A MULTIPLICITY ONE THEOREM FOR GL$_n$ AND SL$_n$ OVER COMPLETE DISCRETE VALUATION RINGS

SHIV PRAKASH PATEL AND POOJA SINGLA

ABSTRACT. Let $\mathfrak{o}$ be the ring of integers of a non-archimedean local field with maximal ideal $\mathfrak{p}$ and finite residue field of characteristic $p$. For $\ell \geq 1$, let $\mathfrak{o}_\ell = \mathfrak{o}/(\mathfrak{p}^\ell)$, $G(\mathfrak{o}_\ell) = \text{GL}_n(\mathfrak{o}_\ell)$ or $\text{SL}_n(\mathfrak{o}_\ell)$ and $U(\mathfrak{o}_\ell) \subset G(\mathfrak{o}_\ell)$ be the subgroup of upper triangular unipotent matrices. We prove that the induced representation $\text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta)$ of $G(\mathfrak{o}_\ell)$ obtained from a non-degenerate character $\theta$ of $U(\mathfrak{o}_\ell)$ is multiplicity free. This is analogous to the multiplicity one theorem in Gelfand-Graev representation for Chevalley groups. We also prove that for all $G(\mathfrak{o}_\ell)$, except for $\text{SL}_n(\mathfrak{o}_\ell)$ with even $p$ or $p | n$, an irreducible representation of $G(\mathfrak{o}_\ell)$ is a constituent of $\text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta)$ if and only if it is regular.

1. Introduction

Let $F$ be a non-archimedean local field and $\mathfrak{o}$ the ring of integers of $F$ such that residue field $k$ is finite of characteristic $p$ and $|k| = q$. Let $\mathfrak{p}$ be the prime ideal of $\mathfrak{o}$ and $\mathfrak{p}$ a generator of $\mathfrak{p}$. Let $G$ be a split reductive group scheme defined over $\mathfrak{o}$ and $G(\mathfrak{o})$ denotes the set of $\mathfrak{o}$-points of $G$. For $\ell \geq 1$, let $\mathfrak{o}_\ell = \mathfrak{o}/(\mathfrak{p}^\ell)$. The groups $G(\mathfrak{o}_\ell)$ are profinite therefore for every continuous finite dimensional irreducible representation of $G(\mathfrak{o})$ factors through the principal congruence quotient $G(\mathfrak{o}_\ell)$. So we shall only be dealing with finite dimensional complex representations of the groups $G(\mathfrak{o}_\ell)$. Let $U$ be the unipotent radical of a Borel subgroup $B$ of $G$. Fix $\theta : U(\mathfrak{o}_\ell) \to C^\times$ a “non-degenerate” character of $U(\mathfrak{o}_\ell)$. An explicit description of a non-degenerate character of $U(\mathfrak{o}_\ell) = U_n(\mathfrak{o}_\ell)$ is given below for $G = \text{GL}_n$ and $\text{SL}_n$.

Definition 1.1. An irreducible representation $\pi$ of $G(\mathfrak{o}_\ell)$ is said to admit a Whittaker model if $\pi$ is a subrepresentation of the induced representation space $\text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta)$, where $\text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta) = \{f : G(\mathfrak{o}_\ell) \to C | f(ug) = \theta(u)f(g) \text{ for all } u \in U(\mathfrak{o}_\ell), g \in G(\mathfrak{o}_\ell)\}$ on which $G(\mathfrak{o}_\ell)$ acts by right translation.

Using Frobenius reciprocity we have $\text{Hom}_{G(\mathfrak{o}_\ell)}(\pi, \text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta)) \cong \text{Hom}_{U(\mathfrak{o}_\ell)}(\pi U(\mathfrak{o}_\ell), \theta)$. We say that $\pi$ has at most one Whittaker model if the dimension $\text{dim}_{C} \text{Hom}_{G(\mathfrak{o}_\ell)}(\pi, \text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta))$ is at most one. We say the representation $\text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta)$ is multiplicity free, if every irreducible representation of $G(\mathfrak{o}_\ell)$ has at most one Whittaker model. The main questions which we consider are the following.

Question 1.2. (a) Which irreducible representations of $G(\mathfrak{o}_\ell)$ admit a Whittaker model?

(b) What is the dimension of the intertwiner space $\text{Hom}_{G(\mathfrak{o}_\ell)}(\pi, \text{Ind}^{G(\mathfrak{o}_\ell)}_{U(\mathfrak{o}_\ell)}(\theta))$ for a representation $\pi$ of $G(\mathfrak{o}_\ell)$?

These questions have already appeared in literature in many contexts. For example, for $\ell = 1$ and $G = \text{GL}_n$, one knows from the work of Gelfand and Graev [1,2] that for a non-degenerate character $\theta$
of $U_n(F)$, the induced representation $\text{Ind}_{U_0(F)}^{G(F)}(\theta)$ (now a days also called Gelfand-Graev representation) is multiplicity free and further every cuspidal representation of $GL_n(F)$ has a Whittaker model. These results were infact proved by them more generally for finite Chevalley groups. For $GL_n(F)$, where $F$ is a local field, Shalika [10] Theorem 1.6, 2.1] proved that every (smooth) irreducible representation has at most one Whittaker model and generalized this to split and quasi-split groups as well. The case of local fields has several applications to Fourier coefficients of automorphic forms on $G$.

Let $G = GL_n$ or $SL_n$ and $U(\mathfrak{o}_\ell)$ be the group of upper triangular unipotent matrices in $G(\mathfrak{o}_\ell)$. A non-trivial one dimensional representation $\varphi : \mathfrak{o}_\ell \rightarrow \mathbb{C}^\times$ such that $\varphi(\sigma^{-1}e_\ell) \neq 1$ is called a non-degenerate character of $\mathfrak{o}_\ell$. Define a one dimensional representation $\theta : U(\mathfrak{o}_\ell) \rightarrow \mathbb{C}^\times$ by
\[
\theta((x_{ij})) := \varphi(x_{12} + x_{33} + \cdots + x_{(n-1)n}).
\]
Any one dimensional representation of $U(\mathfrak{o}_\ell)$ of the form $\theta$ corresponding to a non-degenerate character of $\mathfrak{o}_\ell$ will be called "non-degenerate" character of $U(\mathfrak{o}_\ell)$. This is a natural generalization of a non-degenerate character of $U_n(F)$ as known in the literature. For a non-degenerate character $\theta$ of $U(\mathfrak{o}_\ell)$, we consider the representation space $\text{Ind}_{U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\theta)$ and prove the following.

**Theorem 1.3.** For $G = GL_n$ or $SL_n$, the $G(\mathfrak{o}_\ell)$-representation space $\text{Ind}_{U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\theta)$ is multiplicity free for every non-degenerate character $\theta$ of $U(\mathfrak{o}_\ell)$.

