We fix exactly and uniquely the infrared structure of the full gluon propagator in QCD, not solving explicitly the corresponding dynamical equation of motion. By construction, this structure is an infinite sum over all possible severe (i.e., more singular than $1/q^2$) infrared singularities. It reflects the zero momentum modes enhancement effect in the true QCD vacuum, which is due to the self-interaction of massless gluons. Its existence automatically exhibits a characteristic mass (the so-called mass gap). It is responsible for the scale of nonperturbative dynamics in the true QCD ground state. The theory of distributions, complemented by the dimensional regularization method, allows one to put the severe infrared singularities under firm mathematical control. By an infrared renormalization of a mass gap only, the infrared structure of the full gluon propagator is exactly reduced to the simplest severe infrared singularity, the famous $(q^2)^{-2}$. Thus we have exactly established the interaction between quarks (concerning its pure gluon (i.e., nonlinear) contribution) up to its unimportant perturbative part. This also makes it possible for the first time to formulate the gluon confinement criterion and intrinsically nonperturbative phase in QCD in a manifestly gauge-invariant ways.

PACS numbers: PACS numbers: 11.15.Tk, 12.38.Lg

I. INTRODUCTION

To say today that QCD is a nonperturbative (NP) theory is almost a tautology. The problem is how to define it exactly, since we know for sure that QCD has a perturbative (PT) phase as well because of asymptotic freedom (AF) \cite{1}. In order to define exactly the NP phase in QCD, let us start with one of the main objects in the Yang-Mills (YM) sector. The two-point Green’s function, describing the full gluon propagator, is (using Euclidean signature here and everywhere below)

$$D_{\mu\nu}(q) = i \left\{ T_{\mu\nu}(q) d(q^2, \xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}, \quad (1.1)$$

where $\xi$ is the gauge fixing parameter ($\xi = 0$ – Landau gauge, $\xi = 1$ – Feynman gauge) and $T_{\mu\nu}(q) = g_{\mu\nu} - (q_\mu q_\nu/q^2) = g_{\mu\nu} - L_{\mu\nu}(q)$. Evidently, $T_{\mu\nu}(q)$ is the transverse (physical) component of the full gluon propagator, while $L_{\mu\nu}(q)$ is its longitudinal (unphysical) one. The free gluon propagator is obtained by simply setting the full gluon form factor $d(q^2, \xi) = 1$ in Eq. (1.1). The dynamical equation of motion for the full gluon propagator is the so-called gluon Schwinger-Dyson (SD) equation, which is part of the whole SD system of dynamical equations of motion \cite{1}. The solutions of the gluon SD equation are supposed to reflect the complexity of the quantum structure of the QCD ground state. Precisely this determines one of the central roles of the full gluon propagator in the SD system of equations. The SD equation for the full gluon propagator is a highly nonlinear system of four-dimensional integrals, containing many different, unknown in general, propagators and vertices, which, in their turn, satisfy too complicated integral equations, containing different scattering amplitudes and kernels, so there is no hope for exact solution(s). However, in any case the solutions of this equation can be distinguished from each other by their behavior in the infrared (IR) limit, describing thus many (several) different types of quantum excitations and fluctuations of gluon field configurations in the QCD vacuum. The ultraviolet (UV) limit of these solutions is uniquely fixed by AF.

The IR asymptotics of the full gluon propagator can be either singular or smooth. However, the smooth behavior of the full gluon propagator (1.1) is possible only in one exceptional covariant gauge - the Landau gauge ($\xi = 0$) \cite{2}, i.e., it is a gauge artifact solution in this case. Being thus a gauge artifact, it can be related to none of the physical phenomena such as quark and gluon confinement or dynamical breakdown of chiral symmetry (DBCS), which are, by definition, manifestly gauge-invariant. To our best knowledge, beyond the covariant gauges, other than the Landau gauge, the smooth behavior is not known. Anyway, nobody knows how to relate the smooth asymptotics in any
covariant gauge to color confinement phenomenon, DBCS, etc. For example, it does not provide a linearly rising potential between heavy quarks "seen" by lattice QCD simulations \[ \text{[3]} \]. Hence we will not discuss it in what follows, though a solution with smooth asymptotics may exist as a formal one to the gluon SD equation.

Thus we are left with the IR singular behavior of the full gluon propagator only, which is possible in any gauge (in principle, the free gluon propagator can be also used in any gauge. The Feynman gauge free gluon propagator in the IR has been used by Gribov \[ \text{[4]} \] in order to investigate quark confinement within precisely the SD system of equations approach). The only problem is to decide which type of the IR singularities is to be accounted for. The free gluon propagator (see Eq. (1.1) with \( d(q^2) = 1 \)) has an exact power-type \( 1/q^2 \) IR singularity. So the IR singularities as much singular as \( 1/q^2 \) as \( q^2 \rightarrow 0 \) will be called PT IR singularities. The IR singularities which are more severe than the above-mentioned exact power-type IR singularity of the free gluon propagator will be called NP IR singularities, i.e., they are more severe than \( 1/q^2 \) as \( q^2 \rightarrow 0 \). They should be summarized (accumulated) into the full gluon propagator and described effectively correctly by its structure in the deep IR domain. Let us remind that for a long time from the very beginning of QCD it has been already well known that the QCD vacuum is really beset with the severe (or equivalently NP) IR singularities if standard PT is applied \[ \text{[1, 5, 6, 7, 8, 9, 10, 11, 12]} \]. "But it is to just this violent IR behavior that we must look for the key to the low energy and large distance hadron phenomena. In particular, the absence of quarks and other colored objects can only be understood in terms of the IR divergences in the self-energy of a color bearing objects" \[ \text{[10]} \]. It is worth emphasizing that in Ref. \[ \text{[13]} \] it is explicitly shown how the severe IR singularities inevitably appear in the QCD vacuum, providing thus the basis for the zero momentum modes enhancement (ZMME) effect there. So it is intrinsically peculiar to the true QCD ground state due to the self-interaction of massless gluons. Precisely this effect is reflected by the appearance of the severe IR singularities in the gluon propagator.

