Non-formality of
the odd dimensional framed little balls operads

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Abstract

We prove that the chain operad of the framed little balls (or disks) operad is not formal as a non-symmetric operad over the rationals if the dimension of their balls is odd and greater than 4.

1 Introduction

A chain operad \( O \) is said to be formal if \( O \) and the homology operad \( H_*(O) \) is connected by a chain of weak equivalences (see section 2 for the precise definition).

The formality of the chain little balls operad is a very important property. It is first discovered by D.Tamarkin in the 2 dimensional case, and applied to the proof of deformation quantization, and generalized by M.Kontsevich to the arbitrary dimension (see [10] for a detailed description of Kontsevich’s argument). The formality is also used to compute the homology of knot spaces and more generally, embedding spaces. The framed little balls operad is a cousin of the little balls operad, which encodes rotations of balls, and appears in many areas. For example, it plays an important role in the embedding calculus of T.Goodwille, J.Klein, M.Weiss, in particular in the case that involving manifolds are not parallelizable (see [22]).

J.Giansiracusa and P.Salvatore [21, Theorem.A] proved the formality of framed little 2-balls (disks) operad, and they questioned whether the framed little balls operads of higher dimensions are formal.

In this paper, we give an answer to this question for odd dimensions greater than 4.

Theorem 1.1. Let \( d \) be an odd number greater than 4. If the coefficients are the rational numbers, the \( d \)-dimensional chain framed little balls operad is not formal as a non-symmetric operad.

See Theorem 3.1 for the precise statement. It is clear that non-formality as a non-symmetric operad implies non-formality as a symmetric operad.

2 Preliminary

We shall recall basic definitions and fix notations.

- In the present paper, the coefficients of all modules are the rational numbers. In particular, All homology groups are supposed to have the rational coefficients.
The term "operad" means non-symmetric operad. A \((\text{non-symmetric})\) operad in a symmetric monoidal category \((\mathcal{C}, \otimes, 1)\) consists of a sequence of objects \(O(0), O(1), \ldots, O(n), \ldots\) in \(\mathcal{C}\) and a set of partial compositions \(\circ_i : O(m) \otimes O(n) \to O(m + n - 1)\). which satisfies axioms of associativity and unity (see [19] section 2 or [2] Variants 1.2). A morphism of operads \(f : O \to \mathcal{P}\) is a sequence of morphisms \(\{f_n : O(n) \to \mathcal{P}(n)\}_{n \geq 0}\) compatible with the partial compositions.

We call \(O(n)\) the part of arity \(n\) of \(O\). If \(\mathcal{C}\) is the monoidal category \(\text{Top}\) of topological spaces, continuous maps, and the cartesian product (resp. \(\text{CH}\) of chain complex, chain maps, and the tensor product over \(\mathbb{Q}\)), we call an operad in \(\mathcal{C}\) a topological operad (resp. a chain operad). The singular chain functor \(C_* : \text{Top} \to \text{CH}\) (with the rational coefficients) assigns a chain operad to a topological operad.

A morphism \(f : O \to \mathcal{P}\) of topological operads (resp. chain operad) is said to be a \textit{weak equivalence} if \(f_n : O(n) \to \mathcal{P}(n)\) is a weak homotopy equivalence (resp. a quasi-isomorphism) for each \(n \geq 0\). A chain operad \(O\) is said to be \textit{formal} if there exists a chain of weak equivalences of operads connecting \(O\) and \(H_*(O)\):

\[
\begin{array}{cccc}
O & \xleftarrow{\theta} & O_1 & \cdots \xrightarrow{\theta} & O_N & \xrightarrow{\theta} & H_*(O),
\end{array}
\]

where \(H_*(O)\) is a chain operad given by \(H_*(O)(n) = H_*(O(n))\) with the zero differential.

We shall deal with the Kontsevich operad \(K_d\) and the choose two operad \(S_d\) (see [14] section 4 or [13] section 2). Let \(d\) be a positive integer and \(S^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}\) be the standard unit \((d-1)\)-sphere in the Euclidean space \(\mathbb{R}^d\). We put

\[
S_d(n) = \prod_{1 \leq i < j \leq n} S^{d-1}.
\]

Let \(F_n(\mathbb{R}^d)\) denote the space of ordered configurations of \(n\)-points in \(\mathbb{R}^d\). We define a map \(\theta : F_n(\mathbb{R}^d) \to S_d(n)\) by

\[
\theta(x_1, \ldots, x_n) = \left( \frac{x_j - x_i}{|x_j - x_i|} \right)_{i,j}
\]

In other words, the \((i,j)\)-component of \(\theta(x_1, \ldots, x_n)\) is the direction vector from \(x_i\) to \(x_j\). The space \(K_d(n)\) is defined to be the closure of the image of \(\theta\) in \(S_d(n)\).

