DISTRIBUTIONS OF ARITHMETIC PROGRESSIONS IN PIATETSKI-SHAPIRO SEQUENCE

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Abstract. It is known that for all $\alpha \in (1, 2)$ and all integers $k \geq 3$ and $r \geq 1$, there exist infinitely many $n \in \mathbb{N}$ such that the sequence $((n + rj)^\alpha)_{j=0}^{k-1}$ is an arithmetic progression of length $k$. In this paper, we show that the asymptotic density of all the above $n$ is equal to $1/(k-1)$. Although the common difference $r$ is arbitrarily fixed in the above result, we also examine the case when $r$ is not fixed. Furthermore, we also examine the number of the above $n$ that are contained in a short interval.

1. Introduction

Notations.—A subset $A$ of $\mathbb{N} := \{1, 2, \ldots\}$ is naturally identified with an increasing sequence of $\mathbb{N}$. For $N \in \mathbb{N}$, the notation $[N]$ denotes the finite set $\{1, 2, \ldots, N\}$; for $x \in \mathbb{R}$, the notation $\lfloor x \rfloor$ denotes the integer part of $x$.

Motivation and introduction.—We say that a set $P \subset \mathbb{N}$ is an arithmetic progression (AP) of length $k$ (for short, $k$-AP) if there exist $s, r \in \mathbb{N}$ such that

$$P = \{s + rj : j = 0, 1, \ldots, k-1\}.$$ 

APs are a simple mathematical structure of $\mathbb{N}$, and many researchers are interested in whether a given set $A \subset \mathbb{N}$ contains long APs or not. Szemerédi’s theorem is a famous theorem that guarantees the existence of APs: every subset of $\mathbb{N}$ with positive upper density contains arbitrarily long APs, where we say that $A \subset \mathbb{N}$ has positive upper density if

$$\lim_{N \to \infty} \frac{\#(A \cap [N])}{N} > 0.$$ 

Hence, it has been studied whether subsets of $\mathbb{N}$ with asymptotic density zero contain long APs or not. For example, Green and Tao proved that the set of all primes contains arbitrarily long APs. Also, Frantzikinakis and Wierdl and the authors proved that for every $\alpha \in (1, 2)$ and every integer $k \geq 3$, there exists a $k$-AP $P$ such that

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For non-integral $\alpha > 1$, the sequences $PS(\alpha) := ([n^\alpha])_{n=1}^\infty$ are called Piatetski-Shapiro sequences, which have been also studied on mathematical structures other than APs \cite{7,15}.

Although the existence of a $k$-AP $P$ with (C) was proved in \cite{3,4}, it was not known how many $k$-APs with (C) exist. In this paper, we do further analysis than \cite{3,4}, namely, investigate distributions of APs contained in $PS(\alpha)$ when $\alpha \in (1, 2)$. First, in fixing both the common difference $r$ and length $k$ of an AP $P$, let us consider whether $([n^\alpha])_{n \in P}$ is an AP or not.

\textbf{Theorem 1.1.} For all $\alpha \in (1, 2)$ and all integers $k \geq 3$ and $r \geq 1$,
\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \#\{n \in [N] : ([n + rj]^\alpha)_{j=0}^{k-1} \text{ is an AP} \} = \frac{1}{k-1}.
\end{equation}

We generalize the function $x^\alpha$ to functions $f$ with certain properties in Section 2, and examine the convergence speed of (1.1) more precisely in Section 5. Next, let us consider the case when the common difference $r$ is not fixed.

\textbf{Theorem 1.2.} For all $\alpha \in (1, 2)$ and all integers $k \geq 3$, there exist $A_{\alpha,k}, B_{\alpha,k} > 0$ and $N_{\alpha,k} \in \mathbb{N}$ such that for all integers $N \geq N_{\alpha,k}$,
\begin{equation}
A_{\alpha,k}N^{2-\alpha/2} \leq \#\{P \subset [N] : P \text{ and } ([n^\alpha])_{n \in P} \text{ are } k\text{-APs} \} \leq B_{\alpha,k}N^{2-\alpha/2}.
\end{equation}

Since the number of $k$-APs contained in $[N]$ is about $N^2/(2(k-1))$, the asymptotic density of the set in (1.2) is zero. We give explicit values of $A_{\alpha,k}$ and $B_{\alpha,k}$ in Section 4.

Related work on Piatetski-Shapiro sequences.—Piatetski-Shapiro \cite{7} proved that for every $\alpha \in (1, 12/11)$, the sequence $([n^\alpha])_{n=1}^\infty$ contains infinitely many primes. For non-integral $\alpha > 1$, primes of the form $[n^\alpha]$ are called Piatetski-Shapiro primes. It is known that the range $(1, 12/11)$ can be improved to $(1, 243/205)$ \cite{8}. Actually, Piatetski-Shapiro \cite{7} proved a stronger statement, namely, the prime number theorem on Piatetski-Shapiro sequences: for all $\alpha \in (1, 12/11)$,
\[\#\{n \in [1, x] : [n^\alpha] \text{ is prime} \} \sim \frac{x}{\alpha \log x} \quad (x \to \infty).\]

Rivat and Sargos \cite{9} improved the range $(1, 12/11)$ to $(1, 2817/2426)$. Similar formulas are also known for square-free $[n^\alpha]$ \cite{10,12}, cube-free $[n^\alpha]$ \cite{13}, and $[n^\alpha] \equiv c \mod m$ \cite{14,15} when $\alpha$ lies in certain ranges.

As already stated, it is known \cite{3,4} that for every $\alpha \in (1, 2)$, $PS(\alpha)$ contains arbitrarily long APs. Actually, the following stronger statement holds \cite{4}: for every $\alpha \in (1, 2)$, every integer $k \geq 3$, and every $A \subset \mathbb{N}$ with positive upper density, there exists a $k$-AP $P \subset A$ with (C). Recently, some researchers studied APs of Piatetski-Shapiro...
primes with fixed exponent. Mirek [16] proved that for every \( \alpha \in (1,72/71) \), the set of all Piatetski-Shapiro primes with exponent \( \alpha \) contains infinitely many 3-APs. Li and Pan [17] claimed that for every integer \( k \geq 3 \), there exists \( \alpha_k > 1 \) such that for every \( \alpha \in (1,\alpha_k) \) the set of all Piatetski-Shapiro primes with exponent \( \alpha \) contains infinitely many \( k \)-APs.

Looking at 3-APs in the viewpoint of Diophantine equations, one can characterize a 3-AP \( \{x < y < z\} \) by the equation \( x + z = 2y \). Hence, for every \( \alpha \in (1,2) \), the equation \( x + z = 2y \) has infinitely many solutions even if \( x < y < z \) are elements of \( \text{PS}(\alpha) := \{[n^\alpha] : n \in \mathbb{N}\} \). Thus, it is natural to consider another Diophantine equation. Ref. [18] mentioned that for every \( \alpha \in (1,2) \), the equation \( x + y = z \) has infinitely many solutions in \( \text{PS}(\alpha) \). Glasscock [18] showed that if the equation \( y = ax + b \) with real \( a > b \geq 0 \) has infinitely many solutions in \( \mathbb{N} \), then, for Lebesgue-a.e. \( \alpha \in (1,2) \), the equation \( y = ax + b \) has infinitely many solutions in \( \text{PS}(\alpha) \).

2. Results

First, let us define asymptotic notations. Suppose that for some \( x_0 > 0 \), complex-valued functions \( f, f_1 \) and \( f_2 \) and non-negative-valued functions \( h, h_1 \) and \( h_2 \) are defined on the interval \([x_0,\infty)\). We write

- \( f_1(x) = f_2(x) + O(h(x)) \) as \( x \to \infty \) if there exists \( C > 0 \) such that \( |f_1(x) - f_2(x)| \leq C h(x) \) for every sufficiently large \( x > 0 \);
- \( f(x) \ll h(x) \) as \( x \to \infty \) if \( f(x) = O(h(x)) \) as \( x \to \infty \);
- \( h_1(x) \asymp h_2(x) \) as \( x \to \infty \) if \( h_1(x) \ll h_2(x) \) and \( h_2(x) \ll h_1(x) \) as \( x \to \infty \).

The symbol “\( x \to \infty \)” is often omitted. Also, the above \( C \) is called an implicit constant. If implicit constants depend on parameters \( a_1,\ldots,a_n \), we often write \( f_1(x) = f_2(x) + O_{a_1,\ldots,a_n}(h(x)) \), \( f(x) \ll_{a_1,\ldots,a_n} h(x) \), and \( h_1(x) \asymp_{a_1,\ldots,a_n} h_2(x) \) to emphasize the dependence.

From now on, we assume that a differentiable function \( f \) satisfies \( \inf_{x \geq 1} f'(x) \geq 1 \) in order to make the sequence \((f(n))_{n=1}^\infty\) an increasing sequence. However, this assumption is not essential in any proofs of theorems.

**Theorem 2.1.** Let \( f : [1,\infty) \to \mathbb{R} \) be a twice differentiable function satisfying that

1. The second derivative \( f''(x) \) vanishes as \( x \to \infty \);
2. \( (f(n), f'(n))_{n=1}^\infty \) is uniformly distributed modulo 1;
3. \( \inf_{x \geq 1} f'(x) \geq 1 \).

Then, for all integers \( k \geq 3 \) and \( r \geq 1 \),

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n \in [N] : ([f(n + rj)]_{j=0}^{k-1} \text{ is an AP} \} = \frac{1}{k-1}.
\]
When a function \( f \) is given, it can be easily checked that the second derivative vanishes as \( x \to \infty \), but it is not so easy to check the uniform distribution. However, there is a useful lemma to verify that \( (\langle f(n), f'(n) \rangle)_{n=1}^{\infty} \) is uniformly distributed modulo 1 (see Lemma 3.2). Using Lemma 3.2, one can apply Theorem 2.1 to the following functions by ignoring the first finite terms of \( \langle \lfloor f(n) \rfloor \rangle_{n=1}^{\infty} \) if necessary:

\[
(2.1) \quad x \log x, \quad x^\alpha, \quad \frac{x^2}{(\log x)^\beta}, \quad \frac{x^2}{(\log \log x)^\gamma},
\]

where \( \alpha \in (1, 2) \) and \( \beta, \gamma > 0 \). In particular, Theorem 2.1 implies Theorem 1.1.

**Theorem 2.2.** Let \( f : [1, \infty) \to \mathbb{R} \) be a twice differentiable function satisfying that

1. The second derivative \( f''(x) \) decreases, and vanishes as \( x \to \infty \);
2. \( \lim_{x \to \infty} x^2 f'''(x) = \infty \);
3. For every \( \delta \in (0, 1) \), there exists \( c(\delta) > 0 \) such that every sufficiently large \( x \geq 1 \) satisfies \( f''(x) \leq f''(\delta x) \leq c(\delta) f''(x) \);
4. \( (f(n))_{n=1}^{\infty} \) and \( (f'(n))_{n=1}^{\infty} \) are uniformly distributed modulo 1;
5. \( \inf_{x \geq 1} f'(x) \geq 1 \).

Then, for every integer \( k \geq 3 \),

\[
(2.2) \quad \# \{ P \subset [N] : P and (\lfloor f(n) \rfloor)_{n \in P} are k-APs \} \asymp_{c(\cdot), k} N f''(N)^{-1/2} \quad (N \to \infty).
\]

One can also apply Theorem 2.2 to the functions (2.1). In particular, Theorem 2.2 implies Theorem 1.2. The implicit constants of (2.2) only depend on \( c(\cdot) \) and \( k \). This fact is seen in Section 4 by giving explicit values of the implicit constants. For special \( c(\cdot) \), the explicit values can be simplified, e.g., the case when \( f(x) = x^\alpha \) with \( \alpha \in (1, 2) \). For details, see Remarks 4.3 and 4.2.

