A 2-CATEGORY OF CHRONOLOGICAL COBORDISMS AND ODD KHOVANOV HOMOLOGY

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Abstract. We create a framework for odd Khovanov homology in the spirit of Bar-Natan’s construction for the ordinary Khovanov homology. Namely, we express the cube of resolutions of a link diagram as a diagram in a certain 2-category of chronological cobordisms and show that it is 2-commutative: the composition of 2-morphisms along any 3-dimensional subcube is trivial. This allows us to create a chain complex, whose homotopy type modulo certain relations is a link invariant. Both the original and the odd Khovanov homology can be recovered from this construction by applying certain strict 2-functors. We describe other possible choices of functors, including the one that covers both homology theories and another generalizing dotted cobordisms to the odd setting. Our construction works as well for tangles and is functorial with respect to tangle cobordisms up to sign.

1. Introduction

The Khovanov homology [Kh99] opened to knot theorists a new and interesting world of powerful invariants, of which knot polynomials are only shadows. For instance, the Euler characteristic of the Khovanov homology is the Jones polynomial of a link. It did not take much time to prove usefulness of these invariants. For instance, the Khovanov homology detects the unknot [KM12] and unlinks [HN12], although the problem is still open for the Jones polynomial. The Lee deformation [Le05] leads to a spectral sequence, from which J. Rasmussen extracted a lower bound for the knot genus, giving a combinatorial proof of Milnor Conjecture [Ra04]. This raised a question, whether there were other link homology theories categorifying the Jones polynomial. D. Bar-Natan [BN05] described a very general construction that produces link homology for rank two Frobenius algebra satisfying some additional relations. Then M. Khovanov classified all theories that arise from Frobenius systems [Kh04], proving that the Bar-Natan’s theory of dotted cobordisms is universal.

When it seemed categorifications of the Jones polynomial were well understood, P. Ozsváth, J. Rasmussen and Z. Szabó published a paper with a distinct construction [ORS13] based on a projective TQFT. Their invariant also categorifies the Jones polynomial, but the algebra used in the construction is not cocommutative and even not coa ssociative. They call it odd Khovanov homology, because of similarity with the original construction, which we now refer to as even. Both theories agree modulo 2, but they are not equivalent over \( \mathbb{Z} \). In particular, results of A. Shumakovitch [Sh11] provide examples of pairs of knots that can be distinguished by one theory but not by the other. Moreover, it was proved by J. Bloom that the odd Khovanov homology is mutation invariant [Blo10], generalizing the similar result by S. Wehrli for even Khovanov homology with \( \mathbb{Z}_2 \) coefficients [Wh10].

Key words and phrases. categorification, chronology, cobordism, covering homology, dotted cobordisms, Frobenius algebra, functoriality, Khovanov homology, knot, link, odd homology, planar algebra, tangle.
Beyond the differences, both theories are constructed in a very similar way. First, given a link diagram $D$ with $n$ crossings we create $2^n$ pictures, by resolving each crossing horizontally (type 0 resolution) or vertically (type 1 resolution):

The picture of crossing highways is placed to the right to help to remember the naming convention: a resolution of a crossing can be seen as leaving one highway by turning right (assuming the traffic is on the right side). In type 0 we leave the lower highway, while in type 1 the upper one. We place all such pictures in vertices of an $n$-dimensional cube and decorate its edges with certain cobordisms. This cube commutes and by applying a TQFT functor we obtain a commuting cube of abelian groups and homomorphisms, which can be collapsed to a chain complex (after changing signs of some maps). On the other hand, a projective TQFT from [ORS13] produces a cube that commutes only up to signs, which has to be fixed before collapsing. This can be always done, although it is kind of a mystery, why this is possible.

The last step is exactly why the odd theory does not fit into Bar-Natan’s framework. The latter starts with a cube of resolutions and cobordisms and invariance is proved at this level, before applying a TQFT functor. The author extended this framework using cobordisms with an additional structure, a chronology [Pu08], which is a framed Morse function $\tau: W \to I$ that separates critical points [Ig87]. Isotopies of these functions equip the category of chronological cobordisms with a structure of a 2-category and we can express the projective functor from [ORS13] as a strict 2-functor. By translating Bar-Natan’s construction into this new framework, we were able to show invariance of the complex built from chronological cobordisms. Applying different 2-functors recovers both the even and odd Khovanov homology. In particular, it follows from contractibility of certain loops in the space of framed functions that in the odd theory we can always distribute signs over edges of the cube to make it commute. In addition to that, we have found several theories with parameters, especially the covering homology $H^\text{cov}(L)$. It is a sequence of graded modules over ring of truncated polynomials $\mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$, from which we can obtain both even and odd Khovanov homology as illustrated below:

The specializations should be made at the level of chains. This construction was first described in [Pu08]. Another example is given by chronological cobordisms with dots that generalizes the universal Bar-Natan’s theory to the odd setting. By an analogy to the even case it is proved to be universal, see Theorem 11.9. A motivation was to find an odd analog of Lee’s deformation, but this goal has not been reached.

A connection with categorified quantum groups. The existence of covering homology theory fits nicely with recent discoveries regarding odd categorifications of quantum groups. It is known that the even Khovanov homology can be recovered from categorical representations of categorified $U_q(\frak{sl}_2)$ [We10] and it is suspected that the same holds for the odd
A support came with a discovery of odd nilHecke algebras [EKL12, KKT11], which categorifies the negative half of $U_q(\mathfrak{sl}_2)$. They appeared to be connected with the Lie superalgebra $U_q(\mathfrak{osp}_{1|2})$. Both are covered by a Kac-Moody algebra $U_{q,\pi}$ introduced by Clark, Hill and Wang [CHW13, HW12], where $\pi$ is a formal parameter with $\pi^2 = 1$. The relationship is illustrated below:

$$
\begin{array}{c}
U_q(\mathfrak{sl}_2) \\
\pi=1
\end{array}
\quad
\begin{array}{c}
U_q(\mathfrak{osp}_{1|2}) \\
\pi=-1
\end{array}
$$

Recently, A. Lauda and A. Ellis categorified the covering algebra $U_{q,\pi}$, using graded supermonoidal categories, in which the relation $(f \otimes \text{id}) \circ (\text{id} \otimes g) = (\text{id} \otimes g) \circ (f \otimes \text{id})$ holds up to a sign in a coherent way [EL13]. It is expected that this categorification leads to homologies covering both odd and even homology theories and the author believes that the covering Khovanov homology described in this paper is one of them.

**Outline.** We start the paper with a picture showing a construction for the Khovanov complex for the trefoil knot, see Fig. 1. We hope it will serve as a motivation for the next two sections, where we define chronological cobordisms and analyze changes of chronologies. Section 3 describes the 2-category of chronological cobordisms of any dimension and explains a symmetric monoidal structure induced by a disjoint sum. Then in Section 4 we restrict to dimension two and find a solution for *chronological relations*: permuting two critical points corresponds to scaling a cobordism by an invertible scalar.

Details of the construction of the generalized Khovanov complex for a tangle diagram are given in Sections 5 and 6. The former deals with link diagram only, whereas the latter describes how to extend the construction to tangles in the spirit of Bar-Natan, using planar algebras. Unfortunately, the functors forming a planar algebra of chronological cobordisms are not strict, so that we cannot combine complexes for tangles in the naive way. This issue is partially resolved in Section 7, where we prove invariance of the generalized complex under Reidemeister moves. Section 8 contains several straightforward properties of the complex.

The next few sections describe the algebraic part. We recover both odd and even Khovanov homology in Section 9. The covering homology is defined in Section 10, in which we define a chronological version of a Frobenius system. Similarly to the ordinary case, a chronological Frobenius system induces a TQFT 2-functor from the category of chronological cobordisms to a 2-category of graded symmetric bimodules. They are analyzed in the next section. In particular, we describe dotted cobordisms and their algebra, proving it is universal among all Frobenius systems fitting into our framework.

Section 12 contains several remarks and constructions related to this paper, but not fully explored. We prove, following [BN05], that our construction is functorial up to ‘sign’, where a sign is understood as an invertible scalar in degree 0. Then we analyze a choice in defining chronological relations: there is one type of changes for which the associated coefficients are defined only up to a scalar $XY$. An argument due to C. Seed shows the whole construction is independent of this choice. Finally, we analyze a possible connection of our construction to the one based on $\mathfrak{sl}_2$ foams [Ca09]. We suppose there is a parallel theory of chronological foams, closely connected to our construction.
The construction of chronological cobordisms utilizes the theory of framed functions, which is an interesting generalization of Morse theory. It is described in Appendix A following [Ig87]. In particular, we describe all singularities of these functions up to codimension two.

The paper uses also several 2-categorical constructions, including semi-strict monoidal structure and braiding. These are included in Appendix B.

Acknowledgement. This paper would never be written without help of many people. The problem of creating a framework for odd Khovanov homology was suggested by Dror Bar-Natan while the author was on the University of Toronto. The dotted algebra was understood with the help of Anna Beliakova, when the author visited her in Zürich. Several ideas used to clarify the original construction came out after discussions with Aaron Lauda, Mikhail Khovanov, Józef Przytycki and Alexander Shumakovitch. The author is also thankful to Alexander Ellis, Maria Hempel, Vasilly Manturov, Cotton Seed and John Baldwin for interesting discussions and remarks.

2. The picture

We begin, as promised in the introduction, with describing elements of the big diagram in Fig. 1. In the next few sections we will create a framework for this picture.

Knot. In the left top corner we can see a diagram $D$ of the left-handed trefoil with enumerated crossings. Each crossing is equipped with an arrow oriented in such a way that it connects the two arcs in the type 0 resolution (there are two choices of such an arrow). This piece of information does not appear in the construction of the even Khovanov complex [Kh99, BN05], but it is crucial for the odd Khovanov complex [ORS13].

Vertices in the cube. Most of the picture is occupied by resolutions of the diagram $D$, placed in vertices of a three-dimensional cube. The diagram $D_\xi$ at a vertex $\xi = (\xi_1, \xi_2, \xi_3)$ is obtained from $D$ by replacing $i$-th crossing with the resolution of type $\xi_i$. The cube is drawn slant, to have all diagrams grouped in columns with respect to the weight of the vertex $|\xi| := \xi_1 + \xi_2 + \xi_3$.

Edges in the cube. Edges are encoded by sequences $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ with exactly one $\zeta_i$ being a star *. The star indicates direction of the edge: replacing it with 0 or 1 results in the source or the target vertex respectively. Choose an edge $\zeta: \xi \to \xi'$ and let $U$ be a small neighborhood of the $i$-th crossing, where $\zeta_i = *$. It is decorated with a unique cobordism $D_\zeta \subset \mathbb{R}^2 \times I$ that has only one critical point: a cylinder $(D-U) \times I$ with a saddle $\boxed{\cup \cap}$ inserted over $U$. We equip this cobordism with a height function $h: D_\zeta \to I$. The small arrow over the crossing determines a framing, hence an orientation of the saddle (see Appendix A). For simplicity we represent the cobordism by its input together with an arrow, which determines both the place and the orientation of the saddle. This is the same arrow that decorates the $i$-th crossing in the diagram of the knot. A 3D picture of the cobordism decorating the edge $0*0$ is given in the left-bottom corner.

An underlying diagram with holes. The two paragraphs above can be unified by a single construction, which also explains how to create the cube for any link diagram $D$. Take the diagram $D$ and remove a small neighborhood of each crossing, obtaining a new diagram $D$. For instance, the trefoil diagram

\footnote{This is an example of a planar diagram, see Section 6.}
produces a diagram with three holes. Copy the numbers associated to crossings to the holes — this gives an ordering of them. The picture $D_\xi$ at a vertex $\xi$ is obtained from $D_\bullet$ by filling the holes with resolutions, type $\xi_i$ at the $i$-th hole. To obtain the cobordism $D_\zeta$ associated to an edge $\zeta$, where $\zeta_i = \ast$, copy the arrow from $i$-th crossing to the $i$-th hole. For a 3D picture, take a product $D_\bullet \times I$ and insert into the $i$-th hole\(^2\) either a pair of vertical rectangles, when $\zeta_i \neq \ast$, or a saddle for $\zeta_i = \ast$ with a framing induced by the small arrow over the crossing, see Fig. 2.

**Faces.** Placing arrows in two holes of $D_\bullet$ results in a cobordism with two saddles. We can think of it as a description of a face of the cube. There are two height functions, depending which of the two saddle points is below the other. Hence, faces of the cube do not commute. This is different from the even construction [BN05], but not very far away from the odd one [ORS13]. As in the latter case, we measure this obstruction to commutativity with a 2-cochain, defined as follows. Take the two-arrow description of a face and disregard all circles that are not touched by any of the two arrows. What remains is one of the pictures listed in Tab. 1. We gathered all such configurations into groups labeled with some monomials from

\(^2\) A hole becomes a tube in $D_\bullet \times I$. 

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**Figure 1.** The Khovanov bracket for the trefoil.
a commutative ring $k := \mathbb{Z}[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)$ (they are explained in Section 4). They define a 2-cochain $\psi \in C^2(I^3; k^*)$, where $k^*$ is the group of invertible elements in $k$. Here one must be careful with the two configurations under $Z$ — the value of $\psi$ is either $Z$ or $Z^{-1}$, depending on the orientation of the face:

$$\psi \left( \begin{array}{c} \text{merge} \\ \text{split} \\ \text{merge} \end{array} \right) = Z, \quad \text{but} \quad \psi \left( \begin{array}{c} \text{split} \\ \text{merge} \\ \text{split} \end{array} \right) = Z^{-1}. $$

We call $\psi$ the *commutativity* 2-cochain.

**Coefficients on edges.** Some edges in Fig. 1 are labeled with elements of $k$, describing a 1-cochain $\epsilon \in C^1(I^3; k^*)$ (take 1 if no coefficient is present). The product of these elements around each face $S$ is equal to $-\psi(S)$, i.e. $\psi = -d\epsilon$. Such a cochain $\epsilon$ is called a *sign assignment*, following [ORS13]. It exists for any link diagram and, in some sense, it is unique (see Section 5).

**Complex.** The bottom line in Fig. 1 shows a sequence of objects and maps between them. This is the *Khovanov bracket* of the trefoil: think of the objects $C^i$ as columns of the diagrams above and the maps $d^i$ as bundles of arrows between the columns. We give more meaning to this in Section 5, showing that $(C, d)$ is a chain complex.

**A word about tangles.** In the same manner we can create a cube of resolutions for a tangle diagram, using cobordisms with corners. However, it has to be explained what we mean by a 2-cochain $\psi$ in this case, as faces are more complicated. This is done in Section 6.

## 3. Cobordisms and Chronologies

We start creating the framework for Fig. 1 by describing a 2-category of chronological cobordisms. Recall that an $(n+1)$-manifold $W$ is a *cobordism* between two oriented $n$-manifolds $\Sigma_0$ and $\Sigma_1$, if it has $\Sigma_0 \sqcup -\Sigma_1$ as a boundary (the minus sign stands for the opposite orientation of $\Sigma_1$). We will often write $W_{in}$ and $W_{out}$ for the components of $\partial W$ identified with $\Sigma_0$ and $\Sigma_1$ respectively and call them the *input* and the *output* of $W$.

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3 A brief introduction to the theory of 2-categories is included in Appendix B.
Definition 3.1. A chronological cobordism is a cobordism $W$ together with an oriented Morse function $\tau: W \to I$ that separates critical points, for which $\tau^{-1}(0) = W_{\text{in}}$ and $\tau^{-1}(1) = W_{\text{out}}$. A homotopy of $\tau$ in the space of oriented Igusa functions is called a change of a chronology.

We now explain some notions from the definition, referring for details to Appendix A. An Igusa function $f: W \to I$ is allowed to have two types of critical points:

- $A_1$ or Morse singularities, characterized by the property that the Hessian $\text{Hess}_p(f)$ of $f$ is nondegenerate, and
- $A_2$ or birth-death singularities, for which the Hessian has a one dimensional kernel $\mathcal{N}(p)$, on which the third derivative of $f$ does not vanish.

Choose a Riemannian metric on $W$. For a critical point $p$ we denote by $E^\pm(p)$ the positive or negative eigenspace of the Hessian $\text{Hess}_p(f): T_pW \to T_pW$. A choice of orientations for negative eigenspaces over all critical points is called an orientation of $f$. We denote the space of oriented Igusa functions on $W$ by $\text{Fun}^{\text{or}}(W)$.

A generic function $f: W \to I$ has only singularities of the first type, but we need the second kind for generic homotopies. Higher singularities are unnecessary for higher homotopies, if we equip these functions with framing, i.e. a choice of a basis for each $E^-(p)$, see Theorem A.3. The space of oriented functions can be seen as a quotient of this space, as explained at the end of Appendix A. This space may no longer be contractible, but it is simply connected.
Definition 3.2. Chronological cobordisms \((W, \tau)\) and \((W', \tau')\) are equivalent, if there exists a diffeomorphism \(\varphi: W \to W'\) compatible with boundaries, i.e. the diagram below commutes

\[
\begin{array}{ccc}
\Sigma_0 & \stackrel{\varphi}{\longrightarrow} & \Sigma_1 \\
W \downarrow & & \downarrow W' \\
\end{array}
\]

such that \(\tau\) and \(\tau' \circ \varphi\) are homotopic in the space of oriented separating Morse functions.\(^4\)

It is worth to notice, that if \(\tau\) is a chronology on \(W\), then \(\tau \circ \varphi^{-1}\) is a chronology on \(W'\) and similarly for changes of chronologies. Hence, we can identify equivalent cobordisms. In particular, we can reparametrize the interval \(I\) as long as the endpoints are fixed.

The notion of equivalence makes it possible to glue chronological cobordisms:\(^5\) if \((W, \tau)\) is a cobordism from \(\Sigma_0\) to \(\Sigma_1\) and \((W', \tau')\) is a cobordism from \(\Sigma_1\) to \(\Sigma_2\), one can glue them together along the diffeomorphism \(W_{\text{out}} \approx \Sigma_1 \approx W_{\text{in}}\) to obtain a cobordism \((WW', \tau'')\) with a chronology given by concatenation:

\[
\tau''(p) = \begin{cases} 
\frac{1}{2} \tau(p), & \text{for } p \in W, \\
\frac{1}{2} (\tau'(p) + 1), & \text{for } p \in W'.
\end{cases}
\]

Such a function might not be smooth, but we can always find a smooth one homotopic to it. This gives an associative and unital operation on cobordisms (up to equivalence), where units are given by cylinders \(\Sigma \times I\) with the simplest chronology — the projection on \(I\).

If \(H, H': W \times I \to I\) are changes of chronologies such that \(H_1 = H'_0\), their concatenation

\[
(H \star H')(p, t) := \begin{cases} 
H(p, 2t), & 0 \leq t \leq 1/2, \\
H'(p, 2t - 1), & 1/2 \leq t \leq 1,
\end{cases}
\]

is another change of a chronology on \(W\). Again, the result may require to be smoothed. Hence, we will consider changes of chronologies only up to higher homotopies.

Finally, we can juxtapose changes occurring on different regions of a cobordism. Formally, if \(H\) and \(H'\) are changes of chronologies on \(W\) and \(W'\) respectively and cobordisms \(W\) and \(W'\) can be glued together, there is a change of a chronology on \(WW'\) induced by the map

\[
(H \circ H')(p, t) := \begin{cases} 
H(p, t), & p \in W, \\
H'(p, t), & p \in W'.
\end{cases}
\]

It is clear from the above formulas that concatenation and juxtaposition of changes of chronologies commute. Hence, chronological cobordisms form a 2-category (see Appendix B).

