On the notions of energy tensors in tetrad-affine gravity

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Abstract

We are concerned with the precise modalities by which mathematical constructions related to energy-tensors can be adapted to a tetrad-affine setting. We show that, for fairly general gauge field theories formulated in that setting, two notions of energy tensor—the canonical tensor and the stress-energy tensor—exactly coincide with no need for tweaking. Moreover we show how both notions of energy-tensor can be naturally extended to include the gravitational field itself, represented by a couple constituted by the tetrad and a spinor connection. Then we examine the on-shell divergences of these tensors in relation to the issue of local energy-conservation in the presence of torsion.

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Introduction

In Lagrangian field theory \[1, 2, 3, 4, 5, 6\] one has a precise mathematical construction yielding a ‘canonical energy-tensor’ associated with each field sector. Such tensors are related to conservation laws by generalizations of the classical Noether theorem, which constitutes the basis for physical interpretation. When the considered field theory is formulated over a curved Lorentzian background then one has the further notion of ‘stress-energy tensor’ \[4, 7, 8, 9\], whose relation with the canonical energy-tensor is known as the Belinfante-Rosenfeld formula \[10, 11, 2\]. In various concretely interesting cases the two said notions yield tensors which turn out to be different just by a numerical coefficient and, possibly, by a needed symmetrization.

In particular, the notion of energy tensor for the gravitational field has been variously debated in the literature \[12, 13, 14, 15, 16, 17\]. Recent results \[18\] suggest that that role should be played by the Ricci tensor. On the other hand, precise covariant constructions \[19\] show that the Ricci tensor is to be seen as the canonical tensor of the gravitational field.

In this paper we are interested in applying the general formalism of Lagrangian field theory in the context where a gauge theory is coupled with tetrad-affine gravity—indeed we regard that as the most natural and convenient setting. This also yields a canonical tensor for the gravitational field that, again, turns out to be essentially the Ricci tensor. It should be stressed, however, that we do not aim at a detailed discussion of the possible physical interpretations of the ensuing mathematical notions,\(^1\) which are introduced just as natural extensions of usual notions.

Tetrad gravity \[20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\] has been introduced and studied mainly as a convenient ‘non-holonomic coordinate’ formalism, but it is interesting to note that the tetrad \(\theta\) acquires a neat geometric meaning if it is viewed as an isomorphism between the tangent bundle \(T M\) of the spacetime manifold \(M\) and a further vector bundle \(H\) over \(M\) whose fibers are endowed with a Lorentz metric \(g\)—i.e. an \(\text{SO}(1,3)\)-bundle. Moreover such \(H\) is naturally generated by the spinor bundle needed for the description of Dirac fields, so that it does not actually constitute an \(ad \ hoc\) unphysical assumption; this result is specially well expressed in the context of 2-spinor geometry \[31, 32, 33, 34, 35, 36, 37\]. Now \(\theta\) transforms \(g\) into a spacetime metric; moreover a metric connection \(\Gamma\) of \(H\) is transformed by \(\theta\) into a metric spacetime connection. Thus the couple \((\theta, \Gamma)\) can be regarded as representing the gravitational field, according to what we may call the ‘tetrad-affine representation’. Note that the spacetime structures, in this view, are derived, non-fundamental quantities. Though the spacetime metric also determines the Levi-Civita (symmetric) connection, the spacetime connection corresponding to \(\Gamma\) has non-zero torsion, which turns out to interact with spin...
fields. Torsion is then unavoidable, but not a fundamental field, since it can be essentially expressed as the covariant differential \( \nabla \) of \( \theta \) with respect to \( \Gamma \). Furthermore, we observe that \( \Gamma \) can be essentially regarded as the spinor connection, as shown by eq. 7.

In general, field theory topics can be most rigorously addressed in the context of a formulation exploiting jet bundle geometry [42, 1, 43, 3, 5, 44, 6]. In this presentation, however, we will skip some technicalities of that kind, limiting ourselves to plain coordinate expressions, even though a mathematically exigent reader might regard some statements as not sufficiently justified.

