CONFORMAL FOLIATIONS ON LIE GROUPS
AND COMPLEX-VALUED HARMONIC MORPHISMS

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Abstract. We study left-invariant foliations $F$ on Riemannian Lie groups $G$ generated by a subgroup $K$. We are interested in such foliations which are conformal and with minimal leaves of codimension two. We classify such foliations $F$ when the subgroup $K$ is one of the important $\text{SU}(2) \times \text{SU}(2)$, $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$, $\text{SU}(2) \times \text{SO}(2)$ or $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$. By this we yield new multi-dimensional families of Lie groups $G$ carrying such foliations in each case. These foliations $F$ produce local complex-valued harmonic morphisms on the corresponding Lie group $G$.

1. Introduction

We study Riemannian Lie groups $G$ equipped with a conformal foliation $F$ generated by the left-translations of a subgroup $K$ of codimension two. We are interested in such foliations with minimal leaves and hence inducing local complex-valued harmonic morphisms. We prove the following result.

**Theorem 1.1.** Let $K$ be a semisimple Lie subgroup of $G$ of codimension two and $F$ be the left-invariant foliation on $G$ generated by $K$. Then $F$ is both conformal and Riemannian.

Theorem 1.1 motivates the following conjecture, which constitutes the main theme of this work.

**Conjecture 1.2.** Let $K$ be a Lie subgroup of $G$ generating a left-invariant conformal foliation $F$ of $G$ of codimension two. If $K$ is semisimple then the foliation $F$ is minimal. If $K$ is semisimple and compact then $F$ is totally geodesic.

Our principal aim is to investigate this conjecture and show that it holds in the four important cases when $K$ is one of the Lie subgroups $\text{SU}(2) \times \text{SU}(2)$, $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$, $\text{SO}(2) \times \text{SU}(2)$ or $\text{SO}(2) \times \text{SL}_2(\mathbb{R})$ of $G$. In each of these cases, we yield multi-dimensional families of Riemannian Lie groups $G$ solving this interesting geometric problem i.e. carrying conformal foliations with minimal leaves of codimension two.

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2. Preliminaries

Let $M$ and $N$ be two manifolds of dimensions $m$ and $n$, respectively. A Riemannian metric $g$ on $M$ gives rise to the notion of a Laplacian on $(M, g)$ and real-valued harmonic functions $f : (M, g) \to \mathbb{R}$. This can be generalised to the concept of harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds, which are solutions to a semi-linear system of partial differential equations, see [2].

**Definition 2.1.** A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is harmonic.

The following characterisation of harmonic morphisms between Riemannian manifolds is due to B. Fuglede and T. Ishihara. For the definition of horizontal (weak) conformality we refer to [2].

**Theorem 2.2.** [3, 6] A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

Let $(M, g)$ be a Riemannian manifold, $\mathcal{V}$ be an integrable distribution on $M$ and denote by $\mathcal{H}$ its orthogonal complement distribution on $M$. As customary, we also use $\mathcal{V}$ and $\mathcal{H}$ to denote the orthogonal projections onto the corresponding subbundles of $TM$ and denote by $\mathcal{F}$ the foliation tangent to $\mathcal{V}$. Then the second fundamental form for $\mathcal{V}$ is given by

$$B^\mathcal{V}(E, F) = \frac{1}{2} \mathcal{H}(\nabla_E F + \nabla_F E) \quad (E, F \in \mathcal{V}),$$

while the second fundamental form for $\mathcal{H}$ satisfies

$$B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).$$

The foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is said to be conformal if there is a vector field $V \in \mathcal{V}$ such that

$$B^\mathcal{H} = g \otimes V,$$

and $\mathcal{F}$ is said to be Riemannian if $V = 0$. Furthermore, $\mathcal{F}$ is said to be minimal if trace $B^\mathcal{V} = 0$ and totally geodesic if $B^\mathcal{V} = 0$. This is equivalent to the leaves of $\mathcal{F}$ being minimal and totally geodesic submanifolds of $M$, respectively.

It is well-known that the fibres of a horizontally conformal map (resp. Riemannian submersion) give rise to a conformal foliation (resp. Riemannian foliation). Conversely, the leaves of any conformal foliation (resp. Riemannian foliation) are locally the fibres of a horizontally conformal map (resp. Riemannian submersion), see [2].

The next result of Baird and Eells gives the theory of harmonic morphisms, with values in a surface, a strong geometric flavour.
Theorem 2.3. [1] Let \( \phi : (M^m, g) \to (N^2, h) \) be a horizontally conformal submersion from a Riemannian manifold to a surface. Then \( \phi \) is harmonic if and only if \( \phi \) has minimal fibres.

3. LEFT-INVARIANT FOLIATIONS OF CODIMENSION 2

Let \((G, g)\) be a Lie group equipped with a left-invariant Riemannian metric \(g\) and \(K\) be a subgroup of \(G\). Let \(\mathfrak{k}\) and \(\mathfrak{g}\) be the Lie algebras of \(K\) and \(G\), respectively. Let \(\mathfrak{m}\) be the orthogonal complement of \(\mathfrak{k}\) in \(\mathfrak{g}\) with respect to the Riemannian metric \(g\) on \(G\). By \(V\) we denote the integrable distribution generated by \(\mathfrak{k}\) and by \(H\) its orthogonal distribution given by \(\mathfrak{m}\). Further let \(F\) be the foliation of \(G\) induced by \(V\). For this situation we have the following result.

