Wick’s Theorem for non-symmetric normal ordered products and contractions

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Abstract

We consider arbitrary splits of field operators into two parts; $\psi = \psi^+ + \psi^-$, and use the corresponding definition of normal ordering introduced in [1]. In this case the normal ordered products and contractions have none of the special symmetry properties assumed in existing proofs of Wick’s theorem. Despite this, we prove that Wick’s theorem still holds in its usual form as long as the contraction is a $c$-number. Wick’s theorem is thus shown to be much more general than existing derivations suggest, and we discuss possible simplifying applications of this result.

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I. INTRODUCTION

Wick’s theorem is at the core of perturbative calculations in the canonical operator approach to quantum field theory: on expansion of the S-matrix, it is used to relate time ordered products of field operators to normal ordered products and contractions [2]. Within the traditional definition of normal ordered products, the interaction picture field operators, $\psi(x)$, are split into positive $\psi^{(0+)}$ and negative $\psi^{(0-)}$ energy waves containing annihilation and creation operators respectively;

$$\psi(x) = \psi^{(0+)}(x) + \psi^{(0-)}(x). \quad (1.1)$$

Normal ordering, $N^{(0)}[\cdot]$, then places the annihilation (creation) operators to the right (left), otherwise leaving the order of the operators unchanged. Hence the vacuum expectation values of normal ordered products are guaranteed to vanish,

$$\langle 0 | N^{(0)}[\cdot] | 0 \rangle = 0. \quad (1.2)$$

Calculations therefore proceed in a straightforward manner by simply taking the vacuum expectation value of the operator identity called Wick’s theorem.

In the literature, Wick’s theorem has been proved either through the use of a generating functional [3–5], or by the manipulation of products of operators [2,6–8]. In both cases, an essential ingredient of the proof is the fact that the normal ordered products defined above are symmetric in the following sense. Let us denote the normal ordered product of $n$ (possibly different) field operators $\psi_i := \psi(x_i)$ evaluated at different space time points $x_i$ by

$$N^{(0)}_{1,2,\ldots,n} := N^{(0)}[\psi_1 \psi_2 \ldots \psi_n]. \quad (1.3)$$

Then since $\left[\psi^{(0+)}(x), \psi^{(0+)}(x')\right]_\sigma = \left[\psi^{(0-)}(x), \psi^{(0-)}(x')\right]_\sigma = 0 \ \forall x, x'$ (where $\sigma = +1$ for bosonic operators, and $\sigma = -1$ for fermionic ones), it follows that

$$N^{(0)}_{1,2,\ldots,n} = (-1)^p N^{(0)}_{a,b,\ldots,t} \ \forall n > 0. \quad (1.4)$$

Here $a, b, \ldots, t$ is any permutation of $1, 2, \ldots, n$ and $p$ is the number of times pairs of fermionic fields are interchanged in moving from an order $1, 2, \ldots, n$ to an $a, b, \ldots, t$ order of fields. Normal ordered products satisfying (1.4) are said to be symmetric. The reason why this symmetry is central to the existing derivations of Wick’s theorem will be explained in detail in section 3 below where we work with the generating functional approach. However, the following comments may help illustrate the reason. Recall that Wick’s theorem relates time ordered products, normal ordered products and contractions. One way to prove this theorem [2,7] is to consider initially a given order of times. The resulting (single) contribution from the time ordered product may then shown to be equivalent to a combination of normal ordered products and contractions. If these normal ordered products and contractions are symmetric, then each contribution to the time ordered product takes the same form, and thus Wick’s theorem is proved [2,7]. However, if they are not symmetric this is no longer the case (see section 3).

In general, however, normal ordering with respect to annihilation and creation operators does not simplify calculations. As an example, consider how, in the presence of symmetry
breaking, one must normal order with respect to new annihilation and creation operators related by a Bogoliubov transformation to the ones appearing in the field \([6,7]\). This is equivalent to choosing to split the field in a different way from \((1.1)\). A more recent example \([1]\) was provided by a system in thermal equilibrium at a temperature \(T\). There the thermal expectation value of a normal ordered product of creation and annihilation operators \((\hat{a}^\dagger \text{ and } \hat{a})\) no longer vanishes in the way that it did at \(T = 0\):

\[
\ll N^{(0)}[\hat{a}, \hat{a}^\dagger] \gg = n_{\text{be}}, n_{\text{fd}} \neq 0.
\]

(Here \(n_{\text{be}}, n_{\text{fd}}\) are the Bose-Einstein and Fermi-Dirac distributions respectively.) As a result, perturbative calculations at finite \(T\) would not seem to be simplified by the use of Wick’s theorem. This point has caused some confusion in the literature \([7,9,10]\) and it was addressed in \([1]\). There it was shown that Wick’s theorem could be used in the usual way at finite temperature provided one split the field in a different way from the usual one into creation and annihilation operators \((1.1)\).

Thus in general one must consider arbitrary splits of the field operators. The precise split used will reflect the physical environment of the problem if it is to simplify practical calculations in the usual way. We denote the arbitrary split by

\[
\psi(x) = \psi^+(x) + \psi^-(x),
\]

where \(\psi^+(x)\) and \(\psi^-(x)\) are unspecified in this paper, so that we can accommodate any physical problem. We still call the two parts ‘positive’ and ‘negative’ even though the split is general and no longer necessarily into positive and negative energy waves. The corresponding (natural) generalisation of normal ordering is then expressed in terms of \(\psi^+(x)\) and \(\psi^-(x)\), and is defined to be

- **Normal ordering** of products of fields, \(N[\bullet]\), places all the ‘positive’ parts to the right and the ‘negative’ parts to the left, otherwise leaving the order of the fields unchanged. A sign change occurs whenever two fermion field operators are interchanged.

