A stability analysis for the Korteweg-de Vries equation

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Abstract

In this paper the stability of the Korteweg-de Vries (KdV) equation is investigated. It is shown analytically and numerically that small perturbations of solutions of the KdV-equation introduce effects of dispersion, hence the perturbation propagates with a different velocity than the unperturbed solution. This effect is investigated analytically by formulating a differential equation for perturbations of solutions of the KdV-equation. This differential equation is solved generally using an Inverse Scattering Technique (IST) using the continuous part of the spectrum of the Schrödinger equation. It is shown explicitly that the perturbation consist of two parts. The first part represents the time-evolution of the perturbation only. The second part represents the interaction between the perturbation and the unperturbed solution. It is shown explicitly that singular non-dispersive solutions of the KdV-equation are unstable.

1 Introduction

Since the discovery of the IST, a great number of nonlinear differential equations have been solved using this technique (see for instance ref. and the references therein). Most of the scientific effort has been focused on finding soliton solutions for these equations. As firstly observed by Scott-Russell in 1834, solitons have the property that they maintain their shape over long time-scales. From this observation the conclusion can be drawn that solitons which are observed in nature are stable solutions.

In this paper we focus on the KdV-equation. The IST approach, as introduced by Gardner, Greene, Kruskal and Miura (GGMT), uses the inverse problem of the Schrödinger equation to solve the KdV-equation. In their approach a solution of the KdV-equation can be regarded as a time-dependent potential of the Schrödinger equation. GGMT have presented a method to compute the time-dependence of the S-matrix that is used to solve the inverse problem of the Schrödinger equation. Furthermore, they have shown that soliton solutions can be constructed using the discrete part of the spectrum of the Schrödinger equation.

Recently, it was discovered, (by solving the KdV-equation using the continuous part of the spectrum of the Schrödinger equation) that the KdV-equation has also singular solutions. It is also shown in ref. that the time-evolution of these singular solutions can be associated with a positive Lyapunov exponent. This implies that singular solutions of the KdV-equation can exhibit unstable behavior.

It is remarkable that although the KdV-equation is one of the better studied equations in the field of mathematical physics, the matter of the stability of this equation has not been investigated. The classical argument that explains that solitons maintain their shape over long time-scales is the presence of the nonlinear term in the KdV-equation that cancels the effects of dispersion. However in nature, solitons are always contaminated with noise and it is not clear that when noise is added to a pure soliton solution, the soliton retains its identity. This matter is related to the stability properties of solutions of the KdV-equation. In this paper we show that for the KdV-equation solitons possess stable behavior if the propagation velocity of the perturbation differs from the perturbation velocity of the unperturbed soliton. In the case of the singular solution this is also the case, but in contrast to the soliton case, the amplitude of the perturbation can grow dramatically. This implies that the shape of the singular solution changes.

This paper has the following structure. In Sec.2 we discuss the balance between nonlinearity and dispersion for non-dispersive solutions of the KdV-equation. It is shown that for small perturbations this balance is disturbed, and that the perturbation has a different propagation velocity than the unperturbed solution. In Sec.3, we derive a general expression for a perturbation of a solution of the KdV-equation. It is shown that the time-evolution of the perturbation consists of two parts. The first part represents the time-evolution of the perturbation in absence of the unperturbed solution. The second part represents the interaction between the soliton and the perturbation. In Sec.4, we give an expansion of the time-evolution of the perturbation in a series solution. This enables us to formulate an expression for the stability of solutions of the KdV-equation. In Sec.5, an numerical example is given to illustrate that the results that are derived in previous sections are also valid in more general cases. The results are discussed in Sec.6. Technical matter concerning the Gelfand-Levitan-Marchenko equations are added in a appendix.
2 Nonlinearity versus dispersion

Consider the KdV-equation:
\[
\begin{align*}
  u_{xxx} - 6uu_x + u_t &= 0 \\
  u(x,t = 0) &= u_0(x)
\end{align*}
\]  

(1)

In Eq. (1), \(u_0(x)\) represents the initial condition. Non-dispersive solutions of Eq. (1) can be obtained by searching for solutions \(u(x,t) = u(x-ct) \equiv u(z)\). In this special case Eq. (1) reduces to:
\[
  u''' - 6uu' - cu' = 0
\]

(2)

The KdV-equation (1) is in the special case of non-dispersive solutions reduced to a third-order ordinary nonlinear differential equation. In Eq. (2), the derivative \(u'\) stands for \(\frac{d}{dz}u(z)\). We can reformulate Eq. (2) in the following way:
\[
  \partial_z (u'' - 3u^2 - cu) = 0
\]

(3)

By integrating Eq. (3), we find that Eq. (2) is equivalent with:
\[
  u'' - 3u^2 - cu + m = 0
\]

(4)

In Eq. (4), the constant \(m \in \mathbb{R}\) is an arbitrary integration constant. We can perform a further simplification by multiplying both sides of Eq. (4) with a factor \(u'\). The total result can be formulated in the following way:
\[
  \partial_z \left( \frac{1}{2} (u')^2 - u^3 - \frac{1}{2} cu^2 + mu \right) = 0
\]

(5)

By performing a further integration we find that Eq. (5) is equivalent with:
\[
  \frac{1}{2} (u')^2 - u^3 - \frac{1}{2} cu^2 + mu + n = 0
\]

(6)

In Eq. (6), the constant \(n \in \mathbb{R}\) is another arbitrary integration constant. In the special case that \(m = n = 0\), Eq. (6) is equivalent with:
\[
  u' = \pm u \sqrt{2u + c}
\]

(7)

We have obtained that in the case of non-dispersive solutions the KdV-equation (1) is equivalent with Eq. (7). Eq. (7) is an ordinary first-order nonlinear differential equation that can be solved directly. This leads to the following result:
\[
  u(z) = \frac{4c}{D_0} e^{\pm \sqrt{c}z} \left( 1 - \frac{2}{D_0} e^{\pm \sqrt{c}z} \right)^2
\]