Theorem 3.1. Let \(K\) be a semisimple Lie subgroup of \(G\) of codimension two and \(F\) be the left-invariant foliation on \(G\) generated by \(K\). Then \(F\) is both conformal and Riemannian.

Proof. Since the subgroup \(K\) is semisimple we know that its Lie algebra \(\mathfrak{k}\) satisfies \([\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}\). It then follows from Remark 3.2 of [5] that the adjoint action of \(V = [V, V]\) has no \(H\)-component. The statement is an immediate consequence of this fact. \(\square\)

The result of Theorem 3.1 is our main motivation for the following.

Conjecture 3.2. Let \(K\) be a Lie subgroup of \(G\) generating a left-invariant conformal foliation \(F\) of \(G\) of codimension two. If \(K\) is semisimple then the foliation \(F\) is minimal. If \(K\) is semisimple and compact then \(F\) is totally geodesic.

We will investigate this conjecture in the four interesting cases when \(K\) is one of the Lie subgroups \(\text{SU}(2) \times \text{SU}(2)\), \(\text{SU}(2) \times \text{SL}_2(\mathbb{R})\), \(\text{SU}(2) \times \text{SO}(2)\) or \(\text{SL}_2(\mathbb{R}) \times \text{SO}(2)\) of \(G\). The following table gives an explanation for our different choices.

| \(K\)               | compact            | non-compact       |
|---------------------|--------------------|-------------------|
| semisimple          | \(\text{SU}(2) \times \text{SU}(2)\) | \(\text{SU}(2) \times \text{SL}_2(\mathbb{R})\) |
| non-semisimple      | \(\text{SU}(2) \times \text{SO}(2)\) | \(\text{SL}_2(\mathbb{R}) \times \text{SO}(2)\) |

4. THE CASE OF \(K = \text{SU}(2) \times \text{SU}(2)\) IN \(G^8\).

Let \((G, g)\) be an eight-dimensional Riemannian Lie group and \(K\) be its compact semisimple subgroup \(\text{SU}(2) \times \text{SU}(2)\) equipped with its standard Riemannian metric induced by the corresponding Killing form. Let \(F\) be the left-invariant foliation on \(G\) generated by the Lie subalgebra \(\mathfrak{k} = \mathfrak{su}(2) \times \mathfrak{su}(2)\) of \(\mathfrak{g}\). Let \(\{A, B, C, R, S, T, X, Y\}\) be an orthonormal basis for the Lie algebra \(\mathfrak{g}\) of \(G\) such that the two copies of \(\mathfrak{su}(2)\) are generated by \(\{A, B, C\}\) and \(\{R, S, T\}\), respectively. For these we have the following standard Lie bracket relations

\[ [A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \]
\[ [R, S] = 2T, \quad [T, R] = 2S, \quad [S, T] = 2R. \]

In this case we have the following result.

**Proposition 4.1.** Let \( G \) be an eight-dimensional Riemannian Lie group and \( K \) be its six-dimensional Lie subgroup \( \text{SU}(2) \times \text{SU}(2) \) equipped with its standard Riemannian metric induced by the Killing form. Then the Lie bracket relations for \( g \) take the following form

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = 2R, \\
[A, X] &= -b_{11} B - c_{11} C, \quad [A, Y] = -b_{21} B - c_{21} C, \\
[B, X] &= b_{11} A - c_{12} C, \quad [B, Y] = b_{21} A - c_{22} C, \\
[C, X] &= c_{11} A + c_{12} B, \quad [C, Y] = c_{21} A + c_{22} B, \\
[R, X] &= -s_{14} S - t_{14} T, \quad [R, Y] = -s_{24} S - t_{24} T, \\
[S, X] &= s_{14} R - t_{15} T, \quad [S, Y] = s_{24} R - t_{25} T, \\
[T, X] &= t_{14} R + t_{15} S, \quad [T, Y] = t_{24} R + t_{25} S, \\
[X, Y] &= \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T,
\end{align*}
\]

where

\[
\begin{pmatrix}
\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} \\ \rho c_{11} + b_{11} c_{22} - b_{21} c_{12} \\ -\rho b_{11} + c_{11} c_{22} - c_{12} c_{21} \\ -\rho t_{15} + s_{14} t_{24} - s_{24} t_{14} \\ \rho t_{14} + s_{14} t_{25} - s_{24} t_{15} \\ -\rho s_{14} + t_{14} t_{25} - t_{15} t_{24}
\end{pmatrix}.
\]

**Proof.** Since \( K = \text{SU}(2) \times \text{SU}(2) \) is semisimple we know from the proof of Theorem 3.1 that the adjoint action of \( \mathcal{V} = [\mathcal{V}, \mathcal{V}] \) has no \( \mathcal{H} \)-component. This tells us that the corresponding Lie bracket relations take the form