1 Tetraddrive gravity

If \((e_a)\) is an orthonormal frame of \( H \) then the tetradd can be expressed as \( \theta = \theta^\lambda_a dx^a \otimes e_\lambda \), where the components \( \theta^\lambda_a \) have the physical dimension of a length. We will use shorthand

\[
|\theta| \equiv \det \theta = \frac{1}{16} \varepsilon^{abcd} \varepsilon_{\lambda \mu \nu \rho} \theta^\lambda_a \theta^\mu_b \theta^\nu_c \theta^\rho_d ,
\]

\[
\tilde{\theta}_\lambda = \partial |\theta| / \partial \theta^\lambda_a = \frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda \mu \nu \rho} \theta^\mu_b \theta^\nu_c \theta^\rho_d ,
\]

\[
\tilde{\theta}^a_\lambda = \partial |\theta| / \partial \theta^a_\lambda = \frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda \mu \nu \rho} \theta^\mu_a \theta^\nu_d \theta^\rho_c ,
\]

\[
\tilde{\theta}^a_\lambda = \partial |\theta| / \partial \theta^a_\lambda = \frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda \mu \nu \rho} \theta^\mu_a \theta^\nu_d \theta^\rho_c .
\]

We observe that the above quantities are well-defined also if \( \theta \) is degenerate; if \( \theta \) is invertible then \( (\theta^{-1})^a_\lambda = \tilde{\theta}^a_\lambda / |\theta| \).

We denote the components of the metric and of a connection of \( H \) by \( g_{ab} \) and \( \Gamma^\lambda_a \) respectively, and the induced spacetime quantities by

\[
g_{ab} \equiv \theta^a_\lambda \theta^b_\lambda g_{\lambda \mu} \quad , \quad \Gamma^c_a \equiv \theta^c_\lambda ( - \partial_a \theta^\lambda_b + \Gamma^\lambda_a \theta^\mu_b ) , \quad (1)
\]

where \( (\theta^{-1})^a_\lambda = \theta^a_\lambda \equiv g^{ab} g_{\lambda \mu} \theta^\mu_b \). Then \( \theta \) can be regarded as a ‘square root of the metric’, and we also get \( |\theta| \equiv \det \theta = \sqrt{\det g} \). The condition that the tetradd be covariantly constant characterizes a connection of the spacetime manifold which turns out to be metric, but does not coincide with the standard spacetime connection since it is not symmetric (for the connection coefficients we use the sign convention yielding \( \nabla_a dx^c = \Gamma^c_a (dx^b) \)). The torsion is expressed as

\[
T^c_a = \Gamma^c_a - \Gamma^c_b ( \partial_a \theta^\lambda_b + \theta^\mu_b \Gamma^\lambda_{a \mu} ) . \quad (2)
\]

Locally, we write the Lagrangian density of a field theory as \( \ell \, dx^a \), where \( \ell \) is a function of the fields and their first derivatives. For the gravitational field we set

\[
\ell_{grav} = \frac{1}{4 \ell^2} R^{\lambda \mu a b} \tilde{\theta}^a_\lambda \tilde{\theta}^b_\lambda + \frac{1}{2 \ell^2} R^{\lambda \mu a b} \theta^a_\lambda \theta^b_\lambda \theta^\mu_a \theta^\mu_b |\theta| , \quad (3)
\]

where

\[
\Gamma^\lambda_{a \mu} \equiv \partial_a \theta^\lambda_b + \theta^\mu_b \Gamma^\lambda_{a \mu} ,
\]

\[
R^{\lambda \mu a b} = ( - \partial_b \Gamma^\lambda_{a \mu} + \Gamma^\lambda_{b \mu} \Gamma^\lambda_{a \mu} ) g^{\mu a} .
\]

If \( \theta \) is non-degenerate then \( R^{\lambda \mu a b} \theta^a_\lambda \theta^b_\lambda \theta^\mu_a \theta^\mu_b \) coincides with the scalar curvature of the spacetime connection, but note that the above Lagrangian density is well-defined also in the degenerate case.

