UNIFORM ANDERSON LOCALIZATION,
UNIMODAL EIGENSTATES AND SIMPLE SPECTRA
IN A CLASS OF “HAARSH” DETERMINISTIC POTENTIALS

VICTOR CHULAEVSKY

Abstract. We study a particular class of families of multi-dimensional lattice Schrödinger operators with deterministic (including quasi-periodic) potentials generated by the "hull" given by an orthogonal series over the Haar wavelet basis on the torus, of arbitrary dimension, with expansion coefficients considered as independent parameters. In the strong disorder regime, we prove Anderson localization for generic operator families, using a variant of the Multi-Scale Analysis, and show that all localized eigenfunctions are unimodal and feature uniform exponential decay away from their respective localization centers. Using the Klein–Molchanov argument and a variant of the Minami estimate for deterministic potentials, we prove the simplicity of the spectrum in our model.

NOTE: This text completes our earlier manuscript (math-ph/0907.1494), originally uploaded in 2009 and revised in 2011, which is kept in arXiv in a reduced form, merely to avoid broken references in earlier works. Compared to [math-ph/0907.1494], we add the results on unimodality of the eigenstates, uniform dynamical localization, and simplicity of p.p. spectra.

Compared to earlier versions of this preprint, the presentation has been adapted to the future extension of the main results (uniform localization, unimodality of the eigenfunctions) to the multi-particle Anderson Hamiltonians with a nontrivial interaction between the particles, which we plan to publish in a forthcoming paper.

1. Introduction. The model and the main results.

We study spectral properties of finite-difference operators, usually called discrete (or lattice) Schrödinger operators (DSO), of the form

\begin{equation}
(H(\omega; \vartheta)f)(x) = \sum_{y: \|y-x\|=1} f(y) + gV(x; \omega; \vartheta)f(x), \quad x, y \in \mathbb{Z}^d, \quad g \in \mathbb{R},
\end{equation}

where \( \omega \) and \( \vartheta \) are parameters, the role of which we explain below.

In mathematical modeling of disordered quantum systems, it makes more sense to study not an individual operator, but an entire family \( H(\omega) \) labeled by the points of the phase space of a dynamical system on some probability space. Moreover, it is often convenient (but not always necessary) to assume ergodicity of the dynamical system in question. The usual approach to the notion of ergodic ensemble of operators in \( \ell^2(\mathbb{Z}^d) \) is as follows: one considers an ergodic dynamical system \( T \) with discrete time \( \mathbb{Z}^d \), \( d \geq 1 \), on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and a measurable mapping

Date: May 11, 2014.
$H$ of the space $\Omega$ into the space of operators (for example, bounded) acting in the Hilbert space $\mathcal{H} = l^2(\mathbb{Z}^d)$ and satisfying for every $x \in \mathbb{Z}^d$:

$$H(T^x(\omega)) = U^{-x} H(\omega) U^x,$$

where $(U^x f)(y) = f(y - x)$ are the conventional, unitary shift operators. In particular, the DSO (1.1) is obtained by setting $H(\omega) = \Delta + V(x; \omega)$, where $(\Delta f)(x) = \sum_{|y-x|=1} f(y)$, and $V(\cdot; \omega)$ is the operator of multiplication by the function

$$x \mapsto V(x; \omega) = v(T^x \omega),$$

(1.2)

where the function $v : \Omega \to \mathbb{R}$ will be called the hull of the potential $V$.

A rich and interesting class of quasi-periodic potentials, e.g., in one dimension, is obtained when $\Omega$ is the torus $\mathbb{T}^1$ endowed with the Haar measure $\mathbb{P}$, and the dynamical system on $\Omega$ is given by $T^x : \omega \mapsto \omega + x\alpha$, $\omega \in \mathbb{T}^1$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. This dynamical system is well-known to be ergodic. Taking a function $v : \mathbb{T}^1 \to \mathbb{R}$, we can define an ergodic family of quasi-periodic potentials $V : \mathbb{Z} \to \mathbb{R}$ by $V(x; \omega) := v(T^x \omega)$. Multi-dimensional quasi-periodic potentials on $\mathbb{Z}^d$ can be constructed in a similar way (with the help of $d$ incommensurate frequency vectors $\alpha^j \in \mathbb{R}^d, j = 1, \ldots, d$).

In the case where $v(\omega) = g \cos(2\pi \omega)$, $g \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the DSO $H(\omega)$ with the potential of the form (1.2) is called Almost Mathieu or Harper’s operator.

Sinai [31] and Fröhlich et al. [26] proved Anderson localization for a class of the DSO with the “cosine-like” potential; more precisely, the hull $v : \mathbb{T}^1 \to \mathbb{R}$ was assumed to be of the class $C(\mathbb{T}^1)$ and have exactly two extrema, both non-degenerate. Operators with several basic frequencies (i.e., $\omega \in \mathbb{T}^\nu$, $\nu > 1$) were studied in [13] ($\nu = 2$), and later in a cycle of papers by Bourgain, Goldstein and Schlag, for various dynamical systems on a torus $\Omega = \mathbb{T}^\nu$, $\nu \leq 2$, where the hull $v(\omega)$ was assumed analytic; see, e.g., [2], [4], [3]. More recently, Chan [6] used a parameter exclusion technique (different from ours) to establish the localization for quasi-periodic operators with sufficiently non-degenerate hull $v \in C^3(\mathbb{T}^1)$.

Note that the number of rigorous results on Anderson localization for almost-periodic and, more generally, deterministic families of random operators remains rather limited, particularly in dimension $d > 1$, compared to the wealth of results for Schrödinger-type operators with IID or weakly correlated random potentials.

Among recent results most closely related to the topic of the present paper, we refer to the works by Damanik and Gan [18, 19] who proved uniform localization for a class of one-dimensional operators with limit-periodic potential.

◊ In the present paper, we consider parametric families of hulls on the phase space $\Omega$, $\{v(\cdot; \vartheta), \vartheta \in \Theta\}$, labeled by elements $\vartheta$ of an auxiliary set $\Theta$ which we endow with the structure of a probability space; the construction is described in Sect. 1.4 and 2. It is this specific construction which allows us to prove our main result on genuinely uniform Anderson localization for typical values of $\vartheta \in \Theta$ (see Theorem 1 in Sect. 1.6). We encapsulate the main requirement for the underlying dynamical system, generating the deterministic random potential, in one mild condition—"Uniform Power-law Aperiodicity" (UPA); cf. (1.3) in Sect. 1.1).

◊ It is to be emphasized that the ergodicity of the dynamical system is not required per se for our proof of localization. However, in the case where $T$ is generated by the shifts of the torus $\mathbb{T}^d$, aperiodicity implies topological transitivity, hence ergodicity of $T$. In fact, for the toral shifts, the condition (UPA) reads as the Diophantine condition on the frequency vectors.
Our class of models features unusually strong localization properties, similar to those of the celebrated Maryland model, discovered and studied by physicists Fishman et al. \cite{23}. The potential in the Maryland model is quasi-periodic and generated by the analytic hull

$$\omega \mapsto g \tan \pi \omega, \quad \omega \in T^1 \equiv [0, 1) \subset \mathbb{R} \hookrightarrow \mathbb{C},$$

which admits a meromorphic continuation to the complex plane. Its restriction to $\mathbb{R}$ is strictly monotone on the period (between two consecutive poles), and this ultimately results in complete absence of “resonances” between distant sites on the lattice $\mathbb{Z}^d$. In turn, this gives rise to the exponentially localized eigenstates which are unimodal, i.e., cannot have multiple “peaks”.

The notion of a ”peak” actually makes sense for the disorder amplitude $|g| \gg 1$: in this case, the Maryland operator has an orthonormal eigenbasis of exponentially fast decaying eigenfunctions $\psi_x$, labeled in a non-ambiguous and natural way by the points $x \in \mathbb{Z}^d$ so that

$$\inf_{x \in \mathbb{Z}^d} |\psi_x(x)|^2 \geq 1 - f(|g|) > \frac{1}{4}, \quad f(|g|) \to 0 \text{ as } |g| \to \infty.$$

In other words, for $|g| \gg 1$, the eigenbasis for $H(\omega)$ is a small-norm perturbation of the standard delta-basis in $\ell^2(\mathbb{Z}^d)$; this would be, of course, an event of probability 0 for the random Anderson Hamiltonians.

Another particularity of the Maryland model, rigorously proven in independent mathematical works by Figotin and Pastur \cite{25} and by Simon \cite{32}, is the non-perturbative complete exponential localization: it occurs for any, arbitrarily small amplitude of disorder $|g| > 0$. With the exception for this particular feature, the ”unimodal”, uniform exponential localization was extended by Bellissard et al. \cite{1} to the class of meromorphic hulls with a real period, strictly monotone on the period. The proof in [1] is a linear version of the KAM (Kolmogorov–Arnold–Moser) method, which requires the parameter $|g|^{-1}$ to be small enough for the inductive procedure to succeed, so it remains yet unknown if the complete localization occurs in the BLS-class for arbitrarily weak disorder.

The class of deterministic Anderson models considered in this paper features the same complete unimodality of the eigenbasis, i.e., genuinely uniform decay of all eigenfunctions, and not just semi-uniform (often referenced to as the SULE property: Semi-Uniformly Localized Eigenfunctions). This class also has important particularities:

1. The class of the underlying dynamical systems, representing the disorder from the traditional point of view, is not limited to quasi-periodic or, more generally, almost-periodic systems. This is explained by the fact that the ”dynamical disorder” plays here a subordinate, indeed minor role in the localization, while the dominant role is given to the ”parametric disorder”, responsible for the decay of eigenfunctions.

2. The uniform decay of eigenfunctions occurs for all phase points of the dynamical system, and not just Lebesgue-almost all, as in many quasi-periodic systems, e.g., for the Almost Mathieu operators. On the other hand, it occurs only for a subset of the parameter set, labeling the hulls. The measure of the excluded subset decays as $|g| \to \infty$. In other words, we prove localization for a.e. $\vartheta \in \Theta$ and all $\omega \in \Omega$, but with $|g| \geq g^*(\vartheta)$. 
The hulls under consideration are, speaking pictorially, “made out of flat pieces” (viz. composed of Haar wavelets), while in most models, one usually had to make special efforts to avoid “flat” components of the random or deterministic hulls. Albeit the hulls ultimately become non-flat, they are piecewise-constant at every step of the inductive approximation procedure, and this is precisely what gives rise to the uniform exponential localization.

In this work, as in [9], we often use the term random, sometimes putting it in quotes, and this might create the illusion that the operators with deterministic – e.g., quasi-periodic – potentials, considered here, are somehow perturbed by a masterly hidden random noise. We do not add, or otherwise introduce, any IID or weakly correlated noise in the potential, which always remains deterministic, with stochastic properties\(^1\) induced exclusively by the underlying dynamical system. For example, if \(\{T^x, x \in \mathbb{Z}^d\}\) is generated by incommensurate shifts of the torus, the obtained potentials are always quasi-periodic, thus feature the weakest possible ergodic properties. Yet, it is true that many techniques used in the proof of localization come from the conventional theory of random Anderson Hamiltonians.

One particularly important advantage of the probabilistic language and tools is that we can prove Minami-type estimates, of all orders, for generic deterministic operator ensembles. Combined with the Klein–Molchanov argument (cf. [29]), this results in the proof of simplicity of the pure point spectrum, for every (and not just a.e.) phase point of the underlying dynamical system. To the best of the author’s knowledge, this is the first result of such kind for a large class of deterministic operators. It is not related to the unimodality of the eigenstates; in a forthcoming work, following the path laid down in [9], we will extend it to a more general class of deterministic DSO with hulls of any finite smoothness, where the respective Hamiltonians feature only the SULE property, and the eigenstates are not unimodal.

Our main results are presented in Sect. 1.6.

Technically speaking, the most tedious analysis is required to establish analogs of the Wegner estimate, and infer from them the unusual – uniform – lower bounds on the “small denominators”, or “resonances”. Once such bounds are obtained, the derivation of the Anderson localization becomes quite simple and “soft” (cf. Sect. 6); the reader will see that it is actually simpler than for the Anderson Hamiltonians with IID random potential.

1. Requirements for the dynamical system. For the sake of clarity, we consider in this paper only the case where \(\Omega = T^\nu, \nu \geq 1\), and it is convenient to define the distance \(\text{dist}_\Omega[\omega', \omega'']\) as follows: for \(\omega' = (\omega'_1, \ldots, \omega'_\nu)\) and \(\omega'' = (\omega''_1, \ldots, \omega''_\nu)\),

\[
\text{dist}_\Omega[\omega', \omega''] := \max_{1 \leq i \leq \nu} \text{dist}_{T^1}[\omega'_i, \omega''_i],
\]

where \(\text{dist}_{T^1}[\cdot , \cdot]\) is the conventional distance on \(T^1 = \mathbb{R}^1/\mathbb{Z}^1\). With this definition, the diameter of a cube of side length \(r\) in \(T^\nu\) equals \(r\), for any dimension \(\nu \geq 1\). The main reason for the choice of the phase space \(\Omega = T^\nu\) is that the parametric families of ensembles of potentials \(V(x; \omega; \vartheta)\) are fairly explicit in this case, and this allows one to construct quasi-periodic operators.

\(^1\)As the matter of fact, we do not make use of any stochastic properties of the underlying dynamics, other than the “Uniform Power-law Aperiodicity” (cf. (1.3)).
We assume that the underlying dynamical system $T$ (generating the potential) satisfies the condition of Uniform Power-law Aperiodicity:

\begin{align*}
\text{(UPA)} & \quad \exists A, C_A \in \mathbb{N}^* \quad \forall \omega \in \Omega \quad \forall x, y \in \mathbb{Z}^\nu \text{ such that } x \neq y \\
\text{dist}_\Omega(T^x\omega, T^y\omega) & \geq C_A^{-1}|x - y|^{-A}, \quad (1.3)
\end{align*}

and the condition of tempered local divergence of trajectories:

\begin{align*}
\text{(DIV)} & \quad \exists A', C_{A'} \in \mathbb{N}^* \quad \forall \omega, \omega' \in \Omega \quad \forall x \in \mathbb{Z}^\nu \setminus \{0\} \\
\text{dist}_\Omega(T^x\omega, T^x\omega') & \leq C_{A'} |x|^A \text{dist}_\Omega(\omega, \omega'). \quad (1.4)
\end{align*}

Remark 1. It is not difficult to see that both (UPA) and (DIV) rule out strongly mixing dynamical systems like the hyperbolic toral automorphisms (while the skew shifts of tori are still allowed). This certainly looks quite surprising, but it has to be emphasized that our proof is oriented towards the dynamical systems with the weakest stochasticity. In a manner of speaking, we actually need that the dynamical system “do not interfere” with the “randomness” provided by the parametric shifts of tori (hence, with $\omega \in \Omega$ fixed), for any cube $B_L(u)$, the values $\{V(x; \omega; \nu) := v(T^x\omega; \nu), x \in B_L(u)\}$, are (conditionally) independent and admit individual (conditional) probability densities $\rho_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}_L)$, satisfying

\begin{align*}
\text{ess sup} \|\rho_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}_L)\|_\infty & \leq C'' L^{B \ln L}, \quad C'' \in (0, +\infty). \quad (1.5)
\end{align*}

It is readily seen that for the scaled random variables $(\omega; \nu) \mapsto gV(x; \omega; \nu)$ the assumption (1.5) implies

\begin{align*}
\text{ess sup} \|\rho_{gv,x}(\cdot | \mathfrak{F} \times \mathfrak{B}_L)\|_\infty & \leq C'' g^{-1} L^{B \ln L}, \quad C'' \in (0, +\infty). \quad (1.6)
\end{align*}

This property allows us to prove satisfactory analogs of the Wegner (cf. Sect. 3.1) and Minami (cf. Sect. 9) estimates in finite cubes of any size $L$.  