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = 2R, \\
[A, X] &= a_{11} A + a_{12} B + a_{13} C + a_{14} R + a_{15} S + a_{16} T, \\
[A, Y] &= a_{21} A + a_{22} B + a_{23} C + a_{24} R + a_{25} S + a_{26} T, \\
[B, X] &= b_{11} A + b_{12} B + b_{13} C + b_{14} R + b_{15} S + b_{16} T, \\
[B, Y] &= b_{21} A + b_{22} B + b_{23} C + b_{24} R + b_{25} S + b_{26} T, \\
[C, X] &= c_{11} A + c_{12} B + c_{13} C + c_{14} R + c_{15} S + c_{16} T, \\
[C, Y] &= c_{21} A + c_{22} B + c_{23} C + c_{24} R + c_{25} S + c_{26} T, \\
[R, X] &= r_{11} A + r_{12} B + r_{13} C + r_{14} R + r_{15} S + r_{16} T, \\
[R, Y] &= r_{21} A + r_{22} B + r_{23} C + r_{24} R + r_{25} S + r_{26} T, \\
[S, X] &= s_{11} A + s_{12} B + s_{13} C + s_{14} R + s_{15} S + s_{16} T, \\
[S, Y] &= s_{21} A + s_{22} B + s_{23} C + s_{24} R + s_{25} S + s_{26} T, \\
[T, X] &= t_{11} A + t_{12} B + t_{13} C + t_{14} R + t_{15} S + t_{16} T,
\end{align*}
\]
\[ [T, Y] = t_{21}A + t_{22}B + t_{23}C + t_{24}R + t_{25}S + t_{26}T, \]
\[ [X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T. \]

The Jacobi identities involving the vector fields \( A, B, C, X \in g \) provide us with the following interesting identities

\[
0 = [[A, B], X] + [[X, A], B] + [[B, X], A] = 2 ((c_{11} + a_{13}) A + (c_{12} + b_{13}) B + (c_{13} - a_{11} - b_{12}) C + c_{14} R + c_{15} S + c_{16} T),
\]
\[
0 = [[C, A], X] + [[X, C], A] + [[A, X], C] = 2 ((b_{11} + a_{12}) A + (b_{12} - a_{11} - c_{13}) B + (b_{13} + c_{12}) C + b_{14} R + b_{15} S + b_{16} T),
\]
\[
0 = [[B, C], X] + [[X, B], C] + [[C, X], B] = 2 ((a_{11} - b_{12} - c_{13}) A + (a_{12} + b_{11}) B + (a_{13} + c_{11}) C + a_{14} R + a_{15} S + a_{16} T).
\]

From these we can immediately see that
\[
\begin{pmatrix} a_{14} \\ a_{15} \\ a_{16} \end{pmatrix} = 0,
\]
\[
\begin{pmatrix} b_{14} \\ b_{15} \\ b_{16} \end{pmatrix} = 0,
\]
\[
\begin{pmatrix} c_{14} \\ c_{15} \\ c_{16} \end{pmatrix} = 0.
\]

Then by the symmetry in \( X \) and \( Y \) we see that this holds for \( a_i, b_i, c_i \) where \( i \in \{24, 25, 26\} \). Additionally, we see by the symmetry in \( R, S, T \) this also holds for \( r_j, s_j, t_j \) for \( j \in \{11, 12, 13, 21, 22, 23\} \). We also see that these Jacobi identities are satisfied according to the following system of equations

\[
\begin{pmatrix} a_{11} - b_{12} - c_{13} \\ a_{11} - b_{12} + c_{13} \\ a_{11} + b_{12} - c_{13} \\ a_{12} + b_{11} \\ a_{13} + c_{11} \\ b_{13} + c_{12} \end{pmatrix} = 0.
\]

Solving this system we obtain

\[
\begin{pmatrix} a_{11} \\ b_{12} \\ c_{13} \end{pmatrix} = 0, \quad \begin{pmatrix} a_{12} \\ a_{13} \\ b_{13} \end{pmatrix} = \begin{pmatrix} -b_{11} \\ -c_{11} \\ -c_{12} \end{pmatrix}.
\]
Due to the symmetry in both $X$ and $Y$ we yield that the Jacobi identities involving $A, B, C, Y$ are equivalent to the following system of equations.

\[
\begin{pmatrix}
  a_{21} - b_{22} - c_{23} \\
  a_{21} - b_{22} + c_{23} \\
  a_{21} + b_{22} - c_{23} \\
  a_{22} + b_{21} \\
  a_{23} + c_{21} \\
  b_{23} + c_{22}
\end{pmatrix} = 0.
\]

Solving this we yield the following results from satisfying the Jacobi identities

\[
\begin{pmatrix}
  a_{21} \\
  b_{22} \\
  c_{23}
\end{pmatrix} = 0, \quad
\begin{pmatrix}
  a_{22} \\
  a_{23} \\
  b_{23}
\end{pmatrix} = \begin{pmatrix}
  -b_{21} \\
  -c_{21} \\
  -c_{22}
\end{pmatrix}.
\]

(4.2)

By the symmetry in $A, B, C$ and $R, S, T$ we have that the Jacobi identities involving $R, S, T, X$ are equivalent to the following system of equations

\[
\begin{pmatrix}
  r_{14} - s_{15} - t_{16} \\
  r_{14} - s_{15} + t_{16} \\
  r_{14} + s_{15} - t_{16} \\
  r_{15} + s_{14} \\
  r_{16} + t_{14} \\
  s_{16} + t_{15}
\end{pmatrix} = 0.
\]