The proof of this theorem is included in Section 2.2. The present consideration of the irreducible representations of $GL_n(\mathfrak{o}_\ell)$ with $\ell \geq 2$ has appeared in the work of Hill [4, Proposition 5.7] where he proved that the space $\text{Ind}_{U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)}(\theta)$ is multiplicity free for even $\ell$ leaving the case when $\ell$ is an odd number. Hill also proved that for $GL_n(\mathfrak{o}_\ell)$ an irreducible representation admits a Whittaker model if and only it is regular. Furthermore, for $\ell$ odd and $G(\mathfrak{o}_\ell) = GL_n(\mathfrak{o}_\ell)$ Hill [4, Proposition 5.7] managed to prove that certain class of irreducible regular representations of $GL_n(\mathfrak{o}_\ell)$ has at most one Whittaker model (see also Takase [15], where he notes that Hill’s arguments work only for the semi-simple split regular case and not for all split regular as stated in [4, Proposition 5.7]). We briefly recall the definition of regular representations of $G(\mathfrak{o}_\ell)$, for more details, see Section 3 and also [5].

Let $\mathfrak{g}$ denote the Lie algebra scheme of $G$ and, for every positive integer $r$, let $K'_r = \text{Ker}(G(\mathfrak{o}_\ell) \mapsto G(\mathfrak{o}^r/\mathfrak{o}^{r+1}))$ be the $r$-th principal congruence subgroup of $G(\mathfrak{o}_\ell)$. There are $G(\mathfrak{k})$-equivariant isomorphisms $K'_r/K'_r^{r+1} \cong (\mathfrak{g}(\mathfrak{k}), +)$ and a $G(\mathfrak{k})$-equivariant isomorphism $x \mapsto \varphi_x$ between $\mathfrak{g}(\mathfrak{k})$ and its Pontryagin dual $\mathfrak{g}(\mathfrak{k})^\vee = \text{Hom}_{\mathbb{C}}(\mathfrak{g}(\mathfrak{k}), \mathbb{C}^\times)$ (see [6, Section 2.3]). An element $x \in \mathfrak{g}(\mathfrak{k})$ is called regular(cyclic) if the characteristic polynomial of $x$ is equal to its minimal polynomial. A one dimensional representation $\varphi_x \in \mathfrak{g}(\mathfrak{k})^\vee$ is called regular if $x$ is regular element.

**Definition 1.4** (Regular representations). An irreducible representation $\rho$ of $G(\mathfrak{o}_\ell)$ is called regular if the orbit of its restriction to $K'_{\ell}^{-1} \cong \mathfrak{g}(\mathfrak{k})$ consists of regular one dimensional representations.

It should be noted that the construction of regular representation, when $\ell$ is odd or $G = SL_n$, was not completely known to Hill. Recently, for $p$-odd the construction of all regular representations for $SL_n(\mathfrak{o}_\ell)$ with $p \nmid n$ and $GL_n(\mathfrak{o}_\ell)$ has appeared in [3]. For general $p$, an independent construction of regular representations of $GL_n(\mathfrak{o}_\ell)$ is given by Stassen-Stevens [14]. The regular representations have also been constructed when $G$ is a unitary, orthogonal or symplectic group for odd $p$, see [6,11]. The case of $G = SL_n$ with $p \mid n$ is not yet completely known even for $n = 2$, see [3] for known results regarding this. In this article, we complete Hill’s results for $GL_n(\mathfrak{o}_\ell)$ with $\ell$ odd as well as generalise these to $SL_n(\mathfrak{o}_\ell)$ for $p \nmid n$. In particular, we prove the following.

**Theorem 1.5.** Let $G(\mathfrak{o}_\ell)$ be either $SL_n(\mathfrak{o}_\ell)$ with $(p, 2) = (p, n) = 1$ or $GL_n(\mathfrak{o}_\ell)$ and $\ell \geq 2$. 
(1) An irreducible representation of $G(o_l)$ admits a Whittaker model if and only if it is regular.

(2) Every irreducible representation of $G(o_l)$ has at most one Whittaker model.

The proof of this theorem is given in Section 4 and depends on the description and construction of the regular representations as given in [6][14], see also Section 3 where we recall briefly this construction. The proof of the second part is quite indirect. Usually for a proof of multiplicity one theorems one expects to use the trick of Gelfand-Kazdan for which one requires a good description of double cosets $U(o_l)\backslash G(o_l)/U(o_l)$. In the present case such a description is known only for $G = GL_2(o_l)$ and $GL_3(o_l)$ (see [14]) and is non-trivial. This description is not known in general and is believed to be quite complicated. Therefore our proof completely avoids the approach using Gelfand-Kazdan trick but is more along the ideas of Hill [Loc. cit.]

To prove the second part of Theorem 1.6, we count the sum of dimensions of all regular representation. This can be achieved following the steps carefully in the construction of the regular representations. In our case, it turns out that this sum is equal to the dimension of the Whittaker space, i.e. dimension of the induced space $\text{Ind}_{U(o_l)}^{G(o_l)}$. At last in Section 5, we discuss the example of $GL_2(o_l)$ in some detail. The construction of irreducible representations of $GL_2(o_l)$ has already appeared in [8][13][14]. In case of $GL_2(\mathbb{F}_q)$, it is well known that an irreducible representation of $GL_2(\mathbb{F}_q)$ admits a Whittaker model if and only if it has dimension greater than one. We generalize this to $GL_2(o_l)$ as follows. An irreducible representation $\rho$ of $G(o_l)$ is called primitive if the orbit of its restriction to $K_{e-1}$ does not contain a one dimensional representation $\varphi_e \in \mathfrak{g}(k)^{\vee}$ where $x$ is a scalar matrix. In other words a representation $\rho$ is primitive if neither $\rho$ nor any of the twists of $\rho$ by a one dimensional representation is obtained as a pull back of an irreducible of $GL_n(o_{-1})$. It is to be noted that by definition every regular representation is primitive but converse need not to be true. However for $GL_2(o_l)$ it is well known that an irreducible representation of regular if and only if it is primitive (see Section 5). This in particular combined with Theorem 1.6 gives the following result regarding GL2.

**Theorem 1.6.** An irreducible representation of $GL_2(o_l)$ for $\ell \geq 2$ admits a Whittaker model if and only if it is primitive.

2. **Regular elements of g(o_l)**

In this case, we collect some facts regarding regular elements of $g(o_l)$, where either $G = SL_n$ with $(p, n) = (p, 2) = 1$ or $G = GL_n$ and $g$ denote the Lie algebra scheme of $G$.

For any $i \leq r$, there exists a natural projection $\rho_{r,i} : o_r \to o_i$ which is a ring homomorphism. By applying entry wise, we obtain a projection $\rho_{r,i} : g(o_l) \to g(o_i)$. For any $x \in g(o_i)$, the image $\rho_{r,i}(x)$ is denoted by $\check{x}$.

**Definition 2.1** (Regular element of $g(o_l)$). An element $x \in g(o_l) \subseteq M_n(o_l)$ is called regular if there exists an element $v \in o_r^{\text{re}}$ such that the set $\{v; x(v), x^2(v), \ldots, x^{n-1}(v)\}$ is a basis of free $o_r$-module $o_r^{\text{re}}$. The vector $v$ is called a cyclic vector of $x$.

The following lemmas summarize the properties of regular matrices in $M_n(o_l)$ and $sl_n(o_l)$ that we require in the sequel.