To get an intuition of the partial composition of \(S_d\), we shall define a map

\[
(- \circ_i -) : F_m(\mathbb{R}^d) \times F_n(\mathbb{R}^d) \to S_d(m + n - 1)
\]

for \(i = 1, \ldots, m\). Take \((x_1, \ldots, x_m) \in F_m(\mathbb{R}^d)\) and \((y_1, \ldots, y_n) \in F_n(\mathbb{R}^d)\). Intuitively speaking, \((x_1, \ldots, x_m) \circ_i (y_1, \ldots, y_n)\) is represented by the configuration made from \(x_1, \ldots, x_m\) by replacing \(x_i\) with \(y_1, \ldots, y_n\) which are infinitesimally rescaled. More precisely, we first put

\[
(w_1', \ldots, w_{m+n-1}') = (x_1, \ldots, x_{i-1}, x_i + ry_1, \ldots x_i + ry_n, x_{i+1}, \ldots, x_n)
\]

where \(ry_k\) means scalar multiplication by a positive number \(r\). Note that if \(r\) is sufficiently small, \((w_1', \ldots, w_{m+n-1}')\) belongs to \(F_{m+n-1}(\mathbb{R}^d)\). Then, we set

\[
(x_1, \ldots, x_m) \circ_i (y_1, \ldots, y_n) = \lim_{r \to 0} \theta(w_1', \ldots, w_{m+n-1}').
\]
Non-formality

Note that the limits of direction vectors of \( w_1^r, \ldots, w_{m+n-1}^r \) depend only on the direction vectors of \( x_1, \ldots, x_m \) and of \( y_1, \ldots, y_n \), so the above map \((- \circ_i -)\) is naturally extended to a map \((- \circ_i -) : \mathcal{S}_d(m) \times \mathcal{S}_d(n) \rightarrow \mathcal{S}_d(m + n - 1)\), which gives \( \mathcal{S}_d \) a structure of an operad (see [13] section 2 for an explicit formula of this partial composition, where the choose two operad is denoted by \( B_n \)). It is known that the restriction of this partial composition to \( \mathcal{K}_d \) and we endow \( \mathcal{K}_d \) this structure of a sub-operad of \( \mathcal{S}_d \). (see [13] Theorem 4.5)). \( \mathcal{K}_d \) and the framed little balls operad \( \mathcal{D}_d \) is known to be weak equivalent as topological operads for each \( d \geq 1 \). In other words, \( \mathcal{K}_d \) and \( \mathcal{D}_d \) are connected by a chain of weak equivalences. We use (the framed version of) \( \mathcal{K}_d \) instead of \( \mathcal{D}_d \) because \( \mathcal{K}_d \) becomes a multiplicative operad, see below.

- The framed version of \( \mathcal{K}_d \) and \( \mathcal{S}_d \) is given as a semidirect product with the rotation group \( SO(d) \). A notion of a semidirect product of an operad is introduced by Wahl and Salvatore [7, Definition 2.1]. Let \( \mathcal{O} \) be a topological operad and \( G \) be a topological group. Suppose each \( \mathcal{O}(n) \) has a \( G \)-action which satisfies some compatibility conditions (see ibid.). We define a topological operad \( \mathcal{O} \rtimes G \) by

\[
\mathcal{O} \rtimes G(n) = \mathcal{O}(n) \times G^n
\]

for \( (x, g_1, \ldots, g_m) \in \mathcal{O} \times G(m), (y, h_1, \ldots, h_n) \in \mathcal{O} \times G(n) \). The most important example of semi-direct products is the framed little balls operad which is the semi-direct product with respect to the natural action of the rotation group \( SO(d) \) on the \( d \)-dimensional framed little balls operad \( \mathcal{D}_d \) (see Example 2.2 of ibid.). \( \mathcal{K}_d \) and \( \mathcal{S}_d \) also have natural actions of \( SO(d) \) such that the inclusion \( \mathcal{K}_d \rightarrow \mathcal{S}_d \) preserves these actions. For \( \mathcal{O} = \mathcal{D}_d, \mathcal{K}_d, \) or \( \mathcal{S}_d \), we put \( f\mathcal{O} = \mathcal{O} \rtimes SO(d) \). We call \( f\mathcal{K}_d \) the framed Kontsevich operad. \( f\mathcal{K}_d \) and the framed little balls operad \( f\mathcal{D}_d \) are known to be weak equivalent as topological operads (see [13] section 3).