Corollary 3.3. There is a strict 2-category of chronological cobordisms \(n\text{ChCob}\) with oriented manifolds of dimension \((n - 1)\) as objects, equivalence classes of chronological cobordisms as morphisms and homotopy classes of changes of chronologies as 2-morphisms. Composition of morphisms is induced by gluing and compositions of 2-morphisms are given by concatenation (vertical composition) and juxtaposition (horizontal composition).

\(^4\)We are allowed to deform not only the function \(f\), but also the chosen Riemannian structure on \(W\). As shown in [Ig87], all Riemannian structures can be related by such deformations.

\(^5\) A smooth structure on a gluing is well-defined only up to a diffeomorphism.
Remark 3.4. Because $\text{Fun}_{\text{or}}(W)$ is simply connected, one might wrongly expect that there is at most one 2-morphism between any two chronological cobordisms. This is not the case, because morphisms are not cobordisms themselves, but their equivalence classes. Hence, a 2-automorphism of $(W, \tau)$ can be represented by an open path of oriented functions.

Remark 3.5. For a chronological cobordism $W$ the set of critical points $\text{crit}(W)$ is linearly ordered by the chronology: we write $x < y$ if $\tau(x) < \tau(y)$. This order is invariant under equivalence of cobordisms, but it is affected by changes of chronologies.

One of the important operations on cobordisms is the disjoint sum. For chronological cobordisms it has to be defined carefully: with the naive definition one might get two critical points at the same level, what is prohibited. Instead, we have to shift critical points of the left or the right cobordism below critical points of the other one, obtaining a ‘left’ and a ‘right’ disjoint sum, denoted by $\sqcup$ or $\sqcup'$ respectively (see Fig. 3). Formally, we equip $(W, \tau) \sqcup(W', \tau')$ with a chronology

$$\tau_r(p) := \begin{cases} \beta_1^{1/2}(\tau(p)), & p \in W; \\ \beta_0^{1/2}(\tau'(p)), & p \in W', \end{cases}$$

where $\beta_a^b: I \to I$ is a pertubed ‘bump function’: an increasing function which is very close to 0 on the interval $[0, a]$ and very close to 1 on $[b, 1]$. The chronology $\tau_r$ on $(W, \tau) \sqcup(W', \tau')$ is defined in a similar way. Finally, the formula (6) can be naturally extended for changes of chronologies — replace $p$ with a pair $(p, t)$.

This is the first place where we can see that chronological cobordisms indeed require a richer structure than just a category: the disjoint sums defined above are functorial up to a change of chronology $\sigma: (W \sqcup W', \tau_r) \Rightarrow (W \sqcup W', \tau_\ell)$ that pulls $W$ below $W'$ with respect to the chronology on $W \sqcup W'$. This is done by a linear interpolation of chronologies $\sigma_t(p) := (1 - t)\tau_r + t\tau_\ell$.

Theorem 3.6. The 2-category $\text{nChCob}$ is Gray monoidal. The monoidal product is induced by the right disjoint sum $\sqcup'$ and the unit is given by the empty manifold $\emptyset$.

Proof. We have to check conditions from Definition B.6. First, $\sqcup'$ is cubical. Indeed, the conditions from Definition B.4 are trivially satisfied, as $\sigma: W \sqcup W' \Rightarrow W \sqcup W'$ does nothing if either $W$ or $W'$ has no critical points. Commutativity of the square (133) is given by a homotopy

$$h_s = \sigma|_{[0, s]} \ast ((1 - s)\alpha \sqcup \beta + s\alpha \sqcup \beta) \ast \sigma|_{[s, 1]}$$
where $\sigma|_{[a,b]}$ is a restriction of $\sigma$ to $t \in [a,b]$. The homotopy $h_*$ first shifts $W$ and $W'$ a bit towards their final position, then it applies the changes $\alpha$ and $\beta$ on appropriate levels, and after that it shifts $W$ and $W'$ further to their final positions. Finally, commutativity of (134) follows easily: the two changes $(\sigma \circ \text{id}) \star \sigma$ and $(\sigma \circ \text{id}) \star \sigma$ are homotopic by a linear interpolation.

Next, unitarity condition for $\emptyset$ is trivial and what remains is to check commutativity of the square (135). This follows directly from the way $\uplus$ is defined: although $W \uplus (W' \uplus W''')$ and $(W \uplus W') \uplus W'''$ may have different chronologies, the chronologies are homotopic by a reparametrization of the target interval $I$.

The ordinary category of cobordisms is not only monoidal, but it possesses a symmetry induced by a family of permutation diffeomorphisms $c: \Sigma_1 \sqcup \Sigma_0 \rightarrow \Sigma_0 \sqcup \Sigma_1$. Namely, take a cylinder $(\Sigma_0 \sqcup \Sigma_1) \times I$ with the standard inclusion as its input and the diffeomorphism $c$ as its output (see the picture to the side). In case of chronological cobordisms, these permutation cylinders form natural transformations between unary functors $C \uplus (\_)$ and $(\_) \uplus C$, where $C$ stands for any cylinder. This suggests the permutation cylinders equip $\text{nChCob}$ with a strict symmetry, see Definition B.7. Indeed, commutativity of the triangle (136) follows easily from this construction.

**Corollary 3.7.** The Gray monoidal category $(\text{nChCob}, \uplus, \emptyset)$ has a strict symmetry induced by permutation diffeomorphisms $c: \Sigma_1 \sqcup \Sigma_0 \rightarrow \Sigma_0 \sqcup \Sigma_1$.

We end this section with a partial presentation of $\text{2ChCob}$.

**Proposition 3.8.** $\text{2ChCob}$ is a symmetric Gray monoidal category with objects freely generated by a circle $\mathbb{S}^1$ and morphisms freely generated by the following five cobordisms:

$$
\begin{align*}
\text{a merge} & \quad \text{a split} & \quad \text{a birth} & \quad \text{a positive death} & \quad \text{a negative death} \\
\begin{tikzpicture}
\end{tikzpicture} & \begin{tikzpicture}
\end{tikzpicture} & \begin{tikzpicture}
\end{tikzpicture} & \begin{tikzpicture}
\end{tikzpicture} & \begin{tikzpicture}
\end{tikzpicture}
\end{align*}
$$

with a twist $\begin{tikzpicture}
\end{tikzpicture}$ acting as a strict symmetry.

One should read the pictures above from bottom to top: the bottom circles form the input of a cobordism, the top ones form the output and the height function determines a chronology. Orientations of critical points are visualized by arrows.

**Proof.** Every 1-dimensional manifold is a family of circles, so that objects of $\text{2ChCob}$ are freely generated under the disjoin sum by a single circle $\mathbb{S}^1$. Since all orientation preserving diffeomorphisms of $\mathbb{S}^1$ are isotopic to the identity, chronological cobordisms with no critical points are generated by a permutation of two circles, the symmetry of the monoidal structure. Morse theory provides a description of cobordisms with a single critical point and, since the order of critical points is fixed, the proposition follows.
4. Embedded cobordisms and linearization

It is not difficult to analyze 2-morphisms in \(2\text{ChCob}\), but the topological data is too little for our construction. We will instead use cobordisms embedded in \(\mathbb{D}^2 \times I\), including cobordisms with corners — they are necessary to construct the generalized Khovanov bracket for tangles. These cobordisms have a natural Riemannian structure induced from the ambient space.

**Definition 4.1.** Let \(\text{ChCob}^3(k)\) be the 2-category defined as follows.

1. Objects are families of disjoint circles and \(k\) intervals in a disk \(\mathbb{D}^2\), the latter touching \(\partial \mathbb{D}^2\) transversely at their endpoints.
2. A morphism is a surface \(W \subset \mathbb{D}^2 \times I\), such that the restriction \(pr|_W\) of the projection \(pr : \mathbb{D}^2 \times I \to I\) to \(W\) is a separative Morse function. We call it a *chronology* on \(W\). As before, we orient critical points of \(pr|_W\). Moreover, we assume that \(W\) is transverse to \(\partial (\mathbb{D}^2 \times I)\) and that \(\partial W\) consists of three parts: the input \(W \cap (\mathbb{D}^2 \times \{0\})\) of \(W\), the output \(W \cap (\mathbb{D}^2 \times \{1\})\) and \(2k\) vertical lines \(W \cap (\partial \mathbb{D}^2 \times I)\).
3. Finally, a 2-morphism is an *admissible* diffeotopy \(\varphi : (\mathbb{D}^2 \times I) \times I \to \mathbb{D}^2 \times I\), i.e. the one that fixes boundary points and at every moment \(t \in I\) the restriction \(pr|_{\varphi(t)(W)}\) is an Igusa function.

We call \(\text{ChCob}^3(k)\) the 2-category of embedded chronological cobordisms.

We will usually identify cobordisms related by diffeotopies \(\varphi_t\) for which \(pr|_{\varphi_t(W)}\) is separative Morse at every moment \(t \in I\). In particular, this holds for the following families of deformations:

- **level-preserving** diffeotopies: \(pr \circ \varphi_t = pr\) for every \(t \in I\),
- **vertical** diffeotopies: \(\varphi_t(p, z) = (p, h_t(z))\) for some diffeotopy \(h_t\) of the interval \(I\).

Hence, we can freely reparametrize both \(\mathbb{D}^2\) and \(I\). Another important family consists of locally vertical diffeotopies.

**Definition 4.2.** Let \(W \subset \mathbb{D}^2 \times I\) be an embedded chronological cobordism and \(C_1, \ldots, C_r\) a family of disjoint and unnested vertical tubes, intersecting \(W\) in vertical lines. We say that a diffeotopy \(\varphi_i\) is *locally vertical* with respect to these tubes if it is vertical inside each tube \(C_i\), but fixes all points outside them (except very small neighborhoods of the tubes).

The requirement that \(W\) intersects each \(C_i\) in vertical lines implies that \(\varphi_t\) cannot create critical points. Moreover, each interpolation \((1-t)\varphi_1 + t\ id\) induces a chronology on \(W\), so that locally vertical diffeotopies can be ‘straightened up’.

**Proposition 4.3.** Let \(\varphi_t\) and \(\varphi'_t\) be diffeotopies locally vertical with respect to the same family of tubes. If \(\varphi_1 = \varphi'_1\), then they are homotopic in the space of admissible diffeotopies. In particular, a locally vertical diffeotopy \(\varphi_t\) satisfying \(\varphi_1 = \text{id}\) is trivial.

**Proof.** Take a linear homotopy \(h_{s,v} := v\varphi_s + (1-v)\varphi'_s\). Because both \(\varphi_s\) and \(\varphi'_s\) are locally vertical, each \(h_{s,v}\) is a diffeomorphism of \(\mathbb{D}^2 \times I\) such that the restriction to \(pr|_{h_{s,v}(W)}\) is a Morse function. \(\square\)

The proposition above makes it possible to define a *disjoint sum* in \(\text{ChCob}^3(0)\) (more general operations on all categories \(\text{ChCob}^3(k)\) are defined in Section 6). Take two cobordisms \(W\) and \(W'\), shrink them in the horizontal direction and insert into \(\mathbb{D}^2 \times I\), one next to the other. As before, we push \(W\) upwards and \(W'\) downwards — this is done by a locally...
vertical diffeotopy. This defines the right disjoint sum $W \uplus W'$ and similarly we have the left one $W \uplus W'$. They are related by a locally vertical diffeotopy $\sigma_{W,W'} : W \uplus W' \Rightarrow W \uplus W'$, which pushes $W$ upwards and $W'$ downwards. Two examples for $\sigma$ and its inverse are illustrated below.

(Same image as before)

Similarly as for abstract cobordisms, the changes $\sigma_{W,W'}$ equips $\uplus$ with a cubical structure.

**Corollary 4.4.** $\text{ChCob}^3(0)$ is a Gray monoidal category, with a monoidal structure given by the right disjoint sum $\uplus$.

**Remark 4.5.** This monoidal structure is strictly braided (see Definition B.7) with a braiding induced by twists $\otimes$ and $\times$. We will not use this fact in our paper.

From now we will restrict to $\text{ChCob}^3(0)$. If we forget the embedded structure, it has the same objects and morphisms as $2\text{ChCob}$. Hence, Proposition 3.8 describes the low-level structure of $\text{ChCob}^3(0)$ pretty well. We will refer to deaths as clockwise or anticlockwise, comparing their orientations with the standard orientation of $\mathbb{D}^2$. This makes it independent of the orientation of a cobordism. Finally, using Cerf theory we can describe changes of chronologies in terms of generators and relation (see Appendix A for details). There are two elementary types of them: creations/annihilations and permutations.

A creation is a diffeotopy with a unique $A_2$ singularity. It creates a pair of critical points of indices $i$ and $i+1$, the point with the higher index lying over the other. Inverting the time of the change results in an annihilation of the points. All three possible cases for $\text{ChCob}^3(0)$ are presented below.

(Same image as before)

An orientation of a death is determined by the monotonicity condition for $d^3\tau$ at an $A_2$ singularity: take the arrow at the saddle and rotate it towards the vertical cylinder.

Permutations are realized by diffeotopies having at some moment two critical points at one level. The disjoint sum permutations $\sigma_{W,W'}$ are the only permutation changes modeled on disconnected cobordisms and the others permute two saddles as shown to the right, which can be connected in five different ways illustrated by the following diagrams.
In each diagram circles visualize a regular section just below the singular one and saddle points are represented by the thick lines. By orienting them we can encode orientations of the points. An example is given in Fig. 4. These changes preserve types of critical points (a merge is always a merge and a split is always a split) except two cases.

A $\times$-change is encoded by a diagram with two circles joined by two parallel arrows. Each of the two related chronologies decompose the cobordism into a merge followed by a split (the cobordism looks like a fattened cross, hence a name). Hence, each arrow represents a merge in one decomposition, but a split in the other.

A $\Diamond$-change is described by a diagram with a unique circle, one arrow inside and one outside the circle. It relates two chronologies decomposing a cobordism into a split followed by a merge that joins the two circles back (it has a shape of a diamond). Again, each arrow orients a merge in one of the cobordisms, but a split in the other. We will remember which arrow is inside and which is outside the circle, but not their exact placement, i.e. the outer arrow can be either over or below the circle. Hence, we reduced this case to two configurations, distinguished by the intersection number of the inner arrow with the outer one (i.e. by a direction of the outer arrow, assuming the inner one points upwards).

**Remark 4.6.** There is a $\mathbb{Z} \times \mathbb{Z}$-valued multiplicative degree function defined on chronological cobordisms

$$\text{deg}(M, \tau) = (\#\text{births} - \#\text{merges}, \#\text{deaths} - \#\text{splits}),$$

preserved by changes of chronologies. Indeed, permutations preserve the numbers of critical points of each type, whereas creations and annihilations preserve the differences above.

**Linearization.** The 2-category $\text{ChCob}^3(0)$ is not good for homological constructions and we will ‘linearize’ it. More precisely, choose a commutative ring $R$ and assume there is a function $\iota : \text{2Mor}(\text{ChCob}^3(0)) \to R$ that is multiplicative with respect to both compositions of 2-morphisms. Define a 2-category $\text{RChCob}^3(0)$ as follows:

1. the set of objects is not changed and it consists of families of circles in $\mathbb{D}^2$,
2. morphisms are finite linear combinations of chronological cobordisms

$$r_1W_1 + \ldots + r_kW_k$$

with $r_i \in R$, modulo chronological relations $W' = \iota(\varphi)W$, one for each 2-morphism $\varphi : W \to W'$ in $\text{ChCob}^3(0)$, and
3. 2-morphisms are elements of $R$ acting by a multiplication, i.e. there is a 2-morphism $r : W \Rightarrow W'$ whenever $r \cdot W = W'$.

We extend the composition of cobordisms to formal sums in a linear way. Both vertical and horizontal compositions of 2-morphisms are given by multiplication in $R$. A function $\iota$
mentioned above induces a monoidal 2-functor $\iota : \text{ChCob}^3(0) \to R\text{ChCob}^3_\iota(0)$, which we denote by the same letter. We want it to be rich enough to support the construction of odd Khovanov homology. We start with a few observations.

**Lemma 4.7.** We can assume $\iota$ maps creations and annihilations into equalities.

*Proof.* Each of the three creations (10) involves different generators. Hence, we can force the coefficients associated to them to be 1 by scaling a birth and deaths. □

**Lemma 4.8.** The coefficient $\iota^W_{W'} := \iota(\sigma_{W,W'} : W \uparrow W' \Rightarrow W \downarrow W')$ associated to a disjoint permutation depends only on degrees of $W$ and $W'$. In particular, there is a function

$$\lambda : \mathbb{Z} \times \mathbb{Z} \to R, \quad (a, b, c, d) \mapsto X^{ac}Y^{bd}Z^{ad-bc},$$

such that $\iota^W_{W'} = \lambda(\deg W, \deg W')$. The coefficients $X, Y, Z \in R$ are some invertible elements and $X^2 = Y^2 = 1$.

*Proof.* If such a function exists, then both $X$ and $Y$ squared must be equal to 1, because the inverse of $\sigma_{W,W}$ is given by itself conjugated with a twist. To see that $\iota^W_{W'}$ depends only on degrees of $W$ and $W'$, it is enough to show that permutations of any cobordism with a merge and with a birth have inverse coefficients — this follows immediately from triviality of the following change of a chronology:

$$\ldots \Rightarrow \begin{array}{c} W \\
\vdots \\
\end{array} \Rightarrow \begin{array}{c} W \\
\vdots \\
\end{array} \Rightarrow \begin{array}{c} W \\
\vdots \\
\end{array} \Rightarrow \begin{array}{c} W \\
\vdots \\
\end{array} \Rightarrow \ldots$$

where $W$ is any cobordism. The same holds for a split and any death. □

The above implies the following table of coefficients for disjoint permutations $\sigma_{W,W'}$ between cobordisms with a single critical point.

| $W$ \(\backslash\) $W'$ | birth | merge | split | death |
|---|---|---|---|---|
| birth | $X$ | $X$ | $Z^{-1}$ | $Z$ |
| merge | $X$ | $X$ | $Z$ | $Z^{-1}$ |
| split | $Z$ | $Z^{-1}$ | $Y$ | $Y$ |
| death | $Z^{-1}$ | $Z$ | $Y$ | $Y$ |

For instance, we have the following equalities

$$X = \quad \text{and} \quad Z =$$

It remains to define $\iota$ for the remaining permutation changes encoded with diagrams (11). We use the coefficients from Tab. 1 on page 7.

**Proposition 4.9.** Choose invertible elements $X, Y, Z \in R$ such that $X^2 = Y^2 = 1$ and define $\iota$ on generating changes of chronologies by the following rules:

1. creations and annihilations are sent to equalities,
(2) a $\times$-change is sent into $Y$, if the arrows point to the same circle, and into $X$ otherwise,

(3) a $\hat{\diamond}$-change with a diagram in which the inner arrow is oriented upwards is sent into $1$ or $XY$ if the outer arrow is oriented to the left or to the right respectively, and

(4) the coefficient associated to any other change permuting two critical points of degrees $(a, b)$ and $(c, d)$ is given by $\lambda(a, b, c, d) = X^{ac}Y^{bd}Z^{ad-bc}$.

Then $\iota: \text{ChCob}^3(0) \to R\text{ChCob}^3(0)$ is a well-defined 2-functor.

