Vertex adjacencies in the set covering polyhedron

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Abstract

We describe adjacency of the extreme points of the (unbounded version of the) set covering polyhedron, similar to the description given by Chvátal for the stable set polytope. We find sufficient conditions and show that these are also necessary when the underlying matrix is row circular. We apply our findings to show a new infinite family of minimally nonideal matrices.

Keywords: polyhedral combinatorics set covering polyhedron extreme points adjacency

1 Introduction

In 1975, Chvátal (1975) gave a characterization of the adjacency of extreme points of the stable set polytope of a graph $G$, STAB$(G)$: the characteristic vectors of two stable sets $W$ and $W'$ of $G$ are adjacent in STAB$(G)$ if and only if the subgraph of $G$ induced by $(W \setminus W') \cup (W' \setminus W)$ is connected.

STAB$(G)$ is one of the most studied polyhedra related to set packing problems, and it is quite natural to try to extend Chvátal’s result to other settings. This was done by several authors, such as Hausmann and Korte (1978), Ikebe and Tamura (1995), and Alfakih and Murty (1998); sometimes in the context of the simplex method or related to the Hirsch conjecture, e.g., Matsui and Tamura (1995), Michini and Sassano (2013). See also Michini (2012) and references therein.

Papadimitriou (1978) observed the difficulty of the adjacency problem for the traveling salesman polytope, later Chung (1980) obtained a similar result for the set covering polytope, whereas Matsui (1995) showed the NP-completeness of the non-adjacency problem for the set covering polytope even though the matrix involved has exactly three ones per row. Thus, in contrast to the case of STAB$(G)$, it is unlikely that a simple characterization of the adjacency of extreme points of the set covering polyhedron may be given.

Nevertheless, in this work we go one step beyond the usual sufficient condition of connectivity of a certain graph, and give another condition which is also sufficient for adjacency of extreme points of the (unbounded version of the) set covering polyhedron, showing that these two conditions are also necessary in the case of row circular matrices. That is, the adjacency problem is polynomial for these matrices.

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This paper is organized as follows. After some preliminary comments on the setting and notation in Section 2, in Section 3 we present the auxiliary graph which we associate with each pair of extreme points of the set covering polyhedron defined by a binary matrix $A$, and the two conditions for adjacency. The sufficiency of these conditions is proved in Theorem 4.3 of Section 4. In Section 5 we prove that they are also necessary when $A$ is a row circular matrix (Theorem 5.12), and give an example (Example 5.15) showing that our sufficient conditions are not always necessary even for circulant matrices. In Section 6 we show that when the row sum of $A$ is constantly two, our findings reduce to those of Chvátal (1975). Finally, in Section 7 we apply our results to obtain a new infinite family of minimally nonideal matrices based on known minimally nonideal circulant matrices.

2 Notation and background

Let us start by establishing some notation, definitions and known results.

We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$; by $\mathbb{Z}$ the set of integers; by $\mathbb{R}$ the set of real numbers; by $\mathbb{B}$ the set of binary numbers, $\mathbb{B} = \{0, 1\}$; and by $\mathbb{I}_{n}$ the set $\{1, 2, \ldots, n\}$.

$e_{1}, \ldots, e_{n}$ denote the vectors in the canonical basis of $\mathbb{R}^{n}$. $0_{n}$ and $1_{n}$ denote the vectors in $\mathbb{R}^{n}$ with all zeroes and all ones (resp.), dropping the subindex $n$ if the dimension is clear from the context.

The scalar product in $\mathbb{R}^{n}$ is denoted by a dot, so that, e.g., $x \cdot e_{i} = x_{i}$ for $x \in \mathbb{R}^{n}$. Given $x$ and $y$ in $\mathbb{R}^{n}$, we say that $x$ dominates $y$, and write $x \geq y$, if $x_{i} \geq y_{i}$ for all $i \in \mathbb{I}_{n}$. $|X|$ denotes the cardinality of the finite set $X$.

The support of $x \in \mathbb{R}^{n}$ is the set $\text{supp}(x) = \{i \in \mathbb{I}_{n} : x_{i} \neq 0\}$. Conversely, given $X \subset \mathbb{I}_{n}$, its characteristic vector, $\chi_{n}(X) \in \mathbb{B}^{n}$, is defined as

$$e_{i} \cdot \chi_{n}(X) = \begin{cases} 1 & \text{if } i \in X, \\ 0 & \text{otherwise}, \end{cases}$$

so that $\text{supp}(\chi_{n}(X)) = X$.

A convex combination of the points $x^{1}, \ldots, x^{\ell}$ of $\mathbb{R}^{n}$ is a point of the form $\sum_{k=1}^{\ell} \lambda_{k}x^{k}$, where $\sum_{k=1}^{\ell} \lambda_{k} = 1$ and $\lambda_{k} \geq 0$ for $k = 1, \ldots, \ell$. The combination is strict if all $x^{k}$ are different and $0 < \lambda_{k} < 1$ for all $k = 1, \ldots, \ell$.

The following result is well known and we will use it to prove adjacency:

**Proposition 2.1.** Suppose $P = \{x \in \mathbb{R}^{n} : Ax \geq b, x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$ has non-negative entries and $b \geq 0$. If $v$ and $v'$ are different extreme points of $P$, the following are equivalent:

(a) $v$ and $v'$ are adjacent in $P$.

(b) If a strict convex combination $\sum_{k=1}^{\ell} \lambda_{k}x^{k}$ of points of $P$ belongs to the segment $S$ with endpoints $v$ and $v'$, then $x^{k} \in S$ for all $k = 1, \ldots, \ell$.

(c) If $y = \sum_{k=1}^{\ell} \lambda_{k}u^{k}$ is a strict convex combination of extreme points $u^{1}, \ldots, u^{\ell}$ of $P$ and $y \leq \frac{1}{2}(v + v')$, then for all $k = 1, \ldots, \ell$ either $u^{k} = v$ or $u^{k} = v'$.
Given the binary matrix $A \in \mathbb{B}^{m \times n}$, the set covering polyhedron, $Q^*(A)$, is the convex hull of non-negative integer solutions of $Ax \geq 1$,

$$Q^*(A) = \text{conv}\{x \in \mathbb{Z}^n : Ax \geq 1, x \geq 0\},$$

(2.1)

and we denote by $Q(A)$ the linear relaxation

$$Q(A) = \{x \in \mathbb{R}^n : Ax \geq 1, x \geq 0\}.$$  (2.2)

We will write simply $Q^*$ or $Q$, omitting $A$, if the matrix is clear from the context.

The following is a variant of Proposition 2.1, which we will use to prove non-adjacency:

**Proposition 2.2.** Let $A \in \mathbb{B}^{m \times n}$, let $Q^*(A)$ and $Q(A)$ be defined as in (2.2) and (2.1), and let $v$ and $v'$ be extreme points of $Q^*(A)$.

Suppose $d$ and $d'$ in $\mathbb{B}^n$ satisfy:

- $0 \leq d \leq v$ and $0 \leq d' \leq v'$,
- $v \cdot d' = 0$,
- $x = v - d + d'$ and $x' = v' - d' + d$ are elements of $Q(A) \cap \mathbb{B}^n$ (and therefore of $Q^*(A)$),
- $x \neq v'$.

Then $v$ and $v'$ are not adjacent in $Q^*(A)$.

**Proof.** We notice first that $0 \leq (v-d) \cdot d' \leq v \cdot d' = 0$, which implies $(v-d) \cdot d' = 0$. If $x = v$ we would have

\[
0 = v \cdot d' = x \cdot d' = (v - d + d') \cdot d' = d' \cdot d' > 0,
\]

a contradiction. Thus, $x \neq v$.

Given that a segment with endpoints in $\mathbb{B}^n$ cannot contain other binary points, it follows that $x$ cannot belong to the segment with endpoints $v$ and $v'$. The result now follows from Proposition 2.1(b), since $\frac{1}{2}(x + x') = \frac{1}{2}(v + v')$. □

In the sequel we assume that the binary matrix $A$ associated with the set covering polyhedra in (2.2) and (2.1) verifies the following assumptions:

**Assumptions 2.3.** $A \in \mathbb{B}^{m \times n}$ satisfies:

- has no dominating rows,
- has between 2 and $n-1$ ones per row,
- has no column of all ones or of all zeroes.
$C_t$ denotes the support of the $t$-th row of $A$, and we let $C = \{C_1, \ldots, C_m\}$. Assumptions 2.3 imply that $2 \leq |C| \leq n - 1$ for every $C \in C$, that $\bigcup_{C \in C} C = I_n$, and that $C$ is a clutter in the nomenclature of Cornuéjols and Novick (1994).

The extreme points of $Q^*(A)$ are the binary extreme points of $Q(A)$. They form the blocker of $A$, $b(A)$, and their supports are the minimal transversals of $C$, which we denote by $T$. That is, $T \in T$ if and only if $T \cap C \neq \emptyset$ for all $C \in C$ and if $R \subset T$ with $R \cap C \neq \emptyset$ for all $C \in C$ then $R = T$. Notice that

- $C \cap T \neq \emptyset$ for all $C \in C$ and $T \in T$.
- For every $T \in T$ and $p \in T$, there exists $C \in C$ such that $C \cap T = \{p\}$.

As $b(b(A)) = A$ if $A$ is a clutter matrix, we also have:

- For every $C \in C$ and $p \in C$ there exists $T \in T$ such that $C \cap T = \{p\}$.

**Remark 2.4.** Notice that Assumptions 2.3 also imply that $T$ has properties similar to those of $C$: $2 \leq |T| \leq n - 1$ for every $T \in T$, and $\bigcup_{T \in T} T = I_n$.

## 3 The auxiliary graph

The following definitions are essential in this paper.

