Quantum aspects of black holes
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Chapter 1

Quantum aspects of black holes

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1.1 Introduction

At the most fundamental level, black holes are genuine quantum objects. This holds irrespective of the fact that direct quantum effects can only be observed for small black holes – black holes that cannot be formed by stellar collapse. For this reason the discussion in this chapter will be of more theoretical nature. But even a black hole as gigantic as the galactic black hole will in the far future (if the Universe will not recollapse) be dominated by quantum effects and eventually evaporate. It is, however, possible that small black holes have been created in the very early Universe. For such primordial black holes quantum effects can be of direct observational significance in the present Universe. I shall thus devote my last section to a brief discussion of their relevance. In the first three sections I shall, however, give an introduction to the key theoretical developments – black-hole mechanics, Hawking radiation, and the interpretation of the black-hole entropy.

In my discussion I shall draw heavily from my review article Kiefer (1999) where many technical details can be found. Other general references include the comprehensive book by Frolov and Novikov (1998), Wald (2001), Hehl et al (1998), as well as the article by Bekenstein (1980) and the book by Thorne (1994).

1.2 The laws of black-hole mechanics

It is a most amazing fact that black holes obey uniqueness theorems (Heusler 1996). If an object collapses to form a black hole, a stationary state is
reached asymptotically. One can prove within the Einstein-Maxwell theory that stationary black holes are uniquely characterised by only three parameters: Mass $M$, angular momentum $J \equiv Ma$, and electric charge $q$. In this sense, black holes are objects much simpler than ordinary stars – given these parameters, they all look the same. All other degrees of freedom that might have been initially present have thus been radiated away during the collapse, e.g. in the form of electromagnetic or gravitational radiation. Since the latter constitute some form of “hair”, one refers to the content of these theorems as black holes have no hair. The three parameters are associated with conservation laws at spatial infinity. In principle, one can thus decide about the nature of a black hole far away from the hole itself, without having to approach it. In astrophysical situations, the two parameters $M$ and $J$ suffice, since a charged object would rapidly discharge. The corresponding solution of Einstein’s equations is called the Kerr solution (Kerr-Newman in the presence of charge). Stationary black holes are axially symmetric with spherical symmetry being obtained as a special case for $J = 0$.

In the presence of other fields, the uniqueness theorems do not always hold, see e.g. Núñez et al (1998). This is in particular the case in the presence of nonabelian gauge fields. In addition to charges at infinity, such “coloured black holes” have to be characterised by additional variables, and it is necessary to approach the hole to determine them. The physical reason for the occurrence of such solutions is the nonlinear character of these gauge fields. Fields in regions closer to the black hole (that would otherwise be swallowed by the hole) are tied to fields far away from the hole (that would otherwise be radiated away) to reach an equilibrium situation. In most examples this equilibrium is, however, unstable and the corresponding black-hole solution does not represent a physical solution. Since classical nonabelian fields have never been observed (the description of objects such as quarks necessarily needs quantised gauge fields which, due to confinement, have no macroscopic limits), they will not be taken into account in the following discussion.

In 1971, Stephen Hawking could prove an important theorem about stationary black holes – that their area can never decrease with time. More precisely, he showed that

For a predictable black hole satisfying $R_{ab}k^ak^b \geq 0$ for all null $k^a$, the surface area of the future event horizon never decreases with time.

A ‘predictable’ black hole is one for which the cosmic censorship hypothesis holds – this is thus a major assumption for the area law. Cosmic censorship states that all black holes occurring in nature have an event horizon, so that the singularity cannot be observed for far-away observers (the singularity is not “naked”). I emphasise that the time asymmetry in
this theorem comes into play because a statement is made about the future horizon, not the past horizon; the analogous statement for white holes would then be that the past event horizon never increases. I also emphasise that the area law only holds in the classical theory, not in the quantum theory (see section section 1.3).

The area law seems to exhibit a close formal analogy to the Second Law of thermodynamics – there the entropy can never decrease with time (for a closed system). However, the conceptual difference could not be more pronounced: while the Second Law is related to statistical behaviour, the area law is just a theorem in differential geometry. That the area law is in fact directly related to the Second Law will become clear in the course of this section.

Further support for this analogy is given by the existence of analogies to the other laws of thermodynamics. The Zeroth Law states that there is a quantity, the temperature, that is constant on a body in thermal equilibrium. Does there exist an analogous quantity for a black hole? One can in fact prove that the surface gravity $\kappa$ is constant over the event horizon (Wald 1984). For a Kerr black hole, $\kappa$ is given by

$$\kappa = \frac{\sqrt{(GM)^2 - a^2}}{2GMr_+} \xrightarrow{a \to 0} \frac{1}{4GM} = \frac{GM}{R_0^2},$$  \hspace{1cm} (1.1)

where $r_+$ denotes the location of the event horizon. One recognises in the Schwarzschild limit the well-known expression for the Newtonian gravitational acceleration. ($R_0 \equiv 2GM$ there denotes the Schwarzschild radius). One can show for a static black hole that $\kappa$ is the limiting force that must be exerted at infinity to hold a unit test mass in place when approaching the horizon. This justifies the name surface gravity.