1.2. The Local Variation Bound. We often work with lattice cubes $B_L(u) := \{x \in \mathbb{Z}^d : |x - u| \leq L\}$, $L \geq 0$; for $y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$, $|y|$ stands for the max-norm, $|y| := \max_i |y_i|.$

Following [10], we introduce now a hypothesis on the random field $v : \Omega \times \Theta \rightarrow \mathbb{R}$ on $\Omega$, relative to the probability space $(\Theta, \mathfrak{B}, P^\Theta)$, which is logically independent of the particular construction given in Sect. 1.4. Later we will show that it holds true for the hulls constructed with the help of the rantelette expansions in Sect. 1.4.

\begin{align*}
\text{(LVB): } & \text{Let } v : \Omega \times \Theta \rightarrow \mathbb{R} \text{ be a measurable function on the product probability space } (\Omega \times \Theta, \mathfrak{F} \times \mathfrak{B}, P \times P^\Theta). \text{ There exists a family of sub sigma-algebras } \mathfrak{B}_L \subset \mathfrak{B}, \text{ } L \in \mathbb{N}^*, \text{ such that, conditional on } \mathfrak{F} \times \mathfrak{B}_L \text{ (hence, with } \omega \in \Omega \text{ fixed), for any cube } B_L(u), \text{ the values } \{V(x; \omega; \nu) := v(T^x\omega; \nu), x \in B_L(u)\}, \text{ are (conditionally)} \\
& \text{independent and admit individual (conditional) probability densities } \rho_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}_L), \text{satisfying} \\
& \text{ess sup} \|\rho_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}_L)\|_\infty \leq C'' L^{B \ln L}, \quad C'' \in (0, +\infty). \quad (1.5)
\end{align*}
1.3. Lattice cubes and local Hamiltonians. Given a DSO $H = \Delta + gV$, where $V : \mathbb{Z}^d \to \mathbb{R}$ and $g > 0$, and a proper subset $\Lambda \subseteq \mathbb{Z}^d$, we consider the restriction $H_\Lambda$ of $H$ to $\Lambda$ defined as follows: $H_\Lambda = 1_\Lambda H 1_\Lambda \upharpoonright \ell^2(\Lambda)$; here the indicator function $1_\Lambda$ is identified with the multiplication operator by this function, and also with the natural orthogonal projection from $\ell^2(\mathbb{Z})$ onto $\ell^2(\Lambda)$. $H_\Lambda$ is usually considered as the discrete analog of the Schrödinger operator with Dirichlet boundary conditions, acting on functions $\psi$ vanishing outside $\Lambda$.

1.4. Randelette expansions: An informal discussion. In Ref. [9] we introduced parametric families of ergodic ensembles of operators \{H(\omega; \vartheta), \omega \in \Omega\} depending upon a parameter $\vartheta \in \Theta$ in an auxiliary space $\Theta$. As shows [9], it is convenient to endow $\Theta$ with the structure of a probability space, $(\Theta, \mathcal{B}, \mathbb{P}^\Theta)$, in such a way that $\vartheta$ be, in fact, an infinite family of IID random variables on $\Theta$, providing an infinite number of auxiliary independent parameters allowing to vary the hull $v(\omega; \vartheta)$ locally in the phase space $\Omega$. We called such parametric families grand ensembles.

The above description is, of course, too general. In the framework of the DSO, we proposed in [9] a more specific construction where $H(\omega; \vartheta) = H_0 + V(\cdot; \omega; \vartheta)$, with $V(x; \omega; \vartheta) = V(T^x\omega; \vartheta)$ and

$$v(\omega; \vartheta) = \sum_{n=0}^{\infty} a_n \sum_{k=1}^{K_n} \vartheta_{n,k} \varphi_{n,k}(\omega),$$

(1.7)

where $\{\vartheta_{n,k}, n \geq 0, 1 \leq k \leq K_n\}$ are IID random variables on $\Theta$, and $\varphi_{n,k} := (\varphi_{n,k}), n \geq 0, 1 \leq k \leq K_n < \infty$, are some functions on the phase space $\Omega$ of the underlying dynamical system $T^x$. Series of the form (1.7) were called in [9] randelette expansions, referring to the "random" nature of the expansion coefficients and to the shape of $\varphi_{n,k}$ reminding the wavelets ("ondelettes", in French).

Putting the amplitude of the function $\varphi_{n,k}$ essentially in the "generation" coefficient $a_n$, it is natural to assume that $|\varphi_{n,k}(\omega)|$ are uniformly bounded in $(n, k)$. Further, in order to control the potential $V(T^x\omega; \vartheta)$ at any lattice site $x \in \mathbb{Z}^d$ or, equivalently, at every point $\omega \in \Omega$, it is natural to require that for every $n \geq 1$, $\Omega$ be covered by the union of the sets where at least one function $\varphi_{n,k}$ is nonzero (and, preferably, not too small).

In the next subsection, we make a specific choice for $\{a_n\}$ and $\{\varphi_{n,k}\}$.

Notice that the dynamics $T^x$ leaves $\vartheta$ invariant.

1.5. Lacunary “haarsh” randelette expansions. A very particular and interesting case is where the randellethes are simply Haar wavelets with coefficients considered, formally, as independent random variables relative to an auxiliary probability space $(\Theta, \mathcal{B}, \mathbb{P}^\Theta)$. For example, if $\Omega = T^1 = \mathbb{R}/\mathbb{Z}$, for $n = 0$ we set $K_0 = 1$, $\varphi_{0,1}(\omega) = 1$, and for $n \geq 1$, $1 \leq k \leq K_n = 2^n$,

$$\varphi_{n,k}(\omega) = 1_{C_{n,k}^+}(\omega) - 1_{C_{n,k}^-}(\omega),$$

(1.8)

where

$$C_{n,k}^+ = \left[ \frac{k-1}{2^n}, \frac{k-1}{2^n} + \frac{1}{2^{n+1}} \right], \quad C_{n,k}^- = C_{n,k}^+ + \frac{1}{2^{n+1}},$$

so

$$\mathrm{supp} \varphi_{n,k} = C_{n,k} := C_{n,k}^+ \cup C_{n,k}^-.$$

(1.9)
On the torus $T^\nu$ with $\nu > 1$, the functions $\varphi_{n,k}$ are tensor products of the one-dimensional Haar’s wavelets, and $C_{n,k} := \text{supp} \varphi_{n,k}$ are cubes in $T^\nu$ of side length $2^{-n}$, of the form
\[ C_{n,k} = \bigotimes_{j=1}^{\nu} \left[ \frac{k_j}{2^n}, \frac{k_j + 1}{2^n} \right]; \]
they define a partition of $T^\nu$, which we denote by $\mathcal{C}_n$.

Furthermore, each of these cubes is partitioned into $2^n$ sub-cubes of side length $2^{-n-1}$, $\{C_{n,k,i}, i = 1, \ldots, 2^n\}$, on which $\varphi_{n,k}$ takes a constant value $\pm 1$; we denote this value by $s_{n,k}(\omega) \in \{-1, +1\}$, so that
\[ \varphi_{n,k}(\omega) = s_{n,k}(\omega) \mathbf{1}_{C_{n,k}}(\omega). \] (1.10)
Clearly, the cubes $C_{n,k,i}$ are elements of the finer partition $\mathcal{C}_{n+1}$. Indeed, similar to (1.8), we have
\[ C_{n,k,i} = \bigotimes_{j=1}^{\nu} \left[ \frac{k_j}{2^{n+1}}, \frac{k_j + l_{j,i}}{2^{n+1}} \right], \quad l_{j,i} \in \{0, 1\}, \] (1.11)
where the combinations of the shifts $l_{j,i}$ determine $\text{sign}(s_{n,k}(\cdot))$.

Next, consider a family of IID random variables $\vartheta_{n,k}$ on an auxiliary probability space $(\Omega, \mathcal{B}, \mathbb{P}^{\theta})$, uniformly distributed in $[0, 1]$.

Finally, let
\[ a_n = 2^{-2bn^2}, \quad n \geq 1, \quad b > 0, \] (1.12)
with $b > 0$ to be specified later, and define a function $\nu(\omega; \vartheta)$ on $\Omega \times \Theta$,
\[ \nu : (\omega; \vartheta) \mapsto \sum_{n=0}^{\infty} a_n \sum_{k=1}^{K_n} \vartheta_{n,k} \varphi_{n,k}(\omega), \] (1.13)
which can be viewed as a family of functions $\nu_{\theta}(\cdot) = \nu(\cdot; \vartheta) : T^\nu \to \mathbb{R}$, parameterized by $\vartheta \in \Theta$, or as a particular case of a “random” series of functions, expanded over the given system of functions $\varphi_{n,k}$ with “random” coefficients. It is to be emphasized that the orthogonality of the system $\{\varphi_{n,k}\}$ is not important for our construction and results; for example, one could simply set $\varphi_{n,k} = \mathbf{1}_{C_{n,k}}$, and this would even result in slightly simpler proofs.

We will call the expansions of the form (1.13) “haarsch” randelette expansions, referring to Haar’s (“Haarsche,” in German) wavelets and to the “harsh” nature of the resulting potentials. Constructing a potential “out of flat pieces” is rather unusual in the framework of the localization theory, where all efforts were usually made to avoid flatness of the potential. Yet, with an infinite number of flat components $\vartheta_{n,k} \varphi_{n,k}(\omega)$, each modulated by its own parameter $\vartheta_{n,k}$, we proved earlier (cf. [7–9]) an analog of Wegner bound [33] for the respective grand ensembles $\{H(\omega; \vartheta), \omega \in \Omega, \vartheta \in \Theta\}$.

The extremely rapid decay of coefficients $a_n$ (the generation amplitudes), making the series “lacunary”, is required for the proof of unimodality and of uniform decay of eigenfunctions. With generation amplitudes behaving like $a_n \sim 2^{-bn}$, the tail series $\epsilon_{N+1} = \sum_{n \geq N+1} a_n$ is comparable to $a_N$, while we need $\epsilon_{N+1} \ll |a_N|$.

We use the term “lacunary” for the following reason: instead of the series $\sum_{n \geq 0} a_n \cdot (\cdots)$ over all generations $n$, say, with $a_n \sim 2^{-bn}$, we could consider a series of the form $\sum_{j=0}^{\infty} a_{n_j} \cdot (\cdots)$, with $a_n = e^{-bn}$ and a sequence $\{n_j\}$ growing fast enough; for example, $n_j = bj^2$. Such series are usually called lacunary.
Building on the techniques from [9], we prove Anderson localization for generic lacunary "haarsch" potentials of large amplitude, under the mild assumptions (UPA) (cf. (1.3)) and (DIV) (cf. (1.4)). In particular, our results imply uniform Anderson localization for a class of quasi-periodic potentials with Diophantine frequencies.

Apparently, there is no hope to establish Anderson localization for a reasonably rich class of quasi-periodic operators without the assumption of strong disorder, even in one dimension, as shows the well-known example of the Almost Mathieu operator $H(\omega) = \Delta + g \cos(\alpha x + \omega)$ with Diophantine frequency $\alpha$, featuring pure a.c. spectrum for $|g| < 2$. The approach based on the Lifshitz tails asymptotics at "extreme" energies does not apply here.

1.6. Main results.

**Theorem 1.** Consider a family of lattice Schrödinger operators in $\ell^2(\mathbb{Z}^d)$, $H(\omega; \vartheta) = \Delta + gV(x; \omega; \vartheta)$, where $V(x; \omega; \vartheta) = v(T^x \omega; \vartheta)$ with $v(\omega; \vartheta)$ given by the expansion (1.13), and the dynamical system $T^x$ satisfies conditions (UPA) and (DIV) (cf. (1.3), (1.4)) for some $A, A', C, C_A, C_A' \in \mathbb{N}^*$.

Then there exists $g_0 = g_0(C, A, C', A', d, \nu) \in (0, +\infty)$ such that for any $g \geq g_0$, there exists a subset $\Theta(\infty)(g) \subset \Theta$ with $\mathbb{P}^\Theta \{ \Theta(\infty)(g) \} \geq 1 - e^{-c_1 \ln^{1/2} g}$ and with the following property: if $\vartheta \in \Theta(\infty)(g)$, then for any $\omega \in \Omega$:

(A) $H(\omega; \vartheta)$ has pure point spectrum;

(B) for any $x \in \mathbb{Z}^d$, there is exactly one eigenfunction $\psi_x(\cdot; \omega; \vartheta)$ such that

$$|\psi_x(x; \omega; \vartheta)|^2 > 1/2,$$

i.e., $\psi_x$ has the "localization center" $x$, so the localization centers establish a bijection between the elements of the eigenbasis $\{\psi_x(\cdot; \omega; \vartheta)\}$ and the lattice $\mathbb{Z}^d$;

(C) for all $x \in \mathbb{Z}^d$, the eigenfunctions $\psi_x$ decay uniformly away from their respective localization centers:

$$\forall y \in \mathbb{Z}^d \quad |\psi_x(y; \omega; \vartheta)| \leq e^{-m|y-x|}, m = m(g, C, A) \to \infty \text{ as } g \to +\infty.$$

In Sect. 8 we establish uniform pointwise dynamical localization for the operators $H(\omega; \vartheta)$ with $\vartheta \in \Theta(\infty)(g)$ and any $\omega \in \Omega$ (cf. Theorem 7).

A direct analog of Theorem 5.2 proven in [9] is the following

**Theorem 2.** Fix a finite interval $I \subset \mathbb{R}$. Then for some $B > 0$, any $\omega \in \Omega$, any integer $J \geq 1$ and some $C_J \in (0, +\infty)$

$$\mathbb{P}^\Theta \{ \vartheta : \text{Tr} \Pi_I (H_{B\ln(I)}(\omega; \vartheta)) \geq J \} \leq C_J L^{JB \ln L} |I|^J$$

and, denoting $\mathbb{P}^{\Omega \times \Theta} : = \mathbb{P} \times \mathbb{P}^\Theta$,

$$\mathbb{P}^{\Omega \times \Theta} \{ (\omega, \vartheta) : \text{Tr} \Pi_I (H_{B\ln(I)}(\omega; \vartheta)) \geq J \} \leq C_J L^{JB \ln L} |I|^J.$$

Clearly, $J = 1$ leads to a Wegner-type estimate. Theorem 2 is proved in Sect. 9.1.

We also prove a variant of Theorem 2 deterministic in $\omega \in \Omega$:

**Theorem 3.** Consider a sequence $L_j = (L_0)^{2^j} \in \mathbb{N}$, $L_0 > 1$. Under the assumptions and with notations of Theorem 1, for any $g \geq g_0$, there exists a subset $\Theta_M(\infty)(g) \subset \Theta(\infty)(g)$ of measure $\mathbb{P}^\Theta \{ \Theta(\infty)(g) \} \geq 1 - e^{-c_2 \ln^{1/2} g}$, and numbers
0 < B' < \tilde{b} < \infty \text{ such that for any } \vartheta \in \Theta_M^{(\infty)}(g) \text{ and all } \omega \in \Omega \text{, for any interval } I \text{ of length } |I| \leq L_j \tilde{b} \ln L_j \text{,}

\begin{equation}
\mathbb{P}^{\Omega} \{ \text{ Tr } \Pi_I(H_{B\nu L_j}(\omega; \vartheta)) \geq 2 \} \leq C_2' L_j 2^{B'} \ln L_j |I|^2. \tag{1.17}
\end{equation}

Here the subscript "M" in \( \Theta_M^{(\infty)}(g) \) refers to the Minami estimate. The explicit values of the parameters \( \tilde{b} \) and \( B' \) will be given in Sect. 9. Using the Klein–Molchanov argument \[29\], we infer from (1.17) the simplicity of spectra of the operators \( H(\omega; \vartheta) \) for all \( \vartheta \in \Theta_M^{(\infty)}(g) \) and every \( \omega \in \Omega \):

**Theorem 4.** Under the assumptions and with notations of Theorem 3, for any \( g \geq g_0 \), any \( \vartheta \in \Theta_M^{(\infty)}(g) \), and all \( \omega \in \Omega \), \( H(\omega; \vartheta) \) has \textit{simple} pure point spectrum.

2. Randellelltes and separation bounds for the potential

2.1. Relations between the key parameters. In what follows, we often use parameters \( A, C, A', B, b \) and some others; for the reader’s convenience, below are given the conditions they have to satisfy:

\[
\begin{array}{c}
b \geq \max \left( \frac{sd+A+4A'}{10A}, 2 \right), \\
A \ln L_0 > |\ln C_A| + 2 \ln 2 \\
B = 800 b A^2 / \ln 2 \\
\beta_0(g) = e^{-c_2 \ln^{1/2} g}
\end{array}
\]

(2.1)

2.2. Boundaries and partitions. Given a lattice subset \( \Lambda \subseteq \mathbb{Z}^d \) with non-empty complement \( \Lambda^c \), introduce its internal, external, and the so-called edge boundary:

\[
\begin{align*}
\partial^{-} \Lambda &= \{ x \in \Lambda : \text{ dist}(x, \Lambda^c) = 1 \}, \\
\partial^{+} \Lambda &= \{ x \in \Lambda^c : \text{ dist}(x, \Lambda) = 1 \} \equiv \partial^{-} \Lambda^c, \\
\partial \Lambda &= \{ (x, y) \in \partial^{-} \Lambda \times \partial^{+} \Lambda : |x - y| = 1 \}.
\end{align*}
\]

(2.2)

Next, consider the phase space \( \Omega \) which we always assume to be the torus \( T^\nu \) of dimension \( \nu \geq 1 \): \( T^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu \cong [0, 1)^\nu \). For each \( n \geq 0 \), we have introduced the family of \( K_n = 2^{n K} \) adjacent cubes \( C_{n,k}, k = 1, \ldots, K_n \), of side length \( 2^{-n} \), and the functions \( \varphi_{n,k} \) with supp \( \varphi_{n,k} = C_{n,k} \).

For every \( n \geq 0 \), the supports \( \{ C_{n,k} = \text{ supp } \varphi_{n,k}, 1 \leq k \leq K_n \} \) naturally define a partition of the phase space \( \Omega \):

\[
C_n = \{ C_{n,k}, 1 \leq k \leq K_n \}.
\]

These partitions form a monotone sequence: \( C_{n+1} \prec C_n \), i.e., each element of \( C_n \) is a union of some elements of the partition \( C_{n+1} \).