It is clear also that any deviation of the full gluon propagator from the free one in the IR, automatically requires an introduction of the corresponding mass scale parameter, responsible for the nontrivial dynamics in the IR region, the so-called mass gap (see below). This is important, since there is none explicitly present in the QCD Lagrangian (current quark mass cannot be considered as a mass gap, since it is not the renormalization group invariant). Of course, such gluon field configurations, which are to be described by the severely IR structure of the full gluon propagator, can be only of dynamical origin. The only dynamical mechanism in QCD which can produce such configurations in the vacuum, is the self-interaction of massless gluons. So the above-mentioned mass gap appears on dynamical ground. Let us remind that precisely this self-interaction in the UV limit leads to AF.

The main purpose of this Letter is to establish exactly the deep IR structure of the full gluon propagator, not solving the gluon SD equation directly, which is a formidable task, anyway. On this basis we will be able to derive the gluon confinement criterion in a manifestly gauge-invariant way. However, it is convenient first to emphasize the distribution nature of the severe (i.e., NP) IR singularities. For this purpose, in the next section we will explicitly introduce a few useful formulae from the distribution theory (DT) \[ \text{[14]} \], complemented by the dimensional regularization (DR) method \[ \text{[15]} \], which are crucial in our investigation.

### II. IR DIMENSIONAL REGULARIZATION WITHIN THE DISTRIBUTION THEORY

In general, all the Green’s functions in QCD are generalized functions, i.e., they are distributions. This is true especially for the NP IR singularities due to the self-interaction of massless gluons in the QCD vacuum. They present a rather broad and important class of functions with algebraic singularities, i.e., functions with nonsummable singularities at isolated points \[ \text{[14]} \] (at zero in our case). Roughly speaking, this means that all relations involving distributions should be considered under corresponding integrals, taking into account the smoothness properties of the corresponding class of test functions (for example, \( \varphi(q) \) below). In principle, any regularization scheme (i.e., how to parameterize the severe IR singularities and thereby to put them under control) can be used; it should, however, be compatible with DT \[ \text{[14]} \].

Let us consider the positive definite \((P > 0)\) squared (quadratic) Euclidean form \( P(q) = q_1^2 + q_2^2 + q_3^2 + \ldots + q_n^2 = q^2 \), where \( n \) is the number of the components. The generalized function (distribution) \( P^\lambda(q) \), where \( \lambda \) is, in general, an arbitrary complex number, is defined as \( P^\lambda(q) = \int_{P > 0} P^\lambda(q) \varphi(q) d^nq \). At \( \text{Re} \lambda \geq 0 \) this integral is convergent and is an analytic function of \( \lambda \). Analytical continuation to the region \( \text{Re} \lambda < 0 \) shows that it has a simple pole at points \[ \text{[14]} \]

\[
\lambda = -\frac{n}{2} - k, \quad k = 0, 1, 2, 3, \ldots
\]  

(2.1)

In order to actually define the system of the SD equations in the deep IR domain, it is necessary to introduce the IR regularization parameter \( \epsilon \), defined as \( D = n + 2\epsilon \), \( \epsilon \rightarrow 0^+ \) within a gauge-invariant DR method \[ \text{[15]} \]. As a result, all the Green’s functions and "bare" parameters should be regularized with respect to \( \epsilon \) (see next sections), which is
to be set to zero at the end of the computations. The structure of the NP IR singularities is then determined (when
n is even number) as follows [14]:

\[(q^2)^\lambda = \frac{C^{(k)}}{\lambda + (D/2) + k} + \text{finite terms}, \quad (2.2)\]

where the residue is

\[C^{(k)}_{-1} = \frac{\pi^{n/2}}{2^{2k}k!\Gamma((n/2) + k)} \times L^k \delta^n(q) \quad (2.3)\]

with \(L = (\partial^2/\partial q_0^2) + (\partial^2/\partial q_i^2) + \ldots + (\partial^2/\partial q_{n-1}^2)\).

Thus the regularization of the NP IR singularities (2.2) is nothing but the so-called Laurent expansion that is
dimensionally regularized. Let us underline its most remarkable feature. The order of singularity does not depend on
\(\lambda, n \), and \(k\). In terms of the IR regularization parameter \(\epsilon\), it is always a simple pole \(1/\epsilon\). This means that all power
terms in Eq. (2.2) will have the same singularity, i.e.,

\[(q^2)^{-\frac{\lambda}{2}} = \frac{1}{\epsilon} C^{(k)}_{-1} + \text{finite terms}, \quad \epsilon \rightarrow 0^+, \quad (2.4)\]

where we can put \(D = n\) now (i.e., after introducing this expansion). By "finite terms" here and everywhere a
number of necessary subtractions under corresponding integrals is understood. However, the residue at a pole will be
dramatically changed from one power singularity to another. This means different solutions to the whole system of the
SD equations for different set of numbers \(\lambda, n\), and \(k\). Different solutions mean, in turn, different vacua. Thus in this
picture different vacua are to be labelled by two independent numbers: the exponent \(\lambda, n\), and \(k\). At given number of
\(D(= n)\) the exponent \(\lambda\) is always negative being integer if \(D(= n)\) is an even number or fractional if \(D(= n)\) is an
odd number. The number \(k\) is always integer and positive and precisely it determines the corresponding residue at
the pole, see Eq. (2.3). It would be not surprising if these numbers were somehow related to the nontrivial topology
of the nD QCD vacuum.

