Two Magnetic Impurities with Arbitrary Spins in Open Boundary $t - J$ Model

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Abstract

From the open boundary $t - J$ model, an impurity model is constructed in which magnetic impurities of arbitrary spins are coupled to the edges of the strongly correlated electron system. The boundary $R$ matrices are given explicitly. The interaction parameters between magnetic impurities and electrons are related to the potentials of the impurities to preserve the integrability of the system. The Hamiltonian of the impurity model is diagonalized exactly. The integral equations of the ground state are derived and the ground state properties are discussed in details. We discuss also the string solutions of the Bethe ansatz equations, which describe the bound states of the charges and spins. By minimizing the thermodynamic potential we get the thermodynamic Bethe ansatz equations. The finite size correction of the free energy contributed by the magnetic impurities is obtained explicitly. The properties of the system at some special limits are discussed and the boundary bound states are obtained.

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I. INTRODUCTION

Strongly correlated systems have been of great interest over the past decade because of their importance as the most fundamental systems related to the theory of high- $T_c$ superconductors \cite{1,2} and these electron systems pose many theoretical challenges due to their essentially nonperturbative nature. The Hubbard chain and $t − J$ model are frequently invoked as the basic models of the one-dimensional strongly electron systems. When the repulsion energy $U$ for two electrons located on the same atom is much larger than the bandwidth of the electrons, the Hubbard model reduces to the $t − J$ model with an occupation of one electron per site. Then there are three states per site in the $t − J$ model. Also the low-energy excitations of the three-band Hubbard model can be changed into an effective one-band $t − J$ model \cite{3,4}. The $t − J$ Hamiltonian is one of the most attractive model to explain high temperature superconductivity without any phonon pairing mechanism \cite{6,7}.

Recently great progress has been made for the one-dimensional systems \cite{8}, in particular for the quantum impurity problems such as Kondo problem and tunneling in quantum wires. The theory of the Kondo effect \cite{9} was developed in the early 1960s to explain the puzzle of the resistance of some metals, that is, the resistance start to increase as the metal is cooled below a certain temperature. This is due to that the local moments on the impurity atoms having an antiferromagnetic coupling to the spins of the conduction electrons and the coupling becomes stronger as the temperature falls. Now we know the impurities play an important role in the strongly correlated electron compounds. Even a small amount of defects may change the properties of the electron systems.

Exact results, even for a simplified model exploring just some features of the system, are always useful and provide a testing ground for approaches intended for the full problem of higher order of complexity. Therefore it is of interest to construct integrable systems of the strongly correlated electron systems including impurities. Indeed, there has been a long successful history to study the effects of the impurities in the many-body quantum systems within the framework of the integrable systems \cite{10,11}, although the impurity usually destroys the integrability of the model when it is introduced to an exactly solved model. Andrei and Johannesson incorporated a magnetic impurity of the arbitrary spin into the isotropic spin-$\frac{1}{2}$ Heisenberg chain with integrability preserved. The results were extended to the Babujian-Takhtajan spin chain in Refs. \cite{15}. Bédőfig, Eßler and Frahm \cite{16} solved the integrable model with the impurity coupled with periodic $t − J$ chain \cite{17,18}, in which the impurity is introduced through a local vertex. Schlottmann and Zvyagin introduced the impurity into supersymmetric $t − J$ model \cite{20} via its scattering matrix with the itinerant electrons \cite{21,22}. The Hamiltonian of the system and other conserved currents can be constructed in principle by the transfer matrix. Zvyagin get that the low field impurity behavior for the periodic correlated electron chain coincides with that in the open chain up to mesoscopic corrections \cite{23}. They also discussed the magnetic impurities embedded in the Hubbard model \cite{24} and a finite concentration of magnetic impurities embedded in one-dimensional lattice via scattering matrices \cite{25,26}. The quantum impurity problem was also discussed by the use of the bosonization technique \cite{27}.

The study of completely integrable quantum spin open chains \cite{28} is also an interesting subject. It deals with the system in a finite interval. The pioneering works of Cherednik and Sklyanin \cite{33,34} are the starting point for the one dimensional exactly solved models where there is a variant of the usual quantum inverse scattering method. The so-called reflection equations appeared as a new ingredient to the Yang-Baxter equations for describing the systems on a finite interval with independent boundary conditions on each end. Using this method, a lot of integrable models have arisen \cite{34,36} such as the Hofstadter problem and the reaction diffusion equations. The quantum-group-invariant transfer matrices and the Hamiltonians can be given by the certain limits of the open-boundary conditions \cite{37,38,39,40}. Some models have been solved with non-trivial generalizations of the methods developed by periodic and twisted boundary conditions.

In 1993, Foerster and Karowski \cite{41} proposed the $sp_{kl}(2,1)$-invariant $t − J$ model with the quantum-group-invariant open boundary conditions and it was generalized to the $SU_q(n)$-invariant chains by de Vega and González-Ruiz in Ref. \cite{42}. González-Ruiz discussed also the open boundary supersymmetric $t − J$ model commuting with the number operator $n$ and $S^z$ \cite{42}. Recently, Zhou
and Batchelor \cite{43} studied the open boundary $t - J$ model via an analytic treatment of the Bethe ansatz equations. Fan, Hou and Shi \cite{44} formulated the eigenvalues and eigenvectors of the $t - J$ model with reflecting boundary conditions by the graded quantum inverse scattering method. The zero-temperature boundary effects in an open $t - J$ chain with boundary fields were discussed by Essler \cite{45}.

Now we know that the quantum inverse scattering method provides us also a very useful technique to construct the impurity model with the preserving of the integrability and this scheme has been adapted in Ref. \cite{16,21,24}. Very recently, the impurity model related to the $t - J$ model was studied also in Ref. \cite{46}. But, when the impurities are embedded in the system with periodic boundary conditions, some unphysical terms must be present in the Hamiltonian to keep the integrability though they may be irrelevant. Notice the fact that the impurities cut the one-dimensional system into 'pieces' when they are introduced and (then) the open boundary systems are formed with the impurities at the ends of the systems. In fact, Gonzalez-Ruiz \cite{42} has pointed out that the boundary terms could take into account impurities or magnetic fields located at the boundaries in 1994. So, the integrable impurity models \cite{47–52} may be studied with the use of open boundary conditions.