Solving this we then yield the following results from satisfying the Jacobi identities

\[
\begin{pmatrix}
  r_{14} \\
  s_{15} \\
  t_{16}
\end{pmatrix} = 0, \quad
\begin{pmatrix}
  r_{15} \\
  r_{16} \\
  s_{16}
\end{pmatrix} = \begin{pmatrix}
  -s_{14} \\
  -t_{14} \\
  -t_{15}
\end{pmatrix}.
\]

(4.3)

Again by the symmetry in $X$ and $Y$ we see the Jacobi identities involving $R, S, T, Y$ are equivalent to the following system of equations

\[
\begin{pmatrix}
  r_{24} - s_{25} - t_{26} \\
  r_{24} - s_{25} + t_{26} \\
  r_{24} + s_{25} - t_{26} \\
  r_{25} + s_{24} \\
  r_{26} + t_{24} \\
  s_{26} + t_{25}
\end{pmatrix} = 0.
\]

Solving this we then obtain the following results from satisfying the Jacobi identities

\[
\begin{pmatrix}
  r_{24} \\
  s_{25} \\
  t_{26}
\end{pmatrix} = 0, \quad
\begin{pmatrix}
  r_{25} \\
  r_{26} \\
  s_{26}
\end{pmatrix} = \begin{pmatrix}
  -s_{24} \\
  -t_{24} \\
  -t_{25}
\end{pmatrix}.
\]

(4.4)

Now considering the Jacobi identities involving both $X$ and $Y$ we see that

\[
0 = [[A, X], Y] + [[Y, A], X] + [[X, Y], A]
= (\rho b_{11} - c_{11} c_{22} + c_{21} c_{12} + 2\theta_3) B + (\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} - 2\theta_2) C,
\]
0 = [[[B, X], Y] + [[Y, B], X] + [[X, Y], B]
= -(ρb_{11} - c_{11}c_{22} + c_{21}c_{12} + 2θ_3) A + (ρc_{12} - b_{11}c_{21} + b_{21}c_{11} + 2θ_1) C.
Satisfying these Jacobi identities is equivalent to the following system of equations

\[
\begin{pmatrix}
ρc_{12} - b_{11}c_{21} + b_{21}c_{11} + 2θ_1 \\
ρc_{11} + b_{11}c_{22} - b_{21}c_{12} - 2θ_2 \\
ρb_{11} - c_{11}c_{22} + c_{21}c_{12} + 2θ_3
\end{pmatrix} = 0,
\]
from which we get

\[
\begin{pmatrix}
θ_1 \\
θ_2 \\
θ_3
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-ρc_{12} + b_{11}c_{21} - b_{21}c_{11} \\
ρc_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-ρb_{11} + c_{11}c_{22} - c_{12}c_{21}
\end{pmatrix}.
\]
Then by the symmetry of the triples (A, B, C) and (R, S, T) we find that

\[
\begin{pmatrix}
θ_4 \\
θ_5 \\
θ_6
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-ρt_{15} + s_{14}t_{24} - s_{24}t_{14} \\
ρt_{14} + s_{14}t_{25} - s_{24}t_{15} \\
-ρs_{14} + t_{14}t_{25} - s_{15}t_{24}
\end{pmatrix}.
\]

These calculations provide us with the stated result.

The following Theorem 4.2 supports the case of Conjecture 3.2 that when the semisimple subgroup \(K\) is compact the resulting foliation is totally geodesic. Together with Proposition 4.1, this provides us with a new 13-dimensional family of 8-dimensional Lie groups carrying a conformal foliation with minimal leaves of codimension two.

**Theorem 4.2.** Let \(G\) be an eight-dimensional Riemannian Lie group carrying a left-invariant distribution \(\mathcal{V}\), generated by the Lie algebra \(\mathfrak{su}(2) \times \mathfrak{su}(2)\) of the subgroup \(\text{SU}(2) \times \text{SU}(2)\). Then the resulting foliation \(\mathcal{F}\) tangent to \(\mathcal{V}\) is conformal, Riemannian and totally geodesic.

**Proof.** It immediately follows from Theorem 3.1 that the foliation \(\mathcal{F}\) is conformal and Riemannian, thus it suffices to show that \(\mathcal{F}\) is totally geodesic. For smooth vector fields \(A, B \in \mathcal{V}\) in the vertical distribution we have

\[
B^\mathcal{V}(A, B) = \frac{1}{2} \mathcal{H}(\nabla_A B + \nabla_B A)
= \frac{1}{2} \left( (\langle [X, A], B \rangle + \langle [X, B], A \rangle) X \\
+ (\langle [Y, A], B \rangle + \langle [Y, B], A \rangle) Y \right).
\]

Evaluating this for the basis elements in \(\mathcal{V}\) and recalling the simplifications made in Proposition 4.1 we determine

\[
B^\mathcal{V}(A, A) = -(a_{11}X + a_{21}Y) = 0,
\]
\[
B^\mathcal{V}(A, B) = \frac{1}{2}((a_{12} + b_{11})X + (a_{22} + b_{21})Y) = 0,
\]
\[
B^\mathcal{V}(A, C) = \frac{1}{2}((a_{13} + c_{11})X + (a_{23} + c_{21})Y) = 0,
\]
From this we see that the second fundamental form $B^\nu$ of the vertical distribution vanishes. Thus the foliation $\mathcal{F}$ is totally geodesic. □