With a tentative formal relation between surface gravity and temperature, and between area and entropy, the question arises whether a First Law of thermodynamics can be proved. This can in fact be done and the result for a Kerr-Newman black hole is

$$\mathrm{d}M = \frac{\kappa}{8\pi G} \mathrm{d}A + \Omega_H \mathrm{d}J + \Phi \mathrm{d}q,$$  \hspace{1cm} (1.2)

where $A, \Omega_H, \Phi$ denote the area of the event horizon, the angular velocity of the black hole, and the electrostatic potential, respectively. This relation can be obtained by conceptually different methods: A physical process version whereby a stationary black hole is altered by infinitesimal physical processes, and an equilibrium state version whereby the areas of two stationary black-hole solutions to Einstein’s equations are compared. Both methods lead to the same result (1.2).

Since $M$ is the energy of the black hole, (1.2) is the analogue of the First Law of thermodynamics given by

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N.$$  \hspace{1cm} (1.3)
‘Modern’ derivations of (1.2) make use of both Hamiltonian and Lagrangian methods of general relativity. For example, the First Law follows from an arbitrary diffeomorphism invariant theory of gravity whose field equations can be derived from a Lagrangian.

What about the Third Law of thermodynamics? A ‘physical process version’ was proved by Israel – it is impossible to reach \( \kappa = 0 \) in a finite number of steps, although it is unclear whether this is true under all circumstances (Farrugia and Hajicek 1979). This corresponds to the ‘Nernst version’ of the Third Law. The stronger ‘Planck version’, which states that the entropy goes to zero (or a material-dependent constant) if the temperature approaches zero, does not seem to hold. The above analogies are summarised in table 1.1.

| Law       | Thermodynamics                                      | Stationary Black Holes                           |
|-----------|-----------------------------------------------------|--------------------------------------------------|
| Zeroth    | \( T \) constant on a body in thermal equilibrium  | \( \kappa \) constant on the horizon of a black hole |
| First     | \( dE = TdS - pdV + \mu dN \)                      | \( dM = \frac{\kappa}{8\pi G}dA + \Omega dJ + \Phi dq \) |
| Second    | \( dS \geq 0 \)                                    | \( dA \geq 0 \)                                  |
| Third     | \( T = 0 \) cannot be reached                      | \( \kappa = 0 \) cannot be reached               |

Table 1.1.

The identification of the horizon area with the entropy for a black hole can be obtained from a conceptually different point of view. If a box with, say, thermal radiation of entropy \( S \) is thrown into the black hole, it seems as if the Second Law could be violated, since the black hole is characterised only by mass, angular momentum, and charge, but nothing else. The demonstration that the Second Law is fulfilled leads immediately to the concept of a black-hole entropy, as will be discussed now (Bekenstein 1980; Sexl and Urbantke 1983).

Consider a box with thermal radiation of mass \( m \) and temperature \( T \) lowered from a spaceship far away from a spherically-symmetric black hole towards the hole (figure 1.1). As an idealisation, both the rope and the walls are assumed to have negligible mass. At a coordinate distance \( r \) from the black hole, the energy of the box is given by

\[
E_r = m \sqrt{1 - \frac{2GM}{r}} \rightarrow r_0 \quad 0 .
\]

If the box is lowered down to the horizon, the energy gain is thus given by \( m \). The box is then opened and thermal radiation of mass \( \delta m \) escapes into
The laws of black-hole mechanics

The key to the resolution of this apparent paradox lies in the observation that the box must be big enough to contain the wavelength of the enclosed radiation. This, in turn, leads to a lower limit on the distance that the box can approach the horizon. Therefore, only part of $\delta m$ can be transformed into work, as I shall show now.

According to Wien’s law, one must have a linear extension of the box
of at least
\[ \lambda_{\text{max}} \approx \frac{\hbar}{k_B T}. \] (1.5)

I emphasise that at this stage Planck’s constant \( \hbar \) comes into play. The box can then be lowered down to the coordinate distance \( \delta r \) (assumed to be \( \ll 2G M \)) from the black hole, where according to the Schwarzschild metric the relation between \( \delta r \) and \( \lambda_{\text{max}} \) is
\[
\lambda_{\text{max}} \approx \int_{2GM}^{2GM+\delta r} \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}} dr \approx 2\sqrt{2GM\delta r} \quad \implies \quad \delta r \approx \frac{\lambda_{\text{max}}^2}{8GM}.
\]

According to (1.4), the energy of the box at \( r = 2GM + \delta r \) is
\[
E_{2GM+\delta r} = m\sqrt{1 - \frac{2GM}{2GM + \delta r}} \approx m\frac{\lambda_{\text{max}}}{4GM} \approx \frac{m\hbar}{4Gk_BT M}.
\]

