Quantum geometry and stability of the fractional quantum Hall effect in the Hofstadter model

David Bauer,1 T. S. Jackson,1 and Rahul Roy1

1Department of Physics and Astronomy, University of California at Los Angeles, 475 Portola Plaza, Los Angeles, California 90095, USA
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We investigate conditions influencing the stability of fractionalized phases in topologically non-trivial lattice systems (“fractional Chern insulators” or FCIs) in the specific context of the Hofstadter model, which offers a well-defined limit (that of vanishing magnetic flux $\phi = \phi_0/N$ through a fundamental lattice plaquette) in which the single-particle Hamiltonian converges to one describing continuum Landau levels. Investigations of other FCI models have focused largely on the effects of band dispersion and nonuniform Berry curvature, but in the Hofstadter model these vanish exponentially in $N$. For $N \gtrsim 15$, we instead find that a geometric criterion related to the trace of the quantum metric is the dominant factor influencing the stability of fractionalized phases. We show that this criterion converges to its continuum limit much more slowly, as a polynomial in $1/N$, and present numerical simulations showing that the many-body gap depends on it monotonically. This correlation holds across several FCI states with different topological orders.

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The possibility of realizing fractional quantum Hall (FQH) phases in topologically non-trivial lattice systems with broken time-reversal symmetry has attracted much theoretical interest as a potential route to the realization of exotic FQH phenomena without the experimental requirement of a large external magnetic field; recent progress is reviewed in [1, 2]. The majority of our theoretical understanding of the fractional quantum Hall effect (FQHE) (e.g., its stability to excitations in the single-mode approximation [3]) has been framed in the context of continuum Landau levels, which occupy a special and highly atypical point in the space of all single-particle Hamiltonians. It is a matter of both theoretical and experimental interest to identify which of the special properties of Landau levels (LLs), which offer optimal band-geometric conditions for FQH states, are essential to the stability of the FQHE.

The connection between band topology and Hall conductance is well-known, but recent literature has investigated the role of band geometry in FQH states. Geometric (i.e., non-topological) properties of bands enter when one considers the algebra of band-projected density operators [4–8], following the continuum calculation done in [3]; in particular, the Fubini-Study metric on the band Hilbert space of bands arises naturally when one considers a long-wavelength expansion of this algebra [8].

The Hofstadter model [9] provides an ideal setting for studying the interplay between band geometry and FQH physics, due to the fact that it offers a controlled limit (that of vanishing flux per plaquette) which converges on continuum LLs, which offer optimal band-geometric conditions for FQH states (in a sense to be made precise below). Most single-particle quantities of interest (in particular, measures of band geometry defined below) may be computed analytically in this limit. A considerable literature on the FQHE in the Hofstadter model exists [10–17]; we note that, although the Hofstadter model involves a net magnetic flux, numerical studies [18, 19] have shown FQH states to be adiabatically connected to those in models with no net flux (“fractional Chern insulators” or FCIs). Hofstadter bands have been experimentally realized using cold atoms in optical lattices [20–22] and graphene superlattices [23, 24].

In the present work, we investigate conditions proposed in [8] for the stability of FQH phases in the context of the Hofstadter model. We show that momentum-space fluctuations of the Berry curvature and quantum metric have exponential convergence to their LL limit, while a third, nontrivial condition related to the trace of the quantum metric has slower, polynomial convergence. Numerical exact diagonalization studies of several different FQH states show that the many-body gap is monotonically dependent on this third criterion, even when momentum-space fluctuations are negligible.

Geometry of Chern bands.— In momentum space, we write the band Hamiltonian of a lattice model as $H(k) = \sum_{\alpha=1}^{N} E_{\alpha}(k)|\mathbf{k},\alpha\rangle\langle \mathbf{k},\alpha|$, where $\alpha = 1, \ldots, N$ indexes the bands. A band is parameterized by its Bloch function $u_\alpha^\mathbf{k}(k)$, an $N$-component vector in terms of which $|\mathbf{k},\alpha\rangle = \sum_{\mathbf{b}=1}^{N} u_b^\alpha(k)|\mathbf{b}\rangle$, where $|\mathbf{b}\rangle$ is the Fourier transform of the $\mathbf{b}$th basis orbital. Topological order in a band is indicated by nonzero values of a corresponding topological invariant, which for two-dimensional systems with broken time-reversal symmetry is the first Chern number