(8)

In Eq. (8) the constant \(D_0 \in \mathbb{R}\) can be chosen arbitrarily and has to be determined by the initial condition. If the constant \(D_0\) is negative, the solution (8) can be written as a soliton solution:
\[
  u(z) = -\frac{c}{2} \text{sech}^2 (z \sqrt{c} + x_0)
\]

(9)

If the constant \(D_0\) is positive, this reduction does not take place. It follows from Eq. (8) that in this case the solution of the KdV-equation has a singularity, but the solution propagates without dispersion. It has been argued in ref. [4] that these singular solutions can be associated with a time-evolution having a positive Lyapunov exponent. This leads to the conclusion that apart from the stable soliton solutions, the KdV-equation has also unstable solutions.
is added to a non-dispersive solution of the KdV-equation, the balance between the dispersion and non-linearity is disturbed. In order to investigate this effect we rewrite the KdV-equation in the following way:

\[ u_t = -cu_x + (cu_x - u_{xxx}) + 6uu_x \quad (10) \]

If the last two terms on the right-hand side of Eq. (10) are removed, we obtain:

\[ u_t = -cu_x \quad (11) \]

Eq. (11) has the non-dispersive solution \( u(x,t) = g(x-ct) \). If the KdV-equation has non-dispersive solutions, the second and the third term on the right-hand side of Eq. (10) cancel each other. The \( cu_x - u_{xxx} \) term on the right-hand side of Eq. (10) describes the dispersion of the solution \( u(x,t) \). The \( 6uu_x \) term on the right-hand side describes the effects of the nonlinearity.

The solid line in Fig.1a the solid line represents a soliton at time \( t = 0 \). The dashed line in Fig.1a, represents the soliton that is contaminated with a 10% amplitude error. In Fig.1b, the balance between the effects of nonlinearity and the effects of dispersion is shown for the unperturbed soliton. The short-dashed line in Fig.1b represent the effect of dispersion as given by the \( cu_x - u_{xxx} \) term on the right-hand side of Eq. (10). The long-dashed line describes the nonlinearity as given by the \( 6uu_x \) term on the right-hand side of Eq. (10). The sum of these curves are given by the solid line in Fig.1b. It follows from Fig.1b that the dispersion and nonlinearity are in balance. This is consistent with the fact that the soliton propagates with velocity \( c \) while maintaining its shape.

If the soliton is contaminated with a small amplitude error, the dispersion and nonlinearity are no longer in balance. In Fig.1c, the examples of Fig.1b are repeated for the contaminated soliton shown by the dashed line in Fig.1a. Similarly as in Fig.1b, the short-dashed line in Fig.1c represents the effect of dispersion as given by the \( cu_x - u_{xxx} \) term on the right-hand side of Eq. (10). The long-dashed line describes the nonlinearity as given by the \( 6uu_x \) term on the right-hand side of Eq. (10). The solid line in Fig.1c represents the sum of the dispersion and the nonlinearity. An imbalance between the dispersion and the nonlinearity is introduced due to the amplification of the nonlinearity. As a result of the fact that the nonlinearity and the dispersion no longer cancel, Eq. (11) (which is only valid for non-dispersive solutions of the KdV-equation), does not describe the time-evolution of the perturbed solution. The propagation effects of the contaminated soliton is given by Eq. (11). The imbalance between the effects of dispersion and nonlinearity is given by the solid line Fig.1c, which is equal to the sum of the last two terms on the right-hand side of Eq. (10). In Fig.1d, the time-derivative of the perturbation is plotted. It turns out that the time-derivative of the perturbation is not equal to a constant times the space derivative of the perturbation. This implies that the time-evolution of the perturbation is dispersive. In the following sections this behavior is investigated analytically. Finally, in Fig.1e, the finite difference solution of contaminated soliton is given after at \( t = 2.2 \) sec. It is observed that as a consequence of the imbalance between the nonlinearity and dispersion, the perturbation propagates with a different velocity than the unperturbed soliton, and during the course of time, the contaminated soliton takes its most natural form.

From this experiment, we can conclude that as a result of the imbalance between the effects of nonlinearity and dispersion, the perturbation of a solution of the KdV-equation propagates with a different velocity. As a result of this the soliton and the perturbation separate during the course of time.

In Fig.2ab the examples of Fig.1 are repeated for the singular solution. The solid line in Fig.2a describes the singular solution of the KdV-equation at \( t = 0 \). The short-dashed line in Fig.2a describes the effects of dispersion as given by the \( cu_x - u_{xxx} \) term on the right-hand side of Eq. (10). The long-dashed line describes the effects of nonlinearity as described by the \( 6uu_x \) term on the right-hand side of Eq. (10). Similarly as for the soliton, it turns out that also for the singular solution effects of dispersion and nonlinearity are in balance. If a small amplitude error is made, this balance is no longer present. In Fig.2b, the effects of dispersion and nonlinearity are plotted. The solid line in Fig.2b represents the sum of the nonlinearity and dispersion in the contaminated case. It turns out that the effects on nonlinearity and dispersion are no longer in balance. As a result of this, the perturbation propagates in the opposite direction the unperturbed wave.