**Theorem 2.3.** For all \( \alpha \in (1, 2) \) and all integers \( k \geq 3 \) and \( r \geq 1 \),

\[
\frac{1}{N} \# \{ n \in [N] : (\lfloor (n + rj)^\alpha \rfloor)_{j=0}^{k-1} is an AP \}
= \frac{1}{k-1} + O_{\alpha, k, r}(F(N)) \quad (N \to \infty),
\]

where

\[
F(x) := \begin{cases}
  x^{(\alpha-2)/6}(\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in (1, 5/4) \cup [11/6, 2), \\
  x^{(\alpha-3)/14}(\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in [5/4, 3/2), \\
  (x^{(\alpha-3)/14} + x^{-\alpha/6})(\log x)^{1/2} & \alpha \in [3/2, 11/6).
\end{cases}
\]

Theorem 2.3 gives an upper bound of the convergence speed of (1.1). We show an extended statement (Proposition 5.1) in Section 5, which can be applied to a short interval \([N, N+L]\). Theorem 2.3 is derived from the extended statement.
Theorem 2.4. For all $\alpha \in (1, 2)$ and integers $k \geq 3$ and $r \geq 1$, there exists $c = c(\alpha, k, r) > 0$ satisfying the following condition: For every $x \geq 1$, the interval $[x, x + cx^{2-\alpha}]$ must contain $n \in \mathbb{N}$ such that $((n + rj)^{\alpha})_{j=0}^{k-1}$ is an AP.

At glance, the length $cx^{2-\alpha}$ of the interval in Theorem 2.2 is strange because the growth rate $O_{\alpha, k, r}(x^{2-\alpha})$ becomes smaller when $\alpha$ increases. However, for all $\alpha \in (1, 2)$ and all integers $k \geq 4$ and $r \geq 1$, the growth rate $O_{\alpha, k, r}(x^{2-\alpha})$ is best in a certain meaning. When $k = 3$, we expect that the length $cx^{2-\alpha}$ can be replaced with $cx^{1-\alpha/2}$.

For details, see Appendix A.

3. Criterion for uniform distribution

To prove main theorems, let us define a uniform distribution modulo 1. Unless there is confusion, $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ of $x \in \mathbb{R}$, and $\mathbf{0}$ denotes the zero vector $(0, \ldots, 0) \in \mathbb{R}^d$. Also, for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, define the notation $\{x\} = (\{x_1\}, \{x_2\}, \ldots, \{x_d\})$.

Let $(x_n)_{n=1}^\infty$ be a sequence of $\mathbb{R}^d$. We say that $(x_n)_{n=1}^\infty$ is uniformly distributed modulo 1 if every convex set $\mathcal{C} \subset [0, 1)^d$ satisfies that

$$
\lim_{N \to \infty} \frac{1}{N} \# \{n \in [N] : \{x_n\} \in \mathcal{C} \} = \mu(\mathcal{C}),
$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^d$. It is known that $(x_n)_{n=1}^\infty$ is uniformly distributed modulo 1 if and only if

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \exp(2\pi i \langle h, x_n \rangle) = 0
$$

for all $h \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. Due to this equivalence, if a sequence $((x_{1,n}, \ldots, x_{d,n}))_{n=1}^\infty$ is uniformly distributed modulo 1, then

- so is the sequence $((r_1 x_{1,n}, \ldots, r_d x_{d,n}))_{n=1}^\infty$ for every $r \in (\mathbb{Z} \setminus \{0\})^d$,
- so is the sequence $(x_{i,n})_{n=1}^\infty$ for every $i = 1, 2, \ldots, d$.

For details, see [19] Theorem 6.2.

Next, we give a sufficient condition for a sequence $(f(n))_{n=1}^\infty$ to be uniformly distributed modulo 1.

Lemma 3.1. Let $f : [1, \infty) \to \mathbb{R}$ be $k$-times differentiable on the interval $(x_0, \infty)$ for some integer $k \geq 1$ and some $x_0 \geq 1$. If $f^{(k)}(x)$ vanishes monotonically as $x \to \infty$ and if $x | f^{(k)}(x) |$ diverges to positive infinity as $x \to \infty$, then $(f(n))_{n=1}^\infty$ is uniformly distributed modulo 1.
Proof. See [19, Theorem 3.5].

Lemma 3.1 is a statement on one-dimensional sequences. However, we need to investigate whether a two-dimensional sequence is uniformly distributed modulo 1 in Section 4. Hence, let us prove a sufficient condition for a sequence \((f(n), f'(n))\) to be uniformly distributed modulo 1.

**Lemma 3.2.** Let \(f : [1, \infty) \to \mathbb{R}\) be \((k + 1)\)-times differentiable on the interval \((x_0, \infty)\) for some integer \(k \geq 2\) and some \(x_0 \geq 1\). If 
\[ f^{(k)}(x) + qf^{(k+1)}(x) \text{ vanishes monotonically as } x \to \infty \text{ for every } q \in \mathbb{Q} \text{ and if } x \left| f^{(k)}(x) + qf^{(k+1)}(x) \right| \text{ diverges to positive infinity as } x \to \infty \text{ for every } q \in \mathbb{Q}, \]
then \((f(n), f'(n))\) is uniformly distributed modulo 1.

**Proof.** Assume the assumption of Lemma 3.2. We show that for all \((h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\},\)
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(h_1 f(n) + h_2 f'(n)) = 0,
\]
where \(e(x) := e^{2\pi i x}\). First, consider the case \(h_1 \neq 0\). Thanks to the assumption with \(q = h_2/h_1\), Lemma 3.1 implies that \((f(n) + (h_2/h_1)f'(n))\) is uniformly distributed modulo 1: for every \(h \in \mathbb{Z} \setminus \{0\},\)
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e\left(h\left(f(n) + (h_2/h_1)f'(n)\right)\right) = 0.
\]
Putting \(h = h_2\) in the above equation, we obtain (3.3).

Next, consider the case \(h_1 = 0\). The \((k - 1)\)-times differentiable function \(f'\) satisfies that \(f^{(k)}(x)\) vanishes monotonically as \(x \to \infty\) and \(x \left| f^{(k)}(x) \right|\) diverges to positive infinity as \(x \to \infty\) thanks to the assumption with \(q = 0\). Thus, Lemma 3.1 implies that \((f'(n))\) is uniformly distributed modulo 1: for every \(h \in \mathbb{Z} \setminus \{0\},\)
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(hf'(n)) = 0.
\]
Putting \(h = h_2 \neq 0\) in the above equation, we obtain (3.3).

Lemma 3.2 yields the following corollary.

**Corollary 3.3.** For every \(\alpha \in (1, 2)\), the sequence \((n^\alpha, \alpha n^{\alpha-1})\) is uniformly distributed modulo 1.
Proof. Let $\alpha \in (1, 2)$, $f(x) = x^\alpha$, $q \in \mathbb{Q}$, and $g(x) = f(x) + qf'(x)$. Then
\[ g''(x) = \alpha(\alpha - 1)(x^{\alpha - 2} + q(\alpha - 2)x^{\alpha - 3}), \]
\[ g'''(x) = \alpha(\alpha - 1)(\alpha - 2)x^{\alpha - 3}(1 + q(\alpha - 3)x^{-1}). \]

Since $g'''(x) < 0$ for every sufficiently large $x > 0$, we find that $g''(x)$ vanishes monotonically as $x \to \infty$. Also, $x |g''(x)|$ diverges to positive infinity as $x \to \infty$. Therefore, Lemma 3.2 implies that $(\langle n^\alpha, \alpha n^{\alpha - 1} \rangle)_{n=1}^\infty$ is uniformly distributed modulo 1. □

Lemma 3.2 can be applied to many functions. For instance, Lemma 3.2 with $k = 2$ implies that for every function $f$ in (2.1), the sequence $(\langle f(n), f'(n) \rangle)_{n=3}^\infty$ is uniformly distributed modulo 1. This fact can be proved in the same way as the proof of Corollary 3.3.

4. Proofs of Theorems 2.1 and 2.2

First, we begin with the proof of Theorem 2.1 which is a basis of subsequent proofs.

Proof of Theorem 2.1 Let $k \geq 3$ and $r \geq 1$ be integers. Taylor’s theorem implies that for every $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, k - 1\}$ there exists $\theta = \theta(n, j) \in (n, n + rj)$ such that
\[ f(n + rj) = \lfloor f(n) \rfloor + j \lfloor rf'(n) \rfloor + \delta_0(n, j, \theta) = \lfloor f(n) \rfloor + j(\lfloor rf'(n) \rfloor + 1) + \delta_1(n, j, \theta), \]
\[ \delta_0(n, j, \theta) = \{ f(n) \} + j\{ rf'(n) \} + \frac{(rj)^2}{2} f''(\theta), \]
\[ \delta_1(n, j, \theta) = \{ f(n) \} + j(\{ rf'(n) \} - 1) + \frac{(rj)^2}{2} f''(\theta). \]

Let $\varepsilon \in (0, 1/2)$ be arbitrary. Thanks to (A1), we can take $x_0 \geq 1$ such that every $x \geq x_0$ satisfies
\[ |f''(x)| \leq \frac{2\varepsilon}{(k - 1)^2r^2} < \frac{1}{4r^2}. \]

Then, for all integers $n \geq x_0$ and $j \in \{0, 1, \ldots, k - 1\}$,
\[ \frac{(rj)^2}{2} |f''(\theta)| \leq \frac{r^2(k - 1)^2}{2} \cdot \frac{2\varepsilon}{(k - 1)^2r^2} = \varepsilon. \]

Let us show that
\[ \lim_{N \to \infty} \frac{1}{N} \# \{ n \in [N] : (\lfloor f(n + rj) \rfloor)_{j=0}^{k-1} \text{ is an AP} \} \geq \frac{(1 - 2\varepsilon)^2}{k - 1}. \]
Define the two sets $B_0$ and $B_1$ as
\begin{align}
B_0 &= \{ n \in \mathbb{N} : \{ f(n) \} > \varepsilon, \ \{ f(n) \} + (k-1)\{ rf'(n) \} < 1 - \varepsilon \}, \\
B_1 &= \{ n \in \mathbb{N} : \{ f(n) \} < 1 - \varepsilon, \ \{ f(n) \} + (k-1)(\{ rf'(n) \} - 1) > \varepsilon \}.
\end{align}

Then $B_0 \cap B_1 = \emptyset$, since if not then some $n \in B_0 \cap B_1$ satisfies
\[2 < k - 1 + \varepsilon < \{ f(n) \} + (k-1)\{ rf'(n) \} < 1 - \varepsilon < 1,\]
which is a contradiction. Now, all $i \in \{0, 1\}$, $n \in B_i \cap [x_0, \infty)$, and $j \in \{0, 1, \ldots, k-1\}$ satisfy that
\[0 = \varepsilon - \varepsilon < \delta_i(n, j, \theta) < 1 - \varepsilon + \varepsilon = 1\]
and thus
\[\lfloor f(n + rj) \rfloor = \lfloor f(n) \rfloor + j(\lfloor rf'(n) \rfloor + i).\]

This implies the inclusion relation
\begin{equation}
(B_0 \cup B_1) \cap [x_0, \infty) \subset \{ n \in \mathbb{N} : \{ f(n + rj) \}_{j=0}^{k-1} \text{ is an AP} \}.
\end{equation}

Since $((f(n), rf'(n)))_{n=1}^\infty$ is also uniformly distributed modulo 1 thanks to (A2), it turns out that
\begin{align}
\lim_{N \to \infty} \frac{\#(B_0 \cap [N])}{N} &= \int \int_{x+(k-1)y < 1-\varepsilon \quad \varepsilon < x < 1, \quad 0 \leq y < 1} 1 \, dx \, dy = \frac{(1 - 2\varepsilon)^2}{2(k-1)}, \\
\lim_{N \to \infty} \frac{\#(B_1 \cap [N])}{N} &= \int \int_{x+(k-1)(y-1) > \varepsilon \quad 0 \leq x < 1-\varepsilon, \quad 0 \leq y < 1} 1 \, dx \, dy = \frac{(1 - 2\varepsilon)^2}{2(k-1)}.
\end{align}

Therefore, using (4.4) and $B_0 \cap B_1 = \emptyset$, we conclude that
\begin{align}
\lim_{N \to \infty} \frac{1}{N} \# \{ n \in [N] : \{ f(n + rj) \}_{j=0}^{k-1} \text{ is an AP} \} \\
&\geq \lim_{N \to \infty} \left( \frac{\#(B_0 \cap [x_0, N])}{N} + \frac{\#(B_1 \cap [x_0, N])}{N} \right) \\
&= \lim_{N \to \infty} \left( \frac{\#(B_0 \cap [N])}{N} + \frac{\#(B_1 \cap [N])}{N} \right) = \frac{(1 - 2\varepsilon)^2}{k-1}.
\end{align}

Next, let us show that
\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \# \{ n \in [N] : \{ f(n + rj) \}_{j=0}^{k-1} \text{ is an AP} \} \leq \frac{1 + 2\varepsilon}{k-1}.
\end{equation}
Take an arbitrary $m \geq x_0$ such that $([f(m + rj)]_{j=0}^{k-1})$ is an AP. Taylor’s theorem and (4.1) imply that
\[
[f(m + r)] - [f(m)] > f(m + r) - f(m) - 1 = rf'(m) + \frac{r^2}{2}f''(\eta) - 1
\]
\[
> rf'(m) - 2,
\]
\[
[f(m + r)] - [f(m)] < f(m + r) - f(m) + 1 = rf'(m) + \frac{r^2}{2}f''(\eta) + 1
\]
\[
< rf'(m) + 2
\]
for some $\eta \in (m, m + r)$. Thus, $\gamma := [f(m + r)] - [f(m)] - [rf'(m)] \in \{-1, 0, 1, 2\}$. Here $\gamma \neq -1$ for the following reason. Suppose $\gamma = -1$. Since $([f(m + rj)]_{j=0}^{k-1})$ is an AP, we have $[f(m + 2r)] - [f(m)] = 2([rf'(m)] - 1)$. However, Taylor’s theorem and (4.1) imply that
\[
2([rf'(m)] - 1) = [f(m + 2r)] - [f(m)] > f(m + 2r) - f(m) - 1
\]
\[
= 2rf'(m) + \frac{(2r)^2}{2}f''(\xi) - 1 > 2rf'(m) - 2
\]
for some $\xi \in (m, m + 2r)$, which yields the contradiction $\{rf'(m)\} < 0$. Thus, $\gamma \neq -1$. Also, $\gamma \neq 2$ in the same way. Therefore, $\gamma \in \{0, 1\}.