5. The case of $K = \text{SU}(2) \times \text{SL}_2(\mathbb{R})$ in $G^8$.

Let $(G, g)$ be an 8-dimensional Riemannian Lie group and $K$ be its 6-dimensional non-compact semisimple subgroup $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$ equipped with its standard Riemannian metric. Further let $\mathcal{F}$ be the left-invariant foliation on $G$ generated by the Lie subalgebra $\mathfrak{k} = \mathfrak{su}(2) \times \mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{g}$. Let $\{A, B, C, R, S, T, X, Y\}$ be an orthonormal basis for the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{su}(2)$ is generated by $\{A, B, C\}$ and $\mathfrak{sl}_2(\mathbb{R})$ by $\{R, S, T\}$,

$$
B^\nu(A, R) = -\frac{1}{2}(a_{14} + r_{11})X + (a_{24} + r_{21})Y = 0, \\
B^\nu(A, S) = -\frac{1}{2}(a_{15} + s_{11})X + (a_{25} + s_{21})Y = 0, \\
B^\nu(A, T) = -\frac{1}{2}(a_{16} + t_{11})X + (a_{26} + t_{21})Y = 0, \\
B^\nu(B, B) = -(b_{12}X + b_{22}Y) = 0, \\
B^\nu(B, C) = -\frac{1}{2}((b_{13} + c_{12})X + (b_{23} + c_{22})Y) = 0, \\
B^\nu(B, R) = -\frac{1}{2}((b_{14} + r_{12})X + (b_{24} + r_{22})Y) = 0, \\
B^\nu(B, S) = -\frac{1}{2}((b_{15} + s_{12})X + (b_{25} + s_{22})Y) = 0, \\
B^\nu(B, T) = -\frac{1}{2}((b_{16} + t_{12})X + (b_{26} + t_{22})Y) = 0, \\
B^\nu(C, C) = -(c_{13}X + c_{23}Y) = 0, \\
B^\nu(C, R) = -\frac{1}{2}((c_{14} + r_{13})X + (c_{24} + r_{23})Y) = 0, \\
B^\nu(C, S) = -\frac{1}{2}((c_{15} + s_{13})X + (c_{25} + s_{23})Y) = 0, \\
B^\nu(C, T) = -\frac{1}{2}((c_{16} + t_{13})X + (c_{26} + t_{23})Y) = 0, \\
B^\nu(R, R) = -(r_{14}X + r_{24}Y) = 0, \\
B^\nu(R, S) = -\frac{1}{2}((r_{15} + s_{14})X + (r_{25} + s_{24})Y) = 0, \\
B^\nu(R, T) = -\frac{1}{2}((r_{16} + t_{14})X + (r_{26} + t_{24})Y) = 0, \\
B^\nu(S, S) = -(s_{15}X + s_{25}Y) = 0, \\
B^\nu(S, T) = -\frac{1}{2}((s_{16} + t_{15})X + (s_{26} + t_{25})Y) = 0, \\
B^\nu(T, T) = -(t_{16}X + t_{26}Y) = 0.
$$
respectively, with their standard Lie bracket relations

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = -2R.
\end{align*}
\]

This case might look much the same as that of Section 4, but the minus in the last equation changes everything. Both the subgroups are semisimple, but \( \text{SU}(2) \times \text{SU}(2) \) is compact and \( \text{SU}(2) \times \text{SL}_2(\mathbb{R}) \) is not. For the latter case we have the next result.

**Proposition 5.1.** Let \( G \) be an eight-dimensional Riemannian Lie group and \( K \) be its six-dimensional Lie subgroup \( \text{SU}(2) \times \text{SL}_2(\mathbb{R}) \) equipped with its standard Riemannian metric. Then the Lie bracket relations for \( \mathfrak{g} \) take the following form

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = -2R, \\
[A, X] &= -b_{11} B - c_{11} C, \quad [A, Y] = -b_{21} B - c_{21} C, \\
[B, X] &= b_{11} A - c_{12} C, \quad [B, Y] = b_{21} A - c_{22} C, \\
[C, X] &= c_{11} A + c_{12} B, \quad [C, Y] = c_{21} A + c_{22} B, \\
[R, X] &= -s_{14} S - t_{14} T, \quad [R, Y] = -s_{24} S - t_{24} T, \\
[S, X] &= s_{14} R - t_{15} T, \quad [S, Y] = s_{24} R - t_{25} T, \\
[T, X] &= t_{14} R + t_{15} S, \quad [T, Y] = t_{24} R + t_{25} S, \\
[X, Y] &= \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T,
\end{align*}
\]

where

\[
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
-\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} \\
\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} \\
-\rho b_{11} + c_{11} c_{22} - c_{12} c_{21} \\
-\rho t_{15} - s_{14} t_{24} + s_{24} t_{14} \\
-\rho t_{14} - s_{14} t_{25} + s_{24} t_{15} \\
\rho s_{14} - t_{14} t_{25} + t_{15} t_{24}
\end{bmatrix}
\cdot
\]

**Proof.** The arguments needed here are exactly the same as already provided in the proof of Proposition 4.1, for the details see [9].

The following Theorem 5.2 supports the case of Conjecture 3.2 that when the subgroup \( K \) is semisimple then the resulting foliation is minimal. Together with Proposition 5.1, this gives a new 13-dimensional family of 8-dimensional Lie groups carrying a conformal foliation with minimal leaves of codimension two.