Concluding, let us note that the structure of the severe IR singularities in Euclidean space is much simpler than in
Minkowski space, where kinematical (unphysical) singularities due to the light cone also exist [10]. In this case it is
rather difficult to untangle them correctly from the dynamical singularities, the only ones which are important for the
calculation of any physical observable. Also the consideration is much more complicated in configuration space [14].
That is why we always prefer to work in momentum space (where propagators do not depend explicitly on the number
of dimensions) with Euclidean signature. We also prefer to work in covariant gauges in order to avoid peculiarities of
the noncovariant gauges [10, 16], for example how to untangle the gauge pole from the dynamical one.

III. THE GENERAL STRUCTURE OF THE FULL GLUON PROPAGATOR

For the above-mentioned purpose, namely, how to define the NP phase in QCD, it is convenient to begin with the
exact decomposition of the full gluon form factor in Eq. (1.1) as follows:

\[d(q^2) = d(q^2) - d^{PT}(q^2) + d^{PT}(q^2) = d^{NP}(q^2) + d^{PT}(q^2), \quad (3.1)\]

where, for simplicity, the dependence on the gauge fixing parameter is omitted. In fact, this formal equation represents
one unknown function (the full gluon form factor) as an exact sum of the two other unknown functions, which can be
always done. So at this stage there is no approximation made (only exact algebraic manipulations). We would like to
let the PT part of this exact decomposition to be responsible for the known UV asymptotics (since it is fixed by AF) of
the full gluon propagator, while the NP part is chosen to be responsible for its unknown yet IR asymptotics. It is worth
emphasizing that in realistic models of the full gluon propagator, the NP part usually correctly reproduces its deep
IR asymptotics, determining thus the strong intrinsic influence of the IR properties of the theory on its NP dynamics.
Evidently, the decomposition (3.1) represents an exact subtraction of the PT contribution at the fundamental gluon
level, and consequently both terms in the right-hand-side of Eq. (3.1) are determined in the whole momentum range,
\((0, \infty)\). Let us emphasize that the exact gluon form factor \(d(q^2)\) being also NP, nevertheless, is "contaminated" by
the PT contributions, while \(d^{NP}(q^2)\) due to the subtraction (3.1) is free of them, i.e., it is truly NP.
Substituting the exact decomposition (3.1) into the full gluon propagator (1.1), one obtains

\[ D_{\mu\nu}(q) = D_{\mu\nu}^{INP}(q) + D_{\mu\nu}^{PT}(q), \]  

where

\[ D_{\mu\nu}^{INP}(q) = iT_{\mu\nu}(q)d^{NP}(q^2)\frac{1}{q^2} = iT_{\mu\nu}(q)d^{INP}(q^2), \]

\[ D_{\mu\nu}^{PT}(q) = iT_{\mu\nu}(q)d^{PT}(q^2) + \xi L_{\mu\nu}(q)\frac{1}{q^2}. \]

Here the superscript "INP" is the short-hand notation for intrinsically NP. Its definition will given below. The exact decomposition (3.2) has a remarkable feature. The explicit gauge dependence of the full gluon propagator is exactly shifted from its INP part to its PT part. In other words, we want the INP part to be always transverse, while leaving the PT part to be of arbitrary gauge. This exact separation will have also a dynamical ground. It is clear also that the PT part of the full gluon propagator is, by definition, as much singular as the free gluon propagator’s exact power-type IR singularity. This is the first reason why the longitudinal part of the full gluon propagator, which has the same exact IR singularity, has been shifted to its PT part, and the existence of which is determined by AF.

As was mentioned above, we want the INP gluon form factor to be responsible for the deep IR structure of the full gluon propagator, which is saturated by the severe IR singularities. So there is a problem how to take them into account analytically in terms of the full gluon propagator. For this aim, it is convenient to introduce the auxiliary INP gluon form factor as follows:

\[ d_{\lambda_k}^{INP}(q^2, \Delta^2) = (\Delta^2)^{-\lambda_k-1}(q^2)^{\lambda_k} f_{\lambda_k}(q^2), \]

where the exponent \( \lambda_k \) is, in general, an arbitrary complex number with \( Re \lambda_k < 0 \) (see below). The mass squared parameter \( \Delta^2 \) (the above-mentioned mass gap) is responsible for the scale of NP dynamics in the IR region in our approach. The functions \( f_{\lambda_k}(q^2) \) are, by definition, dimensionless, regular at zero, and otherwise remaining arbitrary, but preserving AF in the UV limit. And finally the number \( k \) is a positive integer, i.e., \( k = 0, 1, 2, 3, \ldots \) (see above). Evidently, a real INP gluon form factor \( d_{\lambda_k}^{INP}(q^2) \), which now should depend on the mass gap as well, i.e., \( d_{\lambda_k}^{INP}(q^2, \Delta^2) \), is a sum over all \( d_{\lambda_k}^{INP}(q^2, \Delta^2) \).

However, this is not the whole story yet. Since we are especially interested in the deep IR structure of the full gluon propagator, the arbitrary functions \( f_{\lambda_k}(q^2) \) should be also expanded around zero in the form of the Taylor series in powers of \( q^2 \), i.e,

\[ f_{\lambda_k}(q^2) = \sum_{m=0}^{[-\lambda_k]-(n/2)} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0) + \sum_{m=\lfloor -\lambda_k \rfloor -(n/2)+1}^{\infty} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0), \]

where \( [-\lambda_k] \) denotes its integer number and \( n \) is a number of the components in the Euclidean squared form \( q^2 \) (see above). Also

\[ f_{\lambda_k}^{(m)}(0) = \frac{d^m f_{\lambda_k}(q^2)}{d(q^2)^m}\bigg|_{q^2=0}. \]