Define the two sets $C_0$ and $C_1$ as
\[
C_0 = \{ n \in \mathbb{N} : \{f(n)\} + (k-1)\{rf'(n)\} < 1 + \varepsilon \},
\]
\[
C_1 = \{ n \in \mathbb{N} : \{f(n)\} + (k-1)\{rf'(n)\} - 1 \geq -\varepsilon \}.
\]
Let us show $m \in C_0 \cup C_1$. Since $([f(m + rj)]_{j=0}^{k-1})$ is an AP, it follows that
\[
[f(m + rj)] = \lfloor f(m) \rfloor + j([rf'(m)] + \gamma)
\]
for all $j \in \{0, 1, \ldots, k-1\}$. Moreover,
\[
f(m + rj) = \lfloor f(m) \rfloor + j([rf'(m)] + \gamma) + \delta_\gamma(m, j, \theta),
\]
and thus $\delta_\gamma(m, j, \theta) = \{f(m + rj)\}$. This yields that
\begin{align*}
\text{Case } \gamma = 0 & \quad \{f(m)\} + (k-1)\{rf'(m)\} - \varepsilon \leq \delta_0(m, j, \theta) < 1, \\
\text{Case } \gamma = 1 & \quad \{f(m)\} + (k-1)\{rf'(m)\} - 1 + \varepsilon \geq \delta_1(m, j, \theta) \geq 0,
\end{align*}
whence $m \in C_0 \cup C_1$. Therefore, we obtain the inclusion relation
\[
\{ n \in [x_0, \infty) : ([f(n + rj)]_{j=0}^{k-1}) \text{ is an AP} \} \subset C_0 \cup C_1.
\]
Since \((f(n), rf'(n))\) is also uniformly distributed modulo 1 thanks to (A2), it turns out that
\[
\lim_{N \to \infty} \frac{\#(C_0 \cap [N])}{N} = \int \int \mathbb{1}_{x+(k-1)y < 1, 0 \leq x, y < 1} 1 dxdy = \frac{1}{2(k-1)} + \frac{\varepsilon}{k-1} = \frac{1 + 2\varepsilon}{2(k-1)},
\]
\[
(4.9)\]
\[
\lim_{N \to \infty} \frac{\#(C_1 \cap [N])}{N} = \int \int \mathbb{1}_{x+(k-1)y \geq \varepsilon, 0 \leq x, y < 1} 1 dxdy = \frac{1}{2(k-1)} + \frac{\varepsilon}{k-1} = \frac{1 + 2\varepsilon}{2(k-1)}.
\]
Therefore, using (4.8), we conclude that
\[
\lim_{N \to \infty} \frac{\# \{ n \in [N] : \lfloor f(n + rj) \rfloor j=0^{k-1} \text{ is an AP} \}}{N} = \lim_{N \to \infty} \frac{\#(C_0 \cap [N])}{N} + \frac{\#(C_1 \cap [N])}{N} = 1 + \frac{2\varepsilon}{k-1}.
\]
Finally, once letting \(\varepsilon \to +0\) in (4.2) and (4.6), Theorem 2.1 follows.

Next, to prove Theorem 2.2, we show the following lemma.

**Lemma 4.1.** If \(N_m, R_m \in \mathbb{N}\) diverge to positive infinity as \(m \to \infty\) and if \((x_n)_{n=1}^{\infty}\) and \((y_n)_{n=1}^{\infty}\) are uniformly distributed modulo 1, then for every convex set \(C \subset [0,1]^2\),
\[
\lim_{m \to \infty} \frac{\# \{ (n, r) \in [N_m] \times [R_m] : (\{x_n\}, \{r y_n\}) \in C \}}{N_m R_m} = \mu(C),
\]
where \(\mu(C)\) denotes the area of \(C\).

**Proof.** If the following criterion holds, Lemma 4.1 follows in the same way as Weyl’s theorem on uniform distribution. **Weyl’s criterion:** for every \((h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\},
\[
(4.10)\]
\[
\lim_{m \to \infty} \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \sum_{r=1}^{R_m} e(h_1 x_n + h_2 r y_n) = 0,
\]
where \(e(x) := e^{2\pi i x}\). Hence, let us show Weyl’s criterion. First, we show (4.10) when \(h_2 \neq 0\). Take arbitrary \(\varepsilon > 0\) and \(\delta \in (0, 1/2)\). Since \((y_n)_{n=1}^{\infty}\) is uniformly distributed modulo 1, so is \((h_2 y_n)_{n=1}^{\infty}\). Thus, there exists \(m_0 \in \mathbb{N}\) such that for every integer
\[ m \geq m_0, \]
\[
\frac{1}{N_m} \sum_{n \in [N_m]} \frac{1}{|\sin(\pi h_2 y_n)|} = \frac{1}{N_m} \sum_{n \in [N_m]} \frac{1}{\sin(\pi \{h_2 y_n\})} \]
\[
\leq \int_{\delta}^{1-\delta} \frac{dx}{\sin(\pi x)} + \varepsilon, \tag{4.11}
\]
\[
\frac{1}{N_m} \sum_{n \in [N_m]} 1 \leq 2\delta + \varepsilon. \tag{4.12}
\]

Using the inequalities (4.11) and (4.12), for every integer \( m \geq m_0 \) we have

\[
\frac{1}{R_m N_m} \sum_{n=1}^{N_m} \sum_{r=1}^{R_m} e(h_1 x_n + h_2 r y_n) \leq \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \left| 1 - e(h_2 R_m y_n) \right| \]
\[
= \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \left| \frac{\sin(\pi h_2 R_m y_n)}{\sin(\pi h_2 y_n)} \right| \leq \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \min \left\{ R_m, \frac{1}{|\sin(\pi h_2 y_n)|} \right\} \]
\[
\leq \frac{1}{N_m} \sum_{n \in [N_m]} 1 + \frac{1}{N_m R_m} \sum_{n \in [N_m]} \frac{1}{|\sin(\pi h_2 y_n)|} \]
\[
\leq 2\delta + \varepsilon + \frac{1}{R_m} \left( \int_{\delta}^{1-\delta} \frac{dx}{\sin(\pi x)} + \varepsilon \right). \]

Since \( R_m \) diverges to positive infinity as \( m \to \infty \), it follows that

\[
\lim_{m \to \infty} \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \sum_{r=1}^{R_m} e(h_1 x_n + h_2 r y_n) \leq 2\delta + \varepsilon.
\]

The arbitrariness of \( \varepsilon > 0 \) and \( \delta \in (0, 1/2) \) yields (4.10).

Finally, let us show (4.10) when \( h_2 = 0 \) and \( h_1 \neq 0 \). Since \( (x_n)_{n=1}^{\infty} \) is uniformly distributed modulo 1, we have

\[
\lim_{m \to \infty} \frac{1}{N_m R_m} \sum_{n=1}^{N_m} \sum_{r=1}^{R_m} e(h_1 x_n + h_2 r y_n) = \lim_{m \to \infty} \frac{1}{N_m} \sum_{n=1}^{N_m} e(h_1 x_n) = 0,
\]

whence (4.10) follows. \( \square \)
Now, let us show Theorem 2.2. Since (2.2) consists of the following inequalities:

\[
\liminf_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs}\}}{N f''(N)^{-1/2}} > 0,
\]

\[
\limsup_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs}\}}{N f''(N)^{-1/2}} < \infty,
\]

we prove the above inequalities. Also, note that \( f''(x) > 0 \) for every sufficiently large \( x > 0 \) because of (B2).

**Proof of Theorem 2.2 (liminf).** Let \( k \geq 3 \) be an integer and \( N \in \mathbb{N} \) be sufficiently large. Take arbitrary \( \varepsilon \in (0, 1) \) and \( 0 < \delta_1 < \cdots < \delta_s < \delta_{s+1} = 1 \). Put

\[
R_i = R_i(N) = \left\lfloor \frac{(2\varepsilon)^{1/2} f''(\delta_i N)^{-1/2}}{k-1} \right\rfloor
\]

for \( i \in [s] \) and \( N \in \mathbb{N} \). Then every \( x \geq \delta_i N \) and every \( r \in [R_i] \) satisfy

\[
0 < f''(x) \leq f''(\delta_i N) \leq \frac{2\varepsilon}{(k-1)^2 R_i^2} \leq \frac{2\varepsilon}{(k-1)^2 r^2}.
\]

Now, the following inequality holds:

\[
\#\{P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs}\}
\]

\[
\geq \#\{(n, r) \in [N - (k-1)R_s] \times [R_s] : ([f(n + rj)]_{j=0}^{k-1} \text{ is an AP})
\]

\[
\geq \#\{(n, r) \in [N] \times [R_s] : ([f(n + rj)]_{j=0}^{k-1} \text{ is an AP}) - (k-1)R_s^2
\]

\[
\geq \sum_{i=1}^{s} \#\{(n, r) \in (\delta_i N, \delta_{i+1} N) \times [R_i] : ([f(n + rj)]_{j=0}^{k-1} \text{ is an AP})
\]

\[
- (k-1)R_s^2.
\]

For \( r \in \mathbb{N} \), define the disjoint sets \( B_{0,r} \) and \( B_{1,r} \) as

\[
B_{0,r} = \{n \in \mathbb{N} : \{f(n)\} + (k-1)\{rf'(n)\} < 1 - \varepsilon\},
\]

\[
B_{1,r} = \{n \in \mathbb{N} : \{f(n)\} < 1 - \varepsilon, \{f(n)\} + (k-1)(\{rf'(n)\} - 1) \geq 0\}.
\]
Due to (4.13), the same argument as the proof of Theorem 2.1 implies that for all $n \in (B_{0,r} \cup B_{1,r}) \cap [\delta_i N, \infty)$ and $r \in [R_i]$, the sequence $\binom{f(n + r_j)}{j=0}^{k-1}$ is an AP. Thus,

$$
\#\{P \subset [N] : P \text{ and } (\lfloor f(n) \rfloor)_{n \in P} \text{ are } k-\text{APs}\} \quad \text{\cite{4.14}} \\
\geq \sum_{i=1}^{\delta_i} \frac{\# \{(n, r) \in (\delta_i N, \delta_i+1 N] \times [R_i] : (\lfloor f(n + r_j) \rfloor)_{j=0}^{k-1} \text{ is an AP}\}}{N f''(N)^{-1/2}} - \frac{(k - 1)R^2_s}{N f''(N)^{-1/2}} \\
\geq \sum_{i=1}^{\delta_i} \frac{\# \{(n, r) \in (\delta_i N, \delta_i+1 N] \times [R_i] : n \in B_{0,r} \cup B_{1,r}\}}{N f''(N)^{-1/2}} - \frac{(k - 1)R^2_s}{N f''(N)^{-1/2}}.
$$

