Finite Energy One-Half Monopole Solutions of the SU(2) Yang-Mills-Higgs Theory

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Abstract

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Acknowledgements
We present finite energy SU(2) Yang-Mills-Higgs particles of one-half topological charge.

The 't Hooft Abelian magnetic fields of these solutions at spatial infinity correspond to the magnetic field of a positive one-half magnetic monopole located at \( r = 0 \) and a semi-infinite Dirac string singularity located on one half of the \( z \)-axis which carries a magnetic flux of \( \frac{2\pi}{g} \) going into the center of the sphere at infinity. Hence the net magnetic charge is zero.

The non-Abelian solutions possess gauge potentials that are singular at only one point at large distances, elsewhere they are regular.
The SU(2) YMH Theory

- The SU(2) YMH Lagrangian is given by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{2} D^\mu \Phi^a D_\mu \Phi^a - \frac{1}{4} \lambda (\Phi^a \Phi^a - \mu^2)^2. \]  

(1)

Here \( \mu \) is the Higgs field mass, \( \lambda \) is the strength of the Higgs potential. The vacuum expectation value of the Higgs field is \( \xi = \mu / \sqrt{\lambda} \).

- The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

\[ D_\mu \Phi^a = \partial_\mu \Phi^a + g \epsilon^{abc} A^b_\mu \Phi^c, \]
\[ F^{a \mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu, \]  

(2)

where \( g \) is the gauge field coupling constant. The metric used is \( g_{\mu \nu} = (- + ++) \). The SU(2) internal group indices \( a, b, c = 1, 2, 3 \) and the space-time indices are \( \mu, \nu, \alpha = 0, 1, 2, \) and \( 3 \) in Minkowski space.
The SU(2) YMH Theory (cont.)

- The equations of motion that follow from the Lagrangian (1) are

\[ D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + g \epsilon^{abc} A^{b\mu} F^c_{\mu\nu} = g \epsilon^{abc} \Phi^b D_\nu \Phi^c, \]

\[ D^\mu D_\mu \Phi^a = \lambda \Phi^a (\Phi^b \Phi^b - \xi^2), \]  \hspace{1cm} (3)

and the Bogomol'nyi equation holds in the limit of vanishing \( \mu \) and \( \lambda \),

\[ B_i^a \pm D_i \Phi^a = 0. \]  \hspace{1cm} (4)

- The electromagnetic field tensor proposed by 't Hooft [1] is

\[ F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \frac{1}{g} \epsilon^{abc} \hat{\Phi}^a D^\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c, \]

\[ = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g} \epsilon^{abc} \hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c, \]  \hspace{1cm} (5)

where \( A_\mu = \hat{\Phi}^a A^a_\mu \), the Higgs unit vector, \( \hat{\Phi}^a = \Phi^a / |\Phi| \), and the Higgs field magnitude \( |\Phi| = \sqrt{\Phi^a \Phi^a} \).
The SU(2) YMH Theory (cont.)

- Eq. (5) can be decomposed into the gauge part and the Higgs part respectively,

\[ G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\frac{1}{g} \epsilon^{abc} \hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c. \]  

(6)

- Hence the decomposed magnetic field is

\[ B_i = -\frac{1}{2} \epsilon_{ijk} F_{jk} = B^G_i + B^H_i. \]  

(7)

The net magnetic charge of the system is

\[ M = \frac{1}{4\pi} \int \partial^i B_i \, d^3x = \frac{1}{4\pi} \oint d^2\sigma_i \, B_i. \]  

(8)
The topological magnetic current \([12]\)

\[
k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_{abc} \partial^\mu \hat{\Phi}^a \partial^\rho \hat{\Phi}^b \partial^\sigma \hat{\Phi}^c,
\]

(9)

is also the topological current density of the system. Hence the corresponding conserved topological magnetic charge is

\[
M_H = \frac{1}{g} \int d^3x \ k_0 = \frac{1}{8\pi g} \int \epsilon_{ijk} \epsilon^{abc} \partial_i \left( \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right) d^3x
\]

\[
= \frac{1}{8\pi} \int d^2\sigma_i \left( \frac{1}{g} \epsilon_{ijk} \epsilon^{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right)
\]

\[
= \frac{1}{4\pi} \int d^2\sigma_i \ B_i^H.
\]

