REGULAR HOMOTOPY CLASSES OF LOCALLY GENERIC MAPPINGS

ANDRÁS JUHÁSZ

Abstract. In this paper we generalize the notion of regular homotopy of immersions of a closed connected n-manifold into \( \mathbb{R}^{2n-1} \) to locally generic mappings. The main result is that if \( n = 2 \) then two mappings with singularities are regularly homotopic if and only if they have the same number of cross-cap (or Whitney-umbrella) singularities. As an application, we get a description of the path-components of the space of those immersions of a surface into \( \mathbb{R}^4 \) whose projections into \( \mathbb{R}^3 \) are locally generic.

1. Introduction

Our work was motivated by the paper of U. Pinkall [4], which classifies immersions of compact surfaces into \( \mathbb{R}^3 \) up to regular homotopy, allowing diffeomorphisms of the source manifold \( M^2 \). So two immersions \( f, g: M^2 \rightarrow \mathbb{R}^3 \) are considered equivalent if there is a diffeomorphism \( \varphi \) of \( M^2 \) such that \( f = g \circ \varphi \). This notion is different from regular homotopy, yielding an interesting classification of immersed surfaces using the Arf invariant. That paper also gives generators for the abelian semigroup of immersed surfaces with the connected sum operation. Professor András Szücs asked me what happens with Pinkall’s classification if we allow cross-cap (also called Whitney-umbrella) singularities. The notion of regular homotopy has to be revised and, unlike for immersions, for singular maps it turns out that all natural definitions are equivalent. In fact, we prove that for singular mappings (i.e. not immersions) of a closed connected surface the number of cross-caps totally determines the regular homotopy class, diffeomorphisms of the source manifold are not needed. Thus the approach of U. Pinkall and the classical regular homotopy classification give the same result for singular maps.

In the final section of our paper we present an application of the above result to the study of the path components of the space of those immersions of a closed connected surface into \( \mathbb{R}^4 \) whose projections into \( \mathbb{R}^3 \) are locally generic, i.e. may have cross-cap singularities.

It is a natural question if Theorem 2.6 generalizes to higher dimensions, for locally generic maps of a closed n-manifold \( M^n \) into \( \mathbb{R}^{2n-1} \). Our methods of proof for Theorem 2.6 do not seem to work if \( n > 2 \) since they rely heavily on the results of surface topology. However, I could prove the general result in the case when \( n > 3 \) and \( M^n \) is 2-connected. I will publish this in a separate paper. I want to emphasize that the results of Section 3 easily generalize for any closed manifold \( M^n \) provided that the generalization of Theorem 2.6 holds true for \( M^n \).

1991 Mathematics Subject Classification. 57R45; 58K30; 57R42.

Key words and phrases. immersion, locally generic mapping, regular homotopy, cross-cap singularity.

Research partially supported by OTKA grant no. T037735.
I would like to take this opportunity to express my gratitude to Professor András Szűcs for drawing my attention to this problem and for his constant support and encouragement. I would also like to thank Professor Balázs Csikós who read the first version of this paper and suggested several improvements.

2. The main result

Let us start by introducing the notion of a locally generic mapping of a closed $n$-manifold $M^n$ into a $(2n - 1)$-manifold $N^{2n - 1}$ for $n \geq 2$.

**Definition 2.1.** $f : M^n \to N^{2n - 1}$ is called locally generic if it is an immersion except for cross-cap singularities. The set of singular points of $f$ in $M^n$ is denoted by $S(f)$. ($|S(f)| < \infty$ because $M^n$ is compact.) Let us denote by $\text{Lgen}(M^n, N^{2n - 1})$ the subspace of locally generic mappings in $C^\infty(M^n, N^{2n - 1})$ endowed with the $C^\infty$-topology.

Recall that a map $f : M^n \to N^{2n - 1}$ is called generic (or stable), if it is an immersion with normal crossings except in a finite set of points, moreover the singular points of $f$ are non-multiple cross-cap points. This explains our terminology. Whitney proved that the set of stable maps is dense open in $C^\infty(M^n, \mathbb{R}^{2n - 1})$ with respect to the $C^\infty$-topology. Studying the double point set of a generic mapping close to the locally generic mapping $f$ in the $C^\infty$-topology, one can easily verify that for $M^n$ closed $|S(f)|$ is an even integer. (For the cross-caps are precisely endpoints of double-point curves.)

**Definition 2.2.** Two locally generic mappings $f, g : M^n \to N^{2n - 1}$ are called regularly homotopic (denoted by $f \sim g$) if there is a smooth mapping $H : M^n \times [0, 1] \to N^{2n - 1}$ such that $H_t$ is locally generic for each $t \in [0, 1]$, moreover $H_0 = f$ and $H_1 = g$. Here $H_t(x) = H(x, t)$ for $x \in M^n$ and $t \in [0, 1]$.

The following definition will be especially useful in the case $n = 2$.

**Definition 2.3.** Two locally generic mappings $f, g : M^n \to N^{2n - 1}$ are called image-homotopic if there is a diffeomorphism $\varphi$ of $M^n$ such that $f \circ \varphi$ is regularly homotopic to $g$. We denote this by $\text{Im}(f) \sim \text{Im}(g)$.

**Proposition 2.4.** If $f \sim g$ or just $\text{Im}(f) \sim \text{Im}(g)$, then $|S(f)| = |S(g)|$. In fact, if $H : M^n \times [0, 1] \to N^{2n - 1}$ is a regular homotopy, then $|S(H_t)| = |S(H_0)|$ for every $t \in [0, 1]$.

**Proof.** If $\varphi$ is a diffeomorphism of $M^n$, then $S(f) = \varphi(S(f \circ \varphi))$, so $|S(f)| = |S(f \circ \varphi)|$. Thus we can suppose that $f \sim g$. From the definition of stability it is clear that every locally generic mapping $h$ has a neighborhood $U_h$ in the Whitney $C^\infty$ topology such that for every $h' \in U_h$ we have $|S(h')| = |S(h)|$ (because every $p \in M^n$ has a neighborhood $U$ such that $h|U$ is equivalent to $h'|U$ and since $M^n$ is closed, see [11], p. 72]). Thus the function $|S(\cdot)| : \text{Lgen}(M^n, N^{2n - 1}) \to \mathbb{Z}$ is locally constant. So if $H$ is a regular homotopy connecting $f$ and $g$, this implies that $H_t$ is a continuous path in $\text{Lgen}(M^n, N^{2n - 1})$, along which $|S(H_t)|$ is constant.

