We consistently develop a recently proposed scheme of matrix extensions of dispersionless integrable systems in the general case of multidimensional hierarchies, concentrating on the case of dimension $d \geq 4$. We present extended Lax pairs, Lax–Sato equations, matrix equations on the background of vector fields, and the dressing scheme. Reductions, the construction of solutions, and connections to geometry are discussed. We separately consider the case of an Abelian extension, for which the Riemann–Hilbert equations of the dressing scheme are explicitly solvable and give an analogue of the Penrose formula in curved space.

Keywords: dispersionless integrable system, gauge field, self-dual Yang–Mills equations

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1. Introduction

Multidimensional dispersionless integrable systems correspond to Lax pairs of the type

$$[X_1, X_2] = 0,$$

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i \partial_{x_i} + F_0 \partial_{\lambda},$$

$$X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i \partial_{x_i} + G_0 \partial_{\lambda},$$

where $\lambda$ is a “spectral parameter” and the functions $F_k$ and $G_k$ are holomorphic in $\lambda$ and depend on the variables $t_1$, $t_2$, and $x_n$. We consider polynomials (or Laurent polynomials) in $\lambda$. This class includes dispersionless limits of integrable (2+1)-dimensional equations (the dispersionless Kadomtsev–Petviashvili equation, the dispersionless 2DTL hierarchy) and equations of complex relativity, integrable by the twistor method (Plebański heavenly equations, hyper-Kähler hierarchies, etc.).

Recently, we proposed [1]–[3] a scheme of a matrix extension of Lax pairs (1), leading to gauge covariant Lax pairs of the type

$$\nabla X_1 = X_1 + A_1,$$

$$\nabla X_2 = X_2 + A_2,$$

where $A_1$ and $A_2$ are matrix-valued functions of space–time variables holomorphic in $\lambda$ (polynomials or Laurent polynomials).
Lax pairs of this structure were already present in the seminal work of Zakharov and Shabat [4], where it was noted that the commutation relation splits into a (scalar) vector field part and a Lie algebraic part, which gives the equations similar to the self-dual Yang–Mills (SDYM) equations, but on some background whose origin was not clear at the moment.

Indeed, the commutator of two covariant vector fields also has the form of a covariant vector field: it contains a vector field part and a matrix (Lie algebraic) part,

\[ \nabla_{X_1}, \nabla_{X_2} = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]. \]

The compatibility condition contains the vector fields part \((N+2)\text{-dimensional dispersionless system)}\)

\[ [X_1, X_2] = 0 \]

and the matrix part (matrix equations on a dispersionless background)

\[ X_1 A_2 - X_2 A_1 + [A_1, A_2] = 0. \]

To define a closed system of matrix equations for the coefficients of the polynomials (meromorphic functions) \(A_1\) and \(A_2\), the holomorphic structure of \(A_1\) and \(A_2\) (the order of zeroes and poles) must be consistent with the holomorphic structure of vector fields, and the gauge must be fixed (it is also possible to consider gauge-invariant matrix equations without fixing the gauge). The scheme of matrix extension that we discuss below allows constructing prolongation terms \(A_1\) and \(A_2\) that have the correct structure and satisfy the compatibility condition, thus giving a solution of matrix equations on the dispersionless background.

The compatibility conditions for Lax pair (1) imply the (local) existence of common solutions of the linear equations

\[ X_1 \psi_i = 0, \quad X_2 \psi_i = 0. \]

These equations, according to the Frobenius theorem, have \(N+1\) functionally independent solutions \(\psi_i(\lambda, t_1, t_2, x)\), the “wave functions” of linear operators of the background dispersionless integrable system. The general solution is

\[ \psi = f(\psi_0, \ldots, \psi_N). \]

We introduce a matrix-valued wave function for the extended system

\[ \nabla_{X_1} \Phi_0 = (X_1 + A_1) \Phi_0 = 0, \quad \nabla_{X_2} \Phi_0 = (X_2 + A_2) \Phi_0 = 0. \]

Locally, \(\Phi\) can be considered a series in \(\lambda\) with matrix coefficients. The general solution of the linear equations is of the form

\[ \Phi = \Phi_0 F(\psi_0, \ldots, \psi_N), \]

where \(F\) is a matrix-valued complex analytic function of \(N+1\) variables.