From the simple examples in this section, we can conclude that if nondispersive solutions of the KdV-equation are contaminated with small perturbations, dispersion effects are introduced. The imbalance between the dispersion and nonlinearity is in the coordinate frame that moves with the unperturbed solution very close to the x-derivative of the perturbation. This implies that the perturbation has a non-zero velocity in this reference frame so that the perturbation separates with time from the unperturbed solution.
3 The stability of localized solutions

The result in the previous section indicates that perturbations on the initial condition of the KdV-equation propagate with a different velocity than the unperturbed solution. In this section the effects of different perturbations are investigated in a more general fashion. If a perturbation $u(x, t) \rightarrow u(x, t) + f(x, t)$ is substituted in Eq. (14), the following differential equation for the perturbation $f(x, t)$ can be derived:

$$f_{xxx} - 6 (uf_x + fxu + ffx_x) + ft = 0$$  
$$f(x, t = 0) = f_0(x)$$ (12)

Eq. (12) represents a differential equation for the perturbation $f(x, t)$, which depends on the unperturbed solution of the KdV-equation $u(x, t)$. Eq. (12) can be solved using an inverse scattering technique if a satisfying Lax-pair is constructed. However, because of the fact that the perturbed solution $u(x, t) + f(x, t)$ satisfies the KdV-equation, the solution $f(x, t)$ of Eq. (12) can be computed directly using the techniques described in ref. [6].

In Appendix A, analytical expressions for the solution $u_0(x)$ undergoes a perturbation:

$$R(k, t = 0) \rightarrow R(k, t = 0) + \hat{R}(k, t = 0)$$ (13)

In Eq. (13), $R(k, t)$ describes the reflection coefficient corresponding to the initial condition. Since the relation between the reflection coefficient and the potential function is nonlinear, the perturbation of the reflection coefficient $\hat{R}(k, t = 0)$ cannot be associated with the spectral reflection coefficient corresponding to the initial condition $f_0(x)$. However, we can construct any initial condition $f_0(x)$, by imposing special conditions on the spectral reflection coefficient $\hat{R}(k, t)$. With this we mean, the we can compute and analyze the time-evolution of different classes of perturbations $f(x, t)$, by changing the analytical structure of $\hat{R}(k, t)$. If both the unperturbed reflection coefficient and the perturbation of the reflection coefficient are rational functions of the wavenumber, analytic expressions for $f(x, t)$ can be obtained.

Suppose $R(k, t)$ is a spectral reflection coefficient that can be associated with the unperturbed solution $u(x, t)$. In Appendix A, analytical expressions for the solution $u(x, t)$ are derived if the reflection coefficient $R(k, t)$ is a rational function of the wavenumber. From the unperturbed reflection coefficient $R(k, t)$, we can construct a kernel $K(x, x, t)$ (Appendix A):

$$K(x, x, t) = \frac{D'(x, t)}{D(x, t)}$$ (14)

In Eq. (14) stands the prime for the derivative with respect to the space-coordinate $x$. The determinant $D(x, t)$ in Eq. (14) is given by:

$$D(x, t) = \det \left\{ \delta_{ij} - (p_i + p_j)^{-1} R e^{2i(p_i x + 4p_j t)} \right\}$$ (15)

In Eq. (15), $p_i$ are the poles and $R_i$ the residues of the unperturbed reflection coefficient $R(k, t)$. Solutions of the KdV-equation can be derived by taking the following derivative:

$$u(x, t) = -2 \frac{d}{dx} K(x, x, t)$$ (16)

If the reflection coefficient $R(k, t)$ is contaminated with a small perturbation $\hat{R}(k, t)$, Eq. (15) contains the poles and residues of both the unperturbed reflection coefficient $R(k, t)$ and the reflection coefficient corresponding to the perturbation $\hat{R}(k, t)$. It is shown in ref. [6], that if the reflection coefficient $R(k, t)$ undergoes a perturbation as given in Eq. (13), the determinant $D(x, t)$ can be expanded into the following series:

$$D(x, t) \rightarrow D(x, t) + E(x, t)$$ (17)

In Eq. (17), $D(x, t)$ is the determinant (15) in absence of perturbations. The effect of the perturbation is expressed in the determinant $E(x, t)$. If Eq. (17) is substituted in Eq. (14), we obtain the following result:

$$K(x, x, t) \rightarrow \frac{D'(x, t) + E'(x, t)}{D(x, t) + E(x, t)}$$ (18)
Using some basic algebra, the kernel associated with the unperturbed solution \( u(x, t) \) can be separated from the right-hand side of Eq.(19):

\[
K(x, x, t) \to \frac{\mathcal{D}'(x, t)}{\mathcal{D}(x, t)} + \frac{\mathcal{D}(x, t)\mathcal{E}'(x, t) - \mathcal{D}'(x, t)\mathcal{E}(x, t)}{\mathcal{D}(x, t)[\mathcal{D}(x, t) + \mathcal{E}(x, t)]} \tag{19}
\]

The term \( \mathcal{D}'(x, t)/\mathcal{D}(x, t) \) in Eq.(19) can be identified with the time-evolution of the unperturbed problem. From Eq.(19), we can identify an expression for the perturbation \( f(x, t) \):

\[
f(x, t) = -2 \frac{d}{dx} \left\{ \frac{\mathcal{D}(x, t)\mathcal{E}'(x, t) - \mathcal{D}'(x, t)\mathcal{E}(x, t)}{\mathcal{D}(x, t)[\mathcal{D}(x, t) + \mathcal{E}(x, t)]} \right\} \tag{20}
\]

It should be realized that in the determinant \( \mathcal{E}(x, t) \) the poles and residues of both the reflection coefficients \( R(k, t) \) and \( \overline{R}(k, t) \) are present. It follows from Eq.(19) that the perturbation \( f(x, t) \) is large with respect to the unperturbed solution if the denominator \( \mathcal{D}(x, t)[\mathcal{D}(x, t) + \mathcal{E}(x, t)] \) of Eq.(20) is small. As illustrated in the following examples, in this case the perturbation \( f(x, t) \) can dominate the total solution of the KdV-equation.