- We will use McClure-Smith’s procedure which produces cosimplicial objects \( \mathcal{O}^* \) from a multiplicative operad (see [10]). Let \( \mathcal{A} \) denote the associative operad. (We consider the unital version so that \( \mathcal{A}(0) \) is a point). A multiplicative operad is a morphism from \( \mathcal{A} \) to an operad \( \mathcal{O} \) and a morphism of multiplicative operads is the same as a morphism of the under category. We can associate a cosimplicial space \( \mathcal{O}^* \) to a multiplicative operad \( f : \mathcal{A} \rightarrow \mathcal{O} \) as follows. Let \( \mu, e \in \mathcal{O}(2) \) be the image of the unique element of \( \mathcal{A}(2) \) and \( e \in \mathcal{O}(0) \) be the image of the unique element of \( \mathcal{A}(0) \). We put \( \mathcal{O}^n = \mathcal{O}(n) \) for each integer \( n \geq 0 \), and we define the coface \( d^i : \mathcal{O}^n \rightarrow \mathcal{O}^{n+1} \) and codegeneracy \( s^i : \mathcal{O}^n \rightarrow \mathcal{O}^{n-1} \) by

\[
d^i(x) = \mu \circ_2 x, \quad d^{n+1}(x) = \mu \circ_1 x, \quad d^i(x) = x \circ_i \mu \quad (1 \leq i \leq n)
\]

\[
s^i(x) = x \circ_i e \quad (1 \leq i \leq n)
\]

The main advantage of the Kontsevich operad to the little balls operad is that it has a structure of multiplicative operad. We set \( v_0 = (1, 0, \ldots, 0) \) as the base point of \( S^{d-1} \). The map \( \mathcal{A}(2) \rightarrow \mathcal{S}_d(2) = S^{d-1} \) taking the value on \( v_0 \) uniquely extends to a morphism \( \mathcal{A} \rightarrow \mathcal{S}_d \). This morphism factors through \( \mathcal{K}_d \). We regard \( \mathcal{S}_d \) and \( \mathcal{K}_d \) as multiplicative operads with these morphisms (see [13] [13]). For \( \mathcal{O} = \mathcal{S}_d, \mathcal{K}_d \), we also regard \( f\mathcal{O} \) as a multiplicative operad with the composition \( \mathcal{A} \rightarrow \mathcal{O} \subset f\mathcal{O} \). Let \( \widetilde{\text{Tot}} \) (resp. \( \text{Tot} \)) denote the totalization (resp. homotopy
We deal with cosimplicial objects $S^*$, $K^*$, $fS^*$, $fK^*$. We use one more cosimplicial object $S^* \times S^{d-1}$. We set $(S^* \times S^{d-1})^n = S^n \times (S^{d-1})^n$. For $(x, v_1, \ldots, v_n) \in (S^* \times S^{d-1})^n$, we set

$$d^0(x, v_1, \ldots, v_n) = (v_0 \circ_1 x, \mu, v_1, \ldots, v_n)$$
$$d^i(x, v_1, \ldots, v_n) = (x \circ_1 v_i, v_1, \ldots, v_i, v_{i+1}, \ldots, v_n)$$
$$d^n(x, v_1, \ldots, v_n) = (v_0 \circ_2 x, v_1, \ldots, v_n, \mu)$$

Here, we use the identification $S^{d-1} = S(2)$.

We can identify the (homotopy) totalizations of the cosimplicial objects $S^*$, $fS^*$, and $(S \times S^{d-1})^*$ as in the following lemma. This lemma was proved by Salvatore.

**Lemma 2.1** (Corollary 10 of [13]). For any integer $d \geq 2$, we have weak homotopy equivalences:

$$\text{Tot}(S^*) \simeq \Omega^2 S^{d-1}, \quad \text{Tot}(S^* \times S^{d-1}) \simeq P\Omega S^{d-1}, \quad \text{Tot}(fS^*) \simeq \Omega(SO(d - 1)).$$

Here $\Omega$ denotes a based loop space and $P$ denotes a based path space.

