On Bounded Positive Stationary Solutions for a Nonlocal Fisher-KPP Equation

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Abstract

We study the existence of stationary solutions for a nonlocal version of the Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation. The main motivation is a recent study by Berestycki et al. [Nonlinearity 22 (2009), pp. 2813–2844] where the nonlocal FKPP equation has been studied and it was shown for the spatial domain $\mathbb{R}$ and sufficiently small nonlocality that there are only two bounded non-negative stationary solutions. Here we generalize this result to $\mathbb{R}^d$ using a different approach. In particular, an abstract perturbation argument is used in suitable weighted Sobolev spaces. One aim of the alternative strategy is that it can eventually be generalized to obtain persistence results for hyperbolic invariant sets for other nonlocal evolution equations on unbounded domains with small nonlocality.

Keywords: Fisher-KPP equation, FKPP, nonlocal convolution operator, steady state, stationary solution, weighted Sobolev space, implicit function theorem, perturbation theory.

1 Introduction

The (local) Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation is given by

$$\frac{\partial u}{\partial t} = \Delta u + \mu u(1-u), \quad x \in \mathbb{R}^d, \quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}, \quad u = u(x, t),$$

(1)

where $\mu > 0$ is a parameter and $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian. Originally the equation was studied by Fisher [20] and Kolmogorov-Petrovskii-Piscounov [30] for $d = 1$ with a focus on traveling waves connecting the two homogeneous stationary (or steady) states $u \equiv 0$ and $u \equiv 1$. The FKPP equation has been studied extensively as a standard model for invasion waves in mathematical biology [42, 39] and for propagation into unstable states in physics [16], often with a focus on the wave speed [41, 5].

This paper is directly motivated by the results of Berestycki et al. [7] who considered a nonlocal version of FKPP equation

$$\frac{\partial u}{\partial t} = \Delta u + \mu u(1 - \phi \ast u),$$

(2)
where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a kernel for the convolution

$$(\phi * u)(x) := \int_{\mathbb{R}^d} u(x - y)\phi(y) \, dy = \int_{\mathbb{R}^d} u(y)\phi(x - y) \, dy.$$  

The term $\phi * u$ models nonlocal saturation or competition effects [31]. The nonlocal FKPP equation (2) has been studied from various perspectives [1, 8, 9, 17, 23, 25, 40] with a focus on stability analysis of steady states and the existence of traveling waves; there are also several results available for bounded domains [22, 43] instead of $\mathbb{R}^d$. Other types of nonlocality may arise instead of the nonlinear term $u(\phi * u)$ [11, 10, 44], generalizations of (2) have been considered [16, 47] as well as the nonlocal bistable case [2]. Furthermore, there are several other different nonlocal versions of the FKPP equation involving time delay [3, 48] or nonlocal diffusion via fractional operators [13, 18, 36].

In this paper we focus on the analysis of stationary solutions for the FKPP equation (2) which satisfy

$$0 = \Delta u + \mu u(1 - \phi * u), \quad x \in \mathbb{R}^d, \ u : \mathbb{R}^d \to \mathbb{R}, \ x \mapsto u(x). \tag{3}$$

The main goal is to understand certain subclasses of classical solutions $u \in C^2_b(\mathbb{R}^d, \mathbb{R}) =: C^2_b$ which satisfy (3) pointwise. However, we shall need weaker solution spaces to infer properties about the relevant classical solutions. It is assumed in [7] that the kernel satisfies

$$\phi \geq 0, \quad \phi(0) > 0, \quad \nabla \phi \in C_b(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \phi(x) \, dx = 1, \quad \int_{\mathbb{R}} x^2\phi(x) \, dx < \infty. \quad (4)$$

Observe that $u \equiv 0$ is a solution of (3) and since $\int_{\mathbb{R}^d} \phi \, dx = 1$ it follows that $u \equiv 1$ is also always a solution of (3). Note that a Gaussian probability density and a (suitably extended) exponential probability density satisfy the assumptions (4). Considering $\tilde{x} := \sigma x$, a function $\tilde{u}(\tilde{x}) := u(\tilde{x}/\sigma) = u(x)$ and kernel $\phi_{\sigma}(\tilde{x}) := \frac{1}{\sigma^d}\phi(\tilde{x}/\sigma) = \frac{1}{\sigma^d}\phi(x)$ one finds, upon dropping the tildes and letting $\mu = \sigma^2$, that

$$0 = \Delta u + u(1 - \phi_{\sigma} * u) =: F(u, \sigma), \tag{5}$$

Hence, upon a space rescaling, the equations (3) and (5) are equivalent if $\mu \neq 0$ and $\sigma \neq 0$. We shall work with the version (5) from now on. The kernel $\phi_{\sigma}$ converges to a delta-distribution $\lim_{\sigma \to 0} \phi_{\sigma}(x) = \delta(x)$. In the limit $\sigma = 0$ for (5) one recovers the standard elliptic (local) FKPP steady state problem

$$0 = \Delta u + u(1 - u), \tag{6}$$

which again has the homogeneous steady-states $u \equiv 0$ and $u \equiv 1$. Note that the formal limits $\mu = 0$ and $\sigma = 0$ do not coincide, if the spatial scaling is disregarded, since

$$\text{(5) for } \sigma \to 0 \Rightarrow 0 = \Delta u + u(1 - u) \quad \not\Rightarrow \quad \text{(3) for } \mu \to 0 \Rightarrow 0 = \Delta u.$$  

For the one-dimensional case, the following result for the nonlocal FKPP-equation is known:
Theorem 1.1 (\([7], \, d = 1\)). Suppose the assumptions (4) hold. There exists \(\sigma_0 > 0\) \((\mu_0 > 0)\) such that for \(\sigma \in [0, \sigma_0] \,(\mu \in (0, \mu_0))\) the only bounded non-negative classical solutions of the stationary nonlocal FKPP equation (5) \((respectively \, 3)\) are \(u \equiv 0\) and \(u \equiv 1\).

Theorem 1.1 is a persistence result which shows that if the nonlocal effect is sufficiently small then there are no additional non-negative bounded solutions beyond the two trivial ones. The proof by Berestycki et al. \([7]\) uses a combination of a-priori estimates, explicit Taylor expansion, approximation on finite regions and several integral estimates.

In this paper we generalize Theorem 1.1 to arbitrary dimensions also lifting several assumptions on the kernel \(\phi\). This main result is stated in Section 2. We note that our proof does not use the approach in \([7]\). We use a perturbation technique involving the implicit function theorem in suitable function spaces, bifurcation theory and knowledge about the limiting equation (6) for \(\sigma = 0\); the strategy of the proof is outlined in Section 2.

2 The Main Result

Instead of the assumptions (4) we shall require that

(A) \(\phi \in L^1(\mathbb{R}^d), \, \phi \geq 0\) with \(\int_{\mathbb{R}^d} \phi(x) \, dx = 1\).

We note that some of the assumptions (4) have also been removed in \([26]\) for the case \(d = 1\). However, to remove any form of integrability or boundedness assumption of the kernel \(\phi\) seems to be very difficult, if not impossible.

Theorem 2.1 \((d \in \mathbb{N})\). Suppose (A) holds and \(u\), with \(0 \leq u \leq K_u\) for a constant \(K_u > 0\), is a bounded non-negative classical solution of the stationary nonlocal FKPP equation (\([5]\)) \((respectively \, 3)\), i.e. \(u\) is non-negative and \(u \in C^2_b(\mathbb{R}^d)\). Then there exists \(\sigma_0 > 0\) \((\mu_0 > 0)\) such that for \(\sigma \in [0, \sigma_0] \,(\mu \in (0, \mu_0))\) the only bounded non-negative classical solutions are \(u \equiv 0\) and \(u \equiv 1\).

We point out that in Theorem 2.1 the order of quantifiers implies that the constant \(\sigma_0\) \((respectively \, \mu_0 > 0)\) may depend upon \(K_u > 0\). We outline, on a formal level, the main steps of the perturbation argument:

(S1) The local problem: The case \(\sigma = 0\) is well understood and we collect the relevant results later in this section. In particular, it is known that the only bounded non-negative solutions of the (local) FKPP equation (6) are \(u \equiv 0\) and \(u \equiv 1\).

(S2) Function spaces: To analyze the problem we utilize spaces for weak solutions to infer results about classical solutions. In particular, one would like to choose Banach spaces which include the homogeneous stationary solutions and are adapted to the linear and nonlinear parts of the mapping \(F(u, \sigma)\) induced by the nonlocal FKPP equation (5). We use suitably weighted Sobolev spaces and their intersections in this paper; see Section 3.
(S3) **Regular points:** It turns out that the solution \((u, \sigma) = (1, 0)\) can be viewed as a regular point of \(F\), i.e. the Fréchet derivative \((D_u F)_{(1,0)}\) is invertible as a linear map in the spaces chosen in (S2). Then the implicit function theorem shows that the solution branch \((u, \sigma) = (1, 0)\) is locally unique for sufficiently small \(\sigma\); see Section 4.

(S4) **Special points:** The solution \((u, \sigma) = (0, 0)\) cannot be treated directly using the implicit function theorem. Hence it requires a special technique. We use results about purely oscillatory solutions to show that any possible bifurcating solutions near the special point must change sign; see Section 5.

(S5) **Additional branches:** Using an argument from bifurcation theory we show that there are no additional branches of non-negative bounded solutions outside of neighborhoods of the two states \((u, \sigma) = (0, 0)\) and \((u, \sigma) = (1, 0)\) for sufficiently small \(\sigma_0 > 0\); see also Figure 1(a)-(b). The details of this argument can be found in Section 6.