In the following examples, we consider the case in which the unperturbed solution of the KdV-equation has one single pole \( (p = i\beta) \) and one residue \( (R = id) \). The unperturbed solution of the KdV-equation has either soliton-like behavior as in Eq.(18) or singular behavior depending on the position of the pole and the residue. In the following experiment, we choose in case of the soliton \( d = -1 \) and \( \beta = 1 \). We can illustrate the effects of perturbations on the soliton by adding a certain number of poles and residues to the unperturbed determinant \( \mathcal{D}(x, t) \). In the lower panel of Fig.3a, the time-evolution of a contaminated soliton \( u(x, t) \) is plotted. As a reference, in the upper panel of Fig.3a the time-evolution of the unperturbed soliton is given. It can be observed from Fig.3a that the effects of the contamination either spreads out, or travel with a different velocity. This implies that after a certain amount of time the unperturbed soliton and the effects of the perturbation separate. This is more clear in Fig.3b. The short-dashed contamination either spreads out, or travel with a different velocity. This implies that after a certain amount of time the unperturbed soliton is given. It can be observed from Fig.3a that the effects of the perturbation \( f(x, t) \) can dominate the total solution of the KdV-equation.

The situation dramatically changes if we chose \( d = 0.01 \) and \( \beta = 1 \) and keep the positions of all the other 9 poles and residues which specify the perturbation constant. By choosing \( d = 0.01 \) and \( \beta = 1 \), the unperturbed determinant \( \mathcal{D}(x, t) \) can by equal to zero for certain values of \( x \) and \( t \). Because of the analytical structure of the determinant \( \mathcal{D}(x, t) \), the perturbation \( \mathcal{E}(x, t) \) has in this case also zeros for certain values of \( x \) and \( t \). The time-evolution is in this case plotted in the lower panel of Fig.4. This result has to be compared with the time-evolution of the unperturbed case which is given in the upper panel in Fig.4. It follows that in this special case, the perturbed solution consists of two different branches. The first branch propagates with a similar propagation velocity as the unperturbed solution but it has large amplitude fluctuations on the characteristic of the unperturbed solution. The second part propagates with a different velocity than the unperturbed solution. In the following section, we derive analytical expressions for these different parts of the perturbation. We want to remark that a small perturbation of a singular solution at \( t = 0 \) can always be constructed by choosing the positions of the poles and residues properly. It follows from the structure of equation (20) that these small perturbations can grow when \( \mathcal{D}(x, t)[\mathcal{D}(x, t) + \mathcal{E}(x, t)] \) is small.

From the results in this section we can conclude that the soliton exhibits stable behavior whereas the singular solution is unstable. Eq.(20) gives a general expression for the perturbation of a solution of the KdV-equation. The perturbation is small with respect to the unperturbed solution, if the unperturbed solution has no poles close to the origin in the complex plane since the dominator \( \mathcal{D}(x, t) + \mathcal{E}(x, t) \) can not be zero. In contrast to this, the perturbation is large with respect to the unperturbed solution if the dominator \( \mathcal{D}(x, t) + \mathcal{E}(x, t) \) in Eq.(20) is nearly singular. This is the case if the unperturbed problem has poles close to the origin in the complex plane. This means that the singular solution is sensitive for noise. Moreover, for both the soliton and the singular solution the propagation velocity of the perturbation, which determines whether the perturbation separates from the unperturbed solution, is a crucial parameter for the stability. The propagation velocity and the amplitude of the perturbation depend of the position of the poles and residues of the perturbation. In the following section we study this in more detail.
4 A series solution for \( f(x, t) \)

In order to analyze the behavior of the perturbation \( f(x, t) \), it is convenient to formulate this function as a series solution. Our starting point is the Marchenko equation without bound states in the wave-number domain as given in Appendix A. If the reflection coefficient undergoes a perturbation (13), we find the following relation.

\[
F(k, x, t) = 1 + \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dk' \left\{ \frac{R(k', t = 0) + \overline{R}(k', t = 0)}{k' + k + i\epsilon} \right\} F(k', x, t) \exp[2i(k'x + 4{k'}^3t)]
\]

\[= 1 + \int_{-\infty}^{\infty} C(k, k', t) F(k', x, t) dk' + \int_{-\infty}^{\infty} \overline{C}(k, k', t) F(k', x, t) dk' \tag{21}\]

The function \( F(k, x, t) \) is related to the kernel \( K(x, y, t) \) by the following Fourier transform:

\[
K(x, y, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} dk e^{-ik(x-y)} (F(k, x, t) - 1) \tag{22}\]

Furthermore, the kernel \( C(k', k, t) \) in Eq.(21) is given by:

\[
C(k, k', t) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \frac{R(k', t = 0) e^{2i(k'x + 4{k'}^3t)}}{k' + k + i\epsilon} \tag{23}\]

The kernel \( \overline{C}(k', k, t) \) in Eq.(21) is defined by:

\[
\overline{C}(k, k', t) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \frac{\overline{R}(k', t = 0) e^{2i(k'x + 4{k'}^3t)}}{k' + k + i\epsilon} \tag{24}\]

Eq.(21) can be represented schematically using a Dyson’s representation as given if Fig.5. Using an iteration technique, the Dyson’s series of Fig.5, can be expanded. The result is given in Fig.6. It is observed from Fig.6, that the total expression for the function \( F(k, x, t) \) consists of three parts. The first part can be identified with all diagrams consisting of solid dots only. This series of diagrams represents the time-evolution of the unperturbed solution. The second series consists of diagrams having open dots only. This series of diagrams represents the time-evolution the perturbation in absence on the unperturbed solution. The remaining diagrams consist of a combination of both solid and open dots. This series of diagrams represents the interaction between the unperturbed solution and the perturbation. We can formally solve the Dyson’s equation by iteration. The solution is shown by the diagrams in Fig.6, and is given by:

\[
F(k, x, t) = 1 + \int_{-\infty}^{\infty} C(k', k, t) dk' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, k', t) C(k', k'', t) dk' dk'' + \cdots
\]

\[+ \int_{-\infty}^{\infty} \overline{C}(k', k, t) dk' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{C}(k, k', t) \overline{C}(k', k'', t) dk' dk'' + \cdots
\]

\[+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, k', t) \overline{C}(k', k'', t) dk' dk'' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{C}(k, k', t) C(k', k'', t) dk' dk'' + \cdots \tag{25}\]