Recalling that according to (1.2) the formal temperature of the black hole, \( T_{\text{BH}} \), is proportional to the surface gravity \( \kappa = 1/(4GM) \), the energy of the box before opening is
\[
E_{2GM+\delta r}^{(\text{before})} \approx m \frac{T_{\text{BH}}}{T},
\]
while after opening it is
\[
E_{2GM+\delta r}^{(\text{after})} \approx (m - \delta m) \frac{T_{\text{BH}}}{T}.
\]

The degree of efficiency of transforming thermal radiation into work is thus given by
\[
\eta \approx \left( \frac{\delta m - \delta m \frac{T_{\text{BH}}}{T}}{\delta m} \right) = 1 - \frac{T_{\text{BH}}}{T} < 1,
\]
which is the well-known Carnot limit for the efficiency of heat engines. From the First Law (1.2) one then finds for the entropy of the black hole \( S_{\text{BH}} \propto A = 16\pi(GM)^2 \). It is this agreement of conceptually different approaches to black-hole thermodynamics that gives confidence into the physical meaning of these concepts. In the next section I shall show how all these formal results can be physically interpreted in the context of quantum theory.

### 1.3 Hawking radiation

We have already seen in the gedankenexperiment discussed in the last section that \( \hbar \) enters the scene, see (1.5). That Planck’s constant has to
play a role, can be seen also from the First Law (1.2). Since \( T_{BH} dS_{BH} = \kappa/(8\pi G) dA \), one must have

\[
T_{BH} = \frac{\kappa}{G\zeta}, \quad S_{BH} = \frac{\zeta A}{8\pi}
\]

with an undetermined factor \( \zeta \). What is the dimension of \( \zeta \)? Since \( S_{BH} \) has the dimension of Boltzmann’s constant \( k_B \), \( k_B/\zeta \) must have the dimension of a length squared. There is, however, only one fundamental length available, the Planck length

\[
l_p = \sqrt{\frac{G\hbar}{c}} \approx 10^{-33} \text{ cm.} \tag{1.6}
\]

(For string theory, this may be replaced by the fundamental string length.) Therefore,

\[
T_{BH} \propto \frac{\hbar c}{k_B}, \quad S_{BH} \propto k_B A \frac{G}{\hbar c}. \tag{1.7}
\]

The determination of the precise factors in (1.7) was achieved in the pioneering paper by Hawking (1975). The key ingredient in his discussion is the behaviour of quantum fields on the background of an object collapsing to form a black hole. Similar to the situation of an external electric field (Schwinger effect), there is no uniquely defined notion of vacuum. This leads to the occurrence of particle creation. The peculiarity of the black-hole case is the thermal distribution of the created particles.

There exists an analogous effect already in flat spacetime, discussed by Unruh (1976), following work by Fulling (1973) and Davies (1975). In the following I shall briefly describe this effect.

Whereas all inertial observers in Minkowski space agree on the notion of vacuum (and therefore on particles), this no longer holds for non-inertial observers. Consider an observer who is uniformly accelerating along the \( X \)-direction in (1+1)-dimensional Minkowski spacetime (figure 1.2). The Minkowski cartesian coordinates are labelled here by upper-case letters. The orbit of this observer is the hyperbola shown in figure 1.2. One recognises that, as in the Kruskal diagram for the Schwarzschild metric, the observer encounters a horizon (here an “acceleration horizon”). There is, however, no singularity behind this horizon. The region I is a globally hyperbolic spacetime on its own – the so-called Rindler spacetime. This spacetime can be described by coordinates \((\tau, \rho)\) which are connected to the cartesian coordinates via the coordinate transformation

\[
\begin{pmatrix}
T \\
X
\end{pmatrix}
= \rho
\begin{pmatrix}
sinh a\tau \\
cosh a\tau
\end{pmatrix}, \tag{1.8}
\]

where \( a \) is a constant (the orbit in figure 1.2 describes an observer with acceleration \( a \), who has \( \rho = 1/a \)).
Since
\[ ds^2 = dT^2 - dX^2 = a^2 \rho^2 d\tau^2 - d\rho^2 , \tag{1.9} \]
the orbits \( \rho = \text{constant} \) are also orbits of a timelike Killing field \( \partial/\partial \tau \). It is clear that \( \tau \) corresponds to the external Schwarzschild coordinate \( t \) and that \( \rho \) corresponds to \( r \). As in the Kruskal case, \( \partial/\partial \tau \) becomes spacelike in regions II and IV.

