LOCAL INDEX FORMULA FOR THE QUANTUM DOUBLE SUSPENSION

PARTHA SARATHI CHAKRABORTY AND BIPUL SAURABH

Abstract. Our understanding of local index formula in noncommutative geometry is stalled for a while because we do not have more than one explicit computation, namely that of Connes for quantum SU(2) and do not understand the meaning of the various multilinear functionals involved in the formula. In such a situation further progress in understanding necessitates more explicit computations and here we execute the second explicit computation for the quantum double suspension, a construction inspired by the Toeplitz extension. More specifically we compute local index formula for the quantum double suspensions of $C(S^2)$ and the noncommutative 2-torus.

1. INTRODUCTION

In the spectral formulation of noncommutative geometry, Connes ([6]) specifies a noncommutative geometric ‘space’ by a triple consisting of a Hilbert space $\mathcal{H}$, an involutive subalgebra $\mathcal{A}$ of the algebra of bounded operators on $\mathcal{H}$ and a self adjoint operator $D$ with compact resolvent. The algebra $\mathcal{A}$ and the operator $D$ are tied up by the requirement that the commutators $[D, \mathcal{A}]$ give rise to bounded operators. Such a triple is called a spectral triple or an unbounded $K$-cycle. Often it is required that the spectral triple satisfies further conditions ([8]). The conditions of regularity and discrete dimension spectrum was introduced by Connes and Moscovici in [9]. More specifically a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be regular if both $\mathcal{A}$ and $[D, \mathcal{A}]$ are in the domains of $\delta^n$ for all $n \geq 0$, where $\delta$ is the derivation $\| D \| \cdot \cdot$.

One says that the spectral triple has dimension spectrum $\mathcal{G}$, if for every element $b$ in the smallest algebra $\mathcal{B}$ containing $\mathcal{A}$, $[D, \mathcal{A}]$ and closed under the derivation $\delta$, the associated zeta function $\zeta_b(z) = Tr b |D|^{-z}$ a priori defined on the right half plane $\Re(z) > p$ admits a meromorphic extension to the

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whole complex plane with poles contained in $\mathcal{S}$. They arrived at these conditions in their efforts to give an expression for the Chern character in terms of local data. The local Chern character constructed in (9) is a cocycle in the total complex of the b-B bicomplex and therefore given by a sum of several multilinear functionals $\phi_n$'s. At this point one sees a departure from the classical world in the sense that for the canonical spectral triple associated with a Riemannian spin manifold most of these terms vanish (b) page 231 in (9)). This indicates that the terms \{\phi_n\} appearing in the local Chern character are offshoots of noncommutativity and may exhibit features of the noncommutative world not visible in the commutative side of the story. This calls for further exploration of the local Chern character.

Indeed during 2000-2006 we saw a lot of activity around the local index formula (LIF). Most of these ([1],[2],[3]) were concerned with extending the formula itself to the framework of semi-finite spectral triples, while Connes ([10]) gave a demonstration of the formula for the spectral triple constructed in ([4]). This computation of Connes is our point of departure. One might wonder what is the point of the article ([10])? Should one interpret it as just a one time calculation or as an invitation to make explicit computations in various instances to gain further insight on the terms appearing in the formula. We take the second interpretation. But unfortunately in all these years we have only one ([18]) more explicit computation of the local index formula. That too essentially follows from the arguments of Connes and the terms that contribute to the residue cocycle in ([18]) are not different from those found by Connes. Therefore, so far, effectively we have only one explicit computation of the local index formula. In another attempt, ([17]) did a careful analysis of ([10]) and extended the proof of regularity and discreteness of dimension spectrum to the case of odd dimensional quantum spheres, but did not quite compute the formula. In such a situation it looked imperative to try and compute in some more cases. The first stumbling block is establishing regularity and discreteness of the dimension spectrum. That was achieved in ([5]), through the construction of quantum double suspension (QDS) of spectral triples hitherto known for $C^*$-algebras ([16]). Chakraborty and Sundar also identified the hypothesis of weak heat kernel asymptotic expansion (to be abbreviated as WHKAE) that allows one to conclude stability of the hypothesis of regularity and dimension spectrum under QDS. More specifically it was shown that if one starts with a spectral triple that is regular and has got discrete dimension spectrum satisfying WHKAE then its quantum double suspension is regular, has discrete dimension spectrum and also satisfies WHKAE. The computation in ([10]) can also be seen in this light as the QDS of the canonical spectral triple of the circle. But ([5]) fell short of the actual description of the multilinear functionals involved in the local
Chern character and that brings us to the contents of this paper, namely, explicit computation of the LIF for the quantum double suspension. In the terminology of \([5]\) the articles \([10,17]\) considered LIF for \(\Sigma^2C(S^1)\), the quantum double suspension of the circle and its iterations. Here we consider LIF for the quantum double suspension of two dimensional manifolds. More specifically we take up two cases, one classical and one noncommutative. The classical case we consider is that of the two sphere and the noncommutative case tackled is the noncommutative two torus. Thus this article can be seen as the second computation after that of \([10]\). In section two we recall the local Chern character, the main object of interest in the LIF. It is a cocycle in the total complex of the \(b-B\)-bicomplex and depending upon the parity of the spectral triple is given by a finite sequence of multilinear functionals \(\{\phi_{2n}\}\) or \(\{\phi_{2n+1}\}\). For our purpose we express \(\phi_n\)'s in terms of some other functional \(\psi_1(0)\)'s. Let \(\Sigma^2\psi_1(0)\) denote the corresponding quantity for the quantum double suspended spectral triple. In section three we obtain expressions for these. Finally in section four we apply these in two concrete situations. We take up the case of quantum double suspension of the two sphere and the QDS of the noncommutative two torus. We explicitly describe \(\Sigma^2\phi_n\)'s, the multilinear functionals involved in the LIF. Section five contains concluding remarks.

2. LOCAL INDEX FORMULA

Let \((A,\mathcal{H},D)\) be a \(p^+\) summable regular spectral triple. We denote by \(d\), \(\delta\) and \(\nabla\) the commutator with \(D\), \(|D|\) and \(D^2\) respectively i.e. \(da = [D,a]\), \(\delta a = |D|a\), and \(\nabla a = [D^2,a]\). For \(k \in \mathbb{N}^n\), define \(|k| = \sum_{i=1}^n k_i\). We will now state the Connes–Moscovici local index theorem for a regular odd spectral triple with discrete dimension spectrum.

**Theorem 2.1** \([9]\). Let \((A,\mathcal{H},D)\) be a regular \(p^+\) summable odd spectral triple. Assume further that the dimension spectrum is discrete with finite multiplicity. Define, for \(n\) odd,

\[
\phi_n(a_0, \ldots, a_n) = \sum_{k_j \geq 0} c_{n,k} \text{Res}_{z=0} \left( \Gamma(|k| + \frac{n}{2} + z) \text{Trace} \left( a_0 \nabla^{k_1}(da_1) \cdots \nabla^{k_n}(da_n)|D|^{-n-2|k|-2z} \right) \right),
\]

where \(a_j \in A\), \(k = (k_1, \ldots, k_n) \in \mathbb{N}^n\) and \(c_{n,k}\) are given by

\[
c_{n,k} = \sqrt{2i(-1)^{|k|}} \left( \prod_{j=1}^n k_j \prod_{j=1}^n (k_1 + \ldots + k_j + j) \right)^{-1}
\]

Then \(\phi_n\) is zero except for finitely many \(n\)'s and \((\phi_1,\phi_3,\ldots)\) is a \((b,B)\)-cocycle.
Furthermore, the cohomology class of this cocycle in $HC^{\text{odd}}(A)$ is same as the Chern character of $(A, \mathcal{H}, D)$, in particular, for $[u] \in K_1(A)$, one has

$$\langle (\phi_n), [u] \rangle = \langle [Ch^F], [u] \rangle.$$  

Corollary 2.2. If in addition we assume that the dimension spectrum is discrete and simple, then the cocycle $\phi_n$ in the above theorem is given by

$$\phi_n(a_0, \ldots, a_n) := \sum_k c_{n,k} \text{Res}_{z=0} \text{Trace} \left( a_0 \nabla^{k_1} (da_1) \cdots \nabla^{k_n} (da_n) |D|^{-n-2|k|-2z} \right),$$

where $a_j \in A$, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $c_{n,k}$ are given by

$$c_{n,k} = (-1)^{|k|} \sqrt{2i} \left( \prod_{j=1}^{n} k_j \right)^{n} (k_1 + \cdots + k_j + j)^{-1} \Gamma(|k| + \frac{n}{2}).$$

For a regular even spectral triple with discrete dimension spectrum, we state the Connes–Moscovici local index theorem as follows:

**Theorem 2.3** ([13]). Let $(A, \mathcal{H}, D, \gamma)$ be a regular $p^+$ summable even spectral triple. Assume further that the dimension spectrum is discrete with finite multiplicity. Define, for $n$ even,

$$\phi_0(a_0) = \text{Res}_{z=0} \left( \Gamma(z) \text{Trace} (\gamma a_0 |D|^{-2z}) \right).$$

$$\phi_n(a_0, \ldots, a_n) = \sum_{k \geq 0} c_{n,k} \text{Res}_{z=0} \left( \Gamma(|k| + \frac{n}{2} + z) \text{Trace} \left( \gamma a_0 \nabla^{k_1} (da_1) \cdots \nabla^{k_n} (da_n) |D|^{-n-2|k|-2z} \right) \right),$$

where $a_j \in A$, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $c_{n,k}$ are given by

$$c_{n,k} = \sqrt{2i} (-1)^{|k|} \left( \prod_{j=1}^{n} k_j \right)^{n} (k_1 + \cdots + k_j + j)^{-1}.$$  

Then $\phi_n$ is zero except for finitely many $n$‘s and $(\phi_0, \phi_2, \ldots)$ is a $(b, B)$-cocycle.

