Traffic Characterization of LTI Event-triggered Control Systems: a Formal Approach

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Abstract—Unnecessary communication and computation in the periodic execution of control tasks lead to over-provisioning in hardware design or underexploitation in hardware utilization) in control applications, such as networked control systems. To address these issues, researchers have proposed a new class of strategies, named event-driven strategies. Despite of their beneficiary effects, matters like task scheduling and appropriate dimensioning of communication components have become more complicated with respect to traditional periodic strategies. In this paper, we present a formal approach to derive an abstracted system that captures the sampling behavior of a family of event-triggered strategies for the case of LTI systems. This structure, termed power quotient system, approximately simulates the sampling behavior of the aperiodic control system. Furthermore, the resulting quotient system is equivalent to a timed automaton. In the construction of the abstraction, the state space is confined to a finite number of convex regions, each of which represents a mode in the quotient system. An LMI-based technique is deployed to derive a sampling time interval associated to each region. Finally, a reachability analysis is leveraged to find the transitions of the quotient system.

Index Terms—Event-triggered control, timed automata, LMI, formal methods, reachability analysis.

I. INTRODUCTION

In networked control systems, particularly over wireless or shared channels, the scarcity of communication resources makes the application of traditional control strategies with periodic sampling inefficient. Alternative approaches with aperiodic sampling, such as event-triggered control (ETC) and self-triggered control (STC), have been recently proposed to reduce network usage. The underlying idea in these approaches is to take into account the dynamics of control systems in the sampling procedure, to only use the communication channel when strictly necessary.

In these approaches, control actions are executed at the instants that a specified condition, the so-called triggering mechanism is violated. In ETC, an intelligent sensory system continuously monitors the states of the control plant and determines the instants when the triggering condition is violated [1]. ETC has been also used under other names, such as interrupt-based feedback control [2], Lebesgue sampling [3], asynchronous sampling [4], state-triggered feedback [5], and level crossing sampling [6]. The necessity of an intelligent sensory system in ETC motivated researchers to propose another class of aperiodic approaches, namely STC [8]. In STC, the controller is responsible for the determination of sampling instants, i.e., the next sampling instant is computed at the current sampling instant by the controller [7], [9], [10]. However, the robustness of STC strategies against disturbances is a concern because of the open-loop nature of the control action implementation between two consecutive sampling instants (notice that this is not an issue in ETC strategies which continuously supervise the plant). The assumption of availability of states in the feedback path is addressed in an extension to output-feedback event-triggered strategies in [11].

Most of the aforementioned approaches focus in decreasing the (computation and communication) resource utilization of real-time control systems. Nonetheless, there is a lack of tools to translate such approaches to frameworks that can be exploited by computer engineers in designing real-time systems. Providing such frameworks shall enable the reduction of hardware over-provisioning in the design process.

Several controller/scheduler co-design approaches of real-time systems can be found in the literature, e.g. feedback modification to task attributes [12]–[15], anytime controllers [16], [17], and event-based control and scheduling [18], [19]. In [12]–[15], the principles of feedback control theory were used to integrate scheduling and control in real-time system designs. [16] and [17] deployed the concept of anytime controllers in their co-design approaches in which there is a trade off between control performance and computation complexity. The problems of resource utilization and resource distribution have been jointly addressed in [18], [19]. These approaches include designing of a control law, a scheduler, and an event generator. The control law enhances the performance while the scheduler and event generator improve the efficiency of the resource usage. In the present paper we take an alternative approach providing a decoupling between controller design, event-triggered implementation and the scheduling design. Such decoupling between control and scheduling has the benefits of increased scalability and versatility of networked controller designs, with respect to monolithic co-design approaches as those mentioned earlier.

While in traditional periodic implementations of controllers the decoupling between control and scheduling is naturally provided by the period defining the frequency at which the control loop must be updated, in ETC (and STC) the matter is more involved. The triggering mechanism in ETC results in aperiodic control systems, in which sampling is a function of time and/or states, and thus no single parameter provides the necessary information for appropriate scheduling. In this study, we take one step further to complement a family of ETC strategies by abstracting their sampling behavior into simple structures that can be employed by real-time engineers.
for scheduling (and possibly network dimensioning). The objective of this paper is thus to model the traffic generated by aperiodic ETC control systems, understood as all possible traces of sampling times. We achieve this by mapping the initial control system (which is infinite-state) into a type of quotient system (which is finite-state) called power quotient system. We show that the power quotient system is in fact a timed automaton [21], and that it captures the original timing behavior accurately. The latter is shown by establishing an approximate simulation relation between the abstraction and the original system.

We consider linear time invariant (LTI) systems with state feedback laws and sample-and-hold control actions implemented in an ETC fashion as in [20]. Our procedural technique is inspired by a state-dependent sampling proposed by [22] where the \( \beta \)-stability [23] of the system origin is guaranteed based on an LMI-based approach derived from Lyapunov-Razumikhin stability conditions [23]. These conditions are developed based on the delayed nature of the system (related to the sample-and-hold nature of such systems) suggested by [22]. Inspired by [22], we employ a two-step approach to compute sampling intervals associated to states: first, to remove the spatial dependencies, the state space is partitioned (abstracted) to a finite number of convex polyhedral cones (pointed at the origin); then, to remove the temporal dependencies, each conic region is considered as a discrete state (mode). Then, in order to derive the transitions of the quotient system, time intervals captures all possible inter-sample times of the original system. We achieve this by mapping the traces of sampling times. We achieve this by mapping the traces of sampling times.

A. Mathematical Preliminaries

In what follows, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, \( \mathbb{R}^+ \) and \( \mathbb{R}_+^n \) denote the positive and nonnegative reals, respectively, \( \mathbb{N} \) is the set of positive integers, and \( \mathbb{R}_+^n \) is the set of all closed intervals \( [a, b] \) such that \( a, b \in \mathbb{R}^+ \) and \( a \leq b \). For any set \( S, 2^S \) denotes the set of all subsets of \( S \), i.e. the power set of \( S \). \( M_{m \times n} \) and \( M_n \) are the set of all \( m \times n \) real-valued matrices and the set of all \( n \times n \) real-valued symmetric matrices, respectively. For a matrix \( M, M \preceq 0 \) (or \( M \succeq 0 \)) means \( M \) is a negative (or positive) semidefinite matrix and \( M \succeq 0 \) indicates \( M \) is a positive definite matrix. \( |x| \) indicates the largest integer not greater than \( x \in \mathbb{R} \) and \( |y| \) denotes the Euclidean norm of a vector \( y \in \mathbb{R}^n \). Given two sets \( Z_a \) and \( Z_b \), every relation \( Q \subseteq Z_a \times Z_b \) admits \( Q^{-1} = \{(z_b, z_a) \in Z_b \times Z_a \mid (z_a, z_b) \in Q\} \) as its inverse relation. When \( Q \subseteq Z \times Z \) is an equivalence relation on a set \( Z \), \( |z| \) denotes the equivalence class of \( z \in Z \) and \( Z/Q \) denotes the set of all equivalence classes.