We shall give a proof of Lemma 2.1 which is slightly different from [13], [14] as it helps the reader understand the rest of the paper. Let $\Delta^n_k$ be the standard simplicial simplex. In other words, $\Delta^n_k$ is the set of weakly order preserving maps $[n] \to [k]$ where $[k] = \{0, 1, \ldots, k\}$ with the usual order. Let $S^n = \Delta^n / \partial \Delta^n$ be the simplicial n-sphere with the basepoint represented by $\partial \Delta^n$ and $D^2 = \Delta^2 / (\Delta_0^2 \cup \Delta_2^2)$ be the pointed simplicial set with the basepoint represented by $\Delta_0^2 \cup \Delta_2^2$. For a pointed simplicial set $K$, and a pointed topological space $T$, Let $T^K$ denote the cosimplicial space defined by

$$(T^K)^n = Map_* (K_n, T),$$

where $Map_*$ denotes the mapping space of pointed topological spaces and $K_n$ is considered as a discrete pointed space. It is well known that the cosimplicial space $T^K$ has the following property:

**Lemma 2.2.** For any pointed simplicial set $K$ and any pointed topological space $T$, (1) there exists a functorial homeomorphism $\text{Tot}(T^K) \cong Map_* ([K], T)$, and (2) the canonical map $\text{Tot}(T^K) \to \text{Tot}(T^K)$ is a weak homotopy equivalence. Here $[K]$ denotes the geometric realization.

To prove Lemma 2.1 we shall define two isomorphisms of cosimplicial spaces:

$$S^* \simeq (S^{d-1}) S^2, \quad S^* \times S^{d-1} \cong (S^{d-1}) D^2.$$ 

The first isomorphism is given as follows. The $n$-simplices of $S^2$ which are not the base point naturally correspond to the (weakly) order-preserving surjections $f : [n] \to$
Here, the right arrow is induced by the map $SO_n$. We have a natural fiber sequence of cosimplicial spaces based cone over the circle $P$. The right space is weak homotopy equivalent to $\Lambda$. The corresponding sequence for homotopy totalization is a homotopy fiber sequence:

$$\text{Tot}((S^1)^d) \cong \text{Tot}((S^1)^d) \cong \text{Map}_*(S^d, S^d-1) \cong \Omega^2(S^d-1).$$

We have a natural fiber sequence of cosimplicial spaces

$$\{SO(d-1)\}^1 \rightarrow fS^1_d \rightarrow S^d \rightarrow S^d-1.$$

Here, the right arrow is induced by the map $SO(d) \rightarrow S^d-1$ which assigns each matrix its first column. This right arrow is a levelwise fibration of cosimplicial spaces so the corresponding sequence for homotopy totalization is a homotopy fiber sequence:

$$\text{Tot}((SO(d-1))^1) \rightarrow \text{Tot}(fS^1_d) \rightarrow \text{Tot}(S^d) \rightarrow S^d-1).$$

The right space is weak homotopy equivalent to $P\Omega(S^d-1)$ which is contractible so the left arrow is weak homotopy equivalence. This and Lemma 2.2 implies the last equivalence in Lemma 2.1. The following lemma is well-known.

**Lemma 2.3.** Let $d = 2m + 1$ be an odd number greater than 2.

1. There exist isomorphisms of rational homology algebras:

   $$H_*(SO(d)) \cong \bigwedge(\beta_1, \ldots, \beta_m), \quad H_*(SO(d-1)) \cong \bigwedge(\beta_1, \ldots, \beta_{m-1}, e)$$

   Here, $\bigwedge$ denotes the free anti-commutative algebra, and we set $\deg \beta_i = 4i - 1$, $\deg e = 2m - 1$.

2. There exist isomorphisms of rational homology algebras:

   $$H_*(\Omega(SO(d))) \cong \mathbb{Q}[\gamma_1, \ldots, \gamma_m], \quad H_*(\Omega(SO(d-1))) \cong \mathbb{Q}[\gamma_1, \ldots, \gamma_{m-1}, f]$$

   Here, $\mathbb{Q}[\cdots]$ denotes the free commutative algebra, and we set $\deg \gamma_i = 4i - 2$, $\deg f = 2m - 2$.