Furthermore, the cohomology class of this cocycle in $HC^{\text{even}}(A)$ is same as the Chern character of $(A, \mathcal{H}, D, \gamma)$, in particular, for $[p] \in K_0(A)$, one has

$$\langle (\phi_n), [p] \rangle = \langle [Ch^F], [p] \rangle.$$  

Corollary 2.4. If in addition we assume that the dimension spectrum is discrete and simple, then the cocycle $\phi_n$ in the above theorem is given by

$$\phi_0(a_0) = \text{Res}_{z=0} \left( \Gamma(z) \text{Trace} (\gamma a_0 |D|^{-2z}) \right).$$

$$\phi_n(a_0, \ldots, a_n) = \sum_k c_{n,k} \text{Res}_{z=0} \text{Trace} \left( a_0 \nabla^{k_1} (da_1) \cdots \nabla^{k_n} (da_n) |D|^{-n-2|k|-2z} \right),$$

where $a_j \in A$, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $c_{n,k}$ are given by

$$c_{n,k} = (-1)^{|k|} \sqrt{2i} \left( \prod_{j=1}^{n} k_j \right)^{n} (k_1 + \cdots + k_j + j)^{-1} \Gamma(|k| + \frac{n}{2}).$$
By rearranging the terms, we write the linear functionals $\phi_n$ as sum of some other linear functionals which will be more tractable for our purpose. Note that

$$\nabla^n(T) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n+k}(T)|D|^{n-k}. $$

Using these equations, we get

$$|D|^n T = \sum_{k=0}^{n} \binom{n}{k} \delta^k(T)|D|^{n-k}. $$

Using these equations, we get

$$\nabla^{k_1}(T_1)\nabla^{k_2}(T_2)$$

$$= \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} 2^{k_1-j_1} 2^{k_2-j_2} \binom{k_1}{j_1} \binom{k_2}{j_2} \delta^{k_1+j_1}(T_1)\delta^{k_2+j_2}(T_2)|D|^{k_1-j_1+k_2-j_2}. $$

Hence

$$a_0\nabla^{k_1}(a_1)\nabla^{k_2}(a_2)\ldots \nabla^{k_n}(a_n)|D|^{-n-2|k|-2z} = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \ldots \sum_{j_n=0}^{k_n} \frac{\delta^{k_1+j_1}(a_1)\delta^{k_2+j_2}(a_2)\ldots \delta^{k_n+j_n}(a_n)|D|^{-n-|k|-|\ell|-2z}}{\delta^{k_1+j_1}(a_1)\delta^{k_2+j_2}(a_2)\ldots \delta^{k_n+j_n}(a_n)|D|^{-n-|k|-|\ell|-2z}}.$$
where $B_n^x$ is given by

$$B_n^x = \sum_{k_1=\lfloor x_1/2 \rfloor}^{x_1} \sum_{s_1=0}^{\lfloor (x_2-s_1)/2 \rfloor} \sum_{s_2=0}^{x_2} \cdots \sum_{s_n=0}^{\lfloor (x_n-s_{n-1})/2 \rfloor} \sum_{k_n=\lfloor (x_n-s_{n-1})/2 \rfloor}^{x_n} c_{n,k} 2^{2k-|x|+s} \prod_{i=1}^{n} (x_i - k_i - s_{i-1}) (S_i)$$

Define for $x \in \mathbb{N}^n$ and $k \in \mathbb{N}$,

$$\psi^k_x(a_0, a_1, \ldots, a_n) = \text{Res}_{z=(n+|x|+k)/2} \text{Trace} \left( a_0 \delta_x^0(a_1) \cdots \delta_x^n(a_n) |D|^{-2x} \right)$$

Then $\phi_n$ can be written as

$$\phi_n(a_0, a_1, \ldots, a_n) = \sum_{x_i=0}^{\infty} B_n^x \psi_x^0(a_0, da_1, \ldots, da_n) \quad (2.1)$$

Note that if $n + |x| + k > p$ then $\psi^k_x(a_0, a_1, \ldots, a_n) = 0$ for all $(a_0, a_1, \ldots, a_n)$.

So, the sum given above is a finite sum. For an even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$, one has

$$\phi_n(a_0, a_1, \ldots, a_n) = \sum_{x_i=0}^{\infty} B_n^x \psi_x^0(\gamma a_0, da_1, \ldots, da_n) \quad (2.2)$$

### 3. LOCAL INDEX FORMULA FOR QUANTUM DOUBLE SUSPENSION

Let us fix some notations. We denote the left shift on $L^2(\mathbb{N})$ by $S$ which is defined on the standard orthonormal basis $(e_n)$ as $Se_n = e_{n-1}$. For $n < 0$, we denote by $S^n$ the operator $S^{n|n|}$. Let $p$ be the projection $|e_0\rangle \langle e_0|$. The number operator on $L^2(\mathbb{N})$ is denoted by $N$ and defined as $Ne_n := ne_n$. The Toeplitz algebra is denoted by $\mathcal{T}$.

**Definition 3.1.** Let $A$ be a unital $C^*$-algebra. Then quantum double suspension of $A$ denoted by $\Sigma^2 A$ is defined as the $C^*$-algebra generated by $A \otimes p$ and $1 \otimes S$ in $A \otimes \mathcal{T}$.

Let $\mathcal{A}$ be a dense $*$-subalgebra of a $C^*$-algebra $A$. Define

$$\Sigma^2(\mathcal{A}) = \text{span} \left\{ a \otimes k, 1 \otimes S^n : a \in \mathcal{A}, k \in \mathcal{S}(L^2(\mathbb{N})), n \in \mathbb{Z} \right\}$$

where $\mathcal{S}(L^2(\mathbb{N})) := \left\{ (a_{mn}) : \sum_{m,n}(1+m+n)^p |a_{mn}| < \infty \text{ for } p \in \mathbb{N} \right\}$. Clearly $\Sigma^2(\mathcal{A})$ is a dense $*$ subalgebra of $\Sigma^2(\mathcal{A})$.

**Definition 3.2.** Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd spectral triple and denote the sign of $D$ by $F$. Then the spectral triple $(\Sigma_{alg}^2(\mathcal{A}), \mathcal{H} \otimes L^2(\mathbb{N}), \Sigma^2(D) := ((F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. For an even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ where $\gamma$ is the grading operator, the spectral triple $(\Sigma_{alg}^2(\mathcal{A}), \mathcal{H} \otimes L^2(\mathbb{N}), \Sigma^2(D) := ((F \otimes 1)(|D| \otimes 1 + 1 \otimes N), \gamma \otimes 1)$ is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$. 
Remark 3.3. It should be emphasized that $\Sigma^2(D)$ can not be replaced by operators like $D \otimes I + I \otimes N$ for the simple reason that the latter operator do not have compact resolvent.

We use $D_0$ to denote the Dirac operator $\Sigma^2 D$. We denote by $d_0$ and $\delta_0$ the commutator with $D_0$ and $|D_0|$ respectively. Given a multilinear functional $\phi$ defined on $A$, we denote by $\Sigma^2 \phi$, the corresponding multilinear functional defined on $\Sigma^2 A$. Now we proceed to our main aim that is to describe local index formula $f$ or the quantum double suspension spectral triple in terms of linear functionals appearing in local index formula for original spectral triple. To approach the problem, we need to put an extra condition namely WHKAE on the spectral triple.

Let $\phi : (0, \infty) \rightarrow \mathbb{C}$ be a continuous function. We say that $\phi$ has an asymptotic expansion near 0 if there exists a sequence of complex numbers $(a_r)_0^\infty$ such that given $N$ there exist $\epsilon, M$ such that if $t \in (0, \infty)$

$$\left| \phi(t) - \sum_{r=0}^{N} a_r t^r \right| \leq Mt^{N+1}$$

We write $\phi(t) \sim \sum_{r=0}^{\infty} a_r t^r$ as $t \rightarrow 0^+$. Note that the coefficients $a_r$ are unique. For,

$$a_N = \lim_{t \rightarrow 0^+} \frac{\phi(t) - \sum_{r=0}^{N} a_r t^r}{t^N}.$$ 

If $\phi(t) \sim \sum_{r=0}^{\infty} a_r t^r$ as $t \rightarrow 0^+$ then $\phi$ can be extended continuously to $[0, \infty)$ simply by letting $\phi(0) := a_0$.

Definition 3.4. Let $(A, \mathcal{H}, D)$ be a $p^+$-summable odd spectral triple for a $C^*$-algebra $A$ where $A$ is a dense $*$-subalgebra of $A$. We say that the spectral triple has the weak heat kernel asymptotic expansion property (WHKAE) of dimension $p$ if there exists a $*$-subalgebra $B \subset \mathcal{L}(\mathcal{H})$ such that

1. $B$ contains $A$.
2. The unbounded derivation $\delta := ||D||_*$ leaves $B$ invariant. Also the unbounded derivation $d := [D, \cdot]$ maps $A$ into $B$.
3. $B$ is invariant under the left multiplication by $F := \text{sign} D$.
4. For every $b \in B$, the function $\tau_{p, b}(t) : [0, \infty) \rightarrow \mathbb{C}$ defined by $\tau_{p, b}(t) = t^p Tr \left( be^{-t|D|} \right)$ has an asymptotic expansion.

Remark 3.5. (1) If the algebra $A$ is unital and the representation of $A$ on $\mathcal{H}$ is unital then condition 3 can be replaced by the condition $F \in B$.

(2) In case of an even spectral triple, we further demand $B$ to be invariant under left multiplication by the grading operator $\gamma$. If the algebra $A$
unital and the representation of $\mathcal{A}$ on $\mathcal{H}$ is unital then this condition can be replaced by the condition $\gamma \in \mathcal{B}$.

It is known that a spectral triple (odd or even) with WHKAE property is regular and has simple dimension spectrum.

**Proposition 3.6** (Theorem 3.2, [5]). Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p^+$-summable odd spectral triple with WHKAE property of dimension $p$. Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular and it has finite simple dimension spectrum contained in $\{1, 2, \cdots, p\}$.

**Remark 3.7.** Above result holds for an even spectral triple. Similar proof will work.

We state some results which relates the co-efficients of asymptotic expansion of the functions $t^p \text{Trace}(b^e - t |D|)$ and $t^p \text{Trace}(b^e - t^2 |D|^2)$ with residues of the zeta functions $\text{Trace}(b^e |D|^{-z})$ and $\text{Trace}(b^e |D|^{-2z})$.

**Proposition 3.8.** If $t^p \text{Trace}(b^e - t |D|) \sim \sum_{r=0}^{\infty} b_r t^r$, then

1. $\text{Res}_{z=k} \text{Trace}(b^e |D|^{-z}) = \frac{t^{p-k}}{\Gamma(k)}$, for $k \in \{1, 2, \cdots, p\}$.
2. $\text{Res}_{z=k} \text{Trace}(b^e |D|^{-2z}) = \frac{b_{p-2k}}{2^{2k} \Gamma(k)}$, for $k \in \{1/2, 1/2, \cdots, p/2\}$.
3. $\text{Res}_{z=0} z^{-1} \text{Trace}(b^e |D|^{-z}) = \text{Trace}(b^e |D|^{-z})_{z=0} = b_p$.
4. $\text{Res}_{z=0} z^{-1} \text{Trace}(b^e |D|^{-2z}) = \text{Trace}(b^e |D|^{-2z})_{z=0} = 2b_p$.

*Proof:* It follows easily from remark (3.3) [5].

**Proposition 3.9.** If $t^p \text{Trace}(b^e - t |D|) \sim \sum_{r=0}^{\infty} b_r t^r$ and $t^p \text{Trace}(b^e - t^2 |D|^2) \sim \sum_{r=0}^{\infty} b'_r t^r$, then

$$b_r = \frac{1}{\sqrt{\pi}} 2^{p-r} \Gamma(\frac{p-r+1}{2}) b'_{r} \quad \text{for} \quad r \in \{0, 1, \cdots, p\}.$$

*Proof:* For the proof, see proposition 3.5 [5].

The following proposition shows that WHKAE property is preserved under double suspension which ensures that the quantum double suspension spectral triple is regular and has simple dimension spectrum. We will consider odd spectral triple first.

**Proposition 3.10** (Theorem 4.5, [5]). Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd spectral triple with weak heat kernel expansion property of dimension $p$. Assume that the algebra $\mathcal{A}$ is unital and the representation on $\mathcal{H}$ is unital. Then the spectral triple $(\Sigma^2_{\text{alg}}(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the weak heat kernel expansion property with dimension $p + 1$. 

Proof: We will give sketch of the proof. For more detail, see proposition 4.5 [4].

Let $B$ be a $*$ subalgebra of $L(H)$ for which (1) - (4) of definition 3.4 holds. Define

$$\Sigma^2 B := \text{span} \{ b \otimes k, 1 \otimes S^n, F \otimes S^n : b \in B, k \in S(\ell^2(\mathbb{N})), n \in \mathbb{Z} \}$$

where $S(\ell^2(\mathbb{N})) := \{ (a_{mn}) : \sum_{m,n} (1 + m + n)^p |a_{mn}| < \infty \}$.

