EXACT SOLUTIONS FOR WAVE PROPAGATION IN BIREFRINGENT OPTICAL FIBERS

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Abstract

We carry out a group-theoretical study of the pair of nonlinear Schrödinger equations describing the propagation of waves in nonlinear birefringent optical fibers. We exploit the symmetry algebra associated with these equations to provide examples of specific exact solutions. Among them, we obtain the soliton profile, which is related to the coordinate translations and the constant change of phase.

1 Introduction

The propagation of optical pulses in nonlinear birefringent fibers is described by the pair of nonlinear Schrödinger equations [1]

\[ \Delta_1 = i u_x + u_{tt} + kv + (\alpha |u|^2 + \beta |v|^2)u = 0, \]

\[ \Delta_2 = i v_x + v_{tt} + ku + (\alpha |v|^2 + \beta |u|^2)v = 0, \]

where \( u = u(x,t) \) and \( v = v(x,t) \) are the circularly polarized components of the optical field, \( x \) and \( t \) denote the (normalized) longitudinal coordinate of the fiber and the time variable, respectively, \( k \) is the birefringence parameter and the coefficients \( \alpha \) and \( \beta \) are responsible for the nonlinear properties of the fiber [2, 3, 4]. Performing the change of variables \( x \rightarrow \alpha x, \quad t \rightarrow \mu t, \) with \( \mu^2 = \pm \frac{1}{2} \alpha \), Eqs.(1.1) take the form

\[ i u_x \pm \frac{1}{2} u_{tt} + k'v + (|u|^2 + \sigma |v|^2)u = 0, \]
$$i v_x \pm \frac{1}{2} v u + k' u + (|v|^2 + \sigma|u|^2) v = 0,$$

where \( k' = k/\alpha \), +\((-)\) holds in the anomalous (normal) dispersion regime and \( \sigma = \frac{\beta}{\alpha} = \frac{1+B}{1-B} \), \( B \) being the third-order susceptibility coefficient \([2, 3, 4]\). For \( \alpha = \beta \) (\( \sigma = 1 \)) and \( k = 0 \), the system (1.1) has an infinite set of constants of motion and may be solved by the inverse scattering method \([4]\).

In general, i.e. for \( \alpha \neq \beta \) and \( k \neq 0 \), the system (1.1) is not integrable by inverse scattering. In this case, Eqs.(1.1) possess three constants of motion only \([4]\). On the other hand, it is well-known that a powerful tool for handling both integrable and nonintegrable differential equations is represented by the so-called symmetry approach \([5]\). This method, which is based on the Lie group theory, consists essentially in looking for symmetry transformations that reduce the equations under consideration to certain ordinary differential equations, each of them comes from an invariant quantity associated with a given symmetry allowed by the system. Following this idea, in this work we apply the symmetry approach to Eqs.(1.1). We display examples of new exact solutions in both the cases \( \alpha = \beta \) and \( \alpha \neq \beta \). In this regard, we observe that the birefringent parameter \( k \) involved in Eqs.(1.1) is real. However, the symmetry algebra found for \( \alpha = \beta \) does not depend on \( k \). This fact has suggested us to study the system also for imaginary values of \( k \). In such a situation, at least when \( u = v \), \( k \) can be interpreted as the loss coefficient of the fiber \([7]\). Anyway, we have obtained an interesting exact solution for \( u \neq v \) as well. This solution is derived from the Galilean boost. Another important aspect of the symmetry reduction technique is the determination of the infinitesimal operator which is responsible for the soliton profile and the periodic configuration.

In Sec. 2 we outline the method of symmetry reduction and obtain the symmetry algebra and the corresponding symmetry group related to Eqs.(1.1). Sec. 3 contains examples of specific exact solutions, while in Sec. 4 some concluding remarks are reported.

2 The method of symmetry reduction
2.1 a) The symmetry algebra

The method of symmetry reduction (SR) consists of an application of the Lie group theory to reduce Eqs. (1.1) to a system of ordinary differential equations.