In this paper, we construct a Hamiltonian of the impurity model within the framework of the open boundary $t - J$ chain. The magnetic impurities are coupled to the edges of the strongly correlated electron system and have arbitrary spins. The boundary $R$ matrices are given explicitly and satisfy the reflection equation. They are compatible with the Yang-Baxter equation in the bulk. These guarantee the integrability of our model including the impurities with the arbitrary spins. The interaction parameters of the magnetic impurities with the electrons are parameterized by the constants which relate to the potentials of the impurities in order to contain the integrability of the system. The Hamiltonian with the impurities is diagonalized exactly by the use of the Bethe ansatz method and the Bethe ansatz equations are obtained explicitly via the quantum inverse scattering method. The integral equations of the ground state are derived and the properties of the ground state are discussed in detail. The finite size corrections of the ground state energy due to the impurities are obtained. The distribution functions of the rapidities are given. The string solutions of the Bethe ansatz equations are discussed and the coupled integral equations of the excited state are obtained. By minimizing the thermodynamical potential of the system with impurities, we get the thermodynamical Bethe ansatz equations. The finite size correction of the free energy is obtained for the two magnetic impurities. We study also the properties of the system in the high-temperature and the low temperature limits. The boundary bound states are constructed in our model.

The paper is divided into eight sections. In section 2 we give the Hamiltonian of our impurity model where the generalized “permutation operator” is presented. In section 3 the integrability conditions are discussed and the boundary $R$ matrices are obtained. Our Hamiltonian is diagonalized exactly and the Bethe ansatz equations of the impurity model are obtained in section 4. In section 5 the coupled integral equations of the ground state are derived and the ground state properties of the system are discussed. In section 6, the string solutions of the Bethe ansatz equations are discussed and the integral equations of the excited states are obtained. We get also the thermodynamic Bethe ansatz equations. The finite size correction of the free energy contributed by the two magnetic impurities is obtained. In section 7, the properties of the system at the high-temperature and the low temperature limits are discussed and the boundary bound states are constructed. Conclusions follow in section 8. In order to complete the four integrable cases of the $t - J$ model, some boundary $R$ matrices and the Bethe ansatz equations are given in Appendix A and B, respectively.

II. THE HAMILTONIAN OF THE IMPURITY MODEL

As is well known, the $t - J$ model describes electrons with spin $\frac{1}{2}$ on an one-dimensional lattice and permits the nearest-neighbor hopping ($t$) of the electrons. The large on-site Coulomb repulsion makes that the double-occupancy of every site impossible. There are two types of interactions
between the electrons on the nearest neighbor sites. One is the spin exchange interaction \( (J) \) and the other is the charge interaction \( (V) \) independent of spin. Our starting point is the following Hamiltonian

\[
H = -t \sum_{j=1}^{G-1} \sum_{\sigma = \uparrow, \downarrow} (C_{j \sigma}^+ C_{j+1 \sigma} + C_{j+1 \sigma}^+ C_{j \sigma}) + J \sum_{j=1}^{G-1} \mathbf{S}_j \cdot \mathbf{S}_{j+1} + V \sum_{j=1}^{G-1} n_j n_{j+1} + J_{\sigma} \mathbf{S}_d \cdot \mathbf{d}_a + V_a n_1 + J_b \mathbf{S}_G \cdot \mathbf{d}_b + V_b n_G,
\]

where \( C_{j \sigma}^+ (C_{j \sigma}) \) is the creation (annihilation) operator of a conduction electron with spin \( \sigma \) on the site \( j \); \( \mathbf{S}_j = \frac{1}{2} \sum_{\sigma, \sigma'} C_{j \sigma}^+ C_{j \sigma'} \sigma \sigma' \) is the spin operator of the conduction electron; \( d_{a,b} \) are the spin operators of the impurities; \( J_{\sigma, a,b}, V_{\sigma, a,b} \) are the interaction constants of the impurities with the electrons and the scattering potentials of the magnetic impurities, respectively; \( n_j = C_{j \uparrow}^+ C_{j \uparrow} + C_{j \downarrow}^+ C_{j \downarrow} \) is the number operator of the conduction electrons; \( G \) is the length (or site number) of the system. For the bulk of the system, the scattering matrices \( S \) of the electrons satisfy the Yang-Baxter equation:

\[
S_{12}(k_1, k_2) S_{13}(k_1, k_3) S_{23}(k_2, k_3) = S_{23}(k_2, k_3) S_{13}(k_1, k_3) S_{12}(k_1, k_2)
\]

which gives the factorization condition for the integrability of the free boundary and periodic boundary \( t - J \) models. For the open boundary system, the reflection equation \([33, 28]\) should also be satisfied, i.e.

\[
S_{12}(k_1, k_2) \frac{1}{R}(k_1) S_{12}(k_1, -k_2) \frac{2}{R}(k_2) = \frac{2}{R}(k_2) S_{12}(k_1, -k_2) \frac{1}{R}(k_1) S_{12}(k_1, k_2),
\]

where the boundary \( R \) matrices are written down as

\[
\frac{1}{R}(k_1) = R(k_1) \otimes i d_{V_2}, \quad \frac{2}{R}(k_2) = i d_{V_1} \otimes R(k_2)
\]

for matrix \( R \in \text{End}(V) \).

To consider magnetic impurities of arbitrary spins, we define an operator \( P_d \) as

\[
P_d = \frac{1}{\sqrt{1 + 4t(t+1)}} (1 + 4S \cdot d)
\]

where \( S \) and \( d \) are the spin operators of the electron and impurity, respectively. It has the property \( P_d^2 = 1 \) and degenerates to the ordinary permutation operator when the impurity spin \( d \) takes its value \( l = 1/2 \). We call \( P_d \) the generalized “permutation operator”. The Hamiltonian may be written in the form

\[
H = -\sum_{j=1}^{N} (T_j^+ T_j^-) + \sum_{j=1}^{N} (K_j + K_{a_j} \delta_{x_j,1} + K_{b_j} \delta_{x_j,G})
\]

where we have set \( t = 1 \). The translation operators \( T_j^\pm \) are defined as \([33]\)

\[
T_j^\pm \Psi(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j \pm 1, \ldots, x_N)
\]

where \( \Psi(x_1, \ldots, x_j, \ldots, x_N) \) is the wave function of \( N \) conduction electrons. The operator \( K_j \) acts on the wave function \( \Psi(x_1, \ldots, x_j, \ldots, x_N) \) in the form

\[
K_j \Psi(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} \delta_{x_j, x_i+1} K_{ij} \Psi(x_1, x_2, \ldots, x_N)
\]

where \( K_{ij} = V - \frac{t}{4} + \frac{J}{4} P_{ij} \) denotes the interactions between the conduction electrons. The permutation operator \( P_{ij} \) permutes the spins of the \( i \)-th and \( j \)-th electrons. The interactions between the impurities with arbitrary spins and the electrons are given by the relations.
\[ K_{aj} = V_a - \frac{J_a}{4} + \frac{J_a \sqrt{1 + 4s_a (s_a + 1)}}{4} P_{aj}, \]
\[ K_{bj} = V_b - \frac{J_b}{4} + \frac{J_b \sqrt{1 + 4s_b (s_b + 1)}}{4} P_{bj}, \]

with
\[ P_{aj} = \frac{1}{\sqrt{1 + 4s_a (s_a + 1)}} (1 + 4S_j \cdot d_a), \]
\[ P_{bj} = \frac{1}{\sqrt{1 + 4s_b (s_b + 1)}} (1 + 4S_j \cdot d_b) \]

where \( S_j \) are the spin operators of the conduction electrons; \( d_a \) and \( d_b \) are the spin operators of the magnetic impurities with the arbitrary spin values \( s_a \) and \( s_b \) respectively. The boundary \( R \) matrices of our impurity model will be given explicitly in the following section.