**Definition 3.1.** Given the matrix $A \in \mathbb{B}^{m \times n}$ and the associated covering $C$ as described in the previous section, let $v$ and $v'$ be distinct extreme points of $Q^*$, and set

$$V = \text{supp} \ v, \quad V' = \text{supp} \ v', \quad S(v, v') = V \setminus V', \quad S(v', v) = V' \setminus V.$$  

We construct a simple undirected auxiliary graph $G = G(v, v')$, depending on $v$ and $v'$, by the following setup:

- the set of vertices of $G$ is $V \triangle V' = S(v, v') \cup S(v', v)$,
- $G$ is bipartite with bipartitions $S(v, v')$ and $S(v', v)$,
- $p \in S(v, v')$ and $p' \in S(v', v)$ are neighbors in $G$ if there exists $C_t \in C$ such that $C_t \cap V = \{p\}$ and $C_t \cap V' = \{p'\}$.

**Remark 3.2.** If $v$ and $v'$ are distinct extreme points of $Q^*$, then $S(v, v') \neq \emptyset$ since we cannot have $v \leq v'$.

The characterization of adjacency of extreme points of $Q^*$ will be given in terms of properties of $G$. In particular, we will need the following definition:

**Definition 3.3.** A bipartite graph is partly-connected if it is not connected and one of the bipartitions is completely contained in a component of the graph.

In this paper we will be concerned mainly with the following conditions:

**Conditions for adjacency 3.4.** If $G$ is the bipartite graph associated with the extreme points $v$ and $v'$ of $Q^*$, we consider the following conditions:
C1. $G$ is connected.

C2. $G$ is partly-connected and $(\text{supp } v) \cap (\text{supp } v') = \emptyset$.

Remark 3.5. Definition 3.3 implies that if a bipartite graph has a bipartition with just one point, then the graph is either connected or partly-connected. However, when $G$ is the graph associated with the extreme points $v$ and $v'$ of $Q^*$, if supp $v$ and supp $v'$ are disjoint (as in Condition C2), then the bipartitions of $G$ are these supports and each of them has at least two points (see Remark 2.4).

4 Sufficiency of the conditions

In this section we show that the conditions in 3.4 are sufficient for adjacency of extreme points in $Q^*$.

Lemma 4.1. Let $v$ and $v'$ be two distinct extreme points of $Q^*$ and let $G$ be the associated graph. Suppose $y = \sum_{k=1}^{\ell} \lambda_k u^k$ is a strict convex combination of extreme points $u^1, \ldots, u^\ell$ of $Q^*$ such that $y \leq \frac{1}{2} (v + v')$.

Then, for each $h = 1, \ldots, \ell$ we have:

(a) $\text{supp } u^h \subseteq (\text{supp } v) \cup (\text{supp } v')$.

(b) $|\{p, p'\} \cap (\text{supp } u^h)| = 1$ whenever $p$ and $p'$ are neighbors in $G$.

Proof. Let us write $z = \frac{1}{2} (v + v')$.

If $q \in \text{supp } u^h$ we must have $u^h \cdot e_q > 0$ and therefore $y \cdot e_q > 0$ since the convex combination for $y$ is strict. Hence $z \cdot e_q > 0$ as $z \geq y$. Therefore, $q \in (\text{supp } v) \cup (\text{supp } v')$.

For the second part, let $a$ be a row of $A$ such that (3.1) holds with $C = \text{supp } a$, $V = \text{supp } v$ and $V' = \text{supp } v'$. Since $\text{supp } u^h \subseteq (\text{supp } v) \cup (\text{supp } v')$, by (3.1) it follows that $p$ and $p'$ are the only elements of $C$ which can belong to $\text{supp } u^h$.

Then, from $\{p, p'\} \subset C$ we conclude that

$$\{p, p'\} \cap (\text{supp } u^h) = C \cap (\text{supp } u^h).$$

Moreover, by (3.1) we have $a \cdot z = 1$, and since $z \geq y$, it follows that $a \cdot y \leq 1$.

On the other hand, as $y \in Q^*$, $a \cdot y \geq 1$ and therefore $a \cdot y = 1$. Similarly, since $y$ is a strict convex combination of the points $u^1, \ldots, u^\ell$ and $a \cdot u^k \geq 1$ for all $k = 1, \ldots, \ell$, we have

$$a \cdot u^k = 1 \quad \text{for all } k = 1, \ldots, \ell,$$

and so, in particular, $|C \cap (\text{supp } u^h)| = 1$, proving the lemma.

Lemma 4.2. Under the hypothesis on $v$, $v'$, $y$ and $u^1, \ldots, u^\ell$ of Lemma 4.1, let us further assume that either of the conditions in 3.4 holds.

Then, for all $k = 1, \ldots, \ell$, either $u^k = v$ or $u^k = v'$.

Proof. Let us consider firstly the case where $G$ is connected, and let us show that, for each $k = 1, \ldots, \ell$, either $\text{supp } u^k \subseteq \text{supp } v$ or $\text{supp } u^k \subseteq \text{supp } v'$, which will imply the result as $v$, $v'$ and $u^k$ are binary extreme points of $Q^*$.

On the contrary, suppose there exist $p \in S(u^k, v')$ and $p' \in S(u^k, v)$. 

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By Lemma 4.1(a), supp $u^k \subseteq (\text{supp } v) \cup (\text{supp } v')$, and so we must have $p \in S(v, v')$ and $p' \in S(v', v)$, that is, $p$ and $p'$ are vertices of $G$.

As $G$ is connected and bipartite, there exists an alternating path joining $p$ and $p'$, let us say $(p = p_1, p_1', p_2, p_2', \ldots, p_h' = p')$, where $p_i \in S(v, v')$ and $p_i' \in S(v', v)$ for $i = 1, \ldots, h$. Using repeatedly Lemma 4.1(b), we see that $p = p_1 \in \text{supp } u^k$, $p_1' \notin \text{supp } u^k$, $p_2 \in \text{supp } u^k$, and so on, i.e., $p_i \in \text{supp } u^k$ and $p_i' \notin \text{supp } u^k$ for all $i = 1, \ldots, h$. In particular, we have $p_h' \notin \text{supp } u^k$, which contradicts $p_h' = p' \in \text{supp } u^k$.

Let us suppose now that $G$ is partly-connected and $(\text{supp } v) \cap (\text{supp } v') = \emptyset$. Without loss of generality, assume that supp $v'$ is contained in a component of $G$. The result will follow if we show that either supp $u^k \subseteq \text{supp } v$ or supp $v' \subseteq \text{supp } u^k$, again since $v$, $v'$ and $u^k$ are binary extreme points of $Q^*$.

If $(\text{supp } u^k) \cap (\text{supp } v') = \emptyset$, we must have supp $u^k \subseteq \text{supp } v$ as supp $u^k \subseteq (\text{supp } v) \cup (\text{supp } v')$ by Lemma 4.1(a). Otherwise, let $p' \in (\text{supp } u^k) \cap (\text{supp } v')$. Since supp $v'$ is contained in a component of $G$, for each $q' \in \text{supp } v'$ there exists an alternating path $(p' = p_1', p_1, \ldots, p_h = q')$, where $p_i' \in S(v', v) = \text{supp } v'$ and $p_i \in S(v, v') = \text{supp } v$ for $i = 1, \ldots, h$. Again from Lemma 4.1(b), it follows that $p_i' \in \text{supp } u^k$ and $p_i \notin \text{supp } u^k$ for all $i = 1, \ldots, h$, and so in particular $q' = p_h' \in \text{supp } u^k$. Since this holds for any $q' \in \text{supp } v'$, we conclude that supp $v' \subseteq \text{supp } u^k$.

Using Proposition 2.1 and Lemma 4.2, we obtain the main result of this section:

**Theorem 4.3.** If $v$ and $v'$ are extreme points of $Q^*$ and any of the conditions in 3.4 holds, then $v$ and $v'$ are adjacent in $Q^*$.

## 5 Necessity of the conditions for row circular matrices

In this section we show that the reciprocal of Theorem 4.3 holds if we restrict ourselves to coverings associated with circular arc intervals, and also show that this is no longer true if we consider the similar class of coverings given by circulant matrices.

For $n \in \mathbb{N}$, $n \geq 2$, we consider $n$ points on a circle, labeled $1, 2, \ldots, n$ clockwise (say). Each pair $i$ and $j$ of these points defines two arcs in the circle, each of them containing both $i$ and $j$: one starting in $i$ and going clockwise to $j$, denoted by $[i, j]_n$, and another starting at $j$ and going clockwise to $i$, denoted by $[j, i]_n$. That is, we define the directed circular arc $[i, j]_n$ by:

$$[i, j]_n = \begin{cases} 
\{i, \ldots, j\} & \text{if } i \leq j, \\
\{i, \ldots, n\} \cup \{1, \ldots, j\} & \text{if } j < i.
\end{cases}$$

We consider a covering $C = \{C_1, \ldots, C_m\}$ of $I_n = \{1, 2, \ldots, n\}$, where each $C_t$ is an arc of the form $[\alpha_t, \beta_t]_n$ and $2 \leq |C_t| \leq n - 1$. In particular, we assume $\alpha_t \neq \beta_t$. For simplicity we also ask $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, and use the convention that $C_t = C_s$ if $t \notin I_n$ and $s \in I_n$ with $t \equiv s \pmod{m}$.

Notice that we may have, say, $\alpha_1 = 1$ and $\beta_m = n$, in which case the “circular arcs” turn out to be just “intervals”.
Associated with the covering \( C \), we consider the matrix \( A \in \mathbb{R}^{n \times n} \) whose rows are \( \chi_n(C_1), \ldots, \chi_n(C_m) \). In each row of \( A \) either the ones are all consecutive or the zeroes are all consecutive: following Bartholdi et al. (1980), we call such binary matrices row circular. We will also assume that \( A \) satisfies Assumptions 2.3.

As previously, we let \( Q = \{ x \in \mathbb{R}^n : Ax \geq 1, x \geq 0 \} \), and denote by \( Q^* \) the convex hull of the integer points in \( Q \).