Note that this reduces to the usual definition, \(N^{(0)}[\bullet]\), when the split is into the annihilation and creation operators appearing in the field. We have used the \((0)\) superscript to denote this traditional and very special case. So, for example, in the two-point case one has

\[
N[\psi_1\psi_2] := N_{1,2} = \psi^+_1\psi^+_2 + \psi^-_1\psi^-_2 + \sigma\psi^-_2\psi^+_1 + \psi^-_1\psi^+_2.
\]

Such a generalisation of normal ordering was successfully used for thermal field theory in \([1]\), and is equivalent to discussions of normal ordering involving Bogoliubov transformations and symmetry breaking \([6,7]\). Contractions are then defined in the usual way through the two-point time ordered products which, being independent of the split, take the canonical form:

\[
T_{1,2} := T[\psi_1\psi_2] = \psi_1\psi_2\theta(t_1 - t_2) + \sigma\psi_2\psi_1\theta(t_2 - t_1).
\]

(In general the \(\theta\) functions may be defined for complex times and hence be contour dependent – as for example in Euclidean quantum field theory and thermal field theory \([3,11]\). Thus
the time ordered products are also contour dependent. However, the analysis presented here holds regardless of contour.) Using (1.7) and (1.8) the contraction $D_{1,2}$ is:

$$D_{1,2} = D[\psi_1, \psi_2] := T_{1,2} - N_{1,2}$$

$$= \theta(t_1 - t_2) \left[ \psi_1^+, \psi_2^- \right]_\sigma$$

$$- \theta(t_2 - t_1) \left\{ \left[ \psi_1^+, \psi_2^+ \right]_\sigma + \left[ \psi_1^-, \psi_2^+ \right]_\sigma + \left[ \psi_1^-, \psi_2^- \right]_\sigma \right\}. \quad (1.10)$$

Now the key point is that for such arbitrary splits, the contraction and two-point normal product are not generally symmetric: from (1.7) and (1.10) the condition for symmetry is that for all times $t_1, t_2$

$$D_{1,2} = \sigma D_{2,1}, \quad N_{1,2} = \sigma N_{2,1} \iff \left[ \psi_1^+, \psi_2^+ \right]_\sigma + \left[ \psi_1^-, \psi_2^- \right]_\sigma = 0, \quad (1.11)$$

which is not necessarily satisfied. If these two-point relations are true, then all higher order normal ordered products are symmetric in the sense of (1.4). This is only easy to prove if one has already proved that Wick’s theorem holds independent of the symmetry of all normal ordered products!

It is conceivable that in some situations, for example systems out of equilibrium, calculations may be simplified by the use of a split for which the normal ordered products are not symmetric (i.e. (1.11) is not satisfied). Indeed, out of equilibrium it may be possible to find a split such that (1.11) is true for specific times, but without time-translation symmetry it is doubtful that this split will ensure (1.11) will be true for all times. Wick’s theorem, which relates normal ordered products (1.7), time ordered products (1.8), and contractions (1.10), has only been proved for the special case where the normal ordered products and contractions are symmetric. The question this paper asks then is: Does the usual form of Wick’s theorem still hold when normal ordered products are not symmetric?

Our answer is yes. Wick’s theorem is in fact more general than the existing proofs suggest, namely: if the contraction is a c-number (and hence the split is linear in creation and annihilation operators), Wick’s theorem holds for both symmetric and non-symmetric normal ordered products. Possible practical applications of this result, in particular to time dependent non-equilibrium systems, are discussed in the conclusions.

In section 2 we derive the generating functional for Wick’s theorem in terms of the generalised definition of normal ordering. We stress the places in which the non-symmetry of products means that our derivation differs from others existing in the literature. In section 3 we use the generating functional to obtain Wick’s theorem. For symmetric products the proof is immediate – for non-symmetric ones it is considerably more complicated. We finally summarise our work and also comment on a claim made in the original paper on thermal field theory by Matsubara [9]. Applications for our work are then discussed.

II. THE GENERATING FUNCTIONAL FOR WICK’S THEOREM

In this section we derive the generating functional for Wick’s Theorem – that is, we prove the identity
\[ T \left[ \exp \left\{ -i \int d^4 x \, j(x) \psi(x) \right\} \right] = N \left[ \exp \left\{ -i \int d^4 x \, j(x) \psi(x) \right\} \right] \times \exp \left\{ -\frac{1}{2} \int d^4 x \, d^4 y \, j(x) D[\psi(x) \psi(y)] j(y) \right\}. \] (2.1)

Here \( \int d^4 x = \int dt \int d^3 \mathbf{x} \), and the precise range of integration does not affect our arguments. For simplicity, we just consider bosonic fields. (Comments regarding fermionic fields will be made later.) Thus the \( j(x) \)'s are \( c \)-number sources and since Wick’s theorem is used in perturbation theory, the field operators \( \psi(x) \) are in the interaction picture. The normal ordered products \( N[\cdot] \) and contraction \( D[\psi(x) \psi(y)] \) have been defined above in terms of the arbitrary split of the fields into positive and negative parts; \( \psi(x) = \psi^+(x) + \psi^-(x) \).

Following the discussion in the previous section, we therefore make no assumptions about their symmetry properties. The first part of the proof presented here follows closely that of [3,4].

Recall that for two operators \( A \) and \( B \)
\[ e^{A+B} = e^A e^B e^{\frac{1}{2}[B,A]} \] (2.2)
if and only if
\[ [A,[A,B]] = [B,[A,B]] = 0. \] (2.3)

Recall, too, that time ordering puts the larger times to the left, and so for example in the case of two operators \( A(t_1) \) and \( A(t_2) \) with \( t_2 > t_1 \),
\[ T \left[ e^{A(t_1)+A(t_2)} \right] = e^{A(t_2)} e^{A(t_1)} = T \left[ e^{A(t_1)} e^{A(t_2)} \right]. \] (twooporder) (2.4)

We comment that a similar relation holds for any operation which places the two operators in a given order. This will be used for normal ordered products below in equation (2.11).