(10)

The magnetic charge \(M_H\) is the total magnetic charge of the system if and only if the gauge field is non singular \([13]\). If the gauge field is singular and carries Dirac string monopoles \(M_G\), then the total magnetic charge is \(M = M_G + M_H\).
In the electrically neutral BPS limit when the Higgs potential vanishes, the energy is a minimum, [14]

\[ E_{\text{min}} = \mp \int \partial_i (B_i^a \Phi^a) \, d^3x + \int \frac{1}{2} (B_i^a \pm D_i \Phi^a)^2 \, d^3x = \mp \int \partial_i (B_i^a \Phi^a) \, d^3x = \frac{4\pi \xi}{g} M_H. \]  

(11)

Hence the dimensionless minimum total energy is \( M_H \).

For non-BPS solution, its energy must be greater than that of Eq. (11). The dimensionless value is given by

\[ E = \frac{g}{8\pi \xi} \int \{ B_i^a B_i^a + D_i \Phi^a D_i \Phi^a + \frac{\lambda}{2} (\Phi^a \Phi^a - \xi^2)^2 \} \, d^3x \geq M_H. \]  

(12)
The SU(2) YMH Theory

The magnetic ansatz is [8]

$$gA^a_i = -\frac{\hat{n}_\phi^a}{r} \left\{ \psi_1(r, \theta)\hat{\theta}_i - R_1(r, \theta)\hat{r}_i \right\} + \frac{1}{r \sin \theta} \left\{ P_1(r, \theta)\hat{n}_\theta^a - P_2(r, \theta)\hat{n}_r^a \right\} \hat{\phi}_i,$$

$$gA_0^a = 0, \quad g\Phi^a = \Phi_1(r, \theta) \hat{n}_r^a + \Phi_2(r, \theta)\hat{n}_\theta^a,$$

(13)

where $P_1(r, \theta) = \sin \theta \psi_2(r, \theta)$ and $P_2(r, \theta) = \sin \theta R_2(r, \theta)$. The spatial and isospin unit vectors are respectively,

$$\hat{r}_i = \sin \theta \cos \phi \delta_{i1} + \sin \theta \sin \phi \delta_{i2} + \cos \theta \delta_{i3},$$

$$\hat{\theta}_i = \cos \theta \cos \phi \delta_{i1} + \cos \theta \sin \phi \delta_{i2} - \sin \theta \delta_{i3},$$

$$\hat{\phi}_i = -\sin \phi \delta_{i1} + \cos \phi \delta_{i2},$$

(14)

$$\hat{n}_r^a = \sin \theta \cos n\phi \delta_{1}^a + \sin \theta \sin n\phi \delta_{2}^a + \cos \theta \delta_{3}^a,$$

$$\hat{n}_\theta^a = \cos \theta \cos n\phi \delta_{1}^a + \cos \theta \sin n\phi \delta_{2}^a - \sin \theta \delta_{3}^a,$$

$$\hat{n}_\phi^a = -\sin n\phi \delta_{1}^a + \cos n\phi \delta_{2}^a; \quad \text{where} \quad n = 1.$$

(15)
The magnetic ansatz (13) is form invariant under the gauge transformation

\[ \omega = \exp \left( \frac{i}{2} \sigma^a \hat{n}_a \phi f(r, \theta) \right), \quad \sigma^a = \text{Pauli matrices} \quad (16) \]

and the transformed gauge potential and Higgs field take the form,

\[ gA_i' = -\frac{1}{r} \{ \psi_1 - \partial_\theta f \} \hat{n}_\phi \hat{\theta}_i + \frac{1}{r} \{ R_1 + r \partial_r f \} \hat{n}_\phi \hat{r}_i \]

\[ + \frac{1}{r \sin \theta} \{ P_1 \cos f + P_2 \sin f + n[\sin \theta - \sin(f + \theta)] \} \hat{n}_\theta \hat{\phi}_i \]

\[ - \frac{1}{r \sin \theta} \{ P_2 \cos f - P_1 \sin f + n[\cos \theta - \cos(f + \theta)] \} \hat{n}_r \hat{\phi}_i, \]

\[ gA_0' = 0, \]

\[ g\Phi' = (\Phi_1 \cos f + \Phi_2 \sin f) \hat{n}_r + (\Phi_2 \cos f - \Phi_1 \sin f) \hat{n}_\theta. \quad (17) \]
The SU(2) YMH Theory (cont.)