**Lemma 2.2.** Let $x \in M_n(o_l)$ and let $r \geq 1$. The following are equivalent.

1. The element $x \in M_n(o_l)$ is regular.
2. The projection of $x$ to $M_n(o_i)$ for every $1 \leq i \leq r$ is regular.
3. The centralizer $C_{M_n(o_l)}(x) = \{y \in M_n(o_l) | xy = yx\}$ of $x$ in $M_n(o_l)$ is abelian.
4. $C_{M_n(o_l)}(x) = o_r[x]$, where $o_r[x]$ is a $o_r$-subalgebra of $M_n(o_l)$ generated by $[I, x, x^2, \ldots, x^r]$. 

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Lemma 2.3. An element \( x \in M_n(\mathbb{F}_q) \) is regular if and only if characteristic polynomial of \( x \) is equal to its minimal polynomial.

Proof. The proof of this follows from the definition of regular \( x \in M_n(\mathbb{F}_q) \) (appears in for example, Suprunenko-Tyshkevich [15, Theorem 5]). \( \square \)

Recall \( \mathbb{F}_q \). We use \( d_g \) to denote the dimension of \( \mathbb{F}_q \)-vector space \( g(\mathbb{F}_q) \). For \( x \in g(\mathbb{F}_q) \), the dimension of the centralizer algebra \( C_{g(\mathbb{F}_q)}(x) = \{ y \in g(\mathbb{F}_q) \mid xy = yx \} \) as \( \mathbb{F}_q \)-vector space is denoted by \( d_{g(\mathbb{F}_q)}(x) \). Note that by Lemmas 2.2 and 2.3 for every regular \( x \in M_n(\mathbb{F}_q) \) and \( y \in sl_n(\mathbb{F}_q) \) we have \( d_{M_n(\mathbb{F}_q)}(x) = n \) and \( d_{sl_n(\mathbb{F}_q)}(y) = n - 1 \).

Lemma 2.4. Let \( x \in g(\mathbb{F}_q) \) be a regular matrix and \( C_{g(\mathbb{F}_q)}(x) = \{ y \in g(\mathbb{F}_q) \mid xy = yx \} \) be the centralizer of \( x \) in \( g(\mathbb{F}_q) \). Then \( |C_{g(\mathbb{F}_q)}(x)| = q^{d_{g(\mathbb{F}_q)}(x)} \).

Proof. For \( g = M_n(\mathbb{F}_q) \) and any regular \( x \in M_n(\mathbb{F}_q) \), by Lemmas 2.2 and 2.3 we have \( |C_{M_n(\mathbb{F}_q)}(x)| = q^{d_{M_n(\mathbb{F}_q)}(x)} = q^{d_{g(\mathbb{F}_q)}(x)} \). For \( g = sl_n \) and \( x \) regular, by Lemmas 2.2 and 2.3 we note that \( |C_{sl_n}(x)| = q^{d_{sl_n}(x)} = q^{d_{g(\mathbb{F}_q)}(x)} \). \( \square \)

Definition 2.5. (canonical-regular element) Let \( x \in M_n(\mathbb{F}_q) \) be of the following form,

\[
(2) \quad x = \begin{pmatrix}
0 & 0 & \cdots & 0 & x_1 \\
1 & 0 & \cdots & 0 & x_2 \\
0 & 1 & \cdots & 0 & x_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & x_n
\end{pmatrix}
\]

where \( x_i \in \mathbb{F}_q \) for \( 1 \leq i \leq n \). Then \( x \in M_n(\mathbb{F}_q) \) is regular and is called canonical-regular element.

Lemma 2.6. The following are true for the set of regular elements of \( g(\mathbb{F}_q) \).

1. The set of regular elements in \( g(\mathbb{F}_q) \) is invariant under the conjugation action of \( G(\mathbb{F}_q) \).
2. Every conjugacy class of regular elements of \( g(\mathbb{F}_q) \) contains a unique canonical-regular element.
3. The number of regular conjugacy classes of \( g(\mathbb{F}_q) \) is \( q^{d_{g(\mathbb{F}_q)}(x)} \), where \( x \in g(\mathbb{F}_q) \) is any regular matrix.

Proof. The proof of (1) follows by Lemma 2.2. A proof of (2) and (3) for \( G = GL_n \) and \( g = M_n \) follows from McDonald [7, p.417-419] (see also Hill [4, Remark 3.10]).

For \( G = SL_n, g = sl_n \) and \( p \nmid n \), we first note that for \( x_1, x_2 \in sl_n(\mathbb{F}_q) \), there exists \( g \in GL_n(\mathbb{F}_q) \) such that \( g_1 x_1 g^{-1} = x_2 \) if and only if there exists \( g' \in SL_n(\mathbb{F}_q) \) such that \( g' x_1 g'^{-1} = x_2 \). As \( p \nmid n \), therefore by Hensel’s Lemma, for every \( g \in GL_n(\mathbb{F}_q) \), there exists \( z \in \mathbb{F}_q \) such that \( z = \det(g)^{1/n} \). Consider the scalar matrix \( a = \frac{1}{\det(g)^{1/n}}(I_n) \in GL_n(\mathbb{F}_q) \) and \( g' = ga \in SL_n(\mathbb{F}_q) \) then \( g_1 a g^{-1} = a_2 \) implies \( g' a (g')^{-1} = a_2 \). This in particular implies that a \( GL_n(\mathbb{F}_q) \)-similarity class in \( sl_n(\mathbb{F}_q) \) is also a similarity class for \( SL_n(\mathbb{F}_q) \). From this (2) and (3) for \( g = sl_n \) follow easily from the corresponding results for \( g = M_n \). \( \square \)

Lemma 2.7. Let \( x \in M_n(\mathbb{F}_q) \) be canonical-regular. Then \( C_{g(\mathbb{F}_q)}(x) \cap U(\mathbb{F}_q) = C_{G(\mathbb{F}_q)}(x) \cap U(\mathbb{F}_q) = [I_n] \)

Proof. A proof of this follows from Hill [4, Theorem 3.6, Corollary 3.7]. \( \square \)
Proof. For canonical-regular matrix $x$, we note that for every $1 \leq i \leq n - 1$, there exists $1 \leq j, k \leq n$ such that $j > k$ and $(j, k)^{th}$ entry of $x'$ is non-trivial. Now result follows by the fact that $C_{G_{0}(n)}(x) \subseteq C_{\hat{G}_{0}(n)}(x)$, by Lemma 2.2 the set $C_{G_{0}(n)}(x)$ is generated by $\{I, x, x^{2}, \ldots, x^{n-1}\}$ as $O_{r}$-algebra and for every $y \in U_{r}(n)$, the $(j, k)^{th}$ entry of $y$ is zero for $j > k$.

Lemma 2.8. Let $x = (x_{ij}) \in M_{n}(O_{r})$ be such that $x_{ij} = 1$ for all $i, j$ such that $i = j + 1$ and $x_{ij} = 0$ for all $i, j$ such that $i - j \geq 2$. Then $x$ is regular.

Proof. By Lemma 2.2 it is enough to prove that $\bar{x}$ is regular for all above $x$. For that it is easy to see that $v = (1, 0, 0, \ldots, 0) \in O_{1}^{\infty}$ satisfies that $\{v, xv, x^{2}v, \ldots, x^{n-1}v\}$ generates the $n$-dimensional vectors space $O_{1}^{\infty}$.