Consider the LHS of (2.1),
\[ T \left[ e^{-i \int dt \int d^3 \mathbf{x} \, j(x) \psi(x)} \right] := T \left[ e^{-i \int dt \, O(t)} \right]. \] (2.5)

Let \( t_i \) and \( t_f \) denote the initial and final time. Divide the total time interval into \( N \) equal intervals with \( \Delta t = \frac{t_f-t_i}{N} \) and
\[ t_f = t_N > t_{N-1} > \ldots > t_1 > t_0 = t_i. \] (2.6)

By definition,
\[ \int_{t_i}^{t_f} dt O(t) = \lim_{N \rightarrow \infty} \Delta t \sum_{j=1}^{N} O(t_j). \] (2.7)

So here,
\[ T \left[ e^{-i \int_{t_i}^{t_f} dt O(t)} \right] = \lim_{N \rightarrow \infty} T \left[ e^{-i \Delta t \sum_{j=1}^{N} O(t_j)} \right] = \lim_{N \rightarrow \infty} \left[ e^{-i \Delta t O(t_N)} e^{-i \Delta t O(t_{N-1})} \ldots e^{-i \Delta t O(t_1)} \right] \] (2.8)
\[ = \lim_{N \rightarrow \infty} \left[ e^{-i \Delta t \sum_{k=1}^{N} O(t_k)} e^{-\frac{1}{2} \Delta t^2 \sum_{1<k<l<N} [O(t_k),O(t_l)]} \right]. \] (2.9)
In going from (2.8) to (2.9) we have used (2.2), which is legitimate since $[O(t_i), O(t_k)]$ is a $c$-number. Note also that in this exponential $t_i > t_k$. Thus using the definition of $O(t)$, (2.9) gives

$$
T \left[ e^{-i \int d^4 x j(x) \psi(x)} \right] = e^{-i \int d^4 x j(x) \psi(x)} e^{-\frac{1}{2} \int d^4 x \int d^4 y \theta(x_0 - y_0) j(x) [\psi(x) \psi(y)] j(y)}. \tag{2.10}
$$

In order to relate the time ordered product in (2.10) to a normal ordered product, consider an arbitrary split of the fields into ‘positive’ and ‘negative’ parts introduced in (1.6) and the corresponding definition normal ordering given in the previous section. Using (2.2), the first exponential on the RHS of (2.10) may be re-written as

$$
e^{-i \int d^4 x j(x) \psi(x)}
= e^{-i \int d^4 x j(x) \left\{ \psi^{-}(x) + \psi^{+}(x) \right\}}
= e^{-i \int d^4 x j(x) \psi^{-}(x)} e^{-i \int d^4 x j(x) \psi^{+}(x)}
\times e^{\frac{i}{2} \int d^4 x \int d^4 y j(x) \left[ \psi^{-}(x), \psi^{+}(y) \right] j(y)}
= N \left[ e^{-i \int d^4 x j(x) \psi(x)} \right] e^{\frac{i}{2} \int d^4 x \int d^4 y j(x) \left[ \psi^{-}(x), \psi^{+}(y) \right] j(y)}. \tag{2.11}
$$

In obtaining (2.11) we have used (2.4) for normal ordered products, and (2.2), which requires that

$$\left[ \psi^{\pm}, \left[ \psi^{+}, \psi^{-} \right] \right] = 0. \tag{2.12}
$$

This condition is satisfied if $\psi^{+}$ and $\psi^{-}$ are linear in annihilation and creation operators, and in turn it ensures that the contraction is a $c$-number. Substituting (2.11) into (2.10) gives

$$
T \left[ e^{-i \int d^4 x j(x) \psi(x)} \right] = N \left[ e^{-i \int d^4 x j(x) \psi(x)} \right]
\times e^{\frac{i}{2} \int d^4 x \int d^4 y j(x) \left\{ \theta(x_0 - y_0) [\psi(x) \psi(y)] - [\psi^{-}(x), \psi^{+}(y)] \right\} j(y)}. \tag{2.13}
$$

Our proof now diverges from that given in the usual texts [3,4] since we are considering the possibility of having an arbitrary (unsymmetric) split of the fields rather than a (symmetric) one into annihilation and creation operators. Look at the part in curly brackets in the second exponential of (2.13); it is

$$
\theta(x_0 - y_0) [\psi(x), \psi(y)] - [\psi^{-}(x), \psi^{+}(y)]
= \left[ \psi^{+}(y), \psi^{-}(x) \right] + \theta(x_0 - y_0) \left\{ \left[ \psi^{+}(x), \psi^{+}(y) \right] + \left[ \psi^{-}(x), \psi^{+}(y) \right] \right\}
+ \left[ \psi^{+}(x), \psi^{-}(y) \right] + \left[ \psi^{-}(x), \psi^{-}(y) \right]
= \theta(y_0 - x_0) \left[ \psi^{+}(y), \psi^{-}(x) \right] - \theta(x_0 - y_0) \left\{ \left[ \psi^{+}(y), \psi^{+}(x) \right]
+ \left[ \psi^{-}(y), \psi^{+}(x) \right] + \left[ \psi^{-}(y), \psi^{-}(x) \right] \right\}
= D [\psi(y) \psi(x)], \tag{2.14}
$$

where we have used the definition of the contraction in (1.10).
Thus, now substitute (2.14) into the final exponential of (2.13), relabel $x \leftrightarrow y$ and use the fact that the $j(x), j(y)$ are $c$-numbers to switch their positions. This enables (2.13) to be re-written as

$$
T \left[ \exp \{-i \int d^4 x \, j(x) \psi(x) \} \right] = N \left[ \exp \{-i \int d^4 x \, j(x) \psi(x) \} \right] 
\times \exp \{- \frac{1}{2} \int d^4 x \, d^4 y \, j(x) \, D[\psi(x) \psi(y)] j(y) \},
$$

(2.15)

thus proving (2.1). This equation is also true for fermionic operators. There the sources are Grassmann variables, and the contraction, defined in (1.10), contains anti-commutators.

III. WICK’S THEOREM FOR NON-SYMMETRIC PRODUCTS

Wick’s theorem is obtained by functionally differentiating (2.15) with respect to the sources $j(x)$. Observe that we need only work with an even number of fields as the perturbation is about the expectation value of the fields, and hence the expectation value of an odd number of fields will always vanish.