The general Higgs fields in the spherical and the rectangular coordinate systems are respectively

\[ g\Phi^a = \Phi_1(x) \hat{n}_r + \Phi_2(x)\hat{n}_\theta + \Phi_3(x)\hat{n}_\phi \]
\[ = \tilde{\Phi}_1(x) \delta^{a1} + \tilde{\Phi}_2(x) \delta^{a2} + \tilde{\Phi}_3(x) \delta^{a3}, \]

(18)

\[ \tilde{\Phi}_1 = \sin \theta \cos n\phi \Phi_1 + \cos \theta \cos n\phi \Phi_2 - \sin n\phi \Phi_3 = |\Phi| \sin \alpha \cos \beta \]
\[ \tilde{\Phi}_2 = \sin \theta \sin n\phi \Phi_1 + \cos \theta \sin n\phi \Phi_2 + \cos n\phi \Phi_3 = |\Phi| \sin \alpha \sin \beta \]
\[ \tilde{\Phi}_3 = \cos \theta \Phi_1 - \sin \theta \Phi_2 = |\Phi| \cos \alpha. \]

(19)

The axially symmetric Higgs unit vector in rectangular coordinate system is

\[ \hat{\Phi}^a = \sin \alpha \cos \beta \delta^{a1} + \sin \alpha \sin \beta \delta^{a2} + \cos \alpha \delta^{a3}, \beta = n\phi, \]

(20)

\[ \cos \alpha = g \cos \theta - h \sin \theta, \quad g = \frac{\Phi_1}{|\Phi|}, \quad h = \frac{\Phi_2}{|\Phi|}. \]

(21)
Hence the Higgs part and the gauge part of the ’t Hooft magnetic field (7) become

\[
gB^H_i = -n\epsilon_{ijk} \partial^j \cos \alpha \partial^k \phi
\]
\[
gB^G_i = -n\epsilon_{ijk} \partial^j \cos \kappa \partial_k \phi, \quad \cos \kappa = \frac{1}{n} (hP_1 - gP_2),
\]

and the ’t Hooft’s magnetic field is

\[
gB_i = -n\epsilon_{ijk} \partial^j (\cos \alpha + \cos \kappa) \partial_k \phi = -\epsilon_{ijk} \partial^j A_k,
\]

where \(A_i\) is the ’t Hooft’s gauge potential.

The orientation of the magnetic field can be plotted by using the vector plot of the magnetic field unit vector,

\[
\hat{B}_i = \frac{-\partial_\theta (\cos \alpha + \cos \kappa) \hat{r}_i + r \partial_r (\cos \alpha + \cos \kappa) \hat{\theta}_i}{\sqrt{[r \partial_r (\cos \alpha + \cos \kappa)]^2 + [\partial_\theta (\cos \alpha + \cos \kappa)]^2}}.
\]
We also note that the Higgs field (13) and the gauge transformed Higgs field (17) can be simplified to

\[ \Phi^a = |\Phi|(\cos(\alpha - \theta) \hat{n}_r^a + \sin(\alpha - \theta)\hat{n}_\theta^a), \]
\[ \Phi'^a = |\Phi|(\cos(\alpha' - \theta) \hat{n}_r^a + \sin(\alpha' - \theta)\hat{n}_\theta^a), \quad \alpha' = \alpha - f. \] (26)

At spatial infinity, all non-Abelian components of the gauge potential vanish and the electromagnetic field tends to

\[ F^a_{\mu\nu} \bigg|_{r \to \infty} = \{ \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g} \epsilon^{cde} \hat{\Phi}^c \partial_\mu \hat{\Phi}^d \partial_\nu \hat{\Phi}^e \} \hat{\Phi}^a = F_{\mu\nu} \hat{\Phi}^a, \] (27)

where \( F_{\mu\nu} \) is the 't Hooft electromagnetic field.