3. Construction of Regular Representations of $G(O_{r})$

In this section we follow [6, 14] to give a brief outline of the construction of regular representations of $G(O_{r})$, where either $G = SL_{n}$ with $(p, n) = (p, 2) = 1$ or $G = GL_{n}$.

Recall regular representations of $G(O_{r})$ is defined as below. For any $i \leq \ell$, there exists a natural projection $\rho_{(\ell)} : O_{r} \to O_{r}$ which is a ring homomorphism. By applying entry wise, we obtain a surjective group homomorphism $\rho_{(\ell)} : G(O_{r}) \to G(O_{r})$. For any $x \in G(O_{r})$, the image $\rho_{(\ell)}(x)$ is denoted by $\hat{x}$. Let $K_{\ell}^{r} = \ker(\rho_{(\ell)})$. These are called $\ell$-th congruence subgroups of $G(O_{r})$. For any $i \in \mathbb{N}$, let $M_{n}(O_{r})$ be the set of $n \times n$ matrices with entries from $O_{r}$. It is easy to note that for $i \geq \ell/2$, the group $K_{\ell}^{r}$ is isomorphic to an abelian additive subgroup, say $g(O_{r-\ell})$, of $M_{n}(O_{r-\ell})$. Next we fix a one dimensional representation $\varphi : O_{r} \to C^{*}$ such that $\varphi(\sigma^{i-1}O_{r}) \neq 1$. For any $i \leq \ell/2$ and $x \in g(O_{r})$, let $\hat{x} \in g(O_{r})$ be an arbitrary lift of $x$ satisfying $\rho_{(\ell)}(\hat{x}) = x$. Define $\varphi_{x} : I + \sigma^{i-1}g(O_{r}) \to C^{*}$ by

$$\varphi_{x}(I + \sigma^{i-1}y) = \varphi(\sigma^{i-1}tr(\hat{xy})),$$

for all $I + \sigma^{i-1}y \in K_{\ell}^{r-\ell}$. Then $\varphi_{x}$ is easily seen to be a well defined one dimensional representation of $K_{\ell}^{r-\ell}$. Further it is easy to see that for $i \geq \ell/2$ the following duality of abelian group $K_{\ell}^{r}$ and $g(O_{r-\ell})$ holds.

(3) $g(O_{r-\ell}) \cong K_{\ell}^{r} : x \mapsto \varphi_{x}$ where, $\varphi_{x}(y) = \varphi(\sigma^{i-1}tr(\hat{xy}))$, where $\hat{x} \in g(O_{r})$ is an arbitrary lift of $x \in g(O_{r-\ell})$. We say a one dimensional representation $\varphi_{x} \in K_{\ell}^{r-\ell}$ for $i \geq \ell/2$ is regular if and only if $x \in M_{n}(O_{r-\ell/2})$ is a regular matrix.

Lemma 3.1 (Characterisation of Regular representation). An irreducible representation $\rho \in G(O_{r})$ is regular if and only if its restriction to $K_{\ell}^{r}$ for any $i \geq \ell/2$ is a direct sum of regular characters.

Proof. We note that by above discussion, every regular one dimensional representation of $K_{\ell}^{r}$ for $i \geq \ell/2$ is of the form $\varphi_{x}$ for some regular $x \in g(O_{r-\ell})$. By Lemma 2.2 $x \in g(O_{r-\ell})$ is regular if and only if $\hat{x}$ is and therefore result follows by the definition of regular representation.

For a group $G$, the set of all inequivalent irreducible representations of $G$ is denoted by $\text{Irr}(G)$. The set of one dimensional representation of an abelian group $A$ is also denoted by $\hat{A}$. For a normal subgroup $N$ of $G$ and an irreducible representation $\phi$ of $N$, the set of all $\rho \in \text{Irr}(G)$ such that the restriction $\rho|_{N}$ of $\rho$ to $N$ has $\phi$ as a non-trivial constituent is denoted by $\text{Irr}(G \mid \phi)$.

Now we summarize the steps of construction of regular representations of $G(O_{r})$. For $G(O_{r})$ with odd $p$ we will outline the method as given in [6] and we follow [14] for $GL_{n}(O_{r})$ with $p = 2$. It might be
useful to look at Figures 1 and 2 while going through the steps of construction. For more details on this see loc cit. The construction in both cases is based on the tools of abstract Clifford theory and orbits. See [5, Chapter 6] for general results and their proofs regarding Clifford theory and [12, Theorem 2.1] for the precise statements related to Clifford theory that are used in this article. The main point of the construction is that in good cases, for any regular one dimensional representation \( \phi_x \) of \( \phi_{\ell}^{\ell/2} \), the inertia group \( I_{\ell}(\varphi_e) = \{ g \in G(\ell) \mid \varphi_e^g \equiv \varphi_e \} \) although non-abelian but still is nice in the sense that it is a product of an abelian group with a congruence subgroup and therefore all the representations of the inertia group lying above \( \varphi_e \) can be constructed. This combined with Clifford theory gives the construction of all regular irreducible representations of \( G(\ell) \).

3.1. \( \ell = 2m \). The construction in this case is easy. So we give almost all the details towards the proof. Let \( \varphi_x \in \phi_{\ell}^{\ell} \) be a regular one dimensional representation of \( K_{\ell}^m \) for \( x \in g(\ell) \). Then the following are true.

E.1 Let \( I_{\ell}(\varphi_e) = \{ g \in G(\ell) \mid \varphi_e^g \equiv \varphi_e \} \) be the inertia group of \( \varphi_e \) in \( G(\ell) \).

1. Then \( I_{\ell}(\varphi_e) = C_{G_{\ell}^{\ell}}(\xi)K_{\ell}^m \), where \( \bar{x} \in \varphi_e \) is any lift of \( x \) to \( g(\ell) \).
2. Although the inertia group \( I_{\ell}(\varphi_e) \) is not abelian but the quotient \( I_{\ell}(\varphi_e)/K_{\ell}^m \equiv C_{G_{\ell}^{\ell}}(\bar{x}) \) is abelian by Lemma 2.2.

E.2 Let \( \delta \) be a one dimensional representation of \( C_{G_{\ell}^{\ell}}(\bar{x}) \) extending \( \varphi_2 : K_{\ell}^{m-1}C_{G_{\ell}^{\ell}}(\bar{x}) \). The existence of this \( \delta \) follows easily because the group \( C_{G_{\ell}^{\ell}}(\bar{x}) \) is abelian. Define \( \tilde{\varphi}_e : C_{G_{\ell}^{\ell}}(\bar{x}) K_{\ell}^m \rightarrow \mathbb{C}^x \) by \( \tilde{\varphi}_e(ab) = \delta(a)\varphi_e(b) \) for all \( a \in C_{G_{\ell}^{\ell}}(\bar{x}) \) and \( b \in K_{\ell}^m \). It can be easily shown that \( \tilde{\varphi}_e \) is a well defined one dimensional representation of \( I_{\ell}(\varphi_e) \) that extends the representation \( \varphi_e \).