It is important to point out that the steps (S1)-(S5) have been designed with a view towards other persistence problems arising in nonlocal evolution equations; this extension is discussed in Section 7. Let us also remark that the main technical complications arise due to the unbounded domain \(\mathbb{R}^d\) and the convolution term \(\phi_{\sigma} \ast u\) which substantially restrict the type of spaces one may use in (S2).

\[
\|u\|_X \quad (a) \quad (b) \quad (c) \quad (d) \quad u \equiv 1 \quad \sigma \quad u \equiv 0
\]

Figure 1: Sketch of possible bifurcation scenarios in \((\|u\|_X, \sigma)\)-space. The thick lines mark the homogeneous stationary states \(u \equiv 0\) and \(u \equiv 1\). The thin curves indicate possible bifurcation curves. The cases (a)-(b) are impossible as they would violate the result for the local problem \(\sigma = 0\). The case (c) near \((u, \sigma) = (1, 0)\) will be shown to be impossible using the implicit function theorem. We are going to show that the only solutions that could potentially bifurcate near \((u, \sigma) = (0, 0)\) are solutions which change sign.

Regardless of these complications, it is always key to understand the step (S1) for a perturbation argument. In particular, consider the PDE

\[
0 = \Delta u + uf(u), \quad x \in \mathbb{R}^d, u : \mathbb{R}^d \to \mathbb{R}, x \mapsto u(x). \tag{7}
\]

Suppose \(f : \mathbb{R} \to \mathbb{R}\) is \(C^2\). The following result is known:

**Theorem 2.2.** *(see e.g. [6, 31]*) Suppose \(f(u) < 0\) for all \(u \in \mathbb{R}^+\) with \(u \geq \beta_0\) for some \(\beta_0 > 0\) and \(f'(u) < 0\) for all \(u \in \mathbb{R}^+ = (0, +\infty)\). Then there exists a unique positive solution \(u^* \in C^2(\mathbb{R}^d, \mathbb{R}^+)\) for (7) such that \(\inf_{x \in \mathbb{R}^d} u^*(x) > 0\).
Further variations and generalizations of the previous result are also known \[4, 46\]. For \(f(u) = 1 - u\) one may just apply Theorem 2.2 with \(\beta_0 = 1\) – see also \[15\] – which yields the following:

**Lemma 2.3.** The only non-negative solutions of (6) are \(u \equiv 0\) and \(u \equiv 1\).

In particular, consider a bifurcation diagram in the space \(\{ (\sigma, \|u\|_X) \}\), where the norm \(\| \cdot \|_X\) is defined in the next section, as shown in Figure 1. The idea is to show that there cannot be any solution branches tangent to or crossing the vertical segments \(\{ 0 \} \times (0, 1)\) and \(\{ 0 \} \times (1, \infty)\). However, this does not exclude bifurcating solution branches from \((\sigma, u) = (0, 0)\) or \((\sigma, u) = (0, 1)\).

### 3 Function Spaces

To analyze the local behavior of solutions one possibility is to consider the problem via an abstract nonlinear map. Let \(X, Y\) be Banach spaces and \(I = [0, \sigma_0] \subset \mathbb{R}\) be an interval for some \(\sigma_0 > 0\) chosen sufficiently small. Then the mapping \(F : X \times I \to Y\) given by

$$F(u, \sigma) = \Delta u + u(1 - \phi_{\sigma} \ast u)$$

(8)

has as a zero set \(\{ (u, \sigma) \in X \times I : F(u, \sigma) = 0 \}\) the stationary solutions in \(X\) of the nonlocal FKPP equation. A choice of function spaces \(X\) and \(Y\) is required to carry out the analysis explicitly. It seems natural to consider Sobolev spaces \(W^{k,p}(\mathbb{R}^d)\) for \(X\) and \(Y\). However, \(u(x) \equiv 1\) and other nonzero constants do not belong to \(W^{k,p}(\mathbb{R}^d)\). Since we eventually want to compute a (Fréchet) derivative of \(F\) at \((u, \sigma) = (1, 0)\) the standard Sobolev spaces do not suffice. Another natural option would be Hölder spaces \(C^{k+\gamma}(\mathbb{R}^d)\) for \(k \in \mathbb{N}_0\) and \(\gamma \in (0, 1)\). The norm in \(C^{k+\gamma}(\mathbb{R}^d)\) is given by

$$\|u\|_{C^{k+\gamma}(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\infty} + \langle u \rangle_{C^{k+\gamma}(\mathbb{R}^d)},$$

where \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)\) is a multi-index with \(\alpha_i \in \mathbb{N}_0, |\alpha| = \sum_{i=1}^d \alpha_i\) and

$$\|D^\alpha u\|_{\infty} = \sup_{x \in \mathbb{R}^d} |D^\alpha u(x)|, \quad \langle u \rangle_{C^{k+\gamma}(\mathbb{R}^d)} = \sum_{|\alpha| = k} \sup_{x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma}.$$

\(C^{k+\gamma}(\mathbb{R}^d)\) does contain the constants but in this case there are problems with the convolution term \(\phi_{\sigma} \ast u\) which leads to complications regarding continuity properties of the mapping \(F\). In this paper we use weighted Sobolev spaces to avoid these problems. However, there could certainly be other good choices which we do not consider here.

Let \(w \in L^1(\mathbb{R}^d)\) denote a positive weight function and define the weighted Sobolev space \(W^{k,p}(\mathbb{R}^d; w)\) for \(p \in (1, \infty)\) as

$$W^{k,p}(\mathbb{R}^d; w) = \{ u : w^{1/p}(D^\alpha u) \in L^p(\mathbb{R}^d) \text{ for all } |\alpha| \leq k \}.$$
With the norm
\[ \|u\|_{k,p,w} := \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^d} |D^\alpha u(x)|^p w(x) \, dx \right)^{1/p} \]
the space \( W^{k,p}(\mathbb{R}^d; w) \) is a Banach space [32]. Note that for \( k = 0 \) we just have \( L^p(\mathbb{R}^d; w) := W^{0,p}(\mathbb{R}^d; w) \) with norm \( \| \cdot \|_{p,w} \). If \( p = 2 \) we shall also use the standard notation \( H^k(\mathbb{R}^d; w) := W^{k,2}(\mathbb{R}^d; w) \) for the Hilbert space case. We start by fixing the main spaces
\[ X = W^{k+2,p}(\mathbb{R}^d; w_X) \cap W^{k,2p}(\mathbb{R}^d; w_X) \quad \text{and} \quad Y = W^{k,p}(\mathbb{R}^d; w_Y) \quad (9) \]
for \( k \in \mathbb{N}_0 \) and the norm on \( X \) is given by
\[ \|u\|_X = \max\{\|u\|_{k+2,p,w_X}, \|u\|_{k,2p,w_X}\}. \]
With a view towards bounded classical solutions, we observe that one always has \( C_b^2(\mathbb{R}^d) \subset X \). For concreteness, we make a choice of weights as well as exponent \( p \).

(B) Let \( \| \cdot \| \) denote the usual Euclidean norm on \( \mathbb{R}^d \) and let \( l_X, l_Y \) be constants such that \( \frac{1}{2} < l_X \leq l_Y < \infty \). For the spaces \( X \) and \( Y \) make the choice
\[ p = 2, \quad w_X(x) = \frac{1}{(1 + \|x\|^2)^{l_X}}, \quad w_Y(x) = \frac{1}{(1 + \|x\|^2)^{l_Y}}. \quad (10) \]
For (10) we have the pointwise estimate \( w_Y(x) \leq w_X(x) \) for all \( x \in \mathbb{R}^d \) which implies that one may order the associated norms
\[ \|u\|_{k,p,w_Y} \leq \|u\|_{k,p,w_X}. \quad (11) \]
The choice of spaces requires some explanation. To use the intersection of two weighted spaces for \( X \) is convenient to gain continuity of \( F \) as the nonlinear term \( u(\phi_\sigma \ast u) \) is essentially a product of two terms. The particular choice (B) is going to allow us to use some known results about \( H^{k+2}(\mathbb{R}^d; w_X) \) which are helpful to shorten the proof. Appendix [C] contains a sketch how one may approach the case (9) for more general \( p, w_X \) and \( w_Y \) and avoid using results about \( H^{k+2}(\mathbb{R}^d; w_X) \). It seems important to include this outline as we shall explain in the remarks after Proposition 4.2.

Remark: The choice (B) is an auxiliary tool for the analysis and the proof of Theorem 2.1. Essentially the idea is analogous to existence and regularity theory for elliptic equations with smooth input data where weak solution spaces are an analytic tool but do not appear in the final result.

Before we establish the continuity and differentiability properties of (8) one may check that \( F : X \to Y \) is indeed a well-defined map.