A fundamental observation that we employ in what follows is that the ordered pair \((\mathbb{R}^+, d_H)\) is a metric space:

**Definition 1.** (Metric [29]) Consider a set \( T, d : T \times T \rightarrow \mathbb{R}_+ \cup \{\infty\} \) is a metric (or a distance function) if the following three conditions are satisfied \( \forall x, y, z \in T : \)

1) \( d(x, y) = d(y, x) \),
2) \( d(x, y) = 0 \Leftrightarrow x = y \),
3) \( d(x, y) \leq d(x, z) + d(y, z) \).

The ordered pair \((T, d)\) is said to be a metric space.

**Definition 2.** (Hausdorff Distance [29]) Assume \( X \) and \( Y \) are two non-empty subsets of a metric space \((T, d)\). The Hausdorff distance \( d_H(X, Y) \) is given by:

\[
\max \{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \}.
\]

We also employ the framework from [28] to establish relations between different systems. Some relevant notions from that framework are summarized in the following:

**Definition 3.** (System [28]) A system is a sextuple \((X, X_0, U, \rightarrow, Y, H)\) consisting of:

- a set of states \( X \);
- a set of initial states \( X_0 \subseteq X \);
- a set of inputs \( U \);
- a transition relation \( \rightarrow \subseteq X \times U \times X \);
- a set of outputs \( Y \);
- an output map \( H : X \rightarrow Y \).

The term finite-state (or infinite-state) system indicates \( X \) is a finite (or infinite) set. We employ the shorthand \( S = (X, U, \rightarrow) \) to denote a system when \( X = X_0 = Y \) and \( H : X \rightarrow X \) is the identity map.

**Definition 4.** (Metric System [28]) A system \( S \) is said to be a metric system if the set of outputs \( Y \) is equipped with a metric \( d : Y \times Y \rightarrow \mathbb{R}^+_0 \).

**Definition 5.** (Approximate Simulation Relation [28]) Consider two metric systems \( S_a \) and \( S_b \) with \( Y_a = Y_b \), and let \( \epsilon \in \mathbb{R}^+_0 \). A relation \( R \subseteq X_a \times X_b \) is an \( \epsilon \)-approximate simulation relation from \( S_a \) to \( S_b \) if the following three conditions are satisfied:

1) \( \forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0} \) such that \( (x_{a0}, x_{b0}) \in R \);
2) \( \forall (x_a, x_b) \in R \) we have \( d(H_a(x_a), H_b(x_b)) \leq \epsilon \);
Lemma 1. \(\forall (x_a, x_b) \in R \text{ such that } (x_a, u_a, x'_a) \in \rightarrow \text{ in } S_a \implies S(x_b, u_b, x'_b) \in \rightarrow \text{ in } S_b \text{ satisfying } (x'_a, x'_b) \in R.\)

We denote the existence of an \(\epsilon\)-approximate simulation relation from \(S_a\) to \(S_b\) by \(S_a \leq \epsilon S_b\), and say that \(S_b\) \(\epsilon\)-approximately simulates \(S_a\). Whenever \(\epsilon = 0\), the inequality \(d(H_a(x_a), H_b(x_b)) \leq \epsilon\) implies \(H_a(x_a) = H_b(x_b)\) and the resulting relation is called \((exact)\) simulation relation.

Finally, we propose a new notion of quotient system (see e.g. [23] for the traditional definition) to suit our needs:

**Definition 6. (Power Quotient System)** Let \(S = (X, X_0, U, \rightarrow, Y, H)\) be a system and \(R\) be an equivalence relation on \((\text{Power Quotient System})\) \(S\). The power quotient of \(S\) by \(R\), denoted by \(S/R\), is the system \((X/R, X_0/R, U/R, \rightarrow_R, Y/R, H/R)\) consisting of:

- \(X/R = X/R;\)
- \(X_0/R = \{x | x \in X | x \cap 0 \neq \emptyset\};\)
- \(U/R = U;\)
- \(x/R, u, x'_R \in \rightarrow \text{ if } \exists (x, u, x') \in \rightarrow \text{ in } S \text{ with } x \in x/R \text{ and } x \in x'_R;\)
- \(Y/R \subset 2^Y;\)
- \(H/R(x/R) = \bigcup_{x \in x/R} H(x).\)

In the following lemma, we provide a new result establishing the relationship between a power quotient system and the original system.

**Lemma 1.** Let \(S\) be a metric system, \(R\) be an equivalence relation on \(X\), and let the metric system \(S/R\) be the power quotient system of \(S\) by \(R\). For any \(\epsilon \geq \max_{x \in X, x' \in X} d(H(x), H(x'/R)), \text{ with } d\text{ the Hausdorff distance over the set } 2^Y, \text{ the power quotient } S/R \text{ \(\epsilon\)-approximately simulates } S, \text{ i.e. } S \leq \epsilon S/R.\)

**Proof.** Let us consider the candidate simulation relation: \(R_s = X \times X/R, \text{ where } (x, x/R) \in R_s \text{ if and only if } x \in x/R.\) From Definition 6, the three conditions in Definition 5 immediately follow.

Note that we are using the fact that for a given set \(Y, Y \subset 2^Y\) in order to use the Hausdorff distance as a common metric between the output sets of the power quotient system and the concrete system.

Let \(X\) be a set of finitely many real-valued variables (clocks). Consider \(\prec \in \{>, \geq, <, \leq\}\), a clock constraint \(C(X)\) is a conjunctive formula of atomic constraints \(x \sim k\) or \(x - y \sim k\) for \(x, y \in X\), and \(k \in \mathbb{R}^+\) (note that a timed automaton is a particular class of hybrid automata for which guard and invariant sets are in the form of clock constraints). A clock valuation (or assignment) is a mapping of the form \(X \rightarrow \mathbb{R}^+_0\). Let \(u\) be a clock valuation and \(g \in C(X)\), then \(u \models g\) denotes \(u\) satisfies \(g\). For \(d \in \mathbb{R}^+_0\), \((u + d)(x) := u(x) + d, \forall x \in X.\) Assume \(x' \subseteq X, u(x')\) is the clock assignment that maps all clocks in \(x'\) to \(0\) (a vector with all entries equal to zero which has the same size as \(x'\)) and keeps the other clocks in \(C \setminus x'\) same as \(u\).