Functionally differentiate (2.15) with respect to the sources $2n$ times and then set every $j(x_a) = j_a$ to zero. Since time-ordered products of fields evaluated at the same time points are undefined, the indices $a$ refer to different space-time points and are therefore all different. The LHS of (2.15) gives

$$
T \left[ \delta \delta j_1 \cdots \delta j_{2n} \exp \{-ij \psi \} \right]_{j_a=0} = T \left[ \delta \delta j_1 \cdots \delta j_{2n} \frac{(-ij \psi)^{2n}}{(2n)!} \right]
$$

(3.1)

$$
= \sum_{\text{perm} \{a\}} (-1)^p T_{a_1,a_2,\ldots,a_{2n}}
$$

(3.2)

$$
= T_{1,2,\ldots,2n}.
$$

(3.3)

In (3.2) the sum takes $a_1, \ldots, a_{2n}$ through all permutations of $1, \ldots, 2n$;

$$
\text{perm} \{a\} = \left( \begin{array}{cccc}
1 & 2 & 3 & \ldots & 2n \\
\begin{array}{c}
a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{2n}
\end{array}
\end{array} \right)
$$

(3.4)

and $p$ is the number of times one has to interchange pairs of fermions in moving from $1, \ldots, 2n$ to $a_1, \ldots, a_{2n}$ order. The last identity (going from (3.2) to (3.3)) follows from the fact that by definition time ordered products are symmetric in their indices, and that there are $(2n)!$ permutations of the numbers $1, 2, \ldots, 2n$. This type of manipulation will not hold for normal ordered products and contractions since they are not, in general, symmetric in their indices.

Carrying out a similar operation for the RHS of (2.15) gives a general form of Wick’s theorem for non-symmetric products;

$$
T_{1,2,\ldots,2n} = \sum_{m=0}^{n} \sum_{\text{perm} \{a\}} (-1)^p \left[ \frac{N_{a_1,a_2,\ldots,a_{2m}}}{(2m)!} \cdot \frac{1}{(n-m)!} \cdot \left( \prod_{j=m+1}^{n} \frac{1}{2} D_{a_{2j-1},a_{2j}} \right) \right].
$$

(3.5)
In the traditional split of the fields into annihilation and creation operators, the normal ordered products \( N^{(0)}_{1,2} \) and contraction \( D^{(0)}_{1,2} \) are symmetric as was discussed in section 1. As a result, the sum over permutations on the RHS of (3.5) can easily be evaluated and thus (3.5) rapidly reduces to the well known form of Wick’s theorem. For two bosonic fields this is

\[
T_{1,2} = N^{(0)}_{1,2} + D^{(0)}_{1,2},
\]  

(3.6)

and for four bosonic fields

\[
T_{1,2,3,4} = N^{(0)}_{1,2,3,4} + D^{(0)}_{1,2,3,4} + D^{(0)}_{1,3}N^{(0)}_{2,4} + D^{(0)}_{1,4}N^{(0)}_{2,3} + D^{(0)}_{2,3}N^{(0)}_{1,4} +
\]

\[
D^{(0)}_{2,4}N^{(0)}_{1,3} + D^{(0)}_{3,4}N^{(0)}_{1,2} + D^{(0)}_{1,2}D^{(0)}_{3,4} + D^{(0)}_{1,3}D^{(0)}_{2,4} + D^{(0)}_{1,4}D^{(0)}_{2,3}.
\]

(3.7)

Higher order relations follow the same pattern. Note that in the above expressions we have chosen to arrange the indices on any one given normal ordered product and contraction such that the smaller indices are to the left of the larger ones. In the case of symmetric products, this ordering is of course not important. However, it will also be used below where, starting from (3.5), we show that the above expressions (equations (3.6) and (3.7)) hold even for non-symmetric products \( N \) and \( D \), as long as the contraction is a c-number. We again consider bosonic fields. The results below can be extended to fermionic fields by inserting a minus sign every time the positions of two fermionic fields are interchanged. The following relations and lemmas are useful for the proof.

**A. Lemmas and relations**

**Lemma 1** Two point normal ordered products and contractions are related by

\[
N_{2,1} - N_{1,2} = D_{1,2} - D_{2,1}.
\]

(3.8)

**Proof:** By definition of the contraction (1.9)

\[
D_{1,2} - D_{2,1} = \theta_{1,2} \left[ \psi_1^+, \psi_2^- \right] - \theta_{2,1} \left[ \left[ \psi_1^+, \psi_2^- \right] + \left[ \psi_1^-, \psi_2^+ \right] + \left[ \psi_1^-, \psi_2^- \right] \right] 
\]

\[
- \theta_{2,1} \left[ \psi_2^+, \psi_1^- \right] + \theta_{1,2} \left[ \left[ \psi_2^+, \psi_1^- \right] + \left[ \psi_2^-, \psi_1^+ \right] + \left[ \psi_2^-, \psi_1^- \right] \right] 
\]

\[
= \left[ \psi_2^+, \psi_1^+ \right] + \left[ \psi_2^-, \psi_1^- \right] 
\]

\[
= N_{2,1} - N_{1,2},
\]

(3.9)

where \( \theta_{1,2} = \theta(t_1 - t_2) \) and where the last line follows directly from the definition of normal ordering (1.7).

