Endpoint Sobolev Theory for the Muskat Equation

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Abstract: This paper is devoted to the study of solutions with critical regularity for the two-dimensional Muskat equation. We prove that the Cauchy problem is well-posed on the endpoint Sobolev space of $L^2$ functions with three-half derivative in $L^2$. This result is optimal with respect to the scaling of the equation. One well-known difficulty is that one cannot define a flow map such that the lifespan is bounded from below on bounded subsets of this critical Sobolev space. To overcome this, we estimate the solutions for a norm which depends on the initial data themselves, using the weighted fractional Laplacians introduced in our previous works. Our proof is the first in which a null-type structure is identified for the Muskat equation, allowing to compensate for the degeneracy of the parabolic behavior for large slopes.

1. Introduction

This is the main in a sequence of three papers on the Cauchy problem for the Muskat equation for solutions with critical regularity. The goal of the present paper is to obtain a definitive result, that is to solve the Cauchy problem in the endpoint Sobolev space

$$H^{3/2}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) ; \int_{\mathbb{R}} (1 + |\xi|^4)^{3/2} |\hat{f}(\xi)|^2 \, d\xi < +\infty \right\}.$$

The study of the Cauchy problem for the Muskat equation begun two decades ago, but with many roots in earlier works. Inspired by the analysis of free boundary flows, different approaches succeeded to establish local well-posedness results for smooth enough initial data (see [6,7,31,33,51,59]). In the last several years, this problem was extensively studied. There are now many different proofs that the Cauchy problem is well-posed, locally in time, on the sub-critical Sobolev spaces $H^s(\mathbb{R})$ with $s > 3/2$ (see [2,24,28,36,44,46]).

Another important component of the background is that the Muskat equation has a compact formulation in terms of singular integrals (see Caflisch et al. [51], Córdoba
and Gancedo [33]). Besides its esthetic aspect, it allows to apply tools at interface of harmonic analysis and nonlinear partial differential equations. One can think of the circle of methods centering around the study of the Hilbert transform and Riesz potentials, or Besov and Triebel-Lizorkin spaces, which can be said to be at the root of the present analysis. In this direction, we are strongly influenced by many earlier works about global existence results in critical spaces under some smallness assumptions (see in particular [5,15,23,26,27,35,38]).

Building upon the above mentioned papers, we develop here an approach which allows to resolve several difficult questions. In particular, the analysis presented here if the first in which a null-type structure is exhibited for the Muskat equation. This provides a substantial clarification of the structure of the nonlinearity, which is the key to compensate for the degeneracy of the parabolic behavior for large slopes.

The results proven in this article are of two types: global existence results valid under a smallness assumption, or local well-posedness results for arbitrary initial data. For our subject, the last are the most important. To explain this, we first of all recall that Castro, Córdoba, Fefferman, Gancedo and López-Fernández proved that the Cauchy problem is not well posed globally in time for some large data ([17–19]). A profound consequence of this fact is that one cannot define a flow map such that the lifespan is bounded from below on bounded subsets of $H^{\frac{3}{2}}(\mathbb{R})$ (otherwise, we would obtain a global existence result for any initial data in $H^{\frac{3}{2}}(\mathbb{R})$ by a scaling argument; using that if $f = f(t,x)$ solves the Muskat equation, so does the function $\lambda^{-\frac{1}{2}} f(\lambda t, \lambda x)$). This is one of the reasons why it is rather surprising that the Cauchy problem is well-posed for large data on the endpoint Sobolev space $H^{\frac{3}{2}}(\mathbb{R})$. The same situation appears for various dispersive equations, however here the problem is considerably more difficult since the Muskat equation is fully-nonlinear.

The discussion in the previous paragraph shows that one cannot rely on classical Sobolev energy estimates; since the time of existence must depend on the profile of the initial data and not only on its Sobolev norm. The problem therefore arises of estimating the solutions of the Muskat equation with a norm whose definition involves the initial data themselves. In fact, it is already a challenging question to define these norms. For that, we have introduced in [4] some weighted fractional Laplacians which are tailored to the fractional characteristic of the Muskat equation, following previous works in [10–13,47]. We will review the main properties of these operators in Sect. 3. We will push this idea further by introducing some time-dependent frequency decompositions which also depend on the solution.

The study of the well-posedness of the Cauchy problem for the Muskat equation on the endpoint Sobolev space $H^{\frac{3}{2}}(\mathbb{R})$ faces two other serious obstacles. Firstly, an obvious difficulty is that $H^{\frac{3}{2}}(\mathbb{R})$ is not embedded in the space of Lipschitz functions. As a consequence, the spatial derivative is not controlled in $L^\infty$-norm. This raises interesting new difficulties since $L^\infty$-norms are ubiquitous in classical nonlinear estimates. Secondly, the Muskat equation is a degenerate parabolic equation for solutions which are not controlled in Lipschitz norm (this statement will be given a rigorous meaning in Sect. 4, see Remark 4.5). As mentioned above, the only way to overcome this difficulty is to identify a null-type structure in the nonlinearity. In addition, one has to perform a paradifferential-type analysis which is compatible with both the weighted fractional Laplacians and this null-type structure. This is the main problem whose solution is the subject of this article.
We will come back to these three points in Sect. 1.3 after having introduced some notations and stated our main result. In Sect. 1.3, we will also compare the main results of this article with our previous ones.

1.1. Main result. The Muskat equation dictates the dynamics of the interface $\Sigma$ separating two fluids having different densities $\rho_1$ and $\rho_2$, obeying Darcy’s law and evolving under the restoring force of gravity. Hereafter we assume that $\rho_2 > \rho_1$ (this corresponds to the stable physical case where the heaviest fluid is located underneath the free surface) and we further assume that $\rho_2 - \rho_1 = 2$ (without loss of generality).

On the supposition that the fluids are two-dimensional and that the interface is a graph, $\Sigma$ is of the form $\Sigma(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}; \ y = f(t, x)\}$ $(t \geq 0)$, where the free surface elevation $f$ is an unknown. In the Córdoba-Gancedo formulation [33], the evolution equation for $f$ simplifies to

$$\partial_t f = \frac{1}{\pi} \frac{1}{p v} \int_{\mathbb{R}} \frac{\Delta_\alpha f_1}{1 + (\Delta_\alpha f)^2} \ d\alpha,$$

where the integral is understood as a principal value integral\(^1\), and $\Delta_\alpha f$ is a slope, defined by the quotient

$$\Delta_\alpha f(t, x) = \frac{f(t, x) - f(t, x - \alpha)}{\alpha}.$$

1 Since it is not immediately obvious that the right-hand side is well defined, we refer to the discussion in paragraph Sect. 4.1 where we recall that the Muskat equation can be written under the form $\partial_t f + |D| f = T(f) f$ where $T(f) f$ is given by a well-defined integral.

Recall that the following notations for Sobolev and homogeneous Sobolev norms:

$$\|u\|_{\dot{H}^\sigma}^2 := (2\pi)^{-1} \int_{\mathbb{R}} |\xi|^{2\sigma} |\hat{u}(\xi)|^2 \ d\xi,$$

$$\|u\|_{H^\sigma} := (2\pi)^{-1} \int_{\mathbb{R}} (1 + |\xi|^2)^\sigma |\hat{u}(\xi)|^2 \ d\xi,$$

where $\hat{u}$ is the Fourier transform of $u$.

Theorem 1.1. i) For any initial data $f_0$ in $H^{3/2}(\mathbb{R})$, there exists a time $T > 0$ such that the Cauchy problem for the Muskat Eq. (1) has a unique solution $f$ in the space

$$X^{3/2}(T) = \left\{ f \in C^0([0, T]; H^{3/2}(\mathbb{R})): \int_0^T \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial x f)^2} \ dx \ dt < \infty \right\}.$$

ii) Moreover, there exists a positive constant $\varepsilon_0$ such that, for any initial data $f_0$ in $H^{3/2}(\mathbb{R})$ satisfying

$$\|f_0\|_{H^{3/2}} \leq \varepsilon_0,$$

the Cauchy problem for (1) has a unique global solution $f$ in $X^{3/2}(+\infty)$.
1.2. Comparison with previous works. The study of the well-posedness of the Cauchy problem for various partial differential equations has attracted a lot of attention in the last decades.

For the Schrödinger equation, which is the prototypical example of a semi-linear dispersive equation, the study of Cauchy problem in the energy critical case goes back to the works of Cazenave and Weissler [20, 21] and culminates with the global existence results of Bourgain [8], Grillakis [40] and Colliander, Keel, Staffilani, Takaoka and Tao [25]. By contrast with sub-critical problems, the time of existence given by the local theory in [20, 21] depends on the profile of the data and not only on its energy. Consequently, the conservation of the energy is not sufficient to obtain global existence results. Detailed historical accounts of the subject can be found in [54]. We also refer to the recent paper by Merle, Raphäel, Rodnianski and Szeftel [45] which establishes an unexpected blow-up result for supercritical defocusing nonlinear Schrödinger equations.

If instead of a semi-linear equation, one considers a quasi-linear problem, then the scaling is not necessarily the only relevant criteria. One key result in this direction is the resolution of the bounded $L^2$ curvature conjecture by Klainerman, Rodnianski and Szeftel [43]. As for hyperbolic or dispersive equations, the study of critical problems for parabolic equations has a very rich litterature. Consider the equation

$$
\partial_t \theta + u \cdot \nabla \theta + (-\Delta)\frac{\alpha}{2} \theta = 0 \quad \text{with} \quad u = \nabla \perp (-\Delta)^{-\frac{1}{2}} \theta,
$$

which is a dissipative version of the classical surface quasi-geostrophic equation introduced by Constantin-Majda-Tabak [29]. In the critical case $\alpha = 1$, the global in time well-posedness has been proved by Kiselev-Nazarov-Volberg [42], Caffarelli-Vasseur [14] and Constantin-Vicol [30] (see also [41, 48, 52, 57]). For the Muskat equation, the nonlinear term is considerably more complicated. However, Cameron proved in [15] that one can apply the method introduced by Kiselev-Nazarov-Volberg to prove the existence of global in time solutions whenever the product of the maximal and minimal slopes is less than 1.

For the Muskat equation, the study of the critical Sobolev problem was initiated by Córdoba and Lazar in [35]. They proved a global existence result for initial data $f_0$ in $H^\frac{5}{2} (\mathbb{R})$ such that $(1 + \|\partial_x f_0\|_{L^\infty}) \|f_0\|_{H^\frac{1}{2}} \ll 1$ (see also [3, 38]). Their main technical result is a key a priori estimate in the critical space $\dot{H}^\frac{3}{2} (\mathbb{R})$, which reads:

$$
\frac{d}{dt} \|f\|^2_{\dot{H}^\frac{3}{2}} + \int_\mathbb{R} \frac{(\partial_{x x} f)^2}{1 + (\partial_x f)^2} \, dx \lesssim \left( \|f\|_{\dot{H}^\frac{3}{2}} + \|f\|^2_{\dot{H}^\frac{3}{2}} \right) \|f\|^2_{\dot{H}^\frac{1}{2}}.
$$

This estimate is easy to state, however its proof is very delicate. Córdoba and Lazar introduce in [35] a tricky reformulation of the Muskat equation in terms of oscillatory integrals, together with the systematic use of Besov spaces.

For the sake of comparison, it might be useful for the reader to explain how to get a global existence result out of the inequality (4). Here the idea is quite simple: one wants to absorb the right-hand side by the left-hand side. This requires to control from below the denominator $1 + (\partial_x f)^2$. For that purpose, one uses an integration by parts argument that goes back to the work of Córdoba and Gancedo [34] (see also [15, 38]). This gives an estimate of the form (see [3])

$$
\frac{d}{dt} \|\partial_x f\|_{L^\infty} \lesssim \|f\|^2_{H^\frac{3}{2}}.
$$
By combining the latter inequality with (4), one obtains that there exists $c_0$ small enough, so that

$$
\frac{d}{dt} \|f\|_{\dot{H}^{3/2}}^2 + c_0 \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \, dx \leq 0,
$$

(5)

which gives the wanted control of the $L^\infty_t(\dot{H}^{3/2})$-norm.

Eventually, we discuss the main result in our previous paper [3], where we proved that the Cauchy problem is well-posed in the space $W^{1,\infty}(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$ for arbitrary large data (together with a global in time regularity result under a smallness assumption in $\dot{H}^{3/2}(\mathbb{R})$). The key difference between the former large data result and other known results for other critical parabolic problems is that we know that, for the Muskat equation, there exist some solutions with large data which do not exist globally in time (as proved by Castro, Córdoba, Fefferman, Gancedo and López-Fernández ([17–19]). Consequently, one cannot solve the Cauchy problem for a time $T$ which depends only on the norm of $f_0$ in $W^{1,\infty}(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$. Otherwise, this would yield a global existence result for any initial data by a scaling argument. Recently, Chen, Xu and the second author [22] proved local well-posedness for the muskat problem with $C^1$ data.

As already said, the study of critical problems is a very vast subject of which we have barely scratched the surface in this paragraph. There are many related results. Let us mention papers by Vázquez [58], for the fractional porous media equation, as well as Granero-Belinchón and Scrobogna [39] and Scrobogna [50] for quadratic approximation of the Hele-Shaw equation (with or without surface tension). We also refer the reader to Abedin and Schwab [1], who obtained recently the existence of a modulus of continuity for the Muskat equation by means of Krylov-Safonov estimates. Let us also mention that the existence and (possible) non-uniqueness of some weak-solutions in the unstable regime has also been thoroughly studied (see [9,16,32,37,49,53]).

1.3. Strategy of the proof. A key difficulty is that the coercive quantity which appears in the left-hand side of the Córdoba-Lazar’s critical estimate (4), that is

$$
\int_0^T \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \, dx \, dt,
$$

(6)

is insufficient to control the $\|\cdot\|_{L^\infty_t \dot{H}^{3/2}}$-norm of $f$. In sharp contrast with the previous works in [3,15,35,38], in this paper we make no extra assumption on the initial data which would allow us to control from above the denominator $1 + (\partial_x f)^2$.

By comparison with the strategy recalled in the previous paragraph, this leads to a serious difficulty since the parabolic feature degenerates in this case. Thus a major ingredient in the proof of Theorem 1.1, and another object of the paper, is to prove that it is sufficient to control the quantity (6). To do so, we will uncover some key cancellations showing that one can divide the most singular terms by the possibly large factor $1 + (\partial_x f)^2$. This is the null-type structure for the Muskat equation which will be systematically studied in the following.

In the rest of this introduction, we will explain how one can overcome the two main difficulties mentioned above; namely, the fact that the time of existence must depend on the initial data and the degeneracy in the parabolic behavior.
A weak null-type property Our first idea will consist in proving a (weak) null-type property. By this we mean the proof of an inequality of the form (4) but with the key difference that the right-hand side does not involve the $\|\cdot\|_{H^2}$-norm of $f$, but the weaker coercive quantity given by (6).

More precisely, we will prove that

$$\frac{d}{dt} \| f \|_{H^2}^2 + \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \, dx \lesssim \left( 1 + \| f \|_{H^2}^7 \right) \| f \|_{H^2} \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \, dx.$$  \hspace{1cm} (7)

This estimate will imply at once statement (ii) in Theorem 1.1, about the global existence result under a smallness assumption. Indeed, this part is perturbative in character, and one can absorb the right-hand side of (7) by the left-hand side, provided that $\| f \|_{H^2}$ is small enough.

However, the estimate (7) is insufficient to prove the main result, which is statement (i) in Theorem 1.1. Indeed, as we have explained before, for the latter purpose, one cannot rely entirely on an estimate of the critical $\|\cdot\|_{H^2}$-norm. This will force us to prove a strong null-type property, whose proof is considerably more involved.

Weighted fractional Laplacians Before going any further, we need to define the norm that we are going to estimate. To do so, we pause to introduce a weighted version of the classical fractional Laplacian.

Given a real number $s$ and a function $\phi : [0, +\infty) \to [1, +\infty)$ satisfying a doubling condition (this means that there is $c_0 > 0$ such that $\phi(2r) \leq c_0 \phi(r)$ for any $r \geq 0$), we introduce the weighted fractional Laplacian $|D|^{s, \phi}$ as the Fourier multiplier with symbol $|\xi|^{s} \phi(|\xi|)$. More precisely,

$$\mathcal{F}(|D|^{s, \phi} f)(\xi) = |\xi|^{s} \phi(|\xi|) \mathcal{F}(f)(\xi).$$

Moreover, we define the space

$$\mathcal{H}^{s, \phi}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : |D|^{s, \phi} f \in L^2(\mathbb{R}) \}. \hspace{1cm} (8)$$

A weighted fractional Laplacian $|D|^{s, \phi}$, as the name indicates, attempts to be a slight modulation of the usual fractional Laplacian $|D|^s = (-\partial_{xx})^{s/2}$. We will consider special weight functions $\phi$ depending on some extra functions $\kappa : [0, \infty) \to [1, \infty)$, of the form

$$\phi(\lambda) = \int_0^{+\infty} \frac{1 - \cos(h)}{h^2} \kappa \left( \frac{\lambda}{h} \right) \, dh, \quad \text{for } \lambda \geq 0. \hspace{1cm} (9)$$

Let us clarify the assumptions on $\kappa$ which will be needed. We say that a function $\kappa : [0, +\infty) \to [1, +\infty)$ is an admissible weight if it satisfies the following conditions:

(H1) $\kappa$ is increasing;

(H2) there is a positive constant $c_0$ such that $\kappa(2r) \leq c_0 \kappa(r)$ for any $r \geq 0$;

(H3) the function $r \mapsto \kappa(r)/\log(4 + r)$ is decreasing on $[0, \infty)$.

In our first paper [4], we studied the Muskat equation in the spaces $\mathcal{H}^{s, \log(+\|\xi\|^a)}(\mathbb{R})$ for some $a \geq \frac{1}{2}$, which permitted to solve the Cauchy problem for non-Lipschitz initial data, following earlier works by Deng et al. [36], Cameron [15], Córdoba and Lazar [35], Gancedo and Lazar [38] which allowed arbitrary large slopes. The main difficulty in [4] is that the assumption of finite slopes is difficult to dispense of. Indeed, classical nonlinear estimates require to control the $L^\infty$-norm of some factors, which amounts for
the Muskat problem to control the $L^\infty$-norm of the slopes $\Delta_\alpha f$, equivalent to control the Lipschitz norm of $f$. Loosely speaking, the assumption $a \geq \frac{1}{3}$ might be understood as the minimal assumption in terms of weighted fractional Laplacian to overcome the degeneracy in the parabolic feature by an interpolation argument. In this paper, we will perform a much more difficult analysis which will allow us to remove completely the assumption $a \geq \frac{1}{3}$.

**Weighted Sobolev estimates** As already explained, for large data, the time of existence must depend on the initial data themselves and not only on its critical Sobolev norm. To overcome this problem, our strategy is to estimate the solution for a stronger norm whose definition involves the initial data themselves. To do so, we use the following elementary result (which follows from Lemma 3 in [4]).

**Lemma 1.2.** For all $f_0$ in $H^3_2(\mathbb{R})$, there exists an admissible weight $\kappa$ satisfying $\lim_{r \to +\infty} \kappa(r) = \infty$ and such that $f_0$ belongs to $\mathcal{H}^3_{2,\phi}(\mathbb{R})$ where $\phi$ is given by (9).

Then, we are in position to state an improvement of statement i) in Theorem 1.1 which asserts that, whenever one controls a bigger norm than the critical one, the time of existence depends only on the norm. Recall that $\mathcal{H}^3_{2,\phi}(\mathbb{R})$ is defined by (8).

**Theorem 1.3.** Consider a real number $M_0 > 0$ and a function $\phi$ given by (9) for some admissible weight satisfying assumptions (H1)–(H3) with $\lim_{r \to +\infty} \kappa(r) = \infty$. Then there exists a time $T_0 > 0$ depending on $M_0$, $\kappa$ such that, for any initial data $f_0$ in $\mathcal{H}^3_{2,\phi}(\mathbb{R})$ satisfying $\|f_0\|_{\mathcal{H}^3_{2,\phi}} \leq M_0$, the Cauchy problem for (1) has a unique solution in the space

$$\mathcal{X}^3_{2,\phi}(T_0) := \left\{ f \in C^0([0, T_0]; \mathcal{H}^3_{2,\phi}(\mathbb{R})); \int_0^{T_0} \int_{\mathbb{R}} \frac{|D|^{2,\phi} f|^2}{1 + (\partial_x f)^2} \, dx \, dt < \infty \right\}.$$ 

Notice that statement i) in Theorem 1.1 follows at once from Lemma 1.2 and the above result.

A strong null-type property To prove Theorem 1.3, we shall prove an estimate in weight Sobolev space $\mathcal{H}^3_{2,\phi}(\mathbb{R})$ which extends (7) in several direction. Namely, given an initial data $f_0$ and an admissible weight $\phi$ such that $f_0$ belongs to $\mathcal{H}^3_{2,\phi}(\mathbb{R})$, we shall prove that the following norms

$$A(t) := \|f(t)\|^2_{L^2} + \|D|^{3,\phi} f(t)\|^2_{L^2}, \quad B(t) := \int_{\mathbb{R}} \frac{|D|^{2,\phi} f(t,x)|^2}{1 + (\partial_x f(t,x))^2} \, dx$$

satisfy

$$\frac{d}{dt} A + B \leq C_0 \left( 1 + A^4 \right) \left( \log(4 + B)^{\frac{1}{2}} B^{\frac{3}{2}} + \frac{B}{\kappa(B^{\frac{1}{2}})} + \frac{B}{\kappa(B^{\frac{1}{2}})} \right) + \mathcal{G}(A), \quad (10)$$

for some positive constant $C_0$ and some non-decreasing function $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}_+$.

The proof of (10) relies on several new ideas. Firstly we shall systematically decompose the integrals in $\alpha$ into two parts: short scales $|\alpha| \leq \lambda$ and large scales $|\alpha| \geq \lambda$. This is of course a classical idea, but here we are using it in a novel way tailored to our subject.
Indeed, we will allow the frequency threshold to be a time dependent function (in the end, we select $\lambda = \lambda(t) = B(t)^{\frac{1}{2}}$). Also, the reason to introduce this decomposition is that, for any admissible weight $\kappa$ and any $\beta$ in $(0, 1]$, there is a positive constant $C_\beta$ such that, for all $\lambda > 0$ and all $\xi \in \mathbb{R}$,

$$\min\{\lambda|\xi|, 1\}^\beta \leq C_\beta \frac{\kappa(|\xi|)}{\kappa(1/\lambda)}.$$

Secondly, we shall uncover in the nonlinearity a special structure which allows to divide the most singular terms by the possibly large factor $1 + (\partial_x f)^2$. To give this vague sentence a rigorous meaning, let us introduce the operator $T(f)$ defined by

$$T(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha.$$

This operator describes the nonlinearity in the Muskat equation since the latter can be written under the form (see Sect. 4)

$$\partial_t f + |D| f = T(f) f.$$

We shall prove the following key estimate (see Proposition 5.8),

$$\left\| \left\{ |D|^{1+\phi} \cdot T(f) \right\}(g), h \right\| \lesssim \left( 1 + \left\| |D|^{3/2} \phi f \right\|_{L^2} \right)^2 \left( \left\| |D|^{3/2} g \right\|_{L^2} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} + \left\| |D|^{3/2} \phi f \right\|_{L^2} \left\| \frac{g_{xx}}{\langle f_x \rangle} \right\|_{L^2} \right) \left\| \frac{h}{\langle f_x \rangle} \right\|_{L^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2$-scalar product. This will allow us to commute derivatives with the Muskat equation. In addition, we shall prove two similar results required to analyze the convective term and a remainder term (see Proposition 5.1 and Proposition 5.6).

