Gravity localization in non-minimally coupled scalar thick braneworlds with a Gauss-Bonnet term

D. Malagón–Morejón\textsuperscript{1}, A. Herrera–Aguilar\textsuperscript{2} and I. Quiros\textsuperscript{3}

\textsuperscript{(1,2)}Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo. Edificio C–3, Ciudad Universitaria, C.P. 58040, Morelia, Michoacán, México
\textsuperscript{3}División de Ciencias e Ingeniería de la Universidad de Guanajuato, A.P. 150, 37150, León, Guanajuato, México
E-mail: malagon@ifm.umich.mx\textsuperscript{1}, herrera@ifm.umich.mx\textsuperscript{2}, iquiros@fisica.ugto.mx\textsuperscript{3}

Abstract. We consider a warped five-dimensional thick braneworld with a four-dimensional Poincaré invariant space-time in the framework of scalar matter non-minimally coupled to gravity plus a Gauss-Bonnet term in the bulk. Scalar field and higher curvature corrections to the background equations as well as the perturbed equations are shown. A relationship between 4-dimensional and 5-dimensional Planck masses is studied in general terms. By imposing finiteness of the 4-dimensional Planck mass and regularity of the geometry, the localization properties of the tensor modes of the first order perturbed geometry are analyzed for an important class of solutions motivated by models with scalar fields which are minimally coupled to gravity. In order to study the gravity localization properties for this model, the normalizability condition for the lowest level of the tensor fluctuations is analyzed. We see that for the class of solutions examined, gravity in 4 dimensions is recovered if the curvature invariants are regular and Planck masses are finite.

1. Introduction
In contrast to Kaluza-Klein picture where extra dimensions are compactified, the braneworld models [1, 2, 3] allow an infinite extra dimension. These models requires that Standard Model fields are trapped on a 4-dimensional surface, called a 3-brane. Since gravity can propagate through all dimensions, the first important question concerning to braneworld models is to check whether they give back standard 4-dimensional gravity on the brane.

The thin braneworld models [2] have the disadvantage that the curvature is singular at the location of the wall. Two ways of smoothing out the singular character of the brane configurations are: by replacing the delta functions in the action of the system by a self-interacting scalar field [4, 5, 6, 7] or by modifying the Einstein-Hilbert action itself [8, 9].

In this work we study a 5-dimensional thick braneworld modeled by a smooth scalar domain wall non-minimally coupled to gravity with a Gauss-Bonnet term on the bulk, in the framework of conformally flat geometries. We analyze for an important class of solutions the relationship among: localization properties of the zero tensor mode, finiteness of 4-dimensional Planck mass and smoothness of the geometry.
2. The model
Consider a thick braneworld described by the following 5-dimensional action

\[ S = \int d^5x \sqrt{|g|} \left\{ -\frac{L(\varphi)R}{2\kappa} - \alpha' R_{\text{GB}}^2 + \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right\}, \]  

where \( A, B = 0, \ldots, 3, 5 \).

The constant \( \alpha' > 0 \) and \( \kappa \approx 1/M^3 \), where \( M \) is the 5-dimensional Planck mass, \( L(\varphi) \) is the coupling between scalar field \( \varphi \) and gravity and the term \( R_{\text{GB}}^2 \) is the Gauss-Bonnet term in 5 dimensions

\[ R_{\text{GB}}^2 = R^{ABCD} R_{ABCD} - 4 R^{AB} R_{AB} + R^2. \]

Einstein equations for this model are

\[ L R_{AB} = \kappa \tau_{AB} + \nabla_A \nabla_B L + \frac{1}{3} g_{AB} \Box L - \epsilon Q_{AB}, \]

where \( \epsilon = 2\alpha' \kappa \).

The reduced energy-momentum tensor \( \tau_{AB} \) corresponds to the scalar matter content on the bulk and takes the form:

\[ \tau_{AB} = \partial_A \varphi \partial_B \varphi - \frac{2}{3} g_{AB} V(\varphi), \]

on the other hand, the term \( Q_{AB} \) is called Lanczos tensor and represents the corrections of Gauss-Bonnet term to Einstein equations, which can be written in the form

\[ Q_{AB} = \frac{1}{3} g_{AB} R_{\text{GB}}^2 - 2 R R_{AB} + 4 R_{AC} R_C^B + 4 R^{CD} R_{ACBD} - 2 R_{ACDE} R_B^{\ CDE}. \]