**Definition 7. (Timed Automaton [27])** A timed automaton is a sextuple \(T\mathcal{A} = (L, L_0, \mathcal{A}, X, E, \mathcal{I})\) where:

- \(L\) is a finite set of locations (discrete modes);
- \(L_0\) is the initial location;
- \(\mathcal{A}\) is a finite set of actions;
- \(X\) is a finite set of real-valued variables (clocks);
- \(E \subseteq L \times C(X) \times \mathcal{A} \times 2^X \times L\) is a finite set of edges;
- \(\mathcal{I} : L \rightarrow C(X)\) is an invariant map.

A state of \(T\mathcal{A}\) is a pair of location and clock assignment \((l, u)\) for which the transition map \(\rightarrow\) is as follows:

1) continuous transition: \((l, u) \xrightarrow{\mathcal{A}} (l, u + d)\) if \(u \models \mathcal{I}(l)\) and \((u + d) \models \mathcal{I}(l)\) for a non-negative real number \(d \in \mathbb{R}^+_0;\)
2) discrete transition: \((l, u) \xrightarrow{\mathcal{E}} (l', u')\) if \(l \xrightarrow{\mathcal{E}} l', u \models \mathcal{E}\) and \(u' = u[r]\) and \(u' \models \mathcal{I}(l').\)

**B. LTI Event-Triggered Control System**

Consider linear time invariant (LTI) systems without disturbances given by:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad \xi(t) \in \mathbb{R}^n, \quad v(t) \in \mathbb{R}^m
\]

with linear state-feedback laws implemented in a sample-and-hold manner:

\[
v(t) = v(t_k) = K\xi(t_k), \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.
\]

Let us denote by \(\xi_x(t) \in \mathbb{R}^n\) the solution of \((1)\) for \(t \in [t_k, t_{k+1})\) with \(\xi_x(t_k) = x\) as its initial condition. Furthermore, let us denote by \(e_x(t) := x - \xi_x(t)\) the virtual error introduced by the sampling mechanism. We consider the event triggering approach proposed in [20] which enforces sampling instants given by:

\[
t_{k+1} = \min\{t > t_k | e_x(t)^2 \geq \alpha(\xi_x(t))^2\}.
\]

Let the sampling period associated to a state \(\xi(t_k) = x\) be denoted by \(\tau(x) := t_{k+1} - t_k\). Then, the values of \(\xi_x\) and \(e_x\) can be expressed in terms of \(x\), for \(\sigma \in [0, t_{k+1} - t_k]\) as follows:

\[
\xi_x(t_k + \sigma) = \Lambda(\sigma)x,
\]

\[
e_x(t_k + \sigma) = [I - \Lambda(\sigma)]x
\]

where

\[
\Lambda(\sigma) = [I + \int_0^\sigma e^{Ar}dr(A + BK)].
\]

Thus, substituting \((4)\) and \((5)\) in \((3)\), the following expression for the state-dependent sampling period is obtained:

\[
\tau(x) = \min\{\sigma > 0 | x^T\Phi(\sigma)x = 0\},
\]

where

\[
\Phi(\sigma) = [I - \Lambda^T(\sigma)][I - \Lambda(\sigma)] - \alpha\Lambda^T(\sigma)\Lambda(\sigma).
\]
C. Problem Statement

Consider the system:

$$S = (X, X_0, U, \rightarrow, Y, H)$$

where

- $X = \mathbb{R}^n$;
- $X_0 = \mathbb{R}^n$;
- $U = \emptyset$, i.e. the system is autonomous;
- $\rightarrow \subseteq X \times U \times X$ such that $\forall x, x' \in X : (x, x') \in \rightarrow$ iff $\xi(x) = x'$;
- $Y \subseteq \mathbb{R}^+$;
- $H : \mathbb{R}^n \to \mathbb{R}^+$ where $H(x) = \tau(x)$.

The system $S$ generates as output sequences all possible sequences of inter-sampling intervals that the system \([1,2]\) with triggering condition \([3]\) can exhibit. Note in particular that $S$ is an infinite-state system.

**Problem 1.** We seek to construct power quotient systems $S/P$ based on adequately designed equivalence relations $P$ defined over the state set $X$ of $S$.

In particular, we propose to construct the system $S/P$ as follows:

$$S/P = (X/P, X_0/P, U/P, \rightarrow/P, Y/P, H/P)$$

where

- $X/P = \mathbb{R}^n/P := \{R_1, \ldots, R_q\}$;
- $X_0/P = \mathbb{R}^n/P$;
- $U/P = \emptyset$, i.e. the system is autonomous;
- $(x/P, x'/P) \in \rightarrow/P$ if $\forall x \in x/P, \exists x' \in x'/P$ such that $\xi(x) = x'$;
- $Y/P \subseteq 2^Y \subset \mathbb{R}^+$;
- $H/P(x/P) = \min_{x/P} H(x), \max_{x/P} H(x) := [\bar{x}_P, \bar{x}_P]$.

The remaining question is now how to: select an appropriate equivalence relation $P$, compute the respective intervals $[\bar{x}_P, \bar{x}_P]$, and determine when there is a transition between a pair of abstract states $(x/P, x'/P)$. These three questions, and the respective constructions, are addressed in the following section.

### III. Construction of the Abstraction

In this section, we address the construction of $S/P$, and more specifically $X/P$, $H/P$, and $\rightarrow/P$. The set $X/P$ and map $H/P$ are constructed employing a two-step approach inspired by the work in \([22]\). Nonetheless, some modifications are introduced to make their approach suitable to our goals. In \([22]\) the authors compute minimum inter-sample times, $\tau_S$, for conic regions and Lyapunov-Razumikhin stability conditions. We adapt their approach (in our Lemma \([2]\) and Theorem \([1]\)) to apply instead to the ISS based triggering mechanism in \([7,8]\). Additionally, we modify the approach to also provide upper limits, $\bar{\tau}_S$, on the regional inter-sample times (see our Lemma \([3]\) and Theorem \([2]\)). We also propose, inspired by the work on stability analysis of switched systems using multiple Lyapunov functions \([30]\), a modification to the way the S-procedure is employed in \([22]\) for $n$-dimensional state spaces with $n \geq 3$ (see Theorems \([1]\) and \([2]\)). Finally, we develop an extension to the work in \([27]\) for reachability analysis of dynamical systems in order to construct the transition relation $\rightarrow/P$.

### A. State Set

First, an important observation from the state-dependent sampling period \([7]\) is pointed out. This observation is the cornerstone in removing spatial dependencies from \([7]\) and mapping the infinite-state system $S$ to a finite-state system $S/P$.

**Remark 1.** Excluding the origin, all the states which lie on a line, that goes through the origin, have the same inter-sample time, i.e., $\tau(x) = \tau(\lambda x)$, $\forall \lambda \neq 0$ \([22]\).

