Clifford algebras and universal sets of quantum gates

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In this paper is shown an application of Clifford algebras to the construction of computationally universal sets of quantum gates for \( n \)-qubit systems. It is based on the well-known application of Lie algebras together with the especially simple commutation law for Clifford algebras, which states that all basic elements either commute or anticommute.

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I. INTRODUCTION

In this paper is discussed an algebraic approach to the construction of computationally universal sets of quantum gates. A quantum gate \( U \) for a system of \( n \) qubits is a unitary \( 2^n \times 2^n \) matrix. It is possible to write \( U = e^{iH} \), where \( H \) is the Hermitian \( 2^n \times 2^n \) matrix.

A set of quantum gates \( U_k \) is (computationally) universal if any unitary matrix can be obtained with given pre-
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plex Clifford algebra with \( 2^n \n \times \n \) a unitary \( 2^n \times 2^n \) matrix. It is possible to write

\[
U \equiv C \tau H_k \text{ for a system of } 2^n \text{ qubits.}
\]

The extra Hermitian matrix is

\[
H = \Gamma_1 \Gamma_2 \cdots \Gamma_n - 2\delta_{I1}\]

where \( \Gamma_1 \) is unit matrix and \( 2^{2n} \)-different products of \( \Gamma_k \) generate a basis of \( \mathbb{C}(2^n \times 2^n) \).

The 2n matrices \( \Gamma_k \) are not enough for proof of universality, because we may not use arbitrary products of \( \Gamma_k \), but only commutators. In this paper it is shown that by using commutators of \( \Gamma_k \), it is possible to generate only the \( (2n^2+n) \)-dimensional subspace, but it is enough to add only one element \( \Gamma_n \) and the new set is universal, i.e., it generates a full \( 4^n \)-dimensional space \( \mathbb{C}(2^n) \).

All 2n matrices \( \Gamma_k \) may be chosen to be Hermitian and the full complex algebra was used for simplification. The extra Hermitian matrix is

\[
\Gamma_n = iI_{123} \equiv iI_1 \Gamma_2 \Gamma_3 \text{ or } I_{1234}, \text{ or any such product of three or four different } \Gamma_k.
\]

A constructive proof of universality using the language of the Clifford algebras is based on a simple commuta-
tion law of \( 4^n \) basic elements: they either commute or anticommute, because any such element is a product of up to \( 2n \) \( \Gamma_k \). Direct construction of any \( 2^n \times 2^n \) matrix \( \Gamma_n \equiv \prod_{k=1}^{2n} \Gamma_k \) of the Clifford basis by commutators of \( 2n+1 \) initial elements is shown below in Sec. II, theorem 1.

The question about universality is widely investigated \( \equiv \mathbb{C}(2^n) \), but the method discussed in the present work has some special properties. Construction of a universal set of gates uses only infinitesimal and continuous symmetries of group \( U(2^n) \) and does not require such discrete operations as permutations of qubits or basic vectors related to the “classical limit of quantum circuits.” The properties of discrete, binary transformations of qubits simply emerge here from the structure of infinitesimal transformations of Hilbert space, i.e., directly from Hamiltonians, cf. \( \equiv \mathbb{C}(2^n) \).

II. CLIFFORD ALGEBRAS

A. General definitions

For \( n \)-dimensional vector space with a quadratic form (metric) \( g(\vec{x}) \), the Clifford algebra \( \mathbb{C} \) is a formal way to represent a square root of \(-g(\vec{x}) \equiv \mathbb{C}(\vec{x}) \) or, more formally, \(-g(\vec{x}) \equiv \mathbb{C} \) where \( \mathbb{C} \) is the unit of algebra \( \mathbb{C} \). The vector space corresponds to the \( n \)-dimensional subspace \( \mathbb{V} \) of \( \mathbb{R} \) \( \vec{x} \mapsto \mathbb{V} \equiv \sum_{l=0}^{n-1} x_l \mathbb{e}_l \), where \( \mathbb{e}_l \in \mathbb{V} \subset \mathbb{C} \). From

\[
\mathbb{V} \equiv \sum_{l=0}^{n-1} x_l \mathbb{e}_l \equiv \sum_{l=0}^{n-1} g_{ij} x_i x_j
\]

follow the main properties of the generators \( \mathbb{e}_l \) of the Clifford algebra:

\[
\{ \mathbb{e}_i, \mathbb{e}_j \} \equiv \mathbb{e}_i \mathbb{e}_j + \mathbb{e}_j \mathbb{e}_i = -2g_{ij}.
\]