### 3 Proof of Theorem 1.1

As the framed little balls operads are weak equivalent to the framed Kontsevich operads, Theorem 1.1 is equivalent to the following statement which we will prove in the rest of the paper.

**Theorem 3.1.** If $d$ is odd and greater than 4, The chain operad $C_*(fK_d)$ of the framed $d$-dimensional Kontsevich operad is not formal.
Non-formality

In the rest of paper, \( d = 2m + 1 \) denotes an odd number greater than 4. The following lemma is a key in the proof of Theorem 3.1. For \( \mathcal{O} = \mathcal{K}_d \) or \( \mathcal{S}_d \), let \( E_{p,q}^r(f\mathcal{O}) \) denote the \( E^r \)-page of Bousfield-Kan spectral sequence associated to the cosimplicial chain complex \( C_*((f\mathcal{O})^\bullet) \) (Note that \( E_{p,q}^r \) is the part of cosimplicial degree \( -p \), chain degree \( q \)). \( E_{p,q}^r(f\mathcal{O}) \) has a natural structure of a Gerstenhaber algebra.

**Lemma 3.2.** Let \( \mathcal{O} \) be \( \mathcal{K}_d \) or \( \mathcal{S}_d \).

1. There exists an isomorphism of algebras
   \[
   E_{p,q}^2(f\mathcal{O}) \cong \bigoplus_{p_1 + p_2 = p, q_1 + q_2 = q} HH_{p_1,q_1}(H_\bullet(\mathcal{O})) \otimes H_{p_2,q_2}(cB^d)
   \]
   which is natural with respect to the inclusion \( i : \mathcal{K}_d \to \mathcal{S} \). Here, \( HH(H_\bullet(\mathcal{O})) \) denotes the Hochschild homology of \( H_\bullet(\mathcal{O}) \) considered as a chain operad with the zero differential (see [1, 12, 13, 14] for the definition of Hochschild cohomology), and \( cB^d \) denotes the coar complex for the coalgebra \( H_\bullet(SO(d)) \) with Alexander-Whitney diagonal. \( H_\bullet(\mathcal{O}) \) is the total homology with the natural bi-grading.

2. Let \( \bar{\beta}_m \) be the image of the element \( \beta_m \) by the natural map \( H_{4m-1}(SO(d)) \to H_{-1,4m-1}(cB^d) \). When we regard \( 1 \otimes \bar{\beta}_m \) as an element of \( E^2(f\mathcal{O}) \) under the isomorphism of (1), \( d^2(1 \otimes \bar{\beta}_m) \) is non-zero in \( E^2(f\mathcal{O}) \).

**Proof.** The proof of part 1 is easy. By definition, \( E_{2,*}^2(f\mathcal{O}) \cong HH_{2,*}(H_\bullet(\mathcal{O})) \). The action of the Hopf algebra \( H_\bullet(SO(d)) \) on \( H_\bullet(\mathcal{O}(n)) \) is trivial by the degree reason for any odd \( d \), see [14, Theorem 6.5], so the semidirect product splits on the homology level. This easily implies \( HH_{2,*}(H_\bullet(\mathcal{O})) \cong HH_{2,*}(H_\bullet(\mathcal{O})) \otimes H_{2,*}(cB^d) \).

We shall show the part 2 for \( \mathcal{O} = \mathcal{S}_d \). Note that the partial compositions of \( \mathcal{S}_d \) is defined by the diagonal map of \( S^{d-1} \). This fact and the formality of the rational singular chain coalgebra \( C_\bullet(S^{d-1}) \) implies the formality of the rational singular chain multiplicative operad \( C_\bullet(\mathcal{S}_d) \). This multiplicative operad formality implies the Bousfield-Kan spectral sequence associated to the cosimplicial space \( \mathcal{S}_d \) collapses at \( E^2 \), see the proof of Theorem 1.4 in [23]. This spectral sequence converges to the homology \( H_\bullet(Tot(\mathcal{S}_d)) \). By these observations and Lemma 2.1, we have an isomorphism

\[
HH_{2,*}(H_\bullet(\mathcal{S}_d)) \cong H_\bullet(\Omega^2 S^{d-1})
\]