Proof. We have to check that $\iota$ defined as above is consistent with relations between elementary changes of chronologies. These are described by codimension 2 singularities and can be gathered into four groups:

1. changes supported on different regions commute (123),
2. a change followed by its inverse is trivial (124),
3. coherence relations between permutations and creations or annihilations (125), the top row, and
4. the two ways of reversing an order of three critical points describe the same change of a chronology (125), the bottom row.

Coherence of $\iota$ with the first group of relations follows from the commutativity of $R$, whereas relations from the second group holds because $\lambda(a, b, c, d)\lambda(c, d, a, b) = 1$ and the coefficients assigned to $\times$- and $\hat{\diamond}$-changes are square roots of 1. The third group of relations is satisfied due to Lemma 4.8. Each relation from the last group is an equality between two changes, each modeled by a locally vertical diffeotopy with respect to three tubes around critical points $a$, $b$ and $c$ in a cobordism $W$. Therefore, we have to check that $\iota$ preserves commutativity of hexagons

$$
\begin{array}{c}
W(a<b<c) & \overset{\lor}{\longrightarrow} & W(b<a<c) \\
W(b<a<c) & \overset{\lor}{\longrightarrow} & W(b<c<a) \\
W(a<c<b) & \overset{\lor}{\longrightarrow} & W(c<b<a) \\
W(a<c<b) & \overset{\lor}{\longrightarrow} & W(c<a<b)
\end{array}
$$

where $W(b<a<c)$ is a cobordism obtained from $W$ by placing its critical points in the specified order and each arrow represents an elementary permutation change.

If $W$ is disjoint, say $a$ is in a component $W_1$ and $b, c$ in a component $W_2$, it is enough to notice that both $W_2(b<c)$ and $W_2(c<b)$ have the same degree — permutating $W_1$ with both of them results in the same coefficient.

If $W$ is connected, then all critical points have index 1. Take a look at the singular level containing all of them. The choice for $\iota$ we made satisfies

$$
\iota\left(\begin{array}{cc}
A & B \\
\end{array}\right) = \iota\left(\begin{array}{cc}
A & \lor B \\
\end{array}\right)
$$

for any way we insert saddles into holes $A$ and $B$, so that the following modification of the cobordism $W$

$$
\begin{array}{c}
\lor \\
\rightarrow \\
\lor \\
\lor
\end{array}
$$
Table 2. Singular levels of homotopies relating permutation changes. Numbers below each diagram indicate how many times various permutations occur: $\times$-changes with parallel or opposite arrows (the first group), $\diamond$-changes with outer arrows oriented to the left or to the right (the second group) and the other changes grouped by the value of $\iota$ (respectively $X$, $Y$ and $Z$). Different sequences correspond to different orientations of saddle points.

does not affect values of $\iota$ for the two changes being related. It remains to check four more cases listed in Tab. 2. The numbers below each diagram indicate how many times a particular elementary change occurs when we go around the hexagon (16). In each case, the product of values of $\iota$ is equal to 1.

Remark 4.10. We will usually omit the subscript, writing $R\text{ChCob}^3(0)$ for the linearized category. If the choice of $\iota$ is important, we will write $R\text{ChCob}^3_{abc}(0)$ for the quotient by chronological relations with parameters $X$, $Y$ and $Z$ set to $a$, $b$ and $c$ accordingly.

Corollary 4.11. The following rules for reversing orientations hold:

\begin{equation}
\begin{align*}
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
= X
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
= Y
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
= Y
\end{align*}
\end{equation}

Proof. The last rule follows from the following change

\begin{equation}
\begin{align*}
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\end{align*}
\end{equation}

and the first one from

\begin{equation}
\begin{align*}
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\Rightarrow
\begin{multlined}
\begin{tikzpicture}
\node (a) at (0,0) [simple] {}; \\
\node (b) at (1,0) [simple] {}; \\
\node (c) at (0.5,0.866) [simple] {}; \\
\end{tikzpicture}
\end{multlined}
\end{align*}
\end{equation}

Reversing an orientation of a split is done in a similar way.

As explained in Remark 3.4, even if the space of admissible diffeotopies is simply connected, there are 2-automorphisms of chronological cobordisms that are not trivial. In the case of $R\text{ChCob}^3(0)$ there are cobordisms that are annihilated by $1 - XY$: a disjoint sum of two spheres (permute the components) and a twice punctured torus (reverse orientations of both saddle points and rotate the cobordism). In fact nothing else can happen.
Theorem 4.12. Let \( R = \mathbb{Z}[X, Y, Z^\pm 1]/(X^2 = Y^2 = 1) \) and choose an embedded chronological cobordism \( W \) in \( R\text{ChCob}^3(0) \). Then

\[
\text{Aut}(W) = \begin{cases} 
\{1\}, & \text{if } g(W) = 0 \text{ and } W \text{ has at most one spherical component}, \\
\{1, XY\}, & \text{otherwise}.
\end{cases}
\]

A proof of this result uses a chronological version of Frobenius algebras and is postponed to Section 10. a chronological version of Frobenius algebras.

5. Khovanov complex

Now we go back to Figure 1. We can first see it as a diagram \( I(D) \) in the 2-category \( \text{ChCob}^3(0) \): vertices are 1-manifolds (resolutions of the diagram \( D \)), edges are chronological cobordisms between these manifolds and faces are decorated with changes of chronologies. This diagram commutes in the 2-categorical sense: a composition of 2-morphisms along any 3-dimensional subcube is trivial, i.e.

\[
\begin{array}{c}
100 \\
\downarrow \\
010 \\
\downarrow \\
000
\end{array}
\quad =
\begin{array}{c}
110 \\
\downarrow \\
101 \\
\downarrow \\
111
\end{array}
\]

This follows from Proposition 4.3, as the two changes are locally vertical with respect to small tubes around the crossings of \( D \).

Apply the 2-functor \( \iota : \text{ChCob}^3(0) \to R\text{ChCob}^3(0) \) from Section 4. It preserves vertices and edges, but faces are now decorated with elements of the ring \( R \) according to Tab. 1 from page 7. We used this table in Section 2 to define a 2-cochain \( \psi \in C^2(I^n; U(R)) \), where \( U(R) \) is the group of invertible elements in \( R \) and \( n \) is the number of crossings in \( D \). Recall that a 1-cochain \( \epsilon \in C^1(I^p; U(R)) \) is called a sign assignment if \( d\epsilon = -\psi \). This means the corrected cube \( \bar{I}(D, \epsilon) \) anticommutes, where \( \bar{I}(D, \epsilon) \) has the same vertices as \( I(D) \), but for an edge \( \zeta \) one has \( \bar{I}(D, \epsilon)(\zeta) = \epsilon(\zeta) \cdot I(D)(\zeta) \). Existence of such a cochain follows easily.

Proposition 5.1. The cochain \( \psi \) is a cocycle for any link diagram \( D \). Hence, \(-\psi = d\epsilon \) for some sign assignment \( \epsilon \).

Proof. The 2-commutativity of faces (22) of any 3-dimensional subcube in \( I(D) \) implies that \( d(-\psi) = d\psi = 1 \). The existence of \( \epsilon \) follows from the contractibility of \( I^n \).

At this point we can forget about the 2-categorical structure of \( R\text{ChCob}^3(0) \) (but it will appear again in the next section). Following [BN05] we define the Khovanov bracket in the additive closure \( \text{Mat}(R\text{ChCob}^3(0)) \).

Definition 5.2. Choose a commutative ring \( R \) and let \( C \) be an \( R \)-linear category, i.e. the sets of morphisms \( \text{Mor}(A, B) \) are modules over \( R \) and the composition of morphisms is bilinear. The additive closure \( \text{Mat}(C) \) of \( C \) is a category defined as follows:

- objects are formal direct sums \( \bigoplus_{i=1}^{n} C_i \) of objects from \( C \),
The composition of morphisms in the additive closure of a category.

The component \((F \circ G)_{21}\) is indicated by solid lines.

- a morphism \(F: \bigoplus_{i=1}^n A_i \to \bigoplus_{j=1}^m B_j\) is a matrix \((F_{ij}: A_j \to B_i)\) of morphisms from \(C\),

- the composition of morphisms \(F \circ G\) mimics the formula for a product of matrices

\[
(F \circ G)_{ij} := \sum_k F_{ik} \circ G_{kj}.
\]

This category is \(R\)-linear with a natural action of \(R\) and addition defined as addition of matrices: \((F + G)_{ij} := F_{ij} + G_{ij}\).

We can represent objects of \(\text{Mat}(C)\) by finite sequences (vectors) of objects in \(C\) and morphisms between such sequences by bundles\(^6\) (matrices) of morphisms in \(C\), see Fig. 5. It means each column in Fig. 1 forms a single object \(C^i\), as indicated by dotted arrows going downwards, and all edges between two columns form a single morphism \(d: C^i \to C^{i+1}\). Because every square in \(\mathcal{I}(D, e)\) anticommutes, \(d^2 = 0\).

There is one more ingredient to Fig. 1: the numbers in curly brackets along the dotted arrows. This notation is usually reserved for degree shifts and this is not an exception.

**Definition 5.3.** Choose an abelian group \(G\). We say an \(R\)-linear category \(C\) is \(G\)-graded, if

1. for any objects \(A, B\) the set \(\text{Mor}(A, B)\) is a \(G\)-graded \(R\)-module such that \(\text{id}_A\) is homogeneous in degree \(0\) for any object \(A\),
2. the degree function is additive with respect to composition: \(\deg(f \circ g) = \deg f + \deg g\), for homogeneous \(f\) and \(g\), and
3. there is a degree shift functor \(\text{Ob}(C) \times G \ni (A, m) \mapsto A\{m\} \in \text{Ob}(C)\) preserving morphisms, i.e. \(\text{Mor}(A\{m\}, B\{n\}) = \text{Mor}(A, B)\), but degrees are changed: if a morphism \(f \in \text{Mor}(A, B)\) has degree \(d\), then \(\deg f = d + n - m\) if seen as an element of \(\text{Mor}(A\{m\}, B\{n\})\).

We have already defined a \(Z \times Z\)-valued degree function for chronological cobordisms (12). Here, we will collapse it to a \(Z\)-grading, by summing up both numbers, so that \(\deg W = \chi(W)\) is the Euler characteristic of a cobordism \(W\). Degree shifts are introduced artificially: add formal objects \(\Sigma\{m\}\) for every 1-manifold \(\Sigma\) and \(m \in \mathbb{Z}\), and extend the degree map as in the definition above:

\[
\deg W := \chi(W) + n - m \quad \text{for} \quad W: \Sigma_0\{m\} \to \Sigma_1\{n\}.
\]

\(^6\) In the colloquial sense, not the mathematical one.
All cobordisms in the cube of resolutions have degree $-1$. Hence, taking $C^n\{i\}$ results in a complex with a degree 0 differential.

This is the last piece of the construction. Below we summarize everything, giving a full definition of the bracket.

**Definition 5.4.** Given a link diagram $D$ with $n$ crossings construct its cube of resolutions $\mathcal{I}(D)$ and choose a sign assignment $\epsilon$. The *generalized Khovanov bracket* of $D$ is a chain complex $[D]_\epsilon$ with:

\[
[D]_\epsilon^i := \bigoplus_{\|\xi\| = i} D_{\xi}\{i\},
\]

\[
d_\epsilon|_{D_{\xi}} := \sum_{\zeta: \xi \to \xi'} \epsilon(\zeta) \cdot D_{\zeta},
\]

where $\|\xi\| := \xi_1 + \ldots + \xi_n$ is the weight of a vertex $\xi$.

**Corollary 5.5.** The sequence $(C, d)$ at the bottom line of Fig. 1 is a chain complex.

*Proof.* This follows from anticommutativity of the corrected cube $\mathcal{I}(D, \epsilon)$. □

There are a few choices involved in the construction of $[D]_\epsilon$: an order of crossings, arrows at the crossings and the sign assignment $\epsilon$. We will now show that different choices lead to isomorphic complexes.

Changing an order of crossings results in a different parametrization of the cube $\mathcal{I}(D)$, but the chain objects $[D]_\epsilon^i$ are direct sums of the same objects and similarly for the differential. Different orientation of the arrows over crossings can be compensated by an edge assignment.

**Lemma 5.6.** Let $D_1, D_2$ be diagrams of a link $L$ with $n$ crossings, which differ only in directions of arrows over crossings. Then for any sign assignment $\epsilon_1$ for $\mathcal{I}(D_1)$ there exists a sign assignment $\epsilon_2$ for $\mathcal{I}(D_2)$ such that $\mathcal{I}(D_1, \epsilon_1) = \mathcal{I}(D_2, \epsilon_2)$.

*Proof.* Without loss of generality we may assume $D_1$ and $D_2$ differ only in the direction of the arrow at an $i$-th crossing. Reversing the arrow changes orientation of critical points of cobordisms at edges $\zeta$ with $\zeta_i = *$. Let $\psi_i$ be the commutativity cocycle of the cube $\mathcal{I}(D_i)$. For a sign assignment $\epsilon_1$ for $\mathcal{I}(D_1)$ we define

\[
\epsilon_2(\zeta) := \begin{cases} 
\epsilon_1(\zeta), & \text{if } \zeta_i \neq *, \\
X\epsilon_1(\zeta), & \text{if } \zeta_i = * \text{ and } D_{\zeta} \text{ is a merge}, \\
Y\epsilon_1(\zeta), & \text{if } \zeta_i = * \text{ and } D_{\zeta} \text{ is a split}.
\end{cases}
\]

A direct computation shows that $d\epsilon_2 = -\psi_2$. Hence, $\epsilon_2$ is the desired sign assignment. □

Finally, different sign assignments lead to isomorphic cubes, where by an isomorphism of two $n$-dimensional cubes $F$ and $G$ we mean a family of isomorphisms $\eta_\zeta: F_\zeta \to G_\zeta$ such that the following square commutes

\[
\begin{array}{ccc}
F_\xi & \xrightarrow{\eta_\xi} & G_\xi \\
\downarrow & & \downarrow \\
F_\zeta & \xrightarrow{\eta_\zeta} & G_\zeta \\
\end{array}
\]

for every edge $\zeta: \xi \to \xi'$. 
Lemma 5.7. Choose a link diagram $D$ with $n$ crossings and let $\epsilon$ and $\epsilon'$ be two sign assignments for $I(D)$. Then the cubes $I(D, \epsilon)$ and $I(D, \epsilon')$ are isomorphic.

Proof. The equality $d\epsilon = -\psi = d\epsilon'$ and contractibility of $I^n$ implies that $\epsilon' = d\eta \cdot \epsilon$ for some 0-cochain $\eta \in C^0(I^n; U(R))$. Define the isomorphism $f : I(D, \epsilon) \to I(D, \epsilon')$ by setting $f_\xi := \eta(\xi) \cdot \text{id}$. □

An isomorphism of cubes induces an isomorphism of complexes.

Corollary 5.8. The isomorphism class of the generalized Khovanov bracket $[D]_\epsilon$ depends only on the diagram $D$ of a link.

The generalized bracket, even up to chain homotopies, is not a link invariant, but it is not very far from it. To construct an invariant, we have to take an oriented diagram and shift degrees (both the internal grading and the homological one) according to the number of positive and negative crossings.

Definition 5.9. Let $D$ be an oriented link diagram with $n_+$ positive and $n_-$ negative crossings. The generalized Khovanov complex $\text{Kh}(D)$ of $D$ is obtained from the bracket $[D]_\epsilon$ by the degree shifts $\text{Kh}(D) := [D][-n_-]_{n_+ - 2n_-}$, i.e. $\text{Kh}'(D) = [D]^{-n_-}_{n_+ - 2n_-}$.

6. Tangles and planar algebras

Tangles have a rich algebraic structure called a planar algebra [Jo99]: they can be combined together to produce larger tangles, by connecting some of their endpoints. We will now define this algebraic structure following [BN05].

Definition 6.1. A planar arc diagram $D$ with $d$ inputs is a disk $\mathbb{D}^2$ missing $d$ smaller disks $\mathbb{D}^2_i$, together with an embedding of disjoint circles and closed intervals, transverse to all boundaries. Each boundary component carries a basepoint and meets an even number of intervals. Say $D$ is oriented if the embedded circles and intervals are oriented. Both oriented and non-oriented planar arc diagrams are considered up to planar isotopies.

We can compose planar arc diagrams by placing one of them in a hole of another. This composition is associative: when composing more than two diagrams, the final result does not depend in which order they are composed.

Definition 6.2. A planar algebra $\mathcal{P}$ is a collection of sets $\mathcal{P}(k)$ together with an operator

$$D : \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_s) \to \mathcal{P}(k)$$

defined for each planar arc diagram $D$, such that their composition is associative and radial diagrams (i.e. those with a single input and radially embedded intervals) correspond to identity maps. An oriented planar algebra is defined similarly, using oriented planar arc diagrams.

Example 6.3. Given a planar arc diagram $D$ we can insert into its holes some tangle diagrams, creating another tangle diagram. This results in a map

$$D : \mathcal{T}^0(k_1) \times \cdots \times \mathcal{T}^0(k_s) \to \mathcal{T}^0(k),$$

where $\mathcal{T}^0(k)$ is the set of all tangle diagrams with $2k$ endpoints embedded in a disk $\mathbb{D}^2$ with a basepoint on its boundary (so that this operation is well-defined). Because Reidemeister
moves are local, we can replace $\mathcal{T}^0(k)$ with sets of tangles $\mathcal{T}(k)$ and the operation induced by $D$ is still well-defined. In a similar way oriented diagrams allow us to combine oriented tangles. Here, we group tangles (or tangle diagrams) into sets $\mathcal{T}_+(\vec{s})$ (respectively $\mathcal{T}_+(\vec{s})$), labeled with finite sequences $\vec{s}$ of $+$’s and $-$’s encoding orientation of the endpoints.

**Definition 6.4.** A morphism of planar algebras $\Phi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is a collection of morphisms $\Phi_k: \mathcal{P}_1(k) \rightarrow \mathcal{P}_2(k)$ commuting with planar operators, i.e.

$$D \circ (\Phi_{k_1}, \ldots, \Phi_{k_s}) = \Phi_k \circ D,$$

for every operator $D$. In a similar way one defines morphisms of oriented planar algebras.

**Example 6.5.** There is a natural morphism from the planar algebra of tangle diagrams to the planar algebra of tangles that maps a tangle diagram into the tangle it represents.

We will now construct the Khovanov complex for a tangle diagram $T$ with $k$ endpoints. As mentioned in Section 2, we can construct the cube of resolutions $\mathcal{I}(T)$ using cobordisms with corners $\text{ChCob}^3(k)$. These 2-categories also form a planar algebra, where the cubical functor

$$D: \text{ChCob}^3(k_1) \times \ldots \text{ChCob}^3(k_d) \rightarrow \text{ChCob}^3(k)$$

associated to a planar arc diagram $D$ is defined in the following way.

- The object $D(\Sigma_1, \ldots, \Sigma_d)$ is defined in the same way as a composition of tangle diagrams: simply insert the pictures $\Sigma_i$ into holes of $D$.
- For cobordisms take a curtain diagram $D \times I$ (see Fig. 6 for an example) and insert them into the missing tubes. Here, one has to do the same trick as with the disjoint sum — to shift all critical points, placing the critical points of the first cobordism at the top, below the critical points of the second one and so on.
- Finally, for changes of chronologies $\alpha_i: W_i \Rightarrow W'_i$ there is an induced change

$$D(\alpha_1, \ldots, \alpha_d): D(W_1, \ldots, W_d) \Rightarrow D(W'_1, \ldots, W'_d)$$

defined as a composition $\bar{\alpha}_1 \circ \ldots \circ \bar{\alpha}_d$, where $\bar{\alpha}_i = D(id_{W_1}, \ldots, \alpha_i, \ldots, id_{W_d})$ is given by $\alpha_i$ in the $i$-th hole (reparametrized accordingly) and fixed beyond it. Simply speaking, all changes $\alpha_i$ are applied at the same time, but on different regions.\(^7\)

It follows directly from Proposition 4.3 that the functors defined above are cubical. They extend naturally to cubes in $\text{ChCob}^3(k)$.