Based on Remark \([1]\) and the fact that a convex polyhedral cone (pointed at the origin) is the union of an infinite number of rays, a convenient approach to partition the state space is via abstracting it into a finite number of convex polyhedral cones (pointed at the origin) $R_s$, where $s \in \{1, \ldots, q\}$ and $\bigcup_{s=1}^q R_s = \mathbb{R}^n$ (see Figure \([1]\)).

The state space abstraction technique proposed by \([22]\), called isotropic covering, is briefly explained in the following. Generalized spherical coordinates $x \in \mathbb{R}^n : (r, \theta_1, \ldots, \theta_{n-1})$ are used for the abstraction purpose, where $r = |x|$ and $\theta_1, \ldots, \theta_{n-1}$ are the corresponding angular coordinates of $x$. Each angular coordinate is divided into $m$ equidistant intervals (note $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$ and $\theta_{n-1} \in [-\pi, \pi]$). Hence, the number of conic regions, $q$, is equal to $m^{(n-1)}$ (see \([22]\) for more details on the state space abstraction technique). However, Remark \([1]\) also suggests that it suffices to only consider half of the state space since $x$ and $-x$ behave the same in \([7]\). Therefore, one can consider half of the state space (for example, by assuming $\theta_{n-1} \in [0, \pi]$), and then appropriately map the results to the other half of the state space. In this case, $q$ is equal to $2 \times m^{(n-1)}$.

### B. Output Map

We focus now on the construction of the output map $H/P$ (and the output set $Y/P$ along the way). We propose a constructive method to find a time interval $[\bar{\tau}_S, \bar{\tau}_S]$ associated to each region $R_s$ such that $\forall x \in R_s : \tau(x) \in [\bar{\tau}_S, \bar{\tau}_S]$, with $\tau(\cdot)$ as in \([7]\). In order to find $\bar{\tau}_S$ and $\bar{\tau}_S$, we employ the approach proposed in \([26]\) to construct a convex polytope around the matrix $\Phi(\sigma)$ (this is possible since $\Phi(\sigma)$ is continuous on a compact set $[0, t]$ where $t > 0$ is an arbitrary instant and $\sigma \in [0, t]$).

First, we illustrate how one can derive $\bar{\tau}_S$. The time interval $[0, \bar{\tau}_S]$ determines a regional interval of time in which no triggering is enabled. The main idea of the following lemma is to construct a finite set of matrices $\Phi_{k,\kappa}$, with $\kappa \in K_s$, (a finite set of indices), such that:

$$x^T \Phi_{k,\kappa} x \leq 0, \forall \kappa \in K_s \Rightarrow (x^T \Phi(\sigma)x \leq 0, \forall \sigma \in [0, \bar{\tau}_S]).$$

In what follows, the parameters used in Lemma \([2]\) are given. The vector $x \in \mathbb{R}^n$ is a state. $R_s \subseteq \mathbb{R}^n$ denotes an equivalence class on $\mathbb{R}^n$ where $R_s = \{x \in \mathbb{R}^2 : x^T Q_s x \geq 0\}$, $Q_s =$
describes how to find a suitable $\bar{\sigma}$ for the purpose of reducing the conservativeness involved in the polytopic embedding of $\hat{\Phi}(\sigma)$.

**Lemma 2.** Consider a time limit $\tau_s \in (0, \bar{\sigma}]$. If the condition $x^T \hat{\Phi}(i,j),x \leq 0$ is satisfied for $\forall (i,j) \in K_s = \{(0, \ldots, N_{\text{conv}}) \times \{0, \ldots, [\tau_s/n]\}\}$, then, $\forall \sigma \in [0, \tau_s]$, $x^T \Phi(\sigma)x \leq 0$ with $\Phi$ defined in (8) and

$$\Phi_{(i,j),s} = \bar{\Phi}_{(i,j),s} + \nu I,$$

$$\bar{\Phi}_{(i,j),s} = \begin{dcases}
\sum_{k=0}^{\lfloor \frac{\tau_s}{\tau} \rfloor} L_{k,j} \left(\frac{\tau}{\tau}\right)^k & \text{if } j < \frac{\tau_s}{\tau}, \\
\sum_{k=0}^{\lfloor \frac{\tau_s}{\tau} \rfloor} L_{k,j} \left(\tau_s - \frac{\tau_s}{\tau}\right)^k & \text{otherwise},
\end{dcases}$$

$$L_{0,j} = I - \Pi_{1,j} - \Pi_{1,j}^T + (1 - \alpha)\Pi_{1,j}^T \Pi_{1,j},$$

$$L_{1,j} = \left[(1 - \alpha)\Pi_{1,j}^T - I\right]\Pi_{1,j} + \Pi_{2,j}(1 - \alpha)\Pi_{1,j} - I,$$

$$L_{k\geq2,j} = \left[(1 - \alpha)\Pi_{1,j}^T - I\right]\Pi_{2,j} + \Pi_{2,j} \left(\Pi_{1,j}^T\right)^T [1 - (1 - \alpha)\Pi_{1,j}^T - I] + (1 - \alpha)\Pi_{2,j}\left(\sum_{i=1}^{k-1} [A_{i-1}^T A_{i-1}]^T \frac{A_{i-1}}{i!} \right)\Pi_{2,j},$$

and

$$\nu \geq \max_{\sigma \in [0, \tau], \tau \in [0, l-1]} \lambda_{\text{min}}(\Phi(\sigma + \tau) - \bar{\Phi}_{\text{conv},r}(\sigma)),$$

with

$$\bar{\Phi}_{\text{conv},r}(\sigma) = \sum_{k=0}^{N_{\text{conv}}} L_{k,r} \sigma^k.$$  

**Proof.** See Appendix.

In Theorem 1, the S-procedure is used along with Lemma 2 to regionally reduce the conservativeness involved in $\tau_s$.

**Theorem 1** (Regional Lower Bound Approximation). Consider the inter-sampling time set $\{\tau_1, \ldots, \tau_q\}$ satisfying $0 < \tau_s \leq \bar{\sigma}$ and matrices $\Phi_{s,s}$ satisfying $\Phi_{s,s} \preceq 0$, $\forall s \in \{1, \ldots, q\}$, $\kappa = (i,j) \in K_s = \{(0, \ldots, N_{\text{conv}}) \times \{0, \ldots, [\frac{\tau_s}{\tau}]\}\}$. If there exist scalars $\varepsilon_{s,s} \geq 0$ (for $n = 2$) or symmetric matrices $U_{s,s}$ with nonnegative entries (for $n \geq 3$) such that LMI

$$\Phi_{s,s} + \varepsilon_{s,s} \tau_s \preceq 0$$

or

$$\Phi_{s,s} + E_s^T U_{s,s} E_s \preceq 0$$

are satisfied, then, the inter-sample time $[\tau_s]$ of the control system $[12]$ is regionally bounded from below by $\bar{\tau}_s$, $\forall x \in R_s$.