$$c_1 = A_{BZ}(B_\alpha)/2\pi,$$

where $A_{BZ}$ is the area of the Brillouin zone (BZ), $\langle \cdots \rangle$ denotes the average over the BZ, normalized so $\langle 1 \rangle = 1$, and the Berry curvature [25, 26] of the band $\alpha$ in terms of Bloch functions is

$$B_\alpha(k) = -i \sum_{b=1}^{N} \left( \frac{\partial u_b^\alpha}{\partial k_x} \frac{\partial u_b^\alpha}{\partial k_y} - \frac{\partial u_b^\alpha}{\partial k_y} \frac{\partial u_b^\alpha}{\partial k_x} \right).$$

One may also define a “quantum metric” on the BZ by introducing the Fubini-Study metric on the band Hilbert space [27] and using $u_b^\alpha(k)$ to pull it back to the BZ, yielding
Observables of Berry curvature are reviewed in [28]; the quantum metric is experimentally accessible through current noise measurements [29].

In the present work, we consider geometric conditions for the stability of FCIs involving $B_\alpha$ and $g^\alpha$ which were identified in [8], refining arguments made in [1, 6]. These are that (i) the Berry curvature (2) is constant over the BZ, (ii) all components of the quantum metric (3) are constant over the BZ, and (iii) the following quantity

\[ D(k) \equiv \det g^\alpha(k) - \frac{B_\alpha(k)^2}{4} \geq 0 \]  

vanishes everywhere, i.e. the inequality (4) is saturated. In [8] a similar inequality involving the trace of the metric

\[ T(k) \equiv \text{tr} g^\alpha(k) - |B_\alpha(k)| \geq 0 \]  

was proved, and it was shown that $T(k) = 0$ automatically implies $D(k) = 0$.

**Hofstadter model at small flux.** — The quantum dynamics of an electron on a two-dimensional square lattice in the presence of a transverse magnetic field was first studied by Harper [30], and its self-similar band structure was further elucidated by Azbel [31]. With these caveats, for brevity we refer to the following model as the Hofstadter model [9]:

\[ H = -i \sum_{ij} c_i^\dagger c_j \exp \left(2\pi i \int_{r_i}^{r_j} A \cdot d\ell \right) + \text{h.c.} \, , \]

with $i, j$ indexing sites on the square lattice. In the Landau gauge, $A = Bx \hat{y}$ and all eigenstates have trivial $y$ dependence $\psi(x = na, y) = e^{i k_y y} \psi_n$, with the remaining $x$ dependence obeying the discrete Harper equation

\[ \psi_{n+1} + \psi_{n-1} + 2 \cos(2\pi \phi n - k_y) \psi_n = \varepsilon \psi_n, \]  

where $\varepsilon$ is the energy in units of $t$ and $\phi$ is the magnetic flux per lattice cell in units of the flux quantum $\phi_0 = \hbar/e$. For rational $\phi = p/q$, (7) is invariant under $n \to n + q$ and one may apply Bloch’s theorem over the enlarged magnetic unit cell consisting of $q \times 1$ lattice plaquettes.

In [17], the Hofstadter model was studied perturbatively for $\phi$ near rational values. We recall some results relevant to the case of interest, $\phi = 1/N$. In the large-$N$ limit, the Harper equation (7) may be approximated by a second-order differential equation in $x = n/N$ which permits a WKB approximation. To lowest order, this reproduces the Schrödinger equation for Landau levels in the Landau gauge, with lattice effects manifesting at higher orders in $N$. The unitary operator which transforms continuum Landau levels into Hofstadter Bloch eigenfunctions is

\[ U^\dagger = \exp \left[ \frac{1}{96N} \left( \frac{1}{\pi} + \frac{\pi^2}{128N^2} \right) (a^{\dagger 4} - a^4) \right. \]

\[ \left. + \frac{1}{320N^2} (a^{\dagger 5} - a^4 a^5) + O \left( \frac{1}{N^3} \right) \right], \]

where $a, a^\dagger$ are the bosonic raising and lowering operators for Landau levels. The lattice potential breaks the full rotational symmetry of the continuum problem, but angular momentum non-conserving terms are exponentially small in $N$, as are gauge-dependent terms (see Sec. VI of [17]).