Lemma 3.1. Suppose (A)-(B) hold then the map \( F : X \times I \to Y \) is well-defined, i.e., if \( (u, \sigma) \in X \times I \) then \( F(u, \sigma) \in Y \).
Proof. By the triangle inequality and (11) we have
\[ \|F(u, \sigma)\|_Y = \|\Delta u + u(1 - \phi \ast u)\|_{k,p;w_Y} \leq \|\Delta u\|_{k,p;w_X} + \|u\|_{k,p;w_X} + \|u(\phi \ast u)\|_{k,p;w_X} \]
and the first two terms are finite since \( u \in W^{k+2,p}(\mathbb{R}^d; w_X) \). For the third term we have
\[ \|u(\phi \ast u)\|_{k,p;w_X} = \sum_{|\alpha| \leq k} \|(D^\alpha u)(\phi \ast u) + u(\phi \ast D^\alpha u)\|_{p;w_X}^p \leq \sum_{|\alpha| \leq k} \|\|D^\alpha u\|_{p;w_X} + \|u(\phi \ast D^\alpha u)\|_{p;w_X}\|^p. \]
Furthermore, by Cauchy-Schwarz it follows that
\[ \|(D^\alpha u)(\phi \ast u)\|_{p;w_X} \leq \|D^\alpha u\|_{2p;w_X} \|\phi \ast u\|_{2p;w_X}, \quad (12) \]
\[ \|u(\phi \ast D^\alpha u)\|_{p;w_X} \leq \|u\|_{2p;w_X} \|\phi \ast D^\alpha u\|_{2p;w_X}. \quad (13) \]

Upper bounds for the terms \( \|\phi \ast u\|_{2p;w_X} \) and \( \|\phi \ast D^\alpha u\|_{2p;w_X} \) from (12)-(13) can be obtained by using the generalized Young’s inequality from Appendix A and the assumption \( \|\phi\|_1 = 1 \) from (A)
\[ \|D^\alpha u\|_{2p;w_X} \|\phi \ast u\|_{2p;w_X} \leq \|D^\alpha u\|_{2p;w_X} \|u\|_{2p;w_X}, \quad (14) \]
\[ \|u\|_{2p;w_X} \|\phi \ast D^\alpha u\|_{2p;w_X} \leq \|u\|_{2p;w_X} \|D^\alpha u\|_{2p;w_X}. \quad (15) \]
Since \( u \in X \) by assumption the result follows. \( \square \)

Lemma 3.2. Suppose (A)-(B) hold then the mapping \( F : X \times I \to Y \) is continuous in \( \sigma \) and continuously differentiable in \( u \) with Fréchet derivative
\[ (D_u F)(u, \sigma) \Delta U = \Delta U + (1 - \phi \ast u)U - u(\phi \ast U). \]

Proof. For continuity in \( \sigma \) we just focus on continuity at \( \sigma = 0 \), the case \( \sigma > 0 \) is easily checked. We have
\[ \|F(u, \sigma) - F(u, 0)\|_Y^p = \|u(\phi \ast u - u)\|_{k,p;w_Y}^p \leq \sum_{|\alpha| \leq k} \|\|D^\alpha u(\phi \ast u - u)\|_{p;w_X} + \|u(\phi \ast D^\alpha u - D^\alpha u)\|_{p;w_X}\|^p. \]
Similar to the proof of Lemma 3.1 the Cauchy-Schwarz inequality yields that
\[ \|D^\alpha u(\phi \ast u - u)\|_{p;w_X} \leq \|D^\alpha u\|_{2p;w_X} \|\phi \ast u - u\|_{2p;w_X}, \quad (16) \]
\[ \|u(\phi \ast D^\alpha u - D^\alpha u)\|_{p;w_X} \leq \|u\|_{2p;w_X} \|\phi \ast D^\alpha u - D^\alpha u\|_{2p;w_X}. \quad (17) \]
Recall that there is strong convergence \( \phi \ast v \to v \) for \( v \in L^r(\mathbb{R}^d) \) with \( 1 \leq r < \infty \) [35, p.64] as \( \sigma \to 0 \); the proof can be adapted to yield strong convergence in \( L^r(\mathbb{R}^d; w_X) \) as shown in Appendix A Setting \( r = 2p \) yields that \( \|F(u, \sigma) - F(u, 0)\|_Y \to 0 \) as \( \sigma \to 0 \) which yields continuity at \( \sigma = 0 \). Next, fix \( \sigma \in I \) and let \( u, v \in X \) then we obtain for continuity in \( u \) that
\[ \|F(u, \sigma) - F(v, \sigma)\|_Y \leq \|\Delta(u - v)\|_Y + \|u - v\|_Y + \|u(\phi \ast u) - v(\phi \ast u)\|_Y, \]
where the first two terms in the right-hand side can be made small since \( X \subset Y \). For the last term a direct calculation shows that

\[
\|u(\phi_\sigma * u) - v(\phi_\sigma * v)\|_Y \leq \|(u - v)(\phi_\sigma * u)\|_Y + \|v(\phi_\sigma * (u - v))\|_Y,
\]

where both summands can be made small by using Cauchy-Schwarz and the generalized Young inequality as in Lemma 3.1 which implies continuity in \( u \). Calculating the Gâteaux derivative in \( u \) yields

\[
(\nabla_u F)[U] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [F(u + \epsilon U, \sigma) - F(u, \sigma)] = \Delta U - u(\phi_\sigma * U) + (1 - \phi_\sigma * u)U.
\]

To show that the Gâteaux derivative coincides with the Fréchet derivative, i.e. \( \nabla_u F = D_u F \), we have to verify continuity of \( \nabla_u F \) in \( u \) [12, p.47]. We have

\[
\|(\nabla_u F)[U] - (\nabla_v F)[U]\|_Y \leq \|(u - v)(\phi_\sigma * U)\|_Y + \|v(\phi_\sigma * (u - v))U\|_Y
\]

and the same argument as for proving continuity of \( F \) in \( u \) can be applied.

Based on the proof it is now more evident why the construction of \( X \) is essentially enforced by the nonlinear structure of the nonlocal FKPP equation. For example, for \( k = 0 \) and \( p = 2 \) the quadratic term \( u^2 \) for \( \sigma = 0 \) indicates that the space \( L^4(\mathbb{R}^d; w_X) \) would be a good choice. Observe also that one does not have to make the precise choice (B) to prove Lemma 3.2. In fact, what is required is the weaker assumption that there exists a positive constant \( K > 0 \), possibly dependent upon \( p \) and \( k \), such that

\[
\|u\|_{k,p; w_Y} \leq K \|u\|_{k,p; w_X}
\]

for a choice of weights for which Young’s inequality still holds. Hence the key parts for continuity and differentiability properties of \( F \) in Lemma 3.2 do not depend upon the concrete choice (10) but only on constructing an intersection of weighted Sobolev spaces for \( X \) adapted to the nonlinearity with sufficiently ‘nice’ weights.

4 The Implicit Function Theorem

The next goal is to apply the implicit function theorem; see [12, p. 148] for a detailed statement. In particular, we want to consider neighborhoods of \( (u, \sigma) = (1, 0) \) and \( (u, \sigma) = (0, 0) \). Since \( w \in L^1(\mathbb{R}^d) \) the non-zero constants belong to \( X \). Substituting the stationary solutions \( (u, \sigma) = (1, 0) \) and \( (u, \sigma) = (0, 0) \) into Lemma 3.2 yields:

**Lemma 4.1.** The linearized operators acting on \( U \in X \) with \( U = U(x), x \in \mathbb{R}^d \), are

\[
\mathcal{L}_0 := (D_u F)_{(0,0)}U = \Delta U + U, \tag{19}
\]
\[
\mathcal{L}_1 := (D_u F)_{(1,0)}U = \Delta U - U. \tag{20}
\]
For the one-dimensional case $d = 1$, one gets the ordinary differential equations (ODEs)

\[
(\text{ODE})_0 : \begin{cases}
U' = V, \\
V' = -U,
\end{cases} \quad \text{and} \quad (\text{ODE})_1 : \begin{cases}
U' = V, \\
V' = U.
\end{cases}
\]

Clearly, $(\text{ODE})_1$ has a saddle-point at the origin and hence $U'' - U = 0$ has no other bounded solutions except $U \equiv 0$. However, $(\text{ODE})_0$ has a center equilibrium with infinitely many bounded non-zero periodic solutions. This indicates that $\mathcal{L}_0$ does not have the required inverse to apply the implicit function theorem. Hence we are going to treat the neighborhood of the zero branch $u \equiv 0$ separately in Section 5.

For $\mathcal{L}_1$ on $\mathbb{R}^d$ it is tempting to consider the Fourier transform

\[
\hat{u}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix^T\xi} u(x) \, dx, \quad \xi \in \mathbb{R}^d,
\]

where $(\cdot)^T$ denotes the transpose and apply it to $\mathcal{L}_1 U = 0$. This yields

\[
(1 + \|\xi\|^2) \hat{U}(\xi) = 0,
\]

where $\| \cdot \|$ denotes the Euclidean norm. From (21) it follows that $\hat{U} = 0$ which implies that the nullspace of $\mathcal{L}_1$ only contains the zero solution. However, this calculation assumes that $u \in L^p(\mathbb{R}^d)$ for some suitable $p$, e.g. $p = 1$ or $p = 2$, whereas we have to work in weighted spaces. To illustrate the problem, we consider the one-dimensional case which can be solved explicitly. If $d = 1$ we have $0 = \mathcal{L}_1 U = U'' - U$ so that the general solution is

\[
U(x) = c_1 e^{-x} + c_2 e^x \quad \text{for constants } c_1, c_2 \in \mathbb{R}.
\]

If we would choose $w(x) = e^{-x^2}$ it follows that for constants $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in (0, \infty)$, we also have

\[
\int_{\mathbb{R}} e^{\alpha_1 x} e^{-\alpha_2 x^2} \, dx = \frac{\alpha_1^2}{\alpha_2} \sqrt{\frac{\pi}{\alpha_2}} < \infty.
\]

For the case $p = 2$, $k = 0$ the last calculation yields that $\mathcal{L}_1 U = 0$ has non-zero solutions in $H^2(\mathbb{R}; e^{-x^2}) \cap L^4(\mathbb{R}; e^{-x^4})$ which implies that $\mathcal{L}_1 : X \to Y$ is not invertible. For the case $w(x) = (1 + x^2)^{-1}$ it is straightforward to calculate that the integration

\[
\int_{\mathbb{R}} \frac{U(x)^2}{1 + x^2} \, dx = \int_{\mathbb{R}} \frac{(c_1 e^{-x} + c_2 e^x)^2}{1 + x^2} \, dx
\]

implies that $U(x) w^{1/2}(x) \in L^2(\mathbb{R})$ if and only if $c_1 = 0 = c_2$. Therefore, $\mathcal{L}_1 U = 0$ on $H^2(\mathbb{R}; (1 + x^2)^{-1}) \cap L^4(\mathbb{R}; (1 + x^2)^{-1})$ if and only if $U \equiv 0$ which implies that $\mathcal{L}_1$ has trivial nullspace when $w(x) = (1 + x^2)^{-1}$ is used. Hence, the choice of weight function is crucial if we want to apply the implicit function theorem at the constant solution $(u, \sigma) = (1, 0)$.