As a result, we will be left with the finite sum of power terms with an exponent decreasing by unity starting from \( -\lambda_k \). All other remaining terms from the Taylor expansion (3.6), starting from the term having already a PT IR singularity (the second sum in Eq. (3.6)), should be shifted to the PT part of the full gluon propagator in Eq. (3.2). So the INP part of the full gluon form factor becomes

\[ d_{\lambda_k}^{INP}(q^2, \Delta^2) = \sum_{\lambda_k} d_{\lambda_k}^{INP}(q^2, \Delta^2) = \sum_{\lambda_k} (\Delta^2)^{-\lambda_k-1}(q^2)^{\lambda_k} \sum_{m=0}^{[-\lambda_k]-(n/2)} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0), \]

while the piece which is to be shifted to the PT part (3.4) of the full gluon propagator (3.2) can be shown to have only PT IR singularities with respect to the gluon momentum, indeed [13].
IV. 4D QCD

We are particularly interested in 4D QCD (i.e., \( n = 4 \)), which is a realistic dynamical theory of strong interactions not only at the fundamental quark-gluon level, but at the hadronic level as well. Let us discuss the gluon propagator (3.2) in more detail for 4D QCD, i.e., QCD itself. On account of the expansion (3.8) and Eq. (2.1) at \( n = 4 \) with the obvious identification \( \lambda_k \equiv \lambda \), its INP part becomes

\[
d^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} d_k^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)_{-1+k}(q^2)^{-2-k} \sum_{m=0}^{k} \frac{(q^2)^m}{m!} f_k^{(m)}(0),
\]

and \( f_k^{(0)}(0) = f_k(0) \). Obviously, in this case the subscript ”\( \lambda_k \)” should be replaced by the subscript ”\( k \)”, since \( \lambda_k = -2-k \) for \( k = 0, 1, 2, \ldots \). Thus \( d^{INP}(q^2, \Delta^2) \) describes the true (physical) NP vacuum of QCD, while \( d_k^{INP}(q^2, \Delta^2) \) describes the auxiliary ones, and the former is an infinite sum of the latter ones. The expansion (4.1) is obviously the Laurent expansion in the inverse powers of the gluon momentum squared, which every term ends at the simplest NP IR singularity \((q^2)^{-2}\). The only physical quantity (apart from the mass gap, of course) which can appear in this expansion is the coupling constant squared in the corresponding powers. In QCD it is dimensionless and is evidently included into the \( f_k \) functions. Let us note in advance that all the finite numerical factors and constants (for example, the coupling constant) play no independent role in the presence of a mass gap.

It is instructive to show explicitly expansions for a few first different \( d_k^{INP}(q^2, \Delta^2) \) namely

\[
\begin{align*}
d_0^{INP}(q^2, \Delta^2) &= \Delta^2 f_0(0)/(q^2)^2, \\
d_1^{INP}(q^2, \Delta^2) &= (\Delta^2)^2 f_1(0)/(q^2)^3 + (\Delta^2)^2 f_1^{(1)}(0)/(q^2)^2, \\
d_2^{INP}(q^2, \Delta^2) &= (\Delta^2)^3 f_2(0)/(q^2)^4 + (\Delta^2)^3 f_2^{(1)}(0)/(q^2)^3 + \frac{1}{2} (\Delta^2)^3 f_2^{(2)}(0)/(q^2)^2, \\
d_3^{INP}(q^2, \Delta^2) &= (\Delta^2)^4 f_3(0)/(q^2)^5 + (\Delta^2)^4 f_3^{(1)}(0)/(q^2)^4 + \frac{1}{2} (\Delta^2)^4 f_3^{(2)}(0)/(q^2)^3 + \frac{1}{6} (\Delta^2)^4 f_3^{(3)}(0)/(q^2)^2,
\end{align*}
\]

and so on. Apparently, there is no way that such kind of an infinite series (as it is present in expansion (4.1), on account of the relations (4.2)) could be summed up into the finite functions, i.e., functions which could be regular at zero. That is why the above-mentioned smooth gluon propagator is, in general, very unlikely to exist. Let also note in advance that the simplest NP IR singularity \((q^2)^{-2}\) is present in each expansion, which emphasizes its special and important role (see below).

The expansion (4.1), on account of the relations (4.2), can be equivalently written down as follows:

\[
d^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (q^2)^{-2-k} \sum_{m=0}^{\infty} \frac{1}{m!} (\Delta^2)^{k+m+1} f_k^{(m)}(0) = \sum_{k=0}^{\infty} (q^2)^{-2-k} (\Delta^2)^{k+1} \sum_{m=0}^{\infty} \frac{1}{m!} \phi_{k,m}(0),
\]

where we use the relation

\[
f_k^{(m)}(0) = (\Delta^2)^{-m} \phi_{k,m}(0),
\]

which obviously follows from the relation (3.7), since all \( f_k^{(m)}(0) \) have the dimensions of the inverse mass squared in powers of \( m \), i.e., \( [f_k^{(m)}(0)] = [\Delta^{-2m}] = [\Delta^2]^{-m} \). Here \( \phi_{k,m}(0) \) are dimensionless quantities, by definition. This expansion explicitly shows that the coefficient at each NP IR singularity is an infinite series itself. It also shows that we can analyze the IR properties of the INP part of the full gluon form factor in terms of the mass gap \( \Delta^2 \) and the dimensionless quantities \( \phi_{k,m}(0) \) only, which is very convenient from a technical point of view (see below). This form of the Laurent expansion shows also clearly the dynamical context of the INP part of the full gluon propagator.

The regularization of the NP IR singularities in QCD is determined by the Laurent expansion (2.4) at \( n = 4 \) as follows:

\[
(q^2)^{-2-k} = \frac{1}{\epsilon} a(k) [\delta^4(q)]^{(k)} + f.t., \quad \epsilon \to 0^+,
\]

(4.5)
where \(a(k)\) is a finite constant depending only on \(k\) and \([\delta^k(q)]^{(k)}\) represents the \(k\)th derivative of the \(\delta\)-function (see Eq. (2.3)). We point out that after introducing this expansion everywhere one can fix the number of dimensions, indeed, i.e., put \(D = n = 4\) for QCD without any further problems, since there will be no other severe IR singularities with respect to \(\epsilon\) as it goes to zero, but those explicitly shown in this expansion. Thus, as it follows from the Laurent expansion (4.5), any power-type NP IR singularity, including the simplest one, scales as \(1/\epsilon\) as it goes to zero. Just this plays a crucial role in the IR renormalization of the theory within our approach.