Given \( n \geq 0 \), for each \( \omega \in \Omega \) we denote by \( \bar{k}_n(\omega) \) the unique index such that

\[
\omega \in C_{n,\bar{k}_n(\omega)}.
\]

(2.3)

2.3. Piecewise-constant approximants of the hull. For each \( N \geq 0 \), introduce the approximant of \( v(\omega; \vartheta) \) given by (1.7):

\[
v_N(\omega; \vartheta) = \sum_{n=0}^{N} a_n \sum_{k=1}^{K_n} \vartheta_{n,k} \varphi_{n,k}(\omega),
\]

(2.4)
the truncated potential $V_N$ and the truncated Hamiltonian $H^{(N)}$:

$$V_N(x; \omega; \vartheta) := v_N(T^\omega; \vartheta), \quad H^{(N)} := \Delta + V_N.$$  \hfill (2.5)

With $b \geq 2$ (which follows from (2.1)), for any $N \geq 0$ we have

$$\sum_{n \geq N+1} a_n = \sum_{n \geq N+1} 2^{-bn^2} = 2^{-b(2N+1)} \sum_{i \geq 0} 2^{-b(N+i)^2+i(2N+1)^2} \leq 2^{-b(2N+1)} a_N \sum_{i \geq 0} 2^{-i} \leq \frac{1}{2} 2^{-2bN} a_N,$$  \hfill (2.6)

so the norm $\|v - v_N\|_\infty := \sup_{\omega \in \Omega} \|v - v_N\|_{L^\infty(\Omega)}$ can be bounded as follows:

$$\|v - v_N\|_\infty \leq \frac{1}{2} 2^{-2bN} a_N.$$  \hfill (2.7)

Owing to (2.6), the RHS is much smaller than the width ($a_N$) of the distribution of random coefficients $a_N \vartheta_{N,k}$, $1 \leq k \leq K_N$ (recall: $\vartheta_{N,k} \sim \text{Unif}[0,1]$). Set

$$\tilde{n}(L) = \tilde{n}(L, A, C_A) := 1 + \left[ \frac{4A \ln L - \ln(C_A/2)}{\ln 2} \right]$$  \hfill (2.8)

and observe that, for $L$ large enough so $|\ln C_A| + 2 \ln 2 < A \ln L$,

$$\frac{3A}{2} \ln L < \tilde{n}(L) < \frac{5A}{2} \ln 2 L, \quad L^{-5A} < 2^{-\tilde{n}(L)} < L^{-3A}.$$  \hfill (2.9)

Further, set

$$\tilde{N}(L) = \tilde{n}(L^4),$$  \hfill (2.10)

then we have

$$\tilde{N}(L) = \tilde{A} \ln L, \quad \tilde{A} = \tilde{A}(A, C_A) \in \left[ \frac{12A}{\ln 2}, \frac{20A}{\ln 2} \right] \subset [17A, 29A]$$  \hfill (2.11)

and

$$L^{-20A} < 2^{-\tilde{N}(L)} < L^{-12A}.$$  \hfill (2.12)

The condition $A \ln L_0 > |\ln C_A| + 2 \ln 2$ will be always assumed below (cf. (2.1)). Then for any $u \in \mathbb{Z}^d$ and any $\omega \in \Omega$, all the points of the finite trajectory $\{T^\omega x, x \in B_L(u)\}$ are separated by the elements of the partition $\mathcal{C}_{N(L)}$, since by (UPA) and the first LHS inequality in (2.9), we have

$$\frac{1}{2} \text{dist}_\Omega(T^\omega x, T^\omega y) \geq \frac{1}{2} C_A^{-1} (L^4)^{-A} > 2^{-\tilde{N}(L)}.$$  \hfill (2.13)

**Lemma 2.1.** Under the assumptions (UPA) and (DIV), the bound (LVB) holds true with $C'' = 1$ and $B = 800 bA^2 / \ln 2$.

**Proof.** Fix any integer $L \geq 1$ and let $\mathfrak{B}_L$ be the sigma-algebra generated by the random variables $\{\vartheta_{n,k}, n \neq \tilde{N}(L), 1 \leq k \leq K_N\}$. By (2.13), all the points of the finite trajectory $\{T^\omega x, x \in B_{L}(u)\}$ are separated by the elements of the partition $\mathcal{C}_{\tilde{N}(L)}$, so each value $v(T^\omega x; \vartheta)$ has the form (we set for brevity $N = \tilde{N}(L)$)

$$v(T^\omega x; \vartheta) = \sum_{n \notin N} \sum_{k=1}^{K_N} a_n \vartheta_{n,k} \varphi_{n,k}(T^\omega x) + \sum_{k=1}^{K_N} a_{\tilde{N}_k} \vartheta_{n,k} \varphi_{\tilde{N}_k}(T^\omega x)$$  \hfill (2.14)

$$= \zeta(v) + a_{\tilde{N}_k} \vartheta_{\tilde{N}_k,\tilde{N}_q}(T^\omega x) g_{\tilde{N},\tilde{N}_q}(T^\omega x), \quad g_{\tilde{N},\tilde{N}_q}(T^\omega x) \in \{1, -1\},$$
where $\zeta(\vartheta)$ is $\mathfrak{B}_L$-measurable. Since $\vartheta_{\tilde{N}, \tilde{k}} \sim \text{Unif}([0,1])$ and $s_{\tilde{N}, \tilde{k}}(T^\omega) = \pm 1$, the second term in the above RHS has probability density bounded by

$$a_n^{-1} = 2^{2b\tilde{N}^2} \leq \exp\left\{\ln 2 \cdot 2b \frac{(20A)^2 \ln^2 L}{\ln^2 2}\right\} = L^{B \ln L}$$

with

$$B = 800 b A^2 / \ln 2,$$  \hspace{1cm} (2.15)

and it is independent of $\mathfrak{B}_L$. This proves the claim. $\square$

3. Wegner-type bounds and spectral spacings

We will use a sequence of integers (length scales) $L_j, j \geq 0$, defined as follows: given an integer $L_0 \geq 2$, we set

$$L_j := L_{j-1}^2 = (L_0)^{2^j}, \quad j = 1, 2, \ldots$$  \hspace{1cm} (3.1)

A number of our formulae and estimates involve the cubes of size $L_4^j$; in view of the above definition, $L_4^j = (L_{j+1})^2 = L_{j+2}$, and the role of the quantities $L_4^j$ will become clear at the final stage of localization analysis, by the end of Sect. 6.1.

In addition, in the proof of uniform exponential decay of eigenfunctions away from their "localization centers", we will also use the length scale $L_{-1} = 0$.  \hspace{1cm} (3.2)

In Sect. 4.1, we will introduce a function $g \mapsto L_0(g)$, providing for $g$ large enough the value of the initial length scale suitable for the scale induction. Here $g > 0$ is the amplitude parameter in the potential $gV$ in (1.1). Specifically, we will show, in the proof of Lemma 4.1, that it suffices to set, with some $c_1 > 0$,

$$L_0(g) = \left\lfloor e^{c_1 \ln^{1/2} g} \right\rfloor.$$  \hspace{1cm} (3.3)

As a result, the length scales suitable for our scaling scheme become functions of $g$: $L_j = L_j(g) = (L_0(g))^{2^j}$. Next, given $g > 0$, set

$$\delta_j(g) = \beta_j(g) a_{\tilde{N}(L_j)},$$

$$\beta_j(g) = 2^{-2b\tilde{N}(L_j)}.$$  \hspace{1cm} (3.4)

Here the function $L \mapsto \tilde{N}(L)$ is defined in (2.8). It follows that (cf. (2.1))

$$\beta_0(g) \leq e^{-c_2 \ln^{1/2} g},$$  \hspace{1cm} (3.5)

with some $c_2 > 0$ which will be specified later (in the proof of Lemma 4.1). Observe that, owing to (2.9), we have

$$\delta_j = 2^{-2b\tilde{N}(L_j)} a_{\tilde{N}(L_j)} < L_j^{-\lambda} a_{\tilde{N}(L_j)} \leq L_j^{-\lambda} \leq L_0^{-C \gamma^2 \ln L_0},$$  \hspace{1cm} (3.6)

so that $\sum_{j \geq 0} \delta_j < \infty$. Moreover, $\sum_{j \geq 0} \delta_j \to 0$ as $L_0 \to \infty$.

3.1. The Wegner-type bound. As was said in Sect. 1.6, the particular case of Theorem 2 with $J = 1$ gives a Wegner-type bound; it is non-uniform in the size of the cube $\text{B}_L(u)$, but sufficient for the purposes of the scale induction in Sect. 6.
3.2. Parametric control of spectral spacings. Consider a finite cube $B = B_L(u) \subset \mathbb{Z}^d$ and the operator $H_B = \Delta_B + gV$. If $g$ is large enough, then the values of the potential $\{V(x), x \in B\}$ can be considered as (satisfactory) approximations to the eigenvalues $E^B_j$ of operator $H_B$, by virtue of the min-max principle. In particular, if all the values of the potential in $B$ are distinct and $g$ is large enough, then all spectral spacings $|E^B_i - E^B_j|$ of $H_B$ are bounded from below by $C(V)g$.

A similar lower bound holds for all pairs of disjoint cubes $B_\ell(u), B_\ell(v)$ inside a larger cube $B_L(w)$. Specifically, if all the values $\{V(x), x \in B_L(w)\}$ are distinct and $g$ is large enough, then $|E^B_\ell(u) - E^B_\ell(v)| \geq C(V)g > 0$. In other words, the distance between the two spectra (as subsets of $\mathbb{R}$) satisfies

$$\text{dist} \left[ \Sigma(H_{B_\ell(u)}), \Sigma(H_{B_\ell(v)}) \right] \geq \text{Const}(V)g > 0.$$ 

Here and below, we denote the spectrum of a finite-dimensional operator $H$ by $\Sigma(H)$. In the case where $H = H_{B_L(u)}$, we will write for brevity $\Sigma(B_L(u))$.

However, it is clear that such a simple control of inter-spectral spacings is impossible at an arbitrarily large scale, once $g$ is fixed.

In the traditional Multi-Scale Analysis of random operators, inter-spectral spacings are controlled in a probabilistic way, using the Wegner bound or its variants. The main raison d'être of the auxiliary measurable space $\Theta$ in the framework of the grand ensembles (cf. [9]) is precisely to mimic, to a certain extent, the Wegner-type bounds and to assess the spectral spacings for generic hulls $v : \Omega \to \mathbb{R}$.

Quite naturally, some hulls labeled by $\vartheta \in \Theta$ have to be excluded, essentially for the same reasons that some samples of the IID random potentials have to be excluded in the proof of localization: for example, setting all $\vartheta_{n,k} = 0$, we get $V(x; \omega; \vartheta) \equiv 0$, hence the operator $H = \Delta$ with a.c. spectrum.

The above discussion suggests, and the analysis carried out below actually confirms that, although the structure of probability (or, more generally, measure) space on the set of auxiliary parameters $\vartheta_{n,k}$ is a very convenient tool, it can be replaced by the structure of a metric space. The unwanted values of parameters are covered by small balls, since the conditions required for a successful application of the MSA procedure, in terms of the potentials and matrix elements of the resolvents, have the form of inequalities. One particular advantage of the probabilistic language is the possibility to adapt the conventional Wegner estimate in a straightforward way.

The role of Sect. 4 and 5 is to establish the crucial, and quite unusual, property of the operators $H_{B_\ell(x)}(\omega, \vartheta)$: for any ”good” $\vartheta \in \Theta$ (this notion will be made precise), and for any (not just $\mathbb{P}$-a.e.) $\omega \in \Omega$, the spectra of the operators $H_{B_\ell(x)}(\omega, \vartheta)$, $H_{B_\ell(y)}(\omega, \vartheta)$ in disjoint cubes with $|x - y| \leq L^4$ cannot be “dangerously close” to each other, so the usual small denominators, appearing in the perturbative scaling analysis, are never excessively small.

Pictorially, there are no resonances in our model, exactly as in the Maryland model and its generalizations studied in [1].

4. Separation of local spectra: initial scale

We work with the DSO $H_B$ in cubes $B = B_L(x) \subset \mathbb{Z}^d$, with Dirichlet boundary conditions: $H_B = 1_B H 1_B | \ell^2(B)$. With $L = 0$, $H_{B_n(x)}$ is the multiplication by $gV(x)$. 
Lemma 4.1. Assume the condition (UPA), with fixed the parameters $A, C, \lambda$, and fix the decay exponent $b > 0$ in the definition of the sequence $\{a_n\}$ (cf. (1.12)). Then there exist constants $c_1, c_2 > 0$ with the following properties:

For all $g > 0$ large enough there exists an integer $L_0 = L_0(g) \geq e^{c_1 \ln^{1/2} g}$ (cf. (3.3)) and a positive number $\beta_0(g) \leq e^{-c_2 \ln^{1/2} g}$ (cf. (3.5)) such that for any $\omega \in \Omega$, any $u \in \mathbb{Z}^d$, with $\delta_0 = \delta_0(g) a_{\tilde{N}(L_0)}$ (cf. (3.4)) one has

$$\mathbb{P}^\Theta \left\{ \text{Sep} \left[ g \mathcal{V}_{\tilde{N}(L_0)}(\cdot; \omega; \vartheta), B_{L_0}(u) \right] < 5\delta_0 \right\} \leq C L_0^{4d} \beta_0(g)$$

(4.3)

and

$$\mathbb{P}^\Theta \left\{ \text{Sep} \left[ g \mathcal{V}(\cdot; \omega; \vartheta), B_{L_0}(u) \right] < 4\delta_0 \right\} \leq C L_0^{4d} \beta_0(g).$$

(4.4)

Consequently, for any $m \geq 1$ there exists $g_0 = g_0(m) \in (0, +\infty)$ such that for all $g \geq g_0(m)$, the estimates (4.3) and (4.4) hold true, with $4\delta_0 \geq 16d e^{4m}$.

Equivalently, one can say that (4.3)–(4.4) hold true for sufficiently large $g > 0$ with $4\delta_0 \geq 16d e^{4m(g)}$, where $m(g) \to +\infty$ as $g \to +\infty$.

Proof. 1. Estimates for the truncated potential. Setting for brevity $B = B_{L_0}(u)$, we have for any $\tilde{N} \geq 1$ and $s > 0$

$$\mathbb{P}^\Theta \left\{ \min_{x \neq y \in \tilde{B}} |g \mathcal{V}_{\tilde{N}}(x; \omega; \vartheta) - g \mathcal{V}_{\tilde{N}}(y; \omega; \vartheta)| < gs \right\} \leq \frac{1}{2} \left( |\tilde{B}| - 1 \right) \max_{x \neq y \in \tilde{B}} \mathbb{P}^\Theta \left\{ |\mathcal{V}_{\tilde{N}}(x; \omega; \vartheta) - \mathcal{V}_{\tilde{N}}(y; \omega; \vartheta)| < s \right\}. \quad (4.5)$$

Given $L_0 \geq 2$, let $\tilde{N} = \tilde{N}(L_0)$, with $L \mapsto \tilde{N}(L)$ defined in (2.10). Fix $x \in \tilde{B}$, $\omega \in \Omega$; $\mathcal{V}(x; \omega; \vartheta) \equiv \mathcal{V}(T^x \omega; \vartheta)$ is a random variable of the form (1.7), and (cf. (2.14))

$$v_{\tilde{N}}(T^x \omega; \vartheta) = a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x)) + \sum_{n \in \tilde{N}} a_n(\vartheta, \tilde{k}_{\tilde{N}}(x))$$

$$= a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x)) + \tilde{\vartheta}_{\tilde{N}, x, \omega}(\vartheta),$$

(4.6)

where $\tilde{\vartheta}_{\tilde{N}, x, \omega}$ is a sum of random variables (relative to $(\Theta, \mathbb{P}^\Theta)$), independent of $a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x))$. By construction, $\tilde{\vartheta}_{\tilde{N}, \tilde{k}_{\tilde{N}}(x)} \sim \text{Unif}([0, 1])$, so $a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x))$ admits the probability density bounded by $a_{\tilde{N}}^{-1}$, and so does the sum $a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x)) + \tilde{\vartheta}_{\tilde{N}, x, \omega}(\vartheta)$.