**Remark 2.5.** We have defined the notions of regular homotopy and image-homotopy between two locally generic maps $f$ and $g$. Proposition 2.4 implies that if $f$ and $g$ are immersions then they are regularly homotopic, resp. image-homotopic as immersions iff they are those as locally generic maps.
If \( H: M^2 \times [0, 1] \to \mathbb{R}^3 \) is a regular homotopy between two locally generic mappings \( f = H_0 \) and \( g = H_1 \) and \( k = |S(f)| = |S(g)| \), then there exists curves \( \gamma_1, \ldots, \gamma_k: [0, 1] \to M^2 \) such that \( S(H_t) = \{ \gamma_j(t): 1 \leq j \leq k \} \) for every \( t \in [0, 1] \). We define the bijection \( i_H: S(f) \to S(g) \) the following way: for \( 1 \leq j \leq k \) let \( i_H(\gamma_j(0)) = \gamma_j(1) \).

Now we can state the main result of this paper yielding a converse of Proposition 2.4 for singular mappings.

**Theorem 2.6.** Let \( M^2 \) be a closed connected surface and suppose that \( f, g: M^2 \to \mathbb{R}^3 \) are locally generic mappings with \( |S(f)| = |S(g)| > 0 \). Then \( f \sim g \). Moreover for any bijection \( i: S(f) \to S(g) \) there exists a regular homotopy \( H \) connecting \( f \) and \( g \) such that \( i_H = i \).

Let us now list a few interesting corollaries of this. A surprising consequence of Theorem 2.6 is the following: If \( U \) is the standard locally generic mapping of \( S^2 \) into \( \mathbb{R}^3 \) with two cross-cap points then for any two immersions \( f, g: M^2 \to \mathbb{R}^3 \) the connected sums \( f \# U \) and \( g \# U \) become regularly homotopic as locally generic maps!

Another consequence of our theorem is that, given a closed connected surface \( M^2 \), we can easily produce a full list of representatives of all regular homotopy classes of locally generic maps \( M^2 \to \mathbb{R}^3 \). (We only do this for singular maps, for immersions see [4].) First suppose that \( M^2 \) is orientable. If we denote by \( i_M \) the standard embedding of \( M^2 \) into \( \mathbb{R}^3 \) then

\[
i_M \# U \# \ldots \# U
\]

is a representative for the class of locally generic maps with \( 2n > 0 \) singular points. Now suppose that \( M^2 \) is a non-orientable surface of genus \( g \). We denote the Boy surface by \( B \). Now

\[
B \# \ldots \# B \# U \# \ldots \# U
\]

is a locally generic mapping of \( M^2 \) into \( \mathbb{R}^3 \) with \( 2n \) cross-cap points.

Perhaps the following construction can be visualized more easily: Let us denote by \( V \) the well-known locally generic mapping of \( \mathbb{R}P^2 \) into \( \mathbb{R}^3 \) having two singular points \( (V \sim B \# U) \). Then any singular locally generic mapping is regularly homotopic to one of the form \( i_N \# U \# \ldots \# U \# V \# \ldots \# V \), where \( N^2 \) is orientable. (The left side of Figure 1 depicts \( A_2 \# U \# U \), where \( A_2 \) is the orientable surface of genus 2. The right side of Figure 1 illustrates \( B \# B \# B \# U \# U \sim V \# V \# V \).

Pinkall determined the abelian semigroup \( H \) of immersed surfaces in \( \mathbb{R}^3 \) with the connected sum operation (see [4]). If we consider the extended semigroup \( \tilde{H} \) of locally generic surfaces, then only \( U \) is needed as a new generator with the following new relations (using Pinkall’s notation): \( S \# U = T \# U \) and \( B \# U = \overline{B} \# U \).

Note that \( \tilde{H} \setminus H \) is a sub-semigroup of \( \tilde{H} \) and it easily follows from Theorem 2.6 that it is isomorphic to \( J \oplus \mathbb{Z}_+ \), where \( J \) denotes the semigroup of (closed connected) surfaces. As a corollary we may conclude that the Grothendieck group of \( \tilde{H} \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \).
3. Proof of the main result

The purpose of this section is to prove Theorem 2.6. First we recall the classification of immersions of an arbitrary two-dimensional manifold \( F \) into \( \mathbb{R}^3 \) using Hirsch-Smale theory.

**Theorem 3.1.** There is a 1-1 correspondence between the regular homotopy classes of immersions of a surface \( F \) into \( \mathbb{R}^3 \) denoted by \( \text{Imm}(F^2, \mathbb{R}^3) \) and \( H^1(F^2; \mathbb{Z}_2) \).

**Proof.** By Hirsch (see [4]) there is a weak homotopy equivalence between the space \( \text{Imm}(F^2, \mathbb{R}^3) \) and \( \Gamma(\mu) \). Here \( \Gamma(\mu) \) denotes the space of sections of the vector bundle \( \mu = \text{MONO}(TF^2, \mathbb{R}^3) \) over \( F^2 \) whose fiber over \( p \in F^2 \) consists of all linear injections from \( T_p F \) to \( \mathbb{R}^3 \). Thus there is a bijection between the regular homotopy classes \( \pi_0(\text{Imm}(F^2, \mathbb{R}^3)) \) and \( \pi_0(\Gamma(\mu)) \). Fix an arbitrary Riemannian metric on \( F^2 \) and let \( \mu' \) be the bundle over \( F^2 \) whose general fiber over \( p \) is the space of orthogonal injections of \( T_p F \) into \( \mathbb{R}^3 \). Then the inclusion of \( \mu' \) into \( \mu \) is a fiber homotopy equivalence (see [4, p. 426]), thus \( \pi_0(\Gamma(\mu)) = \pi_0(\Gamma(\mu')) \). Fixing a section \( s \in \Gamma(\mu') \) every section \( t \in \Gamma(\mu') \) can be obtained by the action of a unique element of \( C(F^2, SO(3)) \) on \( s \). Thus \( \Gamma(\mu') \) is homeomorphic to \( C(F^2, SO(3)) \), yielding \( \pi_0(\Gamma(\mu)) = [F^2, SO(3)] \). Since \( SO(3) \) is homeomorphic to \( RP^3 \), it follows from obstruction theory that

\[
[F^2, SO(3)] = [F^2, RP^3] = [F^2, RP^{\infty}] = [F^2, K(\mathbb{Z}_2, 1)] = H^1(F^2; \mathbb{Z}_2),
\]

where \( K(\mathbb{Z}_2, 1) = RP^{\infty} \) is an Eilenberg-MacLane space. \( \square \)