By so doing, we will get that, for any time dependent function $\lambda = \lambda(t)$,

$$\frac{d}{dt} \left\| |D|^{3/2} \phi f \right\|_{L^2}^2 + \left\| \frac{|D|^{2/2} \phi f}{\langle f_x \rangle} \right\|_{L^2}^2 \lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} \left( 1 + \left\| |D|^{3/2} \phi f \right\|_{L^2}^7 \right) \left\| |D|^{3/2} \phi f \right\|_{L^2} \left\| \frac{|D|^{2/2} \phi f}{\langle f_x \rangle} \right\|_{L^2}^2 \left( 1 + \left\| |D|^{3/2} \phi f \right\|_{L^2}^5 \right) \left\| |D|^{3/2} \phi f \right\|_{L^2} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{|D|^{2} \phi f}{\langle f_x \rangle} \right\|_{L^2} \left( 1 + \left\| |D|^{3/2} \phi f \right\|_{L^2}^2 \right) \left\| |D|^{3/2} \phi f \right\|_{L^2} \left\| \frac{|D|^{2/2} \phi f}{\langle f_x \rangle} \right\|_{L^2},$$

where $f_x = \partial_x f$, $f_{xx} = \partial_{xx} f$ and $\langle f_x \rangle = \sqrt{1 + f_{xx}^2}$. Notice that (11) implies (7) by choosing $\kappa = 1$ (then $\phi$ is a constant function) and letting $\lambda$ goes to $+\infty$.

Once (11) will be established, the derivation of our main estimate (10) will require to prove commutator estimates and interpolation inequalities which are also compatible with our singular weights.

One can mention in this introduction the following estimate (see Lemma 2.2) about the Hilbert transform,

$$\int \frac{(Hg)^2}{1 + h^2} \, dx \lesssim \left( 1 + \|h\|_{H^2}^2 \right) \int \frac{g^2}{1 + h^2} \, dx.$$

(12)
In the same vein, we shall prove that (see Proposition 3.6)
\[
\left\| D^{\sigma, \phi} \left( \frac{g}{\sqrt{1+h^2}} \right) - \frac{1}{\sqrt{1+h^2}} |D|^{\sigma, \phi} g \right\|_{L^2} \lesssim \left\| D^{\frac{1}{2}, \phi} h \right\|_{L^2} \left\| \frac{g}{\sqrt{1+h^2}} \right\|_{\dot{H}^\sigma}.
\] (13)

**Uniqueness and propagation of regularity** Once a priori estimates will be granted, the end of the proof will rely on two additional ingredients. Firstly, a contraction estimate for the difference of two solutions. Namely, we shall prove (see Proposition 6.6) that if \(f_1\) and \(f_2\) solve the Muskat equation, then
\[
\frac{d}{dt} \left\| (f_1 - f_2)(t) \right\|_{\dot{H}^{\frac{1}{2}}} \leq G(M) \log \left( 2 + \sum_{k=1}^{2} \left\| \frac{\partial_{xx} f_k}{\langle \partial_x f_k \rangle} (t) \right\|_{L^2} \right) \left\| (f_1 - f_2)(t) \right\|_{\dot{H}^{\frac{1}{2}}},
\]
where
\[
M := \sup_{t \in [0,T]} \left\{ \left\| f_1(t) \right\|_{\mathcal{H}^{\frac{3}{2}, \phi}} + \left\| f_2(t) \right\|_{\mathcal{H}^{\frac{3}{2}, \phi}} \right\}.
\]
Since \(M\) involves a stronger norm than the one of \(C^0([0,T]; \mathcal{H}^{\frac{3}{2}}(\mathbb{R}))\), to close the argument and obtain an unconditional uniqueness result in the space \(X^{\frac{3}{2}}(\mathbb{R})\) introduced in the statement of Theorem 1.1, we shall prove an enhanced regularity result. Namely, we shall prove (see Proposition 6.7) that for any solution in the space \(X^{\frac{3}{2}}(T)\), if the initial data belongs to \(\mathcal{H}^{\frac{3}{2}, \phi}(\mathbb{R})\) for some \(\phi\), then the solution \(f\) belongs to \(C^0([0,T]; \mathcal{H}^{\frac{3}{2}, \phi}(\mathbb{R}))\), after a possible shrinking of the time interval.

**1.4. Plan of the paper.** The remainder of the paper is organized as follows. Sect. 2 reviews many inequalities which will be used throughout the article: Hardy, Hardy-Littlewood-Sobolev, Minkowski, embedding of Sobolev spaces in Triebel-Lizorkin spaces. We also prove in Sect. 2.4 the estimate (12). In Sect. 3 we recall various results about weighted fractional Laplacians that we borrow from our previous paper [4], along with the key commutator estimate (13). In Sect. 4, we recall from [2] a paradifferential analysis of the Muskat equation, and apply it to obtain an energy estimate. The main task is then to estimate the various commutators and remainder terms, to obtain (11). This is done in Sect. 5, which is the core of this paper. Once (11) is proved, we conclude the proof in Sect. 6 by using interpolation inequalities and a fair amount of bookkeeping.

**1.5. Notations.** Triebel-Lizorkin spaces are recalled in Sect. 2.5. We refer to Sect. 2.4 for the definitions of the Hilbert transform and Riesz potentials. More definitions and notations about the weighted fractional Laplacians are given in Sect. 3.

In addition:

1. We denote partial derivatives using subscripts: \(f_x = \partial_x f\) and \(f_{xx} = \partial_{xx} f\).
2. If \(A, B\) are nonnegative quantities, we use \(A \lesssim B\) to denote the estimate \(A \leq CB\) for some constant \(C\) depending only on fixed quantities, and \(A \sim B\) to denote the estimate \(A \lesssim B \lesssim A\).
Given two operators $A$ and $B$, we denote by $[A, B]$ the commutator $A \circ B - B \circ A$.

Given a normed space $X$ and a function $\varphi = \varphi(t, x)$ defined on $[0, T] \times \mathbb{R}$ with values in $X$, we denote by $\varphi(t)$ the function $x \mapsto \varphi(t, x)$. In the same vein, we use $\|\varphi\|_X$ as a compact notation for the time dependent function $t \mapsto \|\varphi(t)\|_X$.

All functions are assumed to be real-valued. However, in many computations, we will use the complex-modulus notation: for instance, we will write $|\Delta_1 \alpha|^2$ or $|\alpha|^2$ instead of $(\Delta_1 \alpha)^2$ or $\alpha^2$, since we think that this might help to read certain formulas.

2. Preliminaries

We begin by discussing some classical inequalities which are basic to the analysis developed below. We also prove in Sect. 2.4 a new estimate about the Hilbert transform.

2.1. Minkowski’s inequality. Suppose that $(S_1, \mu_1)$ and $(S_2, \mu_2)$ are two $\sigma$-finite measure spaces and $F : S_1 \times S_2 \to \mathbb{R}$ is measurable. Then

$$\left( \int_{S_2} \left( \int_{S_1} F(x, y) \mu_1(\text{d}x) \right)^p \mu_2(\text{d}y) \right)^{\frac{1}{p}} \leq \left( \int_{S_1} \left( \int_{S_2} |F(x, y)|^p \mu_2(\text{d}y) \right) \mu_1(\text{d}x) \right)^{\frac{1}{p}}. \quad (14)$$

2.2. Hardy’s inequality. Consider two real numbers $r \geq 1$ and $s < r - 1$. For any non-negative function $f$, there holds (see [60, page 20])

$$\int_0^\infty \left( \frac{1}{x} \int_0^x f(\tau) \, d\tau \right)^r x^s \, dx \leq \left( \frac{r}{r - s - 1} \right)^r \int_0^\infty f(x)^r x^s \, dx. \quad (15)$$

In particular, one has,

$$\int_{\mathbb{R}} \left( \int_0^x f(\tau) \, d\tau \right)^r |x|^s \, dx \leq \left( \frac{r}{r - s - 1} \right)^r \int_{\mathbb{R}} f(x)^r |x|^s \, dx. \quad (16)$$

2.3. Riesz potentials. The Riesz potentials are the operators

$$I_s f(x) = \int \frac{f(y)}{|x - y|^{1-s}} \, dy, \quad 0 < s < 1.$$ 

The Hardy-Littlewood-Sobolev lemma states that these operators map $L^p \to L^q$, for

$$1 < p < \frac{1}{s} \quad \text{such that} \quad s + \frac{1}{q} = \frac{1}{p}. \quad (17)$$

The previous result will be used throughout the paper. In particular, we will make extensive use of the fact that, for $0 < s < 1$, the Fourier multiplier $|D|^{-s}$ is a multiple of $I_s$:

$$|D|^{-s} = c_s I_s, \quad (18)$$

for some constant $c_s$. 
2.4. Hilbert transform. The Hilbert transform $\mathcal{H}$ is defined by:

$$\mathcal{H}u(x) = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{u(y)}{x-y} \, dy.$$  \hspace{1cm} (19)

This operator is ubiquitous in the study of the Muskat equation.

For later purposes, we record in this paragraph two commutator estimates for the Hilbert transform. Firstly, we will need to estimate the commutator of $\mathcal{H}$ with a function in $\dot{H}^1_{\text{loc}}(\mathbb{R})$. Since $\dot{H}^1_{\text{loc}}(\mathbb{R})$ is embedded into $C^1_2(\mathbb{R})$ and since the Hilbert transform is an operator of order 0 (bounded from $H^\mu(\mathbb{R})$ to itself for any $\mu \in \mathbb{R}$), we expect the commutator to be an operator of order $-1/2$. The next result (which is Lemma 3.4 in [4]) states that this is indeed the case when the commutator is acting on an element of $\dot{H}^{-1/2}_{\text{loc}}(\mathbb{R})$.

**Lemma 2.1.** [from [4]] For all $g_1$ in $\dot{H}^1(\mathbb{R})$ and all $g_2$ in $\dot{H}^{1/2}_{\text{loc}}(\mathbb{R})$, there holds

$$\|\mathcal{H}, g_1\|_{L^2} \lesssim \|g_1\|_{\dot{H}^1} \|g_2\|_{\dot{H}^{1/2}}.$$  \hspace{1cm} (21)

We will also need a lemma which generalizes the boundedness of the Hilbert transform on $L^2(\mathbb{R})$.

**Lemma 2.2.** Consider a function $\gamma : \mathbb{R} \to [0, 1]$ such that, for any $z_1, z_2$ in $\mathbb{R}$,

$$|\gamma(z_1) - \gamma(z_2)| \leq C \gamma(z_2) |z_1 - z_2|, \hspace{1cm} (20)$$

for some absolute constant $C$. Then, for all function $g$ in $L^2(\mathbb{R})$ and all function $h$ in $\dot{H}^{1/2}_{\text{loc}}(\mathbb{R})$, there holds

$$\int |\gamma(h)\mathcal{H}g|^2 \, dx \lesssim C \left( 1 + \|h\|_{\dot{H}^{1/2}_{\text{loc}}}^2 \right) \int |\gamma(h)g|^2 \, dx. \hspace{1cm} (21)$$

**Remark 2.3.** i) We will apply this result with $\gamma(z) = \langle z \rangle^{-1} = (1 + z^2)^{-1/2}$.  
ii) A classical result states that

$$\|\mathcal{H}|u\|_{L^p} \leq C \|b\|_{\text{BMO}} \|u\|_{L^p}, \hspace{1cm} 1 < p < \infty.$$  

This implies that the left-hand side of (21) is bounded by

$$\|\gamma(h)g\|_{L^2} + \|\gamma(h)\|_{\text{BMO}} \|g\|_{L^2}.$$  

Compared to the latter inequality, the key point is that the right-hand side of (21) does not involve the $L^2$-norm of $g$, but the one of $\gamma(h)g$. This is crucial here since we do not assume that $\gamma(h)$ is not bounded from below.

**Proof.** Set $u = \gamma(h)g$ and write

$$\int |\gamma(h)\mathcal{H}g|^2 \, dx \sim \int \left( \int \gamma(h)(x) \frac{g(y)}{x-y} \, dy \right)^2 \, dx$$

$$\lesssim \int |\mathcal{H}u|^2 \, dx + \int \left( \int |u(y)| |h(x) - h(y)| \frac{dy}{|x-y|} \right)^2 \, dx,$$

where we have used (20) in the last inequality.
The first term in the right-hand side above is bounded by \( \|u\|_{L^2}^2 \) while the Hölder’s inequality implies that the second term is estimated by

\[
\left( \int \left( \int \frac{|u(y)|^{\frac{3}{2}}}{|x-y|^{\frac{1}{2}}} \, dy \right)^4 \, dx \right)^{\frac{1}{3}} \times \left( \int \left( \frac{|h(x) - h(y)|}{|x-y|^{\frac{1}{2}}} \right)^3 \, dy \, dx \right)^{\frac{2}{3}}.
\]

Using the boundedness of the Riesz potentials on the Lebesgue spaces (see (17)), we get that the first factor is estimated by

\[
\left( \int \left( \int \frac{|u(y)|^{\frac{3}{2}}}{|x-y|^{\frac{1}{2}}} \, dy \right)^4 \, dx \right)^{\frac{1}{3}} \lesssim \|u\|_{L^2}^2.
\]

On the other hand, the Sobolev embedding \( \dot{H}^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow \dot{W}^{1,3}(\mathbb{R}) \) (see (25) and (26) below) implies that

\[
\left( \int \left( \int \frac{|h(x) - h(y)|}{|x-y|^{\frac{1}{2}}} \right)^3 \, dy \, dx \right)^{\frac{2}{3}} \lesssim \|h\|_{\dot{H}^{\frac{1}{2}}}^2.
\]

This proves (21). \( \square \)

2.5. Triebel-Lizorkin norms. For easy reading, we review some elementary results about Triebel-Lizorkin spaces following Triebel (see [55, Sect. 2.3.5] and [56, Section 3]).

Given a function \( f : \mathbb{R} \to \mathbb{R} \), an integer \( m \in \mathbb{N} \setminus \{0\} \) and a real number \( h \in \mathbb{R} \), we define the finite differences \( \delta_h^m f \) as follows:

\[
\delta_h f(x) = f(x) - f(x-h), \quad \delta_h^{m+1} f = \delta_h (\delta_h^m f).
\]

**Definition 2.4.** Given an integer \( m \in \mathbb{N} \setminus \{0\} \), a real number \( s \in [m - 1, m) \) and \( (p, q) \in [1, \infty)^2 \), the homogeneous Triebel-Lizorkin space \( \dot{F}^s_{p,q}(\mathbb{R}) \) consists of those tempered distributions \( f \) whose Fourier transform is integrable on a neighborhood of the origin and such that

\[
\| f \|_{\dot{F}^s_{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\delta_h^m f(x)|^q \frac{dh}{|h|^{1+qs}} \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}} < +\infty.
\]

We will extensively use the embedding of homogeneous Sobolev spaces in Triebel-Lizorkin spaces. Firstly, recall that \( \| \cdot \|_{\dot{H}^s} \) and \( \| \cdot \|_{\dot{F}^s_{2,2}} \) are equivalent. Moreover, for \( s \in (0, 1) \),

\[
\|u\|_{\dot{H}^s}^2 = \frac{1}{4\pi c(s)} \|u\|_{\dot{F}^s_{2,2}}^2 \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1 - \cos(h)}{|h|^{1+2s}} \, dh.
\]

In addition, for any \( 2 < p < \infty \) and any \( q \leq \infty \), if \( t - \frac{1}{2} = s - \frac{1}{p} \) then

\[
\| f \|_{\dot{F}^s_{p,q}(\mathbb{R})} \lesssim \| f \|_{\dot{H}^t(\mathbb{R})}.
\]

\( \square \)
This includes the Sobolev embedding
\[ \dot{H}^t(\mathbb{R}) \hookrightarrow L^{1-2t}(\mathbb{R}) \quad \text{for} \quad 0 \leq t < \frac{1}{2}. \]  
(25)

Recall also that, for \( s \in (0, 1) \) and \( 1 \leq p < +\infty \), fractional Sobolev spaces \( \dot{W}^{s,p}(\mathbb{R}) \) are associated with the Gagliardo semi-norms
\[ \| u \|_{\dot{W}^{s,p}}^p := \iint_{\mathbb{R} \times \mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^{sp}} \, dx \, dy. \]  
(26)

2.6. Several inequalities. We gather in this paragraph several inequalities which will be used repeatedly. We also want to mention them in this preliminary section because their proofs are good examples of the kind of techniques that will be used later on.

We begin with the three following elementary estimates:
\[ \iint_{\mathbb{R}^2} |\Delta_\alpha f|^2 \, d\alpha \, dx \sim \| f \|_{\dot{H}^{\frac{1}{2}}}^2, \]  
(27)

\[ \iint_{\mathbb{R}^2} |\Delta_\alpha f - f_x|^2 \frac{d\alpha}{|\alpha|^2} \, dx \sim \| f \|_{\dot{H}^{\frac{3}{2}}}^2, \]  
(28)

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\Delta_\alpha f|^2 \, d\alpha \right)^2 \, dx \lesssim \| f \|_{\dot{H}^{\frac{3}{2}}}^4. \]  
(29)

The estimates (27) and (29) follow immediately from the Sobolev embeddings in Triebel-Lizorkin spaces (see (23) and (24)), while (28) is proved by using the Fourier transform.

Next, we prove more technical estimates for the following quantities:
\[ \Delta_\alpha f - f_x \quad \text{or} \quad \Delta_\alpha f - \Delta_{-\alpha} f. \]

We are going to prove estimates valid for frequencies large enough (or equivalently, estimates valid when we integrate only on sets where the physical variable \( |\alpha| \) is smaller than some threshold \( \lambda \)).

Definition 2.5. Given a real number \( \alpha \), we define the operators \( E_\alpha \) and \( S_\alpha \) by
\[ E_\alpha(f)(x) = \Delta_\alpha f(x) - f_x(x) = \frac{f(x) - f(x - \alpha)}{\alpha} - f_x(x), \]
\[ S_\alpha(f)(x) = \Delta_\alpha f(x) - \Delta_{-\alpha} f(x) = \frac{2f(x) - f(x - \alpha) - f(x + \alpha)}{\alpha}. \]

Lemma 2.6. i) Consider five real numbers \((\gamma, a_1, a_2, a_3, \beta)\) satisfying
\[ a_1 \in \mathbb{R}, \quad \gamma \geq 2, \quad 1 < a_2 < 1 + \gamma, \quad a_3 = a_1 + \frac{a_2 - 1}{\gamma}, \quad \beta = \frac{2 + 2\gamma - 2a_2}{\gamma}. \]

Then, for any \( \lambda > 0 \) and any \( f \in \mathcal{S}(\mathbb{R}) \), the following inequalities hold:
\[ \int_{|\alpha| \leq \lambda} \| E_\alpha(f) \|_{\dot{H}^{a_1}}^\gamma \frac{d\alpha}{|\alpha|^{a_2}} \lesssim \left( \int \min\{\lambda, |\xi|, 1\}^\beta |\xi|^{2a_3} |\hat{f}(\xi)|^2 \, d\xi \right)^\frac{\gamma}{2}, \]  
(30)

\[ \int_{|\alpha| \leq \lambda} \| S_\alpha(f) \|_{\dot{H}^{a_1}}^\gamma \frac{d\alpha}{|\alpha|^{a_2}} \lesssim \left( \int \min\{\lambda, |\xi|, 1\}^\beta |\xi|^{2a_3} |\hat{f}(\xi)|^2 \, d\xi \right)^\frac{\gamma}{2}, \]  
(31)
and

\[ \int_{|\alpha| \leq \lambda} \| \delta_\alpha (f) \|_{H^{a_1}}^\gamma \frac{d\alpha}{|\alpha|^{a_2}} \lesssim \left( \int \min\{\lambda, |\xi|, 1\}^{\beta} |\xi|^{2a_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{\gamma}{2}}. \tag{32} \]

**Proof.** This proof is based on the Fourier transform. The key point is that

\[ \| E_\alpha (f) \|_{H^{a_1}}^2 = \int |\xi|^{2a_1} \left| 1 - e^{-i \alpha \xi} - i \alpha \frac{2}{\alpha^2} |\hat{f}(\xi)|^2 \right| d\xi \lesssim \int |\xi|^{2+2a_1} \min\{1, |\alpha| |\xi|\} |\hat{f}(\xi)|^2 d\xi, \tag{33} \]

\[ \| S_\alpha (f) \|_{H^{a_1}}^2 \lesssim \int |\xi|^{2+2a_1} \min\{1, |\alpha| |\xi|\} |\hat{f}(\xi)|^2 d\xi, \]

\[ \| \delta_\alpha (f) \|_{H^{a_1}}^2 \lesssim \int |\xi|^{2a_1} \min\{1, |\alpha| |\xi|\} |\hat{f}(\xi)|^2 d\xi. \]

i) Let us prove (30) and (31). Directly from (33) we have

\[ Q^\gamma := \int_{|\alpha| \leq \lambda} \| E_\alpha (f) \|_{H^{a_1}}^\gamma \frac{d\alpha}{|\alpha|^{a_2}} + \int_{|\alpha| \leq \lambda} \| S_\alpha (f) \|_{H^{a_1}}^\gamma \frac{d\alpha}{|\alpha|^{a_2}} \lesssim \int \left( \int_{|\xi| \leq \lambda} |\xi|^{2+2a_1} \min\{1, |\alpha| |\xi|\} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{\gamma}{2}} d\alpha. \]

We raise both sides of the previous inequality to the power $\gamma^{-1}$, to obtain

\[ Q \leq \left( \int \left( \int_{|\alpha| \leq \lambda} \min\{1, |\alpha| |\xi|\} |\xi|^{2+2a_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{\gamma}{2}} d\alpha \right)^{2 - \frac{\gamma}{2}}. \]

Now we are in position to apply Minkowski’s inequality. Since $\gamma / 2 \geq 1$, we obtain that

\[ Q \lesssim \left( \int_{|\alpha| \leq \lambda} \left( \int_{|\alpha| \leq \lambda} \min\{1, |\alpha| |\xi|\}^{\gamma} \frac{d\alpha}{|\alpha|^{a_2}} \right)^{\frac{\gamma}{2}} |\xi|^{2+2a_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{\gamma}{2}}. \tag{34} \]

Now, by an elementary change of variable, we have

\[ \int_{|\alpha| \leq \lambda} \min\{1, |\alpha| |\xi|\}^{\gamma} \frac{d\alpha}{|\alpha|^{a_2}} = |\xi|^{a_2-1} \int_{|\alpha| \leq \lambda |\xi|} \min\{1, |\alpha|\}^{\gamma} \frac{d\alpha}{|\alpha|^{a_2}} \lesssim |\xi|^{a_2-1} \min\{1, \lambda |\xi|\}^{1+\gamma-a_2}. \]

It follows that

\[ \left( \int_{|\alpha| \leq \lambda} \min\{1, |\alpha| |\xi|\}^{\gamma} \frac{d\alpha}{|\alpha|^{a_2}} \right)^{\frac{\gamma}{2}} \lesssim |\xi|^{\frac{2(a_2-1)}{\gamma}} \times \min\{1, \lambda |\xi|\}^{\frac{2+2\gamma-2a_2}{\gamma}}. \tag{35} \]

Then the wanted estimates (30) and (31) follow from the previous bound and (34).
ii) It remains to prove (32). Starting from (33) and repeating the arguments used to infer (34), we get
\[
\int_{|\alpha| \leq \lambda} \| \delta_{\alpha} (f) \|_{H^{a_1}} \, d\alpha \lesssim \left( \int_{\mathbb{R}} \left( \int_{|\alpha| \leq \lambda} \min \{ 1, |\alpha| |\xi| \}^\gamma \frac{d\alpha}{|\alpha|^{\beta_2}} \right)^{\frac{\gamma}{\gamma-2}} |\xi|^{2a_1} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{\gamma}{2}},
\]
which implies (32).

\[\Box\]

3. Weighted Fractional Laplacians

This section is devoted to the study of weighted fractional Laplacians, which is our pivotal tool to study the Muskat equation in critical spaces. The first paragraph is a review consisting of various notations and results about these operators, following our previous paper [4]. After these preliminaries, in § 3.2 we pass to the proof of a key commutator estimate, which generalizes in some sense Lemma 2.2 about the Hilbert transform.

3.1. Definition and characterization. Following our previous works, we will work with the following weighted fractional Laplacians.