**Lemma 2** Let \( X \) and \( Y \) be arbitrary operators. Then if the contraction is a c-number,

\[
N \left[ X \psi_1 \psi_2 Y \right] = N \left[ X \psi_2 \psi_1 Y \right] + (D_{2,1} - D_{1,2})N \left[ XY \right].
\]

(3.10)
Proof: By definition
\[ N \left[ X \psi^+_1 \psi^-_2 Y \right] = N \left[ X \psi^+_2 \psi^-_1 Y \right] \quad \forall X, Y. \] (3.11)

Therefore
\[
N \left[ X \psi_1 \psi_2 Y \right] = N \left[ X \psi^+_1 \psi^+_2 Y \right] + N \left[ X \psi^-_1 \psi^-_2 Y \right] \\
+ N \left[ X \psi^+_1 \psi^-_2 Y \right] + N \left[ X \psi^-_1 \psi^+_2 Y \right] \\
= N \left[ X \psi^+_2 \psi^+_1 Y \right] + N \left[ X \psi^-_2 \psi^+_1 Y \right] \\
+ N \left[ X \psi^-_2 \psi^-_1 Y \right] + N \left[ X \psi^+_2 \psi^-_1 Y \right] \\
+ N \left[ X \left( \psi^+_1, \psi^+_2 \right) Y \right] + N \left[ X \left( \psi^-_1, \psi^-_2 \right) Y \right] \\
= N \left[ X \psi_2 \psi_1 Y \right] + (D_{1,2} - D_{2,1})N \left[ XY \right],
\] (3.12)

provided \( [\psi^+_1, \psi^+_2] \) and \( [\psi^-_1, \psi^-_2] \) are c-numbers. This implies that both \( \psi^+ \) and \( \psi^- \) are linear in the creation and annihilation operators, and hence that the contraction is a c-number.

Relation 3 The number of ways of filling \( p \) indistinguishable bags each with exactly two marbles labelled 1, 2, \ldots, 2q is precisely
\[
[(2q)(2q - 1)][(2q - 2)(2q - 3)] \ldots [(2q - 2p + 2)(2q - 2p + 1)] \frac{1}{p!} \frac{1}{2^p} = \frac{(2q)!}{(2q - 2p)!2^p p!}.
\] (3.13)

Consequence 3.1 Let labels \( a_i \)'s take values from 1, 2, 3, \ldots, 2q. The number of distinct configurations of \( p \) contractions \( D_{a_1,a_2} D_{a_3,a_4} \ldots D_{a_{2p-1},a_{2p}} \) with unordered indices is given by (3.13).

Consequence 3.2 For a product of \( p \) contractions and one normal ordered product, with a total number of 2q indices; \( N_{a_1,a_2,\ldots,a_{2q-2p}} D_{a_{2q-2p+1},a_{2q-2p+2}} \ldots D_{a_{2q-1},a_{2q}} \), the number of distinct configurations with unordered indices is again given by (3.13).

Using these, we now show that (3.6), (3.7) and the usual higher order version of Wick's theorem holds even for non-symmetric normal ordered products and contractions, as long as the contraction is a c-number.

B. Proof for \( n = 1 \)

This is the easiest case. For \( n = 1 \), (3.5) gives
\[
T_{1,2} = \frac{1}{2}(N_{1,2} + N_{2,1} + D_{1,2} + D_{2,1}).
\] (3.14)

Substituting (3.8) into (3.14) gives
showing that for } n = 1 \text{ (3.6) holds even for non-symmetric normal ordered products and contractions. (This result is of course consistent with the definition of the contraction; from (1.9), } D_{1,2} := T_{1,2} - N_{1,2} \text{ and similarly } D_{2,1} := T_{2,1} - N_{2,1} = T_{1,2} - N_{2,1}. \text{ Adding these two equations gives (3.14).}

\textbf{C. Proof for } n = 2

For } n = 2, \text{ (3.5) gives}

\[ T_{1,2,3,4} = \sum_{\text{perm} \{a\}} \sum_{m=0}^{2} \frac{N_{a_1,a_2,...,a_m}}{(2m)!} \frac{1}{(2-m)!} \left( \prod_{j=m+1}^{2} \frac{1}{2} D_{a_j-1,a_j} \right) \]

\[ = \sum_{\text{perm} \{a\}} \left( \frac{1}{4!} N_{a_1,a_2,a_3,a_4} + \frac{1}{2!} N_{a_1,a_2} D_{a_3,a_4} + \frac{1}{2!} D_{a_1,a_2} D_{a_3,a_4} \right) \]

\[ = \sum_{\text{perm} \{a\}} \frac{1}{4!} S_{a_1,a_2,a_3,a_4} \]

\[ T_{1,2} = N_{1,2} + D_{1,2} \quad (3.15) \]

\[ \text{where in the first term of (3.16) } m = 2, \text{ in the second } m = 1, \text{ in the third } m = 0, \text{ and } S_{a_1,a_2,a_3,a_4} \text{ will be defined later. Provided we choose } S_{a_1,a_2,a_3,a_4} \text{ correctly (see below), proof that } T_{1,2,3,4} \text{ is given by the usual expression (i.e. (3.7)) even for non-symmetric } N \text{ and } D \text{ will then follow directly from (3.17) if we can show that } S_{a_1,a_2,a_3,a_4} \text{ is symmetric under interchange of any two indices.}

To do that, consider initially the second term of (3.16); } \sum_{\text{perm} \{a\}} 6 N_{a_1,a_2} D_{a_3,a_4}. \text{ The first important point to notice is that because of the sum over permutations, the order in which the } a_i \text{ indices are written is irrelevant; for example}

\[ \sum_{\text{perm} \{a\}} N_{a_1,a_2} D_{a_3,a_4} = \sum_{\text{perm} \{a\}} N_{a_1,a_3} D_{a_2,a_4}. \quad (3.18) \]

Secondly, the factor of 6 can be understood in the following way. From consequence 3.2 we know that for a product of one normal ordered product and one contraction with a total of } 2q = 4 \text{ indices, there are } 4!/2!2 = 6 \text{ distinct configurations with unordered indices. Thus we can re-write the second term of (3.16) as a sum over the 6 distinct configurations with unordered indices:}

\[ \sum_{\text{perm} \{a\}} 6 N_{a_1,a_2} D_{a_3,a_4} = \sum_{\text{perm} \{a\}} \left[ D_{a_1,a_2} N_{a_3,a_4} + D_{a_1,a_3} N_{a_2,a_4} + D_{a_1,a_4} N_{a_2,a_3} + D_{a_2,a_3} N_{a_1,a_4} + D_{a_2,a_4} N_{a_1,a_3} + D_{a_3,a_4} N_{a_1,a_2} \right]. \quad (3.19) \]

Note that since the indices on the contractions and normal ordered products are unordered, there was still some freedom in writing down the above expression. We have made the
particular choice to write the \( a_i \) indices on each individual \( N \) and each individual \( D \) with the smallest value of \( i \) to the left. The reason for this will become clear later.