Given the structure of the subsets in the covering \( C \), for any extreme point \( v \) of \( Q^* \) and \( p \in V = \text{supp} v \), we may find \( t_-(v,p) \in I_m \) and \( t_+(v,p) \in I_m \) such that
\[
C_t \cap V = \{ p \} \quad \text{if and only if} \quad t \in [t_-(v,p), t_+(v,p)]_m. \tag{5.1}
\]

We also notice that by Assumptions 2.3, we have \( m = |C| \geq 2 \) and \( 2 \leq v \cdot 1 \leq n - 1 \).

**Remark 5.1.** If (3.1) is satisfied, that is, \( p \leftrightarrow p' \), then exactly one of \([p,p']_n\) or \([p',p]_n\) is such that its intersection with \( V \cup V' \) is \( \{ p,p' \} \), as each of \( V \) and \( V' \) have at least two elements.

**Example 5.2.** The circulant matrix associated with \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) is defined as
\[
\mathcal{C}(c) = \mathcal{C}(c_1, \ldots, c_n) = \begin{bmatrix}
c_1 & c_2 & \cdots & c_n \\
c_n & c_1 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_2 & c_3 & \cdots & c_1
\end{bmatrix},
\]
where each row is a right rotation (shift) of the previous one.

A particularly interesting case of row circular matrices is that of the consecutive ones circulant matrices \( \mathcal{C}_n^k \), where the sets in the covering \( C \) are of the form
\[
C_t = \{ t, t + 1, \ldots, t + k - 1 \}, \quad t \in \mathbb{I}_n,
\]
and the sums are taken modulo \( n \) with values in \( \mathbb{I}_n \), so that \( \mathcal{C}_n^k \) is also circulant. For example,
\[
\mathcal{C}_2^2 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix} = \mathcal{C}(1,1,0).
\]

In this section we will also use the following convention:

**Notation 5.3.** For \( v \in \mathbb{R}^n \) we will write \( \text{supp} v = V = \{ p_1, p_2, \ldots, p_r \} \), with \( p_1 < p_2 < \cdots < p_r \); and for \( h \not\equiv i \mod r \) we let \( p_h = p_i \) with \( i \equiv h \mod r \). Similarly, for \( v' \in \mathbb{R}^n \) we will write \( \text{supp} v' = V' = \{ p'_1, \ldots, p'_{r'} \} \) with \( p'_1 < \cdots < p'_{r'} \), etc. Finally, we denote by \( p \leftrightarrow p' \) and \( p \not\leftrightarrow p' \) whether \( p \in S(v,v') \) and \( p' \in S(v',v) \) are neighbors in \( G \) or not (resp.).

Let us state now some simple results.

**Lemma 5.4.** Suppose \( C_t \in C \) and \( v \) is an extreme point of \( Q^* \). Then
\[
1 \leq |C_t \cap V| \leq 2.
\]

Moreover, if \( |C_t \cap V| = 2 \), then \( C_t \cap V = \{ p_i, p_{i+1} \} \) for some \( i \).
Proof. \(|C_t \cap V| \geq 1\) since \(V\) is a transversal. If \(|C_t \cap V| \geq 3\), we would have \([p_i-1, p_i+1]_n \subseteq C_t\) for some \(i\) \((p_i-1 \neq p_i+1)\) and, since \(V\) is minimal, there exists \(C_s \in \mathcal{C}\) such that \(C_s \cap V = \{p_i\}\). But since \(C_t\) and \(C_s\) are circular arcs, we must have \(C_s \subseteq C_t\), contradicting that \(A\) has no dominating rows.

The last part follows from the fact that \(C_t\) is a circular arc. \(\square\)

**Remark 5.5.** If \(v\) is an extreme point of \(Q^*\) and \(p_i \in \text{supp} v\), then \(p_i-1 \in C_{t_{-}(v,p_i)}\), although we may have \(p_i+1 \notin C_{t_{-}(v,p_i)}\) (unlike the case for \(\mathcal{C}_n^0\)).

**Lemma 5.6.** Let \(v\) and \(v'\) be extreme points of \(Q^*\), and let \(p_i \in S(v,v')\) be an isolated vertex of \(G(v,v')\). If \(C_t \in \mathcal{C}\) and \(C_t \cap V = \{p_i\}\), then \(|C_t \cap S(v',v)| = 2\).

In particular, \(|S(v',v)| \geq 2\).

**Proof.** As \(V\) is a transversal, there exists \(p_j' \in C_t \cap V\), but since \(p_i\) has no neighbors in \(G(v,v')\), \(C_t \cap V\) must contain another element \(p_k'\). Notice that any element of \(C_t \cap V\) must be in \(S(v',v)\) because \(C_t \cap V = \{p_i\}\) and \(p_i \in S(v,v')\). Thus, \(C_t \cap S(v',v) = C_t \cap V\). The result now follows from Lemma 5.4. \(\square\)

**Lemma 5.7.** Let \(v\) and \(v'\) be extreme points of \(Q^*\), and suppose \([p_j',p_j'+1]_n \subseteq [p_i,p_{i+1}]_n\).

Then, if \(p_i \leftrightarrow p_j'-1\) we must have \(p_i \leftrightarrow p_j'\). Similarly, if \(p_{i+1} \leftrightarrow p_j'+2\) then \(p_{i+1} \leftrightarrow p_j'+1\).

**Proof.** Assume \(p_i \leftrightarrow p_j'-1\). As \([p_j',p_j'+1]_n \subseteq [p_i,p_{i+1}]_n\), observe that if \(C_t \in \mathcal{C}\) satisfies \(\{p_i,p_j'+1\} \subseteq C_t\), then \(\{p_{i+1},p_j'\} \cap C_t \neq \emptyset\). Thus, we have \(p_i \leftrightarrow p_j'+1\), and so \(p_j'-1 \neq p_j'+1\).

Since \(p_j'-1 \neq p_j'+1\), we must have \(p_j'+1 \notin [p_i,p_j']_n\), because otherwise we would have \([p_j'-1,p_j'+1]_n \subseteq [p_i,p_{i+1}]_n\), and so \(C_t \cap V = \emptyset\) for any \(C_t \in \mathcal{C}\) satisfying \(C_t \cap V' = \{p_j'\}\) (at least one of such \(C_t\) exists), contradicting that \(V\) is a transversal. As \(p_i\) is different from \(p_j'-1\) and \(p_j'\) \((p_i \leftrightarrow p_j'-1)\), observe that \(p_j'-1 \notin [p_i,p_j']_n\) is equivalent to \(p_j' \notin [p_j'-1,p_j']_n\).

Notice that \([p_j'-1,p_j']_n \cap V = \{p_i\}\) and \([p_j'-1,p_j']_n \cap V' = \{p_j'\}\) because \(p_i \leftrightarrow p_j'-1\) and \(p_j' \notin [p_j'-1,p_j']_n\) (see Remark 5.1). It follows that \([p_j'-1,p_j']_n \cap V = \{p_i\}\) and \([p_j',p_j'+1-1]_n \cap V = \emptyset\) since \([p_j',p_j'+1-1]_n \subseteq [p_i,p_{i+1} - 1]_n\).

Then, if \(C_t \in \mathcal{C}\) is such that \(C_t \cap V' = \{p_j'\}\), we must have \(C_t \cap V = \{p_i\}\) because \(C_t\) is a circular arc, and the proof follows from the definition of the edges in \(G(v,v')\). \(\square\)

**Lemma 5.8.** Let \(v\) and \(v'\) be extreme points of \(Q^*\), and suppose \(p_i \in S(v,v')\), and \(p_j' \in S(v',v)\) are such that

\[
[p_j',p_i]_n \cap V = \{p_i\} \quad \text{and} \quad [p_j',p_i]_n \cap V' = \{p_j'\}.
\]

Then:

(a) If \(r = t_{-}(v',p_j'+1)\) and \(p_{i+1} \in C_t\), then \(C_t \cap \{p_j',p_{i+1}\} \neq \emptyset\) for all \(C_t \in \mathcal{C}\) such that \(p_i \in C_t\).
Lemma 5.8. Suppose $p_i \in C_t = [\alpha_t, \beta_t]_n$, and $p'_j \notin C_t$. Then, recalling Remark 3.2, we have $\alpha_t \in [\alpha_r, p_i]_n \subseteq [\alpha_r, p'_j + 1]_n \subseteq C_r$, and therefore $p_{i+1} \in C_t$.

Similarly, if $p'_j \in C_t$ and $p_i \notin C_t$, then $\beta_t \in [p'_j, \beta_s]_n \subseteq [p_{i-1}, \beta_s]_n \subseteq C_s$, and so $p'_{j-1} \in C_t$.

Lemma 5.9. Suppose the graph $G$ associated with the extreme points $v$ and $v'$ of $Q^*$ has no edges. Then, $v$ and $v'$ are not adjacent in $Q^*$.

Proof. Recalling that $G$ has at least two vertices (one in $S(v, v')$ and another in $S(v', v)$ by Remark 3.2), let us fix $p_i \in S(v, v')$ and $C_t \in C$ satisfying $C_t \cap V = \{p_i\}$. Then, by Lemma 5.6 we have $|C_t \cap S(v', v)| = 2$ and $C_t \cap S(v', v) = C_t \cap V'$. Due to this and the fact that $C_t$ is a circular arc, observe that we can always pick $p'_j$ in $C_t \cap S(v', v) = C_t \cap V'$ so that the hypotheses of Lemma 5.8 are satisfied (interchanging $v$ and $v'$ when $[\alpha_t, p_i] \cap V' = \emptyset$). Thus, in the rest of this proof we assume that we can apply Lemma 5.8.