**Theorem 5.2.** Let \( G \) be an eight-dimensional Riemannian Lie group carrying a left-invariant distribution \( \mathcal{V} \), generated by the Lie algebra \( \mathfrak{su}(2) \times \mathfrak{sl}_2(\mathbb{R}) \) of the subgroup \( \text{SU}(2) \times \text{SL}_2(\mathbb{R}) \). Then the resulting foliation \( \mathcal{F} \) tangent to \( \mathcal{V} \) is conformal, Riemannian and minimal. Furthermore \( \mathcal{F} \) is totally geodesic if and only if \( s_{14} = s_{24} = t_{14} = t_{24} = 0 \).
Proof. It immediately follows from Theorem 3.1 that $\mathcal{F}$ is conformal and Riemannian. Now employing the Koszul formula, for the Levi-Civita connection, we obtain

\[
\text{trace } B^\nu = \mathcal{H}(\nabla_A A + \cdots + \nabla_T T) = ((\nabla_A A, X) + \cdots + (\nabla_T T, X))X + ((\nabla_A A, Y) + \cdots + (\nabla_T T, Y))Y
\]

\[
= (\{(X, A), A\} + \cdots + \{(X, T), T\})X + (\{(Y, A), A\} + \cdots + \{(Y, T), T\})Y
\]

\[
= -(a_{11} + b_{12} + c_{13} + r_{14} + s_{15} + t_{16})X
\]

\[
- (a_{21} + b_{22} + c_{23} + r_{24} + s_{25} + t_{26})Y.
\]

Therefore, by the simplifications made to the Lie bracket relations in Proposition 5.1, we see that $\mathcal{F}$ is clearly minimal i.e. trace $B^\nu = 0$.

We now check when the foliation $\mathcal{F}$ is totally geodesic by using the same method as in the proof of Theorem 4.2. We find that all but the two following evaluations to be zero.

\[
B^\nu(R, S) = -s_{14}X - s_{24}Y,
\]

\[
B^\nu(R, T) = -t_{14}X - t_{24}Y.
\]

This proves the statement. \qed

6. The Case of $K = \text{SU}(2) \times \text{SO}(2)$ in $G^6$.

Let $(G, g)$ be a six-dimensional Riemannian Lie group and $K$ be its compact subgroup $\text{SU}(2) \times \text{SO}(2)$ equipped with its standard Riemannian metric. Further let $\mathcal{F}$ be the left-invariant foliation on $G$ generated by the Lie subalgebra $\mathfrak{k} = \text{su}(2) \times \text{so}(2)$ of $\mathfrak{g}$. Let $\{A, B, C, T, X, Y\}$ be an orthonormal basis for the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{su}(2)$ is generated by $\{A, B, C\}$ and $\mathfrak{so}(2)$ by $\{T\}$, respectively. For the Lie algebra $\mathfrak{k}$ of $K$ we have the following standard non-vanishing Lie bracket relations

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A.
\]

In this case the subalgebra $\mathfrak{k}$ is not semisimple since $\mathfrak{so}(2)$ is abelian, therefore we find that Remark 3.2 of [5] only applies to $\mathfrak{su}(2)$, the semisimple component of $\mathfrak{k}$. Due to this the remaining Lie bracket relations, given by $\text{ad}_Y(\mathcal{H})$ and $\text{ad}_H(\mathcal{H})$, are of the following form:

\[
[A, X] = a_{11}A + a_{12}B + a_{13}C + a_{14}T,
\]

\[
[A, Y] = a_{21}A + a_{22}B + a_{23}C + a_{24}T,
\]

\[
[B, X] = b_{11}A + b_{12}B + b_{13}C + b_{14}T,
\]

\[
[B, Y] = b_{21}A + b_{22}B + b_{23}C + b_{24}T,
\]

\[
[C, X] = c_{11}A + c_{12}B + c_{13}C + c_{14}T,
\]

\[
[C, Y] = c_{21}A + c_{22}B + c_{23}C + c_{24}T,
\]

\[
[T, X] = x_1X + y_1Y + t_{11}A + t_{12}B + t_{13}C + t_{14}T,
\]

\[
[T, Y] = x_2X + y_2Y + t_{21}A + t_{22}B + t_{23}C + t_{24}T.
\]
$[X,Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 T.$

By a method similar to the one used in the case when $K$ is semisimple, we find that the Lie bracket relations involving $X$ and $Y$ can be greatly simplified.

