The elliptic hypergeometric functions associated to the configuration space of points on an elliptic curve I: Twisted cycles

By

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Abstract

We consider the Euler type integral associated to the configuration space of points on an elliptic curve, which is an analogue of the hypergeometric function associated to the configuration space of points on a projective line. We calculate the twisted homology group with coefficients in the local system associated to a power function \( g^\alpha \) of an elliptic function \( g \), and the related intersection form. Applying these calculations, we describe the connection matrices representing the linear isomorphisms induced from analytic continuations of the functions defined by integrations of \( g^\alpha \) over twisted cycles.

1. Introduction

In many mathematical contexts, hypergeometric functions often have (or are defined by) Euler type integrals. In fact, the hypergeometric function associated to the configuration space of points on \( \mathbb{P}^1 \) has the following integral representation:

\[
\int_{x_i}^{x_j} (t-x_1)^{\alpha_1}(t-x_2)^{\alpha_2}\cdots(t-x_n)^{\alpha_n}dt.
\]

In this paper, instead of \( \mathbb{P}^1 \), we shall consider the configuration space of points on an elliptic curve.

Let \( C \) be an elliptic curve defined by \( \mathbb{C}/\Gamma \), where \( \Gamma := \mathbb{Z}_1 \oplus \mathbb{Z}_2 \) is a lattice generated by \( \omega_1 \) and \( \omega_2 \). We assume \( \Im \frac{\omega_2}{\omega_1} > 0 \). We shall denote a point on \( C \) by an equivalence class \( \sigma \in C \) for \( x \in \mathbb{C} \). Our main purpose is to study the definite integrals of the multi-valued function \( g^\alpha \), where \( \alpha \in \mathbb{C} \setminus \left( \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \right) \) is a fixed complex number and

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Here $\sigma$ is Weierstrass’ $\sigma$ function with the periods $\omega_1$ and $\omega_2$. Given $x_0$, $x_1$ and $x_2$, the function $g$ depends only on the equivalence class $\overline{I} \in \mathbb{C}$. So this function $g$ is a meromorphic function on $C$, which is a counterpart of rational functions on $\mathbb{P}^1$, with a pole of the third order at $\overline{\Omega}$ and zeros of the first order at $\overline{x_0, x_1, x_2}$. Thus the function $g^\omega$ has branch points at $\overline{x_0, x_1, x_2, \Omega} \in \mathbb{C}$, which is analogous to the case of the multi-valued function $t^{-3\omega}(t-x_0)^\omega(t-x_1)^\omega(t-x_2)^\omega$ on $\mathbb{P}^1$. However, $g$ may depend on a representative $x_0$ (resp. $x_1$, $x_2$) of the equivalence class $\overline{x_0}$ (resp. $\overline{x_1, x_2}$). We consider a subgroup $W$ of $\Gamma^3$ defined by

$$W := \{ (\Pi_0, \Pi_1, \Pi_2) \in \Gamma^3 \mid \Pi_0 + \Pi_1 + \Pi_2 = 0 \}$$

and a $W$-invariant subspace $L$ of $C^3$ defined by

$$L := \{ (x_0, x_1, x_2) \in C^3 \mid x_0 + x_1 + x_2 = 0 \}.$$

Put $S := L/W$. Thus, $g$ is a meromorphic function on $C \times S$. For $q = (x_0, x_1, x_2) \in S$, we denote by $X_q$ the punctured elliptic curve, that is, $C$ minus four points $\overline{x_0, x_1, x_2}$ and $\overline{\Omega}$. The integral of $g^\omega dt$ over a path $\Xi_{\mu, q}$ (mentioned below) lying on $X_q$

$$(1.1) \quad F_{\mu}(q) := \int_{\Xi_{\mu, q}} g^\omega dt$$

is a multi-valued function with respect to $q$ defined over the space $S$ of configuration with constraint (defined later), which is a subset of $S$.

This function $F_{\mu}$ has singularities: Put $D^{ij} := \{ (x_0, x_1, x_2) \in S \mid x_i - x_j \in \Gamma \}$ and $D^{ij}_\infty := \{ (x_0, x_1, x_2) \in S \mid x_i \in \Gamma \}$; when $q$ belongs to $D^{ij}$, the topological type of $X_q$ differs from that of $C$ minus four points; when $q$ belongs to $D^{ij}_\infty$, the integrand $g^\omega$ is divergent. Hence we define the domain $S$ of the function $F_{\mu}$ by removing the singular loci $D^{ij}$ and $D^{ij}_\infty$ from $S$:

$$(1.2) \quad S := S \setminus \left( \bigcup_{i \neq j} D^{ij} \cup \bigcup_i D^{ij}_\infty \right).$$