The field equation for the scalar field is

\[ \Box \varphi + \frac{1}{2\kappa} L R \varphi + \frac{\partial V}{\partial \varphi} = 0, \quad \text{where} \quad L \varphi = \frac{dL}{d\varphi}. \]

As in [10] let us consider a warped metric in conformally flat coordinates

\[ ds^2 = a^2(w) [\eta_{\mu\nu} dx^\mu dx^\nu - dw^2], \]

the variable \( w \) is the bulk coordinate and \( \eta_{\mu\nu} \) is the 4-dimensional Minkowski metric. In our work we consider the simplest case where the scalar field depends only on the bulk coordinate \( w \) and \( a(w) = a(-w) \).

3. Localization of gravity
In order to study the localization of gravity let us consider the linear tensor fluctuations of the background metric (6)

\[ ds_p^2 = \left[ a^2(w) \eta_{AB} + H_{AB} \right] dx^A dx^B, \quad \text{where} \quad H_{AB} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}. \]

The tensor \( h_{\mu\nu} \) is a divergence-less and trace-less rank-two tensor in the 4-dimensional Poincaré invariant space-time [10, 11], the tensor fluctuation equation is

\[ q h''_{\mu\nu} + (3Hq + q') h'_{\mu\nu} - \left[ q + \left( \frac{q - L}{2H} \right)' \right] \Box h_{\mu\nu} = 0, \]

where \( q = \frac{L}{\dot{\varphi}} \), \( \dot{\varphi} = \frac{d\varphi}{dw} \) and \( q = L - \frac{4\kappa}{a^2} H^2 \). If we redefine the field \( h_{\mu\nu} \) as \( \Psi_{\mu\nu} = \sqrt{s(w)} h_{\mu\nu} \) the equation (8) changes to:

\[ \Psi''_{\mu\nu} - \frac{(\sqrt{s})''}{\sqrt{s}} \Psi_{\mu\nu} - \frac{r}{s} \Box \Psi_{\mu\nu} = 0, \]
where \( s(w) = a^3q \) and \( r(w) = a^3\left(q + \frac{(q-L)v}{2w} \right) \).

With the aim to study the spectrum of the metric fluctuations we propose a separation of variables given by \( \Psi_{\mu\nu} = \psi(w)\chi_{\mu\nu}(x) \), then equation (9) is separated in two equations:

\[
\Box\chi_{\mu\nu} + m^2\chi_{\mu\nu} = 0, \tag{10}
\]

\[
\psi'' - \frac{(\sqrt{s})''}{\sqrt{s}}\psi + m^2\frac{r}{s}\psi = 0. \tag{11}
\]

Equation (10) describes the 4-dimensional massive tensorial modes and (11) defines the localization properties of \( \Psi_{\mu\nu} \). On the other hand, a sector of gravity is localized on the brane if the associated zero mode fluctuation is normalizable, in other words, the norm for the tensor zero mode fluctuations has to be finite

\[
\langle \psi_0 | \psi_0 \rangle = \int_{-\infty}^{\infty} \frac{r}{s} \psi_0^2 dw < \infty. \tag{12}
\]

The lowest mass eigenstate of (11) is \( \psi_0 = \sqrt{s} \), then we obtain:

\[
\langle \psi_0 | \psi_0 \rangle = \int_{-\infty}^{\infty} a^3 q dw - 4 \epsilon [a'_{-\infty}]^2 + 8 \epsilon \int_{-\infty}^{\infty} \frac{a^2}{a'} dw. \tag{13}
\]

It is interesting to ask: what is the relationship between the 4-dimensional Planck mass and the normalizability condition (12)?

4. Planck masses

For a space-time that take the following form:

\[
ds^2 = a^2(w)[g_{\mu\nu}(x)dx^\mu dx^\nu - dw^2], \tag{14}
\]

where \( g_{\mu\nu}(x) \) is an arbitrary 4-dimensional metric, the relationship between 4-dimensional Planck mass and 5-dimensional Planck mass can be obtained if we make a dimensional reduction by integrating (1) with respect to the \( w \) coordinate. Thus, the 5-dimensional theory (1) is reduced to a 4-dimensional Einstein-Hilbert theory plus the corrections that comes from the scalar matter and higher curvature terms of the bulk

\[
S_4 \simeq M_{Pl}^2 \int d^4x \sqrt{|g_4|} R_4 + \ldots, \tag{15}
\]

yielding the following expression for \( M_{Pl} \)

\[
M_{Pl} \simeq M^3 \int_{-\infty}^{\infty} a^3(w) \left[ L(\varphi) + \frac{4\epsilon}{a^2}(H^2 + 2\dot{H}') \right] dw = M^3 \int_{-\infty}^{\infty} a^3(w) q dw + 8 M^3 \epsilon [a'_{-\infty}]^2.
\]