\(^7\) A change defined in this way might not be generic.
Corollary 6.6. The function $T \mapsto I(T)$ is a morphism of planar algebras, i.e.
\begin{equation}
I(T) = D(I(T_1), \ldots, I(T_d))
\end{equation}
for a planar arc diagram $D$ and tangle diagrams $T_1, \ldots, T_d$ with an appropriate number of endpoints.

The main problem when defining the Khovanov complex for tangles is to understand the commutativity cocycle $\psi$. For instance, a single saddle is a part of both a merge and a split:

\begin{equation}
\begin{array}{c}
\text{split} \\
\text{merge}
\end{array}
\end{equation}

and any of the diagrams in Tab. 1 is a closure of two saddles. Therefore, a coefficient associated to a change of a chronology cannot be a single element of $R$, but is must be a gadget that returns such an element after all corners and vertical boundaries are connected in pairs.

Definition 6.7. A closure planar diagram is a planar arc diagram with one input (hence, it is an annulus) and embedded intervals only with endpoints on the input boundary.

Denote by $\mathcal{CPO}(k)$ the set of all closure planar diagrams. If $T$ is a tangle diagram with $2k$ endpoints and $D \in \mathcal{CPO}(k)$ is a closure operator, then $D(T)$ is a link. A diagram $D \in \mathcal{CPO}(k)$ induces a strict 2-functor
\begin{equation}
\text{ChCob}^3(k) \to \text{ChCob}^3(0) \to \text{RChCob}^3(0),
\end{equation}
what suggests the commutativity cochain $\psi$ takes values in $R(k) := \{f : \mathcal{CPO}(k) \to R\}$, the ring of all functions from the set of closure planar operators to $R$. To compute $\psi(S)(D)$ identify the picture $D(S)$ in Tab. 1 on page 7 (an example is given in Tab. 3). It follows immediately that $\psi$ is a cocycle and that $[T]_\epsilon$, up to an isomorphism, does not depend on a sign assignment $\epsilon$.

---

\footnote{A cubical functor taking one argument is automatically strict.}
Remark 6.8. For simplicity, the linearization of $\text{ChCob}^3(k)$ with coefficients in $R(k)$ will be written as $R\text{ChCob}^3(k)$.

Remark 6.9. It can be shown that categories $R\text{ChCob}^3(k)$ form a planar algebra. However, there is no analogue of Corollary 6.6 for signed cubes: planar operators are only cubical functors and as such they do not preserve anticommutativity. In particular, we cannot use planar operators to combine generalized brackets together as it was done in [BN05].

7. Proof of Invariance

The Khovanov complex $\text{Kh}(T)$ is not a tangle invariant. For example, it depends on the number of crossings in a chosen diagram. This dependence disappears after passing to the homotopy category of complexes and imposing modified versions of Bar-Natan’s $S$, $T$ and $4Tu$ relations [BN05], explained below.

- $(S)$ The $S$ relation replaces with 0 all cobordisms that have a sphere as a connected component. The number and orientations of critical points do not matter.

- $(T)$ The $T$ relation allows us to remove a standard torus component at a cost of multiplying the cobordism with $Z(X+Y)$. Here, the standard torus is defined as a torus with four critical points and an arrow at the merge pointing to the circle originating on the left hand side of the split. The death is oriented anticlockwise.

- $(4Tu)$ The four tube relation $4Tu$ involves four cobordisms going from two circles to two circles. Each of them consists of a tube and two cups, but the position of the tube is different in each picture: for the first two cobordisms the tube is a vertical cylinder over one of the two circles, while in the remaining two cases it connects either the input or the output circles. Notice the choice of framings for saddle points and heights of caps (the left caps are smaller than the right ones). Again, all deaths are oriented anticlockwise.

The relations above, especially $T$ and $4Tu$, are local. This means all other critical points can appear only above or below the pictures shown. Notice that all of them are homogeneous: the degree of the standard torus is zero and each cobordism involved in $4Tu$ has degree $(-1,-1)$. Thence, these relations are coherent with changes of chronologies. Let $R\text{ChCob}^3_{R}(k)$ be the quotient of $R\text{ChCob}^3(k)$ by these relations.

Theorem 7.1. Let $T$ be a tangle with $2k$ endpoints. The homotopy type of the generalized Khovanov complex $\text{Kh}(T)$, regarded as an object of $\text{Kom}(R\text{ChCob}^3_{R}(k))$, is an invariant of $T$, i.e. complexes for two tangle diagrams related by any of the Reidemeister moves are chain homotopy equivalent.

We will first prove this theorem locally, for the tangles defining Reidemeister moves. Using the planar algebra of chronological cobordisms we will then extend the homotopy equivalences to complexes for bigger tangles. Proofs will be done on pictures of cobordisms and for simplicity we omit some details, keeping in mind the following conventions:

1. all deaths are oriented anticlockwise,
(2) arrows orienting saddles are directed either to the right or to the front, so that we can cancel a merge or a split with a birth or a death respectively on its right hand side at no cost, while a left-hand cancellation costs a multiplication by $X$ or $Y$, 

(3) basepoints should be chosen in the same place for all pictures involved in each proof. In addition, we will not write degree shifts.

**Definition 7.2.** We say that a chain complex $D$ is a strong deformation retract of a chain complex $C$ if there are chain maps $f: D \to C$ and $g: C \to D$ such that $gf = id$ and $fg - id = dh + hd$ for a homotopy $h$ such that $hf = 0$. The chain map $f$ is called an inclusion into a deformation retract.

**Lemma 7.3.** The bracket $\langle \rangle \{1\}$ is a strong deformation retract of $\langle \rangle$. Hence, $\text{Kh}(\langle \rangle)$ and $\text{Kh}(\langle \rangle)$ are homotopy equivalent for any orientation of the tangle.

**Proof.** The second statement follows from the first one, because no matter how the tangle is oriented, its crossing is positive. Consider now the diagram to the right. Rows together with morphisms pointing to the right represent the Khovanov brackets $\langle \rangle$ (the top row) and $\langle \rangle$ (the bottom row), whereas the morphisms pointing to the left are chain homotopies in these complexes (zero at the top and a curtain with a birth at the bottom). The coefficient $\epsilon$ comes from a sign assignment — although we could take $\epsilon = 1$, this more general situation turns out to be useful when extending the proof to the global case.

Vertical arrows define chain maps $f: \langle \rangle \to \langle \rangle$ and $g: \langle \rangle \to \langle \rangle$, which is obvious for $g$, but requires the following short computation for $f$:

\[
\epsilon^{-1} df^0 = XY - YZ = YZ - YZ = 0.
\]

When the degree shifts are applied, both $f$ and $g$ have degree $0$. The relation $T$ implies $gf = id$:

\[
g^0 f^0 = YZ^{-1} - XY = (Y(X + Y) - XY) = id,
\]

whereas $4Tu$ makes $f^0 g^0 - id = hd$:

\[
0 = Z + Z - X - Y
\]

\[\text{We will often omit the composition sign } \circ.\]
After expanding $f^0g^0$ we can see that the last cobordism should appear with the coefficient $-XY$. The equality holds, because the cobordism has a handle, hence, it is annihilated by $(1-XY)$. Together with $dh = -id = f^1g^1-id$ (remove the birth), this shows that the maps $f$ and $g$ are mutually inverse homotopy equivalences. To finish the proof, notice that $hf = 0$ trivially. \hfill $\Box$

**Remark 7.4.** Suppose $\bowtie$ is a part of a bigger tangle diagram $T$ and consider the corrected cube of resolutions $I(T, \epsilon)$. If we replace edges $d_\zeta$ corresponding to the crossing in $\bowtie$ with homotopies $h_\zeta$ defined as in Lemma 7.3 (this reverses directions of the edges), the new cube still anticommutes. Indeed, as $d_\zeta$ is always a merge and $h_\zeta$ is a birth, checking anticommutativity reduces to comparing the following squares.

![Squares](39)

Whatever the saddle is, the relation between the top and the bottom cobordism in the left square is exactly the same as the relation between the left and the right cobordism in the right square. Because we corrected $h_\zeta$ with the inverse of the coefficient for $d_\zeta$, coefficients along the curly arrows are the same and anticommutativity of one of the squares implies anticommutativity of the other.

**Lemma 7.5.** The bracket $\llbracket \bowtie \rrbracket \{1\}$ is a strong deformation retract of $\llbracket \bowtie \rrbracket \llbracket 1\rrbracket$. Hence, $\text{Kh}(\bowtie)$ and $\text{Kh}(\bowtie\bowtie)$ are homotopy equivalent for any orientation of the tangles involved.

**Proof.** As before, the second claim follows from the first one, as the two crossings in $\bowtie\bowtie$ have different signs for any orientation of the tangle. The first sentence follows from the diagram in Fig. 7. As before, $\epsilon$’s come from some sign assignment (so that the lower square anticommutes). The nontrivial components of $f$ and $g$ are chosen as compositions $f^0 := h_{s_1}d_{1*}$ and $g^0 := d_{0*}h_{0*}$. Again, both $f$ and $g$ have degree 0 after the degree shifts are applied.

The morphisms $f$ and $g$ are chain maps: the equalities $df = 0$ and $gd = 0$ are either trivial or they follow easily from the chronological relations. The relation $S$ makes both $gf = id$ and $hf = 0$ and it remains to show that $h$ is a chain homotopy between $fg$ and the identity morphism. The only nontrivial case is in the middle, were we have to check the matrix equality

$$
\begin{pmatrix}
    f^0 & 0 \\
    g^0 & I
\end{pmatrix}
- 
\begin{pmatrix}
    id & 0 \\
    0 & id
\end{pmatrix}
= 
\begin{pmatrix}
    h_{s_1}d_{1*} + d_{0*}h_{0*} & h_{s_1}d_{1*} \\
    d_{0*}h_{0*} & 0
\end{pmatrix}.
$$

It follows from definitions of $f^0$ and $g^0$ and the \textit{4Tu} relation:

$$
0 = \begin{array}{c}
0 \\
Z \\
+ \\
Z \\
- \\
X \\
- \\
Y
\end{array}
$$
The coefficient $X$ at the first term appears, because the birth is cancelled with a merge from the left side. The same happens in the last two terms, but in the third one we also have to reverse an orientation of the lower merge. Finally, to modify the second term, we first used chronological relations and then anticommutativity of the lower square in Fig. 7 (erase the caps to see compositions of differentials).

**Remark 7.6.** As before, if we take a cube for a bigger tangle diagram the homotopies $h_0$ and $h_1$ anticommute with edges corresponding to other crossings than the two involved in the second Reidemeister move. This can be shown similarly as in Remark 7.4: the two homotopies are paired with edges that are always a merge or a split, however we close the diagram.

The case of the third move is the simplest one, although it deals with the largest complex. This is because it can be derived from the invariance under the second move, as it is done in the case of the Kauffman bracket. For this, we need one property of mapping cone complexes.

**Definition 7.7.** The mapping cone of a chain map $\psi: C \to D$ is the chain complex $C(\psi)$ defined as

$$C(\psi)^i := C^{i+1} \oplus D^i, \quad d = \begin{pmatrix} -d_C & 0 \\ \psi & d_D \end{pmatrix}$$
Lemma 7.8. The homotopy type of a mapping cone is preserved under compositions with inclusions into strong deformation retracts. More precisely, given a pair of strong deformation retracts

\[ C_a \xleftarrow{g_a} f_a \xrightarrow{D_a} \quad \text{and} \quad C_b \xleftarrow{g_b} f_b \xrightarrow{D_b} \]

and a chain map \( \psi: C_a \to C_b \), the mapping cones \( C(\psi f_a) \) and \( C(f_b \psi) \) are strong deformation retracts of \( C(\psi) \).

**Proof.** Let \( h \) be the homotopy associated to the retract \( D \). The morphisms

\[ \bar{f}_a: C(\psi f_a) \to C(\psi) \quad \bar{f}_a = \begin{pmatrix} f_a & 0 \\ 0 & \text{id} \end{pmatrix}, \]

\[ \bar{g}_a: C(\psi) \to C(\psi f_a) \quad \bar{g}_a = \begin{pmatrix} g_a & 0 \\ -\psi h & \text{id} \end{pmatrix}, \]

\[ \bar{h}: C(\psi) \to C(\psi)[1] \quad \bar{h} = \begin{pmatrix} -h & 0 \\ 0 & 0 \end{pmatrix}, \]

fit into a diagram with commuting squares:

\[ C(\psi f_a) : \quad \cdots \quad D_a^r \oplus C_{b}^{r-1} \xrightarrow{d} D_{a}^{r+1} \oplus C_{b}^{r} \xrightarrow{d} \cdots \]

\[ C(\psi) : \quad \cdots \quad C_{a}^{r} \oplus C_{b}^{r-1} \xleftarrow{d_h} C_{a}^{r+1} \oplus C_{b}^{r} \xrightarrow{d_h} \cdots \]

A quick computation shows \( \bar{h}\bar{f}_a = 0 \), \( \bar{g}_a\bar{f}_a = \text{id} \) and \( \bar{f}_a\bar{g}_a - \text{id} = d\bar{h} + \bar{h}d \), proving \( C(\psi f_a) \) is a strong deformation retract of \( C(\psi) \). The other case is shown in a similar way. \( \square \)

Lemma 7.9. The complexes \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) and \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) are homotopy equivalent and so are \( \text{Kh}(\begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix}) \) and \( \text{Kh}(\begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix}) \) for any orientation of tangles.

**Proof.** Again, the second part follows from the first one, because \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) and \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) have same crossings, regardless of orientation. The complex \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) is a mapping cone of the chain map \( \Psi = \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} : \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \to \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) given by the four vertical morphisms below.

\[ \begin{array}{ccc}
\begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} & \xrightarrow{\Psi} & \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \\
\begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} & \xrightarrow{\Psi} & \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \\
\begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} & \xrightarrow{\Psi} & \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \\
\end{array} \]

Consider the chain map \( f \) from the proof of Lemma 7.5. It is an inclusion into a strong deformation retract and Lemma 7.8 implies \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) is homotopy equivalent to the mapping cone of \( \Psi_L = \Psi \circ f \) given in Fig 8. For the same argument \( \begin{smallmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{smallmatrix} \) is homotopy equivalent to
ΨR. Since tangle diagrams \( \Psi_L \) and \( \Psi_R \) are isotopic, the mapping cone complexes \( C(\Psi_L) \) and \( C(\Psi_R) \) are isomorphic.

**Proof of Theorem 7.1.** It remains to show, that the above local proofs extend to diagrams of bigger tangles. Each case follows the same pattern. Assume there is a chain map \( \psi: \text{Kh}(T_1) \to \text{Kh}(T_2) \) defined for whatever sign assignments were chosen to construct the complexes. Choose a tangle \( T \) and a planar arc diagram \( D \) with two inputs, and construct a corrected cube \( I(D(T, T_1), \epsilon_1) \) using some sign assignment \( \epsilon_1 \). We can collapse it partially to obtain a cube of complexes as in Fig. 9. Namely, a resolution \( T_\xi \) of the tangle \( T \) picks a subcube \( I(D(T_\xi, T_1), \epsilon_1|_\xi) \), which collapses to the complex \( \text{Kh}(D(T_\xi, T_1)) \). Put these complexes in vertices of an \( n \)-dimensional cube, where \( n \) is the number of crossings of \( T \). Since the original cube \( I(D(T, T_1), \epsilon_1) \) anticommutes, the edge morphisms corresponding to changing resolutions of \( T \) induce ‘anti-chain’ maps

\[
 d_\zeta: \text{Kh}(D(T_\xi, T_1)) \to \text{Kh}(D(T_\xi', T_1)),
\]

i.e. morphisms that anticommute with differentials.

We can do the same with the tangle \( T_2 \), obtaining a cube of complexes \( \text{Kh}((D(T_\xi, T_2))) \). Because planar operators with one input are strict 2-functors, \( \text{Kh}(D(T_\xi, T_1)) = D(T_\xi, \text{Kh}(T_1)) \) and there are chain maps

\[
 D(T_\xi, \psi): \text{Kh}(D(T_\xi, T_1)) \to \text{Kh}(D(T_\xi, T_2)),
\]

one for each resolution \( T_\xi \). Hence, we have two cubes of complexes and a morphism between them. Collapsing these cubes (while taking care about homological grading of complexes in vertices\footnote{This can be achieved for instance by shifting a homological degree of a complex \( \text{Kh}(D(T_\xi, T_1)) \) by \(-|\xi|\) and then taking a direct sum of complexes over all vertices.}) results in the complexes \( \text{Kh}(D(T, T_1)) \). If the chain maps \( D(T_\xi, \psi) \) commute with the edge morphisms \( d_\zeta \), they induce a chain map \( \Psi: \text{Kh}(D(T, T_1)) \to \text{Kh}(D(T, T_2)) \).

In particular, if all \( \psi \) are homotopy equivalences, so is \( \Psi \).

There is nothing to do for the second Reidemeister move. If a tangle diagram \( T \) can be reduced to \( T' \) by this move, consider \( I(T') \) as a subcube of \( I(T) \). Remark 7.6 implies that both homotopies \( h_{0*} \) and \( h_{1*} \) from Lemma 7.5 anticommute with edge morphisms from \( I(T') \), so that the morphisms \( f^0 := h_{1*}d_{1*} \) and \( g^0 := d_{0*}h_{0*} \) commute with them.
Figure 9. The cube of complexes for a tangle $T$ induced by a planar diagram $D$ and a tangle $\bowtie$ with 2 crossings. Each arrow is a degree 0 morphism that anticommutes with differentials.

Invariance under the third Reidemeister move follows from the same argument as the one used to prove Lemma 7.9: the chain map $f$ from the previous paragraph is again an inclusion into a strong deformation retract.

The first Reidemeister move is the most challenging one. As before, choose a diagram $T$ that can be reduced to $T'$ by this move, and consider $I(T')$ as a subcube of $I(T)$. The morphisms $f$ and $g$ does not commute with the edge morphisms of $I(T')$: for an edge $\zeta: \xi \to \xi'$ decorated with a morphism $d_\zeta$ we have

\begin{equation}
    d_\zeta g_\xi = \lambda(\deg d_\zeta, \deg(a \text{ birth})) g_\xi d_\zeta
\end{equation}

and similarly for $f$. To fix it, we define a 0-cochain $\eta \in C^0(I^{n-1}; U(R))$ in the following way. Pick any oriented path in $I(T')$ from the origin $(0, \cdots, 0)$ to a vertex $\xi$. It represents some chronological cobordism $W$, whose degree $\deg W$ depends only on $\xi$, but not on the path. Define $\eta(\xi) := \lambda(\deg W, \deg(a \text{ birth}))$. Then

\begin{equation}
    \eta(\xi') g_\xi d_\zeta = \eta(\xi) \lambda(\deg d_\zeta, \deg(a \text{ birth})) g_\xi d_\zeta = \eta(\xi) d_\zeta g_\xi.
\end{equation}

Hence, $\eta g$ commutes with edge morphisms. In a similar way we show that $\eta^{-1} f$ induces a chain map.