We introduce an integration path $\Xi_{\mu, q}$ of (1.1), the so-called twisted cycle. In case of projective space, Aomoto theory ([5]) tells us that a path of an Euler type integral can be regarded as a homology class with coefficients in the local system defined by the multi-valuedness of the integrand. In principle, this interpretation is valid for other cases. In our case, the local system $L_{X_q}$ is defined by the multi-valuedness of $g^\omega$ and we denote by $\{ \Xi_{\mu, q} \}_\mu$ generators of $H_1(X_q, L_{X_q})$. In Section 3, we give the concrete description of $\{ \Xi_{\mu, q} \}_\mu$ (Theorem 3.1). In Section 4, we calculate the intersection form among the twisted cycles (Theorem 4.1).
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The function $F^\mu$ on $S$ can be seen as a flat section of a certain vector bundle over $S$ or, equivalently, a solution of a certain linear differential equation ([8]):
The sheaf $\hat{\mathcal{S}}$ of germs of functions defined by $C$-linear combinations of $\{F^\mu\}_\mu$ is locally constant and $\hat{\mathcal{S}} \otimes \mathcal{O}_S$ is a flat (trivial) vector bundle over $S$. The analytic continuation along a path $\gamma$ in $S$ induces the linear isomorphism between the stalks of $\hat{\mathcal{S}}$ over the initial point and the terminal point of $\gamma$; the corresponding matrix is said to be the connection matrix. In Section 5, we consider the paths in $S$ connecting two points $(\omega_i^2, \omega_j^2, \omega_k^2)$ and $(\omega_{\tau(i)}^2, \omega_{\tau(j)}^2, \omega_{\tau(k)}^2)$, where $\tau$ is a transposition of two of three letters $i$, $j$, $k$. For these paths, we calculate the connection matrices (Theorem 5.1).

Finally, we briefly mention the relation with other recent works. T. Mano and H. Watanabe study the Riemann-Wirtinger integrals ([10], [13]); the definite integrals of power products of an exponential function and theta functions, which are similar to our integrals (1.1). H. Watanabe considers in [11] a fixed configuration; the two-torsion points are removed from elliptic curves, which are deformed by the moduli parameter. He focuses the action of modular transformations on such elliptic curves and studies that action on (co)homology groups of the elliptic curves minus the two-torsion points with coefficients in certain local systems corresponding to the integrands of Riemann-Wirtinger integrals. Using the Riemann-Wirtinger integral representation of Gauß’s hypergeometric function, he interprets the monodromy transformations of the hypergeometric function into the modular transformations. T. Mano considers in [9] the special solutions to the monodromy-preserving deformation of Fuchsian differential equations on an elliptic curve which are described Riemann-Wirtinger integrals. Their integrands have $n$ branch points on an elliptic curve. He calculates the monodromy corresponding to interchange two of $n$ branch points.

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2. The local system

Let $C$ be an elliptic curve defined by $\mathbb{C}/\Gamma$, where $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a lattice generated by $\omega_1$ and $\omega_2$. We assume $\Im \omega_1 > 0$. We shall denote a point on $C$ by an equivalence class $[x] \in C$ represented by $x \in \mathbb{C}$. We consider a subgroup $W$ of $\Gamma^3$ defined by

$$W := \{ (\Pi_0, \Pi_1, \Pi_2) \in \Gamma^3 \mid \Pi_0 + \Pi_1 + \Pi_2 = 0 \}$$

and a $W$-invariant subspace $L$ of $\mathbb{C}^3$ defined by

$$L := \{ (x_0, x_1, x_2) \in \mathbb{C}^3 \mid x_0 + x_1 + x_2 = 0 \}.$$
Put \( \mathcal{S} := L/W \). For \( q = (x_0, x_1, x_2) \in \mathcal{S} \), we denote by \( X_q \) the punctured elliptic curve, that is, \( C \) minus four points \( \bar{x}_0, \bar{x}_1, \bar{x}_2 \) and \( \bar{0} \). We assume that these four points are distinct. (By the definition in (1.2), this condition is equivalent to \( q \in S \).) We shall introduce a local system \( \mathcal{L}_{X_q} \) which reflects the data of multi-valuedness of the function \( g^\alpha \) defined as follows. Let \( g \) be the following holomorphic function on \( X_q \):

\[
g := \frac{\sigma(t - x_0)\sigma(t - x_1)\sigma(t - x_2)}{\sigma_3(t)\sigma(x_0)\sigma(x_1)\sigma(x_2)}.
\]