**Definition 3.1.** Consider $s \in [0, +\infty)$ and a function $\phi : [0, \infty) \to [1, \infty)$ satisfying a doubling condition (such that $\phi(2r) \leq c_0 \phi(r)$ for any $r \geq 0$ and some constant $c_0 > 0$). Then $|D|^{s, \phi}$ denotes the Fourier multiplier with symbol $|\xi|^s \phi(|\xi|)$. More precisely,
\[
\mathcal{F}(|D|^{s, \phi} f)(\xi) = |\xi|^s \phi(|\xi|) \mathcal{F}(f)(\xi).
\]
Moreover, we define the space
\[
\mathcal{H}^{s, \phi}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : |D|^{s, \phi} f \in L^2(\mathbb{R}) \},
\]
equipped with the norm
\[
\| f \|_{\mathcal{H}^{s, \phi}} := \| f \|_{L^2} + \left( \int_{\mathbb{R}} |\xi|^{2s} (\phi(|\xi|))^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

We shall consider some special functions $\phi$ which depend on some extra function $\kappa : [0, \infty) \to [1, \infty)$, of the form
\[
\phi(\lambda) = \int_{0}^{\infty} \frac{1 - \cos(h)}{h^2} \kappa \left( \frac{\lambda}{h} \right) \, dh, \quad \text{for } \lambda \geq 0.
\]
Let us clarify the assumptions on $\kappa$ which will be needed in the sequel.

**Definition 3.2.** We say that a function $\kappa : [0, \infty) \to [1, \infty)$ is an admissible weight if it satisfies the following three conditions:

(H1) $\kappa$ is increasing;
(H2) there is a positive constant $c_0$ such that $\kappa(2r) \leq c_0 \kappa(r)$ for any $r \geq 0$;
(H3) the function \( r \mapsto \kappa(r)/\log(4+r) \) is decreasing on \([r_0, \infty)\), for some \( r_0 > 0 \) large enough.

In [4, Prop. 2.7], it is proved that \( \phi \) and \( \kappa \) are equivalent. More precisely, we have the following

**Proposition 3.3** (from [4]) Assume that \( \phi \) is as defined in (36) for some admissible weight \( \kappa \). Then there exist two constants \( c, C > 0 \) such that, for all \( \lambda \geq 0 \),

\[
    ck(\lambda) \leq \phi(\lambda) \leq Ck(\lambda). \tag{37}
\]

In particular, \( \phi \) is also an admissible weight.

For later applications, we make some preliminary remarks about admissible weights.

**Lemma 3.4.** i) For any \( \sigma > 0 \), there exists a positive constant \( C \) such that, for any real numbers \( 0 < r \leq \mu \),

\[
    r^\sigma \kappa \left( \frac{1}{r} \right) \leq C \sigma^\sigma \mu^\sigma \kappa \left( \frac{1}{\mu} \right). \tag{38}
\]

ii) For any \( \beta \) in \((0, 1]\), there is a positive constant \( C_\beta \) such that, for all \( \lambda > 0 \) and all \( \xi \in \mathbb{R} \),

\[
    \min \left\{ \lambda \left| \xi \right|, 1 \right\}^\beta \leq C_\beta \kappa \left( \frac{1}{\left| \xi \right|} \right) \kappa \left( \frac{1}{\lambda} \right). \tag{39}
\]

**Proof.** To prove statement i) we use a decomposition into three factors:

\[
    r^\sigma \kappa \left( \frac{1}{r} \right) = \frac{\kappa \left( \frac{1}{r} \right)}{\log(4+1/r)} \times \left[ r^\sigma \log \left( \frac{1}{\sigma} + \frac{1}{r} \right) \right] \times \frac{\log(4+1/r)}{\log \left( e^{\frac{1}{\sigma}} + \frac{1}{r} \right)}.
\]

The first factor is an increasing function of \( r \) by assumption on \( \kappa \) (see Definition 3.2); the second factor is also increasing, as can be verified by computing its derivative, and the third factor is a bounded function on \((0, +\infty)\).

Now, notice that the inequality in statement ii) is obvious if \( |\lambda \xi| \geq 1 \) since \( \kappa \) is increasing. When \( |\lambda \xi| \leq 1 \), it follows directly from i). \( \square \)

The next proposition clarifies the links between \( |D|^{s, \phi} \) and the function \( \kappa \) (we refer to Lemma 2.6 and Proposition 2.7 in [4] for the proof).

**Proposition 3.5** (from [4]) Assume that \( \phi \) is as defined in (36) for some admissible weight \( \kappa \).

i) For all \( g \in S(\mathbb{R}) \), there holds

\[
    |D|^{1, \phi}g(x) = \frac{1}{4} \int_\mathbb{R} \frac{2g(x) - g(x + \alpha) - g(x - \alpha)}{\alpha^2} \kappa \left( \frac{1}{|\alpha|} \right) d\alpha. \tag{40}
\]

ii) Given \( g \in S(\mathbb{R}) \), define the semi-norm

\[
    \| g \|_{s, \kappa} := \left( \int_{\mathbb{R}^2} \left| 2g(x) - g(x + \alpha) - g(x - \alpha) \right|^2 \left( \frac{1}{|\alpha|^s} \kappa \left( \frac{1}{|\alpha|} \right) \right)^2 \frac{d\alpha}{|\alpha|} \right)^{1/2}.
\]

Then, for all \( 1 < s < 2 \), there exist two constants \( c, C > 0 \) such that, for all \( g \in S(\mathbb{R}) \),

\[
    c \int_\mathbb{R} |D|^{s, \phi}g(x)|^2 \, dx \leq \| g \|_{s, \kappa}^2 \leq C \int_\mathbb{R} |D|^{s, \phi}g(x)|^2 \, dx.
\]
3.2. A commutator estimate. Here is the main result of this section.

**Proposition 3.6.** Let \( \phi \) be as defined in (36) for some admissible weight \( \kappa \). Consider a function \( \gamma : \mathbb{R} \to [0, 1] \) such that, for any \( x, y \) in \( \mathbb{R} \),

\[
|\gamma(x) - \gamma(y)| \leq C \gamma(y) |x - y|,
\]

for some absolute constant \( C \). Then, for any \( \sigma \) \( \in (0, 1/2) \), and \( h, g \in S(\mathbb{R}) \), there holds

\[
\left\| D^{\sigma, \phi} (\gamma(h)g) - \gamma(h) |D|^{\sigma, \phi} g \right\|_{L^2} \lesssim_C \left\| D^{\sigma, \phi} h \right\|_{L^2} \| \gamma(h)g \|_{\dot{H}^\sigma}.
\]

**Remark 3.7.** As mentioned in Remark 2.3, the key point is that the right-hand side does not involve the \( L^2 \)-norm of \( g \) but the one of \( \gamma(h)g \). This is crucial here since we do not assume that \( \gamma(h) \) is not bounded from below.

**Proof.** The first step of the proof consists in proving the following inequality:

\[
\left| D^{\sigma, \phi} (\gamma(h)g) (x) - (\gamma(h) |D|^{\sigma, \phi} g)(x) \right| \lesssim \int |\delta_t h(x)| \kappa \left( \frac{1}{|z|} \right) (\gamma(h)g)(x - z) \frac{dz}{|z|^{1+\sigma}}.
\]

To do so, we begin by applying the representation formula (40) (resp. (18)) for the operator \( |D|^{1, \phi} \) (resp. \( |D|^{-\alpha} \) with \( 0 < \alpha < 1 \)). Since

\[
|D|^{\sigma, \phi} = |D|^{1, \phi} |D|^{-(1-\sigma)},
\]

it follows that

\[
|D|^{\sigma, \phi} \eta(x)
= \frac{1}{4} \int_{\mathbb{R}} 2 |D|^{-(1-\sigma)} \eta(x) - |D|^{-(1-\sigma)} \eta(x + t) - |D|^{-(1-\sigma)} \eta(x - t) \kappa \left( \frac{1}{|t|} \right) dt \\
= \frac{c_{1-\sigma}}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{2}{|x - z|^{\sigma}} - \frac{1}{|x + t - z|^{\sigma}} - \frac{1}{|x - t - z|^{\sigma}} \right) \frac{1}{t^2} \kappa \left( \frac{1}{|t|} \right) \eta(z) dz dt.
\]

Set \( u = \gamma(h)g \). It follows from the assumption (41) and the triangle inequality that

\[
|g(z)(\gamma(h(z)) - \gamma(h(x)))| \lesssim |u(z)| |h(z) - h(x)|.
\]

Consequently, we get

\[
\left| D^{\sigma, \phi} (\gamma(h)g) (x) - (\gamma(h) |D|^{\sigma, \phi} g)(x) \right| \lesssim \int K(x, z) |u(z)| |h(z) - h(x)| dz,
\]

where

\[
K(y) = \int_{\mathbb{R}} \left( \frac{2}{|y|^{\sigma}} - \frac{1}{|y + t|^{\sigma}} - \frac{1}{|y - t|^{\sigma}} \right) \frac{1}{t^2} \kappa \left( \frac{1}{|t|} \right) dt.
\]

Now, to obtain the claim (43) (with \( z \) replaced by \( x - z \)) it is sufficient to prove that

\[
K(y) \lesssim \kappa \left( \frac{1}{|y|} \right) \frac{1}{|y|^{1+\sigma}}.
\]
To do so, we begin by applying (39) to get that
\[ \kappa \left( \frac{1}{|t|} \right) \lesssim \kappa \left( \frac{1}{|y|} \right) \left( 1 + |y|^\frac{\sigma}{2} |t|^{-\frac{\sigma}{2}} \right). \] (45)

Thus, we find that, for any \( y \in \mathbb{R} \setminus \{0\}, \)
\[ K(y) \lesssim \kappa \left( \frac{1}{|y|} \right) \int \left| \frac{2}{|y|^\sigma} - \frac{1}{|y + t|^\sigma} - \frac{1}{|y - t|^\sigma} \right| \left( 1 + |y|^\frac{\sigma}{2} |t|^{-\frac{\sigma}{2}} \right) \frac{1}{|t|^2} \, dt. \] (46)

Now, the wanted estimate (44) follows from the fact that, for any \( \beta \) in \([0, 1),\)
\[ \int \left| \frac{2}{|y|^\sigma} - \frac{1}{|y + t|^\sigma} - \frac{1}{|y - t|^\sigma} \right| \frac{dt}{|t|^{2+\beta}} = c(\sigma, \beta) |y|^{1+\sigma+\beta}, \]
as can be verified by an immediate homogeneity argument. This completes the proof of the claim (43).

Now, using (18) and Cauchy-Schwarz inequality, we have
\[ I := \int ||D|^{\sigma, \phi} (\gamma(h) g) - (\gamma(h) |D|^{\sigma, \phi} g)|^2 \, dx \]
\[ \lesssim \int \left( \int |\delta_z h(x)| \kappa \left( \frac{1}{|z|} \right) u(x - z) \frac{dz}{|z|^{1+\sigma}} \right)^2 \, dx \]
\[ \lesssim \int (|D|^{-\frac{1}{2}+2\sigma} (u^2)(x) \left( \int |\delta_z h(x)|^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{|z|^2} \right) dx \]
\[ \lesssim \| |D|^{-\frac{1}{2}+2\sigma} (u^2) \|_{L^2} \int \| \delta_z h(\cdot) \|_{L^4}^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{|z|^2}. \]

Then, by using the boundedness of Riesz potentials on Lebesgue spaces (see (17)) and the Sobolev embedding \( \dot{H}^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R}) \) (see (25)), we conclude that
\[ I \lesssim \| u \|_{\dot{H}^{2\sigma}}^2 \int \| \delta_z h(\cdot) \|_{\dot{H}^{\frac{1}{4}}}^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{|z|^3} \]
\[ \lesssim \| u \|_{\dot{H}^{2\sigma}}^2 \int |\xi|^\frac{1}{2} \left( \int \min\{|\xi|, |z|, 1\}^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{|z|^2} \right) |\hat{h}(\xi)|^2 \, d\xi. \]

The estimate (39) already used above to find (44) also implies that
\[ \int \min\{|\xi|, |z|, 1\}^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{|z|^2} \lesssim |\xi|^\frac{1}{2} \kappa \left( |\xi| \right) \]
\[ \lesssim \| u \|_{\dot{H}^{2\sigma}}^2 \int |\xi|^\frac{1}{2} \kappa \left( |\xi| \right) \frac{dx}{|\xi|^2} \lesssim \| u \|_{\dot{H}^{2\sigma}}^2 \| D^{\frac{1}{2}, \phi} h \|_{L^2}^2. \]

where we have used the equivalence \( \phi \sim \kappa \) (see (37)). This implies (42), which completes the proof. \( \Box \)
4. Preliminaries About the Muskat Equation

In this section, we recall some simple properties about the nonlinearity in the Muskat equation. This will serve to rigorously justify the computations in the sequel. Firstly, this will clarify the meaning of the integrals with respect to the variable $\alpha$. And secondly, we will exploit these results to introduce approximate Cauchy problems which are well-posed globally in time.

4.1. About the nonlinearity. Write

$$\frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} = \partial_x \Delta_\alpha f - \partial_x \Delta_\alpha f \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2}. \tag{47}$$

Notice that the Hilbert transform (see (19)) satisfies

$$\mathcal{H} u(x) = -\frac{1}{\pi} \text{pv} \int\limits_{\mathbb{R}} \Delta_\alpha u \, d\alpha.$$

In addition, the fractional laplacian $|D| = (-\partial_{xx})^{1/2}$ satisfies $|D| = \partial_x \mathcal{H}$. Consequently, it follows from (47) that

$$\partial_t f + |D| f = T(f) f, \tag{48}$$

where $T(f)$ is the operator defined by

$$T(f) g = -\frac{1}{\pi} \int\limits_{\mathbb{R}} (\partial_x \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha. \tag{49}$$

Then we have the following elementary result, which is Proposition 2.3 in [2].

**Proposition 4.1** (from [2])

(i) For all $f$ in $\dot{H}^1(\mathbb{R})$ and $g$ in $\dot{H}^{3/2}(\mathbb{R})$, the function

$$\alpha \mapsto \Delta_\alpha g x \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2}$$

belongs to $L^1_\alpha(\mathbb{R}; L^2_x(\mathbb{R}))$. Therefore, $T(f) g$ belongs to $L^2(\mathbb{R})$. Moreover, there is a positive constant $C$ such that

$$\|T(f) g\|_{L^2} \leq C \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{3/2}}. \tag{50}$$

(ii) For all $\delta \in [0, 1/2)$, there is a positive constant $C$ such that, for all functions $f_1, f_2$ in $\dot{H}^{1-\delta}(\mathbb{R}) \cap \dot{H}^{3/2+\delta}(\mathbb{R})$,

$$\|(T(f_1) - T(f_2)) f_2\|_{L^2} \leq C \|f_1 - f_2\|_{\dot{H}^{1-\delta}} \|f_2\|_{\dot{H}^{3/2+\delta}}. \tag{51}$$

In particular,

$$\|T(f) f\|_{L^2} \leq C \|f\|_{\dot{H}^{3/2}} \|f_2\|_{\dot{H}^{7/4}}.$$

(iii) The map $f \mapsto T(f) f$ is locally Lipschitz from $\dot{H}^1(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$ to $L^2(\mathbb{R})$. 
4.2. Approximate Cauchy problems. We now take up the second question raised at the beginning of this section, that of defining approximate Muskat equations. The goal is to introduce equations for which the Cauchy problem is easily studied, and whose solutions are expected to converge to solutions of the original Muskat equation. There are many possible ways to do so. The next proposition introduces a parabolic regularization of the Muskat equation, depending on some viscosity parameter \( \varepsilon > 0 \), which is convenient for our purposes. The following result states that these equations are well-posed for any fixed \( \varepsilon > 0 \). Our main task in the next sections will be to obtain estimates which are uniform in \( \varepsilon \).

**Proposition 4.2.** Consider an initial data \( f_0 \) in \( H^{3/2}(\mathbb{R}) \). For any \( \varepsilon > 0 \), the Cauchy problem

\[
\partial_t f + \varepsilon |D|^8 f + |D| f = T(f) f ; \quad f|_{t=0} = f_0,
\]

has a unique solution \( f \) in \( C^0(0, +\infty); H^{3/2}(\mathbb{R}) \cap L^2(0, +\infty); \dot{H}^2(\mathbb{R}) \). In addition \( f \) is \( C^\infty \) on \((0, +\infty) \times \mathbb{R}\).

**Proof.** Given an initial data \( f_0 \) in \( H^{3/2}(\mathbb{R}) \), the existence of a unique solution \( f \) locally in time follows at once from Proposition 4.1, by applying the classical methods for nonlinear parabolic equations. Then, to prove that the solution \( f \) exists globally in time, it will suffice to estimate its \( H^{3/2} \) norm.

We recall first that the square of the \( L^2 \)-norm is a Lyapunov functional. Indeed, by [27, Section 2], we have

\[
\int_{\mathbb{R}} (|D| f - T(f) f) f(x) \, dx = \int_{\mathbb{R}^2} \log \left( \frac{1}{1 + \alpha^2} \right) \, dx \, d\alpha \geq 0.
\]

This implies that

\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{L^2} + \varepsilon \| f \|^2_{\dot{H}^4} \leq 0.
\]  

(53)

Now we multiply the equation (52) by \( |D|^3 \) and use Plancherel’s identity to obtain the following Sobolev energy estimate:

\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{\dot{H}^{3/2}} + \varepsilon \| f \|^2_{\dot{H}^1} + \| f \|^2_{\dot{H}^2} \leq \| f \|_{\dot{H}^{3/4}} \| T(f) f \|_{L^2}.
\]

Now, apply (50) to obtain

\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{\dot{H}^{3/2}} + \varepsilon \| f \|^2_{\dot{H}^1} + \| f \|^2_{\dot{H}^2} \leq C \| f \|_{\dot{H}^{3/4}} \| f \|_{\dot{H}^1} \| f \|_{\dot{H}^{3/2}}.
\]

One may estimate the right-hand side above by means of the classical interpolation inequality in Sobolev spaces. This gives

\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{\dot{H}^{3/2}} + \varepsilon \| f \|^2_{\dot{H}^1} + \| f \|^2_{\dot{H}^2} \lesssim \| f \|^2_{L^2} \| f \|_{\dot{H}^1}.
\]

Hence, using Young’s inequality,

\[
\frac{d}{dt} \| f \|^2_{\dot{H}^{3/2}} \lesssim \varepsilon^{-1} \| f \|^4_{L^2}.
\]  

(54)

By combining (53) and (54), we get the wanted Sobolev estimates, which implies that, for any fixed \( \varepsilon > 0 \), the solution exists globally in time. \( \square \)
4.3. A reformulation of the Muskat equation. In this paragraph, we recall a key decomposition of the Muskat equation, as introduced in [2].

**Proposition 4.3** (from [2]) i) The Muskat equation (1) can be written under the form

\[
\partial_t f + W(f)\partial_x f + \frac{1}{1 + f_x^2} |D| f = R(f) f, \tag{55}
\]

where the coefficient \( W(f) \) the operator \( R(f) \) are defined by

\[
W(f) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathcal{O}(\alpha, \cdot)}{\alpha} \, d\alpha,
\]

\[
R(f) g := -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \left( \mathcal{E}(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) \, d\alpha
\]

\[
+ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x g(\cdot - \alpha)}{\alpha} \mathcal{O}(\alpha, \cdot) \, d\alpha,
\]

with

\[
\mathcal{O}(\alpha, \cdot) = \frac{1}{2} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} - \frac{1}{2} \frac{(\Delta_{-\alpha} f)^2}{1 + (\Delta_{-\alpha} f)^2}, \tag{57}
\]

\[
\mathcal{E}(\alpha, \cdot) = \frac{1}{2} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} + \frac{1}{2} \frac{(\Delta_{-\alpha} f)^2}{1 + (\Delta_{-\alpha} f)^2}. \tag{58}
\]

ii) Similarly, the approximate Muskat problem (52) is equivalent to

\[
\partial_t f + \varepsilon |D|^8 f + W(f)\partial_x f + \frac{1}{1 + f_x^2} |D| f = R(f) f. \tag{59}
\]

**Proof.** Recall from Sect. 4.1 that the Muskat equation (1) reads

\[
\partial_t f + |D| f = T(f) f,
\]

where \( T(f) \) is the operator defined by

\[
T(f) g = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha.
\]

Then splits the coefficient

\[
F_\alpha := \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2}
\]

into its odd and even components \( \mathcal{O}(\alpha, \cdot) \) and \( \mathcal{E}(\alpha, \cdot) \), as defined in the statement of the proposition. Since \( \Delta_\alpha f(x) \) converges to \( f_x(x) \) when \( \alpha \) goes to 0, we further decompose \( \mathcal{E}(\alpha, \cdot) \) as

\[
\mathcal{E}(\alpha, \cdot) = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} + \left( \mathcal{E}(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right).
\]

It follows that

\[
T(f) g = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} |D| g - W(f)\partial_x g + R(f) g, \tag{60}
\]

which implies the wanted result (55). The same argument proves also the second statement. \( \square \)
The main application that we will work out of the previous formulation is an \(L^2\)-energy estimate for the weighted fractional derivatives \(|D|^{\frac{3}{2}} f\).

**Corollary 4.4.** Consider a function \(\phi\) defined by (36), for some admissible weight \(\kappa\). For any \(\varepsilon \in [0, 1]\) and for any smooth solution \(f\) of the approximate Muskat equation (52), there holds

\[
\frac{1}{2} \frac{d}{dt} \|D^{\frac{3}{2}} \phi f\|_{L^2}^2 + \int_{\mathbb{R}} \frac{|D^{\frac{3}{2}} \phi f|^2}{1 + (\partial_x f)^2} \, dx + \varepsilon \|f\|_{H^4}^2 = (I) + (II) + (III)
\]

with

\[
(I) := -\frac{1}{2} \left( [\mathcal{H}, W(f)] |D^{\frac{3}{2}} \phi f, |D^{\frac{3}{2}} f] \right),
\]

\[
(II) := \langle R(f)(|D|^{1, \phi} f), |D|^{2, \phi} f \rangle,
\]

\[
(III) := \left( [\mathcal{H}^{1, \phi}, \mathcal{T}(f)] f, |D|^{2, \phi} f \right).
\]

where \([A, B]\) denotes the commutator \(A \circ B - B \circ A\) of two operators.

**Remark 4.5.** i) Hereafter, we say that \(f = f(t, x)\) is a smooth function if it satisfies the conclusion of Proposition 4.2, that is:

\[f \in C^0([0, +\infty); H^3(\mathbb{R})) \cap L^2([0, +\infty); \dot{H}^2(\mathbb{R})) \cap C^\infty((0, +\infty) \times \mathbb{R})\]

ii) As explained in Sect. 1.3, observe that one does not control \(\int |D|^{2, \phi} f|^2 \, dx\). Instead, we merely control

\[\int_{\mathbb{R}} \frac{|D|^{2, \phi} f|^2}{1 + (\partial_x f)^2} \, dx\]

The key point is that \(1 + (\partial_x f)^2\) is not bounded from above for solutions \(f\) which are \(L^\infty\) in times with values in \(H^3(\mathbb{R})\). This is why the parabolic feature of the Muskat equation degenerates in the critical regime.

**Proof.** By commuting \(|D|^{1, \phi} f\) with the equation (48) and then using the decomposition (60) with \(g = |D|^{1, \phi} f\), we find that

\[
(\partial_t + W(f)\partial_x + \frac{1}{1 + f_x^2} |D| + \varepsilon |D|^8 - R(f)) |D|^{1, \phi} f = [|D|^{1, \phi}, \mathcal{T}(f)] f. \quad (61)
\]

Then multiply the equation (61) by \(|D|^{2, \phi} f\) and use Plancherel’s identity to write

\[
\langle \partial_t |D|^{1, \phi} f, |D|^{2, \phi} f \rangle = \frac{1}{2} \frac{d}{dt} \|D^{\frac{3}{2}} \phi f\|_{L^2}^2.
\]

If follows that

\[
\frac{1}{2} \frac{d}{dt} \|D^{\frac{3}{2}} f\|_{L^2}^2 + \varepsilon \|f\|_{H^4}^2 + \int_{\mathbb{R}} \frac{|D^{\frac{3}{2}} \phi f|^2}{1 + (\partial_x f)^2} \, dx
\]

\[- = \langle W(f)\partial_x |D|^{1, \phi} f, |D|^{2, \phi} f \rangle + (II) + (III),
\]

where (II) and (III) are as defined in the statement.