In [17], the WKB wavefunctions were used to compute the leading-order corrections to uniform Berry curvature as

\[ B(k) = \frac{N}{2\pi} + c N^2 e^{-\tilde{\sigma}N} \left[ \cos(Nk_x) + \cos(Nk_y) \right], \]

where $\tilde{\sigma}, c$ are nonessential dimensionless constants with numerical values of $O(1)$ reflecting details of the perturbation scheme. The remarkable feature of (9) is that Berry curvature fluctuations vanish exponentially fast in $N$, meaning that the band geometry of the Hofstadter model converges rapidly to the LL limit. As we will now show, similar behavior characterizes the quantum metric.

**Perturbative quantum metric.** — Computation of the quantum metric from the WKB wavefunctions proceeds analogously to the computation done for the Berry curvature in Appendix A of [17]. We find the fluctuation $\tilde{g}_{xx} = g_{xx}(k_x) - g_{xx}(0)$ to be

\[ \tilde{g}_{xx} \equiv \frac{2P^2}{\text{Erf}(\alpha)} \sqrt{\pi e} N \left( \frac{5}{6} - \frac{2\alpha}{\sqrt{2\pi N}} + \frac{\alpha^2}{\pi N} \right) \times e^{-\tilde{\sigma}N} \cos(Nk_x), \]

where $\alpha, P$ are other dimensionless constants of $O(1)$ [32]. As with the Berry curvature and dispersion, $k$ dependent terms in the quantum metric are exponentially small in $N$. We encounter the same issue faced in [17], namely that the Landau gauge wavefunctions give no information about fluctuations in $k_y$. Lattice symmetries constrain the full form of (10) to depend on some combination of $\cos(Nk_x) + \cos(Nk_y)$ and $\cos(Nk_x) \cos(Nk_y)$ [32]. While the Berry curvature depends only on the former, quantities involving the metric depend on both; numerically we find that (10) provides an accurate bound on the amplitude of fluctuations, regardless of their $k$ dependence (see Fig. 1 (a) and Supplemental Material [32]).

Because the metric inequalities (4) and (5) are integrated over the BZ, they may be computed more economically by using (8) to generate perturbative eigenstates. We recall that states in the LL problem are spanned by two commuting free boson algebras generated by $a, a^\dagger, b, b^\dagger$, where the first pair act as ladder operators for the LLs (as in (8)) and the second pair raise and lower angular momentum within a LL. In terms
of these, the complexified position operators are $\tilde{x} = x + i\tilde{y} = \sqrt{2i(a - ib)}$, $\tilde{z} = \tilde{x} - i\tilde{y} = -\sqrt{2i(a^* + ib)}$. Components of the quantum metric may then be computed as

$$g_{ij} = \text{tr} \left[ \tilde{P}_0 \tilde{r}_i (1 - \tilde{P}_0) \tilde{r}_j \tilde{P}_0 \right],$$

(11)

where $|\tilde{0}\rangle = U\uparrow |0\rangle$ is the perturbative ground state generated by (8). Note that (11) is equivalent to its momentum space formulation, (3). We obtain

$$\langle g_{xx} \rangle = \langle g_{yy} \rangle = 1 + \frac{1}{96} \frac{\pi^2}{N^2} + \frac{36 - \sqrt{6}}{211} \frac{\pi^4}{N^4} + \frac{298 - 27\sqrt{6}}{214} \frac{\pi^4}{N^4} + O\left(\frac{1}{N^5}\right),$$

(12)

and

$$\langle g_{xy} \rangle = 0 + O(1/N^5).$$

Although the unitary operator which generates the ground state is only known to $O(1/N^2)$, contributions from higher order terms in $U\uparrow$ only enter the above expressions at $O(1/N^5)$ and above [32]. Because the BZ average of the curvature is always exactly quantized to the Chern number and the RMS fluctuations of all quantities are exponentially small, we find

$$\langle T \rangle = 2 \langle g_{xx} \rangle - 1;$$

(13)

$$\langle D \rangle = \langle g_{xx} \rangle^2 - 1/4.$$  

(14)

Unlike the momentum-space fluctuations of the metric components and Berry curvature, the trace and determinant inequalities have a much slower asymptotic decay which is polynomial in $1/N$. Therefore, at large $N$ these conditions should be the dominant factors dictating the stability of FQH-like phases. In practice, this regime occurs for $N$ larger than $\sim 10$: as shown in Fig. 1 (b), the numerically computed BZ integrals of these quantities converge rapidly to their asymptotic values.