From now on \( M^2 \) denotes the closed connected surface mentioned in the statement of Theorem 2.6. If \( f \) and \( g \) are locally generic mappings of \( M^2 \) into \( \mathbb{R}^3 \) with \( |S(f)| = |S(g)| \), then according to the lemma of homogeneity there exists a diffeotopy \( \{ \varphi_t: t \in [0, 1] \} \) of \( M^2 \) such that \( \varphi_0 = \text{id}_{M^2} \) and \( \varphi_1(S(g)) = S(f) \). Since \( f \circ \varphi_1 \) provides a regular homotopy between \( f \) and \( f \circ \varphi_1 \), it is sufficient to prove Theorem 2.6 in the case \( S(f) = S(g) \). Let \( S(f) = S(g) = \{ p_1, \ldots, p_k \} \), where \( k = |S(f)| > 0 \) is an even integer. For each \( p_i \) choose a sufficiently small open neighborhood \( D_i \) diffeomorphic to an open 2-disc such that \( f|D_i \) has the canonical form \( (x_1^2, x_2, x_1x_2) \) in an appropriate pair of local coordinate-systems centered at \( p_i \) and \( f(p_i) \). Similarly \( g|D_i \) should have the same canonical form in another pair of local coordinate-systems. Assume moreover that the discs \( D_1, \ldots, D_k \) are pairwise disjoint. Denote by \( A \) the disjoint union \( \bigsqcup_{i=1}^k D_i \), then \( F^2 = M^2 \setminus A \) is a two-manifold with boundary.

![Figure 1.](image-url)
Lemma 3.2. Suppose that \( f \) and \( g \) are locally generic mappings of the surface \( M^2 \) into \( \mathbb{R}^3 \) such that \( S(f) = S(g) \). Choose open discs \( D_1, \ldots, D_k \subset M^2 \) centered at the points of \( S(f) \) as above. Define \( F^2 = M^2 \setminus \bigsqcup_{i=1}^k D_i \). Then there exists a diffeomorphism \( d \) of the pair \((M^2, F^2)\) such that the immersions \((f|F^2) \circ d \) and \( g|F^2 \) are regularly homotopic and \( d \) permutes \( S(f) \). Moreover there is a diffeotopy \( d_t \) of \( M^2 \) with \( d_0 = \text{id}_{M^2} \) and \( d_1 = d \).

Proof. According to Theorem 3.1 the regular homotopy classes of \((f|F^2) \) and \((g|F^2) \) correspond to cohomology classes \( \alpha, \beta \in H^1(F^2; \mathbb{Z}_2) \). (These will be shown to be non-zero later.) We construct a diffeomorphism \( d \) of the pair \((M^2, F^2)\) such that for the induced automorphism \( d^* \) of \( H^1(F^2; \mathbb{Z}_2) \) it holds that \( d^*(\alpha) = \beta \). Using Theorem 3.1 again this gives the required result \((f|F^2) \circ d \sim g|F^2 \).

We first note that

\[
(3.1) \quad H_1(F^2; \mathbb{Z}_2) = H_1(M^2; \mathbb{Z}_2) \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2,
\]

as can be seen from the exact sequence of the pair \((M^2, F^2)\). (Recall that \( k = \left| S(f) \right| \).) For each \( i \), \( 1 \leq i \leq k \) choose an embedded curve \( \gamma_i \) in \( F^2 \) around \( D_i \). Denote its homology class by \([\gamma_i] = c_i\). The classes \( c_1, \ldots, c_{k-1} \) can be chosen for the generators of the \((k-1)\mathbb{Z}_2\) summands in \(3.1\) (Note that \( c_1 + \cdots + c_k = 0 \) since \( \partial F^2 = \gamma_1 \cup \cdots \cup \gamma_k \)). According to Pinkall [4] we have that \([\alpha, c_i] = 1 \) and \([\beta, c_i] = 1 \) for every \( i \in \{1, \ldots, k\} \), since \( \gamma_i \) has a neighborhood homeomorphic to \( S^1 \times [0, 1] \) which is mapped by \( f \) and also by \( g \) into a "figure eight \times [0, 1]". Now we have two cases according to the orientability of \( M^2 \).

If \( M^2 \) is orientable of genus \( g \) we denote the standard generators of \( H_1(M^2; \mathbb{Z}_2) \) by \( a_1, b_1, \ldots, a_g, b_g \), and choose embedded curves \( \varphi_1, \psi_1, \ldots, \varphi_g, \psi_g \) in \( F^2 \) representing them. Define

\[
H_a = \{1 \leq i \leq g \mid \langle \alpha, a_i \rangle \neq \langle \beta, a_i \rangle \},
\]

and

\[
H_b = \{1 \leq i \leq g \mid \langle \alpha, b_i \rangle \neq \langle \beta, b_i \rangle \}.
\]

There is a simple (i.e. embedded) closed curve \( \delta \) in \( F^2 \) that for each \( i \), \( 1 \leq i \leq g \) intersects transversally in one point the curve \( \varphi_i \) if \( i \in H_a \) and is disjoint from \( \varphi_i \) if \( i \notin H_a \), moreover \( \delta \) intersects transversally in one point \( \psi_i \) if \( i \in H_b \) and is disjoint from \( \psi_i \) if \( i \notin H_b \). Note that the homology class of \( \delta \) will be

\[
(3.2) \quad \left( \sum_{i \in H_a} b_i \right) + \left( \sum_{i \in H_b} a_i \right).
\]

Such a \( \delta \) exists because in \( H_1(M^2; \mathbb{Z}_2) \) any class can be represented by a simple curve, and a simple curve representing the class \(3.2\) can be arranged to be transversal to all the curves \( \varphi_j \) and \( \psi_j \) for \( j = 1, \ldots, k \) and to intersect each of them at most in one point. Now choose two points on \( \delta \) very close to each other so that none of the curves \( \varphi_i \) and \( \psi_i \) for \( 1 \leq i \leq g \) intersects the shorter arc \( \delta^* \) between them. Thus the following equalities hold:

\[
(3.3) \quad |\delta \cap \varphi_i| = \begin{cases} 1 & \text{if } i \in H_a \\ 0 & \text{if } i \notin H_a \end{cases}
\]