The analogy with Kruskal becomes even more transparent if the Schwarzschild metric is expanded around the horizon at \( r = 2GM \). Introducing \( \rho^2/(8GM) = r - 2GM \) and recalling (1.1), one has
\[ ds^2 \approx \kappa^2 \rho^2 dt^2 - d\rho^2 - \frac{1}{4\kappa^2} d\Omega^2 . \tag{1.10} \]
Comparison with (1.9) shows that the first two terms on the right-hand side of (1.10) correspond exactly to the Rindler spacetime (1.9) with the acceleration \( a \) replaced by the surface gravity \( \kappa \). The last term in (1.10) describes a two-sphere with radius \( (2\kappa)^{-1} \).

\[ ^1 \text{It is this term that is responsible for the non-vanishing curvature of (1.10) compared to} \]
How does the accelerating observer experience the standard Minkowski vacuum $|0\rangle_M$? The key point is that the vacuum is a global state correlating regions I and III in figure 1.2 (similar to Einstein-Podolsky-Rosen correlations), but that the accelerated observer is restricted to region I. Considering for simplicity the case of a massless scalar field, the global vacuum state comprising the regions I and II can be written in the form

$$|0\rangle_M = \prod_{\omega} \sqrt{1 - e^{-2\pi \omega a^{-1}}} \sum_{n} e^{-n \pi \omega a^{-1}} |n_{\omega}^I \rangle \otimes |n_{\omega}^II \rangle,$$

where $|n_{\omega}^I \rangle$ and $|n_{\omega}^II \rangle$ are $n$-particle states with frequency $\omega = |k|$ in regions I and II, respectively. The expression (1.11) is an example for the Schmidt expansion of two entangled quantum systems, see e.g. Giulini et al. (1996); note also the analogy of (1.11) with a BCS-state in the theory of superconductivity.

For an observer restricted to region I, the state (1.11) cannot be distinguished, by operators with support in I only, from a density matrix that is found from (1.11) by tracing out all degrees of freedom in region II,

$$\rho_I \equiv \text{Tr}_{\text{II}} |0\rangle_M \langle 0|_M = \prod_{\omega} \left(1 - e^{-2\pi \omega a^{-1}}\right) \sum_{n} e^{-2n \pi \omega a^{-1}} |n_{\omega}^I \rangle \langle n_{\omega}^I|.$$

(1.12)

Note that the density matrix $\rho_I$ has exactly the form corresponding to a thermal canonical ensemble with temperature

$$T_U = \frac{\hbar a}{2\pi k_B} \approx 4 \times 10^{-23} a \left[\frac{\text{cm}}{s^2}\right] \text{K}.$$  

(1.13)

An observer who is accelerating uniformly through Minkowski space thus sees a thermal distribution of particles. This is an important manifestation of the non-uniqueness of the vacuum state in quantum field theory, even for flat spacetime. A more detailed discussion invoking models of particle detectors confirms this result.

The “Unruh temperature” (1.13), although being very small for most accelerations, might be observable for electrons in storage rings where spin precession is used as ‘detector’ (Leinaas 2001). Due to the circular nature of the accelerator, the spectrum of the observed particles is then, however, not thermal. Since this would complicate the direct comparison with the Hawking effect, there exist other proposals to measure (1.13), for example with ultraintense lasers (Chen and Tajima 1999).

I shall now turn to the case of black holes. From the form of the line element near the horizon, (1.10), one can already anticipate that – according to the equivalence principle – a black hole radiates with temperature the flat-space metric (1.4) whose extension into the (neglected) other dimensions would be just $-dy^2 - dZ^2$. 

Hawking radiation
in which $a$ is replaced by $\kappa$. This is in fact what Hawking (1975) found. The temperature reads

$$T_{\text{BH}} = \frac{\hbar \kappa}{2\pi k_B}. \quad (1.14)$$

For the total luminosity of the black hole one finds

$$L = -\frac{dM}{dt} = \frac{1}{2\pi} \sum_{l=0}^{\infty} (2l + 1) \int_0^\infty d\omega \frac{\Gamma_{\omega l}}{e^{2\omega / k_B \kappa} - 1}. \quad (1.15)$$

The term $\Gamma_{\omega l}$ – called ‘greybody factor’ because it encodes a deviation from the black-body spectrum – takes into account the fact that some of the particle modes are back-scattered into the black hole through the effect of spacetime curvature.

For the special case of the Schwarzschild metric where $\kappa = (4GM)^{-1}$, (1.14) becomes

$$T_{\text{BH}} = \frac{\hbar}{8\pi Gk_B M} \approx 10^{-6} \frac{M_\odot}{M} \text{ K}. \quad (1.16)$$

For solar-mass black holes (and even more so for the galactic black hole), this is of course utterly negligible – the black hole absorbs much more from the ubiquitous 3K-microwave background radiation than it radiates itself.

One can, however, estimate the lifetime of a black hole by making the plausible assumption that the decrease in mass is equal to the energy radiated to infinity, and using Stefan-Boltzmann’s law:

$$\frac{dM}{dt} \propto -AT_{\text{BH}}^4 \propto -M^2 \times \left( \frac{1}{M} \right)^4 = -\frac{1}{M^2},$$

which, when integrated, yields

$$t(M) \propto (M^3_0 - M^3) \approx M^3_0. \quad (1.17)$$

Here $M_0$ is the initial mass, and it has been assumed that after the evaporation $M \ll M_0$. Very roughly, the lifetime of a black hole is thus given by

$$\tau_{\text{BH}} \approx \left( \frac{M_0}{m_p} \right)^3 t_p \approx 10^{65} \left( \frac{M_0}{M_\odot} \right)^3 \text{ years} \quad (1.18)$$

($m_p$ and $t_p$ denote Planck mass and Planck time: $m_p = \hbar/l_p$, $t_p = l_p/c$.)