Moreover, this isomorphism is an isomorphism of Gerstenhaber algebras (see the proof of Corollary 1.6 in [23]). Let \( \{ - , - \} \) denote Gerstenhaber bracket. It is well-known that \( H_\bullet(\Omega^2 S^{d-1}) \) is generated by two elements \( x, \{ x, x \} \) with deg \( x = d - 3 \) as a graded commutative algebra. Under the above isomorphism, \( x \) corresponds to an element \( \alpha \in H_{d-1}(\mathcal{S}_d(2)) \subset HH_{-2,d-1}(H_\bullet(\mathcal{S}_d)) \) and \( \{ \alpha, \alpha \} \) corresponds to \( \{ \alpha, \alpha \} \in H_{2d-2}(\mathcal{O}(3)) \). Look at the isomorphism of part 1

\[
E_{2,*}^2(f\mathcal{S}_d) \cong HH_{2,*}(H_\bullet(\mathcal{S}_d)) \otimes H_{2,*}(cB^d)
\]

The right hand side of this isomorphism is generated by \( \alpha \otimes 1, \{ \alpha, \alpha \} \otimes 1, 1 \otimes \bar{\beta}_1, \ldots, 1 \otimes \bar{\beta}_m \). Here \( \bar{\beta}_i \) is the image of the element \( \beta_i \) by the natural map \( H_{4i-1}(SO(d)) \to H_{-1,4i-1}(cB^d) \). Note that there exists morphisms of cosimplicial spaces

\[
\mathcal{S}_d^* \to f\mathcal{S}_d^* \leftrightarrow SO(d-1)^{S^1}.
\]

Here the left morphism is induced by the inclusion to unity \( * \to SO(d) \) and the right one by the inclusion \( SO(d-1) \subset SO(d) \) (see the proof of Lemma 2.1). These morphisms induce morphisms of spectral sequences

\[
E^r_{p,q}(\mathcal{S}_d^*) \to E^r_{p,q}(f\mathcal{S}_d^*) \leftrightarrow E^r_{p,q}(SO(d-1)^{S^1})
\]
As \(\alpha \otimes 1\) and \(\{\alpha, \alpha\} \otimes 1\) comes from the left hand side, they are cycles in any page of the spectral sequence. \(E^r_*,(fS^n_k)\) converges to \(H_*(\Omega(SO(d-1)))\) as \(E^p_{0,q} = 0\) if \(q < -d\frac{d-1}{2}\) and \(d\frac{d-1}{2} > 1\) (this is the point we use \(d \geq 5\)). The total degree of \(\{\alpha, \alpha\} \otimes 1\) is odd and those of the rest of the generators are even, and the homology \(H_*(\Omega(SO(d-1)))\) is non zero only in even degree. These facts imply that \(\{\alpha, \alpha\} \otimes 1\) must be a boundary in some page. Suppose \(\{\alpha, \alpha\} \otimes 1\) is a boundary in \(E^r\)-page but not in \(E^{r-1}\)-page. Take an element \(y\) such that \(d^r(y) = \{\alpha, \alpha\} \otimes 1\). We may write \(y = y_1 + k \otimes \beta_m\) where \(y_1\) is a polynomial of \(1 \otimes \beta_1, 1 \otimes \beta_2, \ldots, 1 \otimes \beta_{m-1}\) and \(k\) is a scalar. As \(SO(d-1)^S\) is the usual cosimplicial cobar complex of \(SO(d-1)\), by Lemma 2.3, \(1 \otimes \beta_1, \ldots, 1 \otimes \beta_{m-1}\) come from the right hand side and they are cycles in any page. This implies \(d^r(y_1) = 0\) and \(kd^r(1 \otimes \beta_m) = \{\alpha, \alpha\} \otimes 1\). As the bidegrees of \(1 \otimes \beta_m\), \(\{\alpha, \alpha\} \otimes 1\), and \(d^r\) is \((-1,4m-1), (-3,4m)\) and \((-r,r-1)\) respectively, we conclude \(r = 2\), and we have \(d^2(1 \otimes \beta_m) \neq 0\).

By the naturality of the spectral sequences for the inclusion \(K_d \subset S_d\), the result for \(O = S_d\) immediately implies the \(d^2(1 \otimes \beta_m) \neq 0\) for \(O = K_d\).

By the part 2 of Lemma 3.2, we immediately see \(fK_d\) is not formal as a multiplicative operad (see the proof of Theorem 1.4 of [23]). To obtain the genuine non-formality, we need additional technical arguments. We shall define an obstruction to formality.