**FIG. 1.** (color online). (a) RMS fluctuation over the BZ of the trace of the quantum metric (points) versus the asymptotic expansion (10) (blue line). (b) Numerical integral over the BZ of the trace inequality (5) (points) versus the expansion (13) (blue line), as a function of $N$. The thin gray line shows the expansion from panel (a) on the same scale. Plots of other single-particle quantities are similar and given in the Supplemental Material [32].

**Many-body computations.**— We investigate the degree to which the above band-geometric criteria influence the stability of an FCI state by adding repulsive interactions and numerically computing the many-body gap for different values of $\phi = 1/N$. Lowering the flux per plaquette changes the relative strength of the interaction, since the size of the system increases while the number of particles remains constant. In order to compare gaps at different $N$, the strength of the interaction potential must be scaled with $N$ as follows.

In the spirit of the WKB analysis, let $\psi_n, n = 1, \ldots, N$ be a wavefunction defined on the lattice and $\psi(x)$ be its continuum approximation in the $N \gg 1$ limit, with $x = n/N$. Under the (nontrivial) assumption that $\psi_n$ has support over almost all tight-binding orbitals, the requirement that both $\psi_n$ and $\psi(x)$ be normalized to unity dictates the scaling $\psi_n \sim \sqrt{N}\psi(x)$. Consideration of the matrix elements of a two-body $\delta$-function interaction $V(x) = V_0 \delta(x)$ shows that

$$\langle \psi^3, \psi^4 | \hat{V} | \psi^3, \psi^2 \rangle = V_0 \int dx \psi^3(x)^* \psi^4(x)^* \psi^3(x) \psi^2(x)$$

$$\sim NV_0 \sum_n \psi_n^4 \psi_n^3 \psi_n^2 \psi_n^2,$$

so that the strength of the corresponding discrete on-site interaction should scale as $V_{\text{disc}} \sim V_0 N^2$. Similar considerations show that for the two-body nearest-neighbor repulsion used to stabilize the fermionic Laughlin state and for the three-body delta-function interaction stabilizing the Moore-Read state, the leading scaling should be $V_{\text{disc}} \sim V_0 N^2$.

In order to study the influence of band geometry on FCI phases, we carried out exact numerical diagonalization of a repulsive interaction Hamiltonian projected to the lowest Hofstadter band for $N_p = 8$ particles partially filling the lowest band. To study the bosonic Laughlin state at $\nu = 1/2$, we used a lattice of $4 \times 4$ unit cells and a two-body on-site repulsion; for the fermionic Laughlin state at $\nu = 1/3$ we used a $3 \times 6$ lattice with a two-body nearest-neighbor repulsion, and for the bosonic Moore-Read state at $\nu = 1$ we used a $1 \times 8$ lattice with a three-body on-site repulsion. For geometric reasons, we used system sizes of the form $N = m^2$ up to $N = 169$ for the bosonic Laughlin state, $N = 6m^2$ up to $N = 294$ for the fermionic Laughlin state, and $N = 2m^2$ up to $N = 162$ for the bosonic Moore-Read state.

For each system, we verified that the many-body ground state had nontrivial topological order corresponding to the appropriate FQH state via properties of energy and entanglement spectra. We required that the many-body spectrum exhibit a quasidegenerate ground state, with the gap $\Delta$ to excited states larger than the spread in ground-state energies. We compute the particle entanglement spectrum by tracing out four particles from the density matrix formed by an equal superposition of all ground states, and required that this spectrum be gapped, with the counting of eigenvalues below the gap in each momentum sector given by the counting rules derived in [33, 34] for the corresponding FQH state.

In Fig. 2 we plot the scaled gaps as a function of $\langle T \rangle$, the Brillouin zone average of the trace inequality (5). We note a clear correlation between an increasing value of the gap and increasing saturation of the inequality, with the gap continuing to increase even for values of $N$ for which BZ fluctuations of
curvature are negligible. Dependence of the scaled gaps on \( \langle D \rangle \) is similar [32].