**III. THE BOUNDARY R MATRIX AND INTEGRABILITY CONDITIONS**

The boundary \( R \) matrices satisfy the reflecting Yang-Baxter equation for the open boundary system and the reflection equation is compatible with the factorization condition. These give the integrability of the one-dimensional many body system. In our model, the boundary \( R \) matrix should take the form
\[ R = \exp(i\phi) \frac{q - ic - i (2l + 1) P_d/2}{q + ic + i (2l + 1) P_d/2}, \]

where \( P_d \) is the generalized “permutation operator”, \( q = \frac{1}{2} \tan \frac{k}{2} \) and \( c \) is an arbitrary constant. Here we have set \( J = 2 \) and \( V = 3/2 \). The other integrable cases will be given in the Appendix. Put \( K_{a(b),j} = m + n P_d \). From the boundary Yang-Baxter relation we have that
\[ 2n \left( \frac{2}{2l + 1} q^2 + \frac{2}{2l + 1} c^2 - \frac{2l + 1}{2} \right) \tan \frac{k}{2} \]
\[ + q \left[ (m - 1)^2 - n^2 \right] \tan^2 \frac{k}{2} + q \left[ (m + 1)^2 - n^2 \right] = 0. \]

This relation gives the restricting condition imposed on the coupling constants of our impurity model. It should be satisfied to maintain the integrability of the system when the magnetic impurities with the arbitrary spins are introduced to the \( t - J \) model by coupling the impurities to the edges of the system. From the above relation, we have that
\[ m = \frac{(2l + 1)^2 + 1 - 4c^2}{4 (l \pm c) (l \mp c + 1)}, \]
\[ n = \frac{2l + 1}{2 (l \pm c) (l \mp c + 1)}. \]

The phase factor in the boundary \( R \) matrix related the magnetic impurity at the left edge of the system is given by
The phase factor $\exp [i\varphi_a(k)]$, corresponding to the scattering of the electron with the impurity at the right edge of the system, is given by the substituting $b$ for $a$. Without loss of generality, the interaction parameters $J_{a,b}$ and $V_{a,b}$ take the following forms:

\[
J_a = \frac{2}{(s_a + c_a)(s_a - c_a + 1)},
\]

\[
V_a = -\frac{4c_a^2 - (2s_a + 1)^2 - 3}{4(s_a + c_a)(s_a - c_a + 1)}.
\]

The parameters $J_b$ and $V_b$ are given similarly by the substitution of index $a$ by index $b$ in the above relations. By the use of these parameterizations of the interaction constants $J$, the phase factors take the form: $\exp [i\varphi_a(k)] = \exp [i\varphi_b(k)] = -\exp (ik)$. Then, the boundary $R$ matrices for the impurity model with the arbitrary spins can be expressed as

\[
R_a(k_j, \sigma_j) = -\exp (ik_j) \frac{2q_j - 2ic_a - i(2s_a + 1)P_{aj}}{2q_j + 2ic_a + i(2s_a + 1)P_{aj}},
\]

\[
R_b(-k_j, \sigma_j) = -\exp [-ik_j (2G + 1)] \frac{2q_j - 2ic_b - i(2s_b + 1)P_{bj}}{2q_j + 2ic_b + i(2s_b + 1)P_{bj}}
\]

with $q_j = \frac{k}{2} \tan \frac{k_j}{2}$, respectively. Here the operators $P_{aj}$ and $P_{bj}$ are the generalized “permutation operators” and have the expressions (8) and (6).

IV. BETHE ANSATZ EQUATIONS OF THE IMPURITY MODEL

Firstly, we can write down the wave function $\Psi_{\sigma_1, \sigma_2, \ldots, \sigma_N}(x_1, x_2, \ldots, x_N)$ in the region $0 \leq x_{Q1} \leq x_{Q2} \leq \cdots \leq x_{QN} \leq G - 1$ as

\[
\Psi_{\sigma_1, \sigma_2, \ldots, \sigma_N}(x_1, x_2, \ldots, x_N) = \sum_{P} \sum_{r_1, r_2, \ldots, r_N} \varepsilon_P \varepsilon_r A_{\sigma_{Q1}, \sigma_{Q2}, \ldots, \sigma_{QN}}(r_{PQ1}k_{PQ1}, r_{PQ2}k_{PQ2}, \ldots, r_{PQN}k_{PQN}) \cdot \exp [i \sum_{j=1}^{N} r_{pj}k_{pj}x_j],
\]

where the coefficients $A_{\sigma_{Q1}, \sigma_{Q2}, \ldots, \sigma_{QN}}(r_{PQ1}k_{PQ1}, r_{PQ2}k_{PQ2}, \ldots, r_{PQN}k_{PQN})$ are also dependent on the spins of magnetic impurities which are suppressed for brevity, and $\varepsilon_P = 1(-1)$, when the parity of $P$ is even(odd), $\varepsilon_r = \prod_{j=1}^{N} r$ in which $r$ takes the value $+1$ or $-1$. The coefficients $A$ are related by the scattering matrices. The boundary $R$ matrix satisfies the reflecting Yang-Baxter equation (3). The scattering matrices in the bulk satisfy the Yang-Baxter equation (2). By the use of the standard Bethe ansatz method, we can diagonalize the Hamiltonian with the two magnetic impurities [17] [18]. The boundary $R$ matrices can be cast into the forms:

\[
R_a(k_j, \sigma_j) = -\exp (ik_j) \frac{q_j - ic_a + i(s_a + 1/2)}{q_j + ic_a - i(s_a + 1/2)} S_{j0}(k_j, k_0),
\]

\[
R_b(-k_j, \sigma_j) = -\exp [-ik_j (2G + 1)] \frac{q_j - ic_b + i(s_b + 1/2)}{q_j + ic_b - i(s_b + 1/2)} S_{jN+1}(k_j, k_{N+1}).
\]
where

\[ S_{j0}(k_j, k_0) = -\frac{q_j - q_0 - i(s_a + 1/2) P_{\alpha j}}{q_j - q_0 + i(s_a + 1/2)} \]