A fundamental step of the SR procedure is to obtain the Lie point symmetries of Eqs. (1.1): in other words, the symmetry algebra \( L \) and the corresponding symmetry group \( G \) of the equations under investigation. Then, we can build up solutions that are invariants under some specific subgroup \( G_0 \) of \( G \). The SR can be carried out via the determination of the invariants of \( G_0 \). Invariants are furnished by the partial differential equations

\[
V_j I(x, t, u, v, u^*, v^*) = 0, \quad j = 1, 2, ..., n,
\]

where \( \{V_j\} \) is a basis of the Lie algebra \( L_0 \) of \( G_0 \) and \( n \) is the number of the independent elements (infinitesimal operators) \( V_j \) of \( L_0 \). Once the invariants related to \( G_0 \) are known, Eqs.(1.1) can be written in terms of them. In such a way, we are led to a set of reduced equations which may yield exact solutions to the original system (1.1).

The Lie point symmetries of Eqs. (1.1) can be found by resorting to the standard technique outlined in [6]. Precisely, let us introduce the vector field

\[
V = \xi_1 \partial_x + \xi_2 \partial_t + \xi_3 \partial_u + \xi_4 \partial_{u^*} + \xi_5 \partial_v + \xi_6 \partial_{v^*},
\]

where \( \xi_j (j = 1, 2, ..., 6) \) are functions which depend in general on \( x, t, u, u^*, v, v^* \), and \( \partial_x = \frac{\partial}{\partial x} \), and so on. A local group of transformations \( G \) is a symmetry group for Eqs.(1.1) if and only if

\[
pr^{(2)}V[\Delta_j] = 0, \quad pr^{(2)}V[\Delta_j^*] = 0, \quad j = 1, 2,
\]

whenever \( \Delta_j = 0, \quad \Delta_j^* = 0 \) for every generator of \( G \), where \( pr^{(2)}V \) is the second prolongation of \( V \).

The conditions (2.3) constitute a set of constraints in the form of partial differential equations which enable us to obtain the coefficients \( \xi_j \). The calculations have been performed in part by using the symbolic language REDUCE [8]. We have achieved the following results.
**Case I :** $\alpha \neq \beta$

The symmetry algebra is defined by four elements, namely:

\[
V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = i(u\partial_u - u^*\partial_{u^*} + v\partial_v - v^*\partial_{v^*}), \\
V_4 = x\partial_t + \frac{i}{2}t(u\partial_u - u^*\partial_{u^*} + v\partial_v - v^*\partial_{v^*}),
\]

where the nonvanishing commutation relations are

\[
[V_1, V_4] = V_2, \quad [V_2, V_4] = \frac{1}{2}V_3.
\]

**Case II :** $\alpha = \beta$

The symmetry algebra is of the $sl(3,\mathbb{C})$ type. It is defined by eight elements: four of them coincide with the previous ones, while the others are given by

\[
V_5 = x\partial_x + \frac{1}{2}t\partial_t - \frac{1}{2}(u\partial_u + u^*\partial_{u^*} + v\partial_v + v^*\partial_{v^*}) + ikx(v\partial_u - v^*\partial_{u^*} + u\partial_v - u^*\partial_{v^*}),
\]

\[
V_6 = i(v\partial_u - v^*\partial_{u^*} + u\partial_v - u^*\partial_{v^*}),
\]

\[
V_7 = (v\partial_u + v^*\partial_{u^*} - u\partial_v - u^*\partial_{v^*})\cos 2kx - i(u\partial_u - u^*\partial_{u^*} - v\partial_v + v^*\partial_{v^*})\sin 2kx,
\]

\[
V_8 = (v\partial_u + v^*\partial_{u^*} - u\partial_v - u^*\partial_{v^*})\sin 2kx + i(u\partial_u - u^*\partial_{u^*} - v\partial_v + v^*\partial_{v^*})\cos 2kx.
\]

The nonvanishing commutation relations fulfilled by $V_1, \ldots, V_8$, are (2.5) together with

\[
[V_1, V_5] = V_1 + kV_6, \quad [V_1, V_7] = -2kV_8, \quad [V_1, V_8] = 2kV_7, \\
[V_2, V_5] = \frac{1}{2}V_2, \quad [V_5, V_4] = \frac{1}{2}V_4.
\]

and

\[
[V_6, V_7] = 2V_8, \quad [V_7, V_8] = 2V_6, \quad [V_8, V_6] = 2V_7
\]