Let \( g_{ij} \) be diagonal and \( g_{ii} = -1 \) (the case \( g_{ii} = 0 \) is not considered here, but see \( \equiv \mathbb{C}(\vec{x}) \)). Then,

\[
\mathbb{e}_i \mathbb{e}_j = -\mathbb{e}_j \mathbb{e}_i \quad (i \neq j), \quad (2.2a)
\]

\[
\mathbb{e}_i^2 = \pm 1. \quad (2.2b)
\]

It is clearer from Eq. (2.2) that it is possible to generate no more than \( 2^n \) different products of up to \( n \mathbb{e}_i \). A linear span of all the products is a full algebra \( \mathbb{C} \). Let us use the notations \( \mathbb{e}_i = \mathbb{e}_{i_{123}} \) \( \equiv \mathbb{e}_{i_1} \mathbb{e}_{i_2} \cdots \mathbb{e}_{i_k} \), where \( k \) is the number of multipliers or the order of \( \mathbb{e}_i \), \( k = \mathcal{N}(\mathbb{e}_i) \).

If there are no algebraic relations other than Eq. (2.2), then the algebra has a maximal dimension \( 2^n \) and is called the universal Clifford algebra, \( \mathbb{C}(g) \), because for any other Clifford algebra \( \mathbb{C} \) with the same metric \( g(\vec{x}) \) there is a homomorphism \( \mathbb{C}(g) \to \mathbb{C} \) (see Ref. \( \equiv \mathbb{C}(\vec{x}) \)).
Let us use the notation $\mathcal{C}(l, m)$ for the diagonal metric Eq. (2.3) with $l$ pluses and $m$ minuses in Eq. (2.2b), i.e., for pseudo-Euclidean (Minkowski) space $\mathbb{R}^{l,m}$. There is a special notation for Euclidean space: $\mathcal{C}(n) \equiv \mathcal{C}(n,0)$ and $\mathcal{C}_n(n) \equiv \mathcal{C}(0, n)$.

Complexification of any Clifford algebra $\mathcal{C}(l, m)$ with $l + m = n$ is the same complex algebra $\mathcal{C}(n, \mathbb{C})$, because all signs in Eq. (2.2b) may be “adjusted” by the substitution $\epsilon_k \rightarrow i \epsilon_k$.

Let us denote $e_0^2 \equiv \sqrt{e_0^2}, i.e.,$ if $e_0^2 = 1$, then $e_0^2 = e_1$, but if $e_0^2 = -1$, then $e_0^2 = i e_1$ and so always $(e_0^2)^2 = 1$.

B. Matrix representations

All complex Clifford algebras in even dimension $\mathcal{C}(2n, \mathbb{C})$ are isomorphic with a full algebra of $2^n \times 2^n$ complex matrices $\mathbb{C}$). The simplest case $\mathcal{C}(2, \mathbb{C})$ is the Pauli algebra. Matrices $\sigma_x$ and $\sigma_y$ can be chosen as generators $e_0, e_1$ and $\sigma_z$ is $i e_0 e_1 = \sigma_{01}$.

The Pauli algebra is four-dimensional complex algebra and can also be considered as eight-dimensional real algebra, $\mathcal{C}_4$. Prevalent applications of Clifford algebras in the theory of NMR quantum computation $\mathbb{C}$) are based on a real representation $\mathcal{C}_4(3)$ rather than on a complex one $\mathcal{C}(2, \mathbb{C})$, discussed in the present work. These two approaches are very close, but may be different in some of the details.

There is simple recursive construction of the complex Clifford algebra with an even number of generators $\mathcal{C}(2n, \mathbb{C})$ with $\mathcal{C}(2, \mathbb{C})$. For $n = 1$, it is the Pauli algebra, and if there is some algebra $\mathcal{C}(2n, \mathbb{C})$ for $n \geq 1$, then

$$\mathcal{C}(2n+2, \mathbb{C}) \equiv \mathcal{C}(2n, \mathbb{C}) \otimes \mathcal{C}(2, \mathbb{C}).$$

The proof of Eq. (2.3) is as follows: if $e_0, \ldots, e_{2n-1}$ are 2$n$ generators of $\mathcal{C}(2n, \mathbb{C})$ then $1_2 \otimes e_0$ and $1_2 \otimes e_1$ together with 2$n$ elements $e_k \otimes e_0^{1}$ are 2$n+2$ generators of $\mathcal{C}(2n+2, \mathbb{C})$.