However there is no unique way of representing the Abelian electromagnetic field in the region at finite values of \( r \) [15]. One proposal was given by 't Hooft as in Eq. (5).

Another proposal which is less singular was given by Bogomol’nyi [3] and Faddeev [16],

\[ B_i = B_i^a \left( \frac{\Phi^a}{\xi} \right), \quad E_i = E_i^a \left( \frac{\Phi^a}{\xi} \right). \] (28)
We start by analysing four seemingly different types of one-half monopole charge solutions that we label as Type A1, A2, B1, and B2.

The profile functions of the non-Abelian gauge potentials (13) of the four types of one-half monopole solution at $r$ infinity are given respectively by,

(Type A1) \begin{align*}
\psi_1 &= \frac{1}{2}, \quad P_1 = \sin \theta - \frac{1}{2} \sin \frac{1}{2} \theta(1 + \cos \theta), \\
R_1 &= 0, \quad P_2 = \cos \theta - \frac{1}{2} \cos \frac{1}{2} \theta(1 + \cos \theta), \\
\Phi_1 &= \xi \cos \frac{1}{2} \theta, \quad \Phi_2 = -\xi \sin \frac{1}{2} \theta.
\end{align*}

(Type A2) \begin{align*}
\psi_1 &= -\frac{1}{2}, \quad P_1 = \sin \theta - \frac{1}{2} \sin \frac{3}{2} \theta(1 + \cos \theta), \\
R_1 &= 0, \quad P_2 = \cos \theta - \frac{1}{2} \cos \frac{3}{2} \theta(1 + \cos \theta), \\
\Phi_1 &= \xi \cos \frac{3}{2} \theta, \quad \Phi_2 = -\xi \sin \frac{3}{2} \theta.
\end{align*}

(Type B1) \begin{align*}
\psi_1 &= \frac{1}{2}, \quad P_1 = \sin \theta - \frac{1}{2} \cos \frac{1}{2} \theta(1 - \cos \theta), \\
R_1 &= \xi \cos \frac{1}{2} \theta, \quad \Phi_2 = -\xi \sin \frac{1}{2} \theta.
\end{align*}
The Exact Asymptotic One-Half Monopole Solutions (cont.)

\[
\begin{align*}
R_1 &= 0, \quad P_2 = \cos \theta + \frac{1}{2} \sin \frac{1}{2} \theta (1 - \cos \theta), \\
\Phi_1 &= \xi \sin \frac{1}{2} \theta, \quad \Phi_2 = \xi \cos \frac{1}{2} \theta. \quad (31)
\end{align*}
\]

(Type B2) \[
\begin{align*}
\psi_1 &= -\frac{1}{2}, \quad P_1 = \sin \theta + \frac{1}{2} \cos \frac{3}{2} \theta (1 - \cos \theta), \\
R_1 &= 0, \quad P_2 = \cos \theta - \frac{1}{2} \sin \frac{3}{2} \theta (1 - \cos \theta), \\
\Phi_1 &= -\xi \sin \frac{3}{2} \theta, \quad \Phi_2 = -\xi \cos \frac{3}{2} \theta. \quad (32)
\end{align*}
\]

- The Type A (Type B) solutions possess 't Hooft's gauge potential, which is singular along the positive (negative) z-axis,

\[
\mathcal{A}_i = \left\{ \frac{\cos \theta \pm 1}{2r \sin \theta} \right\} \hat{\phi}_i. \quad (33)
\]
The Higgs magnetic fields are given by

\[ gB^H_i = \frac{1}{2} \sin\left(\frac{1}{2} \theta\right) \frac{\hat{r}_i}{\sin \theta}, \quad \text{Type A solutions,} \quad (34) \]

\[ gB^H_i = \frac{1}{2} \cos\left(\frac{1}{2} \theta\right) \frac{\hat{r}_i}{\sin \theta}, \quad \text{Type B solutions,} \quad (35) \]

and the net 't Hooft’s magnetic fields are given by

\[ gB_i = \frac{\hat{r}_i}{2r^2} - 2\pi \delta(x_1) \delta(x_2) \theta(x_3) \delta^3_i, \quad \text{Type A solutions,} \quad (36) \]

\[ gB_i = \frac{\hat{r}_i}{2r^2} + 2\pi \delta(x_1) \delta(x_2) \theta(-x_3) \delta^3_i, \quad \text{Type B solutions.} \quad (37) \]