Let $r = t - (v', p'_{j+1})$, and let us see that $p_{i+1} \in C_r$. This is true if $p_{i+1} \in [\alpha_r, p'_{j+1}]_n$. Otherwise, if $p'_{j+1} \in [\alpha_r, p_{i+1} - 1]_n$, we must have $p'_{j+1} \in S(v', v)$ (a contradiction to the hypotheses of Lemma 5.8), and therefore, since $G$ has no edges, either $\{p_i, p_{i+1}\}$ or $\{p_{i+1}, p_{i+2}\}$ is contained in $C_r$. At any rate, $p_{i+1} \in C_r$.

Similarly, if $s = t + (v, p_{i-1})$ then $p'_{j-1} \in C_s$, either because it is in the arc $[p_{i-1}, \beta_s]_n$, or because $p_{i-1} \in S(v, v')$ and then either $\{p'_{j-1}, p'_j\}$ or $\{p'_{j-2}, p'_{j-1}\}$ are subsets of $C_s$.

Therefore, the conclusions of Lemma 5.8 are satisfied, which implies the sets

$$X = (V \setminus \{p_i\}) \cup \{p'_j\} \quad \text{and} \quad X' = (V' \setminus \{p'_j\}) \cup \{p_i\}$$

are transversals. Observe that $X$ cannot coincide with either $V'$ or $V$ since we are exchanging just one element of $S(v', v)$, but $|S(v', v)| \geq 2$ by Lemma 5.6.

The lemma follows now from Proposition 2.2 defining $x = \chi_n(X)$, $x' = \chi_n(X')$, $d = \chi_n(\{p_i\})$ and $d' = \chi_n(\{p'_j\})$.

Lemma 5.10. Let $v$ and $v'$ be extreme points of $Q^*$. Then:

(a) If $p_i \leftrightarrow p'_j$, we must have $p'_j \in C_i$ for some $t \in [t_-(v, p_i), t_+(v, p_i)]_m$.

(b) If $p_i \leftrightarrow p'_j$ and $p_h \leftrightarrow p'_j$, we must have either $h = i - 1$ or $h = i + 1$.

(c) The vertices of $G(v, v')$ have degree at most 2. If $p_i \leftrightarrow p'_j$, $p_{i+1} \leftrightarrow p'_j$ and $|V| > 2$, we must have $p'_j \in [p_i, p_{i+1}]_n$, and there are no other elements of $V'$ (or $V$) in that arc.

Notice that if $|V| = 2$ then any $p'_j$ could be in either $[p_1, p_2]_n$ or $[p_2, p_1]_n$.

(d) Each component of $G(v, v')$ must be either an alternating cycle or an alternating path (including isolated points).

(e) But, if a component of $G(v, v')$ is a cycle, then it is the disjoint union of $V$ and $V'$. In particular, $G(v, v')$ is connected.

---

Notice that we are not excluding the possibilities $p_{i-1} = p_{i+1}$ or $p'_{j-2} = p'_j$. 

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Proof. (a) follows from the definition of the edges of $G(v, v')$ and (5.1).

For (b), let us assume that if $C_s$ and $C_r$ are such that

$$C_s \cap V = \{p_i\}, \quad C_s \cap V' = \{p'_j\}, \quad C_r \cap V = \{p_h\}, \quad C_r \cap V' = \{p'_j\},$$

then $C_s \cup C_r = [\alpha_s, \beta_s]_n$ intersects $V$ only at $p_i$ and $p_h$, and hence these are consecutive elements of $V$. This shows (b). (c) follows similarly.

(d) follows from (c), and (e) follows from (b) and (e).

\[ \square \]

**Lemma 5.11.** Under the hypothesis of Lemma 5.10, suppose $p_i$ and $p_{i+1}$ are in the same component $F$ of $G(v, v')$ but do not have a common neighbor in $S(v', v)$.

Then:

(a) $|V| > 2$.

(b) $V \subset F$ and $V \cap V' = \emptyset$.

(c) If $h \neq i$, there exists a unique $p'_k \in V'$ such that $p'_k \in [p_h + 1, p_{h+1} - 1]_n$.

Furthermore, $p_h \leftrightarrow p'_k$ and $p_{h+1} \leftrightarrow p'_k$.

(d) $|V' \setminus F| \leq 1$ and $|V| - |V'| \leq 1$.

(e) By interchanging $v$ and $v'$ if necessary, either $V' \subset F$ and $G(v, v')$ satisfies $C_1$, or $G(v, v')$ satisfies $C_2$ and $|V| = |V'|$.

Proof. (a) follows from Lemma 5.10(d).

By Lemma 5.10(c), the path from $p_i$ to $p_{i+1}$ must contain every point of $V$ and these are in $S(v, v')$, showing (b).

(c) follows from Lemma 5.10.

Suppose now $p'_j \in V' \setminus F$ and let us show that $p'_{j-1}$ and $p'_{j+1}$ are in $F$.

From (c), we know that

$$p'_j \in [p_i + 1, p_{i+1} - 1]_n.$$

We also notice that $p_{i+2} \neq p_i$ (by (a)), and therefore (b) and (c) tell us that there is a unique $p'_k \in V'$ connected to both $p_{i+1}$ and $p_{i+2}$ with $p'_k \in [p_{i+1}, p_{i+2}]_n$, which implies $p'_{j+1} \in [p'_j + 1, p'_k]_n$. So, if $p'_{j+1} = p'_k$ we have $p'_{j+1} \in F$, and otherwise

$$p'_{j+1} \in [p'_j + 1, p_{i+1} - 1]_n.$$

Now, let $C_t \in \mathcal{C}$ be such that $C_t \cap V' = \{p'_{j+1}\}$. Then, we must have

$$C_t \subset [p'_j + 1, p'_k - 1]_n. \quad (5.2)$$

Being $V$ a transversal, it must intersect $C_t$, which implies $C_t \cap V = \{p_{i+1}\}$ by (5.2) and the fact that $[p'_j + 1, p'_k - 1]_n \subset [p_i + 1, p_{i+2} - 1]_n$. Thus $p_{i+1} \leftrightarrow p'_{j+1}$, and so $p'_{j+1} \in F$.

Similarly, we may show that $p'_{j-1} \in F$.

We cannot have $\{p'_{j-1}, p'_j, p'_{j+1}\} \subset [p_i + 1, p_{i+1} - 1]_n$, since that would imply that there exists $C_t \in \mathcal{C}$ such that $C_t \cap V = \emptyset$ (this would hold for any $C_t \in \mathcal{C}$ such that $C_t \cap V' = \{p'_j\}$), contradicting that $V$ is a transversal. This proves (d).

Thus, there are at most two points of $V'$ between $p_i$ and $p_{i+1}$. If there are none, then $V' \subset F$, $G(v, v')$ is connected and therefore satisfies Condition $C_1$. 

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If there are two points, say \( p_j \) and \( p_{j+1} \). Lemma 5.7 shows that \( p_i \leftrightarrow p_j \) and \( p_{i+1} \leftrightarrow p_{j+1} \), and we may interchange \( v \) and \( v' \). Finally, let us suppose there is exactly one point, say \( p_j' \). If \( p_j' \) is a neighbor of either \( p_i \) or \( p_{i+1} \), then \( \mathcal{G}(v, v') \) is an alternating path. Otherwise, \( \mathcal{G}(v, v') \) consists of the isolated vertex \( p_j' \) and an alternating path connecting \( p_i \) with \( p_{i+1} \), so it satisfies Condition C2 and \(|V| = |V'|\).

**Theorem 5.12.** Let \( v \) and \( v' \) be distinct extreme points of \( Q^* \) and let \( \mathcal{G} \) be the associated graph. If none of the conditions in 3.4 is satisfied, then \( v \) and \( v' \) are not adjacent in \( Q^* \).

**Proof.** If \( \mathcal{G} \) has no edges the result follows from Lemma 5.9, so we next assume that \( \mathcal{G} \) contains at least one edge.

Let \( F \) be a component containing an edge of \( \mathcal{G} \). Since \( \mathcal{G} \) is not connected, by Lemma 5.10 we know that \( F \) must be an alternating path.

In order to show that \( v \) and \( v' \) are not adjacent we will use Proposition 2.2.

Proof. For doing this, we let \( R = V \cap V' \) and define \( D \) and \( T \) by

\[
D = F \cap V, \quad T = V \setminus (R \cup D).
\]

Similarly, we define \( D' \) and \( T' \) by

\[
D' = F \cap V', \quad T' = V' \setminus (R \cup D').
\]

Finally, we define \( X \) and \( X' \) by

\[
X = R \cup T \cup D', \quad X' = R \cup T' \cup D.
\]

Our first aim is to prove that \( X \) and \( X' \) are transversals, and we notice that it is enough to prove this only for \( X \), given the symmetry of the definitions in (5.3).

We need to show that \( C_t \cap X \neq \emptyset \) for every \( C_t \in \mathcal{C} \). Since \( V = R \cup T \cup D \) is a transversal, it will be enough to consider just the case where \( C_t \) intersects \( V \) at some point of \( D \).

So let us take \( C_t \in \mathcal{C} \), and assuming \( p_i \in C_t \cap D \), let us prove that \( C_t \cap X \neq \emptyset \).

If \( p_i \) is connected (in \( \mathcal{G} \)) to two elements of \( S(v', v) \) (and hence of \( D' \)), then by Lemma 5.10 we may assume that these are of the form \( p_j' \) and \( p_{j+1}' \) and that \( p_i \in [p_j', p_{j+1}'] \). Since \( p_i \in C_t \) and \( C_t \) intersects \( V' \), we necessarily have \( \{p_j', p_{j+1}'\} \cap C_t \neq \emptyset \), and so \( \emptyset \neq \{p_j', p_{j+1}'\} \cap C_t \subseteq D' \cap C_t \subseteq X \cap C_t \).

Suppose now that \( p_i \) is a leaf of the path \( F \), and let \( p_j' \in S(v', v) \) be its only neighbor. Observe that if \( p_j' \in C_t \) we are done, because \( p_j' \in D' \subseteq X \). Then, in what follows we assume that \( p_j' \notin C_t \).