It is easy to verify that the extended linear equations are equivalent to

\[ (X_1 \Phi) \Phi^{-1} = -A_1, \quad (X_2 \Phi) \Phi^{-1} = -A_2, \]

where \(A_1\) and \(A_2\) are polynomials in \(\lambda\) with matrix coefficients (Laurent polynomials, meromorphic functions). This is a characteristic analytic property of \(\Phi\), important for the algebraic definition of the hierarchy (Lax–Sato equations) and for the construction of solutions by means of the Riemann–Hilbert (RH) problem.
To construct $\Phi$, we can consider the matrix RH problem
\[
\Phi_{\text{in}} = \Phi_{\text{out}} R(\psi_0, \ldots, \psi_N),
\]
defined on some curve in the complex plane (we usually consider the unit circle), where $\psi_i(\lambda, t)$ are wave functions of the dispersionless Lax pair. This problem ensures the analyticity of the functions $(X_1 \Phi)(\Phi^{-1}$ and $(X_2 \Phi)(\Phi^{-1}$, thus leading to an extended Lax pair and defining the matrix prolongation functions $A_1$ and $A_2$.

In this paper, we consistently develop the scheme of a matrix extension of dispersionless integrable systems in the general case of multidimensional hierarchies with vector fields that are polynomial in the spectral variable, mainly in dimensions $d \geq 4$. We briefly describe the picture of the basic dispersionless hierarchy. On the background of the basic hierarchy, we construct the matrix extension associated with the introduction of gauge covariant vector fields. We present extended Lax pairs, Lax–Sato equations, matrix equations on the background of vector fields, and the dressing scheme. Reductions and the construction of solutions are discussed. We consider the case of an Abelian extension separately, for which the RH equations of the dressing scheme are explicitly solvable and give an analogue of the Penrose formula in curved space.

2. Basic hierarchy

A general $(N + 2)$-dimensional hierarchy $(N \geq 1)$ with polynomial vector fields [5], [6] is defined by the generating relation
\[
(\Omega_0) = (J_0^{-1} d\Psi^0 \wedge d\Psi^1 \wedge \cdots \wedge d\Psi^N) = 0. \tag{3}
\]
where $\Psi^0, \ldots, \Psi^N$ are the series
\[
\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n (t^1, \ldots, t^N) \lambda^{-n}, \tag{4}
\]
\[
\Psi^k = \sum_{n=0}^{\infty} t^n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^k_n (t^1, \ldots, t^N) (\Psi^0)^{-n}, \tag{5}
\]
$1 \leq k \leq N$, $t^k = (t^k_1, \ldots, t^k_N)$, $(\cdot)$ is a projection to negative powers of $\lambda$, and $J_0$ is the determinant of the Jacobian matrix $J$,
\[
J_0 = \det J, \quad J_{ij} = \partial_i \Psi^j, \quad 0 \leq i, j \leq N, \tag{6}
\]
where $\partial_0 = \partial/\partial \lambda$, $\partial_k = \partial/\partial x^k$ ($1 \leq k \leq N$), $x^k = t^k_0$. Generating relation (3) is equivalent to the set of Lax–Sato equations
\[
\partial_{k+1}^{n} \Psi = \sum_{i=0}^{N} ((J^{-1})_{ki}(\Psi^0)^n)_+ \partial_i \Psi, \quad 0 \leq n \leq \infty, \quad 1 \leq k \leq N, \tag{7}
\]
where $\Psi = (\Psi^0, \ldots, \Psi^N)$, $(\cdot)_+$ is a projection to nonnegative powers of $\lambda$. The first flows of the hierarchy are
\[
\partial_{k+1}^{n} \Psi = \left( \lambda \partial_k - \sum_{p=1}^{N} (\partial_k u_p) \partial_p - (\partial_k u_0) \partial_0 \right) \Psi, \quad 0 < k \leq N, \tag{8}
\]
where $u_0 = \Psi^0_1$, $u_k = \Psi^k_1$, $1 \leq k \leq N$. The compatibility condition for any pair of linear equations (8) (e.g., with $\partial^2_{q}$ and $\partial^2_{k}$, $k \neq q$) implies a closed nonlinear $(N + 2)$-dimensional system of PDEs for the set of functions $u_k, u_0$, which can be written in the form
\[
\partial^2_{k} \partial_q u - \partial^2_{q} \partial_k u + [\partial_k \partial_q u, \partial_q] = (\partial_k u_0) \partial_q - (\partial_q u_0) \partial_k, \tag{9}
\]
where $\partial^2_{k} \partial_q u_0 - \partial^2_{q} \partial_k u_0 + (\partial_k \partial_q u_0) - (\partial_q \partial_k u_0) = 0$, $u = \sum_{p=1}^{N} u_p \partial_p$ is a vector field,
In the case $N = 2$, we have two vector fields (8) [5],
\begin{align*}
\nabla_1^2 &= \partial_1^2 - \lambda \partial_1 + (\partial_1 u_1) \partial_1 + (\partial_1 u_2) \partial_2 + (\partial_1 u_0) \partial_\lambda, \\
\nabla_2^2 &= \partial_2^2 - \lambda \partial_2 + (\partial_2 u_1) \partial_1 + (\partial_2 u_2) \partial_2 + (\partial_2 u_0) \partial_\lambda,
\end{align*}
(10)
giving a Lax pair for a 4-dimensional closed set of second-order equations for three functions $u_0, u_1, u_2$ with independent variables $t_1, t_2, x^1, x^2$:
\begin{align*}
Q u_2 &= \partial_2 u_0, \\
Q u_1 &= -\partial_1 u_0, \\
(\partial_1^2 + (\partial_1 u_1) \partial_1 + (\partial_1 u_2) \partial_2) \partial_2 u_0 &= (\partial_2^2 + (\partial_2 u_1) \partial_1 + (\partial_2 u_2) \partial_2) \partial_1 u_0,
\end{align*}
(11)
where the linear operator $Q$ is
\begin{equation}
Q = \partial_2^2 \partial_1 - \partial_1^2 \partial_2 + (\partial_2 u_1) \partial_1 \partial_1 - (\partial_1 u_2) \partial_2 \partial_2 - ((\partial_1 u_1) - (\partial_2 u_2)) \partial_1 \partial_2.
\end{equation}
(12)
This system can be easily rewritten in the equivalent form of two third-order equations for the functions $u_1$ and $u_2$ used in the work [7], where it was demonstrated that it describes the general local form of a self-dual conformal structure (the complex analytic case or the real case with neutral signature, modulo coordinate transformations). The linear operator $Q$ represents the conformal structure in the form of a symmetric bivector (the inverse metric), and the corresponding metric (representative of the conformal structure) is given by
\begin{equation}
g = dt_1^2 dx^1 - dt_1^1 dx^2 - (\partial_2 u_1) dt_1^2 dt_1^2 - ((\partial_1 u_1) - (\partial_2 u_2)) dt_1^2 dt_1^1 + (\partial_1 u_2) dt_1^1 dt_1^1.
\end{equation}
(13)