![Graphs of scaled many-body gaps as a function of BZ-averaged trace inequality (5) for FQH-like states at various values of \( N \). Insets show behavior of points at larger values of \( N \). (a) Laughlin state of \( N_p = 8 \) bosons at \( \nu = 1/2 \). (b) Laughlin state of \( N_p = 8 \) fermions at \( \nu = 1/3 \). (c) Moore-Read state of \( N_p = 8 \) bosons at \( \nu = 1 \).](image)

**Discussion.**— The Hofstadter model provides an ideal laboratory for studying FCI phenomena: the existence of a controlled limit in which its spectrum converges to continuum LLs, which makes the model amenable to perturbative expansion in \( 1/N \) and allows a controlled study of the relationship between band geometry and FQHE physics. In this paper, we have shown that there is a natural distinction between effects that are nonperturbatively small in \( 1/N \), such as the BZ fluctuations of the Berry curvature and quantum metric, and effects which have a perturbative expansion in \( 1/N \), such as the trace and determinant conditions. The behavior of the many-body gaps obtained via exact diagonalization indicates that the latter effects dominate for large \( N \). In the future, it would be interesting to study models where the role of the determinant inequality can be isolated.

The Hofstadter model has been realized experimentally in optical lattices of cold atoms [20–22] and graphene superlattices [23, 24]. The geometrical criteria for the stability of Chern bands are readily computable single-particle properties that act as a meaningful proxy for the many-body gap, and hence may serve as a useful guide in selecting experimental parameters and couplings which are most favorable to the existence of a stable FQH state. In this sense, the present work complements [35], in which the effects of band geometry on the many-body gap were measured by varying the couplings of several FCI models. In that study, BZ fluctuations of the Berry curvature were found to have the largest influence on the gap, with the trace condition a subdominant factor.

In this paper we have presented a perturbative treatment of the band geometry of the Hofstadter model at \( \phi = 1/N \) in the limit \( N \to \infty \), but our analysis may be extended to the most general case considered in [17], namely \( \phi = P/Q + M/N \) with \( N \gg M \). Our own single- and many-particle computations suggest that the case \( \phi = M/N \) is qualitatively similar to the results discussed above, with all quantities taking values which interpolate among those obtained at \( M = 1 \). Different physics is encountered when perturbing around nonzero \( P/Q \), since the lowest band now has Chern number \( c_1 = Q > 1 \). FCI states in bands with \( c_1 > 1 \) may be mapped onto multilayer FQH states [36], albeit with significant distinctions [37–39]. The specific case of the Hofstadter model (and generalizations) at \( \phi = P/Q \) has been examined in [16, 40–42].

One could also extend our analysis to excited bands of the Hofstadter model, corresponding to higher LLs. Finally, obtaining an analytic formula for the quantitative dependence of the gap (or other many-body observables) on band-geometric parameters remains an important open problem.

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Metric oscillations

In this section we give details on the calculation of the asymptotic $N$ dependence of the amplitude of BZ fluctuations of metric components, using the WKB approximation to the Bloch wavefunction of the lowest Hofstadter band [17, 43].

As explained in the main text, we approximate the discrete Harper equation by a second-order differential equation, which we then solve with a WKB expansion in the small parameter $1/N$. The WKB solution is unphysical in the region of the turning points, and in the standard WKB treatment the problem is treated by linearizing the effective cosine potential in this region and matching the WKB solution at the boundaries. In our case, the large-$N$ limit corresponds to the situation in which the turning points collapse toward the minimum of the potential to form a single second-order turning point. We use a quadratic approximation of the potential in this region, and use the WKB solution only in the classically forbidden region.