E.3 Let \( \rho \in \text{Irr}(G(\ell) \mid \varphi_e) \) be a regular representation of \( G(\ell) \), then there exists an extension \( \tilde{\varphi}_e \) of \( \varphi_e \) to \( I_{\ell}(\varphi_e) \) such that \( \rho \equiv \text{Ind}_{I_{\ell}(\varphi_e)}^{G(\ell)}(\tilde{\varphi}_e) \). This follows by E.1, E.2 combined with Clifford theory.

E.4 We have \( |\text{Irr}(G(\ell) \mid \varphi_e)| = |C_{G_{\ell}^{\ell}}(\bar{x})| \).

E.5 Every \( \rho \in \text{Irr}(G(\ell) \mid \varphi_e) \) has dimension \( \frac{|G(\ell)|}{|C_{G_{\ell}^{\ell}}(\bar{x})||K_{\ell}^m|} \).

3.2. \( \ell = 2m + 1 \). The construction for this case is involved as compared to \( \ell = 2m \) case. So here we highlight what is important for us and refer the reader to [Loc. cit.] for more details. Let \( \varphi_e \in K_{\ell}^{m+1} \) be a regular one dimensional representation of \( K_{\ell}^{m+1} \) for \( x \in g(\ell) \). Let \( \bar{x} \in g(\ell) \) be a lift of \( x \) to \( g(\ell) \). For \( p \neq 2 \), either \( G = \text{SL}_n \) with \( p \nmid n \) or \( G = \text{GL}_n \). For \( p = 2 \), we consider \( G = \text{GL}_n \).

O.1 Define groups \( H_{\ell}^p \) and \( R_{\ell} \) as follows.

\[
H_{\ell}^p = \begin{cases} 
K_{\ell}^p, & p \neq 2 \\
(K_{\ell}^p \cap C_{G_{\ell}^{\ell}}(\bar{x}))K_{\ell}^p, & p = 2.
\end{cases}
\]

\[
R_{\ell} = \begin{cases} 
(K_{\ell}^p \cap C_{G_{\ell}^{\ell}}(\bar{x}))K_{\ell}^{m+1}, & p \neq 2 \\
(K_{\ell}^p \cap C_{G_{\ell}^{\ell}}(\bar{x}))K_{\ell}^{m+1}, & p = 2.
\end{cases}
\]

O.2 The following are true for groups \( H_{\ell}^p \) and \( R_{\ell} \).

1. The group \( R_{\ell} \) is a normal subgroup of \( H_{\ell}^m \).
(2) The quotient group $R/\mathcal{K}_{m+1}$ is abelian.

(3) The one dimensional representation $\varphi_x \in \hat{\mathcal{K}}_{m+1}$ extends to $R$

O.3 Each extension of $\varphi_x$ to $R$ satisfies the following:

(1) Each extension of $\varphi_x$ to $R$ is stable under $\mathcal{K}_m$.

(2) Each extension of $\varphi_x$ to $R$ determines a non-degenerate bilinear form on $H_{m}/R$ and $J_x$ is a well chosen maximal isotropic space with respect to this bilinear form. Parallel to the method of construction of “Heisenberg” groups, every extension of $\varphi_x$ to $R$ determines a unique irreducible representation of $H_{m}$ of dimension $q^{\frac{(d-g_{m/2})}{2}}$, see [6, Section 3] for $p \neq 2$ and [14, Section 3] for $p = 2$ for more details.

(3) Let $\bar{\varphi}_x$ be an extension of $\varphi_x$ to $R$ and $\sigma \in \text{Irr}(H_{m}/\varphi_x)$ be unique irreducible representation determined by $\bar{\varphi}_x$. Then,

$$\sigma |_{R} = \bar{\varphi}_x + \cdots + \bar{\varphi}_x.$$ 

O.4 Let $I_{G(\varphi)} = \{ g \in G(\varphi) \mid \sigma^g \cong \sigma \}$ be the inertia groups of $\sigma \in \text{Irr}(H_{m}/\varphi_x)$. Then the following are true:
We deduce that for matrices for which the entries in the strictly upper triangular part belong to $U_{\text{procity Hom}}$. Section 4.2.

Proof. Every $\sigma \in \text{Irr}(U_p^g)$ extends to the inertia group $I_{G_{\text{procity Hom}}}$. In particular, every such extension induces irreducibly to $G_{\text{procity Hom}}$ and gives rise to an irreducible regular representation of $G_{\text{procity Hom}}$. This is slightly technical part and depends on the structure of groups $R_\ell$, $J_\ell$, $H_\ell^m$ and $I_{G_{\text{procity Hom}}}$. 

O.5 Combining above all, there exists a bijection $\text{Irr}(G_{\text{procity Hom}}) \leftrightarrow C_{G_{\text{procity Hom}}}(x)$.

O.6 Every $\rho \in \text{Irr}(G_{\text{procity Hom}})$ for $\sigma \in \text{Irr}(K_{\text{procity Hom}}) \varphi_x$ has dimension $q^d_{\rho_{\text{procity Hom}}} = \frac{q^d_{\rho_{\text{procity Hom}}}}{|C_{G_{\text{procity Hom}}}(x)|}$. 

O.7 Altogether, we have the following:

1) $|\text{Irr}(G_{\text{procity Hom}}) \varphi_x| = q^d_{\rho_{\text{procity Hom}}} |C_{G_{\text{procity Hom}}}(x)|$.

2) Every $\rho \in \text{Irr}(G_{\text{procity Hom}}) \varphi_x$ has dimension $q^d_{\rho_{\text{procity Hom}}} = \frac{|G_{\text{procity Hom}}|}{|C_{G_{\text{procity Hom}}}(x)|}$. 

4. Proof of Theorems 1.3 and 1.5

In this section first we prove Theorem 1.3 in Section 4.1 and then use this to prove Theorem 1.5 in Section 4.2.

4.1. Proof of Theorem 1.5 Throughout this section we assume that $G$ satisfies that either $G = SL_n$ with $(p, n) = (p, 2) = 1$ or $G = GL_n$. For $1 \leq k < \ell$, $U(\sigma^k_0)$ for the group of unipotent upper triangular matrices for which the entries in the strictly upper triangular part belong to $\sigma^k_0$.

Theorem 4.1. An irreducible representation $\pi$ of $G(\ell)$ admits a Whittaker model if and only if it is regular.

Proof. Assume that $\pi$ admits a Whittaker model then $\text{Hom}_{G(\ell)}(\pi, \text{Ind}_{U(\ell)}^{G(\ell)}(\theta)) \neq 0$. By Frobenius reciprocity $\text{Hom}_{U(\ell)}(\pi, \theta) \neq 0$. In particular, $\text{Hom}_{U(\sigma^{\ell-1} \circ \ell)}(\pi, \theta) \neq 0$. Since $U(\sigma^{\ell-1} \circ \ell) \subset K_{\ell}$ and $\pi|_{K_{\ell-1}} = \oplus \varphi_x$ where $x$ runs over a certain conjugacy class in $g(\mathbb{F}_q)$ determined by $\pi$ and $m_x$ is multiplicity of $\varphi_x$ in $\pi|_{K_{\ell-1}}$. We will prove that the conjugacy class of $x$ appearing in $\pi|_{K_{\ell-1}}$ is a conjugacy class of regular elements and this will prove that $\pi$ is a regular representation.