The galactic black hole thus has a lifetime of about $3 \times 10^{85}$ years! If in the early universe primordial black holes with $M_0 \approx 5 \times 10^{14}$ g were created, they would evaporate at the present age of the universe, see section 1.5.
A very detailed investigation into black-hole evaporation was made by Page (1976). He found that for $M \gg 10^{17}$ g the power emitted from an (uncharged, non-rotating) black hole is

$$P \approx 2.28 \times 10^{-54} L_\odot \left(\frac{M}{M_\odot}\right)^{-2},$$

81.4% of which is in neutrinos (he considered only electron- and muon-neutrinos), 16.7% in photons, and 1.9% in gravitons, assuming that there are no other massless particles. Since a black hole emits all existing particles in Nature, this result would of be changed by the existence of massless supersymmetric or other particles. In the range $5 \times 10^{14}$ g $\ll M \ll 10^{17}$ g, Page found

$$P \approx 6.3 \times 10^{16} \left(\frac{M}{10^{15} \text{g}}\right)^{-2} \text{erg s}^{-1},$$

45% of which is in electrons and positrons, 45% in neutrinos, 9% in photons, and 1% in gravitons. Massive particles with mass $m$ are only suppressed if $k_B T_{BH} < m$. For $M < 5 \times 10^{14}$ g also higher-mass particles are emitted.

All of the above derivations use the approximation where the spacetime background remains classical. In a theory of quantum gravity, however, such a picture cannot be maintained. Since the black hole becomes hotter while radiating, see (1.16), its mass will eventually enter the quantum-gravity domain $M \approx m_p$, where the semiclassical approximation breaks down. The evaporation then enters the realm of speculation. As an intermediate step one might consider the heuristic ‘semiclassical’ Einstein equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G \langle T_{ab} \rangle, \quad (1.19)$$

where on the right-hand side the quantum expectation value of the energy-momentum tensor appears. The evaluation of $\langle T_{ab} \rangle$ – which requires regularisation and renormalisation – is a difficult subject on its own (Frolov and Novikov 1998). The renormalised $\langle T_{ab} \rangle$ is essentially unique (its ambiguities can be absorbed in coupling constants) if certain sensible requirements are imposed. Evaluating the components of the renormalised $\langle T_{ab} \rangle$ near the horizon, one finds that there is a flux of negative energy into the hole. Clearly this leads to a decrease of the black hole’s mass. These negative energies are a typical quantum effect and are well-known from the – accurately measured – Casimir effect. This occurrence of negative energies

\[ This limit is referred to as the semiclassical approximation to quantum gravity (see e.g. Kiefer 1994). \]
is also responsible for the breakdown of the classical area law discussed in section 1.2.

The negative flux near the horizon lies also at the heart of the 'pictorial' representation of Hawking radiation that is often used, see e.g. Parikh and Wilczek (2000). In vacuum, virtual pairs of particles are created and destroyed. However, close to the horizon, one partner of this virtual pair might fall into the black hole, thereby liberating the other partner to become a real particle and escaping to infinity as Hawking radiation. The global quantum field exhibits quantum entanglement between the in-and outside of the black hole, similar to the case of the accelerated observer discussed above.

I want to end this section by giving the explicit expressions for the Hawking temperature (1.14) in the case of rotating and charged black holes. For the Kerr solution, one has

\[ k_B T_{BH} = \frac{\hbar}{2 \pi} = 2 \left( 1 + \frac{M}{\sqrt{M^2 - a^2}} \right)^{-1} \frac{\hbar}{8 \pi M} \leq \frac{\hbar}{8 \pi M}. \]  
(1.20)

Rotation thus reduces the Hawking temperature. For the Reissner-Nordström solution (describing a charged spherically-symmetric black hole) one has

\[ k_B T_{BH} = \frac{\hbar}{8 \pi M} \left( 1 - \frac{(Gq)^4}{r_+^4} \right) < \frac{\hbar}{8 \pi M}. \]  
(1.21)

Thus, also electric charge reduces the Hawking temperature. For an extremal black hole, \( r_+ = GM = \sqrt{G|q|} \), and thus \( T_{BH} = 0 \).

### 1.4 Interpretation of entropy

We have seen in the last section that – if quantum theory is taken into account – black holes emit thermal radiation with the temperature (1.14). Consequently, the laws of black-hole mechanics discussed in section 1.2 have indeed a physical interpretation as thermodynamical laws – black holes are thermodynamical systems.