Now carry out a similar procedure for the third term of (3.16) involving the product of 2 contractions: the 3 terms arise from the 3 distinct configurations of two contractions with unordered indices (see consequence 3.1), and we then chose to write the indices on each individual contraction such that the smallest value of \( i \) is to the left. Thus \( S_{a_1,a_2,a_3,a_4} \) in (3.17) is defined by

\[
S_{a_1,a_2,a_3,a_4} := N_{a_1,a_2,a_3,a_4} + D_{a_1,a_2}N_{a_3,a_4} + D_{a_1,a_3}N_{a_2,a_4} + D_{a_1,a_4}N_{a_2,a_3} + \\
D_{a_2,a_3}N_{a_1,a_4} + D_{a_2,a_4}N_{a_1,a_3} + D_{a_2,a_4}N_{a_1,a_2} + D_{a_1,a_2}D_{a_3,a_4} + D_{a_1,a_3}D_{a_2,a_4} + D_{a_1,a_4}D_{a_2,a_3}.
\] (3.20)

Hence we see that \( S_{a_1,a_2,a_3,a_4} \) is given by the RHS of (3.7) where instead of the symmetric normal ordered products and contractions one uses the more general non-symmetric normal ordered products and contractions: this was the reason why (3.16) was decomposed as above.

To show that \( S_{a_1,a_2,a_3,a_4} \) is symmetric under the interchange of any two indices, consider a particular order of the indices, say \( a, b, c, d \) and then the effect of \( a \leftrightarrow b \) say. Now

\[
S_{a,b,c,d} = N_{a,b,c,d} + D_{a,b}N_{c,d} + D_{c,d}(N_{a,b} + D_{a,b}) + [D_{a,c}N_{b,d} + D_{b,c}N_{a,d}] + [D_{a,d}N_{b,c} + D_{b,d}N_{a,c}] + [D_{a,c}D_{b,d} + D_{a,d}D_{b,c}].
\] (3.21)

The pairs of terms in square brackets are invariant under interchange \( a \leftrightarrow b \). Now use (3.8) and (3.10) to show that the first line of (3.21) is also invariant under \( a \leftrightarrow b \):

\[
N_{a,b,c,d} + D_{a,b}N_{c,d} + D_{c,d}(N_{a,b} + D_{a,b}) = N_{b,a,c,d} + (D_{b,a} - D_{a,b})N_{c,d} + D_{a,b}N_{c,d} + D_{c,d}(N_{b,a} + D_{b,a}) = N_{b,a,c,d} + D_{b,a}N_{c,d} + D_{c,d}(N_{b,a} + D_{b,a}).
\] (3.22)

Thus \( S_{a,b,c,d} = S_{b,a,c,d} \). One can prove more generally that \( S_{a,b,c,d} \) is invariant under the interchange of any two adjacent indices and hence under any of the 4! permutations of the indices. Thus from (3.18) and (3.21) it follows that

\[
T_{1,2,3,4} = N_{1,2,3,4} + D_{1,2}N_{3,4} + D_{1,3}N_{2,4} + D_{1,4}N_{2,3} + D_{2,3}N_{1,4} + \\
D_{2,4}N_{1,3} + D_{3,4}N_{1,2} + D_{1,2}D_{3,4} + D_{1,3}D_{2,4} + D_{1,4}D_{2,3}.
\] (3.23)

Hence, both at the 2- and 4-point level, the usual form of Wick’s theorem holds even if the contraction and normal ordered product are not symmetric.

D. Proof for arbitrary \( n \)

At the \( n \)th order, the proof is a generalisation of that given above for \( n = 2 \). However, before proceeding, we use some of the expressions encountered in the previous subsection to motivate the introduction of two new pieces of notation.
The first enables us to write the terms appearing in square brackets on the RHS of (3.19) in a much more compact way. Recall that in this square bracket (which we now denote by $Q$), the $i$ subscript on each normal ordered product and on each contraction is arranged in increasing order going from left to right. The simplest case is when $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 4$, then $Q$ is given by

\[ Q = D_{1,2}N_{3,4} + D_{1,3}N_{2,4} + D_{1,4}N_{2,3} + D_{2,3}N_{1,4} + D_{2,4}N_{1,3} + D_{3,4}N_{1,2}. \]  

(3.24)

However, we can re-write this as

\[ Q = \sum_{\text{perm } \{b\} \in B} D_{b_1,b_2}N_{b_3,b_4} \]  

(3.25)

where the sum over permutations of $\{b\}$ is restricted to include only those permutations which are an element of the set $B$ defined by

\[ B = \{\text{perm } \{b\} : b_1 < b_2, b_3 < b_4\}. \]  

(3.26)

(It is possible to check that (3.25) indeed reduces to (3.24).) If we now return to the case in which the $a_i$’s are arbitrary then $Q$ is given by

\[ Q = \sum_{\text{perm } \{b\} \in B} D_{a_{b_1},a_{b_2}}N_{a_{b_3},a_{b_4}} \]  

(3.27)

where $B$ is defined as in (3.26).

We now introduce the second piece of notation whose aim is to enable one to re-write expressions such as (3.27) in a slightly simpler way (with fewer subscripts). Define

\[ N(a_1, a_2, \ldots, a_2n) := N_{a_1,a_2,\ldots,a_{2n}}, \]
\[ D(a_1, a_2) := D_{a_1,a_2}. \]  

(3.28)

As an example of the use of this notation, use (3.27) and (3.28) to re-write (3.19) as

\[ \sum_{\text{perm } \{a\}} 6D(a_1, a_2) N(a_3, a_4) = \sum_{\text{perm } \{a\}} \sum_{\text{perm } \{b\} \in B} D(a_{b_1}, a_{b_2}) N(a_{b_3}, a_{b_4}). \]  

(3.29)

We now continue with the proof of Wick’s theorem to $n$th order.