We will consider the case \( p_j' \in [p_{i-1}, p_i]_n \), the case \( p_j' \in [p_i, p_{i+1}]_n \) being similar. We notice this implies the hypotheses of Lemma 5.8 are satisfied because \( p_i \leftrightarrow p_j' \) (see Remark 5.1).

Observe that \( p_i \) and \( p_{i+1} \) cannot have a common neighbor in \( \mathcal{G} \), because that and \( p_j' \in [p_{i-1}, p_i]_n \), would imply \( p_{i-1} = p_{i+1} \) (\( V = \{p_i, p_{i+1}\} \)), and in that case \( \mathcal{G} \) would be connected or partly-connected. In consequence, from Lemma 5.11 we conclude that \( p_{i+1} \notin D \), because \( \mathcal{G} \) does not satisfy the conditions in 3.4.

Recalling that \( p_j' \notin C_t \), \( p_i \in C_t \) and the hypotheses of Lemma 5.8 are satisfied, we also notice that \( p_{j+1}' \in C_t \). Then, if \( p_{i+1} \in [p_i, p_{j+1}]_n \), we have \( p_{i+1} \in C_t \), and so \( p_{i+1} \in C_t \cap X \) because \( p_{i+1} \notin D \).
Lemma 5.8. To deduce that Remark 6.4) and there must applies. Lemma 5.11: disjoint supports and even path (type 1 in the Table 1
\begin{equation*}
\begin{array}{c|c|c|c}
\text{type} & V & V' & \text{component/s} \\
\hline
1 & \{1, 7, 13\} & \{6, 8, 14, 15\} & 15, 1, 6, 7, 8, 13, 14 \text{ (even path)} \\
2 & \{6, 12, 15\} & \{5, 11, 14\} & 14, 15, 5, 6, 11, 12 \text{ (odd path)} \\
3 & \{6, 12, 15\} & \{5, 8, 14\} & (5, 6, 8, 12, 14, 15) \text{ (cycle)} \\
4 & \{6, 12, 15\} & \{3, 9, 14\} & 15, 3, 6, 9, 12 \text{ (even path) + 14 (isol. pt.)} \\
5 & \{6, 12, 15\} & \{6, 8, 14, 15\} & 8, 12, 14 \text{ (even path)} \\
6 & \{6, 12, 15\} & \{6, 11, 15\} & 11, 12 \text{ (odd path)} \\
\end{array}
\end{equation*}

Table 1: Examples showing each of the six possible behaviors of $G(v, v')$ for adjacent extreme points $v$ and $v'$ of $Q^*$ ($A = \mathcal{F}^R_0$).

We now apply Lemma 5.8 to show that $X \cap C_t \neq \emptyset$ when $p_{i+1} \notin [p_i, p'_{j+1}]$. So let us show that $p_{i+1} \in C_r$ for $r = t\cdots(v', p'_{j+1})$. Since $p_{i+1} \notin [p_i, p'_{j+1}]_n$ and $[p'_{j}, p_i]_n \cap V' = \{p'_{j}\}$, we have $p'_{j+1} \in [p_i + 1, p_{i+1} - 1]_n$. Thus $V \cap C_r \subseteq \{p_i, p_{i+1}\}$ by Lemma 5.4. It follows that $p_{i+1} \in C_r$ because we cannot have $V \cap C_r = \{p_i\}$ (recall that $p'_j$ is only neighbor of $p_i$, and so $p_i \not\leftrightarrow p'_{j+1}$). Since $p_i \in C_t$, we can now apply Lemma 5.8 to deduce that $C_t \cap \{p'_j, p_{i+1}\} \neq \emptyset$. Recalling that $p'_j \in D'$ and $p_{i+1} \not\in D$, we conclude that $C_t \cap X \neq \emptyset$. This completes the proof of the fact that $X$ is a transversal.

Finally, we set $d = \chi_n(D)$, $d' = \chi_n(D')$, $x = v - d + d'$, $x' = v' - d + d$, so that $X = \supp x$ and $X' = \supp x'$. Notice that $d, d' > 0$ since $D, D' \neq \emptyset$, $d \neq v$ and $d' \neq v'$ since $T \cup R, T' \cup R \neq \emptyset$ (otherwise $G$ would be connected or partly-connected), and $x$ is different from either $v$ (since $D \neq \emptyset$ and $D \cap X = \emptyset$) or $v'$ (since otherwise we would have $T = T' = \emptyset$, and then $G$ would be connected). The result follows now from Proposition 2.2. 

Example 5.13. There are six types of graphs $G(v, v')$ for adjacent extreme points $v$ and $v'$ of $Q^*(A)$ when $A$ is row circular. Among consecutive ones circulant matrices (see Example 5.2), $\mathcal{F}^R_0$ is one of the smallest exhibiting all of these types as shown in Table 1: disjoint supports and even path (type 1 in the table), odd path (type 2), cycle (type 3) or partly-connected (type 4); and then overlapping supports and even path (type 5) or odd path (type 6).

See also Remark 6.4 below for the case $\mathcal{F}^R_0$. 

Table 1 also exhibits a simple consequence of our discussions:

Corollary 5.14. If $v$ and $v'$ are adjacent extreme points of $Q^*$, then the cardinality of their supports differ by one at most.

Proof. If $G(v, v')$ is connected, then it is either an alternating path or an alternating cycle (Lemma 5.10), and the cardinals of $V$ and $V'$ differ at most by one. If $G(v, v')$ is partly-connected, $V \cap V' = \emptyset$, and $V$ is contained in the component $F$, $F$ must be an alternating path (again by Lemma 5.10) and there must be two consecutive vertices in $V$ (of the form $p_i$ and $p_{i+1}$) with no common neighbor, so that Lemma 5.11 applies. 

The previous corollary is also a consequence of a technique by Bartholdi et al. (1980), which Eisenbrand et al. (2008) employed to show that if $A$ is row circular, then the slices \{X \in Q(A) : 1 \cdot x = \beta \} are integral polytopes for $\beta \in \mathbb{Z}$.
When the matrix $A$ is not row circular, the behavior of the graphs $G(v, v')$ may be quite different, as shown by the following example.

**Example 5.15.** Let us consider the circulant matrix (see Example 5.2)

$$A = C(1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0) \in \mathbb{R}^{13 \times 13},$$

which is the line-point incidence matrix of a non-degenerate finite projective plane of order 3, and so it is a circulant matrix not isomorphic to any $C_k^m$ (or any other row circular matrix).

It turns out that if $v$ and $v'$ are adjacent extreme points of $Q^*(A)$, then their supports cannot be disjoint, and the components of the graph $G(v, v')$ are isomorphic to a complete bipartite graph: either $K_{1,1}$ (one edge), or $K_{2,1}$ (path with two edges), or $K_{3,1}$, or $K_{3,3}$. In particular, the vertices of $G(v, v')$ may have degree more than 2 (compare with Lemma 5.10).

For instance, consider the extreme point $v$ with support $V = \{6, 10, 11, 13\}$, and the following choices for an adjacent extreme point $v'$:

- $v'$ with support $V' = \{5, 9, 10, 12\}$. Then $G(v, v')$ is isomorphic to $K_{3,3}$.
- $v'$ with support $V' = \{4, 5, 9, 10, 11, 13\}$. In this case, $G(v, v')$ is isomorphic to $K_{3,1}$.
- $v'$ with support $V' = \{5, 7, 8, 10, 12, 13\}$. In this case $G(v, v')$ has two components, each isomorphic to $K_{2,1}$, so it is not partly-connected (and the supports of $v$ and $v'$ are not disjoint).

Moreover, the support of the extreme points of $Q^*(A)$ have cardinality either 4 or 6, so that the conclusions of Corollary 5.14 do not hold (for instance, the previous choice of $v$ and the second choice for $v'$).

### 6 Adjacency in the case of the edge-vertex incidence matrix of a graph

In this section we restrict our study to the case where every row of the matrix $A \in \mathbb{B}^{m \times n}$ has exactly two ones.

Under these circumstances, we may think of $A$ as being the edge-vertex incidence matrix of a simple undirected graph $G$ with $n$ vertices and $m$ edges. That is,

$$a_{ep} = \begin{cases} 
1 & \text{if the edge } e \text{ is incident upon } p, \\
0 & \text{otherwise.}
\end{cases}$$

Thus, we may consider the stable set polytope of $G$, $\text{STAB}(G)$, defined by

$$\text{STAB}(G) = \text{conv} \{x \in \mathbb{B}^n : Ax \leq 1, x \geq 0\}$$

Chvátal (1975) proved:

**Theorem 6.1.** Let

- $G$ be a graph on $n$ vertices,
• \( w \) and \( w' \) be two extreme points of \( \text{STAB}(G) \), with corresponding supports \( W = \text{supp} \, w \) and \( W' = \text{supp} \, w' \).

Then \( w \) and \( w' \) are adjacent in \( \text{STAB}(G) \) if and only if the subgraph of \( G \) induced by \( W \Delta W' \) is connected.

By “flipping” the coordinates, \( x \rightarrow \varphi(x) = 1 - x \), \( \text{STAB}(G) \) maps to \( Q_0^* = \varphi(\text{STAB}(G)) = Q^*(A) \cap [0, 1]^n \).

Furthermore, as \( \varphi \) is linear and an involution, there is a one-to-one correspondence between faces of \( \text{STAB}(G) \) and those of \( Q_0^* \), in particular, \( \varphi \) preserves extreme points and their adjacencies.

Although \( \varphi \) determines a bijection between the extreme points of \( \text{STAB}(G) \) and \( Q_0^* \), it is only a one-to-one correspondence between extreme points of \( Q^*(A) \) and those of \( \text{STAB}(G) \). However, it has some useful properties:

**Lemma 6.2.** Suppose

- \( v \) and \( v' \) are extreme points of \( Q^*(A) \) with corresponding supports \( V \) and \( V' \),
- \( w = \varphi(v) \), \( w' = \varphi(v') \), \( W = \text{supp} \, w \), \( W' = \text{supp} \, w' \).