The vector fields $V_1, \ldots, V_8$ are the generators of the infinitesimal symmetries transformations of Eqs.(1.1). These are the coordinate translations and the Galilean boost ($V_4$), which are common to both the cases I and II. Moreover, for $\alpha = \beta$ Eqs. (1.1) admit the additional symmetry $su(2,\mathbb{C})$ which is expressed by the generators $V_6$, $V_7$, $V_8$ satisfying the commutation rules (2.8)\footnote{It is noteworthy that for $\alpha \neq \beta$, the symmetry algebra turns out to be independent from the...}.
b) The group transformations

By integrating the infinitesimal operators $V_1, \ldots, V_8$, we provide the group transformations that leave Eqs.(1.1) invariant. These are, respectively:

$$V_1: \quad \tilde{t} = t, \quad \tilde{x} = x + \lambda, \quad \tilde{u} = u, \quad \tilde{v} = v, \quad (2.9a)$$

$$V_2: \quad \tilde{t} = t + \lambda, \quad \tilde{x} = x, \quad \tilde{u} = u, \quad \tilde{v} = v, \quad (2.9b)$$

$$V_3: \quad \tilde{x} = x, \quad \tilde{t} = t, \quad \tilde{u} = u e^{i\lambda}, \quad \tilde{v} = v e^{i\lambda}, \quad (2.9c)$$

$$V_4: \quad \tilde{x} = x, \quad \tilde{t} = t + \lambda \tilde{x}, \quad \tilde{u} = u e^{\frac{i}{2}(\lambda \tilde{t} - \frac{1}{2}\lambda^2 \tilde{x})}, \quad \tilde{v} = v e^{\frac{i}{2}(\lambda \tilde{t} - \frac{1}{2}\lambda^2 \tilde{x})}, \quad (2.9d)$$

$$V_5: \quad \tilde{x} = e^{\lambda} x, \quad \tilde{t} = e^{\lambda/2} t, \quad \left( \begin{array}{c} \tilde{u} \\ i \tilde{v} \end{array} \right) = \sqrt{e^{2\lambda}} \left( \begin{array}{cc} \cos k(\tilde{x} - x) & \sin k(\tilde{x} - x) \\ -\sin k(\tilde{x} - x) & \cos k(\tilde{x} - x) \end{array} \right) \left( \begin{array}{c} u \\ iv \end{array} \right), \quad (2.9e)$$

$$V_6: \quad \tilde{x} = x, \quad \tilde{t} = t, \quad \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) = e^{i\lambda \sigma_1} \left( \begin{array}{c} u \\ v \end{array} \right), \quad (2.9f)$$

$$V_7: \quad \tilde{x} = x, \quad \tilde{t} = t, \quad \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) = e^{i\lambda(\cos 2kx \sigma_2 - \sin 2kx \sigma_3)} \left( \begin{array}{c} u \\ v \end{array} \right), \quad (2.9g)$$

$$V_8: \quad \tilde{x} = x, \quad \tilde{t} = t, \quad \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) = e^{i\lambda(\sin 2kx \sigma_2 + \cos 2kx \sigma_3)} \left( \begin{array}{c} u \\ v \end{array} \right), \quad (2.9h)$$

Conversely, for $\alpha = \beta$ and $k$ such that $\text{Im} k \neq 0$, the symmetry algebra is defined by the vector fields $V_1, V_2, V_3, V_4$ and $V_6$. In other words, the presence in Eqs.(1.1) of a non-real coefficient in front of $u$ and $v$ changes the symmetry algebra, which is no longer of the $sl(3,C)$ type. This fact is connected with the loss of the integrability property of Eqs.(1.1) for $\alpha = \beta$ and $k$ such that $\text{Im} k \neq 0$. 

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where \( \lambda \) and \( \sigma_1, \sigma_2, \sigma_3 \) are the group parameter and the Pauli matrices, respectively. In deriving (2.9), we have used the initial conditions \( \tilde{x}(\lambda) \mid_{\lambda=0} = x, \quad \tilde{t}(\lambda) \mid_{\lambda=0} = t, \quad \tilde{u}(\lambda) \mid_{\lambda=0} = u, \quad \tilde{v}(\lambda) \mid_{\lambda=0} = v. \)