The solutions therefore carry a positive one-half monopole at the origin and the semi-infinite Dirac string singularity carries the other opposite half of the magnetic monopole charge. Hence the net magnetic charge of the configuration is zero.
The Numerical Calculations

- The numerical one-half monopole solutions are constructed by using the exact asymptotic solutions at large distances (29) - (32) for the Type A1, A2, B1, and B2 solutions respectively and by fixing the boundary conditions for all the profile functions (13) along the z-axis and near \( r = 0 \).

- In order to avoid the singularity of \( R_2(r, \theta) \), we choose to perform our numerical analysis with the functions,

\[
P_1(r, \theta) = \psi_2(r, \theta) \sin \theta, \quad P_2(r, \theta) = R_2(r, \theta) \sin \theta.
\]

- Near \( r = 0 \), we have the common trivial vacuum solution for all the four solutions. The asymptotic solutions and boundary conditions at small distances that will give rise to finite energy solutions are

\[
\psi_1 = P_1 = R_1 = P_2 = 0, \quad \Phi_1 = \xi_0 \cos \theta, \quad \Phi_2 = -\xi_0 \sin \theta,
\]

\[
\sin \theta \Phi_1(0, \theta) + \cos \theta \Phi_2(0, \theta) = 0,
\]

\[
\partial_r (\cos \theta \Phi_1(r, \theta) - \sin \theta \Phi_2(r, \theta))|_{r=0} = 0.
\]
The Numerical Calculations (cont.)

The boundary conditions imposed along the positive z-axis for the profile functions (13) of the Type A solutions are

\[
\partial_\theta \Phi_1(r, \theta)|_{\theta=0} = 0, \quad \Phi_2(r, 0) = 0, \quad \partial_\theta \psi_1(r, \theta)|_{\theta=0} = 0, \\
R_1(r, 0) = 0, \quad P_1(r, 0) = 0, \quad P_2(r, 0) = 0.
\]  

(41)

Along the negative z-axis, the boundary conditions imposed are

\[
\Phi_1(r, \pi) = 0, \quad \partial_\theta \Phi_2(r, \theta)|_{\theta=\pi} = 0, \quad \partial_\theta \psi_1(r, \theta)|_{\theta=\pi} = 0, \\
R_1(r, \pi) = 0, \quad P_1(r, \pi) = 0, \quad \partial_\theta P_2(r, \theta)|_{\theta=\pi} = 0.
\]  

(42)

The boundary conditions imposed along the positive z-axis for the profile functions (13) of the Type B solutions are

\[
\Phi_1(r, 0) = 0, \quad \partial_\theta \Phi_2(r, \theta)|_{\theta=0} = 0, \quad \partial_\theta \psi_1(r, \theta)|_{\theta=0} = 0, \\
R_1(r, 0) = 0, \quad P_1(r, 0) = 0, \quad \partial_\theta P_2(r, \theta)|_{\theta=0} = 0,
\]  

(43)

and along the negative z-axis, the boundary conditions imposed are

\[
\partial_\theta \Phi_1(r, \theta)|_{\theta=\pi} = 0, \quad \Phi_2(r, \pi) = 0, \quad \partial_\theta \psi_1(r, \theta)|_{\theta=\pi} = 0, \\
R_1(r, \pi) = 0, \quad P_1(r, \pi) = 0, \quad P_2(r, \pi) = 0.
\]  

(44)
The Numerical Calculations (cont.)

We set $\xi = 1$, $g = 1$ and $0 \leq \lambda = \mu \leq 12$. The numerical solutions connecting the asymptotic solutions (29)-(32) at large $r$ to the trivial vacuum solution (39) at small $r$ and subjected to the boundary conditions (40)-(42) for the Type A solutions and the boundary conditions (40), (43)-(44) for the Type B solutions together with the gauge fixing condition [8]

$$r \partial_r R_1 - \partial_\theta \psi_1 = 0,$$  \hspace{0.5cm} (45)

are solved using the Maple 12 and MatLab R2009a softwares [19].