To conclude, we notice that, since \(\partial_x = -\mathcal{H}|D|\) and since \(\mathcal{H}^* = -\mathcal{H}\), one has

\[
\langle W(f)\partial_x |D|^{1, \phi} f, |D|^{2, \phi} f \rangle = \frac{1}{2} \left( [\mathcal{H}, W(f)] |D|^{2, \phi} f, |D|^{2, \phi} f \right) = -(I). \quad (62)
\]

This completes the proof. \(\square\)
5. Paralinearization of the Muskat Equation

With these preliminaries established, we start the analysis of the nonlinearity in the Muskat equation. More precisely, we want to estimate the three terms which appear in Corollary 4.4, in terms of the quantities 

\[ \sup_{t \in [0,T]} \| D^{3,\phi} f(t) \|_{L^2}, \quad \int_0^T \int_{\mathbb{R}} \frac{|D|^{2,\phi} f(t,x)|^2}{1 + (\partial_x f(t,x))^2} \, dx \, dt. \]

To achieve this goal, as the title of this section attempts to indicate, we shall develop several tools needed to commute the weighted fractional Laplacians with the Muskat equation. As we have explained in the introduction, the main difficulty is to exhibit a null-type structure.

Let us introduce three notations which will be used continually in this section.

(i) We use the bracket notation:

\[ \langle a \rangle = \sqrt{1 + a^2}. \]

To compensate for the degeneracy of the parabolic behavior for large slopes mentioned in Remark 4.5, we want to prove estimates which involves the \( L^2(\frac{dx}{\langle f_x \rangle}) \)-norm (instead of the \( L^2(dx) \)-norm).

(ii) A key ingredient of our analysis is to decompose the frequency space. To do so, we will use the following notation. Given a parameter \( \lambda > 0 \), we set

\[ \| f \|_{\dot{H}^{3/2}_\lambda} := \left( \int_{\mathbb{R}} \min(\lambda|\xi|, 1) \frac{1}{\mathfrak{m}^2} |\xi|^3 |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}. \]  

(63)

(iii) To truncate the frequency domain, it will be convenient to consider a smooth bump function \( \chi : \mathbb{R} \to [0, 1] \) satisfying \( \chi(x) = \chi(-x) \) and such that \( \chi(x) = 1 \) for \( x \in [-1, 1] \) and \( \chi(x) = 0 \) if \( |x| \geq 2 \).

5.1. The commutator with the Hilbert transform. Our first result will allow us to estimate the term (I) which appears in Corollary 4.4.

Proposition 5.1. For any \( \lambda > 0 \) and any smooth functions \( f, g, h \), there holds

\[ \| [\mathcal{H}, W(f)](g_x), h \| \lesssim \frac{1}{\sqrt{\lambda}} \| f \|_{\dot{H}^{3/2}_\lambda} \| g \|_{\dot{H}^{1/2}_\lambda} \| h \|_{L^2} \]

\[ + \left(1 + \| f \|_{\dot{H}^{3/2}_\lambda}^7\right) \| f \|_{\dot{H}^{3/2}_\lambda} \left\| \frac{g_x}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{h}{\langle f_x \rangle} \right\|_{L^2}. \]

(64)

Remark 5.2. Assume that \( \phi \) is as defined in (36) for some admissible weight \( \kappa \). Then it follows from (39) and the previous inequality that

\[ \| [\mathcal{H}, W(f)](g_x), h \| \lesssim \frac{1}{\sqrt{\lambda}} \| f \|_{\dot{H}^{3/2}_\lambda} \| g \|_{\dot{H}^{1/2}_\lambda} \| h \|_{L^2} \]

\[ + \kappa \left(\frac{1}{\lambda}\right)^{-1} \left(1 + \| f \|_{\dot{H}^{3/2}_\lambda}^7\right) \| D |^{2,\phi} f \|_{L^2} \left\| \frac{g_x}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{h}{\langle f_x \rangle} \right\|_{L^2}. \]
Proof. One can write the function $O(\alpha, x)$ (see (57)) as follows:

$$O(\alpha, \cdot) = A_\alpha(f) (\Delta_\alpha f - \Delta_{-\alpha} f) \quad \text{where}$$

$$A_\alpha(f) = \frac{1}{2} \frac{\Delta_\alpha f + \Delta_{-\alpha} f}{(1 + (\Delta_\alpha f)^2)(1 + (\Delta_{-\alpha} f)^2)}.$$  \hfill (65)

Then, by definition of $W(f)$ (see (56)), for any $\lambda > 0$, one has $W(f) = W_{1, \lambda}(f) + W_{2, \lambda}(f)$ with

$$W_{1, \lambda}(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{O(\alpha, \cdot)}{\alpha} \chi \left( \frac{\alpha}{\lambda} \right) d\alpha,$$

$$W_{2, \lambda}(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{O(\alpha, \cdot)}{\alpha} \left( 1 - \chi \left( \frac{\alpha}{\lambda} \right) \right) d\alpha.$$

Then, the Cauchy-Schwarz inequality implies that

$$|\langle [\mathcal{H}, W(f)](g_x), h \rangle| \leq \| (f_x) \langle [\mathcal{H}, W_{1, \lambda}(f)](g_x) \rangle \|_{L^2} \left\| h \right\|_{\langle f_x \rangle L^2}$$

$$+ \left\| [\mathcal{H}, W_{2, \lambda}(f)](g_x) \right\|_{L^2} \| h \|_{L^2}.$$ \hfill (66)

We claim that

$$\| [\mathcal{H}, W_{2, \lambda}(f)](g_x) \|_{L^2} \lesssim \frac{1}{\sqrt{\lambda}} \| f \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{\dot{H}^{\frac{3}{2}}},$$ \hfill (67)

and

$$\left\| (f_x) \langle [\mathcal{H}, W_{1, \lambda}(f)](g_x) \rangle \right\|_{L^2} \lesssim \left( 1 + \| f \|_{\dot{H}^{\frac{3}{2}}}^7 \right) \| f \|_{\dot{H}^{\frac{3}{2}}} \left\| g_x \right\|_{\langle f_x \rangle L^2}.$$ \hfill (68)

The wanted conclusion (64) will follow at once from (66), (67) and (68).

Step 1: proof of the claim (67). We use Lemma 2.1 which gives an estimate for the commutator of the Hilbert transform and a function in $\dot{H}^1(\mathbb{R})$. It implies that

$$\| [\mathcal{H}, W_{2, \lambda}(f)](g_x) \|_{L^2} \leq \| W_{2, \lambda}(f) \|_{\dot{H}^1} \| g \|_{\dot{H}^{\frac{3}{2}}}. \hfill (69)$$

By definition of $W_{2, \lambda}(f)$, we have

$$\| W_{2, \lambda}(f) \|_{\dot{H}^1} \lesssim \int \left( \int_{\mathbb{R}} \frac{|\partial_x O(\alpha, x)|}{|\alpha|} \left( 1 - \chi \left( \frac{\alpha}{\lambda} \right) \right) d\alpha \right)^2 dx.$$ \hfill (70)

Now, directly from the definition of $O$, we have

$$| \partial_x O(x, \alpha) | \lesssim | \Delta_\alpha f_x | + | \Delta_{-\alpha} f_x |.$$ \hfill (71)
Observe that, by symmetry\(^2\) it is sufficient to consider the contribution of \(\Delta_\alpha f_x\). Consequently, using the Cauchy-Schwarz inequality together with the bound (27), we get
\[
\| W_{2,\lambda}(f) \|_{H^1}^2 \lesssim \int \left( \int_{|\alpha| \geq \lambda} \frac{|f_x(x) - f_x(x - \alpha)|}{|\alpha|^2} \, d\alpha \right)^2 \, dx \\
\lesssim \frac{1}{\lambda} \int \int \frac{|f_x(x) - f_x(x - \alpha)|^2}{|\alpha|^2} \, d\alpha \, dx \\
\lesssim \frac{1}{\lambda} \| f \|_{H^{\frac{3}{2}}}^2.
\]
So the desired result (67) follows from (69) and the previous inequality.

Step 2: proof of the claim (68). This is the most delicate task. Directly from the definition of the Hilbert transform, one has
\[
\| \langle f_x \rangle [\mathcal{H}, W_{1,\lambda}(f)](g_x) \|_{L^2}^2 = \frac{1}{\pi^2} \left( \int a(x, y) \frac{W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)}{x - y} \tilde{g}(y) \, dy \right)^2 \, dx
\]
with
\[
\tilde{g}(y) = \frac{g_x(y)}{\langle f_x(y) \rangle} \quad \text{and} \quad a(x, y) = \langle f_x(x) \rangle \langle f_x(y) \rangle.
\]
By applying Holder’s inequality together with the representation formula (18), we find
\[
\| \langle f_x \rangle [\mathcal{H}, W_{1,\lambda}(f)](g) \|_{L^2}^2 \\
\lesssim \int \left( \int a(x, y)^5 |W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)|^5 \frac{dy}{|x - y|^2} \right)^{\frac{2}{5}} \left( |D|^{-\frac{1}{2}} (|\tilde{g}|^5)(x) \right)^{\frac{8}{5}} \, dx \\
\lesssim \left( \int \int a(x, y)^5 |W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)|^5 \frac{dy \, dx}{|x - y|^2} \right)^{\frac{2}{5}} \left( \int |D|^{-\frac{1}{2}} (|\tilde{g}|^5)(x) \right)^{\frac{8}{5}} \, dx \\
\lesssim \left( \int \int a(x, y)^5 |W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)|^5 \frac{dy \, dx}{|x - y|^2} \right)^{\frac{2}{5}} \| \tilde{g} \|_{L^2}^2,
\]
where we have used (17) to write
\[
\| |D|^{-\frac{1}{2}} h \|_{L^\frac{8}{5}} \lesssim \| h \|_{L^\frac{8}{5}}.
\]
So, it is enough to show that
\[
\int \int a(x, y)^5 |W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)|^5 \frac{dy \, dx}{|x - y|^2} \lesssim \left( 1 + \| f \|_{H^{\frac{3}{2}}} \right) \| f \|_{H^\frac{3}{2}}^5. \tag{71}
\]
\(^2\) Let us mention that we will make extensive of this symmetry to handle the contributions of terms involving \(\Delta_{-\alpha}\), without explicitly recalling this argument.
One has

\[ a(x, y)|W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)| \]
\[ \leq \frac{a(x, y)}{\pi} \left| \int A_\alpha(f)(x) (S_\alpha f(x) - S_\alpha f(y)) \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{\alpha} \right| + \frac{a(x, y)}{\pi} \int (A_\alpha(f)(x) - A_\alpha(f)(y)) S_\alpha f(y) \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{\alpha} \].

We further decompose \( A_\alpha(f) \) according to \( A_{0}(f) + (A_\alpha(f) - A_0(f)) \), with

\[ A_0 = A_\alpha|_{\alpha=0} = \frac{f_x}{(1 + f_x^2)^2}. \]

Then

\[ a(x, y)|W_{1,\lambda}(f)(x) - W_{1,\lambda}(f)(y)| \leq \frac{1}{\pi} (I_1(x, y) + I_2(x, y) + I_3(x, y)), \]

where

\[ I_1(x, y) = a(x, y)A_0(f)(x) \left| \int (S_\alpha f(x) - S_\alpha f(y)) \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{\alpha} \right|, \]
\[ I_2(x, y) = a(x, y) \int |A_\alpha(f)(x) - A_0(f)(x)| |S_\alpha f(x) - S_\alpha f(y)| \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{|\alpha|}, \]
\[ I_3(x, y) = a(x, y) \int |A_\alpha(f)(x) - A_\alpha(f)(y)||S_\alpha f(y)| \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{|\alpha|}. \]

**Step 2.1: estimate of the contribution of I₁.** We begin by estimating

\[ \iint_{\mathbb{R}^2} I_1(x, y)^5 \frac{dx\,dy}{|x - y|^2}. \]

**Lemma 5.3.** Set

\[ \theta_\lambda(\alpha) = \text{sign}(\alpha) \int_{|\alpha|}^{\infty} \chi \left( \frac{s}{\lambda} \right) \frac{ds}{s^2} \]

and define

\[ \mathcal{H}_\lambda h(x) = h \ast \theta_\lambda(x) = \int_{\mathbb{R}} h(x - \alpha)\theta_\lambda(\alpha)\,d\alpha. \]

Then

\[ \int_{\mathbb{R}} S_\alpha f(x) \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{\alpha} = 2\mathcal{H}_\lambda f_x(x). \]
Proof. Recall that
\[ S_\alpha(f)(x) = \Delta_\alpha f(x) - \Delta_{-\alpha} f(x) = \frac{2f(x) - f(x - \alpha) - f(x + \alpha)}{\alpha}. \]

By a parity argument, we have
\[ \hat{R} S_\alpha f(x) \chi(\alpha \lambda) = 2 \int_0^\infty (2f(x) - f(x - \alpha) - f(x + \alpha)) \chi(\alpha \lambda) \frac{d\alpha}{\alpha^2}. \]

Now, on \( \mathbb{R}_+ \), by definition of \( \theta_\lambda(\alpha) \), we have
\[ \partial_\alpha \theta_\lambda = -\chi(\alpha/\lambda)/\alpha^2. \]

Hence, integrating by parts, we obtain
\[ \hat{R} S_\alpha f(x) \chi(\alpha \lambda) = 2 \int_0^\infty \partial_\alpha (2f(x) - f(x - \alpha) - f(x + \alpha)) \theta_\lambda(\alpha) d\alpha = 2 \int_0^\infty (f_x(x - \alpha) - f_x(x + \alpha)) \theta_\lambda(\alpha) d\alpha. \]

Since \( \theta_\lambda(-\alpha) = -\theta_\lambda(\alpha) \), this gives the wanted formula. \( \Box \)

On the other hand, remembering that
\[ a(x, y) = \langle f_x(x) \rangle \langle f_x(y) \rangle \quad \text{and} \quad A_0(x) = \frac{f_x(x)}{(f_x(x))^4}, \]
we get
\[ a(x, y) A_0(x) \leq \frac{\langle f_x(y) \rangle}{1 + f_x(x)^2} \lesssim |f_x(x) - f_x(y)| + 1. \]

Consequently, we deduce from the definition of \( I_1 \) and the previous lemma that
\[ I_1(x, y) \lesssim |f_x(x) - f_x(y)| \| \mathcal{H}_\lambda f_x(x) - \mathcal{H}_\lambda f_x(y) \| + |\mathcal{H}_\lambda f_x(x) - \mathcal{H}_\lambda f_x(y)|. \]

So, using Hölder’s inequality and the characterization of the fractional Sobolev spaces in terms of Gagliardo semi-norms (see (26)), we have
\[ \iint \left| I_1(x, y) \right|^5 \frac{dx \, dy}{|x - y|^2} \lesssim \| \mathcal{H}_\lambda f_x \|^5_{\dot{W}^{1,10}} \| f_x \|^5_{\dot{W}^{1,10}} + \| \mathcal{H}_\lambda f_x \|^5_{\dot{W}^{1/2,5}}. \]

Then, using Sobolev’s embeddings, we conclude that
\[ \iint \left| I_1(x, y) \right|^5 \frac{dx \, dy}{|x - y|^2} \lesssim (\| f \|^5_{L^5} + 1) \| \mathcal{H}_\lambda f_x \|^5_{H^{1/2}}. \]

It remains to estimate the \( H^{1/2} \)-norm of \( \mathcal{H}_\lambda f_x \). To do so, we will use the following inequality.

**Lemma 5.4.** There exists a constant \( C > 0 \) such that, for any \( \lambda \),
\[ |\hat{\theta}_\lambda(\xi)| \lesssim \min\{1, |\xi|/\lambda\}. \]
Proof. One has
\[ |\hat{\theta}_\lambda(\xi)| = |\int \theta_\lambda(\alpha) \sin(\alpha \xi) \, d\alpha| \]
\[ \leq \left| \int \chi(\xi \alpha)\theta_\lambda(\alpha) \sin(\alpha \xi) \, d\alpha \right| + \left| \int (1 - \chi(\xi \alpha))\theta_\lambda(\alpha) \sin(\alpha \xi) \, d\alpha \right|. \]
Since \(|\sin(\alpha \xi)| \leq |\alpha \xi|\) and \(\xi \sin(\alpha \xi) = -\partial_\alpha \cos(\alpha \xi)\),
\[ |\hat{\theta}_\lambda(\xi)| \leq \frac{1}{|\xi|} \frac{|\lambda|}{|\alpha|^2} \int \frac{1}{|\xi \alpha|^2} \, d\alpha \]
\[ + \frac{1}{|\xi \alpha|^2} \int \frac{1}{|\alpha|^2} \, d\alpha \]
\[ \leq \frac{1}{|\xi|} \min\left\{ \frac{1}{|\xi|}, \lambda \right\} , \]
equivalent to (73). \(\square\)

Since \(\hat{\mathcal{H}}_\lambda f_x(\xi) = i \xi \hat{f}(\xi)\hat{\theta}_\lambda(\xi)\), the above lemma implies that
\[ \|\mathcal{H}_\lambda f_x\|_{\dot{H}^{\frac{3}{2}}} \leq \left( \int |\xi|^3 (\min\{1, |\xi| |\lambda|\})^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq \|f\|_{\dot{H}^{\frac{3}{2}}}^{\frac{3}{2}}, \]
where recall that \(\|f\|_{\dot{H}^{\frac{3}{2}}}\) is defined by (63). Therefore, we obtain from (72) that
\[ \iint I_1(x, y)^5 \frac{d x \, d y}{|x - y|^2} \lesssim \left( 1 + \|f\|_{\dot{H}^{\frac{3}{2}}}^5 \right) \|f\|_{\dot{H}^{\frac{3}{2}}}^5. \] (74)
This concludes the analysis of this term.

Step 2.2: estimate of the contribution of \(I_2\). We now estimate
\[ \iint I_2(x, y)^5 \frac{d x \, d y}{|x - y|^2} . \]
Recall that
\[ I_2(x, y) = a(x, y) \int |A_\alpha(f)(x) - A_0(f)(x)| |S_\alpha f(x) - S_\alpha f(y)| \chi \left( \frac{\alpha}{x} \right) \frac{d \alpha}{|\alpha|} , \]
with \(a(x, y) = \langle f_x(x) \rangle \langle f_x(y) \rangle\) and
\[ A_\alpha(f) = \frac{1}{2} \frac{\Delta_\alpha f + \Delta_{-\alpha} f}{(1 + (\Delta_\alpha f)^2)(1 + (\Delta_{-\alpha} f)^2)}, \quad A_0 = \frac{f_x}{(1 + f_x^2)^2} . \]
We begin by estimating the factor \(A_\alpha(f)(x) - A_0(f)(x)\) by means of the following
Lemma 5.5. For any \((x, x_1, x_2)\), the quantity
\[
M = \left| \frac{x_1 + x_2}{2(1 + x_1^2)(1 + x_2^2)} - \frac{x}{(1 + x^2)^2} \right|
\]
satisfies
\[
M \lesssim \frac{1}{\langle x \rangle^2} \left( |x_1 - x|^2 + |x_2 - x|^2 + |x_1 - x| + |x_2 - x| \right) . \tag{75}
\]

Proof. Set \(M_0 = 2(1 + x_1^2)(1 + x_2^2)(1 + x^2)^2\). Then
\[
M = \left| \frac{1}{M_0} \left( (x_1 + x_2)(1 + x^2)^2 - 2x(1 + x_1^2)(1 + x_2^2) \right) \right| \leq \frac{|x_1 + x_2 - 2x|}{M_0} + \frac{|(x_1 + x_2)x^2 - x(x_1^2 + x_2^2)|}{M_0} + \frac{|(x_1 + x_2)x^4 - 2xx_1^2x_2^2|}{M_0} . \tag{76}
\]
The first two factors are clearly bounded by \((1 + x^2)^{-3} (|x_1 - x| + |x_2 - x|)\). To bound the last one, set \(h_1 = x - x_1, h_2 = x - x_2\) and write
\[
(x_1 + x_2)x^4 - 2xx_1^2x_2^2 = x_1x(x_1h_2^2 + 2x_1x_2h_2 + h_1(x_2^2 + h_2^2 + 2x_2h_2)) .
\]
This implies that the last term in the right-hand side of (76) is bounded by the right-hand side of (75). This completes the proof. \(\square\)

Recall that \(E_\alpha(f) = \Delta_\alpha f - f_\alpha\). It follows from the previous lemma that
\[
|A_\alpha(f) - A_0(f)| \leq \frac{C}{\langle f_\alpha \rangle^2} \left( |E_\alpha(f)| + |E_\alpha(f)|^2 + |E_{-\alpha}(f)| + |E_{-\alpha}(f)|^2 \right) .
\]

As above, by using the change of variables \(\alpha \rightarrow -\alpha\) and a parity argument, it is sufficient to handle the contributions of \(E_\alpha f\). We have
\[
I_2(x, y) \lesssim \langle f_\alpha(y) \rangle \langle f_\alpha(x) \rangle^{-1} \int \left( |E_\alpha f(x)| + |E_\alpha f(x)|^2 \right) |S_\alpha f(x) - S_\alpha f(y)| \chi \left( \frac{\alpha}{\lambda} \right) \frac{d\alpha}{|\alpha|} \lesssim (1 + |f_\alpha(x) - f_\alpha(y)|) \int_{|\alpha| \leq 2\lambda} \left( |E_\alpha f(x)| + |E_\alpha f(x)|^2 \right) |S_\alpha(x) - S_\alpha(y)| \frac{d\alpha}{|\alpha|} ,
\]
where we have used the fact that \(\langle f_\alpha(y) \rangle \langle f_\alpha(x) \rangle^{-1} \lesssim 1 + |f_\alpha(x) - f_\alpha(y)|\). By Hölder and Minkowski’s inequalities, the term \(\iint I_2(x, y)^5 \frac{dx\,dy}{|x-y|^2}\) is bounded by
\[
\left[ \int_{|\alpha| \leq 2\lambda} \left( \|E_\alpha(f)\|_{L^{10}} + \|E_\alpha(f)\|_{L^{20}}^2 \right) \|S_\alpha(f)\|_{F^{10.5}_{10,10}} \frac{d\alpha}{|\alpha|} \right]^5 + \|f_\alpha\|_{W^{10,10}}^5 \int_{|\alpha| \leq 2\lambda} \left( \|E_\alpha(f)\|_{L^{10}} + \|E_\alpha(f)\|_{L^{20}}^2 \right) \|S_\alpha(f)\|_{F^{10}_{20,10}} \frac{d\alpha}{|\alpha|} \right]^5 .
\]
Hence, the Sobolev embeddings (24), (25) and (26) imply that
\[
\|h\|_{L^4} \lesssim \|h\|_{H^{\frac{3}{2}}}^{\frac{3}{2}}, \quad \|h\|_{H^3} \lesssim \|h\|_{H^{\frac{9}{2}}}, \quad \|h\|_{L^4} \lesssim \|h\|_{H^{\frac{19}{10}}}, \\
\|f_x\|_{W^{1,10}} \lesssim \|f\|_{H^{\frac{3}{2}}}, \quad \|h\|_{F^{10,10}} \lesssim \|h\|_{H^{\frac{11}{2}}}, \quad \|h\|_{F^{3,10}} \lesssim \|h\|_{H^{\frac{19}{5}}},
\]
Consequently, using again Hölder’s inequality,
\[
\int \int I_2(x, y)^5 \frac{dx\,dy}{|x - y|^2} \lesssim \left( \int \int (\|E_\alpha(f)\|^2_{H^{\frac{3}{2}}} + \|E_\alpha(f)\|^4_{H^{\frac{9}{2}}}) \frac{d\alpha}{|\alpha|^\frac{6}{5}} \right)^2 \left( \int \int \|S_\alpha(f)\|^2_{H^{\frac{19}{10}}} \frac{d\alpha}{|\alpha|^\frac{9}{10}} \right)^2 \|f\|_{H^{\frac{19}{5}}}^{\frac{5}{2}}.
\]
So, it follows from Lemma 2.6 that
\[
\int \int I_2(x, y)^5 \frac{dx\,dy}{|x - y|^2} \lesssim \left( 1 + \|f\|_{H^{\frac{19}{5}}}^{10} \right) \|f\|_{H^{\frac{19}{5}}}^{\frac{10}{3}}. \tag{77}
\]
**Step 2.3: Estimate of the contribution of I_3.** We now pass to the estimate of the last term
\[
\int \int I_3(x, y)^5 \frac{dx\,dy}{|x - y|^2}.
\]
Here we use the following variant of the inequality given by Lemma 5.5:
\[
\left| \frac{x_1 + x_2}{(1 + x_1^2)(1 + x_2^2)} - \frac{y_1 + y_2}{(1 + y_1^2)(1 + y_2^2)} \right| \lesssim \sum_{j=1}^{2} |x_j - y_j| + |x_j - y_j|^3.
\]
as can be verified by using arguments parallel to those used to prove Lemma 5.5.