Using the identity \( \frac{\sigma'(t)}{\sigma(t)} = \zeta(t) \) (Weierstraß’s \( \zeta \) function), we have \( \frac{dg}{g} = (\zeta(t - x_0) + \zeta(t - x_1) + \zeta(t - x_2) - 3\zeta(t))dt \). We shall consider the following sheaf \( \mathcal{L}_{X_q} \):

\[
\mathcal{L}_{X_q} := \text{Ker} \left( d - \frac{dg}{g} : \mathcal{O}_{X_q} \longrightarrow \Omega^1_{X_q} \right),
\]

where \( \alpha \in \mathbb{C} \setminus \left\{ \frac{1}{3}, \frac{1}{2}, 2 \right\} \) is a fixed complex number. This sheaf has no global section on \( X_q \). We take the open part \( \hat{X}_q \) in \( X_q \):

\[
\hat{X}_q := C \setminus (\bar{0}x_0 \cup \bar{0}x_1 \cup \bar{0}x_2),
\]

where \( \bar{0}x_i \) is the image of the segment \([0, x_i]\) in \( \mathbb{C} \) through the natural projection \( \mathbb{C} \longrightarrow C \). We should modify the segment \([0, x_i]\) in its homotopy class in such a way that the images of the segments do not have non-trivial intersection (nor self-intersection) except for the one at \( \bar{0} \).

**Lemma 2.1.** Regardless of the choice of a representative \((x_0, x_1, x_2)\), \( \hat{X}_q \) is homotopic to the bouquet \( S^1 \vee S^1 \).

**Proof.** The restriction to \([0, x_0] \cup [0, x_1] \cup [0, x_2]\) of the projection is a homeomorphism. We have a neighborhood \( \mathcal{O} \) of \( \bar{0}x_0 \cup \bar{0}x_1 \cup \bar{0}x_2 \) and a neighborhood \( O \) of \([0, x_0] \cup [0, x_1] \cup [0, x_2]\) such that the restriction of the projection to \( O : O \longrightarrow \mathcal{O} \) is also a homeomorphism because the projection is a covering map. Here \( O \setminus (\bar{0}x_0 \cup [0, x_1] \cup [0, x_2]) \) is homeomorphic to \( O \setminus \bar{0} \). So \( \mathcal{O} \setminus (\bar{0}x_0 \cup \bar{0}x_1 \cup \bar{0}x_2) \) is homeomorphic to \( \mathcal{O} \setminus \bar{0} \). Then, \( C \setminus (\bar{0}x_0 \cup \bar{0}x_1 \cup \bar{0}x_2) \) is homeomorphic to \( C \setminus \bar{0} \), which is homotopic to \( S^1 \vee S^1 \).

**Lemma 2.2.**

1. The sheaf \( \mathcal{L}_{X_q} \) is a local system of rank 1 over \( \mathbb{C} \) whose local sections are regarded as single-valued branches of some constant multiple of \( g^\alpha \).
2. The sheaf \( \mathcal{L}_{X_q} \) has non-zero global sections on \( \hat{X}_q \).
Proof.
1. Any local section of $L_{X_q}$ is a local solution of the first-order ordinary linear differential equation $\left( d - \alpha \frac{d}{\partial y} \right) \varsigma = 0$. Any local solution of this equation is a constant multiple of a single-valued branch of $g^\alpha$. Hence, sections of $L_{X_q}$ on a small enough open set form a one dimensional vector space, and such a section is a single-valued branch of some constant multiple of $g^\alpha$.

2. We shall prove that a single-valued solution of the equation $\left( d - \alpha \frac{d}{\partial y} \right) \varsigma = 0$ on $\tilde{X}_q$ does exist. The logarithm of any solution is given by an integral of $\alpha \frac{d}{\partial y}$ over a path. The single-valuedness is equivalent to the condition that the integral over any closed loop in $\tilde{X}_q$ vanishes. Due to Lemma 2.1, $\tilde{X}_q$ is homotopic to $S^1 \vee S^1$. We can pick two loops $l_{\omega_1}, l_{\omega_2}$ on $\tilde{X}_q$ such that the inclusion map $l_{\omega_1} \cup l_{\omega_2} \hookrightarrow \tilde{X}_q$ is a deformation retract. To prove the single-valuedness, it suffices to verify the vanishing of integrals $I_i$ given by

$$I_i(x_0, x_1, x_2) := \int_{l_{\omega_i}} \frac{dg}{g} = \int_{l_{\omega_i}} (\zeta(t - x_0) + \zeta(t - x_1) + \zeta(t - x_2) - 3\zeta(t)) dt.$$ 

Using the identity $\zeta'(t) = -\varphi(t)$, we have $\frac{\partial I_i}{\partial x_j} = \int_{l_{\omega_i}} \varphi(t - x_j) dt = \int_{l_{\omega_i}} \varphi(t) dt$. This is a constant, which we denote by $\pi_i$. Then the equality $I_i(x_0, x_1, x_2) = \pi_i x_0 + \pi_i x_1 + \pi_i x_2 + I_i(0, 0, 0)$ holds. Since $x_0 + x_1 + x_2 = 0$ (by the assumption) and $I_i(0, 0, 0) = 0$ (by its definition), the integrals $I_i(x_0, x_1, x_2)$ vanish, as desired.