A natural strategy is to prove that $\mathcal{L}_1 : X \to Y$ has trivial nullspace for suitable classes of $w_X, w_Y, p$ and then show that nullspace($\mathcal{L}_1$) = {0} implies that $\mathcal{L}_1$ is invertible in a rather large class of weighted spaces. For our choice of $w_X, w_Y, p$ given in (B) one may directly use previous results.
Proposition 4.2. For the choice (10), i.e., under assumption (B)

\[ p = 2, \quad w_X(x) = \frac{1}{(1 + \|x\|^2)^{1/2}}, \quad w_Y(x) = \frac{1}{(1 + \|x\|^2)^{1/2}}, \]  

(23)

with \( \frac{1}{2} < l_X \leq l_Y < \infty \), the operator \( L_1 : X \to Y \) is invertible with bounded inverse.

Proof. Apply [24, Thm. 4.2, see p.58] which makes the argument based upon the Fourier transform precise for the weighted spaces defined via (23); the result [24, Thm. 4.2, see p.58] uses that the symbol in (21) has no zeros which is easily checked.

It is important to note that we would have to establish similar results as Proposition 4.2 for persistence results in other types of nonlocal PDEs again, as the linearized problem may change. Therefore, we provide an outline in Appendix C how invertibility can be established via a more direct approach.

5 The Zero Solution

As discussed in the previous section, one has to treat \( L_0 \) separately. First, we apply Taylor’s Theorem (in \( u \)) to \( F : X \to Y \), which yields

\[ F(u, \sigma) = F(0, \sigma) + D_u F_{(0,\sigma)} u + R(u), \]  

(24)

where the remainder \( R(u) \) satisfies \( R(u) = \mathcal{O}(\|u\|_{X}^2) \). Note that for the nonlocal FKPP equation the remainder is exact and given by \( R(u) = -u(\phi \ast u) \). However, we shall only need the existence of a bound for the remainder when \( \|u\|_X \) is small; one may calculate this bound efficiently also for other equations, not only FKPP, by using the Lagrange or integral forms of the remainder (see e.g. [14, Ch.5]). Since we always have \( F(0, \sigma) = 0 \) it follows from (24) that

\[ F(u, \sigma) = D_u F_{(0,\sigma)} u + R(u) = \Delta u + u + R(u). \]  

(25)

Let \( Z := X \cap L^\infty(R^d) \) with norm \( \| \cdot \|_Z = \max\{\| \cdot \|_X, \| \cdot \|_\infty\} \) and observe that we may make the last term in (25) small if \( u \) is in a neighborhood of \( u = 0 \) i.e. when \( \|u\|_Z \) is small. We start by proving a one-dimensional result to illustrate the main idea.

Proposition 5.1. Let \( d = 1 \) then there exists \( \sigma_0 > 0 \) and a ball \( B \subset Z \) centered at \( u = 0 \) such that \( B \times [0, \sigma_0] \) does not contain any non-negative bounded classical solution \( u \neq 0 \), \( u \in C^2_0 \), to \( F(u, \sigma) = 0 \).

Proof. Fix \( \epsilon > 0 \). Consider any non-negative bounded classical solution \( (u, \sigma) \in B \times (0, \sigma_0] \) with \( u \neq 0 \) and define \( \eta := \phi \ast u \) then we obtain

\[ F(u, \sigma) = u'' + u(1 - \eta), \]

where \( |\eta(x)| < \epsilon \) holds for all \( x \in \mathbb{R} \) as \( \|u\|_Z \) can be made small; note that this step would have worked with a general small remainder for a Taylor expansion. Considering \( F(u, \sigma) = 0 \) leads to the two-dimensional non-autonomous vector field

\[ \begin{align*}
\frac{du}{dt}(x) &= v(x), \\
\frac{dv}{dt}(x) &= -u(x)(1 - \eta(x)).
\end{align*} \]  

(26)
We argue by contradiction and suppose there exists a nonzero solution \((u, v)\) to (26) such
that \(u(x) \geq 0\) for all \(x \in \mathbb{R}\) and there exists some \(x\) where \(u(x) > 0\). We have the pointwise estimate

\[-u(x)(1 + \epsilon) < v'(x) < -u(x)(1 - \epsilon)\] (27)

for all \(x \in \mathbb{R}\). A phase plane analysis can now be carried out. Since \(u(x) > 0\) for some \(x \in \mathbb{R}\) we may take this value as \(x = 0\) without loss of generality and consider

\((u(0), v(0)) \in \{(0, \infty) \times (-\infty, \infty)\} - \{(0, 0)\}.

Furthermore, if \(v(0) \leq 0\) then we immediately get that \(v(x) < 0\) for some \(x > 0\) close to zero. If \(v(x) < 0\) it follows that \(u\) will decay until we have \(u(x^*) < 0\) for some \(x^* > 0\) which yields a contradiction. Hence, we are left with the case \(v(0) > 0\). However, if \(v(x) > 0\) then \(u\) increases so the estimate (27) can be applied to show that \(v\) has to cross \(\{v = 0\}\) into the region where \(v(x) < 0\). This finishes the proof. \(\square\)

The key ideas of the last proof are that we just look at solutions inside a small neighborhood of the origin \((u, \sigma) = (0, 0)\) in \(X \times I\) and that any nonzero solution \(u\) must change sign for an equation which is controlled by solutions of

\[u'' + u(1 \pm \epsilon) = 0.\]

To generalize this idea we briefly recall some classical theory for linear homogeneous partial differential operators [29, 28] focusing just on the operator given by

\[\mathcal{L}_0^\alpha U := \Delta U + (1 + \alpha)U, \quad \alpha \in \mathbb{R}, \quad |\alpha| < \epsilon\]

for sufficiently small fixed \(\epsilon\) with \(0 < \epsilon < 1\).

Remark: We are going to develop the following ideas in slightly more generality than strictly necessary here to illustrate that the approach is applicable to a much broader class of problems.

A solution \(U = U(x)\) of \(\mathcal{L}_0^\alpha U = 0\) is called an exponential solution if it is of the form

\[U(x) = f(x)e^{i\xi^T x}, \quad \text{for } \xi \in \mathbb{C}^d,\] (28)

where \(f\) is a polynomial. We say that a solution of the form (28) is a simple exponential solution if \(f \equiv 1\) and \(\xi \neq 0\). Furthermore, we call a simple exponential solution purely oscillatory if \(\xi \in \mathbb{R}^d\). The next result is a slightly modified version of [29, Thm. 7.3.6, p.185].

**Proposition 5.2.** The closed linear hull of simple exponential solutions to \(\mathcal{L}_0^\alpha U = 0\) in the space \(C^\infty(\mathbb{R}^d)\) yields all solutions to \(\mathcal{L}_0^\alpha U = 0\) in \(C^\infty(\mathbb{R}^d)\).

**Proof.** To show that the closed linear hull of exponential solutions contains all solutions follows verbatim from [29, Thm. 7.3.6] as \(\mathcal{L}_0^\alpha\) is a special case of the elliptic operators covered. To restrict the class to simple exponential solutions, one observes that the symbol of \(\mathcal{L}_0^\alpha\) has no multiple factors which makes the remark in [28, p.39] applicable such that \(f(x) \equiv 1\) for all exponential solutions. Since we restrict to nonzero solutions it follows that \(\xi \neq 0\) can be assumed as well since \(\mathcal{L}_0^\alpha U = 0\) has no other constant solutions except \(U \equiv 0\). \(\square\)
The next result is not necessary to prove the main result of this paper but it is very interesting to understand the structure of solutions to $L^a_0 U = 0$ in weighted spaces.

**Proposition 5.3.** Suppose (B) holds with weight function $w_X \in L^1(\mathbb{R}^d)$. Suppose $L^a_0 U = 0$ and $U$ is in the linear hull of simple exponential solutions. Then $U \in X$ if and only if $U$ is in the linear hull of purely oscillatory solutions.

*Proof.* See Appendix [x].

The next result, albeit straightforward, is a crucial observation to deal with solutions near the point $(u, \sigma) = (0,0)$ in the proof of Theorem 2.1.

**Lemma 5.4.** There exists a purely oscillatory solution of $L^a_0 U = 0$ such that $U$ changes sign i.e. there exist two points $x^-, x^+ \in \mathbb{R}^d$ such that $U(x^-) < 0$ and $U(x^+) > 0$.

*Proof.* It is easy to see that $U(x) = \cos(\sqrt{1 + \alpha x_1})$. In fact, there are many other trigonometric functions which would work as well.

The last proof also showed that $L^a_0 : X \to Y$ is not invertible as it does have a non-trivial nullspace; cf. Appendix [x]. The next step is to exclude the existence of non-negative bounded solutions $u \not\equiv 0$ for sufficiently small nonlocality near the special point $(u, \sigma) = (0,0)$ for arbitrary dimension $d$.

**Proposition 5.5.** There exists $\sigma_0 > 0$ and a ball $B \subset Z$ centered at $u = 0$ such that $B \times [0, \sigma_0]$ does not contain any non-negative bounded classical solution $u \not\equiv 0$, $u \in C^2_b(\mathbb{R}^d)$, to $F(u, \sigma) = 0$.