We follow the discussion and notation of Appendix A of [17], providing more details where necessary. Introducing the continuous variable $x = n/N$, the Harper equation in the classically forbidden region is

$$\psi\left(x + \frac{1}{N}\right) + \psi\left(x - \frac{1}{N}\right) = 2 \cosh[\tilde{p}(x)] \psi(x), \quad (15)$$

with $\cosh \tilde{p}(x) = -\varepsilon/2 - \cos(2\pi x)$. Here we have chosen $k_y = 0$, since it becomes equivalent to translations in $x$ in this gauge. With the equation in this form, we can apply the WKB ansatz $\psi(x) = \exp[N S_0(x) + S_1(x)]$. Expanding to first order in $1/N$ and comparing coefficients yields

$$S_0'(x) = \pm \tilde{p}(x) \Rightarrow S_0(x) = \pm \int_{y_0}^{x} \tilde{p}(y) \, dy;$$

$$S_1(x) = -\frac{1}{2} \log \left[ \sinh \tilde{p}(x) \right],$$

or

$$\psi_{\exp}^{\pm}(x) = \frac{1}{\sqrt{\sinh \tilde{p}(x)}} \exp \left[ \pm N \int_{y_0}^{x} \tilde{p}(y) \, dy \right], \quad (16)$$

where $\psi_{\exp}$ denotes the exponentially decaying part of the WKB solution. Because the turning points coincide in the large-$N$ limit, the region corresponding to the oscillatory solution vanishes.

Turning points occur at $\beta = (1/2\pi) \cos^{-1}(-\varepsilon/2 - 1)$ with $\beta \approx 1/\sqrt{2\pi N}$ for $N \gg 1$. In the region near the turning points, we replace the difference equation (15) with a differential equation

$$\frac{1}{N^2} \psi''(x) + \left[ (\varepsilon + 2) + 2 \cos(2\pi x) \right] \psi(x) = 0. \quad (17)$$

Expanding the cosine around $x = 0$ and introducing rescaled variables $a = \frac{N}{\pi} (\varepsilon + 4)$ and $\xi(x) = 2\sqrt{\pi N x}$ yields

$$\psi''(\xi) - \left[ \frac{1}{4} \xi^2 + a \right] \psi(\xi) = 0, \quad (18)$$

the solutions of which are parabolic cylinder functions $U(a, \xi), V(a, \xi)$. The limit $N \to \infty$ corresponds to $a \to -1/2$, due to the fact that $\varepsilon \sim -4 - 2\pi N + \ldots$; in this limit, (18) reduces to the Schrödinger equation for a quantum harmonic oscillator in its ground state, i.e. the lowest Landau level.

We define the Bloch wavefunction piecewise in three regions of the unit cell, with

$$u_k(x) = \begin{cases} 
AU(a, x) + BV(a, x), & 0 \leq x < \alpha \beta, \\
C\psi_{\exp}^+(x) + D\psi_{\exp}^-(x), & \alpha \beta \leq x < 1 - \alpha \beta, \\
FU(a, x) + GV(a, x), & 1 - \alpha \beta \leq x < 1.
\end{cases} \quad (19)$$

The second region corresponds to the classically forbidden region, and the first and third regions are in the vicinity of the second order classical turning point (near integer $x$). There is some freedom in choosing the width of the WKB region, expressed via the free parameter $\alpha$. We note that, as explained in [17], the choice of this value has little effect on behavior of the final result for large enough values of $\alpha$, and we follow the authors of that paper in setting $\alpha = 2.3$.

Matching of the wavefunctions at the boundaries of the three regions is done using the limiting forms of the WKB and parabolic cylinder solutions and the reflection relation

$$\begin{pmatrix} U(a, -\xi) \\ V(a, -\xi) \end{pmatrix} = \begin{pmatrix} -\sin(\pi a) & \pi \frac{\sin(\pi a)}{\Gamma(\frac{3}{2}-a)} \\ \cos(\pi a) & \Gamma(\frac{3}{2}-a) \end{pmatrix} \begin{pmatrix} U(a, \xi) \\ V(a, \xi) \end{pmatrix}. \quad (20)$$

The resulting Bloch wavefunction is
with a an overall normalization factor and
\[ \tilde{\sigma} = \int_{\beta}^{1-\beta} \cosh^{-1} \left( \frac{-\varepsilon}{2} - \cos(2\pi t) \right) dt, \] (20)
which depends implicitly on \( N \) through both \( \beta \) and \( \varepsilon \). Using Simpson’s rule and integrating numerically, we obtain the approximate \( N \) dependence as
\[ \tilde{\sigma} \approx 1.166 - \frac{0.208}{\sqrt{N}} - \frac{2.227}{N} + \ldots \] (21)
Finally, note that in (19) the \( k_y \) dependence has been inserted by hand, under the assumption of continuous translational dependence. As a consequence, we cannot recover any information about fluctuations in the \( k_y \) direction and must appeal to symmetry arguments to reconstruct the full functional dependence from the \( k_x \) behavior. This is the same difficulty that was encountered in [17] for the dispersion and Berry curvature.