Independent variations of the fields \( \delta^\lambda_a \) and \( \Gamma^\lambda_{a \mu} \) then yield the Euler-Lagrange operator components

\[
(\delta \ell_{grav})^\lambda_a = \frac{1}{4 \ell^2} \tilde{\theta}^a_\lambda R^{\lambda \mu a b} , \quad (4)
\]

\[
(\delta \ell_{grav})^a_{\lambda \mu} = - \frac{1}{4 \ell^2} \partial_c \theta^c_\lambda \theta^\lambda_a \theta^\mu_b , \quad (5)
\]

where \( E^a_\lambda \) is the Einstein tensor (not symmetric in this context).

2 Gauge field theories in tetraddrive gravity

A spin-zero ‘matter field’ in a gauge theory is a section of some vector bundle whose fibers are not ‘soldered’ to spacetime. A field with non-zero spin can be seen as a section of a similar bundle tensorIALIZED by a spin bundle; we denote its components by \( \phi^{i \alpha} \), where \( \alpha \) is the spin-related index (which may represent a sequence of ordinary spinor indices). The adjoint field \( \bar{\phi}^{\alpha} \) can be regarded as an independent section of the dual bundle.

The matter fields interact with a gauge field \( A_i^\alpha_a \) that is a connection of the ‘unsoldered’ bundle. Usually \( A \) is assumed to preserve some fiber structure and is accordingly valued into the appropriate Lie algebra, so one uses components \( A^i_a \), but we won’t need to deal with such restriction explicitly—it is not difficult to see that the arguments presented here work seamlessly with respect to the needed restriction.

The covariant derivative of the matter field has the expression

\[
\nabla_i \phi^{i \alpha} = \partial_i \phi^{i \alpha} - A_i^i \phi^{i \alpha} - \omega^{\alpha \beta}_{\lambda \mu} \phi^{i \beta} ,
\]

where the ‘spinor connection’ \( \omega^{\alpha \beta}_{\lambda \mu} \) is related to \( \Gamma^\lambda_{a \mu} \) by a linear relation of the type

\[
\omega^{\alpha \beta}_{\lambda \mu} = G^{i \alpha}_{\beta \lambda \mu} \Gamma^i_{a \mu} .
\]

The coefficients \( G^{i \alpha}_{\beta \lambda \mu} \) can be expressed as combinations of Kronecker deltas in the case of integer spin, while Dirac matrices are involved for semi-integer spin. In particular, for spin one-half we have

\[
\omega^{\alpha \beta}_{\lambda \mu} = \frac{1}{4} \Gamma^\lambda_{a \mu} ( \gamma^\lambda \gamma^\mu )^{\alpha \beta} , \quad (6)
\]

which can be inverted as

\[
\Gamma^\lambda_{a \mu} = \frac{1}{2} \mathrm{Tr} ( \gamma^\lambda \omega_b \gamma_b ) . \quad (7)
\]