Similarly, we decompose

$$v_{\tilde{N}}(T^y \omega; \vartheta) = a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(y)) + \tilde{\vartheta}_{\tilde{N}, y, \omega}(\vartheta).$$

(4.7)

By definition of $\tilde{N} = \tilde{N}(L_0) = \tilde{v}(L_0^4)$, the elements of the partition $\mathcal{C}_{\tilde{N}}$ separate the points $T^x \omega$ and $T^y \omega$, for all $x, y \in B_{L_0}(0)$, $x \neq y$, thus $\tilde{k}_{\tilde{N}}(x) \neq \tilde{k}_{\tilde{N}}(y)$, and $a_{\tilde{N}}(\vartheta, \tilde{k}_{\tilde{N}}(x))$ is independent of $\tilde{\vartheta}_{\tilde{N}, \tilde{k}_{\tilde{N}}(y)}$.
Denote $X(\vartheta) = \vartheta \tilde{N}, \tilde{k}_N(x), Y(\vartheta) = \vartheta \tilde{N}, \tilde{k}_N(x)$ ($\omega$ is fixed and omitted); then
\[ p^\Theta \left\{ \left| v_N(T^x; \omega; \vartheta) - v_N(T^y; \omega; \vartheta) \right| \leq t \right\} = \mathbb{A}_L^a \left\{ \left| X - Y \right| \leq a_N^{-1} t \right\}
\]
where the random variable (we omit again its parameter $\omega$ which is fixed)
\[ Z(\vartheta) = a_N^{-1} \left( \tilde{g}_{N,y,\omega}(\vartheta) - \tilde{g}_{N,x,\omega}(\vartheta) \right)
\]
is $\mathbb{A}_L^a$-measurable, so we have, $p^\Theta$-a.s.,
\[ p^\Theta \left\{ \left| X - X \right| \leq a_N^{-1} t \right\} \leq \sup_{s \in \mathbb{R}} p^\Theta \left\{ \left| X - Y \right| \leq a_N^{-1} t \right\}
\]
\[ \leq 2a_N^{-1} t,
\]
since $X \sim \text{Unif}[0,1]$ and $Y$ is independent of $X$, so $X - Y$ has density $\leq 1$. Thus
\[ p^\Theta \left\{ \left| v_N(T^x; \omega; \vartheta) - v_N(T^y; \omega; \vartheta) \right| \leq t \right\} = \mathbb{E}^\Theta \left[ p^\Theta \left\{ \left| v_N(T^x; \omega; \vartheta) - v_N(T^y; \omega; \vartheta) \right| \leq t \right\} \right]
\]
\[ \leq 2a_N^{-1} t.
\]
Recalling (4.5), we conclude that
\[ p^\Theta \left\{ \min_{x \neq y \in \mathbb{B}} \left| gV_N(x; \omega; \vartheta) - gV_N(y; \omega; \vartheta) \right| < S \right\} \leq C L_0^{4d} a^{-1} N. \tag{4.8}
\]

2. Perturbation estimates. Let $\beta > 0$ (a suitable value of $\beta$ will be specified later) and
\[ s = 5\beta a_N.
\]
Then we infer from (4.8) that
\[ p^\Theta \left\{ \text{Sep} \left[ gV_N, B_L^a \right] < 5g \beta a_N \right\} \leq C' L_0^{4d} \beta. \tag{4.9}
\]
Let
\[ \Theta^{-1}(g, \omega) := \left\{ \vartheta \in \Theta : \text{Sep} \left[ gV_N, B_L^a \right] \geq 5g \beta a_N \right\}, \tag{4.10}
\]
then by (4.9), we have
\[ p^\Theta \left\{ \Theta^{-1}(g, \omega) \right\} \geq 1 - C' L_0^{4d} \beta. \tag{4.11}
\]
On the other hand, $\| gV - gV_N \|_\infty \leq \frac{1}{2} g 2^{-2N} a_N$ (cf. (2.7)). Now set
\[ \beta = \beta_0(L_0) := 2^{-2N}(L_0), \tag{4.12}
\]
then
\[ \| gV - gV_N \|_\infty \leq \frac{1}{2} g 2^{-2N} a_N = \frac{1}{2} g \beta a_N = \frac{1}{2} g \delta_0.
\]
Thus for any $\vartheta \in \Theta^{-1}(g, \omega)$, we have, by triangle inequality,
\[ \text{Sep} \left[ gV, B_L^a \right] \geq \text{Sep} \left[ gV_N, B_L^a \right] - 2\| gV - gV_N \|_\infty
\]
\[ \geq 5g \delta_0 - \frac{1}{2} g \delta_0
\]
\[ = 4g \delta_0 = 4g 2^{-2N} a_N. \tag{4.13}
\]
Further, we need the quantity \( \text{Sep} \left[ gV, B_{L_4} \right] \) to be large, viz.

\[
\text{Sep} \left[ gV, B_{L_4} \right] \geq 16d e^{4m}, \ m \geq 1. \tag{4.14}
\]

On the account of the lower bound (4.13), it suffices that

\[ 4g 2^{-2b \bar{N}} a_{\bar{N}} \geq 16d e^{4m}, \quad \text{where} \quad a_{\bar{N}(L_0)} = 2^{-b \bar{N}^2(L_0)}. \]

Consequently, given the numbers \( g > 0, m \geq 1, \) we set

\[
L_0(g) = L_0(g, m) := \max \left\{ l_0 \in \mathbb{N} : 4d e^{4m} 2^b \bar{N}^2(L_0) + 2b \bar{N}(L_0) \leq l_0 \right\}, \tag{4.15}
\]

\[
\beta_0(g) = \beta_0(g, m) := \beta(L_0(g, m)).
\]

Then it is readily seen that, for any fixed \( m, \)

\[
\lim_{g \to \infty} L_0(g) = +\infty, \quad \lim_{g \to +\infty} \beta_0(g) = 0. \tag{4.16}
\]

Indeed, recall that \( \bar{N}(L_0) \leq 29A \ln L_0 \) (cf. (2.11)); since \( \bar{N}^2 > 2 \bar{N} \) for \( \bar{N} > 1, \) we have

\[
b \bar{N}^2(L_0) + 2b \bar{N}(L_0) \leq 2b \bar{N}^2(L_0) \leq 2 \cdot (29A \ln L_0)^2.
\]

Therefore, are admissible in (4.15) the integers \( L_0 \) such that

\[
\ln^2 L_0 \leq \frac{\ln g - \ln (4de^{4m})}{2b(29A)^2}.
\]

For \( g \) large enough, so \( \frac{1}{2} \ln g \geq \ln(4de^{4m}), \) the above RHS is bigger than \( \frac{\ln g}{58A \sqrt{b}}, \)

thus for such \( g > 0, \) the maximum \( L_0(g) \) in (4.15) satisfies the lower bound

\[
\ln L_0(g) \geq c_1 \ln^{1/2} g, \quad c_1 = c_1(A, b) := \frac{1}{58A \sqrt{b}}. \tag{4.17}
\]

One can transform it into a formal definition, setting \( L_0(g) := \lfloor e^{c_1 \ln^{1/2} g} \rfloor. \)

The quantity \( \beta(L_0), \) with \( L_0 = L_0(g), \) becomes a function of \( g, \) and we have

\[
\beta_0(g) = \beta(L_0(g)) = 2^{-b \bar{N}(L_0(g))} = 2^{-4b \ln L_0(g)} \leq e^{-c_2 \ln^{1/2} g}, \tag{4.18}
\]

with \( c_2 = c_2(b, A, C_A) \geq 68Ab \) (recall that by (2.11), \( \tilde{A} \geq 17A \)).

Collecting (4.9), (4.10), (4.11) and (2.12), we obtain

\[
\mathbb{P}^\Theta \left\{ \Theta^{(-1)}(g, \omega) \right\} \equiv \mathbb{P}^\Theta \left\{ \text{Sep} \left[ gV_{\bar{N}}, B_{L_4} \right] \geq 5g \delta_0 \right\} \geq 1 - C L_0^{8d} \beta_0(g) \geq 1 - C L_0^{-12bA + 8d}. \tag{4.19}
\]

Since the inequality \( \text{Sep} \left[ gV_{\bar{N}}, B_{L_4} \right] \geq 5g \delta_0 \) implies \( \text{Sep} \left[ gV, B_{L_4} \right] \geq 4g \delta_0 \) (cf. (4.13)), both asserted bounds (4.3)–(4.4) follow from (4.19).

For any \( m \geq 1, \) there is a sufficiently large \( g_*(m) > 0 \) such that for \( g \geq g_*(m), \)

\[
\text{Sep} \left[ gV, B_{L_4} \right] \geq 4g \delta_0 \geq 16d e^{4m}, \tag{4.20}
\]

since

\[
g \delta_0(g) \geq g \ln g^{1/2} + g^{1-o(1)}, \quad \text{as} \ g \to \infty.
\]

One can start with \( m \geq 1 \) and find an appropriate lower threshold \( g_*(m) \) for \( g, \) or start with \( g > 0 \) large enough and define

\[
m = m(g) := \frac{1}{4} \ln \frac{g \delta_0(g)}{4d} \quad \text{as} \ g \to +\infty + \infty. \tag{4.21}
\]
4.2. Separation of finite trajectories. Introduce some geometrical objects related to the length scales $L_j$, $j \geq 0$. First, let

$$R_j = \frac{1}{6} C_\nu^{-1} L_j^{-4A}$$

(recall that $A, C_\nu \in \mathbb{N}^+$), and cover the torus $\Omega$ redundantly by the union of $N_{R_j} := (R_j)^{-v}$ cubes $Q_{3R_j}(\omega_i), i \in \mathbb{Z}^d \cap [1, N_{R_j}]$, of radius $3R_j$ and with centers of the form

$$\omega_i = [l_1 R_j, \ldots, l_d R_j], l_1, \ldots, l_d \in [0, (2R_j)^{-1} - 1].$$

The order of numbering can be arbitrary. Next, decompose each cube $Q_{3R_j}(\omega_i)$ into a union of $3^v$ neighboring sub-cubes $Q_{R_j}(\omega_i')$ of radius $R_j$, which we number starting from the central sub-cube, $Q_{R_j}(\omega_{i,1})$. Observe that the collection of all central sub-cubes $Q_{i,1}(R_j)$ covers the torus $\Omega$, and $\omega_{i,1} \equiv \omega_i$.

Similarly, cover the torus $\Omega$ by adjacent cubes $Q_{r_j}(\omega_i^\nu)$ of radius

$$r_j = C_\nu^{-1} L_j^{-4A'} R_j = (6C_\nu C_\nu A L_j^{4A+4A'})^{-1}.$$

Lemma 4.2. (See Fig. 1.) Fix $j \geq 0$ and consider $B_{L_j}(0) \subset \mathbb{R}^d$. Fix any point $z \in B_{L_j}(0)$ and a cube $Q_{r_j}(\omega_i^\nu)$. If $T^z \omega_i^\nu \in Q_{R_j}(\omega_{i,j})$ for some $i_0 = i_0(i, z)$, then

$$T^z(Q_{r_j}(\omega_i^\nu)) \subset Q_{3R_j}(\omega_i) \equiv Q_{3R_j}(\omega_{i,1}).$$
Proof. For any \( \omega \in Q_{r_j}(\omega_i') \), we have \( \text{dist}(\omega_i', \omega) \leq r_j \), thus by (DIV) and (4.23),
\[
\text{dist}(T^z \omega_i', T^z \omega) \leq C_{A'}(L_j^4)A' \text{dist}(\omega_i', \omega) \leq C_A L_j^{4A'} r_j = R_j.
\]
(4.25)
By assumption,
\[
\text{dist}(T^z \omega_i', \omega_{i,1}) \equiv \text{dist}(T^z \omega_i', \omega_i') \leq R_j,
\]
(4.26)
therefore, by (4.25) and (4.26),
\[
\text{dist}(T^z \omega, \omega_{i,1}) \leq \text{dist}(T^z \omega, T^z \omega_i') + \text{dist}(T^z \omega_i', \omega_{i,1}) \leq R_j + R_j < 3R_j,
\]
yielding the assertion (4.24). \( \square \)

For each \( j \geq -1 \), define the integers
\[
\tilde{N}_j = \begin{cases} \tilde{N}(L_j), & j \geq 0 \\ \tilde{N}(L_0), & j = -1 \end{cases}, \quad L_j = \begin{cases} (12C_A C_{A'})^\nu L_j^{4\nu(A+A')}, & j \geq 0 \\ (12C_A C_{A'})^\nu L_0^{4\nu(A+A')}, & j = -1 \end{cases}
\]
(4.27)
with \( L \mapsto \tilde{N}(L) = O(\ln L) \) defined in (2.10).

**Corollary 4.1.** Fix any integer \( j \geq 0 \). There exists a finite collection of points, \( T_j = \{ \tau_{j,l}, 1 \leq l \leq L_j' \leq L_j \} \), and a measurable partition of \( \Omega = T^\nu \), \( P_j = \{ P_{j,l} \supseteq \tau_{j,l}, 1 \leq l \leq L_j' \} \), such that

- any cube \( Q_{r_j}(\omega_i') \) is covered by at most \( 2^\nu \) elements of \( P_j \);
- for every \( z \in B_{L_j}(0) \), the image \( T^z P_{j,l} \) is covered by exactly one element of the partition \( C_{\tilde{N}(L_j)+1} \).

**Proof.** For notational brevity, let \( n = \tilde{N}(L_j) \). Fix a cube \( Q_{r_j}(\omega_i') \subset \Omega \) and any \( z \in B_{L_j}(0) \). Consider the image \( T^z Q_{r_j}(\omega_i') \). By Lemma 4.2, it is covered by one cube \( Q_{3R_j}(\omega_i') \), with some \( i_0 = i_0(n, i, j) \). Since by (4.22) and (2.13) we have
\[
\text{diam} \ 3R_j(\omega_i') = 6R_j = C_A^{-1}n^4 \nu(A) \leq 2^{-\nu}(\tilde{N}(L_j)-1) = 2^{n-1},
\]
the image \( T^z Q_{r_j}(\omega_i') \) is covered by at most \( 2^\nu \) adjacent cubes of side length \( 2^{n-1} \) – the elements of the partition \( C_{n+1} \), which are sub-cubes of \( Q_{3R_j}(\omega_i') \). Following the notation introduced in Sect. 1.5 (cf. (1.11)), denote these cubes by \( C_{n,k_0,\nu}, l = 1, \ldots, 2^\nu \). (Recall that, by definition, for each pair \( (n, k_0) \), the cube \( C_{n,k_0} = \text{supp} \varphi_{n,k_0} \) is partitioned in to the sub-cubes \( C_{n,k_0}, i = 1, \ldots, 2^\nu \), of side length \( 2^{-n-1} \), on which the Haar’s wavele \( \varphi_{n,k_0} \) takes constant value \( \pm 1 \).

Now the required partition \( P_j \) can be formed by taking all the non-empty intersections of the form \( T^{-z} C_{n,k_0,\nu} \cap Q_{r_j}(\omega_i') \), \( l = 1, \ldots, 2^\nu \), for all \( i_0 \). For the collection \( T_j \), it suffices to pick exactly one point from each set \( P_{j,l} \), and denote it by \( \tau_{j,l} \). Since the number of cubes \( Q_{r_j}(\omega_i') \subset T^\nu \) of diameter \( r_j \) is bounded by \( r_j^{-\nu} = (6C_A C_{A'} L_j^{4\nu(A+A')})^{-\nu} \), we have card \( T_j \leq (12C_A C_{A'})^\nu L_j^{4\nu(A+A')} = L_j \), as asserted. \( \square \)

Now define the operator-valued mappings
\[
\eta_j^n_{\tilde{N}_j} : \omega \mapsto \begin{cases} H^n_{B_{L_j}(0)}(\omega; \theta) \uparrow \ell^2(B_{L_j}(0)), & j \geq 0 \\ gV^n_{\tilde{N}(L_0)}(\omega; \theta) \uparrow \ell^2(B_{L_0}(0)), & j = -1 \end{cases}
\]
(4.28)
In the above formula, \( gV^n_{\tilde{N}(L_0)}(\omega; \theta) \) is the truncated potential on \( B_{L_0}(0) \), identified with the multiplication operator by this potential.
4.1. Fix any $j \geq -1$. For any fixed $\vartheta \in \Theta$, the mapping $\mathcal{N}_{j,\vartheta}$, defined in (4.28), is piecewise-constant on $\Omega$.