We have,

$$\text{Hom}_{U(\sigma^{\ell-1} \circ \ell)}(\pi, \theta) = \bigoplus_x m_x \text{Hom}_{U(\sigma^{\ell-1} \circ \ell)}(\varphi_x, \theta).$$

Thus there exists $x \in g(\mathbb{F}_q)$ such that $\text{Hom}_{U(\sigma^{\ell-1} \circ \ell)}(\varphi_x, \theta) \neq 0$. We identify $U(\sigma^{\ell-1} \circ \ell)$ with the strictly upper triangular matrices in $g(\mathbb{F}_q)$. By the definition of $\theta$ and the fact that $\text{Hom}_{U(\sigma^{\ell-1} \circ \ell)}(\varphi_x, \theta) \neq 0$, we must have that for every strictly upper triangular matrix $y \in g(\mathbb{F}_q)$ such that $\tilde{y} = (y_{ij}) \in g(\mathbb{F}_q)$,

$$\varphi_x(t + \sigma^{\ell-1} y) = \varphi(t(\text{tr}(y))) = \varphi(y_{12} + 2y_{23} + \cdots + y_{(n-1)n}).$$

We deduce that

$$x = \begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$
where the entries with * means that it is possibly non-zero. Thus by Lemma 2.3, the matrix $x \in \mathfrak{g}(\mathbb{F}_q)$ is a regular matrix.

Now we prove that a regular representation admits a Whittaker model. This will be proved separately when $\ell$ is even and odd.

Proof of the converse of Theorem 2.7 for $G(\mathbb{F}_\ell)$ for even $\ell$. Let $\ell = 2m$. Since $\pi$ is a regular representation of $G(\mathbb{F}_\ell)$, by E.3, there exists a regular one dimensional representation $\varphi_x$ of $K_\ell^m$ and its extension $\bar{\varphi}_x$ to $I_{G(\mathbb{F}_\ell)}(\varphi_x)$ such that

$$\pi \cong \text{Ind}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x).$$

Our aim is prove that $\text{Hom}_{G(\mathbb{F}_\ell)}(\pi, \text{Ind}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x)) \neq 0$. By Lemma 2.6, we can assume that $x$ is canonical-regular. Write $I(x)$ for the inertia group $I_{G(\mathbb{F}_\ell)}(\varphi_x)$. Using Frobenius reciprocity we have

$$\text{Hom}_{G(\mathbb{F}_\ell)}(\pi, \text{Ind}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x)) = \text{Hom}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}(\text{Res}_{I(x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x), \text{Ind}_{I(x)}^{I_{G(\mathbb{F}_\ell)}(\varphi_x)}(\bar{\varphi}_x), \theta).$$

Further by Mackey theory and Frobenius reciprocity we obtain the following

$$\text{Hom}_{I(x)}(\text{Res}_{I(x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x), \theta) = \bigoplus_{g \in E.3} \text{Hom}_{I(x)}\left(\text{Ind}_{I(x)}^{I_{G(\mathbb{F}_\ell)}(\varphi_x)}(\bar{\varphi}_x), \theta\right)$$

Therefore, to prove $\text{Hom}_{G(\mathbb{F}_\ell)}(\pi, \text{Ind}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}^{G(\mathbb{F}_\ell)}(\bar{\varphi}_x)) \neq 0$ it is enough to prove that $\text{Hom}_{I_{G(\mathbb{F}_\ell)}(\varphi_x)}(\bar{\varphi}_x, \theta) \neq 0$ and this we prove in the next lemma.

Lemma 4.2. Let $x \in \mathfrak{g}(\mathbb{F}_m)$ be a canonical-regular element. The following are true.

1. The intersection $U(\mathbb{F}_\ell) \cap I_{G(\mathbb{F}_\ell)}(\varphi_x) = U(\sigma^m \mathbb{F}_\ell)$.
2. We have $\text{Hom}_{I(\sigma^m \mathbb{F}_\ell)}(\bar{\varphi}_x, \theta) \neq 0$.

Proof. Since $I_{G(\mathbb{F}_\ell)}(\varphi_x) = C_{G(\mathbb{F}_\ell)}(\bar{x})K_\ell^m$ for any lift $\bar{x}$ of $x$, it is clear that $U(\sigma^m \mathbb{F}_\ell) \subseteq U(\mathbb{F}_\ell) \cap I_{G(\mathbb{F}_\ell)}(\varphi_x)$, so we proceed to prove the other side inclusion. The element $x$ is canonical-regular, assume that $\bar{x} \in \mathfrak{g}(\mathbb{F}_\ell)$ is also a canonical regular lift of $x$. Therefore by Lemma 2.7, we have $U(\mathbb{F}_\ell) \cap C_{G(\mathbb{F}_\ell)}(\bar{x}) = \{1\}$. This implies, any $v \in U(\mathbb{F}_\ell) \cap I_{G(\mathbb{F}_\ell)}(\varphi_x)$ satisfies $v \equiv 1 \mod(\sigma^m \mathbb{F}_\ell)$. This along with $v \in U(\mathbb{F}_\ell)$ gives $v \in U(\sigma^m \mathbb{F}_\ell)$. Therefore (1) follows.

The proof of (2) amounts to say that the restriction of $\bar{\varphi}_x$ to $U(\sigma^m \mathbb{F}_\ell)$ is same as the restriction of $\theta$ to $U(\sigma^m \mathbb{F}_\ell)$. Which is clear since $x$ is canonical regular and then the trace of $xy$ is $(y_2 + y_3 + \cdots + y_{n-1})$ for $y = (y_i) \in U(\sigma^m \mathbb{F}_\ell)$. This proves part (2).

Proof of the converse of Theorem 2.7 for $G(\mathbb{F}_\ell)$ for odd $\ell$. Let $\ell = 2m + 1$. Since $\pi$ is a regular representation of $G(\mathbb{F}_\ell)$ there exists an irreducible representation $\sigma$ of $H_\ell^m$ which lies over a regular one dimensional representation $\varphi_x$ of $K_\ell^{m+1}$ and $\sigma$ extends to the inertia group $I_{G(\mathbb{F}_\ell)}(\sigma)$ written as $\bar{\sigma}$ with the property that

$$\pi \cong \text{Ind}_{I_{G(\mathbb{F}_\ell)}(\sigma)}^{G(\mathbb{F}_\ell)}(\bar{\sigma}).$$

Without loss of generality, we can assume that $x$ is canonical-regular. Write $I(\sigma)$ for the inertia group $I_{G(\mathbb{F}_\ell)}(\sigma)$. As in the even case, by using Frobenius reciprocity and Mackey theory we get

$$\text{Hom}_{G(\mathbb{F}_\ell)}(\pi, \text{Ind}_{I(\sigma)}(\bar{\sigma})) = \text{Hom}_{I(\sigma)}(\text{Res}_{I(\sigma)}^{G(\mathbb{F}_\ell)}(\bar{\sigma}), \text{Ind}_{I(\sigma)}(\bar{\sigma}), \theta) = \bigoplus_{g \in E.3} \text{Hom}_{I(\sigma)}(\text{Ind}_{I(\sigma)}(\bar{\sigma}), \theta)$$


Therefore to prove $\text{Hom}_{G_0}(\pi, \text{Ind}^{G_0}_{U_0}(\rho)) \neq 0$, it is enough to prove that $\text{Hom}_{U(\sigma, \theta)}(\sigma, \theta) \neq 0$ and this we prove in next two lemmas. 