**Definition 3.3.** Let \(O\) be a chain operad such that the homology \(H_*(O)\) is isomorphic to \(H_*(fK_d)\). Let \(\nu \in O(2)_0\) be a 0-cycle which represents a generator of \(H_0(O(2)) \cong \mathbb{Q}\). Let \(g \in O(1)_{4m-1}\) be a \(4m-1\)-cycle which represents a primitive generator of \(H_*(O(1)) \cong H_*(SO(d))\). (As the diagonal \(SO(d) \rightarrow SO(d) \times SO(d)\) and the multiplications of the group \(SO(d) \times SO(d) \rightarrow SO(d)\) can be constructed in terms of the partial compositions of \(fK_d\), \(H_*(O)\) has an intrinsic Hopf algebra structure unique up to a scalar multiple and the term “primitive” makes sense for an element of \(H_*(O)\) independently of a particular isomorphism \(H_*(O) \cong H_*(fK_d)\).) As \([g]\) is primitive, we have

\[
[g \circ \nu] = [\nu \circ 2 g + \nu \circ 1 g].
\]

We can pick an element \(h \in O(2)_{4m}\) such that

\[
dh = \nu \circ 2 g + \nu \circ 1 g - g \circ \nu.
\]

As \([\nu]\) represents an associative multiplication, we can pick an element \(\xi \in O(3)_1\) such that

\[
d\xi = \nu \circ 2 \nu - \nu \circ 1 \nu.
\]

Then, we define an element \(\omega = \omega_1(\nu,g) \in O(3)_{4m}\) by

\[
\omega_1 = \nu \circ 2 h - h \circ 1 \nu + h \circ 2 \nu - \nu \circ 1 h
\]

\[
\omega_2 = g \circ \xi + \xi \circ 1 g + \xi \circ 2 g + \xi \circ 3 g
\]

Using \([\nu]\), we may define the Hochschild complex of \(H_*(O)\). Its differential \(\delta_\nu : H_*(O(n)) \rightarrow H_*(O(n+1))\) is given by \(\delta_\nu([x]) = [\nu \circ 2 x + \sum_{i=1}^{n-1} (-1)^n x_{o1} \nu + (-1)^n \nu \circ 1 x]\)

**Lemma 3.4.** Under the above notations, \(\omega\) is a cycle and the corresponding class \([\omega] \in H_{4m}(O(3))/\delta_\nu H_{4m}(O(2))\) is independent of choice of \(h\) and \(\xi\).

**Proof.** We first show \(\omega\) is a cycle.

\[
d\omega_1 = \nu \circ 2 dh - dh \circ 1 \nu + dh \circ 2 \nu - \nu \circ 1 dh
\]

\[
= \nu \circ 2 (\nu \circ 2 g + \nu \circ 1 g - g \circ \nu) - (\nu \circ 2 g + \nu \circ 1 g - g \circ \nu) \circ 1 \nu
\]

\[
+ (\nu \circ 2 g + \nu \circ 1 g - g \circ \nu) \circ 2 \nu - \nu \circ 1 (\nu \circ 2 g + \nu \circ 1 g - g \circ \nu)
\]
By the associativity of partial composition, we have some equalities. For example, we have \( \nu \circ_2 (\nu \circ_2 g) = (\nu \circ_2 \nu) \circ_3 g \) and \((\nu \circ_2 g) \circ_1 \nu = (\nu \circ_1 \nu) \circ_3 g \). By using these and similar equalities, we have
\[
d\omega_1 = -g \circ (\nu \circ_2 \nu - \nu \circ_1 \nu) + (\nu \circ_2 \nu - \nu \circ_1 \nu) \circ_1 g \\
+ (\nu \circ_2 \nu - \nu \circ_1 \nu) \circ_2 g + (\nu \circ_2 \nu - \nu \circ_1 \nu) \circ_3 g \\
= -g \circ d\xi + (d\xi) \circ_1 g + (d\xi) \circ_2 g + (d\xi) \circ_3 g \\
= d\omega_2
\]
Thus, we have \( d\omega = d(\omega_1 - \omega_2) = 0 \).