**Proof.** See appendix.

Now, we illustrate how one can find $\bar{\tau}_s$, the upper bound approximation of the regional inter-sample time. To do so, we change the triggering condition (3) to $x^T \Phi(\sigma)x \geq 0$. The underlying ideas in Theorem 2 remain the same as in Lemma 3. Theorem 1, we embed $\Phi$ in a polytope with finite vertices (see Lemma 3) and use the S-procedure to derive the regional upper bound $\tilde{\tau}_s$ of the inter-sample time $\tau(\cdot)$ (see Theorem 2).

**Lemma 3.** Consider a time limit $\bar{\tau}_s \in [\tau_s, \sigma]$. If the condition $x^T \bar{\Phi}(i,j),s x \geq 0$ is satisfied $\forall (i,j) \in K_s = \{(0, \ldots, N_{\text{conv}}) \times \{0, \ldots, \lfloor \frac{\tau_s}{\tau} \rfloor\}\}$, then, $\forall \sigma \in [\bar{\tau}_s, \sigma]$, $x^T \Phi(\sigma)x \geq 0$ with $\Phi$ defined in (8) and

$$\bar{\Phi}_{(i,j),s} = \tilde{\Phi}_{(i,j),s} + \tilde{\nu} I,$$

$$\tilde{\Phi}_{(i,j),s} = \begin{dcases}
\sum_{k=0}^{\lfloor \frac{\tau_s}{\tau} \rfloor} L_{k,j} \left(\frac{\tau}{\tau}\right)^k & \text{if } j < \frac{\tau_s}{\tau}, \\
\sum_{k=0}^{\lfloor \frac{\tau_s}{\tau} \rfloor} L_{k,j} \left(\tau_s - \frac{\tau_s}{\tau}\right)^k & \text{otherwise},
\end{dcases}$$

$$\tilde{\nu} \leq \max_{\sigma' \in [\bar{\tau}_s, \sigma]} \lambda_{\text{min}}(\Phi(\sigma' + \tau) - \bar{\Phi}_{\text{conv},r}(\sigma')),$$

and $L_{k,j}$ is given by (70).

**Proof.** See appendix.

**Theorem 2** (Regional Upper Bound Approximation). Consider the inter-sampling time set $\{\bar{\tau}_1, \ldots, \bar{\tau}_q\}$ satisfying $\bar{\tau}_s < \bar{\tau}_s \leq \bar{\sigma}$ and matrices $\Phi_{s,s}$ satisfying $x^T \Phi_{s,s} x \geq 0$, $\forall s \in \{1, \ldots, q\}$, $\kappa = (i,j) \in K_s = \{(0, \ldots, N_{\text{conv}}) \times \{0, \ldots, \lfloor \frac{\tau_s}{\tau} \rfloor\}\}$. If there exist scalars $\varepsilon_{s,s} \geq 0$ (for $n = 2$) or symmetric matrices $U_{s,s}$ with nonnegative entries (for $n \geq 3$) such that LMI

$$\tilde{\Phi}_{s,s} - \bar{\Phi}_{s,s} \tau_s \preceq 0$$

or

$$\tilde{\Phi}_{s,s} - E_s^T U_{s,s} E_s \preceq 0$$

are satisfied, then, the inter-sample time $[\bar{\tau}_s]$ of the control system $[12]$ is regionally bounded from above by $\bar{\tau}_s$, $\forall x \in R_s$.

**Proof.** See appendix.
are satisfied, then, the inter-sampling time \( k \) of the control system \( X(\tau) \) is regionally bounded from above by \( \bar{\tau}_s \), \( \forall x \in \mathcal{R}_s \).

**Proof.** See appendix.

We now suggest a procedural algorithm to use Theorems 1 and 2 to compute \( \bar{\tau}_s \) and \( \bar{\tau}_s \). First, derive \( \nu \) using (13). Consider \( s = 1 \) in Theorem 1. Implement a line search on \( \tau^* \) in the interval \([0, \bar{\tau}]\) to find the lower bound on the inter-sampling time in the whole state space. Partition the state space into \( q \) regions and again use Theorem 1 to find \( \{\tau_1, \ldots, \tau_q\} \).

This optimization problem is a line search on \( \tau^* \) in the interval \([\tau^*, \bar{\tau}]\) combined with LMI feasibility problems on \( \bar{x}_{\mathcal{R}_s} \) or \( \bar{U}_{\mathcal{R}_s} \). In order to find \( \bar{\tau}_s \), first, derive \( \nu \) using (13) and then, use Theorem 2. This problem is a combination of a line search on \( \bar{\tau}_s \) in the time interval \([\tau^*, \bar{\tau}]\) with LMI feasibility problems on \( \bar{x}_{\mathcal{R}_s} \) or \( \bar{U}_{\mathcal{R}_s} \). Finally, we introduce an algorithmic guideline to find a suitable \( \bar{\tau}_s \) in Remark 2.

**Remark 2.** To the best of our knowledge, there is no such approach to find \( \bar{\tau}_s \) in a well-established manner. However, one can increase the value of \( \bar{\tau}_s \) by using Theorems 1 and 2 to satisfy \( \bar{\tau}_s < \bar{\tau} \).

**C. Transition Relation**

In this subsection, we address the construction of the transition map \( \mathcal{P} \). To do so, each conic region is considered as a discrete mode of \( S_{\mathcal{P}} \) and a reachability analysis is deployed to derive all possible transitions in \( S_{\mathcal{P}} \). Adapted from (27), the reachability analysis constructs an outer approximations of the reachable set of each conic region \( \mathcal{R}_s \) in its corresponding time interval \([\tau_s, \bar{\tau}_s]\). In other words, we seek to compute the reachable set of \( \xi_s(t) = \Lambda(t)x, \forall t \in [\tau_s, \bar{\tau}_s], \forall x \in \mathcal{R}_s \), and \( \forall s \in \{1, \ldots, q\} \) where \( \Lambda(\cdot) \) is given by (6). Note that, because of the state feedback law, the sample-and-hold system (4) is a time-invariant autonomous system between two consecutive sampling instants. Fortunately, (27) proposed a technique to derive an approximation of a time-invariant autonomous system which is used in this study. In what follows, we sketch the procedure of how the technique proposed in (27) can be used in our favor to derive the transitions in \( S_{\mathcal{P}} \) (see (27) for more details).

Before going any further, let us define the term **flow pipe** employed in this subsection that is adapted from (27). Consider \( X(0) \) is a suitable initial set defined in the conic region \( \mathcal{R}_s \).