8. Basic properties

Directly from its definition the generalized Khovanov bracket satisfies the following properties, similar to the rules of the Kauffman bracket:

(KB1) $\llbracket \emptyset \rrbracket = \emptyset$,
(KB2) $\llbracket T \sqcup T' \rrbracket = \llbracket T \rrbracket \sqcup \llbracket T' \rrbracket$, if $T'$ has no crossings, and
(KB3) $\llbracket \times \rrbracket = C(\llbracket \times \rrbracket; \llbracket \times \rrbracket \to \llbracket \times \rrbracket \{1\}) [-1]$.

In the last property, the symbols $\times$, $\bowtie$ and $\bowtie$ represent three tangle diagrams that are identical except the indicated region and the morphism $\llbracket \times \rrbracket$ is induced by edge maps in the cube $I(\times)$ at which the resolution of the distinguished crossing is changed.

The property (KB3) implies a long exact sequence of generalized Khovanov complexes that mimics the Jones skein relation.

Proposition 8.1. There is an exact sequence of complexes

\begin{equation}
    0 \longrightarrow \text{Kh}(\times \{1\}) \longrightarrow \text{Kh}(\times \{2\}) \longrightarrow \text{Kh}(\times \{-2\}) \longrightarrow \text{Kh}(\times \{-1\}) \longrightarrow 0.
\end{equation}
Proof. The property (KB3) for diagrams $\times$ and $\times$ implies the following sequences are exact:

\begin{align*}
0 & \rightarrow \llbracket \gamma \rrbracket [-1]\{1\} \rightarrow \llbracket \times \rrbracket \rightarrow \llbracket \gamma \rrbracket \rightarrow 0, \\
0 & \rightarrow \llbracket \times \rrbracket [-1]\{1\} \rightarrow \llbracket \times \rrbracket \rightarrow \llbracket \gamma \rrbracket \rightarrow 0.
\end{align*}

Gluing them together results in an exact sequence

\begin{align*}
0 & \rightarrow \llbracket \gamma \rrbracket [-2]\{1\} \rightarrow \llbracket \times \rrbracket [-1] \rightarrow \llbracket \times \rrbracket \{1\} \rightarrow \llbracket \gamma \rrbracket \{1\} \rightarrow 0,
\end{align*}

which is the same as (49) up to grading shifts. □

Next, the generalized Khovanov complex $\text{Kh}(T)$ depends on the orientation of $T$ in a well understood way.

**Proposition 8.2.** Choose an oriented tangle $T$. Denote by $-T$ the same tangle with reversed orientation of all its components and by $T'$ the tangle when only the orientation of a single component $T_0$ is reversed. Then

\begin{align*}
\text{Kh}(-T)^r & \cong \text{Kh}(T)^r, \\
\text{Kh}(T')^r & \cong \text{Kh}(T)^{r-2\ell}\{-6\ell\},
\end{align*}

where $\ell = \text{lk}(T, T_0)$ is the linking number of the component $T_0$ in $T$.

**Proof.** For (53) it is enough to see that after reversing orientation of all components the signs of crossings are the same. If we reverse the orientation only of one component $T_0$, then the crossings of $T_0$ with other components change signs, so that $n_+(T') = n_+(T) - 2\ell$ and $n_-(T') = n_-(T) + 2\ell$. □

Complexes of a tangle $T$ and its mirror image $T^*$ are closely related as well. Namely, the cubes $\mathcal{I}(T^*)$ and $\mathcal{I}(T)$ are `symmetric in time’, which means the former cube looks like the latter, when we proceed from the terminal vertex $(1, \ldots, 1)$ to the initial one $(0, \ldots, 0)$. Formally, this operation comes from a duality 2-functor on $\text{ChCob}^3(k)$ induced by a vertical flip of $D^2 \times I$, i.e. the map $(q, z) \mapsto (q, 1 - z)$. Denote this operation as $(\_)^*$. To make it well-defined, we have to decide how to orient critical points of a cobordism $W^*$.

The operation $(\_)^*$ flips the degree of a cobordism: $\deg W^* = (b, a)$ if $\deg W = (a, b)$. Hence, in the linearized case, the roles of $X$ and $Y$ are exchanged, and similarly for $Z$ and $Z^{-1}$. For instance, permuting two merges costs $Y$. We make it coherent with $\Diamond$-permutations as follows. Color each region in the complement of $W$ black or white, so that regions with same color do not meet. Rotate a framing arrow in $W^*$ clockwise, if the region below it (when in $W$) is white, and anticlockwise otherwise:
Since we want the duality functor to be coherent with annihilations and creations, this convention leaves no choice for births and deaths:

\[
\begin{align*}
\left( \begin{array}{c}
- & 1 \\
1 & -
\end{array} \right)^* &= \\
\left( \begin{array}{c}
- & 2 \\
2 & -
\end{array} \right)^* &= \\
\left( \begin{array}{c}
- & 3 \\
3 & -
\end{array} \right)^* &= \\
\left( \begin{array}{c}
C & w \\
Y & w
\end{array} \right)^* &= Y \quad \left( \begin{array}{c}
C & w \\
Y & w
\end{array} \right)^* = Y \\
\left( \begin{array}{c}
C & w \\
Y & w
\end{array} \right)^* &= Y \\
\end{align*}
\]

The functor \( (\_)^* : R\text{ChCob}^3_{XYZ}(k) \to R\text{ChCob}^3_{YXZ^{-1}}(k) \) defined in such a way is now coherent will all chronological relations, as well as with relations \( S, T \) and \( 4Tu \). We extend it to categories of complexes, by reflecting the homological grading, i.e. we set \((C^*)^i := C^{-i}\) for a complex \( C \) in \( \text{Kom}(R\text{ChCob}^3(k)) \).

**Proposition 8.3.** The generalized Khovanov complexes of a tangle \( T \) and its mirror image \( T^* \) are dual to each other, i.e. \( \text{Kh}_{XYZ}(T^*) \cong \text{Kh}_{YXZ^{-1}}(T)^* \), where \( \text{Kh}_{abc} \) stands for a Khovanov complex constructed in the 2-category \( R\text{ChCob}_{\ell}^3(k) \) with chronology change coefficients \( X, Y \) and \( Z \) set to \( a, b \) and \( c \) respectively.

**Proof.** Choose a diagram of \( T \) with \( n \) enumerated crossings and arrows over them. To obtain a diagram for \( T^* \) replace first each crossing \( \times \) with the opposite one \( \times \). Then rotate the arrows over crossings using the same convention as for \((\_)^*\): color regions black and white and rotate an arrow anticlockwise, when it is placed over white regions, and clockwise otherwise. With this choice of diagrams \( \mathcal{I}(T^*) = \mathcal{I}(T)^* \), which follows directly from the construction of a cube of resolutions. Moreover, a sign assignment \( \epsilon \in C^1(I^n;U(R)) \) for \( \mathcal{I}(T) \) is automatically a sign assignment for \( \mathcal{I}(T)^* \): this is obvious except faces for which \( \psi \) evaluates to \( Z \) or \( Z^{-1} \), where the equality \( \psi^*(S^*) = \psi(S) \) holds, because dualizing a cube reverses its orientation. Therefore, \(([T]_\epsilon)^*[n] = [T^*]_\epsilon \) and the proposition follows. \( \square \)

9. Homology

Although the complex \( \text{Kh}(T) \) is an invariant of the tangle \( T \), it is a difficult problem to determine whether two complexes in \( R\text{ChCob}_{\ell}^3 \) are homotopy equivalent. One can obtain a partial answer, by applying a functor \( \mathcal{F} : R\text{ChCob}_{\ell}^3 \to A \) to some abelian category \( A \). Such a functor extends naturally to categories of complexes \( \mathcal{F} : \text{Kom}(R\text{ChCob}_{\ell}^3) \to \text{Kom}(A) \) and the homology \( H(\mathcal{F}\text{Kh}(T)) \) is an invariant of the tangle \( T \).

For simplicity, we will consider only functors \( \mathcal{F} : R\text{ChCob}_{\ell}^3(0) \to \text{Mod}_R \), producing invariants of links. If we restrict to \( \mathbb{Z} \times \mathbb{Z} \)-graded \( R \)-modules and \( \mathcal{F} \) preserves degrees of morphisms, then homology groups \( H^i(\mathcal{F}\text{Kh}(T)) \) are \( \mathbb{Z} \)-graded (recall, that in Kh we collapse the \( \mathbb{Z} \times \mathbb{Z} \)-grading into the \( \mathbb{Z} \)-grading, by sending \((a,b)\) into \( a+b \)).

**Even Khovanov homology.** Let \( R = \mathbb{Z} \) and put \( X = Y = Z = 1 \). This reduces \( R\text{ChCob}_{\ell}^3 \) to the category of ordinary cobordisms, so that all invariants described in \( [BN05] \) can be computed from \( \text{Kh}(L) \). In particular, we can take a functor \( \mathcal{F}^{cv} \) that sends a family of \( s \) circles in \( \mathbb{D}^2 \) into an \( s \)-folded tensor product\(^{12} \) \( A^{\otimes s} \) of a rank 2 module \( A = \mathbb{Z}v_+ \oplus \mathbb{Z}v_- \),

\(^{12}\) Strictly speaking, one should think of \( A^{\otimes s} \) as an orderless tensor product, which makes sense in any symmetric monoidal category. Otherwise, \( \mathcal{F}^{cv} \) is defined on objects only up to an isomorphism, since it requires ordering of circles. However, there is a canonical isomorphism induced by the symmetric structure, and coherence result for symmetric monoidal categories implies it is unique. The same issue arises in other examples of functors described in this paper.
graded with \( \deg v_+ = (1, 0) \) and \( \deg v_- = (0, -1) \). For cobordisms we define \( \mathcal{F}^{ev} \) as below

\[
\mathcal{F}^{ev} \left( \begin{array}{c}
\end{array} \right) : A \otimes A \to A, \\
\begin{cases}
v_+ \otimes v_+ \mapsto v_+, \\
v_+ \otimes v_- \mapsto v_-, \\
v_- \otimes v_+ \mapsto v_-,
\end{cases}
\]

\[
\mathcal{F}^{ev} \left( \begin{array}{c}
\end{array} \right) : A \to A \otimes A, \\
\begin{cases}
v_+ \mapsto v_- \otimes v_+ + v_+ \otimes v_-, \\
v_- \mapsto v_- \otimes v_-,
\end{cases}
\]

\[
\mathcal{F}^{ev} \left( \begin{array}{c}
\end{array} \right) : R \to A, \\
\begin{cases}
1 \mapsto v_+,
\end{cases}
\]

\[
\mathcal{F}^{ev} \left( \begin{array}{c}
\end{array} \right) : A \to R, \\
\begin{cases}
v_+ \mapsto 0, \\
v_- \mapsto 1.
\end{cases}
\]

The above turns \( A \) into a Frobenius algebra, so that \( \mathcal{F}^{ev} \) is well-defined. Compatibility with the three relations \( S, T \) and \( 4Tu \) is easy to check [BN05]. The resulting homology \( H^{ev}(L) := H(\mathcal{F}^{ev}\text{Kh}(L)) \) is the categorification of the Jones polynomial from [Kh99].

**Odd Khovanov homology.** We choose as before \( R = \mathbb{Z} \) and \( X = \mathbb{Z} = 1 \), but let \( Y = -1 \). This choice provides a framework for the odd Khovanov homology [ORS13]. The functor \( \mathcal{F}^{odd} : R\text{ChCob}^3/\ell(0) \to \text{Mod}_\mathbb{Z} \) associates to a family of \( s \) circles in \( \mathbb{D}^2 \) the exterior algebra \( \Lambda_s := \bigwedge^* \mathbb{Z}\langle a_1, \ldots, a_s \rangle \) with one generator \( a_i \) for each circle. Merging two circles identifies appropriate generators, while a split translates into a map

\[
\Lambda_s/(a_i - a_j) \ni [w] \mapsto (a_i - a_j) \wedge w \in \Lambda_s,
\]

assuming the \( i \)-th circle in the target configuration is to the left of the framing arrow and the \( j \)-th one is to the right. A birth is an inclusion of algebras and an anticlockwise death of an \( i \)-th circle is the Kronecker delta function \( a_j \mapsto \delta_{i,j} \) wedged with identity, i.e. it strips off \( a_i \) from the element \( w \) from the left hand side, if it is present, or sends \( w \) to 0 otherwise.

One can directly check that \( \mathcal{F}^{odd} \) defined in this way is a strict 2-functor. It is shown in [ORS13] that \( H^{odd}(L) := H(\mathcal{F}^{odd}\text{Kh}(L)) \) is an invariant of a link \( L \). The group \( \Lambda_s \) is graded with an element \( a_i \wedge \cdots \wedge a_{i_r} \) in degree \( s - 2r \), which makes \( \mathcal{F}^{odd} \) a degree-preserving functor. Both a sphere and a torus evaluate to zero (\( a_i - a_j \) becomes 0 after merging \( i \)-th and \( j \)-th circles) and \( 4Tu \) follows from the table below.

|      | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} | \begin{tabular}{c} \end{tabular} |
|------|---|---|---|---|---|---|
| 1    | 0 | 0 | 0 | 0 |
| \( a_1 \) | 0 | 1 | 1 | 0 |
| \( a_2 \) | 1 | 0 | 1 | 0 |
| \( a_1 \wedge a_2 \) | \(-a_1\) | \( a_2 \) | 0 | \( a_2 - a_1 \) |

Therefore, invariance of \( \mathcal{F}^{odd}\text{Kh}(L) \) also follows from Theorem 7.1.

**10. Chronological Frobenius algebras**

A natural target for a functor \( \mathcal{F} \) is a 2-category of symmetric bimodules over a ring \( S \), graded by some abelian group \( G \), with 2-morphisms given by multiplication with elements
of \( S \): the existence of a 2-morphism \( s: f \Rightarrow g \) implies \( s \cdot f = g \). This 2-category has an interesting symmetric Gray monoidal structure given by a graded tensor product. It is defined for modules in a usual way, but for homogeneous morphisms we put
\[
(f \otimes g)(m \otimes n) := \lambda_G(\deg g, \deg m)f(m) \otimes g(n),
\]
for some function \( \lambda_G: G \times G \to U_0(S) \), where \( G \) is the grading group and \( U_0(S) \) is the group of invertible elements of \( S \) in degree 0. With this definition, we have
\[
\lambda_G(\deg f, \deg g')(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g),
\]
so the tensor product is not a strict 2-functor. The following proposition follows directly from the definition of a cubical functor.

**Proposition 10.1.** The graded tensor product \( (f, g) \mapsto f \otimes g \) is a cubical functor if and only if \( \lambda_G: G \times G \to U_0(S) \) is a normalized cocycle, i.e. for any \( g, h, k \in G \) one has
\[
\lambda_G(g, h)\lambda_G(g, h + k) = \lambda_G(h, k)\lambda_G(g + h, k)
\]
and \( \lambda_G(g, h) = 1 \) if either \( g = 0 \) and \( h = 0 \).

**Example 10.2.** The 2-category \( \text{Mod}_S \) of \( \mathbb{Z} \times \mathbb{Z} \)-graded modules over a commutative ring \( S = \mathbb{Z}[X, Y, Z \pm 1]/(X^2 = Y^2 = 1) \) is monoidal in the above sense with \( \lambda \) defined as in (13).

There is a nice graphical interpretation of the formulas (60) and (61). As usual, we represent a homomorphism \( f: M \to N \) by a box labeled \( f \) with two legs: one at the bottom, labeled with \( M \), and one at the top, labeled with \( N \). Composition of morphism is given by placing the boxes one over the other and a tensor product of homomorphisms by placing them side by side, the left on the higher level than the right one. Then we have the following relation for homogeneous morphisms \( f \) and \( g \):
\[
(f \circ g)(m \otimes n) = \lambda_G(\deg f, \deg g)n \otimes m.
\]
For example, (60) follows from the following simple calculation, where we represent an element \( m \in M \) of a module \( M \) by a box with no input (think of it as a homomorphisms \( S \to M \) taking 1 to \( m \)):
If the ring $S$ is $G$-graded, we must be careful with what we mean by commutativity, linearity, etc. We provide below definitions of some of these basic properties.

1. A $G$-graded ring $S$ is **commutative** if $ab = \lambda_G(\deg a, \deg b)ba$ for all homogeneous elements $a, b \in S$.
2. A $G$-graded bimodule $M$ over a commutative ring $S$ is **symmetric** if one has $am = \lambda_G(\deg a, \deg m)ma$ for homogeneous elements $a \in S$ and $m \in M$.
3. A homogeneous function $f: M \to N$ between $G$-graded bimodules is **right linear** if $f(ma) = f(m)a$, but **left linear** if $f(am) = \lambda_G(\deg f, \deg a)af(m)$ for a homogeneous element $a \in S$.

If we think of linearity as a commutativity of a map $f$ with the action of $S$, then the last definition follows easily from the graphical calculus (notice that the actions of $S$ are degree 0 maps):

\[
\begin{align*}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (m) at (0,-1) {$m$};
  \node (f) at (1.5,0) {$f$};
  \node (ah) at (2.5,0) {};\node (bh) at (2.5,-1) {};

  \draw[->] (a) -- (f); \draw[->] (m) -- (f);
  \draw[->] (f) -- (ah); \draw[->] (f) -- (bh);
\end{tikzpicture}
\end{align*}
\]

(67)

With these conventions we can define a tensor product of $G$-graded bimodules $M \otimes_S N$ in the usual way, with actions of $S$ given as $a(m \otimes n) := (am) \otimes n$ and $(m \otimes n)a := m \otimes (na)$.

By an analogy to ordinary cobordisms, a functor $\mathcal{F}: R\text{ChCob}^3(0) \to \text{Mod}_S$ is determined by the pair $(\mathcal{F}(\emptyset), \mathcal{F}(\bigcirc))$, a variant of a Frobenius system over $R$.

**Definition 10.3.** Choose an abelian group $G$, a commutative $G$-graded ring $R$ and a category of $G$-graded symmetric bimodules $\text{Mod}_R$ with a graded tensor product (60) and a symmetry (65) defined by some normalized cocycle $\lambda_G: G \times G \to U_0(R)$. A **chronological Frobenius system** in this category is a pair $(S, A)$ consisting of a graded ring $S$ with a degree 0 ring homomorphism $R \to S$ and a symmetric $S$-bimodule $A$ together with four operations:

- a unit $\eta: S \to A$,
- a counit $\epsilon: A \to S$,
- a multiplication $\mu: A \otimes_S A \to A$, and
- a comultiplication $\Delta: A \to A \otimes_S A$,

with homogeneous decompositions $\mu = \sum_{g \in G} \mu_g$ and $\Delta = \sum_{g \in G} \Delta_g$, $\deg \mu_g = \deg \Delta_g = g$, subject to the following conditions:

\[
\begin{align*}
(\lambda_G(\deg g, h)\mu_g \circ (\text{id} \otimes \mu_h)), \\
(\Delta_g \otimes \text{id}) \circ \Delta_h &= \lambda_G(g, h)(\text{id} \otimes \Delta_g) \circ \Delta_h, \\
\mu \circ (\eta \otimes \text{id}) &= \text{id}, \\
(\epsilon \otimes \text{id}) \circ \Delta &= \text{id}, \\
\mu_g \circ c &= \lambda_G(g, g)\mu_g, \\
&\quad c \circ \Delta_g = \lambda_G(g, g)\Delta_g, \\
(\mu_h \otimes \text{id}) \circ (\text{id} \otimes \Delta_g) &= \lambda_G(g, h)\Delta_g \circ \mu_h = (\text{id} \otimes \mu_h) \circ (\Delta_g \otimes \text{id}).
\end{align*}
\]

We call $A$ a **chronological Frobenius algebra** over $S$. We say the algebra $A$ is **homogeneous**, if its operations are homogeneous (but they may have nontrivial degrees).
The conditions for a chronological Frobenius algebra reflect the chronological relations: (68), (69) and (72) reflect the permutation changes with connected singular sets, (70) mimics the creation and the annihilation changes, and (71) is the orientation reversion.