\[
|\delta \cap \psi_i| = \begin{cases} 1 & \text{for } i \in H_b \\ 0 & \text{for } i \notin H_b \end{cases}
\]
On the other hand if $d$ avoids the curves $\varphi_i$ and $\psi_i$ for $i = 1, \ldots, g$ as well as the discs $D_i$ for $i = 3, \ldots, k$. This can be done since $M^2 \setminus \bigcup_{i=1}^{2g} (\varphi_i \cup \psi_i)$ is path-connected. From now on we will denote this new simple curve on $M^2$ by $\delta$ (see Figure 2). Note that the equalities still hold. Now choose a tubular neighborhood $T$ of $\delta$ such that $D_1, D_2 \subset T$ and for $i \in H_a$ the curve $\varphi_i$ and for $j \in H_b$ the curve $\psi_j$ intersects $T$ in a line segment. (See the left side of Figure 3) We also select a slightly wider tubular neighborhood $T' \supset T$. Define $d$ on $T$ to be a rotation of $T = S^1 \times [-1, 1]$ by $180^\circ$ interchanging $D_1$ and $D_2$ and also interchanging $p_1$ and $p_2$. The diffeomorphism $d$ acts identically on $M^2 \setminus T'$. On $T' \setminus T$, which is homeomorphic to $S^1 \times ([-2, 2] \setminus [-1, 1])$, define $d$ as the rotation of $S^1 \times \{s\}$ by $(2 - |s|) \times 180^\circ$ for $s \in [-2, 2] \setminus [-1, 1]$ (see the right side of Figure 3). The diffeomorphism $d$ is diffeotopic to $id_{M^2}$: construct $d_t : M^2 \longrightarrow M^2$ similarly to $d$, just take a rotation by $180^\circ \cdot t$ instead of rotating by $180^\circ$ such that $d_t(p_1) \in \delta^*$ for every $t \in [0, 1]$.

For $i \notin H_a$ it is clear that $d$ is the identity on the image of $\varphi_i$, thus $d_*(a_i) = a_i$. On the other hand if $i \in H_a$, then $d \circ \varphi_i$ is homologous to the connected sum of $\varphi_i$ and $\gamma_1$ (surrounding $D_1$), thus $d_*(a_i) = a_i + c_1$. Similarly for $i \notin H_b$ we have $d_*(b_i) = b_i$, and if $i \in H_b$ then $d_*(b_i) = b_i + c_1$. Finally it holds that $d_*(c_2) = c_1$, $d_*(c_1) = c_2$ and if $i > 2$ then $d_*(c_i) = c_i$. (If $k = 2$ then $d_*(c_1) = c_2 = c_1$.) This can be verified by looking at the action of $d$ on the curves $\gamma_1, \ldots, \gamma_k$. Since $d_*$ permutes the generators $c_1, \ldots, c_{k-1}$, for $1 \leq i \leq k - 1$ we have $(\alpha, d_*(c_i)) = 1$. (Recall that $(\alpha, c_1) = 1$ and $(\beta, c_i) = 1$ for every $i$.) Thus $\langle \alpha, d_*(c_i) \rangle = \langle \beta, c_i \rangle$. By the choice of $H_a$ we see that for $i \in H_a$ it holds that $\langle \alpha, d_*(a_i) \rangle = \langle \alpha, a_i + c_1 \rangle = \langle \alpha, a_i \rangle$.

\[ \delta^* \cap \varphi_i = \delta^* \cap \psi_i = \emptyset \quad \text{for} \quad 1 \leq i \leq g. \]
\[ \langle \alpha, a_1 \rangle + 1 = \langle \beta, a_i \rangle \] and for \( i \notin H_a \) we have that \( \langle \alpha, d_*(a_i) \rangle = \langle \alpha, a_i \rangle = \langle \beta, a_i \rangle \). A similar argument holds for \( b_1, \ldots, b_g \). Since \( a_1, b_1, \ldots, a_g, b_g \) and \( c_1, \ldots, c_{k-1} \) form a basis of \( H_1(F^2; \mathbb{Z}_2) \), we have shown that \( \langle d^*(\alpha), x \rangle = \langle \alpha, d_*(x) \rangle = \langle \beta, x \rangle \) for every \( x \in H_1(F^2; \mathbb{Z}_2) \). Thus \( d^*(\alpha) = \beta \). Hence \( f \circ d \) is regularly homotopic to \( \phi \). As required. Also \( d \) satisfies \( d(S(f)) = S(f) \).

Now suppose that \( M^2 \) is a non-orientable surface of genus \( g \), i.e. a sphere with \( g \) Moebius bands. Choose a curve \( \varphi_i \) in \( F^2 \) on the \( i \)-th Moebius band representing its homology generator for \( 1 \leq i \leq g \). Then \( b_1 = \left[ \varphi_1 \right], \ldots, b_g = \left[ \varphi_g \right] \) together with \( c_1, \ldots, c_{k-1} \) is the standard basis of \( H_1(F^2; \mathbb{Z}_2) \). Analogously to the orientable case let

\[ H = \{ 1 \leq i \leq g : \langle \alpha, b_i \rangle \neq \langle \beta, b_i \rangle \}, \]

and similarly it is sufficient to construct a diffeomorphism \( d \) of the pair \( (M^2, F^2) \) with \( d_*(c_i) = c_i \) for \( 1 \leq i \leq k \), \( d_*(b_i) = b_i + c_1 \) for \( i \in H \) and \( d_*(b_i) = b_i \) for \( i \notin H \).

It is easy to show that for any fix \( 1 \leq j \leq g \) we can find a diffeomorphism \( d_j \) of the pair \( (M^2, F^2) \) such that \( d_{j*}(b_j) = b_j + c_1 \) and that \( d_{j*} \) is identical on every other homology-generator. (Then \( \prod_{j \in H} d_j \) is a good choice for \( d \).)