*Proof.* Fix $\epsilon > 0$. Let $u \in B$ and $u \not\equiv 0$. If $u \geq 0$ then it follows $u(x) > 0$ for all $x \in \mathbb{R}^d$ by [x] Lem. 2.1. We argue by contradiction and suppose that there exists a solution $u(x) > 0$ for all $x \in \mathbb{R}^d$. In particular, this implies $u(0) > 0$. Observe that

$$0 = \Delta u + u(1 - \phi_\sigma * u) \geq \Delta u + u(1 - \epsilon)$$

holds with $0 < \epsilon < 1$ if $\|u\|_Z$ is sufficiently small, which we can achieve by shrinking $B$. Define the differential operator

$$L^\epsilon v := \Delta v + v(1 - \epsilon), \quad \text{with } 0 < \epsilon < 1.$$  

Now we choose a special radially symmetric oscillatory solution $\bar{u}^\epsilon$ with $\bar{u}^\epsilon(0) > 0$ which, for some $\delta_3 > \delta_2 > \delta_1 > 0$, obeys

$$\bar{u}^\epsilon(x) > 0 \text{ for } x \in \{\|x\| < \delta_1\} \cup \{\delta_2 < \|x\| \leq \delta_3\}$$

and

$$\bar{u}^\epsilon(x) < 0 \text{ for } x \in \{\delta_1 < \|x\| < \delta_2\}.$$  

To construct such a solution we may just take the solution from the proof Lemma 5.4 for dimension $d = 1$. For $d \geq 2$, we note that $L^\epsilon v = 0$ is just Helmholtz’s equation and a suitable
radial solution is given by scaled hyperspherical Bessel functions; see e.g. [19]. We may scale the solution \( \bar{u}^\epsilon \) by a positive constant, say \( \kappa > 0 \), and still denote it \( \bar{u}^\epsilon \), such that
\[
\bar{u}^\epsilon(x) > u(x) \quad \text{for} \quad x \in \{ \delta_2 + \delta < \|x\| < \delta_3 \},
\]
for some small constant \( \delta > 0 \), and a suitable \( \delta_3 \) such that \( \delta_2 + \delta < \delta_3 \) and the conditions (30)-(31) still hold. By making \( \kappa > \delta > 0 \) sufficiently large and using smoothness of \( u \) and \( \bar{u}^\epsilon \) it follows that the mapping (30)-(31) still hold. By making \( \kappa > \delta > 0 \) sufficiently large and using smoothness of \( u \) and \( \bar{u}^\epsilon \) it follows that the mapping (30)-(31) still hold. By making \( \kappa > \delta > 0 \) sufficiently large and using smoothness of \( u \) and \( \bar{u}^\epsilon \) yield the existence of a closed embedded \( (d - 1) \)-dimensional manifold \( \Gamma \), which consists of the zeros of \( u - \bar{u}^\epsilon \) in \( \{ \delta_2 < \|x\| < \delta_2 + \delta \} \), i.e., \( u(x) = \bar{u}^\epsilon(x) \) for \( x \in \Gamma \). In fact, \( \Gamma \) is smooth [33, Thm 8.2].

Furthermore, consider an arbitrary nonzero unit length vector \( v \in \mathbb{R}^d \) and the function \( \rho(\lambda) := (u - \bar{u}^\epsilon)(\lambda v) \) for \( \lambda \in (\delta_2, \delta_2 + \delta) \). By continuity, there exists a point \( p = \lambda v \in \Gamma \) such that \( \rho(p) = 0 \). Upon making \( \delta > 0 \) sufficiently small, and increasing \( \kappa > 0 \) if necessary, it follows that there exists a unique such point \( p \). Hence, it follows that \( \Gamma \) is a sphere. Using (29) we apply the comparison principle [35, Thm. 2.2.4] to \( L^\epsilon \) for the domain \( \Omega \) with boundary \( \partial \Omega = \Gamma \) which implies
\[
u(x) \geq \bar{u}^\epsilon(x) \quad \text{for all} \quad x \in \Omega.
\]
However, we can scale \( \bar{u}^\epsilon \) by a positive constant so that \( \bar{u}^\epsilon(x) > u(x) \) for some \( x \in \{ \|x\| < \delta_1 \} \). This contradiction to (33) concludes the proof.

We could have proved Proposition 5.5 slightly quicker and bypassed some of the preliminary discussion of the oscillatory solution space. However, it is important to point out the structure of special points. In particular, the linearized problem \( D_u F(0,0) u = 0 \), and a suitable small perturbation of it, only contain solutions outside of the class we are interested in (‘bounded non-negative classical solutions’). This observation facilitates the comparison approach and is a much more general strategy applicable to problems beyond the FKPP equation.

6 Proof of the Main Result

Recall that we try to establish that the only bounded non-negative classical solutions to the nonlocal FKPP equation for small nonlocality are \( u \equiv 0 \) and \( u \equiv 1 \).

Proof. (of Theorem 2.1) By Proposition 4.2 and the implicit function theorem applied to \( F : X \times I \to Y \) at \( (u, \sigma) = (1, 0) \) it follows that there exist balls \( B(1, r_1) \subset X \) and \( B(0, r_0) \subset I \) with some radii \( r_{0,1} > 0 \) and exactly one continuous map \( T : B(0, r_0) \to B(1, r_1) \) such that
\[
T(0) = 1 \quad \text{and} \quad F(T(\sigma), \sigma) = 0 \quad \text{on} \quad B(0, r_0).
\]
Since \( F(1, \sigma) = 0 \) gives a solution branch \( u \equiv 1 \) the uniqueness of \( T \) implies that \( T(\sigma) \equiv 1 \) for \( \sigma \in B(0, r_0) \). Hence, in a sufficiently small neighborhood \( N_1 \) of \( (u, \sigma) = (1, 0) \) only the homogeneous solution \( u \equiv 1 \) exists.

In a neighborhood \( N_0 \) of \( (u, \sigma) = (0, 0) \) we may apply Proposition 5.5 to conclude that the only branch of bounded non-negative classical solutions to \( F(u, \sigma) = 0 \) is given by \( u \equiv 0 \).
Hence it remains to consider the possibility of solutions outside of neighborhoods of the two trivial solution branches.

We argue by contradiction. Suppose there exists a sequence of bounded non-negative solutions \((u_j, \sigma_j)\) such that
\[
(u_j, \sigma_j) \in X \times I - (\mathcal{N}_0 \cup \mathcal{N}_1), \quad u_j \in C_b^2(\mathbb{R}^d), \quad u_j \leq K_u \text{ for all } j \in \mathbb{N},
\]
and we have
\[
F(u_j, \sigma_j) = 0, \quad \sigma_j \to 0 \text{ as } j \to \infty.
\]
Note that the sequence \(\{u_j\}\) is bounded in \(X_1 = W^{k+2,p}(\mathbb{R}^d; w_X)\) for \(k = 0\) and \(p = 2\), i.e., in \(X_1 = H^2(\mathbb{R}^d; w_X)\). Due to the theory of weighted Sobolev spaces and their associated compact embeddings [27, Prop. 2.5] we have that the embedding \(W^{k+2,2}(\mathbb{R}^d; w_X) \hookrightarrow W^{k+1,2}(\mathbb{R}^d; w_X)\) is compact. It follows, upon passing to a subsequence, \(u_j \to u_\infty\) for some \(u_\infty \in \bar{X}_1 := H^1(\mathbb{R}^d; w_X)\). By continuity of \(F\) from Lemma 3.2, it follows that \(u_\infty\) is a weak solution to the local problem
\[
F(u, 0) = 0
\]
i.e. for any test function \(\psi \in C_c^\infty(\mathbb{R}^d)\) we have
\[
- \int_{\mathbb{R}^d} \sum_{j=1}^d \frac{\partial u_\infty}{\partial x_j}(x) \frac{\partial \psi}{\partial x_j}(x) \, dx + \int_{\mathbb{R}^d} u_\infty(x)(1 - u_\infty(x))\psi(x) \, dx.
\]
Furthermore, \(u_\infty \geq 0\) almost everywhere as \(u_j(x) \geq 0\) for all \(x \in \mathbb{R}^d, j \in \mathbb{N}\). By elliptic regularity we can obtain that \(u_\infty \in C^2(\mathbb{R}^d)\). Therefore, the pair \((u_\infty, 0) \in X \times I - (\mathcal{N}_0 \cup \mathcal{N}_1)\) yields a non-negative solution to the local FKPP equation, which does not coincide with the two constant solutions. This contradicts Lemma 2.3 and the main result follows.

7 Outlook

From the details of the proof in Section 6 it is already evident that we have not shown the strongest possible result that our methods allow. For example, Theorem 2.2 for \(\sigma = 0\) holds for a much wider class of equations beyond the local FKPP equation. Also standard results about the Laplacian, which we used, do hold for much more general general elliptic operators. Hence one could rephrase our main theorem for certain nonlocal PDEs of the form
\[
0 = Au + f(u, \phi_\sigma * u), \quad x \in \mathbb{R}^d, \quad u = u(x),
\]
where \(A\) is a suitable generalization of the Laplacian and \(f\) is sufficiently smooth function which generalizes \(f(u, \phi_\sigma * u) = u(1 - \phi_\sigma * u)\). However, it seemed more accessible to illustrate the overall strategy on an example first. The generalization will be considered in future work.

We briefly point out how the steps (S1)-(S5) described in Section 2 have to be modified in an attempt to obtain results for (34) or even more general persistence theorems for solutions of nonlocal problems. The step (S1) should always be the starting point for a local perturbation argument. In particular, the relevant solution spaces should be fully understood for the local problem. (S2) is based upon a suitable choice of function spaces which are adapted
to the nonlinearity of the problem and to obtain continuity of the solution mapping. For example, it may be necessary to use other types of intersections of two or more weighted Sobolev spaces. Based upon the well-chosen construction of the spaces in (S2) the application of the implicit function theorem at regular points makes (S3) a relatively standard step. Nevertheless, if there are no previous results available for the invertibility of the class of linear operators under consideration, then a direct argument to show invertibility is often highly nontrivial; see Appendix C. If special points occur then one has to find a modification of – or alternative to – the step (S4). For our case, the key idea is that the linearized problem at the special point, as well as a small perturbation of this linear problem, do not contain any non-negative bounded solutions we are interested in. Hence we have been able to use a comparison argument to exclude the existence of a certain class of solutions near the special point. The step (S5) requires a good understanding of the local problem and is expected to be quite standard for most problems.