**Definition 8. (Flow Pipe)** The set of reachable states or the flow pipe from an initial set \( X(0) \) in the time interval \([\tau_s, \bar{\tau}_s]\) is denoted by:

\[
X_{[\tau_s, \bar{\tau}_s]}(X(0)) = \bigcup_{t \in [\tau_s, \bar{\tau}_s]} X_t(X(0))
\]

where \( X_t(X(0)) \) denotes the reachable set at time \( t \) from \( X(0) \), given by:

\[
X_t(X(0)) = \{ \xi_{x_0}(t) \mid x_0 \in X(0) \}.
\]

Let us introduce how to define a suitable \( X \) in each \( \mathcal{R}_s \). We have observed that the states that lie on a line going through the origin have an identical triggering behavior (see Remark 1). Furthermore, these states are mapped to another line that goes through the origin (since the state evolution (4) is a linear map on \( x \)). Moreover, the image of a convex polytope under a linear map is another convex polytope. Therefore, one can consider \( X(0) \) as a polytope for which its vertices lie on the extreme rays of \( \mathcal{R}_s \) and \( \mathcal{R}_s \neq 0 \).

Given \( X(0) \) in \( \mathcal{R}_s \) with corresponding \( \bar{\tau}_s \) and \( \bar{\tau}_s \), our first objective is to derive a polytopic outer approximation \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \) to the flow pipe \( X_{[\tau_s, \bar{\tau}_s]}(X(0)) \), i.e.

\[
X_{[\tau_s, \bar{\tau}_s]}(X(0)) \subseteq \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)).
\]

The overall procedure to construct \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \) is presented in the sequence. First, the time interval \([\tau_s, \bar{\tau}_s]\) is divided into \( f \) subintervals to reduce the conservatism in the polytopic approximation \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \). The set \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \) is therefore computed by a sequence of convex polytopes \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \forall f \in \{1, \ldots, f\} \), and \( \tau_i \) is the time interval between two consecutive sampling instants.

\[
\hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) = \bigcup_{f} \hat{X}_{[\tau_i, \bar{\tau}_i]}(X(0)).
\]

The procedure to derive the outer approximation \( \hat{X}_{[\tau_s, \bar{\tau}_s]}(X(0)) \) of each flow pipe segment \( X_{[\tau_s, \tau_i]}(X(0)) \) as follows. Let \( Poly(C, d) \) represent the convex polytope \( P = \{ x \mid Cx \geq d \} \) where \( C \in \mathbb{R}^{n \times n} \) and \( d \in \mathbb{R}^n \). Let \( c^f_i \) be the \( i \)-th row of \( C \) where \( c^f_i \in \mathbb{R}^n \) (note that \( c^f_i \) is in fact the normal vector of \( i \)-th face of \( Poly(C, d) \) pointing toward outside of the polytope. \( \delta(P) \) denotes the set of vertices of \( P \). Furthermore, denote the convex hull of a finite number of points \( \Omega \) by \( CH(\Omega) \). Assume \( V_i(X(0)) \) represents the set of system trajectories evolved from the vertices of \( X(0) \) at the instant \( t \).

Consider \( Poly(C', d') \) represents an initial approximation of \( X_{[\tau_i, \tau_{i+1}]}(X(0)) \) denoted by \( W_{[\tau_i, \tau_{i+1}]}(X(0)) \) and given by:

\[
W_{[\tau_i, \tau_{i+1}]}(X(0)) = CH(V_i(X(0)) \cup V_{i+1}(X(0))).
\]

We fix \( C' \) and search for a vector \( d' \in \mathbb{R}^n \) such that \( Poly(C', d') \supseteq X_{[\tau_i, \tau_{i+1}]}(X(0)) \) and the volume of \( Poly(C', d') \) is minimal. In other words, one needs to solve the following non-convex optimization problem in the vector-valued decision variable \( d' \):

\[
Poly(C', d') = \min_{d'} \text{volume}[Poly(C', d')]
\]

s.t. \( X_{[\tau_i, \tau_{i+1}]} \subseteq Poly(C', d'). \)

Note that fixing \( C' \) implies that each face of \( Poly(C', d') \) is parallel to a face of \( Poly(C', d) \). A tractable approach to avoid solving the non-convex optimization problem (25) is by solving the following non-convex optimization in the scalar-valued decision variable \( d^* \):

\[
d^*_i = \max_{x_0 \in X(0)} c_i^f \Lambda(t)x_0
\]

s.t. \( x_0 \in X(0) \)

\( t \in [\tau_i, \tau_{i+1}] \),

where \( d^*_i \) is the \( i \)-th entry of \( d^* \), \( i \in \{1, \ldots, g\} \), and \( \Lambda(\cdot) \) is given by (6). Note that (26) is a line search on the evolution of...
Refining the conic regions of a given abstraction

Remark 3. In order to derive the transitions in \( S_{/P} \), the intersection between the flow pipe corresponding to each region \( R_s \) and the initial conic regions need to be found. This can be done by solving the following feasibility problem for each pair of conic regions \( (R_s, R_{s'}) \):

\[
\begin{align*}
\text{Feas} & : C_{f,s} \bar{x} \leq d_{f,s}' \quad \forall \bar{x} \in \mathbb{R}^n | C_{f,s} \bar{x} \leq d' \end{align*}
\]  

where \( \{\bar{x} \in \mathbb{R}^n | C_{f,s} \bar{x} \leq d'\} \) represents the outer approximation of the \( f \)-th flow pipe segment associated with \( R_s \), and \( \{\bar{x} \in \mathbb{R}^n | E_{s,s'} \bar{x} \geq 0\} \) is the initial conic region where \( s' \in \{1, \ldots, q\} \). The feasibility of (27) for any value of \( f \in \{1, \ldots, \bar{f}\} \) indicates that there is a transition from the mode \( s \) to the mode \( s' \) in \( S_{/P} \).

IV. MAIN RESULTS

A. \( \varepsilon \)-Approximate Simulation Relation

Based on Lemma 1, the metric system \( S_{/P} \) \( \varepsilon \)-approximately simulates \( S \) where \( \varepsilon = \max d_H(y, y') \), \( y = H(x) \in Y, y' = H_{/P}(x') \in Y//P, \forall (x, x') \in P \), and \( d_H(\cdot, \cdot) \) is the Hausdorff distance.

Remark 3. Refining the conic regions of a given abstraction into a larger number of conic regions \( R_s \), \( q \), the length of the sampling intervals in each region \( |\bar{r}_s - r_s| \) cannot increase. Thus, increasing \( q \) results in abstractions with a new precision \( \varepsilon' \leq \varepsilon \) where \( \varepsilon' \) is computed based on the refined state space abstraction. Note that this is not necessarily the case if the partition of the state-space in the new abstraction is not a refinement of the original abstraction.

However, as previously mentioned, there is an exponential dependency between \( \hat{m} \) (the number of subdivisions considered for each angular coordinates in the state space abstraction) and \( q \) (the number of resulting conic regions). Hence, refining the conic partitions by increasing \( \hat{m} \) comes at the price of an exponential explosion in computation for high-dimensional problems.