(18)

\[ S_{jN+1}(k_j, k_{N+1}) = -\frac{q_j - q_{N+1} - i(s_b + 1/2) P_{\beta j}}{q_j - q_{N+1} + i(s_b + 1/2)} \]

(19)

with \( q_0 = ic_a \) and \( q_{N+1} = ic_b \). Notice that the scattering matrix in the bulk has the form

\[ S_{jl}(k_j, k_l) = -\frac{q_j - q_l - iP_{jl}}{q_j - q_l + i}, \quad (j, l = 1, 2, \ldots, N) \]

(20)

where \( q_j = \frac{1}{2} \tan \frac{k_j}{2} \). The present case (\( J = 2 \) and \( V = 3/2 \)) corresponds to triplet scattering and not to the ordinary \( t - J \) model. The symmetry of this model is SU(3), rather than a graded FFB super-algebra which corresponds to the traditional \( t - J \) model. In this case [17], \( \chi^2 = V/2 + J/8 = 1 \) and \( \chi^4 = V/2 - 3J/8 = 0 \). These quantities are interchanged in Ref. [17], but this was corrected in Ref. [8]. If we set

\[ S_{\tau l}(\lambda) = -\frac{\lambda - q_l - iP_{\tau l}}{\lambda - q_l + i}, \quad (l = 1, 2, \ldots, N) \]

\[ S_{\tau 0}(\lambda) = -\frac{2\lambda - 2ic_a - i(2s_a + 1) P_{\alpha \tau}}{2\lambda - 2ic_a + i(2s_a + 1)}, \]

\[ S_{\tau N+1}(\lambda) = -\frac{2\lambda - 2ic_b - i(2s_b + 1) P_{\beta \tau}}{2\lambda - 2ic_b + i(2s_b + 1)}, \]

(21)

then we get the equation

\[ Tr \left[ T(\lambda) T^{-1}(\cdot \lambda) \right] \bigg|_{\lambda=q_j} \Phi = \exp \left( 2ik_j G \right) \frac{i - q_j^2}{i/2 + q_j} \frac{q_j + ic_a - i(s_a + 1/2)}{q_j - ic_a + i(s_a + 1/2)} \]

\[ \frac{q_j + ic_b - i(s_b + 1/2)}{q_j - ic_b + i(s_b + 1/2)} \Phi \]

(22)

where \( \Phi \) is the eigenstate of the system and \( T(\lambda) \) is defined as

\[ T(\lambda) = S_{\tau j}(\lambda) S_{\tau 0}(\lambda) S_{\tau 1}(\lambda) \cdots S_{\tau j-1}(\lambda) S_{\tau j+1}(\lambda) \cdots S_{\tau N+1}(\lambda). \]

(23)

Then we have the following Bethe ansatz equations:

\[ e \left( 2q_j \right)^{2G} \prod_{\beta=1}^{M} e \left( 2q_j - 2\lambda_{\beta} \right) e \left( 2q_j + 2\lambda_{\beta} \right) \]

\[ = e \left( \frac{2q_j}{2c_\alpha + 2s_a + 1} \right) e \left( \frac{2q_j}{2c_\beta + 2s_b + 1} \right) \cdot \prod_{l=1(l\neq j)}^{N} e \left( q_j - q_l \right) e \left( q_j + q_l \right), \]

(24)

\[ e \left( \frac{\lambda_\alpha}{s_a + c_\alpha} \right) e \left( \frac{-\lambda_\alpha}{s_b + c_b} \right) e \left( \frac{\lambda_\alpha}{s_a - c_\alpha} \right) e \left( -\frac{\lambda_\alpha}{s_b - c_b} \right) \]

\[ \cdot \prod_{l=1}^{N} e \left( 2\lambda_\alpha - 2q_l \right) e \left( 2\lambda_\alpha + 2q_l \right) \]

\[ = \prod_{\beta=1(\beta \neq \alpha)}^{M} e \left( \lambda_\alpha - \lambda_{\beta} \right) e \left( \lambda_\alpha + \lambda_{\beta} \right), \]

(25)

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where we have used the notation $e(x) = (x + i) / (x - i)$. Notice that $e(\pm \infty) = 1$. For the case of $J = -2$ and $V = -3/2$, only the energy changes sign. The Bethe ansatz equations for this case are the same as for antiferromagnetic coupling.

V. GROUND STATE PROPERTIES

The eigenvalue of the Hamiltonian can be written down as

$$E = 2N - \sum_{j=1}^{N} \frac{1}{q_j^2 + \frac{1}{4}}$$

with $q_j = \frac{1}{2} \tan \frac{k_j}{2}$. In order to obtain the ground state properties for the case of the triplet state scattering we should find out the solutions of the above Bethe ansatz equations. The sets of the rapidities $\{q_j\}$ and $\{\lambda_\alpha\}$ have real and the complex solutions. Complex solutions of the $\lambda_\alpha$ correspond to the excited states and will be discussed in section 6.

For $J = 2$ and $V = 3/2$, the model is the $SU(3)$ invariant $t-J$ model where electrons in triplet states are scattered but not those in singlet states. For the bulk this model is isomorphic to the spin 1 Heisenberg chain with $SU(3)$ invariance. It has no graded super-algebra and corresponds to three bosonic degrees of freedom. All rapidities are real for the ground state. Now we take the thermodynamic limit and introduce the distribution functions $\rho(q)$ and $\sigma(\lambda)$. The corresponding ones of the holes are denoted by $\rho^h(q)$ and $\sigma^h(\lambda)$. So the integral equations describing the ground state of the impurity model can be written as

$$\frac{1}{2} G \left[ a(\lambda, s_a + c_a) + a(\lambda, s_b + c_b) + a(\lambda, s_a - c_a) + a(\lambda, s_b - c_b) + a(\frac{\lambda}{2}) \right]$$

$$+ \frac{1}{2} \int d\rho(q) \left[ a(\lambda - q, \frac{1}{2}) + a(\lambda + q, \frac{1}{2}) \right]$$

$$= \sigma(\lambda) + \sigma^h(\lambda) + \frac{1}{2} \int d\lambda' \sigma(\lambda') \left[ a(\lambda - \lambda', 1) + a(\lambda + \lambda', 1) \right],$$

where $a(\lambda, \eta) = \frac{1}{\sqrt{\lambda^2 + \eta^2}}$. The ground state properties can be obtained from the above coupled integral equations. The details are as follows.