Then:

(a) The graph \( G(v, v') \) associated with \( v \) and \( v' \), as given in Definition 3.1, is the same as the subgraph of \( G \) induced by \( W \Delta W' \) (mentioned in Theorem 6.1).

(b) \( v \) and \( v' \) are adjacent in \( Q^*(A) \) if and only if \( w \) and \( w' \) are adjacent in \( \text{STAB}(G) \).

**Proof.** The first part is immediate once we realize that \( W = I_n \setminus V \), \( W' = I_n \setminus V' \), and \( W \Delta W' = V \Delta V' \).

For the second part, let \( S \) be the segment joining \( v \) and \( v' \), so that \( x \in S \) implies \( 0 \leq x \leq 1 \), that is, \( S \subset [0, 1]^n \).

If \( v \) and \( v' \) are adjacent in \( Q^*(A) \), there exists a half-space \( H \) such that

\[
H \cap Q^*(A) = S. \tag{6.1}
\]

Since \( S \subset [0, 1]^n \),

\[
H \cap Q_0^* = S, \tag{6.2}
\]

which implies that \( v \) and \( v' \) are adjacent in \( Q_0^* \), and therefore \( w \) and \( w' \) are adjacent in \( \text{STAB}(G) \).

Conversely, if \( w \) and \( w' \) are adjacent in \( \text{STAB}(G) \) then \( v \) and \( v' \) are adjacent in \( Q_0^* \), there exists a half-space \( H \) satisfying (6.2) and therefore (6.1), which implies that \( v \) and \( v' \) are adjacent in \( Q^*(A) \).
The following is a consequence of Chvátal’s Theorem 6.1 and the previous Lemma:

**Theorem 6.3.** Suppose that every row of \( A \in \mathbb{B}^{m \times n} \) has exactly two ones. Then, the extreme points \( v \) and \( v' \) of \( Q^* \) are adjacent if and only if the graph \( G(v, v') \) is connected, that is, if and only if Condition C1 is satisfied.

**Remark 6.4.** Notice that Theorems 6.3 and 4.3 imply that Condition C2 can never happen if there are exactly two ones per row in \( A \).

7 Minimally nonideal matrices

As previously, let us assume that \( A \in \mathbb{B}^{m \times n} \) has no dominating rows. Recalling the definitions of \( Q(A) \) and \( Q^*(A) \) in (2.2) and (2.1), we say that \( A \) is ideal if \( Q(A) = Q^*(A) \), and it is minimally nonideal (mni for short) if it is not ideal but \( Q(A) \cap \{ x \in \mathbb{R}^n : x_i = 0 \} \) and \( Q(A) \cap \{ x \in \mathbb{R}^n : x_i = 1 \} \) are integral polyhedra for all \( i \in \mathbb{I}_n \).

There are still several interesting open questions regarding mni matrices. On one hand, there is no good characterization of them and many studies revolve around Lehman’s fundamental ideas (Lehman, 1979b,a, 1990). On the other hand, few infinite families of mni matrices are known: \( C_2^n \) for odd \( n \), the matrices corresponding to degenerate finite projective planes, the family described by Wang (2011), as well as all of the corresponding blockers of these families.

Cornuéjols and Novick (1994) stated that, for \( n \) odd and greater than 9, it is always possible to add to \( C_2^n \) one row so that the resulting matrix is still minimally nonideal, obtaining another infinite family of mni matrices. In this section we will apply our findings to prove this result, showing in addition other more elaborate infinite families of mni matrices based on the family \( C_2^n \).

**Definition 7.1.** If a binary matrix \( A \) with no dominating rows and \( n \) columns contains a row submatrix \( A_1 \in \mathbb{B}^{n \times n} \) which is nonsingular and has \( r \) (where \( r \geq 2 \)) ones per row and column, and the other rows of \( A \) have more than \( r \) ones, then \( A_1 \) is called a core of \( A \).

Notice that if \( A \) has a core then it is unique (up to the permutation of rows). On the other hand, \( A \) may coincide with its core.

Denoting by \( J_t \) the matrix associated with the degenerate projective plane with \( t + 1 \) points and lines, we summarize some of Lehman’s results on mni matrices and their consequences (Lehman, 1979b,a, 1990) in the next two theorems:

**Theorem 7.2.** If \( A \in \mathbb{B}^{m \times n} \) is a mni matrix, then \( Q(A) \) has a unique fractional extreme point and \( b(A) \) is mni.

**Theorem 7.3.** Let \( A \in \mathbb{B}^{m \times n} \) be a mni matrix which is not isomorphic to \( J_t \) for any \( t \geq 2 \). Then \( A \) has a core, say \( A_1 \), and \( b(A) \) has a core, say \( B_1 \), such that:

(a) \( A_1 1 = r 1 \) and \( B_1 1 = s 1 \).

(b) The rows of \( A_1 \) and \( B_1 \) may be permuted so that

\[
A_1 B_1^T = J + (rs - n) I,
\]

where \( J \) is the matrix of all ones and \( I \) is the identity matrix.
(c) \( f^* = \frac{1}{r} \mathbf{1} \) is a fractional extreme point of \( Q(A) \) and \( Q(A_1) \).

(d) \( f^* \) is in exactly \( n \) edges of \( Q(A) \) (resp. \( Q(A_1) \)), so that \( f^* \) is adjacent, in both \( Q(A) \) and \( Q(A_1) \), to exactly \( n \) extreme points which make up the rows of \( B_1 \).

(e) \( x \cdot \mathbf{1} \geq s \) defines a facet of \( Q^*(A) \), and \( Q(A) \cap \{ x \in \mathbb{R}^n : x \cdot \mathbf{1} \geq s \} = Q^*(A) \).

Even if \( A_1 \) and \( B_1 \) satisfy (7.1), \( Q(A_1) \) may have several fractional extreme points. For example, taking \( n = 14, r = 5, s = 3, A = A_1 = \psi_{14}^0 \), we may see that \( Q(A_1) \) has 3 fractional extreme points.

On the other hand, a mni matrix may have a core which is not mni. For example, \( B = \mathbf{b}(\psi_{17}^5) \) is mni and its core \( B_1 \) is not mni since \( Q(B_1) \) has 69 fractional extreme points.

Lütolf and Margot (1998, Lemma 2.7 and Lemma 2.8) proved similar results without the mni assumption:

**Lemma 7.4.** Suppose that \( A \) has core \( A_1 \) and \( \mathbf{b}(A) \) has core \( B_1 \) so that (7.1) holds. Then:

(a) \( f^* = \frac{1}{r} \mathbf{1} \) is a fractional extreme point of \( Q(A) \) and \( Q(A_1) \).

(b) \( f^* \) is adjacent in \( Q(A) \) and \( Q(A_1) \) to exactly the \( n \) rows of \( B_1 \).

(c) \( \mathbf{b}(A_1) \) has core \( B_1 \).

(d) If \( Q(A) \) has just one fractional extreme point, then \( A \) must be mni.

Despite the “minimal” in mni, a mni matrix may have a row submatrix which is also mni. Cornuéjols and Novick (1994), and later Lütolf and Margot (1998), used this fact to construct many new mni matrices by adding rows to known ones. Of interest to us here is the possibility of adding one or more rows to \( \psi_n^s \), which is a mni matrix for \( n \) odd, to obtain another mni matrix.

One of the main tools for studying the extreme points obtained after the addition of an inequality is the following variant of Lemma 8 in Fukuda and Prodon (1996):

**Proposition 7.5.** Let \( A \in \mathbb{R}^{m \times n} \) be a matrix with non-negative entries, and suppose \( P = \{ x \in \mathbb{R}^n : Ax \geq b, x \geq 0 \} \) is a full dimensional polyhedron. Let us further assume that the inequality \( a \cdot x \geq c \) is independent of those defining \( P \), where \( a \geq 0 \) and \( c > 0 \).

Then, any extreme point \( v \) of the polyhedron \( P' = P \cap \{ x \in \mathbb{R}^n : a \cdot x \geq c \} \) must satisfy one (and only one) of the following:

- \( v \) is an extreme point of \( P \) satisfying \( a \cdot v \geq c \),
- \( v \) is a convex combination \( v = \alpha w + (1 - \alpha)w' \) of adjacent extreme points \( w \) and \( w' \) of \( P \), satisfying \( a \cdot w > c, a \cdot w' < c, \) and \( a \cdot v = c \), that is, \( \alpha = (c - a \cdot w')/(a \cdot w - a \cdot w') \),
- \( v = w + \beta a_h \) for some extreme point \( w \) of \( P, \beta > 0 \) and \( h \in \mathbb{N} \), such that \( \{ w + \gamma a_h : \gamma \geq 0 \} \) is an (infinite) edge of \( P, a \cdot w < c \) and \( a \cdot v = c \), that is, \( \beta = (c - a \cdot w)/a_h \) (necessarily \( a_h \neq 0 \)).
Suppose the mni matrix $A$ has core $A_1$, $B = b(A)$ has core $B_1$, and (7.1) is satisfied. Let $B$ be the set consisting of the fractional extreme point $f^*$ and its binary neighbors in $Q(A)$ (Theorem 7.3). Suppose furthermore that the binary matrix $M$ has more than $r$ ones per row, and we add to $A$ the rows of $M$ obtaining the matrix $E$, which has no dominating rows. Schematically,

$$E = \begin{bmatrix} A \\ M \end{bmatrix}.$$

(7.2)

Then, we have:

**Lemma 7.6.** If $Mu \geq 1$ for all $u \in B$, and any extreme point of $Q(E)$ which is not in $B$ is binary and has more than $s$ ones, then $E$ is mni.