- **Step 1.**

For a given value of $m$, the RHS of (3.5) can be re-written as a sum over distinct configurations of $N$’s and $D$’s, with the indices on each individual $N$ and each individual $D$ arranged in increasing order going from left to right (compare with (3.29)). To do that, use Lemma (4):

**Lemma 4** The following identity holds

\[ \]
\[
\frac{(2n)!}{(2m)!((n-m)!2^{n-m})} \sum_{\text{perm } \{a\}} N(a_1, a_2, \ldots, a_{2m}) \prod_{j=m+1}^{n} D(a_{2j-1}, a_{2j})
\]
\[
= \sum_{\text{perm } \{a\}} \sum_{\text{perm } \{b\} \in B} N(a_{b_1}, a_{b_2}, \ldots, a_{b_{2m}}) \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}})
\]
\[
(3.30)
\]

where
\[
\text{perm } \{b\} = \begin{pmatrix}
1 & 2 & 3 & \cdots & 2n \\
b_1 & b_2 & b_3 & \cdots & b_{2n}
\end{pmatrix}
\]
\[
(3.31)
\]

and the restriction on the sum of permutations of \{b\} is such that 1) it arranges the indices on each individual normal ordered product and on each individual contraction in increasing order going from left to right, and 2) it orders the contractions by size of the leading element so as to avoid repetition of \(D_{1,2}D_{3,4} \ldots \) as \(D_{3,4}D_{1,2} \ldots\):
\[
B = \{\text{perm } \{b\} : \begin{bmatrix}
b_1 < b_2 \ldots < b_{2m} \\
b_{2m+1} < b_{2m+2} \\
b_{2m+3} < b_{2m+4} \\
\vdots \\
b_{2n-1} < b_{2n}
\end{bmatrix} ; b_{2m+1} < b_{2m+3} < \ldots < b_{2n-1}\}.
\]
\[
(3.32)
\]

**Proof:** From Relation (3), for a product of \(n - m\) contractions and one normal ordered product with a total of \(2n\) indices, there are \((2n)!/(2m)!((n-m)!2^{n-m})\) distinct configurations with unordered indices. These distinct configurations are represented by the sum over permutations of the \(b\) labels. On a given contraction or normal ordered product, the order of the indices is arbitrary, but above the choice has been made by the restriction.

- **Step 2.**
  Using Lemma (4), re-write the RHS of (3.5) as
  \[
  T_{1,2,\ldots,2n} = \frac{1}{(2n)!} \sum_{\text{perm } \{a\}} \sum_{m=0}^{n} \sum_{\text{perm } \{b\} \in B} N(a_{b_1}, a_{b_2}, \ldots, a_{b_{2m}}) \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}})
  \]
  \[
  =: \frac{1}{(2n)!} \sum_{\text{perm } \{a\}} S(\{a\}).
  \]
  \[
  (3.33)
  \]
  This is a generalisation of (3.20) to arbitrary \(n\).

- **Step 3.**
  Use Lemmas (1) and (2) to show that

**Lemma 5** For any two permutations, \(\{a\}\) and \(\{a'\}\)
\[
S(\{a\}) = S(\{a'\}).
\]
\[
(3.34)
\]
Proof: Consider two permutations related by \( \{a\} \rightarrow \{a'\} \) with

\[
\begin{align*}
    a_i &= a'_{i+1} \\
    a_{i+1} &= a'_i \\
    a_j &= a'_j \quad \forall j \neq i, i + 1.
\end{align*}
\]  

(3.35)

The only non-zero contributions to \( \Delta := S(\{a\}) - S(\{a'\}) \) arise when both the indices \( a_i \) and \( a_{i+1} \) are either on the same normal ordered product, or on the same contraction. The reason is that when these indices occur on different normal ordered products or contractions, the restriction on the sum of permutations of \( \{b\} \) orders the terms in exactly the same way in both the \( \{a\} \) and \( \{a'\} \) permutation thus making \( \Delta \) vanish. (These terms are the generalisation of those in square brackets in (3.21).)

Thus we may write

\[
S(\{a\}) - S(\{a'\}) = \sum_{m=0}^{n} \sum'_{\text{perm } \{b\} \in B} \left[ N(a_{b_1}, a_{b_2}, \ldots, a_{b_{2m}}) - N(a'_{b_1}, a'_{b_2}, \ldots, a'_{b_{2m}}) \right] \times \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}}) \]  

(3.36)

\[
+ \sum_{m=0}^{n} \sum''_{\text{perm } \{b\} \in B} N(a_{b_1}, a_{b_2}, \ldots, a_{b_{2m}}) \times \left[ D(a_i, a_{i+1}) - D(a'_i, a'_{i+1}) \right] \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}}). 
\]  

(3.37)

Here the prime on the sum in (3.36) means that the sum over permutations of \( \{b\} \) is further restricted by the condition that \((i, i + 1) \in \{b_1, b_2, \ldots, b_{2m}\}\). (Thus it guarantees that the indices \( a_i \) and \( a_{i+1} \) only occur on the \( N \) and not on the \( D \)'s.) In (3.37), the double prime means that this sum is further restricted to have \( \{i, i + 1\} = \{b_{2k-1}, b_{2k}\} \) for some \( k > m \), and this is the reason for which the contribution with \((j = k)\) is excluded in the product. (Hence the double prime on the sum ensures that the indices \( a_i \) and \( a_{i+1} \) both occur on a single \( D \).)