If both \( R(k, t = 0) \) and \( \overline{R}(k, t = 0) \) are rational functions of the wave-number, the integrations in the series (25) can be carried out analytically by performing a contour integration in \( C^+ \). This is justified by the fact that the reflection coefficient \( R(k, t = 0) \) \( \to O(1/k) \) if \( k \to \infty \). The poles of the denominator of Eq.(25) are all situated in \( C^- \) so the only contribution to the integrals of Eq.(25) comes from the poles of \( R(k, t = 0) \) which are situated in \( C^+ \). We write Eq.(25) in the following way:

\[
F(x, k, t) = 1 = F_{un}(x, k, t) + F_{pe}(x, k, t) + F_{int}(x, k, t) \tag{26}\]

The function \( F_{un}(x, k, t) \) only contains contributions of \( R(k, t) \) and can be identified with the time-evolution of the unperturbed solution (solid-dot diagrams). The function \( f(x, t) \) consists of contributions of both \( F_{pe}(x, k, t) \) and \( F_{int}(x, k, t) \). The term \( F_{pe}(x, k) \) only contains contributions of \( \overline{R}(k, t) \) and can be identified with the time-evolution of the perturbation in the absence of \( u(x, t) \) (open-dot diagrams). The term \( F_{int}(x, k, t) \) contains contributions of both \( R(k, t) \) and \( \overline{R}(k, t) \) and represents the interaction between the unperturbed solution \( u(x, t) \) and the perturbation (diagrams consisting of solid-dots and open-dots). In the following we analyze these three terms separately.
4.1 The time-evolution of the non-interaction terms

Suppose the unperturbed reflection coefficient \( R(k, t) \) has \( N \) poles. It follows from Eq. (23) and Eq. (24) that the contribution to the function \( F(k, x, t) \) from the terms containing only the reflection coefficient \( R(k, t) \) (solid-dot diagrams) is equal to:

\[
F_{un}(x, k, t) = \sum_{i=1}^{N} \frac{R_i}{k + p_i} e^{2i(p_i x + 4p_i^3 t)} + \sum_{i,j=1}^{N} \frac{R_i R_j}{(k + p_i)(p_i + p_j)} e^{2i((p_i + p_j) x + 4(p_i^3 + p_j^3) t)}
\]

\[
\sum_{i,j,l=1}^{N} \frac{R_i R_j R_l}{(p_j + p_l)(p_i + p_j)} e^{2i((p_i + p_j + p_l) x + 4(p_i^3 + p_j^3 + p_l^3) t)} \ldots
\]

where \( p_i \) and \( R_i \) are the poles and residues of the unperturbed reflection coefficient. If the Fourier transform (22) is now performed on the unperturbed part of \( F_{un}(x, k, t) \), we obtain the following expression for the kernel \( K_{un}(x, y, t) \):

\[
K_{un}(x, y, t) = i \sum_{i=1}^{N} R_i e^{2i p_i (x+y) + 8i p_i^3 t} + \sum_{i,j=1}^{N} \frac{R_i R_j}{p_i + p_j} e^{i p_i (x+y)} e^{2i(p_i + p_j) x} e^{8i(p_i^3 + p_j^3) t} + \ldots
\]

After putting \( y = x \) and taking the derivative:

\[
u_{un}(x, t) = -2 \frac{d}{dx} K_{un}(x, x, t),
\]

the following expression for the unperturbed solution is obtained:

\[
u_{un}(x, t) = 4 \sum_{i=1}^{N} R_i p_i e^{2i(p_i x + 4p_i^3 t)} + 4 \sum_{i,j=1}^{N} R_i R_j e^{2i((p_i + p_j) x + 4(p_i^3 + p_j^3) t)} + \ldots
\]

This result is already obtained in ref. [3]. From this result we can conclude that a general solution of the KdV-equation can be expanded in an infinite series of exponential basis functions.

In a similar manner we can derive the time-evolution of the contributions to the Dyson’s series in Fig. 6 for all the terms that can be identified with the perturbation only (open-dot diagrams). Suppose the perturbation on the reflection coefficient \( \mathcal{R}(k, t) \) has \( M \) poles, it follows using a similar argumentation as for the evaluation of the term \( u_{un}(x, k, t) \) that:

\[
u_{pe}(x, t) = 4 \sum_{i=1}^{M} \mathcal{R}_i \mathcal{P}_i e^{2i(\mathcal{P}_i x + \mathcal{P}_i^3 t)} + 4 \sum_{i,j=1}^{M} \mathcal{R}_i \mathcal{R}_j e^{2i((\mathcal{P}_i + \mathcal{P}_j) x + 4(\mathcal{P}_i^3 + \mathcal{P}_j^3) t)} + \ldots
\]

\[
4 \sum_{i,j,l=1}^{M} \frac{(\mathcal{R}_i \mathcal{R}_j \mathcal{R}_l)(\mathcal{P}_i + \mathcal{P}_j + \mathcal{P}_l)}{\mathcal{P}_i + \mathcal{P}_j} e^{2i((\mathcal{P}_i + \mathcal{P}_j + \mathcal{P}_l) x + 4(\mathcal{P}_i^3 + \mathcal{P}_j^3 + \mathcal{P}_l^3) t)} + \ldots
\]