As the last remark, we stress again that the assumption of a sufficiently small nonlocality is crucial. On bounded domains it is well-known that the stationary solution $u \equiv 1$ of the nonlocal FKPP equation may bifurcate into other bounded non-negative solutions for suitably large nonlocality \[22\]. The same result is likely to be true for the unbounded domain case. In fact, numerical simulations \[7, \text{Fig.1}\] naturally lead to the conjecture that the state $u \equiv 1$ may undergo a (‘supercritical Hopf-type’) bifurcation with an exchange of stability to oscillations. It is important future work to investigate this bifurcation point in more detail as the numerical simulations \[7, \text{Fig.1}\] also show that the associated traveling wave between $u \equiv 0$ and $u \equiv 1$ changes from an equilibrium-to-equilibrium (E-to-E) heteroclinic front to an equilibrium-to-periodic (E-to-P) heteroclinic front. In fact, very recently the existence of oscillatory bounded solutions for sufficiently large nonlocality has been proven \[26\] in the one-dimensional case. Furthermore, we also refer to recent work on a similar problem in a related nonlocal reaction-diffusion system \[41\].

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A Convolutions in Weighted Spaces

For convenience we record here a few facts about convolution operators on weighted spaces. A standard result is the following:

**Theorem A.1.** (Generalized Young’s Inequality, \[21, \text{p.13}\]) Let $(S, \mu)$ be a sigma-finite measure space, and let $1 \leq p \leq \infty$ and $C > 0$. Suppose $\kappa(x,y)$ is a measurable function on $S \times S$ such that

$$\int_S |\kappa(x,y)| \, d\mu(y) \leq C \quad \text{for all } x \in S,$$

$$\int_S |\kappa(x,y)| \, d\mu(x) \leq C \quad \text{for all } y \in S,$$

and suppose $f \in L^p(S)$. Then the function

$$Tf(x) := \int_S \kappa(x,y)f(y) \, d\mu(y)$$

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is well-defined almost everywhere and \( Tf \in L^p(S) \). Furthermore, we have the estimate

\[
\|Tf\|_p \leq C\|f\|_p. \tag{35}
\]

We can apply the generalized Young’s inequality to the space

\[
S = \mathbb{R}^d, \quad d\mu(x) := w(x)dx, \quad w(x) = \frac{1}{(1 + \|x\|^2)^l}
\]

for \( \frac{1}{2} < l < \infty \) as discussed in Section 3. Furthermore, suppose we only assume that \( \phi \in L^1(\mathbb{R}^d) \) (i.e. it may not be normalized to \( \|\phi\| = 1 \)) and set

\[
\kappa(x, y) := \phi_\sigma(x - y), \quad \text{where} \quad \phi_\sigma(x) = \frac{1}{\sigma^d} \phi\left(\frac{x}{\sigma}\right).
\]

Then one is lead to the following direct calculation

\[
\int_S \phi_\sigma(x - y) \, d\mu(y) = \int_{\mathbb{R}^d} \phi_\sigma(x)w(x - y) \, dx = \frac{1}{\sigma^d} \int_{\mathbb{R}^d} \frac{\phi(\tilde{x})}{(1 + \|\sigma \tilde{x} - y\|^2)^l} \sigma^d \, d\tilde{x} \leq \|\phi\|_1. \tag{36}
\]

Remark: Note that we crucially use here that our choice of weight is quite ‘tame’ in the sense that it is absolutely continuous with respect to standard Lebesgue measure. Singular and/or growing weight functions would present a problem which would have to be compensated by additional regularity assumptions on the kernel \( \phi \).

Hence, we may take \( C = 1 = \|\phi\|_1 \) in Theorem A.1 and the previous discussion to obtain a weighted version of the more standard Young’s inequality.

**Proposition A.2.** (Young’s inequality, see also [21, p.14]) If \( \phi \in L^1(\mathbb{R}^d) \) and \( f \in L^p(\mathbb{R}^d, w_X) \) for \( 1 \leq p < \infty \) then \( f * \phi_\sigma \in L^p(\mathbb{R}^d, w_X) \) and

\[
\|f * \phi_\sigma\|_{p,w_X} \leq \|\phi\|_1 \|f\|_{p,w_X}.
\]

**Proof.** Note that \( \phi \in L^1(\mathbb{R}^d) \) implies that \( \phi_\sigma \in L^1(\mathbb{R}^d; w_X) \) for each \( \sigma > 0 \). Thus, applying Theorem A.1 and the calculation (36) yield the result.

**Theorem A.3.** ([35, p.64]) Let \( \phi \in L^1(\mathbb{R}^d) \) with \( \|\phi\| = 1 \). Let \( f \in L^p(\mathbb{R}^d, w_X) \) for some \( p \in [1, \infty) \) then \( \phi_\sigma * f \to f \) in \( L^p(\mathbb{R}^d; w_X) \) for \( \sigma \to 0 \).

**Proof.** The proof from [35, p.65–66] applies verbatim as we have a weighted version of Young’s inequality given in Proposition A.2 for our choice of weight function \( w_X \) in [10].

It is clear that Proposition A.2 and Theorem A.3 would also hold true for more general classes of weight functions which include as a subset the family \( w_X \), parametrized by the choice of \( l \).
B Purely Oscillatory Solutions

In this section we provide a proof of the result stated in Proposition 5.3 which we re-state here for convenience:

**Proposition B.1.** Suppose (B) holds with weight function $w_X \in L^1(\mathbb{R}^d)$. Suppose $\mathcal{L}_0^0 U = 0$ and $U$ is in the linear hull of simple exponential solutions. Then $U \in X$ if and only if $U$ is in the linear hull of purely oscillatory solutions.

**Proof.** If $U$ is in the linear hull of purely oscillatory solutions it easily follows that $U \in X$ by using the global boundedness of a finite sum of sines and cosines together with the polynomial weight function $w_X$. Therefore, it remains to show the converse.

Suppose $U \in X$ and $\mathcal{L}_0^0 U = 0$. Then elliptic regularity [28, Cor. 11.4.13] implies that $U \in C^\infty(\mathbb{R}^d)$; in fact, $U$ is even real analytic. Proposition 5.2 yields that $U$ can be expressed as a linear combination of exponential solutions

$$U(x) = \sum_{k=1}^{K} a_k e^{i k^T x}$$

for some $a_k \in \mathbb{C}$, a natural number $K > 0$, $\xi_k \in \mathbb{C}^d$ are distinct vectors and the series converges pointwise for each $x \in \mathbb{R}^d$. Note that the sum (37) has a finite number of terms since we assumed that $U$ is in the linear hull and we do not have to take a closure. A solution is purely oscillatory if $\xi_k \in \mathbb{R}^d$ for all $k \in \mathbb{N}$. We argue by contradiction and suppose there exists at least one vector $\xi_k \notin \mathbb{R}^d$. Define the following set of indices

$$\mathcal{K} := \{k \in \{1, 2, \ldots, K\} : \xi_k \notin \mathbb{R}^d\}$$

and denote a maximum growth or decay exponent index pair by $(k^*, l^*)$, i.e., $|\text{Im}((\xi_k)^{l^*})| \geq |\text{Im}((\xi_l)^{l^*})|$ for all $k \in \mathcal{K}$ and $l \in \{1, 2, \ldots, d\}$. As a first case, suppose that the pair $(k^*, l^*)$ is unique.

Without loss of generality we are going to assume that $\text{Im}(\xi_k)^{l^*} > 0$ since a coordinate transformation of the form $x_{l^*} \rightarrow -x_{l^*}$ is going to cover the case when $\text{Im}(\xi_k)^{l^*} < 0$. Furthermore, we may assume without loss of generality that $l^* = 1$, respectively $k^* = 1$, since a permutation of the coordinate indices, respectively the summands, can always be applied. Let $(\xi_1)_1 = \mu - \lambda i$ with $\lambda > 0$, $\mu \in \mathbb{R}$ and focus on a summand of $U(x)$ given by

$$a_1 e^{i k^T x} = a_1 e^{\lambda x_1} e^{\mu i x_1} e^{i((\xi_1)_2 x_2 + \cdots + (\xi_1)_d x_d)} = a_1 e^{\lambda x_1} s(x).$$

Observe that $\dim(\{x \in \mathbb{R}^d : s(x) = 0\}) \leq d - 1$ as the zero set consists of the zeros of a product of trigonometric functions. Fix some $\delta > 0$ and use the last observation and the fact that $\lim_{x_1 \rightarrow \infty} e^{\lambda x_1} / w_X(x) = +\infty$ to conclude the existence of a sequence of disjoint balls $B(y_m, \delta)$ with centers $y_m \in \mathbb{R}^d$, $\|y_m\| \rightarrow \infty$, and radius $\delta$ such that

$$\left| a_1 e^{\lambda x_1} s(x) + \sum_{k=2}^{K} a_k e^{i k^T x} \right|^p w_X(x) > 1 \quad \text{for } x \in B(y_m, \delta). \tag{38}$$
Note that the lower bound 1 is just chosen for convenience and any other fixed positive constant would work as well. As a last step, we shall show that \( U \not\in L^p(\mathbb{R}^d; w_X) \) for \( 1 \leq p < \infty \). We have
\[
\|U\|_{L^p(\mathbb{R}^d; w_X)}^p \geq \sum_{m=1}^{\infty} \int_{B(y_m, \delta)} \left| \sum_{k=1}^{K} a_k e^{i \xi_k^T x} \right|^p w(x) \, dx \\
\geq \sum_{m=1}^{\infty} \int_{B(y_m, \delta)} dx = \sum_{m=1}^{\infty} \frac{\pi^{d/2} \delta^d}{\Gamma(\frac{d}{2} + 1)} = +\infty.
\]

Therefore, it clearly follows that \( U \not\in X \) as \( X \subset L^p(\mathbb{R}^d, w_X) \). This concludes the case when the maximum growth or decay exponent index pair by \((k^*, l^*)\) is unique.