Equations (2.9a)-(2.9h) tell us that if \( u = f(x, t), \quad v = g(x, t) \) is a solution of the system (1.1), so are

\[
\begin{align*}
  u^{(1)} &= f(x - \lambda, t), \\
  v^{(1)} &= g(x - \lambda, t), \\
  u^{(2)} &= f(x, t - \lambda), \\
  v^{(2)} &= g(x, t - \lambda), \\
  u^{(3)} &= e^{i\lambda} f(x, t), \\
  v^{(3)} &= e^{i\lambda} g(x, t), \\
  u^{(4)} &= f(x, t - \lambda x)e^{\frac{i}{2}(\lambda t - \frac{1}{2}\lambda^2 x)}, \\
  v^{(4)} &= g(x, t - \lambda x)e^{\frac{i}{2}(\lambda t - \frac{1}{2}\lambda^2 x)}, \\
  u^{(5)} &= e^{-\frac{i}{2}} \left\{ f(e^{-\lambda x}, e^{-\frac{i}{2}t}) \cos [kx(e^\lambda - 1)] + ig(e^{-\lambda x}, e^{-\frac{i}{2}t}) \sin [kx(e^\lambda - 1)] \right\}, \\
  v^{(5)} &= e^{-\frac{i}{2}} \left\{ g(e^{-\lambda x}, e^{-\frac{i}{2}t}) \cos [kx(e^\lambda - 1)] + if(e^{-\lambda x}, e^{-\frac{i}{2}t}) \sin [kx(e^\lambda - 1)] \right\}, \\
  u^{(6)} &= f(x, t) \cos \lambda + ig(x, t) \sin \lambda, \\
  v^{(6)} &= if(x, t) \sin \lambda + g(x, t) \cos \lambda, \\
  u^{(7)} &= (\cos \lambda - i \sin \lambda \sin 2kx) f(x, t) + (\sin \lambda \cos 2kx) g(x, t), \\
  v^{(7)} &= (-\sin \lambda \cos 2kx) f(x, t) + (\cos \lambda + i \sin \lambda \sin 2kx) g(x, t), \\
  u^{(8)} &= (\cos \lambda + i \sin \lambda \cos 2kx) f(x, t) + (\sin \lambda \sin 2kx) g(x, t), \\
  v^{(8)} &= (-\sin \lambda \sin 2kx) f(x, t) + (\cos \lambda - i \sin \lambda \cos 2kx) g(x, t).
\end{align*}
\]
3 Exact solutions

As we have already mentioned, the method of symmetry reduction of a partial differential equation amounts essentially to finding the invariants (symmetry variables) of a given subgroup of the symmetry group admitted by the equation under consideration. A basis set of invariants for the generators \( V_j \) can be obtained by solving Eq.(2.1). Alternatively, one can resort to a direct equivalent procedure by using the group transformations (2.9).

The invariants can be exploited to provide exact solutions to Eqs.(1.1). By way of example, in this Section we shall deal with the invariants related to the symmetry operators

\( a \) \( V_0 = V_1 + V_2 + V_3 \), \( b \) \( V_4 \), \( c \) \( V_1 + V_4 \) and \( d \) \( V_5 \).

Case a) A set of invariants related to \( V_0 \) is

\[
y = \tilde{t} - \tilde{x} = t - x,
\]

\[
U = \tilde{u}e^{-i\tilde{x}} = ue^{-ix}, \quad W = \tilde{v}e^{-i\tilde{x}} = ve^{-ix},
\]

\[
U_1 = \tilde{u}e^{-i\tilde{u}} = ue^{-i\tilde{t}}, \quad W_1 = \tilde{v}e^{-i\tilde{v}} = ve^{-i\tilde{t}}.
\] (3.1)

Inserting, for instance,

\[
u(x,t) = U(y)e^{ix}, \quad v(x,t) = W(y)e^{ix}, \quad (3.2)
\]

into Eqs.(1.1) (for \( \alpha = \beta \)) gives the pair of (ordinary) reduced equations

\[
iU'' + U - U'' - kW - \alpha(|U|^2 + |W|^2)U = 0 \quad (3.3a)
\]

\[
iW' + W - W'' - kW - \alpha(|U|^2 + |W|^2)W = 0 \quad (3.3b)
\]

where

\[
U = U(y), \quad W = W(y), \quad U' = \frac{dU}{dy}, \quad \text{and} \quad W' = \frac{dW}{dy}.
\]