From the First Law (1.2), one can therefore also infer the expression for the black-hole entropy. From \( dM = T_{BH} dS_{BH} \) one finds the 'Bekenstein-Hawking entropy'

\[ S_{BH} = \frac{k_B A}{4G \hbar}, \]  
(1.22)

in which the unknown factor in (1.7) has now been fixed. For the special case of a Schwarzschild black hole, this yields

\[ S_{BH} = \frac{k_B \pi R_0^2 \hbar}{G \hbar}. \]  
(1.23)
Interpretation of entropy

It can easily be estimated that $S_{BH}$ is much bigger than the entropy of the star that collapsed to form the black hole. The entropy of the sun, for example, is $S_\odot \approx 10^{57} k_B$, whereas the entropy of a solar-mass black hole is about $10^{77} k_B$, which is twenty orders of magnitude larger! For the galactic black hole, the entropy is $S_{GBH} \approx 10^{80} k_B$ which is one hundred times the entropy of the Universe. (Under the “entropy of the Universe” I understand the entropy of the present Universe up to the Hubble radius without taking black holes into account. It is dominated by the entropy of the cosmic microwave background radiation.)

Can a physical interpretation of this huge discrepancy be given? Up to now the laws of black-hole mechanics are only phenomenological thermodynamical laws. The central open question therefore is: Can $S_{BH}$ be derived from quantum-statistical considerations? This would mean that $S_{BH}$ could be calculated from a Gibbs-type formula according to

$$S_{BH} \approx -k_B \text{Tr}(\rho \ln \rho) \equiv S_{SM},$$

where $\rho$ denotes an appropriate density matrix; $S_{BH}$ would then somehow correspond to the number of quantum microstates that are consistent with the macrostate of the black hole that is – according to the no-hair theorem – uniquely characterised by mass, angular momentum, and charge. Some important questions are:

- Does $S_{BH}$ correspond to states hidden behind the horizon?
- Or does $S_{BH}$ correspond to the number of possible initial states?
- What are the microscopic degrees of freedom?
- Where are they located (if at all)?
- Can one understand the universality of the result?
- What happens with $S_{BH}$ after the black hole has evaporated?
- Is the entropy a “one loop” or a “tree level” effect?

The attempts to calculate $S_{BH}$ by state counting are usually done in the ‘one-loop limit’ of quantum field theory in curved spacetime – this is the limit where gravity is classical but non-gravitational fields are fully quantum, and it is the limit where the Hawking radiation (1.14) has been derived. The expression (1.22) can already be calculated from the so-called ‘tree level’ of the theory, where only the gravitational degrees of freedom are taken into account. Usually a saddle-point approximation for a euclidean path integral is being performed. Such derivations are, however, equivalent to derivations within classical thermodynamics, cf. Wald (2001).

If the entropy (1.22) is to make sense, there should be a generalised Second Law of thermodynamics according to

$$\frac{d}{dt}(S_{BH} + S_M) \geq 0,$$
where $S_M$ denotes all contributions to non-gravitational entropy. The validity of (1.25), although far from being proven in general, has been shown in a variety of gedanken experiments. One of the most instructive of such experiments was devised by Unruh and Wald. It makes use of the box shown in figure 1.1 that is adiabatically lowered towards a (spherically symmetric) black hole.

At asymptotic infinity $r \to \infty$, the black-hole radiation is given by (1.14). However, for finite $r$ the temperature is modified by the occurrence of a redshift factor $\chi(r) \equiv (1 - 2GM/r)^{1/2}$ in the denominator. Since the box is not in free fall, it is accelerated with an acceleration $a$. From the relation (Wald 1984)

$$\kappa = \lim_{r \to R_0} (a\chi),$$

(1.26)

one has

$$T_{BH}(r) = \frac{\hbar \kappa}{2\pi k_B \chi(r)} \quad r \to R_0 \quad \frac{\hbar a}{2\pi k_B},$$

(1.27)

which is just the Unruh temperature (1.13)! This means that a freely falling observer near the horizon observes no radiation at all, and the whole effect (1.27) comes from the observer (or box) being non-inertial with acceleration $a$.

The analysis of Unruh and Wald, which is a generalisation of the gedankenexperiment discussed at the end of section 1.2, shows that the entropy of the black hole increases at least by the entropy of the Unruh radiation displaced at the floating point – this is the point where the gravitational force (pointing downwards) and the buoyancy force from the Unruh radiation (1.27) are in equilibrium. Interestingly, it is just the application of ‘Archimedes’principle’ to this situation that rescues the generalised Second Law (1.25).

An inertial, i.e. free-falling, observer does not see any Unruh radiation. How does he interpret the above result? For him the box is accelerated and therefore the interior of the box fills up with negative energy and pressure – a typical quantum effect that occurs if a ‘mirror’ is accelerated through the vacuum. The ‘floating point’ is then reached after this negative energy is so large that the total energy of the box is zero.