The reduction of the hierarchy to volume-preserving flows $J_0 = 1$ corresponds to divergence-free vector fields. For vector fields (10), it leads to the introduction of the potential $\Theta$, $u_1 = \Theta_y$, $u_2 = -\Theta_x$, $x = x^1$, $y = x^2$. After the identification $z = t_1^1$, $w = t_1^2$, $\varphi = u_0$, we obtain the Dunajski system generalizing the second heavenly equation [8]
\begin{align*}
\Theta_{wx} - \Theta_{xy} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2 &= \varphi, \\
\varphi_{wx} - \varphi_{yz} + \Theta_{yy} \varphi_{xx} + \Theta_{xx} \varphi_{yy} - 2 \Theta_{xy} \varphi_{xy} &= 0.
\end{align*}
(14)

A further reduction to the linearly degenerate case $\varphi = 0$ gives the second heavenly equation [9]
\begin{equation}
\Theta_{wx} - \Theta_{zy} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2 = 0.
\end{equation}
(15)
The operator $Q$ defining the conformal structure is of the same form for the Dunajski system and the second heavenly equation,
\begin{equation}
Q = \partial_x \partial_x - \partial_x \partial_y + \Theta_{yy} \partial_x \partial_x + \Theta_{xx} \partial_y \partial_y - 2 \Theta_{xy} \partial_x \partial_y.
\end{equation}

It coincides with the linearization of the second heavenly equation. Accordingly, the metric is given by
\begin{equation}
g = dw \, dy - dz \, dy - \Theta_{yy} \, dw^2 - 2 \Theta_{xy} \, dw \, dz = \Theta_{xx} \, dz^2.
\end{equation}
(16)
2.1. Dressing scheme. The dressing scheme for hierarchy (3), (7) can be formulated in terms of an \((N+1)\)-component nonlinear RH problem on the unit circle \(S\) in the complex plane of the variable \(\lambda\):

\[
\begin{align*}
\Psi^0_{\text{in}} &= F_0(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \ldots, \Psi^N_{\text{out}}), \\
\Psi^k_{\text{in}} &= F_k(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \ldots, \Psi^N_{\text{out}}), \quad 1 \leq k \leq N,
\end{align*}
\] (17)

where the functions \(\Psi^0_{\text{in}}(\lambda, t)\) and \(\Psi^k_{\text{in}}(\lambda, t)\) are analytic inside the unit circle, and the functions \(\Psi^0_{\text{out}}(\lambda, t)\) and \(\Psi^k_{\text{out}}(\lambda, t)\) are analytic outside the unit circle and have an expansion of form (4), (5). The functions \(F_0\) and \(F_k\) are assumed to define a diffeomorphism in \(\mathbb{C}^{N+1}\), \(F \in \text{Diff}(N+1)\); they represent the dressing data. In compact form, problem (17) can be written as

\[
\Psi_{\text{in}} = F(\Psi_{\text{out}}).
\] (18)