More precisely, let the collection $\mathcal{T}_j$ and the partition $\mathcal{P}_j$ be defined as in Corollary 4.1. Then $\mathcal{N}_{j,\vartheta}$ is constant on each element $\mathcal{P}_{j,i}$ of $\mathcal{P}_j$. Thus the operator-valued function $\mathcal{N}_{j,\vartheta}$ takes on $\Omega$ only a finite number of values,

$$\mathcal{N}_{j,\vartheta}(\mathcal{T}_{j,i}), \quad 1 \leq l \leq L_{j}', \leq L_j. \quad (4.29)$$

**Proof.** Fix $j \geq -1$ and let $\mathcal{N}_j$ be given by (4.27). By Corollary 4.1, the truncated hull $v_{\mathcal{N}_j}$ is constant on each element of the partition $\mathcal{P}_j$, and so is, therefore, the function $\mathcal{N}_{j,\vartheta}$, since the kinetic energy operator $\Delta$ (present in $H_{B_1^j(\omega; \vartheta)}$ for $j \geq 0$) is constant in $\omega \in \Omega$ (and in $\vartheta \in \Theta$). This proves the claim. \hfill \Box

4.3. **Separation bounds uniform in $\omega \in \Omega$.**

**Lemma 4.3.** For all $g > 0$ large enough, there is a measurable subset $\Theta^{-1}(g) \subset \Theta$ with

$$\mathbb{P}_\Theta \left\{ \Theta^{-1}(g) \right\} \geq 1 - CL_0^{-12bA + 8d + 4\omega(A + A')} \quad (4.30)$$

such that for any $\vartheta \in \Theta^{-1}(g)$ and every $\omega \in \Omega$, one has

$$\text{Sep}[g \mathcal{V}, L_0^4] \geq 4g_\delta_0.$$

**Proof.** Consider the sets

$$\Theta \supset P_{-1,l,1} \supset \tau_{-1,l}, \quad l = 1, \ldots, L_{-1}' \leq L_{-1},$$

introduced in the Sect. 4.2. By Lemma 4.3, the function

$$\mathcal{N}_{-1,\vartheta} : \omega \mapsto g \mathcal{V}_{\tau_{-1,l}}(\cdot; \vartheta) \mid B_{L_0^4}(0)$$

is constant on each $P_{-1,l,1}$, so if the required separation bound holds true for each phase point $\tau_{-1,l}, l \in [1, L_{-1}]$, then it also holds for every $\omega \in \Omega$. By Lemma 4.1 and Eqn. (4.19), which apply to any $\omega \in \Omega$, including of course $\tau_{-1,l}$,

$$\mathbb{P}_\Theta \left\{ \text{Sep}[g \mathcal{V}; \tau_{-1,l}; \vartheta], L_0^4 \right\} \leq CL_0^{-12bA + 8d}. \quad (4.31)$$

By (4.27), $L_{-1}' \leq L_{-1} = \text{Const}L_0^{4\omega(A + A')}$, yielding (4.30). \hfill \Box

4.4. **Uniform separation bounds for spectra: The initial scale.**

**Definition 4.1.** Given $E \in \mathbb{R}$ and a DSO $H_{B_L}(x)$, the cube $B_L(x)$ is called $E$-non-resonant ($E$-NR) if the following bound holds:

$$\text{dist} \left[ \Sigma(H_{B_L(x)}), E \right] \geq g_\delta_j.$$ 

Otherwise, it is called $E$-resonant ($E$-R).

**Definition 4.2.** Given a DSO $H_{B_L}(x)$ in a cube $B_L(x)$, we say that $B_L(x)$ is $(E, m)$-non-singular ($(E, m)$-NS), with $E \in \mathbb{R}$, $m > 0$, if for any $y \in \partial^- B_L(x)$ (cf. the definition of $\partial^-$ in (2.2))

$$|G_{B_L}(x, y; E)| \leq \left\{ \begin{array}{ll}
(3L)^{-d} e^{-\gamma(mL)}, & \text{if } L \geq 1, \\
(2d)^{-d} e^{-\gamma(mL)}, & \text{if } L = 0,
\end{array} \right. \quad (4.31)$$
where
\[
\gamma(m, L) := \begin{cases} 
  m(1 + L^{-1/8})L, & \text{if } L \geq 1, \\
  2m, & \text{if } L = 0.
\end{cases}
\] (4.32)

Otherwise, \(B_L(u)\) is called \((E, m)\)-singular \(\{(E, m)\}-S\).

**Lemma 4.5.** Let the subset \(\Theta^{(-1)}(g) \subset \Theta\) be defined as in Lemma 4.4. For any \(m \geq 1\), there exist an integer \(L_0 = L_0(m) \geq 2\) and a real number \(g_*(m) > 0\) such that for \(g \geq g_*(m)\) and for any \(\vartheta \in \Theta^{(-1)}(g)\), any \(\omega \in \Omega\), any \(u \in \mathbb{Z}\) and any \(E \in \mathbb{R}\), there is at most one single-site cube \(\{x\} = B_0(x) \subset B_{L_0}(u)\) which is \((E, m)\)-S.

**Proof.** It follows from the definition of the subset \(\Theta^{(-1)}(g)\) (cf. Lemma 4.4 and (4.30)) that, for an arbitrarily large \(m \geq 1\) and all \(g > 0\) large enough, so that \(m(g) \geq m\) with \(m(g)\) defined as in (4.21), for all \(x, y \in B_{L_0}(0)\) with \(x \neq y\),
\[
|gV(x; \omega; \vartheta) - gV(y; \omega; \vartheta)| \geq 4g\delta_0 \geq 16d\varepsilon^m,
\]
thus there is no pair of points \(x, y \in B_{L_0}(u), x \neq y, \) such that
\[
\max \{ |gV(x; \omega; \vartheta) - E|, |gV(y; \omega; \vartheta) - E| \} < 2g\delta_0.
\] (4.33)

Given any \(x \neq y \in B_{L_0}(u)\), for at least one point \(z \in \{x, y\}\), we have
\[
\|G_{B_0(z)}(E)\| = |(gV(z; \omega; \vartheta) - E)^{-1}| \leq (2g\delta_0)^{-1} \leq (8d)^{-1}e^{-4m} < (2d)^{-1}e^{-\gamma(m, 0)},
\]
yielding the \((E, m)\)-NS property of \(B_0(z)\). Hence no pair of distinct single-site cubes \(B_0(x), B_0(y) \subset B_{L_0}(u)\) can be \((E, m)\)-S for the same value of \(E \in \mathbb{R}\). □

Decay of the Green functions in the balls of radius \(L_0 > 0\) can be assessed with the help of a variant of the Combes–Thomas estimate (cf. [14], [21]) adapted to large spectral gaps.

**Proposition 4.1** (Cf. [12, Theorem 2.3.4]). Suppose that for some \(E \in \mathbb{R}\), one has \(\text{dist}(E, \sigma(H_{B_L(u)})) \geq \eta > 4d\). Then for any \(x, y \in B_L(u)\)
\[
|G_{B_L(u)}(x, y; E)| \leq 2\eta^{-1}e^{-\mu|x-y|} < e^{-\mu|x-y|}
\] (4.34)

with
\[
\mu = \frac{1}{2} \ln \frac{\eta}{4d}.
\] (4.35)

Consequently, for large \(g > 0\), any cube \(B_{L_0}(u)\) which is \(E\)-NR is also \((E, m)\)-NS.

**Proof.** The first assertion (4.34) is proved in [12]. If the cube \(B_{L_0}(u)\) is \(E\)-NR, then \(\text{dist}(E, \Sigma(H_{B_L(u)})) \geq \eta > 0\), with \(\eta = g\delta_0 \geq 4d\varepsilon^m > 4d\), so (4.34) implies
\[
|G_{B_L(u)}(x, y; E)| \leq e^{-\mu|x-y|},
\] (4.36)

where
\[
\mu = \frac{1}{2} \ln \frac{4d\varepsilon^m}{4d} = 2m = \gamma(m, L_0) + m(L_0 - L_0^{7/8}).
\]

For \(L_0 \geq 3\), one has \(L_0 - L_0^{7/8} > L_0/2\); for \(g > 0\) large and \(L_0 = L_0(g)\), the latter condition is fulfilled. Further, for \(|x - y| = L_0\) and \(g\) large enough, we have
\[
e^{-\mu|x-y|} \leq e^{-\gamma(m, L_0)}e^{-mL_0/2} \leq (3L_0)^{-1}e^{-\gamma(m, L_0)},
\]
thus \(B_{L_0}(u)\) is \((E, m)\)-NS. □
Lemma 4.6. Let be given real numbers \( m \geq 1 \) and sufficiently large \( g > 0 \), so that \( m(g) \geq m \) (with \( m(g) \) defined in (4.21)). Let \( L_0 = L_0(g) \), \( \delta_0 = \delta_0(g) \). Then for any \((\omega, \vartheta) \in \Omega \times \Theta^{(-1)}(g)\), any \( u \in \mathbb{Z} \) and any \( E \in \mathbb{R} \), there is no pair of disjoint \((E, m)\)-S cubes \( B_{L_0}(x), B_{L_0}(y) \subset B_{L_0}(u)\).

Proof. Consider any cube \( B_{L_0}(u)\). Suppose that some cube \( B_{L_0}(x) \subset B_{L_0}(u) \) is \((E, m)\)-S; then it be \( E \)-R, for otherwise it would be \((E, m)\)-NS, by Proposition 4.1. Therefore,

\[
\exists x_0 \in B_{L_0}(x) \mid |gV(x_0) - E| \leq 4g\delta_0. \tag{4.37}
\]

Consider any cube \( B_{L_0}(y) \subset B_{L_0}(u) \) disjoint from \( B_{L_0}(x) \). For any \( \vartheta \in \Theta^{(-1)}(g) \) and any \( \omega \in \Omega \),

\[
\min_{z \in B_{L_0}(y)} |gV(z; \vartheta; \omega) - gV(z; \vartheta; \omega)| \geq 4g\delta_0, \tag{4.38}
\]

so by the triangle inequality combined with (4.37) and (4.38),

\[
\min_{z \in B_{L_0}(y)} |gV(z; \vartheta; \omega) - E| \geq 4g\delta_0 - g\delta_0 = 3g\delta_0.
\]

Further, by the min-max principle, considering \( H \) as a perturbation of \( gV \) by \( H_0 = \Delta \), we have

\[
dist \left( \Sigma(H_{B_{L_0}(y)}), E \right) \geq \dist \left( \Sigma((gV)_{B_{L_0}(u)}), E \right) - \| \Delta \| \geq 3g\delta_0 - 2d \geq \eta
\]

with

\[
\eta = 2g\delta_0 \geq 8de^{4m},
\]

so that \( \frac{1}{2} \ln \frac{e}{\eta} \geq 2m \). Now the Combes-Thomas estimate (cf. (4.34)-(4.35)) implies

\[
\max_{z \in \partial B_{L_0}(y)} |G_{B_{L_0}(y)}(y, z; \omega; \vartheta)| \leq e^{-2mL_0} \leq (3L_0)^{-1}e^{-\gamma(m, L_0)L_0},
\]

for \( L_0 \) large enough, hence \( B_{L_0}(y) \) is \((E, m)\)-NS. The assertion follows. \( \square \)

5. Separation of local spectra: arbitrary scale

We will use the following notation: by writing \( \langle \Lambda', \Lambda'' \rangle \subset \Lambda \), we mean that \( \Lambda', \Lambda'' \subset \Lambda \) and \( \Lambda' \cap \Lambda'' = \emptyset \). Let

\[
D(L, \omega, \theta; x, y) = \dist \left( \Sigma(H_{B_{L}(x)}(\omega, \vartheta)), \Sigma(H_{B_{L}(y)}(\omega, \vartheta)) \right), \tag{5.1}
\]

\[
D(L, \omega, \theta) = \min_{(B_L(x), B_L(y)) \subset B_L(0)} D(L, \omega, \theta; x, y), \tag{5.2}
\]

\[
D(L, \theta) = \inf_{\omega \in \Omega} D(L, \omega, \theta), \tag{5.3}
\]

and, for \( N \geq 1 \),

\[
D^{(N)}(L, \omega, \theta; x, y) = \dist \left( \Sigma(H_{B_{L}(x)}^{(N)}(\omega, \vartheta)), \Sigma(H_{B_{L}(y)}^{(N)}(\omega, \vartheta)) \right), \tag{5.4}
\]

\[
D^{(N)}(L, \omega, \theta) = \min_{(B_L(x), B_L(y)) \subset B_{L}(0)} D^{(N)}(L, \omega, \theta; x, y), \tag{5.5}
\]

\[
D^{(N)}(L, \theta) = \inf_{\omega \in \Omega} D^{(N)}(L, \omega, \theta). \tag{5.6}
\]
5.1. Spectral separation estimates for the local Hamiltonians.

**Corollary 5.1.** Fix \( j \geq 0 \). Using the notations of Lemma 4.3 and (5.4)–(5.6), with \( \tilde{N}_j = \tilde{N}(L_j) \) (cf. (2.8)), assume that for each \( \tau_{j,l} \) one has (cf. (3.4))

\[
D(\tilde{N}_j)(L_j, \tau_{j,l}, \vartheta) \geq 5g\delta_j.
\]  

(5.7)

Then for the non-truncated operators, one has the uniform lower bound

\[
D(L_j, \vartheta) = \inf_{\omega \in \Omega} D(L_j, \omega, \vartheta) \geq 4g\delta_j.
\]  

(5.8)

**Proof.** By Lemma 4.3, the condition (5.7) implies \( D(\tilde{N}_j)(L_j, \omega, \vartheta) \geq 5g\delta_j \) for all \( \omega \in \Omega \). Further, by (2.7), we have

\[
\|H_{B_{L_j}(u)}(\omega; \vartheta) - H_{B_{L_j}(u)}^{(\tilde{N}_j)}(\omega; \vartheta)\| \leq \frac{1}{2}g^{2}2^{-2 \tilde{N}_j}a_{\tilde{N}_j} = \frac{1}{2}g\delta_j,
\]

so by the min-max principle, the perturbations \( |E_{\vartheta}^{\tilde{N}_j} - E_i| \) of the respective eigenvalues \( E_i \in \Sigma(H_{B_{L_j}(u)}(\omega; \vartheta)) \) induced by the approximation of \( H_{B_{L_j}(u)}(\omega; \vartheta) \) by \( H_{B_{L_j}(u)}^{(\tilde{N}_j)}(\omega; \vartheta) \), does not exceed \( \frac{1}{2}g\delta_j \). Consequently, if for a pair of eigenvalues of \( H_{B_{L_j}(u)}^{(\tilde{N}_j)}(\omega; \vartheta) \) we have \( |E_{\vartheta}^{\tilde{N}_j} - E_{\vartheta}^{\tilde{N}_j}| \geq 5g\delta_j \) (which follows from (5.7) for all \( \omega \)), then

\[
|E_{\vartheta} - E_{\vartheta}| \geq |E_{\vartheta}^{\tilde{N}_j} - E_{\vartheta}^{\tilde{N}_j}| - \frac{1}{2}g\delta_j \geq 5g\delta_j - g\delta_j = 4g\delta_j.
\]

Thus (5.7) implies (5.8).

We see that in order to guarantee a lower bound on \( D(L_j, \vartheta) \), it suffices to estimate a finite number of \( \vartheta \)-probabilities for the approximants of order \( \tilde{N}_j \) and \( \omega \in \{ \tau_{j,l}, 1 \leq l \leq L_j \} \), where \( L_j = L_j \). This task is performed in Sect. 5.2.

5.2. Exclusion of bad \( \vartheta \)-sets by the Wegner-type estimate.

**Lemma 5.1.** Under the assumptions (UPA) and (DIV), for any \( b > b_\ast := (8d + 4\nu A + 4\nu A^\prime)/(10A) \) and \( L_0 \) large enough

\[
P^{\vartheta}\left\{ \inf_{\omega \in \Omega} D(L_j, \vartheta) < 4g\delta_j \right\} \leq L_j^{-4b_\ast}. \tag{5.9}
\]

**Proof.** Fix \( j \geq 0 \) and let \( \tilde{N}_j = \tilde{N}(L_j) = O(\ln L_j) \) be given by (2.8). Further, fix a pair of disjoint cubes \( B_{L_j}(x), B_{L_j}(y) \subset B_{L_j}(0) \) and consider the operators \( H_{B_{L_j}(x)}^{(\tilde{N}_j)}(\omega), H_{B_{L_j}(y)}^{(\tilde{N}_j)}(\omega) \). Recall that all the points of any finite trajectory of the form \( \{ T^s \omega, z \in B_{L_j}(0) \} \} \) are separated by the elements of the partition \( C_{\tilde{N}_j} \). Such a separation occurs in particular for \( \{ T^s \omega, z \in B_{L_j}(x) \cup B_{L_j}(y) \} \), thus conditional on the sigma-algebra \( B_{\tilde{N}_j} \) generated by \( \{ \vartheta_{n,k}, n \neq \tilde{N}_j \} \), and with fixed \( \omega \in \Omega \), the probability distribution of the potential \( V_{\tilde{N}_j}(z; \omega, \vartheta) \) generated by the truncated hull \( v_{\tilde{N}_j} \), gives rise to the sample of independent random variables (relative to the probability space \( \Theta \), and not \( \Omega ! \))

\[
V_{\tilde{N}_j}(\omega, \vartheta) := \{ v_{\tilde{N}_j}(T^s \omega; \vartheta), z \in B_{L_j}(0) \}; \tag{5.10}
\]
each of them is uniformly distributed in its individual interval \([c_z, c_z + a_{N_j}]\), with 
\(c_z = c_z(\omega, \vartheta)\) determined by the random (in \(\vartheta\)) amplitudes \(\vartheta_{n,k}\), from generations 
with \(n < N_j\). Therefore, conditional on \(B_{N_j}\), the independent random variables 
listed in (5.10) have individual probability densities, uniformly bounded by \(a_{N_j}^{-1}\). As 
a result, conditional on \(B_{L_j}\), the operators \(H_{B_{L_j}(z)}^{(N)}\) and \(H_{B_{L_j}(y)}^{(N)}\) are independent, 
and for every fixed \(\tau, l \in \mathcal{T}_j\), by Theorem 3 with \(J = 1\) (Wegner-type bound),

\[
P^{\Theta} \left\{ D^{(N)}(L_j, \tau, l, \vartheta; x, y) \leq 5g\delta_j \right\} \\
= P^{\Theta} \left\{ \mathbb{P} \left( \sum_{\tau} H_{B_{L_j}(z)}^{(N)}(\tau, l; \vartheta), \sum_{\tau} H_{B_{L_j}(y)}^{(N)}(\tau, l; \vartheta) \right) \leq 5g\delta_j | B_{L_j} \right\} \\
\leq \sup_{\lambda \in \mathbb{R}} \operatorname{ess sup} P^{\Theta} \left\{ \left| \sum_{\tau} H_{B_{L_j}(z)}^{(N)}(\tau, l; \vartheta), \lambda \right| \leq 5g\delta_j | B_{L_j} \right\} \\
\leq 3^{2d} L_j^{-d} a_{N_j}^{-1} 5\delta_j \cdot g \cdot g^{-1}.
\]