Then $\Sigma^2 B$ be a $*$ subalgebra of $L(H \otimes \ell^2(\mathbb{N}))$ for which (1) - (4) of definition 3.4 satisfied for the spectral triple $(\Sigma^2_{alg}(A), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$, taking $\Sigma^2 B$ as a bigger algebra. □

For $b \in B$ and $m \in \mathbb{N}$, define

$$\zeta^{(m)}_D(b) = \text{Res}_{z=m/2} \text{Trace} (b |D|^{-2z}).$$

Clearly $\zeta^{(m)}_D$ is a linear functional on $B$. The linear functionals $\psi^k_x$ and $\phi_n$ defined in the previous section can be expressed in terms of the linear functionals $\{ \zeta^{(m)}_D, m \in \mathbb{N} \}$. Hence local index formula can also be written in these functionals.

Therefore to compute local index formula, it is enough to compute $\zeta^{(m)}_D$ on a essential subspace of $B$.

**Definition 3.11.** A subspace $B_{ess}$ of $B$ is called an essential subspace if there exists another subspace $B_0$ with the following properties:

1. $B = B_{ess} \oplus B_0$.
2. For all $b \in B_0$, Trace $(be^{-t|D|})$ has asymptotic power series expansion.

It follows from remark 3.8 that $\zeta^{(m)}_D(b) = 0$ for all $b \in B_0$ and $m \in \mathbb{N}$. Hence it is enough to evaluate the linear functionals $\zeta^{(m)}_D$ on an essential subspace of $B$.

**Proposition 3.12.** Let $B_{ess}$ be an essential subspace of $B$. Then

$$\Sigma^2 B_{ess} = \text{span} \{ b \otimes k, 1 \otimes 1, F \otimes 1 : b \in B_{ess}, k \in S(\ell^2(\mathbb{N})) \}.$$ is an essential subspace of $\Sigma^2 B$.

To prove the proposition, we need to show that for $k \in S(\ell^2(\mathbb{N}))$, Trace $(ke^{-tN})$ has asymptotic expansion near 0. Although exact expression for asymptotic expansion of Trace $(ke^{-tN})$ is not necessary here, we derive it for further use. For $k \in S(\ell^2(\mathbb{N}))$, define

$$\varphi_r(k) = \frac{(-1)^r}{r!} \sum_{i=0}^{\infty} k_{ii}^r.$$ where, $k_{ii} = \langle ke_i, e_i \rangle$ and $\{ e_i \}_{i=0}^{\infty}$ is the standard orthonormal basis of $\ell^2(\mathbb{N})$. Observe that the sum given above converges as $k \in S(\ell^2(\mathbb{N}))$ and hence, $\varphi_r(k)$ is well-defined linear functional on $S(\ell^2(\mathbb{N}))$. 

Proposition 3.13. For $k \in \mathcal{S}(\ell^2(\mathbb{N}))$,

$$\text{Trace} (k e^{-tN}) \sim \sum_{r=0}^{\infty} \varphi_r(k)t^r.$$  

Proof: It is easy to verify the expression for $k = |e_i\rangle\langle e_j|$. By linearity, it holds for all finite rank operators. For $k \in \mathcal{S}(\ell^2(\mathbb{N}))$, the finite rank operators $k_n := \sum_{i=0}^{n} k_{ij} |e_i\rangle\langle e_j|$ converges to $k$ as $n \to \infty$. Thus $\lim_{n \to \infty} \phi_r(k_n) = \phi_r(k)$. \qed

Proof: (of proposition 3.12) Let $B_0$ be a complementary subspace of $B_{ess}$ such that $\text{Trace} (be^{-tD}) = 0$ for all $b \in B_0$. Define $\Sigma^2B_0 = \text{span} \{b \otimes k, 1 \otimes S^n, F \otimes S^n \mid b \in B_0, k \in \mathcal{S}(\ell^2(\mathbb{N})), n \in \mathbb{Z} - \{0\}\}$

Clearly $\Sigma^2B = \Sigma^2B_{ess} \oplus \Sigma^2B_0$. For $n \in \mathbb{Z} - \{0\}$, $\text{Trace} ((1 \otimes S^n)e^{-t(D_0)}) = 0$ and $\text{Trace} ((F \otimes S^n)e^{-t(D_0)}) = 0$. Now for $b \in B_0$, it follows from remark 3.12 that $\text{Trace} (be^{-tD})$ has an asymptotic expansion near 0. Also, from proposition 3.13 it follows that $\text{Trace} (ke^{-t|N|})$ has an asymptotic expansion near 0. Therefore, $\text{Trace} ((b \otimes k)e^{-t(D_0)}) = \text{Trace} (be^{-tD}) \text{Trace} (ke^{-t|N|})$ has asymptotic expansion near 0. This completes the proof. \qed

Expansion of the linear functionals $\zeta_D^{(m)}$ and $\Sigma^2 \phi_x^{(m)}$ involves many residues of the form $\zeta_D^{(m)}$. So, removing unnecessary terms using structure of the algebra $B$ will be very crucial for avoiding difficult calculations. Along this direction, let $I_0 \subset I_1 \subset \cdots \subset I_p = B$ be ideals in $B$ such that

1. $a \in I_\ell \Rightarrow da \in I_\ell$ and $\delta(a) \in I_\ell$,
2. for every $a \in I_\ell$, $t^\ell \text{Trace} (ae^{-tD})$ has asymptotic expansion near 0.

Set $J_\ell = I_\ell \otimes \mathcal{S}(\ell^2(\mathbb{N}))$, for $\ell \leq p$ and $J_{p+1} = \Sigma^2B$. Then $J_0 \subset J_1 \subset \cdots$, $J_p \subset J_{p+1}$ are ideals in $\Sigma^2B$ with the following property.

1. $a \in J_\ell \Rightarrow da \in J_\ell$ and $\delta_0(a) \in J_\ell$,
2. for every $a \in J_\ell$, $t^\ell \text{Trace} (ae^{-tD})$ has asymptotic expansion near 0.

Note that for $b \in I_\ell$, $\text{Res}_{z = \frac{k}{\ell}} \text{Trace} (b|D|^{-2z}) = 0$ and for $b \in J_\ell$, $\text{Res}_{z = \frac{k}{\ell}} \text{Trace} (b|D_0|^{-2z}) = 0$ for $k > \ell$.

Proposition 3.14. For $b \in I_m, k \in \mathcal{S}(\ell^2(\mathbb{N}))$ and $s \in \{1, 2, \cdots, m\}$, one has

$$\zeta_D^{(s)}(b \otimes k) = \frac{1}{\Gamma(s)} \sum_{y=0}^{m-s} \Gamma(s+y) \zeta_D^{(s+y)}(b) \varphi_y(k).$$
Proof: Let $t^m \text{Trace} \left( be^{-t|D|} \right) \sim \sum_{r=0}^{\infty} b_r t^r$. Then

$$t^m \text{Trace} \left( (b \otimes k) e^{-t|D_0|} \right) = t^m \text{Trace} \left( be^{-t|D|} \right) \text{Trace} \left( ke^{-tN} \right).$$

$$\sim \sum_{r=0}^{\infty} b_r t^r \sum_{r=0}^{\infty} \varphi_r(k) t^r.$$

$$\sim \sum_{r=0}^{\infty} c_r t^r.$$

where $c_r = \sum_{y=0}^{r} \varphi_y(k) b_{r-y}$. Hence for $s \in \{1, 2, \ldots, m\}$, we get

$$\zeta_D^{(s)}(b \otimes k) = \text{Res}_{z=\frac{2}{s}} \text{Trace} \left( (b \otimes k) |D_0|^{-2z} \right).$$

$$= \frac{1}{2\Gamma(r)} c_{m-s}.$$

$$= \frac{1}{2\Gamma(s)} \sum_{y=0}^{m-s} \varphi_y(k) b_{m-(s+y)}.$$

$$= \frac{1}{\Gamma(s)} \sum_{y=0}^{m-s} \Gamma(s+y) \zeta_D^{(s+y)}(b) \varphi_y(k).$$

Observe that number of terms in the expression of $\zeta_D^{(s)}(b \otimes k)$ is depending on $m$.

Let

$$(3.3)$$

$$t^p \text{Trace} \left( e^{-t|D|} \right) \sim \sum_{r=0}^{\infty} u_r t^r, \quad t^p \text{Trace} \left( Fe^{-t|D|} \right) \sim \sum_{r=0}^{\infty} v_r t^r, \quad t \text{Trace} \left( e^{-tN} \right) \sim \sum_{r=0}^{\infty} n_r t^r.$$

Proposition 3.15. For $s \in \{2, 3, \ldots, p+1\}$, one has

$$\zeta_D^{(s)}(1) = \frac{1}{\Gamma(s)} \sum_{y=0}^{p+1-s} \Gamma(s+y) \zeta_D^{(s+y)}(1)n_y, \quad \zeta_D^{(s)}(F \otimes 1) = \frac{1}{\Gamma(s)} \sum_{y=0}^{p+1-s} \Gamma(s+y) \zeta_D^{(s+y)}(F)n_y.$$

Proof: We have

$$t^{p+1} \text{Trace} \left( e^{-t|D_0|} \right) = t^p \text{Trace} \left( e^{-t|D|} \right) \text{Trace} \left( e^{-tN} \right).$$

$$\sim \sum_{r=0}^{\infty} u_r t^r \sum_{r=0}^{\infty} n_r t^r.$$

$$\sim \sum_{r=0}^{\infty} u_r t^r.$$
where $\tilde{u}_r = \sum_{y=0}^{r} n_y b_{r-y}$. Hence for $s \in \{2, 3, \cdots, p + 1\}$, we have
\[
\zeta^{(s)}_{D_0}(1) = \text{Res}_{z=\frac{s}{2}} \text{Trace} (|D_0|^{-2z}).
\]
\[
= \frac{1}{2\Gamma(s)} \tilde{u}_{p+1-s}.
\]
\[
= \frac{1}{2\Gamma(s)} \sum_{y=0}^{p+1-s} n_y u_{p+1-(s+y)}.
\]
\[
= \frac{1}{\Gamma(s)} \sum_{y=0}^{p+1-s} \Gamma(s+y) \zeta^{(s+y)}_{D}(1) n_y.
\]
Similar calculation will prove that
\[
\zeta^{(s)}_{D_0}(F \otimes 1) = \frac{1}{\Gamma(s)} \sum_{y=0}^{p+1-s} \Gamma(s+y) \zeta^{(s+y)}_{D}(F) n_y.
\]

**Proposition 3.16.** Let $u_p, v_p$ and $n_i$’s be given by equation (3.3). Then one has
\[
\zeta^{(1)}_{D_0}(1) = n_0 u_p + \sum_{y=1}^{p} \Gamma(y) \zeta^{(y)}_{D}(1)n_y, \quad \zeta^{(1)}_{D_0}(F \otimes 1) = n_0 v_p + \sum_{y=1}^{p} \Gamma(y) \zeta^{(y)}_{D}(F)n_y.
\]

**Proof:** Let $\tilde{u}_{p+1}$ is as in the previous proposition. Then
\[
\zeta^{(1)}_{D_0}(1) = \text{Res}_{z=\frac{s}{2}} \text{Trace} (|D_0|^{-2z}).
\]
\[
= \frac{1}{2\Gamma(1)} \tilde{u}_{p+1}.
\]
\[
= \frac{1}{2} \sum_{y=0}^{p} n_y u_{p+1-y}.
\]
\[
= n_0 u_p + \sum_{y=1}^{p} \Gamma(y) \zeta^{(y)}_{D}(1)n_y.
\]
Similar calculations will prove the other part of the assertion. 