B. Timed-Automaton Representation

We demonstrate now that \( S_{/P} \) is equivalent to a \( TA \). Regardless of \( TA \)’s uncountable state space (related to its clock variables), it has been shown that its reachability analysis is decidable [21]. This fact makes \( TA \) a powerful tool to model real-world systems, such as real-time systems, where discrete transitions are coupled with timing constraints. Several tools, exploring the attractive features of \( TA \), have been developed for verification and synthesis [31] and [32].

In the following, the underlying relation between states and outputs of \( S_{/P} \) is presented to show that \( S_{/P} \) is equivalent to a \( TA \). Intuitively, the corresponding output \( y_{/P} \in Y//P \) of a state \( x_{/P} \in X_{/P} \) indicates that \( S_{/P} \):

1) remains at \( x_{/P} \) during the time interval \([0, \bar{r}_{x_{/P}}]\),
2) possibly leaves \( x_{/P} \) during the time interval \([\bar{r}_{x_{/P}}, \bar{r}_{x_{/P}}]\), and
3) is forced to leave \( x_{/P} \) at \( \bar{r}_{x_{/P}}\).

Based on Definition 7 and the given intuition, \( S_{/P} \) is equivalent to a \( TA = (L, L_0, Act, X, E, Inv) \) where:

- the set of locations \( L := \{l_1, \ldots, l_q\} \);
- the initial location \( L_0 := l_i \) such that \( \xi(0) = l_i \in R_i \);
- the set of actions \( Act = \{s\} \) is an arbitrary labeling of discrete transitions (or edges);
- the clock variable \( X = \{\bar{x}\} \);
- the set of edges \( E \) is derived from reachability analysis: each edge \((l_s, g, a, r, l_{s'}) \in E \) is associated with a guard set \( g = \{x | \bar{r}_s \leq x \leq \bar{r}_{s'}\}, \forall l_s \in \{1, \ldots, q\} \).

V. NUMERICAL EXAMPLE

In this section, we illustrate the results of this study. Consider a linear system, employed in the example in [20], with state feedback law that is given by:

\[
\begin{align*}
\ddot{\xi} &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nu, \\
\nu &= [1 - 4] \xi.
\end{align*}
\]  

We set the triggering relaxation coefficient \( \alpha = 0.05 \), the order of polynomial approximation \( N_{\text{conv}} = 5 \), the number of polytopic subdivisions \( l = 100 \), the number of angular subdivisions \( \hat{m} = 10 \) (hence, the number of conic regions is \( q = 2 \times \hat{m}^{(n-1)} = 2 \times 10^{(2-1)} = 20 \)), the upper bound of the inter-sample interval \( \bar{\sigma} = 1 \text{sec} \), and the time step of the flow pipe segments is 0.01 sec.

Figure 3 depicts \( \bar{r}_s \) and \( \bar{r}_s \) for the closed loop system given by (28) using Theorem 1 and Theorem 2, respectively. In Figure 4, another representation of \( \bar{r}_s \) and \( \bar{r}_s \) is given in the state space. Based on the Hausdorff distance and the derived boundaries, one gets that in this example the precision of the abstraction is \( \varepsilon = 0.119 \).

Note that by increasing \( \hat{m} \): \( \bar{r}_s \) and \( \bar{r}_s \) will possibly shift upward and downward, respectively. By doing so, one can
derive tighter bounds for the given event-triggered control system. Thus, a more precise $\epsilon$-approximate simulation relation from $S$ to $\mathcal{S}/P$ can be achieved. Figure 5 shows a comparison between two different state space abstractions with $\bar{m} = 10$ and 100. Evidently, partitioning half of the state space to 100 regions instead of 10 regions leads to tighter bounds where $\epsilon' = 0.021$.

Figure 6 illustrates the validity of the theoretical bounds that we found for $\tau_s$ (black line) and $\bar{\tau}_s$ (red line). The asterisks represent the inter-sample times sequence during 5 sec simulation of the event-triggered control system.

In Figure 7 a schematic representation of the discrete transition of the resulting $\mathcal{T}_A$ is provided. Each asterisk represents a transition from mode $i$ (or $\mathcal{R}_i$) to mode $j$ (or $\mathcal{R}_j$) where $i$ and $j$ are the horizontal and vertical coordinates of the asterisk, respectively.

VI. CONCLUSION

In this paper, we proposed an LMI-based approach combined with reachability analysis to derive a timed-automaton that captures the sampling behavior of LTI systems controlled with an event-triggered control strategy. It has been shown that there exists an $\epsilon$-approximate simulation relation from the control system to the derived timed-automaton using Hausdorff distance as a metric. The proposed approach relies on two major parts: an abstraction technique that enables to find regional lower and upper bounds of sampling intervals for a finite number of convex regions (which cover the state space); and a reachability analysis that helps to derive all possible transitions between these convex regions. Capturing the timing property of such control systems in the context of a timed-automaton provides a theoretical basis for schedulability analysis. As a result, an interesting research direction to extend the results of this study is to exploit the already existing theory of timed-automaton for formal verification of real-time systems with the considered execution strategy. Another line of research that is worth exploring is extending the results of this study to propose co-design approaches that are less conservative when the considered ETC strategy is
deployed using approaches such as network calculus (or real-time calculus).

**APPENDIX**

**Proof of Lemma 2.** First, we divide the time interval $[0, \bar{\tau}]$ into $l$ subdivisions and assume $\sigma \leq \bar{\tau}_s$ is inside one of these subdivisions. The aim of this step is to make preparations to compute a precise estimation of $\Phi(\cdot)$ by building $l$ small convex embeddings around it instead of building an over conservative one. Second, we compute a polynomial approximation of $\Phi(\cdot)$ over the chosen subdivision interval. Third, we bound the error of polynomial approximation with a constant term. Fourth, we build a convex polytope that contains the polynomial approximation plus the error term, using the method proposed in [26]. The procedural implementation of these four steps is given in the following:

First: we divide the time interval $[0, \bar{\tau}]$ into $l$ subdivisions and pick a $\sigma \in [0, \bar{\tau}_s]$. There exists $j \in \{0, \ldots, \lfloor \frac{\bar{\tau}}{\bar{\tau}_s} \rfloor \}$ such that
\[ j \frac{\bar{\tau}}{\bar{\tau}_s} \leq \sigma \leq (j+1) \frac{\bar{\tau}}{\bar{\tau}_s} \]
Then, we define $\sigma' = \sigma - j \frac{\bar{\tau}}{\bar{\tau}_s} (\sigma' \in [0, \chi]$, with $\chi = \frac{\bar{\tau}}{\bar{\tau}_s}$ for $j < \lfloor \frac{\bar{\tau}}{\bar{\tau}_s} \rfloor$ and $\chi = \bar{\tau} - j \frac{\bar{\tau}}{\bar{\tau}_s}$ otherwise).