Direct construction of $\mathcal{C}(2n, \mathbb{C})$ is $\mathbb{C}$)

$$\Gamma_{2k} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_x \otimes \sigma_y \otimes \sigma_z \cdots \otimes \sigma_z,$$

$$\Gamma_{2k+1} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \cdots \otimes \sigma_y,$$

with $e_l \equiv \Gamma_l, e_l^2 = \mathbb{1}, \forall l = 0, \ldots, 2n-1$. More generally, algebraic properties of elements $e_l$ used in the paper are the same for different matrix representations $e_l \equiv M \Gamma_l M^{-1},$ where $M \in \text{SU}(2^n)$.

C. Spin groups

Most known physical applications of Clifford algebras are due to spin groups. The group has 2:1 homomorphism with an orthogonal (or pseudo-orthogonal) group and is related to the Dirac equation $\mathbb{3}$ and the transformation properties of wave functions in quantum mechanics.

Each element $x \in \mathbb{V}$ (see the definition of $\mathbb{V}$ above in Sec. II A) has an inverse $x^{-1} = -x/g(x)$ if $g(x) \neq 0$. All possible products of even number of such elements with $|g| = 1$ is the spin group. It is Spin$(n)$ for $\mathcal{C}(n)$ and for $\mathcal{C}_n(n)$. The group has 2:1 homomorphism with SO$(n)$. For $s \in \text{Spin}(n)$ an element of SO$(n)$ is represented as $r_s x : x \mapsto s x s^{-1}$ (4).

Because only products of an even number of elements of $\mathcal{C}(n)$ are used in the definition of Spin$(n)$, the group is a subset of even subalgebra $\mathcal{C}^e(n) \subset \mathcal{C}(n)$. In the Euclidean case, $\mathcal{C}^e(n)$ is isomorphic with $\mathcal{C}(n-1)$ and due to the property Spin$(n+1)$ may be defined as a subset of $\mathcal{C}(n)$.

Construction of the Spin$(n+1)$ group from $\mathcal{C}(n)$ is sometimes called the spin group (4). Spin$(n) \equiv \text{Spin}(n+1)$.

Let us consider $(n+1)$-dimensional space $\lambda \mathbb{1} \oplus \mathbb{V}$, i.e., combinations $y = \lambda + x, x \in \mathbb{V}$. Let $\Delta(y) \equiv \lambda^2 + g(x)$. The elements have an inverse $(\lambda + x)^{-1} = (\lambda - x)/\Delta(y)$ if $\Delta(y) \neq 0$. Products of any number of such elements with $|\Delta| = 1$ is the Spin$(n)$ group (4).

The group Spin$(n)$ is 2:1 homorphic with SO$(n+1)$. For $s \in \text{Spin}(n)$, an element of SO$(n+1)$ is represented as $r_s y : y \mapsto sy(s')^{-1}$, where $y = y_0 + \sum_{j=0}^{n-1} y_j e_j$ and ($')$ is the algebra automorphism defined with basis elements $e_j = (-1)^{j(N)} e_j$ (4).

D. Lie algebras and Clifford algebras

Clifford algebra is Lie algebra with respect to a bracket operation $[a, b] \equiv ab - ba$ (3). Here we prove a result that is necessary for the construction of a universal set of gates.

Theorem 1. Let $\mathcal{C}(n, \mathbb{C})$ be the Clifford algebra and $n$ be even. There are enough $n$ generators $e_k, k = 0, \ldots, n-1$ and any element $e_l$ with $N(I) = 3$ or $N(I) = 4$ to generate elements of any order only using commutators of these $n + 1$ elements.

A proof of this result has several steps.

(i) If there are $n$ elements $e_0, \ldots, e_{n-1}$, it is possible by using commutators to generate also all elements of second order, i.e., $[e_i, e_j] = 2e_i e_j \equiv 2e_{ij}$.