By taking into account of the above distribution functions, the energy per site of the system can be expressed as

$$\frac{E}{G} = \frac{2N}{G} - 2\pi \int d\rho(q) a\left(q, \frac{1}{2}\right)$$

for the case of $J = 2$ and $V = 3/2$. The number of the particles per site is

$$\frac{N}{G} = \int d\rho(q).$$

The magnetization of the impurity model can be written as
\[
\frac{S_z}{G} = \frac{1}{2} \int dq \rho(q) - \int d\lambda \sigma(\lambda) + \frac{l_a + l_b}{G} 
\]

where \(l_a = 1 - s_a, 2 - s_a, \cdots, s_a\) and \(l_b = 1 - s_b, 2 - s_b, \cdots, s_b\). \(s_a\) and \(s_b\) are the spins of the impurities located at the two ends of the one-dimensional lattice system. The Fourier transformations of the Bethe ansatz equations (27) and (28) give that

\[
e^{-|\omega|/2} + e^{-|\omega|/2} \bar{\sigma}(\omega) = \bar{\rho}(\omega) \left( 1 + e^{-|\omega|} \right) + \bar{\rho}^h(\omega) + \frac{1}{2G} \tilde{\rho}(\omega),
\]

\[
\tilde{\rho}(\omega) e^{-|\omega|/2} + \frac{1}{2G} \tilde{\sigma}(\omega) = \tilde{\sigma}(\omega) \left( 1 + e^{-|\omega|} \right) + \tilde{\sigma}^h(\omega),
\]

where the tilde denotes the Fourier transform and the functions \(\tilde{\rho}(\omega)\) and \(\tilde{\sigma}(\omega)\) are the Fourier transforms of \(\rho_G(q)\) and \(\sigma_G(\lambda)\) given by

\[
\rho_G(q) = a \left( q, c_a + s_a + \frac{1}{2} \right) + a \left( q, c_b + s_b + \frac{1}{2} \right) - a \left( q, \frac{1}{2} \right),
\]

\[
\sigma_G(\lambda) = a (\lambda, s_a + c_a) + a (\lambda, s_b + c_b) + a (\lambda, s_a - c_a) + a (\lambda, s_b - c_b) + a \left( \lambda, \frac{1}{2} \right),
\]

which are the finite size corrections due to the magnetic impurities and the open boundary conditions. Now we first consider the half-filled band of the ferromagnetic case of \(\sigma(\lambda) = 0\). The nonzero distribution functions are

\[
\bar{\rho}(\omega) = \frac{1}{2 \cosh \frac{\omega}{2}} - \frac{1}{2G} \frac{\bar{\rho}_G(\omega)}{1 + e^{-|\omega|}},
\]

\[
\tilde{\sigma}^h(\omega) = \frac{e^{-|\omega|/2}}{2 \cosh \frac{\omega}{2}} + \frac{1}{2G} \left( \tilde{\sigma}(\omega) - \frac{\bar{\rho}_G(\omega)}{2} \right)
\]

and the energy is

\[
E/G = 1 - 2 \ln 2.
\]

The finite size correction to the energy in this case is

\[
E_{fin}^0 = \ln 2 + \frac{1}{2} \int_0^\infty \frac{\tilde{\sigma}_G(\omega)}{\cosh \frac{\omega}{2}} d\omega,
\]

contributed by the impurities, and the open boundary term gives the correction \(-\ln 2\). For the nonmagnetic case, when the number per site is 2/3, the distribution functions take the form:

\[
\bar{\rho}(\omega) = \frac{2 \cosh \frac{\omega}{2}}{1 + 2 \cosh \omega} + \frac{1}{2G} \frac{e^{\omega/2}}{1 + 2 \cosh \omega} \left\{ \tilde{\sigma}_G(\omega) - 2 \bar{\rho}_G(\omega) \cosh \frac{\omega}{2} \right\},
\]

\[
\tilde{\sigma}(\omega) = \frac{1}{1 + 2 \cosh \omega} + \frac{1}{2G} \frac{e^{\omega/2}}{1 + 2 \cosh \omega} \left\{ 2 \tilde{\sigma}_G(\omega) \cosh \frac{\omega}{2} - \bar{\rho}_G(\omega) \right\}
\]

with the energy being

\[
\frac{E}{G} = \frac{4}{3} - \frac{\pi \sqrt{3}}{9} - \ln 3,
\]

and the finite size correction of the energy is

\[
E_{fin}^1 = \frac{2 \pi \sqrt{3}}{9} \ln 2 - \frac{1}{2} \int_{-\infty}^\infty \frac{\tilde{\sigma}_G(\omega) - 2 \bar{\rho}_G(\omega) \cosh \frac{\omega}{2}}{1 + 2 \cosh \omega} d\omega.
\]

This is the contribution of the magnetic impurities and the open boundary terms give the correction \(-2\pi \sqrt{3}/9\).
VI. EXCITED STATE

The eigenstates of the model are specified by two sets of rapidities, \( \{q_j\} \) and \( \{\lambda_j\} \), for the charges and the spins, respectively. In this section, we will also restrict the discussions on the case of \( J = 2 \) and \( V = 3/2 \). In the thermodynamic limit these rapidities are classified according to the string hypothesis:

\[
q_n^{n,j} = q_n^j + \frac{i}{2} (n + 1 - 2j), \quad j = 1, 2, \ldots, n
\]

\[
\lambda_n^{n,j} = \lambda_n^j + \frac{i}{2} (n + 1 - 2j), \quad j = 1, 2, \ldots, n
\]

(44)

where \( n = 1, 2, \ldots, +\infty \). These strings describe charge and spin boundstates, respectively. The real parameters \( q_n^j \) and \( \lambda_n^j \) are related to the momentum of the center of mass of the boundstate. Their distribution functions are \( \rho_n(q) \), \( \sigma_n(\lambda) \) and the ones corresponding to the holes are \( \rho_n^h(q) \) and \( \sigma_n^h(\lambda) \). By using the notations

\[
[n] f(k) = \int_{-\infty}^{\infty} a \left( k - k', \frac{n}{2} \right) f(k') dk' \quad \text{for} \; n \neq 0,
\]

(45)

\[
[0] f(k) \equiv f(k),
\]

the integral equations can be cast into the following compact forms:

\[
a \left( q, \frac{n}{2} \right) + \sum_{m=1}^{\infty} \sum_{j=0}^{m-1} [n - m + 1 + 2j] \sigma_m(q) + \frac{1}{2G} \rho_n^G(q)
\]

\[
= \rho_n^h(q) + \sum_{m=1}^{\infty} A_{nm} \rho_m(q),
\]

(46)

\[
\frac{1}{2G} \sigma_n^G(\lambda) + \sum_{m=1}^{\infty} \sum_{j=0}^{m-1} [n - m + 1 + 2j] \rho_m(\lambda) = \sigma_n^h(\lambda) + \sum_{m=1}^{\infty} A_{nm} \sigma_m(\lambda)
\]