For this end modify a small arc of \( \varphi_j \) using a homology such that it still lies in \( F^2 \) but gets close to \( D_1 \) and remains disjoint from all the other \( \varphi_i \) for \( i \neq j \). (We shall call this modified curve \( \varphi_{j'} \) also.) This is possible since \( M^2 \setminus \bigcup_{i=1}^g \varphi_i \) is path-connected. Denote by \( T \) a tubular neighborhood of \( \varphi_j \) in \( M^2 \) containing \( D_1 \) and disjoint from \( D_i \) if \( i > 1 \) and from \( \varphi_i \) if \( i \neq j \). (See Figure 4.) Then \( T \) is homeomorphic to the Moebius band. Also choose a slightly larger tubular neighborhood \( T' \) of \( T \) with similar properties. Now think of \( T \) as a rectangle with the vertical sides identified in the opposite direction and with \( D_1 \) in its center. Let \( d_j \) be the reflection of the rectangle \( T \) into its horizontal central axis going through the center of \( D_1 \) which is \( p_1 \). Then \( d_j \) induces an orientation-preserving diffeomorphism (a rotation) of \( \partial T = S^1 \), which can be extended to \( T \setminus T = S^1 \times [0, 1] \) being identical on \( \partial T' = S^1 \times \{ 1 \} \) as we have already seen. Finally \( d_j \) is identical on \( M^2 \setminus T' \). This \( d_j \) maps the curve \( \varphi_j \) (which is the horizontal central line in the rectangle except that it avoids \( D_1 \) (see Figure 4)) to a curve homologous to the connected sum of \( \varphi_j \) and \( \gamma_1 \), thus \( d_{j*}(b_j) = b_j + c_1 \). Since \( \varphi_i \) for \( i \neq j \) and \( \gamma_i \) for \( i > 1 \) are fixed by \( d_j \), it satisfies the required conditions. Concerning \( c_1 \) we have \( d_{j*}(c_1) = -c_1 = c_1 \) since we are working with mod 2 coefficients.
The diffeomorphism $d_j$ of $M^2$ is diffeotopic to $id_{M^2}$: Think of the Möbius band $T$ as the factor space $S^1 \times [0, 1]/\tau \sim (p \times \{1\})$ (thus we identify the opposite points of one boundary component of an annulus). In this model define $(d_j)_t$ on $T$ as the diffeomorphism induced by the rotation of the annulus by $180^\circ \cdot t$ degrees. On $T \setminus T'$ define $(d_j)_t$ as before. Finally on $M^2 \setminus T'$ the diffeomorphism $(d_j)_t$ is the identity mapping. Then $d_j = (d_j)_1$ and $(d_j)_0 = id_{M^2}$ as required.

As a consequence of the above proof we obtain the following proposition (for the definition of the mapping class group see [7]):

**Corollary 3.3.** Suppose that $D^2$ is a surface of genus $g$ with $k > 1$ boundary components, where $k$ is even. We denote, like as before, the homology classes represented by the boundary components of $D^2$ in $H_1(D^2; \mathbb{Z}_2)$ by $c_1, \ldots, c_k$. The mapping class group $M(D^2)$ acts on the set $S = \{ \alpha \in H_1(D^2; \mathbb{Z}_2) : (\alpha, c_i) = 1, 1 \leq i \leq k - 1 \}$. If $\alpha, \gamma \in S$ then $(\alpha, c_k) = (\alpha, c_1 + \cdots + c_{k-1}) = 1$ since $k$ is even.) If $M^2$ is a closed surface of genus $g$ then there is a homomorphism $m: M(D^2) \rightarrow M(M^2)$ obtained by “filling in the holes”. Then $\text{ker}(m)$ acts transitively on $S$.

**Lemma 3.4.** Let $g$ and $h$ be locally generic mappings of $M^2$ into $\mathbb{R}^3$ such that $S(g) = S(h)$ and $g|F^2 \sim h|F^2$, where $F^2$ is the complement of a small open neighborhood of the common singular set. Then $g \sim h$.

**Proof.** Recall that $A = \bigsqcup_{i=1}^k D_i$. Since for each $i$ it holds that $h|D_i$ and $g|D_i$ have canonical forms in appropriate coordinate-systems, there is a regular homotopy $H$ between $h$ and a locally generic mapping $\tilde{h}$ such that for each $t \in [0, 1]$ we have $S(H_t) = S(h)$ and that $\tilde{h}|A = g|A$. Thus $H\mid(F^2 \times [0, 1])$ is a regular homotopy between the immersions $h|F^2$ and $\tilde{h}|F^2$ showing that $\tilde{h}|F^2 \sim g|F^2$. So we can suppose that $g|A = h|A$.

Let $H$ be a regular homotopy between $g|F^2$ and $h|F^2$. For every $i = 1, \ldots, k$ fix a smaller concentric closed disc $B_i$ in $D_i$ (hence $p_i \in B_i \subset D_i$). Finally set $F_i = F^2 \setminus \left( \bigsqcup_{i=1}^k D_i \right)$. We will define recursively a sequence of regular homotopies $H^i: F_i \times [0, 1] \rightarrow \mathbb{R}^3$ connecting $g_i = g|F_i$ and $h_i = h|F_i$ for $i = 0, \ldots, k$ with the property $H^0 = H$. Suppose that we have constructed $H^i$ for $i < j$. Let $q$ be a point in $\partial D_j$. For $t \in [0, 1]$ there is a one-parameter family of elements $M_t \in GL(\mathbb{R}, 3)$ with $M_t d_q H^j = d_q H^j$ and a vector $v_t \in \mathbb{R}^3$ with $M_t(H^j(q) + v_t) = H^j(q)$. Here $d_q H^j$ denotes the differential of the mapping $H^j$ at the point $q$. Now define $H^j(q) = H^j(q) + v_t$. Since $g(q) = h(q)$ and $d_q g = d_q h$, we have that $M_1 = \text{id}_{\mathbb{R}^3}$ and $v_1 = 0$, thus $H^j(q) = H^j(q)$. With this transformation of $H^j$ we have achieved that $H^j(q) = H^j(q)$ and $d_q H^j = d_q H^j$ for every $t \in [0, 1]$. 

![Diagram](image-url)
boundary and normal derivatives. Since deform the whole disk at the same time so as to induce the given deformation on the boundary of the disk and the normal derivatives along the boundary, then we can choose the diffeotopy \( \varphi \) on page 245 of [2] or Theorem 2.1 in [6]), which intuitively states the following: We have seen using the lemma of homogeneity that it is sufficient to prove Theorem 2.6 under the assumption \( S(f) = S(g) \). By Lemma 3.2 there exists a diffeomorphism \( d \) such that \( (f \circ d)|F^2 \sim g|F^2 \) and \( S(f \circ d) = S(f) \). Now applying Lemma 3.2 to \( h = f \circ d \) and \( g \) we get \( f \circ d \sim g \). But \( f \circ d \) is a regular homotopy between \( f \) and \( g \) proving that \( f \sim g \).