It follows that
\[
I_3(x, y) \lesssim \int \frac{a(x, y)}{\langle \Delta_\alpha f(y) \rangle^2} N_\alpha(x, y) |\Delta_\alpha f(y) - \Delta_{-\alpha} f(y)| \frac{d\alpha}{|\alpha|},
\]
where
\[
N_\alpha(x, y) = |\Delta_\alpha f(x) - \Delta_\alpha f(y)| + |\Delta_\alpha f(x) - \Delta_\alpha f(y)|^3. \tag{78}
\]
Thanks to
\[
\frac{a(x, y)}{\langle \Delta_\alpha f(y) \rangle^2} \lesssim (1 + |f_x(x) - f_x(y)|)(1 + (f_x(y) - \Delta_\alpha f(y))^2)
\]
it follows that
\[
I_3(x, y) \lesssim \int \frac{N_\alpha(x, y) M_\alpha(y) d\alpha}{|\alpha|} + |f_x(x) - f_x(y)| \int \frac{N_\alpha(x, y) M_\alpha(y) d\alpha}{|\alpha|},
\]
with
\[
M_\alpha(y) = \left(1 + (f_x(y) - \Delta_\alpha f(y))^2\right) |\Delta_\alpha f(y) - \Delta_{-\alpha} f(y)|. \tag{79}
\]

Once again, we argue using Holder’s inequality and Minkowski’s inequality, to deduce

\[
\iint I_3(x, y)^5 \frac{dx \, dy}{|x - y|^2} \leq \int \left( \iint |M_\alpha(y)|^5 |N_\alpha(x, y)|^5 \frac{dx \, dy}{|x - y|^2} \right)^\frac{1}{5} \frac{d\alpha}{|\alpha|}^5
+ \|f_x\|_{W^{10,10}}^5 \left( \int \left( \iint |M_\alpha(y)|^{10} |N_\alpha(x, y)|^{10} \frac{dx \, dy}{|x - y|^2} \right)^\frac{1}{10} \frac{d\alpha}{|\alpha|}^5 \right)
+ \|f\|_{\dot{H}^{\frac{1}{2}}}^5 \left( \int \left( \iint |N_\alpha(x, y)|^{10} \frac{dx \, dy}{|x - y|^2} \right)^2 \frac{d\alpha}{|\alpha|}^5 \right).
\]

By (79) and Sobolev inequality, for any \(p > 2\), we have

\[
\|M_\alpha\|_{L^p} \lesssim \|\Delta_\alpha f - \Delta_{-\alpha} f\|_{L^p} + \|\Delta_\alpha f - \Delta_{-\alpha} f\|_{L^{2p}}^2 + \|\Delta_\alpha f - f_x\|_{L^{4p}}^4
\lesssim \|\Delta_\alpha f - \Delta_{-\alpha} f\|_{\dot{H}^{\frac{1}{2}(1-\frac{1}{p})}} + \|\Delta_\alpha f - \Delta_{-\alpha} f\|_{\dot{H}^{\frac{1}{2}(1-\frac{1}{4p})}}^2
+ \|\Delta_\alpha f - f_x\|_{\dot{H}^{\frac{1}{2}(1-\frac{1}{4p})}}^4.
\]

On the other hand, by (78), for any \(p > 2\), we have

\[
\left( \int \left( \iint |N_\alpha(x, y)|^p \frac{dx}{|x - y|^2} \right)^\frac{1}{p} \right) \lesssim \|\Delta_\alpha f\|_{F^{p,1}_{2p}} + \|\Delta_\alpha f\|_{F^{1,1}_{6p,3p}}^{\frac{1}{p}}
\lesssim \|\Delta_\alpha f\|_{\dot{H}^{\frac{1}{2}(1+\frac{1}{p})}} + \|\Delta_\alpha f\|_{\dot{H}^{\frac{1}{2}(1+\frac{1}{4p})}}^{\frac{1}{p}}.
\]

So, after some extra bookkeeping, by applying Lemma 2.6, we get that

\[
\iint I_3(x, y)^5 \frac{dx \, dy}{|x - y|^2} \lesssim \left(1 + \|f\|_{\dot{H}^{\frac{1}{2}}}^{30}\right) \|f\|_{\dot{H}^{\frac{1}{2}}}^{\frac{5}{3}}. \tag{80}
\]

Therefore, we have proved (71) which completes the proof of the proposition. \(\square\)
5.2. Estimate of the remainder term. We now consider the remainder term $R(f)(g)$ which appears in Proposition 4.3. Our goal is to estimate the term (II) in Corollary 4.4.

**Proposition 5.6.** For any $\lambda > 0$ and any smooth functions $f$ and $g$, there holds

$$\|\langle R(f)(g), h \rangle \| \lesssim \left( 1 + \|f\|_{H^{3/2}}^2 \right) \|f\|_{H^{3/2}} \frac{\|g\|}{\|f\|_{H^{3/2}}} \frac{\|h\|_{L^2}}{\|f\|_{H^{3/2}}} + \frac{1}{\sqrt{\lambda}} \|g\|_{H^{3/2}} \|f\|_{H^{3/2}} \|h\|_{L^2}.$$ 

**Remark 5.7.** In particular, it follows from (39) that

$$\|\langle R(f)(g), h \rangle \| \lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} \left( 1 + \|f\|_{H^{3/2}}^2 \right) \|D^\phi f\|_{L^2} \frac{\|g\|}{\|f\|_{H^{3/2}}} \frac{\|h\|_{L^2}}{\|f\|_{H^{3/2}}} + \frac{1}{\sqrt{\lambda}} \|g\|_{H^{3/2}} \|f\|_{H^{3/2}} \|h\|_{L^2}.$$ 

**Proof.** Recall that

$$R(f)g := -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta \alpha) \left( E(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) d\alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x g(\cdot - \alpha)}{\alpha} O(\alpha, \cdot) d\alpha.$$ 

By symmetry, one has

$$\int_{\mathbb{R}} \frac{g_x(x)}{\alpha} \left( E(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) d\alpha = 0.$$ 

Hence, $R(f)g$ simplifies to

$$R(f)g = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g_x(\cdot - \alpha)}{\alpha} \left( E(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) d\alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x g(\cdot - \alpha)}{\alpha} O(\alpha, \cdot) d\alpha = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g_x(\cdot - \alpha)}{\alpha} \left( \frac{(\Delta \alpha f)^2}{1 + (\Delta \alpha f)^2} - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) d\alpha.$$ 

Introduce the function $B : \mathbb{R} \to \mathbb{R}$ defined by

$$B(u) = \frac{u^2}{1 + u^2}$$

and then decompose $R(f, g) = R_{1, \lambda} + R_{2, \lambda}$ with

$$R_{1, \lambda} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g_x(\cdot - \alpha)}{\alpha} (B(\Delta \alpha f) - B(\partial_x f)) \chi\left( \frac{\alpha}{\lambda} \right) d\alpha,$$

$$R_{2, \lambda} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g_x(\cdot - \alpha)}{\alpha} (B(\Delta \alpha f) - B(\partial_x f)) \left( 1 - \chi\left( \frac{\alpha}{\lambda} \right) \right) d\alpha.$$ 

We have

$$\|\langle R(f)g, h \rangle \| \leq \langle f_x \rangle \| R_{1, \lambda} \|_{L^2} \frac{\|h\|_{L^2}}{\langle f_x \rangle} + \| R_{2, \lambda} \|_{L^2} \|h\|_{L^2}.$$ 

We will estimate these two terms separately.
Step 1: estimate of $R_{2,\lambda}$. Note that

$$g_x(x - \alpha) = \partial_\alpha(\delta_\alpha g(x)).$$

Then, by integrating by parts in $\alpha$, we get that

$$|R_{2,\lambda}(x)| \lesssim \int_{|\alpha| \geq \lambda} \left| \delta_\alpha g(x) \right| \left| \Theta_\lambda(x, \alpha) \right| \ d\alpha \quad \text{where}$$

$$\Theta_\lambda(x, \alpha) := \partial_\alpha \left[ \frac{1}{\alpha} \left( (B(f_x(x)) - B(\Delta_\alpha f(x))) \left( 1 - \chi\left(\frac{\alpha}{\lambda}\right)\right) \right) \right].$$

It follows from the Cauchy-Schwarz inequality that

$$|R_{2,\lambda}(x)|^2 \leq \left( \int_{|\alpha| \geq \lambda} \frac{\delta_\alpha g(x)}{|\alpha|} \left| \Theta_\lambda(x, \alpha) \right| \ d\alpha \right)^2 \leq \frac{1}{\lambda} \left( \int_{\mathbb{R}} |\delta_\alpha g(x)|^2 \frac{d\alpha}{|\alpha|} \right) \left( \int_{\mathbb{R}} |\alpha|^\frac{5}{2} \left| \Theta_\lambda(x, \alpha) \right|^2 \ d\alpha \right).$$

By using again the Cauchy-Schwarz inequality,

$$\left\| R_{2,\lambda} \right\|_{L^2} \leq \frac{1}{\sqrt{\lambda}} \|g\left\|_{F_{4,2}}^4 \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\alpha|^\frac{5}{2} \left| \Theta_\lambda(x, \alpha) \right|^2 \ d\alpha \right) \ dx \right)^\frac{1}{4}. \quad (81)$$

We now have to estimate $|\Theta_\lambda(x, \alpha)|$. Since $|B'(u)| \leq 1$ for all $u \in \mathbb{R}$, we have

$$|B(\Delta_\alpha f) - B(f_x)| \leq |\Delta_\alpha f - f_x|,$$

$$|\partial_\alpha(B(\Delta_\alpha f) - B(f_x))| = |B'(\Delta_\alpha f) \partial_\alpha \Delta_\alpha f| \leq |\partial_\alpha \Delta_\alpha f|.$$

Consequently,

$$|\Theta_\lambda(x, \alpha)| \lesssim \left[ \frac{1}{\alpha^2} + \frac{1}{\alpha \lambda} \left| \chi'(\frac{\alpha}{\lambda}) \right| \right] |\Delta_\alpha f - f_x| + \frac{1}{\alpha} |\partial_\alpha \Delta_\alpha f| \lesssim \frac{1}{\alpha^2} |\Delta_\alpha f - f_x| + \frac{1}{\alpha} |\partial_\alpha \Delta_\alpha f|,$$

where we have used that $\alpha \sim \lambda$ on the support of $\chi'(\alpha/\lambda)$. Now, verify that

$$\partial_\alpha \Delta_\alpha f = \frac{f(x) - f(x - \alpha) - \alpha f_x(x - \alpha)}{\alpha^2} = -\frac{1}{\alpha^2} \int_0^\alpha \left( f_x(x - y) - f_x(x) \right) \ dy + \frac{1}{\alpha} \left( f_x(x) - f_x(x - \alpha) \right).$$

On the other hand, we have

$$\left| \frac{1}{\alpha^2} |\Delta_\alpha f - f_x| \leq \frac{1}{\alpha^2} \left| \frac{1}{\alpha} \int_0^\alpha |f_x(x - y) - f_x(x)| \ dy \right|.$$
By combining the previous estimates and using \((a + b)^2 \leq 2a^2 + 2b^2\), we conclude
\[
|\alpha|^{\frac{5}{2}} |\Theta_\lambda(x, \alpha)|^2 \lesssim \frac{1}{|\alpha|^{3/2}} \left| \frac{1}{\alpha} \int_0^\alpha |f_x(x - y) - f_x(x)| \, dy \right|^2 + \frac{1}{|\alpha|^{3/2}} |f_x(x) - f_x(x - \alpha)|^2.
\]
Therefore, it follows from Hardy’s inequality (16) that
\[
\int_\mathbb{R} |\alpha|^{\frac{5}{2}} |\Theta_\lambda(x, \alpha)|^2 \, d\alpha \lesssim \int_\mathbb{R} \frac{|\delta_\alpha f_x(x)|^2 \, d\alpha}{|\alpha|^{2}}.
\]
Consequently
\[
\left( \int_\mathbb{R} \left( \int_\mathbb{R} |\alpha|^{\frac{5}{2}} |\Theta_\lambda(x, \alpha)|^2 \, d\alpha \right)^2 \, dx \right)^{\frac{1}{4}} \lesssim \|f_x\|_{\dot{H}^{\frac{1}{4}}_{4,2}}.
\]
By plugging this in (81), we get
\[
\|R_{2,\lambda}\|_{L^2} \leq \frac{1}{\sqrt{\lambda}} \|g\|_{\dot{H}^{\frac{1}{2}}_{4,2}} \|f_x\|_{\dot{H}^{\frac{1}{2}}_{4,2}}.
\]
Then the Sobolev embedding \(\dot{H}^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow \dot{F}^{\frac{1}{4}}_{4,2}(\mathbb{R})\) implies that
\[
\|R_{2,\lambda}\|_{L^2} \leq \frac{1}{\sqrt{\lambda}} \|g\|_{\dot{H}^{\frac{1}{2}}_{4,2}} \|f\|_{\dot{H}^{\frac{1}{2}}_{4,2}}^2.
\]

**Step 2: estimate of \(R_{1,\lambda}\).** Notice that
\[
B(u) - B(v) = \frac{u^2 - v^2}{(1 + u^2)(1 + v^2)} = \frac{(u - v)(u - v + 2v)}{(1 + u^2)(1 + v^2)} \leq \frac{|u - v| + |u - v|^2}{\langle u \rangle^2}.
\]
Then, since \(E_\alpha(f) = \Delta_\alpha f - f_x\),
\[
|R_{1,\lambda}(x)| \lesssim \int_{|\alpha| \leq 2\lambda} |g(x - \alpha)| \frac{|E_\alpha(f)(x)|^2 + |E_\alpha(f)(x)| \, d\alpha}{\langle f_x(x) \rangle^2 |\alpha|}.
\]
Now introduce the function \(\tilde{g} = |g_x|/\langle f_x \rangle\). Directly from the previous inequality, we have
\[
|\langle f_x(x) \rangle R_{1,\lambda}(x)| \lesssim \int_{|\alpha| \leq 2\lambda} \langle f_x(x - \alpha) \rangle |\tilde{g}(x - \alpha)| \frac{|E_\alpha(f)(x)|^2 + |E_\alpha(f)(x)| \, d\alpha}{\langle f_x(x) \rangle} |\alpha|.
\]
Since
\[
\frac{\langle f_x(x - \alpha) \rangle}{\langle f_x(x) \rangle} \leq 1 + \frac{\langle f_x(x - \alpha) \rangle}{\langle f_x(x) \rangle} - 1 \lesssim 1 + |f_x(x) - f_x(x - \alpha)|,
\]
we have
\[ \| \langle f_x(x) \rangle R_{1,\lambda}(x) \|_{L^2} \lesssim \int_{|\alpha| \leq 2\lambda} |\tilde{g}(x - \alpha)| W_\alpha(x) \frac{d\alpha}{|\alpha|}. \]

with
\[ W_\alpha(x) = (1 + |\delta_\alpha f_x(x)|) \left( |E_\alpha(f)(x)|^2 + |E_\alpha(f)(x)| \right). \]

Using Hölder’s inequality, we infer that
\[ \| \langle f_x(x) \rangle R_{1,\lambda}(x) \|_{L^2} \lesssim \left( |D|^{-\frac{1}{4}} (\tilde{g}^3(x)) \right)^{\frac{3}{2}} \left( \int_{|\alpha| \leq 2\lambda} W_\alpha(x)^3 \frac{d\alpha}{|\alpha|^2} \right)^{\frac{1}{3}}. \]

Using successively the Hölder inequality, the boundedness of Riesz potentials on Lebesgue spaces (see (17)) and the Minkowski’s inequality, we obtain that
\[ \int_{\mathbb{R}} W_\alpha(x)^6 \, dx \lesssim \int_{\mathbb{R}} |E_\alpha(f)(x)|^6 \, dx + \int_{\mathbb{R}} |E_\alpha(f)(x)|^{18} \, dx + \int_{\mathbb{R}} |\delta_\alpha f_x(x)|^{18} \, dx, \]
we conclude that
\[ \int_{\mathbb{R}} W_\alpha(x)^6 \, dx \lesssim \| E_\alpha(f) \|_{H^{\frac{3}{2}}}^6 + \| E_\alpha(f) \|_{H^{\frac{9}{4}}}^{18} + \| \delta_\alpha(f_x) \|_{H^{\frac{9}{4}}}^{18}. \]

Therefore,
\[ \| \langle f_x(x) \rangle R_{1,\lambda} \|_{L^2} \lesssim \| \tilde{g} \|_{L^2} \left( \int_{|\alpha| \leq 2\lambda} \left[ \| E_\alpha(f) \|_{H^{\frac{3}{2}}}^3 + \| E_\alpha(f) \|_{H^{\frac{9}{4}}}^9 + \| \delta_\alpha(f_x) \|_{H^{\frac{9}{4}}}^9 \right] \frac{d\alpha}{|\alpha|^2} \right)^{\frac{1}{3}}. \]

We are now in position to apply Lemma 2.6, which implies that
\[ \| \langle f_x(x) \rangle R_{1,\lambda} \|_{L^2} \lesssim \| \tilde{g} \|_{L^2} \left( 1 + \| f \|_{H^{\frac{3}{2}}}^2 \right) \| f \|_{H^{\frac{3}{2}}}. \]

This completes the proof. □
5.3. Commutator with the nonlinearity. Eventually, we want to estimate the term \((III)\) which appears in Corollary 4.4 (this the most delicate step). The next proposition contains the key estimate which will allow us to commute arbitrary weighted fractional Laplacians with \(T(f)\).

**Proposition 5.8.** Assume that \(\phi\) is as defined in (36) for some admissible weight \(\kappa\). Then, for any \(\lambda > 0\) and any smooth functions \(f\) and \(g\), there holds

\[
\left\| \left[ |D|^{1, \phi}, T(f) \right](g), h \right\|
\leq \left(1 + \| |D|^{3, \phi} f \|_{L^2}^5 \right) \left( \| |D|^{3, \phi} g \|_{L^2} \right) + \| |D|^{3, \phi} f \|_{L^2} \| g_{xx} \|_{L^2} \| h \|_{L^2}.
\]

(83)

**Proof.** In view of the obvious inequality

\[
\left\| \left[ |D|^{1, \phi}, T(f) \right] g, h \right\| \leq \| f \|_{L^2} \left\| |D|^{1, \phi}, T(f) \right\| g \| h \|_{L^2},
\]

it is enough to prove that

\[
\| f \|_{L^2} \left\| |D|^{1, \phi}, T(f) \right\| g \| h \|_{L^2} \leq \left(1 + \| |D|^{3, \phi} f \|_{L^2}^5 \right) \left( \| |D|^{3, \phi} g \|_{L^2} \right) + \| |D|^{3, \phi} f \|_{L^2} \| g_{xx} \|_{L^2} \| h \|_{L^2}.
\]

(84)

Recall that the operator \(T(f)\) is defined by

\[
T(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\Delta_\alpha g_\alpha) F_\alpha \, d\alpha \quad \text{where} \quad F_\alpha = \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2}.
\]

Let us introduce

\[
\Gamma_\alpha := |D|^{1, \phi} [F_\alpha \Delta_\alpha g_\alpha] - F_\alpha |D|^{1, \phi} [\Delta_\alpha g_\alpha] - \Delta_\alpha g_\alpha |D|^{1, \phi} [F_\alpha].
\]

With this notation, we have

\[
\left[ |D|^{1, \phi}, T(f) \right](g) = -\frac{1}{\pi} \int_{\mathbb{R}} \Delta_\alpha g_\alpha |D|^{1, \phi} F_\alpha \, d\alpha - \frac{1}{\pi} \int_{\mathbb{R}} \Gamma_\alpha \, d\alpha.
\]

By definition of \(T(f)\) we have
\[\| \langle f_x \rangle \left[ |D|^{1.\phi}, T(f) \right] (g) \|_{L^2} \lesssim (I) + (II) \]

where

\[(I) := \| \langle f_x \rangle \int |\Delta_{\alpha} g_x | \left| |D|^{1.\phi} F_{\alpha} \right| \, d\alpha \|_{L^2},\]

\[(II) := \| \langle f_x \rangle \int |\Gamma_{\alpha} | \, d\alpha \|_{L^2} .\]

(85)

**Step 1: estimate of \((I)\).** By Holder’s inequality and Minkowski’s inequality, one has

\[(I) \leq (I_A) \times (I_B) \]

\[(I_A) := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\Delta_{\alpha} g_x (x) | \frac{3}{2} \langle f_x (x) \rangle^{-\frac{3}{2}} |\alpha|^{-\frac{1}{2}} \, d\alpha \right)^{\frac{4}{5}} \, dx \right)^{\frac{1}{6}},\]

\[(I_B) := \left( \int \langle f_x (x) \rangle^{\frac{6}{5}} \left| |D|^{1.\phi} F_{\alpha} (x) \right|^3 |\alpha| \, d\alpha \, dx \right)^{\frac{1}{3}} .\]

Set

\[u(x) = \frac{|g_{xx}(x)|}{\langle f_x (x) \rangle} .\]

Since

\[|\Delta_{\alpha} g_x (x) | \leq \int_0^\alpha u(x - \tau) \langle f_x (x - \tau) \rangle \, d\tau \]

\[\leq \left| \int_0^\alpha u(x - \tau) \langle f_x (x - \tau) \rangle - \langle f_x (x) \rangle \, d\tau \right| + \left| \int_0^\alpha u(x - \tau) \, d\tau \right| \langle f_x (x) \rangle \]

\[\lesssim \int_0^\alpha u(x - \tau) \frac{10}{\tau} \, d\tau \left| \int_0^\alpha |\delta_\tau f_x (x) |^{10} \, d\tau \right|^{\frac{3}{30}} + \int_0^\alpha u(x - \tau) \, d\tau \left| \langle f_x (x) \rangle \right| ,\]

we have

\[|\Delta_{\alpha} g_x (x) | \frac{3}{2} \langle f_x (x) \rangle^{-\frac{3}{2}} \]

\[\lesssim \int_0^\alpha u(x - \tau) \frac{10}{\tau} \, d\tau \left| \int_0^\alpha |\delta_\tau f_x (x) |^{10} \, d\tau \right|^{\frac{3}{30}} + \int_0^\alpha u(x - \tau) \, d\tau \left| \langle f_x (x) \rangle \right|^{\frac{3}{2}} \]

\[\lesssim \int_0^\alpha u(x - \tau) \frac{10}{\tau} \, d\tau \left| \int_0^\alpha |\delta_\tau f_x (x) |^{10} \, d\tau \right|^{\frac{3}{30}} + \int_0^\alpha u(x - \tau) \, d\tau \left| \langle f_x (x) \rangle \right| .\]