When \((k^*, l^*)\) is not unique, we have a finite number of index pairs which could be maximal and two cases may occur. First, if \(k^*\) is unique then we may have dominating terms with exponentials of the form \(a_k \exp(-\lambda x_1 + \lambda x_l)(\cdots)\) for some \(l \in \{2, 3, \ldots, d\}\). The construction of the balls in (38) still works as the linear subspace \(\{x \in \mathbb{R}^d : x_1 = x_l\}\) has dimension \(d - 1\) i.e. we can always arrange the balls \(B(y_m, \delta)\) such that \(\{x \in \mathbb{R}^d : x_1 = x_l\} \cap B(y_m, \delta) = \emptyset\). If \(k^*\) is not unique, a similar argument applies.

If the sum (39) has terms of the form \(a_k e^{i \xi_k^T x} + a_j e^{i \xi_j^T x}\), where \(a_j = \overline{a_k}\) and \(\xi_j = \overline{\xi_k} \not\in \mathbb{R}^d\) are complex conjugates (as the function \(U\) has to be real-valued), then the first term leads to \(a_k e^{\lambda x_1}(\cdots)\) while the second term contributes \(\overline{a_k} e^{-\lambda x_1}(\cdots)\) for some coordinate \(x_1\). Suppose without loss of generality that \(\lambda > 0\) then we may always choose \(x_l\) sufficiently large, and avoid the zeros of the multiplying trigonometric factors, to construct balls as above.

If there are terms of the form \(e^{\lambda x_1 + ax_2 + \cdots} - e^{\lambda x_1 + bx_2 + \cdots}\) with \(\lambda, a, b \in \mathbb{R}\) then there is the possibility of cancellations. However, since we know that the vectors \(\xi_k\) are distinct, we may assume that \(a \neq b\). Hence, we have terms of the form \(e^{\lambda x_1} s_a(x) - e^{\lambda x_1} s_b(x)\), where the zero sets of the trigonometric functions \(s_a\) and \(s_b\) each have dimension \(d - 1\). In particular, the sets where \(e^{\lambda x_1} s_a(x)\) and \(-e^{\lambda x_1} s_b(x)\) are both positive have full dimension \(d\) and repeat periodically due to the periodicity of the trigonometric prefactors \(s_a\) and \(s_b\).

Indeed, the same arguments apply to all cases where the growth and decay indices are not unique, since \(K\) is finite and the \(\xi_k\) are distinct vectors, we can always construct the balls \(B(y_m, \delta)\) to make the polynomially weighted \(L^p\)-norm of \(U\) arbitrarily large with a finite number of balls. Hence we may conclude that \(\xi_k \in \mathbb{R}^d\) for all \(k\) if \(U \in X\). \(\Box\)

The next step is to generalize Proposition [B.1] to the case of the closure of the linear hull of exponential solutions. If \(U\) is in the closure of the linear hull of exponential solutions it can be written as
\[
U(x) = \lim_{K \to \infty} \sum_{k=1}^{K} a_k e^{i \xi_k^T x} = \sum_{k=1}^{\infty} a_k e^{i \xi_k^T x},
\]
where the convergence for limit as \(K \to \infty\) is understood in a suitable local space, e.g. \(L^p_{\text{loc}}\), as discussed in [28, p.3-39]. Unfortunately, convergence in the natural semi-norms in \(L^p_{\text{loc}}\) does not seem to lead to any convergence results in the weighted space, i.e., there may not even be convergence in \(L^p(\mathbb{R}^d; w_X)\). Although the next statement looks quite natural, it does not seem easy to prove so we leave it here as a conjecture for future work.

**Conjecture B.2.** Proposition [B.1] holds when the linear hull is replaced by its closure.
The problem in generalizing the proof to the closure of the linear hull is that one has to take care of potential intricate cancellation effects in the infinite series involved, as well as dealing with zeros of infinite sums of trigonometric functions.

C Invertibility - A Direct Proof

Recall that we have used in Proposition 4.2 a particular choice of weights \( w_X, w_Y \) and exponent \( p = 2 \) to use a known result about invertibility of the linear operator
\[
\mathcal{L}_1 : X \rightarrow Y, \quad \mathcal{L}_1 U = \Delta U - U,
\]
where \( X \) and \( Y \) are chosen as
\[
X = W^{k+2,p}(\mathbb{R}^d; w_X) \cap W^{k,2p}(\mathbb{R}^d; w_X) \quad \text{and} \quad Y = W^{k,p}(\mathbb{R}^d; w_Y)
\]
for some \( k \in \mathbb{N}_0 \) and the norm on \( X \) is given by
\[
\|u\|_X = \max\{\|u\|_{k+2,p;w_X}, \|u\|_{k,2p;w_X}\}.
\]

Here we sketch the main arguments how a more direct proof of invertibility could be carried out. The basic assumption is that the positive weight functions \( w_X, w_Y \in L^1(\mathbb{R}^d) \) are chosen such that \( \mathcal{L}_1 \) is a well-defined linear operator and \( 1 < p < \infty \). If we restrict the class of weight function to exclude exponential growth then we obtain a trivial nullspace for \( \mathcal{L}_1 \).

Lemma C.1. There exist weight functions \( w_X, w_Y \in L^1(\mathbb{R}^d) \) such that the following statement holds: \( \mathcal{L}_1 U = 0 \) for \( U \in X \) if and only if \( U \equiv 0 \).

Proof. (Sketch of an alternative to Proposition 4.2) We argue by contradiction and suppose we may find a nonzero solution \( U \). The idea of the proof is similar to the proof of Proposition 5.2. By a result on homogeneous linear PDE [28, Thm 10.5.1, p.39] we may write any solution of \( \Delta U - U = 0 \) in \( X \), for any choice of \( w \in L^1(\mathbb{R}^d) \), by a limit of basic exponential solutions,
\[
U(x) = \sum_{j=1}^{\infty} f_k(x)e^{i\xi_k^T x},
\]
where the sequences \( f_k(x) \) are polynomials and \( \xi_k \in \mathbb{C}^d \) are vectors. If there exists a component of \( \xi_k \) with nonzero imaginary part then \( U(x) \) does have a component behaving like \( e^{-x} \) or \( e^x \) and we have seen in Section 4 as well as Appendix 3 that we may choose a weight function with polynomial decay as \( \|x\| \rightarrow \infty \) so that \( U \not\in X \). Hence, it follows that \( \xi_k \in \mathbb{C}^d \). If \( f_k(x) \) is a non-constant polynomial for some \( k \), then we may choose a positive weight function \( w \in L^1(\mathbb{R}^d) \) such that \( |f_k(x)|^p w(x) \rightarrow +\infty \) as \( \|x\| \rightarrow \infty \). In this case, if follows that \( U \not\in X \). Therefore, \( f_k(x) = a_k \) for some constants \( a_k \) and the possible solutions are all purely oscillatory or homogeneous. Obviously we may assume for each fixed \( k \) that \( \xi_{k_1} \neq \xi_{k_2} \) for \( k_1 \neq k_2 \). Applying the differential operator to \( U \) yields
\[
0 = (\mathcal{L}_1 U)(x) = \sum_{j=1}^{\infty} a_k (\|x\| - 1) e^{i\xi_k^T x}.
\]
Since $\xi_{k_1} \neq \xi_{k_2}$ for $k_1 \neq k_2$ it follows from (11) that $a_k(\|\xi_k\|^2 + 1) = 0$ for all $k \in \mathbb{N}$. Thus, we get that that $a_k = 0$ for all $k$. Therefore, it follows that $U \equiv 0$ providing the required contradiction. 

It is important to note that the last proof does not employ the particular choice (B) given in (10) for the weight functions and the Sobolev space exponent. Weight functions with rather general polynomial growth and $p \in (1, \infty)$ suffice not only to prove Lemma C.1 but also to establish continuity and differentiability properties of $F$ as long as (18) holds; see also Section 3. Based on the knowledge of the nullspace we may now proceed to investigate invertibility.