Now we pick a non-zero global section $\varsigma$ over $\tilde{X}_q$, and fix it once for all in the sequel. For a path $l : I \rightarrow X_q$ with the initial point in $\tilde{X}_q$, we denote by $\varsigma_l$ the analytic continuation of $\varsigma$ along $l$.

Remark 1. Note that $X_q$ is homotopic to the bouquet $S^1 \vee \cdots \vee S^1$.

In fact, we can pick five loops $l_0, l_1, l_2, l_{\omega_1}, l_{\omega_2}$ such that the inclusion map $l_0 \cup l_1 \cup l_2 \cup l_{\omega_1} \cup l_{\omega_2} \hookrightarrow X_q$ is a deformation retract. The loops $l_{\omega_1}, l_{\omega_2}$ are the same ones as in the proof of Lemma 2.2, while the loop $l_i$ ($i = 0, 1, 2$) is a closed loop in $X_q$ around the point $\varpi_i$. The representation $\rho : \pi_1(X_q) \rightarrow \mathbb{C}^*$ corresponding to the local system $L_{X_q}$ is given by the formulae $\rho(l_i) = e^{2\pi \sqrt{-1} \alpha}$.

3. The twisted homology

We shall find generators of the twisted homology group $H_1(X_q, L_{X_q})$. The $k$-chain group $C_k(X_q, L_{X_q})$ with coefficients in $L_{X_q}$ is the complex vector space with the basis $\{ \sigma \otimes s_\sigma \}$, where $\sigma : \Delta \rightarrow X_q$ is a singular $k$-simplex and $s_\sigma$ is a section of $\sigma^{-1} L_{X_q}$ over $\sigma$. The first boundary operator $\partial : C_1(X_q, L_{X_q}) \rightarrow$
The other boundary operators are given in similar fashion. Then the complex \((C_\bullet(X_q, \mathcal{L}_{X_q}), \partial)\) defines the 1-st twisted homology group \(H_1(X_q, \mathcal{L}_{X_q})\) in the usual manner. For two loops \(l_{w_1}, l_{w_2}\) in the proof of Lemma 2.2, we denote by \(\Xi_{w_1, q}\) (resp. \(\Xi_{w_2, q}\)) the twisted cycle \(l_{w_1} \otimes \mathcal{S}_{w_1}\) (resp. \(l_{w_2} \otimes \mathcal{S}_{w_2}\)), where \(\epsilon\) is given in Section 2.

We introduce other twisted cycles \(\Xi_{(i), j, k}\), which are the regularizations of the segments \(\overline{x_i x_j}\). We denote by \(S_\epsilon(x_i)\) a closed loop homotopic to the \(\mathcal{C}\)-circle centered at \(x_i\). Let \(x_i^\pm\) (resp. \(x_i^-\)) be the intersection point of the circle \(S_\epsilon(x_i)\) (resp. \(S_i(x_j)\)) and the segment \([x_i, x_j]\). The regularizations of the segments \(\overline{x_i x_j}\) is defined by

\[
\Xi_{(i), j, k} := \frac{1}{c-1}S_\epsilon(x_i) \otimes S_\epsilon(x_i) + [x_i^+, x_i^-] \otimes \mathcal{S}_{x_i x_j} - \frac{1}{c-1}S_\epsilon(x_j) \otimes S_\epsilon(x_j),
\]

where \(\epsilon\) is a sufficiently small positive number; we shall indicate by \(-\) an image through the natural projection \(\mathcal{C} \longrightarrow C\). Note that if \(\epsilon\) is sufficiently small, this defines a homology class, independently of \(\epsilon\).

**Theorem 3.1** (generators of \(H_1\)). The twist cycles \(\Xi_{(01), q}\), \(\Xi_{(12), q}\), \(\Xi_{(20), q}\), \(\Xi_{w_1, q}\), \(\Xi_{w_2, q}\) generate \(H_1(X_q, \mathcal{L}_{X_q})\), and we have the relation \(\Xi_{(01), q} + \Xi_{(12), q} + \Xi_{(20), q} = 0\) in \(H_1(X_q, \mathcal{L}_{X_q})\) which defines the four dimensional vector space \(H_1(X_q, \mathcal{L}_{X_q})\):

\[
H_1(X_q, \mathcal{L}_{X_q}) \cong \frac{\mathcal{C} \Xi_{(01), q} \oplus \mathcal{C} \Xi_{(12), q} \oplus \mathcal{C} \Xi_{(20), q} \oplus \mathcal{C} \Xi_{w_1, q} \oplus \mathcal{C} \Xi_{w_2, q}}{\mathcal{C} (\Xi_{(01), q} + \Xi_{(12), q} + \Xi_{(20), q})}.
\]