**Lemma 4.3.** For $\ell = 2m + 1$, consider the set $\mathcal{S} = \text{Irr}(U(\sigma^m \sigma_f) | \theta(U(\sigma^m \sigma_f)))$. The following are true.

1. The set $\mathcal{S}$ consists of one dimensional representations of $U(\sigma^m \sigma_f)$.
2. If $x \in g(o_m)$ is a canonical-regular element and $\sigma \in \text{Irr}(\mathcal{H}_f | \varphi_x)$. Then, $\sigma|_{U(\sigma^m \sigma_f)} \cong \bigoplus \chi \in \mathcal{S} \chi$.

**Proof.** The one dimensional representation $\theta(U(\sigma^m \sigma_f))$ extends $\theta(U(\sigma^m \sigma_f))$ clearly to $U(\sigma^m \sigma_f)$. The quotient group $\frac{U(\sigma^m \sigma_f)}{U(\sigma^m \sigma_f)}$ is abelian, so (1) follows from Clifford theory. For (2), consider the normal subgroup $R_x = K^m \cap C_{G_0}(x)$ of $K^m$. By O.3, for $\sigma \in \text{Irr}(K^m | \varphi_x)$, there exists an extension $\bar{\varphi}_x$ of $\varphi_x$ to $R_x$ such that $\sigma|_{R_x} = \bar{\varphi}_x + \cdots + \bar{\varphi}_x$ and $\sigma$ is unique irreducible representation of $K^m$ lying above $\bar{\varphi}_x$.

Consider the hierarchy of groups given in Figure 3. By Lemma 2.7, we have $R_x \cap U(\sigma^m \sigma_f) = U(\sigma^m \sigma_f)$. Since $x$ is canonical-regular, we have $\bar{\varphi}_x|_{U(\sigma^m \sigma_f)} = \varphi_x|_{U(\sigma^m \sigma_f)} = \theta(U(\sigma^m \sigma_f))$ and $\bar{\varphi}_x$ is stable under $U(\sigma^m \sigma_f)$, so the map $\bar{\varphi}_x \circ \theta : (R_x)(U(\sigma^m \sigma_f)) \to \mathbb{C}^\times$ defined by $(\bar{\varphi}_x \circ \theta)(xy) = \bar{\varphi}_x(x)\theta(y)$ for all $x \in R_x$ and $y \in U(\sigma^m \sigma_f)$ is a well defined one dimensional representation of $(R_x)(U(\sigma^m \sigma_f))$. Further, the one dimensional representation $\bar{\varphi}_x \circ \theta$ extends $\bar{\varphi}_x$. The quotient group $(R_x)(U(\sigma^m \sigma_f))/R_x$ is easily seen to be abelian of order $q^{d_{\sigma^m \sigma_f}(1)}$. Therefore by Clifford theory, the set $\text{Irr}((R_x)(U(\sigma^m \sigma_f)) | \bar{\varphi}_x)$ consists of exactly $q^{d_{\sigma^m \sigma_f}(1)}$-many distinct one dimensional representations.

We claim that any $\chi \in \text{Irr}((R_x)(U(\sigma^m \sigma_f)) | \bar{\varphi}_x)$ satisfies $\text{Ind}^{R_x}_{(R_x)(U(\sigma^m \sigma_f))}(\chi) \cong \sigma$. Recall $\sigma$ is a unique irreducible representation of $\mathcal{H}_f$ lying above $\bar{\varphi}_x$ and therefore we must have $\langle \sigma, \text{Ind}^{R_x}_{(R_x)(U(\sigma^m \sigma_f))}(\chi) \rangle \neq 0$ for every $\chi \in \text{Irr}((R_x)(U(\sigma^m \sigma_f)) | \bar{\varphi}_x)$. Now the claim follows because

$$\dim(\sigma) = \dim(\text{Ind}^{R_x}_{(R_x)(U(\sigma^m \sigma_f))}(\chi)) = q^{d_{\sigma^m \sigma_f}(1)}$$

for every $\chi \in \text{Irr}((R_x)(U(\sigma^m \sigma_f)) | \bar{\varphi}_x)$. 

**Lemma 4.4.** Let $x \in g(o_m)$ be a canonical-regular element and $\sigma \in \text{Irr}(\mathcal{H}_f | \varphi_x)$. Then:

1. The intersection $U(o_2) \cap I(\sigma) = U(\sigma^m \sigma_f)$.
2. We have, $\text{Hom}_{U(\sigma^m \sigma_f)}(\sigma, \theta) = \text{Hom}_{U(\sigma^m \sigma_f)}(\sigma, \theta) = 0$.

**Proof.** For (1), we note that by O.4, $I(\sigma) = I_{G_0}(\varphi_x)$. The rest of the proof is similar to Lemma 4.3.1. For (2), note that the dimension of $\sigma$ is $q^{d_{\sigma^m \sigma_f}(1)}$. By Lemma 4.3, we have $\sigma|_{U(\sigma^m \sigma_f)}$ is direct sum of $q^{d_{\sigma^m \sigma_f}(1)}$ distinct one dimensional representations of $U(\sigma^m \sigma_f)$. We also know that $\sigma|_{U(\sigma^m \sigma_f)}$ is isomorphic to direct sum of $q^{d_{\sigma^m \sigma_f}(1)}$ times $\varphi_x$. Therefore all the one dimensional representations of $\sigma|_{U(\sigma^m \sigma_f)}$ are extensions of $\varphi_x|_{U(\sigma^m \sigma_f)}$. Note that the index $[U(\sigma^m \sigma_f) : U(\sigma^m \sigma_f)] = q^{d_{\sigma^m \sigma_f}(1)} = q^{d_{\sigma^m \sigma_f}(1)}$ which is the same as the dimension of $\sigma$. Thus $\sigma|_{U(\sigma^m \sigma_f)}$ is direct sum of all the possible extensions of $\varphi_x|_{U(\sigma^m \sigma_f)}$ to
Theorem 4.5. The sum of the dimensions of all inequivalent irreducible regular representations of \( G(\mathfrak{o}_\ell) \) is equal to the dimension of the induced representation \( \text{Ind}_{U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \theta \).

Proof. Let \( R \) be the set of isomorphism classes of irreducible regular representations of \( G(\mathfrak{o}_\ell) \) and \( \bar{x} \in \mathfrak{g}(\mathbb{F}_q) \) be a regular element. First of all we claim that,

\[
\sum_{\pi \in R} \dim(\pi) = \begin{cases} 
q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}|(\mathfrak{g}(\mathfrak{o}_m)|), & \text{for } \ell = 2m \\
q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m} \cdot q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}|(\mathfrak{g}(\mathfrak{o}_m)|}}, & \text{for } \ell = 2m + 1.
\end{cases}
\]

We prove this claim for cases \( \ell = 2m \) and \( \ell = 2m + 1 \) separately.