Thus, \((I_A)\) is bounded from above by

\[\left( \int_{\mathbb{R}} \left( \int_0^\alpha u(x - \tau) \frac{10}{\tau} \, d\tau \left| |\alpha|^{-\frac{1}{2}} \, d\alpha \right|^4 \right)^{\frac{1}{6}} \right)\]

\[+ \left( \int_{\mathbb{R}} \left( \int_0^\alpha u(x - \tau) \frac{10}{\tau} \, d\tau \left| \int_0^\alpha |\delta_\tau f_x (x) |^{10} \, d\tau \right|^{\frac{3}{30}} |\alpha|^{-\frac{1}{2}} \, d\alpha \right)^4 \right)^{\frac{1}{6}} .\]
This implies that

\[
(I_A) \lesssim \left( \int_{\mathbb{R}} \left( \int_0^\alpha u(x - \tau) \frac{1}{2} d\tau \right) \left| \alpha^{-\frac{1}{2}} d\alpha \right|^4 \right)^{\frac{1}{6}} 
+ \left( \int \left( \int \left| \int_0^\alpha u(x - \tau) \frac{1}{2} d\tau \left| \alpha^{-\frac{1}{3}} d\alpha \right|^{10} \right|^{\frac{10}{9}} \right)^{\frac{1}{10}} d\alpha \right) 
\times \left( \int \left( \int_0^\alpha \left| \delta_\tau f_\alpha(x) \right|^{10} d\tau \right)^{\frac{3}{2}} \left| \alpha^{-2} d\alpha \right| \right)^{\frac{1}{15}}.
\]

Now we use Hardy’s inequality (see (16)) to write, for any function \( u_1 \geq 0, \)

\[
\int_0^\alpha \left| \int_0^{u_1(x - \delta)} d\delta \right|^p \left| \alpha^{-\beta} d\alpha \right| \lesssim \int v_1(x - \alpha)^p |\alpha^{-\beta} d\alpha \quad (\forall \ p \geq 1, \ \forall \beta > 0),
\]

and

\[
\int_0^\alpha \left| \int_0^{\delta_\tau f_\alpha(x)} d\tau \right|^{\frac{3}{2}} \left| \alpha^{-2} d\alpha \right| \lesssim \int |\delta_\alpha f_\alpha(x)|^{15} |\alpha^{-2} d\alpha |.
\]

Using also the representation formula (18), it follows that

\[
(I_A) \lesssim \left\| D^{-\frac{1}{2}} (u_3^{\frac{3}{2}}) \right\|_{L^4}^\frac{2}{3} \left\| D^{-\frac{1}{2}} (u_5^{\frac{5}{3}}) \right\|_{L^6}^\frac{3}{2} \left\| f \right\|_{\dot{W}^{15,15}}.
\]

Now, by using the boundedness of Riesz potentials on Lebesgue spaces (see (17)) and Sobolev inequality,

\[
(I_A) \lesssim \left\| u_3^{\frac{3}{2}} \right\|_{L^4}^\frac{2}{3} + \left\| u_5^{\frac{5}{3}} \right\|_{L^6}^\frac{3}{2} \left\| f \right\|_{\dot{H}^{\frac{3}{2}}}.
\]

By definition of \( u, \) this means that

\[
(I_A) \lesssim (1 + \left\| f \right\|_{\dot{H}^{\frac{3}{2}}}^\frac{3}{2}) \left\| \frac{g_{xx}}{(f_x)} \right\|_{L^2}.
\]

This completes the analysis of \( (I_A). \)

As regards the second integral \( (I_B), \) we will prove that

\[
(I_B) \lesssim \left\| D^\frac{3}{2} f \right\|_{L^2}^2 \left( 1 + \left\| D^\frac{3}{2} f \right\|_{L^2}^2 \right).
\]

Recall that

\[
(I_B) := \left( \int \int (f_\alpha(x))^6 \left| D^\frac{1}{2} F_\alpha(x) \right| |\alpha| d\alpha dx \right)^\frac{1}{3}.
\]
To prove (90), we will bound the integrand, starting from

$$\langle f_x(x) \rangle^2 \left| D^{1, \phi} F_\alpha(x) \right| \lesssim \langle f_x(x) \rangle^2 \left| D^{1, \phi} F_\alpha(x) \right| - 2 \frac{\Delta_\alpha f(x)}{\langle \Delta_\alpha f(x) \rangle^4} \left| D^{1, \phi} \Delta_\alpha f(x) \right|$$

$$+ \frac{\langle f_x(x) \rangle^2}{\langle \Delta_\alpha f(x) \rangle^2} \left| D^{1, \phi} \Delta_\alpha f(x) \right|.$$

Then, we want to use the formula (40), which states that

$$|D^{1, \phi} U(x) = \frac{1}{4} \int_{\mathbb{R}} 2U(x) - U(x + z) - U(x - z) \frac{1}{z^2} \kappa \left( \frac{1}{|z|} \right) dz.$$

For the sake of shortness, set

$$d\mu(z) = \kappa \left( \frac{1}{|z|} \right) \frac{dz}{z^2},$$

and observe that $2U(x) - U(x + z) - U(x - z) = \delta_z \delta_{-z} U(x)$. By combining this with the fact that $E_\alpha(f) = \Delta_\alpha f - f_x$ and the elementary inequality

$$\langle f_x(x) \rangle^2 \lesssim \langle \Delta_\alpha f(x) \rangle^2 (1 + |E_\alpha(f)(x)|^2),$$

we deduce that

$$\langle f_x(x) \rangle^2 \left| D^{1, \phi} F_\alpha(x) \right|$$

$$\lesssim (1 + |E_\alpha(f)(x)|^2) \int \langle \Delta_\alpha f(x) \rangle^2 \left| \delta_z \delta_{-z} F_\alpha(x) - 2 \frac{\Delta_\alpha f(x)}{\langle \Delta_\alpha f(x) \rangle^4} \delta_z \delta_{-z} \Delta_\alpha f(x) \right| d\mu(z)$$

$$+ (1 + |E_\alpha(f)(x)|^2) \left| D^{1, \phi} \Delta_\alpha f(x) \right|.$$

To estimate the latter integral, we pause to establish an elementary inequality which will allow us to bound the term

$$\delta_z \delta_{-z} F_\alpha(x) - 2 \frac{\Delta_\alpha f(x)}{\langle \Delta_\alpha f(x) \rangle^4} \delta_z \delta_{-z} \Delta_\alpha f(x).$$

**Lemma 5.9.** There exists a constant $C > 0$ such that, for any triple of real numbers $(x_1, x_2, x_3)$, the quantity

$$\Sigma(x_1, x_2, x_3) = \frac{2x_1^2}{1 + x_1^2} - \frac{x_2^2}{1 + x_2^2} - \frac{x_3^2}{1 + x_3^2}$$

satisfies

$$\left| \Sigma(x_1, x_2, x_3) - \frac{2x_1}{\langle x_1 \rangle^4} (2x_1 - x_2 - x_3) \right|$$

$$\leq \frac{C}{\langle x_1 \rangle^2} \left( |x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_1 - x_2|^3 + |x_1 - x_3|^3 \right).$$  (91)
Proof. By an elementary calculation, one verifies that
\[ \Sigma(x_1, x_2, x_3) = \frac{1}{(x_1)^2 (x_2)^2 (x_3)^2} \left( (1 + x_2^2)(x_1^2 - x_3^2) + (1 + x_3^2)(x_1^2 - x_2^2) \right). \]
This yields the identity
\[
\Sigma(x_1, x_2, x_3) - \frac{2x_1}{(x_1)^4} (2x_1 - x_2 - x_3)
= \frac{\sigma_1(x_1, x_2, x_3)}{(x_1)^2 (x_2)^2 (x_3)^2} + \left( \frac{x_2 + x_1}{(x_1)^2 (x_2)^2} - \frac{2x_1}{(x_1)^4} \right) (2x_1 - x_2 - x_3)
= \frac{\sigma_1(x_1, x_2, x_3)}{(x_1)^2 (x_2)^2 (x_3)^2} + \frac{(x_2 - x_1)(2x_1 - x_2 - x_3)}{(x_1)^2 (x_2)^2}
\]
\[
+ \frac{2x_1(x_1 + x_2)(x_1 - x_2)(2x_1 - x_2 - x_3)}{(x_1)^4 (x_2)^2},
\]
with
\[ \sigma_1(x_1, x_2, x_3) = (x_1 x_2 + x_2 x_3 + x_3 x_1 - 1)(x_2 - x_3)(x_1 - x_3). \]
Now we apply the triangle inequality together with $2ab \leq a^2 + b^2$ to get
\[ |x_1 x_2 + x_2 x_3 + x_3 x_1 - 1| \lesssim (1 + |x_1 - x_2|)(x_2)^2 (x_3)^2, \]
together with $|x_1(x_1 + x_2)| \lesssim (x_1)^2 (x_2)^2$. As a result, we have
\[
\left| \Sigma(x_1, x_2, x_3) - \frac{2x_1}{(x_1)^4} (2x_1 - x_2 - x_3) \right|
\lesssim \frac{1}{(x_1)^2} (1 + |x_1 - x_2|) |x_2 - x_3| |x_1 - x_3| + \frac{1}{(x_1)^2} |x_2 - x_1| |2x_1 - x_2 - x_3|,
\]
which implies (91). This completes the proof. \(\square\)

Consequently we deduce,
\[
\langle f(x) \rangle^2 \langle D \rangle^{1,\phi} F_\alpha(x) \lesssim \left( 1 + |E_\alpha(f)(x)|^2 \right) \int \left( |\delta_z \Delta_\alpha f(x)|^2 + |\delta_z \Delta_\alpha f(x)|^3 \right) d\mu(z)
+ \left( 1 + |E_\alpha(f)(x)|^2 \right) \langle D \rangle^{1,\phi} \Delta_\alpha f(x).
\]
Thus,
\[
(I_B)^3 = \iint \langle f(x) \rangle^6 \langle |D|^{1,\phi} F_\alpha(x) \rangle^3 |\alpha| \, d\alpha \, dx
\lesssim \iint \left( \int \left( 1 + |E_\alpha(f)(x)|^2 \right) |\delta_z \Delta_\alpha f(x)|^2 d\mu(z) \right)^3 |\alpha| \, d\alpha \, dx
+ \iint \left( \int \left( 1 + |E_\alpha(f)(x)|^2 \right) |\delta_z \Delta_\alpha f(x)|^3 d\mu(z) \right)^3 |\alpha| \, d\alpha \, dx
+ \iint \left( 1 + |E_\alpha(f)(x)|^6 \right) \langle |D|^{1,\phi} \Delta_\alpha f(x) \rangle^3 |\alpha| \, d\alpha \, dx.
\]
Then, by using the Minkowski’s inequality and Holder’s inequality,

\[
(I_B) \lesssim \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^6}^6 |\alpha| \, d\alpha \right)^{\frac{1}{3}} \, d\mu(z) + \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^9}^9 |\alpha| \, d\alpha \right)^{\frac{1}{3}} \, d\mu(z)
\]

\[
+ \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^3}^{12} |\alpha|^4 \, d\alpha \right)^{\frac{1}{6}} \, d\mu(z) \left( \int \| E_\alpha(f) \|_{L^3}^{12} \frac{d\alpha}{\alpha^2} \right)^{\frac{1}{6}}
\]

\[
+ \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^3}^{18} |\alpha|^4 \, d\alpha \right)^{\frac{1}{6}} \, d\mu(z) \left( \int \| E_\alpha(f) \|_{L^9}^{12} \frac{d\alpha}{\alpha^2} \right)^{\frac{1}{6}}
\]

\[
+ \left( \int \| D^{1,\phi} \Delta_\alpha f \|_{L^3}^3 |\alpha| \, d\alpha \right)^{\frac{1}{3}}
\]

\[
+ \left( \int \| D^{1,\phi} \Delta_\alpha f \|_{L^6}^6 |\alpha|^4 \, d\alpha \right)^{\frac{1}{6}} \left( \int \| E_\alpha(f) \|_{L^3}^{12} \frac{d\alpha}{\alpha^2} \right)^{\frac{1}{6}}
\].

By Sobolev’s inequality and Lemma 2.6, we have

\[
\int \left( \int \| \Delta_\alpha \delta_z f \|_{L^6}^6 |\alpha| \, d\alpha \right)^{\frac{1}{3}} \, d\mu(z) \lesssim \int \left( \int \| \Delta_\alpha \delta_z |D^{1,\phi} f \|_{L^3}^6 |\alpha| \, d\alpha \right)^{\frac{1}{3}} \, d\mu(z)
\]

\[
\lesssim \int \int |\xi|^2 |F(\delta_z f)(\xi)|^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{z^2}
\]

\[
\lesssim \int \int |\xi|^2 \min\{|z|, |\xi|, 1\}^2 |\hat{f}(\xi)|^2 \kappa \left( \frac{1}{|z|} \right) \frac{dz}{z^2}
\]

\[
\lesssim \| D \|^2 \phi f \|_{L^2}^2,
\]

where we have used again the equivalence \( \phi \sim \kappa \) to obtain the last inequality. By the same token, we have

\[
\int \left( \int \| \Delta_\alpha \delta_z f \|_{L^9}^9 |\alpha| \, d\alpha \right)^{\frac{1}{3}} \, d\mu(z) \lesssim \int \left( \int |\xi|^\frac{7}{3} \min\{||\xi||z|, 1\}^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{2}{3}} \, d\mu(z)
\]

\[
\lesssim \left( \int |\xi|^\frac{3}{5} \kappa(|\xi|)^\frac{2}{3} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{2}{3}} \tag{14}, \tag{39}
\]

\[
\lesssim \| D \|^2 \phi f \|_{L^2}^3.
\]

Similarly, we also obtain that one can estimate by \( \| D \|^2 \phi f \|_{L^2} \) the five following quantities:

\[
\left( \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^{12}}^{12} |\alpha|^4 \, d\alpha \right)^{\frac{1}{6}} \, d\mu(z) \right)^{\frac{2}{3}}, \quad \left( \int \| D^{1,\phi} \Delta_\alpha f \|_{L^3}^3 |\alpha| \, d\alpha \right)^{\frac{1}{3}},
\]

\[
\left( \int \left( \int \| \Delta_\alpha \delta_z f \|_{L^{18}}^{18} |\alpha|^4 \, d\alpha \right)^{\frac{1}{6}} \, d\mu(z) \right)^{\frac{3}{3}} \quad \left( \int \| D^{1,\phi} \Delta_\alpha f \|_{L^6}^6 |\alpha|^4 \, d\alpha \right)^{\frac{1}{12}}
\]

\[
\left( \int \| E_\alpha(f) \|_{L^{12}}^{12} \frac{d\alpha}{\alpha^2} \right)^{\frac{1}{6}}.
\]
Thus,

\[(I_B) \lesssim \|D \frac{3}{2} \phi f\|_{L^2}^2 \left( 1 + \|D \frac{3}{2} \phi f\|_{L^2}^3 \right),\]

which proves (90).

By combining (88) and (90), we conclude that the term (I) which appears in (85) satisfies

\[(I) \lesssim \|D \frac{3}{2} \phi f\|_{L^2}^2 \left( 1 + \|D \frac{3}{2} \phi f\|_{L^2}^4 \right) \frac{g_{xx}}{\langle f_x \rangle L^2},\]

which completes the analysis of (I).

Step 2.2: estimate of (II). It remains to estimate the second term in (85). We want to prove that

\[\langle f_x \rangle |\Gamma_\alpha(x)| \lesssim \langle \delta_\alpha F_\alpha(x) \rangle |\delta_\alpha \Gamma_\alpha(x)\rangle \kappa \left( \frac{1}{|z|} \right) \frac{dz}{z^2}.\]

Using again the estimate (see (82))

\[\left| \frac{x_1^2}{1 + x_1^2} - \frac{x_2^2}{1 + x_2^2} \right| \lesssim \frac{1}{1 + x_1^2} \left( |x_1 - x_2|^2 + |x_1 - x_2| \right),\]

we have

\[|\delta z F_\alpha(x)| \lesssim \langle \Delta_\alpha f(x) \rangle^{-2} |\delta z \Delta_\alpha f(x)\rangle^2 + \langle \Delta_\alpha f(x) \rangle^{-2} |\delta z \Delta_\alpha f(x)\rangle.\]

Combining this with

\[\langle f_x(x) \rangle^2 \langle \Delta_\alpha f(x) \rangle^{-2} \lesssim 1 + |E_\alpha(f)(x)|^2,\]

one gets

\[\langle f_x(x) \rangle |\Gamma_\alpha(x)| \lesssim \left( 1 + |f_x(x) - \Delta_\alpha f(x)|^2 \right) \langle f_x(x) \rangle^{-1} |\delta z \Delta_\alpha f(x)\rangle \times \times (1 + |\delta z \Delta_\alpha f(x)|) |\delta z \Delta_\alpha g_{\alpha x}(x)| \kappa \left( \frac{1}{|z|} \right) \frac{dz}{z^2}.\]

Set

\[u(x) = \frac{|f_{xx}(x)|}{\langle f_x(x) \rangle}.\]
One has
\[
|\delta_z \Delta_\alpha f(x)| \leq \frac{1}{|\alpha|} \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) \langle f_x(x - \tau - z_1) \rangle \, dz_1 \, d\tau \right|
\]
\[
\leq \frac{1}{|\alpha|} \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) |\langle f_x(x - \tau - z_1) \rangle - \langle f_x(x - \tau) \rangle| \, dz_1 \, d\tau \right|
+ \frac{1}{|\alpha|} \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) |\langle f_x(x - \tau) \rangle - \langle f_x(x) \rangle| \, dz_1 \, d\tau \right|
+ \frac{\langle f_x(x) \rangle}{|\alpha|} \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) \, dz_1 \, d\tau \right|.
\]

Then, thanks to \(|\langle x_1 \rangle - \langle x_2 \rangle| \lesssim |x_1 - x_2|\), we infer that
\[
\langle f_x(x) \rangle^{-1} |\delta_z \Delta_\alpha f(x)| \lesssim |z| \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) |\delta_z f_x(x - \tau)| \, dz_1 \, d\tau \right|
+ |z| \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) |\delta_\tau f_x(x)\rangle | \, dz_1 \, d\tau \right|
\]
\[
\lesssim |z| \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) \frac{10}{\Psi} \, dz_1 \, d\tau \right|^\frac{9}{10} J_\alpha(x, z),
\]
where
\[
J_\alpha(x, z) = 1 + \left| \int_0^\alpha |\delta_\tau f_x(x)|^{10} \, d\tau \right|^\frac{1}{10}
+ \left| \int_0^\alpha \int_0^z |\delta_z f_x(x - \tau)|^{10} \, dz_1 \, d\tau \right|^\frac{1}{10}.
\]

Set
\[
\Phi_\alpha(x, z) = J_\alpha(x, z) \left( 1 + |f_x(x) - \Delta_\alpha f(x)|^2 \right) (1 + |\delta_z \Delta_\alpha f(x)|) |\delta_z \Delta_\alpha g_\alpha(x)|.
\]

Then, repeating arguments already used before, we find that
\[
\langle f_x(x) \rangle \int \left| \Gamma_\alpha(x) \right| \, d\alpha
\]
\[
\lesssim \int_{|\alpha| \leq \lambda} \left| \int_0^\alpha \int_0^z u(x - \tau - z_1) \frac{10}{\Psi} \, dz_1 \, d\tau \right|^\frac{9}{10} \Phi_\alpha(x, z) \kappa \left( \frac{1}{|z|} \right) \left| \alpha \right|^{\frac{dz}{\alpha}} \, d\alpha
\]
\[
\lesssim \left( \int_{|\alpha| \leq \lambda} \left| \alpha \right|^{\frac{1}{18}} |z|^{\frac{1}{18}} \int_0^\alpha \int_0^z u(x - \tau - z_1) \frac{10}{\Psi} \, dz_1 \, d\tau \right)^{\frac{9}{10}} \frac{dz \, d\alpha}{|z|^{\frac{1}{18}} |\alpha|^{\frac{17}{18}}}
\]
\[
\times \left( \int \Phi_\alpha(x, z)^{10} \kappa \left( \frac{1}{|z|} \right) \left| \alpha \right|^{\frac{17}{18}} \frac{dz \, d\alpha}{|z|^{\frac{1}{18}} |\alpha|^{\frac{17}{18}}} \right)^{\frac{1}{10}}.
\]

By Hardy’s inequality (see (16)),
\[
\int \left| \alpha \right|^{\frac{1}{18}} |z|^{\frac{1}{18}} \int_0^\alpha \int_0^z u(x - \tau - z_1) \frac{10}{\Psi} \, dz_1 \, d\tau \, dz \, d\alpha
\]
\[
\lesssim \int \left| \alpha \right|^{\frac{1}{18}} |z|^{\frac{1}{18}} \int_0^\alpha u(x - \alpha - z) \frac{10}{\Psi} \, dz \, d\alpha
\]
\[
= c_1 \int \left| \alpha \right|^{\frac{1}{18}} |D|^{\frac{1}{18}} (u^{10}) (x - \alpha) \frac{d\alpha}{|\alpha|} = c_2 |D|^{1 - \frac{1}{\Psi}} (u^{10})(x).
\]
Consequently we deduce,

\[
\langle f_x(x) \rangle \int |\Gamma_\alpha(x)| \, d\alpha \lesssim \left( |D|^{-\frac{1}{9}} (u^{10})_{\frac{10}{39}}(x) \right)^{\frac{9}{10}} \left( \iint \Phi_\alpha(x, z) 10^k \frac{1}{|z|} \left| \alpha \right|^{\frac{17}{2}} \frac{dz \, d\alpha}{|z|^\frac{3}{2}} \right)^\frac{1}{10}.
\]

By Cauchy-Schwarz, Minkowski and Hardy-Littlewood-Sobolev,

\[
\left\| \langle f_x \rangle \int |\Gamma_\alpha| \, d\alpha \right\|_{L^2} \lesssim \left\| \left( |D|^{-\frac{1}{9}} (u^{10})_{\frac{10}{39}}(x) \right)^{\frac{9}{10}} \left( \iint \Phi_\alpha(\cdot, z) 10^k \frac{1}{|z|} \left| \alpha \right|^{\frac{17}{2}} \frac{dz \, d\alpha}{|z|^\frac{3}{2}} \right)^\frac{1}{10} \right\|_{L^{10}} \lesssim \left\| u \right\|_{L^2} \left( \iint \left| \Phi_\alpha(\cdot, z) \right| 10^k \frac{1}{|z|} \left| \alpha \right|^{\frac{17}{2}} \frac{dz \, d\alpha}{|z|^\frac{3}{2}} \right)^\frac{1}{10}.
\]

We now estimate the $L^{10}$-norm of $\Phi_\alpha(\cdot, z)$. Note that

\[
|\delta_z \Delta_\alpha f(x)| + |f_x(x) - \Delta_\alpha f(x)| \lesssim J_\alpha(x, z).
\]

So,

\[
\Phi_\alpha(x, z) \lesssim J_\alpha(x, z) + |z|^4 |\delta_z \Delta_\alpha g_x(x)| \lesssim |\alpha|^{-1} |\delta_z \delta_\alpha g_x(x)| \left( 1 + \int_0^\alpha |\delta_\tau f_x(x)|^{10} \, d\tau \right)^\frac{4}{10} + \left( \int_0^\alpha \int_0^{z_1} |\delta_{z_1} f_x(x - \tau)|^{10} \, dz_1 \, d\tau \right)^\frac{4}{10}.
\]

Then,

\[
\left| \alpha \right|^{10} \left\| \Phi_\alpha(\cdot, z) \right\|_{L^{10}} \lesssim \left\| \delta_z \delta_\alpha g_x \right\|_{L^{10}}^{10} + \left\| \delta_z \delta_\alpha g_x \right\|_{L^{20}}^{10} \left( \int_0^{z_1} \left| \delta_{z_1} f_x \right|_{L^{80}}^{40} \, dz_1 + \int_0^\alpha \left| \delta_\tau f_x \right|_{L^{80}}^{40} \, d\tau \right) \lesssim \left\| \delta_z \delta_\alpha g_x \right\|_{H^{\frac{5}{2}}}^{10} + \left\| \delta_z \delta_\alpha g_x \right\|_{H^{\frac{20}}}{\frac{9}{50}} \left( \int_0^{z_1} \left| \delta_{z_1} f_x \right|_{H^{\frac{80}}}{\frac{39}{80}} \, dz_1 + \int_0^\alpha \left| \delta_\tau f_x \right|_{H^{\frac{80}}}{\frac{39}{80}} \, d\tau \right).}
\]
Thus,
\[
\int \left\| \Phi_\alpha(\cdot, z) \right\|_{L^2_{10}}^{10} \kappa \left( \frac{1}{|z|} \right)^{10} |\alpha|^{\frac{17}{2}} \frac{d\alpha}{|z|^{\frac{3}{2}}} \\
\leq \int \left[ \int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \kappa \left( \frac{1}{|z|} \right)^{10} \frac{d\alpha}{|z|^{\frac{3}{2}}} \right] \frac{dz}{|z|^{\frac{3}{2}}} \frac{d\alpha}{|z|^{\frac{3}{2}}} \\
+ \int \left[ \int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \frac{d\alpha}{|z|^{\frac{3}{2}}} \right] \int_0^z \left\| \delta_{z_1} f_x(\cdot) \right\|_{\dot{H}^{\frac{39}{30}}}^{40} \frac{dz_1}{|z_1|} \kappa \left( \frac{1}{|z|} \right)^{10} \frac{d\alpha}{|z|^{\frac{3}{2}}} \\
+ \int \left[ \int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \frac{d\alpha}{|z|^{\frac{3}{2}}} \right] \int_0^\alpha \left\| \delta_\tau f_x(\cdot) \right\|_{\dot{H}^{\frac{39}{30}}}^{40} \frac{d\tau}{|\alpha|^{\frac{3}{2}}} \frac{d\alpha}{|\alpha|^{\frac{3}{2}}}.
\]

Proceeding as in the proof of Lemma 2.6, we get
\[
\int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \kappa \left( \frac{1}{|z|} \right)^{10} \frac{dz}{|z|^{\frac{3}{2}}} \lesssim \left( \int \left\| \xi \right\|_{\dot{H}^{\frac{20}{3}}}^{20} \kappa \left( \frac{1}{|z|} \right)^{2} |\mathcal{F}(\delta_\alpha g)(\xi)|^2 \, d\xi \right)^{5} \\
\lesssim \left\| \delta_\alpha \right\|_{L^2} \left\| D^{\frac{20}{39}} \phi g \right\|_{L^2}^{10},
\]
\[
\int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \kappa \left( \frac{1}{|z|} \right)^{10} \frac{dz}{|z|^{\frac{3}{2}}} \lesssim \left( \int \left\| \xi \right\|_{\dot{H}^{\frac{39}{30}}}^{39} \kappa \left( \frac{1}{|z|} \right)^{2} |\mathcal{F}(\delta_\alpha g)(\xi)|^2 \, d\xi \right)^{5} \\
\lesssim \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10},
\]
and
\[
\int \left\| \delta_\alpha \delta_z g_x(\cdot) \right\|_{\dot{H}^{\frac{10}{3}}}^{10} \frac{d\alpha}{|\alpha|^{\frac{3}{2}}} \lesssim \left( \int \left\| \xi \right\|_{\dot{H}^{\frac{39}{30}}}^{39} \min\{|z|, |\xi|, 1\}^2 |\mathcal{F} g(\xi)|^2 \, d\xi \right)^{5} \\
\lesssim \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10} \kappa \left( \frac{1}{|z|} \right)^{-10}.
\]
Here we have used the fact that \(\min\{|z|, |\xi|, 1\}^2 \lesssim \phi(|\xi|)^2 \kappa \left( \frac{1}{|z|} \right)^{-2}\) in the last inequality.