**Proposition 6.1.** Let $G$ be a six-dimensional Riemannian Lie group and $K$ its four-dimensional subgroup $SU(2) \times SO(2)$. Then the Lie bracket relations for the Lie algebra $\mathfrak{g}$ of $G$ can be written as

$[A,B] = 2C,$
$[C,A] = 2B,$
$[B,C] = 2A,$
$[A,X] = -b_{11} B - c_{11} C,$
$[A,Y] = -b_{21} B - c_{21} C,$
$[B,X] = c_{11} A + c_{12} B,$
$[B,Y] = c_{21} A + c_{22} B,$

$[T,X] = x_1 X + y_1 Y + \frac{1}{2}((-c_{12} x_1 - c_{22} y_1) A + (c_{11} x_1 + c_{21} y_1) B + (-b_{11} x_1 - b_{21} y_1) C),$
$[T,Y] = x_2 X - x_1 Y + \frac{1}{2}((-c_{12} x_2 + c_{22} y_1) A + (c_{11} x_2 - c_{21} x_1) B + (-b_{11} x_2 + b_{21} x_1) C) - \rho T,$
$[X,Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 T,$

where

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} \\
\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} \\
-\rho b_{11} + c_{11} c_{22} - c_{12} c_{21}
\end{pmatrix}.
$$

**Proof.** The arguments needed here are exactly the same as already provided in the proof of Proposition 4.1, for the details see [9]. □

Proposition 6.1 and the next result provide a new 8-dimensional family of 6-dimensional Lie groups carrying a conformal foliation with minimal leaves of codimension two.

**Theorem 6.2.** Let $G$ be a six-dimensional Riemannian Lie group carrying a left-invariant distribution $\mathcal{V}$ generated by the Lie algebra $\mathfrak{su}(2) \times \mathfrak{so}(2)$ of the subgroup $SU(2) \times SO(2)$. The resulting foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is both conformal and Riemannian if and only if $y_1 + x_2 = 0$ and $x_1 = 0$. Furthermore $\mathcal{F}$ is minimal if and only if $\rho = 0$, and totally geodesic if and only if

$$
\rho = y_1 b_{11} = y_1 b_{21} = y_1 c_{11} = y_1 c_{12} = y_1 c_{21} = y_1 c_{22} = 0.
$$

**Proof.** We begin by noting that since $K$ is not semisimple the condition of conformality is not immediately given. Evaluating the second fundamental form $B^H$ of the horizontal distribution we find that

$$
B^H(X,X) = x_1 T, \quad B^H(Y,Y) = -x_1 T,
$$

where $B^H$ is the horizontal second fundamental form.

$\Box$
From this we determine that $\mathcal{F}$ is conformal if and only if $y_1 + x_2 = 0$ and $x_1 = 0$. This clearly implies that $\mathcal{F}$ is Riemannian in those cases.

Next we check whether the foliation $\mathcal{F}$ is totally geodesic or not, by evaluating the second fundamental form of the vertical distribution $\mathcal{V}$. Here we get

$$B^\mathcal{V}(A,A) = 0, \quad B^\mathcal{V}(A,B) = 0, \quad B^\mathcal{V}(A,C) = 0,$$

$$B^\mathcal{V}(A,T) = \frac{y_1}{4} (c_{22} X - c_{12} Y), \quad B^\mathcal{V}(B,B) = 0, \quad B^\mathcal{V}(B,C) = 0,$$

$$B^\mathcal{V}(B,T) = \frac{y_1}{4} (-c_{21} X + c_{11} Y), \quad B^\mathcal{V}(C,C) = 0,$$

$$B^\mathcal{V}(C,T) = \frac{y_1}{4} (b_{21} X - b_{11} Y), \quad B^\mathcal{V}(T,T) = \rho Y.$$

From these results it is immediately clear that $\mathcal{F}$ is minimal when $\rho = 0$. In addition, we see that the foliation is totally geodesic if and only if $\rho = y_1 b_{11} = y_1 b_{21} = y_1 c_{11} = y_1 c_{12} = y_1 c_{21} = y_1 c_{22} = 0$.

\[ \square \]

7. The case of $K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ in $G^6$.

Let $(G, g)$ be a six-dimensional Riemannian Lie group and $K$ be its non-compact subgroup $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ equipped with its standard Riemannian metric. Further let $\mathcal{F}$ be the left-invariant foliation on $G$ generated by the Lie subalgebra $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{so}(2)$ of $\mathfrak{g}$. Let $\{A,B,C,T,X,Y\}$ be an orthonormal basis for the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{sl}_2(\mathbb{R})$ is generated by $\{A,B,C\}$ and $\mathfrak{so}(2)$ by $\{T\}$, respectively. For the Lie algebra $\mathfrak{k}$ of $K$ we have the following standard non-vanishing Lie bracket relations

$$[A,B] = 2C, \quad [C,A] = 2B, \quad [B,C] = -2A.$$

The subgroup $K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ is not semisimple. Due to this the remaining bracket relations, given by $\text{ad}_Y(\mathcal{H})$ and $\text{ad}_Y(\mathcal{H})$, are of the form

$$[A,X] = a_{11} A + a_{12} B + a_{13} C + a_{14} T,$$

$$[A,Y] = a_{21} A + a_{22} B + a_{23} C + a_{24} T,$$

$$[B,X] = b_{11} A + b_{12} B + b_{13} C + b_{14} T,$$

$$[B,Y] = b_{21} A + b_{22} B + b_{23} C + b_{24} T,$$

$$[C,X] = c_{11} A + c_{12} B + c_{13} C + c_{14} T,$$

$$[C,Y] = c_{21} A + c_{22} B + c_{23} C + c_{24} T,$$

$$[T,X] = x_1 X + y_1 Y + t_{11} A + t_{12} B + t_{13} C + t_{14} T,$$

$$[T,Y] = x_2 X + y_2 Y + t_{21} A + t_{22} B + t_{23} C + t_{24} T,$$

$$[X,Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 T.$$

Using exactly the same method as the one for Proposition 6.1, the remaining Lie bracket relations involving $X$ and $Y$ can be simplified as follows,
Proposition 7.1. Let $G$ be a six-dimensional Riemannian Lie group and $K$ be the four-dimensional subgroup $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ of $G$. Then the Lie brackets for $g$ take the form