**Proposition 10.4.** Choose a homogeneous chronological Frobenius system \((S, A)\) in the category of \(\mathbb{Z} \times \mathbb{Z}\)-graded modules \(\text{Mod}_R\) with the cocycle \(\lambda\) defined as in (13). Then there is a unique functor \(\mathcal{F}_A: R\text{ChCob}^3(0) \to \text{Mod}_S\) that sends a family of circles in \(\mathbb{D}^2\) to a tensor product\(^{13}\) \(A^{\otimes n}\) and

\[
\mathcal{F}_A \left( \begin{array}{c}
\vspace{10pt}
\end{array} \right) = \left( \mu: A \otimes A \to A \right), \quad \mathcal{F}_A \left( \begin{array}{c}
\vspace{10pt}
\end{array} \right) = \left( \eta: S \to A \right),
\]

\[
\mathcal{F}_A \left( \begin{array}{c}
\vspace{10pt}
\end{array} \right) = \left( \Delta: A \to A \otimes A \right), \quad \mathcal{F}_A \left( \begin{array}{c}
\vspace{10pt}
\end{array} \right) = \left( \epsilon: A \to S \right).
\]

**Proof.** We have to check that \(\mathcal{F}_A\) preserves relations induced by changes of chronologies. Most of them follows from (61) and conditions (68) – (72), with the exception of \(\times\)- and \(\diamond\)-changes. The former follows from (71), as an \(\times\)-change adds a twist on one side of the cobordism. In the latter both cobordisms are equivalent, so it is enough to show that \(XY - 1\) annihilates \(\mu \circ \Delta\). This follows from (71):

\[
\mu \circ \Delta = XY(\mu \circ c) \circ (c \circ \Delta) = XY\mu \circ c^2 \circ \Delta = XY\mu \circ \Delta.
\]

\[\square\]

**Remark 10.5.** The above proposition can be extended to \(G\)-graded algebras. For that, we need a group homomorphism \(\psi: \mathbb{Z} \times \mathbb{Z} \to G\) such that \(\deg(i(r)) = \psi(\deg r)\) for \(r \in R\), where \(i: R \to S\) is the homomorphism of rings from Definition 10.3. These should agree with the tensor product structure, i.e. \(\lambda_G(\psi(a, b), \psi(c, d)) = i(\lambda(a, b, c, d))\). A homogeneous chronological Frobenius algebra \((A, \mu, \Delta, \eta, \epsilon)\) in \(\text{Mod}_S\) induces a functor \(\mathcal{F}_A\), if its operations have degrees \(\psi(-1, 0), \psi(0, -1), \psi(1, 0)\) and \(\psi(0, 1)\) respectively.

**Example 10.6** (Covering Khovanov homology). Let \(R = \mathbb{Z}[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)\) be a ring of truncated polynomials and take \(A = Rv_+ \oplus Rv_-\). As before, we grade \(A\) by setting \(\deg v_+ = (1, 0)\) and \(\deg v_- = (0, -1)\). We equip it with the following operations

\[
\mu: A \otimes A \to A,
\]

\[
\Delta: A \to A \otimes A,
\]

\[
\eta: R \to A,
\]

\[
\epsilon: A \to R.
\]

One can directly check that conditions (68) – (72) hold. The induced functor \(\mathcal{F}_{\text{cov}}\) clearly satisfies the sphere relation and a direct calculation shows that a standard torus evaluates to \(Z(X + Y)\). Finally, the \(4Tu\) relation follows from the table below.

\[\text{As explained in the footnote on page 32 we should think of it as an orderless tensor product. Otherwise } \mathcal{F}_A \text{ is defined on objects only up to a canonical isomorphism.}\]
Therefore, this algebra defines a functor $F: R\text{ChCob}^3 \to \text{Mod}_R$. We call the invariant $H^{\text{cov}}(L) := H(\mathcal{F}^{\text{cov}}\text{Kh}(L))$ the covering Khovanov homology of the link $L$.

A $R$-module structure on $Z$ is fully determined by actions of formal variables $X, Y, Z \in R$. If all three act by identities, we will write $Z^{\text{ev}}$, and if $Y$ acts by minus identity, we will write $Z^{\text{odd}}$. The following proposition explains the name covering homology.

**Proposition 10.7.** For any link $L$ there are isomorphisms

\[(80) \quad H^{\text{ev}}(L) \cong H^{\text{cov}}(L; Z^{\text{ev}}) \quad \text{and} \quad H^{\text{odd}}(L) \cong H^{\text{cov}}(L; Z^{\text{odd}}), \]

where $H^{\text{cov}}(L; M) := H(\mathcal{F}^{\text{cov}}\text{Kh}(L) \otimes M)$ for any $R$-module $M$.

**Proof.** The first isomorphism follows directly from the construction: replacing $X, Y$ and $Z$ with $1$’s in the definition of the algebra $A$ results in the Khovanov algebra. For the second one it is enough to show that functors $\mathcal{F}^{\text{cov}}(\_ \otimes Z^{\text{odd}})$ and $\mathcal{F}^{\text{odd}}$ are equivalent. This follows from applying an isomorphism $i: S_{gr}(A) \otimes Z^{\text{odd}} \to \Lambda_s$ that sends any $v_+$ into $1$ and $v_-$, corresponding to the circle labeled with $i$, into $a_i$. Comparing the two definitions, one can easily see that $\mathcal{F}^{\text{odd}}(M) = i \circ (\mathcal{F}^{\text{cov}}(M) \otimes Z^{\text{odd}}) \circ i^{-1}$ for any generating cobordism $M$. \hfill $\square$

The above proposition is an example of a more general operation called a base change: given a chronological Frobenius system $(S, A)$ in $\text{Mod}_R$ and a symmetric $R$-module $S'$, which is also a ring, together with a degree zero ring homomorphism $S \to S'$ that agree with the $R$-module structures, the pair $(S', A')$ with $A' := A \otimes_S S'$ is another chronological Frobenius system, called a base change of $(S, A)$. Clearly, $H(\mathcal{F}_{A'}) \cong H(\mathcal{F}_A; S')$.

**Example 10.8.** One of the consequences of the $4Tu$ relation is the following equality

\[(81) \quad Z(X + Y) \begin{array}{c} \includegraphics[width=1cm]{tu_relation} \end{array} = \begin{array}{c} \includegraphics[width=1cm]{handle0} + \includegraphics[width=1cm]{handle1} \end{array}, \]

called a neck-cutting relation. Assume $Z = 1$. If $X + Y = 0$, we can use this relation to move handles freely between components of a cobordism (up to multiplication with $X$). A similar theory over the two-element field $\mathbb{F}_2$ was analyzed in [BN05], suggesting we have found its lift to $Z$ in the odd setting. Namely, we have an algebra $A_H := \text{Mor}(\bigcirc, \bigcirc)$ over the ring $R_H := \mathbb{Z}[H, X]/(2H, X^2 - 1)$, where $H$ has degree $(-1, -1)$ and represents a handle. Unfortunately, this is a torsion element, as it is annihilated by $1 - XY = 1 + X^2 = 2$. One can check that $A_H$ is free module generated by $v_+$ and $v_-$ in degrees $(1, 0)$ and $(0, -1)$.
respectively, with multiplication and comultiplication given by formulas

\begin{align}
(82) \quad \mu: A_H \otimes A_H & \to A_H, \\
& \begin{cases}
v_+ \otimes v_+ \mapsto v_+, & v_- \otimes v_+ \mapsto Xv_-, \\
v_+ \otimes v_- \mapsto v_-, & v_- \otimes v_- \mapsto Hv_-
\end{cases}
\end{align}

\begin{align}
(83) \quad \Delta: A_H & \to A_H \otimes A_H, \\
& \begin{cases}
v_+ \mapsto v_- \otimes v_+ - Xv_+ \otimes v_- + Hv_+ \otimes v_-, \\
v_- \mapsto v_- \otimes v_-
\end{cases}
\end{align}

The generator $v_+$ is represented by a death followed by a birth and $v_-$ by a vertical cylinder. In tensor products, each $v_+$ is represented by a birth and all other circles are boundaries of a single component built from splits only (or a single death, if there is no $v_-$). See [BN05] for details.

A more general algebra is described in the next section. We will now use two chronological Frobenius algebras to analyze 2-morphisms in $RChCob^3(0)$. Both are obtained from the universal algebra $A_U$, defined at the end of the next section, by certain base changes.

**Proof of Theorem 4.12.** Recall that $R = \mathbb{Z}[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)$ is the ring of truncated polynomials. Given a chronological cobordism $W$ we want to compute the group $\text{Aut}(W)$ of all 2-morphisms $r: W \Rightarrow W$, i.e. elements $r \in \mathfrak{k}$ such that $(1 - r)W = 0$.\(^\text{14}\)

We will first show that $\text{Aut}(W)$ is a subgroup of $\{1, XY\}$. For that, take a graded commutative ring $R_1 = \mathbb{R}[c, f, t]/(X - Y, cft - 1)$ with $\deg c = \deg f = (1, 1)$ and $\deg t = (-2, -2)$, and consider a chronological Frobenius algebra $A_1 = R_1 v_+ \oplus R_1 v_-$ with the following operations:

\begin{align}
(84) \quad \mu: A_1 \otimes A_1 & \to A_1, \\
& \begin{cases}
v_+ \otimes v_+ \mapsto v_+, & v_- \otimes v_+ \mapsto XZv_-, \\
v_+ \otimes v_- \mapsto v_-, & v_- \otimes v_- \mapsto tv_+
\end{cases}
\end{align}

\begin{align}
(85) \quad \Delta: A_1 & \to A_1 \otimes A_1, \\
& \begin{cases}
v_+ \mapsto ftv_+ \otimes v_+ + Z^2fv_- \otimes v_-, \\
v_- \mapsto ftv_+ \otimes v_- + XZ^{-1}ftv_- \otimes v_+
\end{cases}
\end{align}

\begin{align}
(86) \quad \eta: R_1 & \to A_1, \\
& 1 \mapsto v_+
\end{align}

\begin{align}
(87) \quad \epsilon: A_1 & \to R_1, \\
& v_+ \mapsto XZc, \\
v_- \mapsto 0.
\end{align}

Notice, that all $c$, $f$, and $t$ are invertible and polynomials in $Z$ are not zero divisors. It follows $\mathcal{F}_1(W)$ is not a zero divisor for a closed surfaces $W$, since $\mu(\Delta(v_+)) = (1 + Z^2)ftv_+$.\(^\text{15}\) This shows $\mathcal{F}_1(\text{Aut}(W)) = \{1\}$ for a closed cobordism $W$, so that $\text{Aut}(W)$ is a subgroup of $\{1, XY\}$. If $\partial W \neq \emptyset$, create a closed surface $\hat{W}$ by capping its boundary components with births and deaths. Then every 2-morphism $r: W \Rightarrow W$ extends to $\hat{W}$ (by juxtaposition with the identity 2-morphims on components of $\hat{W} - W$), which forces $\mathcal{F}_1(r) = 1$.

Now assume $W$ has no handles and at most one spherical component. Choose $R_2 := R[c, t]/((c^2, (XY - 1)t)$ with $\deg c = (1, 1)$ and $\deg t = (-2, -2)$. Consider a chronological

\(^\text{14}\) All such 2-morphism are invertible, as they represent changes of a chronology on $W$.

\(^\text{15}\) $\mathcal{F}_1(W)$ is a power of $(1 + Z^2)ft$, possibly scaled by an invertible monomial in $X$ and $Z$. 
Frobenius algebra $A_2 = R_2v_+ \oplus R_2v_-$ with the following operations:

\[
\begin{align*}
\mu &: A_2 \otimes A_2 \to A_2, \\
&\begin{cases}
v_+ \otimes v_+ \mapsto v_+ + XZv_- , \\
v_+ \otimes v_- \mapsto v_- , \\
v_- \otimes v_+ \mapsto v_- , \\
v_- \otimes v_- \mapsto Z^2 v_+ ,
\end{cases}
\end{align*}
\]

\[
\Delta &: A_2 \to A_2 \otimes A_2, \\
&\begin{cases}
v_+ \mapsto ctv_+ \otimes v_+ + Zv_+ \otimes v_+ + c v_+ \otimes v_- , \\
v_- \mapsto tv_+ \otimes v_+ + ctv_+ \otimes v_+ + YZ^{-1} v_- \otimes v_+ + Z^2 v_- \otimes v_- ,
\end{cases}
\]

\[
\eta &: R_2 \to A_2, \\
&\begin{cases}
1 \mapsto v_+ ,
\end{cases}
\]

\[
\epsilon &: A_2 \to R_2, \\
&\begin{cases}
v_+ \mapsto 1 ,
\end{cases}
\]

In particular, a sphere is mapped by the induced functor $F_2$ into $-c$. Create $\hat{W}$ by capping some inputs and outputs of $W$, so that each component has exactly one boundary circle, except a unique spherical component, if it exists. As before, it is enough to show that $\hat{W}$ has only trivial 2-endomorphisms. This follows easily: $\hat{W}$ induces a homomorphism $F_2(\hat{W}) : A \otimes k \to A \otimes \ell$ that takes $(v_-) \otimes k$ into $(v_+) \otimes \ell$ or $-c(v_+) \otimes \ell$, possibly multiplied by a monomial in $X$, $Y$ and $Z$. This ends the proof, as none of $r \in k$ annihilates $c$.  

11. Dotted cobordisms

A very generic example of a chronological Frobenius algebra is given by the tautological functor $\text{Mor}(\Sigma, -)$, where $\Sigma$ is any object of $R\text{ChCob}^3(0)$.

**Proposition 11.1.** Given an object $\Sigma \in R\text{ChCob}^3(0)$, the group of morphisms $\text{Mor}(\Sigma, \emptyset)$ is a ring with multiplication induced by the right disjoint sum and $\text{Mor}(\Sigma, \emptyset)$ is a chronological Frobenius algebra over $\text{Mor}(\Sigma, \emptyset)$.

The case $\Sigma = \emptyset$ was analyzed in Example 10.8, with the assumption $X + Y = 0$. Here, $\text{Mor}(\emptyset, \emptyset)$ was a free rank 2 module over the ring $R_H \cong \text{Mor}(\emptyset, \emptyset)$, but in general, the rank of $\text{Mor}(\Sigma, \emptyset)$ over $\text{Mor}(\Sigma, \emptyset)$ is infinite. The neck-cutting relation (81) motivates the following construction.

**Definition 11.2.** The 2-category $R\text{ChCob}^3$ consists of chronological cobordism marked with dots on regular levels. A single dot has a degree $(-1, -1)$ and two dots cannot lie on the same level. In addition to chronological relations we impose the following three:

\[
\begin{align*}
(S) \quad &\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = 0 ,
\end{align*}
\]

\[
\begin{align*}
(D) \quad &\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = 1 ,
\end{align*}
\]

\[
\begin{align*}
(N) \quad &\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{---}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} .
\end{align*}
\]

where all deaths are oriented anticlockwise.

Dots are part of the chronological structure and one can think of them as ‘infinitesimal’ handles, which are ‘frozen’, so that a dot is not annihilated by $1 - XY$. But a cobordism with two dots on one component is, because permuting two dots costs $XY$. All relations are homogeneous, hence coherent with changes of chronologies. Even more: the neck cutting
relation \( N \) together with the cubical structure of the disjoint sum determines all coefficients for changes of chronologies, except the \( \diamond \)-change. For example,

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
+ Z^2
+ \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
- \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= X
+ XZ^2
- X
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= X
+ XZ^2
- X
= X
\end{align*}
\]

where we moved dots in the middle pictures from the birth to the top by the cost of \( Z^2 \). Dotted cobordisms satisfy also the other relations from \( \mathbb{R} \text{ChCob}^3_{/\ell} \).

Lemma 11.3. Relations \( T \) and \( 4Tu \) follow from \( S \), \( D \) and \( N \). Therefore, there is a natural functor \( \mathbb{R} \text{ChCob}^3_{/\ell} \to \mathbb{R} \text{ChCob}^3 \).

Proof. For the \( T \) relation take a standard torus and cut its handle. In the resulting expression, one term has a sphere as its component and the other two can be reduced to dotted spheres by changing chronologies:

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
- \begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
= (XZ + YZ)
\end{align*}
\]

The \( 4Tu \) relation is proved in a similar way, by cutting the unique tube in each term. Again, by changing chronologies we can reduce each term to four caps, with left caps smaller than the right ones, possibly with a two-dotted sphere in the middle:

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= X
+ Y
- XYZ
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline=-2pt]
\draw (0,0) circle (0.5cm);
\draw (0,1) circle (0.5cm);
\filldraw (0,0) circle (0.1cm);
\filldraw (0,1) circle (0.1cm);
\end{tikzpicture}
&= Z
+ Z
- Z
\end{align*}
\]
Because a two-dotted sphere is annihilated by \((XY - 1)\), the sum of right hand sides of (93) and (94) is equal to the sum of right hand sides of (95) and (96).

\[X = X + Z - XYZ,\]

\[Y = Y + Z - XYZ.\]

The additive closure \(\text{Mat}(RCh\text{Cob}^{3}(0))\) is equivalent to a category of finitely generated free graded symmetric bimodules over a certain ring. This follows from the proposition below.

**Proposition 11.4 (Delooping).** The following two morphisms

\[\emptyset \{ -1 \} \to \emptyset \{ -1 \} \oplus \emptyset \{ +1 \} \to \emptyset \{ +1 \},\]

form a pair of inverse isomorphisms in the additive closure \(\text{Mat}(RCh\text{Cob}^{3}).\)

**Proof.** Call the left map \(f\) and the right one \(g\). The equality \(g \circ f = \text{id}\) is exactly the neck-cutting relation \(N\). The other composition is the identity \(2 \times 2\) matrix, which follows directly from relations \(D\) and \(S\).

**Corollary 11.5.** The tautological functor \(\text{Mor}(\emptyset, \_): RCh\text{Cob}^{3}(0) \to \text{Mod}_{R'}\) is full and faithful, where \(R' := \text{Mor}(\emptyset, \emptyset)\). Hence, we can identify \(RCh\text{Cob}^{3}(0)\) with the category of finitely generated free graded symmetric \(\text{Mor}(\emptyset, \emptyset)\)-bimodules.

We will now compute a presentation of the ring \(\text{Mor}(\emptyset, \emptyset)\).