**Proof.** In view of Remark 1, \(H_1(X_q, \mathcal{L}_{X_q})\) coincides with the first homology of the following complex:

\[
0 \longrightarrow \mathcal{C} \xrightarrow{\partial} \bigoplus_{\mu \in \{0, 1, 2, \omega_1, \omega_2\}} \mathcal{C} \mathcal{L}_\mu \otimes \mathcal{S}_\mu \longrightarrow 0,
\]
where \( \partial(l_i \otimes s_i) = c - 1 \) and \( \partial(l_\omega \otimes s_{\omega i}) = 0 \). The first homology of it is \( \text{Ker} \partial \), which is generated by

\[
\left\{ \frac{1}{c - 1} l_i \otimes s_i - \frac{1}{c - 1} l_j \otimes s_j, \ l_\omega \otimes s_{\omega i} \right\}_{(i,j) \in \{(01), (12), (20)\}, k = 1, 2}.
\]

The chain \( \frac{1}{c - 1} l_i \otimes s_i - \frac{1}{c - 1} l_j \otimes s_j \) is a cycle homologous to \( \Xi_{(ij),q} \), whence the assertion. \( \square \)

4. The intersection form

The intersection form on the twisted cycles is a bilinear form between \( H_1(X_q, \mathcal{L}_{X_q}) \) and \( H_1(X_q, \mathcal{L}_{X_q}) \), where \( \mathcal{L}_{X_q} \) is the local system dual to \( \mathcal{L}_{X_q} \), i.e., the local system whose local sections are generated by \( g^{-\alpha} \). To describe the intersection form explicitly, we shall take the generators \( \Xi^{(01),q}, \Xi^{(12),q}, \Xi^{(20),q} \), and \( \Xi^{q,1,q} \) of \( H_1(X_q, \mathcal{L}_{X_q}) \) which are given by replacing \( \zeta \) (resp. \( \frac{1}{\zeta} \)) in the definitions of \( \Xi_{(01),q}, \Xi_{(12),q}, \Xi_{(20),q}, \Xi_{\omega_1,q} \), and \( \Xi_{\omega_2,q} \), respectively:

\[
\zeta \longleftrightarrow \frac{1}{\zeta}, \quad c \longleftrightarrow \frac{1}{c}; \quad \Xi = \sum_{\gamma} a_\gamma(c) \gamma \otimes c_{\gamma} \longleftrightarrow \Xi^\vee = \sum_{\gamma} a_\gamma \left( \frac{1}{c} \right) \gamma \otimes \frac{1}{c_{\gamma}},
\]

where \( \gamma \otimes c_{\gamma} \) is a twisted singular 1-simplex and \( a_\gamma(X) \in \mathbb{C}(X) \). (Here, \( \mathbb{C}(X) \) is the rational function field over \( \mathbb{C} \) in one valuable \( X \).

We define the ordering on the index set \( J := \{(01), (12), (20), \omega_1, \omega_2\} \) of the generators of \( H_1 \):

\[
(01) \prec (12) \prec (20) \prec \omega_1 \prec \omega_2.
\]

Now we have

**Theorem 4.1** (Intersection form). We assume that

1. \( l_\omega \) is \( \frac{1}{c - 1} l_i \otimes \frac{1}{c - 1} l_j \), \( l_\omega \) is \( \frac{1}{c - 1} l_i \otimes l_j \),
2. \( x_0, x_1 \) and \( x_2 \) belong to the interior of the parallelogram spanned by the two segments \( \left[ \frac{1}{c - 1}, \frac{1}{c - 1} \right], \left[ \frac{1}{c - 1}, \frac{1}{c - 1} \right] \),
3. \( \text{arg} x_0 < \text{arg} x_1 < \text{arg} x_2 \).

Then the intersection form between \( H_1(X_q, \mathcal{L}_{X_q}) \) and \( H_1(X_q, \mathcal{L}_{X_q}^\vee) \) is given by the matrix

\[
[(\Xi_{\mu,q}, \Xi_{\nu,q}^\vee)]_{\mu,\nu \in J} = \begin{bmatrix}
-\frac{c + 1}{c - 1} & -\frac{1}{c - 1} & 0 & 0 \\
-\frac{1}{c - 1} & -\frac{c + 1}{c - 1} & \frac{1}{c - 1} & 0 \\
0 & \frac{1}{c - 1} & -\frac{c + 1}{c - 1} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}.
\]

In particular, the \( (2,2) \)-cofactor of this matrix is equal to

\[
\frac{c + 1}{(c - 1)^2}.
\]
Proof. The intersection number $\langle \Xi_{\mu,q}, \Xi^\nu_{\nu,q} \rangle$ can be calculated by summing up the local intersection numbers at each of the intersection points of the supports of $\Xi_{\mu,q}$ and of $\Xi^\nu_{\nu,q}$. The local intersection number $\langle \gamma_1 \otimes \gamma_1, \gamma_2 \otimes \frac{1}{\gamma_2(p)} \rangle_p$ at the intersection point $p$ of their supports is the product of the ordinary local intersection number $\langle \gamma_1, \gamma_2 \rangle_p$ and the ratio $\frac{1}{\gamma_2(p)}$.