The above calculations shows that the linear functionals $\left\{ \zeta^{(k)}_{D_0} \right\}_{k=1}^{p+1}$ on $\Sigma^2 \mathcal{B}$ can be computed by the linear functionals $\left\{ \zeta^{(k)}_{D} \right\}_{k=1}^{p}$ on $\mathcal{B}$ and two values $\zeta^{(1)}_{D_0}(1)$ and $\zeta^{(1)}_{D_0}(F \otimes 1)$ (or $u_p$ and $v_p$). We will now put these data altogether and derive exact expression of local index formula for the spectral triple $(\Sigma^2 \mathcal{B}, \mathcal{H} \otimes \ell^2(\mathbb{N}), D_0)$. For $x \in \mathbb{N}^n$ and $\ell \in \mathbb{N}$, define
\[
\sum^2 \psi^\ell_{\delta_0}(\tilde{b}_0, \tilde{b}_1, \cdots, \tilde{b}_n) = \text{Res}_{z=-(n+|x|+\ell)/2} \text{Trace} \left( \tilde{b}_0 \delta^x_{D_0}(\tilde{b}_1) \cdots \delta^x_{D_0}(\tilde{b}_n) |D_0|^{-2z} \right)
\]
where $\tilde{b}_i \in \Sigma^2 \mathcal{B}$. To evaluate this, it is enough to take elements of the type $a \otimes k$ and $1 \otimes S^n$ which we commonly write in the form $a \otimes c$. Let $a^{(1)} = da, a^{(2)} =$
Lemma 3.17. Let \( x \in \mathbb{N}^n \). If for some \( i \in \{0, 1, \cdots, n\} \) and for some \( m \in \{1, 2, \cdots, p\} \), \( a_i \otimes c_i \in J_m \) then

\[
\Sigma^2 \psi_2^0(a_0 \otimes c_0, d_0(a_1 \otimes c_1), \cdots, d_0(a_n \otimes c_n)) = \sum_{1 \leq j_1 \leq 3 \atop 1 \leq j_n \leq 3} \sum_{1 \leq j_n \leq 3 \atop 1 \leq j_n \leq 3} \prod_{i=1}^n \frac{(n+|x|+k)}{\Gamma(n+|x|)} \psi_r^{(k+|x|+|r|)}(a_0, a_1^{(j_1)}, \cdots, a_n^{(j_n)}) \varphi_j(c_0 \prod_{i=1}^n \delta_{N}^{x_i-r_i}(c_i^{(j_i)})).
\]

Proof: It is easy to see that \( \delta_0^0(a \otimes c) = \sum_{r=0}^n \binom{n}{r} \delta^r(a) \otimes \delta_N^{n-r}(c) \). Using this, we get

\[
\Sigma^2 \psi_2^0(a_0 \otimes c_0, d_0(a_1 \otimes c_1), \cdots, d_0(a_n \otimes c_n)) = \text{Res}_{z=(n+|x|)/2} \text{Trace} \left( a_0 \otimes c_0 \prod_{i=0}^n \delta_0^0 d_0(a_i \otimes c_i) |D_0|^{-2z} \right) = \sum_{1 \leq j_1 \leq 3 \atop 1 \leq j_n \leq 3} \sum_{1 \leq j_n \leq 3 \atop 1 \leq j_n \leq 3} \prod_{i=1}^n \frac{(n+|x|+k)}{\Gamma(n+|x|)} \psi_r^{(k+|x|+|r|)}(a_0, a_1^{(j_1)}, \cdots, a_n^{(j_n)}) \varphi_j(c_0 \prod_{i=1}^n \delta_{N}^{x_i-r_i}(c_i^{(j_i)})).
\]

\[
\boxempty
\]

Proposition 3.18. Let \( x \in \mathbb{N}^n \). For \( m_0, m_1, \cdots, m_n \in \mathbb{Z} \) such that \( \sum_{i=0}^n m_i = 0 \), one has

\[
\Sigma^2 \psi_2^0(1 \otimes S^{m_0}, d_0(1 \otimes S^{m_1}), \cdots, d_0(1 \otimes S^{m_n})) = \prod_{i=1}^n m_i^{x_i+1} \zeta_{D_0}^{(n+|x|)}(F \otimes 1).
\]
Proof: Observe that \( d_0(1 \otimes S^m) = -mF \otimes S^m \) and \( \delta^r_0 d_0(1 \otimes S^m) = -m^{r+1}F \otimes S^m \). Hence we have

\[
\Sigma^2 \psi^{0}(1 \otimes S^{m_0}, d_0(1 \otimes S^{m_1}), \ldots, d_0(1 \otimes S^{m_n}))
\]

\[
= \text{Res}_{z=(n+|x|)/2} \text{Trace} \left( 1 \otimes S^{m_0} \prod_{i=0}^{n} \delta^r_0 d_0(1 \otimes S^{m_i}) |D_0|^{-2z} \right)
\]

\[
= (-1)^n \prod_{i=1}^{n} m^{x_i+1} \text{Res}_{z=(n+|x|)/2} \text{Trace} \left( (F^n \otimes S \sum_{i=0}^{n} m_i) |D_0|^{-2z} \right)
\]

\[
= - \prod_{i=1}^{n} m^{x_i+1} \text{Res}_{z=(n+|x|)/2} \text{Trace} \left( (F \otimes 1) |D_0|^{-2z} \right)
\]

Note that since we are taking odd spectral triple, \( n \) is odd in this case.

It is easy to see that \( \sum_{i=0}^{n} m_i \neq 0 \), then \( \Sigma^2 \psi^{0}(1 \otimes S^{m_0}, d_0(1 \otimes S^{m_1}), \ldots, d_0(1 \otimes S^{m_n})) = 0 \). Putting these results in equation (24), we get the local index formula for the spectral triple \((\Sigma^2 \mathcal{A}, \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2 D)\).

Now we will consider even spectral triple \((\mathcal{A}, \mathcal{H}, D, \gamma)\) and its quantum double suspension \((\Sigma^2 \mathcal{A}, \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2 D, \gamma \otimes 1)\). In this case, we define

\[
\Sigma^2 \mathcal{B} := \text{span} \{ b \otimes k, 1, F \otimes S^n, \gamma \otimes S^n, \gamma F \otimes S^n : b \in \mathcal{B}, k \in \mathcal{S}(\ell^2(\mathbb{N})), n \in \mathbb{Z} \}.
\]

where \( \mathcal{B} \) is a \( * \)-subalgebra of \( \mathcal{L}(\mathcal{H}) \) for which all conditions for WHKAE property of spectral triple \((\mathcal{A}, \mathcal{H}, D, \gamma)\) holds. Since, \( F \otimes S^n \) and \( \gamma F \otimes S^n \) are odd operators, \( \zeta^{(m)}_{D_0} (F \otimes S^n) = 0 \) and \( \zeta^{(m)}_{D_0} (\gamma F \otimes S^n) = 0 \) for all \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \). Using this, one can easily check that if \( \mathcal{B}_{ess} \) is an essential subspace of \( \mathcal{B} \) then \( \Sigma^2 \mathcal{B}_{ess} \) defined below will be an essential subspace of \( \Sigma^2 \mathcal{B} \).

\[
\Sigma^2 \mathcal{B}_{ess} = \text{span} \{ b \otimes k, 1, \gamma \otimes 1 : b \in \mathcal{B}_{ess}, k \in \mathcal{S}(\ell^2(\mathbb{N})) \}.
\]

Let

\[
t^p \text{Trace} (\gamma e^{-t|D|}) \sim \sum_{r=0}^{\infty} w_r t^r.
\]

Then lemma 3.14 and 3.16 will follow with \( F \) and \( v_r \) replaced by \( \gamma \) and \( w_r \) respectively. For local index formula, we state the following results. Since proof of these results are similar to that in odd case, we omit it.
Lemma 3.19. Let \( x \in \mathbb{N}^n \) where \( n \) is positive even integer. If for some \( i \in \{0,1,\ldots,n\} \) and for some \( m \in \{1,2,\ldots,p\} \), \( a_i \otimes c_i \in J_m \) then

\[
\Sigma^2 \psi^0_x((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), \ldots, d_0(a_n \otimes c_n))
\]

\[
= \sum_{1 \leq j_1 \leq 3} \sum_{0 \leq r_1 \leq x_1} \sum_{1 \leq j_n \leq 3} \sum_{0 \leq r_n \leq x_n} \prod_{k=0}^{m-|x|} \frac{n!}{r_n!} \prod_{i=1}^{n} \Gamma(n + |x| + k) \Gamma(n + |x|)
\]

\[
\psi(x^{|x|} \gamma)^{0,1}(1, 0, a_1^{(j_1)}, \ldots, a_n^{(j_n)}) \varphi(c_0 \prod_{i=1}^{n} \delta_{n_{r_i}}(c_{(j_i)})).
\]

Proposition 3.20. Let \( x \in \mathbb{N}^n \) where \( n \) is positive even integer. For \( m_0, m_1, \ldots, m_n \in \mathbb{Z} \) such that \( \sum_{i=0}^{n} m_i = 0 \),

\[
\Sigma^2 \psi^0_x((\gamma \otimes 1)(1 \otimes S^{m_0}), d_0(1 \otimes S^{m_1}), \ldots, d_0(1 \otimes S^{m_n})) = - \prod_{i=1}^{n} m_i \gamma_{s_{D_0}}(1)(1 \otimes 1).
\]

For \( \sum_{i=0}^{n} m_i \neq 0 \), it is easy to check that \( \Sigma^2 \psi^0_x((\gamma \otimes 1)(1 \otimes S^{m_0}), d_0(1 \otimes S^{m_1}), \ldots, d_0(1 \otimes S^{m_n})) = 0 \). Putting these results in equation (2.2), we can compute the functionals \( \Sigma^2 \phi_0 \) for all \( n > 0 \) and even. Moreover, we need to calculate \( \Sigma^2 \phi(\hat{b}) := \text{Res}_{z=0} z^{-1} \text{Trace}(\hat{b} | D_0|^{-2z}) \) where \( \hat{b} \in \Sigma^2 \mathcal{E} \).

Proposition 3.21. For \( b \in I_m \) and \( s \in \{1,2,\ldots,m\} \), one has

\[
\Sigma^2 \phi_0(b \otimes k) = \phi_0(b) \varphi_0(k) + \frac{1}{\Gamma(s)} \sum_{y=1}^{m} \Gamma(y) \zeta(y, b) \varphi_y(k).
\]

Proposition 3.22. Let \( w_p \) and \( w_{p+1} \) be given by equation (3.3). Assume that \( w_p, u_{p+1} \) and \( n_i \)’s are given by (3.3). Then one has

\[
\Sigma^2 \phi_0(1) = 2n_0 u_{p+1} + 2n_1 u_p + \sum_{y=1}^{p} \gamma y \zeta(y, 1)n_y.
\]

\[
\Sigma^2 \phi_0(\gamma \otimes 1) = 2n_0 u_{p+1} + 2n_1 u_p + \sum_{y=1}^{p} \gamma y \zeta(y, \gamma)n_y.
\]

Putting these results in equation (2.2), we get the local index formula for the even spectral triple \( (\Sigma_{alg}^2(A), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D), \gamma \otimes 1) \).