Noting that when \( m = 0 \) there are no contributions to (3.36) (since it involves no normal ordered products), and using Lemma (2), (3.36) may be re-written as

\[
\sum_{m=0}^{n} \sum'_{\text{perm } \{b\} \in B} \left[ N(a_{b_1}, a_{b_2}, \ldots, a_i, a_{i+1}, \ldots, a_{b_{2m}}) - N(a'_{b_1}, a'_{b_2}, \ldots, a'_i, a'_{i+1}, \ldots, a'_{b_{2m}}) \right] \times \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}}) 
\]

\[
= \sum_{m=1}^{n} \sum_{\text{perm } \{c\} \in B} N(a_{c_1}, a_{c_2}, \ldots, a_{c_{2m-2}}) \left[ D(a_{i+1}, a_i) - D(a_{i+1}, a_{i+1}) \right] \prod_{j=m}^{n-1} D(a_{c_{2j-1}}, a_{c_{2j}}) 
\]

\[
= \sum_{p=0}^{n-1} \sum_{\text{perm } \{c\} \in B} N(a_{c_1}, \ldots, a_{c_{2p}}) \left[ D(a_{i+1}, a_i) - D(a_{i}, a_{i+1}) \right] \prod_{j=p+1}^{n-1} D(a_{c_{2j-1}}, a_{c_{2j}}) 
\]  

(3.38)
where $p = m - 1$, $c_j = b_j$ for $j < i$ and $c_j = b_{j+2}$ for $j \geq i$.

The second term of equation (3.37) vanishes for $m = n$ (as then it contains no contractions) and thus it can be re-written as

$$
\sum_{m=0}^{n} \sum_{\text{perm } \{b\} \in B} N(a_{b_1}, a_{b_2}, \ldots, a_{b_{2n}}) 
\times \left[ D(a_i, a_{i+1}) - D(a_i', a_{i+1}') \right]
\prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}})
\times \prod_{j=m+1}^{n} D(a_{b_{2j-1}}, a_{b_{2j}}).
$$

(3.39)

Recalling the definition of the $c$’s, it is now possible to see that (3.38) and (3.39) just differ by a sign, and thus from (3.37) we have proved that $\Delta = S(\{a\}) - S(\{a'\}) = 0$. Hence $\Delta$ vanishes for any two permutations.

- **Step 4.**

Since $S(\{a\}) = S(\{a'\})$ for any two permutations $\{a\}$ and $\{a'\}$, then from (3.33) one has

$$
T_{1,2,\ldots,2n} = \frac{1}{(2n)!} \sum_{\text{perm } \{a\}} S(\{a\})
= S(\{a\})
$$

(3.40)

(wickns)

where $S(\{a\})$ is just given by the usual form of Wick’s theorem with the ordered indices. This finishes the proof that Wick’s theorem holds in its usual form for non-symmetric normal ordered products and contractions, *provided* the contraction is a $c$-number.

**IV. CONCLUSIONS**

In this paper we have worked with a general split of the field operators into two parts and the corresponding non-symmetric normal ordered products and contractions. We have shown that, somewhat surprisingly, Wick’s theorem still holds in its usual form for such non-symmetric products as long as the contraction is a $c$-number. As a result, Wick’s theorem – an important calculational tool in field theory – is more general than was originally thought.

We now comment on another interesting idea inspired by the discussion of Wick’s theorem in the original paper on thermal field theory by Matsubara [9]. There, the definition of the contraction given for bosonic relativistic phonon field is *always symmetric* for all splits, and consistent with another definition of normal ordering $N(T)[\bullet]$;

$$
N^{(T)}[\psi_1 \psi_2 \ldots \psi_n] = N[T[\psi_1 \psi_2 \ldots \psi_n]].
$$

(nomat)
Note that $N^{(T)}[\bullet]$ is distinct from $N[\bullet]$, the definition of normal ordering which was given in section 1 and was used throughout this paper: by definition $N^{(T)}$ is always symmetric in the sense of (1.4) because of the way it is built out of the time-ordered product (hence our designation with a $(T)$ superscript [12]). It therefore gives a corresponding symmetric contraction

$$D^{(T)}[\psi_1, \psi_2] = T_{12} - N^{(T)}_{12} = \theta(t_1 - t_2)[\psi_1^+, \psi_2^-]_\sigma + \sigma \theta(t_2 - t_1)[\psi_2^+, \psi_1^-]_\sigma. \quad (4.2)$$

It is clear that there are in fact many possible splits of the field which may be used to define normal ordering, and even then the definition of $N^{(T)}$ shows that one can think of other definitions of normal ordering different from the $N$ given in section 1. Whichever definition is used, we have shown that even if the normal ordered product is not symmetric and the contraction is a $c$-number, Wick’s theorem can be used so that calculations are not too cumbersome.

In practice the only definite cases we know of have symmetric normal ordered products. Symmetry breaking problems expressed in terms of Bogoliubov transformations [6,7] and equilibrium thermal field theory, whether using path-ordered methods [1] or Thermo Field Dynamics [13], all use our definition of $N$ but have symmetric products. It is easy to see why symmetric products are so common; it is because the most useful normal ordered product is one which has zero expectation value. Thus taking the expectation value of Wick’s theorem implies that the $c$-number contraction must then be symmetric (since it is then equal to the expectation value of the two-point time-ordered function). In turn the operator version of Wick’s theorem now shows that the normal ordered operator must itself be symmetric.

However, there is at least one situation where we might envisage the appearance of non-symmetric products, and that is non-equilibrium field theory. In this case the environment is changing with time so we should expect the split, and hence the normal ordered products, also to change in time. It is then not at all obvious whether one can ever define a normal ordered products of fields at arbitrary times which have zero physical expectation values. Thus, reversing the argument of the previous paragraph, it is not at all clear that symmetric products will simplify the calculations, and therefore it is reasonable that non-symmetric products may prove to be a useful tool.

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[12] Only by using $N(T)$ rather than our $N$ for the normal ordering can the derivation of Wick’s theorem given in the appendix of Matsubara [9] be of use, as it is only given for a particular time ordering and requires symmetry arguments if one is to apply it for all possible time orderings. However, the definition of normal ordering for phonons given in the main text of [9] is in fact our $N[\bullet]$! Thus it is not consistent with the definition of the contraction in the main text, nor with the proof of Wick theorem proof in the appendix, whereas the $N(T)$ definition (4.1) is consistent.
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