In the case of $\mu = \nu = (ij) \in \{(01), (12), (20)\}$, we change the representative of the homology class $\Xi^\nu_{\nu,q}$ in such a way that its support meets that of $\Xi_{\nu,q}$ at two points $p_i, p_j$, which belong to the supports of $S_i(x_i) \otimes S_{\nu}(x_i)$ and of $S_j(x_j) \otimes S_{\nu}(x_j)$, respectively. Then we have

$$\langle \Xi_{\mu,q}, \Xi^\nu_{\nu,q} \rangle = \frac{1}{c - 1} \times (-1) \times \frac{\langle S_{\nu}(x_i) \rangle(p_i)}{\langle x_i + \epsilon, x_j - \epsilon \rangle(p_i)} = \frac{1}{c - 1} \times (-1) \times \frac{\langle S_{\nu}(x_j) \rangle(p_j)}{\langle x_i + \epsilon, x_j - \epsilon \rangle(p_j)}$$

$$= \frac{1}{c - 1} \times (-1) \times c - \frac{1}{c - 1} \times 1 \times 1 = - \frac{c + 1}{c - 1}.$$

\[\text{In the case of } \mu \neq \nu, \mu, \nu \in \{(01), (12), (20)\}:
\]

We change the representative of the homology class $\Xi^\nu_{(ij),q}$ in such a way that its support meets that of $\Xi_{(ij),q}$ at a point $p_j$ on the support of $S_j(x_j) \otimes S_{(ij),q}$. Then we have

$$\langle \Xi_{(01),q}, \Xi^\nu_{(12),q} \rangle = \langle \Xi_{(12),q}, \Xi^\nu_{(20),q} \rangle = \langle \Xi_{(20),q}, \Xi^\nu_{(01),q} \rangle$$

$$= - \frac{1}{c - 1} \times (-1) \times \frac{\langle S_{(ij)} \rangle(p_j)}{\langle x_i + \epsilon, x_j - \epsilon \rangle(p_j)} = \frac{1}{c - 1}.$$

$$\langle \Xi_{(12),q}, \Xi^\nu_{(01),q} \rangle = \langle \Xi_{(20),q}, \Xi^\nu_{(12),q} \rangle = \langle \Xi_{(01),q}, \Xi^\nu_{(20),q} \rangle$$

$$= \frac{1}{c - 1} \times 1 \times \frac{\langle S_{(ij)} \rangle(p_j)}{\langle x_i + \epsilon, x_j - \epsilon \rangle(p_j)} = \frac{c}{c - 1}.$$
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In the case of $\mu, \nu \in \{\omega_1, \omega_2\}$:

The ordinary intersection form $(\bullet, \bullet) : H_1(X_q, \mathbb{C}) \times H_1(X_q, \mathbb{C}) \rightarrow \mathbb{C}$ is given by $[(l_{\omega_i}, l_{\omega_j})]_{i,j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\frac{\omega_i(p)}{\omega_j(p)} = 1$ at the intersection point $p$ of the supports of $l_{\omega_i}$ and of $l_{\omega_j}$. Hence,

$$\left[ (\Xi_{\omega_i,q}^\vee, \Xi_{\omega_j,q}^\vee) \right]_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

In the other cases:

We can change the representative of the homology class $\Xi_{\nu,q}^\vee$ in such a way that its support does not meet that of $\Xi_{\mu,q}$. \hfill \Box

5. The connection matrix

Put $D^{ij} := \{(x_0, x_1, x_2) \in S \mid x_i - x_j \in \Gamma \} \ (i, j = 0, 1, 2)$ and $D^\infty := \{(x_0, x_1, x_2) \in S \mid x_i \in \Gamma \} \ (i = 0, 1, 2)$. We denote the space of configuration with constraint by $S := S \setminus \left( \bigcup_{i \neq j} D^{ij} \cup \bigcup_i D^\infty \right)$ and the point $(\frac{\omega_1}{\omega_2}, \frac{\omega_2}{\omega_1}, \frac{\omega_0}{\omega_1})$ on $S$ by $q_{(i,j,k)}$, where $\omega_0 := - (\omega_1 + \omega_2)$. The integral of $g^\alpha dt$ over a path $\Xi_{\mu,q}$ lying on $X_q$

$$F_\mu(q) := \int_{\Xi_{\mu,q}} g^\alpha dt$$

is a multi-valued function with respect to $q$ defined over the space $S$. Let $S$ be the sheaf of germs of the functions on $S$ defined by $\mathbb{C}$-linear combinations of $\{F_\mu\}_{\mu \in \{(01), (20), \omega_1, \omega_2\}}$.