It is straightforward to demonstrate that problem (17) implies the analyticity of the differential form

\[
\Omega_0 = J_0^{-1} d\Psi^0 \wedge d\Psi^1 \wedge \cdots \wedge d\Psi^N
\] (19)

in the complex plane (where the independent variables of the differential include all the times \(t\) and the spectral variable \(\lambda\)) and also implies generating relation (3), thus defining a solution of the hierarchy. Considering a reduction to the group of volume-preserving diffeomorphisms \(F \in \text{SDiff}(N+1)\), we obtain a reduction of the general hierarchy (3) to the case \(J_0 = 1\) (divergence-free vector fields),

\[
(d\Psi^0 \wedge d\Psi^1 \wedge \cdots \wedge d\Psi^N)_- = 0.
\]

3. Matrix extension

To construct the matrix extension of an involutive distribution of vector fields that are polynomial in the spectral parameter and have a basis \(X_i\), we introduce a matrix-valued function \(\Phi\) having the property that all the functions \((X_i\Phi)\Phi^{-1}\) are polynomial in the spectral variable. The existence of such functions in terms of series is implied by the compatibility of the Lax–Sato equations. They can be constructed analytically using the RH problem (see below). The function \(\Phi\) is assumed to be bounded and invertible in the spectral plane and analytic in some neighborhood of infinity. The extended linear problems are then given by the relations

\[
X_i\Phi = ((X_i\Phi)\Phi^{-1})_+ \Phi,
\] (20)

where \(((X_i\Phi)\Phi^{-1})_+\) is the polynomial part of a Laurent series at infinity, containing a finite number of matrix fields. The extended vector fields are

\[
\nabla_{X_i} = X_i - ((X_i\Phi)\Phi^{-1})_+.
\]

We consider the canonically normalized \(\Phi\) as a series

\[
\Phi = I + \sum_{n=1}^{\infty} \Phi_n(t)\lambda^{-n}.
\] (21)

For polynomial vector fields of form (1), relations (20) become

\[
\begin{align*}
\partial_1 \Phi &= (-F + ((F\Phi)\Phi^{-1})_+)\Phi, \\
\partial_2 \Phi &= (-G + ((G\Phi)\Phi^{-1})_+)\Phi,
\end{align*}
\] (22)
where
\[
F = \sum_{i=1}^{N} F_i \partial x_i + F_0 \partial \lambda,
\]
\[
G = \sum_{i=1}^{N} G_i \partial x_i + G_0 \partial \lambda.
\]

These relations can be regarded as Lax–Sato equations defining the evolution of the series \( \Phi \) with matrix coefficients depending on the variables \( x_1, \ldots, x_N \) with respect to the times \( t_1, t_2 \).

**Proposition 1.** For commuting polynomial vector fields \( X_1 \) and \( X_2 \) of form (1), Lax–Sato equations (22) for series (21) are compatible.

**Proof.** Compatibility means that cross-derivatives over times are equal by virtue of Eqs. (22):
\[
\Delta = \partial_{t_1}(-G + ((G\Phi)\Phi^{-1})_+)\Phi - \partial_{t_2}(-F + ((F\Phi)\Phi^{-1})_+)\Phi = 0.
\]

On one hand,
\[
\Delta = \{\partial_{t_1}((G\Phi)\Phi^{-1})_+ - \partial_{t_2}((F\Phi)\Phi^{-1})_+ + [((G\Phi)\Phi^{-1})_+, ((F\Phi)\Phi^{-1})_+]\} \Phi
\]
and \( \Delta \Phi^{-1} \) contains only nonnegative powers of \( \lambda \), \( (\Delta \Phi^{-1})_- = 0 \). On the other hand, relations (22) can be evidently rewritten as
\[
\partial_{t_1} \Phi = -((F\Phi)\Phi^{-1})_- \Phi,
\]
\[
\partial_{t_2} \Phi = -((G\Phi)\Phi^{-1})_- \Phi.
\]

Then
\[
-\Delta = \{((\partial_{t_1} (G\Phi)\Phi^{-1})_- - (\partial_{t_2} (F\Phi)\Phi^{-1})_- + [((G\Phi)\Phi^{-1})_-, ((F\Phi)\Phi^{-1})_-])\} \Phi,
\]
and \( \Delta \Phi^{-1} \) contains only negative powers of \( \lambda \), \( (\Delta \Phi^{-1})_+ = 0 \), whence \( \Delta = 0 \).