Therefore, by applying Lemma 2.6 together with Hardy and Minkowski inequalities, we obtain
\[
\int \left\| \Phi_\alpha(\cdot, z) \right\|_{L^{10}}^{10} \kappa \left( \frac{1}{|z|} \right)^{10} |\alpha|^{\frac{17}{2}} \frac{d\alpha}{|z|^{\frac{3}{2}}} \\
\lesssim \int \left\| \delta_\alpha \right\|_{L^2} \left\| D^{\frac{20}{39}} \phi g \right\|_{L^2}^{10} \frac{d\alpha}{|\alpha|^{\frac{3}{2}}} + \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10} \int \left\| \delta_\alpha \right\|_{L^2} \left\| D^{\frac{40}{39}} \phi g \right\|_{L^2} \frac{dz}{|z|^{\frac{3}{2}}} \\
+ \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10} \int \left\| \delta_\alpha \right\|_{L^2} \left\| D^{\frac{40}{39}} \phi g \right\|_{L^2} \frac{d\alpha}{|\alpha|^{\frac{3}{2}}} \\
\lesssim \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10} + \left\| D^{\frac{3}{2}} \phi g \right\|_{L^2}^{10} \| f \|_{H^{\frac{3}{2}}}^{40}. \]
The proof of (92) is now in reach. Indeed, altogether, the previous estimates imply that
\[
\int \Phi_\alpha (\cdot, z) \|L_{10}\kappa \left( \frac{1}{|z|} \right)^{10} |\alpha|^{1/2} \frac{d\alpha dz}{|z|^{3/2}} \lesssim \|D\|^{3/2} \|g\|_{L^2} \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^{40} \right).
\]
Therefore,
\[
\left\| \langle f_x \rangle \int \Gamma_\alpha \, d\alpha \right\|_{L^2} \lesssim \|u\|_{L^2} \|D\|^{3/2} \|g\|_{L^2} \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^4 \right)
\]
equivalent to the wanted result (92).
This completes the proof. □

6. Proof of the Main Theorem

6.1. The main estimate. By combining Corollary 4.4 with Propositions 5.1, 5.6 and 5.8, we obtain our main weighted Sobolev energy estimate.

**Proposition 6.1.** Consider a function \( \phi \) defined by (36), for some admissible weight \( \kappa \). Then, the following property holds: For all \( \varepsilon \in (0, 1) \), for all smooth solution \( f \) of the approximate Muskat Eq. (52), and for all positive function \( \lambda = \lambda(t) \),
\[
\frac{d}{dt} \|D\|^{3/2} \|f\|_{L^2}^2 + \int_\mathbb{R} \frac{|D|^{2/3} f|^2}{1 + (\partial_x f)^2} \, dx + \varepsilon \|f\|_{H^4}^2
\]
\[
\lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^7 \right) \|D\|^{3/2} \|f\|_{L^2} \int_\mathbb{R} \frac{|D|^{2/3} f|^2}{1 + (\partial_x f)^2} \, dx
\]
\[
+ \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^5 \right) \|D\|^{3/2} \|f\|_{L^2} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{|D|^{2/3} f}{\langle f_x \rangle} \right\|_{L^2}
\]
\[
+ \frac{1}{\sqrt{\lambda}} \left( 1 + \|D\|^{3/2} \|f\|_{L^2} \right) \|D\|^{3/2} \|f\|_{L^2} \|D\|^{2/3} \|f\|_{L^2}.
\]

**Proof.** It follows from Corollary 4.4 with Propositions 5.1, 5.6 and 5.8 that
\[
\frac{d}{dt} \|D\|^{3/2} \|f\|_{L^2}^2 + \int_\mathbb{R} \frac{|D|^{2/3} f|^2}{1 + (\partial_x f)^2} \, dx + \varepsilon \|f\|_{H^4}^2
\]
\[
\lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^7 \right) \|D\|^{3/2} \|f\|_{L^2} \int_\mathbb{R} \frac{|D|^{2/3} f|^2}{1 + (\partial_x f)^2} \, dx
\]
\[
+ \kappa \left( \frac{1}{\lambda} \right)^{-1} \left( 1 + \|D\|^{3/2} \|f\|_{L^2}^5 \right) \|D\|^{3/2} \|f\|_{L^2} \left\| \frac{|D|^{2/3} f}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{\mathcal{H}(|D|^{2/3} f)}{\langle f_x \rangle} \right\|_{L^2}
\]
\[
+ \left( 1 + \|D\|^{3/2} \|f\|_{L^2} \right) \|D\|^{3/2} \|f\|_{L^2} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{|D|^{2/3} f}{\langle f_x \rangle} \right\|_{L^2}
\]
\[
+ \frac{1}{\sqrt{\lambda}} \left( 1 + \|D\|^{3/2} \|f\|_{L^2} \right) \|D\|^{3/2} \|f\|_{L^2} \|D\|^{2/3} \|f\|_{L^2}.
\]
Combining this with Lemma 2.2, which implies that
\[
\int \frac{(\mathcal{H}g)^2}{1 + h^2} \, dx \lesssim \left( 1 + \|h\|_{L^2}^2 \right) \int \frac{g^2}{1 + h^2} \, dx,
\]
on one obtains the wanted result. □
6.2. Uniform global in time estimates under a smallness assumption. Set $\phi = 1$ and let $\lambda$ goes to $+\infty$ in the main estimate given by Proposition 6.1, to obtain
\[
\frac{d}{dt} \| f \|_{\dot{H}^{1/2}}^2 + \int_\mathbb{R} \frac{1}{1+(\partial_x f)^2} \| f \|_{\dot{H}^1}^2 \, dx + \varepsilon \| f \|_{\dot{H}^{1/2}}^2 \leq \left( 1 + \| f \|_{\dot{H}^1}^2 \right) \| f \|_{\dot{H}^{1/2}}^2 \int_\mathbb{R} \frac{1}{1+(\partial_x f)^2} \, dx.
\]
Then it is very easy to prove a uniform estimate under a smallness assumption on the initial data (see for instance [4, Sect. 3.3] or [3, Sect. 2.5]). We obtain the following

**Proposition 6.2.** There exists a positive constant $c$ such that the following property holds. For all initial data $f_0$ in $H^{3/2}(\mathbb{R})$ satisfying
\[
\| f_0 \|_{\dot{H}^{3/2}} \leq c,
\]
and for all $\varepsilon \in (0, 1]$, the solution $f$ to the approximate Cauchy problem (52) satisfies
\[
\sup_{t \in [0, +\infty)} \left\{ \| f(t) \|_{\dot{H}^{3/2}}^2 + \int_0^t \int_\mathbb{R} \frac{(\partial_x f)^2}{1+(\partial_x f)^2} \, dx \, dt \right\} \leq 2 \| f_0 \|_{\dot{H}^{3/2}}^2.
\]

6.3. Uniform local in time estimates for arbitrary initial data. This is the most delicate step which requires to use the full strength of Proposition 6.1. We will prove the following

**Proposition 6.3.** Consider a function $\phi$ defined by (36), for some admissible weight $\kappa$ which in addition satisfies
\[
\lim_{r \to +\infty} \kappa(r) = +\infty.
\]
Then for any $M_0 > 0$ there exists $T_0 > 0$ depending on $M_0, \phi$ such that the following properties holds. For all $\varepsilon \in (0, 1]$ and for all smooth solution $f$ of the approximate Muskat Eq. (52), if
\[
\| f_0 \|^2_{L^2} + \| D)^{3/2, \phi} f_0 \|_{L^2}^2 \leq M_0,
\]
then,
\[
\sup_{t \in [0, T_0]} \left\{ \| f(t) \|_{L^2}^2 + \| D)^{3/2, \phi} f(t) \|_{L^2}^2 \right\} + \int_0^{T_0} \int_\mathbb{R} \frac{|D)^{2, \phi} f(t, x)|^2}{1 + f_x(t, x)^2} \, dx \, dt \leq 4M_0.
\]

**Proof.** We want to estimate the following quantities:
\[
A(t) = \| f(t) \|_{L^2}^2 + \| D)^{3/2, \phi} f(t) \|_{L^2}^2, \quad B(t) = \int_\mathbb{R} \frac{|D)^{2, \phi} f(t, x)|^2}{1 + (\partial_x f(t, x))^2} \, dx.
\]
As already seen in the proof of Proposition 4.2, the $L^2$-norm is decreasing:
\[
\| f(t) \|_{L^2}^2 \leq \| f_0 \|_{L^2}^2.
\]

(95)
On the other hand, notice that we have the obvious bound \( \| f \|_{H^\frac{3}{2}} \leq \sqrt{A} \). Hence, the main estimate, as given by Proposition 6.1, implies that

\[
\frac{d}{dt} A + B 
\leq \kappa \left( \frac{1}{\lambda} \right)^{-1} (1 + A)^4 B + (1 + A)^3 B^\frac{1}{2} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} + \frac{1}{\sqrt{\lambda}} (1 + A)^\frac{3}{2} \left\| D^{2.\phi} f \right\|_{L^2}.
\]

(96)

Our next task presents itself: we must estimate

\[
\left\| D^{2.\phi} f \right\|_{L^2}, \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2}
\]

(97)
in terms of \( A \) and \( B \).

To estimate the three other terms in (97), we shall use some interpolation inequalities adapted to our problem. We begin with the following

**Lemma 6.4.** The following inequalities hold:

\[
\| f_x \|_{L^\infty} \lesssim (1 + A) \log (2 + B) \frac{1}{2},
\]

(98)

\[
\left\| D^{2.\phi} f \right\|_{L^2} \lesssim (1 + A) \log (2 + B) \frac{1}{2} B^\frac{1}{2}.
\]

(99)

**Proof.** Let us prove (98). Recall the classical interpolation inequality

\[
\| f_x \|_{L^\infty} \lesssim (1 + A) \log (2 + \| f \|_{H^3}) \frac{1}{2} \| f \|_{H^\frac{3}{2}}.
\]

(100)

For the sake of completeness, recall that (100) is proved by writing

\[
\| f_x \|_{L^\infty} \leq \int |\xi| \left| \hat{f}(\xi) \right| d\xi 
\]

\[
\lesssim \| f \|_{L^2} + \int_{1 \leq |\xi| \leq \lambda_0} |\xi| \left| \hat{f}(\xi) \right| d\xi + \int_{|\xi| \geq \lambda_0} |\xi| \left| \hat{f}(\xi) \right| d\xi 
\]

\[
\lesssim \| f \|_{L^2} + (\log \lambda_0)^\frac{1}{2} \| f \|_{H^\frac{3}{2}} + \lambda_0^{-\frac{1}{2}} \| f \|_{H^2},
\]

and then choosing \( \lambda_0 = (2 + \| f \|_{H^2})^2 \).

On the other hand, directly from the definition of \( B = \int_{\mathbb{R}} \left| D^{2.\phi} f(x) \right|^2 \frac{1}{1 + f_x(x)^2} \ dx \), we have

\[
\| f \|_{H^2} \lesssim \left\| D^{2.\phi} f \right\|_{L^2} \leq (1 + \| f_x \|_{L^\infty}) B^\frac{1}{2} \leq 1 + \| f_x \|_{L^\infty}^2 + B.
\]

(101)

By combining (100) and (101), we obtain

\[
\| f_x \|_{L^n} \lesssim (1 + \| f \|_{H^2}) \log (3 + \| f_x \|_{L^\infty}^2 + B) \frac{1}{2},
\]

which in turn implies that,

\[
\| f_x \|_{L^n} \lesssim (1 + \| f \|_{H^2}^2) \log \left( 2 + \int_{\mathbb{R}} \left| D^{2.\phi} f(x) \right|^2 \frac{1}{1 + f_x(x)^2} \ dx \right) \frac{1}{2}.
\]

This proves (98). Also, by plugging the bound (98) in (101) we get (99). \( \square \)
It follows from (96) and the previous estimate that
\[
\frac{d}{dt} A + B \lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} (1 + A)^4 B + (1 + A)^3 B^{\frac{1}{2}} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} + \frac{1}{\sqrt{\lambda}} (1 + A)^2 \log (4 + B)^{\frac{1}{2}} B^{\frac{1}{2}}.
\]
Choosing \( \lambda = B^{-\frac{1}{2}} \),
\[
\frac{d}{dt} A + B \lesssim \left( 1 + A^4 \right) \left( \log (4 + B)^{\frac{1}{2}} B^{\frac{3}{2}} + B/\kappa (B^{\frac{1}{2}}) \right) + \left( 1 + A^3 \right) B^{\frac{1}{2}} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2}.
\]

Lemma 6.5. There exists a non-decreasing function \( \mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that
\[
\left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \lesssim \left( 1 + A \right) B^{\frac{1}{2}} / \kappa \left( B^{\frac{1}{20}} \right) + \mathcal{G} (A).
\]

Proof. For this proof, the key inequality is given by Proposition 3.6, which implies that, for any \( \sigma \in (0, 1/2) \),
\[
\left\| |D|^{\sigma} \left( \frac{g}{\sqrt{1 + h^2}} \right) - \frac{1}{\sqrt{1 + h^2}} |D|^{\sigma} g \right\|_{L^2} \lesssim \left\| |D|^{\frac{1}{2}} \phi h \right\|_{L^2} \left\| \frac{g}{\sqrt{1 + h^2}} \right\|_{H^\sigma}.
\]

We will work out two applications of the above estimate. Firstly we notice that the previous inequality holds for any function \( \phi \) given by (36), for some admissible weight \( \kappa \). In particular, it holds for \( \phi \equiv 1. \) Since \( \partial_{xx} = - |D|^\frac{1}{2} |D|^\frac{7}{2} \), the latter result implies that
\[
\left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \lesssim \left( 1 + \left\| f \right\|_{\dot{H}^\frac{3}{2}} \right) \left\| |D|^\frac{1}{2} \left( \frac{|D|^\frac{7}{2} f}{\langle f_x \rangle} \right) \right\|_{L^2}.
\]

Secondly, we will also use the fact that (104) implies that
\[
\left\| |D|^\frac{1}{2} \phi \left( \frac{|D|^\frac{7}{2} f}{\langle f_x \rangle} \right) \right\|_{L^2} \lesssim B^{\frac{1}{2}} + \left\| |D|^\frac{3}{2} \phi f \right\|_{L^2} \left\| |D|^\frac{1}{2} \left( \frac{|D|^\frac{7}{2} f}{\langle f_x \rangle} \right) \right\|_{L^2}.
\]

Then, to derive the wanted estimate (103), we must compare the two following quantities:
\[
X = \left\| |D|^\frac{1}{2} \left( \frac{|D|^\frac{7}{2} f}{\langle f_x \rangle} \right) \right\|_{L^2} \quad \text{and} \quad X_\phi = \left\| |D|^\frac{1}{2} \phi \left( \frac{|D|^\frac{7}{2} f}{\langle f_x \rangle} \right) \right\|_{L^2}.
\]

Firstly, we use the elementary inequality: for any \( \lambda > 0 \),
\[
\left\| |D|^\frac{1}{2} u \right\|_{L^2}^2 \lesssim \sqrt{\lambda} \left\| u \right\|_{L^2}^2 + \frac{1}{\kappa (\lambda)^2} \left\| |D|^\frac{1}{2} \phi u \right\|_{L^2}^2,
\]
which is proved as above, by using Plancherel’s identity, dividing the integral into two parts and using the equivalence \( \phi \sim \kappa \). Now the previous estimate implies at once that
\[
X \lesssim X_\phi^\frac{100}{1} \left\| |D|^\frac{7}{2} f \right\|_{L^2} + X_\phi / \kappa \left( X_\phi^\frac{1}{20} \right).
\]
So, it follows from (106) that

\[ X_\phi \lesssim B^{\frac{1}{2}} + X_\phi^{\frac{1}{2}} \| |D|^2 f\|_{L^2} + \| |D|^3 \phi f\|_{L^2} + \| |D|^2 \phi f\|_{L^2} X_\phi / \kappa \left( X_\phi^{\frac{25}{28}} \right). \]

Since \( \| |D|^2 f\|_{L^2}^2 \lesssim \| f\|_{H^\frac{3}{2}}^2 \| f\|_{H^2} \), by using Young’s inequality one infers that

\[ X_\phi \lesssim B^{\frac{1}{2}} + \| f\|_{H^\frac{3}{2}}^\frac{50}{3} \| f\|_{H^\frac{3}{2}}^\frac{50}{3} \| |D|^3 \phi f\|_{L^2}^{\frac{100}{99}} + \| |D|^3 \phi f\|_{L^2} X_\phi / \kappa \left( X_\phi^{\frac{25}{28}} \right). \]

Since \( \kappa(r) \to \infty \) as \( r \to \infty \), we infer that there exists a non-decreasing function \( G: \mathbb{R}_+ \to \mathbb{R}_+ \) so that

\[ X_\phi \lesssim B^{\frac{1}{2}} + \| f\|_{H^\frac{3}{2}}^{\frac{20}{33}} + G \left( \| |D|^3 \phi f\|_{L^2} \right). \]

Combining this with (107), we obtain

\[ X \lesssim B^{\frac{1}{2}} / \kappa \left( B^{\frac{1}{20}} \right) + \| f\|_{H^\frac{3}{2}}^{\frac{20}{33}} + G \left( \| |D|^3 \phi f\|_{L^2} \right), \quad (108) \]

and hence

\[ \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \lesssim \left( 1 + \| f\|_{H^\frac{3}{2}}^\frac{7}{33} \right) B^{\frac{1}{2}} / \kappa \left( B^{\frac{1}{20}} \right) + \| f\|_{H^\frac{3}{2}}^{\frac{7}{33}} + G \left( \| |D|^3 \phi f\|_{L^2} \right). \quad (109) \]

This means that

\[ \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \lesssim \left( 1 + A \right) B^{\frac{1}{2}} / \kappa \left( B^{\frac{1}{20}} \right) + \| f\|_{H^\frac{3}{2}}^{\frac{7}{33}} + G \left( A \right). \]

Then the wanted result follows from the bound (99). This proves the lemma. \( \square \)

It follows from (102) and (103) that

\[ \frac{d}{dt} A + B \leq C \left( 1 + A^4 \right) \left( \log(4 + B)^{\frac{1}{2}} B^{\frac{3}{2}} + B / \kappa (B^{\frac{1}{2}}) + B / \kappa (B^{\frac{1}{20}}) \right) + G \left( A \right). \]

Since \( \lim_{r \to +\infty} \kappa(r) = +\infty \), it follows that, up to modifying the function \( G \), we have

\[ \frac{d}{dt} A + \frac{1}{2} B \leq G \left( A \right). \]

Now the proposition follows at once from a continuity argument. \( \square \)
6.4. Contraction estimate. To conclude the proof we need an estimate for the difference of two solutions. This will serve us to prove uniqueness results as well as to prove that the approximate solutions of the Muskat equations defined above form a Cauchy sequence and hence they converge strongly.