I want to conclude this section with some speculations about the final stages of black-hole evolution and the information-loss problem. The point is that – in the semiclassical approximation used by Hawking – the radiation of a black hole seems to be purely thermal. If the black hole evaporates completely and leaves only thermal radiation behind, one would have a conflict with established principles in quantum theory: Any initial state (in particular a pure state) would evolve into a mixed state. In ordinary
quantum theory, because of the unitary evolution of the total system, this cannot happen. Formally, $\text{Tr} \rho^2$ remains constant under the von Neumann equation; the same is true for the entropy $S_{SM} = -k_B \text{Tr}(\rho \ln \rho)$: For a unitarily evolving system, there is no increase in entropy. If these laws are violated during black-hole evaporation, information would be destroyed. This is indeed the speculation that Hawking made after his discovery of black-hole radiation. The attitudes towards this information-loss problem can be roughly divided into the following classes,

- The information is indeed lost during black-hole evaporation, and the quantum-mechanical Liouville equation is replaced by an equation of the form
  \[ \rho \rightarrow \$ \rho \neq S\rho S^\dagger. \]  
  \[ (1.28) \]

- The full evolution is in fact unitary; the black-hole radiation contains subtle quantum correlations that cannot be seen in the semiclassical approximation.
- The black hole does not evaporate completely, but leaves a 'remnant' with mass in the order of the Planck mass that carries the whole information.

In my opinion, the information-loss problem is only a pseudoproblem. Already in the original calculation of Hawking (1975) only pure states appear. Reference to thermal radiation is being made because the particle number operator in the final pure state possesses an exact Planckian distribution. As has been shown in Kiefer (2001) the coupling of this pure state (a squeezed state in quantum-optics language) to its natural environment produces a thermal ensemble for the Hawking radiation, which constitutes an open quantum system, after this environment is traced out. The thermal nature of this radiation is thus a consequence of decoherence (Giulini et al. 1996).

There exist many attempts to derive the Bekenstein-Hawking entropy within approaches to quantum gravity, see e.g. Kiefer (1999) and Wald (2001) for more details and references. Examples are the derivations within superstring theory (counting of states referring to microscopic objects called D-branes), canonical quantum gravity, Sakharov’s induced gravity, conformal field theories, and others. Although many of these look very promising, a final consensus has not yet been reached.

1.5 Primordial black holes

Can the above discussed quantum effects of black holes be observed? As has already been mentioned, black holes formed by stellar collapse are much too heavy to exhibit quantum behaviour. To form smaller black holes one
needs higher densities which can only occur under the extreme situations of the early Universe. Such primordial black holes can originate in the radiation-dominated phase during which no stars or other objects can be formed.

Consider for simplicity a spherically-symmetric region with radius \( R \) and density \( \rho = \rho_c + \delta \rho \) embedded in a flat Universe with the critical density \( \rho_c \), cf. Carr (1985). For spherical symmetry the inner region is not affected by matter in the surrounding part of the Universe, so it will behave like a closed Friedmann Universe (since its density is overcritical), i.e., the expansion of this region will come to a halt at some stage, followed by a collapse. In order to reach a complete collapse, the (absolute value of the) potential energy, \( V \), at the time of maximal expansion has to exceed the inner energy, \( U \), given by the pressure \( p \). I.e.,

\[
V \sim \frac{GM^2}{R} \sim G\rho^2 R^5 \gtrsim pR^3.
\]

(1.29)

If the equation of state reads \( p = wR \) (\( w = 1/3 \) for radiation dominance), this gives

\[
R \gtrsim \sqrt{w} \frac{1}{\sqrt{G\rho}}.
\]

(1.30)

The lower bound for \( R \) is thus just given by the Jeans length. There also exists an upper bound. The reason is that \( R \) must be smaller than the curvature radius (given by \( 1/\sqrt{G\rho} \)) of the overdense region at the moment of collapse. Otherwise the region would contain a compact three-sphere which is topologically disconnected from the rest of the Universe. This case would then not lead to a black hole within our Universe. Using \( \rho \sim \rho_c \sim H^2/G \), where \( H \) denotes the Hubble parameter of the background flat Universe, one has the condition

\[
H^{-1} \gtrsim R \gtrsim \sqrt{w}H^{-1},
\]

(1.31)

evaluated at the time of collapse, for the formation of a black hole. This relation can be rewritten also as a condition referring to any initial time of interest (Carr 1985). In particular, one is often interested in the time where the fluctuation enters the horizon in the radiation-dominated Universe. This is illustrated in figure 1.3, where the presence of a possible inflationary phase at earlier times is also shown.

At horizon entry one gets, denoting \( \delta \equiv \delta \rho/\rho_c \),

\[
1 \gtrsim \delta_{\text{enter}} \gtrsim 0.3.
\]

(1.32)

\(^3\text{In theories with large extra dimensions it is imaginable that quantum effects of black holes can be seen at ordinary accelerators, see Dimopoulos and Landsberg 2001.}\)
Figure 1.3. Time development of a physical scale $\lambda(t)$ and the Hubble horizon $H^{-1}(t)$. During an inflationary phase $H^{-1}(t)$ remains approximately constant. After the end of inflation ($a_f$) the horizon $H^{-1}(t)$ increases faster than any scale. Therefore $\lambda_k$ enters the horizon again at $t_{k,\text{enter}}$ in the radiation- (or matter-) dominated phase.

This is, however, only a rough estimate. Numerical calculations give instead the bigger value $\delta_{\text{min}} \approx 0.7$ (Niemeyer and Jedamzik 1999).