The absolute value of the second term of (4.15) is bounded above by

$$
\frac{(k - 1)R^2_s}{N f''(N)^{-1/2}} \leq \frac{(k - 1)}{N f''(N)^{-1/2}} \cdot \frac{2\epsilon f''(\delta_s N)^{-1}}{(k - 1)^2} \\
\leq \frac{1}{N f''(N)^{-1/2}} \cdot \frac{2\epsilon f''(N)^{-1}}{k - 1} \leq \frac{1}{N f''(N)^{1/2}} \xrightarrow{N \to \infty} 0.
$$
Also, the following inequality holds:

\[
\frac{\#\{(n, r) \in (\delta_i N, \delta_{i+1} N] \times [R_i] : n \in B_{0,r} \cup B_{1,r}\}}{N f''(N)^{-1/2}} \geq \frac{\#\{(n, r) \in (\delta_i N, \delta_{i+1} N] \times [R_i] : n \in B_{0,r} \cup B_{1,r}\}}{c(\delta_i)^{1/2} N f''(\delta_i N)^{-1/2}} \\
= \frac{\#\{(n, r) \in [1, \delta_{i+1} N] \times [R_i] : n \in B_{0,r} \cup B_{1,r}\}}{c(\delta_i)^{1/2} N f''(\delta_i N)^{-1/2}} - \frac{\#\{(n, r) \in (\delta_i N, \delta_{i+1} N] \times [R_i] : n \in B_{0,r} \cup B_{1,r}\}}{c(\delta_i)^{1/2} N f''(\delta_i N)^{-1/2}} \cdot \frac{c(\delta_i)^{-1/2} \delta_{i+1} R_i}{\delta_i N R_i} \cdot \frac{f''(\delta_i N)^{-1/2}}{f''(\delta_i N)^{-1/2}} \\
\xrightarrow{N \to \infty} (\frac{(1-\varepsilon)^2}{2(k-1)} + \frac{(1-\varepsilon)^2}{2(k-1)}) \cdot \frac{(2\varepsilon)^{1/2} c(\delta_i)^{-1/2}}{k-1} (\delta_{i+1} - \delta_i) \\
= \frac{(1-\varepsilon)^2 (2\varepsilon)^{1/2}}{(k-1)^2} \cdot c(\delta_i)^{-1/2} (\delta_{i+1} - \delta_i).
\]

Therefore, letting \( N \to \infty \) in (4.15), we obtain

\[
\lim_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs}\}}{N f''(N)^{-1/2}} \geq \frac{(1-\varepsilon)^2 (2\varepsilon)^{1/2}}{(k-1)^2} \sum_{i=1}^{s} c(\delta_i)^{-1/2} (\delta_{i+1} - \delta_i) > 0.
\]

\[\square\]

**Remark 4.2.** Let us consider the special case \( f(x) = x^\alpha \) with \( \alpha \in (1, 2) \). Then we can take \( c(\delta) \) in (B3) as \( c(\delta) = \delta^{\alpha-2} \). Thus,

\[
\lim_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([n^\alpha])_{n \in P} \text{ are } k-\text{APs}\}}{\alpha(\alpha-1))^{-1/2} N^{2-\alpha/2}} \geq \frac{(1-\varepsilon)^2 (2\varepsilon)^{1/2}}{(k-1)^2} \sum_{i=1}^{s} \delta_i^{1-\alpha/2} (\delta_{i+1} - \delta_i).
\]
Since $\varepsilon \in (0, 1)$ and $0 < \delta_1 < \cdots < \delta_s < \delta_{s+1} = 1$ are arbitrary, we obtain
\[
\lim_{N \to \infty} \frac{\# \{ P \subset [N] : P \text{ and } ([n^\alpha])_{n \in P} \text{ are } k\text{-APs} \} }{N^{2-\alpha/2}} \geq \frac{C}{(k-1)^2(\alpha(\alpha-1))^{1/2}} \int_0^1 x^{1-\alpha/2} dx = \frac{2C}{(k-1)^2(\alpha(\alpha-1))^{1/2}(4-\alpha)} =: \tilde{A}_{a,k},
\]
where $C = \max_{0 \leq x \leq 1} (1-x)^2 (2x)^{1/2} = 16\sqrt{10}/125$. Therefore, the constant $A_{a,k}$ in Theorem 1.2 is an arbitrary value in the interval $(0, \tilde{A}_{a,k})$.

**Proof of Theorem 2.2** (limsup). Let $k \geq 3$ be an integer and take an arbitrary $\beta > 1$. Due to (B2), we can take $N_0 \in \mathbb{N}$ such that every integer $n \geq N_0$ satisfies $1 + (k-1)R(n)/n < \beta$, where
\[
R(n) := \left( \frac{2c(1/\beta)}{k-2} \right)^{1/2} f''(n)^{-1/2}.
\]
First, we show that if $([f(n+rj)])_{j=0}^{k-1}$ is an AP with $n \geq N_0$ and $r \geq 1$, then $r < R(n)$ by contradiction. Suppose that for some $n_0 \geq N_0$ and $r_0 \geq 1$, the sequence $([f(n_0+r_0j)])_{j=0}^{k-1}$ was an AP and $r_0 \geq R_0 := R(n_0)$. Since the derivative of the function
\[
(f(n+(k-1)r) - f(n+(k-2)r)) - (f(n+r) - f(n))
\]
of $r$ is
\[
(k-1)f'(n+(k-1)r) - (k-2)f'(n+(k-2)r) - f'(n+r) > 0
\]
due to $f''(x) > 0$, the function (4.16) of $r$ increases. Recalling that $([f(n_0+r_0j)])_{j=0}^{k-1}$ are APs, we have
\[
2 > (f(n_0+(k-1)r_0) - f(n_0+(k-2)r_0)) - (f(n_0+r_0) - f(n_0)) \overset{(a)}{\geq} (f(n_0+(k-1)R_0) - f(n_0+(k-2)R_0)) - (f(n_0+R_0) - f(n_0)) \overset{(b)}{=} (k-2)R_0 f''(n_0 + \theta + \theta') \overset{(c)}{=} (k-2)R_0^2 f''(n_0 + \theta + \theta')
\]
where the monotonicity of the function (4.16) and the inequality $R_0 \geq r_0$ have been used to obtain $(a)$; and the mean value theorem has been used to obtain $(b)$; and $\theta \in (0, (k-2)R_0)$ and $\theta' \in (0, R_0)$. Put $\beta_0 = 1 + (k-1)R_0/n_0$. Since the inequality
\[ \beta_0 < \beta \] holds due to \( n_0 \geq N_0 \), it follows that

\[
f''(n_0 + \theta + \theta') \geq f''(n_0 + (k - 1)R_0) = f''(\beta n_0) \geq f''(\beta n_0) \geq c(1/\beta)^{-1} f''(n_0). \tag{B1} \]

(4.18)

Thus, (4.17) and (4.18) yield that

\[ 2 > 2c(1/\beta) \frac{f''(n_0 + \theta + \theta')}{f''(n_0)} \geq 2c(1/\beta)^{-1} \frac{f''(n_0)}{f''(n_0)} = 2, \]

which is a contradiction. Therefore, if \( (|f(n + rj)|)^{k-1}_{j=0} \) is an AP with \( n \geq N_0 \) and \( r \geq 1 \), then \( r < R(n) \).

Next, we show Theorem 2.2 (limsup). Let \( N \in \mathbb{N} \) be sufficiently large. Since the inequality

\[ \# \{ P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs} \} \]

\[ \leq \# \{ (n, r) \in [N] \times [N] : (\lfloor f(n + rj) \rfloor)^{k-1}_{j=0} \text{ is an AP} \} \]

\[ \leq \# \{ (n, r) \in [N_0] \times [N] \} + \# \{ (n, r) \in [N_0, N] \times [N] : r < R(n) \} \]

\[ \leq N_0 N + \sum_{n=N_0}^{N} R(n) \]

holds, it follows that

\[ \lim_{N \to \infty} \frac{\# \{ P \subset [N] : P \text{ and } ([f(n)])_{n \in P} \text{ are } k-\text{APs} \}}{N f''(N)^{-1/2}} \]

\[ \leq \lim_{N \to \infty} \frac{f''(N)^{1/2}}{N} \sum_{n=N_0}^{N} R(n) \leq \lim_{N \to \infty} f''(N)^{1/2} R(N) = \left( \frac{2c(1/\beta)}{k-2} \right)^{1/2} < \infty. \]

□
Remark 4.3. Let us consider the special case \( f(x) = x^\alpha \) with \( \alpha \in (1, 2) \). Then we can take \( c(\delta) \) in (B3) as \( c(\delta) = \delta^{\alpha-2} \). Thus,

\[
\lim_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([n^\alpha])_{n \in P} \text{ are } k\text{-APs}\}}{(\alpha(\alpha - 1))^{-1/2}N^{2-\alpha/2}} \\
\leq \lim_{N \to \infty} \frac{f''(N)^{1/2}}{N} \sum_{n=1}^{N} \left( \frac{2c(1/\beta)}{k - 2} \right)^{1/2} f''(n)^{-1/2} \\
= \lim_{N \to \infty} N^{\alpha/2 - 2} \sum_{n=1}^{N} \left( \frac{2\beta^{2-\alpha}}{k - 2} \right)^{1/2} n^{1-\alpha/2} \\
= \lim_{N \to \infty} N^{\alpha/2 - 2} \left( \frac{2\beta^{2-\alpha}}{k - 2} \right)^{1/2} N^{2-\alpha/2} \left( \frac{2}{2-\alpha/2} \right) \frac{2}{4-\alpha}.
\]

The arbitrariness of \( \beta > 1 \) yields

\[
\lim_{N \to \infty} \frac{\#\{P \subset [N] : P \text{ and } ([n^\alpha])_{n \in P} \text{ are } k\text{-APs}\}}{N^{2-\alpha/2}} \\
\leq \left( \frac{2}{\alpha(\alpha - 1)(k - 2)} \right)^{1/2} \frac{2}{4-\alpha} =: \tilde{B}_{\alpha,k}.
\]

Therefore, the constant \( B_{\alpha,k} \) in Theorem 1.2 is an arbitrary value in the interval \((\tilde{B}_{\alpha,k}, \infty)\).

5. Further analysis: discrepancy and short intervals

In this section, let us show Theorem 2.2 and Theorem 2.3. These theorems follow from the following proposition.

Proposition 5.1. Let \( \alpha \in (1, 2) \) and \( c > 0 \), and let \( k \geq 3 \) and \( r \geq 1 \) be integers. Then, for all integers \( 1 \leq L \leq cN \),

\[
\frac{1}{L} \#\{n \in [N, N + L] : ((n + rj)^\alpha)_{j=0}^{k-1} \text{ is an AP}\} = \frac{1}{k-1} + E(N, L),
\]

where

\[
E(N, L) \ll_{\alpha, k, r, c} \begin{cases} 
\frac{N^{(\alpha-2)/6}(\log N)^{1/2}}{L^{1/2}} + \frac{N^{(2-\alpha)/2}/L^{1/2}}{L^{1/2}} & \alpha \in (1, 5/4) \cup [11/6, 2), \\
\frac{N^{(\alpha-3)/14}(\log N)^{1/2}}{L^{1/2}} + \frac{N^{(2-\alpha)/2}/L^{1/2}}{L^{1/2}} & \alpha \in [5/4, 3/2), \\
\frac{N^{(\alpha-3)/14} + N^{(3-\alpha)/6}/L^{1/2}(\log N)^{1/2}}{L^{1/2}} & \alpha \in [3/2, 11/6),
\end{cases}
\]

as \( N \to \infty \); and \( L \) may depend on \( N \).
Proposition 5.1 is an asymptotic formula for the number of integers \( n \geq 1 \) in short intervals such that \((\lfloor (n + r_j)^\alpha \rfloor)_{j=0}^{k-1}\) is an AP. We prove Proposition 5.1 at the end of this section. Note that (5.1) is not effective when \( L = L(N) \) is sufficiently smaller than \( N \). This is because in the case, the error term in Proposition 5.1 is larger than the leading term. Before proving Proposition 5.1 let us show Theorem 2.3 and Theorem 2.4 by using Proposition 5.1.