Since the number of all pairs \(x, y \in B_{L_j}(0)\) is bounded by \(|B_{L_j}(0)|^2 / 2 \leq 3^{2d} L_j^{8d} / 2\), we obtain (cf. (5.5))

\[
P^{\Theta} \left\{ D^{(N)}(L_j, \tau, l, \vartheta) \leq 5g\delta_j \right\} \leq \frac{1}{2} 3^{2d} L_j^{8d} a_{N_j}^{-1} 5\delta_j.
\]

Further,

\[
P^{\Theta} \left\{ \min_{l} D^{(N)}(L_j, \tau, l, \vartheta) < 5g\delta_j \right\} \leq L_j \max_{l} P^{\Theta} \left\{ D^{(N)}(L_j, \tau, l, \vartheta) < 5g\delta_j \right\}
\leq C(d, \nu) L_j^{8d+4\nu A+4\nu A'} a_{N_j}^{-1} \delta_j.
\]

By Corollary 5.1, we conclude that

\[
P^{\Theta} \left\{ \inf_{\omega \in \Omega} D(L_j, \vartheta) < 4g\delta_j \right\} \leq P^{\Theta} \left\{ \min_{l} D^{(N)}(L_j, \tau, l, \vartheta) < 5g\delta_j \right\}
\leq C_1(d, \nu) L_j^{8d+4\nu A+4\nu A'} a_{N_j}^{-1} \delta_j.
\]

By construction (cf. (3.4)),

\[
\delta_j = 2^{-2b N_j} a_{N_j} \leq C''(L_j^{4})^{-3b A} a_{N_j}.
\]

By our assumption, \(b = \epsilon + (8d + 4\nu A + 4\nu A')/(10A)\), \(\epsilon > 0\), thus for large \(L_0\),

\[
P^{\Theta} \left\{ \inf_{\omega \in \Omega} D(L_j, \vartheta) < 4g\delta_j \right\} \leq C_2(d, \nu) L_j^{8d+4\nu A+4\nu A'-12b A}
\leq C_2(d, \nu) L_j^{-\epsilon} \cdot L_j^{-2b A} \leq L_j^{-b A},
\]

which proves (5.9). \(\square\)

Now define the sets

\[
\Theta^{(j)}(g) := \left\{ \vartheta \in \Theta : \inf_{\omega \in \Omega} D(L_j, \vartheta) \geq 4g\delta_j \right\}, \quad j \geq 0,
\]

\[
\Theta^{(\infty)}(g) := \cap_{j \geq -1} \Theta^{(j)}(g).
\]

(5.11)
Corollary 5.2. Under the assumptions (UPA) and (DIV), for any \( b > b_* := (8d + 4νA + 4νA')/(10A) \) and \( L_0 \) large enough (or for \( b > 0 \) large enough),

\[
\forall j \geq 0 \quad \mathbb{P}^\Theta \left\{ \Theta^{(j)}(g) \right\} \geq 1 - L_j^{-bA}
\]

and, therefore, owing to the estimate (4.11),

\[
\mathbb{P}^\Theta \left\{ \Theta^{(\infty)}(g) \right\} \geq 1 - \text{Const} \, e^{-c' \ln^{1/2} g} \xrightarrow{g \to \infty} 1.
\]

Proof. The estimate (5.12) follows directly from Lemma 5.1 and the definition (5.11) of the set \( \Theta^{(j)}(g) \). The second assertion (5.13) follows from (5.12) by a simple calculation, for \( g \) large enough, since \( \sum_{j \geq 1} L_j^{-bA} < \infty \), and \( L_j(g) = (L_0(g))^2^j \to \infty \) as \( g \to \infty \).

5.3. Sparseness of resonant cubes. Recall (cf. Definition 4.1) that, given \( E \in \mathbb{R} \) and a DSO \( H_{B_{L_j}(x)} \), the cube \( B_{L_j}(x) \) is called \( E \)-resonant if

\[
\text{dist} \left[ \Sigma(H_{B_{L_j}(x)}), E \right] < g\delta_j.
\]

Taking into account Corollary 5.2, we come to an important conclusion: for any ”good” value of \( \vartheta \) and every (not just P-a.e. !) \( \omega \in \Omega \), the \( E \)-R cubes are sparse:

Corollary 5.3. For \( g \) large enough and any \( (\omega, \vartheta) \in \Omega \times \Theta^{(\infty)}(g) \), for each \( j \geq 0 \) and any \( E \in \mathbb{R} \), there is no pair of disjoint \( E \)-R cubes \( B_{L_j}(x), B_{L_j}(y) \subset B_{L_j}(0) \).

Proof. Assume otherwise; then for some disjoint cubes in \( B_{L_j}(0) \)

\[
\text{dist} \left[ \Sigma(H_{B_{L_j}(x)}(\omega; \vartheta)), \Sigma(H_{B_{L_j}(y)}(\omega; \vartheta)) \right] < 4g\delta_j,
\]

which is impossible for \( \vartheta \in \Theta^{(j)}(g) \), due to (5.11).

6. Simplified scale induction for deterministic operators

Now we can start collecting the fruits of the tedious analysis of eigenvalue concentration for the local Hamiltonians \( H_{B_{L_j}(\omega; \vartheta)} \), performed in the previous sections.

6.1. Decay of the Green functions in finite cubes.

Definition 6.1. Let \( L \geq \ell \geq 0 \) be integers and \( q \in (0, 1) \). Consider a finite set \( \Lambda \subset \mathbb{Z}^d \) such that \( \Lambda \supset B_{L+1}(u) \). A function \( f : \Lambda \to \mathbb{R} \) is called \((\ell, q)\)-dominated in \( B_{L}(u) \) if for any cube \( B_{\ell}(x) \subset B_{L}(u) \) one has

\[
|f(x)| \leq q \max_{y : |y-u| \leq \ell+1} |f(y)|.
\]

Below we use the notation \( \mathcal{M}(f, \Lambda) := \max_{x \in \Lambda} |f(x)| \).

The motivation for this definition comes from the following observation.

Lemma 6.1. Consider a cube \( B = B_{L}(u) \subset B_{L+1}(u) \subset \Lambda \subset \mathbb{Z}^d \), \( L \geq \ell \geq 0 \), \( u \in \mathbb{Z}^d \), and the operator \( H_\Lambda = -\Delta_\Lambda + gV \) in \( \ell^2(\Lambda) \) with fixed potential \( V \).Fix \( E \in \mathbb{R} \) and let \( \psi \in \ell^2(\Lambda) \) be a normalized eigenfunction of \( H_\Lambda \) with eigenvalue \( E \). If every cube \( B_{\ell}(x) \subset B \) is \((E, m)\)-NS for some \( m \geq 1 \), then the function \( x \mapsto |\psi(x)| \) (bounded by 1) is \((\ell, q)\)-dominated in \( B \), with \( q = e^{-\gamma(m, \ell)} \).
Proof. By the Geometric Resolvent Inequality for the eigenfunctions (cf. [21]),

$$|\psi(x)| \leq |B_\ell(x)| \max_{y:|y-x|=\ell} |G_{B_\ell}(x, y; E)| \max_{y:|y-x|=\ell+1} |\psi(y)|.$$  

The assumed \((E, m)\)-NS property of \(B_\ell(x) \subset B\) implies that, for \(\ell \geq 1\), the two maxima figuring in the RHS are bounded, respectively, by \(|B_\ell(x)|^{-1} e^{-\gamma(m, \ell)}\) and by \(\|\psi\|_\infty\). This proves the claim. \(\square\)

Lemma 6.2 (Cf. Lemma 4 in [10]). Consider a cube \(B = B_{L_{k+1}},\; k \geq 0\), and \(H_B\) with fixed potential \(V\). Pick \(x_0, y_0 \in B\) with \(|x_0 - y_0| > L_k\), and fix \(E \in \mathbb{R}\). Suppose that \(B\) is \(E\)-NR and every cube \(B_{L_k}(x) \subset B\) is \((E, m)\)-NS for some \(m \geq 1\). Then the function

\[
f_{y_0} : x \mapsto |G_B(x, y_0; E)|
\]

is \((L_k, q)\)-dominated in \(B\), with \(q = e^{-\gamma(m, L_k)}\), and bounded by \(e^{L_k}\).

The proof is similar to that of Lemma 6.1 and will be omitted; the upper bound on \(f_{y_0}\) follows, of course, from the \(E\)-NR property.

Lemma 6.3 (Cf. Lemma 2 in [10]). Suppose that a function \(f : \Lambda \to \mathbb{R}_+\), with \(\mathbb{Z}^d \supset \Lambda \supset B_{L_{k+1}}(x)\), is \((\ell, q)\)-dominated in \(B_{L_k}(x)\). Then

\[
|f(x)| \leq q^{(\frac{\ell+1}{\ell+2})} M(f, \Lambda) \leq q^{(\frac{\ell+1}{\ell+2})} M(f, \Lambda).
\]

We omit the proof, the details of which can be found in Refs. [10] and [12].

Definition 6.2. A cube \(B_{L_{j+1}}(u), j \geq 0\), is called \(m\)-bad, if for some \(E \in \mathbb{R}\), it contains at least two disjoint \((E, m)\)-S cubes of radius \(L_j\). Otherwise, it is called \(m\)-good.

The following statement is a (simpler) variant of Lemma 4.2 in [16]; similar results have been used in numerous papers using the Multi-Scale Analysis; cf. e.g., Lemma 4.4 in [15], or Theorem 10.14 and a stronger Theorem 10.20 in the review [21] by Kirsch, or Theorems 2.4.1, 2.4.3 and Lemma 2.4.4 in [12], or Lemma 5 in [11]. For these reasons, and for brevity, we omit the proof.

Lemma 6.4. For \(m \geq 1\) and \(L_0\) large enough, if a cube \(B_{L_{j+1}}(u), j \geq 0\), is \(m\)-good and \(E\)-NR for some \(E \in \mathbb{R}\), then it is \((E, m)\)-NS.

Introduce the following property which will be proved by scale induction:

Sparse(\(L_{-1}\)) : For all \(\vartheta \in \Theta^\infty(g), \omega \in \Omega, E \in \mathbb{R}\) and \(u \in \mathbb{Z}^d\), the cube \(B_{L_{j}}(u)\) contains no pair of disjoint \((E, m; \omega; \vartheta)\)-S cubes of radius \(L_{j}\).

Recall that we set \(L_{-1} = 0\); it is convenient to formulate in a special way the property Sparse(\(L_{-1}\)) \(\equiv\) Sparse(0):

Sparse(0) : For all \(\vartheta \in \Theta^\infty(g), \omega \in \Omega, E \in \mathbb{R}\) and \(u \in \mathbb{Z}^d\), there is at most one point \(x \in B_{L_{\emptyset}}(u)\) such that the single-site cube \(B_{0}(x)\) is \((E, m; \omega; \vartheta)\)-S.

The property Sparse(\(L_{j}\)) could be formulated in an equivalent way, where only the cubes \(B_{L_{j}}(0)\) (centered at the origin) are considered, since \(H_{B_{L_{j}}(0)}(\omega; \vartheta) = H_{B_{L_{j}}}(0)(\omega; \vartheta)\).
Anticipating the discussion in Sect. 7, we can say that the "exceptional" sites mentioned in Sparse(0) will be the centers of localization of unimodal eigenfunctions, with eigenvalues $E$ “close” to $gV(T^x; \vartheta)$: $E = gV(T^x; \vartheta) + O(\|\Delta\|)$.

For $g$ large enough (i.e., with $m = m(g) \gg 1$), the property Sparse($L_{-1}$) $\equiv$ Sparse(0) follows directly from Lemma 4.5, since $E^{(\infty)}(g) \subset \Theta(1)(g)$.

Theorem 5. Assume that Sparse($L_j$) holds for some $j \geq 0$. Then Sparse($L_{j+1}$) also holds true. Consequently, Sparse($L_0$) implies Sparse($L_j$) for all $j \geq 0$.

Proof. Fix any $\vartheta \in G(\infty)$, any $u \in \mathbb{Z}^d$ and any $E \in \mathbb{R}$. Consider the cube $B_{L_{j+1}}(u)$. By definition of the set $\Theta^{(j+1)}(g) \supset \Theta^{(\infty)}(g)$ and Corollary 5.3, there is at most one $E$-$R$ cube $B_{L_{j+1}}(v) \subset B_{L_{j+1}}^{(\infty)}(u)$. Let us show by contraposition that there can be no pair of disjoint $(E, m)$-S cubes $B_{L_{j+1}}(x)$, $B_{L_{j+1}}(y) \subset B_{L_{j+1}}^{(\infty)}(u)$.

Assume otherwise; then one of these cubes – w.l.o.g., let it be $B_{L_{j+1}}(x)$ – must be $E$-$NR$. Then by Lemma 6.4, the cube $B_{L_{j+1}}(x)$ must contain two disjoint $(E, m)$-S cubes of radius $L_1$, which contradicts the hypothesis Sparse($L_j$).

The property Sparse($L_j$), established at all scales $L_j$, $j \geq -1$, uniformly in $\omega \in \Omega$, is a stronger – deterministic – analog of the well-known probabilistic "double-singularity" bound for the pairs of $(E, m)$-S cubes, which represents the final result of the variable-energy MSA for random operators (cf., e.g., [16]).

6.2. From the MSA to strong dynamical localization. It would not be difficult now to infer from the results of the deterministic Multi-Scale Analysis, carried out in the previous subsection, strong dynamical localization for the operators $H(\omega; \vartheta)$.

Recall that for random Schrödinger operators the derivation of dynamical localization from the MSA bounds was obtained by Germinet and De Bièvre [27], by Damanik and Stollmann [17] (in a stronger form) and by Germinet and Klein [28] (in yet a stronger form, and for a larger class of random operators).

However, it will be even easier to infer in Sect. 8 pointwise and uniform in $\omega \in \Omega$ dynamical localization from the uniform (and not just semi-uniform, as in the theory of random Anderson Hamiltonians) decay of all eigenfunctions, proven in Sect. 7.

7. Uniform localization and unimodal eigenstates

Definition 7.1. Let $\psi \in \ell^2(\mathbb{Z}^d)$. A point $x \in \mathbb{Z}^d$ is called a localization center for $\psi$ iff $|\psi(x)| = \|\psi\|_{\infty}$.

Definition 7.2. A normalized eigenfunction $\psi$ of a DSO $H$ is called uniformly $m$-localized if
(a) $\psi$ has a localization center $\hat{x}$ such that $|\psi(\hat{x})|^2 > \frac{1}{2}$;
(b) $\forall y \in \mathbb{Z}^d \setminus \{\hat{x}\}$, one has $|\psi(y)| \leq e^{-m|x-y|}$.

When the value $m$ is irrelevant, we will simply say that $\psi$ is uniformly localized.

Sometimes we will refer to (a) as the unimodality property of $\psi$.

Note that every normalized eigenfunction in $\mathbb{Z}^d$ admits a non-empty but finite set of its localization centers; it will be denoted by $X(\psi)$. As shows assertion (A) of Lemma 7.1 below, with no loss of generality, we can restrict our analysis to the situation where the localization center is unique, so we will write $\hat{x}(\psi)$.

Lemma 7.1. (A) Any uniformly localized eigenfunction $\psi$ of a DSO $H$ has a unique localization center.
(B) Let \( \{ \psi_i, i \in \mathcal{I} \} \), \( \mathcal{I} \subset \mathbb{N} \), be an orthonormal family of uniformly localized eigenfunctions of a given DSO \( H \). Then for any \( x \in \mathbb{Z}^d \), there is at most one eigenfunction \( \psi \) with localization center \( x \).