It remains to show that the above regular homotopy \( H \) joining \( f \) and \( g \) can be chosen in such a way that it defines a prescribed bijection \( i: S(f) \rightarrow S(g) \), that is \( i_H = i \). Recall that the bijection \( i_H: S(f) \rightarrow S(g) \) depends only on the choice of the diffeotopy \( \varphi_t \) mentioned before Lemma 3.2 that clearly might induce any prescribed bijection between \( S(f) \) and \( S(g) \). If \( M^2 \) is orientable, then the diffeomorphism \( d: M^2 \rightarrow M^2 \) of Lemma 3.2 swaps the singular points \( p_1 \) and \( p_2 \) (i.e. \( d(p_1) = p_2 \), \( d(p_2) = p_1 \) and \( d(p_i) = p_i \) for \( i > 2 \)), and in the non-orientable
case $d(p_i) = p_i$ for $1 \leq i \leq k$. Finally the homotopy constructed in Lemma 3.3 between $f \circ d$ and $g$ is a singularity fixing homotopy in the sense of Definition 3.4. This completes the proof of Theorem 2.6.\hfill \Box

The converse of Lemma 3.3 is true only in the following form:

**Proposition 3.5.** Suppose that $g$ and $h$ are locally generic mappings of $M^2$ into $\mathbb{R}^3$ such that $S(g) = S(h)$. Denote by $F^2$ the complement of a small open neighborhood of the common singular set. Then $g \sim h$ implies that $\text{Im}(g|F^2) \sim \text{Im}(h|F^2)$.

**Proof.** Let $H$ be a regular homotopy connecting $g$ and $h$. Then

$$S(H_t) = \{ p_1(t), \ldots, p_k(t) \},$$

where $p_i(t)$ is a smooth curve in $M^2$. The lemma of homogeneity gives a diffeomopy $\varphi_t$ of $M^2$ such that $\varphi_t = id_{M^2}$ and $\varphi_t(S(H_0)) = S(H_t)$ for every $t \in [0,1]$. The homotopy $G_t = H_t \circ \varphi_t$ has the property that $S(G_t) = S(G_0)$ for $t \in [0,1]$ and connects $g$ with $h \circ \varphi_1$. Since $\varphi_1$ permutes $S(g)$ (because $\varphi_1(S(g)) = \varphi_1(S(H_0)) = S(H_1) = S(g)$) we can choose $\varphi_t$ to map $F^2$ onto itself. $G_t|F^2$ is a regular homotopy between the immersions $g|F^2$ and $(h|F^2) \circ (\varphi_1|F^2)$ which means by definition that $\text{Im}(g|F^2) \sim \text{Im}(h|F^2)$.\hfill \Box

**Remark 3.6.** Modify Definition 2.2 of regular homotopy the following way:

**Definition 3.7.** Locally generic mappings $f, g : M^2 \to \mathbb{R}^3$ are regularly homotopic through a singularity fixing homotopy – notation $f \sim_s g$ – if $S(f) = S(g)$ and there exists a smooth mapping $H : M^2 \times [0,1] \to \mathbb{R}^3$ such that $H_0 = f$ and $H_1 = g$ and for every $t \in [0,1]$ the mapping $H_t$ is locally generic with $S(H_t) = S(f)$. (That is the singular points are kept fixed.)

This gives a modification of the definition of image-homotopic maps:

**Definition 3.8.** Locally generic mappings $f, g : M^2 \to \mathbb{R}^3$ are image homotopic through a singularity fixing homotopy – notation $\text{Im}(f) \sim_s \text{Im}(g)$ – if $S(f) = S(g)$ and there is a $d : M^2 \to M^2$ diffeomorphism such that $f \circ d \sim_s g$. Note that $d(S(f)) = S(g) = S(f)$ and $d$ can permute the points of $S(f)$.

Suppose that $S(f)$ or $S(g)$ is non-empty. The arguments above show that $\text{Im}(f) \sim_s \text{Im}(g)$ if and only if $|S(f)| = |S(g)|$. To prove this we only have to use diffeomorphisms instead of diffeotopies since Lemma 3.3 remains true using the new definition.

On the other hand $f \sim_s g$ implies that the immersions $f|(M^2 \setminus S(f))$ and $g|(M^2 \setminus S(f))$ are regularly homotopic. But there are locally generic mappings $f, g : M^2 \to \mathbb{R}^3$ satisfying $S(f) = S(g)$ such that $f \sim g$ but $f|(M^2 \setminus S(f)) \sim g|(M^2 \setminus S(f))$. Take for example $M^2 = RP^2$ and choose two arbitrary points $p, q \in RP^2$. Denote $RP^2 \setminus \{ p, q \}$ by $F^2$. Using the notations of Lemma 3.2 we have that $H_1(F^2; \mathbb{Z}_2) = (f_1, c_1)$. Define the cohomology classes $\alpha, \beta \in H^1(F^2; \mathbb{Z}_2)$ by the equalities $\alpha(f_1) = 0, \beta(f_1) = 1$ and $\alpha(c_1) = \beta(c_1) = 1$. Then there exist locally generic mappings $f, g : RP^2 \to \mathbb{R}^3$ satisfying $S(f) = S(g) = \{ p, q \}$ such that $f|F^2$ and $g|F^2$ correspond to $\alpha$ and $\beta$ using the bijection of Theorem 3.4. Denote by $U$ the locally generic mapping of $S^2$ to $\mathbb{R}^3$ with singular points $p$ and $q$ (this is unique up to singularity fixing homotopy). $B$ is the famous Boy surface, $\overline{B}$ is the mirror image of $B$ (see [1]). Then the connected sums $f = B \# U$ and $g = \overline{B} \# U$ satisfy the above conditions. Clearly $f|F^2 \sim g|F^2$, thus $f \sim_s g$. This provides examples of locally generic mappings $f$ and $g$ such that $\text{Im}(f) \sim_s \text{Im}(g)$, but $f \sim_s g$. 
Definition 4.3. Two immersions $F,G \in \text{Imm}(M^2, \mathbb{R}^4)$ are $\pi$-homotopic (denoted by $F \sim_{\pi} G$) if and only if $e(F) = e(G)$, where $e(F)$ denotes the (twisted) Euler-number of the normal bundle of the immersion $F$. It is clear that if $F \sim_{\pi} G$ then $e(F) = e(G)$ and $\vert S(\pi \circ F) \vert = \vert S(\pi \circ G) \vert$. We are going to prove that if $S(\pi \circ F)$ (and $S(\pi \circ G)$) are non-empty then the converse also holds.

First let us recall that two immersions $F,G \in \text{Imm}(M^2, \mathbb{R}^4)$ are regularly homotopic if and only if $e(F) = e(G)$, where $e(F)$ denotes the (twisted) Euler-number of the normal bundle of the immersion $F$. It is clear that if $F \sim_G G$ then $e(F) = e(G)$ and $\vert S(\pi \circ F) \vert = \vert S(\pi \circ G) \vert$. We are going to prove that if $S(\pi \circ F)$ (and $S(\pi \circ G)$) are non-empty then the converse also holds.