**Proof of Theorem 2.3 assuming Proposition 5.1.** Let \( \alpha \in (1,2) \), and let \( k \geq 3 \) and \( r \geq 1 \) be integers. Also, define the set \( Q \) as

\[
Q = \{ n \in \mathbb{N} : (\lfloor (n + r j)^\alpha \rfloor)_{j=0}^{k-1} \text{ is an AP} \}.
\]

Then Proposition 5.1 implies that

\[
\frac{\#(Q \cap [x, 2x])}{N} = \frac{1}{k-1} + E(x, 2x),
\]

(5.3) \( E(x, 2x) \ll_{\alpha, k, r} \begin{cases} x(\alpha - 2)/6 (\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in (1, 5/4) \cup [11/6, 2), \\ x(\alpha - 3)/14 (\log x)^{1/2} + x^{(1-\alpha)/2} & \alpha \in [5/4, 3/2), \\ x^{(\alpha - 3)/14} + x^{-\alpha/6} (\log x)^{1/2} & \alpha \in [3/2, 11/6). \end{cases} \)

The right-hand side in (5.3) is equal to the function \( F(x) \) defined in Theorem 2.3. Let \( N \in \mathbb{N} \) be sufficiently large and take \( M \in \mathbb{N} \) with \( 2^M \leq N < 2^{M+1} \). This yields that

\[
\frac{\#(Q \cap [N])}{N} \geq \frac{1}{N} \sum_{m=1}^{M} \#(Q \cap [2^{-m}N, 2^{1-m}N]) \\
= \sum_{m=1}^{M} \frac{2^{-m}}{k-1} + O_{\alpha, k, r} \left( \sum_{m=1}^{M} 2^{-m} F(2^{-m}N) \right) = \frac{1}{k-1} + O_{\alpha, k, r}(F(N)) \\
= \frac{1}{k-1} + O_{\alpha, k, r}(1/N + F(N)).
\]

Similarly, it follows that

\[
\frac{\#(Q \cap [N])}{N} \leq \frac{1}{N} \sum_{m=1}^{M+1} \#(Q \cap [2^{-m}N, 2^{1-m}N]) \\
= \sum_{m=1}^{M+1} \frac{2^{-m}}{k-1} + O_{\alpha, k, r} \left( \sum_{m=1}^{M+1} 2^{-m} F(2^{-m}N) \right) = \frac{1}{k-1} + O_{\alpha, k, r}(F(N)) \\
= \frac{1}{k-1} + O_{\alpha, k, r}(1/N + F(N)).
\]

Therefore, Theorem 2.2 holds. \( \square \)
Proof of Theorem 2.4 assuming Proposition 5.1. Let $\alpha \in (1, 2)$, and let $k \geq 3$ and $r \geq 1$ be integers. Define the set $Q$ as (5.2). Thanks to Proposition 5.1, there exists a constant $C > 0$ such that for every sufficiently large $N \in \mathbb{N}$ and every $L \in [N]$,

\[(5.4) \quad \left| \frac{\#(Q \cap [N, N + L])}{L} - \frac{1}{k - 1} \right| \leq CF(N, L),\]

where $F(N, L)$ is defined as the right-hand side in (5.1). Putting $L = \lceil 4C^2(k - 1)^2N^{2-\alpha} \rceil$, we have

$$F(N, L) \leq N(2-\alpha)/2L^{1/2} + o(1) \leq \frac{1}{2C(k - 1)} + o(1) < \frac{1}{C(k - 1)}$$

for every sufficiently large $N \in \mathbb{N}$. Therefore, for every sufficiently large $N \in \mathbb{N}$, the left-hand side in (5.4) is less than $1/(k - 1)$, whence $\#(Q \cap [N, N + L]) > 0$. \hfill $\Box$

To prove Proposition 5.1, we need to estimate the convergence speed of (3.1) for a uniformly distributed sequence. For this purpose, let us define two kinds of discrepancies. For a sequence $(x_n)_{n=1}^N$, define the discrepancy $D_N$ and isotropic discrepancy $J_N$ as

$$D_N = D(x_1, \ldots, x_N) = \sup_{0 \leq a_i < b_i \leq 1} \left| \frac{\#\{n \in [N] : x_n \in \prod_{i=1}^d [a_i, b_i]\}}{N} \right| - \prod_{i=1}^d (b_i - a_i), \hspace{1cm} J_N = J(x_1, \ldots, x_N) = \sup_{C \subset [0,1]^d} \left| \frac{\#\{n \in [N] : x_n \in C\}}{N} - \mu(C) \right|,$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^d$. Although the inequality $D_N \leq J_N$ is trivial, the following reverse inequality holds [19, Theorem 1.6, Chapter 2]:

\[(5.5) \quad J_N \leq (4d\sqrt{d} + 1)D_N^{1/d}\]

for every $d, N \in \mathbb{N}$ and $x_1, \ldots, x_N \in \mathbb{R}^d$. Thanks to (5.5), it suffices to give an upper bound of the discrepancy in order to estimate the convergence speed of (3.1).

Now, the following inequality is useful to evaluate discrepancies.

**Lemma 5.2** (Koksma and Szüs [19]). For all $d, L, H \in \mathbb{N}$ and $x_1, \ldots, x_L \in \mathbb{R}^d$,

$$D(x_1, \ldots, x_L) \ll_d \frac{1}{H} + \sum_{0 < \|h\|_\infty \leq H} \frac{1}{u(h)} \left| \frac{1}{L} \sum_{n=1}^L e(\langle h, x_n \rangle) \right|,$$

where $u(h) := \prod_{i=1}^d \max\{1, |h_i|\}$ and $e(x) := e^{2\pi ix}$.
This inequality is sometimes referred to as the Erdős-Turán-Koksma inequality. Thanks to Lemma 5.2, it suffices to evaluate exponential sums in order to find upper bounds of discrepancies. Next, let us state the following lemmas that are used to evaluate exponential sums.

**Lemma 5.3** (Kusmin-Landau). Let $I$ be an interval of $\mathbb{R}$, and $f : I \to \mathbb{R}$ be a $C^1$ function such that $f'$ is monotonic. If $\lambda_1 > 0$ satisfies that $\lambda_1 \leq \min\{|f'(x) - n| : n \in \mathbb{Z}\}$ for all $x \in I$, then

$$\sum_{n \in I \cap \mathbb{Z}} e(f(n)) \ll \lambda_1^{-1}.$$ 

**Lemma 5.4** (van der Corput). Let $I$ be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a $C^2$ function, and let $c \geq 1$. If $\lambda_2 > 0$ satisfies that $\lambda_2 \leq |f''(x)| \leq c\lambda_2$ for all $x \in I$, then

$$\sum_{n \in I \cap \mathbb{Z}} e(f(n)) \ll_c |I| \lambda_2^{1/2} + \lambda_2^{-1/2},$$

where $|I|$ denotes the length of the interval $I$.

**Lemma 5.5** (Sargos-Gritsenko). Let $I$ be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a $C^3$ function, and let $0 < c_1 < c_2$. If $\lambda_3 \in (0,1)$ satisfies that $c_1\lambda_3 \leq |f'''(x)| \leq c_2\lambda_3$ for all $x \in I$, then

$$\sum_{n \in I \cap \mathbb{Z}} e(f(n)) \ll_{c_1,c_2} |I| \lambda_3^{1/6} + \lambda_3^{-1/3}.$$ 

Lemmas 5.3 and 5.4 are called the first and second derivative tests, respectively. One can see their proofs in [20, Theorems 2.1 and 2.2]. Lemma 5.5 was shown by Sargos [21] and Grisenko [22] independently. Using Lemmas 5.3–5.5, we evaluate discrepancies.

**Lemma 5.6.** Let $\alpha \in (1,2)$, $r \in \mathbb{N}$, and $c > 0$. For all $L, N \in \mathbb{N}$, define $D(N, L)$ as the discrepancy of the sequence $((N + n)^{\alpha}, c(N + n)^{\alpha-1})_{n=0}^{N-1}$. Then, for all integers $1 \leq L \leq cN$,

$$D(N, L) \ll_{\alpha,r,c} \begin{cases} N^{(\alpha-2)/3} \log N + N^{2-\alpha}/L & \alpha \in (1,5/4) \cup [11/6,2), \\
N^{(\alpha-3)/7} \log N + N^{2-\alpha}/L & \alpha \in [5/4,3/2), \\
(N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L) \log N & \alpha \in [3/2,11/6) \\
\end{cases}$$

as $N \to \infty$, where $L$ may depend on $N$. 
Proof. Let \( f(x) = (N + x)^\alpha \). Lemma 5.2 implies that for all \( L, N, H \in \mathbb{N} \),

\[
D(N, L) \ll \frac{1}{H} + \sum_{|h_1|, |h_2| \leq H \atop (h_1, h_2) \neq (0,0)} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 r f'(n)) \right|.
\]

Letting \( N \in \mathbb{N} \) be sufficiently large, we evaluate the right-hand side above in two ways.

**Step 1.** Let us show that for all integers \( 1 \leq L \leq cN \),

\[
(5.6) \quad D(N, L) \ll_{\alpha, r, c} N^{(\alpha-2)/3} \log N + N^{2-\alpha}/L.
\]

Put \( H = \lfloor N^{(2-\alpha)/3} \rfloor \). Consider the case when \( |h_1|, |h_2| \leq H \) and \( h_1 \neq 0 \). When \( x \in [N, N + L - 1] \), the function \( g(x) = h_1 f(x) + h_2 r f'(x) \) satisfies that

\[
|g''(x)| \leq |h_1| f''(x)(1 + rH |f''(x)/f''(x)|) \\
\ll |h_1| N^{\alpha-2}(1 + rH/N) \ll_{\alpha, r} |h_1| N^{\alpha-2} ; \\
|g''(x)| \geq |h_1| f''(x)(1 - rH |f''(x)/f''(x)|) \\
\gg_{\alpha} |h_1| (N + L)^{\alpha-2}(1 - rH/N) \gg_{\alpha, r, c} |h_1| N^{\alpha-2}.
\]

Thus, Lemma 5.4 implies that

\[
\frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 r f'(n)) \ll_{\alpha, r, c} |h_1|^{1/2} N^{(\alpha-2)/2} + |h_1|^{-1/2} N^{(2-\alpha)/2}/L.
\]

Therefore, it follows that

\[
\sum_{|h_1|, |h_2| \leq H \atop h_1 \neq 0} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 r f'(n)) \right| \\
\ll_{\alpha, r, c} \sum_{|h_1|, |h_2| \leq H \atop h_1 \neq 0} \frac{|h_1|^{1/2} N^{(\alpha-2)/2} + |h_1|^{-1/2} N^{(2-\alpha)/2}/L}{u(h_1, h_2)} \\
\ll \left( \sum_{h_2=1}^{H} \frac{1}{h_2} + 1 \right) \sum_{h_1=1}^{H} \left( h_1^{-1/2} N^{(\alpha-2)/2} + h_1^{-3/2} N^{(2-\alpha)/2}/L \right) \\
\ll (\log H)(H^{1/2} N^{(\alpha-2)/2} + N^{(2-\alpha)/2}/L) \ll (N^{(\alpha-2)/3} + N^{(2-\alpha)/2}/L) \log N.
\]
Next, consider the case when \(1 \leq |h_2| \leq H\) and \(h_1 = 0\). When \(x \in [N, N + L - 1]\), the function \(g(x) = h_2rf'(x)\) satisfies that

\[
|g'(x)| = r|h_2|f''(x) \ll_r H^{\alpha-2} \leq N^{(2/3)(\alpha-2)},
\]
\[
|g'(x)| \gg_\alpha |h_2|(N + L)^{\alpha-2} \gg_r |h_2|N^{\alpha-2}.
\]

Due to (5.7), we may assume that \(|g'(x)| \leq 1/2\) for all \(x \in [N, N + L - 1]\), which yields that \(\min\{|g'(x) - m| : m \in \mathbb{Z}\} = |g'(x)|\) for all \(x \in [N, N + L - 1]\). Thus, Lemma 5.3 implies that

\[
\frac{1}{L} \sum_{n=0}^{L-1} e(h_2rf'(n)) \ll_{\alpha,r,c} |h_2|^{-1} N^{2-\alpha}/L.
\]

Therefore, it follows that

\[
\sum_{1 \leq |h_2| \leq H} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1f(n) + h_2rf'(n)) \right| \ll_{\alpha,r,c} \sum_{1 \leq |h_2| \leq H} \frac{|h_2|^{-1} N^{2-\alpha}/L}{|h_2|} \ll N^{2-\alpha}/L.
\]

Summarizing the above two cases, we have

\[
D(N, L) \ll \frac{1}{H} + \sum_{|h_1|, |h_2| \leq H} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1f(n) + h_2rf'(n)) \right| \ll_{\alpha,r,c} N^{(\alpha-2)/3} + (N^{(\alpha-2)/3} + N^{(2-\alpha)/2}/L) \log N + N^{2-\alpha}/L
\]

\[
\ll_\alpha N^{(\alpha-2)/3} \log N + N^{2-\alpha}/L,
\]

which is just (5.6).