In this section, we shall describe the linear isomorphisms between stalks of $S$ over $q_{(012)}$, $q_{(210)}$ and $q_{(102)}$, of which the corresponding matrices are called
the connection matrices, induced by the analytic continuations of $F_\mu$ along four paths (given later) connecting two of $q_{(012)}$, $q_{(210)}$, $q_{(102)}$ on $S$. These linear isomorphisms are identified with the isomorphisms between the homology groups \( \{H_1(X_q, \mathcal{L}_{X_q})\} \), which are induced by translating homology classes along the paths. For a path $\gamma : I \to S$, the pull back of the fibration $\bigcup X_q \to S$ is a trivial one. Then we have the isomorphism between the homology groups of the fibers over $\gamma(0)$ and $\gamma(1)$. Put $\mathcal{H}(q) := H_1(X_q, \mathcal{L}_{X_q})$ for brevity.

We consider the paths on $S$ connecting $q_{(012)}$ and $q_{(\tau_{(ij)}(0), \tau_{(ij)}(1), \tau_{(ij)}(2))}$, where $\tau_{(ij)}$ is the transposition (belonging to the symmetric group $\mathfrak{S}_3$); $\tau_{(ij)}(i) = j$, $\tau_{(ij)}(j) = i$ and $\tau_{(ij)}(k) = k$ for $k \neq i, j$. Put

$$q_{(ij)}^{m_1, m_2} := \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix} + \frac{m_1 \omega_1 + m_2 \omega_2}{4} e_i - \frac{m_1 \omega_1 + m_2 \omega_2}{4} e_j,$$

where \( \{e_0, e_1, e_2\} \) is the standard basis of $\mathbb{C}^3$. We denote by $\lambda_{(ij)}^{m_1, m_2}$ a path on the subset

$$\left\{ (x_0, x_1, x_2) \in \overline{S} \mid x_k = \frac{\omega_k}{2} \ (k \neq i, j) \right\}$$

from $q_{(012)}$ to $q_{(\tau_{(ij)}(0), \tau_{(ij)}(1), \tau_{(ij)}(2))}$ via $q_{(ij)}^{m_1, m_2}$:

$$\lambda_{(ij)}^{m_1, m_2}(s) := \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix} + \frac{m_1 \omega_1 + m_2 \omega_2}{2} s e_i - \frac{m_1 \omega_1 + m_2 \omega_2}{2} s e_j, \quad 0 \leq s \leq 1.$$

The subset $\left\{ (x_0, x_1, x_2) \in \overline{S} \mid x_k = \frac{\omega_k}{2} \ (k \neq i, j) \right\}$ meets the singular loci at six points; for example, in case of $(i, j) = (0, 2)$, it meets $D_{12}$ at four points $q_{(02)}$, $q_{(02)}$, $q_{(02)}$, $q_{(02)}$, $q_{(02)}$, and $q_{(02)}$ at the same point $q_{(02)}$, and meets both $D_{12}$ and $D_{\infty}$ at the same point $q_{(02)}$. Thus, the path $\gamma_{(ij)}^{m_1, m_2}$ is defined by slightly deforming the path $\lambda_{(ij)}^{m_1, m_2}$ such that $\gamma_{(ij)}^{m_1, m_2}$ avoids $q_{(ij)}^{m_1, m_2}$. Note that the path $\gamma_{(ij)}^{m_1, m_2}$ can be realized as a composition of $\gamma_{(02)}^{1,0}$, $\gamma_{(20)}^{1,0}$, $\gamma_{(01)}^{0,1}$ and $\gamma_{(01)}^{0,-1}$ up to homotopy. For example, $\gamma_{(01)}^{0,1}$ is homotopic to $\gamma_{(01)}^{0,1} \circ \gamma_{(01)}^{0,1} \circ \gamma_{(01)}^{0,1} \circ \gamma_{(01)}^{0,1} \circ \gamma_{(01)}^{0,1} \circ \gamma_{(01)}^{0,1}$. (The following figure depicts the behavior of the $x_0$-component $x_{0, (ij)}^{m_1, m_2}(s)$ of a representative of $\gamma_{(ij)}^{m_1, m_2}(s)$, where $q_{0, (ij)}^{m_1, m_2}$ denotes the $x_0$-component of a representative of $q_{(ij)}^{m_1, m_2}$.)
We calculate the linear isomorphism \( \gamma_{(ij)}^{m_1, m_2} \) between \( \mathcal{H}(q_{(012)}) \) and \( \mathcal{H}(q_{(\tau_{(ij)(0)}, \tau_{(ij)(1)}, \tau_{(ij)(2)})}) \) by the twisted Picard-Lefschetz formula ([7]):