**Proof.** (A) By Definition 7.1, \( |\psi(x)| \) takes the constant value \( \|\psi\|_{\infty} \) at all its localization centers \( x \in \hat{X}(\psi) \neq \emptyset \). It follows from the condition (a) of the uniform localization that \( |\psi(x)|^2 > \frac{1}{2} \) for \( x \in \hat{X}(\psi) \). By normalization,

\[
1 = \sum_{y \in \hat{X}(\psi)} |\psi(y)|^2 + \sum_{y \notin \hat{X}(\psi)} |\psi(y)|^2 \geq |\hat{X}(\psi)| \cdot |\psi(x)|^2 > \frac{1}{2} |\hat{X}(\psi)|,
\]

yielding \( |\hat{X}(\psi)| < 2 \).

(B) Assume otherwise, and let \( \phi, \psi \) be orthogonal, normalized, uniformly localized eigenfunctions of \( H \) with localization center \( x \), and let \( \chi = 1_{\mathbb{Z}^d \setminus \{x\}} \). Then we have \( \|\chi\phi\|_2^2, \|\chi\psi\|_2^2 < 1/2 \), thus by Cauchy–Schwarz inequality,

\[
|\langle \phi, \psi \rangle| = |\phi(x)\psi(x) + \sum_{y \neq x} \phi(y)\psi(y)| \geq |\phi(x)| \cdot |\psi(x)| - |\sum_{y \neq x} \phi(y)\psi(y)| > \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0,
\]

so that \( \phi \) and \( \psi \) are not orthogonal; this contradiction proves the claim. \( \square \)

In view of Lemma 7.1, given an eigenbasis \( \{ \psi_i, i \in \mathcal{I} \} \) of uniformly localized eigenfunctions of a DSO \( H \), we can associate with each localization center \( \hat{x} \) of some uniformly localized eigenfunction \( \psi_i \) a unique eigenvalue \( \hat{\lambda} = \hat{\lambda}(\hat{x}) \) – the one of the eigenfunction \( \psi_i \). In Sect. 9, we will show that, for typical \( \vartheta \) and all \( \omega \in \Omega \), the mapping \( \hat{x} \mapsto \hat{\lambda}(\hat{x}) \) is actually a bijection, since the spectrum of \( H(\omega; \vartheta) \) for such \( \vartheta \) is simple (and pure point).

To prove that every \( x \in \mathbb{Z}^d \) is a localization center for some eigenfunction of \( H(\omega; \vartheta) \) (cf. Theorem 6), we will need the following simple auxiliary result, valid for any DSO, regardless of the form of its potential.

**Lemma 7.2.** Let \( \psi \) be a normalized eigenfunction of a DSO \( H \), and let \( \hat{x} \) be any of its localization centers. Then for any \( L \in \mathbb{N} \), the cube \( B_L(\hat{x}) \) is \( (\hat{\lambda}(\hat{x}), m) \)-S.

**Proof.** Fix an eigenfunction \( \psi \) with localization center \( \hat{x} \) and assume otherwise. Since \( \gamma(m, L) > 0 \) and \( q := e^{-\gamma(m, L)} < 1 \), Lemma 6.1 implies

\[
\|\psi\|_{\infty} = |\psi(\hat{x})| \leq e^{-\gamma(m, L)L} \max_{y \in \partial \hat{B}_L(\hat{x})} |\psi(y)| \leq q \|\psi\|_{\infty},
\]

thus \( \|\psi\|_{\infty} = 0 \), which is impossible, since \( \|\psi\|_2 > 0 \). \( \square \)

**Lemma 7.3.** Consider a DSO \( H \) and assume that Sparse(L) holds true for all \( j \geq -1 \), and \( L_0 \geq 11 \). If, in addition, \( m > 0 \) is large enough, so that

\[
\sum_{r \geq 1} (3r)^d e^{-mr} \leq \frac{1}{2}, \tag{7.1}
\]

then every normalized eigenfunction \( \psi \) of \( H \), with localization center \( \hat{x} \), is uniformly \( m \)-localized at \( \hat{x} \). Furthermore, any polynomially bounded solution \( \Psi \) to the equation \( H(\omega; \vartheta)\Psi = E\Psi \) decays exponentially fast, thus \( \|\Psi\|_2 < \infty \); consequently, \( H(\omega; \vartheta) \) has pure point spectrum.
Proof. \textbf{Step 1.} Fix an eigenfunction $\psi$ with $\|\psi\|_2 = 1$, $\hat{x} \in \hat{X}(\psi)$, $H \psi = \lambda \psi$, and assume first that $R := |y - \hat{x}| \in [1, L_1]$, $L_1 = L_0^2 < L_0^4$. By Lemma 7.2, the cube $B_0(\hat{x}) = \{\hat{x}\}$ is $(\lambda, m)$-S. Therefore, by Sparse(0), for all $u$ with $|x - u| \in [1, L_1]$, the single-site cubes $B_0(u) = \{u\}$ are $(\lambda, m)$-NS. Fix any $y$ with $1 \leq R := |\hat{x} - y| \leq L_1$ and set $r := R - 1$. Each single-site cube $B_0(u) \subset B_r(y)$ is $(\lambda, m)$-NS, so by Lemma 6.3 (where one has to set $L = r$, $l = 0$), combined with Lemma 6.1, we have

$$|\psi(y)| \leq e^{-\gamma(m,0)\delta_{Lj}^{Lj,2}} \|\psi\|_\infty \leq e^{-\gamma(m,0)(r+1)} \leq e^{-2m|y - \hat{x}|}.$$  

Using (7.1) and the crude estimate card($u : |u| = r \leq (2r+1)^d \leq (3r)^d$, we obtain

$$\sum_{y \in B_{Lj}(\hat{x}) \setminus \{\hat{x}\}} |\psi(y)|^2 \leq \sum_{r=1}^{L_1} (3r)^d e^{-4mr}. \quad \text{(7.2)}$$

\textbf{Step 2.} Now let $R := |y - \hat{x}| > L_1$. The complement of $B_{Lj}(\hat{x})$ is covered by the disjoint annuli:

$$Z \setminus B_{Lj}(\hat{x}) = \bigcup_{j \geq 2} A_j, \quad A_j := B_{Lj}(\hat{x}) \setminus B_{Lj-1}(\hat{x}).$$

Fix $j \geq 2$ and $y \in A_j$, so $R > L_{j-1}$. Since $B_{Lj-1}(\hat{x})$ is $(\lambda, m)$-S, every cube $B_{Lj-2}(-u) \subset B_{R-Lj-2}(y) \subset B_{Lj-2}(\hat{x}) \setminus B_{Lj-1}(\hat{x})$, being disjoint from $B_{Lj-2}(\hat{x})$, must be $(\lambda, m)$-NS, owing to Sparse(Lj−2). By Lemma 6.1 and Lemma 6.3, with $||\psi||_\infty \leq 1$,

$$|\psi(y)| \leq \sum_{r=1}^{L_1} e^{-m(1+Lj^{-1/8})^{Lj-2} \frac{(R-Lj-2)-2Lj-2}{Lj^{1/2+1}}} \|\psi\|_\infty \leq e^{-mR \left(1+Lj^{-1/8}\right)^{\frac{1-3Lj^{-1/2}}{1+Lj^{-1/2}}} \leq e^{-mR \left(1+L^{-1/8}\right)^{\frac{1-3L^{-1/2}}{1+L^{-1/2}}} \leq e^{-mR \left(1+L^{-1/8}\right)^{\frac{1-3L^{-1/2}}{1+L^{-1/2}}}} \leq e^{-mR},$$

provided that $11 \leq L_0 \leq L_{j-1}$, as shows an elementary numerical calculation.\footnote{It suffices that $L_0^{7/8} \geq 8$, and actually $11^{7/8} > 8$.} Since $|A_j| \lesssim (2L_j + 1)^d \leq (3L_j)^d$, we obtain, with $R > L_j$,

$$\sum_{y \in A_j} |\psi(y)|^2 \leq (3L_j)^d \left(e^{-mL_j}\right)^2. \quad \text{(7.3)}$$

Collecting (7.2), (7.3) and (7.1), we conclude that

$$\sum_{y \neq \hat{x}} |\psi(y)|^2 \leq \sum_{r=1}^{L_1} (3r)^d e^{-4mr} + \sum_{j \geq 2} (3L_j)^d e^{-2mL_j} \leq e^{-m} \sum_{r=1}^{\infty} (3r)^d e^{-mr} \leq \frac{1}{2} e^{-m} < \frac{1}{2}$$

Therefore, $|\psi(\hat{x})|^2 > 1/2$, so $\psi$ is uniformly $m$-localized at $\hat{x}$.

For the proof of the second assertion, it suffices to repeat Step 2, but replace the uniform bound $||\psi||_\infty \leq 1$ by $|\psi(z)| \leq C(|z| + 1)^a$, $z \in \mathbb{Z}^d$, and also replace $\hat{x}$ by any point $\hat{y}$ where $\Psi(\hat{y}) \neq 0$. This still gives an exponential upper bound on
$\{\Psi(y)\} \in B_{L_2}(\tilde{y}) \setminus B_{L_2}(\tilde{y})$, for $j$ large enough.\footnote{In both cases (normalized eigenfunctions and polynomially bounded generalised eigenfunctions), the argument we use is well-known and goes back to [16, 22].} It is well-known (cf. e.g., [21]) that for spectrally-a.e. $E \in \Sigma(H)$, a DSO $H$ has a polynomially bounded generalized eigenfunction with eigenvalue $E$. Therefore, $H(\omega; \vartheta)$ has pure point spectrum for all $(\omega, \vartheta) \in \Omega \times \Theta^{(\infty)}(g)$. \hfill \qed

The following statement marks the end of the proof of Theorem 1.

**Theorem 6.** For all sufficiently large $m \geq 1$ and $g \geq g_*(m)$ large enough, so that in particular (7.1) holds true, for any $(\vartheta, \omega) \in \Theta^{(\infty)}(g) \times \Omega$, the operator $H(\omega; \vartheta)$ has an eigenbasis of uniformly $m$-localized eigenfunctions $\psi_x$, uniquely labeled by their respective localization centers:

$$\forall x \in \mathcal{Z}, \hat{X}(\psi_x) = \{x\}, |\psi_x(x)|^2 > 1/2.$$ 

For any $x \in \mathcal{Z}$ there is exactly one eigenfunction of $H(\omega; \vartheta)$ localized at $x$.

**Proof.** By Lemma 7.3, for all $g$ large enough, $H(\omega; \vartheta)$ has an eigenbasis of uniformly $m$-localized eigenfunctions $\psi_k$, $k = 1, 2, \ldots$. Therefore, by Lemma 7.1, each $\psi_k$ admits a unique localization center $\hat{x}(\psi_k)$. It remains to show that each point $x \in \mathcal{Z}^d$ is the localization center for exactly one eigenfunction.

Pick any $x \in \mathcal{Z}^d$; then we have by the Parseval identity:

$$1 = \sum_k |\psi_k(x)|^2 = \sum_{k : x \in \hat{X}(\psi_k)} |\psi_k(x)|^2 + \sum_{k : x \notin \hat{X}(\psi_k)} |\psi_k(x)|^2 =: S_1 + S_2.$$ 

By assertion (B) of Lemma 7.1, distinct uniformly $m$-localized eigenfunctions have distinct localization centers, thus

$$S_2 = \sum_{k : x \notin \hat{X}(\psi_k)} |\psi_k(x)|^2 \leq \sum_{r=1}^{\infty} \sum_{k : |x - \hat{x}(\psi_k)| = r} e^{-2mr} \leq \sum_{r=1}^{\infty} (3r)^d e^{-2mr} \leq e^{-m} 1/2 < 1 \implies S_1 > 0.$$ 

Hence $1 \geq |\{k : x \in \hat{X}(\psi_k)\}| > 0$ for any $x \in \mathcal{Z}^d$, so $|\{k : x \in \hat{X}(\psi_k)\}| = 1$, and there exists a bijection between the elements $\psi_k$ of the eigenbasis of uniformly $m$-localized, unimodal eigenfunctions and the lattice $\mathcal{Z}^d$. \hfill \qed

### 8. Uniform Dynamical Localization

Below we use the standard Dirac’s "bra-ket" notation $\langle \phi | H | \psi \rangle$ for the scalar product of $\phi$ and $H\psi$ in the Hilbert space $l^2(\mathcal{Z}^d)$.

**Theorem 7.** For all $g > 0$ large enough, all $\vartheta \in \Theta^{(\infty)}(g)$, for any $\omega \in \Omega$ and all $x, y \in \mathcal{Z}^d$, for any continuous function $\phi : \mathbb{R} \to \mathbb{C}$ with $\|\phi\|_{\infty} \leq 1$,

$$|\langle 1_x | \phi(H(\omega; \vartheta)) | 1_y \rangle| \leq \text{Const}(d) |x - y|^d e^{-m|x - y|}.$$ 

**Proof.** By functional calculus, we have the following identity, assuming that the series in the RHS of (8.1) converges absolutely:

$$\langle 1_x | \phi(H) | 1_y \rangle = \sum_{z \in \mathcal{Z}^d} \langle 1_x | \psi_z \rangle \phi(\lambda_z) \langle \psi_z | 1_y \rangle, \quad (8.1)$$
so it suffices to prove convergence of the series

\[ \| \phi \| \infty \sum_{z \in \mathbb{Z}^d} |\langle 1_x | \psi_z \rangle \langle \psi_z | 1_y \rangle| \leq \sum_{z \in \mathbb{Z}^d} |\langle 1_x | \psi_z \rangle \langle \psi_z | 1_y \rangle|. \]

By Theorem 6, we have \(|\psi_z(x)| \leq e^{-m|x-z|}\) and \(|\psi_z(y)| \leq e^{-m|z-y|}\), with the decay exponent \(m \geq m_\ast (g) \to +\infty\) as \(g \to +\infty\), so that

\[ \sum_{z \in \mathbb{Z}^d} |\langle 1_x | \psi_z \rangle \langle \psi_z | 1_y \rangle| \leq \sum_{z \in \mathbb{Z}^d} e^{-m|x-z| - m|z-y|}. \]

Let \(R = |x - y|\). For any \(z \notin B_{2R}(x)\), setting \(n = |z - x| \geq 2R + 1\), we have

\[ |z - x| + |z - y| \geq n + \text{dist}(z, B_{R}(x)) \geq 2n - R, \]

since \(y \in B_{R}(x) \subset B_{2R}(x) \neq z\). Furthermore,

\[ \forall n > R \quad \text{card}\{z \in \mathbb{Z}^d : |z - x| = n\} \leq C(d)n^{d-1}, \]

thus

\[ \sum_{z \in B_{2R}(x)} e^{-m|x-z| - m|z-y|} \leq \sum_{n > 2R} C(d)n^{d-1}e^{-m(2n-R)} \leq C(d)R^d e^{-2mR}. \]

For \(z \in B_{2R}(x)\) (indeed, for any \(z \in \mathbb{Z}^d\) one can use a simpler bound: by the triangle inequality, \(|z - x| + |z - y| \geq |x - y| = R\). Therefore,

\[ \sum_{z \in B_{2R}(x)} e^{-m|x-z| - m|z-y|} \leq e^{-mR}|B_{2R}(x)| \leq C'' R^d e^{-mR}. \]

Finally,

\[ |\langle 1_x | \phi(H) | 1_y \rangle| \leq \text{Const}(d) |x - y|^{d} e^{-m|x-z|}. \]

\[ \square \]

The standard form of dynamical localization is obtained with the functions \(\phi = \phi_t : \lambda \mapsto e^{-i\lambda t}, t \in \mathbb{R}\).

9. Minami-type estimates. Simplicity of spectra

9.1. Spectral spacings in large cubes. Recall the generalized Minami estimate [30] proven in Refs [5, 20].

**Theorem 8** (Cf. [5, 20]). Let \(H_{B_L,(\omega)}(\vartheta)\) be a random DSO relative to some probability space \((\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\mathbb{P}})\), with IID random potential \(V(x; \vartheta)\). Assume that the common probability distribution of the random variables \(V(x; \vartheta)\) has a bounded density \(\rho\). Then for any finite interval \(I \subset \mathbb{R}\) one has

\[ \tilde{\mathbb{P}} \{ \text{Tr} \Pi_I (H_{B_L,(\omega)}(\vartheta)) \geq J \} \leq \left( \frac{\| \rho \|_\infty}{J!} \right)^J |I|^J. \]

**Theorem 2** is actually an adaptation of Theorem 5.2 from [9] to the ”haarsh” deterministic potentials.

**Proof of Theorem 2.** We prove first (1.15): for any fixed \(\omega \in \Omega\),

\[ \mathbb{P}^{\vartheta} \left\{ \text{Tr} \Pi_I (H_{B_L,(\omega)}(\vartheta)) \geq J \right\} \leq C_J L^{IB} \mathbb{L}^{IB} |I|^J. \]

To this end, consider the sigma-algebra \(\mathcal{H}_L\), figuring in (LVB). Conditional on \(\mathcal{F} \times \mathcal{B}_L\) (hence, with fixed \(\omega\)), the values of the potential \(v(T^\omega \cdot \vartheta)\) with \(x \in B_L(0)\) become independent and admit probability densities \(\rho_{x,L}\) with \(\| \rho_{x,L} \|_\infty \leq \)
Now the assertion follows from Theorem 8 applied to the operators $H_{B_{L_j}(0)}(\omega; \vartheta)$, with fixed $\omega$ and subject to the conditional measure $P^\Theta \{ \cdot | B_{L_j} \}$ with respect to $\vartheta \in \Theta$:

$$P^\Theta \left\{ \text{Tr } \Pi_f(H_{B_{L_j}(0)}(\omega; \vartheta)) \geq J \right\}$$

$$= E^\Theta \left[ P^\Theta \left\{ \text{Tr } \Pi_f(H_{B_{L_j}(0)}(\omega; \vartheta)) \geq J | \mathcal{F} \times B_N \right\} \right]$$

$$\leq \frac{1}{J} \left( \pi C'' L_j^{B \ln L_j} \right)^J |J|^J,$$

which proves the first assertion (1.15).