V. IR RENORMALIZATION OF A MASS GAP

In the presence of such severe IR singularities (4.5), all the "bare" parameters (dimensional or not, does not matter) should, in principle, depend on \(\epsilon\) as well, i.e., they become IR regularized. Let us thus introduce the following relation

\[ \Delta^2 = X_\Delta(\epsilon) \bar{\Delta}^2, \quad \epsilon \to 0^+, \]  

(5.1)

where \(X_\Delta(\epsilon)\) is the corresponding IR multiplicative renormalization (IRMR) constant. Here and below, the quantities with an overbar are, by definition, IR renormalized, i.e., they exist as \(\epsilon\) goes to zero. However, in the above-mentioned paper \(^13\) it has been shown that neither the QCD coupling constant squared nor the gauge fixing parameter are to be IR renormalized, i.e., they are IR finite from the very beginning. As was mentioned above, they can appear only in \(\varphi^{(m)}(0)\) quantities, i.e., in \(\varphi_{k,m}(0)\) ones. So these quantities also are IR finite from the very beginning, which means that we can put \(\varphi_{k,m}(0) = \bar{\varphi}_{k,m}(0)\). This is so indeed, since the rest in these quantities is simply the product of the numerical factors like \(\pi\)’s in negative powers, eigenvalues of the color group generators (we are not considering the numbers of different colors and flavors as free parameters of the theory), etc.

We already know that all the NP IR singularities, which can appear in the full gluon propagator scale as \(1/\epsilon\) with respect to \(\epsilon\) (see Eq. (4.5)). So let us introduce the so-called IR convergence condition as follows:

\[ X_\Delta^{1+k}(\epsilon) = \epsilon \bar{A}_k, \quad \epsilon \to 0^+, \]  

(5.2)

where we put \(\bar{A}_k = \bar{A}_k / \sum_{m=0}^{\infty} (1/m!) \bar{\varphi}_{k,m}(0)\), for convenience. The cancellation of the NP IR singularities with respect to \(\epsilon\) then will be guaranteed in Eq. (4.3), and one obtains the finite (nonzero) result in the \(\epsilon \to 0^+\) limit for every \(k = 0, 1, 2, 3, \ldots\). Here \(A_k\) and \(\bar{A}_k\) are some arbitrary, but finite constants, not depending on \(\epsilon\) as it goes to zero.

It makes sense to emphasize now that this IR convergence condition should be valid at any \(k\), in particular at \(k = 0\), so it follows \(X_\Delta(\epsilon) = \epsilon \bar{A}_0, \quad \epsilon \to 0^+\), which means that in this case one is able to establish an explicit solution for the mass gap’s IRMR constant. Thus the mass gap is IR renormalized as follows:

\[ \Delta^2 = \epsilon \bar{\Delta}^2, \quad \epsilon \to 0^+, \]  

(5.3)

where we include an arbitrary but finite constant \(\bar{A}_0\) into the IR renormalized mass gap \(\bar{\Delta}^2\), and retaining the same notation, for simplicity. This means that in what follows we can put it to unity, not losing generality, i.e., \(\bar{A}_0 = 1\).

It is instructive to rewrite the expansion (4.3) as follows:

\[ d^{\text{INP}}(q^2, \Delta^2) = \Delta^2(q^2)^{-2} \sum_{m=0}^{\infty} \frac{1}{m!} \bar{\varphi}_{0,m}(0) + \sum_{k=1}^{\infty} (q^2)^{-2-k}(\Delta^2)^{k+1} \sum_{m=0}^{\infty} \frac{1}{m!} \bar{\varphi}_{k,m}(0). \]  

(5.4)

Taking into account now the above-described scaling with respect to \(\epsilon\), the INP part of the full gluon form factor effectively becomes

\[ d^{\text{INP}}(q^2, \Delta^2) = \Delta^2(q^2)^{-2} \sum_{m=0}^{\infty} \frac{1}{m!} \bar{\varphi}_{0,m}(0), \]  

(5.5)

where finite dimensionless quantities \(\bar{\varphi}_{0,m}(0)\) contain the IR finite coupling constant squared (in different powers, of course, so the whole expansion in \(m\) includes all powers in the coupling constant squared). It also contains the different combinations of the finite numerical factors only. Evidently, no other terms, explicitly shown in the expansion (5.4) as the second sum, will survive in the \(\epsilon \to 0^+\) limit. They become terms of the order of \(\epsilon\), at least, in this limit (they
start from \((\Delta^2)^2 \sim \epsilon^2\), while \((q^2)^{-2-k}\) always scales as \(1/\epsilon\). In other words, in the Laurent expansion (5.4) only the terms which contain the simplest NP IR singularity with respect to the gluon momentum \((q^2)^{-2}\) will survive as \(\epsilon \to 0^+\). In this case all terms (apart from the simplest NP IR singularity) will be additionally suppressed in positive powers of \(\epsilon\). The IR renormalization of the mass gap \(\Delta^2\) in accordance with Eq. (5.3) cancels \(1/\epsilon\), which comes from the regularization of \((q^2)^{-2}\). The so-called "f.t." terms in the Laurent expansion (4.5) become terms of the order of \(\epsilon\), at least, so here and everywhere they vanish in the \(\epsilon \to 0^+\) limit.