The second order equations of motion (3) which are reduced to six partial differential equations with the ansatz (13) are then transformed into a system of nonlinear equations using the finite difference approximation.

The system of nonlinear equations are discretized on a non-equidistant grid of size $90 \times 80$ covering the integration regions $0 \leq \bar{x} \leq 1$ and $0 \leq \theta \leq \pi$. Here $\bar{x} = \frac{r}{r+1}$ is the finite interval compactified coordinate. The partial derivative with respect to the radial coordinate is then replaced accordingly by $\partial_r \rightarrow (1 - \bar{x})^2 \partial_{\bar{x}}$ and $\frac{\partial^2}{\partial r^2} \rightarrow (1 - \bar{x})^4 \frac{\partial^2}{\partial \bar{x}^2} - 2(1 - \bar{x})^3 \frac{\partial}{\partial \bar{x}}$.

The numerical overall error estimate is $10^{-4}$. 

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The Numerical Calculations

The Exact Asymptotic One-Half Monopole Solutions
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The SU(2) YMH Theory
The One-Half Monopole Solutions

Abstract
The Magnetic Dipole Moment and Magnetic Charge

- The profile functions, $\psi_1, P_1, R_1, P_2, \Phi_1,$ and $\Phi_2$ of the one-half monopole solutions are regular functions of $r$ and $\theta$. However the function $R_2 = \frac{P_2}{\sin\theta}$ possesses only one singular point at infinity (negative $z$-axis - Type A, positive $z$-axis - Type B).

- The 't Hooft’s gauge potential at large $r$, tends to

$$A_i = (\cos\alpha + \cos\kappa) \partial_i \phi |_{r \to \infty} = \frac{\hat{\phi}_i}{r \sin\theta} \left( \frac{1}{2} (\cos \theta \pm 1) + \frac{F_G(\theta)}{r} \right), \quad (46)$$

$$F_G(\theta) = r(h(P_1 - \sin \theta) - g(P_2 - \cos \theta) - \frac{1}{2}(\cos \theta \pm 1)) |_{r \to \infty}, \quad (47)$$

for the Type A and Type B solutions respectively.

- From the graphs of $F_G(\theta)$ versus angle $\theta$, we find that $F_G(\theta) = \mu_m \sin^2 \theta$, where $\mu_m$ is the dimensionless magnetic dipole moment of the one-half monopole and is non vanishing for all values of $\lambda$.

- In the limit when $\lambda = 0$, $\mu_m = \pm 1.32$ for the Type 1 solutions and $\mu_m = \pm 1.04$ for the Type 2 solutions. Figure 1 (a).
Figure 1: (a) Plot of $\mu_m$ versus $\lambda^{1/2}$. (b) Plot of $M$ versus $\bar{x}$. (UH = upper hemisphere, LH = lower hemisphere). Plots of $E$ versus (c) $\lambda^{1/2}$ and (d) $\ln(1 + \lambda)$. Here $g = \xi = 1.$
The Magnetic Dipole Moment and Magnetic Charge

- The net magnetic charge $M$ (8) of the one-half monopole configurations are plotted numerically versus the compactified axis, $\bar{x}$, when $\lambda = \xi = 1$. Figure 1 (b). We notice that $M = \frac{1}{2}$ when $r \geq 4$ and its value vanishes as $r$ tends to zero.

- Also plotted in Figure 1 (b) are the magnetic charges covered by the upper hemisphere and lower hemisphere versus $\bar{x}$. The plots show that the magnetic charge at the origin is ‘heavier’ on the side of the $z$-axis away from the Dirac string.

- Using the definition (28) by Bogomol’nyi [3] and Faddeev [16], we can write

$$M = \frac{1}{4\pi} \int \partial^i B_i \ d^3x = \int M \ d\theta \ dr, \quad M = \frac{1}{2} r^2 \sin \theta \{\partial^i B_i\}, \quad (48)$$

where $M$ is the magnetic charge density. Figure 2.
Figure 2: 3D surface and contour plots of the magnetic charge density $M$ of the (a) Type $B1$ and (b) Type $B2$ one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$. 
### Table 1: Table of the dimensionless magnetic dipole moment $\mu_m$ and dimensionless total energy $E$ of the Type 1 and Type 2 solutions for different values of $\lambda$. The magnetic dipole moment $\mu_m$ is positive for the Type A solutions and negative for the Type B solutions.