**Proof.** Since $Q(E) \subseteq Q(A)$, if $v \in Q(E)$ is an extreme point of $Q(A)$, then it is also an extreme point of $Q(E)$. Thus, the elements of $B$ are extreme points of $Q(E)$ because $Mu \geq 1$ for $u \in B$. Since any extreme point of $Q(E)$ which is not in $B$ is binary, we conclude that $Q(E)$ has just one fractional extreme point.

By Lemma 7.4, it is enough to show now that $E$ has core $A_1$ and $b(E)$ has core $B_1$. The first condition is clear ($M$ has more than $r$ ones per row), and the second one follows from the fact that the rows of $b(E)$ are exactly the binary extreme points of $Q(E)$, and that any extreme point of $Q(E)$ which is not in $B$ has more than $s$ ones.

The following result relates adjacency in $Q(A)$ with adjacency in $Q^*(A)$ when $A$ is mni.

**Corollary 7.7.** Let $A$ be a mni matrix not isomorphic to any $\mathcal{F}_t$ ($t \geq 2$), and suppose its core $A_1$ and the core $B_1$ of its blocker satisfy (7.1). Let $f^* = \frac{1}{t}1$, and let $v$ and $v'$ be binary extreme points of $Q(A)$. Then:

(a) If $\max\{v \cdot 1, v' \cdot 1\} > s$, then $v$ and $v'$ are adjacent in $Q(A)$ if and only if they are adjacent in $Q^*(A)$.

(b) If $v \cdot 1 = v' \cdot 1 = s$, $v$ and $v'$ are always adjacent in $Q^*(A)$, and they are adjacent in $Q(A)$ if and only if the union of their supports is not $I_n$.

**Proof.** (a) follows from Theorem 7.3(e) and Proposition 7.5.

For the first part of (b), we notice that $v$ satisfies with equality $n - 1$ of the inequalities corresponding to the rows of $A_1$, as it is adjacent to $f^*$. Since this is also true for $v'$, $v$ and $v'$ satisfy tightly $n - 2$ inequalities coming from $A_1$ and the equality $v \cdot 1 = s$ which defines a facet of $Q^*(A)$ and is linearly independent with those of $A_1$ (as $f^*$ does not satisfy it).

For the last part of (b), let $y = \sum_{k=1}^\ell \lambda_k u^k$ be a strict convex combination of extreme points of $Q(A)$, and suppose $\frac{1}{\ell} (v + v') \geq y$. If $v$ and $v'$ have a common null coordinate, say $v_h = v'_h = 0$, then $y_h = 0$ and $f^* \neq u^k$ for all $k = 1, \ldots, \ell$. Therefore $u^k$ is a binary extreme point of $Q(A)$ and of $Q^*(A)$, and Proposition 2.1 allows us to conclude that $v$ and $v'$ are adjacent in $Q(A)$ if and only if they are adjacent in $Q^*(A)$.

Finally, if $(\text{supp } v) \cup (\text{supp } v') = I_n$, we have $\frac{1}{t} (v + v') \geq f^*$ (since $r \geq 2$, see Definition 7.1), and then by Proposition 2.1 we conclude that $v$ and $v'$ are not adjacent in $Q(A)$. 

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In the remainder of the section we will focus our attention on the matrix $C_n^2$ with $n$ odd. $C_n^2$ coincides with its core, having exactly 2 ones per row and per column, and the core of $b(C_n^2)$ has $s = (n + 1)/2$ ones per row and per column. For simplicity of notation, in what follows we consider:

- $n$ odd, $s = (n + 1)/2$,
- if the matrix $A$ is not written explicitly, $Q$ will stand for $Q(C_n^2)$, and,
- once and for all we write $f^* = \frac{1}{2} \cdot 1$.

Let us start with some simple properties.

**Lemma 7.8.** $v \in B^n$ is an extreme point of $Q$ if and only if it does not have two consecutive zeroes and does not have three consecutive ones (cyclically).

*Proof.* For the “only if” part we notice that a binary extreme point cannot have two consecutive zeroes, since otherwise it would not be a transversal, and cannot have three consecutive ones cyclically, since changing the middle 1 to 0 still yields a transversal.

The “if” part is similar. If $v$ does not have two consecutive zeroes, then it is in $Q$. If $v$ were not minimal, we could diminish a coordinate from 1 to 0, still staying in $Q$, let us call $w$ this new point. Since $v$ doesn’t have three consecutive ones, $w$ must have two consecutive zeroes, but then $w \not\in Q$.

**Lemma 7.9.** Let $v$ and $v'$ be two distinct binary extreme points of $Q$, and let $G$ be the associated graph (Definition 3.1).

- If $p \in S(v, v')$, $p' \in S(v', v)$ and $p \leftrightarrow p'$ in $G$, then $p' = p \pm 1$.
- If $i, i + 1, \ldots, j$ is a path of $G$, then $i$ and $j$ belong to the same bipartition of $G$ if and only if $[i, j]_v$ has odd cardinality.
- If $i$ and $j$ belong to the same component of $G$, then either $i, i + 1, \ldots, j$ or $j, j + 1, \ldots, i$ is a path of $G$, but not both.
- $G$ has no cycles.
- If $\max \{v \cdot 1, v' \cdot 1\} > s$, then $v$ and $v'$ are adjacent in $Q$ if and only if $G$ is a path. Moreover, this path must be even if $v \cdot 1 \neq v' \cdot 1$.

*Proof.* Most of the statements are clear considering that $n$ is odd.

For the last item, Corollary 7.7(a) tells us that adjacency in $Q$ of $v$ and $v'$ is equivalent to their adjacency in $Q^*$, and Theorem 6.3 tells us that this is equivalent to $G$ being a path because $G$ has no cycles.

Finally, if $v \cdot 1 \neq v' \cdot 1$, by Corollary 5.14 one of $|S(v, v')|$ and $|S(v', v)|$ is odd and the other even, so that the path $G$ has an even number of edges.

**Lemma 7.10.** For $n \geq 9$ odd, let $\{i, j, l\} \subset I_n$ be such that $i < j < l$, $j - i \geq 3$ odd, $l - j \geq 3$ odd, and either $i \neq 1$ or $l \neq n$. Let $v$ and $v'$ be two distinct binary extreme points of $Q$ and let $G$ be the associated graph. If $v'_i = v'_j = v'_l = 0$, then any component of $G$ contains at most one element of the set $\{i, j, l\}$.
Proof. Let us observe first that the cardinalities of $[i,j]_n$, $[j,l]_n$ and $[l,j]_n$ are even as $j-i$, $l-j$ and $n$ are odd.

To prove the lemma, assume by contradiction that a component of $\mathcal{G}$ contains two elements of the set $\{i,j,l\}$. Without loss of generality, let us say these are $i$ and $j$. Now, since $v'_i = v'_j = 0$, we have $i,j \in S(v',v')$, and so $i$ and $j$ are in the same bipartition of $\mathcal{G}$. Using also the fact that the cardinality of $[i,j]_n$ is even, from Lemma 7.9 we conclude that $j,j+1,\ldots,i$ is a path of $\mathcal{G}$. As $l$ is in this path and $[j,l]_n$ has an even number of elements, using again Lemma 7.9 we conclude that $l \in S(v',v)$, but this contradicts $v'_i = 0$.

We are now ready to prove the following claim by Cornucójols and Novick (1994).

**Proposition 7.11.** Under the hypotheses of Lemma 7.10, let $a \in \mathbb{B}^n$ be the characteristic vector of the set $\{i,j,l\} \subset \mathbb{B}$. Then, the matrix $E$ obtained by adding to $\mathcal{G}_n^2$ the row vector $a$ is mni.

**Proof.** By Lemma 7.6 it will be enough to show that:

(a) If $B$ is the set consisting of $f^*$ and its neighbors in $Q$, then $a \cdot u \geq 1$ for all $u \in B$.

(b) Any extreme point of $Q(E)$ not in $B$ is binary and has more than $s = (n+1)/2$ ones.

To show (a), notice that $a \cdot f^* = 3/2 > 1$. On the other hand, if $u$ is in the core of $b(\mathcal{G}_n^2)$, then it is a rotation (or shift) of

$$(1,1,0,1,0,1,0,1,0,1,0,1,0),$$

and therefore $a \cdot u \geq 1$, as the cardinalities of $[i,j]_n$, $[j,l]_n$ and $[l,j]_n$ are even.

To show (b) we rely on Proposition 7.5.

Suppose $v$ is an extreme point of $Q(E)$ which is a convex combination of the adjacent extreme points $w$ and $w'$ of $Q$,

$$v = \alpha w + (1-\alpha)w',$$

with

$$a \cdot v = 1, \quad a \cdot w > 1, \quad a \cdot w' < 1. \quad (7.3)$$

Since $a \cdot f^* = 3/2$ and the neighbors $u$ of $f^*$ in $Q$ satisfy $a \cdot u \geq 1$, $f^*$ is different from both $w$ and $w'$. Thus, $w$ and $w'$ are binary. In particular,

$$a \cdot w' = 0,$$

and by the first part we have $w' \notin B$, which in turn implies that $w' \cdot 1 > s$.

Now, since $w$ and $w'$ are adjacent in $Q$, by Lemma 7.9 the graph $\mathcal{G}$ is a path (in particular, connected), and so by Lemma 7.10, we must have

$$a \cdot w \leq 1,$$

contradicting (7.3). Thus, the second possibility described in Proposition 7.5 cannot happen in the case of $Q(E) = Q \cap \{x \in \mathbb{R}^n : a \cdot x \geq 1\}$. 