VI. ZMME QUANTUM MODEL OF THE QCD GROUND STATE

The true QCD ground state is a very complicated confining medium, containing many types of gluon field configurations, components, ingredients and objects of different nature. Its dynamical and topological complexity means that its structure can be organized at both the quantum and classical levels \[1,17\]. Our quantum, dynamical model of the true QCD true vacuum is based on the existence and the importance of such kind of the NP excitations and fluctuations of gluon field configurations which are due to the self-interaction of massless gluons only, without explicitly involving some extra degrees of freedom. They are to be summarized (accumulated) into the INP part of the full gluon propagator, and are to be effectively correctly described by its strongly singular structure in the deep IR domain (for simplicity, we will call them as singular gluon field configurations). At this stage, it is difficult to identify actually which type of gauge field configurations can be behind the singular gluon field configurations in the QCD ground state (i.e., to identify relevant field configurations: chromomagnetic, self-dual, stochastic, etc.). However, if these gauge field configurations can be absorbed into the gluon propagator (i.e., if they can be considered as solutions to the corresponding SD equation), then its severe IR singular behavior is a common feature for all of them. Being thus a general phenomenon, the existence and the importance of quantum excitations and fluctuations of the severely IR degrees of freedom inevitably lead to the ZMME effect in the QCD ground state. That is why we call our model of the QCD ground state as the ZMME quantum model, or simply zero modes enhancement (ZME, since we work always in the momentum space). For preliminary investigation of this model see our papers \[18,19\] and references therein.

Our approach to the QCD true ground state, based on the ZMME phenomenon there, in terms of the gluon propagator, can be analytically formulated as follows:

\[
D_{\mu \nu}(q) = D_{\mu \nu}^{INP}(q, \Delta) + D_{\mu \nu}^{PT}(q),
\]

where the INP part of the full gluon propagator before the IR renormalization is

\[
D_{\mu \nu}^{INP}(q, \Delta) = iT_{\mu \nu}(q)d^{INP}(q^2, \Delta^2) = iT_{\mu \nu}(q) \times \left[ \Delta^2 \bar{A}_0(q^2)^{-2} + \sum_{k=1}^{\infty} (\Delta^2)^{1+k} a_k(q^2)^{-2-k} \right],
\]

where \(a_k = (\bar{A}_k/\bar{A}_k)\) and we used Eq. (5.4), on account of the relation (5.2) for the coefficients. In fact, Eq. (6.2) is already partially IR renormalized with respect to the coefficients, but not with respect to the mass gap \(\Delta^2\) and the NP IR singularities \((q^2)^{-2-k}\). After the IR renormalization it effectively becomes

\[
D_{\mu \nu}^{INP}(q, \Delta) = iT_{\mu \nu}(q)d^{INP}(q^2, \Delta^2) = iT_{\mu \nu}(q)\Delta^2(q^2)^{-2},
\]

since only the first term in Eq.(6.2) survives in the \(\epsilon \to 0^+\) limit, as was explained above. An arbitrary but finite constant \(\bar{A}_0\) has been included into the mass gap with retaining the same notation, for convenience. The PT part of the full gluon propagator in any case remains undetermined (see remarks below).

A. Confinement criterion for gluons

It is worth discussing the properties of the obtained solution for the full gluon propagator in more detail. It is already clear that by the IR renormalization of the mass gap only, we can render the full gluon propagator IR finite from the very beginning, t.e., to put \(D(q) \equiv \bar{D}(q) = D_{\mu \nu}^{INP}(q) + D_{\mu \nu}^{PT}(q)\) (this has been rigorously proven in Ref. \[13\]). In principle, two different cases should be considered due to the distribution nature of the simplest NP IR singularity\((q^2)^{-2}\), which saturates its INP part in Eq. (6.3).
I. If there is the explicit integration over the gluon momentum, then from Eq. (6.3) it follows

$$D^{\text{INP}}(q) \equiv D^{\text{INP}}_{\mu \nu}(q, \Delta) = i T_{\mu \nu}(q) \Delta^2 \pi^2 \delta^4(q), \quad (6.4)$$

i.e., in this case we have to replace the NP IR singularity \((q^2)^{-2}\) in Eq. (6.3) by its \(\delta\)-type regularization (4.5) at \(k = 0\). We also always should take into account the relation (5.3) for the IR renormalization of the mass gap in order to express all relations in terms of the IR renormalized quantities. The \(\delta\)-type regularization is valid even for the multi-loop skeleton diagrams, where the number of independent loops is equal to the number of the gluon propagators. In the multi-loop skeleton diagrams, where these numbers do not coincide (for example, in the diagrams containing three or four-gluon proper vertices), the general regularization (4.5) should be used (i.e., derivatives of the \(\delta\)-functions), and not the product of the \(\delta\)-functions at the same point, which has no mathematical meaning in the DT sense [14].

II. If there is no explicit integration over the gluon momentum, then the full gluon propagator is reduced to

$$D(q) \equiv D_{\mu \nu}(q) = D_{\mu \nu}^{\text{PT}}(q)$$

in the \(\epsilon \to 0^+\) limit, since in this case the function \((q^2)^{-2}\) in Eq. (6.3) cannot be treated as the distribution. Only the relation (5.3) again comes out into the play. So the INP part of the full gluon propagator (6.3) in this case disappears as \(\epsilon\) as \(\epsilon \to 0^+\), namely

$$D^{\text{INP}}_{\mu \nu}(q, \Delta^2) \sim \epsilon, \quad \epsilon \to 0^+. \quad (6.5)$$

This means that any amplitude (more precisely its INP part, see below) for any number of soft-gluon emissions (no integration over their momenta) will vanish in the IR limit in our picture. In other words, there are no transverse gluons in the IR, i.e., at large distances (small momenta) there is no possibility to observe gluons experimentally as free particles. So this behavior can be treated as the gluon confinement criterion (see also Ref. [12]), and it supports the consistency of the exact solution (6.3) for the INP part of the full gluon propagator. Evidently, this behavior does not explicitly depend on the gauge choice in the full gluon propagator, i.e., it is a manifestly gauge-invariant as it should be, in principle. Concluding this Sect., it is worth underlining that the gluon confinement criterion (6.5) is valid in the general case as well, i.e., before explicitly showing that the QCD coupling constant squared and the gauge fixing parameter are IR finite [13].