We observe from this result that both the unperturbed solution and the time-evolution of the non-interaction elements have a similar analytical structure. We remark that the solution \( u_{un}(x, t) \) corresponds to a series expansion of the terms in Eq. (13) which is only the determinant \( D(x, t) \) is present. We can conclude that both the unperturbed solution and the perturbation evolve as an infinite series of non-dispersive solutions in time. However, the spectral components of the perturbation \( u_{pe}(x, t) \) generally travel with a different velocity than \( u_{un}(x, t) \). This is already observed in Fig. 1.
and Fig.2. In Fig.1c it is shown that a perturbation of a nondispersive solution of the KdV-equation introduces a dispersion effect, and in Fig.1d, it is shown that this results in a different propagation velocity of perturbation. If the unperturbed solution is a localized non-dispersive function, both the unperturbed solution and the perturbation travel with a different velocity depending on the position of the poles of the perturbation. The velocity of every spectral component of the perturbation is determined by its corresponding pole position. In the following we examine the behavior of the interaction terms.

4.2 The time-evolution of the interaction terms

The function $F_{int}(x, k, t)$ corresponding to the interaction term contains contributions of both the unperturbed reflection coefficient $R(k, t)$ and the perturbation of the reflection coefficient $\overline{R}(k, t)$ in Fig.6, it consists of all the diagrams having both solid and open dots. An analytic expression of all these diagrams is given by the following equation:

$$F_{int}(x, k, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, k', t) \overline{C}(k', t) dk' dk'' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{C}(k, k', t) C(k'', t) dk' dk'' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, k', t) \overline{C}(k', t) C(k'', t) \overline{C}(k'', t) dk' dk'' + \cdots$$ (32)

Since, both $R(k, t)$ and $\overline{R}(k, t)$ are rational functions of the wave-number, we can carry out the integrations in Eq.(32) analytically. If we proceed in a similar manner as for the non-interaction terms, we obtain the following result for the time-evolution of the interaction terms:

$$u_{int}(x, t) = 8 \sum_{i=1}^{N} \sum_{j=1}^{M} R_{i} \overline{R}_{j} e^{2i((p_{i}+\overline{p}_{j})x+4(p_{i}^{2}+\overline{p}_{j}^{2})t)}$$

$$+ 12 \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{l=1}^{N} R_{i} \overline{R}_{j} \overline{R}_{l} (p_{i} + \overline{p}_{j} + \overline{p}_{l}) (p_{i} + \overline{p}_{j} + \overline{p}_{l}) e^{2i((p_{i}+\overline{p}_{j}+\overline{p}_{l})x+4(p_{i}^{2}+\overline{p}_{j}^{2}+\overline{p}_{l}^{2})t)}$$

(33)

As witnessed from the previous section, it follows from Eq.(33) that the interaction term is large with respect to the unperturbed solution, if the unperturbed solution has poles close to the origin in the complex plane. The analytical structure of Eq.(30), Eq.(31) and Eq.(33) enables us to formulate an expression for the amplitude behavior of the perturbation $f(x, t)$. If the perturbation does not have poles close to the origin in the complex plane and if the perturbation is small with respect to the unperturbed problem at $t = 0$, it follows from Eq.(33) and Eq.(31) that the term $u_{pe}(x, t)$ remains small with respect to $u_{un}(x, t)$. However, as already concluded in the previous section, if the unperturbed solution has singularities, the structure of the interaction term (33) introduces necessarily singularities in the time-evolution of the perturbed problem. The physical meaning of $u_{int}(x, t)$ is visualized in Fig.4. The example given in Fig.4 only differs from the example given in Fig.3, by the position of the poles and residues that characterize the unperturbed solution. This implies that the function $u_{pe}(x, t)$ in Fig.4 does not differ from that in Fig.3. However, due to the large amplitudes in Fig.4, $u_{pe}(x, t)$ is small with respect to $u_{un}(x, t)$. In the upper panel of Fig.4, the unperturbed singular solution is plotted. In the lower panel of Fig.4, the perturbed solution is plotted. As remarked in Sec.3, in this special case, the perturbation consists of two parts. One part propagates over a different characteristic than $u_{un}(x, t)$. The second part propagates over the characteristic of $u_{un}(x, t)$ and is responsible for large amplitude fluctuations on the characteristic of the unperturbed solution. It is easy to see that an interaction term as given by Eq.(33) introduces large amplitude fluctuations. If we assume that the unperturbed solution consist of one pole ($p_{i} = p$) and one residue, than we find at the characteristic $x = -4p^{2}t$:

$$u_{int}(x = -4p^{2}t, t) = 8 \sum_{j=1}^{M} R_{i} \overline{R}_{j} e^{2i(p_{j}^{2}+\overline{p}_{j}^{2})t} + \text{h.o.t.}$$ (34)
From this result it follows that the interaction term $f_{int}(x,t)$ introduces fluctuations at the characteristic of the unperturbed solutions. This result explains the behavior of the perturbed singular solution in Fig.4. It is observed in this figure that certain spectral components of the noise travel with a different velocity, and that at the characteristic of the unperturbed solution large amplitude fluctuations occur. This is the result of the presence of the interaction term $u_{int}(x,t)$ which has the same magnitude as the unperturbed solution $u_{un}(x,t)$. The interaction term is large with respect to the unperturbed solution if $|\mathcal{R}_i| \ll p_j$ for all possible $\mathcal{R}_i$ and $p_j$. If this condition is satisfied, the amplitude of the perturbation $f(x,t)$ is small with respect to the amplitude of the unperturbed solution.