Having obtained the wavefunctions, we can take derivatives and calculate eq. (3) for the component \( g_{xx} \) of the metric, with the inner product over the band index replaced by an integral over \( x \). We make the assumption (made in [17] and justified by numerical evidence) that the dominant oscillatory contribution to the metric comes from the WKB piece of the wavefunction, as the parabolic cylinder solution roughly corresponds to the continuum cyclotron solution for which the metric is uniform.

In the second region, the \( k_x \) derivative of \( u_k(x) \)
\[ \partial_k u = -iN A e^{-ikN x} \left( x \psi_{\text{exp}} - e^{-\sigma} e^{-ikN (1 + x)} \psi^\dagger_{\text{exp}} \right). \] (22)
Focusing only on the \( k \) dependent terms in the metric and keeping only the lowest order exponential \( N \) dependence,
\[ \tilde{g}_{xx} = g_{xx}(k_x) - g_{xx}(0) \\
= 2A^2 N^2 I_0 e^{-\sigma N} \cos(Nk_x) + O(e^{-2\sigma N}), \] (23)
with
\[ I_0 = \int_{\alpha\beta}^{1-\alpha\beta} dx \frac{x(x+1)}{\sinh \rho(x)} \] (24)
\[ = \int_{\alpha\beta}^{1-\alpha\beta} dx \frac{x(x+1)}{\sinh \rho(x)} \] (25)
We follow [17] in approximating \( \sinh \rho(x) \)^{-1/2} by its average over the BZ, \( P \approx 1/2 \), so that
\[ I_0 = P^2 \int_{\alpha\beta}^{1-\alpha\beta} dx \frac{x(x+1)}{\sinh \rho(x)} \]
\[ \approx P^2 \left( \frac{5}{6} - \frac{2\pi}{N} \alpha + \frac{2\alpha^2}{2\pi N} \right), \] (26)
truncated to \( O(1/N) \). The normalization factor \( A \) is, to leading order,
\[ A = \sqrt{\frac{1}{N \text{ Erf}(\alpha)}} \sqrt{\frac{\pi}{e}}, \] (27)
giving
\[ \tilde{g}_{xx} = \frac{2P^2}{\text{Erf}(\alpha)} \sqrt{\frac{\pi}{e} N} \left( \frac{5}{6} - \frac{2\alpha}{2\sqrt{2\pi N}} + \frac{\alpha^2}{\pi N} \right) \times e^{-\eta N} \cos(Nk_x). \] (28)
This expression is plotted along with the corresponding numerical result in Fig. 1 (a). As with the Berry curvature and dispersion, \( k \)-dependent terms in the quantum metric are exponentially small in \( N \).