Suppose that $F,G: M^2 \to \mathbb{R}^4$ are immersions. From Proposition 4.2 it is clear that for almost every projection $\pi \in \text{Proj}(\mathbb{R}^4, \mathbb{R}^3)$ both $f = \pi \circ F$ and $g = \pi \circ G$ are locally generic. Fix such a projection $\pi$.

We are going to define the sign of every point in $S(f)$ (and in $S(g)$). (An equivalent definition can be found in [4]). Take a cross-cap point $p \in S(f)$. Choose orientations of $\mathbb{R}^4$ and of $\mathbb{R}^3$. We will define the sign of $p$ as follows.

Fix local coordinates $(x_1, x_2)$ on a neighborhood $U_p$ of $p$ and $(y_1, y_2, y_3)$ centered at $f(p)$ such that $f$ has the following normal form:

$$y_1 \circ f = x_1^2, \quad y_2 \circ f = x_2, \quad y_3 \circ f = x_1x_2.$$  

Suppose that $U_p$ is so small that $F|U_p$ is an embedding. The sign of $p$ will depend only on $F|U_p$ (thus the definition is local). Set $D_\varepsilon = \{y_1^2 + y_2^2 + y_3^2 \leq \varepsilon \} \subset \mathbb{R}^3$ and choose $\varepsilon > 0$ sufficiently small such that $\bar{D}_\varepsilon = f^{-1}(D_\varepsilon) \subset U_p$. The set $\bar{D}_\varepsilon$ is a closed disc neighborhood of $p$ in $M^2$ and $\partial \bar{D}_\varepsilon = f^{-1}(\partial D_\varepsilon)$ (see Lemma 2.2 in [4]). Let $L$ be the closure of the double point set of $f|\bar{D}_\varepsilon$, i.e. $L = \{x \in \bar{D}_\varepsilon : f^{-1}(f(x)) \cap \bar{D}_\varepsilon \neq \{x\} \cup \{p\}$. Then $L$ is a one-dimensional smooth submanifold of $\bar{D}_\varepsilon$ and $L \cap \partial \bar{D}_\varepsilon$ consists of two points $p_1$ and $p_2$. We fix an orientation of $\partial \bar{D}_\varepsilon$.
and take an oriented base \((u)\) (resp. \((v)\)) of the tangent space \(T_p(\partial \mathcal{D}_e)\) (resp. \(T_p(\partial \mathcal{D}_c)\)). Then \(\langle df_p(u), df_p(v)\rangle\) is a base of \(T_p(\partial \mathcal{D}_e)\), where \(q = f(p_1) = f(p_2)\). We may assume that \(\langle df_p(u), df_p(v), \xi\rangle\) is a positive basis of \(T_{\partial \mathcal{D}_c} \mathbb{R}^3\), where \(\xi\) is the outward normal vector of \(\partial \mathcal{D}_c\), exchanging \(p_1\) and \(p_2\) if necessary. Now orient \(L\) from \(p_2\) to \(p_1\).

Denote by \(\nu\) a positive basis vector of \(T_pL\). (If \(M^2\) possess a Riemannian metric, then choose \(\nu\) to be a unit-vector. This way \(\nu\) is unique up to the orientation of \(\mathbb{R}^3\).) Orient \(\text{ker} \pi\) in such a way that together with the orientation of \(\text{Im} \pi = \mathbb{R}^3\) we obtain the fixed orientation of \(\mathbb{R}^4 = \text{Im} \pi \oplus \text{ker} \pi\). Using this direct sum decomposition of \(\mathbb{R}^4\) the mapping \(F: M^2 \to \mathbb{R}^4\) can be written in the form \(F = (\pi \circ F, F^\ast) = (f, F^\ast)\).

Since \(F\) is an immersion at the point \(p \in S(f)\), we have \(dF^\ast(\nu) \neq 0\). After all this preparation we can now define the sign of \(p\).

**Definition 4.4.** The cross-cap point \(p\) is positive if \(dF^\ast(\nu) > 0\) and is negative if \(dF^\ast(\nu) < 0\). We denote by \(p(F)\) (resp. \(n(F)\)) the number of positive (resp. negative) cross-cap singularities of the locally generic mapping \(F = \pi \circ f: M^2 \to \mathbb{R}^3\).

The following proposition is a special case of Proposition 2.5 in [5].

**Proposition 4.5.** Suppose that \(\mathbb{R}^4\) and \(\mathbb{R}^3\) are oriented. Then we always have \(e(F) = p(F) - n(F)\), where \(e(F) \in \mathbb{Z}\) is the (twisted) Euler-number of the normal bundle of the immersion \(F\).

Thus the immersions \(F\) and \(G\) are regularly homotopic if and only if \(p(F) - n(F) = e(F) = e(G) = p(G) - n(G)\).

On the other hand Theorem 2.4 states that if \(S(f)\) and \(S(g)\) are non-empty then \(f \sim g\) if and only if
\[
P(F) + n(F) = |S(f)| = |S(g)| = p(G) + n(G).
\]
Comparing the preceding two chains of equations we have that if both \(f\) and \(g\) are singular then \([F \sim G \text{ and } f \sim g] \Leftrightarrow [p(F) = p(G) \text{ and } n(F) = n(G)]\). The following theorem implies that in this case we can even find a regular homotopy between \(F\) and \(G\) whose projection is a regular homotopy between \(f\) and \(g\), i.e. \(F \sim_G G\).

**Theorem 4.6.** Suppose that \(M^2\) is a closed connected surface, \(F, G: M^2 \to \mathbb{R}^4\) are immersions and \(\pi: \mathbb{R}^4 \to \mathbb{R}^3\) is a projection such that \(f = \pi \circ F\) and \(g = \pi \circ G\) are both locally generic and singular. Then the following are equivalent:

1. There exists a regular homotopy \(H: M^2 \times [0, 1] \to \mathbb{R}^4\) between \(F\) and \(G\) such that \(\tilde{H} = \pi \circ H\) is a regular homotopy between \(F\) and \(G\), i.e. \(F \sim_G G\).

2. The numbers of positive and negative cross-caps of \(f\) and \(g\) are the same, i.e. \(p(F) = p(G)\) and \(n(F) = n(G)\).

**Proof.** First we prove the implication (1) \(\Rightarrow\) (2). In this case \(f \sim g\), thus using Proposition 2.4 we have that \(|S(f)| = |S(g)|\). From Definition 4.4 it is clear that the signs of the singular points do not change during a regular homotopy. (Here we did not use the assumption that \(|S(f)| > 0, |S(g)| > 0\).