5 Numerical example

In this section the results that are obtained analytically in the previous sections are illustrated numerically in the case of a reflection coefficient consisting of an infinite number of poles and residues. In this section we give an numerical example in to illustrate the stability of the soliton. In a discrete representation the KdV-equation takes the following form:

$$u^{n+1}_i = u^n_i + \Delta t \left\{ \frac{u^{n+3}_{i+1} - 3u^{n+1}_i + 3u^n_{i-1} - u^n_{i-3}}{(2\Delta x)^3} - \frac{6u^n_{i+1}u^n_i - 6u^n_iu^n_{i-1}}{2\Delta x} \right\}$$  \hspace{1cm} (35)

In Eq.(35), the solution of the KdV-equation at time $i\Delta t$ and position $n\Delta x$ is given by $u^n_i$. In Eq.(35), $\Delta t$ represents the time-step and $\Delta x$ represents the distance between two grid-points. The discrete KdV-equation (35) is solved numerically using the fourth-order Runge-Kutta scheme as given in ref.9.

In the numerical example that follows, the KdV-equation is solved on a line-segment of a total length of $8\pi$. The initial condition given in Fig.7a consists of the standard soliton which is used in the previous sections ($\beta = 1$ and $d = -1$). The soliton is contaminated with a noise-function having a maximum amplitude of 10% of the maximum amplitude of the soliton. In Fig.7b, the soliton after a propagation-time of 1.1 sec is given, it can be seen that the contamination has started propagating out of the soliton. This process continues, and at $t = 2.2$ sec, the contamination has virtually propagated out of the soliton.

This experiment is a numerical confirmation of the experiments of the previous sections. It reflects the case that small perturbations propagate out of the unperturbed soliton so that the noise separates form the unperturbed solution. This is also the reason why we observe solitons in nature. In the example of Fig.7 the spectral contents of the perturbation are chosen in such a way that the soliton and all the spectral components of the perturbation travel in such a way that the unperturbed solution and the perturbation separate during the course of time. In a real physical situation solitons are always contaminated with noise at time $t = 0$. If we observe solitons in nature, it follows from this paper that the spectral components of the perturbation have a small amplitude for low frequencies. As a result of this, the soliton and the perturbation separate during the course of time and the soliton is “born”. If the spectral components of the noise function are of the same magnitude than the spectral components of the soliton, the noise propagates with a similar velocity as the soliton, and hence we can conclude that no soliton is created.

6 Conclusions

In this paper the stability of the KdV-equation is investigated. In particular, attention is paid to the stability of localized solutions. Is Sec.2, the effects of nonlinearity and dispersion are investigated. For non-dispersive solutions of the KdV-equation these effects have to be in balance. It is pointed out in Sec.2 that in the cases of small perturbations the balance the dispersion and the nonlinearity is disturbed. As a result of this an additional dispersion effect is introduced, which generates the “force” that separates the noise from the unperturbed solution.

In Sec.3, inverse scattering techniques are used to formulate an analytical expression which describes the behavior of perturbations of solutions of the KdV-equation. From the examples of Sec.3, it can be concluded for soliton-like solutions of the KdV-equation, that stability implies that noise either propagates out of the unperturbed solution, or that the noise spreads out so that the amplitude is reduced. Furthermore, it is observed for singular solutions that although the noise is propagating partly out of the unperturbed solution, large amplitude variations contaminate the perturbed solution.

This behavior is examined in Sec.4, by expanding the perturbation into a series-solution. It is concluded in Sec.4 that the behavior of the perturbation $f(x,t)$ of the KdV-equation is strongly correlated to the behavior of the
unperturbed solution. If the unperturbed problem has singularities the amplitude of the perturbation is in the same order of magnitude as the amplitude of the unperturbed problem. This result explains the large amplitude variations from which the perturbed singular singular problems suffers. Furthermore, in Sec.4, a criterion for the position of the poles and residues of the perturbation is given to posses a stable behavior of the soliton.

Lastly, in Sec.5 we give a numerical example is given to illustrate the stability of the soliton. The perturbation used in Sec.5 has an infinite number of poles and residues. We observe that in the numerical case the same conclusions can be drawn as in cases of analytical perturbations: the soliton possesses stable behavior if the noise is propagating out of the soliton.

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Appendix A: The inverse problem for rational reflection coefficients

In this appendix a brief formulation of the inverse problem for rational reflection coefficients based upon the formulation of Sabatier [7] is given. For a detailed treatment of the mathematics we refer to the book of Chadan and Sabatier [8]. Our starting-point is the following equation:

$$ F_\pm(k, x, t) - 1 = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{1 - T(k', t)F_\mp(k', x, t)}{k' + k + i\epsilon} dk' $$
$$ + \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{R_\pm(k', t)F_\mp(k', x, t)\exp[\pm 2ik'x]}{k' + k + i\epsilon} dk' $$

(A-1)

In Eq. (A-1) $F_+(k, x)$ is defined for $x > 0$, and $F_-(k, x, t)$ for $x < 0$, $k \in \mathbb{C}$. The function $F_\pm(k, x, t)$ is defined by:

$$ F_\pm(k, x, t) = \exp[\mp ikx]f_\pm(k, x, t) $$

(A-2)

The Jost solutions $f_\pm(k, x, t)$ are those solutions of the Schrödinger equation satisfying the following boundary conditions:

$$ f_+(k, x, t) : \lim_{x \to \infty} e^{-ikx}f_+(k, x, t) = 1 $$
$$ f_-(k, x, t) : \lim_{x \to -\infty} e^{ikx}f_-(k, x, t) = 1 $$

(A-3) (A-4)

They satisfy the following integral equations:

$$ f_+(k, x, t) = e^{ikx} - \int_{x}^{\infty} \frac{\sin k(x - y)}{k} V(y, t)f_+(k, y, t) dy $$
$$ f_-(k, x, t) = e^{-ikx} - \int_{-\infty}^{x} \frac{\sin k(x - y)}{k} V(y, t)f_-(k, y, t) dy $$

(A-5) (A-6)