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Suppose now \( v \) is an extreme point of \( Q(E) \) of the form

\[
v = w + \beta e_h,
\]

where \( w \) is an extreme point of \( Q \) satisfying \( a \cdot w < 1 \), and \( a \cdot v = 1 \). Once again, \( w \) cannot be either \( f^* \) or any of its neighbors. It follows that \( w \) is binary and \( s < w \cdot 1 \). Then, we have \( a \cdot w = 0 \), which implies \( \beta = 1 \) since \( a \cdot v = 1 \). Thus, we conclude that \( v \) is binary, and

\[
s < w \cdot 1 \leq v \cdot 1,
\]

proving (b). \( \square \)

One would hope that it is possible to add a circulant matrix \( M \) instead of just a single row, but this is not true in general. For instance, if we add to \( A = \mathcal{E}_3^2 \) the matrix

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

the resulting matrix \( E \) is not mni, whereas, by Proposition 7.11, adding just one row of \( M \) we obtain a mni matrix.

Let us see that we may obtain systematically a mni matrix by adding to \( \mathcal{E}_3^2 \) all the rows of a circulant matrix.

For \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), let us denote by \( \rho^h(x) \) the rotation (or shift) of \( x \) in \( h \) spaces to the right:

\[
\rho^h(x) = (x_{n-h+1}, \ldots, x_n, x_1, \ldots, x_{n-h}).
\]

If \( n = 3\nu \), let

\[
a^1 = (1, 0, 0, 1, 0, 0, \ldots, 1, 0, 0), \quad a^2 = \rho^1(a^1), \quad a^3 = \rho^2(a^1),
\]

that is, \( a^1 \) consists of \( \nu \) groups of the form \((1, 0, 0)\), and let \( M \) be the matrix with rows \( a^1, a^2, a^3 \).

We would like to show next:

**Theorem 7.12.** Let \( n = 3\nu \) be odd, where \( \nu \geq 3 \). If \( E \) is the matrix obtained from \( A = \mathcal{E}_3^2 \) by appending the rows of \( M \) (as in (7.2)), then \( E \) is mni.

Thus, varying \( \nu \) we obtain an infinite family of mni matrices.

The proof will be based on several lemmata, in which we preserve the notation.

**Lemma 7.13.** For \( i = 1, 2, 3 \) let

\[
w^i = 1 - a^i,
\]

so that, for example, \( w^1 = (0, 1, 1, 0, 1, 1, \ldots, 0, 1, 1) \), and let \( \mathcal{W} = \{w^1, w^2, w^3\} \).

Then
(a) \( w^i \) is an extreme point of \( Q \).

(b) \( w^i \cdot 1 = 2\nu \) and any other extreme point \( v \) of \( Q \) not in \( W \) satisfies \( v \cdot 1 \leq 2\nu - 1 \).

(c) \( w^i \cdot a^i = 0 \).

(d) If \( i \neq j \), \( w^i \) and \( w^j \) are not adjacent in \( Q \).

(e) \( v \cdot a^i \geq 1 \) for every extreme point \( v \) of \( Q \) different from \( w^i \).

Therefore, an extreme point \( v \) of \( Q \) is in \( Q(E) \) if and only if \( v \notin W \).

**Proof.** Notice that \( f^* \cdot a^i = \nu/2 > 1 \) and \( f^* \cdot 1 = 3\nu/2 < 2\nu - 1 \) as \( \nu \geq 3 \).

(a) and (b) follow from Lemma 7.8, and (c) follows from the definition of \( w^i \) in (7.5). 

Since \( w^i \cdot 1 = w^j \cdot 1 = 2\nu > s = (3\nu + 1)/2 \) and \( G(w^i, w^j) \) consists of \( \nu \) disjoint arcs, we may use Corollary 7.7 and Theorem 6.3 to show (d).

For (e), if \( v \) is binary and \( v \cdot a^i = 0 \), then \( v \) is dominated by \( w^i \) and so they must coincide as both are extreme points of \( Q \).

---

**Lemma 7.14.** Let

\[
    u^1 = (0, 1, 0, 1, 0, 1, 0, 1, 1, \ldots, 0, 1, 1),
\]

that is, \( u^1 \) starts with the group \((0, 1, 0, 1, 0, 1)\) followed by \( \nu - 2 \) groups of the form \((0, 1, 1)\), and let

\[
    u^j = \rho^{j-1}(u^1) \quad \text{for} \quad j = 2, \ldots, n.
\]

Then, for fixed \( i \), the extreme points of \( Q \) adjacent to \( w^i \) are

\[
    u^i, u^{i+3}, \ldots, u^{i+3(\nu-1)},
\]

which are the only extreme points of \( Q \) in the hyperplane \( \{ x \in \mathbb{R}^n : x \cdot a^i = 1 \} \).

**Proof.** Let us start by observing that the points in (7.6) are indeed extreme points of \( Q \) by Lemma 7.8, and they satisfy the equality \( x \cdot a^i = 1 \).

Let us see that they are the only ones in the hyperplane \( \{ x \in \mathbb{R}^n : x \cdot a^i = 1 \} \).

Suppose \( v \) is an extreme point such that \( v \cdot a^i = 1 \). After eventually some rotations, we may assume that \( i = 1 \) and \( v_1 = 1 \). Then, \( v_4 = v_7 = \cdots = v_{1+3(\nu-1)} = 0 \), and these zeroes must be surrounded by ones (Lemma 7.8).

Thus \( v \) is of the form:

\[
    (1, ?, 1, 0, 1, 0, 1, 0, 1, \ldots, 0, 1, ?),
\]

and since it cannot have three consecutive ones (again by Lemma 7.8), we have:

\[
    v = (1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0) = u^{n-2} = u^{1+3(\nu-1)}.
\]

To see that these points are adjacent to \( u^i \), we observe that it is enough to show this for \( u^i \), since \( \rho^3(w^i) = w^i \). Even more, without loss of generality we may restrict ourselves to showing that \( u^i \) and \( w^i \) are adjacent in \( Q \).

Now, \( G(u^1, u^i) \) consists of the path \((3, 4, 5)\), with \( S(w^i, u^1) = \{3, 5\} \) and \( S(u^i, w^i) = \{4\} \), so \( u^1 \) and \( w^i \) are adjacent.
Suppose now that $u$ is an extreme point of $Q$ adjacent to $w^1$ in $Q$. By Lemma 7.13, $u \notin \{w^2, w^3\}$ and $u \cdot 1 = 2\nu - 1$, so that $G(u, w^1)$ reduces to an even path by Lemma 7.9. Since the ones in $w^1$ come in pairs and $u \cdot 1 < w^1 \cdot 1$, this path must be of the form $j, j + 1, j + 2 \pmod n$, with $\{j, j + 2\} = S(u, w^1)$ and $\{j + 1\} = S(u, w^3)$. Hence, $u$ is obtained from $w^1$ by replacing a group $(1, 1, 0, 1, 1)$ by $(1, 0, 1, 0, 1)$, so $u = \rho^{3k}(w^1)$ for some $k$. \hfill \Box

For future reference, we notice that
\[
\begin{align*}
   u^1 + e_3 + e_5 &= w^1 + e_4, \\
   u^2 + e_{j+2} + e_{j+4} &= w^1 + e_{j+3},
\end{align*}
\]
where, as usual, the sums in the indices are to be understood modulo $n$.

**Lemma 7.15.** If $E$ is the matrix $G_n^2$ to which we have appended the rows $a^1$, $a^2$ and $a^3$ (defined in (7.4)), then the extreme points of $Q(E)$ are those of $Q(G_n^2)$ except for $w^1$, $w^2$ and $w^3$ (defined in (7.5)).

**Proof.** By Lemma 7.13, every extreme point of $Q$ not in $W = \{w^1, w^2, w^3\}$ is an extreme point of $Q(E)$.

Conversely, let us see that if we add one row at a time then no new extreme points are created, and the points in $W$ are the only extreme points that are eliminated.

In the first place, let us consider the intersection of $Q$ with the half-space \( \{x \in \mathbb{R}^n : a^i \cdot x \geq 1\} \). If new extreme points are created, then they should come from the intersection of the hyperplane $\{x \in \mathbb{R}^n : a^i \cdot x = 1\}$ with an edge (Proposition 7.5). This edge should be incident to an extreme point $v$ satisfying $a^i \cdot v < 1$, and therefore $v = w^1$ is an endpoint of the edge (Lemma 7.13). Given that $w^1$ is adjacent only to the extreme points in (7.6), and these points belong to $\{x \in \mathbb{R}^n : a^i \cdot x = 1\}$, any new extreme point must come from the intersection of an (infinite) edge of the form $\{w^1 + \gamma e_h : \gamma \geq 0\}$ with the hyperplane $\{x \in \mathbb{R}^n : a^i \cdot x = 1\}$. Since $a^i \cdot w^1 = 0$, this intersection must be of the form $\{w^1 + e_h\}$ with $h \in \text{supp } a^i$. Hence, $h = 3k + i$ for some $k$, and so, setting $j = h - 3$ and using (7.7), we have
\[
w^1 + e_h = w^1 + e_{j+3} = w^1 + e_{j+2} + e_{j+4}.
\]
It follows that $w^1 + e_h$ dominates $w^j$, and then it cannot be an extreme point of $Q \cap \{x \in \mathbb{R}^n : a^i \cdot x \geq 1\}$. Thus, we conclude that no new extreme point is created and the extreme points of $Q \cap \{x \in \mathbb{R}^n : a^i \cdot x \geq 1\}$ are the extreme points of $Q$ except for $w^1$.

Finally, since $a^i \cdot w^j = \nu > 1$ for $j \neq i$, observe that the addition of the constraint $a^i \cdot x \geq 1$ does not modify the adjacency relations for $w^1 (j \neq i)$ in the resulting polyhedron, which are still given by Lemma 7.14. Then, we can repeat the argument above each time we add a new constraint. This completes the proof. \hfill \Box

**Proof of Theorem 7.12.** The previous Lemmata show that, except for the points in $W = \{w^1, w^2, w^3\}$ (defined in (7.5)), the extreme points of $Q$ and $Q(E)$ coincide, so that Lemma 7.6 yields that $E$ is mni. \hfill \Box
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