**Proposition 6.6.** Consider a function φ defined by (36), for some admissible weight κ which in addition satisfies

\[ \lim_{r \to +\infty} \kappa(r) = +\infty. \]

Then there exists a non-decreasing function \( G : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the following property holds. For any \( \varepsilon \in [0, 1] \), any \( T \in (0, 1] \) and any couple of solutions \( f_1, f_2 \) of the approximate Muskat Eq. (52), defined in the time interval \([0, T]\) with initial data \( f_{1,0}, f_{2,0} \) respectively, satisfying

\[
\sup_{t \in [0, T]} \| f_k(t) \|_{H^\frac{3}{2}}^2 \leq M < \infty, \quad \int_0^T \int \frac{(\partial_{xx} f_k)^2}{1 + (\partial_x f_k)^2} \, dx \, dt < \infty \quad (k = 1, 2),
\]

the difference \( g = f_1 - f_2 \) is estimated by

\[
\frac{d}{dt} \| g(t) \|_{H^\frac{1}{2}} \leq G(M) \log \left( 2 + \sum_{k=1}^2 \left\| \frac{\partial_{xx} f_k}{\langle \partial_x f_k \rangle} (t) \right\|_{L^2} \right) \| g(t) \|_{H^\frac{1}{2}}. \quad (110)
\]

**Proof of Proposition 6.6** Using the decomposition of \( T(f_k) f_k \), we find that the difference \( g = f_1 - f_2 \) satisfies

\[
\partial_t g + W(f_1) \partial_x g + \frac{|D| g}{1 + (\partial_x f_1)^2} = R(f_1)(g) + (T(f_2 + g) - T(f_2)) f_2.
\]

We take the \( L^2 \)-scalar product of this equation with \( |D| g \) to get

\[
\frac{1}{2} \frac{d}{dt} \| g \|_{H^\frac{1}{2}}^2 + \int \frac{(|D| g)^2}{1 + (\partial_x f_1)^2} \, dx \leq |\langle W(f_1) \partial_x g, |D| g \rangle| + |\langle R(f_1)(g), |D| g \rangle| + |\langle (T(f_2 + g) - T(f_2)) f_2, |D| g \rangle|.
\]

By combining the previous estimate with the one that we obtain by interchanging the role of \( f_1 \) and \( f_2 \), we find that

\[
\frac{d}{dt} \| g \|_{H^\frac{1}{2}}^2 + \int \left( \frac{(|D| g)^2}{1 + (\partial_x f_1)^2} + \frac{(|D| g)^2}{1 + (\partial_x f_2)^2} \right) \, dx \leq (I) + (II) + (III) \quad (112)
\]

with

\[
(I) = |\langle W(f_1) \partial_x g, |D| g \rangle| + |\langle W(f_2) \partial_x g, |D| g \rangle|,
(II) = |\langle R(f_1)(g), |D| g \rangle| + |\langle R(f_2)(g), |D| g \rangle|,
(III) = |\langle (T(f_2 + g) - T(f_2)) f_2, |D| g \rangle| + |\langle (T(f_1 - g) - T(f_1)) f_1, |D| g \rangle|.
\]
Set
\[ A(t) = \left\| f_1(t) \right\|_{H^\frac{1}{2}}^2 + \left\| f_2(t) \right\|_{H^\frac{1}{2}}^2, \]
\[ B(t) = \int \frac{(\partial_{xx} f_1(t, x))^2}{1 + (\partial_x f_1(t, x))^2} \, dx + \int \frac{(\partial_{xx} f_2(t, x))^2}{1 + (\partial_x f_2(t, x))^2} \, dx. \]

We claim that for any positive constant \( \lambda \in (0, 1) \), there holds
\[
|I| + |(II)| + |(III)| \lesssim \frac{1}{\sqrt{\lambda}} (1 + A)^2 \log(2 + B)^{\frac{1}{2}} \left\| g \right\|_{\dot{H}^\frac{1}{2}}^2 \sum_{k=1}^2 \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2}^2 \\
+ \frac{1}{\kappa(1/\lambda)} \left( 1 + A^4 \right) A^{\frac{1}{2}} \sum_{k=1}^2 \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2}^2.
\]
(113)

Assume that this is true. Since \( \lim_{r \to +\infty} \kappa(r) = +\infty \), we can choose \( \lambda = \lambda(A, \kappa) \) small enough such that
\[
|I| + |(II)| + |(III)| \leq \mathcal{G}(A) \log(2 + B) \left\| g \right\|_{\dot{H}^\frac{1}{2}}^2 + \frac{1}{2} \sum_{k=1}^2 \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2}^2.
\]

So, by (112),
\[
\frac{d}{dt} \left\| g \right\|_{\dot{H}^\frac{1}{2}}^2 \leq \mathcal{G}(A) \log(2 + B) \left\| g \right\|_{\dot{H}^\frac{1}{2}}^2.
\]

Then the wanted estimate (111) follows at once.

\textit{Step 1: estimates of (I) and (II).} It follows from (62) that
\[
|\langle W(f_k) \partial_x g, |D| g \rangle| = \frac{1}{2} \left| \left[ \mathcal{H}, W(f_k) \right] |D| g, |D| g \right|.
\]

Therefore, we deduce from Proposition 5.1 that for any \( \lambda \in (0, 1) \)
\[
|I| \lesssim \frac{1}{\sqrt{\lambda}} A^{\frac{1}{2}} \left\| g \right\|_{\dot{H}^\frac{1}{2}} \left\| |D| g \right\|_{L^2} + \frac{1}{\kappa(1/\lambda)} \sum_{k=1}^2 \left( 1 + A^7 \right) A^{\frac{1}{2}} \left\| \frac{g_x}{\partial_x f_k} \right\|_{L^2} \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2}.
\]

Similarly, it follows from Proposition 5.6 that for any \( \lambda \in (0, 1) \)
\[
|(II)| \lesssim \frac{1}{\sqrt{\lambda}} A \left\| g \right\|_{\dot{H}^\frac{1}{2}} \left\| |D| g \right\|_{L^2} + \frac{1}{\kappa(1/\lambda)} \sum_{k=1}^2 \left( 1 + A \right) A^{\frac{1}{2}} \left\| \frac{g_x}{\partial_x f_k} \right\|_{L^2} \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2}.
\]

Notice that
\[
\left\| |D| g \right\|_{L^2} \leq \left( 1 + \left\| \partial_x f_k \right\|_{L^\infty} \right)^{\frac{1}{2}} \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2} \lesssim (1 + A) \log(2 + B)^{\frac{1}{2}} \left\| \frac{D}{\partial_x f_k} g \right\|_{L^2},
\]
where we have used the interpolation estimate (99).
On the other hand, it follows from Lemma 2.2 that
\[ \left\| \frac{g_x}{\partial_x f_k} \right\|_{L^2} \lesssim \left( 1 + \| f_k \|_{\dot{H}^2} \right) \left\| \frac{|D| g}{\partial_x f_k} \right\|_{L^2}. \]

Consequently,
\[
\left| (I) \right| + \left| (II) \right| \lesssim \frac{1}{\sqrt{\lambda}} (1 + A)^2 \log(2 + B) \frac{1}{2} \| g \|_{\dot{H}^2} \sum_{k=1}^{2} \left\| \frac{|D| g}{\partial_x f_k} \right\|_{L^2} + \frac{1}{\kappa (1/\lambda)} A^{1/2} \sum_{k=1}^{2} \left\| \frac{|D| g}{\partial_x f_k} \right\|_{L^2}.
\]

By using the Young’s inequality, we see that \( |(I)| + |(II)| \) is estimated by the right-hand side of (113).

**Step 2: estimate of (III).** Dividing and multiplying by \( \langle \partial_x f_k \rangle \) and then using the Cauchy-Schwarz inequality, we have
\[
\left| (III) \right| \leq \| \langle \partial_x f_2 \rangle (T(f_2 + g) - T(f_2)) f_2 \|_{L^2} \left\| \frac{|D| g}{\partial_x f_2} \right\|_{L^2} + \| \langle \partial_x f_1 \rangle (T(f_1 - g) - T(f_1)) f_1 \|_{L^2} \left\| \frac{|D| g}{\partial_x f_1} \right\|_{L^2}.
\]

Consider three functions \( h_1, h_2, h \). We want to estimate
\[ \Lambda := \langle \partial_x h_1 \rangle (T(h_1) - T(h_2)) h. \]

Recall that
\[ T(h_k)h = -\frac{1}{\pi} \int_{\mathbb{R}} \langle \partial_x \Delta_\alpha h \rangle - \left( \Delta_\alpha h_k \right)^2 d\alpha. \]

Using the obvious estimate
\[ \left| \frac{x_1^2}{1 + x_1^2} - \frac{x_2^2}{1 + x_2^2} \right| \leq \frac{1}{\sqrt{1 + x_1^2}} |x_1 - x_2|, \]
we have
\[
\left| \Lambda \right| \lesssim \int \frac{\langle \partial_x h_1 \rangle}{\langle \Delta_\alpha h_1 \rangle} |\Delta_\alpha (h_1 - h_2)||\Delta_\alpha h_x| d\alpha \\
\lesssim \int (1 + |\partial_x h_1 - \Delta_\alpha h_1|) |\Delta_\alpha (h_1 - h_2)||\Delta_\alpha h_x| d\alpha.
\]

Consequently, using the Cauchy-Schwarz inequality, we find that the \( L^2 \)-norm of \( \Lambda \) is bounded by
\[
\left( \frac{1}{\int |\Delta_\alpha (h_1 - h_2)|^2 d\alpha dx} \right)^{1/2} \left( \frac{1}{\int |\Delta_\alpha h_x|^2 d\alpha dx} \right)^{1/2} + \left( \frac{1}{\int |\Delta_\alpha h_1 - \partial_x h_1|^2 d\alpha dx} \right)^{1/2} \\
\left( \frac{1}{\int \delta_\alpha (h_1 - h_2)^2 |\alpha|^{1/4} d\alpha dx} \right)^{1/2} \left( \frac{1}{\int \delta_\alpha h_x |\alpha|^{1/4} |\alpha| d\alpha} \right)^{1/2}.
\]
and hence, in view of the embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow \dot{F}^{1/4}_{4,4}(\mathbb{R})$ (see (24)) and the estimate (28), we conclude that
\[
\|\Lambda\|_{L^2} \lesssim \left(1 + \|h_1\|_{\dot{H}^{1/2}}\right) \|h_1 - h_2\|_{\dot{H}^{1/2}} \|h\|_{\dot{H}^{1/2}}.
\]
This implies that
\[
|\langle III \rangle| \leq 2 \sum_{k=1}^{2} \left(1 + A^1_k\right) A^1_k \|g\|_{\dot{H}^{1/2}} \left\|\frac{|D| g}{(\partial_x f_k)}\right\|_{L^2}.
\]
This proves that $\langle III \rangle$ is estimated by the right-hand side of (113), which completes the proof. □

6.5. Enhanced regularity. Eventually, we need to prove a result which asserts that one can always improve the regularity of a solution with values in the endpoint Sobolev space in order to gain the existence of a weight.

Proposition 6.7. Let $f \in C^0([0, T]; H^{3/2}(\mathbb{R}))$ be a solution of the Muskat Eq. (1) such that $f_{xx}/(f_x)$ belongs to $L^2((0, T) \times \mathbb{R})$. Assume that initially $f_0$ belongs to $H^{3/2}$, where $f_0$ is given by (37) for some admissible weight $\kappa_0$. Then there exists $T_0 < T$ such that
\[
\sup_{t \in [0, T_0]} \left\|D^{3/2} f(t)\right\|_{L^2}^2 + 2 \int_0^{T_0} \int_\mathbb{R} \frac{|D|^{2, \phi} f|^2}{1 + (\partial_x f)^2} \, dx \, dt < \infty.
\]

Proof. Consider a function $\phi$ defined by (36), for some admissible weight $\kappa$ with $\sup_{r \geq 0} \kappa(r) < \infty$ and such that $\kappa \lesssim \kappa_0$. Introduce $f_\varepsilon = f \ast \rho_\varepsilon$ where $\rho_\varepsilon$ is the standard mollifier in $\mathbb{R}$. Proceeding as in the proof of Corollary 4.4 (that is by commuting $|D|^{1, \phi}$ with the Muskat equation and then by taking the $L^2$-scalar product with $|D|^{2, \phi} f_\varepsilon \ast \rho_\varepsilon$), we deduce from Fatou’s lemma that, for any $T_0 \in (0, T]$,\[
\sup_{t \in [0, T_0]} \left\|D^{3/2} f(t)\right\|_{L^2}^2 + 2 \int_0^{T_0} \int_\mathbb{R} \frac{|D|^{2, \phi} f|^2}{1 + (\partial_x f)^2} \, dx \, dt \leq \left\|D^{3/2} f(0)\right\|_{L^2}^2 + 2 \sum_{k=1}^{3} I_k(T_0),
\]
where
\[
I_1(T_0) = \limsup_{\varepsilon \to 0} \int_0^{T_0} \left\|\left(W(f) \mathcal{H} |D|^{2, \phi} f, |D|^{2, \phi} f_\varepsilon \ast \rho_\varepsilon\right)\right\| \, d\tau,
\]
\[
I_2(T_0) = \limsup_{\varepsilon \to 0} \int_0^{T_0} \left\|\left(R(f)(|D|^{1, \phi} f), |D|^{2, \phi} f_\varepsilon \ast \rho_\varepsilon\right)\right\| \, d\tau,
\]
\[
I_3(T_0) = \limsup_{\varepsilon \to 0} \int_0^{T_0} \left\|\left(|D|^{1, \phi}, T(f)\right) f, |D|^{2, \phi} f_\varepsilon \ast \rho_\varepsilon\right\| \, d\tau.
\]
By combining the estimates given by Propositions 5.1, 5.6 and 5.8, we obtain

\[
I_2(T_0) + I_3(T_0) + \lim_{\varepsilon \to 0} \int_0^{T_0} \left\| \left( W(f)\mathcal{H} |D|^{2,\phi} f_\varepsilon, |D|^{2,\phi} f_\varepsilon \right) \right\| \, d\tau \\
\lesssim \kappa \left( \frac{1}{\lambda} \right)^{-1} \int_0^{T_0} \left( 1 + \left\| |D|^{3,\phi} f \right\|_{L^2}^7 \left\| |D|^{3,\phi} f \right\|_{L^2} \right) \left\| |D|^{2,\phi} f \right\|_{L^2}^2 \, d\tau \\
+ \int_0^{T_0} \left( 1 + \left\| |D|^{3,\phi} f \right\|_{L^2}^5 \left\| |D|^{3,\phi} f \right\|_{L^2} \right) \left\| |D|^{2,\phi} f \right\|_{L^2} \left\| |D|^{2,\phi} f \right\|_{L^2} \, d\tau \\
+ \frac{1}{\sqrt{\lambda}} \int_0^{T_0} \left( 1 + \left\| |D|^{3,\phi} f \right\|_{L^2} \left\| |D|^{3,\phi} f \right\|_{L^2} \left\| |D|^{2,\phi} f \right\|_{L^2} \right) \, d\tau,
\]

for any \( \lambda \in (0, 1) \).

Now, we will prove that

\[
\lim_{\varepsilon \to 0} \sup \int_0^{T_0} \left| \left( \left( W(f)\mathcal{H} \star \rho_\varepsilon - W(f)\mathcal{H} \star \rho_\varepsilon \right) , g \star \rho_\varepsilon \right) \right| \, d\tau = 0, \quad (115)
\]

with \( g = |D|^{2,\phi} f \). It is enough to show that

\[
\lim_{\varepsilon \to 0} \sup \int_0^{T_0} \left\| (f_\varepsilon) \left( W(f)\mathcal{H} \star \rho_\varepsilon - W(f)\mathcal{H} \star \rho_\varepsilon \right) \right\|_{L^2}^2 \, d\tau = 0. \quad (116)
\]

To see this, we set \( \tilde{g} = |\mathcal{H}(g)|/\langle f_\varepsilon \rangle \) and repeat the computations used to derive (70). This gives

\[
\int_0^{T_0} \left\| (f_\varepsilon) \left( W(f)\mathcal{H} \star \rho_\varepsilon - W(f)\mathcal{H} \star \rho_\varepsilon \right) \right\|_{L^2}^2 \, d\tau \\
\lesssim \int_0^{T_0} \left( \int_{|y| \leq \varepsilon} (f_\varepsilon(x) \langle f_\varepsilon(x - y) \rangle W(f)(x) - W(f)(x - y)|\tilde{g}(x - y) \, dy \right)^2 \, dx \, d\tau \\
\lesssim \int_0^{T_0} \left( \int_{|x - y| \leq \varepsilon} (f_\varepsilon(x))^5 (f_\varepsilon(y))^5 |W(f)(x) - W(f)(y)|^5 \frac{dy \, dx}{|x - y|^2} \right)^2 \, d\tau \| \tilde{g} \|_{L^2((0,T) \times \mathbb{R})}^2.
\]

The next step is a reprise of one argument in the proof of Proposition 5.1. More precisely, it follows from the proof of (71) that

\[
\lim_{\varepsilon \to 0} \int_0^{T_0} \left( \int_{|x - y| \leq \varepsilon} (f_\varepsilon(x))^5 (f_\varepsilon(y))^5 |W(f)(x) - W(f)(y)|^5 \frac{dy \, dx}{|x - y|^2} \right)^2 \, d\tau = 0.
\]

This implies (115).

Now, we set

\[
A(T_0) = \sup_{t \in [0,T_0]} \left\{ \left\| |D|^{3,\phi} f(t) \right\|_{L^2}^2 + \left\| f(t) \right\|_{L^2}^2 \right\}, \quad B(T_0) = \int_0^{T_0} \left\| |D|^{2,\phi} f \right\|_{L^2}^2 \, d\tau.
\]
We previous estimates imply that, for any $\lambda \in (0, 1)$,
\[
A(T_0) + B(T_0) \lesssim \| D^{\frac{3}{2}} \phi (0) \|_{L^2}^2 + \| f(0) \|_{L^2}^2 + \left(1 + A(T_0)^4\right) \kappa \left(\frac{1}{\lambda}\right)^{-1} B(T_0) + \left(1 + A(T_0)^3\right) \int_0^{T_0} \left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \left\| \frac{D^{2, \phi} f}{\langle f_x \rangle} \right\|_{L^2} \, d\tau + \frac{1}{\sqrt{\lambda}} \left(1 + A(T_0)^2\right) \int_0^{T_0} \left\| D^{2, \phi} f \right\|_{L^2} \, d\tau.
\]

On the other hand, by (99) and (103), for any $\delta \in (0, 1)$, we have a non-decreasing function $G_{\kappa_0} : \mathbb{R}_+ \to \mathbb{R}_+$ (depending only on $\kappa_0$) such that
\[
\left\| \frac{f_{xx}}{\langle f_x \rangle} \right\|_{L^2} \leq \delta \left\| \frac{D^{2, \phi} f}{\langle f_x \rangle} \right\|_{L^2} + G_{\kappa_0} \left( A(T_0) + \frac{1}{\delta} \right),
\]
\[
\left\| D^{2, \phi} f \right\|_{L^2} \leq \delta \left\| \frac{D^{2, \phi} f}{\langle f_x \rangle} \right\|_{L^2}^2 + G_{\kappa_0} \left( A(T_0) + \frac{1}{\delta} \right).
\]

It follows that, up to modifying the function $G_{\kappa_0}$,
\[
A(T_0) + B(T_0) \leq C \left( \left\| D^{3, \kappa_0} f(0) \right\|_{L^2}^2 + \| f(0) \|_{L^2}^2 \right) + T_0 G_{\kappa_0} \left( A(T_0) + \frac{1}{\delta} + \frac{1}{\lambda} \right) + \left( C \left(1 + A(T_0)^4\right) \kappa \left(\frac{1}{\lambda}\right)^{-1} + C \delta \left(1 + A(T_0)^3\right) + C \frac{\delta}{\sqrt{\lambda}} \left(1 + A(T_0)^2\right) \right) B(T_0, \phi).
\]

Set
\[
M_0 = 2C \left( \left\| D^{3, \kappa_0} f(0) \right\|_{L^2}^2 + \| f(0) \|_{L^2}^2 \right).
\]

By successively choosing $\lambda = \lambda(A(T_0, \phi)) \in (0, 1)$ small enough, and then $\delta = \delta(A(T_0, \phi)) \in (0, 1)$ small enough, one can guarantee that
\[
C \left(1 + A(T_0)^4\right) \kappa \left(\frac{1}{\lambda}\right)^{-1} + C \delta \left(1 + A(T_0)^3\right) + C \frac{\delta}{\sqrt{\lambda}} \left(1 + A(T_0)^2\right) \leq \frac{1}{2}.
\]

Now, we deduce that
\[
A(T_0) + \frac{1}{2} B(T_0) \leq \frac{M_0}{2} + T_0 G_{\kappa_0} \left( A(T_0) \right).
\]

Then we fix $T_0 > 0$ such that
\[
T_0 G_{\kappa_0} (M_0) = \frac{M_0}{2}.
\]

With this special choice, it follows that
\[
A(T_0) + \frac{1}{2} B(T_0) \leq M_0.
\]

Now, letting $\kappa \uparrow \kappa_0$, we find that
\[
\sup_{t \in [0, T_0]} \left\| D^{\frac{3}{2}, \kappa_0} f(t) \right\|_{L^2}^2 + \int_0^{T_0} \int_\mathbb{R} \left\| \frac{|D^{2, \kappa_0} f|^2}{1 + (\partial_x \phi)^2} \right\|_{L^2} \, dx \, d\tau < \infty.
\]

Since $\phi_0 \sim \kappa_0$, this completes the proof. □
The following is as a consequence of Proposition 6.7.

**Corollary 6.8.** Let \((f_0, n)_n\) be equi-integrable in \(H^{3/2}(\mathbb{R})\). Then, there exists a unique solution \(f_n\) of the Muskat Eq. (1) with initial data \(f_0, n\) in \([0, T_0]\) for some \(T_0 > 0\) and all \(n\), such that

\[
\sup_{n \in \mathbb{N}} \left( \sup_{t \in [0, T_0]} \left\| f_n(t) \right\|_{H^{3/2}}^2 + \int_0^{T_0} \int_{\mathbb{R}} \frac{|\partial_{xx} f_n|^2}{1 + (\partial_x f)^2} \, dx \, dt \right) < \infty,
\]

and

\[
\sup_{n \in \mathbb{N}} \left( \sup_{t \in [0, T_0]} \left\| (f_n - f_n \ast \rho_\varepsilon)(t) \right\|_{H^{3/2}}^2 + \int_0^{T_0} \int_{\mathbb{R}} \frac{|\partial_{xx} (f_n - f_n \ast \rho_\varepsilon)|^2}{1 + (\partial_x f_n)^2} \, dx \, dt \right) \to 0 \quad \varepsilon \to 0.
\]

This proof is simple and left to the reader.

### 6.6. End of the proof

We are now in position to complete the proof of Theorems 1.1 and 1.3. These two theorems contain two different results: an existence result and a uniqueness result. We discuss these two parts separately.

**Uniqueness.** Firstly, we notice that the estimate (111) in the Proposition 6.6 applies also for the Muskat equation (1) (that is for the approximate equation with \(\varepsilon = 0\)). This proves immediately that the solutions of the Muskat equation are unique in the space of functions satisfying (110) (the fact that the size of the time interval is assumed to be less than 1 is not a limitation since one can always divide an arbitrary interval into intervals of length less than one). This proves the uniqueness part of Theorem 1.3. In view of Proposition 6.7, this also proves the uniqueness part of Theorem 1.1.

**Existence.** On the other hand, one can apply Proposition 6.6 to infer the existence of a solution to the Muskat equation. To do so, consider the approximate solutions of the approximate Muskat equation (52), whose existence is given by Proposition 4.2. In the previous paragraphs, we have proved uniform bounds for these solutions (see Proposition 6.2 and Proposition 6.3). Then Proposition 6.6 implies that these approximate solutions forms a Cauchy sequence in the space of bounded functions with values in \(\dot{H}^{1/2}(\mathbb{R})\). Now, we claim that one has the following inequality

\[
\| u \|_{\dot{H}^{3/2}}^2 \lesssim \| u \|_{\dot{H}^{3/2}} \| D^{3/2} \phi \|_{L^2} + \phi \left( \left( \frac{\| u \|_{\dot{H}^{3/2}}}{{\| u \|_{\dot{H}^{1/2}}}^{1/2}} \right)^{-2} \| D^{3/2} \phi \|_{L^2}^2 \right).
\]

This is proved by writing

\[
\| u \|_{\dot{H}^{3/2}}^2 \lesssim \int_{|\xi| \leq R} R^2 |\xi| \left| \hat{u}(\xi) \right|^2 \, d\xi + \int_{|\xi| \geq R} |\xi|^{-3} \frac{\phi(2|\xi|)}{\phi(R)^2} \left| \hat{u}(\xi) \right|^2 \, d\xi,
\]

and then by choosing

\[
R = \left( \frac{\| D^{3/2} \phi \|_{L^2}}{\| u \|_{\dot{H}^{3/2}}} \right)^{1/2} \gtrsim \left( \frac{\| u \|_{\dot{H}^{3/2}}}{{\| u \|_{\dot{H}^{1/2}}}^{1/2}} \right)^{1/2}.
\]
We deduce from the previous inequality that the sequence of approximate solutions is also a Cauchy sequence in $C^0([0, T_0]; H^{3/2}(\mathbb{R}))$. Now, in view of statement iii) in Proposition 4.1 one can pass to the limit in the nonlinearity to prove that the limit of the Cauchy sequence satisfies the Muskat equation. This proves the existence part of Theorem 1.1. In fact, in view of Proposition 6.7, this also implies the existence part of Theorem 1.3.

Theorems 1.1 and 1.3 are therefore proved.

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