subsection INP phase in QCD

For the sake of self-consistency and transparency of our approach to low-energy QCD, it is convenient to discuss in more detail what we mean by the INP phase in QCD. The INP part of the full gluon propagator is, in general, given in Eqs. (6.4) and (6.5) for the above-discussed two different cases. Let us remind that within our approach all the severe IR singularities of the dynamical origin possible in QCD are to be incorporated into the full gluon propagator and are to be effectively correctly described by its INP part. So all other QCD Green’s functions can be considered as regular functions with respect to all the gluon momenta involved. If, nevertheless, they might be singular, then it would require a completely different investigation, anyway.

In principle, all other fundamental quantities in QCD could be formally decomposed similar to the decomposition (6.1) for the full gluon propagator. Evidently, this should be done for quantities, which explicitly depend on the gluon momenta, i.e., proper vertices. The decomposition does not make any sense for the coupling constant, quark and ghost propagators, since they do not depend on the gluon momentum. This is also true for the quark- and ghost-gluon proper vertices, since they depend on the quark and ghost momenta, which are completely independent from the gluon momentum, playing the role of the momentum transfer in these vertices. So the only proper vertices which makes sense to decompose are the pure gluon vertices, since they crucially depend on all the gluon momenta involved.

However, we define the INP phase in QCD in more general terms, which includes the decomposition of the full gluon propagator only as follows:

(i) It is always transverse, i.e., it depends only on physical degrees of freedom of gauge bosons.

(ii) Before the IR renormalization, the presence of the NP IR singularities \((q^2)^{-2-k}\), \(k = 0, 1, 2, 3, ...\) is only possible.

(iii). After the IR renormalization, the INP part of the full gluon propagator is fully saturated by the simplest NP IR singularity and all other NP IR singularities will be additionally suppressed in the \(\epsilon \to 0^+\) limit.

(iv). There is an inevitable dependence on the mass gap \(\Delta^2\), so that when it formally goes to zero, then the INP phase vanishes, while the PT phase survives.

This definition implies that the INP part of any multi-loop skeleton diagram in QCD should contain only the INP parts of all the corresponding gluon propagators. At the same time, the PT part of any multi-loop skeleton diagram always remains of arbitrary gauge. It may even contain the terms, where the NP IR singularities are present along with the PT IR ones as well (the so-called general PT term). Let us also remind, that at the level of a single full gluon propagator, its PT part is defined as to be of arbitrary gauge and is as much singular as \(1/q^2\). The difference
between the NP IR singularities \((q^2)^{-2-k}\) and the PT IR singularity \((q^2)^{-1}\) is that the latter is not defined by its own Laurent expansion (4.5) that is dimensionally regularized like the former ones. That is why it does not require the IR renormalization program itself.

VII. DISCUSSION

Evidently, the ZMME mechanism for quark confinement is nothing but the well forgotten IR slavery (IRS) one, which can be equivalently referred to as a strong coupling regime \([1]\). Indeed, at the very beginning of QCD it was expressed a general idea \([5, 6, 7, 8, 9, 10, 11, 12]\) that the quantum excitations of the IR degrees of freedom, because of self-interaction of massless gluons in the QCD vacuum, made it only possible to understand confinement, DCSB and other NP effects. In other words, the importance of the deep IR structure of the true QCD vacuum has been emphasized as well as its relevance to quark confinement, DCSB, etc., and the other way around. This development was stopped by the wide-spread wrong opinion that the severe IR singularities cannot be put under control. We have explicitly shown that the correct mathematical theory of quantum YM physical theory is the theory of distributions (theory of generalized functions) \([14]\), complemented by the DR method \([15]\). They provide a correct treatment of these severe IR singularities without any problems. Thus we come back to the old idea but on a new basis that is why it becomes new (“new is well forgotten old”). In other words, we only put the IRS mechanism of quark confinement on a firm mathematical ground provided by DT.

There are a lot of direct and indirect evidences in favor of this behavior, Eq. (6.3) \([13]\). Let us note only that as a possible IR asymptotics of the full gluon propagator in different covariant and non-covariant gauges, it has been already successfully investigated in Refs. \([20, 21, 22, 23, 24, 25]\). Following the pioneering paper of Pagels \([12]\), we always used it in our preliminary investigation of the quark confinement problem (see Refs. \([18, 19]\) and references therein). It is worth emphasizing, however, that Eq. (6.3) is an exact result, and expresses not only the deep IR asymptotics of the full gluon propagator. Just this behavior \((6.3)\) in the continuous theory provides the linear rising potential (indicative of confinement) between heavy quarks "seen" by lattice simulations \([3]\).

VIII. CONCLUSIONS

Emphasizing the highly nontrivial structure of the true QCD ground state in the deep IR region, one can conclude:

1). The self-interaction of massless gluons is only responsible for the ZMME effect in the true QCD vacuum, which in its turn, is to be taken into account by the deep IR structure of the full gluon propagator.

2). The full gluon propagator thus is inevitably more singular in the IR than its free counterpart.

3). This requires the existence of a mass gap, which is responsible for the NP dynamics in the QCD vacuum. It appears on dynamical ground due to the self-interaction of massless gluons only.

4). We decompose algebraically (i.e., exactly) the full gluon propagator as a sum of its INP and PT parts. We additionally distinguish between them dynamically by the different character of the IR singularities in each part.

5). We have exactly established the general IR structure of the full gluon propagator in QCD as an infinite sum over all possible NP IR singularities (see the expansion (4.1) or equivalently (4.3)).