$$[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = -2A.$$  

$$[A, X] = b_{11}B + c_{11}C, \quad [A, Y] = b_{21}B + c_{21}C,$$
$$[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C,$$
$$[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B,$$

$$[T, X] = x_1X + y_1Y + \frac{1}{2}((-c_{12}x_1 - c_{22}y_1)A + (-c_{11}x_1 - c_{21}y_1)B + (b_{11}x_1 + b_{21}y_1)C),$$
$$[T, Y] = x_2X - x_1Y + \frac{1}{2}((c_{22}x_1 - c_{12}x_2)A + (c_{21}x_1 - c_{11}x_2)B + (-b_{21}x_1 + b_{11}x_2)C) - \rho T,$$
$$[X, Y] = \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4T,$$

where

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\rho c_{12} - b_{11}c_{21} + b_{21}c_{11} \\ -\rho c_{11} - b_{11}c_{22} + b_{21}c_{12} \\ \rho b_{11} - c_{11}c_{22} + c_{21}c_{12} \end{pmatrix}.$$  

Proof. The arguments needed here are exactly the same as already provided in the proof of Proposition 4.1, for the details see [9]. 

Proposition 7.1 and the next result provide a new 8-dimensional family of 6-dimensional Lie groups carrying a conformal foliation with minimal leaves of codimension two.

Theorem 7.2. Let $G$ be a six-dimensional Riemannian Lie group carrying a left-invariant distribution $\mathcal{V}$ generated by the Lie algebra $\text{sl}_2(\mathbb{R}) \times \text{SO}(2)$ of the subgroup $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$. Then the resulting foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is both conformal and Riemannian if and only if $x_2 + y_1 = 0$ and $x_1 = 0$. Furthermore $\mathcal{F}$ is minimal if and only if $\rho = 0$, and totally geodesic if and only if

$$\rho = b_{11} = b_{21} = c_{11} = c_{21} = y_1c_{22} = y_1c_{12} = 0.$$

Proof. We start by determining when the foliation $\mathcal{F}$ is conformal. Calculating the second fundamental form of the horizontal distribution $\mathcal{H}$ we find that

$$B^\mathcal{H}(X, X) = x_1T, \quad B^\mathcal{H}(Y, Y) = -x_1T,$$

$$B^\mathcal{H}(X, Y) = \frac{1}{2}(y_1 + x_2)T.$$

From this it is clear that $\mathcal{F}$ is conformal exactly when $y_1 + x_2 = 0$ and $x_1 = 0$. It immediately follows that in these cases the foliation $\mathcal{F}$ is also Riemannian.
Evaluating the second fundamental form $B^V$ for the basis elements in the vertical distribution $V$ we observe that
\[
B^V(A, A) = 0, \quad B^V(A, B) = -b_{11} X - b_{21} Y, \\
B^V(A, C) = -c_{11} X - c_{21} Y, \quad B^V(A, T) = \frac{y_1}{4} (c_{22} X - c_{12} Y), \\
B^V(B, B) = 0, \quad B^V(B, C) = 0, \quad B^V(B, T) = \frac{y_1}{4} (c_{21} X - c_{11} Y), \\
B^V(C, C) = 0, \quad B^V(C, T) = \frac{y_1}{4} (-b_{21} X + b_{11} Y), \quad B^V(T, T) = \rho Y.
\]
It is immediately apparent that $\mathcal{F}$ is minimal if and only if $\rho = 0$. In addition to this, we find that $\mathcal{F}$ is totally geodesic exactly when
\[
\rho = b_{11} = b_{21} = c_{11} = c_{21} = y_1 c_{22} = y_1 c_{12} = 0.
\]

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References

[1] P. Baird and J. Eells, *A conservation law for harmonic maps*, Geometry Symposium Utrecht 1980, Lecture Notes in Mathematics 894, 1-25, Springer (1981).
[2] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, The London Mathematical Society Monographs 29, Oxford University Press (2003).
[3] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier 28 (1978), 107-144.
[4] S. Gudmundsson, *The Bibliography of Harmonic Morphisms*, www.matematik.lu.se/matematiklu/personal/sigma/harmonic/bibliography.html
[5] S. Gudmundsson, M. Svensson, *Harmonic morphisms from four-dimensional Lie groups*, J. Geom. Phys. 83 (2014), 1-11.
[6] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. 19 (1979), 215-229.
[7] C. G. J. Jacobi, *Über eine Lösung der partiellen Differentialgleichung $\Delta(V) = 0$*, J. Reine Angew. Math. 36 (1848), 113-134.
[8] V. Ottosson *Conformal foliations on Lie groups producing harmonic morphisms*, Bachelor’s thesis, Lund University (2019), http://www.matematik.lu.se/matematiklu/personal/sigma/students/Victor-Ottosson-BSc.pdf
[9] T. Turner, *Minimal and conformal foliations of codimension two on Riemannian Lie groups*, Master’s thesis, Lund University (2020), http://www.matematik.lu.se/matematiklu/personal/sigma/students/Thomas-Turner-MSc.pdf
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