Taking from (1.31) $R \approx \sqrt{wH^{-1}}$, one gets for the initial mass of a primordial black hole (PBH),

$$M_{PBH} = \frac{4\pi}{3} \rho R^3 \approx \frac{4\pi}{3} \rho_c (1 + \delta) w^{3/2} H^{-3} \approx w^{3/2} M_H ,$$

(1.33)

where $M_H \equiv (4\pi/3) \rho_c H^{-3}$ denotes the mass inside the horizon. Since $M_{PBH}$ is of the order of this horizon mass, a collapsing region will form a black hole practically immediately after horizon entry. Using the relation $M_H = t/G$, valid for a radiation dominated Universe, one gets from (1.33) the quantitative estimate

$$M_{PBH}[\text{g}] \approx 10^{38} t[\text{s}] .$$

(1.34)

This means that one can create Planck-mass black holes at the Planck time, and PBHs with $M_{PBH} \approx 5 \times 10^{14}\text{g}$ at $t \approx 5 \times 10^{-24}\text{s}$. The latter value is important, since according to (1.18) black holes with masses smaller than $M_{PBH} \approx 5 \times 10^{14}\text{g}$ have by now evaporated due to Hawking radiation. PBHs with bigger mass are still present today. At $t \approx 10^{-5}\text{s}$ one can create a solar-mass black hole and at $t \approx 10\text{s}$ (the time of nucleosynthesis) one could form a PBH with the mass of the galactic black hole. The initial
mass can of course increase through accretion, but it turns out that this is negligible in most circumstances (Carr 1985).

In the presence of an inflationary phase in the early Universe, all PBHs produced before the end of inflation are diluted away. This gives the bound

$$M_{PBH} > M_H(T_{RH}) \approx \frac{n_p^3}{10.88 T_{RH}^2} \sim 1\text{g} ,$$

if for the reheating temperature $T_{RH}$ a value of $10^{16}\text{GeV}$ is chosen.

According to the numerical calculations by Niemeyer and Jedamzik (1999), there exists a whole spectrum of initial masses,

$$M_{PBH} = KM_H(\delta - \delta_{\text{min}})^\gamma ,$$

a relation that is reminiscent of the theory of critical phenomena. This may change some of the quantitative conclusions.

To calculate the production rate of PBHs one needs an initial spectrum of fluctuations. This is usually taken to be of a Gaussian form, as predicted by most inflationary models (cf. Liddle and Lyth 2000). Therefore, there exists always a nonvanishing probability that the density contrast is high enough to form a black hole, even if the maximum of the Gaussian corresponds to a small value. One can then calculate the mass ratio (compared to the total mass) of regions which will develop into PBHs with mass $M_{PBH} \gtrsim M$, see, e.g., Bringmann et al (2001), Sec. 2, for details. This mass ratio, given by

$$\alpha(M) := \frac{\rho_{PBH,M}}{\rho_r} \approx \Omega_{PBH,M} \equiv \frac{\rho_{PBH,M}}{\rho_c} ,$$

where $\rho_r$ is the radiation density, is then compared with observation. This, in turn, gives a constraint on the theoretically calculated initial spectrum. Table 1.2 presents various observational constraints on $\alpha$ (see Green and Liddle 1997). The corresponding maximal value for each $\alpha$ is shown for the various constraints in figure 1.4.

Constraints arise either from Hawking radiation or from the gravitational contribution of PBHs to the present Universe (last entry). PBHs with initial mass of about $5 \times 10^{15}\text{g}$ evaporate “today”. (They release about $10^{30}\text{erg}$ in the last second.) From observations of the $\gamma$-ray background one can find the constraint given in the table. It corresponds to an upper limit of about $10^4$ PBHs per cubic parsec or $\Omega_{PBH,0} < 10^{-8}$. One can also try to observe directly the final evaporation event of a single PBH. This gives an upper limit of about $4.4 \times 10^5$ events per cubic parsec per year.

Given these observational constraints, one can then calculate the ensuing constraints on the primordial spectrum. The gravitational constraint $\Omega_{PBH,0} < 1$ gives surprisingly strong restrictions (cf. Bringmann et al 2001). For a scale-free spectrum of the form $\propto k^n$, as it is usually discussed
Table 1.2. Constraints on the mass fraction $\alpha(M) := \frac{\rho_{PBH,M}}{\rho_r} \approx \Omega_{PBH,M}$ of primordial black holes at their time of formation (Green and Liddle 1997).

![Figure 1.4](image)

Figure 1.4. Strongest constraints on the initial PBH mass fraction. The numbers correspond to the various entries in table 1.2.

for inflationary models, one finds restrictions on $n$ that are comparable to the limits obtained by large-scale observations (anisotropy spectrum of the cosmic microwave background radiation). Since these restrictions come from observational constraints referring to much smaller scales, they constitute an important complementary test.

The question whether PBHs really exist in nature has thus not yet been settled. Their presence would be of an importance that could hardly be overestimated. They would give the unique opportunity to study the quantum effects of black holes and could yield the crucial key for the construction of a final theory of quantum gravity.
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