Remark 3.23. We computed linear functionals for finite linear combination of elements from the set \( \{b \otimes \mathcal{S}(\ell^2(\mathbb{N})), 1 \otimes S^n, F \otimes S^n : b \in B, n \in \mathbb{Z}\} \). We can extend these linear functionals to infinite linear combination of these elements in which co-efficients are rapidly decreasing. Hence we can apply these results to the spectral triple with topological weak heat kernel expansion property defined as in [5].
4. SOME EXAMPLES

4.1. **Local index formula for** \( \Sigma^2C(S^2) \). Consider the 2-dimensional sphere \( S^2 \) with usual orientation. The usual spherical coordinates on \( S^2 \) are:

\[
p = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in S^2.
\]

The poles are \( N = (0, 0, 1) \) and \( S = (0, 0, -1) \). Let \( U_N = S^2 - \{N\} \) and \( U_S = S^2 - \{S\} \) be the two charts of \( S^2 \). Consider the stereographic projections \( p \mapsto z \) from \( U_N \) to \( \mathbb{C} \) and \( p \mapsto \xi \) from \( U_S \) to \( \mathbb{C} \) given by

\[
z := e^{-i\phi} \cot \frac{\theta}{2}, \quad \xi := e^{i\phi} \tan \frac{\theta}{2},
\]

such that \( z = \frac{1}{\xi} \) on \( U_N \cap U_S \). Let \( L^+ \) be the tautological line bundle coming from \( S^2 \cong \mathbb{C}P^1 \). Let \( L^- \) be the dual line bundle of \( L^+ \). One can show that \( L^+ \) and \( L^- \) are two nonisomorphic nontrivial complex line bundles. Define \( L = L^+ \oplus L^- \). Denote by \( \Gamma(S^2, L), \Gamma(S^2, L^+) \) and \( \Gamma(S^2, L^-) \) the sets of smooth sections of the bundles \( L, L^+ \) and \( L^- \) respectively. A spinor on \( S^2 \) (smooth section of the bundle \( L \)) is given by two pairs of smooth functions

\[
\left( \begin{array}{c}
\psi^+_N(z, \bar{z}) \\
\bar{\psi}^-_N(z, \bar{z})
\end{array} \right) \text{ on } U_N, \quad \left( \begin{array}{c}
\psi^+_S(\xi, \bar{\xi}) \\
\bar{\psi}^-_S(\xi, \bar{\xi})
\end{array} \right) \text{ on } U_S
\]

satisfying the following properties:

1. \( \psi^+_N(z, \bar{z}) = (\frac{z}{|z|^2}) \bar{\psi}^+_S(z^{-1}, \bar{z}^{-1}) \) for \( z \neq 0 \).
2. \( \psi^-_N(z, \bar{z}) = (\frac{1}{z}) \bar{\psi}^-_S(z^{-1}, \bar{z}^{-1}) \) for \( z \neq 0 \).
3. \( \psi^+_N \) and \( \psi^-_N \) are regular at \( z = 0 \).
4. \( \psi^+_S \) and \( \psi^-_S \) are regular at \( \xi = 0 \).

Note that \( \psi^+_N \) on \( U_N \) and \( \psi^+_S \) on \( U_S \) gives a smooth section of the bundle \( L^+ \) and \( \bar{\psi}^-_N \) on \( U_N \) and \( \bar{\psi}^-_S \) on \( U_S \) gives a smooth section of the bundle \( L^- \). One can show that \( \Gamma(S^2, L) = \Gamma(S^2, L^+) \oplus \Gamma(S^2, L^-) \). The scalar product on \( \Gamma(S^2, L^+) \) is given by

\[
\langle \phi, \psi \rangle = \int_{S^2} \langle \phi(p), \psi(p) \rangle_p \nu_g,
\]

where \( \langle ., . \rangle_p \) is the standard scalar product on \( \mathbb{C}^2 \) and \( \nu_g \) is the Riemannian volume form on \( S^2 \). On completion in the norm \( \|\phi\| = \sqrt{\langle \phi, \phi \rangle} \), we get the Hilbert space \( \mathcal{H}^+ := L^2(S^2, L^+) \), the \( L^2 \)-spinor of the bundle \( L^+ \). In a similar way one can construct the Hilbert space \( \mathcal{H}^- := L^2(S^2, L^-) \), the \( L^2 \)-spinor of the bundle \( L^- \) and \( \mathcal{H} := L^2(S^2, L) \), the \( L^2 \)-spinor of the bundle \( L \). It is easy to see that \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \). Let \( \gamma \) be the grading on \( \mathcal{H} \) associated with this decomposition i.e. \( \gamma(h) = h \) if \( h \in \mathcal{H}^+ \) and \( \gamma(h) = -h \) if \( h \in \mathcal{H}^- \). We represent \( C^\infty(S^2) \) on
\( \mathcal{H} \) by multiplication operators. Let \( D \) be the Dirac operator associated with the Levi-Civita connection. It is given over \( U_N \) by

\[
-i \begin{pmatrix}
0, & (1 + z \bar{z}) \frac{\partial}{\partial z} - \frac{1}{2} z, \\
(1 + z \bar{z}) \frac{\partial}{\partial \bar{z}} - \frac{1}{2} \bar{z}, & 0
\end{pmatrix}.
\]

A similar expression is valid over \( U_S \) by replacing \( z \) and \( \bar{z} \) by \( \xi \) and \( \bar{\xi} \) respectively and by changing overall \((-i)\) factor to \( i \). Then \((C^\infty(S^2), \mathcal{H}, D, \gamma)\) is the classical even spectral triple of \( C(S^2) \). For a complete description, we refer the reader to Varilly ([19], page 98-102). Chakraborty and Sundar ([5], page 15) showed that this spectral triple has the WHKAE property with dimension 2 and hence it is regular with dimension spectrum contained in \( \{1, 2\} \). It follows from Proposition 3.10 that the quantum double suspension spectral triple \((\Sigma^2(C^\infty(S^2)), \mathcal{H} \otimes l^2(\mathbb{N}), D_0 := \Sigma^2 D, \gamma \otimes 1)\) also has the WHKAE property and its dimension spectrum is contained in \( \{1, 2, 3\} \). Our aim is to give an explicit description of the local index formula for this spectral triple. We will use symbol calculus as our main tool (see [14], [19]).

Since the dimension spectrum is contained in \( \{1, 2, 3\} \), it follows from equation (2.2) that \( \Sigma^2 \phi_{2n} = 0 \) for \( n > 1 \). So, we need to compute \( \Sigma^2 \phi_0 \) and \( \Sigma^2 \phi_2 \) only. We first find the asymptotic expansion of \( t^2 \text{Trace} (\gamma a e^{-t^2 D^2}) \) for all \( a \in C^\infty(S^2) \).

**Proposition 4.1.** For all \( a \in C^\infty(S^2) \), one has

\[
t^2 \text{Trace} (\gamma a e^{-t^2 D^2}) \sim 0.
\]

**Proof:** The symbol attached with the Dirac operator \( D \) over a local chart is

\[
p^D(x, \xi) = \begin{pmatrix}
0 & \xi \\
\xi & 0
\end{pmatrix} + \frac{i}{2} \begin{pmatrix}
0 & x_1 - ix_2 \\
x_1 + ix_2 & 0
\end{pmatrix}.
\]

Hence symbol for the pseudodifferential operator \( D^2 \) over the same chart will be given by

\[
p^{D^2}(x, \xi) = \begin{pmatrix} |\xi|^2 & 0 \\
0 & |\xi|^2
\end{pmatrix} + \frac{i}{2} \begin{pmatrix}
\xi_1 x_1 + \xi_2 x_2 & 0 \\
0 & \xi_1 x_1 + \xi_2 x_2
\end{pmatrix} + \begin{pmatrix} 1 - |x|^2 / 4 & 0 \\
0 & 1 - |x|^2 / 4
\end{pmatrix}.
\]

Let \( p_k(x, \xi) \) be homogeneous polynomials in variable \( \xi \) of degree \( k \) such that

\[
p^{D^2}(x, \xi) = \sum_{k=0}^{2} p_k(x, \xi).
\]

Then

\[
p_0(x, \xi) = \begin{pmatrix} 1 - |x|^2 / 4 & 0 \\
0 & 1 - |x|^2 / 4
\end{pmatrix}, \quad p_1(x, \xi) = \frac{i}{2} \begin{pmatrix}
\xi_1 x_1 + \xi_2 x_2 & 0 \\
0 & \xi_1 x_1 + \xi_2 x_2
\end{pmatrix},
\]

and the leading symbol is

\[
p_2(x, \xi) = \begin{pmatrix} |\xi|^2 & 0 \\
0 & |\xi|^2
\end{pmatrix}.
\]
Note that the leading symbol $p_2(x, \psi)$ is positive definite and is a scalar matrix. Let $K(t, x, y)$ be the kernel of $e^{-tD^2}$. Using Lemma 1.7.4 in [14], we get

$$K(t, x, x) \sim \sum_{n=0}^{\infty} t^{\frac{n-2}{2}} e_n(x) \quad \text{as} \quad t \to 0^+.$$ 

Exact expression for $e_n(x)$ is given in [14 page 54]. Since $p_k(x, \xi)$ is a scalar matrix for all $k \in \{0, 1, 2\}$, $e_n(x)$ will be a scalar matrix. It follows from (Lemma 1.7.7, [14]) that the kernel of $\gamma ae^{-tD^2}$ is $\gamma aK(t, x, y)$. Hence

$$t \text{Trace} (\gamma ae^{-tD^2}) = t \int_{S^2} \text{Trace} (\gamma a(x)K(t, x, x)) \nu_g$$

$$\sim t \sum_{n=0}^{\infty} t^{\frac{n-2}{2}} \int_{S^2} \text{Trace} (\gamma a(x)e_n(x)) \nu_g$$

$$\sim \sum_{n=0}^{\infty} t^{\frac{n}{2}} \int_{S^2} \text{Trace} (\gamma a(x)e_n(x)) \nu_g.$$ 

Therefore

$$t^2 \text{Trace} (\gamma ae^{-t^2D^2}) \sim \sum_{n=0}^{\infty} t^n \int_{S^2} \text{Trace} (\gamma a(x)e_n(x)) \nu_g \sim 0,$$ 

as $a(x)e_n(x)$ is a scalar matrix and $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$ \hfill $\square$

For a pseudo-differential operator $A$, let Wres$(A)$ denote the Wodzicki residue of $A$ (see [19], page 57). Let $\{a^{(1)} = [D, a], a^{(2)} = Fa, a^{(3)} = aF\}$.