(47)

where the terms with the factor \( 1/(2G) \) are the contributions provided by the magnetic impurities and the boundary terms. The operator \( A_{nm} \) is defined as

\[
A_{nm} \equiv [n - m] + 2 [n - m] + 2 [n - m + 2] + 2 [n - m + 4] + \cdots + 2 [n + m - 2] + [n + m]
\]

(48)

and

\[
\rho_n^G(q) \equiv a \left( q, \frac{n}{2} \right) - \sum_{j=0}^{n-1} \left\{ a \left( q, s_a + c_a - j + \frac{n}{2} \right) + a \left( q, s_b + c_b - j + \frac{n}{2} \right) \right\},
\]

(49)

\[
\sigma_n^G(\lambda) = a \left( \lambda, \frac{n}{2} \right) + \sum_{j=0}^{n-1} \left\{ a \left( \lambda, s_a + c_a - j + \frac{n-1}{2} \right) + a \left( \lambda, s_b + c_b - j + \frac{n-1}{2} \right) \right. \right.
\]

\[
+ a \left( \lambda, s_a - c_a - j + \frac{n-1}{2} \right) + a \left( \lambda, s_b - c_b - j + \frac{n-1}{2} \right) \right\}
\]

(50)
Now the number of the electrons per site is expressed by
\[ \frac{N}{G} = \sum_{m=1}^{\infty} m \int \rho_m(q) \, dq. \] (51)

By minimizing the thermodynamic potential we get the thermodynamical Bethe ansatz equations:
\[ \frac{n(2 - A - H)}{T} - 2\pi a \left( k, \frac{n}{2} \right) - \ln(1 + \xi_n) + \sum_{m=1}^{\infty} A_{nm} \ln \left( 1 + \xi_m^{-1} \right) \]
\[ - \sum_{m=1}^{\infty} \sum_{j=0}^{n-1} [m - n + 2j] \ln(1 + \eta_m^{-1}) = 0, \] (52)

\[ \frac{2Hn}{T} - \ln(1 + \eta_n) + \sum_{m=1}^{\infty} A_{nn} \ln \left( 1 + \eta_m^{-1} \right) \]
\[ - \sum_{m=1}^{\infty} \sum_{j=0}^{n-1} [m - n + 2j] \ln(1 + \xi_m^{-1}) = 0, \] (53)

where we have set that
\[ \eta_n(k) = \frac{\sigma_n^h(k)}{\sigma_n(k)}, \quad \xi_n(k) = \frac{\rho_n^b(k)}{\rho_n(k)}. \]

Of course, the above expressions can be changed into the several other forms as the ordinary system in one dimensional lattice system. The impurities located at the ends of the system do not change the free energy for the bulk [22,56]. The finite size correction of the free energy contributed by the boundary term is
\[ F_{bou} = -\frac{T}{2} \sum_{n=1}^{\infty} [n] \ln \{ 1 + \xi_n^{-1}(0) \} - \frac{T}{2} \sum_{n=1}^{\infty} [n] \ln \{ 1 + \eta_n^{-1}(0) \}. \] (54)

The free energy due to the two magnetic impurities with the spins \( s_a, s_b \) is
\[ F_{imp} = -\frac{T}{2} \sum_{n=1}^{\infty} \int dq \rho_{n,imp}^G \ln \left( 1 + \xi_n^{-1} \right) - 2H \left( l_a + l_b \right) \]
\[ - \frac{T}{2} \sum_{n=1}^{\infty} \int d\lambda \sigma_{n,imp}^G \ln \left( 1 + \eta_n^{-1} \right), \] (55)

where \( l_a = 1 - s_a, 2 - s_a, \ldots, s_a; \quad l_b = 1 - s_b, 2 - s_b, \ldots, s_b; \) and
\[ \rho_{n,imp}^G(q) = -\sum_{j=0}^{n-1} \left\{ a \left( q, s_a + c_a - j + \frac{n}{2} \right) + a \left( q, s_b + c_b - j + \frac{n}{2} \right) \right\}, \] (56)

\[ \sigma_{n,imp}^G(\lambda) = \sum_{r=\pm 1, l=a,b, j=0}^{n-1} a \left( \lambda, s_l + rc_l - j + \frac{n-1}{2} \right). \] (57)

The interesting observation that the finite size correction due to the magnetic impurities is finite even if the temperature \( T \to 0 \). The properties of the system at the high-temperature limit and the low temperature limit can be discussed by the use of the thermal Bethe ansatz equations. And the boundary bound states exist in our model. These will be studied in the following section.
VII. SPECIAL LIMITS AND THE BOUNDARY BOUND STATES

In order to discuss the low temperature properties of the model and the impurity free energy at high-temperature, we may change the thermal Bethe ansatz equations into the forms:

\[
\frac{2\pi ([0] + [2])^{-1} a(k, \frac{1}{2})}{T} + \hat{G} \ln \frac{1 + \eta_2^{-1}}{1 + \xi_1^{-1}} + \ln \xi_1 = 0, \quad (58)
\]

\[
\ln \xi_n = \hat{G} \ln \frac{(1 + \xi_{n-1}) (1 + \xi_{n+1})}{(1 + \eta_{n-1}) (1 + \eta_{n+1})}, \quad n > 1, \quad (59)
\]

\[
\frac{2\pi ([0] + [2])^{-1} a(k, \frac{1}{2})}{T} - \hat{G} \ln (\xi_2 \eta_2) + \ln (\xi_1 \eta_1) = 0, \quad (60)
\]

\[
\ln \eta_n = \hat{G} \ln \frac{(1 + \eta_{n-1})(1 + \eta_{n+1})}{(1 + \xi_{n-1})(1 + \xi_{n+1})}, \quad n > 1, \quad (61)
\]

\[
\lim_{n \to \infty} (n + 1) \ln \frac{1 + \xi_n}{1 + \eta_n} - n \ln \frac{1 + \xi_{n+1}}{1 + \eta_{n+1}} = \frac{A - 2 + H/2}{T}, \quad (62)
\]

\[
\lim_{n \to \infty} \left( n + 1 \ln \frac{1 + \eta_n}{1 + \xi_n} - n \ln \frac{1 + \eta_{n+1}}{1 + \xi_{n+1}} \right) = -\frac{H}{T}, \quad (63)
\]

with the use of the formulae \( A_{1m} - \hat{G} A_{2m} = \delta_{1m} \), \( A_{nm} - \hat{G} (A_{n-1 m} + A_{n+1 m}) = \delta_{nm} \) \((n > 1)\), and \([n + 1] A_{nm} - [n] A_{n+1} m = 0\) if \(m < n\) or \(-[m + 1] - [m - 1]\) if \(m > n\) where \(\hat{G} = [1]/([0] + [2])\). Furthermore, we have that \(\lim_{n \to \infty} (\ln \xi_n)/n = (2 - A - H/2)/T\) and \(\lim_{n \to \infty} (\ln \eta_n)/n = H/T\) where \(H\) is the magnetic field and \(A\) is the chemical potential of the system. Define the symbol function \(\text{sign}(a)\) as

\[
\text{sign}(a) = \begin{cases} 
1 & a > 0 \\
0 & a = 0 \\
-1 & a < 0 
\end{cases}
\]

