum spectral radius of uniform hypergraphs with given number of pendant edges. Linear Multilinear Algebra. 2019;67(7):1392–1403.

[25] Zhang W, Kang L, Shan E, Bai Y. The spectra of uniform hypertrees. Linear Algebra Appl. 2017;533:84–94.