Planck mass \( M_{Pl} \) is closely related with \( \langle \psi_0 | \psi_0 \rangle \), but a extra term \( \int_{-\infty}^{\infty} \frac{a^2}{a'} dw \) arises in (13). A clearer relation between \( M_{Pl} \) and \( \langle \psi_0 | \psi_0 \rangle \) can be obtained if we analyze the smoothness conditions of the curvature invariants.

5. Smoothness of geometry

A realistic thick braneworld model should not have singularities in the geometry. Let us assume that the curvature invariants and warp factor \( a \) are regular in the whole space-time and consider the class of solutions where asymptotically

\[
a(w \to \infty) \simeq \frac{1}{w^\gamma}. \tag{16}
\]
For (6) the curvature invariants are:

\[ R = \frac{4}{a^2} \left[ 2H' + 3H^2 \right] \]
\[ R^{AB} R_{AB} = \frac{4}{a^4} \left[ 5H'^2 + 9H^4 + 6H'H^2 \right], \]
\[ R^{ABCD} R_{ABCD} = \frac{4}{a^4} \left[ 4H'^2 + 6H^4 \right]. \]

Calculating the asymptotic behavior of the previous expressions at \( w \to \infty \) we get:

\[ R \simeq w^{2(\gamma - 1)}, \quad R^{AB} R_{AB} \simeq R^{ABCD} R_{ABCD} \simeq w^{4(\gamma - 1)}. \]

Thus, by imposing smoothness at infinity, the values of the exponent \( \gamma \) are restricted to the interval \( 0 < \gamma \leq 1 \). Under this restriction for \( \gamma \) we obtain

\[ M_P^2 \simeq M^3 \int_{w_{\infty}}^{\infty} a^3 \, L \, dw + \cdots \quad \text{and} \quad \langle \psi_0 | \psi_0 \rangle \simeq \int_{w_{\infty}}^{\infty} a^3 \, L \, dw + \cdots, \]

where \( \cdots \) denotes finite terms and the interval \((w_{\infty}, \infty)\) formally represents the range where the approximation (16) is valid.

6. Conclusions
We have presented a thick braneworld model where the matter field is coupled non-minimally to the Einstein-Hilbert term, in addition to this, there is a Gauss-Bonnet term on the bulk. We compute the 4-dimensional Planck mass in terms of 5-dimensional Planck mass and quantities related with the matter and the geometry. In contrast with theories where the matter field is minimally coupled to a Einstein-Hilbert term, in our model \( M_P \) depend explicitly on the matter content of the bulk. A relation among smoothness of geometry, finiteness of 4-dimensional Planck mass and localization of tensorial modes was studied for an important class of solutions. We obtained that if the geometry is regular, finite 4-dimensional Planck mass implies localization of gravity on the brane.

7. Acknowledgements
This research was supported by grants CIC–4.16 and CONACYT 60060–J. DMM acknowledge a PhD grant from CONACYT. AHA and IQ thanks SNI for support.

References
[1] Gogberashvili M 2002 Int. J. Mod. Phys. D 11 1635-38
[2] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370-73
[3] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690-93
[4] Gremm M 2000 Phys. Lett. B 478 434-38
[5] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008
[6] Barbosa-Cendejas N, Herrera-Aguilar A, Reyes Santos M A and Schubert C 2008 Phys. Rev. D 77 126013
[7] Barbosa-Cendejas N, Herrera-Aguilar A, Nucamendi U and Quiros I 2007 Mass hierarchy and mass gap on thick branes with Poincare symmetry Preprint arXiv:0712.3098 [hep-th]
[8] Giovannini M 2001 Phys. Rev. D 64 124004
[9] Corradini O and Kakushadze Z 2000 Phys. Lett. B 494 302-10
[10] Giovannini M 2001 Phys. Rev. D 64 064023
[11] Csaki C, Erlich J, Hollowood T J, Shirman Y 2000 Nucl. Phys. B 581 309-38