Then, in the high-temperature limit, the free energy of the impurities is

\[
F_{\text{imp}} = \frac{T}{2} \ln 2 \sum_{l=a,b} \sum_{r_1 = 0, \pm 1} \left\{ \text{sign} \left( s_l + c_l + \frac{1}{2} + r_1 \right) \\
- \sum_{r_2 = \pm 1} \text{sign} \left( s_l + r_2 c_l + r_1 \right) \right\}. \quad (64)
\]

The boundary term give the contribution \(F_{\text{bound}} = -3T \ln 2\). The above result means that the entropy of the model related to the impurities is dependent on the interacting parameters of the impurities with the electrons and the spins of the impurities. At the low temperature limit, from the thermal Bethe ansatz equations, we get that \(\xi_n\) have the different asymptotic properties for \(n < 2\) and \(n \geq 2\). It means that \(\xi_n = \xi f = 2\) and this gives that \(\tau = 4/ (f + 2) = 1\). By considering the free energy expression in the bulk, we find that at low temperature, the specific heat behaves as \(C_{\text{bulk}} \sim T\). Since the asymptotic property of the term \(\ln(1 + \eta_{n-1})\) is different from the one of the term \(\ln(1 + \xi_{n-1})\) at the low temperature limit, the impurities and the open boundary term
contributions to the specific heat at low temperature are nonlinear at the temperature. Although it is difficult to get the exact expression of the specific heat, we know that this nonlinear relation with the temperature is related to the interacting parameters and the spins of the impurities.

The boundary bound states can be formed in the present model in the charge sectors. When $|s_a + c_a| < \frac{1}{2}$, the boundary bound state can be denoted by

$$q^3,0 = \pm i \left( s_a + c_a + \frac{1}{2} \right),$$

$$q^3,1 = \pm \frac{i}{2} \left( s_a + c_a - \frac{1}{2} \right),$$

$$q^3,2 = \pm \frac{i}{2} \left( s_a + c_a + \frac{3}{2} \right)$$

in the charge sectors. It carries the energy:

$$E^{(3)}_{edg} = \frac{3(2s_a + 2c_a + 1)^2(2s_a^2 + 2c_a^2 + 4s_ac_a + 2s_a + 2c_a - 5)}{(s_a + c_a)(s_a + c_a + 1)(2s_a + 2c_a - 3)(2s_a + 2c_a + 5)}.$$ (66)

Obviously, when we substitute the subindex $a$ by $b$, we also get the bound state with three imaginary modes localized at the end of the system. The total momentum of the bound states are zero. Another kind of the boundary bound state with five imaginary modes can be expressed as

$$q^5,0 = \pm i \left( s_a + c_a + \frac{1}{2} \right),$$

$$q^5,1 = \pm \frac{i}{2} \left( 3s_a + 3c_a + \frac{1}{2} \right),$$

$$q^5,2 = \pm \frac{i}{2} \left( 3s_a + 3c_a + \frac{5}{2} \right),$$

$$q^5,3 = \pm i \left( s_a + c_a + \frac{3}{2} \right),$$

$$q^5,4 = \pm i \left( s_a + c_a - \frac{1}{2} \right)$$

when $-\frac{1}{2} < s_a + c_a < -\frac{1}{6}$. It has the energy

$$E^{(5)}_{edg} = \frac{5 \left( 72(s_a + c_a)^4 + 144(s_a + c_a)^3 - 58(s_a + c_a)^2 - 130(s_a + c_a) + 11 \right)}{(s_a + c_a - 1)(s_a + c_a + 2)(6s_a + 6c_a - 1)(6s_a + 6c_a + 7)}.$$ (68)

The imaginary modes with the subindex $b$ also form the boundary bound state. The momentum of the state, which is the sum of the ones of the imaginary modes, is also zero. These bound states are at the ends of the system.

VIII. CONCLUDING REMARKS

Recently, in an interesting work [23], Zvyagin get that the low-energy magnetic behaviors of an impurity in a chain with periodic boundary conditions and with open boundary conditions coincide up to mesoscopic corrections of order of $G^{-1}$ and it was shown that this property is independent of the impurity position in the open chain. From the above discussions, we get the same conclusion. The impurity part of the Hamiltonian for the open boundary condition in Ref. [23] is

$$H_{imp} = |K_{\alpha\beta}/(1+a_0^2)|(J^\alpha_0 J^\beta_1 + H.c.)$$ (69)
with the “weak link” \((1 + u_0^2)^{-1} < 1\) where \(K_{\alpha\beta} = \text{Str} J^\alpha J^\beta\) and \(J_0^{\alpha_1\cdots\alpha_9}\) are the generators of the supersymmetric algebra \(sl(1\vert 2)\). In our model, for the open boundary conditions, the impurity part of the Hamiltonian is

\[
H_{\text{imp}} = J_a S_a \cdot d_a + V_a n_1 + J_b S_G \cdot d_b + V_b n_G.
\]

(70)

\(J_{a,b}\) and \(V_{a,b}\) depend on the interacting parameters \(c_{a,b}\) and the spins \(s_{a,b}\) of the impurities (See relations (11, 12)). The finite size corrections of the ground state due to the impurities are obtained and the properties of the system at the high-temperature limit and the low temperature limit are studied in this paper. We also discussed the boundary bound states which are localized at the edges of the system.

To conclude, we have constructed from the open boundary \(t - J\) model a Hamiltonian with the two magnetic impurities which have arbitrary spins. The impurities are coupled to the ends of the system. The interaction parameters of the impurities with the electrons are parameterized by the constants which related on the potentials of the impurities in order to preserve the integrability of the system. The boundary \(R\) matrices of the impurity model are given explicitly and satisfy the reflection equation. In fact, our impurity model has also the following \(R\) matrices,

\[
R_a(k_j, \sigma_j) = -\exp [ik_j \mp i\theta_a (\pi - k_j)] \frac{2q_j - 2i c_a - i(2s_a + 1) P_{aj}}{2q_j + 2i c_a + i(2s_a + 1) P_{aj}},
\]

(71)

\[
R_b(-k_j, \sigma_j) = -\exp [-ik_j (2G + 1) \mp i\theta_b (\pi - k_j)] \frac{2q_j - 2i c_b - i(2s_b + 1) P_{bj}}{2q_j + 2i c_b + i(2s_b + 1) P_{bj}}
\]