**Lemma C.2.** There exists a weight function $w \in L^1(\mathbb{R}^d)$ such that $L_1 : X \rightarrow Y$ is invertible.

**Proof.** (Sketch of an alternative to Proposition 4.2) For the proof we shall need as auxiliary spaces the Hölder spaces $C^{k+\gamma}(\mathbb{R}^d)$ for $k \in \mathbb{N}_0$ and $\gamma \in (0,1)$ as defined in Section 3. By Lemma C.1 we know that $L_1U = 0$ for $U \in X$ implies that $U \equiv 0$. Since

$$\|u\|_X = \max\{\|u\|_{k+2,p,w}, \|u\|_{k,2;p,w}\} \leq \|u\|_{k+2,p,w} + \|u\|_{k,2;p,w} \leq 2\|u\|_{C^{k+2+\gamma}(\mathbb{R}^d)} \max_{\rho \in (p,2p)} \|w\|_{L^1(\mathbb{R}^d)}^{1/\rho}$$

it follows that $C^{k+2+\gamma}(\mathbb{R}^d) \subset X$; see also [37]. Hence $L_1U = 0$ for $U \in C^{k+2+\gamma}(\mathbb{R}^d)$ implies that $U \equiv 0$ so that $L_1$ has trivial nullspace as an operator

$$L_1 : C^{k+2+\gamma}(\mathbb{R}^d) \rightarrow C^{k+\gamma}(\mathbb{R}^d).$$

Applying Mukhamadiev’s Theorem [38] it follows that $L_1 : C^{k+2+\gamma}(\mathbb{R}^d) \rightarrow C^{k+\gamma}(\mathbb{R}^d)$ is invertible. To show invertibility on $X$ an approximation argument can be used similar to the strategy in [37]. Let $\kappa \in C_c^\infty(\mathbb{R}^d)$ with $\|\kappa\|_{L^1(\mathbb{R}^d)} = 1$ and consider $\kappa_\epsilon(x) = \epsilon^{-d}\kappa(x/\epsilon)$ so that $\|\kappa_\epsilon\|_{L^1(\mathbb{R}^d)} = \|\kappa\|_{L^1(\mathbb{R}^d)}$. Let $f \in L^p(\mathbb{R}^d; w)$ and define $f_\epsilon := \kappa_\epsilon * f$. Then a standard analysis result ([35], p.64], see also Appendix A) implies that

$$f_\epsilon \in C^\infty(\mathbb{R}^d) \quad \text{and} \quad f_\epsilon \rightarrow f \text{ in } L^p(\mathbb{R}^d; w).$$

Due to the invertibility of $L_1 : C^{k+2+\gamma}(\mathbb{R}^d) \rightarrow C^{k+\gamma}(\mathbb{R}^d)$ it follows that the equation

$$L_1u = f_\epsilon$$

has a unique solution $u_\epsilon \in C^{k+2+\gamma}(\mathbb{R}^d)$ for each $\epsilon > 0$. The desired result will follow if one can show that $u_\epsilon$ converges to some function $u_0 \in X$ as $\epsilon \rightarrow 0$ but the convergence essentially follows from a suitable coercivity estimate for the operator $L_1$ as discussed in [37].

The main point of the last proof is that we have picked a more complicated weak solution space $X$ but the invertibility of the restricted operator onto Hölder space, in combination with an approximation argument, can be used to yield invertibility for $L_1 : X \rightarrow Y$. Note that one still has to make some restrictions on $w_X, w_Y, p$ but that no explicit choice is necessary if a direct investigation of the linearized operator is considered.
References

[1] M. Alfaro and J. Coville. Rapid travelling waves in the nonlocal Fisher equation connect two unstable states. *Appl. Math. Lett.*, 25(12):2095–2099, 2012.

[2] M. Alfaro, J. Coville, and G. Raoul. Bistable travelling waves for nonlocal reaction diffusion equations. arXiv:1303.3554v1, pages 1–16, 2013.

[3] P. Ashwin, M.V. Bartuccelli, T.J. Bridges, and S.A. Gourley. Travelling fronts for the KPP equation with spatio-temporal delay. *Zeitschr. Angewand. Math. Phys.*, 53(1):103–122, 2002.

[4] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. I. Periodic framework. *J. Eur. Math. Soc.*, 7(2):173–213, 2005.

[5] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. II. General domains. *J. Amer. Math. Soc.*, 23:1–32, 2010.

[6] H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model: I - Species persistence. *J. Math. Biol.*, 51(1):75–113, 2005.

[7] H. Berestycki, G. Nadin, B. Perthame, and L. Ryzhik. The non-local Fisher-KPP equation: travelling waves and steady states. *Nonlinearity*, 22:2813–2844, 2009.

[8] N.F. Britton. Aggregation and the competitive exclusion principle. *J. Theoret. Biol.*, 136:57–66, 1989.

[9] N.F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. *SIAM J. Appl. Math.*, 50(6):1663–1688, 1990.

[10] J. Coville and L. Dupaigne. Propagation speed of travelling fronts in non local reaction-diffusion equations. *Nonl. Anal. Theor. Meth. Appl.*, 60(5):797–819, 2005.

[11] J. Coville and L. Dupaigne. On a non-local equation arising in population dynamics. *Proc. R. Soc. Edinburgh A*, 137(4):727–756, 2007.

[12] K. Deimling. *Nonlinear Functional Analysis*. Dover, Mineola, NY, 2010.

[13] D. del Castillo-Negrete, B.A. Carreras, and V.E. Lynch. Front dynamics in reaction-diffusion systems with Levy flights: a fractional diffusion approach. *Phys. Rev. Lett.*, 91(1):018302, 2003.

[14] R.M. Dudley and R. Norvaiša. *Concrete Functional Calculus*. Springer, 2011.

[15] Y. Du, and L. Ma. Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions. *J. London Math. Soc.*, 64(2):107–121, 2001.

[16] U. Ebert and W. van Saarloos. Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts. *Physica D*, 146:1–99, 2000.
[17] J. Fang and X.Q. Zhao. Monotone wavefronts of the nonlocal Fisher-KPP equation. *Nonlinearity*, 24(11):3043–3054, 2011.

[18] P. Felmer and M. Yangari. Fast propagation for fractional KPP equations with slowly decaying initial conditions. *SIAM J. Math. Anal.*, 45(2):662–678, 2013.

[19] J.-J. Feng, L. Huang, and S.-J. Yang. Solutions of Laplace equation in $n$-dimensional space. *Commun. Theor. Phys.*, 56(4):623–625, 2011.

[20] R.A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:353–369, 1937.

[21] G. Folland. *Introduction to Partial Differential Equations*. Princeton University Press, 1976.

[22] J. Furter and M. Grinfeld. Local vs. non-local interactions in population dynamics. *J. Math. Biol.*, 27:65–80, 1989.

[23] S. Genieys, V. Volpert, and P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. *Math. Model. Nat. Phenom.*, 1(1):63–80, 2006.

[24] S.G. Gindikin and L.R. Volevich. *Distributions and Convolutions Equations*. Gordon and Breach Science Publishers, 1990.

[25] S.A. Gourley. Travelling front solutions of a nonlocal Fisher equation. *J. Math. Biol.*, 41(3):272–284, 2000.

[26] F. Hamel and L. Ryzhik. On the nonlocal Fisher-KPP equation: steady states, spreading speed and global bounds. *arXiv:1307.3001*, pages 1–47, 2013.

[27] P.S. Harrington and A. Raich. Sobolev spaces and elliptic theory on unbounded domains. *arXiv:1209.4044v1*, pages 1–47, 2012.

[28] L. Hörmander. *The Analysis of Linear Partial Differential Operators II*. Springer, 1983.

[29] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer, 1990.

[30] A. Kolmogorov, I. Petrovskii, and N. Piscounov. A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. In V.M. Tikhomirov, editor, *Selected Works of A. N. Kolmogorov I*, pages 248–270. Kluwer, 1991. Translated by V. M. Volosov from Bull. Moscow Univ., Math. Mech. 1, 1–25, 1937.

[31] L. Kong and W. Shen. Positive stationary solutions and spreading speeds of KPP equations in locally spatially inhomogeneous media. *Meth. Applic. Appl. Anal.*, 18(4):427–456, 2011.

[32] A. Kufner. *Weighted Sobolev Spaces*. Teubner, 1980.
[33] J.M. Lee. *Introduction to Smooth Manifolds*. Springer, 2006.

[34] R. Lefever and O. Dejeune. On the origin of tiger bush. *Bull. Math. Biol.*, 59(2):263–294, 1997.

[35] E.H. Lieb and M. Loss. *Analysis*. AMS, 2nd edition, 2001.

[36] R. Mancinelli, D. Vergni, and A. Vulpiani. Front propagation in reactive systems with anomalous diffusion. *Physica D*, 185(3):175–195, 2003.

[37] O.A. Mitina and V.M. Tyurin. On the invertibility of linear partial differential operators in Hölder and Sobolev spaces. *Sbornik Math.*, 194(5):733–744, 2003.

[38] E.M. Mukhamadiev. On invertibility of elliptic partial differential operators. *Dokl. Akad. Nauk SSSR*, 205:1292–1295, 1972. English transl. in Soviet Math. Dokl. 13, 1972.

[39] J.D. Murray. *Mathematical Biology I: An Introduction*. Springer, 3rd edition, 2002.

[40] G. Nadin, B. Perthame, and M. Tang. Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation. *Comptes Rendus Math.*, 349(9):553–557, 2011.

[41] G. Nadin, L. Rossi, L. Ryzhik, and B. Perthame. Wave-like solutions for nonlocal reaction-diffusion equations: a toy model. *Math. Model. Nat. Phen.*, 8(3):33–41, 2013.

[42] A. Okubo. *Diffusion and Ecological Problems*. Springer, 1980.

[43] B. Perthame and B. Génieys. Concentration in the nonlocal Fisher equation: the Hamilton-Jacobi limit. *Math. Mod. Nat. Phenom.*, 2(4):135–151, 2007.

[44] B. Perthame and P.E. Souganidis. Front propagation for a jump process arising in spatial ecology. *Discrete Contin. Dyn. Syst.*, 13(5):1235–1246, 2005.

[45] P. Pucci and J. Serrin. *The Maximum Principle*. Birkhäuser, 2007.

[46] L. Rossi. Non-existence of positive solutions of fully nonlinear elliptic equations in unbounded domains. *Comm. Pure Appl. Anal.*, 7(1):125–141, 2008.

[47] Z.C. Wang, W.T. Li, and S. Ruan. Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay. *J. Differential Equat.*, 238(1):153–200, 2007.

[48] X. Zou. Delay induced traveling wave fronts in reaction diffusion equations of KPP-Fisher type. *J. Comp. Appl. Math.*, 146(2):309–321, 2002.