6). The next step is to regularize them correctly, i.e., to use the Laurent expansion (4.5) that is dimensionally regularized.

7). We emphasize once more that the IR renormalization program is based on an important observation that the NP IR singularities \((q^2)^{-2-k}\), being distributions, always scale as \(1/\epsilon\), not depending on the power of the singularity \(k\), i.e., \((q^2)^{-2-k} \sim 1/\epsilon\). It is easy to understand that otherwise none of the IR renormalization program in the INP phase of QCD and QCD itself would be possible.

8). The IR renormalization of the initial mass gap (Eq. (5.3)) is only needed in order to fix uniquely and exactly the IR structure of the full gluon propagator in QCD. It is saturated by the simplest NP IR singularity, the famous \((q^2)^{-2}\), Eq. (6.3). The mass gap gains contributions from all orders of PT in the QCD coupling constant squared, which remains IR finite.

9). On this basis, we have formulated the ZMME model of the true QCD ground state. Due to the distribution nature of the NP IR singularities, two different types of the IR renormalization of the INP part of the full gluon propagator are required, preserving, nevertheless, its IR finiteness.

10). This makes it possible to establish the gluon confinement criterion in a manifestly gauge-invariant way in Eq. (6.5).

11). In the same way, we define exactly the INP phase in QCD at the fundamental gluon level. The corresponding decomposition of the full gluon propagator is only needed in order to firmly control the IR region in QCD within our approach (only it contains explicitly the mass gap).
Our general conclusions are:
I. The NP structure of the true QCD ground state is to be described by the IR structure of the full gluon propagator.
II. In its turn, it is an infinite sum over all possible NP IR singularities. Due to their distribution nature, any solution to the gluon SD equation has to be always present in the form of the corresponding Laurent expansion. It makes it possible to control both, the whole expansion as well as its each term, by the correct use of DT.
III. The mass gap responsible for the NP dynamics in the true QCD ground state is required. This is important, since there is none in the QCD Lagrangian.
IV. Complemented by the DR method, DT puts the treatment of the NP IR singularities on a firm mathematical ground. So there is no place for theoretical uncertainties. The wide-spread opinion that they cannot be controlled is not justified.
V. This makes it possible to fix uniquely the IR structure of the full gluon propagator in QCD, not solving directly the corresponding SD equation itself. Thus we have exactly established the interaction between quarks (concerning its pure gluon (i.e., nonlinear) contribution up to its unimportant PT part).
VI. This somehow astonishingly radical result has been achieved at the expense of the PT part of the full gluon propagator. It remains of arbitrary covariant gauge and its functional dependence cannot be determined.

However, let us note in advance that our theory (which we call INP QCD) will be additionally defined by the subtraction of all types of the PT contributions in order to completely decouple it from QCD as a whole (the first step in these subtractions has been already done in Eq. (3.1)). To discuss this point and many other interesting features of our approach in detail is, of course, beyond the scope of this Letter.

A financial support from HAS-JINR Scientific Collaboration Fund (P. Levai) and Hungarian Scientific Fund OTKA T30171 (K. Toth) is to be acknowledged.

[1] W. Marciano, H. Pagels, Phys. Rep. C 36 (1978) 137.
[2] L. von Smekal, A. Hauck, R. Alkofer, Annals Phys. 267 (1998) 1; K.-I. Kondo, hep-th/0303251.
[3] K. D. Born et al., Phys. Lett. B 329 (1994) 325; V. M. Miller et al., Phys. Lett. B 335 (1994) 71.
[4] V.N. Gribov, Eur. Phys. J. C 10 (1999) 71.
[5] D.J. Gross, F. Wilczek, Phys. Rev. D 8 (1973) 3633.
[6] J. Kogut, L. Susskind, Phys. Rev. D 9 (1974) 3501.
[7] H. Fritzsch, M. Gell-Mann, H. Leutwyler, Phys. Lett. B 47 (1973) 365.
[8] S. Weinberg, Phys. Rev. Lett. 31 (1973) 494.
[9] H. Georgi, S. Glashow, Phys. Rev. Lett. 32 (1974) 438.
[10] L. Susskind, J. Kogut, Phys. Rep. 23C (1976) 348.
[11] J.L. Gervais, A. Neveu, Phys. Rep. 23C (1976) 240.
[12] H. Pagels, Phys. Rev. D 15 (1977) 2991.
[13] V. Gogohia, hep-ph/0311061.
[14] I.M. Gel'fand, G.E. Shilov, Generalized Functions, (AP, New York, 1968), Vol. I.
[15] G. 't Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189.
[16] G. Leibbrandt, Noncovariant Gauges (WS, Singapore, 1994);
   A. Bassetto, G. Nardelli, R. Soldati, Yang-Mills Theories in Algebraic Non Covariant Gauges (WS, Singapore, 1991).
[17] V. Rubakov, Classical Theory of Gauge Fields, (Princeton University Press, 2002).
[18] V. Gogohia, Gy. Kluge, Phys. Rev. D 62 (2000) 076008.
[19] V. Gogohia, Gy. Kluge, H. Toki, T. Sakai, Phys. Lett. B 453 (1999) 281.
[20] S. Mandelstam, Phys. Rev. D 20 (1979) 3223.
[21] M. Baker, J.S. Ball, F. Zachariasen, Nucl. Phys. B 186 (1981) 531, 560.
[22] N. Brown, M.R. Pennington, Phys. Rev. D 39 (1989) 2723.
[23] D. Atkinson et al., J. Math. Phys. 25 (1984) 2095.
[24] L.G. Vachnadze, N.A. Kiknadze, A.A. Khelashvili, Sov. J. Teor. Math. Phys. 102 (1995) 47.
[25] A.I. Alekseev, B.A. Arbusov, Mod. Phys. Lett. A 13 (1998) 1747.