Now, let us look for solutions to Eqs.(3.3) of the type

\[
U(y) = p(y)e^{iy}, \quad W = q(y)e^{iy} \quad (3.4)
\]
where \( p, q \) are real functions of \( y \) and \( \gamma, \delta \) are real constants. In doing so, Eqs.(3.3) yield

\[
(2\gamma - 1)p' + kq \sin (\delta - \gamma)y = 0,
\]

\[
(2\delta - 1)q' - kp \sin (\delta - \gamma)y = 0,
\]

\[
p'' + (\gamma - \gamma^2 - 1)p + kq \cos (\delta - \gamma)y + \alpha(p^2 + q^2)p = 0,
\]

\[
q'' + (\delta - \delta^2 - 1)q + kp \cos (\delta - \gamma)y + \alpha(p^2 + q^2)q = 0.
\]

Equations (3.5) produce some interesting configurations of Eqs.(1.1), such as the soliton profile and solutions expressed in terms of the elliptic Jacobi functions \( \text{sn}(\cdot) \).

To this aim, let us choose \( \gamma = \delta = \frac{1}{2} \). Then, by defining \( z = p + iq = \rho e^{i\pi/4} \) (\( \rho = |z| \)), we have

\[
\rho'^2 = \left( \frac{3}{4} - k \right) \rho^2 - \frac{\alpha}{2} \rho^4 + c,
\]

where \( \rho' = \frac{dz}{dy} \) and \( c \) is a constant of integration. The soliton profile comes from (3.6) for \( c = 0 \), by taking \( \alpha > 0, \ k < \frac{3}{4} \). Precisely

\[
p = q = \frac{1}{\sqrt{2}}\rho = \sqrt{\left( \frac{3}{4} - k \right)} \frac{1}{\alpha} \text{sech} \left[ \sqrt{\frac{3}{4} - k} (y - y_0) \right],
\]

where \( y_0 \) is an arbitrary constant. With the help of (3.7), from (3.2) we obtain

\[
u = v = e^{\frac{x}{2} + (t+x)}p(t - x),
\]

with \( p(t - x) \) given by (3.7).

We notice that for \( c = 0, \ \alpha < 0, \ k > \frac{3}{4} \), Eq.(3.6) leads to the solution

\[
p = q = \frac{\rho}{\sqrt{2}} = \sqrt{\left( \frac{3}{4} - k \right)} \frac{1}{\alpha} \sec \left[ \sqrt{\frac{3}{4} - k} (y - y_0) \right].
\]

In this case, we loose the soliton character of the profile. (The onset of the soliton depends on the parameters \( \alpha \) and \( k \)).
Another interesting solution linked to the symmetry operator \( V_0 \) arises for \( c = k - 3/4 - |\alpha|/2 > 0 \) and \( \alpha < 0, \ k > \frac{3}{4} \). Indeed, from Eqs. (3.6), (3.4) and (3.2) we have

\[
u = v = \frac{1}{\sqrt{2}} e^{\frac{\pi}{4}(t-x)} \text{sn} \left[ \sqrt{c} (t - x), h \right], \tag{3.10}\]

where \( \text{sn}(\cdot) \) is the sinus elliptic function of modulus \( h = \sqrt{\frac{|\alpha|}{2c}} \).

**Case b)** By using the basis of invariants

\[
\bar{x} = x, \quad U(x) = \bar{u} e^{-\frac{ix^2}{4\bar{x}}} = ue^{-it^2/4x},
\]

\[
W(x) = \bar{v} e^{-\frac{ix^2}{4\bar{x}}} = ve^{-it^2/4x}, \tag{3.11} \]

associated with the vector field \( V_4 \), from Eqs.(1.1) we find the pair of reduced equations

\[
i \left( U' + \frac{1}{2x} U \right) + kW + \left( \alpha|U|^2 + \beta|W|^2 \right) U = 0, \tag{3.12a}\]

\[
i \left( W' + \frac{1}{2x} W \right) + kW + \left( \alpha|W|^2 + \beta|U|^2 \right) W = 0, \tag{3.12b}\]

where \( U' = \frac{dU}{dx}, \ W' = \frac{dW}{dx} \). For \( \alpha = \beta \) and assuming that \( k \) is real, Eqs.(3.12) can be solved in the following way. First, let us divide (3.12a) and (3.12b) by \( U \) and \( W \), respectively. Second, let us subtract the resulting equation corresponding to (3.12a) from that corresponding to (3.12b). Then, we obtain

\[
i (WU' - UW') + k \left( W^2 - U^2 \right) = 0. \tag{3.13}\]