(ii) If there are all elements of second order and an element of third order, for example $e_{012}$, it is possible to generate any element of third order, i.e., $2e_0 e_1 = [e_{012}, e_{02}], 2e_{01m} = [e_{01m}, e_{1m}], 2e_{0mn} = [e_{0mn}, e_{0p}], 2e_{0mn} = [e_{0nm}, e_{0p}].$

(iii) Analogously, if there is any element of order $2k+1$, it is possible to generate any element of the same order using no more than $2k + 1$ commutators with elements $e_{ij}$. (iv) If we have all elements of third order, it is possible to generate any element of fourth order, $2e_{ijk} = [e_{ijk}, e_i].$

(v) Analogously, if we have all elements with the order $N(I) = 2k + 1$, it is possible to generate any element of
order $2k + 2$, $2\mathbf{c}_{I,l} = [\mathbf{c}_I, \mathbf{c}_l]$, where $l \notin I$.

(vi) If we have an element of fourth order, it is possible to
generate some element of third order, $2\mathbf{c}_{ijk} = [\mathbf{c}_{ijk}, \mathbf{c}_l]$
(and so any element of third and fourth order).

(vii) Analogously, if we have an element of order $2k + 2$,
it is possible to generate some element of order $2k + 1$ [and
so any element with the order $2k + 1$ or $2k + 2$ as in the
steps (iii) and (v)].

(viii) We have all elements with order less than or equal
to $2k$, $k \geq 2$ due to steps (i), (ii), and (iv) and we can
prove the theorem by recursion: by using a commutator
of an element with order $2k - 1$ and an element with or-der
3 it is possible to generate an element of order $2k + 2$
and so any elements of order $2k + 1$ or $2k + 2$, as in the
step (vii).

Note 1. Instead of elements $\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}$, it is possi-
ble to use $\mathbf{c}_0$ together with $n - 1$ elements $\mathbf{c}_{1-I}$:
$[\mathbf{c}_0, \mathbf{c}_n] = 2\mathbf{c}_1, ..., [\mathbf{c}_{n-1}, \mathbf{c}_{n-1}] = 2\mathbf{c}_l$.

Note 2. If $n$ is odd, it is impossible to generate only an
element with the order $n$, because due to step (vii) of re-
cursion it would be generated only from an even element
with the order $n - 1$, but there are no such elements. So
in this case we need $n + 2$ elements, the extra one being
$\mathbf{c}_0, ..., \mathbf{c}_{n-1}$.

Note 3. If we use only $n$ generators $\mathbf{c}_l$, then together
with $n(n-1)/2$ commutators $[\mathbf{c}_k, \mathbf{c}_l] = 2\mathbf{c}_{kl}$, $k \neq j$,
it is possible to generate $n + n(n-1)/2 = n(n+1)/2$ el-
ements, because, as may be checked directly, any new
commutators may not generate an element with order
more than 2. It is the Lie algebra of the Spoin($n$)
group, because products of $\exp(\epsilon \mathbf{c}_k)$ $\approx \mathbb{I} + \epsilon \mathbf{c}_k$ belong to
that group and the dimension of the group is the same,
$\text{dim Spoin}(n) \equiv n(n+1)/2$. Despite the fact that only elements $\mathbf{c}_1$, $\mathcal{N}(I) \leq 2$ belong to the Lie
algebra, all $4^n$ elements $\mathbf{c}_1$, $\mathcal{N}(I) \leq n$ of $\mathfrak{sl}(n)$ belong to
the Lie group Spoin($n$) by definition and so a linear span
of these elements is the full Clifford algebra.

Note 4. The theorem was proved rather for the more
general case of the Lie algebra of the complex Lie group
GL($N$, $\mathbb{C}$), $N = 2^{n/2}$ of all matrices $M$, $\det(M) \neq 0$, than
for the unitary group $U(N) \subset \text{GL}(N, \mathbb{C})$. The proof for
the Lie algebra $\mathfrak{u}(N)$ of the unitary group $U(N)$ is di-
rectly implied. It is sufficient to choose the initial ma-
trices in $\mathfrak{u}(N)$ for a given representation, after which
the Lie brackets may produce only matrices in $\mathfrak{u}(N)$ for each
step of the proof.