Now we are going to prove the implication (2) \(\Rightarrow\) (1). Since \(p(F) = p(G)\) and \(n(F) = n(G)\) there exists a bijection \(i: S(f) \to S(g)\) that preserves the signs of the
cross-cap points. By Theorem 2.6 there is a regular homotopy \( \tilde{H} \) between \( f \) and \( g \) such that \( \pi \circ H = \tilde{H} \) as follows: Choose an arbitrary Riemannian metric on \( M^2 \). In the paragraph preceding Definition 4.4 we saw that in this case for every \( t \in [0, 1] \) and \( p \in S(\tilde{H}_t) \) there is a unique positive unit-vector \( \nu_t(p) \in T_{p}M^2 \) tangent to the double-point curve \( L \) crossing \( p \). The singular sets \( S(\tilde{H}_t) \) for \( t \in [0, 1] \) define curves \( \gamma_1, \ldots, \gamma_k \) on \( M^2 \) such that \( S(\tilde{H}_t) = \{ \gamma_j(t) : 1 \leq j \leq k \} \) for every \( t \in [0, 1] \). The points \( \gamma_i(0) \) and \( \gamma_i(1) \) have the same sign (1 \( \leq i \leq k \)). Suppose for example that for a fix \( i \) both \( \gamma_i(0) \) and \( \gamma_i(1) \) are positive cross-cap points. Introduce the notation \( \nu_t(\gamma_i(t)) \), then \( dF^*(\nu_t(0)) > 0 \) and \( dG^*(\nu_t(1)) > 0 \). (Here \( F^* \) denotes the fourth coordinate function of \( F \) in \( \mathbb{R}^4 \).) Using the Levi-Civita connection of the Riemannian manifold \( M^2 \) we may consider the exponential mapping on \( M^2 \). Since \([0, 1] \) is compact, there exists \( \varepsilon > 0 \) such that for every \( t \in [0, 1] \) the mapping \( h : [-\varepsilon, \varepsilon] \times [0, 1] \rightarrow M^2 \) satisfying
\[
h(s, t) = \exp_{\gamma_i(t)}(s \cdot \nu_t(t))
\]
is defined and for every \( t \in [0, 1] \) the mapping \( h_t(s) = h(s, t) \) is an embedding of \([-\varepsilon, \varepsilon]\) into \( M^2 \) \((h_t \) is a geodetic curve). Define the function \( H_t^* \) on \( \text{Im} \, h_t \) using the following formula:
\[
H_t^*(h_t(s)) = (1 - t) \cdot F^*(h_0(s)) + t \cdot G^*(h_1(s))
\]
for \( s \in [-\varepsilon, \varepsilon] \). Note that \( H_t^* \mid \text{Im} \, h_0 = F^* \mid \text{Im} \, h_0 \) and \( H_t^* \mid \text{Im} \, h_1 = G^* \mid \text{Im} \, h_1 \). Thus we can extend \( H^* \) to an open neighborhood of \( \cup_{t \in [0, 1]} (\text{Im} \, h_t \times \{t\}) \) in \( M^2 \times [0, 1] \) as a smooth function. From the construction of \( H_t^* \) it is clear that
\[
dH_t^*(\nu_t(t)) = (1 - t) \cdot dF^*(\nu_t(0)) + t \cdot dG^*(\nu_t(1)) > 0,
\]
which implies that the mapping \( H_t = (\tilde{H}_t, H_t^*) \) is an immersion at the point \( \gamma_i(t) \) for every \( t \in [0, 1] \). Repeat the preceding extension process for every \( 1 \leq i \leq k \) and afterward extend the obtained \( H^* \) to the whole cylinder \( M^2 \times [0, 1] \) in such a way that \( H^*_0 = F^* \) and \( H^*_1 = G^* \). The mapping \( H = (\tilde{H}, H^*) \) is a regular homotopy connecting \( F \) and \( G \) whose projection is \( \tilde{H} \).

Putting together our previous results we obtain the following theorem:

**Theorem 4.7.** If \( F, G \in \text{Imm}_\pi(M^2, \mathbb{R}^4) \) then
\[
F \sim_\pi G \iff [F \sim G \text{ and } \pi \circ F \sim \pi \circ G].
\]

**Proof.** First we suppose that \( S(\pi \circ F) \) or \( S(\pi \circ G) \) is non-empty. Theorem 4.7 states that \( F \sim_\pi G \iff [p(F) = p(G) \text{ and } n(F) = n(G)] \). We have seen in the paragraph preceding Theorem 4.7 that \( [p(F) = p(G) \text{ and } n(F) = n(G)] \iff [F \sim G \text{ and } \pi \circ F \sim \pi \circ G] \).

Now we consider the case when both \( S(\pi \circ F) \) and \( S(\pi \circ G) \) are empty, i.e. \( f = \pi \circ F \) and \( g = \pi \circ G \) are immersions. If \( f \sim g \) then any regular homotopy connecting \( f \) and \( g \) can be lifted to a regular homotopy between \( F \) and \( G \), thus \( F \sim_\pi G \). This proves the implication \( F \sim_\pi G \iff [F \sim G \text{ and } \pi \circ F \sim \pi \circ G] \). The other implication is trivial. \( \square \)
References

1. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag New York Inc., 1973.
2. M. W. Hirsch, *Immersion of manifolds*, Trans. Am. Math. Soc. 93 (1959), 242–276.
3. J. Mather, *Generic projections*, Ann. of Math. 98 (1973), 226–245.
4. U. Pinkall, *Regular homotopy classes of immersed surfaces*, Topology 24 (1985), 421–434.
5. O. Saeki and K. Sakuma, *Immersed n-manifolds in $\mathbb{R}^{2n}$ and the double points of their generic projections into $\mathbb{R}^{2n-1}$*, Trans. Amer. Math. Soc. 348 (1996), 2585–2606.
6. S. Smale, *A classification of immersions of the two sphere*, Trans. Am. Math. Soc. 90 (1958), 281–290.
7. J. Stillwell, *Topology and combinatorial group theory*, Springer-Verlag New York Inc., 1980.
8. H. Whitney, *The singularities of a smooth n-manifold in $(2n - 1)$-space*, Ann. of Math. 45 (1944), 247–293.

Department of Analysis, Eötvös Loránd University, Pázmány Péter sétány 1/C, Budapest, Hungary 1117

E-mail address: juhasz.6@dpg.hu