By introducing now

\[
U = \frac{1}{2}(A + B), \quad W = \frac{1}{2}(A - B) \tag{3.14}\]

into Eq.(3.13), we can determine \( B \) in terms of \( A \), namely

\[
B = Ae^{-2i(kx + \delta_0)}, \tag{3.15}\]

where \( \delta_0 \) is a constant of integration. Hence, putting the quantities (3.14) into Eq.(3.12a) and taking account of (3.15), after some manipulations we arrive at the solutions

\[
u = \frac{a}{\sqrt{x}} e^{\frac{i\pi}{4x} - \delta_0} x^{i\alpha x^2} \cos (kx + \delta_0), \tag{3.16a}\]
\[ v = i \frac{a}{\sqrt{x}} e^{i \left( \frac{x^2}{4x} - \delta_0 \right)} x^{ia} \sin (kx + \delta_0), \quad (3.16b) \]

where \( a \) is a real constant. The physical role of these configurations remains to be explored. Here we limit ourselves to observe that the 'mass density' \(|u|^2 + |v|^2\) is of the Coulomb-type in the \( x \)-variable and is time independent. Moreover, the quantities (3.16) resemble the singular solution to the nonlinear Schrödinger equation expressed by formula (3.10) of ref.9.

An interesting class of exact solutions related to the generator \( V_4 \) can be found for \( \alpha \neq \beta \) and \( k = ik_0 \), where \( k_0 \) is a real number.

In this case, let us look for solutions to Eqs.(3.12) of the type

\[ U = \rho e^{i\theta}, \quad W = \rho e^{i\gamma}, \quad (3.17) \]

where \( \rho, \theta \) and \( \gamma \) are real functions of \( x \). Inserting (3.17) into Eqs.(3.12) yields

\[ \frac{\rho'}{\rho} + \frac{1}{2x} + k_0 \cos (\gamma - \theta) = 0, \quad (3.18a) \]
\[ \theta' + k_0 \sin (\gamma - \theta) - (\alpha + \beta) \rho^2 = 0, \quad (3.18b) \]
\[ \gamma' - k_0 \sin (\gamma - \theta) - (\alpha + \beta) \rho^2 = 0, \quad (3.18c) \]

By integrating this system and using (3.11), we give the following pair of exact solutions to Eqs.(1.1):

\[ u = \rho e^{i(\frac{x^2}{4x} + \theta)}, \quad v = u e^{i \arcsin \left[ \text{sech} 2k_0 (x_0 - x) \right]}, \quad (3.19) \]

where

\[ \rho = \sqrt{\frac{\cosh 2k_0 (x_0 - x)}{x}}, \]
\[ \theta = \frac{1}{2} \arcsin \left[ \cosh 2k_0 (x_0 - x) \right] + (\alpha + \beta) c^2 \int_{x_1}^{x} \frac{\cosh 2k_0 (x_0 - x')}{x'} dx', \quad (3.20) \]

and \( c, x_0, x_1 \), are real constants.

At this point, let us deal with the special choice \( \theta = \gamma \). Then, Eqs.(3.18) lead to the solution

\[ u = v = c \frac{e^{-k_0 x}}{\sqrt{x}} e^{i \frac{x^2}{4x}} e^{i (\alpha + \beta) c^2 \int_{x_1}^{x} \frac{-2k_0 x'}{x'} dx'}, \quad (3.21) \]
where \( c \) and \( x_1 \) are constants. (For \( x_1 \to \infty \), the integral at the r.h.s. of (3.21) becomes \(-E_1(2kx)\), where \( E_1(\cdot) \) is the exponential integral function \([10]\)). The ‘mass density’ corresponding to the solutions (3.20) and (3.21) is

\[
|u|^2 + |v|^2 = 2c^2 \frac{\cosh 2k_0(x_0 - x)}{x} \quad \text{and} \quad |u|^2 = |v|^2 = c^2 \frac{e^{-2k_0x}}{x}.
\] (3.22)

We point out that for \( k = 0 \), the mass densities (3.22) behave as \( 1/x \). Consequently, the presence of the parameter \( k = ik_0 \) induces a change of the Coulomb-like mass density.