For \( \ell = 2m \), by E.4 and E.5, the sum of dimensions of all inequivalent irreducible regular representations lying over the one dimensional representation \( \varphi_x \) for any regular \( x \in \mathfrak{g}(\mathfrak{o}_m) \) is the index \( [G(\mathfrak{o}_\ell) : K^{\mathfrak{g}}_\ell] = |G(\mathfrak{o}_m)| \). We recall that, the number of conjugacy classes of regular elements in \( \mathfrak{g}(\mathfrak{o}_m) \) is \( q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}|(\mathfrak{g}(\mathfrak{o}_m)|} \) (by Lemma 2.6). Therefore \( \sum_{\pi \in R} \dim(\pi) = q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}|(\mathfrak{g}(\mathfrak{o}_m)|} \) for \( \ell = 2m \).

For \( \ell = 2m + 1 \), by O.7, the sum of dimensions of all inequivalent irreducible regular representations lying over the one dimensional representation \( \varphi_x \) of \( K^{\mathfrak{g}}_\ell \) for any regular \( x \in \mathfrak{g}(\mathfrak{o}_m) \) is \( q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}|(\mathfrak{g}(\mathfrak{o}_m)|}} \). Multiplying this with the number of regular conjugacy classes of \( \mathfrak{g}(\mathfrak{o}_m) \), we obtain our claim even in this case.

Next, we proceed to prove that the dimension of \( \text{Ind}_{U(\mathfrak{o}_\ell)}^{G(\mathfrak{o}_\ell)} \theta \) is same as \( \sum_{\pi \in R} \dim(\pi) \) for all \( \ell \). We note that \( |U(\mathfrak{o}_\ell)| = q^{d_{\mathfrak{g}(\mathfrak{o}_\ell)^m}} \) for all \( \ell \). Consider the short exact sequence,

\[
1 \to K^{\mathfrak{g}}_\ell \to G(\mathfrak{o}_\ell) \to G(\mathfrak{o}_m) \to 1.
\]
Since, dimension of $\text{Ind}_{U(\mathcal{O}_2)}^{G(\mathcal{O}_2)} \theta$ is equal to $|\mathcal{G}(\mathcal{O}_2)|$ we obtain the result by substituting $|\mathcal{G}(\mathcal{O}_2)|$ and $|U(\mathcal{O}_2)|$ from above.

4.2. Proof of Theorem 1.3. Now we are in a position to complete the proof of Theorem 1.3. It is to be noted that for $\ell = 1$, Theorem 1.3 follows from the work of Gelfand and Graev [1,2]. Further for $\ell \geq 2$, Theorem 1.3 follows directly from Theorem 1.5 for the groups $G = \text{GL}_n$ or $G = \text{SL}_n$ with $(p,n) = (p,2) = 1$. For $G = \text{SL}_n$ with $p \mid n$ or $p = 2$, we note that $\text{Ind}_{U(\mathcal{O}_2)}^{\text{GL}_n(\mathcal{O}_2)}(\theta)$ is multiplicity free because $\text{Ind}_{U(\mathcal{O}_2)}^{\text{GL}_n(\mathcal{O}_2)}(\theta) \cong \text{Ind}_{U(\mathcal{O}_2)}^{\text{SL}_n(\mathcal{O}_2)}(\theta)$ and $\text{Ind}_{U(\mathcal{O}_2)}^{\text{GL}_n(\mathcal{O}_2)}(\theta)$ is multiplicity free by Theorem 1.5.

5. An Example: $\text{GL}_2$

In this section, we discuss the example of $\text{GL}_2$ in some detail and prove Theorem 1.6.

Let $G(\mathcal{O}_2) = \text{GL}_2(\mathcal{O}_2)$ with the residue field $\mathcal{O}_2/\mathcal{O} = \mathbb{F}_q$. Recall, the image of any $x \in M_2(\mathcal{O}_2)$ in $M_2(\mathcal{O}_2)$ via natural projection map is denoted by $\bar{x}$. Then $K_{\ell}^0 \cong M_2(\mathbb{F}_q)$ via $1 + \pi^\ell y \mapsto \bar{y}$.

Let $\phi : \mathbb{F}_q \to \mathbb{C}^\times$ be a non-trivial additive one dimensional representation of $\mathbb{F}_q$. For any $x \in M_2(\mathbb{F}_q)$, define $\phi_x : M_2(\mathbb{F}_q) \to \mathbb{C}^\times$ by $\phi_x(y) = \text{Tr}(xy)$. Then it is easy to see that the map $x \mapsto \phi_x$ gives an isomorphism $M_2(\mathbb{F}_q) \cong K_{\ell}^0$. This in particular gives an isomorphism $M_2(\mathbb{F}_q) \cong K_{\ell}^0$ via $x \mapsto \phi_x$, where $\phi_x(1 + \pi^\ell y) = \phi_x(\bar{y})$.

The group $\text{GL}_2(\mathcal{O}_2)$ acts on $K_{\ell}^0$ via conjugation and therefore acts on $K_{\ell}^0$ as follows. For $\alpha \in \text{GL}_2(\mathcal{O}_2)$ and $\phi_x \in K_{\ell}^0$, we have

$$\phi_x(1 + \pi^\ell y) = \phi_x((1 + \pi^\ell y)\alpha^{-1})) = \text{Tr}(x(\bar{y}\bar{\alpha}^{-1})) = \phi_{\alpha^{-1}x}(1 + \pi^\ell y).$$

Thus the action of $\text{GL}_2(\mathcal{O}_2)$ on $K_{\ell}^0$ transforms to conjugation (inverse) action of $\text{GL}_2(\mathbb{F}_q)$ on $M_2(\mathbb{F}_q)$.

Let $\rho$ be an irreducible representation of $\text{GL}_2(\mathcal{O}_2)$. Then by Clifford theory, the restriction of $\rho$ to the normal subgroup $K_{\ell}^0$ must be direct sum (with multiplicity) of all conjugates of some fixed one dimensional representation $\varphi_x$. The following is an elementary fact about the elements of $M_2(\mathbb{F}_q)$.

Lemma 5.1. Any $x \in M_2(\mathbb{F}_q)$ is either a scalar matrix or a cyclic matrix.

Therefore every one dimensional representation $\varphi_x$ of $K_{\ell}^0$ is such that either $x$ is regular or scalar. In particular, a $\rho \in \text{Irr}(\text{GL}_2(\mathcal{O}_2))$ is regular if and only if it is primitive. Therefore, Theorem 1.6 holds for $\text{GL}_2(\mathcal{O}_2)$.

Remark 5.2. The Lemma 5.1 does not hold for $M_n(\mathbb{F}_q)$ for $n \geq 3$ and therefore it is easy to see that Theorem 1.6 does not extend to $\text{GL}_n(\mathcal{O}_2)$ for any $n \geq 3$.

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Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi - 110016, INDIA.

E-mail address: shivprakashpatel@gmail.com

Department of Mathematics, Indian Institute of Science (IISc), Bangalore - 560012, INDIA.

E-mail address: pooja@iisc.ac.in