Thus our variable \( \Gamma \) could be regarded as the spinor connection, namely the gravitational field can be equivalently represented as the couple \( (\theta, \omega) \).
The Klein-Gordon Lagrangian, written in the form
\[ \ell_\phi = \frac{1}{2m^2} \psi \dot{\psi} - \frac{1}{2} m^2 \psi^2 \theta^2 \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta 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holds true for the energy tensors of the gauge and gravitational fields, provided that we use the right notion of ‘covariant derivative’ of such fields. Various arguments [19, 41] clearly indicate that the role of the covariant derivative of a connection is to be taken up by the exterior covariant differential of the connection with respect to itself, that is minus its curvature tensor. Namely we insert $\nabla_b A^i_{\ell j} \equiv (d[A]A)_{bc}^i$ into
\begin{equation}
(U_{\text{gauge}})_\lambda^i = \ell_{\text{gauge}} \theta^\lambda_a - \theta^\lambda_a \nabla_a A^i_{\ell j} \frac{\partial \ell_{\text{gauge}}}{\partial (\partial_a A^i_{\ell j})}
\end{equation}
and obtain the stated identity. As for the gravitational field $(\theta, \Gamma)$, since $\ell_{\text{grav}}$ is independent of the derivatives of the theta we get
\begin{equation}
(U_{\text{grav}})_\lambda^i = \ell_{\text{grav}} \theta^\lambda_a - \nabla_a \Gamma^\mu_{\nu a} \frac{\partial \ell_{\text{grav}}}{\partial (\partial_a \Gamma^\mu_{\nu a})} \theta^\lambda_a |\theta|-1 = \\
= -\frac{1}{2\ell} \theta^\lambda_a - \frac{1}{2\ell} |\theta| \theta^\lambda_a = \\
= \left( R^b_{\mu \rho} \theta^\lambda_a - \frac{1}{2} R \theta^\lambda_a \right) = (U_{\text{grav}})_\lambda^i .
\end{equation}

More generally, one may wish to consider a Lagrangian that also depends on the derivatives of the tetrad. Then the question arises if one can generalize the construction of the canonical energy-tensor to this case. Without being involved in technical details, we state that two ways turns out to be legitimate, the difference between them being in the way in which the action of a vector field on $M$ is properly lifted. Essentially, both ways eventually lead to an expression of the type $U_b^a = \ell^a_b - D_b \theta^\lambda_a P^{a,c,\lambda}$, where $D_b$ is a suitable differential operator. One construction yields just $D_b \theta^\lambda_a = \nabla_b \theta^\lambda_a = 0$. More interestingly, the other construction determines $D_b \theta^\lambda_a$ to be—somewhat similarly to the connection—the covariant differential of the theta, that is essentially the torsion. Namely one gets
\begin{equation}
U_b^a = \ell^a_b - \theta^\lambda_a \theta^\lambda_c T^e_{cb} , \quad \theta^\lambda_c \theta^\lambda_a \equiv \partial \ell/\partial \theta^\lambda_c ,
\end{equation}

For example one may consider the standard ‘ghost Lagrangian’, that in terms of the tetrad can be written as
\begin{equation}
\ell_{\text{ghost}} \equiv g^{\lambda \mu} \theta^\alpha_{\lambda \mu} \chi_i \nabla^\lambda \chi_i |\theta| - \frac{1}{2\ell} f_i f^i |\theta| ,
\end{equation}
\begin{equation}
f_i \equiv |\theta|^{-1} g^{\lambda \mu} \partial_a (\theta^\alpha_{\lambda \mu} |\theta| A^a) .
\end{equation}

Here $\chi_i$ and $\chi_i$ are the ghost and anti-ghost fields, $\xi$ is a constant, and the index $i$ denotes components in the appropriate Lie algebra. Then the ‘gauge fixing Lagrangian’ $\ell_{\text{fix}} \equiv -f_i f^i |\theta|/(2\ell)$ introduces into the total canonical energy tensor, constructed in the above described way, a term which is linear in the torsion. Similarly the stress-energy tensor gets a term that can be expressed as the ‘variational derivative’ of $\ell_{\text{fix}}$ with respect to $\theta$; on turn this can be expressed in terms of the torsion, through somewhat intricate computations.

4 Field equations

Besides eq. (11), the field equations obtained from the variations of $\ell_{\text{tot}}$ with respect to $\theta$, $\Gamma$, $A$, $\phi$ and $\phi$ yield, respectively, the gravitational equation, the torsion equation, the non-Abelian generalization of second Maxwell equation, and the generalization of either the Klein-Gordon equation or the Dirac equation.