**3.1. Matrix dressing on the background.** The function \( \Phi \) for matrix extension of the basic hierarchy (3), (7) can be constructed using a matrix RH problem
\[
\Phi_{\text{in}} = \Phi_{\text{out}} R(\Psi^0, \ldots, \Psi^N), \tag{23}
\]
defined on some curve in the complex plane (we usually consider the unit circle). Here, \( \Psi^0, \ldots, \Psi^N \) are wave functions of the basic hierarchy taken on the curve.

We assume that the solution \( \Phi \) of the RH problem is bounded and invertible, and that the normalization to the unit matrix at infinity fixes the gauge and leads to a closed system of equations for matrix coefficients of the extended Lax pairs. The dispersionless hierarchy corresponding to the integrable distribution with the basis \( X_i \) plays the role of a background.

For a canonically normalized matrix-valued function \( \Phi \), the expansion at infinity is of the form (21)
\[
\Phi_{\text{out}} = I + \sum_{n=1}^{\infty} \Phi_n(t)\lambda^{-n}.
\]

It is sometimes useful to retain the gauge freedom and consider solutions of the RH problem that have the expansion
\[
\tilde{\Phi} = \tilde{\Phi}_0 + \sum_{n=1}^{\infty} \tilde{\Phi}_n(t)\lambda^{-n}. \tag{24}
\]
The canonically normalized function is then expressed as \( \Phi = \tilde{\Phi}_0^{-1} \tilde{\Phi} \).
From the matrix RH problem we deduce the analyticity of the functions \((X_i \Phi) \Phi^{-1}\) and of the matrix-valued form
\[
\Omega = \Omega_0 \wedge d\Phi \cdot \Phi^{-1},
\]
leading to an additional generating relation for the extended hierarchy:
\[
(\Omega_0 \wedge d\Phi \cdot \Phi^{-1})_+ = 0. \tag{25}
\]
Taken together, generating relations (3) and (25) imply the polynomiality of the series \((X_i \Phi) \Phi^{-1}\) and of the Lax–Sato equations for the series (21), (4), (5), which define the evolution of these series.

In the linearly degenerate case \(\Psi^0 = \lambda\), the vector fields do not contain the derivative with respect to \(\lambda\) and the dressing data are manifestly dependent on the parameter of the curve; it suffices to assume that the curve is continuous, and solutions can also be constructed using the \(\bar{\partial}\) problem [10]
\[
\tilde{\partial}\Phi = \Phi R(\lambda, \bar{\lambda}, \Psi^1, \ldots, \Psi^N),
\]
which considerably extends the possibilities to construct explicit solutions.