| $\lambda$ | Type 1 $\mu_m$ | Type 2 $\mu_m$ | Type 1 $E$ | Type 2 $E$ |
|-----------|----------------|----------------|------------|------------|
| 0         | ±1.3203        | ±1.0383        | 0.5090     | 0.5245     |
| 0.04      | ±1.2173        | ±0.9592        | 0.5360     | 0.5518     |
| 0.20      | ±1.1480        | ±0.9073        | 0.5589     | 0.5747     |
| 0.40      | ±1.1124        | ±0.8801        | 0.5723     | 0.5883     |
| 0.60      | ±1.0904        | ±0.8631        | 0.5813     | 0.5973     |
| 0.80      | ±1.0743        | ±0.8507        | 0.5881     | 0.6041     |
| 1.00      | ±1.0616        | ±0.8409        | 0.5936     | 0.6098     |
| 2.00      | ±1.0211        | ±0.8095        | 0.6124     | 0.6286     |
| 4.00      | ±0.9793        | ±0.7772        | 0.6332     | 0.6497     |
| 8.00      | ±0.9367        | ±0.7443        | 0.6561     | 0.6728     |
| 12.00     | ±0.9242        | ±0.7297        | 0.6702     | 0.6862     |
From Eq. (24), the magnetic field lines contour plots of the one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$ are shown in Figure 3.

The direction of the magnetic field lines are shown in the vector field plot of the magnetic field unit vector $\hat{B}_i$ (25) of the one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$. The * indicates the location of the one-half monopole. The presence of a one-half monopole sitting at $r = 0$ is once again indicated by the location of the Higgs field’s node marked * on the Higgs field’s vector plots along the $x$-$z$ plane at $y = 0$. Figure 3.

The 3D surface and contour plots of the modulus of the Higgs field $|\Phi|$ of the one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$, show that there is a point zero of the Higgs field modulus at $r = 0$ where the one-half monopole is located. Figure 4.
The modulus of the Higgs field at large $r$ tends to

$$g|\Phi(r, \theta)|_{r \to \infty} = (\xi - \frac{F_H(\theta)}{r}). \quad (49)$$

We find that $F_H(\theta)$ is a non vanishing constant $c_1$ only when $\lambda = 0$. $c_1 = -0.43$ (Type 1) and $c_1 = -0.63$ (Type 2). For non vanishing values of $\lambda$, $c_1 = 0$ for the one-half monopole solutions.
Figure 3: Contour plot of the 't Hooft magnetic field lines, unit vector field plot of the 't Hooft magnetic field and Higgs field vector plots of the (a) Type B1 and (b) Type B2 solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$. The * indicates the location of the one-half monopole.
Figure 4: The 3D surface and contour plots of the Higgs field modulus $|\Phi|$ of the (a) Type $B_1$ and (b) Type $B_2$ one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$. 
The Energy Density and Total Energy

- The total dimensionless energy (12) of the Type 1 one-half monopole solutions is 0.509 and that of the Type 2 solutions is 0.525 when $\lambda = 0$. Their total energies are higher than the BPS total energy of a one-half monopole which is $\frac{1}{2}$.

- The total energies of the one-half monopole solutions are plotted versus $\lambda^{1/2}$ and $\ln(1 + \lambda)$ for values $0 \geq \lambda \geq 12$. Figure 1 (c), (d). For a particular value of $\lambda$, the Type 2 solutions possess higher total energy compared to the Type 1 solutions.

- The total dimensionless energy (12) when $\xi=g=1$, can be written as

$$E = \int \mathcal{E} \, d\theta \, dr, \quad \mathcal{E} = \frac{1}{4} r^2 \sin \theta \{ B^a_i B^a_i + D_i \Phi^a D_i \Phi^a + \frac{\lambda}{2} (\Phi^a \Phi^a - \xi^2)^2 \}, \quad (50)$$

where $\mathcal{E}$ is the energy density. The 3D surface and contour line plots of $\mathcal{E}$ of the one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = 1$ are shown in Figure 5. The shape of the one-half monopole is that of a rugby ball.
Figure 5: 3D surface and contour plots of the energy density $\mathcal{E}$ of the (a) Type $B1$ and (b) Type $B2$ one-half monopole solutions along the $x$-$z$ plane at $y = 0$ when $\lambda = \xi = 1$. 
From our analysis of the four one-half monopole solutions, we are able to conclude that the Type B solutions are exact 180° rotation of the z-axis about $r = 0$ of the respective Type A solutions in 3D space. Hence they are gauge equivalent.