To prove (1.16), one can repeat the above argument, but replace $P^\Theta \{ \cdot \}$ by the product measure $P^\Omega \times \Theta \{ \cdot \}$ and apply the standard identity $E^\Omega \times \Theta \left[ P^\Omega \times \Theta \{ \cdot | \mathcal{F} \times B_L \right\} \right]$. Conditioning on $\mathcal{F}$ is equivalent to fixing $\omega \in \Omega$, so we can make use of the first assertion, valid for each $\omega \in \Omega$.

For the proof of Theorem 4, we also need a bound deterministic in $\omega \in \Omega$; it will be proved only for sufficiently small intervals $I_j$. The next statement establishes a lower bound on the spacings $\text{Sep}_i^\omega [H_{B_{L_j}(0)}(\omega; \vartheta)]$ (cf. Sect. 4) uniform in $\omega \in \Omega$.

**Theorem 9.** Let the parameter $b > 0$ in the definition of the sequence $\{a_n\}_{n \geq 0}$ (cf. (1.12)) be large enough, so that $4A^2b - 2(B + 4A + 4A') > 1$. Then there exists $\tilde{\omega} = \tilde{\omega}(g)$ such that for all $j \geq \tilde{\omega}$, one has

$$P^\Theta \left\{ \inf_{\omega \in \Omega} \text{Sep}_i^\omega [H_{B_{L_j}(0)}(\omega; \vartheta)] \leq g\delta_j \right\} \leq L_j^{-\ln L_j}. \quad (9.3)$$

**Proof.** Fix $j \geq 0$, let $\tilde{N}_j = \tilde{N}(L_j, A, C)$ (cf. (2.8)), and $B_j = B_{L_j}(0)$.

Consider first $H^{(\tilde{N}_j)}(\omega; \vartheta)$ with the truncated potential $V_{\tilde{N}_j}(x; \omega; \vartheta) = v_{\tilde{N}_j}(T^x; \omega; \vartheta)$. Next, cover $\Omega$ by the sets $P_{j,l} \ni \tau_{j,l}$, $1 \leq l \leq L_j = \text{Const} L_j^{4A+4A'}$, introduced in Sect. 5.1 (cf. Lemma 4.3). Let $I$ be an interval of length $4\delta_j$. By (9.2) with $\omega = \tau_{j,l}$, we have:

$$P^\Theta \left\{ \min_{\omega \in \Omega} \text{Tr } \Pi_f(H^{(\tilde{N}_j)}(\tau_{j,l}; \vartheta)) \geq 2 \right\} \leq L_j C_j L_j^{2B \ln L_j} \delta_j^2 \leq C_j L_j^{2B' \ln L_j} \delta_j^2.$$

$H^{(\tilde{N}_j)}(\omega; \vartheta)$ is constant in $\omega$ on each set $P_l$, and $\cup_l P_{j,l} = \Omega$, so the above bound implies that

$$P^\Theta \left\{ \sup_{\omega \in \Omega} \text{Tr } \Pi_f(H^{(\tilde{N}_j)}(\omega; \vartheta)) \geq 2 \right\} \leq C_j L_j^{2B' \ln L_j} \delta_j^2. \quad (9.4)$$

Further, $\| H(\omega; \vartheta) \| \leq Cg$, thus $\Sigma(\omega; \vartheta) \subset I(\vartheta) = [-gE^*, gE^*]$, for some $E^* \in (0, +\infty)$, so for our purposes, it suffices to analyse only the sub-intervals of $I(\vartheta)$.

Next, cover the interval $I(\vartheta)$ redundantly by $K_j := \left\lfloor \frac{2gE^*}{2\delta_j} \right\rfloor + 1 \leq Cg\delta_j^{-1}$ sub-intervals of length $4\delta_j$,

$$I_{j,k} := [-gE^* + 2k\delta_j, -gE^* + (2k + 4)\delta_j], \quad k = 0, 1, ..., K_j - 1.$$

Then every subinterval of length $2\delta_j$ of $I(\vartheta)$ is covered by at least one of these intervals $I_{j,k}$. Thus the $P^\Theta$-probability that at least one interval of length $2\delta_j$ in $\mathbb{R}$

$C'' L_j^{B \ln L_j}$. Now the assertion follows from Theorem 8 applied to the operators $H_{B_{L_j}(0)}(\omega; \vartheta)$, with fixed $\omega$ and subject to the conditional measure $P^\Theta \{ \cdot | B_{L_j} \}$ with respect to $\vartheta \in \Theta$:

$$P^\Theta \left\{ \text{Tr } \Pi_f(H_{B_{L_j}(0)}(\omega; \vartheta)) \geq J \right\}$$

$$= E^\Theta \left[ P^\Theta \left\{ \text{Tr } \Pi_f(H_{B_{L_j}(0)}(\omega; \vartheta)) \geq J | \mathcal{F} \times B_N \right\} \right]$$

$$\leq \frac{1}{J} \left( \pi C'' L_j^{B \ln L_j} \right)^J |J|^J,$$
contains for some $\omega \in \Omega$ at least two eigenvalues of $H_{B_j}^{(S_j)}(\omega; \vartheta)$, is bounded by (cf. the definition of $\delta_j$ in (3.4))

$$K J C_j L_j^{2B'} \ln L_j \delta_j^2 \leq \frac{C g \delta_j^2}{\delta_j} L_j^{2B'} \ln L_j \leq C g L_j^{-(4A^2b-2B')} \ln L_j \leq g L_j^{-\ln L_j}$$

provided that $4A^2b - 2(B + 4A + 4A') > 1$, and $j \geq j_0(g)$, for $j_0(g)$ large enough.

Finally, by the min-max principle, the eigenvalue perturbations, $|E_i - E_j^{(S_j)}|$, are bounded by $\|H_{B_j} - H_j^{(S_j)}\| \leq g \|v - v_{S_j}\|_{\infty} \leq \frac{1}{2} g \delta_j$, thus, by the triangle inequality,

$$\text{Sep}[H_{B_j}] \geq \text{Sep}[H_j^{(S_j)}] - 2g \|v - v_{S_j}\|_{\infty} \geq 2g \delta_j - g \delta_j = g \delta_j.$$ 

So, we have proved the following implication: denoting $I_s = [s, s + 4k \delta_j]$,

$$\sup_{\omega \in \Omega} \sup_{I_s \subset I(\vartheta)} \text{Tr} \Pi_{I_s}(H_j^{(S_j)}(\omega; \vartheta)) < 2 \Rightarrow \inf_{\omega \in \Omega} \text{Sep}[H_{B_{L_j}(0)}(\omega; \vartheta)] \geq g \delta_j.$$ 

Now the assertion follows from the estimate (9.4).

\[ \square \]

**Remark 2.** The requirement $j \geq j_0(g)$ in Theorem 9 and in Corollary 9.1 becomes unnecessary for even larger $b > 0$, i.e., for $b$ large enough, one can set $j_0 = 0$.

Introduce the sets (here the subscript ”M” stands for ”Minami”)

$$\Theta_M^{(j)}(g) := \left\{ \inf_{\omega \in \Omega} \text{Sep}[H_{B_{L_j}(0)}(\omega; \vartheta)] \geq \delta_j \right\} \cap \Theta^{(\infty)}(g),$$

$$\Theta_M^{(\infty)}(g) := \bigcap_{j \geq j_0(g)} \Theta_M^{(j)}(g).$$

**Corollary 9.1.** Under the assumptions of Theorem 9, for any $j \geq j_0(g)$,

$$\mathbb{P}\left\{ \Theta_M^{(j)}(g) \right\} \geq 1 - L_j^{-\ln L_j}$$

and therefore, if $g$ is large enough, owing to (5.13), one has

$$\mathbb{P}\left\{ \Theta_M^{(\infty)}(g) \right\} \underset{g \rightarrow +\infty}{\longrightarrow} 1.$$ 

**Proof.** The first assertion follows directly from Theorem 9 and the definition (9.5) of $\Theta_M^{(j)}(g)$. With $g$ large enough (hence, $L_j = L_j(g)$ large enough), the bound (9.7) follows from (9.6) by an elementary calculation, since $\sum_{j} L_j^{-\ln L_j} < \infty$. \[ \square \]

9.2. The Klein–Molchanov argument. Proof of Theorem 4.

**Lemma 9.1** (Cf. Lemma 1 in [29]). Let $E$ be an eigenvalue of the discrete Schrödinger operator $H = -\Delta + V$ in $l^2(\mathbb{Z}^d)$ with two linearly independent eigenfunctions $\varphi_1, \varphi_2 \in l^2(\mathbb{Z}^d)$ such that for some $\beta > d/2$ and some function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $f(r) \leq C\Gamma_r^{-\beta}$, $\beta > d/2$, one has

$$\forall x \in \mathbb{Z}^d \quad |\varphi_j(x)| \leq f(|x|) \quad j = 1, 2.$$ 

Then there exists $C \in (0, +\infty)$ such that, setting $\epsilon_L := Cf(L)L^{d/2} > 0$ and $I_L = [E - \epsilon_L, E + \epsilon_L]$ one has $\text{Tr} P_H(H_L) \geq 2$ for all sufficiently large $L$. 

The main application is to the case where \( \phi_i \) decay exponentially fast, so Lemma 9.1 implies that for \( L \) large enough, there are at least two eigenvalues \( \lambda_1, \lambda_2 \) of the operator \( H_L \) with \( \max\{|E - \lambda_1|, |E - \lambda_2|\} \leq \frac{1}{2} e^{-cL} \), for some \( c > 0 \), hence with \( |\lambda_1 - \lambda_2| \leq e^{-cL} \).

**Proof of Theorem 4.** Assume otherwise and fix any any \( \vartheta \in \Theta^{(\infty)}(g) \). Since by construction \( \Theta^{(\infty)}(g) \subset \Theta^{(\infty)}(g) \), \( H(\omega; \vartheta) \) has pure point spectrum for every \( \omega \in \Omega \), and by Lemma 7.3, all its eigenfunctions decay exponentially fast.

Further, by construction of \( \Theta^{(\infty)}(g) \), for any \( \omega \in \Omega \) and all \( j \) large enough, all spectral spacings of \( H_{B_{L,j}(0)}(\omega; \vartheta) \) are bounded from below by \( \delta_j \geq L_j^{-\text{Const} \ln L_j} = e^{-\text{Const} \ln^2 L_j} > e^{-cL_j} \) for any \( c > 0 \) and all \( L_j \) large enough. This contradicts Lemma 9.1 and proves the claim. \( \square \)

**Acknowledgements.** It is a pleasure to thank Yakov Grigor’evich Sinai, Misha Goldstein and Abel Klein for numerous fruitful discussions of localization mechanisms in deterministic disordered media; Tom Spencer for numerous discussions and warm hospitality during my stay at the IAS in 2012; Günter Stolz, Yulia Karpeshina and Roman Shterenberg for stimulating discussions and warm hospitality during my stay at the University of Alabama at Birmingham in 2012.

**References**

[1] J. Bellissard, R. Lima, and E. Scopola, *Localization in \( \nu \)-dimensional incommensurate structures*, Commun. Math. Phys. **88** (1983), 465–477.

[2] J. Bourgain and M. Goldstein, *On nonperturbative localization with quasiperiodic potentials*, Annals of Math. **152** (2000), no. 1, 835–879.

[3] J. Bourgain and W. Schlag, *Anderson localization for Schrödinger operators on \( \mathbb{Z} \) with strongly mixing potential*, Commun. Math. Phys. **215** (2001), 143–175.

[4] J. Bourgain, M. Goldstein, and W. Schlag, *Anderson localization for Schrödinger operators on \( \mathbb{Z} \) with potential generated by skew-shift*, Commun. Math. Phys. **220** (2001), 583–621.

[5] J. Bellissard, P. Hislop, and G. Stolz, *Correlation estimates in the Anderson model*, J. Stat. Phys. **129** (2007), 649–662.

[6] J. Chan, *Method of variations of potential of quasi-periodic Schrödinger equations*, Geom. Funct. Anal. **17** (2007), 1416–1478.

[7] V. Chulaevsky, *Grand ensembles of deterministic operators. I. Randelette expansions and Wegner-type estimates*, 2001, Preprint, Université de Reims.

[8] V. Chulaevsky, *Wegner-Stollmann type estimates for some lattice quantum systems*, Adv. Math. Phys. **447** (2007), 17–28.

[9] V. Chulaevsky, *Anderson localization for generic deterministic potentials*, J. Funct. Anal. **262** (2011), 1230–1250.

[10] V. Chulaevsky, *Direct scaling analysis of localization in single-particle quantum systems on graphs with diagonal disorder*, Math. Phys. Anal. Geom. **15** (2012), 361–399.

[11] V. Chulaevsky, *From fixed-energy MSA to dynamical localization: An elementary path*, To appear in: J. Stat. Phys. **15** (2012), 361–399.

[12] V. Chulaevsky and Y. Suhov, *Multiscale Analysis for Random Quantum Systems with Interaction*, Progress in Mathematical Physics, vol. 65, Birkhäuser Inc., Boston, 2013.

[13] V. Chulaevsky and Ya. G. Sinai, *Anderson localization for the 1-D discrete Schrödinger operator with two-frequency potential*, Commun. Math. Phys. **125** (1989), 91–112.

[14] J.-M. Combes and L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators*, Commun. Math. Phys. **34** (1973), 251–270.

[15] H. von Dreifus, *On effect of randomness in ferromagnetic models and Schrödinger operators*, PhD thesis, New York University, New York (1987).
[16] H. von Dreifus and A. Klein, A new proof of localization in the Anderson tight binding model, Commun. Math. Phys. 124 (1989), 285–299.

[17] D. Damanik and P. Stollmann, Multi-scale analysis implies strong dynamical localization, Geom. Funct. Anal. 11 (2001), no. 1, 11–29.

[18] D. Damanik and Z. Gan, Limit-periodic Schrödinger operators with uniformly localized eigenfunctions, J. d’analyse Math. 115 (2011), 33–49.

[19] D. Damanik and Z. Gan, Limit-periodic Schrödinger operators on $\mathbb{Z}^d$: Uniform localization, 2012, arXiv:math-ph/1207.5881.

[20] G. M. Graf and A. Vaghi, A remark on the estimate of a determinant by Minami, Lett. Math. Phys. 79 (2007), 17–22.

[21] W. Kirsch, An Invitation to Random Schrödinger Operators, Panoramas et Synthèses 25 (2008).

[22] J. Fröhlich, F. Martinelli, T. Spencer, and E. Scoppola, Constructive proof of localization in the Anderson tight-binding model, Commun. Math. Phys. 101 (1985), 21–46.

[23] S. Fishman, D. Grempel, and R. Prange, Localization in a $d$-dimensional incommensurate structure, Phys. Rev. B 194 (1984), 4272–4276.

[24] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, Constructive proof of localization in the Anderson tight binding model, Commun. Math. Phys. 101 (1985), 21–46.

[25] A. Figotin and L. Pastur, An exactly solvable model of a multidimensional incommensurate structure, Commun. Math. Phys. 95 (1984), 401–425.

[26] J. Fröhlich, T. Spencer, and P. Wittwer, Localization for a class of one dimensional quasi-periodic Schrödinger operators, Commun. Math. Phys. 132 (1990), 5–25.

[27] F. Germinet and S. De Bièvre, Dynamical Localization for Discrete and Continuous Random Schrödinger Operators, Commun. Math. Phys. 194 (1998), 323–341.

[28] F. Germinet and A. Klein, Bootstrap Multi-Scale Analysis and localization in random media, Commun. Math. Phys. 222 (2001), 415–448.

[29] A. Klein and S. Molchanov, Simplicity of eigenvalues in the Anderson model, J. Stat. Phys. 122 (2006), no. 1, 95–99.

[30] N. Minami, Local fluctuation of the spectrum of a multidimensional Anderson tight-binding model, Commun. Math. Phys. 177 (1996), 709–725.

[31] Ya. G. Sinai, Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential, J. Statist. Phys. 46 (1987), 861–909.

[32] B. Simon, Almost periodic Schrödinger operators. IV: The Maryland model, An. Phys. 159 (1985), 157–183.

[33] F. Wegner, Bounds on the density of states in disordered systems, Z. Phys. B. Condensed Matter 44 (1981), 9–15.