3.2. Lax–Sato equations and extended vector fields. The second generating relation (25) gives the Lax–Sato equations for the \(\Phi\) series on the vector field background
\[
\begin{align*}
p^k n \Psi &= V^k_n (\lambda) \Psi, \tag{26} \\
p^k n \Phi &= (V^k_n (\lambda) - ((V^k_n (\lambda) \Phi) \cdot \Phi^{-1})_+ + ) \Phi, \tag{27}
\end{align*}
\]
where the vector fields \(V^k_n (\lambda)\) are defined by formula (7). Accordingly, the extended vector fields are expressed as
\[
\nabla^k_n = \partial^k_n - V^k_n (\lambda) + ((V^k_n (\lambda) \Phi) \cdot \Phi^{-1})_+ , \tag{28}
\]
and in the gauge-invariant form, as
\[
\tilde{\nabla}^k_n = \partial^k_n - V^k_n (\lambda) - ((\partial^k_n - V^k_n (\lambda)) \Phi) \cdot \Phi^{-1})_+. \tag{29}
\]
The Lax–Sato equations for the first flows are given by formulas (8) and (27),
\[
\begin{align*}
\partial^k_1 \Phi &= (\lambda \partial_k - (\partial_k \tilde{u}) - (\tilde{u} \partial_k u_0) \partial_\lambda - (\partial_k \Phi_1)) \Phi, & 0 < k \leq N, \tag{30}
\end{align*}
\]
where \(\tilde{u}\) is a vector field, \(\tilde{u} = \sum_{p=1}^N u_p \tilde{\partial}_p\) (see also (9)), and the corresponding extended vector fields are
\[
\nabla^k_1 = \partial^k_1 - \lambda \partial_k + (\partial_k \tilde{u}) + (\partial_k u_0) \partial_\lambda + (\partial_k \Phi_1), \tag{31}
\]
and in the gauge-invariant form,
\[
\begin{align*}
\nabla^k_1 &= \partial^k_1 - \lambda \partial_k + (\partial_k \tilde{u}) + (\partial_k u_0) \partial_\lambda + \lambda A_k + B_k,
A_k &= -(\partial_k \Phi_0) \Phi_0^{-1}, \tag{32} \\
B_k &= (\partial_k \Phi_1) \Phi_0^{-1} - (\partial_k \Phi_0) \Phi_0^{-1} \Phi_1 \Phi_0^{-1} - ((\partial_k \tilde{u}) + (\partial_k \Phi_1)) \Phi_0^{-1}.
\end{align*}
\]
Substituting \(\Phi\) in form (21) in the Lax–Sato equations (30), we obtain recursion relations expressing all the coefficients of the series through the first coefficient \(\Phi_1\) and the scalar functions \(u_0, \ldots, u_n\)
\[
\partial_k \Phi_{m+1} = (\partial_k + (\partial_k \tilde{u})) \Phi_m - (m - 1)(\partial_k u_0) \Phi_{m-1} + (\partial_k \Phi_1) \Phi_m.
\]
The commutation relation for each pair of extended vector fields (31) gives an \((N + 2)\)-dimensional dispersionless equation (9) and a matrix \((N + 2)\)-dimensional equation of the SDYM type on the background of its solution:

\[
(\partial_t^1 + (\partial_t \hat{u}))\partial_q \Phi_1 - (\partial_t^2 + (\partial_t \hat{u}))\partial_k \Phi_1 + [(\partial_k \Phi_1), (\partial_q \Phi_1)] = 0. \tag{33}
\]

In the gauge-invariant form,

\[
[\nabla_k, \nabla_q] = 0,
[\nabla_k^1, \nabla_q^2] = (\partial_k u_0)\nabla_q - (\partial_q u_0)\nabla_k,
[\nabla_k^2, \nabla_q] = [\nabla_k^1, \nabla_k],
\]

where \(\nabla_k^p = \partial_k^p + (\partial_k \hat{u}) + B_k \) and \(\nabla_p = \partial_p + A_p, 1 \leq p \leq N\), and we have three equations for four matrix functions \(A_k, A_q, B_k, B_q\) on the background of the basic dispersionless system.

For the trivial background \(\hat{u} = 0\), we obtain a set of consistent 4-dimensional SDYM equations in the space of \(2N\) variables

\[
\partial^k \partial_q \Phi_1 - \partial^q \partial_k \Phi_1 + [(\partial_k \Phi_1), (\partial_q \Phi_1)] = 0,
\]

and in the gauge-invariant form,

\[
[\nabla_k, \nabla_q] = 0,
[\nabla_k^1, \nabla_q^2] = 0,
[\nabla_k^2, \nabla_q] = [\nabla_k^1, \nabla_k],
\]

where \(\nabla_k^p = \partial_k^p + B_k, \nabla_p = \partial_p + A_p, 1 \leq p \leq N\).

The case \(N = 2\) was considered in [1], where it was demonstrated that the equations on the background admit a very direct and natural geometric interpretation. In that case, we have two extended vector fields (31),

\[
\nabla_1^1 = \partial_1^1 - \lambda \hat{t}_1 + (\partial_1 u_1) \hat{t}_1 + (\partial_1 u_2) \hat{t}_2 + (\partial_1 u_0) \hat{t}_0 + (\partial_1 \Phi_1),
\nabla_1^2 = \partial_1^2 - \lambda \hat{t}_2 + (\partial_2 u_1) \hat{t}_1 + (\partial_2 u_2) \hat{t}_2 + (\partial_2 u_0) \hat{t}_0 + (\partial_2 \Phi_1),
\]

giving a Lax pair for a 4-dimensional equation with independent variables \(t_1^1, t_1^2, x_1, x_2\),

\[
(\partial_t^1 + (\partial_t \hat{u}))\partial_2 \Phi_1 - (\partial_t^2 + (\partial_t \hat{u}))\partial_1 \Phi_1 + [(\partial_1 \Phi_1), (\partial_2 \Phi_1)] = 0, \tag{34}
\]

which is the general local form of the SDYM equations on the background of a self-dual conformal structure [7] (for the neutral signature, modulo coordinate transformations and a gauge). In terms of the linear operator \(Q\) in (12), which represents the conformal structure in the form of a symmetric bivector (inverse metric), this equation can be written as

\[
Q \Phi_1 = [(\partial_1 \Phi_1), (\partial_2 \Phi_1)],
\]

with the corresponding metric (representing the self-dual conformal structure) given by (13).