**Proposition 11.6.** There is an isomorphism of graded commutative rings

\[\text{Mor}(\emptyset, \emptyset) \cong R_{\bullet} := R[h, t]/((XY - 1)t, (XY - 1)h),\]

where \(\deg h = (-1, -1)\) and \(\deg t = (-2, -2)\), such that

\[\emptyset \to h \quad \text{and} \quad \emptyset \to XZt + h^2.\]

**Proof.** It is enough to show that the above defines a homomorphism, because if it exists, it is clearly invertible. For that, we start with constructing a graded monoidal functor \(F_{\bullet}: RCh\text{Cob}^{3}(0) \to \text{Mod}_{R_{\bullet}}\). Take a free rank two symmetric bimodule \(A_{\bullet} = R_{\bullet}v_{+} \oplus R_{\bullet}v_{-}\) with \(\deg v_{+} = (1, 0)\) and \(\deg v_{-} = (0, -1)\) as usual. This module is a chronological Frobenius
algebra with operations

\begin{align*}
\mu &: A_\bullet \otimes A_\bullet \to A_\bullet, & \begin{cases} v_+ \otimes v_+ \mapsto v_+ , \\
v_- \otimes v_- \mapsto XZv_- ,
\end{cases} \\
\Delta &: A_\bullet \to A_\bullet \otimes A_\bullet, & \begin{cases} v_+ \mapsto v_- \otimes v_+ + YZv_+ \otimes v_- - YZ^{-1}hv_+ \otimes v_+ , \\
v_- \mapsto v_- \otimes v_- + Z^{-2}tv_+ \otimes v_+ ,
\end{cases} \\
\eta &: R_\bullet \to A_\bullet, & \begin{cases} 1 \mapsto v_+ ,
\end{cases} \\
\epsilon &: A_\bullet \to R_\bullet, & \begin{cases} v_+ \mapsto 0 , \\
v_- \mapsto 1
\end{cases}
\end{align*}

These tell us how to define \( F_\bullet \) on all generators except one, a cylinder decorated with a dot. Associate to it the following homomorphism:

\begin{align*}
\theta &: A_\bullet \to A_\bullet, & \begin{cases} v_+ \mapsto v_- , \\
v_- \mapsto XZ^{-1}(tv_+ + hv_-) = v_+txZ + v_-h .
\end{cases}
\end{align*}

Clearly, \( \epsilon \circ \eta = 0 \) and \( \epsilon \circ \theta \circ \eta = 1 \), so that \( F_\bullet \) preserves relations \( S \) and \( T \). It remains to show that \( F_\bullet \) is also coherent with the neck-cutting relation \( N \). This follows from computing the terms on the right hand side of \( N \):

\begin{align*}
&\begin{tikzpicture}
  \node (a) at (0,0) {\square};
  \node (b) at (0,-1) {\circ};
  \node (c) at (0,-2) {\circ};
\end{tikzpicture} \\
&\begin{tikzpicture}
  \node (a) at (0,0) {\square};
  \node (b) at (0,-1) {\square};
\end{tikzpicture} \\
&\begin{tikzpicture}
  \node (a) at (0,0) {\square};
  \node (b) at (0,-1) {\circle};
\end{tikzpicture}
\end{align*}

Summing the first two and subtracting the last homomorphism results in the identity on \( A_\bullet \). The functor \( F_\bullet \) induces a homomorphism \( \varphi : \text{Mor}(\emptyset, \emptyset) \to R_\bullet \) by associating an element from the ring to any closed surface with dots. In particular, we compute

\begin{align*}
\varphi \left( \begin{tikzpicture}
  \node (a) at (0,0) {\square};
  \node (b) at (0,-1) {\square};
\end{tikzpicture} \right) = h \quad \text{and} \quad \varphi \left( \begin{tikzpicture}
  \node (a) at (0,0) {\square};
  \node (b) at (0,-1) {\circle};
\end{tikzpicture} \right) = XZt + h^2,
\end{align*}

which is the desired homomorphism. \( \square \)

**Remark 11.7.** Similarly to the even case, dotted cobordisms lead us to a deformation of odd theory, although both \( t \) and \( h \) are torsion elements: \( 2t = 2h = 0 \) if \( XY = -1 \). In particular, we cannot set \( t = 1 \) to obtain Lee deformation, unless we work with \( \mathbb{Z}_2 \) coefficients.

The homology theory defined by the algebra \( A_\bullet \) is universal: it carries the most information among all chronological Frobenius algebras producing link homology. The proof follows the argument from \([Kh04]\) and it is based on the following observation.

Given a chronological Frobenius algebra \( A \) over a ring \( R \) and an invertible element \( y \in A \) in degree \((1,0)\), we can twist the coalgebra by \( y \):

\begin{align*}
\epsilon'(a) &:= \epsilon(ya), \\
\Delta'(a) &:= \Delta(y^{-1}a).
\end{align*}
If $\Delta$ and $\epsilon$ are homogeneous, so are their twisted version $\Delta'$ and $\epsilon'$. The degrees are not changed. Because $\deg y = -\deg \mu$, there is an equality $\Delta(y^{-1}a) = y^{-1}\Delta(a)$:

\[
\begin{array}{c}
\Delta \\
\mu \\
y^{-1}
\end{array}
\begin{array}{c}
= Z^{-1}
\end{array}
\begin{array}{c}
\mu \\
\Delta \\
y^{-1}
\end{array}
\begin{array}{c}
\Delta
\end{array}
\]

Lemma 11.8 (cf. [Kh04]). Assume that $F$ and $F'$ are two functors induced by an algebra $A$ and its twisted version $A'$. Then the complexes $F\text{Kh}(L)$ and $F'\text{Kh}(L')$ are isomorphic.

Proof. Consider cubes $FI(L, \epsilon)$ and $F'I(L, \epsilon)$, both corrected by a sign assignment $\epsilon$. They have the same $R$-modules in vertices and the only difference is in edges labeled with comultiplications. The isomorphism is constructed inductively, starting with the identity homomorphism on the initial vertex $(0, \ldots, 0)$ and applying the following rule at every face:

\[
\begin{array}{c}
FI \xrightarrow{f} F'I \\
\mu \\
FI' \xleftarrow{f} F'I'
\end{array}
\]

where in the case of a split we multiply by $y^{-1}$ the element from the copy of $A$ corresponding to the circle that appears to the left of the split. □

Choose a $\mathbb{Z} \times \mathbb{Z}$-graded commutative ring

\[
R_U := R[a, c, e, f, t, h]/\left( (XY - 1)h, (XY - 1)t, af - ce, \right.
\]

where $\deg a = \deg e = (0, 0)$, $\deg c = \deg f = (1, 1)$, $\deg h = (-1, -1)$ and $\deg t = (-2, -2)$. The element $XY - 1$ annihilates not only polynomials in $h$ and $t$, but also $c^2$ and $f^2$. Consider a rank two chronological Frobenius agebra $A_U$ over $R_U$ with the following operations:

\[
\begin{align*}
\mu(v_+ \otimes v_+) &= v_+, & \mu(v_- \otimes v_-) &= XZv_-,
\mu(v_+ \otimes v_-) &= v_-, & \mu(v_- \otimes v_+) &= hv_- + tv_+,
\end{align*}
\]

\[
\begin{align*}
\Delta(v_+) &= (ft - YZ^{-1}eh)v_+ \otimes v_+ + e(v_- \otimes v_+ + YZv_+ \otimes v_-) + Z^2f v_- \otimes v_-,
\Delta(v_-) &= Z^{-2}ev_+ \otimes v_+ + ft(YZ^{-1}v_- \otimes v_+ + v_+ \otimes v_-) + (e + fh)v_- \otimes v_-,
\end{align*}
\]

\[
\begin{align*}
\eta(1) &= v_+,
\epsilon(v_+) &= a,
\epsilon(v_-) &= -c.
\end{align*}
\]
It is a graded version of the system \((R_4, A_4)\) in [Kh04]. If \((R', A')\) is another rank two homogeneous chronological Frobenius system in \(\text{Mod}_R\), there is a unique degree zero ring homomorphism \(R_U \rightarrow R'\) such that \(A' \cong A \otimes_{R_U} R'\). This is proved in the same way as Proposition 4 in [Kh04].

An element \(y = ev_+ + YZf v_- \in A_U\) is invertible and in degree \((1, 0)\), with an inverse \(y^{-1} = (a + ch)v_+ - YZcv_-\). Twisting by it results in the dotted algebra \(A_\bullet\), which proves the following universality property of dotted cobordisms.

**Theorem 11.9.** Any homogeneous rank two chronological Frobenius system \((R, A)\) is obtained from \((R_\bullet, A_\bullet)\) by a base change and a twist. In particular, \(H(\mathcal{F}_\bullet \text{Kh}(L))\) is the most general link homology theory in our framework.

### 12. Odds and ends

**Tangle cobordisms.** Let \(\text{Cob}^4(k)\) be the category of tangles with \(2k\) endpoints and tangle cobordisms between them, i.e. surfaces \(W \subset \mathbb{D}^3 \times I\) with its boundary decomposing into the input and the output tangles \(T_i \subset \mathbb{D}^3 \times \{i\}, i = 0, 1\), and vertical lines on \(\partial \mathbb{D}^3 \times I\). In particular, cobordisms between empty links are 2-knots.

There is a presentation of \(\text{Cob}^4(k)\) due to Carter and Saito [CS98] using *movies*: sequences of sections of \(W\) cutting the cobordism into simple pieces, each with at most one singularity. There are nine singularities, corresponding to nine generators: the three Reidemeister moves (each represents two generators), a saddle move, a birth, and a death (see Fig. 10). They are subject to a number of relations, called *movie moves*, that represent isotopic cobordisms, see [CS98].

The even Khovanov homology \(H^{ev}(L)\) was proven to be functorial up to sign [Ja04, Kh02, BN05], and corrected later to a functor [CMW09, Bla10]. This means there is a chain map \(\text{Kh}(W) : \text{Kh}(L_0) \rightarrow \text{Kh}(L_1)\) for any surface \(W \subset \mathbb{R}^3 \times I\) with \(L_0\) and \(L_1\) as its boundary, \(L_i \subset \mathbb{R}^3 \times \{i\}\).

Functoriality up to ‘sign’ of the generalized Khovanov complex \(\text{Kh}(\_\_\_\_\_\_)\), where by a ‘sign’ we mean any degree 0 invertible element of the ring \(R\), follows easily from [BN05]. Indeed, the two categories \(\mathbb{P}\text{Kom}(R\text{ChCob}^3\mathcal{I}(k))\) and \(\mathbb{P}\text{Kom}(\text{Cob}^3\mathcal{I}(k))\) are equal, where the letter \(\mathbb{P}\) stands for *projectivisation*: it identifies morphisms that are related, up to homotopy, by an invertible element of \(R\). Hence, it is enough to find chain maps in \(\text{Kom}(R\text{ChCob}^3\mathcal{I}(k))\) representing \(\text{Kh}^{ev}(W)\). This is done as below.

- To Reidemeister moves \(\text{Kh}(\_\_\_\_\_\_)\) assigns one of the homotopy equivalences from Theorem 7.1.
- If \(W\) is a saddle move, take the chain map \([\times] : [\sim] \rightarrow [\times\times]\{1\}\) obtained from a cube for the tangle \(\times\).
- Births and deaths are sent to chain maps induced by births and anticlockwise deaths respectively. Namely, consider a morphism of anticommutative cubes \(b : \mathcal{I}(T, \epsilon) \rightarrow \mathcal{I}(T \cup O, \epsilon)\) with each component \(b_\xi\) given by a birth. They do not commute with edge morphisms of the cubes, but we can fix it by the same argument we used in the proof of Theorem 7.1: scale \(b_\xi\) by \(\lambda((1, 0), \deg W)\), where \(W \subset \mathbb{D}^2 \times I\) is a cobordism given by any path from the initial vertex \((0, \ldots, 0)\) to \(\xi\). In a similar way we define the chain map for a death.
Corollary 12.1. The above defines a functor $\text{Kh}: \text{Cob}^4(k) \to \mathbb{P}\text{Kom}(R\text{ChCob}^3_R(k))$ that assigns to a tangle $T$ the generalized Khovanov complex $\text{Kh}(T)$ and to a tangle cobordism $W$ a chain map $\text{Kh}(W): \text{Kh}(T) \to \text{Kh}(T^\prime)$, defined up to an invertible scalar.

It is not obvious how functoriality should be understood for odd homology. For instance, consider a cobordism $W: \bigcirc \bigcirc \Rightarrow \emptyset$ from the two component unlink to an empty diagram given by two deaths. Depending on how we decompose $W$ into simple pieces (i.e. which link component vanishes first), we obtain two chain maps that differ by $Y$. One can try to show $\text{Kh}$ is a weak 2-functor, where movie moves are 2-morphisms in $\text{Cob}^4(k)$. However, this approach requires to understand higher singularities of embedded cobordisms.

$\Diamond$-change revisited. The choice we used to assign a coefficient for a $\Diamond$-change is not the only one. We might as well assign 1 to the diagram with the outer arrow pointing to the left and $XY$ for the other case, and $\iota$ would still be consisted with all relations between elementary changes of chronologies. This does not change homology, however, due to the following argument pointed out by C. Seed.

Take a link $L$ and rotate it by 180 degrees. Its new diagram looks almost like the old one, producing the same cube of resolutions, with one exception: arrows in $\Diamond$-faces have different relative orientations, see Fig. 11. Yet, the rotation can be realized by a finite number of Reidemeister moves, showing the two complexes are isomorphic.

Corollary 12.2. Setting $\iota\left(\begin{array}{c}\bigcirc \\
\bigcirc \end{array}\right) = XY$ and $\iota\left(\begin{array}{c}\bigcirc \\
\bigcirc \end{array}\right) = 1$ does not change the homotopy class of the Khovanov complex $\text{Kh}(T)$.

Proof. Color regions of $\mathbb{D}^2 - T$ black and white as we did in Proposition 8.3. Reversing arrows lying over black regions, while keeping a sign assignment fixed, results in changing the values of $\iota$ on $\Diamond$-changes. We can now rotate the diagram by 180 degrees to recover its original decoration. \qed
In fact, the only condition for \( \iota \) to be coherent with relations between changes of chronologies is that the quotient of its values on the two \( \Diamond \)-changes is equal to \( XY \). Hence we can set

\[
\iota \left( \begin{array}{c}
\begin{array}{c}
\scriptstyle \downarrow \\
\scriptstyle C
\end{array}
\end{array}\right) = \alpha X \quad \iota \left( \begin{array}{c}
\begin{array}{c}
\scriptstyle \uparrow \\
\scriptstyle C
\end{array}
\end{array}\right) = \alpha Y
\]

for some invertible element \( \alpha \in R \). This new parameter is useless from the point of view of Frobenius algebras: it will give only an additional restriction, that \( \alpha X - 1 \) and \( \alpha Y - 1 \) annihilate \( \mu \circ \Delta \). However, it may be used to produce odd versions of nested homology theories (the two cobordisms related by a \( \Diamond \)-change are diffeomorphic, but not isotopic), see [SW10, BW10].

**Rotating arrows and \( \mathfrak{sl}(2) \) foams.** In the original construction of odd Khovanov homology, the small arrow over a crossing frames not only the negative eigenspace \( E^-(p) \) of a saddle point \( p \), but also its positive eigenspace \( E^+(p) \) and the latter is used to distinguish between the two output circles of a split. Because of the convention that every arrow rotates clockwise when going up, one framing arrow is enough.

If we allow an arrow to rotate in any direction, i.e. when we orient both \( E^-(p) \) and \( E^+(p) \) independently, we will create a richer category with two versions of each generating cobordism. It is not difficult to find out chronological relations: the coefficients assigned to changes do not depend on how the arrows rotate, except \( \times \)- and \( \Diamond \)-changes, in which cases the coefficients are multiplied by \( Y \), if the arrows rotate in different directions, see Tab. 4.

**Remark 12.3.** This is not the most general solution. For instance, one can assign different coefficients to changes permutating merges that are differently oriented. In the most general case one obtains a system of nine independent parameters.

A choice of how a single arrow rotates introduces another data to the construction of the generalized Khovanov complex. The isomorphism class of the complex does not depend on this additional chain, which follows from the commutativity of the following square:
Table 4. Coefficients assigned to $\times$- and $\Diamond$-permutation, when each arrow can rotate either clockwise or anticlockwise. The symbols $\triangleleft c$ and $\triangle$ stand for two alternative ways of rotating an arrow and one has to make the same choice (left/top or right/bottom) for both arrows.

where the left vertical cobordism is an isomorphism in $R\text{ChCob}^3$ and its inverse is given by the same picture, but with different orientations of critical points:

and similarly for the other composition. The vertical morphisms are homogeneous in degree 0, which implies they commute with all other edge morphisms in the cubes. Hence, (118) induces an isomorphism between complexes obtained from two diagrams of a tangle, that differ only in the way a single arrow rotates.

The author was encouraged to investigate rotations of arrows by M. Hempel, who computed several circular movies for the odd theory and noticed, that if arrows over crossings with opposite signs rotate differently, movies consisting of Reidemeister II moves induce identity chain maps. This suggests a connection with $\mathfrak{sl}(2)$ foams, i.e. singular cobordisms with two types of saddle points, one for positive and one for negative crossings.

Appendix A. Framed functions

Definition A.1. An Igusa function is a smooth function $f: W \to \mathbb{R}$, such that at every point $p \in W$ one of the following conditions holds:

IF1: $p$ is regular, i.e. the derivative $df_p$ does not vanish, or

IF2: $f$ has a Morse singularity (or $A_1$ singularity) at $p$, i.e. $df_p = 0$ but the Hessian $\text{Hess}_p(f)$ is nondegenerate, or

IF3: $f$ has a birth-death singularity (or $A_2$ singularity) at $p$, i.e. $df_p = 0$ and $\text{Hess}_p(f)$ has a 1-dimensional kernel $N(p) \subset T_pW$, but $d^3f_p$ is nonzero on $N(p)$. 
In the last two situations \( f \) has one of the following local models:

\[
\begin{align*}
(120) & \quad f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots x_n^2, \\
(121) & \quad f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots x_{n-1}^2 + x_n^3
\end{align*}
\]

and in the latter case the nullspace \( N(p) \) of \( \text{Hess}_p(f) \) is spanned by \( \frac{\partial}{\partial x_n} \). The number \( k = \mu(p) \) is called the index of \( p \).

Igusa functions arise naturally if one considers homotopies between smooth functions: a generic function on \( W \) is Morse (conditions IF1 and IF2) and separative (critical points lie on different levels), but a space of such functions is not even connected. However, a transversality argument implies a generic homotopy \( f_t \) is separative Morse except finitely many moments \( 0 < t_1 < \cdots < t_k < 1 \), at which either two critical points are permuted or a birth-death singularity occurs [Ce68]. We can visualize these critical moments, by drawing the singular locus \( S(f) := \{ (f_t(x), t) \mid x \in \text{crit}(f_t) \} \). In the situations above one obtains the graphs presented below.

\[
\begin{align*}
(122) & \quad \begin{cases}
\text{a permutation} & \\
\text{a creation} & \\
\text{an annihilation}
\end{cases}
\end{align*}
\]

Cusps represent \( A_2 \)-singularities and labels indicate indices of critical points.

Consider now homotopies as paths in the space of Igusa functions and pick a generic homotopy \( f_{t,s} \) between two such generic paths. Using the transversality argument again we can show that for a fixed \( s \) the path \( t \mapsto f_{t,s} \) is generic, except perhaps finitely many values of \( s \), and these exceptional situations can be organized into three groups.

Case I Two events can occur at the same time \( t_i \). For example, we have homotopies

\[
\begin{align*}
(123) & \quad \begin{cases}
\text{Case I} & \\
\text{Case II} & \\
\text{Case III} &
\end{cases}
\end{align*}
\]

where dashed lines indicate singular values of \( t \). See also Fig. 12 for singular locuses of the left two homotopies.
Case II  A non-transverse event occurs, i.e. the singular set is not transverse to some level set \( \{ t = a \} \). Up to direction of the change, there are three such homotopies

\[
\begin{array}{c}
\xrightarrow{\text{Case II}} \\
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\text{Case II}} \\
\end{array}
\]

and their singular locuses are shown in Fig. 13.