**Step 2.** Let us show that for all \(\alpha \in (1, 11/6)\) and all integers \(1 \leq L \leq cN\),

\[
D(N, L) \ll_{\alpha,r,c} \begin{cases} 
N^{(\alpha-3)/7} \log N + N^{2-\alpha}/L & \alpha \in (1, 3/2), \\
(N^{(\alpha-3)/7} + N^{(3-\alpha)/3}/L) \log N & \alpha \in [3/2, 11/6).
\end{cases}
\]
Put \( H = \lfloor N^{(3-\alpha)/7} \rfloor \). Consider the case when \( |h_1|, |h_2| \leq H \) and \( h_1 \neq 0 \). When \( x \in [N, N + L - 1] \), the function \( g(x) = h_1 f(x) + h_2 f'(x) \) satisfies that

\[
|g'''(x)| \leq |h_1 f'''(x)| (1 + rH |f'''(x)/f''(x)|) \ll |h_1| N^{\alpha - 3} (1 + rH/N) \ll_{\alpha, r} |h_1| N^{\alpha - 3},
\]

\[
|g'''(x)| \geq |h_1| f'''(x) (1 - rH |f'''(x)/f''(x)|) \gg_{\alpha} |h_1| (N + L)^{\alpha - 3} (1 - rH/N) \gg_{\alpha, r, c} |h_1| N^{\alpha - 3}.
\]

Since \( 0 < |h_1| N^{\alpha - 3} \leq H N^{\alpha - 3} \leq N^{(6/7)(\alpha - 3)} < 1 \) (\( N \) sufficiently large), Lemma 5.5 implies that

\[
\frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 f'(n)) \ll_{\alpha, r, c} |h_1|^{1/6} N^{(\alpha - 3)/6} + |h_1|^{-1/3} N^{(3-\alpha)/3} / L.
\]

Therefore, it follows that

\[
\sum_{|h_1|,|h_2| \leq H \atop h_1 \neq 0} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 f'(n)) \right| \ll_{\alpha, r, c} \sum_{|h_1|,|h_2| \leq H \atop h_1 \neq 0} \frac{|h_1|^{1/6} N^{(\alpha - 3)/6} + |h_1|^{-1/3} N^{(3-\alpha)/3} / L}{u(h_1, h_2)}
\]

\[
\ll \left( \sum_{h_2=1}^{H} \frac{1}{h_2^2} + 1 \right) \sum_{h_1=1}^{H} (h_1^{-5/6} N^{(\alpha - 3)/6} + h_1^{-4/3} N^{(3-\alpha)/3} / L)
\]

\[
\ll (\log H) (H^{1/6} N^{(\alpha - 3)/6} + N^{(3-\alpha)/3} / L) \ll (N^{(\alpha - 3)/7} + N^{(3-\alpha)/3} / L) \log N.
\]

Next, consider the case when \( 1 \leq |h_2| \leq H \) and \( h_1 = 0 \). When \( x \in [N, N + L - 1] \), the function \( g(x) = h_2 f'(x) \) satisfies that

(5.9) \[
|g'(x)| \ll r H N^{\alpha - 2} \leq N^{(6\alpha - 11)/7},
\]

\[
|g'(x)| \gg_{\alpha, c} |h_2| N^{\alpha - 2}.
\]

Due to (5.9) and \( \alpha \in (1, 11/6) \), we may assume that \( |g'(x)| \leq 1/2 \) for all \( x \in [N, N + L - 1] \), which yields that \( \min\{|g'(x) - m| : m \in \mathbb{Z}\} = |g'(x)| \) for all \( x \in [N, N + L - 1] \).

From the same calculation as Step 1, it follows that

\[
\sum_{1 \leq |h_2| \leq H \atop h_1 = 0} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} e(h_1 f(n) + h_2 f'(n)) \right| \ll_{\alpha, r, c} N^{2 - \alpha} / L.
\]
Summarizing the above two cases, we have

\[ D(N, L) \ll \frac{1}{H} + \sum_{\substack{|h_1|, |h_2| \leq H \\ (h_1, h_2) \neq (0, 0)}} \frac{1}{u(h_1, h_2)} \left| \frac{1}{L} \sum_{n=0}^{L-1} \epsilon(h_1 f(n) + h_2 f'(n)) \right| \]

\[ \ll_{\alpha, r, c} N^{\alpha - 3/7} + (N^{\alpha - 3/7} + N^{3 - \alpha / 3})/L \log N + N^{-2\alpha / L} \]

\[ \ll_{\alpha} \begin{cases} N^{\alpha - 3/7} \log N + N^{-2\alpha / L} & \alpha \in (1, 3/2), \\ \left(N^{\alpha - 3/7} + N^{3 - \alpha / 3}\right)/L \log N & \alpha \in [3/2, 11/6], \end{cases} \]

which is just (5.8).

Finally, combining (5.6) and (5.8), we obtain Lemma 5.6.

\[
\text{Proof of Proposition 5.1.} \quad \text{Let } f(x) = x^\alpha \quad \text{and} \quad \varepsilon = \varepsilon(N) = \frac{r^2(k - 1)^2}{2} f''(N) \in (0, 1/2); \]

let \( N \) be sufficiently large; let \( D(N, L) \) and \( J(N, L) \) be the discrepancy and isotropic discrepancy of the sequence \( \{f(N + n), r f'(N + n)\}_{n=0}^{L-1} \), respectively. Also, define the set \( Q \) as (5.2). Recall the proof of Theorem 2.1. Defining the sets \( B_0, B_1, C_0, \) and \( C_1 \) by (4.3) and (4.7), we have the inclusion relations \( (B_0 \cup B_1) \cap [N, \infty) \subset Q \) and \( Q \cap [N, \infty) \subset C_0 \cup C_1 \). Therefore,

\[
\frac{\#(Q \cap [N, N + L])}{L} \geq \frac{\#((B_0 \cup B_1) \cap [N, N + L])}{L} \]

\[
= \frac{\#(B_0 \cap [N, N + L])}{L} + \frac{\#(B_1 \cap [N, N + L])}{L} \overset{\text{(a)}}{\geq} \frac{(1 - 2\varepsilon)^2}{k - 1} - 2J(N, L),
\]

\[
\frac{\#(Q \cap [N, N + L])}{L} \leq \frac{\#((C_0 \cup C_1) \cap [N, N + L])}{L} \overset{\text{(a)}}{\leq} \frac{1 + 2\varepsilon}{k - 1} + 2J(N, L),
\]

where the integral calculations (4.5) and (4.9) have been used to obtain (a). The above inequalities yield that

\[
\frac{\#(Q \cap [N, N + L])}{L} - \frac{1}{k - 1} \leq \frac{4\varepsilon}{k - 1} + 2J(N, L).
\]
Using the inequality (5.5), we obtain
\[
\left| \frac{|Q \cap [N, N + L]|}{L} - \frac{1}{k-1} \right| \leq \frac{4\varepsilon}{k-1} + 2(8\sqrt{2} + 1)D(N, L)^{1/2} \\
\ll_{\alpha, k, r} N^{\alpha-2} + D(N, L)^{1/2}
\]
\[
\ll_{\alpha, r, c} \begin{cases} 
N^{(\alpha-2)/6}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in (1, 5/4) \cup [11/6, 2), \\
N^{(\alpha-3)/14}(\log N)^{1/2} + N^{(2-\alpha)/2}/L^{1/2} & \alpha \in [5/4, 3/2), \\
(N^{(\alpha-3)/14} + N^{(3-\alpha)/6}/L^{1/2})(\log N)^{1/2} & \alpha \in [3/2, 11/6),
\end{cases}
\]
where the inequality \((x + y)^{1/2} \leq x^{1/2} + y^{1/2}\) for \(x, y \geq 0\) has been used to obtain the last inequality.

6. Future work

We have investigated distributions of APs contained in a Piatetski-Shapiro sequence. Since it is known that the Piatetski-Shapiro sequence with exponent \(\alpha \in (1, 2)\) contains mathematical structures other than APs, e.g., solutions to \(x + y = z\) [18], it might be also interesting to investigate the distribution of such solutions. As other natural questions, we have the positive-density version and prime-number version.

**Question 6.1** (Positive-density version). Let \(\alpha \in (1, 2)\), and let \(A \subset \mathbb{N}\) be a set with positive density and \(k \geq 3\) be an integer. Then does
\[
(6.1) \quad \# \{ P \subset A \cap [N] : P \text{ and } ([n^\alpha]_{n \in P} \text{ are k-APs}) \sim N^{2-\alpha/2} \quad (N \to \infty)
\]
hold?

**Question 6.2** (Prime-number version). How about the case when \(A\) in Question 6.1 is replaced with the set of all primes? In this case, what is suitable as the right-hand side in (6.1)?

Also, we have not proved the convergence in the proof of Theorem 1.2. However, the left-hand side in (1.2) divided by \(N^{2-\alpha/2}\) probably converges as \(N \to \infty\). It is also a future work.

**Question 6.3** (Asymptotic formula for \(\alpha\)). Fix a sufficiently large \(N \in \mathbb{N}\) and integers \(k \geq 3\) and \(r \geq 1\). Let
\[
D_{N,k,r}(\alpha) = \frac{1}{N}\# \{ n \in [N] : ([n + rj]^\alpha)_{j=0}^{k-1} \text{ is an AP} \}.
\]
What is an asymptotic formula of \(D_{N,k,r}(\alpha)\) when \(\alpha\) runs over the interval (1, 2)?
Figure 1. The behavior of $D_{N,k,r}(\alpha)$ for $(N,k,r) \in \{100, 1000\} \times \{3, 4\} \times \{1, 2\}$. The abscissa and ordinate denote values of $\alpha$ and $D_{N,k,r}(\alpha)$, respectively.

Figure 1 illustrates the behavior of $D_{N,k,r}(\alpha)$ by numerical computation, where the points $(\alpha, D_{N,k,r}(\alpha))$ are plotted for all $\alpha \in \{1 + 0.001i : i = 0, 1, \ldots, 1000\}$. In view of this figure, $D_{N,k,r}(\alpha)$ would be approximated by the sum of continuous waves and discrete errors. In order to theoretically observe a phenomenon like this figure, it is probably needed to further investigate the distribution of the sequence $(n^\alpha, \alpha n^{\alpha-1})_{n=1}^N$ modulo 1.

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Appendix A. Optimality of the growth rate $O_{\alpha,k,r}(x^{2-\alpha})$

Throughout this appendix, let $f(x) = x^\alpha$. For $\alpha \in (1, 2)$ and $x \geq 1$ and integers $k \geq 3$ and $r \geq 1$, let us define the minimum length $L_{\alpha,k,r}(x)$ as

$$L_{\alpha,k,r}(x) = \min\{y \geq 0 : \exists n \in [x, x+y] \cap \mathbb{N}, (\lfloor f(n + rj) \rfloor)_{j=0}^{k-1} \text{ is an AP}\}.$$

Thanks to Theorem 2.3, there exists $c = c(\alpha, k, r) > 0$ such that for all $x \geq 1$

$$L_{\alpha,k,r}(x) \leq cx^{2-\alpha}.$$
In this appendix, we show that the length $cx^{2-\alpha}$ is best for $k \geq 4$ in the following meaning.