**Case c)** A basis of invariants related to the symmetry operator \( V_1 + V_4 \) is given by

\[
\eta = \tilde{t} - \frac{1}{2} \tilde{x}^2 = t - \frac{1}{2} x^2,
\]

\[
U(\eta) = \tilde{u} e^{-\frac{i}{2} \eta(x + \frac{1}{2} \tilde{x}^2)} = u e^{-\frac{i}{2} \eta(x + \frac{1}{2} x^2)},
\]

\[
W(\eta) = \tilde{v} e^{-\frac{i}{2} \eta(x + \frac{1}{2} \tilde{x}^2)} = v e^{-\frac{i}{2} \eta(x + \frac{1}{2} x^2)}.
\] (3.23)

By assuming \( \alpha = \beta \), and setting (3.23) into Eqs.(1.1), we get the reduced system

\[
U'' - \frac{1}{2} \eta U + k W + \alpha (|U|^2 + |W|^2) U = 0, \quad (3.24a)
\]

\[
W'' - \frac{1}{2} \eta W + k U + \alpha (|W|^2 + |U|^2) W = 0, \quad (3.24b)
\]

where \( U' = \frac{dU}{d\eta} \). By requiring that \( U \) and \( W \) are real functions and \( U = W \), Eqs.(3.24) lead to a special case of the second Painlevé equation \([11]\), i.e.

\[
\frac{d^2 \psi}{dz^2} = z \psi + 2 \psi^3, \quad (3.25)
\]

where \( z = 2 \frac{1}{2} (\frac{1}{2} \eta - k) \) and \( \psi = 2 \frac{1}{2} \sqrt{-\alpha} U \).

**Case d)** A set of invariants arising from the generator \( V_5 \) is

\[
\xi = \frac{\tilde{x}}{\tilde{t}^2} = \frac{x}{t^2},
\]

\[
I = \sqrt{\xi}(\tilde{u} \sin k\tilde{x} + i\tilde{v} \cos k\tilde{x}) = \sqrt{x}(u \sin kx + iv \cos kx),
\]

\[
J = \sqrt{\xi}(\tilde{u} \cos k\tilde{x} - i\tilde{v} \sin k\tilde{x}) = \sqrt{x}(u \cos kx - iv \sin kx), \quad (3.26)
\]
from which

\[ K = \tilde{x}(|\tilde{u}|^2 + |\tilde{v}|^2) = x(|u|^2 + |v|^2).\]

Equations (3.26) imply

\[ u = \frac{1}{\sqrt{x}} (I \sin kx + J \cos kx), \quad (3.27a) \]

\[ v = \frac{i}{\sqrt{x}} (-I \cos kx + J \sin kx). \quad (3.27b) \]

Then, substitution from (3.27) into Eqs.(1.1) (for \( \alpha = \beta \)) yields

\[ \frac{1}{2i}(I - 2\xi I') + 6\xi^2 I' + 4\xi^3 I'' + \alpha \left(||I|^2 + |J|^2\right) I = 0, \quad (3.28a) \]

\[ \frac{1}{2i}(J - 2\xi J') + 6\xi^2 J' + 4\xi^3 J'' + \alpha \left(||I|^2 + |J|^2\right) J = 0, \quad (3.28b) \]

where \( I = I(\xi), \ J = J(\xi), \ I' = \frac{dI}{d\xi} \) and \( J' = \frac{dJ}{d\xi} \). A simple solution to the system (3.28) can be found supposing that \( I(\xi) \) and \( J(\xi) \) are real functions. In such a case, Eqs.(3.28) can be easily integrated. We have

\[ u = \sqrt{-\frac{2}{\alpha}} \frac{\sin(kx + \phi)}{t}, \]

\[ v = -i \sqrt{-\frac{2}{\alpha}} \frac{\cos(kx + \phi)}{t}, \quad (3.29) \]

from (3.27), where \( \phi \) is a constant and \( \alpha \) may take both positive and negative values.

4 Conclusions

We have found some exact solutions to the system of equations (1.1) describing the propagation of waves in birefringent optical fibers. These equations have been analyzed in the framework of the Lie group theory. We have determined the associated symmetry algebra and the corresponding group transformations that leave Eqs.(1.1) invariant. Explicit configurations have been obtained both in the integrable and in the nonintegrable case. The subgroup (of the symmetry group) responsible for the soliton profile has been provided. Not all the configurations arising from the
symmetry approach have an evident physical meaning. This problem remains open. Notwithstanding, the knowledge of exact solutions to Eqs. (1.1) may be used with benefit as a guide for the development of perturbative techniques or for numerical calculations. Furthermore, the existence of exact solutions could be a challenge for trying to create new experimental patterns.

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