For the second heavenly equation (15), which describes the general self-dual vacuum solutions of the Einstein equations (in the real case with neutral signature), we have

\[
(\partial_w \partial_x + \partial_y \partial_y + \Theta_{xy} \partial_x \partial_y + \Theta_{xx} \partial_y - 2 \Theta_{xy} \partial_x \partial_y) \Phi_1 = [(\partial_1 \Phi_1), (\partial_2 \Phi_1)].
\]

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The metric of the background space is given by expression (16):

\[ g = dw \, dx + dz \, dy - \Theta_{yy} \, dw^2 + 2\Theta_{xy} \, dw \, dz - \Theta_{xx} \, dz^2. \]

Such a simple geometric interpretation can hardly be expected to extend to the case \( N > 2 \). First, there is no geometric interpretation of general dispersionless equations (9), it is only clear that they correspond to some special set of commutation relations of \( 2N \) vector fields in a \( 2N \)-dimensional space (if we consider all pairs of indices). However, the reduced multidimensional linearly degenerate case with Hamiltonian vector fields corresponds to hyper-Kähler hierarchies [11], which do have a geometric origin. This could possibly be a good starting point for the geometric interpretation of multidimensional matrix equations on a dispersionless background. In the hyper-Kähler case, the dimension is even, \( N = 2M \), and covariant vector fields (8) are of the form

\[
\nabla^k_1 = \partial^k_1 - \lambda \partial_k + \{(\partial_k \Theta), \ldots \}_x + (\partial_k \Phi_1),
\]

where

\[
\{f, \ldots \}_x = \sum_{p=1}^{M} (\partial_{M+p} f) \partial_p - (\partial_p f) \partial_{M+p},
\]

\[
\{f, g\}_x = \sum_{p=1}^{M} (\partial_{M+p} f) (\partial_p g) - (\partial_p f) (\partial_{M+p} g).
\]

Equations (9) reduce to the first equations of the hyper-Kähler hierarchy

\[
\partial^k_q \Theta - \partial^q_k \Theta + \{(\partial_k \Theta, \partial_q \Theta)\}_x = 0.
\]

For \( M = 1 \), we obtain the second heavenly equation (15). Matrix equations (33) on the background of the hyper-Kähler hierarchy take the form

\[
\partial^k_1 \partial_q \Phi_1 - \partial^q_k \partial_1 \Phi_1 + \{(\partial_k \Theta, \partial_q \Phi_1\}_x - \{(\partial_q \Theta, \partial_k \Phi_1)\}_x = [(\partial_q \Phi_1), (\partial_k \Phi_1)].
\]

### 3.3. Abelian case and the Penrose formula.

In the Abelian case, Eqs. (33) become linear, their left-hand side represents the action of second-order linear differential operators on the scalar function \( \phi_1 \) (we use \( \phi \) instead of \( \Phi \) in the Abelian case). These equations can nevertheless be of interest for several reasons. First, in three and four dimensions, where we have an interpretation of the equations in terms of gauge fields, the Abelian case corresponds to electromagnetic fields on a geometric background and could be of interest in and of itself. Second, the arising linear operators are connected with the symbol of linearization of the basic dispersionless equations and can be useful in studying the stability of solutions and the singularities of these equations. For example, for the second heavenly equation (15), the linear operator \( Q \) is exactly the linearization of the equation. And finally, scalar RH problem (23) can be solved explicitly, and as a result we obtain an analogue of the Penrose formula in curved space. If we know some solution of the dispersionless system together with the general wave function on the unit circle, this formula gives a solution of the corresponding linear equations on the background, depending on an arbitrary function of \( N + 1 \) variables.