It is well known that the functions $f_\pm(k, x, t)$ and therefore also the functions $F_\pm(k, x, t)$ are holomorphic in $\mathbb{C}^+$ [3]. The potential $V(x, t)$ has to be in the Faddeev class $L_1^+$:

$$ \int_{-\infty}^{\infty} (1 + |x|)|V(x, t)| < \infty $$

(A-7)

The scattering coefficients $R_+(k, t), R_-(k, t), T(k, t)$ are defined by the asymptotic behavior of the physical solution of the Schrödinger equation:

$$ \psi_1(k, x, t) \sim \begin{cases} e^{ikx} + R_+(k, t)e^{-ikx} & x < 0 \\ T(k, t)e^{ikx} & x \to +\infty \end{cases} $$

(A-8)

$$ \psi_2(k, x, t) \sim \begin{cases} T(k, t)e^{-ikx} & x < 0 \\ e^{-ikx} + R_-(k, t)e^{ikx} & x \to +\infty \end{cases} $$

(A-9)

In the case of rational reflection coefficients they take the following form [3]:

$$ R_+(k, t) = \frac{P(-k)}{\prod_{j=1}^{q}(\lambda_j - k)} \prod_{\mu_i \in M^+} \frac{\mu_i + k}{\mu_i - k} \prod_{\lambda_i \in L^+} \frac{\lambda_i + k}{\lambda_i - k} $$

(A-10)

$$ T(k, t) = \frac{\prod_{j=1}^{q}(\mu_j + k)}{\prod_{j=1}^{q}(\lambda_j + k)} $$

(A-11)

$$ R_-(k, t) = \frac{P(k)}{\prod_{j=1}^{q}(\lambda_j - k)} \prod_{\mu_i \in M^-} \frac{\mu_i - k}{\mu_i + k} \prod_{\lambda_i \in L^-} \frac{\lambda_i - k}{\lambda_i + k} $$

(A-12)

Following Sabatier [3], the degree of the polynomial $P(k)$ has to be smaller than $q$. Further, Im $\mu_i > 0$ except if $\mu_i = 0$, Im $\lambda_l < 0$. The transmission coefficient $T(k, t)$ is supposed to be an irreducible fraction, and the sets $M^+$,
$M^-, L^+, L^-$ contain numbers $\neq 0$. If the potential is real then $\mu_k$, $\lambda_k$ are pure imaginary. If $\mu_k, \lambda_k$ are not pure imaginary then there exists $-\mu_k^*, -\lambda_k^*$. It can be shown that $T(k)$ is meromorphic in $\mathbb{C}^+$ and if there are poles they are in Im $k$. If there are no bound states, $T(k, t)F_\pm(k, x, t)$ is holomorphic in $\mathbb{C}^+$ and the first integral of (A-1) is zero. If $T(k, t)F_\pm(k, x, t)$ is holomorphic in $\mathbb{C}^+$ and all the poles $p_i$ of $R_+(k, t)$ are simple, the integral (A-1) can be solved by contour integration in the upper-half plane. The result is:

$$F_+(k, x, t) - 1 = \sum_{p_j \in \mathcal{P}} \frac{R_j F_+(p_j, x, t)e^{2i(p_j x - 4p_j^3 t)}}{p_j + k}$$  \hspace{1cm} (A-13)

The time-evolution of the residues that is used in equation (A-13), is given in ref.[4]. Equation (A-13) can be solved by letting take $k$ the values of the discrete poles $p \in \mathcal{P}$. We then obtain a linear set of algebraic equations that determine $F_+(p_j, x, t)$ for all values of $j$. This set of equations can be solved making use of Cramer’s rule. We obtain after resubstituting the result in equation (A-13), using equation (22) and putting $x = y$:

$$K_+(x, x, t) = \frac{\mathcal{D}_+(x, t)}{\mathcal{D}_+(x, t)}$$  \hspace{1cm} (A-14)

where:

$$\mathcal{D}_+(x, t) = \det\{\delta_{ij} - (p_i + p_j)^{-1}R_j e^{2i(p_j x - 4p_j^3 t)}\}$$  \hspace{1cm} (A-15)

and $\mathcal{D}_+(x, t)$ is the derivative of $\mathcal{D}_+(x, t)$ with respect to $x$. 

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Captions for figures

Figure 1: A: Unperturbed soliton (solid line) and perturbed soliton (dashed line). B: the dispersion of the unperturbed soliton (short-dashed), the nonlinearity of the unperturbed soliton (long-dashed line) and the sum of the dispersion and nonlinearity (solid line). C: Similar as for Fig.1b, but now for the perturbed soliton. D: The time-derivative of the perturbation. E: Finite-difference solution of the perturbed soliton at $t = 2.2$ sec.

Figure 2: A: Singular solution (solid line), the dispersion (short-dashed line) and the nonlinearity (long-dashed line). B: The dispersion in case of a 10% amplitude error (short-dashed line). The nonlinearity in case of a 10% amplitude error (long-dashed line) and the sum of the nonlinearity and the dispersion (solid-line).

Figure 3: A: Time-evolution of a soliton in case of a perturbation consisting of nine additional poles and residues (lower panel) and the time-evolution of the unperturbed soliton, (upper panel). B: The initial condition (short-dashed), a the solution at $t = 1$ (long-dashed line) and the unperturbed soliton (solid-line).

Figure 4: Time-evolution is case of a singular solution contaminated with the same perturbation as in the previous figure (lower panel) and the time-evolution of the unperturbed singular solution (upper panel).

Figure 5: Dyson’s representation for the integral equation (21).

Figure 6: Dyson’s representation of the series expansion of Fig. 5.

Figure 7: A: Plot of the soliton contaminated with a perturbation consisting of an infinite number of poles and residues. B: The initial condition, the solution at $t=1.1$ sec and the solution at time $t = 2.2$ sec.