Second: we use the relation $\int_{0}^{a+b} e^{A r} d r = e^{a e^{A b}} - e^{a e^{A 0}}$ to simplify $\Lambda(\sigma)$ given in (6) as following:

\[ \Lambda(\sigma) = I + M_j(A + BK) + \int_{0}^{\sigma'} e^{A r} d r N_j(A + BK) \]
where $M_j$ and $N_j$ as in (12). Then, by defining two new matrices $\Pi_{1,j}$ and $\Pi_{2,j}$ as in (11), we can rewrite (29) in the following form:

\[ \Lambda(\sigma) = \Pi_{1,j} + \int_{0}^{\sigma'} e^{A r} d r \Pi_{2,j}. \]

Afterwards, we replace $\int_{0}^{\sigma'} e^{A r} d r$ by its $N_{\text{conv}} - th$ order Taylor series expansion to approximate $\Phi$ given by (8), using:

\[ \int_{0}^{\sigma'} e^{A r} d r \simeq \sum_{i=1}^{N_{\text{conv}}} \Lambda^{i-1} \frac{\sigma}{2^i}. \]

Note that $N_{\text{conv}} + 1$ is the number of vertices we consider for polytopic embedding according to time. The Taylor series expansion of $\Phi$ without approximation is given by $\sum_{k=0}^{\infty} L_{k,j} \sigma^k$ with $L_{k,j}$ defined in (10).

Third: in this step, we intend to bound the error that is caused by the $N_{\text{conv}}$-th order approximation of $\Phi$. One can derive the $N_{\text{conv}}$-th order approximation of $\Phi$ on the time interval $[j \frac{\bar{\tau}}{\bar{\tau}_s}, (j+1) \frac{\bar{\tau}}{\bar{\tau}_s}]$ using $\Phi_{N_{\text{conv}}, j}(\sigma')$ given in (14).

Now, let’s call the approximation error $R_{N_{\text{conv}}, j}(\sigma')$ where

\[ R_{N_{\text{conv}}, j}(\sigma') = \Phi(\sigma) - \Phi_{N_{\text{conv}}, j}(\sigma'). \]

Then, we need to compute a bound with a scalar $\nu$ independent of $\sigma'$ such that

\[ R_{N_{\text{conv}}, j}(\sigma') \leq \nu I. \]

By doing so, we are able to imply $x^T \Phi(\sigma)x \leq 0$ from $x^T (\Phi_{N_{\text{conv}}, j}(\sigma') + \nu I)x \leq 0$. Since $R_{N_{\text{conv}}, j}$ is symmetric, we have $R_{N_{\text{conv}}, j}(\sigma') \preceq \lambda_{\text{max}}(\sigma') I$, where $\lambda_{\text{max}}(\sigma')$ is the maximal eigenvalue of $R_{N_{\text{conv}}, j}(\sigma')$. The procedure to find $\nu$ is given by (13).

Fourth: Since the function $\Phi_{N_{\text{conv}}, j}(\sigma') + \nu I : [0, \chi] \rightarrow \mathcal{M}_n(\mathbb{R})$ is a matrix exponential function, one can approximate it by its convex polytope (Lemma 1). Then, by showing $x^T \Phi_{(i,j), s}x \leq 0$, $\forall (i,j) \in \mathcal{K}_s = \{ (0, \ldots, N_{\text{conv}}) \times \{0, \ldots, \lfloor \frac{\bar{\tau}}{\bar{\tau}_s} \rfloor \} \}$, with $\Phi_{(i,j), s} = \sum_{k=0}^{i} L_{k,j} \chi^k + \nu I$, we provide that $x^T (\Phi_{N_{\text{conv}}, j}(\sigma') + \nu I)x \leq 0$ and as a result, we have $x^T \Phi(\sigma)x \leq 0$, $\forall \sigma \in [0, \bar{\tau}_s]$.

**Proof of Theorem 1.** Assume $x \in \mathbb{R}^n$, hence, $\exists \bar{\tau}_s \subseteq \mathbb{R}^n : x \in \mathcal{R}_s$ and $\tau(x) \geq \bar{\tau}_s$. Using a lossless (if $n = 2$) or a lossy (if $n \geq 3$) version of $S$-procedure, one can see that for any $\kappa \in \mathcal{K}_s$ the condition $x^T \Phi_{N_{\text{conv}}, j}(\sigma') + \nu I)x \leq 0$, $\forall \sigma \in [0, \bar{\tau}_s]$.

**Proof of Lemma 3.** The proof of Lemma 3 is the same as Lemma 2 with some minor changes that are explained in the following. The interval $[0, \bar{\tau}]$ is divided to $l$ subdivisions. Let $\sigma \in [\bar{\tau}_s, \bar{\tau}]$, $\exists j \in \{ \{0, \ldots, l\} - l \}$ such that
\[ j \frac{\bar{\tau}}{\bar{\tau}_s} \leq \sigma \leq (j+1) \frac{\bar{\tau}}{\bar{\tau}_s} \]
Define $\sigma' = \sigma - j \frac{\bar{\tau}}{\bar{\tau}_s} (\sigma' \in [0, \chi]$, with $\chi = \frac{\bar{\tau}}{\bar{\tau}_s}$ for $j < \lfloor \frac{\bar{\tau}}{\bar{\tau}_s} \rfloor$ and $\chi = \bar{\tau} - j \frac{\bar{\tau}}{\bar{\tau}_s}$ otherwise).

**Proof of Theorem 2.** Consider $x \in \mathbb{R}^n$, hence, $\exists \bar{\tau}_s \subseteq \mathbb{R}^n : x \in \mathcal{R}_s$ and $\tau(x) \leq \bar{\tau}_s$. Using a lossless (if $n = 2$) or a lossy (if $n \geq 3$) version of $S$-procedure, it follows $\forall \sigma \in \mathcal{K}_s$, $x^T \Phi_{N_{\text{conv}}, j}(\sigma') + \nu I)x \leq 0$, $\forall \sigma \in [\bar{\tau}_s, \bar{\tau}]$, $\exists \bar{\tau}_s \subseteq \mathbb{R}^n : x \in \mathcal{R}_s$ and $\tau(x) \leq \bar{\tau}_s$. Using a lossless (if $n = 2$) or a lossy (if $n \geq 3$) version of $S$-procedure, it follows $\forall \sigma \in \mathcal{K}_s$, $x^T \Phi_{N_{\text{conv}}, j}(\sigma') + \nu I)x \leq 0$, $\forall \sigma \in [\bar{\tau}_s, \bar{\tau}]$, $\exists \bar{\tau}_s \subseteq \mathbb{R}^n : x \in \mathcal{R}_s$ and $\tau(x) \leq \bar{\tau}_s$.

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