We have found two different one-half monopole, the Type 1 and Type 2, as they are not gauge equivalent along the boundary at $\theta = \pi$ for the Type A solutions and along the boundary at $\theta = 0$ for the Type B solutions. These solutions are distinct in the region where the energy of the one-half monopole is concentrated.
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References

[1] G. 't Hooft, Nucl. Phys. B79, 276 (1974); A.M. Polyakov, Sov. Phys. JETP 41, 988 (1975); Phys. Lett. B59, 82 (1975); JETP Lett. 20, 194 (1974).

[2] E.B. Bogomol'nyi and M.S. Marinov, Sov. J. Nucl. Phys. 23, 357 (1976).

[3] M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975); E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24, 449 (1976).

[4] C. Rebbi and P. Rossi, Phys. Rev. D22, 2010 (1980); R.S. Ward, Commun. Math. Phys. 79, 317 (1981); P. Forgacs, Z. Horvarth and L. Palla, Phys. Lett. B99, 232 (1981); Nucl. Phys. B192, 141 (1981); M.K. Prasad, Commun. Math. Phys. 80, 137 (1981); M.K. Prasad and P. Rossi, Phys. Rev. D24, 2182 (1981).

[5] E.J. Weinberg and A.H. Guth, Phys. Rev. D14, 1660 (1976).

[6] Rosy Teh and K.M. Wong, J. Math. Phys. 46, 082301 (2005); Int. J. Mod. Phys. A20, 4291 (2005).
References (cont.)

[7] P.M. Sutcliffe, Int. J. Mod. Phys. A12, 4663 (1997); C.J. Houghton, N.S. Manton and P.M. Sutcliffe, Nucl.Phys. B510, 507 (1998).

[8] B. Kleihaus and J. Kunz, Phys. Rev. D61, 025003 (2000); B. Kleihaus, J. Kunz, and Y. Shnir, Phys. Lett. B570, 237, (2003); B. Kleihaus, J. Kunz, and Y. Shnir, Phys. Rev. D68, 101701 (2003); Phys. Rev. D70, 065010 (2004).

[9] Rosy Teh, K.G. Lim and P.W. Koh, AIP Conference Proceedings Volume 1150, 424 (2009).

[10] E. Harikumar, I. Mitra, and H.S. Sharatchandra, Phys. Lett. B557, 303 (2003).

[11] Rosy Teh and K.M. Wong, Half-Monopole and Multimonopole, Int. J. Mod. Phys. A20, 2195 (2005).

[12] N.S. Manton, Nucl. Phys. (N.Y.) B126, 525 (1977).

[13] J. Arafune, P.G.O. Freund, and C.J. Goebel, J. Math. Phys. 16, 433 (1975).
[14] A. Actor, Rev. Mod. Phys. 51, 461 (1979).

[15] S. Coleman, New Phenomena in Subnuclear Physics, Proc. 1975 Int. School of Physics ‘Ettore Majorana’, ed A Zichichi, New York Plenum, 297 (1975).

[16] L.D. Faddeev, Nonlocal, Nonlinear and Nonrenormalisable Field Theories, Proc. Int. Symp., Alushta, Dubna: Joint Institute for Nuclear Research, 207 (1976); Lett. Math. Phys. 1, 289 (1976).

[18] D.G. Boulware et al., Phys. Rev. D14, 2708 (1976).

[19] K.G. Lim, Rosy Teh and K.M. Wong, J. Phys. G: Nucl. Part. Phys. 39, 025002 (2012).

[20] Rosy Teh, B.L. Ng, and K.M. Wong, Particles of One-Half Topological Charge, ArXiv: submit/0409918 [hep-th] 3 Feb 2012.