**Proposition A.1.** For all $\alpha \in (1,2)$ and all integers $k \geq 4$ and $r \geq 1$,

$$\lim_{x \to \infty} \frac{L_{\alpha,k,r}(x)}{x^{2-\alpha}} \geq \frac{k-3}{\alpha(\alpha-1)r(k-1)}. \quad (A.1)$$

**Proof.** Let $k \geq 4$ and $r \geq 1$ be integers, and let $\alpha \in (1,2)$ and $\beta \in (0, k-3)$. Since $(rf'(N))_{n=1}^{\infty}$ is uniformly distributed modulo 1 and the inequality $1/(k-1) < 1 - (\beta + 1)/(k-1)$ holds, there exist infinitely many $N \in \mathbb{N}$ such that

$$\frac{1}{k-1} \leq \{rf'(N)\} \leq 1 - \frac{\beta + 1}{k-1}. \quad (A.2)$$

Take a sufficiently large $N \in \mathbb{N}$ that satisfies (A.2) and

$$\varepsilon = \varepsilon(N) := \frac{(k-1)^2r^2}{2} f''(N) < 1. \quad (A.3)$$

Also, take an integer $n \in [0, L_{\alpha,k,r}(N)]$ such that $\{(N + n + rj \alpha)\}_{j=0}^{k-1}$ is an AP. Then it follows that $N + n \in C_0 \cup C_1$ in the same way as the proof of Theorem 2.1 where the sets $C_0$ and $C_1$ are defined as

$$C_0 = \{m \in \mathbb{N} : \{f(m)\} + (k-1)\{rf'(m)\} < 1\},$$

$$C_1 = \{m \in \mathbb{N} : \{f(m)\} + (k-1)(\{rf'(m)\} - 1) \geq -\varepsilon\}. \quad (A.4)$$

Note that $f''$ is an decreasing function and the inequalities $f''(N) > 0$ and (A.3) hold in proving $N + n \in C_0 \cup C_1$.

If $N + n \in C_1$, then

$$1 - \frac{1 + \varepsilon}{k-1} n^{+n \in C_1} \leq \{rf'(N + n)\} \leq \{rf'(N)\} + nrf''(N)$$

$$\leq 1 - \frac{\beta + 1}{k-1} + rL_{\alpha,k,r}(N)f''(N),$$

whence $L_{\alpha,k,r}(N)f''(N) \geq (\beta - \varepsilon)/r(k-1)$. If $N + n \in C_0$, then

$$\{rf'(N + n)\} n^{+n \in C_0} \leq \frac{1}{k-1} \leq \{rf'(N)\} \leq 1 - \frac{\beta + 1}{k-1},$$

$$\frac{\beta + 1}{k-1} \leq rf'(N + n) - rf'(N) \leq rnf''(N) \leq rL_{\alpha,k,r}(N)f''(N),$$

whence $L_{\alpha,k,r}(N)f''(N) \geq \beta/r(k-1)$. Since $\varepsilon = \varepsilon(N)$ vanishes as $N \to \infty$, the above inequalities imply

$$\lim_{x \to \infty} \frac{L_{\alpha,k,r}(x)}{x^{2-\alpha}} \geq \frac{\beta}{\alpha(\alpha-1)r(k-1)}. \quad (A.1)$$
Proposition A.2. For all $\alpha \in (1, 2)$ and $r \geq N$, 
\[
\lim_{x \to \infty} \frac{L_{\alpha,3,r}(x)}{x^{1-\alpha/2}} \geq \frac{1 + \sqrt{2}}{\sqrt{\alpha(\alpha - 1)r}}.
\]

To prove Proposition A.2, we need to choose infinitely many $N \in \mathbb{N}$ with certain properties instead of (A.1). For this purpose, let us show the following lemmas.

Lemma A.3. Let $\alpha \in (1, 2)$ and $r \in \mathbb{N}$. Then there exist infinitely many $N \in \mathbb{N}$ such that $0 \leq \{f'(N)\} - 1/2r < f''(N - 1)$.

Proof. Take an arbitrary $N \in \mathbb{N}$ such that $f''(N) < 1/2r$ and $N f''(2N) > 1$. Since the inequality $f'(2N) - f'(N) > N f''(2N) > 1$ holds, some $m \in \mathbb{Z}$ satisfies $f''(N) < 1/2r + m < f''(2N)$. Also, the sequence $(f'(N + n))_{n=0}^\infty$ increases and the difference $f'(N + n + 1) - f'(N + n)$ is bounded above by $f''(N) < 1/2r$. Thus, we can take the minimum $n \in [N]$ such that $f'(N + n - 1) < 1/2r + m \leq f'(N + n) < 1 + m$. Then it follows that
\[
0 \leq \{f'(N + n)\} - 1/2r < f'(N + n) - f'(N + n - 1) < f''(N + n - 1).
\]

The arbitrariness of $N$ implies Lemma A.3.

Lemma A.4. Let $\alpha \in (1, 2)$ and $r \in \mathbb{N}$. For all $c_0 > 2r^{1/2}$ and $c_1 > r^{-1/2}$, there exist infinitely many $N \in \mathbb{N}$ such that $0 \leq \{f'(N)\} - 1/2r < c_0 f''(N - 1)^{1/2}$ and $\{f(N)\} < c_1 f''(N)^{1/2}$.

Proof. Let $c_0 > 2r^{1/2}$ and $c_1 > r^{-1/2}$. Take a sufficiently large $N \in \mathbb{N}$ such that $0 \leq \{f'(N)\} - 1/2r < f''(N - 1)$ (see Lemma A.3). Also, take $s \in [2r]$ such that $-1/2r < \{f(N)\} - s/2r \leq 0$. Defining $n_m = 2rm - s$ and $x_m = f(N + n_m) - m - n_m [f'(N)]$ for $m \in [M + 1]$, we verify the following facts.

1. $0 < x_{m+1} - x_m < 2r f''(N - 1) + 4r^2(M + 1) f''(N)$ for all $m \in [M]$.
2. $x_{M+1} - x_1 > 2r M^2 f''(N + 2r(M + 1)) - 2r^2 f''(N)$.
3. $-1/2r < x_1 - [f(N)] < 2r f''(N - 1) + 2r^2 f''(N)$. 

Letting $\beta \to k - 3$, we obtain [A.1].
Fact (1):

\[
x_{m+1} - x_m > 2rf'(N) - 1 - 2r[f'(N)] = 2r\{f'(N)\} - 1 \geq 0,
\]
\[
x_{m+1} - x_m < 2rf'(N + n_{m+1}) - 1 - 2r[f'(N)]
\]
\[
< 2r(f'(N) + n_{m+1}f''(N)) - 1 - 2r[f'(N)]
\]
\[
< 2r\{f'(N)\} - 1 + 2rn_{m+1}f''(N)
\]
\[
< 2rf''(N - 1) + 4r^2(M + 1)f''(N).
\]

Fact (2):

\[
x_{M+1} - x_1 > f(N + n_{M+1}) - f(N + n_1) - M - 2rM[f'(N)]
\]
\[
= f(N + n_{M+1}) - f(N + n_1) - 2rMf'(N) + M(2r\{f'(N)\} - 1)
\]
\[
\geq f(N + n_{M+1}) - f(N + n_1) - 2rMf'(N)
\]
\[
> \left(f(N) + n_{M+1}f'(N) + \frac{n_{M+1}^2}{2}f''(N + n_{M+1})\right)
\]
\[
- \left(f(N) + n_1f'(N) + \frac{n_1^2}{2}f''(N)\right) - 2rMf'(N)
\]
\[
= \frac{n_{M+1}^2}{2}f''(N + n_{M+1}) - \frac{n_1^2}{2}f''(N)
\]
\[
> 2r^2M^2f''(N + 2r(M + 1)) - 2r^2f''(N).
\]
Case (1): The sufficiently large $N$ satisfies that

\[
\begin{align*}
&x_1 - \lfloor f(N) \rfloor = f(N + n_1) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\
&= f(N) + n_1 f'(N) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor - 1 \\
&\geq \{f(N)\} + n_1 / {2r} - 1 = \{f(N)\} - s / {2r} > -1 / {2r}, \\
&x_1 - \lfloor f(N) \rfloor = f(N) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\
&< f(N) + n_1 f'(N) + \frac{n_1^2}{2} f''(N) - 1 - n_1 \lfloor f'(N) \rfloor - \lfloor f(N) \rfloor \\
&= \{f(N)\} + n_1 \{f'(N)\} + \frac{n_1^2}{2} f''(N) - 1 \\
&< \{f(N)\} + n_1 (1 / {2r} + f''(N - 1)) + \frac{n_1^2}{2} f''(N) - 1 \\
&< \{f(N)\} - s / {2r} + n_1 f''(N - 1) + \frac{n_1^2}{2} f''(N) \\
&\leq n_1 f''(N - 1) + \frac{n_1^2}{2} f''(N) < 2r f''(N - 1) + 2r^2 f''(N).
\end{align*}
\]

Now, we have the following two cases:

1. $x_1 - \lfloor f(N) \rfloor \geq 0$,
2. $x_1 - \lfloor f(N) \rfloor < 0$.

Case (1): The sufficiently large $N$ satisfies that

- $\{f(N + n_1)\} = \{x_1\} \leq 2r f''(N - 1) + 2r^2 f''(N) < c_0 f''(N + n_1)^{1/2}$,
- $\{f'(N + n_1)\} < \{f'(N)\} + n_1 f''(N) < 1 / {2r} + f''(N - 1) + 2r f''(N)$
  \[< 1 / {2r} + c_1 f''(N + n_1)^{1/2},\]
- $\{f'(N + n_1)\} > \{f'(N)\} \geq 1 / {2r}$.

Case (2): Take $1 < \beta < \beta' = \min \{c_0 / {2r}^{1/2}, c_1 / {r}^{-1/2}\}$ and put $M = [\beta f''(N)^{-1/2} / {2r^{3/2}}] = O(N^{1 - \alpha / 2})$. Since the sufficiently large $N$ satisfies

\[
x_{M + 1} - x_1 > \frac{f''(N + 2r(M + 1))}{2r f''(N)} - 2r^2 f''(N) \\
= \frac{\beta}{2r} \left( \frac{N}{N + 2r(M + 1)} \right)^{2 - \alpha} - 2r^2 f''(N) > \frac{1}{2r},
\]
we can take the minimum \( m \in [M] \) such that \( x_{m+1} - \lfloor f(N) \rfloor \geq 0 \). Then the sufficiently large \( N \) satisfies that

\[
\{ f(N + n_{m+1}) \} = \{ x_{m+1} - x_m \} < x_{m+1} - x_m \leq 2r f''(N - 1) + 4r(f(M + 1) f''(N) \\
< 2r^{1/2} \beta f''(N + n_{m+1})^{1/2} \leq c_0 f''(N + n_{m+1})^{1/2},
\]

\[
\{ f'(N + n_{m+1}) \} < \{ f'(N) \} + n_{m+1} f''(N) < 1/2r + f''(N - 1) + 2r(M + 1) f''(N) \\
< 1/2r + r^{-1/2} \beta f''(N + n_{m+1})^{1/2} \leq 1/2r + c_1 f''(N + n_{m+1})^{1/2},
\]

\[
\{ f'(N + n_{m+1}) \} > \{ f'(N) \} \geq 1/2r.
\]

Therefore, Lemma \[A.4\] holds. \( \square \)

**Proof of Proposition** \[A.2\]. Let \( f(x) = x^a, \) \( c_0 > 2r^{1/2}, \) \( c_1 > r^{-1/2}, \) and \( 0 < c_2 < c_1 + \sqrt{c_1^2 + 1/r} \). Thanks to Lemma \[A.4\], we can take a sufficiently large \( N \in \mathbb{N} \) such that

1. \( \{ f(N) \} < c_0 f''(N)^{1/2}, \)
2. \( 0 \leq \{ f'(N) \} - 1/2r < c_1 f''(N)^{1/2}. \)

Set \( \varepsilon = \varepsilon(N) = 2r^{2} f''(N) < 1, \) which is just \[A.3\] with \( k = 3. \) We show that \( L_{a,3,r}(N) > c_2 f''(N)^{-1/2} \) by contradiction. Suppose that \( L_{a,3,r}(N) \leq c_2 f''(N)^{-1/2}. \) Take an integer \( n \in [0, L_{a,3,r}(N)] \) such that \( \{ [f(N + n + rj)] \}_{j=0}^{2} \) is an AP. Then it follows that \( N + n \in C_0 \cup C_1 \) in the same way as the proof of Theorem \[2.1\] where the sets \( C_0 \) and \( C_1 \) are defined by \[A.4\]. If \( N + n \in C_0, \) then the inequalities (2) and \( n \leq L_{a,3,r}(N) \leq c_2 f''(N)^{-1/2} \) yield that

\[
\{ r f'(N + n) \} < 1/2 \leq \{ r f'(N) \} < 1/2 + r c_1 f''(N)^{1/2},
\]

\[
1/2 - r c_1 f''(N)^{1/2} < r f'(N + n) - r f'(N) \leq r n f''(N) \leq r c_2 f''(N)^{1/2},
\]

which is a contradiction because \( N \) is sufficiently large.