**Proposition 4.2.** For $(j_1, j_2) \neq (1, 1)$, one has

$$\text{Wres}(\gamma a_1^{(j_1)}a_2^{(j_2)}|D|^{-2}) = 0.$$ 

**Proof:** We will show the result for $j_1 = 2, j_2 = 1$. Other cases will follow by similar calculations. The principal symbol attached to the $\Psi DO$ $D$ and $D^2$ are $\sigma^D(x, \xi) := \begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix}$ and $\sigma^{D^2}(x, \xi) := \begin{bmatrix} |\xi|^2 & 0 \\ 0 & |\xi|^2 \end{bmatrix}$ respectively. Then the principal symbol $\sigma^F(x, \xi)$ of the operator $F = D|D|^{-1}$ is $\begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix} \begin{bmatrix} |\xi|^{-1} & 0 \\ 0 & |\xi|^{-1} \end{bmatrix} = \begin{bmatrix} 0 & |\xi|^{-1} \\ |\xi|^{-1} & 0 \end{bmatrix}.$ Moreover, since $[D, a] = -ic(da)$ where $c$ denote the clifford action, the principal symbol for $[D, a]$ is $-i \begin{bmatrix} 0 & \frac{\partial a}{\partial x_1} - \frac{\partial a}{\partial x_2} \\ \frac{\partial a}{\partial x_1} - \frac{\partial a}{\partial x_2} & 0 \end{bmatrix}.$ Further, as a multiplication operator, $a_0$ is $\Psi DO$ of order 0 with principal symbol $\sigma^{a_0}(x, \xi) = \begin{bmatrix} a_0(x) & 0 \\ 0 & a_0(x) \end{bmatrix}.$ Hence the principal symbol attached to the $\Psi DO \gamma a_1^{(1)}a_2^{(2)}|D|^{-2}$ is given by

$$\sigma(x, \xi) = -i \begin{bmatrix} a_0(x)a_1(x)\left(\frac{\partial a_0}{\partial x_1} + \frac{\partial a_1}{\partial x_2}\right)|\xi|^{-3} \\ 0 \\ -a_0(x)a_1(x)\left(\frac{\partial a_2}{\partial x_1} - i\frac{\partial a_2}{\partial x_2}\right)|\xi|^{-3} \end{bmatrix}.$$
For $|\xi|=1$, $\text{Trace} (\sigma(x, \xi)) = -2a_0(x)a_1(x)(\xi_1 \frac{\partial a_2}{\partial x_1} - \xi_2 \frac{\partial a_2}{\partial x_2})$. Hence the Wodzicki residue density of the ΨDO $\gamma a_0 a_1^{(j_1)} a_2^{(j_2)} |D|^{-2}$ at the point $x$ is

$$\int_{|\xi|=1} \text{Trace} (\sigma(x, \xi)) = 0.$$ 

Hence

$$\text{Wres}(\gamma a_0 a_1^{(j_1)} a_2^{(j_2)} |D|^{-2}) = 0.$$ 

This proves the claim. \qed

**Proposition 4.3.** The Wodzicki residue of the operator $\gamma a_0 a_1 a_2 |D|^{-2}$ is given by

$$\text{Wres}(\gamma a_0 a_1 a_2 |D|^{-2}) = -4\pi i \int_{S^2} a_0 da_1 \wedge da_2.$$ 

**Proof:** The principal symbol attached to the ΨDO $\gamma a_0 a_1 a_2 |D|^{-2}$ is given by

$$\sigma(x, \xi) = -\left( a_0(x)(\frac{\partial a_2}{\partial x_1} - i \frac{\partial a_2}{\partial x_2}) \frac{\partial a_2}{\partial x_2} + i \frac{\partial a_2}{\partial x_1} \right) \frac{\partial a_2}{\partial x_2} \left( a_0(x)(\frac{\partial a_2}{\partial x_1} + i \frac{\partial a_2}{\partial x_2}) \frac{\partial a_2}{\partial x_1} - i \frac{\partial a_2}{\partial x_2} \right) \right).$$

For $|\xi|=1$, $\text{Trace} (\sigma(x, \xi)) = -2ia_0(\frac{\partial a_1}{\partial x_1} \frac{\partial a_2}{\partial x_2} - \frac{\partial a_1}{\partial x_2} \frac{\partial a_2}{\partial x_1}).$ Hence

$$\text{Wres}(\gamma a_0 a_1 a_2 |D|^{-2}) = \int_{S^2} \left( \int_{|\xi|=1} -2ia_0 \left( \frac{\partial a_1}{\partial x_1} \frac{\partial a_2}{\partial x_2} - \frac{\partial a_1}{\partial x_2} \frac{\partial a_2}{\partial x_1} \right) \sigma \right) dx_1 \wedge dx_2$$

$$= -4\pi i \int_{S^2} a_0 da_1 \wedge da_2.$$ 

This proves the assertion. \qed

Let $B$ be the algebra of pseudo-differential operators of order 0. Then $B$ is a $^*-$subalgebra of $L(H)$ for which all conditions for the WHKAE property of the spectral triple $(C^\infty(S^2), H, D, \gamma)$ holds (see section 4.3, [5]). Let $J_0 = I_1 = \{0\}$ and $I_2 = B$. Then $J_0 = J_1 = \{0\} \subset J_2 = B \otimes S(\ell^2(\mathbb{N})) \subset J_3 = \Sigma^2 B$ are ideals in $\Sigma^2 C(S^2)$ such that

1. $a \in J_\ell \implies d_0(a) \in J_\ell$ and $\delta_0(a) \in J_\ell$,
2. for every $a \in J_\ell$, $t^\ell \text{Trace} (ae^{-t|D_0|})$ has an asymptotic expansion near 0.

**Lemma 4.4.** For $\hat{a} \in \Sigma^2 C^\infty(S^2)$, one has

$$\Sigma^2 \delta_0((\gamma \otimes 1)\hat{a}) = 0.$$ 

**Proof:** Let $\hat{a} = a \otimes k$ where $a \in C^\infty(S^2)$ and $k \in S(\ell^2(\mathbb{N}))$. Using Proposition [4.1] and Proposition [3.9] we get

$$t^2 \text{Trace} ((\gamma \otimes 1)(a \otimes k)e^{-t|D_0|}) = t^2 \text{Trace} (\gamma ae^{-t|D|}) \text{Trace} (ke^{-tN}) \sim 0. \quad (4.5)$$
It follows from Proposition 3.8 that
\[
\Sigma^2 \phi_0(a \otimes k) = \text{Res}_{z=0}(z^{-1} \text{Trace} \left( (\gamma a \otimes k) |D_0|^{-2z} \right)) = 0.
\]
For \(\tilde{a} = 1 \otimes S^n\) where \(n \in \mathbb{Z}\), we have
\[
\text{Trace} \left( (\gamma \otimes 1)(1 \otimes S^n)e^{-t|D_0|} \right) = \text{Trace} \left( \gamma e^{-t|D|} \right) \text{Trace} \left( S^n e^{-tN} \right) = 0.
\]
This completes the proof. \(\square\)

**Lemma 4.5.** For \(m_1, m_2, m_3 \in \mathbb{N}\), one has
\[
\Sigma^2 \phi_2(1 \otimes S^{m_1}, 1 \otimes S^{m_2}, 1 \otimes S^{m_3}) = 0.
\]

**Proof:** Observe that \(d_0(1 \otimes S^n) = -m F \otimes S^n\) and \(\delta_0^2 d_0(1 \otimes S^n) = -m^{x+1} F \otimes S^n\). Hence we have
\[
\Sigma^2 \psi_x^{(0)}((\gamma \otimes 1)(1 \otimes S^{m_0}), d_0(1 \otimes S^{m_1}), d_0(1 \otimes S^{m_2}))
\]
\[
= \text{Res}_{z=(2+|x|)/2} \text{Trace} \left( (\gamma \otimes S^{m_0}) \delta_0^x d_0(1 \otimes S^{m_1}) \delta_0^{x_2} d_0(1 \otimes S^{m_2}) |D_0|^{-2z} \right)
\]
\[
= m_1^{x_1+1} m_2^{x_2+1} \text{Res}_{z=(2+|x|)/2} \text{Trace} \left( (\gamma \otimes S^{m_0}) |D_0|^{-2z} \right).
\]
Since \(\text{Trace} \left( (\gamma \otimes S^{m_0}) |D_0|^{-2z} \right) = \text{Trace} \left( \gamma e^{-t|D|} \right) \text{Trace} \left( S^{m_0} e^{-tN} \right) = 0\), by Proposition 3.8 we have \(\text{Res}_{z=(2+|x|)/2} \text{Trace} \left( (\gamma \otimes S^{m_0}) |D_0|^{-2z} \right) = 0\) which further implies that \(\Sigma^2 \psi_x^{(0)}((\gamma \otimes 1)(1 \otimes S^{m_0}), d_0(1 \otimes S^{m_1}), d_0(1 \otimes S^{m_2})) = 0\). Now by applying equation (2.2), we get
\[
\Sigma^2 \phi_2(1 \otimes S^{m_1}, 1 \otimes S^{m_2}, 1 \otimes S^{m_3}) = 0.
\]
\(\square\)

**Theorem 4.6.** If for some \(i \in \{0, 1, 2\}\), \(a_i \otimes c_i \in C^\infty(S^{2}) \otimes S(\ell^2(\mathbb{N}))\) then
\[
\Sigma^2 \phi_2((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) = -\frac{i^2}{\sqrt{2\pi}} \text{Trace}(c_0 c_1 c_2) \int_{S^2} a_0 da_1 \land da_2.
\]

**Proof:** Let \(c^{(1)} = c\), \(c^{(2)} = Nc\) and \(c^{(3)} = -cN\). Then
\[
d_0(a \otimes c) = a^{(1)} \otimes c^{(1)} + a^{(2)} \otimes c^{(2)} + a^{(3)} \otimes c^{(3)}.
\]
Using this and the fact that \(a_i \otimes c_i \in C^\infty(S^{2}) \otimes S(\ell^2(\mathbb{N}))\) for some \(i \in \{0, 1, 2\}\), we conclude that the operator \((\gamma \otimes 1)(a_0 \otimes c_0) \delta_0^x d_0(a_1 \otimes c_1) \delta_0^{x_2} d_0(a_2 \otimes c_2)\) is in the ideal \(J_2\). Therefore the function \(\ell^2 \text{Trace} \left( (\gamma \otimes 1)(a_0 \otimes c_0) \delta_0^x (a_1 \otimes c_1) \delta_0^{x_2} (a_2 \otimes c_2) e^{-t|D_0|} \right)\) has an asymptotic expansion property near 0 and hence it follows from equation (2.2) that for \(x \neq (0, 0)\),
\[
\Sigma^2 \psi_x^{(0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) = 0.
\]
For \( x = (0, 0) \), we have, by Proposition 4.2,
\[
\sum^2 \psi^{(0)}_{(0,0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2))
\]
\[
= \text{Res}_{z=1} \text{Trace} ((\gamma a_0 \otimes c_0)d_0(a_1 \otimes c_1)d_0(a_3 \otimes c_3) |D_0|^{-2z})
\]
\[
= \sum_{j_1=1}^{3} \sum_{j_2=1}^{3} \psi^{(0)}_{(0,0)}(\gamma a_0, a^{(j_1)}_1, a^{(j_2)}_2)\varphi_0(c_0c^{(j_1)}_1c^{(j_2)}_2)
\]
\[
= \psi^{(0)}_{(0,0)}(\gamma a_0, da_1, da_2) \text{Trace} (c_0c_1c_2).
\]

By Connes’ Trace Theorem, (see Theorem 1, [7]), we have
\[
\text{Res}_{z=1} \text{Trace} (A |D|^{-2z}) = 2\text{Tr}^+(A) = \frac{1}{4\pi^2} \text{Wres} A,
\]
where \( A \) is \( \Psi \)DO of order \(-2\) and \( \text{Tr}^+ \) denotes the Dixmier trace. Note that \( \gamma a_0a^{(j_1)}_1a^{(j_2)}_2 \) is a \( \Psi \)DO of order \( 0 \) and hence \( \gamma a_0a^{(j_1)}_1a^{(j_2)}_2 |D|^{-2} \) is \( \Psi \)DO of order \(-2\). By Proposition 4.3 we get
\[
\sum^2 \psi^{(0)}_{(0,0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2))
\]
\[
= 2\text{Tr}^+(\gamma a_0da_1da_2 |D|^{-2}) \text{Trace} (c_0c_1c_2)
\]
\[
= \frac{1}{4\pi^2} \text{Wres}(\gamma a_0da_1da_2 |D|^{-2}) \text{Trace} (c_0c_1c_2)
\]
\[
= -\frac{i}{\pi} \text{Trace} (c_0c_1c_2) \int_{S^2} a_0da_1 \wedge da_2.
\]

By equation (2.2), we have
\[
\sum^2 \phi_2((a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2))
\]
\[
= \sum_{x \in \mathbb{N}^2} B^2 \Sigma^2 \psi^{(0)}_{x}(((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2))
\]
\[
= B^2 \Sigma^2 \psi^{(0)}_{(0,0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2))
\]
\[
= -\frac{i}{\sqrt{2\pi}} \text{Trace} (c_0c_1c_2) \int_{S^2} a_0da_1 \wedge da_2.
\]

This completes the proof. \( \square \)

### 4.2. Local index formula for quantum double suspension of non-commutative torus.

Let us recall the definition of non-commutative torus. Throughout we assume that \( \theta \) is irrational.