It should be mentioned that there are two traditions for
representations of $\mathfrak{u}(N)$. In physical applications, Her-
mitian matrices $H$ are used, the Lie brackets are $[i[a, b]]$, and
the unitary matrices are represented as $U = \exp(-i\tau H)$
due to relations with Hamiltonians and a quantum ver-
sion of Poisson brackets $[\mathfrak{p}, \mathfrak{p}]$. In Eq. (2), elements
$\mathbf{c}_1 = I_1$, $\mathbf{c}_012$, and $\mathbf{c}_{0123}$ (and $i\mathbf{c}_{kl}$, see Note 1), i.e., all $\mathbf{c}_l$,
are Hermitian. In more general mathematical applica-
tions, $\mathfrak{u}(N)$ are skew-Hermitian matrices $A^\dagger = -A$ and
“$i$” multipliers are not present in the expressions for the
commutators and the exponents $[\mathfrak{e}]$, because $A \triangleq iH$.

III. APPLICATION TO QUANTUM GATES

A. Universal set of quantum gates

Now let us discuss the construction of universal gates
more directly. Instead of Lie algebra $\mathfrak{u}(2^n)$, we should
work with Lie group $U(2^n)$. Then an element $\mathbf{c}_l^T$
corresponds to a unitary gate $U_l^T = \exp(i\mathbf{c}_l^T)$. One of the
advantages of elements $\mathbf{c}_l^T$ is the analytical expression for the
exponent:

$$U_l^T = e^{i\pi \mathbf{c}_l^T} = \cos(\tau) + i \sin(\tau) \mathbf{c}_l^T.$$  (3.1)

Equation (3.1) is valid for any operator with the property
$\mathbf{c}^2 = \mathbb{I}$ and it is true for all $4^n$ basis elements $\mathbf{c}_l^T$.

It is also possible to combine due to Eq. (3.1) to combine
the approach with infinitesimal parameters $\tau$ and an approach
with irrational parameters $\mathbf{c}_l^T$. The smaller $\tau$
is, the higher is the precision in generation of arbitrary
unitary gates in $U(N)$. Due to Eq. (3.1), accuracy may
be arbitrarily high if we use gates $U_l = e^{i\mathbf{c}_l^T}$ with ir-
ratonal $\tau/N$ because for any $\tau$ there exists the natural
number $N$ and $\tau < \epsilon$: $U_l^\tau = (U_l)^N$. It should be
mentioned that the unitary gates do not necessarily have
irrational coefficients even if $\tau/N$ is irrational, for example
$U_l = 0.8 + 0.6\mathbf{c}_l^T$.

Yet another advantage of the elements $\mathbf{c}_l^T$ is a simpler
expression for “commutator gate”. In the usual case $[\mathfrak{e}]$
and the expression has precision $O(1,5)$. For elements
$\mathbf{c}_l^T$, there is an exact construction. If $H_I = \mathbf{c}_l^T$ and
$H_J = \mathbf{c}_l^T$, then either $[H_I, H_J] = 0$ or $[H_I, H_J] = 2H_I H_J$. The
first case is trivial and for the second case due to Eq. (3.1),

$$e^{i\tau \mathbf{c}_l^T} [H_I, H_J] / 2 = e^{i\tau H_I} e^{i\tau H_J} e^{-i\tau H_I}.$$  

After construction of the basis of Hermitean matrices
$H_I = \mathbf{c}_l^T$, it is possible to use an expression

$$e^{\sum \alpha_i H_I} \approx \left( e^{\pi \sum \alpha_i H_I} \right)^N \approx \left( \prod e^{\frac{\pi \alpha_i H_I}} \right)^N = \left( \prod U_{I_l}^{\alpha_i} \right)^N.$$  

The expression has accuracy $O(\sum \alpha_i^2 / N)$.

The approach to a universal set of gates $U$ is more con-
venient and constructive if we know the Hermitean matrix
$H$, $H^\tau = e^{i\tau H}$. It is not a principal limitation, because
for physical realizations we should know the Hamiltonian
to construct the gates. It is also related to the universal
quantum simulation $[\mathfrak{p}]$ in which $H$ is the Hamiltonian
and $\tau$ is a real continuous parameter, the time of “application.”
The description with an exponent may be even more complete, because by using $H$ it is possible to find a unique $U = \exp(iH)$, but using $U$ it is not always possible to restore $H$ because there are many $H$’s for the same $U$. A simple example is $U = i\sigma_\alpha \otimes \sigma_\beta$ with two arbitrary Pauli matrices: $U = e^{i\pi (\sigma_\alpha \otimes 1 + 1 \otimes \sigma_\beta)} = e^{i\pi \sigma_\alpha \otimes \sigma_\beta}$.