Next, consider the case \( N + n \in C_1. \) Then the inequalities (2) and \( n \leq L_{a,3,r}(N) \leq c_2 f''(N)^{-1/2} \) yield that

\[
1 - \frac{\{ f(N + n) \} + \varepsilon}{2} \leq \{ r f'(N + n) \} \leq \{ r f'(N) \} + r n f''(N) \\
< 1/2 + r c_1 f''(N)^{1/2} + r c_2 f''(N)^{1/2},
\]

whence \( \{ f(N + n) \} > 1 - \varepsilon - 2r(c_1 + c_2) f''(N)^{1/2}. \) Since Taylor’s theorem implies that

\[
f(N + n) = f(N) + n f'(N) + \frac{n^2}{2} f''(N + \theta)
\]
for some $\theta \in [0, n]$, the inequalities (1) and $n \leq L_{\alpha, 3, r}(N) \leq c_2 f''(N)^{-1/2}$ yield that
\[
\{nf'(N)\} \leq \{n/2r\} + nc_1 f''(N)^{1/2} \leq 1 - 1/2r + c_1c_2,
\]
\[
0 \leq \frac{n^2}{2} f''(N + \theta) \leq \frac{c_2^2 f''(N)^{-1}}{2} f''(N) = c_2^2/2,
\]
\[
\{f(N + n)\} \leq \{f(N)\} + \{nf'(N)\} + \frac{n^2}{2} f''(N + \theta) < c_0 f''(N)^{1/2} + (1 - 1/2r + c_1c_2) + c_2^2/2.
\]
Thus,
\[
1 - 2r^2 f''(N) - 2r(c_1 + c_2) f''(N)^{1/2} < \{f(N + n)\}
\]
\[
< c_0 f''(N)^{1/2} + (1 - 1/2r + c_1c_2) + c_2^2/2,
\]
(A.5) \quad \frac{1}{2} r - c_1c_2 - c_2^2/2 < 2r^2 f''(N) + (c_0 + 2r(c_1 + c_2)) f''(N)^{1/2}.
\]
Since the assumption $0 < c_2 < c_1 + \sqrt{c_1^2 + 1/r}$ implies $1/2r - c_1c_2 - c_2^2/2 > 0$, the inequality (A.5) is a contradiction because $N$ is sufficiently large. Therefore,
\[
\lim_{x \to \infty} \frac{L_{\alpha, 3, r}(x)}{x^{1-\alpha/2}} \geq \frac{c_2}{\sqrt{\alpha}(\alpha - 1)}.
\]
Finally, letting $c_2 \to c_1 + \sqrt{c_1^2 + 1/r}$ and $c_1 \to r^{-1/2}$, we obtain Proposition A.2.

Next, let us state the following proposition that supports $L_{\alpha, 3, r}(x) = O_{\alpha, r}(x^{1-\alpha/2})$.

**Proposition A.5.** Let $\alpha \in (1/2, 1)$ and $r \in \mathbb{N}$, and let $w(N)$ be an arbitrary positive-valued function such that $N^{\alpha/2-1}w(N) \to 0$ and $w(N) \to \infty$ as $N \to \infty$. Then
\[
\lim_{M \to \infty} \frac{\#\{N \in [M] : L_{\alpha, 3, r}(N) \leq N^{1-\alpha/2}w(N)\}}{M} = 1.
\]

**Proof.** For $N, L \in \mathbb{N}$, define $D(N, L)$ as the discrepancy of the sequence $(f(N + n))_{n=0}^{L-1}$. Let $L = L(N) = \lceil N^{1-\alpha/2}w(N) \rceil$ and $H = H(N) = \lceil N^{(2-\alpha)/3} \rceil$. Lemma A.2 with $d = 1$ and Lemma A.4 imply that
\[
D(N, L) \ll \frac{1}{H} + \frac{1}{L} \sum_{h=1}^{L} \frac{1}{h} \sum_{n=0}^{L-1} e(hf(N + n)) \ll_{\alpha} \frac{1}{H} + H^{1/2}N^{\alpha/2-1} + \frac{N^{1-\alpha/2}}{L} \ll N^{(\alpha-2)/3} + 1/w(N).
Now, let $\varepsilon \in (0, 1/6)$ be arbitrary. Defining the sets $V_0$, $V_1$, $B_0$, $B_1$, and $V$ as

$$V_0 = \{ N \in \mathbb{N} : \{rf'(N)\} < 1/2 - 3\varepsilon \},$$

$$V_1 = \{ N \in \mathbb{N} : 1/2 + \varepsilon < \{rf'(N)\} < 1 - \varepsilon \},$$

$$B_0 = \{ m \in \mathbb{N} : \{f(m)\} + 2\{rf'(m)\} < 1 - \varepsilon \},$$

$$B_1 = \{ m \in \mathbb{N} : \{f(m)\} < 1 - \varepsilon, \{f(m)\} + 2(\{rf'(m)\} - 1) \geq 0 \},$$

$$V = \{ N \in \mathbb{N} : L_{\alpha,3,r}(N) \leq N^{1-\alpha/2}w(N) \},$$

we show that $(V_0 \cup V_1) \cap [x_0, \infty) \subset V$ for a sufficiently large $x_0 > 0$. First, assume $N \in V_0 \cap [x_0, \infty)$. Then the set $W_0 := \{ n \in [0, L - 1] : \varepsilon < \{f(N + n)\} < 3\varepsilon \}$ satisfies

$$\#W_0/L = 2\varepsilon + O(D(N, L)) = 2\varepsilon + O(N^{(\alpha-2)/3} + 1/w(N)) > 0,$$

since $N \geq x_0$ is sufficiently large. Take an element $n \in W_0 \neq \emptyset$. Then the assumptions $N \in V_0$ and $N^{\alpha/2-1}w(N) \to 0$ imply that

$$\{rf'(N + n)\} \leq \{rf'(N)\} + rnf''(N) < 1/2 - 3\varepsilon + r\alpha(\alpha - 1)(L - 1)N^{\alpha - 2}$$

$$< 1/2 - 3\varepsilon + r\alpha(\alpha - 1)N^{\alpha/2-1}w(N) < 1/2 - 2\varepsilon,$$

since $N \geq x_0$ is sufficiently large. Thus,

$$\{f(N + n)\} + 2\{rf'(N + n)\} < 3\varepsilon + 2(1/2 - 2\varepsilon) = 1 - \varepsilon,$$

which is just $N + n \in B_0$. Therefore, $(\{f(N + n + rj)\})_{j=0}^2$ is an AP (see the proof of Theorem 2.1). Since the inequality $L_{\alpha,3,r}(N) \leq n < L$ holds, it turns out that $N \in V$.

Next, assume $N \in V_1 \cap [x_0, \infty)$. The set $W_1 := \{ n \in [0, L - 1] : 1 - 2\varepsilon < \{f(N + n)\} < 1 - \varepsilon \}$ is also not empty in the same way as $W_0 \neq \emptyset$. Take an element $n \in W_1$. Since the difference $rf'(N + n) - rf'(N)$ is bounded above by

$$rnf''(N) \leq r\alpha(\alpha - 1)(L - 1)N^{\alpha - 2} < r\alpha(\alpha - 1)N^{\alpha/2-1}w(N) < \varepsilon$$

and the inequality $1/2 + \varepsilon < \{rf'(N)\} \leq 1 - \varepsilon$ holds, we have $\{rf'(N + n)\} \geq \{rf'(N)\} > 1/2 + \varepsilon$. Hence, $\{f(N + n)\} < 1 - \varepsilon$ and

$$\{f(N + n)\} + 2(\{rf'(N + n)\} - 1) > 1 - 2\varepsilon + 2(1/2 + \varepsilon - 1) = 0,$$

which are just $N + n \in B_1$. Therefore, $(\{f(N + n + rj)\})_{j=0}^2$ is an AP (see the proof of Theorem 2.1). Since the inequality $L_{\alpha,3,r}(N) \leq n < L$ holds, it turns out that $N \in V$.

The inclusion relation $(V_0 \cup V_1) \cap [x_0, \infty) \subset V$ has been proved above. Since the sequence $(rf'(N))_{N=1}^\infty$ is uniformly distributed modulo 1 and the sets $V_0$ and $V_1$ are
disjoint, it follows that
\[
\lim_{M \to \infty} \frac{\#(V \cap [M])}{M} \geq \lim_{M \to \infty} \frac{\#(V_0 \cap [M])}{M} + \lim_{M \to \infty} \frac{\#(V_1 \cap [M])}{M} \\
\geq \left(\frac{1}{2} - 3\varepsilon\right) + \left(\frac{1}{2} - 2\varepsilon\right) = 1 - 5\varepsilon.
\]
Letting \(\varepsilon \to +0\), we obtain Proposition A.5 \(\Box\)

REFERENCES

[1] E. Szemerédi. On sets of integers containing no \(k\) elements in arithmetic progression, Acta Arith. 27 (1975), 199–245.
[2] B. Green and T. Tao. The primes contain arbitrarily long arithmetic progressions, Ann. of Math. 167 (2008), 481–547.
[3] N. Frantzikinakis and M. Wierdl. A Hardy field extension of Szemerédi’s theorem, Adv. Math. 222 (2009), 1–43.
[4] K. Saito and Y. Yoshida. Arithmetic progressions in the graphs of slightly curved sequences, J. Integer Seq. 22, 19.2.1 (2019).
[5] P. Dénes. Über die Diophantische Gleichung \(x^l + y^l = cz^l\), Acta Math. 88 (1952), 241–251.
[6] H. Darmon and L. Merel. Winding quotients and some variants of Fermat’s last theorem, J. Reine Angew. Math. 490 (1997), 81–100.
[7] I. I. Piatetski-Shapiro. On the distribution of prime numbers in sequences of the form \([f(n)]\), Mat. Sbornik N.S. 33 (75) (1953), 559–566.
[8] J. Rivat and J. Wu. Prime numbers of the form \([n^c]\), Glasg. Math. J. 43 (2001), 237–254.
[9] J. Rivat and P. Sargs. Nombres premiers de la forme \([n^c]\), Canad. J. of Math. 53 (2001), 414–433.
[10] R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski, and A. J. Weingartner. Piatetski-Shapiro sequences, Acta Arith. 157 (2013), 37–68.
[11] X. D. Cao and W. G. Zhai. The distribution of square-free numbers of the form \([n^c]\), J. Théor. Nr. Bordx. 10 (1998), 287–299.
[12] X. D. Cao and W. G. Zhai. The distribution of square-free numbers of the form \([n^c]\), II, Acta Math. Sinica (Chin. Ser.) 51 (2008), 1187–1194.
[13] J.-M. Deshouillers. A remark on cube-free numbers in Segal-Piatetski-Shapiro sequences, Hardy-Ramanujan J. 41 (2019), 127–132.
[14] J.-M. Deshouillers. Sur la répartition des nombres \([n^c]\) dans les progressions arithmétiques, C. R. Acad. Sci. Paris Sér. A 277 (1973), 647–650.
[15] J. F. Morgenbesser. The sum of digits of \([n^c]\), Acta Arith. 148 (2011), 367–393.
[16] M. Mirek. Roth’s theorem in the Piatetski-Shapiro primes, Rev. Mat. Iberoam. 31 (2015), 617–656.
[17] H. Li and H. Pan. The Green-Tao theorem for Piatetski-Shapiro primes, preprint, 2019. Available at https://arxiv.org/abs/1901.09372
[18] D. Glasscock. Solutions to certain linear equations in Piatetski-Shapiro sequences, Acta Arith. 177 (2017), 39–52.
[19] L. Kuipers and H. Niederreiter, Uniform distribution of sequences. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. xiv+390 pp.
[20] S.W. Graham and G. Kolesnik, Van der Corput’s method of exponential sums, London Mathematical Society Lecture Note Series 126, Cambridge University Press, Cambridge, 1991. vi+120 pp.
[21] P. Sargos. Points entiers au voisinage d’une courbe, sommes trigonométriques courtes et paires d’exposants, Proc. Lond. Math. Soc. 70 (1995), 285–312.
[22] S. A. Gritsenko. Estimates of trigonometric sums by the third derivative, Mat. Zametki 60 (1996), 383–389 [Math. Notes 60 (1996), 283–287].

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