The gravitational equation is
\begin{equation}
0 = T_{\text{tot}} \equiv T_{\text{grav}} + T_{\text{gauge}} + T_{\text{matter}} ,
\end{equation}
where $T_{\text{matter}}$ is either $T_\phi$ or $T_{\bar{\phi}}$.

The other field equations—in a somewhat concise form—can be written in the K-G case as
\begin{equation}
0 = -\frac{1}{c^2} \partial_{\mu \lambda} T^e_{bc} \phi^e + 2g_{ab} c_{\rho \beta} \partial_{\mu \beta} (\phi_{\alpha \omega} \nabla_b^\beta \phi_{\beta}^\beta - \nabla_b^\beta \phi_{\alpha \omega} \phi_{\beta}^\beta) ,
\end{equation}
\begin{equation}
= (d[A]S^a)_{i} + \frac{1}{2} |\theta| g_{ab} (\phi_{\alpha \omega} \nabla_b^\beta \phi_{\beta}^\beta - \nabla_b^\beta \phi_{\alpha \omega} \phi_{\beta}^\beta) ,
\end{equation}
\begin{equation}
\equiv (d[\Gamma \otimes A] \nabla \phi)_{\alpha \omega} + m^2 \phi_{\alpha \omega} |\theta| ,
\end{equation}
\begin{equation}
\equiv (d[\Gamma \otimes A] \nabla \phi)_{\alpha} + m^2 \phi_{\alpha} |\theta| .
\end{equation}

Here the $\star$ stands for the ‘Hodge isomorphism’ of exterior forms, namely
\begin{equation}
\star F^{abj}_{i} = \star g^{a c} g^{b d} |\theta| F_{c d j} , \quad \star \nabla \phi^{\alpha a} \phi = g^{a b} |\theta| \nabla_b \phi^{\alpha a} ,
\end{equation}
and $d[A]$ and $d[\Gamma \otimes A]$ are the exterior covariant differentials with respect to the connections indicated between brackets. A generalized version of the so-called ‘replacement principle’ states that these differ from the usual ‘covariant divergences’ by torsion terms [19, 41]. In fact we have the identities
\begin{equation}
\nabla_a \xi^{a \iota} = (d[\Gamma \otimes A] \xi)_{i} - T_{ab} \xi^{a \iota} ,
\end{equation}
\begin{equation}
2 \nabla_a \xi^{a \iota} = (d[\Gamma \otimes A] \xi)_{i} - \frac{1}{2} \xi^{a \iota} T_{ab} - \xi_{ba} T^{a b} ,
\end{equation}
where $\xi$ is a $(4-r)$-form ($r = 1, 2$) valued in a vector bundle and $K$ is a connection of that same bundle.

Eq. (15) is the torsion equation; eq. (19) is the ‘second Maxwell equation’; eqs. (20) and (21) are the ‘Klein-Gordon equations’ for $\phi$ and $\bar{\phi}$.

In the Dirac case we find the field equations
\begin{equation}
0 = -\frac{1}{c^2} \bar{\psi}^{abc} T_{bc}^e \theta^e + \frac{1}{2} \bar{\psi}^{a b} \psi \psi_{a b} (\gamma_{\lambda} \wedge \gamma_{\mu} \wedge \gamma_{\rho}) \gamma_{\rho} \psi_{\beta}^\beta ,
\end{equation}
\begin{equation}
0 = (d[A] \psi)_{j} + i \bar{\psi}_{a} \psi_{a} \gamma^{\alpha \beta} \psi_{\beta}^\beta ,
\end{equation}
\begin{equation}
0 = -i \nabla \bar{\psi}_{j} + m \psi_{j} + \frac{1}{2} \psi_{j} \tau_{\lambda} \gamma_{\lambda} \psi_{\beta}^\beta |\theta| ,
\end{equation}
\begin{equation}
0 = (i \nabla \psi_{j} - m \psi_{j} + \frac{1}{2} \tau_{\lambda} \gamma_{\lambda} \psi_{\beta}^\beta |\theta| ,
\end{equation}
where $\tau_{\lambda} \equiv \theta^\lambda_a T^b_{ab}$. Eqs. (24) and (25) are the Dirac equations with torsion.