**B. Two-qubit quantum gates**

Let us consider a full basis $U_{l}$ for space $U_{2}$, $l = 1, \ldots, 2n - 1$, and $i\epsilon_{012}$:

\begin{align}
\epsilon_{0} &= 1 \otimes \ldots \otimes 1 \\
\epsilon_{2k,2k+1} &= 1 \otimes \ldots \otimes 1 \otimes \sigma_{x} \otimes 1 \otimes \ldots \otimes 1 \otimes 1,
\end{align}

(3.2a, b, c, d)

with $k = 0, \ldots, n - 1$ or $n - 2$. The elements were discussed in *Note 1*, and it was shown that they generate the full Lie algebra $u(2n)$.

**C. Nonuniversal set of quantum gates**

In [2], an interesting question was raised, asking which sets of gates are not universal (and why).

Products of gates $U_{l}^k = e^{i\pi \epsilon_{l}} = \cos(\tau) + i\tau \sin(\tau)$ generate a group $\text{Spin}(2n + 1) \equiv \text{Spin}(2n) \subset U(2n)$ due to *Note 3*. It is an interesting example of nonuniversality when only one extra gate like $e^{i\pi \epsilon_{012}}$ may produce a universal set with an exponential improvement from a subgroup $\text{dim Spin}(2n) = n(2n + 1)$ to a full group $\text{dim } U(2n) = 2^{2n}$.

This result is more important if the extra gate $e^{i\pi \epsilon_{l}}$ with $N(I) = 3$ or $N(I) = 4$ has a different physical structure from the gates with $N(I) = 1$ and $N(I) = 2$. It is not clear from Eq. (2) with the extra gate generated by Eq. (32) that the simple one-qubit gate. But this is not so for physical systems with natural Clifford and spin structure.

A possible reason is the Schrödinger equation for $n$ particles without interaction [3]: $i\hbar (\partial \psi / \partial t) = \frac{1}{2} \sum_{a=1}^{2n} (\Delta_{N} / m_{a}) \psi_{a}$, or using $m_{a} = m$ and the Laplacian $\Delta_{N}$ with $N = vn$ variables, it is possible to write for stationary solutions with total energy $E$, $$(\Delta_{N} + \lambda^{2}) \psi(x_{0}, \ldots, x_{N-1}) = 0,$$

(3.3)

where $\lambda \equiv \sqrt{2mE} / \hbar$. Let the dimension of one particle motion be $v = 2$ for simplicity, $N = 2n$.

Let us consider a full basis $\phi_{p}(x) \equiv e^{i(p,x)}$ on Hilbert space $\mathcal{L}$ of wave functions $\psi \in \mathcal{L}$. Here $p, x \in R^{N}$ and $(p, x)$ is the scalar product. The plane waves $\phi_{p}$ correspond to $n$ particles with definite momenta. If $O \in \text{SO}(N)$, then a transformation defined on the basis as $\Sigma_{l} : \phi_{p} \rightarrow e^{i\pi \epsilon_{l}} \phi_{p}$ is a symmetry of Eq. (3.3). It is an analog of the classical transition between two configurations with the same total kinetic energy in “billiard balls” conservative logic [4].

The general Dirac operator [4] is the first-order differential operator $\mathcal{D}_{N} = \sum_{k=0}^{N-1} i\epsilon_{k} (\partial / \partial x_{k})$ with a property $\mathcal{D}_{N}^{2} = -\Delta_{N}$. If to use the Dirac operator for factorization of Eq. (3.3),

$$(\mathcal{D}_{N} - \lambda) (\mathcal{D}_{N} + \lambda) \psi(x_{0}, \ldots, x_{N-1}) = 0,$$

(3.4)

then each component of $\Psi$ is a solution of Eq. (3.3) and the action of the Spin($N$) group on $\Psi$ corresponds to SO($N$) symmetry $\Sigma_{l}$ described above and it has some analog in the classical physics of billiard balls. A Spoin($N$) group is represented less directly, but it can be considered as a symmetry between two stationary solutions with different total energies.

The example above shows that it is possible to find some classical correspondence for elements $\epsilon_{l}$, $N(l) = 2$ of the spin group and maybe for generators $N(l) = 1$ of the spoin group, but the special element with $N(l) = 3$ does not have some allusion with classical physics.

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