\[
\gamma_{(ij)}^{m_1, m_2} \ast (\Xi) = \Xi + \frac{(-c - 1)}{\Delta_{(ij)}^{(m_1, m_2)}(q_{(012)})} \Delta_{(ij)}^{(m_1, m_2)} \left( q_{(\tau_{(ij)(0)}, \tau_{(ij)(1)}, \tau_{(ij)(2)})} \right),
\]

where \( \Xi \) is a twisted cycle over \( C \) minus the four points \( \{0, \omega_1, \omega_2, \omega_2\} \) and \( \Delta_{(ij)}^{(m_1, m_2)}(q) \) be the cycle vanishing at the singular point \( q_{(ij)}^{m_1, m_2} \), as \( q \) tends to \( q_{(ij)}^{m_1, m_2} \) along the path \( \lambda_{(ij)}^{m_1, m_2} \). We have the following:

**Proposition 5.1** (vanishing cycles).

\[
\begin{align*}
\Delta_{(02)}^{(0, 0)}(q_{(012)}) &= -\Xi_{(20)}q_{(012)} - \Xi_{\omega_1 q_{(012)}} - \Xi_{\omega_2 q_{(012)}}, \\
\Delta_{(02)}^{(0, -1)}(q_{(012)}) &= -\Xi_{(20)}q_{(012)} - \Xi_{\omega_1 q_{(012)}}, \\
\Delta_{(01)}^{(0, -1)}(q_{(012)}) &= \Xi_{(01)}q_{(012)} - \Xi_{\omega_1 q_{(012)}} - \Xi_{\omega_2 q_{(012)}}, \\
\Delta_{(01)}^{(0, 0)}(q_{(012)}) &= \Xi_{(01)}q_{(012)} - \Xi_{\omega_1 q_{(012)}},
\end{align*}
\]

**Proof.** The vanishing cycle \( \Delta_{(ij)}^{(m_1, m_2)}(q_{(012)}) \) is the regularization of the segment \( x_{(ij)}^{(m_1, m_2)}(0), x_{(ij)}^{(m_1, m_2)}(1) \). Replacing it by the linear combination of \( \Xi_{(01)}q_{(012)} \), \( \Xi_{(20)}q_{(012)} \), \( \Xi_{\omega_1 q_{(012)}} \) and \( \Xi_{\omega_2 q_{(012)}} \), we obtain the assertion. (See the following figures, in which the dotted lines indicate the vanishing cycles.)
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We have the common basis for $H(q_{(012)})$ and $H(q_{(τ_{ij}(0), τ_{ij}(1), τ_{ij}(2))})$:

$Ξ(kl) := \frac{1}{c - 1} S_e^{\frac{ω_k}{2}} S_e(\frac{ω_l}{2})$

$Ξ_{ω_k} := l_{ω_k} \otimes s_{ω_k}$,

and we define the ordering on the index set $J' := \{(01), (20), ω_1, ω_2\}$ of this basis:

$(01) \prec (20) \prec ω_1 \prec ω_2$.

Now we have the following:

**Theorem 5.1 (connection matrices).**

Let $M(γ_{(ij)}^{m_1, m_2}) = \left[ m(γ_{(ij)}^{m_1, m_2})_{μ, ν} \right]_{μ, ν ∈ J'}$ be the $4 \times 4$-matrix which represents the linear isomorphism $\left( γ_{(ij)}^{m_1, m_2} \right)_* (Ξ_μ)$ = $\sum_ν m(γ_{(ij)}^{m_1, m_2})_{μ, ν} Ξ_ν$. Then,

$\begin{align*}
M(γ_{(02)}^{-1, 0}) &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & c & -c & c + 1 \\
c & -c - 1 & c & c + 1 \\
c & -c - 1 & c & c + 2
\end{bmatrix},
M(γ_{(02)}^{1, 0}) &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
c & -c & c - 1 & 0 \\
0 & 0 & 1 & 0 \\
c & -c - 1 & c - 1 & 1
\end{bmatrix},
M(γ_{(01)}^{0, -1}) &= \begin{bmatrix}
-c & 1 & -c + 1 & c - 1 \\
0 & 0 & 0 & 0 \\
c + 1 & -1 & c & c + 1 \\
c + 1 & -1 & c & c + 2
\end{bmatrix},
M(γ_{(01)}^{0, 1}) &= \begin{bmatrix}
-c & 1 & 0 & c - 1 \\
0 & 1 & 0 & 0 \\
c + 1 & -1 & 1 & -c + 1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\end{align*}$

**Proof.** Substituting the result of Proposition 5.1 for the twisted Picard-Lefschetz formula, we obtain the assertion.
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