**Definition 4.7.** The \( C^* \)-algebra \( A_\theta \) is defined as the universal \( C^* \)-algebra generated by two unitaries \( u \) and \( v \) such that \( uv = e^{2\pi i\theta} vu \).

Define the operators \( U \) and \( V \) on \( \mathcal{H} := l^2(\mathbb{Z}^2) \) as follows:
\[
Ue_{m,n} := e_{m+1,n}
\]
\[
Ve_{m,n} := e^{-2\pi in\theta}e_{m,n+1}
\]
where \( \{e_{m,n}\} \) denotes the standard orthonormal basis of \( \ell^2(\mathbb{Z}^2) \). It is well known that \( u \mapsto U \) and \( v \mapsto V \) gives a faithful representation of the \( C^* \)-algebra \( A_\theta \).

For a function \( f(m,n) \) on \( \mathbb{Z}^2 \), define the operator \( T_f \) as \( T_f e_{m,n} := f(m,n)e_{m,n} \).

The group \( \mathbb{Z}^2 \) acts on the algebra of functions as follows: For \( x = (a,b) \in \mathbb{Z}^2 \) and \( f(m,n) \), define \( x.f := f(m-a,n-b) \). We denote \((0,1)\) by \( e_1 \) and \((0,1)\) by \( e_2 \). Let \( \mathcal{A}_\theta \) be the \( * \)-algebra generated by \( u \) and \( v \). We consider direct sum representation of \( \mathcal{A}_\theta \) on \( \mathcal{H} \oplus \mathcal{H} \). Define \( \mathcal{D} := \left[ \begin{array}{cc} 0 & T_{m-in} \\ T_{m+in} & 0 \end{array} \right] \) and the grading operator \( \gamma := \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \).

Now note the following commutation relations:

\[
UT_f := T_{e_1}fU \\
VT_f := T_{e_2}fV.
\]

It follows that \( [T_{m+in}, U^\alpha V^\beta] = (\alpha + i\beta)U^\alpha V^\beta \) and \( [T_{m-in}, U^\alpha V^\beta] = (\alpha - i\beta)U^\alpha V^\beta \).

Let us define \( \mathcal{D}_k := \text{span}\{T_pU^\alpha V^\beta : \alpha, \beta \in \mathbb{Z}, \text{\ } p \text{ is a polynomial of degree} \lesssim k\} \) and let \( \mathcal{D} := \cup \mathcal{D}_k \). Denote by \( \Delta \) the unbounded operator \( T_{m^2+n^2} \).

It follows from proposition 4.13 [3] and proposition 4.14 [3] that \( (\mathcal{D}, \Delta) \) is a differential pair of analytic dimension 2 with heat kernel expansion property.

Now, by proposition 4.15 [3], the spectral triple \( (\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D, \gamma) \) has WHKAE property with dimension 2. Consider its quantum double suspension spectral triple \( (\Sigma^2(\mathcal{A}_\theta), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2 D, \gamma \otimes 1) \). It follows from proposition 5.10 that its dimension spectrum is contained in \( \{1, 2, 3\} \) and hence \( \Sigma^2 \phi_{2n} = 0 \) for \( n > 1 \). Therefore to compute LIF for this spectral triple, we need to compute \( \Sigma^2 \phi_0 \) and \( \Sigma^2 \phi_2 \).

We will first find asymptotic expansion of \( t^2 \text{Trace} (e^{-t^2 \Delta}) \).

\[
t^2 \text{Trace} (e^{-t^2 \Delta}) = 2t^2 \sum_{m,n \in \mathbb{Z}} e^{-t^2(m^2+n^2)}.
\]

\[
= 2(t \sum_{m \in \mathbb{Z}} e^{-t^2 m^2})(t \sum_{n \in \mathbb{Z}} e^{-t^2 n^2}).
\]

\[
= 2\pi \sqrt{\int_{-\infty}^{\infty} e^{-ax^2}dx} = \sqrt{\frac{\pi}{a}}. \tag{4.6}
\]

**Proposition 4.8.** For \( a_0, a_1, a_2 \in \mathcal{A}_\theta \), one has

1. \( \text{Res}_{z=1}(\text{Trace}(\gamma a_0d(a_1)F a_2 | D|^{-2z})) = 0. \)
2. \( \text{Res}_{z=1}(\text{Trace}(\gamma a_0d(a_1)a_2 F | D|^{-2z})) = 0. \)
3. \( \text{Res}_{z=1}(\text{Trace}(\gamma a_0F a_1d(a_2) | D|^{-2z}) = 0. \)
4. \( \text{Res}_{z=1}(\text{Trace}(\gamma a_0a_1 F d(a_2) | D|^{-2z})) = 0. \)

**Proof:** We will show the claim for the first case. Other cases will follow by similar calculation.

Let \( a_i = u^\alpha v^\beta_i \) for \( i \in \{0, 1, 2\} \). Then

\[
[D, a_1] = \begin{pmatrix}
0 & [T_{m-in}, U^\alpha V^\beta] \\
[T_{m+in}, U^\alpha] & 0
\end{pmatrix}
\]

\[
\text{Res}_{z=1}(\text{Trace}(\gamma a_0d(a_1)F a_2 | D|^{-2z})) = 0.
\]
where \( \beta = \gamma a \) has simple poles at \( \text{Trace} \sum_{j=0}^{\infty} \binom{1/2}{j} D \nabla^j(a) |D|^{-2j} \).

Hence

\[
\text{Res}_{z=0} \text{Trace} (\gamma a_0 da_1 F_{a_2} |D|^{-2-2z}) = \text{Res}_{z=0} \text{Trace} \left( \sum_{j=0}^{\infty} \binom{1/2}{j} \gamma a_0 da_1 D \nabla^j(a_2) |D|^{-3-2j-2z} \right).
\]

Since \( T_{m-in} \Delta^j(a_2) \) and \( T_{m+in} \Delta^j(a_2) \) are elements of \( D_{j+1}, \) the asymptotic expansion near 0. This implies that \( \text{Trace} (\gamma a_0 da_1 D \nabla^j(a_2) |D|^{-2j}) \) has simple poles at \( \{1/2, \ldots, (j + 3)/2\} \). Hence for \( j \neq 0 \), \( \text{Res}_{z=0} \text{Trace} (\gamma a_0 da_1 D \nabla^j(a_2) |D|^{-3-2j-2z}) = 0 \). For \( j = 0 \), we have

\[
\gamma a_0 da_1 D a_2 |D|^{-3-2z} = \left( (\alpha_1 - i \beta_1) U^{\alpha_0} V^{\beta_0} U^{\alpha_1} V^{\beta_1} T_{m-in} U^{\alpha_2} V^{\beta_2} + (\alpha_1 + i \beta_1) U^{\alpha_0} V^{\beta_0} U^{\alpha_1} V^{\beta_1} T_{m+in} U^{\alpha_2} V^{\beta_2} \right) |D|^{-3-2z}
\]

where \( c_1, c_2 \) are constants depending on \( \alpha_1, \beta_1 \) and \( \theta \). It is clear that if \( (\alpha_0 + \alpha_1 + \alpha_2, \beta_0 + \beta_1 + \beta_2) \neq (0, 0) \), \( \text{Trace} (\gamma a_0 da_1 F_{a_2} |D|^{-3-2z}) = 0 \). For \( (\alpha_0 + \alpha_1 + \alpha_2, \beta_0 + \beta_1 + \beta_2) = (0, 0) \), we have

\[
\text{Res}_{z=0} \text{Trace} \left( T_{m-in} - i(n-\beta_2) (T \sqrt{m^2 + n^2})^{-3-2z} \right)
\]

\[
= \sum_{(m,n) \in \mathbb{Z}^2} (-\alpha_2 + i \beta_2) \left( \sqrt{m^2 + n^2} \right)^{-3-2z}
\]

\[
= (-\alpha_2 + i \beta_2) \text{Res}_{z=3/2} \sum_{(m,n) \in \mathbb{Z}^2} \left( \sqrt{m^2 + n^2} \right)^{-2z}
\]

\[
= 0.
\]

Therefore

\[
\text{Res}_{z=1} \text{Trace} (\gamma a_0 da_1 F_{a_2} |D|^{-2z}) = \text{Res}_{z=0} \text{Trace} (\gamma a_0 da_1 F_{a_2} |D|^{-2-2z}).
\]

\[
= \text{Res}_{z=0} \text{Trace} \left( \sum_{j=0}^{\infty} \binom{1/2}{j} \gamma a_0 da_1 D \nabla^j(a_2) |D|^{-3-2j-2z} \right).
\]
= 0. □

Proposition 4.9. For \( a_0, a_1, a_2 \in \mathcal{A}_\theta \), one has

1. \( \text{Res}_{z=1} \text{Trace}(\gamma a_0 F a_1 F a_2 |D|^{-2z}) = 0. \)
2. \( \text{Res}_{z=1} \text{Trace}(\gamma a_0 F a_1 a_2 F |D|^{-2z}) = 0. \)
3. \( \text{Res}_{z=1} \text{Trace}(\gamma a_0 a_1 F a_2 F |D|^{-2z}) = 0. \)
4. \( \text{Res}_{z=1} \text{Trace}(\gamma a_0 a_1 F^2 a_2 |D|^{-2z}) = 0. \)

Proof: Let \( a_i = u^{\alpha_i} v^{\beta_i} \) for \( i \in \{0, 1, 2\} \). Note that

\[
Fa_1 = D |D|^{-1} a_1 \sim \sum_{j=0}^{\infty} \left(-\frac{1}{2}ight)^j D^{\nabla^j(a_1)} |D|^{-1-2j}.
\]