Thus, in the Abelian case, we have a scalar series (21)

\[
\phi = 1 + \sum_{n=1}^{\infty} \phi_n(t) \lambda^{-n},
\]
The Lax–Sato equations for the first flows are

\[ \partial_k^k \phi = (\lambda \partial_k - (\partial_k \hat{u}) - (\partial_k u_0) \partial_\lambda - (\partial_k \phi_1)) \phi, \quad 0 < k \leq N, \]  

and Eqs. (33) become linear:

\[ ((\partial_k^k + (\partial_k \hat{u})) \partial_q - (\partial_q^k + (\partial_q \hat{u})) \partial_k) \phi_1 = 0. \]  

A special solution of Lax–Sato equations (37) (and of the whole set of Abelian Lax–Sato equations (27)) is given by the Jacobian \( J_0 \) (6) (see [12] for more details)

\[ \partial_k^k J_0 = \left( \lambda \partial_k - (\partial_k \hat{u}) - (\partial_k u_0) \partial_\lambda - \left( \partial_k \sum_{i=1}^N \partial_i u_i \right) \right) J_0, \quad 0 < k \leq N, \]

and the function \( \phi_1 = \sum_{i=1}^N \partial_i u_i \) (the divergence of the vector field \( \hat{u} \)) is a special solution of Eqs. (38).

In the scalar case

\[ \phi_{\text{in}} = \phi_{\text{out}} R(\Psi^0, \ldots, \Psi^N), \]

RH problem (23) is solved in the standard way after logarithm

\[ \ln \phi_{\text{in}} - \ln \phi_{\text{out}} = r(\Psi^0, \ldots, \Psi^N), \quad R = e^r, \]

by the explicit formula

\[ \ln \phi = \frac{1}{2\pi i} \oint \frac{r(\Psi^0, \ldots, \Psi^N)}{\mu - \lambda} d\mu. \]

Accordingly,

\[ \phi = \exp \left( \frac{1}{2\pi i} \oint \frac{r(\Psi^0, \ldots, \Psi^N)}{\mu - \lambda} d\mu \right), \]

and we have

\[ \phi_1 = -\frac{1}{2\pi i} \oint r(\Psi^0, \ldots, \Psi^N) d\mu. \]  

(39)

Here, \( r(\Psi^0, \ldots, \Psi^N) \) is an arbitrary complex analytic function of its arguments, which represents the general solution (wave function) of linear equations (37) defined on the unit circle in the complex plane. Any wave function \( \Psi(t, \lambda) \) defined on the unit circle (or another closed curve) gives a solution of Eqs. (38),

\[ \phi_1 = -\frac{1}{2\pi i} \oint \Psi(t, \mu) d\mu. \]  

(40)

as can easily be verified directly. Indeed,

\[ ((\partial_k^k + (\partial_k \hat{u})) \partial_q - (\partial_q^k + (\partial_q \hat{u})) \partial_k) \Psi = ((\partial_q u_0) \partial_k - (\partial_k u_0) \partial_q) \partial_\lambda \Psi, \]

and the integration over \( \lambda \) along a closed contour gives zero, and hence expression (40) is a solution of Eqs. (38).

For \( N = 2 \), we have one linear equation (38)

\[ Q \phi_1 = 0, \]

\[ Q = \partial_w \partial_x - \partial_x \partial_y + (\partial_y u_1) \partial_x \partial_y - (\partial_x u_2) \partial_y \partial_y - ((\partial_x u_1) - (\partial_y u_2)) \partial_x \partial_y, \]

(41)
with the operator $Q$ in (12) corresponding to a symmetric bivector of the self-dual conformal structure; the notation for times corresponds to that in Eq. (14). This equation represents the general local form of the Abelian SDYM equations (self-dual equations for the electromagnetic field) on the background of a conformal structure with neutral signature in a special gauge. The corresponding metric (representative of the conformal structure) is given by formula (13):

$$g = dw dx - dz dy - (\partial_y u_1) dw dw - ((\partial_x u_1) - (\partial_y u_2)) dz dw + (\partial_x u_2) dz dz.$$

Solutions of Eq. (41) are given by formula (39) with $N = 2$:

$$\phi_1 = -\frac{1}{2\pi i} \int r(\Psi^0, \Psi^1, \Psi^2) \, d\mu. \quad (42)$$

For the trivial background

$$\hat{u} = 0, \quad \Psi^0 = \lambda, \quad \Psi^1 = \sum_{n=0}^{\infty} t^1_n \lambda^n, \quad \Psi^2 = \sum_{n=0}^{\infty} t^2_n \lambda^n,$$

Eq. (41) reduces to a 4-dimensional wave equation for neutral signature with constant coefficients:

$$(\partial_w \partial_x - \partial_z \partial_y) \phi_1 = 0,$$

Formula (42) then takes the form

$$\phi_1 = -\frac{1}{2\pi i} \int r(\lambda, \lambda z + x, \lambda w + y) \, d\mu.$$

This formula is easily recognized as a version of the Penrose formula [13], [14] for solutions of the wave equation written for the neutral signature. Thus, formula (42) can be regarded as a generalization of the Penrose formula to the case of operators associated with the self-dual conformal structure.

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