\footnote{Exterior form components $\xi^{a}$, $\xi^{a b}$ with higher indices are to be intended relatively to frames $i(\partial x_a) d^4 x$, $i(\partial x_a \wedge \partial x) d^4 x$ etc.}
5 Divergences

In the standard, torsion-free formulation of General Relativity, the stress-energy tensor in the right-hand side of the Einstein equation is divergence-free (namely when the field equations are taken into account). This well-known result \[1\] \[2\] is a consequence of the naturality of the Lagrangian, and holds in particular for gauge theories provided that the stress-energy tensor contains the contributions of the matter field and the gauge field \[19\]. This property is interpreted as local energy-conservation.

In the presence of torsion the situation is more intricate. The gravitational equation \( T_{\alpha} = 0 \) implies \( \nabla_{\alpha}(T_{\text{grav}})_{\Lambda} = 0 \), but the single contributions have non-vanishing divergence. In particular we remark that the ‘Einstein tensor’ appearing in eq. (4) and eq. (12) is not divergence-free; actually

\[
\nabla_{a}(T_{\text{grav}})_{\Lambda} = \frac{1}{c^4} \psi_{\Lambda} (T_{ca} R_{b}^{a} - \frac{1}{2} T_{ad} R_{bc}^{ad}) .
\]

Hence we expect that the on-shell divergence of \( T_{\text{gauge}} + T_{\text{matter}} \) depends on the torsion linearly. Indeed this can be checked, by not-so-short computations. The vanishing of \( \nabla_{a}(T_{\alpha})_{\Lambda} \) expressed in terms of the exterior covariant differential can be regarded as an ‘integrability condition’ for the gravitational equation. In the K-G case we obtain

\[
\nabla_{a}(T_{\text{gauge}})_{\Lambda} = \frac{1}{c^4} \psi_{\Lambda} \left( T_{ac} R_{b}^{a} T^{c}_{be} - \frac{1}{2} T_{ad} R_{bc}^{ad} \right) + \frac{1}{2} \phi_{\alpha i} \left( \nabla_{e} \phi_{\alpha i} - \nabla_{i} \phi_{\alpha e} \right) .
\]

In the Dirac case we obtain

\[
\nabla_{a}(T_{\text{gauge}})_{\Lambda} = \frac{1}{c^4} \psi_{\Lambda} \left( T_{ac} R_{b}^{a} T^{c}_{be} - \frac{1}{2} T_{ad} R_{bc}^{ad} \right) + \frac{1}{2} \phi_{\alpha i} \left( \nabla_{e} \phi_{\alpha i} - \nabla_{i} \phi_{\alpha e} \right) .
\]

The last term in the above equation can be further elaborated. By Clifford algebra we get

\[
\gamma^{b} R_{ab} + R_{ab} \gamma^{b} = -\frac{1}{2} R_{[abcd]} \gamma^{b} \gamma^{c} \gamma^{d} ,
\]

and \( R_{[abcd]} \), vanishing in the torsion-free situation, can be expressed in terms of the exterior covariant differential \( d \Gamma / T \), which is essentially the right-hand side of the first Bianchi equation with torsion.

6 Conclusions

Offered results support the view that the tetrad-affine representation of gravity is natural and convenient under various respects. In a gauge field theory coupled with gravity there is essentially one energy-tensor for each sector. The total energy-tensor is divergence-free, while the single contributions are not—on account of the torsion. The torsion itself is unavoidable in this setting, but it should be regarded as a ‘byproduct’ rather than a fundamental, independent field.

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\textsuperscript{5} As previously observed, in this paper we are not involved with detailed discussions about physical interpretations of the presented mathematical notions.
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