Let \( h', h'' \) (resp. \( \xi', \xi'' \)) be two elements satisfying the condition of \( h \) (resp. \( \xi \)) in Definition 3.3. In the rest of the proof, \( \omega' = \omega'_1 - \omega'_2 \) and \( \omega'' = \omega''_1 - \omega''_2 \) denote the elements defined by using \( (h', \xi') \) and \( (h'', \xi'') \) as in Definition 3.3 respectively. Put \( h = h' - h'' \). As \( h \) is a cycle, the class of
\[
\omega'_1 - \omega''_1 = \nu \circ_2 h - \nu \circ_1 \nu + \nu \circ_2 \nu - \nu \circ_1 \nu
\]
is the image of \( [h] \) by \( \delta_1 \) and belongs to \( \delta_1H_4m(\mathcal{O}(2)) \). As \( H_1(\mathcal{O}(3)) \cong H_1(fK_d(3)) = 0 \) and \( \xi' - \xi'' \in \mathcal{O}(3) \) is a cycle, it is a boundary. So \( \omega'_1 - \omega''_1 \) is also a boundary. Thus \( \omega' - \omega'' \) represents the zero class in \( H_4m(\mathcal{O}(3))/\delta_1H_4m(\mathcal{O}(2)) \). In other words, \( [\omega'] = [\omega''] \).

The class \( [\omega] \) may depend on the choices of \( \nu \) and \( g \). We must take care about these choices in the following proof. We use a model category of chain operads. For general theory of model categories, see [4]. It is known that the category of chain operads has a model category structure where weak equivalences are the same as those given in section 2 (see [23] Theorem 2.1) or [13] Theorem 1.1, see also [3] for a model category of symmetric operads).

**Proof of Theorem 3.7** Set \( \mathcal{O} = C_*(fK_d) \). Let \( \nu \in \mathcal{O}(2) \) be the 0-cycle represented by the image of the unique point by the structure morphism \( A \to fK_d \) and \( g \in \mathcal{O}(1)_{4m-1} \) be arbitrary \( 4m-1 \)-cycle satisfying the condition of Definition 3.3. As \( \nu \) is strictly associative, we may take zero as \( \xi \). In this case, we see \( [\omega(\nu, g)] = d_2(1 \otimes \beta_{4m}) \) (up to non-zero scalar multiplication) by unwinding the definition of the differential of the spectral sequence. So by the part 2 of Lemma 3.2 we see \( [\omega] \in H_{4m}(\mathcal{O}(3))/\delta_1H_{4m}(\mathcal{O}(2)) \) is non-zero for this choice of \( \nu \). Let \( \mathcal{A}_\infty \) be the Stasheff’s associahedral chain operad. \( \mathcal{A}_\infty \) is a cofibrant operad and has a set of generators \( \{ \nu_i \mid i \geq 1 \} \) (see REFERENCE). We define a morphism of operad \( f : \mathcal{A}_\infty \to \mathcal{O} \) by \( \nu_1 \to \nu \) and taking the other generators to zeros. By a functorial factorization of the model category, \( f \) is factorized as \( \mathcal{A}_\infty \to \mathcal{P} \cong \mathcal{O} \). As \( \mathcal{A}_\infty \) is cofibrant, so is \( \mathcal{P} \).

Suppose \( \mathcal{O} \) is formal. In other words, \( \mathcal{O} \) and \( H_*(\mathcal{O}) \) is connected by a chain of weak equivalences of operads. As \( \mathcal{P} \) is cofibrant (and any chain operad is fibrant), by the theory of model categories, there exists a weak equivalence \( q : \mathcal{P} \to H_*(\mathcal{O}) \). Let \( g' \in \mathcal{P}(1) \) be a cycle satisfying the condition of Definition 3.3 for \( \mathcal{P} \). We have isomorphisms
\[
H_*(\mathcal{O}) \cong H_*(\mathcal{P}) \cong H_*(H_*(\mathcal{O})) \cong H_*(\mathcal{O})
\]
\[
\omega(\nu, p(g')) \cong \omega(\nu(i(\nu_1)), g') \cong \omega(\nu(q(\nu_1)), q(g'))
\]
By these isomorphisms, \( [\omega(\nu, p(g'))] \) corresponds to \( [\omega(q(\nu_1), q(g'))] \). As we show in the above, \( [\omega(\nu, p(g'))] \) is non-zero. On the other hand, the differential of \( H_*(\mathcal{O}) \) is zero, we
may choose zeros as $h$ and $\xi$ in the definition of $\omega(q(\nu_1), q(g'))$. Hence $\omega(q(\nu_1), q(g'))$ represents zero in $H_{4m}(O(3))/\delta_{q(\nu_1)}H_{4m}(O(2))$. This is a contradiction. □

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