Using this expansion together with the identity \(|D|^{-2z} a |D|^{-2z} \sim \sum_{j=0}^{\infty} \left(\frac{j}{2}\right) \nabla^j(a) |D|^{-2j}\), we get

\[
\text{Res}_{z=1}(\text{Trace}(\gamma a_0 F a_1 F a_2 |D|^{-2z})) = \text{Res}_{z=0}(\text{Trace}(\gamma a_0 F a_1 F a_2 |D|^{-2z-2z})) = \text{Res}_{z=0} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left(-\frac{1}{2}\right)^j \left(1 + j\right) \gamma a_0 D^{\nabla^j(a_1)} D^{\nabla^l(a_2)} |D|^{-4-2j-2l-2z}.
\]

Since \( \gamma a_0 D^{\nabla^j(a_1)} D^{\nabla^l(a_2)} \) is element of \( \mathcal{D}_{j+l+2, \ell^j+4} \) \( \text{Trace}(\gamma a_0 D^{\nabla^j(a_1)} D^{\nabla^l(a_2)} e^{-2iD^2}) \) has asymptotic expansion near 0. This implies that \( \text{Trace}(\gamma a_0 D^{\nabla^j(a_1)} D^{\nabla^l(a_2)} |D|^{-2z}) \) has simple poles at \( \{1/2, \cdots, (j + l + 4)/2\} \). Hence for \( (j, l) \neq (0, 0) \),

\[
\text{Res}_{z=0} \text{Trace}(\gamma a_0 D^{\nabla^j(a_1)} D^{\nabla^l(a_2)} |D|^{-4-2j-2l-2z}) = 0.
\]

Therefore

\[
\text{Res}_{z=1}(\text{Trace}(\gamma a_0 F a_1 F a_2 |D|^{-2z})) = \text{Res}_{z=0}(\text{Trace}(\gamma a_0 D a_1 D a_2 |D|^{-4-2z})) = \text{Res}_{z=0} \left(\gamma a_0 a_1 a_2 \left[\begin{array}{cc}
T_{m^2+n^2} & 0 \\
0 & T_{m^2+n^2}\end{array}\right] \right) |D|^{-4-2z} = \left\{\begin{array}{cc}
U_{\alpha_1} V |\beta| & 0 \\
0 & -U_{\alpha} V |\beta|\end{array}\right\} |D|^{-2-2z}
\]

\[
\text{Res}_{z=0} \text{Trace}(\left\{\begin{array}{cc}
U_{\alpha_1} V |\beta| & 0 \\
0 & -U_{\alpha} V |\beta|\end{array}\right\} |D|^{-2-2z}) = 0.
\]

Note that in second step in above calculation, we removed some terms which had zero residues. Other parts of the claim will follow by this result. One can write \( \gamma a_0 F a_1 a_2 F, \gamma a_0 a_1 F a_2 F \) and \( \gamma a_0 a_1 F^2 a_2 \) in the form \( \gamma b_0 F b_1 F b_2 \). and then use this result. □
Proposition 4.10. If $a_i = U^{\alpha_i}V^{\beta_i}$ for $i \in \{0, 1, 2\}$ then

$$\text{Res}_{z=1} \text{Trace} \left( \gamma a_0 da_1 da_2 |D|^{-2z} \right) = \begin{cases} 0 & \text{if } (|\alpha|, |\beta|) \neq 0, \\
4\pi i(\alpha_1 \beta_2 - \alpha_2 \beta_1)e^{-2\pi i\theta(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)} & \text{otherwise} \end{cases}$$

Proof: Note that

$$\gamma da_1 da_2 = \begin{bmatrix}
(\alpha_1 - i\beta_1)(\alpha_2 + i\beta_2)U^{\alpha_1}V^{\beta_0}U^{\alpha_2}V^{\beta_2} & 0 \\
0 & -(\alpha_1 + i\beta_1)(\alpha_2 - i\beta_2)U^{\alpha_0}V^{\beta_0}U^{\alpha_1}V^{\beta_1}U^{\alpha_2}V^{\beta_2}
\end{bmatrix}$$

$$= e^{-2\pi i\theta(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)} \begin{bmatrix}
(\alpha_1 - i\beta_1)(\alpha_2 + i\beta_2)U^{\alpha_1}V^{\beta_0} & 0 \\
0 & -(\alpha_1 + i\beta_1)(\alpha_2 - i\beta_2)U^{\alpha_0}V^{\beta_0}
\end{bmatrix}$$

Hence if $(|\alpha|, |\beta|) \neq 0$, $\text{Res}_{z=1} \text{Trace} \left( \gamma a_0 da_1 da_2 |D|^{-2z} \right) = 0$. For $(|\alpha|, |\beta|) = 0$, we have

$$\text{Res}_{z=1} \text{Trace} \left( \gamma a_0 da_1 da_2 |D|^{-2z} \right) = 2i(\alpha_1 \beta_2 - \alpha_2 \beta_1)e^{-2\pi i\theta(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)}\text{Res}_{z=1} \text{Trace} (|\Delta|^{-z}) = 4\pi i(\alpha_1 \beta_2 - \alpha_2 \beta_1)e^{-2\pi i\theta(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)}.$$ (by equation 4.9 and proposition 4.8)

□

To compute LIF for quantum double suspension of non-commutative torus, it is enough to take elements of the form $1 \otimes S^n$ and $a \otimes k$ where $a = u^a v^b$ and $k \in S(\ell^2(\mathbb{N}))$.

Lemma 4.11. For $\tilde{a} \in \Sigma^2 A_\theta$, one has

$$\Sigma^2 \phi_0 (\gamma \otimes 1) \tilde{a} = 0.$$ 

Proof: Let $\tilde{a} = a \otimes k$ where $a = u^a v^b \in A_\theta$ and $k \in S(\ell^2(\mathbb{N}))$. We have

$$t^2 \text{Trace} \left( (\gamma \otimes 1)(a \otimes k)e^{-t|D|} \right) = t^2 \text{Trace} \left( \gamma u^a v^b e^{-t|D|} \right) \text{Trace} \left( ke^{-tN} \right) \sim 0,$$

For $\tilde{a} = 1 \otimes S^n$ where $n \in \mathbb{Z}$, $\text{Trace} \left( (\gamma \otimes 1)(1 \otimes S^n)e^{-t|D|} \right) = 0$. It follows from proposition 3.8 that $\Sigma^2 \phi_0((\gamma \otimes 1) \tilde{a}) = \text{Res}_{z=0} (z^{-1} \text{Trace} ((\gamma \otimes 1) \tilde{a} |D|^{-2z})) = 0$. This completes the proof. □

Lemma 4.12. For $m_1, m_2, m_3 \in \mathbb{N}$, one has

$$\Sigma^2 \phi_2 (1 \otimes S^{m_1}, 1 \otimes S^{m_2}, 1 \otimes S^{m_3}) = 0.$$ 

Proof: Proof is exactly same as proof of lemma 4.5 □

Let $a^{(1)} = [D, a], a^{(2)} = Fa, a^{(3)} = aF$ and $c^{(1)} = c, c^{(2)} = Nc, c^{(3)} = -cN.$
Theorem 4.13. Let $a_i = u^{\alpha_i}v^{\beta_i}$ for $i \in \{0, 1, 2\}$. If for some $i \in \{0, 1, 2\}$, $a_i \otimes c_i \in \mathcal{A}_0 \otimes \mathcal{S}(\ell^2(\mathbb{N}))$ then

$$
\Sigma^2 \phi_2((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) \\
= \begin{cases} 
0 & \text{if } (|\alpha|, |\beta|) \neq 0, \\
2\sqrt{2\pi}i^2(\alpha_1 \beta_2 - \alpha_2 \beta_1)e^{-2\pi i(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)} \text{Trace}(c_0c_1c_2) & \text{otherwise}
\end{cases}
$$

Proof: Proof is very similar to proof of theorem 4.6 and we will proceed along the same line. By the same reasoning, we get that for $x \neq (0, 0)$,

$$
\Sigma^2 \psi_x^{(0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) = 0.
$$

For $x = (0, 0)$, we have

$$
\Sigma^2 \psi_{(0,0)}^{(0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) \\
= \text{Res}_{z=1} \text{Trace}((\gamma a_0 \otimes c_0)d_0(a_1 \otimes c_1)d_0(a_3 \otimes c_3)|D_0|^{-2z}). \\
= \sum_{j_1=1}^{3} \sum_{j_2=1}^{3} \psi_{(0,0)}^{(0)}(\gamma a_0, a_1^{(j_1)}, a_2^{(j_2)})\psi_0(c_0c_1c_2^{(j_2)}). \\
= \psi_{(0,0)}^{(0)}(\gamma a_0, da_1, da_2) \text{Trace}(c_0c_1c_2) \quad \text{(by proposition 4.8, 4.9, 4.10)}
$$

By equation (2.2), we have

$$
\Sigma^2 \psi_2((a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) \\
= \sum_{x \in \mathbb{N}^2} B_x^2 \Sigma^2 \psi_x^{(0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) \\
= B_{(0,0)}^2 \Sigma^2 \psi_{(0,0)}^{(0)}((\gamma \otimes 1)(a_0 \otimes c_0), d_0(a_1 \otimes c_1), d_0(a_2 \otimes c_2)) \\
= \begin{cases} 
0 & \text{if } (|\alpha|, |\beta|) \neq 0, \\
2\sqrt{2\pi}i^2(\alpha_1 \beta_2 - \alpha_2 \beta_1)e^{-2\pi i(\alpha_1 \beta_0 + \alpha_2 \beta_0 + \alpha_2 \beta_1)} \text{Trace}(c_0c_1c_2) & \text{otherwise}
\end{cases}
$$

This completes the proof. \qed

5. CONCLUDING REMARKS

In conclusion we would like to say the following.

1. So far we have very few instances of actual computations with the LIF. The first demonstration was by Connes [10]. He considered the case of quantum $SU(2)$ which is nothing but the quantum double suspension of the circle. Later Pal and Sundar [17] looked at quantum odd spheres. These are nothing but iterated quantum double suspensions of the two sphere and the noncommutative two torus. According to Connes [11]
to understand a concept or result one may take a two step procedure. In the first step one makes many computations and then the computations are explained conceptually. Here we are really executing the first step. Only after a good number of computations we will understand the meaning of individual terms in the formula.

(2) Here in these computations many linear functionals appearing in LIF are zero and evaluation of nonzero functionals involve only top residues. Next one would probably aim computation of the same for QDS of C*-algebra of continuous functions on higher dimensional spheres or on surfaces of genus more than one. In these cases, residues other than the top ones may contribute and will add an extra bit of complications which has to be resolved. One approach could be to compute linear functionals up to a co-boundary which will remove some unnecessary terms.

(3) In case of odd spectral triple, we need the linear functionals $\zeta^{(m)}_D$ on $B$ and two values $\zeta^{(1)}_{D_0}(1)$ and $\zeta^{(1)}_{D_0}(F \otimes 1)$ or $u_p$ and $v_p$ defined in section 3 to compute the linear functionals $\zeta^{(m)}_{D_0}$ on $\Sigma^2 B$. Therefore we can carry this process further and get LIF for iterated QDS of $A$ as soon as we know the two values $\zeta^{(1)}_{D_0}(1)$ and $\zeta^{(1)}_{D_0}(F \otimes 1)$ at each stage of iteration. Similarly one can iterate the process and get LIF in case of even spectral triple also.

(4) We used WHKAE property very crucially in our calculations. Can we drop this property and still compute LIF for QDS spectral triple?

(5) In another direction one could pursue investigations along the lines of [12, 13]. So far noncommutative tori at various dimensions were the only tractable examples where one could compute with the machinery of local index formula. Now with these models are also available for similar analysis and one would like to investigate them along these lines.

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