(72)

with \(q_j = \frac{1}{2} \tan \frac{k_j}{2}\). Here the interaction parameters are denoted by

\[
J_a = \frac{2}{(s_a - c_a)(s_a + c_a + 1)},
\]

\[
V_a = -\frac{4c_a^2 - (2s_a + 1)^2 - 3}{4(s_a - c_a)(s_a + c_a + 1)}.
\]

The function \(\theta\) is defined as

\[
\theta_a (k) = \frac{1}{i} \ln \left( \frac{2c_a^2 - 2s_a^2 - 2s_a}{\sqrt{(2c_a^2 - 2s_a^2 - 2s_a) \cos k + 2s_a^2 - 2c_a^2 + 2s_a + 1 + 2ic_a \sin k}} \right).
\]

The parameters \(J_b, V_b\) and the function \(\theta_b (k)\) can be obtained from the above relations with the substitutions of the index \(b\) for \(a\). By comparing the expressions of \(J_{a,b}\) and \(V_{a,b}\) with the ones in section 3, we know that the different expressions of the boundary \(R\) matrices are because of the transformations \(c_a \to -c_a\) and \(c_b \to -c_b\). This reminds us the \(R\) matrices of the impurity model can be written down as the other forms with similar substitutions.

The Hamiltonian including the arbitrary spin impurities is diagonalized exactly by the Bethe ansatz method. The coupled integral equations of the ground state are derived and the ground state properties are discussed in details. The distribution functions of the rapidities are given also. Furthermore, we study the string solution of the Bethe ansatz equations. The integral equations of the excited state are obtained. By minimizing the thermodynamic potential, we get the thermodynamic Bethe ansatz equations. The finite size correction due to the magnetic impurities is obtained for the free energy. It is interesting that this correction is finite when the temperature \(T \to 0\), but have the value related to the magnetization of the impurities and the external magnetic field. We discussed also the properties of the system at the high-temperature limit and the low temperature limit. Two kinds of the boundary bound states are constructed also for the case of \(J = 2\) and \(V = 3/2\). Finally, we point out the impurity contributions are derived in the above discussion under the thermodynamic limits since the Bethe ansatz equations.
can be changed into the coupled integral equations and there solved by a Fourier transform. So, it is an interesting problem to discuss the impurity effects for the finite lattice case. de Vega and Woynarovich have given a method to calculate the leading-order finite size corrections to the ground state energy. By using the finite size scaling technique, the case of some other integrable models such as Hubbard model and Heisenberg spin chain are also studied. It is worth studying the impurity effects in different sectors of the present model further and its critical properties are also interesting topics for further discussions.

APPENDIX A. THE BOUNDARY R MATRICES

When the interacting parameters $J_{a,b}$ and $V_{a,b}$ take the forms

$$J_a = \frac{2}{(s_a - c_a)(s_a + c_a + 1)},$$

(A.1)

$$V_a = \frac{4c_a^2 - (2s_a + 1)^2 + 1}{4(s_a - c_a)(s_a + c_a + 1)}$$

(A.2)

for $J = 2$, $V = -1/2$, the boundary $R$ matrices for the impurity model with the arbitrary spins can be written down as

$$R_a(k_j, \sigma_j) = \exp \left( ik_j \right) \frac{2q_j - 2i c_a - i (2s_a + 1) P_{aj}}{2q_j + 2i c_a + i (2s_a + 1) P_{aj}},$$

(A.3)

$$R_b(-k_j, \sigma_j) = \exp \left[ -ik_j (2G + 1) \right] \frac{2q_j - 2i c_b - i (2s_b + 1) P_{b_j}}{2q_j + 2i c_b + i (2s_b + 1) P_{b_j}}$$

(A.4)

with $q = \frac{1}{2} \cot \frac{k}{2}$. When $J = -2$ and $V = 1/2$, $J_a$ and $V_a$ are

$$J_a = \frac{2}{(s_a + c_a)(s_a + c_a + 1)},$$

(A.5)

$$V_a = \frac{4c_a^2 - (2s_a + 1)^2 + 1}{4(s_a + c_a)(s_a + c_a + 1)}.$$  

(A.6)

When $J = -2$ and $V = -3/2$, we have that

$$J_a = \frac{2}{(s_a + c_a)(s_a - c_a + 1)},$$

(A.7)

$$V_a = \frac{4c_a^2 - (2s_a + 1)^2 - 3}{4(s_a + c_a)(s_a - c_a + 1)}.$$  

(A.8)

The boundary $R$ matrices are same as the relations [13-14].

APPENDIX B. THE BETHE ANSATZ EQUATIONS

From the standard Bethe ansatz procedure, we get the following Bethe ansatz equations:
\[
\exp(2G_{ik}) \frac{\cot \frac{k_j}{2} + 2ic_a + i(2s_a + 1) \cot \frac{k_j}{2} + 2ic_b + i(2s_b + 1)}{
\cot \frac{k_j}{2} - 2ic_a - i(2s_a + 1) \cot \frac{k_j}{2} - 2ic_b - i(2s_b + 1)}
\]

\[
= \prod_{\beta=1}^{M} \frac{\cot \frac{k_j}{2} - 2\lambda_\beta + i \cot \frac{k_j}{2} + 2\lambda_\beta + i}{\cot \frac{k_j}{2} - 2\lambda_\beta - i \cot \frac{k_j}{2} + 2\lambda_\beta - i}, \quad (j = 1, 2, \cdots, N) \quad (B.1)
\]

\[
\lambda_\alpha + ic_a + is_a \lambda_\alpha + ic_b + is_b \lambda_\alpha - ic_a + is_a \lambda_\alpha - ic_b + is_b
\]

\[
\lambda_\alpha + is_a \lambda_\alpha + ic_b - is_b \lambda_\alpha - ic_a - is_a \lambda_\alpha - ic_b - is_b
\]

\[
= \prod_{l=1}^{N} \frac{2\lambda_\alpha - \cot \frac{k_j}{2} + i 2\lambda_\alpha + \cot \frac{k_j}{2} + i}{2\lambda_\alpha - \cot \frac{k_j}{2} - i 2\lambda_\alpha + \cot \frac{k_j}{2} - i}
\]

\[
= \prod_{\beta=1(\beta \neq \alpha)}^{M} \frac{\lambda_\alpha - \lambda_\beta + i \lambda_\alpha + \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i \lambda_\alpha + \lambda_\beta - i}, \quad (\alpha = 1, 2, \cdots, M) \quad (B.2)
\]

for the case of \(J = 2\) and \(V = -1/2\). When \(J = -2\) and \(V = 1/2\), the Bethe ansatz equations are the same as the above ones, but the energy changes sign. The Bethe ansatz equations for the case of \(J = -2\) and \(V = -3/2\) are the same as those in [24][23] and the energy changes sign also.
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