**Metric trace average**

In this section, we provide details on the perturbative calculation of BZ-averaged band-geometric quantities. In [17] it was found that, working perturbatively in \( 1/N \), the eigenstates of the large-\( N \) Hofstadter model are given by superpositions of continuum LL states, with the unitary operator \( U^\dagger \) describing the change of basis given by (8), which we reproduce here:
\[ U^\dagger = \exp \left[ \left( \frac{\pi}{96} + \frac{\pi^2}{128N^2} \right) (a^{14} - a^4) \right. \]
\[ + \frac{1}{320N^2} (a^{15} - a^{-1} a^5) + O \left( \frac{1}{N^3} \right) \] (29)
Expanding \( U^\dagger \) in a Taylor series, we find that the Hofstadter ground state \( |\tilde{0}\rangle \) is given by
\[ |\tilde{0}\rangle = U^\dagger |0\rangle = \left[ 1 - 12 \left( \frac{c_1}{N} \right)^2 \right] |0\rangle \]
\[ + 2\sqrt{6} \left( \frac{c_1}{N} + \frac{c_2}{N^2} \right) |4\rangle \]
\[ + 12\sqrt{10} \left( \frac{c_1}{N} \right)^2 |8\rangle + O \left( \frac{1}{N^3} \right), \] (30)
with \( c_1 = \pi/96, c_2 = \pi^2/128 \). This state is normalized to unity, again up to terms of \( O(1/N^3) \); terms entering at \( O(1/N^3) \) in the exponent of (29) are eighth-order in the boson operators \( a, a^\dagger \) and will only contribute to the coefficient of \( |8\rangle \). We now seek to compute BZ averages of metric components, using (11):
\[ g_{ij} = \text{tr} \left[ P_0 \hat{r}_i (1 - P_0) \hat{r}_j P_0 \right]. \] (31)
where \( P_0 \) is the projector onto the Hofstadter ground state, which may be expressed in terms of LL projectors using (30). The quantum-mechanical position operators are linear combinations of the LL ladder operators \( a, a^\dagger \) and angular momentum raising and lowering operators \( b^\dagger, b \), namely
\[ \hat{x} = i \sqrt{3} (a - a^\dagger - ib^\dagger), \] (32)
\[ \hat{y} = \frac{1}{\sqrt{3}} (a + a^\dagger + ib^\dagger). \] (33)
In (31) these operators appear sandwiched between the ground-state projector and its complement. This simplifies...
the problem considerably: because $b, b^\dagger$ do not mix LLs, no term containing them contributes in (31), and we may neglect the angular momentum degree of freedom and work purely in the space of LL indices (as the notation of (30) suggests). Furthermore, because $|\bar{0}\rangle$ only has support on LLs $|0\rangle, |4\rangle$ and $|8\rangle$, terms of the form $aa$ and $a^\dagger a^\dagger$ vanish as well. The only matrix element that contributes to (31) is

$$
(0|a^\dagger a|\bar{0}) = \frac{1}{96} \pi^2 N^2 + \frac{36 - 6\sqrt{6}}{211} \pi^3 N^3
+ \frac{298 - 27\sqrt{6}}{214} \pi^4 N^4 + O\left(\frac{1}{N^5}\right); \quad (34)
$$

the leading-order term not included in (30) is of the form $c_3/N^3|8\rangle$, which is not mixed with any term of lower order in $N$ by the operator $a^\dagger a$.

Eq. (34) suffices to compute the averages of all elements of $g_{ij}$, as described in (12). Although $g_{xx}$ and $g_{yy}$ do not have the same $k$ dependence, symmetry requires that their BZ averages be the same. In calculating the asymptotic forms of the trace and determinant conditions, we additionally make use of the fact that all higher moments of all quantities (considered as distributions over the BZ) vanish to any order in $1/N$, due to the fact that our analysis of the WKB wavefunctions shows that the amplitude of all $k$ dependence falls off exponentially in $N$. We obtain excellent agreement with numerics, as shown in Fig. 1.

As an additional consistency check, one may eliminate $\langle g_{xx}\rangle$ from the system of equations (13), (14) to obtain $4\langle D \rangle - \langle T \rangle (\langle T \rangle + 2) = 0$; this relation is plotted in Fig. 3. All neglected terms, namely $\langle g_{iy}\rangle$ and BZ fluctuations of $B$ and $g$, vanish exponentially in $N$.

![Fig. 3](image3.png)

**Fig. 3.** (color online). Plot of the residuals in the implicit relationship between (13) and (14). Points are obtained from numerically integrated values of $\langle T \rangle$ and $\langle D \rangle$. The blue line is a plot of $e^{-2N}$ as a guide to the eye.

### Additional plots of gap data

![Fig. 4](image4.png)

**Fig. 4.** Scaled many-body gap as a function of determinant inequality for the Laughlin state of $N_p = 8$ bosons at $\nu = 1/2$. $\langle D \rangle = 0$ corresponds to the saturation of the inequality.

![Fig. 5](image5.png)

**Fig. 5.** Scaled many-body gap as a function of determinant inequality for the Laughlin state of $N_p = 8$ fermions at $\nu = 1/3$.

![Fig. 6](image6.png)

**Fig. 6.** Scaled many-body gap as a function of determinant inequality for the Moore-Read state of $N_p = 8$ bosons at $\nu = 1$. 