Case III  Either three Morse singularities or an \( A_2 \)-singularity and a Morse one meet at the same critical level. There are three types of such homotopies

\[
\begin{array}{c}
\xrightarrow{\text{Case III}} \\
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\text{Case III}} \\
\end{array}
\]

with singular locuses of two of them visualized in Fig. 14 (the case of an annihilation is symmetric to the one of a creation).

The space of Igusa functions is not simply connected, which is manifested by the lack of the dove tail singularity in the list above. Indeed, this singularity is modeled by a biquadratic
polynomial and as such it cannot appear. We can fix this issue by adding a framing\textsuperscript{16} In fact, the space of framed functions is contractible [Ig87, Lu09, EM11], but we will not use this fact in this paper. The following definition comes from [EM11].

First, given a Riemannian metric on $W$ and a critical point $p \in W$ of an Igusa function $f : W \to I$, we can see its Hessian as a linear map $\text{Hess}_p(f) : T_pW \to T_pW$. Denote by $E^-(p)$ and $E^+(p)$ its negative and positive eigenspaces respectively. The first has dimension equal to the index $\mu(p)$ of the point $p$.

**Definition A.2.** Let $f : W \to \mathbb{R}$ be an Igusa function. A framing on $f$ is a choice of a Riemannian metric on $W$ and an orthonormal frame $v_1, \ldots, v_{\mu(p)}$ of $E^-(p)$ at every critical point $p$. If $p$ is an $A_2$ singularity, we add an extra vector $v_{\mu(p)+1} \in N(p)$ in the positive direction of $d^3\tau$.

The topology on the space of framed functions $\text{Fun}^f(W)$ was described indirectly in [Ig87] by constructing a simplicial complex homotopy equivalent to this space. Here we only remind how homotopies look like, following [EM11].

Choose a smooth function $f : W \times I^m \to I$ such that each slice $f_B : W \to I$ for $B \in I^m$ is an Igusa function. Denote by $V \subset W \times I^m$ the set of critical points of all slice functions $f_B$ and let $\Sigma$ be the subset of all $A_2$ points. Generically, $V$ is an $m$-dimensional submanifold of $W \times I^m$ and $\Sigma$ has codimension 1 in $V$. Furthermore, $V$ is transverse to each slice $W \times \{B\}$ at the set $V - \Sigma$. Let $V - \Sigma = V^0 \cup \cdots \cup V^n$ and $\Sigma = \Sigma^0 \cup \cdots \cup \Sigma^{n-1}$ be decompositions of $V - \Sigma$ and $\Sigma$ with respect to the index. Then

- $\Sigma^r$ is the intersection of closures of $V^r$ and $V^{r+1}$, and
- for $z = (p, \underline{t}) \in V^r$ one has $T_pW = E^-(z) \oplus E^+(z)$, and
- for $z = (p, \underline{t}) \in \Sigma^r$ one has $T_pW = E^-(z) \oplus N(z) \oplus E^+(z)$,

where $E^\pm(z)$ is the positive of negative eigenspace of $\text{Hess}_p(f_B)$ and $N(z)$ is its nullspace. It follows that for $z_0 \in \Sigma^r$ and $z \in V^r$

\begin{equation}
\lim_{z \to z_0} E^+(z) = N(z_0) \oplus E^+(z_0) \quad \text{and} \quad \lim_{z \to z_0} E^-(z) = E^-(z_0),
\end{equation}

whereas for $z_0 \in \Sigma^r$ and $z \in V^{r+1}$

\begin{equation}
\lim_{z \to z_0} E^-(z) = E^-(z_0) \oplus N(z_0) \quad \text{and} \quad \lim_{z \to z_0} E^+(z) = E^+(z_0).
\end{equation}

\textsuperscript{16} Framed functions were introduced to overcome the problem of lost information, when replacing a manifold with a Morse function: although a Morse function decomposes $W$ into cells, one cannot build $W$ back, unless a parametrization of each cell is given. This is the additional information a framing provides [Ig87].
Figure 15. A cancelation of framed $A_1$ points.

A framing on $f: W \times I^m \to I$ forms a collection of sections $(v_1, \ldots, v_n)$, where each $v_r$ is defined only over the union $\Sigma^{r-1} \cup V^r \cup \cdots \cup V^{n-1} \cup V^n$, such that $v_r(z) \in N(z)$ for $z \in \Sigma^{r-1}$ and at $z \in V^r \cup \Sigma^r$ the vectors $v_1(z), \ldots, v_r(z)$ form an orthonormal frame of $E^-(z)$. In particular, when we approach a birth-death singularity, framings of canceling points agree with the framing of the limiting point, see Fig. 15. For more details see [EM11].

**Theorem A.3** (cf. [EM11, Lu09]). For any compact manifold $W$, the space of framed Igusa functions $\text{Fun}^{fr}(W)$ is contractible.

There is a natural action of $SO(k)$ on the set of all framings of a critical point of index $k$. The quotient by this action, one per each critical point, results in a much smaller space of functions, which is still simply connected.

**Definition A.4.** An orientation of an Igusa function is a choice of an orientation of the negative eigenspace $E^-(p)$ at every critical point $p$. The space of oriented Igusa functions on $W$ will be denoted by $\text{Fun}^{or}(W)$.

**Theorem A.5.** $\text{Fun}^{or}(W)$ is simply connected for any compact manifold $W$.

**Proof.** Consider the canonical projection $\pi: \text{Fun}^{fr}(W) \to \text{Fun}^{or}(W)$. It is easy to see that it has connected fibers (a product of $SO(k)$’s). Hence, if we can show it has a path-lifting property, then any loop $\gamma$ can be lifted to a loop up to reparametrization (lift $\gamma$ as a path and connect its endpoints in a fiber). Then a contracting homotopy upstairs descends to a contracting homotopy of $\gamma$.

Pick a path $\gamma: [0,1] \to \text{Fun}^{or}(W)$. The compactness of $[0,1]$ implies the existence of a sequence $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\gamma|_{[t_{i-1},t_i]}$ looks like one of the homotopies listed in (122). Since $\pi$ has connected fibers, it is enough to lift each of the three homotopies.

- If $\gamma$ has only Morse singularities, for each critical point of $\gamma(0)$ choose any framing with a given orientation and transport it along the path.
- If $\gamma$ has a birth singularity of index $k$ at $p$, pick any framing at this point agreeing with its orientation. Then transport it along the path of points with index $k$ and for the path of index $k+1$ add to the framing the additional vector coming from the nullspace $N(p)$.
- For a death singularity do the same but with the time reversed.

Hence, every path in $\text{Fun}^{or}(W)$ lifts to $\text{Fun}^{fr}(W)$.

**Remark A.6.** The group $SO(k)$ is not simply connected, what gives a choice for a path connecting endpoints of the lift. In particular, $\pi_2(\text{Fun}^{or}(W))$ may be nontrivial. This is not a problem for us, as we never go beyond $\pi_1(\text{Fun}^{or}(W))$ in this paper.
Appendix B. 2-categories

This section provides basic definitions from the theory of 2-categories [Be67, Gr74] and monoidal structures on them [BaNe95, KV94]. The shortest way to define a 2-category is to say that it is a category enriched over Cat. This means the following:

- for every two objects $A$ and $B$ there is a category of morphisms $\text{Mor}(A, B)$ (morphism of this category are called 2-morphisms),
- the composition is given by functors $\circ_{A,B,C}: \text{Mor}(B, C) \times \text{Mor}(A, B) \to \text{Mor}(A, C)$,
- identity morphisms are picked by functors $\mathbb{I}_A: * \to \text{Mor}(A, A)$, where * stands for a category with a single object * and a single morphism id$_*$,
- associativity and unitarity axioms are replaced with invertible 2-morphisms $\rho_f: f \circ \text{id}_A \Rightarrow f$, $\lambda_f: \text{id}_B \circ f \Rightarrow f$ and $\alpha_{f,g,h}: f \circ (g \circ h) \Rightarrow (f \circ g) \circ h$ for any $f \in \text{Mor}(A, B)$, $g \in \text{Mor}(B, C)$ and $h \in \text{Mor}(C, D)$, satisfying MacLane’s coherence conditions [ML98].

A 2-category is strict, if all $\alpha$, $\rho$ and $\lambda$ are identities. Otherwise, it is weak.

If we represent objects by points on a plane and 1-morphisms by oriented edges, then 2-morphisms decorate regions. With this interpretation, a picture of a typical 2-morphisms looks as follows:

\begin{equation}
\begin{array}{c}
A \xrightarrow{f} B \\
\alpha \downarrow \quad g
\end{array}
\end{equation}

Example B.1. The category of small categories $\text{Cat}$ is a strict 2-category. Indeed, the set of functors $\text{Fun}(C, D)$ is in fact a category, with natural transformations as morphisms of functors. This 2-category is strict: all natural isomorphisms $\alpha$, $\rho$ and $\lambda$ are equalities.

There are two ways of composing 2-morphisms: a vertical composition, induced by the internal composition in morphism categories $\text{Mor}(A, B)$

\begin{equation}
\begin{array}{c}
A \xrightarrow{f} B \\
\alpha \downarrow \quad g
\end{array} = \begin{array}{c}
A \xrightarrow{f} B \\
\beta \downarrow \quad g
\end{array}
\end{equation}

and a horizontal composition, given by the composition functors $\circ_{A,B,C}$

\begin{equation}
\begin{array}{c}
A \xrightarrow{\alpha} B \\
\beta \downarrow \quad g
\end{array} = \begin{array}{c}
A \xrightarrow{\beta \circ \alpha} B \\
\beta \downarrow \quad g
\end{array}
\end{equation}
Moreover, the two ways of composing 2-morphisms are compatible, which means that the diagram below

\[(131)\]

produces the same 2-morphism no matter whether we first compose the 2-morphisms vertically or horizontally. In other words,

\[(132)\]

This property, called the interchange law, together with the obvious associativity and unitarity axioms, is another way how to define a 2-category [Be67].

**Example B.2.** Chronological cobordisms form a strict 2-category:
- objects are smooth oriented manifolds,
- morphisms are cobordisms with chronologies,
- 2-morphisms are homotopy classes of changes of chronologies.

The vertical composition of 2-morphisms is given by concatenation of homotopies, whereas the horizontal one by juxtaposition. A routine check shows both operations are compatible, i.e. the interchange law holds.

The higher structure of 2-categories affects a notion of a functor: we no longer assume that it preserves identities nor compositions of morphisms. Instead, both properties should hold up to some 2-morphisms, which are part of a data, subject to some coherence relations.\(^{17}\)

**Definition B.3.** A functor \(F: C \to D\) between 2-categories consists of a function of objects \(F_0: \text{Ob}C \to \text{Ob}D\), a collection of functors \(F_{A,B}: \text{Mor}(A,B) \to \text{Mor}(FA,FB)\) and 2-morphisms \(\iota_A: \text{id}_{FA} \Rightarrow F(\text{id}_A)\) and \(\varphi_{f,g}: F(f) \circ F(g) \Rightarrow F(f \circ g)\) satisfying some coherence relations. A functor \(F\) is strict, if both 2-morphisms are equalities.

A famous result states that every 2-category can be strictified: every 2-category is equivalent to some strict 2-category. Hence, we do not have to care about weak 2-categories. On the other hand, this does not apply to functors: there are functors between strict 2-categories that cannot be replaced by strict ones. However, most functors used in this paper will be strict, with the only exception of cubical functors [GPS95].

**Definition B.4.** A functor \(F: C_1 \times \cdots \times C_r \to D\) between strict 2-categories\(^ {18}\) is cubical, if

1. \(F(\text{id}_{A_1}, \ldots, \text{id}_{A_r}) = \text{id}_{F(A_1,\ldots,A_r)}\), and
2. \(F(f_1 \circ g_1, \ldots, f_r \circ g_r) = F(f_1, g_1) \circ \cdots \circ F(f_r, g_r)\) if there is \(k\) such that \(g_i = \text{id}\) and \(f_j = \text{id}\) for all \(i < k < j\).

In other words, \(\iota\) is the identity 2-morphism and so is \(\varphi\), unless we have to ‘permute’ nontrivial morphisms \(f\) and \(g'\).

\(^{17}\) See [Be67] for details. The most general definition does not even assume invertibility of \(\iota\) and \(\varphi\), but we will never need such functors.

\(^{18}\) There is also a more general notion of a cubical functor between weak 2-categories.
In the case of a cubical functor, the coherence relations mentioned in Definition B.3 reduce to two commuting diagrams of 2-morphisms

\[
\begin{align*}
F(f) \circ F(g) &\xrightarrow{F(\alpha) \circ F(\beta)} F(f') \circ F(g') \\
F(f \circ g) &\xrightarrow{F(\alpha \circ \beta)} F(f' \circ g')
\end{align*}
\]

(133)

\[
\begin{align*}
F(f) \circ F(g) \circ F(h) &\xrightarrow{\varphi \circ \text{id}} F(f \circ g) \circ F(h) \\
F(f) \circ F(g \circ h) &\xrightarrow{\varphi} F(f \circ g \circ h)
\end{align*}
\]

(134)

where we used a shortcut notation \( f = (f_1, \ldots, f_r) \) for morphisms in a product of 2-categories and similarly for 2-morphisms. The latter condition has the following interpretation when \( r = 2 \): whenever we have three pairs of morphisms, passing from a composition of values of \( F \) on them to the value of \( F \) on their composition requires two ‘transpositions’ of ‘inner’ arguments and it can be done in two different ways. The condition (134) says, it does not matter which way we choose.

**Example B.5.** The right disjoint sum is a cubical functor \( \text{ChCob} \times \text{ChCob} \to \text{ChCob} \), whereas the left one is cocubical (i.e. \( \varphi \) in Definition B.4 is identity if for some \( k \) we have \( f_j = \text{id} \) and \( g_i = \text{id} \) for \( i > k > j \)).

Now we are ready to define monoidal 2-categories.

**Definition B.6.** A Gray monoidal structure on a strict 2-category \( C \) consists of a cubical functor \( \otimes: C \times C \to C \) and a unit object \( I \in C \) such that both \( I \otimes (\_\_\_) \) and \( (\_\_) \otimes I \) are identity 2-functors and the functor \( \otimes \) is associative, i.e. the following square commutes

\[
\begin{array}{ccc}
C \times C \times C & \xrightarrow{id \times \otimes} & C \times C \\
\otimes \times \text{id} & & \otimes \\
C \times C & \xrightarrow{\otimes} & C
\end{array}
\]

(135)

A Gray monoidal structure is the analogue of the strict monoidal one in the world of ordinary categories: there is a more general definition of a (weak) monoidal 2-category, but each such category is equivalent (in a monoidal sense) to a Gray-monoidal one [GPS95]. Because of that, it is sometimes called a semi-strict monoidal 2-category [BaNe95, La05].

It is much harder to describe braiding in a monoidal 2-category: writing down all coherence conditions takes usually a few pages [BaNe95, KV94]. Since we will never use this notion in such generality, we provide here a very simplified version, with all 2-morphisms being identities. That is why we call this a strict braiding.
Definition B.7. A strict braiding in a Gray monoidal category \((\mathcal{C}, \otimes, I)\) is a collection of isomorphisms \(\sigma_{A,B}: A \otimes B \to B \otimes A\) such that each \(\sigma_{A,-}\) and \(\sigma_{-,B}\) is a natural transformation and the triangle below commutes

\[
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{id \otimes \sigma_{B,C}} & A \otimes C \otimes B \\
\sigma_{A \otimes B,C} & \downarrow & \sigma_{A,C \otimes B} \\
C \otimes A \otimes B & \xrightarrow{\sigma_{A,B,C}} & \end{array}
\]

for any object \(C\). If in addition \(\sigma_{A,B} \circ \sigma_{B,A} = \text{id}\), we call \(\sigma\) a strict symmetry.

A natural transformation \(\eta: F \to G\) in a 2-categorical setting means a little more than a commutativity of squares. Indeed, it should be coherent with 2-morphisms, what can be translated as the following equality of 2-morphisms

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(\alpha)} & F(B) \\
\eta_X & \downarrow & \eta_Y \\
G(A) & \xrightarrow{G(\alpha)} & G(B) \\
\end{array} = \begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\eta_X & \downarrow & \eta_Y \\
G(A) & \xrightarrow{G(f)} & G(B) \\
\end{array}
\]

for any 2-morphism \(\alpha: f \Rightarrow f'\).

Example B.8. The 2-category \(\text{ChCob}\) of chronological cobordisms is a strictly symmetric Gray monoidal category, with a product given by the right disjoint sum, the empty manifold \(\emptyset\) as a unit object and a permutation cylinder as a symmetry. On the other hand, the 2-category \(\text{ChCob}^3(\emptyset)\) of cobordisms embedded in \(\mathbb{D}^2 \times I\) is strictly braided.

References

[BaNe95] J.C. Baez, M. Neuchl, Higher dimensional algebra I: braided monoidal 2-categories, Advances in Mathematics 121:196–244, 1996. E-print: arXiv:q-alg/9511013.

[BN05] D. Bar-Natan, Khovanov homology for tangles and cobordisms, Geom. Topol. 9:1443–1499, 2005. E-print: arXiv:math/0410495.

[BN07] D. Bar-Natan, Fast Khovanov homology computations, J. Knot Theory Ramifications 16:243–256, 2007. E-print: arXiv:math/0606318.

[BW10] A. Beliakova, E. Wagner, On link homology theories from extended cobordisms, Quantum topology 4(1):379–398, 2009. E-print: arXiv:0910.5050.

[Be67] J. Bénabou, Introduction to bicategories, Lecture Notes in Mathematics Vol. 47. Springer-Verlag, Berlin-New York, 1967.

[Bla10] C. Blanchet, An oriented model for Khovanov homology, J. Knot Theory Ramifications 19:291–312, 2010.

[Blo10] J. M. Bloom, Odd Khovanov homology is mutation invariant, Math. Res. Lett. 17(1):1–10, 2010. E-print: arXiv:0903.3746.

[Ca09] C. Caprau, The universal \(sl(2)\) cohomology via webs and foams, Topology and its Applications 156:1684–1702, 2009. E-print: arXiv:0802.2848.
[Na07] G. Naot, The universal $sl_2$ link homology theory, PhD Thesis, Toronto, 2007. E-print: arXiv:0706.3680.

[ORS13] P. Ozsvath, J. Rasmussen, Z. Szabo, Odd Khovanov homology, 2007 Alg. Geom. Top. 13:1465–1488, 2013. E-print: arXiv:0710.4300.

[Pu08] K.K. Putyra, Cobordisms with chronologies and a generalized Khovanov complex, Masters’ Thesis, Jagiellonian University, 2008. E-print: arXiv:1004.0889.

[Ra04] J. Rasmussen, Khovanov homology and the slice genus, Inventiones mathematicae, 182(2):419–447, 2010. E-print: arXiv:math/0402131.

[Sh11] A. Shumakovitch, Patterns in odd Khovanov homology, J. Knot Theory Ramifications, 20:203–222, 2011. E-print: arXiv:1101.5607.

[SW10] C. Stroppel, B. Webster, 2-block Springer fibers: convolution algebras and coherent sheaves, Commentarii Mathematici Helvetici 2010. E-print: arXiv:0802.1943.

[We10] B. Webster, Knot invariants and higher representation theory, 2010. Preprint: arXiv:1309.3796.

[Wh10] S. M. Wehrli, Mutation invariance of Khovanov homology over $F_2$, Quantum Topology, 1(2):111–128, 2010. E-print: arXiv:0904.3401.

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